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ASYMPTOTIC PROPERTIES OF HARMONIC MEASURES ON HOMOGENEOUS TREES

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Abstract. Let $\mathrm{Aff}(\mathbb{T})$ be the group of isometries of a homogeneous tree $\mathbb{T}$ fixing an end of its boundary. Given a probability measure on $\mathrm{Aff}(\mathbb{T})$ we consider an associated random process on the tree. It is known that under suitable hypothesis this random process converges to the boundary of the tree defining a harmonic measure there. In this paper we study the asymptotic behaviour of this measure.

1. Introduction. Let $\mathbb{T}$ be a homogeneous tree. We denote by $\mathrm{Aff}(\mathbb{T})$ the group of affine transformations of the tree $\mathbb{T}$, that is, the group of isometries of the tree that fix an end ω of the boundary. The group $\mathrm{Aff}(\mathbb{T})$ is an analogue of the real affine group acting on the hyperbolic plane $\mathbb{H}^2$ by isometries and fixing a boundary point. However, due to the graph structure of the tree, which is less rigid than $\mathbb{H}^2$, the study of the affine group of the tree is much more difficult. This group contains on the one hand the affine group of p-adic numbers $\mathrm{Aff}(\mathbb{Q}_p)$ (i.e. the group of matrices of the form $\left[\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right]$, where a, b are p-adic numbers and a is non-zero), is which in some sense similar to $\mathrm{Aff}(\mathbb{R})$, but on the other hand it contains groups having completely different structure like the lamplighter group or automata groups (see [3] for further information on the structure of $\mathrm{Aff}(\mathbb{T})$).

Given a probability measure μ on $\mathrm{Aff}(\mathbb{T})$ we consider the right random walk on $\mathrm{Aff}(\mathbb{T})$, i.e. the sequence of random variables $R_n = X_1 \cdots X_n$, where X_i are i.i.d. with law μ. Choosing a point $o \in \mathbb{T}$ one can define the random process $R_n \cdot o$ on the tree. It has been proved by Cartwright, Kaimanovich and Woess [3] that under some mild conditions (the most important being the 'drift'), which will be explained in detail in Section 2, this random process converges a.s. to some random element of the boundary $\partial^*\mathbb{T} = \partial\mathbb{T} \setminus \{\omega\}$. We denote the law of this random variable by ν. Then ν is the harmonic measure of the random walk, i.e.

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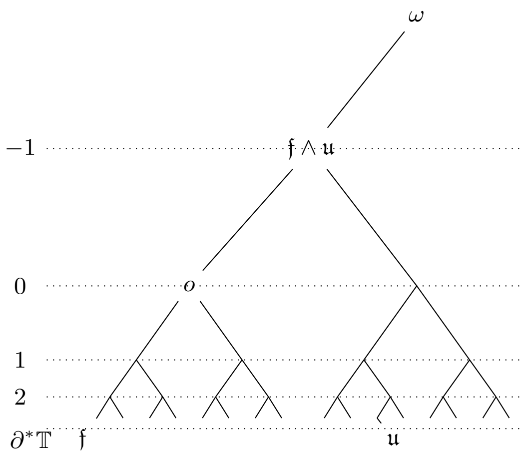
$$\nu(f) = \mu * \nu(f) = \int\limits_{\mathrm{Aff}(\mathbb{T})} \int\limits_{\partial^*\mathbb{T}} f(\gamma\mathfrak{u})\, \nu(d\mathfrak{u})\, \mu(d\gamma)$$

for any $f \in C(\partial^*\mathbb{T})$. Moreover, under some further hypothesis the Poisson boundary has also been identified. Recently Brofferio [1] has proved the renewal theorem for this random walk.

The main goal of this paper is to describe in more detail the asymptotic properties of the harmonic measure ν. More precisely, we fix a point $\mathfrak{f}$ of the boundary, associate with it a distance $|\cdot|$ and we prove that $\nu(\{\mathfrak{u} \in \partial^*\mathbb{T} : |\mathfrak{u}| > t\})$ for large t is of the order of $t^{-\alpha}$ for some $\alpha > 0$. For this purpose we adapt to our setting the techniques introduced by Kesten [9] (see also Goldie [5]), who studied stationary measures on $\mathbb{R}$ for random walks on the affine group of $\mathbb{R}$. Our main result is stated in Theorem 4.1.

2. The affine group of a tree

2.1. A tree. A *homogeneous tree* $\mathbb{T} = \mathbb{T}_{q+1}$ of degree $q+1$ is a connected graph without cycles whose vertices have exactly $q+1$ neighbours each. For any couple of vertices x and y there exists exactly one sequence $x = x_0, x_1, \ldots, x_k = y$ of successive vertices without repetition, denoted by $\overline{xy}$. In this situation we say that the distance between x and y is equal to k and we write $d(x,y) = k$. A *geodesic ray* is an infinite sequence $x_0, x_1, x_2, \ldots$ of successive neighbours without repetition. Two rays are equivalent if they differ only by finitely many vertices. An *end* is an equivalence class of this relation, and the set of all ends will be denoted by $\partial\mathbb{T}$. For $\mathfrak{u} \in \partial\mathbb{T}$ and $x \in \mathbb{T}$ there exists a unique geodesic ray $\overline{x\mathfrak{u}}$ which represents $\mathfrak{u}$.



We choose and fix once for all an end ω and define $\partial^*\mathbb{T} = \partial\mathbb{T} \setminus \{\omega\}$. For $x, y \in \mathbb{T} \cup \partial^*\mathbb{T}$ we denote by $x \wedge y$ the first common vertex of $\overline{x\omega}$ and $\overline{y\omega}$, i.e. $x \wedge y = z$ if $\overline{x\omega} \cap \overline{y\omega} = \overline{z\omega}$. On $\mathbb{T} \cup \partial^*\mathbb{T}$ we have a partial order associated with the end ω: $x \succeq y$ if $x = x \wedge y$. We may imagine the oriented tree as a genealogical tree where ω is a mythical ancestor, and every vertex has one ancestor and q children.

Let us fix a reference vertex o in $\mathbb{T}$ called the *origin*. The *height function* h from $\mathbb{T}$ to $\mathbb{Z}$ is

$$h(x) := d(x, x \wedge o) - d(o, x \wedge o),$$

also known as the *Busemann function*.

For any three vertices $x, y, z \in \mathbb{T}$ we have $x \wedge y \in \overline{y\omega}$ and $z \wedge y \in \overline{y\omega}$, which implies $x \wedge y \succeq z \wedge y$ or $z \wedge y \succeq x \wedge y$. Therefore we get $x \wedge y \wedge z = x \wedge y$ or $x \wedge y \wedge z = y \wedge z$. For $x \succeq y$ we have $h(x) \le h(y)$ and $h(x \wedge y) \ge h(x \wedge y \wedge z) = \min(h(x \wedge z), h(z \wedge y))$. The function h induces an ultra-metric distance Θ on $\mathbb{T} \cup \partial^* \mathbb{T}$: for $x, y \in \mathbb{T} \cup \partial^* \mathbb{T}$ we define

$$\Theta(x, y) := \begin{cases} q^{-h(x \wedge y)} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

2.2. The affine group. Every isometry of $(\mathbb{T}, d)$ has a natural extension to $\partial^* \mathbb{T}$ so we can define the affine group of the tree $\mathbb{T}$ as the stabilizer of ω,

$$\mathrm{Aff}(\mathbb{T}) := \{g \in \mathrm{Aut}(\mathbb{T}) : g\omega = \omega\}.$$

Then $\mathrm{Aff}(\mathbb{T})$ is the subgroup of all isometries which preserve the order. The group $\mathrm{Aff}(\mathbb{T})$ is equipped with the topology of pointwise convergence. The neighbourhood base of the identity consists of sets of the form $G_{x_1} \cap \cdots \cap G_{x_k}$, where $G_x = \{g \in \mathrm{Aff}(\mathbb{T}) : gx = x\}$. The base of an arbitrary element g consists of sets of the form $g(G_{x_1} \cap \cdots \cap G_{x_k})$. Since G_x is open and compact, $\mathrm{Aff}(\mathbb{T})$ is a locally compact totally disconnected group.

All elements of the affine group preserve the order and the distance, therefore

$$h(x) - h(y) = h(gx) - h(gy)$$

for any couple $x, y \in \mathbb{T}$ and $g \in \mathrm{Aff}(\mathbb{T})$. So we may define a homomorphism ϕ of $\mathrm{Aff}(\mathbb{T})$ into $\mathbb{Z}$ by

$$\phi(g) = h(gx) - h(x) = h(go),$$

and by the remark above the definition does not depend on the particular choice of x and o. Moreover,

$$\Theta(gx, gy) = q^{-h(gx \wedge gy)} = q^{-\phi(g)} \Theta(x, y).$$

The horocyclic group of the tree is the subgroup of the affine group that fixes the height

$$\mathrm{Hor}(\mathbb{T}) := \ker \phi = \{g \in \mathrm{Aff}(\mathbb{T}) : h(gx) = h(x),\ \forall x \in \mathbb{T}\}.$$

Let us fix a $\sigma \in \mathrm{Aff}(\mathbb{T})$ such that $\phi(\sigma) = 1$. Every element $g \in \mathrm{Aff}(\mathbb{T})$ has a unique decomposition as a product of an element of the horocyclic group and a power of σ,

$$g = (g\sigma^{-\phi(g)})\sigma^{\phi(g)}.$$

Thus we get a decomposition of the affine group into a semidirect product of $\mathrm{Hor}(\mathbb{T})$ and $\mathbb{Z}$:

$$\mathrm{Hor}(\mathbb{T}) \rtimes \mathbb{Z} \cong \mathrm{Aff}(\mathbb{T}), \quad (\beta, m) \mapsto \beta\sigma^m,$$

where the action of $\mathbb{Z}$ on $\mathrm{Hor}(\mathbb{T})$ is given by $m\beta = m(\beta) := \sigma^m\beta\sigma^{-m}$, and multiplication in the affine group is given by

$$\begin{aligned}(\beta_1, m_1)(\beta_2, m_2) &= \beta_1\sigma^{m_1}\beta_2\sigma^{m_2} = \beta_1\sigma^{m_1}\beta_2\sigma^{-m_1}\sigma^{m_1+m_2}\\ &= (\beta_1 m_1\beta_2, m_1 + m_2).\end{aligned}$$

Notice that the decomposition of $\mathrm{Aff}(\mathbb{T})$ depends on the choice of the element σ.

We say that a subgroup Γ of $\mathrm{Aff}(\mathbb{T})$ is *exceptional* if $\Gamma \subseteq \mathrm{Hor}(\mathbb{T})$ or if Γ fixes an element of $\partial^*\mathbb{T}$. In this paper we will always consider closed and nonexceptional subgroups Γ. It has been shown [3] that Γ is nonexceptional if and only if it is unimodular. In this case the *limit set* $\partial\Gamma$ of Γ, i.e. the set of accumulation points of an orbit Γo in $\partial\mathbb{T}$, is uncountable and $\omega \in \partial\Gamma$. Moreover, for $\mathfrak{u} \in \partial\Gamma \setminus \{\omega\}$ the orbit $\Gamma\mathfrak{u}$ is dense in $\partial\Gamma$ (see [3]).

2.3. Length functions. Notice that there exists a unique $\mathfrak{f} = \mathfrak{f}^\sigma \in \partial^*\mathbb{T}$ such that $\sigma(\mathfrak{f}^\sigma) = \mathfrak{f}^\sigma$. Indeed, for $c = o \wedge \sigma o$ the sequence $c, \sigma c, \sigma^2 c, \ldots$ represents the unique end of $\partial^*\mathbb{T}$ fixed by σ.

Then σ acts by translation on $\overline{\mathfrak{f}\omega}$. For the sake of simplicity we suppose that $o \in \overline{\mathfrak{f}\omega}$. We consider two length functions: one on the boundary $\partial^*\mathbb{T}$, $|\mathfrak{u}| = \Theta(\mathfrak{u}, \mathfrak{f})$, and the second one on the affine group, $\|\gamma\| = \Theta(\gamma\mathfrak{f}, \mathfrak{f})$. Observe that the group $\mathbb{Z}$ is contained in the kernel of $\|\cdot\|$ and for any $\gamma = (\beta, m) \in \mathrm{Aff}(\mathbb{T})$ we have $\|\gamma\| = \|\beta\|$.

LEMMA 2.1. *For any element $\gamma = (\beta, m) \in \mathrm{Aff}(\mathbb{T})$ we have*

$$\frac{1}{2}\, d(\gamma o, o) \leq \log_q^+ \|\beta\| + |m| \leq 2d(\gamma o, o).$$

Proof. Notice that

$$\log_q \|\beta\| = \log_q \Theta(\beta\mathfrak{f}, \mathfrak{f}) = \log_q \Theta(\beta\sigma^m\mathfrak{f}, \mathfrak{f}) = \log_q \Theta(\gamma\mathfrak{f}, \mathfrak{f}) = -h(\gamma\mathfrak{f} \wedge \mathfrak{f}).$$

CASE 1. If $\log_q \|\beta\| \leq 0$ then $h(\gamma\mathfrak{f} \wedge \mathfrak{f}) \geq 0$. This means that the geodesic $\overline{(\beta, m)\mathfrak{f}\omega}$ connects with $\overline{\mathfrak{f}\omega}$ below the center o. Since $\gamma o \in \overline{\gamma\mathfrak{f}\omega}$ we get $d(\gamma o, o) = |m|$ and the statement is now obvious.

CASE 2. If $\log_q \|\beta\| > 0$ then the point $c = \gamma\mathfrak{f} \wedge \mathfrak{f}$ is above the center o. Since $d(\beta o, \gamma o) = |m|$ we get

$$d((\beta, m)o, o) \leq d(\beta o, o) + d((\beta, m)o, \beta o) = 2\log_q \|\beta\| + |m|.$$

On the other hand,

$$\begin{aligned}\log_q \|\beta\| + |m| &= 2d(c,o) - d(c,o) + d(\beta o,(\beta,m)o)\\ &\le 2d(c,o) + |d(c,o) - d(\beta o,(\beta,m)o)|\\ &= 2d(c,o) + |d(c,\beta o) - d(\beta o,(\beta,m)o)|\\ &\le 2d(c,o) + d(c,(\beta,m)o) \le 2d(o,(\beta,m)o),\end{aligned}$$

which finishes the proof of the lemma. ■

3. Random walk on $\mathrm{Aff}(\mathbb{T})$**.** Let μ be a probability measure on $\mathrm{Aff}(\mathbb{T})$. We will assume that the closed semigroup Γ generated by the support of μ is non-exceptional. For simplicity we will also assume that $\phi(\Gamma) = \mathbb{Z}$.

Let $(X_n)_{n\ge1}$ be a sequence of i.i.d. Γ-valued random variables with law μ. The *right random walk* on Γ with law μ is a sequence of random variables

$$R_0 = \mathrm{Id}, \qquad R_{n+1} = R_n X_{n+1}.$$

We denote by $\overline{\mu}$ the image of the measure μ on $\mathbb{Z}$, i.e. $\overline{\mu}(k) = \mu(\phi^{-1}\{k\})$. Then

$$\phi(R_n) = \phi(X_1) + \cdots + \phi(X_n)$$

is a sum of i.i.d. random variables with law $\overline{\mu}$. If the measure $\overline{\mu}$ has a first moment then we denote by m_1 its mean,

$$m_1 = \sum_{k\in\mathbb{Z}} k\overline{\mu}(k) = \int_{\Gamma} \phi(\gamma)\,\mu(d\gamma).$$

It has been proved by Cartwright, Kaimanovich and Woess [3] that if the projected random walk on $\mathbb{Z}$ has a drift, i.e. $m_1 > 0$ and $\int_\Gamma d(\gamma o, o)\,\mu(d\gamma) < \infty$, then $(R_n o)$ converges almost surely to some random variable $\mathbf{R}$ with values in the boundary of the tree $\partial^*\mathbb{T}$. Moreover, it is known that $\mathbf{R}$ does not depend on the choice of the starting point. We write ν for the law of the limit $\mathbf{R}$. As the random variable $X_1^{-1}\mathbf{R}$ is the limit of $(X_2\cdots X_n o)$ it has the same law as $\mathbf{R}$,

$$X\mathbf{R} \stackrel{d}{=} \mathbf{R} \quad \text{for independent } X \text{ and } \mathbf{R}.$$

This means that the measure ν is μ-*stationary*, i.e. $\mu * \nu = \nu$, where

$$\mu * \nu(f) = \int_{\Gamma}\int_{\partial\Gamma} f(\gamma \mathfrak{u})\,\nu(d\mathfrak{u})\,\mu(d\gamma)$$

for $f \in C(\partial\Gamma)$. Thus, for such a function f, the function

$$g(\gamma) = \int_{\partial\Gamma} f(\gamma\mathfrak{u})\,\nu(d\mathfrak{u})$$

is bounded and μ-*harmonic*, i.e.

$$g * \mu(\alpha) = \int_{\Gamma} g(\alpha\gamma)\,\mu(d\gamma) = g(\alpha) \quad \text{for every } \alpha \in \Gamma.$$

Conversely, if the measure μ is sufficiently nice then all bounded harmonic functions g are of this form for some f.

THEOREM 3.1 ([3]). *Let μ be a probability measure on a closed subgroup Γ of* $\mathrm{Aff}(\mathbb{T})$. *Suppose that μ is irreducible, spread out and has finite first moment, $\int_\Gamma d(\gamma o, o)\,\mu(d\gamma) < \infty$. If $m_1 > 0$, then the space $(\partial^*\mathbb{T}, \nu)$ is the Poisson boundary of (Γ, μ), i.e. every bounded harmonic function g on the closed subgroup Γ is of the form*

$$g(\gamma) = \int_{\partial^*\Gamma} f(\gamma\mathfrak{u})\,\nu(d\mathfrak{u})$$

for some $f \in L^\infty(\partial\Gamma)$.

4. Asymptotic behaviour of ν. Our main result is the following:

THEOREM 4.1. *Let (Q, M) be a Γ-valued random variable with law μ. Assume:*

(4.2) $\quad \mathbb{E}[M] = m_1 > 0$;

(4.3) $\quad$ *there is $\alpha > 0$ such that* $\mathbb{E}[q^{-\alpha M}] = \sum_{n\in\mathbb{Z}} q^{\alpha n}\mathbb{P}[M = -n] = 1$;

(4.4) $\quad \mathbb{E}[\|Q\|^\alpha] < \infty$;

(4.5) $\quad \mathbb{E}[-Mq^{-\alpha M}] = m_2 \in (0, \infty)$.

Then

$$\lim_{k\to\infty} q^{\alpha k}\nu\{\mathfrak{u} \in \partial^*\mathbb{T} : |\mathfrak{u}| > q^k\} = C_+,$$

where C_+ is positive and given by the formula

$$C_+ = \frac{1}{m_2}\sum_{n\in\mathbb{Z}}(\mathbb{P}[|\mathbf{R}| \ge q^n] - \mathbb{P}[|M\mathbf{R}| \ge q^n])q^{\alpha n}. \tag{4.6}$$

The result above is an analogue of Kesten's Theorem [9]. He studied a Markov process on the real line generated by a random walk on the real affine group. However, his proof was complicated and it was later simplified by Grincevičjus [6] and Goldie [5]. Here we follow their approach.

Notice that condition (4.4) implies $\mathbb{E}[\log_q^+\|Q\|] < \infty$, so in view of Lemma 2.1, hypotheses (4.2) and (4.4) imply existence of the limit random variable $\mathbf{R}$.

For simplicity, we write $k\mathfrak{u}$ for $\sigma^k\mathfrak{u}$. In the proof we will use the following lemma:

LEMAT 4.7. *Under assumptions* (4.2)–(4.5) *we have*

$$\sum_{k\in\mathbb{Z}} q^{\alpha k}|\mathbb{P}[|\mathbf{R}| \ge q^k] - \mathbb{P}[|M\mathbf{R}| \ge q^k]| < \infty.$$

Proof. For (Q,M) and $\mathbf{R}$ independent random variables with law μ and ν respectively we will show that

$$\mathbb{E}\big[\big||(Q,M)\mathbf{R}|^\alpha - |M\mathbf{R}|^\alpha\big|\big] < \infty. \tag{4.8}$$

Notice that

$$\begin{aligned}|(Q,M)\mathbf{R}| &= \varTheta(Q\sigma^M\mathbf{R},\mathfrak{f}) \le \varTheta(Q\sigma^M\mathbf{R},Q\mathfrak{f}) \vee \varTheta(Q\mathfrak{f},\mathfrak{f})\\ &= \varTheta(\sigma^M\mathbf{R},\mathfrak{f}) \vee \varTheta(Q\mathfrak{f},\mathfrak{f}) = |M\mathbf{R}| \vee \|Q\|,\end{aligned}$$

hence

$$|(Q,M)\mathbf{R}|^\alpha \le |M\mathbf{R}|^\alpha \vee \|Q\|^\alpha \le |M\mathbf{R}|^\alpha + \|Q\|^\alpha. \tag{4.9}$$

Similarly we estimate $|M\mathbf{R}|$:

$$\begin{aligned}|M\mathbf{R}| &= \varTheta(\sigma^M\mathbf{R},\mathfrak{f}) \le \varTheta(\sigma^M\mathbf{R},Q^{-1}\mathfrak{f}) \vee \varTheta(Q^{-1}\mathfrak{f},\mathfrak{f})\\ &= \varTheta(Q\sigma^M\mathbf{R},\mathfrak{f}) \vee \varTheta(\mathfrak{f},Q\mathfrak{f}) = |(Q,M)\mathbf{R}| \vee \|Q\|\end{aligned}$$

and

$$|M\mathbf{R}|^\alpha \le |(Q,M)\mathbf{R}|^\alpha \vee \|Q\|^\alpha \le |(Q,M)\mathbf{R}|^\alpha + \|Q\|^\alpha. \tag{4.10}$$

Comparing (4.9) with (4.10) we obtain

$$\big||(Q,M)\mathbf{R}|^\alpha - |M\mathbf{R}|^\alpha\big| \le \|Q\|^\alpha$$

and (4.4) immediately implies (4.8).

Consider the real-valued random variables $X = |(Q,M)\mathbf{R}|$ and $Y = |M\mathbf{R}|$. Then

$$\begin{aligned}&\sum_k q^{\alpha k}|\mathbb{P}[X \ge q^k] - \mathbb{P}[Y \ge q^k]|\\ &\qquad\le \sum_k q^{\alpha k}(\mathbb{P}[X \ge q^k > Y] + \mathbb{P}[Y \ge q^k > X])\end{aligned}$$

but

$$\begin{aligned}&\sum_{k\in\mathbb{Z}} q^{\alpha k}\mathbb{P}[X \ge q^k > Y] = \sum_{k\in\mathbb{Z}} q^{\alpha k}\mathbb{E}[\mathbf{1}_{[X\ge q^k > Y]}] = \mathbb{E}\Big[\sum_{Y<q^k\le X} q^{\alpha k}\Big]\\ &= \mathbb{E}\Big[\sum_{k=\log_q Y+1}^{\log_q X} (q^\alpha)^k\Big] = \mathbb{E}\Big[q^{\alpha(\log_q Y+1)}\,\frac{1-q^{\alpha(\log_q X-\log_q Y)}}{1-q^\alpha}\,\mathbf{1}_{[Y<X]}\Big]\\ &= \frac{q^\alpha}{q^\alpha-1}\,\mathbb{E}[(X^\alpha - Y^\alpha)\mathbf{1}_{[X>Y]}].\end{aligned}$$

Exactly in the same way we obtain

$$\sum_{k\in\mathbb{Z}} q^{\alpha k}\mathbb{P}[Y \ge q^k > X] = \frac{q^\alpha}{q^\alpha-1}\,\mathbb{E}[(X^\alpha - Y^\alpha)\mathbf{1}_{[Y>X]}]$$

and finally

$$\sum_k q^{\alpha k}|\mathbb{P}[X \geq q^k] - \mathbb{P}[Y \geq q^k]| \leq \frac{q^\alpha}{q^\alpha - 1}\,\mathbb{E}|X^\alpha - Y^\alpha|,$$

and by (4.8) the above value is finite. Since X and $|\mathbf{R}|$ have the same law, we conclude the proof. ■

Proof of Theorem 4.1: existence of the limit. Let $\Pi_k = M_1 + \cdots + M_k$ and define a probability measure $\breve{\overline{\mu}}$ by $\breve{\overline{\mu}}(n) = \overline{\mu}(-n)$. Observe that

$$\begin{aligned}
\mathbb{P}[|\mathbf{R}| \geq q^k] &= \sum_{j=1}^{n}(\mathbb{P}[|\Pi_{j-1}\mathbf{R}| \geq q^k] - \mathbb{P}[|\Pi_j\mathbf{R}| \geq q^k]) + \mathbb{P}[|\Pi_n\mathbf{R}| \geq q^k] \\
&= \sum_{j=1}^{n}(\mathbb{P}[|\Pi_{j-1}\mathbf{R}| \geq q^k] - \mathbb{P}[|\Pi_{j-1}M\mathbf{R}| \geq q^k]) + \mathbb{P}[|\Pi_n\mathbf{R}| \geq q^k] \\
&= \sum_{j=1}^{n}\sum_{u\in\mathbb{Z}}(\mathbb{P}[|\mathbf{R}| \geq q^{k-u}] - \mathbb{P}[|M\mathbf{R}| \geq q^{k-u}])\mathbb{P}[\Pi_{j-1} = -u] + \mathbb{P}[|\Pi_n\mathbf{R}| \geq q^k] \\
&= \sum_{j=0}^{n-1} f * \breve{\overline{\mu}}^{*j}(k) + \mathbb{P}[|\Pi_n\mathbf{R}| \geq q^k],
\end{aligned}$$

where

$$f(n) = \mathbb{P}[|\mathbf{R}| \geq q^n] - \mathbb{P}[|M\mathbf{R}| \geq q^n].$$

Therefore

$$\begin{aligned}
q^{\alpha k}\mathbb{P}[|\mathbf{R}| \geq q^k] &= \sum_{j=0}^{n-1} q^{\alpha k} f * \breve{\overline{\mu}}^{*j}(k) + q^{\alpha k}\mathbb{P}[|\Pi_n\mathbf{R}| \geq q^k] \\
&= f_\alpha * \sum_{j=0}^{n-1}\mu_\alpha^{*j}(k) + q^{\alpha k}\mathbb{P}[|\Pi_n\mathbf{R}| \geq q^k],
\end{aligned} \tag{4.11}$$

for $\mu_\alpha(n) = q^{\alpha n}\breve{\overline{\mu}}(n)$ and $f_\alpha(n) = q^{\alpha n} f(n)$. By (4.3), μ_α is a probability measure on $\mathbb{Z}$, and according to (4.5) it has a positive first moment m_2.

In view of Lemma 4.7, the function f_α is integrable, hence by the Lebesgue theorem

$$\lim_{n\to\infty} f_\alpha * \sum_{j=0}^{n-1}\mu_\alpha^{*j}(k) = f_\alpha * \sum_{j=0}^{\infty}\mu_\alpha^{*j}(k).$$

Moreover, since $|\Pi_n\mathbf{R}| = q^{-(M_1+\cdots+M_n)}|\mathbf{R}|$ and by (4.2), $\lim_{n\to\infty}\sum_{i=1}^n M_i = \infty$ a.e.,

$$\lim_{n\to\infty} q^{\alpha k}\mathbb{P}[|\Pi_n\mathbf{R}| \geq q^k] = 0.$$

So, taking the limit in (4.11) as n goes to ∞, we obtain a renewal equation:

$$q^{\alpha k}\mathbb{P}[|\mathbf{R}| \geq q^k] = f_\alpha * \sum_{j=0}^{\infty} \mu_\alpha^{*j}(k). \tag{4.12}$$

Therefore by the renewal theorem (Feller [4, Chapter XI, (9.2)]) we have

$$\lim_{k\to\infty} q^{\alpha k}\mathbb{P}[|\mathbf{R}| \geq q^k] = \lim_{k\to\infty} f_\alpha * \sum_{j=0}^{\infty} \mu_\alpha^{*j}(k) = \frac{1}{m_2}\sum_{n\in\mathbb{Z}} f_\alpha(n).$$

Finally, we obtain

$$\lim_{k\to\infty} \nu\{\mathfrak{u} \in \partial^*\mathbb{T} : |\mathfrak{u}| > q^k\}q^{\alpha k} = C_+ \tag{4.13}$$

where C_+ is as in (4.6). ∎

In order to prove that C_+ is positive we will use the techniques introduced by Grincevičjus [6], [7]. The crucial step is the following well-known fact:

LEMAT 4.14 (Feller [4, Chapter XII, (5.13)]). *Suppose that X_i are i.i.d. random variables on $\mathbb{Z}$ with negative expectation and there exists a positive constant α such that $\mathbb{E}[e^{\alpha X_1}] = 1$ and $\mathbb{E}[X_1 e^{\alpha X_1}] > 0$. Then*

$$\lim_{n\to\infty} e^{\alpha t}\mathbb{P}\Big[\max_n \sum_{i=1}^{n} X_i > t\Big] = C_0$$

and the constant C_0 is strictly positive.

For simplicity we will introduce some notation. First we define the partial products

$$R_n = (Q_1, M_1)\cdot \ldots \cdot (Q_n, M_n),$$
$$R_{j,n} = (Q_{j+1}, M_{j+1})\cdot \ldots \cdot (Q_n, M_n);$$

then of course $R_n = R_j R_{j,n}$. Next we define the limit of the partial products

$$R_{j,\infty} = \lim_{n\to\infty} R_{j,n}\mathfrak{f}.$$

In view of [3], the limit above is well defined and has the same distribution as $\mathbf{R}$. Moreover $\mathbf{R} = R_j R_{j,\infty}$.

Proof of Theorem 4.1: positivity of the limiting constant. The proof formally follows the same scheme as that of Lemma 4.19 in [2].

Fix some $\mathfrak{u}_0 \in \operatorname{supp}\nu$. Then for any open ball U with centre $\mathfrak{u}_0$ and radius δ we have $\varepsilon := \mathbb{P}[\mathbf{R} \in U] > 0$ and we can write

$$\begin{aligned}\mathbb{P}[\inf_{\mathfrak{u}\in U}|R_n\mathfrak{u}|>q^k \text{ for some } n]&=\sum_n \mathbb{P}[\max_{i<n}\inf_{\mathfrak{u}\in U}|R_i\mathfrak{u}|\le q^k \text{ and } \inf_{\mathfrak{u}\in U}|R_n\mathfrak{u}|>q^k]\\
&=\frac{1}{\varepsilon}\sum_n \mathbb{P}[\max_{i<n}\inf_{\mathfrak{u}\in U}|R_i\mathfrak{u}|\le q^k \text{ and } \inf_{\mathfrak{u}\in U}|R_n\mathfrak{u}|>q^k]\mathbb{P}[R_{n,\infty}\in U]\\
&=\frac{1}{\varepsilon}\sum_n \mathbb{P}[\max_{i<n}\inf_{\mathfrak{u}\in U}|R_i\mathfrak{u}|\le q^k \text{ and } \inf_{\mathfrak{u}\in U}|R_n\mathfrak{u}|>q^k \text{ and } R_{n,\infty}\in U]\\
&\le\frac{1}{\varepsilon}\sum_n \mathbb{P}[\max_{i<n}\inf_{\mathfrak{u}\in U}|R_i\mathfrak{u}|\le q^k \text{ and } \inf_{\mathfrak{u}\in U}|R_n\mathfrak{u}|>q^k \text{ and } |\mathbf{R}|>q^k]\\
&\le\frac{1}{\varepsilon}\mathbb{P}[|\mathbf{R}|>q^k].\end{aligned}$$

Let

$$U_n=\Theta(R_n\mathfrak{u}_0,R_{n-1}\mathfrak{u}_0).$$

For sufficiently large k we have

$$\begin{aligned}\mathbb{P}[|\mathbf{R}|>q^k]&\ge\varepsilon\mathbb{P}[\inf_{\mathfrak{u}\in U}|R_n\mathfrak{u}|>q^k \text{ for some } n]\\
&\ge\varepsilon\mathbb{P}[|R_n\mathfrak{u}_0|-q^{-\Pi_n}\delta>q^k \text{ for some } n]\\
&\ge\varepsilon\mathbb{P}[U_n-(q^{-\Pi_n}+q^{-\Pi_{n-1}})\delta>2q^k \text{ for some } n]\\
&=\varepsilon\mathbb{P}[q^{-\Pi_{n-1}}(\Theta((Q_n,M_n)\mathfrak{u}_0,\mathfrak{u}_0)-(q^{-M_n}+1)\delta)>2q^k \text{ for some } n]\\
&\ge\varepsilon\mathbb{P}[\Theta((Q,M)\mathfrak{u}_0,\mathfrak{u}_0)-(q^{-M}+1)\delta>\eta]\mathbb{P}[\max_n q^{-\Pi_n}>2q^k/\eta]\\
&\ge C_0\mathbb{P}[\Theta((Q,M)\mathfrak{u}_0,\mathfrak{u}_0)-(q^{-M}+1)\delta>\eta]q^{-\alpha k}.\end{aligned}$$

It is enough to find η and δ which guarantee the positivity of the constant above. Since the group generated by μ is non-exceptional (Q,M) does not fix $\mathfrak{u}_0$ and we can find positive numbers η and θ such that

$$\mathbb{P}[\Theta((Q,M)\mathfrak{u}_0,\mathfrak{u}_0)>2\eta]=\theta>0.$$

Moreover, for sufficiently large N we have

$$\mathbb{P}[q^{-M}\ge N]\le\theta/2.$$

If we set $\delta=\eta/(N+1)$ we get

$$\begin{aligned}\mathbb{P}[\Theta((Q,M)\mathfrak{u}_0,\mathfrak{u}_0)&-(q^{-M}+1)\delta>\eta]\\
&\ge\mathbb{P}[\Theta((Q,M)\mathfrak{u}_0,\mathfrak{u}_0)>2\eta \text{ and } q^{-M}<N]\\
&\ge\mathbb{P}[\Theta((Q,M)\mathfrak{u}_0,\mathfrak{u}_0)>2\eta]-\mathbb{P}[q^{-M}\ge N]\ge\theta/2,\end{aligned}$$

which finishes the proof. ■

5. Examples. In this section we present two examples of concrete groups acting on homogeneous trees and being non-exceptional subgroups of the affine group of the tree. Next we formulate our main result in terms of these groups.

5.1. p-adic affine group. For a given prime number p, every rational number w can be written in the form

$$w = p^k \frac{a}{b}$$

where $a, b, k \in \mathbb{Z}$, $(a, b) = 1$ and $p \nmid ab$. For such w we define its evaluation

$$\upsilon_p(w) := k$$

and the ultra-metric norm

$$|w|_p := p^{-\upsilon_p(w)}.$$

The field $\mathbb{Q}_p$ of p-adic rationals is the completion of $\mathbb{Q}$ equipped with the ultra-metric distance $d(u, w) = |u - w|_p$. There exists a relationship between the p-adic rationals and an oriented tree of degree $p+1$. Observe that every ball $D(u, p^k)$ with centre u and radius p^k contains p balls with radius p^{k-1} and centres $u + ip^k$, where $0 \le i \le p-1$. Hence the set of all balls equipped with natural order given by inclusion forms a homogeneous tree of degree p.

If we take the origin $o := D(0, 1)$ then the corresponding height function h will be given by

$$h(D(u, p^k)) = -k.$$

We may identify isometrically the boundary $\partial^*\mathbb{T}$ of the tree with $\mathbb{Q}_p$: the decreasing sequence $\{D(u, p^{-k})\}_{k \in \mathbb{N}}$ corresponds to $u = \bigcap_{k\in\mathbb{N}} D(u, p^{-k})$ in $\mathbb{Q}_p$. We denote by $\mathrm{Aff}(\mathbb{Q}_p)$ the set of all p-adic affine mappings, i.e. of the form $u \mapsto gu = au + t$, for $g = \langle t, a\rangle$, where $a \in \mathbb{Q}_p^*$ and $t \in \mathbb{Q}_p$. Then the group can be realized as the group of matrices

$$\mathrm{Aff}(\mathbb{Q}_p) = \left\{ \begin{bmatrix} a & t \\ 0 & 1 \end{bmatrix} : a \in \mathbb{Q}_p^* \text{ and } t \in \mathbb{Q}_p \right\}.$$

All the elements of $\mathrm{Aff}(\mathbb{Q}_p)$ map balls onto balls preserving the inclusion order, so they constitute a closed and non-exceptional subgroup of $\mathrm{Aff}(\mathbb{T})$. The mapping $\langle t, a\rangle$ transforms a ball $D(0, 1)$ onto some ball with radius $|a|_p$, hence $h(\langle t, a\rangle) = \upsilon_p(a)$. If we take $\sigma = \langle 0, p\rangle$, then its unique fixed point $\mathfrak{f}^\sigma$ is 0, and with such a choice of σ we get the following decomposition:

$$\mathrm{Aff}(\mathbb{Q}_p) = \mathrm{Hor}(\mathbb{Q}_p) \rtimes \mathbb{Z}, \qquad \langle t, a\rangle = (\langle t, ap^{-k}\rangle, k), \qquad \text{where}$$

$$\mathrm{Hor}(\mathbb{Q}_p) = \left\{ \begin{bmatrix} a & t \\ 0 & 1 \end{bmatrix} \in \mathrm{Aff}(\mathbb{Q}_p) : |a| = 1 \right\}.$$

Notice that $|u| = |u|_p$ and $\|\langle t, ap^{-k}\rangle\| = |\langle t, ap^{-k}\rangle 0| = |t|_p$. Now we can apply Theorem 4.1 to the p-adic affine group and obtain

Corollary 5.1. *Let A, T, R be $\mathbb{Q}_p$-valued random variables with $A \neq 0$. Suppose that R and (T, A) are independent and satisfy $AR + T \stackrel{d}{=} R$. If*

- $\mathbb{E}[\upsilon_p(A)] < 0$;
- $\mathbb{E}[|A|_p^\alpha] = 1$ *for some positive* α;

- $\mathbb{E}[|T|_p^\alpha] < \infty$;
- $\mathbb{E}[|v_p(A)||A|_p] < \infty$;
- $\mathbb{P}[Au + T = u] < 1$ *for any* $u \in \mathbb{Q}_p$;
- $\mathbb{P}[v_p(A) = 1] > 0$,

then

$$\lim_{k\to\infty} \mathbb{P}[|R|_p > p^k] p^{\alpha k} = C > 0.$$

5.2. Lamplighter group. Another example of a subgroup of $\mathrm{Aff}(\mathbb{T})$ is the *lamplighter group*

$$G = \mathbb{Z}_q \wr \mathbb{Z},$$

i.e.

$$G = \Sigma_q \rtimes \mathbb{Z}$$

where $\Sigma_q = \{\eta \in \mathbb{Z}_q^{\mathbb{Z}} : \operatorname{supp}\eta \text{ is finite}\}$ is a group with pointwise multiplication, and $\mathbb{Z}$ acts on Σ_q by the formula $k(\eta)(i) = \eta(i-k)$. For any $k \in \mathbb{Z}$ we can introduce an equivalence relation $\overset{k}{\sim}$ on Σ_q such that $\eta \overset{k}{\sim} \xi$ if and only if $\eta(i) = \xi(i)$ for $i \le k$. We will denote by η_k the equivalence class of an element η with respect to the relation $\overset{k}{\sim}$. In a natural way we can identify the tree $\mathbb{T}$ of degree $q+1$ with the set

$$\{\eta_k : \eta \in \Sigma_q \text{ and } k \in \mathbb{Z}\}$$

such that η_{k+1} is an ancestor of η_k. Then the lamplighter group G acts on $\mathbb{T}_q$ by isometries:

$$(\eta, n)\xi_k = (\eta(n\xi))_{k+n}.$$

The bottom boundary $\partial^*\mathbb{T}$ can be identified with the set

$$\{\tau \in \mathbb{Z}_q^{\mathbb{Z}} : \operatorname{supp}\tau \subseteq (k, \infty) \text{ for some } k \in \mathbb{Z}\}.$$

Then taking $o = \mathbf{0}_0$ (the equivalence class of the constantly zero function with respect to the relation $\overset{0}{\sim}$) and $\sigma := (\mathbf{0}, -1)$ we get $\mathfrak{f} = \mathbf{0}$. Since the random variables $\Phi_n = (\Upsilon_1, N_1)\cdots(\Upsilon_n, N_n)o$ converge almost surely in both pointwise and Θ topology to the same limit, by Kaimanovich [8] $(\partial^*\mathbb{T}, \nu)$ is the Poisson boundary of (Γ, μ), where ν is given by

$$\nu(U) = \mathbb{P}[\lim_{n\to\infty} \Phi_n \in U].$$

Every (η, n) we can be written as $(\eta, 0)(\mathbf{0}, 1)^n$, so $\phi((\eta, n)) = n$. Then

$$\Theta(f, g) = q^{-\min\{k: f(k) \ne g(k)\}},$$

so $|f| = q^{-\min\{k: f(k)\ne 0\}}$ and $\|(\eta, 0)\| = |\eta|$.

Applying Theorem 4.1 we obtain

Corollary 5.2. *If a random variable* (Υ, N) *with law* μ *satisfies*

- $\mathbb{E}[N] > 0$;
- *there is* $\alpha > 0$ *such that* $\mathbb{E}[q^{-\alpha N}] = 1$;

- $\mathbb{E}[q^{-\min\{k:\Upsilon(k)\neq 0\}\alpha}] < \infty$;
- $\mathbb{E}[Nq^{-\alpha N}] = m \in (0,\infty)$;
- $\mathbb{P}[(\Upsilon, N)f = f] < 1$ *for any* $f \in \partial^*\mathbb{T}$;
- $\mathbb{P}[N = 1] > 0$;

then

$$\lim_{k\to\infty} \nu(\{f : \operatorname{supp} f \not\subseteq (-k,\infty)\})q^{\alpha k} = C > 0.$$

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