Introduction

The title of this dissertation contains two terms: *stability analysis* and *vector optimization*. *Stability analysis* is the study of how the output of a model varies as a function of input data and the model parameters. It is a prerequisite for the correct model building in a general setting (cf. e.g. Babuška, Hlaváček and Chleboun [13], Eslami [58], Frank [62], Wierzbicki [152]). Stability analysis is investigated for phenomena modelled by ordinary or partial differential equations (cf. e.g. Malanowski [107], Sokołowski and Zolesio [142], Sokołowski and Żochowski [141]). Stability analysis is extensively studied in scalar optimization (cf. e.g. Bonnans and Shapiro [39], Dontchev and Zolezzi [57], Pallaschke and Rolewicz [118]). For the classical problems of linear algebra, e.g. stability of solutions to systems of linear equations and the eigenvalue problem was investigated e.g. by Lewis [102] and Roussellet and Chenais [138].

From the mathematical viewpoint, stability analysis relies on investigation of continuity or/and Lipschitz (Hölder) continuity properties of solutions. Traditionally, investigation of differentiability properties of solutions is called sensitivity analysis (cf. e.g. Fiacco [61]). In optimization, sensitivity analysis constitutes a natural source of nonsmooth mappings such as optimal value functions and optimal solution mappings which are of interest in nonsmooth analysis (cf. e.g. Kiwiel [89]).

Vector optimization or *multiple objective optimization* is gaining momentum in development of its theory and applications. It has its origin primarily in economics, in equilibrium and welfare theories. The most common and natural necessity to optimize multiple objectives arises in social setting when individuals are trying to maximize their benefit, which often leads to competition. Nowadays, vector optimization is exploited also in solving engineering problems.

The underlying concept in vector optimization is the concept of *efficient* (or *nondominated*) point. Let Y be a topological vector space with a closed convex pointed cone $\mathcal{K} \subset Y$. Let $C \subset Y$ be a subset of Y. An element $y \in C$ is *efficient*, written $y \in E(C)$ (also $E_{\mathcal{K}}(C)$), if $(y - \mathcal{K}) \cap C = \{y\}$.

Let X be a topological space. Let $f : X \to Y$ be a mapping and A be a subset of X. The vector optimization problem

$$(P) \qquad \begin{array}{l} \min_{\mathcal{K}} f(x) \\ \text{subject to } x \in A \end{array}$$

consists in finding the set $E(f, A) = E(f(A))$ called the *efficient* (or *nondominated*) point *set* of (P) and the *solution set* $S(f, A) = \{x \in A : f(x) \in E(f, A)\}$. In the following we often refer to problem (P) as the *original problem* or the *unperturbed problem.* The space X is called the *decision space* and Y is called the *outcome space.*

Let U be a topological space. We embed problem (P) into a family (P_u) of vector optimization problems parametrized by a parameter $u \in U$,

$$(P_u) \qquad \begin{array}{l} \min_{\mathcal{K}} f(u,x) \\ \text{subject to } x \in A(u) \end{array}$$

where $f : U \times X \to Y$ is the parametrized objective function and $A(u) \subset X$ is the parametrized feasible subset of X. The sets $A(u)$ give rise to the *feasible* set-valued mapping $\mathcal{A} : U \rightrightarrows X$, $\mathcal{A}(u) = A(u)$, $\mathcal{A}(u_0) = A$. Problem (P) corresponds to a given parameter value $u_0 \in U$.

The *performance* set-valued mapping $\mathcal{P} : U \rightrightarrows Y$ is defined as $\mathcal{P}(u) = E(f(u,\cdot), A(u))$, $\mathcal{P}(u_0) = E(f, A)$, and the *solution* set-valued mapping $\mathcal{S} : U \rightrightarrows X$ is defined as $\mathcal{S}(u) = S(f(u,\cdot), A(u))$ and $\mathcal{S}(u_0) = S(f, A)$.

Our aim is to perform a systematic study of stability properties of the performance mapping $\mathcal{P}$ and the solution mapping $\mathcal{S}$. We focus on conditions ensuring Hausdorff, Lipschitz and Hölder behaviour of $\mathcal{P}$ and $\mathcal{S}$ with respect to the parameter u. To enlarge the applicability of the results we do not assume any particular form of the feasible set and we tend to avoid as much as possible compactness assumptions which are frequently over-used (see e.g. [148]).

Convergence and rates of convergence of solutions to perturbed optimization problems are one of crucial topics of stability analysis in optimization both from the theoretical and numerical viewpoints. For scalar optimization these topics were investigated by many authors (see e.g., [2, 56, 86, 103, 112, 113, 118, 132, 153, 154] and many others). An exhaustive survey of the current state of research is given in the books by Bonnans and Shapiro [39], Dontchev and Zolezzi [57], Pallaschke and Rolewicz [118]. In vector optimization the results on Lipschitz continuity of solutions are scarse and refer only to some classes of problems (cf. e.g. [47], [48], [49] for the linear case and [37], [50] for the convex case).

A characteristic feature of vector optimization problems is that the outcome spaces are equipped with partial orderings which are not linear in general. These partial orders are generated by cones whose properties play an important role in existence results and optimality conditions. To derive stability results we make use of two new concepts pertaining to sets and cones in the outcome space, namely the *containment property*, introduced in [21], and the *strict efficiency*, introduced in [17].

The *containment property* (CP) is used to study upper semicontinuities (in the sense of Hausdorff, Lipschitz, or Hölder) of efficient points (cf. [16, 21]) under perturbation of a set. This property can be viewed as a variant of the domination property (DP) appearing frequently in the context of stability of solutions to finite-dimensional parametric vector optimization problems. To study upper Hölder continuity of efficient points and solutions to (P) we introduce the *containment rate* of a set with respect to a cone, which is a real-valued function of a scalar argument and characterizes the containment property (CP).

Strict efficiency is introduced in [31, 18] to study lower (Hausdorff, Hölder) semicontinuities of efficient points. In normed spaces, strict efficiency is implied by the super efficiency in the sense of Borwein and Zhuang [42]. To study lower Hölder continuity of efficient points and solutions to (P) we define the *modulus of strict efficiency* ([18]). In vector optimization the concept of strict efficiency leads to the notion of *sharp* and *weak sharp solutions* (local and global) ([27]). Both sharp and weak sharp solutions can be viewed as vector counterparts of sharp (and weak sharp) minimality and growth conditions appearing in scalar optimization (cf. [39], [38], [43]).

The organization of the book is as follows. In Chapter 2 we investigate the strict efficiency and the modulus of strict efficiency. Special attention is paid to strict efficiency in the finite-dimensional case.

In Chapter 3 we derive sufficient conditions for lower Hausdorff semicontinuity of the efficient point set-valued mapping $\mathcal{E}$, $\mathcal{E}(u) = E(\mathcal{C}(u))$, where $\mathcal{C} : U \rightrightarrows Y$ is a given set-valued mapping.

In Chapter 4 we formulate conditions for lower Hölder continuity and lower-pseudo-Hölder continuity of $\mathcal{E}$.

In Chapter 5, the containment property (CP) and the containment rate are investigated. Special attention is paid to the finite-dimensional linear and convex cases. In Chapter 6, by using the containment property we prove sufficient conditions for upper Hausdorff continuity of efficient points and in Chapter 7 the containment rate is used to investigate upper Hölder continuity and upper pseudo-Hölder behaviour of $\mathcal{E}$. We apply the results obtained to formulate sufficient conditions for the Hölder continuity of the performance set-valued mapping $\mathcal{P}$ for parametric problems (P_u).

In Chapter 8 we define ϕ-sharp and weak ϕ-sharp solutions to (P). When applied to scalar optimization problems the notions of ϕ-sharp and weak ϕ-sharp solutions reduce to the notions of sharp and weak sharp minima, respectively, introduced by Studniarski and Ward [147], Burke and Ferris [44]. Sharp and weak sharp minima were used e.g. by Attouch and Wets [7], Bonnans and Shapiro [39] to derive stability results for parametric problems.

In Chapter 9, basing on properties of ε-solutions to vector optimization problems we define well-posedness of (P). We investigate relationships between well-posedness of (P) and sharpness or weak sharpness of solutions. In classes of well-posed problems we investigate upper Hausdorff semicontinuity and upper Lipschitz (Hölder) continuity of the solution mapping $\mathcal{S}$, $\mathcal{S}(u) = S(f, A(u))$. By exploiting the notions of local sharp and local weak sharp solutions we prove Hölder calmness of $\mathcal{S}$.

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1. PRELIMINARIES

The general framework of our developments are Hausdorff topological vector spaces (t.v.s.) over the field $\mathbb{R}$ of real numbers. A linear space Y is a topological vector space if Y is equipped with a topology compatible with the linear space structure, that is, both linear space operations $(y_1, y_2) \to y_1 + y_2$, $y_1, y_2 \in Y$, and $(r, y) \to ry$, $r \in \mathbb{R}$, $y \in Y$, are continuous on their domains, $Y \times Y$ and $\mathbb{R} \times Y$, respectively. It is a consequence of the continuity of the linear space operations that the topological structure of Y is determined by a base of neighbourhoods of the origin.

If $\mathcal{V}$ is a base of neighbourhoods of the origin, then for each $V \in \mathcal{V}$,

(i) V is absorbing, i.e., for any $y \in Y$ there is some $\overline{\lambda} > 0$ such that $\lambda y \in V$ for any $0 \leq \lambda \leq \overline{\lambda}$,

(ii) there exists a balanced neighbourhood $\overline{V} \subset V$, i.e., for all $v \in \overline{V}$, $\lambda v \in \overline{V}$ whenever $|\lambda| \leq 1$,

(iii) there exists $W \in \mathcal{V}$ such that $W + W \subset V$.

A topological space is *Hausdorff* (or separated) if any two distinct points have disjoint neighbourhoods. If $\mathcal{V}$ is a base of neighbourhoods in a topological vector space Y, then Y is a Hausdorff space if and only if $\bigcap_{V \in \mathcal{V}} V = \{0\}$. We use the standard notations $\mathrm{cl}(\cdot)$, $\mathrm{int}(\cdot)$, and $\partial(\cdot)$ for closure, interior, and boundary, respectively.

Let C be a subset of Y. We say that C is *convex* if $\lambda x + (1 - \lambda) y \in C$ for all $x, y \in C$ and $0 \leq \lambda \leq 1$. The (linear) segment $[a, b]$ with end-points $a \in Y$ and $b \in Y$ is given as

$$[a, b] = \{z \in Y : z = \lambda x + (1 - \lambda) y,\ 0 \leq \lambda \leq 1\}.$$

For any nonempty subsets C and D of Y the *algebraic sum* of C and D is defined as

$$C + D = \{c + d : c \in C,\ d \in D\}$$

and the *algebraic difference* of C and D is defined as

$$C - D = \{c - d : c \in C,\ d \in D\}.$$

Moreover, the algebraic sum and difference are empty if any of the sets C and D is empty. For any subset C of Y and $\lambda \in \mathbb{R}$,

$$\lambda C = \{\lambda y : y \in C\}.$$

By a *locally convex space* we mean a topological vector space with a base of convex neighbourhoods of the origin. A locally convex space Y has a base $\mathcal{V}$ of neighbourhoods of the origin with the following properties:

(i) if $V \in \mathcal{V}$ and $\lambda \neq 0$, then $\lambda V \in \mathcal{V}$,
(ii) each $V \in \mathcal{V}$ is absolutely convex (i.e., balanced and convex).

Let Y^* be the topological dual of Y, i.e., the space of all continuous functionals defined on Y. If Y is a Hausdorff locally convex space, then Y^* separates points, i.e., for any two different points $y_1, y_2 \in Y$ there exists $f \in Y^*$ such that $f(y_1) \neq f(y_2)$ (see e.g. Holmes [78, Cor. 11.E]).

1.1. Cones in topological vector spaces

In this section we collect basic facts about cones. A subset $\mathcal{K}$ of a vector space Y is a *cone* if

$$y \in \mathcal{K} \text{ and } \lambda \geq 0 \ \Rightarrow\ \lambda y \in \mathcal{K}.$$

By definition, each nonempty cone contains the origin and $\{0\}$ is the trivial cone. A convex cone is a cone which is a convex subset of Y. A cone $\mathcal{K}$ is *pointed* if $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$.

DEFINITION 1.1.1. Let $\{0\} \neq \mathcal{K} \subset Y$ be a convex cone. A nonempty convex subset Θ of $\mathcal{K}$ is a *base* of $\mathcal{K}$ if $0 \notin \operatorname{cl}\Theta$ and $\mathcal{K} = \bigcup\{\lambda\Theta : \lambda \geq 0\}$.

A based cone is necessarily pointed and convex. The example below shows that Definition 1.1.1 does not ensure the uniqueness of the representation of elements of $\mathcal{K}$ via elements of a base.

EXAMPLE 1.1.1. Let $Y = \mathbb{R}^2$, $\mathcal{K} = \mathbb{R}^2_+$. The set

$$\Theta = \mathcal{K} \cap \{(y_1, y_2) : -y_1 + 2 \leq y_2 \leq -y_1 + 4\}$$

is a base of $\mathcal{K}$. Each $0 \neq k \in \mathcal{K}$ can be represented as $(k_1, k_2) = \lambda(y_1, y_2)$, where $\lambda > 0$ and $(y_1, y_2) \in \Theta$. It is enough to take any λ satisfying $(k_1 + k_2)/4 \leq \lambda \leq (k_1 + k_2)/2$.

Conditions ensuring uniqueness of representation are given in the following proposition.

PROPOSITION 1.1.1 (Peressini [122], Jahn [82]). *Let Y be a vector space. Let $\mathcal{K}$ be a convex cone in Y and let Θ be a nonempty convex subset of $\mathcal{K}$. The following conditions are equivalent:*

(i) *each nonzero element $y \in \mathcal{K}$ has a unique representation of the form $y = \lambda\theta$, where $\lambda > 0$, $\theta \in \Theta$,*
(ii) *$\mathcal{K} = \bigcup\{\lambda\Theta : \lambda \geq 0\}$ and the smallest linear manifold in Y containing Θ does not contain* 0.

Proof. If (i) holds, then $\mathcal{K} = \bigcup\{\lambda\Theta : \lambda \geq 0\}$. The smallest linear manifold containing Θ is $L = \{\mu\theta + (1-\mu)\theta' : \theta, \theta' \in \Theta,\ \mu \in \mathbb{R}\}$. If $0 \in L$, there would be $\mu_0 > 1$ and $\theta_0, \theta_0' \in \Theta$ such that $\mu_0\theta_0 = (\mu_0 - 1)\theta_0'$, contrary to (i).

To show uniqueness in (i), suppose on the contrary that $\lambda\theta = \lambda'\theta'$ for $\theta, \theta' \in \Theta$, and positive reals λ, λ', $\lambda \neq \lambda'$. Then

$$0 = \frac{1}{\lambda - \lambda'}\{\lambda\theta - \lambda'\theta'\} \in L$$

contrary to (ii). ■

In some textbooks the base of a cone is defined as a nonempty convex subset of the cone satisfying condition (i) of Proposition 1.1.1 (see e.g. [82, 83, 85, 122]). If Θ satisfies that condition, then $0 \notin \Theta$.

In locally convex spaces, any based convex cone has a base satisfying condition (i) of Proposition 1.1.1.

PROPOSITION 1.1.2. *Let Y be a locally convex Hausdorff topological vector space and let $\mathcal{K}$ be a convex cone in Y with a base Θ. There exists another base Θ_1 of $\mathcal{K}$ such that $\Theta_1 = f^{-1}(1) \cap \mathcal{K}$, where f is a continuous linear functional on Y satisfying condition* (i) *of Proposition* 1.1.1.

Proof. Since $0 \notin \operatorname{cl}\Theta$, there exists a convex 0-neighbourhood V in Y such that $V \cap \operatorname{cl}\Theta = \emptyset$. By separation arguments (see e.g. Holmes [78, Th. 11.E, 12.F]), there exists a continuous linear functional f on Y such that $f(\theta) > 0$ for $\theta \in \Theta$. Hence, $\Theta_1 = f^{-1}(1) \cap \mathcal{K}$ is a base of $\mathcal{K}$ which satisfies condition (i) of Proposition 1.1.1. ■

We say that a subset C of Y is *$\mathcal{K}$-closed* if $C + \mathcal{K}$ is closed, and C is *$\mathcal{K}$-convex* if $C + \mathcal{K}$ is convex.

For any cone $\mathcal{K} \subset Y$, its *dual* $\mathcal{K}^* \subset Y^*$ of Y^* is defined as

$$\mathcal{K}^* = \{f \in Y^* : f(y) \geq 0 \text{ for all } y \in \mathcal{K}\}.$$

The dual cone $\mathcal{K}^*$ is nonempty and weak* closed. To see the latter suppose that f_ω is a net of functionals from $\mathcal{K}^*$ converging weak* to f. Then $f_\omega(y)$ converges to $f(y)$ for all $y \in Y$, in particular, $f_\omega(k)$ converges to $f(k)$ for any $k \in \mathcal{K}$. This entails $f(k) \geq 0$ for all $k \in \mathcal{K}$ since $f_\omega(k) \geq 0$ for all ω and all $k \in \mathcal{K}$.

For any subset C of a topological vector space Y the *polar* $C^\circ \subset Y^*$ of C is defined as

$$C^\circ = \{f \in Y^* : f(y) \leq 1 \text{ for all } y \in C\}.$$

The polar is nonempty since $0 \in C^\circ$, and weak* closed. We have $\mathcal{K}^* = -\mathcal{K}^\circ$. In the same way, for any subset $C \subset Y^*$, we define the polar $C^\circ \subset Y$ as

$$C^\circ = \{y \in Y : f(y) \leq 1 \text{ for all } f \in C\}.$$

The *bipolar* $C^{\circ\circ} \subset Y$ of a subset $C \subset Y$ is

$$C^{\circ\circ} = \{y \in Y : f(y) \leq 1 \text{ for all } f \in C^\circ\}.$$

If C is a subset of a locally convex space Y, then

$$C^{\circ\circ} = \operatorname{cl}((\operatorname{conv}\{0 \cup C\}),$$

where conv stands for convex hull (cf. Holmes [78, Th. 12.C]). Hence, the bidual cone $\mathcal{K}^{**}$,

$$\mathcal{K}^{**} = \{y \in Y : f(y) \geq 0 \text{ for } f \in \mathcal{K}^*\},$$

is convex and weakly closed, and in locally convex spaces $\mathcal{K} = \mathcal{K}^{**}$ if and only if $\mathcal{K}$ is convex and weakly closed (in normed spaces cf. Kurcyusz [98, Lemma 8.6]).

A topological linear space Y is said to be a *Mackey space* (cf. e.g. [85]) if $B^\circ \subset Y$ is a 0-neighbourhood in Y whenever $B \subset Y^*$ is a convex and weak* compact subset of Y.

THEOREM 1.1.1 (Jameson [85, Th. 3.8.6]). *Let $\mathcal{K}$ be a convex cone in a locally convex topological space Y. Then*

(i) *if $\mathcal{K}$ has an interior point, then $\mathcal{K}^*$ has a weak* compact base,*
(ii) *if Y is a Mackey space, $\mathcal{K}$ is closed and $\mathcal{K}^*$ has a weak* compact base, then $\mathcal{K}$ has an interior point.*

Proof. (i) Let $e \in \operatorname{int} \mathcal{K}$ and let $\Theta = \{f \in \mathcal{K}^* : f(e) = 1\}$. Then Θ is a base of $\mathcal{K}^*$. Now $\mathcal{K} - e$ is a 0-neighbourhood in Y and hence $(\mathcal{K} - e)^*$ is weak* compact. The result follows since Θ is a weak* closed subset of $(\mathcal{K} - e)^*$.

(ii) Suppose that Θ is a weak* compact base of $\mathcal{K}^*$. There is an element y_0 of Y such that $f(y_0) \geq 1$ for $f \in \Theta$. Since Y is a Mackey space, Θ° is a 0-neighbourhood in Y. For $y \in \Theta^\circ$ and $f \in \Theta$, $f(y_0 + y) \geq 0$, so $y_0 + y \in \mathcal{K}^{**} = \mathcal{K}$. Hence, $y_0 + \Theta^\circ \subset \mathcal{K}$. ■

Below we give an example of a cone with empty interior such that $\mathcal{K}^*$ has a bounded and closed base in the norm topology.

EXAMPLE 1.1.2 (Jameson [85, p. 123]). Let $Y = c_0$ be the space of real sequences converging to zero with the usual cone c_0^+ of nonnegative elements. Then c_0^+ has no interior points, and $(c_0^+)^*$ is the usual nonnegative cone ℓ_1^+ in ℓ_1. The set of sequences $\{\xi_n\} \subset (c_0^+)^*$ such that $\sum \xi_n = 1$ is a base for $(c_0^+)^*$ that is bounded and closed in the norm topology.

The set
$$\mathcal{K}^{*i} = \{f \in \mathcal{K}^* : f(y) > 0 \text{ for all } y \in \mathcal{K} \setminus \{0\}\}$$
is called the *quasi-interior* of $\mathcal{K}^*$. Note that $\mathcal{K}^{*i}$ may be empty. The set
$$\mathcal{K}^i = \{y \in Y : f(y) > 0 \text{ for all } f \in \mathcal{K}^* \setminus \{0\}\}$$
is called the *quasi-interior* of $\mathcal{K}$ (cf. e.g. [140, 122]). In locally convex spaces, $\mathcal{K}^i \subset \mathcal{K} \setminus \{0\}$, and if $\operatorname{int} \mathcal{K} \neq \emptyset$, then $\operatorname{int} \mathcal{K} = \mathcal{K}^i$. Moreover, by Lemma 5.5 of [46],
$$\mathcal{K} = \{y \in Y : f(y) \geq 0 \text{ for all } f \in \mathcal{K}^{*i}\}.$$
Indeed, suppose that $y \notin \mathcal{K}$. Since Y is locally convex, there exists $f \in \mathcal{K}^*$ such that $f(y) < 0$. Let $g \in \mathcal{K}^{*i}$. By choosing $\alpha > 0$ such that $f(y) + \alpha g(y) < 0$ we get $h = f + \alpha \cdot g \in \mathcal{K}^{*i}$ and $h(y) < 0$.

EXAMPLE 1.1.3 (Peressini [122, Ex. 3.7b, p. 27]). Let $Y = B[a, b]$ be the set of all bounded, real-valued functions on the interval $\langle a, b \rangle$ and
$$\mathcal{K} = \{f \in B[a, b] : f(y) \geq 0 \text{ for all } y \in [a, b]\}.$$
The quasi-interior $\mathcal{K}^{*i}$ of $\mathcal{K}$ is empty.

Necessary and sufficient conditions for $\mathcal{K}^{*i}$ to be nonempty were given by Dauer and Gallagher in [46].

PROPOSITION 1.1.3 (Dauer and Gallagher [46]). *Let Y be a topological vector space and let $\mathcal{K}$ be a convex cone in Y. Then $\mathcal{K}^{*i}$ is nonempty if and only if there exists an open convex subset Q in Y satisfying*

(i) $0 \notin Q$,
(ii) $\mathcal{K} \subset \operatorname{cone}(Q) = \bigcup\{\lambda Q : \lambda \geq 0\}$.

Proof. If $\mathcal{K}^{*i} \neq \emptyset$, then the set $Q = \{y \in Y : f(y) > 0\}$, $f \in \mathcal{K}^{*i}$, satisfies (i) and (ii).

Let Q be a subset of Y satisfying (i) and (ii). Since $0 \notin Q$, by separation arguments (see [139, p. 58]), there exists $f \in Y^*$ such that $f(0) < f(q)$ for $q \in Q$. Thus, $f(q) > 0$ for all $q \in Q$. From (ii) it follows that $f \in \mathcal{K}^{*i}$. ∎

By Proposition 1.1.3, for any convex cone $\mathcal{K}$ in a locally convex space Y, $\mathcal{K}^{*i}$ is nonempty if and only if $\mathcal{K}$ is based. If Y is separable and $\mathcal{K}$ is closed convex and pointed, then $\mathcal{K}^{*i}$ is nonempty (see [94, Thm. 2.1]).

Let C be a subset of a linear space Y. The set

$$\operatorname{core} C = \{z \in C : \forall y \in Y\ \exists \overline{\lambda} > 0 \text{ with } z + \lambda y \in C \text{ for } 0 \leq \lambda \leq \overline{\lambda}\}$$

is called the *algebraic interior* or the *core* of C. For any cone $\mathcal{K}$ in a linear vector space Y, the fact that $\operatorname{core} \mathcal{K} \neq \emptyset$ implies that $\mathcal{K}$ is *reproducing*, i.e., $\mathcal{K} - \mathcal{K} = Y$ (see Lemma 1.13 of [82] and [83]).

THEOREM 1.1.2 (Jahn [82, Lemmas 1.25, 1.26]). *Let $\mathcal{K}$ be a closed convex cone in a topological vector space Y with $\mathcal{K}^* \neq \{0\}$. Then*

(i) $\operatorname{core} \mathcal{K} \subset \mathcal{K}^i$,
(ii) *if Y^* separates points of Y and $\mathcal{K}^{*i} \neq \emptyset$, then* $\operatorname{core} \mathcal{K}^* \subset \mathcal{K}^{*i}$.

Proof. (i) Let $k \in \operatorname{core} \mathcal{K}$. Thus, $k \in \mathcal{K}$ and for any $y \in Y$ there exists $\overline{\lambda} > 0$ with $k + \lambda y \in \mathcal{K}$ for $0 \leq \lambda \leq \overline{\lambda}$. Hence, for any $f \in \mathcal{K}^* \setminus \{0\}$, $f(k + \lambda y) \geq 0$ for any $0 \leq \lambda \leq \overline{\lambda}$. Since $f \in \mathcal{K}^* \setminus \{0\}$, there exists $y_0 \in Y$ with $f(y_0) < 0$ and we get $f(k) \geq -\overline{\lambda} f(y_0) > 0$. Hence, $f(k) > 0$.

(ii) Let $f \in \operatorname{core} \mathcal{K}^*$. Thus, $f \in \mathcal{K}^*$ and for any $g \in Y^*$ there exists $\overline{\lambda} > 0$ with $f + \lambda g \in \mathcal{K}^*$ for $0 \leq \lambda \leq \overline{\lambda}$. Hence, $(f + \lambda g)y \geq 0$ for any $y \in \mathcal{K}$ and any $0 \leq \lambda \leq \overline{\lambda}$. By taking any $g_0 \in Y^*$ with $g_0(y) < 0$ we get $f(y) \geq -\overline{\lambda} g_0(y) > 0$. Hence, $f(y) > 0$. ∎

When $\mathcal{K}^* = \{0\}$ Theorem 1.1.2 is not true; to see this it is enough to take $\mathcal{K} = Y$. As shown in [82, Lemma 1.27], in any linear vector space Y, the cone $\mathcal{K}^*$ is pointed whenever $\operatorname{core} \mathcal{K} \neq \emptyset$. Then, by Theorem 1.1.2, $\mathcal{K}^*$ is based. Moreover, if $\operatorname{core} \mathcal{K}^* \neq \emptyset$, then $\mathcal{K}$ is based (see [78, Theorem I.5C]).

PROPOSITION 1.1.4. *Let Y be a locally convex topological vector space and let $\mathcal{K}$ be a closed convex cone in Y. If $\mathcal{K}^i \neq \emptyset$, and $\mathcal{K}^*$ is nontrivial, then $\mathcal{K}^*$ has a base.*

Proof. Let $y_0 \in \mathcal{K}^i$. Then the set

$$\Theta^* = \{\theta^* \in \mathcal{K}^* : \theta^*(y_0) = 1\} \tag{1.1}$$

is a base of $\mathcal{K}^*$. It is convex, weak* closed, $0 \notin w^*\text{-cl}\, \Theta^*$, where w^*-cl stands for the weak* closure. Moreover, for any $0 \neq f \in \mathcal{K}^*$, we have $f(y_0) = \lambda_f \neq 0$, and $f/\lambda_f \in \Theta^*$. ∎

In the following we refer to any base of the form (1.1) as a *standard base*. By Theorem 1.1.2, $\operatorname{core} \mathcal{K} \subset \mathcal{K}^i$, and by Proposition 1.1.4, if $\operatorname{core} \mathcal{K} \neq \emptyset$ and $\mathcal{K}^* \neq \{0\}$, then $\mathcal{K}^*$ is based. By similar arguments, $\mathcal{K}^{*i}$ is always based.

1.2. Basic concepts of efficiency

Let Y be a topological vector space and let $\mathcal{K}$ be a closed convex cone in Y. The *ordering relation* $\preceq$ (we write also $\preceq_{\mathcal{K}}$) in Y associated with $\mathcal{K}$ is defined as

$$y_1 \preceq_{\mathcal{K}} y_2 \;\Leftrightarrow\; y_1 - y_2 \in \mathcal{K}.$$

The relation $\preceq_{\mathcal{K}}$ is reflexive and transitive, and it is antisymmetric if and only if $\mathcal{K}$ is *pointed*, i.e., $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$. Let C be a subset of Y. An element $y \in C$ is *efficient* (or *nondominated*) for C with respect to $\mathcal{K}$, written $y \in E(C)$ (or $y \in E_{\mathcal{K}}(C)$), if $C \cap (y - \mathcal{K}) \subset \mathcal{K}$. When $\mathcal{K}$ is pointed, an element $y \in C$ is efficient if $C \cap (y - \mathcal{K}) = \{y\}$. When $\operatorname{int}\mathcal{K} \neq \emptyset$ we say that an element $y \in C$ is *weakly efficient*, and we write $y \in WE(C)$ (or $y \in WE_{\mathcal{K}}(C)$), if $C \cap (y - \operatorname{int}\mathcal{K}) = \emptyset$. Clearly, $E(C) \subset WE(C)$.

An element $y \in C$ is *locally efficient* (or *locally nondominated*) in C with respect to $\mathcal{K}$, and we write $y \in LE(C)$ (or $y \in LE_{\mathcal{K}}(C)$), if there exists a 0-neighbourhood V in Y such that $y \in E_{\mathcal{K}}(C \cap (y + V))$. If $C \subset Y$ is a convex subset of Y, then

$$E_{\mathcal{K}}(C) = LE_{\mathcal{K}}(C). \tag{1.2}$$

To see this, suppose that $y_0 \notin E_{\mathcal{K}}(C)$. There exists $y_1 \in C$ such that $y_1 - y_0 \in -\mathcal{K}$. By convexity, $\lambda y_0 + (1-\lambda) y_1 \subset C \cap (y_0 - \mathcal{K})$, $0 \le \lambda \le 1$, and $\lambda y_0 + (1-\lambda) y_1 \in V$ for $0 \le \lambda \le \overline{\lambda} \le 1$. Hence, $y_0 \notin E_{\mathcal{K}}(C \cap V)$.

A well-known fact is that the compactness of C implies that $E(C) \neq \emptyset$. Numerous attempts have been made to weaken the compactness requirement (see e.g. [145], [40], [36], [149]).

We will use the following fundamental existence theorem.

Theorem 1.2.1 ([83, Th.6.5]). *Let C be a nonempty subset of a real locally convex space Y. If C is weakly compact, then for every closed convex cone $\mathcal{K}$ in Y the set C has at least one efficient point with respect to the partial ordering induced by $\mathcal{K}$.*

1.3. Vector optimization problems

Let X and Y be Hausdorff topological vector spaces. Let $\mathcal{K}$ be a closed convex cone in Y. We consider the vector optimization problem

$$(P) \qquad \begin{array}{l} \min_{\mathcal{K}} f(x) \\ \text{subject to } x \in A, \end{array}$$

where $f : X \to Y$ is a mapping and A is a subset of X.

The set $E(f, A)$ of (global) efficient points to (P) (we write also $E_{\mathcal{K}}(f, A)$) is defined as $E(f, A) := E(f(A))$. The set

$$S(f, A) := \{x \in A : f(x) \in E(f, A)\}$$

(we write also $S_{\mathcal{K}}(f, A)$) is the set of (global) solutions to (P) (see Jahn [82, 83], Luc [105]). Clearly, $S(f, A) = A \cap f^{-1}(E(f, A))$.

An element $x \in A$ is a *local solution* to (P), $x \in LS(f, A)$ (we write also $x \in LS_{\mathcal{K}}(f, A)$), if there exists a 0-neighbourhood Q in X such that $x \in A \cap f^{-1}(E(f(A \cap Q)))$.

An element $y \in f(A)$ is a *locally efficient point* for (P), $y \in LE(f,A)$ (we write also $y \in LE_{\mathcal{K}}(f,A)$), if there exists a 0-neighbourhood W such that $y \in E_{\mathcal{K}}(f(A) \cap W)$. In general, $LS(f,A)$ differs from $A \cap f^{-1}(LE(f,A))$.

PROPOSITION 1.3.1. *Let X and Y be Hausdorff topological vector spaces and let $\mathcal{K}$ be a closed convex cone in Y. Let A be a subset of X and $f : X \to Y$ be continuous on A. Then*

$$A \cap f^{-1}(LE_{\mathcal{K}}(f,A)) \subset LS_{\mathcal{K}}(f,A).$$

Proof. Let $x_0 \in A \cap f^{-1}(LE_{\mathcal{K}}(f,A))$. Then $f(x_0) \in LE_{\mathcal{K}}(f(A))$ and there exists a 0-neighbourhood W in Y such that $(f(A) \cap (f(x_0) + W) - f(x_0)) \cap (-\mathcal{K}) \subset \mathcal{K}$. By the continuity of f, there exists a 0-neighbourhood Q in X such that $f(x_0+Q) \subset f(x_0)+W$. Hence, $f((x_0+Q) \cap A) \subset f(x_0+Q) \cap f(A) \subset (f(x_0)+W) \cap f(A)$, and

$$(f((x_0)+Q) \cap A) - f(x_0)) \cap (-\mathcal{K}) \subset \mathcal{K},$$

which means that $x_0 \in LS_{\mathcal{K}}(f,A)$. ∎

The opposite inclusion to that of Proposition 1.3.1 does not hold in general.

If $Y = \mathbb{R}^m$ and $A \subset \mathbb{R}^n$ is given as the solution set to a finite system of equations and/or inequalities and the mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ is given as

$$f = (f_1, \ldots, f_m),$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$, $1 \le i \le m$, are (scalar) criteria (objectives), problem (P) takes the form of a *multicriteria optimization problem*

$$(MOP) \qquad \begin{array}{l} \min_{\mathcal{K}} \ (f_1, \ldots, f_m) \\ \text{subject to} \\ x \in A = \{x \in \mathbb{R}^n : g_i(x) \le b_i,\ i \in I,\ h_j(x) = d_j,\ j \in J\}, \end{array}$$

where I and J are finite systems of indices, $g_i : \mathbb{R}^n \to \mathbb{R}$ and $b_i \in \mathbb{R}$ for $i \in I$, $h_j : \mathbb{R}^n \to \mathbb{R}$ and $d_j \in \mathbb{R}$ for $j \in J$.

In the literature there exist a number of definitions of *properly efficient* points (and solutions) for (P) and (MOP). Properly efficient points are efficient points which satisfy additional conditions in order to eliminate some undesirable behaviour (e.g. the unbounded growth of trade-off coefficients). The definitions of properly efficient points were originally proposed by Geoffrion [65] and Kuhn and Tucker [96]. In the finite-dimensional setting properly efficient points were also investigated by Benson [34], Hartley [71] and Henig [73]. The definition of proper efficiency proposed by Henig in [73] can be naturally generalized to the infinite-dimensional setting. The definitions of proper efficiency in infinite dimensions were also proposed by Borwein [40, 41] and Borwein and Zhuang [42]. The relationships between different notions of proper efficiency were elucidated in [70].

Dual problems to (P) and (MOP) were proposed by many authors. For a survey of the existing approaches and generalizations we refer to Song [143] and the references therein.

Parametric problems related to (P) were investigated on different levels of generality. Convergence of sequences of efficient point sets $E(C_n)$ was investigated by Mighlierina and Molho [110, 111]. The construction of polarities was exploited in proving different type of convergence of efficient point sets by Dolecki [53, 54], Dolecki and Malivert [55], Malivert [108]. $\mathcal{K}$-semicontinuities of efficient sets were investigated by Sterna-Karwat [144] and Sterna-Karwat and Penot [120, 121].

Bibliographical note. Classic textbooks on topological vector spaces are e.g. Alexiewicz [1], Schaefer [140], Robertson and Robertson [127]. The books by Peressini [122] and Jameson [85] are devoted to ordered topological vector spaces. Presentations of different aspects of the theory of set-valued mappings can be found e.g. in books by Berge [35], Aubin and Frankowska [11], Kuratowski [97]. The theory of vector optimization in topological vector spaces with numerous extensions is presented in the books by Jahn [82, 83], Luc [105], Hyers, Isac and Rassias [79], Gopfert, Riahi, Tammer and Zalinescu [68].

2. STRICT EFFICIENCY

In this chapter we introduce the concept of strict efficiency and the modulus of strict efficiency. These concepts constitute main ingredients of sufficient conditions for the lower semicontinuity and lower Hölder (and lower pseudo-Hölder) continuity of efficient points formulated in Chapters 3 and 4. Strict efficiency can be viewed as a kind of proper efficiency (cf. e.g. [42, 73]). We show that strict efficiency is weaker than the proper efficiency in the sense of Henig [73] and weaker than the super efficiency as defined by Borwein and Zhuang [42]. The question of density of proper efficient points in the set of all efficient points was addressed by many authors (cf. e.g. [3, 32, 42, 46, 63, 67]). Based on those results we get density results for strictly efficient points.

In Section 2.1 we define strong proper efficiency which is stronger than Henig proper efficiency. In Section 2.2 we introduce the notion of strict efficiency; we investigate properties of strictly efficient points and we provide a characterization of strict efficiency in terms of nets. In Section 2.3 we investigate strict efficiency for convex sets. In Section 2.4 we define the modulus of strict efficiency and we prove characterizations of strict efficiency in terms of properties of the modulus of strict efficiency.

2.1. Strong proper efficiency

Let Y be a Hausdorff topological vector space and let $\mathcal{K}$ be a closed convex pointed cone in Y. Let C be a subset of Y.

Definition 2.1.1. A point $y_0 \in C$ is *strongly properly efficient* (see [16]), $y_0 \in SPE(C)$, if there exists a closed convex cone $\mathcal{K}_0$, $\mathcal{K}_0 \neq Y$, $\operatorname{int}\mathcal{K}_0 \neq \emptyset$, $\mathcal{K} \setminus \{0\} \subset \operatorname{int}\mathcal{K}_0$, such that for each 0-neighbourhood W there exists a 0-neighbourhood O such that

$$(\mathcal{K} \setminus W) + O \subset \mathcal{K}_0, \tag{2.1}$$

and $y_0 \in E_{\mathcal{K}_0}(C)$.

Recall that a cone $\mathcal{K}$ has a base Θ if Θ is convex, $0 \notin \operatorname{cl}\Theta$, where cl stands for closure, and $\mathcal{K} = \operatorname{cone}(\Theta)$. For any 0-neighbourhood V we put

$$\mathcal{K}_d(V) = \operatorname{cone}(\Theta + V).$$

Proposition 2.1.1. *Let $\mathcal{K} \subset Y$ be a closed convex cone with a base Θ and let $\mathcal{K}_0$ be a closed convex cone, $\mathcal{K}_0 \neq Y$, $\operatorname{int}\mathcal{K}_0 \neq \emptyset$, $\mathcal{K} \setminus \{0\} \subset \operatorname{int}\mathcal{K}_0$. If $\mathcal{K}_0$ satisfies* (2.1), *then*

$$\mathcal{K}_d(V) \subset \mathcal{K}_0 \tag{2.2}$$

for some 0-neighbourhood V.

Proof. Since $0 \notin \operatorname{cl} \Theta$, there exists a 0-neighbourhood W such that $\Theta \cap W = \emptyset$. By (2.1), there exists a 0-neighbourhood O such that $\Theta + O \subset \mathcal{K}_0$, or $\mathcal{K}_d(O) = \operatorname{cone}(\Theta + O) \subset \mathcal{K}_0$. ∎

PROPOSITION 2.1.2. *Let $\mathcal{K}$ be a closed convex cone in Y with a topologically bounded base Θ. For any 0-neighbourhood V, the cone $\mathcal{K}_d(V)$ satisfies condition (2.1), i.e., for each 0-neighbourhood W there exists a 0-neighbourhood O such that*

$$(\mathcal{K} \setminus W) + O \subset \mathcal{K}_d(V). \tag{2.3}$$

Proof. Let W be a 0-neighbourhood. Since Θ is topologically bounded, there exists $\overline{\lambda} > 0$ such that $\lambda\Theta \subset W$ for $0 \le \lambda \le \overline{\lambda}$ and for $x \in \mathcal{K} \setminus W$ we have $x = \lambda_x \theta_x$, where $\lambda_x > \overline{\lambda}$. Moreover, there exists a 0-neighbourhood O such that $O \subset \overline{\lambda} V$. Hence

$$x + O \subset \lambda_x \theta_x + \overline{\lambda} V = \lambda_x \left(\theta_x + \frac{\overline{\lambda}}{\lambda_x} V \right) \subset \operatorname{cone}(\Theta + V). \ \blacksquare$$

In Proposition 2.1.2, the boundedness of Θ is important as shown by the example below.

EXAMPLE 2.1.1. Let $Y = \ell^\infty$, and $\mathcal{K} = \ell^\infty_+$. The functional $f(x) = \sum_{n=1}^\infty x_n/2^n$ has the property that $f(x) > 0$ for $x \in \mathcal{K} \setminus \{0\}$. Hence, the set

$$\Theta = \{x \in \mathcal{K} : f(x) = 1\}$$

is a base of $\mathcal{K}$. It is unbounded since the sequence $(x_k) \subset \Theta$,

$$x_k = (0, \ldots, 0, \underbrace{2^k}_{k\text{th position}}, 0, \ldots),$$

is unbounded and the condition (2.3) is not satisfied. To see this take a sequence $(y_k) \subset \mathcal{K} \setminus W$, $W = \{x \in \ell^\infty : \sup_n |x_n| < 1\}$ and (q_k), where

$$y_k = \frac{1}{k} x_k, \quad \text{and} \quad q_k = \Big(0, \ldots, 0, \underbrace{\frac{1}{k}}_{k\text{th position}}, 0, \ldots\Big).$$

Now, $y_k + q_k \notin \operatorname{cone}(\Theta + V)$ for any 0-neighbourhood V contained in $\overline{V} = \{x \in \ell^\infty : \sup_n |x_n| < 1\}$, since

$$z_k = y_k + q_k = \frac{1}{k} x_k + q_k = \frac{1}{k} [x_k + p_k],$$

where $p_k = (0, \ldots, 0, \underbrace{1}_{k\text{th position}}, 0, \ldots)$. The main feature here is that y_k has the representation $y_k = \lambda_k \theta_k$ with (λ_k) tending to zero.

COROLLARY 2.1.1. *Let $\mathcal{K}$ be a closed convex cone with a topologically bounded base Θ in a locally convex space Y and let C be a subset of Y. The following conditions are equivalent:*

(i) $y \in SPE(C)$,
(ii) $y \in E_{\operatorname{cl} \mathcal{K}_d(V)}(C)$, *where V is a convex 0-neighbourhood.*

Proof. (ii)⇒(i). If $y \in E_{\operatorname{cl} \mathcal{K}_d(V)}(C)$, by Proposition 2.1.2, $\operatorname{cl} \mathcal{K}_d(V)$ satisfies condition (2.1), and hence $y \in SPE(C)$.

(i)⇒(ii). Let $y \in SPE(C)$. Then $y \in E_{\mathcal{K}_0}(C)$, where $\mathcal{K}_0$ satisfies (2.1). By Proposition 2.1.1, there exists a 0-neighbourhood V such that (2.2) holds, and hence $y \in E_{\mathrm{cl}\,\mathcal{K}_d(V)}(C)$. ∎

Let us note that in any locally convex space, for all sufficiently small neighbourhoods V, $\mathcal{K}_d(V)$ is pointed, which may not be the case for $\mathrm{cl}\,\mathcal{K}_d(V)$.

2.2. Strict efficiency

Let $\mathcal{K}$ be a closed convex pointed cone in a Hausdorff topological vector space Y. Let C be a subset of Y.

DEFINITION 2.2.1 ([17, 18]). A point $y_0 \in C$ is *strictly efficient*, $y_0 \in StE(C)$ (we write also $StE_{\mathcal{K}}(C)$), if for any 0-neighbourhood W there exists a 0-neighbourhood O such that

$$((C \setminus (y_0 + W)) + O) \cap (y_0 - \mathcal{K}) = \emptyset. \tag{2.4}$$

Equivalently

$$(C - y_0) \cap (O - \mathcal{K}) \subset W. \tag{2.5}$$

Each strictly efficient point is efficient,

$$StE(C) \subset E(C).$$

Indeed, if $y_0 \notin E(C)$, there exists $y \in C$, $y \neq y_0$, such that $y \in (C - y_0) \cap (-\mathcal{K})$. On the other hand, there exists a 0-neighbourhood $\overline{W}$ such that $y \notin y_0 + \overline{W}$. Hence $y_0 \notin StE(C)$.

If $\mathcal{K}_1 \subset \mathcal{K}$ for a closed convex cone $\mathcal{K}_1$, then $StE_{\mathcal{K}}(C) \subset StE_{\mathcal{K}_1}(C)$.

The following proposition establishes the relationship between strongly properly efficient points and strictly efficient points.

PROPOSITION 2.2.1. *For any subset C of Y we have*

$$SPE(C) \subset StE(C).$$

Proof. Let $y_0 \in SPE(C)$ and let W be a 0-neighbourhood. By (2.1), there exists a 0-neighbourhood O such that $(\mathcal{K} \setminus W) + O \subset \mathcal{K}_0$. Let W_1 be a 0-neighbourhood such that $W_1 + W_1 \subset W$. By O_1 we denote a 0-neighbourhood such that $(\mathcal{K} \setminus W_1) + O_1 \subset \mathcal{K}_0$.

We claim that $(C - y_0) \cap (O_1 \cap W_1 - \mathcal{K}) \subset W$. Indeed, take any $z \in (C - y_0) \cap (O_1 \cap W_1 - \mathcal{K})$. Hence,

$$z = y - y_0 = q - k, \quad \text{where} \quad y \in C,\, q \in O_1 \cap W_1,\, k \in \mathcal{K}.$$

If $z \notin W$, we would have $k \in \mathcal{K} \setminus W_1$ and by (2.1), $-k - q = y - y_0 \in -\mathcal{K}_0$, which would contradict the strong proper efficiency of y_0. This proves that $y_0 \in StE(C)$. ∎

Strict efficiency can be characterized via upper Hausdorff semicontinuity (for the definition see the beginning of Chapter 3) of the section mapping $Sec_C : Y \rightrightarrows Y$, $Sec_C(y) = C_y = C \cap (y - \mathcal{K})$ (cf. also Th. 2 and Corollaries 1 and 2 of [31]).

PROPOSITION 2.2.2. *Let $\mathcal{K}$ be a closed convex pointed cone in a Hausdorff topological vector space Y. Let C be a subset of Y. An element $y_0 \in E(C)$ is strictly efficient if and only if Sec_C is upper Hausdorff semicontinuous at y_0.*

Proof. It is enough to note that $Sec_c(y_0) = \{y_0\}$. Then the strict efficiency of y_0 can be equivalently rewritten as

$$Sec_C(y) \subset Sec_C(y_0) + W \quad \text{for any } y \in y_0 + O,$$

which amounts to the upper Hausdorff semicontinuity of Sec_C at y_0. ■

Recall that a cone $\mathcal{K}$ is *normal* in a topological vector space Y if there exists a basis $\mathcal{V}$ of neighbourhoods of Y such that $(O + K) \cap (O - \mathcal{K}) = O$ for any $O \in \mathcal{V}$.

PROPOSITION 2.2.3. *If $\mathcal{K}$ is normal, then $0 \in StE(\mathcal{K})$.*

Proof. Since $\mathcal{K}$ is normal, for each 0-neighbourhood W, there exists a 0-neighbourhood O such that $(O + \mathcal{K}) \cap (O - \mathcal{K}) \subset W$ and hence $\mathcal{K} \cap (O - \mathcal{K}) \subset W$. ■

The following proposition gives a characterization of strict efficiency in terms of nets.

PROPOSITION 2.2.4. *Let C be a subset of the space Y and $y_0 \in E(C)$. The following are equivalent:*

(i) $y_0 \in StE(C)$,

(ii) *for any nets (x_α), (y_α) such that $(x_\alpha) \subset C$, $y_\alpha \in x_\alpha + \mathcal{K}$ and $y_\alpha \to y_0$, we have $x_\alpha \to y_0$.*

Proof. Suppose on the contrary that there exist two nets (x_α), (y_α) such that $(x_\alpha) \subset C$, $y_\alpha \to y_0$, $x_\alpha \preceq_K y_\alpha$, and x_α does not tend to y_0. This means that there exists a 0-neighbourhood $\overline{W}$ such that for a certain subnet $(x_\beta) \subset (x_\alpha)$ we have $x_\beta - y_0 \notin \overline{W}$. On the other hand, $y_\beta = x_\beta + c_\beta$ for some $c_\beta \in \mathcal{K}$, or

$$x_\beta - y_0 = y_\beta - y_0 - c_\beta.$$

Since (y_β) tends to y_0, for each 0-neighbourhood V we have $y_\beta - y_0 \in V$ for $\beta \geq \beta_v$. Hence, (x_{β_v}) forms a subnet of (x_β) and $x_{\beta_v} - y_0 \in (C - y_0) \cap (V - \mathcal{K})$, but $x_{\beta_v} - y_0 \notin \overline{W}$, which contradicts the strict efficiency of y_0.

Suppose now that $y_0 \notin StE(C)$. There exists a 0-neighbourhood $\overline{W}$ such that for each 0-neighbourhood V one can find $x_v \in C$, $q_v \in V$, $c_v \in \mathcal{K}$ such that

$$x_v - y_0 = q_v - c_v,$$

where q_v tends to zero and $x_v - y_0 \notin \overline{W}$. Moreover, $x_v + c_v = q_v + y_0$, i.e., $x_v \preceq_K y_v = q_v + y_0$, and $\{y_v\}$ tends to y_0 but $\{x_v\}$ does not. This contradicts (ii). ■

By Propositions 2.2.3, 2.2.4 and Proposition 1.3 of [122] we get the following corollary.

COROLLARY 2.2.1. *$\mathcal{K}$ is normal if and only if $0 \in StE(\mathcal{K})$.*

Below we determine $StE(C)$ for C in some finite-dimensional and infinite-dimensional spaces.

EXAMPLE 2.2.1. 1. Let $Y = \mathbb{R}^2$ and $\mathcal{K} = \mathbb{R}^2_+$. Let

$$C = \{(y_1, y_2) : y_2 \geq e^{y_1}\} \cup \{(y_1, y_2) : y_2 \geq y_1\}.$$

Clearly, $E(C) = \{(y_1, y_2) : y_2 \geq y_1,\ y_1 \geq 0\}$ and $StE(C) = E(C)$. For

$$C = \{(y_1, y_2) : y_2 \geq e^{y_1}\} \cup \mathbb{R}^2_+$$

we get $E(C) = \{0\}$ and $StE(C) = \emptyset$.

2. Let $Y = \ell^\infty$, and $\mathcal{K} = \ell^\infty_+$ be the natural ordering cone, $\mathcal{K} = \{x = (x_n) \in \ell^\infty : x_n \geq 0,\ n \geq 1\}$. Let

$$C = \{x \in \ell^\infty : \|x\|_\infty \leq 1\}.$$

We have $y_0 = (-1, -1 \ldots, -1, \ldots) \in E(C)$ and $y_0 \in StE(C)$. To see the latter we need to show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y \in (C - y_0) \cap (Q - \mathcal{K})$, where $Q = \{q \in \ell^\infty : \|q\|_\infty < \delta\}$, we have $\|y\|_\infty < \varepsilon$. Indeed, let $y - y_0 = q - k$, where $y \in C$, $q \in Q$, $k \in \mathcal{K}$. Since $\|y_0 + q - k\|_\infty \leq 1$ we have $k^n \preceq q^n$ for all $n \geq 1$ and consequently

$$|q^n - k^n| \leq q^n + k^n \leq 2q^n,$$

which means that it is enough to take $\delta = \varepsilon/2$.

3. As previously, let $Y = \ell^\infty$ and $\mathcal{K} = \ell^\infty_+$. Let

$$C = \{x \in \ell^\infty : f(x) = 0\}$$

where f is the continuous linear functional $f(x) = \sum_{n=1}^\infty x_n/2^n$. The set C is a subspace, $E(C) = C$ and $StE(C) = \emptyset$. First we show that $0 \notin StE(C)$. Consider the sequence $(y_k) \subset C$ defined as

$$y_k = (1/k, 0, \ldots 0, \underbrace{-2^{k-1}/k}_{k\text{th position}}, 0, \ldots).$$

We have $y_k = q_k - c_k$, where

$$q_k = (1/k, 0, \ldots), \qquad c_k = (0, \ldots, 0, \underbrace{\ 2^{k-1}/k\ }_{k\text{th position}}, 0, \ldots) \in \mathcal{K},$$

and $\|q_k\|_\infty = 1/k$, $\|y_k\|_\infty = 2^{k-1}/k \geq 1$. According to Proposition 2.2.4, $0 \notin StE(C)$. To see that $y \notin StE(C)$ for any $y \in C$, consider the sequence $(z_k) \subset C$, $z_k = y_k + y$. It is enough to observe that $z_k - y = q_k - c_k$ and to apply Proposition 2.2.4.

The following theorem provides conditions for the inclusion $E(C) \subset StE(C)$ to hold.

THEOREM 2.2.1. *Let Y be a locally convex space and let $\mathcal{K}$ be a closed convex pointed cone in Y. If C is a weakly compact subset in Y, then*

$$E(C) \subset StE(C).$$

Proof. Let $y_0 \notin StE(C)$. There exists a 0-neighbourhood $\overline{W}$ such that for any 0-neighbourhood Q one can find $z_q \in C$, $z_q - y_0 \notin \overline{W}$ such that

$$z_q - y_0 = q - k_q, \quad \text{where} \quad q \in Q,\ k_q \in \mathcal{K}.$$

Since C is weakly compact, (z_q) contains a weakly convergent subnet with limit point $z_0 \in C$, $z_0 \neq y_0$. Since $\mathcal{K}$ is weakly closed, the corresponding subnet of (k_q) converges to a nonzero $k_0 \in \mathcal{K}$ and $z_0 - y_0 = -k_0$, which proves that $y_0 \notin E(C)$. ■

When $Y = (Y, \|\cdot\|)$ is a normed space with open unit ball B_Y, the strict efficiency can be rewritten as follows: $y_0 \in C$ is strictly efficient if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(C - y_0) \cap (\delta B_Y - \mathcal{K}) \subset \varepsilon B_Y.$$

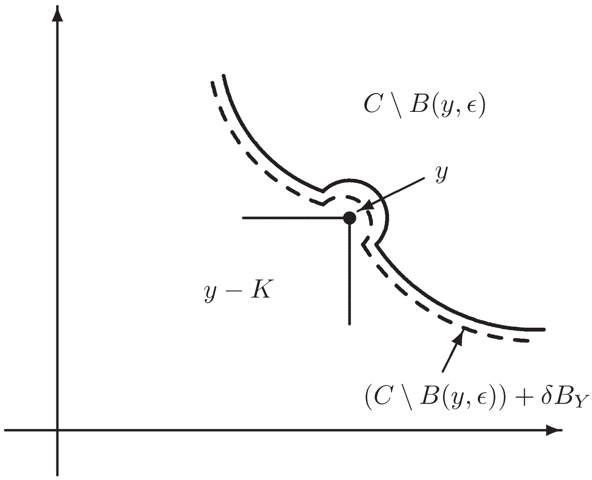

Fig. 2.1 Strict efficiency of $y \in C$

Now we establish the relationship between strict efficiency and proper Henig efficiency.

We say that $y_0 \in C$ is *proper Henig efficient*, [72], $y_0 \in HE(C)$, if there exists a closed convex cone $\Omega \subset Y$, $\Omega \neq Y$, $\mathcal{K} \setminus \{0\} \subset \operatorname{int} \Omega$ such that $y_0 \in E_\Omega(C)$.

THEOREM 2.2.2. *Let $Y = (Y, \|\cdot\|)$ be a normed space and let $\mathcal{K}$ be a closed convex and pointed cone in Y. For any subset C of Y,*

$$HE(C) \subset StE(C).$$

Proof. Suppose that $y_0 \notin StE(C)$. There exists $\varepsilon_0 > 0$ and sequences $(y_n) \subset C$, $(k_n) \subset \mathcal{K}$, $(b_n) \subset B_Y$ such that for all $n \geq 1$,

$$y_n - y_0 = \frac{1}{n} b_n - k_n, \qquad \|y_n - y_0\| > \varepsilon_0.$$

Hence, $d(y_n - y_0, -\mathcal{K}) \to 0$. Consequently, $y_0 \notin E_\Omega(C)$ for any cone $\Omega \subset Y$ with $\mathcal{K} \setminus \{0\} \subset \operatorname{int} \Omega$, which proves that $y_0 \notin HE(C)$. ■

In general, the inclusion $StE(C) \subset HE(C)$ does not hold as shown by the following example.

EXAMPLE 2.2.2. Let $Y = \mathbb{R}^2$ and $\mathcal{K} = \mathbb{R}^2_+$. For the set $C = \operatorname{cl} B_Y$ we have

$$E(C) = \{(y_1, y_2) : -1 \leq y_1 \leq 1,\ y_2 = -\sqrt{1 - y_1^2}\},$$

$E(C) = StE(C)$ and $HE(C) = E(C) \setminus \{(-1, 0), (0, -1)\}$.

We say that $y_0 \in C$ is *super efficient* [42], $y_0 \in SE(C)$, if there exists a number $M > 0$ such that

$$\operatorname{cl} \operatorname{cone}(C - y_0) \cap (B_Y - \mathcal{K}) \subset M B_Y.$$

THEOREM 2.2.3. *For any subset C of a normed space $(Y, \|\cdot\|)$,*

$$SE(C) \subset StE(C).$$

Proof. Suppose that $y_0 \notin StE(C)$. There exists $\varepsilon_0 > 0$ such that for each $n \geq 1$,

$$((C - y_0) \setminus \varepsilon_0 B_Y) \cap \left(\frac{1}{n} B_Y - \mathcal{K}\right) \neq \emptyset,$$

and one can choose $y_n \in C$ such that

$$y_n - y_0 = \frac{1}{n}(b_n - k_n), \qquad \|y_n - y_0\| > \varepsilon_0,$$

where $b_n \in B_Y$, $k_n \in \mathcal{K}$. Consequently,

$$n(y_n - y_0) = b_n - k_n \quad \text{and} \quad \|n(y_n - y_0)\| \to \infty,$$

which proves that $y_0 \notin SE(C)$. ■

THEOREM 2.2.4. *Let $(Y, \|\cdot\|)$ be a normed space and let $\mathcal{K}$ be a closed convex pointed cone in Y with a bounded base Θ. For any subset C of Y,*

$$SPE(C) = SE(C).$$

Proof. If $y_0 \in SPE(C)$, by Proposition 2.1.1, there exists $\varepsilon > 0$ such that

$$(C - y_0) \cap (-\mathcal{K}_d(\varepsilon)) = \{0\},$$

where, as previously, $\mathcal{K}_d(\varepsilon) = \operatorname{cone}(\Theta + \varepsilon B_Y)$. Thus, $\operatorname{cone}(C - y_0) \cap (\varepsilon B_Y - \Theta) = \emptyset$. Now, by the same arguments as those used in the proof of Proposition 3.4 of [42], we conclude that $y_0 \in StE(C)$.

Suppose now that $y_0 \notin SPE(C)$. By Proposition 2.1.1, for any $\varepsilon > 0$,

$$(C - y_0) \cap [-\operatorname{cone}(\Theta + \varepsilon B_Y)] \neq \emptyset.$$

Equivalently, $\operatorname{cone}(C - y_0) \cap (-\Theta + \varepsilon B_Y) \neq \emptyset$. By the same arguments as those used in the proof of Theorem 4.1 of [70], $y_0 \notin StE(C)$, which completes the proof. ■

Now we introduce local strict efficiency. Let $C \subset Y$ be a subset of a Hausdorff topological vector space Y.

DEFINITION 2.2.2. An element $y_0 \in C$ is a *local strictly efficient point*, $y_0 \in LStE(C)$, if there exists a 0-neighbourhood V in Y such that $y_0 \in StE(C \cap (y_0 + V))$, i.e., for each 0-neighbourhood W there exists a 0-neighbourhood O such that

$$(C \cap (y_0 + V) \setminus (y_0 + W)) \cap ((y_0 + O) - \mathcal{K}) = \emptyset.$$

Equivalently,

$$(C - y_0) \cap V \cap (O - \mathcal{K}) \subset W.$$

For instance, if

$$C = \{(y_1, y_2) : y_2 \geq e^{y_1}\} \cup \mathbb{R}^2_+$$

as in Example 2.2.1, then $E(C) = \{0\}$ and 0 is a local strictly efficient point.

Clearly,

$$StE(C) \subset LStE(C) \subset LE(C).$$

For the set $C \subset \mathbb{R}^2_+$,

$$C = \{(y_1, y_2) : 0 < y_1 \leq 1, \quad 0 \leq y_2 \leq 1\} \cup \{(0,1)\},$$

and $\mathcal{K} = \mathbb{R}^2_+$, we have $LE(C) = E(C) = \{(0,1)\}$, $LStE(C) = StE(C) = \emptyset$.

2.3. Strict efficiency for convex sets

Example 2.2.1 shows that $StE(C)$ may differ from $E(C)$. In some instances we can prove the equality $E(C) = StE(C)$ for convex sets C.

THEOREM 2.3.1. *Let $(Y, \|\cdot\|)$ be a normed space and let $\mathcal{K} \subset Y$ be a closed convex pointed cone with a weakly compact base Θ. Let C be a closed convex subset of Y. Then*

$$E(C) \subset StE(C).$$

Proof. Suppose that $y_0 \notin StE(C)$. There exist $\varepsilon_0 > 0$ and a sequence $(y_n) \subset C$ such that

$$y_n = y_0 + \frac{1}{n}\, b_n - \alpha_n \theta_n, \qquad \|y_n - y_0\| > \varepsilon_0 \quad \text{for } n \geq 1, \tag{2.6}$$

where $b_n \in B_Y$, $\theta_n \in \Theta$, and $\alpha_n > 0$. Since Θ is bounded we have

$$\|\theta\| \leq \varepsilon_0/2 \quad \text{for any } \theta \in \Theta.$$

Moreover, $\alpha_n \geq 1$ for all n sufficiently large since

$$\varepsilon_0 \leq \|y_n - y_0\| \leq \frac{1}{n}\, \|b_n\| + \alpha_n\, \frac{\varepsilon_0}{2} \leq \frac{\varepsilon_0}{2}\, (1 + \alpha_n)$$

for all n sufficiently large.

In view of the convexity of C, for $0 < \lambda_n = 1/\alpha_n \leq 1$ we get

$$z_n = \lambda_n y_n + (1 - \lambda_n) y_0 = y_0 + \lambda_n\, 1/n\, b_n - \theta_n \in C.$$

Without loosing generality we can assume that (θ_n) weakly converges to $0 \neq \theta_0 \in \Theta$ and consequently, (z_n) weakly converges to $z_0 = y_0 - \theta_0 \in C$, which contradicts the efficiency of y_0. ∎

In the infinite-dimensional case, weak compactness of the base Θ is a restrictive assumption. We can relax this assumption by imposing more restrictions on C.

We say that a closed convex subset C of a normed space Y is *uniformly rotund* (cf. e.g. Holmes [78, p. 162]) if there exists a nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, $\phi(0) = 0$, $\phi(t) > 0$ for $t > 0$ such that for any $y_1, y_2 \in C$ we have

$$\frac{1}{2}\, (y_1 + y_2) + \phi(\|y_1 - y_2\|) B_Y \subset C.$$

Then we can prove the following theorem.

THEOREM 2.3.2 (cf. [110]). *Let $\mathcal{K}$ be a closed convex pointed cone in a normed space Y. Let C be a uniformly rotund subset of Y. Then*

$$E(C) \subset StE(C).$$

Proof. By contradiction, suppose that there exists $y_0 \in E(C) \setminus StE(C)$. There exist $\varepsilon_0 > 0$ and a sequence $(y_n) \subset C$ such that for $n \geq 1$,

$$y_n = y_0 + q_n - k_n,$$

where $(q_n) \subset Y$, $q_n \to 0$, $(k_n) \subset \mathcal{K}$, $\|q_n - k_n\| > \varepsilon_0$. Then

$$d\left(\frac{1}{2}(y_n - y_0), -\mathcal{K}\right) \to 0$$

and

$$d\left(\frac{1}{2}(y_n - y_0), Y \setminus C\right) \to 0,$$

since $y_0 \in E(C)$, which contradicts the uniform rotundity of C. ■

As a consequence of Theorem 2.3.2, in the spaces L^p, $p \in (1, \infty)$, we have

$$E(\operatorname{cl} B_{L^p}) = StE(\operatorname{cl} B_{L^p}).$$

Corollary 2.3.1. *Let C be a closed convex subset of $\mathbb{R}^m$ and let $\mathcal{K}$ be a closed convex pointed cone in $\mathbb{R}^m$. Then $E(C) = StE(C)$.*

Proof. Follows from Proposition 2.3.1 since in finite-dimensional spaces any closed convex pointed cone has a compact base. ■

It is known that $E(C)$ is closed for closed convex subsets C of $\mathbb{R}^2$ and $\mathcal{K} = \mathbb{R}^2_+$. This is no longer true in $\mathbb{R}^3$. Hence, by Corollary 2.3.1, we deduce that $StE(C)$ may not be closed even when C is a closed and convex subset of $\mathbb{R}^3$.

Example 2.3.1 ([3]). Let $Y = \mathbb{R}^3$, $\mathcal{K} = \mathbb{R}^3_+$ and let $D \subset \mathbb{R}^3$,

$$D = \{(x, y, 1) : (x-1)^2 + (y-1)^2 = 1,\ 0 \leq x, y \leq 1\}.$$

Let $C = \operatorname{conv}(D \cup \{(1, 0, 0)\})$. The point $(1, 0, 1)$ is not efficient but $(1, 0, 1) \in \operatorname{cl} E(C)$.

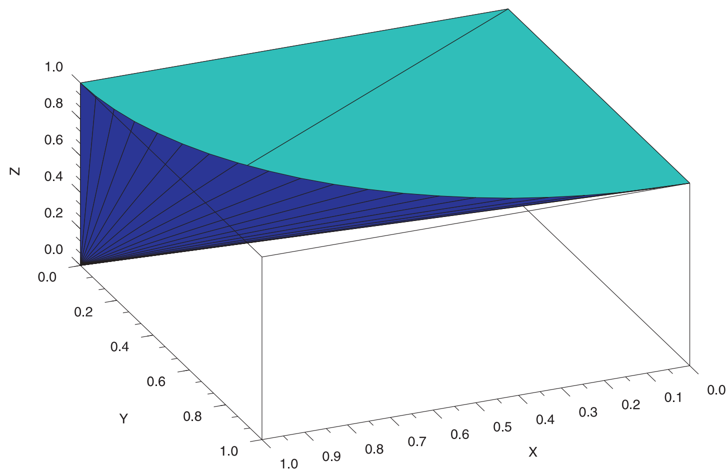

Fig. 2.2 The set C from Example 2.3.1

We close this section by showing that for convex sets C, the equality $LStE(C) = StE(C)$ holds.

Proposition 2.3.1. *Let $Y = (Y, \|\cdot\|)$ be a normed space with a closed convex pointed cone $\mathcal{K}$. If C is a convex subset of Y, then*

$$LStE(C) = StE(C).$$

Proof. We need to show that $LStE(C) \subset StE(C)$. Take any $y_0 \notin StE(C)$. By definition, there exist an $\varepsilon_0 > 0$ and $(y_n) \subset C$ such that

$$y_n - y_0 \in \frac{1}{n} B_Y - \mathcal{K}, \quad \|y_n - y_0\| > \varepsilon_0 \quad \text{for } n \geq 1.$$

Since C is convex, $z_n = y_0 + \lambda(y_n - y_0) \in C$ for any $0 \leq \lambda \leq 1$.

For any $0 \leq \lambda \leq 1$,

$$z_n - y_0 = \frac{\lambda \varepsilon_0}{\|y_n - y_0\|} (y_n - y_0) \in \frac{\lambda \varepsilon_0}{n} B_Y - \mathcal{K}.$$

Moreover, for any 0-neighbourhood V we get $z_n - y_0 = \lambda(y_n - y_0) \in (C - y_0) \cap V$ for $\lambda > 0$ small enough, which proves that $y_0 \notin LStE(C)$. ■

2.4. Modulus of strict efficiency

In this section $Y = (Y, \|\cdot\|)$ is a normed space with open unit ball B_Y and $\mathcal{K}$ is a closed convex pointed cone in Y.

Let C be a subset of Y. Recall that $y_0 \in StE(C)$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(C \setminus (y_0 + \varepsilon B_Y)) \cap ((y_0 + \delta B_Y) - \mathcal{K}) = \emptyset.$$

For any $y \in Y$ put

$$\|y\|_- = d(y, -\mathcal{K}),$$

where for any $y \in Y$ and any subset D of Y, $d(y, D) = \inf\{\|y - d\| : d \in D\}$. For any $r > 0$,

$$\|y\|_- \geq r \iff (y + rB_Y) \cap (-\mathcal{K}) = \emptyset.$$

Definition 2.4.1. Let C be a subset of Y and $y_0 \in C$. The function $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ defined as

$$\nu(\varepsilon) = \inf\{\|z - y_0\|_- : z \in C \setminus (y_0 + \varepsilon B_Y)\}.$$

is called *the modulus of strict efficiency of* y_0 with respect to C and $\mathcal{K}$.

A function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is *admissible* if ϕ is nondecreasing, $\phi(t) > 0$ for $t > 0$ and $\phi(0) = 0$.

Proposition 2.4.1 (cf. also [155]). *Let $\mathcal{K}$ be a closed convex pointed cone in a normed space $Y = (Y, \|\cdot\|)$. Let C be a subset of Y and let $y_0 \in C$ be a nonisolated point of C. Then $y_0 \in StE(C)$ if and only if*

$$\nu(\|y - y_0\|) \leq \|y - y_0\|_- \quad \textit{for } y \in C,$$

where $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ *is an admissible function of the form*

$$\nu(\varepsilon) = \inf\{\|z - y_0\|_- : z \in C \setminus (y_0 + \varepsilon B_Y)\}.$$

Proof. Clearly, ν is nondecreasing and $\nu(0) = 0$. Take any $y \in C$, $y \neq y_0$. Hence, $y \in C \setminus (y_0 + \varepsilon B_Y)$ for some $\varepsilon > 0$. By the strict efficiency of y_0, there exists $\delta > 0$ such that $y - y_0 \notin \delta B_Y - \mathcal{K}$. Hence,

$$0 < \delta \leq \nu(\varepsilon) \leq \nu(\|y - y_0\|) \leq \|y - y_0\|_-.$$

On the other hand, take any $\varepsilon > 0$ and $y \in C \setminus (y_0 + \varepsilon B_Y)$. Hence,

$$0 < \delta := \nu(\varepsilon) \leq \nu(\|y - y_0\|) \leq \|y - y_0\|_-,$$

which proves that $y_0 \in StE(C)$. ■

In what follows we shall consider strictly efficient points with some specific forms of ν. To stress the role of ν we say that $y_0 \in C$ is ν-strictly efficient and we write $y_0 \in StE^\nu(C)$. Hence, equivalently, $y_0 \in StE^\nu(C)$ if

$$(y - y_0) \cap (\nu(\|y - y_0\|)B_Y - \mathcal{K}) = \emptyset \quad \text{for } y \in C, \quad y \neq y_0.$$

In particular, an element $y_0 \in C$ is *strictly efficient of order* $q > 0$, $y_0 \in StE^q(C)$, if there exists a constant $\beta > 0$ such that $\nu(\cdot) = \beta(\cdot)^q$.

In Definition 2.2.2 we defined local strictly efficient points $y_0 \in LStE(C)$. Equivalently, y_0 is a local ν-strictly efficient point of C, $y_0 \in LStE^\nu(C)$, if and only if there exists a constant $t_s > 0$ such that

$$\nu(\|y - y_0\|) \leq \|y - y_0\|_- \quad \text{for } y \in C \cap (y_0 + t_s B_Y).$$

Or

$$y - y_0 \notin \nu(\|y - y_0\|)B_Y - \mathcal{K} \quad \text{for } y \in C \cap (y_0 + t_s B_Y), y \neq y_0.$$

Similarly, $y_0 \in C$ is a *local strictly efficient point of order* q, $y_0 \in LStE^q(C)$, if $y_0 \in LStE^\nu(C)$ with $\nu(\cdot) = \beta(\cdot)^q$ for some $\beta > 0$.

A $y_0 \in C$ is a *local proper Henig efficient point*, $y_0 \in LHE(C)$, if there exists a closed convex cone Ω, $\mathcal{K} \setminus \{0\} \subset \operatorname{int} \Omega$, such that $y_0 \in LE_\Omega(C)$.

Below we show that under some assumptions, local proper Henig efficient points coincide with local strictly efficient points of order 1.

Recall that a vector $d \in Y$ is *tangent* to the set C at $y_0 \in \operatorname{cl} C$ if there exist a sequence $(d_n) \subset Y$, $d_n \to d$, and a sequence $(t_n) \subset \mathbb{R}$, $t_n \downarrow 0$, such that $y_0 + t_n d_n \in C$. The cone $T_C(y_0)$ of all tangent vectors to C at y_0 is called the *Bouligand tangent cone.*

We start with the following characterization of local proper Henig efficient points.

PROPOSITION 2.4.2. *Let* Y *be a normed space and let* $\mathcal{K}$ *be a closed convex pointed cone in* Y *with a compact base* Θ. *Let* C *be a subset of* Y *and* $y_0 \in C$. *Then* $y_0 \in LHE(C)$ *if and only if*

$$T_C(y_0) \cap (-\mathcal{K}) = \{0\}.$$

Proof. Suppose that there exists a nonzero vector $d \in T_C(y_0) \cap (-\mathcal{K})$. There exist sequences $(d_n) \subset Y$, $d_n \to d$, and $(t_n) \subset \mathbb{R}_+$, $t_n \downarrow 0$, such that

$$y_0 + t_n d_n = y_n \in C.$$

Hence, for any 0-neighbourhood V in Y and any closed convex cone $\Omega \subset Y$ with $\mathcal{K}\setminus\{0\} \subset \operatorname{int}\Omega$, we get $t_n d_n \in \Omega$ for all n sufficiently large and

$$y_n \in (y_0 - \Omega) \cap (C \cap (y_0 + V)) \quad \text{for all } n \text{ sufficiently large.}$$

Conversely, suppose that $y_0 \notin LHE(C)$. For the closed convex cone $\Omega^n = \operatorname{cl}\operatorname{cone}(\Theta + \frac{1}{n}B_Y)$, $n \geq 1$, there exists $y_n \in C$ such that $y_n - y_0 \in \frac{1}{n}B_Y$ and $y_n \in y_0 - \Omega^n$. Hence,

$$y_n = y_0 - \lambda_n\left(\theta_n + \frac{1}{n}\, b_n\right), \qquad \text{where } \theta_n \in \Theta_n,\ b_n \in B_Y,\ \lambda_n > 0.$$

Since $y_n \to y_0$, we must have $\lambda_n \to 0$ and

$$\frac{1}{\lambda_n}\,(y_n - y_0) = -\theta_n - \frac{1}{n}\, b_n.$$

Without loss of generality we can assume that $\theta_n \to \theta \in \Theta$, $\theta \neq 0$. Consequently,

$$\frac{1}{\lambda_n}\,(y_n - y_0) = -\theta_n - \frac{1}{n}\, b_n \to -\theta$$

and $-\theta \in T_C(y_0) \cap (-\mathcal{K})$, which is a contradiction. ■

Now we are in a position to prove the following theorem.

THEOREM 2.4.1. *Let $\mathcal{K}$ be a closed convex pointed cone in a normed space Y with a compact base Θ. For any subset $C \subset Y$ we have*

$$LHE(C) = LStE^1(C).$$

Proof. By Proposition 2.4.2, it is enough to show that $y_0 \in LStE^1(C)$ if and only if $T_C(y_0) \cap (-\mathcal{K}) = \{0\}$.

By contradiction, suppose that there exists $d \in T_C(y_0) \cap (-\mathcal{K})$, $\|d\| = 1$. There exists a sequence $(y_n) \subset C$, $y_n \to y_0$, such that

$$\frac{y_n - y_0}{\|y_n - y_0\|} \to d$$

and hence, for any $c > 0$,

$$\frac{y_n - y_0}{\|y_n - y_0\|} \in d + cB_Y \qquad \text{for all } n \text{ sufficiently large.}$$

In other words,

$$y_n - y_0 \in \|y_n - y_0\| d + c\|y_n - y_0\| B_Y, \qquad \text{where } d \in -\mathcal{K},$$

i.e. $\|y_n - y_0\|_- < c\|y_n - y_0\|$, which means that $y_0 \notin LStE^1(C)$.

Suppose now that $y_0 \notin LStE^1(C)$. For each $n \geq 1$ there exists $y_n \in C \cap (y_0 + \frac{1}{n}B_Y)$, $y_n \neq y_0$, such that

$$y_n - y_0 = \frac{1}{n}\|y_n - y_0\| b_n - d_n, \qquad \text{where } b_n \in B_Y,\ d_n \in \mathcal{K}.$$

Moreover, for any $n \geq 1$ we have $d_n = \lambda_n \theta_n$ with $\lambda_n > 0$ and $\theta_n \in \Theta$. Clearly, $\lambda_n \to 0$. The sequence $(\lambda_n / \|y_n - y_0\|)$ is bounded since

$$\frac{y_n - y_0}{\|y_n - y_0\|} = \frac{1}{n}\, b_n - \frac{\lambda_n}{\|y_n - y_0\|}\,\theta_n$$

and without loosing generality we can assume that $\left(\frac{\lambda_n}{\|y_n-y_0\|}\theta_n\right) \to d \in \mathcal{K}$, $d \neq 0$. Hence,

$$\frac{y_n - y_0}{\|y_n - y_0\|} \to -d \in T_C(y_0) \cap (-\mathcal{K}). \ \blacksquare$$

As a corollary from Theorem 2.4.1 we obtain the following characterization of local strict efficiency of order 1.

COROLLARY 2.4.1. *Let Y be a normed space and let $\mathcal{K}$ be a closed convex pointed cone in Y with a compact base Θ. Let C be a subset of Y and $y_0 \in C$. Then $y_0 \in LStE^1(C)$ if and only if*

$$T_C(y_0) \cap (-\mathcal{K}) = \{0\}.$$

In finite-dimensional spaces, Corollary 2.4.1 takes the following form.

COROLLARY 2.4.2. *Let $\mathcal{K}$ be a closed convex pointed cone in $\mathbb{R}^m$. Let C be a subset of $\mathbb{R}^m$ and $y_0 \in C$. Then $y_0 \in LStE^1(C)$ if and only if*

$$T_C(y_0) \cap (-\mathcal{K}) = \{0\}.$$

In the example below we calculate moduli of strict efficiency for efficient points for the closed unit ball in $\mathbb{R}^2$.

EXAMPLE 2.4.1. Let $Y = \mathbb{R}^2$ with the Euclidean norm, $\mathcal{K} = \mathbb{R}^2_+$ and $C = \operatorname{cl} B_Y$. By Theorem 2.3.1, $E(C) = StE(C)$. For $\eta = (-1, 0) \in E(C)$ and any $y = (y_1, y_2) \in C$, $y \neq y_0$ we have

$$d(y-\eta, -\mathcal{K}) = \|y-\eta\|_- = \begin{cases} \|y-\eta\| & \text{for } y_2 \geq 0, \\ 1 + y_2 & \text{for } y_2 \leq 0. \end{cases}$$

Hence, $y_0 = (-1, 0) \in LStE^2(C)$ since for $y \in y_0 + B_Y$,

$$1 + y_1 = \frac{1}{2}(2 + 2y_1) \geq \frac{1}{2}((1 + y_1^2)^2 + y_2^2) = \frac{1}{2}\|y - (-1, 0)\|^2,$$

and

$$d(y - y_0, -\mathcal{K}) \geq \min\left\{\|y - y_0\|, \frac{1}{2}\|y - y_0\|^2\|\right\} = \frac{1}{2}\|y - y_0\|^2.$$

Analogously, $(0, -1) \in LStE^2(C)$. For other $\eta = (\eta_1, \eta_2) \in E(C)$, $\eta \neq (-1, 0)$, $\eta \neq (0, -1)$ by Theorem 2.4.1, $\eta \in LStE^1(C)$. Indeed, put $f(x) := -\sqrt{1 - x^2}$ for $0 < x < 1$. For any $z = (z_1, z_2) \in C$,

$$d(z - \eta, -\mathcal{K}) \geq \frac{1}{\sqrt{1 + (f'(\eta_1))^2}}\|z - \eta\| \quad \text{for } z_1 \geq \eta_1,\ z_2 \leq \eta_2$$

and

$$d(z - \eta, -\mathcal{K}) \geq \frac{1}{\sqrt{1 + (f'(\eta_2))^2}}\|z - \eta\| \quad \text{for } z_1 \leq \eta_1,\ z_2 \geq \eta_2.$$

Thus, $d(z - \eta, -\mathcal{K}) \geq \beta\|z - \eta\|$ for $\eta \neq (-1, 0)$, $\eta \neq (0, -1)$ with $\beta = 1/\sqrt{1 + \max\{(f'(\eta_1))^2, (f'(\eta_2))^2\}}$.

3. LOWER CONTINUITY OF EFFICIENT POINTS UNDER PERTURBATIONS OF A SET

The questions of lower semicontinuity of efficient points arise in many problems, for instance, in investigation of the solvability of vector variational inequalities and in duality theory. The results obtained in this chapter can be directly applied to stability of vector optimization problems.

In infinite-dimensional spaces, lower semicontinuity of efficient points was investigated by several authors, e.g., by Attouch and Riahi [5], Penot and Sterna-Karwat [121], the present author [18], and in finite-dimensional spaces by Gorokhovik and Rachkovski [69], Tanino, Nakayama and Sawaragi [148].

In finite-dimensional spaces, the key requirement which allows us to prove lower semicontinuity of efficient points under perturbations is the density of properly efficient points in the set of efficient points (see e.g. [69]). Under some additional assumptions, e.g. under convexity of the original set C, the density of properly (strictly) efficient points in the set of all efficient points is not needed for the lower semicontinuity of efficient points under perturbations (see the results below and e.g. [109]).

In Section 3.1 we prove our main results (Theorems 3.1.1 and 3.1.2) providing sufficient conditions for lower semicontinuity of efficient points under perturbations. The key requirement is the density of strictly efficient points defined in Chapter 2 in the set $E(C)$. In Theorem 3.1.4 we get rid of the above density requirement by assuming that 0 is a strictly efficient point of $\mathcal{K}$. In Section 3.2 we prove several variants of our main results for set-valued mappings taking values in normed spaces (Theorems 3.2.3, 3.2.2, 3.2.6).

There exist many ways of dealing with perturbations whenever they appear. We express perturbations by set-valued mapping $\mathcal{C} : U \rightrightarrows Y$ defined on a space of perturbations U. For any set-valued mapping we define its domain and graph as follows:

$$\operatorname{dom}\mathcal{C} = \{u \in U : \mathcal{C}(u) \neq \emptyset\}, \qquad \operatorname{graph}\mathcal{C} = \{(u, y) \in U \times Y : y \in \mathcal{C}(u)\}.$$

A set-valued mapping $\mathcal{C} : U \rightrightarrows Y$ is:

- *upper Hausdorff semicontinuous* at u_0 if for every 0-neighbourhood W in Y there exists a neighbourhood U_0 of u_0 such that $\mathcal{C}(u) \subset \mathcal{C}(u_0) + W$ for $u \in U_0$,
- *lower semicontinuous* at $(u_0, y_0) \in \operatorname{graph}\mathcal{C}$ if for any 0-neighbourhood W there exists a neighbourhood U_0 of u_0 such that $(y_0 + W) \cap \mathcal{C}(u) \neq \emptyset$ for all $u \in U_0$,
- *lower uniformly semicontinuous on a subset* $X_0 \subset \mathcal{C}(u_0)$ if for any 0-neighbourhood W there exists a neighbourhood U_0 of u_0 such that for every $x_0 \in X_0$ we have $(x_0 + W) \cap \mathcal{C}(u) \neq \emptyset$ for all $u \in U_0$,

- *lower semicontinuous* at u_0 if for any 0-neighbourhood W and any $y_0 \in \mathcal{C}(u_0)$ there exists a neighbourhood U_0 of u_0 such that $(y_0 + W) \cap \mathcal{C}(u) \neq \emptyset$ for all $u \in U_0$,
- *lower Hausdorff semicontinuous* at u_0 if it is uniformly lower continuous on $\mathcal{C}(u_0)$, i.e., for any 0-neighbourhood W there exists a neighbourhood U_0 of u_0 such that $\mathcal{C}(u) \subset \mathcal{C}(u_0) + W$ for all $u \in U_0$,
- *Hausdorff continuous* at u_0 if it is lower and upper Hausdorff continuous at u_0.

Following Nikodem [117] we define $\mathcal{K}$-Hausdorff semicontinuities. Let $\mathcal{C}_K : U \rightrightarrows Y$ be a set-valued mapping defined as

$$\mathcal{C}_K(u) = \mathcal{C}(u) + \mathcal{K}, \qquad u \in U.$$

We say that $\mathcal{C} : U \rightrightarrows Y$ is:

- $\mathcal{K}$-*upper Hausdorff semicontinuous* at u_0 if $\mathcal{C}_K$ is upper Hausdorff semicontinuous at u_0, i.e., for every 0-neighbourhood W there exists a neighbourhood U_0 of u_0 such that $\mathcal{C}(u) \subset \mathcal{C}(u_0) + W + \mathcal{K}$ for $u \in U_0$,
- $\mathcal{K}$-*lower Hausdorff semicontinuous* at u_0 if $\mathcal{C}_K$ is lower Hausdorff semicontinuous at u_0, i.e., for every 0-neighbourhood W there exists a neighbourhood U_0 of u_0 such that $\mathcal{C}(u_0) \subset \mathcal{C}(u) + W + \mathcal{K}$ for $u \in U_0$,
- $\mathcal{K}$-*lower semicontinuous* at u_0 (cf. [120]) if $\mathcal{C}_K$ is lower semicontinuous at u_0, i.e., for every $y_0 \in \mathcal{C}(u_0)$ and every 0-neighbourhood W there exists a neighbourhood U_0 of u_0 such that $\mathcal{C}(u) \cap (y_0 + W - \mathcal{K}) \neq \emptyset$ for $u \in U_0$.

Here we adopt the standard definitions of lower and upper semicontinuities as defined by Kuratowski [97]. In the context of vector optimization $\mathcal{K}$-semicontinuities of efficient points (C) under perturbation of C were investigated in [144], [120], [121].

Let X be a topological space. A function $f : X \to Y$ is $\mathcal{K}$-*lower continuous* at x_0 if for each 0-neighbourhood W in Y there exists a neighbourhood O of x_0 in X such that $f(x) \in f(x_0) + W + \mathcal{K}$ for all $x \in O$. Analogously, $f : X \to Y$ is $\mathcal{K}$-*upper continuous* at x_0 if for each 0-neighbourhood W in Y there exists a neighbourhood O of x_0 in X such that $f(x) \in f(x_0) + W - \mathcal{K}$ for all $x \in O$ (see also [72], [106]).

3.1. Sufficient conditions for lower semicontinuity of efficient points

In this section we give sufficient conditions for the lower semicontinuity of the efficient point set $E(C)$ when C is subjected to perturbations. We study properties of the *efficient point set-valued mapping* $\mathcal{E} : U \rightrightarrows Y$ defined as

$$\mathcal{E}(u) = E_{\mathcal{K}}(C(u)),$$

where perturbations of C are defined by a set-valued mapping $\mathcal{C} : U \rightrightarrows Y$, $\mathcal{C}(u) = C(u)$, $\mathcal{C}(u_0) = C$. For parametric vector optimization problems

$$(P_u) \qquad \begin{array}{l} \min_{\mathcal{K}} f(u,x) \\ \text{subject to } x \in A(u), \end{array}$$

the performance set-valued mapping $\mathcal{P}$ defined in Introduction is the efficient point set-valued mapping $\mathcal{E}$ with $C(u) = f(u, A(u))$. Recall that the *domination property* (*DP*) holds for C (cf. [105]) if

$$C \subset E(C) + \mathcal{K}.$$

In Chapter 5 we will discuss the domination property and its variants in a more detailed way.

THEOREM 3.1.1. *Let Y be a Hausdorff topological vector space and let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y. Let $u_0 \in \operatorname{dom} \mathcal{C}$ and let $y_0 \in E(C)$. If*

(i)

$$y_0 \in \operatorname{cl} StE(C), \tag{3.1}$$

(ii) *(DP) holds for all $C(u)$ in a certain neighbourhood U_0 of u_0,*

(iii) *$\mathcal{C}$ is $\mathcal{K}$-lower semicontinuous and upper Hausdorff semicontinuous at $u_0 \in \operatorname{dom} \mathcal{C}$,*

then $\mathcal{E}$ is lower semicontinuous at $(u_0, y_0) \in \operatorname{graph} \mathcal{E}$.

Proof. Note first that $u_0 \in \operatorname{int} \operatorname{dom} \mathcal{E}$. Indeed, since $C \neq \emptyset$ and $\mathcal{C}$ is $\mathcal{K}$-lower semicontinuous at $u_0 \in \operatorname{dom} \mathcal{C}$ we must have $C(u) \neq \emptyset$ for u in some neighbourhood U_1 of u_0 and hence by (*DP*), $E(C(u)) \neq \emptyset$ for $u \in U_1 \cap U_0$.

Let W be a 0-neighbourhood, and let W_1, W_2 be 0-neighbourhoods such that $W_1 + W_1 \subset W$ and $W_2 + W_2 \subset W_1$. By (3.1), there exists $y \in StE(C)$, $y \in y_0 + W_2$. By strict efficiency of y, there exists a 0-neighbourhood O such that $((C \setminus (y+W_2))+O) \cap (y-\mathcal{K}) = \emptyset$. Therefore,

$$((C \setminus (y + W_2)) + O_1) \cap (y + O_1 - \mathcal{K}) = \emptyset \tag{3.2}$$

for any 0-neighbourhood O_1 such that $O_1 + O_1 \subset O$.

Let $u \in U_0 \cap U_1$. By the $\mathcal{K}$-lower semicontinuity of $\mathcal{C}$, for each $u \in U_1$ there is $z \in C(u)$ satisfying

$$z \in (y + O_1 \cap W_2 - \mathcal{K}) \cap C(u).$$

Consequently, $z - \mathcal{K} \subset y + O_1 \cap W_2 - \mathcal{K}$ and in view of (3.2),

$$(z - \mathcal{K}) \cap ((C \setminus (y + W_2)) + O_1) = \emptyset.$$

By the upper Hausdorff semicontinuity of $\mathcal{C}$,

$$C(u) \subset C + O_1 \cap W_2 \subset ((C \setminus (y + W_2)) + O_1 \cap W_2) \cup (y + W_1).$$

Consequently,

$$(z - \mathcal{K}) \cap C(u) \subset y + W_1 \subset y_0 + W.$$

By (*DP*), there exists $\eta \in E(C(u))$ such that

$$\eta \in (z - \mathcal{K}) \cap C(u) \subset y_0 + W,$$

which completes the proof. ∎

Note that in the proof we use $\mathcal{K}$-lower semicontinuity of $\mathcal{C}$ only in the vicinity of y_0. Moreover, (ii) can be replaced by a slightly weaker condition

(ii)′ $C(u) \subset \operatorname{cl} E(C(u)) + \mathcal{K}$ for all $u \in U_0$.

THEOREM 3.1.2. *Let $\mathcal{K}$ be a closed convex pointed cone in Y and $u_0 \in \operatorname{dom}\mathcal{C}$. Assume that*

$$E(C) \subset \operatorname{cl} StE(C), \tag{3.3}$$

and (DP) holds for all $C(u)$ in a certain neighbourhood U_0 of u_0. If $\mathcal{C}$ is $\mathcal{K}$-lower semicontinuous at u_0 and upper Hausdorff semicontinuous at u_0, then $\mathcal{E}$ is lower semicontinuous at $u_0 \in \operatorname{dom}\mathcal{E}$.

In view of Proposition 2.2.1, by Theorem 3.1.2, we obtain the following result which generalizes Theorem 3.1 of [16].

THEOREM 3.1.3. *Let $\mathcal{K}$ be a closed convex pointed cone in Y and $u_0 \in \operatorname{dom}\mathcal{C}$. If*

$$E(C) \subset \operatorname{cl} SPE(C), \tag{3.4}$$

$\mathcal{C}$ is upper Hausdorff semicontinuous at u_0 and $\mathcal{K}$-lower semicontinuous at u_0 and (DP) holds for all $C(u)$ in some neighbourhood of u_0, then $\mathcal{E}$ is lower semicontinuous at $u_0 \in \operatorname{dom}\mathcal{E}$.

Sufficient conditions for lower semicontinuity of efficient points can also be given by assuming that 0 is a strictly efficient point of $\mathcal{K}$, which, by Corollary 2.2.1, amounts to saying that $\mathcal{K}$ is normal. We have the following result.

THEOREM 3.1.4. *Let $\mathcal{K} \subset Y$ be a closed convex normal cone in Y. Assume that C is closed, $\operatorname{cl} E(C)$ is compact, and (DP) holds for all $\mathcal{C}(u)$ in a certain neighbourhood U_0 of $u_0 \in \operatorname{dom}\mathcal{C}$. If $\mathcal{C}$ is $\mathcal{K}$-lower semicontinuous and upper Hausdorff semicontinuous at u_0, then $\mathcal{E}$ is lower semicontinuous at $u_0 \in \operatorname{dom}\mathcal{E}$.*

Proof. Let $y_0 \in E(C)$. We start by showing that, under our assumptions, for any 0-neighbourhood W there exists a 0-neighbourhood V such that

$$(((E(C) + \mathcal{K}) \setminus (y_0 + W)) + V) \cap (y_0 - \mathcal{K}) = \emptyset. \tag{3.5}$$

To see this, suppose on the contrary that there exists a 0-neighbourhood W such that for any 0-neighbourhood V there exists $v \in V$ such that

$$y_0 - k_v = \eta_v + k_v^1 + q_v = z_v + q_v,$$

where $k_v, k_v^1 \in \mathcal{K}$, $\eta_v \in E(C)$, $z_v = \eta_v + k_v^1 \notin y_0 + W$, and the net (q_v) tends to 0. Since $\operatorname{cl} E(C)$ is compact, the net (η_v) contains a convergent subnet. Without loss of generality we may assume that the net itself converges to a certain $\eta \in C(u)$. Consequently,

$$y_0 - \eta = \lim_v (k_v + k_v^1), \tag{3.6}$$

and, since $\mathcal{K}$ is closed, $y_0 - \eta \in \mathcal{K}$, which implies that $y_0 = \eta$. By (3.6), $\lim_v (k_v + k_v^1) = 0$, and, since $\mathcal{K}$ is normal, by Proposition 1.3, p. 62 of [122], (k_v) and (k_v^1) both tend to zero. By taking any 0-neighbourhood W_1 such that $W_1 + W_1 \subset W$, one can find a 0-neighbourhood V_0 such that for all $V \subset V_0$ we have $\eta_v + k_v^1 \subset \eta + W_1 + W_1 \subset y_0 + W$, which contradicts the assumption that $\eta_v + k_v^1 \notin y_0 + W$. This proves (3.5).

Let W_1 be a 0-neighbourhood such that $W_1 + W_1 \subset W$. By (3.5), there exists a 0-neighbourhood V_1 such that for any 0-neighbourhood V_2, $V_2 + V_2 \subset V_1$, we have

$$(((E(C) + \mathcal{K}) \setminus (y_0 + W_1)) + V_2) \cap ((y_0 + V_2) - \mathcal{K}) = \emptyset.$$

On the other hand, since (DP) holds for C,

$$C + V_2 \cap W_1 \subset (((E(C) + \mathcal{K}) \setminus (y_0 + W_1)) + V_2 \cap W_1) \cup (y_0 + W).$$

There exists a neighbourhood U_1 of u_0 such that

$$C(u) \subset (((E(C) + \mathcal{K}) \setminus (y_0 + W_1)) + V_2 \cap W_1) \cup (y_0 + W) \tag{3.7}$$

for $u \in U_1$. Moreover, there exists a neighbourhood U_2 of u_0 such that

$$(y_0 + V_2 \cap W_1 - \mathcal{K}) \cap C(u) \neq \emptyset,$$

for $u \in U_2$. Hence, for $u \in U_2$ there exists $y_u \in C(u) \cap (y_0 + V_2 \cap W_1 - \mathcal{K})$ and

$$y_u - \mathcal{K} \subset y_0 + V_2 \cap W_1 - \mathcal{K}.$$

Since $y_u \in V_2 \cap W_1 \subset V_2$, by (3.5),

$$(y_u - \mathcal{K}) \cap [((E(C) + \mathcal{K}) \setminus (y_0 + W_1)) + V_2 \cap W_1] = \emptyset.$$

By (3.7) and by (DP), for $u \in U_0 \cap U_1 \cap U_2$ there exists $\eta_u \in E(C(u))$ such that

$$\eta_u \in (y_u - \mathcal{K}) \cap C(u) \subset (y_0 + W). \tag{3.8}$$

This completes the proof. ■

In view of Theorems 1.2.1 and 2.2.1 we obtain the following variant of Theorem 3.1.2.

THEOREM 3.1.5. *Let Y be a locally convex space and let $\mathcal{K}$ be a closed convex pointed cone in Y. Assume that there exists a neighbourhood U_0 of u_0 such that all $C(u)$ are nonempty and weakly compact for $u \in U_0$. If $\mathcal{C}$ is upper Hausdorff semicontinuous and $\mathcal{K}$-lower semicontinuous at $u_0 \in \operatorname{dom} \mathcal{C}$, then $\mathcal{E}$ is lower semicontinuous at $u_0 \in \operatorname{dom} \mathcal{E}$.*

Proof. It is enough to note that by Theorem 1.2.1, (DP) holds for all $C(u)$, $u \in U_0$. ■

3.2. Lower semicontinuity of efficient points in normed spaces

Let $Y = (Y, \|\cdot\|)$ be a real normed linear space with open unit ball B_Y.

DEFINITION 3.2.1 ([92], [93]). We say that a cone $\mathcal{K} \subset Y$ *allows plastering* $\mathcal{K}_0$, where $\mathcal{K}_0$ is another closed convex pointed cone, if there exists a constant $\delta > 0$ such that for each $k \in \mathcal{K}$,

$$k + \delta \|k\| B_Y \subset \mathcal{K}_0.$$

PROPOSITION 3.2.1. *Let $\mathcal{K}$ be a closed convex pointed cone in Y. The following are equivalent:*

(i) *there exists a closed convex pointed cone $\mathcal{K}_0$ satisfying condition* (2.1),
(ii) *$\mathcal{K}$ allows plastering $\mathcal{K}_0$,*
(iii) *$\mathcal{K}$ has a bounded base.*

Proof. (i)⇔(ii). If $\mathcal{K}$ allows plastering $\mathcal{K}_0$, then $\operatorname{int} \mathcal{K}_0 \neq \emptyset$, $\mathcal{K} \setminus \{0\} \subset \operatorname{int} \mathcal{K}_0$. For any $\varepsilon > 0$ and any $k \in \mathcal{K}$ with $\|k\| \geq \varepsilon$ we have $k + \delta\varepsilon B_Y \subset \mathcal{K}_0$ and $\mathcal{K}_0$ satisfies condition (2.1).

Suppose now that $\mathcal{K}_0$ satisfies condition (2.1). There exists $\delta > 0$ such that for $k \in \mathcal{K}$, $\|k\| \geq 1$, we have

$$k + \delta B_Y \subset \mathcal{K}_0.$$

Hence, for any $k \in \mathcal{K}$, $k/\|k\| + \delta B_Y \subset \mathcal{K}_0$ and consequently, $k + b\|k\| B_Y \subset \mathcal{K}_0$, which means that $\mathcal{K}$ allows plastering $\mathcal{K}_0$.

(ii)⇒(iii). Suppose that $\mathcal{K}$ allows plastering $\mathcal{K}_0$. This means that there exists a continuous linear functional $f \in \mathcal{K}_0^+$ which is strictly uniformly positive on $\mathcal{K}$, i.e. there exists $\delta > 0$ such that

$$f(x) \geq \delta\|x\| \quad \text{for } x \in \mathcal{K}.$$

The set $\Theta = \{x \in \mathcal{K} : f(x) = 1\}$ is clearly bounded, closed and convex, $0 \notin \Theta$, and $\mathcal{K} = \mathrm{cone}(\Theta)$.

(iii)⇒(ii). For the proof of this part see Krasnosel'skiĭ [92]. ■

Let $\mathcal{K}_\alpha$ be a *Bishop–Phelps cone*, i.e.,

$$\mathcal{K}_\alpha = \{y \in Y : f(y) \geq \alpha\|y\|\|f\|\},$$

where f is a continuous linear functional on Y and $0 < \alpha < 1$. This is a closed convex pointed cone. If it is nontrivial, then $\mathcal{K}_\alpha$ has a bounded base

$$\Theta = \{z \in \mathcal{K} : f(z) = 1\}.$$

The following holds true.

PROPOSITION 3.2.2. *Let Y be a normed space, C a nonempty subset of Y and $y_0 \in E_{\mathcal{K}_\alpha}(C)$. If there exists $\beta < \alpha$ such that $y_0 \in E_{\mathcal{K}_\beta}(C)$, then $y_0 \in SPE_{\mathcal{K}_\alpha}(C)$.*

Proof. By Proposition 3.2.1, the cone $\mathcal{K}_\beta$ satisfies condition (2.1). Moreover, for $z \in \mathcal{K}_\alpha$, $\|z\| \geq \varepsilon$, and any $y \in Y$ we have

$$\begin{aligned} f(z+y) = f(z) + f(y) &\geq \alpha\|f\| \cdot \|z\| + f(y) \\ &\geq \alpha\|z+y\| \cdot \|f\| - \alpha\|f\| \cdot \|y\| - \|f\| \cdot \|y\| \\ &\geq \|f\| \cdot \|z+y\| \left[\alpha - \frac{(\alpha+1)\|y\|}{\varepsilon - \|y\|}\right]. \end{aligned}$$

To have $\alpha - (\alpha+1)\|y\|/(\varepsilon - \|y\|) > \beta$ we choose

$$\|y\| < \frac{(\alpha-\beta)\varepsilon}{2\alpha+1-\beta}. \quad ■$$

By Proposition 3.2.2, $\mathcal{K}_\alpha$ allows plastering $\mathcal{K}_\beta$, $\beta < \alpha$, $b = (\alpha-\beta)/(2\alpha+1-\beta)$.

For Bishop–Phelps cones, the following well known result [125] gives sufficient conditions for the domination property to hold.

THEOREM 3.2.1. *Let Y be a Banach space and C a nonempty closed subset of Y. If there exists a functional f on Y such that $\inf f(C) > -\infty$, then for any $y \in C$ there exists $y_0 \in C$ such that $y_0 \in y - \mathcal{K}_\alpha$ and $y_0 \in E(C)$.*

By Theorem 3.2.1 and Proposition 3.2.2 we obtain the following stability result.

THEOREM 3.2.2. *Let Y be a Banach space and $C \neq \emptyset$. Assume that there exists a neighbourhood U_0 of u_0 such that all the sets $C(u)$ are closed and $\inf_{y\in C(u)} f(y) > -\infty$. If*

$$E_{\mathcal{K}_\alpha}(C) \subset \mathrm{cl}\Big(\bigcup_{\beta<\alpha} E_{\mathcal{K}_\beta}(C)\Big), \tag{3.9}$$

and $\mathcal{C}$ *is* $\mathcal{K}_\alpha$*-lower semicontinuous and upper Hausdorff semicontinuous at* $u_0 \in \operatorname{dom} \mathcal{C}$, *then* $\mathcal{E}$ *is lower semicontinuous at* $u_0 \in \operatorname{dom} \mathcal{E}$.

Proof. Follows from Theorem 3.2.1, Proposition 3.2.2, and Theorem 3.1.3. ∎

Theorem 3.2.2 can be viewed as a variant of the stability result proved in [5].

In normed spaces we have the following variant of Theorem 3.1.3.

THEOREM 3.2.3. *Let* Y *be a normed space and* $\mathcal{K}$ *a closed convex pointed cone in* Y. *Let* $u_0 \in \operatorname{dom} \mathcal{C}$ *and* $y_0 \in E(C)$. *Suppose that*

$$y_0 \in \operatorname{cl} SE(C), \tag{3.10}$$

and (*DP*) *holds for all* $C(u)$ *in a certain neighbourhood* U_0 *of* u_0. *If* $\mathcal{C}$ *is* $\mathcal{K}$*-lower semicontinuous at* $(u_0, y_0) \in \operatorname{graph} \mathcal{C}$ *and upper Hausdorff semicontinuous at* u_0, *then* $\mathcal{E}$ *is lower semicontinuous at* $(u_0, y_0) \in \operatorname{graph} \mathcal{E}$.

Proof. By Theorem 2.2.3, each super efficient point is strictly efficient, and by Theorem 3.1.1, the assertion follows. ∎

Conditions (3.1) of Theorem 3.1.1, (3.4) of Theorem 3.1.3 and (3.10) of Theorem 3.2.3 are density type requirements. The density property has been investigated on different levels of generality and for different notions of proper minimality (e.g., [42], [46], [123], [82]). Here we make use of the result of Borwein and Zhuang [42].

We say that a subset C of Y is $\mathcal{K}$*-lower bounded* if there is a constant $M > 0$ such that

$$C \subset MB_Y + \mathcal{K}.$$

A subset $C \subset Y$ is $\mathcal{K}$-lower bounded if either it is topologically bounded, i.e., $C \subset MB_Y$ for some positive constant $M > 0$, or there exists an element $z_0 \in Y$ such that $y - z_0 \in \mathcal{K}$ for all $y \in C$.

THEOREM 3.2.4 (Borwein and Zhuang [42]). *Let* Y *be a Banach space,* $\mathcal{K} \subset Y$ *a closed convex pointed cone and* $C \subset Y$ *a nonempty subset. Assume that* $\mathcal{K}$ *has a closed and bounded base* Θ. *If either of the following conditions is satisfied, then* $SE(C)$ *is norm-dense in the nonempty set* $E(C)$:

(i) C *is weakly compact,*
(ii) C *is weakly closed and* $\mathcal{K}$*-lower bounded while* Θ *is weakly compact.*

For convex sets condition (ii) follows from the condition

(ii)′ C is convex and closed and $\mathcal{K}$-lower bounded while Θ is weakly compact.

By Theorems 3.2.4 and 3.1.2 we obtain the following result.

THEOREM 3.2.5. *Let* Y *be a Banach space and let* $\mathcal{K}$ *be a closed convex pointed cone in* Y. *Assume that* $\mathcal{K}$ *has a closed and bounded base* Θ. *Let* $\mathcal{C}$ *be upper Hausdorff semicontinuous and* $\mathcal{K}$*-lower semicontinuous at* $u_0 \in \operatorname{dom} \mathcal{C}$ *and suppose* (*DP*) *holds for all* $C(u)$ *in a certain neighbourhood of* u_0. *If either of the following conditions is satisfied, then* $\mathcal{E}$ *is lower semicontinuous at* $u_0 \in \operatorname{dom} \mathcal{E}$:

(i) C *is weakly compact,*
(ii) C *is weakly closed and* $\mathcal{K}$*-lower bounded while* Θ *is weakly compact.*

In view of Theorems 2.3.1 and 2.3.2, we obtain the following results.

THEOREM 3.2.6. *Let $\mathcal{K}$ be a closed convex cone with a weakly compact base in a normed space Y. Let $\mathcal{C}$ be upper Hausdorff semicontinuous and $\mathcal{K}$-lower semicontinuous at $u_0 \in \operatorname{dom}\mathcal{C}$. If C is closed and convex and (DP) holds for all $C(u)$ in a certain neighbourhood of u_0, then $\mathcal{E}$ is lower semicontinuous at $u_0 \in \operatorname{dom}\mathcal{E}$.*

THEOREM 3.2.7. *Let $\mathcal{K}$ be a closed convex pointed cone in a normed space Y. Let $\mathcal{C}$ be upper Hausdorff semicontinuous and $\mathcal{K}$-lower semicontinuous at $u_0 \in \operatorname{dom}\mathcal{C}$. If C is uniformly rotund and (DP) holds for all $C(u)$ in a certain neighbourhood of u_0, then $\mathcal{E}$ is lower semicontinuous at $u_0 \in \operatorname{dom}\mathcal{E}$.*

We close this section with sufficient conditions for lower Hausdorff semicontinuity of the efficient point set-valued mapping in which we exploit the (global) modulus of minimality.

DEFINITION 3.2.2. The function $\operatorname{mod} : \mathbb{R}_+ \to \mathbb{R}_+$ defined as

$$\operatorname{mod}(\varepsilon) = \inf\{\nu_\eta(\varepsilon) : \eta \in E(C)\}$$

is called the *modulus of strict efficiency of C.*

We have

$$\operatorname{mod}(\varepsilon) = \inf\{\|z-\eta\|_- : z \in C \setminus B(E(C),\varepsilon),\ \eta \in E(C)\}.$$

THEOREM 3.2.8. *Let Y be a normed space and let $\mathcal{K}$ be a closed convex pointed cone in Y. Assume that $\mathcal{C} : U \rightrightarrows Y$ is a set-valued mapping defined on a normed space U and $u_0 \in \operatorname{dom}\mathcal{C}$. If*

(i) $\operatorname{mod}_C(\varepsilon) > 0$,
(ii) *(DP) holds for all $C(u)$ in some neighbourhood U_1 of u_0,*
(iii) *$\mathcal{C}$ is Hausdorff continuous at $u_0 \in \operatorname{dom}\mathcal{C}$,*

then $\mathcal{E}$ is lower Hausdorff semicontinuous at u_0.

Proof. Fix any $\varepsilon > 0$, and $y \in E(C)$. By Proposition 2.4.1, $y \in StE(C)$, and

$$((C \setminus (y + \tfrac{1}{2}\varepsilon B_Y)) + \operatorname{mod}(\tfrac{1}{2}\varepsilon)B_Y) \cap (y - \mathcal{K}) = \emptyset.$$

Let $r(\varepsilon) = \min\{\operatorname{mod}(\varepsilon), \frac{1}{2}\varepsilon\}$. Hence,

$$((C \setminus (y + \tfrac{1}{2}\varepsilon B_Y)) + \tfrac{1}{2}r(\tfrac{1}{2}\varepsilon)B_Y) \cap (y + \tfrac{1}{2}r(\tfrac{1}{2}\varepsilon)B_Y - \mathcal{K}) = \emptyset. \tag{3.11}$$

By the upper Hausdorff semicontinuity of $\mathcal{C}$, for $u \in U_0$,

$$\begin{aligned} C(u) &\subset C + \tfrac{1}{2}r(\tfrac{1}{2}\varepsilon)B_Y \\ &\subset ((C \setminus (y + \tfrac{1}{2}\varepsilon B_Y)) + \tfrac{1}{2}r(\tfrac{1}{2}\varepsilon)B_Y) \cup (y + (\tfrac{1}{2}r(\tfrac{1}{2}\varepsilon) + \tfrac{1}{2}\varepsilon)B_Y)), \end{aligned} \tag{3.12}$$

and by the lower Hausdorff semicontinuity of $\mathcal{C}$, for $u \in U_2$ there exists $y_1 \in C(u)$ such that

$$y_1 \in y + \tfrac{1}{2}r(\tfrac{1}{2}\varepsilon)B_Y, \qquad y_1 - \mathcal{K} \subset y + \tfrac{1}{2}r(\tfrac{1}{2}\varepsilon)B_Y - \mathcal{K}.$$

By (3.11),

$$(y_1 - \mathcal{K}) \cap ((C(u) \setminus (y + \tfrac{1}{2}\varepsilon \cdot B_Y)) + \tfrac{1}{2}r(\tfrac{1}{2}\varepsilon) \cdot B_Y) = \emptyset.$$

Now, by (3.12), for $u \in U_2$,

$$(y_1 - \mathcal{K}) \cap C(u) \subset y + (\tfrac{1}{2}r(\tfrac{1}{2}\varepsilon) + \tfrac{1}{2}\varepsilon)B_Y.$$

Since (DP) holds for $C(u)$, for $u \in U_1$ there exists $\eta_1 \in \mathcal{E}(u)$, $u \in U_1 \cap U_2$, such that

$$\eta_1 \subset (y_1 - \mathcal{K}) \cap C(u) \subset y + (\tfrac{1}{2}r(\tfrac{1}{2}\varepsilon) + \tfrac{1}{2}\varepsilon)B_Y,$$

and since $r(\frac{1}{2}\varepsilon) \leq \frac{1}{4}\varepsilon$,

$$\eta_1 \in y + \tfrac{5}{8}\varepsilon B_Y \subset y + \varepsilon B_Y.$$

This means that $E(C) \subset \mathcal{E}(u) + \varepsilon B_Y$ for $u \in U_1 \cap U_2$, which completes the proof. ∎

4. LOWER HÖLDER CONTINUITY OF EFFICIENT POINTS UNDER PERTURBATIONS OF A SET

In this chapter we formulate sufficient conditions for lower Hölder continuity and lower pseudo-Hölder continuity of $\mathcal{E}$ at $u_0 \in \operatorname{dom} \mathcal{E}$ and at $(u_0, y_0) \in \operatorname{graph} \mathcal{E}$, respectively. Based on an auxiliary proposition we also derive criteria for Hölder continuity and pseudo-Hölder continuity of $\mathcal{E}$.

Recall that $\mathcal{C} : U \rightrightarrows Y$ is a set-valued mapping, $\mathcal{C}(u_0) = C$ and $\mathcal{C}(u) = C(u)$ and $\mathcal{E} : U \rightrightarrows Y$ is the efficient point set-valued mapping, $\mathcal{E}(u_0) = E(C)$ and $\mathcal{E}(u) = E(C(u))$.

Let $U = (U, \|\cdot\|)$ and $Y = (Y, \|\cdot\|)$ be normed spaces with open unit balls B_U and B_Y, respectively. We say that a set-valued mapping $\mathcal{C} : U \rightrightarrows Y$ is:

- *upper Hölder continuous of order* $q > 0$ *at* $u_0 \in \operatorname{dom} \mathcal{C}$ with constants $L > 0$ and $t > 0$ if
$$\mathcal{C}(u) \subset \mathcal{C}(u_0) + L\|u - u_0\|^q B_Y \quad \text{for } u \in u_0 + tB_U,$$
- *lower Hölder continuous of order* $q > 0$ *at* $u_0 \in \operatorname{dom} \mathcal{C}$ with constants $L > 0$ and $t > 0$ if
$$\mathcal{C}(u_0) \subset \mathcal{C}(u) + L\|u - u_0\|^q B_Y \quad \text{for } u \in u_0 + tB_U,$$
- *Hölder continuous of order* $q > 0$ at $u_0 \in \operatorname{dom} \mathcal{C}$ if it is upper and lower Hölder continuous of order q at u_0,
- *Hölder continuous of order* $q > 0$ *around* $u_0 \in \operatorname{dom} \mathcal{C}$ with constants $L > 0$ and $t > 0$ if
$$\mathcal{C}(u') \subset \mathcal{C}(u) + L\|u' - u\|^q B_Y \quad \text{for } u', u \in u_0 + tB_U,$$
- *upper pseudo-Hölder* (or *Hölder calm*) *of order* $q > 0$ *at* $(u_0, y_0) \in \operatorname{graph} \mathcal{C}$ with 0-neighbourhood V_0 and positive constants $L > 0$, $t > 0$ if
$$\mathcal{C}(u) \cap V_0 \subset \mathcal{C}(u_0) + L\|u - u_0\|^q B_Y \quad \text{for } u \in u_0 + tB_U,$$
- *lower pseudo-Hölder of order* $q > 0$ *at* $(u_0, y_0) \in \operatorname{graph} \mathcal{C}$ with 0-neighbourhood V_0 and positive constants $L > 0$, $t > 0$ if
$$\mathcal{C}(u_0) \cap V_0 \subset \mathcal{C}(u) + L\|u - u_0\|^q B_Y \quad \text{for } u \in u_0 + tB_U,$$
- *pseudo-Hölder of order* $q > 0$ *at* $(u_0, y_0) \in \operatorname{graph} \mathcal{C}$ with 0-neighbourhood V_0 and positive constants $L > 0$, $t > 0$ if it is upper and lower pseudo-Hölder $(u_0, y_0) \in \operatorname{graph} \mathcal{C}$ with 0-neighbourhood V_0 and positive constants $L > 0$, $t > 0$,
- *pseudo-Hölder of order* $q > 0$ *around* $(u_0, y_0) \in \operatorname{graph} \mathcal{C}$ with 0-neighbourhood V_0 and positive constants $L > 0$, $t > 0$ if
$$\mathcal{C}(u') \cap V_0 \subset \mathcal{C}(u) + L\|u' - u\|^q B_Y \quad \text{for } u', u \in u_0 + tB_U.$$

We say that any of the above properties holds for $\mathcal{C}$ *in the sense of Lipschitz* if it holds in the sense of Hölder with $q = 1$. Pseudo-Lipschitzness around $(u_0, y_0) \in \operatorname{graph} \mathcal{C}$ was introduced in [11]. Upper Lipschizness was introduced in [128, 130, 131]. Clearly, if $\mathcal{C}$ is Hölder continuous around $u_0 \in \operatorname{dom} \mathcal{C}$, then $\mathcal{C}$ is upper and lower Hölder continuous at u_0. If $\mathcal{C}$ is pseudo-Hölder continuous around $u_0 \in \operatorname{dom} \mathcal{C}$, then $\mathcal{C}$ is upper and lower pseudo-Hölder continuous at u_0. For $q = 1$ the upper pseudo-Hölder continuity reduces to calmness (see [75], [91]). Criteria for calmness of set-valued mappings can be found, e.g., in [74]. For instance, if $S(y) = [-s(y), s(y)]$, where $s(y) = 1 + \sqrt{|y|}$, $y \in \mathbb{R}$, then S is not calm at $(0, 1)$ (see [91]), but it is Hölder calm at $(0, 1)$ with order $1/2$.

The following proposition will be often used in what follows.

Proposition 4.0.3. *Let $U = (U, \|\cdot\|)$ and $Y = (Y, \|\cdot\|)$ be normed spaces. For any set-valued mapping $\mathcal{C} : U \rightrightarrows Y$ the following equivalences hold true:*

(i) *$\mathcal{C}$ is Hölder around $u_0 \in \operatorname{dom} \mathcal{C}$ if and only if it is uniformly upper Hölder on some neighbourhood U_0 of u_0,*
(ii) *$\mathcal{C}$ is Hölder around $u_0 \in \operatorname{dom} \mathcal{C}$ if and only if it is uniformly lower Hölder on some neighbourhood U_0 of u_0,*
(iii) *$\mathcal{C}$ is pseudo-Hölder around $(u_0, y_0) \in \operatorname{graph} \mathcal{C}$ if and only if it is uniformly upper pseudo-Hölder at $(u_0, y_0) \in \operatorname{graph} \mathcal{C}$ on a neighbourhood U_0 of u_0,*
(iv) *$\mathcal{C}$ is pseudo-Hölder around $(u_0, y_0) \in \operatorname{graph} \mathcal{C}$ if and only if it is uniformly lower pseudo-Hölder at $(u_0, y_0) \in \operatorname{graph} \mathcal{C}$ on some neighbourhood U_0 of u_0.*

Proof. It is enough to note that for any set-valued mapping $\mathcal{C} : U \rightrightarrows Y$, $\mathcal{C}$ is uniformly upper (resp. lower) Hölder on a subset $U_0 \subset U$ if there exist $L_c > 0$ and $t_c > 0$ such that for any $\overline{u} \in U_0$,

$$\mathcal{C}(u) \subset \mathcal{C}(\overline{u}) + L_c\|u - \overline{u}\|B_Y \quad \text{for } u \in \overline{u} + t_c B_U,$$

(resp.

$$\mathcal{C}(\overline{u}) \subset \mathcal{C}(u) + L_c\|u - \overline{u}\|B_Y \quad \text{for } u \in \overline{u} + t_c B_U.)$$

Let us prove (ii). Assume that there exists $t > 0$ such that for $u \in u' + tB_U$ we have

$$\mathcal{C}(u') \subset \mathcal{C}(u) + L_c\|u - u'\|B_Y \quad \text{for } u \in u' + tB_U.$$

Hence, by taking $u, u' \in u_0 + (t/2)B_U$ we get $u - u' \in tB_U$ and the conclusion follows.

Moreover, $\mathcal{C}$ is uniformly upper (lower) pseudo-Hölder at $(u_0, y_0) \in \operatorname{dom} \mathcal{C}$ on a subset $U_0 \subset U$ if there exist a 0-neighbourhood V and constants $L_c > 0$, $t_c > 0$ such that for any $\overline{u} \in U_0$,

$$\mathcal{C}(u) \cap (y_0 + V) \subset \mathcal{C}(\overline{u}) + L_c\|u - \overline{u}\|B_Y \quad \text{for } u \in \overline{u} + t_c B_U,$$

(resp.

$$\mathcal{C}(\overline{u}) \cap (y_0 + V) \subset \mathcal{C}(u) + L_c\|u - \overline{u}\|B_Y \quad \text{for } u \in \overline{u} + t_c B_U.)$$

Let us prove (iv). Let $y_0 \in C(u_0)$. Assume that $\mathcal{C}$ is uniformly lower pseudo-Hölder continuous at $(u_0, y_0) \in \operatorname{graph} C$. There exist a 0-neighbourhood V in Y and $t > 0$ such that for $u \in u' + tB_U$ we have

$$\mathcal{C}(u') \cap (y_0 + V) \subset \mathcal{C}(u) + L_c\|u - u'\|B_Y \quad \text{for } u \in u' + tB_U.$$

Hence, by taking $u, u' \in u_0 + (t/2)B_U$ we get $u - u' \in tB_U$ and the conclusion follows. ∎

4.1. Lower Hölder continuity of efficient points

The main result of this section provides sufficient conditions for lower Hölder continuity of the efficient point set-valued mapping $\mathcal{E}$.

THEOREM 4.1.1. *Let $\mathcal{K}$ be a closed convex pointed cone in a normed space Y and let C be a subset in Y. Assume that*

(i) *there exist $\beta > 0$ and $q \geq 1$ such that*
$$\|y - \overline{y}\|_{-} \geq \beta\|y - \overline{y}\|^q \quad \text{for all } \overline{y} \in E(C),\ y \in C,$$
(ii) *$\mathcal{C}$ is Hölder continuous of order $p \geq 1$ at $u_0 \in \operatorname{dom}\mathcal{C}$ with constants $L_c > 0$ and $0 < t_c < 1$,*
(iii) *(DP) holds for all $C(u)$, $u \in u_0 + t_c B_U$.*

Then $\mathcal{E}$ is lower Hölder continuous of order p/q at $u_0 \in \operatorname{dom}\mathcal{E}$. Precisely,
$$E(C) \subset E(C(u)) + (L_c + (2L_c/\beta)^{1/q})\|u - u_0\|^{p/q} B_Y$$
for $u \in u_0 + t_c B_U$.

Proof. Take any $u \in u_0 + t_c B_U$ and $y_0 \in E(C)$. By (ii), there exists $z \in C(u)$ such that
$$z - y_0 \in L_c\|u - u_0\|^p B_Y.$$
If $z \in E(C(u))$, the conclusion follows. If $z \notin E(C(u))$, by (iii), there exists $z_0 \in E(C(u))$ such that $z_0 \in z - \mathcal{K}$. Again by (ii), there exists $y \in C$ such that $z_0 - y \in L_c\|u - u_0\|^p B_Y$. Therefore,
$$y - y_0 = (y - z_0) + (z_0 - z) + (z - y_0) \in 2L_c\|u - u_0\|^p B_Y - \mathcal{K}.$$
On the other hand, by (i),
$$y - y_0 \notin \beta\|y - y_0\|^q B_Y - \mathcal{K},$$
which entails that $\beta\|y - y_0\|^q \leq 2L_c\|u - u_0\|^p$ and therefore
$$\|y - y_0\| \leq (2L_c/\beta)^{1/q}\|u - u_0\|^{p/q}.$$
Finally,
$$\|y_0 - z_0\| \leq \|y - y_0\| + \|y - z_0\| \leq (L_c + (2L_c/\beta)^{1/q})\|u - u_0\|^{p/q},$$
which completes the proof. ∎

In view of Proposition 4.0.3, Theorem 4.1.1 leads to the following conditions for Hölder continuity of $\mathcal{E}$ around u_0.

THEOREM 4.1.2. *Let $\mathcal{K}$ be a closed convex pointed cone in a normed space Y and let C be a subset in Y. Assume that*

(i) *there exist $0 < t < 1$, $\beta > 0$ and $q \geq 1$ such that*
$$\|z - \overline{z}\|_{-} \geq \beta\|z - \overline{z}\|^q \quad \text{for all } \overline{z} \in E(C(u)),\ z \in C(u),\ u \in u_0 + tB_U,$$
(ii) *$\mathcal{C}$ is Hölder continuous of order $p \geq 1$ around $u_0 \in \operatorname{dom}\mathcal{C}$ with constants $L_c > 0$ and t,*
(iii) *(DP) holds for all $C(u)$, $u \in u_0 + tB_U$.*

Then $\mathcal{E}$ is Hölder continuous of order p/q around $u_0 \in \operatorname{dom}\mathcal{E}$. Precisely,

$$E(C(u')) \subset E(C(u)) + (L_c + (2L_c/\beta)^{1/q})\|u - u'\|^{p/q} B_Y$$

for $u, u' \in u_0 + (t/4)B_U$.

Proof. By Theorem 4.1.1, for any $u' \in u_0 + (t/2)B_U$,

$$E(C(u')) \subset E(C(u)) + (L_c + (2L_c/\beta)^{1/q})\|u - u'\|^{p/q} B_Y$$

for $u \in u' + (t/2)B_U$. This means that $\mathcal{E}$ is uniformly lower Hölder continuous on $B(u_0, t/2)$. Hence, by taking any $u, u' \in u_0 + (t/4)B_U$ we get $u - u' \in (t/2)B_U$ and the conclusion follows. ∎

The following corollary is an immediate consequence of Theorem 1.2.1.

COROLLARY 4.1.1. *Let $\mathcal{K}$ be a closed convex pointed cone in a normed space Y and let $C(u)$ be nonempty weakly compact subsets of Y for all u in some neighbourhood of u_0. If*

(i) *there exist $\beta > 0$ and $q \geq 1$ such that*

$$\|y - \overline{y}\|_- \geq \beta \|y - \overline{y}\|^q \quad \text{for all } \overline{y} \in E(C),\ y \in C,$$

(ii) *$\mathcal{C}$ is Hölder continuous of order $p \geq 1$ at $u_0 \in \operatorname{dom}\mathcal{C}$ with constants $L_c > 0$ and $0 < t_c < 1$,*

then $\mathcal{E}$ is lower Hölder continuous of order p/q at $u_0 \in \operatorname{dom}\mathcal{E}$.

Now we apply Theorem 4.1.1 to parametric vector optimization problems

$$(P_u) \qquad \begin{array}{l} \min_{\mathcal{K}} f(u, x) \\ \text{subject to } x \in A(u). \end{array}$$

For $u = u_0$ we obtain problem (P),

$$(P) \qquad \begin{array}{l} \min_{\mathcal{K}} f(x) \\ \text{subject to } x \in A. \end{array}$$

We formulate sufficient conditions for lower Hölder continuity of the performance set-valued mapping $\mathcal{P} : U \rightrightarrows Y$,

$$\mathcal{P}(u) = E(f(u, \cdot), A(u))$$

at $u_0 \in \operatorname{dom}\mathcal{P}$.

To this end we need a technical lemma. Let $f : X \to Y$ be a mapping from a normed space X into a normed space Y. We say that f is *Lipschitz on a subset $D \subset X$ with constant* $L_f > 0$ if

$$\|f(x') - f(x)\| \leq L_f \|x - x'\| \quad \text{for } x, x' \in D. \tag{4.1}$$

In particular, f is *Lipschitz around* x_0 if f satisfies (4.1) for $D = x_0 + t_f B_X$, where $t_f > 0$.

We say that $f : U \times X \to Y$ is *Lipschitz around* $\{u_0\} \times D$ with constants $L_f > 0$ and $t_f > 0$ if

$$\|f(u', x') - f(u, x)\| \leq L_f(\|u' - u\| + \|x' - x\|) \tag{4.2}$$

for all $x', x \in D$ and $u', u \in u_0 + t_f B_U$. In particular, f is *Lipschitz around* (u_0, x_0) if f satisfies (4.2) around $\{u_0\} \times D$, where D is a neighbourhood of x_0.

Let $\mathcal{A} : U \rightrightarrows Y$ be a set-valued mapping, $\mathcal{A}(u) = A(u)$, $\mathcal{A}(u_0) = A$. The image of $\mathcal{A}$ under a mapping $f : X \to Y$ is defined as $\mathcal{A}_f : U \rightrightarrows Y$, $\mathcal{A}_f(u) = f(A(u))$, $\mathcal{A}_f(u_0) = f(A)$. Clearly, $\operatorname{dom} \mathcal{A}_f = \operatorname{dom} \mathcal{A}$.

PROPOSITION 4.1.1. *Let X and Y be normed spaces. Let $f : X \to Y$ be Lipschitz on X with constant $L_f > 0$.*

- (i) *If $\mathcal{A}$ is lower Hölder continuous at $u_0 \in \operatorname{dom} \mathcal{A}$ of order $p > 0$ with constants $L_a > 0$ and $t_a > 0$, then $\mathcal{A}_f$ is lower Hölder continuous at $u_0 \in \operatorname{dom} \mathcal{A}$ of order $p > 0$ with constants $L_f L_a > 0$ and $t_a > 0$.*
- (ii) *If $\mathcal{A}$ is upper Hölder continuous at $u_0 \in \operatorname{dom} \mathcal{A}$ of order $p > 0$ with constants $L_a > 0$ and $t_a > 0$, then $\mathcal{A}_f$ is upper Hölder continuous at $u_0 \in \operatorname{dom} \mathcal{A}$ of order $p > 0$ with constants $L_f L_a > 0$ and $t_a > 0$.*
- (iii) *If $\mathcal{A}$ is Hölder continuous at $u_0 \in \operatorname{dom} \mathcal{A}$ of order $p > 0$ with constants $L_a > 0$ and $t_a > 0$, then $\mathcal{A}_f$ is Hölder continuous at $u_0 \in \operatorname{dom} \mathcal{A}$ of order $p > 0$ with constants $L_f L_a > 0$ and $t_a > 0$.*

In view of Proposition 4.1.1 and Theorem 4.1.1 we obtain the following result.

THEOREM 4.1.3. *Let X and Y be normed spaces and let $\mathcal{K}$ be a closed convex pointed cone in Y. Assume that*

- (i) *there exists $\beta > 0$ and $q \geq 1$ such that*
$$\|f(x) - f(\overline{x})\|_- \geq \beta \|f(x) - f(\overline{x})\|^q \quad \text{for all } \overline{x} \in S(f, A),\ x \in A,$$
- (ii) *f is Lipschitz on X with constant $L_f > 0$, $\mathcal{A}$ is Hölder continuous of order $p \geq 1$ at $u_0 \in \operatorname{dom} \mathcal{A}$ with constants $L_a > 0$ and $0 < t < 1$,*
- (iii) *(DP) holds for all $f(A(u))$, $u \in u_0 + tB_U$.*

Then $\mathcal{P}$ is lower Hölder continuous of order p/q at $u_0 \in \operatorname{dom} \mathcal{P}$. Precisely,
$$E(f, A) \subset E(f, A(u)) + (L_f L_a + (2L_f L_a/\beta)^{1/q}) \|u - u_0\|^{p/q} B_Y \quad \text{for } u \in B(u_0, t).$$

4.2. Lower pseudo-Hölder continuity of efficient points

In the present section we give sufficient conditions for lower pseudo-Hölder continuity of $\mathcal{E}$ at $(u_0, y_0) \in \operatorname{graph} \mathcal{E}$.

THEOREM 4.2.1. *Let $\mathcal{K}$ be a closed convex pointed cone in a normed space Y and let C be a subset in Y. Let $y_0 \in E(C)$. Assume that*

- (i) *there exist $\beta > 0$ and $q \geq 1$ and a 0-neighbourhood V such that*
$$\|y - \overline{y}\|_- \geq \beta \|y - \overline{y}\|^q \quad \text{for all } \overline{y} \in E(C) \cap (y_0 + V),\ y \in C,$$
- (ii) *$\mathcal{C}$ is lower pseudo-Hölder continuous of order $p \geq 1$ at $(u_0, y_0) \in \operatorname{graph} \mathcal{C}$ with 0-neighbourhood V and constants $L_c > 0$, $0 < t_c < 1$ and upper Hölder continuous of order $p \geq 1$ at $u_0 \in \operatorname{dom} \mathcal{C}$ with constants $L_c > 0$, $0 < t_c < 1$,*
- (iii) *(DP) holds for all $C(u)$, $u \in u_0 + t_c B_U$.*

Then $\mathcal{E}$ is lower pseudo-Hölder continuous of order p/q at $(u_0, y_0) \in \operatorname{graph}\mathcal{E}$. Precisely,

$$E(C) \cap (y_0 + V) \subset E(C(u)) + (L_c + (2L_c/\beta)^{1/q})\|u - u_0\|^{p/q} B_Y$$

for $u \in u_0 + t_c B_U$.

Proof. Take any $u \in u_0 + t_c B_U$ and $\overline{y} \in E(C) \cap (y_0 + V)$. By (ii), there exists $z \in C(u)$ such that

$$z - \overline{y} \in L_c \|u - u_0\|^p B_Y.$$

If $z \in E(C(u))$, the conclusion follows. Otherwise, by (iii), there exists $\overline{z} \in E(C(u))$ such that $\overline{z} \in z - \mathcal{K}$. Again by (ii), there exists $y \in C$ such that $\overline{z} - y \in L_c \|u - u_0\|^p B_Y$. Therefore,

$$y - \overline{y} = (y - \overline{z}) + (\overline{z} - z) + (z - \overline{y}) \in 2L_c B_Y - \mathcal{K}.$$

On the other hand, by (i),

$$y - \overline{y} \notin \beta \|y - \overline{y}\|^q B_Y - \mathcal{K},$$

which gives that $\beta \|y - \overline{y}\|^q \le 2L_c \|u - u_0\|^p$ and therefore

$$\|y - \overline{y}\| \le (2L_c/\beta)^{1/q} \|u - u_0\|^{p/q}.$$

Finally,

$$\|\overline{y} - \overline{z}\| \le \|y - \overline{y}\| + \|y - \overline{z}\| \le (L_c + (2L_c/\beta)^{1/q})\|u - u_0\|^{p/q},$$

which completes the proof. ■

By condition (i) of Theorem 4.2.1, all $\overline{y} \in E(C) \cap (y_0 + V)$ are globally strictly efficient of order q with the same constant β.

Since lower pseudo-Hölder continuity is of local character the question arises whether we can prove lower pseudo-Hölder continuity of $\mathcal{E}$ at (u_0, y_0) by assuming condition (i) for local strictly efficient points. To this end we need the following definition.

Let $C \subset Y$ be a subset of Y.

Definition 4.2.1. The *local domination property* (*LDP*) holds for C at $y_0 \in Y$ if there exists a 0-neighbourhood V such that for any $y \in C \cap (y_0 + V)$ there exists $\eta \in E(C) \cap (y_0 + V)$ such that

$$\eta \in y - \mathcal{K}.$$

(*DP*) is equivalent to (*LDP*) with $V = Y$. Note that whenever (*DP*) holds for C, any $y \in C \cap (y_0 + V)$ is dominated by some $\eta \in E(C)$ but in general $\eta \notin E(C) \cap (y_0 + V)$.

By using (*LDP*) we formulate the following theorem.

Theorem 4.2.2. *Let $\mathcal{K}$ be a closed convex pointed cone in a normed space $(Y, \|\cdot\|)$. Let C be a subset in Y and let $y_0 \in E(C)$. Assume that*

(i) *there exist constants $\beta > 0$, $q \ge 1$, $t_s > 0$ and a 0-neighbourhood V such that*
$$\|y - \overline{y}\|_- \ge \beta \|y - \overline{y}\|^q \quad \textit{for all } \overline{y} \in E(C) \cap (y_0 + V),\ y \in C \cap (\overline{y} + t_s B_Y),$$
(ii) *$\mathcal{C}$ is pseudo-Hölder continuous of order $p \ge 1$ at $(u_0, y_0) \in \operatorname{graph}\mathcal{C}$ with 0-neighbourhood V and constants $L_c > 0$, $0 < t_c < 1$,*

(iii) *(LDP) holds for all $C(u)$, $u \in u_0 + t_c B_U$ at y_0 with a neighbourhood $\overline{V} \subset V \cap \frac{1}{2} t_s B_Y$.*

Then $\mathcal{E}$ is lower pseudo-Hölder continuous of order p/q at $(u_0, y_0) \in \operatorname{graph} \mathcal{E}$. Precisely, there exists a 0-neighbourhood $\widetilde{V} \subset \overline{V}$ such that

$$E(C) \cap (y_0 + \widetilde{V}) \subset E(C(u)) + (L_c + (2L_c/\beta)^{1/q})\|u - u_0\|^{p/q} B_Y$$

for $u \in u_0 + t_c B_U$.

Proof. Take any $u \in u_0 + t_c B_U$. Let $\widetilde{V}$ be any 0-neighbourhood satisfying $\widetilde{V} + L_c t_c \subset \overline{V}$. Let $\overline{y} \in E(C) \cap (y_0 + \widetilde{V})$. By (ii), there exists $z \in C(u)$ such that

$$z - \overline{y} \in L_c \|u - u_0\|^p B_Y.$$

Clearly, $z - y_0 \subset \widetilde{V} + L_c t_c B_Y \subset \overline{V}$. By (iii), there exists $\overline{z} \in E(C(u)) \cap (y_0 + \overline{V})$ such that $\overline{z} \in z - \mathcal{K}$. Since $\overline{z} - y_0 \in \overline{V} \subset V$, by (ii), there exists $y \in C$ such that

$$\overline{z} - y \in L_c \|u - u_0\|^p B_Y$$

and $y - y_0 = (y - \overline{z}) + (\overline{z} - y_0) \in L_c t_c B_Y + \overline{V}$. If $y = \overline{y}$, the conclusion follows. So, assume that $y \neq \overline{y}$. We have

$$y - \overline{y} = (y - \overline{z}) + (\overline{z} - z) + (z - \overline{y}) \in 2L_c B_Y - \mathcal{K}$$

and $y - \overline{y} = (y - y_0) + (y_0 - \overline{y}) \in L_c t_c B_Y + \overline{V} + \widetilde{V} \subset \overline{V} + \overline{V} \subset t_s B_Y$. Hence, by (i),

$$y - \overline{y} \notin \beta \|y - \overline{y}\|^q B_Y - \mathcal{K},$$

which yields the inequality $\beta \|y - \overline{y}\|^q \leq 2L_c \|u - u_0\|^p$ and therefore

$$\|y - \overline{y}\| \leq (2L_c/\beta)^{1/q} \|u - u_0\|^{p/q}.$$

Finally,

$$\|\overline{y} - \overline{z}\| \leq \|y - \overline{y}\| + \|y - \overline{z}\| \leq (L_c + (2L_c/\beta)^{1/q})\|u - u_0\|^{p/q},$$

which completes the proof. ∎

4.3. Pseudo-Hölder continuity of efficient points

In this section we formulate sufficient conditions for pseudo-Hölder continuity of efficient points under perturbations of sets.

THEOREM 4.3.1. *Let $\mathcal{K}$ be a closed convex pointed cone in a normed space Y. Let C be a nonempty subset in Y and $y_0 \in E(C)$. Assume that*

(i) *there exist a 0-neighbourhood V and constants $0 < t < 1$, $\beta > 0$, $q \geq 1$, $t_s > 0$ such that*

$$\|z - \overline{z}\|_- \geq \beta \|z - \overline{z}\|^q \quad \text{for } \overline{z} \in E(C(u)) \cap (y_0 + V),\ z \in C(u) \cap (\overline{z} + t_s B_Y),\ u \in u_0 + t B_U,$$

(ii) *$\mathcal{C}$ is Hölder continuous of order $p \geq 1$ around $u_0 \in \operatorname{dom} \mathcal{C}$ with constants $L_c > 0$ and t,*

(iii) *(LDP) holds for all $C(u)$ for $u \in u_0 + t B_U$ with a 0-neighbourhood $\overline{V} \subset \frac{1}{2} t_s B_Y$.*

Then $\mathcal{E}$ is pseudo-Hölder continuous of order p/q at $(u_0, y_0) \in \operatorname{graph} \mathcal{E}$. Precisely, there *exists a* 0*-neighbourhood* $\widetilde{V}$ *such that*

$$E(C(u')) \cap (y_0 + \widetilde{V}) \subset E(C(u)) + (L_c + (2L_c/\beta)^{1/q})\|u' - u\|^{p/q} B_Y$$

for $u, u' \in u_0 + t/4B_U$.

Proof. It is enough to note that under the assumptions, for any $u' \in u_0 + t/2B_U$,

$$E(C(u')) \cap (y_0 + \widetilde{V}) \subset E(C(u)) + (L_c + (2L_c/\beta)^{1/q})\|u - u'\|^{p/q} B_Y$$

for $u \in u' + t/2B_U$. This means that $\mathcal{E}$ is uniformly lower pseudo-Hölder at $(u_0, y_0) \in \operatorname{graph} \mathcal{E}$. The conclusion follows by Proposition 4.0.3. ∎

In particular, Theorem 4.3.1 gives rise to the following conditions for upper pseudo-Hölder continuity of $\mathcal{E}$ at $(u_0, y_0) \in \operatorname{graph} \mathcal{E}$.

THEOREM 4.3.2. *Let* $\mathcal{K}$ *be a closed convex pointed cone in a normed space* Y. *Let* C *be a subset in* Y *and* $y_0 \in E(C)$. *Assume that*

(i) *there exist a* 0*-neighbourhood* V *and constants* $0 < t < 1$, $\beta > 0$, $q \geq 1$, $t_s > 0$ *such that*

$$\|z - \bar{z}\|_- \geq \beta \|z - \bar{z}\|^q \quad \text{for } \bar{z} \in E(C(u)) \cap (y_0 + V),\ z \in C(u) \cap (\bar{z} + t_s B_Y),\ u \in u_0 + tB_U,$$

(ii) $\mathcal{C}$ *is Hölder continuous of order* $p \geq 1$ *at* $u_0 \in \operatorname{dom} \mathcal{C}$ *with constants* $L_c > 0$ *and* t,

(iii) *(LDP) holds for* C *with a* 0*-neighbourhood* $\overline{V} \subset \frac{1}{2} t_s B_Y$.

Then $\mathcal{E}$ *is upper pseudo-Hölder continuous of order* p/q *at* $(u_0, y_0) \in \operatorname{graph} \mathcal{E}$. *Precisely, there exists a* 0*-neighbourhood* $\widetilde{V}$ *such that*

$$E(C(u)) \cap (y_0 + \widetilde{V}) \subset E(C) + (L_c + (2L_c/\beta)^{1/q})\|u - u_0\|^{p/q} B_Y$$

for $u \in u_0 + tB_U$.

5. CONTAINMENT PROPERTY

Let C be a subset of a Hausdorff topological vector space Y equipped with a closed convex pointed cone $\mathcal{K}$. The domination property (*DP*) holds for C if $C \subset E(C) + \mathcal{K}$. Conditions ensuring the domination property can be found in [72, 106, 124, 149]. For a vector optimization problem

$$(P) \quad \begin{array}{l} \min_{\mathcal{K}} f(x) \\ \text{subject to } x \in A \end{array}$$

the domination property (*DP*) holds if $f(A) \subset E(f, A) + \mathcal{K}$. It says that for each $x \in A$ there exists $x_0 \in S(f, A)$ such that $f(x) - f(x_0) \in \mathcal{K}$. Let us note that if $f : X \to \mathbb{R}$, the set $E_{\mathbb{R}_+}(f, A)$ consists of at most a single element and the domination property holds whenever the solution set is nonempty. This one-dimensional fact was generalized to finite-dimensional spaces $Y = \mathbb{R}^m$ by Henig [72] who proved that for $\mathcal{K}$-convex and $\mathcal{K}$-closed sets C the domination property (*DP*) is equivalent to $E(C) \neq \emptyset$.

5.1. Containment property

Let Y be a Hausdorff topological vector space and let $\mathcal{K}$ be a closed convex pointed cone in Y. Let C be a subset of Y. For any 0-neighbourhood W in Y, define

$$C(W) := C \setminus (E(C) + W).$$

Definition 5.1.1 ([16]). We say that the *containment property* (*CP*) holds for C if for every 0-neighbourhood W there exists a 0-neighbourhood O such that

$$C(W) + O \subset E(C) + \mathcal{K}. \tag{5.1}$$

Clearly, if $C \neq \emptyset$ and (*CP*) holds for C, then $E(C) \neq \emptyset$ and

$$C \subset \operatorname{cl} E(C) + \mathcal{K}, \tag{5.2}$$

where $\operatorname{cl}(\cdot)$ stands for the closure of a set. Indeed, if $y \in C \setminus \operatorname{cl} E(C)$ there exists a 0-neighbourhood W such that $y \notin E(C) + W$ and hence, by (*CP*), $y \in E(C) + \mathcal{K}$. In Section 5.1.2 we give examples of sets for which (*CP*) does not hold.

Proposition 5.1.1. *Let* $\operatorname{int} \mathcal{K} \neq \emptyset$ *and let* C *be a subset of* Y. *If* (*CP*) *holds for* C, *then* $WE(C) \subset \operatorname{cl} E(C)$.

Proof. On the contrary, suppose that there is $y \in WE(C) \setminus \operatorname{cl} E(C)$. Hence, $(y - \operatorname{int} \mathcal{K}) \cap C = \emptyset$ and

$$(y - \operatorname{int} \mathcal{K}) \cap (E(C) + \mathcal{K}) = \emptyset. \tag{$*$}$$

Since $y \notin \mathrm{cl}\, E(C)$ and Y is Hausdorff, by (CP), there exists a 0-neighbourhood O in Y such that $y + O \subset E(C) + \mathcal{K}$ and consequently $(y - \mathrm{int}\, \mathcal{K}) \cap (E(C) + \mathcal{K}) \neq \emptyset$, which contradicts $(*)$. ∎

If C is closed, $WE(C)$ is closed (Theorem 1.1 of [105], p. 136), and hence $\mathrm{cl}\, E(C) \subset WE(C)$. Hence, by Proposition 5.1.1 we obtain the following corollary.

COROLLARY 5.1.1. *Let C be a closed subset of Y. Assume that* $\mathrm{int}\, \mathcal{K} \neq \emptyset$. *If (CP) holds for C, then $WE(C) = \mathrm{cl}\, E(C)$. If (CP) holds for C and $E(C) = WE(C)$, then (DP) holds for C.*

PROPOSITION 5.1.2. *Let* $\mathrm{int}\, \mathcal{K} \neq \emptyset$ *and let C be a nonempty compact subset of Y. The following conditions are equivalent*:

(i) *(CP) holds for C,*
(ii) $\mathrm{cl}\, E(C) = WE(C)$.

Proof. (ii)⇒(i). In view of compactness of C, by Theorem 1 of [40], (DP) holds for C. Let W be a 0-neighbourhood. Take any $y \in C(W)$. Since $y \notin WE(C)$, by (DP), there exist $k_1 \in \mathrm{int}\, \mathcal{K}$, $k \in \mathcal{K}$, and $\eta \in E(C)$ such that $y = \eta + \overline{k}$, $\overline{k} = k_1 + k \in \mathrm{int}\, \mathcal{K}$. Hence, for any $y \in C(W)$ there exists a 0-neighbourhood O_y such that $y + \overline{k} + O_y \subset E(C) + \mathcal{K}$. The family $\{O_y\}_{y \in C(W)}$ forms a covering of $C(W)$. Since $C(W)$ is compact, this covering contains a finite subcovering $O_1, \ldots, O_r$ and by putting $O = \bigcap_{i=1}^{r} O_r$, (i) follows.

(i)⇒(ii). Follows from Corollary 5.1.1. ∎

The following proposition gives a characterization of (CP) whenever $\mathrm{int}\, \mathcal{K} \neq \emptyset$.

PROPOSITION 5.1.3. *Let $\mathcal{K}$ be a closed convex pointed cone in Y with* $\mathrm{int}\, \mathcal{K} \neq \emptyset$, *and let C be a subset of Y. The following statements are equivalent:*

(i) *(CP) holds for C,*
(ii) *for each 0-neighbourhood W there exists a 0-neighbourhood O such that*:

(C) *for any $y \in C(W)$ there is $\eta \in E(C)$ satisfying*

$$(y - \eta) + O \subset \mathcal{K}. \tag{5.3}$$

Proof. (i)⇒(ii). For any 0-neighbourhood O define

$$\mathcal{K}_O = \{k \in \mathcal{K} : k + O \subset \mathcal{K}\}.$$

Clearly, $\mathrm{int}\, \mathcal{K} = \bigcup_{O \in \mathcal{N}} \mathcal{K}_O$. We show that for any 0-neighbourhood Q there exists a 0-neighbourhood O such that

$$(E(C) + \mathcal{K})_Q \subset E(C) + \mathcal{K}_O, \tag{5.4}$$

where $(E(C) + \mathcal{K})_Q = \{y \in Y : y + Q \subset E(C) + \mathcal{K}\}$. Indeed, let $c \in (E(C) + \mathcal{K})_Q$. This means that $c + Q \subset E(C) + \mathcal{K}$. Since $0 \in \mathrm{cl}(-\mathcal{K})$, for any 0-neighbourhood Q there exists a 0-neighbourhood O such that $Q \cap (-\mathcal{K}_O) \neq \emptyset$. Thus there exists $q \in Q \cap (-\mathcal{K}_O)$ such that $c + q \in E(C) + \mathcal{K}$, i.e., $c \in E(C) + \mathcal{K}_O$ By (i), for each 0-neighbourhood W there exists a 0-neighbourhood Q such that for any $y \in C(W)$, $y \in (E(C) + \mathcal{K})_Q$, and by (5.4), for some 0-neighbourhood O, $y \in E(C) + \mathcal{K}_O$.

(ii)⇒(i). Obvious. ∎

Although in Definition 5.1.1 we do not assume explicitly that $\operatorname{int}\mathcal{K} \neq \emptyset$, this assumption is essential for the characterization of (CP) given in Proposition 5.1.3. In turn, Proposition 5.1.3 is exploited in stability theorems of next sections. However, in some important spaces, the cones of nonnegative elements may have empty interiors. This is the case, for example, in the space of integrable functions $L^p(\Omega)$, $1 \leq p < \infty$, for the cone $\mathcal{K}_{L^p(\Omega)}$ of nonnegative elements

$$\mathcal{K}_{L^p(\Omega)} = \{f \in L^p(\Omega) : f \geq 0 \text{ almost everywhere in } \Omega\},$$

as well as in the space ℓ^p, $1 \leq p < \infty$, of summable sequences $s = (s_i)$ for the cone

$$\mathcal{K}_{\ell^p(\Omega)} = \{s \in \ell^p : s_i \geq 0\}$$

(see [82]).

5.1.1. Containment property in normed spaces. Let $Y = (Y, \|\cdot\|)$ be a normed space with open unit ball B_Y. For any subset C of Y, set $d(y, C) = \inf\{\|y - c\| : c \in C\}$, $B(C, \varepsilon) = \{y \in Y : d(y, C) < \varepsilon\}$. For $\varepsilon > 0$ put

$$C(\varepsilon) := C \setminus B(E(C), \varepsilon).$$

Then (CP) holds for C if for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$C(\varepsilon) + \delta B_Y \subset E(C) + \mathcal{K}.$$

Let $(Y, \|\cdot\|)$ be a Banach space and let $f \in Y^*$, $\|f\| = 1$. For any $0 < \alpha \leq 1$ the cone

$$\mathcal{K}_\alpha = \{y \in Y : f(y) \geq \alpha\|y\|\}$$

is the *Bishop–Phelps cone* (cf. Section 3.2 and Definition 2.9 of [124]). It is a closed convex pointed cone with nonempty interior $\operatorname{int}\mathcal{K}_\alpha = \{y \in Y : f(y) > \alpha\|y\|\}$. Moreover, $\mathcal{K}_\alpha$ has a bounded base $\Theta = \{k \in \mathcal{K}_\alpha : f(k) = 1\}$. Bishop–Phelps cones were investigated e.g. in [123], where it is shown that in normed spaces for any convex cone Ω with a closed bounded base there exist an equivalent norm and a functional f such that Ω can be represented as a Bishop–Phelps cone.

THEOREM 5.1.1. *Let C be a convex subset of Y. The following statements are equivalent:*

(i) *(CP) holds for C with respect to $\mathcal{K}_\alpha$,*
(ii) *for each $\varepsilon > 0$ there exists $1 > \beta > \alpha$ such that $C(\varepsilon) \subset E_{\mathcal{K}_\alpha}(C) + \mathcal{K}_\beta$.*

Proof. (i)$\Rightarrow$(ii). Let $\varepsilon > 0$. By (CP), there exists $\delta > 0$ such that

$$C(\varepsilon) + \delta B_Y \subset E_{\mathcal{K}_\alpha}(C) + \mathcal{K}_\alpha.$$

Since C is convex, for any $y \in C(\varepsilon)$ and $\eta \in E_{\mathcal{K}_\alpha}(C)$,

$$z = \eta + \frac{\varepsilon}{\|y - \eta\|}(y - \eta) \in C, \qquad \|z - \eta\| = \varepsilon.$$

By Proposition 5.1.3, there exists $\eta \in E_{\mathcal{K}_\alpha}(C)$ such that $z - \eta \pm w \subset \mathcal{K}_\alpha$ for any $\|w\| < \delta$. Consequently, $f(z - \eta \pm w) \geq \alpha\|f\|\,\|z - \eta \pm w\|$ and

$$f(z - \eta) - |f(w)| \geq \alpha\varepsilon\|f\| - \alpha\delta\|f\|.$$

Hence

$$f(z - \eta) \geq \alpha\varepsilon\|f\| - \alpha\delta\|f\| + \delta \sup_{w \in \delta B_Y} |f(w/\delta)|,$$

and

$$f(z-\eta) \geq \alpha\varepsilon\|f\| - \alpha\delta\|f\| + \delta\|f\| = \varepsilon\|f\|(\alpha - \alpha\delta/\varepsilon + \delta/\varepsilon).$$

By taking $\beta = \alpha + (\delta/\varepsilon)(1-\alpha)$ we obtain (ii).

(ii)$\Rightarrow$(i). Let $\varepsilon > 0$. By (ii), there exists $\beta > \alpha$ such that $C(\varepsilon) \subset E_{\mathcal{K}_\alpha}(C) + \mathcal{K}_\beta$. Hence, for any $y \in C(\varepsilon)$ there exists $\eta \in E_{\mathcal{K}_\alpha}(C)$ such that

$$f(y-\eta) \geq \beta\|f\|\,\|y-\eta\|.$$

For any $w \in Y$, we have $f(y - \eta_y - w) = f(y - \eta_y) - f(w) \geq \beta\|f\|\,\|y - \eta_y\| - f(w)$, and consequently

$$\begin{aligned} f(y-\eta-w) &\geq \beta\|f\|\,\|y-\eta-w+w\| - \|f\|\,\|w\| \\ &\geq \|f\|\,\|y-\eta-w\|\left[\beta - \frac{\beta\|w\|+\|w\|}{\|y-\eta-w\|}\right] \\ &\geq \|f\|\,\|y-\eta-w\|\left[\beta - \frac{\beta\|w\|+\|w\|}{\varepsilon-\|w\|}\right]. \end{aligned}$$

By taking

$$\|w\| < \frac{\varepsilon(\beta-\alpha)}{2\beta-\alpha+1}$$

we obtain

$$\|f\|\,\|y-\eta-w\|\left[\beta - \frac{\beta\|w\|+\|w\|}{\varepsilon-\|w\|}\right] \leq \beta-\alpha$$

and consequently $f(y-\eta-w) \geq \alpha\|f\|\,\|y-\eta-w\|$, which implies (CP). ■

5.1.2. Containment property in finite-dimensional spaces. Let $Y = (\mathbb{R}^m, \|\cdot\|)$ be the m-dimensional space. Let $\mathcal{K}$ be a closed convex cone in Y. If $\mathcal{K}$ is pointed it admits a compact base (see [123]).

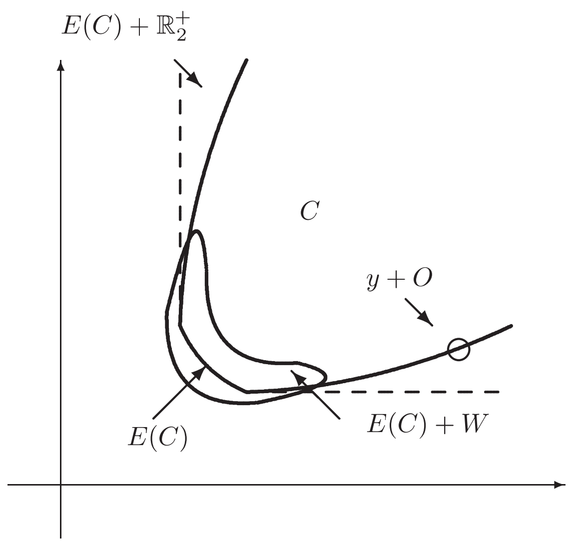

Fig. 5.1. Containment property for the set C with respect to the nonnegative cone $\mathbb{R}^2_+$

Let $C \subset \mathbb{R}^m$. Note that $E(C)$ need not be closed even if C is convex and closed (cf. [3]). Hence, even for closed convex sets of a finite-dimensional space, (CP) does not imply (DP). We start by investigating relationships between the two properties.

THEOREM 5.1.2. *Let $\mathcal{K}$ be a closed convex pointed cone in $\mathbb{R}^m$ with $\operatorname{int}\mathcal{K} \neq \emptyset$. Let C be a closed convex subset of $\mathbb{R}^m$ such that $\operatorname{cl} E(C)$ is compact. If $\operatorname{cl} E(C) = WE(C)$ and (DP) holds for C, then (CP) holds for C.*

Proof. The set $\operatorname{cl} E(C) + \mathcal{K}$ is closed and convex, since $\operatorname{cl} E(C)$ is compact and $C + \mathcal{K} = \operatorname{cl} E(C) + \mathcal{K}$.

Suppose on the contrary that (CP) does not hold for C. There exist $\varepsilon_0 > 0$ and sequences (z_n), (y_n) such that $z_n \in C(\varepsilon_0)$, $y_n \in B(z_n, 1/n)$, and $y_n \notin \operatorname{cl} E(C) + \mathcal{K}$. By (DP), $z_n = \eta_n + k_n$, where $\eta_n \in E(C)$, $k_n \in \mathcal{K}$, $\|k_n\| > \varepsilon_0$.

Let Θ be a compact base of $\mathcal{K}$. We have $M_0 \leq \|\theta\| \leq M$ for any $\theta \in \Theta$ and some $M_0, M > 0$. Moreover, $k_n = \lambda_n \theta_n$ with $\lambda_n > 0$ and $\theta_n \in \Theta$. Since $\varepsilon_0 < \|z_n - \eta_n\| = \lambda_n\|\theta_n\| \leq \lambda_n M$, the sequence (β_n), $\beta_n = 1/\lambda_n$, is bounded. We can assume that $0 < \beta_n \leq 1$. By convexity of C,

$$\eta_n + \theta_n = \beta_n z_n + (1 - \beta_n)\eta_n \in A.$$

Since $\operatorname{cl} E(C)$ is compact, (η_n) contains a convergent subsequence with limit point $\eta \in \operatorname{cl} E(C)$. We can assume that (η_n) converges to $\eta \in C$ and (θ_n) converges to $\theta \in \Theta$. The sequence (r_n), $r_n = \eta_n + \theta_n$, tends to $r = \eta + \theta \in C$. Clearly, $r \notin \operatorname{cl} E(C)$.

We must have $r \in WE(C)$. Indeed, if $(r - \operatorname{int}\mathcal{K}) \cap C \neq \emptyset$, then $r = y + k$, where $y \in C$ and $k \in \operatorname{int}\mathcal{K}$. Hence, $k + \widetilde{\varepsilon} B_Y \subset \mathcal{K}$ for some $\widetilde{\varepsilon} > 0$ and

$$z_n = r + (r_n - r) + (\lambda_n - 1)\theta_n = y + k + (r_n - r) + (\lambda_n - 1)\theta_n = y + k_n,$$

where $k_n \in k + (\widetilde{\varepsilon}/2)B_Y \subset \mathcal{K}$ for all n sufficiently large. Consequently, $y_n = z_n + (y_n - z_n) = y + p_n$, $p_n \in k + (\widetilde{\varepsilon}/3)B_Y \subset \mathcal{K}$ for all n sufficiently large, which contradicts the choice of y_n. Hence, $r \in WE(C) \setminus \operatorname{cl} E(C)$, which is impossible. ■

One can easily give examples showing that in the above proposition the equality $\operatorname{cl} E(C) = WE(C)$ cannot be dropped.

EXAMPLE 5.1.1. Let $\mathcal{K} = \mathbb{R}^2_+ = \{(y_1, y_2) : y_1, y_2 \geq 0\}$ and

$$C = \{(y_1, y_2) : 0 \leq y_1 \leq 1,\ 0 \leq y_2 \leq 1\}.$$

Here $E(C) = \{(0,0)\}$, $WE(C) = \{(y_1, y_2) \in C : y_1 = 0 \text{ or } y_2 = 0\}$, (DP) holds for C and (CP) does not.

Note that convexity and closedness of C cannot be weakened respectively to $\mathcal{K}$-convexity and $\mathcal{K}$-closedness. The following theorem provides a further refinement of the above theorem.

THEOREM 5.1.3 ([34, 72], see also [105]). *Let $\mathcal{K}$ be a closed convex cone in $\mathbb{R}^m$. Let C be a $\mathcal{K}$-convex and $\mathcal{K}$-closed subset of $\mathbb{R}^m$. The following statements are equivalent:*

(i) *(DP) holds for C,*
(ii) *$E(C) \neq \emptyset$.*

As a consequence of this result we obtain the following corollary.

Corollary 5.1.2. *Let $\mathcal{K}$ be a closed convex pointed cone in $\mathbb{R}^m$ with* $\operatorname{int}\mathcal{K} \neq \emptyset$. *Let C be a closed convex subset of $\mathbb{R}^m$ with* $\operatorname{cl} E(C)$ *compact. The following conditions are equivalent:*

(i) $E(C) \neq \emptyset$, $\operatorname{cl} E(C) = WE(C)$,

(ii) *(CP) holds for C.*

Proof. This follows from Theorem 5.1.2 and Corollary 3 of [72]. ∎

Consider now the case where $C \subset \mathbb{R}^m$ is *polyhedral*, i.e., C is the solution set to a system of a finite number of linear inequalities,

$$C = \{y \in \mathbb{R}^m : \langle b_i, y\rangle \le c_i,\ i \in I\}. \tag{5.5}$$

In this case we prove an analogue of Theorem 5.1.2 without assuming compactness of $E(C)$. The *recession cone* $\operatorname{Rec}(C)$ of C is given by the system of homogeneous inequalities,

$$\operatorname{Rec}(C) = \{y \in \mathbb{R}^m : \langle b_i, y\rangle \le 0,\ i \in I\},$$

and $E(C) \neq \emptyset$ if and only if $\operatorname{Rec}(C) \cap (-\mathcal{K}) = \{0\}$ (Th. 3.18 of Ch. 1 of [105]).

To make the presentation self-contained we prove closedness of $E(C)$ and of $E(C)+\mathcal{K}$ whenever C is a polyhedral set. Usually, the closedness of $E(C)$ is proved as a consequence of the scalarization of linear multiple objective optimization problems with polyhedral cones. Here we prove the closedness of $E(C)$ directly for any closed convex cone $\mathcal{K}$. Recall that the *lineality space* $\ell(\mathcal{K})$ of $\mathcal{K}$ is defined as $\ell(\mathcal{K}) = \mathcal{K} \cap (-\mathcal{K})$.

Proposition 5.1.4. *If C is a polyhedral subset of $\mathbb{R}^m$ given by* (5.5) *and $\mathcal{K} \subset \mathbb{R}^m$ is a closed convex cone, then $E(C)$ is closed.*

Proof. Suppose on the contrary that $E(C)$ is not closed. There exists a sequence of efficient points $(\eta_n) \subset E(C)$ which converges to $\eta \in C$ and $\eta \notin E(C)$. Hence, there is an $\overline{\eta} \in C$ such that $\eta - \overline{\eta} \in \mathcal{K} \setminus \ell(\mathcal{K})$.

Passing to a subsequence if necessary, one can find a subset $I_1 \subset I$ such that

$$\langle b_i, \eta_n\rangle = c_i, \quad i \in I_1, \quad \text{and} \quad \langle b_i, \eta_n\rangle < c_i, \quad i \in I \setminus I_1.$$

Hence, $\langle b_i, \eta\rangle = c_i$ and $\langle b_i, \eta\rangle \ge \langle b_i, \overline{\eta}\rangle$ for $i \in I_1$. Moreover, $\langle b_i, \overline{\eta}\rangle > \langle b_i, \eta\rangle$ for some $i \in I \setminus I_1$ since otherwise $0 \neq -k = \overline{\eta} - \eta \in \operatorname{Rec}(C)$. Thus, there are two index subsets $I_2, I_3 \subset I$ with $I_3 \neq \emptyset$ such that

$$\langle b_i, \overline{\eta} - \eta\rangle \le 0, \quad i \in I_2 \supset I_1, \quad \text{and} \quad \langle b_i, \overline{\eta} - \eta\rangle > 0, \quad i \in I_3.$$

For each $n \ge 1$ put

$$\gamma_n = \min_{i \in I_3} \frac{c_i - \langle b_i, \eta_n\rangle}{\langle b_i, \overline{\eta} - \eta\rangle} > 0,$$

and consider $w_n = \eta_n + \gamma_n(\overline{\eta} - \eta)$. We have $w_n \in C$ and $w_n - \eta_n \in (-\mathcal{K}) \setminus \ell(\mathcal{K})$. This contradicts the efficiency of η_n. ∎

Proposition 5.1.5. *For any polyhedral set $C \subset \mathbb{R}^m$ given by* (5.5) *and any closed convex pointed cone $\mathcal{K}$ in $\mathbb{R}^m$ the set $E(C) + \mathcal{K}$ is closed.*

Proof. If $E(C) = \emptyset$, the set $E(C)+\mathcal{K}$ is empty, hence closed. Assume that $E(C) \neq \emptyset$ and let $\Theta \subset \mathcal{K}$ be a base of $\mathcal{K}$.

Consider any convergent sequence $(z_n) \subset E(C) + \mathcal{K}$, $\lim_n z_n = z$. We have $z_n = x_n + \lambda_n \theta_n$, where $x_n \in E(C)$, $\theta_n \in \Theta$ and $\lambda_n \geq 0$. In view of the compactness of Θ, without loss of generality, we may assume that the sequence (θ_n) converges to $\theta \in \Theta$.

We start by showing that under our assumptions, (λ_n) contains a bounded subsequence. Indeed, if $\lambda_n \to +\infty$, then

$$\frac{1}{\lambda_n}(x_n + \lambda_n \theta_n) = \frac{1}{\lambda_n} x_n + \theta_n \to 0,$$

and $\lim_n \frac{1}{\lambda_n} x_n = -\theta$ since $\theta_n \to \theta \neq 0$. On the other hand,

$$\left\langle b_i, \frac{1}{\lambda_n} x_n \right\rangle \leq \frac{1}{\lambda_n} c_i, \qquad i \in I,$$

and, by passing to the limit, $\langle b_i, -\theta \rangle \leq 0$, i.e., $-\theta \in \mathrm{Rec}(C) \cap (-\mathcal{K})$, which contradicts the assumption that $E(C) \neq \emptyset$ (see the remark above).

Consequently, (λ_n) contains a convergent subsequence (λ_{n_ℓ}), $\lambda_{n_\ell} \to \lambda \geq 0$. Moreover, $\lambda_{n_\ell}\theta_{n_\ell} \to k \in \mathcal{K}$ and $x_{n_\ell} \to x \in E(C)$ since $E(C)$ is closed by Proposition 5.1.4. Finally, $z = x + k \in E(C) + \mathcal{K}$. ∎

If $E(C) = WE(C)$ and (DP) holds for C, then

$$C \subset WE(C) + \operatorname{int} \mathcal{K} \cup \{0\}. \tag{5.6}$$

THEOREM 5.1.4. *Let $\mathcal{K}$ be a closed convex pointed cone in $\mathbb{R}^m$. Let $C \subset \mathbb{R}^m$ be a polyhedral set of the form* (5.5). *The following statements are equivalent:*

(i) *(DP) holds for C and $E(C) = WE(C)$,*
(ii) *(CP) holds for C.*

Proof. The implication (ii)⇒(i) is immediate. To prove that (i)⇒(ii) suppose on the contrary that (CP) does not hold for C. There exist $\varepsilon_0 > 0$ and a sequence $(y_n) \subset C(\varepsilon_0)$ such that $B(y_n, 1/n) \cap (C + \mathcal{K})^c \neq \emptyset$. Consequently, one can choose a sequence $(z_n) \subset \partial(E(C) + \mathcal{K})$, where $\partial(\cdot)$ stands for the boundary, with $\lim_n (y_n - z_n) = 0$. If $z_n \in C$ for at least one $n \geq 1$, then $z_n \in WE(C) \setminus E(C)$, a contradiction. Hence, $z_n \notin C$ for all $n \geq 1$ and

$$(z_n - \mathcal{K}) \cap (E(C) + \mathcal{K}) \subset \partial(E(C) + \mathcal{K}). \tag{5.7}$$

By Proposition 5.1.5, $E(C) + \mathcal{K}$ is closed, and hence, $z_n = \eta_n + \lambda_n \theta_n$, where $\eta_n \in E(C)$, $\theta_n \in \Theta$ and $\lambda_n \geq 0$. Moreover, since there exists $M > 0$ such that $\|\theta\| \leq M$, we have

$$\lambda_n M \geq \lambda_n \|\theta_n\| = \|z_n - \eta_n\| > \varepsilon_0$$

and $\lambda_n > \varepsilon_0 / M$. We can assume that $\lambda_n > 1$.

Since $z_n \notin C$, there is a subset I_1 of the index set I such that

$$\langle b_i, z_n \rangle > c_i \quad \text{for } i \in I_1 \quad \text{and} \quad \langle b_i, z_n \rangle \leq c_i \quad \text{for } i \in I \setminus I_1.$$

We claim that there exist an infinite subset $N_1 \subset N$ and an index $i \in I_1$ such that $\langle b_i, \eta_n \rangle = c_i$ for $n \in N_1$. Indeed, if $\langle b_i, \eta_n \rangle < c_i$ for all $n \geq 1$ and $i \in I_1$, then

$$\beta_n = \frac{1}{2} \min_{i \in I_1} \frac{c_i - \langle b_i, \eta_n \rangle}{\langle b_i, \theta_n \rangle} > 0.$$

Clearly, $\lambda_n > (c_i - \langle b_i, \eta_n\rangle)/\langle b_i, \theta_n\rangle > \beta_n$ for all $i \in I_1$ and $n \geq 1$. There is a subset $I_2 \subset I$ such that $\langle b_i, \theta_n\rangle > 0$ for $i \in I_2$, and $\langle b_i, \theta_n\rangle \leq 0$ for $i \in I \setminus I_2$, where $I_1 \subset I_2$. Hence,

$$\langle b_i, z_n - (\lambda_n - \beta_n)\theta_n\rangle = \begin{cases} \langle b_i, \eta_n\rangle + \beta_n\langle b_i, \theta\rangle \leq \langle b_i, \eta_n\rangle + \frac{c_i - \langle b_i, \eta_n\rangle}{\langle b_i, \theta_n\rangle}\langle b_i, \theta_n\rangle = c_i, & i \in I_1, \\ \langle b_i, \eta_n\rangle + \beta_n\langle b_i, \theta_n\rangle \leq \langle b_i, \eta_n\rangle + \lambda_n\langle b_i, \theta_n\rangle \leq c_i, & i \in I_2 \setminus I_1, \\ \langle b_i, \eta_n\rangle \leq c_i, \quad i \in I \setminus I_2. \end{cases}$$

This means that $w_n = z_n - (\lambda_n - \beta_n)\theta_n \in C \cap (z_n - \mathcal{K})$, and by (5.6),

$$w_n \in E(C) + \operatorname{int}\mathcal{K} \subset \operatorname{int}(E(C) + \mathcal{K}),$$

contrary to (5.7). This proves that $\langle b_i, \eta_n\rangle = c_i$ for some $i \in I_1$ and $n \in N_1 \subset N$.

By letting $H_i = \{y \in \mathbb{R}^m : \langle b_i, y\rangle = c_i\}$ we get

$$\|y_n - z_n\| \geq d(z_n, H_i) = \frac{\langle b_i, z_n\rangle - c_i}{\sqrt{(b_i)^2}} = \frac{\lambda_n\langle b_i, \theta_n\rangle}{\sqrt{(b_i)^2}},$$

which implies that $\lambda_n \to 0$. This is a contradiction. ∎

THEOREM 5.1.5. *Let $\mathcal{K}$ be a closed convex pointed cone in $\mathbb{R}^m$. Let $C \subset \mathbb{R}^m$ be a polyhedral set of the form* (5.5). *The following statements are equivalent:*

(i) $\operatorname{Rec}(C) \cap (-\mathcal{K}) = \{0\}$ *and* $E(C) = WE(C)$,
(ii) (*CP*) *holds for* C.

Proof. See [31]. ∎

5.2. Dual containment property

In this section we define the dual containment property (*DCP*) which in some instances provides a dual characterization of (*CP*).

Let $\mathcal{K}$ be a closed convex pointed cone in a locally convex space Y and let $\mathcal{K}^*$ be its dual with base Θ^*. Let C be a subset of Y.

DEFINITION 5.2.1. The *dual containment property* (*DCP*) holds for C with respect to Θ^* if for every 0-neighbourhood W there exists $\delta > 0$ for which the following condition holds:

(C1) for each $y \in C(W)$ there exists $\eta_y \in E(C)$ satisfying

$$\theta^*(y - \eta_y) > \delta \quad \text{for each } \theta^* \in \Theta^*.$$

Note that if $\theta^*(y - \eta_y) > \delta$ for some positive $\delta > 0$ and all $\theta^* \in \Theta^*$, then $y - \eta_y \in \mathcal{K}^i$, where $\mathcal{K}^i$ is defined in Section 1.1. In the spaces ℓ^p, $L^p(\Omega)$, $p \geq 1$, the quasi-interior $\mathcal{K}^i_+$ of the positive cone $\mathcal{K}_+$,

$$\mathcal{K}^i_+ = \{k \in \mathcal{K}_+ : f(k) > 0 \text{ for } f \in \mathcal{K}^*_+ \setminus \{0\}\}.$$

coincides with the set of weak order units (see [122, p. 184]), i.e., for any $y_0 \in \mathcal{K}^i_+$ and any $y \in \mathcal{K}_+$, $y \neq 0$, there exists $z \in \mathcal{K}_+$, $z \neq 0$, such that $z \preceq y_0$ and $z \preceq y$. Characterizations of quasi-interiors of cones of nonnegative elements are given by Peressini (see [122, Ex. 4.4, p. 186]).

EXAMPLE 5.2.1. 1. Let $Y = \mathbb{R}^m$, $\mathcal{K} \subset Y$ be a closed convex pointed cone. For any convex set C in Y, core(C) coincides with $\operatorname{int} C$. Hence, for $\mathcal{K} = \{(y_1, y_2) : y_1 \geq 0,\ y_1 = y_2\}$ we get $\mathcal{K}^* = \{(f_1, f_2) : f_2 \geq -f_1\}$ and $\mathcal{K}^i = \emptyset$.

2. For any $p \in [1, \infty)$ consider the sequence space ℓ^p of sequences $s = (s_i)$ with real terms,

$$\ell^p = \Big\{ s = (s_i) : \sum_{i=1}^{\infty} |s_i|^p < \infty \Big\},$$

with the natural ordering cone

$$\ell^p_+ = \{s = (s_i) \in \ell^p : s_i \geq 0\}.$$

The ordering cone ℓ^p_+ has empty topological interior and empty algebraic interior, core(ℓ^p_+) $= \emptyset$. But $(\ell^p_+)^i = \{s = (s_i) \in \ell^p : s_i > 0\}$.

3. For any $p \in [1, \infty)$, consider the space of all Lebesgue p-integrable functions $f : \Omega \to \mathbb{R}$ with the natural ordering cone

$$L^p_+ = \{f \in L^p : f(x) \geq 0 \text{ almost everywhere on } \Omega\}.$$

The topological interior $\operatorname{int}(L^p_+)$ and core$(L^p)_+$ are both empty but $(L^p_+)^i \neq \emptyset$. To see this recall that

$$(L^p_+)^i = \Big\{ f \in L^p : \int_\Omega fg\, d\mu > 0 \text{ for all } g \in L^q_+ \setminus \{0\} \Big\},$$

$1/p + 1/q = 1$, and

$$(L^p_+)^i = \{f \in L^p : f(x) > 0 \text{ almost everywhere on } \Omega\}.$$

We say that *the dual containment property* (*DCP*) *holds for* C if there exists a base Θ^* of $\mathcal{K}^*$ such that (*DCP*) holds for C with respect to Θ^*. If $\operatorname{int} \mathcal{K} \neq \emptyset$ and $e \in \operatorname{int} \mathcal{K}$, then $\Theta^* = \{f \in \mathcal{K}^* : f(e) = 1\}$ (see Theorem 1.1.1 of Section 1.1) is a base of $\mathcal{K}^*$. Let $y_0 \in \mathcal{K}^i$. Recall that the standard base of $\mathcal{K}^*$ related to y_0 has the form

$$\Theta^*(y_0) = \{\theta^* \in \mathcal{K}^* : \theta^*(y_0) = 1\}. \tag{5.8}$$

We have the following proposition.

PROPOSITION 5.2.1. *Let Y be a Hausdorff topological vector space with a closed convex cone $\mathcal{K} \subset Y$. Assume that* (*DCP*) *holds for C with respect to a standard base $\Theta^*(y_0)$ of $\mathcal{K}^*$, $y_0 \in \mathcal{K}^i$. Then*

(i) (*DCP*) *holds for C with respect to any standard base $\Theta^*(\overline{y})$ of $\mathcal{K}^*$, $\overline{y} \in \mathcal{K}^i$, where $y_0 \in \varrho \cdot \overline{y} + \mathcal{K}$, $\varrho > 0$,*

(ii) *if $\Theta^*(y_0)$ is bounded,* (*DCP*) *holds for C with respect to any standard base $\Theta^*(\overline{y})$, $\overline{y} \in \mathcal{K}^i$, of $\mathcal{K}^*$.*

Proof. (i) For each $\theta^* \in \Theta^*(\overline{y})$ there is $\theta_0^* \in \Theta^*(y_0)$ such that

$$\theta^*(k) = \theta^*(y_0)\theta_0^*(k) \quad \text{ for all } k \in \mathcal{K}. \tag{5.9}$$

Since $y_0 = \varrho \cdot \overline{y} + k_0$, $k_0 \in \mathcal{K}$, we get $\theta^*(y_0) = \varrho + \theta^*(k_0) \geq \varrho$. Hence, $\theta^*(k) = \theta^*(y_0)\theta_0^*(k) \geq \varrho\theta_0^*(k)$ and the conclusion follows.

(ii) By (5.9), $1 = \theta_0^*(\overline{y})\theta^*(y_0)$. Since $\Theta^*(y_0)$ is bounded, there exists $m_0 > 0$ such that $\theta_0^*(\overline{y}) \le m_0$ and $\theta^*(y_0) = 1/\theta_0^*(\overline{y}) \ge 1/m_0$ for some $m_0 > 0$ and, as previously, the conclusion follows. ∎

In locally convex spaces, if (*DCP*) holds for C, then

$$C \subset \operatorname{cl} E(C) + \mathcal{K}. \tag{5.10}$$

Indeed, if $y \in C \setminus \operatorname{cl} E(C)$ there exists $\varepsilon > 0$ such that $y \notin B(E(C), \varepsilon)$. By (*DCP*), there exist $\eta \in E(C)$ and $\delta > 0$ such that $\theta^*(y - \eta) > \delta$ for each $\theta^* \in \Theta^*$ and hence $y - \eta \in \mathcal{K}^i \subset \mathcal{K}$.

When Y is an order complete vector lattice of efficient type (see [140, Ch. V, p. 213]), any point $k \in \mathcal{K}^i$ is proved to be a quasi-interior point of $\mathcal{K}$, where $k \in \mathcal{K}$ is said to be a *quasi-interior point* of $\mathcal{K}$ if the order interval $[0, k]$ is a total subset of Y in the sense that its linear hull is dense in Y (see Schaefer [140, Ch. V. 8, Th. 7.7], and Peressini [122, Ch. 4.4]). Moreover, each $k \in \mathcal{K}^i$ is a *weak order unit* (see [122]), i.e., for each $y \in \mathcal{K}$ there exists $z \in \mathcal{K}$ with $z \preceq_{\mathcal{K}} y$ and $z \preceq_{\mathcal{K}} k$.

EXAMPLE 5.2.2. Let $Y = (\mathbb{R}^2, \|\cdot\|)$ and let $\mathcal{K} = \{(y_1, y_2) : y_1 \ge 0\}$. Let $C = \{(y_1, y_2) : |y_1| + |y_2| \le 1\}$. We have $\mathcal{K}^* = \{(f_1, f_2) : f_1 \ge 0,\ f_2 = 0\}$ and $E(C) = \{(-1, 0)\}$. Consider $\Theta^* = \{(f_1, f_2) \in \mathcal{K}^* : f_1 = 1\}$. Take $\varepsilon > 0$. For any $(y_1, y_2) \in C(\varepsilon)$ we have $y_1 \ge -1 + \sqrt{\varepsilon/2}$ and hence, for any $\theta^* \in \Theta^*$, we have $\theta^*(y_1 + 1, y_2) = y_1 + 1 \ge \sqrt{\varepsilon/2} = \delta$ and (*DCP*) holds.

EXAMPLE 5.2.3. Let Y, $\mathcal{K}$, and Θ^* be as in the previous example. Let $C = \{(y_1, y_2) : \max\{|y_1|, |y_2|\} \le 1\} \setminus \{(y_1, y_2) : y_1 = 1, -1 < y_2 \le 1\}$. We have $E(C) = \{(-1, -1)\}$, (*DCP*) does not hold for Θ^* since for $y_n = (-1 + 1/n, 1) \in C$ we have $\theta^*(y_n - (-1, -1)) = 1/n \to 0$.

EXAMPLE 5.2.4. Let $Y = \ell_1$ and $\mathcal{K} = \ell_1^+$. We have $(\ell_1^+)^i = \{y = (y_i) \in \ell_1 : y_1 > 0\}$. Take $y_0 = (1/i^2) \in (\ell_1^+)^i$. Let $\Theta^* \subset (\ell_1^+)^*$ be a base of $(\ell_1^+)^*$ of the form

$$\Theta^* = \{\theta \in \mathcal{K}^* : \theta^*(y_0) = 1\}.$$

Let $y_1 = 2y_0 + (0, 1, 0, \ldots)$, $y_2 = 3y_0$. Taking $C = \operatorname{conv}(y_0, y_1, y_2)$, where conv stands for convex hull, we have $E(C) = \{y_0\}$ and for any $y \in C$, $y = \lambda_0 y_0 + \lambda_1 y_1 + \lambda_2 y_2$, $\lambda_i \ge 0$, $i = 0, 1, 2$, $\lambda_0 + \lambda_1 + \lambda_2 = 1$. For any $\varepsilon > 0$,

$$C(\varepsilon) = \{y \in C : \|y - y_0\| > \varepsilon\} = \{y \in C : \lambda_1 \pi^2/6 + \lambda_1 + 2\lambda_2 \pi^2/6 > \varepsilon\}.$$

For any $\theta^* = (\theta_i) \in \Theta^*$ and $y \in C(\varepsilon)$ we have

$$\theta^*(y - y_0) = \theta^*(\lambda_1 y_0 + \lambda_1(0, 1, 0, \ldots) + 2\lambda_2 y_0) = \lambda_1 + \theta_2 \lambda_1 + 2\lambda_2 \ge \lambda_1 + \lambda_2 > 3\varepsilon/\pi^2 = \delta,$$

which proves that (*DCP*) holds for C.

Let $y_0 \in \mathcal{K}^i$ and let $\Theta^*(y_0)$ be the standard base of the dual cone $\mathcal{K}^*$. If (*DCP*) holds for the base $\Theta^*(y_0)$, condition (C1) can be rewritten as

(C2) for each $y \in C(W)$ there exists $\eta_y \in E(C)$ satisfying

$$y - \eta_y - \delta y_0 \in \mathcal{K}^i.$$

PROPOSITION 5.2.2. *Let Y be a locally convex space and let $\mathcal{K} \subset Y$ be a closed convex cone with* $\operatorname{int} \mathcal{K} \neq \emptyset$. *For any subset C of Y, (CP) is equivalent to (DCP).*

Proof. Let W be a 0-neighbourhood. By (*CP*), there exists a 0-neighbourhood O such that for each $y \in C(W)$,

$$y - \eta_y + O \subset \mathcal{K} \quad \text{for some } \eta_y \in E(C).$$

Take any $y_0 \in \mathcal{K}^i = \operatorname{int} \mathcal{K}$. Since O can be assumed to be radial, $-\delta y_0 \in O$ for some $\delta > 0$ and $y - \eta_y - \delta y_0 \in \mathcal{K}$, which means that (*DCP*) holds for C.

To see the converse implication, note that by Theorem 1.1.1, $\mathcal{K}^*$ has a weak* compact, hence bounded base Θ^*. By Proposition 5.2.1, (*DCP*) holds for Θ^*. ■

PROPOSITION 5.2.3. *Let Y be a locally convex space and let $\mathcal{K}$ be a closed convex cone in Y. Let $\mathcal{K}^*$ have a bounded base Θ^*. If (DCP) holds for C, then* $\operatorname{int} \mathcal{K} \neq \emptyset$.

Proof. Let W be a 0-neighbourhood. By (*DCP*), there exists $\delta > 0$ such that for each $y \in C(W)$ there is $\eta_y \in E(C)$ such that $\theta^*(y - \eta_y) > \delta$ for $\theta^* \in \Theta^*$. Since Θ^* is bounded there exists a 0-neighbourhood Q such that for any $\theta^* \in \Theta^*$ we have $-\delta/2 < \theta^*(q) < \delta/2$ for $q \in Q$. Consequently, $\theta^*(y - \eta_y + q) > \delta/2$ for any $\theta^* \in \Theta^*$, which proves that $y - \eta_y + Q \in \mathcal{K}$. ■

5.3. Containment rate

Numerous concepts in functional analysis can be characterized by constants and functions of a single real variable. For instance, by using the modulus of convexity $\delta_X(\varepsilon)$ due to Clarkson [45],

$$\delta_X(\varepsilon) = \inf\{1 - \|\tfrac{1}{2}(x + y)\| : x, y \in B_X,\ \|x - y\| \geq \varepsilon\}$$

one can characterize strict convexity and uniform rotundity of the unit ball B_X in the space X. In the present section we define the containment rate (cf. [19, 20]) which is a nondecreasing function of a single variable. The containment rate is used to characterize the containment property. The properties of the containment rate are used in the next chapters to investigate Lipschitz and/or Hölder behaviour of efficient points under perturbations. Similar approaches have been applied in many other domains (see e.g. [12, 51, 80, 81, 119, 113]).

Let $Y = (Y, \|\cdot\|)$ be a normed space and let $\mathcal{K}$ be a closed convex pointed cone in Y. Recall that for any subset C of Y and any $\varepsilon > 0$, the ball of radius ε around C is $B(C, \varepsilon) = \{y \in Y : d(y, C) < \varepsilon\}$, and $C(\varepsilon) = C \setminus B(E(C), \varepsilon)$, and the containment property (*CP*) holds for C if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$C(\varepsilon) + \delta B_Y \subset E(C) + \mathcal{K}. \tag{5.11}$$

Recall that $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is an *admissible* function, i.e. ϕ is nondecreasing, $\phi(t) > 0$ for $t > 0$ and $\phi(0) = 0$.

The following immediate observation is the starting point for our considerations in this section: if there exists an admissible function ϕ such that for each $y \in C$ there exists

$\eta \in E(C)$ satisfying

$$y - \eta + \phi(d(y, E(C)))B_Y \subset \mathcal{K}, \tag{5.12}$$

then (CP) holds for C. Indeed, if we take any $\varepsilon > 0$ and $y \in C(\varepsilon)$, then by taking $\delta := \phi(\varepsilon) \leq \phi(d(y, E(C)))$ we immediately get (5.11).

Below we give a construction of an admissible function ϕ which provides a characterization of (CP).

We start with the definition of the containment function for a closed convex pointed cone $\mathcal{K}$ in Y.

Definition 5.3.1 ([19]). The function $\operatorname{cont} : \mathcal{K} \to \mathbb{R}_+$ defined as

$$\operatorname{cont}(k) = \sup\{r \geq 0 : k + rB_Y \subset \mathcal{K}\}$$

is called the *primal cone containment function.*

The supremum in the above definition is attained since $\mathcal{K}$ is closed. The function cont is positively homogeneous and superlinear and

$$\operatorname{dom} \operatorname{cont} = \{k \in \mathcal{K} : \operatorname{cont}(k) > -\infty\} = \mathcal{K}.$$

Clearly, $\operatorname{cont}(k) \leq \|k\|$ for any $k \in \mathcal{K}$ and $\operatorname{cont} \equiv 0$ whenever $\operatorname{int} \mathcal{K} = \emptyset$. For $k \in \mathcal{K}$ we have $\operatorname{cont}(k) = -\Delta_{\mathcal{K}}(k)$, where $\Delta_{\mathcal{K}}(y) = d(y, \mathcal{K}) - d(y, Y \setminus \mathcal{K})$, $y \in Y$. The function $\Delta_{\mathcal{K}}$ was introduced in [76, 77] to derive optimality conditions in nonsmooth optimization. It was also used in [155] as a scalarizing function for vector optimization problems.

Let C be a subset of Y and let $y \in Y$. Recall that the set

$$C_y = C \cap (y - \mathcal{K})$$

is the section of C with respect to $\mathcal{K}$ and y (cf. Section 2.2).

Definition 5.3.2 ([19, 20]). The function $\mu : Y \to \mathbb{R}$ defined as

$$\mu(y) = \sup\{\operatorname{cont}(y - \eta) : \eta \in E(C)_y\} \tag{5.13}$$

is the *containment rate* of y with respect to C and $\mathcal{K}$.

For any $y \in Y$ put

$$\|y\|_+ = d(y, Y \setminus \mathcal{K}).$$

For any $r \geq 0$,

$$\|y\|_+ \geq r \Leftrightarrow y + rB_Y \subset \mathcal{K}.$$

Hence, for $k \in \mathcal{K}$ we have $\operatorname{cont}(k) = \|k\|_+$ and

$$\mu(y) = \sup\{\|y - \eta\|_+ : y \in E(C)_y\}.$$

We have

$$\operatorname{dom} \mu = \{y \in Y : \mu(y) > -\infty\} = E(C) + \mathcal{K}.$$

Clearly, $\mu(y) = 0$ for $y \in E(C)$. If $\operatorname{int} \mathcal{K} \neq \emptyset$ and $y \in E(C) + \mathcal{K}$ we have $\mu(y) \geq 0$ and moreover, $\mu(y) = 0$ if and only if $y \in WE(C)$ (see Proposition 5.3.6 below).

The value $\mu(y)$ gives the maximal radius r such that $k + rB_Y \subset \mathcal{K}$ for all $k \in y - E(C)_y$. In this sense $\mu(y)$ measures the deviation from efficiency for y.

DEFINITION 5.3.3 ([19, 20]). The function $\delta : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty, -\infty\}$ defined as

$$\delta(\varepsilon) = \inf\{\mu(y) : y \in C(\varepsilon)\}$$

is the *containment rate* of C with respect to $\mathcal{K}$.

The domain of δ is

$$\operatorname{dom} \delta = \{\varepsilon \in \mathbb{R}_+ : \delta(\varepsilon) < \infty\} = \{\varepsilon \in \mathbb{R}_+ : C(\varepsilon) \neq \emptyset\}.$$

Below we prove that δ is an admissible function if and only if (*CP*) holds for C. We start with conditions ensuring that the supremum in the definition of the function μ is attained.

PROPOSITION 5.3.1. *Let $Y = (Y, \|\cdot\|)$ be a normed space. Let $\mathcal{K}$ be a closed convex pointed cone in Y and let C be a subset of Y. Let $y \in E(C) + \mathcal{K}$. If $E(C)_y$ is weakly compact, then there exists $\eta_y \in E(C)$ such that $y - \eta_y + \mu(y)B_Y \subset \mathcal{K}$.*

Proof. Let $y \in E(C) + \mathcal{K}$. For each $n \geq 1$, we have $y = \eta_n + k_n$, where $\eta_n \in E(C)_y$ and $k_n + \operatorname{cont}(k_n)B_Y \subset \mathcal{K}$ satisfy

$$\operatorname{cont}(k_n) \leq \mu(y) \quad \text{and} \quad \operatorname{cont}(k_n) > \mu(y) - 1/n.$$

Since $E(C)_y$ is weakly compact, there exists a weakly convergent subsequence (η_{n_m}) with limit point $\eta_0 \in E(C)_y$. Consequently, $k_{n_m} = y - \eta_{n_m}$ converges weakly to some $k_0 \in \mathcal{K}$ and $y = \eta_0 + k_0$.

To complete the proof we show that $k_0 + \mu(y)B_Y \subset \mathcal{K}$. On the contrary, if $k_0 + \mu(y)b \notin \mathcal{K}$ for some $b \in B_Y$, then by separation arguments

$$f(k_0 + \mu(y)b) < 0 < f(k) \quad \text{for } k \in \mathcal{K},$$

for some $f \in \mathcal{K}^*$. Since $k_{n_m} \xrightarrow{w} k_0$ and $(\operatorname{cont}(k_{n_m}) - \mu(y))b \to 0$, we would have

$$f(k_{n_m} + \operatorname{cont}(k_{n_m})b) = f(k_0 + \mu(y)b) + f(k_{n_m} - k_0) + f((\operatorname{cont}(k_{n_m}) - \mu(y))b) < 0,$$

which contradicts the fact that $k_{n_m} + \operatorname{cont}(k_{n_m})B_Y \subset \mathcal{K}$. ∎

The assertion of Proposition 5.3.1 can also be obtained as a consequence of the Weierstrass theorem on existence of infimum over compact sets. To this end it is enough to note that $\|y - \cdot\|_+$ is a weakly lower semicontinuous function.

Following [42] we say that $R_\sigma(C)$ is the *generalized recession cone* of a set $C \subset Y$ if

$$R_\sigma(C) = \{v \in Y : \text{there exist } \lambda_n > 0 \text{ with } \lambda_n \to 0 \text{ and } c_n \in C \text{ such that } \lambda_n c_n \text{ tends weakly to } v\}.$$

A set $C \subset Y$ is $\mathcal{K}$-*lower bounded* if there is a constant $M > 0$ such that

$$C \subset MB_Y + \mathcal{K}.$$

If $C \subset Y$ is $\mathcal{K}$-lower bounded, then $R_\sigma(C) \subset \mathcal{K}$ (see [42]).

PROPOSITION 5.3.2. *Under any of the conditions:*

(i) $E(C)$ *is weakly compact,*
(ii) $E(C)$ *is $\mathcal{K}$-lower bounded and weakly closed and $\mathcal{K}$ has a weakly compact base,*

the sections $E(C)_y$ are weakly compact for $y \in E(C) + \mathcal{K}$.

Proof. Let $y \in E(C) + \mathcal{K}$. For each $n \geq 1$ there is a representation $y = \eta_n + k_n$ with $\eta_n \in E(C)$, $k_n + \operatorname{cont}(k_n)B_Y \subset \mathcal{K}$ satisfying

$$\operatorname{cont}(k_n) \leq \mu(y) \quad \text{and} \quad \operatorname{cont}(k_n) > \mu(y) - 1/n.$$

We start by proving that under any of the conditions (i) or (ii) the sequences (η_n) and (k_n) contain convergent subsequences with limit points η_0 and k_0, respectively, and

$$y = \eta_0 + k_0. \tag{5.14}$$

If (i) holds, then (η_n) contains a weakly convergent subsequence. We can assume that (η_n) weakly converges to some $\eta_0 \in E(C)$. Since $\mathcal{K}$ is closed and convex, the sequence (k_n), $k_n = y - \eta_n$, converges weakly to $k_0 \in \mathcal{K}$ and $y = \eta_0 + k_0$.

Suppose now that (ii) holds and Θ is a weakly compact base of $\mathcal{K}$. Then $k_n = \lambda_n \theta_n$, where $\lambda_n \geq 0$ and $(\theta_n) \subset \Theta$ contains a weakly convergent subsequence. We can assume that (θ_n) converges to $\theta_0 \in \Theta$. If $\lambda_n \to \infty$, then

$$\frac{1}{\lambda_n}(\eta_n - y) \xrightarrow{w} -\theta_0$$

and $-\theta_0 \in R_\sigma(E(C)) \cap (-\mathcal{K})$, which contradicts the $\mathcal{K}$-lower boundedness of $E(C)$. Hence, (λ_n) is bounded and (k_n) weakly converges to some $k_0 = \lambda_0\theta_0 \in \mathcal{K}$. Consequently, $\eta_n = y - k_n$ converges weakly to some $\eta_0 \in E(C)$ and we get (5.14). ∎

Now we are in a position to prove the main propositions of this section.

PROPOSITION 5.3.3. *Let Y be a normed space and let $\mathcal{K}$ be a closed convex pointed cone with* $\operatorname{int}\mathcal{K} \neq \emptyset$. *Let C be a nonempty subset of Y. The following are equivalent:*

(i) (CP) *holds for* C,
(ii) $\delta : \operatorname{dom}\delta \to \mathbb{R}_+$ *is an admissible function.*

Proof. (i)⇒(ii). Clearly, δ is nondecreasing and $\delta(0) = 0$. By Proposition 5.1.3, for any $\varepsilon \in \operatorname{dom}\delta$ there exists $\gamma > 0$ such that for $y \in C(\varepsilon) \neq \emptyset$ one can find $\eta \in E(C)$ satisfying $(y - \eta) + \gamma B_Y \subset \mathcal{K}$. Consequently, $\mu(y) \geq \gamma$ and $\delta(\varepsilon) \geq \gamma > 0$.

(ii)⇒(i). Let $\varepsilon \in \operatorname{dom}\delta$, $\varepsilon > 0$. Hence, $\delta(\varepsilon) = \gamma > 0$ and $\mu(y) \geq \gamma$ for any $y \in C(\varepsilon)$, which means there exists $\eta \in E(C)$ such that $(y - \eta) + (\gamma/2)B_Y \subset \mathcal{K}$. Thus, (CP) holds for C. ∎

PROPOSITION 5.3.4. *Let $\mathcal{K}$ be a closed convex pointed cone in a normed space Y with* $\operatorname{int}\mathcal{K} \neq \emptyset$. *Let C be a nonempty subset of Y and assume (CP) holds for C. If all the sections $E(C)_y$ for $y \in E(C) + \mathcal{K}$ are weakly compact then for any $\varepsilon > 0$,*

(i) $C(\varepsilon) + \delta(\varepsilon)B_Y \subset E(C) + \mathcal{K}$,
(ii) *for all $\varepsilon > 0$ and for each $y \in C(\varepsilon)$ there exists $\eta \in E(C)$ such that $y - \eta + \delta(\varepsilon)B_Y \subset \mathcal{K}$.*

Proof. (ii) follows directly from Proposition 5.3.1. (i) follows from (ii). ∎

In the example below we calculate $\mu(y)$ for y from the closed unit ball.

EXAMPLE 5.3.1. Let $Y = \mathbb{R}^2$, $\mathcal{K} = \mathbb{R}^2_+$ and $C = \operatorname{cl} B_Y$. Clearly, (DP) and (CP) hold for C and

$$E(C) = \{(\eta_1, \eta_2) \in C : \eta_2 = -\sqrt{1-\eta_1^2},\ -1 \le \eta_1 \le 0\}.$$

For any representation of $(0,0)$ in the form $(0,0) = \eta + k_\eta$, where $\eta \in E(C)$, $k_\eta \in \mathcal{K}$, we have $\eta = (\eta_1, \eta_2) \in E(C)_{(0,0)} = E(C)$ and

$$\operatorname{cont}(k_\eta) = \min\{-\eta_1, \sqrt{1-\eta_1^2}\} = \begin{cases} \sqrt{1-\eta_1^2} & \text{for } -1 \le \eta_1 \le -1/\sqrt{2}, \\ -\eta_1 & \text{for } -1/\sqrt{2} \le \eta_1 \le 0, \end{cases}$$

and $\mu((0,0)) = \sup_{\{-1\le\eta_1\le 0\}} \operatorname{cont}(k_\eta) = 1/\sqrt{2}$. For $y \in C$ with $y_2 \ge 0$ we have

$$E(C)_{(y_1,y_2)} = \{(\eta_1,\eta_2) : \eta_2 = -\sqrt{1-\eta_1^2},\ -1 \le \eta_1 \le \min\{0, y_1\}\}$$

and

$$\mu(y) = \max_{\{-1\le\eta_1\le\min\{0,y_1\}\}} \operatorname{cont}(k_\eta) = \max_{\{-1\le\eta_1\le\min\{0,y_1\}\}} \min\{y_1 - \eta_1, y_2 + \sqrt{1-\eta_1}\}.$$

For $y \in C$ with $y_2 < 0$ we have

$$E(C)_{(y_1,y_2)} = \{(\eta_1,\eta_2) : \eta_2 = -\sqrt{1-\eta_1^2},\ -\sqrt{1-y_2^2} \le \eta_1 \le \min\{0, y_1\}\}$$

and

$$\begin{aligned}\mu(y) &= \max_{\{-\sqrt{1-y_2^2}\le\eta_1\le\min\{0,y_1\}\}} \operatorname{cont}(k_\eta) \\ &= \max_{\{-\sqrt{1-y_2^2}\le\eta_1\le\min\{0,y_1\}\}} \min\{y_1 - \eta_1, y_2 + \sqrt{1-\eta_1^2}\}.\end{aligned}$$

We close this section with characterizations of (DP) and weak efficiency in terms od δ and μ, respectively.

PROPOSITION 5.3.5. *Let Y be a normed space and let $\mathcal{K}$ be a closed convex pointed cone. Let C be a nonempty subset of Y with $E(C)$ nonempty and closed. The following statements are equivalent:*

(i) *(DP) holds for C,*
(ii) *$\delta(\varepsilon) \ge 0$ for all $\varepsilon \in \operatorname{dom}\delta$.*

Proof. (ii)⇒(i). Suppose that (DP) does not hold for C. There exists $y \in C$ which cannot be represented in the form $y = \eta + k$, where $\eta \in E(C)$ and $k \in \mathcal{K}$. Hence, $\mu(y) = -\infty$. By closedness of $E(C)$, $y \in C(\varepsilon)$ for some $\varepsilon > 0$. Consequently, $\delta(\varepsilon) = -\infty$, which contradicts (ii).

(i)⇒(ii). By (DP), for each $y \in C$ we have $y = \eta + k$ where $\eta \in E(C)$ and $k \in \mathcal{K}$. Hence, $\mu(y) \ge 0$ and (ii) follows. ∎

PROPOSITION 5.3.6. *Let Y be a normed space and let $\mathcal{K}$ be a closed convex cone in Y with $\operatorname{int}\mathcal{K} \ne \emptyset$. Let C be a nonempty subset of Y and assume (DP) holds for C. The following are equivalent:*

(i) *$\mu(y) = 0$,*
(ii) *$y \in WE(C)$.*

Proof. (i)⇒(ii). By (i), any representation of y in the form $y = \eta + k$, where $\eta \in E(C)$ and $k \in \mathcal{K}$, satisfies $k \in \partial\mathcal{K}$, which means that $C \cap (y - \operatorname{int} \mathcal{K}) = \emptyset$, i.e., $y \in WE(C)$.

(ii)⇒(i). If $\mu(y) \geq \alpha > 0$, then $y = \eta + k$ with $\eta \in E(C)$ $k + \alpha B_Y \subset \mathcal{K}$ which implies that $y \notin WE(C)$. ∎

5.4. Dual containment rate

Let $\mathcal{K}$ be a closed convex pointed cone in a normed space $(Y, \|\cdot\|)$ with the dual $\mathcal{K}^* \subset Y^*$. Let Θ^* be a base of $\mathcal{K}^*$.

Definition 5.4.1 ([20]). The function $\operatorname{dcont}_{\Theta^*} : \mathcal{K} \to \mathbb{R}_+$ defined as

$$\operatorname{dcont}_{\Theta^*}(k) = \inf\{\theta^*(k) : \theta^* \in \Theta^*\}$$

is called the Θ^*-*dual cone containment function.*

If it is clear from the context which base Θ^* is used, we omit the index Θ^* in the notation. The terminology "primal cone containment function" and "dual cone containment function" is motivated by the fact that in some instances these functions yield a pair of dual linear programming problems.

Let C be a subset of Y and $y \in Y$. Recall that $E(C)_y = E(C) \cap (y - \mathcal{K})$.

Definition 5.4.2 ([20]). The function $\nu : Y \to \mathbb{R} \cup \{\pm\infty\}$ defined as

$$\nu(y) = \sup\{\operatorname{dcont}_{\Theta^*}(y - \eta) : \eta \in E(C)_y\}$$

is the *dual containment rate* of y with respect to C and $\mathcal{K}$.

It follows directly from the definition that $\{y \in Y : \nu(y) > -\infty\} = E(C) + \mathcal{K}$ and $\nu(y) \geq 0$ for $y \in E(C) + \mathcal{K}$.

Definition 5.4.3 ([20]). The function $d : \mathbb{R}_+ \to \mathbb{R}$ defined as

$$d(\varepsilon) = \inf\{\nu(y) : y \in C(\varepsilon)\}$$

is the *dual containment rate* of C with respect to $\mathcal{K}$.

Proposition 5.4.1. *Let $(Y, \|\cdot\|)$ be a normed space with a closed convex pointed cone $\mathcal{K}$ and let $\mathcal{K}^* \subset Y^*$ be its dual cone with base Θ^*. Let C be a subset of Y with $E(C)_y$ weakly compact for $y \in E(C) + \mathcal{K}$. For any $y \in E(C) + \mathcal{K}$ there exists $\eta_y \in E(C)$ such that*

$$\nu(y) = \operatorname{dcont}_{\Theta^*}(y - \eta_y) = \inf\{\theta^*(y - \eta_y) : \theta^* \in \Theta^*\}.$$

Proof. Let $y \in E(C) + \mathcal{K}$. Clearly, $\operatorname{dcont}_{\Theta^*}(y - \eta) \leq \nu(y)$ for any $\eta \in E(C)_y$ and for each $\varrho > 0$ there exists $\eta_\varrho \in E(C)_y$ such that for any $\theta^* \in \Theta^*$,

$$\theta^*(y - \eta_\varrho) \geq \operatorname{dcont}_{\Theta^*}(y - \eta_\varrho) > \nu(y) - \varrho.$$

The net (η_ϱ) contains a weakly convergent subnet; we can assume that (η_ϱ) itself converges weakly to $\eta_y \in E(C)_y$. Since $\mathcal{K}$ is weakly closed, the net (k_ϱ), $k_\varrho = y - \eta_\varrho$, tends to some $k_y \in \mathcal{K}$ and $y = \eta_y + k_y$. Thus, $\operatorname{dcont}_{\Theta^*}(y - \eta_y) \geq \nu(y)$, which completes the proof. ∎

PROPOSITION 5.4.2. *Let $(Y, \|\cdot\|)$ be a normed space and let C be a subset of Y. Let $\mathcal{K}$ be a closed convex pointed cone in Y and let $\mathcal{K}^*$ be its dual with a base Θ^*. The following conditions are equivalent:*

(i) *(DCP) holds for C,*
(ii) *$d(\varepsilon) > 0$ for each $\varepsilon > 0$.*

Proof. (i)$\Rightarrow$(ii). Take any $\varepsilon > 0$ and $y \in C(\varepsilon)$. By (DCP), there exist $\delta > 0$ and $\eta_y \in E(C)$ such that $\mathrm{dcont}_{\Theta^*}(y - \eta_y) \geq \delta$. Hence,

$$\nu(y) = \sup\{\mathrm{dcont}_{\Theta^*}(y - \eta) : \eta \in E(C)_y\} \geq \delta,$$

and $d(\varepsilon) = \inf\{\nu(y) : y \in C(\varepsilon)\} \geq \delta > 0$.

(ii)$\Rightarrow$(i). Let $d(\varepsilon) = \alpha > 0$. For each $y \in C(\varepsilon)$,

$$\nu(y) = \sup\{\mathrm{dcont}_{\Theta^*}(y - \eta) : \eta \in E(C)_y\} \geq \alpha,$$

and consequently, $\mathrm{dcont}_{\Theta^*}(y - \eta_y) > \alpha/2$ for some $\eta_y \in E(C)_y$, i.e., (DCP) holds. ∎

PROPOSITION 5.4.3. *Let $\mathcal{K}$ be a closed convex pointed cone in a topological vector space Y with $\mathcal{K}^i \neq \emptyset$. If Θ_1^* and Θ_2^* are any two bases of the form (5.8) with $y_1, y_2 \in \mathcal{K}^i$ such that $y_2 \in ry_1 + \mathcal{K}$, where $r > 0$, then there exists $\gamma > 0$ with*

$$\mathrm{dcont}_{\Theta_1^*}(k) \geq \gamma\, \mathrm{dcont}_{\Theta_2^*}(k).$$

Proof. Let

$$\Theta_1^* = \{\theta_1^* \in \mathcal{K}^* : \theta_1^*(y_1) = 1\}, \qquad \Theta_2^* = \{\theta_2^* \in \mathcal{K}^* : \theta_2^*(y_2) = 1\},$$

where $y_1, y_2 \in \mathcal{K}^i$. For any $k \in \mathcal{K}$ and $\theta_1^* \in \Theta_1^*$, there exists $\overline{\theta}_2^* \in \Theta_2^*$ such that $\theta_1^*(k) = \theta_1^*(y_2)\overline{\theta}_2^*(k)$ with $\theta_1^*(y_2) > 0$. Hence,

$$\theta_1^*(k) \geq \theta_1^*(y_2) \inf_{\overline{\theta}_2^* \in \Theta_2^*} \overline{\theta}_2^*(k) \geq \theta_1^*(y_2) \inf_{\theta_2^* \in \Theta_2^*} \theta_2^*(k),$$

and

$$\inf_{\theta_1^* \in \Theta_1^*} \theta_1^*(k) \geq \inf_{\theta_1^* \in \Theta_1^*} \theta_1^*(y_2) \inf_{\theta_2^* \in \Theta_2^*} \theta_2^*(k), \tag{5.15}$$

Since $y_2 \in ry_1 + \mathcal{K}$, by putting $\gamma := \inf_{\theta_1^* \in \Theta_1^*} \theta_1^*(y_2) > 0$ we get the assertion. ∎

EXAMPLE 5.4.1. Let $Y = (\mathbb{R}^m, \|\cdot\|_\infty)$, $\mathcal{K} = \mathbb{R}^m_+$. According to Definition 5.3.1,

$$(LP) \qquad \mathrm{cont}(k) = \begin{array}[t]{l} \max\ r \\ \text{subject to} \\ k_i - r \geq 0, \quad i = 1, \ldots, m. \end{array}$$

In view of Definition 5.4.1,

$$(DP) \qquad \mathrm{dcont}(k) = \begin{array}[t]{l} \min\ c_1k_1 + \cdots + c_mk_m \\ \text{subject to} \\ c_1 + \cdots + c_m = 1 \\ c_i \geq 0, \quad i = 1, \ldots, m. \end{array}$$

By linear programming duality, $\mathrm{dcont}(k) \geq \mathrm{cont}(k)$ for $k \in \mathcal{K}$.

Let Y be a Banach space and $\mathcal{K}^i \neq \emptyset$. Consider a standard base of $\mathcal{K}^*$,

$$\Theta^* = \{\theta^* \in \mathcal{K}^* : \theta^*(y_0) = 1\}, \qquad \text{where } y_0 \in \mathcal{K}^i.$$

For any $k \in \mathcal{K}$, the problem of finding

$$\text{dcont}(k) = \inf\{\theta^*(k) : \theta^*(y_0) = 1,\ \theta^* \in \mathcal{K}^*\} \tag{5.16}$$

can be viewed as an *infinite-dimensional linear programming problem.* By applying the duality theory (see e.g. Barbu and Precupanu [15, Ch. 3, par. 3, p. 233]) the dual takes the form

$$\sup\{r \in \mathbb{R} : k - ry_0 \in \mathcal{K}\}, \tag{5.17}$$

(compare also [15, Ch. 3, Th. 3.4, p. 235]). Thus, (5.17) and (5.16) form a pair of dual problems and by Proposition 2.1, Ch. 3, p. 197 of [15], we have

$$0 \le \sup\{r \in \mathbb{R} : k - ry_0 \in \mathcal{K}\} \le \inf\{\theta^*(k) : \theta^*(y_0) = 1,\ \theta^* \in \mathcal{K}^*\} = \overline{r}.$$

The function

$$q(k) = \sup\{r > 0 : r^{-1}k \in y_0 + \mathcal{K}\}$$

has also been considered in other context (see Namioka [115]). It is superlinear and its hypograph

$$\text{hgraph}(q) = \{(k, r) : q(k) \ge r\}$$

is a cone in $Y \times \mathbb{R}$.

Below we give an example of an problem with $\overline{r} = 0$.

EXAMPLE 5.4.2. Let $p > 1$, $Y = \ell^p$, $\mathcal{K} = \ell^p_+$. As observed before,

$$(\ell^p_+)^i = \{(s_i) \in \ell^p : s_i > 0 \text{ for each } i \ge 1\}.$$

By taking $y_0 = (1/i^2)$ and $k_0 = (1/i^3)$ we see that for any $r > 0$ there exists an index i_0 such that $1/i^3 - r/i^2 < 0$ for $i > i_0$ and hence $\overline{r} = 0$.

5.5. Containment rate for convex sets

In this section we investigate the containment rate $\delta(\cdot)$ for convex sets. Define

$$CEQ(\varepsilon) = \{y \in C : d(y, E(C)) = \varepsilon\}.$$

LEMMA 5.5.1. *Let $\mathcal{K}$ be closed convex cone in Y with $\operatorname{int}\mathcal{K} \neq \emptyset$. Let C be a convex subset of Y with weakly compact sections $E(C)_y$ for $y \in E(C) + \mathcal{K}$. Then*

$$\delta(\varepsilon) = \inf\{\mu(y) : y \in CEQ(\varepsilon)\}. \tag{5.18}$$

Proof. Clearly $\delta(\varepsilon) \le \inf\{\mu(y) : y \in CEQ(\varepsilon)\}$. If $\delta(\varepsilon) < \inf\{\mu(y) : y \in CEQ(\varepsilon)\} = e$, then $\mu(\overline{y}) < e$ for a certain $\overline{y} \in C$, $d(\overline{y}, E(C)) > \varepsilon$. In view of Proposition 5.3.1, $\overline{y} = \eta_y + k_y$, $k_y + \mu(\overline{y})B_Y \subset \mathcal{K}$.

Since $[\eta_y, y] \subset C$, one can find $z \in CEQ(\varepsilon)$, $z = \lambda\eta_y + (1-\lambda)y$. Hence, $z = \eta_y + (1-\lambda)k_y = \eta_y + k_z$, $k_z = (1-\lambda)k_y$, $k_z + (1-\lambda)\mu(\overline{y})B_Y \subset \mathcal{K}$ and $\mu(\overline{y}) \ge (1-\lambda)\mu(\overline{y}) = \mu(z) \ge e$, contrary to the choice of $\overline{y}$. ∎

LEMMA 5.5.2. *Let $\mathcal{K}$ be closed convex cone in Y with $\operatorname{int}\mathcal{K} \neq \emptyset$. Let C be a convex subset of Y with weakly compact sections $E(C)_y$ for $y \in E(C) + \mathcal{K}$. Then for any $0 \le \beta \le 1$,*

$$\mu(y(\beta)) = \beta\mu(y),$$

where $y = \eta_y + k_y$, $\eta_y \in E(C)$, $k_y + \mu(y)B_Y \in \mathcal{K}$ and $y(\beta) = \eta_y + \beta \cdot k_y$.

Proof. Let $y \in E(C)$. By Proposition 5.3.1, $y = \eta_y + k_y$, where $\eta_y \in E(C)$ and $k_y + \mu(y)B_Y \subset \mathcal{K}$. Since $\beta k_y + \beta\mu(y)B_Y \subset \mathcal{K}$ for any $\beta \geq 0$, we have $\mu(y(\beta)) \geq \beta\mu(y)$. If $\mu(y(\beta)) > \beta\mu(y)$, then $y(\beta) = \eta + k$, where $k + \mu(y(\beta))B_Y \subset \mathcal{K}$. Then for $0 \leq \beta \leq 1$,

$$\overline{k} = y - \eta = y - y(\beta) + y(\beta) - \eta = (1-\beta)k_y + k \in \mathcal{K}$$

and $\operatorname{cont}(\overline{k}) \geq (1-\beta) + \mu(y(\beta)) > \mu(y)$, contrary to the definition of $\mu(y)$. ■

Applying Lemmas 5.5.1 and 5.5.2 we prove the concavity of the containment rate μ and the quasi-convexity of δ.

PROPOSITION 5.5.1. *Let $\mathcal{K}$ be closed convex cone in Y with* $\operatorname{int}\mathcal{K} \neq \emptyset$. *Let C be a convex subset of Y and let (DP) hold for C. If $E(C)_y$ are weakly compact for $y \in E(C) + \mathcal{K}$, the containment rate μ is concave on $E(C) + \mathcal{K}$.*

Proof. Let $y_1, y_2 \in E(C) + \mathcal{K}$ and $0 \leq \lambda \leq 1$. By Proposition 5.3.1, there exist $\eta_1, \eta_2 \in E(C)$ such that

$$y_1 - \eta_1 + \mu(y_1)B_Y \subset \mathcal{K} \quad \text{and} \quad y_2 - \eta_2 + \mu(y_2)B_Y \subset \mathcal{K}.$$

Since $\mathcal{K}$ is convex,

$$y(\lambda) - \eta(\lambda) + (\lambda\mu(y_1) + (1-\lambda)\mu(y_2))B_Y \subset \mathcal{K},$$

where $y(\lambda) = \lambda y_1 + (1-\lambda)y_2$, $\eta(\lambda) = \lambda\eta_1 + (1-\lambda)\eta_2$. Since C is convex and (*DP*) holds for C, $E(C) + \mathcal{K}$ is convex and $\eta(\lambda) = \eta + k$, where $\eta \in E(C)$ and $k \in \mathcal{K}$, and consequently,

$$y(\lambda) - \eta + (\lambda\mu(y_1) + (1-\lambda)\mu(y_2))B_Y \subset \mathcal{K},$$

which proves the concavity of μ. ■

COROLLARY 5.5.1. *Under the assumptions of Proposition* 5.5.1 *the function μ is locally Lipschitz and weakly upper semicontinuous on* $E(C) + \operatorname{int}\mathcal{K}$.

Proof. See Theorem 10 of [66]. ■

Now we are in a position to prove the quasi-convexity of δ.

THEOREM 5.5.1. *Let $\mathcal{K}$ be a closed convex pointed cone in a normed space $(Y, \|\cdot\|)$ with* $\operatorname{int}\mathcal{K} \neq \emptyset$. *Let C be a convex subset of Y and let (DP) hold for C. If $E(C)_y$ are weakly compact for $y \in E(C) + \mathcal{K}$, then δ is quasiconvex on* $\operatorname{dom}\delta$.

Proof. By Lemma 5.5.1, $\delta(\varepsilon) = \inf\{\mu(y) : y \in CEQ(\varepsilon)\}$. Let $\varepsilon_1, \varepsilon_2 \in \operatorname{dom}\delta$, $\varepsilon_2 < \varepsilon_1$. For any $\alpha > 0$ there is $y_\alpha \in CEQ(\varepsilon_1)$ such that $\mu(y_\alpha) < \delta(\varepsilon_1) + \alpha$. In view of Proposition 5.3.1, there is $\eta_\alpha \in E(C)$ with $y_\alpha - \eta_\alpha + \mu(y_\alpha) \in \mathcal{K}$.

Let $0 \leq \lambda \leq 1$. Since the distance function $d(\cdot, E(C))$ is continuous, there exists $0 \leq \overline{\lambda} \leq 1$ such that $d(\overline{\lambda}y_\alpha + (1-\overline{\lambda})\eta_\alpha, E(C)) = \lambda\varepsilon_1 + (1-\lambda)\varepsilon_2$. By Lemma 5.5.2, $\mu(\overline{\lambda}y_\alpha + (1-\overline{\lambda})\eta_\alpha) = \overline{\lambda}\mu(y_\alpha)$. Hence,

$$\begin{aligned}\delta(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2) &= \inf\{\mu(y) : y \in CEQ(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2)\}\\ &\leq \mu(\overline{\lambda}y_\alpha + (1-\overline{\lambda})\eta_\alpha) = \overline{\lambda}\mu(y_\alpha) < \delta(\varepsilon_1) + \alpha.\end{aligned}$$

Since $\alpha > 0$ is arbitrary and δ is nondecreasing we get $\delta(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2) \leq \max\{\delta(\varepsilon_1), \delta(\varepsilon_2)\}$. ■

6. UPPER HAUSDORFF SEMICONTINUITY OF EFFICIENT POINTS

In this chapter we derive criteria for upper Hausdorff semicontinuity of the efficient point set $E_{\mathcal{K}}(C)$ of a given subset C of a space Y with respect to a closed convex pointed cone $\mathcal{K} \subset Y$ when C is subjected to perturbations.

Perturbations u belong to a topological space U and are handled by a set-valued mapping $\mathcal{C} : U \rightrightarrows Y$ taking values in a topological Hausdorff vector space Y, $\mathcal{C}(u) = C(u)$, $\mathcal{C}(u_0) = C$. Recall that by $\mathcal{E} : U \rightrightarrows Y$, we denote the efficient point set-valued mapping defined as

$$\mathcal{E}(u) = E(C(u)).$$

Upper Hausdorff semicontinuity of $\mathcal{P}$ enters into stability results of the solution mapping $\mathcal{S}$. This aspect will be discussed in detail in Chapter 9.

In Section 6.1 we derive sufficient conditions for upper Hausdorff semicontinuity of efficient points (Theorems 6.1.1, 6.1.3) for a cone $\mathcal{K}$ with nonempty interior with the help of the containment property introduced in Section 5.1. In Section 6.2, by applying the results from Section 6.1 to the mapping $\mathcal{C}(u) = f(u, A(u))$ we derive sufficient conditions for upper Hausdorff continuity of the performance mapping $\mathcal{P}$ to parametric vector optimization problems of the form (P_u).

6.1. Sufficient conditions for upper Hausdorff semicontinuity of efficient points

Let U be a topological space (space of parameters) and let Y be a Hausdorff topological vector space. Let $\mathcal{K}$ be a closed convex pointed cone in Y.

Let $\mathcal{C} : U \rightrightarrows Y$ be a set-valued mapping, $\mathcal{C}(u) = C(u)$, $\mathcal{C}(u_0) = C$.

According to the notation introduced in Section 5.1, for any 0-neighbourhood W,

$$C(W) = (C \setminus E(C)) + W.$$

We start with the main result of this section.

THEOREM 6.1.1 ([21]). *Let U be a topological space and let Y be a Hausdorff topological vector space. Let $\mathcal{K}$ be a closed convex pointed cone in Y with* $\operatorname{int}\mathcal{K} \neq \emptyset$. *Assume that*

(i) *$\mathcal{C}$ is upper Hausdorff semicontinuous at $u_0 \in \operatorname{dom}\mathcal{C}$ and $\mathcal{K}$-lower semicontinuous at u_0, uniformly on $E(C)$,*

(ii) *(CP) holds for C.*

Then $\mathcal{E}$ is upper Hausdorff semicontinuous at $u_0 \in \operatorname{dom}\mathcal{C}$.

Proof. Let W_1, W be 0-neighbourhoods such that $W_1 + W_1 \subset W$. By Proposition 5.1.3, there exists a 0-neighbourhood O such that for any $y \in C(W_1)$ there exists $\eta \in E(C)$ satisfying

$$(y - \eta) + O \subset \mathcal{K}. \tag{6.1}$$

Let O_1 be a 0-neighbourhood such that $O_1 + O_1 \subset O$. By (i), there exists a neighbourhood U_0 of u_0 such that

$$C(u) \subset C + W_1 \cap O_1, \quad (\eta + O_1 - \mathcal{K}) \cap C(u) \neq \emptyset \quad \text{for } u \in U_0. \tag{6.2}$$

Take any $u \in U_0$. If $\mathcal{E}(u) = \emptyset$, the conclusion follows. Hence, suppose that $\mathcal{E}(u) \neq \emptyset$ and $\overline{z} \in \mathcal{E}(u)$. By (6.2) there is $y \in C$ such that $\overline{z} - y \in W_1 \cap O_1$.

If $y \notin E(C) + W_1$, then $y \in C(W_1)$ and by (6.1) there exists $\eta \in E(C)$ such that

$$y - \eta + O \subset \mathcal{K}.$$

Moreover, by (6.2), there exists $z \in C(u)$ such that $z - \eta \in O_1 - \mathcal{K}$ and so $z = \overline{z}$ since otherwise

$$\overline{z} - z = (\overline{z} - y) + (y - \eta) + (\eta - z) \in W_1 \cap O_1 + (y - \eta) + O_1 + \mathcal{K} \subset (y - \eta) + O \subset \mathcal{K},$$

which is impossible since $\overline{z} \in E(C(u))$.

If $y \in E(C) + W_1$, then $\overline{z} \in E(C) + W$, which finishes the proof. ■

Below we give an example showing that the uniform $\mathcal{K}$-lower semicontinuity assumption is essential in Theorem 6.1.1.

Example 6.1.1. Let $U = \mathrm{cl}\{1/n : n = 1, \ldots\}$ with natural topology and $u_0 = 0$ and let $\mathcal{C} : U \rightrightarrows \mathbb{R}^2$ be defined as follows:

$$\mathcal{C}(0) = C := \{(y_1, y_2) : y_2 = -y_1\} \cup \bigcup_{k=1}^{\infty} (k, -k+1),$$

$$\mathcal{C}(1/n) = C(1/n) := \{(y_1, y_2) : y_2 = -y_1 + 1/n,\ -n \leq y_1 \leq n\} \cup \bigcup_{k=1}^{\infty} (k, -k+1).$$

Now $E(C) = \{(y_1, y_2) : y_2 = -y_1\}$ and

$$E(C(1/n)) = \{(y_1, y_2) : y_2 = -y_1 + 1/n, \ \ -n \leq y_1 \leq n\} \cup \bigcup_{k=n+1}^{\infty} (k, -k+1).$$

Theorem 6.1.2. *Let U be a topological space and let Y be a Hausdorff topological vector space. Let $\mathcal{K}$ be a closed convex pointed cone in Y with $\mathrm{int}\,\mathcal{K} \neq \emptyset$. If $\mathcal{C}$ is Hausdorff continuous at $u_0 \in \mathrm{dom}\,\mathcal{C}$ and (CP) holds for C, then $\mathcal{E}$ is upper Hausdorff semicontinuous at $u_0 \in \mathrm{dom}\,\mathcal{C}$.*

By Proposition 5.1.2, we obtain the following corollary.

Corollary 6.1.1. *Let U be a topological space and let Y be a Hausdorff topological vector space. Let $\mathcal{K}$ be a closed convex pointed cone in Y with $\mathrm{int}\,\mathcal{K} \neq \emptyset$. Let C be a compact subset of Y and $\mathrm{cl}\,E(C) = WE(C)$. If $\mathcal{C}$ is Hausdorff continuous at $u_0 \in \mathrm{dom}\,\mathcal{C}$, then $\mathcal{E}$ is upper Hausdorff semicontinuous at $u_0 \in \mathrm{dom}\,\mathcal{E}$.*

In the proof of Theorem 6.1.1 we make use of Proposition 5.1.3 which holds true when $\operatorname{int}\mathcal{K} \neq \emptyset$. There are numerous examples of cones satisfying this condition. For instance, the cone $\mathbb{R}^m_+$ of nonnegative elements in $\mathbb{R}^m$ as well the cones of nonnegative elements in the spaces below have nonempty interiors.

EXAMPLE 6.1.2. 1. In the space ℓ^∞ of sequences $s = (s_i)$ with real terms,

$$\ell^\infty = \{s = (s_i) : \sup_{i\in\mathbb{N}} |s_i| < \infty\}$$

the cone

$$\ell^\infty_+ = \{s = (s_i) \in \ell^\infty : s_i \geq 0\}$$

has nonempty interior.

2. In the space $L^\infty(\Omega)$ of essentially bounded functions $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ with $\operatorname{ess\,sup}_{x\in\Omega} |f(x)| < \infty$ the natural ordering cone

$$L^\infty(\Omega) = \{f \in L^\infty(\Omega) : f(x) \geq 0 \text{ almost everywhere on } \Omega\}$$

has nonempty interior.

A subset F of Y^* is *equicontinuous* ([78, 12.D]) if for any $\varepsilon > 0$ there exists a 0-neighbourhood W such that $|f(W)| < \varepsilon$ for any $f \in F$. Equivalently, there exists a balanced 0-neighbourhood W such that $f(W) \leq 1$ for each $f \in F$. According to the definition of the polar set A° of a given set A, F is equicontinuous if and only if $F \subset W^\circ$ for a balanced 0-neighbourhood W. By the Banach–Alaoglu theorem, W° is relatively weak* compact. When Y is a normed linear space, $F \subset Y^*$ is equicontinuous if and only if it is bounded in the norm topology of Y^*.

Now we formulate a variant of Theorem 6.1.1 with the help of the dual containment property (DCP), which can be applied to cones $\mathcal{K}$ which are not pointed.

THEOREM 6.1.3. *Let U be a topological space and let Y be a Hausdorff locally convex topological vector space. Let $\mathcal{K} \subset Y$ be a closed convex cone in Y and let $\mathcal{K}^*$ have an equicontinuous base Θ^*. If*

(i) *$\mathcal{C}$ is upper Hausdorff semicontinuous at $u_0 \in \operatorname{dom}\mathcal{C}$ and $\mathcal{K}$-lower semicontinuous at u_0, uniformly on $E(C)$,*
(ii) *(DCP) holds for C,*

then the set-valued mapping $\mathcal{E}$ is upper Hausdorff semicontinuous at $u_0 \in \operatorname{dom}\mathcal{E}$.

Proof. Follows from Theorem 6.1.1 and Proposition 5.2.2. ∎

The following example shows that Theorem 6.1.3 cannot be applied to some cones in finite-dimensional spaces.

EXAMPLE 6.1.3. Let $\mathcal{K}$ be a convex closed cone in $\mathbb{R}^n$ with empty interior. Then $\mathcal{K}^*$ has no base since the set $\mathcal{K}^T = \{y \in \mathcal{K}^* : y \cdot x = 0 \text{ for each } x \in \mathcal{K}\}$ is a nontrivial linear subspace contained in $\mathcal{K}^*$.

The assumption of equicontinuity of the base Θ^* is restrictive. The cone of nonnegative elements in $L^p(\Omega)$, $1 < p < \infty$, does not have an equicontinuous base since it does not have a bounded base (see [46]).

6.2. Upper Hausdorff semicontinuity of the performance mapping for parametric vector optimization problems

In this section we apply Theorems 6.1.1 and 6.1.2 to prove the upper Hausdorff semicontinuity of the performance set-valued mapping $\mathcal{P}$ for parametric vector optimization problems

$$(P_u) \quad \begin{array}{l} \min_{\mathcal{K}} f(x) \\ \text{subject to } x \in A(u). \end{array}$$

We start with two technical propositions.

PROPOSITION 6.2.1. *Let U be a topological space and let X and Y be Hausdorff topological vector spaces. If a set-valued mapping $\mathcal{A} : U \rightrightarrows Y$ is upper Hausdorff semicontinuous at $u_0 \in \operatorname{dom} \mathcal{A}$, and $f : X \to Y$ is uniformly continuous on $\mathcal{A}(u_0)$, then $\mathcal{A}_f : U \rightrightarrows Y$, $\mathcal{A}_f(u) = f(\mathcal{A}(u))$, is upper Hausdorff semicontinuous at $u_0 \in \operatorname{dom} \mathcal{A}_f$.*

Proof. Let W be a 0-neighbourhood in Y. There exists a 0-neighbourhood Q in X such that $f(x+Q) \subset f(x) + W$ for $x \in \mathcal{A}(u_0)$. Thus, $f(\mathcal{A}(u_0) + Q) \subset f(\mathcal{A}(u_0) + W$. By the upper Hausdorff semicontinuity of $\mathcal{A}$, there exists a neighbourhood U_0 of u_0 such that $\mathcal{A}(u) \subset \mathcal{A}(u_0) + Q$ for $u \in U_0$. Consequently, $f(\mathcal{A}(u)) \subset f(\mathcal{A}(u_0)) + W$ for $u \in U_0$. ■

PROPOSITION 6.2.2. *Let U be a topological space and let X and Y be Hausdorff topological vector spaces. If $f : X \to Y$ is a (uniformly) upper semicontinuous function, and a set-valued mapping $\mathcal{A} : U \rightrightarrows Y$ is lower (Hausdorff) semicontinuous at u_0, then $\mathcal{A}_f$ is lower (Hausdorff) semicontinuous at $u_0 \in \operatorname{dom} \mathcal{A}_f$.*

Proof. Let W be a 0-neighbourhood in Y. There exists a 0-neighbourhood Q in X such that $f(x+Q) \subset f(x)+W$ for $x \in \mathcal{A}(u_0)$. In view of the lower semicontinuity of $\mathcal{A}$, there exists a neighbourhood U_0 of u_0 such that $(x+Q) \cap \mathcal{A}(u) \neq \emptyset$ for $u \in U_0$. By putting $x_u \in (x+Q) \cap \mathcal{A}(u)$ for $u \in U_0$, we get $f(x_u) \in \mathcal{C}(u) \cap (f(x) + W)$ for $u \in U_0$. ■

By Theorem 6.1.2, we get the following stability result for problems (P_u) with (P_{u_0}) being (P). Let $\mathcal{A} : U \rightrightarrows Y$ be a set-valued mapping, $\mathcal{A}(u) = A(u)$, $\mathcal{A}(u_0) = A$.

THEOREM 6.2.1. *Let U be a topological space and let Y be a Hausdorff topological vector space. Let $\mathcal{K}$ be a closed convex pointed cone in Y with $\operatorname{int} \mathcal{K} \neq \emptyset$. Let $f : X \to Y$ be a uniformly continuous function on A and $\mathcal{A}$ be Hausdorff continuous at $u_0 \in \operatorname{dom} \mathcal{A}$. If (CP) holds for $f(A)$, then $\mathcal{P}$ is upper Hausdorff semicontinuous at $u_0 \in \operatorname{dom} \mathcal{P}$.*

Sufficient conditions for upper Hausdorff semicontinuity of the set-valued mapping $\mathcal{A} : U \rightrightarrows X$,

$$\mathcal{A}(u) = \{x \in X : G(x) \cap (u - \Omega) \neq \emptyset\},$$

where $G : X \rightrightarrows Y$ and $\Omega \subset Y$ is a closed convex and pointed cone in U, were investigated by many authors. In particular, when G is a single-valued mapping,

$$\mathcal{A}(u) = \{x \in X : G(x) \preceq_{\Omega} u\}.$$

Continuity properties of this mapping depend heavily on the properties of the cone Ω. In the case where $\operatorname{int} \Omega \neq \emptyset$, C-lower semicontinuity was investigated by Ferro [59, 60]. For cones with possibly empty interiors, continuity of $\mathcal{A}$ was investigated by Muselli [114].

6.2.1. Multiobjective optimization problems. In this section we consider multiobjective optimization problems

$$(MOP) \qquad \begin{array}{l} \min_{\mathcal{K}} f(x) \\ \text{subject to } x \in A, \end{array}$$

where $f = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$, $A \subset \mathbb{R}^n$ and $\mathcal{K} \subset \mathbb{R}^m$ is a closed convex pointed cone.

THEOREM 6.2.2. *Assume that f_i, $i = 1, \dots, m$, are linear functions and*

$$A = \{x \in \mathbb{R}^n : \langle b_i, x\rangle \le c_i,\ i \in I\}.$$

If $E(f, A) \neq \emptyset$ and $E(f, A) = WE(f, A)$, then (CP) holds for $f(A)$.

Proof. It is enough to observe that $f(A)$ is a polyhedral set and apply Theorem 5.1.4 and Corollary 3 of [72]. ■

THEOREM 6.2.3. *Suppose that f_i, $i = 1, \dots, m$, are linear, $A \subset \mathbb{R}^n$ is convex, and $E(f, A) \neq \emptyset$. If $E(f, A)$ is compact, then (CP) holds for $f(A)$.*

Proof. Note that $f(A)$ is convex and apply Corollary 5.1.2. ■

Consider parametric multiobjective problems

$$(MOP_u) \qquad \begin{array}{l} \min_{\mathcal{K}}\ f(x) \\ \text{subject to } x \in A(u), \end{array}$$

where $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous. Let U be a topological space and $\mathcal{A} : U \rightrightarrows \mathbb{R}^n$ be a set-valued mapping, $\mathcal{A}(u) = A(u)$, $\mathcal{A}(u_0) = A$.

We apply Theorem 6.1.1 to the above parametric problem. We start with the following stability results.

THEOREM 6.2.4. *Let $f = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping and let $\mathcal{A} : U \rightrightarrows \mathbb{R}^n$ be a set-valued mapping given by*

$$\mathcal{A}(u) = \{x \in \mathbb{R}^n : g_j(u, x) \le 0,\ j \in J\},$$

where, for each $j \in J$, the function $g_j(u_0, \cdot) : \mathbb{R}^n \to \mathbb{R}$ is convex. If

- *$\mathcal{A}_f : U \rightrightarrows \mathbb{R}^m$, $\mathcal{A}_f(u) = f(u, A(u))$, is Hausdorff continuous at $u_0 \in \operatorname{dom} \mathcal{A}$,*
- *$E(f(A))$ is nonempty and compact, $E(f(A)) = WE(f(A))$,*

then $\mathcal{E}$ is upper Hausdorff semicontinuous at $u_0 \in \operatorname{dom} \mathcal{E}$.

Proof. Since f is linear with respect to x and $g_j(u_0, \cdot)$, $j \in J$, are convex, the set $\mathcal{A}_f(u_0) = f(A)$ is convex. By Theorems 5.1.2 and 6.2.3, (CP) holds for $f(A)$. By Theorem 6.1.1, the conclusion follows. ■

To close this section let us note that set-valued mappings $\mathcal{A} : U \rightrightarrows \mathbb{R}^n$ given by

$$\mathcal{A}(u) = \{x \in \mathbb{R}^n : g_j(u, x) \le 0,\ j \in J\}, \tag{6.3}$$

where, for each $j \in J$, $g_j : U \times \mathbb{R}^n \to \mathbb{R}$ is a linear function with respect to x, $g_j(u, x) = \langle b_j(u), x\rangle - c_j(u)$, $j \in J$, $b_j : U \to \mathbb{R}^n$, $c_j : U \to \mathbb{R}$, were investigated e.g. in [14].

THEOREM 6.2.5. *Let* $f = (f_1, \dots, f_m) : U \times \mathbb{R}^n \to \mathbb{R}^m$ *be a linear function of* $x \in \mathbb{R}^n$ *and let* $\mathcal{A} : U \rightrightarrows \mathbb{R}^n$ *be a feasible set mapping given by*

$$\mathcal{A}(u) = \{x \in \mathbb{R}^n : g_j(u, x) \leq 0,\ j \in J\},$$

where, for each $j \in J$, $g_j : U \times \mathbb{R}^n \to \mathbb{R}$ *is a linear function with respect to* x, $g_j(u, x) = \langle b_j(u), x \rangle - c_j(u)$, $j \in J$, $b_j : U \to \mathbb{R}^n$, $c_j : U \to \mathbb{R}$. *If*

- $\mathcal{A} : U \rightrightarrows \mathbb{R}^n$ *is upper and lower Hausdorff semicontinuous at* $u_0 \in \operatorname{dom} \mathcal{A}$,
- $E(f(A))$ *is nonempty, and* $E(f(A)) = WE(f(A))$,

then $\mathcal{E}$ *is upper Hausdorff semicontinuous at* $u_0 \in \operatorname{dom} \mathcal{E}$.

Proof. Follows from Theorem 6.1.2 and Propositions 6.2.1, 6.2.2. ∎

7. UPPER HÖLDER CONTINUITY OF EFFICIENT POINTS WITH RESPECT TO PERTURBATIONS OF A SET

In this chapter we derive criteria for upper Hölder continuity and calmness of the efficient point sets $E(C(u))$. These properties appear in many contexts of optimization theory and sensitivity analysis (see e.g. [100, 101, 56, 64, 91]). Criteria for calmness of some set-valued mappings are given in [74, 75]. Upper Hölder continuity of order q and Hölder calmness of the set-valued mapping $\mathcal{E}$ at u_0 provide an estimate of the distance of any efficient point of the perturbed problem (P_u) to the efficient point set of (P_{u_0}) via the distance of the perturbations, $\|u - u_0\|^q$. Hence, the upper Hölder property is of interest whenever it is impossible or too difficult to deal with the original problem and one wants to know the magnitude of the error made by accepting a solution of a perturbed problem as a solution of the original problem. For instance, numerical representation of problems leads to perturbations due to finite precision. As a particular case we obtain conditions for the upper Lipschitz continuity of efficient points. The upper Lipschitz property (upper Hölder property with $q = 1$) has already appeared in investigation of stability of various problems (see e.g. [128, 130, 131]).

In Sections 4.1 and 4.2 we investigate upper Hölder continuity and Hölder calmness of $E(C(u))$ at a given point u_0. The main requirement we impose is that for small arguments the containment rate δ is a sufficiently fast growing function.

In Section 4.3 we apply the results obtained in Sections 4.1 and 4.2 to investigate Lipschitzness and Hölder properties of the performance set-valued mapping $\mathcal{P}$ for parametric vector optimization problems.

7.1. Upper Hölder continuity of efficient points

Let $U = (U, \|\cdot\|)$ and $Y = (Y, \|\cdot\|)$ be normed spaces and let $\mathcal{C} : U \rightrightarrows Y$ be a set-valued mapping, $\mathcal{C}(u) = C(u)$, $\mathcal{C}(u_0) = C$.

In this section we prove sufficient conditions for upper Hölder continuity of the efficient point set-valued mapping $\mathcal{E} : U \rightrightarrows Y$,

$$\mathcal{E}(u) = E(C(u)).$$

At the beginning of this chapter we indicated some situations where upper Hölder continuity has a natural significance. One more example comes from parametric vector optimization. Theorem 6.4 of [16] and Theorem 6.2 of [17] reveal the importance of upper type continuities of the performance set-valued mapping $\mathcal{P}$ in ensuring the continuity of solutions to parametric vector optimization problems.

We start with sufficient conditions for upper Hölder continuity of the efficient point set-valued mapping $\mathcal{E}$.

THEOREM 7.1.1. *Let $Y = (Y, \|\cdot\|)$ and $U = (U, \|\cdot\|)$ be normed spaces. Let $\mathcal{K}$ be a closed convex pointed cone in Y with $\operatorname{int}\mathcal{K} \neq \emptyset$. If*

(i) *$\mathcal{C} : U \rightrightarrows Y$ is Hölder continuous of order $p \geq 1$ at $u_0 \in \operatorname{dom}\mathcal{C}$ with constants $L_c > 0$ and $0 < t_c < 1$,*
(ii) *the sections $E(C)_y$ are weakly compact for $y \in E(C) + \mathcal{K}$,*
(iii) *the containment rate δ of the set C satisfies the following condition: for any $\varepsilon \in \operatorname{dom}\delta$,*

$$\delta(\varepsilon) \geq \alpha\varepsilon^q \quad \text{for some } \alpha > 0 \text{ and } q \geq 1,$$

then $\mathcal{E}$ is upper Hölder continuous of order p/q at $u_0 \in \operatorname{dom}\mathcal{E}$. Precisely,

$$E(C(u)) \subset E(C) + (L_c + (2L_c/\alpha)^{1/q})\|u - u_0\|^{p/q} B_Y$$

for all $u \in u_0 + t_c B_U$.

Proof. Take any $\overline{y} \in E(C(u))$, $u \in u_0 + t_c B_U$. By (i), there exists $z \in C$ such that

$$\|\overline{y} - z\| \leq L_c\|u - u_0\|^p.$$

If $z \in E(C)$, the conclusion follows. If

$$d(z, E(C)) > \varepsilon_0 := (2L_c/\alpha)^{1/q}\|u - u_0\|^{p/q},$$

then by (ii) and Proposition 5.3.4, there is $\eta \in E(C)$ such that

$$z - \eta + \delta(\varepsilon_0)B_Y \subset \mathcal{K}$$

and by (iii), $\delta(\varepsilon_0) \geq 2L_c\|u - u_0\|^p$. By (i), there is $y \in C(u)$ such that

$$\|y - \eta\| \leq L_c\|u - u_0\|^p.$$

and so $y = \overline{y}$ since otherwise

$$\overline{y} - y = (\overline{y} - z) + (z - \eta) + (\eta - y) \in (z - \eta) + 2L_c\|u - u_0\|^p B_Y \subset \mathcal{K},$$

which contradicts the fact that $\overline{y} \in E(C(u))$. If

$$d(z, E(C)) \leq (2L_c/\alpha)^{1/q}\|u - u_0\|^{p/q},$$

then for $u \in u_0 + t_c B_U$ we get

$$d(\overline{y}, E(C)) \leq \|\overline{y} - z\| + d(z, E(C)) \leq (L_c + (2L_c/\alpha)^{1/q})\|u - u_0\|^{p/q},$$

which completes the proof. ∎

By applying Proposition 4.0.3 we obtain the following conditions for Hölder continuity of $\mathcal{E}$.

THEOREM 7.1.2. *Let $Y = (Y, \|\cdot\|)$ and $U = (U, \|\cdot\|)$ be normed spaces. Let $\mathcal{K}$ be a closed convex pointed cone in Y with $\operatorname{int}\mathcal{K} \neq \emptyset$. If*

(i) $\mathcal{C} : U \rightrightarrows Y$ *is Hölder continuous of order* $p \geq 1$ *around* $u_0 \in \operatorname{dom} \mathcal{C}$ *with constants* $L_c > 0$ *and* $0 < t < 1$,

(ii) *for all* $u \in u_0 + tB_U$ *the sections* $E(C(u))_z$ *are weakly compact for* $z \in E(C(u)) + \mathcal{K}$,

(iii) *all the containment rates* δ *of the sets* $C(u)$ *with* $u \in u_0 + tB_U$ *satisfy the condition: for any* $\varepsilon \in \operatorname{dom} \delta$,

$$\delta(\varepsilon) \geq \alpha \varepsilon^q \quad \textit{for some } \alpha > 0 \textit{ and } q \geq 1,$$

then $\mathcal{E}$ *is Hölder continuous of order* p/q *around* $u_0 \in \operatorname{dom} \mathcal{E}$. *Precisely,*

$$E(C(u)) \subset E(C(u')) + (L_c + (2L_c/\alpha)^{1/q})\|u - u'\|^{p/q} B_Y$$

for all $u, u' \in u_0 + (t/4)B_U$.

Proof. It is enough to note that under the above assumptions, for every $u' \in u_0 + (t/2)B_U$,

$$E(C(u)) \subset E(C(u')) + (L_c + (2L_c/\alpha)^{1/q})\|u - u'\|^{p/q} B_Y$$

for $u \in u' + (t/2)B_U$. This means that $\mathcal{E}$ is uniformly upper Hölder continuous at $u' \in u_0 + (t/2)B_U$ and by Proposition 4.0.3, the conclusion follows. ∎

Corollary 7.1.1. *Let* $Y = (Y, \|\cdot\|)$ *and* $U = (U, \|\cdot\|)$ *be normed spaces. Let* $\mathcal{K}$ *be a closed convex pointed cone in* Y *with* $\operatorname{int} \mathcal{K} \neq \emptyset$. *Let* $\mathcal{C}$ *be Hölder continuous of order* $p \geq 1$ *at* $u_0 \in \operatorname{dom} \mathcal{C}$ *with constants* $L_c > 0$ *and* $t_c > 0$. *Suppose that one of the following conditions hold:*

(i) $E(C)$ *is weakly compact,*

(ii) $E(C)$ *is* $\mathcal{K}$*-lower bounded and weakly closed and* $\mathcal{K}$ *has a weakly compact base.*

If the containment rate δ *of* C *satisfies the condition: for any* $\varepsilon > 0$,

$$\delta(\varepsilon) \geq \alpha \varepsilon^q \quad \textit{for some } q \geq 1 \textit{ and } \alpha > 0,$$

then the efficient point set-valued mapping $\mathcal{E}$ *is upper Hölder continuous of order* p/q *at* $u_0 \in \operatorname{dom} \mathcal{E}$ *with constant* $L_c + (2L_c/\alpha)^{1/q}$ *and order* p/q.

Proof. This follows from Theorem 7.1.1 and Proposition 5.3.2. ∎

Corollary 7.1.2. *Let* $Y = (Y, \|\cdot\|)$, $U = (U, \|\cdot\|)$ *be normed spaces. Let* $\mathcal{K}$ *be a closed convex pointed cone in a normed space* Y *with* $\operatorname{int} \mathcal{K} \neq \emptyset$. *Let* $\mathcal{C}$ *be Lipschitz continuous at* $u_0 \in \operatorname{dom} \mathcal{C}$ *with constants* $L_c > 0$ *and* $t_c > 0$. *Suppose that one of the following conditions holds:*

(i) $E(C)$ *is weakly compact,*

(ii) $E(C)$ *is* $\mathcal{K}$*-lower bounded and weakly closed and* $\mathcal{K}$ *has a weakly compact base.*

If the containment rate δ *of* C *satisfies the condition: for any* $\varepsilon > 0$,

$$\delta(\varepsilon) \geq \alpha \varepsilon \quad \textit{for some } \alpha > 0,$$

the efficient point set-valued mapping $\mathcal{E}$ *is upper Lipschitz continuous at* $u_0 \in \operatorname{dom} \mathcal{E}$ *with constant* $L_c + 2L_c/\alpha$.

Proof. This follows from Theorem 7.1.1 and Proposition 5.3.2. ∎

7.2. Hölder calmness of efficient points

The results of the previous section are of global character in the sense that they refer to the behaviour of the whole set $E(C)$ as a function of the parameter u.

In the present section we formulate sufficient conditions for upper pseudo-Hölder continuity (Hölder calmness) of the set-valued mapping $\mathcal{E}$.

Let $y_0 \in E(C)$ and $t_r > 0$.

Definition 7.2.1. The function $\delta_{t_r} : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\delta_{t_r}(\varepsilon) = \inf\{\mu(y) : y \in C \cap (y_0 + t_r B_Y) \setminus E(C) + \varepsilon B_Y\}$$

is called the *local containment rate* of C at $y_0 \in E(C)$ with respect to $\mathcal{K}$.

Note that the only difference between the local containment rate δ_{t_r} and the global containment rate δ is that now the infimum is taken over all $y \in C \cap (y_0 + t_r B_Y)$. Hence, for any $\varepsilon \in \operatorname{dom} \delta_{t_r}$,

$$\delta_{t_r}(\varepsilon) \geq \delta(\varepsilon).$$

Theorem 7.2.1. *Let $Y = (Y, \|\cdot\|)$ and $U = (U, \|\cdot\|)$ be normed spaces. Let $\mathcal{K}$ be a closed convex pointed cone in Y with $\operatorname{int} \mathcal{K} \neq \emptyset$ and $y_0 \in E(C)$. If*

(i) *$\mathcal{C}$ is upper pseudo-Hölder continuous of order $p \geq 1$ with 0-neighbourhood V at $(u_0, y_0) \in \operatorname{graph} \mathcal{C}$ and constants $L_c > 0$, $t_c > 0$ and $\mathcal{C}$ is lower Hölder continuous of order $p \geq 1$ at $u_0 \in \operatorname{dom} \mathcal{C}$ with constants $L_c > 0$, t_c,*

(ii) *there exists a constant $t_r > 0$ such that the sections $E(C)_y$ for $y \in C \cap (y_0 + t_r B_Y)$ are weakly compact,*

(iii) *for any $\varepsilon > 0$ the local containment rate δ_{t_r} satisfies the condition*

$$\delta_{t_r}(\varepsilon) \geq \alpha \varepsilon^q \quad \text{for some } \alpha > 0,\ q \geq 1,$$

then the set-valued mapping $\mathcal{E}$ is upper pseudo-Hölder (Hölder calm) of order p/q at $(u_0, y_0) \in \operatorname{graph} \mathcal{E}$. Precisely, there exists $t_v > 0$ such that

$$E(C(u)) \cap (y_0 + t_v B_Y) \subset E(C) + (L_c + (2L_c/\alpha)^{1/q}) \|u - u_0\|^{p/q} B_Y$$

for all $u \in u_0 + t_c B_U$.

Proof. The proof follows the lines of the proof of Theorem 7.1.1. Let $t_v > 0$ be any number satisfying $(L_c t_c + t_v) B_Y \subset V \subset t_r B_Y$. Take any $\overline{y} \in E(C(u)) \cap (y_0 + t_v B_Y)$, $u \in u_0 + t_c B_U$. By (i), there is $z \in C$ such that $\|\overline{y} - z\| \leq L_c \|u - u_0\|^p$. Moreover, $z - y_0 = (z - y) + (y - y_0) \in (L_c t_c + t_v) B_Y \subset t_r B_Y$. If $z \in E(C)$, the conclusion follows. If

$$d(z, E(C)) > (2L_c/\alpha)^{1/q} \|u - u_0\|^{p/q},$$

there is $\eta \in E(C)$ such that $z - \eta + \mu(z) B_Y \subset \mathcal{K}$. By (iii),

$$\mu(z) \geq \delta_{t_r}((2L_c/\alpha)^{1/q} \|u - u_0\|^{p/q}) \geq 2L_c \|u - u_0\|^p.$$

By (i), there is $y \in C(u)$ such that $\|\eta - y\| \leq L_c \|u - u_0\|^p$ and so $y = \overline{y}$ since otherwise

$$\overline{y} - y = (\overline{y} - z) + (z - \eta) + (\eta - y) \in \mathcal{K},$$

which is impossible since $\overline{y} \in E(C(u))$. If

$$d(z, E(C)) \leq (2L_c/\alpha)^{1/q}\|u - u_0\|^{p/q},$$

then

$$d(\overline{y}, E(C)) \leq \|\overline{y} - z\| + d(z, E(C)) \leq (L_c + (2L_c/\alpha)^{1/q})\|u - u_0\|^{p/q},$$

which completes the proof. ■

7.3. Upper Hölder continuity of efficient points to vector optimization problems

In the present section we apply Theorems 7.1.1 and 7.2.1 to parametric vector optimization problems (P_u),

$$(P_u) \quad \begin{array}{l} \min_{\mathcal{K}} f(u, x) \\ \text{subject to } x \in A(u). \end{array}$$

For $u = u_0$ we obtain problem (P),

$$(P) \quad \begin{array}{l} \min_{\mathcal{K}} f(x) \\ \text{subject to } x \in A. \end{array}$$

We formulate sufficient conditions for upper Hölder and upper pseudo-Hölder continuity of the performance set-valued mapping $\mathcal{P} : U \rightrightarrows Y$,

$$\mathcal{P}(u) = E(f(u, \cdot), A(u))$$

at $u_0 \in \operatorname{dom} \mathcal{P}$.

Based on Proposition 4.1.1 and Theorem 7.1.1 we obtain the following result.

THEOREM 7.3.1. *Let $Y = (Y, \|\cdot\|)$ and $U = (U, \|\cdot\|)$ be normed spaces. Let $\mathcal{K}$ be a closed convex pointed cone in Y with $\operatorname{int} \mathcal{K} \neq \emptyset$. Let $f : X \to Y$ safisfy the Lipschitz condition* (4.1) *on X with constant $L_f > 0$. If*

(i) *$\mathcal{A} : U \rightrightarrows X$ is Hölder continuous of order $p \geq 1$ at $u_0 \in \operatorname{dom} \mathcal{A}$ with constants $L_a > 0$ and $0 < t_a < 1$,*
(ii) *for $y \in f(A)$ the sections $E(f, A)_y$ are weakly compact,*
(iii) *for $\varepsilon \in \operatorname{dom} \delta$ the containment rate δ of the set $f(A)$ satisfies the condition*

$$\delta(\varepsilon) \geq \alpha\varepsilon^q \quad \text{for certain } \alpha > 0 \text{ and } q \geq 1,$$

then $\mathcal{P}$ is upper Hölder continuous of order p/q at $u_0 \in \operatorname{dom} \mathcal{P}$. Precisely,

$$E(f, A(u)) \subset E(f, A) + (L_f L_a + (2L_f L_a/\alpha)^{1/q})\|u - u_0\|^{p/q} B_Y$$

for all $u \in u_0 + t_a B_U$.

Below we define ϕ-strong domination property ϕ-(SDP) which allows us to prove sufficient conditions for the upper Hölder continuity of $\mathcal{P}$ without the assumption that all sections $E(f, A)_y$, $y \in f(A)$ are weakly compact.

Let $C \subset Y$ be a subset of a normed space Y.

Definition 7.3.1. We say that the *ϕ-strong domination property ϕ-(SDP)* holds for C if for each $y \in C$ there exists $\eta \in E(C)$ such that

$$y \succeq_{\mathcal{K}} \eta + \phi(\|y-\eta\|)B_Y, \quad \text{i.e.,} \quad y - \eta + \phi(\|y-\eta\|)B_Y \subset \mathcal{K},$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is an admissible function. In particular, we say that the *strong domination property of order $q > 0$* holds for C if ϕ-(*SCP*) holds for C with $\phi(\cdot) = \alpha(\cdot)^q$, where $\alpha > 0$.

Accordingly, we say that ϕ-strong domination property ϕ-(*SDP*) holds for (P) if the ϕ-strong domination property ϕ-(*SDP*) holds for $f(A)$, i.e. for each $x \in A$ there exists $\overline{x} \in S(f, A)$ such that

$$f(x) \succeq_{\mathcal{K}} f(\overline{x}) + \phi(\|f(x) - f(\overline{x})\|)B_Y, \quad \text{i.e.,} \quad f(x) - f(\overline{x}) + \phi(\|f(x) - f(\overline{x})\|)B_Y \subset \mathcal{K},$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is an admissible function. In particular, we say that the *strong domination property of order $q > 0$* holds for (P) if ϕ-(*SCP*) holds for (P) with $\phi(\cdot) = \alpha(\cdot)^q$, where $\alpha > 0$.

In other words,

$$\|f(x) - f(\overline{x})\|_+ \geq \alpha\|f(x) - f(\overline{x})\|^q,$$

where $\|\cdot\|_+ = d(\cdot, \mathcal{K}^c)$, and D^c denote the complement of D. If $f(A)$ is uniformly rotund with an admissible function ϕ (see Section 2.3) and the sections $f(A)_y$, $y \in f(A)$, are compact, then ϕ-(SDP) holds for (P).

Proposition 7.3.1. *Let $X = (X, \|\cdot\|)$ and $Y = (Y, \|\cdot\|)$ be normed spaces. Let $\mathcal{K}$ be a closed convex pointed cone in Y with $\operatorname{int}\mathcal{K} \neq \emptyset$. If ϕ-(SDP) holds for (P), then (CP) holds for $f(A)$ and $\delta(\varepsilon) \geq \phi(\varepsilon)$ for any $\varepsilon \in \operatorname{dom}\delta$.*

Proof. Take $0 < \varepsilon \in \operatorname{dom}\delta$ and $x \in A$ such that $d(f(x), E(f, A)) \geq \varepsilon$. Since ϕ is nondecreasing, by ϕ-(*SDP*), there exists $\overline{x} \in S(f, A)$ such that

$$f(x) - f(\overline{x}) + \phi(\varepsilon)B_Y \subset f(x) - f(\overline{x}) + \phi(\|f(x) - f(\overline{x})\|)B_Y \subset \mathcal{K},$$

which, by Proposition 5.1.3, amounts to saying that (*CP*) holds for $f(A)$. Moreover,

$$\|f(x) - f(\overline{x})\|_+ \geq \phi(\|f(x) - f(\overline{x})\|).$$

Consequently, $\mu(f(x)) \geq \phi(\|f(x) - f(\overline{x})\|) \geq \phi(\varepsilon)$ and $\delta(\varepsilon) \geq \phi(\varepsilon)$. ∎

Theorem 7.3.2. *Let $X = (X, \|\cdot\|)$, $Y = (Y, \|\cdot\|)$, $U = (U, \|\cdot\|)$ be normed spaces. Let $\mathcal{K}$ be a closed convex pointed cone in Y with $\operatorname{int}\mathcal{K} \neq \emptyset$. Let $f : X \to Y$ be a Lipschitz mapping with constant $L_f > 0$. If*

(i) *$\mathcal{A}$ is Hölder continuous at $u_0 \in \operatorname{dom}\mathcal{A}$ of order $p \geq 1$ with constants $L_a > 0$ and $t_a > 0$,*
(ii) *(SDP) of order $q \geq 1$ with constant $\alpha > 0$ holds for (P),*

then the performance set-valued mapping $\mathcal{P}$ is upper Hölder continuous at $u_0 \in \operatorname{dom}\mathcal{P}$ of order p/q with constants $L_f L_a + (2L_f L_a/\alpha)^{p/q}$ and $t_a > 0$.

Proof. Take any $\overline{y} = f(\overline{x}) \in E(f, A(u))$, $u \in u_0 + t_a B_U$. By (i), there exists $z \in A$ such that

$$\|\overline{x} - z\| \leq L_a\|u - u_0\|^p,$$

and by the Lipschitzness of f, $\|f(\overline{x}) - f(z)\| \leq L_f L_a \|u - u_0\|^p$. If $z \in S(f, A)$, the conclusion follows. Otherwise, by (ii), there exists $\overline{z} \in S(f, A)$ such that

$$f(z) - f(\overline{z}) + \alpha \|f(z) - f(\overline{z})\|^q B_Y \subset \mathcal{K}.$$

If $\alpha \|f(z) - f(\overline{z})\|^q > 2L_f L_a \|u - u_0\|^p$, then by (i), there exists $x \in A(u)$ such that $\|f(x) - f(\overline{z})\| \leq L_f L_a \|u - u_0\|^p$ and so $f(x) = f(\overline{x})$ since otherwise

$$\begin{aligned} f(\overline{x}) - f(x) &= (f(\overline{x}) - f(z)) + (f(z) - f(\overline{z})) + (f(\overline{z}) - f(x)) \\ &\in (f(z) - f(\overline{z})) + 2L_a \|u - u_0\|^p B_Y \subset \mathcal{K}, \end{aligned}$$

contradicting the fact that $y \in E(f, A(u))$. If

$$\alpha \|f(z) - f(\overline{z})\|^q \leq 2L_f L_c \|u - u_0\|^p,$$

then for $u \in u_0 + t_a B_U$ we get

$$\begin{aligned} d(\overline{y}, E(f, A)) &\leq \|\overline{y} - f(\overline{z})\| \leq \|\overline{y} - f(z)\| + \|f(z) - f(\overline{z})\| \\ &\leq (L_f L_a + (2L_f L_a/\alpha)^{1/q}) \|u - u_0\|^{p/q} \end{aligned}$$

which completes the proof. ∎

8. SHARP AND FIRM SOLUTIONS TO VECTOR OPTIMIZATION PROBLEMS

In this chapter we introduce ϕ-sharp and weak ϕ-sharp solutions (local and global) to problem (P). When applied to scalar optimization problems, the concept of weak ϕ-sharp solutions reduces to the concept of weak sharp minima due to Polyak [126]. In scalar optimization weak sharp minima were also investigated via growth conditions, e.g. by Burke and Deng [43], Burke and Ferris [44], Henrion, Jourani and Outrata [74], Ng and Zheng [116], Studniarski and Ward [147], Ward [150, 151]. Weak sharp minima play an important role in deriving conditions for Hölder calmness of solutions in scalar parametric optimization (see e.g. [39, 100, 101]). In the next chapter we will investigate stability for ϕ-sharp and weak ϕ-sharp solutions.

8.1. Sharp solutions

Let $X = (X, \|\cdot\|)$ and $Y = (Y, \|\cdot\|)$ be normed spaces with open unit balls B_X and B_Y, respectively, and let $\mathcal{K} \subset Y$ be a closed convex pointed cone. Consider a vector optimization problem

$$(P) \qquad \begin{array}{l} \min_{\mathcal{K}} f(x) \\ \text{subject to } x \in A. \end{array}$$

Let $\phi, \nu : \mathbb{R}_+ \to \mathbb{R}_+$ be admissible functions. Recall that $y_0 = f(x_0) \in f(A)$ is a ν-strictly efficient point to (P) if

$$f(x) - f(x_0) \notin \nu(\|f(x) - f(x_0)\|)B_Y - \mathcal{K} \quad \text{for } x \in A,\ f(x) \neq f(x_0).$$

For any $\eta \in f(A)$ put

$$S_\eta := \{x \in A : f(x) = \eta\}.$$

Definition 8.1.1. We say that $x_0 \in A$, $f(x_0) = \eta$, is a *ϕ-sharp solution*, $x_0 \in Sh^\phi(f, A)$, if

$$f(x) - f(x_0) \notin \phi(\|x - x_0\|)B_Y - \mathcal{K} \quad \text{for } x \in A \setminus S_\eta. \tag{8.1}$$

Moreover, $x_0 \in A$ is *sharp of order* $q > 0$, $x_0 \in Sh^q(f, A)$, if x_0 is ϕ-sharp with $\phi(\cdot) = \tau\|\cdot\|^q$, where $\tau > 0$.

For any $y \in Y$ put

$$\|y\|_- = d(y, -\mathcal{K}).$$

In Proposition 2.4.1 we have shown that $y_0 \in StE(f(A))$ iff there exists an admissible function $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\nu(\|y-y_0\|) \leq \|y-y_0\|_- \quad \text{for all } y \in f(A),$$

and ν can be chosen in the form

$$\nu(\varepsilon) = \inf\{\|z-y_0\|_- : z \in f(A) \setminus (y_0 + \varepsilon B_Y)\}.$$

Equivalently, $y_0 \in StE^\nu(f(A))$ iff

$$(y-y_0) \cap (\nu(\|y-y_0\|)B_Y - \mathcal{K}) = \emptyset \quad \text{for } y \in f(A) \setminus \{y_0\}. \tag{8.2}$$

As defined in Section 2.4, $y_0 \in f(A)$ is a locally ν-strictly efficient point, $y_0 \in LStE^\nu(f(A))$, if there exists a neighbourhood V of zero in Y such that

$$(y-y_0) \cap (\nu(\|y-y_0\|)B_Y - \mathcal{K}) = \emptyset \quad \text{for } y \in f(A) \cap (y_0+V) \setminus \{y_0\}.$$

In particular, $y_0 \in f(A)$ is a *locally strictly efficient point of order* $q > 0$, $y_0 \in LStE^q(f(A))$, if there exists a constant $\beta > 0$ such that $y_0 \in LStE^\phi(f(A))$ with $\phi(\cdot) = \beta(\cdot)^q$, i.e.,

$$\beta\|y-y_0\|^q \leq \|y-y_0\|_- \quad \text{for } y \in f(A) \cap (y_0+V).$$

Or, in other words, $y_0 \in f(A)$ is a local sharp minimum of order $q > 0$ (cf. [147]) of the function $\|\cdot - y_0\|_-$ over the set $f(A)$. We put $StE^\nu(f,A) := StE^\nu(f(A))$.

Let us note that if $f(A)$ is uniformly rotund (see Section 2.3) with an admissible function ν, then $E(f,A) = StE^\nu(f,A)$. Indeed, suppose there exists $x_0 \in E(f,A)$, $f(x_0) = \eta$, such that $x_0 \notin StE^\nu(f,A)$. There exists $x \in A \setminus S_\eta$ satisfying $f(x) - f(x_0) \in \nu(\|f(x)-f(x_0)\|)B_Y - \mathcal{K}$. Hence, there exist $0 \neq b \in B_Y$ and $0 \neq k \in \mathcal{K}$ such that $\frac{1}{2}(f(x)+f(x_0)) = f(x_0) - \nu(\|f(x)-f(x_0)\|)b - k$. In view of the uniform rotundity of $f(A)$, this entails that there exists $\widetilde{x} \in A \setminus S_\eta$ such that $f(\tilde{x}) \in f(x_0) - \mathcal{K}$, which contradicts the fact that $x_0 \in E(f,A)$.

Equivalently, the relation (8.1) can be rephrased as

$$\|f(x)-f(x_0)\|_- \geq \phi(\|x-x_0\|) \quad \text{for } x \in A \setminus S_\eta. \tag{8.3}$$

Each sharp solution is a solution. Indeed, if $y_0 = f(x_0)$, $x_0 \in A$, is a sharp solution, then by (8.1),

$$f(x) - f(x_0) \notin -\mathcal{K} \quad \text{for } x \in A,\ f(x) \neq f(x_0).$$

The relationship between sharp solutions and strictly efficient points is clarified in the next proposition.

Proposition 8.1.1. *Let $\mathcal{K}$ be a closed convex pointed cone in a normed space Y. Let $f : X \to Y$ be a Lipschitz mapping on A with constant $L_f > 0$. If $x_0 \in Sh^\phi(f,A)$, then $f(x_0) \in StE^\nu(f,A)$ with $\nu(\cdot) = \phi(\frac{1}{L_f}\cdot)$.*

Proof. Let $x_0 \in Sh^\phi(f,A)$ and $f(x_0) = \eta$. Hence,

$$f(x) - f(x_0) \notin \phi(\|x-x_0\|)B_Y - \mathcal{K} \quad \text{for } x \in A \setminus S_\eta.$$

Since $\|f(x)-f(x_0)\| \leq L_f\|x-x_0\|$ and ϕ is nondecreasing, $\phi(\frac{1}{L_f}\|f(x)-f(x_0)\|) \leq \phi(\|x-x_0\|)$ and

$$f(x) - f(x_0) \notin \phi\left(\frac{1}{L_f}\|f(x)-f(x_0)\|\right)B_Y - \mathcal{K} \quad \text{for } x \in A \setminus S_\eta,$$

which proves that $\eta = f(x_0)$ is ν-strictly efficient with $\nu(\cdot) = \phi(\frac{1}{L_f}\cdot)$. ∎

In view of Proposition 8.1.1,

$$Sh^{\phi}(f,A) \subset A \cap f^{-1}(StE^{\nu}(f,A)) \quad \text{with} \quad \nu(\cdot) = \phi\left(\frac{1}{L_f}\,\cdot\right).$$

In particular, it follows from Proposition 8.1.1 that if f is Lipschitz on A with constant $L_f > 0$ and $x_0 \in Sh^q(f,A)$ with constant τ, then $f(x_0) \in StE^q(f,A)$ with constant

$$\beta = \tau / L_f^q.$$

Definition 8.1.2. We say that $x_0 \in A$ with $f(x_0) = \eta$ is a *local ϕ-sharp solution* to (P), $x_0 \in LSh^{\phi}(f,A)$, if there exists $r > 0$ such that

$$f(x) - f(x_0) \notin \phi(\|x - x_0\|)B_Y - \mathcal{K} \quad \text{for } x \in A \cap (x_0 + rB_X),\ x \notin S_{\eta}.$$

Any local ϕ-sharp solution $x_0 \in LSh^{\phi}(f,A)$, where $\phi(t) = \tau t^q$ for $t \in \mathbb{R}_+$ with $\tau > 0$ and $q > 0$ is called a *local sharp solution of order q* (cf. Jiménez [87, 88] for $S_{\eta} = \{x_0\}$).

Clearly, each global sharp solution is a local sharp solution. We prove the converse for $\mathcal{K}$-convex functions.

Recall that $f : X \to Y$ is *$\mathcal{K}$-convex* on X if for any $\lambda \in [0,1]$ and $x, x' \in X$,

$$f(\lambda x + (1-\lambda)x') \in \lambda f(x) + (1-\lambda)f(x') - \mathcal{K} \quad \text{for any } \lambda \in [0,1],\ x, x' \in X.$$

Note that if A is convex and f is $\mathcal{K}$-convex on A, then the sets S_{η} with $\eta \in E(f,A)$ are convex. Indeed, for any $x, x' \in S_{\eta}$,

$$f(\lambda x + (1-\lambda)x') \in \eta - \mathcal{K}$$

and so $f(\lambda x + (1-\lambda)x') = \eta$ since $\eta \in E(f,A)$.

Proposition 8.1.2. *Let A be convex and let f be $\mathcal{K}$-convex. Let $x_0 \in S_{\eta}$. If $x_0 \in LSh^1(f,A)$ with constant $\tau > 0$, then $x_0 \in Sh^1(f,A)$ with constant τ.*

Proof. Suppose on the contrary that x_0 is not a global sharp solution of order 1 with constant τ. There exists $x \in A \setminus S_{\eta}$ such that

$$f(x) - f(x_0) \in \tau\|x - x_0\|B_Y - \mathcal{K}.$$

Let $\lambda \in [0,1]$. Set $x(\lambda) := \lambda x + (1-\lambda)x_0$. For any $r > 0$ there is $\lambda \in [0,1]$ such that $x(\lambda) \in B(x_0, r)$ and by the convexity assumptions

$$f(x(\lambda)) - f(x_0) \in \lambda(f(x) - f(x_0)) - \mathcal{K} \subset \tau\lambda\|x - x_0\|B_Y - \mathcal{K} = \tau\|x(\lambda) - x_0\|B_Y - \mathcal{K},$$

which proves that x_0 is not a local sharp solution of order 1 with constant τ. ■

Below we give an example of problem (P) with sharp solutions.

Example 8.1.1. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}^2$ and $\mathcal{K} = \mathbb{R}^2_+$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given as

$$f(x_1, x_2) = (x_1^2 + x_2^2, \exp(x_1) + x_2)$$

and $A = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \le x_1 \le 1,\quad 0 \le x_2 \le 1\}$. Then $(0,1) \in E(f,A)$ and $(0,0) \in S(f,A)$ and $(0,0) \in Sh^2(f,A)$ with constant $\tau = 0.5$, i.e.

$$\|f(x) - f(0,0)\|_- \ge 0.5\|x - (0,0)\|^2.$$

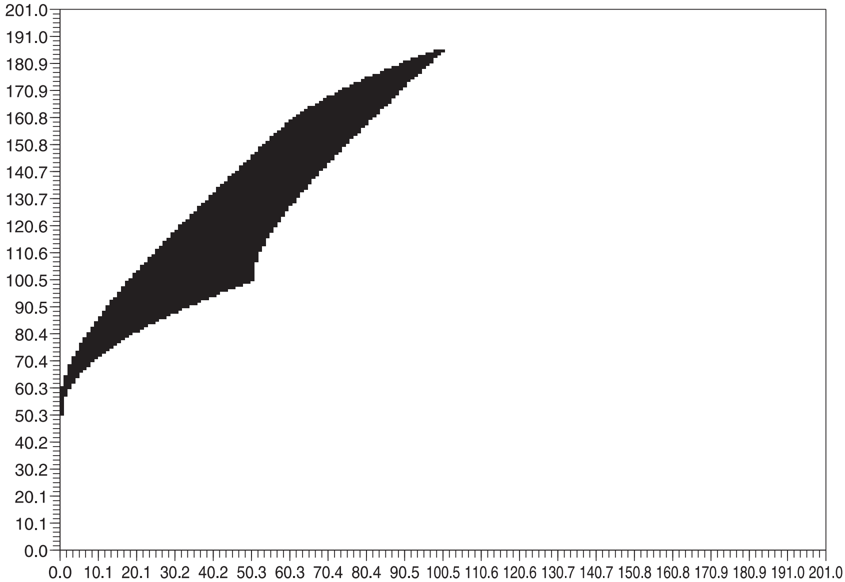

Fig. 8.1 The set $f(A)$ from Example 8.1.1

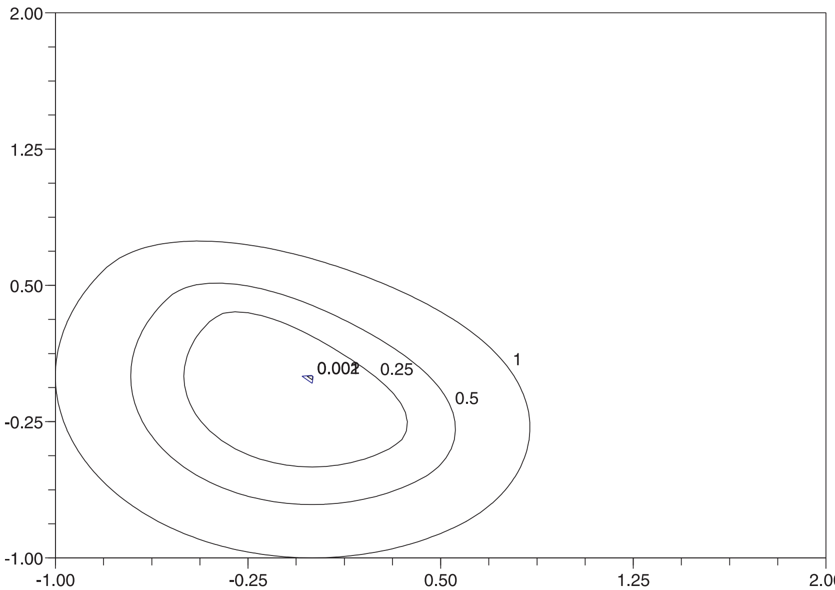

Fig. 8.2 The level sets of the function $\|f(x) - f(0,0)\|_-$ in Example 8.1.1

We define directional differentiability of f at x_0 in the direction u via the contingent derivative

$$f'(x_0; u) = \lim_{(t,v)\to(0^+,u)} \frac{f(x_0 + tv) - f(x_0)}{t}$$

and we say that f is directionally differentiable at x_0 if f is directionally differentiable at x_0 in any direction $v \in X$.

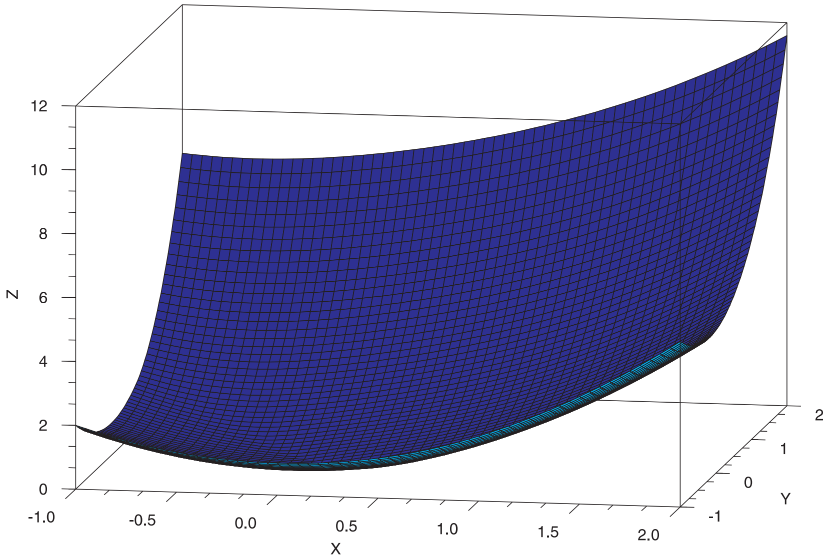

Fig. 8.3 The graph of the function $\|f(x) - f(0,0)\|_{-} - 0.5\|x - (0,0)\|^2$ in Example 8.1.1

The following proposition provides sufficient conditions for sharp solutions in terms of contingent directional derivatives.

PROPOSITION 8.1.3. *Let X be a finite-dimensional space. Let f be directionally differentiable at $x_0 \in A$, $f(x_0) = \eta$. If, for any tangent direction $0 \neq v \in T_{A \setminus S_\eta}(x_0)$,*

$$f'(x_0; v) \notin \tau \operatorname{cl} B_Y - \mathcal{K},$$

then x_0 is a local sharp solution of order 1 to (P) with constant $\tau > 0$.

Conversely, if $x_0 \in A$ is a local sharp solution of order 1 with constant $\tau > 0$, then for any tangent direction $v \in T_{A \setminus S_\eta}(x_0)$, $v \neq 0$,

$$f'(x_0; v) \notin \tau B_Y - \mathcal{K}.$$

Proof. Suppose that x_0, $f(x_0) = \eta$, is not a local sharp solution with constant $\tau > 0$. For each $n \geq 1$ there exists $x_n \in A \cap B(x_0, 1/n)$, $x_n \notin S_\eta$, $x_n \to x_0$, such that

$$f(x_n) - f(x_0) \in \tau \|x_n - x_0\| B_Y - \mathcal{K}.$$

Putting $v_n := (x_n - x_0)/\|x_n - x_0\|$ we get $v_n \to v \in T_{A \setminus S_\eta}(x_0)$, $v \neq 0$, and

$$\frac{f(x_n) - f(x_0)}{\|x_n - x_0\|} \in \tau B_Y - \mathcal{K}, \quad \text{i.e.} \quad f'(x_0; v) \in \tau \operatorname{cl} B_Y - \mathcal{K}.$$

To prove the second assertion suppose that there exists $v \in T_{A \setminus S_\eta}(x_0)$, $v \neq 0$, such that $f(x_0; v) \in \tau B_Y - \mathcal{K}$. Clearly, we may suppose that $\|v\| = 1$. There exists a sequence $(x_n) \subset A \setminus S_\alpha$, $x_n \to x_0$ such that by putting $v_n := (x_n - x_0)/\|x_n - x_0\|$ and $t_n := \|x_n - x_0\|$ we get $v_n \to v \in T_{A \setminus \widetilde{S}_\alpha}(x_0)$. Moreover, $f(x_0 + t_n v_n) - f(x_0) \in \tau t_n B_Y - \mathcal{K}$ for all n sufficiently large, which contradicts the sharp efficiency of x_0. ■

COROLLARY 8.1.1. *Let X be a finite-dimensional space and let f be directionally differentiable at $x_0 \in A$ with $f(x_0) = \eta$. Then x_0 is a local sharp solution of order* 1 *to* (P) *if and only if for any $v \in T_{A \setminus S_\eta}$, $v \neq 0$,*

$$f'(x_0; v) \notin -\mathcal{K}.$$

Proof. The proof of the "if" part is the same as the proof of the "if" part of Proposition 8.1.3 with $\tau = 1/n$.

To complete the proof, assume that there exists $v \in T_{A \setminus \widetilde{S}_\alpha}$, $v \neq 0$, such that

$$f'(x_0; v) = k_0 \in -\mathcal{K}.$$

The remaining part of the proof follows the lines of the second part of the proof of Theorem 4.1 of [88]. ■

Now we discuss the relationships between local sharp solutions and local Henig proper solutions.

Recall that $\eta \in E(f, A)$ is a *local Henig proper efficient point* for (P) if there exist a closed convex cone $\Omega \subset Y$, $\operatorname{int} \Omega \neq \emptyset$, $\mathcal{K} \setminus \{0\} \subset \operatorname{int} \Omega$ and $\varrho > 0$ such that

$$(f(x) - \eta) \cap (-\Omega) = \{0\} \quad \text{for } x \in A \cap B(x_0, \varrho).$$

Moreover, $x_0 \in S(f, A)$, $f(x_0) = \eta$, is a *local Henig proper solution* to (P) if η is a local Henig proper efficient point for (P).

PROPOSITION 8.1.4. *Let $\mathcal{K}$ be a closed convex cone with a compact base Θ.*

- (i) *$\eta \in E(f, A)$ is a local Henig proper efficient point for (P) if and only if η is a local strictly efficient point of order* 1.
- (ii) *Let f be locally Lipschitz around $x_0 \in A$. If x_0 is a local sharp solution of order* 1 *to (P), then x_0 is a local Henig proper solution.*

Proof. (i) Suppose that η is not a local strictly efficient point of order 1 to (P). For each $n \geq 1$ there exists $x_n \in A \setminus S_\eta$, $x_n \to x_0$ such that

$$f(x_n) - f(x_0) \in \frac{1}{n} \|f(x_n) - f(x_0)\| B_Y - \mathcal{K},$$

i.e., there exist $\lambda_n > 0$ and $\theta_n \in \Theta$ such that

$$f(x_n) - f(x_0) = \frac{1}{n} \|f(x_n) - f(x_0)\| b_n - \lambda_n \theta_n \quad \text{for some } b_n \in B_Y. \tag{8.4}$$

Hence,

$$\frac{f(x_n) - f(x_0)}{\|f(x_n) - f(x_0)\|} = \frac{1}{n} b_n - \frac{\lambda_n}{\|f(x_n) - f(x_0)\|} \theta_n.$$

Since Θ is bounded, $\|\theta_n\| \leq M$ for some $M > 0$ and

$$1 \leq \frac{1}{n} + \frac{\lambda_n}{\|f(x_n) - f(x_0)\|} M$$

and consequently, for all n sufficiently large,

$$\frac{\lambda_n}{\|f(x_n) - f(x_0)\|} \geq \frac{1}{2M}.$$

This proves $\|f(x_n)-f(x_0)\|/\lambda_n \le 2M$ and $\varepsilon_n := \|f(x_n)-f(x_0)\|/(n\lambda_n) \to 0$. Finally,

$$f(x_n)-f(x_0) = -\lambda_n(\varepsilon_n(-b_n)+\theta_n)$$

which proves that η is not a local Henig proper efficient point.

Suppose now that η is not a local Henig proper efficient point. For each $n \ge 1$ there exists $x_n \in A$, $f(x_n) \ne f(x_0)$, $x_n \to x_0$, such that

$$f(x_n)-f(x_0) \in -\operatorname{cone}\Big(\frac{1}{n}\,B_Y+\Theta\Big),$$

i.e., there exist $\lambda_n > 0$ and $\theta_n \in \Theta$ such that

$$f(x_n)-f(x_0) = \frac{\lambda_n}{n}\,b_n - \lambda_n\theta_n, \qquad \text{where } b_n \in B_Y. \tag{8.5}$$

Hence,

$$\frac{f(x_n)-f(x_0)}{\lambda_n} = \frac{1}{n}\,b_n - \theta_n,$$

and since Θ is compact, we can assume that $\theta_n \to \theta_0 \in \Theta$, $\theta_0 \ne 0$ and

$$v_n := \frac{f(x_n)-f(x_0)}{\lambda_n} \to -\theta_0.$$

This proves that there exists $M > 0$ such that $\|f(x_n)-f(x_0)\|/\lambda_n \ge M$ and consequently

$$\frac{\lambda_n}{\|f(x_n)-f(x_0)\|} \le \frac{1}{M}.$$

Hence, $\varepsilon_n := \frac{\lambda_n}{n\|f(x_n)-f(x_0)\|} \to 0$ and by (8.5),

$$f(x_n)-f(x_0) = \varepsilon_n\|f(x_n)-f(x_0)\|b_n - k_n, \qquad \text{where } k_n \in \mathcal{K}.$$

This proves that η is not a local strictly efficient point.

(ii) If $x_0 \in A$, $f(x_0)=\eta$, is a local sharp solution of order 1 to (P), then by Proposition 8.1.1, η is a local strictly efficient point of order 1, and by part (i), η is a local Henig proper solution to (P). ∎

8.2. Weak sharp solutions

In the present section we discuss weak sharp solutions to (P) and growth conditions for vector-valued functions. Let us note that one can easily generalize the definitions given below to ϕ-weak sharp solutions and ϕ-growth conditions, where ϕ is an admissible function. In view of further applications we limit our attention to functions ϕ of the form $\phi(\cdot) = \tau(\cdot)^q$, where $\tau > 0$ and $q > 0$ are given constants.

Recall that $S_\eta = \{x \in A : f(x) = \eta\}$.

Definition 8.2.1. We say that $x_0 \in A$ with $f(x_0) = \eta$ is a (*global*) *weak sharp solution of order* $q > 0$ to (P), $x_0 \in Wh^q(f,A)$, if there exists $\tau > 0$ such that

$$f(x)-f(x_0) \notin \tau(d(x,S_\eta))^q B_Y - \mathcal{K} \quad \text{ for } x \in A \setminus S_\eta. \tag{8.6}$$

Relation (8.6) can be rewritten as

$$\|f(x)-f(x_0)\|_- \ge \tau(d(x,S_\eta))^q \quad \text{ for } x \in A \setminus S_\eta. \tag{8.7}$$

Each weak sharp solution to (P) is a solution to (P). If $x_0 \in Sh^q(f,A)$, then $x_0 \in Wh^q(f,A)$. If $x_0 \in Wh^q(f,A)$, then $S_\eta = \{x \in A : f(x) = f(x_0) = \eta\} \subset Wh^q(f,A)$. Moreover, if $x_0 \in Wh^q(f,A)$, then

$$f(x) - f(x_0) \notin \tau(d(x,S(f,A))^q B_Y - \mathcal{K} \quad \text{for } x \in A \setminus S(f,A). \tag{8.8}$$

In the case where $f_0 : X \to \mathbb{R}$ is a real-valued function, with the notation $m_0 = \inf\{f_0(x) : x \in A_0\}$, $x_0 \in S(f_0,A_0) = \{x \in A_0 : f_0(x) = m_0\}$, relation (8.6) takes the form

$$f_0(x) \geq m_0 + \tau(d(x,S(f_0,A_0)))^q \quad \text{for } x \in A_0,$$

which means that $S(f_0,A_0)$ is the set (global) weak sharp minima of order q of f_0 over A_0 as defined e.g. in [43, 116, 147].

Definition 8.2.2. We say that the *global growth condition of order* $q > 0$ holds for problem (P) on $\overline{S} \subset S(f,A)$ if there exists $\tau > 0$ such that for any $\overline{x} \in \overline{S}$ and $x \in A \setminus S(f,A)$ we have

$$(f(x) - f(\overline{x})) \cap (\tau(d(x,S(f,A)))^q B_Y - \mathcal{K}) = \emptyset. \tag{8.9}$$

Note first that if the global growth condition of order q holds for $\overline{S} \subset S(f,A)$, then for any $\overline{x} \in \overline{S}$,

$$S_\eta = \{x \in A : f(x) = f(\overline{x}) = \eta\} \subset \overline{S}.$$

Moreover, the global growth condition holds for (P) on $S(f,A)$ iff for any $\overline{x} \in S(f,A)$,

$$f(x) - f(\overline{x}) \notin \tau(d(x,S(f,A))^q B_Y - \mathcal{K} \quad \text{for } x \in A \setminus S(f,A). \tag{8.10}$$

The following proposition establishes the relationship between global weak sharp solutions and the global growth condition.

Proposition 8.2.1. *Let X and Y be normed spaces and let $\mathcal{K}$ be a closed convex pointed cone in Y. If there exists a subset $\overline{S} \subset S(f,A)$ such that all $\overline{x} \in \overline{S}$ are global weak sharp solutions to (P) of order q with constant $\tau > 0$, then the global growth condition of order q holds for (P) on $\overline{S}$ with constant τ.*

Proof. This follows immediately from the observation that for any $\overline{x} \in \overline{S}$,

$$S_\eta = \{x \in A : f(x) = f(\overline{x}) = \eta\} \subset \overline{S}$$

and hence

$$f(x) - f(\overline{x}) \notin \tau(d(x,S(f,A)))^q B_Y - \mathcal{K} \quad \text{for } x \in A \setminus \overline{S},$$

which proves the assertion. ∎

Local versions of the above notions can be obtained in several ways. The definitions given below are shaped so as to be versatile for applications presented in the next sections.

Definition 8.2.3. We say that $x_0 \in A$, $f(x_0) = \eta$, is a *local weak sharp solution of order* $q > 0$ to (P), $x_0 \in LWh^q(f,A)$, if there exist a 0-neighbourhood V in X and constant $\tau > 0$ such that for $x \in A \cap (x_0 + V)$, $x \notin S_\eta$,

$$(f(x) - f(x_0)) \cap (\tau(d(x,S_\eta))^q B_Y - \mathcal{K}) = \emptyset.$$

Clearly, each local sharp solution of order q to (P) is a local weak sharp solution of order q to (P) and each local weak sharp solution of order q to (P) is a local solution to (P). Or, equivalently, $x_0 \in A$ is a local weak sharp solution to (P) iff x_0 is a local weak sharp minimum ([43, 116, 147]) of the function $\|f(\cdot) - f(x_0)\|_-$ over A.

Definition 8.2.4. *The (local) growth condition of order* $q > 0$ holds for (P) on $\overline{S} \subset S(f, A)$ if there exist a 0-neighbourhood V in X and $\tau > 0$ such that for any $\overline{x} \in \overline{S}$ and $x \in A \cap (\overline{x} + V)$, $x \notin \overline{S}$, we have

$$(f(x) - f(\overline{x})) \cap (\tau(d(x, S(f, A)))^q B_Y - \mathcal{K}) = \emptyset.$$

Moreover, we say that the *local growth condition of order q holds for* (P) *around* $x_0 \in S(f, A)$ if there exists a 0-neighbourhood V in X and a constant $\tau > 0$ such that for any $\overline{x} \in \overline{S} = S(f, A) \cap (x_0 + V)$ and any $x \in A \cap (\overline{x} + V)$ we have

$$\tau(d(x, S(f, A)))^q \leq \|f(x) - f(\overline{x})\|_-.$$

Or equivalently, for $x \in A \cap (\overline{x} + V)$, $x \notin \overline{S}$,

$$f(x) - f(\overline{x}) \notin \tau(d(x, S(f, A)))^q B_Y - \mathcal{K}.$$

This means that each $\overline{x} \in S(f, A) \cap (x_0 + V)$ is a local weak sharp minimum of order q (cf. [43, 116, 147]) of the function $\|f(\cdot) - f(\overline{x})\|_-$ over A with the same constant $\tau > 0$.

Consider now the scalar case with $f_0 : X \to \mathbb{R}$, $\mathcal{K}_+ = \mathbb{R}_+$, and $m_0 = f_0(x_0) = \inf\{f_0(x) : x \in A_0\}$. Then, by definition, the growth condition of order $q > 0$ holds for f_0 on a subset $\overline{S} \subset S(f_0, A_0)$, $f_0(\overline{S}) = m_0$, if there is a neighbourhood V of zero in X and a constant $\tau > 0$ such that

$$f_0(x) \geq m_0 + \tau d(x, S(f_0, A_0))^q \quad \text{for } x \in A \cap (\overline{S} + V) \tag{8.11}$$

which means that each $\overline{x} \in \overline{S}$ is a local weak sharp minimum of order q of f_0 over A_0.

Recall ([39, Ch. 3.1, Def. 3.1]) that the growth condition of order $q > 0$ holds for a real-valued function f_0 on $\overline{S} \subset S(f_0, A_0)$ if there exist a constant $\tau > 0$ and a neighbourhood V of zero in X such that

$$f_0(x) \geq m_0 + \alpha d(x, \overline{S})^q \quad \text{for } x \in A \cap (\overline{S} + V). \tag{8.12}$$

Thus, if $\overline{S} = S(f, A)$ conditions (8.11) and (8.12) coincide.

The question of relationships between well-posedness and weak sharp solutions will be addressed in the next chapter.

Proposition 8.2.2. *Let* $f : X \to Y$ *be a Lipschitz mapping on* X *with constant* $L_f > 0$. *If* $x_0 \in A$ *is a weak sharp solution of order* q *with constant* $\tau > 0$, *then* $f(x_0)$ *is a strictly efficient point of order* q *with constant* $\beta = \tau / L_f^q$.

Proof. By definition, if $x_0 \in S(f, A)$, $f(x_0) = \eta$, is a weak sharp solution of order q with constant τ, then $(f(x) - f(x_0) \cap \tau(d(x, S_\eta))^q B_Y - \mathcal{K}) = \emptyset$ for any $x \in A \setminus S_\eta$. Since f is Lipschitz on X, $\|f(x) - f(x_0)\| \leq L_f \|x - x_0\|$. Consequently, $\|f(x) - \eta\| \leq L_f d(x, S_\eta)$, and

$$f(x) - \eta \notin \frac{\tau}{L^q} \|f(x) - \eta\| B_Y - \mathcal{K} \quad \text{for } x \in A,\ f(x) \neq \eta,$$

which proves that $\eta \in StE^q(f, A)$ with constant τ / L^q. ∎

In the theorem below we prove lower Hölder continuity of the performance set-valued mapping $\mathcal{P}$ at a given $u_0 \in \operatorname{dom} \mathcal{P}$ for a family of parametric problems of the form

$$(P_u) \qquad \begin{array}{l} \min_{\mathcal{K}} f(x) \\ \text{subject to } x \in A(u). \end{array}$$

Let $\mathcal{A} : U \rightrightarrows X$ be a set-valued mapping defined on a normed space U, $\mathcal{A}(u) = A(u)$, $\mathcal{A}(u_0) = A$.

THEOREM 8.2.1. *Let $Y = (Y, \|\cdot\|)$ be a normed space and let $\mathcal{K}$ be a closed convex pointed cone in Y. If*

(i) *all $\overline{x} \in S(f, A)$ are weak sharp solutions of order $q \geq 1$ with constant $\tau > 0$,*
(ii) *there exists $0 < t < 1$ such that (DP) holds for all $f(A(u))$, $u \in u_0 + tB_U$,*
(iii) *$\mathcal{A}$ is Hölder continuous of order $p \geq 1$ with constants $L_a > 0$ and t at $u_0 \in \operatorname{dom} \mathcal{A}$ and f is Lipschitz on X with constant $L_f > 0$,*

then $\mathcal{P}$ is lower Hölder continuous of order p/q at $u_0 \in \operatorname{dom} \mathcal{P}$, i.e.

$$E(f, A) \subset E(f, A(u)) + (L_f L_a + (2L_f^q L_a/\tau)^{1/q})\|u - u_0\|^{p/q} B_Y$$

for $u \in u_0 + tB_U$.

Proof. Note first that under our assumptions the set-valued mapping $\mathcal{A}_f$ is lower and upper Hölder continuous of order p at $u_0 \in \operatorname{dom} \mathcal{A}$. Now, it is enough to observe that by Proposition 8.2.2, if all the solutions $S(f, A)$ are weak sharp of order $q \geq 1$, with constant $\tau > 0$, then all $\eta \in E(f, A)$ are strictly efficient of order q with constant τ. The conclusion follows from Theorem 4.1.1. ∎

Note that we can specify the above result for parametric vector optimization problems in the same way as in Theorem 4.1.3.

THEOREM 8.2.2. *Let X and Y be normed spaces and let $\mathcal{K}$ be a closed convex pointed cone in Y. Assume that*

(i) *there exist $\tau > 0$ and $q \geq 1$ such that for any $\overline{x} \in S(f, A)$,*

$$f(x) - f(\overline{x}) \notin \tau(d(x, S_\eta))^q B_Y - \mathcal{K} \quad \text{for } x \in A \setminus S_\eta,$$

(ii) *f is Lipschitz on X with constant $L_f > 0$, $\mathcal{A}$ is Hölder continuous of order $p \geq 1$ at $u_0 \in \operatorname{dom} \mathcal{A}$ with constants $L_a > 0$ and $0 < t < 1$,*
(iii) *(DP) holds for all $f(A(u))$ and $u \in B(u_0, t)$.*

Then $\mathcal{P}$ is lower Hölder continuous of order p/q at $u_0 \in \operatorname{dom} \mathcal{P}$ and

$$E(f, A) \subset E(f, A(u)) + L_f(L_a + (L_a/\tau)^{1/q})\|u - u_0\|^{p/q} B_Y$$

for $u \in B(u_0, t)$.

In Theorem 7.3.2 we derived conditions for the upper Hölder continuity of $\mathcal{P}$ with the help of the (*SDP*) property. In deriving the stability conditions for different type of continuities we can relax the (*SDP*) (or (*CP*)) property by imposing stronger assumptions on solutions (sharpness, weak sharpness).

Below we prove the upper Hölder continuity of $\mathcal{P}$ by assuming that all the solutions to all (P_u) in some neighbourhood of u_0 are weak sharp with the same constant. Note that in the result below we do not assume that $\operatorname{int}\mathcal{K} \neq \emptyset$.

THEOREM 8.2.3. *Let $X = (X, \|\cdot\|)$, $Y = (Y, \|\cdot\|)$, $U = (U, \|\cdot\|)$ be normed spaces. Let $\mathcal{K}$ be a closed convex pointed cone in Y with $\operatorname{int}\mathcal{K} \neq \emptyset$. Let $f : X \to Y$ be a Lipschitz mapping with constant $L_f > 0$. If*

(i) *$\mathcal{A}$ is Hölder continuous at $u_0 \in \operatorname{dom}\mathcal{A}$ of order $p \geq 1$ with constants $L_a > 0$ and $t > 0$,*
(ii) *(DP) holds for (P),*
(iii) *all $\overline{z} \in S(f, A(u))$ for $u \in B(u_0, t)$ are weak sharp of order $q \geq 1$ with the same constant τ, i.e.*
$$f(z) - f(\overline{z}) \notin \tau(d(z, S_{f(\overline{z})}(u)))^q B_Y - \mathcal{K} \quad \text{for } z \in A(u),\ z \notin S_{f(\overline{z})}(u),$$
where $S_{f(\overline{z})}(u) = \{z \in S(f, A(u)) : f(z) = f(\overline{z})\}$,

then the performance set-valued mapping $\mathcal{P}$ is upper Hölder continuous at $u_0 \in \operatorname{dom}\mathcal{P}$ of order p/q with constants $L_f(L_a + (2L_aL_f/\tau)^{1/q})$ and $t > 0$.

Proof. Take any $\overline{y} = f(\overline{z}) \in E(f, A(u))$, $u \in u_0 + t_aB_U$. By (i), there exists $x \in A$ such that
$$\|\overline{z} - x\| \leq L_a\|u - u_0\|^p$$
and by the Lipschitz property $\|f(\overline{z}) - f(x)\| \leq L_fL_a\|u - u_0\|^p$. If $x \in S(f, A)$, the conclusion follows. Otherwise, by (ii), there exists $\overline{x} \in S(f, A)$, $f(\overline{x}) \neq f(x)$, such that $f(\overline{x}) \in f(x) - \mathcal{K}$. By (i), there exists $z \in A(u)$ such that $\|\overline{x} - z\| \leq L_a\|u - u_0\|^p$ and $\|f(\overline{x}) - f(z)\| \leq L_fL_a\|u - u_0\|^p$. If $f(z) = f(\overline{z})$, the conclusion follows. Otherwise,
$$f(z) - f(\overline{z}) \in 2L_fL_a\|u - u_0\|^p - \mathcal{K}$$
and since by Proposition 8.2.2, $f(\overline{z})$ is a strictly efficient point of order q for (P_u) with constant τ/L_f^q, we obtain
$$f(z) - f(\overline{z}) \notin \frac{\tau}{L_f^q}\|f(z) - f(\overline{z})\|^q B_Y - \mathcal{K}.$$
Hence,
$$\|f(z) - f(\overline{z})\| \leq L_f(2L_aL_f/\tau)^{1/q}\|u - u_0\|^{p/q}$$
and consequently
$$f(\overline{z}) - f(\overline{x}) = (f(\overline{z}) - f(z)) + (f(z) - f(\overline{x})) \in L_f(L_a + (2L_aL_f/\tau)^{1/q})\|u - u_0\|^{p/q}. \ \blacksquare$$

8.3. Firm solutions

In a series of publications Attouch and Wets [6]–[8] developed an approach to investigating quantitative stability of variational systems as defined by Rockafellar and Wets [133]. In [6] Lipschitz and Hölder continuities are investigated for ϕ-local minimizers to parametric scalar minimization problems. Given a function $f_0 : X \to \mathbb{R}$ an element $x_0 \in X$ is called a ϕ-*local minimizer* of f_0 if $f_0(x) \geq f_0(x_0) + \phi(\|x - x_0\|)$ for all x in some ball around x_0 and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is an admissible function, i.e. ϕ is nondecreasing, $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$.

In this section we generalize the above idea to vector-valued functions by defining ϕ-firm solutions to vector optimization problems. We exploit this notion to investigate Hölder behaviour of the performance set-valued mapping $\mathcal{P}$.

Let $f : X \to Y$ be a mapping and A be a subset of X. Consider a vector optimization problem

$$(P) \qquad \begin{array}{l} \min_{\mathcal{K}} f(x) \\ \text{subject to } x \in A. \end{array}$$

In Definition 7.3.1 we defined ϕ-strong containment property. Now we define its analog for problem (P). Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be an admissible function.

DEFINITION 8.3.1. We say that the efficient point set $E(f, A)$ to (P) is ϕ-*dominated* if ϕ-(SDP) holds for $f(A)$, i.e., if for each $x \in A$ there exists $\overline{x} \in S(f, A)$ such that

$$f(x) \succeq_{\mathcal{K}} f(\overline{x}) + \phi(\|f(x) - f(\overline{x})\|)B_Y, \text{ i.e., } f(x) - f(\overline{x}) + \phi(\|f(x) - f(\overline{x})\|)B_Y \subset \mathcal{K}.$$

Moreover, $E(f, A)$ is *dominated of order* $q > 0$ if $E(f, A)$ is ϕ-dominated with $\phi(\cdot) = \alpha(\cdot)^q$ with some $\alpha > 0$.

DEFINITION 8.3.2. The solution set $S(f, A)$ to (P) is called ϕ-*firm* or ϕ-*dominated* if for each $x \in A$ there exists $\overline{x} \in S(f, A)$ such that

$$f(x) \succeq_{\mathcal{K}} f(\overline{x}) + \phi(\|x - \overline{x}\|)B_Y, \quad \text{i.e.,} \quad f(x) - f(\overline{x}) + \phi(\|x - \overline{x}\|)B_Y \subset \mathcal{K}.$$

In particular, $S(f, A)$ is *firm of order* q if $S(f, A)$ is ϕ-firm with $\phi(\cdot) = \varrho(\cdot)^q$ with some $\varrho > 0$, i.e., for each $x \in A$ there exists $\overline{x} \in S(f, A)$ such that

$$f(x) - f(\overline{x}) + \varrho\|x - \overline{x}\|^q B_Y \subset \mathcal{K}.$$

PROPOSITION 8.3.1. *Let* $X = (X, \|\cdot\|)$ *and* $Y = (Y, \|\cdot\|)$ *be normed spaces. Let* $\mathcal{K}$ *be a closed convex pointed cone in* Y *with* $\operatorname{int}\mathcal{K} \neq \emptyset$. *Let* $f : X \to Y$ *be a Lipschitz mapping with constant* $L_f > 0$. *If* $S(f, A)$ *is* ϕ-*firm, then* $E(f, A)$ *is* μ-*dominated with* $\mu(\cdot) = \phi(\frac{1}{L_f}\ \cdot)$.

Proof. By assumption, for each $x \in A$ there exists $\overline{x} \in S(f, A)$ such that

$$f(x) - f(\overline{x}) + \phi(\|x - \overline{x}\|)B_Y \subset \mathcal{K}.$$

Since $\|f(x) - f(\overline{x})\| \leq L_f\|x - \overline{x}\|$ and ϕ is nondecreasing, $\phi(\frac{1}{L_f}\|f(x) - f(\overline{x})\|) \leq \phi(\|x - \overline{x}\|)$ and

$$f(x) - f(\overline{x}) + \phi\left(\frac{1}{L_f}\|f(x) - f(\overline{x})\|\right)B_Y \subset f(x) - f(\overline{x}) + \phi(\|x - \overline{x}\|)B_Y \subset \mathcal{K},$$

which proves the assertion. ■

In particular, if f is Lipschitz on A with constant $L_f > 0$ and the solution set $S(f, A)$ is firm of order q with constant $\varrho > 0$, then $E(f, A)$ is dominated of order q with constant ϱ/L_f^q.

Let C be a subset of Y. Recall that the domination property (DP) holds for C if $C \subset E(C) + \mathcal{K}$, and the domination property (DP) holds for (P) if (DP) holds for $f(A)$, i.e., for each $x \in A$ there is $\overline{x} \in S(f, A)$ such that $f(x) - f(\overline{x}) \subset \mathcal{K}$.

Let $\operatorname{int} \mathcal{K} \neq \emptyset$. We say that the *(global) strong domination property* (SDP) of order $q > 0$ holds for C if there exists $\varrho > 0$ such that for each $y \in C$ there exists $\eta \in E(C)$ such that

$$y - \eta - \varrho \|y - \eta\|^q B_Y \subset \mathcal{K}. \tag{8.13}$$

We say that the (*local*) *strong domination property* ($LSDP$) of order $q > 0$ holds for C around $y_0 \in C$ if there exist a neighbourhood W of zero in Y and $\varrho > 0$ such that for each $y \in C \cap (y_0 + W)$ there exists $\eta \in E(C) \cap (y_0 + W)$ such that (8.13) holds.

To cast the notions of ϕ-firm (or firm of order q) solutions (see Definitions 8.3.2) into the framework of variants of the domination property we say that *the (global) firm domination property* (FDP) *of order* $q > 0$ *holds for* (P) if the solution set $S(f, A)$ is firm of order q, i.e., there exists a constant $\varrho > 0$ such that for each $x \in A \setminus S(f, A)$ there exists $\overline{x} \in S(f, A)$ with

$$f(x) - f(\overline{x}) - \varrho \|x - \overline{x}\|^q B_Y \subset \mathcal{K}. \tag{8.14}$$

Equivalently, (FDP) of order q holds for (P) iff there exists $\varrho > 0$ such that for each $x \in A \setminus S(f, A)$ there exists $\overline{x} \in S(f, A)$ such that

$$\varrho \|x - \overline{x}\|^q \leq \|f(x) - f(\overline{x})\|_+,$$

where $\|\cdot\|_+ = d(\cdot, \mathcal{K}^c)$ and D^c denotes the complement of a subset D. If f is Lipschitz on X with constant $L_f > 0$ and (FDP) of order q with constant $\varrho > 0$ holds for (P), then (SDP) of order q with constant ϱ / L_f holds for (P) (cf. Definition 7.3.1 and (8.13)).

DEFINITION 8.3.3 ([19]). Let $\operatorname{int} \mathcal{K} \neq \emptyset$. We say that *the* (*local*) *firm domination property* ($LFDP$) *of order* $q > 0$ *holds for* (P) *around* $x_0 \in A$ if there exist a 0-neighbourhood V in X and $\varrho > 0$ such that for each $x \in A \cap (x_0 + V)$ there exists $\overline{x} \in S(f, A) \cap (x_0 + V)$ with

$$f(x) - f(\overline{x}) + \varrho \|x - \overline{x}\|^q B_Y \subset \mathcal{K}.$$

Equivalently, ($LFDP$) of order q holds for (P) around $x_0 \in A$ iff there exist a neighbourhood V of zero in X and $\varrho > 0$ such that for each $x \in A \cap (x_0 + V)$, there is $\overline{x} \in S(f, A) \cap (x_0 + V)$ with

$$\varrho \|x - \overline{x}\|^q \leq \|f(x) - f(\overline{x})\|_+. \tag{8.15}$$

If $f_0 : X \to \mathbb{R}$, $\mathcal{K}_+ = \mathbb{R}_+$, and $m_0 = f_0(x_0) = \inf\{f_0(x) : x \in A_0\}$, then, by definition, ($LFDP$) of order q holds around $x_0 \in A_0$ if there are a 0-neighbourhood V in X and $\varrho > 0$ such that for any $x \in A_0 \cap (x_0 + V)$, there is $\overline{x} \in S(f_0, A_0) \cap (x_0 + V)$ satisfying

$$f_0(x) \geq m_0 + \varrho \|x - \overline{x}\|^q \geq m_0 + \varrho d(x, S(f_0, A_0))^q, \tag{8.16}$$

which means that x_0 is a local weak sharp minimum of order q of f_0 over A_0 (cf. [43, 116]). Note that (8.16) coincides with (8.11) for $S = \{x_0\}$, which means that for scalar-valued functions the growth condition of order q around x_0 coincides with the local firm domination property of order q around x_0.

It is worth noticing that, in general, if $(LFDP)$ holds around $x_0 \in A$ with a neighbourhood V, then it may not hold around x_0 with a smaller neighbourhood $V_1 \subset V$.

EXAMPLE 8.3.1. Let $Y = \mathbb{R}^2$, $\mathcal{K} = \mathbb{R}^2_+$, $f = \mathrm{id}$ and $A \subset \mathbb{R}^2$ is the union of three segments of the form

$$A = [(-10, 1/2), (-1, 1)] \cup [(-1, 1), (0, 0)] \cup [(0, 0), (20, 1)].$$

We have $(0,0) \in S(\mathrm{id}, A)$. $(LFDP)$ holds around $(0,0)$ with $V = 11B_Y$, but not with $V = 5B_Y$, since $(-1,1) \in 5B_Y$ and there is no $s \in S(\mathrm{id}, A) \cap 5B_Y$ such that (8.15) holds.

EXAMPLE 8.3.2. Let $Y = \ell^\infty$, $f = \mathrm{id}$, and let $\mathcal{K} = \ell^\infty_+$. Consider

$$A = \{y \in \ell^\infty : 0 \leq f(y) \leq 1\},$$

where f is the continuous linear functional given by $f(y) = \sum_{n=1}^\infty y_n/2^n$. We have $E(\mathrm{id}, A) = \{y \in A : f(y) = 0\}$ and the strong domination property of order one holds for A. It has been shown in [20] that $StE(A) = \emptyset$.

9. STABILITY OF SOLUTIONS

In this chapter we investigate Hausdorff, Hölder and pseudo-Hölder continuities of solutions to parametric vector optimization problems. To this end we propose several definitions of well-posedness for vector optimization problems. These definitions are based on properties of ε-solutions to vector optimization problems (cf. [50, 52, 99, 104]).

The notion of well-posedness and its various generalizations appear to be very fruitful in scalar optimization, especially in stability analysis. Well-posedness plays an important rule in establishing convergence of algorithms for solving scalar optimization problems.

In vector optimization there is no commonly accepted definition of well-posed problem. Some attempts in this direction have been already made by Miglierina and Molho [110] and the present author [21–23].

In Section 9.1, on the basis of continuity properties of ε-solution mappings we define well-posed vector optimization problems. We establish relationships between well-posedness, sharp and weak sharp solutions. In Section 9.2 we give sufficient conditions for the solution set-valued mapping $\mathcal{S}$ to be upper Hausdorff semicontinuous (Theorem 9.2.1). In Section 9.3 we prove lower Lipschitz continuity (Theorems 9.3.1, 9.3.3) of $\mathcal{S}$. In Section 9.4 we formulate sufficient conditions for upper Lipschitz continuity of $\mathcal{S}$ (Theorems 9.4.1–9.4.3). In Section 9.5 lower Hölder and lower pseudo-Hölder continuities of $\mathcal{S}$ are investigated. In Section 9.6 upper Hölder and upper pseudo-Hölder continuities of $\mathcal{S}$ are investigated (Theorem 9.C.1) as well as Hölder calmness (Theorem 9.6.2).

Let Y be a Hausdorff topological vector space ordered by a partial ordering relation $\preceq_{\mathcal{K}}$ generated by a closed convex pointed cone $\mathcal{K}$ (see Section 1.2). Let X and U be topological spaces. Let $f : X \to Y$ and $A \subset X$. We consider vector optimization problems

$$(P) \qquad \begin{array}{l} \min_{\mathcal{K}} f(x) \\ \text{subject to } x \in A \end{array}$$

and the family (P_u) of parametric vector optimization problems parametrized by a parameter $u \in U$,

$$(P_u) \qquad \begin{array}{l} \min_{\mathcal{K}} f(x) \\ \text{subject to } x \in A(u) \end{array}$$

with $A(u_0) = A$. It is worth noticing that the results of the present chapter can be easily generalized to parametric problems (P_u) with parametrized mapping f.

In relation to Propositions 6.2.1 and 6.2.2 we have the following technical result.

THEOREM 9.0.1. *Let X, U be topological spaces and let Y be a Hausdorff topological vector space. Let $f : X \to Y$ be a $\mathcal{K}$-upper continuous (respectively, $\mathcal{K}$-lower continuous)* *mapping and let $\mathcal{A} : U \rightrightarrows X$ be lower semicontinuous at $u_0 \in \operatorname{dom} \mathcal{A}$. Then the set-valued mapping $(\mathcal{A}_f : U) \rightrightarrows (Y)$, $\mathcal{A}_f(u) = f(\mathcal{A}(u))$ for $u \in U$, is sup-lower continuous (respectively, inf-lower continuous) at $u_0 \in \operatorname{dom} \mathcal{A}$.*

Proof. Let $y_0 \in \mathcal{A}_f(u_0)$. Choose any open 0-neighbourhoood Q in Y. There exists an $x_0 \in \mathcal{A}(u_0)$ such that $f(x_0) = y_0$ and, by the upper continuity of f (respectively, lower continuity of f), there exists an open neighbourhood W of x_0 such that $f(W) \subset y_0 + Q - \mathcal{K}$ (respectively, $f(W) \subset y_0 + Q + \mathcal{K}$). Since $\mathcal{A}$ is lower semicontinuous at u_0, there exists a neighbourhood U of u_0 such that $W \cap \mathcal{A}(u) \neq \emptyset$ for $u \in U$. Now, by taking any $x \in \mathcal{A}(u)$, $x \in W$, $u \in U$, we obtain $f(x) \in \mathcal{F}\mathcal{A}(u)$, $f(x) \in y_0 + Q - \mathcal{K}$ (respectively, $f(x) \in y_0 + Q + \mathcal{K}$) and hence $(y_0 + Q - \mathcal{K}) \cap \mathcal{A}_f(u) \neq \emptyset$ (respectively, $(y_0 + Q + \mathcal{K}) \cap \mathcal{F}\mathcal{A}(u) \neq \emptyset$) for $u \in U$. ∎

9.1. Well-posed vector optimization problems

Let X and Y be Hausdorff topological vector spaces and let $\mathcal{K}$ be a closed convex pointed cone in Y with $\operatorname{int} \mathcal{K} \neq \emptyset$. Basing ourselves on the continuity properties of ε-solutions to a vector optimization problem

$$(P) \qquad \begin{array}{l} \min_{\mathcal{K}} f(x) \\ \text{subject to } x \in A \end{array}$$

we introduce several concepts of well-posedness for (P). To this end we exploit ε-solutions to (P) as defined e.g. in [99] and [104].

DEFINITION 9.1.1. Let $\varepsilon \in \mathcal{K}$. A point $\underline{x} \in A$ is an *ε-Pareto solution* to (P) if there is no $x \in A$ such that $f(\underline{x}) - \varepsilon - f(x) \in \mathcal{K} \setminus \{0\}$.

We denote by $S_\varepsilon(f, A)$ the set of all ε-solutions to (P) and by $E^\varepsilon(f, A)$ the set of all ε-points for (P) (i.e. the image of $S^\varepsilon(f, A)$ under f). Thus, $S^\varepsilon(f, A) = A \cap f^{-1}(E^\varepsilon(f, A))$.

Let $K_0 = \operatorname{int} \mathcal{K} \cup \{0\}$ and $\eta \in E(f, A)$. Let $\Pi^\eta : K_0 \rightrightarrows X$ be the set-valued mapping defined as

$$\Pi^\eta(\varepsilon) := \{x \in A : \eta + \varepsilon - f(x) \in \mathcal{K}\}.$$

The set-valued mapping Π^η is called the *η-ε-solution mapping.* We have

$$\Pi^\eta(\varepsilon) = A \cap f^{-1}(\eta + \varepsilon - \mathcal{K}).$$

Moreover, $\Pi^\eta(0) = \{x \in S(f, A) : f(x) = \eta\} = S_\eta$ and $\bigcup_{\eta \in E(f,A)} \Pi^\eta(0) = S(f, A)$. The sets $\Pi^\eta(\varepsilon)$ were used in [4] to investigate some stability properties of sequences of vector optimization problems.

Let $\Pi : K_0 \rightrightarrows X$ be the set-valued mapping defined as

$$\Pi(\varepsilon) = \bigcup_{\eta \in E(f,A)} \Pi^\eta(\varepsilon) = \{x \in A : f(x) \in E(f, A) + \varepsilon - \mathcal{K}\}.$$

It is called the *ε-solution mapping.* We have

$$\Pi(\varepsilon) = A \cap f^{-1}(E(f, A) + \varepsilon - \mathcal{K}).$$

Moreover, $\Pi(\varepsilon) \subset S^\varepsilon(f, A)$ and $\Pi(0) = S(f, A)$.

We start with the following definition of well-posedness of (P) in normed spaces X and Y.

DEFINITION 9.1.2. Problem (P) is *Hausdorff well-posed* if

(i) $E(f,A) \neq \emptyset$,
(ii) the ε-solution mapping Π is upper Hausdorff semicontinuous at $0 \in \operatorname{dom} \Pi$, i.e. for any $M > 0$ there exists $t > 0$ such that

$$\Pi(\varepsilon) \subset S(f,A) + MB_X \quad \text{for } \varepsilon \in K_0 \cap tB_Y.$$

DEFINITION 9.1.3. Let $\eta \in E(f,A)$. Problem (P) is *η-Hausdorff well-posed* if the η-ε-solution mapping Π^η is upper Hausdorff semicontinuous at $0 \in \operatorname{dom} \Pi^\eta$, i.e. for any $M > 0$ there exists $t > 0$ such that

$$\Pi(\varepsilon) \subset S_\eta + MB_X \quad \text{for } \varepsilon \in K_0 \cap tB_Y.$$

DEFINITION 9.1.4. Let $(x_n) \subset A$ be a sequence of feasible elements. It is a *minimizing sequence* for (P) if for each $n \geq 1$ there exist $y_n \in \mathcal{K}$, $\lim_n y_n = 0$, and $\eta_n \in E(f,A)$ such that $f(x_n) \preceq_{\mathcal{K}} \eta_n + y_n$.

The following proposition gives a characterization of Hausdorff well-posedness in terms of minimizing sequences.

PROPOSITION 9.1.1. *Let X and Y be normed spaces and let $\mathcal{K}$ be a closed convex pointed cone in Y. The following conditions are equivalent:*

(i) *(P) is Hausdorff well-posed,*
(ii) *$E(f,A) \neq \emptyset$, and for any minimizing sequence $(x_n) \subset A$ and every 0-neighbourhood W in X,*

$$x_n \in S(f,A) + W \quad \textit{for all } n \textit{ sufficiently large.}$$

Proof. Follows directly from the definitions. ■

The following proposition establishes the relationships between well-posedness, ϕ-sharp, and weak ϕ-sharp solutions.

PROPOSITION 9.1.2. *Let X and Y be normed spaces and let $\mathcal{K}$ be a closed convex pointed cone in Y with* $\operatorname{int} \mathcal{K} \neq \emptyset$. *Let $\eta \in E(f,A)$.*

(i) *If $S_\eta \cap Sh^\phi(f,A) \neq \emptyset$, then (P) is η-Hausdorff well-posed. Moreover, if $S_\eta = \{x_0\}$, then (P) is η-Hausdorff well-posed if and only if $x_0 \in Sh^\phi(f,A)$.*
(ii) *If $S(f,A) = Sh^\phi(f,A)$ (i.e. all solutions are ϕ-sharp with the same function ϕ), then (P) is Hausdorff well-posed.*
(iii) *(P) is Hausdorff well-posed if and only if the global ϕ-growth condition holds for (P), i.e. for any $\overline{x} \in S(f,A)$,*

$$f(x) - f(\overline{x}) \notin \phi(d(x, S(f,A)))B_Y - \mathcal{K} \quad \textit{for } x \in A \setminus S(f,A).$$

Proof. (i) Suppose that Π^η is not upper Hausdorff semicontinuous at $0 \in \operatorname{dom} \Pi^\eta$. There exists $M_0 > 0$ such that for all $n \geq 1$ one can find $\varepsilon_n \in K_0 \cap (1/n)B_Y$ and $z_n \in \Pi^\eta(\varepsilon_n)$ such that $z_n \in \Pi^\eta(\varepsilon_n)$ and $d(z_n, S_\eta) \geq M_0$. Thus, for any $\overline{x} \in S_\eta$,

$$f(z_n) - f(\overline{x}) \in \varepsilon_n - \mathcal{K} \subset \frac{1}{n} B_Y - \mathcal{K}.$$

This proves that no $\overline{x} \in S_\eta$ is ϕ-sharp since $\phi(\|z_n - \overline{x}\|) \geq \phi(M_0) \geq 1/n$.

(ii) Suppose that (P) is not Hausdorff well-posed. There exists $M_0 > 0$ such that for all $n \geq 1$ there are $\varepsilon_n \in K_0 \in (1/n)B_Y$ and $z_n \in \Pi(\varepsilon_n)$ such that $d(z_n, S(f,A)) \geq M_0$. Thus, there exists $x_n \in S(f,A)$ such that

$$f(z_n) - f(x_n) \in \varepsilon_n - \mathcal{K} \subset \frac{1}{n} B_Y - \mathcal{K}.$$

This proves that x_n is not ϕ-sharp since $\phi(\|z_n - x_n\|) \geq \phi(M_0) \geq 1/n$.

(iii) The proof is similar to (ii). ∎

With the definitions introduced below we can characterize global sharp and weak sharp solutions of order q to (P).

Definition 9.1.5. Problem (P) is *Hölder well-posed* of order $q > 0$ if

(i) $E(f,A) \neq \emptyset$,

(ii) the ε-solution mapping Π is upper Hölder of order $q > 0$ at $0 \in \operatorname{dom} \Pi$, i.e. there exist constants $L > 0$ and $t > 0$ such that

$$A \cap f^{-1}(E(f,A) + \varepsilon - \mathcal{K}) \subset S(f,A) + L\|\varepsilon\|^q B_X.$$

We say that (P) is *Lipschitz well-posed* if (P) is Hölder well-posed with $q = 1$.

Definition 9.1.6. Let $\eta \in E(f,A)$. Problem (P) is *η-Hölder well-posed* of order $q > 0$ if the η-ε-solution mapping Π^η is upper Hölder of order $q > 0$ at $0 \in \operatorname{dom} \Pi^\eta$, i.e. there exist constants $L > 0$ and $t > 0$ such that

$$A \cap f^{-1}(\eta + \varepsilon - \mathcal{K}) \subset S_\eta + L\|\varepsilon\|^q B_X.$$

We say that (P) is *η-Lipschitz well-posed* if (P) is η-Hölder well-posed with $q = 1$.

The following proposition establishes the relationships between sharp solutions and well-posedness introduced in Definitions 9.1.5 and 9.1.6. Recall that $S_\eta = A \cap f^{-1}(\eta)$.

Proposition 9.1.3. *Let X and Y be normed spaces and let $\mathcal{K}$ be a closed convex pointed cone in Y with $\operatorname{int} \mathcal{K} \neq \emptyset$. Let $\eta \in E(f,A)$.*

(i) *If $S_\eta \cap Sh^q(f,A) \neq \emptyset$, then (P) is η-Hölder well-posed of order $1/q$. Moreover, if $S_\eta = \{x_0\}$, then (P) is η-Hölder well-posed if and only if $x_0 \in Sh^q(f,A)$.*

(ii) *If all $\overline{x} \in S(f,A)$ are sharp of order q with constant $\tau > 0$, then (P) is Hölder well-posed of order $1/q$.*

Proof. By definition, Π^η is upper Hölder of order $1/q$ at $0 \in \operatorname{dom} \Pi^\eta$ if there are constants $L > 0$ and $t > 0$ such that

$$A \cap f^{-1}(\eta + \varepsilon - \mathcal{K}) \subset S_\eta + L\|\varepsilon\|^{1/q} B_X \quad \text{for } \varepsilon \in K_0 \cap tB_X.$$

(i) Suppose now that Π^η is not upper Hölder of order $1/q$ at $0 \in \operatorname{dom} \Pi^\eta$. For each $n \geq 1$ there exist $\varepsilon_n \in K_0 \cap (1/n)B_Y$ and $x_n \in A \cap f^{-1}(\eta + \varepsilon_n - \mathcal{K})$ such that $d(x_n, S_\eta) > n\|\varepsilon_n\|^{1/q}$. Hence, $\|x_n - x_0\|^q > n^q \|\varepsilon_n\|$ for any $x_0 \in S_\eta$ and

$$f(x_n) - f(x_0) \in \frac{1}{n^q} \|x_n - x_0\|^q \frac{\varepsilon_n}{\frac{1}{n^q}\|x_n - x_0\|^q} - \mathcal{K} \subset \frac{1}{n^q} \|x_n - x_0\|^q B_Y - \mathcal{K},$$

which proves that $S_\eta \cap Sh^q(f,A) = \emptyset$.

To see the second part of (i) suppose on the contrary that x_0 is not sharp of order q. For each $n \geq 1$ there exists $x_n \in A \setminus S_\eta$ such that

$$f(x_n) - f(x_0) \in \frac{1}{n} \|x_n - x_0\|^q B_Y - \mathcal{K}.$$

By taking any $\varepsilon \in \operatorname{int} \mathcal{K}$, $\|\varepsilon\| = 1$, and $\lambda > 0$ such that $B_Y \subset \lambda\varepsilon - \mathcal{K}$ we get

$$f(x_n) - f(x_0) \in \frac{\lambda}{n} \|x_n - x_0\|^q \varepsilon - \mathcal{K},$$

which means that $x_n \in \Pi^\eta(\frac{\lambda}{n}\|x_n - x_0\|^q\varepsilon)$. On the other hand,

$$\|x_n - x_0\| = d(x_n, S_\eta) \not\leq (\lambda/n)^{1/q}\|x_n - x_0\|.$$

(ii) Suppose on the contrary that Π is not upper Hölder of order $1/q$ at $0 \in \operatorname{dom} \Pi$. For each $n \geq 1$ there exist $\varepsilon_n \in K_0 \cap (1/n)B_Y$ and $z_n \in A \cap f^{-1}(E(f, A) + \varepsilon_n - \mathcal{K})$ such that $d(z_n, S) > n\|\varepsilon_n\|^{1/q}$. Thus, there exists $x_n \in S(f, A)$ such that

$$f(z_n) - f(x_n) \in \varepsilon_n - \mathcal{K}.$$

On the other hand, $\|z_n - x_n\| \geq d(z_n, S(f, A))$ and

$$\frac{1}{n^q}\|z_n - x_n\|^q > \|\varepsilon_n\|.$$

Hence, $b_n := \frac{\varepsilon_n}{\frac{1}{n^q}\|z_n - x_n\|^q} \in B_Y$ and

$$f(z_n) - f(x_n) \in \frac{1}{n^q}\|z_n - x_n\|^q B_Y - \mathcal{K}, \qquad f(z_n) \neq f(x_n),$$

which contradicts the assumption that all $\overline{x} \in S(f, A)$ are sharp of order q with the same constant. ∎

Analogously, the following proposition establishes the relationships between well-posedness of (P) and weakly sharp solutions to (P).

Proposition 9.1.4. *Let X and Y be normed spaces and let $\mathcal{K}$ be a closed convex pointed cone in Y with $\operatorname{int} \mathcal{K} \neq \emptyset$. Let $\eta \in E(f, A)$.*

(i) *$S_\eta \cap Wh^q(f, A) \neq \emptyset$ if and only if (P) is η-Hölder well-posed of order $1/q$.*
(ii) *(P) is Hölder well-posed of order $1/q$ if and only if the global growth condition holds for (P) on $S(f, A)$, i.e. there exists a constant $\tau > 0$ such that for all $\overline{x} \in S(f, A)$,*

$$f(x) - f(\overline{x}) \notin \tau(d(x, S(f, A)))^q B_Y - \mathcal{K} \quad \text{for } x \in A \setminus S(f, A).$$

Proof. (i) The proof of this part is analogous to the proof of Proposition 9.1.3.

(ii) Suppose that Π is not upper Hölder of order $1/q$ at $0 \in \operatorname{dom} \Pi$. For each $n \geq 1$ there exist $\varepsilon_n \in K_0 \cap (1/n)B_Y$ and $z_n \in A \cap f^{-1}(E(f, A) + \varepsilon_n - \mathcal{K})$ such that $d(z_n, S(f, A)) > n\|\varepsilon_n\|^{1/q}$. Hence, $z_n \notin S(f, A)$ and there exists $x_n \in S(f, A)$ such that $f(z_n) - f(x_n) \in \varepsilon_n - \mathcal{K}$ and

$$f(z_n) - f(x_n) \in \frac{1}{n^q}\, d(z_n, S(f, A))^q B_Y - \mathcal{K},$$

which contradicts the assumption.

To see the converse, suppose on the contrary that for each $n \geq 1$ one can find $x_n \in S(f, A)$ such that there exists $z_n \in A \setminus S(f, A)$ such that

$$f(z_n) - f(x_n) \in \frac{1}{n} d(z_n, S(f, A))^q B_Y - \mathcal{K}.$$

Since there exist $\varepsilon_0 \in \operatorname{int} \mathcal{K}$, $\|\varepsilon_0\| = 1$, and $\lambda > 0$ such that $B_Y \subset \lambda\varepsilon_0 - \mathcal{K}$, we get

$$f(z_n) - f(x_n) \in \frac{\lambda}{n} d(z_n, S(f, A))^q \varepsilon_0 - \mathcal{K}.$$

Hence, $z_n \in \Pi(\frac{\lambda}{n} d(z_n, S(f, A))^q \varepsilon_0)$. But $d(z_n, S(f, A)) \not\leq (\lambda/n)^{1/q} d(z_n, S(f, A))$ and (P) is not Hölder well-posed of order $1/q$. ■

Now we consider local well-posedness of (P).

Definition 9.1.7. Problem (P) is *Hölder calm well-posed* of order $q > 0$ at $x_0 \in S(f, A)$ if the ε-solution mapping Π is Hölder calm of order $q > 0$ at $(0, x_0) \in \operatorname{graph} \Pi$, i.e. there exist $r > 0$, $L > 0$ and $t > 0$ such that

$$\Pi(\varepsilon) \cap (x_0 + rB_X) \subset \Pi(0) + L\|\varepsilon\|^q B_X$$

for $\varepsilon \in K_0 \cap tB_Y$. We say that (P) is *calm well-posed* at $x_0 \in S(f, A)$ if (P) is Hölder calm well-posed at x_0 with $q = 1$.

Definition 9.1.8. Problem (P) is *η-Hölder calm well-posed* of order $q > 0$ at $x_0 \in S_\eta$ if the η-ε-solution mapping Π^η is Hölder calm of order $q > 0$ at $(0, x_0) \in \operatorname{graph} \Pi^\eta$, i.e. there exist $r > 0$, $L > 0$ and $t > 0$ such that

$$\Pi^\eta(\varepsilon) \cap (x_0 + rB_X) \subset \Pi^\eta(0) + L\|\varepsilon\|^q B_X$$

for $\varepsilon \in K_0 \cap tB_Y$. We say that (P) is *η-calm well-posed* at $x_0 \in S_\eta$ if (P) is η-Hölder calm well-posed of order $q = 1$ at x_0.

Now we address the question of relationships between local well-posedness, local sharp and local weak sharp solutions. Recall that $x_0 \in A$ is a *local sharp solution of order* $q > 0$ *to* (P), $x_0 \in LSh^q(f, A)$, if one can find a 0-neighbourhood V in X and constant $\tau > 0$ such that

$$(f(x) - f(x_0)) \cap (\tau\|x - x_0\|^q B_Y - \mathcal{K}) = \emptyset \quad \text{for all } x \in A \cap (x_0 + V),\ f(x) \neq f(x_0).$$

Equivalently, $x_0 \in LSh^q(f, A)$ iff there is a 0-neighbourhood V in X such that

$$\tau\|x - x_0\|^q \leq \|f(x) - f(x_0)\|_- \quad \text{for all } x \in A \cap (x_0 + V),\ f(x) \neq f(x_0).$$

Or, $x_0 \in LSh^q(f, A)$ iff x_0 is a local sharp minimum of order q of the function $\|f(\cdot) - f(x_0)\|_-$ over A (cf. [147]).

Moreover, $x_0 \in LWh^q(f, A)$, $f(x_0) = \eta$, if there exist a 0-neighbourhood V in X and $\tau > 0$ such that

$$f(x) - f(x_0) \notin \tau(d(x, S_\eta))^q B_Y - \mathcal{K} \quad \text{for } x \in A \cap (x_0 + V),\ x \notin S_\eta.$$

Proposition 9.1.5. *Let $\mathcal{K}$ be a closed convex pointed cone in a normed space $(Y, \|\cdot\|)$ with $\operatorname{int} \mathcal{K} \neq \emptyset$. Let $\eta \in E(f, A)$.*

(i) *(P) is η-Hölder calm of order $1/q$ at $(0, x_0) \in \operatorname{graph} \Pi$ (Definition 9.1.8) if and only if $x_0 \in LWh^q(f, A)$.*

(ii) (P) *is Hölder calm of order* $1/q$ *at* $(0, x_0) \in \operatorname{graph} \Pi$ *(Definition 9.1.7) if and only if there exists a* 0*-neighbourhood* V *such that the local growth condition of order* q *holds for* (P) *on* $\overline{S} = S(f, A) \cap (x_0 + V)$ *(cf. Definition 8.2.4).*

Proof. (i) By definition, Π^η is Hölder calm of order $1/q$ at $(0, x_0) \in \operatorname{graph} \Pi^\eta$ if there are a 0-neighbourhood V in X and constants $L > 0$ and $t > 0$ such that

$$A \cap f^{-1}(\eta + \varepsilon - \mathcal{K}) \cap (x_0 + V) \subset S_\eta + L\|\varepsilon\|^{1/q} B_X \quad \text{for } \varepsilon \in K_0 \cap tB_Y.$$

Suppose on the contrary that $x_0 \notin LWh^q(f, A)$, i.e., for each $n \geq 1$ there are $z_n \in A \cap (x_0 + \frac{1}{n} B_X)$, $f(z_n) \neq f(x_0)$, such that

$$f(z_n) - f(x_0) \in \frac{1}{n} (d(z_n, S_\eta))^q B_Y - \mathcal{K}.$$

Since there exist $\varepsilon_0 \in \operatorname{int} \mathcal{K}$, $\|\varepsilon_0\| = 1$, and $\lambda > 0$ such that $B_Y \subset \lambda\varepsilon_0 - \mathcal{K}$ we get

$$f(z_n) \in f(x_0) + \frac{\lambda}{n} (d(z_n, S_\eta))^q \varepsilon_0 - \mathcal{K}.$$

Hence, $z_n \in \Pi^\eta(\frac{\lambda}{n}(d(z_n, S_\eta))^q \varepsilon_0)$, but $d(z_n, S_\eta) \not\leq L(\lambda/n)^{1/q} d(z_n, S_\eta)$, which means that Π^η is not Hölder calm of order $1/q$ at $(0, x_0) \in \operatorname{graph} \Pi^\eta$.

(ii) By definition, Π is Hölder calm of order $1/q$ at $(0, x_0) \in \operatorname{graph} \Pi$ if there are a 0-neighbourhood V in X and constants $L > 0$ and $t > 0$ such that

$$A \cap f^{-1}(E(f, A) + \varepsilon - \mathcal{K}) \cap (x_0 + V) \subset S(f, A) + L\|\varepsilon\|^{1/q} B_X \quad \text{for } \varepsilon \in K_0 \cap tB_Y.$$

Now, suppose on the contrary that the local growth condition of order q does not hold for (P) around $x_0 \in S(f, A)$, i.e. for each $n \geq 1$ one can find $x_n \in S(f, A) \cap (x_0 + \frac{1}{n} B_X)$ and $z_n \in A \cap (x_n + \frac{1}{n} B_X)$, $f(z_n) \neq f(x_n)$, such that

$$f(z_n) - f(x_n) \in \frac{1}{n} (d(z_n, S(f, A)))^q B_Y - \mathcal{K}.$$

By taking $\varepsilon_0 \in \operatorname{int} \mathcal{K}$, $\|\varepsilon_0\| = 1$, and $\lambda > 0$ such that $B_Y \subset \lambda\varepsilon_0 - \mathcal{K}$ we get

$$f(z_n) = f(x_n) + \frac{\lambda}{n} (d(z_n, S(f, A)))^q \varepsilon - \mathcal{K}.$$

Hence, $z_n \in \Pi(\frac{\lambda}{n}(d(z_n, S(f, A)))^q \varepsilon_0) \cap (x_0 + \frac{2}{n} B_Y)$ but

$$d(z_n, S(f, A)) \not\leq L\left(\frac{\lambda}{n}\right)^{1/q} d(z_n, S(f, A)),$$

which means that Π is not Hölder calm of order $1/q$ at $(0, x_0) \in \operatorname{graph} \Pi$.

For the converse suppose that (P) is not Hölder calm of order $1/q$. For each $n \geq 1$ there exist $\varepsilon_n \in K_0 \cap \frac{1}{n} B_Y$ and $z_n \in \Pi(\varepsilon_n) \cap (x_0 + \frac{1}{n} B_X)$ such that

$$d(z_n, S(f, A)) \geq n\|\varepsilon_n\|^{1/q}.$$

Hence, there exists $x_n \in S(f, A)$ such that $f(z_n) \in f(x_n) + \varepsilon_n - \mathcal{K}$ and thus

$$f(z_n) - f(x_n) \in \frac{1}{n^q} (d(z_n, S(f, A))^q B_Y - \mathcal{K},$$

which proves that the local growth condition does not hold for (P) around x_0. ∎

Analogously we can prove the local counterpart of Proposition 9.1.3.

PROPOSITION 9.1.6. *Let $\mathcal{K}$ be a closed convex pointed cone in a normed space $(Y, \|\cdot\|)$ with* $\operatorname{int}\mathcal{K} \neq \emptyset$. *Let $\eta \in E(f, A)$.*

(i) *If $x_0 \in S_\eta \cap LSh^q(f, A)$, then (P) is η-Hölder calm well-posed at x_0 of order $1/q$. Moreover, if $S_\eta = \{x_0\}$, then (P) is η-Hölder well-posed of order $1/q$ at x_0 if and only if $x_0 \in LSh^q(f, A)$.*

(ii) *If there exists a 0-neighbourhood V such that all $\overline{x} \in S(f, A) \cap (x_0 + V)$ are local sharp of order q with the same constant, then (P) is Hölder calm well-posed at x_0 of order $1/q$.*

Proof. (i) The proof is similar to the proof of Proposition 9.1.3(i).

(ii) Since each local sharp solution is a local weak sharp solution, the conclusion follows from Proposition 9.1.5(ii). ■

9.1.1. Conditions for well-posedness in the outcome space. In this section we investigate relationships between well-posedness of (P), strictly efficient points and local strictly efficient points to (P).

As previously, $K^0 = \operatorname{int}\mathcal{K} \cup \{0\}$ and $\varepsilon \in K^0$. Recall that $y_0 \in C$ is *ε-efficient* [99], $y_0 \in \varepsilon\text{-}E(C)$, if

$$(y_0 - \varepsilon - \mathcal{K}) \cap C = \emptyset.$$

Let C be a subset of a Hausdorff topological vector space Y. According to Definition 2.2.1, an element $y_0 \in C$ is a *strictly efficient point*, $y_0 \in StE(C)$, if for every 0-neighbourhood W in Y there exists a 0-neighbourhood O in Y such that

$$C \cap (y_0 + O - \mathcal{K}) \subset y_0 + W.$$

Let $\eta \in E(C)$. Let $\widetilde{\Pi}^\eta : K_0 \rightrightarrows Y$ be defined as

$$\widetilde{\Pi}^\eta(\varepsilon) := \{y \in C : \eta + \varepsilon - y \in \mathcal{K}\}. \tag{9.1}$$

Thus, $\widetilde{\Pi}^\eta$ is the η-ε-solution mapping Π^η for $f = \mathrm{id}$ and $A = C$ and

$$\widetilde{\Pi}^\eta(\varepsilon) = C \cap (\eta + \varepsilon - \mathcal{K}).$$

Let $\widetilde{\Pi} : \mathcal{K} \rightrightarrows Y$ be defined as

$$\widetilde{\Pi}(\varepsilon) := \{y \in C : E(C) + \varepsilon - y \in \mathcal{K}\}. \tag{9.2}$$

In other words,

$$\widetilde{\Pi}(\varepsilon) = C \cap (E(C) + \varepsilon - \mathcal{K})$$

and $\widetilde{\Pi}$ is the ε-solution mapping Π for $f = \mathrm{id}$ and $A = C$.

The following proposition establishes the relationship between upper Hausdorff semicontinuity of $\widetilde{\Pi}$ or $\widetilde{\Pi}^\eta$ and strictly efficient points.

PROPOSITION 9.1.7. *Let X and Y be Hausdorff topological vector spaces and let $\mathcal{K}$ be a closed convex pointed cone in Y with* $\operatorname{int}\mathcal{K} \neq \emptyset$. *Let C be a subset of Y and let $\eta \in E(C)$.*

(i) *$\widetilde{\Pi}^\eta$ is upper Hausdorff semicontinuous at $\varepsilon = 0$ if and only if $\eta \in StE(C)$.*

(ii) *If all $\eta \in E(C)$ are uniformly strictly efficient in the sense that for any 0-neighbourhood W there exists a 0-neighbourhood O such that for any $\eta \in E(C)$*

$$C \cap (\eta + O - \mathcal{K}) \subset \eta + W,$$

then $\widetilde{\Pi}$ is upper Hausdorff semicontinuous at $\varepsilon = 0$.

Proof. (i) Let $\eta \in StE(C)$ and let W be a 0-neighbourhood in Y. There exists a 0-neighbourhood O in Y such that

$$C \cap (\eta + O - \mathcal{K}) \subset \eta + W.$$

Hence, $C \cap (\eta + \varepsilon - \mathcal{K}) \subset \eta + W$ for any $\varepsilon \in O \cap K_0$, which proves that Π^η is upper Hausdorff semicontinuous at $\varepsilon = 0$. In particular, for $\varepsilon = 0$ we have $C \cap (\eta - \mathcal{K}) = \{\eta\}$.

Suppose now that $\widetilde{\Pi}^\eta$ is upper Hausdorff semicontinuous at $\varepsilon = 0$ and take any 0-neighbourhood W in Y. There exists a 0-neighbourhood O such that

$$\widetilde{\Pi}^\eta(\varepsilon) = C \cap (\eta + \varepsilon - \mathcal{K}) \subset \eta + W \quad \text{for } \varepsilon \in O \cap K_0.$$

Take any $0 \neq \varepsilon \in O \cap K_0$. There exists a 0-neighbourhood $\bar{O}$ in Y such that $\bar{O} \subset \varepsilon - \mathcal{K}$ and hence $C \cap (\eta + \bar{O} - \mathcal{K}) \subset \eta + W$, which completes the proof of the first assertion.

(ii) Let W be a 0-neighbourhood in Y. By the uniform strict efficiency of all $\eta \in E(C)$, there exists a 0-neighbourhood O in Y such that

$$C \cap (\eta + O - \mathcal{K}) \subset \eta + W \quad \text{for any } \eta \in E(C).$$

Hence, for any $\varepsilon \in O \cap K_0$,

$$C \cap (\eta + \varepsilon - \mathcal{K}) \subset \eta + W \quad \text{for any } \eta \in E(C)$$

and consequently for any $\varepsilon \in O \cap K_0$,

$$C \cap (E(C) + \varepsilon - \mathcal{K}) = \bigcup_{\eta \in E(C)} C \cap (\eta + \varepsilon - \mathcal{K}) \subset E(C) + W,$$

which proves that $\widetilde{\Pi}$ is upper Hausdorff semicontinuous at $\varepsilon = 0$. In particular, for $\varepsilon = 0$ we have $C \cap (E(C) - \mathcal{K}) = E(C)$. ■

Proposition 9.1.8. *Let X and Y be normed spaces and let $\mathcal{K}$ be a closed convex pointed cone in Y with $\operatorname{int} \mathcal{K} \neq \emptyset$. Let $C \subset Y$ and $\eta \in E(C)$.*

(i) *$\widetilde{\Pi}^\eta$ is upper Hölder of order $1/q$, $q > 0$, at $\varepsilon = 0$ if and only if $\eta \in StE^q(C)$.*
(ii) *If all $\eta \in E(C)$ are strictly efficient of order $q > 0$ with the same constant β, then $\widetilde{\Pi}$ is upper Hölder of order $1/q$ at $\varepsilon = 0$.*

Proof. (i) Suppose that $\eta \notin StE^q(f, A)$. For each $n \geq 1$ there are $y_n \in C$, $b_n \in B_Y$, $k_n \in \mathcal{K}$ such that

$$y_n - \eta = \frac{1}{n} \|y_n - \eta\|^q b_n - k_n.$$

Since $\operatorname{int} \mathcal{K} \neq \emptyset$, there is $\varepsilon_0 \in \operatorname{int} \mathcal{K}$ such that $B_Y \subset \varepsilon_0 - \mathcal{K}$. Hence,

$$y_n - \eta = \frac{1}{n} \|y_n - \eta\|^q \varepsilon_0 - \ell_n, \quad \text{where } \ell_n \in \mathcal{K}.$$

This means that $y_n \in \widetilde{\Pi}^\eta(\frac{1}{n}\|y_n - \eta\|^q \varepsilon_0)$. On the other hand, $\|y_n - \eta\| \not\leq \frac{1}{n^{1/q}} \|y_n - \eta\|$, which proves that $\widetilde{\Pi}^\eta$ is not upper Hölder of order $1/q$.

(ii) The proof is similar. ■

Proposition 9.1.9. *Let $\mathcal{K}$ be a closed convex pointed cone in a normed space $(Y, \|\cdot\|)$ and $\operatorname{int} \mathcal{K} \neq \emptyset$. Let $\eta \in E(C)$. If $\widetilde{\Pi}^\eta$ is Hölder calm of order $1/q$ at $(0, \eta) \in \operatorname{graph} \widetilde{\Pi}^\eta$, then $\eta \in LStE^q(C)$.*

Proof. By definition, $\widetilde{\Pi}^\eta$ is Hölder calm of order $1/q$ at $(0,\eta) \in \operatorname{graph} \widetilde{\Pi}^\eta$ if there are a neighbourhood V of zero in Y and constants $t > 0$, $L > 0$ such that

$$C \cap (\eta + \varepsilon - \mathcal{K}) \cap (\eta + V) \subset \eta + L\|\varepsilon\|^{1/q} B_Y \quad \text{ for } \varepsilon \in K_0 \cap tB_Y.$$

Suppose that $\eta \notin LStE^q(C)$. For each $n \geq 1$ one can find $y_n \in C \cap (\eta + \frac{1}{n} B_Y)$ such that $\frac{1}{n}\|y_n - \eta\|^q > \|y_n - \eta\|_-$. This means that

$$y_n - \eta \in \frac{1}{n}\,\|y_n - \eta\|^q B_Y - \mathcal{K},$$

i.e., $y_n - \eta = \frac{1}{n}\|y_n - \eta\|^q b_n - k_n$ with $b_n \in B_Y$, $k_n \in \mathcal{K}$. Take any $\varepsilon \in \operatorname{int}\mathcal{K}$, $\|\varepsilon\| = 1$. Since $b_n \in \lambda\varepsilon - \mathcal{K}$, for all $n \geq 1$ and a certain $\lambda > 0$, we get

$$y_n = \eta + \frac{\lambda}{n}\,\|y_n - \eta\|^q \varepsilon - \ell_n, \qquad \ell_n \in \mathcal{K}.$$

Hence, $y_n \in \widetilde{\Pi}^\eta(\frac{\lambda}{n}\|y_n - \eta\|^q \varepsilon)$, and $y_n - \eta \notin \frac{\lambda L}{n}\|y_n - \eta\| B_Y$, which means that $\widetilde{\Pi}^\eta$ is not Hölder calm of order $1/q$ at $(0,\eta) \in \operatorname{graph} \widetilde{\Pi}^\eta$. ∎

PROPOSITION 9.1.10. *Let C be a subset of a Hausdorff topological space Y. If (DP) holds for C, then $\widetilde{\Pi}$ is $\mathcal{K}$-upper Hausdorff semicontinuous at $\varepsilon = 0$.*

Proof. It is enough to observe that $\widetilde{\Pi}(\varepsilon) \subset \widetilde{\Pi}(0) + \mathcal{K}$. ∎

9.2. Hausdorff continuity of solutions

In the following sections we provide sufficient conditions for Hausdorff, Lipschitz and Hölder continuities of the solution mapping $\mathcal{S}$. To formulate these conditions we appeal to the notions of sharpness and weak sharpness of solutions to (P) and/or (P_u). In view of the results of the previous sections analogous conditions can be formulated with the help of well-posedness.

In this section we investigate upper and lower Hausdorff continuities of $\mathcal{S}$ at u_0. The main assumptions are the containment property and the well-posedness in the sense defined in previous sections.

THEOREM 9.2.1. *Let X and U be topological spaces and let Y be a Hausdorff topological vector space. Let $\mathcal{K}$ be a closed convex pointed cone in Y with $\operatorname{int}\mathcal{K} \neq \emptyset$. If*

(i) *$f : X \to Y$ is uniformly continuous on X,*
(ii) *$\mathcal{A} : U \rightrightarrows X$ is Hausdorff continuous at $u_0 \in \operatorname{dom}\mathcal{A}$,*
(iii) *(P) is Hausdorff well-posed,*
(iv) *(CP) holds for $f(A)$,*

then $\mathcal{S}$ is upper Hausdorff semicontinuous at $u_0 \in \operatorname{dom}\mathcal{S}$.

Proof. Let V be 0-neighbourhood in X. Let V_1 be a 0-neighbourhood in Y such that $V_1 + V_1 \subset V$. By the well-posedness of (P), there exists a 0-neighbourhood W such that

$$\Pi(\varepsilon) \subset \Pi(0) + V_1 \quad \text{ for } \varepsilon \in W \cap K_0.$$

Since $\Pi(0) = S(f, A)$, the above inclusion can be rephrased as

$$A \cap f^{-1}(E(f,A) + W \cap K_0 - \mathcal{K}) \subset S(f,A) + V_1. \tag{9.3}$$

Let W_1 be a 0-neighbourhood in Y such that $W_1 + W_1 \subset W$ and let W_2 be a 0-neighbourhood in Y such that $W_2 \subset W \cap K_0 - \mathcal{K}$. By (CP), Proposition 5.1.3, there exists a 0-neighbourhood O in Y such that for any $x \in A$ with $f(x) \notin E(f,A) + W_2$ there exists $\overline{x} \in S(f,A)$ such that

$$f(x) - f(\overline{x}) + O \subset \mathcal{K}.$$

Let O_1 be a 0-neighbourhood in Y such that $O_1 + O_1 \subset O$. By the uniform continuity of f on X, there exists a 0-neighbourhood O_2 in X such that

$$f(x + O_2) \subset f(x) + O_1 \quad \text{for all } x \in X.$$

Moreover, by the Hausdorff continuity of $\mathcal{A}$, there exists a neighbourhood U_0 of u_0 such that

$$A \subset A(u) + V_1 \cap O_2, \qquad A(u) \subset A + V_1 \cap O_2.$$

Take any $\overline{z} \in S(f, A(u))$ for $u \in U_0$. There exists $x \in A$ such that $x \in \overline{z} + V_1 \cap O_2$. Consequently, $f(x) \in f(\overline{z}) + O_1$.

If $f(x) \notin E(f,A) + W_2 - \mathcal{K}$, then $f(x) \notin E(f,A) + W_2$ and by (CP), there exists $\overline{x} \in S(f,A)$ such that

$$f(x) - f(\overline{x}) + O \subset \mathcal{K}.$$

By the Hausdorff continuity of $\mathcal{A}$, there exists $z \in A(u)$ such that $z \in \overline{x} + V_1 \cap O_2$. Hence, $f(z) \in f(\overline{x}) + O_1$ and so $f(z) = f(\overline{z})$ since otherwise

$$f(z) - f(\overline{z}) \in (f(z) - f(\overline{x})) + (f(\overline{x}) - f(x)) + (f(x) - f(\overline{z})) \subset f(\overline{x}) - f(x) + O \subset -\mathcal{K},$$

which is impossible because $\overline{z} \in S(f, A(u))$. If $f(x) \in E(f,A) + W_2 - \mathcal{K}$, by (9.3), $x \in S(f,A) + V_1$ and

$$\overline{z} \in x + V_1 \cap O_2 \subset S(f,A) + V_1 + V_1 \cap O_2 \subset S(f,A) + V,$$

which completes the proof. ∎

The following examples show that well-posedness does not imply the containment property of the set $f(A)$.

EXAMPLE 9.2.1. Let us consider problem (P) (see Figure 9.2) with $\mathcal{K} = \mathbb{R}^2_+$, and $f : \mathbb{R} \to \mathbb{R}^2$,

$$f(x) = \begin{cases} (x, e^{1-x}) & \text{if } x \geq 1, \\ (x, x^2) & \text{if } 0 \leq x \leq 1, \end{cases}$$

under the constraint $x \geq 0$.

In Example 9.2.1 problem (P) is Hausdorff well-posed but the set $f(A)$ does not have the containment property (CP). In a simple modification presented below the set $f(A)$ has the containment property.

EXAMPLE 9.2.2. Let us consider the vector optimization problem (see Figure 9.2) with $\mathcal{K} = \mathbb{R}^2_+$ and $f : \mathbb{R} \to \mathbb{R}^2$ of the form

$$f(x) = \begin{cases} (x, \frac{1}{2} + \frac{1}{2}e^{1-x}) & \text{if } x \geq 1, \\ (x, x^2) & \text{if } 0 \leq x \leq 1, \end{cases}$$

under the constraints ≥ 0.

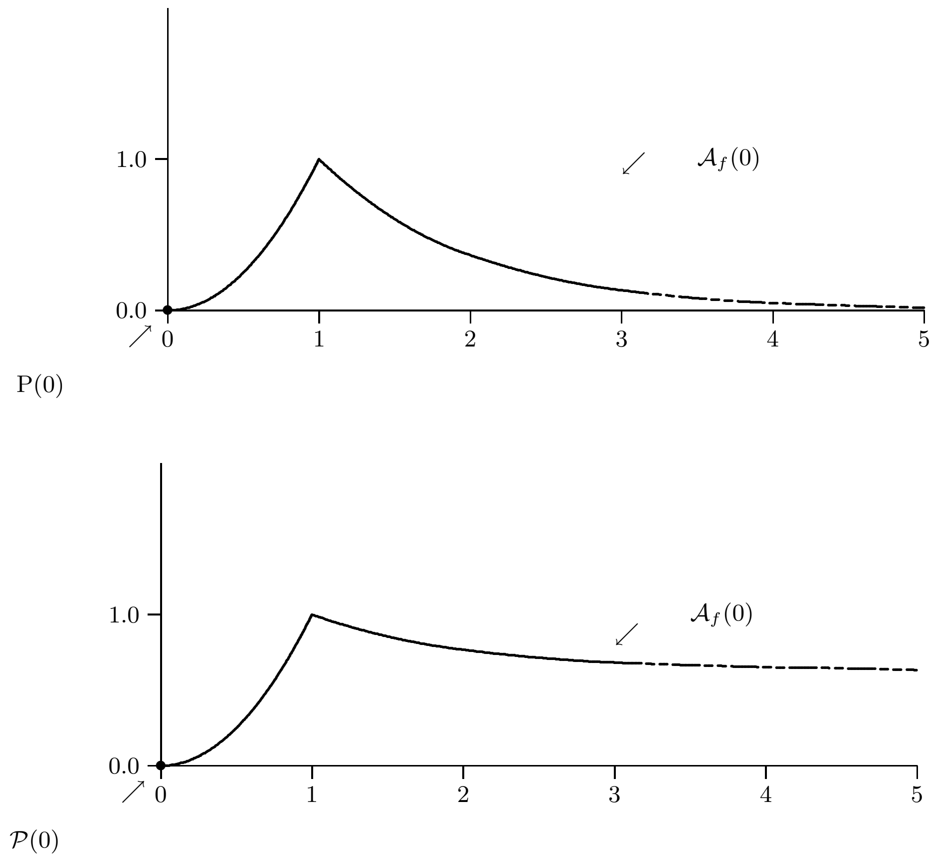

Fig. 9.2

THEOREM 9.2.2. *Let X and U be topological spaces and let Y be a Hausdorff topological vector space. Let $\mathcal{K}$ be a closed convex pointed cone in Y with* $\operatorname{int}\mathcal{K} \neq \emptyset$. *If*

(i) *$f : X \to Y$ is uniformly continuous on X, and $\mathcal{A} : U \rightrightarrows X$ is Hausdorff continuous at $u_0 \in \operatorname{dom}\mathcal{A}$,*

(ii) *there exists a neighbourhood U_0 of u_0 such that all (P_u) for $u \in U_0$ are uniformly Hausdorff well-posed in the sense that for any 0-neighbourhood V in X there exists a 0-neighbourhood W in Y such that*

$$A(u) \cap f^{-1}(E(f, A(u)) + W \cap K_0 - \mathcal{K}) \subset S(f, A(u)) + V \quad \text{for all } u \in U_0,$$

(iii) *(CP) holds uniformly for $f(A(u))$, $u \in U_0$ in the sense that for any 0-neighbourhood W in Y there exists a 0-neighbourhood O in Y such that for any $u \in U_0$ and $z \in A(u)$ $f(z) \notin E(f, A(u)) + W$ there exists $\overline{z} \in S(f, A(u))$ such that*

$$f(z) - f(\overline{z}) + O \subset \mathcal{K},$$

then $\mathcal{S}$ is lower Hausdorff semicontinuous at $u_0 \in \operatorname{dom}\mathcal{A}$.

Proof. Let V be a 0-neighbourhood in X. Let V_1 be a 0-neighbourhood in Y such that $V_1 + V_1 \subset V$. By the (uniform) well-posedness of (P_u), there exists a 0-neighbourhood W such that

$$A(u) \cap f^{-1}(E(f, A(u)) + W \cap K_0 - \mathcal{K}) \subset S(f, A(u)) + V_1 \tag{9.4}$$

for $u \in U_0$.

Let W_1 be a 0-neighbourhood in Y such that $W_1 + W_1 \subset W$ and let W_2 be a 0-neighbourhood in Y such that $W_2 \subset W \cap K_0 - \mathcal{K}$. By (CP) and Proposition 5.1.3, there exists a 0-neighbourhood O in Y such that for any $z \in A(u)$ with $f(z) \notin E(f, A(u)) + W_2$ there exists $\overline{z} \in S(f, A(u))$ such that

$$f(z) - f(\overline{z}) + O \subset \mathcal{K}.$$

Let O_1 be a 0-neighbourhood in Y such that $O_1 + O_1 \subset O$. By the uniform continuity of f on X, there exists a 0-neighbourhood O_2 in X such that

$$f(x + O_2) \subset f(x) + O_1 \quad \text{for all } x \in X.$$

Moreover, by the Hausdorff continuity of $\mathcal{A}$, there exists a neighbourhood U_1 of u_0 such that

$$A \subset A(u) + V_1 \cap O_2, \qquad A(u) \subset A + V_1 \cap O_2$$

for $u \in U_0 \cap U_1$. Take any $\overline{x} \in S(f, A)$ and $u \in U_0 \cap U_1$. There exists $z \in A(u)$ such that $z \in \overline{x} + V_1 \cap O_2$. Consequently, $f(z) \in f(\overline{x}) + O_1$.

If $f(z) \notin E(f, A(u)) + W_2 - \mathcal{K}$, then $f(z) \notin E(f, A(u)) + W_2$. By (CP), there exists $\overline{z} \in S(f, A(u))$ such that

$$f(z) - f(\overline{z}) + O \subset \mathcal{K}$$

and by the Hausdorff continuity of $\mathcal{A}$, there exists $x \in A$ such that $x \in \overline{z} + V_1 \cap O_2$. Consequently, $f(x) \in f(\overline{z}) + O_1$ and

$$f(x) - f(\overline{x}) \in (f(x) - f(\overline{z})) + (f(\overline{z}) - f(z)) + (f(z) - f(\overline{x}) \subset f(\overline{z}) - f(z) + O \subset -\mathcal{K},$$

which contradicts the fact that $\overline{x} \in S(f, A)$.

Hence, $f(z) \in E(f, A(u)) + W_2 - \mathcal{K}$. Then by (9.4), $z \in S(f, A(u)) + V_1$. This implies that

$$\overline{x} \in z + V_1 \cap O_2 \subset S(f, A(u)) + V_1 + V_1 \cap O_2 \subset S(f, A(u)) + V,$$

which completes the proof. ∎

9.3. Lower Lipschitzness of solutions

In this section we derive sufficient conditions for lower Lipschitz continuity of $\mathcal{S}(u) = S(f, A(u))$ at $(u_0, x_0) \in \operatorname{graph} \mathcal{S}$ and at $u_0 \in \operatorname{dom} \mathcal{S}$. By assuming that x_0 is sharp of order 1 we prove lower Lipschitzness of $\mathcal{S}$ at $(u_0, x_0) \in \operatorname{graph} \mathcal{S}$. Correspondingly, to obtain lower Lipschitzness of $\mathcal{S}$ $u_0 \in \operatorname{dom} \mathcal{S}$ we assume that all $x_0 \in S(f, A)$ are sharp of order 1 with the same constant τ.

Recall that for any $\eta \in E(f, A)$,

$$S_\eta := \{x \in S(f, A) : f(x) = \eta\}.$$

Correspondingly, for any $u \in U$ and $\eta \in E(f, A(u))$,

$$S_\eta(u) = \{z \in S(f, A(u)) : f(x) = \eta\}.$$

THEOREM 9.3.1. *Let $f : X \to Y$ be Lipschitz with constant $L_f > 0$. Assume that*

(i) *$\mathcal{A} : U \rightrightarrows X$ is Lipschitz at $u_0 \in \operatorname{dom} \mathcal{A}$ with constants $L_a > 0$, $t > 0$,*
(ii) *(DP) holds for all (P_u), $u \in B(u_0, t)$,*
(iii) *all $x_0 \in S(f, A)$ are global sharp solutions to (P) of order 1 with the same constant $\tau > 0$, i.e. for any $\eta \in E(f, A)$ and $x_0 \in S(f, A)$,*

$$f(x) - f(x_0) \notin \tau \|x - x_0\| B_Y - \mathcal{K} \quad \text{for } x \in A \setminus S_\eta.$$

Then $\mathcal{P}$ is lower Lipschitz at $u_0 \in \operatorname{dom} \mathcal{P}$, i.e.,

$$E(f, A) \in E(f, A(u)) + (L_f L_a + 2L_f^2 L_a/\tau)\|u - u_0\| B_Y \quad \text{for } u \in B(u_0, t).$$

Moreover, if instead of (iii) *we assume that*

(iv) *all $\bar{z} \in S(f, A(u))$ for $u \in B(u_0, t)$ are global sharp solutions to (P_u) of order 1 with the same constant $\tau > 0$, i.e. for any $\eta \in E(f, A(u))$,*

$$f(z) - f(\bar{z}) \notin \tau \|z - \bar{z}\| B_Y - \mathcal{K} \quad \text{for } z \in A(u) \setminus S_\eta(u).$$

then $\mathcal{S}$ is lower Lipschitz at $u_0 \in \operatorname{dom} \mathcal{S}$. Precisely,

$$S(f, A) \subset S(f, A(u)) + (2L_f L_a/\tau + L_a)\|u - u_0\| B_Y \quad \text{for } u \in B(u_0, t).$$

Proof. We start by proving lower Lipschitz continuity of $\mathcal{S}$ at $u_0 \in \operatorname{dom} \mathcal{S}$. Note first that by (ii), $S(f, A(u)) \neq \emptyset$ for $u \in B(u_0, t)$, i.e. $u_0 \in \operatorname{int} \operatorname{dom} \mathcal{S}$. Take any $x_0 \in S(f, A)$ and $u \in B(u_0, t)$. By (i), there is $z \in A(u)$ such that

$$\|x_0 - z\| \leq L_a \|u - u_0\|.$$

If $z \in S(f, A(u))$, the conclusion follows. Otherwise, by (DP), there exists $\bar{z} \in S(f, A(u))$ such that $f(\bar{z}) \in f(z) - \mathcal{K}$ and $f(z) \neq f(\bar{z})$. If $\|z - \bar{z}\| \leq \frac{2L_a L_f}{\tau}\|u - u_0\|$, then

$$\|x_0 - \bar{z}\| \leq (L_a + 2L_a L_f/\tau)\|u - u_0\|$$

and the conclusion follows. So, assume that

$$\|z - \bar{z}\| > \frac{2L_a L_f}{\tau} \|u - u_0\|. \tag{9.5}$$

By (iv), $\bar{z} \in S(f, A(u))$ is a global sharp solution to (P_u). Since $f(z) \neq f(\bar{z})$ we have

$$f(z) - f(\bar{z}) \notin \tau \|z - \bar{z}\| B_Y - \mathcal{K}.$$

By (i), there exists $x \in A$ such that $\|\bar{z} - x\| \leq L_a \|u - u_0\|$ and

$$\|f(\bar{z}) - f(x)\| \leq L_f L_a \|u - u_0\| \quad \text{and} \quad \|f(z) - f(x_0)\| \leq L_f L_a \|u - u_0\|.$$

Hence, in view of (9.5),

$$\begin{aligned} \|f(x_0) - f(x)\| &\geq \|f(z) - f(\bar{z})\| - \|f(x) - f(\bar{z})\| - \|f(z) - f(x_0)\| \\ &\geq \tau \|z - \bar{z}\| - 2L_a L_f \|u - u_0\| > 0, \end{aligned}$$

which proves that $f(x) \neq f(x_0)$. Hence, since x_0 is a global sharp solution to (P),

$$f(x) - f(x_0) \notin \tau \|x - x_0\| B_Y - \mathcal{K}. \tag{9.6}$$

On the other hand,

$$f(x) - f(x_0) = (f(x) - f(\bar{z})) + (f(\bar{z}) - f(z)) + (f(z) - f(x_0)) \in 2L_f L_a \|u - u_0\| B_Y - \mathcal{K}. \tag{9.7}$$

By (9.6) and (9.7),

$$\|x - x_0\| \leq \frac{2L_f L_a}{\tau} \|u - u_0\|.$$

Consequently,

$$\|x_0 - \bar{z}\| \leq \|x_0 - x\| + \|x - \bar{z}\| \leq (L_a + 2L_f L_a/\tau)\|u - u_0\|,$$

which proves the assertion.

To prove that $\mathcal{P}$ is lower Lipschitz at $u_0 \in \operatorname{dom}\mathcal{P}$ take any $\eta \in E(f, A)$ and $u \in B(u_0, t)$. There exists $\bar{x} \in S(f, A)$ such that $f(\bar{x}) = \eta$. By (i), there exists $z \in A(u)$ such that

$$\|\bar{x} - z\| \leq L_a\|u - u_0\| \quad \text{and} \quad \|f(\bar{x}) - f(z)\| \leq L_f L_a \|u - u_0\|.$$

If $z \in S(f, A(u))$, then $f(z) \in E(f, A(u)$ and the conclusion follows. Otherwise, there exists $\bar{z} \in S(f, A(u))$ such that $f(\bar{z}) \in f(z) - \mathcal{K}$ and $f(\bar{z}) \neq f(z)$.

By (i), there exists $x \in A$ such that

$$\|x - \bar{z}\| \leq L_a\|u - u_0\| \quad \text{and} \quad \|f(x) - f(\bar{z})\| \leq L_f L_a \|u - u_0\|.$$

If $f(x) = f(\bar{x})$, the conclusion follows. If $f(x) \neq f(\bar{x})$, by (iii) and by Proposition 8.1.1,

$$f(x) - f(\bar{x}) \notin \frac{\tau}{L_f} \|f(x) - f(\bar{x})\| B_Y - \mathcal{K}.$$

On the other hand, as before,

$$f(x) - f(\bar{x}) = (f(x) - f(\bar{z})) + (f(\bar{z}) - f(z)) + (f(z) - f(\bar{x})) \in 2L_f L_a \|u - u_0\| B_Y - \mathcal{K}.$$

This proves that

$$\|f(x) - f(\bar{x})\| \leq \frac{2L_a L_f^2}{\tau} \|u - u_0\|$$

and consequently

$$\|f(\bar{x}) - f(\bar{z})\| \leq \|f(\bar{x} - f(x)\| + \|f(x) - f(\bar{z})\| \leq (L_f L_a + 2L_f^2 L_a/\tau)\|u - u_0\|,$$

which proves the assertion. ■

Remark 9.3.1. 1. The first assertion of Theorem 9.3.1 can be deduced from Theorem 4.1.3 and hence assumption (iii) of Theorem 9.3.1 can be weakened by assuming that all $\eta \in E(f, A)$ are strictly efficient points of order 1 with the same constant β. Then the conclusion is that $\mathcal{P}$ is lower Lipschitz continuous at $u_0 \in \operatorname{dom}\mathcal{P}$, i.e.

$$E(f, A) \subset E(f, A(u)) + (L_f L_a + 2L_f L_a/\beta)\|u - u_0\| B_Y \quad \text{for } u \in B(u_0, t).$$

2. Moreover, if a given $\eta \in E(f, A)$ is strictly efficient of order 1 with constant $\beta > 0$, then $\mathcal{P}$ is lower Lipschitz continuous at $(u_0, \eta) \in \operatorname{graph}\mathcal{P}$, i.e.

$$\eta \in E(f, A(u)) + (L_f L_a + 2L_f L_a/\beta)\|u - u_0\| B_Y \quad \text{for } u \in B(u_0, t).$$

Clearly, the constants β appearing in the above estimates may be different.

We say that $x_0 \in S(f, A)$ is *strongly sharp of order* $q > 0$ if there exists a constant $\tau > 0$ such that

$$f(x) - f(x_0) \notin \tau \|x - x_0\| B_Y - \mathcal{K} \quad \text{for } x \in A,\ x \neq x_0. \tag{9.8}$$

This condition implies that $f(x) \neq f(x_0)$ for $x \neq x_0$. Hence, each strongly sharp solution is sharp and $S_\eta = \{x_0\}$, where $f(x_0) = \eta$. With this notion we can prove the following variant of Theorem 9.3.1.

THEOREM 9.3.2. *Let $f : X \to Y$ be Lipschitz with constant $L_f > 0$. Assume that*

(i) *$\mathcal{A} : U \rightrightarrows X$ is Lipschitz at $u_0 \in \operatorname{dom} \mathcal{A}$ with constants $L_a > 0$, $t > 0$,*
(ii) *(DP) holds for all (P_u), $u \in B(u_0, t)$,*
(iii) *each $x_0 \in S(f, A)$ is a global strongly sharp solution of order 1 to (P) with constant $\tau > 0$.*

Then $\mathcal{P}$ is lower Lipschitz at $u_0 \in \operatorname{dom} \mathcal{P}$,i.e.,

$$E(f, A) \in E(f, A(u)) + (2L_f^2 L_a/\tau + L_f L_a)\|u - u_0\| B_Y \quad \textit{for any } u \in B(u_0, t)$$

and $\mathcal{S}$ is lower Lipschitz at $u_0 \in \operatorname{dom} \mathcal{S}$, i.e.,

$$S(f, A) \subset S(f, A(u)) + (2L_f L_a/\tau + L_a)\|u - u_0\| B_X \quad \textit{for any } u \in B(u_0, t).$$

Proof. In view of Theorem 9.3.1 we only need to prove the lower Lipschitz continuity of $\mathcal{S}$. Take any $x_0 \in S(f, A)$ and $u \in B(u_0, t)$. By (i), there is $z \in A(u)$ such that

$$\|x_0 - z\| \leq L_a \|u - u_0\|.$$

If $z \in S(f, A(u))$, the conclusion follows. Otherwise, by (DP), there exists $\bar{z} \in S(f, A(u))$ such that $f(\bar{z}) \in f(z) - \mathcal{K}$ and $f(z) \neq f(\bar{z})$. By (i), there exists $x \in A$ such that

$$\|\bar{z} - x\| \leq L_a \|u - u_0\|,$$

and

$$\|f(\bar{z}) - f(x)\| \leq L_f L_a \|u - u_0\| \quad \text{and} \quad \|f(z) - f(x_0)\| \leq L_f L_a \|u - u_0\|.$$

If $x = x_0$, the conclusion follows. Hence, assume that $x \neq x_0$. Since x_0 is a global strongly sharp solution to (P),

$$f(x) - f(x_0) \notin \tau \|x - x_0\| B_Y - \mathcal{K}. \tag{9.9}$$

On the other hand,

$$\begin{aligned} f(x) - f(x_0) &= (f(x) - f(\bar{z})) + (f(\bar{z}) - f(z)) + (f(z) - f(x_0)) \\ &\in 2L_f L_a \|u - u_0\| B_Y - \mathcal{K} \end{aligned} \tag{9.10}$$

By (9.9) and (9.10),

$$\|x - x_0\| \leq \frac{2L_f L_a}{\tau} \|u - u_0\|.$$

Consequently,

$$\|x_0 - \bar{z}\| \leq \|x_0 - x\| + \|x - \bar{z}\| \leq (L_a + 2L_f L_a/\tau)\|u - u_0\|,$$

which proves the assertion. ■

By assuming weak sharpness of solutions to (P) we get the following result.

THEOREM 9.3.3. *Let $f : X \to Y$ be Lipschitz with constant $L_f > 0$. Assume that*

(i) *$\mathcal{A}$ is Lipschitz at $u_0 \in \operatorname{dom}\mathcal{A}$ with constants $L_a > 0$ and $t > 0$,*
(ii) *(DP) holds for (P_u), $u \in B(u_0, t)$,*
(iii) *all $\overline{z} \in S(f, A(u))$ for $u \in B(u_0, t)$ are weak sharp solutions to (P_u) of order 1 with constant $\tau > 0$.*

Then $\mathcal{S}$ is lower Lipschitz at $u_0 \in \operatorname{dom}\mathcal{S}$. Precisely,

$$S(f, A) \subset S(f, A(u)) + (L_a + 2L_f L_a + 2L_a L_f/\tau)\|u - u_0\|B_X \quad \textit{for } u \in B(u_0, t).$$

Proof. Let $\overline{x} \in S(f, A)$ and $u \in B(u_0, t)$. By Theorem 8.2.2, there exists $\overline{z} \in S(f, A(u))$ such that

$$\|f(\overline{x}) - f(\overline{z})\| \leq (L_f L_a + 2L_f^2 L_a/\tau)\|u - u_0\|.$$

By (i), there exists $z \in A(u)$ such that

$$\|\overline{x} - z\| \leq L_a\|u - u_0\| \quad \text{and} \quad \|f(\overline{x}) - f(z)\| \leq L_f L_a\|u - u_0\|.$$

If $z \in S(f, A(u))$, the conclusion follows. Suppose that $z \notin S(f, A(u))$. We have

$$f(z) - f(\overline{z}) = (f(z) - f(\overline{x})) + (f(\overline{x}) - f(\overline{z})) \in (2L_f L_a + 2L_f^2 L_a/\tau)\|u - u_0\|B_Y.$$

On the other hand, since $\overline{z} \in S(f, A(u))$ is weakly sharp, $f(\overline{z}) = \eta$ and $f(z) \neq f(\overline{z})$,

$$f(z) - f(\overline{z}) \notin \tau d(z, S_\eta(u))B_Y - \mathcal{K},$$

where $S_\eta(u) = \{z \in S(f, A(u)) : f(z) = \eta\}$. Consequently,

$$d(\overline{x}, S(f, A(u)) \leq d(\overline{x}, S_\eta(u)) \leq d(\overline{x}, z) + d(z, S_\eta(u)) \leq (L_a + 2L_f L_a + 2L_a L_f^2/\tau)\|u - u_0\|. \blacksquare$$

9.4. Upper Lipschitzness of solutions

In this section making use of sharp and weak sharp solutions we prove upper Lipschitzness of $\mathcal{S}$.

THEOREM 9.4.1. *Let $f : X \to Y$ be Lipschitz with constant $L_f > 0$. Assume that*

(i) *$\mathcal{A}$ is Lipschitz at $u_0 \in \operatorname{dom}\mathcal{A}$ with constants $L_a > 0$ and $t > 0$,*
(ii) *(DP) holds for (P),*
(iii) *all $\overline{z} \in S(f, A(u))$ for $u \in B(u_0, t)$ are sharp solutions to (P_u) of order 1 with constant $\tau > 0$.*

Then

- *$\mathcal{S}$ is upper Lipschitz at $u_0 \in \operatorname{dom}\mathcal{S}$, i.e.,*

$$S(f, A(u)) \subset S(f, A) + (L_a + 2L_a L_f/\tau)\|u - u_0\|B_X \quad \textit{for } u \in B(u_0, t),$$

- *$\mathcal{P}$ is upper Lipschitz at $u_0 \in \operatorname{dom}\mathcal{P}$, i.e.,*

$$E(f, A(u)) \subset E(f, A) + (L_f L_a + 2L_a L_f^2/\tau)\|u - u_0\|B_Y \quad \textit{for } u \in B(u_0, t).$$

Proof. Let $\overline{z} \in S(f, A(u))$, $u \in B(u_0, t)$. By the upper Lipschitzness of $\mathcal{A}$, there exists $x \in A$ such that

$$\|x - \overline{z}\| \leq L_a\|u - u_0\|.$$

If $x \in S(f, A)$, the conclusion follows. Otherwise, by (DP), there exists $\overline{x} \in S(f, A)$ such that $f(\overline{x}) \in f(x) - \mathcal{K}$ and $f(x) \neq f(\overline{x})$.

If $\|x - \overline{x}\| \leq \frac{2L_f L_a}{\tau}\|u - u_0\|$, the conclusion follows. Otherwise, $\|x - \overline{x}\| > \frac{2L_f L_a}{\tau}\|u - u_0\|$. By the lower Lipschitzness of $\mathcal{A}$, there exists $z \in A(u)$ such that

$$\|\overline{x} - z\| \leq L_a \|u - u_0\|.$$

Since f is Lipschitz,

$$\begin{aligned} f(z) - f(\overline{z}) &= (f(z) - f(\overline{x})) + (f(\overline{x}) - f(x)) + (f(x) - f(\overline{z})) \\ &\in 2L_f L_a \|u - u_0\| B_Y - \mathcal{K}. \end{aligned} \tag{9.11}$$

Moreover,

$$\begin{aligned} \|f(z) - f(\overline{z})\| &\geq \|f(x) - f(\overline{x})\| - \|f(x) - f(\overline{z})\| - \|f(\overline{x}) - f(z)\| \\ &\geq \tau\|x - \overline{x}\| - 2L_f L_a \|u - u_0\| > 0, \end{aligned} \tag{9.12}$$

which proves that $f(z) \neq f(\overline{z})$, and since $\overline{z} \in S(f, A(u))$ is a sharp solution to (P_u) we get

$$f(z) - f(\overline{z}) \notin \tau\|z - \overline{z}\| B_Y - \mathcal{K}. \tag{9.13}$$

By (9.11) and (9.13), $\|z - \overline{z}\| \leq \frac{2L_f L_a}{\tau}\|u - u_0\|$ and finally

$$\|\overline{z} - \overline{x}\| \leq \|\overline{z} - x\| + \|z - \overline{z}\| \leq (L_a + 2L_f L_a/\tau)\|u - u_0\|.$$

To see the second assertion, take any $\eta \in E(f, A(u))$. There exists $\overline{z} \in S(f, A(u))$ such that $\eta = f(\overline{z})$. By (i), there exists $x \in A$ such that $\|\overline{z} - x\| \leq L_a\|u - u_0\|$. If $x \in S(f, A)$, the conclusion follows. If $x \notin S(f, A)$, by (ii), there exists $\overline{x} \in S(f, A)$ such that $f(\overline{x}) \in f(x) - \mathcal{K}$ and $f(x) \neq f(\overline{x})$. By (i), there exists $z \in A(u)$ such that $\|z - \overline{x}\| \leq L_a\|u - u_0\|$. If $f(\overline{z}) = f(z)$, the conclusion follows. Otherwise,

$$f(z) - f(\overline{z}) = (f(z) - f(\overline{x})) + (f(\overline{x}) - f(x)) + (f(x) - f(\overline{z})) \in 2L_f L_a \|u - u_0\| B_Y - \mathcal{K}$$

and since $\overline{z} \in S(f, A(u))$ is a sharp solution to (P_u),

$$f(z) - f(\overline{z}) \notin \frac{\tau}{L_f}\|f(z) - f(\overline{z})\| B_Y - \mathcal{K}.$$

Consequently, $\|f(z) - f(\overline{z})\| \leq \frac{2L_f^2 L_a}{\tau}\|u - u_0\|$ and

$$\begin{aligned} f(\overline{x}) - f(\overline{z}) &= (f(\overline{x}) - f(z)) + (f(z) - f(\overline{z})) \\ &\in (L_f L_a + 2L_f^2 L_a/\tau)\|u - u_0\| B_Y. \ \blacksquare \end{aligned}$$

Recall that (*SDP*) of order 1 with constant $\alpha > 0$ holds for (P) if for any $x \in A$ there exists $\overline{x} \in S(f, A)$ such that

$$f(x) - f(\overline{x}) + \alpha\|f(x) - f(\overline{x})\| B_Y \subset \mathcal{K}.$$

By using the strong domination property (*SDP*) of order 1 we can prove the following variant of Theorem 9.4.1 for closed convex pointed cones with nonempty interior.

THEOREM 9.4.2. *Let $\mathcal{K}$ be a closed convex pointed cone with* $\operatorname{int}\mathcal{K} \neq \emptyset$. *Let $f : X \to Y$ be Lipschitz with constant $L_f > 0$. Assume that*

(i) *$\mathcal{A}$ is Lipschitz at $u_0 \in \operatorname{dom}\mathcal{A}$ with constants $L_a > 0$ and $t > 0$,*

(ii) *(SDP) of order 1 with constant $\alpha > 0$ holds for (P).*

Then $\mathcal{P}$ is upper Lipschitz at $u_0 \in \operatorname{dom}\mathcal{P}$, i.e.,

$$E(f, A(u)) \subset E(f, A) + (L_f L_a + 2L_a L_f/\alpha)\|u - u_0\|B_Y \qquad \text{for } u \in B(u_0, t).$$

If moreover,

(iii) *all $\overline{x} \in S(f, A)$ are sharp of order 1 with constant $\tau > 0$,*

then $\mathcal{S}$ is upper Lipschitz at $u_0 \in \operatorname{dom}\mathcal{S}$, i.e.,

$$S(f, A(u)) \subset S(f, A) + (L_a + 2L_a L_f^2/\alpha\tau)\|u - u_0\|B_X \qquad \text{for } u \in B(u_0, t).$$

Proof. To see the first assertion, take any $\eta \in E(f, A(u))$, $u \in B(u_0, t)$. There exists $\overline{z} \in S(f, A(u))$ such that $\eta = f(\overline{z})$. By (i), there exists $x \in A$ such that

$$\|x - \overline{z}\| \leq L_a\|u - u_0\|.$$

If $x \in S(f, A)$, then $\|f(\overline{z}) - f(x)\| \leq L_f L_a\|u - u_0\|$ and the conclusion follows. Otherwise, by (*SDP*), there exists $\overline{x} \in S(f, A)$ with $f(x) \neq f(\overline{x})$ such that

$$f(x) - f(\overline{x}) + \alpha\|f(x) - f(\overline{x})\|B_Y \subset \mathcal{K}.$$

By (i), there exists $z \in A(u)$ such that $\|z - \overline{x}\| \leq L_a\|u - u_0\|$. If $z \in S(f, A(u))$, the conclusion follows. If $z \notin S(f, A(u))$, then

$$\|f(x) - f(\overline{x})\| \leq \frac{2L_f L_a}{\alpha}\|u - u_0\|$$

since otherwise

$$\begin{aligned} f(z) - f(\overline{z}) &= f(z) - f(\overline{x}) + (f(\overline{x}) - f(x)) + (f(x) - f(\overline{z})) \\ &\in (f(\overline{x}) - f(x)) + 2L_a L_f B_Y \\ &\subset (f(\overline{x}) - f(x)) + \alpha\|f(x) - f(\overline{x})\|B_Y \\ &\subset -\mathcal{K}, \end{aligned}$$

which contradicts the fact that $\overline{z} \in S(f, A(u))$. Finally,

$$f(\overline{z}) - f(\overline{x}) = (f(\overline{z}) - f(x)) + (f(x) - f(\overline{x}) \in (L_f L_a + 2L_f L_a/\alpha)\|u - u_0\|B_Y,$$

which proves the first assertion.

To prove the second assertion take any $\overline{z} \in S(f, A(u))$, $u \in B(u_0, t)$. By (i), there exists $x \in A$ such that

$$\|x - \overline{z}\| \leq L_a\|u - u_0\|.$$

If $x \in S(f, A)$, the conclusion follows. Otherwise, by (*SDP*), there exists $\overline{x} \in S(f, A)$ with $f(x) \neq f(\overline{x})$ such that

$$f(x) - f(\overline{x}) + \alpha\|f(x) - f(\overline{x})\|B_Y \subset \mathcal{K}.$$

In the same way as above we argue that

$$f(x) - f(\overline{x}) \in \frac{2L_f L_a}{\alpha}\|u - u_0\|B_Y.$$

Since $\overline{x}$ is a global sharp solution of order 1 to (P) and $f(x) \neq f(\overline{x})$,

$$f(x) - f(\overline{x}) \notin \frac{\tau}{L_f} \|x - \overline{x}\| B_Y - \mathcal{K}$$

and consequently $\|x - \overline{x}\| \leq \frac{2L_f^2 L_a}{\alpha\tau} \|u - u_0\|$. Hence,

$$\|\overline{z} - \overline{x}\| \leq \|\overline{z} - x\| + \|x - \overline{x}\| \leq (L_a + 2L_f^2 L_a/\alpha\tau)\|u - u_0\|. \blacksquare$$

Making use of weakly sharp solutions we obtain the following result.

THEOREM 9.4.3. *Let $f : X \to Y$ be Lipschitz with constant $L_f > 0$. Assume that*

(i) *$\mathcal{A}$ is Lipschitz at $u_0 \in \operatorname{dom} \mathcal{A}$ with constants $L_a > 0$ and $t > 0$,*
(ii) *(DP) holds for (P_u) and $u \in B(u_0, t)$,*
(iii) *all $x \in S(f, A)$ are weakly sharp solutions to (P) of order 1 with constant $\tau > 0$.*

Then $\mathcal{S}$ is upper Lipschitz at $u_0 \in \operatorname{dom} \mathcal{S}$, i.e. for any $u \in B(u_0, t)$,

$$S(f, A(u)) \subset S(f, A) + (L_a + 2L_f L_a + 2L_a^2 L_f/\tau)\|u - u_0\| B_X.$$

Proof. Let $\overline{z} \in S(f, A(u))$, $u \in U_0$. By Theorem 8.2.3, there exists $\overline{x} \in S(f, A)$ such that

$$f(\overline{z}) - f(\overline{x}) \in (L_a L_f + 2L_a L_f^2/\tau)\|u - u_0\| B_Y.$$

By the upper Lipschitzness of $\mathcal{A}$, there exists $x \in A$ such that

$$\|\overline{z} - x\| \leq L_a \|u - u_0\| \quad \text{and} \quad \|f(\overline{z}) - f(x)\| \leq L_f L_a \|u - u_0\|.$$

If $x \in S(f, A)$, the conclusion follows. Otherwise,

$$\begin{aligned} f(x) - f(\overline{x}) &= (f(x) - f(\overline{z})) + (f(\overline{z}) - f(\overline{z}) \\ &\in (2L_f L_a + 2L_a L_f^2/\tau)\|u - u_0\| B_Y. \end{aligned}$$

On the other hand, since $\overline{x} \in S(f, A)$ is a global weakly sharp solution of order 1 with $f(\overline{x}) = \eta$ and $f(x) \neq f(\overline{x})$,

$$f(x) - f(\overline{x}) \notin \tau d(x, S_\eta) B_Y - \mathcal{K}.$$

Consequently, $d(x, S_\eta) \leq (2L_f L_a + 2L_a L_f^2/\tau)\|u - u_0\|$ and

$$\begin{aligned} d(\overline{z}, S(f, A)) &\leq d(\overline{z}, S_\eta) \leq d(\overline{z}, x) + d(x, S_\eta) \\ &\leq (L_a + 2L_f L_a + 2L_a L_f^2/\tau)\|u - u_0\|. \blacksquare \end{aligned}$$

9.5. Lower Hölder and lower pseudo-Hölder continuity of solutions

In this section we investigate lower Hölder continuity of the solution mapping $\mathcal{S}$ at $u_0 \in \operatorname{dom} \mathcal{S}$ and lower pseudo-Hölder continuity of $\mathcal{S}$ at $(u_0, x_0) \in \operatorname{graph} \mathcal{S}$. The spaces X, Y and U are assumed to be normed spaces with open unit balls B_X, B_Y and B_U, respectively.

Recall that for a set-valued mapping $\mathcal{A} : U \rightrightarrows X$, $\mathcal{A}(u) = A(u)$, $\mathcal{A}(u_0) = A$, and $f : X \to Y$ the set-valued mapping $\mathcal{A}_f : U \rightrightarrows Y$ is given by

$$\mathcal{A}_f(u) = f(A(u)), \quad \mathcal{A}_f(u_0) = f(A). \tag{9.14}$$

THEOREM 9.5.1. *Let $\mathcal{K}$ be a closed convex pointed cone in Y. Assume that*

(i) *there exists $0 < t < 1$ such that all $\bar{z} \in S(f, A(u))$ for $u \in B(u_0, t)$ are sharp solutions to (P_u) of order $q \geq 1$ with constant $\tau > 0$, i.e.,*

$$f(z) - f(\bar{z}) \notin \tau\|z - \bar{z}\|^q B_Y - \mathcal{K} \quad \text{for } z \in A(u),\ f(z) \neq f(\bar{z}),$$

(ii) *$f : X \to Y$ is Lipschitz on X with constant $L_f > 0$ and $\mathcal{A}$ is Hölder continuous of order $p \geq 1$ at $u_0 \in \operatorname{dom} \mathcal{A}$ with constants $L_a > 0$ and $0 < t < 1$,*

(iii) *(DP) holds for all $f(A(u))$ and $u \in B(u_0, t)$.*

Then $\mathcal{S}$ is lower Hölder continuous of order $\frac{p}{q}$ at $u_0 \in \operatorname{dom} \mathcal{S}$. Precisely,

$$S(f, A) \subset S(f, A(u)) + (L_a + (2L_aL_f/\tau)^{1/q})\|u - u_0\|^{p/q} B_X$$

for $u \in B(u_0, t_a)$.

Proof. Take $u \in B(u_0, t)$ and $\bar{x} \in S(f, A)$. By (ii), there exists $z \in A(u)$ such that

$$\|\bar{x} - z\| \leq L_a\|u - u_0\|^p \quad \text{and} \quad \|f(\bar{x}) - f(z)\| \leq L_fL_a\|u - u_0\|^p.$$

If $z \in S(f, A(u))$, the assertion follows. If $z \notin S(f, A(u))$, then by (iii), there exists $\bar{z} \in S(f, A(u))$ such that $f(\bar{z}) \in f(z) - \mathcal{K}$. If $\|z - \bar{z}\| \leq (2L_fL_a/\tau)^{1/q}\|u - u_0\|^{p/q}$, the conclusion follows. Hence, assume that

$$\tau\|z - \bar{z}\|^q > 2L_fL_a\|u - u_0\|^p.$$

By (ii), there exists $x \in A$ such that

$$\|x - \bar{z}\| \leq L_a\|u - u_0\|^p \quad \text{and} \quad \|f(x) - f(\bar{z})\| \leq L_fL_a\|u - u_0\|^p.$$

Since $\bar{z} \in Sh^q(f, A(u))$ and $f(z) \neq f(\bar{z})$ we have

$$f(z) - f(\bar{z}) \notin \tau\|z - \bar{z}\|^q B_Y - \mathcal{K}$$

and

$$\begin{aligned}\|f(x) - f(\bar{x})\| &\geq \|f(z) - f(\bar{z})\| - \|f(x) - f(\bar{z})\| - \|f(z) - f(\bar{x})\| \\ &\geq \tau\|z - \bar{z}\|^q - 2L_fL_a\|u - u_0\|^p > 0.\end{aligned}$$

This proves that $f(x) \neq f(\bar{x})$ and in view of the fact that $\bar{x} \in Sh^q(f, A)$ we get

$$f(x) - f(\bar{x}) \notin \tau\|x - \bar{x}\|^q B_Y - \mathcal{K}. \tag{9.15}$$

On the other hand,

$$f(x) - f(\bar{x}) = (f(x) - f(\bar{z})) + (f(\bar{z}) - f(z)) + (f(z) - f(\bar{x})) \in 2L_fL_a\|u - u_0\|^p - \mathcal{K},$$

which together with (9.15) leads to the inequality

$$\|x - \bar{x}\| \leq (2L_fL_a/\tau)^{1/q}.$$

Finally,

$$\|\bar{x} - \bar{z}\| \leq \|x - \bar{z}\| + \|x - \bar{x}\| \leq (L_a + (2L_fL_a/\tau)^{1/q})\|u - u_0\|^{p/q},$$

which proves the assertion. ∎

Now we prove sufficient conditions for lower pseudo-Hölder continuity of $\mathcal{S}$ at $(u_0, x_0) \in \operatorname{graph} \mathcal{S}$.

THEOREM 9.5.2. *Let $\mathcal{K}$ be a closed convex pointed cone in Y. Let $x_0 \in S(f, A)$ and $f(x_0) = \eta$. Assume that*

(i) *there exists $0 < t_a < 1$ such that all $\bar{z} \in S(f, A(u) \cap (x_0 + V)$ for $u \in B(u_0, t_a)$ are local sharp solutions to (P_u) of order $q \geq 1$ with constants $\tau > 0$ and $t_s > 0$, i.e.,*

$$f(z) - f(\bar{z}) \notin \tau \|z - \bar{z}\|^q B_Y - \mathcal{K} \quad \text{for } z \in A(u) \cap (\bar{z} + t_s B_X),\ f(z) \neq f(\bar{z}),$$

(ii) *$f : X \to Y$ is Lipschitz around x_0 with constant $L_f > 0$ and $\mathcal{A}$ is pseudo-Hölder continuous of order $p \geq 1$ at $(u_0, x_0) \in \operatorname{graph} \mathcal{A}$ with 0-neighbourhood V and constants L_a and t_a,*

(iii) *(LDP) holds for all $f(A(u))$ and $u \in B(u_0, t_a)$.*

Then $\mathcal{S}$ is lower pseudo-Hölder continuous of order p/q at $(u_0, x_0) \in \operatorname{graph} \mathcal{S}$. Precisely,

$$S(f, A) \cap (x_0 + V) \subset S(f, A(u)) + (L_a + (2L_a L_f/\tau)^{1/q}) \|u - u_0\|^{p/q} B_X$$

for $u \in B(u_0, t)$ with $t = \min\{t_a, t_s\}$.

Proof. Take $u \in B(u_0, t)$ and $\bar{x} \in S(f, A) \cap (x_0 + V)$. By (ii), in view of the lower pseudo-Hölder continuity of $\mathcal{A}$, there exists $z \in A(u)$ such that

$$\|\bar{x} - z\| \leq L_a \|u - u_0\|^p \quad \text{and} \quad \|f(\bar{x}) - f(z)\| \leq L_f L_a \|u - u_0\|^p.$$

If $z \in S(f, A(u))$, the assertion follows. If $z \notin S(f, A(u))$, then by (iii), there exists $\bar{z} \in S(f, A(u))$ such that $f(\bar{z}) \in f(z) - \mathcal{K}$. If $\|z - \bar{z}\| \leq (2L_f L_a/\tau)^{1/q} \|u - u_0\|^{p/q}$, the conclusion follows. Hence, assume that

$$\tau \|z - \bar{z}\|^q > 2L_f L_a \|u - u_0\|^p.$$

By the upper Hölder continuity of $\mathcal{A}$, there exists $x \in A$ such that

$$\|x - \bar{z}\| \leq L_a \|u - u_0\|^p \quad \text{and} \quad \|f(x) - f(\bar{z})\| \leq L_f L_a \|u - u_0\|^p.$$

Since $\bar{z} \in Sh^q(f, A(u))$ and $f(z) \neq f(\bar{z})$ we have

$$f(z) - f(\bar{z}) \notin \tau \|z - \bar{z}\|^q B_Y - \mathcal{K}$$

and

$$\begin{aligned} \|f(x) - f(\bar{x})\| &\geq \|f(z) - f(\bar{z})\| - \|f(x) - f(\bar{z})\| - \|f(z) - f(\bar{x})\| \\ &\geq \tau \|z - \bar{z}\|^q - 2L_f L_a \|u - u_0\|^p > 0. \end{aligned}$$

This proves that $f(x) \neq f(\bar{x})$ and in view of the fact that $\bar{x} \in Sh^q(f, A)$ we get

$$f(x) - f(\bar{x}) \notin \tau \|x - \bar{x}\|^q B_Y - \mathcal{K}. \tag{9.16}$$

On the other hand,

$$f(x) - f(\bar{x}) = (f(x) - f(\bar{z})) + (f(\bar{z}) - f(z)) + (f(z) - f(\bar{x})) \in 2L_f L_a \|u - u_0\|^p - \mathcal{K},$$

which together with (9.16) leads to the inequality

$$\|x - \bar{x}\| \leq (2L_f L_a/\tau)^{1/q}.$$

Finally,

$$\|\bar{x} - \bar{z}\| \leq \|x - \bar{z}\| + \|x - \bar{x}\| \leq (L_a + (2L_f L_a/\tau)^{1/q}) \|u - u_0\|^{p/q},$$

which proves the assertion. ■

9.6. Upper Hölder continuity and Hölder calmness of solutions to parametric problems

In this section we investigate Hölder calmness of $\mathcal{S}$ at $(u_0, x_0) \in \operatorname{graph} \mathcal{S}$. The spaces X, Y and U are assumed to be normed spaces with open unit balls B_X, B_Y and B_U, respectively. Analogous results for scalar optimization problems were obtained by Bonnans and Shapiro ([39, Sec. 4.4.2]) and Bonnans and Ioffe [38].

Recall that for a set-valued mapping $\mathcal{A} : U \rightrightarrows X$, $\mathcal{A}(u) = A(u)$, $\mathcal{A}(u_0) = A$, and a mapping $f : X \to Y$ the set-valued mapping $\mathcal{A}_f : U \rightrightarrows Y$ is given by

$$\mathcal{A}_f(u) = f(A(u)), \qquad \mathcal{A}_f(u_0) = f(A). \tag{9.17}$$

We start with the result on Hölder calmness of $\mathcal{P}$.

Theorem 9.6.1. *Let $\mathcal{K}$ be a closed convex pointed cone in Y with $\operatorname{int} \mathcal{K} \neq \emptyset$. Let $x_0 \in S(f, A)$. Assume that*

(i) *$\mathcal{A}_f$ given by (9.17) is pseudo-Lipschitz of order $p \geq 1$ at $(u_0, f(x_0)) \in \operatorname{graph} \mathcal{A}$ with a neighbourhood W of zero in Y, $W \subset t_f B_X$, and constants $L_a > 0$ and $t > 0$,*

(ii) *the local strong domination property $(LSDP)$ of order $q \geq 1$ holds for $f(A)$ around $f(x_0)$ with the neighbourhood $\frac{1}{2}W$ and constant $\alpha > 0$.*

Then $\mathcal{P}$ is Hölder calm at $(u_0, f(x_0)) \in \operatorname{graph} \mathcal{P}$. Precisely, there is a neighbourhood $\overline{W}$ of zero in Y such that

$$E(f, A(u)) \cap (f(x_0) + \overline{W}) \subset E(f, A)) + L_f(L_a + (2L_f L_a/\alpha)^{1/q})\|u - u_0\|^{p/q}$$

for $u \in u_0 + tB_U$.

Proof. Let $\overline{W}$ be a neighbourhood of zero in Y such that $\overline{W} + L_a t B_Y \subset W$. Take any $f(x) \in E(f, A(u)) \cap (f(x_0) + \overline{W})$, $u \in u_0 + tB_U$. By the pseudo-Lipschitzness of $\mathcal{A}$ at $(u_0, f(x_0)) \in \operatorname{graph} \mathcal{A}$, there exists $z \in A$ such that

$$\|f(x) - f(z)\| \leq L_a \|u - u_0\|^p.$$

Clearly, $f(z) \in f(x_0) + W$. By $(LSDP)$ of order $q \geq 1$ around $f(x_0)$, there exists $\overline{z} \in S(f, A)$ such that

$$\alpha \|f(z) - f(\overline{z})\|^q \leq \|f(z) - f(\overline{z})\|_+.$$

By the lower pseudo-Lipschitzness of $\mathcal{A}$ at $(u_0, f(x_0)) \in \operatorname{graph} \mathcal{A}$, there exists $\overline{x} \in A(u)$ such that

$$\|f(\overline{x}) - f(\overline{z})\| \leq L_a \|u - u_0\|^p,$$

We have $f(\overline{x}) - f(x) = [f(\overline{z}) - f(z)] + w$, where

$$w = [f(\overline{x}) - f(\overline{z})] + [f(z) - f(x)] \quad \text{and} \quad \|w\| \leq 2L_a \|u - u_0\|^p.$$

Hence $\|w\| > \|f(z) - f(\overline{z})\|_+$ since otherwise

$$f(x) - f(\overline{x}) = [f(z) - f(\overline{z})] + w \in \mathcal{K},$$

contrary to the efficiency of $f(x)$ over $f(A(u))$. Consequently,

$$\alpha \|f(z) - f(\overline{z})\|^q \leq \|w\| \leq 2L_a \|u - u_0\|^p,$$

and

$$\|f(z) - f(\overline{z})\| \le (2L_a/\alpha)^{1/q}\|u - u_0\|^{p/q}. \tag{9.18}$$

Hence,

$$\|f(x) - f(\overline{z})\| \le \|f(x) - f(z)\| + \|f(z) - f(\overline{z})\| \le (L_a + (2L_a/\alpha)^{1/m})\|u - u_0\|^{p/m},$$

which proves the assertion. ∎

THEOREM 9.6.2. *Let $\mathcal{K}$ be a closed convex pointed cone in Y with $\operatorname{int}\mathcal{K} \neq \emptyset$. Let $x_0 \in S(f, A)$ and let $f : X \to Y$ be locally Lipschitz at x_0 with constants $L_f > 0$ and $t > 0$. Assume that*

(i) *$\mathcal{A}$ is pseudo-Lipschitz at $(u_0, x_0) \in \operatorname{graph}\mathcal{A}$ with neighbourhood V of zero in X, $V \subset tB_X$, and constants $L_a > 0$ and t,*
(ii) *$(LFDP)$ holds around x_0 with the neighbourhood $\frac{1}{2}V$ and constant $\alpha > 0$,*
(iii) *the growth condition of order $q > 1$ holds around x_0 with the neighbourhood V and constant $\tau > 0$.*

Then $\mathcal{S}$ is calm of order $1/q$ at $(u_0, x_0) \in \operatorname{graph}\mathcal{S}$. Precisely,

$$S(f, A(u)) \cap (x_0 + \lambda V) \subset S(f, A)) + (L_a + (2L_f^2 L_a/\alpha\tau)^{1/q})\|u - u_0\|^{1/q}B_X$$

for $u \in B(u_0, t)$ and a certain $0 < \lambda < 1/2$.

Proof. By taking t small enough, we can choose $0 < \lambda < \frac{1}{2}$ such that $\lambda V + tL_aB_X \subset \frac{1}{2}V$. Take any $x \in S(f, A(u)) \cap (x_0 + \lambda V)$, $u \in u_0 + tB_U$. By (i), there exists $z \in A$ such that

$$\|x - z\| \le L_a\|u - u_0\|.$$

We have $z - x_0 = (z - x) + (x - x_0) \in tL_aB_X + \lambda V \subset \frac{1}{2}V$. By Lipschitzness of f,

$$\|f(x) - f(z)\| \le L_f L_a\|u - u_0\|\,. \tag{9.19}$$

Since $(LFDP)$ holds around x_0, there exists $\overline{z} \in S(f, A) \cap (x_0 + \frac{1}{2}V)$ such that

$$\alpha\|z - \overline{z}\| \le \|f(z) - f(\overline{z})\|_+.$$

By (i), there exists $\overline{x} \in A(u)$ such that

$$\|\overline{x} - \overline{z}\| \le L_A\|u - u_0\|,$$

and $\overline{x} - x_0 = (\overline{x} - \overline{z}) + (\overline{z} - x_0) \in t_cL_AB_X + \frac{1}{2}V \subset V$. We have $f(\overline{x}) - f(x) = [f(\overline{z}) - f(z)] + w$, where $w = [f(\overline{x}) - f(\overline{z})] + [f(z) - f(x)]$. By Lipschitzness of f,

$$\|w\| \le 2L_f L_a\|u - u_0\|.$$

Since $x \in \mathcal{S}(u)$, we have $\|w\| > \|f(z) - f(\overline{z})\|_+$ and thus,

$$\alpha\|f(z) - f(\overline{z})\| \le \alpha L_f\|z - \overline{z}\| \le L_f\|w\| \le 2L_f^2 L_a\|u - u_0\|.$$

Hence,

$$\|f(z) - f(\overline{z})\| \le \frac{2L_f^2 L_a}{\alpha}\,\|u - u_0\|,$$

or equivalently,

$$f(z) - f(\overline{z}) \in \frac{2L_f^2 L_a}{\alpha}\,\|u - u_0\|B_Y.$$

On the other hand, $z-\overline{z}=(z-x_0)+(x_0-\overline{z})\in\frac{1}{2}V+\frac{1}{2}V\subset V$, and since the growth condition of order $q\geq 1$ holds for f around x_0 we have

$$f(z)-f(\overline{z})\notin \tau d(z,S(f,A))^q B_Y-\mathcal{K}\,.$$

Thus,

$$\frac{2L_f^2 L_c}{\tau}\,\|u-u_0\|B_Y\not\subset \tau d(z,S(f,A))^q B_Y-\mathcal{K},$$

and consequently

$$\frac{2L_f^2 L_a}{\alpha}\,\|u-u_0\|B_Y\not\subset \tau d(z,S(f,A))^q B_Y,$$

which means that

$$d(z,S(f,A))^q\leq \frac{2L_f^2 L_a}{\alpha\tau}\,\|u-u_0\|$$

or $d(z,S(f,A))\leq (2L_f^2L_a/\alpha\tau)^{1/q}\|u-u_0\|^{1/q}$. Finally,

$$d(x,S(f,A))\leq \|x-z\|+d(z,S(f,A))\leq (L_a+(2L_f^2L_a/\alpha\tau)^{1/q})\|u-u_0\|^{1/q}.\ \blacksquare$$

THEOREM 9.6.3. *Let $\mathcal{K}$ be a closed convex pointed cone in Y with $\operatorname{int}\mathcal{K}\neq\emptyset$. Let $x_0\in S(f,A)$ and let $f:X\to Y$ be locally Lipschitz on $x_0+t_fB_X$ with constants L_f. Assume that*

(i) *$\mathcal{A}:U\rightrightarrows X$ is pseudo-Lipschitz at $(u_0,x_0)\in\operatorname{graph}\mathcal{A}$ with neighbourhood V of zero in X, $V\subset t_fB_X$,*
(ii) *the local firm strong domination property holds around x_0 with the neighbourhood $\frac{1}{2}V$,*
(iii) *(P) is Hölder calm well-posed at x_0 of order $1/m$, $m>1$.*

Then $\mathcal{S}$ is Hölder calm of order $1/m$ at $(u_0,x_0)\in\operatorname{graph}\mathcal{S}$.

Proof. Follows directly from Proposition 9.1.4 and Theorem 9.6.2. $\blacksquare$

With $V=X$ we obtain

COROLLARY 9.6.1. *Let $\mathcal{K}$ be a closed convex pointed cone in Y with $\operatorname{int}\mathcal{K}\neq\emptyset$. Let $f:X\to Y$ be locally Lipschitz. Assume that*

(i) *$\mathcal{A}$ is Lipschitz around $u_0\in\operatorname{dom}A$,*
(ii) *the (global) firm domination property holds for (P),*
(iii) *(P) is upper Hölder well-posed of order $1/m$, $m>1$.*

Then $\mathcal{S}$ is upper Hölder of order $1/m$ at u_0.

9.7. Hölder continuity of the solution mapping $\mathcal{S}$

In this section we formulate conditions for Hölder continuity of $\mathcal{S}$ provided that problems (P_u) satisfy the growth condition of order $q\geq 1$. For scalar optimization problems similar results were obtained by Bonnans and Shapiro ([39, Sec. 4.4.2]) and Bonnans and Ioffe [38].

THEOREM 9.7.1. *Let $\mathcal{K}$ be a closed convex pointed cone in Y. Let $f : X \to Y$ be Lipschitz with constant $L_f > 0$. Assume that*

(i) *$\mathcal{A} : U \rightrightarrows X$ is Hölder of order $p > 0$ around $u_0 \in \operatorname{dom} \mathcal{A}$ with constants $L_a > 0$ and $0 < t < 1$,*
(ii) *(DP) holds for (P_u) with $u \in B(u_0, t)$,*
(iii) *the global growth condition of order $q \geq 1$ holds for all (P_u) on $S(f, A(u))$ with constant $\tau > 0$.*

Then $\mathcal{S}$ is Hölder of order p/q at $u_0 \in \operatorname{dom} \mathcal{S}$. Precisely,

$$S(f, A(u)) \subset S(f, A(u')) + (L_a + (2L_f^{q+1} L_a/\tau)^{1/q})\|u - u'\|^{p/q} B_X$$

for $u, u' \in u_0 + (t/4)B_U$.

Proof. The proof follows from Proposition 4.0.3, by observing that under the assumptions $\mathcal{S}$ is uniformly lower Hölder of order p/q at any $u' \in u_0 + (t/2)B_Y$. ■

THEOREM 9.7.2. *Let $\mathcal{K}$ be a closed convex pointed cone in Y with $\operatorname{int} \mathcal{K} \neq \emptyset$. Let $x_0 \in S(f, A)$ and let $f : X \to Y$ be locally Lipschitz on $x_0 + t_f B_X$ with constants L_f. Assume that*

(i) *$\mathcal{A}$ is pseudo-Lipschitz at $(u_0, x_0) \in \operatorname{graph} \mathcal{A}$ with neighbourhood V of zero in X, $V \subset t_f B_X$,*
(ii) *the local firm domination property holds for (P) around x_0 with a neighbourhood Q, $Q + Q \subset V$,*
(iii) *(P) is Hölder calm well-posed at x_0 of order $1/m$, $m \geq 1$.*

Then $\mathcal{S}$ is Hölder calm of order $1/m$ at $(u_0, x_0) \in \operatorname{graph} \mathcal{S}$.

Proof. Follows directly from Proposition 9.1.4 and Theorem 9.6.2. ■

Final remarks

Our aim was to provide sufficient conditions for semi- and pseudo-continuitites in the sense of Lipschitz and/or Hölder for the set-valued mappings $\mathcal{P}$ and $\mathcal{S}$. We focused on formulating sufficient conditions which are as weak as possible in order to make them applicable to a wide class of problems. As a result we have not assumed any particular form of description of the feasible set A. In the literature there exist numerous results which provide conditions guaranteeing Lipschitz and/or Hölder behaviour of the feasible set depending on parameters. This is the reason why we did not tackle this problem here.

An important aspect of the results presented here is that in many cases we are able to determine Lipschitz constants when investigating Lipschitz (or Hölder) behaviour of $\mathcal{P}$ and $\mathcal{S}$. This fact is of importance in investigating conditioning of vector optimization problems. From the material of Chapter 8 we can deduce that strict efficiency and sharp as well as weakly sharp solutions are essential for stability of solutions. Moreover, the greater the constant β related to strict efficiency and the constant τ related to sharp (or weakly sharp) solutions, the greater the corresponding Lipschitz constants for $\mathcal{P}$ and $\mathcal{S}$.

It is an open problem to provide sufficient and necessary conditions for sharp solutions (and strictly efficient points) of higher orders as well as to analyse these notions from the point of view of general extremality scheme as proposed by Kruger [95].

Postscriptum:

> *Si les circonstances arrivent à être surmontées, être vaincues, la nature transporte la lutte du dehors au dedans et fait peu à peu changer assez notre cœur pour qu'il désire autre chose. . .*
>
> Marcel Proust, *A l'ombre des jeunes filles*

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