

Introduction

The theory of function spaces is a well established mathematical theory in its own right. Its purpose is best described by the title of H. Triebel's beautiful essay *How to measure smoothness*, which forms Chapter I of his monograph [79]. The theory of function spaces has, however, always had close relations to other mathematical fields such as the theory of (partial) differential operators, potential theory, or approximation theory, to mention just a few.

The function spaces we are interested in appeared in their generality for the first time in the work of A. Beurling and J. Deny [8], [9] (see also [17]) on Dirichlet spaces. In general, they are contained neither in the Besov B_{pq}^s or Triebel–Lizorkin F_{pq}^s scales nor in the classes of anisotropic spaces considered so far.

By definition a *Dirichlet space* (on \mathbb{R}^n for simplicity) is a pair $(\mathcal{F}, \mathcal{E})$ consisting of a space $\mathcal{F} \subset L_2(\mathbb{R}^n)$ of real-valued functions and a symmetric quadratic form $\mathcal{E} : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}$ which is closed, densely defined, non-negative, and satisfies the following contraction condition:

$$\text{if } u \in \mathcal{F} \text{ then } v := (0 \vee u) \wedge 1 \in \mathcal{F} \text{ and } \mathcal{E}(v, v) \leq \mathcal{E}(u, u).$$

All translation invariant (symmetric) Dirichlet forms (on \mathbb{R}^n) are given by

$$\mathcal{E}^\psi(u, v) = \int_{\mathbb{R}^n} \psi(\xi) \widehat{u}(\xi) \overline{\widehat{v}(\xi)} d\xi, \quad u, v \in \mathcal{S}(\mathbb{R}^n),$$

where $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuous negative definite function, i.e. $\psi(0) \geq 0$ and for all $t > 0$ the function $\xi \mapsto e^{-t\psi(\xi)}$ is positive definite in the usual sense. The domain \mathcal{F}^ψ of \mathcal{E}^ψ is then given by

$$\mathcal{F}^\psi := H_2^{\psi, 1}(\mathbb{R}^n) := \left\{ u \in L_2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + \psi(\xi)) |\widehat{u}(\xi)|^2 d\xi < \infty \right\}.$$

It is well known that we can associate with ψ (or with $(\mathcal{F}^\psi, \mathcal{E}^\psi)$) the operator semigroup $(T_t^{(2)})_{t \geq 0}$ on $L_2(\mathbb{R}^n)$ defined by

$$(1) \quad T_t^{(2)} u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{-t\psi(\xi)} \widehat{u}(\xi) d\xi = \int_{\mathbb{R}^n} u(x - y) \mu_t(dy),$$

where $(\mu_t)_{t \geq 0}$ is a vaguely continuous convolution semigroup of sub-probability measures on \mathbb{R}^n with Fourier transform $\widehat{\mu}_t(\xi) = (2\pi)^{-n/2} e^{-t\psi(\xi)}$.

The generator $(A^{(2)}, D(A^{(2)}))$ of the semigroup $(T_t^{(2)})_{t \geq 0}$ is given by

$$A^{(2)} u(x) = -\psi(D)u(x) = -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \psi(\xi) \widehat{u}(\xi) d\xi$$

with domain

$$H_2^{\psi,2}(\mathbb{R}^n) := \left\{ u \in L_2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + \psi(\xi))^2 |\widehat{u}(\xi)|^2 d\xi < \infty \right\}.$$

Let us mention that the measures μ_t are also the transition probabilities for a Lévy process $(X_t)_{t \geq 0}$ and therefore we have

$$(2) \quad E(e^{iX_t\xi}) = e^{-t\psi(\xi)}.$$

Thus ψ is also a characteristic exponent of a Lévy process.

Since Lévy processes are infinitely divisible, the function ψ has a *Lévy–Khinchin representation*

$$\psi(\xi) = c + i \sum_{j=1}^n b_j \xi_j + \sum_{j,l=1}^n q_{jl} \xi_j \xi_l + \int_{\mathbb{R}^n \setminus \{0\}} \left(1 - e^{-ix \cdot \xi} - \frac{ix \cdot \xi}{1 + |x|^2} \right) \frac{1 + |x|^2}{|x|^2} \mu(dx)$$

which in the case of a real-valued function reduces to

$$(3) \quad \psi(\xi) = c + \sum_{j,l=1}^n q_{jl} \xi_j \xi_l + \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos(x \cdot \xi)) \frac{1 + |x|^2}{|x|^2} \mu(dx).$$

From (3) we may derive a further representation of \mathcal{E}^ψ , namely its *Beurling–Deny representation*

$$\begin{aligned} \mathcal{E}^\psi(u, v) &= \int_{\mathbb{R}^n} cu(x)v(x) dx + \sum_{j,l=1}^n \int_{\mathbb{R}^n} q_{jl} \frac{\partial u(x)}{\partial x_j} \frac{\partial v(x)}{\partial x_l} dx \\ &\quad + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (u(x+y) - u(x))(v(x+y) - v(y)) J(dx, dy) \end{aligned}$$

where $J(dx, dy)$ is a suitable symmetric measure on $\mathbb{R}^n \times \mathbb{R}^n$ which does not charge the diagonal.

Let us assume in what follows that $c = 0$ and $q_{jl} = 0$ for $j, l = 1, \dots, n$. Then we have (on a suitable subspace of \mathcal{F}^ψ) the following equivalent descriptions of the scalar product in the Hilbert (function) space $(\mathcal{F}^\psi, \mathcal{E}_1^\psi)$, $\mathcal{E}_1^\psi = \mathcal{E}^\psi + (\cdot, \cdot)_{L_2}$:

$$\mathcal{E}_1^\psi(u, u) = \int_{\mathbb{R}^n} (1 + \psi(\xi)) |\widehat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |u(x)|^2 dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (u(x+y) - u(x))^2 J(dx, dy).$$

Of course, we also have some abstract characterizations of this scalar product, for example we find

$$\mathcal{E}_1^\psi(u, u) = \|u\|_{L_2(\mathbb{R}^n)}^2 + \|(-A^{(2)})^{1/2}u\|_{L_2(\mathbb{R}^n)}^2.$$

One can also use the *carré du champ* operator which is a bilinear operator and can be defined for $u, v \in D(A^{(2)})$ such that $u \cdot v \in D(A^{(2)})$ by

$$\Gamma(u, v) := A^{(2)}(u \cdot v) - uA^{(2)}v - vA^{(2)}u.$$

The scalar product is then given by

$$\mathcal{E}_1^\psi(u, u) = \|u\|_{L_2(\mathbb{R}^n)}^2 + \|\Gamma(u, u)\|_{L_1(\mathbb{R}^n)}.$$

The carré du champ operator was introduced by J.-P. Roth [66] (see also P.-A. Meyer [60]), and has been thoroughly explored in the monograph [13] of N. Bouleau and F. Hirsch.

The main reason for introducing Dirichlet forms was to give an axiomatic approach to potential theory starting with the notion of energy. Having this fact in mind one should not be surprised that within the framework of Dirichlet spaces many potential-theoretical considerations can be done. Notions like capacities, energy, (equilibrium) potentials, reduced functions, and balayage are best studied. We refer to the classical monograph by M. Fukushima [25] and mention also the books [13], [29], [57] and [71].

Once the function space $H_2^{\psi,1}(\mathbb{R}^n)$ is understood to be a “good” space for potential-theoretic questions it is natural to extend $H_2^{\psi,1}(\mathbb{R}^n)$ to a scale of spaces in order to handle operators derived from $\psi(D)$ in an L_p -context.

Just as in the theory of Sobolev spaces, it is clear that different representations of \mathcal{E}_1^ψ will, in general, lead to different and non-equivalent L_p -norms. For a concrete problem but quite general Dirichlet forms or L_2 -sub-Markovian semigroups, such extensions are discussed in the paper [34] of F. Hirsch. Some related considerations, mainly in infinite dimensions, can be found in the papers by D. Feyel and A. de la Pradelle [21], [23] (see also [20]).

Although we feel that from the mathematical point of view investigating L_p -variants of the spaces $H_2^{\psi,1}(\mathbb{R}^n)$ does not need any justification—it is interesting and non-trivial mathematics in itself—let us point out some important arguments for a systematic approach to these spaces.

Our starting point is formula (2) telling us that continuous negative definite functions are closely related to Lévy processes. As a matter of fact, every reasonable Feller process with state space \mathbb{R}^n is characterized by a family (parametrized by \mathbb{R}^n) of continuous negative definite functions. More precisely, following [47] (see also [70]), we find for the Feller process $((X_t)_{t \geq 0}, P^x)_{x \in \mathbb{R}^n}$ that

$$-q(x, \xi) = \lim_{t \rightarrow 0} \frac{E^x(e^{i(X_t - x) \cdot \xi}) - 1}{t}$$

is the symbol of the generator of the semigroup

$$T_t u(x) = E^x(u(X_t))$$

associated with $((X_t)_{t \geq 0}, P^x)_{x \in \mathbb{R}^n}$, i.e. on $C_0^\infty(\mathbb{R}^n)$ we have

$$(4) \quad Au(x) = -q(x, D)u(x) = -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} q(x, \xi) \widehat{u}(\xi) d\xi.$$

Moreover, $\xi \mapsto q(x, \xi)$ is for each $x \in \mathbb{R}^n$ a continuous negative definite function. Note that this result complements a theorem of P. Courrège [14] which states that on $C_0^\infty(\mathbb{R}^n)$ the generator of a Feller semigroup has necessarily the structure (4).

Now, assuming for example that $q(x, \xi) \sim \psi(\xi)$ where $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is a fixed continuous negative definite function, one should expect that the operator $q(x, D)$ behaves like a perturbation of $\psi(D)$. Hence the scales of spaces associated with ψ should play for $q(x, D)$ the same role as Sobolev or Besov and Triebel–Lizorkin spaces do for elliptic operators in the classical situation, i.e. for operators with symbol $q(x, \xi) \sim |\xi|^{2m}$. We can, however, also ask the converse question: when does an operator given by (4) (defined on $C_0^\infty(\mathbb{R}^n)$) admit an extension to a generator of a Markov process? It is a cornerstone of the modern theory of stochastic processes that with each (regular) Dirichlet form one can associate

a stochastic process. This result is originally due to M. Fukushima [24]. Constructions of stochastic processes starting with $-q(x, D)$ either in the Hilbert space situation (i.e., Dirichlet space case) or in the Feller situation (i.e. in $C_\infty(\mathbb{R}^n)$, the space of continuous functions vanishing at infinity) were first obtained by one of the present authors and subsequently extended in a series of papers by W. Hoh [35]–[38]; see also [39]. We refer as well to more recent (and special) considerations due to F. Baldus [5] and V. Kolokoltsov [56]. In fact, even operators of variable order of differentiability were handled, notably in the papers [37] of W. Hoh, [55] of K. Kikuchi and A. Negoro, [62] of A. Negoro and [51] of H. G. Leopold and one of the present authors. In some of these cases spaces of variable order of differentiation are needed. The interested reader should also consult the survey [52] by G. A. Kaljabin and P. I. Lizorkin on function spaces of generalized smoothness.

From the probabilistic point of view, working with processes associated with Dirichlet spaces has a disadvantage. The process is only defined up to an exceptional set, i.e. a set of capacity zero. It seems that H. Kaneko [53] was the first who proposed to use an L_p -setting to overcome this difficulty. He considered certain L_p -Bessel potential spaces associated with a sub-Markovian semigroup. With such an L_p -Bessel potential space there is always associated a capacity, but for p sufficiently large it might happen that the only exceptional set, i.e. the only set of capacity zero, is the empty set.

In this case no problems occur when constructing the associated Markov process. Thus it is clear that L_p -variants of Dirichlet spaces are of greater interest for probabilistic reasons.

This paper is devoted to the study of Bessel potential spaces associated with a given real-valued continuous negative definite function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$. It can be regarded as the second paper in a series of papers “Function spaces associated with continuous negative definite functions”. As a first part one should consider Section 4.10 in [49] where analogues to the Hörmander spaces $B_{k,p}(\mathbb{R}^n)$ (see L. Hörmander [42]) are discussed. Further work on different scales of function spaces associated with a continuous negative definite function is in preparation.

Before briefly describing the content of this paper let us make a remark about the style. There are at least four groups of colleagues we want to reach with our investigations: specialists in the theory of function spaces, colleagues working on pseudo-differential operators, potential theorists, and probabilists working in stochastic processes. Since this paper is intended to be readily accessible to all of them, we have taken the liberty to include more background material than usual and to give detailed proofs also in situations where we could have referred to an analogue in the literature.

In Sections 1.1 to 1.3 only known background material is collected. We recall basic properties of continuous negative definite functions and convolution semigroups, introduce Bernstein functions, and collect basic facts on one-parameter semigroups. Section 1.4 is devoted to subordination in the sense of S. Bochner and includes some new material related to norm estimates for the perturbed subordinate generator. In Section 1.5 we introduce abstract Bessel potential spaces. They are obtained as the image of L_p -spaces under the Γ -transform of a given L_p -sub-Markovian semigroup $(T_t^{(p)})_{t \geq 0}$. This construction is well known, but it seems that in [19] for the first time systematic use is made

of the fact that the I -transform is a special case of subordination. Again we take this point of view. This allows us to identify the introduced abstract Bessel potential spaces as domains of certain operators, namely $(\text{id} - A^{(p)})^{r/2}$, $r > 0$, $A^{(p)}$ being the generator of $(T_t^{(p)})_{t \geq 0}$, and we show in the general case the equivalence of norms associated with Riesz and Bessel potentials. Our proofs rely mainly on a functional calculus of generators of semigroups related to subordination.

Chapter 2 deals with ψ -Bessel potential spaces, i.e. Bessel potential spaces associated with a fixed continuous negative definite function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$. Of course, we could introduce these spaces in an abstract way along the lines of Section 1.5, but our aim is to identify these spaces with concrete function spaces.

The first and obvious attempt to define these spaces would be to take all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|\mathcal{F}^{-1}[(1 + \psi(\cdot))^{r/2}\widehat{u}] | L_p(\mathbb{R}^n)\| < \infty$$

is finite. The problem is, however, that in general ψ is a continuous but not differentiable function, hence $(1 + \psi(\cdot))^{r/2}\widehat{u}$ is a priori not well defined. We overcome this difficulty in Section 2.1 where we introduce the space $H_p^{\psi,2}(\mathbb{R}^n)$ by making use of the Lévy–Khinchin formula to decompose ψ . This makes it possible to identify

$$H_p^{\psi,2}(\mathbb{R}^n) := \{u \in L_p(\mathbb{R}^n) : \|\mathcal{F}^{-1}[(1 + \psi(\cdot))\widehat{u}] | L_p(\mathbb{R}^n)\| < \infty\}$$

with the domain of definition of the operator $-\psi(D)$ as generator of the L_p -sub-Markovian semigroup $(T_t^{(p)})_{t \geq 0}$ given by (1). This enables us in Section 2.2 to introduce the scale $H_p^{\psi,s}(\mathbb{R}^n)$, $1 < p < \infty$ and $s \in \mathbb{R}$, first by using the functional calculus for the operator $-\psi(D)$ and then by identifying this space with the closure of $\mathcal{S}(\mathbb{R}^n)$ with respect to the norm $\|\mathcal{F}^{-1}[(1 + \psi(\cdot))^{s/2}\widehat{u}] | L_p(\mathbb{R}^n)\|$.

We prove elementary properties of these spaces and give a characterization of the dual space. Note that these scales contain both the classical Bessel potential spaces (we just have to take $\psi(\xi) = |\xi|^2$) and the classical anisotropic Bessel potential spaces associated with the anisotropic distance function $\sqrt{\psi}$ where $\psi(\xi) = |\xi_1|^{2/a_1} + \dots + |\xi_n|^{2/a_n}$ for $a_k \geq 1$, $k = 1, \dots, n$.

However, we want to emphasize that due to our examples of continuous negative definite functions (see Section 1.1, in particular Examples 1.1.15 and 1.1.16), the class under consideration is much larger (even than the classes studied in [52] and in [61]) and contains function spaces not considered so far.

Section 2.3 collects the embedding results and in Section 2.4 we calculate complex interpolation spaces. In Section 2.5 we make use of the fact that the semigroup $(T_t^{(p)})_{t \geq 0}$ and the operators $(\text{id} - A^{(p)})^{-r/2}$, $r > 0$, are positivity preserving. Therefore we can associate a capacity $\text{cap}_{r,p}^\psi$ with each of the spaces $H_p^{\psi,r}(\mathbb{R}^n)$, $r > 0$. This capacity enables us to consider (r, p) -quasi-continuous modifications of elements $u \in H_p^{\psi,r}(\mathbb{R}^n)$, and we show that each $u \in H_p^{\psi,r}(\mathbb{R}^n)$ has a unique quasi-continuous modification (up to (r, p) -quasi-everywhere equality). Further, we obtain comparison results for $\text{cap}_{r_1,p_1}^{\psi_1}$ and $\text{cap}_{r_2,p_2}^{\psi_2}$ based on embedding theorems.

Following H. Triebel [78], [79], for a normed space X we denote by $\|x | X\|$ the norm of the vector $x \in X$. The rest of the notation is standard.

1. Auxiliary results

In this chapter we collect auxiliary material for later purposes. Section 1.1 presents results related to continuous negative definite functions and convolution semigroups, while Section 1.2 gives results on Bernstein functions and convolution semigroups supported in $[0, \infty)$. In Section 1.3 we collect basic facts on one-parameter semigroups. In these sections no new results are contained.

Section 1.4 treats subordination in the sense of Bochner and related functional calculi for generators of semigroups. Starting from Proposition 1.4.9 our results seem to be new.

In Section 1.5 we introduce abstract Bessel potential spaces. The main results in this section are essentially all new.

1.1. Negative definite functions and convolution semigroups. The concept and definition of negative definite functions is due to I. J. Schoenberg [72], who introduced it in connection with isometric embeddings of metric spaces into Hilbert spaces. For \mathbb{R}^n his result may be stated as follows:

Let d be a metric on \mathbb{R}^n . In order that the metric space (\mathbb{R}^n, d) is isometric to a Hilbert space $(\mathbb{R}^n, (\cdot, \cdot))$ it is necessary and sufficient that for all $m \in \mathbb{N}$, all points $\xi^0, \xi^1, \dots, \xi^m \in \mathbb{R}^n$, and all $c_j \in \mathbb{R}$, $1 \leq j \leq m$, the inequality

$$\sum_{k,l=1}^m (d^2(\xi^0, \xi^k) + d^2(\xi^0, \xi^l) - d^2(\xi^k, \xi^l))c_k c_l \geq 0$$

is satisfied. Setting $\psi(\xi - \eta) := d^2(\xi, \eta)$ we find

$$\sum_{k,l=1}^m (\psi(\xi^0 - \xi^k) + \psi(\xi^0 - \xi^l) - \psi(\xi^k - \xi^l))c_k c_l \geq 0.$$

In particular, using the symmetry of d and replacing $\xi^0 - \xi^k$ by ξ^k , we have

$$(1.1.1) \quad \sum_{k,l=1}^m (\psi(\xi^k) + \psi(\xi^l) - \psi(\xi^l - \xi^k))c_k c_l \geq 0.$$

Hence, given an even function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying for all $m \in \mathbb{N}_0$, $\xi^k \in \mathbb{R}^n$, $1 \leq k \leq m$, and $c_k \in \mathbb{R}$, $1 \leq k \leq m$, the inequality (1.1.1), one can expect that $(\xi, \eta) \mapsto \psi^{1/2}(\xi - \eta)$ behaves like a metric.

The following presentation is based on the monograph [7] by C. Berg and G. Forst (see also [49]).

Recall that a function $g : \mathbb{R}^n \rightarrow \mathbb{C}$ is called *positive definite* if for any $k \in \mathbb{N}$ and any vectors $\xi^1, \dots, \xi^k \in \mathbb{R}^n$ the matrix $(g(\xi^j - \xi^l))_{j,l=1,\dots,k}$ is positive Hermitian, that is,

$$\sum_{j,l=1}^k g(\xi^j - \xi^l) z_j \bar{z}_l \geq 0$$

for any numbers $z_1, \dots, z_k \in \mathbb{C}$.

It is not hard to see that the Fourier transform $\hat{\mu}$ of a bounded Borel measure μ on \mathbb{R}^n is a positive definite function.

Bochner's theorem states the converse: given a continuous positive definite function g on \mathbb{R}^n there exists a bounded Borel measure μ with Fourier transform $\widehat{\mu} = g$. Note that for a fixed $x \in \mathbb{R}^n$ the function $\xi \mapsto e^{-ix \cdot \xi}$ is a continuous positive definite function.

We are now able to give the definition of a negative definite function.

DEFINITION 1.1.1. A function $\psi : \mathbb{R}^n \rightarrow \mathbb{C}$ is called *negative definite* if for any $k \in \mathbb{N}$ and all vectors $\xi^1, \dots, \xi^k \in \mathbb{R}^n$ the matrix $(\psi(\xi^j) + \overline{\psi(\xi^l)} - \psi(\xi^j - \xi^l))_{j,l=1,\dots,k}$ is positive Hermitian.

The following result is known as Schoenberg's theorem.

THEOREM 1.1.2. A continuous function $\psi : \mathbb{R}^n \rightarrow \mathbb{C}$ is negative definite if, and only if, $\psi(0) \geq 0$ and for all $t > 0$ the function $\xi \mapsto e^{-t\psi(\xi)}$ is continuous and positive definite.

Let us first state elementary properties of continuous negative definite functions.

Clearly, the set of all continuous negative definite functions is a convex cone which is closed under locally uniform convergence.

LEMMA 1.1.3. (i) If $\psi : \mathbb{R}^n \rightarrow \mathbb{C}$ is a continuous negative definite function then for all $\xi, \eta \in \mathbb{R}^n$,

$$\sqrt{|\psi(\xi + \eta)|} \leq \sqrt{|\psi(\xi)|} + \sqrt{|\psi(\eta)|}$$

and

$$|\psi(\xi) + \psi(\eta) - \psi(\xi \pm \eta)| \leq 2(\operatorname{Re} \psi(\xi))^{1/2} \cdot (\operatorname{Re} \psi(\eta))^{1/2}.$$

(ii) If $\psi : \mathbb{R}^n \rightarrow \mathbb{C}$ is a continuous negative definite function then

$$|\psi(\xi)| \leq c_\psi(1 + |\xi|^2) \quad \text{for all } \xi \in \mathbb{R}^n,$$

where $c_\psi = 2 \sup_{|\eta| \leq 1} |\psi(\eta)|$.

The next results indicate the closeness of continuous negative definite functions to metrics, in fact to slowly varying metrics in the sense of L. Hörmander [43].

LEMMA 1.1.4. Let $\psi : \mathbb{R}^n \rightarrow \mathbb{C}$ be a continuous negative definite function. Then

$$(1.1.2) \quad \frac{1 + |\psi(\xi)|}{1 + |\psi(\eta)|} \leq 2(1 + |\psi(\xi - \eta)|) \quad \text{for any } \xi, \eta \in \mathbb{R}^n.$$

Moreover, we have

$$(1.1.3) \quad 1 + |\psi(\xi \pm \eta)| \leq (1 + |\psi(\xi)|)(1 + \sqrt{|\psi(\eta)|})^2 \quad \text{for any } \xi, \eta \in \mathbb{R}^n.$$

The validity of (1.1.2) was first noted by W. Hoh [35], whereas the inequality (1.1.3) is due to R. L. Schilling (see also [49]).

Every continuous negative definite function admits a Lévy–Khinchin representation:

THEOREM 1.1.5. If $\psi : \mathbb{R}^n \rightarrow \mathbb{C}$ is a continuous negative definite function then there exist: a constant $c > 0$, a vector $d \in \mathbb{R}^n$, a symmetric positive semidefinite quadratic form q on \mathbb{R}^n and a finite measure μ on $\mathbb{R}^n \setminus \{0\}$ such that

$$(1.1.4) \quad \psi(\xi) = c + i(d \cdot \xi) + q(\xi) + \int_{\mathbb{R}^n \setminus \{0\}} \left(1 - e^{-ix \cdot \xi} - \frac{ix \cdot \xi}{1 + |x|^2} \right) \frac{1 + |x|^2}{|x|^2} \mu(dx).$$

The quadruple (c, d, q, μ) is uniquely determined by ψ .

Conversely, given (c, d, q, μ) as above, the function ψ defined by (1.1.4) is continuous and negative definite.

Let us state the Lévy–Khinchin formula for real-valued continuous negative definite functions explicitly.

COROLLARY 1.1.6. *Let $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a real-valued continuous negative definite function. Then we have the representation*

$$(1.1.5) \quad \psi(\xi) = c + q(\xi) + \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos(x \cdot \xi)) \frac{1 + |x|^2}{|x|^2} \mu(dx)$$

with c , q , and μ as in Theorem 1.1.5. In addition, μ is a symmetric measure.

Instead of the measure μ it is often convenient to use the Lévy measure associated with ψ , i.e. the measure

$$\nu(dx) := \frac{1 + |x|^2}{|x|^2} \mu(dx).$$

Thus ν is a Radon measure on $\mathbb{R}^n \setminus \{0\}$ satisfying the integrability condition

$$\int_{\mathbb{R}^n \setminus \{0\}} (|x|^2 \wedge 1) \nu(dx) < \infty.$$

The following result is due to W. Hoh [36, Proposition 2.1] and it relates the smoothness of ψ to integrability properties of ν .

THEOREM 1.1.7. *Let $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous negative definite function with Lévy–Khinchin representation (1.1.5). Suppose that for $2 \leq l \leq m$ all absolute moments of the Lévy measure ν exist, i.e.*

$$M_l := \int_{\mathbb{R}^n \setminus \{0\}} |x|^l \nu(dx) < \infty, \quad 2 \leq l \leq m.$$

Then ψ is of class $C^m(\mathbb{R}^n)$ and for $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq m$, we have the estimate

$$(1.1.6) \quad |\partial^\alpha \psi(\xi)| \leq c_{|\alpha|} \cdot \begin{cases} \psi(\xi), & |\alpha| = 0, \\ \psi^{1/2}(\xi), & |\alpha| = 1, \\ 1, & |\alpha| \geq 2, \end{cases}$$

with $c_0 = 1$, $c_1 = (2M_2)^{1/2} + 2\lambda^{1/2}$, $c_2 = M_2 + 2\lambda$ and $c_l = M_l$ for $3 \leq l \leq m$, where λ is the maximal eigenvalue of the quadratic form q in (1.1.5).

For our purposes it is more convenient to restrict ourselves to continuous negative definite functions $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ of the form

$$(1.1.7) \quad \psi(\xi) = \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos(x \cdot \xi)) \nu(dx),$$

where ν is a Lévy measure on $\mathbb{R}^n \setminus \{0\}$.

COROLLARY 1.1.8. *Suppose that the Lévy measure ν associated with the continuous negative definite function ψ from (1.1.7) has its support in a bounded set, i.e. $\text{supp } \nu \subset \overline{B(0, R)}$ for some $R > 0$. Then the function ψ is infinitely differentiable and the function itself, as well as all its partial derivatives, are of at most quadratic growth.*

Moreover, we have the obvious

LEMMA 1.1.9. *Let ψ be a continuous negative definite function given by (1.1.7). If the support of the Lévy measure ν satisfies $\text{supp } \nu \subset B^c(0, R)$ for some $R > 0$, then ψ is a bounded continuous function.*

From Corollary 1.1.8 and Lemma 1.1.9 we find that every continuous negative definite function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ with representation (1.1.7) has a decomposition $\psi = \psi_R + \tilde{\psi}_R$, $R > 0$, into continuous negative definite functions ψ_R and $\tilde{\psi}_R$ such that ψ_R is infinitely often differentiable and ψ_R as well as its partial derivatives are at most of quadratic growth, and $\tilde{\psi}_R$ is bounded and continuous. In fact, we just have to define

$$(1.1.8) \quad \psi_R(\xi) = \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos(x \cdot \xi)) \chi_{B(0, R)}(x) \nu(dx),$$

$$(1.1.9) \quad \tilde{\psi}_R(\xi) = \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos(x \cdot \xi)) \chi_{B^c(0, R)}(x) \nu(dx).$$

For the following it is important to note that there is a one-to-one correspondence between convolution semigroups of sub-probability measures on \mathbb{R}^n and continuous negative definite functions.

DEFINITION 1.1.10. A family $(\mu_t)_{t \geq 0}$ of sub-probability measures on \mathbb{R}^n is called a *convolution semigroup* if the following conditions are satisfied:

- (i) $\mu_t * \mu_s = \mu_{t+s}$ for any $s, t > 0$ and $\mu_0 = \varepsilon_0$ (Dirac measure);
- (ii) $\mu_t \rightarrow \varepsilon_0$ vaguely for $t \rightarrow 0$.

Note that some authors do not require the normalization $\mu_0 = \varepsilon_0$.

THEOREM 1.1.11. *For every convolution semigroup $(\mu_t)_{t \geq 0}$ of sub-probability measures on \mathbb{R}^n there exists a unique continuous negative definite function $\psi : \mathbb{R}^n \rightarrow \mathbb{C}$ such that*

$$(1.1.10) \quad \widehat{\mu}_t(\xi) = (2\pi)^{-n/2} e^{-t\psi(\xi)}, \quad t \geq 0 \text{ and } \xi \in \mathbb{R}^n.$$

Conversely, given a continuous negative definite function $\psi : \mathbb{R}^n \rightarrow \mathbb{C}$ there exists a unique convolution semigroup $(\mu_t)_{t \geq 0}$ on \mathbb{R}^n such that (1.1.10) holds.

EXAMPLE 1.1.12. For any $\alpha, \beta \in (0, 1]$ the functions $\xi \mapsto |\xi|^{2\alpha}$ and $\xi \mapsto |\xi|^{2\alpha} + |\xi|^{2\beta}$, $\xi \in \mathbb{R}^n$, are continuous and negative definite.

EXAMPLE 1.1.13. Let a_1, \dots, a_n be real numbers such that $a_k \geq 1$ for $k = 1, \dots, n$. The function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$(1.1.11) \quad \psi(\xi) = |\xi_1|^{2/a_1} + \dots + |\xi_n|^{2/a_n}$$

is continuous and negative definite. This is a simple consequence of the previous example and elementary properties of continuous negative functions.

The function $\sqrt{\psi}$, where ψ is given by (1.1.11), is a so-called *anisotropic distance function*; see for example H.-J. Schmeisser and H. Triebel [73, Subsection 4.2.1] and M. Yamazaki [83]. We will return to this example in Remark 2.2.12.

EXAMPLE 1.1.14. From the Lévy–Khinchin formula we deduce immediately that

- (i) any symmetric positive semidefinite quadratic form q on \mathbb{R}^n ,
- (ii) any function of the form $\xi \mapsto i\ell \cdot \xi$, $\ell \in \mathbb{R}^n$,

- (iii) the functions $\xi \mapsto 1 - e^{-ih \cdot \xi}$ and $\xi \mapsto 1 - \cos(h \cdot \xi)$ with $h \in \mathbb{R}^n$,
 (iv) any combination of (i)–(iii)

are continuous and negative definite functions.

EXAMPLE 1.1.15. Fix any $\lambda \in (0, 2)$ and choose $M = M(\lambda) \in \mathbb{N}$ such that $M > 2/(2-\lambda)$. Then the measure

$$\nu(dx) := \sum_{j=1}^{\infty} 2^{\lambda M^j - j} \varepsilon_{2^{-M^j}}(dx)$$

is easily seen to be a Lévy measure. Therefore the function $\psi : \mathbb{R} \rightarrow \mathbb{R}$,

$$(1.1.12) \quad \psi(\xi) := \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos(x\xi)) \nu(dx) = \sum_{j=1}^{\infty} 2^{\lambda M^j - j} (1 - \cos(2^{-M^j} \xi)),$$

is continuous and negative definite. It enjoys the following properties:

$$(1.1.13) \quad \liminf_{|\xi| \rightarrow \infty} \psi(\xi) = 0;$$

$$(1.1.14) \quad \limsup_{|\xi| \rightarrow \infty} \frac{\psi(\xi)}{|\xi|^{\lambda - \varrho}} = \infty \quad \text{for } \varrho > 0;$$

$$(1.1.15) \quad \lim_{|\xi| \rightarrow \infty} \frac{\psi(\xi)}{|\xi|^{\lambda + \varrho}} = 0 \quad \text{for } \varrho > 0.$$

Proof. We will need the following elementary inequalities:

$$(1.1.16) \quad \frac{t^2}{2} \left(1 - \frac{t^2}{12}\right)^+ \leq 1 - \cos(t) \leq \frac{t^2}{2}$$

and

$$(1.1.17) \quad 1 - \cos(t) \leq 2^{1-\lambda} |t|^\lambda.$$

Consider the sequence $\{2\pi 2^{M^k}\}_{k \in \mathbb{N}}$. Clearly, $\lim_{k \rightarrow \infty} 2\pi 2^{M^k} = \infty$ and $2^{M^k} 2^{-M^j} \in \mathbb{N}$ for $j \leq k$. Thus, using (1.1.16) and the fact that $\lambda < 2$, we get

$$\begin{aligned} \psi(2\pi 2^{M^k}) &= \sum_{j=1}^{\infty} 2^{\lambda M^j - j} (1 - \cos(2\pi 2^{M^k} 2^{-M^j})) \\ &= \sum_{j>k} 2^{\lambda M^j - j} (1 - \cos(2\pi 2^{M^k} 2^{-M^j})) \leq 2\pi^2 \sum_{j>k} 2^{(\lambda-2)M^j - j} 2^{2M^k} \\ &\leq 2\pi^2 \sum_{j>k} 2^{(\lambda-2)M^{k+1} - j} 2^{2M^k} = 2\pi^2 2^{(M\lambda - 2M + 2)M^k} \sum_{j>k} 2^{-j}. \end{aligned}$$

By the choice of $M = M(\lambda)$ we have $M\lambda - 2M + 2 \leq 0$, thus $\lim_{k \rightarrow \infty} \psi(2\pi 2^{M^k}) = 0$, and (1.1.13) follows.

Using (1.1.17) we find for any $\xi \in \mathbb{R}^n$ that

$$\psi(\xi) = \sum_{j=1}^{\infty} 2^{\lambda M^j - j} (1 - \cos(2^{-M^j} \xi)) \leq 2^{1-\lambda} \sum_{j=1}^{\infty} 2^{\lambda M^j - j} 2^{-\lambda M^j} |\xi|^\lambda = 2^{1-\lambda} |\xi|^\lambda,$$

which proves (1.1.15). For $\varrho > 0$, $\xi \in \mathbb{R}^n$ and any $k \in \mathbb{N}$ we have, by (1.1.16),

$$\begin{aligned} \frac{\psi(\xi)}{|\xi|^{\lambda-\varrho}} &= |\xi|^{-\lambda+\varrho} \sum_{j=1}^{\infty} 2^{\lambda M^j - j} (1 - \cos(2^{-M^j} \xi)) \\ &\geq |\xi|^{-\lambda+\varrho} 2^{\lambda M^k - k} (1 - \cos(2^{-M^k} \xi)) \geq \frac{1}{2} |\xi|^{2-\lambda+\varrho} 2^{(\lambda-2)M^k - k} \left(1 - \frac{\xi^2}{12} 2^{-2M^k}\right). \end{aligned}$$

Setting $\xi_k := 2^{M^k}$ yields

$$\frac{\psi(\xi_k)}{|\xi_k|^{\lambda-\varrho}} \geq \frac{1}{2} 2^{\varrho M^k - k} \left(1 - \frac{1}{12}\right) \rightarrow \infty \quad \text{as } k \rightarrow \infty,$$

and (1.1.14) follows. ■

EXAMPLE 1.1.16. Pick $0 < \kappa < \lambda < 2$ and denote by $\psi_\lambda(\xi)$ the function constructed in Example 1.1.15. Then

$$(1.1.18) \quad \psi(\xi) := \psi_\lambda(\xi) + |\xi|^\kappa$$

is a continuous negative function that oscillates for $|\xi| \rightarrow \infty$ between the curves $\xi \mapsto |\xi|^\kappa$ and $\xi \mapsto 2|\xi|^\lambda$. Moreover, $\psi(\xi) = \mathcal{O}(|\xi|^\lambda)$ as $|\xi| \rightarrow \infty$.

EXAMPLE 1.1.17. Further examples of continuous negative definite functions with prescribed asymptotic behaviour can be constructed along the lines of Lemma 3.4 in [40].

1.2. Bernstein functions and convolution semigroups supported in $[0, \infty)$. Convolution semigroups of measures $(\eta_t)_{t \geq 0}$ supported in $[0, \infty)$, i.e. with $\text{supp } \eta_t \subset [0, \infty)$, are of particular interest. It turns out that they are better described by their (one-sided) Laplace transforms $\mathcal{L}(\eta_t)$ than by their Fourier transforms.

We need some preparation. Again we refer to the monograph [7] of C. Berg and G. Forst as a standard reference; see also again [49].

DEFINITION 1.2.1. (i) An infinitely often differentiable function $f : (0, \infty) \rightarrow \mathbb{R}$ is said to be *completely monotone* if

$$(-1)^k f^{(k)} \geq 0 \quad \text{for all } k \in \mathbb{N}.$$

(ii) An infinitely often differentiable function $f : (0, \infty) \rightarrow \mathbb{R}$ with continuous extension to $[0, \infty)$ is called a *Bernstein function* if

$$f \geq 0 \quad \text{and} \quad (-1)^k f^{(k)} \leq 0 \quad \text{for all } k \in \mathbb{N}.$$

These two classes of functions are closely related.

THEOREM 1.2.2. *For a function $f : (0, \infty) \rightarrow \mathbb{R}$ the following assertions are equivalent:*

- (i) *f is a Bernstein function;*
- (ii) *$f \geq 0$ and for all $t > 0$ the function $x \mapsto e^{-tf(x)}$ is completely monotone.*

Bernstein functions have a representation formula which is analogous to the Lévy–Khinchin formula.

THEOREM 1.2.3. *Let f be a Bernstein function. Then there exist constants $a, b \geq 0$ and a measure μ on $(0, \infty)$ satisfying*

$$(1.2.1) \quad \int_{0+}^{\infty} \frac{s}{1+s} \mu(ds) < \infty$$

such that

$$(1.2.2) \quad f(x) = a + bx + \int_{0+}^{\infty} (1 - e^{-xs}) \mu(ds), \quad x > 0.$$

The triple (a, b, μ) is uniquely determined by f .

Conversely, if $a, b \geq 0$ and μ is a measure satisfying (1.2.1), then (1.2.2) defines a Bernstein function.

There is a one-to-one correspondence between Bernstein functions and convolution semigroups on $[0, \infty)$.

THEOREM 1.2.4. *Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a Bernstein function. Then there exists a unique convolution semigroup $(\eta_t)_{t \geq 0}$ supported in $[0, \infty)$ such that*

$$(1.2.3) \quad \mathcal{L}(\eta_t)(x) = \int_{0-}^{\infty} e^{-sx} \eta_t(ds) = e^{-tf(x)}, \quad x > 0 \quad \text{and} \quad t > 0.$$

Conversely, for any convolution semigroup $(\eta_t)_{t \geq 0}$ supported in $[0, \infty)$ there exists a unique Bernstein function f such that (1.2.3) holds.

It is not difficult to see that every Bernstein function f extends to the half-plane $\operatorname{Re} z \geq 0$. From this we may deduce one of the most important properties of Bernstein functions: they operate on negative definite functions.

PROPOSITION 1.2.5. *For any Bernstein function f and any continuous negative definite function $\psi : \mathbb{R}^n \rightarrow \mathbb{C}$ the function $f \circ \psi$ is again continuous and negative definite.*

Now let ψ and f be as in Proposition 1.2.5. Since $f \circ \psi$ is a continuous negative definite function on \mathbb{R}^n , there exists a convolution semigroup $(\mu_t^f)_{t \geq 0}$ associated with $f \circ \psi$.

THEOREM 1.2.6. *Let $\psi : \mathbb{R}^n \rightarrow \mathbb{C}$ be a continuous negative definite function with associated convolution semigroup $(\mu_t)_{t \geq 0}$ on \mathbb{R}^n . Let f be a Bernstein function with associated semigroup $(\eta_t)_{t \geq 0}$ supported in $[0, \infty)$. The convolution semigroup $(\mu_t^f)_{t \geq 0}$ on \mathbb{R}^n associated with the continuous negative definite function $f \circ \psi$ is given by*

$$(1.2.4) \quad \int_{\mathbb{R}^n} \varphi(x) \mu_t^f(dx) = \int_{0-}^{\infty} \int_{\mathbb{R}^n} \varphi(x) \mu_s(dx) \eta_t(ds), \quad \varphi \in C_0^\infty(\mathbb{R}^n).$$

REMARK 1.2.7. Instead of (1.2.4) we will sometimes write

$$(1.2.5) \quad \mu_t^f = \int_{0-}^{\infty} \mu_s \eta_t(ds) \quad \text{vaguely.}$$

DEFINITION 1.2.8. In the situation of Theorem 1.2.6 we call the convolution semigroup $(\mu_t^f)_{t \geq 0}$ the semigroup *subordinate* (in the sense of Bochner) to $(\mu_t)_{t \geq 0}$ (with respect to $(\eta_t)_{t \geq 0}$). The convolution semigroup $(\eta_t)_{t \geq 0}$ is sometimes called a *subordinator*.

EXAMPLE 1.2.9. The function $x \mapsto a$, $a \geq 0$, is a Bernstein function, as is the function $x \mapsto bx$, $b \geq 0$. The associated semigroups are $(e^{-at}\varepsilon_0)_{t \geq 0}$ and $(\varepsilon_{bt})_{t \geq 0}$, respectively.

EXAMPLE 1.2.10. If $s \geq 0$ the function $f(x) = 1 - e^{-xs}$ is a Bernstein function. It corresponds to the *Poisson semigroup* with jumps of size s , i.e.

$$\eta_t = \sum_{k=0}^{\infty} e^{-t} \frac{t^k}{k!} \varepsilon_{sk}, \quad t \geq 0.$$

EXAMPLE 1.2.11. The function $f(x) = \log(1+x)$ is a Bernstein function. Note that

$$\log(1+x) = \int_0^{\infty} (1 - e^{-xs}) s^{-1} e^{-s} ds, \quad x > 0.$$

The semigroup associated with this Bernstein function is called the Γ -semigroup; it is given by $\eta_t = g_t(\cdot)\lambda^{(1)}$ where

$$g_t(x) = \chi_{(0,\infty)}(x) \frac{1}{\Gamma(t)} x^{t-1} e^{-x}.$$

Clearly, $x \mapsto \frac{1}{2} \log(1+x)$ is also a Bernstein function with corresponding convolution semigroup

$$\eta_t(ds) = \chi_{(0,\infty)}(s) \frac{1}{\Gamma(t/2)} s^{t/2-1} e^{-s} \lambda^{(1)}(ds).$$

We call this semigroup the *modified Γ -semigroup*. It will become of great importance later on.

EXAMPLE 1.2.12. For $\alpha \in [0, 1]$ the function $f_\alpha(x) = x^\alpha$ is a Bernstein function. For $\alpha = 0$ or $\alpha = 1$ this is obvious, for $\alpha \in (0, 1)$ we note

$$x^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} (1 - e^{-xs}) s^{-\alpha-1} ds, \quad x \geq 0.$$

The corresponding semigroup is called the *one-sided stable semigroup of order α* and is denoted by $(\sigma_t^\alpha)_{t \geq 0}$. Only for $\alpha = 1/2$ a closed expression for σ_t^α is known:

$$\sigma_t^{1/2}(ds) = \chi_{(0,\infty)}(s) \frac{1}{\sqrt{4\pi}} t s^{-3/2} e^{-t^2/(2s)} \lambda^{(1)}(ds).$$

Often we will work with a subclass of Bernstein functions.

DEFINITION 1.2.13. A function $f : (0, \infty) \rightarrow \mathbb{R}$ is called a *complete Bernstein function* if there exists a Bernstein function g such that

$$f(x) = x^2 \mathcal{L}(g)(x).$$

THEOREM 1.2.14. *A complete Bernstein function is itself a Bernstein function. Moreover, the following assertions are equivalent:*

- (i) f is a complete Bernstein function.
- (ii) f is a Bernstein function having the representation

$$f(x) = a + bx + \int_{0+}^{\infty} (1 - e^{-sx}) \mu(ds), \quad x > 0,$$

where a and b are non-negative constants and the measure μ is given by

$$\mu(ds) = m(s)\lambda^{(1)}(ds), \quad m(s) = \int_{0+}^{\infty} e^{-ts} \tau(dt), \quad s > 0,$$

where τ is a measure on $(0, \infty)$ satisfying

$$\int_{0+}^{\infty} \frac{1}{t(t+1)} \tau(dt) < \infty.$$

(iii) f has the representation

$$f(x) = a + bx + \int_{0+}^{\infty} \frac{x}{\lambda + x} \varrho(d\lambda)$$

with a measure ϱ satisfying

$$\int_{0+}^{\infty} \frac{1}{1 + \lambda} \varrho(d\lambda) < \infty.$$

(iv) f has the representation

$$f(x) = a + bx + \int_{0+}^{\infty} \frac{x}{1 + tx} \tilde{\mu}(dt)$$

with a measure $\tilde{\mu}$ on $(0, \infty)$ satisfying

$$\int_{0+}^{\infty} \frac{1}{1 + t} \tilde{\mu}(dt) < \infty.$$

REMARK 1.2.15. There are several additional characterizations for complete Bernstein functions (see [67], [69] or [49]). The fourth assertion in Theorem 1.2.14 is taken from F. Hirsch [33].

EXAMPLE 1.2.16. The following functions are complete Bernstein functions:

$$\begin{aligned} s^\alpha &= \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty \frac{s}{s+r} r^{\alpha-1} dr, \quad 0 < \alpha < 1; \\ \frac{s}{s+\lambda} &= \int_0^\infty \frac{s}{s+r} \varepsilon_\lambda(dr), \quad \lambda > 0; \\ \log(1+s) &= \int_0^\infty \frac{s}{s+r} \chi_{[1,\infty)}(r) \frac{dr}{r}. \end{aligned}$$

1.3. One-parameter operator semigroups. Let $(\mu_t)_{t \geq 0}$ be a convolution semigroup of sub-probability measures on \mathbb{R}^n . For $t \geq 0$ we define the operator

$$T_t u(x) = \int_{\mathbb{R}^n} u(x-y) \mu_t(dy) = \mu_t * u(x).$$

Obviously, T_t is defined for all $u \in \mathcal{S}(\mathbb{R}^n)$. We find, using the convolution theorem,

$$(1.3.1) \quad (T_t u)^\wedge(\xi) = (2\pi)^{n/2} \hat{u}(\xi) \hat{\mu}_t(\xi) = \hat{u}(\xi) e^{-t\psi(\xi)},$$

where $\psi : \mathbb{R}^n \rightarrow \mathbb{C}$ is the continuous negative definite function associated with $(\mu_t)_{t \geq 0}$. It is easy to prove that T_t extends from $\mathcal{S}(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$, $1 < p < \infty$, as well as to $C_\infty(\mathbb{R}^n)$. These extensions will also be denoted by T_t . For the moment we will write $(X, \|\cdot\|_X)$ for any of the above Banach spaces.

The (extended) operators T_t , $t \geq 0$, have the following properties on $(X, \|\cdot\|_X)$:

- (i) $T_{t+s} = T_t \circ T_s$ and $T_0 = \text{id}$;
- (ii) $\lim_{t \rightarrow 0} \|T_t u - u\|_X = 0$;
- (iii) $\|T_t u\|_X \leq \|u\|_X$.

Furthermore, we have in the case of the spaces $L_p(\mathbb{R}^n)$,

$$(1.3.2) \quad 0 \leq u \leq 1 \quad \text{a.e. implies} \quad 0 \leq T_t u \leq 1 \quad \text{a.e.,}$$

and in the context of $C_\infty(\mathbb{R}^n)$,

$$(1.3.3) \quad 0 \leq u \leq 1 \quad \text{implies} \quad 0 \leq T_t u \leq 1.$$

DEFINITION 1.3.1. A family of linear operators $(T_t)_{t \geq 0}$ on a Banach space $(X, \|\cdot\|_X)$ is called a *strongly continuous contraction semigroup* if the conditions (i)–(iii) are satisfied.

Since we are only considering either convolution semigroups of sub-probability measures or strongly continuous contraction semigroups we will sometimes write just “semigroups” for short.

DEFINITION 1.3.2. (i) A strongly continuous contraction semigroup on $L_p(\mathbb{R}^n)$, $1 \leq p < \infty$, is called an *L_p -sub-Markovian semigroup* if (1.3.2) is satisfied.

(ii) A strongly continuous contraction semigroup on $C_\infty(\mathbb{R}^n)$ satisfying (1.3.3) is called a *Feller semigroup*.

From our introductory considerations we conclude that a family $(T_t)_{t \geq 0}$ of operators defined on a small space can be extended to different Banach spaces as a strongly continuous contraction semigroup. In particular, extensions to the spaces $L_p(\mathbb{R}^n)$, $1 \leq p < \infty$, will be denoted by $(T_t^{(p)})_{t \geq 0}$ and the extension to $C_\infty(\mathbb{R}^n)$ will be denoted by $(T_t^{(\infty)})_{t \geq 0}$.

Most important is the definition of the generator of a semigroup.

DEFINITION 1.3.3. Let $(T_t)_{t \geq 0}$ be strongly continuous contraction semigroup on a Banach space $(X, \|\cdot\|_X)$. Its (*infinitesimal*) *generator* is the operator

$$Au := \lim_{t \rightarrow 0} \frac{T_t u - u}{t} \quad (\text{strong limit})$$

with domain

$$D(A) := \left\{ u \in X : \lim_{t \rightarrow 0} \frac{T_t u - u}{t} \text{ exists strongly in } X \right\}.$$

The generator is always a densely defined closed operator which is *dissipative*, i.e. the inequality

$$\lambda \|u\|_X \leq \|(\lambda - A)u\|_X$$

is satisfied for all $\lambda > 0$ and all $u \in D(A)$.

A major problem is to determine the domain $D(A)$ of A . In particular, for L_p -sub-Markovian semigroups it is interesting to characterize $D(A)$ in terms of function spaces.

It is also possible to define the *weak generator* of $(T_t)_{t \geq 0}$ by

$$A_w u := \text{w-}\lim_{t \rightarrow 0} \frac{T_t u - u}{t} \quad (\text{weak limit})$$

with domain

$$D(A_w) := \left\{ u \in X : \text{w-}\lim_{t \rightarrow 0} \frac{T_t u - u}{t} \text{ exists} \right\}.$$

However, this does not lead to a new object as the following theorem shows (see A. Pazy [64, Theorem 1.3, p. 43]).

THEOREM 1.3.4. *Let $(T_t)_{t \geq 0}$ be a strongly continuous contraction semigroup on a Banach space $(X, \|\cdot\|)$. Then its weak generator coincides with its strong generator.*

A strongly continuous semigroup $(T_t)_{t \geq 0}$ on $L_2(\mathbb{R}^n)$ is called *symmetric* if

$$(T_t u, v)_{L_2(\mathbb{R}^n)} = (u, T_t v)_{L_2(\mathbb{R}^n)} \quad \text{for all } u, v \in L_2(\mathbb{R}^n).$$

THEOREM 1.3.5. *Let $(T_t^{(2)})_{t \geq 0}$ be a symmetric sub-Markovian semigroup on $L_2(\mathbb{R}^n)$. Then it extends from $L_2(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$ to a sub-Markovian semigroup $(T_t^{(p)})_{t \geq 0}$ on $L_p(\mathbb{R}^n)$, $p \in [1, \infty)$.*

For a proof of this result see for example E. B. Davies [16].

A semigroup is called *analytic* if $t \mapsto T_t u$ admits an analytic extension $z \mapsto T_z u$ to some sector $S_{\theta, d_0} := \{z \in \mathbb{C} : \arg(z - d_0) < \theta\}$. A result of E. M. Stein [75] says that in the case of a symmetric sub-Markovian semigroup $(T_t^{(2)})_{t \geq 0}$ on $L_2(\mathbb{R}^n)$ this semigroup as well as its extensions to $L_p(\mathbb{R}^n)$, $1 < p < \infty$, are analytic. For every analytic semigroup $(T_t)_{t \geq 0}$ on a Banach space the following regularization result holds:

$$T_t u \in \bigcap_{k \geq 0} D(A^k), \quad u \in X.$$

Here $D(A^k)$ is the domain of the k th power of the generator of $(A, D(A))$.

We want to characterize the generators of Feller semigroups and L_p -sub-Markovian semigroups.

First we recall a version of the classical Hille–Yosida theorem (see S. Ethier and T. Kurtz [18, p. 16]).

THEOREM 1.3.6. *A linear operator on a Banach space $(X, \|\cdot\|)$ is closable and its closure \bar{A} is the generator of a strongly continuous contraction semigroup on X if, and only if, the following three conditions are satisfied:*

- (i) $D(A) \subset X$ is dense;
- (ii) A is dissipative;
- (iii) for some $\lambda > 0$ the range $R(\lambda - A)$ of $\lambda - A$ is dense in X .

For Feller semigroups we have the following characterization (see [18, p. 165]).

THEOREM 1.3.7. *A linear operator $(A^{(\infty)}, D(A^{(\infty)}))$ on $C_\infty(\mathbb{R}^n)$ is closable and its closure is the generator of a Feller semigroup if, and only if, the following three conditions are satisfied:*

- (i) $D(A^{(\infty)}) \subset C_\infty(\mathbb{R}^n)$ is dense;

(ii) $A^{(\infty)}$ satisfies the positive maximum principle, i.e.

$$u(x_0) = \sup_{x \in \mathbb{R}^n} u(x) \geq 0 \quad \text{implies} \quad A^{(\infty)}u(x_0) \leq 0;$$

(iii) for some $\lambda > 0$ the range $R(\lambda - A^{(\infty)})$ of $\lambda - A^{(\infty)}$ is dense in $C_\infty(\mathbb{R}^n)$.

REMARK 1.3.8. Theorem 1.3.7 is often called the Hille–Yosida–Ray theorem.

DEFINITION 1.3.9. A closed linear operator $(A^{(p)}, D(A^{(p)}))$ on $L_p(\mathbb{R}^n)$, $1 < p < \infty$, is called an L_p -Dirichlet operator if

$$\int_{\mathbb{R}^n} A^{(p)}u((u-1)^+)^{p-1} dx \leq 0$$

for all $u \in D(A^{(p)})$.

REMARK 1.3.10. In the case of a self-adjoint operator on $L_2(\mathbb{R}^n)$ the notion of Dirichlet operator was introduced by N. Bouleau and F. Hirsch [12] and for non-symmetric operators in $L_2(\mathbb{R}^n)$ it is due to Z.-M. Ma and M. Röckner [57]. The L_p -analogue was introduced by the second named author in [48] (see also [49]).

THEOREM 1.3.11. Let $(A^{(p)}, D(A^{(p)}))$ be an L_p -Dirichlet operator which is the generator of a strongly continuous contraction semigroup $(T_t^{(p)})_{t \geq 0}$ on $L_p(\mathbb{R}^n)$. Then $(T_t^{(p)})_{t \geq 0}$ is sub-Markovian.

Conversely, if $(A^{(p)}, D(A^{(p)}))$ is the generator of a sub-Markovian semigroup $(T_t^{(p)})_{t \geq 0}$ on $L_p(\mathbb{R}^n)$ then $(A^{(p)}, D(A^{(p)}))$ is an L_p -Dirichlet operator.

The resolvent $(R_\lambda^A)_{\lambda > 0}$ of a generator $(A, D(A))$ of a strongly continuous contraction semigroup $(T_t)_{t \geq 0}$ on a Banach space $(X, \|\cdot\|)$ is defined by

$$R_\lambda^A u = (\lambda - A)^{-1}u,$$

and we have the representation

$$R_\lambda^A u = \int_0^\infty e^{-\lambda t} T_t u dt.$$

1.4. Subordination in the sense of Bochner and a functional calculus for generators. We have already used in Section 1.2, Proposition 1.2.5–Remark 1.2.7, Bernstein functions in order to obtain new negative definite functions, and thus new convolution semigroups, from a given one.

Let us now take formula (1.2.5) as our starting point in order to treat subordination of contraction semigroups.

Denote by $(X, \|\cdot\|)$ some Banach space of functions, which will be in later sections $L_p(\mathbb{R}^n)$, $1 \leq p < \infty$, or $C_\infty(\mathbb{R}^n)$, and let $(T_t)_{t \geq 0}$ be a strongly continuous contraction semigroup on X with generator $(A, D(A))$. We denote by $(\eta_t)_{t \geq 0}$ a convolution semigroup of sub-probability measures supported in $[0, \infty)$ which is associated with the Bernstein function f of the form (1.2.2) (see Section 1.2). Define

$$(1.4.1) \quad T_t^f u := \int_0^\infty T_s u \eta_t(ds), \quad t \geq 0, u \in X,$$

where the right-hand side is given by a Bochner integral. Since $\|T_t u\|_X \leq \|u\|_X$ and since η_t is a sub-probability measure, (1.4.1) is well defined. Moreover, it is not hard to see that $(T_t^f)_{t \geq 0}$ is a strongly continuous contraction semigroup and that T_t^f is again a sub-Markovian or Feller operator whenever $(T_t)_{t \geq 0}$ is.

DEFINITION 1.4.1. Let $(T_t)_{t \geq 0}$ be a strongly continuous contraction semigroup on the Banach space $(X, \|\cdot\|, X)$ and let $(\eta_t)_{t \geq 0}$ be a subordinator with associated Bernstein function f . Then the semigroup $(T_t^f)_{t \geq 0}$ given by (1.4.1) is called *subordinate to $(T_t)_{t \geq 0}$* (with respect to the Bernstein function f). Its generator will be denoted by $(A^f, D(A^f))$.

The notion of subordination essentially goes back to S. Bochner (see [10] and [11]).

The next theorem describes the structure of A^f . It is due to R. S. Phillips [65].

THEOREM 1.4.2. Let $(T_t)_{t \geq 0}$ be a strongly continuous contraction semigroup on X and $(T_t^f)_{t \geq 0}$ be the semigroup obtained by subordination with respect to the Bernstein function

$$f(x) = a + bx + \int_{0+}^{\infty} (1 - e^{-xs}) \mu(ds).$$

Then

$$(1.4.2) \quad A^f u = -au + bAu + \int_{0+}^{\infty} (T_s u - u) \mu(ds), \quad u \in D(A),$$

and, for each $k \in \mathbb{N}$, the set $D(A^k)$ is an operator core for $(A^f, D(A^f))$, i.e., $\overline{A^f | D(A^k)} = A^f$ in the sense of closed operators.

Little is known, in general, about $D(A^f)$. Clearly, $D(A^f) = X$ if either A or f is bounded (in this case A^f is bounded). Apart from this trivial case,

$$D(A^f) = D(A) \quad \text{if, and only if,} \quad b = \lim_{x \rightarrow \infty} \frac{f(x)}{x} \neq 0$$

(cf. [68] and [69]). More information is available if we restrict ourselves to the class of complete Bernstein functions (see Definition 1.2.13).

THEOREM 1.4.3. Let $(T_t)_{t \geq 0}$ be a strongly continuous contraction semigroup on X and $(T_t^f)_{t \geq 0}$ be the semigroup obtained by subordination with respect to the complete Bernstein function

$$f(x) = a + bx + \int_{0+}^{\infty} \frac{x}{x + \lambda} \varrho(d\lambda).$$

Then

$$(1.4.3) \quad A^f u = -au + bAu + \int_{0+}^{\infty} AR_{\lambda}^A u \varrho(d\lambda), \quad u \in D(A).$$

Moreover, if $b \neq 0$ then $D(A^f) = D(A)$, and if $b = 0$ then

$$\begin{aligned} D(A^f) &= \left\{ u \in X : \lim_{k \rightarrow \infty} \int_0^k AR_{\lambda}^A u \varrho(d\lambda) \text{ exists (weakly) in } X \right\} \\ &= \left\{ u \in X : \lim_{k \rightarrow \infty} \int_0^{\infty} (T_t u - u) m_k(t) dt \text{ exists (weakly) in } X \right\} \end{aligned}$$

where $m_k(t) = \int_0^k e^{-st} \tau(ds)$ in the notation of Theorem 1.2.14(ii).

REMARK 1.4.4. Variants of Theorem 1.4.3 are due to F. Hirsch [32], C. Berg, K. Boyadzhiev and R. deLaubenfels [6] (see also [69]).

Formula (1.4.3) is a generalization of Balakrishnan's formula for fractional powers of order $\alpha \in (0, 1)$ (see A. V. Balakrishnan [4] or K. Yosida [84, Section IX.11]). Indeed, we have

$$(1.4.4) \quad A^{f_\alpha} u = -(-A)^\alpha u = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^{\alpha-1} A R_\lambda^A u \, d\lambda,$$

$f_\alpha(x) = x^\alpha$, and it is known that in this case the Bernstein functional calculus is in accordance with the classical Dunford–Taylor integral. This carries over in the following sense (see [68]).

THEOREM 1.4.5. *Let f be a complete Bernstein function and $(A, D(A))$ be the generator of some strongly continuous contraction semigroup on X . Then*

$$A^f = -f(-A)$$

in the sense that for $\lambda > 0$ and $\varepsilon > 0$,

$$(1.4.5) \quad R_\lambda^{(A-\varepsilon \text{id})^f} = (\lambda - (A - \varepsilon \text{id})^f)^{-1} = (\lambda + f(\varepsilon \text{id} - A))^{-1},$$

where the two left-hand side members are the resolvent of $(A - \varepsilon \text{id})^f$, whereas the right-hand side is defined as the Dunford–Taylor integral for $g(x) = 1/(\lambda + f(-x))$ of the unbounded operator $A - \varepsilon \text{id}$. Both sides in (1.4.5) converge as $\varepsilon \rightarrow 0$.

The representation for $(A^f, D(A^f))$ obtained in Theorem 1.4.3 allows us to derive a functional calculus for complete Bernstein functions. From now on we will write A^f or $-f(-A)$ whichever notation seems to be more appropriate.

THEOREM 1.4.6. *Let $(T_t)_{t \geq 0}$ be a strongly continuous contraction semigroup on X and let f, g be two complete Bernstein functions. Then the following identities are true in the sense of closed operators:*

(i) $(cf)(-A) = c f(-A)$, $c \geq 0$;

(ii) $(f + g)(-A) = f(-A) + g(-A)$;

(iii) $(f \circ g)(-A) = f(g(-A))$;

(iv) $(fg)(-A) = f(-A) \circ g(-A) = g(-A) \circ f(-A)$ whenever the product fg is a complete Bernstein function;

(v) $\varphi(x) := x/f(x)$ is a complete Bernstein function, hence $A = -f(-A) \circ \varphi(-A) = -\varphi(-A) \circ f(-A)$.

If f_1, \dots, f_N are complete Bernstein functions, so is $f^{1/N} := f_1^{1/N} \cdot \dots \cdot f_N^{1/N}$, and we can define

$$(1.4.6) \quad f(-A) := (f_1^{1/N}(-A) \cdot \dots \cdot f_N^{1/N}(-A))^N$$

in the sense of integer powers of closed operators. This definition is independent of N and extends (i)–(iv) to the algebra generated by the complete Bernstein functions.

REMARK 1.4.7. Theorem 1.4.6 is due to F. Hirsch [32] and to [69].

For later purposes we need to have a closer look at fractional powers $(-A)^\gamma$, $\gamma \geq 0$, for generators of strongly continuous contraction semigroups. Many important results in

this field are due to U. Westphal [81], [82]. In particular, it should be noted that for $\alpha, \gamma > 0$ we have (as an identity for closed operators)

$$(-A)^\alpha \circ (-A)^\gamma = (-A)^{\alpha+\gamma}$$

(compare also with (1.4.6)), and this equality has an appropriate extension to $\alpha, \gamma \in \mathbb{R}$. Moreover, there is a generalization of A. V. Balakrishnan's formula (1.4.4): see A. V. Balakrishnan [4] or U. Westphal [81], [82].

THEOREM 1.4.8. *Let $(A, D(A))$ be the generator of a strongly continuous contraction semigroup $(T_t)_{t \geq 0}$ with resolvent $(R_\lambda^A)_{\lambda > 0}$ on the Banach space X . For $m < \gamma < m + 1$, $m \in \mathbb{N}$, and $u \in D(A^{m+1})$ we have*

$$(-A)^\gamma u = \frac{\sin \pi(\gamma - m)}{\pi} \int_0^\infty \lambda^{\gamma-m-1} R_\lambda^A A^{m+1} u \, d\lambda.$$

We close this section with some perturbation results related to the functional calculus discussed above. One should note that a variant of the following proposition was obtained by T. Ando [3] for certain matrices A instead of generators of semigroups.

PROPOSITION 1.4.9. *Let A, B be two generators of strongly continuous contraction semigroups on X such that their difference $A - B$ is a bounded operator (extended to the whole of X). Then for any complete Bernstein function f we have*

$$\|f(-A) - f(-B)\| \leq 3f(\|A - B\|)$$

where $\|\cdot\|$ denotes the operator norm for bounded operators acting on the Banach space $(X, \|\cdot\|_X)$.

Proof. We have, for $\lambda > 0$ and $u \in X$,

$$\begin{aligned} (-A)R_\lambda^A u - (-B)R_\lambda^B u &= u - \lambda R_\lambda^A u - u + \lambda R_\lambda^B u \\ &= \lambda R_\lambda^B u - \lambda R_\lambda^A u \\ &= \lambda R_\lambda^A (B - A) R_\lambda^B u. \end{aligned}$$

From the last two lines we get for the operator norms $\|(-A)R_\lambda^A - (-B)R_\lambda^B\| \leq 2$, and for $\lambda > 0$,

$$\|(-A)R_\lambda^A - (-B)R_\lambda^B\| \leq \frac{\|A - B\|}{\lambda}.$$

We will need the following elementary inequalities:

$$2 \leq 3 \frac{\|A - B\|}{\lambda + \|A - B\|} \quad \text{for } \lambda \leq \frac{\|A - B\|}{2}$$

and

$$\frac{\|A - B\|}{\lambda} \leq 3 \frac{\|A - B\|}{\lambda + \|A - B\|} \quad \text{for } \lambda \geq \frac{\|A - B\|}{2}.$$

Using the definition of $f(-A)$ and $f(-B)$ we find

$$\|f(-A) - f(-B)\| \leq b\|A - B\| + \int_{0+}^\infty \|(-A)R_\lambda^A - (-B)R_\lambda^B\| \varrho(d\lambda)$$

$$\begin{aligned}
 &\leq b\|A - B\| + \int_{0+}^{\|A-B\|/2} 2 \varrho(d\lambda) + \int_{\|A-B\|/2}^{\infty} \frac{\|A - B\|}{\lambda} \varrho(d\lambda) \\
 &\leq b\|A - B\| + 3 \int_{0+}^{\|A-B\|/2} \frac{\|A - B\|}{\lambda + \|A - B\|} \varrho(d\lambda) \\
 &\quad + 3 \int_{\|A-B\|/2}^{\infty} \frac{\|A - B\|}{\lambda + \|A - B\|} \varrho(d\lambda) \\
 &\leq 3f(\|A - B\|). \blacksquare
 \end{aligned}$$

Let us apply Proposition 1.4.9 in two important special cases.

COROLLARY 1.4.10. *Let $(A, D(A))$ be the generator of a strongly continuous contraction semigroup. Then $B = A - \text{id}$ also generates a strongly continuous contraction semigroup and for the fractional powers we have*

$$\|(-A)^\alpha - (\text{id} - A)^\alpha\| \leq 3, \quad 0 \leq \alpha \leq 1.$$

REMARK 1.4.11. Note that the constant 3 is not optimal. A direct calculation along the lines of the proof of Proposition 1.4.9 yields the optimal constant 1, i.e.

$$(1.4.7) \quad \|(-A)^\alpha - (\text{id} - A)^\alpha\| \leq 1, \quad 0 \leq \alpha \leq 1.$$

Our next corollary already uses the fact that for a continuous negative definite function ψ the associated pseudo-differential operator $-\psi(D)$ generates a strongly continuous contraction semigroup on $L_p(\mathbb{R}^n)$, $1 \leq p < \infty$, and on $C_\infty(\mathbb{R}^n)$. If ψ is characterized by the Lévy–Khinchin formula (1.1.4) it follows that on $C_0^\infty(\mathbb{R}^n)$ the operator $\psi(D)$ has the following representation as an integro-differential operator:

$$\begin{aligned}
 \psi(D)u(x) &= cu(x) + \sum_{j=1}^n d_j \frac{\partial u(x)}{\partial x_j} - \sum_{j,k=1}^n q_{jk} \frac{\partial^2 u(x)}{\partial x_j \partial x_k} \\
 &\quad + \int_{\mathbb{R}^n \setminus \{0\}} \left(u(x) - u(x-y) - \frac{1}{1+|y|^2} \sum_{j=1}^n y_j \frac{\partial u(x)}{\partial x_j} \right) \frac{1+|y|^2}{|y|^2} \mu(dy).
 \end{aligned}$$

COROLLARY 1.4.12. *Let ψ_1 and ψ_2 be two continuous real-valued negative definite functions with representations*

$$\psi_j(\xi) = \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos(y \cdot \xi)) \nu_j(dy).$$

Assume that the bounded variation norm of the signed measure $\pi := \nu_1 - \nu_2$ is finite, i.e., $\|\pi\|_{\text{BV}} < \infty$. Then

$$\|\psi_1(D) - \psi_2(D)\| \leq 2\|\pi\|_{\text{BV}},$$

and for $0 < \alpha < 1$,

$$\|(\text{id} + \psi_1(D))^\alpha - (\text{id} + \psi_2(D))^\alpha\| \leq 3(2\|\pi\|_{\text{BV}})^\alpha.$$

Proof. From the integro-differential representation of $\psi_j(D)$, $j = 1, 2$, and the fact that ψ_j is real-valued, we deduce that for $u \in C_0^\infty(\mathbb{R}^n)$,

$$(\psi_1(D) - \psi_2(D))u(x) = \int_{\mathbb{R}^n \setminus \{0\}} (u(x) - u(x - y)) \pi(dy).$$

Using now the Minkowski inequality for double integrals we get, for $1 \leq p < \infty$,

$$\begin{aligned} \|(\psi_1(D) - \psi_2(D))u \mid L_p(\mathbb{R}^n)\| &= \left\| \int_{\mathbb{R}^n \setminus \{0\}} (u(\cdot) - u(\cdot - y)) \pi(dy) \mid L_p(\mathbb{R}^n) \right\| \\ &\leq \left\| \int_{\mathbb{R}^n \setminus \{0\}} |u(\cdot) - u(\cdot - y)| \cdot |\pi|(dy) \mid L_p(\mathbb{R}^n) \right\| \\ &\leq \int_{\mathbb{R}^n \setminus \{0\}} \|u(\cdot) - u(\cdot - y) \mid L_p(\mathbb{R}^n)\| \cdot |\pi|(dy) \\ &\leq 2\|u \mid L_p(\mathbb{R}^n)\| \cdot \|\pi\|_{\text{BV}}, \end{aligned}$$

and the obvious modification for the Feller case $(C_\infty(\mathbb{R}^n), \|\cdot\|_{L_\infty})$.

The second part of the assertion now follows directly from Proposition 1.4.9 and the usual density argument for test functions. ■

1.5. Sub-Markovian semigroups and abstract Bessel potential spaces. Let $(X, \|\cdot\|_X)$ be a Banach space of real-valued functions defined on some topological space G . Further, we assume that there exists a natural notion of pointwise inequalities for elements of X . By this we mean that $u \leq v$ has the natural meaning. In the case of L_p -spaces, i.e. the spaces $L_p(G, \mu)$ with a suitable measure μ on $\mathcal{B}(G)$, $u \leq v$ will always mean that $u(x) \leq v(x)$ for μ -almost all $x \in G$.

In accordance with Definition 1.3.2 we call a strongly continuous contraction semigroup $(T_t)_{t \geq 0}$ on X a *sub-Markovian semigroup* if $0 \leq u \leq 1$ implies $0 \leq T_t u \leq 1$ for $u \in X$.

DEFINITION 1.5.1. Let $(X, \|\cdot\|_X)$ be a Banach space. The Γ -transform $(V_r)_{r \geq 0}$ of a sub-Markovian semigroup $(T_t)_{t \geq 0}$ on X is defined by

$$V_r u := \frac{1}{\Gamma(r/2)} \int_0^\infty t^{r/2-1} e^{-t} T_t u dt$$

for $r > 0$ and $V_0 = \text{id}$.

REMARK 1.5.2. Note that

$$\eta_r(ds) := \chi_{[0, \infty)}(s) \frac{1}{\Gamma(r/2)} s^{r/2-1} e^{-s} \lambda^{(1)}(ds)$$

yields a convolution semigroup with support $[0, \infty)$, the so-called *modified Γ -semigroup* (see Example 1.2.11). It is associated with the Bernstein function $f(s) = \frac{1}{2} \log(1 + s)$. Therefore the Γ -transform $(V_r)_{r \geq 0}$ of a sub-Markovian semigroup is again a sub-Markovian semigroup on X , namely the semigroup obtained from $(T_t)_{t \geq 0}$ by subordination in the sense of Bochner with respect to $(\eta_r)_{r \geq 0}$ (or f , respectively); see Section 1.4.

In particular, we have $V_{r_1} \circ V_{r_2} = V_{r_1+r_2}$ and

$$(1.5.1) \quad \|V_r u \mid X\| \leq \|u \mid X\|,$$

i.e. each of the operators V_r , $r \geq 0$, is a contraction.

The following result is taken from [19, Theorem 4.1].

THEOREM 1.5.3. *Let $(X, \|\cdot\|_X)$ be a Banach space and let $(T_t)_{t \geq 0}$ be a sub-Markovian semigroup on X having generator $(A, D(A))$. Then for all $r > 0$ and all $u \in X$,*

$$V_r u = (\text{id} - A)^{-r/2} u.$$

In particular, each of the operators V_r is injective and contractive.

Using the injectivity of the operators V_r , $r \geq 0$, we may define abstract Bessel potential spaces (see also F. Hirsch [34]) by

$$\mathcal{F}_{r,A,X} := V_r(X) \quad \text{and} \quad \|u\|_{\mathcal{F}_{r,A,X}} := \|v\|_X \quad \text{for } u = V_r v.$$

Clearly, $(\mathcal{F}_{r,A,X}, \|\cdot\|_{\mathcal{F}_{r,A,X}})$ is a Banach space which is separable whenever X is.

REMARK 1.5.4. Note that often the general theory requires the regularity of the space $\mathcal{F}_{r,A,X}$, i.e. the density of $C_\infty(G) \cap \mathcal{F}_{r,A,X}$ in both

$$(\mathcal{F}_{r,A,X}, \|\cdot\|_{\mathcal{F}_{r,A,X}}) \quad \text{and} \quad (C_\infty(G), \|\cdot\|_{C_\infty(G)}).$$

In introducing the spaces $\mathcal{F}_{r,A,X}$ we are, of course, motivated by the considerations of P. Malliavin [58] and, in particular, by those of M. Fukushima (and co-workers) [26]–[28] (see also [53]), and the work of I. Shigekawa [54] and [74].

COROLLARY 1.5.5. *In the situation of Theorem 1.5.3 we have $\mathcal{F}_{r,A,X} = D((\text{id} - A)^{r/2})$.*

A proof of this corollary is given in [19, Corollary 4.2].

LEMMA 1.5.6. *Let $(A, D(A))$ be the generator of a sub-Markovian semigroup $(T_t)_{t \geq 0}$ on the Banach space $(X, \|\cdot\|_X)$. For all $r > 0$ and all $u \in D((\text{id} - A)^{r/2}) = \mathcal{F}_{r,A,X}$ we have*

$$(1.5.2) \quad \|u\|_X \leq \|(\text{id} - A)^{r/2} u\|_X.$$

Moreover, if $s, r \geq 0$ then the following continuous embedding holds:

$$(1.5.3) \quad \mathcal{F}_{r+s,A,X} \hookrightarrow \mathcal{F}_{r,A,X}.$$

Proof. Since $\text{id} = (\text{id} - A)^{-r/2}(\text{id} - A)^{r/2} = V_r(\text{id} - A)^{r/2}$ it follows from (1.5.1) that on $\mathcal{F}_{r,A,X}$ one has

$$\|u\|_X = \|V_r(\text{id} - A)^{r/2} u\|_X \leq \|(\text{id} - A)^{r/2} u\|_X$$

and this proves (1.5.2).

Assertion (1.5.3) follows from the semigroup property of $(V_r)_{r \geq 0}$ (see [19, Lemma 5.1.A] for a proof). ■

Let $u \in \mathcal{F}_{r,A,X}$. Then $(\text{id} - A)^{r/2} u \in X$, which yields $\|u\|_{\mathcal{F}_{r,A,X}} = \|(\text{id} - A)^{r/2} u\|_X$. Our aim is to prove the equivalence of the norms

$$\|\cdot\|_{\mathcal{F}_{r,A,X}} \quad \text{and} \quad \|(-A)^{r/2} u\|_X + \|u\|_X.$$

A first result follows from Corollary 1.4.10.

THEOREM 1.5.7. *Let $(A, D(A))$ be the generator of a strongly continuous contraction semigroup on the Banach space $(X, \|\cdot\|_X)$. Let $0 \leq r \leq 1$. Then for all $u \in D(A)$ we*

have

$$(1.5.4) \quad \frac{1}{3}(\|(-A)^r u | X\| + \|u | X\|) \leq \|(\text{id} - A)^r u | X\| \leq \|(-A)^r u | X\| + \|u | X\|.$$

In particular, $D((-A)^r) = D((\text{id} - A)^r)$.

Proof. Using (1.4.7) we find

$$\|(-A)^r u | X\| - \|(\text{id} - A)^r | X\| \leq \|(-A)^r u - (\text{id} - A)^r u | X\| \leq \|u | X\|,$$

which gives, by (1.5.2),

$$\|(-A)^r u | X\| + \|u | X\| \leq \|(\text{id} - A)^r u | X\| + 2\|u | X\| \leq 3\|(\text{id} - A)^r u | X\|.$$

On the other hand, we have

$$\|(\text{id} - A)^r u | X\| - \|(-A)^r u | X\| \leq \|(\text{id} - A)^r u - (-A)^r u | X\| \leq \|u | X\|$$

or

$$\|(\text{id} - A)^r u | X\| \leq \|(-A)^r u | X\| + \|u | X\|,$$

and (1.5.4) follows. Since $D(A) = D(\text{id} - A)$ and since $D((-A)^r)$ and $D((\text{id} - A)^r)$ are obtained by completion of $D(A)$ with respect to the equivalent norms

$$\|(-A)^r u | X\| + \|u | X\| \quad \text{and} \quad \|(\text{id} - A)^r u | X\|,$$

respectively, we have proved the theorem by noting that, according to Theorem 1.4.2, $D(A) = D(\text{id} - A)$ is an operator core for both $(-A)^r$ and $(\text{id} - A)^r$. ■

To extend Theorem 1.5.7 to $k+r$, $k \in \mathbb{N}_0$, $0 \leq r < 1$, we need some preparations. First we recall from Section 1.4 the following results on the functional calculus for generators of strongly continuous contraction semigroups (see Theorem 1.4.6).

For $r, s \geq 0$ we have

$$\begin{aligned} D((-A)^{s+r}) &= \{u \in D((-A)^r) : (-A)^r u \in D((-A)^s)\} \\ &= \{u \in D((-A)^s) : (-A)^s u \in D((-A)^r)\}, \end{aligned}$$

and on $D((-A)^{s+r})$ one has $(-A)^r(-A)^s = (-A)^{r+s} = (-A)^s(-A)^r$.

From the proof of Theorem V.1.2.4, p. 259, in H. Amann [2] we deduce that for any $0 \leq r < s$ one has the estimate

$$\|(\text{id} - A)^r u | X\| \leq \gamma \|(\text{id} - A)^s u | X\|^{r/s} \|u | X\|^{(s-r)/s}$$

for all $u \in D((-A)^s)$ and some constant $\gamma > 0$. Using the inequality

$$a^\lambda b^{1-\lambda} \leq \varepsilon a + \frac{1}{\varepsilon^{\lambda/(1-\lambda)}} b,$$

which holds for $a, b \geq 0$, $\varepsilon > 0$ and $\lambda \in (0, 1)$, we arrive at

$$(1.5.5) \quad \|(\text{id} - A)^r u | X\| \leq \varepsilon \|(\text{id} - A)^s u | X\| + \frac{\gamma^{s/(s-r)}}{\varepsilon^{r/(s-r)}} \|u | X\|,$$

for any $u \in D((\text{id} - A)^s)$.

LEMMA 1.5.8. *Let $(A, D(A))$, $(T_t)_{t \geq 0}$ and $(X, \|\cdot | X\|)$ be as in Theorem 1.5.7. Further, let $k \in \mathbb{N}_0$ and $0 \leq r < 1$. Then for all $u \in D((-A)^{k+1})$ we have*

$$(1.5.6) \quad \|(-A)^{k+r} u | X\| \leq 3 \cdot 2^k \|(\text{id} - A)^{k+r} u | X\|.$$

Proof. We prove (1.5.6) by induction. For $k = 0$ this estimate was proved in Theorem 1.5.7. Assuming (1.5.6) for $k \in \mathbb{N}_0$ fixed we prove that

$$\|(-A)^{k+1+r}u \mid X\| \leq 3 \cdot 2^{k+1} \|(\text{id} - A)^{k+1+r}u \mid X\|$$

is valid for all $u \in D((-A)^{k+2})$. Using the fact that $(\text{id} - A)^{-1}$ is a contraction we find

$$\begin{aligned} \|(-A)^{k+1+r}u \mid X\| &= \|(-A)^{k+r}(-A)u \mid X\| \leq 3 \cdot 2^k \|(\text{id} - A)^{k+r}(-A)u \mid X\| \\ &\leq 3 \cdot 2^k \|(\text{id} - A)^{k+1+r}u \mid X\| + 3 \cdot 2^k \|(\text{id} - A)^{k+r}u \mid X\| \\ &\leq 2 \cdot 3 \cdot 2^k \|(\text{id} - A)^{k+1+r}u \mid X\|, \end{aligned}$$

which proves the lemma. ■

LEMMA 1.5.9. *In the situation of Lemma 1.5.8 we have for all $u \in D((-A)^{k+1})$ the estimate*

$$(1.5.7) \quad \|(\text{id} - A)^{k+r}u \mid X\| \leq c_{k,r} (\|(-A)^{k+r}u \mid X\| + \|u \mid X\|).$$

Proof. Again we use induction and note that the case $k = 0$ was proved in Theorem 1.5.7. Now assuming (1.5.7) we will prove

$$\|(\text{id} - A)^{k+1+r}u \mid X\| \leq c_{k+1,r} (\|(-A)^{k+1+r}u \mid X\| + \|u \mid X\|).$$

For $u \in D((-A)^{k+2})$ it follows that

$$\begin{aligned} \|(\text{id} - A)^{k+1+r}u \mid X\| &= \|(\text{id} - A)^{k+r}(\text{id} - A)u \mid X\| \\ &\leq \|(\text{id} - A)^{k+r}(-A)u \mid X\| + \|(\text{id} - A)^{k+r}u \mid X\| \\ &\leq c_{k,r} \|(-A)^{k+1+r}u \mid X\| + c_{k,r} \| -Au \mid X\| \\ &\quad + \frac{1}{4} \|(\text{id} - A)^{k+1+r}u \mid X\| + c' \|u \mid X\|, \end{aligned}$$

where in the last step we used inequality (1.5.5) with $\varepsilon = 1/4$ noting that $D((-A)^{k+2}) = D((\text{id} - A)^{k+2}) \subset D((\text{id} - A)^{k+1})$.

Thus we obtain

$$\begin{aligned} \frac{3}{4} \|(\text{id} - A)^{k+1+r}u \mid X\| &\leq c_{k,r} \|(-A)^{k+1+r}u \mid X\| + c_{k,r} \| -Au \mid X\| + c' \|u \mid X\| \\ &\leq c_{k,r} \|(-A)^{k+1+r}u \mid X\| + c_{k,r} \|(\text{id} - A)u \mid X\| + c'' \|u \mid X\| \end{aligned}$$

and a further application of (1.5.5) yields

$$\frac{1}{2} \|(\text{id} - A)^{k+1+r}u \mid X\| \leq c_{k,r} \|(-A)^{k+1+r}u \mid X\| + \tilde{c} \|u \mid X\|,$$

which finally proves the lemma. ■

Combining Lemmas 1.5.8 and 1.5.9 we get

THEOREM 1.5.10. *Let $(A, D(A))$, $(T_t)_{t \geq 0}$ and $(X, \|\cdot \mid X\|)$ be as in Theorem 1.5.7. Let $k \in \mathbb{N}_0$ and $0 \leq r < 1$. Then there exist constants $\tilde{\gamma}_{k,r}$ such that for any $u \in D((-A)^{k+1})$,*

$$\begin{aligned} \frac{1}{3 \cdot 2^k + 1} (\|u \mid X\| + \|(-A)^{k+r}u \mid X\|) \\ \leq \|(\text{id} - A)^{k+r}u \mid X\| \leq \tilde{\gamma}_{k,r} (\|u \mid X\| + \|(-A)^{k+r}u \mid X\|). \end{aligned}$$

Furthermore, $D((-A)^{k+r}) = D((\text{id} - A)^{k+r})$.

REMARK 1.5.11. We want to indicate an alternative proof of Theorem 1.5.10 using the following result due to L. Hörmander (cf. K. Yosida [84, p. 79]).

Let A_j , $j = 1, 2$, be two linear operators defined on $D(A_j) \subset X$ into the Banach space $(X, \|\cdot\|_X)$. If A_1 is closed, A_2 is closable, and if $D(A_1) \subset D(A_2)$, then there exists a constant $c > 0$ such that

$$(1.5.8) \quad \|A_2 u\|_X \leq c(\|A_1 u\|_X + \|u\|_X)$$

for all $u \in D(A_1)$.

In order to apply this result one first extends Theorem 1.5.7 to

$$(1.5.9) \quad D((-A)^r) = D((\text{id} - A)^r)$$

for all $r > 0$, which can be done by induction with respect to $k \in \mathbb{N}$, $r \in (0, k]$.

Once (1.5.9) is proved, noting that the operators $((-A)^r, D((-A)^r))$ and $((\text{id} - A)^r, D((\text{id} - A)^r))$ are closed, from (1.5.8) we obtain

$$\|(-A)^r u\|_X \leq c_r(\|(\text{id} - A)^r u\|_X + \|u\|_X)$$

and

$$\|(\text{id} - A)^r u\|_X \leq \tilde{c}_r(\|(-A)^r u\|_X + \|u\|_X),$$

which implies the assertion of Theorem 1.5.10.

2. Bessel potential spaces associated with continuous negative definite functions

This chapter is devoted to a systematic study of Bessel potential spaces associated with a real-valued continuous negative definite function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$. First we identify with a function space $H_p^{\psi,2}(\mathbb{R}^n)$ the L_p -domain of the generator $A^{(p)}$ of the L_p -sub-Markovian semigroup associated with a Lévy process with characteristic exponent ψ . In the next section we extend $H_p^{\psi,2}(\mathbb{R}^n)$ to the scale $H_p^{\psi,s}(\mathbb{R}^n)$, $s \in \mathbb{R}$. In particular, we prove density results, mapping properties of the operators $(\text{id} - A^{(p)})^{r/2}$, $r \in \mathbb{R}$, within this scale, and we characterize the dual spaces. In Section 2.3 embedding theorems are discussed. We consider continuous embeddings $j : H_p^{\psi_1,s}(\mathbb{R}^n) \hookrightarrow H_q^{\psi_2,r}(\mathbb{R}^n)$ and the embedding of $H_p^{\psi,s}(\mathbb{R}^n)$ into the space $C_\infty(\mathbb{R}^n)$.

This yields a characterization of the complex interpolation spaces $[H_{p_0}^{\psi_0,s_0}(\mathbb{R}^n), H_{p_1}^{\psi_1,s_1}(\mathbb{R}^n)]_\theta$. In the final section we make use of the fact that the semigroup $(T_t^{(p)})_{t \geq 0}$ and the operators $(\text{id} - A^{(p)})^{-r/2}$, $r > 0$, are positivity preserving. This enables us to associate a capacity $\text{cap}_{r,p}^\psi$ with each of the spaces $H_p^{\psi,r}(\mathbb{R}^n)$. Using this capacity we may introduce the notion of an (r,p) -quasi-continuous modification of an element $u \in H_p^{\psi,r}(\mathbb{R}^n)$ and we show that each $u \in H_p^{\psi,r}(\mathbb{R}^n)$ has (up to (r,p) -quasi-everywhere equality) a unique quasi-continuous modification. Further, we show a comparison result for $\text{cap}_{r,p}^{\psi_1}$ and $\text{cap}_{r,p}^{\psi_2}$ based on embedding theorems.

2.1. The space $H_p^{\psi,2}(\mathbb{R}^n)$ as a domain of $A^{(p)}$. Let $\mathcal{S}(\mathbb{R}^n, \mathbb{R})$ be the Schwartz space of all real-valued rapidly decreasing C^∞ -functions on \mathbb{R}^n equipped with the usual topology. We denote by $\mathcal{S}'(\mathbb{R}^n, \mathbb{R})$ the space of all real-valued tempered distributions on \mathbb{R}^n . If $\varphi \in \mathcal{S}(\mathbb{R}^n, \mathbb{R})$ then $\widehat{\varphi} = \mathcal{F}\varphi$ and $\mathcal{F}^{-1}\varphi$ are, respectively, the Fourier and inverse Fourier transform of φ . One extends \mathcal{F} and \mathcal{F}^{-1} in the usual way from $\mathcal{S}(\mathbb{R}^n, \mathbb{R})$ to $\mathcal{S}'(\mathbb{R}^n, \mathbb{R})$.

If there is no danger of confusion, we will omit \mathbb{R}^n and/or \mathbb{R} in $\mathcal{S}(\mathbb{R}^n, \mathbb{R})$ and in the other function spaces below.

Let $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a fixed continuous negative definite function with corresponding convolution semigroup $(\mu_t)_{t \geq 0}$. We assume that the function ψ has the Lévy–Khinchin representation

$$(2.1.1) \quad \psi(\xi) = \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos(x \cdot \xi)) \nu(dx)$$

where the Lévy measure ν integrates the function $y \mapsto |y|^2 \wedge 1$.

The sub-Markovian semigroup on $L_p(\mathbb{R}^n)$, $1 \leq p < \infty$, associated with $(\mu_t)_{t \geq 0}$ is denoted by $(T_t^{(p)})_{t \geq 0}$, its generator by $(A^{(p)}, D(A^{(p)}))$, $D(A^{(p)}) \subset L_p(\mathbb{R}^n, \mathbb{R})$.

The Feller semigroup associated with $(\mu_t)_{t \geq 0}$ is denoted by $(T_t^{(\infty)})_{t \geq 0}$ and its generator by $(A^{(\infty)}, D(A^{(\infty)}))$, $D(A^{(\infty)}) \subset C_\infty(\mathbb{R}^n, \mathbb{R})$.

Note that if $1 \leq p, q \leq \infty$ then

$$(2.1.2) \quad T_t^{(p)} u = T_t^{(q)} u \quad \text{for all } u \in \mathcal{S}(\mathbb{R}^n).$$

Since ψ is real-valued, the operator $(A^{(2)}, D(A^{(2)}))$ is self-adjoint, and $(T_t^{(p)})_{t \geq 0}$ is an analytic semigroup by a result of E. M. Stein (cf. Section 1.3).

PROPOSITION 2.1.1. *For $1 < p \leq \infty$ the space $\mathcal{S}(\mathbb{R}^n)$ is contained in $D(A^{(p)})$ and on $\mathcal{S}(\mathbb{R}^n)$ one has*

$$(2.1.3) \quad A^{(p)} u(x) = -\psi(D)u(x) = -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \psi(\xi) \widehat{u}(\xi) d\xi.$$

Proof. First note that from (1.3.1) for $u \in \mathcal{S}(\mathbb{R}^n)$ we get

$$(2.1.4) \quad \frac{T_t^{(p)} u - u}{t} + \psi(D)u = \mathcal{F}^{-1} \left[\left(\frac{e^{-t\psi} - 1}{t} + \psi \right) \widehat{u} \right].$$

Now the inequality

$$\left| \frac{e^{-at} - 1 + at}{t} \right| \leq \frac{1}{2} a^2 t, \quad a > 0, t > 0,$$

and Plancherel's theorem yield

$$\begin{aligned} \left\| \frac{T_t^{(2)} u - u}{t} + \psi(D)u \right\|_{L_2} &= \left\| \left(\frac{e^{-t\psi} - 1}{t} + \psi \right) \widehat{u} \right\|_{L_2} \\ &\leq \frac{1}{2} t \|\psi^2 \widehat{u}\|_{L_2} \leq ct \|(1 + |\cdot|^2)^2 \widehat{u}\|_{L_2}. \end{aligned}$$

Thus,

$$(2.1.5) \quad \lim_{t \rightarrow 0} \left\| \frac{T_t^{(2)} u - u}{t} + \psi(D)u \right\|_{L_2} = 0,$$

since for $u \in \mathcal{S}(\mathbb{R}^n)$ the term $\|(1 + |\cdot|^2)^2 \widehat{u}\|_{L_2}$ is finite. This proves (2.1.3) and $\mathcal{S}(\mathbb{R}^n) \subset D(A^{(2)})$.

Next observe that (2.1.4) also implies

$$\begin{aligned} \left\| \frac{T_t^{(\infty)}u - u}{t} + \psi(D)u \right\| L_\infty &= (2\pi)^{-n/2} \left\| \int_{\mathbb{R}^n} e^{i(\cdot, \xi)} \left(\frac{e^{-t\psi(\xi)} - 1}{t} + \psi(\xi) \right) \widehat{u}(\xi) d\xi \right\| L_\infty \\ &\leq \widetilde{c}t \int_{\mathbb{R}^n} (1 + |\xi|^2)^2 |\widehat{u}(\xi)| d\xi, \end{aligned}$$

which for $u \in \mathcal{S}(\mathbb{R}^n)$ leads to

$$(2.1.6) \quad \lim_{t \rightarrow 0} \left\| \frac{T_t^{(\infty)}u - u}{t} + \psi(D)u \right\| L_\infty = 0,$$

and this in turn gives $\mathcal{S}(\mathbb{R}^n) \subset D(A^{(\infty)})$.

Combining (2.1.5), (2.1.6), and (2.1.2) we find for $2 \leq p < \infty$ and $u \in \mathcal{S}(\mathbb{R}^n)$ that

$$\begin{aligned} \left\| \frac{T_t^{(p)}u - u}{t} + \psi(D)u \right\| L_p &\leq \left\| \frac{T_t^{(p)}u - u}{t} + \psi(D)u \right\| L_\infty^{(p-2)/p} \cdot \left\| \frac{T_t^{(p)}u - u}{t} + \psi(D)u \right\| L_2^{2/p}, \end{aligned}$$

from which we conclude that $\mathcal{S}(\mathbb{R}^n) \subset D(A^{(p)})$ for $2 \leq p < \infty$.

For $1/p + 1/p' = 1$, $1 < p \leq 2$, the equality

$$\int_{\mathbb{R}^n} \left(\frac{T_t^{(p)}u - u}{t} + \psi(D)u \right) \cdot v dx = \int_{\mathbb{R}^n} u \cdot \left(\frac{T_t^{(p')}v - v}{t} + \psi(D)v \right) dx$$

holds for $u, v \in \mathcal{S}(\mathbb{R}^n)$ and implies that $\mathcal{S}(\mathbb{R}^n)$ is in the domain of the weak generator of $(T_t^{(p)})_{t \geq 0}$. But the weak generator coincides with the (strong) generator (see Theorem 1.3.4), and it follows that $\mathcal{S}(\mathbb{R}^n) \subset D(A^{(p)})$ for all $1 < p < \infty$. ■

REMARK 2.1.2. (i) Let $1 \leq p \leq \infty$. From the considerations in Section 1.5 we have

$$(2.1.7) \quad \|u\| L_p(\mathbb{R}^n) \leq \|(\text{id} - A^{(p)})u\| L_p(\mathbb{R}^n) = \|(\text{id} + \psi(D))u\| L_p(\mathbb{R}^n)$$

for any $u \in \mathcal{S}(\mathbb{R}^n)$.

(ii) Let us point out that the case $1 < p < 2$ often requires other techniques and assumptions than the case $p \geq 2$.

In order to introduce a Bessel potential space which will characterize $D(A^{(p)})$ as a function space we need some preparatory considerations.

Suppose that the continuous negative definite function ψ has the representation (2.1.1). As already pointed out in Section 1.1 for $R > 0$ we may decompose ψ into the sum of two continuous negative definite functions by setting

$$(2.1.8) \quad \psi_R(\xi) := \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos(x \cdot \xi)) \chi_{B(0, R)}(x) \nu(dx)$$

and

$$(2.1.9) \quad \widetilde{\psi}_R(\xi) := \psi(\xi) - \psi_R(\xi) = \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos(x \cdot \xi)) \chi_{B^c(0, R)}(x) \nu(dx);$$

see (1.1.8) and (1.1.9).

Note that both functions ψ_R and $\tilde{\psi}_R$ are again continuous and negative definite. We know that ψ_R is infinitely often differentiable and has, together with its partial derivatives, at most quadratic growth. We also know that $\tilde{\psi}_R$ is a bounded continuous negative definite function.

For $u \in \mathcal{S}(\mathbb{R}^n)$ we define the operators

$$\psi_R(D)u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \psi_R(\xi) \widehat{u}(\xi) d\xi$$

and

$$\tilde{\psi}_R(D)u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \tilde{\psi}_R(\xi) \widehat{u}(\xi) d\xi.$$

Thus on $\mathcal{S}(\mathbb{R}^n)$ we have $\psi(D) = \psi_R(D) + \tilde{\psi}_R(D)$. Moreover, using (2.1.9), we find that on $\mathcal{S}(\mathbb{R}^n)$,

$$(2.1.10) \quad \tilde{\psi}_R(D)u(x) = \int_{B^c(0,R)} (u(x) - u(x-y)) \nu(dy).$$

Now, whenever $\|\cdot\|$ is a norm having the property that there exists $\gamma > 0$ with

$$\|u(\cdot + y)\| \leq \gamma \|u(\cdot)\| \quad \text{for all } y \in \mathbb{R}^n, u \in \mathcal{S}(\mathbb{R}^n),$$

we deduce from (2.1.10) that

$$(2.1.11) \quad \|\tilde{\psi}_R(D)u\| \leq \int_{B^c(0,R)} \|u(\cdot) - u(\cdot - y)\| \nu(dy) \leq (1 + \gamma) \nu(B^c(0,R)) \|u\|,$$

i.e., the operator $\tilde{\psi}_R(D)$ extends to a continuous operator from the closure of $\mathcal{S}(\mathbb{R}^n)$ under the norm $\|\cdot\|$ into itself.

COROLLARY 2.1.3. *Let $1 \leq p < \infty$. For any $R > 0$ the operator $\tilde{\psi}_R(D)$ extends to a continuous operator from $L_p(\mathbb{R}^n)$ into itself.*

Since $\psi_R \in C^\infty(\mathbb{R}^n)$ is of at most quadratic growth, it is a multiplier for $\mathcal{S}'(\mathbb{R}^n)$. Consequently, we can extend the operator $\psi_R(D)$ from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ by

$$\psi_R(D)u = \mathcal{F}^{-1}(\psi_R \widehat{u}), \quad u \in \mathcal{S}'(\mathbb{R}^n),$$

where \widehat{u} is considered as a tempered distribution.

Thus we may extend $\psi(D)$ to $L_p(\mathbb{R}^n)$ by the identity

$$\psi(D)u := \mathcal{F}^{-1}(\psi_R \widehat{u}) + \int_{B^c(0,R)} (u(\cdot + y) - u(\cdot)) \nu(dy), \quad u \in L_p(\mathbb{R}^n),$$

and so $\psi(D)u \in \mathcal{S}'(\mathbb{R}^n)$.

It is, therefore, possible to look for all $u \in L_p(\mathbb{R}^n)$ such that $\psi(D)u \in L_p(\mathbb{R}^n)$.

Let us define the family of norms

$$(2.1.12) \quad \|u\|_{\psi,R,p} := \|(\text{id} + \psi_R(D))u\|_{L_p(\mathbb{R}^n)}$$

for those $u \in L_p(\mathbb{R}^n)$ for which (2.1.12) is finite. This is, in particular, the case for $u \in \mathcal{S}(\mathbb{R}^n)$.

Moreover, since ψ_R is a continuous negative definite function we may associate an operator semigroup $(T_t^{(p,R)})_{t \geq 0}$ on $L_p(\mathbb{R}^n)$, $1 < p < \infty$, with ψ_R .

Proposition 2.1.1 implies that $\mathcal{S}(\mathbb{R}^n) \subset D(A^{(p,R)})$ and applying the results from Section 1.5 we conclude that for all $u \in \mathcal{S}(\mathbb{R}^n)$,

$$(2.1.13) \quad \|u \mid L_p(\mathbb{R}^n)\| \leq \|u\|_{\psi,R,p}.$$

The next lemma is an immediate consequence of Corollary 1.4.12 with $\psi_1 = \psi_R$ and $\psi_2 = \psi_S$.

LEMMA 2.1.4. *If $0 < R < S$ then the norms $\|\cdot\|_{\psi,R,p}$ and $\|\cdot\|_{\psi,S,p}$ are equivalent on $\mathcal{S}(\mathbb{R}^n)$.*

Lemma 2.1.4, (2.1.11), (2.1.13), and (2.1.7) prove that for any $R > 0$ and any $u \in \mathcal{S}(\mathbb{R}^n)$ we have the estimates

$$\begin{aligned} \|(\text{id} + \psi(D))u \mid L_p(\mathbb{R}^n)\| &= \|(\text{id} + \psi_R(D))u + \tilde{\psi}_R(D)u \mid L_p(\mathbb{R}^n)\| \\ &\leq c(\|u\|_{\psi,R,p} + \|u \mid L_p(\mathbb{R}^n)\|) \leq 2c\|u\|_{\psi,R,p} \end{aligned}$$

and

$$\begin{aligned} \|u\|_{\psi,R,p} &\leq \|(\text{id} + \psi_R(D))u + \tilde{\psi}_R(D)u \mid L_p(\mathbb{R}^n)\| + \|\tilde{\psi}_R(D)u \mid L_p(\mathbb{R}^n)\| \\ &\leq \|(\text{id} + \psi(D))u \mid L_p(\mathbb{R}^n)\| + c'\|u \mid L_p(\mathbb{R}^n)\| \leq c''\|(\text{id} + \psi(D))u \mid L_p(\mathbb{R}^n)\|. \end{aligned}$$

This motivates the next definition.

DEFINITION 2.1.5. Let $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a real-valued continuous negative definite function. The ψ -Bessel potential space of order 2 with respect to $L_p(\mathbb{R}^n)$ ($1 \leq p < \infty$) is the space

$$H_p^{\psi,2}(\mathbb{R}^n) = \{u \in L_p(\mathbb{R}^n) : \|u \mid H_p^{\psi,2}(\mathbb{R}^n)\| < \infty\}$$

where

$$\|u \mid H_p^{\psi,2}(\mathbb{R}^n)\| := \|(\text{id} + \psi(D))u \mid L_p(\mathbb{R}^n)\|.$$

REMARK 2.1.6. (i) By definition for every $u \in H_p^{\psi,2}(\mathbb{R}^n)$ there exists a unique $f \in L_p(\mathbb{R}^n)$ such that $u + \psi(D)u = f$ and $\|u \mid H_p^{\psi,2}(\mathbb{R}^n)\| = \|f \mid L_p(\mathbb{R}^n)\|$.

(ii) If ψ is of the form (2.1.1), then it is clear that for any $R > 0$ the norms $\|\cdot\|_{\psi,R,p}$ and $\|\cdot \mid H_p^{\psi,2}(\mathbb{R}^n)\|$ are equivalent. Note that $\|\cdot\|_{\psi,R,p} = \|\cdot \mid H_p^{\psi_R,2}(\mathbb{R}^n)\|$ and that $H_p^{\psi,2}(\mathbb{R}^n) = H_p^{\psi_R,2}(\mathbb{R}^n)$ for all $R > 0$.

Moreover, for $u \in \mathcal{S}(\mathbb{R}^n)$ it follows that $u \in H_p^{\psi,2}(\mathbb{R}^n)$, and also

$$\|u \mid H_p^{\psi,2}(\mathbb{R}^n)\| = \|\mathcal{F}^{-1}[(1 + \psi(\cdot))\hat{u}] \mid L_p(\mathbb{R}^n)\|.$$

REMARK 2.1.7. (i) Note that if $\psi(\xi) = |\xi|^2$ or $\psi(\xi) = q(\xi)$ is a non-degenerate, positive definite, symmetric quadratic form, then $H_p^{\psi,2}(\mathbb{R}^n)$ is the classical Sobolev space $W_p^2(\mathbb{R}^n) = H_p^2(\mathbb{R}^n)$.

Moreover, for the continuous negative definite function $\xi \mapsto (1 + |\cdot|^2)^{r/2}$, $0 < r < 2$, we obtain the classical Bessel potential space $H_p^r(\mathbb{R}^n)$.

(ii) Let ψ be a continuous negative definite function of the form (2.1.1) and let $q(\xi)$ be a non-degenerate, positive definite, symmetric quadratic form. Then $\Psi := q + \psi$ is again continuous negative definite, and we have $H_p^{\Psi,2}(\mathbb{R}^n) = H_p^2(\mathbb{R}^n) = W_p^2(\mathbb{R}^n)$.

Indeed, using the integro-differential representation (1.4.8) for $\psi_R(D)$, ψ_R as in (2.1.8), and Taylor's theorem, we find

$$(2.1.14) \quad \|\psi_R(D)u \mid L_p(\mathbb{R}^n)\| \leq \left(\int_{B(0,R) \setminus \{0\}} |x|^2 \nu(dx) \right) \left\| \sum_{j,k=1}^n \frac{\partial^2 u}{\partial x_j \partial x_k} \right\| L_p(\mathbb{R}^n).$$

The continuous injection $W_p^2(\mathbb{R}^n) \hookrightarrow H_p^{\Psi,2}$ follows directly from (2.1.14) and the fact that $\tilde{\psi}_R(D)$ is a bounded operator in $L_p(\mathbb{R}^n)$.

For the converse inclusion we fix some $\varrho > 0$ with $\int_{B(0,\varrho) \setminus \{0\}} |x|^2 \nu(dx) < 1/2$. Then

$$\begin{aligned} \left\| \sum_{j,k=1}^n \frac{\partial^2 u}{\partial x_j \partial x_k} \right\| L_p(\mathbb{R}^n) &\leq \left\| \sum_{j,k=1}^n \frac{\partial^2 u}{\partial x_j \partial x_k} + \psi_\varrho(D)u \right\| L_p(\mathbb{R}^n) + \|\psi_\varrho(D)u \mid L_p(\mathbb{R}^n)\| \\ &\leq \left\| \sum_{j,k=1}^n \frac{\partial^2 u}{\partial x_j \partial x_k} + \psi_\varrho(D)u \right\| L_p(\mathbb{R}^n) \\ &\quad + \frac{1}{2} \left\| \sum_{j,k=1}^n \frac{\partial^2 u}{\partial x_j \partial x_k} \right\| L_p(\mathbb{R}^n), \end{aligned}$$

so that

$$\begin{aligned} \left\| \sum_{j,k=1}^n \frac{\partial^2 u}{\partial x_j \partial x_k} \right\| L_p(\mathbb{R}^n) &\leq 2 \left\| \sum_{j,k=1}^n \frac{\partial^2 u}{\partial x_j \partial x_k} + \psi_\varrho(D)u \right\| L_p(\mathbb{R}^n) \\ &\leq c_\varrho (\|\Psi(D)u \mid L_p(\mathbb{R}^n)\| + \|u \mid L_p(\mathbb{R}^n)\|). \end{aligned}$$

Choosing $r = 1$ and $A = -\Psi(D)$ in Theorem 1.5.7 shows that $H_p^{\Psi,2}(\mathbb{R}^n) \hookrightarrow W_p^2(\mathbb{R}^n)$.

Because of this remark we can without loss of generality restrict ourselves to negative definite functions without quadratic part. We will do so if this helps to avoid cumbersome notation.

(iii) For $p = 2$ the spaces $H_2^{\psi,2}(\mathbb{R}^n)$ are Hilbert spaces and are denoted by $H^{\psi,2}(\mathbb{R}^n)$. The spaces $H^{\psi,2}(\mathbb{R}^n)$ coincide with the analogue of the Hörmander- $B_{k,p}$ -spaces; these are denoted by $B_{\psi,2}^2(\mathbb{R}^n)$ and are discussed in detail in [49, Section 4.10].

EXAMPLE 2.1.8. If $\psi(\xi) = |\xi_1|^{2/a_1} + \dots + |\xi_n|^{2/a_n}$ where $a_1, \dots, a_n \geq 1$ (cf. Example 1.1.13), then $H_p^{\psi,2}$ is the (classical) anisotropic Bessel potential space (of order 2); see [63] and [73, Section 4.2.2].

As usual, for $t \in \mathbb{R}$ we denote by $H^t(\mathbb{R}^n)$ the classical Bessel potential space, i.e. the collection of all $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|u \mid H^t(\mathbb{R}^n)\| = \|\mathcal{F}^{-1}[(1 + |\cdot|^2)^{t/2} \hat{u}] \mid L_2(\mathbb{R}^n)\| < \infty.$$

LEMMA 2.1.9. *The operator $\text{id} + \psi(D)$ maps $\mathcal{S}(\mathbb{R}^n)$ into $\bigcap_{t \geq 0} H^t(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$ and for any $R > 0$ the operator $\text{id} + \psi_R(D)$ maps $\mathcal{S}(\mathbb{R}^n)$ into itself.*

Proof. Let $u \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$. Then

$$\begin{aligned} \|D^\alpha(\text{id} + \psi(D))u \mid L_2(\mathbb{R}^n)\|^2 &= \int_{\mathbb{R}^n} |\xi^\alpha (1 + \psi(\xi))|^2 |\hat{u}(\xi)|^2 d\xi \\ &\leq c \int_{\mathbb{R}^n} (1 + |\xi|^2)^{|\alpha|} (1 + |\xi|^2)^2 |\hat{u}(\xi)|^2 d\xi = c \|u \mid H^{|\alpha|+2}(\mathbb{R}^n)\|^2 \end{aligned}$$

implying $D^\alpha(\text{id} + \psi(D))u \in L_2(\mathbb{R}^n)$. Since $\alpha \in \mathbb{N}_0^n$ was arbitrary, this leads to the first part of the conclusion.

Next let us remark that for $\alpha, \beta \in \mathbb{N}_0^n$ we find

$$\begin{aligned} (2\pi)^{n/2} x^\beta D_x^\alpha \psi_R(D)u(x) &= x^\beta D_x^\alpha \int_{\mathbb{R}^n} e^{ix \cdot \xi} \psi_R(\xi) \widehat{u}(\xi) d\xi \\ &= \int_{\mathbb{R}^n} \xi^\alpha (-i\partial_\xi)^\beta (e^{ix \cdot \xi}) \psi_R(\xi) \widehat{u}(\xi) d\xi \\ &= i^{|\beta|} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \xi^\alpha \partial_\xi^\beta [\psi_R(\xi) \widehat{u}(\xi)] d\xi, \end{aligned}$$

which yields

$$\sup_{x \in \mathbb{R}^n} |x^\beta D_x^\alpha \psi_R(D)u(x)| \leq \widetilde{c} \int_{\mathbb{R}^n} \frac{d\xi}{(1 + |\xi|^2)^{(n+1)/2}} \cdot \sup_{\eta \in \mathbb{R}^n} |\eta^\alpha (1 + |\eta|^2)^{(n+1)/2} \partial_\eta^\beta [\psi_R(\eta) \widehat{u}(\eta)]|.$$

Using the fact that $\widehat{u} \in \mathcal{S}(\mathbb{R}^n)$ and that ψ_R is an infinitely often differentiable function which has, together with all its partial derivatives, at most quadratic growth, we get by standard arguments the second conclusion of the lemma. ■

REMARK 2.1.10. Note that for a general continuous negative definite function we cannot expect $\text{id} + \psi(D)$ to map $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$.

LEMMA 2.1.11. *Let $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous negative definite function and define the function ψ_R , $R > 0$, as in (2.1.8). In addition, we define for $\lambda > 0$ on $\mathcal{S}(\mathbb{R}^n)$ the operator $(\lambda \text{id} + \psi_R(D))^{-1}$ by*

$$(\lambda \text{id} + \psi_R(D))^{-1}u = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \frac{1}{\lambda + \psi_R(\xi)} \widehat{u}(\xi) d\xi.$$

Then the operator $(\lambda \text{id} + \psi_R(D))^{-1}$ maps $\mathcal{S}(\mathbb{R}^n)$ continuously into itself and is the inverse of $\lambda \text{id} + \psi_R(D)$ on $\mathcal{S}(\mathbb{R}^n)$.

Proof. We may argue as in the second part of the proof of Lemma 2.1.9 to get

$$\begin{aligned} \sup_{x \in \mathbb{R}^n} |x^\beta D_x^\alpha (\lambda \text{id} + \psi_R(D))^{-1}u| \\ \leq c \int_{\mathbb{R}^n} \frac{1}{(1 + |\xi|^2)^{(n+1)/2}} d\xi \cdot \sup_{\eta \in \mathbb{R}^n} \left| \eta^\alpha (1 + |\eta|^2)^{(n+1)/2} \partial_\eta^\beta \left(\frac{\widehat{u}(\eta)}{\lambda + \psi_R(\eta)} \right) \right|, \end{aligned}$$

which implies that $(\lambda \text{id} + \psi_R(D))^{-1}$ maps $\mathcal{S}(\mathbb{R}^n)$ continuously into itself. Moreover, for $u \in \mathcal{S}(\mathbb{R}^n)$ we find

$$\begin{aligned} (\lambda \text{id} + \psi_R(D))^{-1}(\lambda \text{id} + \psi_R(D))u &= \mathcal{F}^{-1} \left[\frac{1}{\lambda + \psi_R} \mathcal{F}((\lambda \text{id} + \psi_R(D))u) \right] \\ &= \mathcal{F}^{-1} \left[\frac{1}{\lambda + \psi_R} (\lambda + \psi_R) \widehat{u} \right] = u, \end{aligned}$$

and an analogous calculation shows $(\lambda \text{id} + \psi_R(D))(\lambda \text{id} + \psi_R(D))^{-1}u = u$ on $\mathcal{S}(\mathbb{R}^n)$. ■

REMARK 2.1.12. (i) Recall that the operator $(\text{id} + \psi_R(D))^{-1}$ is also an L_p -contraction by our considerations in Section 1.5 and by Remark 2.1.2.

(ii) From Lemma 2.1.11 it is clear that $\mathcal{S}(\mathbb{R}^n) \subset D(A_R^{(1)})$ where

$$A_R^{(1)}|_{\mathcal{S}(\mathbb{R}^n)} = -\psi_R(D)|_{\mathcal{S}(\mathbb{R}^n)}$$

and $\psi_R(D)$ is defined as above.

By Corollary 2.1.3 we also know that $\tilde{\psi}_R(D)$ is a bounded operator in $L_1(\mathbb{R}^n)$. In addition, since $(-\tilde{\psi}_R(D), \mathcal{S}(\mathbb{R}^n))$ is a pre-generator of a strongly continuous contraction semigroup on $L_p(\mathbb{R}^n)$, $1 < p \leq 2$, we have

$$\|\lambda u + \tilde{\psi}_R(D)u | L_p(\mathbb{R}^n)\| \geq \lambda \|u | L_p(\mathbb{R}^n)\| \quad \text{for } 1 < p \leq 2$$

leading to

$$\|\lambda u + \tilde{\psi}_R(D)u | L_1(\mathbb{R}^n)\| \geq \lambda \|u | L_1(\mathbb{R}^n)\|$$

for all $u, v \in \mathcal{S}(\mathbb{R}^n)$; note that $|v|^p \leq |v| + |v|^2$ for $1 < p \leq 2$. Now it follows by standard perturbation arguments for the $L_1(\mathbb{R}^n)$ -generator $(A^{(1)}, D(A^{(1)}))$ of $(T_t^{(1)})_{t \geq 0}$ (see S. N. Ethier and T. G. Kurtz [18], p. 37) that

$$D(A^{(1)}) = D(A_R^{(1)}) \cap D(-\tilde{\psi}_R(D)) = D(A_R^{(1)}) \supset \mathcal{S}(\mathbb{R}^n),$$

i.e., we have proved that also for $p = 1$ the space $\mathcal{S}(\mathbb{R}^n)$ is contained in $D(A^{(1)})$.

THEOREM 2.1.13. *For $1 \leq p < \infty$ the space $\mathcal{S}(\mathbb{R}^n)$ is dense in $H_p^{\psi,2}(\mathbb{R}^n)$.*

Proof. By Remark 2.1.12 we know for $u \in H_p^{\psi,2}(\mathbb{R}^n)$ and $R > 0$ that $(\text{id} + \psi_R(D))u = f \in L_p(\mathbb{R}^n)$, i.e. $u = (\text{id} + \psi_R(D))^{-1}f$.

Let $(\varphi_j)_{j \in \mathbb{N}}$ be a sequence in $\mathcal{S}(\mathbb{R}^n)$ such that $\lim_{j \rightarrow \infty} \|\varphi_j - f | L_p(\mathbb{R}^n)\| = 0$. Then from Lemma 2.1.11 it follows

$$v_j := \mathcal{F}^{-1}[(1 + \psi_R(\cdot))^{-1}\hat{\varphi}_j] = (\text{id} + \psi_R(D))^{-1}\varphi_j \in \mathcal{S}(\mathbb{R}^n),$$

and $\|u - v_j\|_{\psi,R,p} = \|f - \varphi_j | L_p(\mathbb{R}^n)\|$, which implies that $\lim_{j \rightarrow \infty} \|u - v_j\|_{\psi,R,p} = 0$.

From (2.1.13) we also have $\lim_{j \rightarrow \infty} \|u - v_j | L_p(\mathbb{R}^n)\| = 0$. By the L_p -continuity of the operator $\tilde{\psi}_R(D)$ we obtain $\lim_{j \rightarrow \infty} \|\tilde{\psi}_R(D)u - \tilde{\psi}_R(D)v_j | L_p(\mathbb{R}^n)\| = 0$.

Consequently, by Remark 2.1.6(ii),

$$\lim_{j \rightarrow \infty} \|u - v_j | H_p^{\psi,2}(\mathbb{R}^n)\| \leq \lim_{j \rightarrow \infty} \|u - v_j\|_{\psi,R,p} + \lim_{j \rightarrow \infty} \|\tilde{\psi}_R(D)u - \tilde{\psi}_R(D)v_j | L_p(\mathbb{R}^n)\| = 0$$

and the theorem is proved. ■

We want to strengthen Theorem 2.1.13, namely to prove that $\mathcal{S}(\mathbb{R}^n)$ is an operator core for $A^{(p)}$ and that $D(A^{(p)}) = H_p^{\psi,2}(\mathbb{R}^n)$. For this we need the following auxiliary result. A semigroup version of this result can be found in E. B. Davies [16, Theorem 1.9].

LEMMA 2.1.14. *Let $(A, D(A))$ be the generator of a strongly continuous contraction semigroup $(T_t)_{t \geq 0}$ on a Banach space $(X, \|\cdot\|)$ and denote by $(R_\lambda^A)_{\lambda > 0}$ the resolvent of $(A, D(A))$, i.e. $R_\lambda^A = (\lambda \text{id} - A)^{-1}$. If $\mathcal{D} \subset D(A)$ is a cone such that \mathcal{D} is a dense subset and $R_\lambda^A(\mathcal{D}) \subset \mathcal{D}$ for all $\lambda > 0$, then \mathcal{D} is an operator core for $(A, D(A))$.*

Proof. Write $\|u\|_A = \|u | X\| + \|Au | X\|$ for the graph norm of the operator $(A, D(A))$. We have to prove that $\overline{\mathcal{D}}^{\|\cdot\|_A} = D(A)$.

Since $\mathcal{D} \subset X$ is dense, we find for every $f \in D(A)$ a sequence $(f_k)_{k \in \mathbb{N}}$ such that

$$\lim_{k \rightarrow \infty} \|f - f_k | X\| = 0.$$

Since R_λ^A preserves \mathcal{D} we have $R_\lambda^A f_k \in \mathcal{D}$, so

$$\begin{aligned} \|R_\lambda^A f_k - R_\lambda^A f\|_A &= \|R_\lambda^A f_k - R_\lambda^A f\|_X + \|AR_\lambda^A f_k - AR_\lambda^A f\|_X \\ &\leq \frac{1}{\lambda} \|f_k - f\|_X + 2\|f_k - f\|_X \end{aligned}$$

since $\|AR_\lambda^A\| \leq 2$. It follows that $(R_\lambda^A f_k)_{k \in \mathbb{N}}$ is a sequence in \mathcal{D} which converges in the graph norm $\|\cdot\|_A$ to $R_\lambda^A f$ and $R_\lambda^A f \in \overline{\mathcal{D}}^{\|\cdot\|_A}$. Since \mathcal{D} is a cone, we also have $\lambda R_\lambda^A f \in \overline{\mathcal{D}}^{\|\cdot\|_A}$ and we find

$$\begin{aligned} \|\lambda R_\lambda^A f - f\|_A &= \|\lambda R_\lambda^A f - f\|_X + \|\lambda AR_\lambda^A f - Af\|_X \\ &\leq \int_0^\infty \lambda e^{-\lambda t} \|T_t f - f\|_X dt + \int_0^\infty \lambda e^{-\lambda t} \|T_t Af - Af\|_X dt \\ &= \int_0^\infty e^{-s} (\|T_{s/\lambda} f - f\|_X + \|T_{s/\lambda} Af - Af\|_X) ds. \end{aligned}$$

Using the dominated convergence theorem, and recalling that $f \in D(A)$ and $T_{s/\lambda}$ is a contraction, for $\lambda \rightarrow \infty$ we get

$$\lambda R_\lambda^A f \rightarrow f \in \overline{\mathcal{D}}^{\|\cdot\|_A}$$

where the convergence is with respect to the norm $\|\cdot\|_A$. Hence $D(A) \subset \overline{\mathcal{D}}^{\|\cdot\|_A}$ and the lemma follows. ■

THEOREM 2.1.15. *Let $1 \leq p < \infty$. For any continuous negative definite function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ of the form (2.1.1) we have $D(A^{(p)}) = H_p^{\psi,2}(\mathbb{R}^n)$, and $\mathcal{S}(\mathbb{R}^n)$ is an operator core for $(A^{(p)}, D(A^{(p)}))$.*

Proof. Consider the operator $\psi_R(D)$ where ψ_R is given by (2.1.8). We already know that $-\psi_R(D)$ extends to a generator of a strongly continuous L_p -semigroup which we will denote by $(A_R^{(p)}, D(A_R^{(p)}))$. Lemma 2.1.11 shows that its resolvent has the property that

$$R_\lambda^{A_R^{(p)}}|_{\mathcal{S}(\mathbb{R}^n)} = (\lambda + \psi_R(D))^{-1}$$

and that each of the operators $R_\lambda^{A_R^{(p)}}$ preserves $\mathcal{S}(\mathbb{R}^n)$. Now Lemma 2.1.14 is applicable, showing that $\mathcal{S}(\mathbb{R}^n)$ is a core for $(A_R^{(p)}, D(A_R^{(p)}))$, and because of Theorem 2.1.13 we get

$$D(A_R^{(p)}) = \overline{\mathcal{S}(\mathbb{R}^n)}^{\|\cdot\|_{A_R^{(p)}}} = H_p^{\psi_R,2}(\mathbb{R}^n) = H_p^{\psi,2}(\mathbb{R}^n),$$

where the last equality is essentially Remark 2.1.6(ii). Since $\mathcal{S}(\mathbb{R}^n) \subset L_p(\mathbb{R}^n)$ is dense and since

$$(A^{(p)} - A_R^{(p)})|_{\mathcal{S}(\mathbb{R}^n)} = (\psi_R(D) - \psi(D))|_{\mathcal{S}(\mathbb{R}^n)}$$

is (uniformly) bounded for $R \geq 1$, we also see that $D(A^{(p)}) = H_p^{\psi,2}(\mathbb{R}^n)$.

It remains to show that $\mathcal{S}(\mathbb{R}^n)$ is also a core for $A^{(p)}$. This, however, is clear since $A^{(p)} = A_R^{(p)} + (A^{(p)} - A_R^{(p)})$ and the second summand is a lower order perturbation, in fact it is an L_p -bounded operator. ■

We want to return to Remark 2.1.10 and to Lemma 2.1.9 and investigate the operator $\psi(D)$ on the space $\bigcap_{t \geq 0} H^t(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$.

LEMMA 2.1.16. *Let $f \in \bigcap_{t \geq 0} H^t(\mathbb{R}^n)$, let $\lambda > 0$ and define*

$$u := \mathcal{F}^{-1} \left[\frac{1}{\lambda + \psi(\cdot)} \widehat{f}(\cdot) \right].$$

Then $u \in \bigcap_{t \geq 0} H^t(\mathbb{R}^n)$ and $(\lambda + \psi(D))u = f$.

If, in addition, $f \in L_p(\mathbb{R}^n)$, $1 < p < \infty$, then $u \in \bigcap_{t \geq 0} H^t(\mathbb{R}^n) \cap H_p^{\psi,2}(\mathbb{R}^n)$.

Proof. For $\alpha \in \mathbb{N}_0^n$ we find

$$D^\alpha u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \frac{1}{\lambda + \psi(\xi)} (D^\alpha f)^\wedge(\xi) d\xi,$$

and by Plancherel's theorem we have

$$\begin{aligned} \|D^\alpha u\|_{L_2(\mathbb{R}^n)} &= \|(D^\alpha u)^\wedge\|_{L_2(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \left| \frac{1}{\lambda + \psi(\xi)} (D^\alpha f)^\wedge(\xi) \right|^2 d\xi \right)^{1/2} \\ &\leq \frac{1}{\lambda} \|(D^\alpha f)^\wedge\|_{L_2(\mathbb{R}^n)} = \frac{1}{\lambda} \|D^\alpha f\|_{L_2(\mathbb{R}^n)}, \end{aligned}$$

which implies $u \in \bigcap_{t \geq 0} H^t(\mathbb{R}^n)$.

Further, we have

$$(\lambda + \psi(D))u = \mathcal{F}^{-1}[(\lambda + \psi)\widehat{u}] = \mathcal{F}^{-1} \left[(\lambda + \psi) \frac{1}{\lambda + \psi} \widehat{f} \right] = f.$$

Thus $f \in L_p(\mathbb{R}^n)$ yields $(\lambda + \psi(D))u \in L_p(\mathbb{R}^n)$, i.e. $u \in \bigcap_{t \geq 0} H^t(\mathbb{R}^n) \cap H_p^{\psi,2}(\mathbb{R}^n)$, and this proves the lemma. ■

PROPOSITION 2.1.17. *For $1 < p < \infty$ the space $\bigcap_{t \geq 0} H^t(\mathbb{R}^n) \cap H_p^{\psi,2}(\mathbb{R}^n)$ is contained in $D(A^{(p)})$.*

Proof. For $u \in \bigcap_{t \geq 0} H^t(\mathbb{R}^n) \cap H_p^{\psi,2}(\mathbb{R}^n)$ the norm $\|(1 + |\cdot|)^2 \widehat{u}(\cdot)\|_{L_2(\mathbb{R}^n)}$ is finite and $(1/t)(T_t^{(2)}u - u) + \psi(D)u$ belongs to $L_2(\mathbb{R}^n)$, hence we may argue as in the proof of Proposition 2.1.1.

In fact, all arguments remain true for $u \in H^{\psi,2}(\mathbb{R}^n)$ since then $\psi(\cdot)\widehat{u} \in L_2(\mathbb{R}^n)$.

On the other hand, for $u \in \bigcap_{t \geq 0} H^t(\mathbb{R}^n) \subset C_\infty^\infty(\mathbb{R}^n)$ it follows that $\psi(D)u \in L_\infty(\mathbb{R}^n)$ and therefore we find

$$\begin{aligned} &\left\| \frac{T_t^{(\infty)}u - u}{t} + \psi(D)u \right\|_{L_\infty(\mathbb{R}^n)} \\ &= \left\| \mathcal{F}^{-1} \left[\left(\frac{e^{-t\psi} - 1}{t} + \psi \right) \widehat{u} \right] \right\|_{L_\infty(\mathbb{R}^n)} \\ &= \left\| \mathcal{F}^{-1} \left[\left(\frac{e^{-t\psi} - 1}{t} + \psi \right) (1 + |\cdot|^2)^{-n-2} (1 + |\cdot|^2)^{n+2} \widehat{u} \right] \right\|_{L_\infty(\mathbb{R}^n)}. \end{aligned}$$

Applying elementary properties of the inverse Fourier transform and using Plancherel's

formula we get

$$\begin{aligned}
& \left\| \frac{T_t^{(\infty)}u - u}{t} + \psi(D)u \right\|_{L_\infty(\mathbb{R}^n)} \\
& \leq (2\pi)^{-n/2} \left\| \left(\frac{e^{-t\psi} - 1}{t} + \psi \right) (1 + |\cdot|^2)^{-n-2} \right\|_{L_2(\mathbb{R}^n)} \cdot \|(\text{id} - \Delta)^{n+2}u\|_{L_2(\mathbb{R}^n)} \\
& \leq (2\pi)^{-n/2} \frac{t}{2} \|\psi^2(1 + |\cdot|^2)^{-n-2}\|_{L_2(\mathbb{R}^n)} \cdot \|(\text{id} - \Delta)^{n+2}u\|_{L_2(\mathbb{R}^n)} \\
& \leq ct \|(1 + |\cdot|^2)^2(1 + |\cdot|^2)^{-n-2}\|_{L_2(\mathbb{R}^n)} \cdot \|u\|_{H^{2n+4}(\mathbb{R}^n)},
\end{aligned}$$

which implies

$$\lim_{t \rightarrow 0} \left\| \frac{T_t^{(\infty)}u - u}{t} + \psi(D)u \right\|_{L_\infty(\mathbb{R}^n)} = 0 \quad \text{for } u \in \bigcap_{t \geq 0} H^t(\mathbb{R}^n).$$

For $p \geq 2$ and $u \in \bigcap_{t \geq 0} H^t(\mathbb{R}^n) \cap H_p^{\psi,2}(\mathbb{R}^n)$ the expression $(1/t)(T_t^{(p)}u - u) + \psi(D)u$ belongs to $L_p(\mathbb{R}^n)$, and we may argue again as in Proposition 2.1.1 to see that

$$\lim_{t \rightarrow 0} \left\| \frac{T_t^{(p)}u - u}{t} + \psi(D)u \right\|_{L_p(\mathbb{R}^n)} = 0,$$

and this means that $\bigcap_{t \geq 0} H^t(\mathbb{R}^n) \cap H_p^{\psi,2}(\mathbb{R}^n) \subset D(A^{(p)})$ for $p \geq 2$.

Let now $1 < p \leq 2$, $u \in H_p^{\psi,2}(\mathbb{R}^n)$ and $v \in \mathcal{S}(\mathbb{R}^n)$. First we observe that in this situation $\int_{\mathbb{R}^n} \psi(D)u \cdot v \, dx = \int_{\mathbb{R}^n} u \cdot \psi(D)v \, dx$, and for $1/p + 1/p' = 1$ this leads to

$$\int_{\mathbb{R}^n} \left(\frac{T_t^{(p)}u - u}{t} + \psi(D)u \right) \cdot v \, dx = \int_{\mathbb{R}^n} u \cdot \left(\frac{T_t^{(p')}v - v}{t} + \psi(D)v \right) dx.$$

But now we may argue as in the proof of Proposition 2.1.1 to see that $\bigcap_{t \geq 0} H^t(\mathbb{R}^n) \cap H_p^{\psi,2}(\mathbb{R}^n)$ belongs to the domain of the weak generator of $(T_t^{(p)})_{t \geq 0}$, hence it is a subspace of the domain of the generator. ■

REMARK 2.1.18. Lemma 2.1.16 and Proposition 2.1.17 give an alternative proof for

$$D(A^{(p)}) = H_p^{\psi,2}(\mathbb{R}^n), \quad 1 < p < \infty.$$

We may consider the operator $-\psi(D)$ on $L_p(\mathbb{R}^n)$ with domain $\bigcap_{t \geq 0} H^t(\mathbb{R}^n) \cap H_p^{\psi,2}(\mathbb{R}^n)$ or $\mathcal{S}(\mathbb{R}^n)$ and prove that it extends to a generator of an L_p -sub-Markovian semigroup having domain $H_p^{\psi,2}(\mathbb{R}^n)$. Then it is possible to identify this extension with $(A^{(p)}, D(A^{(p)}))$.

2.2. The spaces $H_p^{\psi,s}(\mathbb{R}^n)$. Let $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a fixed continuous negative definite function. We denote again by $A^{(p)}$, $1 \leq p < \infty$, the generator of the L_p -sub-Markovian semigroup $(T_t^{(p)})_{t \geq 0}$ associated with ψ . From Theorem 2.1.15 we know that $D(A^{(p)}) = H_p^{\psi,2}(\mathbb{R}^n)$.

Let $s \geq 0$. Using the considerations from Section 1.5 we define the abstract Bessel potential space associated with $A^{(p)}$ by

$$H_p^{\psi,s}(\mathbb{R}^n) := \mathcal{F}_{s,A^{(p)},L_p(\mathbb{R}^n)} := (\text{id} - A^{(p)})^{-s/2}(L_p(\mathbb{R}^n)), \quad s > 0, \quad 1 \leq p < \infty.$$

For $s = 0$ we set $H_p^{\psi,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$. The norm on $\mathcal{F}_{s,A^{(p)},L_p(\mathbb{R}^n)}$ is given by

$$\|u | \mathcal{F}_{s,A^{(p)},L_p(\mathbb{R}^n)}\| = \|f | L_p(\mathbb{R}^n)\| \quad \text{for } u = (\text{id} - A^{(p)})^{-s/2} f.$$

From Section 1.5 we know that $H_p^{\psi,s+t}(\mathbb{R}^n) \hookrightarrow H_p^{\psi,s}(\mathbb{R}^n)$ for $s, t \geq 0$.

In what follows we will use the fact that $(\text{id} - A^{(p)})^{-s/2}$ is the operator V_s introduced in Section 1.5 by

$$V_s u = \frac{1}{\Gamma(s/2)} \int_0^\infty t^{s/2-1} e^{-t} T_t^{(p)} u dt$$

(compare Theorem 1.5.3). In particular, we use the fact that $(\text{id} - A^{(p)})^{-s/2}$ is a contraction on $L_p(\mathbb{R}^n)$, and that we have the semigroup property

$$(\text{id} - A^{(p)})^{-s/2} \circ (\text{id} - A^{(p)})^{-t/2} = (\text{id} - A^{(p)})^{-(s+t)/2},$$

which holds on $L_p(\mathbb{R}^n)$ for any $s, t \geq 0$.

The aim of this section is to identify $\mathcal{F}_{s,A^{(p)},L_p(\mathbb{R}^n)}$ with a function space. More precisely, we want to show that $\mathcal{F}_{s,A^{(p)},L_p(\mathbb{R}^n)}$ coincides with the closure of $\mathcal{S}(\mathbb{R}^n)$ with respect to the norm

$$(2.2.1) \quad \|u | H_p^{\psi,s}(\mathbb{R}^n)\| = \|\mathcal{F}^{-1}[(1 + \psi)^{s/2} \hat{u}] | L_p(\mathbb{R}^n)\|.$$

A first result is obtained by using Corollary 1.5.5.

COROLLARY 2.2.1. *For $s \geq 0$, $1 \leq p < \infty$, and a fixed continuous negative definite function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ we have $H_p^{\psi,s}(\mathbb{R}^n) = D((\text{id} - A^{(p)})^{s/2})$.*

In particular, $H_p^{\psi,s}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$.

Another immediate consequence is that for $u \in H_p^{\psi,s}(\mathbb{R}^n)$ we find $(\text{id} - A^{(p)})^{s/2} u \in L_p(\mathbb{R}^n)$ implying that

$$\|u | H_p^{\psi,s}(\mathbb{R}^n)\| = \|(\text{id} - A^{(p)})^{s/2} u | L_p(\mathbb{R}^n)\|.$$

Now we may apply the calculus for fractional powers of generators of contraction semi-groups to the family of operators $(\text{id} - A^{(p)})^{s/2}$, $s \geq 0$.

Let $s, r \geq 0$. As an identity of (closed) operators we have

$$(\text{id} - A^{(p)})^{s/2} \circ (\text{id} - A^{(p)})^{r/2} = (\text{id} - A^{(p)})^{(s+r)/2}$$

and the equality

$$\|u | H_p^{\psi,s+r}(\mathbb{R}^n)\| = \|(\text{id} - A^{(p)})^{r/2} u | H_p^{\psi,s}(\mathbb{R}^n)\|, \quad u \in H_p^{\psi,s+r}(\mathbb{R}^n),$$

holds. Moreover, we have

$$(\text{id} - A^{(p)})^{-s/2} \circ (\text{id} - A^{(p)})^{r/2} = (\text{id} - A^{(p)})^{(r-s)/2}$$

and also the estimate

$$\|(\text{id} - A^{(p)})^{r/2} u | L_p(\mathbb{R}^n)\| \leq \|(\text{id} - A^{(p)})^{(s+r)/2} u | L_p(\mathbb{R}^n)\|.$$

In this concrete setting the abstract Lemma 1.5.6 reads as follows:

COROLLARY 2.2.2. (i) *For all $s, t \geq 0$ we have $H_p^{\psi,s+t}(\mathbb{R}^n) \hookrightarrow H_p^{\psi,s}(\mathbb{R}^n)$, $1 \leq p < \infty$.*

(ii) *For all $s \geq 0$ the operator $(\text{id} - A^{(p)})^{s/2} : H_p^{\psi,s+r}(\mathbb{R}^n) \rightarrow H_p^{\psi,r}(\mathbb{R}^n)$, $1 \leq p < \infty$, is continuous.*

For $k \in \mathbb{N}$ and $u \in \mathcal{S}(\mathbb{R}^n)$ we have the representation

$$(\text{id} - A^{(p)})^k u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} (1 + \psi(\xi))^k \widehat{u}(\xi) d\xi,$$

which implies $\mathcal{S}(\mathbb{R}^n) \subset \bigcap_{t \geq 0} H_p^{\psi, t}(\mathbb{R}^n)$.

By Corollary 2.2.2(ii) we conclude that

$$(\text{id} - A^{(p)})^{s/2} u \in \bigcap_{t \geq 0} H_p^{\psi, t}(\mathbb{R}^n) \quad \text{for } s \geq 0, u \in \mathcal{S}(\mathbb{R}^n).$$

Further, we find that for all $r, s, t \geq 0$ such that $s - r \geq 0$ the operators

$$(2.2.2) \quad (\text{id} - A^{(p)})^{s/2} \circ (\text{id} - A^{(p)})^{-r/2} : H_p^{\psi, t}(\mathbb{R}^n) \rightarrow H_p^{\psi, t+s-r}(\mathbb{R}^n)$$

and

$$(2.2.3) \quad (\text{id} - A^{(p)})^{-r/2} \circ (\text{id} - A^{(p)})^{s/2} : H_p^{\psi, t}(\mathbb{R}^n) \rightarrow H_p^{\psi, t+s-r}(\mathbb{R}^n)$$

are continuous. From (2.2.2) and (2.2.3) we conclude that the operator $(\text{id} - A^{(p)})^{s/2} : H_p^{\psi, s+t}(\mathbb{R}^n) \rightarrow H_p^{\psi, t}(\mathbb{R}^n)$ is injective for $s \geq 0$. For $g \in H_p^{\psi, t}(\mathbb{R}^n)$ there exists $f \in L_p(\mathbb{R}^n)$ such that $g = (\text{id} - A^{(p)})^{-t/2} f$. Setting

$$u := (\text{id} - A^{(p)})^{-s/2} g = (\text{id} - A^{(p)})^{-(t+s)/2} f \in H_p^{\psi, s+t}(\mathbb{R}^n)$$

we find $(\text{id} - A^{(p)})^{s/2} u = g$, i.e., we have proved

COROLLARY 2.2.3. *For $s, t \geq 0$ and $1 \leq p < \infty$ the operator $(\text{id} - A^{(p)})^{s/2} : H_p^{\psi, s+t}(\mathbb{R}^n) \rightarrow H_p^{\psi, t}(\mathbb{R}^n)$ is a bijective continuous operator with continuous inverse.*

PROPOSITION 2.2.4. *For any $s \geq 0$ the space $\mathcal{S}(\mathbb{R}^n)$ is dense in $H_p^{\psi, s}(\mathbb{R}^n)$, $1 \leq p < \infty$.*

Proof. Let $u \in H_p^{\psi, s}(\mathbb{R}^n)$ and set $f := (\text{id} - A^{(p)})^{s/2} u$, so $\|u\|_{H_p^{\psi, s}(\mathbb{R}^n)} = \|f\|_{L_p(\mathbb{R}^n)}$. For $\varepsilon > 0$ there exists $v_\varepsilon \in \mathcal{S}(\mathbb{R}^n)$ such that $\|f - v_\varepsilon\|_{L_p(\mathbb{R}^n)} < \varepsilon$.

Defining $\omega_\varepsilon := (\text{id} - A^{(p)})^{-s/2} v_\varepsilon$ we deduce that $\omega_\varepsilon \in H_p^{\psi, s}(\mathbb{R}^n)$, and

$$\|u - \omega_\varepsilon\|_{H_p^{\psi, s}(\mathbb{R}^n)} = \|f - v_\varepsilon\|_{L_p(\mathbb{R}^n)} < \varepsilon. \quad \blacksquare$$

As an immediate consequence of Proposition 2.2.4 we deduce that $H_p^{\psi, s+t}(\mathbb{R}^n)$ is dense in $H_p^{\psi, s}(\mathbb{R}^n)$ for any $t, s \geq 0$.

THEOREM 2.2.5. *Let $s \geq 0$, $1 \leq p < \infty$, and $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a fixed continuous negative definite function. Then for all $u \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$(2.2.4) \quad (\text{id} - A^{(p)})^s u = \mathcal{F}^{-1}[(1 + \psi)^s \widehat{u}].$$

Proof. Let $u \in \mathcal{S}(\mathbb{R}^n)$, $s = k + \sigma$, $k \in \mathbb{N}_0$ and $0 \leq \sigma < 1$. Using Theorem 1.4.8 we have the representation

$$\begin{aligned} & (\text{id} - A^{(p)})^s u(x) \\ &= \frac{\sin \pi \sigma}{\pi} \int_0^\infty \lambda^{\sigma-1} R_\lambda^{(A^{(p)} - \text{id})} (\text{id} - A^{(p)})^{k+1} u(x) d\lambda \\ &= (2\pi)^{-n/2} \frac{\sin \pi \sigma}{\pi} \int_0^\infty \int_{\mathbb{R}^n} e^{ix \cdot \xi} \lambda^{\sigma-1} (\lambda + (1 + \psi(\xi)))^{-1} (1 + \psi(\xi))^{k+1} \widehat{u}(\xi) d\xi d\lambda \end{aligned}$$

where we used the fact that for $l \in \mathbb{N}_0$ and $u \in \mathcal{S}(\mathbb{R}^n)$ the formulae

$$(2.2.5) \quad (\text{id} - A^{(p)})^l u = \mathcal{F}^{-1}[(1 + \psi)^l \widehat{u}]$$

and

$$R_\lambda^{(A^{(p)} - \text{id})} u = \mathcal{F}^{-1} \left[\left(\frac{1}{\lambda + (1 + \psi)} \right) \widehat{u} \right]$$

hold. Taking into account that $\widehat{u} \in \mathcal{S}(\mathbb{R}^n)$ and that for $a > 0$,

$$\int_0^\infty \frac{\lambda^{\sigma-1}}{\lambda + a} d\lambda = \frac{\Gamma(\sigma)\Gamma(1-\sigma)}{\Gamma(1)} a^{\sigma-1} = \frac{\pi}{\sin \pi \sigma} a^{\sigma-1},$$

we further find

$$\begin{aligned} (\text{id} - A^{(p)})^s u(x) &= (2\pi)^{-n/2} \frac{\sin \pi \sigma}{\pi} \int_{\mathbb{R}^n} e^{ix \cdot \xi} (1 + \psi(\xi))^{k+1} \int_0^\infty \frac{\lambda^{\sigma-1}}{(\lambda + (1 + \psi(\xi)))} d\lambda \widehat{u}(\xi) d\xi \\ &= (2\pi)^{-n/2} \frac{\sin \pi \sigma}{\pi} \frac{\pi}{\sin \pi \sigma} \int_{\mathbb{R}^n} e^{ix \cdot \xi} (1 + \psi(\xi))^{k+\sigma} \widehat{u}(\xi) d\xi \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} (1 + \psi(\xi))^s \widehat{u}(\xi) d\xi. \end{aligned}$$

Since for $s = k \in \mathbb{N}_0$ formula (2.2.4) is already known, the theorem is proved. ■

From (2.2.4) we deduce that for $u \in \mathcal{S}(\mathbb{R}^n)$,

$$\|u | H_p^{\psi,s}(\mathbb{R}^n)\| = \|(\text{id} - A^{(p)})^{s/2} u | L_p(\mathbb{R}^n)\| = \|\mathcal{F}^{-1}[(1 + \psi)^{s/2} \widehat{u}] | L_p(\mathbb{R}^n)\|,$$

which means that (2.2.1) is proved.

Moreover, since $\mathcal{S}(\mathbb{R}^n)$ is dense in $H_p^{\psi,s}(\mathbb{R}^n)$ it follows that $H_p^{\psi,s}(\mathbb{R}^n)$ is the closure of $\mathcal{S}(\mathbb{R}^n)$ with respect to the norm $\|\mathcal{F}^{-1}[(1 + \psi)^{s/2} \widehat{u}] | L_p(\mathbb{R}^n)\|$.

From the proof of Theorem 2.2.5 we get the following

COROLLARY 2.2.6. *Let $s \geq 0$, $1 < p < \infty$, and $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous negative definite function. On $\mathcal{S}(\mathbb{R}^n)$ the operator $(-A^{(p)})^s$ has the representation*

$$(-A^{(p)})^s u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} (\psi(\xi))^s \widehat{u}(\xi) d\xi.$$

Now we may apply Theorem 1.5.10.

THEOREM 2.2.7. *For $u \in \mathcal{S}(\mathbb{R}^n)$ and $1 \leq p < \infty$ the following estimates hold:*

$$(2.2.6) \quad \gamma_0(\|\mathcal{F}^{-1}[\psi^{s/2} \widehat{u}] | L_p(\mathbb{R}^n)\| + \|u | L_p(\mathbb{R}^n)\|) \\ \leq \|u | H_p^{\psi,s}(\mathbb{R}^n)\| \leq \gamma_1(\|\mathcal{F}^{-1}[\psi^{s/2} \widehat{u}] | L_p(\mathbb{R}^n)\| + \|u | L_p(\mathbb{R}^n)\|).$$

By a density argument, (2.2.6) extends to all $u \in H_p^{\psi,s}(\mathbb{R}^n)$ with an appropriate interpretation of $\mathcal{F}^{-1}[\psi^{s/2} \widehat{u}]$ and $\mathcal{F}^{-1}[(1 + \psi)^{s/2} \widehat{u}]$, respectively.

REMARK 2.2.8. Note that (2.2.6) says that for the classical Bessel potential spaces $H_p^s(\mathbb{R}^n)$ (corresponding to $\psi(\xi) = |\xi|^2$),

$$\begin{aligned} \gamma_0(\|(-\Delta)^{s/2} | L_p(\mathbb{R}^n)\| + \|u | L_p(\mathbb{R}^n)\|) \\ \leq \|u | H_p^{\psi,s}(\mathbb{R}^n)\| \leq \gamma_1(\|(-\Delta)^{s/2} | L_p(\mathbb{R}^n)\| + \|u | L_p(\mathbb{R}^n)\|), \end{aligned}$$

i.e. it gives a comparison of the L_p -norms of Riesz potentials and Bessel potentials (compare E. M. Stein [76, Section V.3.2]).

In the previous considerations we have introduced the spaces $H_p^{\psi,s}(\mathbb{R}^n)$ for $s \geq 0$ and we have seen in Proposition 2.2.4 that $\mathcal{S}(\mathbb{R}^n)$ is dense in $H_p^{\psi,s}(\mathbb{R}^n)$.

In what follows we define the spaces $H_p^{\psi,s}(\mathbb{R}^n)$ for $s < 0$, and determine the dual of $H_p^{\psi,s}(\mathbb{R}^n)$ for the whole scale of parameters $s \in \mathbb{R}$.

DEFINITION 2.2.9. Let $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous negative definite function, let $1 \leq p < \infty$ and $s < 0$. The space $H_p^{\psi,s}(\mathbb{R}^n)$ is the closure of $\mathcal{S}(\mathbb{R}^n)$ in the norm

$$(2.2.7) \quad \|u | H_p^{\psi,s}(\mathbb{R}^n)\| = \|\mathcal{F}^{-1}[(1 + \psi)^{s/2}\hat{u}] | L_p(\mathbb{R}^n)\|.$$

Thus we have a scale of Bessel potential spaces $H_p^{\psi,s}(\mathbb{R}^n)$, $s \in \mathbb{R}$, for which $\mathcal{S}(\mathbb{R}^n)$ is a dense subset with respect to the norm (2.2.7).

Consequently, a continuous linear functional on $H_p^{\psi,s}(\mathbb{R}^n)$ can be interpreted in the usual way as an element of $\mathcal{S}'(\mathbb{R}^n)$.

More precisely, $l \in \mathcal{S}'(\mathbb{R}^n)$ belongs to the topological dual space $(H_p^{\psi,s}(\mathbb{R}^n))'$ of $H_p^{\psi,s}(\mathbb{R}^n)$ if, and only if, there exists $c > 0$ such that

$$(2.2.8) \quad |l(\varphi)| \leq c\|\varphi | H_p^{\psi,s}(\mathbb{R}^n)\| \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

The duality assertion must always be understood in this sense.

Recall that all function spaces which are considered here are real.

THEOREM 2.2.10. Let $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous negative definite function, let $s \in \mathbb{R}$, let $1 < p < \infty$ and $1/p + 1/p' = 1$. The topological dual space of $H_p^{\psi,s}(\mathbb{R}^n)$ is the space $H_{p'}^{\psi,-s}(\mathbb{R}^n)$ in the sense that for any $l \in (H_p^{\psi,s}(\mathbb{R}^n))'$ there exists $v \in H_{p'}^{\psi,-s}(\mathbb{R}^n)$ such that for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$(2.2.9) \quad l(\varphi) = \int_{\mathbb{R}^n} \mathcal{F}^{-1}[(1 + \psi)^{-s/2}\hat{v}](x) \cdot \mathcal{F}^{-1}[(1 + \psi)^{s/2}\hat{\varphi}](x) dx.$$

The norm of l is given by $\|v | H_{p'}^{\psi,-s}(\mathbb{R}^n)\|$.

Conversely, for any $v \in H_{p'}^{\psi,-s}(\mathbb{R}^n)$ an element of $(H_p^{\psi,s}(\mathbb{R}^n))'$ is defined by (2.2.9).

Proof. Step 1. We first prove that for any $v \in H_{p'}^{\psi,-s}(\mathbb{R}^n)$ we can define a continuous linear functional by (2.2.9).

Clearly, the integral in (2.2.9) converges, as a simple application of Hölder's inequality shows:

$$|l(\varphi)| \leq \|v | H_{p'}^{\psi,-s}(\mathbb{R}^n)\| \cdot \|\varphi | H_p^{\psi,s}(\mathbb{R}^n)\|.$$

By standard arguments we immediately deduce that the functional l defined by (2.2.9) has norm $\|l\| = \|v | H_{p'}^{\psi,-s}(\mathbb{R}^n)\|$.

Step 2. Conversely, assume that a linear functional l on $H_p^{\psi,s}(\mathbb{R}^n)$ is given and assume that (2.2.8) holds.

On $\mathcal{S}(\mathbb{R}^n)$ we introduce the norm

$$\|\varphi\| = \|\varphi | H_p^{\psi,s}(\mathbb{R}^n)\| = \|\mathcal{F}^{-1}[(1 + \psi)^{s/2}\hat{\varphi}] | L_p(\mathbb{R}^n)\|.$$

The continuous linear functional l on $(\mathcal{S}(\mathbb{R}^n), \|\cdot\|)$ has a representation by a function $w \in L_{p'}(\mathbb{R}^n)$, i.e.

$$(2.2.10) \quad l(\varphi) = \int_{\mathbb{R}^n} \mathcal{F}^{-1}[(1 + \psi)^{s/2} \widehat{\varphi}](x) \cdot w(x) \, dx, \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Since $w \in L_{p'}(\mathbb{R}^n)$, there exists a sequence $(w_n)_{n \in \mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$ such that

$$\|w_n - w\|_{L_{p'}(\mathbb{R}^n)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We remark that $v_n = \mathcal{F}^{-1}[(1 + \psi)^{s/2} \widehat{w}_n] \in H_p^{\psi, -s}(\mathbb{R}^n)$. Moreover, for $m \neq n$,

$$\|\mathcal{F}^{-1}[(1 + \psi)^{s/2} \mathcal{F}(w_n - w_m)]\|_{H_p^{\psi, -s}(\mathbb{R}^n)} = \|w_n - w_m\|_{L_{p'}(\mathbb{R}^n)}$$

and this implies that $(v_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $H_p^{\psi, -s}(\mathbb{R}^n)$. Denote by $v \in H_p^{\psi, -s}(\mathbb{R}^n)$ the limit of $(v_n)_{n \in \mathbb{N}}$ in $H_p^{\psi, -s}(\mathbb{R}^n)$. Since $w_n = \mathcal{F}^{-1}[(1 + \psi)^{-s/2} \widehat{v}_n]$ it follows that

$$(2.2.11) \quad w = \mathcal{F}^{-1}[(1 + \psi)^{-s/2} \widehat{v}] \in L_{p'}(\mathbb{R}^n).$$

But (2.2.11) says that (2.2.10) is in fact exactly the desired formula (2.2.9), and the theorem is proved. ■

REMARK 2.2.11. (i) Note that if $\psi(\xi) = |\xi|^2$ then $H_p^{\psi, s}(\mathbb{R}^n)$ is the classical Bessel potential space $H_p^s(\mathbb{R}^n)$.

(ii) For $p = 2$ the spaces $H_2^{\psi, s}(\mathbb{R}^n)$ are Hilbert spaces and will be denoted by $H^{\psi, s}(\mathbb{R}^n)$. In particular, we write $H^s(\mathbb{R}^n)$ if $\psi(\xi) = |\xi|^2$.

REMARK 2.2.12. (i) If

$$(2.2.12) \quad \psi(\xi) = |\xi_1|^{2/a_1} + \dots + |\xi_n|^{2/a_n}$$

where $a_1, \dots, a_n \geq 1$ (cf. Example 1.1.13), then $H_p^{\psi, s}$ is the (classical) anisotropic Bessel potential space, denoted by $H_p^{s, a}(\mathbb{R}^n)$; see [63] and [73, Section 4.2.2].

(ii) Recall that the classical anisotropic Bessel potential space $H_p^{s, a}(\mathbb{R}^n)$ is defined via an anisotropic distance function associated with $a = (a_1, \dots, a_n)$.

Given $a_1, \dots, a_n \geq 1$, a function $\varrho : \mathbb{R}^n \rightarrow \mathbb{R}$ is called an *anisotropic distance function* associated with $a = (a_1, \dots, a_n)$ if $\varrho(\xi) > 0$ for any $\xi \in \mathbb{R}^n$ and

$$\varrho(t^{a_1} \xi_1, \dots, t^{a_n} \xi_n) = t \varrho(\xi), \quad t > 0, \quad \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n.$$

It is well known that any two anisotropic distance functions ϱ_1 and ϱ_2 associated with a are equivalent in the sense that there exist two constants $c, C > 0$ such that

$$c \varrho_1(\xi) \leq \varrho_2(\xi) \leq C \varrho_1(\xi), \quad \xi \in \mathbb{R}^n.$$

For every $a = (a_1, \dots, a_n)$ there exists an anisotropic distance function $\varrho \in C^\infty(\mathbb{R}^n \setminus \{0\})$.

For fixed $s \in \mathbb{R}$ and $1 < p < \infty$ equivalent anisotropic distance functions generate the same anisotropic Bessel potential space.

The function $\sqrt{\psi}$, where ψ is given in (2.2.12), is an anisotropic distance function.

REMARK 2.2.13. It is not hard to see that the scale $H_p^{\psi, s}(\mathbb{R}^n)$ covers as particular cases all of the so-called generalized Liouville spaces $L_p^{(\mu)}$ from [52, Definition 2.5] where con-

tinuous negative definite functions ψ of the form

$$\psi(\xi)^{s/2} = \mu_1(|\xi_1|) + \dots + \mu_n(|\xi_n|),$$

with appropriate one-variable functions μ_1, \dots, μ_n , are considered.

2.3. Embeddings for the spaces $H_p^{\psi,s}(\mathbb{R}^n)$. Some embeddings for the spaces $H_p^{\psi,s}(\mathbb{R}^n)$ were tacitly stated in the previous section. We will collect them now systematically.

THEOREM 2.3.1. *Let $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous negative definite function, $1 < p < \infty$, and $t \geq 0$. Then for any $s \in \mathbb{R}$,*

$$(2.3.1) \quad H_p^{\psi,s+t}(\mathbb{R}^n) \hookrightarrow H_p^{\psi,s}(\mathbb{R}^n).$$

(If $s \geq 0$, then (2.3.1) also holds for $p = 1$.)

Proof. If $s \geq 0$ and $1 \leq p < \infty$, then the embedding (2.3.1) is just Corollary 2.2.2.

If $s < 0$ and $1 < p < \infty$ we distinguish two cases: $-s - t \geq 0$ and $-s - t < 0$.

Case 1. If $-s - t \geq 0$, then again by Corollary 2.2.2 we have the embedding

$$H_{p'}^{\psi,-s}(\mathbb{R}^n) \hookrightarrow H_{p'}^{\psi,-s-t}(\mathbb{R}^n) \quad \text{where } 1/p + 1/p' = 1,$$

which leads, after applying the duality theorem (see Theorem 2.2.10), to (2.3.1).

Case 2. If $-s - t < 0$, then $H_p^{\psi,s+t}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n)$. Since $H_{p'}^{\psi,-s}(\mathbb{R}^n) \hookrightarrow L_{p'}(\mathbb{R}^n)$ we obtain the embedding (2.3.1) again by applying the duality theorem, and this completes the proof. ■

Let $1 \leq p, q \leq \infty$ and denote by $\mathcal{M}_{p,q}$ the collection of Fourier multipliers of type (p, q) . Recall a tempered distribution $m \in \mathcal{S}'(\mathbb{R}^n)$ is called a *Fourier multiplier* of type (p, q) if

$$\|m\|_{\mathcal{M}_{p,q}} = \sup \left\{ \frac{\|\mathcal{F}^{-1}[m\widehat{\varphi}]\|_{L_q(\mathbb{R}^n)}}{\|\varphi\|_{L_p(\mathbb{R}^n)}} : 0 \neq \varphi \in \mathcal{S}(\mathbb{R}^n) \right\} < \infty.$$

For standard properties of the space $\mathcal{M}_{p,q}$ we refer to L. Hörmander [41]. Let us only mention that

$$\mathcal{F}(L_1(\mathbb{R}^n)) \subset \mathcal{M}_{p,p} \subset \mathcal{M}_{2,2} = L_\infty(\mathbb{R}^n), \quad 1 < p < \infty,$$

and

$$(2.3.2) \quad \mathcal{M}_{1,q} \subset \mathcal{M}_{q',\infty} = \mathcal{F}(L_q(\mathbb{R}^n)) \quad \text{for } 1 < q \leq \infty, \quad 1/q + 1/q' = 1.$$

Moreover, if $q \leq 2$ the space $\mathcal{M}_{p,q}$ contains locally integrable functions.

We are now in a position to state a general embedding theorem for the spaces $H_p^{\psi,s}(\mathbb{R}^n)$ using the notion of Fourier multipliers.

THEOREM 2.3.2. *Let $\psi_1, \psi_2 : \mathbb{R}^n \rightarrow \mathbb{R}$ be two continuous negative definite functions, let $s, r \in \mathbb{R}$ and $1 \leq p, q < \infty$. Then*

$$(2.3.3) \quad H_p^{\psi_1,s}(\mathbb{R}^n) \hookrightarrow H_q^{\psi_2,r}(\mathbb{R}^n)$$

if, and only if,

$$(2.3.4) \quad m := (1 + \psi_2)^{r/2} (1 + \psi_1)^{-s/2} \in \mathcal{M}_{p,q}.$$

Proof. We first assume that (2.3.4) is satisfied. Let $u \in \mathcal{S}(\mathbb{R}^n) \hookrightarrow H_p^{\psi_1, s}(\mathbb{R}^n)$ and $v := \mathcal{F}^{-1}[(1 + \psi_1)^{s/2} \widehat{u}] \in L_p(\mathbb{R}^n)$. Consequently, we have

$$\begin{aligned} \|u\|_{H_q^{\psi_2, r}(\mathbb{R}^n)} &= \|\mathcal{F}^{-1}[(1 + \psi_2)^{r/2} \widehat{u}]\|_{L_q(\mathbb{R}^n)} = \|\mathcal{F}^{-1}[m \widehat{v}]\|_{L_q(\mathbb{R}^n)} \\ &\leq \|m\|_{\mathcal{M}_{p, q}} \cdot \|v\|_{L_p(\mathbb{R}^n)} = c \|u\|_{H_p^{\psi_1, s}(\mathbb{R}^n)}, \end{aligned}$$

which proves, by standard density arguments, the embedding (2.3.3).

Assume now the embedding (2.3.3) and let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\begin{aligned} \|\mathcal{F}^{-1}[m \widehat{\varphi}]\|_{L_q(\mathbb{R}^n)} &= \|\mathcal{F}^{-1}[(1 + \psi_1)^{-s/2} \widehat{\varphi}]\|_{H_q^{\psi_2, r}(\mathbb{R}^n)} \\ &\leq c \|\mathcal{F}^{-1}[(1 + \psi_1)^{-s/2} \widehat{\varphi}]\|_{H_p^{\psi_1, s}(\mathbb{R}^n)} = c \|\varphi\|_{L_p(\mathbb{R}^n)} \end{aligned}$$

and this is (2.3.4). ■

One can easily obtain several embedding results from the above theorem.

COROLLARY 2.3.3. (i) *Let $1 < p < \infty$ and let $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous negative definite function which is unbounded. Then $H_p^{\psi, s}(\mathbb{R}^n) \hookrightarrow H_p^{\psi, r}(\mathbb{R}^n)$ if, and only if, $s \geq r$.*

(ii) *Let $1 < p < \infty$, $s > 0$, and $\psi_1, \psi_2 : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous negative definite functions. Then the embedding $H_p^{\psi_1, s}(\mathbb{R}^n) \hookrightarrow H_p^{\psi_2, s}(\mathbb{R}^n)$ implies that there exists a constant $c > 0$ such that*

$$1 + \psi_2(\xi) \leq c(1 + \psi_1(\xi)), \quad \xi \in \mathbb{R}^n.$$

If, in particular, $H_p^{\psi_1, s}(\mathbb{R}^n) = H_p^{\psi_2, s}(\mathbb{R}^n)$, we have

$$\frac{1}{c}(1 + \psi_1(\xi)) \leq 1 + \psi_2(\xi) \leq c(1 + \psi_1(\xi)).$$

The converse assertions hold for $p = 2$.

Proof. (i) If $s \geq r$ then the assertion was proved in Theorem 2.3.1 even for the case when ψ is bounded.

Assume now $H_p^{\psi, s}(\mathbb{R}^n) \hookrightarrow H_p^{\psi, r}(\mathbb{R}^n)$. By Theorem 2.3.2,

$$m = (1 + \psi)^{(r-s)/2} \in \mathcal{M}_{p, p} \subset L_\infty(\mathbb{R}^n).$$

Since ψ is unbounded, $s \geq r$.

(ii) We only have to use Theorem 2.3.2 and the fact that $\mathcal{M}_{p, p} \subset L_\infty(\mathbb{R}^n)$. ■

We now present a Sobolev type embedding for the space $H_p^{\psi, s}(\mathbb{R}^n)$.

THEOREM 2.3.4. *Let $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a real-valued continuous negative definite function, $1 < p < \infty$, and $s \in \mathbb{R}$. Then*

$$(2.3.5) \quad H_p^{\psi, s}(\mathbb{R}^n) \hookrightarrow C_\infty(\mathbb{R}^n)$$

if, and only if,

$$(2.3.6) \quad \mathcal{F}^{-1}[(1 + \psi)^{-s/2}] \in L_{p'}(\mathbb{R}^n)$$

where $1/p + 1/p' = 1$.

REMARK 2.3.5. Note that according to (2.3.2) condition (2.3.6) is equivalent to the fact that the function $(1 + \psi)^{-s/2}$ is a Fourier multiplier of type (p, ∞) .

Proof of Theorem 2.3.4. First assume that (2.3.6) is satisfied.

Let $M(\cdot) := (1 + \psi(\cdot))^{-s/2}$. Using the definition of a Fourier multiplier of type (p, ∞) , for any $u \in \mathcal{S}(\mathbb{R}^n) \subset H_p^{\psi, s}(\mathbb{R}^n)$ we have

$$\begin{aligned} \|u\|_{L_\infty(\mathbb{R}^n)} &= \|\mathcal{F}^{-1}\{(1 + \psi)^{-s/2} \mathcal{F}(\mathcal{F}^{-1}[(1 + \psi)^{s/2} \widehat{u}])\}\|_{L_\infty(\mathbb{R}^n)} \\ &\leq \|M\|_{\mathcal{M}_{p, \infty}} \cdot \|\mathcal{F}^{-1}[(1 + \psi)^{s/2} \widehat{u}]\|_{L_p(\mathbb{R}^n)} = c \|u\|_{H_p^{\psi, s}(\mathbb{R}^n)}. \end{aligned}$$

Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $H_p^{\psi, s}(\mathbb{R}^n)$ each element $u \in H_p^{\psi, s}(\mathbb{R}^n)$ is a uniform limit of continuous functions. Therefore, it has a continuous representative, and the embedding (2.3.5) follows.

Conversely, if (2.3.5) is satisfied then

$$|u(0)| \leq c \|u\|_{H_p^{\psi, s}(\mathbb{R}^n)}, \quad u \in \mathcal{S}(\mathbb{R}^n).$$

This means that ε_0 (Dirac's distribution) is a continuous linear functional on $H_p^{\psi, s}(\mathbb{R}^n)$, hence by the duality theorem (Theorem 2.2.10),

$$\varepsilon_0 \in H_{p'}^{\psi, -s}(\mathbb{R}^n).$$

By the definition of the space $H_{p'}^{\psi, -s}(\mathbb{R}^n)$, the last condition is equivalent to (2.3.6). ■

REMARK 2.3.6. (i) Theorem 2.3.4 extends the result of L. P. Volevich and B. P. Paneyakh [80, Theorem 13.2] to the the case of general negative definite functions.

(ii) The essential step in the first part of the proof of Theorem 2.3.4 has a nice interpretation from the point of view of operator theory. For $u \in D((\text{id} + \psi(D))^{s/2}) = H_p^{\psi, s}(\mathbb{R}^n)$ it follows that

$$u = ((\text{id} + \psi(D))^{s/2})^{-1} \circ (\text{id} + \psi(D))^{s/2} u,$$

and

$$\|u\|_{L_\infty(\mathbb{R}^n)} \leq \|((\text{id} + \psi(D))^{s/2})^{-1} | L_p(\mathbb{R}^n) \rightarrow L_\infty(\mathbb{R}^n)\| \cdot \|u\|_{H_p^{\psi, s}(\mathbb{R}^n)}.$$

Since $((\text{id} + \psi(D))^{s/2})^{-1} = (\text{id} + \psi(D))^{-s/2} = (R_1^{-\psi(D)})^{s/2}$ (where $R_1^{-\psi(D)}$ is the resolvent of $-\psi(D)$ at 1) we are searching for a bound for $\|((\text{id} + \psi(D))^{s/2})^{-1} | L_p(\mathbb{R}^n) \rightarrow L_\infty(\mathbb{R}^n)\|$. This operator, however, is defined on $\mathcal{S}(\mathbb{R}^n)$ by

$$(\text{id} + \psi(D))^{-s/2} u = \mathcal{F}^{-1}[(1 + \psi)^{-s/2} \widehat{u}],$$

and therefore

$$\begin{aligned} \|(1 + \psi)^{-s/2} | \mathcal{M}_{p, \infty}\| &= \|(\text{id} + \psi(D))^{-s/2} | L_p(\mathbb{R}^n) \rightarrow L_\infty(\mathbb{R}^n)\| \\ &= \|(R_1^{-\psi(D)})^{s/2} | L_p(\mathbb{R}^n) \rightarrow L_\infty(\mathbb{R}^n)\|. \end{aligned}$$

REMARK 2.3.7. Theorem 2.3.4 is the natural extension of the embedding result stated in Theorem 2.3.2. In fact if we put $q = \infty$, $\psi_2 = \psi_1 = \psi$ and $r = 0$ then via the identification $H_\infty^{\psi, 0}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$ condition (2.3.4) becomes

$$m = (1 + \psi)^{-s/2} \in \mathcal{M}_{p, \infty} = \mathcal{F}(L_{p'}(\mathbb{R}^n))$$

and this is exactly (2.3.6).

EXAMPLE 2.3.8. Let $s \in \mathbb{R}$ and $1 < p < \infty$. Recall that if $\psi(\xi) = |\xi|^2$ then $H_p^{\psi, s}(\mathbb{R}^n)$ is the classical Bessel potential space $H_p^s(\mathbb{R}^n)$. Using the considerations from [63, Section 8.1], where the asymptotic behaviour of the Fourier transform of $(1 + |x|^2)^{-s/2}$ is studied

(see in particular formula (6) on p. 291), it is easy to see that (2.3.6) is equivalent to $s > n/p$.

So we recover the classical result that for $1 < p < \infty$ the space $H_p^s(\mathbb{R}^n)$ is embedded in $C_\infty(\mathbb{R}^n)$ if, and only if, $s > n/p$.

EXAMPLE 2.3.9. Let $s \in \mathbb{R}$, $1 < p < \infty$ and let $a_1, \dots, a_n \geq 1$. If ψ is a continuous negative definite function such that the metric $\sqrt{\psi}$ is equivalent to a smooth anisotropic distance function associated with $a = (a_1, \dots, a_n)$, then $H_p^{\psi, s}(\mathbb{R}^n)$ is the classical anisotropic Bessel potential space $H_p^{s, a}(\mathbb{R}^n)$.

Let us remark that in this case (2.3.6) is equivalent to $s > n/p$. Indeed, (2.3.6) means then $\varepsilon_0 \in H_{p'}^{-s, a}(\mathbb{R}^n)$, implying $-s < n(1/p' - 1)$ (with a proof similar to that in the isotropic case) and this is $s > n/p$.

Consequently, we recover the classical result stated for example in [63, Sections 6.3, 5.6.3] (see also [73, Section 4.2.3]).

Let us remark that in [15, formula (3.9)] it was already observed that $s > n/p$ implies (2.3.6) for smooth anisotropic distance functions associated with $a = (a_1, \dots, a_n)$.

We take a closer look at condition (2.3.6). For simplicity write

$$M(x) = (1 + \psi(x))^{-s/2}.$$

A useful criterion for testing (2.3.6) is the so-called *Carlson–Beurling inequality* stated below. A proof can be found for example in [15, Lemma 1.2] where also a more detailed analysis is done.

LEMMA 2.3.10. *Let $1 \leq p' < 2$, let $\kappa = n(1/p' - 1/2)$ and let $N > \kappa$ be an integer. If M is measurable with distributional derivatives $D^\sigma M \in L_2(\mathbb{R}^n)$, $|\sigma| \leq N$, then*

$$\|\mathcal{F}^{-1}M | L_{p'}(\mathbb{R}^n)\| \leq c \|M | L_2(\mathbb{R}^n)\|^{1-\kappa/N} \sum_{|\sigma|=N} \|D^\sigma M | L_2(\mathbb{R}^n)\|^{\kappa/N}.$$

REMARK 2.3.11. Let $0 < p' \leq 2$ and let $K_0 = \{x \in \mathbb{R}^n : |x| \leq 1\}$ and $K_j = \{x \in \mathbb{R}^n : 2^{j-1} \leq |x| \leq 2^j\}$ for $j = 1, 2, \dots$. According to Remark 1.5.2/1 and formula (1.5.2/11) in [78], as a simple application of Hölder’s inequality with exponents $2/(2 - p')$ and $2/p'$ we have

$$(2.3.7) \quad \|\mathcal{F}^{-1}M | L_{p'}(\mathbb{R}^n)\| \leq c \left(\sum_{j=0}^{\infty} \left(2^{jn(1/p'-1/2)} \left(\int_{K_j} |\mathcal{F}^{-1}M(x)|^2 dx \right)^{1/2} \right)^{p'} \right)^{1/p'}.$$

REMARK 2.3.12. Let $0 < p' \leq 2$ and $\kappa = n(1/p' - 1/2)$. According to [78, Subsection 1.5.4] (see also [73, Subsection 1.7.5]), the right-hand side of (2.3.7) is an equivalent quasi-norm (norm if $p' \geq 1$) in the Besov space $B_{2, p'}^\kappa(\mathbb{R}^n)$. Consequently,

$$(2.3.8) \quad \|\mathcal{F}^{-1}M | L_{p'}(\mathbb{R}^n)\| \leq c \|M | B_{2, p'}^\kappa(\mathbb{R}^n)\|.$$

COROLLARY 2.3.13. *If $M = (1 + \psi)^{-s/2} \in B_{2, p'}^\kappa(\mathbb{R}^n)$ where $0 < p' \leq 2$, then $\mathcal{F}^{-1}M \in L_{p'}(\mathbb{R}^n)$.*

PROPOSITION 2.3.14. *If $0 < p' \leq 2$, $t > \kappa$, and if $M \in H^t(\mathbb{R}^n)$, then $\mathcal{F}^{-1}M \in L_{p'}(\mathbb{R}^n)$.*

Proof. Due to our assumption on t there exists the following elementary embedding between Besov spaces (see for example [78, Proposition 2.3.2/2]):

$$(2.3.9) \quad B_{2,2}^t(\mathbb{R}^n) = H^t(\mathbb{R}^n) \hookrightarrow B_{2,p'}^{\kappa}(\mathbb{R}^n),$$

and the proposition is now a simple application of (2.3.8). ■

REMARK 2.3.15. One can prove Proposition 2.3.14 directly, without using the elementary embedding (2.3.9). For this one has to use the equivalence (of Littlewood–Paley type)

$$\|M | H^t(\mathbb{R}^n)\| \sim \left(\sum_{j=0}^{\infty} 2^{2jt} \|\mathcal{F}^{-1}M | L_2(K_j)\|^2 \right)^{1/2}$$

(if $t > 0$) and the inequality (2.3.7).

Let us, finally, give explicit conditions on a continuous negative definite function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ of the form (2.1.1) that guarantee the validity of the continuous embedding $H_p^{\psi,s}(\mathbb{R}^n) \hookrightarrow C_{\infty}(\mathbb{R}^n)$.

THEOREM 2.3.16. *Suppose that $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuous negative definite function with representation (2.1.1) and such that*

$$(2.3.10) \quad 1 + \psi(\xi) \geq c_0(1 + |\xi|^2)^{r_0}, \quad \xi \in \mathbb{R}^n,$$

for some constant $c_0 > 0$ and some $0 < r_0 \leq 1$. Fix $0 < \varepsilon < 1$, choose $1/(2 - \varepsilon) < \theta < 1$, and set

$$p = p_{\varepsilon,\theta} := \frac{1 + \theta\varepsilon}{1 + (\varepsilon - 1)\theta}.$$

Then for this p and its conjugate $p' = 1/\theta + \varepsilon$ we have the following continuous embeddings:

$$H_p^{\psi,\theta n/r_0}(\mathbb{R}^n) \hookrightarrow C_{\infty}(\mathbb{R}^n) \quad \text{and} \quad H_{p'}^{\psi,\theta n/r_0}(\mathbb{R}^n) \hookrightarrow C_{\infty}(\mathbb{R}^n).$$

Proof. Fix $R > 0$ and denote by ψ_R the continuous negative definite function (2.1.8) whose Lévy measure is supported in the ball $B(0, R)$. From the considerations of Section 2.1 we know that the norms $\|\cdot | H_p^{\psi,s}(\mathbb{R}^n)\|$ and $\|\cdot | H_p^{\psi_R,s}(\mathbb{R}^n)\|$ are equivalent for all $s \geq 0$ and any $R > 0$. Therefore, it is enough to prove the continuous embeddings

$$H_p^{\psi_R,\theta n/r_0}(\mathbb{R}^n) \hookrightarrow C_{\infty}(\mathbb{R}^n) \quad \text{and} \quad H_{p'}^{\psi_R,\theta n/r_0}(\mathbb{R}^n) \hookrightarrow C_{\infty}(\mathbb{R}^n).$$

The advantage is, of course, that ψ_R is a smooth function whereas ψ is, in general, only continuous. Moreover, the derivatives of ψ_R can be estimated using (1.1.6).

It is not hard to see that (2.3.10) holds if and only if

$$(2.3.11) \quad 1 + \psi_R(\xi) \geq c_R(1 + |\xi|^2)^{r_0}, \quad \xi \in \mathbb{R}^n,$$

is satisfied with the same r_0 and some constant $c_R > 0$.

In view of Theorem 2.3.4 we have to show that

$$\mathcal{F}^{-1}[(1 + \psi_R)^{\theta n/(2r_0)}] \in L_p(\mathbb{R}^n) \cap L_{p'}(\mathbb{R}^n)$$

where

$$p = \frac{1 + \theta\varepsilon}{1 + (\varepsilon - 1)\theta} \in (2, \infty) \quad \text{and} \quad p' = \frac{1}{\theta} + \varepsilon \in (1, 2).$$

Recall that for two infinitely differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ and for any $\alpha \in \mathbb{N}_0^n$ one has

$$(2.3.12) \quad \partial^\alpha (f \circ g) = \sum_{j=1}^{|\alpha|} f^{(j)}(g(\cdot)) \sum \frac{\alpha!}{\delta_\beta! \delta_\gamma! \cdots \delta_\omega!} \left(\frac{\partial^\beta g(\cdot)}{\beta!} \right)^{\delta_\beta} \cdots \left(\frac{\partial^\omega g(\cdot)}{\omega!} \right)^{\delta_\omega}$$

where the second sum extends over all pairwise different multi-indices $0 \neq \beta, \gamma, \dots, \omega \in \mathbb{N}_0^n$ and all $\delta_\beta, \delta_\gamma, \dots, \delta_\omega \in \mathbb{N}$ such that $\delta_\beta \beta + \delta_\gamma \gamma + \dots + \delta_\omega \omega = \alpha$ and $\delta_\beta + \delta_\gamma + \dots + \delta_\omega = j$.

Since

$$\frac{\theta n}{2r_0} 2r_0 p' = \theta n \left(\frac{1}{\theta} + \varepsilon \right) = n + \theta n \varepsilon > n,$$

we conclude from (2.3.11), (2.3.12), and (1.1.6) that

$$(2.3.13) \quad \partial^\alpha ((1 + \psi_R)^{-\theta n / (2r_0)}) \in L_{p'}(\mathbb{R}^n)$$

for all $\alpha \in \mathbb{N}_0^n$. Since $p' < 2$, we find using the Hausdorff–Young theorem that

$$(2.3.14) \quad \mathcal{F}^{-1}[\partial^\alpha ((1 + \psi_R)^{-\theta n / (2r_0)})] \in L_p(\mathbb{R}^n)$$

for any $\alpha \in \mathbb{N}_0^n$. This shows the embedding $H_{p'}^{\psi_R, \theta n / r_0}(\mathbb{R}^n) \hookrightarrow C_\infty(\mathbb{R}^n)$.

For $m \in \mathbb{N}$ we observe that

$$\begin{aligned} \mathcal{F}^{-1}[(1 + \psi_R)^{-\theta n / (2r_0)}](\xi) &= (1 + |\xi|^2)^{-m} (1 + |\xi|^2)^m \mathcal{F}^{-1}[(1 + \psi_R)^{-\theta n / (2r_0)}](\xi) \\ &= (1 + |\xi|^2)^{-m} \mathcal{F}^{-1}[(1 - \Delta)^m ((1 + \psi_R)^{-\theta n / (2r_0)})](\xi). \end{aligned}$$

An application of Hölder’s inequality gives

$$\begin{aligned} \|\mathcal{F}^{-1}[(1 + \psi_R)^{-\theta n / (2r_0)}] \|_{L_{p'}(\mathbb{R}^n)}^{p'} &= \int_{\mathbb{R}^n} (1 + |\xi|^2)^{-m p'} |\mathcal{F}^{-1}[(1 - \Delta)^m ((1 + \psi_R)^{-\theta n / (2r_0)})](\xi)|^{p'} d\xi \\ &\leq \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^{-m p' p / (p - p')} d\xi \right)^{1 - p' / p} \\ &\quad \times \left(\int_{\mathbb{R}^n} |\mathcal{F}^{-1}[(1 - \Delta)^m ((1 + \psi_R)^{-\theta n / (2r_0)})](\xi)|^p d\xi \right)^{p' / p}. \end{aligned}$$

(Observe that $p' < 2 < p$.) Choosing m large enough we find

$$\begin{aligned} \|\mathcal{F}^{-1}[(1 + \psi_R)^{-\theta n / (2r_0)}] \|_{L_{p'}(\mathbb{R}^n)} &\leq c(m, p, p') \|\mathcal{F}^{-1}[(1 - \Delta)^m ((1 + \psi_R)^{-\theta n / (2r_0)})] \|_{L_p(\mathbb{R}^n)} < \infty \end{aligned}$$

because of (2.3.14). This means that $\mathcal{F}^{-1}[(1 + \psi_R)^{-\theta n / (2r_0)}] \in L_{p'}(\mathbb{R}^n)$, and by Theorem 2.3.4 we find $H_p^{\psi_R, \theta n / r_0}(\mathbb{R}^n) \hookrightarrow C_\infty(\mathbb{R}^n)$. ■

REMARK 2.3.17. Note that the limiting case of Theorem 2.3.16 as $\theta \rightarrow 1/(2 - \varepsilon)$ gives $p = p' = 2$ and

$$H_2^{\psi, n / (2 - \varepsilon) r_0}(\mathbb{R}^n) \hookrightarrow C_\infty(\mathbb{R}^n), \quad 0 < \varepsilon < 1,$$

and letting $\varepsilon \rightarrow 0$ we get

$$H_2^{\psi, s}(\mathbb{R}^n) \hookrightarrow C_\infty(\mathbb{R}^n), \quad s > \frac{n}{2r_0}.$$

This, however, also follows from a direct calculation in $L_2(\mathbb{R}^n)$ involving Plancherel's theorem.

2.4. Complex interpolation. We are now going to present some complex interpolation results for the ψ -Bessel potential spaces.

Let us recall some basic facts on complex interpolation of Banach spaces following the presentation of [77, Section 1.9] where one can find further properties and references; see also [79, Section 1.6.3].

Let $\{X_0, X_1\}$ be an *interpolation couple*, that is to say, there exists a linear complex Hausdorff space H such that both X_0 and X_1 are linearly and continuously embedded in H . Assume that X_0 and X_1 are Banach spaces and set $X = X_0 + X_1$.

Let $G = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ be a strip in the complex plane.

Let $W(G, X)$ be the collection of all functions w defined on \overline{G} with the following properties:

- (α) w is X -continuous in \overline{G} and X -analytic in G with $\sup_{z \in \overline{G}} \|w(z) | X\| < \infty$.
- (β) $w(iy) \in X_0$ and $w(1 + iy) \in X_1$ with $y \in \mathbb{R}$ are continuous with respect to y in the respective Banach spaces.
- (γ) $\|w | W(G, X)\| = \max(\sup_{y \in \mathbb{R}} \|w(iy) | X_0\|, \sup_{y \in \mathbb{R}} \|w(1 + iy) | X_1\|) < \infty$.

By the maximum principle $W(G, X)$ is a Banach space.

DEFINITION 2.4.1. Let $\{X_0, X_1\}$ be an interpolation couple of Banach spaces, let $X = X_0 + X_1$ and let $0 < \theta < 1$. Then

$$[X_0, X_1]_\theta = \{u \in X : \text{there exists } w \in W(G, X) \text{ such that } w(\theta) = u\}$$

and

$$(2.4.1) \quad \|u | [X_0, X_1]_\theta\| = \inf\{\|w | W(G, X)\| : w \in W(G, X), w(\theta) = u\}.$$

An equivalent norm in $[X_0, X_1]_\theta$ is given in the next lemma.

LEMMA 2.4.2. Let $\{X_0, X_1\}$ be an interpolation couple of Banach spaces and let $0 < \theta < 1$. Then

$$(2.4.2) \quad \|u | [X_0, X_1]_\theta\| = \inf_w \left\{ \left(\sup_{y \in \mathbb{R}} \|w(iy) | X_0\| \right)^{1-\theta} \cdot \left(\sup_{y \in \mathbb{R}} \|w(1 + iy) | X_1\| \right)^\theta \right\},$$

the infimum ranging over all $w \in W(G, X)$ such that $w(\theta) = u$.

Proof. Step 1. According to condition (γ) from the definition of the space $W(G, X)$ we have

$$\sup_{y \in \mathbb{R}} \|w(iy) | X_0\|, \sup_{y \in \mathbb{R}} \|w(1 + iy) | X_1\| \leq \|w | W(G, X)\|,$$

and consequently the right-hand side of (2.4.2) does not exceed the left-hand side.

Step 2. In order to prove the converse inequality we observe that $z \mapsto a^{z-\theta} w(z)$, $a > 0$, is an admissible function for the infimum in (2.4.1). Using condition (γ) again, we have

$$(2.4.3) \quad \|u | [X_0, X_1]_\theta\| \leq \max\left\{ a^{-\theta} \sup_{y \in \mathbb{R}} \|w(iy) | X_0\|, a^{1-\theta} \sup_{y \in \mathbb{R}} \|w(1 + iy) | X_1\| \right\}.$$

If now $\|w(1 + iy) | X_1\| \equiv 0$, then $a \rightarrow \infty$ in (2.4.3) yields the desired conclusion. Otherwise we choose

$$a = \frac{\sup_{y \in \mathbb{R}} \|w(iy) | X_0\|}{\sup_{y \in \mathbb{R}} \|w(1 + iy) | X_1\|}$$

in (2.4.3), and deduce that the left-hand side of (2.4.2) does not exceed the right-hand side. ■

REMARK 2.4.3. The above lemma is a general version of Lemma 2.4.6/3 in [78].

EXAMPLE 2.4.4. Let $1 < p_0, p_1 < \infty$, let $0 < \theta < 1$, and let $1/p = (1 - \theta)/p_0 + \theta/p_1$. Then it is well known (see for example [77, Theorem 1.18.4]) that

$$(2.4.4) \quad [L_{p_0}(\mathbb{R}^n), L_{p_1}(\mathbb{R}^n)]_\theta = L_p(\mathbb{R}^n).$$

Applying Lemma 2.4.2 to (2.4.4) we have

$$(2.4.5) \quad \|v | L_p(\mathbb{R}^n)\| = \inf_g \{ (\sup_{y \in \mathbb{R}} \|g(iy) | L_{p_0}(\mathbb{R}^n)\|)^{1-\theta} \cdot (\sup_{y \in \mathbb{R}} \|g(1 + iy) | L_{p_1}(\mathbb{R}^n)\|)^\theta \},$$

the infimum being taken over all $g \in W(G, \mathcal{S}'(\mathbb{R}^n))$ such that $g(\theta) = v$.

Since $-\psi(D)$ generates a sub-Markovian semigroup, the operator $\text{id} + \psi(D)$ is a *positive operator*, i.e., $(-\infty, 0]$ is contained in the resolvent set and we have the estimate

$$\|(\text{id} + \psi(D) - \lambda \text{id})^{-1}\| \leq \frac{c}{1 + |\lambda|}, \quad \lambda \in (-\infty, 0].$$

By standard techniques (cf. H. Amann [2, Example 4.7.3(c), p. 164,] or H. Triebel [77, Theorem 1.15.3, p. 103]), we conclude that the imaginary powers of $\text{id} + \psi(D)$ are locally bounded.

LEMMA 2.4.5. *Let $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous negative definite function and $1 < p < \infty$. Then there exist constants $\gamma \geq 0$ and $C_\gamma \geq 1$ such that*

$$(2.4.6) \quad \|(\text{id} + \psi(D))^{iy} u | L_p(\mathbb{R}^n)\| \leq C_\gamma e^{\gamma|y|} \|u | L_p(\mathbb{R}^n)\|$$

for all $y \in \mathbb{R}$.

We return to our spaces $H_p^{\psi, s}(\mathbb{R}^n)$ and prove the main result of this section.

THEOREM 2.4.6. *Let $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous negative definite function, let $s_0, s_1 \in \mathbb{R}$, let $1 < p_0, p_1 < \infty$, and $0 < \theta < 1$. Set*

$$s = (1 - \theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}.$$

Then

$$[H_{p_0}^{\psi, s_0}(\mathbb{R}^n), H_{p_1}^{\psi, s_1}(\mathbb{R}^n)]_\theta = H_p^{\psi, s}(\mathbb{R}^n).$$

Proof. Let $X = H_{p_0}^{\psi, s_0}(\mathbb{R}^n) + H_{p_1}^{\psi, s_1}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ and $G = \{z \in \mathbb{C} : 0 < \text{Re } z < 1\}$. For simplicity let us write $H_\theta(\mathbb{R}^n) = [H_{p_0}^{\psi, s_0}(\mathbb{R}^n), H_{p_1}^{\psi, s_1}(\mathbb{R}^n)]_\theta$.

Step 1. Let $u \in H_\theta(\mathbb{R}^n)$ and choose any $w \in W(G, X)$ with $w(\theta) = u$. We define on G the function g_w by

$$(2.4.7) \quad g_w(z) = \mathcal{F}^{-1}[e^{(z-\theta)^2} (1 + \psi)^{((1-z)s_0 + zs_1)/2} \mathcal{F}(w(z))].$$

Let us check that g_w satisfies conditions (α) – (γ) for the interpolation couple $L_{p_k}(\mathbb{R}^n)$, $k = 0, 1$. It is clear that g_w is analytic in G , continuous in \overline{G} , and the boundedness condition of (α) is readily verified. Moreover,

$$\begin{aligned} g_w(iy) &= e^{(iy-\theta)^2} (1 + \psi(D))^{iy(s_1-s_0)/2} \mathcal{F}^{-1}[(1 + \psi)^{s_0/2} \mathcal{F}(w(iy))], \\ g_w(1 + iy) &= e^{(iy+1-\theta)^2} (1 + \psi(D))^{iy(s_1-s_0)/2} \mathcal{F}^{-1}[(1 + \psi)^{s_1/2} \mathcal{F}(w(1 + iy))]. \end{aligned}$$

By Lemma 2.4.5 we find for some $\gamma = \gamma(s_0, s_1) \geq 0$ and $C_\gamma \geq 1$ that

$$\begin{aligned} (2.4.8) \quad \|g_w(iy) | L_{p_0}(\mathbb{R}^n)\| &\leq |e^{(iy-\theta)^2}| \cdot \|(1 + \psi(D))^{iy(s_1-s_0)/2}\| \\ &\quad \times \|\mathcal{F}^{-1}[(1 + \psi)^{s_0/2} \mathcal{F}(w(iy)) | L_{p_0}(\mathbb{R}^n)]\| \\ &\leq C_\gamma e^{-y^2+\theta^2} e^{\gamma|y|} \|\mathcal{F}^{-1}[(1 + \psi)^{s_0/2} \mathcal{F}(w(iy)) | L_{p_0}(\mathbb{R}^n)]\| \\ &\leq M_\gamma \|\mathcal{F}^{-1}[(1 + \psi)^{s_0/2} \mathcal{F}(w(iy)) | L_{p_0}(\mathbb{R}^n)]\| \\ &= M_\gamma \|w(iy) | H_{p_0}^{\psi, s_0}(\mathbb{R}^n)\|, \end{aligned}$$

where

$$M_\gamma = \sup_{y \in \mathbb{R}} C_\gamma e^{-y^2+\theta^2} e^{\gamma|y|} < \infty.$$

In particular, $g_w(iy) \in L_{p_0}(\mathbb{R}^n)$. A similar calculation shows $g_w(1 + iy) \in L_{p_1}(\mathbb{R}^n)$ and

$$(2.4.9) \quad \|g_w(1 + iy) | L_{p_1}(\mathbb{R}^n)\| \leq \widetilde{M}_\gamma \|w(1 + iy) | H_{p_1}^{\psi, s_1}(\mathbb{R}^n)\|.$$

Finally, observe that

$$g_w(\theta) = \mathcal{F}^{-1}[(1 + \psi)^{s/2} \mathcal{F}(w(\theta))] =: v.$$

Using the interpolation result between the spaces $L_{p_k}(\mathbb{R}^n)$, $k = 0, 1$ (cf. Example 2.4.4), we find

$$\begin{aligned} \|u | H_{p_1}^{\psi, s_1}(\mathbb{R}^n)\| &= \|\mathcal{F}^{-1}[(1 + \psi)^{s/2} \mathcal{F}(w(\theta))] | L_p(\mathbb{R}^n)\| = \|v | L_p(\mathbb{R}^n)\| \\ &= \inf_{\substack{g \in W(G, S'(\mathbb{R}^n)) \\ g(\theta) = v}} \left\{ \sup_{y \in \mathbb{R}} \|g(iy) | L_{p_0}(\mathbb{R}^n)\|^{1-\theta} \cdot \sup_{y \in \mathbb{R}} \|g(1 + iy) | L_{p_1}(\mathbb{R}^n)\|^\theta \right\} \\ &\leq \sup_{y \in \mathbb{R}} \|g_w(iy) | L_{p_0}(\mathbb{R}^n)\|^{1-\theta} \cdot \sup_{y \in \mathbb{R}} \|g_w(1 + iy) | L_{p_1}(\mathbb{R}^n)\|^\theta, \end{aligned}$$

where g_w is the function (2.4.7)—notice that g_w is an admissible function for the “inf”. By (2.4.8) and (2.4.9) we have

$$\|u | H_p^{\psi, s}(\mathbb{R}^n)\| \leq M_\gamma^{1-\theta} \widetilde{M}_\gamma^\theta \sup_{y \in \mathbb{R}} \|w(iy) | H_{p_0}^{\psi, s_0}(\mathbb{R}^n)\|^{1-\theta} \cdot \sup_{y \in \mathbb{R}} \|w(1 + iy) | H_{p_1}^{\psi, s_1}(\mathbb{R}^n)\|^\theta.$$

Since $w \in W(G, X)$ with $w(\theta) = u$ was arbitrary, we can pass to the infimum over these w , and conclude from Lemma 2.4.2 that

$$\|u | H_p^{\psi, s}(\mathbb{R}^n)\| \leq M_\gamma^{1-\theta} \widetilde{M}_\gamma^\theta \|u | H_\theta\|.$$

This proves $[H_{p_0}^{\psi, s_0}(\mathbb{R}^n), H_{p_1}^{\psi, s_1}(\mathbb{R}^n)]_\theta = H_\theta \hookrightarrow H_p^{\psi, s}(\mathbb{R}^n)$.

Step 2. For the converse inclusion we assume that $u \in H_p^{\psi, s}(\mathbb{R}^n)$. Observe that $(\text{id} + \psi(D))^{s/2} (H_p^{\psi, s}(\mathbb{R}^n)) = L_p(\mathbb{R}^n)$, so $v := \mathcal{F}^{-1}[(1 + \psi)^{s/2} \mathcal{F}u] \in L_p(\mathbb{R}^n)$. Using Example 2.4.4 we get $[L_{p_0}(\mathbb{R}^n), L_{p_1}(\mathbb{R}^n)]_\theta = L_p(\mathbb{R}^n)$, and we may choose an arbitrary $g \in W(G, S'(\mathbb{R}^n))$ with $g(\theta) = v$ and $g(k + iy) \in L_{p_k}(\mathbb{R}^n)$, $k = 0, 1$. In analogy to

(2.4.7) we introduce a function w_g by

$$w_g(z) := \mathcal{F}^{-1}[e^{(z-\theta)^2} (1 + \psi)^{(z-1)s_0/2 - zs_1/2} \mathcal{F}(g(z))].$$

Using calculations similar to those in Step 1, one checks that $w_g(\theta) = u$ and that w_g satisfies conditions (α) – (γ) for the interpolation couple $H_{p_k}^{\psi, s_k}(\mathbb{R}^n)$, $k = 0, 1$. Invoking again Lemmas 2.4.2 and 2.4.5 we find—just as in the first part of the proof—that $H_p^{\psi, s}(\mathbb{R}^n) \hookrightarrow H_\theta = [H_{p_0}^{\psi, s_0}(\mathbb{R}^n), H_{p_1}^{\psi, s_1}(\mathbb{R}^n)]_\theta$. ■

2.5. Capacities and quasi-continuous modifications. So far we have not used the fact that the L_p -sub-Markovian semigroup $(T_t^{(p)})_{t \geq 0}$ associated with a continuous negative definite function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is sub-Markovian and that therefore each of the operators $V_r^{(p)} = (\text{id} - A^{(p)})^{-r/2}$ is positivity preserving, i.e. $u \geq 0$ a.e. implies $T_t^{(p)}u \geq 0$ a.e. and $V_r^{(p)}u \geq 0$ a.e. This property will now be used to study elements of the space $H_p^{\psi, r}(\mathbb{R}^n)$ using capacities. We will associate with the semigroup $(T_t^{(p)})_{t \geq 0}$ a one-parameter family $(\text{cap}_{r,p})_{r > 0}$ of (r, p) -capacities which should in fact be considered as a two-parameter family $(\text{cap}_{r,p}^\psi)_{r > 0, p > 1}$ associated with the real-valued continuous negative definite function ψ .

The concept of (r, p) -capacities was introduced by P. Malliavin in [58] for symmetric sub-Markovian semigroups (see also [59]), and many investigations have been done in the context of sub-Markovian semigroups by M. Fukushima and H. Kaneko (see [26]–[28] and [53]). We will follow these papers to a certain extent and omit some proofs.

For an open set $G \subset \mathbb{R}^n$ we introduce the (r, p) -capacity (associated with a continuous negative definite function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$) by

$$\text{cap}_{r,p}^\psi(G) := \inf\{\|u\| H_p^{\psi, r}(\mathbb{R}^n)\|^p : u \in H_p^{\psi, r}(\mathbb{R}^n) \text{ and } u \geq 1 \text{ a.e. on } G\}.$$

For an arbitrary set $E \subset \mathbb{R}^n$ we put

$$\text{cap}_{r,p}^\psi(E) = \inf\{\text{cap}_{r,p}^\psi(G) : E \subset G \text{ and } G \text{ open}\},$$

which turns $\text{cap}_{r,p}^\psi$ into an *outer capacity*.

REMARK 2.5.1. (i) Note that (r, p) -capacities can be defined for the (abstract) Bessel potential spaces $\mathcal{F}_{r,A,L_p}(\mathbb{R}^n)$ when A is the generator of an L_p -sub-Markovian semigroup; see [28] and [34].

(ii) Whenever it is clear which ψ is meant, we will write $\text{cap}_{r,p}$ instead of $\text{cap}_{r,p}^\psi$.

For a proof of the next result one should consult [28] or the very detailed discussion in the second chapter of [50].

THEOREM 2.5.2. *Let $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous negative definite function, $1 < p < \infty$, and $r > 0$.*

(i) *For any measurable set $E \subset \mathbb{R}^n$ we have $\lambda^{(n)}(E) \leq \text{cap}_{r,p}(E)$, i.e. any measurable set of (r, p) -capacity zero is necessarily of Lebesgue measure zero.*

(ii) *If $E \subset F \subset \mathbb{R}^n$, then $\text{cap}_{r,p}(E) \leq \text{cap}_{r,p}(F)$.*

(iii) *If $r \leq r'$ and $p \leq p'$, then $\text{cap}_{r,p}(E) \leq \text{cap}_{r',p'}(E)$ for all $E \subset \mathbb{R}^n$.*

(iv) For any sequence $(E_j)_{j \in \mathbb{N}}$ of subsets of \mathbb{R}^n we have

$$\text{cap}_{r,p} \left(\bigcup_{j=1}^{\infty} E_j \right) \leq \sum_{j=1}^{\infty} \text{cap}_{r,p}(E_j).$$

(v) For any decreasing sequence $(K_j)_{j \geq 1}$ of compact sets $K_j \subset \mathbb{R}^n$ we have

$$\text{cap}_{r,p} \left(\bigcap_{j=1}^{\infty} K_j \right) = \lim_{j \rightarrow \infty} \text{cap}_{r,p}(K_j) = \inf_{j \geq 1} \text{cap}_{r,p}(K_j).$$

DEFINITION 2.5.3. Let $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous negative definite function, $1 < p < \infty$, $r > 0$, and let $\text{cap}_{r,p}$ be its (r, p) -capacity.

(i) A set $N \subset \mathbb{R}^n$ with $\text{cap}_{r,p}(N) = 0$ is called an (r, p) -exceptional set (with respect to ψ).

(ii) A statement is said to hold (r, p) -quasi-everywhere (with respect to ψ) if there exists an (r, p) -exceptional set N such that the statement holds on $\mathbb{R}^n \setminus N$. We will use the abbreviation (r, p) -q.e. for (r, p) -quasi-everywhere.

(iii) A real-valued function u defined (r, p) -quasi-everywhere on \mathbb{R}^n is called (r, p) -quasi-continuous (with respect to ψ) if for every $\varepsilon > 0$ there exists an open set $G \subset \mathbb{R}^n$ such that $\text{cap}_{r,p}(G) < \varepsilon$ and $u|_{G^c}$ is continuous.

(iv) A function $\tilde{u} \in H_p^{\psi,r}(\mathbb{R}^n)$ is called an (r, p) -quasi-continuous modification of $u \in H_p^{\psi,r}(\mathbb{R}^n)$ if \tilde{u} is quasi-continuous and $u = \tilde{u}$ almost everywhere.

The following results were proved for abstract Bessel potential spaces $\mathcal{F}_{r,A,L_p}(\mathbb{R}^n)$ in [27]; detailed proofs are again given in [50]. We state these results for $\mathcal{F}_{r,A,L_p}(\mathbb{R}^n) = H_p^{\psi,r}(\mathbb{R}^n)$. Note that in [27] the regularity of $\mathcal{F}_{r,A,L_p}(\mathbb{R}^n)$ is required; see Remark 1.5.4.

In the case considered here, i.e. $\mathcal{F}_{r,A,L_p}(\mathbb{R}^n) = H_p^{\psi,r}(\mathbb{R}^n)$, the regularity is guaranteed by Proposition 2.2.4.

PROPOSITION 2.5.4. Let $u \in H_p^{\psi,r}(\mathbb{R}^n)$ be (r, p) -quasi-continuous and $u \geq 0$ a.e. on an open set G . Then $u \geq 0$ (r, p) -q.e. on G .

THEOREM 2.5.5. (i) Each $u \in H_p^{\psi,r}(\mathbb{R}^n)$ admits an (r, p) -quasi-continuous modification \tilde{u} which is unique up to (r, p) -quasi-everywhere equality.

Moreover, for any $\varrho > 0$ we have the following Chebyshev-type inequality:

$$\text{cap}_{r,p}^{\psi}(\{\tilde{u} > \varrho\}) \leq \frac{2}{\varrho^p} \|u\|_{H_p^{\psi,r}(\mathbb{R}^n)}^p, \quad 1 < p < \infty.$$

(ii) Let $(u_k)_{k \in \mathbb{N}}$ be a sequence in $H_p^{\psi,r}(\mathbb{R}^n)$ converging to $u \in H_p^{\psi,r}(\mathbb{R}^n)$. Then there exists a subsequence $(u_{k_l})_{l \in \mathbb{N}}$ such that

$$\lim_{l \rightarrow \infty} u_{k_l}(x) = \tilde{u}(x) \quad (r, p)\text{-q.e.}$$

where again \tilde{u} is the (r, p) -quasi-continuous modification of u .

An important result proved by S. Albeverio and Z. M. Ma in [1] in the abstract Bessel potential situation gives the possibility to obtain \tilde{u} from a kernel representation.

THEOREM 2.5.6. *There exists a kernel $v_r^{(p)}(x, dy)$ such that for all $u \in H_p^{\psi,r}(\mathbb{R}^n)$ given by $u = (\text{id} - A^{(p)})^{-r/2}f$, where $f \in L_p(\mathbb{R}^n)$, an (r, p) -quasi-continuous version of u is given by $x \mapsto \int_{\mathbb{R}^n} f(y) v_r^{(p)}(x, dy)$.*

REMARK 2.5.7. Note that from the formula

$$(\text{id} - A^{(p)})^{-r/2}u = \frac{1}{\Gamma(r/2)} \int_0^\infty t^{r/2-1} e^{-t} T_t u dt$$

it is clear that $(\text{id} - A^{(p)})^{-r/2}$ has a kernel representation.

The importance of Theorem 2.5.6 is that we may use a *single* kernel to get for *each* $u \in H_p^{\psi,r}(\mathbb{R}^n)$ an (r, p) -quasi-continuous representation.

We follow again the general line of [27] and [28].

THEOREM 2.5.8. *For any $A \subset \mathbb{R}^n$ with finite (r, p) -capacity there exists a unique function $u_A \in \{u \in H_p^{\psi,r}(\mathbb{R}^n) : \tilde{u} \geq 1 \text{ (} r, p\text{)-q.e. on } A\}$ which minimizes the norm $\|\cdot\|_{H_p^{\psi,r}(\mathbb{R}^n)}$. The function u_A is non-negative and satisfies*

$$(2.5.1) \quad \text{cap}_{r,p}(A) = \|u_A\|_{H_p^{\psi,r}(\mathbb{R}^n)}^p.$$

DEFINITION 2.5.9. The function u_A satisfying (2.5.1) is called the (r, p) -equilibrium potential of the set A .

Using the existence of the equilibrium potential, one can prove that $\text{cap}_{r,p}$ is in fact a Choquet capacity.

THEOREM 2.5.10. (i) *If a sequence $(u_\nu)_{\nu \in \mathbb{N}}$ of (r, p) -quasi-continuous functions $u_\nu \in H_p^{\psi,r}(\mathbb{R}^n)$ converges to $u \in H_p^{\psi,r}(\mathbb{R}^n)$ in the norm $\|\cdot\|_{H_p^{\psi,r}(\mathbb{R}^n)}$, then a subsequence of $(u_\nu)_{\nu \in \mathbb{N}}$ converges (r, p) -quasi-everywhere to an (r, p) -quasi-continuous modification \tilde{u} of u .*

(ii) *If $(A_\nu)_{\nu \in \mathbb{N}}$ is an increasing family of sets we have*

$$\text{cap}_{r,p} \left(\bigcup_{\nu=1}^\infty A_\nu \right) = \sup_{\nu \in \mathbb{N}} \text{cap}_{r,p}(A_\nu).$$

Clearly a corollary to this theorem is that $\text{cap}_{r,p}^\psi$ is a Choquet capacity.

Let $u \in H_p^{\psi,r}(\mathbb{R}^n)$ and \tilde{u} be an (r, p) -quasi-continuous modification of u . Further, let $E \subset \mathbb{R}^n$ be any set and consider the space

$$H_{p,E}^{\psi,r}(\mathbb{R}^n) := \{u \in H_p^{\psi,r}(\mathbb{R}^n) : \tilde{u} = 0 \text{ (} r, p\text{)-q.e. on } E\}.$$

From Theorem 2.5.5(ii) we deduce that $H_{p,E}^{\psi,r}(\mathbb{R}^n)$ is a closed subspace of $H_p^{\psi,r}(\mathbb{R}^n)$. It is of great interest whether the following (ψ, r, p) -spectral synthesis problem (in the sense of L. Hedberg [31]) is solvable:

PROBLEM 2.5.11. Let $F \subset \mathbb{R}^n$ be a closed set, let $0 < r \leq 1$ and let $u \in H_{p,F}^{\psi,r}(\mathbb{R}^n)$. Does there exist a sequence $(u_k)_{k \in \mathbb{N}}$,

$$u_k \in \hat{H}_p^{\psi,r}(F^c) := \overline{C_0^\infty(F^c)}^{\|\cdot\|_{H_p^{\psi,r}(\mathbb{R}^n)}},$$

with the interpretation $C_0^\infty(F^c) \subset C_0^\infty(\mathbb{R}^n)$, such that $\lim_{k \rightarrow \infty} \|u_k - u\|_{H_p^{\psi,r}(\mathbb{R}^n)} = 0$?

Since in this paper we are mainly interested in properties of the spaces $H_p^{\psi,r}(\mathbb{R}^n)$ as function spaces, we will not discuss the balayage problem in these spaces here.

Let us, finally, consider the question of comparability of (r,p) -capacities associated with different continuous negative definite functions $\psi_j : \mathbb{R}^n \rightarrow \mathbb{R}$, $j = 1, 2$. For the case $p = 2$ and $r = 1$, i.e. the symmetric Dirichlet space situation, such a comparison result is due to J. Hawkes [30]. Stated in our context, his theorem reads as follows:

THEOREM 2.5.12. *Let $\psi_j : \mathbb{R}^n \rightarrow \mathbb{R}$, $j = 1, 2$, be two continuous negative definite functions such that for all $\xi \in \mathbb{R}^n$,*

$$\frac{1}{M} \cdot \frac{1}{\lambda + \psi_1(\xi)} \leq \frac{1}{\lambda + \psi_2(\xi)}$$

holds for some $M > 0$ and some $\lambda > 0$. Then

$$\frac{1}{4} M \text{cap}_{1,2}^{\psi_1}(A) \leq \text{cap}_{1,2}^{\psi_2}(A)$$

for all analytic sets $A \subset \mathbb{R}^n$.

A combination of Theorem 2.5.8, equality (2.5.1), and Theorem 2.5.10(ii) enables us to obtain a comparison result for capacities from embedding results for the spaces $H_{p_1}^{\psi_1,r_1}(\mathbb{R}^n)$ and $H_{p_2}^{\psi_2,r_2}(\mathbb{R}^n)$.

COROLLARY 2.5.13. *Let $\psi_j : \mathbb{R}^n \rightarrow \mathbb{R}$, $j = 1, 2$, be two continuous negative definite functions and $0 < r_1, r_2 < \infty$, $1 < p_1, p_2 < \infty$. If the space $H_{p_2}^{\psi_2,r_2}(\mathbb{R}^n)$ is continuously embedded in the space $H_{p_1}^{\psi_1,r_1}(\mathbb{R}^n)$ and the estimate*

$$(2.5.2) \quad \|u\|_{H_{p_1}^{\psi_1,r_1}(\mathbb{R}^n)} \leq c \|u\|_{H_{p_2}^{\psi_2,r_2}(\mathbb{R}^n)}$$

holds, then for all analytic sets $A \subset \mathbb{R}^n$,

$$\text{cap}_{r_1,p_1}^{\psi_1}(A) \leq c \text{cap}_{r_2,p_2}^{\psi_2}(A).$$

Note that in Section 2.3 we give some conditions in order that (2.5.2) holds. Corollary 2.3.3 in particular generalizes Hawkes' result, i.e. if $1 + \psi_2 \leq c(1 + \psi_1)$ then $c' \text{cap}_{r,2}^{\psi_1} \leq \text{cap}_{r,2}^{\psi_2}$ for any $r > 0$.

An immediate consequence of Theorem 2.3.4 is

COROLLARY 2.5.14. *If $\mathcal{F}^{-1}[(1 + \psi)^{-r/2}] \in L_{p'}(\mathbb{R}^n)$, $1/p + 1/p' = 1$, then $\text{cap}_{r,p}(A) = 0$ implies $A = \emptyset$.*

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