INTRODUCTION

In recent years, the theory of measurability of multifunctions (loosely speaking, set-valued functions) has been developed extensively, with important applications in differential inclusions, mathematical economics, optimal control and optimization (see [1], [3], [16], [21], [29], [33], [36], [37], [38], [45], [46], [57], [89], [90], [91], [94], [98], [100], and elsewhere).

In various problems, one encounters measurability of multifunctions of two variables. Obviously, each multifunction of two variables $x \in X$ and $y \in Y$ may be treated as a multifunction of the single variable $(x, y) \in X \times Y$. The essential difference is the possibility of formulating hypotheses concerning the multifunction in terms of its sectionwise properties. In this case, we can speak about product (sometimes called joint) measurability and superpositional measurability (sup-measurability for short), i.e., roughly speaking, measurability with respect to the product σ-field and measurability of the Carathéodory type superposition $F(x, G(x))$, respectively, where F and G are multifunctions.

In the single valued version, the problem of product measurability and sup-measurability has been studied very extensively in the last 40 years. An overview of some papers in this field can be found in [41]. Far less is known, however, in the multivalued case, although in various fields of mathematics and its applications, the superposition $F(x, G(x))$ occurs frequently (see for instance [1], [3], [21], [46] and [89]).

The difference between sup-measurability and joint measurability is essential. In general, neither of the inclusions between the class of joint measurable multifunctions and the class of sup-measurable multifunctions is true. It is easy to define a joint Lebesgue measurable real function that is not sup-measurable [106]. On the other hand, Grande and Lipiński have given an example of a sup-measurable real function which is not measurable as a function of two variables [44].

Several joint measurability results have been proved for single valued functions of two variables ([40], [41], [17], [18], [23], [24], [78], [80], [84] and others). It is well known that if $(X, \mathcal{M}(X))$ is a measurable space, Y is a separable metric space and Z is a metric space, then a Carathéodory function $f : X \times Y \to Z$ (i.e., loosely speaking a function measurable in the first and continuous in the second variable) is measurable with respect to the product of the σ-field $\mathcal{M}(X)$ and the Borel σ-field of Y. This result was also proved in the case of a multifunction ([111], [116]). Unfortunately, without additional hypotheses, this result cannot be extended to multifunctions with a weaker semicontinuity assumption in place of continuity. Many new features appear in this case which are "hidden" in the single valued theory.

The problem of sup-measurability was for the first time considered by Carathéodory in his book [11]. He formulated a sufficient condition for sup-measurability of a function $f : \mathbb{R}^2 \to \mathbb{R}$, namely, measurability as a function of the first variable for any $y \in \mathbb{R}$ and continuity as a function of the second variable, for almost every $x \in \mathbb{R}$. Certain conditions for sup-measurability of functions in abstract spaces have been presented by Shragin in [106]. Several results on sup-measurability of real functions are given by Grande in [41] and [39].

The purpose of this paper is to prove some new product measurability and sup-measurability results concerning multifunctions.

The present monograph consists of three chapters. Chapter 1 and Chapter 2 are divided into sections: the first one into Sections 1–6, and the second one into Sections 7–11.

In Chapter 1, we collect material that will be used in the next chapters: notation and terminology (Section 1), facts known in the literature (Sections 2 and 3), and facts which are new for multifunctions of one variable (Sections 4, 5 and 6).

In Section 4, we start from the idea of the density of sets in a metric space with respect to some differentiation basis, generating a density topology in this space, then introduce the concept of approximate continuity of multifunctions and prove some basic properties of such multifunctions.

Strong quasi-continuity has been considered in the literature, first by Noiri [88] for functions and then by Neubrunn [85] for multifunctions; there, it meant continuity relative to the α topology of a topological space. In the case of real functions, such strong quasi-continuity coincides with the usual continuity (see [85]).

Strong quasi-continuity of real functions was also considered by Grande in [43] but in a different sense. His definition of strong quasi-continuity is based on the density topology in the space of real numbers. In Section 5, we generalize this notion to the case of multifunctions (in abstract spaces) and show that a multifunction which is strongly quasi-continuous is almost everywhere continuous.

Many steps have been taken toward differential calculus for multifunctions, among them one by Hukuhara [53] and another by Banks and Jacobs [5]. In Section 6, the notion of differentiability is developed, taking advantage of an idea used by Hukuhara to give a definition of differentiability for a reasonably wide class of multifunctions. But the study of differentiability of multifunctions is not the purpose of this paper. We give only some properties needed later on. We consider multifunctions from an interval to a real reflexive normed linear space. In this case, the derivative of a multifunction at a point is a closed convex and bounded set. This is essential for further considerations. The concept of π-differentiability of multifunctions discussed by Banks and Jacobs is presented, taking advantage of Rådström's embedding theorem. In this case the derivative of a multifunction at a point is a continuous linear mapping. (A comparison of the two notions of differentiability is given.) Furthermore, a notion of a derivative multifunction is introduced, making use of the notion of integral given by Banks and Jacobs in [5].

As we are mainly interested in multifunctions of two variables, we study such multifunctions in Chapters 2 and 3.

Chapter 2 is devoted to product measurability of multifunctions. In Section 7, a particular emphasis is put on the possibility of replacing continuity in the second variable of a Carathéodory multifunction by a weaker assumption, keeping product measurability. Among these possibilities, we show that in metric spaces, continuity relative to a certain topology, finer than the metric one, yields product measurability. It also preserves additional features.

Section 8 is concerned with joint measurability of a multifunction in a metric space whose sections are approximately semicontinuous with respect to some differentiation basis. These results were inspired by the results of Grande [41] for real functions. Some new properties arise in the case of multifunctions.

The classical result of Kempisty concerning quasi-continuity of real functions which are quasi-continuous with respect to both variables has been extended to a class of multifunctions [85]. Roughly speaking, the upper (resp. lower) quasi-continuity of a multifunction in the first and both upper quasi-continuity and lower quasi-continuity in the second variable imply its upper (resp. lower) quasi-continuity. By the example of Marcus [79], such a multifunction need not be product measurable.

If, in the notion of a Carathéodory multifunction, we replace the continuity in the second variable by semicontinuity, we obtain a *semi-Carathéodory multifunction*. In general, a multifunction which is semi-Carathéodory need not be product measurable (even if it is compact valued). In Section 9, we show that a lower semi-Carathéodory multifunction which is upper quasicontinuous in the second variable is product measurable.

The situatoin is different for the strong quasi-continuity considered by Grande in [43]. There exists a real function, strongly quasi-continuous in both variables, which is not strongly quasi-continuous (as a function of two variables). But it turns out that such a function is product measurable.

Section 9 is also devoted to the product measurability of a multifunction (in a metric space) which is measurable in the first and both upper strongly quasi-continuous and lower strongly quasi-continuous with respect to a differentiation basis in the second variable.

In Section 10 we introduce a concept of multifunctions (with values in a Banach space) with the (J) property, which may be considered as a multivalued counterpart of the (J) property for real functions given by Lipiński [78]. We show that a multifunction with the (J) property which is a derivative in the second variable is product measurable.

We conclude that chapter by introducing multifunctions having the Scorza-Dragoni properties which have close connections with product measurable multifunctions.

The last chapter, Chapter 3, is concerned with sup-measurability of multifunctions. Shragin [106] introduced a *property of normalization* of functions between Borel measurability and Lebesgue measurability of functions of two variables and proved that any normalized function is sup-measurable. This theorem was generalized by Zygmunt to the case of multifunctions [118], i.e., measurability with respect to the product of a σ-field and the σ-field of Borel sets ensures sup-measurability.

In Chapter 3, we begin with sufficient conditions for sup-measurability of multifunctions which are consequences of theorems of Chapter 2 and Zygmunt's theorem.

Product measurability with respect to a σ-field more general than that required in Zygmunt's theorem need not ensure sup-measurability of a multifunction. We present some ways to reinforce the product measurability with additional assumptions on the sections of the multifunction which do secure its sup-measurability.

It is easy to see that, in some spaces, a compact valued Carathéodory multifunction is sup-measurable. This result can be extended to a general class of multifunctions. It turns out that if the continuity of a Carathéodory multifunction in the second variable is replaced by a more general condition (for instance, R-integrability), then the multifunction will still be sup-measurable.

In general, a multifunction which is semi-Carathéodory need not be sup-measurable (even if it is compact valued). But if a lower semi-Carathéodory multifunction is moreover assumed to be upper quasi-continuous in the second variable, then it is sup-measurable. Furthermore, we show that a multifunction with the (J) property which is a derivative in the second variable is sup-measurable. Finally, some additional density properties of a product measurable multifunction which ensure its sup-measurability are considered.

Definitions, lemmas, theorems, corollaries, examples and remarks are numbered consecutively, but separately within each chapter; thus Theorem 1.2 means the second theorem in Chapter 1. Independently, some important mathematical facts (easy conclusions or known facts) useful later are numbered (also separately within each chapter); thus (2.7) means some statement in Chapter 2.

Proofs are included, as usual, when the assertions are more general than those which have appeared in the literature or when, in my opinion, the result is not known or the proof is simpler than the known one. Otherwise, the reader is referred to the corresponding papers. Numbers in square brackets refer to the bibliography at the end of the monograph.

1. PRELIMINARIES

1. Notations, basic definitions and properties

By means of this chapter, we want to make sure that the reader has become acquainted with the language and useful facts on multifunctions of one variable, needed when we start the main subject in the next chapters. Things will be presented in resonable generality.

We will use standard notations. In particular, the sets of positive integers and real numbers will be denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. $\mathbb{R}^n$ will denote the n-dimensional Euclidean space, $\mathcal{L}(\mathbb{R}^n)$ the σ-field of Lebesgue measurable subsets of $\mathbb{R}^n$ and m_n the Lebesgue measure on $\mathcal{L}(\mathbb{R}^n)$ (we will simply write m instead of m_1). Capital calligraphic letters will usually denote collections, families or classes of sets.

Let S and Z be nonempty sets and let Φ be a mapping which associates to each point $s \in S$ a nonempty set $\Phi(s) \subset Z$. Such a mapping is called a *multifunction* from S to Z, and we write $\Phi : S \rightsquigarrow Z$. As a rule, we will denote functions by f, g, h, ϕ, ψ, etc., and multifunctions by capital letters F, G, H, Φ, Ψ, etc.

The *graph* of a multifunction Φ is defined by

$$\mathrm{Gr}(\Phi) = \{(s, z) \in S \times Z : z \in \Phi(s)\}. \tag{1.1}$$

Let $\mathcal{P}(Z)$ denote the family of all subsets of Z and $\mathcal{P}_0(Z)$ the subfamily of all nonempty subsets of Z. We will sometimes consider a multifunction Φ as a function from S to $\mathcal{P}_0(Z)$. This will always be explicitly indicated in order to avoid vagueness. For instance, the graph of a multifunction Φ from S to Z is a subset of $S \times Z$ (see (1.1)), whereas the graph of a function Φ from S to $\mathcal{P}_0(Z)$ is a subset of $S \times \mathcal{P}_0(Z)$, namely $\{(s, P) \in S \times \mathcal{P}_0(Z) : P = \Phi(s)\}$.

If $\Phi : S \rightsquigarrow Z$ is a multifunction, then for a set $A \subset Z$ two *inverse images* of A under Φ are defined as follows:

$$\Phi^+(A) = \{s \in S : \Phi(s) \subset A\} \quad \text{and} \quad \Phi^-(A) = \{s \in S : \Phi(s) \cap A \neq \emptyset\}. \tag{1.2}$$

One sees immediately that

$$\Phi^-(A) = S \setminus \Phi^+(Z \setminus A) \quad \text{and} \quad \Phi^+(A) = S \setminus \Phi^-(Z \setminus A).$$

Furthermore, if $\mathcal{I}$ is a set of indices and $B_i \subset Z$ for $i \in \mathcal{I}$, then

$$\Phi^-\Big(\bigcup_{i\in\mathcal{I}} B_i\Big) = \bigcup_{i\in\mathcal{I}} \Phi^-(B_i). \tag{1.3}$$

Since always $\Phi^+(A) \subset \Phi^-(A)$ for $A \subset Z$, sometimes $\Phi^+(A)$ and $\Phi^-(A)$ are denoted by $\Phi^s(A)$ and $\Phi^w(A)$ and called *strong* and *weak counterimages* of A, respectively. If Φ is treated as a function, then, as usual,

$$\Phi^{-1}(\mathcal{G}) = \{s \in S : \Phi(s) \in \mathcal{G}\} \quad \text{for } \mathcal{G} \subset \mathcal{P}_0(Z). \tag{1.4}$$

The *image* of a set $B \subset S$ under Φ is defined by

$$\Phi(B) = \bigcup_{b\in B} \Phi(b). \tag{1.5}$$

Any function $\phi : S \to Z$ such that $\phi(s) \in \Phi(s)$ for each $s \in S$ is called a *selection* of the multifunction $\Phi : S \rightsquigarrow Z$.

A function $f : S \to Z$ may be considered as a multifunction assigning to $s \in S$ the singleton $\{f(s)\}$. It is clear that in this case we have $f^+(A) = f^-(A) = f^{-1}(A)$ for $A \subset Z$.

If $(Z, \mathcal{T}(Z))$ is a topological space and $A \subset Z$, then we will use the notations $\mathrm{Int}(A)$, $\mathrm{Cl}(A)$ and $\mathrm{Fr}(A)$ for the interior, closure and boundary of A, respectively. Furthermore, we will denote by $\mathcal{B}(Z)$ the σ-field of Borel subsets of Z and by $\mathcal{F}_\sigma(Z)$ and $\mathcal{G}_\delta(Z)$ the first additive and multiplicative class, respectively, in the Borel hierarchy of subsets of the space $(Z, \mathcal{T}(Z))$. By a *Polish space* we mean a separable space metrizable by a complete metric. If $(Z, \mathcal{T}(Z))$ is metrizable and Z is a continuous image of a Polish space, then we will say that $(Z, \mathcal{T}(Z))$ is a *Suslin space*. We will write (for short) that Z itself is a Polish (resp. Suslin) space.

We also introduce the following notations:

$$\mathcal{C}(Z) = \{A \in \mathcal{P}_0(Z) : A \text{ is closed}\};$$
$$\mathcal{K}(Z) = \{A \in \mathcal{C}(Z) : A \text{ is compact}\};$$
$$\mathcal{C}_b(Z) = \{A \in \mathcal{C}(Z) : A \text{ is bounded}\}, \quad \text{whenever } (Z, \varrho) \text{ is a metric space};$$
$$\mathcal{C}_{bc}(Z) = \{A \in \mathcal{C}_b(Z) : A \text{ is convex}\} \quad \text{and} \quad \mathcal{K}_c(Z) = \{A \in \mathcal{K}(Z) : A \text{ is convex}\},$$

whenever $(Z, \|\cdot\|)$ is a real normed linear space.

If $z_0 \in Z$, then we will use $\mathcal{B}(z_0)$ to denote the neighbourhood filterbase of z_0. The *grill* of $\mathcal{B}(z_0)$ (see [6, p. 12]) will be denoted by $\mathcal{G}(z_0)$; it consists of all sets $A(z_0) \subset Z$ such that $A(z_0) \cap U(z_0) \neq \emptyset$ for each $U(z_0) \in \mathcal{B}(z_0)$, i.e., $z_0 \in \mathrm{Cl}(A(z_0))$.

If (Z, ϱ) is a metric or pseudometric space, $z_0 \in Z$ and $A \subset Z$, then, as usual, we will denote by $B(z_0, r)$ the open ball centred at z_0 with radius $r > 0$ and $B(A, r) = \{z \in Z : \varrho(z, A) < r\}$, where $\varrho(z, A) = \inf\{\varrho(z, y) : y \in A\}$. The topology on Z generated by the metric ϱ will be denoted by $\mathcal{T}_\varrho(Z)$.

If (Z, δ) is a hemimetric space (i.e., δ is a pseudometric which fails to be symmetric), then the open ball will be denoted as in the case of a metric or pseudometric. If interior points and open sets are defined in the usual way for hemimetric space (Z, δ), then the family of all open sets is a topology on the space Z.

2. Continuity of multifunctions

Various definitions of continuity of multifunctions are given in many papers. They all reduce to the usual continuity if a single valued function is considered. We now state two different definitions of continuity for multifunctions which we shall use in this monograph.

Let $(Z, \mathcal{T}(Z))$ be a topological space. The topology on Z allows us to define various topologies on $\mathcal{P}_0(Z)$ and each one yields a corresponding notion of continuity of a multifunction. Following Michael (see [83, Appendix, p. 179]), the *upper* (resp. *lower*) *semifinite topology* on $\mathcal{P}_0(Z)$ is the topology obtained by taking as a basis (resp. sub-basis) for the open sets all collections of the form $\mathcal{U} = \{A \in \mathcal{P}_0(Z) : A \subset G\}$ (resp. $\mathcal{L} = \{A \in \mathcal{P}_0(Z) : A \cap G \neq \emptyset\}$) with $G \in \mathcal{T}(Z)$; we denote it by $\mathcal{T}_{\mathcal{U}}$ (resp. $\mathcal{T}_{\mathcal{L}}$). The *finite* (or *Vietoris*) *topology* on $\mathcal{P}_0(Z)$ is the join of both these topologies and is denoted by $\mathcal{T}_V$.

If we try to adapt to multifunctions the following two equivalent definitions of continuity of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $x_0 \in \mathbb{R}$:

(i) $\forall U(f(x_0))\ \exists U(x_0) \in \mathcal{B}(x_0)\ U(x_0) \subset f^{-1}(U(f(x_0)))$,
(ii) $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$,

then we obtain two notions of continuity which are no longer equivalent. This unfortunate situation led to two concepts of semicontinuity.

Let $(S, \mathcal{T}(S))$ and $(Z, \mathcal{T}(Z))$ be topological spaces. We will call a multifunction $\Phi : S \rightsquigarrow Z$ *upper* (resp. *lower*) *semicontinuous* at a point $s_0 \in S$ if, for any open set $G \subset Z$ such that $\Phi(s_0) \subset G$ (resp. $\Phi(s_0) \cap G \neq \emptyset$), there exists a $U(s_0) \in \mathcal{B}(s_0)$ such that $U(s_0) \subset \Phi^+(G)$ (resp. $U(s_0) \subset \Phi^-(G)$); Φ is called *continuous* at $s_0 \in S$ if it is both upper and lower semicontinuous at s_0.

Φ is called continuous or upper (resp. lower) semicontinuous if it is continuous or upper (resp. lower) semicontinuous at each point $s \in S$.

Note that for a set $G \subset Z$, $\Phi^{-1}(\{A \in \mathcal{P}_0(Z) : A \subset G\}) = \Phi^+(G)$ and $\Phi^{-1}(\{A \in \mathcal{P}_0(Z) : A \cap G \neq \emptyset\}) = \Phi^-(G)$ (see (1.2) and (1.4)). Thus we can say that

(1.6) If $(S, \mathcal{T}(S))$ and $(Z, \mathcal{T}(Z))$ are topological spaces and $s_0 \in S$, then a multifunction $\Phi : S \rightsquigarrow Z$ is upper (resp. lower) semicontinuous at s_0 if and only if the function $\Phi : S \to (\mathcal{P}_0(Z), \mathcal{T}_{\mathcal{U}})$ (resp. $\Phi : S \to (\mathcal{P}_0(Z), \mathcal{T}_{\mathcal{L}})$) is continuous at s_0; Φ is continuous at s_0 if and only if the function $\Phi : S \to (\mathcal{P}_0(Z), \mathcal{T}_V)$ is continuous at s_0.

Note that the definition of continuity or semicontinuity of a multifunction is more handy than the condition (1.6), since we do not need to indicate the topology on $\mathcal{P}_0(Z)$ (the topology on Z is sufficient).

Evidently, in the case of a single valued function the upper semicontinuity and lower semicontinuity as well continuity coincide with the usual notion of continuity.

The next definition of semicontinuity of a multifunction is based on the Hausdorff metric extended to $\mathcal{P}_0(Z)$. If (Z, ϱ) is a metric space, we can introduce the topology on $\mathcal{P}_0(Z)$ generated by the hemimetric h_u defined by

$$h_u(A, B) = \sup\{\varrho(x, A) : x \in B\}, \tag{1.7}$$

called the *upper hemimetric topology* on $\mathcal{P}_0(Z)$, and denoted by $\mathcal{T}_{h_u}$. Dually, we can introduce the *lower hemimetric topology* $\mathcal{T}_{h_l}$ generated by the hemimetric h_l defined by

$$h_l(A,B) = \sup\{\varrho(x,B) : x \in A\}. \tag{1.8}$$

The function h on the product $\mathcal{P}_0(Z) \times \mathcal{P}_0(Z)$ given by

$$h(A,B) = \max\{h_u(A,B), h_l(A,B)\}$$

is a pseudometric on $\mathcal{P}_0(Z)$ and it generates the *Hausdorff topology* on $\mathcal{P}_0(Z)$ denoted by $\mathcal{T}_h$. Of course the space $(\mathcal{C}(Z), h)$ is a metric space. Note that

(1.9) (i) $\mathcal{T}_{h_u} \subset \mathcal{T}_{\mathcal{U}}$ and $\mathcal{T}_{\mathcal{L}} \subset \mathcal{T}_{h_l}$, and the converse inclusions are not true, in general [58, Proposition 4.2.1].

(ii) The topological spaces $(\mathcal{K}(Z), \mathcal{T}_V)$ and $(\mathcal{K}(Z), \mathcal{T}_h)$ are equivalent (see [63, p. 21]).

If $(S, \mathcal{T}(S))$ is a topological space and (Z, ϱ) a metric space, then a multifunction $\Phi : S \rightsquigarrow Z$ is called *hemi-upper* (*h-upper* for short) *semicontinuous* at a point $s_0 \in S$ if, for each $\varepsilon > 0$, there exists a $U(s_0) \in \mathcal{B}(s_0)$ such that $\Phi(s) \subset B(\Phi(s_0), \varepsilon)$ for all $s \in U(s_0)$.

Dually, Φ is called *hemi-lower* (*h-lower* for short) *semicontinuous* at a point $s_0 \in S$ if, for each $\varepsilon > 0$, there exists a $U(s_0) \in \mathcal{B}(s_0)$ such that $\Phi(s_0) \subset B(\Phi(s), \varepsilon)$ for all $s \in U(s_0)$.

Φ is called *hemi-continuous* (*h-continuous* for short) at $s_0 \in S$ if it is both h-upper and h-lower semicontinuous at s_0; Φ is called *h-continuous* if it is h-continuous at each $s \in S$.

Note that in the context of (1.7) and (1.8) we can say that

(1.10) If $(S, \mathcal{T}(S))$ is a topological space and (Z, ϱ) is a metric space, then a multifunction $\Phi : S \rightsquigarrow Z$ is h-upper semicontinuous at a point $s_0 \in S$ if and only if the function $\Phi : S \to (\mathcal{P}_0(Z), h_u)$ is continuous at s_0; that is, for each $\varepsilon > 0$, there exists a $U(s_0) \in \mathcal{B}(s_0)$ such that $h_u(\Phi(s), \Phi(s_0)) < \varepsilon$ for all $s \in U(s_0)$.
Φ is h-lower semicontinuous at $s_0 \in S$ if and only if the function $\Phi : S \to (\mathcal{P}_0(Z), h_l)$ is continuous at s_0; that is, for each $\varepsilon > 0$, there exists a $U(s_0) \in \mathcal{B}(s_0)$ such that $h_l(\Phi(s), \Phi(s_0)) < \varepsilon$ for all $s \in U(s_0)$.

As a consequence of (1.9), we have the following properties.

(1.11) Let $(S, \mathcal{T}(S))$ be a topological space, (Z, ϱ) a metric space and $\Phi : S \rightsquigarrow Z$ a multifunction.

(i) If Φ is upper semicontinuous, then it is h-upper semicontinuous.
(ii) If Φ is h-lower semicontinuous, then it is lower semicontinuous.
(iii) If Φ is compact valued, then its upper (resp. lower) semicontinuity and h-upper (resp. h-lower) semicontinuity are equivalent.

In cases (i) and (ii), the converses are not true.

The definition of equicontinuity of a family of real functions can be extended to multifunctions in the following way. Let $\{\Phi_i\}_{i \in \mathcal{I}}$ be a family of closed valued multifunctions

$\Phi_i : S \rightsquigarrow Z$, $i \in \mathcal{I}$, where $\mathcal{I}$ denotes a set of indices. The family $\{\Phi_i\}_{i\in\mathcal{I}}$ is called *h-lower* (resp. *h-upper*) *equicontinuous* at a point $s_0 \in S$ if, for each $\varepsilon > 0$, there exists an open neighbourhood $U(s_0)$ of s_0 such that $s \in U(s_0)$ implies $\Phi_i(s_0) \subset B(\Phi_i(s), \varepsilon)$ (resp. $(\Phi_i(s) \subset B(\Phi_i(s_0), \varepsilon))$ for each $i \in \mathcal{I}$.

The family $\{\Phi_i\}_{i\in\mathcal{I}}$ is called *h-equicontinuous* if it is both h-upper and h-lower equicontinuous at each $s \in S$.

There are several ways of defining convergence in $\mathcal{P}_0(Z)$ and in consequence its connections with continuity.

If (Z, ϱ) is a metric space, then a sequence $(\Phi_n)_{n\in\mathbb{N}}$ of closed valued multifunctions $\Phi_n : S \rightsquigarrow Z$ is called *converging* to a multifunction $\Phi : S \rightsquigarrow Z$ if for each $s \in S$ the sequence $(\Phi_n(s))_{n\in\mathbb{N}}$ converges to $\Phi(s)$ with respect to the Hausdorff metric h generated by ϱ. We will write $\Phi = h\text{-}\lim_{n\to\infty} \Phi_n$.

It is clear that

(1.12) If $s \in S$ and $\Phi(s) = h\text{-}\lim_{n\to\infty} \Phi_n(s)$ then $\varrho(z, \Phi(s)) = \lim_{n\to\infty} \varrho(z, \Phi_n(s))$ for each $z \in Z$.

Throughout the paper, convergence in the space $\mathcal{C}(Z)$ will be convergence with respect to the Hausdorff metric h.

The set valued notions of limits are rooted in the concepts of lower and upper limits of filtered families of sets (see [6, p. 125]).

Let $(S, \mathcal{T}(S))$ and $(Z, \mathcal{T}(Z))$ be topological spaces. Let $\Phi : S \rightsquigarrow Z$ and $s_0 \in S$. Then $\mathcal{R} = (\Phi(s) : s \in S, \mathcal{B}(s_0))$ forms a filtered family of sets [6, Example 3, p. 126]. The set of all limit points of $\mathcal{R}$ is called the *lower pseudo-limit* of Φ at s_0 and is denoted by $\text{p-}\liminf_{s\to s_0} \Phi(s)$. The set of all cluster points of $\mathcal{R}$ is called the *upper pseudo-limit* of Φ at s_0 and denoted by $\text{p-}\limsup_{s\to s_0} \Phi(s)$ (for the justification of "pseudo" see [6, p. 130]).

It is known [6, Theorems 1 and 1′, p. 127] that

(1.13) (i) $$\text{p-}\limsup_{s\to s_0} \Phi(s) = \bigcap_{U\in\mathcal{B}(s_0)} \mathrm{Cl}\Big(\bigcup_{s\in U} \Phi(s)\Big),$$

(ii) $$\text{p-}\liminf_{s\to s_0} \Phi(s) = \bigcap_{A\in\mathcal{G}(s_0)} \mathrm{Cl}\Big(\bigcup_{s\in A} \Phi(s)\Big).$$

Let $\mathcal{B}$ be a basis of $\mathcal{T}(S)$ and $s_0 \in S$. Let us replace the grill $\mathcal{G}(s_0)$ in (1.13)(ii) by the family

$$\mathcal{A}(s_0) = \{V \in \mathcal{B} : s_0 \in \mathrm{Cl}(V)\} \tag{1.14}$$

and denote the resulting operation by $\text{q-}\liminf_{s\to s_0} \Phi(s)$, i.e.,

$$\text{q-}\liminf_{s\to s_0} \Phi(s) = \bigcap_{V\in\mathcal{A}(s_0)} \mathrm{Cl}\Big(\bigcup_{s\in V} \Phi(s)\Big). \tag{1.15}$$

We have

(1.16) (i) $\text{p-}\liminf_{s\to s_0} \Phi(s) \subset \text{q-}\liminf_{s\to s_0} \Phi(s) \subset \text{p-}\limsup_{s\to s_0} \Phi(s)$.

(ii) If $(Z, \mathcal{T}(Z))$ is regular and Φ is closed valued, then

$$\text{p-}\liminf_{s\to s_0} \Phi(s) = \Phi(s_0) = \text{p-}\limsup_{s\to s_0} \Phi(s)$$

at each continuity point $s_0 \in S$ of Φ (see [76, Theorem 1.5]).

For a multifunction Φ we denote by $D(\Phi)$, $D_l(\Phi)$ and $D_u(\Phi)$ the sets of all its discontinuity, lower discontinuity and upper discontinuity points, respectively. It is evident that

$$\{s_0 \in S : \text{q-}\liminf_{s\to s_0} \Phi(s) \neq \text{p-}\limsup_{s\to s_0} \Phi(s)\} \subset D(\Phi). \tag{1.17}$$

The following lemma will be useful (cf. [64, p. 182]).

LEMMA 1.1. *Let $(S, \mathcal{T}(S))$ be a topological space and let $(Z, \mathcal{T}(Z))$ be a second countable topological space with a base $\mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$. Then for a multifunction $\Phi : S \rightsquigarrow Z$ we have*:

(i) $D_l(\Phi) = \bigcup_{n\in\mathbb{N}}(\Phi^-(B_n) \setminus \mathrm{Int}(\Phi^-(B_n)))$.
(ii) *Let $\mathcal{A} = \{(n_{k,1}, n_{k,2}, \ldots, n_{k,j(k)}) : n_{k,i} \in \mathbb{N}$ for $i = 1, \ldots, j(k)$ and $k \in \mathbb{N}\}$. If Φ is compact valued, then*

$$D_u(\Phi) = \bigcup_{k\in\mathbb{N}} (\Phi^+(V_k) \setminus \mathrm{Int}(\Phi^+(V_k))),$$

where $V_k = \bigcup\{B_{n_{k,i}} : i = 1, \ldots, j(k) \wedge B_{n_{k,i}} \in \mathcal{B}\}$ for $k \in \mathbb{N}$.

3. Measurability of multifunctions

Apart from semicontinuous multifunctions, measurable multifunctions will be very important in the following. Throughout this section we will denote by $(S, \mathcal{M}(S))$ (resp. $(S, \mathcal{M}(S), \mu)$) a measurable (resp. a measure) space (with a nonnegative measure μ on $\mathcal{M}(S)$). A set $N \subset S$ will be called *μ-negligible* if there is an $\mathcal{M}(S)$-measurable set A (i.e. $A \in \mathcal{M}(S)$) such that $N \subset A$ and $\mu(A) = 0$. The measure μ is *complete* if any μ-negligible set $N \subset S$ is $\mathcal{M}(S)$-measurable. The σ-field $\mathcal{M}(S)$ is *complete* if there is a complete measure μ on $\mathcal{M}(S)$.

If $\mathcal{A}$ is a family of sets, then we denote by $\mathbf{S}(\mathcal{A})$ the family of sets obtained from $\mathcal{A}$ by the Suslin operation.

(1.18) If $\mathcal{M}(S)$ is complete with respect to a σ-finite measure, then it is closed under the Suslin operation, i.e., $\mathbf{S}(\mathcal{M}(S)) = \mathcal{M}(S)$ (see [31, 6B(d), 1G and 1H(c)]).

By the completion of $\mathcal{M}(S)$ with respect to a measure μ on $\mathcal{M}(S)$ (*μ-completion* for short) we mean the σ-field $\mathcal{M}_\mu(S)$ generated by $\mathcal{M}(S)$ and the μ-negligible sets in S. The measure μ admits a unique extension to $\mathcal{M}_\mu(S)$. Thus the σ-field $\mathcal{M}_\mu(S)$ is complete.

If $(S, \mathcal{T}(S))$ is a topological space and $\mathcal{M}(S)$ is a σ-field of subsets of S, then a measure μ on $\mathcal{M}(S)$ is called *regular* (resp. *$\mathcal{G}_\delta$-regular*) if, for every $\varepsilon > 0$ and for each $A \in \mathcal{M}(S)$, there is a closed set $A_1 \subset S$ and an open set $A_2 \subset S$ (resp. $A_1 \in \mathcal{F}_\sigma(S)$ and $A_2 \in \mathcal{G}_\delta(S)$) such that $A_1 \subset A \subset A_2$ and for any $B \in \mathcal{M}(S)$ such that $B \subset A_2 \setminus A_1$ we have $\mu(B) < \varepsilon$ (resp. $\mu(B) = 0$). In the case $\mathcal{B}(S) \subset \mathcal{M}(S)$, the measure μ is regular (resp. $\mathcal{G}_\delta$-regular) if $\mu(A_2 \setminus A_1) < \varepsilon$ (resp. $\mu(A_2 \setminus A_1) = 0$).

If $(T, \mathcal{M}(T))$ is also a measurable space, then $\mathcal{M}(S) \otimes \mathcal{M}(T)$ will denote the *product σ-field* in $S \times T$, i.e., the σ-field of subsets of $S \times T$ generated by the family of sets $A \times B$, where $A \in \mathcal{M}(S)$ and $B \in \mathcal{M}(T)$. We shall denote by proj_S the projection map from $S \times T$ to S.

We will say that the pair $((S, \mathcal{M}(S)); (T, \mathcal{T}(T)))$, where $(T, \mathcal{T}(T))$ is a topological space, has the *projection property* if $\mathrm{proj}_S(A) \in \mathcal{M}(S)$ for each $A \in \mathcal{M}(S) \otimes \mathcal{B}(T)$.

If T is a Suslin space and $A \subset S \times T$, then $\mathrm{proj}_S(A) \in \mathbf{S}(\mathcal{M}(S))$ provided $A \in \mathbf{S}(\mathcal{M}(S) \otimes \mathcal{B}(T))$ (see [15]). Therefore, by (1.18) (cf. [20, Theorem 3.4] or [14, Theorem III.23]), we have the following assertion.

(1.19) If T is a Suslin space, then $((S, \mathcal{M}_\mu(S), \mu); T)$, where μ is σ-finite, has the projection property.

The theory of measurability of multifunctions, developed by numerous authors ([4], [12], [14], [20], [49], [52], [54], [65], [94], [99], and others), focuses almost exclusively on multifunctions defined on an abstract measurable space and with values in a metrizable space. We describe measurability of multifunctions without any metrizability assumption.

Let $(S, \mathcal{M}(S))$ be a measurable space, $(Z, \mathcal{T}(Z))$ a topological space, and $\Phi : S \rightsquigarrow Z$ a multifunction. Consider the following properties:

(a) $\Phi^+(G) \in \mathcal{M}(S)$ for each $G \in \mathcal{T}(Z)$;
(b) $\Phi^-(G) \in \mathcal{M}(S)$ for each $G \in \mathcal{T}(Z)$.

It is known (see [71, Proposition 1]) that

(1.20) (i) If $(Z, \mathcal{T}(Z))$ is perfect, then (a) implies (b).
(ii) If $(Z, \mathcal{T}(Z))$ is perfectly normal and Φ is compact valued, then also (b) implies (a).

The example of Kaniewski (see [113, Example 2.4, p. 865]) shows that the compactness of values of the multifunction Φ considered in (1.20)(ii) is essential.

It is natural to say that $\Phi : S \rightsquigarrow Z$ is $\mathcal{M}(S)$-*measurable* if condition (a) is satisfied, and *weakly* $\mathcal{M}(S)$-*measurable* if (b) holds (cf. [49, p. 54]).

It is evident that in the case of a single valued function $f : S \to Z$, the notions of measurability of f and weak measurability of f coincide with the usual notion of measurability of f, i.e., $f^{-1}(G) \in \mathcal{M}(S)$ for any $G \in \mathcal{T}(Z)$.

We can now rephrase property (1.20) as follows.

PROPOSITION 1.2. *If $(S, \mathcal{M}(S))$ is a measurable space, $(Z, \mathcal{T}(Z))$ a perfect topological space, and $\Phi : S \rightsquigarrow Z$ a multifunction, then*

(i) *$\mathcal{M}(S)$-measurability of Φ implies weak $\mathcal{M}(S)$-measurability of Φ.*
(ii) *If $(Z, \mathcal{T}(Z))$ is perfectly normal and Φ has compact values, then $\mathcal{M}(S)$-measurability of Φ and weak $\mathcal{M}(S)$-measurability of Φ are equivalent.*

Excellent sources of information on measurability properties of multifunctions with values in a metric space are the papers of Himmelberg [49] and Castaing and Valadier [14]. We now mention those properties which will be useful later on.

Let (Z, ϱ) be a metric space. For $z \in Z$ and $\Phi : S \rightsquigarrow Z$ we define the function $g_z : S \to \mathbb{R}$ by

$$g_z(s) = \varrho(z, \Phi(s)).$$

Consider the following properties:

(c) For each $z \in Z$ the function g_z is $\mathcal{M}(S)$-measurable;

(d) Φ admits a sequence $(\phi_n)_{n\in\mathbb{N}}$ of $\mathcal{M}(S)$-measurable selections such that $\Phi(s) = \mathrm{Cl}(\{\phi_n(s) : n \in \mathbb{N}\})$ for each $s \in S$ (the sequence $(\phi_n)_{n\in\mathbb{N}}$ is called a *Castaing representation* of Φ).

(e) $\mathrm{Gr}(\Phi) \in \mathcal{M}(S) \otimes \mathcal{B}(Z)$.

PROPOSITION 1.3. *If (Z, ϱ) is separable and $\Phi : S \rightsquigarrow Z$, then*

(i) *Weak $\mathcal{M}(S)$-measurability of Φ is equivalent to* (c) [49, Theorem 3.3].

(ii) *If Φ is complete valued, then weak $\mathcal{M}(S)$-measurability of Φ is equivalent to* (d) [14, Theorem III.9].

(iii) *If Φ is closed valued, then weak $\mathcal{M}(S)$-measurability of Φ implies* (e) [49, Theorem 3.3].

(iv) *If (Z, ϱ) is σ-compact (i.e., $Z = \bigcup_{n\in\mathbb{N}} Z_n$ and Z_n is compact for every $n \in \mathbb{N}$) and Φ is closed valued, then* (a) *and* (b) *are equivalent* [49, Theorem 3.5(ii)].

(v) *If $\mathcal{M}(S)$ is complete with respect to a σ-finite measure, (Z, ϱ) is complete and Φ is closed valued, then* (a)–(e) *are equivalent* [14, Theorem III.30].

(vi) *If Φ is compact valued, then* (a) *and* (b) *are each equivalent to $\mathcal{M}(S)$-measurability of the function $\Phi : S \to (\mathcal{K}(Z), \mathcal{T}_h)$, where h is the Hausdorff metric generated by ϱ* [14, Theorem III.1].

(vii) *If Z is a Polish space and Φ is closed valued, then Φ admits an $\mathcal{M}(S)$-measurable selection* [66].

The following proposition will be applied in the next chapter.

PROPOSITION 1.4 ([71, Proposition 2]). *Let $(S, \mathcal{M}(S))$ be a measurable space and let $(Z, \mathcal{T}(Z))$ be a regular second countable topological space. If $\Phi_1, \Phi_2 : S \rightsquigarrow Z$ are closed valued weakly $\mathcal{M}(S)$-measurable multifunctions, then*

$$\{s \in S : \Phi_1(s) \neq \Phi_2(s)\} \in \mathcal{M}(S).$$

The next proposition on the intersection of closed valued weakly measurable multifunctions will also be useful in the next chapter. The sufficient conditions known earlier involve some compactness assumptions either on Z or on the values of multifunctions.

PROPOSITION 1.5 ([71, Proposition 3]). *Let $(S, \mathcal{M}(S), \mu)$ be a measure space, where μ is σ-finite, and let Z be a Suslin space. Let $\Phi_n : S \rightsquigarrow Z$, for $n \in \mathbb{N}$, be a family of closed valued weakly $\mathcal{M}(S)$-measurable multifunctions such that $\bigcap_{n\in N} \Phi_n(s) \neq \emptyset$ for each $s \in S$. Then the multifunction $\Phi : S \rightsquigarrow Z$ given by*

$$\Phi(s) = \Big(\bigcap_{n\in\mathbb{N}} \Phi_n\Big)(s) = \bigcap_{n\in\mathbb{N}} \Phi_n(s)$$

is $\mathcal{M}_\mu(S)$-measurable.

The projection property of $((S, \mathcal{M}_\mu(S), \mu); Z)$ in the above proposition is essential. We note that the intersection of two weakly $\mathcal{M}(S)$-measurable multifunctions with closed values may not be weakly $\mathcal{M}(S)$-measurable (see [50, Example 2]).

Observe that, by (1.12) and Proposition 1.3(i), the following property is true.

(1.21) If (Z, ϱ) is a separable metric space and $(\Phi_n)_{\in \mathbb{N}}$ is a sequence of closed valued weakly $\mathcal{M}(S)$-measurable multifunctions $\Phi_n : S \rightsquigarrow Z$, $n \in \mathbb{N}$, converging to a multifunction $\Phi : S \rightsquigarrow Z$, then Φ is weakly $\mathcal{M}(S)$-measurable.

Similarly to the case of vector valued functions the strong measurability of multifunctions can be defined. Let $(S, \mathcal{M}(S), \mu)$ be a measurable space, where μ is complete, let $(Z, \|\cdot\|)$ be a reflexive real normed linear space, and let $\Phi : S \rightsquigarrow Z$ be a multifunction with $\Phi(s) \in \mathcal{C}_{bc}(Z)$. Then Φ is said to be *finitely-valued* if it is constant on each of a finite number of disjoint $\mathcal{M}(S)$-measurable sets E_i and equal to $\{\theta\}$ on $S \setminus \bigcup E_i$ (θ is the origin of Z); Φ is said to be a *simple multifunction* if it is finitely-valued and $\mu(\{s \in S : \|\Phi(s)\| > 0\}) < \infty$, where $\|\Phi(S)\| = h(\Phi(s), \{\theta\})$ (h is the Hausdorff metric generated by the norm).

A multifunction Φ is called *countable-valued* if it assumes at most a countable set of values in $\mathcal{C}_{bc}(Z)$, assuming each value different from $\{\theta\}$ on an $\mathcal{M}(S)$-measurable subset of S.

A multifunction $\Phi : S \rightsquigarrow Z$ is called *strongly $\mathcal{M}(S)$-measurable* if there is a sequence of countable-valued multifunctions $(\Phi_n)_{n \in \mathbb{N}}$ such that

$$h\text{-}\lim_{n \to \infty} \Phi_n(s) = \Phi(s)$$

for μ-almost every $s \in S$. If $\mu(S) < \infty$, then we may replace "countable-valued" by "simple".

If $\Phi : S \rightsquigarrow Z$ is strongly $\mathcal{M}(S)$-measurable, then it is weakly $\mathcal{M}(S)$-measurable, but the converse is not true (see [21, Example 3.1, p. 23]). Furthermore (see [21, Proposition 3.3]),

(1.22) If $S = [a, b] \subset \mathbb{R}$, Z is a separable Banach space and $\Phi : S \rightsquigarrow Z$ has values in $\mathcal{K}(Z)$, then $\mathcal{L}(\mathbb{R})$-measurability of Φ and strong $\mathcal{L}(\mathbb{R})$-measurability of Φ are equivalent.

4. Approximate continuity of multifunctions

The notion of approximately continuous function, essential for the concept of density topology, has been studied for real functions of real variable ([22], [35], [34], [72]) and then in various abstract spaces (see [41], [73], [74], [75], [92], [103]). In this section we introduce some concepts of approximate continuity of a multifunction and give some properties of approximately continuous multifunctions which will be essential for the considerations of the next chapters.

Throughout this section we assume that $(S, d, \mathcal{M}(S), \mu)$ is a measure metric space with metric d, with a σ-finite complete and $\mathcal{G}_\delta$-regular measure μ defined on a σ-field $\mathcal{M}(S)$ containing the Borel sets; μ^* will denote the outer measure generated by μ, i.e., $\mu^*(A) = \inf\{\mu(B) : A \subset B \wedge B \in \mathcal{M}(S)\}$ for a set $A \subset S$.

(1.23) Let $\mathcal{F} \subset \mathcal{M}(S)$ be a family of sets with nonempty interiors of positive and finite measure μ, the boundaries of which are μ-negligible. Let $\{I_n\}_{n \in \mathbb{N}} \subset \mathcal{F}$

and $s \in S$. We write $I_n \to s$ if $s \in \mathrm{Int}(I_n)$ for each $n \in \mathbb{N}$ and the diameter of I_n tends to zero as $n \to \infty$.

We assume that for every $s \in S$, there exists a sequence $(I_n)_{n\in\mathbb{N}}$ of sets from $\mathcal{F}$ such that $I_n \to s$.

The pair $(\mathcal{F}, \to)$ then forms *a differentiation basis* for the space $(S, d, \mathcal{M}(S), \mu)$ in Bruckner's terminology [9, p. 30].

Let $A \subset S$ and $s \in S$. The *upper outer density* of the set A at the point s with respect to $\mathcal{F}$ is equal to

$$\limsup_{I_n \to s} \frac{\mu^*(A \cap I_n)}{\mu(I_n)}.$$

Replacing lim sup by lim inf we obtain the *lower outer density* of A at $s \in S$ with respect to $\mathcal{F}$. These densities will be denoted by $D_u^*(A, s)$ and $D_l^*(A, s)$, respectively. If they are equal, their common value will be called the *outer density of* A at s with respect to $\mathcal{F}$ and denoted by $D^*(A, s)$. If $A \in \mathcal{M}(S)$, then the outer density of A at $s \in S$ with respect to $\mathcal{F}$ will be called the *density* of A at s with respect to $\mathcal{F}$ and denoted with no asterisk.

A point $s \in S$ will be called a *density point* of a set $A \subset S$ *with respect to* $\mathcal{F}$ if there exists a $B \in \mathcal{M}(S)$ such that $B \subset A$ and the density of B at s with respect to $\mathcal{F}$ is equal to 1. We will write $D(A, s) = 1$.

We will assume that

(1.24) $\mathcal{F}$ has the *density property*, i.e., $\mu(\{s \in A : D_l^*(A, s) < 1\}) = 0$ for every $A \subset S$.

By the density property of $\mathcal{F}$, it is clear that

(1.25) If μ-almost every point of $A \subset S$ is a density point of A with respect to $\mathcal{F}$, then A is $\mathcal{M}(S)$-measurable.

An $\mathcal{M}(S)$-measurable set will be called *homogeneous with respect to* $\mathcal{F}$ if its density with respect to $\mathcal{F}$ is 1 at each of its points. The space S can be topologized by taking the homogeneous sets with respect to $\mathcal{F}$ as open sets (see [68, p. 251]). This topology will be denoted by $\mathcal{T}_D(S)$ (cf. [109] and [82]). If $A \subset S$, then $\mathcal{T}_D$-$\mathrm{Int}(A)$ will denote the interior of A relative to $\mathcal{T}_D(S)$. Note that $\mathcal{T}_D(S)$ is finer than $\mathcal{T}_d(S)$.

Now we can generalize the notion of approximate continuity to the case of multifunctions. Let $(Z, \mathcal{T}(Z))$ be a topological space.

Definition 1.6. A multifunction $\Phi : S \rightsquigarrow Z$ is called *approximately lower* (resp. *upper*) *semicontinuous at a point* $s_0 \in S$ *with respect to* $\mathcal{F}$ if there is a set $E \in \mathcal{M}(S)$ including s_0 such that $D(E, s_0) = 1$ and the restriction $\Phi|_E$ is lower (resp. upper) semicontinuous at s_0. If Φ is approximately lower (resp. upper) semicontinuous at each point $s \in S$ with respect to $\mathcal{F}$, then it is called *approximately lower* (resp. *upper*) *semicontinuous with respect to* $\mathcal{F}$; Φ is called *approximately continuous with respect to* $\mathcal{F}$ if it is both approximately lower semicontinuous and approximately upper semicontinuous with respect to $\mathcal{F}$.

Remark 1.7. If $S = \mathbb{R}$ and $\mathcal{M}(S) = \mathcal{L}(\mathbb{R})$, then the multifunction Φ will be simply called *approximately lower* (resp. *upper*) *semicontinuous* or *approximately continuous*.

If (Z, ϱ) is a metric space and $\Phi|_E$, in the above definition, is h-lower (resp. h-upper) semicontinuous at $s_0 \in S$ with respect to $\mathcal{F}$, then Φ will be called *approximately h-lower* (resp. *h-upper*) *semicontinuous* at s_0 with respect to $\mathcal{F}$.

Φ is called *approximately h-continuous with respect to $\mathcal{F}$ at s_0* if it is both approximately h-lower semicontinuous and approximately h-upper semicontinuous with respect to $\mathcal{F}$ at s_0; Φ is called *approximately h-continuous with respect to $\mathcal{F}$* if it is approximately h-continuous at every $s \in S$ with respect to $\mathcal{F}$.

It was observed in [35] that a real function of a real variable continuous relative to the density topology in the domain and the usual topology in the range, turns out to be exactly an approximately continuous function, which is also true for multifunctions.

PROPOSITION 1.8. *Let $\Phi : S \rightsquigarrow Z$ be a multifunction and $s_0 \in S$. Then Φ is approximately lower* (*resp. upper*) *semicontinuous at $s_0 \in S$ with respect to $\mathcal{F}$ if and only if Φ is lower* (*resp. upper*) *semicontinuous at $s_0 \in S$ relative to the topology $\mathcal{T}_D(S)$.*

Proof. We only give the proof of the "lower" case; the "upper" case is similar.

To prove sufficiency, let $G \in \mathcal{T}(Z)$ and $\Phi(s_0) \cap G \neq \emptyset$. By the approximate lower semicontinuity of Φ at s_0 with respect to $\mathcal{F}$, there exists an $E \in \mathcal{M}(S)$ such that $s_0 \in E$, $D(E, s_0) = 1$ and $\Phi|_E$ is lower semicontinuous at s_0. Therefore there exists a $U \in \mathcal{B}(s_0)$ such that $E \cap U \subset \Phi^-(G)$. Taking $V = \mathcal{T}_D\text{-Int}(E \cap U)$ we have $V \in \mathcal{T}_D(S)$, $s_0 \in V$ and $V \subset E \cap U \subset \Phi^-(G)$.

The necessity is a straightforward consequence of the lower semicontinuity of Φ at s_0 relative to $\mathcal{T}_D(S)$. ■

Note that if $\Phi : S \rightsquigarrow Z$ is approximately lower (resp. upper) semicontinuous at $s_0 \in S$ with respect to $\mathcal{F}$ and $G \in \mathcal{T}(Z)$ with $s_0 \in \Phi^-(G)$ (resp. $s_0 \in \Phi^+(G)$), then $D(\Phi^-(G), s_0) = 1$ (resp. $D(\Phi^+(G), s_0) = 1$), and hence, by (1.25), we have the following proposition (cf. [69, Theorem 2]).

PROPOSITION 1.9. *If a multifunction $\Phi : S \rightsquigarrow Z$ is μ-almost everywhere approximately lower* (*resp. upper*) *semicontinuous with respect to $\mathcal{F}$, then it is weakly $\mathcal{M}(S)$-measurable* (*resp. $\mathcal{M}(S)$-measurable*).

REMARK 1.10. Let (Z, ϱ) be a metric space.

- (i) If a multifunction $\Phi : S \rightsquigarrow Z$ is approximately h-lower semicontinuous with respect to $\mathcal{F}$, then it is weakly $\mathcal{M}(S)$-measurable, by (1.11)(ii) and Proposition 1.9.
- (ii) If a compact valued multifunction $\Phi : S \rightsquigarrow Z$ is approximately h-upper semicontinuous, then it is $\mathcal{M}(S)$-measurable, by (1.11)(iii) and Proposition 1.9.

DEFINITION 1.11. Let (Z, ϱ) be a metric space, let $\{\Phi_i\}_{i \in \mathcal{I}}$ be a family of closed valued multifunctions $\Phi_i : S \rightsquigarrow Z$ for $i \in \mathcal{I}$ (where $\mathcal{I}$ denotes a set of indices), and let $s \in S$. The family $\{\Phi_i\}_{i \in \mathcal{I}}$ is said to be *approximately h-lower* (resp. *h-upper*) *equicontinuous at $s \in S$ with respect to $\mathcal{F}$* if there exists a set $E(s) \in \mathcal{M}(S)$, including s, such that $D(E(s), s) = 1$ and the family $\{\Phi_i|_{E(s)}\}_{i \in \mathcal{I}}$ is h-lower (resp. h-upper) equicontinuous at $s \in S$; $\{\Phi_i\}_{i \in \mathcal{I}}$ is called *approximately h-lower* (resp. *h-upper*) *equicontinuous with respect to $\mathcal{F}$* if it is approximately h-lower (resp. h-upper) equicontinuous with respect to $\mathcal{F}$ at every $s \in S$.

The family $\{\Phi_i\}_{i\in\mathcal{I}}$ is called *approximately h-equicontinuous with respect to* $\mathcal{F}$ if it is simultaneously approximately h-lower and approximately h-upper equicontinuous with respect to $\mathcal{F}$.

5. Strong quasi-continuity of multifunctions

The quasi-continuity introduced by Kempisty [55] for real functions has been intensively studied. For multifunctions this notion was introduced by Popa [95] and widely considered by many authors, particularly by Neubrunn [86], Ewert [26], [28], and Lipski [27].

From now on let $(S,\mathcal{T}(S))$ and $(Z,\mathcal{T}(Z))$ be topological spaces. Following Neubrunn [86] we say that a multifunction $\Phi : S \rightsquigarrow Z$ is *lower* (resp. *upper*) *quasi-continuous at a point* $s_0 \in S$ if, for each set $G \in \mathcal{T}(Z)$ such that $s_0 \in \Phi^-(G)$ (resp. $s_0 \in \Phi^+(G)$) and for any $U \in \mathcal{B}(s_0)$, there exists a nonempty open set $V \subset U$ such that $V \subset \Phi^-(G)$ (resp. $V \subset \Phi^+(G)$); Φ is said to be *lower* (resp. *upper*) *quasi-continuous* if it is lower (resp. upper) quasi-continuous at each $s \in S$.

Note that for a single valued function the notions of lower quasi-continuity and upper quasi-continuity coincide with quasi-continuity.

A multifunction $\Phi : S \rightsquigarrow Z$ is said to be *quasi-continuous* at a point $s_0 \in S$ if, for arbitrary sets $G \in \mathcal{T}(Z)$ and $H \in \mathcal{T}(Z)$ such that $s_0 \in \Phi^-(G) \cap \Phi^+(H)$ and for every $U \in \mathcal{B}(s_0)$, there exists a nonempty open set $V \subset U$ such that $V \subset \Phi^-(G) \cap \Phi^+(H)$.

It is evident that a quasi-continuous multifunction is both lower quasi-continuous and upper quasi-continuous. The converse is not true (see [85, Example 1.2.7]).

As we know, a multifunction $\Phi : S \rightsquigarrow Z$ is continuous (resp. lower or upper semicontinuous) if and only if it is continuous as a single valued function from S to $\mathcal{P}_0(Z)$ with the finite topology (resp. lower or upper semifinite topology). For quasi-continuity the situation is different (see [85, 1.3.4]).

A set $A \subset S$ is said to be *quasi-open* if there is an open set O such that $O \subset A \subset \mathrm{Cl}(O)$ [77].

It is known (see [85, 1.2.5]) that

(1.26) A multifunction $\Phi : S \rightsquigarrow Z$ is lower (resp. upper) quasi-continuous if and only if for any $G \in \mathcal{T}(Z)$ the set $\Phi^-(G)$ (resp. $\Phi^+(G)$) is quasi-open.

Upper quasi-continuity of a multifunction can be characterized in terms of continuous restrictions. More precisely (see [87, Theorem 1]):

(1.27) If $(S,\mathcal{T}(S))$ is a first countable Hausdorff space, $(Z,\mathcal{T}(Z))$ a second countable space, and $\Phi : S \rightsquigarrow Z$ a compact valued multifunction, then Φ is upper quasi-continuous at a point $s_0 \in S$ if and only if there is a quasi-open set A containing s_0 such that $\Phi|_A$ is upper semicontinuous at s_0.

It may be shown that an analogous characterization of lower quasi-continuity is not possible [87, Example 4].

The following proposition will be useful in the next chapter.

Proposition 1.12. *Let* $(S,\mathcal{T}(S))$ *and* $(Z,\mathcal{T}(Z))$ *be topological spaces.*

(i) *If a multifunction* $\Phi : S \rightsquigarrow Z$ *is lower quasi-continuous at a point* $s_0 \in S$, *then* $\Phi(s_0) \subset \text{p-lim sup}_{s\to s_0} \Phi(s)$.

(ii) *If* $(S,\mathcal{T}(S))$ *is first countable and* $(Z,\mathcal{T}(Z))$ *is regular second countable, and if a multifunction* $\Phi : S \rightsquigarrow Z$ *is compact valued upper quasi-continuous at a point* $s_0 \in S$, *then* $\text{q-lim inf}_{s\to s_0} \Phi(s) \subset \Phi(s_0)$.

Proof. (i) Suppose that $z \in \Phi(s_0)$ and $U \in \mathcal{B}(s_0)$. Fix $G \in \mathcal{B}(z)$. By the lower quasi-continuity of Φ at s_0, for the sets G and U there is a nonempty open set $V \subset U$ such that $V \subset \Phi^-(G)$. Therefore, there is an $s \in V$ with $\Phi(s)\cap G \neq \emptyset$, i.e., $z \in \mathrm{Cl}(\bigcup_{s\in U} \Phi(s))$, and finally $z \in \bigcap_{U\in\mathcal{B}(s_0)} \mathrm{Cl}(\bigcup_{s\in U} \Phi(s))$, which finishes the proof of (i) (see (1.13)(i)).

(ii) Now suppose that $z \notin \Phi(s_0)$. Since the set $\Phi(s_0)$ is closed and the space Z is regular, there are $V \in \mathcal{B}(z)$ and $G \in \mathcal{T}(Z)$ such that $\Phi(s_0) \subset G$ and $G \cap V = \emptyset$. By the upper quasi-continuity of Φ at s_0, in view of (1.27), there is a quasi-open set A containing s_0 such that $\Phi|_A$ is upper semicontinuous at s_0. Thus, there exists a $U \in \mathcal{B}(s_0)$ such that $\Phi(s) \subset G$ for all $s \in U \cap A$. Since $B = U \cap A$ is quasi-open ([77]) and $s_0 \in B$, there is a set $O \in \mathcal{A}(s_0)$ such that $\Phi(s)\cap V = \emptyset$ for each $s \in O$. Thus $z \notin \bigcap_{O\in\mathcal{A}(s_0)} \mathrm{Cl}(\bigcup_{s\in O} \Phi(s))$, and the proof of (ii) is finished (see (1.15)). ∎

From now on let $(S, d, \mathcal{M}(S), \mu)$ be a measure metric space with a differentiation basis $(\mathcal{F}, \to)$ with the density property (see (1.24)), and let $(Z,\mathcal{T}(Z))$ be a topological space.

Definition 1.13. A multifunction $\Phi : S \rightsquigarrow Z$ is called *strongly lower* (resp. *upper*) *quasi-continuous at a point* $s_0 \in S$ *with respect to* $\mathcal{F}$ if, for each $G \in \mathcal{T}(Z)$ such that $s_0 \in \Phi^-(G)$ (resp. $s_0 \in \Phi^+(G)$) and for each $U \in \mathcal{T}_D(S)$ including s_0, there exists a nonempty open set $V \subset S$ such that $V\cap U \neq \emptyset$ and $V \cap U \subset \Phi^-(G)$ (resp. $V \cap U \subset \Phi^+(G)$); Φ is said to be *strongly lower* (resp. *upper*) *quasi-continuous with respect to* $\mathcal{F}$ if it is strongly lower (resp. upper) quasi-continuous with respect to $\mathcal{F}$ at each $s \in S$.

Observe that replacing, in the above definition, the density topology by the topology generated by the metric d, we obtain the notion of lower (resp. upper) quasi-continuity of Φ. Since $\mathcal{T}_d(S)$-open sets are $\mathcal{T}_D(S)$-open, we can say that

(1.28) If a multifunction $\Phi : S \rightsquigarrow Z$ is strongly lower (resp. upper) quasi-continuous with respect to $\mathcal{F}$, then it is lower (resp. upper) quasi-continuous. The converse is not true.

By analogy with the definition of quasi-continuity we define the strong quasi-continuity of a multifunction.

Definition 1.14. A multifunction $\Phi : S \rightsquigarrow Z$ is said to be *strongly quasi-continuous with respect to* $\mathcal{F}$ *at a point* $s_0 \in S$ if, for any $G \in \mathcal{T}(Z)$ and $H \in \mathcal{T}(Z)$ such that $s_0 \in \Phi^-(G) \cap \Phi^+(H)$ and for each $U \in \mathcal{T}_D(S)$ containing s_0, there exists a nonempty open set $V \subset S$ such that $V \cap U \neq \emptyset$ and $V \cap U \subset \Phi^-(G) \cap \Phi^+(H)$.

It is evident that a multifunction $\Phi : S \rightsquigarrow Z$ which is strongly quasi-continuous with respect to $\mathcal{F}$ is quasi-continuous. Furthermore, if Φ is strongly quasi-continuous with respect to $\mathcal{F}$, then it is both strongly lower and strongly upper quasi-continuous with respect to $\mathcal{F}$.

Some connections between the quasi-continuity and the Denjoy property of real functions were considered by Šalát [104]. We now introduce more general properties for multifunctions.

Definition 1.15. A multifunction $\Phi : S \rightsquigarrow Z$ has the D^- (resp. D^+) *property* if for each $G \in \mathcal{T}(Z)$ and each nonempty open set $U \subset S$, the set $U \cap \Phi^-(G)$ (resp. $U \cap \Phi^+(G)$) is either empty or $\mu^*(U \cap \Phi^-(G)) > 0$ (resp. $\mu^*(U \cap \Phi^+(G)) > 0$).

Proposition 1.16. *If a multifunction $\Phi : S \rightsquigarrow Z$ is lower* (*resp. upper*) *quasi-continuous, then Φ has the D^- (resp. D^+) property.*

Proof. Let $G \in \mathcal{T}(Z)$ and let $U \subset S$ be open. By the lower (resp. upper) quasi-continuity of Φ, the set $\Phi^-(G)$ (resp. $\Phi^+(G)$) is quasi-open (see (1.26)). Then $U \cap \Phi^-(G)$ (resp. $U \cap \Phi^+(G)$) is either empty or its interior is nonempty, i.e., $\mu^*(U \cap \Phi^-(G)) > 0$ (resp. $\mu^*(U \cap \Phi^+(G)) > 0$). ∎

Proposition 1.17. *If the space $(Z, \mathcal{T}(Z))$ is regular and second countable*, *and a multifunction $\Phi : S \rightsquigarrow Z$ is strongly lower quasi-continuous with respect to $\mathcal{F}$ and has the D^+ property*, *then $\mu(D_l(\Phi)) = 0$.*

Proof. We first prove that

(1) If $G \in \mathcal{T}(Z)$ and $s \in \Phi^-(G)$, then $D_u(\mathrm{Int}(\Phi^-(G)), s)) > 0$.

Suppose, on the contrary, that there is a $G \in \mathcal{T}(Z)$ with $s \in \Phi^-(G)$ and $D_u(\mathrm{Int}(\Phi^-(G), s)) = 0$. Let $A = S \setminus \Phi^-(G) = \Phi^+(Z \setminus G)$. Then $D_l(\mathrm{Cl}(A), s) = 1 = D(\mathrm{Cl}(A), s)$. We can assume that $A \neq \emptyset$. Since $s \in \Phi^-(G)$, there is a $z \in \Phi(s) \cap G$. By the regularity of Z, there is an open set V including z such that $\mathrm{Cl}(V) \subset G$. Then $s \in \Phi^-(V)$. Let $W = S \setminus \Phi^-(\mathrm{Cl}(V)) = \Phi^+(Z \setminus \mathrm{Cl}(V))$. Then $W \neq \emptyset$, since $A \neq \emptyset$ and $A \subset W$. Therefore, by the D^+ property of Φ, $\mu^*(W) > 0$. Since $\mathrm{Cl}(A) \subset \mathrm{Cl}(W)$ and $D(\mathrm{Cl}(A), s) = 1$, it follows that $D(\mathrm{Cl}(W), s) = 1$. Let $B = \mathcal{T}_D\text{-}\mathrm{Int}(\mathrm{Cl}(W)) \cup \{s\}$. Then $s \in B \in \mathcal{T}_D(S)$. Since Φ is strongly lower quasi-continuous at s with respect to $\mathcal{F}$, for the sets V and B there is a nonempty open set $U \subset S$ such that

(2) $U \cap B \neq \emptyset$ and $U \cap B \subset \Phi^-(V)$.

On the other hand, however, $U \cap B \cap W \neq \emptyset$, i.e., $(U \cap B) \cap (S \setminus \Phi^-(\mathrm{Cl}(V)) \neq \emptyset$, which contradicts (2), i.e., (1) is proved.

Now we prove that $\mu(D_l(\Phi)) = 0$. Suppose, on the contrary, that $\mu^*(D_l(\Phi)) > 0$. Let $\mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$ be a base of $\mathcal{T}(Z)$. Then, by Lemma 1.1(i), there is an $n \in \mathbb{N}$ such that $\mu^*(\Phi^-(B_n) \setminus \mathrm{Int}(\Phi^-(B_n))) > 0$. Let $C = \Phi^-(B_n) \setminus \mathrm{Int}(\Phi^-(B_n))$ and $V = \mathcal{T}_D\text{-}\mathrm{Int}(C)$. Then V is $\mathcal{M}(S)$-measurable and $V \in \mathcal{T}_D(S)$. If $s \in C$, then $s \in \Phi^-(B_n)$, and so $D_u(\mathrm{Int}(\Phi^-(B_n)), s) > 0$, by (1). Since $D(C, s) = 1$, it follows that $C \cap \mathrm{Int}(\Phi^-(B_n)) \neq \emptyset$, which is impossible. ∎

A similar proof works for a dual proposition.

PROPOSITION 1.18. *Let the space $(Z, \mathcal{T}(Z))$ be second countable and normal. If a multifunction $\Phi : S \rightsquigarrow Z$ is compact valued strongly upper quasi-continuous with respect to $\mathcal{F}$ and it has the D^- property, then $\mu(D_u(\Phi)) = 0$.*

By (1.28), Propositions 1.16, 1.17 and 1.18, we have the following proposition (cf. [43, Corollary 3]).

PROPOSITION 1.19. *If the space $(Z, \mathcal{T}(Z))$ is second countable and normal, and if a multifunction $\Phi : S \rightsquigarrow Z$ is compact valued strongly lower quasi-continuous and strongly upper quasi-continuous with respect to $\mathcal{F}$, then Φ is μ-almost everywhere continuous.*

By Propositions 1.19 and 1.9, we have the following corollary.

COROLLARY 1.20. *If the space $(Z, \mathcal{T}(Z))$ is second countable and normal, and if $\Phi : S \rightsquigarrow Z$ is a compact valued multifunction strongly lower quasi-continuous and strongly upper quasi-continuous with respect to $\mathcal{F}$, then Φ is $\mathcal{M}(S)$-measurable.*

REMARK 1.21. It is known that there is a quasi-continuous function $f : [0, 1] \to \mathbb{R}$ which is not Lebesgue measurable [79, (x), p. 49]. So, if we suppose that the multifunction Φ considered in the above corollary is both lower quasi-continuous and upper quasi-continuous, then Φ need not be $\mathcal{M}(S)$-measurable.

6. Derivative multifunctions

The concept of differentiability for multifunctions has been considered by many authors from different points of view (see [5], [19], [45], [53], [81], and others).

Banks and Jacobs reduce differentiability of multifunctions to differentiability of functions in linear normed spaces by the Rådström embedding theorem. Another idea is given by Hukuhara [53]. In this case differentiability of a multifunction at a point, roughly speaking, means the existence of a set which is a limit of a difference quotient.

In this section the notion of differentiability is developed by taking advantage of an idea used by Hukuhara to give a definition of differentiability for a resonably wide class of multifunctions. For this purpose we give a more general definition of differences of sets than that given by Hukuhara. Furthermore, the notion of a derivative multifunction is introduced. In order to get this we use the notion of the integral of a multifunction given by Banks and Jacobs in [5].

Throughout the section, unless otherwise stated, $(Z, \|\cdot\|)$ will denote a real normed linear space with metric ϱ generated by the norm and θ will denote the origin of Z. The symbol $\mathrm{co}(A)$ will denote the convex hull of a set $A \subset Z$.

If $A \subset Z$, $B \subset Z$ and $\lambda \in \mathbb{R}$ then, as usual,

$$A + B = \{a + b : a \in A \wedge b \in B\} \quad \text{and} \quad \lambda A = \{\lambda\, a : a \in A\}.$$

(1.29) The following properties hold:

(i) If A and B are convex, and $\alpha, \beta \geq 0$, then $(\alpha + \beta)A = \alpha A + \beta A$.

(ii) If A and B are closed and convex subsets of Z and $C \subset Z$ is bounded, then $A + C = B + C$ implies $A = B$ [97, Lemma 2].

(iii) If $A_i \in \mathcal{C}_b(Z)$ and $B_i \in \mathcal{C}_b(Z)$ for $i = 1, 2$, then $h(A_1 + A_2, B_1 + B_2) \le h(A_1, B_1) + h(A_2, B_2)$ [19, Lemma 2.2(ii)], where h is the Hausdorff metric generated by the metric ϱ.

(iv) If $(Z, \|\cdot\|)$ is reflexive, $A \in \mathcal{C}_{bc}(Z)$ and $B \in \mathcal{C}_{bc}(Z)$, then $A + B \in \mathcal{C}_{bc}(Z)$ [97, Theorem 2].

(v) If $(Z, \|\cdot\|)$ is reflexive and $A, B, C \in \mathcal{C}_{bc}(Z)$, then $h(A, B) = h(A+C, B+C)$ [97, Lemma 3].

In the results that follow, the requirement that $(Z, \|\cdot\|)$ be reflexive can be replaced by the assumption that $(Z, \|\cdot\|)$ is a Banach space if we agree to deal only with the subcollection $\mathcal{K}_c(Z)$.

If (Z, ϱ) is complete, then $(\mathcal{C}_b(Z), h)$ is also complete (see [62, p. 314]). Therefore Price's inequality [96, (2.9), p. 4]

$$h(\mathrm{co}(A), \mathrm{co}(B)) \le h(A, B)$$

implies that

(1.30) If (Z, ϱ) is complete, then a Cauchy sequence in $\mathcal{C}_{bc}(Z)$ must converge to an element of $\mathcal{C}_{bc}(Z)$.

Now suppose that $(Z, \|\cdot\|)$ is reflexive.

Definition 1.22. Let $A, B \in \mathcal{C}_{bc}(Z)$. We will say the *difference* $A \ominus B$ *is defined* if there exists a set $C \in \mathcal{C}_{bc}(Z)$ such that either $A = B + C$ or $B = A - C$, and we define $A \ominus B$ to be the set C.

The difference $A \ominus B$ is uniquely determined.

Example 1.23. (a) Let $P \in \mathcal{C}_{bc}(Z)$, $A = \alpha P$ and $B = \beta P$, where $\alpha \ge 0$ and $\beta \ge 0$. Put $C = (\alpha - \beta)P$. Then, by (1.29)(i), $A = B + C$ or $B = A - C$ depending on whether $\alpha \ge \beta$ or $\alpha < \beta$. Therefore $A \ominus B$ exists and is equal to C.

(b) If $Z = \mathbb{R}$, $A = [a, x] \subset Z$ and $B = [b, y] \subset Z$, then $A \ominus B$ exists and

$$A \ominus B = [\min\{a - b, x - y\}, \max\{a - b, x - y\}].$$

(c) Let $A = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1 \text{ and } 0 \le y \le 1 - x\}$ and $B = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1 \text{ and } 0 \le y \le \frac{1}{2}(1 - x)\}$. Then $A \ominus B$ does not exist.

Indeed, suppose that there exists $C \in \mathcal{C}_{bc}(\mathbb{R}^2)$ such that $A = B + C$. Since $(0, 1) \in A$, there exist $(a, b) \in B$ and $(c, d) \in C$ such that $(0, 1) = (a + c, b + d)$, where $a \ge 0$. Then $c = -a$ and $d = 1 - b$. On the other hand, $(0, 0) \in B$. Therefore $(0, 0) + (c, d) = (-a, 1 - b) \in A$ and $-a \ge 0$. Hence $a = 0$. Since $(c, d) = (0, 1 - b) \in C$ and $(1, 0) \in B$, we have $(1, 0) + (0, 1 - b) \in A$ and $b = 1$. Therefore, $(a, b) = (0, 1) \notin B$, which is a contradiction.

Now suppose that there exists $C \in \mathcal{C}_{bc}(\mathbb{R}^2)$ such that $B = A - C$. Let $z \in C$. We observe that for every $x \in A$, $x - z \in A - C = B$. Hence, $A - z \subset B$, i.e., some translation of A is contained in B, which is of course impossible.

Remark 1.24. In each case of Example 1, with $Z = \mathbb{R}^n$ in (a), Hukuhara's differences of the relevant sets do not exist, since Hukuhara's difference $A \overset{H}{-} B$ of $A, B \in \mathcal{K}_c(Z)$ exists only if $\mathrm{diam}(A) \ge \mathrm{diam}(B)$.

Let $A, B \in \mathcal{C}_{bc}(Z)$. We write $B \subset_t A$ if, for each $a \in \mathrm{Fr}(A)$, there is $z \in Z$ such that $a \in B + \{z\} \subset A$.

The following is known:

PROPOSITION 1.25 ([70, Theorem 2]). *If $A, B \in \mathcal{C}_{bc}(Z)$, then $A \ominus B$ exists and is equal to a set $C \in \mathcal{C}_{bc}(Z)$ if and only if either $B \subset_t A$ or $A \subset_t B$, and C is a set such that either $A = B + C$ or $B = A - C$, respectively.*

REMARK 1.26. When $A, B \in \mathcal{K}_c(Z)$ we can replace the sets $\mathrm{Fr}(A)$ and $\mathrm{Fr}(B)$ (used in the above proposition) by the respective sets of extreme points, appealing to the Krein–Milman theorem.

It is easy to see that

(1.31) (i) If $A \in \mathcal{C}_{bc}(Z)$ and $z \in Z$, then $(A + \{z\}) \ominus A = \{z\}$. In particular, we have $A \ominus A = \{\theta\}$.

(ii) If $A, B \in \mathcal{C}_{bc}(Y)$ and $A \ominus B$ exists, then

$$A \ominus B = -(B \ominus A) \quad \text{and} \quad A \ominus B = (-B) \ominus (-A).$$

(iii) If $A \ominus B$ exists, then $h(A, B) = \|A \ominus B\|$, where $\|C\| = h(C, \{\theta\})$ for a set $C \subset Z$.

Now we can give a definition of differentiability for multifunctions (cf. [53]). From now on we assume that $I \subset \mathbb{R}$ is an interval.

DEFINITION 1.27. A multifunction $\Phi : I \rightsquigarrow Z$ is said to be *differentiable at a point* $x_0 \in I$ if there exists a set $D\Phi(x_0) \in \mathcal{C}_{bc}(Z)$ such that the limit

$$h\text{-}\lim_{x \to x_0} \frac{\Phi(x) \ominus \Phi(x_0)}{x - x_0}$$

exists and is equal to $D\Phi(x_0)$.

Of course, implicit in the definition of $D\Phi(x_0)$ is the existence of the differences $\Phi(x) \ominus \Phi(x_0)$.

The set $D\Phi(x_0)$ is called the *derivative* of Φ at s_0; Φ is called *differentiable* if it is differentiable at each $x \in I$.

EXAMPLE 1.28. (a) Let B be the closed unit ball in Z and consider the multifunction $\Phi : (0, 2\pi) \rightsquigarrow Z$ defined by the formula $\Phi(\alpha) = (2 + \sin\alpha)B$. Then Φ is differentiable and $D\Phi(\alpha) = (\cos\alpha)B$.

(b) The multifunction $\Phi : [0, 1] \rightsquigarrow \mathbb{R}^2$ defined by

$$\Phi(\alpha) = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1 \wedge 0 \le y \le \alpha - \alpha x\}$$

is not differentiable, since the required differences do not exist.

(c) Let $\Phi : I \rightsquigarrow \mathbb{R}$ be a multifunction with values in $\mathcal{K}_c(\mathbb{R})$. Then $\Phi(x) = [i(x), s(x)]$, where $i(x) = \inf_{x \in I} \Phi(x)$ and $s(x) = \sup_{x \in I} \Phi(x)$. If the functions $i : I \to \mathbb{R}$ and $s : I \to \mathbb{R}$ are differentiable at $x_0 \in I$, then Φ is differentiable at x_0 and

$$D\Phi(x_0) = \begin{cases} [i'(x_0), s'(x_0)] & \text{if } i'(x_0) \le s'(x_0), \\ [s'(x_0), i'(x_0)] & \text{if } i'(x_0) > s'(x_0). \end{cases}$$

However, in general, differentiability of Φ does not imply differentiability of i and s, as the following example shows:

$$\Phi(x) = \begin{cases} [0, x] & \text{if } x \geq 0, \\ [x, 0] & \text{if } x < 0. \end{cases}$$

It is clear that the multifunctions considered in (a) and (c) of Example 1.28 are not differentiable in Hukuhara's sense, because Hukuhara's differences $\Phi(x) \overset{H}{-} \Phi(x_0)$ do not exist.

Proposition 1.29 ([70, Theorem 3]). *If a multifunction $\Phi : I \rightsquigarrow Z$ with $\Phi(x) \in \mathcal{C}_{bc}(Z)$ is differentiable at a point $x_0 \in I$, then Φ is h-continuous at x_0.*

Now we describe the π-differentiability of multifunctions discussed by Banks and Jacobs in [5]. As mentioned at the beginning of this section, this definition makes use of Rådström's embedding theorem (see [97, Theorem 2]): there is a real normed space $\mathcal{V}(Z)$ and an isometric mapping $\pi : \mathcal{C}_{bc}(Z) \to \mathcal{V}(Z)$, where $\mathcal{C}_{bc}(Z)$ is metrized by the Hausdorff metric h, such that $\pi(\mathcal{C}_{bc}(Z))$ is a convex cone in $\mathcal{V}(Z)$ with vertex $\pi(\{\theta\})$. Furthermore, addition in $\mathcal{V}(Z)$ induces addition in $\mathcal{C}_{bc}(Z)$ and multiplication by nonnegative scalars in $\mathcal{V}(Z)$ induces the corresponding operation in $\mathcal{C}_{bc}(Z)$.

The space $\mathcal{V}(Z)$ can be chosen minimal in the sense that if $\mathcal{V}_1(Z)$ is any other real normed linear space into which $\mathcal{C}_{bc}(Z)$ has been embedded in the above fashion, then $\mathcal{V}_1(Z)$ contains a subspace containing $\mathcal{C}_{bc}(Z)$ which is isomorphic to $\mathcal{V}(Z)$.

We describe the space $\mathcal{V}(Z)$ in some detail, since we make use of some of its properties later on.

An equivalence relation $\sim$ is defined on $\mathcal{C}_{bc}(Z) \times \mathcal{C}_{bc}(Z)$ by declaring that $(A, B) \sim (C, D)$ if $A + D = B + C$. The equivalence class containing (A, B) will be denoted by $\langle A, B\rangle$. The space $\mathcal{V}(Z)$ is the quotient space $\mathcal{C}_{bc}(Z) \times \mathcal{C}_{bc}(Z)/{\sim}$, with addition defined by

$$\langle A, B\rangle + \langle C, D\rangle = \langle A + C, B + D\rangle$$

and

$$\alpha\langle A, B\rangle = \begin{cases} \langle \alpha A, \alpha B\rangle & \text{if } \alpha \geq 0, \\ \langle |\alpha| B, |\alpha| A\rangle & \text{if } \alpha < 0. \end{cases}$$

With addition and scalar multiplication defined above the space $\mathcal{V}(Z)$ becomes a linear space. The neutral element $\langle\theta, \theta\rangle$ of $\mathcal{V}(Z)$ is the equivalence class $\{(A, A) : A \in \mathcal{C}_{bc}(Z)\}$.

The embedding $\pi : \mathcal{C}_{bc}(Z) \to \mathcal{V}(Z)$ is given by $\pi(A) = \langle A, \theta\rangle$ for $A \in \mathcal{C}_{bc}(Z)$. We shall denote $\pi(A)$ by $\widehat{A}$ when $A \in \mathcal{C}_{bc}(Z)$, and hence the convex cone $\pi(\mathcal{C}_{bc}(Z))$ by $\widehat{\mathcal{C}}_{bc}(Z)$.

A metric δ on $\mathcal{V}(Z) \times \mathcal{V}(Z)$ is defined by

$$\delta(\langle A, B\rangle, \langle C, D\rangle) = h(A + D, B + C). \tag{1.32}$$

Since δ is translation invariant and positively homogeneous, the relation

$$\|\langle A, B\rangle\| = \delta(\langle A, B\rangle, \langle\theta, \theta\rangle)$$

defines a norm in $\mathcal{V}(Z)$ such that

$$\delta(\langle A, B\rangle, \langle C, D\rangle) = \|\langle A, B\rangle - \langle C, D\rangle\|. \tag{1.33}$$

Note that

(1.34) If $A, B \in \mathcal{C}_{bc}(Z)$ and $A \ominus B$ exists, then

$$\widehat{A} - \widehat{B} = \langle A, \theta\rangle - \langle B, \theta\rangle = \begin{cases} \langle A \ominus B, \theta\rangle & \text{if } B \subset_t A, \\ \langle \theta, B \ominus A\rangle & \text{if } A \subset_t B. \end{cases}$$

Indeed, we have $\langle A, \theta\rangle - \langle B, \theta\rangle = \langle A, B\rangle$. Let $A \ominus B = C$, $C \in \mathcal{C}_{bc}(Z)$. If $B \subset_t A$, then $A = B+C$, and so $\langle A, B\rangle = \langle B+C, B\rangle = \langle C, \theta\rangle = \langle A\ominus B, \theta\rangle$. If $A \subset_t B$, then $B = A-C$, and so $\langle A, B\rangle = \langle A, A-C\rangle = \langle \theta, -C\rangle = \langle \theta, B \ominus A\rangle$, by (1.31)(ii).

We should mention that the space $(\mathcal{V}(Z), \delta)$ need not be complete when (Z, ϱ) is complete (see [20, p. 363]). But since in this case the space $(\mathcal{C}_{bc}(Z), h)$ is complete, so is $(\widehat{\mathcal{C}}_{bc}(Z), \delta)$.

A function $f : V \to W$, where V and W are arbitrary normed linear spaces, is said to be $o(\|\Delta v\|)$ if $\|f(\Delta v)\|/\|\Delta v\| \to 0$ as $\|\Delta v\| \to 0$.

Let $(S, \|\cdot\|)$ be a real linear normed space and let $(Z, \|\cdot\|)$ be a reflexive Banach space. Following Banks and Jacobs [5], a multifunction $\Phi : S \rightsquigarrow Z$ with values in $\mathcal{C}_{bc}(Z)$ is called *π-differentiable at a point* $x_0 \in S$ if the function $\widehat{\Phi} : S \to \mathcal{V}(Z)$ is differentiable at x_0, i.e., there is a continuous linear mapping $\widehat{\Phi}'(x_0) : S \to \mathcal{V}(Z)$ such that

$$\widehat{\Phi}(x) - \widehat{\Phi}(x_0) - \widehat{\Phi}'(x_0)(x - x_0) = o(\|x - x_0\|). \tag{1.35}$$

Φ is *π-differentiable* if it is π-differentiable at every $x \in S$.

If $\widehat{\Phi}'(x_0)(x - x_0) = \langle A_{x-x_0}, B_{x-x_0}\rangle$, where $x - x_0 \in S$ and the sets A_{x-x_0} and B_{x-x_0} belong to $\mathcal{C}_{bc}(Z)$, then, according to (1.35), we have

$$\langle \Phi(x), \Phi(x_0)\rangle - \langle A_{x-x_0}, B_{x-x_0}\rangle = o(\|x - x_0\|).$$

If the space $(S, \|\cdot\|)$ is finite-dimensional with basis $v_1, \ldots, v_n$, then $x - x_0 = \Delta x = \sum_{i=1}^n \Delta x^i v_i$ for $\Delta x \in S$. If $\widehat{\Phi}'(x_0)(v_i) = \langle A_{v_i}, \theta\rangle$, $i = 1, \ldots, n$, then Φ is called *conically differentiable* at x_0 and we have

$$\widehat{\Phi}'(x_0)(\Delta x) = \sum_{i=1}^n \Delta x^i \langle A_{v_i}, \theta\rangle.$$

The following proposition will be essential for the definition of a derivative multifunction.

Proposition 1.30. *Let $(Z, \|\cdot\|)$ be a reflexive Banach space. If a multifunction $\Phi : I \rightsquigarrow Z$ with values in $\mathcal{C}_{bc}(Z)$ is conically differentiable at a point $x_0 \in I$ and the differences $\Phi(x) \ominus \Phi(x_0)$ exist in a neighbourhood $U(x_0)$ of x_0, then Φ is differentiable at x_0 and $D\Phi(x_0) = \Phi'(x_0)$ provided $\widehat{\Phi}'(x_0)(x - x_0) = (x - x_0)\langle \Phi'(x_0), \theta\rangle$, where $\Phi'(x_0) \in \mathcal{C}_{bc}(Z)$.*

Proof. Using (1.32) and (1.33) we have

$$a = h\biggl(\frac{\Phi(x) \ominus \Phi(x_0)}{x - x_0}, \Phi'(x_0)\biggr) = \biggl\|\biggl\langle \frac{\Phi(x) \ominus \Phi(x_0)}{x - x_0}, \theta\biggr\rangle - \langle \Phi'(x_0), \theta\rangle\biggr\|.$$

Suppose that $\Phi(x_0) \subset_t \Phi(x)$. If $x > x_0$, then

$$\begin{aligned} a &= \frac{1}{x - x_0}\, \|\langle \Phi(x) \ominus \Phi(x_0), \theta\rangle - (x - x_0)\langle \Phi'(x_0), \theta\rangle\| \\ &= \frac{1}{x - x_0}\, \|\widehat{\Phi}(x) - \widehat{\Phi}(x_0) - \widehat{\Phi}'(x_0)(x - x_0)\|, \quad \text{by (1.34)}. \end{aligned}$$

The last term tends to 0 as $x \to x_0$, by (1.35). If $x < x_0$, then

$$a = \left\| \frac{1}{x - x_0} \langle \theta, \Phi(x_0) \ominus \Phi(x) \rangle - \langle \Phi'(x_0), \theta \rangle \right\|$$
$$= \frac{1}{|x - x_0|} \| \langle \theta, (-\Phi(x)) \ominus (-\Phi(x_0)) \rangle - (x - x_0) \langle \Phi'(x_0), \theta \rangle \|, \quad \text{by (1.31)(ii)}.$$

Since $-\Phi(x_0) \subset_t -\Phi(x)$,

$$a = \frac{1}{|x - x_0|} \| - \widehat{\Phi}(x_0) - (-\widehat{\Phi}(x)) - \widehat{\Phi}'(x_0)(x - x_0) \|$$

(see (1.34)). Thus, again, $a \to 0$ as $x \to x_0$.

Similar arguments apply to the case $\Phi(x) \subset_t \Phi(x_0)$. ■

REMARK 1.31. Let $\Phi : [-1, 1] \rightsquigarrow \mathbb{R}$ be given by

$$\Phi(x) = x \cdot [-1, 1] = \begin{cases} [-x, x] & \text{if } x \in [0, 1], \\ [x, -x] & \text{if } x \in [-1, 0]. \end{cases}$$

Then $D\Phi(0) = [-1, 1]$. But Φ is not π-differentiable at 0 (see [5, p. 251]). Therefore the converse of Proposition 1.30 is not true.

As mentioned earlier, the completeness of the reflexive real normed linear space $(Z, \|\cdot\|)$ does not imply that the corresponding normed linear space $(\mathcal{V}(Z), \delta)$ is complete, which presents a minor difficulty when considering the integrability of multifunctions with values in $\mathcal{C}_{bc}(Z)$.

Let $\overline{\mathcal{V}}(Z)$ be the completion of $\mathcal{V}(Z)$, which is a Banach space. Following Banks and Jacobs (see [5, p. 266]), we give the definition of integrability for multifunctions with values in $\mathcal{C}_{bc}(Z)$ (based on the definition of Debreu [20]). We also quote some of their results which we shall need later on.

DEFINITION 1.32. We say that a multifunction $\Phi : I \rightsquigarrow Z$ with values in $\mathcal{C}_{bc}(Z)$ is *integrable* (Lebesgue measure m on Lebesgue measurable subsets of I is understood) if the function $\widehat{\Phi} : I \to \overline{\mathcal{V}}(Z)$ is Bochner integrable (in the sense of [25, Definition 17, p. 112]), and the integral of $\widehat{\Phi}$ is denoted by $\int_I \widehat{\Phi}(x)\,dx$ or $\int_a^b \widehat{\Phi}(x)\,dx$, where $[a, b] = I$.

LEMMA 1.33 ([5, Lemmas 5.4 and 5.5]). *Let $(Z, \|\cdot\|)$ be a reflexive Banach space, and let a multifunction $\Phi : I \rightsquigarrow Z$ with values in $\mathcal{C}_{bc}(Z)$ be integrable. Then*

(i) *$\int_I \widehat{\Phi}(x)\,dx$ belongs to the convex cone $\widehat{\mathcal{C}}_{bc}(Z)$.*

(ii) *There is a sequence of measurable simple functions $\widehat{S}_n : I \to \widehat{\mathcal{C}}_{bc}(Z)$ such that $\lim_{n\to\infty} \widehat{S}_n(x) = \widehat{\Phi}(x)$ almost everywhere on I and $\|\widehat{S}_n(x)\| \le \|\widehat{\Phi}(x)\|$ for every $n \in \mathbb{N}$ and $x \in I$. Moreover*, $\lim_{n\to\infty} \int_I \|\widehat{S}_n(x) - \widehat{\Phi}(x)\|\,dx = 0$.

In view of the above lemma it makes sense to introduce the following definition.

DEFINITION 1.34. If a multifunction $\Phi : I \rightsquigarrow Z$ with values in $\mathcal{C}_{bc}(Z)$ is integrable, then we define $\int_I \Phi(x)\,dx$ to be the set $A \in \mathcal{C}_{bc}(Z)$ such that $\int_I \widehat{\Phi}(x)\,dx = \langle A, \theta \rangle$.

Let $\Phi, \Phi_i : I \rightsquigarrow Z$, $i = 1, 2$, be multifunctions with values in $\mathcal{C}_{bc}(Z)$. If these multifunctions are integrable, then

$$\Big\|\Big\langle \int_I \Phi_1(x)\,dx, \theta\Big\rangle - \Big\langle \int_I \Phi_2(x)\,dx, \theta\Big\rangle\Big\| = h\Big(\int_I \Phi_1(x)\,dx, \int_I \Phi_2(x)\,dx\Big)$$

and

$$\|\langle \Phi_1(x), \theta\rangle - \langle \Phi_2(x), \theta\rangle\| = h(\Phi_1(x), \Phi_2(x)).$$

Therefore, by [25, Theorem 20(a), p. 114]), we have

$$h\Big(\int_I \Phi_1(x)\,dx, \int_I \Phi_2(x)\,dx\Big) \le \int_I h(\Phi_1(x), \Phi_2(x))\,dx. \tag{1.36}$$

In particular, $\|\int_I \Phi(x)\,dx\| \le \int_I \|\Phi(x)\|\,dx$.

Let $\Phi : I \rightsquigarrow Z$ be a multifunction; if there exists a Lebesgue integrable function $g : I \to \mathbb{R}$ such that $\|\Phi(x)\| \le g(x)$ almost everywhere in I, then Φ is called *integrably bounded.*

We see from the Bochner theorem [47, Theorem 3.7.4] that

(1.37) If $\Phi : I \rightsquigarrow Z$ with values in $\mathcal{C}_{bc}(Z)$ is strongly $\mathcal{L}(\mathbb{R})$-measurable and integrably bounded, then it is integrable.

A different approach to defining inegrability for multifunctions is given by Hukuhara (see [53] in the case $Z = \mathbb{R}^n$ and compact convex valued multifunctions). This definition is based on the definition of Riemann integral. Starting from Hukuhara's idea of integrability we define R-integrability of multifunctions in a more general case.

Suppose that $(Z, \|\cdot\|)$ is reflexive, $I = [a, b] \subset \mathbb{R}$ and $\Phi : I \rightsquigarrow Z$ is a multifunction with values in $\mathcal{C}_{bc}(Z)$.

Let $\Delta = \{a_0, a_1, \ldots, a_n\}$ be a partition of I and $\lambda(\Delta) = \max_{i=0,\ldots,n-1}\{a_{i+1} - a_i\}$. Let $\mathcal{P}$ denote the family of all pairs (Δ, τ), where $\tau = (x_0, x_1, \ldots, x_{n-1})$ is a sequence of points such that $x_i \in [a_i, a_{i+1}]$ for $i = 0, \ldots, n-1$. Set

$$C(\Delta, \tau) = \sum_{i=0}^{n-1} (a_{i+1} - a_i)\Phi(x_i)$$

for $(\Delta, \tau) \in \mathcal{P}$. Then $C(\Delta, \tau) \in \mathcal{C}_{bc}(Z)$, by (1.29)(iv).

Definition 1.35. We say that a multifunction $\Phi : I \rightsquigarrow Z$ is *R-integrable* (on I) if there exists a set $B \in \mathcal{C}_{bc}(Z)$ such that $C(\Delta, \tau) \to B$ as $\lambda(\Delta) \to 0$, i.e.,

$$\forall \varepsilon > 0\ \exists \eta > 0\ \forall (\Delta, \tau) \in \mathcal{P} \quad [\lambda(\Delta) < \eta \Rightarrow h(C(\Delta, \tau), B) < \varepsilon],$$

and we define $(R)\int_I \Phi(x)\,dx$ to be the set B.

In much the same way as in the case of real functions it can be proved that

(1.38) (i) If $\Phi : I \rightsquigarrow Z$ with values in $\mathcal{C}_{bc}(Z)$ is h-continuous, then it is R-integrable (cf. [53, Section 5]).

(ii) If $\Phi : I \rightsquigarrow Z$ with values in $\mathcal{C}_{bc}(Z)$ is R-integrable, then it is integrable and

$$\int_I \Phi(t)\,dt = (R)\int_I \Phi(x)\,dx.$$

For an integrable multifunction $\Phi : I \rightsquigarrow Z$ with values in $\mathcal{C}_{bc}(Z)$ we define the multifunction $\Psi : I \rightsquigarrow Z$ by

$$x \mapsto \Psi(x) = \int_a^x \Phi(t)\,dt.$$

A simple computation shows that

(1.39) If Φ is integrable and $x_0 \in [a,b]$, then the difference $\Psi(x) \ominus \Psi(x_0)$ exists for every $x \in [a,b]$, and $\int_{x_0}^x \Phi(t)\,dt = \Psi(x) \ominus \Psi(x_0)$.

Indeed, if $x > x_0$, then $\int_a^x \Phi(t)\,dt = \int_a^{x_0} \Phi(t)\,dt + \int_{x_0}^x \Phi(t)\,dt$, and so $\int_{x_0}^x \Phi(t)\,dt = \Psi(x) \ominus \Psi(x_0)$. If $x < x_0$, then $\int_a^{x_0} \Phi(t)\,dt = \int_a^x \Phi(t)\,dt + \int_x^{x_0} \Phi(t)\,dt$, and so $\int_x^{x_0} \Phi(t)\,dt = \Psi(x_0) \ominus \Psi(x)$, that is, $\int_{x_0}^x \Phi(t)\,dt = \Psi(x) \ominus \Psi(x_0)$.

Lemma 1.36. *If a multifunction $\Phi : I \rightsquigarrow Z$ with values in $\mathcal{C}_{bc}(Z)$ is integrable and $\varepsilon > 0$, then the multifunction $\Phi_\varepsilon : I \rightsquigarrow Z$ given by*

$$\Phi_\varepsilon(x) = \int_x^{x+\varepsilon} \Phi(t)\,dt$$

is h-continuous.

Proof. Fix $x_0 \in I$. Then

$$\begin{aligned} h(\Phi_\varepsilon(x_0), \Phi_\varepsilon(x)) &= h\Big(\int_{x_0}^{x_0+\varepsilon} \Phi(t)\,dt, \int_x^{x+\varepsilon} \Phi(t)\,dt\Big) \\ &= h\Big(\int_{x_0}^{x_0+\varepsilon} \Phi(t)\,dt + \int_{x_0+\varepsilon}^{x} \Phi(t)\,dt, \int_{x_0+\varepsilon}^{x} \Phi(t)\,dt + \int_x^{x+\varepsilon} \Phi(t)\,dt\Big), \end{aligned}$$

by (1.29)(v). Thus

$$\begin{aligned} h(\Phi_\varepsilon(x_0), \Phi_\varepsilon(x)) &= h\Big(\int_{x_0}^{x} \Phi(t)\,dt, \int_{x_0+\varepsilon}^{x+\varepsilon} \Phi(t)\,dt\Big) \\ &\le \Big\|\int_{x_0}^{x} \Phi(t)\,dt\Big\| + \Big\|\int_{x_0+\varepsilon}^{x+\varepsilon} \Phi(t)\,dt\Big\| \to 0 \quad \text{as } x \to x_0, \text{ by (1.36).} \ \blacksquare \end{aligned}$$

From now on we suppose that $(Z, \|\cdot\|)$ is a reflexive Banach space. The following result is essential for the definition of a derivative multifunction.

Proposition 1.37 ([5, Theorem 5.3]). *If a multifunction $\Phi : [a,b] \rightsquigarrow Z$ with values in $\mathcal{C}_{bc}(Z)$ is integrable, then the multifunction $\Psi : [a,b] \rightsquigarrow Z$ given by $\Psi(x) = \int_a^x \Phi(t)\,dt$ is conically differentiable almost everywhere on $[a,b]$. Moreover, if $\widehat{\Psi}(x) = \int_a^x \widehat{\Phi}(t)\,dt$, then $\widehat{\Psi}'(x_0)(\Delta x) = \Delta x \widehat{\Phi}(x_0)$ for almost every $x_0 \in [a,b]$.*

Therefore, by (1.39) and Proposition 1.30, the following corollary holds.

Corollary 1.38. *If $\Phi : [a,b] \rightsquigarrow Z$ is an integrable multifunction with values in $\mathcal{C}_{bc}(Z)$, then the multifunction $\Psi : [a,b] \rightsquigarrow Z$ given by $\Psi(x) = \int_a^x \Phi(t)\,dt$, is differentiable almost everywhere on $[a,b]$, and $D\Psi(x_0) = \Phi(x_0)$ for almost every $x_0 \in [a,b]$.*

Similarly to the case of functions we will show

PROPOSITION 1.39. *If a multifunction $\Phi : I \rightsquigarrow Z$ with values in $\mathcal{C}_{bc}(Y)$ is h-continuous, then $D\Psi(x_0) = \Phi(x_0)$ for each $x_0 \in I$.*

Proof. Let $x_0 \in I$ and $\varepsilon > 0$. By h-continuity of Φ at x_0, there is an $\eta > 0$ such that $h(\Phi(x), \Phi(x_0)) < \varepsilon$ whenever $|x - x_0| < \eta$ and $x \in [a, b]$. Note that

$$h\Big(\int_{x_0}^{x} \Phi(t)\,dt, \int_{x_0}^{x} \Phi(x_0)\,dt\Big) = h\Big(\int_{x_0}^{x} \Phi(t)\,dt, (x - x_0)\Phi(x_0)\Big).$$

Furthermore, by (1.36),

$$h\Big(\int_{x_0}^{x} \Phi(t)\,dt, \int_{x_0}^{x} \Phi(x_0)\,dt\Big) \le \int_{x_0}^{x} h(\Phi(t), \Phi(x_0))\,dt < \varepsilon(x - x_0)$$

provided $0 < x - x_0 < \eta$. Since

$$\frac{1}{x - x_0}\int_{x_0}^{x} \Phi(t)\,dt = \frac{\Psi(x) \ominus \Psi(x_0)}{x - x_0} \quad \text{(see (1.39))},$$

it follows that

$$\frac{1}{x - x_0}\, h\Big(\int_{x_0}^{x} \Phi(t)\,dt, (x - x_0)\Phi(x_0)\Big) = h\bigg(\frac{1}{x - x_0}\int_{x_0}^{x} \Phi(t)\,dt, \Phi(x_0)\bigg) < \varepsilon.$$

Hence

$$D\Psi(x_0) = h\text{-}\lim_{x \to x_0^+} \frac{\Psi(x) \ominus \Psi(x_0)}{x - x_0} = \Phi(x_0).$$

Just as above we show that

$$D\Psi(x_0) = h\text{-}\lim_{x \to x_0^-} \frac{\Psi(x) \ominus \Psi(x_0)}{x - x_0} = \Phi(x_0). \ \blacksquare$$

Now we can define the notion of a derivative multifunction.

DEFINITION 1.40. Let $\Phi : I \rightsquigarrow Z$ be an integrable multifunction and $x_0 \in I$. The statement that Φ *is a derivative at* $x_0 \in I$ means that

$$\Phi(x_0) = h\text{-}\lim_{x \to x_0} \frac{1}{x - x_0}\int_{x_0}^{x} \Phi(t)\,dt.$$

The multifunction Φ *is a derivative* if it is a derivative at each point $x \in I$.

By Proposition 1.38, we have

COROLLARY 1.41. *If a multifunction $\Phi : I \rightsquigarrow Z$ with values in $\mathcal{C}_{bc}(Z)$ is h-continuous, then it is a derivative.*

Finally, we show that an approximately h-continuous multifunction is a derivative.

PROPOSITION 1.42. *Let $\Phi : I \rightsquigarrow Z$ be a multifunction with values in $\mathcal{C}_{bc}(Z)$. Suppose that Φ is bounded, i.e., there is a totally bounded set $K \subset Z$ such that $\Phi(x) \subset K$ for each $x \in I$. If Φ is approximately h-continuous, then it is a derivative.*

Proof. By Proposition 1.9, Φ is $\mathcal{L}(\mathbb{R})$-measurable (see Remark 1.10) and, by (1.22), it is strongly $\mathcal{L}(\mathbb{R})$-measurable. Since Φ is integrably bounded, it is integrable on any measurable subset of I, by the Bochner theorem [47, Theorem 3.7.4]. Let $I=[a,b]$. Define the multifunction $\Psi:[a,b]\rightsquigarrow Z$ by

$$\Psi(x)=\int_a^x \Phi(t)\,dt.$$

Let $x_0\in I$. Since Φ is approximately h-continuous at x_0, there exists a measurable set $E\subset I$ with $x_0\in E$ such that $D(E,x_0)=1$ and $\Phi|_E$ is h-continuous at x_0. Suppose $\Delta x>0$ and $x_0+\Delta x\in[a,b]$. Then

$$\Psi(x_0+\Delta x)=\Psi(x_0)+\int_{x_0}^{x_0+\Delta x}\Phi(x)\,dx$$

and thus

$$\Psi(x_0+\Delta x)\ominus\Psi(x_0)=\int_{x_0}^{x_0+\Delta x}\Phi(x)\,dx.$$

Note that

$$\begin{aligned}
h\bigg(\frac{\Psi(x_0+\Delta x)\ominus\Psi(x_0)}{\Delta x},\Phi(x_0)\bigg) &= h\bigg(\frac{1}{\Delta x}\int_{x_0}^{x_0+\Delta x}\Phi(x)\,dx,\Phi(x_0)\bigg)\\
&= h\bigg(\frac{1}{\Delta x}\int_{x_0}^{x_0+\Delta x}\Phi(x)\,dx,\frac{1}{\Delta x}\int_{x_0}^{x_0+\Delta x}\Phi(x_0)\,dx\bigg)\\
&\le \frac{1}{\Delta x}\int_{x_0}^{x_0+\Delta x}h(\Phi(x),\Phi(x_0))\,dx\\
&= \frac{1}{\Delta x}\int_{[x_0,x_0+\Delta x]\cap E}h(\Phi(x),\Phi(x_0))\,dx+\frac{1}{\Delta x}\int_{[x_0,x_0+\Delta x]\setminus E}h(\Phi(x),\Phi(x_0))\,dx.
\end{aligned}$$

As Δx tends to 0, the first term above converges to 0, since Φ is h-continuous on E, and the second is majorized by $\frac{1}{\Delta x}m([x_0,x_0+\Delta x]\setminus E)2\|K\|$, which converges to 0, since $D(I\setminus E,x_0)=0$.

This, together with a similar calculation for $\Delta x<0$ and $x_0+\Delta x\in I$, yields

$$h\bigg(\frac{\Psi(x_0)\ominus\Psi(x_0+\Delta x)}{\Delta x},\Phi(x_0)\bigg)\le\varepsilon,$$

and so $D\Psi(x_0)=\Phi(x_0)$. Hence Φ is a derivative at x_0. ■

2. PRODUCT MEASURABILITY OF MULTIFUNCTIONS OF TWO VARIABLES

7. Carathéodory multifunctions

Let X and Y be nonempty sets, let $F : X \times Y \rightsquigarrow Z$ be a multifunction, and let $(x_0, y_0) \in X \times Y$. Then the multifunction $F_{x_0} : Y \rightsquigarrow Z$ defined by $F_{x_0}(y) = F(x_0, y)$ is called the x_0-*section* of F, and the multifunction $F^{y_0} : X \rightsquigarrow Z$ defined by $F^{y_0}(x) = F(x, y_0)$ is called the y_0-*section* of F.

Similarly, if $E \subset X \times Y$ and $(x_0, y_0) \in X \times Y$, then the set $E_{x_0} = \{y \in Y : (x_0, y) \in E\}$ is called the x_0-*section* of E, and $E^{y_0} = \{x \in X : (x, y_0) \in E\}$ is the y_0-*section* of E.

It is well known that if $f : \mathbb{R}^2 \to \mathbb{R}$ is a Lebesgue measurable function, then the sections f_x and f^y are Lebesgue measurable for almost every $x \in \mathbb{R}$ and almost every $y \in \mathbb{R}$. But the converse is not true even if all sections of f are Lebesgue measurable. There are various sufficient conditions on sections of f ensuring that f is measurable. The most important one (given by Ursell [112]) is the continuity of the sections of f with respect to the first variable and their measurability with respect to the second variable. This result was extended in various ways for functions in spaces more general than $\mathbb{R}$ (see [12, Corollaire 3.1] or [64, Theorem 2, p. 387]). In this section we will consider this topic in the case of multifunctions.

Let $(X, \mathcal{M}(X))$ and $(Y, \mathcal{M}(Y))$ be measurable spaces, and let $(Z, \mathcal{T}(Z))$ be a topological space. A multifunction $F : X \times Y \rightsquigarrow Z$ will be called *product measurable* (resp. *weakly product measurable*) if it is measurable (resp. weakly measurable) with respect to the product σ-field $\mathcal{M}(X) \otimes \mathcal{M}(Y)$ or a more general σ-field in $X \times Y$.

If $(Y, \mathcal{T}(Y))$ is a topological space, then $F : X \times Y \rightsquigarrow Z$ will be called *Carathéodory* (or more precisely $\mathcal{M}(X)$-*Carathéodory*) if the section F^y is $\mathcal{M}(X)$-measurable for every $y \in Y$, and F_x is continuous for every $x \in X$.

A Carathéodory multifunction need not be product measurable, in general.

There are some results on the existence of a Carathéodory selection of a Carathéodory multifunction (some details and a survey of some papers in this field can be found in [59]). It is also known that (under some conditions) the product measurability of a multifunction whose sections with respect to the first variable are lower semicontinuous, is equivalent to the existence of its Castaing representation consisting of Carathéodory functions (see [32, Theorem 1]).

The following result is well known (see [58, Lemma 13.2.3]).

Lemma 2.1. *If* $(X,\mathcal{M}(X))$ *is a measurable space*, (Y,d) *a separable metric space and* (Z,ϱ) *a metric space*, *and if* $f : X\times Y\to Z$ *is a Carathéodory function*, *then* f *is* $\mathcal{M}(X)\otimes\mathcal{B}(Y)$*-measurable.*

As a straightforward consequence of the above lemma and Proposition 1.3(vi) we have the following result (cf. [116, Theorem 2])

Proposition 2.2. *If* $(X,\mathcal{M}(X))$ *is a measurable space*, (Y,d) *a separable metric space and* (Z,ϱ) *a metric space*, *and if* $F : X\times Y \rightsquigarrow Z$ *is a compact valued Carathéodory multifunction*, *then* F *is* $\mathcal{M}(X)\otimes\mathcal{B}(Y)$*-measurable.*

The purpose of this section is to give a generalization of this result.

Theorem 2.3. *Let* $(X,\mathcal{M}(X))$ *be a measurable space. Let* (Y,d) *be a metric space and let* $\mathcal{T}(Y)$ *be a separable topology on* Y *finer than the metric topology. Fix a countable* $\mathcal{T}(Y)$*-dense subset* S *of* Y. *Suppose that each point* $v\in Y$ *has a neighbourhood* $U(v)\in\mathcal{T}(Y)$ *such that*

(i) *for each* $y\in S$, $V(y)=\{v\in Y : y\in U(v)\}\in\mathcal{B}(Y,d)$ *and the family*

$$\mathcal{N}(v)=\{U(v)\cap B(v,2^{-n}) : n\in\mathbb{N}\}$$

forms a filterbase of $\mathcal{T}(Y)$*-neighbourhoods of* v.

If $(Z,\mathcal{T}(Z))$ *is perfectly normal and* $F : X\times Y\rightsquigarrow Z$ *is a multifunction such that* F^y *is* $\mathcal{M}(X)$*-measurable for every* $y\in Y$ *and* F_x *is* $\mathcal{T}(Y)$*-continuous for every* $x\in X$, *then* F *is weakly* $\mathcal{M}(X)\otimes\mathcal{B}(Y,d)$*-measurable.*

Proof. It is sufficient to show that

(1) $$F^+(D)\in\mathcal{M}(X)\otimes\mathcal{B}(Y,d)\quad\text{whenever } D \text{ is a closed subset of } Z.$$

Let D be an arbitrary closed subset of Z. Then, by perfect normality of Z, there exists a sequence $(G_m)_{m\in\mathbb{N}}$ of open subsets of Z such that

(2) $$D=\bigcap_{n\in\mathbb{N}} G_n=\bigcap_{n\in\mathbb{N}}\mathrm{Cl}(G_n)\quad\text{and}\quad \mathrm{Cl}(G_{n+1})\subset G_n\quad\text{for } n\in\mathbb{N}.$$

Let $S=\{y_k\}_{k\in\mathbb{N}}$. We shall prove that

(3) $$F^+(D)=\bigcap_{n\in\mathbb{N}}\bigcup_{k\in\mathbb{N}}(\{x\in X : F(x,y_k)\subset G_n\}\times V_n(y_k)),$$

where $V_n(y_k)=\{v\in Y : y_k\in U(v)\cap B(v,2^{-n})\}$.

Let $(u,v)\in F^+(D)=\{(x,y)\in X\times Y : F(x,y)\subset D\}$. Then, by (2), $F(u,v)\subset G_n$ for every $n\in\mathbb{N}$. Fix $n\in\mathbb{N}$. Since F_u is upper $\mathcal{T}(Y)$-semicontinuous at v, it follows that

(4) there exists a $\mathcal{T}(Y)$-open neighbourhood $W(v)\in\mathcal{N}(v)$ of v such that $F(u,y)\subset G_n$ for all $y\in W(v)$.

Let $K=\{m\in\mathbb{N} : y_m\in W(v)\}$. We put $m_0=\min\{m\in K : v\in V_n(y_m)\}$. Then, by (4), $F(u,y_k)\subset G_n$ for $k=m_0$, which implies $u\in(F^{y_k})^+(G_n)$.

Therefore, the inclusion

$$F^+(D)\subset\bigcap_{n\in\mathbb{N}}\bigcup_{k\in\mathbb{N}}(F^{y_k})^+(G_n)\times V_n(y_k)$$

has been proved. Conversely, suppose, contrary to our claim, that

$$(u,v) \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} (F^{y_k})^+(G_n) \times V_n(y_k), \tag{5}$$

but $(u,v) \notin F^+(D)$. Then $F(u,v) \not\subset D$, and so $F(u,v) \not\subset \bigcap_{m\in\mathbb{N}} \mathrm{Cl}(G_m)$ by (2). Therefore, $F(u,v) \cap (Z \setminus \mathrm{Cl}(G_m)) \neq \emptyset$ for some $m \in \mathbb{N}$.

Thus, by $\mathcal{T}(Y)$-lower semicontinuity of F_u at v,

(6) there is a $\mathcal{T}(Y)$-open neighbourhood $W(v) \in \mathcal{N}(v)$ of v such that

$$F(u,y) \cap (Z \setminus \mathrm{Cl}(G_m)) \neq \emptyset \quad \text{for all } y \in W(v).$$

We see from (5) that to each $n \in \mathbb{N}$ there corresponds an index $k(n) \in \mathbb{N}$ such that $u \in (F^{y_{k(n)}})^+(G_n)$ and $v \in V_n(y_{k(n)})$, i.e.,

$$F(u, y_{k(n)}) \subset G_n \quad \text{and} \quad y_{k(n)} \in U(v) \cap B(v, 2^{-n}). \tag{7}$$

Hence, $\lim_{n\to\infty} y_{k(n)} = v$, and so by (6), there is an $n_0 \in \mathbb{N}$ such that $y_{k(n)} \in W(v)$ and

$$F(u, y_{k(n)}) \cap (Z \setminus \mathrm{Cl}(G_m)) \neq \emptyset \quad \text{for every } n > n_0. \tag{8}$$

By (7) and (2), we arrive at the inclusions

$$F(u, y_{k(n+j)}) \subset G_{n+j} \subset \mathrm{Cl}(G_{n+j}) \subset G_n \quad \text{for } n \in \mathbb{N} \text{ and } j \in \mathbb{N}.$$

Fixing $n = m$, we obtain

$$F(u, y_{k(m+j)}) \subset \mathrm{Cl}(G_{m+j}) \subset G_m \quad \text{for all } j \in \mathbb{N}. \tag{9}$$

Let $j \in \mathbb{N}$ be such that $m + j > n_0$. Then, by (8), we have

$$F(u, y_{k(m+j)}) \cap (Z \setminus \mathrm{Cl}(G_{m+j})) \neq \emptyset,$$

contrary to (9). Thus (3) has been proved.

Observe that $\{x \in X : F(x, y_k) \subset G_n\} \in \mathcal{M}(X)$, because F^{y_k} is $\mathcal{M}(X)$-measurable. Moreover, by assumption (i), $V_n(y_k) \in \mathcal{B}(Y,d)$. Thus, by (3), it is clear that $F^+(D) \in \mathcal{M}(X) \otimes \mathcal{B}(Y,d)$, which proves (1). ∎

Remark 2.4.

(i) If we suppose that the multifunction F considered in Theorem 2.3 is compact valued, then F will be $\mathcal{M}(X) \otimes \mathcal{B}(Y,d)$-measurable, by Proposition 1.2(ii).

(ii) If, in Theorem 2.3, we suppose that the space $(Z, \mathcal{T}(Z))$ is metrizable σ-compact and the multifunction F is closed valued, then F will be $\mathcal{M}(X) \otimes \mathcal{B}(Y,d)$-measurable, by Proposition 1.3(iv).

Below we give two examples of topologies on a metric space (Y,d) fulfilling the requirements of Theorem 2.3. By the first example, it will be clear that if all x-sections of a multifunction F are either right-continuous or left-continuous (in some sense) and all its y-sections are measurable, then F is weakly product measurable.

Example 2.5. Let $(Y, d, \leq)$ be a linearly ordered metric space. We follow Dravecký and Neubrunn [24] in assuming that $(Y, d, \leq)$ has the property $\mathcal{U}$, i.e., $(Y, \leq)$ is a linearly ordered space and there is a countable dense set $S = \{y_n\}_{n\in\mathbb{N}}$ in $(Y, d,)$ such that for any $y \in Y$, we have $y = \lim_{n\to\infty} y_n$, where $y \leq y_n$ for $n \in \mathbb{N}$. Then the topology $\mathcal{T}(Y)$

generated by all open sets in (Y,d) and also by all intervals $I_a = \{y \in Y : y \le a\}$, $a \in Y$, fulfils the assumptions of Theorem 2.3.

Indeed, fix $y \in Y$ and $r > 0$. Then

$$U_r(y) = B(y,r) \cap I_y = \{x \in Y : d(x,y) < r \wedge x \le y\}$$

is a $\mathcal{T}(Y)$-neighbourhood of y.

Let $x \in U_r(y)$. Then $x \in B(y,r)$ and $x \le y$, and so there is an $r_1 > 0$ such that $d(x,y) = r - r_1$. Let $\delta < \min(r - r_1, r_1)$. Then $B(x,\delta) \subset B(y,r)$. Let $n \in \mathbb{N}$ be such that $2^{-n} < \delta$. Then $U_{2^{-n}}(x) \subset U_r(y)$ and $\{U_{2^{-n}}(y)\}_{n\in\mathbb{N}}$ is a filterbase of $\mathcal{T}(Y)$-neighbourhoods of y.

The set S is also $\mathcal{T}(Y)$-dense. It remains to show that

$$V_r(y) = \{z \in Y : y \in U_r(z)\}$$

is a Borel set in (Y,d). First we will show that

(1) If $y_0 \ne y$ and $y_0 \in V_r(y)$, then there exists an $r_1 \in (0,r)$ such that $U_{r_1}(y_0) \subset V_r(y)$.

Suppose, contrary to our claim, that $U_{r_1}(y_0) \not\subset V_r(y)$ for any $0 < r_1 < r$. Let $n \in \mathbb{N}$ be such that $1/n < r$. Then there is a y_n such that $y \le y_n$ and $y_n \in U_{1/n}(y_0) \setminus V_r(y)$, and so, for $n > 1/r$, we have

$$y \le y_n \wedge d(y_n, y_0) < 1/n \wedge y_n \le y_0 \wedge (y_n \le y \vee d(y_n, y) \ge r).$$

If it were true that $d(y_n,y_0) < 1/n$ and $y \le y_n \le y_0$ and $y_n \le y$, we would have $\lim_{n\to\infty} y_n = y_0 = y$, contradicting $y \ne y_0$. Let $d(y_0,y) = \varepsilon$. If it were true that $d(y_n,y_0) < 1/n$ and $d(y_n,y) \ge r$, we would have $r \le d(y_n,y) \le d(y_n,y_0) + d(y_0,y) < 1/n + \varepsilon$. Then $1/n > r - \varepsilon > 0$ for almost every $n \in \mathbb{N}$, which is impossible. This establishes (1).

Our next claim is that

(2) If $y_0 \ne y$ and $y_0 \in V_r(y)$, then there is a $\delta > 0$ such that $B(y_0,\delta) \subset V_r(y)$.

Indeed, according to (1), there is an $r_1 \in (0,r)$ such that $U_{r_1}(y_0) \subset V_r(y)$. Let $\varepsilon = d(y_0,y) < r$ and let $\delta < \min(\varepsilon, r-\varepsilon, r_1)$. If $z \in B(y_0,\delta)$, then either $d(y_0,z) < \delta$ and $z \le y_0$, or $d(y_0,z) < \delta$ and $y_0 \le z$. In the first case, $z \in U_\delta(y_0) \subset V_r(y)$. In the second, $d(z,y) \le d(z,y_0) + d(y_0,y) < \delta + \varepsilon < r - \varepsilon + \varepsilon = r$ and $y \le z$, showing that $z \in V_r(y)$. Combining the two results we conclude that $B(y_0,\delta) \subset V_r(y)$, and (2) is proved.

Thus the set $\{z \in Y : d(z,y) < r \wedge y \le z \wedge y \ne z\}$ is open in (Y,d). Therefore,

$$V_r(y) = \{y\} \cup \{z \in Y : d(z,y) < r \wedge y \le z \wedge y \ne z\} \in \mathcal{F}_\sigma(Y,d) \cap \mathcal{G}_\delta(Y,d),$$

and finally $V_r(y) \in \mathcal{B}(Y,d)$.

Note that the topology $\mathcal{T}(Y)$ in the above example may be viewed as a natural generalization of the Sorgenfrey topology on the real line [114].

By Theorem 2.3, we have the following corollary.

Corollary 2.6. *Let $(X, \mathcal{M}(X))$ be a measurable space and $(Z, \mathcal{T}(Z))$ a perfectly normal topological space. Then a multifunction $F : X \times \mathbb{R} \rightsquigarrow Z$ such that F_x is right-continuous (resp. left-continuous) for every $x \in X$ and F^y is $\mathcal{M}(X)$-measurable for every $y \in Y$, is weakly $\mathcal{M}(X) \otimes \mathcal{B}(\mathbb{R})$-measurable.*

Now we give another example of a topology $\mathcal{T}(Y)$ fulfilling the assumptions of Theorem 2.3.

Example 2.7. Let $(Y, \diamond, d)$ be a topological group whose topology is induced by an invariant distance function d (i.e., $d(\theta, y) = d(v, y \diamond v)$), where θ denotes the neutral element of Y. Furthermore we assume that (Y, d) is separable.

Let $U \subset Y$ be an open set such that θ is an accumulation point of U. Let

$$U_n = (B(\theta, 2^{-n}) \cap U) \cup \{\theta\} \quad \text{and} \quad V_n(y) = y \diamond U_n = \{y \diamond v : v \in U_n\}$$

for any $n \in \mathbb{N}$ and $y \in Y$. Then $\{V_n(y)\}_{n\in\mathbb{N}}$ is a filterbase of neighbourhoods of $y \in Y$, and the topology $\mathcal{T}(Y)$ generated by this base fulfils all requirements of Theorem 2.3.

Indeed, it suffices to prove that $\{U_n\}_{n\in\mathbb{N}}$ is a base of neighbourhoods of θ. We have $U_n \cap U_m = U_{\min(n,m)}$. Let $n \in \mathbb{N}$ and $v \in U_n$. Then, by the definition of $V_n(y)$, there is a $k \in \mathbb{N}$ such that $B(v, 2^{-k}) = v \diamond B(\theta, 2^{-k}) \subset U_n$. Therefore, we conclude that

$$\forall n \in \mathbb{N}\ \forall v \in U_n\ \exists k \in \mathbb{N} \quad V_k(v) \subset U_n.$$

A countable dense subset of (Y, d) is also $\mathcal{T}(Y)$-dense. It remains to show that $V_n(y)$ is a Borel set in (Y, d) for $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$ and let $\Phi : Y \rightsquigarrow Y$ be defined by $\Phi(y) = V_n(y)$. Then Φ is continuous and $\mathrm{Gr}(\Phi) = \{(y, z) : z \in y \diamond U_n\}$ is homeomorphic to $Y \times U_n$. Thus $V_n(y) \in \mathcal{B}(Y, d)$ for each $n \in \mathbb{N}$.

8. Multifunctions with approximately semicontinuous sections

In this section we assume that $(X, d, \mathcal{M}(X), \mu)$ and $(Y, \rho, \mathcal{M}(Y), \nu)$ are measure metric spaces with complete, σ-finite and $\mathcal{G}_\delta$-regular measures μ and ν on the σ-fields $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ containing $\mathcal{B}(X)$ and $\mathcal{B}(Y)$, respectively; $\mu \times \nu$ is the product measure on the σ-field $\mathcal{M}(X) \otimes \mathcal{M}(Y)$, and $\mathcal{M}_{\mu\times\nu}(X \times Y)$ is the $\mu \times \nu$-completion of $\mathcal{M}(X) \otimes \mathcal{M}(Y)$; and $\mathcal{F} \subset \mathcal{M}(X)$ and $\mathcal{G} \subset \mathcal{M}(Y)$ are families of sets (defined as in (1.23)) with the density property (1.24).

Let $B \in \mathcal{M}(X) \otimes \mathcal{M}(Y)$. We will write $B \sqsubset B$ if, for every $(x, y) \in B$, x is a density point of B^y with respect to $\mathcal{F}$ and y is a density point of B_x with respect to $\mathcal{G}$.

The following lemma is known.

Lemma 2.8 ([67, Lemma 2]). *If $A \in \mathcal{M}_{\mu\times\nu}(X \times Y)$, then there is a $B \in \mathcal{M}(X) \otimes \mathcal{M}(Y)$ such that $B \subset A$, $B \sqsubset B$ and $\mu \times \nu(A \setminus B) = 0$.*

The $\mathcal{G}_\delta$-regularity of the measures μ and ν in the above lemma is essential.

Theorem 2.9. *Let (Z, ϱ) be a separable metric space and $F : X \times Y \rightsquigarrow Z$ a closed valued multifunction. If $\{F_x\}_{x\in X}$ is approximately h-equicontinuous with respect to $\mathcal{G}$ and F^y is weakly $\mathcal{M}(X)$-measurable for each $y \in Y$, then F is weakly $\mathcal{M}_{\mu\times\nu}(X \times Y)$-measurable.*

Proof. By Proposition 1.3(i), it suffices to prove that

(1) the real function $g_z(x, y) = \varrho(z, F(x, y))$ is $\mathcal{M}_{\mu\times\nu}(X \times Y)$-measurable for each $z \in Z$.

Fix $z \in Z$. To prove (1) we apply the Davies lemma [17], i.e., it is sufficient to show that for every $\varepsilon > 0$ the family $\mathcal{H}_\varepsilon = \{H \in \mathcal{M}(X) \otimes \mathcal{M}(Y) : \mathrm{osc}_H(g_z) \le \varepsilon\}$ of sets satisfies the following condition:

(D) for every $A \in \mathcal{M}(X) \otimes \mathcal{M}(Y)$ of positive $\mu \times \nu$ measure, there exists an $H \in \mathcal{H}_\varepsilon$ such that $H \subset A$ and $\mu \times \nu(H) > 0$.

Fix $A \in \mathcal{M}(X) \otimes \mathcal{M}(Y)$ with $\mu \times \nu(A) > 0$ and $\varepsilon > 0$. By Lemma 2.8, there is a $B \in \mathcal{M}(X) \otimes \mathcal{M}(Y)$ such that $B \subset A$, $B \sqsubset B$ and $\mu \times \nu(A \setminus B) = 0$.

Let $y_0 \in Y$ be such that $\mu(B^{y_0}) > 0$. Since F^{y_0} is weakly $\mathcal{M}(X)$-measurable, $(g_z)^{y_0}$ is $\mathcal{M}(X)$-measurable. Let $\delta > 0$. By Lusin's theorem, there is a closed set $C \subset X$ such that $(g_z)^{y_0}|_C$ is continuous and $\mu(X \setminus C) < \delta$. Since $\mathcal{F}$ has the density property, μ-almost every point of C is its density point with respect to $\mathcal{F}$. Thus $(g_z)^{y_0}$ is μ-almost everywhere approximately continuous with respect to $\mathcal{F}$. Therefore, there is an $x_0 \in B^{y_0}$ such that $(g_z)^{y_0}$ is approximately continuous at x_0 with respect to $\mathcal{F}$. Thus, there exists a $K \in \mathcal{M}(X)$ such that $D(K, x_0) = 1$ and $|g_z(x, y_0) - g_z(x_0, y_0)| < \varepsilon/4$ for all $x \in K$.

Let $M = K \cap B^{y_0}$. Then $M \in \mathcal{M}(X)$ and $D(M, x_0) = 1$, since $D(K, x_0) = 1$ and $D(B^{y_0}, x_0) = 1$. Furthermore,

$$|g_z(x, y_0) - g_z(x_0, y_0)| < \varepsilon/4 \quad \text{for all } x \in M. \tag{2}$$

On the other hand, by the approximate h-equicontinuity of $\{F_x\}_{x\in X}$ at y_0 with respect to $\mathcal{G}$, there is an $L(y_0) \in \mathcal{M}(Y)$ such that $D(L(y_0), y_0) = 1$ and $\{F_x|_{L(y_0)}\}_{x\in X}$ is h-equicontinuous at y_0. Thus, there is an open set $V(y_0)$ including y_0 such that

$$h(F_x|_{L(y_0)}(y), F_x|_{L(y_0)}(y_0)) < \varepsilon/8 \quad \text{for } x \in X \text{ and } y \in V(y_0). \tag{3}$$

Let $N = L(y_0) \cap V(y_0)$. Then $N \in \mathcal{M}(Y)$ and $D(N, y_0) = 1$. Let $y \in N$. Then, by (3), there is a $z_1 \in F_x(y)$ with $\varrho(z, F_x(y)) + \varepsilon/8 > \varrho(z, z_1)$ and there is a $z_2 \in F_x(y_0)$ with $\varrho(z, F_x(y_0)) + \varepsilon/8 > \varrho(z, z_2)$. Moreover, there is a $z' \in F_x(y)$ with $\varrho(z', z_2) < \varepsilon/8$ and a $z'' \in F_x(y_0)$ with $\varrho(z'', z_1) < \varepsilon/8$. Then

$$\varrho(z, F_x(y)) \le \varrho(z, z') \le \varrho(z, z_2) + \varrho(z_2, z') < \varepsilon/4 + \varrho(z, F_x(y_0)),$$

and

$$\varrho(z, F_x(y_0)) \le \varrho(z, z'') \le \varrho(z, z_1) + \varrho(z_1, z'') < \varepsilon/4 + \varrho(z, F_x(y))$$

for $x \in X$ and $y \in N$. Thus

$$|g_z(x, y) - g_z(x, y_0)| < \varepsilon/4 \quad \text{for } x \in X \text{ and } y \in N. \tag{4}$$

Set $P = M \times N$. We see from (4) and (2) that

$$|g_z(x, y) - g_z(x_0, y_0)| \le |g_z(x, y) - g_z(x, y_0)| + |g_z(x, y_0) - g_z(x_0, y_0)| < \varepsilon/2$$

for every $(x, y) \in P$, and hence $\mathrm{osc}_P(g_z) \le \varepsilon$.

Now let $H = P \cap B$. Since $B \in \mathcal{M}(X) \otimes \mathcal{M}(Y)$ and $P \in \mathcal{M}(X) \otimes \mathcal{M}(Y)$, it follows that $H \in \mathcal{M}(X) \otimes \mathcal{M}(Y)$. Furthermore, $\mu \times \nu(H) > 0$, since $\nu(H_x) > 0$ for μ-almost every $x \in X$. Finally, $H \subset B \subset A$ and $\mathrm{osc}_H(g_z) \le \varepsilon$, which proves (D). ∎

It is known (see [17, Theorem 2]) that if all sections f_x and f^y of a function $f : \mathbb{R}^2 \to \mathbb{R}$ are approximately continuous, then f is of the second Baire class.

In this connection, consider the following example.

EXAMPLE 2.10. Decompose the interval $[0,1] \subset \mathbb{R}$ into two disjoint non-Borel sets A and B and define the multifunction $F : [0,1] \times [0,1] \to \mathbb{R}$ by putting

$$F(x,y) = \begin{cases} [-3,3] & \text{if } x \neq y, \\ [-1,0] & \text{if } x = y \in A \\ [1,2] & \text{if } x = y \in B. \end{cases}$$

Then F is not $\mathcal{B}(\mathbb{R}^2)$-measurable although all its x-sections and y-sections are approximately lower semicontinuous (even lower semicontinuous).

The above example shows that a multifunction $F : X \times Y \rightsquigarrow Z$ (even compact valued) having all x-sections approximately lower semicontinuous with respect to $\mathcal{G}$ and all y-sections approximately lower semicontinuous with respect to $\mathcal{F}$ may by "strange".

Let $\mathcal{F} \times \mathcal{G} = \{E : E = A \times B,\ A \in \mathcal{F},\ B \in \mathcal{G}\}$. For each $P \subset X \times Y$ we define (as in Section 4) the upper and lower outer density of P at $(x,y) \in X \times Y$ with respect to $\mathcal{F} \times \mathcal{G}$, and the density point of P with respect to $\mathcal{F} \times \mathcal{G}$. The family $\mathcal{F} \times \mathcal{G}$ has the density property (see (1.24)), because so do $\mathcal{F}$ and $\mathcal{G}$ (see [9, pp. 5 and 34]).

PROPOSITION 2.11. *Let (Z,ϱ) be a metric space and $F : X \times Y \rightsquigarrow Z$ a multifunction. If F^y is approximately h-lower semicontinuous with respect to $\mathcal{F}$ for each $y \in Y$ and $\{F_x\}_{x\in X}$ is approximately h-lower equicontinuous with respect to $\mathcal{G}$, then F is approximately h-lower semicontinuous with respect to $\mathcal{F} \times \mathcal{G}$.*

Proof. Fix $(x_0,y_0) \in X \times Y$ and $\varepsilon > 0$. Since F^{y_0} is approximately h-lower semicontinuous at x_0 with respect to $\mathcal{F}$, there exists a set $A(x_0) \in \mathcal{M}(X)$ including x_0 such that $D(A(x_0),x_0) = 1$ and $F^{y_0}|_{A(x_0)}$ is h-lower semicontinuous at x_0. Thus, there is an open neighbourhood $U(x_0)$ of x_0 such that

$$F(x_0,y_0) \subset B(F(x,y_0),\varepsilon/2) \quad \text{for all } x \in U(x_0) \cap A(x_0). \tag{1}$$

By the approximate h-lower equicontinuity of $\{F_x\}_{x\in X}$ at y_0 with respect to $\mathcal{G}$, there is a $B(y_0) \in \mathcal{M}(Y)$ including y_0 such that $D(B(y_0),y_0) = 1$ and $\{F_x|_{B(y_0)}\}_{x\in X}$ is h-lower equicontinuous at y_0. Therefore, there is an open neighbourhood $V(y_0)$ of y_0 such that

$$F(x,y_0) \subset B(F(x,y),\varepsilon/2) \quad \text{for } x \in X \text{ and } y \in V(y_0) \cap B(y_0). \tag{2}$$

Let $E(x_0,y_0) = A(x_0) \times B(y_0)$. Then $D(E(x_0,y_0),(x_0,y_0)) = 1$. It is sufficient to show that $F|_{E(x_0,y_0)}$ is h-lower semicontinuous with respect to $\mathcal{F}\times\mathcal{G}$ at (x_0,y_0). Let $W(x_0,y_0) = U(x_0) \times V(y_0)$. Then, by (1) and (2),

$$F(x_0,y_0) \subset B(F(x,y_0),\varepsilon/2) \quad \text{and} \quad F(x,y_0) \subset B(F(x,y),\varepsilon/2)$$

for each $(x,y) \in W(x_0,y_0) \cap E(x_0,y_0)$. Thus, for $(x,y) \in W(x_0,y_0) \cap E(x_0,y_0)$,

$$F(x_0,y_0) \subset B(F(x,y),\varepsilon),$$

i.e., $F|_{E(x_0,y_0)}$ is h-lower semicontinuous at (x_0,y_0). ∎

A similar proof works when we replace "h-lower" by "h-upper" in Proposition 2.11, and we have a dual result.

Proposition 2.12. *Let (Z,ϱ) be a metric space and $F : X\times Y \rightsquigarrow Z$ a multifunction. If F^y is approximately h-upper semicontinuous with respect to $\mathcal{F}$ for every $y\in Y$ and $\{F_x\}_{x\in X}$ is approximately h-upper equicontinuous with respect to $\mathcal{G}$, then F is approximately h-upper semicontinuous with respect to $\mathcal{F}\times\mathcal{G}$.*

Remark 2.13. We see from (1.11)(ii) and Proposition 1.9 that a multifunction F which satisfies the assumptions of Proposition 2.11 is weakly $\mathcal{M}(X)\otimes\mathcal{M}(Y)$-measurable. If we additionally assume that F is compact valued, then it is $\mathcal{M}(X)\otimes\mathcal{M}(Y)$-measurable, by (1.11)(iii) and Proposition 1.9.

Now let $(Z,\mathcal{T}(Z))$ be a topological space. We will show that the approximate lower semicontinuity of all y-sections and upper semicontinuity of all x-sections of a multifunction $F : X\times Y \rightsquigarrow Z$ are sufficient for its product measurability.

We first prove the following proposition.

Proposition 2.14. *Let $F : X\times Y \rightsquigarrow Z$ be a multifunction such that F^y is approximately lower semicontinuous with respect to $\mathcal{F}$ for each $y\in Y$. Then for each $n\in\mathbb{N}$, the multifunction $F_n : X\times Y \rightsquigarrow Z$ defined by*

$$F_n(x,y) = F(x,B(y,2^{-n})) = \bigcup_{v\in B(y,2^{-n})} F(x,v) \tag{2.1}$$

is approximately lower semicontinuous with respect to $\mathcal{F}\times\mathcal{G}$.

Proof. Fix $n\in\mathbb{N}$, $(x,y)\in X\times Y$ and an open set $G\subset Z$ such that $F_n(x,y)\cap G\neq\emptyset$. By (2.1), there exists a $v\in B(y,2^{-n})$ such that $F(x,v)\cap G\neq\emptyset$. Since F^v is approximately lower semicontinuous with respect to $\mathcal{F}$ at x, there is an $E\in\mathcal{M}(X)$ including x such that $D(E,x)=1$ and $F^v|_E$ is lower semicontinuous at x. Therefore, there is an open neighbourhood $U(x)$ of x such that $F(u,v)\cap G\neq\emptyset$ whenever $u\in E\cap U(x)$.

Observe that there exists an $r>0$ such that

$$F(u,v)\subset F(u,B(y_0,2^{-n})) = F_n(u,y_0) \quad \text{for all } u\in U(x) \text{ and } y_0\in B(y,r). \tag{1}$$

Indeed, let $r = 2^{-n}-\rho(v,y)$. Then $r>0$ and for every $t\in B(y,r)$ we have

$$\rho(t,v)\le\rho(t,y)+\rho(y,v) < r+2^{-n}-r = 2^{-n}.$$

Therefore

$$t\in B(y,r) \;\Rightarrow\; v\in B(t,2^{-n}),$$

and the inclusion (1) holds on the set $(E\cap U(x))\times B(y,r)$. Thus $F_n(u,v)\cap G\neq\emptyset$ whenever $(u,v)\in(E\cap U(x))\times B(y,r)$.

Let $V(x,y) = (E\cap U(x))\times B(y,r)$. Then $V(x,y)\in\mathcal{M}(X)\otimes\mathcal{M}(Y)$. Furthermore, $D(V(x,y),(x,y))=1$ and $F_n|_{V(x,y)}$ is lower semicontinuous at (x,y). ∎

Lemma 2.15. *Let $(Z,\mathcal{T}(Z))$ be a regular space and $F : X\times Y\to Z$ a closed valued multifunction such that F_x is upper semicontinuous for each $x\in X$. If $(F_n)_{n\in\mathbb{N}}$ is a sequence of multifunctions from $X\times Y$ to Z defined by (2.1), then for each $(x,y)\in X\times Y$ we have*

$$F(x,y) = \bigcap_{n\in\mathbb{N}} \mathrm{Cl}(F_n(x,y)).$$

Proof. Fix $(x,y) \in X \times Y$. Observe that $F(x,y) \subset F_n(x,y)$ for any $n \in \mathbb{N}$. Therefore

$$F(x,y) \subset \bigcap_{n\in\mathbb{N}} \mathrm{Cl}(F_n(x,y)).$$

Now suppose that $z \in Z \setminus F(x,y)$. Since $F(x,y)$ is closed, there exist an open set $G \subset Z$ and an open neighbourhood $W(z)$ of z such that $F(x,y) \subset G$ and $W(z) \cap G = \emptyset$. By the upper semicontinuity of F_x at y, there exists an $m \in \mathbb{N}$ such that $F(x,v) \subset G$ for each $v \in B(y, 2^{-m})$. Hence,

$$\bigcup_{v\in B(y,2^{-m})} F(x,v) = F_m(x,y) \subset G,$$

and so $W(z) \cap \mathrm{Cl}(F_m(x,y)) = \emptyset$. Thus $z \in Z \setminus \bigcap_{n\in\mathbb{N}} \mathrm{Cl}(F_n(x,y))$, proving the inclusion

$$F(x,y) \supset \bigcap_{n\in\mathbb{N}} \mathrm{Cl}(F_n(x,y)). \ \blacksquare$$

THEOREM 2.16. *Let Z be a Suslin space. If $F : X \times Y \rightsquigarrow Z$ is a closed valued multifunction such that F^y is approximately lower semicontinuous with respect to $\mathcal{F}$ for every $y \in Y$ and F_x is upper semicontinuous for every $x \in X$, then F is $\mathcal{M}_{\mu\times\nu}(X \times Y)$-measurable.*

Proof. Let $(F_n)_{n\in\mathbb{N}}$ be the sequence of multifunctions given by

$$F_n(x,y) = F(x, B(y, 2^{-n})).$$

Then, by Proposition 2.14, F_n is approximately lower semicontinuous with respect to $\mathcal{F} \times \mathcal{G}$ for each $n \in \mathbb{N}$, and hence, according to Proposition 1.9,

(1) each F_n is weakly $\mathcal{M}(X) \otimes \mathcal{M}(Y)$-measurable.

Let $(\overline{F}_n)_{n\in\mathbb{N}}$ be the sequence of multifunctions defined by

$$\overline{F}_n(x,y) = \mathrm{Cl}(F_n(x,y)) \quad \text{for } (x,y) \in X \times Y.$$

Then each $\overline{F}_n$ has closed values, and hence is weakly $\mathcal{M}(X) \otimes \mathcal{M}(Y)$-measurable, by (1). Since the x-sections of F are upper semicontinuous, it follows that

$$F(x,y) = \bigcap_{n\in\mathbb{N}} \mathrm{Cl}(F_n)(x,y) \quad \text{for each } (x,y) \in X \times Y,$$

by Lemma 2.15. Thus Proposition 1.5 finishes the proof. ∎

The following example shows that the upper semicontinuity of x-sections of F in the above theorem cannot be replaced by lower semicontinuity.

EXAMPLE 2.17. Let $E \subset \mathbb{R}^2$ be the Sierpiński set [107], i.e., $E \notin \mathcal{L}(\mathbb{R}^2)$ and for any $y \in \mathbb{R}$ and any $x \in \mathbb{R}$, the sections E^y and E_x have at most two elements. Let $F : \mathbb{R}^2 \rightsquigarrow \mathbb{R}$ be the multifunction given by

$$F(x,y) = \begin{cases} [0,1] & \text{if } (x,y) \notin E, \\ \{0\} & \text{if } (x,y) \in E. \end{cases}$$

Then F is not $\mathcal{L}(\mathbb{R}^2)$-measurable although x-sections and y-sections are lower semicontinuous.

9. Multifunctions with quasi-continuous sections

Let $(X,\mathcal{M}(X))$ be a measurable space and let $(Y,\mathcal{T}(Y))$ and $(Z,\mathcal{T}(Z))$ be topological spaces. A multifunction $F:X\times Y\rightsquigarrow Z$ is called *lower* (resp. *upper*) *semi-Carathéodory* if F^y is $\mathcal{M}(X)$-measurable for each $y\in Y$ and F_x is lower (resp. upper) semicontinuous for each $x\in X$.

If (Z,ϱ) is a metric space, then replacing lower (resp. upper) semicontinuity of F_x in the above definition by h-lower (resp. h-upper) semicontinuity of F_x we obtain the notion of an *h-lower* (resp. *h-upper*) *semi-Carathéodory* multifunction.

Note that $F:X\times Y\rightsquigarrow Z$ is Carathéodory if and only if it is simultaneously lower and upper semi-Carathéodory.

If $F:X\times Y\rightsquigarrow Z$ is given by $F(x,y)=\{f(x,y)\}$, where $f:X\times Y\to Z$ is a function, then F is lower (resp. upper) semi-Carathéodory or Carathéodory if and only if f is a Carathéodory function.

We see from Proposition 2.2 that if (Y,d) is a separable metric space, (Z,ϱ) a metric space, and $F:X\times Y\rightsquigarrow Z$ a compact valued Carathéodory multifunction, then F is $\mathcal{M}(X)\otimes\mathcal{B}(Y)$-measurable.

Example 2.17 shows that a multifunction which is only lower semi-Carathéodory need not be product measurable. It is easy to see that the same is true for upper semi-Carathéodory multifunctions. For instance, the multifunction F in Example 2.17 is lower semi-Carathéodory. But if we transpose the values of F, then F will be upper semi-Carathéodory and still not $\mathcal{L}(\mathbb{R}^2)$-measurable.

One can strengthen the lower semi-Carathéodory assumption to ensure product measurability. For instance, Papageorgiou [94] gives the following result:

THEOREM 2.18. *If $(X,\mathcal{M}(X),\mu)$ is a measure space, where μ is σ-finite, Y is a separable reflexive Banach space, and $F:X\times Y\rightsquigarrow Y$ is a lower semi-Carathéodory multifunction with closed convex values such that the section $F_x:Y\rightsquigarrow Y_\omega$ is upper semicontinuous for every $x\in X$ (where Y_ω denotes Y with the weak topology), then F is $\mathcal{M}_\mu(X)\otimes\mathcal{B}(Y)$-measurable.*

Another possibility is given below.

THEOREM 2.19. *Let $(X,\mathcal{M}(X))$ be a measurable space, Y a Polish space and $(Z,\mathcal{T}(Z))$ a metrizable σ-compact space. Suppose that a lower semi-Carathéodory multifunction $F:X\times Y\rightsquigarrow Z$ with closed values has F_x upper quasi-continuous for each $x\in X$. Then F is $\mathcal{M}(X)\otimes\mathcal{B}(Y)$-measurable.*

Proof. Fix $z\in Z$. By Propositions 1.3(i) and (iv), it is enough to prove that

(1) the real function $g_z(x,y)=\varrho(z,F(x,y))$ is $\mathcal{M}(X)\otimes\mathcal{B}(Y)$-measurable.

Let $B(z,r)\subset Z$ be an open ball centred at z with radius $r>0$ and fix $(x,y)\in X\times Y$. Since F^y is $\mathcal{M}(X)$-measurable, it follows that F^y is weakly $\mathcal{M}(X)$-measurable, by Proposition 1.2(i). Thus, $(F^y)^-(B(z,r))\in\mathcal{M}(X)$. Note that

$$\begin{aligned}(F^y)^-(B(z,r))&=\{x\in X:F^y(x)\cap B(z,r)\neq\emptyset\}=\{x\in X:\rho(z,F^y(x))<r\}\\&=(g_z^y)^{-1}(-\infty,r).\end{aligned}$$

Therefore $(g_z^y)^{-1}(-\infty, r) \in \mathcal{M}(X)$, i.e.,

(2) the y-section of g_z is $\mathcal{M}(X)$-measurable.

By the lower semicontinuity of F_x, we know that $(F_x)^-(B(z,r))$ is an open subset of Y. Since

$$((g_z)_x)^{-1}(-\infty, r) = \{y \in Y : \rho(z, F_x(y)) < r\} = \{y \in Y : F_x(y) \cap B(z,r) \neq \emptyset\} = (F_x)^-(B(z,r)),$$

it follows that $((g_z)_x)^{-1}(-\infty, r)$ is an open subset of Y. Thus

(3) the x-section of g_z is upper semicontinuous.

By the upper quasi-continuity of F_x at y, there exists a quasi-open set $A(y)$ containing y such that $F_x|_{A(y)}$ is upper semicontinuous at y (see (1.27)). Therefore,

(4) there exists a nonempty open set $O(y)$ such that $O(y) \subset A(y) \subset \mathrm{Cl}(O(y))$, $y \in \mathrm{Cl}(O(y))$ and $F_x|_{O(y)\cup\{y\}}$ is continuous at y.

Let $S = \{s_1, s_2, \ldots\}$ be a dense subset of Y. Then, by (4), to each point $(x,y) \in X \times Y$ there corresponds a sequence $(s_n(x,y))_{n\in\mathbb{N}}$ such that

(5) $$s_n(x,y) \in S, \quad \lim_{n\to\infty} s_n(x,y) = y \quad \text{and} \quad \lim_{n\to\infty} g_z(x, s_n(x,y)) = g_z(x,y),$$

since $\lim_{n\to\infty} \rho(z, F(x, s_n(x,y)) = \rho(z, F(x,y))$.

Now define $G : X \rightsquigarrow Y \times \mathbb{R}$ by

$$G(x) = \{(y,r) \in Y \times \mathbb{R} : g_z(x,y) \geq r\}.$$

By (3), it is evident that

(6) $$G(x) \in \mathcal{C}(Y \times \mathbb{R}) \quad \text{for every } x \in X.$$

We will show that

(7) G is weakly $\mathcal{M}(X)$-measurable.

Let $\{q_1, q_2, \ldots\}$ be an enumeration of the rational numbers. Define the sequence of functions $f_{nm} : X \to Y \times \mathbb{R}$ by

$$f_{nm}(x) = (s_n(x,y), \min(q_m, g_z(x, s_n(x,y)))).$$

It is clear that

(8) $f_{nm} : X \to Y \times \mathbb{R}$ is $\mathcal{M}(X)$-measurable and $f_{nm}(x) \in G(x)$ for each $x \in X$ and all $n, m \in \mathbb{N}$.

Thus, $\{f_{nm}(x) : n, m \in \mathbb{N}\} \subset G(x)$ for each $x \in X$, and so, by (6),

(9) $$\mathrm{Cl}(\{f_{nm}(x) : n, m \in \mathbb{N}\}) \subset G(x).$$

Now let $(y,r) \in G(x)$, i.e., $g_z(x,y) \geq r$. We can choose the sequence $(q_m)_{m\in\mathbb{N}}$ so that $q_m \leq g_z(x, s_n(x,y))$ for each $m, n \in \mathbb{N}$, and $\lim_{m\to\infty} q_m = r$. Then $\lim_{n\to\infty} \lim_{m\to\infty} f_{nm}(x) = (y,r)$, and so $(y,r) \in \mathrm{Cl}(\{f_{nm}(x) : n, m \in \mathbb{N}\})$, which, together with (9), gives the equality

(10) $$G(x) = \mathrm{Cl}(\{f_{nm}(x) : n, m \in \mathbb{N}\}).$$

Now (7) is a simple consequence of (8), (10) and Proposition 1.3(ii). Therefore

$$\mathrm{Gr}(G) = \{(x, y, r) \in X \times Y \times \mathbb{R} : (y, r) \in G(x)\} \in \mathcal{M}(X) \otimes \mathcal{B}(Y \times \mathbb{R}),$$

by Proposition 1.3(iii), and thus

$$(\mathrm{Gr}(G))^r = \{(x, y) \in X \times Y : (x, y, r) \in \mathrm{Gr}(G)\} \in \mathcal{M}(X) \otimes \mathcal{B}(Y). \tag{11}$$

Note that

$$\begin{aligned}(\mathrm{Gr}(G))^r &= \{(x, y) \in X \times Y : (y, r) \in G(x)\} = \{(x, y) \in X \times Y : g_z(x, y) \geq r\} \\ &= X \times Y \setminus \{(x, y) \in X \times Y : g_z(x, y) < r\} = X \times Y \setminus g_z^{-1}(-\infty, r).\end{aligned}$$

Therefore, by (11), we have $g_z^{-1}(-\infty, r) \in \mathcal{M}(X) \otimes \mathcal{B}(Y)$, and (1) is proved. ∎

The classical result of Kempisty [55] asserts that a real function of two real variables which is separately quasi-continuous is quasi-continuous as a function of two variables. But such a function may not be product measurable, as shown by Marcus (see [79, (x), p. 49]). Some generalization of the result of Kempisty to the multivalued case was given by Neubrunn (see [85, 4.1.6 and 4.1.5]).

The situation is different for strong quasi-continuity. It is known that there is a function $f : \mathbb{R}^2 \to \mathbb{R}$ having f_x and f^y continuous (and therefore also strongly quasi-continuous), such that the set $D(f)$ of its discontinuity points is of positive m_2 measure (see [43, Theorem 7]). Thus, by Proposition 1.19, f is not strongly quasi-continuous as a function of two variables. But it turns out that it is product measurable.

Now our aim is to show that if a multifunction is measurable in the first variable and both lower and upper strongly quasi-continuous in the second variable, then it is product measurable. For this purpose we introduce some auxiliary multifunctions.

Let $X \neq \emptyset$, let $(Y, \mathcal{T}(Y))$ be a separable topological space with a countable dense set P, and let $(Z, \mathcal{T}(Z))$ be a topological space. We define two multifunctions $G_* : X \times Y \rightsquigarrow Z$ and $G^* : X \times Y \rightsquigarrow Z$ as follows:

$$G_*(x, y) = \text{q-}\liminf_{t \to y \wedge t \in P} (F_x)(t), \tag{2.2}$$

$$G^*(x, y) = \text{p-}\limsup_{t \to y \wedge t \in P} (F_x)(t). \tag{2.3}$$

Proposition 1.12(i) implies

PROPOSITION 2.20. *If $F : X \times Y \rightsquigarrow Z$ is a multifunction such that F_x is lower quasi-continuous for every $x \in X$, then $F(x, y) \subset G^*(x, y)$ for all $(x, y) \in X \times Y$, where G^* is given by* (2.3).

Similarly, Proposition 1.12(ii) yields

PROPOSITION 2.21. *If the space $(Z, \mathcal{T}(Z))$ is regular and second countable, and if $F : X \times Y \rightsquigarrow Z$ is a compact valued multifunction such that F_x is upper quasi-continuous for every $x \in X$, then $G_*(x, y) \subset F(x, y)$ for all $(x, y) \in X \times Y$, where G_* is given by* (2.2).

Now we assume that $(X, \mathcal{M}(X), \mu)$ is a measure space and $(Y, \rho, \mathcal{M}(Y), \nu)$ is a separable metric measure space, where ν is σ-finite and $\mathcal{B}(Y) \subset \mathcal{M}(Y)$. We suppose that $(\mathcal{G}, \to)$ is a differentiation basis of $(Y, \rho, \mathcal{M}(Y), \nu)$ (see (1.23)) with the density property (see (1.24)).

We are now in a position to prove the main theorem of this section.

THEOREM 2.22. *If Z is a Polish space and $F : X \times Y \rightsquigarrow Z$ is a compact valued multifunction such that*

(i) *F^y is weakly $\mathcal{M}(X)$-measurable for each $y \in Y$,*
(ii) *F_x is both lower and upper strongly quasi-continuous with respect to $\mathcal{G}$ for each $x \in X$,*

then F is measurable with respect to the $\mu \times \nu$-completion of $\mathcal{M}(X) \otimes \mathcal{B}(Y)$.

Proof. We first note that

$$\nu(D(F_x)) = 0 \quad \text{for each } x \in X, \tag{1}$$

by assumption (ii) and Proposition 1.19.

Let P be a countable dense subset of Y and let G_* and G^* be defined by (2.2) and (2.3), respectively. Then, by Propositions 2.20 and 2.21,

$$G_*(x,y) \subset F(x,y) \subset G^*(x,y) \quad \text{for all } (x,y) \in X \times Y. \tag{2}$$

Our next step is to show that both G_* and G^* are measurable with respect to the $\mu \times \nu$-completion of $\mathcal{M}(X) \otimes \mathcal{B}(Y)$.

Let $\mathcal{B}$ denote a countable base of Y. We have (see (1.14) and (1.15))

$$G_*(x,y) = \bigcap_{U \in \mathcal{B} \wedge y \in \mathrm{Cl}(U)} \mathrm{Cl}\Big(\bigcup_{t \in U \cap P} F(x,t)\Big).$$

For each $U \in \mathcal{B}$ we define the multifunction $G_U : X \times Y \rightsquigarrow Z$ by

$$G_U(x,y) = \bigcup_{t \in U \cap P} F(x,t),$$

and observe that for each $V \in \mathcal{T}(Z)$ we have

$$\begin{aligned} {G_U}^-(V) &= \Big\{(x,y) : \bigcup_{t \in U \cap P} F(x,t) \cap V \neq \emptyset\Big\} = \bigcup_{t \in U \cap P} (\{x \in X : F(x,t) \cap V \neq \emptyset\} \times Y) \\ &= \bigcup_{t \in U \cap P} ((F^t)^-(V) \times Y) \in \mathcal{M}(X) \otimes \mathcal{B}(Y), \end{aligned}$$

since $U \cap P$ is countable and all sections F^t are weakly $\mathcal{M}(X)$-measurable. Then the multifunction $\overline{G}_U : X \times Y \rightsquigarrow Z$ defined by

$$\overline{G}_U(x,y) = \mathrm{Cl}(G_U(x,y))$$

is $\mathcal{M}(X) \otimes \mathcal{B}(Y)$-measurable. Note that

$$G_*(x,y) = \bigcap \{\overline{G}_U(x,y) : U \in \mathcal{B} \wedge y \in \mathrm{Cl}(U)\}.$$

Now we define the multifunction $H_U : X \times Y \rightsquigarrow Z$ by

$$H_U(x,y) = \begin{cases} \overline{G}_U(x,y) & \text{if } y \in \mathrm{Cl}(U), \\ Z & \text{if } y \notin \mathrm{Cl}(U). \end{cases}$$

Observe that for each $V \in \mathcal{T}(Z)$ we have

$${H_U}^-(V) = (\overline{G}_U)^-(V) \cap (X \times \mathrm{Cl}(U)) \cup (X \times (Y \setminus \mathrm{Cl}(U)) \in \mathcal{M}(X) \otimes \mathcal{B}(Y),$$

since $(\overline{G}_U)^-(V) \in \mathcal{M}(X) \otimes \mathcal{B}(Y)$. Therefore H_U is weakly $\mathcal{M}(X) \otimes \mathcal{B}(Y)$-measurable. Furthermore,

$$G_*(x,y) = \bigcap_{U \in \mathcal{B}} H_U(x,y).$$

Thus, by Proposition 1.5, G_* is measurable with respect to the $\mu \times \nu$-completion of $\mathcal{M}(X) \otimes \mathcal{B}(Y)$; the proof for G^* is analogous.

Now consider the set

$$A = \{(x,y) : G_*(x,y) \neq G^*(x,y)\}.$$

Since G_* and G^* are measurable with respect to the $\mu \times \nu$-completion of $\mathcal{M}(X) \otimes \mathcal{B}(Y)$, it is clear that A belongs to that completion, by Proposition 1.4. By (1), the x-section of A is ν-negligible for each $x \in X$, since $A_x = \{y \in Y : G_*(x,y) \neq G^*(x,y)\} \subset D(F_x)$ (see (1.17)).

Thus A is $\mu \times \nu$-negligible. Furthermore, the double inclusion (2) gives the implication

$$G_*(x,y) = G^*(x,y) \ \Rightarrow \ G_*(x,y) = F(x,y),$$

which guarantees the $\mu \times \nu$-negligibility of the set

$$A_1 = \{(x,y) : G_*(x,y) \neq F(x,y)\} \subset A. \tag{3}$$

Next, let U be an arbitrary open subset of Z. Since G_* is in particular weakly measurable with respect to the $\mu \times \nu$-completion of $\mathcal{M}(X) \otimes \mathcal{B}(Y)$, we can suppose that

$$G_*^-(U) = (B \setminus A_2) \cup A_3, \tag{4}$$

where $B \in \mathcal{M}(X) \otimes \mathcal{B}(Y)$ and the sets A_2 and A_3 are $\mu \times \nu$-negligible.

Note that $F^-(U) = (F^-(U) \cap (X \times Y \setminus A_1)) \cup (F^-(U) \cap A_1)$. Thus, by (3) and (4), we have

$$\begin{aligned} F^-(U) &= ({G_*}^-(U) \cap (X \times Y \setminus A_1)) \cup (F^-(U) \cap A_1) \\ &= [((B \setminus A_2) \cup A_3) \cap (X \times Y \setminus A_1)] \cup (F^-(U) \cap A_1) \\ &= (B \setminus (A_1 \cup A_2)) \cup [A_3 \cap (X \times Y \setminus A_1)] \cup (F^-(U) \cap A_1). \end{aligned}$$

Since $B \in \mathcal{M}(X) \otimes \mathcal{B}(Y)$ and the sets A_i are $\mu \times \nu$-negligible for $i = 1, 2, 3$, Proposition 1.2(ii) finishes the proof. ■

10. Multifunctions whose sections are derivatives

The purpose of this section is to give some sufficient conditions for joint measurability of a multifunction with the (J) property.

The (J) property for real functions of two real variables was introduced by Lipiński [78] and intensively studied by Grande in the case of real functions defined on more general spaces (see [41]). Now we will consider this topic in the case of mltifunctions.

From now on we suppose that $(X, \mathcal{M}(X))$ is a measurable space, $(Z, \|\cdot\|)$ is a reflexive Banach space, and $I \subset \mathbb{R}$ is an interval.

Definition 2.23. We will say that a multifunction $F : X \times I \rightsquigarrow Z$ with values in $\mathcal{C}_{bc}(Z)$ has the (J) *property* if, for each $y \in I$, the section F^y is weakly $\mathcal{M}(X)$-measurable, for each $x \in X$, the section F_x is weakly $\mathcal{L}(\mathbb{R})$-measurable, and for each interval $P \subset I$, the multifunction $\Phi_P : X \rightsquigarrow Z$ given by

$$\Phi_P(x) = \int_P F(x,y)\,dy \tag{2.4}$$

is weakly $\mathcal{M}(X)$-measurable.

Example 2.17 shows that a multifunction with the (J) property need not be product measurable.

PROPOSITION 2.24. *Suppose that the σ-field $\mathcal{M}(X)$ is complete with respect to a σ-finite measure. If the space $(Z, \|\cdot\|)$ is separable and $F : X \times I \rightsquigarrow Z$ is a multifunction with values in $\mathcal{C}_{bc}(Z)$ such that F_x is R-integrable for each $x \in X$ and F^y is weakly $\mathcal{M}(X)$-measurable for each $y \in I$, then F has the* (J) *property.*

Proof. Fix $P = [c,d] \subset I$. We only need to show that the multifunction Φ_P given by (2.4) is weakly $\mathcal{M}(X)$-measurable. Let $y_i = c + i(d-c)/n$ for $i = 0,1,\ldots,n$ and $n \in \mathbb{N}$. If $x \in X$, then, by the R-integrability of F_x, we have

$$(R)\int_P F(x,y)\,dy = h\text{-}\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} F_x(y_i) = h\text{-}\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F^{y_i}(x),$$

and then, appyling (1.38)(ii), we have

$$\Phi_P(x) = h\text{-}\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F^{y_i}(x).$$

Fix $n \in \mathbb{N}$ and define the multifunction $\Phi_n : X \rightsquigarrow Z$ by

$$\Phi_n(x) = \sum_{1=1}^{n} F^{y_i}(x).$$

Then $\Phi_n(x) \in \mathcal{C}_{bc}(Z)$ for $x \in X$ (see (1.29)(iv)). Since F^{y_i} is weakly $\mathcal{M}(X)$-measurable for $i = 0,1,\ldots,n$, so is Φ_n, by Theorem III.40 of [14]. Thus Φ_P is weakly $\mathcal{M}(X)$-measurable, by (1.21). ■

Now we can prove the main theorem of this section.

THEOREM 2.25. *If a multifunction $F : X \times I \rightsquigarrow Z$ with values in $\mathcal{C}_{bc}(Z)$ has the* (J) *property and F_x is a derivative for each $x \in X$, i.e.,*

$$F_x(y) = h\text{-}\lim_{\Delta y \to 0} \frac{1}{\Delta y} \int_y^{y+\Delta y} F_x(t)\,dt \quad \textit{for } y \in I,$$

then F is weakly measurable with respect to the $\mu \times m$-completion of $\mathcal{M}(X) \otimes \mathcal{B}(\mathbb{R})$.

Proof. Fix $n \in \mathbb{N}$ and let $\Delta = \{y_{0,n}, y_{1,n}, \ldots, y_{n,n}\}$ be a partition of I into n equal intervals, i.e., $y_{i,n} - y_{i-1,n} = 1/n$ for $i = 1,\ldots,n$. Set

$$F_n(x,y) = \begin{cases} n\int_{y_{i-1,n}}^{y_{i,n}} F(x,y)\,dy & \text{if } x \in X \text{ and } y \in (y_{i-1,n}, y_{i,n}), \\ \{\theta\} & \text{if } x \in X \text{ and } y = y_{i,n},\ i = 0,1,\ldots,n. \end{cases}$$

Let $\Phi_{i,n} : X \rightsquigarrow Z$, for $i = 1, \ldots, n$, be given by

$$\Phi_{i,n}(x) = \int\limits_{y_{i-1,n}}^{y_{i,n}} F(x, y)\, dy.$$

By the (J) property of F, we see that

$$\text{(1)} \qquad \Phi_{i,n} \text{ is weakly } \mathcal{M}(X)\text{-measurable for each } i = 1, \ldots, n.$$

Define $H_n : X \times \bigcup_{i=1}^{n}(y_{i-1,n}, y_{i,n}) \rightsquigarrow Z$ by

$$H_n(x, y) = \Phi_{i,n}(x).$$

If $V \subset Z$ is open, then, by (1), we have

$$H_n^-(V) = \bigcup_{i=1}^{n} \Phi_{i,n}^-(V) \times (y_{i-1,n}, y_{i,n}) \in \mathcal{M}(X) \otimes \mathcal{B}(\mathbb{R}).$$

Therefore F_n is weakly $\mathcal{M}(X) \otimes \mathcal{B}(\mathbb{R})$-measurable and by (1.21) we only need to show that

$$\text{(2)} \qquad h\text{-}\lim_{n\to\infty} F_n(x, y) = F(x, y) \quad \text{for every } x \in X \text{ and almost every } y \in I.$$

Fix $(x_0, y_0) \in X \times I$ such that $y_0 \neq y_{i,n}$ for $n \in \mathbb{N}$ and $i = 1, \ldots, n$. Choose a sequence $(y_{n(i)})$ such that $y_{n(i)-1} < y_0 < y_{n(i)}$. Since F_{x_0} is a derivative at y_0, it follows that

$$F(x_0, y_0) = h\text{-}\lim_{\Delta y\to 0} \frac{1}{\Delta y} \int\limits_{y_0}^{y_0+\Delta y} F(x_0, y)\, dy.$$

Assume that

$$A_n = \frac{1}{y_0 - y_{n(i)-1}} \int\limits_{y_{n(i)-1}}^{y_0} F(x_0, y)\, dy, \quad B_n = \frac{1}{y_{n(i)} - y_0} \int\limits_{y_0}^{y_{n(i)}} F(x_0, y)\, dy$$

and

$$C_n = \frac{1}{y_{n(i)} - y_{n(i)-1}} \int\limits_{y_{n(i)-1}}^{y_{n(i)}} F(x_0, y)\, dy.$$

Then $h\text{-}\lim_{n\to\infty} A_n = F(x_0, y_0)$ and $h\text{-}\lim_{n\to\infty} B_n = F(x_0, y_0)$. Moreover

$$\begin{aligned} F_n(x_0, y_0) = C_n &= \frac{1}{y_{n(i)} - y_{n(i)-1}} \Big[\int\limits_{y_{n(i)-1}}^{y_0} F(x_0, y)\, dy + \int\limits_{y_0}^{y_{n(i)}} F(x_0, y)\, dy \Big] \\ &= \frac{y_0 - y_{n(i)-1}}{y_{n(i)} - y_{n(i)-1}} A_n + \frac{y_{n(i)} - y_0}{y_{n(i)} - y_{n(i)-1}} B_n. \end{aligned}$$

Let $\alpha_n = \frac{y_0 - y_{n(i)-1}}{y_{n(i)} - y_{n(i)-1}}$. Since the sequence (α_n) is bounded, we can take a subsequence $(\alpha_{n_k})_{k\in\mathbb{N}}$ such that $\alpha_{n_k} \to \alpha_0 \in [0, 1]$. Then

$$h\text{-}\lim_{k\to\infty} C_{n_k} = h\text{-}\lim_{k\to\infty} (\alpha_{n_k} A_{n_k} + (1 - \alpha_{n_k}) B_{n_k}) = \alpha_0 F(x_0, y_0) + (1 - \alpha_0) F(x_0, y_0),$$

and we conclude that

$$h\text{-}\lim_{k\to\infty} C_{n_k} = F(x_0, y_0),$$

since the set $F(x_0, y_0)$ is convex. Therefore any subsequence of $(F_n(x_0, y_0))_{n\in\mathbb{N}}$ converges to $F(x_0, y_0)$, which finishes the proof of (2). ■

REMARK 2.26. If, in Theorem 2.25, we suppose that the measure μ is σ-finite, then the multifunction F will be measurable with respect to the $\mu\times m$-completion of $\mathcal{M}(X)\otimes\mathcal{B}(Y)$, by 1.3(v).

As a straightforward consequence of (1.38)(i), Corollary 1.41, Proposition 2.24 and Theorem 2.25, we have the following corollary (cf. Proposition 2.2).

COROLLARY 2.27. *Let $(X, \mathcal{M}(X), \mu)$ be a measure space, where μ is σ-finite, and let $(Z, \|\cdot\|)$ be separable. If a multifunction $F : X \times I \rightsquigarrow Z$ with values in $\mathcal{C}_{bc}(Z)$ has F_x h-continuous for each $x \in X$ and F^y weakly $\mathcal{M}(X)$-measurable for each $y \in I$, then F is measurable with respect to the $\mu \times m$-completion of $\mathcal{M}(X) \otimes \mathcal{B}(\mathbb{R})$.*

11. The Scorza-Dragoni property of multifunctions

We conclude this chapter by introducing multifunctions having the Scorza-Dragoni property and giving their connections with $\mathcal{M}(X) \otimes \mathcal{B}(Y)$-measurable multifunctions.

G. Scorza-Dragoni [105] showed that every Carathéodory function $f : X \times Y \to Z$ has the property (now called the *Scorza-Dragoni property*) that, given any $\varepsilon > 0$, there is a closed subset X_ε of X with the measure of $X \setminus X_\varepsilon$ less than ε, such that the restriction of f to $X_\varepsilon \times Y$ is continuous. This result was extended in several directions (also to multifunctions), and used e.g. in control theory problems (see [2], [7], [10], [13], [36], [51], [56], [60], [110], [115], and others).

Let $(X, \mathcal{T}(X), \mathcal{M}(X), \mu)$ be a topological measure space and let $(Y, \mathcal{T}(Y))$ and $(Z, \mathcal{T}(Z))$ be topological spaces.

We say that a multifunction $F : X\times Y \rightsquigarrow Z$ has the *upper* (resp. *lower*) *Scorza-Dragoni property* if, given $\varepsilon > 0$, one may find a closed subset X_ε of X such that $\mu(X \setminus X_\varepsilon) < \varepsilon$, and the restriction of F to $X_\varepsilon \times Y$ is upper (resp. lower) semicontinuous. If F has both the upper and lower Scorza-Dragoni property, then we say that F has the *Scorza-Dragoni property*.

If (Z, ϱ) is a metric space, then replacing in the above definition the upper (resp. lower) semicontinuity of the restriction of F by its h-upper (resp. h-lower) semicontinuity, we obtain the *h-upper* (resp. *h-lower*) *Scorza-Dragoni propoerty* and the *h-Scorza-Dragoni property* of F.

Most of the results on the Scorza-Dragoni property of a multifunction F have required that its values are compact and the sections F_x are continuous. In [48] it is shown that, if $(X, \mathcal{T}(X))$ is a locally compact Hausdorff space and μ is a Radon measure on X, Y is a Polish space and (Z, ϱ) is a separable metric space, then a compact valued Carathéodory multifunction has the Scorza-Dragoni property, while a closed valued Carathéodory multifunction only has the lower Scorza-Dragoni property, in general.

The most complete presentation of multifunctions having the Scorza-Dragoni properties is contained in the thesis [117]. In that paper some relations between semi-Carathéodory multifunctions being weakly $\mathcal{M}(X)\otimes\mathcal{B}(Y)$-measurable and having the Scorza-Dragoni property are established.

Theorem 2.28 ([117, Theorem 4.2.5(i)]). *Let $(X,\mathcal{M}(X),\mu)$ be a measure space, where μ is regular and σ-finite, (Y,d) a complete separable metric space and (Z,ϱ) a separable metric space. Let $F: X\times Y \rightsquigarrow Z$ be a closed valued lower semi-Carathéodory multifunction. Then F has the lower Scorza-Dragoni property if and only if F is $\mathcal{M}_\mu(X)\otimes\mathcal{B}(Y)$-measurable.*

The following results are consequences of the above theorem and Theorems 22.18 and 22.19.

Theorem 2.29. *If $(X,\mathcal{M}(X),\mu)$ is a measure space with μ regular and σ-finite, Y is a separable reflexive Banach space, and $F: X\times Y \rightsquigarrow Y$ a lower semi-Carathéodory multifunction with closed convex values such that $F_x: Y\rightsquigarrow Y_\omega$ is upper semicontinuous for each $x\in X$ (where Y_ω denotes Y with the weak topology), then F has the lower Scorza-Dragoni property.*

Theorem 2.30. *Let $(X,\mathcal{M}(X),\mu)$ be a measure space with μ regular and σ-finite, Y a Polish space, and (Z,ϱ) a separable metric space. If $F: X\times Y\rightsquigarrow Z$ is a compact valued lower semi-Carathéodory multifunction such that F_x is upper quasi-continuous for every $x\in X$, then F has the lower Scorza-Dragoni property.*

In the case of an h-lower semi-Carathéodory multifunction an analogue to Theorem 2.28 is not true, in general. Consider the following example.

Example 2.31. Let $I=[0,1]$ and let $F: I\times\mathbb{R}\rightsquigarrow\mathbb{R}^2$ be given by

$$F(x,y)=\{(\alpha,x\alpha):\alpha\in\mathbb{R}\}.$$

Then F is h-lower semi-Carathéodory. It is also $\mathcal{L}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})$-measurable. But, for each $y\in\mathbb{R}$, F^y is h-lower semicontinuous on no subset of I. Therefore, F does not have the h-lower Scorza-Dragoni property.

Theorem 2.32 ([117, Theorem 4.2.5(ii) and (iii)]). *Let $(X,\mathcal{M}(X),\mu)$ be a measure space with μ regular and σ-finite, (Y,d) a complete separable metric space, and (Z,ϱ) a separable metric space. Let $F: X\times Y\rightsquigarrow Z$ be a closed valued h-lower semi-Carathéodory multifunction. Then*

(i) *If F has the h-lower Scorza-Dragoni property, then it is $\mathcal{M}_\mu(X)\otimes\mathcal{B}(Y)$-measurable.*
(ii) *If F is compact valued, then it has the h-lower Scorza-Dragoni property if and only if it is $\mathcal{M}_\mu(X)\otimes\mathcal{B}(Y)$-measurable.*

An analogue of Theorem 2.32(i) is also true for upper semi-Carathéodory multifunctions.

Theorem 2.33 ([117, Theorem 4.2.7(ii)]). *Let $(X,\mathcal{M}(X),\mu)$ be a measure space with μ regular and σ-finite, (Y,d) a complete separable metric space, (Z,ϱ) a separable metric space, and $F: X\times Y\rightsquigarrow Z$ a closed valued upper semi-Carathéodory (resp. h-upper semi-*

Carathéodory) multifunction. If F has the upper (resp. h-upper) Scorza-Dragoni property, then it is $\mathcal{M}_\mu(X) \otimes \mathcal{B}(Y)$-measurable.

Note that the multifunction F in Example 2.31 is both upper semi-Carathéodory and h-upper semi-Carathéodory. But it has neither the h-upper nor the upper Scorza-Dragoni property. In view of this example the problem arises to characterize those upper semi-Carathéodory multifunctions which have the upper Scorza-Dragoni property. A crucial role in solving this problem is played by the Filippov condition [30].

If $(X, \mathcal{M}(X))$ is a complete measurable space, Y is a Polish space and $(Z, \mathcal{T}(Z))$ is a topological space, then any $\mathcal{M}(X) \otimes \mathcal{B}(Y)$-measurable multifunction $F : X \times Y \rightsquigarrow Z$ satisfies the *Filippov condition*, i.e., for each open set $U \subset Y$ and each $V \in \mathcal{T}(Z)$, the set $\{x \in X : F(x, U) \subset V\}$ is $\mathcal{M}(X)$-measurable (see [117, Theorem 4.2.8]). Furthermore, the Filippov condition is a sufficient condition for the Scorza-Dragoni property of a compact valued upper semi-Carathéodory multifunction (see [117, Theorem 4.2.9] or [1, Lemma 5.1]). Finally, in the "upper" case, the following result is true.

THEOREM 2.34 ([117, Conclusion 4.2.10]). *Let $(X, \mathcal{M}(X), \mu)$ be a measure space with μ regular and σ-finite, (Y, d) a complete separable metric space, (Z, ϱ) a separable metric space, and $F : X \times Y \rightsquigarrow Z$ a compact valued upper semi-Carathéodory (resp. h-upper semi-Carathéodory) multifunction. Then F has the upper (or equivalently h-upper) Scorza-Dragoni property if and only if it is $\mathcal{M}_\mu(X) \otimes \mathcal{B}(Y)$-measurable.*

An interesting result on upper semi-Carathéodory multifunctions is given in [102]. If $(X, d, \mathcal{M}(X), \mu)$ is a metric measure space, where μ is σ-finite complete regular and X is locally compact, and if (Y, ρ) and (Z, ϱ) are separable metric spaces, then for every closed valued upper semi-Carathéodory multifunction $F : X \times Y \rightsquigarrow Z$ there is a closed valued multifunction $G : X \times Y \rightsquigarrow Z$ which has the Scorza-Dragoni property and satisfies $G(x, y) \subset F(x, y)$ for μ-almost every $x \in X$ and for all $y \in Y$.

3. SUP-MEASURABILITY OF MULTIFUNCTIONS

Sup-measurability of multifunctions has been considered in the literature (see for example [1], [61], [108], [111], [116] or [118]). The purpose of this chapter is to give some new sufficient conditions for this property.

Let $(X, \mathcal{M}(X))$ be a measurable space and let $(Y, \mathcal{T}(Y))$ and $(Z, \mathcal{T}(Z))$ be topological spaces. If $F : X \times Y \rightsquigarrow Z$ is a multifunction and the superposition of the Carathéodory type $H(x) = F(x, G(x))$ is $\mathcal{M}(X)$-measurable (resp. weakly $\mathcal{M}(X)$-measurable) for every closed valued $\mathcal{M}(X)$-measurable multifunction $G : X \rightsquigarrow Y$, then F is called *$\mathcal{M}(X)$-sup-measurable* (resp. *weakly $\mathcal{M}(X)$-sup-measurable*).

The following theorem is known (see [118, Theorem 1]).

Theorem 3.1. *Let $(X, \mathcal{M}(X), \mu)$ be a measure space with μ σ-finite. Let Y be a Polish space and $(Z, \mathcal{T}(Z))$ a topological space. If $F : X \times Y \rightsquigarrow Z$ is an $\mathcal{M}_\mu(X) \otimes \mathcal{B}(Y)$-measurable multifunction, then it is $\mathcal{M}_\mu(X)$-sup-measurable.*

Note that this is a generalization of Shragin's theorem to the multivalued case (see [106, Theorem 2 and Theorem 6]).

The projection property of the pair $((X, \mathcal{M}_\mu(X)); Y)$ in the above theorem is essential, since F may not be $\mathcal{M}(X)$-sup-measurable.

Example 3.2. Let $X = [0,1]$, $Y = \mathcal{N}$ (the irrational numbers in $(0,1)$) and $Z = \mathbb{R}$. If $K \subset X \times Y$ is closed with $\mathrm{proj}_X(K) \notin \mathcal{B}(X)$ and $F : X \times Y \rightsquigarrow Z$ is given by

$$F(x,y) = \begin{cases} [0,2] & \text{if } (x,y) \in K, \\ [0,1] & \text{if } (x,y) \notin K, \end{cases}$$

then F is $\mathcal{B}(X) \otimes \mathcal{B}(Y)$-measurable and $F^-((1,3)) = K \in \mathcal{B}(X) \otimes \mathcal{B}(Y)$. Define $G : X \rightsquigarrow Y$ by $G(x) = Y$. Then $\mathrm{Gr}(G) = X \times Y$. Set $H(x) = F(x, G(x))$. Then

$$\begin{aligned} H^-((1,3)) &= \{x \in X : F(x, G(x)) \cap (1,3) \neq \emptyset\} \\ &= \{x \in X : F(x,y) \cap (1,3) \neq \emptyset \wedge y \in G(x)\} \\ &= \mathrm{proj}_X \{(x,y) \in X \times Y : F(x,y) \cap (1,3) \neq \emptyset \wedge y \in G(x)\} \\ &= \mathrm{proj}_X (F^-((1,3)) \cap \mathrm{Gr}(G)) = \mathrm{proj}_X (K \cap X \times Y) = \mathrm{proj}_X(K) \notin \mathcal{B}(X), \end{aligned}$$

i.e., F is not weakly $\mathcal{B}(X)$-sup-measurable.

The above example also shows that a weakly $\mathcal{M}(X) \otimes \mathcal{B}(Y)$-measurable multifunction may not be weakly $\mathcal{M}(X)$-sup-measurable.

One can strengthen the weak $\mathcal{M}(X) \otimes \mathcal{B}(Y)$-measurability assumption to ensure weak $\mathcal{M}(X)$-sup-measurability. To see this we need the following proposition.

Proposition 3.3. *Let $(X,\mathcal{M}(X))$ be a measurable space, Y a Polish space and $(Z,\mathcal{T}(Z))$ a topological space. Suppose that $F : X\times Y \rightsquigarrow Z$ is a multifunction such that each section F_x is lower semicontinuous, and for each $\mathcal{M}(X)$-measurable function $h : X \to Y$, the multifunction $H(x) = F(x,h(x))$ is weakly $\mathcal{M}(X)$-measurable. Then F is weakly $\mathcal{M}(X)$-sup-measurable.*

Proof. Let $G : X \rightsquigarrow Y$ be an $\mathcal{M}(X)$-measurable multifunction with closed values. Then G is weakly $\mathcal{M}(X)$-measurable and, by Proposition 1.3(ii), G has a Castaing representation. Thus there is a sequence $(g_n)_{n\in\mathbb{N}}$ of $\mathcal{M}(X)$-measurable functions $g_n : X \to Y$ such that $G(x) = \mathrm{Cl}(\{g_n(x) : n \in \mathbb{N}\})$ for each $x \in X$. Let $H(x) = F(x,G(x))$ and $U \in \mathcal{T}(Z)$. Then

$$\begin{aligned} H^-(U) &= \{x\in X : F(x,G(x))\cap U \neq \emptyset\} \\ &= \Big\{x\in X : \Big(\bigcup_{y\in G(x)} F(x,y)\Big)\cap U \neq \emptyset\Big\} \\ &= \{x\in X : \exists y\in G(x)\ \ F(x,y)\cap U\neq\emptyset\} = \{x\in X : G(x)\cap F_x^-(U)\neq\emptyset\} \\ &= \{x\in X : \mathrm{Cl}(\{g_n(x) : n\in\mathbb{N}\})\cap F_x^-(U)\neq\emptyset\}. \end{aligned}$$

By the lower semicontinuity of F_x, the set $F_x^-(U)$ is open for each $x\in X$. Thus we can omit the closure in the last term of the above expression to obtain

$$H^-(U) = \{x\in X : \{g_n(x) : n\in\mathbb{N}\}\cap F_x^-(U)\neq\emptyset\} = \bigcup_{n\in\mathbb{N}}\{x\in X : F(x,g_n(x))\cap U\neq\emptyset\},$$

and, by assumption, $H^-(U)\in\mathcal{M}(X)$, since g_n is $\mathcal{M}(X)$-measurable for every $n\in\mathbb{N}$. ∎

Theorem 3.4. *Let $(X,\mathcal{M}(X))$ be a measurable space, Y a Polish space and $(Z,\mathcal{T}(Z))$ a topological space. If a multifunction $F : X\times Y\rightsquigarrow Z$ is weakly $\mathcal{M}(X)\otimes\mathcal{B}(Y)$-measurable with F_x lower semicontinuous for each $x\in X$, then F is weakly $\mathcal{M}(X)$-sup-measurable.*

Proof. Let $h : X\to Y$ be an $\mathcal{M}(X)$-measurable function and let $H(x) = F(x,h(x))$. Observe that for each $M\subset Z$ we have

(1) $$H^-(M) = \{x\in X : F(x,h(x))\cap M\neq\emptyset\} = \{x\in X : (x,h(x))\in F^-(M)\}.$$

Let $A\in\mathcal{M}(X)$ and $B\in\mathcal{B}(Y)$. Then

$$\{x\in X : (x,h(x))\in A\times B\} = A\cap h^{-1}(B)\in\mathcal{M}(X),$$

and so

(2) $$\{x\in X : (x,h(x))\in C\}\in\mathcal{M}(X) \quad \text{for each } C\in\mathcal{M}(X)\otimes\mathcal{B}(X).$$

Let $V\in\mathcal{T}(Z)$. By the weak $\mathcal{M}(X)\otimes\mathcal{B}(Y)$-measurability of F, $F^-(V)\in\mathcal{M}(X)\otimes\mathcal{B}(Y)$. Then $H^-(V)\in\mathcal{M}(X)$, by (1) and (2). Thus H is weakly $\mathcal{M}(X)$-measurable and, by Proposition 3.3, F is weakly $\mathcal{M}(X)$-sup-measurable. ∎

As a straightforward consequence of Theorem 3.1 and Proposition 2.2 we have the following corollary (cf. [111] and [116]).

Corollary 3.5. *If $(X,\mathcal{M}(X),\mu)$ is a measure space with μ σ-finite, Y a Polish space, (Z,ϱ) a separable metric space, and $F : X\times Y\rightsquigarrow Z$ a compact valued Carathéodory multifunction, then F is $\mathcal{M}_\mu(X)$-sup-measurable.*

Note that the assumption of compactness of values of F is essential.

EXAMPLE 3.6. Let $X = [0,1]$, $Y = \mathcal{N}$ and $Z = \mathbb{R}^2$. Let $E \subset X$ be a non-$\mathcal{L}(\mathbb{R})$-measurable set. Then $F : X \times Y \rightsquigarrow Z$ given by

$$F(x,y) = \begin{cases} \mathrm{proj}_X^{-1}(x) & \text{if } x \neq y, \\ \mathrm{proj}_X^{-1}(x) & \text{if } x = y \text{ and } x \in E, \\ \{x\} \times [0,1] & \text{if } x = y \text{ and } x \in X \setminus E, \end{cases}$$

is a Carathéodory multifunction. But the multifunction $H(x) = F(x, \{x\})$ is not $\mathcal{L}(\mathbb{R})$-measurable, since $H^+((0,1) \times (0,1)) = E \notin \mathcal{L}(\mathbb{R})$.

It is easy to see that a lower or upper semi-Carathéodory multifunction need not be sup-measurable (even if it is compact valued and the σ-field $\mathcal{M}(X)$ is complete with respect to a σ-finite measure).

EXAMPLE 3.7. Consider $F : \mathbb{R} \times \mathbb{R} \rightsquigarrow \mathbb{R}$ defined by

$$F(x,y) = \begin{cases} [-1,2] & \text{if } x \neq y, \\ [-1,0] & \text{if } x = y \text{ and } x \in A, \\ [1,2] & \text{if } x = y \text{ and } x \in \mathbb{R} \setminus A, \end{cases}$$

where $A \notin \mathcal{L}(\mathbb{R})$. It is clear that F is a lower semi-Carathéodory multifunction. But if $G(x) = \{x\}$ for $x \in \mathbb{R}$, then $H(x) = F(x, G(x))$ is not $\mathcal{L}(\mathbb{R})$-measurable.

One can strengthen the lower semi-Carathéodory assumption to ensure sup-measurability. For instance, by Theorems 2.18 and 3.1 we have the following result.

THEOREM 3.8. *If $(X, \mathcal{M}(X), \mu)$ is a measure space with μ σ-finite, Y a reflexive separable Banach space, and $F : X \times Y \rightsquigarrow Y$ a lower semi-Carathéodory multifunction with closed convex values such that each $F_x : Y \rightsquigarrow Y_\omega$ is upper semicontinuous (where Y_ω denotes Y with the weak topology), then F is $\mathcal{M}_\mu(X)$-sup-measurable.*

By Theorems 2.19 and 3.1, we obtain the following result.

THEOREM 3.9. *Let $(X, \mathcal{M}(X), \mu)$ be a measure space with μ σ-finite. Let Y be a Polish space and $(Z, \mathcal{T}(Z))$ a metrizable σ-compact space. If $F : X \times Y \rightsquigarrow Z$ is a closed valued lower semi-Carathéodory multifunction such that each section F_x is upper quasi-continuous, then F is $\mathcal{M}_\mu(X)$-sup-measurable.*

The next result follows at once from Theorems 2.28 and 3.1.

THEOREM 3.10. *Let $(X, \mathcal{M}(X), \mu)$ be a measure metric space with μ σ-finite and regular. Let Y be a Polish space and (Z, ϱ) a separable metric space. If $F : X \times Y \rightsquigarrow Z$ is a closed valued lower semi-Carathéodory multifunction which has the lower Scorza-Dragoni property, then F is $\mathcal{M}_\mu(X)$-sup-measurable.*

REMARK 3.11. Let $(X, \mathcal{M}(X), \mu)$ be a measure space with μ σ-finite, Y a Polish space and $(Z, \mathcal{T}(Z))$ a perfectly normal topological space. If $F : X \times Y \rightsquigarrow Z$ is a compact valued multifunction fulfilling the assumptions of Theorem 2.3, then F is $\mathcal{M}(X) \otimes \mathcal{B}(Y)$-measurable (see Remark 2.4(i)), and hence also $\mathcal{M}_\mu(X)$-sup-measurable, by Theorem 3.1.

In particular, by Corollary 2.6, we obtain the following result.

PROPOSITION 3.12. *Let $(X, \mathcal{M}(X), \mu)$ be a measure space with μ σ-finite, and $(Z, \mathcal{T}(Z))$ a perfectly normal topological space. If $F : X \times \mathbb{R} \rightsquigarrow Z$ is a compact valued multifunction such that each F_x is right-continuous (resp. left-continuous) and each F^y is $\mathcal{M}(X)$-measurable, then F is $\mathcal{M}_\mu(X)$-sup-measurable.*

It is essential that the x-sections of F in the above proposition are all right-continuous (or all left-continuous).

EXAMPLE 3.13. Let $F : [0,1]^2 \rightsquigarrow \mathbb{R}$ be given by

$$F(x,y) = \begin{cases} [1,2] & \text{if } x \in A \text{ and } y \le x, \\ [1,2] & \text{if } x \in \mathbb{R} \setminus A \text{ and } y < x, \\ \{0\} & \text{in other cases.} \end{cases}$$

where $A \subset [0,1]$ is non-Lebesgue measurable. Then some x-sections of F are right-continuous, others are left-continuous. Furthermore, each y-section is $\mathcal{L}(\mathbb{R})$-measurable. But F is not $\mathcal{L}(\mathbb{R})$-sup-measurable, since $H(x) = F(x, \{x\})$ is not $\mathcal{L}(\mathbb{R})$-measurable.

Note that Proposition 3.12 remains true if we suppose that $(Z, \mathcal{T}(Z))$ is metrizable σ-compact and F is closed valued (see Remark 2.4(ii)).

Now we shall consider the sup-measurability of multifunctions with the (J) property. Example 3.13 shows that such a multifunction may not be sup-measurable. One can strengthen the (J) property assumption to ensure sup-measurability.

Let $(X, \mathcal{M}(X))$ be a measurable space and (Z, ϱ) a separable metric space. Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of closed valued multifunctions $F_n : X \times Y \rightsquigarrow Z$. Observe that

(3.1) If $F = h\text{-}\lim_{n\to\infty} F_n$ and each F_n is $\mathcal{M}(X)$-sup-measurable, then F is weakly $\mathcal{M}(X)$-sup-measurable.

Indeed, let $z \in Z$. By (1.12), we have $\lim_{n\to\infty} \varrho(z, F_n(x,y)) = \varrho(z, F(x,y))$ for each $(x,y) \in X \times Y$. Let $G : X \rightsquigarrow Y$ be $\mathcal{M}(X)$-measurable with closed values. Let $x \in X$, $H_n(x) = F_n(x, G(x))$ for each $n \in \mathbb{N}$, and $H(x) = F(x, G(x))$. It is clear that $\lim_{n\to\infty} \varrho(z, H_n(x)) = \varrho(z, H(x))$. Fix $n \in \mathbb{N}$. Note that F_n being $\mathcal{M}(X)$-sup-measurable implies F_n is weakly $\mathcal{M}(X)$-sup-measurable. Hence H_n is weakly $\mathcal{M}(X)$-measurable. Therefore, by Proposition 1.3 (i), the real function $x \mapsto \varrho(z, H_n(x))$ is $\mathcal{M}(X)$-measurable. Thus the real function $x \mapsto \varrho(z, H(x))$ is $\mathcal{M}(S)$-measurable and, again by Proposition 1.3(i), H is weakly $\mathcal{M}(X)$-measurable.

From now on we assume that $(X, \mathcal{M}(X), \mu)$ is a measure space with μ σ-finite, and $I \subset \mathbb{R}$ is an interval.

THEOREM 3.14. *Let $(Z, \|\cdot\|)$ be a separable Banach space. If a multifunction $F : X \times I \rightsquigarrow Z$ with values in $\mathcal{K}_c(Z)$ has the (J) property and each section F_x is a derivative, then F is $\mathcal{M}_\mu(X)$-sup-measurable.*

Proof. Let $(x,y) \in X \times I$. Since F_x is a derivative at y,

$$F(x,y) = h\text{-}\lim_{\Delta y \to 0} \frac{1}{\Delta y} \int_y^{y+\Delta y} F(x,t)\, dt. \tag{1}$$

For every $n \in \mathbb{N}$ we define $F_n : X \times I \rightsquigarrow Z$ by

$$F_n(x,y) = n \int\limits_{y}^{y+1/n} F(x,t)\,dt.$$

Then $h\text{-}\lim_{n\to\infty} F_n(x,y) = F(x,y)$ for $(x,y) \in X \times Y$, by (1). For fixed $n \in \mathbb{N}$, each section $(F_n)_x$ is continuous, by Lemma 1.36 and (1.11)(iii). Since F has the (J) property, $(F_n)^y$ is $\mathcal{M}(X)$-measurable for every $y \in I$. Thus F_n is a Carathéodory multifunction, and thus, by Corollary 3.5, it is $\mathcal{M}_\mu(X)$-sup-measurable. Then, by (3.1), F is weakly $\mathcal{M}_\mu(X)$-sup-measurable, and hence also $\mathcal{M}_\mu(X)$-measurable, since its values are compact. ∎

In particular, by Proposition 2.24 and the above theorem, we have the following result.

Corollary 3.15. *If $(Z, \|\cdot\|)$ is a separable Banach space and $F : X \times I \rightsquigarrow Z$ is a multifunction with values in $\mathcal{K}_c(Z)$ such that each F_x is an R-integrable derivative and each F^y is $\mathcal{M}(X)$-measurable, then F is $\mathcal{M}_\mu(X)$-sup-measurable.*

Theorem 3.1 implies that each $\mathcal{M}(X) \otimes \mathcal{B}(Y)$-measurable multifunction is $\mathcal{M}_\mu(X)$ -sup-measurable whenever μ is σ-finite and Y is a Polish space. The following example shows that for σ-fields in $X \times Y$ more general than the product $\mathcal{M}(X) \otimes \mathcal{B}(Y)$, this property may not be true.

Example 3.16. Let $X = Y = \mathbb{R}$ and let $E \notin \mathcal{L}(\mathbb{R})$. If $F : \mathbb{R}^2 \rightsquigarrow \mathbb{R}$ is given by

$$F(x,y) = \begin{cases} [0,2] & \text{if } x \neq y, \\ [0,1] & \text{if } x = y \wedge x \in E, \\ \{0\} & \text{if } x = y \wedge x \notin E, \end{cases}$$

then F is $\mathcal{L}(\mathbb{R}^2)$-measurable. But $H(x) = F(x, \{x\})$ is not $\mathcal{L}(\mathbb{R})$-measurable, i.e., F is not $\mathcal{L}(\mathbb{R})$-sup-measurable.

We end this chapter with some results on the sup-measurability of a multifunction which is measurable with respect to a complete σ-field treated as a multifunction of two variables.

Theorem 3.17. *Let (Z, ϱ) be a separable metric space and $F : X \times \mathbb{R} \rightsquigarrow Z$ a closed valued weakly $\mathcal{M}_{\mu\times m}(X \times \mathbb{R})$-measurable multifunction such that each section F_x is weakly $\mathcal{L}(\mathbb{R})$-measurable. If for each open set $V \subset Z$,*

(i) $\quad D_l(F_x^-(V), y) > 2/3$ *and* $D_l(F_x^+(V), y) > 2/3$ *for each* $(x,y) \in X \times \mathbb{R}$,

then F is weakly $\mathcal{M}_\mu(X)$-sup-measurable.

Proof. Let $H : X \rightsquigarrow \mathbb{R}$ be closed valued and $\mathcal{M}_\mu(X)$-measurable. By Proposition 1.3(i), it is sufficient to prove that the real function

(1) $\quad g_z(x) = \varrho(z, F(x, H(x)))$ is $\mathcal{M}_\mu(X)$-measurable for every $z \in Z$.

Fix $z \in Z$. To prove (1), we apply the Davies lemma [17], i.e., it is sufficient to show that, for every $\varepsilon > 0$, the family $\mathcal{D}_\varepsilon = \{D \in \mathcal{M}(X) : \operatorname{osc}_D(g_z) \le \varepsilon\}$ satisfies the following condition:

(D) for every $A \in \mathcal{M}(X)$ of positive measure μ there exists a $D \in \mathcal{D}_\varepsilon$ such that $D \subset A$ and $\mu(D) > 0$.

Fix $\varepsilon > 0$. Let $([a_n, b_n])_{n\in\mathbb{N}}$ be a sequence of intervals with nonnegative rational endpoints such that $b_n - a_n < \varepsilon/4$ for $n \in \mathbb{N}$. Let $A \in \mathcal{M}(X)$ with $\mu(A) > 0$, and put

$$A_n = \{x \in A : a_n \le g_z(x) \le b_n\} \quad \text{for } n \in \mathbb{N}.$$

Then $A = \bigcup_{n\in\mathbb{N}} A_n$. Since $\mu(A) > 0$, there is an $n_0 \in \mathbb{N}$ such that $\mu^*(A_{n_0}) > 0$. Furthermore, $[a_{n_0}, b_{n_0}] \subset [g_z(x) - \varepsilon/2, g_z(x) + \varepsilon/2]$ for $x \in A_{n_0}$ and

$$(2) \qquad g_z(x) = \varrho(z, F(x, H(x)) = \varrho\Big(z, \bigcup_{y\in H(x)} F(x,y)\Big) \in [a_{n_0}, b_{n_0}].$$

Let $f_z : X \times \mathbb{R} \to \mathbb{R}$ be defined by $f_z(x,y) = \varrho(z, F(x,y))$ and let $x_0 \in A_{n_0}$. We put

$$M = \{(x,y) \in X \times \mathbb{R} : |f_z(x,y) - g_z(x_0)| \le \varepsilon/2\}.$$

Observe that f_z is $\mathcal{M}_{\mu\times m}(X\times\mathbb{R})$-measurable, since F is weakly $\mathcal{M}_{\mu\times m}(X\times\mathbb{R})$-measurable. Therefore

$$M = f_z^{-1}([g_z(x_0) - \varepsilon/2, g_z(x_0) + \varepsilon/2]) \in \mathcal{M}_{\mu\times m}(X\times\mathbb{R}).$$

Let $x \in X$. By the weak $\mathcal{L}(\mathbb{R})$-measurability of F_x, the x-section of f_z is $\mathcal{L}(\mathbb{R})$-measurable. Thus $M_x \in \mathcal{L}(\mathbb{R})$ for every $x \in X$. We will show that

(3) for each $\mathcal{M}(X)$-measurable selection h of H, there is a set $C \in \mathcal{M}(X)$ such that $A_{n_0} \subset C$ and $D_u(M_x, h(x) \ge 1/3$ for all $x \in C$.

Note that

$$\begin{aligned} M_x &= \{y \in \mathbb{R} : f_z(x,y) \ge g_z(x_0) - \varepsilon/2\} \cap \{y \in \mathbb{R} : f_z(x,y) \le g_z(x_0) + \varepsilon/2\} \\ &= \mathbb{R} \setminus [(\mathbb{R} \setminus (f_z)_x^{-1}((-\infty, g_z(x_0) - \varepsilon/2))) \cup (\mathbb{R} \setminus (f_z)_x^{-1}((g_z(x_0) + \varepsilon/2, \infty)))] \\ &= \mathbb{R} \setminus [(\mathbb{R} \setminus F_x^-(B(z, g_z(x_0) - \varepsilon/2))) \cup (\mathbb{R} \setminus F_x^+(\mathbb{R} \setminus \mathrm{Cl}(B(z, g_z(x_0) + \varepsilon/2))))]. \end{aligned}$$

By assumption (i), we have $D_l(M_x, y) > 1/3$ for each $y \in \mathbb{R}$. Furthermore,

$$f_z(x,y) \in [a_{n_0}, b_{n_0}] \subset [g_z(x_0) - \varepsilon/2, g_z(x_0) + \varepsilon/2] \quad \text{for } x \in A_{n_0} \text{ and } y \in H(x).$$

In particular, for every $\mathcal{M}(X)$-measurable selection $h : X \to \mathbb{R}$ of H we have

$$\{(x, h(x)) \in X \times \mathbb{R} : x \in A_{n_0}\} \subset M \quad \text{and} \quad D_l(B_x, h(x)) > 1/3 \quad \text{for } x \in A_{n_0}.$$

Let h be an $\mathcal{M}(X)$-measurable selection of H (guaranteed by Proposition 1.3(vii)). Then

$$M \cap \{(x,y) \in X \times \mathbb{R} : y \in B(h(x), 1/n)\} \in \mathcal{M}_{\mu\times m}(X\times\mathbb{R}).$$

Let $n \in \mathbb{N}$ and put

$$B_n = \Big\{x \in A_{n_0} : U \subset B(h(x), 1/n) \wedge h(x) \in U \Rightarrow \frac{m(M_x \cap U)}{m(U)} > \frac{1}{3}\Big\},$$

where $U \subset \mathbb{R}$ is an arbitrary open interval. Then $B_i \subset B_{i+1}$ for $i \in \mathbb{N}$ and $A_{n_0} = \bigcup_{n\in\mathbb{N}} B_n$. Let $i_0 = \min\{i \in \mathbb{N} : \mu^*(B_i) > 0\}$. If $n \ge i_0$ and $x \in B_n$, then

$$M_x \cap B(h(x), 1/n) \in \mathcal{L}(\mathbb{R}) \quad \text{and} \quad m(M_x \cap B(h(x), 1/n)) > \tfrac{1}{3} \cdot m(B(h(x), 1/n)) = 2/(3n).$$

If we put

$$C_n = \left\{ x \in X : m\left(M_x \cap B\left(h(x), \frac{1}{n} \right) \right) > \frac{2}{3n} \right\},$$

then $B_n \subset C_n$ and $C_n \in \mathcal{M}(X)$. Set

$$C = \bigcup_{k \geq i_0} \bigcap_{n \geq k} C_n.$$

Then $A_{n_0} \subset C$, since $B_k \subset \bigcap_{n \geq k} C_n$ for $k \geq i_0$. Furthermore, $C \in \mathcal{M}(X)$ and $D_u(M_x, h(x)) \geq 1/3$ for each $x \in C$. Thus (3) is proved.

Now suppose, on the contrary, that for every $D \in \mathcal{M}(X)$ such that $D \subset A$ and $\mu(D) > 0$ we have $\operatorname{osc}_D g_z > \varepsilon$.

Let $D = A \cap C$. Then $D \in \mathcal{M}(X)$ and $\mu(D) > 0$, since $A_{n_0} \subset A \cap C$ and $\mu^*(A_{n_0}) > 0$. Thus, there is an $x_1 \in D$ such that $|g_z(x_1) - g_z(x_0)| > \varepsilon/2$. We have two possibilities: either $g_z(x_1) > g_z(x_0) + \varepsilon/2$ or $g_z(x_1) < g_z(x_0) - \varepsilon/2$.

Suppose that $g_z(x_1) > g_z(x_0) + \varepsilon/2$. Then

$$g_z(x_1) = \varrho(z, F(x_1, H(x_1))) = \varrho\Big(z, \bigcup_{y \in H(x_1)} F(x_1, y)\Big) > g_z(x_0) + \varepsilon/2. \tag{4}$$

Furthermore,

$$\begin{aligned}\{y \in \mathbb{R} : \varrho(z, F(x_1, y)) > g_z(x_0) + \varepsilon/2\} &= \{y \in \mathbb{R} : f_z(x_1, y) > g_z(x_0) + \varepsilon/2\} \\ &= (f_z)_{x_1}^{-1}((g_z(x_0) + \varepsilon/2, \infty)) = F_{x_1}^{+}(\mathbb{R} \setminus \mathrm{Cl}(B(z, g_z(x_0) + \varepsilon/2))).\end{aligned}$$

Then, by assumption (i) and (4), we have

$$D_l(\{y \in \mathbb{R} : f_z(x_1, y) > g_z(x_0) + \varepsilon/2\}, y) > 2/3 \quad \text{for each } y \in H(x_1),$$

and so

$$D_l(\{y \in \mathbb{R} : f_z(x_1, y) > g_z(x_0) + \varepsilon/2\},\ h(x_1)) > 2/3, \tag{5}$$

because $h(x_1) \in H(x_1)$. Since $x_1 \in C$, by (3) and (5), it follows that

$$M_{x_1} \cap \{y \in \mathbb{R} : f_z(x_1, y) > g_z(x_0) + \varepsilon/2\} \neq \emptyset. \tag{6}$$

Then there is a $t \in \mathbb{R}$ such that

$$|f_z(x_1, t) - g_z(x_0)| \leq \varepsilon/2 \quad \text{and} \quad f_z(x_1, t) > g_z(x_0) + \varepsilon/2,$$

and we have a contradiction.

Now suppose that $g_z(x_1) < g_z(x_0) - \varepsilon/2$. Then

$$g_z(x_1) = \varrho(z, F(x_1, H(x_1)) = \varrho\Big(z, \bigcup_{y \in H(x_1)} F(x_1, y)\Big) < g_z(x_0) - \varepsilon/2.$$

Therefore, there is a $y_1 \in H(x_1)$ such that $\varrho(z, F(x_1, y_1)) < g_z(x_0) - \varepsilon/2$. Furthermore,

$$\begin{aligned}\{y \in \mathbb{R} : \varrho(z, F(x_1, y)) < g_z(x_0) - \varepsilon/2\} &= \{y \in \mathbb{R} : f_z(x_1, y) < g_z(x_0) - \varepsilon/2\} \\ &= F_{x_1}^{-}(B(z, g_z(x_0) - \varepsilon/2)).\end{aligned}$$

Thus, by (i) we have

$$D_l(\{y \in \mathbb{R} : f_z(x_1, y) < g_z(x_0) - \varepsilon/2\}, y_1) > 2/3.$$

The selection h in (3) may be modified if necessary by taking $h(x_1) = y_1$, without changing the set C. Then $D_u(M_{x_1}, y_1) \geq 1/3$, by (3).

As in the proof of (6), we show that

$$\exists t \in M_{x_1} \cap \{y \in \mathbb{R} : f_z(x_1, y) < g_z(x_0) - \varepsilon/2\}.$$

Thus $|f_z(x_1,t) - g_z(x_0)| \leq \varepsilon/2$ and $f_z(x_1,t) < g_z(x_0) - \varepsilon/2$, and again we have a contradiction, which finishes the proof. ■

Observe that by Theorem 3.17 and Propositions 1.8 and 1.9, we have the following result.

Proposition 3.18. *If (Z, ϱ) is a separable metric space and $F : X \times \mathbb{R} \rightsquigarrow Z$ is a closed valued weakly $\mathcal{M}_{\mu\times m}(X \times \mathbb{R})$-measurable multifunction such that F_x is approximately continuous for every $x \in X$, then F is weakly $\mathcal{M}_\mu(X)$-sup-measurable.*

Consider the following example.

Example 3.19. Let $C \subset [0,1]$ be a Cantor set with $m(C) > 0$ and let A be a subset of C such that $A \notin \mathcal{L}(\mathbb{R})$. By Theorem 13.1 of [93], there is a homeomorphism $h : [0,1] \to [0,1]$ such that $h(A) \in \mathcal{L}(\mathbb{R})$ and $m(h(A)) = 0$. Let $B = h(A)$ and define $F : [0,1] \times [0,1] \rightsquigarrow \mathbb{R}$ by

$$F(x,y) = \begin{cases} [0,1] & \text{if } x \in [0,1] \wedge y \notin B, \\ \{0\} & \text{if } x \in [0,1] \wedge y \in B. \end{cases}$$

Then F is $\mathcal{L}(\mathbb{R}^2)$-measurable and F_x is approximately lower semicontinuous for each $x \in [0,1]$. But F is not weakly $\mathcal{L}(\mathbb{R})$-sup-measurable, since the multifunction $G(x) = F(x, \{g(x)\})$, where $g = h^{-1}$, is not $\mathcal{L}(\mathbb{R})$-measurable. Therefore in Proposition 3.18 it is not sufficient to suppose that all the sections F_x are just approximately lower semicontinuous.

If we transpose the values of F, its x-sections will be approximately upper semicontinuous and it will still be $\mathcal{L}(\mathbb{R}^2)$-measurable, but not $\mathcal{L}(\mathbb{R})$-sup-measurable. Again, Proposition 3.18 does not hold if we suppose that all the sections F_x are just approximately upper semicontinuous.

Observe that, by Theorem 3.14 and Proposition 1.42, we have the following corollary.

Corollary 3.20. *Let $(X, \mathcal{M}(X), \mu)$ be a measure space, where μ is σ-finite, and let $(Z, \|\cdot\|)$ be a separable Banach space. Let $F : X \times I \rightsquigarrow Z$ be a bounded multifunction with $F(x,y) \in \mathcal{C}_{bc}(Z)$. If F has the* (J) *property and each x-section of F is approximately continuous, then F is $\mathcal{M}_\mu(X)$-sup-measurable.*

The next corollary follows at once by Theorem 2.9, Proposition 3.18 and (1.11)(iii).

Corollary 3.21. *If $(X, \mathcal{M}(X), \mu)$ is a measure space, where μ is σ-finite, (Z, ϱ) is a separable metric space and $F : X \times \mathbb{R} \rightsquigarrow Z$ is a compact valued multifunction such that $\{F_x\}_{x\in X}$ is approximately h-equicontinuous and F^y is $\mathcal{M}(X)$-measurable for every $y \in \mathbb{R}$, then F is $\mathcal{M}_\mu(X)$-sup-measurable.*

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