Contents

Abstract

In recent years, there have been several studies of various 'approximate' versions of the key notion of amenability, which is defined for all Banach algebras; these studies began with work of Ghahramani and Loy in 2004. The present memoir continues such work: we shall define various notions of approximate amenability, and we shall discuss and extend the known background, which considers the relationships between different versions of approximate amenability. There are a number of open questions on these relationships; these will be considered.

In Chapter 1, we shall give all the relevant definitions and a number of basic results, partly surveying existing work; we shall concentrate on the case of Banach function algebras. In Chapter 2, we shall discuss these properties for the semigroup algebra $\ell^1(S)$ of a semigroup S. In the case where S has only finitely many idempotents, $\ell^1(S)$ is approximately amenable if and only if it is amenable.

In Chapter 3, we shall consider the class of weighted semigroup algebras of the form $\ell^1(\mathbb{N}_\wedge, \omega)$, where $\omega : \mathbb{Z} \to [1, \infty)$ is an arbitrary function. We shall determine necessary and sufficient conditions on ω for these Banach sequence algebras to have each of the various approximate amenability properties that interest us. In this way we shall illuminate the implications between these properties.

In Chapter 4, we shall discuss Segal algebras on $\mathbb{T}$ and on $\mathbb{R}$. It is a conjecture that every proper Segal algebra on $\mathbb{T}$ fails to be approximately amenable; we shall establish this conjecture for a wide class of Segal algebras.

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1. Preliminaries

1.1. Background. In this memoir, we shall be concerned with Banach algebras and derivations into bimodules over these algebras. In this first section, we shall recall some basic properties of Banach algebras to which we shall refer. For full details, see [11].

Let A be an algebra, always over the complex field, $\mathbb{C}$. The algebra formed by adjoining an identity to A (even when A already has an identity) is denoted by $A^\sharp$. A *character* on A is a non-zero homomorphism from A to $\mathbb{C}$; the collection of characters on A will be denoted by Φ_A, and called the *character space* of A. An algebra A *factors* if every element of A is the product of two other elements, and A *factors weakly* if $A = A^2$, that is to say, every element of A is a finite sum of products. Of course, A factors whenever A has a (one-sided) identity.

Let E and F be Banach spaces. Then we denote by $\mathcal{B}(E, F)$ the Banach space of all bounded linear maps from E to F. We write $\mathcal{B}(E)$ for $\mathcal{B}(E, E)$, so that $\mathcal{B}(E)$ is a unital Banach algebra with product the composition of operators. The dual of a Banach space E is denoted by E', and the duality is expressed by

$$(x, \lambda) \mapsto \langle x, \lambda\rangle, \qquad E \times E' \to \mathbb{C}.$$

For $T \in \mathcal{B}(E, F)$, the *dual* T' of T is defined by

$$\langle x, T'\lambda\rangle = \langle Tx, \lambda\rangle \qquad (x \in E,\ \lambda \in F'),$$

so that $T' \in \mathcal{B}(F', E')$. For a subset S of a Banach space E, the *annihilator of* S is

$$S^\circ = \{\lambda \in E' : \lambda | S = 0\}.$$

A closed subspace S of E is *weakly complemented* in E if S° is a complemented subspace in E'. The *projective tensor product* of E with itself is denoted by $E \mathbin{\widehat{\otimes}} E$; the projective norm is

$$\|a\| := \inf\Big\{\sum_{i=1}^{\infty} \|b_i\|\, \|c_i\| : a = \sum_{i=1}^{\infty} b_i \otimes c_i\Big\} \qquad (a \in E \mathbin{\widehat{\otimes}} E).$$

Now let A be an algebra and X an A-bimodule. A *derivation* from A into X is a linear map $D : A \to X$ such that

$$D(ab) = a \cdot Db + Da \cdot b \qquad (a, b \in A).$$

Derivations of the form

$$a \mapsto a \cdot x - x \cdot a, \qquad A \to X,$$

for some $x \in X$ are *inner derivations*. In this case, we say that x *implements* the inner derivation. Let $\varphi \in \Phi_A$. Then a *point derivation at* φ is a linear functional $d : A \to \mathbb{C}$

such that

$$d(ab) = \varphi(a)d(b) + \varphi(b)d(a) \quad (a, b \in A);$$

that is, a derivation $d : A \to \mathbb{C}$, where $\mathbb{C}$ has the A-bimodule actions

$$a \cdot z = z \cdot a = \varphi(a)z \quad (a \in A,\ z \in \mathbb{C}).$$

Let A be a Banach algebra. Then $A^\sharp$ is also a Banach algebra in a standard way which gives the adjoined identity norm one. There is a continuous *product map* $\pi : A \widehat{\otimes} A \to A$, so that π is the continuous linear operator satisfying the constraint that

$$\pi(a \otimes b) = ab \quad (a, b \in A).$$

A Banach space X which is an A-bimodule is a *Banach A-bimodule* if the module operations are continuous; in fact, we shall suppose that

$$\max\{\|a \cdot x\|, \|x \cdot a\|\} \le \|a\|\,\|x\| \quad (a \in A,\ x \in X).$$

For X a Banach A-bimodule, X' is also a Banach A-bimodule for module maps defined by

$$\langle x, \lambda \cdot a \rangle = \langle a \cdot x, \lambda \rangle, \quad \langle x, a \cdot \lambda \rangle = \langle x \cdot a, \lambda \rangle \quad (a \in A,\ x \in X,\ \lambda \in X').$$

The Banach algebra A is a *dual* Banach algebra if there is a $\|\cdot\|$-closed submodule X of A' such that $X' = A$ as a Banach space; this is equivalent to the requirement that X be a Banach space with $X' = A$ and that the product in A be separately $\sigma(A, X)$-continuous.

A [*bounded*] *left approximate identity* for A is a [bounded] net (u_α) such that $\|a - u_\alpha a\| \to 0$ for each $a \in A$; similarly for right and two-sided approximate identities. By Cohen's factorization theorem [11, Theorem 2.9.24], A factors whenever it has a bounded left or right approximate identity. From a pointwise perspective, A has *bounded approximate units* if there is a constant $K > 0$ such that, for each $a \in A$ and $\varepsilon > 0$, there is $u \in A$ such that $\|u\| \le K$ and $\|a - au\| + \|a - ua\| < \varepsilon$. It is standard that this implies that A has a bounded approximate identity [11, §2.9].

Let G be a locally compact group. We shall have occasion to refer to the *group algebra* $L^1(G)$ of G. This is the Banach space

$$\left\{ f : G \to \mathbb{C},\ f \text{ measurable} : \|f\| := \int_G |f(t)|\,\mathrm{d}\mu(t) < \infty \right\},$$

where μ denotes left Haar measure on G and we equate functions that are equal almost everywhere with respect to μ; the product on $L^1(G)$ is defined by

$$(f \star g)(t) = \int_G f(s)g(s^{-1}t)\,\mathrm{d}\mu(s) \quad (t \in G,\ f, g \in L^1(G)),$$

and now $(L^1(G), \star, \|\cdot\|)$ is a Banach algebra. In the case where G is discrete, we write $\ell^1(G)$ for $L^1(G)$. For details, see [11, §3.3], for example.

On purely notational matters, we write $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$, and we set $\mathbb{N}_k = \{1, 2, \ldots, k\}$ $(k \in \mathbb{N})$. For an algebra A, we write $\mathbb{M}_n(A)$ for the algebra of $n \times n$-matrices over A, so that $\mathbb{M}_n(A) = \mathbb{M}_n \otimes A$. Let A be a Banach algebra. When normed as the algebra of operators on the n-fold Cartesian product $(A^\sharp)^{(n)}$, the algebra $\mathbb{M}_n(A)$ is a Banach algebra. We denote the cardinality of a set S by $|S|$.

1.2. Amenability for Banach algebras. The notion of amenability for Banach algebras was introduced by Johnson in 1972 [41], and has been extremely fruitful; after 35 years the consequences continue to be actively studied. There is an extended discussion of this topic in [11, §2.8]. About the same time as Johnson's work appeared, Helemskii and his school in Moscow independently developed the theory of 'topological homology' which gave a more categorical approach to many of the same ideas [38].

Let A be a Banach algebra, and let X be a Banach A-bimodule. The space of continuous derivations from A to X is denoted by $\mathcal{Z}^1(A,X)$, and the space of inner derivations from A to X by $\mathcal{N}^1(A,X)$; the *first* (*continuous*) *cohomology group* of A with coefficients in X is defined to be the quotient space

$$\mathcal{H}^1(A,X) := \mathcal{Z}^1(A,X)/\mathcal{N}^1(A,X).$$

We shall see below that A is amenable if and only if $\mathcal{H}^1(A,X') = \{0\}$ for each Banach A-bimodule X, and that A is contractible if and only if $\mathcal{H}^1(A,X) = \{0\}$ for each Banach A-bimodule X. It is a famous conjecture that every contractible Banach algebra is finite-dimensional. This is true for commutative Banach algebras and for C^*-algebras; see [63] for a fuller discussion. The class of amenable Banach algebras is a far richer study than that of contractible Banach algebras. For example, the determination of which C^*-algebras are amenable, namely the nuclear ones, has been a major strand of operator theory in recent decades; see [11, 64].

One of the most famous results is the following theorem [41]; see also [11, Theorem 5.6.42]. It is the origin of the term 'amenable' for Banach algebras.

THEOREM 1.2.1 (Johnson). *Let G be a locally compact group. Then the group algebra $(L^1(G),\star)$ is an amenable Banach algebra if and only if G is amenable as a locally compact group.* ∎

Let S be a semigroup with semigroup algebra $\ell^1(S)$; see Chapter 2 below for more details of this algebra. The characterization of the semigroups S such that $\ell^1(S)$ is amenable as a Banach algebra is somewhat complicated; such a characterization is given in [13].

Let E be a Banach space. It is a deep and interesting question to determine when the algebra $\mathcal{B}(E)$ of all bounded linear operators on E is amenable. The intuition is that when E is infinite-dimensional then $\mathcal{B}(E)$ is 'too big' to be amenable, but this intuition has recently proved to be false. Let $\mathcal{K}(E)$ denote the closed ideal of compact operators in $\mathcal{B}(E)$. A recent paper of Argyros and Haydon [1] produces an infinite-dimensional Banach space E such that every operator on E has the form $\zeta I_E + T$, where $\zeta \in \mathbb{C}$ and T is a compact operator. Thus $\mathcal{K}(E)$ has codimension 1 in $\mathcal{B}(E)$. This solves the famous 'scalar-plus-compact' problem. It follows from [37] that $\mathcal{K}(E)$ is an amenable Banach algebra, and so $\mathcal{B}(E)$ is also amenable. This is the first known example of an infinite-dimensional Banach space E such that $\mathcal{B}(E)$ is amenable.

Consider the case where E is the standard sequence space ℓ^p, where $p \geq 1$. In the case where $p = 2$, ℓ^2 is a Hilbert space, and it was shown, effectively by Wassermann [68], that the C^*-algebra $\mathcal{B}(H)$ is amenable only if $\dim(H) < \infty$. A completely different proof that $\mathcal{B}(\ell^1)$ is not amenable was given by Read [58], and a synthesis of these two results was given by Ozawa [55]. Finally, building on results in [16], Runde has recently shown that $\mathcal{B}(\ell^p)$ is never amenable for $1 \leq p \leq \infty$ [65].

An apparently easier question is to determine the amenability of $\mathcal{K}(E)$. There are infinite-dimensional spaces E where the result is known: for example $\mathcal{K}(\ell^2)$ is amenable, but $\mathcal{K}(\ell^r \oplus \ell^s)$ is not amenable whenever $r, s \in (1,2) \cup (2,\infty)$ are distinct [37]. Since amenable algebras have a bounded approximate identity, this question is related to approximation properties for E and E' [37]. Nevertheless we still await a full characterization of the Banach spaces E such that $\mathcal{K}(E)$ is amenable.

The related notion of weak amenability was introduced in [3] for commutative algebras, and more generally in [43]. Indeed, by definition, a Banach algebra is *weakly amenable* if $\mathcal{H}^1(A, A') = \{0\}$. For a locally compact group G, the group algebra $L^1(G)$ is always weakly amenable [11, Theorem 5.6.48], and C^*-algebras are always weakly amenable [11, Theorem 5.6.77]. The Banach function algebra $\mathrm{lip}_\alpha(\mathbb{T})$ is not amenable for $0 < \alpha < 1$, but it is weakly amenable for $0 < \alpha \leq 1/2$ [3, Theorem 5.6.14].

For the sake of later comparisons, we recall the following properties of amenable Banach algebras; see [11, §§2.8, 2.9] for the proofs of the statements.

THEOREM 1.2.2. *Let A be a Banach algebra, and let I be a closed ideal of A.*

(i) *Suppose that A is amenable. Then A has a bounded approximate identity, and so $A = A^2$.*
(ii) *Suppose that A is amenable. Then A/I is amenable.*
(iii) *Suppose that A is amenable and that I has a bounded approximate identity. Then I is amenable.*
(iv) *Suppose that A is amenable and that I is weakly complemented. Then I has a bounded approximate identity* (*and so is amenable*).
(v) *Suppose that I and A/I are amenable. Then A is amenable.* ■

DEFINITION 1.2.3. An *approximate diagonal* for A is a net (m_α) in $A \widehat{\otimes} A$ such that, for each $a \in A$,

$$\|a \cdot m_\alpha - m_\alpha \cdot a\| \to 0 \quad \text{and} \quad \|\pi(m_\alpha)a - a\| \to 0.$$

A *bounded approximate diagonal* for A is a bounded net in $A \widehat{\otimes} A$ with the above properties.

One of the most useful characterizations of amenability is the following result from [42]; see also [11, Theorem 2.9.65]. Indeed, essentially all results that determine whether or not Banach algebras in a particular class are amenable rely on this characterization.

THEOREM 1.2.4. *A Banach algebra A is amenable if and only if it possesses a bounded approximate diagonal.* ■

There are two constants that are associated with an amenable Banach algebra that are relevant for us.

DEFINITION 1.2.5. Let A be a Banach algebra.

(i) For A amenable, the *amenability constant* of A is the infimum of the numbers $C \geq 1$ such that A has an approximate diagonal bounded by C.

(ii) For a Banach A-bimodule X and an inner derivation $D : A \to X$, the *implementation constant* of D is the infimum of the norms of the elements $x \in E$ such that x implements the derivation.

For calculations involving the amenability constant of some semigroup algebras, see [13] and [29].

For each $k \in \mathbb{N}$, there is a Banach algebra A, a Banach A-bimodule X, an inner derivation $D : A \to X'$, and $a \in A$ with $\|a\| = 1$ such that $\|\zeta\|$ is at least k whenever $\zeta \in X'$ and $D(a) = a \cdot \zeta - \zeta \cdot a$. The existence of such A, X, D and a follows from the construction in Example 1.5.5 below; certainly the implementation constant of such a derivation D is at least k.

1.3. Basic definitions. We shall be concerned with several variants of the above notion of amenability. It is known that amenable Banach algebras have rather special properties. With this in mind, the authors of [25] introduced several wider classes of Banach algebras by generalizing the notion of amenability to allow for 'approximate versions'. In this memoir we shall continue the investigation of these approximate notions of amenability, in particular by considering when various Banach function algebras connected with harmonic analysis have these properties.

We begin by recalling two standard definitions and by defining several variants, and then we shall give some of the basic consequences of our definitions. Further results in this area are given in the recent paper [9], which in particular contains interesting results on bounded approximate contractibility, together with several illuminating examples.

Definition 1.3.1. Let A be a Banach algebra. Then:

(i) A is *amenable* if, for each Banach A-bimodule X, every continuous derivation $D : A \to X'$ is inner;
(ii) A is *contractible* if, for each Banach A-bimodule X, every continuous derivation $D : A \to X$ is inner;
(iii) A is *weakly amenable* if every continuous derivation $D : A \to A'$ is inner;
(iv) a continuous derivation $D : A \to X$ from A into a Banach A-bimodule X is *approximately inner* if there exists a net (ξ_ν) in X such that
$$D(a) = \lim_\nu (a \cdot \xi_\nu - \xi_\nu \cdot a) \quad (a \in A);$$
(v) A is *approximately amenable* if, for each Banach A-bimodule X, every continuous derivation $D : A \to X'$ is approximately inner;
(vi) A is *approximately contractible* if, for each Banach A-bimodule X, every continuous derivation $D : A \to X$ is approximately inner;
(vii) A is *approximately weakly amenable* if every continuous derivation $D : A \to A'$ is approximately inner;
(viii) A is *boundedly approximately amenable* if, for each Banach A-bimodule X and each continuous derivation $D : A \to X'$, there exist $K > 0$ and a net (ξ_ν) in X' such that $\|b \cdot \xi_\nu - \xi_\nu \cdot b\| \le K\|b\|$ for each $b \in A$ and each ν, and such that
$$D(a) = \lim_\nu (a \cdot \xi_\nu - \xi_\nu \cdot a) \quad (a \in A);$$

(ix) A is *boundedly approximately contractible* if, for each Banach A-bimodule X and each continuous derivation $D : A \to X$, there exist $K > 0$ and a net (ξ_ν) in X such that $\|b \cdot \xi_\nu - \xi_\nu \cdot b\| \leq K\|b\|$ for each $b \in A$ and each ν, and such that

$$D(a) = \lim_\nu (a \cdot \xi_\nu - \xi_\nu \cdot a) \quad (a \in A);$$

(x) A is *pseudo-amenable* if it possesses a (possibly unbounded) approximate diagonal.

The qualifier *sequentially* will be used when the nets (ξ_ν) in the above definitions can in fact be chosen to be sequences.

Remarks. The notion of approximate amenability was introduced in [25].

The two notions of 'approximate amenability' and 'approximate contractibility' are in fact equivalent [14], [26].

We note that, in the bounded variants of the above definitions, it is the net (D_ν) of approximating inner derivations, where

$$D_\nu : b \mapsto b \cdot \xi_\nu - \xi_\nu \cdot b,$$

that is required to be uniformly bounded, not the implementing net (ξ_ν). It is not known whether 'boundedly approximately contractible' and 'boundedly approximately amenable' are equivalent for all Banach algebras. More surprisingly, it is not known whether 'boundedly approximately amenable' and 'approximately amenable' are the same—all the known examples of approximately amenable Banach algebras are, in fact, boundedly approximately contractible.

Suppose that A is amenable. Then A is boundedly approximately contractible with the implementing net bounded [32], [11, Proposition 2.8.59].

Pseudo-amenability was introduced and studied in [28].

Finally, note that, since every inner derivation is continuous, the uniform boundedness principle shows that any sequentially approximately inner derivation is necessarily continuous; whether or not every approximately inner derivation is automatically continuous seems to be a difficult question to answer.

A weakly amenable Banach algebra is not necessarily approximately amenable: as noted above, a group algebra $L^1(G)$ is always weakly amenable, but it is approximately amenable if and only if G is amenable [25, Theorem 3.2]. The Banach sequence algebras $\ell^p(\mathbb{N})$, where $1 \leq p < \infty$, are weakly amenable, but not approximately amenable [14, Theorem 4.1]. Trivially a commutative, approximately amenable Banach algebra is weakly amenable; an example of an approximately amenable Banach algebra which is not weakly amenable is given in [25, Example 6.2].

1.4. Intrinsic characterizations. Analogous to Theorem 1.2.4 there are approximate-diagonal characterization of the above variants of amenability, and we now give these.

Theorem 1.4.1 ([14, Proposition 2.1]). *A Banach algebra A is approximately amenable if and only if, for each $\varepsilon > 0$ and each finite subset S of A, there exist $F \in A \hat{\otimes} A$ and $u, v \in A$ such that $\pi(F) = u + v$, and such that, for each $a \in S$:*

(i) $\|a \cdot F - F \cdot a + u \otimes a - a \otimes v\| < \varepsilon$;
(ii) $\|a - au\| < \varepsilon$, $\|a - va\| < \varepsilon$. ∎

To test for approximate amenability in the case where our Banach algebra is commutative, we can use the following criterion.

Once and for all we set some notation. Let A be an algebra. For $F \in A \otimes A$ and $a \in A$, we define

$$\Delta_a(F) = a \cdot F - F \cdot a + u \otimes a - a \otimes u, \tag{1.4.1}$$

where $u = \pi(F)/2$.

PROPOSITION 1.4.2 ([14, Proposition 2.3]). *Let A be a commutative Banach algebra. Then A is approximately amenable if and only if, for each $\varepsilon > 0$ and each finite subset S of A, there exist $F \in A \otimes A$ and $u \in A$ with $\pi(F) = 2u$, and such that, for each $a \in S$:*

(i) $\|\Delta_a(F)\| < \varepsilon$;
(ii) $\|a - au\| < \varepsilon$. ∎

PROPOSITION 1.4.3. *Let A be a commutative Banach algebra which is boundedly approximately contractible. Then there exists $K > 0$ such that, for each $\varepsilon > 0$ and each finite subset S of A, there exist $F \in A \otimes A$ and $u \in A$ with $\pi(F) = 2u$ and such that, for each $a \in S$ and each $b \in A$:*

(i) $\|\Delta_a(F)\| < \varepsilon$;
(ii) $\|\Delta_b(F)\| \leq K\|b\|$;
(iii) $\|a - au\| < \varepsilon$;
(iv) $\|b - bu\| < K\|b\|$.

Proof. By the contractible case of [25, Theorem 2.1], there are a constant $K > 0$ and a net (M_ν) in $A^\# \widehat{\otimes} A^\#$ such that, for each $a \in A^\#$, we have $a \cdot M_\nu - M_\nu \cdot a \to 0$ and $\|a \cdot M_\nu - M_\nu \cdot a\| \leq K\|a\|$. As in [25, Corollary 2.2], this gives nets (N_ν) in $A \widehat{\otimes} A$ and (F_ν) and (G_ν) in A such that, for each $b \in A$, we have:

(i)′ $a \cdot N_\nu - N_\nu \cdot a + F_\nu \otimes a - a \otimes G_\nu \to 0$;
(ii)′ $\|b \cdot N_\nu - N_\nu \cdot b + F_\nu \otimes b - b \otimes G_\nu\| \leq K\|b\|$;
(iii)′ $a \cdot F_\nu \to a$ and $G_\nu \cdot a \to a$;
(iv)′ $\|b - b \cdot F_\nu\| \leq K\|b\|$, $\|b - G_\nu \cdot b\| \leq K\|b\|$;
(v)′ $\pi(N_\nu) - F_\nu - G_\nu \to 0$.

As in [26], we may suppose that N_ν is symmetric and that $F_\nu = G_\nu$. Finally, at the cost of a slight increase in K, we may suppose that each N_ν lies in $A \otimes A$ and that $\pi(N_\nu) = 2F_\nu$. It is now standard how to obtain the desired F and u. ∎

REMARK. A more explicit version of Proposition 1.4.3 is given in [9, Theorem 2.5]. It is an open question whether or not the converse of the above proposition holds (cf. [26, Theorem 5.4]). A criterion for a Banach algebra to be boundedly approximately amenable is given in [26, Theorem 5.10].

Note that Theorem 1.4.1 and Propositions 1.4.2 and 1.4.3 are all phrased in terms of requiring, for any finite subset S of A, that there exist elements of $A \otimes A$ and A satisfying certain inequalities for all elements of S. As shown in [14], when the requirement fails, it is often two-point sets S that suffice to negate these inequalities. We do not know if this is always the case. This also raises the question as to the status of singleton sets S. Indeed, [14, Proposition 3.6], to be generalized below, broaches this problem.

1.5. Pointwise variations. The following pointwise variants of the above notions were introduced formally by Fereidoun Ghahramani.

DEFINITION 1.5.1. Let A be a Banach algebra. Then:

(i) A is *pointwise amenable at* $a \in A$ if, for each Banach A-bimodule X, every continuous derivation $D : A \to X'$ is *pointwise inner at* a, that is, there exists $\xi \in X'$ such that $D(a) = a \cdot \xi - \xi \cdot a$;
(ii) A is *pointwise approximately amenable at* $a \in A$ if, for each Banach A-bimodule X, every continous derivation $D : A \to X'$ is *pointwise approximately inner at* a, that is, there exists a sequence (ξ_n) in X' such that $D(a) = \lim_{n\to\infty}(a \cdot \xi_n - \xi_n \cdot a)$.
(iii) A is *pointwise* [*approximately*] *amenable* if A is pointwise [approximately] amenable at a for each $a \in A$.

REMARK. Trivially, a commutative, pointwise approximately amenable Banach algebra is weakly amenable.

A small variation of standard arguments in [25] and [26] regarding approximately amenable and approximate amenability shows that 'pointwise approximately amenable' is the same as 'pointwise approximately contractible', and gives the following characterization of pointwise approximate amenability analogous to Theorem 1.4.1.

PROPOSITION 1.5.2. *Let A be Banach algebra, and let $a \in A$. Then A is pointwise approximately amenable at a if and only if, for each $\varepsilon > 0$, there exist $F \in A \otimes A$ and $u, v \in A$ with $\pi(F) = u + v$ such that:*

(i) $\|a \cdot F - F \cdot a + u \otimes a - a \otimes v\| < \varepsilon$;
(ii) $\|a - au\| < \varepsilon$, $\|a - va\| < \varepsilon$.

In the case where A is commutative, we may take $u = v$. ■

COROLLARY 1.5.3. *Let A be a pointwise approximately amenable Banach algebra. Then A has left and right approximate units. In particular,*

$$a \in \overline{aA} \cap \overline{Aa} \quad (a \in A) \quad \textit{and} \quad A = \overline{A^2}. \ ■$$

The following is an unpublished result of Fereidoun Ghahramani.

THEOREM 1.5.4. *Let A be a pointwise amenable, commutative Banach algebra. Then A is approximately amenable.* ■

It is not known whether or not this theorem holds for arbitrary (non-commutative) Banach algebras. Indeed, we do not know of a pointwise amenable Banach algebra which is not already amenable.

One simple way to get an approximately amenable Banach algebra $\mathfrak{A}$ which is not amenable is to take $\mathfrak{A} = c_0(A_n)$, where the A_n are amenable algebras, each with identity of norm one, whose amenability constants are unbounded in n; such examples are constructed in [25, §6] and also in [13]. Then $\mathfrak{A}$ is approximately amenable [25, Example 6.1].

We now indicate how to extend [25, Example 6.2] to obtain an example of an approximately amenable Banach algebra which is not even pointwise amenable.

EXAMPLE 1.5.5. Take $n \in \mathbb{N}$, and consider the algebra $\mathbb{M}_{2^n}$ of $2^n \times 2^n$-matrices over $\mathbb{C}$ with the norm

$$\|(a_{ij})\| = \Big(\sum_{i,j=1}^{2^n} |a_{ij}|^2\Big)^{1/2} \qquad ((a_{ij}) \in \mathbb{M}_{2^n}).$$

Note that the identity in $\mathbb{M}_{2^n}$ has norm $\sqrt{2n}$. Take the matrix $P_n \in \mathbb{M}_{2^n}$ as in [25, Example 6.2]: P_n is 'anti-diagonal' and has non-zero terms in the (i,j)-position only if $i + j = 2^n + 1$, and then $|p_{i,j}| = 1$ in this case. We see that $\|P_n\| = 2^{n/2}$.

Define $\mathfrak{A}_n = \mathbb{M}^{\sharp}_{2^n}$, so that the identity in $\mathfrak{A}_n$ has norm one, and set

$$D_n : B \mapsto P_n B^T - B^T P_n, \qquad \mathfrak{A}_n \to \mathfrak{A}'_n.$$

Then it is shown in [25, Example 6.2] that D_n is an inner derivation with $\|D_n\| = 2$.

Now let A_n be the specific matrix

$$A_n = \frac{1}{(2^n - 1)^{1/2}} \sum_{r=1}^{2^n-1} E_{r,r+1},$$

where the E_{ij} are the matrix units in $\mathbb{M}_{2^n}$, so that $A_n \in \mathfrak{A}_n$ with $\|A_n\| = 1$. Suppose that $Q_n \in \mathbb{M}_{2^n}$ is such that $A_n Q_n = Q_n A_n$. We first note that Q_n is upper-triangular and that Q_n is constant on diagonals. Thus $(Q_n)_{i,j} = 0$ at least when $i = 2^{n-1}+1, \dots, 2^n$ and $j = 2^n + 1 - i$.

Now define $\mathfrak{A} = c_0(\mathfrak{A}_n)$, so that $\mathfrak{A}' = \ell^1(\mathfrak{A}'_n)$, and $\mathfrak{A}$ is approximately amenable by [25, Example 6.2]. Set

$$D : (B_n) \mapsto \frac{1}{n^2}(P_n B_n^T - B_n^T P_n), \qquad \mathfrak{A} \to \mathfrak{A}'.$$

Then D is a continuous derivation that is not inner, and so $\mathfrak{A}$ is not even weakly amenable.

We now *claim* that, in fact, $\mathfrak{A}$ is not pointwise amenable. For consider the specific element $A = (A_n^T/n)$, which belongs to $\mathfrak{A}$, and assume towards a contradiction that D is pointwise inner at A. Then there exists a sequence $R = (R_n)$ in $\mathfrak{A}'$ such that, for each $n \in \mathbb{N}$, we have $D(A_n) = R_n A_n - R_n A_n$. For each $n \in \mathbb{N}$, we see that $R_n - P_n/n^2$ commutes with A_n, and so is zero when $i = 2^{n-1} + 1, \dots, 2^n$ and $j = 2^n + 1 - i$. This shows that R_n takes the value $1/n^2$ in at least $2^{n/2}$ places, and so

$$\|R_n\| \geq 2^{n/4}/n^2 \to \infty \qquad \text{as } n \to \infty.$$

This contradicts the claim that $R = (R_n)$ is an element of $\mathfrak{A}'$, and so $\mathfrak{A}$ is not pointwise amenable. $\diamond$

Here are some specific questions about pointwise amenability that we cannot answer. Let G be a (discrete) group, and consider the group algebra $\ell^1(G)$. For which G is $\ell^1(G)$ pointwise amenable? Maybe this is true for each group G? In particular, is $\ell^1(\mathbb{F}_2)$ pointwise amenable, where $\mathbb{F}_2$ is the free group on two generators?

We note that pseudo-amenability and approximate amenability agree in the presence of a bounded approximate identity, and that an approximately amenable, commutative Banach algebra is pseudo-amenable [28, Proposition 3.2, Corollary 3.4].

1.6. Results concerning ideals. We shall require the following results.

Proposition 1.6.1.

(i) *A Banach algebra A is [pointwise] approximately amenable if and only is $A^\sharp$ is [pointwise] approximately amenable.*
(ii) *Let A be an approximately amenable Banach algebra, and let I be a weakly complemented closed left ideal in A. Then I has a right approximate identity, and so $\overline{I^2} = I$.*
(iii) *Let A be a pointwise approximately amenable Banach algebra, and let I be a weakly complemented closed left ideal in A. Then I has right approximate units, and so $\overline{I^2} = I$.*

Proof. (i) This is the pointwise version of Proposition 2.4 of [25].

(ii) This is Corollary 2.4 of [25].

(iii) This is the pointwise version of (ii). ∎

Corollary 1.6.2. *Each finite-dimensional, pointwise approximately amenable Banach algebra is semisimple, and hence amenable.*

Proof. Let A be a finite-dimensional Banach algebra with radical R. Since A is finite-dimensional, R is nilpotent, and of course R and R^2 are closed ideals in A.

Supposing that A is pointwise approximately amenable, Proposition 1.6.1(iii) shows that $\overline{R^2} = R$, and so $R^2 = R$. Thus $R^n = R$ for each $n \in \mathbb{N}$, and so $R = \{0\}$, showing that A is semisimple, and thus amenable. ∎

Proposition 1.6.3. *Let A be a pointwise amenable, commutative Banach algebra, and let I be a weakly complemented, closed ideal in A. Then I has a bounded approximate identity.*

Proof. Let $\iota : I \to A$ be the natural embedding. We follow the proof of [11, Theorem 2.9.58(ii)] to see that, for each $a \in I$, there exists $Q \in \mathcal{B}(I', A')$ such that $\iota' \circ Q$ is the identity on I' and also

$$Q(a \cdot \lambda) = a \cdot Q(\lambda) \qquad (\lambda \in I');$$

this step uses the pointwise version of [11, Theorem 2.8.60]. The argument of Johnson [11, Theorem 2.9.57] shows that there is $\Phi_a \in A''$ with $a = \Phi_a \cdot a$, whence $Q'(\Phi_a) \cdot a = a$. Now a standard Mazur argument shows there is a bounded sequence (u_n) in I with $u_n a \to a$. This holding for each $a \in I$, [52, Lemma 12] or [17, Theorem 9.7] shows that I has a bounded approximate identity; it is this last step that needs the hypothesis that A is commutative. ∎

It follows from Proposition 1.6.3 that closed ideals of finite codimension in commutative, pointwise amenable Banach algebras have approximate units. In fact being pointwise amenable is an unnecessarily strong hypothesis, as we shall show in Proposition 1.6.6 below. We do not know whether a (weakly) complemented, closed ideal in a pointwise amenable Banach algebra necessarily has a bounded approximate identity.

Remark. Let I be a complemented, closed ideal in a Banach algebra A. In the case where A is amenable, so is I [11, Corollary 2.9.59]. However it does not follow from the fact that A is approximately amenable that I is also approximately amenable; for this see the discussion following [26, Corollary 4.5]. Indeed, we do not know the answer to the following apparently innocuous question. Let I be a closed ideal of finite codimension in a unital, approximately amenable Banach algebra A. Is I also approximately amenable? This is open even when I has codimension two and A is commutative. Fereidoun Ghahramani has pointed out that, if true, this would have far-reaching consequences. For suppose that A is approximately amenable. Then by [26, Proposition 6.1], $A^\sharp \oplus A$ is approximately amenable. But $A \oplus A$ is an ideal of codimension one in $A^\sharp \oplus A$, and so would be approximately amenable. Whether in fact $A \oplus A$ is approximately amenable, given that A is approximately amenable, is an open question.

On a more positive note we have the following.

PROPOSITION 1.6.4. *Let A be a Banach algebra, and let I be a closed ideal in A.*

(i) *Suppose that A is [pointwise] approximately amenable. Then A/I is [pointwise] approximately amenable.*
(ii) *Suppose that A is [pointwise] approximately amenable and that I has a bounded approximate identity. Then I is [pointwise] approximately amenable.*
(iii) *Suppose that I is amenable and that A/I is approximately amenable. Then A is approximately amenable.*

Proof. In the non-pointwise cases, clauses (i) and (iii) are contained in [25, Corollary 2.1], and (ii) is [25, Corollary 2.3]. The pointwise versions are similar. ∎

LEMMA 1.6.5. *Let A be a commutative, unital Banach algebra, and suppose that each maximal ideal of A has approximate units. Let I be a closed ideal of finite codimension in A. Then:*

(i) *I is the intersection of finitely many distinct maximal ideals;*
(ii) *I has approximate units.*

Proof. We use an idea from [70, Theorem 2].

Let I be a closed ideal of finite codimension, and assume inductively that the result holds for all proper ideals properly containing I. Let $\rho : A \to A/I$ be the quotient map, and let R denote the radical of A/I. Since R is finite-dimensional and radical, it is nilpotent, and so, in the case where $R \neq \{0\}$, it follows that R^2 is a closed and proper ideal in R. But then $\rho^{-1}(R)$ is a closed ideal in A such that $\rho^{-1}(R)$ properly contains I and satisfies

$$\overline{\rho^{-1}(R)^2} \subseteq \rho^{-1}(R^2) \neq \rho^{-1}(R),$$

contrary to the hypothesis on I. Thus $R = \{0\}$, and so $A/I \cong \mathbb{C}^n$ for some $n \in \mathbb{N}$. It is immediate that $I = M_1 \cap \cdots \cap M_n$ for some distinct maximal ideals $M_1, \ldots, M_n$, giving (i).

Finally, by the hypothesis on I, we know that $M_1 \cap \cdots \cap M_{n-1}$ has approximate units. Take $a \in I$ and $\varepsilon > 0$. Then there exists $u_1 \in M_1 \cap \cdots \cap M_{n-1}$ with $\|a - au_1\| < \varepsilon$. Since $au_1 \in M_n$, there exists $u_2 \in M_n$ with $\|au_1 - au_1u_2\| < \varepsilon$. Then $u := u_1u_2 \in I$ and $\|a - au\| < 2\varepsilon$. Thus $a \in \overline{aI}$, and so (ii) holds. ∎

Proposition 1.6.6. *Let A be a commutative Banach algebra that is pointwise approximately amenable. Then each closed modular ideal I of finite codimension n is the intersection of n distinct maximal ideals and has approximate units.* ∎

Proof. This is immediate from Lemma 1.6.5 in the case where A is unital. Otherwise $A^\sharp$ is pointwise approximately amenable, and has the stated properties. But a closed modular ideal in A has the form $A \cap J$ for some closed ideal J of $A^\sharp$, and A is of course a (modular) maximal ideal in $A^\sharp$. ∎

We shall also need the following observation.

Proposition 1.6.7. *Let A be a Banach algebra. Then:*

(i) $\mathbb{M}_n(A)$ *is amenable if and only if A is amenable;*
(ii) $\mathbb{M}_n(A)$ *is approximately amenable if and only if A is approximately amenable.*

Proof. (i) This is [13, Theorem 2.7(i)].

(ii) Suppose that $\mathbb{M}_n(A)$ is approximately amenable. Then we follow the proof of [13, Theorem 2.7(i)], replacing Λ by a suitable net (Λ_ν), to see that A is approximately amenable.

Conversely, suppose that A is approximately amenable. We modify slightly the proof of [13, Theorem 2.7(ii)] by replacing the net (u_α) by the net given by [25, Corollary 2.2] (and ignoring the bound estimates). ∎

Let A be a Banach algebra that is approximately weakly amenable. Then $A^\sharp$ is also approximately weakly amenable. However the converse may fail. The group algebra $L^1(G)$ is weakly amenable for any locally compact group G [11, 44]. However for $G = SL(2, \mathbb{R})$, the augmentation ideal $L_0^1(G)$ is not weakly amenable, yet $L_0^1(G)^\sharp$ is weakly amenable [45]. That $L_0^1(G)$ is not approximately weakly amenable in this example is shown in [4].

1.7. Banach function algebras. By a *Banach function algebra* on a locally compact space X we mean an algebra A of functions on X such that A separates the points of X, such that A is a Banach algebra with respect to some norm, and such that the topology on X is the weak topology induced by A.

Let A be a natural Banach function algebra, defined on its locally compact character space Φ_A. Denote by A_{00} the ideal of functions in A of compact support. For $\varphi \in \Phi_A$, set $M_\varphi = \ker \varphi$, and take J_φ to be the ideal of functions $f \in A_{00}$ such that $\varphi \notin \operatorname{supp} f$. As in [11, Definition 4.1.31], A is *strongly regular* if $\overline{A_{00}} = A$ and $\overline{J_\varphi} = M_\varphi$ for each $\varphi \in \Phi_A$; A has *bounded relative units* if, for each $\varphi \in \Phi_A$, there is $m_\varphi > 0$ such that, for each compact subset K of $\Phi_A \setminus \{\varphi\}$, there exists $f \in J_\varphi$ with $f(K) = \{1\}$ and $\|f\| \leq m_\varphi$, and for each compact subset K of Φ_A, there is $f \in A$ with with $f(K) = \{1\}$ and $\|f\| \leq m_\varphi$. The Banach function algebra A is a *Ditkin algebra* if $f \in \overline{fA_{00}}$ for each $f \in A$ and $f \in \overline{fJ_\varphi}$ for each $f \in M_\varphi$; and A is a *strong Ditkin algebra* if M_φ has a bounded approximate identity contained in J_φ for each $\varphi \in \Phi_A$. We ask if various of these Banach function algebras are necessarily approximately amenable.

It is not the case that a pointwise approximately amenable Banach algebra A necessarily satisfies the condition that $A = A^2$, but it may be that every approximately amenable Banach algebra satisfies this condition.

There exist strongly regular, commutative Banach algebras A such that $A = A^2$, but A is not a Ditkin algebra, so that the condition that $A = A^2$ does not necessitate that A is pointwise approximately amenable. Such an example is given by $A = \ell^1(G)$ for a totally ordered η_1-group G, as in [11, Example 2.9.45]. This algebra A satisfies the condition that $A = A^2$ because it has an identity, but it is not pointwise approximately amenable because the augmentation ideal does not have approximate units. However, in this example the Banach algebra A is not separable. We do not know of a commutative, separable Banach algebra, nor even a separable Banach sequence algebra, such that $A = A^2$, but A is not a Ditkin algebra. Consider the example of the semigroup algebra $B = \ell^1(S)$ where $S = (\mathbb{Q}^{+\bullet}, +)$: certainly B is a commutative, separable Banach algebra which is not a Ditkin algebra, but it is not known whether or not $B = B^2$. An interesting example of a proper, unital, uniform algebra which is a strong Ditkin algebra is given by Feinstein in [20]. We do not know whether this example is (pointwise) approximately amenable.

1.8. Banach sequence algebras. By a *Banach sequence algebra* we mean a (commutative) algebra A on a subset S of $\mathbb{Z}$ such that $c_{00}(S) \subset A \subseteq c_0(S)$ and such that A is a Banach algebra under some norm; see [11] for further details. In particular a Banach sequence algebra on S is a Banach function algebra on S. Here the underlying set S will be either $\mathbb{N}$ or $\mathbb{Z}$; generally results stated for $\mathbb{N}$ have an obvious analogue for $\mathbb{Z}$. For convenience we shall often write $a \in A$ as a sequence $a = (\alpha(i))$, and set δ_i for the characteristic function of $\{i\}$ when $i \in S$; we shall eschew using formal sums such as $a = \sum_{i=1}^{\infty} \alpha(i)\delta_i$, as there is no reason for this series to converge in $(A, \|\cdot\|)$ in general.

For example, for each $1 \leq p < \infty$, the space $(\ell^p(\mathbb{N}), \|\cdot\|_p)$ is a Banach sequence algebra on $\mathbb{N}$ with respect to the coordinatewise product.

A Banach sequence algebra A on $\mathbb{N}$ is strongly regular if and only if c_{00} is dense in A, and is a Ditkin algebra if and only if $f \in \overline{fc_{00}}$ for each $f \in A$. It is shown in [14, Corollary 3.5] that a Banach sequence algebra which is a strong Ditkin algebra is even sequentially approximately amenable.

Now let $(A, \|\cdot\|)$ be a Banach sequence algebra on S such that c_{00} is dense in A, so that A is strongly regular. Suppose that A is pointwise approximately amenable. Then by Proposition 1.5.2, for each $a \in A$ and $\varepsilon > 0$, there exists $u \in c_{00}$ such that $\|a - au\| < \varepsilon$. It follows that A is a Ditkin algebra. We ask if the converse to this holds: *Is every Banach sequence algebra which is a Ditkin algebra necessarily pointwise approximately amenable?* We do not have a counter-example to this possibility.

A strongly regular Banach sequence algebra is not necessarily a Ditkin algebra. An example M to show this is given in [11, Example 4.5.33]; the example is due to H. Mirkil. It follows that M is a strongly regular Banach sequence algebra, but M is not pointwise approximately amenable.

Let A be a Banach sequence algebra on $\mathbb{N}$. For each $n \in \mathbb{N}$, we set

$$e_n = \sum_{i=1}^{n} \delta_i = (\overbrace{1, 1 \ldots, 1}^{n}, 0, 0, \ldots),$$

and, for each subset $D \subset \mathbb{N}$, we define $P_D : A \to c_0$ by

$$P_D(a) = \sum_{i \in D} \alpha(i)\delta_i \quad (a = (\alpha(i)) \in A).$$

In the case where D is either finite or cofinite in $\mathbb{N}$, P_D maps into A. Similar definitions apply for subsets of $\mathbb{Z}$.

DEFINITION 1.8.1. Let A be a Banach sequence algebra on S. For $a = (\alpha(i)) \in A$ and $k \in S$, the set

$$\{j \in S : \alpha(j) = \alpha(k)\}$$

is a *level set* of a.

REMARK. Since $A \subset c_0$, the level sets are all finite except possibly for the zero set of a.

Let $a = (\alpha(i)) \in A$ and $F \in c_{00}(S \times S)$; set $\pi(F) = 2u$. Recalling (1.4.1), we have the fundamental identity

$$\Delta_a(F)(i,j) = (\alpha(i) - \alpha(j))F(i,j) + u(i)\alpha(j) - u(j)\alpha(i) \quad (i, j \in S). \tag{1.8.1}$$

This formula is used at several key points in the arguments to follow.

We are seeking to verify the conjecture that every Banach sequence algebra A that is a Ditkin algebra is pointwise approximately amenable; the following theorem achieves this provided that A satisfies an extra hypothesis.

THEOREM 1.8.2. *Let A be a Banach sequence algebra on S such that A is a Ditkin algebra, and let $a \in A$. Suppose that, for each $\varepsilon > 0$, there exists $u \in c_{00}$ such that $\|a - au\| < \varepsilon$ and u is constant on the level sets of a. Then A is pointwise approximately amenable at a.*

Proof. We may suppose that $a \neq 0$. Set $a = (\alpha(i)) \in A$. Fix $\varepsilon \in (0, \|a\|)$, and choose $u \in c_{00}$ such that $\|a - au\| < \varepsilon$ and u is constant on the level sets of a. Then $u \neq 0$. Set $U = \operatorname{supp} u$, and then set $b = (\beta(i)) = a - P_U a \in A$. Since A is a Ditkin algebra, we can choose $v \in c_{00}$ such that

$$2\|u\|\|b - bv\| < \varepsilon;$$

set $V = \operatorname{supp} v$.

We now define our element $F \in c_{00}(\mathbb{N} \times \mathbb{N})$. Firstly, for $i \in \mathbb{N}$, set $F(i,i) = 2u(i)$; this guarantees that $\pi(F) = 2u$.

Off the diagonal of $\mathbb{N} \times \mathbb{N}$, we consider several cases.

(i) Suppose that $i, j \in U$ with $i \neq j$ and $\alpha(i) = \alpha(j)$. Then we choose $F(i,j)$ arbitrarily; in this case $\Delta_a(F)(i,j) = 0$ because we also have $u(i) = u(j)$.

(ii) Suppose that $i, j \in U$ with $\alpha(i) \neq \alpha(j)$. Then, recalling equation (1.8.1), we can specify $F(i,j)$ so that $\Delta_a(F)(i,j) = 0$.

(iii) Suppose that $i \in U$ and $j \in V \setminus U$. Then $u(i) \neq 0$ and $u(j) = 0$, and so $u(i) \neq u(j)$. Since u is constant on the level sets of a, it follows that $\alpha(i) \neq \alpha(j)$. Thus we may choose $F(i,j)$ such that

$$\Delta_a(F)(i,j) = u(i)\alpha(j)(1 - v(j)).$$

(iv) Suppose that $i \in U$ and $j \in S \setminus (U \cup V)$. Set $F(i,j) = 0$, so that, since $v(j) = 0$, we have

$$\Delta_a(F)(i,j) = u(i)\alpha(j) = u(i)\alpha(j)(1 - v(j)).$$

(v) Suppose that $i \in V \setminus U$ and $j \in U$. Again $\alpha(i) \neq \alpha(j)$, and so we may choose $F(i,j)$ such that

$$\Delta_a(F)(i,j) = u(j)\alpha(i)(1 - v(i)).$$

(vi) Suppose that $j \in U$ and $i \in S \setminus (U \cup V)$. Set $F(i,j) = 0$, so that

$$\Delta_a(F)(i,j) = u(j)\alpha(i) = u(j)\alpha(i)(1 - v(i)).$$

(vii) Suppose that $i, j \in S \setminus U$. Set $F(i,j) = 0$, so that $\Delta_a(F)(i,j) = 0$.

Note that certainly $F \in c_{00}(S \times S)$ because $F(i,j) = 0$ when i or j lie outside $U \cup V$.

Since $u(i) = 0$ for $i \notin U$, we see that

$$\Delta_a(F)(i,j) = (u \otimes (b - bv))(i,j) = ((b - bv) \otimes u)(i,j) = 0,$$

except possibly when $i \in U$ and $j \notin U$ or $i \notin U$ and $j \in U$. But for $j \notin U$, we have $\alpha(j) = \beta(j)$, and so, for $(i,j) \in U \times (V \setminus U)$, we see that

$$\Delta_a(F)(i,j) = u(i)(\alpha(j) - \alpha(j)v(j)) = (u \otimes (b - bv))(i,j),$$

and similarly, for $(i,j) \in (V \setminus U) \times U$, we have

$$\Delta_a(F)(i,j) = u(j)(\alpha(i) - \alpha(i)v(i)) = ((b - bv) \otimes u)(i,j).$$

Further, for $i \in U$ and $j \in S \setminus (U \cup V)$, so that $u(j) = v(j) = 0$ and $\alpha(j) = \beta(j)$, we also have

$$\Delta_a(F)(i,j) = (u \otimes (b - bv))(i,j),$$

and similarly for $j \in U$ and $i \in S \setminus (U \cup V)$.

By checking each case, we see that

$$\Delta_a(F) = u \otimes (b - bv) - (b - bv) \otimes u$$

in $c_{00}(S \times S)$. Thus

$$\|\Delta_a(F)\| = \|u \otimes (b - bv) - (b - bv) \otimes u\| \leq 2\|u\|\,\|b - bv\| < \varepsilon.$$

We conclude that F and u together satisfy the conditions of Proposition 1.5.2, and so A is pointwise approximately amenable at a. ∎

COROLLARY 1.8.3. *Let A be a Banach sequence algebra on S such that A is a Ditkin algebra. Then the set of elements $a \in A$ such that A is pointwise approximately amenable at a is dense in A, and every element $a \in A$ is the sum of two elements $b, c \in A$ such that A is pointwise approximately amenable at b and c.*

Proof. Let $a \in A$, and denote the (countable) range of a on S by R. Enumerate the elements of the support of a as $\{r_1, r_2, \ldots\}$, and fix $\varepsilon > 0$.

We shall define inductively a sequence (ε_n). Indeed choose ε_1 so that $\varepsilon_1\|\delta_{r_1}\| < \varepsilon$ and $a(r_1)+\varepsilon_1 \notin R$. Now suppose that $\varepsilon_1, \ldots, \varepsilon_n$ have been chosen. Choose $\varepsilon_{n+1} \notin \{\varepsilon_1, \ldots, \varepsilon_n\}$ so that $\varepsilon_{n+1}\|\delta_{r_{n+1}}\| < \varepsilon/2^{n+1}$ and $a(r_{n+1}) + \varepsilon_{n+1} \notin R \cup \{a(r_1) + \varepsilon_1, \ldots, a(r_n) + \varepsilon_n\}$. Set $b = \sum_{n=1}^{\infty} \varepsilon_n \delta_n$ and $c = a + b$. Then $b \in A$ and $\|b\| < \varepsilon$.

Clearly all the non-zero level sets of both b and c are singletons and so the 'level sets hypothesis' of Theorem 1.8.2 is trivially satisfied for the elements b and c. By the theorem, A is pointwise approximately amenable at b and c. Since $\|a-c\|<\varepsilon$, the result follows. ∎

REMARK 1.8.4. Suppose that the hypothesis of Theorem 1.8.2 holds. For $a=(\alpha(i))\in A$, set $Z(a)=\{j\in S:\alpha(j)=0\}$. Fix $\varepsilon>0$, and choose $u_0\in c_{00}(S)$ such that $\|a-au_0\|<\varepsilon$ and u_0 is constant on the level sets of a. Define

$$u(i)=\begin{cases}u_0(i), & \alpha(i)\neq 0,\\ 0, & \alpha(i)=0.\end{cases}$$

Set $U=\operatorname{supp} u$. Then u satisfies the same conditions as u_0, and in addition $Z(a)\cap U=\emptyset$. Set $b=(\beta(i))=a-P_Ua\in A$. Since A is a Ditkin algebra, we can choose $v\in c_{00}(S)$ such that

$$2\|u\|\|b-bv\|<\varepsilon;$$

set $V=\operatorname{supp} v$. As with u, we may suppose that $Z(a)\cap V=\emptyset$. Then in the construction of F, $\alpha(i)=0$ or $\alpha(j)=0$ only occurs in cases (vi) and (vii), and then $F(i,j)=0$.

The point of this modification is as follows. Since A is regular and $\mathbb{Z}$ is countable, spectral synthesis holds in A. Thus for a closed ideal $I\subset A$, if the element $a\in A$ in fact lies in I, then so do u and v, and $F\in I\otimes I$. Thus we conclude that I is also pointwise approximately amenable at a.

COROLLARY 1.8.5 ([14, Proposition 3.6]). *For $1\le p<\infty$, the Banach sequence algebra ℓ^p is pointwise approximately amenable.*

Proof. The algebra ℓ^p is a Ditkin algebra. Take $a\in\ell^p$. For $n\in\mathbb{N}$, define (finite) subsets B_n of $\mathbb{N}$ by

$$B_n=\{j\in\mathbb{N}:|a(j)|\ge 1/n\},$$

and define u_n to be the characteristic function of B_n, so that u_n is certainly constant on the level sets of a. Then

$$(a-au_n)(m)=\begin{cases}0 & (m\in B_n),\\ a(m) & (m\notin B_n),\end{cases}$$

so that $a-au_n\to 0$ pointwise, and each sequence $a-au_n$ is dominated by $|a|$, so that $\|a-au_n\|_p\to 0$.

Thus the hypotheses of Theorem 1.8.2 are satisfied, and so ℓ^p is pointwise approximately amenable at a. Hence ℓ^p is pointwise approximately amenable. ∎

Theorem 1.8.2 is a substantial generalization of [14, Proposition 3.6]. In that result, it was required that $\|(I-P_C)(a)\|\to 0$ as the finite subsets C expand to $\mathbb{N}$; this shows that the hypothesis of Theorem 1.8.2 holds with u a suitable characteristic function. This latter condition is a stronger requirement than that in Theorem 1.8.2, and, in particular, will be shown to fail for the example A_ω, to be discussed below, for certain weight functions ω; see in particular §3.8.

Note also that the hypothesis of Theorem 1.8.2 implies that c_{00} is dense in A, so that A must be separable.

Unfortunately, the 'level set hypothesis' of Theorem 1.8.2 is not necessary for pointwise approximate amenability. For let $A = A(\mathbb{Z})$, the algebra of Fourier transforms of elements of $(L^1(\mathbb{T}), \star)$. Then A is a Banach sequence algebra on $\mathbb{Z}$, and A is even amenable, and hence pointwise approximately amenable. However the extra hypothesis is not always satisfied. Indeed, take $f \in A$ to be either of the functions constructed in [46, §2] or in [61, Example A]. The *raison d'être* of these constructions will ensure that the hypothesis of Theorem 1.8.2 concerning level sets fails; see also §4.2.

However, for $A = L^1(\mathbb{T})$ we note that by [22, Theorem 1.2] the construction in Theorem 1.8.2 shows that A is pointwise approximately amenable at elements $a \in A$ whose (non-zero) level sets L have the following property: there is a constant $K > 0$ such that, for each L, we have

$$m, n \in L,\ |n| \le |m| \ \Rightarrow\ |n - m| \le K|n|^{1/2}. \tag{1.8.2}$$

Indeed, [61] shows that A is pointwise approximately amenable at a for $A = L^p(\mathbb{T})$, where $1 \le p < 2$, for elements a whose (non-zero) level sets L satisfy the condition that $m, n \in L, |n| \le |m|$ necessitates $|n - m| \le K|n|^{(3p-2)/2p}$. Note that [47] shows that no restriction on the *cardinality* of the level sets will circumvent the barrier that these sets create.

In fact we shall show that a result like the above holds for many Banach sequence algebras. First we give an elementary estimate.

LEMMA 1.8.6. *Let $K_1, K_2, K_3, \delta > 0$, and let $\beta = (\beta_n)$ be a sequence in $\mathbb{C}$. Take $p > 1$ with conjugate index q. Let $m \in \mathbb{N}$, and take $r_1, \dots, r_m \ge 0$. Suppose that $\{r_1, \dots, r_m\}$ satisfies*

$$\sum_{n=1}^{m} |\beta_n|^p r_n \le K_3,$$

and that N is so large that

$$\max_{1 \le n \le m} r_n \le K_1 N^\delta \quad \text{and} \quad \sum_{n=1}^{m} r_n \le K_2 N.$$

Then

$$\sum_{n=1}^{m} |\beta_n| r_n^2 \le K_1 K_2^{1/q} K_3^{1/p} N^{\delta + 1/q}.$$

Proof. We just rearrange the summands to show that the given bound follows from Hölder's inequality. Thus

$$\begin{aligned}\sum_{n=1}^{m} |\beta_n| r_n^2 = \sum_{n=1}^{m} |\beta_n| r_n^{1/p} \cdot r_n^{2-1/p} &\le \Big(\sum_{n=1}^{m} |\beta_n|^p r_n\Big)^{1/p} \Big(\sum_{n=1}^{m} r_n^{(2-1/p)q}\Big)^{1/q} \\ &\le K_3^{1/p} \Big(\sum_{n=1}^{m} r_n\Big)^{1/q} \max_{1 \le n \le m} r_n \\ &\le K_1 K_2^{1/q} K_3^{1/p} N^{\delta + 1/q},\end{aligned}$$

as claimed. ∎

For $N \in \mathbb{N}$, we set

$$u_N(k) = \begin{cases} 1 - \dfrac{|k|}{N+1} & (|k| \le N), \\ 0 & (\text{otherwise}). \end{cases} \qquad v_N(k) = \begin{cases} 1 & (|k| \le N), \\ 0 & (\text{otherwise}). \end{cases}$$

Note that these functions are just the Fourier transforms of the Fejér and Dirichlet kernels on $\mathbb{T}$, respectively.

Definition 1.8.7. Let A be a Banach sequence algebra on $\mathbb{Z}$, let $a \in A$ and take $\delta > 0$. Then a has *δ-small level sets* if there is a constant $K > 0$ such that

$$|i - j| \le K i^{\delta} \quad \text{whenever} \quad \alpha(i) = \alpha(j) \ne 0.$$

Note that no assumption is made about the structure of the possibly infinite zero set of a.

The hypotheses of the next result are satisfied when $A = \widehat{L^p(\mathbb{T})}$ and $p \ge 2$.

Theorem 1.8.8. *Let $(A, \|\cdot\|)$ be a Banach sequence algebra on $\mathbb{Z}$ such that the set $\{\delta_n : n \in \mathbb{Z}\}$ of idempotents is bounded, such that $A \subset \ell^p$ for some $p > 1$, and such that*

$$\lim_{N\to\infty} \|b - bu_N\| = \lim_{N\to\infty} \|b - bv_N\| = 0$$

for each $b \in A$. Let $a \in A$ have δ-small level sets for some $0 < \delta < (q-1)/2q$, where q is the conjugate index to p. Then, for each $\varepsilon > 0$, there exist $N \in \mathbb{N}$ and $F \in c_{00}(\mathbb{Z} \times \mathbb{Z})$ such that $\pi(F) = 2u_N$ and $\|\Delta_a(F)\| < \varepsilon$. In particular, A is pointwise approximately amenable at a.

Proof. Let $\{L_n : n \in \mathbb{N}\}$ be an enumeration of the (non-zero) level sets of a, and, for $n \in \mathbb{N}$, take β_n to be the constant value of a on L_n. Since $a \in \ell^p$, we have

$$K_3 := \sum_{n=1}^{\infty} |\beta_n|^p |L_n| < \infty. \tag{1.8.3}$$

First, choose $N \in \mathbb{N}$ so that $\|a - au_N\| < \varepsilon$. Set $b = a - au_N$; certainly $b \in A$. Now choose $M \in \mathbb{N}$ such that $2\|u_N\|\,\|b - bv_M\| < \varepsilon$. By increasing M if necessary, we may suppose that $\alpha(i) \ne \alpha(j)$ whenever $|i| \le N$, $\alpha(i) \ne 0$, and $|j| > M$.

We now construct $F \in c_{00}(\mathbb{Z} \times \mathbb{Z})$ with the desired properties; the method follows that of Theorem 1.8.2. For convenience, and to parallel the notation of that theorem, set $u = u_N$, $U = \operatorname{supp} u_N = [-N, N]$, $v = v_M$, and $V = \operatorname{supp} v_M = [-M, M]$. Also set $C = \sup_n \|\delta_n\|$.

Next, as before, choose F on the diagonal on $\mathbb{Z} \times \mathbb{Z}$ such that $\pi(F) = 2u$. Then proceed as in steps (i)–(vii) of Theorem 1.8.2, where we found, with u', v' denoting the c_{00}-elements that arose there, that

$$\Delta_a(F) = u' \otimes (b - bv') - (b - bv') \otimes u',$$

where $b = a - au'$. With this in mind, we examine how $\Delta_a(F)$ differs from the present value of

$$H := u \otimes (b - bv) - (b - bv) \otimes u. \tag{1.8.4}$$

First, in (i), where $i, j \in U$ with $i \neq j$ and $\alpha(i) = \alpha(j)$, we obtain

$$\Delta_a(F)(i,j) = (u'(i) - u'(j))\alpha(i), \tag{1.8.5}$$

whereas the expression (1.8.4) gives value 0 since $v(j) = 1$. Suppose that the level sets of a that meet $[-N, N]$ are $L_1, \ldots, L_m$ (relabelling, if necessary), and, for $n = 1, \ldots, m$, set $r_n = |L_n \cap [-N, N]|$. Then Lemma 1.8.6 is applicable, taking $K_1 = K$ from the δ-small level sets hypothesis, with $K_2 = 1$, and with K_3 as in (1.8.3) above. In the case where $i, j \in L_n$, we have $|i - j| \leq KN^\delta$, and so equation (1.8.5) gives

$$|\Delta_a(F)(i,j)| = |\beta_n|\,|u'(i) - u'(j)| = \frac{|\beta_n|}{N+1}|i-j| \leq K|\beta_n|N^{\delta-1}.$$

Noting that $\|\delta_i \otimes \delta_j\| \leq C^2$, we see that the contribution to the new $\|\Delta_a(F)\|$ of all these terms from L_n is at most

$$C^2K|\beta_n|r_n^2N^{\delta-1}.$$

Thus the total contribution to $\|\Delta_a(F)\|$ over all the level sets meeting $[N, N]$ is at most

$$C^2K_3N^{\delta-1}\sum_{n=1}^m |\beta_n|r_n^2 \leq C^2K_1K_2^{1/q}K_3^{1+1/p}KN^{2\delta-1+1/q},$$

by Lemma 1.8.6.

Second, in (iii), where $i \in U$ and $j \in V \setminus U$, so that $v(j) = 1$, there are now two possibilities. If $\alpha(i) \neq \alpha(j)$, which was necessarily the case before, the same choice as before gives 0, as does H, since $v(j) = 1$. But now we have the new possibility that $i, j \in L_n$ for some $n \leq m$, in which case $\Delta_a(F)(i,j) = u'(i)\beta_n$ for any choice of $F(i,j)$. Set $F(i,j) = 0$. Now $N + 1 - |i| \leq KN^\delta$, whence

$$|\Delta_a(F)(i,j)| = \left(1 - \frac{i}{N+1}\right)|\beta_n| \leq |\beta_n|KN^{\delta-1}.$$

All such terms contribute at most

$$C^2KN^{\delta-1}\sum_{n=1}^m |\beta_n|\,|L_n|^2 \leq C^2K_1K_2^{1/q}K_3^{1/p}KN^{2\delta-1+1/q}$$

to the sum $\|\Delta_a(F)\|$. A similar estimate holds for the new possibility in (v).

All other cases considered in Theorem 1.8.2 are unchanged.

It follows that, in our new situation, we have the estimate

$$\|\Delta_a(F) - u \otimes (b - bv) - (b - bv) \otimes u\| \leq C^2K_1K_2^{1/q}K_3^{1/p}KN^{2\delta-1+1/q}.$$

Since $\|u \otimes (b - bv) - (b - bv) \otimes u\| < 2\varepsilon$ and $\delta < (q-1)/2q$, we have $\|\Delta_a(F)\| < 2\varepsilon$ for N sufficiently large, as required. ■

Remark 1.8.9. We do not know whether the constraint that $\delta < (q-1)/2q$ is necessary. However, the following example shows that the above method of proof does not show that A is pointwise approximately amenable at a for each $a \in A$.

Example 1.8.10 (An element with two-point level sets). For $j \in \mathbb{N}$, set $n_j = 2^j$, and select j distinct points in the interval $[2^j + 1, 2^{j+1}]$; enumerate all the latter points in order of increasing magnitude as $\{m_i : i \in \mathbb{N}\}$. Put a mass of i^{-1} at each m_i and n_i. For the resulting sequence a we have $a \in \ell^p$ for each $p > 1$, each non-zero level set of a has exactly two elements, and a fails to have δ-small level sets for any $\delta > 0$ because, for each $\delta > 0$, we have

$$\frac{|m_i - n_i|}{i^\delta} \sim \frac{2^i - i}{i^\delta} \to \infty \quad \text{as } i \to \infty.$$

Take $N \in \mathbb{N}$, say $n_j \leq N < n_{j+1}$. Then the number of points m_i less than n_j is $(j^2+j)/2$. We see that $m_i \leq N \leq n_i$ whenever $j+1 \leq i \leq (j^2+j)/2$, and we have

$$\sum_{\{i\,:\,m_i \leq N \leq n_i\}} a(m_i)u_N(m_i) = \sum_{i=j+1}^{(j^2+j)/2} \frac{1}{i}\left(1 - \frac{m_i}{N+1}\right) \geq \frac{1}{2}\sum_{i=j+1}^{(j^2+j)/2} \frac{1}{i} \sim \log j.$$

Thus the estimates on $\|\Delta_a(F)\|$ used in the above proof cannot work for this a. $\diamond$

EXAMPLE 1.8.11. Let $A(\overline{\mathbb{D}})$ be the usual disc algebra with the uniform norm, and set $A = \{f \in A(\overline{\mathbb{D}}) : f(0) = 0\}$. Then, for $f \in A$, we have

$$f(z) = \sum_{k=1}^{\infty} \widehat{f}(k)z^k \quad (|z| < 1),$$

where

$$\widehat{f}(k) = \frac{1}{2\pi \mathrm{i}} \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}k\theta}\,\mathrm{d}\theta \quad (k \in \mathbb{N}),$$

so that $(\widehat{f}(k)) \in \ell^2 \subset c_0$. Recall that for $f, g \in A$, their *Hadamard product* is

$$(f \star g)(z) = \sum_{k=1}^{\infty} \widehat{f}(k)\widehat{g}(k)z^k \quad (|z| < 1), \tag{1.8.6}$$

and that this can be written as

$$(f \star g)(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\xi\mathrm{e}^{\mathrm{i}\theta})g(\zeta\mathrm{e}^{-\mathrm{i}\theta})\,\mathrm{d}\theta \quad (|z| < 1), \tag{1.8.7}$$

where $\xi\zeta = z$ with $|\xi|, |\zeta| < 1$ [67, p. 158]. By the uniform continuity of f and g on $\overline{\mathbb{D}}$, it follows from equation (1.8.7) that $f \star g \in A$ and that $\|f \star g\| \leq \|f\|\,\|g\|$.

Thus $(A, \star)$ is isomorphic to a Banach sequence algebra. Here c_{00} is certainly dense in A. By equation (1.8.6), $A^2 \subset A^+(\overline{\mathbb{D}})$, the algebra of absolutely convergent Taylor series [11, Example 2.1.13(ii)], so that A^2 is properly dense in A, and hence A does not have a bounded approximate identity. For $k \in \mathbb{Z}$, set $Z^k : z \mapsto z^k$ $(z \in \mathbb{T})$, and for $f \in C(\mathbb{T})$, set $S_n(f) = \sum_{k=n}^{n} \widehat{f}(k)Z^k$. Then it is well-known that there exists $f \in C(\mathbb{T})$ such that $\limsup_{n\to\infty} S_n(f)(1) = \infty$; for example, see [49, §18]. A small modification of the proof gives a function $f \in A$ with this property; such a modification is given explicitly in [72, Theorem VIII.1.14]. Let (n_k) be a strictly increasing sequence in $\mathbb{N}$. Then we may further slightly modify the proof to see that there exists $f \in A$ with $\limsup_{k\to\infty} S_{n_k}(f)(1) = \infty$. It follows that, for each strictly increasing sequence (n_k) in $\mathbb{N}$, there exists $f \in A$ such that $e_{n_k}f$ does not converge to f in A, and so no subsequence of (e_k) can be an approximate identity.

Of course,

$$A = \{f \in C(\mathbb{T}) : \widehat{f}(-n) = 0 \ (n \in \mathbb{Z}^+)\},$$

and it is standard that Cesàro means of continuous functions converge uniformly on $\mathbb{T}$, so by the maximum modulus principle the Fejér kernels define an approximate identity for A. So, for $f \in A$, we have $f \in \overline{fA_{00}}$, and so A is a Ditkin algebra.

Take a sequence $S = (n_k)$ in $\mathbb{N}$ such that $n_{k+1} \geq 2n_k$ $(k \in \mathbb{N})$. Then, by [71, III.E.9], we have $\{(\widehat{f}(n_k)) : f \in A\} = \ell^2$, and so the map $f \mapsto \widehat{f}\,|_S$ is a continuous epimorphism of A onto ℓ^2, whence, by Proposition 1.6.4, A is not approximately amenable.

By Theorem 1.5.4 it follows that A is not pointwise amenable. Is A pointwise approximately amenable? $\diamond$

1.9. On approximate identities. We have the following pointwise variant of [14, Proposition 3.4]; see also [27, Proposition 3.13]. Note that it picks out the estimate which underlies the argument of Theorem 1.8.2.

PROPOSITION 1.9.1. *Let A be a Banach sequence algebra, and let $a \in A$. Suppose that there is $\eta > 0$ such that, for each $\varepsilon > 0$, there exists $u \in c_{00}$ with*

$$\|u\| \geq \eta \quad \textit{and} \quad \|a - au\|\,\|u\| < \varepsilon. \tag{1.9.1}$$

Then A is pointwise approximately amenable at a. ∎

Suppose that a Banach algebra A is approximately amenable. Then Corollary 1.5.3 shows that A has left and right approximate units; it is not known whether A must have approximate units. Indeed, all known examples of approximately amenable algebras have bounded approximate identities. It is known that a boundedly approximately contractible Banach algebra has a bounded approximate identity [8, Corollary 3.4], and results in [9, §2] show certain classes of algebras without bounded approximate identities cannot be approximately amenable.

Both Theorem 1.8.2 and Proposition 1.9.1 posit approximate units with special properties to deduce approximate amenability consequences. In particular, the hypothesis of Proposition 1.9.1 is certainly satisfied if A has bounded approximate units. But then A has a bounded approximate identity [11, §2.9], and so is approximately amenable [14, Corollary 3.5]. In fact, no Banach algebras A are known for which A satisfies condition (1.9.1) of Proposition 1.9.1 without A having a bounded right approximate identity. Regarding this latter, we offer the following observations.

PROPOSITION 1.9.2. *Let A be a Banach algebra with a right approximate identity (u_α). Suppose that there is a net (ε_α) of positive numbers converging to 0 such that, for each $a \in A$,*

$$\sup_\alpha \varepsilon_\alpha^{-1}\|a - au_\alpha\| < \infty. \tag{1.9.2}$$

Suppose further that A has an element which is not a left topological divisor of zero. Then (u_α) is bounded, and A has a right identity. In particular, this latter holds if A also has a left approximate identity satisfying the analogue of (1.9.2).

Proof. For $k \in \mathbb{N}$, define

$$A_k = \{a \in A : \sup_\alpha \varepsilon_\alpha^{-1}\|a - au_\alpha\| \leq k\}.$$

Clearly $\bigcup A_k = A$, and each A_k is closed. Thus some A_m has a non-empty interior, and, being convex and balanced, A_m must contain a neighbourhood of zero. Thus there is $\delta > 0$ such that $\|a\| \le \delta$ necessitates $\|a - au_\alpha\| \le m\varepsilon_\alpha$ for all α. It follows that $\|a - au_\alpha\| \to 0$ uniformly on the unit ball of A. Since A has an element which is not a left topological divisor of zero, there is α_0 such that $(u_\alpha)_{\alpha \ge \alpha_0}$ is bounded. Now [53, Proposition 1] shows that A has a right identity.

Finally, when there is a left approximate identity satisfying the analogue of (1.9.2), [53, Proposition 1] shows that A has an element which is not a left topological divisor of zero. (It follows that in fact A is unital.) ∎

COROLLARY 1.9.3. *Let A be a Banach algebra, and let (u_n) and (v_n) be left and right approximate identities, respectively, such that the two sequences $(\|u_n\| \cdot \|a - au_n\|)$ and $(\|v_n\| \cdot \|a - v_n a\|)$ are bounded for each $a \in A$. Then (u_n) and (v_n) are bounded.*

Proof. Assume towards a contradiction that (u'_k) is an unbounded subsequence of (u_n). Then (u'_k) satisfies the hypothesis of Proposition 1.9.2 on the right with $\varepsilon_k = \|u'_k\|^{-1}$, a contradiction of the theorem. ∎

This does not resolve the problem of whether the situation considered in [14, Proposition 3.4] can actually arise in the absence of a bounded approximate identity. The approximate identity $(u_{(\varepsilon,S)})$ built in the natural way from the elements u for each $\varepsilon > 0$ and each finite $S \subset A$ satisfies the condition that

$$\lim_{(\varepsilon,S)} \|u_{(\varepsilon,S)}\| \cdot \|a - au_{(\varepsilon,S)}\| = 0 \tag{1.9.3}$$

for each $a \in A$. However, there is no apparent reason for this to give (1.9.2).

An approximate identity satisfying (1.9.3) is called *quasi-bounded* in [27, §4].

COROLLARY 1.9.4. *A sequential quasi-bounded approximate identity in a commutative Banach algebra is in fact bounded.* ∎

1.10. Summary of interrelations. The following diagram shows what is known relating these various notions of amenability ([1]).

$$\begin{tikzcd}
\text{Contractible} \arrow[r, shift left] \arrow[d, shift right] & \text{Bdd App Cont} \arrow[l, shift left, "\times"] \arrow[r, shift left] \arrow[d, shift right] & \text{App Cont} \arrow[l, shift left, "\times"] \arrow[d, shift right] \\
\text{Amenable} \arrow[u, shift right, "\times"'] \arrow[r, shift left] \arrow[d, shift right] & \text{Bdd App Amen} \arrow[u, shift right, "??"'] \arrow[l, shift left, "\times"] \arrow[r, shift left] \arrow[dl, "\times"] & \text{App Amen} \arrow[u, shift right, "{[14], [26]}"'] \arrow[l, shift left, "??"] \arrow[d, shift right] \\
\text{Ptwise Amen} \arrow[u, shift right, "??"'] \arrow[rr, shift left] \arrow[urr, "\text{commutative [Theorem 1.5.4]}"'] & & \text{Ptwise App Amen} \arrow[u, shift right, "\times"'] \arrow[ll, shift left, "\times"]
\end{tikzcd}$$

Here $\times$ indicates the fact that there is a counter-example to the relevant implication. The unannotated implications all hold trivially. We are not able to decide whether or not

([1]) Uses Paul Taylor's `diagrams.sty` macros.

any of the undecided ('??') implications in the diagram are valid or not. (We suspect the latter.) However, in the remainder of this paper we shall study a variety of examples; in each case, they will be seen to be consistent with the conjecture that Banach sequence algebras which are Ditkin algebras are always pointwise approximately amenable, and are approximately amenable if and only if they have a bounded approximate identity.

2. Semigroup algebras

2.1. Background. Let S be a semigroup. Thus S is a non-empty set with an associative binary operation, denoted by

$$(s,t) \mapsto st, \qquad S \times S \to S.$$

An element $p \in S$ is an *idempotent* if $p^2 = p$; we write $E(S)$ for the set of idempotents of S. The semigroup is *regular* if, for each $s \in S$, there exists $t \in S$ such that $sts = s$; in this case, st and ts belong to $E(S)$, and $J^2 = J$ for each left or right ideal J in S.

The semigroup S is *amenable* if there exists a mean Λ on $\ell^\infty(S)$ such that Λ is left and right invariant under the natural action of S [40]; S is *right cancellative* if, for all $a, x, y \in S$, $xa = ya$ implies that $x = y$; S is *right weakly cancellative* if, for all $x, y \in S$, the set $\{z \in S : zx = y\}$ is finite.

For a semigroup S, $(\ell^1(S), \star)$ will denote the corresponding semigroup algebra. This algebra is discussed at length in [13]. In particular, it is determined in [13, Theorem 10.12] exactly when $\ell^1(S)$ is amenable. However, it is not known when $\ell^1(S)$ is weakly amenable; for some partial results, see [5] and [31]. It is also not known when $\ell^1(S)$ is approximately amenable or pointwise approximately amenable; for some remarks on the former issue, see [26, §9] and Theorem 2.2.9 below.

The first result is well-known.

Proposition 2.1.1. *Let S be a semigroup such that S is regular and amenable. Suppose further that S is right cancellative. Then S is an amenable group.*

Proof. Since S is regular, it follows that, for each $s \in S$, there exists $e_s \in E(S)$ such that $e_s s = s$. Since S is right cancellative, the element e_s is uniquely defined by this equation.

Since S is amenable, it is left-reversible [56, Proposition (1.23)]; this means that, for each pair $\{s, t\}$ in S, there exists $x \in sS \cap tS$, say $x = sy = tz$ for some $y, z \in S$. Clearly $e_x sy = sy$, and so $e_x s = s$ because S is right cancellative. Thus $e_x = e_s$. Similarly $e_x = e_t$, and so $e_s = e_t$. Thus there is a unique element $e \in S$ such that $es = s \ (s \in S)$.

Let $s \in S$. Then $se^2 = se$, and so $se = s$, again by right cancellativity. Thus e is the identity of S.

Take $s \in S$. By the regularity of S, there exists $t \in S$ with $sts = s$, By replacing t by sts, we may suppose that also $tst = t$. We have $st = ts = e$ by right cancellativity, and so $t = s^{-1} \in S$. Thus S is a group. ∎

Theorem 2.1.2. *Let S be a semigroup such that the semigroup algebra $\ell^1(S)$ is approximately amenable. Then S is regular and amenable.*

Proof. This is [26, Theorem 9.2]. ∎

Example 2.1.3. Let S be the bicyclic semigroup, so that S is the semigroup with identity generated by two elements p and q subject to the relation $pq = e$. By [18], S is regular and amenable. However $\ell^1(S)$ is not approximately amenable [30]. It is also not weakly amenable [6]. $\diamond$

The following result was suggested by [35, Theorem 2.3].

Corollary 2.1.4. *Let S be a semigroup such that $\ell^1(S)$ is approximately amenable. Suppose further that S is right cancellative. Then S is an amenable group, and $\ell^1(S)$ is amenable.* ∎

Note that $\mathbb{N}_\vee$ is weakly cancellative and $\ell^1(\mathbb{N}_\vee)$ is approximately amenable (see below), but $\mathbb{N}_\vee$ is certainly not a group. Thus we cannot replace 'right cancellative' by 'weakly cancellative' in the above corollary.

2.2. The case where $E(S)$ finite. Let S be a semigroup, and suppose that $\ell^1(S)$ is an amenable Banach algebra. Then it was shown in [19] that $E(S)$ is necessarily finite. In this section, we shall consider semigroups S for which $E(S)$ is finite, and ask when $\ell^1(S)$ is approximately amenable. Indeed, we shall show that this occurs if and only if $\ell^1(S)$ is already amenable.

Proposition 2.2.1. *Let S be a semigroup such that $E(S)$ is finite and*

$$S = \bigcup\{pSq : p, q \in E(S)\}. \tag{2.2.1}$$

Suppose that $\ell^1(S)$ has a left approximate identity and a right approximate identity. Then $\ell^1(S)$ has an identity.

Proof. This is [13, Proposition 4.3]. ∎

Corollary 2.2.2. *Let S be a semigroup such that $E(S)$ is finite and $\ell^1(S)$ is approximately amenable. Then $\ell^1(S)$ has an identity.*

Proof. Since $\ell^1(S)$ is approximately amenable, it follows from Corollary 1.5.3 that $\ell^1(S)$ has a left approximate identity and a right approximate identity. By Theorem 2.1.2, S is regular, and so equation (2.2.1) is satisfied. Thus $\ell^1(S)$ has an identity by Proposition 2.2.1. ∎

For example, let $S = (\mathbb{N} \cup \{\infty\}, +)$ with ∞ an absorbent element. Then $E(S)$ is empty, S fails to be regular, and so $\ell^1(S)$ is not approximately amenable by Theorem 2.1.2, and $\ell^1(S)$ has no identity.

Proposition 2.2.3. *Let S be a semigroup such that $E(S)$ is finite, and let J be an ideal in S. Suppose that $\ell^1(S)$ is approximately amenable. Then $\ell^1(J)$ is approximately amenable.*

Proof. Set $I = \ell^1(J)$. Then I is a complemented ideal in $\ell^1(S)$; in particular, I is a left ideal. By Proposition 1.6.1(ii), I has a right approximate identity.

Let T be the opposite semigroup to S, and set $B = \ell^1(T)$. Then B is also approximately amenable. Since I is a left ideal in B, it follows that I has a right approximate identity as a subalgebra of B, and hence I has a left approximate identity as a subalgebra of $\ell^1(S)$.

Certainly $E(J)$ is finite, and J is regular, so that

$$J = \bigcup \{pJq : p, q \in E(J)\}.$$

By Proposition 2.2.1, I has an identity. By Proposition 1.6.4(ii), I is approximately amenable. ∎

We shall use the following structure theorem, which combines Theorems 3.12 and 3.13 in [13], where details of the notation are given. (There are many other standard sources for this result, for example [10] and [40].)

PROPOSITION 2.2.4. *Let S be a regular semigroup such that $E(S)$ is finite. Then S has a principal series*

$$S = J_1 \supsetneq \cdots \supsetneq J_k = K(S),$$

where $J_1, \ldots, J_k$ are ideals in S and $K(S)$ is the minimum ideal in S, and the series is such that each quotient J_i/J_{i+1} for $i = 1, \ldots, k-1$ and J_k is isomorphic as a semigroup to a regular Rees matrix semigroup with a zero of the form $\mathcal{M}^o(G, P, m, n)$ for a certain group G, a sandwich matrix P, and some $m, n \in \mathbb{N}$. ∎

Recalling that J_i/J_{i+1} can be identified with $(J_i \setminus J_{i+1}) \cup \{0\}$, we see that the map

$$\Psi : \{\delta_s + \ell^1(J) : s \in I \setminus J\} \cup \{\ell^1(J)\} \to J_i/J_{i+1},$$

defined by

$$\Psi(s) = \begin{cases} s & (s \in I \setminus J), \\ 0 & (s = \ell^1(J)), \end{cases}$$

clearly extends to an isometric isomorphism $\ell^1(J_i)/\ell^1(J_{i+1}) \to \ell^1(J_i/J_{i+1})$.

COROLLARY 2.2.5. *Let S be a semigroup such that $E(S)$ is finite and $\ell^1(S)$ is approximately amenable. Let S have the above principal series. Then the semigroup algebra of each regular Rees matrix semigroup with a zero that arises as above is unital and approximately amenable.*

Proof. We recall that S is regular because $\ell^1(S)$ is approximately amenable, and so it does have a principal series as above. By Proposition 2.2.3, each algebra $\ell^1(J_i)$ is approximately amenable; by Corollary 2.2.2, each of these algebras has an identity. By Proposition 1.6.4(i), each algebra $\ell^1(J_i)/\ell^1(J_{i+1})$ is approximately amenable, whence so is $\ell^1(J_i/J_{i+1})$. Since $E(J_i/J_{i+1})$ is finite, $\ell^1(J_i/J_{i+1})$ is unital by Corollary 2.2.2. ∎

The construction of the semigroup algebra for a regular Rees matrix semigroup with a zero of the form $\mathcal{M}^o(G, P, m, n)$ is explained in [13, Chapter 4]. A quotient of this algebra by a one-dimensional ideal $\mathbb{C}\delta_o$ is isometrically isomorphic to a Munn algebra of the form

$$\mathcal{M}(\ell^1(G), P, m, n),$$

as explained in [13, p. 62], and this algebra is unital. Take $A = \ell^1(G)$, so that A is a unital algebra with a character (viz., the augmentation character); since $\mathcal{M}(\ell^1(G), P, m, n)$ is also unital, it follows from [13, Proposition 2.16] that our Munn algebra is isomorphic to $\mathbb{M}_n(\ell^1(G))$.

Proposition 2.2.6. *Let G be a group, and let $n \in \mathbb{N}$. Then $\mathbb{M}_n(\ell^1(G))$ is approximately amenable if and only if it is amenable.*

Proof. Suppose that $\mathbb{M}_n(\ell^1(G))$ is approximately amenable. Proposition 1.6.7(ii) shows that $\ell^1(G)$ is approximately amenable. By [26, Theorem 3.2], the algebra $\ell^1(G)$ is approximately amenable if and only if the group G is amenable, and this holds if and only if $\ell^1(G)$ is amenable. Thus in our case $\ell^1(G)$ is amenable. But now, by Proposition 1.6.7(i), $\mathbb{M}_n(\ell^1(G))$ is also amenable. ∎

Corollary 2.2.7. *Let G be a group, and let S be the semigroup $\mathcal{M}^o(G,P,n)$. Then $\ell^1(S)$ is approximately amenable if and only if it is amenable.*

Proof. Suppose that $\ell^1(S)$ is approximately amenable. Then $\ell^1(S)/\mathbb{C}\delta_o$ is approximately amenable by Proposition 1.6.4(i). Since $\ell^1(S)/\mathbb{C}\delta_o$ is isomorphic to $\mathbb{M}_n(\ell^1(G))$, it follows from Proposition 2.2.6 that $\ell^1(S)/\mathbb{C}\delta_o$ is amenable. By [13, Theorem 2.12], $\ell^1(S)$ is amenable. ∎

Theorem 2.2.8. *Let S be a semigroup such that $E(S)$ is finite. Then $\ell^1(S)$ is approximately amenable if and only if it is amenable.*

Proof. Suppose that $\ell^1(S)$ is approximately amenable, so that S is regular. The set $E(S)$ is finite by hypothesis. Let S have a principal series as above. By Corollary 2.2.5, the semigroup algebra of each regular Rees matrix semigroup with a zero that arises is approximately amenable. By Corollary 2.2.7, each of these algebras is amenable, and so each algebra $\ell^1(J_i)/\ell^1(J_{i+1})$ and $\ell^1(J_k)$ is amenable. By Theorem 1.2.2, $\ell^1(S)$ is itself amenable. ∎

In view of [13, Theorem 10.12], we now know exactly when $\ell^1(S)$ is approximately amenable in the special case where $E(S)$ is finite. However the finiteness of $E(S)$ is not necessary for approximate amenability; indeed, $\ell^1(\mathbb{N}_\wedge)$, which is described below, is even boundedly approximately contractible.

In specific situations, however, the finiteness of $E(S)$ is necessary. Recall that the *Brandt semigroup* S over a group G with index set I is the set of elementary $I \times I$ matrices over $G \cup \{0\}$; it equals the Rees matrix semigroup with zero over G with index set I and identity sandwich matrix. Clearly $|E(S)| \geq |I|$. The amenability of $\ell^1(S)$ was first considered in [18].

Theorem 2.2.9 (Pourabbas and Maysami Sadr [57]). *Let S be the Brandt semigroup over the group G with index set I. Then the following are equivalent:*

(i) *$\ell^1(S)$ is amenable;*
(ii) *$\ell^1(S)$ is approximately amenable;*
(iii) *I is finite and G is amenable.* ∎

3. A weighted semigroup algebra

For our first serious 'test case' for the missing implications in the diagram of §1.9, we study some weighted semigroup algebras. In this section, we shall consider a weighted version of $\ell^1(S)$ for a specific semigroup S.

3.1. Basic definitions. Let S be a non-empty set, and let $\omega : S \to [1, \infty)$. Then $\ell^1(S, \omega)$ is the Banach space of all functions $f : S \to \mathbb{C}$ such that

$$\|f\|_\omega = \sum_{s \in S} |f(s)|\omega(s) < \infty,$$

with $\|\cdot\|_\omega$ as the norm. As before, the characteristic function of $\{s\}$ for $s \in S$ will be denoted by δ_s, so that $f = \sum f(s)\delta_s$ $(f \in \ell^1(S, \omega))$.

Similarly, we have

$$\ell^\infty(S, \omega) = \{\lambda : S \to \mathbb{C} : \|f\|_{\omega,\infty} = \sup_{s \in S} |f(s)|\omega(s) < \infty\},$$

and so $(\ell^\infty(S, \omega), \|\cdot\|_{\omega,\infty})$ is a Banach space. The closed subspace consisting of functions $\lambda \in \ell^\infty$ such that, for each $\varepsilon > 0$, there exists a finite subset F of S such that $\sup\{|f(s)|\omega(s) : s \in S \setminus F\} < \varepsilon$ is denoted $c_0(S, \omega)$.

Then $\ell^\infty(S, 1/\omega)$ is the dual of $\ell^1(S, \omega)$, with the duality

$$\langle f, \lambda \rangle = \sum_{s \in S} f(s)\lambda(s) \qquad (f \in \ell^\infty(S, \omega),\ \lambda \in \ell^\infty(S, \omega)).$$

Similarly, $c_0(S, 1/\omega)$ is the predual of $\ell^1(S, \omega)$.

Now let S be a semigroup. A *weight* on S is a function $\omega : S \to [1, \infty)$ such that

$$\omega(st) \le \omega(s)\omega(t) \qquad (s, t \in S).$$

In this case $\ell^1(S, \omega)$ is a Banach algebra with respect to the product specified by the requirement that

$$\delta_s \star \delta_t = \delta_{st} \qquad (s, t \in S).$$

For example, let $\mathbb{N}_\wedge$ be the semigroup which is $\mathbb{N}$ with the semigroup operation $\wedge$, where

$$m \wedge n = \min\{m, n\} \qquad (m, n \in \mathbb{N}).$$

Each element of $\mathbb{N}_\wedge$ is an idempotent. It is well known that $\ell^1(\mathbb{N}_\wedge)$ is weakly amenable, but not amenable. It is shown in [13, Proposition 10.10] and [26, Example 4.6] that $\ell^1(\mathbb{N}_\wedge)$ is boundedly approximately amenable. Here we shall consider weighted versions of $\ell^1(\mathbb{N}_\wedge)$.

Let $\omega : \mathbb{N} \to [1, \infty)$ be any function. Then ω is a weight on the semigroup $\mathbb{N}_\wedge$. Throughout this section, set

$$A_\omega = (\ell^1(\mathbb{N}_\wedge, \omega), \|\cdot\|_\omega),$$

so that $A'_\omega = \ell^\infty(\mathbb{N}, 1/\omega)$ as a Banach space; set $E_\omega = c_0(\mathbb{N}, 1/\omega)$, so that $E'_\omega = A_\omega$ as a Banach space.

Note that $c_{00} \subset A_\omega \subset \ell^1$ and that c_{00} is dense in A_ω, so that A_ω is separable.

Let $a = (\alpha(i)) \in A_\omega$, and take $n \in \mathbb{N}$. Then of course

$$\|\delta_n\|_\omega = \omega(n) \qquad (n \in \mathbb{N}),$$

and we see that

$$a \star \delta_n = \sum_{i=1}^{n} \alpha(i)\delta_i + \Big(\sum_{i=n+1}^{\infty} \alpha(i)\Big)\delta_n.$$

Throughout we shall set

$$T_n a = \sum_{i=n+1}^{\infty} \alpha(i)\delta_i \quad (n \in \mathbb{N},\, a = (\alpha(i)) \in A_\omega), \tag{3.1.1}$$

for the 'tail' after the n-th entry of a.

The action in the dual module A'_ω is defined for $n \in \mathbb{N}$ and $\lambda \in A'_\omega$ by

$$(\delta_n \cdot \lambda)(m) = \begin{cases} \lambda(m) & (m \le n), \\ \lambda(n) & (m > n), \end{cases} \qquad (m \in \mathbb{N}).$$

The projective tensor product $A_\omega \widehat{\otimes} A_\omega$ is identified with the Banach space $\ell^1(\mathbb{N} \times \mathbb{N}, \omega \otimes \omega)$, where

$$(\omega \otimes \omega)(i,j) = \omega(i)\omega(j) \quad (i,j \in \mathbb{N}).$$

The projective tensor norm in $A_\omega \widehat{\otimes} A_\omega$ is also denoted by $\|\cdot\|_\omega$; the canonical product map is $\pi_\omega : A_\omega \widehat{\otimes} A_\omega \to A_\omega$, and the dual of $A_\omega \widehat{\otimes} A_\omega$ is $(A_\omega \widehat{\otimes} A_\omega)'$, identified with $\ell^\infty(\mathbb{N} \times \mathbb{N}, 1/(\omega \otimes \omega))$.

PROPOSITION 3.1.1. *Suppose that* $\lim_{n\to\infty} \omega(n) = \infty$. *Then* E_ω *is a closed submodule of* A'_ω, *and so* A_ω *is a dual Banach algebra.*

Further $c_0(\mathbb{N} \times \mathbb{N}, 1/(\omega \otimes \omega))$ *is a closed submodule of* $(A_\omega \widehat{\otimes} A_\omega)'$, *and so* $A_\omega \widehat{\otimes} A_\omega$ *is a dual module.*

Proof. This is easily verified (cf. [12, Example 9.13]). ■

Let ω be as above. Throughout we set

$$\widetilde{\omega}(n) = \inf\{\omega(i) : i \ge n\} \quad (n \in \mathbb{N}).$$

We fix a strictly increasing subsequence (n_j) of $\mathbb{N}$ as follows. In the case where $\lim_{n\to\infty} \omega(n) = \infty$, so that each infimum above is actually attained, first choose $n_1 \in \mathbb{N}$ so that $\omega(n_1) = \widetilde{\omega}(1)$; having defined n_j, choose $n_{j+1} \in \mathbb{N}$ with $n_{j+1} > n_j$ and so that $\omega(n_{j+1}) = \widetilde{\omega}(n_j + 1)$. We note that, for each $j \in \mathbb{N}$, we have

$$\omega(n_j) \le \widetilde{\omega}(n_j) \le \omega(i) \quad (i \ge n_j). \tag{3.1.2}$$

Otherwise, $\liminf_{n\to\infty} \omega(n) < \infty$, and we take (n_j) to be any strictly increasing sequence in $\mathbb{N}$ such that $\omega(n_j) \to \liminf_{n\to\infty} \omega(n)$.

3.2. The Gel'fand transform. We continue with the above notation. Take $a = (\alpha(i))$ in A_ω, and set

$$\beta(n) = \sum_{i=n}^{\infty} \alpha(i) \quad (n \in \mathbb{N}), \quad \text{and} \quad b = (\beta(n)).$$

The characters on A_ω have the form $a \mapsto \beta(n)$ for $n \in \mathbb{N}$, and so the character space of A_ω is $\mathbb{N}$; the Gel'fand transform of a is just b. In particular,

$$\widehat{\delta}_n = e_n = (\overbrace{1, \dots, 1}^{n}, 0, 0, \dots) \quad (n \in \mathbb{N}).$$

We write B_ω for the algebra $\widehat{A_\omega}$ which is the Gel'fand transform of A_ω, so that B_ω is a strongly regular Banach sequence algebra on $\mathbb{N}$. Given $b \in B_\omega$, the corresponding

$a = (\alpha(i)) \in A_\omega$ with Gel'fand transform equal to b is specified by

$$\alpha(i) = \beta(i) - \beta(i+1) \quad (i \in \mathbb{N}),$$

and so

$$B_\omega = \Big\{ b \in c_0 : \|b\|_\omega = \sum_{i=1}^{\infty} |\beta(i+1) - \beta(i)|\omega(i) < \infty \Big\}.$$

These are exactly the *Feinstein algebras*, first studied in [21]; see also [69]. The norm of b in B_ω in these sources is taken to be $\|b\|_\infty + \|b\|_\omega$, but these norms are equivalent to our norm whenever $\omega \geq 1$. Note that for $a \in A_\omega$ we have $\|a\|_\omega = \|\widehat{a}\|_\omega$ as just defined, so the double usage of $\|\cdot\|_\omega$ should cause no confusion.

In the case where $\omega(n) = 1 \ (n \in \mathbb{N})$, B_ω is the algebra of sequences of *bounded variation*, bv; see [11, Example 4.1.44].

Proposition 3.2.1.

(i) *Let $b \in B_\omega$. Then $\|b\|_\omega \geq \widetilde{\omega}(r)|\beta(r)|$ $(r \in \mathbb{N})$.*
(ii) *Let $F \in B_\omega \otimes B_\omega$. Then $\|F\|_\omega \geq \widetilde{\omega}(r)\widetilde{\omega}(s)|F(r,s)|$ $(r, s \in \mathbb{N})$.*
(iii) *Let $b \in B_\omega$. Then $\lim_{k\to\infty} \|T_{n_k}(b)\|_\omega = 0$.*

Proof. (i) Let $r \in \mathbb{N}$. For each $n > r$, we have

$$\|b\|_\omega \geq \sum_{i=r}^{\infty} |\beta(i+1) - \beta(i)|\omega(i) \geq \widetilde{\omega}(r) \sum_{i=r}^{n} |\beta(i+1) - \beta(i)| \geq \widetilde{\omega}(r)|\beta(n+1) - \beta(r)|,$$

and so the result follows because $\lim_{n\to\infty} \beta(n) = 0$.

(ii) Fix $\varepsilon > 0$, and choose $m \in \mathbb{N}$ and elements $b_1, \ldots, b_m, c_1, \ldots, c_m \in B_\omega$ such that $F = \sum_{j=1}^{m} b_j \otimes c_j$ and $\sum_{j=1}^{m} \|b_j\|_\omega \|c_j\|_\omega \leq \|F\|_\omega + \varepsilon$. Let $r, s \in \mathbb{N}$. Then

$$F(r,s) = \sum_{j=1}^{m} b_j(r) c_j(s),$$

and so, using (i), we have

$$\widetilde{\omega}(r)\widetilde{\omega}(s)|F(r,s)| \leq \sum_{i=1}^{m} \|b_j\|_\omega \|c_j\|_\omega \leq \|F\|_\omega + \varepsilon.$$

This holds for each $\varepsilon > 0$, and so the result follows.

(iii) Take (n_j) as above. In the case where $\lim_{n\to\infty} \omega(n) = \infty$, we have

$$|\beta(n_k + 1)|\omega(n_k) \leq \sum_{i=n_k+1}^{\infty} |\alpha_i|\omega(n_k) \leq \sum_{i=n_k+1}^{\infty} |\alpha_i|\omega(i) \quad (k \in \mathbb{N}),$$

and so $|\beta(n_k+1)|\omega(n_k) \to 0$ as $k \to \infty$. On the other hand, if $\liminf_{n\to\infty} \omega(n) < \infty$, this limit is obvious because $b \in c_0$. Thus

$$\lim_{k\to\infty} \|T_{n_k}(b)\|_\omega = \lim_{k\to\infty} \Big(\sum_{i=n_k+1}^{\infty} |\beta(i+1) - \beta(i)|\omega(i) + |\beta(n_k+1)|\omega(n_k) \Big) = 0,$$

giving (iii). ∎

The following corollary is noted in [21, Corollary 2.5].

Corollary 3.2.2. *The Banach sequence algebra $(B_\omega, \|\cdot\|_\omega)$ is a Ditkin algebra.*

Proof. This is immediate from clause (iii) of Proposition 3.2.1. ■

3.3. An approximate identity. Conditions on ω for A_ω to have a (bounded) approximate identity were first given in [21]. Indeed, with (n_j) as above, define

$$u_j = \delta_{n_j} \quad (j \in \mathbb{N}).$$

For each $a = (\alpha(i)) \in A_\omega$ and $j \in \mathbb{N}$, we have $\|a - a \star u_j\|_\omega = \|\widehat{a} - \widehat{a}e_{n_j}\|_\omega$, and so $\|a - a \star u_j\|_\omega \to 0$ as $k \to \infty$ by Proposition 3.2.1(iii). Thus we have the following result.

Proposition 3.3.1. *For each ω, the sequence (u_j) is an approximate identity for A_ω; the sequence is bounded whenever* $\liminf_{n\to\infty} \omega(n) < \infty$. ■

Suppose that $\lim_{n\to\infty} \omega(n) = \infty$. Then it is easily seen that A_ω does not have a bounded approximate identity. In fact, a slightly stronger remark than this holds true; we recall that each Banach algebra with a bounded approximate identity factors [11, Theorem 2.9.24].

Proposition 3.3.2. *Suppose that* $\lim_{n\to\infty} \omega(n) = \infty$. *Then the algebra A_ω does not factor weakly.*

Proof. Assume towards a contradiction that $A_\omega = A_\omega^2$. Since A_ω is separable, it follows from a theorem of Loy [11, Proposition 2.2.6(i)] that there exist $m \in \mathbb{N}$ and $C > 2$ such that each $a \in A_\omega$ with $\|a\|_\omega = 1$ can be written as $a = \sum_{j=1}^m a_j \star b_j$ with $\sum_{j=1}^m \|a_j\|_\omega \|b_j\|_\omega \leq C$. Choose $n \in \mathbb{N}$ so that $\omega(i) \geq C$ $(i \geq n)$, and write δ_n in the above form. Then

$$1 \leq \sum_{j=1}^m \Big(|\alpha_j(n)| \sum_{i=n}^\infty |\beta_j(i)| + |\beta_j(n)| \sum_{i=n}^\infty |\alpha_j(i)| \Big),$$

and so

$$\begin{aligned} C^2 &\leq \sum_{j=1}^m \Big(|\alpha_j(n)|\omega(n) \sum_{i=n}^\infty |\beta_j(i)|\omega(i) + |\beta_j(n)|\omega(n) \sum_{i=n}^\infty |\alpha_j(i)|\omega(i) \Big) \\ &\leq 2 \sum_{j=1}^m \|a_j\|_\omega \|b_j\|_\omega \leq 2C, \end{aligned}$$

a contradiction. ■

It is noted in [21, Theorem 2.6] that B_ω has bounded relative units if and only if $\liminf_{n\to\infty} \omega(n) < \infty$. The question when closed ideals in B_ω have bounded approximate identities is answered in [69]; sufficient conditions for closed ideals in B_ω to be complemented as Banach spaces are also given therein.

3.4. Weak amenability and amenability. The Banach algebra A_ω is commutative and is spanned by its idempotents. Thus A_ω is always weakly amenable [11, Proposition 2.8.72(i)].

In the case where $\omega = 1$, the algebra A is not amenable because it has infinitely many idempotents [13, Proposition 10.5]. It follows that A_ω is not amenable for any ω [11, Proposition 2.8.64].

In fact, [69] proves, with no supposition that $\omega \geq 1$, that B_ω is amenable if and only if $\sum_{n=1}^{\infty} \omega(n) < \infty$; [69] also gives conditions that are necessary and sufficient for an arbitrary closed ideal I to be amenable—in our case, with $\omega \geq 1$, this happens only if I is finite-dimensional.

It is proved in [13, Example 11.4] that, in the case where $\omega = 1$, the second dual algebra $(A''_\omega, \Box)$ is also weakly amenable.

3.5. Pointwise amenability. In this subsection we shall show that the algebras A_ω are never pointwise amenable.

Proposition 3.5.1. *Suppose that there is a constant $C \geq 1$ and a strictly increasing sequence (m_j) in $\mathbb{N}$ such that $\omega(m_j + 1) \leq C\omega(m_j)$ $(j \in \mathbb{N})$. Then A_ω is not pointwise amenable.*

Proof. By replacing (m_j) by (m_{2j}) and C by C^2, if necessary, we may suppose that $m_{j+1} > m_j + 1$ $(m \in \mathbb{N})$. Set $T = \{m_j + 1 : j \in \mathbb{N}\}$ and

$$\begin{aligned} I &= \{a \in A_\omega : \widehat{a}(n) = 0 \ \ (n \in \mathbb{N} \setminus T)\} \\ &= \Big\{\sum_{j=1}^{\infty} \alpha(m_j)(\delta_{m_j+1} - \delta_{m_j}) : \sum_{j=1}^{\infty} |\alpha(m_j)|\omega(m_j) < \infty\Big\}, \end{aligned}$$

so that I is a closed ideal in A_ω. Then

$$I^\circ = \{\lambda \in A'_\omega : \lambda(m_j + 1) = \lambda(m_j) \ \ (j \in \mathbb{N})\}.$$

We *claim* that I is weakly complemented in A_ω. Indeed, define

$$P : \lambda \mapsto \frac{1}{2}\sum_{j=1}^{\infty}(\lambda(m_j) + \lambda(m_j + 1))(\delta_{m_j} + \delta_{m_j+1}), \qquad A'_\omega \to I^\circ.$$

Clearly P is a linear map, and

$$\begin{aligned} \sup_{n\in\mathbb{N}} \frac{|(P\lambda)(n)|}{\omega(n)} &= \frac{1}{2}\sup_{j\in\mathbb{N}} \frac{|\lambda(m_j)| + |\lambda(m_j+1)|}{\omega(m_j)} \\ &\leq \frac{1}{2}\sup_{j\in\mathbb{N}} \frac{|\lambda(m_j)|}{\omega(m_j)} + \frac{C}{2}\sup_{j\in\mathbb{N}} \frac{|\lambda(m_j+1)|}{\omega(m_j+1)} \leq \frac{C+1}{2}\|\lambda\|_\omega, \end{aligned}$$

so that P is continuous with $\|P\| \leq (C+1)/2$. Since P is a projection onto I°, the subspace I is weakly complemented in A_ω.

We next *claim* that I does not have bounded approximate units. Assume towards a contradiction that I has bounded approximate units; take K such that, for each $a \in I$, there is $u \in I$ with $\|u\|_\omega \leq K$ and $\|a - a \star u\|_\omega < 1$, say

$$u = \sum_{j=1}^{\infty} u(m_j)(\delta_{m_j+1} - \delta_{m_j}).$$

Now choose $k > K + 1$, and set

$$a = \sum_{j=1}^{k} (\delta_{m_j+1} - \delta_{m_j}),$$

so that $a \in I$. Then

$$a - a \star u = \sum_{j=1}^{k} (1 - u(m_j))(\delta_{m_j+1} - \delta_{m_j}),$$

and so $\sum_{j=1}^{k} |1 - u(m_j)|\omega(m_j) < 1$. Thus

$$\|u\|_\omega \geq \sum_{j=1}^{k} |u(m_j)|\omega(n_j) \geq \sum_{j=1}^{k} \omega(m_j) - 1 \geq k - 1 > K,$$

a contradiction. Hence I does not have bounded approximate units. (This is also a consequence of [69, Lemma 5.3].) By Proposition 1.6.3, A_ω is not pointwise amenable. ■

Now suppose that $\lim_{n\to\infty} \omega(n) = \infty$. Then, by a remark above, A_ω itself does not have a bounded approximate identity, and so, directly from Theorem 1.5.4, A_ω is not pointwise amenable.

Thus we have proved the following result.

THEOREM 3.5.2. *For each weight ω, the Banach algebra A_ω is not pointwise amenable.* ■

3.6. Bounded approximate contractibility. In this subsection we shall determine when A_ω is boundedly approximately contractible.

In the case where $\liminf_{n\to\infty} \omega(n) < \infty$, it is easy to give an explicit construction showing that A_ω is sequentially (and hence boundedly) approximately contractible, and hence boundedly approximately amenable (cf. [13, Example 10.10] and [26, Example 4.6]). The result is also a consequence of Theorem 3.10.1 below.

For $n \in \mathbb{N}$, set

$$F_n = \delta_n \otimes \delta_n + \sum_{j=1}^{n} (\delta_j - \delta_{j-1}) \otimes (\delta_j - \delta_{j-1}) \in A_\omega \otimes A_\omega,$$

where $\delta_0 = 0$. Then $\pi(F_n) = 2\delta_n$. Let $a \in A_\omega$. As in [13] and [26], we have

$$\Delta_a(F_n) = \delta_n \otimes T_n a - T_n a \otimes \delta_n \tag{3.6.1}$$

with $T_n a$ as in (3.1.1), and so

$$\|\Delta_a(F_n)\|_\omega \leq 2\omega(n)\|T_n a\|_\omega.$$

Now let the sequence (n_j) be as above, set $C = \liminf_{n\to\infty} \omega(n) \geq 1$, and again set $u_j = \delta_{n_j}$ $(j \in \mathbb{N})$. Let $a \in A_\omega$. By an earlier remark, $\lim_{j\to\infty} a \star u_j = a$. Further,

$$\|\Delta_a(F_{n_j})\|_\omega \leq C\|T_{n_j} a\|_\omega \to 0 \quad \text{as } j \to \infty.$$

This shows that conditions (i) and (iii) of Proposition 1.4.3 are satisfied, and (ii) and (iv) follow by the uniform boundedness principle.

The next result will be essentially superseded by Theorem 3.9.1 below. However, we give a proof here as it clarifies the use of boundedness, and also because, surprisingly, condition (i) of Proposition 1.4.3 plays no rôle. Further the argument here is in terms of A_ω, whereas that of Theorem 3.9.1 is in terms of the algebra B_ω of Gel'fand transforms.

THEOREM 3.6.1. *Suppose that* $\lim_{n\to\infty}\omega(n)=\infty$. *Then there is no constant* $K>0$ *such that, for each* $\varepsilon>0$ *and* $a\in A$, *there exist elements* $u\in A_\omega$ *and* $F\in A_\omega\otimes A_\omega$ *with* $\pi(F)=2u$ *and such that:*

(i) $\|\Delta_{\delta_k}(F)\|_\omega\le K\|\delta_k\|_\omega=K\omega(k)\ (k\in\mathbb{N})$;
(ii) $\|a-a\star u\|_\omega<\varepsilon$.

Proof. Assume towards a contradiction that there is such a constant $K>0$. Take $a\in A$ and $\varepsilon>0$, and corresponding element $F\in A_\omega\otimes A_\omega$.

Temporarily fix $k\in\mathbb{N}$ and set $G=\Delta_{\delta_k}(F)\in A_\omega\otimes A_\omega$. Then

$$\begin{aligned}G=&\sum_{j=1}^{\infty}\Big[\sum_{i=1}^{k-1}F(i,j)\delta_i\otimes\delta_j+\Big(\sum_{i=k}^{\infty}F(i,j)\Big)\delta_k\otimes\delta_j\Big]\\&-\sum_{i=1}^{\infty}\Big[\sum_{j=1}^{k-1}F(i,j)\delta_i\otimes\delta_j+\Big(\sum_{j=k}^{\infty}F(i,j)\Big)\delta_i\otimes\delta_k\Big]\\&-\sum_{j=1}^{\infty}u(j)\delta_k\otimes\delta_j+\sum_{i=1}^{\infty}u(i)\delta_i\otimes\delta_k.\end{aligned}$$

Evaluating the expression for G at the point $(k,s)\in\mathbb{N}\times\mathbb{N}$, we see that

$$G(k,s)=\begin{cases}\displaystyle\sum_{i=k+1}^{\infty}F(i,s)-u(s) & (s<k),\\ \displaystyle\sum_{i=k}^{\infty}F(i,k)-\sum_{j=k}^{\infty}F(k,j) & (s=k),\\ \displaystyle\sum_{i=k}^{\infty}F(i,s)-u(s) & (s>k).\end{cases}\tag{3.6.2}$$

We also note that $u(k)=G(k,k)/2$.

Since $\ell^1(\mathbb{N},\omega)\mathbin{\widehat{\otimes}}\ell^1(\mathbb{N},\omega)=\ell^1(\mathbb{N}\times\mathbb{N},\omega\otimes\omega)$ isometrically, we have

$$\|\Delta_b(F)\|_\omega=\sum_{i=1}^{\infty}\sum_{s=1}^{\infty}|G(i,s)|\omega(i)\omega(s)\le K\|\delta_k\|=K\omega(k),$$

and so

$$\sum_{s=1}^{\infty}|G(k,s)|\omega(s)\le K.\tag{3.6.3}$$

Also,

$$\sum_{s=1}^{\infty}\sum_{i=k}^{\infty}|F(i,s)|\widetilde{\omega}(k)\omega(s)\le\sum_{s=1}^{\infty}\sum_{i=k}^{\infty}|F(i,s)|\omega(i)\omega(s)\le\|F\|_\omega,$$

whence

$$\sum_{s=1}^{\infty}\sum_{i=k}^{\infty}|F(i,s)|\omega(s)\le\frac{\|F\|_\omega}{\widetilde{\omega}(k)}.$$

Similarly,

$$\lim_{k\to\infty}\sum_{j=k}^{\infty}|F(k,j)|\omega(k)\le\frac{\|F\|_\omega}{\widetilde{\omega}(k)}.$$

It follows from (3.6.2) and (3.6.3) that, for each $k\in\mathbb{N}$, we have

$$\|u\|_\omega=\sum_{s=1}^{\infty}|u(s)|\omega(s)\le K+\frac{2\|F\|_\omega}{\widetilde{\omega}(k)}.$$

Thus $\|u\|_\omega\le K$ since $\widetilde{\omega}(k)\to\infty$ as $k\to\infty$.

We conclude that, for each $a\in A$ and $\varepsilon>0$, there exists $u\in A$ with $\|u\|_\omega\le K$ and such that $\|a-a\star u\|_\omega\le\varepsilon$. So A_ω has bounded approximate units. By [11, §2.9], A_ω has a bounded approximate identity, a contradiction of Proposition 3.3.2. ∎

COROLLARY 3.6.2. *The Banach algebra A_ω is boundedly approximately contractible if and only if* $\liminf_{n\to\infty}\omega(n)<\infty$.

Proof. This follows from Proposition 1.4.3 and Theorem 3.6.1. ∎

REMARK. The above result is also a consequence of [9, Theorem 3.3].

3.7. Pointwise approximate amenability

THEOREM 3.7.1. *For each weight ω, the Banach algebra A_ω is pointwise approximately amenable.*

Proof. We shall show that the hypotheses of Theorem 1.8.2 are satisfied.

By Proposition 3.2.2, A_ω is a Ditkin algebra, which confirms one hypothesis of Theorem 1.8.2.

Now take $a\in A$, set $b=\beta(i)=\widehat{a}$, and take $\varepsilon>0$. Choose $n\in\mathbb{N}$ such that

$$\sum_{j=n}^{\infty}|\beta(j+1)-\beta(j)|\omega(j)<\varepsilon. \tag{3.7.1}$$

Next, choose $\eta>0$ with

$$2\eta\sum_{j=1}^{n}\omega(j)<\varepsilon, \tag{3.7.2}$$

and then set $D=\{i\in\mathbb{N}:|\beta(i)|\ge\eta\}$, a finite set. We define

$$\widetilde{\beta}(i)=\begin{cases}\eta\beta(i)/|\beta(i)| & (i\in D),\\ \beta(i) & (i\in\mathbb{N}\setminus D),\end{cases}$$

and then set $\widetilde{b}=(\widetilde{\beta}(i))$. We note that $|\widetilde{\beta}(i)|\le\min\{|\beta(i)|,\eta\}$ $(i\in\mathbb{N})$ and that

$$|\widetilde{\beta}(i)-\widetilde{\beta}(j)|\le\min\{|\beta(i)-\beta(j)|,2\eta\}\quad(i,j\in\mathbb{N}). \tag{3.7.3}$$

In particular, $\widetilde{b}$ is constant on the sets of constancy of b. Finally, we define $u=(u(i))\in\mathbb{C}^{\mathbb{N}}$ by setting

$$(1-u(i))\beta(i)=\widetilde{\beta}(i)\quad(i\in\mathbb{N}),$$

taking $u(i)=0$ in the case where $\beta(i)=0$. We see that $u(i)=0$ $(i\in\mathbb{N}\setminus D)$, and so $u\in c_{00}$. Also we see that u is constant on the level sets of b, and that $b-bu=\widetilde{b}$.

Further, we calculate that

$$\begin{aligned}\|b-bu\|_\omega &= \sum_{j=1}^{n} |\widetilde{\beta}(j+1)-\widetilde{\beta}(j)|\omega(j) + \sum_{j=n+1}^{\infty} |\widetilde{\beta}(j+1)-\widetilde{\beta}(j)|\omega(j) \\ &\le 2\eta \sum_{j=1}^{n}\omega(j) + \sum_{j=n+1}^{\infty} |\beta(j+1)-\beta(j)|\omega(j)\end{aligned}$$

by (3.7.3), and so $\|\widetilde{b}\|_\omega < 2\varepsilon$ by (3.7.2) and (3.7.1).

The result now follows from Theorem 1.8.2. ■

3.8. Convergence of 'tails'. Following Theorem 1.8.2 above, we foreshadowed that the requirement

$$\|(I-P_C)(a)\| \to 0 \quad \text{as the finite subsets } C \text{ expand to } \mathbb{N}, \tag{3.8.1}$$

a condition which was used in [14, Proposition 3.6], was not necessarily satisfied in A_ω for certain weight functions ω. We shall see that the two cases where $\omega(n) = n^\alpha$ $(n \in \mathbb{N})$, and where $\omega(n) = n^{\alpha n}$ $(n \in \mathbb{N})$, for $\alpha > 0$, give contrasting conclusions. We shall work in B_ω.

Take $b \in B_\omega$. First consider the situation when $(I-P_C)(b)$ has the form of zeros, then an 'alternating interval', then an unchanged tail of b:

$$(I-P_C)(b) = (0,\dots,0,b_k,0,b_{k+2},0,\dots,0,\ \overbrace{b_{k+2\ell},b_{k+3\ell},\dots}^{\text{unchanged }(k+2\ell)\text{-tail}}\ ),$$

where $k,\ell \in \mathbb{N}$. (Some of the b_j could be zero as well.) The corresponding element in A_ω is given by

$$a' = (0,\dots,0,\ \overset{k-1}{b_k}\ ,-\overset{k}{b_k},\dots,\ \overset{k+2\ell-1}{b_{k+2\ell}}\ ,\overbrace{a_{k+2\ell},\ a_{k+3\ell},\ \dots}^{\text{unchanged }(k+2\ell)\text{-tail}}).$$

So the original a (with $\widehat{a} = b$) has been modified by setting the first $k-2$ elements to 0, then replacing 2ℓ terms by sums of certain tails.

Then

$$\begin{aligned}\|a'\|_\omega &= \sum_{i=0}^{\ell} |b_{k+2i}|(\omega(k+2i-1)+\omega(k+2i)) + |b_{k+2\ell}|\omega(k+2\ell) + \sum_{i=k+2\ell}^{\infty} |a_i|\omega(i) \\ &\ge \sum_{i=0}^{\ell}\Big|\sum_{j=k+2i}^{\infty} a_j\Big|(\omega(k+2i-1)+\omega(k+2i)).\end{aligned} \tag{3.8.2}$$

In particular, in the case where $\omega(n) = n^\alpha$ for $\alpha > 0$ and $a_n = 1/n^2\omega(n) = n^{-2-\alpha}$, this becomes

$$\begin{aligned}\sum_{i=0}^{\ell}[(k+2i-1)^\alpha + (k+2i)^\alpha] \sum_{j=k+2i}^{\infty} j^{-2-\alpha} &\sim \sum_{i=0}^{\ell} \frac{2(k+2i)^\alpha}{1+\alpha}(k+2i)^{-1-\alpha} \\ &= \frac{2}{1+\alpha}\sum_{i=0}^{\ell}\frac{1}{k+2i} \sim \frac{2^{1-\alpha}}{1+\alpha}\log\Big(\frac{k+2\ell}{k}\Big),\end{aligned}$$

which is arbitrarily large as ℓ increases for each fixed $k \in \mathbb{N}$. In particular, by first increasing k, and then increasing ℓ in terms of k, we see that $\|(I - P_C)b\|_\omega \nrightarrow 0$ as C expands, and so (3.8.1) fails to hold.

On the other hand, consider the case where $\omega(n) = n^{\alpha n}$ for some $\alpha > 0$. Choose $p > 2/\alpha$, so that $i^2\omega(i) \leq \omega(j)$ $(j > i + p)$. Then we have

$$\sum_{i=0}^{\ell}\Big|\sum_{j=k+2i}^{\infty} a_j\Big|(\omega(k+2i-1)+\omega(k+2i)) \leq 2\sum_{i=0}^{\ell}\sum_{j=k+2i}^{\infty}|a_j|\omega(k+2i)$$
$$\leq 2\sum_{i=0}^{\ell}(k+2i)^{-2}\Big\{\sum_{j=k+2i+p}^{\infty}|a_j|\omega(j)\Big\} + 2(p-1)\sum_{j=k}^{\infty}|a_j|\omega(j).$$

Now, given $\varepsilon > 0$, choose $k_0 \in \mathbb{N}$ so large that the k-th tails of a have norm less than ε for each $k > k_0$. Then, for each $k > k_0$, we have

$$\sum_{i=0}^{\ell}\Big|\sum_{j=k+2i}^{\infty} a_j\Big|(\omega(k+2i-1)+\omega(k+2i)) \leq 2\varepsilon\sum_{i=0}^{\ell}(k+2i)^{-2} + 2(p-1)\varepsilon < 2(p+1)\varepsilon.$$

Thus, for k sufficiently large, we have $\|a'\|_\omega < 2(p+3)\varepsilon$ since the last two terms of (3.8.2) converge to zero as $k \to \infty$.

Now given any $(I - P_C)(b)$, by setting alternate entries to zero, we have an element of the above form. What change has this done to our estimates? If

$$(I - P_C)(b) = (\dots, b_{q-1}, b_q, b_{q+1}, \dots) \quad \text{is modified to} \quad (\dots, b_{q-1}, 0, b_{q+1}, \dots),$$

then a' changes from

$$(\dots, b_q - b_{q-1}, b_{q+1} - b_q, b_{q+2} - b_{q+1}, \dots) \quad \text{to} \quad (\dots, -b_{q-1}, b_{q+1}, b_{q+2} - b_{q+1}, \dots).$$

In the norm calculation we are thus replacing

$$|b_q - b_{q-1}|\omega_{q-1} + |b_{q+1} - b_q|\omega_q = |a_{q-1}|\omega_{q-1} + |a_q|\omega_q \tag{3.8.3}$$

by

$$|b_{q-1}|\omega_{q-1} + |b_{q+1}|\omega_q = \Big|\sum_{j=q-1}^{\infty} a_j\Big|\omega_{q-1} + \Big|\sum_{j=q}^{\infty} a_j\Big|\omega_q.$$

This latter need not be larger, but in our estimates we replace this with

$$\sum_{j=q-1}^{\infty}|a_j|\omega_{q-1} + \sum_{j=q}^{\infty}|a_j|\omega_q,$$

which is clearly larger than (3.8.3).

It follows that the 'alternating form' is the worst possible scenario, so that indeed (3.8.1) holds when $\omega(n) = n^{\alpha n}$.

3.9. Approximate amenability. We continue with the above notation concerning A_ω.

THEOREM 3.9.1. *Suppose that* $\lim_{n\to\infty}\omega(n) = \infty$. *Then* A_ω *is not approximately amenable.*

Proof. Let (n_k) be the sequence defined at the end of §3.1 above, so that $\omega(n_k) \to \infty$. Take a subsequence $s(j)$ of (n_k) such that, for each $j \in \mathbb{N}$, we have:

(i) $s_j \geq s_{j-1} + 2$;
(ii) $\omega(s_j) \geq (j+1)^4$.

Here we take $s_0 = 0$; set $\omega(0) = 1$ for convenience.

Define $p, q > 0$ by

$$p \sum_{i=1}^{\infty} \frac{1}{\omega(s_i) i^2} = 1, \qquad q \sum_{i=1}^{\infty} \frac{1}{\omega(s_i + 1) i^2} = 1;$$

since $\omega(s_i + 1) \geq \omega(s_i)$ $(i \in \mathbb{N})$, we have $q \geq p$.

Define sequences $(\beta(j))$ and $(\gamma(j))$ by setting

$$\beta(j) = p \sum_{i=j}^{\infty} \frac{1}{\omega(s_i) i^2} \quad \text{and} \quad \gamma(j) = q \sum_{i=j}^{\infty} \frac{1}{\omega(s_i + 1) i^2}$$

for $j \in \mathbb{N}$. Note that $\beta(1) = \gamma(1) = 1$ and that

$$\beta(i) - \beta(i+1) = \frac{p}{\omega(s_i) i^2}, \qquad \gamma(i) - \gamma(i+1) = \frac{q}{\omega(s_i + 1) i^2} \qquad (i \in \mathbb{N}). \tag{3.9.1}$$

Set $r_i = s_i - s_{i-1}$ $(i \in \mathbb{N})$, and define

$$b = (\overbrace{\beta(1), \ldots, \beta(1)}^{r_1}, \overbrace{\beta(2), \ldots, \beta(2)}^{r_2}, \ldots),$$
$$c = (\overbrace{\gamma(1), \ldots, \gamma(1)}^{r_1+1}, \overbrace{\gamma(2), \ldots, \gamma(2)}^{r_2}, \ldots).$$

(It is only the first block in c that is longer than the corresponding block of b.)

Clearly $\beta(n) \to 0$ and $\gamma(n) \to 0$ as $n \to \infty$, so that both the sequences b and c belong to the space c_0. Further, by (3.9.1), we have $b, c \in B_\omega$.

For $j \in \mathbb{N}$, we have

$$\omega(s_j)\omega(s_{j-1} + 1)\beta(j) \geq \omega(s_j)\omega(s_{j-1}) p \sum_{i=j}^{\infty} \frac{1}{\omega(s_i) i^2} \geq \omega(s_{j-1}) p / j^2 \geq j^2 p \tag{3.9.2}$$

and, similarly,

$$\omega(s_j)\omega(s_j + 1)\gamma(j) \geq \omega(s_j) q / j^2 \geq j^2 q. \tag{3.9.3}$$

Finally, take $\varepsilon > 0$ such that

$$\varepsilon < \min\{p/\pi^2, 1/2\}. \tag{3.9.4}$$

Assume towards a contradiction that A_ω is approximately amenable. Then, by Proposition 1.4.2, there exists $F \in c_{00}(\mathbb{N} \times \mathbb{N})$ such that $u = \pi(F)/2$ and F satisfy

$$\|b - bu\|_\omega < \varepsilon, \qquad \|c - cu\|_\omega < \varepsilon, \qquad \|\Delta_b(F)\|_\omega < \varepsilon, \qquad \|\Delta_c(F)\|_\omega < \varepsilon.$$

In particular $|b(1)(1 - u(1))| \leq \|b - bu\|_\omega < 1/2$, so that $|u(1)| > 1/2$.

Let $j \in \mathbb{N}$. Since b has constant value $\beta(j)$ on the block $\{s_{j-1}+1, \ldots, s_j\}$, we see that

$$\Delta_b(F)(s_j, s_{j-1} + 1) = (u(s_j) - u(s_{j-1} + 1))\beta(j).$$

Similarly, $\gamma(s_j + 1) = \gamma(s_j)$, and so

$$\Delta_c(F)(s_j + 1, s_j) = (u(s_j + 1) - u(s_j))\gamma(j).$$

It follows from Proposition 3.2.1(ii) and (3.1.2) and (3.9.2) that we have

$$\begin{aligned} \varepsilon \geq \|\Delta_b(F)\|_\omega &\geq \omega(s_j)\omega(s_{j-1}+1)|u(s_j) - u(s_{j-1}+1)|\beta(j) \\ &\geq j^2 p|u(s_j) - u(s_{j-1}+1)|. \end{aligned} \tag{3.9.5}$$

Similarly, using (3.9.3) instead of (3.9.2), we see that

$$\varepsilon \geq \|\Delta_c(F)\|_\omega \geq j^2 q|u(s_j+1) - u(s_j)|. \tag{3.9.6}$$

Then (3.9.5), (3.9.6), and (3.9.4) together show that

$$|u(1) - u(s_k)| \leq \frac{2\varepsilon}{p}\sum_{i=1}^{\infty}\frac{1}{i^2} < \frac{1}{3}$$

for each $k \in \mathbb{N}$, whence

$$|u(s_k)| \geq \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \qquad (k \in \mathbb{N}).$$

This contradicts the fact that $u(s_k) = 0$ for all sufficiently large $k \in \mathbb{N}$.

Hence A_ω is not approximately amenable. ■

3.10. Summary. Putting all the above results together yields a complete description of the amenability (and some other) properties of the weighted semigroup algebras A_ω.

THEOREM 3.10.1. *Let $\omega : \mathbb{N} \to [1,\infty)$ be a function, and set $A_\omega = (\ell^1(\mathbb{N}_\wedge, \omega), \star)$. Then the following conditions on ω are equivalent:*

(a) $\liminf_{n\to\infty} \omega(n) < \infty$;
(b) *A_ω is boundedly approximately contractible;*
(c) *A_ω is boundedly approximately amenable;*
(d) *A_ω is approximately amenable;*
(e) *A_ω has a bounded approximate identity;*
(f) *A_ω factors;*
(g) *A_ω factors weakly;*
(h) *A_ω has bounded relative units.*

The Banach algebras A_ω are always weakly amenable and pointwise approximately amenable, but they are never pointwise amenable. ■

In particular, the algebras A_ω do not give counter-examples to any of the questioned implications in the diagram at the end of §1.

Recall that for $\omega : \mathbb{N} \to [1,\infty)$ and $1 \leq p < \infty$, we have

$$\ell^p(\omega) = \{f : \mathbb{N} \to \mathbb{C} : \|f\|^p = \sum |f(i)|^p\omega(i) < \infty\},$$

under pointwise operations.

The following table gives a quick comparison of A_ω with other common sequence algebras. The $\ell^p(\omega)$ results follow from the absence of a bounded approximate identity and Corollary 1.8.5, and the properties of the James algebra follow from Proposition 1.6.1 and [11, Example 4.1.44].

	$\ell^p(\omega)$	A_ω	James
Amenable	never	$\sum \omega_i < \infty$	no
Ptwise amenable	never	never	no
Approx. amenable	never	$\liminf \omega < \infty$	yes
Ptwise approx. amenable	always	always	yes
Bounded ai	never	$\liminf \omega < \infty$	yes

Remark. A consequence of the above characterization is that, if A and B are both pointwise approximately amenable, the same need not be true of $A \cap B$ taken with the maximum of the norms. For take ω_1 and ω_2 to be weights satisfying (a) above, but such that $\omega_1 \vee \omega_2 \to \infty$. Then A_{ω_1} and A_{ω_2} are pointwise approximately amenable, but $A_{\omega_1 \vee \omega_2} = A_{\omega_1} \cap A_{\omega_2}$ is not.

Note that an analogous remark holds for amenability. Indeed, take the continuous weights $t \mapsto e^t$ and $t \mapsto e^{-t}$ on $(\mathbb{R}, +)$. Then $L^1(e^t)$ and $L^1(e^{-t})$ are amenable, yet the algebra $L^1(e^t) \cap L^1(e^{-t}) = L^1(e^{|t|})$ is not [36, Theorem 0], [26, Theorem 8.6].

3.11. A subsidiary example. Changing the semigroup operation on $\mathbb{N}$, we consider the weighted semigroup algebra $A = \ell^1(\mathbb{N}_\vee, \omega)$, where $\mathbb{N}_\vee$ is the set $\mathbb{N}$ with the product

$$m \vee n = \max\{m, n\} \quad (m, n \in \mathbb{N}).$$

Here δ_1 is the identity of A. Take $\varphi \in \Phi_A$, and suppose that $\varphi(\delta_j) = 1$ for some $j \in \mathbb{N}$. For $i > j$, we have $\delta_i \star \delta_j = \delta_i$, and so $\varphi(\delta_i) = 0$. For $i < j$, we have $\delta_i \star \delta_j = \delta_j$, and so $\varphi(\delta_i) = 1$. Thus, if there is a least $j \in \mathbb{N}$ with $\varphi(\delta_{j+1}) = 0$, then

$$\varphi(a) = \sum_{i=1}^{j} \alpha_i \quad (a = (\alpha(i)) \in A).$$

Otherwise $\varphi(\delta_j) = 1$ for all $j \in \mathbb{N}$, and so

$$\varphi(a) = \varphi_\infty(a) = \sum_{i=1}^{\infty} \alpha_i \quad (a = (\alpha(i)) \in A).$$

Thus $\Phi_A = \mathbb{N} \cup \{\infty\}$, with Gel'fand map

$$a \mapsto \Big(\sum_{i=1}^{j} \alpha_i : j \in \mathbb{N} \cup \{\infty\}\Big).$$

Setting $\beta(0) = 0$, the element a with transform $(\beta(i))$ is given by

$$\alpha(i) = \beta(i) - \beta(i-1) \quad (i \in \mathbb{N}).$$

Consider $B = \ker \varphi_\infty$, so that

$$\begin{aligned} \widehat{B} &= \Big\{\beta \in c_0 : \|\beta\| = \sum_{i=1}^{\infty} |\beta(i) - \beta(i-1)|\omega(i) < \infty\Big\} \\ &= \Big\{\beta \in c_0 : |\beta(1)|\omega(1) + \sum_{j=1}^{\infty} |\beta(j+1) - \beta(j)|\omega(j+1) < \infty\Big\}. \end{aligned}$$

Noting that

$$\begin{aligned}\sum_{j=1}^{\infty}|\beta(j+1)-\beta(j)|\omega(j+1) &\leq |\beta(1)|\omega(1)+\sum_{j=1}^{\infty}|\beta(j+1)-\beta(j)|\omega(j+1)\\ &\leq \sum_{j=1}^{\infty}|\beta(j+1)-\beta(j)|(\omega(j+1)+1)\\ &\leq 2\omega(1)\sum_{j=1}^{\infty}|\beta(j+1)-\beta(j)|\omega(j+1),\end{aligned}$$

we see that $B \simeq B_\sigma$, where $\sigma(j) = \omega(j+1)$ $(j \in \mathbb{N})$. But the amenability properties of B_σ have been characterized in Theorem 3.10.1 in terms of the finiteness or otherwise of $\liminf_{n\to\infty}\sigma(n)$, which is exactly the same as the finiteness or otherwise of $\liminf_{n\to\infty}\omega(n)$. Finally, B and $A = B^\sharp$ have the same amenability properties, so we are done.

4. Segal algebras

4.1. Introduction. In this section, we shall consider the amenability properties of some Segal algebras. For earlier discussions on this topic, see [23], [24], and [28]; for more recent results see [9]. In particular, it is proved in [24, Theorem 2.1, Corollary 3.3] that each symmetric Segal algebra on a SIN group and on an amenable group (this includes all Segal algebras on locally compact abelian groups) is approximately weakly amenable.

Here we shall concentrate on Segal algebras on only the locally compact abelian groups $\mathbb{T}$ and $\mathbb{R}$, but for the convenience of the reader, we recall the following general definition taken from [11, Definition 4.5.26]; for a more detailed account, see [24, §1] and [60, §6.2]. For a function f on G, and $a \in G$, we denote by ${}_af$ the function $x \mapsto f(ax)$, $G \to \mathbb{C}$.

DEFINITION 4.1.1. Let G be a locally compact abelian group. A Banach algebra $(S, \|\cdot\|_S)$ is a *Segal algebra* on G if:

(i) S is a dense subalgebra of $(L^1(G), \star, \|\cdot\|_1)$;
(ii) $\|f\|_1 \leq \|f\|_S$ $(f \in S)$;
(iii) S is isometrically translation-invariant, and the map $a \mapsto {}_af$, $G \to S$, is continuous for each $f \in S$.

The Segal algebra S is *proper* if $S \neq L^1(G)$.

For a locally compact abelian group G, we write Γ for the dual group; for $f \in L^1(G)$ the Fourier transform of f on Γ is denoted by $\mathcal{F}(f) = \widehat{f}$, and

$$A(\Gamma) = \{\widehat{f} : f \in L^1(G)\},$$

so that $A(\Gamma)$ is a self-adjoint, regular, natural Banach function algebra on Γ [11, §4.5]. The subalgebra of $A(\Gamma)$ consisting of transforms with compact support is denoted by $A_{00}(\Gamma)$, and, for a Segal algebra S, we write

$$S_{00} := \{f \in S : \widehat{f} \in A_{00}(\Gamma)\}.$$

We shall require the following properties of Segal algebras; see [60, Propositions 6.2.5 and 6.2.8], [59], and [7].

LEMMA 4.1.2. *Let S be a Segal algebra on a locally compact abelian group G. Then:*

(i) *the character space of S is Γ;*
(ii) *$S_{00} = L^1(G)_{00}$ is dense in S;*
(iii) *S is a Ditkin algebra.* ■

Let G be a locally compact abelian group, and take q with $1 \le q < \infty$. Then we define

$$S_q(G) = \{f \in L^1(G) : \widehat{f} \in L^q(\Gamma)\},$$

with the norm

$$|||f|||_q = \|f\|_1 + \|\widehat{f}\|_q \quad (f \in S_q(G)).$$

Clearly $(S_q(G), |||\cdot|||_q)$ is a Segal algebra on G. In particular, the algebra $(S_1(G), |||\cdot|||_1)$ is often called the *Lebesgue–Fourier algebra* of G, and is denoted by $\mathcal{LA}(G)$; see, for example, [24]. The algebras $S_q(G)$ were first studied in [50], and more recently in [34], where they are denoted A_2^q. For example it is shown in [34, Theorem 2] that we have $S_{q_1}(G) \subsetneq S_{q_2}(G)$ whenever $1 \le q_1 < q_2$ and G is non-compact.

Since the Fourier transform $\mathcal{F} : S_q(G) \to L^q(\Gamma)$ is continuous, it induces a continuous operator

$$\mathcal{F} \otimes \mathcal{F} : S_q(G) \mathbin{\widehat{\otimes}} S_q(G) \to L^q(\Gamma) \mathbin{\widehat{\otimes}} L^q(\Gamma);$$

we write $\widehat{F}$ for $(\mathcal{F} \otimes \mathcal{F})(F)$, and when convenient view it as an element of $L^q(\Gamma \times \Gamma)$.

For the remainder of this section, we define and fix

$$\gamma_j = \begin{cases} \dfrac{1}{j^2 + |j|} & (j \in \mathbb{Z} \setminus \{0\}), \\ 0 & (j = 0), \end{cases} \tag{4.1.1}$$

and set $\gamma = (\gamma_j)$. Note that $\gamma_j \searrow 0$ as $j \to \infty$ and that

$$k\gamma_k \le \sum_{j=k+1}^{\infty} \gamma_j \quad (k \in \mathbb{N}). \tag{4.1.2}$$

This latter inequality is the key property of the sequence γ, and dictates the definition of γ in (4.1.1); for example, we cannot replace $1/(j^2 + |j|)$ by $1/(|j|^m + |j|)$ for any $m > 2$.

We are interested in the possible approximate amenability of Segal algebras, and make the following observation in the general abelian situation.

REMARK 4.1.3. Let G be a locally compact abelian group. By [60, Theorem 6.2.38], the Fourier transform maps the Feichtinger algebra $\mathfrak{S}^1(G)$ bicontinuously onto $\mathfrak{S}^1(\Gamma)$. For G compact, so that Γ is discrete, [60, Proposition 6.2.9(ii)] shows that this latter algebra is $\ell^1(\Gamma)$, the two norms from $\mathfrak{S}^1(\Gamma)$ and $\ell^1(\Gamma)$ being equivalent. In particular the map

$$f \mapsto \widehat{f}, \quad \mathfrak{S}^1(G) \to \ell^1(\Gamma),$$

is a Banach algebra isomorphism, where the pointwise product is taken on $\ell^1(\Gamma)$. It follows that $\mathfrak{S}^1(G)$ fails to be approximately amenable whenever G is an infinite, compact abelian group. This result is also shown in [9, Proposition 5.1].

4.2. The approximate amenability of Segal algebras on $\mathbb{T}$. In this subsection, we shall often write $f(\theta)$ for $f(\mathrm{e}^{\mathrm{i}\theta})$, and regard f as a function on $[0, 2\pi]$. Also we normalize Lebesgue measure on $\mathbb{T}$ to have total mass 1, and so

$$\|f\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^p \,\mathrm{d}\theta\right)^{1/p}$$

for $f \in L^p(\mathbb{T})$ and $p \geq 1$. For $1 < p < \infty$, p' will always denote its conjugate index, and we set

$$\widetilde{p} = \max\{2, p'\}.$$

We recall that $L^q(\mathbb{T}) \subset L^p(\mathbb{T}) \subset L^1(\mathbb{T})$ whenever $1 < p < q$, and that $\|f\|_1 \leq \|f\|_p$ when $p \geq 1$ and $f \in L^p(\mathbb{T})$. For $q \geq 1$ the algebras $(L^q(\mathbb{T}), \star, \|\cdot\|_q)$ are Segal algebras on $\mathbb{T}$; they are proper whenever $q > 1$.

It follows from Lemma 4.1.2 that each Segal algebra on $\mathbb{T}$ can be regarded as a Banach sequence algebra on $\mathbb{Z}$ that is a Ditkin algebra. We have already asked whether or not every such Ditkin algebra is necessarily pointwise approximately amenable. By a result of Ghahramani and Zhang [28], all Segal algebras on $\mathbb{T}$ are pseudo-amenable.

Thus we have a *conjecture*, also stated in [28]:

$$\text{Every proper Segal algebra on } \mathbb{T} \text{ fails to be approximately amenable.} \tag{4.2.1}$$

We shall establish the conjecture for a fairly wide class of such Segal algebras on $\mathbb{T}$.

The most obvious Segal algebras on $\mathbb{T}$ are the algebras $(L^p(\mathbb{T}), \star)$, where $p \geq 1$, and $S_q(\mathbb{T})$ for $q \geq 1$. We start by defining a common generalization of these algebras.

Definition 4.2.1. Let $(A, \|\cdot\|_A)$ be a Banach sequence algebra on $\mathbb{Z}$. For $p \geq 1$, set

$$S_{p,A} = \{f \in L^p(\mathbb{T}) : \widehat{f} \in A\},$$

and define

$$|\!|\!| f |\!|\!|_{p,A} = \|f\|_p + \|\widehat{f}\|_A.$$

We shall write $S_{p,q}$ for S_{p,ℓ^q}.

Lemma 4.2.2. *Let $(A, \|\cdot\|_A)$ be a Banach sequence algebra on $\mathbb{Z}$. Suppose that A is such that a function g on $\mathbb{Z}$ belongs to A if and only if $|g|$ belongs to A, and that, in this case, $\|g\| = \|\,|g|\,\|$. Take $p \geq 1$. Then $(S_{p,A}, |\!|\!|\cdot|\!|\!|_{p,A})$ is a Segal algebra on $\mathbb{T}$.*

Proof. That $(S_{p,A}, |\!|\!|\cdot|\!|\!|_{p,A})$ is a Banach algebra is clear, and it is also immediate that $\|f\|_1 \leq |\!|\!| f |\!|\!|_{p,A}$ for $f \in S_{p,A}$. Since $S_{p,A}$ contains the trigonometric polynomials, $S_{p,A}$ is certainly dense in $(L^1(\mathbb{T}), \|\cdot\|_1)$. The space $S_{p,A}$ is clearly translation-invariant, and each translation is an isometry; this follows because the sequence $(e^{\mathrm{i}n\theta} g(n))_{n\in\mathbb{Z}}$ belongs to A whenever $g \in A$ and they have the same norm. Finally, if $t \to 0$, then

$$\|{}_t f - f\|_{p,A} = \|{}_t f - f\|_p + \|(\mathrm{e}^{\mathrm{i}t} - 1)\widehat{f}\|_A \to 0.$$

Thus $(S_{p,A}, |\!|\!|\cdot|\!|\!|_{p,A})$ satisfies the conditions of Definition 4.1.1. ∎

Remark 4.2.3. In fact, given a Segal algebra on $\mathbb{T}$, $\widehat{S}$ is a Banach sequence algebra on $\mathbb{Z}$ with norm inherited from S, and $S = S_{1,\widehat{S}}$ with equivalent norms.

We shall make repeated use of the classical Hausdorff–Young inequality [72, Theorem XII.2.3], which we state here for future reference.

THEOREM 4.2.4 (Hausdorff–Young). *Let $1 \leq p \leq 2$. Then, for each $f \in L^p(\mathbb{T})$, $\widehat{f} \in \ell^{p'}$ and*

$$\|\widehat{f}\|_{p'} \leq \|f\|_p.$$

Further, if $(a_n) \in \ell^p$, then there is $f \in L^{p'}(\mathbb{T})$ with $\widehat{f} = (a_n)$ and

$$\|f\|_{p'} \leq \|\widehat{f}\|_p. \ \blacksquare$$

EXAMPLES 4.2.5. (i) Take $p, q \geq 1$. For $1 < p \leq 2$, we have

$$\|\widehat{f}\|_{p'} \leq \|f\|_p \quad (f \in L^p(\mathbb{T}))$$

by Theorem 4.2.4, so we have $S_{p,q} = S_{p,p'} = L^p(\mathbb{T})$ for $q \geq p'$. If $p \geq 2$, we have

$$\|\widehat{f}\|_2 = \|f\|_2 \leq \|f\|_p \quad (f \in L^p(\mathbb{T})),$$

and so $S_{p,q} = S_{p,2} = L^p(\mathbb{T})$ for $q \geq 2$. Thus every Segal algebra $(L^p(\mathbb{T}), \star)$ for $p \geq 1$ has the form $S_{p,q}$ for suitable q. Further, $S_{1,q} = S_q(\mathbb{T})$, as defined earlier.

For $1 \leq q \leq 2$ and $f \in S_{p,q}$, Theorem 4.2.4 shows that $\|f\|_{q'} \leq \|\widehat{f}\|_q$; by Hölder's inequality, $\|f\|_p \leq \|f\|_{q'}$ when $p \leq q'$. Thus $S_{p,q} \cong \ell^q(\mathbb{Z})$ when $1 \leq q \leq 2$ and $p \leq q'$.

(ii) Partition $\mathbb{N}$ into infinite subsets (T_j) $(j \in \mathbb{N})$, and define

$$A = \Big\{(\alpha(n)) \in c_0 : \lim_{j\to\infty}\Big(j \sum_{n\in T_j} |\alpha(n)|^j\Big)^{1/j} = 0\Big\},$$

with norm the supremum of the sum terms. Clearly $A \not\subset \ell^q$ for any $q \geq 1$, and $\ell^1 \not\subset A$. $\diamond$

We shall seek to determine when the algebras of the form $S_{p,q}$ are approximately amenable and when they are pointwise approximately amenable. As a guide, recall that $S_{1,q} \simeq \ell^q(\mathbb{Z})$ in the case where $1 \leq q \leq 2$, and so, as in [14], $S_{1,q}$ is pointwise approximately amenable, but not approximately amenable. We shall show that no $S_{p,q}$ is approximately amenable, so confirming the conjecture in some special cases. Recall that it follows from Theorem 1.5.4 that $S_{p,q}$ fails to be pointwise amenable whenever it fails to be approximately amenable.

We shall need the following (complex) version of [72, Lemma XII.6.6].

THEOREM 4.2.6. *Take $p > 1$. Suppose that (a_n) is a real-valued sequence with $a_n \searrow 0$ as $n \to \infty$. Then*

$$\sum_{n=1}^{\infty} n^{p-2} a_n^p < \infty \tag{4.2.2}$$

is a necessary and sufficient condition for there to be a function $f \in L^p(\mathbb{T})$ such that $\widehat{f}(n) = a_n$ $(n \in \mathbb{N})$. $\blacksquare$

Actually, [72, Lemma XII.6.6] shows that (4.2.2) ensures that $\sum_{n=1}^{\infty} a_n \cos nx$ and $\sum_{n=1}^{\infty} a_n \sin nx$ define functions in $L^p(\mathbb{T})$, whence $\sum_{n=1}^{\infty} a_n \mathrm{e}^{inx}$ is in $L^p(\mathbb{T})$ (and conversely).

For an element $H \in \ell^r \widehat{\otimes} \ell^r$, we shall write $\|H\|_r$ for its projective norm.

THEOREM 4.2.7. *Let A be a Banach sequence algebra on $\mathbb{Z}$. Take $p > 1$, and suppose that $\ell^r \subset A$ for some $r > \widetilde{p}/2$. Then $S_{p,A}$ is not approximately amenable, and hence not pointwise amenable.*

Proof. We write S for $S_{p,A}$, and $\|\!|\cdot|\!\|$ for $\|\!|\cdot|\!\|_{p,A}$.

We shall again consider the argument of [14, Theorem 4.1].

As noted in Example 4.2.5(i), we have $\|\!|f|\!\| \geq \|\widehat{f}\|_{\widetilde{p}}$ $(f \in S)$. It follows that, for each $F \in S_{00} \otimes S_{00}$, we have

$$\|\!|\Delta_f(F)|\!\|_{S\widehat{\otimes}S} \geq \|\Delta_a(\widehat{F})\|_{\widetilde{p}},$$

where $a = \widehat{f}$.

In the proof of the cited theorem, we found sequences a and b with certain properties. The actual sequences are

$$a = (\gamma_1^\beta, 0, \gamma_2^\beta, 0, \dots) \quad \text{and} \quad b = (0, \gamma_1^\beta, 0, \gamma_2^\beta, \dots);$$

here

$$\beta = 1/\widetilde{p} - \delta \tag{4.2.3}$$

for some appropriately small $\delta > 0$, and $\gamma = (\gamma_j)$ was specified in equation (4.1.1). Since $\gamma_j^\beta \sim j^{-2\beta}$ as $j \to \infty$ and $2\beta\widetilde{p} = 2 - 2\delta\widetilde{p} > 1$ for δ sufficiently small, we have $a, b \in \ell^{\widetilde{p}}$. Since $r > \widetilde{p}/2$, we also have

$$2r\beta = 2r/\widetilde{p} - 2r\delta > 1$$

for $\delta > 0$ sufficiently small, because $2r > \widetilde{p}$. It follows immediately that we may suppose that $\gamma^\beta \in \ell^r \subset A$. Further our estimates imply that $a, b \in \ell^r \subset A$.

As in the cited theorem there exists $\varepsilon > 0$ such that there is no element G in $\ell^{\widetilde{p}}(\mathbb{N}) \otimes \ell^{\widetilde{p}}(\mathbb{N})$ for which all the following inequalities hold:

$$\|\Delta_a(G)\|_{\widetilde{p}} < \varepsilon, \quad \|\Delta_b(G)\|_{\widetilde{p}} < \varepsilon, \tag{4.2.4}$$

$$\|a - a\pi(G)/2\| < \varepsilon, \quad \|b - b\pi(G)/2\| < \varepsilon. \tag{4.2.5}$$

We shall show that $a, b \in \widehat{S}$, say $a = \widehat{g}$ and $b = \widehat{h}$, where $g, h \in S$. Once we have established this, it will follow that, whenever $F \in S_{00} \widehat{\otimes} S_{00}$ satisfies $\|f - f\pi(F)/2\| < \varepsilon$ and $\|g - g\pi(F)/2\| < \varepsilon$, then

$$\|\!|\Delta_g(F)|\!\| + \|\!|\Delta_g(F)|\!\| \geq \varepsilon,$$

and hence the condition for approximate amenability given in Proposition 1.4.2 fails.

It remains to show that the specified elements a and b belong to $\widehat{S}$.

Suppose firstly that $1 < p < 2$, and consider the sequence γ^β. We claim that

$$\sum_{j=1}^{\infty} j^{p-2}\gamma_j^{p\beta} < \infty.$$

In fact $j^{p-2}\gamma_j^{p\beta} \sim j^{-\rho}$ as $n \to \infty$, where

$$\rho := 2p\beta + 2 - p = 2p(1 - 1/p - \delta) + 2 - p = p(1 - 2\delta),$$

and so $\rho > 1$ provided that $\delta < (p-1)/2p$.

In the second case, where $p \geq 2$, we need a slightly different argument. Here we note that, for each $f \in L^p(\mathbb{T})$, we have $f \in L^2(\mathbb{T})$ and $\|\widehat{f}\|_2 \leq \|f\|_p$. Now apply the above argument with $\beta = (1 - 2\delta)/2$. We have

$$j^{p-2}\gamma_j^{p\beta} \sim j^{-(2-p\delta)} \quad \text{as } n \to \infty,$$

and $2 - p\delta > 1$ provided that $\delta < 1/p$, so again we have

$$\sum_{j=1}^{\infty} j^{p-2}\gamma_j^{p\beta} < \infty$$

for suitable $\delta > 0$.

Thus in both cases, provided that $\delta > 0$ is sufficiently small, Theorem 4.2.6 applies to show there exists a function $f \in L^p(\mathbb{T})$ with $\widehat{f}(j) = \gamma_j^\beta$ $(j \in \mathbb{N})$. Thus $f \in S$.

Now define $g, h \in L^p(\mathbb{T})$ by

$$g(z) = f(z^2), \qquad h(z) = zg(z) \qquad (z \in \mathbb{T}).$$

Then $\widehat{g} = a$ and $\widehat{h} = b$, whence $g, h \in S = S_{p,A}$.

Thus we have found elements g and h of $S_{p,A}$ that demonstrate that the conditions for approximate amenability given in Proposition 1.4.2 fail. It follows from Theorem 1.5.4 that S is not pointwise amenable. ■

We note that [9, Example 3.7(e)] shows that $L^p(\mathbb{T})$ is not boundedly approximately amenable for $p > 1$; we strengthen this in the next result which answers in the negative Open Question 2 of [24].

COROLLARY 4.2.8. *Let $p > 1$. Then the Segal algebra $(L^p(\mathbb{T}), \star)$ is not approximately amenable, and hence it is not pointwise amenable.*

Proof. We just note that $S_{p,p'} = L^p(\mathbb{T})$ and that $p' > \widetilde{p}/2$ always holds. ■

There remains the case where $p = 1$ in Theorem 4.2.7. Note that Example 4.2.5(ii) shows that there are Banach sequence algebras A such that $A \not\subset \ell^q$ for any $q \geq 1$.

THEOREM 4.2.9. *Let A be a Banach sequence algebra on $\mathbb{Z}$. Suppose that $\ell^r \subset A \subset \ell^q$ for some $q \geq 1$ and $r > q/2$. Then $S_{1,A}$ is not approximately amenable, and hence it is not pointwise amenable.*

Proof. Again we set $S = S_{1,A}$, and argue as above. For each $f \in S$, certainly we have $\|\!|f|\!\| \geq \|\widehat{f}\|_q$, and so, for each $F \in S_{00} \otimes S_{00}$, we have

$$\|\!|\Delta_f(F)|\!\|_{S\widehat{\otimes}S} \geq \|\Delta_{\widehat{f}}(\widehat{F})\|_q.$$

We now take

$$\beta = 1/q - \delta.$$

We first note that, since $\gamma_j^\beta \searrow 0$ as $j \to \infty$ and γ^β is a convex sequence, there exists $f \in L^1(\mathbb{T})$ with $\widehat{f} = \gamma$ [72, Theorem V.1.5]. Also, since $r > q/2$, we again have

$$2\beta r = 2r/q - 2r\delta > 1$$

for $\delta > 0$ sufficiently small; for each such δ, we have $\gamma \in \ell^r$, and hence $f \in S_{1,A}$.

The proof concludes as before. ■

COROLLARY 4.2.10. *Let $q \geq 1$. Then the Segal algebra S_q is not approximately amenable, and hence it is not pointwise amenable.*

Proof. The rôle of r in the above is to ensure that $\widehat{f} \in A$ for a certain element $f \in L^1(\mathbb{T})$. Here $A = \ell^q$, so to ensure that $\widehat{f} \in A$, we just need $2\beta q = 2 - 2q\delta > 1$; this holds for $\delta > 0$ sufficiently small, and so the result follows. ■

We conclude that, for many proper Segal algebras S on $\mathbb{T}$, it is indeed true that A is not approximately amenable, so supporting the conjecture of Ghahramani and Zhang. We do not know if the constraints imposed in the above theorems are really necessary.

4.3. Segal algebras on $\mathbb{R}$. In this section we shall show that certain Segal algebras on $\mathbb{R}$ are not approximately amenable.

Fix q with $1 \le q < \infty$, and consider the well-known Segal algebras

$$\mathcal{S}_q = \{f \in L^1(\mathbb{R}) : \widehat{f} \in L^q(\widehat{\mathbb{R}})\}$$

on $\mathbb{R}$; the general case was mentioned in §4.1. Of course $\widehat{\mathbb{R}} = \mathbb{R}$, but we need to distinguish between the two copies of the real line. Set

$$A(\widehat{\mathbb{R}}) = \{\widehat{f} : f \in L^1(\mathbb{R})\}.$$

We need some preliminaries to show that the constructions to be given below are well-defined.

Let $\mathcal{S}$ be a Segal algebra on $\mathbb{R}$. For $F \in \mathcal{S}_{00} \otimes \mathcal{S}_{00}$, we have

$$\widehat{F} \in A_{00}(\widehat{\mathbb{R}}) \otimes A_{00}(\widehat{\mathbb{R}}) \subset C_{00}(\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}).$$

Thus for a function $w \in L^q(\widehat{\mathbb{R}})$, we can certainly define

$$\Delta_w(\widehat{F})(x,y) = (w(x) - w(y))\widehat{F}(x,y) + u(x)w(y) - w(x)u(y) \qquad (x, y \in \widehat{\mathbb{R}}),$$

where $u = \pi(\widehat{F})/2$. Further it is clear that, for each $f \in \mathcal{S}$, we have

$$\widehat{\Delta_f(F)} = \Delta_{\widehat{f}}(\widehat{F}).$$

Now let $0 \le \xi \le 1$ denote a (fixed) $C^\infty(\widehat{\mathbb{R}})$ function such that

$$\xi(t) = \begin{cases} 1 & (|t| < 1/4), \\ 0 & (|t| \ge 1/2), \end{cases}$$

and, for $j \in \mathbb{Z}$, denote by ξ_j the translate of ξ centred on j. We continue to take γ as in equation (4.1.1), and write

$$\beta = \frac{1}{q} - \frac{\alpha}{q'},$$

which is a slight modification to the previous version of β. We take $\beta = 1$ in the case where $q = 1$. Take $\alpha > 0$ sufficiently small so that $q\beta = 1 - \alpha q/q' > 1/2$ (there is no constraint on $\alpha > 0$ when $q = 1$), and then define

$$a = \sum_{j=1}^{\infty} \gamma_j^{\beta} \xi_{2j-1}, \qquad b = \sum_{j=1}^{\infty} \gamma_j^{\beta} \xi_{2j}.$$

Clearly $a, b \in L^q(\widehat{\mathbb{R}})$. Also, $a, b \in C^\infty(\widehat{\mathbb{R}})$ and $a, b, a', b' \in C_0(\widehat{\mathbb{R}})$ (where $'$ denotes the derivative), and so $a, b \in A(\widehat{\mathbb{R}})$, say $a = \widehat{f}$ and $b = \widehat{g}$, where $f, g \in L^1(\mathbb{R})$. Indeed, clearly $f, g \in \mathcal{S}_q$.

Finally, take $L^q_{00}(\mathbb{R})$ to be the space of functions in $L^q(\mathbb{R})$ of compact support. For an element $H \in L^r(\widehat{\mathbb{R}}) \mathbin{\widehat{\otimes}} L^r(\widehat{\mathbb{R}})$, we shall write $\|H\|_r$ for its projective norm, as before.

Theorem 4.3.1. *Let $q \ge 1$. Then the Segal algebra $\mathcal{S}_q$ is not approximately amenable, and hence it is not pointwise amenable.*

Proof. We again follow the argument of [14, Theorem 4.1]. Once again, to simplify notation we write $\mathcal{S}$ for $\mathcal{S}_q$ and $\|\!|\cdot|\!\|$ for $\|\!|\cdot|\!\|_q$; as usual, we write $\|\cdot\|_q$ for the norm in $L^q(\widehat{\mathbb{R}})$. We shall show that, for a suitable choice of β and for a certain $\varepsilon > 0$, there is no element $F \in \mathcal{S}_{00} \otimes \mathcal{S}_{00}$ such that both the following inequalities are true:

$$\|\Delta_a(G)\|_q + \|\Delta_b(G)\|_q < \varepsilon; \tag{4.3.1}$$

$$\|a - au\|_q + \|b - bu\|_q < \varepsilon. \tag{4.3.2}$$

Here, the specific functions a and b were defined above, and $G = \widehat{F}$ and $u = \pi(G)/2$. Note this is not quite the same as in [14], as here we cannot adjust on the diagonal, as can be done in the case of sequence algebras.

We also note that

$$\|\!|\Delta_f(F)|\!\|_{\mathcal{S}\widehat{\otimes}\mathcal{S}} \geq \|\Delta_a(G)\|_q$$

and that

$$\|\!|f - f \star v|\!\| \geq \|a - au\|_q,$$

so it is sufficient to work with the quantities a, b and $u = \pi(G)/2 = \widehat{v}$ on $\widehat{\mathbb{R}}$.

By construction, a and b are sums of functions which are supported on pairwise disjoint neighbourhoods of points of $\mathbb{Z}^+$, and a and b have disjoint supports. For $i, j \in \mathbb{N}$ and $w \in L^q(\widehat{\mathbb{R}})$, we write w_j for the restriction of w to $[j - 1/4, j + 1/4]$ and $\Delta_w(i,j)$ for the restriction of $\Delta_w(G)$ to the square $[i - 1/4, i + 1/4] \times [j - 1/4, j + 1/4]$. (Note that the functions w_j need not be continuous even when w is continuous.) As in (1.4.1), we have

$$\Delta_w(G)(i,j)(x,y) = (w_i(x) - w_j(y))G(x,y) + u_i(x)w_j(y) - w_i(x)u_j(y)$$

for $x, y \in \widehat{\mathbb{R}}$.

We may suppose that the $\varepsilon > 0$ to be chosen will satisfy $\varepsilon < 1/2$.

Now assume towards a contradiction that F satisfies

$$\|\!|\Delta_f(F)|\!\|_{\mathcal{S}\widehat{\otimes}\mathcal{S}} + \|\!|\Delta_g(F)|\!\|_{\mathcal{S}\widehat{\otimes}\mathcal{S}} < \varepsilon \quad \text{and} \quad \|\!|f - f \star v|\!\| < \varepsilon, \tag{4.3.3}$$

where $\pi(F) = 2v$. Then (recalling that $u = \widehat{v}$)

$$\frac{1}{2^q} > \|\!|f - f \star v|\!\|^q \geq \|a - au\|_q^q \geq \int_{3/4}^{5/4} |1 - u|^q,$$

and so $1 \geq \|u_1\|_q > 1/2$.

Take $i, j \in \mathbb{N}$, and consider the point $(2i - 1, 2j) \in \mathbb{N} \times \mathbb{N}$. For (x, y) in the rectangle $[2i - 5/4, 2i - 3/4] \times [2j - 1/4, 2j + 1/4]$, we have $\xi_{2i-1}(x) = \xi_{2j}(y) = 1$, and we calculate the values

$$\Delta_a(2i - 1, 2j)(x,y) = \gamma_i^\beta (G(x,y) - u_{2j}(y)),$$

$$\Delta_b(2i - 1, 2j)(x,y) = \gamma_j^\beta (u_{2i-1}(x) - G(x,y)).$$

In the case where $i \leq j$, so that $\gamma_i \geq \gamma_j$, geometrical considerations show that

$$|G(x,y) - u_{2j}(y)|^q \gamma_i^{q\beta} + |u_{2i-1}(x) - G(x,y)|^q \gamma_j^{q\beta} \geq \gamma_j^{q\beta} (|u_{2i-1}(x) - u_{2j}(y)|/2)^q.$$

The points $(2i, 2j - 1)$ taken with $i \leq j - 1$ and $j \geq 2$, so that $\gamma_i \geq \gamma_j$, lead to a similar estimate for (x, y) in the rectangle $[2i - 1/4, 2i + 1/4] \times [2j - 5/4, 2j - 3/4]$.

For $w \in L^q_{00}(\widehat{\mathbb{R}})$ and $\lambda = (\lambda_j \in \ell^1$ with $\lambda_j \geq 0$ $(j \in \mathbb{N})$, we define

$$\Phi_q(\lambda, w)(x,y) = \sum_{j=1}^{\infty} \lambda_j \sum_{i=1}^{j} |w_{2i-1}(x) - w_{2j}(y)|^q + \sum_{j=2}^{\infty} \lambda_j \sum_{i=1}^{j-1} |w_{2i}(x) - w_{2j-1}(y)|^q. \quad (4.3.4)$$

Then, recalling that a and b have disjoint supports, we see that

$$2^q(\|\Delta_f(G)\|_q^q + \|\Delta_g(G)\|_q^q) \geq \|\Phi_q(\gamma^{q\beta}, u)\|_1.$$

Set

$$\theta_q = \inf\{\|\Phi_q(\gamma^{q\beta}, w)\|_1 : w \in L^q_{00}(\widehat{\mathbb{R}}),\ 1/2 \leq \|w_1\|_q \leq 1\}.$$

Note that the infimum is taken over a larger set than $\widehat{\mathcal{S}_{00}}$; this is because we shall be truncating functions and need to stay within the relevant space when we do this. We shall seek to show that $\theta_q > 0$, for then (4.3.1) fails for each $\varepsilon > 0$ sufficiently small, and so $\mathcal{S}$ is not approximately amenable.

Suppose for the moment that $q = 1$, so that $\beta = 1$. In this case, with $\lambda = \gamma$, (4.3.4) gives

$$\|\Phi_1(\gamma, w)\|_1 = \sum_{j=1}^{\infty} \gamma_j \sum_{i=1}^{j} \|w_{2i-1}(x) - w_{2j}(y)\|_1 + \sum_{j=2}^{\infty} \gamma_j \sum_{i=1}^{j-1} \|w_{2i}(x) - w_{2j-1}(y)\|_1. \quad (4.3.5)$$

Consider the values of $\|\Phi_1(\gamma, w)\|_1$ for $\operatorname{supp} w \subset [0, d+1/2]$, for some $d \in \mathbb{N}$ with $d \geq 2$. Indeed, take such w not identically 0 on $[d-1/2, d+1/2]$. We *claim* that, by setting w to zero on this interval, the value of $\|\Phi_1(\gamma, w)\|_1$ is reduced.

To establish this claim, first suppose that $d = 2k+1$ for some $k \in \mathbb{N}$. By the change specified, we first increase each term in the summand

$$\gamma_{k+1} \sum_{i=1}^{k} \|w_{2i}(x) - w_{2k+1}(y)\|_1$$

by at most $\|w_{2k+1}\|_1 \gamma_{k+1}/2$, and so $\|\Phi_1(\gamma, w)\|_1$ increases by at most $k\|w_{2k+1}\|_1\gamma_{k+1}/2$. On the other hand, the term

$$\sum_{j=k+1}^{\infty} \gamma_j \|w_{2k+1}(x) - w_{2j}(y)\|_1 = \sum_{j=k+1}^{\infty} \gamma_j \|w_{2k+1}(x) - 0\|_1 = \Big(\sum_{j=k+1}^{\infty} \gamma_j\Big)\|w_{2k+1}\|$$

becomes 0. The other terms are not affected. However, for each $k \in \mathbb{N}$, we have

$$k\gamma_{k+1} \leq k\gamma_k \leq \sum_{j=k+1}^{\infty} \gamma_j$$

by the key property (4.1.2), and so, in total, the value of $\|\Phi_1(\gamma, w)\|_1$ has been decreased.

The case $d = 2k$ is similar.

By continuing, we see that, subject to the constraints that we have imposed, and in particular that $w \in L^1_{00}$ and $1/2 \leq \|w_1\|_1 \leq 1$, we have

$$\theta_1 \geq \|\Phi_1(\gamma, w_1)\|_1 = \gamma_1 \|w_1\|_1 \geq 1/4.$$

Hence we obtain the required contradiction in the case where $q = 1$.

Now suppose that $q > 1$. We have chosen $\alpha > 0$ so small that $q\beta = 1 - q\alpha/q' > 1/2$. Then

$$\sum_{j=1}^{\infty} j\gamma_j^{1+\alpha} < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \gamma_j^{q\beta} < \infty,$$

and so, in particular, the element $\gamma^{q\beta}$ is a sequence in ℓ^1 which is positive and decreasing.

Note that $(1+\alpha)/q' = 1$, so that $\gamma = \gamma^{\beta} \cdot \gamma^{(1+\alpha)/q'}$. Set

$$\delta = \Big(\sum_{j=1}^{\infty} j\gamma_j^{1+\alpha}\Big)^{1/q'} = \Big(\sum_{j=1}^{\infty}\sum_{i=1}^{j} \gamma_j^{1+\alpha}\Big)^{1/q'}.$$

Fix $u \in \widehat{\mathcal{S}_{00}}$ with $1/2 \le \|u_1\|_q \le 1$. Applying Hölder's inequality to each integral, we have the estimate

$$\begin{aligned} 1/4 &\le \|\Phi_1(\gamma, u)\|_1 \\ &= \sum_{j=1}^{\infty} \gamma_j \sum_{i=1}^{j} \|u_{2i-1}(x) - u_{2j}(y)\|_1 + \sum_{j=2}^{\infty} \gamma_j \sum_{i=1}^{j-1} \|u_{2i}(x) - u_{2j-1}(y)\|_1 \\ &\le \Big(\frac{1}{4}\Big)^{1/q'} \Big[\sum_{j=1}^{\infty} \gamma_j \sum_{i=1}^{j} \|u_{2i-1}(x) - u_{2j}(y)\|_q + \sum_{j=2}^{\infty} \gamma_j \sum_{i=1}^{j-1} \|u_{2i}(x) - u_{2j-1}(y)\|_q\Big]. \end{aligned}$$

Now let (x_r) be the sequence with generic term $\gamma_j^{\beta}\|u_{2i-1} - u_{2j}\|_q$ or $\gamma_j^{\beta}\|u_{2i} - u_{2j-1}\|_q$, and (y_r) the sequence with corresponding generic term $\gamma_j^{(1+\alpha)/q'}$. Applying Hölder's inequality to the sequence $(x_r y_r)$, we find that

$$\frac{1}{4} \le \delta^q \Big[\sum_{j=1}^{\infty} \gamma_j^{q\beta} \sum_{i=1}^{j} \|u_{2i-1}(x) - u_{2j}(y)\|_q^q + \sum_{j=2}^{\infty} \gamma_j^{q\beta} \sum_{i=1}^{j-1} \|u_{2i}(x) - u_{2j-1}(y)\|_q^q\Big].$$

Thus $1/4 \le \delta^q \|\Phi_q(\gamma, u)\|_1$. It follows that $\theta_q \ge 1/(4\delta^q)$, as required.

This concludes the proof of Theorem 4.3.1. ∎

There is another important family of Segal algebras on $\mathbb{R}$, namely $S_p = L^1(\mathbb{R}) \cap L^p(\mathbb{R})$, where $p > 1$ [60, Example 1.5.2 and Proposition 1.5.6]. Here $\|f\|_{S_p} = \|f\|_1 + \|f\|_p$ for $f \in S_p$.

Theorem 4.3.2. *Suppose that $3/2 < p \le 2$. Then $L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ is not approximately amenable, and hence it is not pointwise amenable.*

Proof. Suppose that $1 < p \le 2$, so that $S_p \subset S_{p'}$. Assume for the moment that the functions f, g of Theorem 4.3.1 lie in $L^1(\mathbb{R}) \cap L^p(\mathbb{R})$, and that $\widehat{f}, \widehat{g} \in L^{p'}(\widehat{\mathbb{R}})$. Then the same argument as in Theorem 4.3.1 applies to show that $L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ fails to be approximately amenable.

Now, as before, $\widehat{f}, \widehat{g} \in L^{p'}(\widehat{\mathbb{R}})$ provided that

$$p'\beta = 1 - \alpha p'/p > 1/2,$$

and this latter is true if $\alpha > 0$ is sufficiently small. Write $h = \check{\xi}$. Since $\xi \in C^{\infty}(\mathbb{R})$ and is of compact support, we certainly have $h \in L^p(\mathbb{R})$. But for $t \in \mathbb{R}$, we have

$$f(t) = \Big(\sum_{j=1}^{\infty} \gamma_j^{\beta} \mathrm{e}^{-(2j-1)t\mathrm{i}}\Big) h(t), \qquad g(t) = \mathrm{e}^{-t\mathrm{i}} f(t),$$

so that

$$\|f\|_p, \|g\|_p \le \|h\|_p \|\gamma^p\|_p,$$

which is finite provided that

$$p\beta = p/p' - \alpha > 1/2.$$

This holds for $\alpha > 0$ sufficiently small exactly when $p > 3/2$. Thus $f, g \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ for $p > 3/2$. So we have shown that $L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ is not approximately amenable in the case where $3/2 < p \le 2$. ■

The case where $1 < p \le 3/2$ remains unresolved.

4.4. The pointwise approximate amenability of Segal algebras on $\mathbb{T}$. We have seen that many, perhaps all, proper Segal algebras on $\mathbb{T}$ are not approximately amenable, and hence not pointwise amenable. The only property in this area that they may have is that they are pointwise approximately amenable. We now give some brief remarks concerning when Segal algebras on $\mathbb{T}$ are pointwise approximately amenable.

First consider the algebras $S_{p,q}$. For $1 \le q \le 2$ and $p \le q'$, we have $S_{p,q} = \ell^q$, and so $S_{p,q}$ is pointwise approximately amenable by Corollary 1.8.5.

What about the case where $q > 2$? Theorem 1.8.2 cannot be used for these algebras. For take a specific f as given by Rider in [61, Example A]. Then, again by Theorem 4.2.4,

$$f \in \bigcap_{1 \le p < 2} L^p(\mathbb{T}) \;\Rightarrow\; \widehat{f} \in \bigcap_{q > 2} \ell^q(\mathbb{Z}) \;\Rightarrow\; f \in \bigcap_{q > 2} S_q.$$

But, as noted in §1.8, the function f fails to satisfy the hypothesis of Theorem 1.8.2. By [22, Corollary 3.5], the closed subalgebra generated by f in $L^1(\mathbb{T})$ is not itself a Ditkin algebra, so it is not pointwise approximately amenable.

Now we consider whether or not the Segal algebra $(L^p(\mathbb{T}), \star)$ is pointwise approximately amenable when $p > 1$. By our earlier theorem, this is immediate for the special case where $p = 2$. Unfortunately, it is again the case that Theorem 1.8.2 cannot be used for any value of $p \ne 2$: if $p < 2$, then the above example of Rider shows that the 'level sets problem' closes the door to a solution along these lines, and, if $p > 2$, similar examples of Oberlin [54] and of Bachelis and Gilbert [2] do the same. Thus we do not know whether or not $L^p(\mathbb{T})$ is pointwise approximately amenable for any $p > 1$, save for $p = 2$.

Note that there exist Banach sequence algebras with $\ell^1 \subset A \subset \ell^p$ which are not pointwise approximate amenable. Indeed, take

$$A_p = \Big\{(x_i) \in \ell^p : \sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{i=1}^{n} |x(i+1) - x(i)| < \infty\Big\},$$

with the obvious norm. This satisfies $\ell^1 \subset A_p \subset \ell^p$, but

$$A_p^2 \subset \Big\{(x_i) \in \ell^p : \frac{1}{n} \sum_{i=1}^{n} |x(i+1) - x(i)| \to 0\Big\},$$

and so A_p fails to be pointwise approximately amenable by Corollary 1.5.3 since $\overline{A_p^2} \ne A_p$.

4.5. Postscript. Since obtaining the above results, we have received the fine paper [8], which contains interesting new results on approximate amenability, including the theorem that no proper Segal algebra on $\mathbb{R}^d$ or $\mathbb{T}^d$, $d \in \mathbb{N}$, can be approximately amenable, thus confirming conjecture (4.2.1) in full generality.

5. Open questions

We list here some questions that we believe are open:

1. Are the notions of 'boundedly approximately amenable' and 'approximately amenable' the same? All the known examples of approximately amenable Banach algebras are, in fact, boundedly approximately contractible.
2. Does every approximately amenable Banach algebra have a bounded approximate identity? All known examples do.
3. Is every approximately inner derivation automatically continuous?
4. As shown in [14], when the inequalities characterizing approximate amenability fail, it is often a two-point set that suffices to negate these inequalities. Is this is always the case?
5. Is there a pointwise amenable Banach algebra which is not already amenable?
6. For which groups G is $\ell^1(G)$ pointwise amenable? In particular, is $\ell^1(\mathbb{F}_2)$ pointwise amenable, where $\mathbb{F}_2$ is the free group on two generators?
7. Let I be a closed ideal of finite codimension in a unital, approximately amenable Banach algebra A. Is I also approximately amenable? This is open even when I has codimension two and A is commutative.
8. Is there a commutative, separable Banach algebra, or even a Banach sequence algebra, such that $A = A^2$, but A is not a Ditkin algebra?
9. An interesting example of a proper, unital, uniform algebra which is a strong Ditkin algebra is given by Feinstein in [20]. Is this example (pointwise) approximately amenable?
10. Characterize those semigroups S with $\ell^1(S)$ approximately amenable (and $E(S)$ infinite).
11. Is every Banach sequence algebra which is a Ditkin algebra necessarily pointwise approximately amenable? In particular, what can be said about $S_{p,q}$ for general p and q?

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