COLLOQUIUM MATHEMATICUM

VOL. 118 2010 NO. 2

BOUNDARY BEHAVIOUR OF HOLOMORPHIC FUNCTIONS IN HARDY–SOBOLEV SPACES ON CONVEX DOMAINS IN $\mathbb{C}^n$

BY

MARCO M. PELOSO and HERCULE VALENCOURT (Milano)

This paper is dedicated to the memory of Andrzej Hulanicki

Abstract. We study the boundary behaviour of holomorphic functions in the Hardy–Sobolev spaces $\mathcal{H}^{p,k}(\mathcal{D})$, where $\mathcal{D}$ is a smooth, bounded convex domain of finite type in $\mathbb{C}^n$, by describing the approach regions for such functions. In particular, we extend a phenomenon first discovered by Nagel–Rudin and Shapiro in the case of the unit disk, and later extended by Sueiro to the case of strongly pseudoconvex domains.

In memory of Andrzej Hulanicki (by Marco Peloso). I first met Andrzej Hulanicki in Torino when he visited the Department of Mathematics of the Politecnico, where I had recently started my appointment. I was a young mathematician, with my new Ph.D. degree. Andrzej asked me about my mathematics and then tried to find some area of interest for both of us to work on. He soon invited me to spend an extended period of time in Wrocław and to give a short course to his group of collaborators and colleagues. I gladly accepted. Once in Wrocław I was impressed by Andrzej's hospitality, kindness and by the very pleasant atmosphere that he created in his department. I realized that he took a sincere interest in younger mathematicians with the intent to help them proceed in their careers.

We started discussing some mathematics that eventually developed into a paper written in collaboration with Ewa Damek and Detlef Müller. Although that was our only joint paper, I stayed in contact with Andrzej throughout the years and we exchanged several visits.

I will always remember Andrzej as a very good hearted person, as a friend who helped me at the beginning of my career, and I still thank him for everything.

Introduction. Let $\mathcal{D}$ be a smooth, bounded domain in $\mathbb{C}^n$. For $0 < p \leq \infty$, let $L^p(\mathcal{D})$ denote the Lebesgue space with respect to the volume form, and $L^p(b\mathcal{D})$ be the Lebesgue space on $b\mathcal{D}$ with respect to the induced surface

2010 *Mathematics Subject Classification*: 32A35, 32T25, 46E22.
Key words and phrases: Hardy spaces, finite type domains, convex domains.

DOI: 10.4064/cm118-2-18

measure $d\sigma$. We will denote by $\mathcal{H}(\mathcal{D})$ the space of holomorphic functions on $\mathcal{D}$.

We let $\mathcal{H}^p(\mathcal{D})$ denote the Hardy space of holomorphic functions on $\mathcal{D}$, with norm given by

$$\|f\|^p_{\mathcal{H}^p(\mathcal{D})} := \sup_{0<\varepsilon<\varepsilon_0} \int_{\delta(w)=\varepsilon} |f(w)|^p \, d\sigma_\varepsilon(w),$$

where $\delta(w)$ is the distance from w to $b\mathcal{D}$ and $d\sigma_\varepsilon$ denotes the surface measure on the manifold $\{\delta(w) = \varepsilon\}$. To any $f \in \mathcal{H}^p(\mathcal{D})$ corresponds a unique boundary function in $L^p(b\mathcal{D})$, which we still denote by f, obtained as normal almost everywhere limit, [St]. Thus, we may identify $\mathcal{H}^p(\mathcal{D})$ with a closed subspace of $L^p(b\mathcal{D})$.

Let k be a non-negative integer. In this paper we study the spaces $\mathcal{H}^{p,k} = \mathcal{H}^{p,k}(\mathcal{D})$ of holomorphic functions on $\mathcal{D}$ whose derivatives of order less than or equal to k belong the Hardy space $\mathcal{H}^p(\mathcal{D})$:

$$\mathcal{H}^{p,k} = \{f \in \mathcal{H}(\mathcal{D}) : \partial^\alpha f \in \mathcal{H}^p(\mathcal{D}) \text{ for } |\alpha| \le k\}.$$

In the case of the classical Hardy spaces $\mathcal{H}^p$ and $p \ge 1$ it is well known that a function $f \in \mathcal{H}^p$ converges to its boundary values as z tends to a point ζ on the boundary while varying in the so-called *approach regions* $\mathcal{A} = \mathcal{A}(\zeta)$ whose shape is determined by the geometry of the boundary $b\mathcal{D}$ (see (13) for the definition). For instance, if $\mathcal{D}$ is strongly pseudoconvex the approach region centered at $\zeta \in b\mathcal{D}$ is

$$\mathcal{A}(\zeta) = \{z \in \mathcal{D} : d(z,\zeta) < \delta(z)\},$$

where d is a *natural* pseudometric on the domain $\mathcal{D}$ and $\delta(z)$ denotes the distance of the point z from the boundary $b\mathcal{D}$ (we refer the reader to Section 1 for precise definitions).

These spaces have been intensively studied in the case of the unit ball (see [AhBr]), and in the case of finite type domains ([G] and references therein). Properties of Hardy spaces (i.e. without any condition on the derivatives) on convex domains of finite type have been studied in [KL, BPS, GP, DFi].

In this paper we study the boundary behaviour of functions in $\mathcal{H}^{p,k}$ on convex domains of finite type in $\mathbb{C}^n$.

When $\mathcal{D}$ is a smooth, bounded convex domain of finite type, there exists a natural pseudodistance d_b on $b\mathcal{D}$ (see [Mc]) that makes $b\mathcal{D}$ into a space of homogeneous type.

The geometry of convex domains of finite type was first described by McNeal [Mc]. This description was later applied to the analysis of the mapping properties of the Bergman projection [McS1] and Szegö projection [McS2] by McNeal and Stein. In the case of strongly pseudoconvex domains and finite type domains in $\mathbb{C}^2$ the geometry was determined by canonical vector fields. The natural pseudodistance on $b\mathcal{D}$ was the control distance determined by these vector fields, i.e. the Carnot metric. The situation of convex domains of finite type is much more general. The "weight" of each vector field may vary from point to point, and one needs to take into consideration the different order of contact of complex lines with $b\mathcal{D}$. For these reasons it is natural to consider a *diameter function* $\tau(\zeta,\lambda,r)$, which gives the diameter of the largest one-dimensional disk in the direction of λ with centre at ζ, that fits inside the region $\{z' : \varrho(z') < r\}$. Here ϱ denotes a fixed smooth defining function for $\mathcal{D}$.

Our main result can be stated by saying that if $f \in \mathcal{H}^{p,k}$, then f converges to its boundary values as z tends to the boundary point ζ along an approach region whose shape depends on p and k, besides the geometry of the boundary of $\mathcal{D}$. Our results extend, to the setting of convex domains of finite type, results by Nagel, Rudin and Shapiro [NRSh] in the case of the unit disk, and Sueiro [Su1, Su2] in the case of the unit ball, respectively.

We use the notation $A \lesssim B$ to indicate that $A \le cB$ where the constant c does not depend on the important parameters on which the functions A and B depend. (Typically, the constant c will only depend on the geometry of the domain $\mathcal{D}$.) We use the symbols $\gtrsim$ and $\approx$ with similar, obvious meanings.

1. Basic facts and notation. Let $\mathcal{D}$ be a smooth, bounded convex domain in $\mathbb{C}^n$. A point $\zeta \in b\mathcal{D}$ is said to be *of finite type* if the order of contact of complex lines with $b\mathcal{D}$ at this point is finite (see [BoS] and references therein). The *type* of the point is the least upper bound of the various orders of contact. We say that $\mathcal{D}$ is *of finite type* $T_{\mathcal{D}}$ if every point on $b\mathcal{D}$ is of finite type $\le T_{\mathcal{D}}$ and $T_{\mathcal{D}}$ is the maximum of the types of the points on $b\mathcal{D}$.

Let $\mathcal{D} = \{z \in \mathbb{C}^n : \varrho(z) < 0\}$. There exists $\varepsilon_0 > 0$ such that for $|\varepsilon| \le \varepsilon_0$ the sets $\mathcal{D}_\varepsilon = \{z \in \mathbb{C}^n : \varrho(z) < \varepsilon\}$ are all convex, and the normal projection $\pi : \overline{U} \to b\mathcal{D}$ is well defined and smooth, where $U = \{z \in \mathbb{C}^n : \delta(z) < \varepsilon_0\}$.

The basic geometric facts about convex domains of finite type were first proved by McNeal [Mc] (see also [McS1, McS2, DiFo]). By recalling the results that are involved in the present work we take the opportunity to review the main elements of the construction and set some notation.

For $z \in U$ and $\lambda \in \mathbb{C}^n$ a unit vector, we denote by $\tau(z,\lambda,r)$ the distance from z to the surface $\{z' : \varrho(z') = \varrho(z)+r\}$ along the complex line determined by λ.

For each $z \in U$ and $r < \varepsilon_0$ there exists a special set of coordinates $\{w_1^{z,r},\dots,w_n^{z,r}\}$, which we call *r-extremal*. The first vector $v^{(1)}$ is given by the direction transversal to the boundary, in the sense that the shortest distance from z to the set $\{z' : \varrho(z') = \varrho(z) + r\}$ is realized on the complex line determined by $v^{(1)}$.

The vector $v^{(2)}$ is chosen among the vectors orthogonal to $v^{(1)}$ in such a way that $\tau(z, v^{(2)}, r)$ is maximal. We repeat the same process until we determine an orthonormal basis $\{v^{(1)}, \dots, v^{(n)}\}$. We denote by $(w_1, \dots, w_n)$ the coordinates with respect to this basis. Notice that these coordinates $(w_1, \dots, w_n) = (w_1^{z,r}, \dots, w_n^{z,r})$ depend on z and r. However, the transversal direction w_1 does not depend on r.

For $k = 1, \dots, n$, we set

$$\tau_k(z,r) = \tau(z, v^{(k)}, r), \tag{1}$$

and define the polydisk

$$Q(z,r) = \{w : |w_k| < \tau_k(z,r),\ k = 1, \dots, n\}. \tag{2}$$

Basic relations among these quantities are the following (see [McS2, Prop. 1.1] and also [BPS, Lemma 2.1].

Proposition 1.1. *There exists a constant $C > 0$ depending only on $\mathcal{D}$ such that for any unit vector $\lambda \in \mathbb{C}^n$, $0 < r \le \varepsilon_0$, $z \in U$, and $0 < \delta < 1$ we have:*

(i) $\delta^{1/2}\tau(z,\lambda,r) \lesssim \tau(z,\lambda,\delta r) \lesssim \delta^{1/T_{\mathcal{D}}}\tau(z,\lambda,r)$;
(ii) $\delta^{1/2}Q(z,C^{-1}r) \subset Q(z,\delta r) \subset \delta^{1/T_{\mathcal{D}}}Q(z,Cr)$;
(iii) *if* $w \in Q(z,r)$ *then* $\tau(z,\lambda,r) \approx \tau(w,\lambda,r)$.

We define the quasi-distance $d_b : U \times U \to [0,+\infty)$ by setting

$$d_b(z,w) = \inf\{\delta : w \in Q(z,\delta)\}, \tag{3}$$

and the function d

$$d(z,w) = d_b(z,w) + \delta(z) + \delta(w). \tag{4}$$

Notice that d is initially defined on $U \times U$ and we extend it to $\mathbb{C}^n \times \mathbb{C}^n$ by setting

$$d(z,w) = \psi(\varrho(z))\psi(\varrho(w))d(z,w) + (1-\psi(\varrho(z)))(1-\psi(\varrho(w)))|z-w|,$$

where ψ is a smooth cut-off function on $\mathbb{R}$ such that $\psi(t) = 1$ for $|t| \le \varepsilon_0/2$ and $\psi(t) = 0$ for $|t| \ge \varepsilon_0$.

On the boundary we will use a family of "balls" centred at $\zeta \in b\mathcal{D}$ of radius δ defined as

$$B(\zeta,\delta) = Q(\zeta,\delta) \cap b\mathcal{D}.$$

For any unit vector λ we introduce the differential operator

$$L_\lambda = (\partial_\lambda\varrho)\partial_{x_1} - (\partial_{x_1}\varrho)\partial_\lambda, \tag{5}$$

where $w_1 = x_1 + iy_1$ is the transversal direction fixed earlier. Here, ∂_λ is the standard vector field defined by λ as $\partial_\lambda f = \langle \lambda, df \rangle$, for the real differential df of a smooth function f, and where $\langle\, ,\rangle$ denotes the usual pairing between a one-form and a vector.

Notice that L_λ is always a tangential vector field. If $\lambda \in S^{2n-1}$ is itself tangent to $b\mathcal{D}$, then L_λ is the directional derivative in the direction λ.

For $\Lambda = (\lambda_1, \ldots, \lambda_q)$ a q-list of vectors in S^{2n-1} and $\mu = (\mu_1, \ldots, \mu_q)$ a q-index we set $|\mu| = \mu_1 + \cdots + \mu_n$,

$$L^{\mu}_{\Lambda} = L^{\mu_1}_{\lambda_1} \cdots L^{\mu_q}_{\lambda_q}, \tag{6}$$

and

$$\tau^{\mu}(z, \Lambda, \delta) = \tau(z, \lambda_1, \delta)^{\mu_1} \cdots \tau(z, \lambda_q, \delta)^{\mu_q}. \tag{7}$$

We recall the fundamental estimates for the Szegö kernel and its derivatives [McS2] (called interior estimates of S-type, see [McS2, Def. 4 and Thm. 3.6]). Here, and in the rest of the paper, we denote by $S_{\mathcal{D}}(z, \zeta)$ the Szegö kernel for $\mathcal{D}$.

We have

$$|L^{\mu}_{\Lambda,z} L^{\mu'}_{\Lambda',z'} S_{\mathcal{D}}(z, z')| \lesssim \frac{\tau^{-\mu}(z, \Lambda, \delta)\tau^{-\mu'}(z', \Lambda', \delta)}{|B(\pi(z), \delta)|}, \tag{8}$$

where $\delta = d(z, z')$, $z, z' \in \overline{\mathcal{D}} \times \overline{\mathcal{D}} \setminus \Delta_{b\mathcal{D}}$, $\Delta_{b\mathcal{D}}$ denoting the diagonal on $b\mathcal{D}$. Here and in what follows, we denote by $|E|$ the surface area measure of a measurable set $E \subseteq b\mathcal{D}$, or the Lebesgue measure of E if $E \subseteq \mathcal{D}$.

2. Statement of the main results. In order to describe the boundary behaviour of functions in the Hardy–Sobolev spaces $\mathcal{H}^{p,k}$ we will study the boundedness of a maximal function that can be defined also for non-integral values.

We now recall the definition of an *operator of order* $a = (a_1, \ldots, a_n)$, as introduced in [McS2, Definition 3]. A function $\psi \in C^N(b\mathcal{D})$ is called a *normalized bump function* if $\operatorname{supp} \psi \subseteq B(\zeta_0, r)$ and

$$|L^{\mu}_{\Lambda}\psi(\zeta)| \le \tau^{-\mu}(\zeta_0, \Lambda, r)$$

for all lists Λ and indices μ with $|\mu| \le N$, and for all $\zeta \in B(\zeta_0, r)$.

Given $a = (a_1, \ldots, a_n)$, we are going to use the notation

$$\tau^{a}(z, \delta) = \tau_1(z, \delta)^{a_1} \cdots \tau_n(z, \delta)^{a_n}. \tag{9}$$

Definition 2.1. An operator

$$Tf(\zeta) = \int_{b\mathcal{D}} H(\zeta, \omega) f(\omega)\, d\sigma(\omega)$$

is said to be *of order* $a = (a_1, \ldots, a_n)$ if there exists a family of operators $T_{\varepsilon} f(\zeta) = \int_{b\mathcal{D}} H_{\varepsilon}(\zeta, \omega) f(\omega)\, d\sigma(\omega)$ such that:

(i) for $f \in C^\infty(b\mathcal{D})$, $T_\varepsilon f \to Tf$ in C^∞ as $\varepsilon \to 0$;

(ii) $H_\varepsilon \in C^\infty(b\mathcal{D} \times b\mathcal{D})$;

(iii) for all multi-indices μ, μ' and lists Λ, Λ' we have

$$|L^{\mu}_{\Lambda,\zeta} L^{\mu'}_{\Lambda',\omega} H_\varepsilon(\zeta,\omega)| \lesssim \frac{\tau^a(\zeta,\delta)\tau^{-\mu}(\zeta,\Lambda,\delta)\tau^{-\mu'}(\zeta,\Lambda',\delta)}{|B(\zeta,\delta)|}, \tag{10}$$

where $\delta = d_b(\zeta,\omega)$, and the estimate holds uniformly in ε;

(iv) for each non-negative integer m there exists a positive integer N_m such that for any normalized bump function ψ of order $\geq N_m$ on $B(\zeta_0,\delta)$,

$$\sup_{\zeta \in B(\zeta_0,\delta)} |L^{\mu}_{\Lambda}(T_\varepsilon \psi)(\zeta)| \lesssim \tau^a(\zeta,\delta)\tau^{-\mu}(\zeta,\Lambda,\delta)$$

for all lists Λ and indices μ with $|\mu| \leq m$, uniformly in ε.

Remark 2.2. (a) We notice that typical examples of operators of order 0 are the identity operator and the Szegö projection; see [McS2].

(b) The estimates in the definition of an operator of order a are symmetric in ζ and ω.

(c) Since $\delta^{1/2} \lesssim \tau_j(\zeta,\delta) \lesssim \delta^{1/T_{\mathcal{D}}}$, it follows that an operator of order $a = (a_1,\dots,a_n)$ is also of order $(s_a, 0, \dots, 0)$, where

$$s_a := a_1 + \frac{1}{T_{\mathcal{D}}} \sum_{a_j>0,\, j\neq 1} a_j + \frac{1}{2} \sum_{a_j<0,\, j\neq 1} a_j.$$

Definition 2.3. We say that an operator T of order $a = (a_1,\dots,a_n)$ with kernel H satisfies *interior estimates of S-type of order a* if the approximating kernels $H_\varepsilon(z,\omega)$ extend holomorphically in $z \in \mathcal{D}$ and they satisfy estimates

$$|L^{\mu}_{\Lambda,\zeta} L^{\mu'}_{\Lambda',\omega} H_\varepsilon(z,\omega)| \lesssim \frac{\tau^a(z,\delta)\tau^{-\mu}(z,\Lambda,\delta)\tau^{-\mu'}(z,\Lambda',\delta)}{|Q(\pi(z),\delta)|},$$

uniformly in ε, where $\delta = d(\zeta,\omega)$, for all lists Λ, Λ' and all multi-indices μ, μ'.

On the smooth, bounded, convex domain $\mathcal{D}$ of finite type we introduce approach regions whose shape is clearly determined by the geometry of the boundary. Here, and in the rest of the paper, for $z \in \overline{\mathcal{D}}$ and $\delta > 0$ we set

$$\nu(z,\delta) = \prod_{j=2}^{n} \tau_j(z,\delta), \tag{11}$$

and, for short, for $z \in \mathcal{D}$,

$$\nu(z) = \prod_{j=2}^{n} \tau_j(z,\delta(z)). \tag{12}$$

Notice that $|B(\pi(z),\delta(z))| \approx \delta(z)\nu^2(z)$.

DEFINITION 2.4. Let $1 \le p < \infty$ and let $0 \le s < n/p$. For $\zeta \in b\mathcal{D}$ we define the *tangential approach regions*

$$\mathcal{A}(\zeta) = \{z \in \mathcal{D} : d_b(\pi(z), \zeta)\nu^2(\pi(z), d_b(\pi(z), \zeta)) \le (\delta(z)\nu^2(z))^{1-sp/n}\}. \tag{13}$$

Recall that, here and in what follows, for $z \in \mathcal{D}$ we denote by $\pi(z)$ the unique normal projection of z onto the boundary.

Notice that, if $s = 0$ and $\mathcal{D}$ is a strongly pseudoconvex domain, we recover the classical approach regions for the Hardy spaces $\mathcal{H}^p$, which are independent of p. On the other hand, if $s > 0$, then $\mathcal{A}(\zeta) = \mathcal{A}_{p,s}(\zeta)$, that is, $\mathcal{A}(\zeta)$ depend on p and s. Finally, notice that the inequality in the definition of $\mathcal{A}(\zeta)$ can be written as

$$|B(\pi(z), d_b(\pi(z), \zeta))| \le |B(\pi(z), \delta(z))|^{1-sp/n}.$$

The main results of the present work are the following.

THEOREM 2.5. *Let $\mathcal{D}$ be a smooth, bounded, convex domain of finite type. Let T be an operator of order a with kernel H satisfying interior estimates of S-type, where $a = (s/n, 2s/n, \dots, 2s/n)$ and $0 \le s < n$. Let $1 \le p < \infty$ and $\mathcal{A}$ be defined in* (13). *Define the maximal operator*

$$\mathcal{M}f(\zeta) = \sup_{z\in\mathcal{A}(\zeta)} |(Tf)(z)|.$$

Let $1 \le p < \infty$ and assume that $1 - sp/n > 0$. Then $\mathcal{M}$ is weak-type $(1,1)$, *and for $p > 1$, $\mathcal{M} : L^p(b\mathcal{D}) \to L^p(b\mathcal{D})$ is bounded.*

THEOREM 2.6. *Let $\mathcal{D}$ be a smooth, bounded, convex domain of finite type, $1 \le p < \infty$, $0 \le k < n/p$ and let $f \in \mathcal{H}^{p,k}$. Let $\mathcal{A}(\zeta) = \mathcal{A}_{p,k}(\zeta)$ be the approach region defined in* (13) (*with $s = k$*). *Then*

$$\lim_{\mathcal{A}(\zeta)\ni z\to\zeta\in b\mathcal{D}} f(z) = f(\zeta)$$

exists for a.a. $\zeta \in b\mathcal{D}$. Moreover, the boundary function $f(\zeta)$ so determined belongs to the Sobolev space $W^{k,p}(b\mathcal{D})$.

We recall that the above results hold true when $\mathcal{D}$ is a smooth, bounded strongly pseudoconvex domain (see [Su2]).

3. Proof of Theorem 2.5. We begin by recalling that the maximal operators are defined in terms of the regions $\mathcal{A}_{p,s}$ that depend on s and p. Hence, in order to apply an interpolation argument one would need to prove that the family of operators depend analytically on s and p. For this reason we have chosen to prove the results for $p > 1$ and $p = 1$ directly and independently. We begin with the case $p > 1$ and treat the case $p = 1$ later.

The kernel H of the operator T can be approximated by operators $H_\varepsilon(z, \omega)$ that can be extended holomorphically in $z \in \mathcal{D}$ and that satisfy the estimates in Definition 2.3, uniformly in ε.

We consider the corresponding maximal operator $\mathcal{M}_\varepsilon$ defined in terms of the operator with kernel H_ε and prove the result in the statement of the theorem for such operators. All the estimates in the proof will hold uniformly in ε and hence the result will follow for the operator $\mathcal{M}$ itself.

Therefore, for simplicity of notation we drop the subscript ε from H_ε in what follows.

We set $B_k = B(\pi(z), 2^k\delta(z))$ and

$$I_0 = \int_{B_0} f(\omega)H(z,\omega)\,d\sigma(\omega),$$

and, for $k \geq 1$,

$$I_k = \int_{B_k\setminus B_{k-1}} f(\omega)H(z,\omega)\,d\sigma(\omega).$$

Hence,

$$(Tf)(z) = \sum_{k\geq 0} I_k.$$

For a point $z \in U$ we write $z' = \pi(z)$. We know that I_k depends on $z \in \mathcal{D}$ and

$$|I_k| \leq |B(z', 2^k\delta(z))|^{s/n}\,\frac{1}{|B_k|}\int_{B_k} |f(\omega)|\,d\sigma(\omega).$$

We wish to estimate $|\sum_{k\geq 0} I_k|$. We have three cases.

Case 1. We drop the subscript in $T_{\mathcal{D}}$ for simplicity. Let

$$A_1 = \{k : 2^k\delta(z)\nu^2(z', 2^k\delta(z)) \geq 1\}.$$

Then, for $k \in A_1$, we have $(2^k\delta(z))^{1+\frac{2}{T}(n-1)} \geq c$, that is

$$2^k\delta(z) \geq c'.$$

Therefore,

$$\Big|\sum_{k\in A_1} I_k\Big| = \Big|\sum_{k\in A_1}\int_{B_k\setminus B_{k-1}} f(\omega)H(z,\omega)d\sigma(\omega)\Big| \tag{14}$$
$$\leq \int_{b\mathcal{D}\setminus B(z',c'/2)} |f(\omega)|\,|H(z,\omega)|\,d\sigma(\omega)$$
$$\leq C\int_{b\mathcal{D}} |f(\omega)|\,d\sigma(\omega) \leq C'\|f\|_{L^p(b\mathcal{D})}.$$

Case 2. We let

$$A_2 = \{k : 2^k\delta(z)\nu^2(z', 2^k\delta(z)) < 1\}.$$

Then for $k \in A_2$ we have $(2^k\delta(z))^{n+1} \leq c$, that is,

$$2^k\delta(z) \leq c'.$$

We split this case into two subcases. We define

$$A_2' = \{k : 1 > 2^k\delta(z)\nu^2(z', 2^k\delta(z)) \geq (\delta(z)\nu^2(z))^{1-sp/n}\}.$$

Then for $k \in A_2'$ and $z \in \mathcal{A}(\zeta)$ we have

$$d_b(z',\zeta)\nu^2(\zeta, d_b(z',\zeta)) \leq (\delta(z)\nu^2(z))^{1-sp/n} \leq 2^k\delta(z)\nu^2(z', 2^k\delta(z)). \tag{15}$$

Claim 1. *For $k \in A_2'$ and $z \in \mathcal{A}(\zeta)$ we have*

$$d_b(z',\zeta) \leq C2^k\delta(z),$$

where C does not depend on k or z.

Assume that $2^k\delta(z) \leq d_b(z',\zeta)$ (otherwise there is nothing to prove). This gives

$$\begin{aligned}\nu^2(z', 2^k\delta(z)) &= \nu^2\left(z', d_b(z',\zeta)\,\frac{2^k\delta(z)}{d_b(z',\zeta)}\right)\\ &\leq C\left(\frac{2^k\delta(z)}{d_b(z',\zeta)}\right)^{\frac{2}{T}(n-1)}\nu^2(z', d_b(z',\zeta)).\end{aligned}$$

Using this last inequality in (15) we obtain

$$d_b(z',\zeta)\nu^2(\zeta, d_b(z',\zeta)) \leq C2^k\delta(z)\left(\frac{2^k\delta(z)}{d_b(z',\zeta)}\right)^{\frac{2}{T}(n-1)}\nu^2(z', 2^k\delta(z));$$

that is, $d_b(z',\zeta) \leq C2^k\delta(z)$, which proves the claim.

The claim shows that the ball $\widetilde{B}_k = B(z', C2^k\delta(z))$ contains the point ζ. Then

$$\begin{aligned}\frac{1}{|B_k|}\int_{B_k}|f(\omega)|d\sigma(\omega) &\leq C\frac{1}{|\widetilde{B}_k|}\int_{\widetilde{B}_k}|f(\omega)|\,d\sigma(\omega)\\ &\leq C\sup_{d_b(z',\zeta)<r\leq c'}\frac{1}{|B(z',r)|}\int_{B(z',r)}|f(\omega)|\,d\sigma(\omega)\\ &\leq CM_{b\mathcal{D}}f(\zeta),\end{aligned}$$

where $M_{b\mathcal{D}}$ denotes the Hardy–Littlewood maximal function on $b\mathcal{D}$.

Thus, we have shown that

$$\Big|\sum_{k\in A_2'} I_k\Big| \leq C\sum_{2^k\delta(z)\leq c}|B(z', 2^k\delta(z))|^{s/n}M_{b\mathcal{D}}f(\zeta).$$

We now show that the above series converges. Since $2^k\delta(z) \leq c$ we have

$$|B(z', 2^k\delta(z))|^{s/n} = (2^k\delta(z)\nu^2(z', 2^k\delta(z)))^{s/n} \leq C_{s,n}(2^k\delta(z))^{s/n}$$

and that

$$\sum_{2^k\delta(z)\le c} (2^k\delta(z))^{s/n} \le \delta(z)^{s/n}\,\frac{2^{(k_0+1)s/n}-1}{2^{s/n}-1}$$
$$\le C2^{s/n}\,\frac{(2^{k_0}\delta(z))^{s/n}}{2^{s/n}-1} \le C_{s,n}.$$

Therefore,

$$\Big|\sum_{k\in A_2'} I_k\Big| \le C_{s,n} M_{b\mathcal{D}} f(\zeta). \tag{16}$$

Finally, we consider the set

$$A_2'' = \{k : 2^k\delta(z)\nu^2(z', 2^k\delta(z)) \le (\delta(z)\nu^2(z))^{1-sp/n}\}.$$

In this case, we need to introduce the weight function already considered in [NRSh] and [Su2]. For a point $z \in \mathcal{D}$, we write $z' = \pi(z)$. Let $\varrho \ge 1$. For $z \in \mathcal{D}$ we define the *shadow* of the ball $B(z', \varrho\delta(z))$ to be the set

$$\begin{aligned}\Omega^\varrho(z) &= \Omega^\varrho(z', \varrho\delta(z))\\ &= \{\theta \in b\mathcal{D} : \text{there exists } w \in \mathcal{A}(\theta) \text{ with } \delta(w) = \delta(z) \text{ and } w' \in B(z', \varrho\delta(z))\}.\end{aligned}$$

Next, we define the weight $\eta^\varrho : b\mathcal{D} \times (0, C_{\mathcal{D}}) \to [0, +\infty)$ as

$$\eta^\varrho(\omega, \varrho s) = \inf\Big\{\frac{|B(z', \varrho\delta(z))|}{|\Omega^\varrho(z', \varrho\delta(z))|} : \delta(z) \ge s,\ \omega \in B(z', \varrho\delta(z))\Big\}, \tag{17}$$

where $\omega \in b\mathcal{D}$ and $C_{\mathcal{D}}$ is a constant depending only on $\mathcal{D}$; cf. [NS, Lemma 11] and [Su2, Sect. 1].

We are going to use the above definitions to prove the following statement.

Claim 1. *Set*

$$M_p^{2^k} f(\zeta) = \sup_{z\in\mathcal{A}(\zeta)} \eta^{2^k}(z', 2^k\delta(z))^{1/p}\frac{1}{|B_k|}\int_{B_k} |f(\omega)|\,d\sigma(\omega).$$

Then for any $p > 1$ there exists $A_p > 0$ such that

$$\|M_p^{2^k} f\|_{L^p(b\mathcal{D})} \le A_p\|f\|_{L^p(b\mathcal{D})},$$

where the constant A_p does not depend on k.

We will adapt a result by Sueiro [Su2, Prop. 1.12] to the case of a smooth, bounded domain of finite type. In order to do this, we need a number of preliminary facts.

Our ultimate goal is to estimate the weight η^{2^k} from below, while our first goal is to estimate the measure of the set $\Omega^{2^k}(z, 2^k\delta(z))$.

Let $z \in \mathcal{D}$ be fixed and let $w \in \mathcal{D}$ be such that $\delta(w) = \delta(z)$. We write $w' = \pi(w)$. Notice that $w \in \mathcal{A}(\theta)$ if and only if

$$d_b(\theta, w')\nu^2(\theta, d_b(\theta, w')) \le (\delta(w)\nu^2(w))^{1-sp/n} \ ;$$

that is, if and only if

$$d_b(\theta, w')\nu^2(\theta, d_b(\theta, w')) \le (\delta(z)\nu^2(w', \delta(z)))^{1-sp/n}. \tag{18}$$

Moreover, $w' \in B(z', 2^k\delta(z))$ if and only if $d_b(w', z') \le 2^k\delta(z)$, and from the definition of A''_2 we get

$$d_b(w', z')\nu^2(z', d_b(w', z')) \le c(\delta(z)\nu^2(z))^{1-sp/n}. \tag{19}$$

Notice that we can reformulate (18) as

$$|B(\theta, d_b(\theta, w'))| \le c|B(w', \delta(z))|^{1-sp/n}, \tag{20}$$

and (19) as

$$|B(z', d_b(z', w'))| \le c|B(z', \delta(z))|^{1-sp/n}. \tag{21}$$

We now estimate $|\Omega^{2^k}(z', 2^k\delta(z))|$. We denote this set by $\Omega^{2^k}(z)$ for short.

Let $\theta \in \Omega(z)$ and denote by $w(\theta)$ the point in $\mathcal{A}(\theta)$ as in the definition of $\Omega^{2^k}(z)$, and let $w'(\theta) = \pi(w(\theta))$. Define

$$d(\theta) = \max\{d_b(\theta, w'(\theta)), d_b(z', w'(\theta))\} \quad \text{and} \quad d = \sup_{\theta \in \Omega^{2^k}(z)} d(\theta). \tag{22}$$

Notice that $d = d_z$ depends on the point z.

Then there exists a constant $c_{\mathcal{D}}$ such that $\Omega^{2^k}(z', 2^k\delta(z)) \subseteq B(z', c_{\mathcal{D}}d)$. In fact, for all $\theta \in \Omega^{2^k}(z', 2^k\delta(z))$,

$$d_b(z', \theta) \le C_{\mathcal{D}}(d_b(\theta, w'(\theta)) + d_b(w'(\theta), z')) \le 2C_{\mathcal{D}}d(\theta) \le 2C_{\mathcal{D}}d = c_{\mathcal{D}}d.$$

Claim 2. *For all $\theta \in \Omega^{2^k}(z)$ we have*

$$|B(z', d(\theta))| \le C2^{-k(1+\frac{2}{T}(n-1))(1-sp/n)}|B(z', 2^k\delta(z))|^{1-sp/n}.$$

In order to prove the claim, we distinguish two cases. We first assume that $d_b(\theta, w'(\theta)) \le d_b(z', w'(\theta))$, that is, $d(\theta) = d_b(z', w'(\theta))$. Then (21) gives

$$|B(z', d(\theta))| \le |B(z', \delta(z))|^{1-sp/n}$$

and we just have to use the fact that

$$\delta(z)\nu^2(z, \delta(z)) \le C2^{-k(1+\frac{2}{T}(n-1))}|2^k\delta(z)\nu^2(z', 2^k\delta(z))|.$$

Next, we assume that $d(\theta) = d_b(\theta, w'(\theta))$. Then

$$d_b(z', \theta) \le C_{\mathcal{D}}(d_b(z', w'(\theta)) + d_b(w'(\theta), \theta)) \le 2C_{\mathcal{D}}d_b(w'(\theta), \theta),$$

that is, $z' \in B(\theta, c_{\mathcal{D}}d_b(\theta, w'(\theta)))$. Hence,

$$|B(z', d_b(\theta, w'(\theta)))| \approx |B(\theta, d_b(\theta, w'(\theta)))|.$$

Now, (20) gives

$$|B(z', d_b(\theta, w'(\theta)))| \le |B(w'(\theta), \delta(z))|^{1-sp/n} \le 2^{-k(1+\frac{2}{T}(n-1))(1-sp/n)}|B(w'(\theta), 2^k\delta(z))|^{1-sp/n}.$$

Recalling that $\theta \in \Omega^{2^k}(z)$ implies that $d_b(w'(\theta), z') \le 2^k\delta(z)$, that is, $z' \in B(w'(\theta), 2^k\delta(z))$, we have

$$|B(z', 2^k\delta(z))| \approx |B(w'(\theta), 2^k\delta(z))|.$$

Therefore, by (21),

$$|B(w'(\theta), d_b(\theta, w'(\theta)))| \le 2^{-k(1+\frac{2}{T}(n-1))(1-sp/n)}|B(w'(\theta), 2^k\delta(z))|^{1-sp/n} \lesssim 2^{-k(1+\frac{2}{T}(n-1))(1-sp/n)}|B(z', 2^k\delta(z))|^{1-sp/n}.$$

This proves the claim.

We remark that from the previous discussion we learn that

$$|\Omega^{2^k}(z', 2^k\delta(z))| \le C2^{-k(1+\frac{2}{T}(n-1))(1-sp/n)}|B(z', 2^k\delta(z))|^{1-sp/n}. \tag{23}$$

We are in a position to estimate the weight (17).

Claim 3. *There exist a positive integer $k_0 = k_0(\mathcal{D})$ and $\varepsilon_0 = \varepsilon_0(\mathcal{D}) > 0$ such that for all $z \in \mathcal{D}$ with $\delta(z) < \varepsilon_0$,*

$$\eta^{2^k}(z', 2^k\delta(z)) \gtrsim \frac{|B(z', 2^{k+k_0}\delta(z))|}{|\Omega^{2^k}(z', 2^{k+k_0}\delta(z))|}.$$

Let $z \in \mathcal{D}$ and suppose $\theta \in B(z', 2^k\delta(z))$. Let $\omega \in \Omega^{2^k}(\theta, 2^k\delta(z))$. Let $w = w(\omega)$ with $\delta(w) = \delta(z)$ and $w' \in B(\theta, 2^k\delta(z))$. Then $w' \in B(z', c_1 2^k\delta(z))$, so that $\omega \in \Omega^{2^k}(z', 2^k c_1\delta(z))$.

Therefore, for all $\theta \in B(z', 2^k\delta(z))$,

$$\Omega^{2^k}(\theta, 2^k\delta(z)) \subset \Omega^{2^k}(z', 2^k c_1\delta(z)),$$

so that

$$\inf_{\theta:\, d_b(z',\theta)\le 2^k\delta(z)} \frac{|B(\theta, 2^k\delta(z))|}{|\Omega^{2^k}(\theta, 2^k\delta(z))|} \gtrsim \frac{|B(z', 2^k c_1\delta(z))|}{|\Omega^{2^k}(z', 2^k c_1\delta(z))|}.$$

This proves the claim.

Using (22), it now follows that

$$\eta^{2^k}(z', 2^k\delta(z)) \gtrsim \frac{|B(z', 2^k\delta(z))|}{|B(z', c_1 d)|} \gtrsim \frac{|B(z', 2^k\delta(z))|}{2^{-k(1+\frac{2}{T}(n-1))(1-sp/n)}|B(z', 2^k\delta(z))|^{1-sp/n}},$$

that is,

$$|B(z', 2^k\delta(z))|^{sp/n} \le C2^{-k(1+\frac{2}{T}(n-1))(1-sp/n)}\eta^{2^k}(z', 2^k\delta(z)).$$

Assuming the validity of the previous Claim 1, we have

$$\begin{aligned}\Big|\sum_{k\in A_2''} I_k\Big| &\le \sum_{k\in A_2''} |B(z', 2^k\delta(z))|^{s/n}\,\frac{1}{|B_k|}\int_{B_k}|f(\omega)|\,d\sigma(\omega)\\ &\le \sum_{k\in A_2''} 2^{-k(1+\frac{2}{T}(n-1))(1-sp/n)} M_p^{2^k} f(\zeta).\end{aligned}$$

Combining the above estimates with (14) and (16) we obtain

$$\begin{aligned}|(Tf)(\zeta)| &\le \Big|\sum_k I_k\Big|\\ &\le C\Big(\|f\|_{L^p(b\mathcal{D})} + M_{b\mathcal{D}}f(\zeta) + \sum_k 2^{-k(1+\frac{2}{T}(n-1))(1-sp/n)} M_p^{2^k} f(\zeta)\Big),\end{aligned}$$

so that

$$\begin{aligned}\|\mathcal{M}f\|_{L^p(b\mathcal{D})} &\le C\Big(\sum_k 2^{-k(1+\frac{2}{T}(n-1))(1-sp/n)}\Big)\|f\|_{L^p(b\mathcal{D})}\\ &\le C'\|f\|_{L^p(b\mathcal{D})}\end{aligned}$$

since $1 - sp/n > 0$. Modulo the proof of Claim 1, this proves the theorem for the case $p > 1$.

Now, the proof of Claim 1 is as the one of Thm. 1.7 in [Su2], using the fact that the family of balls $\{B(\zeta, r)\}$ on $b\mathcal{D}$ satisfy the requirements for $b\mathcal{D}$ to be a space of homogeneous type with respect to the surface measure (see [McS2], e.g.).

Finally, in the case $p = 1$ we consider the approach regions

$$\mathcal{A}_{1,s}(\zeta) = \{z \in \mathcal{D} : d_b(\pi(z), \zeta)\nu^2(\pi(z), d_b(\pi(z), \zeta)) \le (\delta(z)\nu^2(z))^{1-s/n}\}.$$

Writing $B_\zeta := B(\zeta, 2C_{\mathcal{D}} d(z', \zeta))$ we decompose

$$\begin{aligned}(Tf)(z) &= \int_{B_\zeta} H(z,\omega)f(\omega)\,d\sigma(\omega) + \int_{b\mathcal{D}\setminus B_\zeta} H(z,\omega)f(\omega)\,d\sigma(\omega)\\ &=: J_1 + J_2.\end{aligned}$$

Using the estimate for the kernel of type $a = (s/n, 2s/n, \ldots, 2s/n)$, we see that

$$\begin{aligned}|H(z,\omega)| &\le C(d(z,\omega)\nu^2(z, d(z,\omega)))^{-(1-s/n)}\\ &\le (\delta(z)\nu^2(z, \delta(z)))^{-(1-s/n)}.\end{aligned}$$

Then for $z \in \mathcal{A}_{1,s}(\zeta)$ we have

$$\begin{aligned}|J_1| &\le (\delta(z)\nu^2(z,\delta(z)))^{-1+s/n} \int_{B_\zeta} |f(\omega)|\,d\sigma(\omega)\\ &\le C\,\frac{1}{|B(\zeta,d_b(\zeta,z'))|}\int_{B_\zeta}|f(\omega)|\,d\sigma(\omega)\\ &\le C\,\frac{1}{|B_\zeta|}\int_{B_\zeta}|f(\omega)|\,d\sigma(\omega)\le CM_{b\mathcal{D}}f(\zeta),\end{aligned}$$

where again $z' = \pi(z)$.

Next, notice that $\omega \in b\mathcal{D} \setminus B(\zeta, 2C_{\mathcal{D}}d(z',\zeta))$ implies that $d_b(\omega, z') \ge cd_b(\omega,\zeta)$. Indeed, since $d_b(\zeta,\omega) \ge 2C_{\mathcal{D}}d_b(z',\zeta)$, and since

$$d_b(\omega,\zeta) \le C_{\mathcal{D}}(d_b(\omega,z') + d_b(z',\zeta)) \le C_{\mathcal{D}}d_b(\omega,z') + \frac{1}{2}\,d_b(\omega,\zeta),$$

the assertion follows with $c = 1/2C_{\mathcal{D}}$. This allows us to majorize

$$|J_2| \le C\int_{b\mathcal{D}} \frac{|f(\omega)|}{(d_b(\zeta,\omega)\nu^2(\omega,d_b(\zeta,\omega)))^{1-s/n}}\,d\sigma(\omega).$$

Now notice that, if $d_b(\zeta,\omega) \ge 1$, the denominator above is bounded from below by a positive constant, so that

$$\int_{b\mathcal{D}\setminus\{d_b(\zeta,\omega)\ge 1\}} \frac{|f(\omega)|}{(d_b(\zeta,\omega)\nu^2(\omega,d_b(\zeta,\omega)))^{1-s/n}}\,d\sigma(\omega) \le c\|f\|_{L^1(b\mathcal{D})}.$$

Otherwise, write

$$\Big\{\omega : d_b(\zeta,\omega)\le 1\} \subseteq \bigcup_{k\ge 0}\{\omega : 2^{-(k+1)} \le d_b(\zeta,\omega) \le 2^{-k}\Big\} = \bigcup_{k\ge 0}\mathcal{C}_k.$$

Then

$$\begin{aligned}&\int_{\{\omega\,:\,d_b(\zeta,\omega)\le 1\}} \frac{|f(\omega)|}{(d_b(\zeta,\omega)\nu^2(\omega,d_b(\zeta,\omega)))^{1-s/n}}\,d\sigma(\omega)\\ &\qquad\le \sum_{k\ge 0}\int_{\mathcal{C}_k}\frac{|f(\omega)|}{(d_b(\zeta,\omega)\nu^2(\omega,d_b(\zeta,\omega)))^{1-s/n}}\,d\sigma(\omega)\\ &\qquad\le C\sum_{k\ge 0}\frac{1}{(2^{-k}\nu^2(\zeta,2^{-k}))^{1-s/n}}\int_{\mathcal{C}_k}|f(\omega)|\,d\sigma(\omega)\\ &\qquad\le C\sum_{k\ge 0}(2^{-k}\nu^2(\zeta,2^{-k}))^{s/n}M_{b\mathcal{D}}f(\zeta)\\ &\qquad\le CM_{b\mathcal{D}}f(\zeta),\end{aligned}$$

since $2^{-k} \le 1$ implies that $\nu^2(\zeta,2^{-k}) \le C'$.

Therefore, $\mathcal{M}f(\zeta) \le C(M_{b\mathcal{D}}f(\zeta) + \|f\|_{L^1(b\mathcal{D})})$. This shows that $\mathcal{M}$ is weak-type $(1,1)$ and we are done. ∎

4. Proof of Theorem 2.6. Let $1 \le p < \infty$ and k a non-negative integer such that $k < n/p$. Let b be a fixed (small) positive number. For $\zeta \in b\mathcal{D}$, let N denote a smooth vector field defined in a tubular neighborhood of the boundary, transversal to the boundary itself. For $\zeta \in b\mathcal{D}$, let $N(\zeta, s)$ be the integral curve of N such that $N(\zeta, 0) = \zeta$ and $N(\zeta, s) \in \mathcal{D}$ for $0 < s \le b$.

Without loss of generality, by induction we may assume that $k = 1$. Then, for $f \in \mathcal{H}^{p,1}$, recalling that $S_{\mathcal{D}}$ is the Szegö kernel, we write

$$\begin{aligned} f(\zeta) - f(N(\zeta, b)) &= -\int_0^b \frac{d}{ds}(f(N(\zeta, s)))\, ds \\ &= \int_0^b \sum_{j=1}^n g_j(N(\zeta, s))\partial_{z_j} f(N(\zeta, s))\, ds \\ &= \sum_{j=1}^n \int_0^b g_j(N(\zeta, s)) \int_{b\mathcal{D}} \partial_{z_j} f(\omega) S_{\mathcal{D}}(N(\zeta, s), \omega)\, d\sigma(\omega)\, ds \\ &= \sum_{j=1}^n \int_{b\mathcal{D}} \partial_{z_j} f(\omega) \int_0^b g_j(N(\zeta, s)) S_{\mathcal{D}}(N(\zeta, s), \omega)\, ds\, d\sigma(\omega). \end{aligned}$$

Therefore, for $z \in \mathcal{D}$, denoting by $\mathcal{P}$ the Szegö projection, we have

$$\begin{aligned} f(z) &= \sum_{j=1}^n \int_{b\mathcal{D}} \partial_{z_j} f(\omega) \int_{b\mathcal{D}} S_{\mathcal{D}}(z, \zeta) \int_0^b g_j(N(\zeta, s)) S_{\mathcal{D}}(N(\zeta, s), \omega)\, ds\, d\sigma(\zeta)\, d\sigma(\omega) \\ &\quad + \mathcal{P}(f(N(\cdot, b))(z) \\ &= \sum_{j=1}^n \int_{b\mathcal{D}} \partial_{z_j} f(\omega) H^{(j)}(\zeta, \omega)\, d\sigma(\omega) + \mathcal{P}(f(N(\cdot, b)))(z). \end{aligned}$$

We now claim that the operators T_j having kernels

$$H^{(j)}(z, \omega) = \int_{b\mathcal{D}} S_{\mathcal{D}}(z, \zeta) \int_0^b g_j(N(\zeta, s)) S_{\mathcal{D}}(N(\zeta, s), \omega)\, ds\, d\sigma(\zeta) \tag{24}$$

are operators of order $a = (1/n, 2/n, \ldots, 2/n)$ satisfying interior estimates of S-type of the same order, in the sense of Definition 2.3.

Assume this fact for the moment, and denote by $\mathcal{M}_j$ the operator defined on $L^p(b\mathcal{D})$ by

$$\mathcal{M}_j g(\zeta) = \sup_{z \in \mathcal{A}(\zeta)} |(T_j g)(z)|.$$

Also, notice that the function $\mathcal{P}(f(N(\cdot, b)))$ is holomorphic in $\mathcal{D}$ and continues smoothly to the boundary $b\mathcal{D}$, and that its sup norm is controlled, say, by the $\mathcal{H}^p$ norm of f.

Then we obtain

$$\sup_{z\in\mathcal{A}(\zeta)} |f(z)| \le \sum_{j=1}^{n} \mathcal{M}_j(\partial_{z_j} f)(\zeta) + \|f\|_{\mathcal{H}^p},$$

so that, if we denote by $f^*(z)$ the left hand side above, we have

$$\begin{aligned}\|f^*\|_{L^p(b\mathcal{D})} &\lesssim \sum_{j=1}^{n} \|\mathcal{M}_j(\partial_{z_j} f)\|_{L^p(b\mathcal{D})} + \|f\|_{\mathcal{H}^p} \lesssim \sum_{j=1}^{n} \|\partial_{z_j} f\|_{L^p(b\mathcal{D})} + \|f\|_{\mathcal{H}^p} \\ &\lesssim \sum_{j=1}^{n} \|\partial_{z_j} f\|_{\mathcal{H}^p} + \|f\|_{\mathcal{H}^p} \lesssim \|f\|_{\mathcal{H}^{p,1}}.\end{aligned}$$

Now, a classical argument finishes the proof, modulo the claim.

We now prove the claim, the proof of which depends essentially on two properties, similar to Propositions 3.1 and 2.3 in [McS2].

It is clear that the kernels $H^{(j)}$ defined in (24) are holomorphic in the the first variable z in $\mathcal{D}$ and have anti-holomorphic extension in the second variable ω in $\mathcal{D}$ as well.

Next, since the Szegö projection is an operator of order 0, by [McS2, Prop. 2.3], it suffices to show that an operator having kernel

$$K(\zeta,\omega) = \int_0^b g(N(\zeta,s)) S_{\mathcal{D}}(N(\zeta,s),\omega)\, ds, \tag{25}$$

where g is a smooth function on $b\mathcal{D}$, is an operator of order $a = (1/n, 2/n, \dots, 2/n)$, since condition (2.1) in [McS2] is clearly satisfied.

Define

$$K_\varepsilon(\zeta,\omega) = \int_\varepsilon^b g(N(\zeta,s)) S_{\mathcal{D}}(N(\zeta,s),\omega)\, ds. \tag{26}$$

We need to show that K_ε satisfies conditions (i)–(iii) in Definition 2.1 and that the corresponding integral operator T_ε satisfies condition (iv) in Definition 2.1, all uniformly in ε.

Let $t = \delta(N(\zeta,s)) + d_b(N(\zeta,s),\omega)$. As observed in [McS2, 3.2], $t \gtrsim s$ and $\tau_j(N(\zeta,s),t) \approx \tau_j(\zeta,t)$, and also, for all s,

$$d_b(\zeta,\omega) \lesssim d_b(N(\zeta,s),\omega).$$

It follows that

$$s + d_b(\zeta,\omega) \lesssim t. \tag{27}$$

Therefore, using the fundamental estimates (8) for the Szegö kernel we obtain

$$(28)\quad |S_{\mathcal{D}}(N(\zeta,s),\omega)| \lesssim (s+d_b(\zeta,\omega))^{-1}\prod_{j=2}^{n}\tau_j(N(\zeta,s),s+d_b(N(\zeta,s),\omega))^{-2}$$
$$\lesssim (s+d_b(\zeta,\omega))^{-1}\prod_{j=2}^{n}\tau_j(\zeta,s+d_b(\zeta,\omega))^{-2}.$$

We write

$$K_\varepsilon(\zeta,\omega) = \int_{\varepsilon}^{d_b(\zeta,\omega)} g(N(\zeta,s))S_{\mathcal{D}}(N(\zeta,s),\omega)\,ds + \int_{d_b(\zeta,\omega)}^{b} g(N(\zeta,s))S_{\mathcal{D}}(N(\zeta,s),\omega)\,ds =: I + II.$$

Notice that, since $\delta^{1/2} \lesssim \tau_j(z,\delta)$, it follows that $\delta \lesssim |B(\zeta,\delta)|^{1/n}$. Hence

$$(29)\quad I \lesssim \int_{\varepsilon}^{d_b(\zeta,\omega)} (s+d_b(\zeta,\omega))^{-1}\prod_{j=2}^{n}\tau_j(\zeta,s+d_b(\zeta,\omega))^{-2}\,ds$$
$$\lesssim [\nu^2(\zeta,d_b(\zeta,\omega))]^{-1}\int_{\varepsilon}^{d_b(\zeta,\omega)} (s+d_b(\zeta,\omega))^{-1}\,ds$$
$$\lesssim |B(\zeta,d_b(\zeta,\omega))|^{-1+1/n},$$

uniformly in $\varepsilon > 0$. On the other hand, for $J \geq \log(b/d_b(\zeta,\omega))$,

$$(30)\quad II \lesssim \int_{d_b(\zeta,\omega)}^{b} (s+d_b(\zeta,\omega))^{-1}\prod_{j=2}^{n}\tau_j(\zeta,s+d_b(\zeta,\omega))^{-2}\,ds$$
$$\lesssim \sum_{k=0}^{J}\int_{2^k d_b(\zeta,\omega)}^{2^{k+1}d_b(\zeta,\omega)} [(s+d_b(\zeta,\omega))\nu^2(\zeta,s+d_b(\zeta,\omega))]^{-1}\,ds$$
$$\lesssim \sum_{k=0}^{J}[(2^k d_b(\zeta,\omega))\nu^2(\zeta,2^k d_b(\zeta,\omega))]^{-1}\int_{2^k d_b(\zeta,\omega)}^{2^{k+1}d_b(\zeta,\omega)} ds$$
$$\lesssim |B(\zeta,d_b(\zeta,\omega))|^{-1}\sum_{k=0}^{J}2^{-k}2^{-2k(n-1)/T}\int_{2^k d_b(\zeta,\omega)}^{2^{k+1}d_b(\zeta,\omega)} ds$$
$$\lesssim |B(\zeta,d_b(\zeta,\omega))|^{-1+1/n}.$$

The estimates (29) and (30) show that K_ε satisfies condition (iii) in Definition 2.1, uniformly in ε. Conditions (i)–(ii) are easily checked. The fact that the corresponding integral operator T_ε satisfies condition (iv) in Definition 2.1, uniformly in ε, follows from the same argument as in [McS2, Lemma 3.2]. This shows that the operator having integral given by (25) is of order $a = (1/n, 2/n, \dots, 2/n)$.

The last part of the statement is clear. ∎

5. Final remarks

Another maximal function. A different, less natural, extension of the tangential approach regions of Nagel–Rudin–Shapiro and Sueiro, in the setting of convex domains of finite type, is to use

$$\widetilde{\mathcal{A}}(\zeta) = \{z \in \mathcal{D} : d_b(\pi(z), \zeta) \le \delta(z)^{\gamma(1-sp/n)}\},$$

where $\gamma \ge 1$. Hence, we define the approach regions whose aperture depends only on the distance to the boundary. Notice that also in this case the regions $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}_{p,s}$ depend on p and on s.

We then set

$$\widetilde{\mathcal{M}}_\gamma f(\zeta) = \sup_{z \in \tilde{\mathcal{A}}(\zeta)} |(Tf)(z)|.$$

Then for the regions $\widetilde{\mathcal{A}}$ and the maximal opertor $\widetilde{\mathcal{M}}_\gamma$ we can easily prove the following results.

Proposition 5.1. *Let $p \ge 1$ and $s \ge 0$ be such that $1 - sp/n > 0$. Let*

$$\gamma \le 1 + \frac{sp}{n}\frac{n-1}{n-sp}\left(1 - \frac{2}{T}\right).$$

Then for all $\zeta \in b\mathcal{D}$ we have $\widetilde{\mathcal{A}} \subseteq \mathcal{A}$ and therefore the maximal operator $\widetilde{\mathcal{M}}_\gamma$ is controlled by the operator $\mathcal{M}$. Therefore, the maximal operator $\widetilde{\mathcal{M}}_\gamma$ is weak-type $(1,1)$ and hence $\widetilde{\mathcal{M}}_\gamma : L^p(b\mathcal{D}) \to L^p(b\mathcal{D})$ is bounded for $p > 1$.

Proof. The proof is somewhat obvious, and we only sketch it. It suffices to prove the result for $\gamma = 1 + \frac{sp}{n}\frac{n-1}{n-sp}\left(1 - \frac{2}{T}\right)$. One notices that

$$\gamma\left(1 - \frac{sp}{n}\right) - 1 = -\frac{sp}{n^2}\left(1 + \frac{2}{T}(n-1)\right) \le 0. \tag{31}$$

In order to prove that $\widetilde{\mathcal{A}}_{p,s} \subseteq \mathcal{A}_{p,s}$ one shows that $d_b(z',\zeta) \le \delta(z)^{\gamma(1-sp/n)}$ implies that

$$d_b(z',\zeta)\nu^2(z', d_b(z',\zeta)) \le (\delta(z)\nu^2(z))^{\gamma(1-sp/n)}. \tag{32}$$

Using the fact that $\delta(z)^{\gamma(1-sp/n)} \le 1$ we have

$$\begin{aligned} d_b(z',\zeta)\nu^2(z', d_b(z',\zeta)) &\le \delta(z)^{\gamma(1-sp/n)}\nu^2(z', \delta(z)^{\gamma(1-sp/n)}) \\ &\le C\delta(z)^{\gamma(1-sp/n)}\delta(z)^{(\gamma(1-sp/n)-1)(n-1)}\nu^2(z', \delta(z)) \\ &= C\delta(z)^{n\gamma(1-sp/n)-(n-1)}\nu^2(z', \delta(z)). \end{aligned}$$

Hence, in order to prove (32) it suffices to prove that

$$\delta(z)^{n\gamma(1-sp/n)-(n-1)}\nu^2(z',\delta(z)) \le \delta(z)^{1-sp/n}\nu^2(z',\delta(z))^{1-sp/n},$$

that is,

$$\delta(z)^{\gamma(n-sp)-n}(\delta(z)\nu^2(z',\delta(z)))^{sp/n} \le 1.$$

But by (31) the above inequality can be rewritten as

$$\delta(z)^{-(sp/n)\frac{2}{T}(1-n)}\nu^2(z',\delta(z))^{sp/n} \le 1.$$

Since $\nu^2(z',\delta(z)) \le C\delta(z)^{\frac{2}{T}(n-1)}$, the above inequality holds true and we are done. ■

Fractional Hardy–Sobolev spaces. It would be interesting to study Hardy–Sobolev spaces $\mathcal{H}^{p,s}$ for $s \ge 0$ not necessarily an integer. These spaces can be defined as follows. When $s \ge 0$, we set $s = [s] + s'$, where $[s]$ denotes the integral part of s and $0 \le s' < 1$. Then we define

$$\mathcal{H}^{p,s} = \Big\{ f \in \mathcal{H}(\mathcal{D}) : \partial^\alpha \Big(\int_{\mathcal{D}} f(w) K_{\mathcal{D}}(\cdot, w)\delta(w)^{-s'}\, dV(w) \Big) \in \mathcal{H}^p(\mathcal{D}) \quad \text{for all } \alpha,\ |\alpha| \le [s] \Big\}.$$

The analysis of these space would require the use of weighted reproducing kernels, in particular of the Bergman kernel for the weight given by powers of $\delta(z)$. This could probably be done using known techniques such as in [A, BeA, DiFFi]. We wish to return to these questions in the future.

Acknowledgments. The first named author wishes to thank Sandrine Grellier for several insightful conversations regarding various topics about these spaces.

It is a pleasure to thank the referee for several suggestions and remarks that helped improve the presentation.

Part of this work was done while the second author held a predoctoral position at the Politecnico di Torino within the project IHP Network HARP. We thank the European Commission and the Network for the support provided.

REFERENCES

[AhBr] P. Ahern and J. Bruna, *Maximal and area integral characterizations of Hardy–Sobolev spaces in the unit ball of* $\mathbb{C}^n$, Rev. Mat. Iberoamer. 4 (1988), 123–153.

[A] M. Andersson, *Integral representation with weights. I*, Math. Ann. 326 (2003), 1–18.

[BeA] B. Berndtsson and M. Andersson, *Henkin–Ramirez formulas with weight factors*, Ann. Inst. Fourier (Grenoble) 32 (1982), no. 3, 91–110.

[BoS] H. P. Boas and E. J. Straube, *On equality of line type and variety type of real hypersurfaces in* $\mathbb{C}^n$, J. Geom. Anal. 2 (1992), 95–98.

[BPS] A. Bonami, M. M. Peloso and F. Symesak, *Factorization of Hardy spaces and Hankel operators on convex domains in* $\mathbb{C}^n$, ibid. 11 (2001), 363–397.

[DFi] F. Di Biase and B. Fischer, *Boundary behaviour of* $\mathcal{H}^p$ *functions on convex domains of finite type in* $\mathbb{C}^n$, Pacific J. Math. 183 (1998), 25–38.

[DiFo] K. D. Diederich and J. E. Fornæss, *Support functions for convex domains of finite type*, Math. Z. 230 (1999), 145–164.

[DiFFi] K. D. Diederich, J. E. Fornæss and B. Fischer, *Hölder estimates on convex domains of finite type*, Math. Z. 232 (1999), 43–61.

[G] S. Grellier, *Hardy–Sobolev spaces of complex tangential derivatives of holomorphic functions in domains of finite type*, Math. Scand. 90 (2002), 232–250.

[GP] S. Grellier and M. M. Peloso, *Decomposition theorems for Hardy spaces on convex domains of finite type*, Illinois J. Math. 46 (2002), 207–232.

[KL] S. G. Krantz and S.-Y. Li, *Area integral characterization for functions in Hardy spaces on domains in* $\mathbb{C}^n$, Complex Var. 32 (1997), 373–399.

[Mc] J. D. McNeal, *Estimates on the Bergman kernels of convex domains*, Adv. Math. 109 (1994), 108–139.

[McS1] J. D. McNeal and E. M. Stein, *Mapping properties of the Bergman projection on convex domains of finite type*, Duke J. Math. 73 (1994), 177–199.

[McS2] —, *The Szegö projection on convex domains*, Math. Z. 224 (1997), 519–553.

[NRSh] A. Nagel, W. Rudin and J. H. Shapiro, *Tangential boundary behavior of functions in Dirichlet-type spaces*, Ann. of Math. 116 (1982), 331–360.

[NS] A. Nagel and E. M. Stein, *On certain maximal functions and approach regions*, Adv. Math. 54 (1984), 83–106.

[St] E. M. Stein, *Boundary Behavior of Holomorphic Functions of Several Complex Variables*, Math. Notes 11, Princeton Univ. Press, Princeton 1972.

[Su1] J. Sueiro, *On maximal functions and Poisson–Szegö integrals*, Trans. Amer. Math. Soc. 298 (1986), 653–669.

[Su2] —, *Tangential boundary limits and exceptional sets for holomorphic functions in Dirichlet-type spaces*, Math. Ann. 286 (1990), 661–678.

Marco M. Peloso, Hercule Valencourt
Dipartimento di Matematica
Università degli Studi di Milano
Via C. Saldini 50, 20133 Milano, Italy
E-mail: marco.peloso@unimi.it
hvalencourt@yahoo.fr

Received 22 April 2009;
revised 6 August 2009 (5219)