1. Introduction

In this paper we consider the problem

$$
(1.1)\qquad
\begin{aligned}
&v_t - \operatorname{div}\mathbb{D}(v) + \nabla p = f && \text{in } \Omega^T = \Omega\times(0,T),\\
&\operatorname{div} v = 0 && \text{in } \Omega^T,\\
&v\cdot\overline{n} = 0 && \text{on } S^T = S\times(0,T),\\
&\overline{n}\cdot\mathbb{D}(v)\cdot\overline{\tau}_\alpha = 0,\quad \alpha=1,2, && \text{on } S^T,\\
&v|_{t=0} = v_0 && \text{in } \Omega,
\end{aligned}
$$

where Ω is a domain with a distinguished axis, v is the velocity of the fluid, p the pressure, f the external force field, ν the constant viscosity coefficient, $\overline{n}$ the unit outward vector normal to the boundary S, and $\overline{\tau}_1$, $\overline{\tau}_2$ unit tangent vectors to S.

By $\mathbb{D}(v)$ we denote the dilatation tensor of the form

$$
(1.2)\qquad \mathbb{D}(v) = \nu\{\partial_{x_i}v_j + \partial_{x_j}v_i\}_{i,j=1,2,3}.
$$

Finally by dot we denote the scalar product in $\mathbb{R}^3$.

The aim of this paper is to prove the existence of regular solutions to problem (1.1) for the r.h.s. (right-hand side) functions from weighted Sobolev spaces, where the weight is a power of the distance from the distinguished axis. We are mainly interested in the case where the power of the distance is negative. The result is necessary in [10], where the existence of a global solution to the Navier–Stokes equations (which is close to the axially symmetric solution) in a bounded cylindrical domain and with boundary slip conditions is proved. Since the behaviour of the axially symmetric solution near the axis of symmetry is described by weighted Sobolev spaces with the weight equal to a negative power of the distance to the axis we have the same property for our global situation. Therefore in this paper we consider problem (1.1) in the spaces described by Theorem 5.3, where the most important case is $\mu = 1$.

To prove the existence of solutions to problem (1.1) we use the existence of weak solutions and the technique of regularizers. Therefore to show regularity of weak solutions we have to consider problem (1.1) locally. Let $\zeta = \zeta(x)$ be a function from a partition of unity. Let $\widetilde{v} = v\zeta$, $\widetilde{p} = p\zeta$, $\widetilde{f} = f\zeta$, $\widetilde{v}_0 = v_0\zeta$. Then instead of (1.1) we have

$$
(1.3)\qquad
\begin{aligned}
&\widetilde{v}_t - \operatorname{div}\mathbb{D}(v) + \nabla\widetilde{p} = \widetilde{f} - \operatorname{div}\mathbb{B}(v,\zeta) - \mathbb{D}(v)\cdot\nabla\zeta + p\nabla\zeta,\\
&\operatorname{div}\widetilde{v} = v\cdot\nabla\zeta,\\
&\widetilde{v}\cdot\overline{n} = 0,\\
&\overline{n}\cdot\mathbb{D}(\widetilde{v})\cdot\overline{\tau}_\alpha = \overline{n}\cdot\mathbb{B}(v,\zeta)\cdot\overline{\tau}_\alpha,\quad \alpha = 1,2,\\
&\widetilde{v}|_{t=0} = \widetilde{v}_0,
\end{aligned}
$$

where

$$\mathbb{B}(v,\zeta)=\{v_i\partial_{x_j}\zeta+v_j\partial_{x_i}\zeta\}_{i,j=1,2,3}.$$

We assume that $\widetilde{\Omega}=\operatorname{supp}\zeta$.

Since we are looking for a solution which has some special properties in a neighbourhood of the axis we distinguish the following neighbourhoods of the partition of unity: near an internal point of $\{\text{distinguished axis}\}\cap\overline{\Omega}$, near the point where the axis meets the boundary S, near a point of S and near an internal point which does not belong to the axis.

Now we examine (1.3) in any of these neighbourhoods. First we consider (1.3) in a neighbourhood of a point from the axis. Since we want to prove existence of solutions to (1.1) in weighted spaces we use the Kondrat'ev theory [2]. Therefore we introduce artificially an angle. Since $S\cap\widetilde{\Omega}=\emptyset$ we write (1.3) in the form

$$\begin{aligned}
&u_t-\operatorname{div}\mathbb{T}(u,q)=g && \text{in } \mathcal{D}_{2\pi}^T=\mathcal{D}_{2\pi}\times(0,T),\\
&\operatorname{div}u=h && \text{in } \mathcal{D}_{2\pi}^T,\\
&u|_{\Gamma_0}=u|_{\Gamma_{2\pi}} && \text{on } \Gamma_0^T=\Gamma_{2\pi}^T=\Gamma_{2\pi}\times(0,T),\\
&\overline{n}\cdot\mathbb{T}(u,q)|_{\Gamma_0}=-\overline{n}\cdot\mathbb{T}(u,q)|_{\Gamma_{2\pi}} && \text{on } \Gamma_0^T=\Gamma_{2\pi}^T,\\
&u|_{t=0}=u_0 && \text{in } \mathcal{D}_{2\pi},\\
&u=0 && \text{on } S_R^T=S_R\times(0,T),
\end{aligned}\tag{1.4}$$

where $u=\widetilde{v}$, $q=\widetilde{p}$, B_R is a ball with radius R, $S_R=\partial B_R$, $\mathbb{T}(u,q)$ is the stress tensor of the form

$$\mathbb{T}(u,q)=\mathbb{D}(u)-qI,\tag{1.5}$$

where I is the unit matrix, $\overline{n}|_{\Gamma_0}=-\overline{n}|_{\Gamma_{2\pi}}$, so the normal vectors in $(1.4)_4$ are different and the remaining notation is described in Section 2.

Since $\mathcal{D}_{2\pi}=\mathbb{R}^3$ the boundary conditions $(1.4)_{3,4}$ are artificial. They are introduced for the purpose of applying the Kondrat'ev method. Since the plane $\Gamma_0=\Gamma_{2\pi}$ is chosen arbitrarily we have to show that the solution is continuous with all derivatives when crossing $\Gamma_0=\Gamma_{2\pi}$. Such a proof will always be given.

In the next step we consider (1.3) in a neighbourhood of the point where the distinguished axis meets S. In this case under the same notation as in (1.4) and after straightening the boundary locally to the plane $x_3=0$, the problem (1.3) takes the form

$$\begin{aligned}
&u_t-\operatorname{div}\mathbb{T}(u,q)=g && \text{in } \mathcal{D}_{2\pi}^T\cap\mathbb{R}_+^3\times(0,T),\\
&\operatorname{div}u=h && \text{in } \mathcal{D}_{2\pi}^T\cap\mathbb{R}_+^3\times(0,T),\\
&u\cdot\overline{n}|_{x_3=0}=k_1 && \text{on } \mathcal{D}_{2\pi}^T\cap\{x:x_3=0\}\times(0,T),\\
&\overline{n}\cdot\mathbb{D}(u)\cdot\overline{\tau}_\alpha|_{x_3=0}=k_{2\alpha},\quad \alpha=1,2, && \text{on } \mathcal{D}_{2\pi}^T\cap\{x:x_3=0\}\times(0,T),\\
&u|_{\Gamma_0}=u|_{\Gamma_{2\pi}}, && \text{on } \Gamma_0^T\cap\mathbb{R}_+^3\times(0,T),\\
&\overline{n}\cdot\mathbb{T}(u,q)|_{\Gamma_0}=\overline{n}\cdot\mathbb{T}(u,q)|_{\Gamma_{2\pi}}, && \text{on } \Gamma_0^T\cap\mathbb{R}_+^3\times(0,T),\\
&u|_{t=0}=u_0 && \text{in } \mathcal{D}_{2\pi}\cap\mathbb{R}_+^3,
\end{aligned}\tag{1.6}$$

where $\mathbb{R}_+^3=\{x\in\mathbb{R}^3:x_3>0\}$.

In a neighbourhood of any point of S, problem (1.3) takes the following form after straightening $S \cap \bar{\bar{\Omega}}$:

$$
\begin{aligned}
&u_t - \operatorname{div} \mathbb{T}(u,q) = g && \text{in } \mathbb{R}^3_+ \times (0,T),\\
&\operatorname{div} u = h && \text{in } \mathbb{R}^3_+ \times (0,T),\\
&u \cdot \overline{n}|_{x_3=0} = k_1 && \text{on } \mathbb{R}^2 \times (0,T),\\
&\overline{n} \cdot \mathbb{D}(u) \cdot \overline{\tau}_\alpha|_{x_3=0} = k_{2\alpha}, \quad \alpha = 1,2, && \text{on } \mathbb{R}^2 \times (0,T),\\
&u|_{t=0} = v_0 && \text{in } \mathbb{R}^3_+.
\end{aligned}
\tag{1.7}
$$

Finally at an internal point we obtain the Cauchy problem

$$
\begin{aligned}
&u_t - \operatorname{div} \mathbb{T}(u,q) = g && \text{in } \mathbb{R}^3 \times (0,T),\\
&\operatorname{div} u = h && \text{in } \mathbb{R}^3 \times (0,T),\\
&u|_{t=0} = u_0 && \text{in } \mathbb{R}^3.
\end{aligned}
\tag{1.8}
$$

We are interested in two cases: $f \in L_{2,-\mu}(\Omega^T)$, $v_0 \in H^1_{-\mu}(\Omega)$, $\mu \in (0,1)$ and $f \in L_{2,-1}(\Omega^T)$, $v_0 \in H^1_{-1}(\Omega)$. The second case must be distinguished because it needs additional considerations. All considerations in this paper base on the L_2-approach.

The main result of this paper is formulated in Theorem 5.3. In the theorem we have to distinguish two cases: $\mu \in (0,1)$ and $\mu = 1$. In the second case the parameter λ in (3.73) is an eigenvalue so problem (3.73) cannot be solved directly. This needs extra considerations (see the proof of Lemma 3.8).

Finally we introduce some additional properties of any solution of problem (1.1). Let us introduce the cylindrical coordinates (see Section 2) such that the z-axis is the distinguished axis. Then we assume

$$
\int_0^{2\pi} u(r,\varphi,z) r \, d\varphi = 0,
\tag{1.9}
$$

for any $r \le \min \operatorname{dist}\{z\text{-axis}, S\}$, $z \in \{\text{distinguished axis}\}$.

2. Notation and auxiliary results

To simplify considerations we introduce the notation:

$$
|u|_{p,Q} = \|u\|_{L_p(Q)}, \qquad Q \in \{\Omega, S, \Omega^T, S^T\}, \qquad p \in [1,\infty],
$$
$$
\|u\|_{k,Q^T} = \|u\|_{W_2^{k,k/2}(Q^T)}, \qquad \|u\|_{k,Q} = \|u\|_{W_2^k(Q)}, \qquad Q \in \{\Omega, S\}, \qquad k \in \mathbb{N},
$$
$$
|u|_{p,\mu,Q} = \|u\|_{L_{p,\mu}(Q)} = \Big(\int_Q |u|^p r^{p\mu} \, dx \Big)^{1/p}, \qquad Q \in \{\Omega, S\}, \qquad p \in [1,\infty),
$$

$\mu \in \mathbb{R}$ and r is the distance from the distinguished axis expressed in the cylindrical coordinates (r,φ,z), where the axis is the z-axis. If $\Omega \subset \mathbb{R}^2$ then r is the distance to the origin of coordinates.

Since we consider problem (1.1) in weighted Sobolev spaces we use the following spaces: $V^k_{p,\mu}(\Omega)$, $k\in\mathbb{N}$, $p\in[1,\infty)$, $\mu\in\mathbb{R}$, with the norm

$$\|u\|_{V^k_{p,\mu}(\Omega)}=\Big(\sum_{|\alpha|\le k}\int_\Omega|D^\alpha_xu|^pr^{p(\mu-(k-|\alpha|))}\,dx\Big)^{1/p}\equiv\|u\|_{k,p,\mu,\Omega},$$

and $V^{k,k/2}_{p,\mu}(\Omega^T)$, $k\in\mathbb{N}$, $p\in[1,\infty)$, $\mu\in\mathbb{R}$, with the norm

$$\|u\|_{V^{k,k/2}_{p,\mu}(\Omega^T)}=\Big(\sum_{2i+|\alpha|\le k}\int_{\Omega^T}|\partial^i_tD^\alpha_xu|^pr^{p(\mu-(k-|\alpha|))}\,dx\,dt\Big)^{1/p}=\|u\|_{k,p,\mu,\Omega^T},$$

where $D^\alpha_x=\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}$, $\alpha=(\alpha_1,\alpha_2,\alpha_3)$, $|\alpha|=\alpha_1+\alpha_2+\alpha_3$.

For these spaces we introduce the spaces of traces $V^{k-2/p}_{p,\mu}(\Omega)$, $V^{k-1/p,k/2-1/2p}_{p,\mu}(S^T)$, $V^{k-1/p}_{p,\mu}(S)$.

For $p=2$ we have the spaces introduced by Kondrat'ev [2]

$$H^k_\mu(\Omega)=V^k_{2,\mu}(\Omega),\qquad H^{k,k/2}_\mu(\Omega^T)=V^{k,k/2}_{2,\mu}(\Omega^T).$$

Moreover, we set

$$\|u\|_{H^k_\mu(\Omega)}=\|u\|_{k,\mu,\Omega},\qquad \|u\|_{H^{k,k/2}_\mu(\Omega^T)}=\|u\|_{k,\mu,\Omega^T}.$$

To prove existence of solutions to problem (1.1) which do not vanish near the axis of symmetry we introduce the spaces $W^k_{p,\mu}(\Omega)$ and $W^{k,k/2}_{p,\mu}(\Omega^T)$ with the norms

$$\|u\|_{W^k_{p,\mu}(\Omega)}=\Big(\sum_{|\alpha|\le k}\int_\Omega|D^\alpha_xu|^pr^{p\mu}\,dx\Big)^{1/p}\equiv|||u|||_{k,p,\mu,\Omega},$$

and

$$\|u\|_{W^{k,k/2}_{p,\mu}(\Omega^T)}=\Big(\sum_{2i+|\alpha|\le k}\int_{\Omega^T}|\partial^i_tD^\alpha_xu|^pr^{p\mu}\,dx\,dt\Big)^{1/p}\equiv\|u\|_{k,p,\mu,\Omega^T}.$$

For the spaces $W^k_{p,\mu}(\Omega)$ and $W^{k,k/2}_{p,\mu}(\Omega^T)$ we have the spaces of traces $W^{k-1/p}_{p,\mu}(S)$, $W^{k-1/p,k/2-1/2p}_{p,\mu}(S^T)$, $W^{k-2/p}_{p,\mu}(\Omega)$, respectively, and the corresponding trace theorems.

Let r,φ be the polar coordinates in the plane; $d_\vartheta\subset\mathbb{R}^2$ the infinite angle $\{r>0, 0<\varphi<\vartheta\}$ with magnitude ϑ; γ_1,γ_2 the sides of d_ϑ described by $\varphi=0$ and $\varphi=\vartheta$, respectively; $\mathcal{D}_\vartheta=d_\vartheta\times\mathbb{R}^1$ the dihedral angle in $\mathbb{R}^3$ with sides $\Gamma_i=\gamma_i\times\mathbb{R}^1$, $i=1,2$, and edge $L=\overline{\Gamma}_1\cap\overline{\Gamma}_2$.

To examine problem (3.2) we introduce some spaces. Let $k\ge0$ be an integer and $\mu\in\mathbb{R}$. By $\overset{\circ}{L}{}^{k,k/2}_{2,\mu}(\mathcal{D}^T_\vartheta)$ we denote the closure of the set of compactly supported smooth functions equal to zero for $t<0$ in the norm

$$\begin{aligned}\|u\|_{\overset{\circ}{L}{}^{k,k/2}_{2,\mu}(\mathcal{D}^T_\vartheta)}=\Big(&\sum_{|\alpha|+2a=k}\int_0^T\int_{\mathcal{D}_\vartheta}|D^\alpha_x\partial^a_tu(x,t)|^2|x'|^{2\mu}\,dx\\&+\sum_{|\alpha|+2a=k-1}\int_{\mathcal{D}_\vartheta}|x'|^{2\mu}\,dx\int_{-\infty}^Tdt\int_{-\infty}^T\frac{|D^\alpha_x\partial^a_tu(x,t)-D^\alpha_x\partial^a_{t'}u(x,t')|^2}{|t-t'|^2}\,dt'\Big)^{1/2},\end{aligned}$$

where $x'=(x_1,x_2)$, $|x'|=\sqrt{x_1^2+x_2^2}=r$. Boundedness of the norm implies $\partial^i_tu|_{t=0}=0$ for $i\le[(k-1)/2]$.

If $T=\infty$ we can express the norm of $\overset{\circ}{L}{}^{k,k/2}_{2,\mu}$ by the Fourier transform of $u(x,t)$ with respect to the variables t,x'' $(x=(x',x''))$, which for an integrable function is given by the formula

$$\overline{u}(x',\xi,\xi_0)=(2\pi)^{-(n-1)/2}\int\limits_{\mathbb{R}^{n-2}}dx''\int\limits_{-\infty}^{\infty}u(x,t)e^{-i(x''\cdot\xi+t\xi_0)}\,dt,$$

where $x\in\mathbb{R}^n$, $x'\in\mathbb{R}^2$, $x''\cdot\xi=x_3\xi_1+\ldots+x_n\xi_{n-2}$. Then the norm of $\overset{\circ}{L}{}^{k,k/2}_{2,\mu}(\mathcal{D}^T_\vartheta)$ is equivalent to the norm

$$\|u\|_{\overset{\circ}{\widetilde{L}}{}^{k,k/2}_{2,\mu}(\mathcal{D}^T_\vartheta)}=\Big[\int\limits_{\mathbb{R}^{n-2}}d\xi\int\limits_{-\infty}^{\infty}d\xi_0\sum_{j=0}^{k}\|\widetilde{u}\|^2_{L^{k-j}_{2,\mu}(d_\vartheta)}(|\xi|^2+|\xi_0|)^j\Big]^{1/2}.$$

We introduce a partition of unity. Let us define two collections of open subsets $\{w^{(k)}\}$ and $\{\Omega^{(k)}\}$, $k\in\mathcal{M}\cup\mathcal{N}$, such that $\overline{w}^{(k)}\subset\Omega^{(k)}$, $\bigcup_k w^{(k)}=\bigcup_k\Omega^{(k)}=\Omega$, $\overline{\Omega^{(k)}}\cap S=\emptyset$ for $k\in\mathcal{M}$ and $\overline{\Omega^{(k)}}\cap S\neq\emptyset$ for $k\in\mathcal{N}$. We assume that $\mathcal{M}=\mathcal{M}_1\cup\mathcal{M}_2$, $\mathcal{N}=\mathcal{N}_1\cup\mathcal{N}_2$ and $\Omega^{(k)}$, $k\in\mathcal{M}_1$, is a neighbourhood of an internal point of $\{z\text{-axis}\}\cap\overline{\Omega}$, $\Omega^{(k)}$, $k\in\mathcal{M}_2$, is a neighbourhood of an internal point which does not belong to the axis, $\Omega^{(k)}$, $k\in\mathcal{N}_1$, is a neighbourhood of the point where the axis meets the boundary S. We know that $\mathcal{N}_1$ consists of two points p_1,p_2, so $\mathcal{N}_1=\{p_1,p_2\}$. Finally $\Omega^{(k)}$, $k\in\mathcal{N}_2$, is a neighbourhood of a point of S. Next we assume that at most N_0 of the $\Omega^{(k)}$ have nonempty intersection, and $\sup_k\operatorname{diam}\Omega^{(k)}\le 2\lambda$ for some $\lambda>0$. Let $\zeta^{(k)}(x)$ be a smooth function such that $0\le\zeta^{(k)}(x)\le 1$, $\zeta^{(k)}(x)=1$ for $x\in w^{(k)}$, $\zeta^{(k)}(x)=0$ for $x\in\Omega\backslash\Omega^{(k)}$ and $|D^\nu_x\zeta^{(k)}(x)|\le c/|\lambda|^\nu$. Then $1\le\sum_k(\zeta^{(k)}(x))^2\le N_0$. Introducing the function $\eta^{(k)}(x)=\zeta^{(k)}(x)/\sum_l(\zeta^{(l)}(x))^2$, we have $\eta^{(k)}(x)=0$ for $x\in\Omega\backslash\Omega^{(k)}$, $\sum_k\eta^{(k)}(x)\zeta^{(k)}(x)=1$ and $|D^\nu_x\eta^{(k)}(x)|\le c/|\lambda|^\nu$. By $\xi^{(k)}$ we denote a fixed internal point of $w^{(k)}$ and $\Omega^{(k)}$ for $k\in\mathcal{M}$, and a point of $\overline{w^{(k)}}\cap S$ and $\overline{\Omega^{(k)}}\cap S$ for $k\in\mathcal{N}$. For $k\in\mathcal{N}_1$, $\xi^{(k)}$ is either p_1 or p_2.

Since we consider a problem invariant with respect to translations and rotations we can introduce a local coordinate system $y=(y_1,y_2,y_3)$ with center at $\xi^{(k)}$ such that for $k\in\mathcal{N}$ the part $\widetilde{S}^{(k)}=S\cap\overline{\Omega^{(k)}}$ of the boundary is described by $y_3=F(y_1,y_2)$. Then we introduce new coordinates by

$$z_i=y_i,\qquad i=1,2,\qquad z_3=y_3-F(y_1,y_2).$$

We will denote this transformation by $\widehat{\Omega}^{(k)}\ni z=\Phi_k(y)$, where $y\in\Omega^{(k)}$ and $\widehat{w}^{(k)}\ni z=\Phi_k(y)$ where $y\in w^{(k)}$. We assume that the sets $\widehat{w}^{(k)}$, $\widehat{\Omega}^{(k)}$ are described in local coordinates at $\xi^{(k)}$ by the inequalities

$$|y_i|<\lambda,\qquad i=1,2,\qquad 0<y_3-F(y_1,y_2)<\lambda,$$
$$|y_i|<2\lambda,\qquad i=1,2,\qquad 0<y_3-F(y_1,y_2)<2\lambda,$$

respectively.

Let $y=Y_k(x)$ be a transformation from the x coordinates to local coordinates y which is the composition of a translation and a rotation. Then we set

$$\widehat{u}^{(k)}(z,t)=u(\Phi_k^{-1}\circ Y_k^{-1}(z),t),\qquad\widetilde{u}^{(k)}(z,t)=\widehat{u}^{(k)}(z,t)\widehat{\zeta}^{(k)}(z,t),\qquad k\in\mathcal{N}.$$

Now we recall some auxiliary results. From [11], Lemma 5.1, we have

Lemma 2.1. *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain. There exists a constant c such that for all $v\in L_2(\Omega)$ with*

$$E_\Omega(v)=\int_\Omega (v_{i,x_j}+v_{j,x_i})^2\,dx<\infty \tag{2.1}$$

we have

$$\|v\|_{1,\Omega}^2\le c(E_\Omega(v)+\|v\|_{0,\Omega}^2). \tag{2.2}$$

3. Existence of solutions to problem (1.4)

To prove existence of solutions to problem (1.4) we introduce a function φ which is the solution to the problem

$$\begin{aligned}
&\Delta\varphi=h_0,\\
&\varphi|_{\Gamma_0}=\varphi|_{\Gamma_{2\pi}},\\
&\frac{\partial\varphi}{\partial n}\bigg|_{\Gamma_0}=\frac{\partial\varphi}{\partial n}\bigg|_{\Gamma_{2\pi}},\\
&\varphi|_{S_{R'}}=0,\\
&\varphi|_{t=0}=0,
\end{aligned}\tag{3.1}$$

where $R'<R$ and we use the fact that $\operatorname{supp}h_0\subset B_{R'}$.

If h_0 is continuous when crossing $\Gamma_0=\Gamma_{2\pi}$, then from (3.1) we see that all first and second order derivatives of φ are also continuous when crossing $\Gamma_0=\Gamma_{2\pi}$.

Let us extend the initial data in (1.4) for $t>0$. Let us denote the extension by $\widetilde u_0$. Introducing a new function $v=u-\widetilde u_0-\nabla\varphi$, where φ is the solution of (3.1) with $h_0=h-\operatorname{div}\widetilde u_0$, we obtain the problem

$$\begin{aligned}
&v_t-\operatorname{div}\mathbb{T}(v,p)=f,\\
&\operatorname{div}v=0,\\
&v|_{\Gamma_0}=v|_{\Gamma_{2\pi}},\\
&\overline{n}\cdot\mathbb{T}(v,p)|_{\Gamma_0}=-\overline{n}\cdot\mathbb{T}(v,p)|_{\Gamma_{2\pi}},\\
&v|_{t=0}=0,\\
&v|_{S_R}=0.
\end{aligned}\tag{3.2}$$

The transmission conditions $(3.2)_{3,4}$ are artificial and by Lemma 3.7 no solution of (3.2) loses regularity when crossing the plane $\Gamma_0=\Gamma_{2\pi}$. Hence in considerations concerning solutions in the whole domain B_R we can omit these conditions. Therefore we also consider the problem

$$\begin{aligned}
&v_t-\operatorname{div}\mathbb{T}(v,p)=f &&\text{in } B_R\times(0,T),\\
&\operatorname{div}v=0 &&\text{in } B_R\times(0,T),\\
&v|_{S_R}=0 &&\text{on } S_R\times(0,T),\\
&v|_{t=0}=0 &&\text{in } B_R.
\end{aligned}\tag{3.2^1}$$

In view of the boundary condition $(3.2^1)_3$ we can extend solutions of problem (3.2^1) to $\mathbb{R}^3$ and treat (3.2^1) as the Cauchy problem only.

To examine problem (3.2^1) we have to introduce weak solutions. To obtain an integral identity we multiply (3.2^1) by a function η and integrate over B_R^T to obtain

$$\int_{B_R^T} v_t\cdot\eta\,dx - \int_{B_R^T} \operatorname{div}\mathbb{T}(v,p)\cdot\eta\,dx = \int_{B_R^T} f\cdot\eta\,dx. \tag{3.3}$$

Let η be such that

$$\begin{aligned}&\operatorname{div}\eta = 0,\\ &\eta|_{t=T} = 0,\\ &\eta|_{S_R^T} = 0.\end{aligned} \tag{3.4}$$

Then integrating by parts in (3.3) we obtain the integral identity

$$-\int_{B_R^T} v\cdot\eta_t\,dx\,dt + \frac{1}{2}\int_{B_R^T} \nabla v\cdot\nabla\eta\,dx\,dt = \int_{B_R^T} f\cdot\eta\,dx\,dt. \tag{3.5}$$

Definition 3.1. By a *weak solution* to problem (3.2^1) we mean a pair of functions (v,p) satisfying the integral identity (3.5) for any function $\eta\in H^1(B_R^T)$ satisfying (3.4) and for $f\in L_2(B_R^T)$.

Now we obtain some estimates for the weak solution. Inserting $\eta = v$ in (3.3), integrating by parts, using the Korn inequality (2.2), the Poincaré and Gronwall inequalities we obtain

$$\int_{B_R} v^2(t)\,dx + \int_0^t dt' \int_{B_R} (|v|^2 + |v_x|^2)\,dx \le c\int_0^t\int_{B_R} |f|^2\,dx\,dt', \tag{3.6}$$

where $t\le T$. Extending v and f by 0 for $t<0$, we obtain

$$\int_{B_R} v^2(t)\,dx + \int_{-\infty}^t dt' \int_{B_R} (|v|^2 + |v_x|^2)\,dx \le c\int_{-\infty}^t\int_{B_R} |f|^2\,dx\,dt'. \tag{3.7}$$

Extending v and f with respect to t up to $t=\infty$, we rewrite (3.7) in the form

$$\int_{B_R} v^2(t)\,dx + \int_{-\infty}^t dt' \int_{B_R} (|v|^2 + |v_x|^2)\,dx \le c\int_{-\infty}^t dt' \int_{B_R} |f|^2\,dx, \quad t\le\infty. \tag{3.8}$$

Besides the energy estimate (3.8) we also need an estimate for v_t. For this purpose inserting $\eta = v_t$ in (3.3) we obtain

$$\int_{B_R} v_t^2\,dx - \int_{B_R} \operatorname{div}\mathbb{T}(v,p)\cdot v_t\,dx = \int_{B_R} f\cdot v_t\,dx. \tag{3.9}$$

Integrating by parts in the second term and using the boundary conditions we obtain

$$\begin{aligned}-\int_{B_R}\operatorname{div}\mathbb{T}(v,p)\cdot v_t\,dx &= \int_{B_R}\mathbb{T}(v,p)\cdot\nabla v_t\,dx = \nu\int_{B_R}\nabla v\cdot\nabla v_t\,dx\\ &= \frac{1}{2}\frac{d}{dt}\nu\int_{B_R}|\nabla v|^2\,dx.\end{aligned}$$

Using it in (3.9), integrating the result with respect to time and using the vanishing of the initial conditions we arrive at

$$\int_{-\infty}^{\infty} dt \int_{B_R} (|v_t|^2 + |\nabla v(t)|^2)\,dx \le c \int_{-\infty}^{\infty} dt \int_{B_R} |f|^2\,dx, \tag{3.10}$$

where we used the extension with respect to time. In view of the definition of the space $\overset{\circ}{L}{}_{2,0}^{1,1/2}(\mathcal{D}_\vartheta \cap B_R)$, (3.7), (3.10) and after extending with respect to x we obtain

$$\begin{aligned} \|v\|^2_{\overset{\circ}{L}{}^{1,1/2}_{2,0}(\mathcal{D}_\vartheta\cap B_R)} &\le c \int_{-\infty}^{\infty} d\xi_0 \int_{\mathbb{R}^1} (\|\widetilde{v}\|^2_{L^1_{2,0}(d_\vartheta)} + \|\widetilde{v}\|^2_{L^0_{2,0}(d_\vartheta)}(|\xi|^2 + |\xi_0|))\,dx \\ &\le c \int_{-\infty}^{\infty} dt \int_{\mathcal{D}_{2\pi}} dx\,(|v|^2 + |v_t|^2 + |\nabla v|^2) \le c \int_{-\infty}^{\infty} dt \int_{\mathcal{D}_{2\pi}} |f|^2\,dx. \end{aligned} \tag{3.11}$$

Next using the definition of the space $L_{2,0}^{1,1/2}(\mathcal{D}_{2\pi}\cap B_R\times(-\infty,\infty))$ we obtain the inequality

$$\int_{\mathcal{D}_{2\pi}\cap B_R} dx \int_{-\infty}^{T} dt \int_{-\infty}^{T} dt'\, \frac{|v(x,t)-v(x,t')|^2}{|t-t'|^2} \le c \int_0^T dt \int_{\mathcal{D}_{2\pi}\cap B_R} dx\,|f|^2. \tag{3.12}$$

Since $v(x,t)=0$ for $t<0$ the expression on the l.h.s. of (3.12) is equal to

$$\int_0^T dt \int_{\mathcal{D}_{2\pi}\cap B_R} dx\, \frac{|v(x,t)|^2}{t} + \int_{\mathcal{D}_{2\pi}\cap B_R} dx \int_0^T dt \int_0^T dt'\, \frac{|v(x,t)-v(x,t')|^2}{|t-t'|^2}. \tag{3.13}$$

Summarizing we obtain

LEMMA 3.2. *Assume that $f \in L_2(B_R \times (0,T))$. Then there exists a weak solution of problem* (3.2^1) *which satisfies the inequality*

$$\begin{aligned} &\int_{\mathcal{D}_{2\pi}\cap B_R} (v^2(T) + |\nabla v(T)|^2)\,dx + \int_0^T dt \int_{\mathcal{D}_{2\pi}\cap B_R} (|v|^2 + |v_t|^2 + |v_x|^2)\,dx \\ &\quad + \int_0^T \int_{\mathcal{D}_{2\pi}\cap B_R} \frac{|v(x,t)|^2}{t}\,dx\,dt + \int_{\mathcal{D}_{2\pi}\cap B_R} dx \int_0^T dt \int_0^T dt' \frac{|v(x,t)-v(x,t')|^2}{|t-t'|^2} \\ &\le c \int_0^T dt \int_{\mathcal{D}_{2\pi}\cap B_R} dx\,|f(x,t)|^2. \end{aligned} \tag{3.14}$$

Existence follows from the Galerkin method.

To show higher regularity of solutions to problem (3.2) in a neighbourhood of the distinguished axis we apply the Kondrat'ev theory (see [2]). For this purpose we write (3.2) in the shortened form

$$\begin{aligned} &v_t - \operatorname{div}\mathbb{T}(v,p) = f, \\ &\operatorname{div} v = 0, \\ &v|_{\Gamma_0} = v|_{\Gamma_{2\pi}}, \\ &\overline{n}\cdot\mathbb{T}(v,p)|_{\Gamma_0} = -\overline{n}\cdot\mathbb{T}(v,p)|_{\Gamma_{2\pi}}, \end{aligned} \tag{3.15}$$

where $\Gamma_0 = \Gamma_{2\pi} = \{x \in \mathbb{R}^3 : x_2 = 0\}$, $\overline{n}|_{\Gamma_0} = (0,-1,0)$, $\overline{n}|_{\Gamma_{2\pi}} = (0,1,0)$. To express the boundary conditions $(3.15)_4$ more explicitly, we calculate

$$\overline{n}\cdot\mathbb{T}(v,p)|_{\Gamma_0} = -\begin{pmatrix} \nu(v_{1,x_2}+v_{2,x_1}) \\ 2\nu v_{2,x_2}-p \\ \nu(v_{3,x_2}+v_{2,x_3}) \end{pmatrix}, \quad \overline{n}\cdot\mathbb{T}(v,p)|_{\Gamma_{2\pi}} = \begin{pmatrix} \nu(v_{1,x_2}+v_{2,x_1}) \\ 2\nu v_{2,x_2}-p \\ \nu(v_{3,x_2}+v_{2,x_3}) \end{pmatrix}.$$

Hence $(3.15)_4$ takes the form

$$\begin{pmatrix} \nu(v_{1,x_2}+v_{2,x_1}) \\ 2\nu v_{2,x_2}-p \\ \nu(v_{3,x_2}+v_{2,x_3}) \end{pmatrix}\Bigg|_{\Gamma_0} = \begin{pmatrix} \nu(v_{1,x_2}+v_{2,x_1}) \\ 2\nu v_{2,x_2}-p \\ \nu(v_{3,x_2}+v_{2,x_3}) \end{pmatrix}\Bigg|_{\Gamma_{2\pi}}. \tag{3.16}$$

In the above considerations the x_3-axis is the distinguished axis.

Applying the Laplace transform with respect to t and the Fourier transform with respect to x_3 to problem (3.15) (see (3.17^1)) yields

$$\begin{aligned}
&s\widetilde{v}_j - \nu(\Delta'\widetilde{v}_j - \xi^2\widetilde{v}_j) + \widetilde{p}_{x_j} = \widetilde{f}_j, \quad j=1,2,\\
&s\widetilde{v}_3 - \nu(\Delta'\widetilde{v}_3 - \xi^2\widetilde{v}_3) + i\xi\widetilde{p} = \widetilde{f}_3,\\
&\widetilde{v}_{1,x_1} + \widetilde{v}_{2,x_2} + i\xi\widetilde{v}_3 = 0,\\
&\widetilde{v}|_{\gamma_0} = \widetilde{v}|_{\gamma_{2\pi}},\\
&\begin{pmatrix} \nu(\widetilde{v}_{1,x_2}+\widetilde{v}_{2,x_1}) \\ 2\nu\widetilde{v}_{2,x_2}-\widetilde{p} \\ \nu(\widetilde{v}_{3,x_2}+i\xi\widetilde{v}_2) \end{pmatrix}\Bigg|_{\gamma_0} = \begin{pmatrix} \nu(\widetilde{v}_{1,x_2}+\widetilde{v}_{2,x_1}) \\ 2\nu\widetilde{v}_{2,x_2}-\widetilde{p} \\ \nu(\widetilde{v}_{3,x_2}+i\xi\widetilde{v}_2) \end{pmatrix}\Bigg|_{\gamma_{2\pi}},
\end{aligned} \tag{3.17}$$

where $\widetilde{v},\widetilde{p}$ are the Laplace–Fourier transforms of v,p defined by

$$u(x,t) = \int_0^\infty ds \int_{-\infty}^\infty d\xi\, e^{ix_3\cdot\xi}\, e^{st}\widetilde{u}(x',\xi,s), \tag{3.17^1}$$

where $s = i\xi_0+\gamma_0$, $\gamma_0>0$, $\Delta' = \partial_{x_1}^2+\partial_{x_2}^2$, $\gamma_0,\gamma_{2\pi}$ are the "sides" of $d_{2\pi}=\mathbb{R}^2$ given by $x_2=0$.

To examine regularity of solutions to (3.17) we divide it into two problems:

$$\begin{aligned}
&-\nu\Delta'\widetilde{v}_j + \widetilde{p}_{x_j} = \widetilde{f}_j - q\widetilde{v}_j, \quad j=1,2, && \text{in } \mathbb{R}^2,\\
&\widetilde{v}_{1,x_1}+\widetilde{v}_{2,x_2} = -i\xi\widetilde{v}_3, && \text{in } \mathbb{R}^2,\\
&\widetilde{v}_j|_{\gamma_0} = \widetilde{v}_j|_{\gamma_{2\pi}}, \quad j=1,2, && x_2=0,\\
&\begin{pmatrix} \widetilde{v}_{1,x_2}+\widetilde{v}_{2,x_1} \\ 2\nu\widetilde{v}_{2,x_2}-\widetilde{p} \end{pmatrix}\Bigg|_{\gamma_0} = \begin{pmatrix} \widetilde{v}_{1,x_2}+\widetilde{v}_{2,x_1} \\ 2\nu\widetilde{v}_{2,x_2}-\widetilde{p} \end{pmatrix}\Bigg|_{\gamma_{2\pi}}, && x_2=0,
\end{aligned} \tag{3.18}$$

where $q=\nu\xi^2+s$ and

$$\begin{aligned}
&-\nu\Delta'\widetilde{v}_3 = \widetilde{f}_3 - q\widetilde{v}_3 - i\xi\widetilde{p} && \text{in } \mathbb{R}^2,\\
&\widetilde{v}_3|_{\gamma_0} = \widetilde{v}_3|_{\gamma_{2\pi}} && x_2=0,\\
&\widetilde{v}_{3,x_2}|_{\gamma_0} = \widetilde{v}_{3,x_2}|_{\gamma_{2\pi}} && x_2=0,
\end{aligned} \tag{3.19}$$

where we used $(3.18)_3$ to obtain the second boundary condition $(3.19)_2$.

To show regularity of weak solutions described by Lemma 3.2 in a neighbourhood of the axis we apply the Kondrat'ev technique (see [2]) because $f\in L_{2,\mu}$, where the weight is a power of the distance from the axis. Since the solution from Lemma 3.2 is examined we use the fact that it has a compact support. Therefore the functions from the r.h.s. of problems (3.18) and (3.19) have compact supports with respect to $x' = (x_1, x_2)$. To simplify the notation we write problems (3.18) and (3.19) in the form

$$
\begin{aligned}
&-\nu \Delta' v + \nabla' p = f && \text{in } \mathbb{R}^2,\\
&\operatorname{div}' v = h && \text{in } \mathbb{R}^2,\\
&v|_{\gamma_0} = v|_{\gamma_{2\pi}}, && x_2 = 0,\\
&(v_{1,x_2} + v_{2,x_1})|_{\gamma_0} = (v_{1,x_2} + v_{2,x_1})|_{\gamma_{2\pi}}, && x_2 = 0,\\
&(2\nu v_{2,x_2} - p)|_{\gamma_0} = (2\nu v_{2,x_2} - p)|_{\gamma_{2\pi}}, && x_2 = 0,
\end{aligned}
\tag{3.20}
$$

where all operators are two-dimensional and

$$
\begin{aligned}
&\Delta' u = f && \text{in } \mathbb{R}^2,\\
&u|_{\gamma_0} = u|_{\gamma_{2\pi}}, && x_2 = 0,\\
&u_{,x_2}|_{\gamma_0} = u_{,x_2}|_{\gamma_{2\pi}}, && x_2 = 0.
\end{aligned}
\tag{3.21}
$$

To obtain an estimate describing the behaviour of solutions of (3.21) in a neighbourhood of the origin we need

Lemma 3.3. *Assume that $f \in L_{2,-1}(\mathbb{R}^2)$ and supp $f \subset B_R$, $R < \infty$. Then there exists a solution of* (3.21) *such that $u \in H^2_{-1}(\mathbb{R}^2)$ and*

$$
\|u\|_{2,-1,\mathbb{R}^2} \le c\|f\|_{0,-1,\mathbb{R}^2}.
\tag{3.22}
$$

Proof. First we obtain solutions to the homogeneous problem (3.21). Expressing the homogeneous problem (3.21) in polar coordinates we have

$$
\begin{aligned}
&\frac{1}{r}\partial_r(ru_{,r}) + \frac{1}{r^2}u_{,\varphi\varphi} = 0,\\
&u|_{\varphi=0} = u|_{\varphi=2\pi},\\
&u_{,\varphi}|_{\varphi=0} = u_{,\varphi}|_{\varphi=2\pi}.
\end{aligned}
\tag{3.23}
$$

A general solution of $(3.23)_1$ has the form

$$
u = r^\alpha(a_1 \sin\alpha\varphi + a_2\cos\alpha\varphi),
\tag{3.24}
$$

where a_1, a_2 are arbitrary constants.

Using (3.24) in $(3.23)_{2,3}$ we obtain $\sin 2\pi\alpha = 0$, $\cos 2\pi\alpha = 1$, so $\alpha = k$, $k = 0, \mp 1, \mp 2, \ldots$

To obtain an estimate we consider the problem

$$
\begin{aligned}
&\frac{1}{r}\partial_r(ru_{,r}) + \frac{1}{r^2}u_{,\varphi\varphi} = f,\\
&u|_{\varphi=0} = u|_{\varphi=2\pi},\\
&u_{,\varphi}|_{\varphi=0} = u_{,\varphi}|_{\varphi=2\pi}.
\end{aligned}
\tag{3.25}
$$

Introducing the new variable $\tau = -\log r$ and the new quantity $v(\tau,\varphi) = u(e^{-\tau},\varphi)$

instead of (3.25) we have

$$(3.26)\qquad \begin{aligned} &v_{,\tau\tau}+v_{,\varphi\varphi}=f\,e^{-2\tau}\equiv F,\\ &v|_{\varphi=0}=v|_{\varphi=2\pi},\\ &v_{,\varphi}|_{\varphi=0}=v_{,\varphi}|_{\varphi=2\pi}. \end{aligned}$$

Applying the Fourier transform (3.49) to (3.26) and putting $\sigma=-i\lambda$ we obtain

$$(3.27)\qquad \begin{aligned} &\sigma^2\widetilde{v}+\widetilde{v}_{,\varphi\varphi}=\widetilde{F},\\ &v|_{\varphi=0}=v|_{\varphi=2\pi},\\ &v_{,\varphi}|_{\varphi=0}=v_{,\varphi}|_{\varphi=2\pi}. \end{aligned}$$

Solving the homogeneous problem (3.27) we write its solutions in the form

$$(3.28)\qquad \widetilde{v}=a_1\sin\sigma\varphi+a_2\cos\sigma\varphi,$$

where a_1,a_2 are arbitrary parameters.

Since $f\in L_{2,-1}(\mathbb{R}^2)$, as in the proof of Lemma 3.6, $\sigma=2$ is an eigenvalue of problem (3.27). Therefore to obtain an estimate we have to find an operator M which annihilates eigenfunctions coresponding to $\sigma=2$,

$$(3.29)\qquad v_1=r^2\sin 2\varphi,\qquad v_2=r^2\cos 2\varphi.$$

The operator has the form

$$M=r\partial_r-2.$$

As in the case of estimate (3.92) we have to use equation $(3.26)_1$ to obtain estimates for all derivatives. Finally we have

$$(3.30)\qquad \|u_{xx}\|_{L_{2,-1}(\mathbb{R}^2)}+\|u_x\|_{L_{2,-2}(\mathbb{R}^2)}\le c(\|f\|_{L_{2,-1}(\mathbb{R}^2)}+\|u_x\|_{L_{2,-1}(\mathbb{R}^2)}+\|u\|_{L_{2,-2}(\mathbb{R}^2)}).$$

The above considerations were necessary to obtain only the estimate (3.30). Since problem (3.21) follows from localizing the original problem we know that f and u have compact supports, so we can add without any restrictions the homogeneous Dirichlet boundary condition

$$(3.31)\qquad u|_{S_R}=0.$$

Moreover the conditions $(3.21)_{2,3}$ are also artificial because there is no loss of regularity when crossing the line $x_2=0$. The conditions were introduced only to apply the Kondrat'ev technique and to obtain the inequality (3.30). Therefore we can also consider the problem

$$(3.32)\qquad \begin{aligned} &\varDelta' u=f &&\text{in } B_R\cap\mathbb{R}^2,\\ &u=0 &&\text{on } S_R\cap\mathbb{R}^2. \end{aligned}$$

From (3.32) we have existence of a unique solution and that nontrivial solutions to the homogeneous problem (3.21) do not exist. Therefore for a sufficiently small r and after applying the Hardy inequality the last two terms from the r.h.s. of (3.30) are absorbed by the terms from the l.h.s. of (3.30). Moreover from (3.32) we obtain the estimate

$$\|u\|_{1,B_R\cap\mathbb{R}^2}\le c\|f\|_{0,B_R\cap\mathbb{R}^2},$$

so finally we have (3.22). This concludes the proof.

Next we have

Lemma 3.4. *Assume that $f \in L_{2,-\mu}(\mathbb{R}^2)$, $\mu \in (0,1)$. Then there exists a solution of* (3.21) *such that $u \in H^2_{-\mu}(\mathbb{R}^2)$ and*

$$\|u\|_{2,-\mu,\mathbb{R}^2} \le c\|f\|_{0,-\mu,\mathbb{R}^2}. \tag{3.33}$$

Proof. The proof is similar to the proof of Lemma 3.3 with the difference that in this case we do not encounter any eigenvalue of problem (3.21) when applying the inverse Fourier transform to the inequality analogous to (3.80). This concludes the proof.

Similarly to Lemma 3.4 we obtain

Lemma 3.5. *Assume that $f \in L_{2,\mu}(\mathbb{R}^2)$, $\mu \in (0,1)$. Then there exists a solution of* (3.21) *such that $u \in H^2_{\mu}(\mathbb{R}^2)$ and*

$$\|u\|_{2,\mu,\mathbb{R}^2} \le c\|f\|_{0,\mu,\mathbb{R}^2}. \tag{3.34}$$

Now we consider the homogeneous problem (3.20):

$$\begin{aligned} &-\nu\Delta' v + \nabla' p = 0,\\ &\operatorname{div}' v = 0,\\ &v|_{\gamma_0} = v|_{\gamma_{2\pi}},\\ &(v_{1,x_2} + v_{2,x_1})|_{\gamma_0} = (v_{1,x_2} + v_{2,x_1})|_{\gamma_{2\pi}},\\ &(2\nu v_{2,x_2} - p)|_{\gamma_0} = (2\nu v_{2,x_2} - p)|_{\gamma_{2\pi}}. \end{aligned} \tag{3.35}$$

For the homogeneous system (3.35) we have

Lemma 3.6. *The spectrum of the homogeneous problem* (3.35) *contains only integer numbers. Then the solutions of* (3.35) *have the form*

$$\begin{aligned} \begin{pmatrix} u_1\\ u_2\\ q\end{pmatrix} = \sum_{k=-\infty}^{\infty}\Bigg[&\begin{pmatrix}(k-1)\gamma\\ (k+1)\delta\\ 4k\gamma\end{pmatrix}\cos(k-1)\varphi + \begin{pmatrix}(k-1)\delta\\ -(k+1)\gamma\\ 4k\delta\end{pmatrix}\sin(k-1)\varphi\\ &+\begin{pmatrix}\alpha\\ \beta\\ 0\end{pmatrix}\cos(k+1)\varphi + \begin{pmatrix}\beta\\ -\alpha\\ 0\end{pmatrix}\sin(k+1)\varphi\Bigg] r^k, \end{aligned} \tag{3.36}$$

where $\alpha,\beta,\gamma,\delta$ are arbitrary parameters.

For $k=0$ we have the solution

$$\begin{aligned} u_1 &= a\cos\varphi + b\sin\varphi,\\ u_2 &= b\cos\varphi - a\sin\varphi,\\ q &= 0, \end{aligned} \tag{3.37}$$

where $a = \alpha - \gamma$, $b = \beta + \delta$. The solution (3.37) is a rigid motion.

Proof. In the polar coordinates $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, the radial and angular components of the velocity have the form

$$u = v_1\cos\varphi + v_2\sin\varphi, \qquad w = -v_1\sin\varphi + v_2\cos\varphi. \tag{3.38}$$

Then equations (3.35) take the form

$$
(3.39)\qquad
\begin{aligned}
&-\nu\left[\frac{1}{r}\partial_r(ru_r)+\frac{1}{r^2}u_{\varphi\varphi}-\frac{1}{r^2}u-\frac{2}{r^2}w_\varphi\right]+p_r=0,\\
&-\nu\left[\frac{1}{r}\partial_r(rw_r)+\frac{1}{r^2}w_{\varphi\varphi}-\frac{1}{r^2}w+\frac{2}{r^2}u_\varphi\right]+\frac{1}{r}p_\varphi=0,\\
&u_r+\frac{1}{r}u+\frac{1}{r}w_\varphi=0.
\end{aligned}
$$

From (3.38) we have

$$
(3.40)\qquad v_1=u\cos\varphi-w\sin\varphi,\qquad v_2=u\sin\varphi+w\cos\varphi,
$$

so the boundary conditions $(3.35)_3$ take the form

$$
(3.41)\qquad
\begin{aligned}
u|_{\varphi=0}&=u|_{\varphi=2\pi},\\
w|_{\varphi=0}&=w|_{\varphi=2\pi}.
\end{aligned}
$$

Finally the boundary conditions $(3.35)_{4,5}$ take the form

$$
(3.42)\qquad
\begin{aligned}
\mathbb{T}_{\varphi r}|_{\varphi=0}&=\mathbb{T}_{\varphi r}|_{\varphi=2\pi},\\
\mathbb{T}_{\varphi\varphi}|_{\varphi=0}&=\mathbb{T}_{\varphi\varphi}|_{\varphi=2\pi},
\end{aligned}
$$

because the normal vector $\overline{n}$ to γ_0 and $\gamma_{2\pi}$ has the direction of the φ coordinate line.

Writing (3.42) more explicitly we obtain

$$
(3.43)\qquad
\begin{aligned}
\left(\frac{1}{r}u_\varphi+w_r-\frac{1}{r}w\right)\bigg|_{\varphi=0}&=\left(\frac{1}{r}u_\varphi+w_r-\frac{1}{r}w\right)\bigg|_{\varphi=2\pi},\\
\left(\frac{2\nu}{r}(w_\varphi+u)-p\right)\bigg|_{\varphi=0}&=\left(\frac{2\nu}{r}(w_\varphi+u)-p\right)\bigg|_{\varphi=2\pi}.
\end{aligned}
$$

If we introduce the new variable

$$
(3.44)\qquad q'=\frac{rp}{\mu}
$$

then problem (3.39), (3.41), (3.43) takes the form

$$
(3.45)\qquad
\begin{aligned}
&-\left[\frac{1}{r}\partial_r(ru_r)+\frac{1}{r^2}u_{\varphi\varphi}-\frac{1}{r^2}u-\frac{2}{r^2}w_\varphi\right]+\partial_r\left(\frac{q'}{r}\right)=0,\\
&-\left[\frac{1}{r}\partial_r(rw_r)+\frac{1}{r^2}w_{\varphi\varphi}-\frac{1}{r^2}w+\frac{2}{r^2}w_\varphi\right]+\frac{1}{r^2}q'_\varphi=0,\\
&u_r+\frac{1}{r}u+\frac{1}{r}w_\varphi=0,\\
&u|_{\varphi=0}=u|_{\varphi=2\pi},\\
&w|_{\varphi=0}=w|_{\varphi=2\pi},\\
&\left(\frac{1}{r}u_\varphi+w_r-\frac{1}{r}w\right)\bigg|_{\varphi=0}=\left(\frac{1}{r}u_\varphi+w_r-\frac{1}{r}w\right)\bigg|_{\varphi=2\pi},\\
&(2(w_\varphi+u)-q')|_{\varphi=0}=(2(w_\varphi+u)-q')|_{\varphi=2\pi}.
\end{aligned}
$$

Let us introduce the new variable

$$
(3.46)\qquad \tau=-\log r
$$

and the quantities

$$u_1(\tau,\varphi)=u(e^{-\tau},\varphi),\quad u_2(\tau,\varphi)=w(e^{-\tau},\varphi),\quad q(\tau,\varphi)=q'(e^{-\tau},\varphi). \tag{3.47}$$

Then problem (3.45) assumes the form

$$\begin{aligned}
&u_{1\tau\tau}+u_{1\varphi\varphi}-u_1-2u_{2\varphi}+q+q_\tau=0 && \text{in } \mathbb{R}^1\times(0,2\pi),\\
&u_{2\tau\tau}+u_{2\varphi\varphi}-u_2+2u_{1\varphi}-q_\varphi=0 && \text{in } \mathbb{R}^1\times(0,2\pi),\\
&-u_{1\tau}+u_{2\varphi}+u_1=0 && \text{in } \mathbb{R}^1\times(0,2\pi),\\
&u_1|_{\varphi=0}=u_1|_{\varphi=2\pi} && \text{on } \mathbb{R}^1,\\
&u_2|_{\varphi=0}=u_2|_{\varphi=2\pi} && \text{on } \mathbb{R}^1,\\
&(-u_{2\tau}+u_{1\varphi}-u_2)|_{\varphi=0}=(-u_{2\tau}+u_{1\varphi}-u_2)|_{\varphi=2\pi} && \text{on } \mathbb{R}^1,\\
&(2(u_{2\varphi}+u_1)-q)|_{\varphi=0}=(2(u_{2\varphi}+u_1)-q)|_{\varphi=2\pi} && \text{on } \mathbb{R}^1.
\end{aligned} \tag{3.48}$$

Applying the Fourier transform

$$u(\tau,\varphi)=\int_{-\infty}^{\infty}e^{i\lambda\tau}\widetilde{u}(\lambda,\varphi)\,d\lambda \tag{3.49}$$

to (3.48) and putting $\sigma=-i\lambda$ we obtain

$$\begin{aligned}
&-\frac{d^2\widetilde{u}_1}{d\varphi^2}+(1-\sigma^2)\widetilde{u}_1+2\frac{d\widetilde{u}_2}{d\varphi}+(\sigma-1)\widetilde{q}=0 && \text{in } (0,2\pi),\\
&-\frac{d^2\widetilde{u}_2}{d\varphi^2}+(1-\sigma^2)\widetilde{u}_2-2\frac{d\widetilde{u}_1}{d\varphi}+\frac{d\widetilde{q}}{d\varphi}=0 && \text{in } (0,2\pi),\\
&\frac{d\widetilde{u}_2}{d\varphi}+(\sigma+1)\widetilde{u}_1=0 && \text{in } (0,2\pi),\\
&\widetilde{u}_1|_{\varphi=0}=\widetilde{u}_1|_{\varphi=2\pi},\\
&\widetilde{u}_2|_{\varphi=0}=\widetilde{u}_2|_{\varphi=2\pi},\\
&\left(\frac{d\widetilde{u}_1}{d\varphi}+(\sigma-1)\widetilde{u}_2\right)\bigg|_{\varphi=0}=\left(\frac{d\widetilde{u}_1}{d\varphi}+(\sigma-1)\widetilde{u}_2\right)\bigg|_{\varphi=2\pi},\\
&\left(2\left(\frac{d\widetilde{u}_2}{d\varphi}+\widetilde{u}_1\right)-\widetilde{q}\right)\bigg|_{\varphi=0}=\left(2\left(\frac{d\widetilde{u}_2}{d\varphi}+\widetilde{u}_1\right)-\widetilde{q}\right)\bigg|_{\varrho=2\pi}.
\end{aligned} \tag{3.50}$$

A general solution to $(3.50)_{1,2,3}$ has the form

$$\begin{aligned}
\begin{pmatrix}\widetilde{u}_1\\ \widetilde{u}_2\\ \widetilde{q}\end{pmatrix}&=\begin{pmatrix}(\sigma-1)\gamma\\ (\sigma+1)\delta\\ 4\sigma\gamma\end{pmatrix}\cos(\sigma-1)\varphi+\begin{pmatrix}(\sigma-1)\delta\\ -(\sigma+1)\gamma\\ 4\sigma\delta\end{pmatrix}\sin(\sigma-1)\varphi\\
&\quad+\begin{pmatrix}\alpha\\ \beta\\ 0\end{pmatrix}\cos(\sigma+1)\varphi+\begin{pmatrix}\beta\\ -\alpha\\ 0\end{pmatrix}\sin(\sigma+1)\varphi,
\end{aligned} \tag{3.51}$$

where $\alpha,\beta,\gamma,\delta$ are arbitrary parameters which should be determined by the boundary conditions $(3.50)_{4–7}$.

The boundary condition $(3.50)_4$ implies

$$\begin{aligned}
&(\sigma-1)\gamma(1-\cos 2\pi(\sigma-1))+\alpha(1-\cos 2\pi(\sigma+1))\\
&\qquad-(\sigma-1)\delta\sin 2\pi(\sigma-1)-\beta\sin 2\pi(\sigma+1)=0.
\end{aligned} \tag{3.52}$$

Condition $(3.50)_5$ gives

$$(3.53)\qquad (\sigma+1)\delta(1-\cos 2\pi(\sigma-1))+\beta(1-\cos 2\pi(\sigma+1)) + (\sigma+1)\gamma\sin 2\pi(\sigma-1)+\alpha\sin 2\pi(\sigma+1)=0.$$

Next $(3.50)_6$ yields

$$(3.54)\qquad 2\sigma(\sigma-1)\delta(1-\cos 2\pi(\sigma-1))+2\sigma\beta(1-\cos 2\pi(\sigma+1)) + 2\sigma(\sigma-1)\gamma\sin 2\pi(\sigma-1)+2\sigma\alpha\sin 2\pi(\sigma+1)=0.$$

Finally $(5.50)_7$ implies

$$(3.55)\qquad -2\sigma(\sigma+1)\gamma(1-\cos 2\pi(\sigma-1))-2\sigma\alpha(1-\cos 2\pi(\sigma+1)) + 2\sigma(\sigma+1)\delta\sin 2\pi(\sigma-1)+2\sigma\beta\sin 2\pi(\sigma+1)=0.$$

Consider the case $\sigma=0$. Then (3.54) and (3.55) disappear. Inserting $\sigma=0$ into (3.52) and (3.53) we see that they are satisfied identically. Then the solution (3.51) takes the form

$$(3.56)\qquad \begin{aligned} \widetilde{u}_1 &= (\alpha-\gamma)\cos\varphi+(\beta+\delta)\sin\varphi = a\cos\varphi+b\sin\varphi,\\ \widetilde{u}_2 &= (\delta+\beta)\cos\varphi+(\gamma-\alpha)\sin\varphi = b\cos\varphi-a\sin\varphi,\\ \widetilde{q} &= 0, \end{aligned}$$

so this solution describes a rigid rotation.

Consider the case $\sigma\neq 0$. Then equations (3.52)–(3.55) imply

$$(3.57)\qquad \alpha(1-\cos 2\pi(\sigma+1))-\beta\sin 2\pi(\sigma+1) + (\sigma-1)\gamma(1-\cos 2\pi(\sigma-1))-(\sigma-1)\delta\sin 2\pi(\sigma-1)=0,$$

$$(3.58)\qquad \alpha\sin 2\pi(\sigma+1)+\beta(1-\cos 2\pi(\sigma+1)) + (\sigma+1)\gamma\sin 2\pi(\sigma-1)+(\sigma+1)\delta(1-\cos 2\pi(\sigma-1))=0,$$

$$(3.59)\qquad \alpha\sin 2\pi(\sigma+1)+\beta(1-\cos 2\pi(\sigma+1)) + (\sigma-1)\gamma\sin 2\pi(\sigma-1)+(\sigma-1)\delta(1-\cos 2\pi(\sigma-1))=0,$$

$$(3.60)\qquad -\alpha(1-\cos 2\pi(\sigma+1))+\beta\sin 2\pi(\sigma+1) -(\sigma+1)\gamma(1-\cos 2\pi(\sigma-1))+(\sigma+1)\delta\sin 2\pi(\sigma-1)=0.$$

Adding (3.57) and (3.60) gives

$$(3.61)\qquad -\gamma(1-\cos 2\pi(\sigma-1))+\delta\sin 2\pi(\sigma-1)=0.$$

Subtracting (3.59) from (3.58) yields

$$(3.62)\qquad \gamma\sin 2\pi(\sigma-1)+\delta(1-\cos 2\pi(\sigma-1))=0.$$

From (3.61) and (3.62) we have the condition for the existence of nontrivial solutions

$$(3.63)\qquad (1-\cos 2\pi(\sigma-1))^2+\sin^2 2\pi(\sigma-1)=0.$$

Hence

$$(3.64)\qquad \cos 2\pi(\sigma-1)=1 \quad \text{and} \quad \sin 2\pi(\sigma-1)=0.$$

The first condition gives

$$(3.65)\qquad \sigma=k+1,\quad k=0,\mp 1,\mp 2,\dots,$$

and the second implies

$$\sigma = 1 + k/2, \quad k = 0, \mp 1, \mp 2, \ldots \tag{3.66}$$

Using (3.64) in (3.57)–(3.60) yields

$$\begin{aligned} &\alpha(1 - \cos 2\pi(\sigma + 1)) - \beta \sin 2\pi(\sigma + 1) = 0, \\ &\alpha(1 - \cos 2\pi(\sigma + 1)) + \beta \sin 2\pi(\sigma + 1) = 0. \end{aligned} \tag{3.67}$$

Hence we also obtain

$$\cos 2\pi(\sigma + 1) = 1 \quad \text{and} \quad \sin 2\pi(\sigma + 1) = 0. \tag{3.68}$$

The first condition implies

$$\sigma = k - 1, \quad k = 0, \mp 1, \mp 2, \ldots, \tag{3.69}$$

and the second gives

$$\sigma = k/2 - 1, \quad k = 0, \mp 1, \mp 2, \ldots \tag{3.70}$$

Conditions (3.65), (3.66), (3.69), (3.70) must be satisfied simultaneously, so σ is an arbitrary integer.

Applying now the considerations from [3] we obtain (3.36). This concludes the proof.

Now we want to explain why we consider such boundary conditions in the problem (3.20). In reality we consider problem (3.20) in the whole space $\mathbb{R}^2$, so the boundary conditions are artificial. But we introduced them because the r.h.s. functions of (3.20) are such that $f \in L_{2,-\mu}(\mathbb{R}^2)$, $g \in H^1_{-\mu}(\mathbb{R}^2)$, $\mu \in [-1, 1]$, where the weight is a power of the distance from the origin. Therefore we expect that solutions of (3.20) should also belong to the same kind of spaces, so $v \in H^2_{-\mu}(\mathbb{R}^2)$ and $p \in H^1_{-\mu}(\mathbb{R}^2)$. These kind of spaces suggest that to prove existence of solutions to problem (3.20) we should apply the Kondrat'ev theory (see [2, 3]). Therefore we need a cone, but our cone is $d_{2\pi} = \mathbb{R}^2$, so the boundary conditions $(3.20)_{3,4,5}$ are chosen on the same line $\gamma_0 = \gamma_{2\pi}$. Since the boundary conditions are artificial for the problem considered in the cylinder they should be chosen in such a way that there would be no loss of regularity on the line $\gamma_0 = \gamma_{2\pi}$. Therefore we need

Lemma 3.7. *Assume that f and h are continuous with all derivatives when crossing the line $\gamma_0 = \gamma_{2\pi}$. Then there is no loss of regularity of solutions to problem* (3.20) *on the line $\gamma_0 = \gamma_{2\pi}$.*

Proof. We have to show that solutions of (3.20) are such that v_{xx} and p_x are continuous when crossing the line $\gamma_0 = \gamma_{2\pi}$. To be specific, we choose

$$\gamma_0 = \gamma_{2\pi} = \{x \in \mathbb{R}^2 : x_2 = 0\}.$$

From $(3.20)_3$ we see that v is continuous on $\gamma_0 = \gamma_{2\pi}$. From $(3.20)_2$ we have $v_{2x_2} = h - v_{1x_1}$, so it is continuous on $\gamma_0 = \gamma_{2\pi}$. From $(3.20)_4$ we see that v_{1x_2} is continuous on $\gamma_0 = \gamma_{2\pi}$. Next we see from $(3.20)_5$ that p is continuous when crossing $\gamma_0 = \gamma_{2\pi}$. Since p, v_{1x_2}, v_{2x_2} are continuous when crossing $\gamma_0 = \gamma_{2\pi}$, so also are p_{x_1}, $v_{1x_2x_1}$, $v_{2x_2x_1}$. Differentiating $(3.20)_2$ with respect to x_2 gives

$$v_{2x_2x_2} = h_{x_2} - v_{1x_1x_2},$$

so we see that $v_{2x_2x_2}$ is also continuous.

Now we write $(3.20)_1$ more explicitly:

$$-\nu(v_{1x_1x_1}+v_{1x_2x_2})+p_{x_1}=f_1,\qquad -\nu(v_{2x_1x_1}+v_{2x_2x_2})+p_{x_2}=f_2. \tag{3.71}$$

Since $v_{2x_2x_2}$ is continuous we deduce from $(3.71)_2$ that so is p_{x_2}. Finally $(3.71)_1$ implies that $v_{1x_2x_2}$ is continuous. Therefore all v_{xx} and p_x are continuous.

To show that higher derivatives of v and p are continuous we differentiate appropriately the equations of (3.20) and repeat the above considerations. This concludes the proof.

Next we have

Lemma 3.8. *Let $f\in L_{2,-1}(\mathbb{R}^2)$, $h\in H^1_{-1}(\mathbb{R}^2)$. Then there exists a solution of* (3.20) *such that $v\in H^2_{-1}(\mathbb{R}^2)$, $p\in H^1_{-1}(\mathbb{R}^2)$ and*

$$\|v\|_{2,-1,d_{2\pi}}+\|p\|_{1,-1,d_{2\pi}}\le c(\|f\|_{0,-1,d_{2\pi}}+\|h\|_{1,-1,d_{2\pi}}), \tag{3.72}$$

where we recall that $d_{2\pi}=\mathbb{R}^2$.

Proof. We apply the Kondrat'ev technique [2]. Repeating all calculations leading to (3.50) for problem (3.20) we obtain

$$\begin{aligned}
&-\frac{d^2\widetilde{u}_1}{d\varphi^2}+(1-\sigma^2)\widetilde{u}_1+2\frac{d\widetilde{u}_2}{d\varphi}+(\sigma-1)\widetilde{q}=\widetilde{F}_1,\\
&-\frac{d^2\widetilde{u}_2}{d\varphi^2}+(1-\sigma^2)\widetilde{u}_2-2\frac{d\widetilde{u}_1}{d\varphi}+\frac{d\widetilde{q}}{d\varphi}=\widetilde{F}_2,\\
&\frac{d\widetilde{u}_2}{d\varphi}+(\sigma+1)\widetilde{u}_1=\widetilde{H},\\
&\widetilde{u}_1|_{\varphi=0}=\widetilde{u}_1|_{\varphi=2\pi},\\
&\widetilde{u}_2|_{\varphi=0}=\widetilde{u}_2|_{\varphi=2\pi},\\
&\bigg(\frac{d\widetilde{u}_1}{d\varphi}+(\sigma-1)\widetilde{u}_2\bigg)\bigg|_{\varphi=0}=\bigg(\frac{d\widetilde{u}_1}{d\varphi}+(\sigma-1)\widetilde{u}_2\bigg)\bigg|_{\varphi=2\pi},\\
&\bigg(2\bigg(\frac{d\widetilde{u}_2}{d\varphi}+\widetilde{u}_1\bigg)-\widetilde{q}\bigg)\bigg|_{\varphi=0}=\bigg(2\bigg(\frac{d\widetilde{u}_2}{d\varphi}+\widetilde{u}_1\bigg)-\widetilde{q}\bigg)\bigg|_{\varphi=2\pi},
\end{aligned} \tag{3.73}$$

where

$$F_i=f_i\,e^{-2\tau},\qquad i=1,2,\qquad H=he^{-\tau}. \tag{3.74}$$

In view of (3.74) the condition $f\in L_{2,-1}(\mathbb{R}^2)$ means

$$\int_{-\infty}^{\infty}\int_0^{2\pi}|F|^2\,e^{2h_0\tau}\,d\tau\,d\varphi\le c\|f\|^2_{L_{2,-1}(\mathbb{R}^2)},$$

where in the notation of Kondrat'ev (see [2]) we have $h_0=2$.

Therefore for the Fourier transform $\widetilde{F}$ we have

$$\int_{-\infty+2i}^{+\infty+2i}\|\widetilde{F}\|^2_{L_2((0,2\pi))}\,d\lambda\le c\|f\|^2_{L_{2,-1}(\mathbb{R}^2)}, \tag{3.75}$$

where $\sigma=-i\lambda$.

The condition $h \in H^1_{-1}(\mathbb{R}^2)$ means

$$\sum_{i_1+i_2\le 1} \int_{-\infty}^{\infty} \int_0^{2\pi} |\partial_\tau^{i_1} \partial_\varphi^{i_2} H|^2 \, e^{2h_0\tau} \, d\tau \, d\varphi \le c\|h\|_{H^1_{-1}(\mathbb{R}^2)}.$$

Hence for the Fourier transform $\widetilde{H}$ we obtain

$$\sum_{s=0}^{1} \int_{-\infty+2i}^{+\infty+2i} |\lambda|^{2s} \|\widetilde{H}\|^2_{H^{1-s}((0,2\pi))} \, d\lambda \le c\|h\|^2_{H^1_{-1}(\mathbb{R}^2)}. \tag{3.76}$$

We write problem (3.73) in the short form

$$\begin{aligned} &L_1(\varphi, i\lambda, \partial_\varphi)\widetilde{v} = \widetilde{F}, \\ &L_2(\varphi, i\lambda, \partial_\varphi)\widetilde{v} = \widetilde{H}, \\ &B(\varphi, i\lambda, \partial_\varphi)\widetilde{v} = 0, \end{aligned} \tag{3.77}$$

where $\widetilde{v} = (\widetilde{u}, \widetilde{q})$, L_1, L_2 are the differential operators $(3.73)_{1,2,3}$ and B are the boundary conditions $(3.73)_{4\text{–}7}$.

From [1] it follows that there exists an operator $R(\lambda)$,

$$R(\lambda) : L_2(0, 2\pi) \times H^1(0, 2\pi) \to H^2(0, 2\pi) \times H^1(0, 2\pi),$$

which is a meromorphic function of λ such that

$$(L_1, L_2, B)R = E, \tag{3.78}$$

where E is the unit matrix.

Using the operator R we can write a solution of (3.77) in the form

$$\widetilde{v}(\lambda, \varphi) = R(\lambda)(\widetilde{F}, \widetilde{H}). \tag{3.79}$$

In [1] it is shown that for any layer $|\mathrm{im}\,\lambda| < c_1$ there exists a number $c_2 = c_2(c_1)$ such that for $|\mathrm{re}\,\lambda| > c_2$ the function $R(\lambda)$ does not have any singular point and we have

$$\begin{aligned} |\lambda|^4 \|R_1(\lambda)(\widetilde{F}, \widetilde{H})\|^2_{L_2(0,2\pi)} &+ \|R_1(\lambda)(\widetilde{F}, \widetilde{H})\|^2_{H^2(0,2\pi)} \\ &+ |\lambda| \|R_2(\lambda)(\widetilde{F}, \widetilde{H})\|^2_{L_2(0,2\pi)} + \|R_2(\lambda)(\widetilde{F}, \widetilde{H})\|^2_{H^1(0,2\pi)} \\ &\le c\|\widetilde{F}\|^2_{L_2(0,2\pi)} + c|\lambda|^2 \|\widetilde{H}\|^2_{L_2(0,2\pi)} + c\|\widetilde{H}\|^2_{H^1(0,2\pi)}, \end{aligned} \tag{3.80}$$

where $\widetilde{u} = R_1(\widetilde{F}, \widetilde{H})$, $\widetilde{q} = R_2(\widetilde{F}, \widetilde{H})$.

In view of the results of Kondrat'ev (see [2], Th. 1.1), if the operator $R(\lambda)$ did not have singular points on the line $\mathrm{im}\,\lambda = h_0 = 2$, then we would integrate (3.80) along the line $\mathrm{im}\ \lambda = h_0$ to obtain the estimate

$$\|u\|^2_{H^2_{-1}(\mathbb{R}^2)} + \|p\|^2_{H_{-1}(\mathbb{R}^2)} \le c(\|f\|^2_{L_{2,-1}(\mathbb{R}^2)} + \|h\|^2_{H^2_{-1}(\mathbb{R}^2)}). \tag{3.81}$$

But unfortunately this is not the case, because $\sigma = -i\lambda = 2$ belongs to the spectrum of problem (3.50).

Moreover, the eigenfunctions of the operator (3.50) corresponding to the eigenvalue $\sigma = 2$ have the form

$$\begin{pmatrix} \widetilde{u}_1 \\ \widetilde{u}_2 \\ \widetilde{q} \end{pmatrix} = \gamma \begin{pmatrix} \cos\varphi \\ -3\sin\varphi \\ 8\cos\varphi \end{pmatrix} + \delta \begin{pmatrix} \sin\varphi \\ 3\cos\varphi \\ 8\sin\varphi \end{pmatrix} + \alpha \begin{pmatrix} \cos 3\varphi \\ -\sin 3\varphi \\ 0 \end{pmatrix} + \beta \begin{pmatrix} \sin 3\varphi \\ \cos 3\varphi \\ 0 \end{pmatrix}, \tag{3.82}$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary parameters.

To apply Remark 1.1 from [2] we have to find an operator of the lowest possible order which annihilates the eigenfunctions (3.82). Let us denote such an operator by $M(r\partial_r, \partial_\varphi) = M(-\partial_\tau, \partial_\varphi)$. On the Fourier transforms (3.49) the operator acts as $M(-i\lambda, \partial_\varphi)$. Continuing we see that $M(-i\lambda, \partial_\varphi)R(i\lambda)(\widetilde{F}, \widetilde{H})$ does not have singular points on the line im $\lambda = 2$, so Theorem 1.1 from [2] can be applied.

Now we find the operator M. Since $\sigma = 2$ we have the following eigenfunctions:

$$\overline{e}_1 = r^2 \begin{pmatrix} \cos\varphi \\ -3\sin\varphi \\ 8\cos\varphi \end{pmatrix}, \quad \overline{e}_2 = r^2 \begin{pmatrix} \sin\varphi \\ 3\cos\varphi \\ 8\sin\varphi \end{pmatrix},$$

$$\overline{e}_3 = r^2 \begin{pmatrix} \cos 3\varphi \\ -\sin 3\varphi \\ 0 \end{pmatrix}, \quad \overline{e}_4 = r^2 \begin{pmatrix} \sin 3\varphi \\ \cos 3\varphi \\ 0 \end{pmatrix}.$$

Hence we are looking for the operator M such that

$$M(\overline{e}_i) = 0, \quad i = 1, 2, 3, 4, \tag{3.83}$$

are satisfied identically.

Therefore we can look for the operator M in the form

$$M_j = a_j r\partial_r + b_j \partial_\varphi + c_j, \quad j = 1, 2, 3, \tag{3.84}$$

where a_j, b_j, c_j, $j = 1, 2, 3$, are constants and (3.83) has the form

$$M(\overline{e}_i) = \sum_{j=1}^{3} M_j \overline{e}_{ji}, \quad i = 1, 2, 3, 4. \tag{3.85}$$

The equation $M(\overline{e}_1) = 0$ implies

$$-(6a_2 + b_1 + 8b_3 + 3c_2)\sin\varphi + (2a_1 + 16a_3 - 3b_2 + c_1 + 8c_3)\cos\varphi = 0, \tag{3.86}$$

so we obtain two equations for the coefficients of M,

$$\begin{aligned} &6a_2 + b_1 + 8b_3 + 3c_2 = 0, \\ &2a_1 + 16a_3 - 3b_2 + c_1 + 8c_3 = 0. \end{aligned} \tag{3.87}$$

The equation $M(\overline{e}_2) = 0$ gives

$$(2a_1 + 16a_3 - 3b_2 + c_1 + 8c_3)\sin\varphi + (6a_2 + b_1 + 8b_3 + 3c_2)\cos\varphi = 0, \tag{3.88}$$

so the same equations as in (3.87) follow.

The condition $M(\overline{e}_3) = 0$ yields

$$-(2a_2 + 3b_1 + c_2)\sin 3\varphi + (2a_1 - 3b_2 + c_1)\cos 3\varphi = 0, \tag{3.89}$$

so we have two equations

$$
\begin{aligned}
&2a_2 + 3b_1 + c_2 = 0,\\
&2a_1 - 3b_2 + c_1 = 0.
\end{aligned}
\tag{3.90}
$$

Finally the equation $M(\overline{e}_4) = 0$ implies

$$
(2a_1 - 3b_2 + c_1)\sin 3\varphi + (2a_2 + 3b_1 + c_2)\cos 3\varphi = 0,
\tag{3.91}
$$

so we also have equations (3.90).

Therefore the operator M is defined by the four equations (3.87) and (3.90).

Using the operator M we apply Remark 1.1 from [2] to obtain

$$
\begin{aligned}
\int\limits_{-\infty+2i}^{+\infty+2i} [|\lambda|^2 \|M^* R_1(\lambda)(\widetilde{F}, \widetilde{H})\|^2_{L_2(0,2\pi)} + \|M^* R_1(\lambda)\widetilde{F}, \widetilde{H})\|^2_{H^1(0,2\pi)}]\, d\lambda&\\
+ \int\limits_{-\infty+2i}^{+\infty+2i} \|M^* R_2(\lambda)(\widetilde{F}, \widetilde{H})\|^2_{L_2(0,2\pi)}\, d\lambda&\\
\le c(\|f\|^2_{L_{2,-1}(\mathbb{R}^2)} + \|h\|^2_{H^1_{-1}(\mathbb{R}^2)})&,
\end{aligned}
\tag{3.92}
$$

where $M^* = M(-i\lambda, \partial_\varphi)$.

Generally we need to obtain an estimate for solutions u and q involving all second derivatives of u and all first derivatives of q. However inequality (3.92) contains the second derivatives of u and the first derivatives of q but in the form of some combinations which in general do not involve the seminorms $\|u_{xx}\|_{L_2}$ and $\|q_x\|_{L_2}$. Therefore to obtain all the second derivatives of u and all the first derivatives of q from (3.92) we have to use different operators M. This is possible because the operator M contains 9 parameters determined by 4 equations only.

Since the operator R_1 gives u and the operator R_2 determines q we can take different operators in different integrals from the l.h.s. of (3.92).

Let us consider the first term on the l.h.s. of (3.92). Since the term contains the first derivatives we have to choose operators M which do not contain any derivative of q. Therefore we choose M such that $a_3 = 0$, $b_3 = 0$, $c_3 = 0$. Then equations (3.87) and (3.90) imply

$$
\begin{aligned}
&6a_2 + b_1 + 3c_2 = 0,\\
&2a_1 - 3b_2 + c_1 = 0,\\
&2a_2 + 3b_1 + c_2 = 0.
\end{aligned}
\tag{3.93}
$$

Solving (3.93) we obtain

$$
b_1 = 0, \qquad a_2 = -\frac{1}{2}c_2, \qquad a_1 = \frac{3}{2}b_2 - \frac{1}{2}c_1.
\tag{3.94}
$$

Hence we can choose 3 independent operators because we have 3 arbitrary parameters c_1, c_2, b_2.

Choosing $c_1 = 1$, $c_2 = 0$, $b_2 = 0$ we obtain the operator $M_1 = -\frac{1}{2}\partial_\tau u_1 + u_1$. Choosing $c_1 = 0$, $c_2 = 0$, $b_2 = 1$ we have $M_2 = \frac{3}{2}\partial_\tau u_1 + \partial_\varphi u_2$, and finally selecting $c_1 = 0$, $c_2 = 1$, $b_2 = 0$, we get $M_3 = -\frac{1}{2}\partial_\tau u_2 + u_2$.

Taking the first derivatives with respect to τ and φ we see that the operators M_1, M_2, M_3 generate the following derivatives:

$$\begin{aligned}&-\frac{1}{2}\partial_\tau^2 u_1+\partial_\tau u_1, && -\frac{1}{2}\partial_\tau\partial_\varphi u_1+\partial_\varphi u_1, && \frac{3}{2}\partial_\tau^2 u_1+\partial_\tau\partial_\varphi u_2,\\ &\frac{3}{2}\partial_\tau\partial_\varphi u_1+\partial_\varphi^2 u_2, && -\frac{1}{2}\partial_\tau^2 u_2+\partial_\tau u_2, && -\frac{1}{2}\partial_\tau\partial_\varphi u_2+\partial_\varphi u_2.\end{aligned}\tag{3.95}$$

Hence we have the following second derivatives:

$$u_{1\tau\tau},\quad u_{1\tau\varphi},\quad u_{2\tau\tau},\quad u_{2\tau\varphi},\quad u_{2\varphi\varphi},$$

so the missing derivative $u_{1\varphi\varphi}$ is calculated from $(3.48)_1$ with nonvanishing r.h.s. in terms of q_τ and q.

To find derivatives of q we have to choose the operator M in the second term on the l.h.s. of (3.92). For this purpose we consider a general operator M. From (3.90) we have

$$a_1=\frac{3}{2}b_2-\frac{1}{2}c_1,\tag{3.96}$$

and from $(3.90)_1$ we get

$$a_2=-\frac{3}{2}b_1-\frac{1}{2}c_1.\tag{3.97}$$

Using (3.96) in $(3.87)_2$ yields

$$a_3=-\frac{1}{2}c_3.\tag{3.98}$$

Finally from $(3.87)_1$ we obtain

$$b_3=b_1.\tag{3.99}$$

Therefore our operator M takes the form

$$\begin{aligned}M&=a_1\partial_\tau u_1+a_2\partial_\tau u_2+a_3\partial_\tau q+b_1\partial_\varphi u_1\\&\quad+b_2\partial_\varphi u_2+b_3\partial_\varphi q+c_1u_1+c_2u_2+c_3q\\&=\left(\frac{3}{2}b_2-\frac{1}{2}c_1\right)\partial_\tau u_1-\left(\frac{3}{2}b_1+c_2\right)\partial_\tau u_2-\frac{1}{2}c_3\partial_\tau q+b_1\partial_\varphi u_1+b_2\partial_\varphi u_2\\&\quad+b_1\partial_\varphi q+c_1u_1+c_2u_2+c_3q.\end{aligned}\tag{3.100}$$

Choosing $c_1=c_2=c_3=b_2=0$, $b_1=1$ we obtain $M_4=-\frac{3}{2}\partial_\tau u_2+\partial_\varphi u_1+\partial_\varphi q$. Choosing $c_1=c_2=b_1=b_2=0$, $c_3=1$ we get $M_5=-\frac{1}{2}\partial_\tau q+q$. Therefore the operators M_4,M_5 give us all derivatives of q.

Summarizing the above considerations we obtain from (3.84) the estimate

$$\begin{aligned}&\|u_{xx}\|^2_{L_{2,-1}(\mathbb{R}^2)}+\|u_x\|^2_{L_{2,-2}(\mathbb{R}^2)}+\|q_x\|^2_{L_{2,-1}(\mathbb{R}^2)}\\&\quad\le c(\|f\|^2_{L_{2,-1}(\mathbb{R}^2)}+\|g\|^2_{H^1_{-1}(\mathbb{R}^2)}+\|u_x\|^2_{L_{2,-1}(\mathbb{R}^2)}+\|q\|^2_{L_{2,-1}(\mathbb{R}^2)}+\|u\|^2_{L_{2,-2}(\mathbb{R}^2)}),\end{aligned}\tag{3.101}$$

where the last three terms on the r.h.s. are of lower order because they can be estimated by suitable terms on the l.h.s.

The estimate (3.101) has a local character so it is obtained for some neighbourhood of the distinguished axis. We assume that the neighbourhood is such that $r<R$ and R is sufficiently small. Then by the Hardy inequality we are able to estimate the last three

terms on the r.h.s. of (3.101) by the terms from the l.h.s. First we consider

$$\int_0^{2\pi} d\varphi \int_0^R |q|^2 r^{-2} r\,dr \le R^\mu \int_0^{2\pi} d\varphi \int_0^R |q|^2 r^{-2-\mu} r\,dr \le cR^\mu \int_0^{2\pi} d\varphi \int_0^R |\partial_r q|^2 r^{-\mu} r\,dr$$
$$\le cR^2 \int_0^{2\pi} d\varphi \int_0^R |\nabla q|^2 r^{-2} r\,dr = cR^2 \|q_x\|^2_{L_{2,-1}(\mathbb{R}^2)},$$

where $\mu > 0$. Similar considerations can be applied to the other terms on the r.h.s. of (3.101). The trick with the weight μ has to be used because otherwise the Hardy inequality does not work.

Finally we obtain (3.64). This concludes the proof.

To apply Lemmas 3.6 and 3.7 for problems (3.18) and (3.19) we have to increase regularity of the weak solution determined by Lemma 3.2.

Since we have to increase regularity with respect to x we have to differentiate problem (3.2^1) with respect to x. But in (3.2^1) we have the boundary condition $(3.2^1)_3$ which could imply difficulties. But problem (3.2^1) follows from the local problem (1.4) hence the boundary condition $(3.2^1)_3$ is chosen artificially because the solution could vanish on $S_{R'}$, $R' < R$. Therefore the solution of problem (3.2^1) can be extended by zero over $\mathbb{R}^3$. Hence differentiating $(3.2^1)_1$ with respect to x, multiplying the result by v_x and integrating over $\mathbb{R}^3 \times (0,T)$ we obtain

$$\|v(t)\|^2_{1,\mathbb{R}^3} + \int_0^T \|v\|^2_{2,\mathbb{R}^3}\,d\tau \le c\int_0^T \|f\|^2_{1,\mathbb{R}^3}\,d\tau, \tag{3.102}$$

where we used the fact that $T < \infty$.

Next we introduce a function φ which is a solution of the problem, where $p_x \in L_2(\mathbb{R}^3 \times (0,T))$,

$$\begin{aligned} &\operatorname{div}\varphi = p_x,\\ &\varphi|_{S_R} = 0. \end{aligned} \tag{3.103}$$

We know (see [4]) that there exists a solution of (3.103) such that $\varphi \in L_2(0,T;H^1(\mathbb{R}^3))$ and we have the estimate

$$\|\varphi\|_{L_2(0,T;H^1(\mathbb{R}^3))} \le c\|p_x\|_{0,\mathbb{R}^3\times(0,T)}. \tag{3.104}$$

Differentiating $(3.2^1)_1$ with respect to x, multiplying the result by φ and integrating over $\mathbb{R}^3$ yields

$$|p_x|^2_{2,\mathbb{R}^3} \le c(|v_t|^2_{2,\mathbb{R}^3} + |v_x|^2_{2,\mathbb{R}^3} + |f|^2_{2,\mathbb{R}^3}). \tag{3.105}$$

Integrating (3.105) with respect to t, using (3.8), (3.10) and (3.102) we obtain

$$\sup_t \|v\|^2_{1,B_R} + \int_0^T (|v_t|^2_{2,B_R} + \|v\|^2_{2,B_R} + \|p\|^2_{1,B_R})\,d\tau \le c\int_0^T \|f\|^2_{1,B_R}\,d\tau. \tag{3.106}$$

Summarizing the above considerations we have proved

LEMMA 3.9. *Assume that* $f \in L_2(0,T;H^1(B_R))$. *Then the weak solution to problem* (3.2^1) *is such that*

$$v \in L_\infty(0,T;H^1(B_R)) \cap H^1(B_R \times (0,T)) \cap L_2(0,T;H^2(B_R)),\ p \in L_2(0,T;H^1(B_R))$$

and the estimate (3.106) *holds.*

Next we have

LEMMA 3.10. *Let* $f \in L_{2,-\mu}(\mathbb{R}^2)$, $h \in H^1_{-\mu}(\mathbb{R}^2)$, $\mu \in (0,1)$, *have compact supports. Then there exists a solution of problem* (3.20) *such that* $v \in H^2_{-\mu}(\mathbb{R}^2)$, $p \in H^1_{-\mu}(\mathbb{R}^2)$ *and*

$$\|v\|_{2,-\mu,\mathbb{R}^2} + \|p\|_{1,-\mu,\mathbb{R}^2} \le c(\|f\|_{0,-\mu,\mathbb{R}^2} + \|h\|_{1,-\mu,\mathbb{R}^2}). \tag{3.107}$$

Proof. We repeat the proof of Lemma 3.8 up to inequality (3.80). Taking the inverse Fourier transform of (3.80) along the line $\operatorname{im}\lambda = 1+\mu$, $\mu \in (0,1)$ we obtain (3.107). This concludes the proof.

Similarly to Lemma 3.10 we prove

LEMMA 3.11. *Let* $f \in L_{2,\mu}(\mathbb{R}^2)$, $h \in H^1_\mu(\mathbb{R}^2)$, $\mu \in (0,1)$, *have compact supports. Then there exists a solution to problem* (3.20) *such that* $v \in H^2_\mu(\mathbb{R}^2)$, $p \in H^1_\mu(\mathbb{R}^2)$ *and*

$$\|v\|_{2,\mu,\mathbb{R}^2} + \|p\|_{1,\mu,\mathbb{R}^2} \le c(\|f\|_{0,\mu,\mathbb{R}^2} + \|h\|_{1,\mu,\mathbb{R}^2}). \tag{3.108}$$

4. Existence of solutions to problem (1.4). Continuation

In this section we prove the estimates necessary to apply the Kondrat'ev theory used in Section 3.

To show regularity of solutions to problem (1.1) in a neighbourhood of the distinguished axis we consider problem (3.2) and Lemmas 3.3–3.11 must be used. Hence we have to consider problem (3.15) in the cylindrical domain $P_R = B_R \cap \{x \in \mathbb{R}^3 : x_3 = 0\} \times \mathbb{R}$ with the x_3-axis as the distinguished axis. To generalize considerations instead of (3.15) we consider the problem

$$\begin{aligned} &v_t - \operatorname{div}\mathbb{T}(v,p) = f && \text{in } P_R \times \mathbb{R}_+,\\ &\operatorname{div} v = h && \text{in } P_R \times \mathbb{R}_+,\\ &v = 0 && \text{on } \partial P_R \times \mathbb{R}_+,\\ &v|_{t=0} = 0 && \text{in } P_R. \end{aligned} \tag{4.1}$$

For (4.1) we have

LEMMA 4.1. *Assume that* $v_t, f \in L_{2,\mu}(P_R \times \mathbb{R}_+)$, $h \in L_2\ (\mathbb{R}_+; H^1_\mu(P_R))$, $\mu \in (0,1)$. *Then there exists a solution of* (4.1) *such that* $v \in H^2_\mu(P_R \times \mathbb{R}_+)$, $p \in L_2(\mathbb{R}_+; H^1_\mu(P_R))$ *and*

$$\begin{aligned} &\|v\|_{H^2_\mu(P_R\times\mathbb{R}_+)} + \|p\|_{L_2(\mathbb{R}_+;H^1_\mu(P_R))}\\ &\qquad \le c(\|f\|_{L_{2,\mu}(P_R\times\mathbb{R}_+)} + \|h\|_{L_2(\mathbb{R}_+;H^1_\mu(P_R))} + \|v_t\|_{L_{2,\mu}(P_R\times\mathbb{R}_+)}). \end{aligned} \tag{4.2}$$

Proof. Applying the Laplace–Fourier transform (3.17^1) to (4.1) yields

$$
\begin{aligned}
&\nu(-\Delta\widetilde{v}_i+\xi^2\widetilde{v}_i)+\frac{\partial\widetilde{p}}{\partial x_i}=\widetilde{f}_i-s\widetilde{v}_i\equiv\widetilde{g}_i,\quad i=1,2, &&\text{in } d_{2\pi},\\
&\nu(-\Delta\widetilde{v}_3+\xi^2\widetilde{v}_3)+i\xi\widetilde{p}=\widetilde{f}_3-s\widetilde{v}_3\equiv\widetilde{g}_3 &&\text{in } d_{2\pi},\\
&\frac{\partial\widetilde{v}_1}{\partial x_1}+\frac{\partial\widetilde{v}_2}{\partial x_2}+i\xi\widetilde{v}_3=\widetilde{h} &&\text{in } d_{2\pi},
\end{aligned}
\tag{4.3}
$$

where v and p vanish outside B_R.

Applying the proof of inequality (3.13) from the proof of Theorem 3.1 in [6] to problem (4.3) we obtain the estimate

$$
\|v_{x_3}\|^2_{L_2(\mathbb{R}_+;H^1_\mu(\mathcal{D}_{2\pi}\times\mathbb{R}_+))}\le c(\|g\|^2_{L_{2,\mu}(\mathcal{D}_{2\pi}\times\mathbb{R}_+)}+\|h\|_{L_2(\mathbb{R}_+;H^1_\mu(\mathcal{D}_{2\pi}))}).
\tag{4.4}
$$

We have to recall that instead of the problem considered in Theorem 2.7 in [6] we consider the problem

$$
\begin{aligned}
&-\Delta u+\nabla q=0 &&\text{in } B_R,\\
&\operatorname{div} u=p &&\text{in } B_R,\\
&u=0 &&\text{on } \partial B_R.
\end{aligned}
\tag{4.5}
$$

Then the same estimate as in (2.6) of [6] follows. Morever, we have to underline that to construct the function ψ from (3.12) in [6] we have to assume that $\psi|_{\varphi=0}=0$. This concludes the proof.

Now we estimate the last norm on the r.h.s. of (4.2)

LEMMA 4.2. *Assume that $h=0$, $f\in L_{2,\mu}(P_R\times\mathbb{R}_+)$, $\mu\in(0,1)$. Then for solutions of (4.1) we have*

$$
\|v_t\|_{L_{2,\mu}(P_R\times\mathbb{R}_+)}\le c\|f\|_{L_{2,\mu}(P_R\times\mathbb{R}_+)}.
\tag{4.6}
$$

Proof. Applying the Laplace transform to (4.1) with $h=0$ we obtain

$$
\begin{aligned}
&s\widetilde{v}-\nu\Delta\widetilde{v}+\nabla\widetilde{p}=\widetilde{f} &&\text{in } P_R,\\
&\operatorname{div}\widetilde{v}=0 &&\text{in } P_R,\\
&\widetilde{v}=0 &&\text{on } \partial P_R.
\end{aligned}
\tag{4.7}
$$

To obtain (4.6) we examine the expression

$$
\int|s|^2\,ds\int_{P_R}|\widetilde{v}|^2|x'|^{2\mu}\,dx.
\tag{4.8}
$$

We examine (4.8) in two cases: $|s||x'|^2\le a$ and $|s||x'|^2\ge a$, where a will be chosen sufficiently large. In the first case we have

$$
\int|s|^2\,ds\int_{P_R\cap\{|x'|^2\le a|s|^{-1}\}}|\widetilde{v}|^2|x'|^{2\mu}\,dx\le a\int|s|^{2-\mu}\,ds\int_{P_R\cap\{|x'|^2\le a|s|^{-1}\}}|\widetilde{v}|^2\,dx.
\tag{4.9}
$$

Multiplying $(4.7)_1$ by $\widetilde{v}$ and integrating over P_R implies

$$
\int_{P_R}(\nu|\nabla\widetilde{v}|^2+|s||\widetilde{v}|^2)\,dx\le\varepsilon|s|^{1-\mu}\int_{P_R}|\widetilde{v}|^2|x'|^{-2\mu}\,dx+\frac{c(\varepsilon)}{|s|^{1-\mu}}\int_{P_R}|\widetilde{f}|^2|x'|^{2\mu}\,dx.
\tag{4.10}
$$

Using the Hardy inequality and $|s||x'|^2 \le a$ we obtain

$$\int_{P_R} (\nu|\nabla\widetilde{v}|^2 + |s||\widetilde{v}|^2)\,dx \le \varepsilon a^{1-\mu} \int_{P_R} |\nabla\widetilde{v}|^2\,dx + \frac{c(\varepsilon)}{|s|^{1-\mu}} \int_{P_R} |\widetilde{f}|^2|x'|^{2\mu}\,dx. \tag{4.11}$$

Hence for ε sufficiently small (4.11) implies

$$|s|^{1-\mu} \int_{P_R} (\nu|\nabla\widetilde{v}|^2 + |s||\widetilde{v}|^2)\,dx \le c \int_{P_R} |\widetilde{f}|^2|x'|^{2\mu}\,dx. \tag{4.12}$$

From (4.8), (4.9) and (4.12) we have (see [9])

$$\int |s|^2\,ds \int_{P_R\cap\{|x'|^2\le a|s|^{-1}\}} |\widetilde{v}|^2|x'|^{2\mu}\,dx \le c\int ds \int_{P_R} |\widetilde{f}|^2|x'|^{2\mu}\,dx. \tag{4.13}$$

Now we consider the case $|s||x'|^2 \ge a$. Let us introduce a function $\widetilde{\varphi}$ which is the solution to the problem

$$\begin{aligned} \Delta\widetilde{\varphi} &= \operatorname{div}(\widetilde{v}|x'|^{2\mu}) && \text{in } P_R,\\ \widetilde{\varphi} &= 0 && \text{on } \partial P_R. \end{aligned} \tag{4.14}$$

Multiplying $(4.7)_1$ by $\overline{\widetilde{v}|x'|^{2\mu} - \nabla\widetilde{\varphi}}$ and integrating the result over P_R gives

$$\int_{P_R} (s\widetilde{v} - \nu\Delta\widetilde{v})(\overline{\widetilde{v}|x'|^{2\mu} - \nabla\widetilde{\varphi}})\,dx = \int_{P_R} \widetilde{f}(\overline{\widetilde{v}|x'|^{2\mu} - \nabla\widetilde{\varphi}})\,dx, \tag{4.15}$$

where we used the fact that $\operatorname{div}(\widetilde{v}|x'|^{2\mu} - \nabla\widetilde{\varphi}) = 0$, the fact that $\widetilde{\varphi}$ vanishes identically outside the ball B_R because $\widetilde{v}$ vanishes outside B_R and $\widetilde{\varphi} = 0$ on $S_R = \partial B_R$ and also on ∂P_R and $\overline{u}$ is the complex conjugate to u.

From (4.15) we have

$$\begin{aligned} &\int_{P_R} (|s||\widetilde{v}|^2 + \nu|\nabla\widetilde{v}|^2)|x'|^{2\mu}\,dx \\ &\quad\le \Big|\int_{P_R} \nabla\widetilde{v}\cdot\overline{\widetilde{v}}\cdot\nabla|x'|^{2\mu}\,dx\Big| + \Big|\int_{P_R} \widetilde{f}\cdot\overline{\widetilde{v}}|x'|^{2\mu}\,dx\Big| + \Big|\int_{P_R} \widetilde{f}\cdot\overline{\nabla\widetilde{\varphi}}\,dx\Big| \\ &\quad\le \varepsilon \int_{P_R} |\nabla\widetilde{v}|^2|x'|^{2\mu}\,dx + c(\varepsilon)\int_{P_R} |\widetilde{v}|^2|x'|^{2\mu-2}\,dx + \varepsilon_1|s|\int_{P_R} |\widetilde{v}|^2|x'|^{2\mu}\,dx \\ &\qquad + \frac{c(\varepsilon_1)}{|s|}\int_{P_R} |\widetilde{f}|^2|x'|^{2\mu}\,dx + \varepsilon_2|s|\int_{P_R} |\nabla\widetilde{\varphi}|^2|x'|^{-2\mu}\,dx + \frac{c(\varepsilon_2)}{|s|}\int_{P_R} |\widetilde{f}|^2|x'|^{2\mu}\,dx. \end{aligned} \tag{4.16}$$

Assuming that ε, ε_1 and ε_2 are sufficiently small we obtain from (4.16) the inequality

$$\begin{aligned} &\int_{P_R} (|s||\widetilde{v}|^2 + \nu|\nabla\widetilde{v}|^2)|x'|^{2\mu}\,dx \\ &\qquad\le c\int_{P_R} |\widetilde{v}|^2|x'|^{2\mu-2}\,dx + \varepsilon|s|\int_{P_R} |\nabla\widetilde{\varphi}|^2|x'|^{-2\mu}\,dx + \frac{c}{|s|}\int_{P_R} |\widetilde{f}^2|x'|^{2\mu}\,dx. \end{aligned} \tag{4.17}$$

The problem (4.14) can be written in the form

$$\begin{aligned} \Delta\widetilde{\varphi} &= \widetilde{v}\cdot\nabla|x'|^{2\mu} && \text{in } P_R,\\ \widetilde{\varphi} &= 0 && \text{on } \partial P_R. \end{aligned} \tag{4.18}$$

The second term on the r.h.s. of (4.17) suggests that $\widetilde{\varphi} \in H^2_{1-\mu}(P_R)$. Therefore for solutions of (4.18) we have the estimate

$$\|\widetilde{\varphi}\|_{H^2_{1-\mu}(P_R)} \le c\|v|x'|^{2\mu-1}\|_{L_{2,1-\mu}(P_R)} \le c\|v\|_{L_{2,\mu}(P_R)}. \tag{4.19}$$

Then the second term from the r.h.s. of (4.17) is estimated by

$$\varepsilon|s|c \int\limits_{P_R} |\widetilde{v}|^2|x'|^{2\mu}\,dx$$

so using it in (4.17) and assuming that ε is sufficiently small we obtain

$$\int\limits_{P_R} (|s||\widetilde{v}|^2 + \nu|\nabla\widetilde{v}|^2)|x'|^{2\mu}\,dx \le c\int\limits_{P_R} |\widetilde{v}|^2|x'|^{2\mu-2}\,dx + \frac{c}{|s|}\int\limits_{P_R} |\widetilde{f}|^2|x'|^{2\mu}\,dx. \tag{4.20}$$

Since we consider the case $|s||x'|^2 \ge a$ the first term on the l.h.s. of (4.20) is estimated from below by

$$\int\limits_{P_R} |s||\widetilde{v}|^2|x'|^{2\mu}\,ds \ge a\int\limits_{P_R} |\widetilde{v}|^2|x'|^{2\mu-2}\,dx.$$

Using this in (4.20) and assuming that a is sufficiently large we obtain from (4.20) the inequality

$$\int ds \int\limits_{P_R\cap\{|x'|^{-2}\le a^{-1}|s|\}} (|s|^2|\widetilde{v}|^2 + \nu|s||\nabla\widetilde{v}|^2)|x'|^{2\mu}\,dx \le c\int ds\int\limits_{P_R} |\widetilde{f}|^2|x'|^{2\mu}\,dx. \tag{4.21}$$

From (4.13) and (4.21) we obtain (4.6). This concludes the proof.

Next we have (see also [9])

LEMMA 4.3. *Assume that* $h = 0$, $g \in L_{2,-\mu_1}(P_R \times \mathbb{R}_+)$, $\mu_1 \in (0,1)$. *Then for solutions of* (4.1) *we have*

$$\|v\|_{L_2(\mathbb{R}_+;H^2_{-\mu_1}(P_R))} + \|p\|_{L_2(\mathbb{R}_+;H^1_{-\mu_1}(P_R))} \le c\|g\|_{L_{2,-\mu_1}(P_R\times\mathbb{R}_+)}, \tag{4.22}$$

where $g = f - v_t$.

Proof. We consider problem (4.1) with $h = 0$. Take a function η such that

$$\begin{aligned} &\operatorname{div}\eta = \varrho,\\ &\eta|_{\partial P_R} = 0. \end{aligned} \tag{4.23}$$

Then multiplying $(4.1)_1$ by η and integrating over P_R yields

$$\nu\int\limits_{P_R} \nabla v\cdot\nabla\eta\,dx = \int\limits_{P_R} g\cdot\eta\,dx + \int\limits_{P_R} p\varrho\,dx. \tag{4.24}$$

Taking $\eta = v|x'|^{-2\mu_1}$ implies

$$\begin{aligned} \nu\int\limits_{P_R} |\nabla v|^2|x'|^{-2\mu_1}\,dx = &\int\limits_{P_R} g\cdot v|x'|^{-2\mu_1}\,dx - \nu\int\limits_{P_R} \nabla v\cdot v\cdot\nabla|x'|^{-2\mu_1}\,dx\\ &+ \int\limits_{P_R} pv\cdot\nabla|x'|^{-2\mu_1}\,dx. \end{aligned} \tag{4.25}$$

Passing to the Fourier transform with respect to x_3 yields

$$(4.26)\quad \int\limits_{B_R(0)} (|\nabla'\widetilde{v}|^2+\xi^2|\widetilde{v}|^2)|x'|^{-2\mu_1}\,dx' \\ \le \varepsilon \int\limits_{B_R(0)} \xi^2|\widetilde{v}|^2|x'|^{-2\mu_1}\,dx' + \frac{c(\varepsilon)}{|\xi|^2}\int\limits_{B_R(0)} |\widetilde{g}|^2|x'|^{-2\mu_1}\,dx' \\ + \varepsilon_1 \int\limits_{B_R(0)} |\nabla'\widetilde{v}|^2|x'|^{-2\mu_1}\,dx + c(\varepsilon_1)\int\limits_{B_R(0)} |\widetilde{v}|^2|x'|^{-2\mu_1-2}\,dx' \\ + \varepsilon_2 \int\limits_{B_R(0)} |\widetilde{p}|^2|x'|^{-2\mu_1}\,dx' + c(\varepsilon_2)\int\limits_{B_R(0)} |\widetilde{v}|^2|x'|^{-2\mu_1-2}\,dx',$$

where $B_R(0)=B_R\cap\{x: x_3=0\}$.

Assuming ε and ε_1 in (4.26) sufficiently small implies

$$(4.27)\quad |\xi|^2 \int\limits_{B_R(0)} (|\nabla'\widetilde{v}|^2+|\xi|^2|\widetilde{v}|^2)|x'|^{-2\mu_1}\,dx' \\ \le c\int\limits_{B_R(0)} |\widetilde{g}|^2|x'|^{-2\mu_1}\,dx' + \varepsilon|\xi|^2\int\limits_{B_R(0)} |\widetilde{p}|^2|x'|^{-2\mu_1}\,dx' + c(\varepsilon)|\xi|^2\int\limits_{B_R(0)} |\widetilde{v}|^2|x'|^{-2\mu_1-2}\,dx'.$$

Now we consider three sets

$$(4.28)\qquad a_1\ge |x'|\,|\xi|,\qquad |x'|\,|\xi|\ge a_2,\qquad a_1\le |x'|\,|\xi|\le a_2,$$

where a_1,a_2 will be chosen later.

Considering the problem

$$(4.29)\qquad \begin{aligned} &-\nu\Delta'\widetilde{v}_i+\partial_{x_i}\widetilde{p}=\widetilde{g}_i-\nu\xi^2\widetilde{v}_i,\quad i=1,2,\\ &\widetilde{v}_{1,x_1}+\widetilde{v}_{2,x_2}=-i\xi\widetilde{v}_3,\\ &\widetilde{v}'|_{S_R(0)}=0,\end{aligned}$$

with the transmission conditions we obtain the estimate

$$(4.30)\quad \|\widetilde{v}'\|^2_{H^2_{-\mu_1}(B_R(0))}+\|\widetilde{p}\|^2_{H^1_{-\mu_1}(B_R(0))} \\ \le c\|\widetilde{g}\|^2_{L_{2,-\mu_1}(B_R(0))}+c\xi^4\|\widetilde{v}\|^2_{L_{2,-\mu_1}(B_R(0))}+c\xi^2\|\widetilde{v}\|^2_{H^1_{-\mu_1}(B_R(0))},$$

where $\widetilde{v}'=(\widetilde{v}_1,\widetilde{v}_2)$ and $S_R(0)=\partial B_R(0)$.

Moreover from the problem

$$(4.31)\qquad \begin{aligned} &-\nu\Delta\widetilde{v}_3=\widetilde{g}_3-\nu\xi^2\widetilde{v}_3-i\xi\widetilde{p},\\ &\widetilde{v}_3|_{S_R(0)}=0,\end{aligned}$$

with the transmission conditions we get the estimate

$$(4.32)\quad \|\widetilde{v}_3\|^2_{H^2_{-\mu_1}(B_R(0))}\le c\|\widetilde{g}\|^2_{L_{2,-\mu_1}(B_R(0))}+c\xi^4\|\widetilde{v}\|^2_{L_{2,-\mu_1}(B_R(0))}+c\xi^2\|\widetilde{p}\|^2_{L_{2,-\mu_1}(B_R(0))}.$$

Therefore (4.30) and (4.32) imply

$$(4.33)\quad \|\widetilde{v}\|^2_{H^2_{-\mu_1}(B_R(0))}+\|\widetilde{p}\|^2_{H^1_{-\mu_1}(B_R(0))} \\ \le c\|\widetilde{g}\|_{L_{2,-\mu_1}(B_R(0))}+c\xi^4\|\widetilde{v}\|^2_{L_{2,-\mu_1}(B_R(0))}+c\xi^2\|\widetilde{v}\|^2_{L_{2,-\mu_1-1}(B_R(0))}+c\xi^2\|\widetilde{p}\|^2_{L_{2,-\mu_1}(B_R(0))},$$

where we used (4.27) in the form

$$|\xi|^2 \int\limits_{B_R(0)} (|\nabla'\widetilde{v}|^2|x'|^{-2\mu_1} + |\widetilde{v}|^2|x'|^{-2\mu_1-2})\,dx'$$
$$\le c \int\limits_{B_R(0)} |\widetilde{g}|^2|x'|^{-2\mu_1}\,dx' + c|\xi|^2 \int\limits_{B_R(0)} |\widetilde{p}|^2|x'|^{-2\mu_1}\,dx' + c|\xi|^2 \int\limits_{B_R(0)} |\widetilde{v}|^2|x'|^{-2\mu_1-2}\,dx'$$

and the last inequality is obtained from (4.27) after applying the Hardy inequality.

Now we consider the case $(4.28)_1$. Hence $|\xi| \le \alpha_1|x'|^{-1}$. Therefore instead of (4.27) and (4.33) we have

$$\xi^2 \int\limits_{B_R(0)} (|\nabla'\widetilde{v}|^2 + \xi^2|\widetilde{v}|^2)|x'|^{-2\mu_1}\,dx \tag{4.34}$$
$$\le c \int\limits_{B_R(0)} |\widetilde{g}|^2|x'|^{-2\mu_1}\,dx + ca_1 \int\limits_{B_R(0)} |\widetilde{p}|^2|x'|^{-2\mu_1-2}\,dx + ca_1 \int\limits_{B_R(0)} |\widetilde{v}|^2|x'|^{-2\mu_1-4}\,dx,$$

and

$$\|\widetilde{v}\|^2_{H^2_{-\mu_1}(B_R(0))} + \|\widetilde{p}\|^2_{H^1_{-\mu_1}(B_R(0))} \tag{4.35}$$
$$\le c\|\widetilde{g}\|^2_{L_{2,-\mu_1}(B_R(0))} + ca_1\|\widetilde{v}\|^2_{L_{2,-\mu_1-2}(B_R(0))} + ca_1\|\widetilde{p}\|^2_{L_{2,-\mu_1-1}(B_R(0))},$$

respectively.

From (4.34) and (4.35) we obtain the following estimate for sufficiently small a_1:

$$\int\limits_{-\infty}^{\infty} d\xi\,[\|\widetilde{v}\|^2_{H^2_{-\mu_1}(B_R(0))} + \xi^2\|\widetilde{v}\|^2_{H^1_{-\mu_1}(B_R(0))} + \xi^4\|\widetilde{v}\|^2_{L_{2,-\mu_1}(B_R(0))} \tag{4.36}$$
$$+ \|\widetilde{p}\|^2_{H^1_{-\mu_1}(B_R(0))} + \xi^2\|\widetilde{p}\|^2_{L_{2,-\mu_1}(B_R(0))}] \le c \int\limits_{-\infty}^{\infty} \|\widetilde{g}\|^2_{L_{2,-\mu_1}(B_R(0))}\,d\xi.$$

Now we consider the case $(4.28)_2$. First we have to obtain an estimate for

$$\xi^2 \int\limits_{B_R(0)} |\widetilde{p}|^2|x'|^{-2\mu_1}\,dx'.$$

For this purpose we consider the problem

$$\begin{aligned} &-\Delta\varphi' + \nabla\eta = 0,\\ &\operatorname{div}\varphi' = p|x'|^{-\mu_1},\\ &\varphi'|_{S_R} = 0, \qquad \varphi'|_{\varphi=0} = 0. \end{aligned} \tag{4.37}$$

We have existence of weak solutions to (4.37) and the estimate

$$\|\varphi'\|^2_{H^1(B_R)} + \|\eta\|^2_{L_2(B_R)} \le c\|p|x'|^{-\mu_1}\|^2_{L_2(B_R)}. \tag{4.38}$$

From the condition $\varphi'|_{\varphi=0} = 0$ we also have the inequality

$$\int\limits_{B_R} |\varphi'|^2|x'|^{-2}\,dx \le c \int\limits_{B_R} |\nabla\varphi'|^2\,dx. \tag{4.39}$$

Introducing $\psi = |x'|^{\mu_1}\varphi'$ we see that $\operatorname{div}(|x'|^{-\mu_1}\psi) = p|x'|^{-\mu_1}$, $\psi|_{\varphi=0} = 0$, $\psi|_{S_R} = 0$.

From (4.38) and (4.39) we obtain

$$(4.40)\qquad \int_{B_R} (|\nabla\psi|^2|x'|^{-2\mu_1}+|\psi|^2|x'|^{-2\mu_1-2})\,dx\le c\int_{B_R}|p|^2|x'|^{-2\mu_1}\,dx.$$

Passing to the Fourier transform with respect to x_3 we get

$$(4.41)\qquad \int_{B_R(0)} (|\nabla'\widetilde\psi|^2|x'|^{-2\mu_1}+\xi^2|\widetilde\psi|^2|x'|^{-2\mu_1}+|\widetilde\psi|^2|x'|^{-2\mu_1-2})\,dx' \le c\int_{B_R(0)}|\widetilde p|^2|x'|^{-2\mu_1}\,dx'.$$

Multiplying $(4.3)_{1,2}$ by $\bar{\widetilde\psi}|x'|^{-2\mu_1}\zeta(|\xi||x'|)$ and integrating the result over $B_R(0)$ we obtain

$$(4.42)\qquad \int_{B_R(0)} \widetilde g\cdot\bar{\widetilde\psi}|x'|^{-2\mu_1}\zeta\,dx'=\nu\int_{B_R(0)}(-\Delta'\widetilde v+\xi^2\widetilde v)\cdot\bar{\widetilde\psi}|x'|^{-2\mu_1}\zeta\,dx' + \int_{B_R(0)}\bigg(\frac{\partial\widetilde p}{\partial x_1}\bar{\widetilde\psi}_1+\frac{\partial\widetilde p}{\partial x_2}\bar{\widetilde\psi}_2+i\xi\widetilde p\bar{\widetilde\psi}_3\bigg)|x'|^{-2\mu_1}\zeta\,dx',$$

where $\zeta=\zeta(t)$ is a smooth function such that $\zeta(t)=0$ for $t\le a_2/2$ and $\zeta(t)=1$ for $t\ge a_2$. Continuing calculations in (4.42) gives

$$(4.43)\qquad \int_{B_R(0)}[\widetilde p(\bar{\widetilde\psi}_1|x'|^{-\mu_1})_{,x_1}+\widetilde p(\bar{\widetilde\psi}_2|x'|^{-\mu_1})_{,x_2}+\widetilde p\overline{i\xi\widetilde\psi_3}|x'|^{-\mu_1}]|x'|^{-\mu_1}\zeta\,dx' + \int_{B_R(0)}[\widetilde p\bar{\widetilde\psi}_1|x'|^{-\mu_1}(|x'|^{-\mu_1}\zeta)_{,x_1}+\widetilde p\bar{\widetilde\psi}_2|x'|^{-\mu_1}(|x'|^{-\mu_1}\zeta)_{,x_2}]\,dx' = \nu\int_{B_R(0)}[\nabla'\widetilde v\cdot\nabla'(\bar{\widetilde\psi}|x'|^{-2\mu_1}\zeta)+\xi^2\widetilde v\cdot\bar{\widetilde\psi}|x'|^{-2\mu_1}\zeta]\,dx'-\int_{B_R(0)}\widetilde g\cdot\bar{\widetilde\psi}|x'|^{-2\mu_1}\zeta\,dx'.$$

From (4.43) we have

$$(4.44)\qquad \int\xi^2\,d\xi\int_{B_R(0)}|\widetilde p|^2|x'|^{-2\mu_1}\zeta\,dx' \le c\int\xi^2\,d\xi\int_{B_R(0)}|\widetilde p||\widetilde\psi|(|x'|^{-2\mu_1-1}\zeta+|x'|^{-2\mu_1}\dot\zeta|\xi|)\,dx' + c\int\xi^2\,d\xi\int_{B_R(0)}[|\nabla'\widetilde v||\nabla\widetilde\psi||x'|^{-2\mu_1}\zeta+|\nabla'\widetilde v||\widetilde\psi||x'|^{-2\mu_1-1}\zeta + |\nabla\widetilde v||\widetilde\psi||x'|^{-2\mu_1}\dot\zeta|\xi|+\xi^2|\widetilde v||\widetilde\psi||x'|^{-2\mu_1}\zeta]\,dx'+c\int\xi^2\,d\xi\int_{B_R(0)}|\widetilde g||\widetilde\psi||x'|^{-2\mu_1}\zeta\,dx'.$$

Continuing we have

$$(4.45)\qquad \int\xi^2\,d\xi\int_{B_R(0)}|\widetilde p|^2|x'|^{-2\mu_1}\zeta\,dx' \le\varepsilon\int\xi^2\,d\xi\int_{B_R(0)}|\widetilde p|^2|x'|^{-2\mu_1}\zeta\,dx'+c(\varepsilon)\int\xi^2\,d\xi\int_{B_R(0)}|\widetilde\psi|^2|x'|^{-2\mu_1-2}\zeta\,dx'$$

$$+c(\varepsilon_1)\int\xi^2\,d\xi\int\limits_{B_R(0)}|\widetilde{p}|^2|x'|^{-2\mu_1}|\dot\zeta|\,dx'+\varepsilon_1\int\xi^4\,d\xi\int\limits_{B_R(0)}|\widetilde{\psi}|^2|x'|^{-2\mu_1}|\dot\zeta|\,dx'$$

$$+\varepsilon_2\int\xi^2\,d\xi\int\limits_{B_R(0)}(|\nabla\widetilde{\psi}|^2|x'|^{-2\mu_1}+|\widetilde{\psi}|^2|x'|^{-2\mu_1-2}+\xi^2|\widetilde{\psi}|^2|x'|^{-2\mu_1})\zeta\,dx'$$

$$+c(\varepsilon_2)\int\xi^2\,d\xi\int\limits_{B_R(0)}(|\nabla'\widetilde{v}|^2+\xi^2|\widetilde{v}|^2)|x'|^{-2\mu_1}\zeta\,dx'$$

$$+\varepsilon_3\int\xi^4\,d\xi\int\limits_{B_R(0)}|\widetilde{\psi}|^2|x'|^{-2\mu_1}|\dot\zeta|\,dx'+c(\varepsilon_3)\int\xi^2\,d\xi\int\limits_{B_R(0)}|\nabla\widetilde{v}|^2|x'|^{-2\mu_1}|\dot\zeta|\,dx'$$

$$+\varepsilon_4\int\xi^4\,d\xi\int\limits_{B_R(0)}|\widetilde{\psi}|^2|x'|^{-2\mu_1}\zeta\,dx'+c(\varepsilon_4)\int d\varphi\int\limits_{B_R(0)}|\widetilde{g}|^2|x'|^{-2\mu_1}\zeta\,dx'.$$

Assuming that $\varepsilon_1,\varepsilon_2,\varepsilon_4$ are sufficiently small we obtain from (4.45) and (4.41) the inequality

$$\begin{aligned}(4.46)\quad &\int\xi^2\,d\xi\int\limits_{B_R(0)}|\widetilde{p}|^2|x'|^{-2\mu_1}\zeta\,dx'\\
&\le c\int\xi^2\,d\xi\int\limits_{B_R(0)}|\widetilde{\psi}|^2|x'|^{-2\mu_1-2}\,dx'+c(\varepsilon_1)\int\xi^2\,d\xi\int\limits_{B_R(0)}|\widetilde{p}|^2|x'|^{-2\mu_1}|\dot\zeta|\,dx'\\
&\quad+\varepsilon_1\int\xi^4\,d\xi\int\limits_{B_R(0)}|\widetilde{\psi}|^2|x'|^{-2\mu_1}|\dot\zeta|\,dx'\\
&\quad+c\int\xi^2\,d\xi\int\limits_{B_R(0)}(|\nabla'\widetilde{v}|^2+\xi^2|\widetilde{v}|^2)|x'|^{-2\mu_1}\zeta\,dx'\\
&\quad+\varepsilon_3\int\xi^4\,d\xi\int\limits_{B_R(0)}|\widetilde{\psi}|^2|x'|^{-2\mu_1}\,dx'+c(\varepsilon_3)\int\xi^2\,d\xi\int\limits_{B_R(0)}|\nabla\widetilde{v}|^2|x'|^{-2\mu_1}|\dot\zeta|\,dx'\\
&\quad+c\int d\xi\int\limits_{B_R(0)}|\widetilde{g}|^2|x'|^{-2\mu_1}\zeta\,dx'.\end{aligned}$$

Adding the term

$$\int\xi^2\,d\xi\int\limits_{B_R(0)}|\widetilde{p}|^2|x'|^{-2\mu_1}(1-\zeta)\,dx'$$

to both sides of (4.46) and using the fact that $|x'|^{-1}\le\frac{1}{a_2}|\xi|$ we obtain

$$\begin{aligned}(4.47)\quad &\int\xi^2\,d\xi\int\limits_{B_R(0)}|\widetilde{p}|^2|x'|^{-2\mu_1}\,dx'\\
&\le\int\xi^2\,d\xi\int\limits_{B_R(0)}|\widetilde{p}|^2|x'|^{-2\mu_1}(1-\zeta)\,dx'+\frac{c(\varepsilon_1)}{a_2}\int\xi^{2+2\mu_1}\,d\xi\int\limits_{B_R(0)}|\widetilde{p}|^2|\dot\zeta|\,dx'\\
&\quad+c\bigg(\frac{1}{a_2}+\varepsilon_1+\varepsilon_3\bigg)\int\xi^4\,d\xi\int\limits_{B_R(0)}|\widetilde{\psi}|^2|x'|^{-2\mu_1}\,dx'\\
&\quad+c\int\xi^2\,d\xi\int\limits_{B_R(0)}(|\nabla'\widetilde{v}|^2+\xi^2|\widetilde{v}|^2)|x'|^{-2\mu_1}\,dx'+c\int d\xi\int\limits_{B_R(0)}|\widetilde{g}|^2|x'|^{-2\mu_1}\,dx'.\end{aligned}$$

Using the fact that $|x'||\xi|\le a_2$ holds in the first term on the r.h.s. of (4.47) and exploiting (4.41) yields

$$
\begin{aligned}
(4.48)\quad &\int\xi^2\,d\xi\int_{B_R(0)}|\widetilde p|^2|x'|^{-2\mu_1}\,dx'\\
&\le c\int d\xi\int_{B_R(0)}|\widetilde p|^2|x'|^{-2\mu_1-2}\,dx'+\frac{c}{a_2}\int\xi^{2+2\mu_1}\,d\xi\int_{B_R(0)}|\widetilde p|^2|\dot\zeta|\,dx'\\
&\quad+c\left(\frac{1}{a_2}+\varepsilon_1+\varepsilon_3\right)\int\xi^2\,d\xi\int_{B_R(0)}|\widetilde p|^2|x'|^{-2\mu_1}\,dx'\\
&\quad+c\int\xi^2\,d\xi\int_{B_R(0)}(|\nabla'\widetilde v|^2+\xi^2|\widetilde v|^2)|x'|^{-2\mu_1}\,dx'+c\int d\xi\int_{B_R(0)}|\widetilde g|^2|x'|^{-2\mu_1}\,dx'.
\end{aligned}
$$

Assuming that $\varepsilon_1,\varepsilon_3$ are sufficiently small and a_2 is sufficiently large we obtain from (4.48) the inequality

$$
\begin{aligned}
(4.49)\quad &\int\xi^2\,d\xi\int_{B_R(0)}|\widetilde p|^2|x'|^{-2\mu_1}\,dx'\\
&\le c\int d\xi\int_{B_R(0)}|\widetilde p|^2|x'|^{-2\mu_1-2}\,dx'+\frac{c}{a_2}\int\xi^{2+2\mu_1}\,d\xi\int_{B_R(0)}|\widetilde p|^2|\dot\zeta|\,dx'\\
&\quad+c\int\xi^2\,d\xi\int_{B_R(0)}(|\nabla'\widetilde v|^2+\xi^2|\widetilde v|^2)|x'|^{-2\mu_1}\,dx'+\int d\xi\int_{B_R(0)}|\widetilde g|^2|x'|^{-2\mu_1}\,dx'.
\end{aligned}
$$

Now we estimate the second term from the r.h.s. of (4.49). The integral is considered in the set where $\dot\zeta\ne 0$ so for $a_2/2\le|x'||\xi|\le a_2$. Multiplying $(4.1)_1$ by v, integrating over P_R, using $\operatorname{div}v=0$ and passing to the Fourier transforms yields

$$
(4.50)\qquad \int_{B_R(0)}(|\nabla'\widetilde v|^2+\xi^2|\widetilde v|^2)\,dx'=\int_{B_R(0)}\widetilde g\cdot\widetilde v\,dx',
$$

so

$$
\begin{aligned}
(4.51)\quad &\int_{B_R(0)}(|\nabla'\widetilde v|^2+|\xi|^2|\widetilde v|^2)\,dx'\\
&\qquad\le\varepsilon|\xi|^{2+2\mu_1}\int_{B_R(0)}|\widetilde v|^2|x'|^{2\mu_1}\,dx'+\frac{c(\varepsilon)}{|\xi|^{2+2\mu_1}}\int_{B_R(0)}|\widetilde g|^2|x'|^{-2\mu_1}\,dx'.
\end{aligned}
$$

Using $|x'||\xi|\le a_2$ and choosing ε sufficiently small we obtain from (4.51) the estimate

$$
(4.52)\qquad \int|\xi|^{2+2\mu_1}\,d\xi\int_{B_R(0)}(|\nabla'\widetilde v|^2+\xi^2|\widetilde v|^2)\,dx'\le c\int d\xi\int_{B_R(0)}|\widetilde g|^2|x'|^{-2\mu_1}\,dx'.
$$

Next we have to obtain a similar estimate for $\widetilde p$. Consider problem (4.5). We have existence of weak solutions to (4.5) and the estimate

$$
(4.53)\qquad \int_{P_R}|\nabla u|^2\,dx+\int_{P_R}|q|^2\,dx\le c\int_{P_R}|p|^2\,dx.
$$

Passing to the Fourier transforms yields

$$\int d\xi \int_{B_R(0)} (|\nabla'\widetilde{u}|^2 + \xi^2|\widetilde{u}|^2)\,dx' \le c\int d\xi \int_{B_R(0)} |\widetilde{p}|^2\,dx'. \tag{4.54}$$

Since we consider the cylindrical domain P_R we can differentiate problem (4.5) with respect to x_3. Therefore using a fractional derivative with respect to x_3 we can obtain instead of (4.54) the estimate

$$\int d\xi\,|\xi|^{2+2\mu_1} \int_{B_r(0)} (|\nabla'\widetilde{u}|^2 + \xi^2|\widetilde{u}|^2)\,dx' \le c\int d\xi\,|\xi|^{2+2\mu_1} \int_{B_R(0)} |\widetilde{p}|^2\,dx'. \tag{4.55}$$

Multiplying (4.3) by $\widetilde{u}\xi^{2+2\mu_1}$ and integrating over $B_R(0)$ and ξ imply

$$\begin{aligned}\int d\xi\,|\xi|^{2+2\mu_1} \int_{B_R(0)} |\widetilde{p}|^2\,dx' \le \int d\xi\,|\xi|^{2+2\mu_1} &\int_{B_R(0)} (\nabla'\widetilde{v}\cdot\nabla'\widetilde{u} + \xi^2\widetilde{v}\cdot\widetilde{u})\,dx' \\ -\int d\xi\,|\xi|^{2+2\mu_1} &\int_{B_R(0)} \widetilde{g}\cdot\widetilde{u}\ \,dx'.\end{aligned} \tag{4.56}$$

Using (4.52) and (4.55) we have

$$\begin{aligned}&\int d\xi\,|\xi|^{2+2\mu_1} \int_{B_R(0)} |\widetilde{p}|^2\,dx' \\ &\le \varepsilon\int d\xi\,|\xi|^{4+4\mu_1} \int_{B_R(0)} |\widetilde{u}|^2|x'|^{2\mu_1}\,dx' + c(\varepsilon)\int d\xi \int_{B_R(0)} |\widetilde{g}|^2|x'|^{-2\mu_1}\,dx'.\end{aligned} \tag{4.57}$$

Using $|x'||\xi| \le a_2$ in the first term on the r.h.s. of (4.57) we estimate it by

$$\varepsilon a_2^{2\mu_1}\int d\xi\,|\xi|^{4+2\mu_1} \int_{B_R(0)} |\widetilde{u}|^2\,dx' \le \varepsilon a_2^{2\mu_1}c\int d\xi\,|\xi|^{2+2\mu_1} \int_{B_R(0)} |\widetilde{p}|^2\,dx'.$$

Therefore for sufficiently small ε we get from (4.57) the estimate

$$\int d\xi\,|\xi|^{2+2\mu_1} \int_{B_R(0)} |\widetilde{p}|^2\,dx' \le c\int d\xi \int_{B_R(0)} |\widetilde{g}|^2|x'|^{-2\mu_1}\,dx'. \tag{4.58}$$

In view of (4.58) the inequality (4.49) assumes the form

$$\begin{aligned}&\int \xi^2\,d\xi \int_{B_R(0)} |\widetilde{p}|^2|x'|^{-2\mu_1}\,dx' \\ &\le c\int d\xi \int_{B_R(0)} |\widetilde{p}|^2|x'|^{-2\mu_1-2}\,dx' + c\int \xi^2\,d\xi \int_{B_R(0)} (|\nabla'\widetilde{v}|^2 + \xi^2|\widetilde{v}|^2)|x'|^{-2\mu_1}\,dx' \\ &\quad + c\int d\xi \int_{B_R(0)} |\widetilde{g}|^2|x'|^{-2\mu_1}\,dx'.\end{aligned} \tag{4.59}$$

Now using (4.59) in (4.27) yields

$$\begin{aligned}&\int d\xi\,|\xi|^2 \int_{B_R(0)} (|\nabla'\widetilde{v}|^2 + |\xi|^2|\widetilde{v}|^2)|x'|^{-2\mu_1}\,dx' \\ &\le \varepsilon\int d\xi \int_{B_R(0)} |\widetilde{p}|^2|x'|^{-2\mu_1-2}\,dx' \\ &\quad + c(\varepsilon)\int d\xi\,\xi^2 \int_{B_R(0)} |\widetilde{v}|^2|x'|^{-2\mu_1-2}\,dx' + c\int d\xi \int_{B_R(0)} |\widetilde{g}|^2|x'|^{-2\mu_1}\,dx'.\end{aligned} \tag{4.60}$$

Since $|x'|^{-1} \le (1/a_2)|\xi|$ the second term on the r.h.s. of (4.60) is estimated by
$$\frac{c(\varepsilon)}{a_2}\int d\xi\,\xi^2 \int\limits_{B_R(0)} |\widetilde{v}|^2|x'|^{-2\mu_1}\,dx'.$$
Assuming that
$$\frac{c(\varepsilon)}{a_2} \le \frac{1}{2}, \tag{4.61}$$
where $c(\varepsilon) \sim \varepsilon^{-\alpha}$, $\alpha > 0$, we obtain from (4.60) the inequality
$$\begin{aligned}\int d\xi\,|\xi|^2 \int\limits_{B_R(0)} (|\nabla'\widetilde{v}|^2 + \xi^2|\widetilde{v}|^2)|x'|^{-2\mu_1}\,dx' \\ \le \varepsilon\int d\xi \int\limits_{B_R(0)} |\widetilde{p}|^2|x'|^{-2\mu_1-2}\,dx' + c\int d\xi \int\limits_{B_R(0)} |\widetilde{g}|^2|x'|^{-2\mu_1}\,dx'.\end{aligned} \tag{4.62}$$
Now using (4.62) in (4.30) yields
$$\int d\xi\,(\|\widetilde{v}'\|^2_{H^2_{-\mu_1}(B_R(0))} + \|\widetilde{p}\|^2_{H^1_{-\mu_1}(B_R(0))}) \le c\int d\xi\,\|\widetilde{g}\|^2_{L_{2,-\mu_1}(B_R(0))}. \tag{4.63}$$
Finally from (4.62), (4.63) and (4.32) we obtain (4.36) for $(4.28)_2$.

Finally we consider the case $(4.28)_3$. In this case we have
$$\begin{aligned}\int d\xi\,\xi^2\int(|\nabla'\widetilde{v}|^2 + \xi^2|\widetilde{v}|^2)|x'|^{-2\mu_1}\,dx' + \int d\xi\,\xi^2 \int\limits_{B_R(0)} |\widetilde{p}|^2|x'|^{-2\mu_1}\,dx' \\ \le \int d\xi\,|\xi|^{2+2\mu_1} \int\limits_{B_R(0)} (|\nabla'\widetilde{v}|^2 + \xi^2|\widetilde{v}|^2)\,dx' + \int d\xi\,\xi^{2+2\mu_1} \int\limits_{B_R(0)} |\widetilde{p}|^2\,dx' \\ \le c\int d\xi \int\limits_{B_R(0)} |\widetilde{g}|^2|x'|^{-2\mu_1}\,dx'.\end{aligned} \tag{4.64}$$
From this estimate we obtain (4.36) for $(4.28)_3$. This concludes the proof.

Finally we prove

LEMMA 4.4. *Assume that* $h = 0$, $f \in L_{2,-\mu}(P_A \times \mathbb{R}_+)$, $\mu \in (0,1)$. *Then for solutions of* (4.1) *we have*
$$\|v_t\|_{L_{2,-\mu}(P_R\times\mathbb{R}_+)} \le c\|f\|_{L_{2,-\mu}(P_R\times\mathbb{R}_+)}. \tag{4.65}$$
Proof. Multiplying $(4.7)_1$ by $\overline{\widetilde{v}}|x'|^{-2\mu}$ and integrating over P_R we obtain
$$\begin{aligned}\int\limits_{P_R} (s|\widetilde{v}|^2 + \nu|\nabla\widetilde{v}|^2)|x'|^{-2\mu}\,dx \\ = \int\limits_{P_R} \widetilde{f}\cdot\overline{\widetilde{v}}|x'|^{-2\mu}\,dx - \nu\int\limits_{P_R} \nabla\widetilde{v}\cdot\overline{\widetilde{v}}\nabla|x'|^{-2\mu}\,dx + \int\limits_{P_R} \widetilde{p}\overline{\widetilde{v}}\cdot\nabla|x'|^{-2\mu}\,dx.\end{aligned} \tag{4.66}$$
Estimating the r.h.s. of (4.66) we get
$$\begin{aligned}\int\limits_{P_R} (|s||\widetilde{v}|^2 + \nu|\nabla\widetilde{v}|^2)|x'|^{-2\mu}\,dx \\ \le \varepsilon\int\limits_{P_R} |s||\widetilde{v}|^2|x'|^{-2\mu}\,dx + c(\varepsilon)\frac{1}{|s|}\int\limits_{P_R} |\widetilde{f}|^2|x'|^{-2\mu}\,dx \\ + \varepsilon_1\int\limits_{P_R} |\nabla\widetilde{v}|^2|x'|^{-2\mu}\,dx + c(\varepsilon_1)\int\limits_{P_R} |\widetilde{v}|^2|x'|^{-2\mu-2}\,dx \\ + c(\varepsilon_2)\int\limits_{P_R} |\widetilde{p}|^2|x'|^{-2\mu}\,dx + \varepsilon_2\int\limits_{P_R} |\widetilde{v}|^2|x'|^{-2\mu-2}\,dx.\end{aligned} \tag{4.67}$$

By the Hardy inequality the last term is estimated by

$$\varepsilon_2 c \int_{P_R} |\nabla\widetilde{v}|^2 |x'|^{-2\mu}\, dx.$$

Then assuming that $\varepsilon, \varepsilon_1$ and ε_2 are sufficiently small we obtain from (4.67) the inequality

$$\begin{aligned}(4.68)\qquad &\int_{P_R} (|s||\widetilde{v}|^2 + \nu|\nabla\widetilde{v}|^2)|x'|^{-2\mu}\, dx \\ &\le \frac{c}{|s|}\int_{P_R} |\widetilde{f}|^2|x'|^{-2\mu}\, dx + c\int_{P_R} |\widetilde{v}|^2|x'|^{-2\mu-2}\, dx + c\int_{P_R} |\widetilde{p}|^2|x'|^{-2\mu}\, dx.\end{aligned}$$

Multiplying (4.68) by $|s|$ and integrating with respect to s yields

$$\begin{aligned}(4.69)\qquad &\int ds\, |s| \int_{P_R} (|s||\widetilde{v}|^2 + \nu|\nabla\widetilde{v}|^2)|x'|^{-2\mu}\, dx \\ &\le c\int ds \int_{P_R} |\widetilde{f}|^2|x'|^{-2\mu}\, dx + c\int ds\, |s| \int_{P_R} |\widetilde{v}|^2|x'|^{-2\mu-2}\, dx + c\int ds\, |s| \int_{P_R} |\widetilde{p}|^2|x'|^{-2\mu}\, dx.\end{aligned}$$

We restrict our considerations to the set

$$(4.70)\qquad |s||x'|^2 \le a_1,$$

where a_1 will be chosen later. Then $|s| \le a_1|x'|^{-2}$ and (4.69) takes the form

$$\begin{aligned}(4.71)\qquad &\int ds\, |s| \int_{P_R} (|s||\widetilde{v}|^2 + \nu|\nabla\widetilde{v}|^2)|x'|^{-2\mu}\, dx \\ &\le c\int ds \int_{P_R} |\widetilde{f}|^2|x'|^{-2\mu}\, dx + ca_1\int ds \int_{P_R} |\widetilde{v}|^2|x'|^{-2\mu-4}\, dx + ca_1\int ds \int_{P_R} |\widetilde{p}|^2|x'|^{-2\mu-2}\, dx.\end{aligned}$$

Let us consider the elliptic problem

$$(4.72)\qquad \begin{aligned} &-\nu\varDelta\widetilde{v} + \nabla\widetilde{p} = -s\widetilde{v} + \widetilde{f} && \text{in } P_R,\\ &\operatorname{div}\widetilde{v} = 0 && \text{in } P_R,\\ &\widetilde{v}|_{\partial P_R} = 0. \end{aligned}$$

In view of the Kondrat'ev theory and (4.71) we obtain

$$\begin{aligned}(4.73)\qquad \|\widetilde{v}\|^2_{H^2_{-\mu}(P_R)} + \|\widetilde{p}\|^2_{H^1_{-\mu}(P_R)} &\le c|s|^2\|\widetilde{v}\|^2_{L_{2,-\mu}(P_R)} + c\|\widetilde{f}\|^2_{L_{2,-\mu}(P_R)} \\ &\le c\|\widetilde{f}\|^2_{L_{2,-\mu}(P_R)} + ca_1\|\widetilde{v}\|^2_{L_{2,-\mu-2}(P_R)} + ca_1\|\widetilde{p}\|^2_{L_{2,-\mu-1}(P_R)}.\end{aligned}$$

For sufficiently small a_1 we get from (4.73) the inequality

$$(4.74)\qquad \|\widetilde{v}\|^2_{H^2_{-\mu}(P_R)} + \|\widetilde{p}\|^2_{H^2_{-\mu}(P_R)} \le c\|\widetilde{f}\|^2_{L_{2,-\mu}(P_R)}.$$

Using (4.74) in (4.71) we obtain (4.65) for (4.70) with a_1 sufficiently small.

Now we consider the set

$$(4.75)\qquad |s||x'|^2 \ge a_2,$$

where a_2 will be chosen sufficiently large.

Let us consider the problem

$$
\begin{aligned}
&\Delta\widetilde{\varphi} = \operatorname{div}(\widetilde{v}|x'|^{-2\mu}) &&\text{in } P_R,\\
&\widetilde{\varphi} = 0 &&\text{on } \partial P_R.
\end{aligned}
\tag{4.76}
$$

Multiplying $(4.7)_1$ by $\overline{\widetilde{v}|x'|^{-2\mu} - \nabla\widetilde{\varphi}}$ and integrating over P_R we obtain

$$
\int_{P_R} (s\widetilde{v} - \nu\Delta\widetilde{v})\cdot(\overline{\widetilde{v}|x'|^{-2\mu} - \nabla\widetilde{\varphi}})\,dx = \int_{P_R} \widetilde{f}\cdot(\overline{\widetilde{v}|x'|^{-2\mu} - \nabla\widetilde{\varphi}})\,dx,
\tag{4.77}
$$

where we used the fact that $\widetilde{v}|x'|^{-2\mu} - \nabla\widetilde{\varphi}$ is divergence free.

Integrating by parts in (4.77) yields

$$
\begin{aligned}
&\int_{P_R} (s|\widetilde{v}|^2 + \nu|\nabla\widetilde{v}|^2)|x'|^{-2\mu}\,dx\\
&\qquad = \nu\int_{P_R} \nabla\widetilde{v}\cdot\overline{\widetilde{v}}\nabla|x'|^{-2\mu}\,dx + \int_{P_R}\widetilde{f}\cdot\overline{\widetilde{v}}|x'|^{-2\mu}\,dx - \int_{P_R} f\cdot\nabla\overline{\widetilde{\varphi}}\,dx.
\end{aligned}
\tag{4.78}
$$

Continuing we have

$$
\begin{aligned}
&\int_{P_R} (|s||\widetilde{v}|^2 + \nu|\nabla\widetilde{v}|^2)|x'|^{-2\mu}\,dx\\
&\quad\le \varepsilon\int_{P_R}|\nabla\widetilde{v}|^2|x'|^{-2\mu}\,dx + c(\varepsilon)\int_{P_R}|\widetilde{v}|^2|x'|^{-2\mu-2}\,dx + \varepsilon_1|s|\int_{P_R}|\widetilde{v}|^2|x'|^{-2\mu}\,dx\\
&\qquad + c(\varepsilon_1)\frac{1}{|s|}\int_{P_R}|\widetilde{f}|^2|x'|^{-2\mu}\,dx + \varepsilon_1|s|\int_{P_R}|\nabla\widetilde{\varphi}|^2|x'|^{2\mu}\,dx.
\end{aligned}
\tag{4.79}
$$

We write problem (4.76) in the form

$$
\begin{aligned}
&\Delta\widetilde{\varphi} = \widetilde{v}\cdot\nabla|x'|^{-2\mu},\\
&\widetilde{\varphi}|_{\partial P_R} = 0,
\end{aligned}
\tag{4.80}
$$

and the corresponding transmission conditions are satisfied.

The last term on the r.h.s. of (4.79) suggests that $\widetilde{\varphi}\in H^2_{1+\mu}(P_R)$. Therefore from (4.80) we have the estimate

$$
\|\widetilde{\varphi}\|_{H^2_{1+\mu}(P_R)} \le c\|\widetilde{v}\cdot\nabla|x'|^{-2\mu}\|_{L_{2,1+\mu}(P_R)} \le c\|\widetilde{v}\|_{L_{2,-\mu}(P_R)}.
\tag{4.81}
$$

Using (4.81) in (4.79) and assuming that $\varepsilon,\varepsilon_1$ are sufficiently small we obtain

$$
\int_{P_R} (|s||\widetilde{v}|^2 + \nu|\nabla\widetilde{v}|^2)|x'|^{-2\mu}\,dx \le c\int_{P_R}|\widetilde{v}|^2|x'|^{-2\mu-2}\,dx + \frac{c}{|s|}\int_{P_R}|\widetilde{f}|^2|x'|^{-2\mu}\,dx.
\tag{4.82}
$$

From (4.75) we have $|s|\ge a_2|x'|^{-2}$ so (4.82) implies

$$
\begin{aligned}
&a_2\int|s|\,ds\int_{P_R}|\widetilde{v}|^2|x'|^{-2\mu-2}\,dx + \int|s|^2\,ds\int_{P_R}|\widetilde{v}|^2|x'|^{-2\mu}\,dx\\
&\qquad + \nu\int|s|\,ds\int_{P_R}|\nabla\widetilde{v}|^2|x'|^{-2\mu}\,dx\\
&\qquad \le c\int|s|\,ds\int_{P_R}|\widetilde{v}|^2|x'|^{-2\mu-2}\,dx + c\int ds\int_{P_R}|\widetilde{f}|^2|x'|^{-2\mu}\,dx.
\end{aligned}
\tag{4.83}
$$

For sufficiently large a_2 we obtain from (4.83) the inequality

$$\int |s|^2\,ds \int_{P_R} |\widetilde{v}|^2|x'|^{-2\mu}\,dx + \nu\int |s|\,ds \int_{P_R} |\nabla\widetilde{v}|^2|x'|^{-2\mu}\,dx \le c\int ds \int_{P_R} |\widetilde{f}|^2|x'|^{-2\mu}\,dx. \tag{4.84}$$

From (4.84) we have (4.65) in the case (4.75).

Finally we consider the case

$$a_1 \le |s||x'|^2 \le a_2. \tag{4.85}$$

Multiplying $(4.7)_1$ by $\overline{\widetilde{v}}$ and integrating over P_R yields

$$\begin{aligned}\int_{P_R} (\nu|\nabla\widetilde{v}|^2 + |s||\widetilde{v}|^2)\,dx &\le \Big|\int_{P_R} \widetilde{f}\cdot\overline{\widetilde{v}}\,dx\Big| \\ &\le \varepsilon|s|^{1+\mu}\int_{P_R} |\widetilde{v}|^2|x'|^{2\mu}\,dx + \frac{c(\varepsilon)}{|s|^{1+\mu}}\int_{P_R} |\widetilde{f}|^2|x'|^{-2\mu}\,dx.\end{aligned} \tag{4.86}$$

In view of (4.85) the first term on the r.h.s. of (4.86) is estimated by

$$\varepsilon a_2|s|\int_{P_R} |\widetilde{v}|^2\,dx.$$

Therefore for sufficiently small ε we obtain from (4.86) the inequality

$$\int |s|^{1+\mu}\,ds \int_{P_R} (\nu|\nabla\widetilde{v}|^2 + |s||\widetilde{v}|^2)\,dx \le c\int ds \int_{P_R} |\widetilde{f}|^2|x'|^{-2\mu}\,dx. \tag{4.87}$$

In view of (4.85) we see that the l.h.s. of (4.87) is estimated from below by

$$\int |s|^{1+\mu}\,ds \int_{P_R} (\nu|\nabla\widetilde{v}|^2 + |s||\widetilde{v}|^2)\,dx \ge a_1\int |s|\,ds \int_{P_R} (\nu|\nabla\widetilde{v}|^2 + |s||\widetilde{v}|^2)|x'|^{-2\mu}\,dx,$$

so we see that (4.65) also holds for (4.85). This concludes the proof.

Corollary 4.5. *Lemmas* 4.3 *and* 4.4 *also hold for* $\mu_1 = 1$.

Now we want to summarize the above results. Let us consider the problem

$$\begin{aligned} &v_t - \operatorname{div}\mathbb{D}(v,p) = g && \text{in } P_R\times\mathbb{R}_+,\\ &\operatorname{div} v = 0 && \text{in } P_R\times\mathbb{R}_+,\\ &v|_S = 0 && \text{on } \partial P_R\times\mathbb{R}_+,\\ &v|_{t=0} = 0 && \text{in } P_R.\end{aligned} \tag{4.88}$$

From Lemmas 3.11, 4.1 and 4.2 we have

Lemma 4.6. *Assume that* $g\in L_{2,\mu}(P_R\times\mathbb{R}_+)$, $\mu\in(0,1)$. *Then there exists a solution to* (4.88) *such that* $v\in H^{2,1}_\mu(P_R\times\mathbb{R}_+)$, $p\in L_2(\mathbb{R}_+;H^1_\mu(P_R))$ *and*

$$\|v\|_{H^{2,1}_\mu(P_R\times\mathbb{R}_+)} + \|p\|_{L_2(\mathbb{R}_+;H^1_\mu(P_R))} \le c\|g\|_{L_{2,\mu}(P_R\times\mathbb{R}_+)}. \tag{4.89}$$

Lemmas 3.10, 4.3 and 4.4 imply

Lemma 4.7. *Assume that* $g\in L_{2,-\mu}(P_R\times\mathbb{R}_+)$, $\mu\in(0,1)$. *Then there exists a solution to* (4.88) *such that* $v\in H^{2,1}_{-\mu}(P_R\times\mathbb{R}_+)$, $p\in L_2(\mathbb{R}_+;H^1_{-\mu}(P_R))$ *and*

$$\|v\|_{H^{2,1}_{-\mu}(P_R\times\mathbb{R}_+)} + \|p\|_{L_2(\mathbb{R}_+;H^1_{-\mu}(P_R))} \le c\|g\|_{L_{2,-\mu}(P_R\times\mathbb{R}_+)}. \tag{4.90}$$

Finally from Lemma 3.8 and Corollary 4.5 we obtain

LEMMA 4.8. *Assume that $g \in L_{2,-1}(P_R \times \mathbb{R}_+)$. Then there exists a solution to* (4.88) *such that $v \in H^{2,1}_{-1}(P_R \times \mathbb{R}_+)$, $p \in L_2(\mathbb{R}_+; H^1_{-1}(P_R))$ and*

$$\|v\|_{H^{2,1}_{-1}(P_R\times\mathbb{R}_+)} + \|p\|_{L_2(\mathbb{R}_+;H^1_{-1}(P_R))} \le c\|g\|_{L_{2,-1}(P_R\times\mathbb{R}_+)}. \tag{4.91}$$

Finally we consider the problem

$$\begin{aligned} &u_t - \operatorname{div}\mathbb{D}(u,q) = f && \text{in } P_R \times \mathbb{R}_+,\\ &\operatorname{div} u = h && \text{in } P_R \times \mathbb{R}_+,\\ &u|_S = b && \text{on } \partial P_R \times \mathbb{R}_+,\\ &u|_{t=0} = u_0 && \text{in } P_R. \end{aligned} \tag{4.92}$$

Using [8] we have

LEMMA 4.9. *Assume that*

(1) $f \in L_{2,-\mu}(P_R \times \mathbb{R}_+)$, $h \in L_2(\mathbb{R}_+; H^1_{-\mu}(P_R))$, $b \in H^{3/2,3/4}_{-\mu}(P_R \times \mathbb{R}_+)$, $u_0 \in H^1_{-\mu}(P_R)$, $\mu \in (0,1]$.

(2) *The following compatibility conditions hold*:

$$\begin{aligned} h|_{t=0} &= \operatorname{div} u_0,\\ b|_{t=0} &= u_0|_S. \end{aligned} \tag{4.93}$$

(3) *The following equality holds*:

$$\partial_t h - \operatorname{div} f = \operatorname{div}\delta + \tau. \tag{4.94}$$

Then there exists a unique solution to problem (4.92) *such that $u \in H^{2,1}_{-\mu}(P_R \times \mathbb{R}_+)$, $g \in L_2(\mathbb{R}_+; H^1_{-\mu}(P_R))$ and*

$$\begin{aligned} &\|u\|_{H^{2,1}_{-\mu}(P_R\times\mathbb{R}_+)} + \|q\|_{L_2(\mathbb{R}_+;H^1_{-\mu}(P_R))}\\ &\quad\le c(\|f\|_{L_{2,-\mu}(P_R\times\mathbb{R}_+)} + \|u_0\|_{H^1_{-\mu}(P_R)} + \|h\|_{L_2(\mathbb{R}_+;H^1_{-\mu}(P_R))}\\ &\qquad + \|b\|_{H^{3/2,3/4}_{-\mu}(S_R\times\mathbb{R}_+)} + \|\delta\|_{L_{2,-\mu}(P_R\times\mathbb{R}_+)} + R^{3/2}\|\tau\|_{L_{2,-\mu}(P_R\times\mathbb{R}_+)}) \end{aligned} \tag{4.95}$$

Proof. Denoting by $\widetilde{f}$ and $\widetilde{u}_0$ extensions of f and u_0 outside B_R we consider the Cauchy problem

$$\begin{aligned} &\partial_1 u^{(1)} - \mu\Delta u^{(1)} = \widetilde{f} && \text{in } \mathbb{R}^3 \times \mathbb{R}_+,\\ &u^{(1)}|_{t=0} = \widetilde{u}_0 && \text{in } \mathbb{R}^3. \end{aligned} \tag{4.96}$$

Using the fundamental solutions for the heat equation we solve (4.96) in the form

$$u^{(1)}(x,t) = \int_0^t \int_{\mathbb{R}^3} \Gamma(x-y,t-\tau)\widetilde{f}(y,\tau)\,dy\,d\tau + \int_{\mathbb{R}^3} \Gamma(x-y)\widetilde{u}_0(y)\,dy. \tag{4.97}$$

Next we define $u^{(2)} = \nabla\varphi$, where φ is the solution of the Dirichlet problem

$$\begin{aligned} &\Delta\varphi = h - \operatorname{div} u^{(1)} \equiv \sigma && \text{in } B_R,\\ &\varphi|_{S_R} = 0 && \text{on } S_R. \end{aligned} \tag{4.98}$$

Let $G(x,y)$ be the Green function for the Dirichlet problem (4.98). Then solving (4.98) we obtain

$$\varphi(x,t)=\int_{B_R} G(x,y)\sigma(y,t)\,dy. \tag{4.99}$$

Writing (4.99) more explicitly implies

$$u^{(2)}(x,t)=\nabla_x\int_{B_R} G(x,y)h(y,t)\,dy+\nabla_x\int_{B_R}\nabla_y G(x,y)u^{(1)}(y,t)\,dy, \tag{4.100}$$

where we used $G|_{S_R}=0$.

Finally we define $u^{(3)}$, $q^{(3)}$ as the solution of the problem

$$\begin{aligned} &\partial_t u^{(3)}-\mu\Delta u^{(3)}+\nabla q^{(3)}=0 && \text{in } B_R\times\mathbb{R}_+,\\ &\operatorname{div} u^{(3)}=0 && \text{in } B_R\times\mathbb{R}_+,\\ &u^{(3)}|_{S_R}=b-u^{(3)}|_{S_R}-u^{(1)}|_{S_R}\equiv d && \text{on } S_R\times\mathbb{R}_+,\\ &u^{(3)}|_{t=0}=0 && \text{in } B_R, \end{aligned} \tag{4.101}$$

where the compatibility condition $(4.93)_1$ was used.

Therefore any solution of (4.92) can be written in the form

$$u=u^{(1)}+u^{(2)}+u^{(3)},\qquad q=q^{(2)}+q^{(3)}, \tag{4.102}$$

where

$$q^{(2)}=\mu(h-\operatorname{div} u^{(1)})-\partial_t\varphi. \tag{4.103}$$

Compatibility conditions (4.93) imply

$$d|_{t=0}=0. \tag{4.104}$$

Next we consider the problem (see [4])

$$\begin{aligned} &\operatorname{div} w=0 && \text{in } B_R\times\mathbb{R}_+,\\ &w|_{S_R}=d && \text{on } S_R\times\mathbb{R}_+,\\ &w|_{t=0}=0 && \text{in } B_R. \end{aligned} \tag{4.105}$$

Let $d\in H^{3/2,3/4}_{-\mu}(S_R\times\mathbb{R}_+)$. Then we can construct a function $w\in H^{2,1}_{-\mu}(B_R\times\mathbb{R}_+)$ which satisfies (4.105) and the estimate

$$\|w\|_{H^{2,1}_{-\mu}(B_R\times\mathbb{R}_+)}\le c\|d\|_{H^{3/2,3/4}_{-\mu}(B_R\times\mathbb{R}_+)}. \tag{4.106}$$

Let us introduce the functions

$$v=u^{(3)}-w,\qquad p=q^{(3)}, \tag{4.107}$$

which are the solutions of the problem

$$\begin{aligned} &\partial_t v-\mu\Delta v+\nabla p=-(w_t-\mu\Delta w)\equiv g,\\ &\operatorname{div} v=0,\\ &v|_S=0,\\ &v|_{t=0}=0. \end{aligned} \tag{4.108}$$

In view of assumptions (1) any solution of (4.96) belongs to $H^{2,1}_{-\mu}(P_R\times\mathbb{R}_+)$ and

$$\|u^{(1)}\|_{H^{2,1}_{-\mu}(P_R\times\mathbb{R}_+)}\le c(\|f\|_{L_{2,-\mu}(P_R\times\mathbb{R}_+)}+c\|u_0\|_{H^1_{-\mu}(P_R)}). \tag{4.109}$$

In view of Lemmas 4.7 and 4.8 we have existence of solutions of (4.108) such that $v\in H^{2,1}_{-\mu}(P_R\times\mathbb{R}_+)$, $p\in L_2(\mathbb{R}_+;H^1_{-\mu}(P_R))$ and

$$\begin{aligned}\|v\|_{H^{2,1}_{-\mu}(P_R\times\mathbb{R}_+)}&+\|p\|_{L_2(\mathbb{R}_+;H^1_{-\mu}(P_R))}\\ &\le c\|g\|_{L_{2,-\mu}(P_R\times\mathbb{R}_+)}\le c\|w\|_{H^{2,1}_{-\mu}(P_R\times\mathbb{R}_+)}\le c\|d\|_{H^{3/2,3/4}_{-\mu}(P_R\times\mathbb{R}_+)}.\end{aligned} \tag{4.110}$$

Now we estimate $u^{(2)}$. From (4.100) we have

$$\|\nabla_x^2u^{(2)}(x,t)\|^2_{L_{2,-\mu}(B_R)}\le c(\|\nabla_xh\|^2_{L_{2,-\mu}(B_R)}+\|\nabla_x^2u^{(1)}(x,t)\|_{L_{2,-\mu}(B_R)}),$$

where ∇_x^2 contains all second derivatives with respect to x. Integrating the above inequality with respect to t yields

$$\|\nabla_x^2u^{(2)}(x,t)\|_{L_{2,-\mu}(B_R\times\mathbb{R}_+)}\le c(\|\nabla_xh\|_{L_{2,-\mu}(B_R\times\mathbb{R}_+)}+\|\nabla_x^2u^{(1)}\|_{L_{2,-\mu}(B_R\times\mathbb{R}_+)}). \tag{4.111}$$

Differentiating (4.100) with respect to t and using (4.94) implies

$$\begin{aligned}&\|\partial_tu^{(2)}\|_{L_{2,-\mu}(B_R\times\mathbb{R}_+)}\\ &\quad\le c\Big\|\nabla_x\int_{B_R}G(x,y)\partial_th(y,t)\,dy\Big\|_{L_{2,-\mu}(B_R\times\mathbb{R}_+)}\\ &\qquad+c\Big\|\nabla_x\int_{B_R}\nabla_yG(x,y)\partial_tu^{(1)}(y,t)\,dy\Big\|_{L_{2,-\mu}(B_R\times\mathbb{R}_+)}\\ &\quad\le c\Big\|\nabla_x\int_{B_R}G(x,y)(\operatorname{div}f+\operatorname{div}\delta+\tau)\,dy\Big\|_{L_{2,-\mu}(B_R\times\mathbb{R}_+)}\\ &\qquad+c\Big\|\nabla_x\int_{B_R}\nabla_yG(x,y)\partial_tu^{(1)}(y,t)\,dy\Big\|_{L_{2,-\mu}(B_R\times\mathbb{R}_+)}\\ &\quad=c\Big\|\nabla_x\int_{B_R}G(x,y)\tau(y,t)\,dy\Big\|_{L_{2,-\mu}(B_R\times\mathbb{R}_+)}\\ &\qquad+c\Big\|\nabla_x\int_{B_R}\nabla_yG(x,y)(-f(y,t)-\delta(y,t)+\partial_tu^{(1)}(y,t))\,dy\Big\|_{L_{2,-\mu}(B_R\times\mathbb{R}_+)}\\ &\quad\equiv c(A_1+A_2).\end{aligned} \tag{4.112}$$

To estimate A_2 we have to apply estimates in $L_{2,-\mu}(B_R)$ for the integral with the singular kernel $K(x-y)=\nabla_x\nabla_xG(x,y)$. For this purpose we can use either Theorem 3.1 from [7] or Theorem 8.1 from [12]. However to apply the theorems the following condition must be satisfied:

$$|Q|^{-1}\int_Q w\,dx|Q|^{-1}\int_Q w^{-1}\,dx\le c, \tag{4.113}$$

which should hold for any cube $Q\subset\mathbb{R}^3$, where c is a constant and $w=|x'|^{-2\mu}$ is the weight function. We see that (4.113) does not hold for $\mu=1$. Therefore to obtain an estimate for A_2 we consider the following Dirichlet problem:

$$\begin{aligned} &\Delta w = -f - \delta + \partial_t u^{(1)} \equiv F \quad &&\text{in } B_R,\\ &w|_{S_R} = 0 &&\text{on } S_R. \end{aligned} \tag{4.114}$$

To obtain the estimate in weighted spaces we add the following transmission conditions to problem (4.114)

$$\begin{aligned} &w|_{\Gamma_0} = w|_{\Gamma_{2\pi}},\\ &\overline{n}\cdot\nabla w|_{\Gamma_0} = -\overline{n}\cdot\nabla w|_{\Gamma_{2\pi}}, \end{aligned} \tag{4.115}$$

where $\overline{n}|_{\Gamma_0} = -\overline{n}|_{\Gamma_{2\pi}}$. Repeating the considerations from Sections 3 and 4 to problem (4.114), (4.115) we obtain

$$\begin{aligned} &\|w\|_{L_2(\mathbb{R}_+;H^2_{-\mu}(B_R))}\\ &\quad \le c\|F\|_{L_2(\mathbb{R}_+;L_{2,-\mu}(B_R))}\\ &\quad \le c(\|f\|_{L_2(\mathbb{R}_+;L_{2,-\mu}(B_R))} + \|\delta\|_{L_2(\mathbb{R}_+;L_{2,-\mu}(B_R))} + \|u_t^{(1)}\|_{L_2(\mathbb{R}_+;L_{2,-\mu}(B_R))}), \end{aligned} \tag{4.116}$$

where the last norm is estimated by (4.109).

Next we obtain an estimate for A_1. To do it we write

$$\begin{aligned} A_1 &= \Big\|\nabla_x \int_{B_R} \partial_{r_y} G(x,y) \int_0^{r_y} \tau\,dr_y\,dy\Big\|_{L_{2,-\mu}(B_R)} \le c\Big(\int_0^R r^{2-2\mu}\,dr\Big)^{1/2}\|\tau\|_{L_2(B_R)}\\ &\le cR^{3/2-\mu}\|\tau\|_{L_2(B_R)} \le cR^{3/2}\|\tau\|_{L_{2,-\mu}(B_R)}, \end{aligned}$$

where $r_y = \sqrt{y_1^2 + y_2^2}$ and the same considerations as above have been used.

Next we estimate

$$\begin{aligned} &\|q^{(2)}\|_{L_2(\mathbb{R}_+;H^1_{-\mu}(B_R))}\\ &\quad \le c(\|h\|_{L_2(\mathbb{R}_+;H^1_{-\mu}(B_R))} + \|u^{(1)}\|_{H^{2,1}_{-\mu}(B_R\times\mathbb{R}_+)} + \|f\|_{L_{2,-\mu}(B_R\times\mathbb{R}_+)}\\ &\qquad + \|\delta\|_{L_{2,-\mu}(B_R\times\mathbb{R}_+)} + R^{3/2}\|\tau\|_{L_{2,-\mu}(B_R\times\mathbb{R}_+)}). \end{aligned} \tag{4.117}$$

Summarizing the above estimates we obtain

$$\begin{aligned} &\|u\|_{H^{2,1}_{-\mu}(B_R\times\mathbb{R}_+)} + \|q\|_{L_2(\mathbb{R}_+;H^1_{-\mu}(B_R))}\\ &\quad \le c(\|u^{(1)}\|_{H^{2,1}_{-\mu}(B_R\times\mathbb{R}_+)} + \|u^{(2)}\|_{H^{2,1}_{-\mu}(B_R\times\mathbb{R}_+)} + \|u^{(3)}\|_{H^{2,1}_{-\mu}(B_R\times\mathbb{R}_+)}\\ &\qquad + \|q^{(2)}\|_{L_2(\mathbb{R}_+;H^1_{-\mu}(B_R))} + \|q^{(3)}\|_{L_2(\mathbb{R}_+;H^1_{-\mu}(B_R))})\\ &\quad \le c(\|f\|_{L_{2,-\mu}(B_R\times\mathbb{R}_+)} + \|u_0\|_{H^1_{-\mu}(B_R)} + \|h\|_{L_2(\mathbb{R}_+;H^1_{-\mu}(B_R))}\\ &\qquad + \|\delta\|_{L_{2,-\mu}(B_R\times\mathbb{R}_+)} + R^{3/2}\|\tau\|_{L_{2,-\mu}(B_R\times\mathbb{R}_+)} + \|d\|_{H^{3/2,3/4}_{-\mu}(S_R\times\mathbb{R}_+)}) \end{aligned} \tag{4.118}$$

where

$$\begin{aligned} &\|d\|_{H^{3/2,3/4}_{-\mu}(S_R\times\mathbb{R}_+)}\\ &\quad \le c(\|b\|_{H^{3/2,3/4}_{-\mu}(S_R\times\mathbb{R}_+)} + \|u^{(2)}\|_{H^{2,1}_{-\mu}(B_R\times\mathbb{R}_+)} + \|u^{(1)}\|_{H^{2,1}_{-\mu}(B_R\times\mathbb{R}_+)}). \end{aligned} \tag{4.119}$$

From (4.118) and (4.119) we obtain (4.95). Uniqueness follows from standard considerations. This concludes the proof.

5. Existence of solutions to (1.1)

Extending the solutions of problem (1.6) for $x_3 < 0$ we obtain a problem similar to (1.4). Therefore we obtain an analogue of Lemma 4.9. Let us denote it by Lemma 4.9′.

Problems (1.7) and (1.8) describe solutions at a positive distance from the axis of symmetry. Therefore to examine them we do not need weighted Sobolev spaces.

First we have

LEMMA 5.1. *Assume*

(1) $g \in L_2(\mathbb{R}^3_+ \times (0,T))$, $h \in L_2(0,T;H^1(\mathbb{R}^3_+))$, $k_1 \in L_2(0,T;H^{3/2}(\mathbb{R}^2))$, $k_2 \in L_2(0,T;H^{1/2}(\mathbb{R}^2))$, $u_0 \in H^1(\mathbb{R}^3_+)$,

(2) *the compatibility conditions*

$$\operatorname{div} u_0 = h|_{t=0}, \qquad u_0 \cdot \overline{u}|_{x_3=0} = k_1,$$

(3) $\partial_t h = \operatorname{div} g + \operatorname{div}\delta + \tau$, *where* $\tau, \delta \in L_2(\mathbb{R}^3_+ \times (0,T))$.

Then there exists a solution to problem (1.7) *such that* $u \in H^{2,1}(\mathbb{R}^3_+ \times (0,T))$, $q \in L_2(0,T;H^1(\mathbb{R}^3_+))$ *and*

$$\begin{aligned}(5.1)\quad &\|u\|_{H^{2,1}(\mathbb{R}^3_+\times(0,T))} + \|q\|_{L_2(0,T;H^1(\mathbb{R}^3_+))} \\ &\le c(\|g\|_{L_2(\mathbb{R}^3_+\times(0,T))} + \|h\|_{L_2(0,T;H^1(\mathbb{R}^3_+))} + \|k_1\|_{H^{3/2,3/4}(\mathbb{R}^2_+\times(0,T))} \\ &\quad + \|k_2\|_{H^{1/2,1/4}(\mathbb{R}^2\times(0,T))} + \|u_0\|_{H^1(\mathbb{R}^3_+)} + R^{3/2}\|\tau\|_{L_2(\mathbb{R}^3_+\times(0,T))} \\ &\quad + \|\delta\|_{L_2(\mathbb{R}^3_+\times(0,T))}).\end{aligned}$$

Similarly we obtain

LEMMA 5.2. *Assume*

(1) $g \in L_2(\mathbb{R}^3 \times (0,T))$, $h \in L_2(0,T;H^1(\mathbb{R}^3))$, $u_0 \in H^1(\mathbb{R}^3)$,

(2) *the compatibility condition*

$$\operatorname{div} u_0 = h|_{t=0},$$

(3) $\partial_t h = \operatorname{div} g + \operatorname{div}\delta + \tau$, $\tau, \delta \in L_2(\mathbb{R}^3 \times (0,T))$.

Then there exists a solution to (1.8) *such that* $u \in H^{2,1}(\mathbb{R}^3 \times (0,T))$, $q \in L_2(0,T;H^1(\mathbb{R}^3))$ *and*

$$\begin{aligned}(5.2)\quad &\|u\|_{H^{2,1}(\mathbb{R}^3\times(0,T))} + \|q\|_{L_2(0,T;H^1(\mathbb{R}^3))} \\ &\le c(\|g\|_{L_2(\mathbb{R}^3\times(0,T))} + \|h\|_{L_2(0,T;H^1(\mathbb{R}^3))} + \|u_0\|_{H^1(\mathbb{R}^3)} \\ &\quad + \|\delta\|_{L_2(\mathbb{R}^3\times(0,T))} + R^{3/2}\|\tau\|_{L_2(\mathbb{R}^3\times(0,T))}),\end{aligned}$$

where R is the diameter of $\operatorname{supp}\zeta$, *where ζ is any function from the relevant partition of unity.*

Considering problems (1.4), (1.6), (1.7), (1.8) with vanishing initial data we prove the existence of solutions to problem (1.1) with vanishing initial data using the technique of regularizers (see e.g. [5]) and Lemmas 4.9, 4.9′, 5.1, 5.2. Next by extending the initial data we obtain

THEOREM 5.3. *Assume that* $f \in L_{2,-\mu}(\Omega^T)$, $v_0 \in H^1_{-\mu}(\Omega)$, $\mu \in (0,1]$. *Then there exists a solution to problem* (1.1) *such that* $v \in H^{2,1}_{-\mu}(\Omega^T)$, $p \in L_2(,T; H^1_{-\mu}(\Omega))$ *and*

$$\|v\|_{H^{2,1}_{-\mu}(\Omega^T)} + \|p\|_{L_2(0,T;H^1_{-\mu}(\Omega))} \le c(\|f\|_{L_{2,-\mu}(\Omega^T)} + \|v_0\|_{H^1_{-\mu}(\Omega)}). \tag{5.3}$$

We have to underline that the pressure is not determined up to an arbitrary constant.

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