COLLOQUIUM MATHEMATICUM

VOL. 118 2010 NO. 2

REGULAR BEHAVIOR AT INFINITY OF STATIONARY MEASURES OF STOCHASTIC RECURSION ON NA GROUPS

BY

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Dedicated to the memory of Andrzej Hulanicki

Abstract. Let N be a simply connected nilpotent Lie group and let $S = N \rtimes (\mathbb{R}^+)^d$ be a semidirect product, $(\mathbb{R}^+)^d$ acting on N by diagonal automorphisms. Let (Q_n, M_n) be a sequence of i.i.d. random variables with values in S. Under natural conditions, including contractivity in the mean, there is a unique stationary measure ν on N for the Markov process $X_n = M_n X_{n-1} + Q_n$. We prove that for an appropriate homogeneous norm on N there is χ_0 such that

$$\lim_{t\to\infty} t^{\chi_0} \nu\{x : |x| > t\} = C > 0.$$

In particular, this applies to classical Poisson kernels on symmetric spaces, bounded homogeneous domains in $\mathbb{C}^n$ or homogeneous manifolds of negative curvature.

1. Introduction. Let $S = N \rtimes A$ be a semidirect product of a simply connected nilpotent Lie group N and an Abelian group $A = (\mathbb{R}^+)^d$ acting on N by diagonal isomorphisms δ_a, i.e.

$$\delta_a(x) = (e^{\lambda_1(\log a)} x_1, \ldots, e^{\lambda_{n_0}(\log a)} x_{n_0}),$$

$x = (x_1, \ldots, x_{n_0}) \in N$, $a \in A$, and $\lambda_1, \ldots, \lambda_{n_0}$, not necessarily distinct, belong to the dual of the Lie algebra $\mathcal{A}$ of A. Various classical objects like symmetric spaces, bounded homogeneous domains in $\mathbb{C}^n$ and homogeneous manifolds of negative curvature admit simply transitive actions of such groups [1, 12, 22, 21, 26]. Given a probability measure μ on S, we study properties of the finite measure ν on N such that $\mu * \nu = \nu$ provided ν exists and it is unique up to a constant (see Section 2.2). Being the stationary measure for the Markov chain on N with the transition kernel $Pf(x) = \int_S f(gx)\, d\mu(g)$, the measure ν appears in various situations interesting both from probabilistic and analytical points of view. In particular, classical Poisson kernels on the spaces mentioned above are of this form.

Existence of ν was proved by A. Raugi [27] under the assumption of finite logarithmic moments of μ and contraction in mean (see Section 2.2). The

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2010 *Mathematics Subject Classification*: Primary 60B15; Secondary 60J50, 22E25.
Key words and phrases: solvable Lie groups, stationary measure, Poisson kernel.

DOI: 10.4064/cm118-2-8

latter means that for every root λ_j,

$$\int_S \lambda_j(\log a)\, d\mu(x,a) < 0. \tag{1.1}$$

Clearly, (1.1) implies existence of the positive Weyl chamber, i.e the cone $\mathcal{A}^{++}$ of $H \in \mathcal{A}$ such that $\lambda_j(H) > 0$ for every j. If $a = \exp(-H)$ for some $H \in \mathcal{A}^{++}$, then

$$\delta_a^n(x) \to e \quad \text{for every } x \in N,$$

and so the action of A on N is contractive. Our aim is to study the behavior of ν at infinity, i.e. the size of

$$\nu\{|x| > t\}$$

when $t \to \infty$ and $|\ |$ is an appropriate norm. We introduce a family of homogeneous norms $|\ |$ with the property that given a norm, there is $\chi_0 > 0$ such that

$$\lim_{t\to\infty} t^{\chi_0}\nu\{x : |x| > t\} = C \tag{1.2}$$

and $C > 0$ under natural assumptions, which means that χ_0 is optimal (see Theorem 2.5).

In the most general situation there is no canonical norm and χ_0. The exponent χ_0 depends on $|\ |$, but all the results are equivalent. However, for $N \rtimes A$ groups with particular root systems (like those acting simply transitively on symmetric spaces) there is a norm that is more intuitive than the others (see Section 2.5).

Let us now discuss some particular cases and existing results. When $A = \mathbb{R}^+$ all the homogeneous norms are equivalent and the behavior of the tail is well understood. If additionally $N = \mathbb{R}$, i.e. S is the "$ax + b$" group, it was observed by Kesten [23] that under natural assumptions there is $\chi > 0$ such that

$$\lim_{t\to\infty} t^{\chi}\nu\{x : |x| > t\} = C > 0, \tag{1.3}$$

$|\ |$ being the absolute value of x. His proof was later on simplified by Grincevičius [15, 16] and Goldie [14]. If N is a homogeneous group with $A = \mathbb{R}^+$ acting on it by dilations, (1.2) was obtained in [5] (see Theorem 2.1 below). Then all the norms are equivalent and χ_0 is unique.

More can be said if a left-invariant second order subelliptic differential operator $\mathcal{L}$ and the related heat semigroup μ_t are considered. Then

$$\mu_t * \nu = \nu$$

for every t, and not only (1.2) holds but also pointwise estimates of ν at infinity have been obtained [6, 9, 10].

Finally, the case when A is multidimensional was treated in [3] but the tail of ν was only estimated from above and below,

$$C_1 t^{-\chi_0} \le \nu\{x : |x| > t\} \le C_2 t^{-\chi_0}, \tag{1.4}$$

except of a very special case when (1.2) was obtained. Estimates (1.4) essentially improve an earlier result of the second author and A. Hulanicki obtained for kernels coming from differential operators [7, 8].

Suppose now that $N = \mathbb{R}^{n_0}$ and let us describe briefly the idea of the estimates we obtain in the paper. We assume that for every root λ_j there is a unique $s_j > 0$ such that

$$\int_S e^{s_j \lambda_j (\log a)}\, d\mu(x,a) = 1. \tag{1.5}$$

Existence of the positive Weyl chamber allows us to write every root as a positive combination of so called simple ones $\eta_1, \ldots, \eta_k$, i.e.

$$\lambda = \sum_{j=1}^{k} \alpha_i \eta_i, \qquad \alpha_i \ge 0,\ \sum_{i=1}^{k} \alpha_i > 0, \tag{1.6}$$

where by a *simple root* we mean a root η that cannot be written as in (1.6) for at least two roots and if $\lambda_j = \alpha\eta$ then $\alpha \ge 1$ (1). Let Δ be the set of roots and Δ_1 be the set of simple roots. Suppose the homogeneous norm is of the form

$$|x| = \max_{\lambda_j \in \Delta} \{|x_j|^{1/d_j}\}. \tag{1.7}$$

We choose d_j's as follows:

- $d_j = 1$ if $\lambda_j \in \Delta_1$,
- $d_j = \alpha$ if $\lambda_j = \alpha\eta$ and $\eta \in \Delta_1$,
- $d_j = \sum_{i=1}^{k} \alpha_i$ if λ_j is of the form (1.6).

Then $|\cdot|$ is subadditive provided $\sum_{i=1}^{k} \alpha_i \ge 1$ for all λ_j (written as in (1.6)) (2). If χ_0 is as in (1.2) then

$$\chi_0 = \min_{\eta_j \in \Delta_1} \{s_j\},$$

and for every nonsimple root λ_j or a simple root with $s_j > \chi_0$,

$$\lim_{t\to\infty} t^{\chi_0} \nu\{x : |x_j|^{1/d_j} > t\} = 0. \tag{1.8}$$

(1) Simple here does not mean simple in the classical sense of [2], but when NA is a symmetric space then the two notions coincide: $k = \dim A$ and all the roots are linear combinations of $\eta_1, \ldots, \eta_k$ with positive integer coefficients.

(2) Otherwise we take dd_j for sufficiently large d.

Moreover, for every simple root λ_j with $s_j = \chi_0$,

$$\lim_{t\to\infty} t^{\chi_0}\nu\{x : |x_j|^{1/d_j} > t\} = C_j. \tag{1.9}$$

Therefore only simple roots with minimal s_j count in (1.2). This phenomenon has a simple explanation. Hypothesis (1.5) implies that $\operatorname{supp}\mu\cap\{(x,a) : a<1\}$ and $\operatorname{supp}\mu\cap\{(x,a) : a>1\}$ are nonempty. Therefore, both contracting and expanding elements are in the support of μ. The stronger the expansion, the smaller s_j is necessary to have (1.5). Of course, (1.8) and (1.9) are not enough for (1.2) and one has to deal with intersections of sets $\{x : |x_j|^{1/d_j} > t\}$, which is explained in Section 3.3.

It is natural to consider more general actions on N than the diagonal one. The asymptotic (1.2) remains valid when $S = N \rtimes AK$, where K is a compact group commuting with A. Then the chosen norm is additionally preserved by K (see the Appendix).

The case when $N = \mathbb{R}^{n_0}$ and there is a group $G \subset GL(n_0)$ acting on it was studied by many authors [11, 18, 19, 24, 25]. Then $S = \mathbb{R}^{n_0} \rtimes G$ and the action of G is assumed to be proximal and irreducible. Let $\bar{\mu}$ be the canonical projection of μ onto G. Then irreducibility means that there is not a finite union of proper subspaces of $\mathbb{R}^{n_0}$ invariant under the action of the support of $\bar{\mu}$. The action is proximal if in the support of $\bar{\mu}$ there is an element with a dominant real eigenvalue (i.e. the corresponding eigenspace is one-dimensional). Here, of course, the action is generally nonproximal and highly reducible.

The paper is organized as follows. In Section 2 we introduce a class of NA groups, a class of norms, we describe previous results and at the end we formulate the Main Theorem 2.5. Section 3.3 contains the scheme of the proof, and Sections 3.4 and 3.5 the details of it.

2. Preliminaries and the Main Theorem

2.1. A class of solvable Lie groups. The semidirect product $S = N \rtimes A$ acts on N in the following way:

$$(x,a)\circ y = x\cdot\delta_a(y) \quad \text{for } (x,a)\in S \text{ and } y\in N.$$

Therefore, the group multiplication in S is given by

$$(x,a)\cdot(y,b) = ((x,a)\circ y, ab). \tag{2.1}$$

Let e (0, I respectively) be the neutral element of S (N, A respectively).

The Lie algebras of A, N, S are denoted by $\mathcal{A}$, $\mathcal{N}$ and $\mathcal{S}$. Then $\mathcal{S} = \mathcal{N}\oplus\mathcal{A}$ and for every $H\in\mathcal{A}$, $\operatorname{ad} H$ preserves $\mathcal{N}$. The exponential maps are global diffeomorphisms both between $\mathcal{N}$ and N, and between $\mathcal{A}$ and A. Their inverses will be denoted by log. Then for any $X\in\mathcal{N}$,

$$\delta_a(\exp(X)) = \exp(e^{\operatorname{ad}(\log a)}X). \tag{2.2}$$

We shall denote the foregoing action of the group A on the Lie algebra $\mathcal{N}$ by the same symbol $\delta_a(X)$.

We shall assume that the action of A on N is diagonalizable. For any λ in the dual $\mathcal{A}^*$ of $\mathcal{A}$ let

$$\mathcal{N}_\lambda = \{X \in \mathcal{N} : [H, X] = \lambda(H)X \text{ for any } H \in \mathcal{A}\}. \tag{2.3}$$

Then, for $\lambda_1, \lambda_2 \in \mathcal{A}^*$,

$$[\mathcal{N}_{\lambda_1}, \mathcal{N}_{\lambda_2}] \subset \mathcal{N}_{\lambda_1+\lambda_2}. \tag{2.4}$$

Moreover, any space $\mathcal{N}_\lambda$ is preserved by the action of the group A, i.e.

$$\delta_a(X) \in \mathcal{N}_\lambda \quad \text{for } X \in \mathcal{N}_\lambda. \tag{2.5}$$

We shall say that λ is a *root* if $\mathcal{N}_\lambda$ is nonempty. The set of all roots will be denoted by Δ. Then

$$\mathcal{N} = \bigoplus_{\lambda \in \Delta} \mathcal{N}_\lambda.$$

All the roots are real and there exists a basis of $\mathcal{N}$, $\{X_1, \ldots, X_{n_0}\}$ ($n_0 = \dim N$), such that for any $H \in \mathcal{A}$,

$$\operatorname{ad}(H)X_j = \lambda_j(H)X_j, \quad j = 1, \ldots, n_0,$$

for some root λ_j. In this notation it may happen that $\lambda_i = \lambda_j$ for $i \neq j$. An element $x \in N$ will be written as

$$x = \exp\Big(\sum_{j=1}^{n_0} x_j X_j\Big) =: (x_1, \ldots, x_{n_0}). \tag{2.6}$$

2.2. Random walks and positive Weyl chamber. Given a probability measure μ on S we define a random walk

$$S_n = (Q_n, M_n) \cdot \ldots \cdot (Q_1, M_1),$$

where (Q_n, M_n) is a sequence of i.i.d. S-valued random variables with distribution μ. The law of S_n is the nth convolution μ^{*n} of μ.

Our aim is to study the N-*component* of S_n, i.e. the Markov chain on N generated by S_n:

$$R_n = \pi_N(S_n) = (Q_n, M_n) \circ R_{n-1}, \quad R_0 = \delta_0, \tag{2.7}$$

where π_N denotes the projection $\pi_N : S \to N$. By π_A we shall denote the analogous projection of S onto $A = S/N$. Let $\mu_A = \pi_A(\mu)$.

We assume that

$$\mathbb{E}[\log^+ \|Q\|] < \infty \tag{2.8}$$

(where $\|\cdot\|$ is the Euclidean norm on N identified with $\mathcal{N}$ via (2.6)) and for every root λ,

$$\mathbb{E}[|\lambda(\log M)|] < \infty \tag{2.9}$$

and there is a unique $s_\lambda > 0$ such that

$$\mathbb{E}[e^{s_\lambda \lambda(\log M)}] = 1. \tag{2.10}$$

As is shown below, (2.10) implies that μ is *mean-contracting*, i.e. for every root λ,

$$\mathbb{E}[\lambda(\log M)] = \int_A \lambda(\log M)\, \mu_A(dM) < 0. \tag{2.11}$$

It was proved by A. Raugi [27] that if (2.8), (2.9) and (2.11) are satisfied, then R_n converges in law to a random variable R independently of the choice of R_0. Moreover, the law ν of R is a unique stationary solution of the stochastic equation

$$\nu = \mu * \nu,$$

where

$$\mu * \nu(f) = \int_S \int_N f(g \circ x)\, \mu(dg)\, \nu(dx),$$

or equivalently

$$R =_d (Q, M) \circ R,$$

where R and (Q, M) are independent with laws ν and μ, respectively.

Notice that the functional

$$\lambda \mapsto -\mathbb{E}[\lambda(\log M)]$$

on $\mathcal{A}^*$ is given by a vector H_1, i.e.

$$\lambda(H_1) = -\mathbb{E}[\lambda(\log M)] > 0.$$

Thus (2.11) implies the existence of a nontrivial positive Weyl chamber

$$\mathcal{A}^{++} = \{H \in \mathcal{A} : \lambda(H) > 0 \text{ for every } \lambda \in \varDelta\}.$$

Define $\mathcal{A}^{--} = -\mathcal{A}^{++}$. Then for every $x \in N$ and $H \in \mathcal{A}^{--}$,

$$\lim_{k\to\infty} \delta^k_{\exp H}(x) = 0,$$

i.e. the action of A on N is contractive. This means that the only semidirect products $S = N \rtimes A$ that possess random walks with the above properties are those with a contractive action of A on N.

Now we are going to show that (2.10) implies (2.11). The function $\psi(s) = \mathbb{E}[e^{s\lambda(\log M)}]$ is well defined for $s \le s_\lambda$, because for $p = s_\lambda/s$, by the Hölder inequality, we have

$$\psi(s) \le (\mathbb{E}[e^{s_\lambda \lambda(\log M)}])^{1/p}.$$

Moreover,

$$\psi''(s) = \mathbb{E}[e^{s\lambda(\log M)}(\lambda(\log M))^2] > 0,$$

and so ψ is convex. Since $\psi(0) = \psi(s_\lambda) = 1$ and ψ is not constant (otherwise s_λ would not be unique), $\psi'(0) = \mathbb{E}[\lambda(\log M)]$ must be negative.

2.3. Asymptotic behavior of R when $\dim A = 1$**.** As was mentioned in the introduction, when the Abelian group A is one-dimensional, the tail of R is well-known. The ideas of Kesten [23], Grincevičius [15] and Goldie [14] were used in [5] to handle the general situation of homogeneous groups, when the group S is a semidirect product of a nilpotent group N and a one-dimensional group of dilations $A = \mathbb{R}^+$:

$$\delta_a(x) = (a^{d_1}x_1, \dots, a^{d_{n_0}}x_{n_0}), \qquad d_j > 0.$$

In this case there are constants c_j such that the norm

$$|x| = \sum_j c_j |x_j|^{1/d_j}$$

is homogeneous and subadditive, i.e. $|\delta_a(x)| = a|x|$ and $|xy| \le |x| + |y|$ for all $a \in \mathbb{R}^+$ and $x, y \in N$ (see [13, 20]) for more details). Then we have the following theorem:

THEOREM 2.1 ([5]). *Let $S = N \rtimes \mathbb{R}^+$ and assume that*

- $\mathbb{E}[\log M] < 0$,
- *there exists $\alpha > 0$ such that $\mathbb{E}[M^\alpha] = 1$,*
- *the law of $\log M$ is nonarithmetic, i.e. $\log M \in a\mathbb{Z}$ for no $a > 0$,*
- $\mathbb{E}[M^\alpha |\log M|] < \infty$,
- $\mathbb{E}[|Q|^\alpha] < \infty$,

Then

$$\lim_{t\to\infty} t^\alpha \mathbb{P}[|R| > t] = C. \tag{2.12}$$

The constant C is nonzero if and only if for every $x \in N$,

$$\mathbb{P}[(Q, M) \circ x = x] < 1.$$

Moreover, for every j there is C_j such that

$$C_j^{-1} t^{-\alpha} \le \mathbb{P}\{|R_j|^{1/d_j} > t\} \le C_j t^{-\alpha}. \tag{2.13}$$

If $N = \mathbb{R}^{n_0}$ then

$$\lim_{t\to\infty} t^\alpha \mathbb{P}\{|R_j|^{1/d_j} > t\} = C_j, \tag{2.14}$$

and C_j is nonzero if and only if for every $x_j \in \mathbb{R}$,

$$\mathbb{P}\{Q_j + M_j x_j = x_j\} < 1.$$

If N is non-Abelian and some further assumptions are satisfied then $C_j = 0$ implies that R_j is bounded a.s.

The above statement requires some comments. The detailed proof of Theorem 2.1 is given in [5] only for the Euclidean case, i.e. when N is Abelian and the norm is the Euclidean norm. However, as is explained in the appendix of [5], it goes along the same lines in the general case. First one proves that for $f \in C_c(N \setminus \{e\})$,

$$\lim_{a\to 0} a^{-\alpha} \int_N f(\delta_a x)\, d\nu(x) = \langle f, \Lambda\rangle \tag{2.15}$$

exists and defines a homogeneous measure Λ, i.e.

$$\langle f, \Lambda\rangle = \int_{\mathbb{R}\times S^1} f(\delta_r\omega)\, \frac{dr}{r^{1+\alpha}}\, d\sigma(\omega),$$

where $S_1 = \{x : |x| = 1\}$ is the unit sphere in the homogeneous norm and $x = \delta_r\omega$ is the related radial decomposition [13]. Moreover, (2.15) extends to bounded functions f such that $0 \notin \operatorname{supp} f$ and the Λ-measure of the set of discontinuities of f is 0. Therefore (2.15) may be applied to $f = 1_{B_1^c}$, the characteristic function of the exterior of the unit ball, which yields (2.12). To prove that C in (2.12) is strictly positive one has to use an argument due to Grincevičius [15, 16] in the "$ax + b$" case. It requires only homogeneity and subadditivity of the norm and generalizes directly to our setting (see e.g. [4, Proposition 2.6]).

For (2.13) one has to pick up two bounded continuous functions ϕ_1, ϕ_2 such that

$$\mathbf{1}_{\{x_i>2\}} \le \phi_1 \le \mathbf{1}_{\{x_i>1\}} \le \phi_2$$

and apply (2.15) to them. Finally, (2.14) and nonvanishing of C_j in the Euclidean case follow directly from the one-dimensional case. The last sentence of the theorem requires some further arguments, which will be omitted.

Notice that the contribution of all "unbounded" coordinates of R to $\mathbb{P}\{|R| > t\}$ is of the same size, provided it is measured by a homogeneous norm.

2.4. Simple roots. Let $\widetilde{\Delta} \subset \mathcal{A}^*$ be a family of functionals such that any two $\lambda_1, \lambda_2 \in \widetilde{\Delta}$ are linearly independent. A root λ_0 will be called *simple* if it cannot be written as a "positive" sum of other roots, i.e. for all possible choices of nonnegative numbers c_λ,

$$\lambda_0 \ne \sum_{\lambda\in\widetilde{\Delta}\setminus\{\lambda_0\}} c_\lambda \lambda.$$

PROPOSITION 2.2. *Let $\widetilde{\Delta}$ be as above and assume that there is $H \in \mathcal{A}$ such that $\lambda(H) > 0$ for every $\lambda \in \widetilde{\Delta}$. Then every $\lambda \in \widetilde{\Delta}$ is a positive combination of simple roots $\Delta_1 = \{\eta_1, \dots, \eta_k\}$, i.e.*

$$\lambda = \sum_{j=1}^{k} \alpha_j \eta_j, \qquad \alpha_j \ge 0. \tag{2.16}$$

Proof. We proceed by induction with respect to n, the number of elements of $\widetilde{\Delta}$. If $n = 1, 2$ then any root is simple. Assume that $\widetilde{\Delta} = \{\lambda_1, , \ldots, \lambda_{n+1}\}$ and λ_{n+1} is not simple. We are going to prove that $\widetilde{\Delta}$ and $\widetilde{\Delta} \setminus \{\lambda_{n+1}\}$ have the same sets of simple roots and so the conclusion will follow by induction. Clearly, removing a root cannot reduce the number of simple roots. Let us show that it also cannot increase the number of simple roots. Assume a contrario that λ_1 is simple in $\widetilde{\Delta} \setminus \{\lambda_{n+1}\}$ and it is not in $\widetilde{\Delta}$. Let $\lambda_1 = \sum_{j=2}^{n+1} \beta_j \lambda_j$ with $\beta_j \geq 0$, $\beta_{n+1} > 0$ and $\lambda_{n+1} = \sum_{j=1}^{n} \alpha_j \lambda_j$, $\alpha_j \geq 0$ and at least two coefficients are strictly positive. We have

$$\lambda_1 = \sum_{j=2}^{n} \beta_j \lambda_j + \beta_{n+1} \Big(\sum_{j=1}^{n} \alpha_j \lambda_j \Big)$$

and so

$$(1 - \beta_{n+1}\alpha_1)\lambda_1 = \sum_{j=2}^{n} (\beta_j + \beta_{n+1}\alpha_j)\lambda_j.$$

Since both λ_1 and the right hand side applied to H are strictly positive, we have $1 - \beta_{n+1}\alpha_1 > 0$. Therefore λ_1 is not simple, which gives the desired contradiction. ∎

REMARK 2.3. Notice that for any family Δ of functionals having a positive Weyl chamber we can define a set of simple roots so that (2.16) holds. To do so we take the set $\widetilde{\Delta}$ of equivalence classes of the relation of "being linearly dependent" and so a simple root is defined up to a multiplicative constant. However, here we will be more precise. We fix an element H_0 of the Weyl chamber and from any equivalence class we will take the element whose value on H_0 is the smallest. The set of simple roots will be denoted Δ_1.

2.5. Homogeneous norms on N. Suppose we are given an n_0-tuple of strictly positive exponents $d_1, \ldots, d_{n_0}$ so that the dilations

$$\sigma_r(x) = (r^{d_1} x_1, \ldots, r^{d_{n_0}} x_{n_0})$$

are automorphisms of N. Then there is a norm on N such that

- $|\cdot|$ is symmetric: $|x^{-1}| = |x|$;
- $|x| = 0$ if and only if $z = 0$;
- $|\sigma_r(x)| = r|x|$ for any $r \in \mathbb{R}^+$.
- $|\cdot|$ is subadditive, i.e. $|x \cdot y| \leq |x| + |y|$.

Homogeneous norms (i.e. satisfying the first three properties) were introduced in [13]. Later on W. Hebisch and A. Sikora [20] suggested a construction that gives a norm that is additionally subadditive (see also Guivarc'h [17] for a similar result). Their construction was extended in [3] to define an appropriate norm on N homogeneous with respect to some one-parameter subgroup of A. Since in this paper we will strongly rely on formulas defining norms, we recall some details for the reader's convenience. The key step is the following lemma:

LEMMA 2.4 ([20]). *Let X_j be as in* (2.6). *If ε is sufficiently small then the rectangle*

$$\Omega = \Big\{X = \sum_i x_i X_i \in \mathcal{N} : |x_i| < \varepsilon\Big\} \tag{2.17}$$

has the property

$$\text{if } \log x, \log y \in \Omega \text{ with } x, y \in N \text{ and } 0 < r < 1 \text{ then } \log(\sigma_r(x)\sigma_{1-r}(y)) \in \Omega. \tag{2.18}$$

The norm defined on N by

$$|x| = \inf\{r : \log(\sigma_{r^{-1}}(x)) \in \Omega\}$$

is homogeneous and subadditive.

The above norm can be explicitly computed:

$$|x| = \max_j \{\overline{c}_j |x_j|^{1/d_j}\} \tag{2.19}$$

for $\overline{c}_j = \varepsilon^{-1/d_j}$. *Notice that here and elsewhere $|x_i|$ is the absolute value of x_i while $|x|$ is the homogeneous norm.*

Now using the above scheme we introduce homogeneous norms adapted to various dilations.

1st norm. Fix $H_0 \in \mathcal{A}^{++}$ such that $\lambda(H_0) \geq 1$ for all roots λ and take dilations

$$\sigma_r(x) = \delta_{\exp(\log r)H_0}(x) = (r^{\lambda_1(H_0)}x_1, \ldots, r^{\lambda_{n_0}(H_0)}x_{n_0}) \tag{2.20}$$

for $r \in \mathbb{R}^+$ and $x \in N$.

Then the exponents of the norm are $d_\lambda = \lambda(H_0)$ and

$$d_\lambda = \sum c_\eta d_\eta \quad \text{if} \quad \lambda = \sum c_\eta \eta. \tag{2.21}$$

The norm (2.20) is a straightforward generalization of the norm considered in Section 2.3. It depends strongly on the choice of H_0 and in general no norm is better than the others. However, for various specific N we may define homogeneous subadditive norms that are scaled in the same way for all simple roots, i.e. there is $d \geq 1$ such that

$$|x| = \max_j \{\overline{c}_j |x_j|^{1/d}\}$$

for $x \in \bigoplus_{\eta \in \Delta_1} \mathcal{N}_\eta$.

2nd norm. Assume $|\Delta_1| = \dim A$. Given $H_1, \ldots, H_k$ dual to $\eta_1, \ldots, \eta_k$ let $H_0 = d(H_1 + \cdots + H_k)$. Then $\eta_j(H_0) = d$, $\lambda(H_0) = d\sum_{j=1}^k \lambda(H_j)$. If NA

is a symmetric space then all the roots are integer combinations of $\eta_1, \ldots, \eta_k$ and so we can take $d = 1$ and

$$|x| = \max_j \{\bar{c}_j |x_j|\}$$

for $x \in \bigoplus_{\eta \in \Delta_1} \mathcal{N}_\eta$.

3rd norm. If $N = \mathbb{R}^{n_0}$ we choose $d \geq 1$ such that for every root $\lambda_j = \sum \alpha_i \eta_i$, $d_j = d \sum \alpha_i \geq 1$. Now given $x \in \mathcal{N}_\lambda$, we put

$$|x| = \begin{cases} |x_j|^{1/d} & \text{if } \lambda_j \in \Delta_1, \\ |x_j|^{1/d_j} & \text{if } \lambda_j = \sum \alpha_i \eta_i. \end{cases}$$

$|\ |$ corresponds to dilations $\delta_r(x) = (r^{d_1} x_1, \ldots, r^{d_{n_0}} x_{n_0})$ and it is subadditive.

4th norm. Assume that N is stratified, i.e. $\mathcal{N} = \bigoplus V_j$ with $[V_1, V_j] = V_{j+1}$. Since δ_a are automorphisms, each V_j is a direct sum of eigenspaces $\mathcal{N}_\lambda$ and if η is simple then $\mathcal{N}_\eta \subset V_1$. We assume that

$$V_1 = \bigoplus_{\eta \in \Delta_1} \mathcal{N}_\eta.$$

Notice that all the other roots are linear combinations of the simple ones with integer coefficients and $\mathcal{N}_\lambda \subset V_j$ if and only if $\sum \alpha_i = j$ provided $\lambda_j = \sum \alpha_i \eta_i$. Writing

$$\delta_r X = r^j \quad \text{if } X \in V_j$$

we obtain automorphic dilations. The corresponding homogeneous norm satisfies

$$d_\eta = 1, \quad d_\lambda = \sum \alpha_j \quad \text{if } \lambda = \sum \alpha_j \eta_j.$$

2.6. Main Theorem. Assume now that we fix dilations and the corresponding homogeneous norm. Given a root λ let d_λ be the exponent corresponding to the eigenspace $\mathcal{N}_\lambda$ and let $\chi_\lambda = s_\lambda d_\lambda$ be the unique positive number such that

$$\mathbb{E}[e^{\chi_\lambda \lambda(\log M)/d_\lambda}] = 1.$$

Sometimes the notation χ_j will be used instead of χ_{λ_j}. Observe that all the roots proportional to λ have the same χ_λ. Let $\chi_0 = \min\{\chi_\lambda : \lambda \in \Delta\}$. We say that λ is *dominant* if it is simple and $\chi_\lambda = \chi_0$. The set of dominant roots will be denoted Δ_{dom}. In Section 3.1 we will prove that $\chi_0 = \min\{\chi_\lambda : \lambda \in \Delta_{\mathrm{dom}}\}$.

For a dominant root λ_0 let

$$I_{\lambda_0} = \{j : \lambda_j \text{ is a multiple of } \lambda_0\},$$
$$\overline{\mathcal{N}}_{\lambda_0} = \operatorname{Lie span}\{X_j\}_{j \in I_{\lambda_0}} = \operatorname{span}\{X_j\}_{j \in I_{\lambda_0}}.$$

Then $\mathcal{N}_{\lambda_0}$ is a Lie subalgebra of $\mathcal{N}$. For any norm defined in the previous section we have the following:

MAIN THEOREM 2.5. *Assume*

(H1) *for every root* λ *there is a unique strictly positive number* χ_λ *such that* $\mathbb{E}[e^{\chi_\lambda \lambda(\log M)/d_\lambda}] = 1$;

(H2) *for every root* λ, $\mathbb{E}[e^{\chi_\lambda \lambda(\log M)/d_\lambda} |\lambda(\log M)|] < \infty$;

(H3) $\mathbb{E}|Q|^{\chi_0} < \infty$;

(H4) *for every root* $\lambda \in \Delta_{\mathrm{dom}}$ *the law of* $\lambda(\log M)$ *is nonarithmetic;*

(H5) *there is* $\lambda \in \Delta_{\mathrm{dom}}$ *such that for every* $X \in \overline{\mathcal{N}}_{\lambda_0}$,

$$\mathbb{P}[\log((Q, M) \circ \exp X)|_{\overline{\mathcal{N}}_{\lambda_0}} = X] < 1.$$

Then there exists a strictly positive number C_1 *such that*

$$\lim_{t\to\infty} t^{\chi_0} \mathbb{P}[|R| > t] = C_1. \tag{2.22}$$

The above theorem improves the Main Theorem B in [3] which says that there is a positive C_1 such that

$$\frac{1}{C_1} t^{-\chi_0} \le \mathbb{P}[|R| > t] \le C_1 t^{-\chi_0} \tag{2.23}$$

for the norm determined by the dilations $\delta_{\exp(\log r)H_0}$. We are going to use (2.23) in the proof. In fact, we will need the second inequality of (2.23) for any of the norms defined above. For that one proves

$$\mathbb{E}|R|^\beta < \infty \quad \text{for every } \beta < \chi_0, \tag{2.24}$$

which follows from the expression (5.7) in [3] for the coordinates of the backward process $(Q_1, M_1) \cdot \ldots \cdot (Q_m, M_m)$ and (3.5) below. Moreover, we prove that the only nonzero contribution to (2.22) comes from the coordinates corresponding to dominant roots (see Lemmas 3.2–3.4 and Corollary 3.5).

COROLLARY 2.6. *Assume that the homogeneous norm is chosen so that* $d_\eta = 1$ *for every simple root* η, *i.e.*

$$\mathbb{E}[e^{\chi_\eta \eta(\log H)}] = 1.$$

Then (2.22) *holds with* $\chi_0 = \min_{\eta\in\Delta_1} \chi_\eta$, *i.e. the nonzero contribution to* (2.22) *is determined by dominant roots with the strongest expansion (see Introduction).*

3. Proof of the Main Theorem

3.1. Dominant roots. First we are going to prove that without any loss of generality we may assume additionally that

(H6) The support of μ_A is not contained in an affine subspace of $\mathcal{A}$.

Indeed, suppose there exists a linear subspace W of $\mathcal{A}$ and a vector v such that $\mathrm{supp}\mu_A \subset W + v$. We take W of minimal dimension. Let $\widetilde{\mu}$ be the image of μ via the map

$$(x, \exp H) \mapsto (x, \exp(H - v)).$$

For $H \in W$ we have

$$\begin{aligned}\delta_{\exp(H+v)}x &= (e^{\lambda_1(H+v)}x_1, \ldots, e^{\lambda_{n_0}(H+v)}x_{n_0})\\ &= (e^{\lambda_1(H)}e^{\lambda_1(v)}x_1, \ldots, e^{\lambda_{n_0}(H)}e^{\lambda_{n_0}(v)}x_{n_0})\end{aligned}$$

and changing coordinates

$$(x_1, \ldots, x_n) \mapsto (e^{\lambda_1(v)}x_1, \ldots, e^{\lambda_{n_0}(v)}x_{n_0}) = (x_1', \ldots, x_n')$$

we have

$$\delta_{\exp(H+v)}x = \delta_{\exp(H)}x'.$$

Eigenspaces are preserved and classes of homogeneous norms satisfying (2.21) are the same. Therefore, we may assume that $S = N \rtimes \exp W$ and that μ_W is not supported by an affine subspace of W.

PROPOSITION 3.1. *If λ is not proportional to a simple root then $\chi_\lambda > \chi_0$ and so $\Delta_{\mathrm{dom}} \subset \Delta_1$.*

Proof. It is enough to prove that

$$\mathbb{E}[e^{\chi_0\lambda(\log M)/d_\lambda}] < 1.$$

Suppose that $\lambda = \sum_{j=1}^m \alpha_j\lambda_j$, $\lambda_1, \ldots, \lambda_m$ being simple and $p_j = d_\lambda/(\alpha_j d_j)$. Then by (2.21), $\sum 1/p_j = 1$, and by the Hölder inequality with parameters p_j,

$$\mathbb{E}\,[e^{\chi_0\lambda(\log M)/d_\lambda}] = \mathbb{E}\Big[\prod_{j=1}^m e^{\chi_0\alpha_j\lambda_j(\log M)/d_\lambda}\Big] \le \prod_{j=1}^m (\mathbb{E}\,[e^{\chi_0\lambda_j(\log M)/d_j}])^{1/p_j} \le 1,$$

and the above product is equal to 1 if and only if each of its factors is 1, i.e. $\chi_j = \chi_0$ and the Hölder inequality applied above is in fact an equality, i.e. for every j, k,

$$e^{\chi_0\lambda_j(\log M)/d_j} = C_{j,k}e^{\chi_0\lambda_k(\log M)/d_k} \quad \mu_A\text{-a.s.}$$

This means

$$\frac{\chi_0}{d_j}\,\lambda_j(\log M) = \log C_{j,k} + \frac{\chi_0}{d_k}\,\lambda_k(\log M) \quad \mu_A\text{-a.s.}$$

on the support of μ, which in view of (H6) is impossible. ∎

3.2. Campbell–Hausdorff formula. The group multiplication in N is given by the Campbell–Hausdorff formula:

$$\exp(X)\cdot\exp(Y) = \exp(X + Y + [X,Y]/2 + \cdots) \quad \text{for } X, Y \in \mathcal{N}. \tag{3.1}$$

Since the Lie algebra $\mathcal{N}$ is nilpotent, the sum above is finite.

We shall use the lower central sequence to obtain a better description of the Campbell–Hausdorff formula [13]. Since A acts by isomorphisms, it preserves the lower central sequence, i.e. we can choose a basis X_j of $\mathcal{N}$ consisting of eigenvectors and such that for every element of the central sequence there is a basis of it consisting of some of the vectors X_j. More precisely, if $(x \cdot y)_i$ denotes the ith coordinate of $x \cdot y$, for $x = \exp(\sum x_i X_i)$, $y = \exp(\sum y_i X_i)$ elements of N, then

$$(3.2) \qquad \begin{aligned} (x\cdot y)_i &= x_i + y_i && \text{for } i \le i_1,\\ (x\cdot y)_i &= x_i + y_i + P_i(x,y) && \text{for } i_{p-1} < i \le i_p, \text{ for } p > 1. \end{aligned}$$

where P_i are polynomials depending on $x_1, \dots, x_{i_{p-1}}, y_1, \dots, y_{i_{p-1}}$ and they can be written as

$$(3.3) \qquad P_i(x,y) = \sum_{\underline{\mathbf{a}},\underline{\mathbf{b}}} c_{\underline{\mathbf{a}},\underline{\mathbf{b}}} P_i^{\underline{\mathbf{a}},\underline{\mathbf{b}}}(x,y) = \sum_{\underline{\mathbf{a}},\underline{\mathbf{b}}} c_{\underline{\mathbf{a}},\underline{\mathbf{b}}} x^{\underline{\mathbf{a}}} y^{\underline{\mathbf{b}}},$$

where $c_{\underline{\mathbf{a}},\underline{\mathbf{b}}}$ are some real numbers, $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ are multi-indices of natural numbers of length i_{p-1}, and

- $0^0 = 1$;
- if $\underline{\mathbf{c}}$ is a multi-index of length i and z is a vector of length at least i (usually it will be longer than i) then
$$z^{\underline{\mathbf{c}}} = \prod_{j \le i} z_j^{\underline{\mathbf{c}}_j}.$$

The above notation will be used also in the rest of the paper. Moreover, we shall strongly rely on the following properties of the Campbell–Hausdorff formula: if $c_{\underline{\mathbf{a}},\underline{\mathbf{b}}}$ is nonzero then

$$(3.4) \qquad \text{both } \underline{\mathbf{a}} \text{ and } \underline{\mathbf{b}} \text{ are nonzero} \quad \text{and} \quad \sum_{j<i} (\underline{\mathbf{a}}_j + \underline{\mathbf{b}}_j)\lambda_j = \lambda_i.$$

In order to prove the last equation we shall use (2.3). Fix $H \in \mathcal{A}$. Then for any $x, y \in N$ we have

$$(\delta_{\exp H}(xy))_i = e^{\lambda_i(H)} (x\cdot y)_i,$$

but on the other hand, by (3.2) and (3.3) we can write

$$\begin{aligned} (\delta_{\exp H}(xy))_i &= (\delta_{\exp H}(x) \cdot \delta_{\exp H}(x))_i \\ &= \sum_{\underline{\mathbf{a}},\underline{\mathbf{b}}} c_{\underline{\mathbf{a}},\underline{\mathbf{b}}} (\delta_{\exp H}(x))^{\underline{\mathbf{a}}} (\delta_{\exp H}(y))^{\underline{\mathbf{b}}} = \sum_{\underline{\mathbf{a}},\underline{\mathbf{b}}} c_{\underline{\mathbf{a}},\underline{\mathbf{b}}} e^{\sum_{j<i}(\underline{\mathbf{a}}_j + \underline{\mathbf{b}}_j)\lambda_j(H)} x^{\underline{\mathbf{a}}} y^{\underline{\mathbf{b}}} \end{aligned}$$

Comparing the last two equations we obtain (3.4). For any norm with exponents satisfying (2.21), we then have

$$(3.5) \qquad \sum_{j<i} (\underline{\mathbf{a}}_j + \underline{\mathbf{b}}_j) d_j = d_i,$$

where $d_j = d_{\lambda_j}$.

3.3. Scheme of the proof and behavior of R_j's. For a dominant root η let $N_\eta = \exp \mathcal{N}_\eta$ and let $S_\eta = \overline{N}_\eta \rtimes \mathbb{R}^+$ be the semidirect product of $\overline{N}_\eta$ and $\mathbb{R}^+$ with the group multiplication

$$(x, b) \cdot (x', b') = (x \cdot \sigma_b(x'), bb'), \qquad x, x' \in \overline{N}_\eta,\ b, b' \in \mathbb{R}^+.$$

Let $|\cdot|_\eta$ be the restriction of $|\cdot|$ to $\overline{N}_\eta$, i.e. $|x|_\eta = |x|$ for $x \in \overline{N}_\eta$; by (2.19)

$$|x|_\eta = \max_{j\in I_\eta}\{\overline{c}_j |x_j|^{1/d_j}\}.$$

For any $x = \exp(\sum_{j=1}^{n_0} x_j X_j) \in N$ let $x|_{\overline{N}_\eta}$ denote its restriction to $\overline{N}_\eta$, i.e.

$$x|_{\overline{N}_\eta} = \exp\Big(\sum_{j\in I_\eta} x_j X_j\Big).$$

In view of (3.4) for any $x, y \in N$ and $\eta \in \Delta_{\mathrm{dom}}$ we have

$$x|_{\overline{N}_\eta} \cdot y|_{\overline{N}_\eta} = (x\cdot y)|_{\overline{N}_\eta}. \tag{3.6}$$

Applying Theorem 2.1 to S_η we obtain

LEMMA 3.2. *For every dominant root η we have*

$$\lim_{t\to\infty} t^{\chi_\eta}\mathbb{P}[|\overline{R}|_\eta > t^{d_\eta}] = C_\eta,$$

where $\overline{R} = R|_{\overline{N}_\eta}$, and $C_\eta > 0$ if (H5) *is satisfied.*

As in Lemma 2.4, we shall write $R = \exp(\sum_{j=1}^{n_0} R_j X_j)$ and $|R_j|$ will be the absolute value of the coordinate $|R_j|$. To deduce the Main Theorem we shall need two more lemmas.

LEMMA 3.3. *If $\chi_j > \chi_0$ then*

$$\lim_{t\to\infty} t^{\chi_0}\mathbb{P}[|R_j|^{1/d_j} > t] = 0.$$

LEMMA 3.4. *If $\chi_j = \chi_i = \chi_0$ but λ_i, λ_j do not belong to I_η for some $\eta \in \Delta_{\mathrm{dom}}$ then*

$$\lim_{t\to\infty} t^{\chi_0}\mathbb{P}[|R_j|^{1/d_j} > t,\ |R_i|^{1/d_i} > t] = 0.$$

COROLLARY 3.5. *Given $\eta \in \Delta_{\mathrm{dom}}$ let*

$$\Omega_{\eta,t} = \{|\overline{R}|_\eta > t,\ \max_{j\notin I_\eta} \overline{c}_j |R_j|^{1/d_j} \le t\}.$$

Then

$$\lim_{t\to\infty} \mathbb{P}(\Omega_{\eta,t}) t^{\chi_0} = C_\eta,$$

and $C_\eta > 0$ if and only if (H5) *holds. Moreover,*

$$\lim_{t\to\infty} \mathbb{P}\Big[\{|R| > t\} \setminus \bigcup_{\eta\in\Delta_{\mathrm{dom}}} \Omega_{\eta,t}\Big] t^{\chi_0} = 0,$$

i.e. the only nonzero contribution to (2.22) *comes from the "cones" $\Omega_{\eta,t}$.*

Proof of the Main Theorem. We write

$$\begin{aligned}\mathbb{P}\big[|R|>t\big] &= \mathbb{P}[\max_j\{\overline{c}_j|R_j|^{1/d_j}\}>t]\\ &= \mathbb{P}[\max_{\eta\in\Delta_{\rm dom}}|\overline{R}_\eta|>t,\ \max_{\lambda_j\notin\bigcup_{\eta\in\Delta_{\rm dom}}I_\eta}\overline{c}_j|R_j|^{1/d_j}>t]\\ &= \sum_{\eta\in\Delta_{\rm dom}}\mathbb{P}[|\overline{R}|_\eta>t]+\sum_{\lambda_j\notin\bigcup_{\lambda\in\Delta_{\rm dom}}I_\lambda}\mathbb{P}[\overline{c}_j|R_j|^{1/d_j}>t]\\ &\quad+\sum_{I,J}C_{I,J}\mathbb{P}[|\overline{R}_\eta|>t,\,\eta\in I,\,\overline{c}_j|R_j|^{1/d_j}>t,\,j\in J],\end{aligned}$$

where the last sum is taken over all sets I and J such that $I\subset\Delta_{\rm dom}$, $J\subset\{j:\lambda_j\notin\bigcup_{\eta\in\Delta_{\rm dom}}I_\eta\}$, $|I|+|J|\ge 2$. The constants $C_{I,J}$ are -1, 1 or 0, and $C_{I,J}=0$ only if $J=\emptyset$ and $I\subset I_\eta$ for some $\eta\in\Delta_{\rm dom}$.

In view of Lemmas 3.2–3.4,

$$\lim_{t\to\infty}t^{\chi_0}\mathbb{P}[|R|>t]=\lim_{t\to\infty}t^{\chi_0}\sum_{\eta\in\Delta_{\rm dom}}\mathbb{P}[|R|_\eta>t].$$

The limit exists and is strictly positive. ∎

3.4. Proofs of Lemmas 3.3 and 3.4. The idea is the same for both lemmas. We start by giving the main steps needed for Lemma 3.4. Let f_0 be a Hölder function on $\mathbb{R}^2$ bounded by 1 and such that $\operatorname{supp} f_0\subset[1/2,\infty)\times[1/2,\infty)$ and $f_0(x)=1$ for $x\in[1,\infty)\times[1,\infty)$. Define a function f on N by $f(x)=f_0(x_i,x_j)$. Given a function h on $\mathbb{R}^2$ we define

$$\widetilde{h}(s,t)=e^{-\chi_0\cdot\frac{d_is+d_jt}{d_i+d_j}}h(s,t).$$

Let

$$g(s,t)=\int_N f_0(e^{d_is}x_i,e^{d_jt}x_j)\,\nu(dx).$$

Then it is enough to prove that

$$\lim_{t\to-\infty}\widetilde{g}(t,t)=0, \tag{3.7}$$

because then

$$\begin{aligned}\lim_{t\to\infty}e^{\chi_0t}\nu\{x:x_i>e^{d_it}\text{ and }x_j>e^{d_jt}\}&\\ &\le\lim_{t\to\infty}e^{\chi_0t}\int_N f_0(e^{-d_it}x_i,e^{-d_jt}x_j)\,\nu(dx)\\ &=\lim_{t\to-\infty}e^{-\chi_0t}g(t,t)=\lim_{t\to-\infty}\widetilde{g}(t,t)=0.\end{aligned}$$

Define a measure μ_0 on $\mathbb{R}^2$ by

$$\mu_0(U)=\mu_A\{M:\lambda_i(\log M)/d_i,\lambda_j(\log M)/d_j\in U\},\qquad U\subset\mathbb{R}^2.$$

Then

$$\int_{\mathbb{R}^2} e^{\chi_0 t}\, d\mu_0(t,s) = \int_{\mathbb{R}^2} e^{\chi_0 s}\, d\mu_0(t,s) = 1.$$

Let

$$\psi(s,t) = \mu_0 * g(s,t) - g(s,t),$$

and

$$\widetilde{\mu} = e^{\chi_0 \cdot \frac{d_i s + d_j t}{d_i + d_j}} \mu_0.$$

We shall prove that

$$\widetilde{\mu}(\mathbb{R}^2) < 1, \tag{3.8}$$

and for every $s', s'' \in \mathbb{R}$,

$$\lim_{t\to-\infty} \widetilde{\psi}(t+s', t+s'') = 0, \tag{3.9}$$

$$\widetilde{g}(s,t) = -\widetilde{G} * \widetilde{\psi}(s,t), \tag{3.10}$$

where $\widetilde{G} = \sum_{n=0}^{\infty} \widetilde{\mu}^{*n}$ is a finite measure. Then (3.7) will follow by the Lebesgue dominated convergence theorem.

For Lemma 3.3 we proceed in an analogous way. Let f_0 be a bounded Hölder function on $\mathbb{R}$ such that $\operatorname{supp} f_0 \subset [1/2, \infty)$ and $f_0(x) = 1$ for $x > 1$. Define a function f on N by $f(x) = f(x_j)$. Let

$$g(t) = \int_N f_0(e^{d_j t} x_j)\, \nu(dx), \qquad \widetilde{g}(t) = e^{-\chi_0 t} g(t).$$

It is enough to prove that

$$\lim_{t\to-\infty} \widetilde{g}(t) = 0, \tag{3.11}$$

because then

$$\begin{aligned}\lim_{t\to\infty} e^{\chi_0 t} \nu\{x : x_j > e^{d_j t}\} &\le \lim_{t\to\infty} e^{\chi_0 t} \int_N f_0(e^{-d_j t} x_j)\, \nu(dx) \\ &= \lim_{t\to-\infty} e^{-\chi_0 t} g(t) = \lim_{t\to-\infty} \widetilde{g}(t) = 0.\end{aligned}$$

Define a measure μ_0 on $\mathbb{R}$ by

$$\mu_0(U) = \mu_A\{M : \lambda_j(\log M)/d_j \in U\}, \qquad U \subset \mathbb{R}.$$

Then

$$\int_{\mathbb{R}} e^{\chi_0 t}\, d\mu_0(t) < 1,$$

i.e. $\widetilde{\mu} = e^{\chi_0 t}\mu$ is a subprobability measure. Let

$$\psi(t) = \mu_0 * g(t) - g(t), \qquad \widetilde{\psi}(t) = e^{-\chi_0 t}\psi(t).$$

We shall prove that for every s,

$$\lim_{t\to-\infty} \widetilde{\psi}(t+s) = 0, \tag{3.12}$$

$$\widetilde{g}(t) = -\widetilde{G} * \widetilde{\psi}(t), \tag{3.13}$$

where $\widetilde{G} = \sum_{n=0}^{\infty} \widetilde{\mu}^{*n}$ is finite. And again (3.11) will follow by dominated convergence.

3.5. Remaining lemmas. Now we are going to prove (3.8)–(3.10). The argument for (3.12) and (3.13) is the same.

LEMMA 3.6. *The function $\widetilde{\psi}$ is continuous, bounded and for every s', s'',*

$$\lim_{t\to-\infty} \widetilde{\psi}(t+s', t+s'') = 0.$$

Proof. First we will prove that the function $\widetilde{g}$ is bounded. For that we use (2.23) and the Hölder inequality with $p = (d_i + d_j)/d_i$, $q = (d_i + d_j)/d_j$:

$$\begin{aligned}
\widetilde{g}(s,t) &= e^{-\chi_0 \cdot \frac{d_i s + d_j t}{d_i + d_j}} \int_N f_0(e^{d_i s} x_i, e^{d_j t} x_j)\, \nu(dx) \\
&\le e^{-\chi_0 \cdot \frac{d_i s + d_j t}{d_i + d_j}} \int_N \mathbf{1}_{\{x_i > \frac{1}{2} e^{-d_i s}\}} \mathbf{1}_{\{x_j > \frac{1}{2} e^{-d_j t}\}}\, \nu(dx) \\
&\le e^{-\chi_0 \cdot \frac{d_i s + d_j t}{d_i + d_j}} \Big(\nu \Big\{ x : x_i > \frac{1}{2}\, e^{-d_i s} \Big\} \Big)^{\frac{d_i}{d_i + d_j}} \cdot \Big(\nu \Big\{ x : x_j > \frac{1}{2}\, e^{-d_j s} \Big\} \Big)^{\frac{d_j}{d_i + d_j}} \\
&\le C.
\end{aligned}$$

Next we will prove that $\widetilde{\mu_0 * g}$ is bounded, using again the Hölder inequality with the same parameters p, q:

$$\begin{aligned}
|\widetilde{\mu_0 * g}(s,t)| &= \Big| e^{-\chi_0 \cdot \frac{d_i s + d_j t}{d_i + d_j}} \int_{\mathbb{R}^2} g(s+s', t+t')\, d\mu_0(s',t') \Big| \\
&= \Big| \int_{\mathbb{R}^2} \widetilde{g}(s+s', t+t') e^{\chi_0 \cdot \frac{d_i s'}{d_i + d_j}} e^{\chi_0 \cdot \frac{d_j t'}{d_i + d_j}}\, d\mu_0(s',t') \Big| \\
&\le C \Big(\int_{\mathbb{R}^2} e^{\chi_0 s'}\, d\mu_0(s',t') \Big)^{1/p} \Big(\int_{\mathbb{R}^2} e^{\chi_0 t'}\, d\mu_0(s',t') \Big)^{1/q} \\
&= C.
\end{aligned}$$

Hence $\widetilde{\psi}$ is bounded. Continuity is obvious. To prove the last part of the lemma assume $\varepsilon d_i, \varepsilon d_j < \chi_0$. We are going to prove a stronger condition that for every $s', s'' \in \mathbb{R}$,

$$I = \sum_{n\in\mathbb{Z}} \sup_{n \le t < n+1} |\widetilde{\psi}(t+s', t+s'')| < \infty, \tag{3.14}$$

which of course implies that $\widetilde{\psi}$ vanishes at $-\infty$.

First we write

$$
\begin{aligned}
&\widetilde{\psi}(t+s_i,t+s_j)\\
&= e^{-\chi_0\cdot\frac{d_i(t+s_i)+d_j(t+s_j)}{d_i+d_j}} \int\limits_N \Big[\int\limits_{\mathbb{R}^2} f_0(e^{d_i(t+s_i+t_i)}x_i, e^{d_j(t+s_j+t_j)}x_j)\,d\mu_0(t_i,t_j)\\
&\qquad\qquad - f_0(e^{d_i(t+s_i)}x_i, e^{d_j(t+s_j)}x_j)\Big]\,\nu(dx)\\
&= e^{-\chi_0 t}e^{-\chi_0\cdot\frac{d_is_i+d_js_j}{d_i+d_j}} \int\limits_S\int\limits_N [f_0(e^{d_i(t+s_i)+\lambda_i(\log a)}x_i, e^{d_j(t+s_j)+\lambda_j(\log a)}x_j)\\
&\qquad\qquad - f_0(e^{d_i(t+s_i)}\pi_i(b\cdot\delta_a(x)), e^{d_j(t+s_j)}\pi_j(b\cdot\delta_a(x)))]\,\mu(da,db)\,\nu(dx)\\
&= e^{-\chi_0 t}e^{-\chi_0\cdot\frac{d_is_i+d_js_j}{d_i+d_j}} \int\limits_S\int\limits_N [f_0(e^{d_i(t+s_i)+\lambda_i(\log a)}x_i, e^{d_j(t+s_j)+\lambda_j(\log a)}x_j)\\
&\qquad\qquad - f_0(e^{d_i(t+s_i)}(e^{\lambda_i(\log a)}x_i + b_i + P_i(b,\delta_a(x))),\\
&\qquad\qquad e^{d_j(t+s_j)}(e^{\lambda_j(\log a)}x_j + b_j + P_j(b,\delta_a(x))))]\,\mu(da,db)\,\nu(dx).
\end{aligned}
$$

We may assume that $f_0(x,y) = h(x)h(y)$ for some ε-Hölder function h on $\mathbb{R}$ such that $h(x)=1$ for $x\ge 1$ and $\operatorname{supp} h\subset(1/2,\infty)$, where $\varepsilon<\min\{\chi_0/d_i,\chi_0/d_j\}$. Then the Hölder condition implies

$$
\begin{aligned}
&|f_0(x_i,x_j)-f_0(y_i,y_j)| = |h(x_j)(h(x_i)-h(y_i)) + h(y_i)(h(x_j)-h(y_j))|\\
&\le C\big(|x_i-y_i|^\varepsilon\cdot(\mathbf{1}_{\{x_i>1/2\}}+\mathbf{1}_{\{y_i>1/2\}}) + |x_j-y_j|^\varepsilon\cdot(\mathbf{1}_{\{x_j>1/2\}}+\mathbf{1}_{\{y_j>1/2\}})\big).
\end{aligned}
$$

Therefore, since s_i and s_j are fixed,

$$
\begin{aligned}
&|\widetilde{\psi}(t+s_i,t+s_j)|\\
&\quad\le C(s_i,s_j)e^{-\chi_0 t}\sum_{k\in\{i,j\}} C(s_k)\int\limits_S\int\limits_N e^{\varepsilon d_k t}(|b_k|^\varepsilon + |P_k(b,\delta_a(x))|^\varepsilon)\\
&\qquad\quad\times(\mathbf{1}_{\{1/2\le e^{d_k(t+s_k)+\lambda_k(\log a)}x_k\}} + \mathbf{1}_{\{1/2\le e^{d_k(t+s_k)}\pi_k(b\cdot\delta_a(x))\}})\,\mu(da,db)\,\nu(dx)
\end{aligned}
$$

and to prove (3.14) we have to estimate three integrals:

$$
\begin{aligned}
I_{1,k}(t) &= e^{-\chi_0 t}\int\limits_S\int\limits_N e^{\varepsilon d_k t}|b_k|^\varepsilon\cdot\mathbf{1}_{\{1/2\le e^{d_k(t+s_k)+\lambda_k(\log a)}x_k\}}\,\mu(da,db)\,\nu(dx),\\
I_{2,k}(t) &= e^{-\chi_0 t}\int\limits_S\int\limits_N e^{\varepsilon d_k t}|P_k(b,\delta_a(x))|^\varepsilon\cdot\mathbf{1}_{\{1/2\le e^{d_k(t+s_k)+\lambda_k(\log a)}x_k\}}\,\mu(da,db)\,\nu(dx),\\
I_{3,k}(t) &= e^{-\chi_0 t}\int\limits_S\int\limits_N e^{\varepsilon d_k t}(|b_k|^\varepsilon + |P_k(b,\delta_a(x))|^\varepsilon)\\
&\qquad\qquad\times\mathbf{1}_{\{1/2\le e^{d_k(t+s_k)}\pi_k(b\cdot\delta_a(x))\}}\,\mu(da,db)\,\nu(dx)
\end{aligned}
$$

for $k\in\{i,j\}$.

We begin with $I_{1,k}(t)$. For $n \le t < n+1$ we have

$$I_{1,k}(t) \le C_1 e^{-(\chi_0-\varepsilon d_k)n} \int_S \int_N |b_k|^\varepsilon \cdot \mathbf{1}_{\{C_2 e^{-nd_k} \le e^{\lambda_k(\log a)+\log|x_k|}\}}\, \mu(da,db)\,\nu(dx)$$

Let $n_0(a,x) = \lfloor -(1/d_k)(\log C_2 + \lambda_k(\log a) + \log|x_k|)\rfloor$. Then

$$\begin{aligned} I_{1,k} &= \sum_{n\in\mathbb{Z}} \sup_{n<t\le n+1} I_{1,k}(t) \le C \int_S \int_N \sum_{n\ge n_0(a,x)} e^{-(\chi_0-\varepsilon d_k)n} |b_k|^\varepsilon\, \mu(da,db)\,\nu(dx) \\ &\le C \int_S \int_N e^{(1/d_k)(\chi_0-\varepsilon d_k)\lambda_k(\log a)} |x_k|^{(1/d_k)(\chi_0-\varepsilon d_k)} |b_k|^\varepsilon\, \mu(da,db)\,\nu(dx) \\ &= C\Big(\int_N |x_k|^{\chi_0/d_k-\varepsilon}\,\nu(dx)\Big)\cdot\Big(\int_S e^{(\chi_0/d_k-\varepsilon)\lambda_k(\log a)}|b_k|^\varepsilon\,\mu(da,db)\Big). \end{aligned}$$

Both integrals are finite: the first one because of (2.23), and for the second one we apply the Hölder inequality with $(\chi_0/d_k-\varepsilon)p = \chi_0/d_k$, $\varepsilon q = \chi_0/d_k$ to obtain

$$\begin{aligned} &\int_S e^{(\chi_0/d_k-\varepsilon)\lambda_k(\log a)}|b_k|^\varepsilon \mu(da,db) \\ &\qquad \le \Big(\int_S e^{(\chi_0/d_k)\lambda_k(\log a)}\,\mu(da,db)\Big)^{1/p}\Big(\int_S |b_k|^{\chi_0/d_k}\,\mu(da,db)\Big)^{1/q} < \infty, \end{aligned}$$

which proves that $I_{1,k}$ is finite. For $I_{2,k} = \sum_{n\in\mathbb{Z}}\sup_{n<t\le n+1} I_{2,k}(t)$, arguing as above we reduce the problem to estimating

$$\begin{aligned} &\int_S\int_N e^{(1/d_k)(\chi_0-\varepsilon d_k)\lambda_k(\log a)}|x_k|^{\chi_0/d_k-\varepsilon}|b^{\underline{\mathbf{a}}}\delta_a(x)^{\underline{\mathbf{b}}}|^\varepsilon\,\mu(da,db)\,\nu(dx) \\ &\qquad \le \int_S\int_N e^{(\chi_0/d_k-\varepsilon)\lambda_k(\log a)}|x_k|^{\chi_0/d_k-\varepsilon}\prod_{r\in A}|b_r|^{\underline{\mathbf{a}}_r\varepsilon} \\ &\qquad\qquad \cdot \prod_{s\in B}(e^{\lambda_s(\log a)\underline{\mathbf{b}}_s\varepsilon}|x_s|^{\underline{\mathbf{b}}_s\varepsilon})\,\nu(dx)\,\mu(da,db), \end{aligned}$$

where

$$A = \{r : \underline{\mathbf{a}}_r \ne 0\} \subset I_{\lambda_k}, \qquad B = \{s : \underline{\mathbf{b}}_s \ne 0\} \subset I_{\lambda_k},$$

because $k \in I_{\lambda_k}$. By (3.4),

$$\frac{1}{d_k}\sum_{r\in A}\underline{\mathbf{a}}_r d_r + \frac{1}{d_k}\sum_{s\in B}\underline{\mathbf{b}}_s d_s = 1. \tag{3.15}$$

First we integrate over N, we apply (2.23) and the Hölder inequality with

$p(\chi_0/d_k-\varepsilon)<\chi_0/d_k$, $p_s\underline{\mathbf{b}}_s\varepsilon<\chi_0/d_s$ and $1/p+\sum_s 1/p_s=1$ to obtain

$$\int_N |x_k|^{\chi_0/d_k-\varepsilon}\prod_{s\in B}|x_s|^{\underline{\mathbf{b}}_s\varepsilon}\,\nu(dx) \le \Big(\int_N |x_k|^{p(\chi_0/d_k-\varepsilon)}\,\nu(dx)\Big)^{1/p}\prod_{s\in B}\Big(\int_N |x_s|^{p_s\underline{\mathbf{b}}_s\varepsilon}\,\nu(dx)\Big)^{1/p_s}<\infty.$$

Such p,p_s exist because by (3.15),

$$\frac{d_k}{\chi_0}\,d_k\Big(\frac{\chi_0}{d_k}-\varepsilon\Big)+\sum_s\frac{\underline{\mathbf{b}}_s\varepsilon d_s}{\chi_0}<1.$$

For the integral on S we apply the Hölder inequality with $p(\chi_0/d_k-\varepsilon)=\chi_0/d_k$, $p_s\underline{\mathbf{b}}_s\varepsilon=\chi_0/d_s$, $q_r\varepsilon\underline{\mathbf{a}}_r=\chi_0/d_r$ (clearly, by (3.15), $1/p+\sum_s 1/p_s+\sum_r 1/q_r=1-\varepsilon d_k/\chi_0+\sum_s\varepsilon\underline{\mathbf{b}}_s d_s/\chi_0+\sum_r\varepsilon\underline{\mathbf{a}}_r d_r/\chi_0=1$) and we obtain

$$\int_S e^{(\chi_0/d_k-\varepsilon)\lambda_k(\log a)}\prod_{r\in A}|b_r|^{\underline{\mathbf{a}}_r\varepsilon}\cdot\prod_{s\in B}e^{\lambda_s(\log a)\underline{\mathbf{b}}_s\varepsilon}\,\mu(da,db)$$
$$\le\Big(\int_S e^{(\chi_0/d_k)\lambda_k(\log a)}\mu(da,db)\Big)^{1/p}\prod_{r\in A}\Big(\int_S |b_r|^{\chi_0/d_r}\,\mu(da,db)\Big)^{1/q_r}\cdot\prod_{s\in B}\Big(\int_S e^{(\chi_0/d_s)\lambda_s(\log a)}\,\mu(da,db)\Big)^{1/p_s},$$

hence $I_{2,k}$ is bounded.

To estimate $I_{3,k}$ we take

$$n_0(a,k,x)=\Big\lfloor-\frac{1}{d_k}(\log C_2+\log|\pi_k(b\cdot\delta_a(x))|)\Big\rfloor$$

and in view of the Campbell–Hausdorff formula we estimate

$$\int_S\int_N(|b_k|^\varepsilon+|P_k(b,\delta_a(x))|^\varepsilon)\pi_k(b\cdot\delta_a(x))^{\chi_0/d_k-\varepsilon}\,\mu(da,db)\,\nu(dx)$$

by the following sum of integrals:

$$\int_S\int_N e^{(\chi_0/d_k-\varepsilon)\lambda_k(\log)}|x_k|^{\chi_0/d_k-\varepsilon}(|b_k|^\varepsilon+|P_k(b,\delta_a(x))|^\varepsilon)\,\mu(da,db)\,\nu(dx)$$
$$+\int_S\int_N|b_k|^{\chi_0/d_k-\varepsilon}(|b_k|^\varepsilon+|P_k(b,\delta_a(x))|^\varepsilon)\,\mu(da,db)\,\nu(dx)$$
$$+\int_S\int_N|P_k(b,\delta_a(x))|^{\chi_0/d_k-\varepsilon}|b_k|^\varepsilon\,\mu(da,db)\,\nu(dx)$$
$$+\int_S\int_N|P_k(b,\delta_a(x))|^{\chi_0/d_k}\,\mu(da,db)\,\nu(dx).$$

To all of them we apply the Hölder inequality in the same way as above. Let us check the last one, i.e.

$$\int_S\int_N \prod_{r\in A} |b_r|^{\underline{\mathbf{a}}_r\chi_0/d_k}\cdot\prod_{s\in B} e^{\underline{\mathbf{b}}_s(\chi_0/d_k)\lambda_s(\log a)}|x_s|^{\underline{\mathbf{b}}_s\chi_0/d_k}\,\nu(dx)\,\mu(da,db)$$

We first integrate over N to obtain

$$\int_N \prod_{s\in B}|x_s|^{\underline{\mathbf{b}}_s\chi_0/d_k}\,\nu(dx)\,\mu(da,db)\le\prod_{s\in B}\Big(\int_N|x_s|^{p_s\underline{\mathbf{b}}_s\chi_0/d_k}\Big)^{1/p_s}\nu(dx)\,\mu(da,db)$$

where p_s are chosen so that $\underline{\mathbf{b}}_s p_s\chi_0/d_k<\chi_0/d_s$ and $\sum 1/p_s=1$, which is possible because by (3.15), $\sum\underline{\mathbf{b}}_s d_s/d_k<1$. For the integral over S we have

$$\int_S\prod_{r\in A}|b_r|^{\underline{\mathbf{a}}_r\chi_0/d_k}\cdot\prod_{s\in B}e^{\underline{\mathbf{b}}_s(\chi_0/d_k)\lambda_s(\log a)}\,\mu(da,db)$$

$$\le\prod_{r\in A}\Big(\int_S|b_r|^{p_r\underline{\mathbf{a}}_r\chi_0/d_k}\,\mu(da,db)\Big)^{1/p_r}\prod_{s\in B}\Big(\int_S e^{q_s\underline{\mathbf{b}}_s(\chi_0/d_k)\lambda_s(\log a)}\,\mu(da,db)\Big)^{1/q_s}$$

with $p_r=d_k/(\underline{\mathbf{a}}_r d_r)$, $q_s=d_k/(\underline{\mathbf{b}}_s d_s)$. ∎

Lemma 3.7. *The measure $\widetilde{\mu}$ is subprobabilistic, i.e. $\widetilde{\mu}(\mathbb{R}^2)<1$.*

Proof.

$$\widetilde{\mu}(\mathbb{R}^2)=\mathbb{E}\big[e^{\frac{\lambda_i(\log M)}{d_i+d_j}}\,e^{\frac{\lambda_j(\log M)}{d_i+d_j}}\big]\le(\mathbb{E}e^{\frac{\lambda_i(\log M)}{d_i}})^{\frac{d_i}{d_i+d_j}}(\mathbb{E}e^{\frac{\lambda_j(\log M)}{d_j}})^{\frac{d_i}{d_i+d_j}}=1$$

and we have equality only if

$$e^{\lambda_i(\log M)/d_i}=Ce^{\lambda_j(\log M)/d_j}$$

on the support of μ_A, which contradicts hypothesis (H6). ∎

Lemma 3.8. *The function $\widetilde{g}$ can be written as*

$$\widetilde{g}(s,t)=-\widetilde{G}*\widetilde{\psi}(s,t),$$

where $\widetilde{G}=\sum_{n=0}^{\infty}\widetilde{\mu}^{*n}$ is a finite measure.

Proof. By definition of ψ the function g satisfies the Poisson equation

$$\mu_0*g(s,t)=g(s,t)+\psi(s,t).$$

Hence

$$\begin{aligned}\mu_0^{*(n+1)}*g(s,t)&=\mu_0^{*n}*(g+\psi)(s,t)\\&=\mu_0^{*(n-1)}*(g+\psi)(s,t)+\mu_0^{*n}*\psi(s,t)=\cdots\\&=g(s,t)+\sum_{k=0}^{n}\mu_0^{*k}*\psi(s,t).\end{aligned}$$

Multiplying both sides by $e^{-\chi_0\cdot\frac{d_is+d_jt}{d_i+d_j}}$ we obtain

$$\widetilde{\mu}^{*(n+1)}*\widetilde{g}(s,t)=\widetilde{g}(s,t)+\widetilde{\mu}^{*n}*\widetilde{\psi}(s,t).\tag{3.16}$$

Indeed,

$$
\begin{aligned}
&e^{-\chi_0\cdot\frac{d_i s+d_j t}{d_i+d_j}}\mu_0^{*k}*\psi(s,t)\\
&= e^{-\chi_0\cdot\frac{d_i s+d_j t}{d_i+d_j}}\int\limits_{\mathbb{R}^2}\dots\int\limits_{\mathbb{R}^2}\psi(s+s_1+\dots+s_k,t+t_1+\dots+t_k)\\
&\qquad\qquad\times\mu_0(ds_1,dt_1)\dots\mu_0(ds_k,dt_k)\\
&= \int\limits_{\mathbb{R}^2}\dots\int\limits_{\mathbb{R}^2} e^{-\chi_0\cdot\frac{d_i(s+s_1+\cdots s_k)+d_j(t+t_1+\cdots t_k)}{d_i+d_j}}\psi(s+s_1+\dots+s_k,t+t_1+\dots+t_k)\\
&\quad\cdot e^{\chi_0\cdot\frac{d_i s_1+d_j t_1}{d_i+d_j}}\mu_0(ds_1,dt_1)\dots e^{\chi_0\cdot\frac{d_i s_k+d_j t_k}{d_i+d_j}}\mu_0(ds_k,dt_k)=\widetilde{\mu}^{*k}*\widetilde{\psi}(s,t).
\end{aligned}
$$

We have

$$|\widetilde{\mu}^{*k}*\widetilde{g}(s,t)|\le|\widetilde{g}|_{\sup}\widetilde{\mu}(\mathbb{R}^2)^k.$$

Hence

$$\lim_{n\to\infty}\widetilde{\mu}^{*(n+1)}*\widetilde{g}(s,t)=0$$

and

$$\widetilde{g}=-\lim_{n\to\infty}\sum_{k=0}^{n}\widetilde{\mu}^{*k}*\widetilde{\psi}=-\widetilde{G}*\widetilde{\psi}$$

with $\widetilde{G}$ being finite. ∎

Proof of Lemma 3.4. By the Lebesgue Theorem,

$$
\begin{aligned}
\lim_{t\to-\infty}\widetilde{g}(t,t) &= -\lim_{t\to-\infty}\widetilde{G}*\widetilde{\psi}(t,t)\\
&= -\lim_{t\to\infty}\int\limits_{\mathbb{R}^2}\widetilde{\psi}(t+s',t+t')\,\widetilde{G}(ds',dt')=0. \ \blacksquare
\end{aligned}
$$

Appendix A. Action of a compact group of automorphisms. Assume now that there is a compact group K acting on $\mathcal{N}$ and commuting with A. Then every eigenspace $\mathcal{N}_\lambda$ is preserved by K. On $S=N\rtimes AK$ we define a random walk as in Section 2.2 with M_j being in AK. As before, R_n converges in law to a random variable R independent of the choice of R_0. The law of R is the unique stationary solution of $\nu=\mu*\nu$.

We define homogeneous norms analogously as before but making them invariant under the action of K. To do this we choose on every $\mathcal{N}_\lambda$ a norm $|\ |_\lambda$ preserved by K and we define the rectangle Ω as

$$\Omega=\Big\{x=\sum x_\lambda:|x_\lambda|_\lambda<\varepsilon\Big\}.$$

Proceeding as in [20] we prove that if ε is sufficiently small then (2.18) holds. Then the norm is

$$|x|=\max_\lambda\{\varepsilon^{-1/d_\lambda}|x_\lambda|^{1/d_\lambda}\}\tag{A.1}$$

and it is invariant under the action of K. Theorem 2.5 is valid with the same proof provided we write everything in terms of the decomposition $x = \sum_\lambda x_\lambda$ and not in terms of coordinates.

Acknowledgements. This research project has been partially supported by Marie Curie Transfer of Knowledge Fellowship *Harmonic Analysis, Nonlinear Analysis and Probability* (contract number MTKD-CT-2004-013389) and by MNiSW grants: N201 012 31/1020 and N N201 393937.

REFERENCES

[1] R. Azencott and E. Wilson, *On homogeneous manifolds with negative curvature, part I*, Trans. Amer. Math. Soc. 215 (1976), 323–362.

[2] N. Bourbaki, *Groupes et algèbres de Lie*, Hermann, Paris, 1968.

[3] D. Buraczewski, *On tails of stationary measures on a class of solvable groups*, Ann. Inst. H. Poincaré Probab. Statist. 43 (2007), 417–440.

[4] D. Buraczewski, E. Damek and Y. Guivarc'h, *Convergence to stable laws for a class of multidimensional stochastic recursions*, Probab. Theory Related Fields, to appear.

[5] D. Buraczewski, E. Damek, Y. Guivarc'h, A. Hulanicki and R. Urban, *Tail-homogeneity of stationary measures for some multidimensional stochastic recursions*, ibid. 145 (2009), 385–420.

[6] D. Buraczewski, E. Damek and A. Hulanicki, *Asymptotic behavior of Poisson kernels on NA groups*, Comm. Partial Differential Equations 31 (2006), 1547–1589.

[7] E. Damek, *Left-invariant degenerate elliptic operators on semidirect extensions of homogeneous groups*, Studia Math. 89 (1988), 169–196.

[8] E. Damek and A. Hulanicki, *Boundaries for left-invariant subelliptic operators on semidirect products of nilpotent and Abelian groups,* J. Reine Angew. Math. 411 (1990), 1–38.

[9] —, —, *Asymptotic behavior of the invariant measure for a diffusion related to an NA group*, Colloq. Math. 104 (2006), 285–309.

[10] E. Damek, A. Hulanicki and R. Urban, *Martin boundary for homogeneous Riemannian manifolds of negative curvature at the bottom of the spectrum,* Rev. Mat. Iberoamer. 17 (2001), 257–293.

[11] B. de Saporta, Y. Guivarc'h and E. Le Page, *On the multidimensional stochastic equation* $Y_{n+1} = A_n Y_n + B_n$, C. R. Math. Acad. Sci. Paris 339 (2004), 499–502.

[12] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*, Clarendon Press, Oxford, 1994.

[13] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, Princeton Univ. Press, Princeton, NJ, 1982.

[14] Ch. M. Goldie, *Implicit renewal theory and tails of solutions of random equations*, Ann. Appl. Probab. 1 (1991), 126–166.

[15] A. K. Grincevičius, *On limit distribution for a random walk on the line*, Lithuanian Math. J. 15 (1975), 580–589.

[16] —, *Products of random affine transformations*, ibid. 20 (1980), 279–282.

[17] Y. Guivarc'h, *Croissance polynomiale et périodes des fonctions harmoniques*, Bull. Soc. Math. France 101 (1973), 333–379.

[18] Y. Guivarc'h, *Heavy tail properties of multidimensional stochastic recursions*, in: Dynamics & Stochastics, IMS Lecture Notes Monogr. Ser. 48, Inst. Math. Statist., Beachwood, OH, 2006, 85–99.

[19] Y. Guivarc'h and É. Le Page, *On the tails of stationary laws for affine stochastic recursions*, preprint, Rennes, 2005.

[20] W. Hebisch and A. Sikora, *A smooth subadditive homogeneous norm on a homogeneous group*, Studia Math. 96 (1990), 231–236.

[21] E. Heintze, *On homogeneous manifolds of negative curvature*, Math. Ann. 211 (1974), 23–34.

[22] S. Helgason, *Differential Geometry and Symmetric Spaces*, Academic Press, New York and London, 1962.

[23] H. Kesten, *Random difference equations and renewal theory for products of random matrices*, Acta Math. 131 (1973), 207–248.

[24] C. Klüppelberg and S. Pergamenchtchikov, *The tail of the stationary distribution of a random coefficient* $AR(q)$ *model*, Ann. Appl. Probab. 14 (2004), 971–1005.

[25] É. Le Page, *Théorème de renouvellement pour les produits de matrices aléatoires et les équations aux différences aléatoires*, in: Séminaires de probabilités Rennes 1983, Univ. Rennes I, Rennes, 1983, 116 pp.

[26] I. I. Piatetski-Shapiro, *Geometry of classical domains and theory of automorphic functions*, Sovremennye Problemy Matematiki, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1961 (in Russian).

[27] A. Raugi, *Fonctions harmoniques sur les groupes localement compacts à base dénombrable*, Bull. Soc. Math. France Mém. 54 (1977).

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Received 20 March 2009;
revised 16 July 2009 (5186)