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HÖRMANDER TYPE MULTIPLIER THEOREM ON COMPLEX IWASAWA AN GROUPS

BY

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Dedicated to the memory of Andrzej Hulanicki

Abstract. We prove that, for a distinguished laplacian on an Iwasawa AN group corresponding to a complex semisimple Lie group, a Hörmander type multiplier theorem holds. Our argument is based on Littlewood–Paley theory.

1. Introduction and preliminaries. Multiplier theorems are a long studied subject. Most of works on multipliers of Hörmander type were done in the polynomial growth setting. Operators on spaces of exponential growth are more difficult; in some cases (like laplacian on non-compact symmetric spaces) only holomorphic functions can give operators which are bounded on L^p, $p \neq 2$.

Currently, there are several known results (starting from [3] and [2]) on solvable groups of exponential growth. However, only [4] and follow-up works give Hörmander type multiplier theorems; other works put additional restrictions on the multiplier so that at infinity the resulting operator is given by convolution with an integrable function. In [4] only distinguished laplacians on (a particular class of) groups of rank 1 are handled. The method of [4] can be extended to a distinguished laplacian on Iwasawa type solvable groups, but the full argument is long and only part of it is written up. This paper presents a different, much simpler argument for a distinguished laplacian on Iwasawa type solvable groups corresponding to complex semisimple groups.

Let G denote a connected, complex semisimple Lie group and $\mathbf{g}$ its Lie algebra. Denote by θ a Cartan involution of $\mathbf{g}$, and write

$$\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$$

for the associated Cartan decomposition. Fix a maximal abelian subspace $\mathbf{a}$

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of $\mathbf{p}$; this determines a root space decomposition

$$\mathbf{g} = \mathbf{g}_0 \oplus \sum_{\alpha \in \Lambda} \mathbf{g}_\alpha,$$

Λ denoting the set of roots of the pair $(\mathbf{g}, \mathbf{a})$. Corresponding to the choice of an ordering of the roots, we have an Iwasawa decomposition

$$\mathbf{g} = \mathbf{a} \oplus \mathbf{n} \oplus \mathbf{k}.$$

Let $G = ANK$ be the corresponding Iwasawa decomposition of G. A distinguished laplacian on AN can be constructed as follows. Let $\pi : \mathbf{g} \to \mathbf{p}$ be the projection (defined by the Cartan decomposition). We define a positive definite form $\tilde{B}$ on $\mathbf{a} \oplus \mathbf{n}$ setting $\tilde{B}(X, Y) = B(\pi X, \pi Y)$ where B is the Killing form on $\mathbf{g}$. Put $n = \dim(AN)$. Choose an orthonormal (with respect to $\tilde{B}$) basis in $\mathbf{a} \oplus \mathbf{n}$, say $\{X_1, \ldots, X_n\}$.

Define the (minus) laplacian L by setting

$$Lf(x) = -\sum_{j=1}^{n} X_j^2$$

where we identify elements of $\mathbf{a} \oplus \mathbf{n}$ with right-invariant vector fields on AN.

L is a densely defined selfadjoint operator on $L^2(AN)$ (integration is with respect to left-invariant Haar measure). For a bounded Borel measurable function m on $[0, \infty)$ we can define the bounded operator $m(L)$ on $L^2(AN)$ using the spectral theorem:

$$m(L) = \int m(\lambda)\, dE(\lambda)$$

where E is the spectral measure of L. It is natural to ask for sufficient conditions on m which imply that $m(L)$ can be extended to $L^p(AN)$, $p \neq 2$.

2. Main theorem

THEOREM 2.1. *If for some function $\psi \in C_c^\infty(\mathbb{R}_+)$, $\psi \neq 0$,*

$$\sup_{t>0} \|\psi m(t\,\cdot\,)\|_{C^{n+1}} < \infty$$

then $m(L)$ is bounded on $L^p(AN)$, $1 < p < \infty$.

REMARK. If the assumption in 2.1 is satisfied by one ψ, then it is satisfied by all ψ.

Let ϕ be a bounded holomorphic function defined for $\Re z > 0$ such that $|\phi(z)| \leq c(|z|/(1 + |z|^2))$ and $\phi(x) > 0$ for positive real x. Put $\phi_k(\lambda) = \phi(2^{-k}\lambda)$. We define a vector-valued operator S_ϕ by the formula

$$S_\phi(f) = \{\phi_k(L) f\}_{k=-\infty}^{\infty}.$$

FACT 2.2. *S_ϕ is bounded from $L^p(dx)$ to $L^p(\ell^2)$.*

Proof. This is a consequence of the holomorphic multiplier theorem from [1] or [5], using classical arguments. ∎

Choose $\psi \in C_c^\infty(\mathbb{R}_+)$ such that $\sum_k \psi(2^k x) = 1$ for all $x > 0$. Let $m_k(\lambda) = \psi(2^{-k}\lambda)m(\lambda)$ and $h_k = \phi_k^{-2} m_k$. Then

$$m(L) = \sum m_k(L) = \sum \phi_k(L) h_k(L) \phi_k(L) = S_\phi^* H S_\phi$$

where H is the bounded operator on $L^2(\ell^2)$ given by the formula

$$H\{f_k\}_{k=-\infty}^{\infty} = \{h_k(L) f_k\}_{k=-\infty}^{\infty}$$

and $S_\phi^* : L^2(\ell^2) \to L^2(dx)$ is the adjoint of S_ϕ.

Thus, to prove Theorem 2.1 we only need to prove that H is bounded on $L^p(dx, \ell^2)$.

LEMMA 2.3. *There exists C such that for all k and f_k,*

$$\|h_k(L)\|_{L^1,L^1} \le C,$$
$$|h_k(L) f_k|(x) \le C \sup_{t>0} \exp(-tL)|f_k|(x).$$

Proof. For $x \in \mathbb{R}^n$ put $\eta_k(x) = h_k(2^k|x|^2)$. The functions η_k are in C_c^{n+1} with uniform bounds on their support and their derivatives so

$$|\widehat{\eta_k}|(y) \le C_1(1 + |y|)^{-n-1}$$

where $\widehat{\ }$ denotes the Fourier transform and C_1 does not depend on k. Next, there is a nonnegative integrable function w such that

$$C_1(1 + |y|)^{-n-1} \le \int_0^\infty w(t) \widehat{e_t}(y)\, dt$$

where $e_t(x) = \exp(-t|x|^2)$. For example, we can take a multiple of $(1+t)^{-3/2}$ as w. Consequently,

$$|h_k(-\Delta)|(x) \le C_1 \int_0^\infty w(t) \exp(t2^{-k}\Delta)(x)\, dt$$

where Δ is the laplacian on $\mathbb{R}^n$. By [3] the last inequality remains valid on AN:

$$|h_k(L)|(x) \le C_1 \int_0^\infty w(t) \exp(-t2^{-k}L)(x)\, dt.$$

Since $\|\exp(t2^{-k}L)\|_{L^1,L^1} = 1$ the first claim follows. To get the second claim we note that

$$
\begin{aligned}
|h_k(L)f_k|(x) &\le C_1 \int_0^\infty w(t)\exp(-t2^{-k}L)|f_k|(x)\,dt \\
&\le C_1 \int_0^\infty w(t)\,dt \sup_{t>0} \exp(-t2^{-k}L)|f_k|(x) = C_2 \sup_{t>0} \exp(-tL)|f_k|(x). \ \blacksquare
\end{aligned}
$$

Now

$$
\sup_k |h_k(L)f_k|(x) \le C \sup_{t>0} \exp(-tL)(\sup_k |f_k|)(x).
$$

Since the semigroup maximal function is bounded on L^p, H is bounded on $L^p(dx,\ell^\infty)$. Next, since

$$
\|h_k(L)\|_{L^1(dx),L^1(dx)} \le C
$$

we have

$$
\begin{aligned}
\|H\{f_k\}_{k=-\infty}^{\infty}\|_{L^1(dx,\ell^1)} &= \Big\|\sum_k |h_k(L)f_k|\Big\|_{L^1(dx)} = \sum_k \|h_k(L)f_k\|_{L^1(dx)} \\
&\le C\sum_k \|f_k\|_{L^1(dx)} = \|\{f_k\}_{k=-\infty}^{\infty}\|_{L^1(dx,\ell^1)}.
\end{aligned}
$$

By analytic interpolation between $L^1(dx,\ell^1)$ and $L^2(dx,\ell^2)$, H is bounded on $L^p(dx,\ell^p)$, $1\le p\le 2$. Again, by interpolation between $L^p(dx,\ell^p)$ and $L^p(dx,\ell^\infty)$, H is bounded on $L^p(dx,\ell^2)$, $1\le p\le 2$. We handle $p>2$ by duality, which ends the proof.

3. Possible improvements and limitations. In Lemma 2.3 it is enough to bound the $H^{(n+1)/2+\varepsilon}$ Sobolev norm of h_n. Since this is the only place where we use regularity of m, the main theorem remains valid if m only satisfies

$$
\sup_{t>0} \|\psi m(t\cdot)\|_{H^s} < \infty
$$

with $s>(n+1)/2$.

Lemma 2.3 (with n replaced by apropriate values like in [2]) remains valid for distinguished laplacians on all (not necessarily complex) Iwasawa AN groups, but the proof is much more complicated.

Since our argument is based on the use of a maximal function it probably cannot be improved to give the expected critical exponent $n/2$. Also, it is probably impossible to get the weak type $(1,1)$ of the multiplier operator preserving the simplicity of the argument.

Finally, let us mention that the related problem of bounding Riesz transforms requires estimates of derivatives of the semigroup kernel, hence a quite different method.

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