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RIESZ TRANSFORMS FOR THE DUNKL ORNSTEIN–UHLENBECK OPERATOR

BY

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Dedicated to the memory of Professor Andrzej Hulanicki

Abstract. We propose a definition of Riesz transforms associated to the Ornstein–Uhlenbeck operator based on the Dunkl Laplacian. In the case related to the group $\mathbb{Z}_2$ it is proved that the Riesz transform is bounded on the corresponding L^p spaces, $1 < p < \infty$.

1. Introduction. In the recent years Riesz transforms in the setting of orthogonal expansions related to general second order differential operators have been intensively studied. In particular, the first and third-named authors proposed a unified approach to this topic [12]. The investigation in the context of differential-difference operators was initiated very recently in [13], where Riesz transforms for the Dunkl harmonic oscillator were defined and studied. The present paper is a continuation of [13]. Now we consider the Ornstein–Uhlenbeck operator based on the Dunkl Laplacian, and define and investigate related Riesz operators. Our results partially contribute to the Dunkl theory, which has gained a considerable interest in various fields of mathematics as well as in theoretical physics during the last years.

Given a finite reflection group $G \subset O(\mathbb{R}^d)$ and a G-invariant nonnegative multiplicity function $k\colon R \to [0,\infty)$ on a root system $R \subset \mathbb{R}^d$ associated with the reflections of G, the Dunkl differential-difference operators T_j^k, $j = 1,\dots,d$, are defined by

$$T_j^k f(x) = \partial_j f(x) + \sum_{\beta\in R_+} k(\beta)\beta_j \,\frac{f(x)-f(\sigma_\beta x)}{\langle \beta, x\rangle}, \qquad f \in C^1(\mathbb{R}^d);$$

here ∂_j is the jth partial derivative, $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product in $\mathbb{R}^d$, R_+ is a fixed positive subsystem of R, and σ_β denotes the reflection in the hyperplane orthogonal to β. The Dunkl operators T_j^k, $j = 1,\dots,d$,

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form a commuting system (this is an important feature, cf. [3]) of first order differential-difference operators, and reduce to ∂_j, $j = 1, \dots, d$, when $k \equiv 0$. Moreover, T_j^k are homogeneous of degree -1 on $\mathcal{P}$, the space of all polynomials in $\mathbb{R}^d$.

In Dunkl's theory the operator

$$\Delta_k = \sum_{j=1}^{d} (T_j^k)^2$$

plays the role of the Euclidean Laplacian (in fact Δ comes into play when $k \equiv 0$). It is homogeneous of degree -2 on $\mathcal{P}$ and symmetric in $L^2(\mathbb{R}^d, w_k)$, where

$$w_k(x) = \prod_{\beta \in R_+} |\langle \beta, x \rangle|^{2k(\beta)},$$

if considered initially on $C_c^\infty(\mathbb{R}^d)$. Note that w_k is G-invariant. For basic facts concerning Dunkl's theory we refer the reader to the survey article by Rösler [15]. There, one can also find a discussion (see [15, Section 3]) and extensive references concerning applications of Dunkl's theory in mathematical physics.

In this article we propose a definition of Riesz transforms associated to the operator

$$L_k = -\Delta_k + 2x \cdot \nabla,$$

which is symmetric with respect to the measure

$$d\mu_k(x) = e^{-\|x\|^2} w_k(x)\, dx, \tag{1.1}$$

and becomes the classical Ornstein–Uhlenbeck operator when $k \equiv 0$. It turns out that L_k (or rather its suitable self-adjoint extension $\mathcal{L}_k$) has a discrete spectrum and the corresponding eigenfunctions are the generalized Hermite polynomials defined and investigated by Rösler [14]. Then the formal definition $R_j^k = \delta_j (\mathcal{L}_k)^{-1/2}$, $j = 1, \dots, d$, with $\delta_j = T_j^k$ being appropriate "derivatives" associated to L_k, rewritten properly in terms of the related expansions, produces L^2-bounded Riesz operators.

In the one-dimensional case of a reflection group isomorphic to $\mathbb{Z}_2$ we study L^p mapping properties of the above Riesz transform in detail. As the main result (Theorem 5.1) we prove that this operator is bounded on the corresponding L^p spaces for $1 < p < \infty$. This can be regarded as a generalization of the one-dimensional L^p results obtained by Muckenhoupt [8, 9] for the conjugate mappings related to classical Hermite and Laguerre expansions. We conjecture that an analogous result holds for arbitrary dimension d.

In the $\mathbb{Z}_2^d$ case we also consider an alternative Dunkl Ornstein–Uhlenbeck operator defined by means of the Dunkl gradient rather than the Euclidean one. This variant of the operator seems to be more natural, at least from the

Riesz transforms theory point of view. In particular, suitably defined Riesz operators are L^2-contractions, which is not the case of R^k_j.

Finally, still in the $\mathbb{Z}_2^d$ case, we obtain the weak type $(1,1)$ estimate for the maximal operator of the semigroup generated by the Dunkl Ornstein–Uhlenbeck operator. This extends the analogous results proved earlier by Sjögren [16] and Dinger [2] in the classical Hermite and Laguerre settings.

The paper is organized as follows. In Section 2 we define, in an appropriate L^2 space, Riesz transforms in the context of the Dunkl Ornstein–Uhlenbeck operator based on the general Dunkl Laplacian. Section 3 introduces the particular Dunkl setting related to the group $\mathbb{Z}_2^d$. In Section 4 we establish the above-mentioned weak type $(1,1)$ estimate for the heat semigroup maximal operator in the $\mathbb{Z}_2^d$ case (Theorem 4.1). Section 5 is devoted to the $\mathbb{Z}_2^d$ Riesz–Dunkl transforms, and the main result of the paper is stated there (Theorem 5.1). In Section 6 we gather several facts from the theory of classical Laguerre expansions needed in the proof of the main result. In particular, we establish L^p-boundedness, $1 < p < \infty$, of the left and right shift operators in the Laguerre setting (Theorem 6.3); this result is new and of independent interest. The proof of Theorem 5.1 is given at the end of Section 6. Eventually, in Section 7 we discuss Riesz operators related to the already mentioned variant of the Dunkl Ornstein–Uhlenbeck operator.

Throughout the paper we use fairly standard notation. Given a multi-index $n \in \mathbb{N}^d$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$, we write $|n| = n_1 + \cdots + n_d$; $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{R}^d$, and e_j the jth coordinate vector in $\mathbb{R}^d$. For a nonnegative weight function w on $\mathbb{R}^d$, we denote by $L^p(\mathbb{R}^d, w)$, $1 \le p < \infty$, the usual Lebesgue spaces related to the measure $dw(x) = w(x)dx$ (we will often abuse the notation slightly and use the same symbol w for the measure induced by a density w). Similarly, when w is a nonnegative weight function on $\mathbb{R}^d_+ = (0,\infty)^d$, we write $L^p(\mathbb{R}^d_+, w)$ for the relevant Lebesgue spaces.

2. The general setting.

Similarly to numerous frameworks discussed in the literature (see [12]), it is reasonable to define, at least formally, the Riesz transforms $R^k_1, \ldots, R^k_d$ associated with L_k as

$$R^k_j = \delta_j(\mathcal{L}_k)^{-1/2}\Pi_0; \tag{2.1}$$

here $\mathcal{L}_k$ is a suitable self-adjoint extension of L_k in $L^2(\mathbb{R}^d, \mu_k)$, Π_0 is a projection annihilating the eigenspace of $\mathcal{L}_k$ corresponding to the eigenvalue 0, and δ_j's are appropriately defined first order differential-difference operators.

In the present setting we define the jth partial derivative δ_j related to L_k by

$$\delta_j = T^k_j.$$

A short calculation shows that the formal adjoint of δ_j in $L^2(\mathbb{R}^d, \mu_k)$ is

$$\delta_j^* = -T_j^k + 2x_j.$$

To be precise, this means that

$$\langle \delta_j f, g\rangle_k = \langle f, \delta_j^* g\rangle_k, \qquad f, g \in C_c^1(\mathbb{R}^d), \tag{2.2}$$

where $\langle\cdot,\cdot\rangle_k$ is the canonical inner product in $L^2(\mathbb{R}^d, \mu_k)$. One of the facts which motivate the definition (2.1) is that, as a direct computation shows,

$$L_k + (d + 2\gamma) = \frac{1}{2}\sum_{j=1}^{d}(\delta_j^*\delta_j + \delta_j\delta_j^*), \qquad \gamma = \sum_{\beta\in R_+} k(\beta).$$

In the setting of Dunkl's general theory Rösler [14] constructed systems of naturally associated multivariable generalized Hermite polynomials H_n^k such that $\{H_n^k : n \in \mathbb{N}^d\}$ is a complete orthogonal system in $L^2(\mathbb{R}^d, \mu_k)$ (cf. [14, Corollary 3.5(i)]). Note that, for $k \equiv 0$ the construction leads to (suitably normalized) classical Hermite polynomials. Moreover, H_n^k are eigenfunctions of L_k,

$$L_k H_n^k = 2|n| H_n^k.$$

From now on we will always consider the generalized Hermite polynomials normalized by dividing them by their $L^2(\mathbb{R}^d, \mu_k)$ norms. For clarity, polynomials of the normalized system will be denoted by $\mathcal{H}_n^k$. The operator

$$\mathcal{L}_k f = \sum_{n\in\mathbb{N}^d} 2|n|\langle f, \mathcal{H}_n^k\rangle_k \mathcal{H}_n^k,$$

defined on the domain

$$\mathrm{Dom}(\mathcal{L}_k) = \Big\{ f \in L^2(\mathbb{R}^d, \mu_k) : \sum_{n\in\mathbb{N}^d} |\, 2|n|\langle f, \mathcal{H}_n^k\rangle_k|^2 < \infty \Big\},$$

is a self-adjoint extension of L_k considered on $C_c^\infty(\mathbb{R}^d)$ as the natural domain (the inclusion $C_c^\infty(\mathbb{R}^d) \subset \mathrm{Dom}(\mathcal{L}_k)$ may be easily verified). The spectrum of $\mathcal{L}_k$ is the discrete set $\{2m : m \in \mathbb{N}\}$, and the spectral decomposition of $\mathcal{L}_k$ is

$$\mathcal{L}_k f = \sum_{m=0}^{\infty} 2m\, \mathcal{P}_m^k f, \qquad f \in \mathrm{Dom}(\mathcal{L}_k),$$

where the spectral projections are

$$\mathcal{P}_m^k f = \sum_{|n|=m} \langle f, \mathcal{H}_n^k\rangle_k \mathcal{H}_n^k.$$

Then, letting Π_0 be the orthogonal projection onto the orthogonal complement of the subspace spanned by the constant function $\mathcal{H}_{(0,\ldots,0)}^k$, we have

$$\mathcal{L}_k^{-1/2}\Pi_0 f = \sum_{m=1}^{\infty} (2m)^{-1/2}\mathcal{P}_m^k f,$$

and this superposition is clearly a bounded operator on $L^2(\mathbb{R}^d, \mu_k)$.

We now furnish the rigorous definition of R_j^k on $L^2(\mathbb{R}^d,\mu_k)$. Let E be the dense subspace of $L^2(\mathbb{R}^d,\mu_k)$ spanned by $\{\mathcal{H}_n^k : n\in\mathbb{N}^d\}$. Note that E precisely consists of all polynomials in $\mathbb{R}^d$. Moreover, E is stable under the action of $\mathcal{L}_k^{-1/2}$, $\mathit{\Pi}_0$, δ_j, δ_j^*, and (2.2) is valid also for $f\in E$. Then for $f\in E$ we may define the Riesz transforms by (2.1), and these are bounded operators on E. Indeed, letting $\widehat{R}_j^k=\delta_j^*\mathcal{L}_k^{-1/2}\mathit{\Pi}_0$ we see that for $f\in E$,

$$\begin{aligned}\|R_j^kf\|^2_{L^2(\mathbb{R}^d,\mu_k)}&\le\|R_j^kf\|^2_{L^2(\mathbb{R}^d,\mu_k)}+\|\widehat{R}_j^kf\|^2_{L^2(\mathbb{R}^d,\mu_k)}\\&=\langle\delta_j^*\delta_j\mathcal{L}_k^{-1/2}\mathit{\Pi}_0f,\mathcal{L}_k^{-1/2}\mathit{\Pi}_0f\rangle_k+\langle\delta_j\delta_j^*\mathcal{L}_k^{-1/2}\mathit{\Pi}_0f,\mathcal{L}_k^{-1/2}\mathit{\Pi}_0f\rangle_k\\&\le\Big\langle\Big(\sum_{i=1}^d(\delta_i^*\delta_i+\delta_i\delta_i^*)\mathcal{L}_k^{-1/2}\mathit{\Pi}_0f\Big),\mathcal{L}_k^{-1/2}\mathit{\Pi}_0f\Big\rangle_k\\&=2\langle(L_k+d+2\gamma)\mathcal{L}_k^{-1/2}\mathit{\Pi}_0f,\mathcal{L}_k^{-1/2}\mathit{\Pi}_0f\rangle_k\\&=2\|\mathit{\Pi}_0f\|^2_{L^2(\mathbb{R}^d,\mu_k)}+2(d+2\gamma)\|\mathcal{L}_k^{-1/2}\mathit{\Pi}_0f\|^2_{L^2(\mathbb{R}^d,\mu_k)}\\&\le(2+d+2\gamma)\|f\|^2_{L^2(\mathbb{R}^d,\mu_k)}.\end{aligned}$$

It follows that the unique extension of R_j^k to the whole $L^2(\mathbb{R}^d,\mu_k)$ is given by

$$R_j^kf=\sum_{|n|>0}(2|n|)^{-1/2}\langle f,\mathcal{H}_n^k\rangle_k\,\delta_j\mathcal{H}_n^k,$$

the series being convergent in $L^2(\mathbb{R}^d,\mu_k)$ and its sum being independent of the order of summation.

3. Preliminaries for the $\mathbb{Z}_2^d$ case. Consider the finite reflection group generated by σ_j, $j=1,\dots,d$,

$$\sigma_j(x_1,\dots,x_j,\dots,x_d)=(x_1,\dots,-x_j,\dots,x_d),$$

and isomorphic to $\mathbb{Z}_2^d=\{0,1\}^d$. The reflection σ_j is in the hyperplane orthogonal to e_j, the jth coordinate vector in $\mathbb{R}^d$. Thus $R=\{\pm\sqrt{2}\,e_j : j=1,\dots,d\}$, $R_+=\{\sqrt{2}\,e_j : j=1,\dots,d\}$, and for a nonnegative multiplicity function $k\colon R\to[0,\infty)$ which is $\mathbb{Z}_2^d$-invariant, only values of k on R_+ are essential. Hence we may think $k=(\alpha_1+1/2,\dots,\alpha_d+1/2)$, $\alpha_j\ge-1/2$. We write $\alpha_j+1/2$ in place of seemingly more appropriate α_j since, for the sake of clarity, it is convenient for us to stick to the notation used in the Laguerre polynomial setting.

In what follows, the symbols T_j^α, δ_j, $\mathit{\Delta}_\alpha$, μ_α, L_α and so on denote the objects introduced in Section 2 and related to the present particular setting. Thus the Dunkl differential-difference operators T_j^α, $j=1,\dots,d$, are now given by

$$T_j^\alpha f(x) = \partial_j f(x) + (\alpha_j + 1/2)\,\frac{f(x) - f(\sigma_j x)}{x_j}, \qquad f \in C^1(\mathbb{R}^d),$$

and the explicit form of the Dunkl Laplacian is

$$\varDelta_\alpha f(x) = \sum_{j=1}^{d} \left(\frac{\partial^2 f}{\partial x_j^2}(x) + \frac{2\alpha_j + 1}{x_j}\,\frac{\partial f}{\partial x_j}(x) - (\alpha_j + 1/2)\,\frac{f(x) - f(\sigma_j x)}{x_j^2} \right).$$

Note that $\varDelta_\alpha$, when restricted to the "even" subspace

$$\{f \in C^2(\mathbb{R}^d) : \forall j = 1, \ldots, d,\ f(x) = f(\sigma_j x)\}, \tag{3.1}$$

coincides with the multi-dimensional Bessel differential operator $\sum_{j=1}^d \big(\partial_j^2 + \frac{2\alpha_j+1}{x_j}\partial_j\big)$, and consequently $L_\alpha = -\varDelta_\alpha + 2x\cdot\nabla$ reduces to the Laguerre-type operator

$$-\varDelta + 2x\cdot\nabla - \sum_{j=1}^{d} \frac{2\alpha_j + 1}{x_j}\,\frac{\partial}{\partial x_j} \tag{3.2}$$

(both operators acting on $\mathbb{R}_+^d$).

The corresponding measure μ_α has a product structure of the form

$$\begin{aligned} d\mu_\alpha(x) &= \prod_{j=1}^{d} |x_j|^{2\alpha_j+1} e^{-x_j^2}\, dx_j \\ &= 2^{-|\alpha|-d/2} e^{-\|x\|^2} \prod_{\beta\in R_+} |\langle \beta, x\rangle|^{2k(\beta)}\, dx, \qquad x \in \mathbb{R}^d; \end{aligned}$$

for simplicity we neglect the constant factor in comparison with (1.1). In dimension one, for the reflection group $\mathbb{Z}_2$ (see [14, Example 3.3(2)]) and the multiplicity parameter $\alpha + 1/2$, $\alpha \ge -1/2$, one obtains as the corresponding (normalized) generalized Hermite polynomials

$$\begin{aligned} \mathcal{H}_{2n}^\alpha(x) &= (-1)^n \left(\frac{n!}{\Gamma(n+\alpha+1)} \right)^{1/2} L_n^\alpha(x^2), \\ \mathcal{H}_{2n+1}^\alpha(x) &= (-1)^n \left(\frac{n!}{\Gamma(n+\alpha+2)} \right)^{1/2} x L_n^{\alpha+1}(x^2), \end{aligned}$$

where $n \in \mathbb{N}$ and L_n^α denotes the Laguerre polynomial of degree n and order α (see [6, p. 76]). Note that these $\mathcal{H}_n^\alpha$ are, up to multiplicative constants, the genuine generalized Hermite polynomials $H_n^{\alpha+1/2}$ on $\mathbb{R}$, as defined and studied by Chihara [1]. For $\alpha = -1/2$ the $\mathcal{H}_n^\alpha$ become the classical (normalized) Hermite polynomials (see [6, p. 81]). In the multi-dimensional setting, corresponding to the group $\mathbb{Z}_2^d$, the generalized Hermite polynomials are obtained by taking tensor products of the one-dimensional $\mathcal{H}_n^\alpha$. Thus for a multi-index $\alpha \in [-1/2, \infty)^d$,

$$\mathcal{H}_n^\alpha(x) = \mathcal{H}_{n_1}^{\alpha_1}(x_1) \cdot \ldots \cdot \mathcal{H}_{n_d}^{\alpha_d}(x_d), \qquad x \in \mathbb{R}^d,\ n \in \mathbb{N}^d.$$

The system $\{\mathcal{H}_n^\alpha : n \in \mathbb{N}^d\}$ is an orthonormal basis in $L^2(\mathbb{R}^d, \mu_\alpha)$ consisting of eigenfunctions of L_α; recall that $L_\alpha \mathcal{H}_n^\alpha = 2|n|\mathcal{H}_n^\alpha$.

4. $\mathbb{Z}_2^d$ heat semigroup maximal operator.

Let $\{T_t^\alpha\}_{t>0}$ be the heat-diffusion semigroup generated by $\mathcal{L}_\alpha$,

$$T_t^\alpha f = \sum_{m=0}^{\infty} e^{-2mt} \mathcal{P}_m^\alpha f, \qquad f \in L^2(\mathbb{R}^d, \mu_\alpha).$$

Then the integral representation of T_t^α is

$$T_t^\alpha f(x) = \int_{\mathbb{R}^d} \mathcal{G}_t^\alpha(x, y) f(y)\, d\mu_\alpha(y), \qquad x \in \mathbb{R}^d,$$

where the heat kernel is expressed in terms of the $\mathcal{H}_n^\alpha$,

$$\mathcal{G}_t^\alpha(x, y) = \sum_{m=0}^{\infty} e^{-2mt} \sum_{|n|=m} \mathcal{H}_n^\alpha(x)\mathcal{H}_n^\alpha(y).$$

The oscillating series defining $\mathcal{G}_t^\alpha(x, y)$ can be summed and we get

$$\mathcal{G}_t^\alpha(x, y) = \sum_{\varepsilon \in \{0,1\}^d} \mathcal{G}_t^{\alpha,\varepsilon}(x, y), \tag{4.1}$$

where the component kernels are given by

$$\begin{aligned}\mathcal{G}_t^{\alpha,\varepsilon}&(x, y) \\ &= \frac{e^{2t|\alpha|}}{(1 - e^{-4t})^d} \exp\biggl(-\frac{1}{e^{4t} - 1}\,(\|x\|^2 + \|y\|^2)\biggr) \prod_{i=1}^{d} (x_i y_i)^{\varepsilon_i}\, \frac{I_{\alpha_i+\varepsilon_i}\bigl(\frac{x_i y_i}{\sinh 2t}\bigr)}{(x_i y_i)^{\alpha_i+\varepsilon_i}},\end{aligned}$$

with I_ν being the modified Bessel function of the first kind and order ν (see [6, Chapter 5]). This formula can be deduced, for instance, from a relation with the setting considered in [13, Section 3] and the facts invoked there. Indeed, it is easy to see that $\mathcal{G}_t^\alpha(x, y) = e^{2t(|\alpha|+d)} e^{(\|x\|^2+\|y\|^2)/2} G_t^\alpha(x, y)$, with $G_t^\alpha(x, y)$ defined in [13, Section 3]. Then the decomposition $G_t^\alpha(x, y) = \sum_{\varepsilon\in\{0,1\}^d} G_t^{\alpha,\varepsilon}(x, y)$, together with the explicit form of $G_t^{\alpha,\varepsilon}(x, y)$, shows (4.1).

Consider the maximal operator $T_*^\alpha f = \sup_{t>0} |T_t^\alpha f|$. By Stein's general maximal theorem [18, p. 73], T_*^α is bounded on $L^p(\mathbb{R}^d, \mu_\alpha)$ for $1 < p \le \infty$. The case $p = 1$ is more subtle. The following theorem is a consequence of Dinger's result [2] in the classical Laguerre setting. In fact, it generalizes analogous multi-dimensional results for classical Hermite [16] and Laguerre [2] settings, which in one dimension were originally obtained by Muckenhoupt [7].

THEOREM 4.1. *Let* $\alpha \in [-1/2, \infty)^d$. *Then* T_*^α *satisfies the weak type* $(1,1)$ *inequality*

$$\mu_\alpha\{x \in \mathbb{R}^d : T_*^\alpha f(x) > \lambda\} \le \frac{C}{\lambda}\, \|f\|_{L^1(\mathbb{R}^d,\mu_\alpha)}, \qquad \lambda > 0.$$

Proof. Denote $\varepsilon_o = (0, \ldots, 0)$. By Soni's inequality [17]

$$I_{\nu+1}(z) < I_\nu(z), \qquad z > 0,\ \nu \ge -1/2,$$

we see that

$$0 < \mathcal{G}_t^\alpha(x,y) \le 2^d \mathcal{G}_t^{\alpha,\varepsilon_o}(x,y), \qquad t > 0,\ x, y \in \mathbb{R}^d.$$

Since both $G_t^{\alpha,\varepsilon_o}(x,y)$ and the density of the measure μ_α are even with respect to each coordinate, it follows that

$$\begin{aligned} 2^{-d} T_*^\alpha f(x) &\le \sup_{t>0} \int_{\mathbb{R}^d} \mathcal{G}_t^{\alpha,\varepsilon_o}(x,y)|f(y)|\, d\mu_\alpha(y) \\ &\le \sum_{\delta\in\{-1,1\}^d} \sup_{t>0} \int_{\mathbb{R}^d_+} \mathcal{G}_t^{\alpha,\varepsilon_o}((|x_1|, \ldots, |x_d|), y)|f_\delta(y)|\, d\mu_\alpha(y) \\ &\equiv \sum_{\delta\in\{-1,1\}^d} T_*^{\alpha,\varepsilon_o}|f_\delta|((|x_1|, \ldots, |x_d|)), \end{aligned}$$

where $f_\delta(x) = f(\delta_1 x_1, \ldots, \delta_d x_d)$. Thus it suffices to show the weak type $(1,1)$ for the maximal operator $T_*^{\alpha,\varepsilon_o}$ in $\mathbb{R}^d_+$. But $T_*^{\alpha,\varepsilon_o}$ is, up to a constant factor and the change of variable $\mathbb{R}^d_+ \ni x \mapsto x^2 \in \mathbb{R}^d_+$, the Laguerre maximal operator considered by Dinger [2]. The relevant weak type $(1,1)$ estimate is stated in [2, Theorem 1]; see also the accompanying comments explaining the validity of the result for any type multi-index. ∎

An important consequence of Theorem 4.1 is that $T_t^\alpha f \to f$ almost everywhere as $t \to 0^+$, for $f \in L^1(\mathbb{R}^d, \mu_\alpha)$.

5. $\mathbb{Z}_2^d$ Riesz transforms.

Recall that our choice of "derivatives" δ_j is motivated by the decomposition

$$L_\alpha + (2|\alpha| + 2d) = \frac{1}{2}\sum_{j=1}^d (\delta_j^*\delta_j + \delta_j\delta_j^*).$$

First we shall see how δ_j's act on $\mathcal{H}_n^\alpha$. It is sufficient to consider the one-dimensional situation and then distinguish between the even and odd cases. Recall that $\delta_j = T_j^\alpha$; in the one-dimensional case we simply write δ in place of δ_1. For $n \in \mathbb{N}$ and $\alpha \ge -1/2$, combining the fact that $\mathcal{H}_{2n}^\alpha$ is an even function with the identity

$$\frac{d}{dy} L_n^\alpha(y) = -L_{n-1}^{\alpha+1}(y) \tag{5.1}$$

(see [6, (4.18.6)]), one easily obtains

$$\delta\mathcal{H}^\alpha_{2n} = \sqrt{4n}\,\mathcal{H}^\alpha_{2n-1};$$

here, and also later on, we use the convention that $\mathcal{H}^\alpha_m \equiv 0 \equiv L^\alpha_m$ if $m = -1$. Similarly, combining the fact that $\mathcal{H}^\alpha_{2n+1}$ is an odd function with (5.1) and the identities

$$(5.2)\qquad \begin{aligned} -yL^{\alpha+2}_{n-1}(y) + (\alpha+1)L^{\alpha+1}_n(y) &= yL^{\alpha+1}_n(y) + (n+1)L^\alpha_{n+1}(y) \\ &= (n+\alpha+1)L^\alpha_n(y), \end{aligned}$$

which in turn can be deduced from (5.1), [6, (4.18.7)] and [6, (4.18.4)], one gets

$$\delta\mathcal{H}^\alpha_{2n+1} = \sqrt{4n+4\alpha+4}\,\mathcal{H}^\alpha_{2n}.$$

Summarizing, in d dimensions, for $n \in \mathbb{N}^d$ and $\alpha \in [-1/2, \infty)^d$ we have

$$\delta_j\mathcal{H}^\alpha_n = m_j(n,\alpha)\mathcal{H}^\alpha_{n-e_j},$$

where

$$m_j(n,\alpha) = \begin{cases} \sqrt{2n_j} & \text{if } n_j \text{ is even,} \\ \sqrt{2n_j + 4\alpha_j + 2} & \text{if } n_j \text{ is odd;} \end{cases}$$

by convention, $\mathcal{H}_{n-e_j} \equiv 0$ if $n_j = 0$. Note that for each j the system $\{\delta_j\mathcal{H}^\alpha_n : n_j \geq 1\}$ is orthogonal in $L^2(\mathbb{R}^d, \mu_\alpha)$.

The rigorous definition of the Riesz transforms on $L^2(\mathbb{R}^d, \mu_\alpha)$ is provided by the orthogonal series

$$(5.3)\qquad R^\alpha_j f = \sum_{|n|>0} \frac{m_j(n,\alpha)}{\sqrt{2|n|}} \langle f, \mathcal{H}^\alpha_n\rangle_\alpha \mathcal{H}^\alpha_{n-e_j},$$

from which the L^2-boundedness can easily be seen directly. Notice, however, that R^α_j is not a contraction on $L^2(\mathbb{R}^d, \mu_\alpha)$ if $\alpha_j > -1/2$ for some j.

Our main result, Theorem 5.1 below, is an extension of Muckenhoupt's L^p results [8, 9] for the conjugate mappings related to classical Hermite and Laguerre expansions.

THEOREM 5.1. *Let $d = 1$ and assume that $\alpha \geq -1/2$. Then for each $1 < p < \infty$ the Riesz transform R^α_1, defined on $L^2(\mathbb{R}, \mu_\alpha)$ by* (5.3)*, extends to a bounded operator on $L^p(\mathbb{R}, \mu_\alpha)$.*

We conjecture that an analogous result holds for arbitrary dimension d and $\alpha \in [-1/2, \infty)^d$, but proving this seems to be a rather difficult task. In contrast with the maximal operator, it is not possible to deduce the result in a straightforward manner from the known results [11] in the Laguerre setting. Nor the technique of square functions used in [11] seems to be suitable in the present context.

The proof of Theorem 5.1 is partially based on known results in the classical Laguerre setting. To show the L^p estimate we split a function into its even and odd parts. Then the Riesz transform of the even part can be identified with the Riesz–Laguerre transform for which the relevant bound is known. Treatment of the odd part is less straightforward. The Riesz operator coincides, up to shift and multiplier operators, with the adjoint of the Riesz–Laguerre transform. Thus to get the desired estimate we need to invoke a suitable multiplier theorem and to establish L^p-boundedness of a shift operator in the Laguerre setting. The next section gathers the above-mentioned auxiliary results. The proof of the main theorem is furnished at the end of Section 6.

6. Laguerre setting results and proof of Theorem 5.1. The one-dimensional setting discussed below is equivalent to the classical Laguerre polynomial setting, from which it emerges by the change of variable $x \mapsto x^2$ on $\mathbb{R}_+$. Thus all relevant definitions and results can be directly translated from the original to "squared" Laguerre setting. In what follows, we always assume that $\alpha \geq -1/2$. The restriction of μ_α to $\mathbb{R}_+$ will be denoted by the same symbol.

Consider the operator (3.2) in dimension one,

$$\mathbb{L}_\alpha = -\frac{d^2}{dx^2} - \frac{2\alpha+1-2x^2}{x}\frac{d}{dx},$$

which is positive and symmetric in $L^2(\mathbb{R}_+, \mu_\alpha)$. The polynomials $L_n^\alpha(x^2)$, $n \in \mathbb{N}$, are eigenfunctions of $\mathbb{L}_\alpha$,

$$\mathbb{L}_\alpha L_n^\alpha(x^2) = 4n L_n^\alpha(x^2),$$

and the set $\{L_n^\alpha(x^2) : n \in \mathbb{N}\}$ forms an orthogonal basis in $L^2(\mathbb{R}_+, \mu_\alpha)$. In what follows, it is convenient to normalize this system in $L^2(\mathbb{R}_+, \mu_\alpha)$ and consider the polynomials

$$\varphi_n^\alpha(x) = \left(\frac{2n!}{\Gamma(n+\alpha+1)}\right)^{1/2} L_n^\alpha(x^2).$$

The definition of the Riesz transform in the "squared" Laguerre setting is inherited from the classical Laguerre setting (see [9] or [11]), and hence is induced by the mapping

$$R_\varphi^\alpha \colon \varphi_n^\alpha \mapsto -\psi_{n-1}^\alpha, \qquad n \in \mathbb{N},$$

where $\psi_{-1}^\alpha \equiv 0$ and $\{\psi_n^\alpha : n \in \mathbb{N}\}$ is another orthonormal basis of $L^2(\mathbb{R}_+, \mu_\alpha)$ consisting of the polynomials

$$\psi_n^\alpha(x) = \left(\frac{2n!}{\Gamma(n+\alpha+2)}\right)^{1/2} x L_n^{\alpha+1}(x^2).$$

By Plancherel's theorem, R_φ^α extends uniquely to a contraction on $L^2(\mathbb{R}_+, \mu_\alpha)$, which we denote by the same symbol. Notice that φ_n^α and ψ_n^α

coincide, up to constant factors independent of n and α, with the generalized Hermite polynomials $\mathcal{H}_{2n}^{\alpha}$ and $\mathcal{H}_{2n+1}^{\alpha}$, respectively.

In view of Muckenhoupt's result [9, Theorem 3(b)] (see also [11, Theorem 13]), we have the following

Theorem 6.1. *Let $\alpha \geq -1/2$ and $1 < p < \infty$. Then*

$$\|R_{\varphi}^{\alpha} f\|_{L^p(\mathbb{R}_+,\,\mu_\alpha)} \leq C\|f\|_{L^p(\mathbb{R}_+,\,\mu_\alpha)},$$

with a constant C independent of $f \in L^2 \cap L^p(\mathbb{R}_+,\,\mu_\alpha)$.

It is immediate that the adjoint operator $(R_{\varphi}^{\alpha})^*$, taken in $L^2(\mathbb{R}_+,\,\mu_\alpha)$, is determined by the mapping

$$R_{\psi}^{\alpha}\colon \psi_n^{\alpha} \mapsto -\varphi_{n+1}^{\alpha}, \qquad n \in \mathbb{N},$$

whose (unique) extension to $L^2(\mathbb{R}_+,\,\mu_\alpha)$ (still denoted by the same symbol) is precisely the adjoint of R_{φ}^{α}. Consequently, by Theorem 6.1 and duality we see that for $1 < p < \infty$,

$$\|R_{\psi}^{\alpha} f\|_{L^p(\mathbb{R}_+,\,\mu_\alpha)} \leq C\|f\|_{L^p(\mathbb{R}_+,\,\mu_\alpha)}, \tag{6.1}$$

with a constant C independent of $f \in L^2 \cap L^p(\mathbb{R}_+,\,\mu_\alpha)$.

The next ingredient that will be needed in the proof of Theorem 5.1 is the multiplier theorem below. It is a direct translation to the "squared" Laguerre setting of [5, Theorem 3.4], after specifying it to one dimension and taking $\beta = 1$.

Theorem 6.2. *Let $1 < p < \infty$ and $\alpha \geq -1/2$. Assume that h is a function analytic in a neighborhood of the origin. Let $\{\xi(n)\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\xi(n) = h(n^{-1})$ for $n \geq n_0 \geq 0$. Then the multiplier operator given by*

$$M_{\xi}\colon \varphi_n^{\alpha} \mapsto \xi(n)\varphi_n^{\alpha}$$

satisfies

$$\|M_{\xi} f\|_{L^p(\mathbb{R}_+,\,\mu_\alpha)} \leq C\|f\|_{L^p(\mathbb{R}_+,\,\mu_\alpha)},$$

with a constant C independent of $f \in L^2 \cap L^p(\mathbb{R}_+,\,\mu_\alpha)$.

Finally, we establish L^p-boundedness of the left and right shift operators related to the system $\{\varphi_n^{\alpha}\}$. Changing the variable leads to the analogous result for the system of (normalized) Laguerre polynomials. This may be regarded as an extension of the result stated in [4, Proposition 3.3(a)].

Theorem 6.3. *Let $\alpha \geq -1/2$ and $1 < p < \infty$. Then the shift operators given by*

$$S_L\colon \varphi_n^{\alpha} \mapsto \varphi_{n-1}^{\alpha}, \qquad S_R\colon \varphi_n^{\alpha} \mapsto \varphi_{n+1}^{\alpha},$$

satisfy

$$\|S_L f\|_{L^p(\mathbb{R}_+,\,\mu_\alpha)} \leq C\|f\|_{L^p(\mathbb{R}_+,\,\mu_\alpha)}, \qquad \|S_R f\|_{L^p(\mathbb{R}_+,\,\mu_\alpha)} \leq C\|f\|_{L^p(\mathbb{R}_+,\,\mu_\alpha)},$$

with a constant C independent of $f \in L^2 \cap L^p(\mathbb{R}_+,\,\mu_\alpha)$.

Proof. First, observe that by duality it suffices to prove the statement only for S_R, the adjoint of S_L in $L^2(\mathbb{R}_+,\, \mu_\alpha)$. Then the estimate we need to justify is

$$\int_0^\infty \Big|\sum_{n=0}^\infty a_n\varphi_{n+1}^\alpha(x)\Big|^p\,d\mu_\alpha(x) \le C\int_0^\infty \Big|\sum_{n=0}^\infty a_n\varphi_n^\alpha(x)\Big|^p\,d\mu_\alpha(x). \tag{6.2}$$

Next notice that by means of Theorem 6.2 the task of showing (6.2) can be reduced to proving the estimate

$$\int_0^\infty \Big|\sum_{n=0}^\infty \frac{n+1}{n+\alpha+1}\,b_nL_{n+1}^\alpha(x^2)\Big|^p\,d\mu_\alpha(x) \le C\int_0^\infty \Big|\sum_{n=0}^\infty b_nL_n^\alpha(x^2)\Big|^p\,d\mu_\alpha(x). \tag{6.3}$$

Indeed, to get (6.2) let $\xi(n) = \sqrt{\frac{n+\alpha+1}{n+1}}$ and apply first (6.3) with $b_n = \xi(n)a_n$ and then use Theorem 6.2 for the multiplier $\xi(n)$.

It remains to prove (6.3). We invoke the formula (see (5.2))

$$\frac{n+1}{n+\alpha+1}\,L_{n+1}^\alpha(y) = L_n^\alpha(y) - \frac{y}{n+\alpha+1}\,L_n^{\alpha+1}(y)$$

and use it to estimate the left-hand side in (6.3). We get

$$\begin{aligned}&\int_0^\infty \Big|\sum_{n=0}^\infty \frac{n+1}{n+\alpha+1}\,b_nL_{n+1}^\alpha(x^2)\Big|^p\,d\mu_\alpha(x)\\ &\quad\le 2^p\int_0^\infty \Big|\sum_{n=0}^\infty b_nL_n^\alpha(x^2)\Big|^p\,d\mu_\alpha(x) + 2^p\int_0^\infty \Big|\sum_{n=0}^\infty \frac{x^2}{n+\alpha+1}\,b_nL_n^{\alpha+1}(x^2)\Big|^p\,d\mu_\alpha(x).\end{aligned}$$

To deal with the last integral we apply the identity (see Koshlyakov's formula [6, p. 94])

$$\frac{x^2}{n+\alpha+1}\,L_n^{\alpha+1}(x^2) = \frac{2}{x^{2\alpha}}\int_0^x L_n^\alpha(y^2)y^{2\alpha+1}\,dy.$$

This produces

$$\begin{aligned}&\int_0^\infty \Big|\sum_{n=0}^\infty \frac{x^2}{n+\alpha+1}\,b_nL_n^{\alpha+1}(x^2)\Big|^p\,d\mu_\alpha(x)\\ &\qquad\qquad = \int_0^\infty \Big|\frac{2}{x^{2\alpha}}\int_0^x\Big(\sum_{n=0}^\infty b_nL_n^\alpha(y^2)\Big)y^{2\alpha+1}\,dy\Big|^p\,d\mu_\alpha(x).\end{aligned}$$

Now the desired estimate is a consequence of weighted Hardy's inequality

$$\int_0^\infty \Big|\int_0^x g(y)\,dy\Big|^p x^{2\alpha(1-p)+1}e^{-x^2}\,dx \le C\int_0^\infty |g(x)|^p x^{(2\alpha+1)(1-p)}e^{-x^2}\,dx. \tag{6.4}$$

But it is known (see for instance [10, Theorem 1]) that a sufficient (and necessary) condition for (6.4) to hold is

$$\sup_{r>0}\Big(\int_r^\infty x^{\alpha(1-p)}e^{-x}\,dx\Big)^{1/p}\Big(\int_0^r x^\alpha e^{x/(p-1)}\,dx\Big)^{1-1/p}<\infty \tag{6.5}$$

(this condition is, by the change of variable $x^2\mapsto x$, equivalent to [10, (1.3)] with suitably chosen weights U,V).

Thus the proof will be finished once we verify (6.5). The decay at $+\infty$ of the integrated expressions in (6.5) is essentially determined by the exponentials. So neglecting the power factors at the price of adding a positive constant to both exponents, we see that when r is large, say $r\ge 1$, the whole expression under the supremum is dominated by a constant. On the other hand, for x close to 0^+ the exponential factors can be neglected. Then taking into account small r and integrating the power factors shows that the expression under the supremum is controlled by a positive power of r. The conclusion follows. ∎

Remark 6.4. The Laguerre setting results of this section hold in fact for a wider range $\alpha>-1$ of the type parameter. This remark concerns in particular Theorem 6.3, and the proof given above is valid also for $\alpha\in(-1,-1/2)$.

We are now in a position to prove Theorem 5.1. Given $f\in L^2\cap L^p(\mathbb{R},\mu_\alpha)$, decompose it into its even and odd parts,

$$f=f_e+f_o.$$

To prove the theorem it is sufficient to show the L^p estimates

$$\|R_1^\alpha f_e\|_{L^p(\mathbb{R},\mu_\alpha)}\le C\|f_e\|_{L^p(\mathbb{R},\mu_\alpha)},\qquad \|R_1^\alpha f_o\|_{L^p(\mathbb{R},\mu_\alpha)}\le C\|f_o\|_{L^p(\mathbb{R},\mu_\alpha)}. \tag{6.6}$$

Since the generalized Hermite polynomial $\mathcal{H}_n^\alpha$ is even if n is even, and odd for n odd, the expansions of f_e and f_o are given only by even and odd $\mathcal{H}_n^\alpha$'s, respectively. Moreover, in view of (5.3), $R_1^\alpha f_e$ is odd and $R_1^\alpha f_o$ is even. Due to these symmetries we consider the operators R_e^α and R_o^α on $L^2(\mathbb{R}_+,\mu_\alpha)$ emerging naturally from restrictions of R_1^α to the subspaces of $L^2(\mathbb{R},\mu_\alpha)$ of even and odd functions, respectively. Clearly, (6.6) will follow once we show suitable L^p estimates for R_e^α and R_o^α.

Observe that by (5.3) we have

$$R_e^\alpha\colon \varphi_n^\alpha\mapsto -\psi_{n-1}^\alpha,\qquad R_o^\alpha\colon \psi_n^\alpha\mapsto\sqrt{\frac{n+\alpha+1}{n+1/2}}\,\varphi_n^\alpha.$$

Thus R_e^α coincides with R_φ^α, and the corresponding L^p estimate is provided by Theorem 6.1. On the other hand, R_o^α is related to the mapping R_ψ^α by means of shift and multiplier operators,

$$R_o^\alpha = M_\xi S_L R_\psi^\alpha, \qquad \xi(n) = -\sqrt{\frac{n+\alpha+1}{n+1/2}}.$$

Consequently, the relevant L^p estimate follows by Theorems 6.2 and 6.3, and (6.1).

The proof of Theorem 5.1 is complete.

7. Alternative $\mathbb{Z}_2^d$ Dunkl Ornstein–Uhlenbeck operator. In this section we consider the "Laplacian"

$$\widetilde{L}_\alpha = -\varDelta_\alpha + 2x \cdot \nabla_\alpha,$$

a variant of the Dunkl Ornstein–Uhlenbeck operator based on the Dunkl gradient

$$\nabla_\alpha = (T_1^\alpha, \dots, T_d^\alpha).$$

This variant seems to be more natural than L_α for defining Riesz transforms, at least in the $\mathbb{Z}_2^d$ case. It turns out that Riesz transforms naturally associated with $\widetilde{L}_\alpha$ are contractions in $L^2(\mathbb{R}^d, \mu_\alpha)$, which is not the case of R_j^α related to L_α. Moreover, the context of $\widetilde{L}_\alpha$ is better related to the classical Laguerre setting, as will be seen below. Similarly to L_α, when restricted to the "even" subspace (3.1), $\widetilde{L}_\alpha$ coincides with the Laguerre-type operator (3.2), and for $\alpha = (-1/2, \dots, -1/2)$ it reduces to the classical Ornstein–Uhlenbeck operator. Below we keep the notation introduced in previous sections.

It is straightforward to check that $\widetilde{L}_\alpha$ admits the decomposition

$$\widetilde{L}_\alpha = \sum_{j=1}^d \delta_j^* \delta_j.$$

In fact, the decomposition $-\varDelta_k + 2x \cdot \nabla_k = \sum_{j=1}^d \delta_j^* \delta_j$, $\nabla_k = (T_1^k, \dots, T_d^k)$, $\delta_j = T_j^k$, also holds in the general setting from Section 2. It follows that $\widetilde{L}_\alpha$ is symmetric and nonnegative in $L^2(\mathbb{R}^d, \mu_\alpha)$. Thus it is reasonable (see [12]) to define formally the Riesz transforms associated with $\widetilde{L}_\alpha$ by

$$\widetilde{R}_j^\alpha = \delta_j (\widetilde{L}_\alpha)^{-1/2} \varPi_0, \qquad j = 1, \dots, d.$$

The multi-dimensional generalized Hermite polynomials are eigenfunctions of $\widetilde{L}_\alpha$,

$$\widetilde{L}_\alpha \mathcal{H}_n^\alpha = \Big(2|n| + \sum_{\{j \,:\, n_j \,\mathrm{odd}\}} (4\alpha_j + 2)\Big) \mathcal{H}_n^\alpha = \Big(\sum_{j=1}^d [m_j(n,\alpha)]^2\Big) \mathcal{H}_n^\alpha.$$

Let $\widetilde{\mathcal{L}}_\alpha$ be the self-adjoint extension of $\widetilde{L}_\alpha$ whose spectral decomposition is given by the $\mathcal{H}_n^\alpha$. Then the rigorous definition of $\widetilde{R}_j^\alpha f$ for f being a (generalized Hermite) polynomial is $\widetilde{R}_j^\alpha = \delta_j \widetilde{\mathcal{L}}_\alpha^{-1/2} \Pi_0$. Rewriting this in terms of the corresponding expansions leads to the orthogonal series

$$\widetilde{R}_j^\alpha f = \sum_{|n|>0} \frac{m_j(n,\alpha)}{\sqrt{\sum_{j=1}^d [m_j(n,\alpha)]^2}} \langle f, \mathcal{H}_n^\alpha \rangle_\alpha \mathcal{H}_{n-e_j}^\alpha, \tag{7.1}$$

which provides a definition of the Riesz operators on $L^2(\mathbb{R}^d, \mu_\alpha)$. Clearly, by Plancherel's theorem the mapping

$$f \mapsto \sqrt{|\widetilde{R}_1^\alpha f|^2 + \cdots + |\widetilde{R}_d^\alpha f|^2}$$

is a contraction on $L^2(\mathbb{R}^d, \mu_\alpha)$, and even an isometry on the orthogonal complement of the constant function $\mathcal{H}_{(0,\ldots,0)}^\alpha$.

We now state an analogue of Theorem 5.1 in the context of $\widetilde{L}_\alpha$.

THEOREM 7.1. *Let $d = 1$ and $\alpha \geq -1/2$. Then for each $1 < p < \infty$ the Riesz transform $\widetilde{R}_1^\alpha$, defined on $L^2(\mathbb{R}, \mu_\alpha)$ by* (7.1)*, extends to a bounded operator on $L^p(\mathbb{R}, \mu_\alpha)$.*

Proof. We proceed as in the proof of Theorem 5.1 and arrive at the operators $\widetilde{R}_e^\alpha$ and $\widetilde{R}_o^\alpha$ acting on $L^2(\mathbb{R}_+, \mu_\alpha)$. The conclusion will follow once we show suitable L^p estimates for these two operators. Notice that by (7.1) we have

$$\widetilde{R}_e^\alpha : \varphi_n^\alpha \mapsto -\psi_{n-1}^\alpha, \qquad \widetilde{R}_o^\alpha : \psi_n^\alpha \mapsto \varphi_n^\alpha.$$

Thus $\widetilde{R}_e^\alpha = R_\varphi^\alpha$ and $\widetilde{R}_o^\alpha = -S_L R_\psi^\alpha$. Now the relevant L^p estimates are consequences of Theorem 6.1, and Theorem 6.3 and (6.1), respectively. ■

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