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RIESZ MEETS SOBOLEV

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Abstract. We show that the L^p boundedness, $p > 2$, of the Riesz transform on a complete non-compact Riemannian manifold with upper and lower Gaussian heat kernel estimates is equivalent to a certain form of Sobolev inequality. We also characterize in such terms the heat kernel gradient upper estimate on manifolds with polynomial growth.

1. Introduction. The present paper may be considered as a companion paper to [2], which gave criteria for the L^p boundedness, for $p > 2$, of the Riesz transform on non-compact Riemannian manifolds. Here we reformulate these criteria in terms of certain Sobolev inequalities. That is, we deduce some L^p to L^p estimates from suitable L^q to L^p estimates, for $q < p$.

Let M be a complete, connected, non-compact Riemannian manifold. The methods of this paper remain valid for other types of spaces endowed with a gradient, a metric which is compatible with this gradient, a measure, and finally an operator associated with the Dirichlet form constructed from the gradient and the measure. An interesting example is a Lie group endowed with a family of left-invariant Hörmander vector fields. We leave the details of such extensions to the reader.

Let d be the geodesic distance on M; denote by $B(x, r)$ the open ball with respect to d with center $x \in M$ and radius $r > 0$.

Denote by μ the Riemannian measure, by $L^p(M, \mu)$, $1 \le p \le \infty$, the corresponding L^p spaces, and let $V(x, r) = \mu(B(x, r))$.

Let Δ be the (non-negative) Laplace–Beltrami operator. One could consider another measure $\tilde{\mu}$ with positive smooth non-zero density with respect to μ, and the associated operator $\Delta_{\tilde{\mu}}$, formally given by

$$(\Delta_{\tilde{\mu}} f, f) = \int_M |\nabla f|^2 \, d\tilde{\mu}.$$

Again, for simplicity, we stick to the standard case.

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Let ∇ be the Riemannian gradient. We can now formally define the Riesz transform operator $\nabla\Delta^{-1/2}$.

Let $p \in (1,\infty)$. The boundedness of the Riesz transform on $L^p(M,\mu)$ reads

$$(R_p) \qquad \| \, |\nabla f| \, \|_p \le C_p \|\Delta^{1/2} f\|_p, \quad \forall f \in \mathbf{C}_0^\infty(M),$$

and if the reverse inequality (RR_p) also holds, one has

$$(E_p) \qquad \| \, |\nabla f| \, \|_p \simeq \|\Delta^{1/2} f\|_p, \quad \forall f \in \mathbf{C}_0^\infty(M).$$

One says that M has the *volume doubling property* if there exists C such that

$$(D) \qquad V(x,2r) \le C V(x,r), \quad \forall r>0,\, x \in M,$$

more precisely if there exist $\nu, C_\nu > 0$ such that

$$(D_\nu) \qquad \frac{V(x,r)}{V(x,s)} \le C_\nu \left(\frac{r}{s}\right)^\nu, \quad \forall r \ge s > 0,\, x \in M.$$

The heat semigroup is the family of operators $(\exp(-t\Delta))_{t>0}$ acting on $L^2(M,\mu)$, with a positive and smooth kernel $p_t(x,y)$ called the heat kernel. We shall consider the following standard heat kernel estimates for manifolds with doubling: the on-diagonal upper estimate,

$$(DUE) \qquad p_t(x,x) \le \frac{C}{V(x,\sqrt{t})}$$

for some $C>0$, all $x \in M$ and $t>0$; the full Gaussian upper estimate,

$$(UE) \qquad p_t(x,y) \le \frac{C}{V(x,\sqrt{t})} \exp\left(-c\,\frac{d^2(x,y)}{t}\right)$$

for some $C,c>0$, all $x,y \in M$ and $t>0$; the upper and lower Gaussian estimates,

$$(LY) \quad \frac{c}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{ct}\right) \le p_t(x,y) \le \frac{C}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{Ct}\right)$$

for some $C,c>0$, all $x,y \in M$ and $t>0$; and finally the gradient upper estimate

$$(G) \qquad |\nabla_x p_t(x,y)| \le \frac{C}{\sqrt{t}\,V(y,\sqrt{t})}$$

for all $x,y \in M$, $t>0$. It is known that, under (D) and (DUE), (G) self-improves into

$$|\nabla_x p_t(x,y)| \le \frac{C}{\sqrt{t}\,V(y,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{Ct}\right)$$

for all $x,y \in M$, $t>0$ (see [15] and also [12, Section 4.4]). It will follow from Proposition 2.1 below that assumption (DUE) is not needed here.

Recall that (DUE) plus (D) implies (UE) ([18, Theorem 1.1], see also [12, Corollary 4.6]), and (UE) plus (G) and (D) implies (LY) ([21]). We shall see in Proposition 2.1 below that (G) implies (DUE), therefore (G) plus (D) implies (LY). Conversely, (LY) implies (D) (see for instance [24, p. 161]).

The following is one of the two main results of [2] (Theorem 1.4 there).

THEOREM 1.1. *Let M be a complete non-compact Riemannian manifold satisfying (D), (DUE), and (G). Then the equivalence (E_p) holds for all $1 < p < \infty$.*

Taking into account Proposition 2.1 below, one can skip condition (DUE), and formulate this result in the following simpler way.

THEOREM 1.2. *Let M be a complete non-compact Riemannian manifold satisfying (D) and (G). Then the equivalence (E_p) holds for all $1 < p < \infty$.*

Let us now introduce an L^p version of (G), namely

$$(G_p) \qquad \| \, |\nabla e^{-t\Delta}| \, \|_{p\to p} \le \frac{C_p}{\sqrt{t}}, \quad \forall t > 0.$$

The other main result of [2] is the following (Theorem 1.3 and Proposition 1.10 there).

THEOREM 1.3. *Let M be a complete non-compact Riemannian manifold satisfying (D) and (LY). Let $p_0 \in (2, \infty]$. The following assertions are equivalent:*

(a) *(R_p) holds for all $p \in (2, p_0)$.*
(b) *(G_p) holds for all $p \in (2, p_0)$.*
(c) *For all $p \in (2, p_0)$, there exists C_p such that*

$$\| \, |\nabla p_t(\cdot, y)| \, \|_p \le \frac{C_p}{\sqrt{t}\,[V(y, \sqrt{t})]^{1-\frac{1}{p}}}, \quad \forall t > 0, \, y \in M.$$

According to Proposition 3.6 below, we will be able to add another equivalent condition to the above list, namely

(d) *For all $p \in (2, p_0)$, there exists C_p such that*

$$\| \, |\nabla f| \, \|_p^2 \le C_p \|f\|_p \|\Delta f\|_p, \quad \forall f \in \mathbf{C}_0^\infty(M).$$

The above two results are the cornerstones of the present paper. Our main results are Theorems 4.1, 4.2 and Corollary 4.3 below. In Theorem 4.1, using Theorem 1.3, we give a necessary and sufficient condition for (R_p) to hold for p in an interval above 2 on manifolds with polynomial volume growth satisfying (D) and (LY), in terms of an L^p-L^q Sobolev type inequality with a gradient in the left-hand side. In Theorem 4.2, we give a necessary and

sufficient condition for (G) on manifolds with polynomial volume growth satisfying a mild local condition, in terms of a multiplicative L^∞ Sobolev type inequality, with a gradient in the left-hand side. In Corollary 4.3 we deduce from Theorems 4.1 and 1.1 that this L^∞ Sobolev inequality alone implies (E_p) on manifolds with polynomial growth and the above local condition.

Here is the plan we will follow. In Section 2, we prove that (G) implies (DUE), together with a similar statement for some related kernels. In Section 3, we give a first version of our results for manifolds with doubling and a polynomial volume upper bound. In Section 4, we assume full polynomial growth and obtain more complete results. Finally, in Section 5, we give applications of our methods to second order elliptic operators in $\mathbb{R}^n$.

2. Gradient estimates imply heat kernel bounds. Note that the following result does not require assumption (D).

PROPOSITION 2.1. *(G) implies (DUE).*

Proof. For $x \in M$, $t > 0$, define

$$K = K(x,t) = V(x,\sqrt{t})p_t(x,x)/2.$$

We claim that

$$p_t(y,x) \geq K/V(x,\sqrt{t})$$

for all $y \in B(x, K\sqrt{t}/C)$. Indeed, according to (G) and the mean value theorem, for such y,

$$|p_t(y,x) - p_t(x,x)| \leq \frac{Cd(y,x)}{\sqrt{t}\,V(x,\sqrt{t})} \leq \frac{C}{\sqrt{t}\,V(x,\sqrt{t})}\,\frac{K\sqrt{t}}{C} = \frac{K}{V(x,\sqrt{t})}.$$

Thus, given the definition of K,

$$p_t(y,x) \geq p_t(x,x) - K/V(x,\sqrt{t}) = K/V(x,\sqrt{t}),$$

hence the claim. Now

$$1 \geq \int_M p_t(y,x)\,d\mu(y) \geq \int_{B(x,K\sqrt{t}/C)} p_t(y,x)\,d\mu(y)$$

$$\geq \int_{B(x,K\sqrt{t}/C)} \frac{K\,d\mu(y)}{V(x,\sqrt{t})} = \frac{KV(x,K\sqrt{t}/C)}{V(x,\sqrt{t})}.$$

If $K \geq C$, this means that $K \leq 1$. Hence

$$K \leq \max(C,1),$$

and

$$p_t(x,x) \leq 2\max(C,1)/V(x,\sqrt{t}), \qquad \forall t > 0,\ x \in M. \ \blacksquare$$

As we noticed in the introduction, the following is a consequence of Proposition 2.1 together with known results.

COROLLARY 2.2. *Assume that M satisfies (G) and (D). Then M satisfies (LY).*

It may be of interest to notice that the assumption in Proposition 2.1 can be replaced by a gradient estimate of some other kernels. Namely, for $a > 0$, denote by $r_t^a(x,y)$ the (positive) kernel of the operator

$$R_t^a = (I + t\Delta)^{-a} = \frac{1}{\Gamma(a)} \int_0^{+\infty} s^{a-1} \exp(-s(I + t\Delta))\, ds.$$

Similarly, for $0 < a < 1$, denote by $p_t^a(x,y)$ the kernel of the operator $P_t^a = \exp(-(t\Delta)^a)$. In the following statement, we assume doubling only for simplicity, otherwise one has to include an additional constant in the outcome.

PROPOSITION 2.3. *Assume (D). Suppose that $q_t^a = r_t^a$ for some $a > 0$ or $q_t^a = p_t^a$ for some $0 < a < 1$. Next assume that M satisfies the gradient upper estimate*

$$(G^a) \qquad |\nabla_x q_t^a(x,y)| \le \frac{C}{\sqrt{t}\, V(y, \sqrt{t})}$$

for all $x, y \in M$, $t > 0$. Then M satisfies (DUE).

Proof. First note that in all cases

$$\int_M q_t^a(x,y)\, d\mu(y) \le 1, \qquad \forall x \in M.$$

Fix $x \in M$, $t > 0$. Define

$$K = K(x,t) = V(x, \sqrt{t}) q_t^a(x,x)/2.$$

Exactly as in the proof of Proposition 2.1, one shows that $K \le \max(C, 1)$. To finish the proof of Proposition 2.3, note that, for $a > 0$,

$$p_t(x,x) = \|p_{t/2}(\cdot, x)\|_2 \le C \|r_{t/2}^{a/2}(\cdot, x)\|_2 = C r_{t/2}^a(x,x).$$

Indeed, write

$$\exp(-t\Delta) = \exp(-t\Delta)(I + t\Delta)^a (I + t\Delta)^{-a},$$

so that

$$p_t(\cdot, x) = \exp(-t\Delta)(I + t\Delta)^a r_t^a(\cdot, x),$$

and since by spectral theory the operator $\exp(-t\Delta)(I + t\Delta)^a$ is uniformly bounded on $L^2(M, \mu)$, the claim is proved.

Similarly, for $0 < a < 1$, writing

$$\exp(-t\Delta) = \exp(-t\Delta) \exp(t\Delta)^a \exp(-(t\Delta)^a),$$

one sees that

$$p_t(x,x) = \|p_{t/2}(\cdot, x)\|_2 \le C \|p_{t/2}^a(\cdot, x)\|_2 = C p_{2^{(1-a)/a} t}^a(x,x). \quad \blacksquare$$

3. Doubling volume. Recall that (LY) implies (D). Thus the doubling volume assumption will be implicit in the first two statements of this section.

THEOREM 3.1. *Let M satisfy (LY). Let $\nu > 0$ be such that*

$$V(x,r) \le Cr^\nu, \quad \forall r > 0,\ x \in M. \tag{3.1}$$

Let $p_0 \in (2,\infty]$. Assume

$$\| \, |\nabla f| \, \|_p \le C_p \|\Delta^{\frac{\nu}{2}(\frac{1}{q}-\frac{1}{p})+\frac{1}{2}} f\|_q, \quad \forall f \in \mathbf{C}_0^\infty(M), \tag{3.2}$$

for all $p \in (2,p_0)$ and some $1 < q < p$. Then (R_p) holds for all $p \in (2,p_0)$.

Proof. Let p be such that $2 < p < p_0$, and $q \in (1,p)$ such that (3.2) holds. Taking $f = p_{2t}(\cdot,y) = \exp(-t\Delta)p_t(\cdot,y)$, $t > 0$, $y \in M$, in (3.2), one obtains

$$\| \, |\nabla p_{2t}(\cdot,y)| \, \|_p \le C_p \|\Delta^{\frac{\nu}{2}(\frac{1}{q}-\frac{1}{p})+\frac{1}{2}} \exp(-t\Delta)p_t(\cdot,y)\|_q,$$

hence, by analyticity of the heat semigroup on $L^q(M,\mu)$,

$$\| \, |\nabla p_{2t}(\cdot,y)| \, \|_p \le Ct^{-\frac{\nu}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}} \|p_t(\cdot,y)\|_q, \quad \forall t > 0,\ y \in M.$$

On the other hand, (UE) yields

$$\|p_t(\cdot,y)\|_q \le \frac{C}{[V(y,\sqrt{t})]^{1-\frac{1}{q}}}, \quad \forall t > 0,\ y \in M.$$

Hence

$$\| \, |\nabla p_{2t}(\cdot,y)| \, \|_p \le \frac{Ct^{-1/2}}{[V(y,\sqrt{t})]^{1-\frac{1}{p}}} \, [t^{-\nu/2}V(y,\sqrt{t})]^{\frac{1}{q}-\frac{1}{p}}, \quad \forall t > 0,\ y \in M,$$

and, according to (3.1), the quantity $t^{-\nu/2}V(y,\sqrt{t})$ is bounded from above, therefore

$$\| \, |\nabla p_{2t}(\cdot,y)| \, \|_p \le \frac{C'}{\sqrt{t}\,[V(y,\sqrt{t})]^{1-\frac{1}{p}}}, \quad \forall t > 0,\ y \in M. \tag{3.3}$$

One concludes by applying [2], namely Theorem 1.3 above. ∎

REMARKS.

• Remember that it follows from [9] that, under the assumptions of Theorems 3.1, (R_p) also holds for $p \in (1,2]$. As a consequence, (RR_p) also holds, therefore assumption (3.2) implies

$$\|\Delta^{1/2} f\|_p \le C_p \|\Delta^{\frac{\nu}{2}(\frac{1}{q}-\frac{1}{p})+\frac{1}{2}} f\|_q, \quad \forall f \in \mathbf{C}_0^\infty(M),$$

hence, by making the change of functions $\Delta^{1/2} f \to f$, we obtain the Sobolev inequality

$$\|f\|_p \le C_p \|\Delta^{\frac{\nu}{2}(\frac{1}{q}-\frac{1}{p})} f\|_q, \quad \forall f \in \mathbf{C}_0^\infty(M).$$

It follows that

$$V(x,r) \ge cr^\nu, \quad \forall r > 0,\ x \in M$$

(see [6]). Thus, in fact, under the assumptions of Theorem 3.1, the volume growth of M has to be polynomial of exponent ν (in particular, ν has to coincide with the topological dimension of M). However, the fact that we do not use explicitly polynomial growth in the proof will allow us below some true excursions in the doubling volume realm.

• An equivalent formulation of (3.2) is

$$\| \, |\nabla f| \, \|_p \leq C_p \| \Delta^{\alpha} f \|_q, \quad \forall f \in \mathbf{C}_0^\infty(M),$$

for all $p \in (2, p_0)$ and some $\alpha > 1/2$, with $q = 1/\big(\frac{1}{p} + \frac{2}{\nu}\big(\alpha - \frac{1}{2}\big)\big)$. In particular, $\alpha = 1$ and $q = p\nu/(\nu + p)$ is a valid choice. See Section 5 below.

• When $p_0 < \infty$, if one assumes

$$\| \, |\nabla f| \, \|_{p_0} \leq C \| \Delta^{\frac{\nu}{2}(\frac{1}{q} - \frac{1}{p_0}) + \frac{1}{2}} f \|_q, \quad \forall f \in \mathbf{C}_0^\infty(M), \tag{3.4}$$

instead of (3.2), one still obtains the same conclusion by interpolation.

• One can also replace (3.2) by the following weaker inequality:

$$\| \, |\nabla f| \, \|_p \leq C_p \| \Delta^{\alpha/2} f \|_{q_1}^{\theta} \| f \|_{q_2}^{1-\theta}, \quad \forall f \in \mathbf{C}_0^\infty(M),$$

where $0 < \theta < 1$, $1 \leq q_1, q_2 \leq \infty$, $\frac{1}{p} < \frac{\theta}{q_1} + \frac{1-\theta}{q_2} < 1$ and

$$\alpha\theta = \nu \left(\frac{\theta}{q_1} + \frac{1-\theta}{q_2} - \frac{1}{p} \right) + 1.$$

Here also, one can take $p = p_0$.

In the next statement, we shall relax the volume upper bound assumption for small radii. This can be useful in situations where the volume growth is polynomial, but with different exponents for small and large radii, say for instance the Heisenberg group endowed with a group invariant Riemannian metric.

We shall say that the *local Riesz inequality* $(R_p)_{\mathrm{loc}}$ holds on M if

$$\| \, |\nabla f| \, \|_p \leq C_p (\| \Delta^{1/2} f \|_p + \| f \|_p), \quad \forall f \in \mathbf{C}_0^\infty(M).$$

This is the case for instance if M has Ricci curvature bounded from below (see [3]).

THEOREM 3.2. *Let M satisfy (LY) and $(R_p)_{\mathrm{loc}}$. Let $\nu > 0$ be such that*

$$V(x, r) \leq C r^{\nu}, \quad \forall r \geq 1, \, x \in M. \tag{3.5}$$

Let $p_0 \in (2, \infty]$. Assume (3.2) for all $p \in (2, p_0)$ and some $1 < q < p$. Then (R_p) holds for all $p \in (2, p_0)$.

Proof. Given (3.5), the same proof as in Theorem 3.1 yields

$$\| \, |\nabla p_{2t}(\cdot, y)| \, \|_p \leq \frac{C}{\sqrt{t} [V(y, \sqrt{t})]^{1 - \frac{1}{p}}}, \quad \forall t \geq 1, \, y \in M.$$

On the other hand, $(R_p)_{\rm loc}$ easily implies, by analyticity of the heat semigroup on $L^p(M,\mu)$,

$$\| \, |\nabla e^{-t\Delta}| \, \|_{p\to p} \le C(1/\sqrt{t}+1),$$

for all $t>0$, hence, following [2, p. 944],

$$\| \, |\nabla p_{2t}(\cdot,y)| \, \|_p \le \frac{C}{\sqrt{t}\,[V(y,\sqrt{t})]^{1-\frac{1}{p}}}, \qquad \forall t\le 1,\ y\in M.$$

One concludes as before. ∎

Remark. One way to ensure (3.5) is to assume (D_ν) and

$$\sup_{x\in M} V(x,1) < +\infty. \tag{3.6}$$

Let us consider now the limit case $p=\infty$ in inequality (3.2).

Theorem 3.3. *Let M satisfy (D), (DUE), and (3.1) for some $\nu>0$. Assume*

$$\| \, |\nabla f| \, \|_\infty \le C\|f\|_q^{1-\frac{\nu+q}{\alpha q}} \, \|\Delta^{\alpha/2} f\|_q^{\frac{\nu+q}{\alpha q}}, \qquad \forall f\in \mathbf{C}_0^\infty(M), \tag{3.7}$$

for some $q\in[1,\infty)$ and some $\alpha>\frac{\nu}{q}+1$. Then (E_p) holds for all $p\in(1,\infty)$.

Proof. Taking again $f=p_{2t}(\cdot,y)=\exp(-t\Delta)p_t(\cdot,y)$, $t>0$, $y\in M$, in (3.7), and using the fact that (UE) yields

$$\|p_t(\cdot,y)\|_q \le \frac{C}{[V(y,\sqrt{t})]^{1-\frac{1}{q}}}, \qquad \forall t>0,\ y\in M, \tag{3.8}$$

one obtains

$$\| \, |\nabla p_{2t}(\cdot,y)| \, \|_\infty \le \frac{C}{[V(y,\sqrt{t})]^{(1-\frac{1}{q})(1-\frac{\nu+q}{\alpha q})}} \|\Delta^{\alpha/2}\exp(-t\Delta)p_t(\cdot,y)\|_q^{\frac{\nu+q}{\alpha q}},$$

hence, by analyticity of the heat semigroup on $L^q(M,\mu)$ (when $q=1$, it follows from (UE), see for instance [14, Lemma 9], [13, Theorem 3.4.8, p. 103] or [23]),

$$\begin{aligned}
\| \, |\nabla p_{2t}(\cdot,y)| \, \|_\infty &\le \frac{C}{[V(y,\sqrt{t})]^{(1-\frac{1}{q})(1-\frac{\nu+q}{\alpha q})}} t^{-\frac{\alpha}{2}(\frac{\nu+q}{\alpha q})} \|p_t(\cdot,y)\|_q^{\frac{\nu+q}{\alpha q}} \\
&\le \frac{C}{[V(y,\sqrt{t})]^{1-\frac{1}{q}}}\, t^{-\frac{\nu+q}{2q}} \\
&\le \frac{C}{\sqrt{t}\,V(y,\sqrt{t})}\,[t^{-\nu/2}V(y,\sqrt{t})]^{1/q}, \qquad \forall t>0,\ y\in M,
\end{aligned}$$

hence, using (3.1),

$$\| \, |\nabla p_{2t}(\cdot,y)| \, \|_\infty \le \frac{C}{\sqrt{t}\,V(y,\sqrt{t})}, \qquad \forall t>0,\ y\in M,$$

that is, (G). One concludes by applying [2], namely Theorem 1.1 above. ∎

THEOREM 3.4. *Assume that M has Ricci curvature bounded from below. Let M satisfy (D), (DUE), (3.5) and (3.7) for some $\nu > 0$, some $q \in [1, \infty)$ and some $\alpha > \frac{\nu}{q} + 1$. Then (E_p) holds for all $p \in (1, \infty)$.*

Proof. Given (3.5), the same proof as in Theorem 3.3 yields

$$\| \, |\nabla p_{2t}(\cdot, y)| \, \|_\infty \leq \frac{C}{\sqrt{t}\, V(y, \sqrt{t})}, \qquad \forall t \geq 1, \, y \in M,$$

that is, (G) for large time. Since M has Ricci curvature bounded from below, it follows from [21] that (G) also holds for small time. One concludes as before. ∎

Note that inequality (3.7) is known in $\mathbb{R}^n$, with $\nu = n$.

REMARK. Let the space $\mathrm{Lip}(M)$ be the completion of $\mathbf{C}_0^\infty(M)$ with respect to the norm

$$\|f\|_{\mathrm{Lip}} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}.$$

It is well-known that if $f \in \mathrm{Lip}(M)$, then f is differentiable almost everywhere and

$$\|f\|_{\mathrm{Lip}} = \| \, |\nabla f| \, \|_\infty.$$

By the reiteration lemma (see for instance [4, Proposition 2.10, p. 316]), inequality (3.7) is equivalent to the embedding

$$[L^q_\alpha, L^q]_{\theta,1} \to \mathrm{Lip},$$

where $[X, Y]_{\theta,r}$ denotes the real interpolation space between X and Y with parameters θ and r, $\theta = (\nu + q)/\alpha q$, and L^q_α is the completion of $\mathbf{C}_0^\infty(M)$ with respect to the norm $\|\Delta^{\alpha/2} f\|_q$. Then it is a well-known fact (see for instance [5, Proposition 3.5.3] and modify it to obtain a version for homogeneous spaces) that

$$[L^q_\alpha, L^q]_{\theta,1} = \Lambda^{q,1}_{\theta\alpha},$$

where the Besov space $\Lambda^{p,q}_\alpha$ is defined via the norm

$$\Lambda^{q,1}_\alpha(f) = \int_0^{+\infty} t^{k - \frac{\alpha}{2}} \|\Delta^k e^{-t\Delta} f\|_q \, \frac{dt}{t}$$

for $k > \alpha/2$. Finally, (3.7) is equivalent to

$$\Lambda^{q,1}_{\frac{\nu}{q}+1} \to \mathrm{Lip}.$$

Let us finally consider the limit case $q = \infty$ in Theorem 3.3. Here, no upper volume bound is needed.

THEOREM 3.5. *Let M satisfy (D) and (DUE). Assume*

$$\| \, |\nabla f| \, \|_\infty \le C \|f\|_\infty^{1-\frac{1}{\alpha}} \|\Delta^{\alpha/2} f\|_\infty^{1/\alpha}, \qquad \forall f \in \mathbf{C}_0^\infty(M), \tag{3.9}$$

for some $\alpha > 1$. Then (E_p) holds for all $p \in (1, \infty)$.

Let us emphasize the particular case $\alpha = 2$ of inequality (3.9):

$$\| \, |\nabla f| \, \|_\infty^2 \le C \|f\|_\infty \|\Delta f\|_\infty, \qquad \forall f \in \mathbf{C}_0^\infty(M). \tag{3.10}$$

Proof. Substituting $\exp(-t\Delta)f$ in (3.9) yields

$$\| \, |\nabla \exp(-t\Delta) f| \, \|_\infty \le C \|\exp(-t\Delta) f\|_\infty^{1-\frac{1}{\alpha}} \|\Delta^{\alpha/2} \exp(-t\Delta) f\|_\infty^{1/\alpha}.$$

Recall that it follows from (DUE) that the heat semigroup is analytic on $L^1(M, \mu)$, hence by duality

$$\|\Delta^{\alpha/2} \exp(-t\Delta) f\|_\infty \le C t^{-\alpha/2} \|f\|_\infty.$$

The heat semigroup being uniformly bounded on $L^\infty(M, \mu)$, one obtains

$$\| \, |\nabla \exp(-t\Delta) f| \, \|_\infty \le C t^{-1/2} \|f\|_\infty,$$

that is, (G_∞), or

$$\sup_{x \in M, \, t > 0} \sqrt{t} \int_M |\nabla_x p_t(x, y)| \, d\mu(y) < \infty.$$

It is well-known and easy to see that (G_∞) together with (D) and (UE) implies (G) (in fact, these conditions are equivalent, because of the already mentioned self-improvement of (G)). One concludes again by applying [2], namely Theorem 1.1 above. ∎

Next we discuss a result which does not require any assumption on the volume growth and which is motivated by (3.10). This result is contained in [16], with a similar argument, in a discrete setting. For another approach to inequality (3.11) below, see [10, Section 4].

PROPOSITION 3.6. *For any $1 \le p \le \infty$, condition (G_p) is equivalent to*

$$\| \, |\nabla f| \, \|_p^2 \le C \|f\|_p \|\Delta f\|_p, \qquad \forall f \in \mathbf{C}_0^\infty(M). \tag{3.11}$$

Proof. To prove that (3.11) implies (G_p) we modify slightly the argument of the proof of Theorem 3.5; namely, we put $\alpha = 2$ and replace L^∞ norm by L^p norm.

To prove the opposite direction, write

$$\nabla (I + t\Delta)^{-1} = \int_0^\infty \nabla \exp(-s(1 + t\Delta)) \, ds.$$

Hence, for suitable f,

$$\| \, |\nabla (I + t\Delta)^{-1} f| \, \|_p \le \int_0^\infty e^{-s} \| \, |\nabla \exp(-st\Delta) f| \, \|_p \, ds.$$

Assuming (G_p), one obtains, for $f \in L^q(M,\mu)$,

$$\| \, |\nabla (I+t\Delta)^{-1} f| \, \|_p \le Ct^{-1/2}\|f\|_p \int_0^\infty s^{-1/2} e^{-s}\, ds = C' t^{-1/2}\|f\|_p.$$

Hence

$$\| \, |\nabla f| \, \|_p \le C' t^{-1/2} \|(I+t\Delta) f\|_p \le C' t^{-1/2} (\|f\|_p + t\|\Delta f\|_p).$$

Taking $t = \|f\|_p \|\Delta f\|_p^{-1}$ yields (3.11). ∎

4. Polynomial volume growth

THEOREM 4.1. *Let $n > 0$. Suppose that M satisfies upper and lower n-dimensional Gaussian estimates*

$$ct^{-n/2} \exp\left(-\frac{d^2(x,y)}{ct}\right) \le p_t(x,y) \le Ct^{-n/2} \exp\left(-\frac{d^2(x,y)}{Ct}\right)$$

for some $C, c > 0$, all $x, y \in M$ and $t > 0$. Let $p_0 \in (2, \infty]$. Then the following are equivalent:

(i)

$$\| \, |\nabla f| \, \|_p \le C_{p,q} \|\Delta^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})+\frac{1}{2}} f\|_q, \qquad \forall f \in \mathbf{C}_0^\infty(M), \tag{4.1}$$

for some $q \in (1,p)$, and all $p \in (2, p_0)$.

(ii) *(R_p) holds for all $p \in (2, p_0)$.*

Proof. Let q and p be such that $1 < q < p < \infty$ and $p > 2$. According to [26], the following Sobolev inequality is a consequence of the upper heat kernel estimate:

$$\|f\|_p \le C \|\Delta^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})} f\|_q, \qquad \forall f \in \mathbf{C}_0^\infty(M),$$

and in particular

$$\|\Delta^{1/2} f\|_p \le C \|\Delta^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})+\frac{1}{2}} f\|_q, \qquad \forall f \in \mathbf{C}_0^\infty(M).$$

Thus (R_p) for some $p > 2$ implies (4.1) for all q such that $1 < q < p$, and in particular (ii) implies (i).

Conversely, observe that the heat kernel estimates imply $V(x,r) \simeq r^n$ for all $r > 0$ and $x \in M$ (see [19, Theorem 3.2]). Therefore Theorem 3.1 applies with $\nu = n$ and shows that (i) implies (ii). ∎

Remarks similar to those after Theorem 3.1 are valid. We add one more.

REMARK. According to [1, Theorem 0.4], under the assumptions of Theorem 4.1, there always exists a p_0 such that (ii) holds. It would be nice to have a proof of this fact using (i).

Again, we shall now consider the limit case $p = \infty$ of inequality (4.1).

We shall have to make local assumptions in order to ensure that the quantity

$$\theta(t) := \sup_{0<u\le t,\, x\in M} u^{n/2} p_u(x,x) = \sup_{0<u\le t} u^{n/2} \|\exp(-u\Delta)\|_{1\to\infty}$$

is finite for some (all) $t > 0$. For instance, a local Sobolev inequality of dimension n is enough, since then $\sup_{\in M} p_t(x,x) \le Ct^{-n/2},\ 0 < t \le 1$. This holds for instance if $\dim M \le n$, M has Ricci curvature bounded from below and satisfies the matching condition to (3.6): $\inf_{x\in M} V(x,1) > 0$.

THEOREM 4.2. *Assume that M has Ricci curvature bounded from below. Let $n \in \mathbb{N}^*$. Assume that*

$$V(x,r) \simeq r^n, \qquad \forall r > 0,\ x \in M. \tag{4.2}$$

Then M satisfies the heat kernel gradient estimate (G), that is,

$$|\nabla_x p_t(x,y)| \le Ct^{-(n+1)/2} \exp\left(-\frac{d^2(x,y)}{Ct}\right), \tag{4.3}$$

for some $C > 0$, all $x, y \in M$ and $t > 0$, if and only if

$$\| \, |\nabla f| \, \|_\infty \le C \|f\|_q^{1-\frac{n+q}{\alpha q}} \|\Delta^{\alpha/2} f\|_q^{\frac{n+q}{\alpha q}}, \qquad \forall f \in \mathbf{C}_0^\infty(M), \tag{4.4}$$

for some (all) $q \in (1,\infty)$ and some (all) $\alpha > \frac{n}{q} + 1$. Moreover, if (4.3) or (4.4) holds, then M satisfies (LY), that is, the upper and lower n-dimensional Gaussian estimates

$$ct^{-n/2} \exp\left(-C\,\frac{d^2(x,y)}{t}\right) \le p_t(x,y) \le Ct^{-n/2} \exp\left(-c\,\frac{d^2(x,y)}{t}\right), \tag{4.5}$$

for some $C, c > 0$, all $x, y \in M$ and $t > 0$.

The following result is a direct consequence of Theorem 4.2 together with [2, Theorem 1.4], that is, Theorem 1.1 above.

COROLLARY 4.3. *Assume that M has Ricci curvature bounded from below, and satisfies (4.2) and (4.4). Then (E_p) holds for all $p \in (1,\infty)$.*

Let us prepare the proof of Theorem 4.2 with two lemmas. The first one is reminiscent of Proposition 2.1: it shows that a certain gradient estimate implies an upper bound of the heat kernel.

LEMMA 4.4. *Assume that M has Ricci curvature bounded from below and let $n > 0$. Assume that, for some $c > 0$,*

$$V(x,r) \ge cr^n \tag{4.6}$$

for all $x \in M$ and $r > 0$. Next suppose that, for some $q \in [1,\infty]$,

$$\| \, |\nabla \exp(-t\Delta)| \, \|_{q\to\infty} \le Ct^{-\frac{n+q}{2q}} \tag{$G^n_{q,\infty}$}$$

for all $t > 0$. Then there exists a constant C' such that

$$\sup_{x\in M} p_t(x,x) \le C' t^{-n/2}$$

for all $t > 0$.

Proof. Set

$$\theta(t) = \sup_{0<u\le t,\, x\in M} u^{n/2} p_u(x,x) = \sup_{0<u\le t} u^{n/2} \|\exp(-u\Delta)\|_{1\to\infty}.$$

Remember that the curvature assumption together with the volume lower bound ensures the finiteness of $\theta(t)$ for all $t > 0$. Using $(G^n_{q,\infty})$ and interpolation, we can write

$$\begin{aligned}\| \, |\nabla \exp(-2s\Delta)| \, \|_{1\to\infty} &\le \| \, |\nabla \exp(-s\Delta)| \, \|_{q\to\infty} \|\exp(-s\Delta)\|_{1\to q} \\ &\le C s^{-\frac{n+q}{2q}} (\theta(s) s^{-n/2})^{1-\frac{1}{q}} = C s^{-(n+1)/2} \theta(s)^{1-\frac{1}{q}}.\end{aligned}$$

For $x \in M$ and $s > 0$, define

$$K = K(s,x) = s^{n/2} p_{2s}(x,x)/2.$$

For all $y \in B_s(x) = B(x, K\sqrt{s}/(C\theta(s)^{1-\frac{1}{q}}))$,

$$\begin{aligned}|p_{2s}(y,x) - p_{2s}(x,x)| &\le d(y,x) \sup_{z\in M} |\nabla p_{2s}(z,x)| \\ &\le d(y,x) \| \, |\nabla \exp(-2s\Delta)| \, \|_{1\to\infty} \\ &\le \frac{K\sqrt{s}}{C\theta(s)^{1-\frac{1}{q}}} \frac{C\theta(s)^{1-\frac{1}{q}}}{s^{(n+1)/2}} = \frac{K}{s^{n/2}},\end{aligned}$$

therefore

$$p_{2s}(y,x) \ge p_{2s}(x,x) - \frac{K}{s^{n/2}} = \frac{2K}{s^{n/2}} - \frac{K}{s^{n/2}} = \frac{K}{s^{n/2}}.$$

Hence

$$\begin{aligned}1 &\ge \int_M p_{2s}(y,x)\, d\mu(y) \ge \int_{B_s(x)} p_{2s}(y,x)\, d\mu(y) \\ &\ge \int_{B_s(x)} \frac{K}{s^{n/2}}\, d\mu(y) \ge c\, \frac{K^{n+1}}{C^n \theta(s)^{n(1-\frac{1}{q})}},\end{aligned}$$

using (4.6) in the last inequality. Since θ is obviously non-decreasing, we also have

$$1 \ge c\, \frac{K^{n+1}}{C^n \theta(2s)^{n(1-\frac{1}{q})}},$$

that is,

$$K(s,x) \le \left(\frac{C^n}{c}\, \theta(2s)^{n(1-\frac{1}{q})} \right)^{1/(n+1)},$$

hence

$$K(s,x) \le \left(\frac{C^n}{c}\, \theta(2t)^{n(1-\frac{1}{q})} \right)^{1/(n+1)}$$

for $0 < s \le t$.

Taking supremum in x and s yields

$$2^{-n/2-1}\theta(2t) \le \left(\frac{C^n}{c}\, \theta(2t)^{n(1-\frac{1}{q})} \right)^{1/(n+1)}.$$

Since $n(1-\frac{1}{q})/(n+1) < 1$, it follows that θ is bounded from above, which proves the claim. ∎

REMARK. One can write a version of the above lemma in the case where $V(x,r) \ge v(r)$ for some doubling function v.

The lemma below yields as a by-product a new proof of inequality (4.4) in $\mathbb{R}^n$. It does not require any volume growth assumption.

LEMMA 4.5. *Let $1 < q < \infty$ and $n > 0$. The following estimates are equivalent:*

(i)

$$(G^n_{q,\infty}) \qquad \| \, |\nabla \exp(-t\Delta)| \, \|_{q\to\infty} \le C t^{-\frac{n+q}{2q}}, \quad \forall t > 0.$$

(ii)

$$\| \, |\nabla (I+t\Delta)^{-\alpha/2}| \, \|_{q\to\infty} \le C_\alpha t^{-\frac{n+q}{2q}},$$

for some (all) $\alpha > \frac{n}{q} + 1$ and all $t > 0$.

(iii)

$$\| \, |\nabla f| \, \|_\infty \le C \|f\|_q^{1-\frac{n+q}{\alpha q}} \, \|\Delta^{\alpha/2} f\|_q^{\frac{n+q}{\alpha q}}, \quad \forall f \in \mathbf{C}_0^\infty(M),$$

for some (all) $\alpha > \frac{n}{q} + 1$, that is, (4.4).

Proof. We shall show that (i)⇒(ii)⇒(iii)⇒(i).

Write

$$\nabla (I+t\Delta)^{-\alpha/2} = \int_0^\infty s^{\frac{\alpha}{2}-1} \nabla \exp(-s(1+t\Delta))\, ds.$$

Hence, for suitable f,

$$\| \, |\nabla (I+t\Delta)^{-\alpha/2} f| \, \|_\infty \le \int_0^\infty s^{\frac{\alpha}{2}-1} e^{-s} \| \, |\nabla \exp(-st\Delta) f| \, \|_\infty \, ds.$$

Assuming $(G^n_{q,\infty})$, one obtains, for $f \in L^q(M,\mu)$,

$$\begin{aligned}\| \, |\nabla(I+t\Delta)^{-\alpha/2} f| \, \|_\infty &\le C \int_0^\infty s^{\frac{\alpha}{2}-1} e^{-s} (ts)^{-\frac{n+q}{2q}} \|f\|_q \, ds \\ &= C t^{-\frac{n+q}{2q}} \|f\|_q \int_0^\infty s^{\frac{\alpha}{2}-\frac{n}{2q}-\frac{3}{2}} e^{-s} \, ds \\ &= C_\alpha t^{-\frac{n+q}{2q}} \|f\|_q,\end{aligned}$$

since $\alpha > \frac{n}{q} + 1$.

Assume (ii), and write

$$\begin{aligned}\| \, |\nabla f| \, \|_\infty &\le C t^{-\frac{n+q}{2q}} \|(I+t\Delta)^{\alpha/2} f\|_q \\ &\le C_\alpha t^{-\frac{n+q}{2q}} (\|f\|_q + \|(t\Delta)^{\alpha/2} f\|_q) \\ &= C t^{-\frac{n+q}{2q}} (\|f\|_q + t^{\alpha/2} \|\Delta^{\alpha/2} f\|_q).\end{aligned}$$

The second inequality relies on the L^p-boundedness of the operator $(I + t\Delta)^{\alpha/2}(I + (t\Delta)^{\alpha/2})^{-1}$ (see [25], or use analyticity).

Taking $t = \|f\|_q^{2/\alpha} \|\Delta^{\alpha/2} f\|_q^{-2/\alpha}$ yields (iii).

Finally, assume (iii). Replacing f by $\exp(-t\Delta)f$, one obtains, by contractivity and analyticity of the heat semigroup on $L^q(M,\mu)$,

$$\begin{aligned}\| \, |\nabla \exp(-t\Delta) f| \, \|_\infty &\le C \|\exp(-t\Delta) f\|_q^{1-\frac{n+q}{\alpha q}} \|\Delta^{\alpha/2} \exp(-t\Delta) f\|_q^{\frac{n+q}{\alpha q}} \\ &\le C t^{-\alpha \frac{n+q}{2\alpha q}} \|f\|_q = C t^{-\frac{n+q}{2q}} \|f\|_q,\end{aligned}$$

that is, (i). ∎

REMARK. The above lemma also holds for $q = 1, \infty$, provided the heat semigroup is analytic on $L^1(M,\mu)$, which is the case, as we already said, if it satisfies Gaussian estimates and (D) holds. Note that, according to Lemma 4.4, this is automatic from (i) under the boundedness from below of the Ricci curvature and (4.2).

Proof of Theorem 4.2. Assume (4.4). By Lemma 4.5, $(G^n_{q,\infty})$ follows, and by Lemma 4.4,

$$\sup_{x \in M} p_t(x,x) = \|\exp(-t\Delta)\|_{1\to\infty} \le C' t^{-n/2} \tag{4.7}$$

for all $t > 0$. The Gaussian upper bound follows:

$$p_t(x,y) \le C t^{-n/2} \exp\left(-\frac{d^2(x,y)}{Ct}\right)$$

for some $C, c > 0$ and all $x, y \in M$ and $t > 0$. By interpolation, (4.7) yields

$$\|\exp(-t\Delta)\|_{1\to q} \le C t^{-n(1-\frac{1}{q})/2}. \tag{4.8}$$

Combining $(G^n_{q,\infty})$ with (4.8) yields

$$\sup_{x,y\in M} |\nabla_x p_t(x,y)| = \| \, |\nabla \exp(-t\Delta)| \, \|_{1\to\infty} \le Ct^{-(n+1)/2}, \tag{4.9}$$

which together with the upper bound yields the Gaussian lower bound

$$ct^{-n/2} \exp\left(-\frac{d^2(x,y)}{ct}\right) \le p_t(x,y).$$

Finally, (4.9) self-improves to (4.3).

Conversely, (4.3) obviously implies $(G^n_{1,\infty})$ and, together with the volume upper bound, $(G^n_{\infty,\infty})$ or, in other words, (G_∞):

$$\sup_{x\in M} \int_M |\nabla_x p_t(x,y)| \, d\mu(y) = \| \, |\nabla \exp(-t\Delta)| \, \|_{\infty\to\infty} \le Ct^{-1/2}.$$

By interpolation, one obtains $(G^n_{q,\infty})$, therefore (4.4), thanks to Lemma 4.5. ∎

Remarks.

• As a consequence of Theorem 4.2, (4.4) implies, using the results in [7],

$$\|f\|_\infty \le C\|f\|_q^{1-\frac{n}{\alpha q}} \|\Delta^{\alpha/2} f\|_q^{\frac{n}{\alpha q}}, \qquad \forall f \in \mathbf{C}_0^\infty(M),$$

for $\alpha > n/q$, $q \in [1,\infty)$, and, using the results in [8],

$$\frac{|f(x)-f(y)|}{[d(x,y)]^{\alpha-\frac{n}{q}}} \le C\|\Delta^{\alpha/2} f\|_q, \qquad \forall x,y \in M,\ f \in \mathbf{C}_0^\infty(M),$$

for $\alpha > n/q$, $q \in [1,\infty)$. It would be interesting to have a direct proof of these two implications.

• According to known results on Riesz transforms (see [2] for references), (4.4) is true for manifolds with non-negative Ricci curvature, Lie groups with polynomial volume growth, cocompact coverings with polynomial volume growth. Again, it would be interesting to have direct proofs.

• It would interesting to study the stability under perturbation of inequalities (4.4) or (4.1), in the light of the result in [11].

5. Applications. Now we consider a uniformly elliptic operator H in divergence form acting on $\mathbb{R}^n$, $n \in \mathbb{N}^*$, that is,

$$Hf = -\sum_{i,j=1}^n \partial_i(a_{ij}\partial_j f)$$

where $a_{ij} \in L^\infty$ for all $1 \le i,j \le n$, and the matrix $(a_{ij}(x))_{1\le i,j\le n}$ is a symmetric matrix with real coefficients, such that

$$\sum_{i,j} a_{ij}(x)\xi_j\xi_i \ge c|\xi|^2 \quad \text{for a.e. } x,\xi \in \mathbb{R}^n,$$

for some $c > 0$. Next let Δ denote the standard non-negative Laplace operator acting on $\mathbb{R}^n$.

It follows from the above uniform ellipticity assumption and the boundedness of the coefficients that

$$|\nabla_H f(x)|^2 = \sum_{i,j} a_{ij}(x)\partial_j f(x)\partial_i f(x) \simeq |\nabla f(x)|^2.$$

We say that H satisfies (R_p) for some $p \in (1,\infty)$ if

$$\| \, |\nabla_H f| \, \|_p \le C_p \|H^{1/2} f\|_p, \quad \forall f \in \mathbf{C}_0^\infty(\mathbb{R}^n),$$

which according to the above remark is equivalent to

$$\| \, |\nabla f| \, \|_p \le C_p \|H^{1/2} f\|_p, \quad \forall f \in \mathbf{C}_0^\infty(\mathbb{R}^n).$$

To avoid technicalities we assume that all coefficients a_{ij}, b_{ij} discussed below are smooth. However, we point out that this assumption can be substantially relaxed.

Recall that the Gaussian estimates do hold for e^{-tH} and that the above framework applies.

The assumption in our first application may be seen as some boundedness for the higher order Riesz transform associated with H.

THEOREM 5.1. *Suppose that*

$$\|\Delta^{\alpha/2} f\|_{q_0} \le C \|H^{\alpha/2} f\|_{q_0}, \quad \forall f \in \mathbf{C}_0^\infty(\mathbb{R}^n), \tag{5.1}$$

for some $\alpha > 1$ and $1 < q_0 < \infty$. If $\alpha < \frac{n}{q_0} + 1$, then H satisfies (R_p) for all $p \in (1, p_0)$, where $p_0 = n/(\frac{n}{q_0} + 1 - \alpha)$, while if $\alpha \ge \frac{n}{q_0} + 1$, then H satisfies (R_p) for all $p \in (1,\infty)$.

Proof. The boundedness of the classical Riesz transform on $L^p(\mathbb{R}^n, dx)$ together with the Sobolev inequality in $\mathbb{R}^n$ implies, for $1 < q_0 < p < \infty$,

$$\| \, |\nabla f| \, \|_p \le C \|\Delta^{\frac{n}{2}(\frac{1}{q_0} - \frac{1}{p}) + \frac{1}{2}} f\|_{q_0} \le C' \|\Delta^{\alpha/2} f\|_{q_0}^{\theta} \|f\|_{q_0}^{1-\theta}, \quad \forall f \in \mathbf{C}_0^\infty(\mathbb{R}^n),$$

as soon as $\alpha \ge n(\frac{1}{q_0} - \frac{1}{p}) + 1$, $\theta \in (0,1]$ being such that $\alpha\theta = n(\frac{1}{q_0} - \frac{1}{p}) + 1$. Now let α and q_0 be such that (5.1) holds. If $\alpha \ge \frac{n}{q_0} + 1$, choose any $p > q_0$. If $\alpha < n/q_0 + 1$, choose p_0 in (q_0, ∞) so that $\alpha = n(\frac{1}{q_0} - \frac{1}{p_0}) + 1$. In both cases,

$$\| \, |\nabla f| \, \|_{p_0} \le C \|H^{\alpha/2} f\|_{q_0}^{\theta} \|f\|_{q_0}^{1-\theta}, \quad \forall f \in \mathbf{C}_0^\infty(\mathbb{R}^n),$$

and (R_p) for $1 < p < p_0$ follows from Theorem 3.1 and the remarks afterwards. ∎

Our next application says that (R_p) also holds for small $L^\infty \cap W^{1,n}$ perturbations of operators with bounded second order Riesz transform.

To state this result we set

$$H_\varepsilon f = Hf + \varepsilon \sum_{i,j} \partial_i b_{ij}(x) \partial_j f,$$

where $H = H_0$ is as above. We do not assume here that the matrix $(b_{ij}(x))_{1\le i,j\le n}$ is positive definite. However, we assume that $b_{ij}\in L^\infty(\mathbb{R}^n, dx)$ and that ε is small enough so that the operator H_ε is uniformly elliptic.

THEOREM 5.2. *Suppose that $b_{ij} \in L^\infty(\mathbb{R}^n, dx)$ for all $1 \le i, j \le n$ and that $\partial_j b_{ij} \in L^n(\mathbb{R}^n, dx)$ for all $1 \le i, j \le n$. Next assume that for some $q_0 < n$,*

$$\|\Delta f\|_{q_0} \le C_{q_0}\|H_0 f\|_{q_0}, \qquad \forall f \in \mathbf{C}_0^\infty(\mathbb{R}^n), \tag{5.2}$$

Then there exists $\gamma > 0$ such that (5.2) holds for H_ε for $\varepsilon < \gamma$ and so (R_p) holds for H_ε for all $\varepsilon < \gamma$ and $1 < p < p_0$ where $\frac{1}{p_0} + \frac{1}{n} = \frac{1}{q_0}$.

Proof. Note that (5.2) is just (5.1) for $\alpha = 2$. We will extend this inequality from H to H_ε for $0 < \varepsilon < \gamma$ and apply Theorem 5.1. To this end, it is enough to show that for some $\gamma > 0$ and all $\varepsilon < \gamma$,

$$\|H_\varepsilon f - H_0 f\|_{p_0} \le \frac{1}{2C_{q_0}}\,\|\Delta f\|_{p_0}. \tag{5.3}$$

Now

$$\|H_\varepsilon f - H_0 f\|_{q_0} \le \varepsilon\Big(\sum_{i,j}\|\partial_i b_{ij}\partial_j f\|_{q_0}\Big).$$

Since

$$\partial_i(b_{ij}\partial_j f) = b_{ij}(\partial_i\partial_j f) + (\partial_i b_{ij})(\partial_j f),$$

one may write

$$\sum_{i,j}\|\partial_i b_{ij}\partial_j f\|_{q_0} \le \max_{i,j}\|b_{ij}\|_\infty \sum_{i,j}\|\partial_i\partial_j f\|_{q_0} + \sum_{i,j}\|(\partial_i b_{ij})(\partial_j f)\|_{q_0},$$

hence

$$\begin{aligned}\sum_{i,j}\|\partial_i b_{ij}\partial_j f\|_{q_0} &\le \max_{i,j}\|b_{ij}\|_\infty \sum_{i,j}\|\partial_i\partial_j f\|_{q_0} + \sum_{i,j}\|(\partial_i b_{ij})(\partial_j f)\|_{q_0}\\ &\le n^2 \max_{i,j}\|b_{ij}\|_\infty \|\partial_i\partial_j \Delta^{-1}\|_{q_0\to q_0}\|\Delta f\|_{q_0} + \sum_{i,j}\|\partial_i b_{ij}\|_n\|\partial_j f\|_{p_0}.\end{aligned}$$

Here we have used the L^{q_0} boundedness of the second order Riesz transform in $\mathbb{R}^n$ and the Hölder inequality $\|fg\|_{q_0} \le \|f\|_n\|g\|_{p_0}$.

Now recall that an inequality similar to (4.1) holds in $\mathbb{R}^n$, that is,

$$\|\partial_j f\|_{p_0} \le C\|\Delta f\|_{q_0}.$$

Therefore

$$\begin{aligned}&\sum_{i,j}\|\partial_i b_{ij}\partial_j f\|_{q_0}\\ &\le \Big(n^2 \max_{i,j}\|b_{ij}\|_\infty\|\partial_i\partial_j\Delta^{-1}\|_{q_0\to q_0} + \max_i\|\partial_i\Delta^{-1}\|_{q_0\to p_0}\sum_{i,j}\|\partial_i b_{ij}\|_n\Big)\|\Delta f\|_{q_0}.\end{aligned}$$

This yields (5.3) with

$$\gamma\Big(n^2 \max_{i,j} \|b_{ij}\|_\infty \|\partial_i\partial_j \Delta^{-1}\|_{q_0\to q_0} + \max_i \|\partial_i \Delta^{-1}\|_{q_0\to p_0} \sum_{i,j} \|\partial_i b_{ij}\|_n\Big) = \frac{1}{2C_{q_0}}. \ \blacksquare$$

The second order Riesz transform bound (5.2) is known for various large classes of operators. We discuss one instance of such class next.

EXAMPLE. Assume the coefficients a_{ij} of H are continuous and periodic with a common period and that $\sum_{i=1}^n \partial_i a_{ij} = 0$ for $1 \le j \le n$. Then

$$\|Hf\|_p \simeq \|\Delta f\|_p, \quad \forall f \in \mathbf{C}_0^\infty(\mathbb{R}^n),$$

for all $1 \le p \le \infty$ (see [17, Theorem 1.3]), so that H satisfies the assumption of Theorem 5.2. In [17], it is proved that (R_p) holds for such H, but the above shows that it also holds for small $L^\infty \cap W^{1,n}$ perturbations of H.

REMARK. It is interesting to compare Theorem 5.1, which proves that boundedness of second order Riesz transform implies boundedness of first order Riesz transform on a larger range on L^p spaces, with the results obtained in [22]. See also [20, (1.26)].

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