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DYNAMICS AND LIEB–ROBINSON ESTIMATES FOR LATTICES OF INTERACTING ANHARMONIC OSCILLATORS

BY

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Dedicated to the memory of Andrzej Hulanicki

Abstract. For a class of infinite lattices of interacting anharmonic oscillators, we study the existence of the dynamics, together with Lieb–Robinson bounds, in a suitable algebra of observables.

1. Introduction. Statement of results. Infinite lattices of nearest-neighbors interacting harmonic oscillators are a usual model in quantum statistical mechanics. Among the objects associated to this model, an important one is the dynamics describing the time evolution of some algebra of observables, related to the lattice. Such dynamics on a lattice was defined by Malyshev–Minlos [M-M] and by Thirring [TH], when the potential is a quadratic form.

We also note that, for bounded Hamiltonian models, Lieb and Robinson have established in [L-R] an estimate, concerning the propagation speed for the correlation between two local observables. These inequalities have been improved more recently, with bounds that are uniform with respect to the dimension of the Hilbert space defined at each site, allowing this dimension to go to infinity. See [N-O-S], where the existence of the dynamics is also proven in some algebra (not the same as in [M-M] or [TH]). See also [H-K], [N-S] for applications of these inequalities, and [R-S] for an analogue in classical mechanics.

More recently, Nachtergaele, Raz, Schlein and Sims [N-R-S-S] have derived Lieb–Robinson type inequalities for lattices of unbounded operators. More precisely, they consider a lattice of harmonic oscillators with quadratic interactions with, moreover, on each site of the lattice, a self-interaction potential in a more general class. More precisely, Lieb–Robinson type inequalities are proved ([N-R-S-S]) for Hamiltonians associated to a finite subset

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Λ of the lattice, and hold uniformly in $|\Lambda|$. However, to the best of our knowledge, the existence of dynamics as $|\Lambda| \to \infty$ is established when the potential is a quadratic form, but not with smaller perturbations.

The aim of this article is twofold. First, we shall take the limit when $|\Lambda|$ goes to infinity. For that, we define a $C^\star$-algebra $\mathcal{W}_2$ which seems to be more convenient, when the perturbation is turned on, than the Weyl algebra defined in [M-M] or in [TH], or than the quasilocal algebra used in [N-O-S]. We prove the existence of a dynamics (defined as a limit when the number of sites goes to infinity) for local and non-local observables in this algebra. Secondly, we are able to perturb the quadratic potential of interaction in a more general way than in [N-R-S-S], with not only self-interacting terms. We allow interactions between sites at arbitrary distance, with an exponential decay of this interaction. In this framework, we also obtain Lieb–Robinson type inequalities, with a bound for the propagation speed of the correlations which is perhaps different from the estimation given in [N-R-S-S] (see the remark after (1.20)).

We consider a one-dimensional lattice $\mathbb{Z}$ in order to simplify the notations. For each subset $\Lambda_n = \{-n, \dots, +n\}$ $(n \geq 1)$ in $\mathbb{Z}$, we define a Hamiltonian H_{Λ_n} in $\mathbb{R}^{\Lambda_n}$ by

$$H_{\Lambda_n} = -\frac{1}{2} \sum_{\lambda \in \Lambda_n} \frac{\partial^2}{\partial x_\lambda^2} + V_{\Lambda_n}, \qquad V_{\Lambda_n} = V_{\Lambda_n}^{\mathrm{quad}} + V_{\Lambda_n}^{\mathrm{pert}}. \tag{1.1}$$

where the potential $V_{\Lambda_n}^{\mathrm{quad}}$ is a positive definite quadratic form on $\mathbb{R}^{\Lambda_n}$, and $V_{\Lambda_n}^{\mathrm{pert}}$ is viewed as a perturbation of $V_{\Lambda_n}^{\mathrm{quad}}$.

The quadratic potential is defined for all n by

$$V_{\Lambda_n}^{\mathrm{quad}}(x) = \frac{a}{2} |x|^2 - b \sum_{\lambda=-n}^{n-1} x_\lambda x_{\lambda+1} \tag{1.2}$$

where a and b are real numbers satisfying $a > 2b > 0$.

Precise hypotheses on the perturbation potential are stated in (H1) and (H2) below. These assumptions imply that $V_{\Lambda_n}^{\mathrm{pert}}$ is the multiplication operator by a real-valued function $v_{\Lambda_n}^{\mathrm{pert}}$ belonging to $C^3(\mathbb{R}^{\Lambda_n})$, and satisfying $v_{\Lambda_n}^{\mathrm{pert}}(x) = o(|x|^2)$ near infinity.

By Kato–Rellich's theorem, the operator H_{Λ_n} defined in (1.1), with the hypotheses (H1) and (H2), is self-adjoint with the same domain as the harmonic oscillator on $\mathbb{R}^{\Lambda_n}$. Hence, we can define the unitary operator $e^{itH_{\Lambda_n}}$ $(t \in \mathbb{R})$.

Thus, the following operator is well-defined:

$$\alpha_{\Lambda_n}^{(t)}(A) = e^{itH_{\Lambda_n}} A e^{-itH_{\Lambda_n}} \tag{1.3}$$

for all $A \in \mathcal{L}(\mathcal{H}_{\Lambda_n})$ (where $\mathcal{H}_{\Lambda_n} = L^2(\mathbb{R}^{\Lambda_n})$) and all $t \in \mathbb{R}$. It is then natural to ask whether this sequence of operators has a limit when n tends to $+\infty$, and for which class of operators A. More precisely, we are looking for a Banach algebra $\mathcal{A}$ satisfying the following conditions:

- The space $\mathcal{L}(L^2(\mathbb{R}^\Lambda))$ (where Λ is a finite subset of $\mathbb{Z}$) is isometrically immersed in the algebra (the elements of $\mathcal{L}(\mathcal{H}_\Lambda)$ are, under this identification, called *local observables* supported in Λ).
- For all local observables A, the limit as n tends to infinity of $\alpha_{\Lambda_n}^{(t)}(A)$, denoted by $\alpha^{(t)}(A)$, exists in $\mathcal{A}$.
- The operator $\alpha^{(t)}$, defined in this procedure for local observables A, may be extended by density to the whole algebra $\mathcal{A}$, and acts in a continuous way.

Several works, related to this issue, have considered the $C^\star$-algebra $\mathcal{A}$ of quasi-local observables. Let us recall its definition (cf. [S]). For each finite subset Λ in $\mathbb{Z}$ set $\mathcal{H}_\Lambda = L^2(\mathbb{R}^\Lambda)$. One notes that if $\Lambda \subset \Lambda'$ then $\mathcal{L}(\mathcal{H}_\Lambda)$ is isometrically immersed in $\mathcal{L}(\mathcal{H}_{\Lambda'})$. Therefore, one may define $\mathcal{A}$ as the completion of the inductive limit of the spaces $\mathcal{L}(\mathcal{H}_\Lambda)$:

$$\mathcal{A} = \overline{\bigcup_{\Lambda\subset\mathbb{Z}} \mathcal{L}(\mathcal{H}_\Lambda)}. \tag{1.4}$$

This algebra is well-adapted in the case of bounded potentials, or when the first order derivatives are bounded (cf. e.g. [N-O-S] for the existence of a dynamics, or [A-C-L-N] for estimates on the decay of the correlations), whereas it might not be suitable for the perturbed quadratic case studied here.

Another algebra, *the Weyl algebra*, is considered by Malyshev–Minlos [M-M] and Thirring [TH]. This algebra fits the unperturbed quadratic case ($V_{\Lambda_n}^{\text{pert}} = 0$), and is defined using the Fock space formalism.

The space $\mathcal{H}$ denotes the symmetrized Fock space $\mathcal{H} = F_s(\ell^2(\mathbb{Z}))$, associated to the Hilbert space $\ell^2(\mathbb{Z})$. For all $\lambda \in \mathbb{Z}$, one defines two self-adjoint operators P_λ and Q_λ in the Fock space, satisfying the same commutation relations as the position and momentum operators in $L^2(\mathbb{R}^n)$. (Note that there are an infinite number of these operators.) For each finite subset Λ of $\mathbb{Z}$, the space $\mathcal{L}(\mathcal{H}_\Lambda)$ (where $\mathcal{H}_\Lambda = L^2(\mathbb{R}^\Lambda)$) is isometrically immersed in $\mathcal{L}(\mathcal{H})$. This identification extends also to unbounded operators. Thus, the multiplication operator by x_λ and the operator $\frac{1}{i}\frac{\partial}{\partial x_\lambda}$ ($\lambda \in \Lambda$) become the two operators Q_λ and P_λ, sometimes denoted by $Q_\lambda^{(0)}$ and $Q_\lambda^{(1)}$:

$$Q_\lambda^{(0)} = Q_\lambda = x_\lambda, \qquad Q_\lambda^{(1)} = P_\lambda = \frac{1}{i}\,\frac{\partial}{\partial x_\lambda}. \tag{1.5}$$

The Fock space formalism allows us to properly define, for all real sequences u and v in $\ell^2(\mathbb{Z})$, an unbounded self-adjoint operator (the *Segal* *operator*), formally defined by

$$\Pi(u,v) = \sum_{\lambda\in\mathbb{Z}} (u_\lambda P_\lambda + v_\lambda Q_\lambda). \tag{1.6}$$

The operators P_λ and Q_λ are not generally defined by (1.5) anymore, but instead, $\Pi(u,v)$ is defined starting from the creation and annihilation operators associated to $\ell^2(\mathbb{Z})$ (see Section 2). The corresponding unitary operator $W(u,v) = e^{i\Pi(u,v)}$ is called a *Weyl operator*.

The Weyl algebra introduced by Malyshev–Minlos [M-M] and Thirring [TH] is the closure in $\mathcal{L}(\mathcal{H})$ of the subspace generated by the operators $W(u,v)$ (u and v being real sequences in $\ell^2(\mathbb{Z})$).

In the purely quadradic case ($V^{\rm pert}_{\Lambda_n} = 0$), for all A in this Weyl algebra, an explicit analysis allows us to define $\alpha^{(t)}_{\Lambda_n}(A)$ properly (even if A is not supported in Λ_n) and to define the limit operator $\alpha^{(t)}(A)$ such that, for all $f \in \mathcal{H}$,

$$\lim_{n\to\infty} \|[\alpha^{(t)}_{\Lambda_n}(A) - \alpha^{(t)}(A)]f\|_{\mathcal{H}} = 0.$$

In order to derive the latter limit, uniform estimates, such as those established in [N-R-S-S], are needed.

Using the Weyl algebra defined above, it is probably difficult to obtain these results when the perturbation potential is turned on. The purpose of this work is then to extend the above results to the quadratic case with perturbations by involving another algebra $\mathcal{W}_2$ included in $\mathcal{L}(\mathcal{H})$. Furthermore, the Lieb–Robinson estimates in [N-R-S-S] are also extended to that framework.

Before giving the definition of $\mathcal{W}_2$, let us mention that the works of Calderón–Vaillancourt [C-V] and Beals [BE] (see also Hörmander [HO]) give an important role to a particular subalgebra of $\mathcal{L}(L^2(\mathbb{R}^n))$ or here, of $\mathcal{L}(L^2(\mathbb{R}^\Lambda))$, for all finite subsets Λ in $\mathbb{Z}$. This subalgebra is the set $OPS^0(\mathbb{R}^\Lambda)$ of pseudo-differential operators on $\mathbb{R}^\Lambda$, associated to symbols that are bounded together with all their derivatives. From Beals [BE], these operators are characterized by the following property, involving the operators $Q^{(0)}_\lambda$ and $Q^{(1)}_\lambda$ defined in (1.5) for all $\lambda \in \Lambda$. An operator A in $\mathcal{L}(L^2(\mathbb{R}^\Lambda))$ is in $OPS^0(\mathbb{R}^\Lambda)$ if, and only if, all the iterated commutators $(\mathrm{ad}\, Q^{k_1}_{\lambda_1}) \dots (\mathrm{ad}\, Q^{k_m}_{\lambda_m})A$, (with $\lambda_1, \dots, \lambda_m$ in Λ, $m \geq 0$, and $k_j \in \{0,1\}$), are bounded in $L^2(\mathbb{R}^\Lambda)$. (The commutators are known, a priori, to map $\mathcal{S}(\mathbb{R}^\Lambda)$ into $\mathcal{S}'(\mathbb{R}^\Lambda)$.)

Replacing Λ by $\mathbb{Z}$, one may analogously define a decreasing sequence of subalgebras $\mathcal{W}_k$ in $\mathcal{L}(\mathcal{H})$ ($k \geq 0$). Set $\mathcal{W}_0 = \mathcal{L}(\mathcal{H})$. We denote by $\mathcal{W}_1$ the set of all A in $\mathcal{W}_0$ such that, for all $\lambda \in \mathbb{Z}$, the commutators $[A, Q_\lambda]$ and $[A, P_\lambda]$

are bounded in $\mathcal{H}$, and the sum in the following norm is finite:

$$\|A\|_{\mathcal{W}_1} = \|A\|_{\mathcal{W}_0} + \sum_{\substack{\lambda\in\mathbb{Z}\\ k=0,1}} \|[A, Q_\lambda^{(k)}]\|_{\mathcal{W}_0}. \tag{1.7}$$

Note that the above commutators are properly defined in Section 2. From now on, the operators $Q_\lambda^{(0)} = Q_\lambda$ and $Q_\lambda^{(1)} = P_\lambda$ are defined through the Fock space formalism, and not by (1.5) anymore.

Let us denote by $\mathcal{W}_2$ the set of all operators $A \in \mathcal{W}_1$ such that the commutators $[Q_\lambda^{(k)}, A]$ belong to $\mathcal{W}_1$ for all λ in $\mathbb{Z}$, and the sum in the norm below is finite:

$$\|A\|_{\mathcal{W}_2} = \|A\|_{\mathcal{W}_1} + \frac{1}{2} \sum_{\substack{(\lambda,\mu)\in\mathbb{Z}^2\\ 0\le j,k\le 1}} \|[[A, Q_\lambda^{(j)}], Q_\mu^{(k)}]\|_{\mathcal{L}(\mathcal{H})}. \tag{1.8}$$

An example. For all u and v in $\ell^1(\mathbb{Z})$, the Weyl operator $W(u,v) = e^{i\Pi(u,v)}$ is in $\mathcal{W}_k$ $(0 \le k \le 2)$.

One might similarly define a sequence of algebras $\mathcal{W}_k$ using iterated commutators. In particular, the intersection of these algebras could correspond to an analogue of OPS^0 in infinite dimensions. Other particular classes of pseudo-differential operators in infinite dimensions are studied by B. Lascar (see [L1], [L2]), or more recently by Ammari–Nier [A-N].

Among all these algebras, from our point of view, it is $\mathcal{W}_2$ that appears to be the most suitable for our study. If A is not supposed to be in $\mathcal{W}_2$, but only in $\mathcal{L}(\mathcal{H})$ and supported on a finite subset E of $\mathbb{Z}$, it appears to be possible to show that, for all f in $\mathcal{H}$, the sequence $\alpha_{\Lambda_n}^{(t)}(A)f$ weakly converges in $\mathcal{H}$. If this limit is denoted by $\alpha^{(t)}(A)f$, it is not clear whether the map $t \mapsto \alpha^{(t)}$ is continuous, neither whether $\alpha^{(t)}$ may be extended to a suitable Banach algebra.

More precise estimates are obtained when the local observable A belongs to $\mathcal{W}_2$. First, let us describe the perturbation potential.

Hypotheses on the perturbation potentials. The operator $V_{\Lambda_n}^{\text{pert}}$ is written as the following sum:

$$V_{\Lambda_n}^{\text{pert}} = \sum_{\lambda\in\Lambda_n} V_\lambda + \sum_{\substack{(\lambda,\mu)\in\Lambda_n^2\\ \lambda\neq\mu}} V_{\lambda\mu}, \tag{1.9}$$

where the operators V_λ and $V_{\lambda\mu}$ are defined for all λ and μ in $\mathbb{Z}$, and satisfy the assumptions below:

(H1) For each pair $(\lambda,\mu) \in \mathbb{Z}^2$ with $\lambda \neq \mu$, $V_{\lambda\mu}$ is multiplication by a C^3 real-valued function $v_{\lambda\mu}$ depending only on the variables x_λ and x_μ. Moreover, denote by $\xi \mapsto \widehat{v_{\lambda\mu}}(\xi)$ the Fourier transform of $v_{\lambda\mu}$ (on $\mathbb{R}^2$ and in the sense of distributions). Then $\xi \mapsto \xi_\lambda^j \xi_\mu^k \widehat{v_{\lambda\mu}}(\xi)$ belongs to $L^1(\mathbb{R}^2)$ for $2 \le j+k \le 3$. Furthermore, there exist $C_0, \gamma_0 > 0$ (not depending on λ and μ) such that

$$\sum_{2\le j+k\le 3} \|\xi_\lambda^j \xi_\mu^k \widehat{v_{\lambda\mu}}\|_{L^1(\mathbb{R}^2)} \le C_0 e^{-\gamma_0|\lambda-\mu|}, \tag{1.10}$$

$$|\nabla v_{\lambda\mu}(0)| \le C_0 e^{-\gamma_0|\lambda-\mu|}. \tag{1.11}$$

(H2) For each λ in $\mathbb{Z}$, V_λ is multiplication by a C^3 real-valued function v_λ depending only on the variable x_λ. If we denote by $\widehat{v_\lambda}$ the Fourier transform of v_λ, then $\xi \mapsto \xi_\lambda^j \widehat{v_\lambda}(\xi)$ is in $L^1(\mathbb{R})$ for $2 \le j \le 3$, and

$$\sum_{2\le j\le 3} \|\xi_\lambda^j \widehat{v_\lambda}\|_{L^1(\mathbb{R})} \le C_0, \qquad |\nabla v_\lambda(0)| \le C_0. \tag{1.12}$$

In particular, in the case of *interactions between nearest neighbors,* one has $V_{\lambda\mu} = 0$ whenever $|\lambda-\mu| \ge 2$. It is then sufficient that the integrals on the l.h.s. of (1.10) and (1.12) are uniformly bounded in λ. In that case, the hypotheses (H1) and (H2) are satisfied for any $\gamma_0 > 0$, and in all the results below, the phrase "for all $\gamma \in (0, \gamma_0)$" has to be replaced by "for all $\gamma > 0$".

For each integer n, the perturbation potential $V_{\Lambda_n}^{\text{pert}}$ and the Hamiltonian H_{Λ_n} are defined by (1.9) and (1.1) respectively. In [N-R-S-S], the authors have only considered the V_λ's. We shall say that an element A of $\mathcal{W}_2$ *has finite support* if there exists a finite subset E in $\mathbb{Z}$ such that A identifies with an element of $\mathcal{L}(\mathcal{H}_E)$. The smallest such set is called the *support* of A and is denoted by $\sigma(A)$.

THEOREM 1.1. *Under the above hypotheses, for all $A \in \mathcal{W}_2$ with finite support, all $t \in \mathbb{R}$, and all n such that Λ_n contains the support of A, the operator $\alpha_{\Lambda_n}^{(t)}(A)$ belongs to $\mathcal{W}_2$. Moreover, there exist $C, M > 0$ not depending on n and t such that*

$$\|\alpha_{\Lambda_n}^{(t)}(A)\|_{\mathcal{W}_2} \le C e^{M|t|} \|A\|_{\mathcal{W}_2}. \tag{1.13}$$

Furthermore, for each $f \in \mathcal{H}$, the sequence $\alpha_{\Lambda_n}^{(t)}(A) f$ strongly converges in $\mathcal{H}$. Denoting the limit by $\alpha^{(t)}(A) f$, the map $t \mapsto \alpha^{(t)}(A) f$ is strongly continuous, the operator $\alpha^{(t)}(A)$ is in $\mathcal{W}_2$, and

$$\|\alpha^{(t)}(A)\|_{\mathcal{W}_2} \le C e^{M|t|} \|A\|_{\mathcal{W}_2}. \tag{1.14}$$

In the first part of this theorem (where n is fixed), one may think that $\alpha_{\Lambda_n}^{(t)}$ acts in the algebra $\mathcal{W}_k$, defined similarly to $\mathcal{W}_1$ and $\mathcal{W}_2$, but with iterated commutators of length k, and for operators supported in Λ_n. (The hypotheses (H1) and (H2) naturally need to be strengthened.) From Beals' characterization, one would deduce a group action of $\alpha_{\Lambda_n}^{(t)}$ on the operators in $OPS^0(\mathbb{R}^\Lambda)$. An alternative approach may be found in the works of Bony (see [BO1] and [BO2]).

Moreover, under the hypotheses of Theorem 1.1, the automorphism $\alpha^{(t)}$ (initially defined for local observables) extends uniquely to the whole algebra $\mathcal{W}_2$ (see below). To this end, we introduce Sobolev-type spaces.

Let $\mathcal{H}^2$ be the subspace of $f \in \mathcal{H}$ such that the following norm is finite:

$$\|f\|_{\mathcal{H}^2} = \|f\|_{\mathcal{H}} + \sup_{\substack{\lambda\in\mathbb{Z}\\ 0\le j\le 1}} \|Q_\lambda^{(j)} f\|_{\mathcal{H}} + \sup_{\substack{(\lambda,\mu)\in\mathbb{Z}^2\\ 0\le j,k\le 1}} \|Q_\lambda^{(j)} Q_\mu^{(k)} f\|_{\mathcal{H}}. \tag{1.15}$$

Since convergence in norm is needed, Theorem 1.1 is now completed with the result below:

THEOREM 1.2. *There exist $C,\gamma,M>0$ with the following properties. For all A in $\mathcal{W}_2$ with finite support $\sigma(A)$, all n such that Λ_n contains $\sigma(A)$, and all $t\in\mathbb{R}$,*

$$\|\alpha_{\Lambda_n}^{(t)}(A) - \alpha^{(t)}(A)\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})} \le Ce^{M|t|}e^{-\gamma d(\sigma(A),\Lambda_n^c)}\|A\|_{\mathcal{W}_2}. \tag{1.16}$$

Moreover,

$$\|\alpha^{(t)}(A)\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})} \le Ce^{M|t|}\|A\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})}. \tag{1.17}$$

The set of all observables with finite support is not dense in $\mathcal{W}_2$. To extend $\alpha^{(t)}$, we shall use, instead of density, the following two results.

THEOREM 1.3. *Let A be in $\mathcal{W}_2$. Then there is a sequence (A_n) in $\mathcal{W}_2$ such that each A_n has finite support, and*

$$\|A_n\|_{\mathcal{W}_2} \le \|A\|_{\mathcal{W}_2}, \qquad \lim_{n\to\infty}\|A_n - A\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})} = 0. \tag{1.18}$$

THEOREM 1.4. *Let (A_n) be a sequence in $\mathcal{W}_2$. Suppose that $\|A_n\|_{\mathcal{W}_2}\le 1$ and there exists $A\in\mathcal{L}(\mathcal{H}^2,\mathcal{H})$ such that $\|A_n - A\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})}\to 0$. Then A may be extended to an element of $\mathcal{L}(\mathcal{H})$ which belongs to $\mathcal{W}_2$ and $\|A\|_{\mathcal{W}_2}\le 1$. Moreover, $A_nf\to Af$ in $\mathcal{H}$ for all $f\in\mathcal{H}$.*

Consequently, we easily deduce from Theorems 1.1–1.4 that $\alpha^{(t)}$ can be extended, in a unique way, to the whole algebra $\mathcal{W}_2$, without any conditions on the finiteness of the supports (see Section 7). The map $\alpha^{(t)}$ is not a $\mathcal{W}_2$ norm preserving map, but it is $\mathcal{L}(\mathcal{H})$ norm preserving. Using this fact, $\alpha^{(t)}$ can be extended to the closure $\overline{\mathcal{W}_2}$ of $\mathcal{W}_2$ in $\mathcal{L}(\mathcal{H})$. Thus, $\alpha^{(t)}$ acts in $\overline{\mathcal{W}_2}$ in a continuous way (for the simple topology) and is norm preserving.

Lieb–Robinson's inequalities. These inequalities, established in [L-R] for bounded Hamiltonians and, more recently, in [N-R-S-S] for quadratic Hamiltonians, express the propagation of the correlation between two observables with separated supports, as a function of time and of distance between the supports.

For all h in $\mathbb{Z}$, let T_h be the map in $\ell^2(\mathbb{Z})$ defined by $(T_h u)_\lambda = u_{\lambda+h}$ for all $u \in \ell^2(\mathbb{Z})$ and $\lambda \in \mathbb{Z}$. With T_h we define a map in the Fock space $\mathcal{H} = F_s(\ell^2(\mathbb{Z}))$, still denoted T_h. For A in $\mathcal{L}(\mathcal{H})$ we set $\tau_h(A) = T_h^{-1} A T_h$.

In our framework, Lieb–Robinson type inequalities have the following form:

THEOREM 1.5. *There exists a real number v_0 with the following property. For any A and B in $\mathcal{W}_2$ with finite supports, any sequence (h_n, t_n) tending to infinity in $\mathbb{Z} \times \mathbb{R}$ with $|h_n| \geq v_0 |t_n|$, and any $f \in \mathcal{H}$,*

$$\lim_{n\to\infty} [\alpha^{(t_n)}(A), \tau_{h_n}(B)] f = 0. \tag{1.19}$$

The infimum V_0 of all the v_0 with the above property defines a kind of propagation speed, which is different from the usual definitions of phase and group velocities (cf. Cohen-Tannoudji [C-T]). In the case of cyclic quadratic potentials (that is, without any perturbation, but obtained by adding to $V_{\Lambda_n}^{\text{quad}}$ of (1.2) an end point interaction potential $-bx_n x_{-n}$), one finds in [N-R-S-S] an estimate of this propagation speed. (In [N-R-S-S] this is written for a multidimensional lattice model.) We shall provide here an alternative estimate of the same type, with an elementary proof, given in Section 4. The analysis of chains of harmonic oscillators with cyclic interactions usually involves the dispersion relation $\omega(\theta) = \sqrt{a - 2b\cos\theta}$ (cf. [C-T]). It is then natural to define a complex version of this relation,

$$\Omega(z) = \sqrt{a - b(z + z^{-1})}, \qquad z \in \mathbb{C} \setminus \{0\}.$$

For any $\gamma > 0$, set

$$M(\gamma) = \sup_{|z|=e^\gamma} |\operatorname{Im} \Omega(z)|.$$

The propagation speed satisfies, in the cyclic quadratic case,

$$V_0 \leq \inf_{\gamma>0} \frac{M(\gamma)}{\gamma}.$$

In a more general case, this estimate is less precise. For all γ in $(0, \gamma_0)$ (with γ_0 as in the hypotheses (H1) and (H2)), we shall define in Proposition 3.4 a positive number S_γ and we shall prove in Section 8 that the propagation speed satisfies

$$V_0 \leq \inf_{0<\gamma<\gamma_0} \frac{2\sqrt{S_\gamma}}{\gamma}. \tag{1.20}$$

The constant V_0 depends only on a and b, together with the norms in $\mathcal{F}L^1(\mathbb{R})$ or $\mathcal{F}L^1(\mathbb{R}^2)$ of the second derivatives of the potentials of perturbation. We then note that if we multiply a, b and the potentials of perturbation by a constant $g > 0$, our estimate on the propagation speed is multiplied by $\sqrt{g}$. It seems that the estimate in [N-R-S-S] did not have this property. It is also possible to give a norm estimate, instead of (1.19). Then we need the Sobolev spaces of Section 2. We shall prove in Section 8 that for $M > 2\sqrt{S_\gamma}$

$$\|[\alpha^{(t)}(A), B]\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})} \leq C(M,\gamma)\|A\|_{\mathcal{W}_2}\|B\|_{\mathcal{W}_2} e^{M|t|} e^{-\gamma d(\sigma(A),\sigma(B))}$$

where $\sigma(A)$ and $\sigma(B)$ are the supports of the local observables A and B.

Section 2 concerns the subalgebra $\mathcal{W}_k$. In Section 3, properties of V_{Λ_n} under the hypotheses (H1) and (H2) are established. Evolution operators, for finite systems on the lattice, are studied in Sections 4–6. Sections 7 and 8 are respectively devoted to perform the limit as n (the number of sites) goes to infinity, and to derive Lieb–Robinson's inequalities.

2. Algebras of operators in the Fock space

Notations on Fock spaces (cf. [R-S]). For any subset E of $\mathbb{Z}$, the symmetrized Fock space associated to the Hilbert space $\ell^2(E)$ will be denoted $\mathcal{H}_E$. When $E = \mathbb{Z}$, this space is still denoted $\mathcal{H}$. The ground state of $\mathcal{H}_E$ is denoted by Ω_E or Ω when $E = \mathbb{Z}$.

If E_1 and E_2 are two disjoint subsets of $\mathbb{Z}$ one may identify $\mathcal{H}_{E_1\cup E_2}$ and $\mathcal{H}_{E_1} \otimes \mathcal{H}_{E_2}$ (the completed tensor product). One may also identify $\Omega_{E_1\cup E_2}$ with $\Omega_{E_1} \otimes \Omega_{E_2}$.

For all real sequences u in $\ell^2(\mathbb{Z})$ we define two unbounded operators $a(u)$ (annihilation operator) and $a^\star(u)$ (creation operator), formal adjoints of each other, and satisfying the following commutation relations:

$$[a(u), a(v)] = [a^\star(u), a^\star(v)] = 0, \qquad [a(u), a^\star(v)] = (u, v),$$

for all u and v in $\ell^2(\mathbb{Z})$.

We denote by $(e_\lambda)_{\lambda\in\mathbb{Z}}$ the canonical basis of $\ell^2(\mathbb{Z})$. Starting from the ground state Ω, and applying successively the creation operators, one defines $a^\star(e_{\lambda_1})\ldots a^\star(e_{\lambda_m})\Omega$, which are orthogonal elements of $\mathcal{H}$. Let $\mathcal{D}$ be the subspace of $\mathcal{H}$ generated by these vectors. It is known that $\mathcal{D}$ is dense in $\mathcal{H}$. The space $\mathcal{D}$ is included in the domain of all $a(u)$ and $a^\star(u)$ ($u \in \ell^2(\mathbb{Z})$). For all f in $\mathcal{D}$ there exists a finite subset $S \subset \mathbb{Z}$ such that f can be written as $f = g \otimes \Omega_{S^c}$ with $g \in \mathcal{H}_S$. We then say that f is *supported in* S.

Next we define the *Segal operator* $\Pi(u, v)$ by

$$\Pi(u,v) = \frac{a(u) + a^\star(u)}{\sqrt{2}} + \frac{a(v) - a^\star(v)}{i\sqrt{2}} \tag{2.1}$$

for all real elements u and v in $\ell^2(\mathbb{Z})$. An element $f \in \mathcal{H}$ is in the domain of $\Pi(u, v)$ if there exists a sequence (f_n) in $\mathcal{D}$ such that f_n converges to f in $\mathcal{H}$, and $\Pi(u, v)f_n$ has a limit in $\mathcal{H}$. Thus, $\Pi(u, v)$ is a self-adjoint operator. The associated *Weyl operator* is $W(u, v) = e^{i\Pi(u,v)}$.

In particular, for each element e_λ in the canonical basis of $\ell^2(\mathbb{Z})$ the Segal operators are denoted

$$(2.2)\qquad Q_\lambda = Q_\lambda^{(0)} = \frac{a(e_\lambda) + a^\star(e_\lambda)}{\sqrt{2}}, \qquad P_\lambda = Q_\lambda^{(1)} = \frac{a(e_\lambda) - a^\star(e_\lambda)}{i\sqrt{2}}.$$

Let us write down an orthonormal basis. We shall limit ourselves to the Hilbert space $\mathcal{H}_{\{\lambda\}}$ associated to a subset of $\mathbb{Z}$ reduced to one element λ. In this space we again use the construction of $\mathcal{D}$ and obtain the basis $(h_n)_{n\geq 0}$, now normalized by setting

$$(2.3)\qquad h_0 = \Omega_{\{\lambda\}}, \qquad h_{j+1} = (j+1)^{-1/2} a^\star(e_\lambda) h_j \qquad (j \geq 0).$$

The space $\mathcal{H}_{\{\lambda\}}$ may be identified with $L^2(\mathbb{R})$ in an isometric way. Then the basis (h_j) becomes the Hermite functions basis, and the operators Q_λ and P_λ respectively become multiplication by x_λ and the operator $\frac{1}{i}\frac{\partial}{\partial_{x_\lambda}}$. Effectuating the completed tensor product, the space $\mathcal{H}_\Lambda$ is similarly identified with $L^2(\mathbb{R}^\Lambda)$ for each finite subset Λ of $\mathbb{Z}$.

For any $E \subset F \subseteq \mathbb{Z}$, and any operator $T \in \mathcal{L}(E)$, we define $i_{EF}(T)$ by

$$(2.4)\qquad i_{EF}(T) = T \otimes I_{F\setminus E},$$

where $I_{F\setminus E}$ is the identity in $\mathcal{H}_{F\setminus E}$. In particular, if $F = \mathbb{Z}$ the operator $i_{E\mathbb{Z}}(T)$ is said to be *supported in* E.

Sobolev spaces. Let us denote by $\mathcal{H}^1$ the set of all $f \in \mathcal{H}$ that belong to the domains of the Segal operators $Q_\lambda = Q_\lambda^{(0)}$ and $P_\lambda = Q_\lambda^{(1)}$ for all $\lambda \in \mathbb{Z}$, and the following norm is finite:

$$(2.5)\qquad \|f\|_{\mathcal{H}^1} = \|f\|_{\mathcal{H}} + \sup_{\substack{\lambda\in\mathbb{Z}\\ 0\leq j\leq 1}} \|Q_\lambda^{(j)} f\|_{\mathcal{H}}.$$

The space $\mathcal{H}^2$ is the set of all $f \in \mathcal{H}^1$ such that $Q_\lambda^{(0)} f$ and $Q_\lambda^{(1)} f$ belong to $\mathcal{H}^1$ for all λ in $\mathbb{Z}$, and with the following norm finite:

$$(2.6)\qquad \|f\|_{\mathcal{H}^2} = \|f\|_{\mathcal{H}^1} + \sup_{\substack{(\lambda,\mu)\in\mathbb{Z}^2\\ 0\leq j,k\leq 1}} \|Q_\lambda^{(j)} Q_\mu^{(k)} f\|_{\mathcal{H}}.$$

These spaces are dense in $\mathcal{H}$ since they contain $\mathcal{D}$. If E is a subset of $\mathbb{Z}$ then the subspace $\mathcal{H}_E^k$ is defined analogously in the corresponding Hilbert space $\mathcal{H}_E$.

Commutators, and spaces with negative orders. For all A in $\mathcal{L}(\mathcal{H})$, $f \in \mathcal{H}^1$ and $\lambda \in \mathbb{Z}$ the map

$$(2.7)\qquad \mathcal{H}^1 \ni g \mapsto \langle A Q_\lambda^{(j)} f, g\rangle - \langle Af,\, Q_\lambda^{(j)} g\rangle, \qquad 0 \leq j \leq 1,$$

is a continuous antilinear map on $\mathcal{H}^1$. We denote by $\mathcal{H}^{-k}$ the anti-dual of $\mathcal{H}^k$ $(0 \leq k \leq 2)$. For any A in $\mathcal{L}(\mathcal{H})$ the map (2.7) is linear and continuous from $\mathcal{H}^1$ to $\mathcal{H}^{-1}$. It is denoted $[A, Q_\lambda^{(j)}]$. One may identify $\mathcal{H}$ with a subspace of $\mathcal{H}^{-1}$, and the latter with a subspace of $\mathcal{H}^{-2}$. Thus, the operators $Q_\lambda^{(j)}$ are bounded from $\mathcal{H}^m$ to $\mathcal{H}^{m-1}$ $(-1 \leq m \leq 2)$, and this allows us to define the iterated commutators $[Q_\lambda^{(j)}, [Q_\mu^{(k)}, A]]$ ($(\lambda, \mu) \in \mathbb{Z}^2$, $0 \leq j, k \leq 1$) as continuous linear maps from $\mathcal{H}^2$ to $\mathcal{H}^{-2}$. This map is also denoted by $(\operatorname{ad} Q_\lambda^{(j)})(\operatorname{ad} Q_\mu^{(k)})A$.

If there is a $C > 0$ satisfying

$$|\langle AQ_\lambda^{(j)} f, g\rangle - \langle Af, Q_\lambda^{(j)} g\rangle| \leq C\|f\|_{\mathcal{H}}\|g\|_{\mathcal{H}}$$

for all f and g in $\mathcal{H}^1$ we shall say that the commutators $[A, Q_\lambda^{(j)}]$ are in $\mathcal{L}(\mathcal{H})$. Then for all f in $\mathcal{H}^1$ there exists an element of $\mathcal{H}$, denoted $[A, Q_\lambda^{(j)}]f$, such that

$$\langle APf,\ g\rangle - \langle Af, Q_\lambda^{(j)} g\rangle = \langle [A, Q_\lambda^{(j)}]f, g\rangle$$

for all g in $\mathcal{H}^1$, and the previously defined operator $[A, Q_\lambda^{(j)}] : \mathcal{H}^1 \to \mathcal{H}$ extends to an element of $\mathcal{L}(\mathcal{H})$. Proceeding similarly, one gives a precise meaning to the statement "the commutator $[[A, Q_\lambda^{(j)}], Q_\mu^{(k)}]$ is in $\mathcal{L}(\mathcal{H})$".

Weyl algebra. We denote by $\mathcal{W}_1$ the set of all A in $\mathcal{L}(\mathcal{H})$ having the commutators $[A, Q_\lambda^{(j)}]$ ($0 \leq j \leq 1$) in $\mathcal{L}(\mathcal{H})$ for all λ in $\mathbb{Z}$, and having the following norm finite:

$$\|A\|_{\mathcal{W}_1} = \|A\|_{\mathcal{L}(\mathcal{H})} + \sum_{\substack{\lambda \in \mathbb{Z} \\ 0 \leq j \leq 1}} \|[A, Q_\lambda^{(j)}]\|_{\mathcal{L}(\mathcal{H})}. \tag{2.8}$$

We denote by $\mathcal{W}_2$ the set of elements A belonging to $\mathcal{W}_1$, having the commutators $[[A, Q_\lambda^{(j)}], Q_\mu^{(k)}]$ in $\mathcal{L}(\mathcal{H})$ for all λ and μ in $\mathbb{Z}$, and having the following norm finite:

$$\|A\|_{\mathcal{W}_2} = \|A\|_{\mathcal{W}_1} + \frac{1}{2} \sum_{\substack{(\lambda,\mu) \in \mathbb{Z}^2 \\ 0 \leq j,k \leq 1}} \|[[A, Q_\lambda^{(j)}], Q_\mu^{(k)}]\|_{\mathcal{L}(\mathcal{H})}. \tag{2.9}$$

We easily verify the next proposition.

Proposition 2.1. *For all $k \leq 2$ the algebra $\mathcal{W}_k$ is a Banach algebra. For all A and B in $\mathcal{W}_k$,*

$$\|AB\|_{\mathcal{W}_k} \leq \|A\|_{\mathcal{W}_k}\|B\|_{\mathcal{W}_k}. \tag{2.10}$$

Each $A \in \mathcal{W}_2$ is bounded in the Sobolev space $\mathcal{H}^2$ and

$$\|A\|_{\mathcal{L}(\mathcal{H}^2, \mathcal{H}^2)} \leq 3\|A\|_{\mathcal{W}_2}. \tag{2.11}$$

Proof of Theorem 1.4. Let (A_n) be a sequence in $\mathcal{W}_2$ and let A in $\mathcal{L}(\mathcal{H}^2, \mathcal{H})$ satisfy

$$\|A_n\|_{\mathcal{W}_2} \leq 1, \qquad \lim_{n \to \infty} \|A_n - A\|_{\mathcal{L}(\mathcal{H}^2, \mathcal{H})} = 0.$$

For each f in $\mathcal{H}^2$, one deduces that $\|Af\| \leq \|f\|$ and A thus extends by density to an element of $\mathcal{L}(\mathcal{H})$ with

$$\|A\|_{\mathcal{L}(\mathcal{H})} \leq \liminf_{n\to\infty} \|A_n\|_{\mathcal{L}(\mathcal{H})}.$$

For all λ in $\mathbb{Z}$, all f and g in $\mathcal{D}$ and any $n \geq 1$ we see that

$$|\langle AQ_\lambda^{(j)} f, g\rangle - \langle Af, Q_\lambda^{(j)} g\rangle| \leq \|[A_n, Q_\lambda^{(j)}]\| \, \|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}} + \varepsilon_n$$

where the sequence ε_n tends to 0. As a consequence,

$$|\langle AQ_\lambda^{(j)} f,\ g\rangle - \langle Af, Q_\lambda^{(j)} g\rangle| \leq \|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}} \liminf_{n\to\infty} \|[A_n, Q_\lambda^{(j)}]\|.$$

Since $\mathcal{D}$ is dense in $\mathcal{H}^1$ this inequality is still valid for all f and g in $\mathcal{H}^1$. With the above definition the commutator $[A, Q_\lambda^{(j)}]$ is thus in $\mathcal{L}(\mathcal{H})$ and

$$\|[A, Q_\lambda^{(j)}]\|_{\mathcal{L}(\mathcal{H})} \leq \liminf_{n\to\infty} \|[A_n, Q_\lambda^{(j)}]\|_{\mathcal{L}(\mathcal{H})}.$$

From Fatou's lemma one deduces

$$\sum_{\substack{\lambda\in\mathbb{Z} \\ 0\leq j\leq 1}} \|[A, Q_\lambda^{(j)}]\|_{\mathcal{L}(\mathcal{H})} \leq \liminf_{n\to\infty} \sum_{\substack{\lambda\in\mathbb{Z} \\ 0\leq j\leq 1}} \|[A_n, Q_\lambda^{(j)}]\|_{\mathcal{L}(\mathcal{H})}.$$

It is similarly derived that the commutator $[[A, Q_\lambda^{(j)}], Q_\mu^{(k)}]$ is in $\mathcal{L}(\mathcal{H})$ for all λ and μ in $\mathbb{Z}$, and

$$\sum_{\substack{(\lambda,\mu)\in\mathbb{Z}^2 \\ 0\leq j,k\leq 1}} \|[[A, Q_\lambda^{(j)}], Q_\mu^{(k)}]\|_{\mathcal{L}(\mathcal{H})} \leq \liminf_{n\to\infty} \sum_{\substack{(\lambda,\mu)\in\mathbb{Z}^2 \\ 0\leq j,k\leq 1}} \|[[A_n, Q_\lambda^{(j)}], Q_\mu^{(k)}]\|_{\mathcal{L}(\mathcal{H})}.$$

Theorem 1.4 is then an easy consequence of these facts. ∎

To derive Theorem 1.3, we shall construct, for any subsets E and F such that $E \subset F \subseteq \mathbb{Z}$, an almost right inverse of the operator i_{EF} defined in (2.4). Let $\Omega_{F\setminus E}$ be the ground state of $F \setminus E$. Let $\pi_{EF} : \mathcal{H}_E \to \mathcal{H}_F$ be the map

$$f \mapsto \pi_{EF}(f) = f \otimes \Omega_{F\setminus E}, \tag{2.12}$$

and let $\pi_{EF}^\star : \mathcal{H}_F \to \mathcal{H}_E$ be the adjoint operator. Note that $\pi_{EF}^\star \pi_{EF} = I$. For all A in $\mathcal{L}(\mathcal{H}_F)$ one defines $\rho_{FE}(A)$ in $\mathcal{L}(\mathcal{H}_E)$ by

$$\rho_{FE}(A) = \pi_{EF}^\star \circ A \circ \pi_{EF}. \tag{2.13}$$

One can easily see that, for each $A \in \mathcal{W}_2$,

$$\|\rho_{FE}(A)\|_{\mathcal{W}_2} \leq \|A\|_{\mathcal{W}_2}. \tag{2.14}$$

Also, if $E \subset F \subset G$ then

$$\rho_{GE} = \rho_{FE} \circ \rho_{GF}. \tag{2.15}$$

We shall study how an operator $A \in \mathcal{L}(\mathcal{H}_F)$ may be approximated by $i_{EF} \circ \rho_{FE}(A)$ when E is a subset of F, both finite.

PROPOSITION 2.2. *There exists $C > 0$ such that, for all finite subsets E and F of $\mathbb{Z}$ with $E \subset F$, and all A in $\mathcal{W}_2$ supported in F,*

$$\|A - i_{EF} \circ \rho_{FE}(A)\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})} \leq C \sum_{\substack{\lambda \in F\setminus E \\ 1\leq j+k\leq 2}} \|(\operatorname{ad} P_\lambda)^j (\operatorname{ad} Q_\lambda)^k A\|_{\mathcal{L}(\mathcal{H})}. \tag{2.6}$$

This proposition is proven in Appendix A. Let us show how it implies Theorem 1.3.

Proof of Theorem 1.3. Let $A \in \mathcal{W}_2$. Set $A_n = i_{\Lambda_n\mathbb{Z}} \circ \rho_{\mathbb{Z}\Lambda_n}(A)$. The A_n are in $\mathcal{W}_2$ with finite supports and $\|A_n\|_{\mathcal{W}_2} \leq \|A\|_{\mathcal{W}_2}$. If $m < n$ then Proposition 2.5 yields

$$\begin{aligned} \|A_m - A_n\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})} &\leq \|\rho_{\Lambda_n\Lambda_m}(A_n) - A_n\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})} \\ &\leq C \sum_{\substack{\lambda\in\mathbb{Z}\setminus\Lambda_m \\ 1\leq j+k\leq 2}} \|(\operatorname{ad} P_\lambda)^j (\operatorname{ad} Q_\lambda)^k A\|_{\mathcal{L}(\mathcal{H})}. \end{aligned}$$

The latter sequence goes to 0 as $m \to \infty$ if $A \in \mathcal{W}_2$. Consequently, the sequence A_n converges, in $\mathcal{L}(\mathcal{H}^2,\mathcal{H})$, to an element $B \in \mathcal{L}(\mathcal{H}^2,\mathcal{H})$. From Theorem 1.4, B is in $\mathcal{W}_2$ and $A_n f$ strongly converges to Bf for all $f \in \mathcal{H}$. Let us check that $B = A$. To this end, let $f, g \in \mathcal{D}$. If Λ_n contains the support of f then $A_n f = \pi_{\Lambda_n\mathbb{Z}}\pi^\star_{\Lambda_n\mathbb{Z}} Af$. Therefore, if Λ_n also contains the support of g then

$$\langle A_n f, g\rangle = \langle \pi_{\Lambda_n\mathbb{Z}}\pi^\star_{\Lambda_n\mathbb{Z}} Af, \pi_{\Lambda_n\mathbb{Z}}\pi_{E_2\Lambda_n}\psi\rangle = \langle \pi^\star_{\Lambda_n\mathbb{Z}} Af, \pi_{E_2\Lambda_n}\psi\rangle = \langle Af, g\rangle.$$

Since $A_n f$ strongly converges to Bf we have $\langle Af, g\rangle = \langle Bf, g\rangle$ for all f and g in $\mathcal{D}$. Since $\mathcal{D}$ is dense in $\mathcal{H}$ the equality $B = A$ is indeed true. As a consequence, A_n converges to A in $\mathcal{L}(\mathcal{H}^2,\mathcal{H})$ and the proof is finished. ■

Proposition 2.5 also implies the following result.

COROLLARY 2.3. *For all A and B in $\mathcal{W}_2$ with finite supports,*

$$\|[A, B]\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})} \leq C\|B\|_{\mathcal{W}_2} \sum_{\substack{\lambda\in\sigma(B) \\ 1\leq j+k\leq 2}} \|(\operatorname{ad} P_\lambda)^j (\operatorname{ad} Q_\lambda)^k A\|_{\mathcal{L}(\mathcal{H})} \tag{2.17}$$

where C does not depend on any of the parameters.

Proof. We make use of the operator ρ_{FE} for $F = \sigma(A) \cup \sigma(B)$ and $E = F \setminus \sigma(B)$. It is known that $\rho_{FE}(A)$ commutes with B since its support does not intersect $\sigma(B)$. Hence

$$\begin{aligned} \|[A, B]\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})} &= \|[A - \rho_{FE}(A), B]\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})} \\ &\leq [\|B\|_{\mathcal{L}(\mathcal{H}^2)} + \|B\|_{\mathcal{L}(\mathcal{H})}]\|A - \rho_{FE}(A)\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})}. \end{aligned}$$

From Proposition 2.1,

$$\|B\|_{\mathcal{L}(\mathcal{H}^2)} + \|B\|_{\mathcal{L}(\mathcal{H})} \leq C\|B\|_{\mathcal{W}_2}.$$

Using Proposition 2.5, we find a constant $C > 0$, which does not depend on any of the parameters, such that (2.17) is satisfied. ■

3. Perturbation potentials and commutators. We have to express the perturbation potentials V_λ and $V_{\lambda\mu}$, satisfying hypotheses (H1) and (H2) of Section 1, as integrals of Weyl operators, and to verify precisely that, under (H1) and (H2), these integrals are convergent and define operators in Sobolev spaces. We shall do the same for the commutators of $V_{\lambda\mu}$ with elements of $\mathcal{W}_1$, or with Segal operators, and for iterated commutators. These norm estimates will be used in the following sections.

Partial Sobolev spaces. The Sobolev spaces defined in Section 2 are not Hilbert spaces. Nevertheless, for any finite subset like Λ_n, the space $\mathcal{H}^k_{\Lambda_n}$ can be endowed with a Hilbert space norm which is equivalent, for each fixed n, to the norm of Section 2. As an example, for $k = 1$, one may set

$$\|f\|^2_{\mathcal{H}^1_{\Lambda_n}} = \sum_{\substack{\lambda\in\Lambda_n\\ j=0,1}} \|Q^{(j)}_\lambda f\|^2_{\mathcal{H}_{\Lambda_n}}.$$

For all n, these norms and those of Section 2 are equivalent but the constant involved in the inequality depends on n.

Let us choose an orthonormal basis $(\varphi_\alpha)_{\alpha\geq 0}$ in the Hilbert space $\mathcal{H}_{\Lambda_n^c}$. We define a map Ψ_α from $\mathcal{H}_{\Lambda_n}$ into $\mathcal{H}$ by $\Psi_\alpha(f) = f\otimes\varphi_\alpha$. The adjoint map from $\mathcal{H}$ to $\mathcal{H}_{\Lambda_n}$ is denoted by $\Psi^\star_\alpha$. For all f in $\mathcal{H}$ we have

$$\|f\|^2 = \sum_{\alpha\geq 0} \|\Psi^\star_\alpha f\|^2_{\mathcal{H}_{\Lambda_n}}.$$

Then we define the space $\mathcal{H}^k(\Lambda_n)$ as the set of all f such that the following norm is finite:

$$\|f\|^2_{\mathcal{H}^k(\Lambda_n)} = \sum_{\alpha\geq 0} \|\Psi^\star_\alpha f\|^2_{\mathcal{H}^k_{\Lambda_n}}. \tag{3.1}$$

Thus, $\mathcal{H}^k \subset \mathcal{H}^k(\Lambda_n) \subset \mathcal{H}$ if $k \geq 0$. When $k = 1$, an element f of $\mathcal{H}$ is in $\mathcal{H}^1$ if it belongs to $\mathcal{H}^1(\Lambda_n)$ and if, for all $\lambda \in \Lambda_n^c$, one has $Q^{(j)}_\lambda f \in \mathcal{H}$, and the sequence $\|Q^{(j)}_\lambda\|_{\mathcal{H}}$ ($\lambda \in \mathbb{Z}$, $j = 0, 1$) are bounded. This property may be used only for fixed n.

Partial Sobolev spaces with negative order. Let $\mathcal{H}^{-k}(\Lambda_n)$ be the anti-dual of $\mathcal{H}^k(\Lambda_n)$ ($k = 1, 2$). Thus

$$\mathcal{H}^2(\Lambda_n) \subset \mathcal{H}^1(\Lambda_n) \subset \mathcal{H} \subset \mathcal{H}^{-1}(\Lambda_n) \subset \mathcal{H}^{-2}(\Lambda_n).$$

If an operator $\Phi \in \mathcal{L}(\mathcal{H}^1_{\Lambda_n}, \mathcal{H})$ satisfies $\langle\Phi f, g\rangle = \langle f, \Phi g\rangle$ for all f and g in $\mathcal{H}^1_{\Lambda_n}$, where $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathcal{H}$, then, for all $f \in \mathcal{H}$, the map $g \mapsto \langle f, \Phi g\rangle$ is an element of $\mathcal{H}^{-1}(\Lambda_n)$ denoted here by Φf. Thus, the operator Q_λ is bounded from $\mathcal{H}^k(\Lambda_n)$ into $\mathcal{H}^{k-1}(\Lambda_n)$ ($-1 \le k \le 2$, $\lambda \in \Lambda_n$). We shall check that similar considerations are also valid for the operators $i[P_\lambda, V_{\Lambda_n}]$. The commutator of these two types of operators is in $\mathcal{L}(\mathcal{H}^1(\Lambda_n), \mathcal{H}^{-1}(\Lambda_n))$.

Perturbation potentials and Weyl operators. If ξ is a real sequence in $\ell^2(\mathbb{Z})$ with finite support then the Segal operator $\Pi(\xi, 0)$ defined in (2.1) is also written as $\sum \xi_\lambda Q_\lambda$. Since the hypotheses on the perturbation potentials involve only the derivatives of order 2 and 3, the following function will appear below:

$$x \mapsto F(x) = e^{ix} - 1 - ix = i^2x^2 \int_0^1 (1-\theta)e^{i\theta x}\, d\theta. \tag{3.2}$$

Let $V_{\lambda_1\lambda_2}$ ($\lambda_1 \ne \lambda_2$) be multiplication by a real function $v_{\lambda_1\lambda_2}$ on $\mathbb{R}^2$. It is an unbounded operator in $L^2(\mathbb{R}^2)$ or, under the identification of these two spaces, in $\mathcal{H}_{\{\lambda_1\lambda_2\}}$. If $v_{\lambda_1\lambda_2}$ satisfes the hypothesis (H1), then

$$\begin{aligned} V_{\lambda_1\lambda_2} = v_{\lambda_1\lambda_2}(0)I &+ \sum_{1\le j\le 2} (\partial_{\lambda_j} v_{\lambda_1\lambda_2})(0) Q_{\lambda_j} \\ &+ (2\pi)^{-2} \int_{\mathbb{R}^2} \widehat{v_{\lambda_1\lambda_2}}(\xi) F(\xi_{\lambda_1} Q_{\lambda_1} + \xi_{\lambda_2} Q_{\lambda_2})\, d\xi. \end{aligned} \tag{3.3}$$

Under the hypothesis (H1) the integral is convergent and defines a bounded operator from $\mathcal{H}^2$ to $\mathcal{H}$.

Commutators. In order to study the commutators of $V_{\lambda_1\lambda_2}$ with other operators, we shall use the following relations, valid for any operators X and A in a Banach space, and for the function F of (3.2):

$$[e^{iX}, A] = i\int_0^1 e^{i\theta X}[X,A]e^{i(1-\theta)X}\, d\theta, \tag{3.4}$$

$$[F(X), A] = i[X,A](e^{iX} - I) + i^2 \int_0^1 (1-\theta)e^{i\theta X}[X,[X,A]]e^{i(1-\theta)X} d\theta. \tag{3.5}$$

Equality (3.5) is first applied with $X = \xi_{\lambda_1} Q_{\lambda_1} + \xi_{\lambda_2} Q_{\lambda_2}$ and $A = P_{\lambda_j}$ ($j = 1, 2$). Using equality (3.3) for $V_{\lambda_1\lambda_2}$ we obtain

$$[P_{\lambda_j}, V_{\lambda_1\lambda_2}] = -i(\partial_{\lambda_j} v_{\lambda_1\lambda_2})(0)I + \sum_{1\le k\le 2} A^{jk}_{\lambda_1\lambda_2} Q_{\lambda_k},$$

$$A^{jk}_{\lambda_1\lambda_2} = (2\pi)^{-2} \int_{\mathbb{R}^2\times[0,1]} \widehat{v_{\lambda_1\lambda_2}}(\xi)\xi_{\lambda_j}\xi_{\lambda_k} e^{i\theta(\xi_{\lambda_1}Q_{\lambda_1} + \xi_{\lambda_2}Q_{\lambda_2})}\, d\xi\, d\theta.$$

Under the assumption (H1), this integral converges and defines an operator $A^{jk}_{\lambda_1\lambda_2}$ in $\mathcal{L}(\mathcal{H})$, with $\mathcal{O}(e^{-\gamma_0|\lambda_1-\lambda_2|})$ norm. Each single site operator is similarly treated. Note that the integrals are then integrals on $\mathbb{R}$. We deduce the following proposition concerning the potential V_{Λ_n} defined in (1.1) and (1.9):

Proposition 3.1. *Under the hypotheses* (H1) *and* (H2), *one may write*

$$[P_\lambda, V_{\Lambda_n}] = -ia^{(n)}_\lambda + \sum_{\mu\in\Lambda_n} W^{(n)}_{\lambda\mu} Q_\mu \tag{3.6}$$

where $a^{(n)}_\lambda$ is a real constant and $W^{(n)}_{\lambda\mu}$ is a bounded operator in $\mathcal{H}$. Moreover, there exists $C_1 > 0$, independent of λ, μ and n, such that

$$|a^{(n)}_\lambda| \le C_1, \qquad \|W^{(n)}_{\lambda\mu}\|_{\mathcal{L}(\mathcal{H})} \le C_1 e^{-\gamma_0|\lambda-\mu|}. \tag{3.7}$$

We can also apply the commutation formula (3.5), still setting $X = \xi_{\lambda_1}Q_{\lambda_1} + \xi_{\lambda_2}Q_{\lambda_2}$, but with $A \in \mathcal{W}_2$. Inserting the expression (3.3) for $V_{\lambda_1\lambda_2}$ and using hypothesis (H1), we obtain the following proposition.

Proposition 3.2. *For all A in $\mathcal{W}_2$, and all λ and μ in $\mathbb{Z}$, the commutator $[A, V_{\lambda\mu}]$ is in $\mathcal{L}(\mathcal{H}^1, \mathcal{H})$. There is $C > 0$, independent of all the parameters, such that*

$$\|[A, V_{\lambda\mu}]\|_{\mathcal{L}(\mathcal{H}^1,\mathcal{H})} \le Ce^{-\gamma_0|\lambda-\mu|} \sum_{1\le j+k\le 2} \|(\operatorname{ad} Q_\lambda)^j (\operatorname{ad} Q_\mu)^k A]\|_{\mathcal{L}(\mathcal{H})}. \tag{3.8}$$

Double commutators. If A, B and X are three operators such that $[X, B]$ is the identity operator up to a multiplicative factor, and if F is the function given by (3.2), then (3.4) and (3.5) imply that

$$[[F(X), B], A] = i^2[X, B] \int_0^1 e^{i\theta X}[X, A]e^{i(1-\theta)X}\, d\theta. \tag{3.9}$$

This formula is applied with $X = \xi_{\lambda_1}Q_{\lambda_1} + \xi_{\lambda_2}Q_{\lambda_2}$, $B = P_{\lambda_j}$ $(j = 1, 2)$ and $A \in \mathcal{L}(\mathcal{H})$ (in particular $A \in \mathcal{W}_1$). Inserting the expression (3.3) for $V_{\lambda_1\lambda_2}$ and using (H1), one gets

$$[[V_{\lambda_1\lambda_2}, P_{\lambda_j}], A] = \sum_{1\le k\le 2} S^{jk}_{\lambda_1\lambda_2}([A, Q_{\lambda_k}]) \tag{3.10}$$

where we have set, for all Φ in $\mathcal{L}(\mathcal{H}^2(\Lambda_n), \mathcal{H}^{-2}(\Lambda_n))$,

$$S^{jk}_{\lambda_1\lambda_2}(\Phi) = (2\pi)^{-2} \int_{\mathbb{R}^2\times[0,1]} \widehat{v_{\lambda_1\lambda_2}}(\xi)\, \xi_{\lambda_j}\xi_{\lambda_k} e^{i\theta X(\xi)} \circ \Phi \circ e^{i(1-\theta)X(\xi)}\, d\xi\, d\theta \tag{3.11}$$

with the notation $X(\xi) = \xi_{\lambda_1}Q_{\lambda_1} + \xi_{\lambda_2}Q_{\lambda_2}$.

Next we shall deduce the following proposition.

PROPOSITION 3.3. *For all* λ *and* μ *in* Λ_n $(n \geq 1)$, *there exists a continuous linear map* $K_{\lambda\mu}$ *from* $\mathcal{L}(\mathcal{H}^2(\Lambda_n), \mathcal{H}^{-2}(\Lambda_n))$ *into itself, leaving the subspaces* $\mathcal{L}(\mathcal{H}^1(\Lambda_n), \mathcal{H}^{-1})(\Lambda_n)$ *and* $\mathcal{L}(\mathcal{H})$ *invariant, such that, for all* A *in* $\mathcal{L}(\mathcal{H})$,

$$[A, [P_\lambda, V_{\Lambda_n}]] = \sum_{\mu\in\Lambda_n} K_{\lambda\mu}([A, Q_\mu]). \tag{3.12}$$

Moreover, when restricted to $\mathcal{L}(\mathcal{H})$, $K_{\lambda\mu}$ *is in* $\mathcal{L}(\mathcal{L}(\mathcal{H}))$, *and there exists* $C_0 > 0$, *independent of* n, λ *and* μ, *such that*

$$\|K_{\lambda\mu}\|_{\mathcal{L}(\mathcal{L}(\mathcal{H}))} \leq C_0 e^{-\gamma_0|\lambda-\mu|}. \tag{3.13}$$

Proof. The operator $\Phi \mapsto S^{jk}_{\lambda_1\lambda_2}(\Phi)$ maps $\mathcal{L}(\mathcal{H}^2(\Lambda_n), \mathcal{H}^{-2}(\Lambda_n))$ into itself. It also maps $\mathcal{L}(\mathcal{H})$ into itself, with norm $\leq C_0 e^{-\gamma_0|\lambda-\mu|}$. For one site potentials V_λ, we define similar operators S_λ such that $[[V_\lambda, P_\lambda], A] = S_\lambda([A, Q_\lambda])$ for all $A \in \mathcal{W}_1$. We then set, for all λ and μ in Λ_n such that $\lambda \neq \mu$,

$$K^{(n)}_{\lambda\mu}(\Phi) = \begin{cases} S^{12}_{\lambda\mu}(\Phi) + S^{21}_{\mu\lambda}(\Phi) & \text{if } |\lambda-\mu| \geq 2, \\ -b\Phi + S^{12}_{\lambda\mu}(\Phi) + S^{21}_{\mu\lambda}(\Phi) & \text{if } |\lambda-\mu| = 1, \end{cases}$$

and if $\lambda = \mu$,

$$K^{(n)}_{\lambda\lambda}(\Phi) = a\Phi + S_\lambda(\Phi) + \sum_{\substack{\mu\in\Lambda_n \\ \mu\neq\lambda}} (S^{11}_{\lambda\mu}(\Phi) + T^{22}_{\mu\lambda}(\Phi)).$$

The equality (3.12) and the estimate (3.13) follow. ∎

In the next proposition, we shall define a constant S_γ which gives, in (1.20), an upper bound for the Lieb–Robinson group velocity. This will be proved in Section 8.

PROPOSITION 3.4. *Under the hypotheses* (H1) *and* (H2), *for all* γ *in* $(0, \gamma_0)$ *(or in* $(0, \infty)$ *in the case of interaction with nearest neighbors), there exists* $S_\gamma > 0$ *such that, for all* n, *and all* λ *and* ν *in* Λ_n,

$$\sum_{\mu\in\Lambda_n} \|K_{\lambda\mu}\|_{\mathcal{L}(\mathcal{L}(\mathcal{H}))} e^{-\gamma|\mu-\nu|} \leq S_\gamma e^{-\gamma|\lambda-\nu|},$$

$$\sum_{\mu\in\Lambda_n} \|W_{\lambda\mu}\|_{\mathcal{L}(\mathcal{H})} e^{-\gamma|\mu-\nu|} \leq S_\gamma e^{-\gamma|\lambda-\nu|},$$

where the $K_{\lambda\mu}$ *are the operators constructed in Proposition* 3.3 *and where the* $W_{\lambda\mu}$ *are those of Proposition* 3.1.

Triple commutators. If X, A, B, C are operators such that $[X, B]$ and $[X, C]$ are equal to the identity operator up to a multiplicative factor, and if F is the function defined by (3.2), then we deduce from (3.9) and

(3.4) that

$$\big[[[F(X),B],A],C\big] = i^2[X,B]\int_0^1 e^{i\theta X}[[X,A],C]e^{i(1-\theta)X}d\theta$$
$$+\, i^3[X,B][X,C]\int_0^1 e^{i\theta X}[X,A]e^{i(1-\theta)X}\,d\theta.$$

We shall apply this formula with $X = \xi_{\lambda_1}Q_{\lambda_1} + \xi_{\lambda_2}Q_{\lambda_2}$, $B = P_{\lambda_j}$ $(j = 1,2)$, $A \in \mathcal{W}_2$ and C being a Segal operator. Inserting the expression of $V_{\lambda_1\lambda_2}$ given in (3.3) and using (H1), we obtain

$$\big[[[V_{\lambda_1\lambda_2},P_{\lambda_j}],A],C\big] = \sum_{1\le k\le 2}\big(S^{jk}_{\lambda_1\lambda_2}([[A,Q_{\lambda_k}],C]) + T^{jk}_{\lambda_1\lambda_2}([[A,Q_{\lambda_k}],C])\big)$$

where $S^{jk}_{\lambda_1\lambda_2}(\Phi)$ is defined in (3.11) and $T^{jk}_{\lambda_1\lambda_2}(\Phi,C)$ is defined by

$$T^{jk}_{\lambda_1\lambda_2}(\Phi,C) = \int_{\mathbb{R}^2\times[0,1]} \widehat{v_{\lambda_1\lambda_2}}(\xi)\xi_{\lambda_j}\xi_{\lambda_k}[X(\xi),C]e^{i\theta X(\xi)}\Phi e^{i(1-\theta)X(\xi)}\,\frac{d\xi\,d\theta}{(2\pi)^2}.$$

If C is a Segal operator (a linear combination of P_λ and Q_λ) then $[X(\xi),C]$ is a constant and the above integral converges by (H1). It is at this point that the hypothesis "$|\xi|^3\widehat{v_{\lambda\mu}}(\xi)$ belongs to $L^1(\mathbb{R}^2)$" is involved. We proceed similarly for all single site operators V_λ. Summing up as in Proposition 3.3, one obtains the next result:

Proposition 3.5. *For all λ and μ in Λ_n $(n \ge 1)$, and every Segal operator Ψ, there exists a map $\Phi \mapsto R_{\lambda\mu}(\Phi,\Psi)$ from $\mathcal{L}(\mathcal{H}^1(\Lambda_n),\mathcal{H}^{-1}(\Lambda_n))$ into itself such that, for all $A \in \mathcal{L}(\mathcal{H})$ supported in Λ_n,*

$$\big[[A,[P_\lambda,V_{\Lambda_n}]],\Psi\big] = \sum_{\mu\in\Lambda_n}\big(K_{\lambda,\mu}([[A,Q_\mu],\Psi]) + R_{\lambda,\mu}([A,Q_\mu],\Psi)\big)$$

where $\Phi \mapsto K_{\lambda,\mu}(\Phi)$ is the map of Proposition 3.3. If Φ is in $\mathcal{L}(\mathcal{H})$ then $R_{\lambda\mu}(\Phi,\Psi)$ is in $\mathcal{L}(\mathcal{H})$. One has $R_{\lambda\mu}(\Phi,Q_\rho) = 0$ for all ρ. Also, $R_{\lambda\mu}(\Phi,P_\rho) = 0$ except when the set $\{\lambda,\mu,\rho\}$ has only two distinct elements ($\lambda = \mu$, or $\lambda = \rho$, or $\mu = \rho$). In the latter case,

$$\|R_{\lambda\mu}(\Phi,P_\rho)\|_{\mathcal{L}(\mathcal{H})} \le C_0 e^{-\gamma_0|\lambda-\mu|}\|\Phi\|_{\mathcal{L}(\mathcal{H})} \quad \text{if } \lambda \ne \mu,$$
$$\|R_{\lambda\mu}(\Phi,P_\rho)\|_{\mathcal{L}(\mathcal{H})} \le C_0 e^{-\gamma_0|\lambda-\rho|}\|\Phi\|_{\mathcal{L}(\mathcal{H})} \quad \text{if } \lambda = \mu.$$

4. Evolution of the position and momentum operators. Using the Fock space notations, the Hamiltonian H_{Λ_n} in (1.1) is written as

$$H_{\Lambda_n} = \sum_{\lambda\in\Lambda_n}\Big[P_\lambda^2 + \frac{a}{2}Q_\lambda^2\Big] - b\sum_{\lambda=-n}^{n-1} Q_\lambda Q_{\lambda+1} + V^{\text{pert}}_{\Lambda_n} \tag{4.1}$$

where the operator $V_{\Lambda_n}^{\text{pert}}$ is expressed as the sum (1.9). The terms in the sum satisfy (H1) and (H2); recall that these two hypotheses are analyzed in Section 3. Let us start by giving the domain of self-adjointness of H_{Λ_n}.

PROPOSITION 4.1. *In the Hilbert space $\mathcal{H}_{\Lambda_n}$, the operator H_{Λ_n} is self-adjoint with domain $\mathcal{H}^2_{\Lambda_n}$. The operator $e^{itH_{\Lambda_n}}$ is bounded in $\mathcal{H}^k_{\Lambda_n}$ ($k = 0,1,2$). The operator $e^{itH_{\Lambda_n}} \otimes I_{\Lambda_n^c}$ is bounded in $\mathcal{H}^k(\Lambda_n)$ defined in Section 3 ($-2 \le k \le 2$).*

Proof. We know that $\mathcal{H}_{\Lambda_n}$ is naturally identified with $L^2(\mathbb{R}^{\Lambda_n}) = L^2(\mathbb{R}^{2n+1})$ in such a way that the operators P_λ and Q_λ become

$$P_\lambda = \frac{1}{i}\frac{\partial}{\partial x_\lambda}, \qquad Q_\lambda = x_\lambda.$$

The spaces $\mathcal{H}^k_{\Lambda_n}$ are then identified with the usual spaces B^k of the theory of globally elliptic operators (cf. Helffer [HE]). When $V_{\Lambda_n}^{\text{pert}} = 0$, the operator H_{Λ_n} is a Schrödinger operator, where the potential is a positive definite quadratic form (if $a > 2b > 0$). In this case, it is well-known that H_{Λ_n} is self-adjoint with domain $B^2 = \mathcal{H}^2_{\Lambda_n}$. Let us show that the addition of $V_{\Lambda_n}^{\text{pert}}$ does not affect this result. With the preceding identification and under our hypotheses, V_λ and $V_{\lambda\mu}$ are multiplications by functions v_λ and $v_{\lambda\mu}$ with second-order derivatives going to 0 at infinity. (These functions are the Fourier transforms of functions in $L^1(\mathbb{R})$ or in $L^1(\mathbb{R}^2)$.) Consequently, the functions $v_\lambda(x_\lambda)/|x_\lambda|^2$ and $v_{\lambda\mu}(x_\lambda, x_\mu)/[|x_\lambda|^2 + |x_\mu|^2]$ go to 0 at infinity. The above proposition thus follows from Kato–Rellich's theorem. As a consequence, $e^{itH_{\Lambda_n}}$ is a well-defined bounded operator in $\mathcal{H}$ and in the domain of H_{Λ_n}, that is, in $\mathcal{H}^2_{\Lambda_n}$. By interpolation it is also bounded in $\mathcal{H}^1_{\Lambda_n}$. The last statement of the proposition comes from (3.1) if $0 \le k \le 2$ and is deduced by duality if $k \le 0$. ∎

Consequently, if $A \in \mathcal{L}(\mathcal{H}^k(\Lambda_n), \mathcal{H}^{k'}(\Lambda_n))$, then the operator

$$\alpha^{(t)}_{\Lambda_n}(A) = (e^{itH_{\Lambda_n}} \otimes I) \circ A \circ (e^{-itH_{\Lambda_n}} \otimes I) \tag{4.2}$$

is also in $\mathcal{L}(\mathcal{H}^k(\Lambda_n), \mathcal{H}^{k'}(\Lambda_n))$. In particular, $\alpha^{(t)}_{\Lambda_n}(Q^{(j)}_\lambda) \in \mathcal{L}(\mathcal{H}^1(\Lambda_n), \mathcal{H})$ ($\lambda \in \Lambda_n$).

PROPOSITION 4.2. *For all λ and μ in Λ_n, there exist C^1 maps $t \mapsto A^{(n)}_{\lambda\mu}(t)$, $t \mapsto B^{(n)}_{\lambda\mu}(t)$, and $t \mapsto R^{(n)}_\lambda(t)$ from $\mathbb{R}$ into $\mathcal{L}(\mathcal{H})$ such that (omitting the superscript n in the expressions)*

$$\alpha^{(t)}_{\Lambda_n}(Q_\lambda) = \sum_{\mu\in\Lambda_n} (A_{\lambda\mu}(t)Q_\mu + B_{\lambda\mu}(t)P_\mu) + R_\lambda(t), \tag{4.3}$$

$$\alpha^{(t)}_{\Lambda_n}(P_\lambda) = \sum_{\mu\in\Lambda_n} (A'_{\lambda\mu}(t)Q_\mu + B'_{\lambda\mu}(t)P_\mu) + R'_\lambda(t). \tag{4.4}$$

Moreover, for all γ *in* $(0, \gamma_0)$*, and all* $M > \sqrt{S_\gamma}$ *(where* S_γ *is the constant of Proposition* 3.4*), there exists* $C > 0$ *such that*

$$\|A_{\lambda\mu}(t)\| + \|B_{\lambda\mu}(t)\| + \|A'_{\lambda\mu}(t)\| + \|B'_{\lambda\mu}(t)\| \leq Ce^{M|t|}e^{-\gamma|\lambda-\mu|}, \tag{4.5}$$

$$\|R_\lambda(t)\| + \|R'_\lambda(t)\| \leq Ce^{M|t|}. \tag{4.6}$$

Proof. First step. We shall study the differential system satisfied by

$$Q_\lambda(t) = \alpha^{(t)}_{\Lambda_n}(Q_\lambda), \qquad P_\lambda(t) = \alpha^{(t)}_{\Lambda_n}(P_\lambda).$$

One observes that $t \mapsto Q_\lambda(t)$ and $t \mapsto P_\lambda(t)$ are C^1 functions from $\mathbb{R}$ into $\mathcal{L}(\mathcal{H}^1(\Lambda_n), \mathcal{H})$ satisfying

$$Q'_\lambda(t) = P_\lambda(t), \qquad P'_\lambda(t) = -i\alpha^{(t)}_{\Lambda_n}([P_\lambda, V_{\Lambda_n}]).$$

With the operators $W^{(n)}_{\lambda\mu}$ and the constant $a^{(n)}_\lambda$ of Proposition 3.1, it follows that

$$P'_\lambda(t) = -a^{(n)}_\lambda - i \sum_{\mu \in \Lambda_n} \alpha^{(t)}_{\Lambda_n}(W_{\lambda\mu}Q_\mu).$$

We define an operator in $\mathcal{L}(\mathcal{H})$ by setting

$$\widetilde{W}_{\lambda\mu}(t) = \alpha^{(t)}_{\Lambda_n}(W^{(n)}_{\lambda\mu}). \tag{4.7}$$

With these notations, the preceding system is written as

$$Q'_\lambda(t) = P_\lambda(t), \qquad P'_\lambda(t) = -a^{(n)}_\lambda - i \sum_{\mu \in \Lambda_n} \widetilde{W}_{\lambda\mu}(t) \circ Q_\mu(t). \tag{4.8}$$

Thus, $t \mapsto (Q_\lambda(t), P_\lambda(t))$ is the unique C^1 map from $\mathbb{R}$ into $\mathcal{L}(\mathcal{H}^1(\Lambda_n), \mathcal{H})$ which solves (4.8) and satisfies $Q_\lambda(0) = Q_\lambda$ and $P_\lambda(0) = P_\lambda$.

Second step. We shall now construct matrices $A_{\lambda\mu}(t), \ldots$ such that the right-hand side of (4.3) is also a solution to the same system (4.8) and satisfies the same initial data. First, we can find an operator-valued matrix $(A^0_{\lambda\mu}(t), A^1_{\lambda\mu}(t))$ in $\mathcal{L}(\mathcal{H})$ which solves

$$\frac{d}{dt}A^0_{\lambda\mu}(t) = A^1_{\lambda\mu}(t), \qquad \frac{d}{dt}A^1_{\lambda\mu}(t) = -i \sum_{\nu \in \Lambda_n} \widetilde{W}_{\lambda\nu}(t) A^0_{\nu\mu}(t), \tag{4.9}$$

$$A^0_{\lambda\mu}(0) = \delta_{\lambda\mu} I, \qquad A^1_{\lambda\mu}(0) = 0.$$

Indeed, from Propositions 3.1 and 3.4 one sees that the hypotheses in Proposition B.1 (Appendix B) are satisfied for all $\gamma \in (0, \gamma_0)$. Thus, there exists a solution of (4.9) satisfying the above initial condition, and also, if $M > \sqrt{S_\gamma}$,

$$\|A^j_{\lambda\mu}(t)\|_{\mathcal{L}(\mathcal{H})} \leq C(M, \gamma)e^{M|t|}e^{-\gamma|\lambda-\mu|}. \tag{4.10}$$

Analogously, we construct an operator-valued matrix $(B^0_{\lambda\mu}(t), B^1_{\lambda\mu}(t))$ which solves the same system (4.9), satisfies the same estimates (4.10) and the following initial conditions:

$$B^0_{\lambda\mu}(0) = 0, \qquad A^1_{\lambda\mu}(0) = \delta_{\lambda\mu} I.$$

From Remark 2 in Appendix B, one may find operators $(R^0_\lambda(t), R^1_\lambda(t))$ in $\mathcal{L}(\mathcal{H})$ which solve

$$\frac{d}{dt} R^0_\lambda(t) = R^1_\lambda(t), \qquad \frac{d}{dt} R^1_\lambda(t) = -i \sum_{\nu\in\Lambda_n} \widetilde{W}_{\lambda\nu}(t) R^0_\nu(t) + i a^{(n)}_\lambda,$$
$$R^0_\lambda(0) = R^1_\lambda(0) = 0,$$
$$\|R^j_\lambda(t)\|_{\mathcal{L}(\mathcal{H})} \le C(M,\gamma) e^{M|t|} \sum_{\mu\in\Lambda_n} e^{-\gamma|\lambda-\mu|} |a_\mu|, \qquad j = 0, 1.$$

We define operators in $\mathcal{L}(\mathcal{H}^1(\Lambda_n), \mathcal{H})$ by

$$\widetilde{Q}^j_\lambda(t) = \sum_{\mu\in\Lambda_n} [A^j_{\lambda\mu}(t) Q_\mu + B^j_{\lambda\mu}(t) P_\mu] + R^j_\lambda(t), \qquad j = 0, 1.$$

These functions satisfy the same system (4.8) as $Q^j_\lambda(t)$, with the same initial conditions $\widetilde{Q}^0_\lambda(0) = Q_\lambda$, $\widetilde{Q}^1_\lambda(0) = P_\lambda$. Uniqueness shows $\widetilde{Q}^0_\lambda(t) = Q_\lambda(t)$ and $\widetilde{Q}^1_\lambda(t) = P_\lambda(t)$, thus the equalities (4.3) and (4.4) hold and the estimates (4.5) and (4.6) are valid. ∎

Example (The cyclic quadratic case). In the case of a positive definite quadratic form potential (without perturbation potentials), it is well-known that the equalities (4.3) and (4.4) are valid with $R_\lambda(t) = 0$ and with the operators $A_{\lambda\mu}(t)$ and $B_{\lambda\mu}(t)$ being real numbers. The following classical proposition summarizes this situation:

Proposition 4.3. *In the case where the potentials V_λ and $V_{\lambda\mu}$ (perturbation potentials) vanish, the operators $\alpha^{(t)}_{\Lambda_n}(Q_\lambda)$ and $\alpha^{(t)}_{\Lambda_n}(P_\lambda)$ satisfy equalities (4.2) and (4.3) where $R^{(n)}_\lambda(t) = 0$ and the $A^{(n)}_{\lambda\mu}(t)$ and $B^{(n)}_{\lambda\mu}(t)$ are real numbers. The matrices $A^{(n)}(t)$ and $B^{(n)}(t)$ are related to the matrix W_n of the quadratic form $V^{\mathrm{quad}}_{\Lambda_n}$ in the canonical basis by the equalities*

$$A^{(n)}(t) = \cos(t\sqrt{W_n}), \qquad B^{(n)}(t) = -\frac{\sin(t\sqrt{W_n})}{\sqrt{W_n}}.$$

One may estimate the matrix elements $A_{\lambda\mu}(t)$ and $A_{\lambda\mu}(t)$ using Proposition 4.2. However, in some cases, the inequalities of Proposition 4.2 together with the Lieb–Robinson inequalities can be strongly improved and explicitly written down. This is precisely the case if the perturbation potential vanishes, and if the quadratic potential takes the following form (with an interaction between the two ends of the linear chain):

$$V_{\Lambda_n}^{\mathrm{cycl}}(x) = \frac{a}{2}\,|x|^2 - b\sum_{\lambda=-n}^{n-1} x_\lambda x_{\lambda+1} - b x_n x_{-n}.$$

In that case, we can make the estimates of Proposition 4.2 more precise if the distance $\mathrm{d}(\lambda,\mu) = |\lambda-\mu|$ is replaced by the cyclic distance on Λ_n, $d_n(\lambda,\mu) = d(\lambda-\mu,(2n+1)\mathbb{Z})$.

These improved estimates follow on from [N-R-S-S] in the cyclic quadratic case. Let us give here a simplified proof of a perhaps less precise type of estimates.

In the cyclic quadratic case, the analysis of chains of oscillators involves the dispersion relations $\omega(\theta) = \sqrt{a - 2b\cos\theta}$ (cf. Cohen-Tannoudji [C-T]). It is natural to give a corresponding complex expression by setting

$$\Omega(z) = \sqrt{a - b(z+z^{-1})}. \tag{4.11}$$

This function is analytic in $\mathbb{C}\setminus\{(-\infty,z_1]\cup[z_2,0]\}$ where z_1 and z_2 are the roots of $bz^2 - az + b = 0$. Note, however, that the function $|\mathrm{Im}\,\Omega(z)|$ is well defined on $\mathbb{C}\setminus\{0\}$. Set, for all $\gamma>0$,

$$M(\gamma) = \sup_{|z|=e^\gamma} |\mathrm{Im}\,\Omega(z)|. \tag{4.12}$$

This function is well defined on $\mathbb{C}\setminus\{0\}$.

Proposition 4.4. *Under the above hypotheses, for all $\gamma>0$ there exists $C(\gamma)>0$, independent of n, such that the matrices $A^{(n)}(t)$ and $B^{(n)}(t)$ of Proposition 4.3 satisfy*

$$|A_{\lambda\mu}^{(n)}(t)| + |B_{\lambda\mu}^{(n)}(t)| + \left|\frac{d}{dt}A_{\lambda\mu}^{(n)}(t)\right| + \left|\frac{d}{dt}B_{\lambda\mu}^{(n)}(t)\right| \le C(\gamma)e^{|t|M(\gamma)}e^{-\gamma d_n(\lambda,\mu)}$$

where $M(\gamma)$ is defined in (4.12) *and $d_n(\lambda,\mu) = d(\lambda-\mu,(2n+1)\mathbb{Z})$.*

Proof. The matrix W_n of the quadratic form $V_{\Lambda_n}^{\mathrm{cycl}}$, and therefore all the matrices $A^{(n)}(t)$ and $B^{(n)}(t)$, are functions of the cyclic shift operator S_n defined in $\mathbb{R}^{\Lambda_n}$ by

$$S_n e_j = \begin{cases} e_{j+1} & \text{if } -n\le j<n,\\ e_{-n} & \text{if } j=n.\end{cases}$$

More precisely, $W_n = aI + bS_n + bS_n^{-1}$ and

$$A^{(n)}(t) = f(S_n,t),\qquad B^{(n)}(t) = g(S_n,t),\qquad C^{(n)}(t) = h(S_n,t),$$

where we have set, using the function $\Omega(z)$ defined in (4.11),

$$\begin{aligned} &f(z,t) = \cos(t\Omega(z)),\qquad g(z,t) = \frac{\sin(t\Omega(z))}{\Omega(z)},\\ &h(z,t) = -\sin(t\Omega(z))\Omega(z).\end{aligned} \tag{4.13}$$

These functions are analytic on $\mathbb{C}\setminus\{0\}$. The proof uses the following elementary lemma:

LEMMA 4.5. *Let S be a unitary operator in a Hilbert space $\mathcal{H}$. Let $f(z,t)$ be the function defined in* (4.11) *and* (4.13) *where $a > 2|b| > 0$. Then one can write, for all $t\in\mathbb{R}$,*

$$f(S,t)=\sum_{k\in\mathbb{Z}} c_k(t)S^k.$$

Moreover, for all $\gamma>0$, $t\in\mathbb{R}$ and $k\in\mathbb{Z}$,

$$|c_k(t)|\le e^{-\gamma|k|}\frac{1}{2\pi}\int_0^{2\pi}|f(e^{\gamma}e^{i\theta},t)|\,d\theta.$$

The same result holds for the functions g and h defined in (4.13).

End of proof of Proposition 4.4. Since $S_n^{2n+1}=I$, the sum in Lemma 4.5 can be written as a finite sum, and

$$A^{(n)}(t)=f(S_n,t)=\sum_{k=0}^{2n}a_k(t)S_n^k,\qquad a_k(t)=\sum_{p\in\mathbb{Z}}c_{k+p(2n+1)}(t),$$

where the $c_j(t)$ are the coefficients of Lemma 4.5. Consequently, if $-n\le\lambda\le\mu\le n$ and $\gamma>0$ then

$$|A^{(n)}_{\lambda\mu}(t)|=|\langle f(S_n,t)e_\lambda,e_\mu\rangle|=|a_{\mu-\lambda}(t)|\le\sum_{p\in\mathbb{Z}}|c_{\mu-\lambda+p(2n+1)}(t)|$$

$$\le\Big[\sum_{p\in\mathbb{Z}}e^{-\gamma|\mu-\lambda+p(2n+1)|}\Big]\frac{1}{2\pi}\int_0^{2\pi}|f(e^{\gamma}e^{i\theta},t)|\,d\theta.$$

There exist $C_1(\gamma)$ and $C_2(\gamma)$, independent of n, such that

$$\sum_{p\in\mathbb{Z}}e^{-\gamma|\mu-\lambda+p(2n+1)|}\le C_1(\gamma)e^{-\gamma d_n(\lambda,\mu)},$$

$$\frac{1}{2\pi}\int_0^{2\pi}|f(e^{\gamma}e^{i\theta},t)|\,d\theta\le C_2(\gamma)e^{|t|M(\gamma)}.$$

where $M(\gamma)$ is defined in (4.12). As a consequence,

$$|A^{(n)}_{\lambda\mu}(t)|\le C_1(\gamma)C_2(\gamma)e^{|t|M(\gamma)}e^{-\gamma d_n(\lambda,\mu)}.$$

Similar estimates for the matrix elements $B^{(n)}_{\lambda\mu}(t)$ together with their derivatives can be obtained. The conclusion of Proposition 4.4 follows. ■

5. Evolution of the commutators. From Proposition 4.1, the commutators $[A,\alpha^{(t)}_{\Lambda_n}(Q_\lambda)]$ and $[A,\alpha^{(t)}_{\Lambda_n}(P_\lambda)]$ are defined as operators mapping $\mathcal{H}^1(\Lambda_n)$ into $\mathcal{H}^{-1}(\Lambda_n)$ for all A in $\mathcal{L}(\mathcal{H})$ supported in Λ_n, and all $t\in\mathbb{R}$.

Proposition 5.1. *For all $A \in \mathcal{W}_1$ supported in Λ_n and all $t \in \mathbb{R}$ the commutators $[A, \alpha_{\Lambda_n}^{(t)}(Q_\lambda^{(j)})]$ are bounded in $\mathcal{H}$ ($\lambda \in \Lambda_n$, $0 \leq j \leq 1$). For all γ in $(0, \gamma_0)$ and all $M > \sqrt{S_\gamma}$ there exists $C(M, \gamma) > 0$ (independent of n) such that*

$$(5.1)\quad \|[A, \alpha_{\Lambda_n}^{(t)}(Q_\lambda^{(j)})]\|_{\mathcal{L}(\mathcal{H})} \leq C(M,\gamma) e^{M|t|} \sum_{\substack{\mu \in \Lambda_n \\ 0 \leq k \leq 1}} e^{-\gamma d(\lambda,\mu)} \|[A, Q_\mu^{(k)}]\|_{\mathcal{L}(\mathcal{H})}.$$

Proof. First step. Assuming first that A is only in $\mathcal{L}(\mathcal{H})$ we shall study the differential system satisfied by the functions

$$(5.2)\qquad \Phi_\lambda^{(j)}(t) = [A, \alpha_{\Lambda_n}^{(t)}(Q_\lambda^{(j)})], \qquad 0 \leq j \leq 1.$$

The Φ_λ^j's are C^1 maps from $\mathbb{R}$ into $\mathcal{L}(\mathcal{H}^1(\Lambda_n), \mathcal{H}^{-1}(\Lambda_n))$ and satisfy

$$\frac{d}{dt}\Phi_\lambda^0(t) = \Phi_\lambda^1(t),$$
$$\frac{d}{dt}\Phi_\lambda^1(t) = -i[A, \alpha_{\Lambda_n}^{(t)}([P_\lambda, V_{\Lambda_n}])] = -i\alpha_{\Lambda_n}^{(t)}([\alpha_{\Lambda_n}^{(-t)}(A), [P_\lambda, V_{\Lambda_n}]]).$$

Using the operators $K_{\lambda\mu}$ of Proposition 3.3, we have

$$[\alpha_{\Lambda_n}^{(-t)}(A), [P_\lambda, V_{\Lambda_n}]] = \sum_{\mu \in \Lambda_n} K_{\lambda\mu}([\alpha_{\Lambda_n}^{(-t)}(A), Q_\mu]).$$

Next define $\widetilde{K}_{\lambda\mu}(t)$, an operator mapping $\mathcal{L}(\mathcal{H}^1(\Lambda_n), \mathcal{H}^{-1}(\Lambda_n))$ into itself, by

$$(5.3)\qquad \widetilde{K}_{\lambda\mu}(t)(\Phi) = \alpha_{\Lambda_n}^{(t)}(K_{\lambda\mu}(\alpha_{\Lambda_n}^{(-t)}\Phi)), \qquad \forall \Phi \in \mathcal{L}(\mathcal{H}^1(\Lambda_n), \mathcal{H}^{-1}(\Lambda_n)).$$

With these notations the system becomes

$$(5.4)\qquad \frac{d}{dt}\Phi_\lambda^0(t) = \Phi_\lambda^1(t), \qquad \frac{d}{dt}\Phi_\lambda^1(t) = -i \sum_{\mu \in \Lambda_n} \widetilde{K}_{\lambda\mu}(t)(\Phi_\mu^0(t)).$$

Summing up, for all A in $\mathcal{L}(\mathcal{H})$ supported in Λ_n, the functions $\Phi_\lambda^j(t)$ defined in (5.2) ($\lambda \in \Lambda_n$) are C^1 from $\mathbb{R}$ to $\mathcal{L}(\mathcal{H}^1\Lambda_n), \mathcal{H}^{-1}(\Lambda_n))$. These maps are bounded independently of t and satisfy (5.4). This is the unique solution to (5.4) having these properties together with

$$(5.5)\qquad \Phi_\lambda^0(0) = [A, Q_\lambda], \qquad \Phi_\lambda^1(0) = [A, P_\lambda].$$

Second step. One can find operator-valued matrices $(A_{\lambda\mu}^0(t), A_{\lambda\mu}^1(t))$ in $\mathcal{L}(\mathcal{L}(\mathcal{H}))$ satisfying

$$(5.6)\qquad \frac{d}{dt}A_{\lambda\mu}^0(t) = A_{\lambda\mu}^1(t), \qquad \frac{d}{dt}A_{\lambda\mu}^1(t) = -i \sum_{\nu \in \Lambda_n} \widetilde{K}_{\lambda\nu}(t) \circ A_{\nu\mu}^0(t),$$

$$(5.7)\qquad A_{\lambda\mu}^0(0) = \delta_{\lambda\mu} I, \qquad A_{\lambda\mu}^1(0) = 0.$$

In (5.6) the composition is now the composition in $\mathcal{L}(\mathcal{L}(\mathcal{H}))$, and in (5.7) the identity operator is the identity in $\mathcal{L}(\mathcal{L}(\mathcal{H}))$. Indeed, for all γ in $(0, \gamma_0)$, the hypotheses in Proposition B.1 are satisfied, by Proposition 3.4. If γ is in $(0, \gamma_0)$ and $M > \sqrt{S_\gamma}$, there exists $C(M, \gamma)$ such that

$$\|A^j_{\lambda\mu}(t)\|_{\mathcal{L}(\mathcal{L}(\mathcal{H}))} \le C(M, \gamma) e^{-\gamma|\lambda-\mu|}. \tag{5.8}$$

We can find, by a similar construction, operator-valued matrices $B^0_{\lambda\mu}(t)$ and $B^1_{\lambda\mu}(t)$ of $\mathcal{L}(\mathcal{L}(\mathcal{H}))$ satisfying the same differential system (5.6) together with the same estimates (5.8) and the new initial conditions

$$B^0_{\lambda\mu}(0) = 0, \qquad B^1_{\lambda\mu}(0) = \delta_{\lambda\mu} I. \tag{5.9}$$

Suppose now that A belongs to $\mathcal{W}_1$ and is supported in Λ_n. The operators $[A, Q_\lambda]$ and $[A, P_\lambda]$ are in $\mathcal{L}(\mathcal{H})$. We then define the operators in $\mathcal{L}(\mathcal{H})$ by

$$\Psi^j_\lambda(t) = \sum_{\mu\in\Lambda_n} \big(A^j_{\lambda\mu}(t)([A, Q_\mu]) + B^j_{\lambda\mu}(t)([A, P_\mu])\big), \qquad j = 0, 1.$$

These functions, taking values in $\mathcal{L}(\mathcal{H})$, satisfy the same differential system (5.4) with the same initial conditions (5.5) as the functions $\Phi^j_\lambda(t)$ (being a priori in $\mathcal{L}(\mathcal{H}^1(\Lambda_n), \mathcal{H}^{-1}(\Lambda_n))$). Uniqueness shows that $\Phi^j_\lambda(t) = \Psi^j_\lambda(t)$. The functions $\Phi^j_\lambda(t)$ defined in (5.2) therefore have the stated properties. ∎

For all λ and μ in Λ_n the commutator $[Q^{(j)}_\lambda, \alpha^{(t)}_{\Lambda_n}(Q^{(k)}_\mu)]$ $(0 \le j, k \le 1)$ is bounded from $\mathcal{H}^1(\Lambda_n)$ into $\mathcal{H}^{-1}(\Lambda_n)$. We shall show that it is an element of $\mathcal{L}(\mathcal{H})$ and we shall estimate its norm.

Proposition 5.2. *Under the hypotheses* (H1) *and* (H2) *of Section* 1, *for all λ and μ in Λ_n, the commutator* $[Q^{(j)}_\lambda, \alpha^{(t)}_{\Lambda_n}(Q^{(k)}_\mu)]$ $(0 \le j, k \le 1)$ *is a bounded operator in $\mathcal{H}$. Moreover, for all γ in* $(0, \gamma_0)$ *and $M > \sqrt{S_\gamma}$, there exists $C(M, \gamma) > 0$* (*independent of n, t, λ and μ*) *such that*

$$\|[Q^{(j)}_\lambda, \alpha^{(t)}_{\Lambda_n}(Q^{(k)}_\mu)]\| \le C(M, \gamma) e^{M|t|} e^{-\gamma d(\lambda,\mu)}, \qquad 0 \le j, k \le 1.$$

Proof. Using the matrices $A^j_{\lambda\mu}(t)$ and $B^j_{\lambda\mu}(t)$ $(j = 0, 1)$ defined in the second step of the proof of Proposition 5.1 one shows that

$$[P_\lambda, \alpha^{(t)}_{\Lambda_n}(Q^{(j)}_\mu)] = A^j_{\lambda\mu}(t)(I), \qquad [Q_\lambda, \alpha^{(t)}_{\Lambda_n}(Q^{(j)}_\mu)] = B^j_{\lambda\mu}(t)(I), \qquad 0 \le j \le 1.$$

The proof uses the same points as those in Proposition 5.1. Then Proposition 5.2 follows from the estimates on these matrices in Proposition B.1. ∎

Let us now consider commutators of length two.

Proposition 5.3. *If A is in $\mathcal{W}_2$, then the commutators* $[[\alpha^{(t)}_{\Lambda_n}(A), Q^{(j_1)}_{\lambda_1}], Q^{(j_2)}_{\lambda_2}]$ *are in $\mathcal{L}(\mathcal{H})$* ($t \in \mathbb{R}$, λ_1 *and* λ_2 *in* Λ_n, $0 \le j_1, j_2 \le 1$). *Moreover, if γ is in* $(0, \gamma_0)$ *and $M > 2\sqrt{S_\gamma}$, then there exists $C = C(M, \gamma)$ such that*

$$\|[[\alpha_{\Lambda_n}^{(t)}(A), Q_{\lambda_1}^{(j_1)}],\ Q_{\lambda_2}^{(j_2)}]\|_{\mathcal{L}(\mathcal{H})} \\ \le Ce^{M|t|}\Big[\sum_{\substack{(\mu_1,\mu_2)\in\Lambda_n^2\\ 0\le k_1,k_2\le 1}} e^{-\gamma[|\lambda_1-\mu_1|+|\lambda_2-\mu_2|]}\|[[A, Q_{\mu_1}^{(k_1)}], Q_{\mu_2}^{(k_2)}]\| \\ + \sum_{\substack{\nu\in\Lambda_n\\ 0\le k\le 1}} e^{-\gamma d(\nu,\{\lambda_1,\lambda_2\})}\|[A, Q_\nu^{(k)}]\|\Big].$$

Proof. First step. Take A in $\mathcal{L}(\mathcal{H})$. We show that the functions defined for all real t by

$$\Phi_{\lambda_1\lambda_2}^{j_1,j_2}(t) = [[A, \alpha_{\Lambda_n}^{(t)}(Q_{\lambda_1}^{(j_1)})],\ \alpha_{\Lambda_n}^{(t)}(Q_{\lambda_2}^{(j_2)})], \quad 0\le j_1, j_2\le 1, \tag{5.10}$$

are C^1 from $\mathbb{R}$ into $\mathcal{L}(\mathcal{H}^2(\Lambda_n), \mathcal{H}^{-2}(\Lambda_n))$ and satisfy the following differential system where the operators $\widetilde{K}_{\lambda\mu}(t)$ are defined in (5.3) and where the operators $R_{\lambda\mu}$ are given by Proposition 3.5:

$$\frac{d}{dt}\Phi_{\lambda_1\lambda_2}^{00}(t) = \Phi_{\lambda_1\lambda_2}^{01}(t) + \Phi_{\lambda_1\lambda_2}^{10}(t), \tag{5.11}$$

$$\frac{d}{dt}\Phi_{\lambda_1\lambda_2}^{10}(t) = \Phi_{\lambda_1\lambda_2}^{11}(t) - i\sum_{\mu_1\in\Lambda_n}\widetilde{K}_{\lambda_1\mu_1}(t)(\Phi_{\mu_1\lambda_2}^{00}(t)), \tag{5.12}$$

$$\frac{d}{dt}\Phi_{\lambda_1\lambda_2}^{01}(t) = \Phi_{\lambda_1\lambda_2}^{11}(t) - i\sum_{\mu_2\in\Lambda_n}\widetilde{K}_{\lambda_2\mu_2}(t)(\Phi_{\lambda_1\mu_2}^{00}(t)), \tag{5.13}$$

$$\frac{d}{dt}\Phi_{\lambda_1\lambda_2}^{11}(t) = -i\sum_{\mu_1\in\Lambda_n}\widetilde{K}_{\lambda_1\mu_1}(t)(\Phi_{\mu_1\lambda_2}^{01}(t)) \\ - i\sum_{\mu_2\in\Lambda_n}\widetilde{K}_{\lambda_2\mu_2}(t)(\Phi_{\lambda_1\mu_2}^{10}(t)) + F_{\lambda_1,\lambda_2}(t), \tag{5.14}$$

$$F_{\lambda_1,\lambda_2}(t) = -\sum_{\mu_1\in\Lambda_n}\alpha_{\Lambda_n}^{(t)}(R_{\lambda_1\mu_1}([\alpha_{\Lambda_n}^{(-t)}(A), Q_{\mu_1}], P_{\lambda_2})). \tag{5.15}$$

The system of functions $\Phi_{\lambda_1\mu_2}^{10}(t)$ is the unique solution to the differential system (5.11)–(5.15) satisfying the initial conditions

$$\Phi_{\lambda_1\lambda_2}^{j_1j_2}(0) = [[A, Q_{\lambda_1}^{(j_1)}], Q_{\lambda_2}^{(j_2)}], \quad 0\le j_1, j_2\le 1. \tag{5.16}$$

Let us give more details for the proof of (5.14). By the differential system satisfied by $\alpha_{\Lambda_n}^{(t)}(Q_\lambda)$ and $\alpha_{\Lambda_n}^{(t)}(Q_\mu)$ (see the first step of the proof of Proposition 4.2), one observes that

$$\frac{d}{dt}\Phi_{\lambda_1\lambda_2}^{11}(t) = -i\alpha_{\Lambda_n}^{(t)}\big([[\alpha_{\Lambda_n}^{(-t)}(A), [P_{\lambda_1}, V_{\Lambda_n}]], P_{\lambda_2}] \\ + [[\alpha_{\Lambda_n}^{(-t)}(A), P_{\lambda_1}], [P_{\lambda_2}, V_{\Lambda_n}]]\big).$$

Using the operators $K_{\lambda\mu}$ of Proposition 3.3, one gets

$$\big[[\alpha_{\Lambda_n}^{(-t)}(A), P_{\lambda_1}], [P_{\lambda_2}, V_{\Lambda_n}]\big] = \sum_{\mu_2 \in \Lambda_n} K_{\lambda_2\mu_2}([[\alpha_{\Lambda_n}^{(-t)}(A), P_{\lambda_1}], Q_{\mu_2}])$$

Also using the operators $R_{\lambda\mu}$ of Proposition 3.5, one sees that

$$\big[[\alpha_{\Lambda_n}^{(-t)}(A), [P_{\lambda_1}, V_{\Lambda_n}]], P_{\lambda_2}\big] = \sum_{\mu_1 \in \Lambda_n} \big(K_{\lambda_1\mu_1}([[\alpha_{\Lambda_n}^{(-t)}(A), Q_{\mu_1}], P_{\lambda_2}]) + R_{\lambda_1\mu_1}([\alpha_{\Lambda_n}^{(-t)}(A), Q_{\mu_1}], P_{\lambda_2})\big).$$

Equalities (5.14) and (5.15) then follow.

Second step. Suppose now that A is in $\mathcal{W}_2$. We shall show that the operators $F_{\lambda_1,\lambda_2}(t)$ defined in (5.15) are in $\mathcal{L}(\mathcal{H})$ and we shall estimate their norms. More precisely, we shall show that if $\gamma \in (0, \gamma_0)$ and $M > \sqrt{S_\gamma}$, then

$$\|F_{\lambda_1,\lambda_2}(t)\|_{\mathcal{L}(\mathcal{H})} \le Ce^{M|t|} \sum_{\substack{\nu \in \Lambda_n \\ 0 \le k \le 1}} e^{-\gamma_0|\lambda_1-\lambda_2| - \gamma d(\nu, \{\lambda_1,\lambda_2\})} \|[A, Q_\nu^{(k)}]\|. \tag{5.17}$$

Indeed, from Proposition 3.5, if $\lambda_1 \ne \lambda_2$, then the sum in (5.15) is reduced to two terms: with $\mu_1 = \lambda_1$ and with $\mu_1 = \lambda_2$. In this case,

$$\|F_{\lambda_1,\lambda_2}(t)\|_{\mathcal{L}(\mathcal{H})} \le Ce^{-\gamma_0|\lambda_1-\lambda_2|}(\|[\alpha_{\Lambda_n}^{(-t)}(A), Q_{\lambda_1}]\|_{\mathcal{L}(\mathcal{H})} + \|[\alpha_{\Lambda_n}^{(-t)}(A), Q_{\lambda_2}]\|_{\mathcal{L}(\mathcal{H})}).$$

If $\lambda_1 = \lambda_2$, then from Proposition 3.5,

$$\|F_{\lambda_1,\lambda_1}(t)\|_{\mathcal{L}(\mathcal{H})} \le C \sum_{\mu_1 \in \Lambda_n} e^{-\gamma_0|\lambda_1-\mu_1|} \|[\alpha_{\Lambda_n}^{(-t)}(A), Q_{\mu_1}]\|_{\mathcal{L}(\mathcal{H})}.$$

In view of Proposition 5.1, if $M > \sqrt{S_\gamma}$ then

$$\|[\alpha_{\Lambda_n}^{(-t)}(A), Q_{\mu_1}]\|_{\mathcal{L}(\mathcal{H})} = \|[A, \alpha_{\Lambda_n}^{(t)}(Q_{\mu_1})]\|_{\mathcal{L}(\mathcal{H})} \le C(M,\gamma)e^{M|t|} \sum_{\substack{\nu \in \Lambda_n \\ 0 \le k \le 1}} e^{-\gamma|\mu_1-\nu|} \|[A, Q_\nu^{(k)}\|_{\mathcal{L}(\mathcal{H})},$$

and the estimates (5.17) are easily deduced.

Third step. If A is in $\mathcal{W}_2$, then the initial data (5.16) are in $\mathcal{L}(\mathcal{H})$. From the remarks below Proposition B.1, if γ is in $(0, \gamma_0)$, the system (5.11)–(5.14) has a solution $\Psi_{\lambda_1\lambda_2}^{j_1j_2}(t)$ in $\mathcal{L}(\mathcal{H})$ satisfying (5.16). Moreover, if $M > 2\sqrt{S_\gamma}$, there exists $C(M,\gamma)$ such that

$$\|\Psi_{\lambda_1\lambda_2}^{j_1j_2}(t)\| \le C(M,\gamma)e^{M|t|} \sum_{\substack{(\mu_1,\mu_2) \in \Lambda_n^2 \\ 0 \le k_1,k_2 \le 1}} e^{-\gamma(|\lambda_1-\mu_1|+|\lambda_2-\mu_2|)} \|[[A, Q_{\mu_1}^{(k_1)}], Q_{\mu_2}^{(k_2)}]\| + C(M,\gamma) \sum_{(\mu_1,\mu_2) \in \Lambda_n^2} e^{-\gamma(|\lambda_1-\mu_1|+|\lambda_2-\mu_2|)} \int_0^t e^{M|t-s|} \|F_{\mu_1,\mu_2}(s)\|\, ds.$$

The conclusion then follows from the estimates of $F_{\mu_1,\mu_2}(s)$ in (5.17).

6. Evolution for a finite number of sites. From Proposition 4.1, the operator $e^{itH_{\Lambda_n}} \otimes I$ is bounded in each $\mathcal{H}^k(\Lambda_n)$. However, the proof of that proposition might suggest that the norm of this operator could depend on n. On the contrary, the next proposition provides a bound independent of n.

PROPOSITION 6.1. *The operator* $e^{itH_{\Lambda_n}} \otimes I$ *is bounded in* $\mathcal{H}^k$ $(0 \le k \le 2)$ *with norm* $\le C_k e^{M_k|t|}$ *where* $C_k, M_k > 0$ *are independent of all the parameters. For all* $A \in \mathcal{L}(\mathcal{H}^k, \mathcal{H})$ $(k = 1, 2)$ *with finite support, if* Λ_n *contains the support of* A, *then*

$$\|\alpha_{\Lambda_n}^{(t)}(A)\|_{\mathcal{L}(\mathcal{H}^k,\mathcal{H})} \le C_k e^{M_k|t|} \|A\|_{\mathcal{L}(\mathcal{H}^k,\mathcal{H})}. \tag{6.1}$$

Proof. Let $f \in \mathcal{H}^1$. From Proposition 4.2, for all $\lambda \in \Lambda_n$,

$$\begin{aligned}\|Q_\lambda(e^{itH_{\Lambda_n}} \otimes I)f\| &= \|\alpha_{\Lambda_n}^{(-t)}(Q_\lambda)f\| \\ &\le \|R_\lambda(-t)f\| + \sum_{\mu\in\Lambda_n} (\|A_{\lambda\mu}^{(n)}(-t)Q_\mu f\| + \|B_{\lambda\mu}^{(n)}(-t)P_\mu f\|).\end{aligned}$$

We deduce from (4.5) and (4.6) that if $\gamma \in (0, \gamma_0)$ and $M_1 > \sqrt{S_\gamma}$ then

$$\|Q_\lambda(e^{itH_{\Lambda_n}} \otimes I)f\| \ \le C_1 e^{M_1|t|} \|f\|_{\mathcal{H}^1}$$

with $C_1 > 1$ independent of n and t. If λ is not in Λ_n then the same inequality is valid since Q_λ commutes with $e^{itH_{\Lambda_n}} \otimes I$. We proceed similarly with the operators P_λ, proving that $\|e^{itH_{\Lambda_n}} \otimes I\|_{\mathcal{L}(\mathcal{H}^1)} \le C_1 e^{M_1|t|}$.

Action in $\mathcal{H}^2$. For all λ_1 and λ_2 in Λ_n we have, from the above,

$$\begin{aligned}\|Q_{\lambda_1}^{(j_1)} Q_{\lambda_2}^{(j_2)}(e^{itH_{\Lambda_n}} \otimes I)f\| &= \|Q_{\lambda_1}^{(j_1)}(e^{itH_{\Lambda_n}} \otimes I)\alpha_{\Lambda_n}^{(-t)}(Q_{\lambda_2}^{(j_2)})f\| \\ &\le C_1 e^{M_1|t|} \|\alpha_{\Lambda_n}^{(-t)}(Q_{\lambda_2}^{(j_2)})f\|_{\mathcal{H}^1}.\end{aligned}$$

One sees that

$$\begin{aligned}\|Q_\mu^{(k)}\alpha_{\Lambda_n}^{(-t)}(Q_{\lambda_2}^{(j_2)})f\| &\le \|[Q_\mu^{(k)}, \alpha_{\Lambda_n}^{(-t)}(Q_{\lambda_2}^{(j_2)})]f\| + \|\alpha_{\Lambda_n}^{(-t)}(Q_{\lambda_2}^{(j_2)})Q_\mu^{(k)}f\| \\ &\le C_1' e^{M_1|t|}(\|f\| + \|Q_\mu^{(k)}f\|_{\mathcal{H}^1})\end{aligned}$$

for all $\mu \in \Lambda_n$.

The above two terms have been estimated using Propositions 5.2 and 5.1 respectively. One deduces (with another constant C_2) that

$$\|Q_{\lambda_1}^{(j_1)}\ Q_{\lambda_2}^{(j_2)}(e^{-itH_{\Lambda_n}} \otimes I)f\| \le C_2 e^{2M_1|t|} \|f\|_{\mathcal{H}^2}.$$

The proof is complete. ■

THEOREM 6.2. *If* A *is in* $\mathcal{W}_k$ *with finite support, and* Λ_n *contains the support of* A, *then* $\alpha_{\Lambda_n}^{(t)}(A)$ *is in* $\mathcal{W}_k$ $(0 \le k \le 2)$. *Moreover, there exist constants* $C_k, M_k > 0$, *independent of* A, n *and* t, *such that*

$$\|\alpha_{\Lambda_n}^{(t)}(A)\|_{\mathcal{W}_k} \le C_k e^{M_k|t|} \|A\|_{\mathcal{W}_k}. \tag{6.2}$$

Proof. The norm in $\mathcal{L}(\mathcal{H})$ is conserved by $\alpha_{\Lambda_n}^{(t)}$. By Proposition 5.1, if $A \in \mathcal{W}_1$ is supported in Λ_n and $\lambda \in \Lambda_n$ then the commutators of A with $\alpha_{\Lambda_n}^{(-t)}(Q_\lambda^{(j)})$ are bounded operators. Thus, if $\lambda \in \Lambda_n$, the commutators of $\alpha_{\Lambda_n}^{(t)}(A)$ with $Q_\lambda^{(j)}$ are bounded operators. Since these commutators vanish when $\lambda \notin \Lambda_n$ it follows that $\alpha_{\Lambda_n}^{(t)}(A)$ is in $\mathcal{W}_1$. If γ is in $(0,\gamma_0)$ and $M_1 > \sqrt{S_\gamma}$, we see that

$$\sum_{\substack{\lambda\in\Lambda_n\\ 0\le j\le 1}} \|[\alpha_{\Lambda_n}^{(t)}(A), Q_\lambda^{(j)}]\| \le C(M_1,\gamma)e^{M|t|} \sum_{\substack{(\lambda,\mu)\in\Lambda_n^2\\ 0\le j,k\le 1}} e^{-\gamma d(\lambda,\mu)}\|[A, Q_\mu^{(k)}]\|$$
$$\le C_1(M_1,\gamma)e^{M|t|}\|A\|_{\mathcal{W}_1} \sup_{\mu\in\mathbb{Z}} \sum_{\lambda\in\mathbb{Z}} e^{-\gamma d(\lambda,\mu)}.$$

Consequently, there are $C_1, M_1 > 0$ such that (6.2) is valid for $k = 1$.

Action in $\mathcal{W}^2$. Proposition 5.3 shows that the commutators written as $[\alpha_{\Lambda_n}^{(t)}(A), Q_{\lambda_1}^{(j_1)}], Q_{\lambda_2}^{(j_2)}]$ are bounded operators and vanish if λ_1 or λ_2 is not in Λ_n. Consequently, $\alpha_{\Lambda_n}^{(t)}(A)$ is in $\mathcal{W}_2$. If γ is in $(0,\gamma_0)$ and $M_2 > 2\sqrt{S_\gamma}$ then Proposition 5.3 implies (6.2) for $k = 2$. ∎

7. Existence of dynamics in the Weyl algebra. The number of sites will now go to infinity. The proofs of Theorems 1.1 and 1.2 on the existence of a limit rely on the description of the difference $\alpha_{\Lambda_m}^{(t)}(A) - \alpha_{\Lambda_n}^{(t)}(A)$.

PROPOSITION 7.1. *There exist $C, M, \gamma > 0$ with the following properties. For all $A \in \mathcal{W}_2$ with finite support, all integers m and n with $0 < m < n$ and such that Λ_m contains the support $\sigma(A)$ of A, and all $t \in \mathbb{R}$,*

$$\|\alpha_{\Lambda_m}^{(t)}(A) - \alpha_{\Lambda_n}^{(t)}(A)\|_{\mathcal{L}(\mathcal{H}^1,\mathcal{H}^0)} \le C\|A\|_{\mathcal{W}_2} e^{M|t|} e^{-\gamma d(\sigma(A),\Lambda_m^c)}. \tag{7.1}$$

Proof. For $m < n$ we denote by V_{mn}^{inter} the potential of interaction between Λ_m and $\Lambda_n \setminus \Lambda_m$:

$$V_{mn}^{\text{inter}}(x) = -bQ_mQ_{m+1} - bQ_{-m}Q_{-m-1} + \sum_{(\lambda,\mu)\in E_{mn}} V_{\lambda\mu}$$

where E_{mn} denotes the set of pairs of sites (λ,μ) such that one of the sites (λ or μ) is in Λ_m, and the other in $\Lambda_n \setminus \Lambda_m$. For all $\theta \in [0,1]$, set

$$H_{mn\theta} = H_{\Lambda_n} - (1-\theta)V_{mn}^{\text{inter}}.$$

One can define a unitary operator by $e^{itH_{mn\theta}}$ and set

$$\alpha_{mn\theta}^{(t)}(A) = (e^{itH_{mn\theta}} \otimes I)A(e^{-itH_{mn\theta}} \otimes I).$$

Thus, if A is supported in Λ_m and $m < n$ then

$$\alpha_{mn1}^{(t)}(A) = \alpha_{\Lambda_n}^{(t)}(A), \quad \alpha_{mn0}^{(t)}(A) = \alpha_{\Lambda_m}^{(t)}(A).$$

The function $\varphi(t,\theta)=\frac{\partial}{\partial\theta}\alpha^{(t)}_{mn\theta}(A)$ satisfies

$$\frac{\partial\varphi}{\partial t}=i[H_{mn\theta},\varphi]+i[V^{\text{inter}}_{mn},\alpha^{(t)}_{mn\theta}(A)],\qquad \varphi(0,\theta)=0.$$

Consequently,

$$\frac{\partial}{\partial\theta}\alpha^{(t)}_{mn\theta}(A)=i\int_0^t\alpha^{(t-s)}_{mn\theta}([V^{\text{inter}}_{mn},\alpha^{(s)}_{mn\theta}(A)])\,ds.$$

One obtains the integral representation

$$\alpha^{(t)}_{\Lambda_n}(A)-\alpha^{(t)}_{\Lambda_m}(A)=i\int_0^t\int_0^1\alpha^{(t-s)}_{mn\theta}([V^{\text{inter}}_{mn},\alpha^{(s)}_{mn\theta}(A)])\,ds\,d\theta.$$

Applying Proposition 6.1 to the operator $H_{mn\theta}$ which satisfies the same hypotheses as H_{Λ_n}, we deduce that there exist $C,M>0$ such that

$$\|\alpha^{(t)}_{\Lambda_n}(A)-\alpha^{(t)}_{\Lambda_m}(A)\|_{\mathcal{L}(\mathcal{H}^1,\mathcal{H}^0)}\le C\int_0^t\int_0^1 e^{M|t-s|}\|[V^{\text{inter}}_{mn},\alpha^{(s)}_{mn\theta}(A)]\|_{\mathcal{L}(\mathcal{H}^1,\mathcal{H})}\,ds\,d\theta$$

for all (λ,μ) in E_{mn}. Applying Proposition 3.2 to the operator $\alpha^{(s)}_{mn\theta}(A)$ belonging in $\mathcal{W}_2$ we obtain

$$\|[V_{\lambda\mu},\alpha^{(s)}_{mn\theta}(A)]\|_{\mathcal{L}(\mathcal{H}^1,\mathcal{H}^0)}\le Ce^{-\gamma_0|\lambda-\mu|}\sum_{1\le j+k\le 2}\|(\operatorname{ad}Q_\lambda)^j(\operatorname{ad}Q_\mu)^k\alpha^{(s)}_{mn\theta}(A)\|.$$

Similarly,

$$\|[Q_mQ_{m+1},\alpha^{(s)}_{mn\theta}(A)]\|_{\mathcal{L}(\mathcal{H}^1,\mathcal{H}^0)}\le C\sum_{1\le j+k\le 2}\|(\operatorname{ad}Q_m)^j(\operatorname{ad}Q_{m+1})^k\alpha^{(s)}_{mn\theta}(A)\|.$$

Summing on the pairs (λ,μ) in E_{mn} we get

$$\|[V^{\text{inter}}_{mn},\alpha^{(s)}_{mn\theta}(A)]\|_{\mathcal{L}(\mathcal{H}^1,\mathcal{H}^0)}\le C\sum_{(\lambda,\mu)\in E_{mn}}e^{-\gamma_0|\lambda-\mu|}\sum_{1\le j+k\le 2}\|(\operatorname{ad}Q_\lambda)^j(\operatorname{ad}Q_\mu)^k\alpha^{(s)}_{mn\theta}(A)\|.$$

Consequently,

$$\|\alpha^{(t)}_{\Lambda_n}(A)-\alpha^{(t)}_{\Lambda_m}(A)\|_{\mathcal{L}(\mathcal{H}^1,\mathcal{H}^0)}\le C\sum_{\substack{(\lambda,\mu)\in E_{mn}\\ 1\le j+k\le 2}}e^{-\gamma_0|\lambda-\mu|}\int_0^t\int_0^1 e^{M|t-s|}\|(\operatorname{ad}Q_\lambda)^j(\operatorname{ad}Q_\mu)^k\alpha^{(s)}_{mn\theta}(A)\|\,ds\,d\theta. \tag{7.2}$$

Proposition 7.1 then follows from the next lemma, which will also be used in Section 8.

Lemma 7.2. *If γ is in $(0,\gamma_0)$ and $M>2\sqrt{S_\gamma}$ then there exists $C(M,\gamma)$ such that for all n, all disjoint sets E_1 and E_2 included in Λ_n, and all $A\in\mathcal{W}_2$*

supported in E_1,

$$\sum_{\substack{(\lambda_1,\lambda_2)\in\Lambda_n\times E_2\\ 1\le\alpha+\beta\le 2}} e^{-\gamma_0|\lambda_1-\lambda_2|}\|(\operatorname{ad} Q_{\lambda_1})^\alpha(\operatorname{ad} Q_{\lambda_2})^\beta\alpha^{(s)}_{mn\theta}(A)\| \le C(M,\gamma)\|A\|_{\mathcal{W}_2}e^{M|s|}e^{-\gamma d(E_1,E_2)}.$$

This lemma can be deduced from Propositions 5.1 and 5.3 applied to the Hamiltonian $H_{mn\theta}$. Proposition 7.1 is a consequence of (7.2) and the lemma, with $E_1=\sigma(A)$ and $E_2=\Lambda_n\setminus\Lambda_m$.

Proof of Theorems 1.1 and 1.2. From Proposition 7.1 the sequence $\alpha^{(t)}_{\Lambda_n}(A)$ is a Cauchy sequence in $\mathcal{L}(\mathcal{H}^2,\mathcal{H})$ and thus converges in $\mathcal{L}(\mathcal{H}^2,\mathcal{H})$ to an element, say $\alpha^{(t)}(A)$. By Proposition 6.2, we have $\|\alpha^{(t)}_{\Lambda_n}(A)\|_{\mathcal{W}_2}\le Ce^{M|t|}\|A\|_{\mathcal{W}_2}$. By Theorem 1.4 the operator $\alpha^{(t)}(A)$ is in $\mathcal{W}_2$ with norm $\le Ce^{M|t|}\|A\|_{\mathcal{W}_2}$, and for all $f\in\mathcal{H}$, the sequence $\alpha^{(t)}_{\Lambda_n}(A)f$ strongly converges to $\alpha^{(t)}(A)f$. The classical continuity of the map $t\mapsto\alpha^{(t)}_{\Lambda_n}(A)f$ for all n and all f, together with the above inequalities, shows the continuity of $t\mapsto\alpha^{(t)}_{\Lambda_n}(A)f$. ■

Extension of $\alpha^{(t)}$ to the algebra $\mathcal{W}_2$. Let A in $\mathcal{W}_2$ have an arbitrary support. From Theorem 1.3 there exists a sequence (A_n) in $\mathcal{W}_2$ with finite support such that

$$\|A_n\|_{\mathcal{W}_2}\le\|A\|_{\mathcal{W}_2},\qquad \lim_{n\to\infty}\|A_n-A\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})}=0.$$

The operator $\alpha^{(t)}(A_n)$ is well-defined, in view of Theorems 1.1 and 1.2, since the A_n have finite support. One has

$$\|\alpha^{(t)}(A_n)\|_{\mathcal{W}_2}\le Ce^{M|t|}\|A_n\|_{\mathcal{W}_2}\le Ce^{M|t|}\|A\|_{\mathcal{W}_2}. \tag{7.3}$$

If $m<n$ then we also see from Theorem 1.2 that

$$\|\alpha^{(t)}(A_n-A_m)\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})}\le Ce^{M|t|}\|A_n-A_m\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})}.$$

The sequence $\alpha^{(t)}(A_n)$ thus converges in $\mathcal{L}(\mathcal{H}^2,\mathcal{H})$ to an element denoted $\alpha^{(t)}(A)$. From (7.3) and Theorem 1.4 this element is in $\mathcal{W}_2$ and

$$\|\alpha^{(t)}(A)\|_{\mathcal{W}_2}\le Ce^{M|t|}\|A\|_{\mathcal{W}_2}.$$

The group $\alpha^{(t)}$ is thus extended to the whole algebra $\mathcal{W}_2$.

8. Lieb–Robinson's inequalities

PROPOSITION 8.1. *For all γ in $(0,\gamma_0)$ and $M>2\sqrt{S_\gamma}$, there exists $C(M,\gamma)>0$ such that, for all A and B in $\mathcal{W}_2$ with finite supports $\sigma(A)$ and $\sigma(B)$, all n such that Λ_n contains $\sigma(A)$ and $\sigma(B)$, and all $t\in\mathbb{R}$,*

$$\|[\alpha_{\Lambda_n}^{(t)}(A), B]\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})} \leq C(M,\gamma)\|A\|_{\mathcal{W}_2}\|B\|_{\mathcal{W}_2} e^{M|t|} e^{-\gamma d(\sigma(A),\sigma(B))}. \tag{8.1}$$

The same inequality is valid with $\alpha_{\Lambda_n}^{(t)}$ *replaced by* $\alpha^{(t)}$.

Proof. From Corollary 2.6 applied to the operators B and $\alpha_{\Lambda_n}^{(t)}(A)$ (both having support in Λ_n), one has

$$\|[\alpha_{\Lambda_n}^{(t)}(A), B]\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})} \leq C\|B\|_{\mathcal{W}_2} \sum_{\substack{\lambda\in\sigma(B)\\ 1\leq j+k\leq 2}} \|(\operatorname{ad} P_\lambda)^j (\operatorname{ad} Q_\lambda)^k (\alpha_{\Lambda_n}^{(t)}(A))\|.$$

Inequality (8.1) then follows by applying Lemma 7.2 to the sets $E_1 = \sigma(A)$ and $E_2 = \sigma(B)$. The analogous inequality for $\alpha^{(t)}(A)$ is then deduced since $\|\alpha_{\Lambda_n}^{(t)}(A) - \alpha^{(t)}(A)\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})}$ tends to 0. ∎

Propagation speed. Set

$$V_0 = \inf_{0<\gamma<\gamma_0} \frac{2\sqrt{S_\gamma}}{\gamma} \tag{8.2}$$

where S_γ is the constant of Proposition 3.4. For the case of interaction with nearest neighbors the infimum is taken on $(0,\infty)$.

Proof of Theorem 1.5. Let A and B in $\mathcal{W}_2$ have finite supports $\sigma(A)$ and $\sigma(B)$. Let (h_n,t_n) be a sequence in $\mathbb{Z}\times\mathbb{R}$ with $|t_n|\to\infty$ and $|h_n|\geq v_1|t_n|$ where $v_1 > V_0$, V_0 as above. Choose $\gamma\in(0,\gamma_0)$ such that $2\sqrt{S_\gamma} < v_1\gamma$. Pick M such that $2\sqrt{S_\gamma} < M < v_1\gamma$. The sequence $M|t_n| - \gamma d(\sigma(A),\sigma(\tau_{h_n}(B)))$ tends to $-\infty$. For all $f\in\mathcal{H}^2$ the inequality (8.1) (with $\alpha_{\Lambda_n}^{(t)}$ replaced with $\alpha^{(t)}$) shows that

$$\lim_{n\to\infty} \|[\alpha^{(t_n)}(A), \tau_{h_n}(B)]f\|_{\mathcal{H}} = 0.$$

This extends by density to all $f\in\mathcal{H}$. ∎

Appendix A. Proof of Proposition 2.5. We shall first prove the conclusion of Proposition 2.5 for E and F with $F\setminus E$ consisting of only one element λ. Operators in $\mathcal{L}(\mathcal{H}_F)$ will be identified, using the map $i_{F\mathbb{Z}}$, with elements of $\mathcal{L}(\mathcal{H})$ supported in F. We denote by $\mathcal{W}_k(F)$ the set of all A in $\mathcal{L}(\mathcal{H}_F)$ such that $i_{F\mathbb{Z}}(A)$ is in $\mathcal{W}_k$.

Proposition A.1. *There exists a constant* $C>0$ *such that, for all finite subsets* E *in* $\mathbb{Z}$ *and* $F = E\cup\{\lambda\}$ *where* $\lambda\in\mathbb{Z}\setminus E$, *and for all* T *in* $\mathcal{W}_2(F)$,

$$\|(T - i_{EF}\circ\rho_{FE}(T))f\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})} \leq C\sum_{1\leq j+k\leq 2} \|(\operatorname{ad} P_\lambda)^j(\operatorname{ad} Q_\lambda)^k T\|_{\mathcal{L}(\mathcal{H})}. \tag{A.1}$$

End of proof of Proposition 2.5. If $E\subset F\subset G$ then $\rho_{GE} = \rho_{FE}\circ\rho_{GF}$ and $i_{EG} = i_{FG}\circ i_{EF}$. Consequently, if $F = E\cup\{\lambda_1,\dots,\lambda_m\}$ then we successively apply Proposition A.1 with the set $E_k = E \cup \{\lambda_1, \dots, \lambda_k\}$ $(1 \le k \le m)$ and $E_0 = E$. We obtain, for all T in $\mathcal{W}_2(F)$,

$$\|T - i_{EF} \circ \rho_{FE}(T)\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})} \le \sum_{k=1}^{m} \|T_k - i_{E_{k-1}E_k} \circ \rho_{E_k E_{k-1}}(T_k)\|_{\mathcal{L}(\mathcal{H}^2,\mathcal{H})}$$

where $T_k = \rho_{FE_k}(T)$. Proposition 2.5 thus follows from Proposition A.1 applied to the operators T_k. ■

Notations. $\Omega_{\{\lambda\}}$ denotes the ground state of the space $\mathcal{H}_{\{\lambda\}}$ associated to the corresponding creation and annihilation operators a_λ and $a_\lambda^\star$. One knows that $\mathcal{H}_{\{\lambda\}}$ is associated with the orthonormal basis $(h_j)_{j\ge 0}$ defined by

$$h_0 = \Omega_{\{\lambda\}}, \qquad h_{j+1} = (j+1)^{-1/2} a_\lambda^\star h_j.$$

If we identify $\mathcal{H}_{\{\lambda\}}$ with $L^2(\mathbb{R})$, this basis is the basis of Hermite's functions and $a_\lambda h_j = \sqrt{j}\, h_{j-1}$ $(j \ge 1)$. We shall use the following notations for the operators belonging to the tensor product $\mathcal{H}_F = \mathcal{H}_E \otimes \mathcal{H}_{\{\lambda\}}$. We set $A = I \otimes a_\lambda$, $A^\star = I \otimes a_\lambda^\star$ and for all $T \in \mathcal{L}(\mathcal{H}_F)$ we set $R(T) = \rho_{FE}(T) \otimes I$ where $\rho(T)$ is defined in Section 2 by $\rho(T) = \pi_{EF}^\star T \pi_{EF}$. Thus $R(T) = i_{EF}\rho_{FE}(T)$. In order to generalize the operator π_{EF} we define, for all $j \ge 0$, a map Ψ_j from $\mathcal{H}_E$ into $\mathcal{H}_F$ by

$$\Psi_j f = f \otimes h_j.$$

We denote by $\Psi_j^\star$ the adjoint operator of $\mathcal{H}_F$ in $\mathcal{H}_E$. With these notations, we can gather some of the usual properties of Hermite's functions in the next lemma:

Lemma A.2. *With these notations one has*

$$\sum_{j=0}^{\infty} \Psi_j \Psi_j^\star = I, \qquad \sum_{j=0}^{\infty} \|\Psi_j^\star f\|_{\mathcal{H}_E}^2 = \|f\|_{\mathcal{H}_F}^2 \qquad \forall f \in \mathcal{H}_F. \tag{A.2}$$

If we denote by $\mathcal{H}^m(E,F)$ $(m \ge 0)$ the partial Sobolev space consisting of $f \in \mathcal{H}_F$ such that

$$\|f\|_{\mathcal{H}^m(E,F)}^2 = \sum_{j=0}^{\infty} (1+j)^m \|\Psi_j^\star f\|_{\mathcal{H}_E}^2 < \infty,$$

then the operator $AA^\star$ with domain $\mathcal{H}^2(E,F)$ is self-adjoint and satisfies $AA^\star \ge I$. For all $\alpha \in \mathbb{R}$ and $j \ge 0$,

$$(AA^\star)^\alpha \Psi_j = (j+1)^\alpha \Psi_j, \qquad \Psi_j^\star (AA^\star)^\alpha = (j+1)^\alpha \Psi_j^\star. \tag{A.3}$$

For all $j \ge 1$,

$$A\Psi_j = \sqrt{j}\, \Psi_{j-1}, \qquad \Psi_j^\star A^\star = \sqrt{j}\, \Psi_{j-1}^\star \tag{A.4}$$

(if $j = 0$ then the right-hand sides are replaced by 0*). For all* $j \geq 0$,

$$A^\star \Psi_j = \sqrt{j+1}\,\Psi_{j+1}, \qquad \Psi_j^\star A = \sqrt{j+1}\,\Psi_{j+1}^\star. \tag{A.5}$$

For each T in $\mathcal{L}(\mathcal{H}_F)$ we define an operator-valued matrix $a_{jk}(T)$ in $\mathcal{L}(\mathcal{H}_E)$ by

$$a_{jk}(T) = \Psi_j^\star T \Psi_k. \tag{A.6}$$

Thus $\pi_{EF} = \Psi_0$ and $\rho_{FE}(T) = A_{00}(T)$. The norm of T in $\mathcal{L}(\mathcal{H}_F)$ can be estimated starting from those of the $a_{jk}(T)$ using the following proposition which is a variant of Schur's Lemma.

Proposition A.3. *Let T be in $\mathcal{L}(\mathcal{H}_F)$. Suppose that there exists $M > 0$ such that, for all $k \geq 0$ and φ in $\mathcal{H}_E$,*

$$\sum_{j\geq 0} \|a_{jk}(T)\varphi\|_{\mathcal{H}_E} \leq M\|\varphi\|_{\mathcal{H}_E}, \qquad \sum_{j\geq 0} \|a_{jk}(T^\star)\varphi\|_{\mathcal{H}_E} \leq M\|\varphi\|_{\mathcal{H}_E}. \tag{A.7}$$

Then $\|T\|_{\mathcal{L}(\mathcal{H}_F)} \leq M$.

Proof. From Lemma A.2, for all f and g in $\mathcal{H}_F$ one gets

$$\langle Tf, g\rangle = \sum_{jk} \langle a_{jk}(T)\Psi_k^\star f, \Psi_j^\star g\rangle.$$

One has

$$|\langle a_{jk}(T)\Psi_k^\star f, \Psi_j^\star g\rangle| \leq \|a_{jk}(T)\Psi_k^\star f\|\, \|\Psi_j^\star g\|.$$

This scalar product can be bounded by

$$|\langle a_{jk}(T)\Psi_k^\star f, \Psi_j^\star g\rangle| \leq \|\Psi_k^\star f\|\, \|a_{jk}(T)^\star \Psi_j^\star g\|.$$

Consequently,

$$|\langle a_{jk}(T)\Psi_k^\star f, \Psi_j^\star g\rangle| \leq (\|a_{jk}(T)\Psi_k^\star f\|\, \|\Psi_k^\star f\|)^{1/2} (\|a_{jk}(T)^\star \Psi_j^\star g\|\, \|\Psi_j^\star g\|)^{1/2}.$$

From Cauchy–Schwarz,

$$|\langle Tf, g\rangle|^2 \leq \Big[\sum_{jk} \|a_{jk}(T)\Psi_k^\star f\|\, \|\Psi_k^\star f\|\Big]\Big[\sum_{jk} \|a_{jk}(T)^\star \Psi_j^\star g\|\, \|\Psi_j^\star g\|\Big].$$

Noticing that $(a_{jk}(T))^\star = a_{kj}(T^\star)$ we obtain

$$|\langle Tf, g\rangle|^2 \leq M^2 \Big[\sum_{k\geq 0} \|\Psi_k^\star f\|^2\Big]\Big[\sum_{j\geq 0} \|\Psi_j^\star g\|^2\Big] \leq M^2 \|f\|_{\mathcal{H}_F}^2 \|g\|_{\mathcal{H}_F}^2.$$

The proof of Proposition A.3 is complete. ∎

Proposition A.4. *Let T be in $\mathcal{W}_1(F)$. Assume that there is $M > 0$ such that for all φ in $\mathcal{H}_E$,*

$$\text{(A.8)} \qquad \sup_{k\geq 0}\sum_{j\geq 0}\frac{\|a_{jk}(T)\varphi\|_{\mathcal{H}_E}}{\sqrt{(j+1)(k+1)}} \leq M\|\varphi\|_{\mathcal{H}_E},$$

$$\text{(A.9)} \qquad \sup_{k\geq 0}\sum_{j\geq 0}\frac{\|a_{jk}(T^\star)\varphi\|_{\mathcal{H}_E}}{\sqrt{(j+1)(k+1)}} \leq M\|\varphi\|_{\mathcal{H}_E}.$$

Then

$$\text{(A.10)} \qquad \|Tf\| \leq M\sqrt{2}\,\|(A_\lambda^\star)^2 f\| + \sqrt{2}\,\|[A_\lambda^\star, T]\|\,\|f\|$$

for all f in $\mathcal{H}^2(E,F)$.

Proof. Set $S = (AA^\star)^{-1/2}T(AA^\star)^{-1/2}$. By Lemma A.2,

$$a_{jk}(S) = \frac{a_{jk}(T)}{\sqrt{(j+1)(k+1)}}.$$

Under the hypotheses of the proposition the operator S is then bounded in $\mathcal{H}_F$ with norm $\leq M$. From Lemma A.2, for all g in $\mathcal{H}_F$,

$$\|(AA^\star)^{-1/2}A^\star g\|^2 = \sum_{j\geq 1}\frac{j}{j+1}\,\|\Psi_{j-1}^\star g\|^2 \geq \frac{1}{2}\sum_{j\geq 0}\|\Psi_j^\star g\|^2 = \frac{1}{2}\,\|g\|^2.$$

Consequently, for all f in $\mathcal{H}^2(E,F)$,

$$\|Tf\| \leq \sqrt{2}\,\|(AA^\star)^{-1/2}A^\star Tf\| \leq \sqrt{2}\,\|[A^\star, T]\|\;\|f\| + \sqrt{2}\,\|(AA^\star)^{-1/2}TA^\star f\|.$$

Since $\|(AA^\star)^{-1/2}\| \leq 1$, we have

$$\begin{aligned}\|(AA^\star)^{-1/2}TA^\star f\| &\leq \|S(AA^\star)^{+1/2}A^\star f\|\\ &\leq M\|(AA^\star)^{+1/2}A^\star f\| = M\|(A^\star)^2 f\|.\end{aligned}$$

Consequently, (A.10) follows. ∎

We shall apply Proposition A.4 to the operator $T - R(T)$ noticing that $R(T)$ commutes with A and $A^\star$. The operator $R(T)$ is chosen such that $a_{00}(T - R(T)) = 0$. Using commutators, we shall estimate all the other elements $a_{jk}(T - R(T))$. This is the purpose of the next proposition.

PROPOSITION A.5. *Under the hypotheses of Proposition* A.1*, for all $k \geq 0$ and φ in $\mathcal{H}_E$,*

$$\text{(A.11)} \qquad \begin{aligned} S_k(T,\varphi) &:= \sum_{j\geq 0}\frac{\|a_{jk}(T - R(T))\varphi\|}{\sqrt{(j+1)(k+1)}}\\ &\leq C\|\varphi\|\sum_{1\leq\alpha+\beta\leq 2}\|(\operatorname{ad}P_\lambda)^\alpha(\operatorname{ad}Q_\lambda)^\beta T\|_{\mathcal{L}(\mathcal{H})},\end{aligned}$$

and a similar estimate holds with T replaced by $T^\star$.

Proof. Estimations of $S_0(T,\varphi)$. We shall prove that

$$\text{(A.12)} \qquad S_0(T,\varphi) \leq \|[A,T]\|\,\|\varphi\|.$$

From (A.5), one sees that for all $j \geq 1$,

$$\sqrt{j}\, a_{j0}(T - R(T)) = \Psi_{j-1}^\star [A, T]\Psi_0.$$

Since $a_{00}(T - R(T)) = 0$, we deduce using (A.2) that

$$\begin{aligned} S_0(T,\varphi) &\leq \sum_{j=1}^{\infty} \frac{1}{\sqrt{j(j+1)}} \|\Psi_{j-1}^\star [A,T]\Psi_0\| \\ &\leq \Big[\sum_{j=1}^{\infty} \frac{1}{j(j+1)}\Big]^{1/2} \Big[\sum_{j=1}^{\infty} \|\Psi_{j-1}^\star [A,T]\Psi_0\varphi\|^2\Big]^{1/2} \\ &\leq \|[A,T]\Psi_0\varphi\| \leq \|[A,T]\|\,\|\varphi\|. \end{aligned}$$

Inequality (A.12) is therefore true.

Recursion between the $S_k(T,\varphi)$. We shall prove that if $k \geq 1$ then

$$\text{(A.13)} \qquad \begin{aligned} S_k(T,\varphi) \leq\ & \frac{k}{k+1} S_{k-1}(T,\varphi) \\ & + \frac{C\|\varphi\|}{k+1} (\|[A,T]\| + \|[A^\star,T]\| + \|[A^\star,[A^\star,T]]\|). \end{aligned}$$

To this end, we use the fact that if $1 \leq j \leq k$ then from (A.4) and (A.5) we have

$$\sqrt{k}\, a_{jk}(T - R(T)) = \sqrt{j}\, a_{j-1,k-1}(T - R(T)) + \Psi_j^\star [T, A^\star]\Psi_{k-1}.$$

If $j = 0$ then the first term above has to be replaced by 0. If $j > k$ then we use

$$\sqrt{j}\, a_{jk}(T - R(T)) = \sqrt{k}\, a_{j-1,k-1}(T - R(T)) + \Psi_{j-1}^\star [A, T]\Psi_k.$$

Then we can write $S_k(T,\varphi) \leq S_k'(T,\varphi) + S_k''(T,\varphi) + S_k'''(T,\varphi)$ where

$$\begin{aligned} S_k'(T,\varphi) &= \sum_{j=1}^{\infty} \inf(\sqrt{j/k}, \sqrt{k/j})\, \frac{\|a_{j-1,k-1}(T - R(T))\varphi\|}{\sqrt{(j+1)(k+1)}} \\ &\leq \frac{k}{k+1} S_{k-1}(T,\varphi), \\ S_k''(T,\varphi) &= \sum_{j=k+1}^{\infty} \frac{\|\Psi_{j-1}^\star [T,A]\Psi_k\varphi\|}{\sqrt{j(j+1)(k+1)}}, \\ S_k'''(T,\varphi) &= \sum_{j=0}^{k} \frac{\|\Psi_j^\star [T,A^\star]\Psi_{k-1}\varphi\|}{\sqrt{(j+1)k(k+1)}}. \end{aligned}$$

From (A.2) and since $\|\Psi_k\varphi\| = \|\varphi\|$,

$$S''_k(T,\varphi) \le \frac{1}{\sqrt{k+1}}\Big[\sum_{j=k+1}^{\infty}\frac{1}{j(j+1)}\Big]^{1/2}\Big[\sum_{j\ge 1}\|\Psi^\star_{j-1}[T,A]\Psi_k\varphi\|^2\Big]^{1/2}$$
$$\le \frac{1}{k+1}\,\|[T,A]\|\,\|\varphi\|.$$

If $k=1$ then we see that $S'''_1(T,\varphi) \le \|[A^\star,T]\|\,\|\varphi\|$. If $k\ge 2$ then the estimation of $S'''_k(T,\varphi)$ involves commutators with length 2. We still have, if $j\le k$,

$$\sqrt{k-1}\,\Psi^\star_j[T,A^\star]\Psi_{k-1} = \sqrt{j}\,\Psi^\star_{j-1}[T,A^\star]\Psi_{k-2} + \Psi^\star_j[[T,A^\star],A^\star]\Psi_{k-2}.$$

Consequently, for $k\ge 2$,

$$S'''_k(T,\varphi) \le \sum_{j=1}^{k}\sqrt{\frac{j}{k-1}}\,\frac{\|\Psi^\star_{j-1}[T,A^\star]\Psi_{k-2}\varphi\|}{\sqrt{(j+1)k(k+1)}}$$
$$+\sum_{j=0}^{k}\frac{\|\Psi^\star_j[[T,A^\star],A^\star]\Psi_{k-2}\varphi\|}{\sqrt{(j+1)(k+1)k(k-1)}}.$$

Using again Cauchy–Schwarz and Lemma A.2, we obtain for $k\ge 2$,

$$S'''_k(T,\varphi) \le \frac{1}{\sqrt{k(k-1)}}\,(\|[A^\star,T]\| + \|[A^\star,[A^\star,T]]\|)\|\varphi\|.$$

Hence we deduce (A.13). Inequality (A.11) follows by iteration of (A.12) and (A.13). Proposition A.1 is a consequence of Propositions A.4 and A.5, and so the proof of Proposition 2.5 is finished.

Appendix B. Differential systems

PROPOSITION B.1. *Suppose that for all λ and μ in Λ_n we are given a continuous map $t\mapsto \Omega_{\lambda\mu}(t)$ from $\mathbb{R}$ into $\mathcal{L}(\mathcal{L}(\mathcal{H}))$. Assume that there are $\gamma>0$ and $S_\gamma>0$ such that, for all λ and ν in Λ_n, and all $t\in\mathbb{R}$,*

$$\text{(B.1)}\qquad \sum_{\mu\in\Lambda_n}\|\Omega_{\lambda\mu}(t)\|_{\mathcal{L}(\mathcal{L}(\mathcal{H}))}e^{-\gamma|\mu-\nu|} \le S_\gamma e^{-\gamma|\lambda-\nu|}.$$

Then, for all $s\in\mathbb{R}$, there exist C^1 functions $t\mapsto A^{(0)}_{\lambda\mu}(t,s)$ and $t\to A^{(1)}_{\lambda\mu}(t,s)$ $((\lambda,\mu)\in\Lambda_n^2)$ from $\mathbb{R}$ into $\mathcal{L}(\mathcal{L}(\mathcal{H}))$ such that

$$\text{(B.2)}\qquad \frac{d}{dt}A^{(0)}_{\lambda\mu}(t,s) = A^{(1)}_{\lambda\mu}(t,s),\qquad \frac{d}{dt}A^{(1)}_{\lambda\mu}(t,s) = \sum_{\nu\in\Lambda_n}\Omega_{\lambda\mu}(t)\circ A^{(0)}_{\nu\mu}(t,s),$$

$$\text{(B.3)}\qquad A^{(0)}_{\lambda\mu}(s,s) = \delta_{\lambda\mu}I,\qquad A^{(1)}_{\lambda\mu}(s,s) = 0$$

(*in* (B.2), *the composition is the one of $\mathcal{L}(\mathcal{L}(\mathcal{H}))$, and in* (B.3) *the identity operator I is the one of $\mathcal{L}(\mathcal{L}(\mathcal{H}))$*). *Moreover, if $M>\sqrt{S_\gamma}$, then there exists*

$C(M,\gamma)>0$ independent of n such that

$$\|A^{(j)}_{\lambda\mu}(t,s)\|_{\mathcal{L}(\mathcal{L}(\mathcal{H}))}\le C(M,\gamma)e^{M|t-s|}e^{-\gamma|\lambda-\mu|}\qquad\forall(\lambda,\mu)\in\Lambda^2_n. \tag{B.4}$$

There are also operator-valued matrices $t\mapsto B^{(0)}_{\lambda\mu}(t,s)$ and $t\mapsto B^{(1)}_{\lambda\mu}(t,s)$ satisfying the same system with the same estimates and the initial conditions

$$B^{(0)}_{\lambda\mu}(s,s)=0,\qquad B^{(1)}_{\lambda\mu}(s,s)=\delta_{\lambda\mu}I. \tag{B.5}$$

Proof. Let $E_{n\gamma}$ be the set of all matrices $A=(A_{\lambda\mu})_{((\lambda,\mu)\in\Lambda^2_n)}$ where each $A_{\lambda\mu}$ is in $\mathcal{L}(\mathcal{L}(\mathcal{H}))$, endowed with the norm

$$\|A\|_{n\gamma}=\sup_{(\lambda,\mu)\in\Lambda^2_n}e^{\gamma|\lambda-\mu|}\|A_{\lambda\mu}\|_{\mathcal{L}(\mathcal{L}(\mathcal{H}))}.$$

The left composition by the operator-valued matrix $\Omega_{\lambda\mu}(t)$ defines a map $\Omega(t)$ in $\mathcal{L}(E_{n\gamma})$ with norm $\le S_\gamma$. For all $\varepsilon>0$ we can endow $E^2_{n\gamma}$ with a norm such that the norm of the operator

$$U(t)=\begin{pmatrix}0 & I\\ \Omega(t) & 0\end{pmatrix}$$

is $\le\sqrt{S_\gamma}(1+\varepsilon)$. The stated result is then valid. ∎

REMARK 1. In the tensor product $E^2_{n\gamma}\otimes E^2_{n\gamma}$ let $V(t)$ be the map defined by $V(t)=U(t)\otimes I+I\otimes U(t)$. For all $\varepsilon>0$ one can endow $E^2_{n\gamma}\otimes E^2_{n\gamma}$ with a norm such that the norm of the map $V(t)$ is $\le 2\sqrt{S_\gamma}\,(1+\varepsilon)$. Consequently, if $M>2\sqrt{S_\gamma}$ and A_0 is in $E^2_{n\gamma}\otimes E^2_{n\gamma}$ then the differential system

$$A'(t)=U(t)A(t),\qquad A(0)=A_0,$$

has a solution taking values in $E^2_{n\gamma}\otimes E^2_{n\gamma}$ and with exponential time growth $e^{M|t|}$.

REMARK 2. If we are also given a continuous function $t\mapsto F_\lambda(t)$ from $\mathbb{R}$ to $\mathcal{L}(\mathcal{H})$ then the family of functions $t\mapsto X^{(j)}_\lambda(t)$ defined by

$$X^{(j)}_\lambda(t)=\sum_{\mu\in\Lambda_n}\int_0^t B^{(j)}_{\lambda\mu}(t,s)(F_\mu(s))\,ds$$

satisfies the differential system

$$\frac{d}{dt}X^{(0)}_\lambda(t)=X^{(1)}_\lambda(t),\qquad \frac{d}{dt}X^{(1)}_\lambda(t)=\sum_{\mu\in\Lambda_n}\Omega_{\lambda\mu}(t)(X^{(0)}_\mu(t))+F_\lambda(t)$$

with the initial conditions $X^{(j)}_\lambda(0)=0$ and the following estimates (for example for $t>0$):

$$\|X^{(j)}_\lambda(t)\|_{\mathcal{L}(\mathcal{H})}\le C(M,\gamma)\sum_{\mu\in\Lambda_n}e^{-\gamma|\lambda-\mu|}\int_0^t e^{M|t-s|}\|F_\mu(s)\|_{\mathcal{L}(\mathcal{H})}\,ds.$$

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