0. Introduction

Let us consider the following stochastic differential equation in a real separable Hilbert space H:

$$(*) \qquad \begin{cases} dX_t = [AX_t + F(X_t)]dt + BdW_t, \\ X_0 = x \in H, \qquad t \geq 0. \end{cases}$$

In this equation:

- A is the infinitesimal generator of a strongly continuous semigroup S_t, $t \geq 0$, of linear bounded operators on H,
- F is a Borel mapping from H to H,
- B is a linear bounded operator from a real separable Hilbert space K to H,
- W_t, $t \geq 0$, is a K-valued standard cylindrical Wiener process.

If the operators Q_t defined by

$$(0.1) \qquad Q_t x = \int_0^t S_s BB^* S_s^* x \, ds, \qquad x \in H,\ t > 0,$$

are nuclear ($\operatorname{tr} Q_t < \infty$), then the process Z given by the formula

$$(0.2) \qquad Z_t^x = Z_t(x) = S_t x + \int_0^t S_{t-s} B \, dW_s, \qquad t \geq 0,$$

is a unique mild solution to the linear equation corresponding to $(*)$ ($F \equiv 0$) (see e.g. [D-Z; S]). The process Z is Gaussian and Markovian and it is called an *Ornstein–Uhlenbeck process* (O-U process for short).

In this paper our basic assumption is

$$(\mathrm{A1}) \qquad \int_0^\infty \operatorname{tr} S_s BB^* S_s^* \, ds < \infty.$$

If (A1) holds, then the Gaussian measure $\mu = \mathcal{N}(0, Q_\infty)$ on H with mean zero and with covariance operator

$$(0.3) \qquad Q_\infty x = \int_0^\infty S_s BB^* S_s^* x \, ds, \qquad x \in H,$$

is an invariant measure for the O-U process Z defined by (0.2). It is well known that under (A1) the transition semigroup (R_t) of Z,

$$R_t \phi(x) := E(\phi(Z_t^x)),$$

is a positivity preserving C_0-semigroup of contractions in $L^p(H,\mu)$, for all $1 \le p < \infty$. An important example is the so-called Malliavin process which is a solution of $(*)$ with $A = -1/2$, $F \equiv 0$ and a Hilbert–Schmidt operator B, and then $Q_\infty = BB^*$. The generator L^M (the Malliavin generator) of its transition semigroup (R_t^M) is known in quantum physics as the Number Operator. Let us recall some remarkable properties of (R_t^M) like hypercontractivity ([N]) and the Logarithmic Sobolev Inequality for L^M ([Gr1], [S1]). Moreover, (R_t^M) is symmetric (in $L^2(H,\mu)$).

Two classes of O-U semigroups (and related semilinear equations) have been intensively studied for many years:

• The first is the class of symmetric O-U semigroups, which is important because of applications in physics. Recall that symmetric transition semigroups correspond to reversible processes.

• The second one is the class of strongly Feller O-U semigroups, which is important in the theory of Kolmogorov equations because of smoothing properties of such semigroups.

The aim of this paper is to investigate the transition semigroup for equation $(*)$ in the spaces $L^p(H,\mu)$. We do not assume that the corresponding O-U semigroup is associated to a Dirichlet form (in particular symmetric) nor do we assume that it has the strong Feller property. Results on O-U processes obtained in several papers [Ch-G, ...] enable us to consider quite a general class of O-U semigroups (Section 2), which also contains the Malliavin semigroup (R_t^M) as well as a certain important subclass of strongly Feller semigroups. Applications to non-reversible systems and recently also to Mathematical Finance ([M]) provide some motivation here.

We make weak assumptions on the nonlinear term, namely our basic assumption on F is

(F1) $F : H \to \operatorname{im} B$ is a Borel function and

$$\int_H \exp(\delta\|B^{-1}F(x)\|^2)\,\mu(dx) < \infty \quad \text{for some } \delta > 0,$$

where B^{-1} means the pseudoinverse of B. By the Fernique theorem, functions F of linear growth satisfy the exponential integrability condition in (F1). Extensions of the Fernique theorem and conditions for (F1) to hold have been given e.g. in [A-Ms-Sh], [A-St] and [L].

Starting from the observation that under (A1) and (F1) for any $T > 0$ equation $(*)$ has a solution X_t^x, $0 \le t \le T$, given by the Girsanov transform (a *Girsanov solution* for short), we define a family $(P_t)_{0\le t\le T}$ of operators on $L^\infty(H,\mu)$ by

$$P_t\varphi(x) = E\varphi(X_t^x), \quad 0 \le t \le T. \tag{0.4}$$

If uniqueness in law holds for $(*)$ (in particular, if F is bounded), then any realization of the martingale solution to $(*)$ is a Markov process and the (P_t) defined above is its transition semigroup. If there is no uniqueness, we use one of solutions to $(*)$, which is constructed on compact intervals $[0,T]$, to define $(P_t^T)_{0\le t\le T}$ in (0.4). But it follows from the properties of the Girsanov transform that for a fixed $t \ge 0$ the operator P_t^T does not depend on T, $T \geq t$ (Remark 1.5a). We show that the operators P_t, $t \geq 0$, form a semigroup and the Girsanov solution X_t^x, $0 \leq t \leq T$, has a quasi Markov property with respect to (P_t) (Remark 1.7), so roughly speaking (P_t) is a transition semigroup for any Girsanov solution to $(*)$.

We prove that, in the case of F bounded, (P_t) is a C_0-semigroup on $L^p(H,\mu)$ for all $1 < p < \infty$, and in the case of F exponentially integrable the same holds for p suitably large. Then we study basic analytic properties of the semigroup (P_t), first for F bounded and then in the general case. We investigate hyperboundedness, Logarithmic Sobolev Inequality (LSI), the domain of the generator and invariant measures with densities w.r.t. (with respect to) μ, obtaining new results for nonsymmetric non-strongly Feller systems. Hyperboundedness and LSI have been investigated mainly for reversible (P_t) or for perturbations of symmetric systems (see [Gr2], [Ba] and references therein). Invariant measures with densities were an object of intensive study starting from the results of [Sh; E], [vV] obtained for a semilinear equation $(*)$ corresponding to the Malliavin process on Wiener space. In both the papers the theory of Fredholm operators was used, and F was assumed either to be bounded ([Sh; E]) or to satisfy a stronger condition than our (F1b) in Section 7 ([vV]). Recently, many results have been obtained for strongly Feller processes ([D-Z; R], [Ch-G; E], [D-Z; E] and references therein, [D-G,2]) and for processes corresponding to Dirichlet forms (e.g. [B-R], [H; E], [B-R-Zh], and references therein) but they do not cover our results even in the case of bounded F.

In view of recent results in [H; P] the hyperboundedness of (P_t) is important for the existence of an invariant measure with density. It is well known ([Gr1], [Gr2]) that in the case of (P_t) symmetric, hyperboundedness and LSI are equivalent but in the nonsymmetric case LSI is a stronger property (see [F; H], [Ch-G; N] and also Sections 2, 5, 6).

The LSI, established in the case of F bounded, enables us to obtain crucial estimates related to (F1). Thanks to these estimates, we prove by approximation that for general F, (P_t) is a hyperbounded C_0-semigroup in $L^p(H,\mu)$ for $p > p_0$, p_0 being given explicitly. As a corollary we get a result on an invariant measure analogous to the previous one for F bounded. In the particular case of $A = -1/2$, a similar result was obtained in [H; E] by the Dirichlet form approach and the hyperboundedness of (P_t) was proved by tedious direct calculations. For gradient systems (see [D-Z; E]), i.e. where P_t is symmetric w.r.t. its own invariant measure, the same L^p-regularity of the invariant density as in Corollary 7.3 has been obtained in a different setting in [L] and [A-Ms-Sh].

Finally, we give a characterization of the domain of the generator L_F, extending the result in [Sh; N] and partially generalizing a result in [D]. An LSI is also proved.

Our main tools are the Girsanov transform and Miyadera perturbation method. The first one gives good estimates for the norm and the second one provides some information about the domain of L_F. Let us mention that the advantage of using the Girsanov transform in the study of the strong Feller property of (P_t) has recently been demonstrated in [Ma-Se].

Roughly speaking, we show that (P_t) has similar properties to those of the corresponding O-U semigroup. Therefore the results obtained in [Ch-G,...] are of basic importance. In Section 2 we extend them to the case where μ need not be a full measure.

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Annotated contents

0. Introduction

1. Preliminaries
It is shown that under (F1), equation $(*)$ has a martingale solution given by the Girsanov transform (Prop. 1.2 and Cor. 1.3) and in particular P_t is well defined on $L^\infty(H,\mu)$. Lemma 1.6 on convergence of Girsanov martingales is proved.

2. The O-U semigroup (R_t) and Gaussian Sobolev spaces
Some properties from [Ch-G; ...] are recalled in a more general setting.

3. The semigroup (P_t) on $L^p(H,\mu)$—existence for bounded F

3.1. Probabilistic approach
Using the Girsanov transform it is shown that (P_t) is a C_0-semigroup on $L^p(H,\mu)$, $1<p<\infty$.
3.2. Analytic approach and equivalence
A C_0-semigroup $(\mathcal{V}_t)$ on $L^2(H,\mu)$ is constructed by a Miyadera perturbation of L (Thm. 3.4) and the equality $\mathcal{V}_t = P_t$ is proved (Thm. 3.6). Consequently, $\mathrm{dom}_2(L_F) = \mathrm{dom}_2(L)$ (where L_F is the generator of (P_t) and L is the corresponding O-U generator).

4. Properties of (P_t)—the case of bounded F

4.1. Hyperboundedness
- "Iff" conditions (Thm. 4.1).
- P_t improves positivity (Prop. 4.2).

4.2. Domains of generators
In L^2 and in L^p, some consequences of results for O-U generators.
4.3. Invariant measures with densities
- Thm. 4.7—existence; uniqueness of the density ϱ; $\varrho>0$, $\varrho\in\bigcap_{p\ge1}L^p(H,\mu)$.
- Exponential convergence of P_t (Prop. 4.8) and $W^{1,2}_{Q_\infty}$-regularity of ϱ (Prop. 4.9).

5. Examples
Simple examples show that, even in the case of $F(x)\equiv b\in\operatorname{im}B$ and under (A1), it may happen that equation $(*)$ has no invariant measure or $(*)$ has a unique invariant measure which is singular with respect to μ. An example of (P_t) hyperbounded for $t\ge t_0$ and non-hyperbounded for $t<t_0$ is also given.

6. Logarithmic Sobolev inequality—the case of bounded F

- "Iff" conditions are given for L_F to satisfy a defective LSI (Thm. 6.1). To prove the LSI for all $1<p<\infty$ some domain consideration is needed, since we only have information about $\mathrm{dom}_2(L_F)$.
- The LSI enables us to obtain some auxiliary estimates for $\|P_t\|_{p\to q}$, corresponding to (F1) (Lem. 6.2 and Cor. 6.3)

7. The semigroup (P_t) on $L^p(H,\mu)$—the case of general F
This section contains the main results.

- Under (F1) with some lower bound on δ (i.e. (F1a)), (P_t) is shown to be a C_0-semigroup on $L^p(H,\mu)$ for $p>p_0$, where p_0 is given explicitly. Moreover, (P_t) is hyperbounded (Thm. 7.1).

In the proof F is approximated by bounded F_n and the estimates obtained in Section 6 are essential.
- As a corollary concerning invariant measures we obtain the statements of Thm. 4.7 and Prop. 4.8 but with a weaker L^p-regularity (Cor. 7.3).
- As another corollary, $\mathrm{dom}_2(L_F)$ is characterized in case the corresponding O-U semigroup (R_t) is symmetric (Cor. 7.5).
- Under assumption (F1b), a bit stronger than (F1a), it is proved that in the case of an arbitrary (nonsymmetric) (R_t) we have $\mathrm{dom}_2(L_F) = \mathrm{dom}_2(L)$ and L_F satisfies a defective LSI for $p \geq 2$ (Thm. 7.4).

Appendix

Technical lemmas on approximation.

1. Preliminaries

We assume that $(\Omega, \mathcal{F}, P)$ is a fixed probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, $W = (W_t)$ is a standard cylindrical K-valued Wiener process w.r.t. $(\mathcal{F}_t)$ and η is an H-valued $\mathcal{F}_0$-measurable random variable. If $\operatorname{tr} Q_t < \infty$, $t > 0$ (see (0.1)), then the process

$$Z_t(\eta) = S_t\eta + \int_0^t S_{t-s}B\, dW_s, \qquad t \geq 0, \tag{1.1}$$

is a unique mild solution to the equation

$$\begin{cases} dZ_t = AZ_t dt + BdW_t, \\ Z_0 = \eta, \end{cases}$$

on the given $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ w.r.t. the fixed Wiener process W. For preliminaries on stochastic integration and equations in Hilbert spaces see e.g. [D-Z; S].

For the equation

$$\begin{cases} dX_t = [AX_t + F(X_t)]dt + Bd\widetilde{W}_t, \\ X_0 = x \in H, \end{cases} \tag{$*$}$$

we consider so-called martingale solutions (see *ibid.*):

Definition 1.1. Fix $x \in H$. If there exist: a probability space $(\widetilde{\Omega}^x, \widetilde{\mathcal{F}}^x, \widetilde{P}^x)$ with a filtration $(\widetilde{\mathcal{F}}_t^x)$ satisfying the usual conditions, a K-valued standard cylindrical Wiener process $\widetilde{W}^x$ relative to $(\widetilde{\mathcal{F}}_t^x)$ and an $(\widetilde{\mathcal{F}}_t^x)$-adapted process $\widetilde{X}^x$ satisfying

$$\widetilde{X}_t^x = S_t x + \int_0^t S_{t-s}F(\widetilde{X}_s^x)\, ds + \int_0^t S_{t-s}B\, d\widetilde{W}_s^x, \qquad t \geq 0,\ \widetilde{P}^x\text{-a.e.}, \tag{1.2}$$

then the process $\widetilde{X}^x$ is called a *martingale solution* to equation $(*)$. More precisely, the *martingale solution* is the sequence $((\widetilde{\Omega}^x, \widetilde{\mathcal{F}}^x, \widetilde{P}^x); (\widetilde{\mathcal{F}}_t^x); \widetilde{W}^x; \widetilde{X}^x)$.

We first prove, by means of the Girsanov theorem, that under our basic assumptions (A1) and (F1) below, equation $(*)$ has a martingale solution. Under condition (A1) the nonlinear term F in $(*)$ is assumed to satisfy

(F1) $F : H \to \operatorname{im} B$ is a Borel function and

$$\int_H \exp(\delta \|B^{-1}F(x)\|^2)\,\mu(dx) < \infty \quad \text{for some } \delta > 0,$$

where $\mu = \mathcal{N}(0, Q_\infty)$ is well defined by (0.3) in view of (A1) and B^{-1} stands for the pseudoinverse of the operator B (see e.g. [D-Z; S], p. 407). (Recall that $\operatorname{dom}(B^{-1}) = \operatorname{im} B$ and for $y \in \operatorname{im} B$, $B^{-1}y = x$, where x is the element in $B^{-1}\{y\}$ of minimal norm. Moreover, $\operatorname{im} B^{-1}$ is orthogonal to $\ker B$.)

For the O-U process (Z_t^x) given by (0.2) and (1.1), define the following processes:

$$\Psi(t,x) := B^{-1}F(Z_t^x), \quad x \in H,\ t \geq 0, \tag{1.3}$$

$$U_t^x = U_t^x(\Psi) := \exp\biggl(\int_0^t \langle \Psi(s,x), dW_s\rangle - \frac{1}{2}\int_0^t \|\Psi(s,x)\|^2\,ds\biggr). \tag{1.4}$$

Proposition 1.2. *Assume* (A1) *and* (F1). *Then for μ-a.a. x,*

$$E(U_t^x) = 1 \quad \textit{for all } t \geq 0.$$

(*Equivalently*, (U_t^x) *is an* $(\mathcal{F}_t)$*-martingale.*)

Proof. Let (Z_t^μ) denote the process (1.1) with the initial distribution μ, i.e. the random variable η in (1.1) has the probability distribution $\mathcal{L}(\eta) = \mu$. Then $\mathcal{L}(Z_t^\mu) = \mu$ for $t \geq 0$ and hence

$$\begin{aligned}\int_H \exp(\delta\|B^{-1}F(x)\|^2)\,\mu(dx) &= E[\exp(\delta\|B^{-1}F(Z_t^\mu)\|^2)] \\ &= \int_H E[\exp(\delta\|B^{-1}F(Z_t^x)\|^2)]\,\mu(dx) \quad \text{for every } t \geq 0,\end{aligned} \tag{1.5}$$

the latter equality being a consequence of the Markov property of Z.

Let Ψ be as in (1.3) and fix $t, T > 0$, $x \in H$. Since the function $s \mapsto \exp s$ is convex we obtain by Jensen's inequality

$$\begin{aligned}\int_t^{t+T} \exp(\delta\|\Psi(s,x)\|^2)\,ds &= T\int_t^{t+T} \exp(\delta\|\Psi(s,x)\|^2)\,\frac{ds}{T} \\ &\geq T\exp\biggl[\frac{\delta}{T}\int_t^{t+T} \|\Psi(s,x)\|^2\,ds\biggr].\end{aligned} \tag{1.6}$$

Note that $(x,s,\omega) \mapsto Z_s^x$ is $\mathcal{B}(H)\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{F}$-measurable, where $\mathcal{B}(H)$ denotes the Borel σ-field in H. Then from (F1), (1.5), (1.6) and the Fubini theorem it follows that

$$\infty > \int_t^{t+T}\int_H E[\exp(\delta\|\Psi(s,x)\|^2)]\,\mu(dx)\,ds \geq T\int_H E\biggl[\exp\biggl(\frac{\delta}{T}\int_t^{t+T}\|\Psi(s,x)\|^2\,ds\biggr)\biggr]\,\mu(dx).$$

Hence, if we take $T = 2\delta$, then for a set $\mathcal{G}_t \in \mathcal{B}(H)$ with $\mu(\mathcal{G}_t) = 1$ the Novikov condition

$$E\biggl[\exp\biggl(\frac{1}{2}\int_t^{t+T}\|\Psi(s,x)\|^2\,ds\biggr)\biggr] < \infty \tag{1.7}$$

holds for $x \in \mathcal{G}_t$. The rest of the proof runs as in [K-S, Cor. 3.5.14] and [D-Z; S, p. 299]. Namely, set $t_k := 2k\delta$, $k = 0, 1, 2, \ldots$, and $\mathcal{G} = \bigcap_k \mathcal{G}_{t_k}$. Then $\mu(\mathcal{G}) = 1$. For a fixed $x \in \mathcal{G}$, it follows from [D-Z; S, Prop. 10.17] or [K-S, Cor. 3.5.13] that for any $k = 0, 1, 2, \ldots$ the process

$$M_t^k := \exp\Bigg(\int_{t_k}^{t} \langle \Psi(s,x), dW_s\rangle - \frac{1}{2}\int_{t_k}^{t} \|\Psi(s,x)\|^2\, ds\Bigg), \qquad t \in [t_k, t_{k+1}],$$

is an $(\mathcal{F}_t)$-martingale. In particular we have $E(M_t^k \mid \mathcal{F}_{t_k}) = 1$. If $t \in [t_m, t_{m+1}]$, then $U_t^x = M_{t_1}^0 \cdot \ldots \cdot M_{t_m}^{m-1} \cdot M_t^m$ and hence $EU_t^x = 1$, which finishes the proof. ■

As an immediate consequence of Proposition 1.2 and the Girsanov theorem [D-Z; S, Thm. 10.14 and p. 300] we obtain

Corollary 1.3. *Assume* (A1) *and* (F1). *Then for μ-a.a. x and any $T > 0$ there exists a martingale solution of equation* $(*)$ *on the interval* $[0, T]$. *Namely, the process* $\widetilde{X}_t^x = Z_t^x$, $t \in [0, T]$, *considered on* $(\Omega, \mathcal{F}, (\mathcal{F}_t), \widetilde{P}_T^x)$, *where*

$$d\widetilde{P}_T^x(\omega) = U_T^x(\omega)dP(\omega), \tag{1.8}$$

is a martingale solution of $(*)$ *relative to the Wiener process* $\widetilde{W}_t^x = W_t - \int_0^t \Psi(s,x)\, ds$, $t \in [0, T]$.

Remark 1.4. Let (A1) be satisfied. If for a constant $c > 0$,

$$\|B^{-1}F(x)\| \le c(1 + \|x\|), \qquad x \in H,$$

then (F1) holds by the Fernique theorem.

Remark 1.5. If for all $t > 0$ the operators Q_t defined by (0.1) are nuclear and

$$\|B^{-1}F\|_\infty := \sup_{x\in H} \|B^{-1}F(x)\| =: \beta < \infty, \tag{F2}$$

then for every $x \in H$ and $T > 0$ equation $(*)$ has a unique-in-law martingale solution on $[0, T]$. (Indeed, for any $x \in H$ the Novikov condition (1.7) holds for $t = 0$ and all $T > 0$, which implies the existence by [D-Z; S, Prop. 10.17]. The uniqueness follows as in [K-S, Cor. 5.3.11].)

Let $B_{\mathrm{b}}(H)$ denote the space of real-valued bounded Borel functions on H. Assume (A1), (F1) and let $\mathcal{G}$ be the set defined in the proof of Proposition 1.2. Then, in view of Proposition 1.2, for $x \in \mathcal{G}$, $t \ge 0$ and $\phi \in B_{\mathrm{b}}(H)$ we can define

$$P_t\phi(x) := E[\phi(Z_t^x)U_t^x]. \tag{1.9}$$

By (1.9), $P_t\phi$ is μ-measurable and

$$\|P_t\phi\|_\infty = \operatorname*{ess\,sup}_{x\in H} |P_t\phi(x)| \le \|\phi\|_\infty,$$

which implies that P_t is a contraction on $L^\infty(H, \mu)$.

Note that by Proposition 1.2 and Corollary 1.3 we have the following equality for any $x \in \mathcal{G}$ any $T > 0$ and $0 \le t \le T$, $\phi \in B_{\mathrm{b}}(H)$:

$$P_t\phi(x) = \widetilde{E}_T^x(\phi(\widetilde{X}_t^x)), \tag{1.9a}$$

where $\widetilde{E}_T^x$ means the expectation w.r.t. the probability measure $\widetilde{P}_T^x$ defined by (1.8) and $\widetilde{X}_t^x$ is a martingale solution of $(*)$ given in Corollary 1.3. Hence $P_t\phi$ corresponds to equation $(*)$.

Remark 1.5a. Even if there is no uniqueness in law of martingale solutions to $(*)$, the RHS (right hand side) of (1.9a) does not depend on T. For $0 < T_1 < T_2$ let $\widetilde{X}^{1,x}$ and $\widetilde{X}^{2,x}$ be the solutions on $[0, T_1]$ and $[0, T_2]$, respectively, starting from $x \in \mathcal{G}$ and constructed by the Girsanov transform on different probability systems. Then the expected values of $\phi(\widetilde{X}_t^{1,x})$ and $\phi(\widetilde{X}_t^{2,x})$ are equal for $0 \le t \le T_1$.

The following simple lemma, about convergence of Girsanov martingales, will be useful in Sections 3 and 7.

Let

$$\Psi_n(t,x) := B^{-1} F_n(Z_t^x) \quad \text{and} \quad U_{n,t}^x := U_t^x(\Psi_n),$$

where U_t^x is given by (1.4).

Lemma 1.6. *Assume* (A1). *Let* F, F_n, $n = 1, 2, \ldots$, *satisfy* (F1) *and*

$$\int_H \|B^{-1}(F_n(x) - F(x))\|^2 \, d\mu \to 0 \quad \text{as } n \to \infty. \tag{1.10}$$

Then for any $T > 0$ *there exists a subsequence* (n_m) *such that*

$$E|U_{n_m,T}^x - U_T^x| \to 0 \quad \text{as } m \to \infty, \text{ for } \mu\text{-a.a. } x. \tag{1.11}$$

Proof. Analogously to (1.5) we have

$$\begin{aligned}\int_H E\Big(\int_0^T \|\Psi_n(t,x) - \Psi(t,x)\|^2 \, dt\Big)\, \mu(dx) &= \int_0^T \int_H E[\|B^{-1}(F_n(Z_t^x) - F(Z_t^x))\|^2]\, \mu(dx)\, dt \\ &= \int_0^T E\|B^{-1}(F_n(Z_t^\mu) - F(Z_t^\mu))\|^2 \, dt \\ &= T \int_H \|B^{-1}(F_n(x) - F(x))\|^2 \, d\mu \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Hence for a subsequence

$$E \int_0^T \|\Psi_{n_m}(t,x) - \Psi(t,x)\|^2 \, dt \to 0 \quad \text{as } m \to \infty, \text{ for } \mu\text{-a.a. } x. \tag{1.12}$$

Fix $x \in H$ such that (1.12) holds. Then

$$\int_0^T \langle \Psi_{n_m}(t,x), dW_t\rangle \to \int_0^T \langle \Psi(t,x), dW_t\rangle \quad \text{in mean square, as } m \to \infty,$$

which implies that $U_{n_m,T}^x \to U_T^x$ in probability as $m \to \infty$. Since $U_{n_m,T}^x$, U_T^x are nonnegative random variables with mean one, (1.11) follows by [D-Z; S, Lem. 10.16]. ■

Remark 1.7. It follows from the general definition that the Girsanow solution $\widetilde{X}_t^x = Z_t^x$, $0 \le t \le T$, defined in Corollary 1.3 is a Markov process with the transition semigroup (P_t) iff

$$\widetilde{E}_T^x(\varphi(Z_t^x) \,|\, \mathcal{F}_s) = P_{t-s}\varphi(Z_s^x) \quad \text{for } 0 \le s \le t \le T,\ \varphi \in B_{\mathrm{b}}(H). \tag{1.13}$$

Since for $0 \le s \le t$, $Z_t^x = Z(t, s; Z_s^x)$ P-a.e., where $Z(t,s;\eta) = S_{t-s}\eta + \int_s^t S_{t-u} B \, dW_u$ with $\mathcal{F}_s$-measurable η, repeating the argument from the proof of [D-Z; S, Thm. 9.9] we see that (1.13) follows from the equality

$$\widetilde{E}_T^x(\varphi(Z(t,s;y)\,|\,\mathcal{F}_s) = P_{t-s}\varphi(y) \quad \text{for } \varphi \in C_{\mathrm{b}}(H),\ y \in H. \tag{1.14}$$

If $F = B\widehat{F}$ with $\widehat{F} \in C_{\mathrm{b}}^1(H,K)$, then (1.14) holds, since $(*)$ has a pathwise unique mild solution. Approximating U_T^x as in Lemma 1.6 (compare the proofs of Theorem 7.1 and Proposition 3.2) one can prove that if F satisfies (F1), then (1.14) holds for μ-a.a. y, for all $0 \le s < t \le T$ and if F satisfies (F2), then (1.14) holds for all $y \in H$. Therefore in the latter case the Girsanov solution to $(*)$ is Markov w.r.t. (P_t). In the former case it has a weaker Markov-like property and one can show that it satisfies (1.13) w.r.t. $\widetilde{P}_T^\nu$, where $\widetilde{P}_T^\nu$ is defined in Remark 4.6 and ν is absolutely continuous w.r.t. μ.

2. The O-U semigroup (R_t) and Gaussian Sobolev spaces

Here we discuss some properties of O-U semigroups to be used in the next sections. Some results from several papers [Ch-G, ...] are reviewed in a more general setting. In contrast to [Ch-G, ...], in the present paper it is not assumed that

$$\ker Q_\infty = \{0\}. \tag{2.1}$$

Recall that the O-U semigroup (R_t) (i.e. the transition semigroup for the O-U process Z in (0.2)) is given by

$$R_t\phi(x) = E\phi(Z_t^x) = \int_H \phi(S_t x + y)\,\mu_t(dy), \qquad \phi \in B_{\mathrm{b}}(H),$$

where $\mu_t = \mathcal{N}(0, Q_t)$ with Q_t as in (0.1). (R_t) is a C_0-semigroup of contractions on $L^p(H,\mu)$ for $1 \le p < \infty$.

We define the following class of cylindrical functions:

$$\mathbb{F}C_{\mathrm{b}}^\infty := \{\varphi : H \to \mathbb{R} : \varphi(x) = f(\langle x, h_1\rangle, \ldots, \langle x, h_m\rangle) \text{ for some } m \in \mathbb{N} \text{ and } h_1, \ldots, h_m \in \operatorname{dom}(A^*),\ f \in C_{\mathrm{b}}^\infty(\mathbb{R}^m)\}. \tag{2.1a}$$

and the differential operator, for $\phi \in \mathbb{F}C_{\mathrm{b}}^\infty$,

$$L^0\phi(x) := \tfrac{1}{2}\operatorname{tr}(QD^2\phi(x)) + \langle Ax, D\phi(x)\rangle, \qquad x \in \operatorname{dom}(A),$$

where $Q := BB^*$ and D denotes the Fréchet derivative.

It was proved in [Ch-G; E, Lem. 1] (see also [Z2]) that under (A1) the generator L of (R_t) in $L^p(H,\mu)$, $1 \le p < \infty$, is the closure of L^0 and moreover $\mathbb{F}C_{\mathrm{b}}^\infty$ is invariant for (R_t).

Assume (A1) and (2.1). Then it was shown in [Ch-G; Q] and [Ch-G; R] that the equality of images

$$\operatorname{im} Q_t^{1/2} = \operatorname{im} Q_\infty^{1/2} \tag{A2}$$

assures many regularizing properties of R_t, for instance (A2) is a sufficient and necessary condition for hypercontractivity of (R_t) in $L^p(H,\mu)$ (see also [F; H]). It was proved in [Ch-G; N] that under (A1) and (2.1), the inclusion

$$\operatorname{im} Q_\infty^{1/2} \subset \operatorname{im} Q^{1/2} \tag{A3}$$

is equivalent to the Log Sobolev Inequality for L (and it is equivalent to the existence of a spectral gap for L). It follows from this section that the same is true without the assumption (2.1).

Let us remark that (A3) is stronger than (A2)—see Lemma 2.1(iv) and Corollary 2.3 below. Note that (A3) is satisfied in two important cases: for the Malliavin process ($Q = Q_\infty$) and for $Q = I$. In the latter case the corresponding semigroup (R_t) is strongly Feller ([D-Z; S, Cor. 9.23]) and it has many regularity properties.

2.1. Preliminaries—the semigroup $(S_0(t))$**.** We first investigate an auxillary semigroup $S_0(t)$ which plays an important role in the study of properties of (R_t) (see [Ch-G; Q], [Ch-G; R]).

It has been proved in [Ch-G; R, Prop. 1] that under (A1) the subspace

$$H_0 := Q_\infty^{1/2}(H)$$

is invariant for the semigroup (S_t):

$$S_t(H_0) \subset H_0 \quad \text{for all } t \geq 0. \tag{2.2}$$

(For the reader's convenience the proof of (2.2) is given in the Appendix.)

We denote by π the orthogonal projection in H onto $\overline{H}_0$, the closure of H_0 in H. (We always consider $\overline{H}_0$ with the scalar product induced from H.) For a mapping T on H let

$$T^\pi := T|_{\overline{H}_0} \tag{2.3}$$

be the restriction of T to $\overline{H}_0$. Obviously, by (2.2),

$$S_t^\pi : \overline{H}_0 \to \overline{H}_0 \quad \text{is a } C_0\text{-semigroup on } \overline{H}_0.$$

Observe that $Q_\infty^{-1/2}$, the pseudoinverse of $Q_\infty^{1/2}$, satisfies

$$\operatorname{im} Q_\infty^{-1/2} = \overline{H}_0, \quad Q_\infty^{-1/2} Q_\infty^{1/2} = \pi, \quad Q_\infty^{1/2} Q_\infty^{-1/2} = I_{H_0}. \tag{2.4}$$

It also follows from (2.2) that

$$S_0(t) := (Q_\infty^{-1/2} S_t Q_\infty^{1/2})^\pi = Q_\infty^{-1/2} S_t^\pi (Q_\infty^{1/2})^\pi \tag{2.5}$$

is a bounded operator from $\overline{H}_0$ to $\overline{H}_0$ with the adjoint operator on $\overline{H}_0$:

$$S_0^*(t) = \overline{Q_\infty^{1/2} (S_t^\pi)^* Q_\infty^{-1/2}}. \tag{2.5a}$$

Below we prove modifications of some propositions from [Ch-G; ...].

LEMMA 2.1 (comp. [Ch-G; R, Prop. 2]). *Assume* (A1).

(i) *For all $t \geq 0$ the following identities hold*:

$$(S_t^\pi)^* = \pi (S_t^*)^\pi, \tag{2.6}$$

$$S_0^*(t)x = Q_\infty^{1/2} S_t^* Q_\infty^{-1/2} x \quad \textit{for } x \in H_0, \tag{2.7}$$

and $S_0^*(t)$ restricted to H_0 is a C_0-semigroup in the norm $\|\cdot\|_{H_0}$ (where $\|x\|_{H_0} := \|Q_\infty^{-1/2} x\|$).

(ii) *For all $t \geq 0$ we have the equality*

$$Q_\infty^{1/2} [I^\pi - S_0(t) S_0^*(t)] Q_\infty^{1/2} y = Q_t y, \quad y \in \overline{H}_0. \tag{2.8}$$

(iii) $(S_0(t))$ *is a C_0-semigroup of contractions on $\overline{H}_0$.*

(iv) *For any* $t > 0$,

$$\|S_0(t)\| < 1 \quad \textit{iff} \ \text{(A2)} \ \textit{holds for } t.$$

Proof. (i) For $x, y \in \overline{H}_0$ we have

$$\langle (S_t^\pi)^* x, y\rangle_{\overline{H}_0} = \langle x, S_t y\rangle_H = \langle S_t^* x, y\rangle_H = \langle \pi S_t^* x, y\rangle_{\overline{H}_0},$$

which yields (2.6). Observe that $Q_\infty^{1/2} y = Q_\infty^{1/2} \pi y$ for $y \in H$, which together with (2.6) gives (2.7). The rest of (i) follows from (2.5a) and (2.4), since $(S_t^\pi)^*$ is a C_0-semigroup on $\overline{H}_0$.

(ii) For $y \in \overline{H}_0$, from (2.7) and (2.4) we obtain

$$Q_\infty^{1/2} S_0(t) S_0^*(t) Q_\infty^{1/2} y = S_t Q_\infty S_t^* y.$$

Since, as a consequence of (A1), we always have

$$Q_\infty - S_t Q_\infty S_t^* = Q_t, \tag{2.8a}$$

the equality (2.8) follows.

(iii) By (2.8) the operator $I - S_0(t) S_0^*(t)$ is nonnegative on $\overline{H}_0$ and hence $\|S_0(t)\| \le 1$. Strong continuity of $S_0(t)$ follows as in [Ch-G; R, Prop. 2].

(iv) Let $V_t := S_0(t) S_0^*(t) : \overline{H}_0 \to \overline{H}_0$ and observe that $\|S_0(t)\| < 1$ iff $I^\pi - V_t$ has a bounded inverse. By (2.8a) we have $Q_t^{1/2}(\overline{H}_0) \subset Q_\infty^{1/2}(H) = Q_\infty^{1/2}(\overline{H}_0)$ and hence $J_t := Q_\infty^{-1/2} (Q_t^{1/2})^\pi$ is a bounded operator from $\overline{H}_0$ to $\overline{H}_0$. Note that by (2.8),

$$I^\pi - V_t = J_t J_t^*.$$

If (A2) holds, then $\ker (Q_t^{1/2})^\pi = \{0\}$ and $Q_t^{-1/2} (Q_\infty^{1/2})^\pi = J_t^{-1}$ is bounded from $\overline{H}_0$ to $\overline{H}_0$. Hence $I^\pi - V_t$ has a bounded inverse. By (2.8) and [D-Z; S, Prop. B1],

$$\operatorname{im} (Q_t^{1/2})^\pi = \operatorname{im} Q_\infty^{1/2} (I^\pi - V_t)^{1/2}$$

and if $I^\pi - V_t$ has a bounded inverse then $Q_t^{1/2}(\overline{H}_0) = Q_\infty^{1/2}(\overline{H}_0)$, which implies (A2). ∎

Remark. With respect to the decomposition $H = \overline{H}_0 \oplus H_0^\perp$ the operator S_t has the form

$$S_t = \begin{bmatrix} S_t^{11} & S_t^{12} \\ 0 & S_t^{22} \end{bmatrix} \quad \text{with } S_t^{11} = S_t^\pi.$$

Recall that $Q := BB^*$. Then

$$\ker Q_\infty^{1/2} \subset \ker Q^{1/2}.$$

Indeed,

$$\langle Q_\infty x, x\rangle = \int_0^\infty \|Q^{1/2} S_t^* x\|^2 \, dt, \qquad x \in H,$$

and hence for $x \in \ker Q_\infty^{1/2}$ we have $Q^{1/2} S_t^* x = 0$ for a.a. $t \in [0, \infty)$ and by continuity $Q^{1/2} x = 0$.

The above inclusion yields

$$Q^{1/2} y = Q^{1/2} \pi y \quad \text{for } y \in H, \tag{2.8b}$$

$$\overline{H}_0 \supset \overline{\operatorname{im} Q^{1/2}} \ (= \overline{\operatorname{im} B}). \tag{2.8c}$$

Let $V := Q^{1/2}Q_\infty^{-1/2}$ be an operator from $\overline{H}_0$ to H with domain H_0. By H_1 we denote $\mathrm{dom}(A_0^*|H_0)$, the domain of A_0^* considered as the generator of the semigroup $(S_0^*(t))$ acting on H_0 (see Lemma 1(i)). Note that by [Da, Thm. 1.9, p. 8], H_1 is a core for A_0^* in $\overline{H}_0$. It is easy to check that $H_1 = Q_\infty^{1/2}(\mathrm{dom}(A^\pi)^*)$, where A^π is the generator of the semigroup (S_t^π). Let $H_2 := Q_\infty^{1/2}(\mathrm{dom}(A^*))$.

Lemma 2.2 (for (a) comp. [Ch-G; Sp, Lem. 1.2]). *Assume* (A1).

(a) *For $x, y \in H_1$ the following* (*Lyapunov*) *equation holds*:

$$-\langle A_0^*x, y\rangle - \langle x, A_0^*y\rangle = \langle Vx, Vy\rangle. \tag{2.9}$$

(b) *If $h \in \mathrm{dom}(A^*)$ then $\pi h \in \mathrm{dom}((A^\pi)^*)$ and $(A^\pi)^*\pi h = \pi A^* h$.*

(c) *$H_2 \subset H_1$, H_2 is invariant for $(S_0^*(t))$ and it is a core for A_0^* in $\overline{H}_0$.*

Proof. (a) The following identity follows from (2.8) for $h, g \in \overline{H}_0$:

$$\langle Q_\infty^{1/2}h, Q_\infty^{1/2}g\rangle - \langle S_0^*(t)Q_\infty^{1/2}h, S_0^*(t)Q_\infty^{1/2}g\rangle_{\overline{H}_0} (= \langle Q_t h, g\rangle) = \int_0^t \langle Q^{1/2}S_s^*h, Q^{1/2}S_s^*g\rangle\, ds.$$

Assume that $Q_\infty^{1/2}h, Q_\infty^{1/2}g \in H_1$. Then we can differentiate at $t = 0$ both sides of the above equality to obtain

$$-\langle A_0^*Q_\infty^{1/2}h, Q_\infty^{1/2}g\rangle_{\overline{H}_0} - \langle Q_\infty^{1/2}h, A_0^*Q_\infty^{1/2}g\rangle_{\overline{H}_0} = \langle Q^{1/2}h, Q^{1/2}g\rangle.$$

Putting $x = Q_\infty^{1/2}h$ and $y = Q_\infty^{1/2}g$ we get (2.9).

(b) By (2.2), $\pi S_t^*\pi^\perp x = 0$ and hence $\pi S_t^*\pi x = \pi S_t^* x$. Therefore for $h \in \mathrm{dom}(A^*)$,

$$t^{-1}[(S_t^\pi)^*(\pi h) - \pi h] = t^{-1}\pi(S_t^*h - h) \underset{t\to 0^+}{\longrightarrow} \pi A^*h,$$

and (b) follows.

(c) Let $g \in H_2$. Then $g = Q_\infty^{1/2}h$, where $h \in \mathrm{dom}(A^*)$ and $Q_\infty^{-1/2}g = \pi h \in \mathrm{dom}((A^\pi)^*)$ by (b). Hence $g \in H_1$, since $A_0^* = Q_\infty^{1/2}(A^\pi)^*Q_\infty^{-1/2}$ on H_0. We have

$$S_0^*(t)g = Q_\infty^{1/2}\pi S_t^*\pi h = Q_\infty^{1/2}\pi S_t^* h = Q_\infty^{1/2}S_t^*h \in H_2.$$

Finally, H_2 is dense in $(H_0, \|\ \|_0)$. Therefore, by [Da, Thm. 1.9], H_2 is a core for A_0^* in H_0 and hence in $\overline{H}_0$. ∎

Corollary 2.3. *Assume* (A1).

(i) *For any $\alpha > 0$ the following conditions are equivalent*:

$$\|Vx\| \ge \alpha\|x\|, \qquad x \in H_0, \tag{2.10}$$

$$\|S_0(t)\| \le \exp(-\alpha^2 t/2) \quad \text{for all } t \ge 0. \tag{2.11}$$

(ii) (*A3*) *is satisfied iff* (2.10) *holds for some $\alpha > 0$.*

Proof. (i) Since H_1 is a core for A_0^* in $\overline{H}_0$, we see from Lemma 2.2 that (2.10) is equivalent to the inequality

$$\langle A_0^*x, x\rangle_{\overline{H}_0} \le -\frac{\alpha^2}{2}\|x\|^2 \quad \text{for } x \in \mathrm{dom}(A_0^*),$$

which is equivalent to (2.11).

(ii) For $y \in H$, putting $x = Q_\infty^{1/2} y$ in (2.10) and using (2.4) and (2.8b) we obtain

$$\|Q^{1/2}y\| \ge \alpha \|Q_\infty^{1/2} y\|. \tag{2.12}$$

Conversely, for $x \in H_0$, setting $y = Q_\infty^{-1/2}x$ in (2.12) we get (2.10). Finally, by e.g. [D-Z; S, Prop. B.1] or [Z1], (A3) is satisfied iff (2.12) holds for some $\alpha > 0$. ■

2.2. The semigroups (R_t) and $(\widehat{R}_t)$. By well known properties of Gaussian measures $\mu(\overline{H}_0) = 1$ and therefore for any $1 \le p \le \infty$ the spaces $L^p(H,\mu)$ and $L^p(\overline{H}_0,\mu)$ are isometrically isomorphic. Indeed, if $\phi, \psi \in L^p(H,\mu)$ and $\phi^\pi(x) = \psi^\pi(x)$ for μ-a.a. $x \in \overline{H}_0$ (ϕ^π as in (2.3)) then $\phi = \psi$ μ-a.e., in particular $\phi(x) = \phi(\pi x)$ for μ-a.a. $x \in H$. Hence e.g. the mappings

$$L^p(H,\mu) \ni \phi \mapsto \phi^\pi \in L^p(\overline{H}_0,\mu) \quad \text{and} \quad L^p(\overline{H}_0,\mu) \ni f \mapsto f\circ\pi \in L^p(H,\mu) \tag{2.13}$$

define the suitable isomorphisms.

If T is a bounded linear operator on $\overline{H}_0$, then the operator $Q_\infty^{1/2} T Q_\infty^{-1/2}$ is bounded on H_0 and therefore it can be uniquely extended to a μ-measurable linear transformation T_{Q_∞} on $\overline{H}_0$ such that

$$\int\limits_{\overline{H}_0} \|T_{Q_\infty} x\|^2 \, \mu(dx) = \operatorname{tr}(Q_\infty^{1/2} T T^* Q_\infty^{1/2}). \tag{2.14}$$

It has been proved in [Ch-G; Q] that the so-called *generalized Mehler formula*

$$\widehat{R}_t f(x) := \int\limits_{\overline{H}_0} f((S_0(t))_{Q_\infty} x + (I - S_0(t)S_0^*(t))_{Q_\infty}^{1/2} y)\, \mu(dy) \tag{2.15}$$
$$\text{for } f \in B_{\mathrm{b}}(\overline{H}_0) \text{ and } \mu\text{-a.a. } x \in \overline{H}_0$$

defines a C_0-semigroup of contractions in all spaces $L^p(\overline{H}_0,\mu)$, $1 \le p < \infty$, and moreover in $L^2(\overline{H}_0,\mu)$,

$$\widehat{R}_t = \Gamma(S_0^*(t)),$$

where Γ is the second quantization operator as defined in [S2] (see also [Ch-G; Q]).

Using (2.14), (2.8) and the equality $\pi Q_t \pi = Q_t$ we obtain from (2.15)

$$\widehat{R}_t f(x) = \int\limits_H f(S_t^\pi x + \pi z)\mu_t(dz) = R_t(f\circ\pi)(x) \tag{2.16}$$
$$\text{for } f \in B_{\mathrm{b}}(\overline{H}_0) \text{ and } x \in \overline{H}_0 \text{ (where } \mu_t = \mathcal{N}(0,Q_t)).$$

In much the same way we get

$$(R_t\varphi)(\pi x) = \widehat{R}_t \varphi^\pi(\pi x) \quad \text{for } \varphi \in B_{\mathrm{b}}(H),\ x \in H. \tag{2.17}$$

Remark. Note that $\widehat{R}_t f(x) = Ef(Z_t^x)$ and $Z_t^x = S_t^\pi x + \int_0^t S_s^\pi B\, dW_s = \pi Z_t^x$, $x \in \overline{H}_0$.

It follows from (2.16), (2.17) and (2.13) that

$$\|R_t\varphi\|_p = \|\widehat{R}_t\varphi^\pi\|_p \quad \text{and} \quad \|R_t\|_{p\to q} = \|\widehat{R}_t\|_{p\to q}. \tag{2.18}$$

Let $\widehat{L}$ be the generator of the semigroup $\widehat{R}_t$ in $L^p(\overline{H}_0,\mu)$. As a consequence of (2.13), (2.16)–(2.18) we find that

- if $\varphi \in \operatorname{dom}_p(L)$, then $\varphi^\pi \in \operatorname{dom}_p(\widehat{L})$ and $\widehat{L}\varphi^\pi(y) = L\varphi(y)$ for μ-a.a. $y \in \overline{H}_0$,
- if $f \in \operatorname{dom}_p(\widehat{L})$, then $f\circ\pi \in \operatorname{dom}_p(L)$ and $L(f\circ\pi)(x) = \widehat{L}f(\pi x)$ for μ-a.a. $x \in H$.

Therefore, if $\widetilde{\mathcal{D}}$ is a core for L in $L^p(H,\mu)$ then $\{\varphi^\pi : \varphi \in \widetilde{\mathcal{D}}\}$ is a core for $\widehat{L}$ in $L^p(\overline{H}_0,\mu)$, and if $\mathcal{D}$ is a core for $\widehat{L}$ in $L^p(\overline{H}_0,\mu)$ then

$$\mathcal{D}_\pi := \{f \circ \pi : f \in \mathcal{D}\} \tag{2.19}$$

is a core for L in $L^p(H,\mu)$. By (2.13) we see that in particular

$$\text{if } \widetilde{\mathcal{D}} \text{ is a core for } L \text{ in } L^p(H,\mu) \text{ then so is } \widetilde{\mathcal{D}}_\pi = \{\varphi \circ \pi : \varphi \in \widetilde{\mathcal{D}}\}. \tag{2.20}$$

2.3. Sobolev spaces. Let

$$\mathbb{F}C^\infty(H) := \{\varphi : H \to \mathbb{R} : \varphi(x) = f(\langle x, h_1\rangle, \ldots, \langle x, h_m\rangle) \text{ for some } m \in \mathbb{N} \text{ and } h_1,\ldots,h_m \in H,\ f \in C^\infty(\mathbb{R}^m)\}.$$

For a bounded selfadjoint operator T in H and $\varphi \in \mathbb{F}C^\infty(H)$ we define

$$D_T\varphi(x) := T^{1/2}\pi D\varphi(x), \tag{2.21}$$

where $D\varphi(x)$ denotes the Fréchet derivative of φ at the point x. Then in particular $D_{Q_\infty}\varphi(x) = Q_\infty^{1/2} D\varphi(x)$ and

$$D_T\varphi(x) = (T^{1/2}Q_\infty^{-1/2})D_{Q_\infty}\varphi(x). \tag{2.22}$$

Note also that by (2.8b) we have

$$D_Q\phi(x) = Q^{1/2}D\phi(x). \tag{2.23}$$

Observe that $D(\psi\circ\pi)(x) = \pi D\psi(\pi x) = D\psi^\pi(\pi x)$, and hence

$$D_T(\psi\circ\pi)(x) = D_T\psi^\pi(\pi x). \tag{2.24}$$

It also follows that if $\varphi(x) = \psi(\pi x)$, where $\psi \in \mathbb{F}C^\infty(H)$, then

$$D_T\varphi(x) = T^{1/2}D\varphi(x).$$

(Indeed, $D\varphi(x) = \pi D\psi(\pi x) = \pi(\pi D\psi(\pi x)) = \pi D\varphi(x)$.)

For $h \in H_0$ we denote by f_h the linear functional on $\overline{H}_0$ defined by $f_h(y) := \langle y, Q_\infty^{-1/2}h\rangle$. If K_0 is a subspace of H_0, we set

$$\mathcal{P}(K_0) := \operatorname{lin}\{1, f_{h_1}\cdot f_{h_2}\cdot\ldots\cdot f_{h_n} : n = 1,2,\ldots,\ h_1,\ldots,h_n \in K_0\}$$

and $\mathcal{P}_\pi(K_0)$ is defined as in (2.19).

Then $\mathcal{P}(K_0)$ and $\mathcal{P}_\pi(K_0)$ are subspaces of $\mathbb{F}C^\infty(\overline{H}_0)$ and $\mathbb{F}C^\infty(H)$, respectively, and their elements may be identified with polynomials of n variables, $n = 0,1,\ldots$ If $\overline{K}_0 = \overline{H}_0$, then $\mathcal{P}(K_0)$ is dense in $L^p(\overline{H}_0,\mu)$ and hence $\mathcal{P}_\pi(K_0)$ is dense in $L^p(H,\mu)$.

For $\varphi \in \mathcal{P}_\pi(H_0)$ and T as in (2.21) we define the first Sobolev norm of φ by

$$\|\varphi\|_{1,p}^p = \int_H |\varphi(x)|^p\,\mu(dx) + \int_H \|D_T\varphi(x)\|_H^p\,\mu(dx)$$

and for $n = 2,3,\ldots,$

$$\|\varphi\|_{n,p}^p = \|\varphi\|_{n-1,p}^p + \int_H \|(T^{1/2}\pi)^{\otimes n}D^n\varphi(x)\|_{H^{\otimes n}}^p\,\mu(dx).$$

The Sobolev space $W_T^{n,p}$ is defined as the completion of $\mathcal{P}_\pi(H_0)$ in the norm $\|\ \|_{n,p}$, $n = 1,2,\ldots$ Note that

$$W_T^{1,p} \text{ is continuously embedded in } L^p(H,\mu) \tag{2.25}$$

iff the operator D_T with domain $\mathcal{P}_\pi(H_0)$ is closable in $L^p(H,\mu)$ and then $W_T^{n,p}$ is continuously embedded in $W_T^{n-1,p}$.

It is well known that (2.25) holds for $T = Q_\infty$, $p > 1$ (see e.g. [W]). In the general case it can be shown as in [G] (see also [Ch-G; M]) that (2.25) holds ($p > 1$) iff the operator $T^{1/2}Q_\infty^{-1/2} : H_0 \to H$ is closable (in $H \times H$). Here we deal with T equal to Q or Q_∞. Finally, note that by (2.24), $W_T^{n,p}$ is isometrically isomorphic to $W_T^{n,p}(\overline{H}_0)$ (the isomorphism (2.13)), $n = 1, 2, \ldots$

2.4. Properties of L and (R_t). First recall the Ito–Wiener decomposition

$$L^2(\overline{H}_0,\mu) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n,$$

where the spaces $\mathcal{H}_n$ are defined as follows: $\mathcal{H}_0 = \mathcal{H}_{\le 0}$ is the space of constants; for $n \ge 1$, $\mathcal{H}_{\le n}$ denotes the closed subspace spanned by all products $f_{h_1} \cdot \ldots \cdot f_{h_m} \in \mathcal{P}(H_0)$ of order $m \le n$ and $\mathcal{H}_n$ is the orthogonal complement of $\mathcal{H}_{\le n-1}$ in $\mathcal{H}_{\le n}$. Let I_n be the orthogonal projection in $L^2(\overline{H}_0,\mu)$ onto $\mathcal{H}_n$. Then for $h \in H_0$,

$$\int_{\overline{H}_0} |I_n(f_h^n)(x)|^2 \, \mu(dx) = n!\|h\|^{2n}, \qquad n = 1, 2, \ldots,$$

and note that $I_1(f_h) = f_h$.

In particular, if $\|h\| = 1$, then $I_n(f_h^n)$, $n = 0, 1, 2, \ldots$, may be identified with the usual Hermite polynomials. It follows by polarization that

$$\mathcal{P}(K_0) = \operatorname{lin}\{I_n(f_h^n) : n = 0, 1, 2, \ldots,\ h \in K_0\}.$$

Let $H_1 := \operatorname{dom}(A_0^*|H_0)$ as in Section 2.1. By [Ch-G; N] the space $\mathcal{P}(H_1)$ is a core for $\widehat{L}$ in $L^p(\overline{H}_0,\mu)$, $1 \le p < \infty$. Hence $\mathcal{P}_\pi(H_1)$ is a core for L in $L^p(H,\mu)$.

Let $\overline{D}_{Q_\infty}$ be the closure of D_{Q_∞} with domain $\mathcal{P}_\pi(H_0)$ in $L^2(H,\mu)$ and $D^*_{Q_\infty}$ be the adjoint of the operator D_{Q_∞} acting in L^2-space. We denote by $\mathcal{G}$ the maximal domain of the operator $D^*_{Q_\infty}A_0^*\overline{D}_{Q_\infty}$, that is, $\phi \in \mathcal{G}$ iff $\phi \in W^{1,2}_{Q_\infty}$, $\overline{D}_{Q_\infty}\phi \in L^2(H,\mu;\operatorname{dom}(A_0^*))$ and $A_0^*\overline{D}_{Q_\infty}\phi \in \operatorname{dom}_2(D^*_{Q_\infty})$. Below we show a basic identity. Part (i) has been proved in [Ch-G; Sp, Thm. 1.6] for exponential functions. Let $\mathbb{F}C_{\mathrm{b}}^\infty$ be as in (2.1a).

Lemma 2.4. *Assume* (A1). *Then*:

(i) $\widehat{L}f = D^*_{Q_\infty}A_0^*D_{Q_\infty}f$ *for* $f \in \mathcal{P}(H_1)$.

(ii) $\mathcal{G} \subset \operatorname{dom}_2(L)$ *and* $L\phi = D^*_{Q_\infty}A_0^*\overline{D}_{Q_\infty}\phi$ *for* $\phi \in \mathcal{G}$.

(iii) $\mathbb{F}C_{\mathrm{b}}^\infty \subset \mathcal{G}$ *and* $L\phi = D^*_{Q_\infty}A_0^*D_{Q_\infty}\phi$ *for* $\phi \in \mathbb{F}C_{\mathrm{b}}^\infty$.

Proof. (i) Let $h \in H_1$. Then by [Ch-G; Q] we have

$$LI_n(f_h^n) = nI_n(f_{A_0^*h} \cdot f_h^{n-1}). \tag{2.25a}$$

On the other hand

$$D_{Q_\infty}(I_n(f_h^n))(x) = nhI_{n-1}(f_h^{n-1})(x) \in H_1$$

and hence $A_0^*D_{Q_\infty}(I_n(f_h^n))$ is an H_0-valued polynomial. Recall that for an H_0-valued polynomial Ψ and $x \in \overline{H}_0$ we have

$$(D^*_{Q_\infty}\Psi)(x) = -\operatorname{tr}(D_{Q_\infty}\Psi)(x) + \langle x, Q_\infty^{-1/2}\Psi(x)\rangle. \tag{2.25b}$$

Therefore taking into account that $\langle x, Q_\infty^{-1/2} A_0^* h\rangle = f_{A_0^* h}(x)$ we obtain

$$\text{(2.25c)}\qquad D_{Q_\infty}^* A_0^* D_{Q_\infty}(I_n(f_h^n)) = -n(n-1)\langle A_0^* h, h\rangle I_{n-2}(f_h^{n-2}) + n f_{A_0^* h} I_{n-1}(f_h^{n-1}) \quad \text{for } n \ge 2,$$

and for $n = 1$, $D_{Q_\infty}^* A_0^* D_{Q_\infty} f_h = f_{A_0^* h} = \widehat{L} f_h$. Then for $f = I_n(f_h^n)$ and $n \ge 2$ the equality (i) follows from (2.25a), (2.25c) and the well known identity for Hermite polynomials:

$$I_n(f_g f_h^{n-1}) = f_g I_{n-1}(f_h^{n-1}) - (n-1)\langle g, h\rangle I_{n-2}(f_h^{n-2}).$$

Thus (i) holds for $f \in \mathcal{P}(H_1)$.

(ii) It follows from (2.9) that $\langle A_0^* x, x\rangle \le 0$ for $x \in H_1$ and hence for $x \in \operatorname{dom}(A_0^*)$, since H_1 is a core for A_0^*. Let $\phi \in \mathcal{G}$. Then by the definition of $\mathcal{G}$, $\overline{D}_{Q_\infty}\phi(x) \in \operatorname{dom}(A_0^*)$ for μ-a.a. x. Therefore

$$\langle D_{Q_\infty}^* A_0^* \overline{D}_{Q_\infty}\phi, \phi\rangle_{L^2} = \int_H \langle A_0^* \overline{D}_{Q_\infty}\phi(x), \overline{D}_{Q_\infty}\phi(x)\rangle\, \mu(dx) \le 0,$$

which means that the operator $D_{Q_\infty}^* A_0^* \overline{D}_{Q_\infty}$ with domain $\mathcal{G}$, denoted by $\mathcal{L}_\mathcal{G}$, is dissipative in $L^2(H,\mu)$ and hence closable (see e.g. [P]). Obviously, $\mathcal{P}_\pi(H_1) \subset \mathcal{G}$. Then by (i),

$$\text{(2.25d)}\qquad \mathcal{L}_\mathcal{G} = L \text{ on } \mathcal{P}_\pi(H_1) \quad\text{and}\quad \mathcal{P}_\pi(H_1) \text{ is a core for } L.$$

Since L generates a C_0-semigroup of contractions, $(\alpha - L)(\mathcal{P}_\pi(H_1))$ is dense in $L^2(H,\mu)$ for $\alpha > 0$ and by (2.25d) so is $(\alpha - \mathcal{L}_\mathcal{G})(\mathcal{G})$. From this, the dissipativity of $\mathcal{L}_\mathcal{G}$ and the Lumer–Phillips theorem (see e.g. [P]) we conclude that $\overline{\mathcal{L}}_\mathcal{G}$ (the closure of $\mathcal{L}_\mathcal{G}$) generates a C_0-semigroup on $L^2(H,\mu)$. Then it follows from (2.25d) and Lemma 3.5 of Section 3 that $\overline{\mathcal{L}}_\mathcal{G} = L$, which implies (ii).

(iii) Let $\phi \in \mathbb{F}C_\mathrm{b}^\infty$. Then $\phi(x) = \psi(\Pi x)$, where Π is a finite-dimensional orthogonal projection in H with $\Pi(H) \subset \operatorname{dom}(A^*)$ and $\psi \in C_\mathrm{b}^\infty(H)$. Therefore,

$$\overline{D}_{Q_\infty}\phi(x) = D_{Q_\infty}\phi(x) = Q_\infty^{1/2}\Pi D\phi(x) \in Q_\infty^{1/2}(\operatorname{dom}(A^*)) \subset H_1,$$

where the first equality holds for μ-a.a. $x \in H$ by well known properties of the Sobolev space $W_{Q_\infty}^{1,2}$ and the inclusion follows from Lemma 2.2(c). Taking into account that $Q_\infty^{1/2}\Pi$ is a bounded operator from H to $(H_0, \|\ \|_{H_0})$, $Q_\infty^{1/2}\Pi(H) \subset H_1$ and A_0^* with domain H_1 is a closed operator in $(H_0, \|\ \|_{H_0})$, we conclude by the closed graph theorem that $A_0^* Q_\infty^{1/2}\Pi$ is a bounded operator from H to $(H_0, \|\ \|_{H_0})$. Therefore, $\Psi := A_0^* D_{Q_\infty}\phi$ is a bounded H_0-valued function on H and for this Ψ the RHS of (2.25b) defines a bounded function on H. The former implies that $\overline{D}_{Q_\infty}\phi \in L^2(H,\mu;\operatorname{dom}(A^*))$, and the latter that $A_0^* \overline{D}_{Q_\infty}\phi \in \operatorname{dom}_2(D_{Q_\infty}^*)$. Thus $\phi \in \mathcal{G}$ and the equality follows from (ii). ∎

The proposition below is a slight modification of [Ch-G; Sp, Prop. 1.7] but we prove it in a different and simpler way.

Proposition 2.5. *Assume* (A1). *Then*:

(i) *For* $\varphi, \psi \in \mathcal{P}_\pi(H_1)$ *or for* $\varphi, \psi \in \mathbb{F}C_\mathrm{b}^\infty$,

$$-\langle L\varphi, \psi\rangle - \langle \varphi, L\psi\rangle = \langle D_Q\varphi, D_Q\psi\rangle.$$

(ii) *The operator* D_Q *with domain* $\mathcal{P}_\pi(H_1)$ *has a unique extension to an operator* $\overline{D}_Q$ *bounded on* $\operatorname{dom}_2(L)$ *endowed with the graph norm. The operator* $\overline{D}_Q$ *is also the unique extension of* D_Q *with domain* $\mathbb{F}C_\mathrm{b}^\infty$.

(iii) *For any* $\varphi \in \mathrm{dom}_2(L)$ *and* $t > 0$,

$$\int_0^t \|\overline{D}_Q R_s\varphi\|_2^2\, ds = \|\varphi\|_2^2 - \|R_t\varphi\|_2^2.$$

Proof. (i) By Lemmas 2.4 and 2.2, for $f \in \mathcal{P}(H_1)$ we have

$$\begin{aligned}-2\langle \widehat{L}f, f\rangle_{L^2(\overline{H}_0,\mu)} &= -2\langle A_0^* D_{Q_\infty} f, D_{Q_\infty} f\rangle = \int_{\overline{H}_0} \|VD_{Q_\infty} f(y)\|_H^2\, \mu(dy)\\ &= \int_H \|Q^{1/2}\pi Df(\pi x)\|_H^2\, \mu(dx) = \int_H \|D_Q(f\circ\pi)(x)\|_H^2\, \mu(dx)\end{aligned}$$

(the last equality being a consequence of (2.24)). Therefore

$$-2\langle L(f\circ\pi), f\circ\pi\rangle_{L^2(H,\mu)} = \|D_Q(f\circ\pi)\|_{L^2(H,\mu:H)}^2,$$

which implies the first part of (i). The second part follows similarly from Lemma 2.4(iii).

(ii) From (i) we have the estimate

$$\|D_Q\varphi\|_2^2 \le 2\|L\varphi\|_2\|\varphi\|_2 \le \|L\varphi\|_2^2 + \|\varphi\|_2^2 \quad \text{for } \varphi \in \mathcal{P}_\pi(H_1).$$

Since $\mathcal{P}_\pi(H_1)$ is dense in $\mathrm{dom}_2(L)$ in the graph norm, D_Q with domain $\mathcal{P}_\pi(H_1)$ can be uniquely extended to an operator $\overline{D}_Q$ defined on $\mathrm{dom}_2(L)$ and L-bounded (i.e. bounded in the norm of the graph of L). It follows from (i) that $D_Q\varphi = \overline{D}_Q\varphi$ for $\varphi \in \mathbb{F}C_{\mathrm{b}}^\infty$ and since $\mathbb{F}C_{\mathrm{b}}^\infty$ is also a core for L we obtain the second statement.

(iii) By (i) and (ii), for $\varphi \in \mathrm{dom}_2(L)$ we have $-2\langle LR_s\varphi, R_s\varphi\rangle = \|\overline{D}_Q R_s\varphi\|_2^2$, and integration from 0 to t gives the equality in (iii). ■

Now we recall some properties of (R_t), which are equivalent to (A3) and analogous to those of $(S_0(t))$ in Corollary 2.3. (2.27) is a spectral gap inequality for L.

Let

$$L_0^2(H,\mu) := \{\phi \in L^2(H,\mu) : \langle \phi, 1\rangle = 0\}.$$

Then $L_0^2(H,\mu)$ is invariant for the semigroup (R_t). The restriction of R_t to $L_0^2(H,\mu)$ will be denoted by R_t^0.

COROLLARY 2.6 (comp. [Ch-G; N, Thm. 3.1]). *Assume* (A1). *Then for any* $\alpha > 0$, *the condition* (2.10) *holds iff*

$$\|R_t^0\|_2 \le \exp(-\alpha^2 t/2) \tag{2.26}$$

and iff

$$\langle -L\phi, \phi\rangle \ge \tfrac{1}{2}\alpha^2\|\phi\|^2, \qquad \phi \in \mathrm{dom}_2(L) \cap L_0^2(H,\mu). \tag{2.27}$$

Moreover, *if* (2.10) *holds then for each* $1 < p < \infty$,

$$\|R_t^0\|_p \le \exp\Big(-\frac{\alpha^2 t}{\max(p,p')}\Big). \tag{2.26a}$$

Proof. (2.26) and (2.27) are equivalent by the properties of contraction semigroups. Suppose (2.10) holds for some $\alpha > 0$. Then for $\varphi \in \mathcal{P}_\pi(H_1) \cap L_2^0(H,\mu)$, by Proposition 2.5(i) and (2.22) we have

$$\langle -L\phi, \phi\rangle = \tfrac{1}{2}\langle VD_{Q_\infty}\phi, VD_{Q_\infty}\phi\rangle \ge \tfrac{1}{2}\alpha^2\|D_{Q_\infty}\phi\|_2^2 \ge \tfrac{1}{2}\alpha^2\|\phi\|_2^2,$$

where the last inequality follows from the well known properties of the Malliavin derivative D_{Q_∞} (see e.g. [W]). Since $\mathcal{P}_\pi(H_1)$ is a core for L, we get (2.27).

For $h \in H_0$, let $\psi_h(x) := \langle x, Q_\infty^{-1/2}h\rangle$. Then $\psi_h \in L_2^0(H,\mu)$, $\psi_h(x) = f_h(\pi x)$ and $\|\psi_h\|^2 = \|h\|^2$. By the definition of R_t we have

$$\begin{aligned} R_t\psi_h(x) &= \int_H \langle S_t x + y, Q_\infty^{-1/2}h\rangle\,\mu(dy) = \langle \pi x, S_t^* Q_\infty^{-1/2}h\rangle + \langle \pi^\perp x, S_t^* Q_\infty^{-1/2}h\rangle \\ &= \langle x, Q_\infty^{-1/2}S_0^*(t)h\rangle + \langle x, \pi^\perp S_t^* Q_\infty^{-1/2}h\rangle, \end{aligned}$$

where $\pi^\perp$ is the projection orthogonal to π. Therefore $\|R_t\psi_h\|^2 = \|S_0^*(t)h\|^2$. Thus (2.26) implies (2.11) and hence (2.10) by Corollary 2.3.

Finally (2.26a) follows from (2.26), the inequality $\|R_t^0\|_\infty \le 1$ and the Riesz–Thorin Interpolation Theorem (see [Ch-G; Sp]). ∎

Now we formulate some consequences of (A3) for the domain of L.

Corollary 2.7. *Assume* (A1) *and* (A3). *Then* $\mathrm{dom}_2(L)$ *is continuously embedded in* $W_{Q_\infty}^{2,2}$ *and*

$$\|D_{Q_\infty}^2\varphi\|_2 \le \frac{2}{a^2}\|L\varphi\|_2, \tag{2.28}$$

where $a = \sup\{\alpha : (2.10)$ *holds*$\}$ *and*

$$\|D_{Q_\infty}^2\varphi\|_2^2 = \int_H \|D_{Q_\infty}^2\varphi(x)\|_{H\otimes H}^2\,\mu(dx).$$

Proof. The close inspection of the proof of [Ch-G; N, Thm. 4.3] shows that for $\varphi \in L_2^0(H,\mu)$ we have the estimate

$$\|L_M L^{-1}\varphi\|_2 \le \frac{1}{a^2}\|\varphi\|_2. \tag{2.29}$$

Indeed, to prove (2.29) first note that

$$\|L^M L^{-1}\|_{L_0^2\to L_0^2} = \sup_{n\ge1}\|L^M L^{-1} I_n\| \tag{2.30}$$

(because for $n \ge 1$ the space $\mathcal{H}_n$ is invariant for L and for L^M). It follows from the properties of L^M $\big($or from (2.25a) with $A_0^* = -\frac{1}{2}I\big)$ that for $\varphi \in \mathcal{P}(H_0)$,

$$L^M I_n\varphi = -\frac{n}{2}I_n\varphi.$$

Hence, taking into account that R_t and L^M commute and by [Ch-G; Q, Lem. 1c and Thm. 1],

$$\|R_t I_n\| \le \exp(-na^2t/2),$$

for $n \ge 1$ we obtain

$$\|L^M L^{-1} I_n\varphi\| \le \int_0^\infty \|R_t L^M I_n\varphi\|\,dt \le \frac{n}{2}\int_0^\infty e^{-na^2t/2}\|I_n\varphi\|\,dt = \frac{1}{a^2}\|I_n\varphi\|.$$

This and (2.30) imply (2.29).

From (2.29), for $\varphi \in \mathrm{dom}_2(L)$ we have

$$\|L_M\varphi\|_2 \le \frac{1}{a^2}\|L\varphi\|_2.$$

This and the known inequality $\|D_{Q_\infty}^2\varphi\|_2 \le 2\|L_M\varphi\|_2$ (see e.g. [Sh; D]) give (2.28). ∎

Finally, the hypercontractivity of R_t and the Log Sobolev Inequality for L are recalled in Sections 4.1 and 6, respectively.

3. The semigroup (P_t) on $L^p(H,\mu)$—existence for bounded F

In this section we assume (A1) and

$$\|B^{-1}F\|_\infty := \sup_{x\in H} \|B^{-1}F(x)\| =: \beta < \infty, \tag{F2}$$

Then, in view of Remark 1.5, for $\phi \in B_b(H)$ and for every $x \in H$, $P_t\phi(x)$ is well defined by formula (1.9) which is recalled below:

$$P_t\phi(x) := E[\phi(Z_t^x)U_t^x] \tag{1.9}$$

(where Z is the O-U process corresponding to $(*)$ and U is the Girsanov martingale given in (1.4)).

3.1. Probabilistic approach

PROPOSITION 3.1. *Let* (A1) *and* (F2) *be satisfied. Then for any* $p \in (1,\infty)$ *the following conditions hold*:

(i) *For every* $t \geq 0$, P_t *has a unique extension* (*still denoted by* P_t) *to a bounded operator from* $L^p(H,\mu)$ *to* $L^p(H,\mu)$. *Moreover*

$$\|P_t\|_{p\to p} \leq \exp\left(\frac{\beta^2}{2(p-1)}t\right). \tag{3.1}$$

(ii) *For* $\phi \in L^p(H,\mu)$, $\|P_t\phi - \phi\|_p \to 0$ *as* $t \to 0$.

Proof. (i) We first prove a more general estimate (3.4) which will be useful in Section 4. Let $\phi \in B_b(H)$ and $1 < r \leq p$. Hölder's inequality with exponents r and $r' = r/(r-1)$ yields

$$\begin{aligned}\|P_t\phi\|_p^p &= \int_H |P_t\phi(x)|^p\,\mu(dx) \leq \int_H \Big[\int_\Omega |\phi(Z_t^x)|U_t^x\,dP\Big]^p\,\mu(dx)\\ &\leq \int_H \Big(\int_\Omega |\phi(Z_t^x)|^r\,dP\Big)^{p/r}\cdot\Big(\int_\Omega (U_t^x)^{r'}\,dP\Big)^{p/r'}\,\mu(dx).\end{aligned} \tag{3.2}$$

Next, from (1.4), (F2) and Proposition 1.2 we have

$$\begin{aligned}E([U_t^x(\Psi)]^{r'}) &= E\bigg(\exp\bigg[\int_0^t \langle r'\Psi, dW_s\rangle - \frac{(r')^2}{2}\int_0^t \|\Psi\|^2\,ds + \frac{r'(r'-1)}{2}\int_0^t \|\Psi\|^2\,ds\bigg]\bigg)\\ &= E\bigg[U_t^x(r'\Psi)\cdot\exp\bigg(\frac{r'(r'-1)}{2}\int_0^t \|\Psi\|^2\,ds\bigg)\bigg]\\ &\leq \exp\Big(\frac{1}{2}\cdot r'(r'-1)\beta^2 t\Big) =: [c_t(r')]^{r'}.\end{aligned} \tag{3.3}$$

Taking into account that $p \geq r$ and the O-U semigroup R_t is a contraction in L^q for $q \geq 1$, from (3.2) and (3.3) we obtain

$$\|P_t\phi\|_p^p \leq [c_t(r')]^p \int_H (R_t(|\phi|^r)(x))^{p/r}\, \mu(dx) \leq [c_t(r')]^p \||\phi|^r\|_{p/r}^{p/r}. \tag{3.4}$$

Therefore

$$\|P_t\phi\|_p \leq \exp\left(\frac{r'-1}{2}\beta^2 t\right)\|\phi\|_p$$

for $\phi \in B_{\mathrm{b}}(H)$ and then for $\phi \in L^p(H)$. Hence

$$\|P_t\phi\|_{p\to p} \leq \exp\left(\frac{r'-1}{2}\beta^2 t\right)$$

for any $r' \geq p' = p/(p-1)$. Setting $r' = p'$, we obtain (3.1).

To prove (ii), first consider $\phi \in C_{\mathrm{b}}^1(H)$. Let

$$\|\phi\|_\infty + \|D\phi\|_\infty =: c_\phi.$$

Then for every $x \in H$, $t \in [0,1]$, by Hölder's inequality and (3.3) we have

$$|P_t\phi(x) - \phi(x)|^2 \leq [E(|\phi(Z_t^x) - \phi(x)|U_t^x)]^2 \leq \exp(\beta^2)\cdot E([\phi(Z_t^x) - \phi(x)]^2). \tag{3.5}$$

Since, by our assumption, $E([\phi(Z_t^x) - \phi(x)]^2) \leq c_\phi^2 E((Z_t^x - x)^2)$ and the process Z_t^x is mean square continuous, estimate (3.5) implies that for all $x \in H$, $P_t\phi(x) \to \phi(x)$ as $t \to 0$. Moreover, $|P_t\phi(x)| \leq \|\phi\|_\infty \leq c_\phi$ for all x.

Therefore, by the Lebesgue Dominated Convergence Theorem, (ii) follows for $\phi \in C_{\mathrm{b}}^1(H)$. For $\phi \in L^p(H,\mu)$ we can choose a sequence $(\phi_n) \subset C_{\mathrm{b}}^1(H)$ converging to ϕ in L^p. The standard estimate

$$\|P_t\phi - \phi\| \leq \|P_t(\phi - \phi_n)\| + \|P_t\phi_n - \phi_n\| + \|\phi_n - \phi\|$$

and (3.1) yield, for $t \in [0,1]$,

$$\|P_t\phi - \phi\|_p \leq \left[1 + \exp\frac{\beta^2}{2(p-1)}\right]\|\phi_n - \phi\|_p + \|P_t\phi_n - \phi_n\|.$$

Hence, for an arbitrary $\varepsilon > 0$ we can take n_ε large enough to make the first term on the RHS less than $\varepsilon/2$ and then for n_ε we can choose $\delta_\varepsilon > 0$ such that for $t \in [0,\delta_\varepsilon]$ the second term is less than $\varepsilon/2$, which finishes the proof. ∎

PROPOSITION 3.2. *Let* (A1) *hold. Assume that* $\widehat{F}, \widehat{F}_n \in B_{\mathrm{b}}(H,K)$, $n = 1,2,\ldots,$ *and* $(\widehat{F}_n)$ *converges* μ*-a.s. and boundedly to* $\widehat{F}$. *Let* $F_n = B\widehat{F}_n$, $F = B\widehat{F}$, *let* $U_{n,t}^x$ *denote the Girsanov martingale corresponding to* F_n *and*

$$P_t^n\phi(x) = E[\phi(Z_t^x)U_{n,t}^x] \quad \textit{for } \phi \in B_{\mathrm{b}}(H),\ x \in H.$$

Then for every $T > 0$ *there exists a subsequence* (n_m) *such that for every* $\phi \in L^p(H,\mu)$,

$$\sup_{0\leq t\leq T} \|P_t^{n_m}\phi - P_t\phi\|_p \to 0 \quad \textit{as } m \to \infty. \tag{3.6}$$

Moreover, if $(\widehat{F}_n)$ *converges pointwise to* $\widehat{F}$, *then for every* $x \in H$,

$$E|U_{n,T}^x - U_T^x| \to 0 \quad \textit{as } n \to \infty$$

and

$$(3.6') \qquad \sup_{0\le t\le T} \|P_t^n\phi - P_t\phi\|_p \to 0 \quad \textit{as } n\to\infty.$$

Proof. Since $B^{-1}B$ is an orthogonal projection, it follows easily that F_n, F satisfy (F2) and (1.10). Then by Lemma 1.6, for some subsequence (n_m),

$$(3.7) \qquad E|U^x_{n_m,T} - U^x_T| \to 0 \quad \text{as } m\to\infty, \text{ for } \mu\text{-a.a. } x.$$

If $\phi \in B_{\mathrm{b}}(H)$, then by Proposition 1.2 and Remark 1.5, for any $x\in H$ we have

$$(3.8) \quad |P_t^n\phi(x) - P_t\phi(x)| \le E|\phi(Z_t^x)\cdot(U^x_{n,t} - U^x_t)| \le \|\phi\|_\infty \cdot E|U^x_{n,T} - U^x_T| \le 2\|\phi\|_\infty.$$

Therefore, for the subsequence satisfying (3.7) we obtain (3.6) from (3.8) and the Lebesgue Dominated Convergence Theorem.

For $\phi\in L^p(H,\mu)$ let $(\phi_m)\subset B_{\mathrm{b}}(H)$ be such that $\phi_m\to\phi$ in L^p. Then for $t\in[0,T]$, Proposition 3.1(i) yields

$$(3.9) \qquad \begin{aligned}\|P_t^n\phi - P_t\phi\|_p &\le \|P_t^n(\phi-\phi_m)\|_p + \|P_t^n\phi_m - P_t\phi_m\|_p + \|P_t(\phi_m-\phi)\|_p\\ &\le 2c(p,T)\|\phi_m-\phi\|_p + \|P_t^n\phi_m - P\phi_m\|_p,\end{aligned}$$

where $c(p,T) = \exp\left(\frac{\gamma^2}{2(p-1)}T\right)$ and $\gamma = \sup_n \|\widehat{F}_n\|_\infty$. From (3.9) we obtain (3.6) by the same argument as in the proof of Proposition 3.1(ii).

Finally, let $\widehat{F}_n(x)\to\widehat{F}(x)$ boundedly for all x. Then (under the notation of Lemma 1.6) for every x, $\Psi_n(t,x)\to\Psi(t,x)$ P-a.s. and boundedly, therefore $U^x_{n,T}\to U^x_T$ in probability and in $L^1(\Omega,\mathcal{F},P)$ by [D-Z; S; Lem. 10.16]. Arguing as before, we obtain (3.6′). ■

COROLLARY 3.3. *Under assumptions* (A1) *and* (F2), *for any* $p\in(1,\infty)$, $(P_t)_{t\ge0}$ *is a* C_0*-semigroup in* $L^p(H,\mu)$.

Proof. By Proposition 3.1 it suffices to show that

$$(3.10) \qquad P_t(P_s\phi)(x) = P_{t+s}\phi(x) \quad \text{for } B_{\mathrm{b}}(H),\ t,s>0,\ \text{for } \mu\text{-a.a. } x.$$

First assume that F is a Lipschitz function. Then equation $(*)$ has a unique mild solution X on the given probability space $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$ w.r.t. the fixed Wiener process W. Therefore, in virtue of Remark 1.5, for any $T>0$,

$$P_t\phi(x) = \widetilde{E}^x_T\phi(\widetilde{X}^x_t) = E\phi(X^x_t), \qquad 0\le t\le T,$$

for $\phi\in B_{\mathrm{b}}(H)$ and $x\in H$. By [D-Z; S, Thm. 9.8, Cor. 9.9] the process X is Markovian and (3.10) holds for all $x\in H$. For every F satisfying (F2) we can find a sequence of bounded Lipschitz functions $\widehat{F}_n : H\to K$, converging μ-a.s. and boundedly to $\widehat{F} := B^{-1}F \in B_{\mathrm{b}}(H,K)$. For fixed $T>0$, by Proposition 3.2 we can choose a subsequence $(P_t^{n_m}, 0\le t\le T)$ satisfying (3.6). By the first part of the proof

$$P_t^{n_m}(P_s^{n_m}\phi) = P_{t+s}^{n_m}\phi, \qquad \phi\in B_{\mathrm{b}}(H),\ m=1,2,\ldots$$

Letting $m\to\infty$, from (3.6) and the uniform boundedness of $\|P_t^{n_m}\|$, $m=1,2,\ldots$, $t\in[0,T]$, we obtain (3.10). ■

3.2. Analytic approach and equivalence. On the other hand, with equation $(*)$ one can associate, at least formally, the differential operator L_F^0 on H given by the formula

$$L_F^0\phi(x) = \tfrac{1}{2}\operatorname{tr}(QD^2\phi(x)) + \langle Ax, D\phi(x)\rangle + \langle F(x), D\phi(x)\rangle, \quad x \in \operatorname{dom}(A), \tag{3.11}$$

for $\phi \in \operatorname{dom}(L_F^0) = \mathbb{F}C_{\mathrm{b}}^\infty$, where $\mathbb{F}C_{\mathrm{b}}^\infty$ is defined in (2.1a). Then $L_F^0\phi = L\phi + G_0\phi$, where L is the generator of the O-U semigroup (R_t) and $G_0\phi(x) = \langle F(x), D\phi(x)\rangle$, $\operatorname{dom}(G_0) = \mathbb{F}C_{\mathrm{b}}^\infty$.

THEOREM 3.4. *Assume* (A1) *and* (F2). *Then the operator L_F^0 is closable in $L^2(H,\mu)$, its closure L_F is the generator of a C_0-semigroup $(\mathcal{V}_t, t \ge 0)$ on $L^2(H,\mu)$ and* $\operatorname{dom}(L_F) = \operatorname{dom}(L)$. *Moreover, $L_F = L + G$, where G is the unique extension of G_0 to an L-bounded operator with domain* $\operatorname{dom}(L)$. *The semigroup $(\mathcal{V}_t)$ is a unique C_0-semigroup on $L^2(H,\mu)$ satisfying*

$$\begin{aligned} \mathcal{V}_t\phi &= R_t\phi + \int_0^t \mathcal{V}_{t-s}GR_s\phi\,ds \\ &= R_t\phi + \int_0^t R_{t-s}G\mathcal{V}_s\phi\,ds \quad \textit{for all } \phi \in \operatorname{dom}(L),\ t \ge 0. \end{aligned} \tag{3.12}$$

Proof. (A version of this theorem has been proved in an unpublished paper [Ch-G; P] and it follows from a result in [V; P] on Miyadera perturbations.) Note that by (2.23) for $\phi \in \mathbb{F}C_{\mathrm{b}}^\infty$ we have $G_0\phi(x) = \langle Q^{-1/2}F(x), D_Q\phi(x)\rangle$. By [D-Z; S, Cor. B.4], we have

$$\|Q^{-1/2}h\| = \|B^{-1}h\| \quad \text{for } h \in \operatorname{im} Q^{1/2} = \operatorname{im} B \tag{3.13}$$

(where $Q^{-1/2}$ is the pseudoinverse of $Q^{1/2}$).

Therefore

$$\|G_0\phi\|_2 \le \|B^{-1}F\|_\infty \cdot \|D_Q\phi\|_2. \tag{3.14}$$

Recall that $\mathbb{F}C_{\mathrm{b}}^\infty$ is a core for L. By Proposition 2.5(ii), D_Q uniquely extends to an L-bounded operator (still denoted by D_Q) with domain $\operatorname{dom}(L)$ and hence so does G_0. Thus, by [V; P, Thm. 1] it is enough to show that there exist $T, \gamma > 0$ such that

$$\gamma < 1 \quad \text{and} \quad \int_0^T \|GR_t\phi\|_2\,dt \le \gamma\|\phi\|_2 \quad \text{for all } \phi \in \operatorname{dom}(L). \tag{3.15}$$

To this end we estimate the integral in (3.15) using the Hölder inequality, (3.14) and finally Proposition 2.5(iii):

$$\begin{aligned} \int_0^T \|GR_t\phi\|_2\,dt &\le \sqrt{T}\Big(\int_0^T \|GR_t\phi\|_2^2\,dt\Big)^{1/2} \\ &\le \sqrt{T}\beta\Big(\int_0^T \|D_QR_t\phi\|_2^2\,dt\Big)^{1/2} \le \sqrt{T}\,\beta\|\phi\|_2. \end{aligned} \tag{3.16}$$

Hence, for T sufficiently small, (3.15) holds and the theorem follows from [V; P, Thm. 1]. ∎

The next theorem shows that both the probabilistic and analytic constructions of the transition semigroup for $(*)$ coincide. We will need the following lemma about general

C_0-semigroups. This fact is known (see e.g. [E]) but, for completeness, we give a simple proof different from that in [E].

Lemma 3.5. *Let $(\mathcal{S}_t)$ and $(\mathcal{T}_t)$ be C_0-semigroups of bounded linear operators on a Banach space E with generators $\mathcal{A}$ and $\mathcal{B}$, respectively. If $\mathcal{D}$ is a core for $\mathcal{A}$ and $\mathcal{A}x = \mathcal{B}x$ for all $x \in \mathcal{D}$, then $\mathcal{S}_t = \mathcal{T}_t$ for all $t \geq 0$.*

Proof. Because $\mathcal{D}$ is a core for $\mathcal{A}$, for every $x \in \mathrm{dom}(\mathcal{A})$ there exists a sequence $(x_n) \subset \mathcal{D}$ such that $x_n \to x$ and $\mathcal{A}x_n \to \mathcal{A}x$ as $n \to \infty$. But $\mathcal{A}x_n = \mathcal{B}x_n$ for all n and from the closedness of $\mathcal{B}$ it follows that $x \in \mathrm{dom}(\mathcal{B})$ and $\mathcal{B}x = \mathcal{A}x$. Consequently, $\mathcal{A} \subset \mathcal{B}$.

Next, for fixed $t > 0$ and $y \in \mathrm{dom}(\mathcal{A})$ consider the function $[0,t] \ni s \mapsto \mathcal{T}_{t-s}\mathcal{S}_s y \in E$, which is differentiable since $\mathcal{S}_s(\mathrm{dom}(\mathcal{A})) \subset \mathrm{dom}(\mathcal{A}) \subset \mathrm{dom}(\mathcal{B})$. Basic properties of C_0-semigroups and $\mathcal{A} \subset \mathcal{B}$ yield

$$\frac{d}{ds}[\mathcal{T}_{t-s}\mathcal{S}_s y] = -\mathcal{B}\mathcal{T}_{t-s}\mathcal{S}_s y + \mathcal{T}_{t-s}\mathcal{A}\mathcal{S}_s y = -\mathcal{T}_{t-s}\mathcal{B}\mathcal{S}_s y + \mathcal{T}_{t-s}\mathcal{B}\mathcal{S}_s y = 0.$$

Therefore, for all $s \in [0,t]$, $\mathcal{T}_{t-s}\mathcal{S}_s y = \mathcal{T}_t y$ and, in particular $\mathcal{S}_t y = \mathcal{T}_t y$. It follows that $\mathcal{S}_t = \mathcal{T}_t$ on $\mathrm{dom}(\mathcal{A})$ and then on E. ■

Theorem 3.6. *Let* (A1) *and* (F2) *hold. Then for every $t \geq 0$,*

$$P_t\phi = \mathcal{V}_t\phi \quad \textit{for all } \phi \in L^2(H,\mu),$$

where $(\mathcal{V}_t)$ is the semigroup given by Theorem 3.4.

Proof. Step 1. Here we assume that $F \in C_{\mathrm{b}}^2(H,H)$. Then $(*)$ has a unique mild solution (X_t^x) on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P, (W_t))$ and for $\psi \in B_{\mathrm{b}}(H)$,

$$P_t\psi(x) = E\psi(X_t^x), \quad x \in H.$$

Let $\mathcal{A}_F$ denote the generator of the semigroup (P_t) in $L^2(H,\mu)$. Because $\mathbb{F}C_{\mathrm{b}}^\infty$ is a core in $L^2(H,\mu)$ for the O-U generator L and, by Theorem 3.4, $\mathrm{dom}(L_F) = \mathrm{dom}(L)$, it follows that $\mathbb{F}C_{\mathrm{b}}^\infty$ is a core for L_F. Therefore, by Lemma 3.5, in order to prove the theorem it is enough to show that

$$\mathcal{A}_F\phi = L_F\phi \quad \text{for } \phi \in \mathbb{F}C_{\mathrm{b}}^\infty, \tag{3.17}$$

which is equivalent to the condition:

$$\text{for } \phi \in \mathbb{F}C_{\mathrm{b}}^\infty, \quad t^{-1}(P_t\phi - \phi) \underset{t\to 0^+}{\longrightarrow} L_F\phi \quad \text{in } L^2(H,\mu).$$

Because we do not assume that $Q := BB^*$ is nuclear, the relevant result on the Kolmogorov equation [D-Z; S, Thm. 9.17], [D-Z; E, Thm. 5.4.2], [Z2] is not directly applicable. We use approximations as in [P-Z], [D-Z; E, Thm. 7.1.1]. Let $A_k := kA(kI - A)^{-1}$, for sufficiently large k, be the Yosida approximation of A. Let (e_n) be an ON basis in H and π_k be the orthogonal projection onto $\mathrm{lin}\{e_1, \ldots, e_k\}$. For each k, let $(X_t^{k,x})$ denote the solution to the stochastic equation

$$\begin{cases} dX_t^k = [A_k X_t^k + F(X_t^k)]dt + \pi_k B dW_t, \\ X_0^k = x. \end{cases} \tag{3.18}$$

Then for any $T > 0$,

$$\sup_{0\leq t\leq T} E[(X_t^{k,x} - X_t^x)^2] \underset{k\to\infty}{\longrightarrow} 0. \tag{3.19}$$

Fix $\varphi \in \mathbb{F}C_{\mathrm{b}}^{\infty}$. Then $\varphi(x) = f(\Pi x)$, where, for a certain natural number N, Π is an N-dimensional orthogonal projection such that $\Pi(H) \subset \mathrm{dom}(A^*)$ and $f \in C_{\mathrm{b}}^{\infty}(H)$. Therefore

$$
\begin{aligned}
D\varphi(x) &= \Pi Df(\Pi x) = \Pi \circ \Pi Df(\Pi x) = \Pi D\varphi(x), \\
D^2\varphi(x) &= \Pi D^2 f(\Pi x)\Pi = \Pi D^2\varphi(x)\Pi.
\end{aligned}
\tag{3.20}
$$

Write

$$
\begin{aligned}
v(t,x) &:= E\varphi(X_t^x) = P_t\varphi(x), \\
v_k(t,x) &:= E\varphi(X_t^{k,x}),
\end{aligned}
\qquad t \geq 0,\ x \in H.
\tag{3.21}
$$

Since $(X_t^{k,x})$ is the strong solution to (3.18), using the Ito lemma [D-Z; S, Thm. 4.17], (3.20) and differentiating with respect to t, we obtain

$$
\begin{aligned}
\frac{\partial v_k}{\partial t}(t,x) = {} & \tfrac{1}{2}E(\mathrm{tr}[\Pi D^2\varphi(X_t^{k,x})\Pi Q_k]) \\
& + E\langle X_t^{k,x}, A_k^*\Pi D\varphi(X_t^{k,x})\rangle + E\langle F(X_t^{k,x}), D\varphi(X_t^{k,x})\rangle,
\end{aligned}
\tag{3.22}
$$

where $Q_k := \pi_k Q \pi_k$.

Because φ is a bounded Lipschitz function, it follows from (3.21), (3.19) and the Lebesgue Dominated Convergence Theorem (LDCT for short) that for $x \in H$,

$$
\sup_{0\leq t\leq T} |v_k(t,x) - v(t,x)| \to 0.
\tag{3.23}
$$

Consider the RHS of (3.22). Note that $A^*\Pi$ is a bounded operator on H and

$$
\begin{aligned}
|\mathrm{tr}\, \Pi D^2\varphi(X_t^{k,x})\Pi Q_k| &\leq (\mathrm{tr}\, \Pi)\cdot \|Q_k\|_{L(H,H)} \cdot \|D^2\varphi(X_t^{k,x})\|_{L(H,H)} \\
&\leq N\|Q\|_{L(H,H)}\|D^2\varphi\|_{\infty}
\end{aligned}
\tag{3.24}
$$

(where $\|D^2\varphi\|_\infty = \sup_{x\in H} \|D^2\varphi(x)\|_{L(H,H)}$). For $t \geq 0$, $x \in H$ define

$$
\begin{aligned}
u(t,x) = {} & \tfrac{1}{2}E(\mathrm{tr}\, \Pi D^2\varphi(X_t^x)\Pi Q) \\
& + E\langle X_t^x, A^*\Pi D\varphi(X_t^x)\rangle + E\langle F(X_t^x), D\varphi(X_t^x)\rangle.
\end{aligned}
\tag{3.25}
$$

Because φ, F, $D\varphi$ and $D^2\varphi$ are bounded Lipschitz mappings, from (3.19), (3.22), (3.24) and LDCT for $x \in H$ we obtain

$$
\sup_{0\leq t\leq T}\left|\frac{\partial v_k}{\partial t}(t,x) - u(t,x)\right| \xrightarrow[k\to\infty]{} 0,
$$

which combined with (3.23) implies that $v(\cdot,x)$ is t-differentiable and

$$
\frac{\partial v}{\partial t}(t,x) = u(t,x), \qquad t \geq 0,\ x \in H.
\tag{3.26}
$$

It is well known that for some constant $c_T > 0$,

$$
\sup_{t\in[0,T]} E((X_t^x)^2) \leq c_T(1 + \|x\|^2)
$$

and therefore by (3.24) and (3.25),

$$
\sup_{t\in[0,T]}\left|\frac{\partial v}{\partial t}(t,x)\right|^2 \leq \widetilde{c}(T,\varphi)(1 + \|x\|^2), \quad x \in H,
$$

where $\widetilde{c}(T,\varphi) > 0$ is a constant independent of x. This combined with (3.26) and LDCT gives

$$\lim_{t\to 0^+}\int_H \left[\frac{v(t,x)-v(0,x)}{t} - u(0,x)\right]^2 d\mu = 0. \tag{3.27}$$

Note that $u(0,x) = L_F^0\varphi(x)$ (see (3.11)). Hence, recalling that $v(t,x) = P_t\varphi(x)$, we can write (3.27) as

$$\lim_{t\to 0^+}\frac{P_t\varphi-\varphi}{t} = L_F\varphi \quad (\text{in } L^2(H,\mu)),$$

which implies (3.17) and completes the proof of Step 1.

Step 2. We proceed as in the proof of Corollary 3.3. If F satisfies (F2), then $\widehat{F} := B^{-1}F \in B_{\rm b}(H,K)$ and one can find a sequence $(\widehat{F}_n)_{n=1}^\infty \subset C_{\rm b}^2(H,K)$ converging to $\widehat{F}$ μ-a.s. and boundedly.

Let $(P_t^n)_{t\ge 0}$ be the semigroup defined in Proposition 3.2, which corresponds to $F_n := B\widehat{F}_n$. Then for $\phi \in B_{\rm b}(H)$,

$$P_t^n\phi(x) = E\phi(X_{n,t}^x),$$

where $(X_{n,t}^x)$ is the mild solution to equation $(*)$ with nonlinear term F_n. According to Theorem 3.4, let $(\mathcal{V}_t^n)$ denote the semigroup with generator $L_{F_n} = L + G^{(n)}$, where $G^{(n)}\phi(x) = \langle F_n(x), D\phi(x)\rangle$ for $\phi\in \mathrm{dom}(L_F^0)$. By (3.14) and (3.16) for any $t>0$ and $\phi \in \mathrm{dom}(L)$ we have

$$\int_0^t \|G^{(n)}R_s\phi\|_2\,ds \le \sqrt{t}\,\widetilde{\beta}\,\|\phi\|_2, \quad \text{where} \quad \widetilde{\beta} := \sup_n \|\widehat{F}_n\|_\infty < \infty, \tag{3.28}$$

$$\begin{aligned}\int_0^t \|(G^{(n)}-G)R_s\phi\|_2\,ds \\ \le \sqrt{t}\int_0^t\int_H \|\widehat{F}_n(x)-\widehat{F}(x)\|^2\|D_QR_s\phi(x)\|^2\,\mu(dx)\,ds \to 0 \quad \text{as } n\to\infty,\end{aligned} \tag{3.29}$$

where the convergence follows from LDCT, since by Section 2,

$$\int_0^t\int_H \|D_QR_s\phi(x)\|^2\,\mu(dx)\,ds \le \|\phi\|_2^2.$$

From (3.28) and (3.29) we conclude, using [V; A, Thm. 1.4], that for every $T>0$ and $\phi\in L^2(H,\mu)$,

$$\sup_{0\le t\le T}\|\mathcal{V}_t^n\phi - \mathcal{V}_t\phi\|_2 \to 0 \quad \text{as } n\to\infty. \tag{3.30}$$

Since by Step 1,

$$\mathcal{V}_t^n\phi = P_t^n\phi, \quad n=1,2,\ldots,\ t\ge 0,\ \phi\in L^2(H,\mu),$$

it follows from (3.30) and (3.6) in Proposition 3.2 that $\mathcal{V}_t\phi = P_t\phi$. ∎

According to Theorem 3.6, in what follows we denote by L_F the generator of the semigroup (P_t).

REMARK 3.7. It is interesting to compare the estimate (3.1) of the norm $\|P_t\|_{2\to 2}$ with the estimate that follows from Theorems 3.4 and 3.6.

Write $u := \sqrt{T}\,\beta$, where T satisfies (3.15) and (3.16). Then $0 < u < 1$. Theorem 3.4 and [V; P, (1.3)] imply

$$\|P_t\|_{2\to 2} \le \frac{1}{1-u}\exp\left[\left(\frac{\beta^2}{u^2}\log\frac{1}{1-u}\right)t\right] =: M_u \exp[\alpha(u)t], \quad t \ge 0. \tag{3.31}$$

Let us find a lower bound of $\alpha(u)$. As $\alpha(u) > 0$, $\lim_{u\to 0^+}\alpha(u) = \infty$ and $\lim_{u\to 1^-}\alpha(u) = \infty$, it follows that $\alpha(\cdot)$ achieves its minimum at some $\overline{u} \in (0,1)$. Then $\frac{d\alpha}{du}(\overline{u}) = 0$ and by an easy computation we have

$$\log(1-\overline{u}) = -\frac{\overline{u}}{2(1-\overline{u})},$$

which yields

$$\alpha(\overline{u}) = \frac{\beta^2}{2\overline{u}(1-\overline{u})} \ge 2\beta^2.$$

Therefore in (3.31)

$$M_u \exp[\alpha(u)t] > \exp(2\beta^2 t),$$

while (3.1) gives

$$\|P_t\|_{2\to 2} \le \exp(\beta^2 t/2).$$

4. Properties of (P_t)—the case of bounded F

4.1. Hyperboundedness. First, recall that for $t > 0$ the condition

$$\operatorname{im} Q_t^{1/2} = \operatorname{im} Q_\infty^{1/2} \tag{A2}$$

is equivalent to

$$\|S_0(t)\| < 1 \quad (\text{where } S_0(t) = Q_\infty^{-1/2} S_t Q_\infty^{1/2}{}_{|\overline{H}_0}) \quad (\text{see Sect. 2}), \tag{4.1}$$

which, finally, is equivalent to the hypercontractivity of R_t. By the result in [Ch-G; Q] and (2.18), for every $p, q \ge 1$,

$$\|R_t\|_{p\to q} = \begin{cases} 1 & \text{if } q \le q(t,p), \\ \infty & \text{if } q > q(t,p), \end{cases} \tag{4.2}$$

where

$$q(t,p) := 1 + \frac{p-1}{\|S_0(t)\|^2}. \tag{4.3}$$

(Recall that always $\|S_0(t)\| \le 1$.)

The theorem below says that the semigroup (P_t) has a similar property with hypercontractivity replaced by hyperboundedness.

THEOREM 4.1. *Assume* (A1) *and* (F2).

(i) *If* (A2) *holds for some* $t_0 > 0$, *then for every* $t \ge t_0$, $p > 1$, $q \ge 1$ *the operator* $P_t : L^p(H,\mu) \to L^q(H,\mu)$ *is bounded for* $q < q(t,p)$ *and unbounded for* $q > q(t,p)$, *where* $q(t,p)$ *is defined by* (4.3).

(ii) *Conversely, if* $P_{t_0} : L^{p_0}(H,\mu) \to L^{q_0}(H,\mu)$ *is bounded for some* $t_0 > 0$ *and* $q_0 > p_0 > 1$, *then* (A2) *holds for all* $t \ge t_0$.

Proof. (i) Fix $p > 1$, $t \ge t_0$ and r such that $1 < r < p$. Let $\varphi \in B_{\mathrm{b}}(H)$ and $q > p$. Then the first inequality in (3.4) with p replaced by q takes the form

$$\|P_t\varphi\|_q \le c_t(r')\|R_t(|\varphi|^r)\|_{q/r}^{1/r}. \tag{4.4}$$

From what has been recalled, R_t is a contraction from $L^{p/r}$ to $L^{q/r}$ for

$$\frac{q}{r} = 1 + \frac{p/r-1}{\|S_0(t)\|^2}.$$

Hence taking

$$q_r := r + \frac{p-r}{\|S_0(t)\|^2},$$

in (4.4) we obtain

$$\|P_t\varphi\|_{q_r} \le c_t(r')\cdot \||\varphi|^r\|_{p/r}^{1/r} = c_t(r')\|\varphi\|_p, \tag{4.5}$$

where by (3.3), $c_t(r') = \exp\big(\frac{\beta^2 t}{2(r-1)}\big)$. Writing $q(t,p)$ and q_r in the form

$$q(t,p) = \frac{p}{\|S_0(t)\|^2} - \bigg(\frac{1}{\|S_0(t)\|^2} - 1\bigg), \qquad q_r = \frac{p}{\|S_0(t)\|^2} - r\bigg(\frac{1}{\|S_0(t)\|^2} - 1\bigg), \tag{4.6}$$

we see that for any $q < q(t,p)$ $(q \ge 1)$, one can find $\varepsilon > 0$ such that for $r_\varepsilon = 1+\varepsilon$, $q_{r_\varepsilon} = q$, which by (4.5) completes the proof of the first part of (i).

(ii) Let $q > r > 1$, $t > 0$, $\varphi \in B_{\mathrm{b}}(H)$. Next, let U_t^x be the Girsanov martingale defined by (1.3), (1.4). Since $U_t^x > 0$, P-a.e., using Hölder's inequality with exponents r and $r' = r/(r-1)$, we have

$$\begin{aligned}\|R_t\varphi\|_q^q &= \int_H \Big|\int_\Omega \varphi(Z_t^x)(U_t^x)^{1/r}(U_t^x)^{-1/r}\,dP\Big|^q \mu(dx) \\ &\le \int_H \Big(\int_\Omega |\varphi(Z_t^x)|^r U_t^x\,dP\Big)^{q/r}\cdot\Big(\int_\Omega (U_t^x)^{-1/(r-1)}\,dP\Big)^{q/r'}\mu(dx) \\ &= \int_H [P_t(|\varphi|^r)]^{q/r}(E[(U_t^x)^{-1/(r-1)}])^{q/r'}\mu(dx).\end{aligned} \tag{4.7}$$

For $v > 0$, in much the same way as in (3.3) we obtain the estimate

$$E[(U_t^x)^{-v}] \le \exp\bigg(\frac{v(v+1)}{2}\beta^2 t\bigg) =: \widetilde{c}_t(v),$$

which combined with (4.7) yields

$$\|R_t\varphi\|_q \le \widetilde{\widetilde{c}}_t(r)\|P_t(|\varphi|^r)\|_{q/r}^{1/r}, \quad \text{where} \tag{4.8}$$

$$\widetilde{\widetilde{c}}_t(r) := \bigg[\widetilde{c}_t\bigg(\frac{1}{r-1}\bigg)\bigg]^{1/r'} = \exp\bigg(\frac{\beta^2 t}{2(r-1)}\bigg).$$

Fix $r > 1$. Setting $t = t_0$, $q = \overline{q} := q_0 r$ in (4.8) and using the notation

$$M := \|P_{t_0}\|_{p_0\to q_0} < \infty, \quad c_r := \widetilde{\widetilde{c}}_{t_0}(r), \quad \overline{p} := p_0 r,$$

we get

$$\|R_{t_0}\varphi\|_{\overline{q}} \le c_r\|P_{t_0}(|\varphi|^r)\|_{q_0}^{1/r} \le c_r M^{1/r}\||\varphi|^r\|_{p_0}^{1/r} = c_r M^{1/r}\|\varphi\|_{\overline{p}}. \tag{4.9}$$

Therefore $\|R_{t_0}\|_{\overline{p}\to\overline{q}} < \infty$, and since $\overline{q} > \overline{p}$, we deduce from (4.2), (4.3) that necessarily $\|S_0(t_0)\| < 1$. This implies that $\|S_0(t)\| < 1$ for all $t \ge t_0$ and hence (ii) follows.

(i) (cont.) Finally, to prove the latter claim of (i), suppose conversely that for $p > 1$, $\|P_t\|_{p\to q} < \infty$ for some $q > q(t,p)$. Then it follows from (4.9) that

$$\|R_t\|_{pr\to qr} < \infty \quad \text{for any } r > 1. \tag{4.10}$$

Since by assumption $\|S_0(t)\| < 1$, we can set

$$\varepsilon := \frac{q - q(t,p)}{2(\|S_0(t)\|^{-2} - 1)} > 0.$$

Thus from (4.6) we have

$$\begin{aligned} q(1+\varepsilon) &= (1+\varepsilon)q(t,p) + (1+\varepsilon)[q - q(t,p)] \\ &> (1+\varepsilon)[p\|S_0(t)\|^{-2} - (\|S_0(t)\|^{-2} - 1)] + 2\varepsilon(\|S_0(t)\|^{-2} - 1) \\ &> p(1+\varepsilon)\|S_0(t)\|^{-2} - (\|S_0(t)\|^{-2} - 1) = q(t, p(1+\varepsilon)). \end{aligned}$$

Therefore, taking $r = 1 + \varepsilon$ in (4.10), we obtain a contradiction with (4.2). ■

Below we show another consequence of (A2): P_t improves positivity. Since in Proposition 4.2 assumption (F2) is replaced by the weaker condition (F1), we treat P_t defined by (1.9) as an operator on $L^\infty(H,\mu)$.

PROPOSITION 4.2. *Assume* (A1), (F1) *and let* (A2) *be satisfied for some* $t_0 > 0$. *Then for each* $t \ge t_0$,

(4.11) *if* $\varphi \in B_{\mathrm{b}}(H)$ *is nonnegative and* $\varphi \not\equiv 0$, *then* $P_t\varphi(x) > 0$ *for* μ-*a.a.* x.

Proof. Because any nonnegative Borel function is a pointwise limit of a nondecreasing sequence of simple nonnegative functions and P_t is a linear and positivity preserving operator, it suffices to prove (4.11) for $\phi = \mathbf{1}_C$, the indicator function of a Borel set C with $\mu(C) > 0$.

Since (A2) holds for $t_0 > 0$, by (4.1) we have

$$\|S_0^*(t)\| < 1 \quad \text{for all } t \ge t_0. \tag{4.12}$$

By the result of [Ch-G; Q], $R_t = \Gamma(S_0^*(t))$, where Γ is the second quantization operator. Hence from (4.12) and [S2, Thm. I.16] we conclude that for $t \ge t_0$, R_t improves positivity, in particular $R_t\mathbf{1}_C(x) > 0$ for μ-a.a. x. Let $\mathcal{G}$ be the set mentioned above (1.9). Then $\mu(\mathcal{G}) = 1$ and for each $x \in \mathcal{G}$,

$$E(U_t^x) = 1 \quad \text{and} \quad \frac{d\widetilde{P}_t^x}{dP}(\omega) = U_t^x(\omega) > 0, \quad P\text{-a.e.},$$

which means the probability measures $\widetilde{P}_t^x$ and P are equivalent. Consequently, for $x \in \mathcal{G}$ if $R_t\mathbf{1}_C(x) = P(Z_t^x \in C) > 0$, then $P_t\mathbf{1}_C(x) = \widetilde{P}_t^x(Z_t^x \in C) > 0$. This finishes the proof. ■

4.2. Domains of generators. In some results of this subsection we assume

$$\operatorname{im} Q_\infty^{1/2} \subset \operatorname{im} Q^{1/2}, \tag{A3}$$

which is stronger than (A2) (see Section 2). Corollaries 4.3 and 4.5(i) below concerning $\mathrm{dom}_2(L_F)$ are immediate consequences of Theorem 3.4 and the analogous results for the O-U generator L, obtained in [Ch-G; N] (see also Proposition 2.5 and Corollary 2.7).

COROLLARY 4.3. (i) *If* (A1), (F2) *hold and the operator* $V := Q^{1/2}Q_\infty^{-1/2}$ *with* $\mathrm{dom}(V) = \mathrm{im}\, Q_\infty^{1/2}$ *is closable, then* $\mathrm{dom}_2(L_F)$ *is continuously embedded in* $W_Q^{1,2}$.

(ii) *If* (A1), (A3), *and* (F2) *are satisfied, then* $\mathrm{dom}_2(L_F)$ *is continuously embedded into* $W_{Q_\infty}^{2,2}$ *and into the Orlicz space* $L^2 \log^r L$ *for* $0 \le r < 2$.

To consider $\mathrm{dom}_p(L_F)$ we need the following assumption:

(F3) $\quad F : H \to \mathrm{im}\, Q_\infty^{1/2}$ is a Borel function and $\beta_0 := \|Q_\infty^{-1/2}F\|_\infty < \infty$.

PROPOSITION 4.4. *Assume* (A1), (A3), *and* (F3). *Then for every* $p \in (1,\infty)$,

(i) $\mathrm{dom}_p(L_F) = \mathrm{dom}_p(L)$ *and in particular*

(ii) $\mathrm{dom}_p(L_F)$ *is continuously embedded in the Orlicz space* $L^p \log^r L$ *for* $0 < r < p$.

Proof. (i) Note that in the notation of Theorem 3.4 by (2.21), (2.22) we have

$$L_F\varphi = L\varphi + G\varphi, \qquad G\varphi(x) = \langle Q_\infty^{-1/2}F(x), D_{Q_\infty}\varphi(x)\rangle, \qquad \varphi \in \mathbb{F}C_{\mathrm{b}}^\infty.$$

Hence

$$\|G\varphi\|_p \le \beta_0 \|D_{Q_\infty}\varphi\|_p, \tag{4.13}$$

which means that G can be uniquely extended to a bounded operator (still denoted by G) acting from $W_{Q_\infty}^{1,p}$ to $L^p(H,\mu)$. By [Ch-G; R, Thm. 1, (15)], under (A3),

$$\|D_{Q_\infty} R_t\varphi\|_p \le c(p)\|(I - V_t)^{-1}V_t\|^{1/2}\|\varphi\|_p, \tag{4.14}$$

where $V_t := S_0(t)S_0^*(t)$. Because (A3) implies that $\|S_0(t)\| \le e^{-t\lambda/2}$ for some $\lambda > 0$ (see Section 2), we have

$$\|(I - V_t)^{-1}V_t\| \le \frac{\|V_t\|}{1 - \|V_t\|} \le \frac{e^{-\lambda t}}{\lambda t e^{-\lambda t}} = \frac{1}{\lambda t}. \tag{4.15}$$

It follows from (4.13)–(4.15) that for any $T > 0$,

$$\int_0^T \|GR_t\varphi\|_p\, dt < \infty \quad \text{and} \quad \mathrm{dom}_p(G) = W_{Q_\infty}^{1,p} \supset \mathrm{dom}_p(L)$$

(the latter is due to [Da; Lem. 3.4, p. 70]). Hence G is a Phillips perturbation of L and (i) follows by [H-Ph; Cor. 1, p. 400 and Thm. 13.5.3]. Next, (i) and [Ch-G; N, Thm. 4.4] imply at once (ii). ∎

If the O-U generator L is symmetric, the L_p-domains of L_F can be characterized explicitly, as a consequence of the corresponding result for L obtained in [D-G,1] for $p = 2$ and in [Ch-G; M], [Ch-G; N] for $1 < p < \infty$. Recall that if $\ker Q_\infty = \{0\}$, the O-U semigroup (R_t) is self-adjoint in $L^2(H,\mu)$ iff

(A4) $\quad Q(\mathrm{dom}(A^*)) \subset \mathrm{dom}(A) \quad$ and $\quad AQx = QA^*x, \quad x \in \mathrm{dom}(A^*)$

(see [Ch-G; S]). Under (A4) the operator $-AQ$ has a Friedrichs extension to a self-adjoint nonnegative operator in H (see *ibid.*).

COROLLARY 4.5. *Assume* (A1) *and let* $\ker Q_\infty = \{0\}$.

(i) *If* (A4) *and* (F2) *are satisfied, then* $\mathrm{dom}_2(L_F) = W^{2,2}_Q \cap W^{1,2}_{-AQ}$.

(ii) *If* (A2), (A4), *and* (F3) *hold, then* $\mathrm{dom}_p(L_F) = W^{2,p}_Q \cap W^{1,p}_{-AQ}$.

Proof. (i) is an easy consequence of Theorem 3.4 and [D-G,1, Thm. 3.2] (see also [Ch-G; M, Cor. 5.4] or [Ch-G; N, Thm. 2.4]) and (ii) follows from this last result, Proposition 4.4 and the fact that under (A4) conditions (A2) and (A3) are equivalent.

4.3. Invariant measures with densities. Recall that a Borel measure ν on H is *invariant* for the semigroup (P_t) if

$$\int_H P_t\varphi(x)\,\nu(dx) = \int_H \varphi(x)\,\nu(dx), \qquad \varphi \in B_{\mathrm{b}}(H),\ t \ge 0. \tag{4.16}$$

We consider only probability invariant measures.

REMARK 4.6. Under the assumptions of Remark 1.5 and notation of Proposition 1.2, it follows from an obvious modification of [K-S, Cor. 5.3.11] that equation $(*)$ has a unique martingale solution $(\widetilde{X}^\nu_t)_{0\le t\le T}$ with initial distribution ν on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \widetilde{P}^\nu_T)$, where

$$\widetilde{P}^\nu_T(d\omega) = \int_H \widetilde{P}^x_T(d\omega)\,\nu(dx)$$

and

$$\widetilde{P}^\nu_T(\widetilde{X}^\nu_t \in C) = \int_H \widetilde{P}^x_T(\widetilde{X}^x_t \in C)\,\nu(dx), \quad C \in \mathcal{B}(H).$$

The latter together with the equality

$$P_t\varphi(x) = \widetilde{E}^x_T(\varphi(\widetilde{X}^x_t)), \qquad 0 \le t \le T,$$

shows that (4.16) holds iff

$$\widetilde{P}^\nu_T(\widetilde{X}^\nu_t \in C) = \nu(C), \qquad C \in B(H),\ 0 \le t \le T.$$

Hence, ν is an invariant measure for (P_t) iff ν is a stationary distribution for $(*)$. If F satisfies (F1) the same holds, except the uniqueness of solution, for ν absolutely continuous w.r.t. μ.

We are concerned with the existence of invariant measures for (P_t) that are absolutely continuous w.r.t. μ. Equivalently, we look for $\varrho \in L^1(H,\mu)$ such that

$$\varrho \ge 0, \quad \|\varrho\|_1 = 1, \quad \int_H (P_t\varphi)\varrho\,d\mu = \int_H \varphi\varrho\,d\mu \quad \text{for } \varphi \in B_{\mathrm{b}}(H),\ t \ge 0. \tag{4.17}$$

Note that if $\varrho \in L^p(H,\mu)$ for some $p \in (1,\infty)$, then (4.17) holds iff $P^*_t\varrho = \varrho$, $t \ge 0$, and hence iff $\varrho \in \mathrm{dom}_p(L^*_F)$ and $L^*_F\varrho = 0$, where L^*_F denotes the generator of the semigroup (P^*_t) on $L^p(H,\mu)$ and (P^*_t) is adjoint to the semigroup (P_t) acting on $L^{p'}(H,\mu)$, $p' = p/(p-1)$.

The theorem below is a counterpart of [Ch-G; E, Thm. 5] which was proved by the compactness method.

THEOREM 4.7. *Assume* (A1) *and* (F2). *If* (A2) *holds for some* $t_0 > 0$, *then*

(a) *there exists an invariant measure* ν *for* (P_t) *which is absolutely continuous w.r.t.* μ;

(b) $\varrho := d\nu/d\mu \in \bigcap_{p\ge1} L^p(H,\mu)$;

(c) $\varrho(x) > 0$ *for* μ*-a.a.* x;

(d) ν *is a unique invariant measure for* (P_t) *in the class of probability measures absolutely continuous w.r.t.* μ.

Proof. Fix $p > 1$ and let $p' = p/(p-1)$. (P_t) is considered in $L^{p'}(H,\mu)$, so (P_t^*) acts in $L^p(H,\nu)$.

(a) The proof of existence is based on a result obtained recently in [H; P]. It follows from (A2) and Theorem 4.1 that for some $q' > p'$, $P_{t_0} : L^{p'}(H,\mu) \to L^{q'}(H,\mu)$ is bounded. Combining this with the observation that $1 \in \ker(I - P_t)$, $t \geq 0$, we deduce from [H; P, Thm. 2.8 and Lem. 2.2] that there exists a nonnegative nonzero $\widetilde{\widetilde{\varrho}} \in \ker(I - P_{t_0}^*)$ in $L^p(H,\mu)$. Then it is easy to verify that

$$\widetilde{\varrho} := \int_0^{t_0} P_s^* \widetilde{\widetilde{\varrho}}\, ds \in \ker(I - P_t^*) \quad \text{ for every } t \geq 0,$$

$\widetilde{\varrho} \geq 0$ and $\widetilde{\varrho} \not\equiv 0$ (see [H; P, Rem. 2.10]). Hence $\varrho := \widetilde{\varrho}/\|\widetilde{\varrho}\|_1 \in L^p(H,\mu)$ and ϱ satisfies (4.17). In particular the measure $d\nu = \varrho d\mu$ is invariant for (P_t), which proves (a).

(c), (d), (b). In this part of the proof we use the result [Da, Thm. 7.3] on irreducible positive semigroups. According to [Da, p. 174], a set $C \in \mathcal{B}(H)$ is called *invariant* for the operator P_t acting on $L^{p'}(H,\mu)$ if for any $f \in L^{p'}(H,\mu)$,

$$\operatorname{supp}(f) \subset C \quad \text{implies} \quad \operatorname{supp}(P_t f) \subset C$$

(with all inclusions up to sets of measure zero).

We will show that under condition (A2) for t_0, the semigroup (P_t) is *irreducible*, that is, the only sets which are invariant for all P_t, $t \geq 0$, are sets of measure zero or one. To this end suppose that $C \in \mathcal{B}(H)$, $\mu(C) > 0$ and C is an invariant set w.r.t. P_t for some $t \geq t_0$. Then $\operatorname{supp}(P_t \mathbf{1}_C) \subset C$. But, by Proposition 4.2, $P_t \mathbf{1}_C(x) > 0$ for μ-a.a. x and hence $\mu(C) = 1$.

Consequently, (P_t^*) acting on $L^p(H,\mu)$ is also irreducible. Indeed, let U be invariant for P_t^* for some $t \geq t_0$, $0 < \mu(U) < 1$. If $0 \leq f \in L^p(H,\mu)$ and $\operatorname{supp}(f) \subset U$ then for every nonnegative $g \in L^{p'}(H,\mu)$ with $\operatorname{supp}(g) \subset H - U$ we get

$$0 = \int_H (P_t^* f) \cdot g \, d\mu = \int_H f \cdot (P_t g)\, d\mu$$

and since f and g are arbitrary we conclude that $\operatorname{supp}(P_t g) \subset H - U$. This contradicts the irreducibility of (P_t).

(P_t^*) is obviously a positive semigroup (i.e. $f \geq 0$ implies $P_t^* f \geq 0$).

In [Da, Thm. 7.3] it is assumed that $(\mathcal{T}_t)$ is a semigroup of contractions but a close inspection of the proof enables us to reformulate this theorem as follows:

THEOREM [Da, Thm. 7.3]. *Let* $(\mathcal{T}_t)$ *be a positive irreducible* C_0*-semigroup on* $L^p(H,\mu)$ *for some* $p \in [1,\infty)$.

(i) *If there exists a nonnegative* $f \not\equiv 0$ *such that* $\mathcal{T}_t f = f$ *for all* $t \geq 0$, *then* $f(x) > 0$ *for* μ*-a.a.* x.

(ii) *If* $\mathcal{K} := \bigcap_{t\geq 0} \ker(I - \mathcal{T}_t)$ *is a sublattice*, *then* $\dim \mathcal{K} \leq 1$.

Therefore, for $\mathcal{T}_t = P_t^*$ the assumptions of the theorem quoted above are satisfied. By (a) we can put $f = \varrho$ in (i). Thus (c) follows.

In the proof of (d) we use an idea from [B-R-Zh, Cor. 2.13]. Since P_t^* is positive, $|P_t^* g| \le P_t^*|g|$ for $g \in B_{\mathrm{b}}(H)$, and hence

$$\|P_t^* g\|_1 \le \int_H P_t^*|g|\,d\mu = \int_H |g| P_t 1\,d\mu = \|g\|_1,$$

which implies that P_t^* can be extended to a contraction $\widetilde{P}_t^*$ on $L^1(H,\mu)$. It is standard to prove that $(\widetilde{P}_t^*)$ is strongly continuous in $L^1(H,\mu)$ since so is (P_t^*) in $L^p(H,\mu)$, $1 < p < \infty$. Consequently, $(\widetilde{P}_t^*)$ is a positive irreducible C_0-semigroup of contractions on $L^1(H,\mu)$, i.e. $\mathcal{T}_t = \widetilde{P}_t^*$ satisfies the assumptions of the theorem above. It follows easily from (4.17) that if $0 \le \varrho \in L^1(H,\mu)$ and ϱ is an invariant density for (P_t) then $\widetilde{P}_t^* \varrho = \varrho$ for all $t \ge 0$. Therefore to prove (d) it is enough to show that

$$\ker(I - \widetilde{P}_t^*) \quad \text{is a sublattice for each } t \ge 0 \tag{4.18}$$

and invoke (ii).

(4.18) follows from [Da, Thm. 7.2] or [B-R-Zh, Cor. 2.13] and we repeat here the latter simple proof for completeness. It is sufficient to show that

$$f \in \ker(I - \widetilde{P}_t^*) \quad \text{implies} \quad f^+ \in \ker(I - \widetilde{P}_t^*). \tag{4.18a}$$

Let $f = \widetilde{P}_t^* f$. Since $\widetilde{P}_t^* f \le \widetilde{P}_t^* f^+$ we have $f \le \widetilde{P}_t^* f^+$ and hence

$$f^+ = \max(f, 0) \le \widetilde{P}_t^* f^+. \tag{4.19}$$

Therefore

$$0 = \int_H f^+(P_t - I)1\,d\mu = \int_H (P_t^* f^+ - f^+)\,d\mu,$$

which combined with (4.19) yields $f^+ = \widetilde{P}_t^* f^+$ and (4.18a) follows. This finishes the proof of (d).

Finally, (b) follows from (d) and the proof of (a). ∎

Proposition 4.8. *If* (A1), (A3) *and* (F2) *are satisfied*, *then all the statements of Theorem* 4.7 *hold and moreover for each* $p \in (1,\infty)$ *there exist constants* $M_p > 0$, $\lambda_p > 0$ *such that*

$$\Big\| P_t\varphi - \int_H \varphi\varrho\,d\mu \Big\|_p \le M_p e^{-\lambda_p t}\|\varphi\|_p \tag{4.20}$$

for all $\varphi \in L^p(H,\mu)$.

Proof. If (A3) holds then by (2.26a) for each $p \in (1,\infty)$ and for a constant $\alpha_p > 0$,

$$\Big\| R_t\varphi - \int_H \varphi\,d\mu \Big\|_p \le e^{-\alpha_p t}\|\varphi\|_p, \qquad \varphi \in L^p(H,\mu).$$

Hence by [H; P, Thm. 3.6], (R_t) satisfies condition (E) of Definition 3.1 *ibid.* Then by [H; P, Prop. 4.5], (E) holds for (P_t) and (4.20) follows again by [H; P, Thm. 3.6]. ∎

The proposition below is some generalization of the result obtained in [Sh; E].

PROPOSITION 4.9. *Assume* (A1), (A3) *and* (F3). *Then*

$$\mathrm{dom}_2(L_F^*) \subset W^{1,2}_{Q_\infty}. \tag{4.21}$$

In particular $\varrho \in W^{1,2}_{Q_\infty}$, *where* ϱ *is the* (P_t)-*invariant density* (*which exists by Theorem* 4.7).

Proof. To show (4.21) we follow the proof of [Sh; E, Thm. 2.1], where a similar inclusion was proved in the case of $L = L^M$, the Malliavin generator. We write $\langle\cdot,\cdot\rangle_2$, $\|\cdot\|_2$ and $\|\cdot\|_{2\to 2}$ for the scalar product and norm in $L^2(H,\mu)$ and the norm of operators on $L^2(H,\mu)$, respectively. The norm in $W^{\alpha,2}_{Q_\infty}$ is denoted by $|\cdot|_\alpha$ and the norm of operators from $W^{\alpha,2}_{Q_\infty}$ to $W^{\beta,2}_{Q_\infty}$ is denoted by $|\cdot|_{\alpha\to\beta}$, $\alpha,\beta\in\mathbb{R}$. Recall that

$$|\varphi|_\alpha = \|(I-L^M)^{\alpha/2}\varphi\|_2, \qquad \alpha\in\mathbb{R},\ \varphi\in W^{\alpha,2}_{Q_\infty}.$$

For $\varphi\in L^2(H,\mu)$, let $I_0\varphi := \langle\varphi,1\rangle_2 1$ and $\varphi_0 := \varphi - I_0\varphi$. Hence φ_0 is the orthogonal projection of φ onto

$$L^2_0(H,\mu) := \mathcal{H}_0^\perp = \{f\in L^2(H,\mu) : \langle f,1\rangle_2 = 0\}$$

(where $\mathcal{H}_0$ is the subspace of constant functions).

Recall that both $\mathcal{H}_0$ and $\mathcal{H}_0^\perp$ are invariant for any O-U semigroup and accordingly write

$$R_t^0 := R_t|_{L^2_0(H,\mu)} \quad\text{and}\quad L_0 := L|_{L^2_0(H,\mu)}.$$

By Corollary 2.6, (R_t^0) is exponentially stable. Hence $\mathcal{J}$, defined as

$$\mathcal{J}\varphi := \int_0^\infty R_t\varphi_0\,dt, \qquad \varphi\in L^2(H,\mu), \tag{4.22}$$

is a bounded operator on $L^2(H,\mu)$ and

$$\mathcal{J}\varphi := -L_0^{-1}\varphi_0 dt, \qquad L\mathcal{J}\varphi = -\varphi_0. \tag{4.23}$$

We first show that for any $\alpha\in\mathbb{R}$,

(4.24) $\mathcal{J} : W^{\alpha,2}_{Q_\infty} \to W^{\alpha+2,2}_{Q_\infty}$ is bounded and $|\mathcal{J}|_{\alpha\to\alpha+2} \le \sigma < \infty$, with σ independent of α.

Indeed, by [Ch-G; Q], L and L_M commute and by Corollary 2.7, (A3) implies

$$\mathrm{dom}_2(L) \subset W^{2,2}_{Q_\infty}. \tag{4.25}$$

Combining this with (4.23) we obtain

$$\begin{aligned}|\mathcal{J}\varphi|_{\alpha+2} &= \|(I-L^M)^{(\alpha+2)/2}L_0^{-1}\varphi_0\|_2 = \|(I-L^M)L_0^{-1}(I-L^M)^{\alpha/2}\varphi_0\|_2\\ &\le \|(I-L^M)L_0^{-1}\|_{2\to 2}\cdot|\varphi_0|_\alpha =: \sigma|\varphi|_\alpha.\end{aligned}$$

Since by (4.25), $\sigma<\infty$, (4.24) follows.

Throughout the rest of the proof we assume that $\psi\in\mathrm{dom}_2(L_F^*)$. Then (4.24) yields

$$|\langle\mathcal{J}\varphi, L_F^*\psi\rangle| \le \sigma\|L_F^*\psi\|_2|\varphi|_{-2}, \quad \varphi\in L^2(H,\mu). \tag{4.26}$$

Since, by Theorem 3.4, $\mathrm{dom}_2(L_F) = \mathrm{dom}_2(L)$, it follows from (4.23) that $\mathcal{J}\varphi \in \mathrm{dom}_2(L_F)$ for $\varphi \in L^2(H,\mu)$. From this and the formula for L_F (Theorem 3.4) we obtain

$$\begin{aligned}\langle \mathcal{J}\varphi, L_F^*\psi\rangle_2 &= \langle L_F\mathcal{J}\varphi, \psi\rangle_2 = \langle L\mathcal{J}\varphi + G\mathcal{J}\varphi, \psi\rangle_2 \\ &= -\langle \varphi, \psi\rangle_2 + \langle I_0\varphi, \psi\rangle_2 + \langle G\mathcal{J}\varphi, \psi\rangle_2\end{aligned}$$

using (4.23) in the last equality. This and (4.13) yield

$$\begin{aligned}|\langle \varphi, \psi\rangle_2| &\le |\langle \mathcal{J}\varphi, L_F^*\psi\rangle_2| + |\langle I_0\varphi, \psi\rangle_2| + |\langle\langle Q_\infty^{-1/2}F(\cdot), \overline{D}_{Q_\infty}\mathcal{J}\varphi(\cdot)\rangle, \psi\rangle_2| \\ &=: J_1 + J_2 + J_3.\end{aligned} \tag{4.27}$$

Then by (4.26) we have

$$J_1 + J_2 \le \sigma\|L_F^*\psi\|_2 \cdot |\varphi|_{-2} + |I_0|_{-2\to 0} \cdot \|\psi\|_2 \cdot |\varphi|_{-2}; \tag{4.28}$$

$$J_3 \le \|Q_\infty^{-1/2}F\|_\infty \cdot \|\overline{D}_{Q_\infty}\mathcal{J}\varphi\|_2 \cdot \|\psi\|_2 \le \beta_0|\mathcal{J}\varphi|_1 \cdot \|\psi\|_2 \le \beta_0\sigma|\varphi|_{-1} \cdot \|\psi\|_2, \tag{4.29}$$

the last inequality being a consequence of (4.24).

Thus, for $\psi \in \mathrm{dom}(L_F^*)$ and $\varphi \in L^2(H,\mu)$ we obtain from (4.27)–(4.29) the estimate

$$|\langle \varphi, \psi\rangle_{L^2}| \le c(\psi) \cdot |\varphi|_{-1},$$

where $c(\psi)$ is a finite constant depending on ψ. This means that $\langle \psi, \cdot\rangle_2$ has an extension to a continuous functional on $W_{Q_\infty}^{-1,2}$, which implies that $\psi \in W_{Q_\infty}^{1,2}$. Hence $\mathrm{dom}(L_F^*) \subset W_{Q_\infty}^{1,2}$. ∎

5. Examples

Example 1 shows that our assumption, $F(H) \subset \mathrm{im}\, B$, is in some sense justified. In Examples 2 and 3 we consider the simplest case of system $(*)$, namely equation $(5^{*\prime})$, which satisfies (A1) and (F2). However, in Example 2 the unique invariant measure for $(5^{*\prime})$ is singular w.r.t. μ and in Example 3 there is no invariant measure for $(5^{*\prime})$. By virtue of Theorem 4.7, in both examples for no $t > 0$ does (A2) hold. Equivalently, for no $t > 0$ can R_t and P_t be hyperbounded in $L^p(H,\mu)$. An example similar to Example 2, but not so explicit, has also been given in [F; L].

Finally, in Example 4 we present a model $(*)$ (with nonconstant F) which satisfies precisely the assumptions of Theorem 4.7. That is, (A2) is satisfied for some $t_0 > 0$ but does not hold for $0 < t < t_0$. Equivalently the corresponding O-U semigroup (R_t) is hypercontractive for $t \ge t_0$ but it is not hyperbounded for $0 < t < t_0$. Such a phenomenon cannot happen when (R_t) is symmetric or H has finite dimension. Moreover, (A3) is not satisfied here. It should be mentioned that Example 4 is of some importance in Mathematical Finance ([M], [Z3]). A shortened version of Example 4 has been given in [Ch-G; N].

As an illustration we first consider the simplest semilinear equation $(*)$ with constant nonlinear term:

$$dX_t = AX_t dt + b dt + B dW_t, \qquad t > 0, \qquad \text{where } b \in H,\ Q := BB^*. \tag{5*}$$

We are mainly interested in invariant measures with densities. Let us recall the following known facts.

PROPOSITION 5.1 ([Ch, Prop. 3.5, Cor. 3.4], [D-Z; E, p. 185]). *There exists an invariant measure ν for* (5*) *iff* (A1) *holds and*

(5.1) *the deterministic equation $\dot{y} = Ay + b$ has an invariant measure ν^1.*

Then $\nu = \nu^1 * \mathcal{N}(0, Q_\infty)$.

COROLLARY 5.2 (see e.g. [Ch, Prop. 6.1]). *Let (S_t) be a stable semigroup (i.e. $\lim_{t\to\infty} S_t x = 0$ for all $x \in H$). Then* (5*) *has an invariant measure iff* (A1) *holds and*

$$\text{(5.2)} \qquad \textit{the improper integral} \quad \int_0^\infty S_t b\, dt := \lim_{T\to\infty} \int_0^T S_t b\, dt \quad \textit{exists.}$$

If ν is an invariant measure for (5*), *then*

$$\nu = \mathcal{N}(a_\infty, Q_\infty), \quad \textit{where} \quad a_\infty := \int_0^\infty S_t b\, dt.$$

(*Note that for stable (S_t),* (5.2) *implies that $b \in \operatorname{im} A$ and $a_\infty = -A^{-1}b$.*)

If (S_t) is stable and (A1), (5.2) hold then, by the Cameron–Martin Theorem (see e.g. [D-Z; S, Thm. 2.21]), $\nu = \mathcal{N}(a_\infty, Q_\infty)$ is absolutely continuous w.r.t. $\mu = \mathcal{N}(0, Q_\infty)$ iff

$$\text{(5.3)} \qquad a_\infty \in \operatorname{im} Q_\infty^{1/2} = H_0$$

and then

$$\varrho_b(x) := \frac{d\nu}{d\mu}(x) = \exp\Big(\langle Q_\infty^{-1/2} a_\infty, Q_\infty^{-1/2} x\rangle - \frac{1}{2}\|Q_\infty^{-1/2} a_\infty\|^2\Big).$$

The last equality implies that for $h \in H_0$,

$$\langle D\varrho_b(x), h\rangle = \langle Q_\infty^{-1/2} a_\infty, Q_\infty^{-1/2} h\rangle \varrho_b(x)$$

and hence $D_{Q_\infty}\varrho_b = \varrho_b Q_\infty^{-1/2} a_\infty$. Therefore $\varrho_b \in W^{1,p}_{Q_\infty}$ for $p > 1$ iff (5.3) holds. Note that under condition (5.3), $\varrho_b \in W^{n,p}_{Q_\infty}$ for all $n \geq 1$ and $p > 1$.

EXAMPLE 1. Let $A = A^*$ be a bounded operator with spectrum in $(-\infty, 0)$. Hence (S_t) is exponentially stable and A^{-1} is bounded. Suppose that Q is nuclear and

$$\text{(5.4)} \qquad QA = AQ.$$

Then the corresponding O-U semigroup is symmetric in $L^2(H, \mu)$. By Corollary 5.2, $\mathcal{N}(a_\infty, Q_\infty)$ is a unique invariant measure for $(*)$. By (5.4), $Q_\infty = \int_0^\infty S(t)QS^*(t)\, dt = \int_0^\infty S(2t)Q\, dt = -\frac{1}{2}A^{-1}Q$. Hence by (5.4),

$$Q_\infty^{1/2} = \frac{1}{\sqrt{2}} Q^{1/2}(-A)^{-1/2},$$

which yields

$$\text{(5.5)} \qquad \operatorname{im} Q_\infty^{1/2} = \operatorname{im} Q^{1/2}.$$

In particular, (A3) holds. By (5.4) we have

$$A^{-1}b = Q^{1/2}h \quad \text{iff} \quad b = AQ^{1/2}h = Q^{1/2}(Ah)$$

and since A is a bijection, we conclude from (5.5) that (5.3) holds iff $b \in \operatorname{im} Q^{1/2}$. Therefore if we look for invariant measures for $(*)$ absolutely continuous w.r.t. μ, our assumption that $F(H) \subset \operatorname{im} B = \operatorname{im} Q^{1/2}$ is justified.

EXAMPLE 2. Here $H = L^2(0,\infty)$, the operator $A = \partial/\partial\theta$ with $\operatorname{dom}(A) = H^1(0,\infty)$ generates the left shift semigroup

$$S(t)x(\theta) = x(t+\theta), \quad x \in H, \quad b(\theta) = \exp(-\theta^2/2), \quad \theta \ge 0,$$

and w is a one-dimensional Wiener process. Consider the particular case of (5*):

$$dX_t = AX_t dt + b dt + b dw_t. \tag{5*$'$}$$

Then $Q = b \otimes b$ and

$$\int_0^\infty \operatorname{tr} S_t Q S_t^* \, dt = \int_0^\infty \|S_t b\|^2 \, dt = \int_0^\infty \int_t^\infty e^{-s^2} \, ds \, dt < \infty.$$

Hence (A1) holds.

Consider (5.2). Note first that for every $\theta \ge 0$, the function $t \mapsto \int_0^t S_s b(\theta) \, ds$ is increasing in t and

$$\int_0^\infty S_s b(\theta) \, ds = \lim_{t\to\infty} \int_0^t S_s b(\theta) \, ds \quad \text{exists.}$$

Then to prove (5.2) it is enough to observe that $\int_0^\infty S_s b(\cdot) \, ds \in L^2(0,\infty)$.

Since (S_t) is stable, by Corollary 5.2, $\mathcal{N}(a_\infty, Q_\infty)$ is a unique invariant measure for (5*$'$).

Finally, suppose that (5.3) holds. By a result in [D-Z; S], $\operatorname{im} Q_\infty^{1/2} = \operatorname{im} \mathcal{L}_\infty$, where

$$\mathcal{L}_\infty : L^2(0,\infty) \to H, \quad \mathcal{L}_\infty u = \int_0^\infty S_s b u(s) \, ds.$$

(Note that by the estimate

$$\int_0^\infty \|S_s b u(s)\| \, ds \le \Big[\int_0^\infty \|S_s b\|^2 \, ds \Big]^{1/2} \cdot \Big[\int_0^\infty u^2(s) \, ds \Big]^{1/2}$$

the operator $\mathcal{L}_\infty$ is well defined.) Therefore $a_\infty \in \operatorname{im} \mathcal{L}_\infty$, which means that for some $u \in L^2(0,\infty)$,

$$\int_0^\infty S_s b \, ds = \int_0^\infty S_s b u(s) \, ds.$$

Hence

$$\int_0^\infty b(s+\theta) \, ds = \int_0^\infty b(s+\theta) u(s) \, ds$$

for a.a. θ and by continuity for all $\theta \in [0,\infty)$. Then for every $\theta \ge 0$ we have

$$0 = \int_0^\infty b(s+\theta)[1-u(s)] \, ds = e^{-\theta^2/2} \int_0^\infty e^{-\theta s}[e^{-s^2/2}(1-u(s))] \, ds. \tag{5.6}$$

Therefore the Laplace transform of the function

$$[0,\infty) \ni s \mapsto e^{-s^2/2}[1-u(s)]$$

vanishes identically, which implies that $u(s) \equiv 1$. But $u(s) \equiv 1 \notin L^2(0,\infty)$, a contradiction. Hence the measures $\mathcal{N}(a_\infty, Q_\infty)$ and $\mathcal{N}(0, Q_\infty)$ *are singular.*

EXAMPLE 3. Consider equation ($5^{*\prime}$) in Example 2, where b is now replaced by

$$\widetilde{b}(\theta) = (\theta+1)^{-3/2}, \qquad \theta \ge 0.$$

Then

$$\int_0^\infty \|S_t\widetilde{b}\|^2\,dt = \int_0^\infty \Big(\int_0^\infty (t+\theta+1)^{-3}\,d\theta\Big)\,dt = \frac{1}{2}\int_0^\infty (t+1)^{-2}\,dt = \frac{1}{2}$$

and (A1) holds. Hence the corresponding O-U process has an invariant measure.

We will show that (5.2) is not satisfied. Conversely, suppose that (5.2) holds. This means that

$$f_t := \int_0^t S_s\widetilde{b}\,ds \quad \text{converges in } L^2(0,\infty), \text{ as } t\to\infty,$$

to some f. Therefore, for some sequence (t_n) with $t_n \to \infty$, we have $f_{t_n}(\theta) \to f(\theta)$ for a.a. $\theta \in [0,\infty)$. Hence

$$f(\theta) = \int_0^\infty (s+\theta+1)^{-3/2}\,ds = 2(\theta+1)^{-1/2}$$

but $f \notin L^2(0,\infty)$, a contradiction. It follows from Corollary 5.2 that now there is no invariant measure for ($5^{*\prime}$).

EXAMPLE 4. Consider the equation

$$\begin{cases} dX_t = [AX_t + bf(X_t)]dt + bdw_t, \\ X_0 = x, \end{cases} \tag{5.7}$$

in the space $H = L^2(0,1)$, where $A = \partial/\partial\theta$ with $\operatorname{dom}(A) = \{x \in H^1(0,1) : x(1) = 0\}$ generates the semigroup (S_t) given by

$$S_t(\theta) = \begin{cases} x(t+\theta) & \text{if } t+\theta \le 1, \\ 0 & \text{if } t+\theta > 1. \end{cases}$$

Let w be a one-dimensional Wiener process, $f \in B_{\mathrm{b}}(H)$ and $b \in H$, $b \not\equiv 0$. Then $Q_\infty = Q_1$ and (A2) holds for $t \ge 1$. Hence for $t \ge 1$ the corresponding O-U semigroup (R_t) is hypercontractive and (P_t) is hyperbounded in $L^p(H,\mu)$, by Theorem 4.1. For simplicity take $b \equiv 1$. Then

$$\mathcal{L}_\infty u = \int_0^\infty S_s b u(s)\,ds = \int_0^1 (S_s\mathbf{1})u(s)\,ds, \qquad u \in L^2(0,\infty).$$

For $u_n(s) := (n+1)s^n\mathbf{1}_{[0,1]}$ we have

$$\mathcal{L}_\infty u_n(\theta) = (1-\theta)^{n+1}, \qquad \theta\in[0,1],\ n = 1,2,\ldots,$$

which implies that $\overline{\operatorname{im} Q_\infty^{1/2}} = H$. In particular, (A3) is not satisfied. By [D-Z; S], $\operatorname{im} Q_t^{1/2} = \operatorname{im} \mathcal{L}_t$, where

$$\mathcal{L}_t u = \int_0^t S_s b u(s)\,ds, \qquad u \in L^2(0,t).$$

For $0 < t < 1$ we have

$$\mathcal{L}_t u(\theta) = \int\limits_0^{t\wedge(1-\theta)} u(s)\,ds, \qquad \theta \in [0,1],$$

and consequently any function in $\operatorname{im}\mathcal{L}_t$ is constant on the interval $[0, 1-t]$. Thus for $0 < s < t < 1$ we have

$$\overline{\operatorname{im} Q_s^{1/2}} \subsetneq \overline{\operatorname{im} Q_t^{1/2}} \subsetneq H$$

and for no $t \in (0,1)$ does (A2) hold. Hence for any $0 < t < 1$, R_t and P_t are not hyperbounded in $L^p(H,\mu)$.

However, all the assumptions of Theorem 4.7 are satisfied and (5.7) has an invariant measure equivalent to $\mu = \mathcal{N}(0, Q_1)$.

6. Logarithmic Sobolev inequality—the case of bounded F

It has been proved in [Ch-G; N] that under (2.1) the O-U generator L satisfies the Logarithmic Sobolev Inequality (LSI, for short) (6.3) below iff

$$\operatorname{im} Q_\infty^{1/2} \subset \operatorname{im} Q^{1/2}. \tag{A3}$$

(Hence, by Section 2, the same is true without assumption (2.1).)

Recall that (A3) is equivalent to the following condition (see e.g. [D-Z; S, Prop. B.1]):

There exists $\alpha > 0$ such that

$$\|Q^{1/2}x\| \ge \alpha \|Q_\infty^{1/2} x\| \quad \text{ for all } x \in H. \tag{$\widetilde{6.1}$}$$

By Corollary 2.3 condition ($\widetilde{6.1}$) holds iff

$$\|Vx\| \ge \alpha\|x\| \quad \text{ for all } x \in H_0 := \operatorname{im}\, Q_\infty^{1/2}, \tag{6.1}$$

where $V = Q^{1/2} Q_\infty^{-1/2}$ with $\operatorname{dom}(V) = H_0$. Define

$$a := \sup\{\alpha > 0 : (6.1) \text{ holds}\}. \tag{6.2}$$

Then a is easily seen to be the maximum, i.e. a is the best constant in the inequality (6.1).

It follows from [Ch-G; N] and Section 2 that if (A3) holds, then for $p > 1$ and $\phi \in \operatorname{dom}_p(L)$,

$$\int\limits_H |\phi(x)|^p \log|\phi(x)|\,\mu(dx) \le \frac{p}{p-1}\cdot\frac{1}{a^2}\langle -L\phi, \phi_p\rangle + \|\phi\|_p^p \log\|\phi\|_p, \tag{6.3}$$

where the constant a is given in (6.2) and

$$\phi_p := \operatorname{sgn}\phi \cdot |\phi|^{p-1}. \tag{6.4}$$

Below we prove that the generator L_F of the semigroup (P_t) enjoys a similar property. We do not use (6.3) for L in the proof but we apply the well known LSI for the Malliavin generator L^M. We still assume (F2), now writing it in the equivalent form (see (3.13))

$$F : H \to \operatorname{im} Q^{1/2} \text{ is a Borel function and } \beta := \|Q^{-1/2}F\|_\infty < \infty. \tag{F2}$$

Note that for $\varphi \in \mathbb{F}C_{\mathrm{b}}^{\infty}$, by Theorem 3.4 and Proposition 2.5(ii),

$$L_F\varphi = L\varphi + G\varphi, \quad \text{where} \quad G\varphi(x) = \langle Q^{-1/2}F(x), D_Q\varphi(x)\rangle.$$

THEOREM 6.1. *Assume* (A1), (F2).

I. *If* (A3) *holds, then for every* $p > 1$ *and* $0 < \varepsilon < 1$,

$$\int_H |\varphi(x)|^p \log|\varphi(x)|\,\mu(dx) \le c_\varepsilon(p)\langle(\gamma_\varepsilon(p) - L_F)\varphi, \varphi_p\rangle + \|\varphi\|_p^p \log\|\varphi\|_p, \qquad \varphi \in \mathrm{dom}_p(L_F), \tag{6.5}$$

where

$$c_\varepsilon(p) = \frac{p}{(p-1)a^2} \cdot (1-\varepsilon)^{-1}, \tag{6.6}$$

$$\gamma_\varepsilon(p) = \frac{\beta^2}{2(p-1)} \cdot \varepsilon^{-1}, \tag{6.7}$$

a, β *are the constants given above and* φ_p *is defined in* (6.4).

II. *Conversely, if* (6.5) *holds for some* $p_0 > 1$ *and constants* $c(p_0) > 0$ *and* $\gamma(p_0) \ge 0$, *then* (A3) *is satisfied.*

Proof of I. We first consider $p \ge 2$.

STEP 1. First (6.5) will be proved for $\varphi \in \mathbb{F}C_{\mathrm{b}}^{\infty}$. Then $\varphi(x) = f(\Pi_m x)$, where Π_m is an orthogonal projection such that for some m, $\dim \Pi_m = m$, $\Pi_m(H) \subset \mathrm{dom}(A^*)$ and $f \in C_{\mathrm{b}}^{\infty}(H)$. It follows by Lemma 2.2(c) that

$$D_{Q_\infty}\varphi(x) = Q_\infty^{1/2}\Pi_m Df(\Pi_m x) = Q_\infty^{1/2}\Pi_m D\varphi(x) \quad \text{is in } \mathrm{dom}(A_0^*|H_0)$$

and by Lemma 2.4(iii),

$$L\varphi = D_{Q_\infty}^* A_0^* D_{Q_\infty}\varphi. \tag{6.8}$$

Observe that for $s \ge 1$ the function $g(y) = \mathrm{sgn}\, y \cdot |y|^s$, $y \in \mathbb{R}$, is differentiable and $g'(y) = s|y|^{s-1}$ (where we adopt the convention that $0^0 = 1$). Therefore, for $p \ge 2$, $\varphi_p := \mathrm{sgn}\,\varphi \cdot |\varphi|^{p-1}$ is Fréchet differentiable and from (6.8) we obtain

$$\begin{aligned}\langle -L\varphi, \varphi_p\rangle &= \langle -A_0^* D_{Q_\infty}\varphi, D_{Q_\infty}\varphi_p\rangle_{L^2(H,\mu;H)} = \langle -A_0^* D_{Q_\infty}\varphi, (p-1)|\varphi|^{p-2} D_{Q_\infty}\varphi\rangle \\ &= \int_H (p-1)|\varphi(x)|^{p-2}\langle -A_0^* D_{Q_\infty}\varphi(x), D_{Q_\infty}\varphi(x)\rangle_H\,\mu(dx).\end{aligned}$$

Since $\langle -2A_0^* x, x\rangle = \langle Vx, Vx\rangle$ for $x \in \mathrm{dom}(A_0^*|H_0)$ and $VD_{Q_\infty}\varphi(x) = D_Q\varphi(x)$ for $\varphi \in \mathbb{F}C_{\mathrm{b}}^{\infty}$ by (2.22), we get

$$\langle -L\varphi, \varphi_p\rangle = \frac{p-1}{2}\langle|\varphi(\cdot)|^{p-2}, \langle D_Q\varphi(\cdot), D_Q\varphi(\cdot)\rangle_H\rangle. \tag{6.9}$$

Similarly, for $r := 1 + p/2$,

$$\|D_Q\varphi_r\|_{L^2(H,\mu;H)}^2 = \frac{p^2}{4}\langle|\varphi(\cdot)|^{p-2}, \langle D_Q\varphi(\cdot), D_Q\varphi(\cdot)\rangle_H\rangle \tag{6.10}$$

and

$$\langle G\varphi_r, \varphi_r\rangle = \int_H \langle Q^{-1/2}F(x), D_Q\varphi_r(x)\rangle\varphi_r(x)\,\mu(dx) = \frac{p}{2}\langle G\varphi, \varphi_p\rangle. \tag{6.11}$$

It follows from (6.9) and (6.10) that

$$\langle -L\varphi, \varphi_p\rangle = \frac{2(p-1)}{p^2}\|D_Q\varphi_r\|^2.$$

From this and (6.11) we have

$$\langle -L_F\varphi, \varphi_p\rangle = \frac{2(p-1)}{p^2}\bigg(\|D_Q\varphi_r\|^2 - \frac{p}{p-1}\langle G\varphi_r, \varphi_r\rangle\bigg). \tag{6.12}$$

By (F2) we can estimate the last term as follows, for $\sigma > 0$:

$$\begin{aligned} \frac{p}{p-1}|\langle G\varphi_r, \varphi_r\rangle| &\le \frac{p\beta}{p-1}\bigg|\bigg\langle \sqrt{\sigma}\,\|D_Q\varphi_r(\cdot)\|_H, \frac{|\varphi_r(\cdot)|}{\sqrt{\sigma}}\bigg\rangle\bigg| \\ &\le \frac{p}{p-1}\cdot\frac{\beta\sigma}{2}\|D_Q\varphi_r\|^2 + \frac{p}{p-1}\cdot\frac{\beta}{2\sigma}\|\varphi_r\|^2. \end{aligned} \tag{6.13}$$

Therefore the RHS of (6.12) can be estimated from below by $c\|D_Q\varphi_r\|^2$ with some constant $c > 0$, independent of φ, if

$$0 < \sigma < \frac{2(p-1)}{p\beta} =: \frac{1}{\widetilde{\beta}}. \tag{6.14}$$

Then, assuming (6.14) and taking into account that $\|\varphi_r\|^2 = \|\varphi\|_p^p$ and $D_Q\varphi_r = VD_{Q_\infty}\varphi_r$ for $\varphi \in \mathbb{F}C_{\mathrm{b}}^\infty$, from (6.12) and (6.13) we obtain

$$\begin{aligned} &\frac{p^2}{2(p-1)}\langle -L_F\varphi, \varphi_p\rangle + \frac{\widetilde{\beta}}{\sigma}\|\varphi\|_p^p \\ &\qquad \ge (1-\widetilde{\beta}\sigma)\|VD_{Q_\infty}\varphi_r\|^2 \ge a^2(1-\widetilde{\beta}\sigma)\|D_{Q_\infty}\varphi_r\|^2, \end{aligned} \tag{6.15}$$

the last inequality being a consequence of (6.2). Since φ_r, $r = p/2+1$, is in $\mathrm{dom}(D_{Q_\infty})$, the well known LSI for the quadratic form $\|D_{Q_\infty}\psi\|^2$ ([Gr1, 2]) with $\psi = \varphi_r$ takes the form

$$\|D_{Q_\infty}\varphi_r\|^2 \ge \frac{p}{2}\Big[\int_H |\varphi|^p \log|\varphi|\,d\mu - \|\varphi\|_p^p \log\|\varphi\|_p\Big].$$

If $0 < \sigma < 1/\widetilde{\beta}$, from this and (6.15) we get (6.5) for $\varphi \in \mathbb{F}C_{\mathrm{b}}^\infty$ with the constants

$$c(\sigma, p) = \frac{p}{(p-1)a^2}[1-\sigma\widetilde{\beta}]^{-1}, \qquad \gamma(\sigma, p) = \frac{\beta}{p\sigma}.$$

Putting $\varepsilon = \sigma\widetilde{\beta}$, we obtain (6.6) and (6.7).

Step 2. Let $\varphi \in \mathrm{dom}_2(L_F)$ be bounded. By Theorem 3.4, $\mathrm{dom}_2(L_F) = \mathrm{dom}_2(L)$ and the graph norms are equivalent. Hence, by Lemma 2 from Appendix, we can approximate φ in the graph norm and μ-a.e. by a sequence $\varphi_n \in \mathbb{F}C_{\mathrm{b}}^\infty$ with $\|\varphi_n\|_\infty \le \|\varphi\|_\infty$. Therefore for any $p \ge 2$ we can pass to the limit in (6.5) for φ_n: using Fatou's lemma to the LHS and LDCT to the RHS, we conclude that φ satisfies (6.5) for each $p \ge 2$.

Step 3. Finally, fix $p \ge 2$ and consider $\mathrm{dom}_p(L_F)$. It follows from the last part of [P, Thm. 5.5, p. 123] that

$$\mathrm{dom}_p(L_F) = \{\varphi \in L^p(H,\mu) \cap \mathrm{dom}_2(L_F) : L_F\varphi \in L^p(H,\mu)\}. \tag{6.16}$$

Let $K_p := \mathrm{dom}_p(L_F) \cap L^\infty(H,\mu)$. Note that for $\varphi \in \mathbb{F}C_{\mathrm{b}}^\infty$, $L_F\varphi \in L^p(H,\mu)$ and hence $\mathbb{F}C_{\mathrm{b}}^\infty \subset K_p$, which implies the density of K_p in $L^p(H,\mu)$. Moreover, because P_t is a bounded operator from L^∞ to L^∞, $t \geq 0$, we have

$$P_t(K_p) \subset K_p, \qquad t \geq 0.$$

Therefore, by [Da, Thm. 1.9] the space K_p is dense in $\mathrm{dom}_p(L_F)$ in the graph norm. By Step 2 and (6.16), the inequality (6.5) holds for $\varphi \in K_p$ and hence, by a limiting argument, for $\varphi \in \mathrm{dom}_p(L_F)$.

Consider now the case $p \in (1, 2)$.

Step 1′. Let $\varphi \in \mathbb{F}C_{\mathrm{b}}^\infty$, $\varphi \geq \delta > 0$. Then $\varphi_s(x) = \varphi^{s-1}(x)$ ($s = p$ or $s = p/2 + 1$) is Fréchet differentiable and we can repeat Step 1 of the proof.

Step 2′. Let $\varphi \in \mathrm{dom}_2(L_F)$ be a nonnegative bounded function. By Lemma 2 in Appendix, we can find a sequence (φ_n) such that $\varphi_n \in \mathbb{F}C_{\mathrm{b}}^\infty$, $0 \leq \varphi_n \leq \|\varphi\|_\infty$ and $\varphi_n \to \varphi$ in the graph norm and μ-a.e. Let $0 < \delta_n \leq 1$ be a sequence of numbers converging to 0. Then $\varphi_n + \delta_n \in \mathbb{F}C_{\mathrm{b}}^\infty$, $0 < \delta_n \leq \varphi_n + \delta_n \leq \|\varphi\|_\infty + 1$ and since $L_F\delta_n = 0$, we have $\varphi_n + \delta_n \to \varphi$ in the graph norm. By Step 1′, (6.5) holds for $\varphi_n + \delta_n$ and a passage to the limit similar to that in Step 2 yields (6.5) for φ.

Step 3′. Recall that $K_2 = \mathrm{dom}_2(L_F) \cap L^\infty(H, \mu)$ is dense in $L^p(H, \mu)$, $1 < p < 2$, and $P_t(K_2) \subset K_2$, $t \geq 0$. Because P_t preserves positivity we conclude from Lemma 1 in Appendix that each φ, $0 \leq \varphi \in \mathrm{dom}_p(L_F)$ can be approximated in the graph norm by $0 \leq \varphi_n \in K_2$. Hence by a limiting argument we show (6.5) for every nonnegative $\varphi \in \mathrm{dom}_p(L_F)$. Finally, using again the fact that P_t preserves positivity, we can apply [Gr2, Cor. 3.10 and Thm. 3.12] to prove (6.5) for all $\varphi \in \mathrm{dom}_p(L_F)$. This completes the proof of Part I.

Proof of II. Let

$$\mathbb{F}_+C_{\mathrm{b}}^\infty := \{\varphi \in \mathbb{F}C_{\mathrm{b}}^\infty : \varphi \geq \delta_\varphi > 0 \text{ for some } \delta_\varphi \in \mathbb{R}\}. \tag{6.17}$$

It follows from (6.12) and (6.13) that for some constants $d_1 > 0$, $d_2 \geq 0$,

$$\langle -L_F\varphi, \varphi_{p_0}\rangle \leq d_1\|D_Q\varphi_{r_0}\|^2 + d_2\|\varphi_{r_0}\|^2 \quad \text{for all } \varphi \in \mathbb{F}_+C_{\mathrm{b}}^\infty \text{ (where } \varphi_{r_0} = \varphi^{p_0/2}).$$

Therefore, by assumption we obtain

$$\int_H \varphi^{p_0}\log\varphi\,d\mu \leq \widetilde{\widetilde{c}}_{p_0}(\|D_Q\varphi_{r_0}\|^2 + \widetilde{\widetilde{\gamma}}_{p_0}\|\varphi_{r_0}\|^2) + \|\varphi\|_{p_0}^{p_0}\log\|\varphi\|_{p_0} \tag{6.18}$$

for some constants $\widetilde{\widetilde{c}}_{p_0} > 0$, $\widetilde{\widetilde{\gamma}}_{p_0} \geq 0$ and all $\varphi \in \mathbb{F}_+C_{\mathrm{b}}^\infty$.

For a fixed $p \in (1, \infty)$ and an arbitrary $\psi \in \mathbb{F}_+C_{\mathrm{b}}^\infty$ we can put $\varphi := \psi^{p/p_0} \in \mathbb{F}_+C_{\mathrm{b}}^\infty$ into (6.18) and then we get the LSI of the form (6.18) with ψ instead of φ and with the index p instead of p_0. This and the equality below (6.11) imply that the O-U generator L satisfies for each $p \in (1, \infty)$ and all $\varphi \in \mathbb{F}_+C_{\mathrm{b}}^\infty$ the defective LSI of the form (6.5) with coefficients $\widetilde{c}(p) > 0$ and $\widetilde{\gamma}(p) \geq 0$ continuous in p. Arguing as in Step 2′ we deduce from Lemma 2 in Appendix that this LSI for L holds for all nonnegative $\varphi \in \mathrm{dom}_p(L)$ and hence, by [Gr2, Cor. 3.10 and Thm. 3.12], for all $\varphi \in \mathrm{dom}_p(L)$. Therefore, it follows as in the proof of [Ch-G; N, Thm. 3.2] that (6.1) holds. Hence (A3) is satisfied. ∎

Lemma 6.2. *Assume* (A1), (A3), (F2) *and let the constant a be given in* (6.2). *If $p \in (1,\infty)$ and $\theta > 0$ are such that*

$$\left(\frac{p}{p-1}\right)^2 \cdot \frac{2}{a^2} < \theta \tag{6.20}$$

then the generator L_F satisfies the LSI (6.5) *with the principal coefficient*

$$c_\theta(p) = \frac{p}{(p-1)a^2}\left[1 - \frac{p\sqrt{2}}{(p-1)a\sqrt{\theta}}\right]^{-1} \tag{6.21}$$

and the local norm

$$\gamma_\theta(p) = \frac{a^2(p-1)}{2p^2}(\kappa_\theta - 1), \tag{6.22}$$

where

$$\kappa_\theta := \int_H \exp(\theta\|Q^{-1/2}F(x)\|^2)\,\mu(dx).$$

Proof. We will now estimate the expression in (6.13) using the Hausdorff–Young (H-Y) inequality: for $s \in \mathbb{R}$, $t > 0$,

$$st \le e^s + t\log t - t.$$

Let $\varphi \in \mathbb{F}C_{\mathrm{b}}^\infty$ and write $\psi := \varphi_r$. Then

$$\langle G\psi, \psi\rangle = \int_H \langle Q^{-1/2}F(x)\psi(x), D_Q\psi(x)\rangle\,\mu(dx) = \langle b\psi, D_Q\psi\rangle, \tag{6.23}$$

where to shorten notation we have set $b(x) := Q^{-1/2}F(x)$. Then

$$\langle b\psi, D_Q\psi\rangle^2 \le \theta^{-1}\|D_Q\psi\|_2^2\|\psi\|_2^2 \int_H \theta\|b(x)\|^2 \cdot \frac{\psi^2(x)}{\|\psi\|_2^2}\,\mu(dx)$$

and applying the H-Y inequality with

$$s = \theta\|b(x)\|^2, \qquad t = \frac{\psi^2(x)}{\|\psi\|^2},$$

yields

$$\langle b\psi, D_Q\psi\rangle^2 \le \theta^{-1}\|D_Q\psi\|^2\|\psi\|^2 \cdot \left[\int_H \exp(\theta\|b(x)\|^2)d\mu + 2\int_H \frac{\psi^2(x)}{\|\psi\|^2}\log\frac{|\psi(x)|}{\|\psi\|}\,d\mu - 1\right].$$

(We omit the subscript 2 in the L^2-norm.) Hence, using the LSI (6.3) with $p = 2$ and taking into account that $\langle -2L\psi, \psi\rangle = \|D_Q\psi\|^2$ and $\kappa := \kappa_\theta$ for brevity, we obtain

$$\begin{aligned}
\langle b\psi, D_Q\psi\rangle^2 &\le \theta^{-1}\|D_Q\psi\|^2\left[(\kappa-1)\|\psi\|^2 + \frac{2}{a^2}\|D_Q\psi\|^2\right]\\
&= \frac{2}{\theta a^2}\|D_Q\psi\|^4 + \frac{\kappa-1}{\theta}\|D_Q\psi\|^2 \cdot \|\psi\|^2\\
&\le \left(\frac{\sqrt{2}}{a\sqrt{\theta}}\|D_Q\psi\|^2 + \frac{a(\kappa-1)}{2\sqrt{2\theta}}\|\psi\|^2\right)^2.
\end{aligned}$$

This and (6.23) give

$$\frac{p}{p-1}|\langle G\psi, \psi\rangle| \le \frac{p\sqrt{2}}{(p-1)a\sqrt{\theta}}\|D_Q\psi\|^2 + \frac{pa(\kappa-1)}{(p-1)2\sqrt{2\theta}}\|\psi\|^2. \tag{6.24}$$

Consequently, the RHS of (6.12) can be estimated from below by $c\|D_Q\varphi_r\|^2$ with some absolute constant $c > 0$ if

$$\sigma_\theta := \frac{p\sqrt{2}}{(p-1)a\sqrt{\theta}} < 1,$$

which is equivalent to (6.20). If (6.2) holds, then proceeding analogously to the proof of Theorem 6.1, from (6.12), (6.24) and the Gross LSI for $\|D_{Q_\infty}\|^2$ we get the inequality

$$\frac{p^2}{2(p-1)}\bigg(\langle -L_F\varphi, \varphi_p\rangle + \frac{a(\kappa-1)}{p\sqrt{2\theta}}\|\varphi\|_p^p\bigg) \geq a^2(1-\sigma_\theta)\frac{p}{2}\Big[\int_H |\varphi|^p \log|\varphi| d\mu - \|\varphi\|_p^p \log\|\varphi\|_p\Big].$$

Consequently, L_F satisfies (6.5) with the principal coefficient $c_\theta(p)$ of the form (6.21).

Finally, by (6.20),

$$\sqrt{\theta} > \frac{\sqrt{2}\,p}{a(p-1)}$$

and hence

$$\frac{a(\kappa_\theta - 1)}{p\sqrt{2\theta}} \leq \frac{a^2(p-1)}{2p^2}(\kappa_\theta - 1).$$

Therefore, (6.5) holds for $\varphi \in \mathbb{F}C_{\mathrm{b}}^\infty$ and p satisfying (6.20) with $c_\theta(p)$ given by (6.21) and $\gamma_\theta(p)$ of the form (6.22).

Then in much the same way as in Step 2 of the proof of Theorem 6.1 we deduce that for each bounded $\varphi \in \mathrm{dom}_2(L_F)$ and each p satisfying (6.20) the inequality (6.5) holds with $c(\theta, p)$ and $\gamma(\theta, p)$ given by (6.21) and (6.22). The proof is completed by repeating Steps 3 and $1'$–$3'$ of the proof of Theorem 6.1. ■

Corollary 6.3 (Auxiliary estimates). *Let* (A1), (A3), (F2) *hold and the constant a be given in* (6.2). *Assume that*

$$\theta > 2/a^2.$$

Let p satisfy (6.20) *and $p' := p/(p-1)$ denote the conjugate exponent of p. Then for $t \geq 0$,*

(a) $\|P_t\|_{p\to p} \leq \exp\left[\dfrac{a^2}{2pp'}(\kappa_\theta - 1)t\right]$,

(b) $\|P_t\|_{p\to q} \leq \exp\left[\dfrac{a^2}{8}(\kappa_\theta - 1)t\right]$,

for $p \leq q \leq \widetilde{q}(t,p)$, where

$$\widetilde{q}(t,p) := 1 + (p-1)\exp\left[a^2\bigg(1 - \frac{p'\sqrt{2}}{a\sqrt{\theta}}\bigg)t\right] \tag{6.25}$$

and κ_θ is defined in Lemma 6.2.

Proof. (a) By Lemma 6.2, the LSI (6.5) holds with $\gamma(\theta, p)$ given by (6.22). Because the function $g : [0,\infty) \to \mathbb{R}$, $g(s) = s\log s$, $s \neq 0$, $g(0) = 0$, is convex, using Jensen's inequality we have

$$\langle (\gamma(\theta,p) - L_F)\varphi, \varphi_p\rangle \geq 0 \quad \text{ for } \varphi \in \mathrm{dom}_p(L_F),$$

which implies that $(L_F - \gamma(\theta,p)I)$ generates a contraction semigroup in $L^p(H,\mu)$ (see [Gr2, Rem. 3.5]). Hence (a) follows.

(b) By an easy calculation, p satisfies (6.20) iff

$$p > \left(1 - \frac{\sqrt{2}}{a\sqrt{\theta}}\right)^{-1}. \tag{6.26}$$

Fix p satisfying (6.26) and for $q \geq p$ define

$$\widetilde{c}_p(q) = \frac{q'}{a^2}\left[1 - \frac{p'\sqrt{2}}{a\sqrt{\theta}}\right]^{-1},$$

where $q' := q/(q-1)$. Then for $q \geq p$,

$$\widetilde{c}_p(q) \geq \widetilde{c}_q(q) = c(\theta,q),$$

where $c(\theta,q)$ is given by (6.21). For $(1-\sqrt{2}/(a\sqrt{\theta}))^{-1} < q < p$ define $\widetilde{c}_p(q) = c(\theta,q)$.

Since $1/(qq') \leq 1/4$, from (6.22) we have

$$\gamma(\theta,q) \leq \frac{a^2}{8}(\kappa_\theta - 1) =: \gamma.$$

Therefore, by Lemma 6.2, for q satisfying (6.26) the LSI (6.5) holds with the principal coefficient $\widetilde{c}_p(q)$ and the local norm $\widetilde{\gamma}(q) \equiv \gamma$. Moreover, $\widetilde{c}_p(\cdot)$ is continuous in $q \in ((1-\sqrt{2}/(a\sqrt{\theta}))^{-1}, \infty)$. Consequently, all the assumptions of [Gr2, Thm. 3.7] are satisfied and one can consider the initial value problem

$$\widetilde{c}_p(q)\frac{dq(t)}{dt} = q(t), \qquad q(0) = p, \qquad t \geq 0. \tag{6.27}$$

Observe that $q(\cdot)$ in (6.27) is an increasing function and hence, using the notation

$$\alpha := a^2\left[1 - \frac{p'\sqrt{2}}{a\sqrt{\theta}}\right],$$

we can write (6.27) in the explicit form

$$\frac{dq}{q-1} = \alpha dt, \qquad q(0) = p, \qquad t \geq 0.$$

Therefore, the solution $q(t,p)$ to (6.27) is given by the formula

$$q(t,p) = (p-1)\exp(\alpha t) + 1 = \widetilde{q}(t,p), \quad t \geq 0,$$

where $\widetilde{q}(t,p)$ is defined by (6.25).

From [Gr2, Thm. 3.7] we conclude that P_t is a bounded operator from $L^p(H,\mu)$ to $L^{\widetilde{q}(t,p)}(H,\mu)$ and $\|P_t\|_{p\to\widetilde{q}(t,p)} \leq \exp M(t,p)$, where

$$M(t,p) = \int_0^t \widetilde{\gamma}(q(s,p))\,ds = t\cdot\gamma = t\cdot\frac{a^2}{8}(\kappa_\theta - 1).$$

Thus (b) follows. ■

7. The semigroup (P_t)—the case of general F

In this section we assume (A1) and (A3). Let a be the constant corresponding to (A3) via (6.2). The nonlinear term F in equation $(*)$ is required to satisfy the following condition (F1a) which is a bit stronger than (F1):

(F1a) $F : H \to \operatorname{im} B$ is a Borel function and

$$\kappa := \int_H \exp(\delta \|B^{-1}F(x)\|^2)\, \mu(dx) < \infty \quad \text{for some } \delta > 2/a^2$$

(where B^{-1} denotes the pseudoinverse of B).

Recall that for $\varphi \in B_{\mathrm{b}}(H)$,

$$P_t\varphi(x) = E(\varphi(Z_t^x)U_t^x) \quad \text{for } \mu\text{-a.a. } x \text{ and all } t \geq 0,$$

where U_t^x is the Girsanov martingale corresponding to F (see (1.9), (1.4)). It is shown in Theorem 7.1 below that under the assumption (F1a), (P_t) is a C_0-semigroup in $L^p(H,\mu)$ for sufficiently large p and (P_t) is hyperbounded. In the proof we approximate F by a suitable sequence (F_n) of functions satisfying (F2) and then we use the auxiliary estimates from Corollary 6.3.

Recall that (see (3.11)) for $\varphi \in \operatorname{dom}(L_F^0) = \mathbb{F}C_{\mathrm{b}}^\infty$,

$$L_F^0\varphi(x) = L\varphi(x) + G_0\varphi(x) = L\varphi(x) + \langle F(x), D\varphi(x)\rangle.$$

(Note that by (F1), $G_0\varphi \in L^q(H,\mu)$ for all $q \in (1,\infty)$.)

THEOREM 7.1. *Assume* (A1), (A3), (F1a) *and let a and (δ,κ) be the constants corresponding to* (6.2) *and* (F1a) *respectively. Then for each $p \in (1,\infty)$ such that*

$$p' := \frac{p}{p-1} < \frac{a\sqrt{\delta}}{\sqrt{2}}, \tag{7.1}$$

we have:

(a) *(P_t) is a C_0-semigroup on $L^p(H,\mu)$ and its generator L_F is an extension of L_F^0. Moreover*,

$$\|P_t\|_{p\to p} \leq \exp\left[\frac{a^2}{2pp'}(\kappa-1)t\right], \quad t \geq 0. \tag{7.2}$$

(b) *For each $t > 0$, P_t is a bounded operator from $L^p(H,\mu)$ to $L^q(H,\mu)$ for $p \leq q \leq q_\delta(t,p)$, where*

$$q_\delta(t,p) = 1 + (p-1)\exp\left[a^2\left(1 - \frac{p'\sqrt{2}}{a\sqrt{\delta}}\right)t\right], \tag{7.3}$$

and in this case

$$\|P_t\|_{p\to q} \leq \exp\left[\frac{a^2}{8}(\kappa-1)t\right].$$

Proof. Clearly, (F1) implies that

$$\int_H \|B^{-1}F(x)\|^2\, \mu(dx) < \infty. \tag{7.4}$$

For $\widehat{F} := B^{-1}F$ define

$$\widehat{F}_n(x) = \begin{cases} \widehat{F}(x) & \text{if } \|\widehat{F}(x)\| \le n, \\ 0 & \text{otherwise,} \end{cases} \qquad F_n(x) = B\widehat{F}_n(x).$$

Then

$$\|B^{-1}F_n(x)\| \le \|\widehat{F}_n(x)\| \le n$$

and in particular F_n satisfies (F2) for each n.

By definition

$$\widehat{F}_n(x) \to \widehat{F}(x) \quad \text{for } \mu\text{-a.a. } x \quad \text{and} \quad \|\widehat{F}_n(x)\| \le \|\widehat{F}(x)\|.$$

Hence by (7.4) and LDCT, the condition (1.10) of Lemma 1.6 is satisfied.

Let $(U^x_{n,t})$ and (U^x_t), $t \ge 0$, be the Girsanov martingales corresponding to F_n and F, respectively (see (1.4) and Lemma 1.6). Let

$$P^n_t\varphi(x) = E(\varphi(Z^x_t)U^x_{n,t}), \qquad \varphi \in B_{\mathrm{b}}(H),$$

be the transition semigroup for equation $(*)$ with nonlinear term F_n. In virtue of Lemma 1.6, for any $T > 0$ one can choose a subsequence (n_m) such that for μ-a.a. x,

$$E|U^x_{n_m,T} - U^x_T| \to 0 \quad \text{as } m \to \infty. \tag{7.5}$$

From the estimate (3.8) (in the proof of Proposition 3.2) and (7.5) we deduce that for μ-a.a. x and for every $\varphi \in B_{\mathrm{b}}(H)$,

$$\sup_{0 \le t \le T} |P^{n_m}_t\varphi(x) - P_t\varphi(x)| \to 0 \quad \text{as } m \to \infty. \tag{7.6}$$

Since for each n, F_n satisfies the condition (F2) and the remaining assumptions of Corollary 6.3 hold for p satisfying (7.1) and $\theta = \delta$ (δ given in (F1a)) we conclude from Corollary 6.3(a) that

$$\|P^n_t\|_{p\to p} \le \exp\left[\frac{a^2}{2pp'}(\kappa^n_\delta - 1)t\right],$$

where

$$\kappa^n_\delta = \int_H \exp(\delta\|Q^{-1/2}F_n(x)\|^2\,\mu(dx) = \int_H \exp(\delta\|B^{-1}F_n(x)\|^2)\,\mu(dx) \le \kappa,$$

with κ given in (F1a). (The second equality holds because $\operatorname{im} B = \operatorname{im} Q^{1/2}$ and $\|Q^{-1/2}y\| = \|B^{-1}y\|$ for $y \in \operatorname{im} B$.)

Therefore

$$\|P^n_t\|_{p\to p} \le \exp\left[\frac{a^2}{2pp'}(\kappa - 1)t\right] =: k(p,t). \tag{7.7}$$

In much the same way we deduce from Corollary 6.3(b) that

$$\|P^n_t\|_{p\to q} \le \exp\left[\frac{a^2}{8}(\kappa - 1)t\right] \tag{7.8}$$

for $p \le q \le q_\delta(t,p)$, where $q_\delta(t,p)$ is given by (7.3). Then (7.6), (7.7) and the Fatou lemma yield for $\varphi \in B_{\mathrm{b}}(H)$,

$$\int_H |P_t\varphi(x)|^p\,\mu(dx) \le \liminf_{m\to\infty} \int_H |P^{n_m}_t\varphi(x)|^p\,\mu(dx) \le (k(p,t))^p\|\varphi\|^p_p,$$

which means that P_t extends to a bounded operator on $L^p(H,\mu)$ and (7.2) holds. Similarly, from (7.6), (7.8) and the Fatou lemma we obtain our assertion (b).

To prove the remaining claims of (a) first note that (7.7) and (7.2) imply that for any $T>0$,

$$\sup_n \sup_{0\le t\le T} \|P_t^n\|_{p\to p} \le k(p,T), \qquad \sup_{0\le t\le T} \|P_t\|_{p\to p} \le k(p,T). \tag{7.9}$$

Then from (7.6), (7.9) and LDCT we obtain, first for $\varphi \in B_{\mathrm{b}}(H)$ and next for $\varphi \in L^p(H,\mu)$,

$$\sup_{0\le t\le T} \|P_t^{n_m}\varphi - P_t\varphi\|_p \underset{m\to\infty}{\longrightarrow} 0. \tag{7.10}$$

(Compare the proof of (3.6) in Proposition 3.2.)

Since by Corollary 3.3 for each n and $\varphi \in L^p(H,\mu)$,

$$P_t^n(P_s^n\varphi) = P_{t+s}^n\varphi, \quad s\ge 0,\ t\ge 0, \quad \text{and} \quad \lim_{t\to 0^+} P_t^n\varphi = \varphi,$$

we deduce easily from (7.9) and (7.10) that (P_t) has the same properties.

It remains to prove that

$$L_F^0 \subset L_F \quad \text{in } L^p(H,\mu). \tag{7.11}$$

Recall that $\mathbb{F}C_{\mathrm{b}}^\infty = \mathrm{dom}(L_F^0) = \mathrm{dom}(L_{F_n}^0)$, $\mathbb{F}C_{\mathrm{b}}^\infty$ as in (2.1a), and by Theorem 3.4 and (6.16) for each $q\in(1,\infty)$ we have $L_{F_n}^0 \subset L_{F_n}$ in $L^q(H,\mu)$. Hence for $\varphi \in \mathbb{F}C_{\mathrm{b}}^\infty$,

$$\begin{aligned}\|L_{F_n}\varphi - L_F^0\varphi\|_p &= \|L_{F_n}^0\varphi - L_F^0\varphi\|_p = \|\langle B^{-1}(F(\cdot)-F_n(\cdot)), BD\varphi(\cdot)\rangle_H\|_p \\ &\le \|B\|\cdot\|D\varphi\|_\infty \cdot \|B^{-1}(F_n-F)\|_p \to 0 \quad \text{as } n\to\infty.\end{aligned} \tag{7.12}$$

Note also that the semigroup property implies that for the subsequence (n_m) in (7.10) we have for all $t\ge 0$ and $\varphi\in L^p(H,\mu)$,

$$\|P_t^{n_m}\varphi - P_t\varphi\|_p \to 0 \quad \text{as } m\to\infty. \tag{7.13}$$

Taking into account (7.7), (7.12) and (7.13), we will have shown (7.11) if we prove the following simple lemma.

LEMMA 7.2. *Let $(\mathcal{T}_t)$, $(\mathcal{T}_t^n)$ be C_0-semigroups on a Banach space E, with generators $\mathcal{A}$, $\mathcal{A}_n$, respectively, $n=1,2,\ldots$ Assume that for some constants $M\ge 1$ and λ_0,*

$$\|\mathcal{T}_t^n\| \le M\exp(\lambda_0 t), \qquad n=1,2,\ldots,$$

and

$$\mathcal{T}_t^n\varphi \underset{n\to\infty}{\longrightarrow} \mathcal{T}_t\varphi \quad \textit{for all } t>0,\ \varphi\in E.$$

If a linear operator $\mathcal{B}$ with domain $\mathrm{dom}(\mathcal{B}) =: \mathcal{D}$ has the properties

$$\mathcal{D}\subset \mathrm{dom}(\mathcal{A}_n) \quad \textit{for every } n, \qquad \lim_n \mathcal{A}_n\varphi = \mathcal{B}\varphi \quad \textit{for } \varphi\in\mathcal{D},$$

then $\mathcal{A}\supset\mathcal{B}$.

Proof. Fix a real $\lambda>\lambda_0$. It follows easily that the resolvent operators satisfy $\mathcal{R}(\lambda,\mathcal{A}_n)\psi \to \mathcal{R}(\lambda,\mathcal{A})\psi$ for $\psi\in E$ and $\|\mathcal{R}(\lambda,\mathcal{A}_n)\|$ are bounded uniformly in n. Therefore for $\varphi\in\mathcal{D}$ we have

$$\lim_n \mathcal{R}(\lambda,\mathcal{A}_n)(\lambda I-\mathcal{B})\varphi = \mathcal{R}(\lambda,\mathcal{A})(\lambda\varphi-\mathcal{B}\varphi)$$

and on the other hand

$$\mathcal{R}(\lambda,\mathcal{A}_n)(\lambda\varphi-\mathcal{B}\varphi)=\varphi+\mathcal{R}(\lambda,\mathcal{A}_n)(\mathcal{A}_n\varphi-\mathcal{B}\varphi)\to\varphi \quad \text{as } n\to\infty.$$

Hence $\varphi\in\operatorname{dom}(\mathcal{A})$ and $\mathcal{A}\varphi=\mathcal{B}\varphi$. ∎

COROLLARY 7.3. *If the assumptions of Theorem* 7.1 *are satisfied, then*

(a) *the semigroup* (P_t) *has an invariant measure* ν *absolutely continuous w.r.t.* μ;
(b) $\varrho:=d\nu/d\mu\in L^{p'}(H,\mu)$ *for all* $p'<a\sqrt{\delta}/\sqrt{2}$;
(c) $\varrho(x)>0$ *for* μ-*a.a.* x;
(d) *for each* p *satisfying* (7.1) *there exist constants* $M_p,\lambda_p>0$ *such that*

$$\Big\|P_t\varphi-\int_H\varphi\varrho\,d\mu\Big\|_p\le M_pe^{-\lambda_pt}\|\varphi\|_p$$

for all $\varphi\in L^p(H,\mu)$, $t>0$;

(e) ν *is a unique* (P_t)-*invariant probability measure which is absolutely continuous w.r.t.* μ.

Proof. (a)–(c), (e) are proved in much the same way as Theorem 4.7 and (d) follows as Proposition 4.8. ∎

THEOREM 7.4. *Assume* (A1), (A3) *and let* $a=\sup\{\alpha>0:(6.1)\text{ holds}\}$. *If*

(F1b) $F:H\to\operatorname{im}Q_\infty^{1/2}$ *is a Borel function and*

$$\widetilde{\kappa}:=\int_H\exp(\delta\|Q_\infty^{-1/2}F(x)\|^2)\,\mu(dx)<\infty \quad \textit{for some } \delta>8/a^4,$$

then

(a) *for every* $p\ge2$, (P_t) *is a* C_0-*semigroup in* $L^p(H,\mu)$ *and its generator* $L_F\supset L_F^0$;
(b) $\operatorname{dom}_2(L_F)=\operatorname{dom}_2(L)$;
(c) $\operatorname{dom}_2(L_F)$ *is continuously embedded into* $W^{2,2}_{Q_\infty}$ *and into the Orlicz space* $L^2\log^rL$ *for* $0\le r<2$;
(d) *for* $p\ge2$, *the generator* L_F *satisfies the* LSI (6.5) *with the principal coefficient* $c(\overline{\delta},p)$ *and the local norm* $\gamma(\overline{\delta},p)$ *as in* (6.21), (6.22) (*respectively*) *with* $\theta=\overline{\delta}=a^2\delta$.

Proof. (a) We will show that (F1a) is satisfied. Note that by (A3) the operator $Q^{-1/2}Q_\infty^{1/2}$ is bounded on H. If $y\in H_Q:=\operatorname{im}Q^{1/2}$, then $z:=Q_\infty^{1/2}Q^{-1/2}y\in H_0$ and by (6.1), $\|Vz\|\ge a\|z\|$, i.e.

$$\frac{1}{a}\|y\|\ge\|Q_\infty^{1/2}Q^{-1/2}y\|.$$

Recall that the image $\operatorname{im}U^{-1}$ of the pseudoinverse of an operator U is orthogonal to $\ker U$, so in particular $\operatorname{im}Q^{-1/2}\subset\overline{H}_Q$ and hence for $x\in H$,

$$\begin{aligned}\sup_{\|y\|\le1}|\langle Q^{-1/2}Q_\infty^{1/2}x,y\rangle|&=\sup_{\|y\|\le1,\,y\in\overline{H}_Q}|\langle Q^{-1/2}Q_\infty^{1/2}x,y\rangle|\\&=\sup_{\|y\|\le1,\,y\in H_Q}|\langle x,Q_\infty^{1/2}Q^{-1/2}y\rangle|\le\frac{1}{a}\|x\|.\end{aligned}$$

Consequently, $\|Q^{-1/2}Q_\infty^{1/2}\|\le1/a$, and hence by (F1b) we have

$$\|Q^{-1/2}F(x)\|^2=\|Q^{-1/2}Q_\infty^{1/2}Q_\infty^{-1/2}F(x)\|^2\le\frac{1}{a^2}\|Q_\infty^{-1/2}F(x)\|^2,$$

which implies that (F1a) holds for $\overline{\delta} = a^2\delta$. Since $\overline{\delta} > 8/a^2$, all the statements of Theorem 7.1 hold for $p \ge 2$ and with $\kappa = \kappa(\overline{\delta})$ as in (F1a). In particular (a) follows.

(b) To prove that $L_F^0 = L + G_0$ with $\mathrm{dom}(L_F^0) = \mathbb{F}C_{\mathrm{b}}^\infty$ has an extension $\widetilde{L}_F$ with $\mathrm{dom}(\widetilde{L}_F) = \mathrm{dom}_2(L)$, which generates a C_0-semigroup on $L^2(H,\mu)$, we proceed similarly to [Sh; N], where perturbations of the Malliavin generator L^M were considered.

We first show that for some constants $0 < \alpha < 1$ and $\sigma > 0$,

$$\|G_0\varphi\|_2 \le \alpha\|L\varphi\|_2 + \sigma\|\varphi\|_2, \quad \varphi \in \mathbb{F}C_{\mathrm{b}}^\infty. \tag{7.14}$$

Let $\widetilde{b}(x) = Q_\infty^{-1/2}F(x)$. Then by the H-Y inequality (see above (6.23)) we have, for $\varphi \in \mathbb{F}C_{\mathrm{b}}^\infty$ such that $\|D_{Q_\infty}\varphi\|_2 \ne 0$,

$$\begin{aligned}\langle\widetilde{b}(x), D_{Q_\infty}\varphi(x)\rangle^2 &\le \delta\|\widetilde{b}(x)\|^2 \cdot \delta^{-1}\|D_{Q_\infty}\varphi(x)\|^2 \\ &\le \delta^{-1}\|D_{Q_\infty}\varphi\|_2^2\bigg(e^{\delta\|\widetilde{b}(x)\|^2} + 2\frac{\|D_{Q_\infty}\varphi(x)\|^2}{\|D_{Q_\infty}\varphi\|_2^2}\log\frac{\|D_{Q_\infty}\varphi(x)\|}{\|D_{Q_\infty}\varphi\|_2} - \frac{\|D_{Q_\infty}\varphi(x)\|^2}{\|D_{Q_\infty}\varphi\|_2^2}\bigg).\end{aligned}$$

This and the LSI for the quadratic form $\|D_{Q_\infty}\widetilde{\psi}\|^2$ of the H-valued function $\widetilde{\psi} = \psi/\|\psi\|$, where $\psi = D_{Q_\infty}\varphi$ ([Sh; C, (2.12)]), yield

$$\|G_0\varphi\|_2^2 \le \delta^{-1}\|D_{Q_\infty}\varphi\|_2^2\bigg(\widetilde{\kappa} + 2\frac{\|D_{Q_\infty}^2\varphi\|_2^2}{\|D_{Q_\infty}\varphi\|_2^2} - 1\bigg). \tag{7.15}$$

By Corollary 2.7 we have the estimate

$$\|D_{Q_\infty}^2\varphi\|_2 \le \frac{2}{a^2}\|L\varphi\|_2,$$

which together with (7.15) implies

$$\|G_0\varphi\|_2^2 \le \frac{8}{\delta a^4}\|L\varphi\|_2^2 + \delta^{-1}(\widetilde{\kappa} - 1)\|D_{Q_\infty}\varphi\|_2^2$$

and hence

$$\|G_0\varphi\|_2 \le \frac{2\sqrt{2}}{a^2\sqrt{\delta}}\|L\varphi\|_2 + \frac{\sqrt{\widetilde{\kappa} - 1}}{\sqrt{\delta}}\|D_{Q_\infty}\varphi\|_2.$$

Since $\delta > 8/a^4$, for some $\varepsilon > 0$ we have

$$\frac{2\sqrt{2}}{a^2\sqrt{\delta}} = 1 - 2\varepsilon.$$

It follows from (4.14), (4.15) that

$$\int_0^1 \|D_{Q_\infty}R_t\varphi\|_2\,dt < \infty$$

and hence by [Da, Lem. 3.4, p. 70], D_{Q_∞} has L-bound zero. Therefore, for σ_ε sufficiently large, (7.14) holds with $\alpha = 1 - \varepsilon$. Because $\mathbb{F}C_{\mathrm{b}}^\infty$ is dense in $\mathrm{dom}_2(L)$ in the graph norm, G_0 can be uniquely extended to an operator G defined on $\mathrm{dom}_2(L)$ and satisfying (7.14) for $\varphi \in \mathrm{dom}_2(L)$.

For $\overline{\delta} = a^2\delta$ and $\kappa(\overline{\delta})$ as in (F1a) let

$$\overline{\gamma} := \frac{a^2}{8}[\kappa(\overline{\delta}) - 1] \quad \text{and} \quad L^{(\overline{\gamma})} := L - \overline{\gamma}I.$$

It follows from the proof of (a) and (7.2) that for each $u \in [0,1]$, the operator $L_{uF} - \overline{\gamma} I$ generates a C_0-semigroup of contractions on $L^2(H,\mu)$ and $L_{uF} \supset L^0_{uF}$. In particular for $\varphi \in \mathbb{F}C^\infty_{\mathrm{b}}$,

$$0 \geq \langle (L^0_{uF} - \overline{\gamma} I)\varphi, \varphi\rangle = \langle L^{(\overline{\gamma})}\varphi, \varphi\rangle + u\langle G_0\varphi, \varphi\rangle. \tag{7.16}$$

Because $\mathbb{F}C^\infty_{\mathrm{b}}$ is a core in L^2 for L (and hence for $L^{(\overline{\gamma})}$) and G_0 satisfies (7.14), inequality (7.16) can be extended to $\mathrm{dom}_2(L)$, i.e.

$$L^{(\overline{\gamma})} + uG \text{ with domain } \mathrm{dom}_2(L) \text{ is dissipative in } L^2(H,\mu),\ u \in [0,1]. \tag{7.17}$$

Clearly,

$$L^{(\overline{\gamma})} \text{ with domain } \mathrm{dom}_2(L) \text{ is } m\text{-dissipative in } L^2(H,\mu). \tag{7.18}$$

Then using [P, Thm. 3.2, p. 81] we conclude from (7.14), (7.17), (7.18) that $L^{(\overline{\gamma})} + G$ defined on $\mathrm{dom}_2(L)$ is m-dissipative, and hence $L+G$, with domain $\mathrm{dom}_2(L)$, generates a C_0-semigroup on $L^2(H,\mu)$. Since for $\varphi \in \mathbb{F}C^\infty_{\mathrm{b}}$,

$$(L+G)\varphi = L^0_F\varphi = L_F\varphi$$

and $\mathbb{F}C^\infty_{\mathrm{b}}$ is a core for $L+G$, as a consequence of Lemma 3.5 we find that $L+G = L_F$, which implies (b).

(c) follows from (b) and the results of [Ch-G; N] (compare Corollary 4.3(iii)).

(d) It is proved in (a) that (F1a) holds with $\overline{\delta} = a^2\delta > 8/a^2$. Let (F_n) be the sequence approximating F defined in the proof of Theorem 7.1.

Applying Lemma 6.2 with $\theta = \overline{\delta}$ to the generators L_{F_n}, we see that for each $p \geq 2$ and $n = 1,2,\ldots$, L_{F_n} satisfies the LSI (6.5) with $c(\overline{\delta},p)$ as in (6.21) and $\gamma(\overline{\delta},p)$ defined as in (6.22), where

$$\kappa(\overline{\delta}) = \int_H \exp(\overline{\delta}\|Q^{-1/2}F(x)\|^2)\,\mu(dx).$$

Therefore $c(\overline{\delta},p)$ and $\gamma(\overline{\delta},p)$ are independent of n. For a fixed $\varphi \in \mathbb{F}C^\infty_{\mathrm{b}}$ and $p \geq 2$, letting $n \to \infty$, we see from (7.12) that L^0_F satisfies (6.5) for all $p \geq 2$ with the above coefficients. Since $L^0_F \subset L_F$ and $\mathbb{F}C^\infty_{\mathrm{b}}$ is a core for L_F in $L^2(H,\mu)$, the proof of (d) is completed by repeating Steps 2 and 3 of the proof of Theorem 6.1. ∎

Corollary 7.5 (The case of symmetric O-U). *Assume* (A1), (2.1), (A3) *and let* $R_t = R_t^*$ *in* $L^2(H,\mu)$. *If* (F1a) *holds with* $\delta > 8/a^2$, *then*

(i) $\mathrm{dom}_2(L_F) = \mathrm{dom}_2(L) = W^{2,2}_Q \cap W^{1,2}_{-AQ}$;

(ii) *statement* (d) *of Theorem* 7.4 *holds.*

Proof. (i) The first equality follows by the same method as in the proof of Theorem 7.4(b) with D_{Q_∞} replaced by D_Q. Now, the above-mentioned LSI for H-valued functions [Sh; C, (2.12)] is a counterpart of (6.3) with $L = -\frac{1}{2}D_Q^* D_Q$, which gives the constant factor $2/a^2$ instead of 2 in the middle term in brackets in (7.15). By [Ch-G; N] we have $\|D^2_Q\varphi\|_2 \leq 2\|L\varphi\|_2$. Therefore G_0 now has L-bound equal to $2\sqrt{2}/(a\sqrt{\delta}) < 1$ and the rest of the proof runs as before.

The second equality of (i) follows from [D-G,1] or [Ch-G; N]. Note that (ii) has actually been shown in the proof of Theorem 7.4(d). ∎

Appendix

Proof of (2.2). By a result in [D-Z; S], $H_0 = \operatorname{im} \mathcal{L}_\infty$, where

$$\mathcal{L}_\infty : L^2((0,\infty); H) \to H, \qquad \mathcal{L}_\infty u = \int_0^\infty S_s Q^{1/2} u(s)\, ds$$

($\mathcal{L}_\infty$ is well defined by (A1)).

For any $u \in L^2((0,\infty); H)$ we have

$$S_t \mathcal{L}_\infty u = \int_0^\infty S_{t+s} Q^{1/2} u(s)\, ds = \mathcal{L}_\infty \widetilde{u}_t, \quad \text{where} \quad \widetilde{u}_t(s) = \begin{cases} 0 & \text{if } 0 \le s \le t, \\ u(s-t) & \text{if } s > t, \end{cases}$$

and hence $S_t(H_0) \subset H_0$. ■

LEMMA 1. *Let $p \ge 1$ be fixed.*

(a) *Let T_t be a positivity preserving strongly continuous semigroup on $L^p(H,\mu)$ with generator $\mathcal{A}$. Let $\mathcal{G}$ be a linear subspace of $L^p(H,\mu)$ such that $\mathcal{G} \subset \operatorname{dom}(\mathcal{A})$, $T_t\mathcal{G} \subset \mathcal{G}$ and $\mathcal{G}_+$ is dense in $L^p_+(H,\mu)$ (where the subscript $+$ means the cone of nonnegative functions in the suitable space). Then $\mathcal{G}_+$ is dense in $\operatorname{dom}_+(\mathcal{A})$ in the graph norm.*

(b) *Moreover, if T_t restricted to $L^\infty(H,\mu)$ is a contraction semigroup on L^∞ and any $\varphi \in L^\infty_+(H,\mu)$ can be approximated in L^p-norm by $(\psi_n)_{n=1}^\infty$ such that*

$$\psi_n \in \mathcal{G}_+, \qquad \|\psi_n\|_\infty \le \|\varphi\|_\infty, \qquad n = 1,2,\ldots, \tag{1}$$

then any $\varphi \in \operatorname{dom}_+(\mathcal{A})$ can be approximated in the graph norm by a sequence $(\varphi_n)_{n=1}$ satisfying (1).

Proof. The lemma is a modification of [Da; Thm. 1.9] and we adapt the proof given there.

To prove (a) let $\varphi \in \operatorname{dom}_+(\mathcal{A})$ and define

$$J_n\varphi = 2^n \int_0^{2^{-n}} T_s\varphi\, ds, \qquad n = 1,2,\ldots$$

Then $J_n\varphi \in \operatorname{dom}_+(\mathcal{A})$. Let $|||\cdot|||$ denote the graph norm in $\operatorname{dom}(\mathcal{A})$. From the strong continuity of T_s it follows that $s \mapsto T_s\varphi$ is continuous in $|||\ |||$ norm. Hence

$$|||J_n\varphi - \varphi||| \to 0 \quad \text{as } n \to \infty. \tag{2}$$

From semigroup properties we have for $0 \le t \le 1$,

$$\begin{aligned} \Big|\Big|\Big| \int_0^t T_s\varphi\, ds \Big|\Big|\Big|^2 &= \Big\| \mathcal{A}\int_0^t T_s\varphi\, ds \Big\|^2 + \Big\| \int_0^t T_s\varphi\, ds \Big\|^2 \\ &\le \|T_t\varphi - \varphi\|^2 + \int_0^1 \|T_s\varphi\|^2\, ds \le 2(M+1)^2\|\varphi\|, \end{aligned} \tag{3}$$

where $M = \sup_{0\le t\le 1} \|T_t\|$.

Let now (ψ_m) be a sequence such that

$$\|\psi_m - \varphi\| \to 0 \quad \text{as } m \to \infty, \qquad \psi_m \in \mathcal{G}_+. \tag{4}$$

The estimate (3) implies that for any fixed n,

$$\|J_n\psi_m - J_n\varphi\| \to 0 \quad \text{as } m \to \infty. \tag{5}$$

Finally, for $\psi \in \operatorname{dom}(\mathcal{A})$ and any fixed n, it follows again from the strong continuity of $s \to T_s\psi$ in $\|\cdot\|$ norm that $J_n\psi$ is the limit in $\|\cdot\|$ norm of the Stieltjes sums:

$$\|J_n\psi - \mathcal{S}_k^n\psi\| \to 0 \quad \text{as } k \to \infty, \quad \text{where} \quad \mathcal{S}_k^n\psi = 2^{n-k}\sum_{j=1}^{2^{k-n}} T_{j/2^k}\psi. \tag{6}$$

Note that if $\psi \in \mathcal{G}_+$, then $\mathcal{S}_k^n\psi \in \mathcal{G}_+$. From (2), (5), and (6) we conclude that φ is the limit in $\|\cdot\|$ norm of some subsequence of $(\mathcal{S}_k^n\psi_m)$ and hence part (a) follows.

To prove (b) note that if the functions ψ_m in (4) satisfy additionally the estimate

$$\|\psi_m\|_\infty \le \|\varphi\|_\infty, \quad m = 1, 2, \ldots,$$

then

$$\|\mathcal{S}_k^n\psi_m\|_\infty \le \|\varphi\|_\infty \quad \text{for all } n, m, k$$

(because T_t is a contraction on L^∞).

Lemma 2. *Let $p \ge 1$. Recall that*

$$\mathbb{F}C_{\mathrm{b}}^\infty := \{\varphi : H \to \mathbb{R} : \varphi(x) = f(\langle x, h_1\rangle, \ldots, \langle x, h_m\rangle) \text{ for some } m \in \mathbb{N} \\ \text{and } h_1, \ldots, h_m \in \operatorname{dom}(A^*),\ f \in C_{\mathrm{b}}^\infty(\mathbb{R}^m)\}.$$

Then for any bounded $\varphi \in \operatorname{dom}_p(L)$ there exists $(\varphi_n)_{n=1}^\infty$ such that

$$\varphi_n \in \mathbb{F}C_{\mathrm{b}}^\infty, \quad \|\varphi_n\|_\infty \le \|\varphi\|_\infty \text{ for all } n, \quad \text{and} \quad \|\varphi_n - \varphi\| \to 0,$$

where $\|\cdot\|$ means the graph norm in $\operatorname{dom}_p(L)$. Moreover, if $\varphi \ge 0$, then one can choose $\varphi_n \ge 0$.

Proof. Obviously $\mathbb{F}C_{\mathrm{b}}^\infty$ is a linear subspace of $L^p(H,\mu)$ and $\mathbb{F}C_{\mathrm{b}}^\infty \subset \operatorname{dom}_p(L)$. Moreover, $R_t(\mathbb{F}C_{\mathrm{b}}^\infty) \subset \mathbb{F}C_{\mathrm{b}}^\infty$ for $t \ge 0$ (see the proof of [G-Ch; E, Lem. 1]). Since R_t preserves positivity and is a contraction on L^∞ we can apply Lemma 1(b). Hence it suffices to prove that for any given $\varphi \in \mathcal{B}_{\mathrm{b}}(H)$, $\varphi \ge 0$, one can find a sequence (φ_n) such that

$$\varphi_n \in \mathbb{F}C_{\mathrm{b}}^\infty, \quad 0 \le \varphi_n \le \|\varphi\|_\infty =: M, \quad \|\varphi_n - \varphi\|_{L^p} \to 0. \tag{7}$$

First, by Lusin's theorem

$$\forall \varepsilon > 0\ \exists C \subset H, C \text{ closed} \quad \mu(H \setminus C) < \varepsilon \quad \text{and} \quad \varphi|_C \text{ is continuous.}$$

Next, by the Tietze–Urysohn theorem $\varphi|_C$ can be extended to a function φ_ε which is continuous on the whole of H and such that $0 \le \varphi_\varepsilon \le M$. Then

$$\|\varphi_\varepsilon - \varphi\|_{L^p} \le 2\varepsilon M. \tag{8}$$

Since $\operatorname{dom}(A^*)$ is dense in H, there exists a sequence $(\Pi_m)_{m=1}^\infty$ of orthogonal projections such that $\dim \Pi_m = m$, $\Pi_m(H) \subset \operatorname{dom}(A^*)$ and $\Pi_m x \to x$ as $m \to \infty$, $x \in H$. If $\psi \ge 0$, $\psi \in C_{\mathrm{b}}(H)$, then

$$\begin{aligned} &\psi_m := \psi(\Pi_m x) \to \psi(x) \quad \text{for all } x \in H, \\ &0 \le \psi_m \le \|\psi\|_\infty, \quad \psi_m(x) = f_m(\langle x, h_1\rangle, \ldots, \langle x, h_m\rangle), \end{aligned} \tag{9}$$

where $f_m \in C_{\mathrm{b}}(\mathbb{R}^m)$, $h_1, \ldots, h_m \in \Pi_m(H)$. Finally, any nonnegative $f \in C_{\mathrm{b}}(\mathbb{R}^m)$ can be pointwise approximated by functions $\widetilde{f}_k \in C_0^\infty(\mathbb{R}^m)$ with $0 \le \widetilde{f}_k \le \|f\|_\infty$, which together with (8) and (9) proves (7). Hence the lemma follows. ■

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