COLLOQUIUM MATHEMATICUM

VOL. 118 2010 NO. 2

A CHARACTERIZATION OF FOURIER TRANSFORMS

BY

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Abstract. The aim of this paper is to show that, in various situations, the only continuous linear (or not) map that transforms a convolution product into a pointwise product is a Fourier transform. We focus on the cyclic groups $\mathbb{Z}/n\mathbb{Z}$, the integers $\mathbb{Z}$, the torus $\mathbb{T}$ and the real line. We also ask a related question for the twisted convolution.

1. Introduction. The aim of this paper is to characterize the Fourier transform by some of its properties. Indeed, the Fourier transform is well known to change a translation into a modulation (multiplication by a character) and vice-versa and to change a convolution into a pointwise product. Moreover, these are some of its main features and are fundamental properties in many of its applications. The aim of this paper is to show that the Fourier transform is, to some extent, uniquely determined by some of these properties.

Before going on, let us introduce some notation. Let G be a locally compact Abelian group with Haar measure ν and let $\hat{G}$ be the dual group. Operations on G will be denoted additively. Let us recall that the convolution on G is defined for $f,g\in L^1(G)$ by

$$f*g(x)=\int_G f(t)g(x-t)\,d\nu(t)$$

(and $f*g\in L^1(G)$) while the Fourier transform is defined by

$$\mathcal{F}(f)(\gamma)=\hat{f}(\gamma)=\int_G f(t)\,\overline{\gamma(t)}\,d\nu(t).$$

We will here mainly focus on the following four cases, $G=\hat{G}=\mathbb{Z}/n\mathbb{Z}$, $G=\mathbb{Z}$ and $\hat{G}=\mathbb{T}$ and vice versa or $G=\hat{G}=\mathbb{R}$; our results will then easily extend to products to such groups.

We will here focus on two types of results. The first ones concerns the characterization of the Fourier transform as being essentially the only con-

2010 *Mathematics Subject Classification*: 42A38, 42A85, 42B10, 43A25.
Key words and phrases: Fourier transform, convolution.

DOI: 10.4064/cm118-2-12

tinuous linear transform that changes a convolution product into a pointwise product. To our knowledge the first results in that direction appear in the work of Lukacs [Lu1, Lu2], pursued in [Em], and an essentially complete result appeared in [Fi] for all LCA groups, under the mild additional constraint that the transform has a reasonable kernel. We will show here that this hypothesis can be removed. Further, a striking result, recently proved by Alekser, Artstein-Avidan, and Milman [AM1, AM2], is that, to some extent, continuity and linearity may be removed as well. More precisely, let us denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz functions on $\mathbb{R}^d$ and by $\mathcal{S}'(\mathbb{R}^d)$ the Schwartz (tempered) distributions.

Theorem (Alekser, Artstein-Avidan, Milman). *Let $T:\mathcal{S}(\mathbb{R}^d)\to\mathcal{S}(\mathbb{R}^d)$ be a mapping that extends to a mapping $T:\mathcal{S}'(\mathbb{R}^d)\to\mathcal{S}'(\mathbb{R}^d)$ that is bijective and such that*

(i) *for every $f\in\mathcal{S}(\mathbb{R}^d)$ and $g\in\mathcal{S}'(\mathbb{R}^d)$, $T(f*g)=T(f)\cdot T(g)$;*
(ii) *for every $f\in\mathcal{S}(\mathbb{R}^d)$ and $g\in\mathcal{S}'(\mathbb{R}^d)$, $T(f\cdot g)=T(f)*T(g)$.*

Then there exists $B\in\mathcal{M}_n(\mathbb{R})$ with $\det B=1$ such that $T(f)=\mathcal{F}(f)\circ B$.

Note that T is not assumed to be linear or continuous. We will adapt the proof of this theorem to obtain an analogous result on the cyclic group. This has the advantage of highlighting the main features which come into the proof. The main difference is that in the above theorem, we assume that T sends smooth functions into smooth functions. In the case of the cyclic group, we do not have such functions at hand and are therefore led to assume some mild continuity; see Theorem 2.2 for a precise statement.

A second set of results has its origin in the work of Cooper [Co1, Co2]. Here one considers the Fourier transform as an intertwining operator between two groups of transforms acting on L^p-spaces. In order to state the precise result, let us define, for $\alpha\in\mathbb{R}$ and f a function on $\mathbb{R}$, $\tau_\alpha f(t)=f(t+\alpha)$. Further, if $\varphi:\mathbb{R}\to\mathbb{R}$, let $M_\alpha^{(\varphi)}f(t)=e^{i\alpha\varphi(t)}f(t)$. It is easy to see that $\mathcal{F}\tau_\alpha=M_\alpha^{(t)}\mathcal{F}$ and $\mathcal{F}M_\alpha^{(-t)}=\tau_\alpha\mathcal{F}$, i.e. the Fourier transform intertwines translations and modulations. The converse is also true. More precisely:

Theorem (Cooper). *Let $T:L^2(\mathbb{R})\to L^2(\mathbb{R})$ be a continuous linear transformation such that there exist two measurable functions $\varphi,\psi:\mathbb{R}\to\mathbb{R}$ for which*

$$T\tau_\alpha=M_\alpha^{(\varphi)}T\quad\text{and}\quad TM_\alpha^{(\psi)}=\tau_\alpha T.$$

Then $\varphi(t)=bt+c$, $\psi(t)=bt+d$ with $b,c,d\in\mathbb{R}$ and $T=\mathcal{F}$.

We will extend this theorem to $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}$.

The article is organized as follows. In the next section, we will prove the results for the groups $G=\mathbb{Z}$ and $G=\mathbb{Z}/n\mathbb{Z}$, while Section 3 is devoted to the cases of $G = \mathbb{T}^d$ and $G = \mathbb{R}^d$. We conclude with some questions concerning the twisted convolution.

Before going on, let us introduce some more notation. If $E \subset G$, we will denote by χ_E the function on G given by $\chi_E(k) = 1$ if $k \in E$ and $\chi_E(k) = 0$ otherwise. The Kronecker symbol is denoted by $\delta_{j,k}$.

2. The cyclic group and the integers. In this section, we consider $G = \mathbb{Z}/n\mathbb{Z}$ or $G = \mathbb{Z}$. We will write $\mathcal{C}(\hat{G})$ for the set of n-periodic sequences when $G = \hat{G} = \mathbb{Z}/n\mathbb{Z}$ or of continuous functions on $\hat{G} = \mathbb{T}$ if $G = \mathbb{Z}$. Our first result is the following:

THEOREM 2.1. *Let $G = \mathbb{Z}/n\mathbb{Z}$ or $G = \mathbb{Z}$. Let T be a continuous linear map $T : L^1(G) \to \mathcal{C}(\hat{G})$ such that $T(f * g) = T(f).T(g)$. Then there exists $E \subset \hat{G}$ and a map $\sigma : \hat{G} \to \hat{G}$ such that, for $f \in L^1(G)$ and almost every $\eta \in \hat{G}$, $T(f)(\eta) = \chi_E(\eta)\widehat{f}(\sigma(\eta))$. Moreover, σ is measurable if $G = \mathbb{Z}$.*

Proof. Let $\delta_k = (\delta_{j,k})_{j\in G} \in L^1(G)$. Then $\delta_k * \delta_l = \delta_{k+l}$, so that

$$T(\delta_{k+l}) = T(\delta_k * \delta_l) = T(\delta_k)T(\delta_l). \tag{2.1}$$

In particular, for each $\eta \in \hat{G}$, the map $\pi_\eta : k \mapsto T(\delta_k)(\eta)$ is a group homomorphism from G to $\mathbb{C}$.

First note $\pi_\eta(0) = \pi_\eta(k)\pi_\eta(-k)$ so that if π_η vanishes somewhere, it vanishes at 0. Conversely $\pi_\eta(k) = \pi_\eta(k)\pi_\eta(0)$ so that if π_η vanishes at 0, it vanishes everywhere. Further, $\pi_\eta(0) = \pi_\eta(0)^2$ so that $\pi_\eta(0) = 0$ or 1.

We will now assume that $\pi_\eta(0) = 1$ and exploit $\pi_\eta(k+1) = \pi_\eta(k)\pi_\eta(1)$ which implies that $\pi_\eta(k) = \pi_\eta(1)^k$. We now need to distinguish two cases:

- If $G = \mathbb{Z}/n\mathbb{Z}$, then $1 = \pi_\eta(0) = \pi_\eta(n) = \pi_\eta(1)^n$, $\pi_\eta(1)$ is an nth root of unity, i.e. $T(\delta_1)(\eta) = e^{2i\pi\sigma(\eta)/n}$ for some $\sigma(\eta) \in \{0, 1 \dots, n-1\} = \mathbb{Z}/n\mathbb{Z}$. It follows that $T(\delta_k)(\eta) = e^{2i\pi k\sigma(\eta)/n}$.
- If $G = \mathbb{Z}$, as T was assumed to be continuous $L^1(G) \to \mathcal{G}(\hat{G})$, there is a constant $C > 0$ such that, for every $f \in L^1(G)$, $\|Tf\|_\infty \le C\|f\|_1$. In particular, for every $k \in \mathbb{Z}$ and every $m \in \hat{G} = \mathbb{T}$,

  $$|\pi_\eta(1)^k| = |[T(\delta_1)(\eta)]^k| = |T(\delta_k)(\eta)| \le C\|\delta_k\|_1 = C,$$

  thus, by letting $k \to \pm\infty$, we see that $\pi_\eta(1)$ is a complex number of modulus 1 (it is not 0 since $\pi_\eta(0) \neq 0$). We may thus write $T(\delta_1)(\eta) = e^{2i\pi\sigma(\eta)}$ for some $\sigma(\eta) \in [0, 1] \simeq \mathbb{T} = \hat{G}$. Moreover, as $\eta \to T(\delta_1)(\eta)$ is measurable, we may assume that σ is measurable as well.

Let us now define $E = \{\eta \in \hat{G} : T(\delta_k)(\eta) = 0\ \forall k \in G\}$. Then, by linearity and continuity of T, for $f \in L^1(G)$,

$$\begin{aligned} Tf(\eta) &= T\Big(\sum_{k\in G} f(k)\delta_k\Big)(\eta) = \sum_{k\in G} f(k)T(\delta_k)(\eta) \\ &= \begin{cases} \displaystyle\sum_{k\in G} f(k)\chi_E(\eta)e^{2i\pi k\sigma(\eta)/n} & \text{if } G=\mathbb{Z}/n\mathbb{Z} \\ \displaystyle\sum_{k\in G} f(k)\chi_E(\eta)e^{2i\pi k\sigma(\eta)} & \text{if } G=\mathbb{Z} \end{cases} \\ &= \chi_E(\eta)\widehat{f}(\sigma(\eta)), \end{aligned}$$

which completes the proof. ∎

Remark. Using tensorization, we may extend the result with no difficulty to $G=\prod_{i\in I}\mathbb{Z}/n_i\mathbb{Z}\times\mathbb{Z}^d$.

We will now adapt the proof of [AM1, AM2] to show that on $\mathbb{Z}_n$, a bijective transform that maps a product into a convolution is essentially a Fourier transform. We will need some notation. We will consider the elements $\mathbf{0}=(0,\dots,0)$ and $\mathbf{1}=(1,\dots,1)$ of $L^1(\mathbb{Z}/n\mathbb{Z})$. Further, if $a\in L^1(\mathbb{Z}/n\mathbb{Z})$ we will write

$$\mathbb{E}[a]=\sum_{j=0}^{n-1}a(j).$$

We can now state the main theorem:

Theorem 2.2. *Let $\mathcal{T}: L^1(\mathbb{Z}/n\mathbb{Z})\to L^1(\mathbb{Z}/n\mathbb{Z})$ be a bijective transformation (not necessarily linear) such that the map $\mathbb{C}\to L^1(\mathbb{Z}/n\mathbb{Z})$, $c\mapsto\mathcal{T}(c\mathbf{1})$, is continuous. Assume that*

(i) *for every $a,b\in L^1(\mathbb{Z}/n\mathbb{Z})$, $\mathcal{T}(a\cdot b)=\mathcal{T}(a)\cdot\mathcal{T}(b)$;*
(ii) *for every $a,b\in L^1(\mathbb{Z}/n\mathbb{Z})$, $\mathcal{T}(a*b)=\mathcal{T}(a)*\mathcal{T}(b)$.*

Then there exists $\eta\in\{1,\dots,n-1\}$ that has no common divisor with n such that either

- *for every $j\in\mathbb{Z}/n\mathbb{Z}$ and every $a\in L^1(\mathbb{Z}/n\mathbb{Z})$, $\mathcal{T}(a)(\eta j)=a(j)$, or*
- *for every $j\in\mathbb{Z}/n\mathbb{Z}$ and every $a\in L^1(\mathbb{Z}/n\mathbb{Z})$, $\mathcal{T}(a)(\eta j)=\overline{a(j)}$.*

Remark. The fact that η has no common divisor with n implies that the map $j\mapsto j\eta$ is a permutation of $\{0,\dots,n-1\}$ so that the map $\mathcal{T}$ is actually fully determined.

Corollary 2.3. *Let $\mathcal{T}: L^1(\mathbb{Z}/n\mathbb{Z})\to L^1(\mathbb{Z}/n\mathbb{Z})$ be a bijective transformation (not necessarily linear) such that the map $\mathbb{C}\to L^1(\mathbb{Z}/n\mathbb{Z})$, $c\mapsto\mathcal{T}(c\mathbf{1})$, is continuous. Assume that*

(i) *for every $a,b\in L^1(\mathbb{Z}/n\mathbb{Z})$, $\mathcal{T}(a\cdot b)=\mathcal{T}(a)*\mathcal{T}(b)$;*
(ii) *for every $a,b\in L^1(\mathbb{Z}/n\mathbb{Z})$, $\mathcal{T}(a*b)=\mathcal{T}(a)\cdot\mathcal{T}(b)$.*

Then there exists $\eta \in \{1,\dots,n-1\}$ that has no common divisor with n such that either $\mathcal{T}(a)(\eta j) = \hat{a}(j)$ for every $j \in \mathbb{Z}/n\mathbb{Z}$ and every $a \in L^1(\mathbb{Z}/n\mathbb{Z})$, or $\mathcal{T}(a)(\eta j) = \overline{\hat{a}(j)}$ for every $j \in \mathbb{Z}/n\mathbb{Z}$ and every $a \in L^1(\mathbb{Z}/n\mathbb{Z})$.

Proof of Corollary 2.3. Apply Theorem 2.2 to $\tilde{\mathcal{T}} = \mathcal{F}^{-1}\mathcal{T}$. ∎

COROLLARY 2.4. *Let $\mathcal{T} : L^1(\mathbb{Z}/n\mathbb{Z}) \to L^1(\mathbb{Z}/n\mathbb{Z})$ be a bijective transformation (not necessarily linear) such that the map $\mathbb{C} \to L^1(\mathbb{Z}/n\mathbb{Z})$, $c \mapsto \mathcal{T}(c\mathbf{1})$, is continuous. Assume that for every $a \in L^1(\mathbb{Z}/n\mathbb{Z})$, $\mathcal{T}^2 a(k) = a(-k)$ and that one of the following two identities holds:*

(i) *for every $a, b \in L^1(\mathbb{Z}/n\mathbb{Z})$, $\mathcal{T}(a.b) = \mathcal{T}(a) * \mathcal{T}(b)$;*
(ii) *for every $a, b \in L^1(\mathbb{Z}/n\mathbb{Z})$, $\mathcal{T}(a * b) = \mathcal{T}(a) \cdot \mathcal{T}(b)$.*

Then there exists $\eta \in \{1,\dots,n-1\}$ that has no common divisor with n such that either $\mathcal{T}(a)(\eta j) = \hat{a}(j)$ for every $j \in \mathbb{Z}/n\mathbb{Z}$ and every $a \in L^1(\mathbb{Z}/n\mathbb{Z})$, or $\mathcal{T}(a)(\eta j) = \overline{\hat{a}(j)}$ for every $j \in \mathbb{Z}/n\mathbb{Z}$ and every $a \in L^1(\mathbb{Z}/n\mathbb{Z})$.

Proof of Corollary 2.4. If $\mathcal{T}^2 a(k) = a(-k)$ then if one of the identities holds, so does the other, so that Corollary 2.3 gives the result. ∎

Proof of Theorem 2.2. The proof is in several steps that are similar to those in [AM2]. The first one consists in identifying the image under $\mathcal{T}$ of some particular elements of $L^1(\mathbb{Z}/n\mathbb{Z})$:

STEP 1. *We have $\mathcal{T}(\delta_0) = \delta_0$, $\mathcal{T}(\mathbf{0}) = \mathbf{0}$ and $\mathcal{T}(\mathbf{1}) = \mathcal{T}(\mathbf{1})$. Moreover, there is a $k \in \{-1, 1\}$ and an $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$ such that, if we define $\beta : \mathbb{C} \to \mathbb{C}$ by $\beta(0) = 0$ and $\beta(c) = (c/|c|)^k |c|^\alpha$ for $c \neq 0$, then $\mathcal{T}(c\mathbf{1}) = \beta(c)\mathbf{1}$.*

Indeed, as $\mathcal{T}(a \cdot b) = \mathcal{T}(a) \cdot \mathcal{T}(b)$, we immediately get the following:

$$\mathcal{T}(c_1 c_2 \mathbf{1}) = \mathcal{T}(c_1\mathbf{1}) \cdot \mathcal{T}(c_2\mathbf{1}) \quad \text{and} \quad \mathcal{T}(c_1\delta_j) = \mathcal{T}(c_1\mathbf{1}) \cdot \mathcal{T}(\delta_j)$$

while from $\mathcal{T}(a * b) = \mathcal{T}(a) * \mathcal{T}(b)$ we deduce that

$$\mathcal{T}(\delta_{j+k}) = \mathcal{T}(\delta_j)\mathcal{T}(\delta_k) \quad \text{and} \quad \mathcal{T}(a) = \mathcal{T}(\delta_0 * a) = \mathcal{T}(\delta_0) * \mathcal{T}(a).$$

Applying this last identity to $a = \mathcal{T}^{-1}(\delta_0)$ we get $\delta_0 = \mathcal{T}(\delta_0) * \delta_0 = \mathcal{T}(\delta_0)$.

Further, $a = a \cdot \mathbf{1}$, thus $\mathcal{T}(a) = \mathcal{T}(a) \cdot \mathcal{T}(\mathbf{1})$, and applying this again to $a = \mathcal{T}^{-1}(b)$, we have $b = b \cdot \mathcal{T}(\mathbf{1})$ for all $b \in \ell^2_n$, thus $\mathcal{T}(\mathbf{1}) = \mathbf{1}$. Similarly, $\mathbf{0} = a \cdot \mathbf{0}$, thus $\mathcal{T}(\mathbf{0}) = \mathcal{T}(a) \cdot \mathcal{T}(\mathbf{0})$, and applying this to $a = \mathcal{T}^{-1}(\mathbf{0})$ we get $\mathcal{T}(\mathbf{0}) = \mathbf{0} \cdot \mathcal{T}(\mathbf{0}) = \mathbf{0}$.

Finally, $\mathbb{E}[a]\mathbf{1} = a * 1$, thus $\mathcal{T}(\mathbb{E}[a]\mathbf{1}) = \mathcal{T}(a * \mathbf{1}) = \mathcal{T}(a) * \mathbf{1} = \mathbb{E}[\mathcal{T}(a)]\mathbf{1}$. As every $c \in \mathbb{C}$ may be written $c = E[(c/n)\mathbf{1}]$, we may define $\beta(c) = E[\mathcal{T}((c/n)\mathbf{1})]$ so that $T(c\mathbf{1}) = \beta(c)\mathbf{1}$. Note that β is continuous since we have assumed that T acts continuously on constants and as $\mathcal{T}$ is one-to-one, so is β. Moreover, β is multiplicative:

$$\beta(c_1c_2)\mathbf{1} = \mathcal{T}(c_1c_2\mathbf{1}) = \mathcal{T}(c_1\mathbf{1}) \cdot \mathcal{T}(c_2\mathbf{1}) = \beta(c_1)\beta(c_2)\mathbf{1}.$$

It is then easy to check that there is a $k \in \{-1,1\}$ and an $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$ such that $\beta(0) = 0$ and $\beta(c) = (c/|c|)^k |c|^\alpha$.

We will now take care of the image of δ_j, $j = 0,\dots,n-1$.

STEP 2. *There is an $\eta \in \{1,\dots,n-1\}$ with no common divisor with n such that $\mathcal{T}(\delta_j) = \delta_{\eta j}$.*

Assume that $k \neq l \in \operatorname{supp}\mathcal{T}(\delta_j)$, thus $\delta_k \cdot \mathcal{T}(\delta_j) \neq \mathbf{0}$ and $\delta_l \cdot \mathcal{T}(\delta_j) \neq \mathbf{0}$. Let $a = \mathcal{T}^{-1}(\delta_k)$, $b = \mathcal{T}^{-1}(\delta_l)$. Then

$$a \cdot \delta_j = \mathcal{T}^{-1}(\delta_k) \cdot \mathcal{T}^{-1}(\mathcal{T}(\delta_j)) = \mathcal{T}^{-1}(\delta_k \cdot \mathcal{T}(\delta_j)) \neq \mathcal{T}^{-1}(\mathbf{0})$$

since $\mathcal{T}$ is one-to-one. From Step 1, we know that $\mathcal{T}^{-1}(\mathbf{0}) = \mathbf{0}$, therefore $a \cdot \delta_j \neq \mathbf{0}$. For the same reason, $b \cdot \delta_j \neq \mathbf{0}$. In particular, $a \cdot b \neq \mathbf{0}$, thus $\mathcal{T}(a \cdot b) \neq \mathbf{0}$. But this contradicts $\mathcal{T}(a \cdot b) = \delta_k \cdot \delta_l$ with $k \neq l$.

It follows that, for each $j \in \{1,\dots,n-1\}$, there exist $c_j \in \mathbb{C} \setminus \{0\}$ and $\sigma(j) \in \{0,\dots,n-1\}$ such that $\mathcal{T}(\delta_j) = c_j \delta_{\sigma(j)}$. But then

$$\mathbf{1} = \mathcal{T}(\mathbf{1}) = \mathcal{T}(\mathbf{1} * \delta_j) = \mathcal{T}(\mathbf{1}) * \mathcal{T}\delta_j = c_j \mathbf{1} * \delta_j = c_j \mathbf{1},$$

thus $c_j = 1$. As $\mathcal{T}$ is one-to-one, it follows that $\sigma(j) \in \{1,\dots,n-1\}$ and that σ is a permutation.

Next,

$$\delta_{\sigma(j+k)} = \mathcal{T}(\delta_{j+k}) = \mathcal{T}(\delta_j * \delta_k) = \mathcal{T}(\delta_j) * \mathcal{T}(\delta_k) = \delta_{\sigma(j)} * \delta_{\sigma(k)} = \delta_{\sigma(j)+\sigma(k)}.$$

Thus $\sigma(j+k) = \sigma(j) + \sigma(k)$ and therefore $\sigma(j) = j\sigma(1)$. Further, the fact that σ is a permutation implies that $\sigma(1)$ has no common divisor with n (Bézout's Theorem).

STEP 3. *Conclusion.*

We can now prove that $\mathcal{T}$ is of the expected form. Fix $j \in \{0,\dots,n-1\}$ and $a \in \ell^2_n$. Let $k = \sigma^{-1}(j)$ so that $\mathcal{T}(\delta_k) = \delta_j$. Then

$$\begin{aligned}\mathcal{T}(a)(j)\delta_j &= \mathcal{T}(a) \cdot \delta_j = \mathcal{T}(a) \cdot \mathcal{T}(\delta_k) = \mathcal{T}(a \cdot \delta_k)\\ &= \mathcal{T}(a(k)\mathbf{1} \cdot \delta_k) = \beta(a(k))\mathbf{1} \cdot \mathcal{T}(\delta_k) = \beta(a(k))\delta_j.\end{aligned}$$

It follows that

$$\mathcal{T}(a)(j) = \beta(a \circ \sigma^{-1}(j)) = \left(\frac{a \circ \sigma^{-1}(j)}{|a \circ \sigma^{-1}(j)|}\right)^k |a \circ \sigma^{-1}(j)|^\alpha.$$

We want to prove that $\alpha = 1$. But

$$\begin{aligned}\mathbb{E}[\mathcal{T}(a)]\mathbf{1} &= \mathcal{T}(a) * \mathbf{1} = \mathcal{T}(a) * \mathcal{T}(\mathbf{1}) = \mathcal{T}(a * \mathbf{1}) = \mathcal{T}(\mathbb{E}[a]\mathbf{1})\\ &= (\mathbb{E}[a]/\overline{\mathbb{E}[a]})^k |\mathbb{E}[a]|^\alpha \mathbf{1}\end{aligned}$$

so that $\mathbb{E}[\mathcal{T}(a)] = (\mathbb{E}[a]/|\mathbb{E}[a]|)^k|\mathbb{E}[a]|^\alpha$ or, in other words,

$$\sum_{l=0}^{n-1}\left(\frac{a(l)}{|a(l)|}\right)^k|a(l)|^\alpha = \sum_{j=0}^{n-1}\left(\frac{a(\sigma^{-1}(j))}{|a(\sigma^{-1}(j))|}\right)^k|a(\sigma^{-1}(j))|^\alpha$$
$$= \left(\frac{\sum_{j=0}^{n-1}a_j}{\sum_{j=0}^{n-1}|a_j|}\right)^k\Big|\sum_{j=0}^{n-1}a_j\Big|^\alpha.$$

If we take $a(0) = 1$, $a(1) = t > 0$ and $a(j) = 0$ for $j = 2,\dots,n-1$, this reduces to $1 + t^\alpha = (1+t)^\alpha$. This implies that $\alpha = 1$ (which is most easily seen by differentiating and letting $t \to 0$). It follows that $\beta(c) = c$ or $\bar{c}$ according to $k = 1$ or -1. ∎

Remark. The proof adapts with no difficulty to any finite Abelian group. To prove the same result on $\mathbb{Z}$, it is best to first compose $\mathcal{T}$ with a Fourier transform and then to adapt the proof in [AM2] from the real line to the torus. We refrain from giving the details here.

The proof given here follows the lines of those given in [AM2] (up to the ordering and the removal of technicalities that are useless in the finite group setting). The main difference is that we need to assume that $\mathcal{T}$ acts continuously on constants. In [AM2] this hypothesis is replaced by the fact that $\mathcal{T}$ sends smooth functions to smooth functions.

Finally, it should also be noted that hypotheses (i) and (ii) are only used when either a or b is either a constant $c\mathbf{1}$ or a Dirac δ_j.

We will conclude this section with a Cooper like theorem. Let us first introduce some notation. For $\varphi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$ and $k \in \mathbb{Z}/n\mathbb{Z}$, we define the following two linear operators $L^1(\mathbb{Z}/n\mathbb{Z}) \to L^1(\mathbb{Z}/n\mathbb{Z})$:

$$\tau_k a(j) = a(j+k) \quad \text{and} \quad M_k^{(\varphi)}a(j) = e^{k\varphi(j)}a(j).$$

Note that actually $\varphi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}/2i\pi\mathbb{Z}$. As is well known, if $\varphi(j) = 2i\pi j/n$ for some $k \in \mathbb{Z}/n\mathbb{Z}$, then $\mathcal{F}\tau_{-k} = M_k^{(\varphi)}\mathcal{F}$ and $\mathcal{F}M_k^{(\varphi)} = \tau_k\mathcal{F}$.

We can now state the following:

Theorem 2.5. *Let $\mathcal{T} : L^1(\mathbb{Z}/n\mathbb{Z}) \to L^1(\mathbb{Z}/n\mathbb{Z})$ be a continuous linear operator such that there exist two maps $\varphi,\psi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$ for which*

$$\mathcal{T}\tau_k = M_k^{(\varphi)}\mathcal{T} \quad \text{and} \quad \mathcal{T}M_k^{(\psi)} = \tau_k\mathcal{T}.$$

Then there exist $k_0, m_0, m_1 \in \mathbb{Z}/n\mathbb{Z}$, $c \in \mathbb{C}$ such that

$$\varphi(j) = \frac{2i\pi}{n}(k_0 j + m_0), \qquad \psi(j) = \frac{2i\pi}{n}(-k_0 j + m_1),$$
$$\mathcal{T}(a)(l) = ce^{2i\pi l m_1/n}\hat{a}(k_0 l + m_0).$$

Proof. Without loss of generality, we may assume that $\mathcal{T}\neq 0$. First note that the conditions are equivalent to

$$(2.2)\quad \text{(a) } \mathcal{T}(\delta_k*a)(l)=e^{-k\varphi(l)}\mathcal{T}(a)(l)\quad\text{and}\quad \text{(b) } \mathcal{T}(e^{-k\psi(\cdot)}a)=\delta_k*\mathcal{T}(a).$$

Note that these two expressions are n-periodic in k so that φ and ψ take their values in $\{0,2i\pi/n,\dots,2i\pi(n-1)/n\}$.

First, (2.2)(a) implies that

$$\mathcal{T}(\delta_j)(l)=\mathcal{T}(\delta_j*\delta_0)(l)=e^{-j\varphi(l)}\mathcal{T}(\delta_0)(l).$$

Next, (2.2)(b) implies that

$$\begin{aligned}e^{-k\psi(j)}\mathcal{T}(\delta_j)(l)&=T(e^{-k\psi(j)}\delta_j)(l)=T(e^{-k\psi(\cdot)}\delta_j)(l)\\&=\delta_k*T(\delta_j)(l)=T(\delta_j)(l-k).\end{aligned}$$

In particular, $\mathcal{T}(\delta_j)(l)=e^{l\psi(j)}\mathcal{T}(\delta_j)(0)$, thus

$$\mathcal{T}(\delta_j)(l)=e^{l\psi(j)-j\varphi(0)}\mathcal{T}(\delta_0)(0).$$

From linearity, we thus infer that for $a\in\ell^2_n$,

$$\mathcal{T}(a)(l)=\sum_{j=0}^{n-1}a(j)\mathcal{T}(\delta_j)(l)=\Big(\sum_{j=0}^{n-1}a(j)e^{l\psi(j)-j\varphi(0)}\Big)\mathcal{T}(\delta_0)(0).$$

As we assumed that $\mathcal{T}\neq 0$, we thus have $\mathcal{T}(\delta_0)(0)\neq 0$. Then (2.2) reads

$$\sum_{j=0}^{n-1}a(j)e^{l\psi(j+k)-(j+k)\varphi(0)}=\sum_{j=0}^{n-1}a(j)e^{l\psi(j)-j\varphi(0)-k\varphi(l)},$$

thus $l\psi(j+k)-(j+k)\varphi(0)=l\psi(j)-j\varphi(0)-k\varphi(l)$ for all $j,k,l\in\mathbb{Z}/n\mathbb{Z}$ (modulo $2i\pi/n$). Taking $k=1$, we get

$$\varphi(l)-\varphi(0)=(\psi(j)-\psi(j+1))l$$

so that φ and ψ are "affine". More precisely, $\varphi(l)=(\psi(0)-\psi(1))l+\varphi(0)$ modulo $2i\pi/n$ and, as φ takes its values in $\frac{2i\pi}{n}\mathbb{Z}/n\mathbb{Z}$, $\varphi(l)=\frac{2i\pi}{n}(k_0l+m_0)$ (modulo $2i\pi/n$) with $k_0,m_0\in\{0,\dots,n-1\}$ and $b\in\mathbb{C}$. Further, $\psi(j+1)=\psi(j)+\varphi(0)-\varphi(1)$, thus $\psi(j)=\psi(0)+j(\varphi(0)-\varphi(1))=\frac{2i\pi}{n}(-k_0j+m_1)$ (again modulo $2i\pi/n$).

We thus conclude that

$$\mathcal{T}(a)(l)=e^{2i\pi\frac{lm_1}{n}}\sum_{j=0}^{n-1}a(j)e^{-2i\pi\frac{k_0l+m_0}{n}j},$$

as expected. ∎

3. The real line and the torus. We now consider the case $G=\mathbb{R}^d$ resp. $G=\mathbb{T}^d$ so that $\hat{G}=\mathbb{R}^d$ resp. $\hat{G}=\mathbb{Z}^d$. To simplify notation, we write $\mathcal{C}(\mathbb{Z}^d)=L^\infty(\mathbb{Z}^d)$.

THEOREM 3.1. *Let $d \geq 1$ be an integer and $G = \mathbb{R}^d$ or $G = \mathbb{T}^d$. Let T be a continuous linear operator $L^1(G) \to \mathcal{C}(\hat{G})$ such that $T(f * g) = T(f)T(g)$. Then there exists a set $E \subset G$ and a function $\varphi : \hat{G} \to \hat{G}$ such that $T(f)(\xi) = \chi_E(\xi)\widehat{f}(\varphi(\xi))$.*

Proof. Let us fix $\xi \in \hat{G}$ and consider the continuous linear functional T_ξ on $L^1(G)$ given by $T_\xi(f) = T(f)(\xi)$. Then there exists a bounded function h_ξ on G such that $T_\xi(f) = \int_G f(t)h_\xi(t)\,dt$. There is no loss of generality in assuming that $h_\xi \neq 0$.

Let now A, B be sets of finite measure. Then Fubini's Theorem implies that

$$\begin{aligned}(3.3)\qquad \int_{A\times B} h_\xi(s+t)\,ds\,dt &= \int_{\mathbb{R}^d} \chi_A * \chi_B(t)h_\xi(t)\,dt = T(\chi_A * \chi_B)(\xi)\\ &= T(\chi_A)(\xi)T(\chi_B)(\xi) = \int_A h_\xi(t)\,dt \int_B h_\xi(t)\,dt.\end{aligned}$$

Now let φ_n be defined on G^2 by

$$\varphi_n(x,y) = \begin{cases} (h_\xi(x+y) - h_\xi(x)h_\xi(y))\chi_{[-n,n]}(x)\chi_{[-n,n]}(y) & \text{if } G = \mathbb{R},\\ h_\xi(x+y) - h_\xi(x)h_\xi(y) & \text{if } G = \mathbb{T}.\end{cases}$$

As φ_n is bounded (since h_ξ is) and has compact support, $\varphi_n \in L^1(G^2)$ and (3.3) implies that

$$\int_{A\times B} \varphi_n(x,y)\,dx\,dy = 0$$

for any sets A, B of finite measure, so that $\varphi_n = 0$ for every n. That is,

$$(3.4)\qquad h_\xi(x+y) = h_\xi(x)h_\xi(y) \quad \text{for almost every } x, y \in G.$$

If h_ξ were continuous, this would imply that $h_\xi(x) = e^{i\langle a_\xi, x\rangle}$ and, by boundedness of h_ξ, that $a_\xi \in \mathbb{R}^d$. We will now overcome this difficulty by introducing

$$H_{\xi,j}(x) = \int_0^x h_\xi(t\mathbf{e}_j)\,dt$$

where $j = 1, \dots, d$ and $\mathbf{e}_j = (\delta_{j,k})_{k=1,\dots,d}$ is the jth vector in the standard basis. Clearly $H_{\xi,j}$ is continuous and satisfies

$$H_{\xi,j}(x)H_{\xi,j}(y) = \int_0^x (H_{\xi,j}(y+t) - H_{\xi,j}(t))\,dt.$$

From this, we immediately deduce that $H_{\xi,j}$ is smooth, that $H'_{\xi,j}(t) = h_\xi(t\mathbf{e}_j)$ almost everywhere and that $H'_{\xi,j}(x+y) = H'_{\xi,j}(x)H'_{\xi,j}(y)$ everywhere. Thus, for almost every $x \in \mathbb{R}$ or $\mathbb{T}$, $h_\xi(x\mathbf{e}_j) = e^{ia_{\xi,j}x}$ with $a_{\xi,j}$ real.

Finally, for $x \in G$,

$$h_\xi(x) = h_\xi(x_1\mathbf{e}_1 + \cdots + x_d\mathbf{e}_d) = h_\xi(x_1\mathbf{e}_1)\cdots h_\xi(x_d\mathbf{e}_d) = e^{i\langle a_\xi, x\rangle}$$

where $a_\xi = (a_{\xi,1}, \ldots, a_{\xi,d})$.

We have thus proved that there exists a map from $\varphi : G \to G$ and a set E such that

$$Tf(\xi) = \chi_E(\xi)\widehat{f}(\varphi(\xi)), \tag{3.5}$$

which completes the proof. ∎

Remark. If T extends to a unitary operator from $L^2(\mathbb{R}^d)$ onto $L^2(\mathbb{R}^d)$ then $E = \mathbb{R}^d$ and $\varphi : G \to G$ is a bijection and is measure preserving, i.e. $|\varphi^{-1}(E)| = |E|$ for every set $E \subset G$ of finite measure. This last fact is a corollary of [Si] (see also [No]).

Note that in this theorem, we have only used the L^1-L^∞ duality to show that the operator is a kernel operator. This can be obtained directly. More precisely, it is a consequence of the following theorem that dates back at least to Gelfand [Ge] and Kantorovich–Vullich [KV] (see also [DP, Theorem 2.2.5] or [AT, Theorem 1.3]):

Theorem 3.2. *Let (Ω_1, μ_1) and (Ω_2, μ_2) be two σ-finite measure spaces. There is a one-to-one correspondence between bounded linear operators $T : L^1(\Omega_1) \to L^\infty(\Omega_2)$ and kernels $k \in L^\infty(\Omega_1 \times \Omega_2)$. This correspondence is given by $T = T_k$ where T_k is defined by*

$$T_k f(\omega) = \int_{\Omega_1} k(\zeta, \omega) f(\zeta)\, d\mu_1(\zeta), \qquad f \in L^1(\Omega_1).$$

It follows that Theorem 3.1 then essentially reduces to the results in [Lu1, Lu2]. However, a non-explicit condition in those papers is that k should be defined everywhere as it is applied to Dirac masses.

4. The twisted convolution. In this section, we consider the case of twisted convolution (for background on this transform we refer to [Fo]). Recall that it is defined for $f, g \in L^1(\mathbb{R}^{2d})$ by

$$f \natural g(x,y) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-s, y-t) g(s,t) e^{i\pi(\langle x,t\rangle - \langle y,s\rangle)}\, ds\, dt.$$

This defines a new $L^1(\mathbb{R}^{2d})$ function. Note also that this operation is non-commutative.

Next, for $p, q \in \mathbb{R}^d$, let us define the following operator that acts on functions on $\mathbb{R}^d$:

$$\rho(p,q)\varphi(x) = e^{2i\pi\langle q,x\rangle + i\pi\langle p,q\rangle}\varphi(x+p).$$

For $f \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$ we define a (bounded linear) operator $L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ by

$$\rho(f)\varphi(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(p,q)\rho(p,q)\varphi(x)\,dp\,dq = \int_{\mathbb{R}^d} K_f(x,y)\varphi(y)\,dy$$

where

$$K_f(x,y) = \int_{\mathbb{R}^d} f(y-x,q)e^{i\pi\langle q,x+y\rangle}\,dq = \mathcal{F}_2^{-1}[f]\biggl(y-x,\frac{x+y}{2}\biggr)$$

and $\mathcal{F}_2$ stands for the Fourier transform in the second variable.

One then checks through a cumbersome computation that $\rho(f \natural g) = \rho(f)\rho(g)$ (here the product stands for composition of operators) or, for the kernels,

$$K_{f\natural g}(x,y) = \int_{\mathbb{R}^d} K_f(x,z)K_g(z,y)\,dz.$$

Question. *To what extent does this characterize the transform $f \mapsto \rho(f)$?*

Acknowledgments. The author wishes to thank S. Alesker for sending him the preprint [AM2] and O. Guedon for pointing out that paper. This work was partially financed by the French ANR project AHPI (ANR-07-BLAN-0247-01).

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Received 26 March 2009;
revised 6 July 2009 (5195)