Introduction

This dissertation is devoted to a thorough investigation of the nonlinear wave equation in canonical form

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) = F(x,y,u(x,y))$$

with a smooth nonlinear function F on the right hand side.

We investigate solutions with distributions or other generalized functions as initial data; thus we must search for solutions in algebras containing the space of distributions which are invariant under nonlinear functions. We use the recent theories of generalized functions (J.-F. Colombeau [1985], Yu. V. Egorov [1990], M. Oberguggenberger [1992]) and particularly the $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras (J.-A. Marti [1998]–[2004], J.-A. Marti, S. P. Nuiro and V. S. Valmorin [1998b]). This study permits one to see the usefulness of algebras of generalized functions in cases where distribution theory turns out to be insufficient.

We search for a generalized solution u, in the sense to be defined later, to the following Cauchy problem (P) and Goursat problem (P'):

$$(P)\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = F(\cdot,\cdot,u), \\ u|_\gamma = \varphi, \\ \left.\dfrac{\partial u}{\partial y}\right|_\gamma = \psi, \end{cases} \qquad (P')\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = F(\cdot,\cdot,u), \\ u|_{(Ox)} = \varphi, \\ u|_\gamma = \psi. \end{cases}$$

Here φ and ψ are one-variable generalized functions. The notation $F(\cdot,\cdot,u)$ extends, with a meaning to be defined later, the expression $(x,y) \mapsto F(x,y,u(x,y))$ in the case where u is a generalized function of two variables x and y.

For the Cauchy problem the data are given along the monotonic curve γ of equation $y = f(x)$. We also study the case where the data are carried on a characteristic curve $\gamma = (Ox)$.

For the Goursat problem the initial values are given along a characteristic curve $C = (Ox)$, and along a monotonic curve γ of equation $x = g(y)$.

Sections 1 and 2 are devoted to the construction of global smooth solutions to both the Cauchy problem and the Goursat problem when the data are smooth. This is achieved by rewriting the differential equation as an integral equation and making a thorough investigation of the method of successive approximations (P. R. Garabedian [1964]). Several improvements to classical methods and results are needed to obtain precise estimates used in the later sections. Namely, the growth in the parameter ε of the families of solutions has to be known to choose the good algebraic structure to solve the regularized problems. So the results of those sections form an essential basis for the construction of generalized solutions.

Section 3 is devoted to the definition of the algebras of generalized functions and to the setup of an algebra of generalized functions, $\mathcal{A}(\mathbb{R}^2)$, adapted to the generalized Cauchy problem. The concept of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras introduced by J.-A. Marti [1998]–[2004] is an improvement and generalization of the algebras of J.-F. Colombeau [1985]. The theory of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras is built on three completely independent algebraic ($\mathcal{C}$ is any subring of generalized numbers and $\mathcal{E}$ any algebra) and topological ($\mathcal{P}$ is any compatible topology on $\mathcal{E}$) parameters and its philosophy is to adjust them to the given problem. These algebras are constructed as factor algebras of infinite products of locally convex topological spaces. In such algebras, we have good tools to deal with many nonlinear differential problems with irregular data (J.-A. Marti and S. P. Nuiro [1999], J.-A. Marti, S. P. Nuiro and V. S. Valmorin [1998a]).

The Colombeau algebra is invariant under superposition with polynomially bounded smooth maps. To cover the case of more general nonlinearities, other variants of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras are needed.

We introduce the notion of an algebra stable under F ($F \in \mathrm{C}^\infty(\mathbb{R}^3, \mathbb{R})$). For any generalized function u, we define the $\mathcal{D}'$-parametric singular spectrum of u (J.-A. Marti [1995], [1998], J.-A. Marti, S. P. Nuiro and V. S. Valmorin [1998b]). These tools allow us to tackle the generalized problems in Sections 4 and 5.

We take up again the formulation of the Cauchy problem but now φ and ψ are generalized functions. We search for a solution u, in $\mathcal{A}(\mathbb{R}^2)$, to this generalized Cauchy problem (P_G). After specifying the meaning of (P_G), we show that, if $\mathcal{A}(\mathbb{R}^2)$ is stable under F, if $\mathcal{A}(\mathbb{R})$ and $\mathcal{A}(\mathbb{R}^2)$ are built on the same ring $\mathcal{C} = A/I$ of generalized constants, and if the data of problem (P_G) satisfy the conditions $\varphi \in \mathcal{A}(\mathbb{R})$, $\psi \in \mathcal{A}(\mathbb{R})$, $f \in \mathrm{C}^\infty(\mathbb{R})$, then problem (P_G) has a unique solution u in $\mathcal{A}(\mathbb{R}^2)$. To prove existence, a representative can be constructed invoking the existence of smooth solutions from Section 1 and proving that it satisfies the required asymptotic estimates. To prove uniqueness, one has to show that the difference of two solutions is asymptotically negligible when this is true of the difference of the data. This again involves estimates as derived in Section 1.

The $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras give an efficient algebraic framework which permits a precise study of solutions. We make a qualitative study of the solutions. We describe their local and microlocal behavior and we study the propagation of their singularities. We show that the parametric singular support of the solution with bounded nonlinear function F is the same as the one of the homogeneous equation ($F = 0$). Then various special cases with distributions as data are studied, notably for $F = 0$ and $f(x) = ax$.

We can study a generalized Goursat problem in the same way. We extend the case of the degenerate Goursat problem solved by V. S. Valmorin [1995a], [1995b] to the general case in which the data are given along the x-axis and along another possibly characteristic curve.

We can then deal with the characteristic problems in Section 6. In that case, the formal calculus of partial derivatives on the manifold carrying the data meets a geometric obstruction which is difficult to get around. For characteristic linear problems, some results on existence, but not uniqueness, are proved in distributional framework (Yu. V. Egorov and M. A. Shubin [1993], L. Hörmander [1983]). Other results are proved (Gårding, Kotake, Leray, Wagschal, Hamada, Dunau) in the complex framework where the solutions may be holomorphic and may have ramifications around characteristic curves issuing from characteristics. However, we do not know any general answer in real analytical or C^∞ cases and for nonlinear problems (as in the present paper). For these cases, and even for linear cases, the characteristic problems are those where we "fall into the holes" of the canonical stratification as defined in the Shih Weishu theory (W. Shih [1986]). Furthermore, Shi Wei Hui [1992] shows that the Cauchy problem is not well posed for the Navier–Stokes equations, on the hyperplane $\{t=0\}$.

We extend some results of J.-A. Marti [2004] to general cases, by approaching some characteristic problems by some families of noncharacteristic problems and by interpreting the results algebraically.

We study the case where the data are given along the characteristic curve $\gamma = (Ox)$. This characteristic irregular Cauchy problem has no smooth solution (not even C^2) even if the data φ and ψ are smooth. We replace it by the family of noncharacteristic problems $(P_\varepsilon)_\varepsilon$ by moving the initial data to the curve γ_ε of equation $y = \varepsilon x$ as data. We also try to give a meaning to the family of solutions by interpreting it as generalized functions belonging to an appropriately defined algebra.

In the case of regular data, if u_ε is a solution to problem (P_ε), the family $(u_\varepsilon)_\varepsilon$ is a representative of a generalized function which belongs to the algebra $\mathcal{A}(\mathbb{R}^2)$. Then $u = [u_\varepsilon]$ is a generalized function that we consider as a generalized solution to the characteristic Cauchy problem (P_C).

We also give a meaning to the characteristic Cauchy problem (P_C) in the case where φ and ψ are themselves irregular data (for example some generalized functions), by replacing it by the family of noncharacteristic problems $(P_{(\varepsilon,\eta)})_{(\varepsilon,\eta)}$ in an appropriate algebra:

$$P_{(\varepsilon,\eta)} \begin{cases} \dfrac{\partial^2 u_{(\varepsilon,\eta)}}{\partial x \partial y}(x,y) = F(x,y,u_{(\varepsilon,\eta)}(x,y)), \\ u_{(\varepsilon,\eta)}(x,\varepsilon x) = \varphi_\eta(x), \\ \dfrac{\partial u_{(\varepsilon,\eta)}}{\partial y}(x,\varepsilon x) = \psi_\eta(x), \end{cases}$$

where $(\varphi_\eta)_\eta$ and $(\psi_\eta)_\eta$ are representatives of φ and ψ.

The parameter ε transforms the given problem into a noncharacteristic one and the parameter η regularizes the data. We build a two-parametric algebra in which the irregular characteristic problem is solved. If $u_{(\varepsilon,\eta)}$ is a solution to problem $P_{(\varepsilon,\eta)}$, the family $(u_{(\varepsilon,\eta)})_{(\varepsilon,\eta)}$ is a representative of a generalized function $u = [u_{(\varepsilon,\eta)}]$ that we consider as a generalized solution to the characteristic Cauchy problem (P_C).

For $F = 0$, we rediscover some results of J.-A. Marti [2004] from a general study with distributions as data.

1. Global smooth solutions to the Cauchy problem

Solution of the Cauchy problem for the semilinear wave equation whose nonlinearity satisfies a global Lipschitz condition, by means of successive approximation techniques, is well known (P. R. Garabedian [1964]). However, for the following study of generalized situation, we will need precise estimates for the case of smooth data, which is not sufficiently detailed in the available literature.

1.1. Formulation of the problem. We search for a solution u to the following Cauchy problem:

$$(P)\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = F(\cdot,\cdot,u), \\ u|_\gamma = \varphi, \\ \dfrac{\partial u}{\partial y}|_\gamma = \psi, \end{cases}$$

where $f, \varphi, \psi : \mathbb{R} \to \mathbb{R}$ are some smooth one-variable functions, γ is the curve of equation $y = f(x)$ and F is smooth in its arguments. In all cases the following hypothesis will be satisfied:

$$(H)\qquad \begin{cases} F \in \mathrm{C}^\infty(\mathbb{R}^3, \mathbb{R}), \\ \forall K \Subset \mathbb{R}^2, \sup_{(x,y)\in K; z\in\mathbb{R}} |\partial_z F(x,y,z)| < \infty, \\ f \text{ is defined and strictly increasing on } \mathbb{R} \text{ with image } \mathbb{R}, \\ \forall x \in \mathbb{R}, f'(x) \neq 0, \end{cases}$$

where the notation $K \Subset \mathbb{R}^2$ means that K is a compact subset of $\mathbb{R}^2$. We denote by (P_∞) the problem which consists in searching for a function $u \in \mathrm{C}^2(\mathbb{R}^2)$ satisfying

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) = F(x,y,u(x,y)), \tag{1.1}$$

$$u(x,f(x)) = \varphi(x), \tag{1.2}$$

$$\frac{\partial u}{\partial y}(x,f(x)) = \psi(x). \tag{1.3}$$

We denote by (P_i) the problem which consists in searching for a function $u \in \mathrm{C}^0(\mathbb{R}^2)$ satisfying

$$u(x,y) = u_0(x,y) - \iint\limits_{D(x,y,f)} F(\xi,\eta,u(\xi,\eta))\,d\xi\,d\eta, \tag{1.4}$$

where

$$u_0(x,y) = \chi(y) - \chi(f(x)) + \varphi(x)$$

and χ denotes a primitive of $\psi \circ f^{-1}$, with

$$D(x,y,f) = \begin{cases} \{(\xi,\eta) : f^{-1}(y) \le \xi \le x, y \le \eta \le f(\xi)\} & \text{if } y \le f(x), \\ \{(\xi,\eta) : x \le \xi \le f^{-1}(y), f(\xi) \le \eta \le y\} & \text{if } y \ge f(x). \end{cases}$$

THEOREM 1. *Let $u \in \mathrm{C}^0(\mathbb{R}^2)$. The function u is a solution to (P_∞) if and only if u is a solution to (P_i).*

Proof. The existence of f^{-1} is ensured by (H). Hypothesis (H) also ensures that the domain $D(x,y,f)$ is bounded. We consider the points $M(x,y)$, $P(f^{-1}(y),y)$, $Q(x,f(x))$, and the domain $D(x,y,f)$ is the "curvilinear triangle" MPQ. If u is solution to (P_∞), suppose that $y \geq f(x)$. We have

$$\iint\limits_{D(x,y,f)} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\,d\xi\,d\eta = \int\limits_{f(x)}^{y} \left(\int\limits_{x}^{f^{-1}(\eta)} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\,d\xi \right) d\eta.$$

Then

$$\begin{aligned}\iint\limits_{D(x,y,f)} \left(\frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\,d\xi \right) d\eta &= \int\limits_{f(x)}^{y} \frac{\partial u}{\partial y}(f^{-1}(\eta),\eta)\,d\eta - \int\limits_{f(x)}^{y} \frac{\partial u}{\partial y}(x,\eta)\,d\eta \\ &= \chi(y) - \chi(f(x)) - u(x,y) + \varphi(x),\end{aligned}$$

where χ denotes a primitive of $\psi \circ f^{-1}$. Then

$$u(x,y) = u_0(x,y) - \iint\limits_{D(x,y,f)} F(\xi,\eta,u(\xi,\eta))\,d\xi\,d\eta,$$

where $u_0(x,y) = \chi(y) - \chi(f(x)) + \varphi(x)$. We obtain the same result if we suppose $y \leq f(x)$. Thus u satisfies (P_i). If u satisfies (P_i), suppose that $y \geq f(x)$; we can write

$$u(x,y) = u_0(x,y) - \int\limits_{x}^{f^{-1}(y)} \left(\int\limits_{f(\xi)}^{y} F(\xi,\eta,u(\xi,\eta))\,d\eta \right) d\xi.$$

As $u \in \mathrm{C}^0(\mathbb{R}^2)$ we have

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial u_0}{\partial x}(x,y) + \int\limits_{f(x)}^{y} F(x,\eta,u(x,\eta))\,d\eta$$

and consequently

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)(x,y) = \frac{\partial^2 u_0}{\partial y \partial x}(x,y) + F(x,y,u(x,y)) = F(x,y,u(x,y)).$$

Let us calculate again $u(x,y)$ in the following way:

$$u(x,y) = u_0(x,y) - \int\limits_{f(x)}^{y} \left(\int\limits_{x}^{f^{-1}(\eta)} F(\xi,\eta,u(\xi,\eta))\,d\xi \right) d\eta.$$

As $u \in \mathrm{C}^0(\mathbb{R}^2)$ we have

$$\frac{\partial u}{\partial y}(x,y) = \frac{\partial u_0}{\partial y}(x,y) - \int\limits_{x}^{f^{-1}(y)} F(\xi,y,u(\xi,y))\,d\xi.$$

Then

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)(x,y) = \frac{\partial^2 u_0}{\partial x \partial y}(x,y) + F(x,y,u(x,y)) = F(x,y,u(x,y)).$$

Finally, the partial derivatives can be exchanged and we have

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) = F(x,y,u(x,y)).$$

Furthermore,

$$\begin{aligned} u(x,f(x)) &= u_0(x,f(x)) = \varphi(x), \\ \frac{\partial u}{\partial y}(x,f(x)) &= \frac{\partial u_0}{\partial y}(x,f(x)) = \psi \circ f^{-1}(f(x)) = \psi(x). \end{aligned}$$

These results are unchanged if we suppose $y \leq f(x)$, so u satisfies (P_∞). If u is of class C^1 then $(x,y) \mapsto F(x,y,u(x,y))$ is of class C^1. Then

$$W : (x,y) \mapsto u_0(x,y) - \int_x^{f^{-1}(y)} \Big(\int_{f(\xi)}^{y} F(\xi,\eta,u(\xi,\eta))\, d\eta \Big)\, d\xi$$

has a partial derivative with respect to x of class C^1, and

$$W : (x,y) \mapsto u_0(x,y) - \int_{f(x)}^{y} \Big(\int_x^{f^{-1}(\eta)} F(\xi,\eta,u(\xi,\eta))\, d\xi \Big)\, d\eta$$

has a partial derivative with respect to y of class C^1. As

$$\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial y}\right)(x,y) = F(x,y,u(x,y)) = \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial x}\right)(x,y)$$

is of class C^1 it follows that $u = W$ is of class C^2. We remark moreover that, if u is of class C^n, then $(x,y) \mapsto F(x,y,u(x,y))$ is of class C^n, therefore

$$W : (x,y) \mapsto u_0(x,y) - \int_x^{f^{-1}(y)} \Big(\int_{f(\xi)}^{y} F(\xi,\eta,u(\xi,\eta))\, d\eta \Big)\, d\xi$$

has a partial derivative with respect to x of class C^n, and

$$W : (x,y) \mapsto u_0(x,y) - \int_{f(x)}^{y} \Big(\int_x^{f^{-1}(\eta)} F(\xi,\eta,u(\xi,\eta))\, d\xi \Big)\, d\eta$$

has a partial derivative with respect to y of class C^n. As

$$\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial y}\right)(x,y) = F(x,y,u(x,y)) = \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial x}\right)(x,y)$$

is of class C^n we conclude that $u = W$ is of class C^{n+1}. By induction, u is therefore of class C^∞. ■

We have, of course, the following corollary.

COROLLARY 2. *If u is a solution to (P_i) (or to (P_∞)), then u belongs to $\mathrm{C}^\infty(\mathbb{R}^2)$.*

Remark 3 (Second order partial derivatives of u; these results will be used in Subsection 4.2). If u is solution to (P_i) we have

$$\frac{\partial u}{\partial x}(x,y)=\frac{\partial u_0}{\partial x}(x,y)+\int_{f(x)}^{y} F(x,\eta,u(x,\eta))\,d\eta.$$

It follows that

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}(x,y)=&\frac{\partial^2 u_0}{\partial x^2}(x,y)-f'(x)F(x,f(x),u(x,f(x)))\\&+\int_{f(x)}^{y}\left(\frac{\partial F}{\partial x}(x,\eta,u(x,\eta))+\frac{\partial F}{\partial z}(x,\eta,u(x,\eta))\,\frac{\partial u}{\partial x}(x,\eta)\right)d\eta.\end{aligned}$$

As

$$\frac{\partial u_0}{\partial x}(x,y)=-f'(x)\psi(x)+\varphi'(x)\quad\text{and}\quad u(x,f(x))=\varphi(x),$$

we find that

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}(x,y)=&-f''(x)\psi(x)-f'(x)\psi'(x)+\varphi\text{''}(x)-f'(x)F(x,f(x),\varphi(x))\\&+\int_{f(x)}^{y}\left(\frac{\partial F}{\partial x}(x,\eta,u(x,\eta))+\frac{\partial F}{\partial z}(x,\eta,u(x,\eta))\,\frac{\partial u}{\partial x}(x,\eta)\right)d\eta.\end{aligned}$$

Let us calculate again $u(x,y)$ in the following way:

$$u(x,y)=u_0(x,y)-\int_{f(x)}^{y}\Big(\int_{x}^{f^{-1}(\eta)} F(\xi,\eta,u(\xi,\eta))\,d\xi\Big)\,d\eta.$$

Starting from

$$\frac{\partial u}{\partial y}(x,y)=\frac{\partial u_0}{\partial y}(x,y)-\int_{x}^{f^{-1}(y)} F(\xi,y,u(\xi,y))\,d\xi,$$

we obtain

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2}(x,y)=&\frac{\partial^2 u_0}{\partial y^2}(x,y)-\left(\frac{1}{f'(f^{-1}(y))}\right)F(f^{-1}(y),y,\varphi(f^{-1}(y)))\\&-\int_{x}^{f^{-1}(y)}\left(\frac{\partial F}{\partial y}(\xi,y,u(\xi,y))+\frac{\partial F}{\partial z}(\xi,y,u(\xi,y))\frac{\partial u}{\partial y}(\xi,y)\right)d\xi.\end{aligned}$$

As $\frac{\partial u_0}{\partial y}(x,y)=\psi(f^{-1}(y))$, we have

$$\frac{\partial^2 u_0}{\partial y^2}(x,y)=\left(\frac{1}{f'(f^{-1}(y))}\right)\psi'(f^{-1}(y)).$$

1.2. Existence and uniqueness of solutions

Theorem 4. *From hypothesis (H) it follows that problem (P_∞) has a unique solution in* $C^\infty(\mathbb{R}^2)$.

Proof. According to Theorem 1, solving problem (P_∞) amounts to solving problem (P_i), that is, searching for $u \in \mathrm{C}^0(\mathbb{R}^2)$ satisfying (1.4). For every compact subset of $\mathbb{R}^2$, we can find $\lambda > 0$, large enough, so that this compact subset is contained in $K_\lambda = [-\lambda, \lambda] \times [f(-\lambda), f(\lambda)]$. Let us assume always that $y \geq f(x)$ and let us make the change of variables $X = x + \lambda$, $Y = y - f(-\lambda)$. The relation (1.4) can be written as

$$u(X-\lambda, Y+f(-\lambda)) = u_0(X-\lambda, Y+f(-\lambda)) - \iint\limits_{D(X-\lambda, Y+f(-\lambda), f)} F(\xi-\lambda, \eta+f(-\lambda), u(\xi-\lambda, \eta+f(-\lambda)))\, d\xi\, d\eta,$$

whose form is

$$U(X,Y) = U_0(X,Y) - \iint\limits_{\mathfrak{D}(X,Y,g)} \mathfrak{F}(\xi, \eta, U(\xi,\eta))\, d\xi\, d\eta, \tag{1.5}$$

with $g(X) = f(X-\lambda) - f(-\lambda)$; K_λ turns into the compact subset $Q_\lambda = [0, 2\lambda] \times [0, g(2\lambda)]$. The equation of (γ) can then be written as $Y = g(X)$ and $g(0) = 0$. So we now have $X \geq 0$ and $Y \geq g(X)$. According to hypothesis (H), we can put

$$m_\lambda = \sup_{(\xi,\eta)\in Q_\lambda;\, z\in\mathbb{R}} \left| \frac{\partial \mathfrak{F}}{\partial z}(\xi, \eta, z) \right|.$$

Let us consider the sequence $(U_n)_{n\in\mathbb{N}}$ of functions defined on $\mathbb{R}^2$ by

$$\forall n \in \mathbb{N}^*, \qquad U_n(X,Y) = U_0(X,Y) - \iint\limits_{\mathfrak{D}(X,Y,g)} \mathfrak{F}(\xi, \eta, U_{n-1}(\xi,\eta))\, d\xi\, d\eta.$$

For every compact subset $H \Subset \mathbb{R}^2$, let us put

$$\|U_0\|_{\infty,H} = \sup_{(x,y)\in H} |U_0(x,y)|.$$

According to the mean value theorem in integral form, we can write

$$\mathfrak{F}(\xi,\eta,t) - \mathfrak{F}(\xi,\eta,r) = (t-r) \int_0^1 \frac{\partial \mathfrak{F}}{\partial z}(\xi, \eta, r + \sigma(t-r))\, d\sigma, \tag{1.6}$$

hence for all $(\xi,\eta) \in \mathfrak{D}(X,Y,g)$,

$$\mathfrak{F}(\xi,\eta,U_0(\xi,\eta)) - \mathfrak{F}(\xi,\eta,0) = U_0(\xi,\eta) \int_0^1 \frac{\partial \mathfrak{F}}{\partial z}(\xi, \eta, \sigma U_0(\xi,\eta))\, d\sigma.$$

So

$$|\mathfrak{F}(\xi,\eta,U_0(\xi,\eta))| \leq |\mathfrak{F}(\xi,\eta,0)| + m_\lambda \|U_0\|_{\infty,Q_\lambda}.$$

Let us put

$$\Phi_\lambda = \|\mathfrak{F}(\cdot,\cdot,0)\|_{\infty,Q_\lambda} + m_\lambda \|U_0\|_{\infty,Q_\lambda},$$
$$\forall n \in \mathbb{N}^*, \qquad V_n = U_n - U_{n-1},$$

which implies

$$V_1(X,Y) = U_1(X,Y) - U_0(X,Y) = -\iint\limits_{\mathfrak{D}(X,Y,g)} \mathfrak{F}(\xi,\eta,U_0(\xi,\eta))\,d\xi\,d\eta,$$

$$|V_1(X,Y)| \le \iint\limits_{\mathfrak{D}(X,Y,g)} |\mathfrak{F}(\xi,\eta,U_0(\xi,\eta))|\,d\xi\,d\eta \le \Phi_\lambda A(X,Y),$$

where $A(X,Y) = \iint_{\mathfrak{D}(X,Y,g)} d\xi\,d\eta$ indicates the area of the domain $\mathfrak{D}(X,Y,g)$. We have

$$\begin{aligned}|V_2(X,Y)| &= |U_2(X,Y) - U_1(X,Y)| \\ &\le \iint\limits_{\mathfrak{D}(X,Y,g)} |\mathfrak{F}(\xi,\eta,U_0(\xi,\eta)) - \mathfrak{F}(\xi,\eta,U_1(\xi,\eta))|\,d\xi\,d\eta.\end{aligned}$$

Then using the relation (1.6), we obtain

$$\begin{aligned}&|\mathfrak{F}(\xi,\eta,U_0(\xi,\eta)) - \mathfrak{F}(\xi,\eta,U_1(\xi,\eta))| \\ &\quad\le |U_0(\xi,\eta) - U_1(\xi,\eta)|\left|\int_0^1 \frac{\partial}{\partial z}\mathfrak{F}(\xi,\eta,U_1(\xi,\eta) + \sigma(U_1(\xi,\eta) - U_0(\xi,\eta)))\,d\sigma\right|,\end{aligned}$$

and consequently

$$|\mathfrak{F}(\xi,\eta,U_0(\xi,\eta)) - \mathfrak{F}(\xi,\eta,U_1(\xi,\eta))| \le |V_1(\xi,\eta)|m_\lambda.$$

From this it may be deduced that

$$|V_2(X,Y)| \le m_\lambda \iint\limits_{\mathfrak{D}(X,Y,g)} |V_1(\xi,\eta)|\,d\xi\,d\eta \le m_\lambda\Phi_\lambda \iint\limits_{\mathfrak{D}(X,Y,g)} A(\xi,\eta)\,d\xi\,d\eta.$$

We can notice that $A(X,Y) \le (2\lambda - X)Y \le (2\lambda)Y$ and then

$$|V_2(X,Y)| \le m_\lambda\Phi_\lambda \int_0^Y \left(\int_0^{2\lambda} 2\lambda\eta\,d\xi\right)d\eta \le m_\lambda\Phi_\lambda(2\lambda)^2Y^2 2^{-1}.$$

Consequently,

$$\forall(\xi,\eta) \in \mathfrak{D}(X,Y,g), \qquad |V_2(\xi,\eta)| \le m_\lambda\Phi_\lambda(2\lambda)^2\eta^2 2^{-1}.$$

By induction, we obtain

$$|V_n(X,Y)| \le m_\lambda^{n-1}\Phi_\lambda\left((2\lambda)^n\,\frac{Y^n}{n!}\right).$$

Hence

$$\|V_n\|_{\infty,Q_\lambda} \le \frac{\Phi_\lambda[(2\lambda)m_\lambda g(2\lambda)]^n}{m_\lambda n!},$$

which ensures the uniform convergence of the series $\sum_{n\ge1} V_n$ on Q_λ and consequently on every compact subset of $\mathbb{R}^2$. From the equality $\sum_{k=1}^n V_k = U_n - U_0$ we deduce that the sequence $(U_n)_{n\in\mathbb{N}}$ converges uniformly on Q_λ to a function U. As every U_n is continuous, the uniform limit U is continuous on every compact subset Q_λ, so on $\mathbb{R}^2$. Let us put

$\varepsilon_n(X,Y) = U(X,Y) - U_n(X,Y)$. Then

$$U(X,Y) - U_0(X,Y) + \iint\limits_{\mathfrak{D}(X,Y,g)} \mathfrak{F}(\xi,\eta,U(\xi,\eta))\,d\xi\,d\eta$$
$$= U(X,Y) - U_n(X,Y) + \Big(U_n(X,Y) - U_0(X,Y) + \iint\limits_{\mathfrak{D}(X,Y,g)} \mathfrak{F}(\xi,\eta,U(\xi,\eta))\,d\xi\,d\eta\Big)$$
$$= \varepsilon_n(X,Y) + \iint\limits_{\mathfrak{D}(X,Y,g)} (\mathfrak{F}(\xi,\eta,U(\xi,\eta)) - \mathfrak{F}(\xi,\eta,U_{n-1}(\xi,\eta)))\,d\xi\,d\eta.$$

As for all $(\xi,\eta) \in \mathfrak{D}(X,Y,g)$,

$$|\mathfrak{F}(\xi,\eta,U(\xi,\eta)) - \mathfrak{F}(\xi,\eta,U_n(\xi,\eta))| \leq |U(\xi,\eta) - U_{n-1}(\xi,\eta)|m_\lambda,$$

the second member above is bounded by

$$\sup_{(X,Y)\in Q_\lambda} |\varepsilon_n(X,Y)| + m_\lambda \sup_{(X,Y)\in Q_\lambda} A(X,Y) \sup_{(X,Y)\in Q_\lambda} |U(X,Y) - U_{n-1}(X,Y)|,$$

that is, by

$$\sup_{(X,Y)\in Q_\lambda} |\varepsilon_n(X,Y)| + m_\lambda 2\lambda g(2\lambda) \sup_{(X,Y)\in Q_\lambda} |\varepsilon_{n-1}(X,Y)|$$

whose limit is 0 when n tends to $+\infty$. It follows that

$$U(X,Y) = U_0(X,Y) - \iint\limits_{\mathfrak{D}(X,Y,g)} \mathfrak{F}(\xi,\eta,U(\xi,\eta))\,d\xi\,d\eta$$

for $(X,Y) \in Q_\lambda \cap \{(X,Y) : Y \geq g(X)\} = Q_\lambda^+$.

Let us show the uniqueness of the solution. Let W be another solution to (1.5). Putting $\Delta = W - U$, we obtain

$$\Delta(X,Y) = \iint\limits_{\mathfrak{D}(X,Y,g)} (-\mathfrak{F}(\xi,\eta,W(\xi,\eta)) + \mathfrak{F}(\xi,\eta,U(\xi,\eta)))\,d\xi\,d\eta.$$

Let $(X,Y) \in Q_\lambda$. As $\mathfrak{D}(X,Y,g) \subset Q_\lambda$, we have

$$|\Delta(X,Y)| \leq \iint\limits_{\mathfrak{D}(X,Y,g)} m_\lambda |W(\xi,\eta) - U(\xi,\eta)|\,d\xi\,d\eta \leq m_\lambda \iint\limits_{\mathfrak{D}(X,Y,g)} |\Delta(\xi,\eta)|\,d\xi\,d\eta.$$

As $Y \geq g(X)$,

$$|\Delta(X,Y)| \leq m_\lambda \int\limits_X^{g^{-1}(Y)} \int\limits_{g(X)}^{Y} |\Delta(\xi,\eta)|\,d\eta\,d\xi \leq m_\lambda \int\limits_0^Y \Big(\int\limits_0^{2\lambda} \sup_{\xi\in[0,2\lambda]} |\Delta(\xi,\eta)|\,d\xi\Big)\,d\eta.$$

For every $Y \in [0, g(2\lambda)]$, let us put

$$E(Y) = \sup_{\xi\in[0,2\lambda]} |\Delta(\xi,Y)|.$$

Then

$$|\Delta(X,Y)| \leq m_\lambda 2\lambda \Big|\int\limits_0^Y E(\eta)\,d\eta\Big|;$$

it follows that

$$\forall Y \in [0, g(2\lambda)], \qquad E(Y) \le m_\lambda 2\lambda \Big| \int_0^Y E(\eta)\, d\eta \Big|.$$

In this way, by applying Gronwall's lemma, we get $E = 0$, hence $\Delta = 0$, which proves the uniqueness of U on Q_λ^+. Then putting $v_\lambda(x, y) = U(x + \lambda, y - f(-\lambda))$, it follows that v_λ is the unique solution to (1.4) on $K_\lambda \cap \{(x, y) : y \ge f(x)\} = K_\lambda^+$.

Now consider the case $y \le f(x)$. Let us make the change of variables $X = -x + \lambda$, $Y = -y + f(\lambda)$. We put $D' = D(-X + \lambda, -Y + f(\lambda), f)$; then

$$\begin{aligned} u(-X + \lambda, -Y + f(\lambda)) &= u_0(-X + \lambda, -Y + f(\lambda)) \\ &\qquad - \iint_{D'} F(-\xi + \lambda, -\eta + f(\lambda), u(-\xi + \lambda, -\eta + f(\lambda)))\, d\xi\, d\eta, \end{aligned}$$

whose form is

$$W(X, Y) = W_0(X, Y) - \iint_{\mathfrak{D}(X,Y,g)} \mathfrak{F}(\xi, \eta, W(\xi, \eta))\, d\xi\, d\eta$$

and $g(X) = f(\lambda) - f(\lambda - X)$; K_λ turns into the compact subset $Q_\lambda = [0, 2\lambda] \times [0, g(2\lambda)]$. As $y \le f(x)$, we have $f(\lambda) - y \ge f(\lambda) - f(x)$. Then $Y \ge f(\lambda) - f(\lambda - X)$, that is to say, $Y \ge g(X)$. So everything boils down to the case $X \ge 0$, $Y \ge g(X)$, with which we can deal as previously. It follows that

$$w_\lambda(x, y) = W(-x + \lambda, -y + f(\lambda))$$

is a solution to (1.4) on

$$K_\lambda \cap \{(x, y) : y \le f(x)\} = K_\lambda^-.$$

From the continuity of U on Q_λ^+ and of W on Q_λ^- we have the continuity of v_λ on K_λ^+ and of w_λ on K_λ^-. Moreover, v_λ and w_λ agree on γ because $v_\lambda(x, f(x)) = w_\lambda(x, f(x)) = \varphi(x)$. Finally, if we put

$$u_\lambda(x, y) = \begin{cases} v_\lambda(x, y) & \text{for } (x, y) \in K_\lambda^+, \\ w_\lambda(x, y) & \text{for } (x, y) \in K_\lambda^-, \end{cases}$$

then u_λ is the unique continuous solution to (P_i) on K_λ.

It remains to prove that the method actually gives a continuous global solution u to (1.4) on $\mathbb{R}^2$, that is, which satisfies (P_i). If $\lambda_2 > \lambda_1$ then $K_{\lambda_1} \subset K_{\lambda_2}$, so, we must prove that $u_{\lambda_2}|_{K_{\lambda_1}} = u_{\lambda_1}$. But for all $(x, y) \in K_{\lambda_2}$,

$$u_{\lambda_2}(x, y) = u_0(x, y) - \iint_{D(x,y,f)} F(\xi, \eta, u_{\lambda_2}(\xi, \eta))\, d\xi\, d\eta$$

and we have this equality, all the more so, for $(x, y) \in K_{\lambda_1}$. So we have

$$u_{\lambda_2}|_{K_{\lambda_1}}(x, y) = u_0(x, y) - \iint_{D(x,y,f)} F(\xi, \eta, u_{\lambda_2|K_{\lambda_1}}(\xi, \eta))\, d\xi\, d\eta.$$

In other words, $u_{\lambda_2}|_{K_{\lambda_1}}$ satisfies (1.4) on K_{λ_1} and so coincides on it with its unique solution u_{λ_1}. For every $(x,y)\in\mathbb{R}^2$ we can thus put

$$u(x,y)=u_\lambda(x,y)=u_0(x,y)-\iint\limits_{D(x,y,f)} F(\xi,\eta,u(\xi,\eta))\,d\xi\,d\eta, \tag{1.7}$$

where u_λ satisfies (1.4) on K_λ and $(x,y)\in K_\lambda$.

The definition of u in (1.7), being independent of the compact subset K_λ, finally gives the unique global solution to (P_i) or (P_∞). ■

In Section 4, we will need the estimates specified by the following result.

Proposition 5. *With the previous notations, for every compact subset $K\Subset\mathbb{R}^2$, there exists a compact subset $K_\lambda\Subset\mathbb{R}^2$ containing K such that*

$$m_\lambda=\sup_{(x,y)\in K_\lambda;t\in\mathbb{R}}\left|\frac{\partial F}{\partial z}(x,y,t)\right|;\qquad \varPhi_\lambda=\|F(\cdot,\cdot,0)\|_{\infty,K_\lambda}+m_\lambda\|u_0\|_{\infty,K_\lambda}; \tag{1.8}$$

$$\|u\|_{\infty,K}\le\|u\|_{\infty,K_\lambda}\le\|u_0\|_{\infty,K_\lambda}+\frac{\varPhi_\lambda}{m_\lambda}\exp[2\lambda m_\lambda(f(\lambda)-f(-\lambda))]. \tag{1.9}$$

Proof. We have clearly

$$m_\lambda=\sup_{(\xi,\eta)\in Q_\lambda;t\in\mathbb{R}}\left|\frac{\partial\mathfrak{F}}{\partial z}(\xi,\eta,t)\right|=\sup_{(x,y)\in K_\lambda;t\in\mathbb{R}}\left|\frac{\partial F}{\partial z}(x,y,t)\right|;$$
$$\varPhi_\lambda=\|\mathfrak{F}(\cdot,\cdot,0)\|_{\infty,Q_\lambda}+m_\lambda\|U_0\|_{\infty,Q_\lambda}=\|F(\cdot,\cdot,0)\|_{\infty,K_\lambda}+m_\lambda\|u_0\|_{\infty,K_\lambda}.$$

Keeping the previous notations, we have

$$u_n(x,y)=u_0(x,y)-\iint\limits_{D(x,y,f)} F(\xi,\eta,u_{n-1}(\xi,\eta))\,d\xi\,d\eta,\qquad n\ge1,$$

$$u_{n,\lambda}(x,y)=\begin{cases} v_{n,\lambda}(x,y) & \text{for } (x,y)\in K_\lambda^+,\\ w_{n,\lambda}(x,y) & \text{for } (x,y)\in K_\lambda^-.\end{cases}.$$

As

$$U_n(X,Y)=U_0(X,Y)-\iint\limits_{\mathfrak{D}(X,Y,g)} \mathfrak{F}(\xi,\eta,U_{n-1}(\xi,\eta))\,d\xi\,d\eta,$$
$$\varPhi_\lambda=\|\mathfrak{F}(\cdot,\cdot,0)\|_{\infty,Q_\lambda}+m_\lambda\|U_0\|_{\infty,Q_\lambda},$$
$$V_n=U_n-U_{n-1},$$

where K_λ is mapped by g into the compact subset $Q_\lambda=[0,2\lambda]\times[0,g(2\lambda)]$. According to the proof of Theorem 4, we have

$$\forall n\in\mathbb{N}^*,\qquad \|V_n\|_{\infty,Q_\lambda}\le\frac{\varPhi_\lambda[m_\lambda(2\lambda)g(2\lambda)]^n}{m_\lambda n!},$$

and consequently,

$$\|U\|_{\infty,Q_\lambda}\le\|U_0\|_{\infty,Q_\lambda}+\sum_{n=1}^{\infty}\|V_n\|_{\infty,Q_\lambda}\le\|U_0\|_{\infty,Q_\lambda}+\frac{\varPhi_\lambda}{m_\lambda}\exp[m_\lambda(2\lambda)g(2\lambda)].$$

Furthermore, $g(2\lambda)=f(\lambda)-f(-\lambda)$. From the relations

$$\begin{cases}\|v_\lambda\|_{\infty,K_\lambda^+}=\|U\|_{\infty,Q_\lambda},\\ \|w_\lambda\|_{\infty,K_\lambda^-}=\|W\|_{\infty,Q_\lambda},\end{cases}\quad \begin{cases}\|u_0\|_{\infty,K_\lambda^+}=\|U_0\|_{\infty,Q_\lambda},\\ \|u_0\|_{\infty,K_\lambda^-}=\|W_0\|_{\infty,Q_\lambda},\end{cases}\quad u_\lambda=\begin{cases} v_\lambda & \text{on } K_\lambda^+,\\ w_\lambda & \text{on } K_\lambda^-,\end{cases}$$

it may be deduced that

$$\|u\|_{\infty,K_\lambda^+} \leq \|u_0\|_{\infty,K_\lambda^+} + \frac{\Phi_\lambda}{m_\lambda}\exp[m_\lambda(2\lambda)(f(\lambda)-f(-\lambda))],$$

and, in the same way,

$$\|u\|_{\infty,K_\lambda^-} \leq \|u_0\|_{\infty,K_\lambda^-} + \frac{\Phi_\lambda}{m_\lambda}\exp[m_\lambda(2\lambda)(f(\lambda)-f(-\lambda))].$$

So

$$\|u\|_{\infty,K_\lambda} \leq \|u_0\|_{\infty,K_\lambda} + \frac{\Phi_\lambda}{m_\lambda}\exp[m_\lambda(2\lambda)(f(\lambda)-f(-\lambda))].$$

As $\|u\|_{\infty,K} \leq \|u\|_{\infty,K_\lambda}$, the previous inequality implies the conclusion (1.9). ■

2. Global smooth solutions to the Goursat problem

2.1. Formulation of the problem. We search for a solution u to the following Goursat problem:

$$(P')\begin{cases} \dfrac{\partial^2 u}{\partial x\partial y} = F(\cdot,\cdot,u), \\ u|_{(Ox)} = \varphi, \\ u|_\gamma = \psi, \end{cases}$$

where $g,\varphi,\psi : \mathbb{R}\to\mathbb{R}$ are some smooth one-variable functions with $\psi(0) = \varphi(g(0))$, γ is the curve of equation $x = g(y)$ and F is smooth in its arguments. In all cases the following hypothesis will be satisfied:

$$(H')\qquad\begin{cases} F\in \mathrm{C}^\infty(\mathbb{R}^3,\mathbb{R}), \\ \forall K \Subset \mathbb{R}^2, \sup_{(x,y)\in K; z\in\mathbb{R}} |\partial_z F(x,y,z)| < \infty, \\ g \text{ is increasing on } \mathbb{R}. \end{cases}$$

We denote by (P'_∞) the problem which consists in searching for a function $u\in \mathrm{C}^2(\mathbb{R})$ satisfying

$$\frac{\partial^2 u}{\partial x\partial y}(x,y) = F(x,y,u(x,y)), \tag{2.1}$$

$$u(x,0) = \varphi(x), \tag{2.2}$$

$$u(g(y),y) = \psi(y). \tag{2.3}$$

We denote by (P'_i) the problem which consists in searching for a function $u\in \mathrm{C}^0(\mathbb{R})$ satisfying

$$u(x,y) = u_0(x,y) + \iint\limits_{D(x,y,g)} F(\xi,\eta,u(\xi,\eta))\,d\xi\,d\eta, \tag{2.4}$$

where

$$u_0(x,y) = \psi(y) + \varphi(x) - \varphi(g(y)),$$

with

$$D(x,y,g) = \begin{cases} \{(\xi,\eta) : g(y) \le \xi \le x, 0 \le \eta \le y\} & \text{if } g(y) \le x \text{ and } 0 \le y, \\ \{(\xi,\eta) : x \le \xi \le g(y), 0 \le \eta \le y\} & \text{if } g(y) \ge x \text{ and } 0 \le y, \\ \{(\xi,\eta) : x \le \xi \le g(y), y \le \eta \le 0\} & \text{if } g(y) \ge x \text{ and } y \le 0, \\ \{(\xi,\eta) : g(y) \le \xi \le x, y \le \eta \le 0\} & \text{if } g(y) \le x \text{ and } y \le 0. \end{cases}$$

THEOREM 6. *Let $u \in \mathrm{C}^0(\mathbb{R}^2)$. The function u is a solution to (P'_∞) if and only if u is a solution to (P'_i).*

Proof. Hypothesis (H') ensures that the domain $D(x,y,g)$ is bounded. We consider the points $M(x,y)$, $N(x,0)$, $P(g(y),y)$, $Q(g(y),0)$. Let us suppose first $0 \le y$ and $g(y) \le x$. Then $D(x,y,g)$ is the rectangle $PQNM$. We have

$$\iint\limits_{D(x,y,g)} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\, d\xi\, d\eta = \int\limits_{g(y)}^{x} \Big(\int\limits_{0}^{y} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\, d\eta \Big)\, d\xi.$$

Then

$$\begin{aligned} \iint\limits_{D(x,y,g)} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\, d\xi\, d\eta &= \int\limits_{g(y)}^{x} \frac{\partial u}{\partial x}(\xi,y)\, d\xi - \int\limits_{g(y)}^{x} \frac{\partial u}{\partial x}(\xi,0))\, d\xi \\ &= [u(\xi,y)]_{g(y)}^{x} - [\varphi(\xi)]_{g(y)}^{x} \\ &= u(x,y) - \psi(y) - \varphi(x) + \varphi(g(y)). \end{aligned}$$

We deduce that

$$u(x,y) = u_0(x,y) + \iint\limits_{D(x,y,g)} F(\xi,\eta,u(\xi,\eta))\, d\xi\, d\eta,$$

where

$$u_0(x,y) = \psi(y) + \varphi(x) - \varphi(g(y)).$$

So we have $u_0(x,0) = \psi(0) + \varphi(x) - \varphi(g(0))$ and

$$u_0(g(y),y) = \psi(y) + \varphi(g(y)) - \varphi(g(y)) = \psi(y).$$

It follows that $u(x,0) = \varphi(x)$ and $u(g(y),y) = \psi(y)$. So u is a solution to (P'_i). To calculate

$$\iint\limits_{D(x,y,g)} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\, d\xi\, d\eta$$

we must consider four cases:

Case (1): $(0 \le y$ and $g(y) \le x)$, Case (2): $(0 \le y$ and $x \le g(y))$,
Case (3): $(y \le 0$ and $x \le g(y))$, Case (4): $(y \le 0$ and $g(y) \le x)$.

Let us briefly consider the other cases.

Case (2): If $0 \le y$ and $x \le g(y)$, then

$$\iint\limits_{D(x,y,g)} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\, d\xi\, d\eta = \int\limits_{x}^{g(y)} \Big(\int\limits_{0}^{y} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\, d\eta \Big)\, d\xi = -\int\limits_{g(y)}^{x} \Big(\int\limits_{0}^{y} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\, d\eta \Big) d\xi.$$

Case (3): If $x \leq g(y)$ and $y \leq 0$, then

$$\iint\limits_{D(x,y,g)} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\, d\xi\, d\eta = \int\limits_{x}^{g(y)} \left(\int\limits_{y}^{0} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\, d\eta \right) d\xi = \int\limits_{g(y)}^{x} \left(\int\limits_{0}^{y} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\, d\eta \right) d\xi.$$

Case (4): If $y \leq 0$ and $g(y) \leq x$, then

$$\iint\limits_{D(x,y,g)} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\, d\xi\, d\eta = \int\limits_{g(y)}^{x} \left(\int\limits_{y}^{0} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\, d\eta \right) d\xi = -\int\limits_{g(y)}^{x} \left(\int\limits_{0}^{y} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\, d\eta \right) d\xi.$$

If u satisfies (P_i'), assume that $g(y) \leq x$ and $0 \leq y$. We can write

$$u(x,y) = u_0(x,y) + \int\limits_{g(y)}^{x} \left(\int\limits_{0}^{y} F(\xi,\eta,u(\xi,\eta))\, d\eta \right) d\xi,$$

so

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial u_0}{\partial x}(x,y) + \int\limits_{0}^{y} F(x,\eta,u(x,\eta))\, d\eta$$

and consequently,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)(x,y) = \frac{\partial^2 u_0}{\partial y \partial x}(x,y) + F(x,y,u(x,y)) = F(x,y,u(x,y)).$$

Let us calculate again $u(x,y)$ in the following way:

$$u(x,y) = u_0(x,y) + \int\limits_{0}^{y} \left(\int\limits_{g(y)}^{x} F(\xi,\eta,u(\xi,\eta))\, d\xi \right) d\eta.$$

We have

$$\frac{\partial u}{\partial y}(x,y) = \frac{\partial u_0}{\partial y}(x,y) + \int\limits_{g(y)}^{x} F(\xi,y,u(\xi,y))\, d\xi - g'(y) \int\limits_{0}^{y} F(g(y),\eta,u(g(y),\eta))\, d\eta,$$

hence

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)(x,y) = \frac{\partial^2 u_0}{\partial x \partial y}(x,y) + F(x,y,u(x,y)) = F(x,y,u(x,y)).$$

Finally, the partial derivatives can be exchanged and we have

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) = F(x,y,u(x,y)).$$

Furthermore $u(g(y),y) = u_0(g(y),y) = \psi(y)$ and $u(x,0) = u_0(x,0) = \varphi(x)$. These results are unchanged if we suppose $x \leq g(y)$ and $0 \leq y$, so u actually satisfies (P_∞'). If u is of class C^1, the function $(x,y) \mapsto F(x,y,u(x,y))$ is of class C^1, so

$$W : (x,y) \mapsto u_0(x,y) + \int\limits_{g(y)}^{x} \left(\int\limits_{0}^{y} F(\xi,\eta,u(\xi,\eta))\, d\eta \right) d\xi$$

has a partial derivative with respect to x of class C^1, and

$$W : (x,y) \mapsto u_0(x,y) + \int\limits_{0}^{y} \left(\int\limits_{g(y)}^{x} F(\xi,\eta,u(\xi,\eta))\, d\xi \right) d\eta,$$

has a partial derivative with respect to y of class C^1. As

$$\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial y}\right)(x,y) = F(x,y,u(x,y)) = \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial x}\right)(x,y),$$

it follows that $u = W$ is of class C^2. We remark moreover that, if u is of class C^n, the function $(x,y) \mapsto F(x,y,u(x,y))$ is of class C^n,

$$W : (x,y) \mapsto u_0(x,y) + \int_{g(y)}^{x} \Big(\int_0^y F(\xi,\eta,u(\xi,\eta))\,d\eta\Big)\,d\xi$$

has a partial derivative with respect to x of class C^n, and

$$W : (x,y) \mapsto u_0(x,y) + \int_0^y \Big(\int_{g(y)}^{x} F(\xi,\eta,u(\xi,\eta))\,d\xi\Big)\,d\eta$$

has a partial derivative with respect to y of class C^n. As

$$\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial y}\right)(x,y) = F(x,y,u(x,y)) = \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial x}\right)(x,y)$$

is of class C^n it follows that $u = W$ is of class C^{n+1}. By induction u is therefore of class C^∞. ∎

We have, of course, the following corollary.

Corollary 7. *If u is a solution to (P_i') (or to (P_∞')), then u belongs to $\mathrm{C}^\infty(\mathbb{R}^2)$.*

Remark 8. (Second order partial derivatives of u; these results will be used in Subsection 5.2). Let us assume that u is a solution to (P_i), $g(y) \le x$ and $0 \le y$. Let us remember that

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial u_0}{\partial x}(x,y) + \int_0^y F(x,\eta,u(x,\eta))\,d\eta.$$

As $\frac{\partial^2 u_0}{\partial x^2}(x,y) = \varphi''(x)$, we find that

$$\frac{\partial^2 u}{\partial x^2}(x,y) = \varphi''(x) + \int_0^y \left(\frac{\partial F}{\partial x}(x,\eta,u(x,\eta)) + \frac{\partial F}{\partial z}(x,\eta,u(x,\eta))\frac{\partial u}{\partial x}(x,\eta)\right)d\eta.$$

We calculate again $u(x,y)$ in the following way:

$$u(x,y) = u_0(x,y) + \int_0^y \Big(\int_{g(y)}^{x} F(\xi,\eta,u(\xi,\eta))\,d\xi\Big)\,d\eta.$$

Starting from

$$\frac{\partial u}{\partial y}(x,y) = \frac{\partial u_0}{\partial y}(x,y) + \int_{g(y)}^{x} F(\xi,y,u(\xi,y))\,d\xi - g'(y)\int_0^y F(g(y),\eta,u(g(y),\eta))\,d\eta$$

we obtain

$$\frac{\partial^2 u}{\partial y^2}(x,y) = \frac{\partial^2 u_0}{\partial y^2}(x,y) - 2g'(y)F(g(y),y,u(g(y),y)) - \int\limits_x^{g(y)} \left(\frac{\partial F}{\partial y}(\xi,y,u(\xi,y)) + \frac{\partial F}{\partial z}(\xi,y,u(\xi,y))\frac{\partial u}{\partial y}(\xi,y) \right) d\xi.$$

Since

$$\frac{\partial u_0}{\partial y}(x,y) = \psi'(y) - [g'(y)]\varphi'(g(y)),$$

hence

$$\frac{\partial^2 u_0}{\partial y^2}(x,y) = \psi''(y) - [g''(y)\varphi'(g(y)) + (g'(y))^2\varphi''(g(y))].$$

2.2. Existence and uniqueness of solutions

THEOREM 9. *Under hypothesis (H'), problem (P'_∞) has a unique solution u in* $\mathrm{C}^\infty(\mathbb{R}^2)$.

Proof. Let us assume that $0 \le y,\ g(y) \le x$. According to Theorem 6, solving problem (P'_∞) amounts to solving problem (P'_i), that is, searching for $u \in \mathrm{C}^0(\mathbb{R}^2)$ satisfying (2.4). For every compact subset of $\mathbb{R}^2$, we can find λ, large enough, so that this compact subset is contained in $K_\lambda = [g(-\lambda), g(\lambda)] \times [-\lambda, \lambda]$. Let us put, in accordance with hypothesis (H'),

$$m_\lambda = \sup_{(\xi,\eta)\in K_\lambda; z\in\mathbb{R}} \left| \frac{\partial F}{\partial z}(\xi,\eta,z) \right|.$$

Let us consider the sequence $(u_n)_{n\in\mathbb{N}}$ of functions defined on $\mathbb{R}^2$ by

$$\forall n \in \mathbb{N}^*, \qquad u_n(x,y) = u_0(x,y) + \iint\limits_{D(x,y,g)} F(\xi,\eta,u_{n-1}(\xi,\eta))\,d\xi\,d\eta.$$

For every compact subset $H \Subset \mathbb{R}^2$, let us put

$$\|u_0\|_{\infty,H} = \sup_{(x,y)\in H} |u_0(x,y)|.$$

According to the mean value theorem in integral form, we can write

$$F(\xi,\eta,t) - F(\xi,\eta,r) = (t-r)\int\limits_0^1 \frac{\partial F}{\partial z}(\xi,\eta,r+\sigma(t-r))\,d\sigma, \tag{2.5}$$

hence, for every $(\xi,\eta) \in D(x,y,g)$, we have

$$F(\xi,\eta,u_0(\xi,\eta)) - F(\xi,\eta,0) = u_0(\xi,\eta)\int\limits_0^1 \frac{\partial F}{\partial z}(\xi,\eta,\sigma u_0(\xi,\eta))\,d\sigma$$

and so

$$|F(\xi,\eta,u_0(\xi,\eta))| \le |F(\xi,\eta,0)| + |u_0(\xi,\eta)|\int\limits_0^1 m_\lambda\,d\sigma \le |F(\xi,\eta,0)| + m_\lambda\|u_0\|_{\infty,K_\lambda}.$$

Let us put

$$\begin{aligned}&\Phi_\lambda = \|F(\cdot,\cdot,0)\|_{\infty,K_\lambda} + m_\lambda \|u_0\|_{\infty,K_\lambda},\\ &\forall n \in \mathbb{N}^*, \qquad V_n = u_n - u_{n-1}.\end{aligned}$$

With these notations we have

$$V_1(x,y) = u_1(x,y) - u_0(x,y) = \iint\limits_{D(x,y,g)} F(\xi,\eta,u_0(\xi,\eta))\,d\xi\,d\eta$$

and so

$$|V_1(x,y)| \le \iint\limits_{D(x,y,g)} |F(\xi,\eta,u_0(\xi,\eta))|\,d\xi\,d\eta \le \Phi_\lambda A(x,y),$$

where $A(x,y) = \iint_{D(x,y,g)} d\xi\,d\eta$ indicates the area of the domain $D(x,y,g)$. Similarly, we also have

$$|V_2(x,y)| = |u_2(x,y) - u_1(x,y)| \le \iint\limits_{D(x,y,g)} |F(\xi,\eta,u_1(\xi,\eta)) - F(\xi,\eta,u_0(\xi,\eta))|\,d\xi\,d\eta.$$

Then using the relation (2.5), we obtain

$$\begin{aligned}&|F(\xi,\eta,u_1(\xi,\eta)) - F(\xi,\eta,u_0(\xi,\eta))|\\ &\qquad \le |u_1(\xi,\eta) - u_0(\xi,\eta)| \left| \int_0^1 \frac{\partial}{\partial z} F(\xi,\eta,u_1(\xi,\eta) + \sigma(u_1(\xi,\eta) - u_0(\xi,\eta)))\,d\sigma \right|\\ &\qquad \le |V_1(\xi,\eta)| m_\lambda.\end{aligned}$$

We deduce that

$$|V_2(x,y)| \le m_\lambda \iint\limits_{D(x,y,g)} |V_1(\xi,\eta)|\,d\xi\,d\eta \le m_\lambda \Phi_\lambda \iint\limits_{D(x,y,g)} A(\xi,\eta)\,d\xi\,d\eta.$$

Putting $2\lambda' = g(\lambda) - g(-\lambda)$, we have $A(x,y) \le 2\lambda' y$ and then

$$|V_2(x,y)| \le m_\lambda \Phi_\lambda \int_0^y \Big(\int_0^{2\lambda} 2\lambda' \eta\,d\xi \Big)\,d\eta \le m_\lambda \Phi_\lambda ((2\lambda')^2 y^2 2^{-1}).$$

Consequently, for all $(\xi,\eta) \in D(x,y,g)$,

$$|V_2(\xi,\eta)| \le m_\lambda \Phi_\lambda \left((2\lambda')^2 \frac{\eta^2}{2} \right).$$

It follows by induction that

$$|V_n(x,y)| \le m_\lambda^{n-1} \Phi_\lambda \left((2\lambda')^n \frac{y^n}{n!} \right).$$

Hence

$$\|V_n\|_{\infty,K_\lambda} \le \frac{\Phi_\lambda [(2\lambda') m_\lambda \lambda]^n}{m_\lambda n!}$$

which ensures the uniform convergence of the series $\sum_{n\ge 1} V_n$ on K_λ, and, consequently, on every compact subset of $\mathbb{R}^2$. We have $\sum_{k=1}^n V_k = u_n - u_0$, so the sequence $(u_n)_{n\in\mathbb{N}}$ converges uniformly on K_λ to a function u. As every u_n is continuous, the uniform limit u is continuous on every compact subset K_λ, so on $\mathbb{R}^2$. Let us put $\varepsilon_n(x,y) = u(x,y) - u_n(x,y)$; then

$$\begin{aligned} u(x,y) - u_0(x,y) - \iint\limits_{D(x,y,g)} F(\xi,\eta,u(\xi,\eta))\,d\xi\,d\eta \\ = u(x,y) - u_n(x,y) + \Big(u_n(x,y) - u_0(x,y) - \iint\limits_{D(x,y,g)} F(\xi,\eta,u(\xi,\eta))\,d\xi\,d\eta\Big) \\ = \varepsilon_n(x,y) - \iint\limits_{D(x,y,g)} (F(\xi,\eta,u(\xi,\eta)) - F(\xi,\eta,u_{n-1}(\xi,\eta)))\,d\xi\,d\eta. \end{aligned}$$

As for every $(\xi,\eta) \in D(x,y,g)$,

$$|F(\xi,\eta,u(\xi,\eta)) - F(\xi,\eta,u_n(\xi,\eta))| \le |u(\xi,\eta) - u_{n-1}(\xi,\eta)|m_\lambda,$$

the second member of the above equality is bounded by

$$\sup_{(x,y)\in K_\lambda} |\varepsilon_n(x,y)| + m_\lambda \sup_{(x,y)\in K_\lambda} A(x,y)[\sup_{(x,y)\in K_\lambda} |u(x,y) - u_{n-1}(x,y)|],$$

that is, by

$$\sup_{(x,y)\in K_\lambda} |\varepsilon_n(x,y)| + m_\lambda 2\lambda'\lambda \sup_{(x,y)\in K_\lambda} |\varepsilon_{n-1}(x,y)|,$$

whose limit is 0 as n tends to $+\infty$. So, it follows that

$$u(x,y) = u_0(x,y) + \iint\limits_{D(x,y,g)} F(\xi,\eta,u(\xi,\eta))\,d\xi\,d\eta$$

for $(x,y) \in K_\lambda \cap \{(x,y) : 0 \le y,\ g(y) \le x\} = K^-_{1,\lambda}$.

To prove uniqueness, let W be another solution to (2.4). Putting $\varDelta = W - u$, we obtain

$$\varDelta(x,y) = \iint\limits_{D(x,y,g)} (F(\xi,\eta,W(\xi,\eta)) - F(\xi,\eta,u(\xi,\eta)))\,d\xi\,d\eta.$$

Let $(x,y) \in K_\lambda$. As $D(x,y,g) \subset K_\lambda$, we have

$$|\varDelta(x,y)| \le \iint\limits_{D(x,y,g)} m_\lambda |W(\xi,\eta) - u(\xi,\eta)| d\xi d\eta \le m_\lambda \iint\limits_{D(x,y,g)} |\varDelta(\xi,\eta)|\,d\xi\,d\eta.$$

As $g(y) \le x$,

$$|\varDelta(x,y)| \le m_\lambda \int\limits_{g(y)}^{x} \int\limits_{0}^{y} |\varDelta(\xi,\eta)|\,d\eta\,d\xi \le m_\lambda \int\limits_0^y \Big(\int\limits_{g(-\lambda)}^{g(\lambda)} \sup_{\xi\in[0,2\lambda]} |\varDelta(\xi,\eta)|\,d\xi\Big)\,d\eta.$$

For every $y \in [0,\lambda]$, put

$$E(y) = \sup_{\xi\in[0,2\lambda]} |\varDelta(\xi,y)|.$$

Then

$$|\varDelta(x,y)| \le m_\lambda 2\lambda' \Big|\int\limits_0^y E(\eta)\,d\eta\Big|;$$

it follows that

$$\forall y \in [0, f(\lambda)], \qquad E(y) \le m_\lambda 2\lambda' \Big| \int_0^y E(\eta)\, d\eta \Big|.$$

In this way, by applying Gronwall's lemma, we get $E = 0$, hence $\Delta = 0$, which proves the uniqueness of u on $K^-_{1,\lambda}$. We write v^-_λ for this solution. Let us assume that $0 \le y$, $x \le g(y)$. We have

$$\iint\limits_{D(x,y,g)} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\, d\xi\, d\eta = \int_x^{g(y)} \Big(\int_0^y \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\, d\eta \Big)\, d\xi = -\int_{g(y)}^x \Big(\int_0^y \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\, d\eta \Big)\, d\xi.$$

We can solve this case in the same way.

In the case $y \le 0$, we make the change of variables $X = -x$, $Y = -y$. Then $Y \ge 0$ and $h(Y) = -g(-Y)$. The compact subset K_λ turns into the compact subset $Q_\lambda = [h(-\lambda), h(\lambda)] \times [-\lambda, \lambda]$ and $h(\lambda) = -g(-\lambda)$. So we now have

$$\begin{cases} g(y) \le x \Leftrightarrow Y \ge h(X); \mathfrak{D}(X,Y,h) = D(-X,-Y,g) = \text{rectangle}(MNQP); \\ g(y) \ge x \Leftrightarrow Y \le h(X); \mathfrak{D}(X,Y,h) = D(-X,-Y,g) = \text{rectangle}(MPQN). \end{cases}$$

If $x \le g(y)$, then

$$\iint\limits_{D(x,y,g)} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\, d\xi\, d\eta = \int_x^{g(y)} \Big(\int_y^0 \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\, d\eta \Big)\, d\xi = \int_{g(y)}^x \Big(\int_0^y \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\, d\eta \Big)\, d\xi.$$

If $g(y) \le x$, then

$$\iint\limits_{D(x,y,g)} \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\, d\xi\, d\eta = \int_{g(y)}^x \Big(\int_y^0 \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\, d\eta \Big)\, d\xi = -\int_{g(y)}^x \Big(\int_0^y \frac{\partial^2 u}{\partial x \partial y}(\xi,\eta)\, d\eta \Big)\, d\xi.$$

The change of variables gives then

$$u(x,y) = u(-X,-Y) = u_0(-X,-Y) + \iint\limits_{D(-X,-Y,g)} F(-\xi,-\eta,u(-\xi,-\eta))\, d\xi\, d\eta,$$

whose form is

$$U(X,Y) = U_0(X,Y) + \iint\limits_{\mathfrak{D}(X,Y,h)} \mathfrak{F}(\xi,\eta,u(\xi,\eta))\, d\xi\, d\eta.$$

We can deal with this case as previously with

$$\|V_n\|_{\infty,K_\lambda} \le \frac{\Phi_\lambda}{m_\lambda} \frac{(2\lambda' m_\lambda \lambda)^n}{n!}, \qquad u(x,y) = U(-x,-y).$$

For existence of a global solution, we have four cases:

- $(0 \le y$ and $g(y) \le x)$,
- $(0 \le y$ and $x \le g(y))$,
- $(y \le 0$ and $x \le g(y))$,
- $(y \le 0$ and $g(y) \le x)$.

Finally, if we put

$$K^-_{1,\lambda} = K_\lambda \cap \{(x,y) : 0 \le y,\, g(y) \le x\}, \qquad K^+_{1,\lambda} = K_\lambda \cap \{(x,y) : 0 \le y,\, x \le g(y)\},$$
$$K^+_{2,\lambda} = K_\lambda \cap \{(x,y) : y \le 0,\, x \le g(y)\}, \qquad K^-_{2,\lambda} = K_\lambda \cap \{(x,y) : y \le 0,\, g(y) \le x\}$$

and if we let

- v^-_λ be the solution on $K^-_{1,\lambda}$,
- v^+_λ be the solution on $K^+_{1,\lambda}$,
- w^-_λ be the solution on $K^-_{2,\lambda}$,
- w^+_λ be the solution on $K^+_{2,\lambda}$,

then we can put

$$u_\lambda(x,y) = \begin{cases} v^-_\lambda(x,y) & \text{for } (x,y) \in K^-_{1,\lambda}, \\ w^+_\lambda(x,y) & \text{for } (x,y) \in K^+_{2,\lambda}, \\ v^+_\lambda(x,y) & \text{for } (x,y) \in K^+_{1,\lambda}, \\ w^-_\lambda(x,y) & \text{for } (x,y) \in K^-_{2,\lambda}. \end{cases} \tag{2.6}$$

Now,

- v^-_λ and v^+_λ agree on γ because $v^-_\lambda(g(y,y)) = v^+_\lambda(g(y,y)) = \psi(y)$,
- w^-_λ and w^+_λ agree on γ because $w^-_\lambda(g(y,y)) = w^+_\lambda(g(y,y)) = \psi(y)$,
- w^-_λ and v^-_λ agree on $(y = 0)$ because $w^-_\lambda(x,0) = v^-_\lambda(x,0) = \varphi(x)$,
- w^+_λ and v^+_λ agree on $(y = 0)$ because $w^+_\lambda(x,0) = v^+_\lambda(x,0) = \varphi(x)$,

which ensures the existence and uniqueness of the solution u_λ on $K_\lambda = K^-_{1,\lambda} \cup K^+_{2,\lambda} \cup K^+_{1,\lambda} \cup K^-_{2,\lambda}$. It remains to prove that the method actually gives a continuous global solution u on $\mathbb{R}^2$, that is, one which satisfies (P'_i). If $\lambda_2 > \lambda_1$ then $K_{\lambda_1} \subset K_{\lambda_2}$; so, we must prove that $u_{\lambda_2}|_{K_{\lambda_1}} = u_{\lambda_1}$. But

$$\forall (x,y) \in K_{\lambda_2}, \qquad u_{\lambda_2}(x,y) = u_0(x,y) + \iint_{D(x,y,g)} F(\xi,\eta,u_{\lambda_2}(\xi,\eta))\, d\xi\, d\eta$$

and we have this equality, all the more so, for $(x,y) \in K_{\lambda_1}$. So we have

$$u_{\lambda_2}|_{K_{\lambda_1}}(x,y) = u_0(x,y) + \iint_{D(x,y,g)} F(\xi,\eta,u_{\lambda_2|K_{\lambda_1}}(\xi,\eta))\, d\xi\, d\eta.$$

In other words, $u_{\lambda_2}|_{K_{\lambda_1}}$ satisfies (2.4) on K_{λ_1} and so coincides on it with its unique solution u_{λ_1}. For every $(x,y) \in \mathbb{R}^2$, we can then put

$$u(x,y) = u_\lambda(x,y) = u_0(x,y) + \iint_{D(x,y,g)} F(\xi,\eta,u(\xi,\eta))\, d\xi\, d\eta \tag{2.7}$$

where u_λ satisfies (2.4) on K_λ and $(x,y) \in K_\lambda$. The definition of u by (2.7), being independent of the compact subset K_λ, finally gives the unique global solution to (P'_i) or (P'_∞). ∎

In Section 5, we will need the estimates specified by the following result.

Proposition 10. *With the previous notations, for every compact subset $K \Subset \mathbb{R}^2$, there exists a compact subset $K_\lambda \Subset \mathbb{R}^2$ containing K such that*

$$m_\lambda = \sup_{(x,y)\in K_\lambda ; t\in\mathbb{R}} \left| \frac{\partial F}{\partial z}(x,y,t) \right|; \qquad \Phi_\lambda = \|F(\cdot,\cdot,0)\|_{\infty,K_\lambda} + m_\lambda \|u_0\|_{\infty,K_\lambda}; \tag{2.8}$$

$$\|u\|_{\infty,K} \le \|u\|_{\infty,K_\lambda} \le \|u_0\|_{\infty,K_\lambda} + \frac{\Phi_\lambda}{m_\lambda} \exp(2\lambda' m_\lambda \lambda). \tag{2.9}$$

Proof. We have

$$u_n(x,y) = u_0(x,y) + \iint_{D(x,y,g)} F(\xi,\eta,u_{n-1}(\xi,\eta))\, d\xi\, d\eta, \qquad n \ge 1,$$

and (2.6) holds. As

$$\Phi_\lambda = \|F(\cdot,\cdot,0)\|_{\infty,K_\lambda} + m_\lambda \|u_{0,\varepsilon}\|_{\infty,K_\lambda}, \qquad V_n = u_n - u_{n-1},$$

according the proof of Theorem 9, we have

$$\forall n \in \mathbb{N}^*, \qquad \|V_n\|_{\infty,K^-_{1,\lambda}} \le m_\lambda^{n-1} \Phi_\lambda \frac{(2\lambda'\lambda)^n}{n!} = \frac{\Phi_\lambda}{m_\lambda} \frac{(2\lambda' m_\lambda \lambda)^n}{n!}$$

and consequently,

$$\|u\|_{\infty,K^-_{1,\lambda}} \le \|u_0\|_{\infty,K^-_{1,\lambda}} + \sum_{n=1}^{\infty} \|V_n\|_{\infty,K^-_{1,\lambda}} \le \|u_0\|_{\infty,K^-_{1,\lambda}} + \frac{\Phi_\lambda}{m_\lambda} \exp(2\lambda' m_\lambda \lambda).$$

We deduce that

$$\|u\|_{\infty,K^-_{1,\lambda}} \le \|u_0\|_{\infty,K^-_{1,\lambda}} + \frac{\Phi_\lambda}{m_\lambda} \exp(2\lambda' m_\lambda \lambda)$$

and similarly

$$\|u\|_{\infty,K^+_{2,\lambda}} \le \|u_0\|_{\infty,K^+_{2,\lambda}} + \frac{\Phi_\lambda}{m_\lambda} \exp(2\lambda' m_\lambda \lambda),$$
$$\|u\|_{\infty,K^+_{1,\lambda}} \le \|u_0\|_{\infty,K^+_{1,\lambda}} + \frac{\Phi_\lambda}{m_\lambda} \exp(2\lambda' m_\lambda \lambda),$$
$$\|u\|_{\infty,K^-_{2,\lambda}} \le \|u_0\|_{\infty,K^-_{2,\lambda}} + \frac{\Phi_\lambda}{m_\lambda} \exp(2\lambda' m_\lambda \lambda).$$

So

$$\|u\|_{\infty,K_\lambda} \le \|u_0\|_{\infty,K_\lambda} + \frac{\Phi_\lambda}{m_\lambda} \exp(2\lambda' m_\lambda \lambda),$$

hence

$$\|u\|_{\infty,K} \le \|u\|_{\infty,K_\lambda} \le \|u_0\|_{\infty,K_\lambda} + \frac{\Phi_\lambda}{m_\lambda} \exp(2\lambda' m_\lambda \lambda). \ ■$$

3. Algebras of generalized functions

Algebras of generalized functions are the most effective tool to solve many nonlinear differential problems with irregular data or characteristic data. To choose an appropriate structure for the Cauchy problem considered, we use the results and notations of J.-A. Marti [1998]–[2004], J.-A. Marti and S. P. Nuiro [1999].

3.1. The sheaves of $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras. Suppose that

- Λ is a set of indices;
- A is a subring of the ring $\mathbb{K}^\Lambda$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$);
- $A_+ = \{(r_\lambda)_\lambda \in A : r_\lambda \geq 0\}$;
- A has the following stability property: whenever $(|s_\lambda|)_\lambda \leq (r_\lambda)_\lambda$ (i.e. for each λ, $|s_\lambda| \leq r_\lambda$) for any pair $((s_\lambda)_\lambda,(r_\lambda)_\lambda) \in \mathbb{K}^\Lambda \times A_+$, it follows that $(s_\lambda)_\lambda \in A$;
- I_A is an ideal of A with the same property;
- $\mathcal{E}$ is a sheaf of $\mathbb{K}$-topological algebras on a topological space X such that for each open set Ω in X, the algebra $\mathcal{E}(\Omega)$ is endowed with a family $\mathcal{P}(\Omega)=(p_i)_{i\in I(\Omega)}$ of seminorms satisfying
$$\forall i \in I(\Omega), \exists (j,k,C) \in I(\Omega)\times I(\Omega)\times \mathbb{R}_+^*, \forall f,g \in \mathcal{E}(\Omega):\ p_i(fg) \leq Cp_j(f)p_k(g);$$
- For any two open subsets Ω_1, Ω_2 of X such that $\Omega_1 \subset \Omega_2$, we have $I(\Omega_1)\subset I(\Omega_2)$ and if ϱ_1^2 is the restriction operator $\mathcal{E}(\Omega_2)\to\mathcal{E}(\Omega_1)$, then, for each $p_i \in \mathcal{P}(\Omega_1)$, the seminorm $\widetilde{p}_i = p_i \circ \varrho_1^2$ extends p_i to $\mathcal{P}(\Omega_2)$;
- For any family $\mathcal{F}=(\Omega_h)_{h\in H}$ of open subsets of X if $\Omega=\bigcup_{h\in H}\Omega_h$, then, for each $p_i \in \mathcal{P}(\Omega)$, $i \in I(\Omega)$, there exists a finite subfamily $\Omega_1,\dots,\Omega_{n(i)}$ of $\mathcal{F}$ and corresponding seminorms $p_1 \in \mathcal{P}(\Omega_1),\dots,p_{n(i)} \in \mathcal{P}(\Omega_{n(i)})$ such that, for each $u \in \mathcal{E}(\Omega)$,
$$p_i(u) \leq p_1(u_{|\Omega_1}) + \cdots + p_{n(i)}(u_{|\Omega_{n(i)}}).$$

Set
$$\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(\Omega) = \{(u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^\Lambda : \forall i \in I(\Omega),\ (p_i(u_\lambda))_\lambda \in A_+\},$$
$$\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega) = \{(u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^\Lambda : \forall i \in I(\Omega),\ (p_i(u_\lambda))_\lambda \in I_A^+\},$$
$$\mathcal{C} = A/I_A.$$

PROPOSITION 11. *If*
$$|A| = \{(|r_\lambda|)_\lambda \in \mathbb{R}_+^\Lambda : (r_\lambda)_\lambda \in A\} \quad \textit{and} \quad |I_A| = \{(|r_\lambda|)_\lambda \in \mathbb{R}_+^\Lambda : (r_\lambda)_\lambda \in I_A\}$$
are respectively subsets of A and I_A then $|A| = A_+$ and $|I_A| = I_A^+$.

PROPOSITION 12 (J.-A. Marti, S. P. Nuiro and V. S. Valmorin [1998a], [1998b]). *$\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$ is a sheaf of subalgebras of the sheaf ε^Λ; $\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}$ is a sheaf of ideals of $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$; the constant sheaf $\mathcal{H}_{(A,\mathbb{K},|\cdot|)}/\mathcal{J}_{(I_A,\mathbb{K},|\cdot|)}$ is exactly the sheaf $\mathcal{C}=A/I_A$.*

DEFINITION 13. A $(\mathcal{C},\mathcal{E},\mathcal{P})$-*algebra* is every factor algebra $\mathcal{A}=\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}$. We denote by $[u_\lambda]$ the class defined by the representative $(u_\lambda)_{\lambda\in\Lambda}$.

REMARK 14. In the context of $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras, it is proved that, if $A = A_+$, then
$$\mathcal{H}_{(A,\mathbb{K},|\cdot|)}/\mathcal{J}_{(I_A,\mathbb{K},|\cdot|)} = A/I_A = \mathcal{C}.$$
But the first term is, in principle, a $(\mathcal{C},\mathbb{K},|.|)$-algebra and the second a ring of generalized constants, which is therefore an algebra. In fact, the following proposition will prove it.

PROPOSITION 15. *If A is a subring of $\mathbb{K}^\Lambda (\mathbb{K}=\mathbb{R}$ or $\mathbb{C})$ with the stability property such that $|A| = A_+$, then A is a $\mathbb{K}$-subalgebra of $\mathbb{K}^\Lambda$.*

Overgenerated rings. In practice, the ring A and the ideal I_A are overgenerated by finite families of elements according to the following definition:

Let $B_p = \{(r_{n,\lambda})_\lambda \in (\mathbb{R}_+^*)^\Lambda : n = 1, \dots, p\}$ and B be the subset of $(\mathbb{R}_+^*)^\Lambda$ consisting of all products, quotients and linear combinations with coefficients in $\mathbb{R}_+^*$ of elements in B_p. Define

$$A = \{(a_\lambda)_\lambda \in \mathbb{K}^\Lambda : \exists (b_\lambda)_\lambda \in B, |a_\lambda| \le b_\lambda\}.$$

It is easy to see that A is a subring of $\mathbb{K}^\Lambda$ with the stability property and moreover $A_+ = |A|$. Then we make the following definition:

Definition 16. In the above situation, we say that A is *overgenerated* by B_p. If I_A is some ideal of A with the same stability property, we can also say that $\mathcal{C} = A/I_A$ is *overgenerated* by B_p.

Example 17. As a "canonical" ideal of A, we can take

$$I_A = \{(a_\lambda)_\lambda \in \mathbb{K}^\Lambda : \forall (b_\lambda)_\lambda \in B, |a_\lambda| \le b_\lambda\}.$$

The association process. We suppose that Λ is left-filtering for the given partial order relation $\prec$. Let us denote by Ω an open subset of X, E a given sheaf of topological $\mathbb{K}$-vector spaces containing $\mathcal{E}$ as a subsheaf, Φ a given map from Λ to $\mathbb{K}$ such that $(\Phi(\lambda))_\lambda = (\Phi_\lambda)_\lambda$ is an element of A. We also suppose that

$$\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega) \subset \{(u_\lambda)_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(\Omega) : \lim_{E(\Omega),\Lambda} u_\lambda = 0\}.$$

Then, for $u = [u_\lambda]$ and $v = [v_\lambda] \in \mathcal{E}(\Omega)$, we define the Φ-E association.

Definition 18. We denote by

$$u \underset{E(\Omega)}{\overset{\Phi}{\approx}} v$$

the *Φ-E association* between u and v defined by

$$\lim_{E(\Omega),\Lambda} \Phi_\lambda(u_\lambda - v_\lambda) = 0.$$

That is to say, for each neighborhood V of 0 for the E-topology, there exists $\lambda_0 \in \Lambda$ such that

$$\lambda \prec \lambda_0 \ \Rightarrow \ \Phi_\lambda(u_\lambda - v_\lambda) \in V.$$

Remark 19. To ensure the independence of the definition from the representatives of u and v, we must verify that if $\lim_{E(\Omega),\Lambda} \Phi_\lambda(w_\lambda) = 0$ for some $(w_\lambda)_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(\Omega)$, then, for any $(i_\lambda)_\lambda \in \mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega)$, $\lim_{E(\Omega),\Lambda} \Phi_\lambda(w_\lambda + i_\lambda) = 0$.

To prove the last condition, it is sufficient to show that $(\Phi_\lambda i_\lambda)_\lambda \in \mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega)$. But for each $i \in I(\Omega)$, we have $p_i(\Phi_\lambda(i_\lambda)) = |\Phi_\lambda| p_i(i_\lambda)$. And, considering the definitions and the stability properties given above, we have $|\Phi_\lambda|_\lambda \in A_+$ and $(p_i(i_\lambda))_\lambda \in I_A^+$. Then we also have $(|\Phi_\lambda| p_i(i_\lambda))_\lambda \in I_A^+$, which proves the required condition.

Remark 20. We can also define an association process between $u = [u_\lambda] \in E(\Omega)$ and $T \in E(\Omega)$ by writing simply

$$u \sim T \ \Leftrightarrow \ \lim_{E(\Omega),\Lambda} u_\lambda = T.$$

Then taking $E = \mathcal{D}'$, $\mathcal{E} = \mathrm{C}^\infty$, $\Lambda =]0,1]$, we find again the association process defined in the literature (J.-F. Colombeau [1985], Yu. V. Egorov [1990]).

Remark 21 (Relationship between ring and injection). It is shown by J.-A. Marti [2003] that a necessary and sufficient condition for the existence of a canonical sheaf morphism of algebras from $\mathcal{E}$ into $\mathcal{A}$ is that A is a ring. If, in addition, $I_A \subset \{(a_\lambda)_\lambda \in A : \lim_\Lambda a_\lambda = 0\}$ and, for each Ω, the $\mathcal{P}(\Omega)$ topology of $\mathcal{E}(\Omega)$ is separated, then this morphism is an injective mapping.

3.2. An algebra adapted to the generalized Cauchy problem. The first step is to link the problem and its data to algebraic and topological parameters that make it possible to build an appropriate ($\mathcal{C}$,$\mathcal{E}$,$\mathcal{P}$)-algebra.

Definition 22. We choose $\mathcal{E} = \mathrm{C}^\infty$, $X = \mathbb{R}^d$ for $d = 1, 2$, $E = \mathcal{D}'$ and $\Lambda =]0,1]$. For every open set Ω, in $\mathbb{R}^d$, $\mathcal{E}(\Omega)$ is endowed with the $\mathcal{P}(\Omega)$ topology of uniform convergence of all derivatives on compact subsets of Ω. This topology may be defined by the family of the seminorms

$$P_{K,l}(u_\varepsilon) = \sup_{|\alpha|\le l} \sup_{x\in K} |D^\alpha u_\varepsilon(x)|$$

with $K \Subset \Omega$ and

$$D^\alpha = \frac{\partial^{\alpha_1+\cdots+\alpha_d}}{\partial z_1^{\alpha_1}\cdots\partial z_d^{\alpha_d}} \qquad \text{for } z = (z_1,\ldots,z_d) \in \Omega, l \in \mathbb{N}, \alpha = (\alpha_1,\ldots,\alpha_d) \in \mathbb{N}^d.$$

We verify that it is compatible with the algebraic structure of $\mathcal{E}(\Omega)$ since

$$\forall K \Subset \Omega, \forall \alpha \in \mathbb{N}^d, \exists C > 0, \forall f, g \in \mathrm{C}^\infty(\Omega), \qquad P_{K,l}(fg) \le P_{K,l}(f)P_{K,l}(g).$$

We put $P_{K,\alpha}(u_\varepsilon) = \sup_{x\in K} |D^\alpha u_\varepsilon(x)|$, so $P_{K,l}(u_\varepsilon) = \sup_{|\alpha|\le l} P_{K,\alpha}(u_\varepsilon)$. Let A be a subring of the ring $\mathbb{R}^\Lambda$ of family of reals with the usual laws. We consider an ideal I_A of A with the stability property. To simplify, we write

$$\mathcal{X} = \mathcal{H}_{(A,\mathrm{C}^\infty,\mathcal{P})}, \qquad \mathcal{N} = \mathcal{J}_{(I_A,\mathrm{C}^\infty,\mathcal{P})}, \qquad \mathcal{A} = \mathcal{X}/\mathcal{N}.$$

We put

$$\mathcal{X}(\Omega) = \{(u_\varepsilon)_\varepsilon \in [\mathrm{C}^\infty(\Omega)]^\Lambda : \forall K \Subset \Omega, \forall l \in \mathbb{N}, (P_{K,l}(u_\varepsilon))_\varepsilon \in A_+\},$$
$$\mathcal{N}(\Omega) = \{(u_\varepsilon)_\varepsilon \in [\mathrm{C}^\infty(\Omega)]^\Lambda : \forall K \Subset \Omega, \forall l \in \mathbb{N}, (P_{K,l}(u_\varepsilon))_\varepsilon \in I_A^+\}.$$

The ring of generalized constants associated with the factor algebra is exactly the factor ring $\mathcal{C} = A/I_A$. Finally, the generalized derivation $D^\alpha : u\ (= [u_\varepsilon]) \mapsto D^\alpha u = [D^\alpha u_\varepsilon]$ provides $\mathcal{A}(\Omega)$ with a differential algebraic structure.

Example 23. If we consider

$$A = \mathbb{R}_M^\Lambda = \{(m_\varepsilon)_\varepsilon \in \mathbb{R}^\Lambda : \exists p \in \mathbb{R}_+^*,\ \exists C \in \mathbb{R}_+^*,\ \exists \mu \in]0,1],\ \forall \varepsilon \in]0,\mu],\ |m_\varepsilon| \le C\varepsilon^{-p}\}$$

and the ideal

$$I_A = \{(m_\varepsilon)_\varepsilon \in \mathbb{R}^\Lambda : \forall q \in \mathbb{R}_+^*,\ \exists D \in \mathbb{R}_+^*,\ \exists \mu \in]0,1],\ \forall \varepsilon \in]0,\mu],\ |m_\varepsilon| \le D\varepsilon^{q}\},$$

then $\mathcal{A}(\mathbb{R}^d) = \mathcal{G}(\mathbb{R}^d)$ is the algebra of Colombeau generalized functions.

If u is a generalized function of the variable $x \in \mathbb{R}^2$ and $F \in \mathrm{C}^\infty(\mathbb{R}^3, \mathbb{R})$, we extend the notation $F(\cdot,\cdot,u)$ in the following way:

Definition 24. Let Ω be an open subset of $\mathbb{R}^2$ and $F \in \mathrm{C}^\infty(\Omega \times \mathbb{R}, \mathbb{R})$. We say that the algebra $\mathcal{A}(\Omega)$ is *stable* under F if the following two conditions are satisfied:

- For each $K \Subset \mathbb{R}^2$, $l \in \mathbb{N}$ and $(u_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\Omega)^{]0,1]}$, there is a positive finite sequence $C_1, \dots, C_l$ such that
$$P_{K,l}(F(\cdot,\cdot,u_\varepsilon)) \le \sum_{i=0}^{l} C_i P_{K,l}^i(u_\varepsilon).$$
- For each $K \Subset \mathbb{R}^2$, $l \in \mathbb{N}$, $(v_\varepsilon)_\varepsilon, (u_\varepsilon)_\varepsilon \in \mathcal{X}(\Omega)$, there is a positive finite sequence $D_1, \dots, D_l$ such that
$$P_{K,l}(F(\cdot,\cdot,v_\varepsilon) - F(\cdot,\cdot,u_\varepsilon)) \le \sum_{j=0}^{l} D_j P_{K,l}^j(v_\varepsilon - u_\varepsilon).$$

Proposition 25. *If $\mathcal{A}(\Omega)$ is stable under F then*:

- *For each $K \Subset \mathbb{R}^2$, $l \in \mathbb{N}$ and $(u_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\Omega)^{]0,1]}$, we have*
$$(P_{K,l}(u_\varepsilon))_\varepsilon \in A_+ \;\Rightarrow\; (P_{K,l}(F(\cdot,\cdot,,u_\varepsilon)))_\varepsilon \in A_+.$$
- *For each $K \Subset \mathbb{R}^2$, $l \in \mathbb{N}$, $(v_\varepsilon)_\varepsilon, (u_\varepsilon)_\varepsilon \in \mathcal{X}(\Omega)$, we have*
$$(P_{K,l}(v_\varepsilon - u_\varepsilon))_\varepsilon \in I_A^+ \;\Rightarrow\; (P_{K,l}(F(\cdot,\cdot,v_\varepsilon) - F(\cdot,\cdot,u_\varepsilon)))_\varepsilon \in I_A^+.$$

Proposition 26. *If $\mathcal{A}(\Omega)$ is stable under F then, for all $(u_\varepsilon)_\varepsilon \in \mathcal{X}(\Omega)$ and $(i_\varepsilon)_\varepsilon \in \mathcal{N}(\Omega)$, we have*
$$(F(\cdot,\cdot,u_\varepsilon))_\varepsilon \in \mathcal{X}(\Omega); \qquad (F(\cdot,\cdot,u_\varepsilon + i_\varepsilon) - F(\cdot,\cdot,u_\varepsilon))_\varepsilon \in \mathcal{N}(\Omega).$$

We shall use the following lemma.

Lemma 27 (Francesco Faà di Bruno's formula). *The nth order derivative of $f \circ u$ can be written*
$$(f \circ u)^{(n)} = \sum_{r=1}^{n} \sum_{\substack{i_1 \ge \cdots \ge i_r \\ i_1 + \cdots + i_r = n}} t_{i_1,\dots,i_r} f^{(r)} \circ u \cdot \prod_{k=1}^{r} u^{(i_k)}$$
where the coefficients $t_{i_1,\dots,i_r}$ are integers.

Proposition 28. *Let $F \in \mathrm{C}^\infty(\mathbb{R}^2, \mathbb{R})$ be defined by $F(x,y,z) = z/(1+z^2)$. Then $\mathcal{A}(\mathbb{R}^2)$ is stable under F.*

Proof. We put
$$f(z) = \frac{z}{1+z^2} \quad \text{and} \quad \Phi_\varepsilon(x,y) = F(x,y,u_\varepsilon(x,y)) = \frac{u_\varepsilon(x,y)}{1+u_\varepsilon^2(x,y)}. \tag{3.1}$$

For each real z we have
$$f(z) = \frac{z}{1+z^2} = \frac{i}{2}\left(\frac{1}{1+iz} - \frac{1}{1-iz}\right).$$

We put
$$g_\alpha(z) = \frac{1}{1+\alpha z}, \qquad \alpha = i \text{ or } \alpha = -i.$$

By induction, for $n \geq 1$ we obtain

$$g_\alpha^{(n)}(z) = \frac{(-1)^n (n!) \alpha^n}{(1+\alpha z)^{n+1}}.$$

We have

$$f^{(n)}(z) = \frac{i}{2}\,(g_i^{(n)}(z) - g_{-i}^{(n)}(z)),$$

and, for $\alpha = i$ or $\alpha = -i$,

$$|g_\alpha^{(n)}(z)| \leq \left| \frac{(-1)^n n! \alpha^n}{(1+\alpha z)^{n+1}} \right| \leq n!\, \frac{|i|^n}{(1+z^2)^{n+1}} \leq n!,$$

so

$$|f^{(n)}(z)| \leq \frac{1}{2}\,(|g_i^{(n)}(z)| + |g_{-i}^{(n)}(z)|) \leq n!.$$

All the successive derivatives of f are therefore bounded on $\mathbb{R}$, and for each integer n,

$$\sup_{z\in\mathbb{R}} |f^{(n)}(z)| \leq n!.$$

Let us show that for each n, there is $C_{r,n} > 0$, $1 \leq r \leq n$, such that

$$P_{K,n}(F(\cdot,\cdot,u_\varepsilon)) \leq \sum_{r=1}^{n} C_{r,n} P_{K,n}^r(u_\varepsilon).$$

In terms of $\Phi_\varepsilon(x,y) = F(x,y,u_\varepsilon(x,y))$, x and y have similar roles, therefore the study of $\partial^n/\partial x^k \partial y^{n-k}\, \Phi_\varepsilon$ is similar to that of $\partial^n/\partial x^{n-k}\partial y^k\, \Phi_\varepsilon$. Thus we can prove the assertion only for $\partial^n/\partial x^n\, \Phi_\varepsilon$ We have

$$\frac{\partial \Phi_\varepsilon}{\partial x}(x,y) = f'(u_\varepsilon(x,y))\, \frac{\partial u_\varepsilon}{\partial x}(x,y),$$

hence

$$\forall K \Subset \mathbb{R}^2, \qquad P_{K,(1,0)}(F(\cdot,\cdot,u_\varepsilon)) \leq P_{K,(1,0)}(u_\varepsilon).$$

Consequently,

$$\forall K \Subset \mathbb{R}^2, \qquad P_{K,1}(F(\cdot,\cdot,u_\varepsilon)) \leq P_{K,1}(u_\varepsilon).$$

For each $K \Subset \mathbb{R}^2$, we have

$$\frac{\partial^2 \Phi_\varepsilon}{\partial x \partial y}(x,y) = f^{(2)}(u_\varepsilon(x,y))\, \frac{\partial u_\varepsilon}{\partial y}(x,y) \frac{\partial u_\varepsilon}{\partial x}(x,y) + f'(u_\varepsilon(x,y)) \frac{\partial^2 u_\varepsilon}{\partial x \partial y}(x,y),$$

hence

$$P_{K,(1,1)}(F(\cdot,\cdot,u_\varepsilon)) \leq 2P_{K,1}^2(u_\varepsilon) + P_{K,2}(u_\varepsilon) \leq 2P_{K,2}^2(u_\varepsilon) + P_{K,2}(u_\varepsilon).$$

We have

$$\frac{\partial^2 \Phi_\varepsilon}{\partial x^2}(x,y) = f^{(2)}(u_\varepsilon(x,y)) \left(\frac{\partial u_\varepsilon}{\partial x} \right)^2 (x,y) + f'(u_\varepsilon(x,y))\, \frac{\partial^2 u_\varepsilon}{\partial x^2}(x,y).$$

Thus

$$P_{K,(2,0)}(F(\cdot,\cdot,u_\varepsilon)) \leq 2P_{K,1}^2(u_\varepsilon) + P_{K,2}(u_\varepsilon) \leq 2P_{K,2}^2(u_\varepsilon) + P_{K,2}(u_\varepsilon).$$

Consequently,

$$\forall K \Subset \mathbb{R}^2, \qquad P_{K,2}(F(\cdot,\cdot,u_\varepsilon) \leq 2P_{K,2}^2(u_\varepsilon) + P_{K,2}(u_\varepsilon).$$

Therefore we have, for $\alpha = n$ and $\beta = 0$,

$$\frac{\partial^n \Phi_\varepsilon}{\partial x^n}(x,y) = \sum_{r=1}^{n} \sum_{\substack{i_1 \geq \cdots \geq i_r \\ i_1 + \cdots + i_r = n}} t_{i_1,\ldots,i_r} f^{(r)}(u_\varepsilon(x,y)) \prod_{k=1}^{r} \frac{\partial^{i_k} u_\varepsilon}{\partial x^{i_k}}(x,y),$$

For all $K \Subset \mathbb{R}^2$, $i_k \in \mathbb{N}$, $i_k \leq n$ and $r \in \mathbb{N}$,

$$\sup_{(x,y)\in K} |f^{(r)}(u_\varepsilon(x,y))| \leq r! \leq n!,$$

therefore

$$\max_{1\leq i_k \leq n} \sup_{(x,y)\in K} |f^{(i_k)}(u_\varepsilon(x,y))| \leq n!.$$

We have

$$\sup_{(x,y)\in K} \left| \frac{\partial^{i_k} u_\varepsilon}{\partial x^{i_k}}(x,y) \right| \leq P_{K,i_k}(u_\varepsilon) \leq P_{K,n}(u_\varepsilon)$$

and

$$\sup_{(x,y)\in K} \left| \prod_{k=1}^{r} \frac{\partial^{i_k} u_\varepsilon}{\partial x^{i_k}}(x,y) \right| \leq P_{K,n}^r(u_\varepsilon),$$

therefore

$$\sup_{(x,y)\in K} \left| t_{i_1,\ldots,i_r} f^{(r)}(u_\varepsilon(x,y)) \prod_{k=1}^{r} \frac{\partial^{i_k} u_\varepsilon}{\partial x^{i_k}}(x,y) \right| \leq t_{i_1,\ldots,i_r} n! P_{K,n}^r(u_\varepsilon).$$

Consequently,

$$\sup_{(x,y)\in K} \left| \frac{\partial^n \Phi_\varepsilon}{\partial x^n}(x,y) \right| \leq \sum_{r=1}^{n} \Big(\sum_{\substack{i_1 \geq \cdots \geq i_r \\ i_1 + \cdots + i_r = n}} t_{i_1,\ldots,i_r} \Big) n! P_{K,n}^r(u_\varepsilon).$$

Let us show that, for all $K \Subset \mathbb{R}^2$, $l \in \mathbb{N}$, and $(v_\varepsilon)_\varepsilon, (u_\varepsilon)_\varepsilon \in \mathcal{X}(\Omega)$, there is a positive number D_l such that

$$P_{K,l}(F(\cdot,\cdot,v_\varepsilon) - F(\cdot,\cdot,u_\varepsilon)) \leq D_l P_{K,l}(v_\varepsilon - u_\varepsilon).$$

First let us show this relation for $l = 0$. For all $K \Subset \mathbb{R}^2$ and $(x,y) \in K$, we have

$$\begin{aligned} g_\alpha(v_\varepsilon(x,y)) - g_\alpha(u_\varepsilon(x,y)) &= \frac{1}{1+\alpha v_\varepsilon(x,y)} - \frac{1}{1+\alpha u_\varepsilon(x,y)} \\ &= \frac{\alpha(u_\varepsilon(x,y) - v_\varepsilon(x,y))}{(1+\alpha v_\varepsilon(x,y))(1+\alpha u_\varepsilon(x,y))}, \end{aligned}$$

so

$$|g_\alpha(v_\varepsilon(x,y)) - g_\alpha(u_\varepsilon(x,y))| \leq \frac{|u_\varepsilon(x,y) - v_\varepsilon(x,y)|}{|1+v_\varepsilon^2(x,y)|\,|1+u_\varepsilon^2(x,y)|} \leq |v_\varepsilon(x,y) - u_\varepsilon(x,y)|,$$

because $\alpha = i$ or $\alpha = -i$. As

$$f(z) = \frac{z}{1+z^2} = \frac{i}{2}\,(g_i(z) - g_{-i}(z)),$$

we have

$$f(v_\varepsilon(x,y)) - f(u_\varepsilon(x,y)) = \frac{i}{2}\left[g_i(v_\varepsilon(x,y)) - g_i(u_\varepsilon(x,y)) - (g_{-i}(v_\varepsilon(x,y)) - g_{-i}(u_\varepsilon(x,y)))\right]$$

and

$$\begin{aligned}|f(v_\varepsilon(x,y)) - f(u_\varepsilon(x,y))| \\ &\le \frac{1}{2}\left[|g_i(v_\varepsilon(x,y)) - g_i(u_\varepsilon(x,y))| + |g_{-i}(v_\varepsilon(x,y)) - g_{-i}(u_\varepsilon(x,y))|\right] \\ &\le |v_\varepsilon(x,y) - u_\varepsilon(x,y)|,\end{aligned}$$

and consequently

$$P_{K,0}(F(\cdot,\cdot,v_\varepsilon) - F(\cdot,\cdot,u_\varepsilon)) \le P_{K,0}(v_\varepsilon - u_\varepsilon).$$

It is sufficient to prove the relation for g_α. For each $K \Subset \mathbb{R}^2$ and $(x,y) \in K$, we have

$$\begin{aligned}\Psi_\varepsilon(x,y) &= g_\alpha(v_\varepsilon(x,y)) - g_\alpha(u_\varepsilon(x,y)) \\ &= \frac{-\alpha}{(1+\alpha v_\varepsilon(x,y))(1+\alpha u_\varepsilon(x,y))}\,(v_\varepsilon(x,y) - u_\varepsilon(x,y))\end{aligned}$$

and

$$|\Psi_\varepsilon(x,y)| \le |g_\alpha(v_\varepsilon(x,y)) - g_\alpha(u_\varepsilon(x,y))| \le |v_\varepsilon(x,y) - u_\varepsilon(x,y)|,$$

so

$$\sup_{(x,y)\in K} |\Psi_\varepsilon(x,y)| \le P_{K,0}(v_\varepsilon - u_\varepsilon).$$

We put

$$h_\varepsilon(x,y) = \frac{-\alpha}{(1+\alpha v_\varepsilon(x,y))(1+\alpha u_\varepsilon(x,y))} = -\alpha g_\alpha(v_\varepsilon(x,y)) g_\alpha(u_\varepsilon(x,y)).$$

As g_α and all the successive derivatives are bounded, for each integer n, $(\partial^n/\partial x^n)h_\varepsilon$ is bounded on K by a polynomial of

$$\|v_\varepsilon\|_{\infty,K}, \|u_\varepsilon\|_{\infty,K}, \left\|\frac{\partial v_\varepsilon}{\partial x}\right\|_{\infty,K}, \left\|\frac{\partial u_\varepsilon}{\partial x}\right\|_{\infty,K}, \dots, \left\|\frac{\partial^n v_\varepsilon}{\partial x^n}\right\|_{\infty,K}, \left\|\frac{\partial^n u_\varepsilon}{\partial x^n}\right\|_{\infty,K},$$

with positive coefficients, which we can write $d_n(K, u_\varepsilon, v_\varepsilon)$. According to Leibniz's rule we have

$$\frac{\partial^n \Psi_\varepsilon}{\partial x^n}(x,y) = -\alpha \sum_{i=0}^{n} C_n^i \, \frac{\partial^i h_\varepsilon}{\partial x^i}(x,y)\, \frac{\partial^{n-i}(v_\varepsilon - u_\varepsilon)}{\partial x^{n-i}}(x,y).$$

Consequently,

$$\begin{aligned}\sup_{(x,y)\in K}\left|\frac{\partial^n \Psi_\varepsilon}{\partial x^n}(x,y)\right| &\le \sum_{i=0}^{n} C_n^i d_i(K,u_\varepsilon,v_\varepsilon) P_{K,n-i}(v_\varepsilon - u_\varepsilon) \\ &\le \Big(\sum_{i=0}^{n} C_n^i d_i(K,u_\varepsilon,v_\varepsilon)\Big) P_{K,n}(v_\varepsilon - u_\varepsilon).\end{aligned}$$

From this, it may be deduced that

$$P_{K,n}(F(\cdot,\cdot,v_\varepsilon) - F(\cdot,\cdot,u_\varepsilon)) \le D_n P_{K,n}(v_\varepsilon - u_\varepsilon). \ \blacksquare$$

3.3. Parametric singular spectrum. We suppose that

$$\mathcal{N}^{\mathcal{A}}_{\mathcal{D}'}(\Omega) = \{(u_\varepsilon) \in \mathcal{X}(\Omega) : \lim_{\varepsilon\to 0} u_\varepsilon = 0 \text{ in } \mathcal{D}'(\Omega)\} \supset \mathcal{N}(\Omega).$$

Then we put

$$\mathcal{D}'_{\mathcal{A}}(\Omega) = \{[u_\varepsilon] \in \mathcal{A}(\Omega) : \exists T \in \mathcal{D}'(\Omega),\ \lim_{\varepsilon \to 0} u_\varepsilon = T \text{ in } \mathcal{D}'(\Omega)\}.$$

$\mathcal{D}'_{\mathcal{A}}(\Omega)$ is clearly well defined because the limit is independent of the chosen representative; indeed,

$$\lim_{\substack{\varepsilon\to 0\\ \mathcal{D}'(\mathbb{R})}} (u_\varepsilon + i_\varepsilon) = \lim_{\substack{\varepsilon\to 0\\ \mathcal{D}'(\mathbb{R})}} u_\varepsilon + \lim_{\substack{\varepsilon\to 0\\ \mathcal{D}'(\mathbb{R})}} i_\varepsilon = \lim_{\substack{\varepsilon\to 0\\ \mathcal{D}'(\mathbb{R})}} u_\varepsilon, \quad \text{since} \quad \lim_{\substack{\varepsilon\to 0\\ \mathcal{D}'(\mathbb{R})}} i_\varepsilon = 0.$$

$\mathcal{D}'_{\mathcal{A}}(\Omega)$ is an $\mathbb{R}$-vector subspace of $\mathcal{A}(\Omega)$. Therefore we can consider the set $\mathcal{O}_{D'_{\mathcal{A}}}$ of all x having a neighborhood V on which u is associated to a distribution:

$$\mathcal{O}_{D'_{\mathcal{A}}}(u) = \{x \in \Omega : \exists V \in \mathcal{V}(x),\, u|_V \in \mathcal{D}'_{\mathcal{A}}(V)\},$$

$\mathcal{V}(x)$ being the set of all neighborhoods of x.

Definition 29. We define the $\mathcal{D}'$-*singular support* of $u \in \mathcal{A}(\Omega)$, denoted sing $\text{supp}_{\mathcal{D}'}(u) = S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u)$, as

$$S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u) = \Omega \setminus \mathcal{O}_{D'_{\mathcal{A}}}(u).$$

Elements of parametric microlocal analysis. Let $u \in \mathcal{A}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. It may happen that $u = [u_\varepsilon]$ is not associated with any distribution in a neighborhood of x, that is, there is no open neighborhood V_x of x for which $\lim_{\varepsilon\to 0} u_\varepsilon|_{V_x}$ belongs to $\mathcal{D}'(V_x)$ (J.-A. Marti [1998], J.-A. Marti, S. P. Nuiro and V. S. Valmorin [1998b]). But in this case, it may happen that some real number r and some neighborhood V_x of x exist such that $\lim_{\varepsilon\to 0} \varepsilon^r u_\varepsilon|_{V_x}$ belongs to $\mathcal{D}'(V_x)$, that is, $[\varepsilon^r u_\varepsilon]$ belongs to $\mathcal{D}'_{\mathcal{A}}(V_x)$, the vector subspace of $\mathcal{A}(V_x)$ whose elements u are associated with some distribution of $\mathcal{D}'(V_x)$ (J.-A. Marti, S. P. Nuiro and V. S. Valmorin [1998b]).

We refer to J.-A. Marti [1995], J.-A. Marti, S. P. Nuiro and V. S. Valmorin [1998b].

Let Ω be an open subset of $\mathbb{R}^d$. For $x \in \Omega$ and $u = [u_\varepsilon] \in \mathcal{A}(\Omega)$, we put

$$N_{\mathcal{D}',x}(u) = \{r \in \mathbb{R}_+ : \exists V_x \in \mathcal{V}(x),\, \lim_{\varepsilon\to 0} \varepsilon^r u_\varepsilon|_{V_x} \in \mathcal{D}'(V_x)\}.$$

We can show that $N_{\mathcal{D}',x}(u)$ does not depend on the chosen representative of u and that if $N_{\mathcal{D}',x}(u)$ contains some $r_0 \in \mathbb{R}_+$, it must contain every $r \geq r_0$. Then one defines the $\mathcal{D}'$-fiber over x as

$$\Sigma_{\mathcal{D}',x}(u) = \mathbb{R}_+ \setminus N_{\mathcal{D}',x}(u).$$

This is either a bounded interval of $\mathbb{R}_+$ of the form $[0, r[$ or $[0, r]$, $\mathbb{R}_+$ itself, or the empty set.

Then we can give the following definition of the parametric singular spectrum of a generalized function:

Definition 30. We define the $\mathcal{D}'$-*parametric singular spectrum* of $u \in \mathcal{A}(\Omega)$ as the following subset of $\Omega \times \mathbb{R}_+$:

$$S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u = \{(x, r) \in \Omega \times \mathbb{R}_+ : r \in \Sigma_{\mathcal{D}',x}(u)\}.$$

Remark 31. We have $\Sigma_{\mathcal{D}',x}(u) = \emptyset$ if, and only if, there exists a neighborhood V_x of x such that

$$\lim_{\varepsilon\to 0} u_\varepsilon|_{V_x} \in \mathcal{D}'(V_x),$$

that is, if, and only if, x does not belong to the $\mathcal{D}'$-singular support of u, $\mathcal{S}^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u)$. It follows that the projection on Ω of $S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u$ is exactly $S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u$.

Theorem 32. *Let $u, v \in \mathcal{A}(\Omega)$. Then*

$$S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u+v) \subset S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u) \cup S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(v).$$

Proof. Let $r \in N_{\mathcal{D}',x}(u) \cap N_{\mathcal{D}',x}(v)$. Then there exist $V_x, W_x \in \mathcal{V}(x)$ such that

$$\lim_{\varepsilon \to 0} \varepsilon^r u_\varepsilon|_{V_x} \in \mathcal{D}'(V_x) \quad \text{and} \quad \lim_{\varepsilon \to 0} \varepsilon^r v_\varepsilon|_{W_x} \in \mathcal{D}'(W_x).$$

From this it may be deduced that

$$\lim_{\varepsilon \to 0} \varepsilon^r (u_\varepsilon + v_\varepsilon)|_{V_x \cap W_x} \in \mathcal{D}'(V_x \cap W_\varepsilon),$$

so $r \in N_{\mathcal{D}',x}(u+v)$ and consequently

$$N_{\mathcal{D}',x}(u) \cap N_{\mathcal{D}',x}(v) \subset N_{\mathcal{D}',x}(u+v).$$

We obtain the result by taking complements in $\mathbb{R}_+$. ∎

Corollary 33. *For any u, u_0, u_1 in $\mathcal{A}(\Omega)$ with*

$$u = u_0 + u_1, \tag{3.2}$$

$$S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u_0) = \emptyset, \tag{3.3}$$

we have

$$S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u) = S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u_1).$$

Proof. The previous theorem and condition (3.3) give

$$S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u) \subset S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u_1).$$

But, as (3.2) implies $u_0 = u - u_1$, we obtain of course the converse inclusion, and thus the result. ∎

Theorem 34. *Let $u \in \mathcal{A}(\Omega)$. Then $S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(D^\alpha u) \subset S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u)$ for all $\alpha \in \mathbb{N}^d$.*

Proof. Let $r \in N_{\mathcal{D}',x}(u)$. There exists $V_x \in \mathcal{V}(x)$ such that

$$\lim_{\varepsilon \to 0} \varepsilon^r u_\varepsilon|_{V_x} = T \in \mathcal{D}'(V_x).$$

The continuity of D^α implies

$$\lim_{\varepsilon \to 0} \varepsilon^r D^\alpha u_\varepsilon|_{V_x} = \lim_{\varepsilon \to 0} D^\alpha \varepsilon^r u_\varepsilon|_{V_x} = D^\alpha T \in \mathcal{D}'(V_x).$$

Thus $N_{\mathcal{D}',x}(u) \subset N_{\mathcal{D}',x}(D^\alpha u)$; we obtain the result by taking complements in $\mathbb{R}_+$. ∎

Theorem 35. *Let $f \in \mathrm{C}^\infty(\Omega)$ and $u \in \mathcal{A}(\Omega)$. Then $S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(fu) \subset S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u)$.*

Proof. Let $r \in N_{\mathcal{D}',x}(u)$. There exists $V_x \in \mathcal{V}(x)$ such that $\lim_{\varepsilon \to 0} \varepsilon^r u_\varepsilon|_{V_x} = T \in \mathcal{D}'(V_x)$, that is, for each $\varphi \in \mathcal{D}(V_x)$,

$$\lim_{\varepsilon \to 0} \int \varepsilon^r u_\varepsilon(x)\varphi(x)\, dx = \langle T, \varphi \rangle.$$

Thus, we have

$$\lim_{\varepsilon \to 0} \int \varepsilon^r (f u_\varepsilon)(x)\varphi(x)\, dx = \lim_{\varepsilon \to 0} \int \varepsilon^r u_\varepsilon(x) f\varphi(x)\, dx = \langle T, f\varphi \rangle = \langle fT, \varphi \rangle.$$

It follows that

$$\lim_{\varepsilon\to 0} \varepsilon^r f u_\varepsilon|_{V_x} = fT \in \mathcal{D}'(V_x)$$

and therefore $r \in N_{\mathcal{D}',x}(fu)$. From $N_{\mathcal{D}',x}(u) \subset N_{\mathcal{D}',x}(fu)$, we can deduce the result. ∎

Corollary 36. *Let $P(D) = \sum_{|\alpha|\le m} C_\alpha D^\alpha$ be a differential polynomial with coefficients in $\mathrm{C}^\infty(\Omega)$. Then $S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(P(D)u) \subset S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u)$ for any $u \in \mathcal{A}(\Omega)$.*

Proof. Write $P(D)u = \sum_{|\alpha|\le m} C_\alpha D^\alpha u$ and apply the previous theorems. ∎

4. Generalized Cauchy problem

4.1. Formulation of the problem. We take up again the formulation of the Cauchy problem posed in Subsection 1.1 in the form

$$(P_G)\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = F(\cdot,\cdot,u), \\ u|_\gamma = \varphi, \\ \left.\dfrac{\partial u}{\partial y}\right|_\gamma = \psi, \end{cases}$$

but now we search for u in the algebra of generalized functions $\mathcal{A}(\mathbb{R}^2)$ defined in the previous section. $\varphi = [\varphi_\varepsilon]$, $\psi = [\psi_\varepsilon]$, $\varphi_\varepsilon, \psi_\varepsilon$: $\mathbb{R} \to \mathbb{R}$ are some smooth one-variable functions, the hypotheses on F and f are kept, $\mathcal{A}(\mathbb{R})$ and $\mathcal{A}(\mathbb{R}^2)$ are built on the same ring of generalized constants, $\mathcal{A}(\mathbb{R}^2)$ is stable under F. We suppose that, for every ε, the problem

$$P_\infty(\varphi_\varepsilon,\psi_\varepsilon)\begin{cases} \dfrac{\partial^2 u_\varepsilon}{\partial x \partial y}(x,y) = F(x,y,u_\varepsilon(x,y)), \\ u_\varepsilon(x,f(x)) = \varphi_\varepsilon(x), \\ \dfrac{\partial u_\varepsilon}{\partial y}(x,f(x)) = \psi_\varepsilon(x), \end{cases}$$

has a solution $u_\varepsilon \in \mathrm{C}^\infty(\mathbb{R}^2)$.

Giving a meaning to (P_G) is first giving a meaning to

$$\frac{\partial^2 u}{\partial x \partial y} = F(\cdot,\cdot,u), \tag{4.1}$$

$$u|_\gamma = \varphi \in \mathcal{A}(\mathbb{R}), \tag{4.2}$$

$$\left.\frac{\partial u}{\partial y}\right|_\gamma = \psi \in \mathcal{A}(\mathbb{R}), \tag{4.3}$$

when $u \in \mathcal{A}(\mathbb{R}^2)$ and γ is the smooth submanifold of $\mathbb{R}^2$ defined by $y = f(x)$. Giving a meaning to (4.1), under the hypothesis that $\mathcal{A}(\mathbb{R}^2)$ is stable under F, amounts to saying that, for a representative $(u_\varepsilon)_\varepsilon$ of u, we must have for all $(i_\varepsilon)_\varepsilon, (j_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$,

$$\left(\frac{\partial^2 (u_\varepsilon + i_\varepsilon)}{\partial x \partial y} - F(\cdot,\cdot,u_\varepsilon + j_\varepsilon)\right)_\varepsilon \in \mathcal{N}(\mathbb{R}^2).$$

As

$$\left(\frac{\partial^2(u_\varepsilon+i_\varepsilon)}{\partial x\partial y}-\frac{\partial^2 u_\varepsilon}{\partial x\partial y}\right)_\varepsilon \in \mathcal{N}(\mathbb{R}^2) \text{ and } (F(\cdot,\cdot,,u_\varepsilon+j_\varepsilon)-F(\cdot,\cdot,u_\varepsilon))_\varepsilon \in \mathcal{N}(\mathbb{R}^2),$$

we must verify that

$$\left(\frac{\partial^2 u_\varepsilon}{\partial x\partial y}-F(\cdot,\cdot,u_\varepsilon)\right)_\varepsilon \in \mathcal{N}(\mathbb{R}^2).$$

Giving a meaning to (4.2) and (4.3) amounts to defining $u|_\gamma$ and $(\partial/\partial y)u|_\gamma$. As γ is a smooth submanifold of $\mathbb{R}^2$ that can be represented by a single map ($\gamma=f(x)$), we can identify $\mathcal{A}(\gamma)$ and $\mathcal{A}(\mathbb{R})$ and so $u|_\gamma$ to the element of $\mathcal{A}(\mathbb{R})$ with representative $(x\mapsto u_\varepsilon(x,f(x)))_\varepsilon$ and we can identify $(\partial/\partial y)u|_\gamma$ to the element of $\mathcal{A}(\mathbb{R})$ with representative $(x\mapsto \frac{\partial u_\varepsilon}{\partial y}(x,f(x)))_\varepsilon$. So (4.2) is equivalent to

$$(x\mapsto ((u_\varepsilon+i_\varepsilon)(x,f(x))-(\varphi_\varepsilon+\alpha_\varepsilon)(x)))_\varepsilon \in \mathcal{N}(\mathbb{R}).$$

(4.3) is equivalent to

$$\left(x\mapsto\left(\left(\frac{\partial(u_\varepsilon+i_\varepsilon)}{\partial y}\right)(x,f(x))-(\psi_\varepsilon+\beta_\varepsilon)(x)\right)\right)_\varepsilon \in \mathcal{N}(\mathbb{R}),$$

for all $(i_\varepsilon)_\varepsilon\in\mathcal{N}(\mathbb{R}^2)$, $(\alpha_\varepsilon)_\varepsilon,(\beta_\varepsilon)_\varepsilon\in\mathcal{N}(\mathbb{R})$. Considering

$$(x\mapsto ((u_\varepsilon+i_\varepsilon)(x,f(x))-u_\varepsilon(x,f(x))))_\varepsilon \in \mathcal{N}(\mathbb{R}),$$
$$(x\mapsto ((\varphi_\varepsilon+\alpha_\varepsilon)(x)-\varphi_\varepsilon(x)))_\varepsilon \in \mathcal{N}(\mathbb{R}),$$
$$\left(x\mapsto\left(\left(\frac{\partial(u_\varepsilon+i_\varepsilon)}{\partial y}\right)(x,f(x))-\frac{\partial u_\varepsilon}{\partial y}(x,f(x))\right)\right)_\varepsilon \in \mathcal{N}(\mathbb{R}),$$
$$(x\mapsto ((\psi_\varepsilon+\beta_\varepsilon)(x)-\psi_\varepsilon(x)))_\varepsilon \in \mathcal{N}(\mathbb{R}),$$
$$(x\mapsto (j_\varepsilon(x)-i_\varepsilon(x,f(x))))_\varepsilon \in \mathcal{N}(\mathbb{R}),$$

this boils down to

$$(x\mapsto (u_\varepsilon(x,f(x))-\varphi_\varepsilon(x)))_\varepsilon \in \mathcal{N}(\mathbb{R}),$$
$$\left(x\mapsto\left(\frac{\partial u_\varepsilon}{\partial y}(x,f(x))-\psi_\varepsilon(x)\right)\right)_\varepsilon \in \mathcal{N}(\mathbb{R}).$$

To sum up, (P_G) has a meaning if, and only if,

$$\begin{cases}\left(\dfrac{\partial^2 u_\varepsilon}{\partial x\partial y}-F(\cdot,\cdot,u_\varepsilon)\right)_\varepsilon \in \mathcal{N}(\mathbb{R}^2),\\ (x\mapsto (u_\varepsilon(x,f(x))-\varphi_\varepsilon(x)))_\varepsilon \in \mathcal{N}(\mathbb{R}),\\ \left(x\mapsto\left(\dfrac{\partial u_\varepsilon}{\partial y}(x,f(x))-\psi_\varepsilon(x)\right)\right)_\varepsilon \in \mathcal{N}(\mathbb{R}).\end{cases}$$

So, if for every ε, u_ε is a solution to $P_\infty(\varphi_\varepsilon,\psi_\varepsilon)$ and if $(u_\varepsilon)_\varepsilon\in\mathcal{X}(\mathbb{R}^2)$ then the relations above are all the more true and $[u_\varepsilon]$ is a solution to (P_G).

4.2. Existence and uniqueness of solutions

Theorem 37. *Suppose that $\mathcal{A}(\mathbb{R}^2)$ is stable under F and $\mathcal{A}(\mathbb{R})$, $\mathcal{A}(\mathbb{R}^2)$ are built on the same ring $\mathcal{C}=A/I$ of generalized constants. Suppose that the data of problem (P_G) satisfy* *the conditions* $\varphi, \psi \in \mathcal{A}(\mathbb{R})$, $f \in \mathrm{C}^\infty(\mathbb{R})$. *Then problem* (P_G) *has a unique solution in* $\mathcal{A}(\mathbb{R}^2)$.

Proof. Let u_ε be the solution to $P_\infty(\varphi_\varepsilon, \psi_\varepsilon)$. According to the previous result, it is enough to prove $(u_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{R}^2)$; then $u = [u_\varepsilon]$ will be a solution to (P_G). We will prove that

$$\forall K \Subset \mathbb{R}^2, \forall l \in \mathbb{N}, \qquad (P_{K,l}(u_\varepsilon))_\varepsilon \in A_+.$$

Proceeding by induction, we first show that

$$\forall K \Subset \mathbb{R}^2, \qquad (P_{K,(0,0)}(u_\varepsilon))_\varepsilon = (\|u_\varepsilon\|_{\infty,K})_\varepsilon \in A_+,$$

that is, the 0th order estimate is satisfied. Put

$$u_{0,\varepsilon}(x,y) = \chi_\varepsilon(y) - \chi_\varepsilon(f(x)) + \varphi_\varepsilon(x)$$

where χ_ε indicates a primitive of $\psi_\varepsilon \circ f^{-1}$. According to Proposition 5, for each $K \Subset \mathbb{R}^2$ there exists $K_\lambda \Subset \mathbb{R}^2$ with $K \subset K_\lambda$ such that

$$\|u_\varepsilon\|_{\infty,K} \le \|u_\varepsilon\|_{\infty,K_\lambda} \le \|u_{0,\varepsilon}\|_{\infty,K_\lambda} + \frac{\Phi_{\lambda,\varepsilon}}{m_\lambda} \exp(2\lambda m_\lambda(f(\lambda) - f(-\lambda))).$$

We have $(\|u_{0,\varepsilon}\|_{\infty,K_\lambda})_\varepsilon \in A$ because $[\varphi_\varepsilon]$ and $[\psi_\varepsilon]$ are elements of $\mathcal{A}(\mathbb{R})$. The constant

$$m_\lambda = \sup_{(x,y)\in K_\lambda; t\in\mathbb{R}} \left| \frac{\partial F}{\partial z}(x,y,t) \right|$$

depends only on F, K_λ, and the constant

$$c(K_\lambda) = \frac{1}{m_\lambda} \exp(2\lambda m_\lambda(f(\lambda) - f(-\lambda)))$$

depends only on F, f, K_λ. We have

$$\Phi_{\lambda,\varepsilon} = \|F(\cdot,\cdot,0)\|_{\infty,K_\lambda} + m_\lambda \|u_{0,\varepsilon}\|_{\infty,K_\lambda}$$

so

$$\begin{aligned} c(K_\lambda)\Phi_{\lambda,\varepsilon} &= \frac{\Phi_{\lambda,\varepsilon}}{m_\lambda} \exp[2\lambda m_\lambda(f(\lambda) - f(-\lambda))] \\ &= c(K_\lambda)\|F(\cdot,\cdot,0)\|_{\infty,K_\lambda} + \exp(2\lambda m_\lambda(f(\lambda) - f(-\lambda)))\|u_{0,\varepsilon}\|_{\infty,K_\lambda}. \end{aligned}$$

Moreover, the constant

$$c_1(K_\lambda) = c(K_\lambda)\|F(\cdot,\cdot,0)\|_{\infty,K_\lambda}$$

depends only on F, K_λ and $c_2(K_\lambda) = \exp(2\lambda m_\lambda(f(\lambda) - f(-\lambda)))$ depends entirely on K_λ, F, f. Consequently,

$$\|u_\varepsilon\|_{\infty,K} \le \|u_\varepsilon\|_{\infty,K_\lambda} \le (1 + c_2(K_\lambda))\|u_{0,\varepsilon}\|_{\infty,K_\lambda} + c_1(K_\lambda).$$

Since $(\|u_{0,\varepsilon}\|_{\infty,K_\lambda})_\varepsilon \in A$, we have

$$((1 + c_2(K_\lambda))\|u_{0,\varepsilon}\|_{\infty,K_\lambda})_\varepsilon \in A$$

(if $(r_\varepsilon)_\varepsilon \in A$ then $(cr_\varepsilon)_\varepsilon \in A$) and as $c_1(K_\lambda)$ is a constant $((1)_\varepsilon \in A)$, we deduce that

$$((1 + c_2(K_\lambda))\|u_{0,\varepsilon}\|_{\infty,K_\lambda} + c_1(K_\lambda))_\varepsilon \in A.$$

A being stable, we have $(\|u_\varepsilon\|_{\infty,K_\lambda})_\varepsilon \in A_+$ and so $(\|u_\varepsilon\|_{\infty,K})_\varepsilon \in A_+$, that is, $(P_{K,0}(u_\varepsilon))_\varepsilon \in A_+$. Let us show that $(P_{K,1}(u_\varepsilon))_\varepsilon \in A_+$. We have

$$\frac{\partial u_\varepsilon}{\partial x}(x,y) = \frac{\partial u_{0,\varepsilon}}{\partial x}(x,y) + \int\limits_{f(x)}^{y} F(x,\eta,u_\varepsilon(x,\eta))\,d\eta,$$

hence

$$P_{K,(1,0)}(u_\varepsilon) \leq \sup_K \left|\frac{\partial u_{0,\varepsilon}}{\partial x}(x,y)\right| + (f(\lambda)-f(-\lambda)) \sup_{K_\lambda} |F(x,\eta,u_\varepsilon(x,\eta))|.$$

$\mathcal{A}(\mathbb{R}^2)$ being stable under F, there exists $C>0$ such that

$$P_{K_\lambda,(0,0)}(F(\cdot,\cdot,u_\varepsilon)) \leq P_{K_\lambda,0}(F(\cdot,\cdot,u_\varepsilon)) \leq C.$$

We have

$$(\|(\partial/\partial x)u_{0,\varepsilon}\|_{\infty,K})_\varepsilon \in A_+$$

because $[\varphi_\varepsilon]$ and $[\psi_\varepsilon]$ are elements of $\mathcal{A}(\mathbb{R})$. So

$$P_{K,(1,0)}(u_\varepsilon) \leq \|(\partial/\partial x)u_{0,\varepsilon}\|_{\infty,K} + C(f(\lambda)-f(-\lambda)).$$

A being stable, we get $(P_{K,(1,0)}(u_\varepsilon))_\varepsilon \in A^+$. We have

$$\frac{\partial u_\varepsilon}{\partial y}(x,y) = \frac{\partial u_{0,\varepsilon}}{\partial y}(x,y) - \int\limits_{x}^{f^{-1}(y)} F(\xi,y,u_\varepsilon(\xi,y))\,d\xi,$$

so

$$P_{K,(0,1)}(u_\varepsilon) \leq \sup_K \left|\frac{\partial u_{0,\varepsilon}}{\partial y}(x,y)\right| + 2\lambda \sup_{K_\lambda} |F(x,\eta,u_\varepsilon(x,\eta))|.$$

We have

$$(\|(\partial/\partial y)u_{0,\varepsilon}\|_{\infty,K})_\varepsilon \in A_+$$

because $[\psi_\varepsilon]$ is element of $\mathcal{A}(\mathbb{R})$; hence

$$P_{K,(0,1)}(u_\varepsilon) \leq \|(\partial/\partial y)u_{0,\varepsilon}\|_{\infty,K} + C2\lambda$$

and so, as previously,

$$(\|(\partial/\partial y)u_\varepsilon\|_{\infty,K\varepsilon})_\varepsilon \in A_+.$$

Now we proceed by induction. Suppose that $(P_{K,l}(u_\varepsilon))_\varepsilon \in A_+$ for every $l \leq n$, and let us show that this implies $(P_{K,n+1}(u_\varepsilon))_\varepsilon \in A_+$. We have $P_{K,n+1} = \max(P_{K,n}, P_{1,n}, P_{2,n}, P_{3,n}, P_{4,n})$ with

$$P_{1,n} = P_{K,(n+1,0)}, \qquad P_{2,n} = P_{K,(0,n+1)},$$
$$P_{3,n} = \sup_{\alpha+\beta=n;\beta\geq 1} P_{K,(\alpha+1,\beta)}, \qquad P_{4,n} = \sup_{\alpha+\beta=n;\alpha\geq 1} P_{K,(\alpha,\beta+1)}.$$

First let us show that $(P_{1,n}(u_\varepsilon))_\varepsilon, (P_{2,n}(u_\varepsilon))_\varepsilon \in A_+$ for every $n \in \mathbb{N}$. We have by successive derivations, for $n \geq 1$,

$$\frac{\partial^{n+1} u_\varepsilon}{\partial x^{n+1}}(x,y) = \frac{\partial^{n+1} u_{0,\varepsilon}}{\partial x^{n+1}}(x,y) - \sum_{j=0}^{n-1} C_n^j f^{(n-j)}(x) \frac{\partial^j}{\partial x^j} F(x,f(x),\varphi_\varepsilon(x)) + \int\limits_{f(x)}^{y} \frac{\partial^n}{\partial x^n} F(x,\eta,u_\varepsilon(x,\eta))\,d\eta.$$

As $K \subset K_\lambda$, we can write

$$\begin{aligned}
&\sup_{(x,y)\in K}\left|\frac{\partial^{n+1}u_\varepsilon}{\partial x^{n+1}}(x,y)\right|\\
&\qquad \le \left\|\frac{\partial^{n+1}u_{0,\varepsilon}}{\partial x^{n+1}}\right\|_{\infty,K} + \sup_{x\in[-\lambda,\lambda]}\sum_{j=0}^{n-1} C_n^j |f^{(n-j)}(x)| \left|\frac{\partial^j}{\partial x^j}F(x,f(x),\varphi_\varepsilon(x))\right|\\
&\qquad\quad + (f(\lambda)-f(-\lambda))\sup_{(x,y)\in K}\left|\frac{\partial^n}{\partial x^n}F(x,y,u_\varepsilon(x,y))\right|.
\end{aligned}$$

We have

$$\sup_{(x,y)\in K}\left|\frac{\partial^n}{\partial x^n}F(x,y,u_\varepsilon(x,y))\right| = P_{K,(n,0)}(F(\cdot,\cdot,u_\varepsilon)) \le P_{K,n}(F(\cdot,\cdot,u_\varepsilon)),$$

and

$$\begin{aligned}
\sup_{x\in[-\lambda,\lambda]}\left|\frac{\partial^j}{\partial x^j}F(x,f(x),\varphi_\varepsilon(x))\right| &\le P_{K,(j,0)}(F(\cdot,\cdot,u_\varepsilon))\\
&\le P_{K,(n,0)}(F(\cdot,\cdot,u_\varepsilon)) \le P_{K,n}(F(\cdot,\cdot,u_\varepsilon)),
\end{aligned}$$

moreover

$$(\|(\partial^{n+1}/\partial x^{n+1})u_{0,\varepsilon}\|_{\infty,K})_\varepsilon \in A_+.$$

According to the stability hypothesis, a simple calculation shows that, for every $K \Subset \mathbb{R}^2$, $(P_{K,(n+1,0)}(u_\varepsilon))_\varepsilon \in A_+$.

Let us show that $(P_{2,n}(u_\varepsilon))_\varepsilon \in A_+$ for every $n \in \mathbb{N}$. We have by successive derivations, for $n \ge 1$,

$$\begin{aligned}
\frac{\partial^{n+1}u_\varepsilon}{\partial y^{n+1}}(x,y) &= \frac{\partial^{n+1}u_{0,\varepsilon}}{\partial y^{n+1}}(x,y) - \int_x^{f^{-1}(y)} \frac{\partial^n}{\partial y^n}F(\xi,y,u_\varepsilon(\xi,y))\,d\xi\\
&\quad - \sum_{j=0}^{n-1} C_n^j (f^{-1})^{(n-j)}(y)\frac{\partial^j}{\partial y^j}F(f^{-1}(y),y,\varphi_\varepsilon(f^{-1}(y))).
\end{aligned}$$

As $K \subset K_\lambda$, we can write

$$\begin{aligned}
&\sup_{(x,y)\in K}\left|\frac{\partial^{n+1}u_\varepsilon}{\partial y^{n+1}}(x,y)\right|\\
&\qquad \le \left\|\frac{\partial^{n+1}u_{0,\varepsilon}}{\partial y^{n+1}}\right\|_{\infty,K} + 2\lambda\sup_{(x,y)\in K}\left|\frac{\partial^n}{\partial y^n}F(x,y,u_\varepsilon(x,y))\right|\\
&\qquad\quad + \sup_{y\in[f(-\lambda),f(\lambda)]}\sum_{j=0}^{n-1} C_n^j |(f^{-1})^{(n-j)}(y)| \left|\frac{\partial^j}{\partial y^j}F(f^{-1}(y),y,\varphi_\varepsilon(f^{-1}(y)))\right|.
\end{aligned}$$

We have

$$\sup_{(x,y)\in K}\left|\frac{\partial^n}{\partial y^n}F(x,y,u_\varepsilon(x,y))\right| = P_{K,(0,n)}(F(\cdot,\cdot,u_\varepsilon)) \le P_{K,n}(F(\cdot,\cdot,u_\varepsilon))$$

and

$$\sup_{y\in[f(-\lambda),f(\lambda)]}\left|\frac{\partial^j}{\partial y^j}F(f^{-1}(y),y,\varphi_\varepsilon(f^{-1}(y)))\right| \leq \sup_{(x,y)\in K}\left|\frac{\partial^i}{\partial y^i}F(x,y,u_\varepsilon(x,y))\right|$$
$$\leq P_{K,i}(F(\cdot,\cdot,u_\varepsilon)) \leq P_{K,n}(F(\cdot,\cdot,u_\varepsilon)).$$

According to the stability hypothesis, a simple calculation shows that, for every $K \Subset \mathbb{R}^2$ and $n \in \mathbb{N}$, $(P_{K,(0,n+1)}(u_\varepsilon))_\varepsilon \in A_+$. For $\alpha+\beta = n$ and $\beta \geq 1$, we now have

$$\begin{aligned}P_{K,(\alpha+1,\beta)}(u_\varepsilon) &= \sup_{(x,y)\in K}|D^{(\alpha+1,\beta)}u_\varepsilon(x,y)| = \sup_{(x,y)\in K}|D^{(\alpha,\beta-1)}D^{(1,1)}u_\varepsilon(x,y)|\\ &= \sup_{(x,y)\in K}|D^{(\alpha,\beta-1)}F(x,y,u_\varepsilon(x,y))| = P_{K,(\alpha,\beta-1)}(F(\cdot,\cdot,u_\varepsilon))\\ &\leq P_{K,n-1}(F(\cdot,\cdot,u_\varepsilon)) \leq P_{K,n}(F(\cdot,\cdot,u_\varepsilon)).\end{aligned}$$

So we finally have

$$P_{3,n}(u_\varepsilon) = \sup_{\alpha+\beta=n;\beta\geq1} P_{K,(\alpha+1,\beta)}(u_\varepsilon) \leq P_{K,n}(F(\cdot,\cdot,u_\varepsilon))$$

and the stability hypothesis ensures that $(P_{3,n}(u_\varepsilon))_\varepsilon \in A_+$. In the same way, for $\alpha+\beta = n$ and $\alpha \geq 1$, we have

$$\begin{aligned}P_{K,(\alpha,\beta+1)}(u_\varepsilon) &= \sup_{(x,y)\in K}|D^{(\alpha,\beta+1)}u_\varepsilon(x,y)| = \sup_{(x,y)\in K}|D^{(\alpha-1,\beta)}D^{(1,1)}u_\varepsilon(x,y)|\\ &= \sup_{(x,y)\in K}|D^{(\alpha-1,\beta)}F(x,y,u_\varepsilon(x,y))| = P_{K,(\alpha-1,\beta)}(F(\cdot,\cdot,u_\varepsilon))\\ &\leq P_{K,n-1}(F(\cdot,\cdot,u_\varepsilon)) \leq P_{K,n}(F(\cdot,\cdot,u_\varepsilon)).\end{aligned}$$

So we have

$$P_{4,n}(u_\varepsilon) = \sup_{\alpha+\beta=n;\alpha\geq1} P_{K,(\alpha,\beta+1)}(u_\varepsilon) \leq P_{K,n}(F(\cdot,\cdot,u_\varepsilon))$$

and the stability hypothesis ensures that $(P_{4,n}(u_\varepsilon))_\varepsilon \in A_+$. Finally, we clearly have $(P_{K,n+1}(u_\varepsilon))_\varepsilon \in A_+$.

Let us show that u is the unique solution to (P_G). Let $v = [v_\varepsilon]$ be another solution to (P_G). There are $(i_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$ and $(\alpha_\varepsilon)_\varepsilon, (\beta_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R})$, such that

$$\begin{cases}\dfrac{\partial^2 v_\varepsilon}{\partial x\partial y}(x,y) = F(x,y,v_\varepsilon(x,y)) + i_\varepsilon(x,y),\\ v_\varepsilon(x,f(x)) = \varphi_\varepsilon(x) + \alpha_\varepsilon(x),\\ \dfrac{\partial v_\varepsilon}{\partial y}(x,f(x)) = \psi_\varepsilon(x) + \beta_\varepsilon(x).\end{cases}$$

The uniqueness of the solution to (P_G) will be a consequence of $(v_\varepsilon - u_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$. It is easy to see that

$$\Big((x,y) \mapsto \iint_{D(x,y,f)} i_\varepsilon(\xi,\eta)\,d\xi\,d\eta\Big)_\varepsilon \in \mathcal{N}(\mathbb{R}^2).$$

So there is $(j_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$ such that

$$v_\varepsilon(x,y) = v_{0,\varepsilon}(x,y) - \iint_{D(x,y,f)} F(\xi,\eta,v_\varepsilon(\xi,\eta))\,d\xi\,d\eta + j_\varepsilon(x,y),$$

with $v_{0,\varepsilon}(x,y) = u_{0,\varepsilon}(x,y) + \theta_\varepsilon(x,y)$, where $\theta_\varepsilon(x,y) = B_\varepsilon(y) - B_\varepsilon(f(x)) + \alpha_\varepsilon(x)$ and B_ε is a primitive of $\beta_\varepsilon \circ f^{-1}$. So $(\theta_\varepsilon)_\varepsilon$ belongs to $\mathcal{N}(\mathbb{R}^2)$. Hence there is $(\sigma_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$ such that

$$v_\varepsilon(x,y) = u_{0,\varepsilon}(x,y) + \sigma_\varepsilon(x,y) - \iint\limits_{D(x,y,f)} F(\xi,\eta,v_\varepsilon(\xi,\eta))\, d\xi\, d\eta.$$

Let us put $w_\varepsilon = v_\varepsilon - u_\varepsilon$ and show that $(w_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$. We have to prove that

$$\forall K \Subset \mathbb{R}^2, \forall n \in \mathbb{N}, \qquad (P_{K,n}(w_\varepsilon))_\varepsilon \in I_A^+.$$

We proceed by induction showing first that $(P_{K,1}(w_\varepsilon))_\varepsilon \in I_A$. We have

$$w_\varepsilon(x,y) = \iint\limits_{D(x,y,f)} (-F(\xi,\eta,v_\varepsilon(\xi,\eta)) + F(\xi,\eta,u_\varepsilon(\xi,\eta)))\, d\xi\, d\eta + \sigma_\varepsilon(x,y),$$

but

$$\begin{aligned} &F(\xi,\eta,v_\varepsilon(\xi,\eta)) - F(\xi,\eta,u_\varepsilon(\xi,\eta)) \\ &\qquad = (v_\varepsilon(\xi,\eta) - u_\varepsilon(\xi,\eta))\bigg(\int_0^1 \frac{\partial F}{\partial z}(\xi,\eta,u_\varepsilon(\xi,\eta) + \theta(v_\varepsilon(\xi,\eta) - u_\varepsilon(\xi,\eta)))\, d\theta\bigg), \end{aligned}$$

so

$$w_\varepsilon(x,y) = -\iint\limits_{D(x,y,f)} w_\varepsilon(\xi,\eta)\bigg(\int_0^1 \frac{\partial F}{\partial z}(\xi,\eta,u_\varepsilon(\xi,\eta) + \theta(w_\varepsilon(\xi,\eta)))\, d\theta\bigg)\, d\xi\, d\eta + \sigma_\varepsilon(x,y).$$

Let $(x,y) \in K_\lambda$. Since $D(x,y,f) \subset K_\lambda$, if $y \geq f(x)$, we have

$$\begin{aligned} |w_\varepsilon(x,y)| &\leq m_\lambda \int_x^{f^{-1}(y)} \int_{f(\xi)}^{y} |w_\varepsilon(\xi,\eta)|\, d\xi\, d\eta + \|\sigma_\varepsilon\|_{\infty,K_\lambda} \\ &\leq m_\lambda \int_{-\lambda}^{+\lambda} \int_{f(x)}^{y} |w_\varepsilon(\xi,\eta)|\, d\xi\, d\eta + \|\sigma_\varepsilon\|_{\infty,K_\lambda}. \end{aligned}$$

Put $e_\varepsilon(y) = \sup_{\xi \in [-\lambda,\lambda]} |w_\varepsilon(\xi,y)|$. Then

$$|w_\varepsilon(x,y)| \leq m_\lambda 2\lambda \int_{f(-\lambda)}^{y} e_\varepsilon(\eta)\, d\eta + \|\sigma_\varepsilon\|_{\infty,k_\lambda}.$$

We deduce that

$$\forall y \in [f(-\lambda), f(\lambda)], \quad \text{if } y \geq f(x), \quad e_\varepsilon(y) \leq m_\lambda 2\lambda \int_{f(-\lambda)}^{y} e_\varepsilon(\eta)\, d\eta + \|\sigma_\varepsilon\|_{\infty,K_\lambda}.$$

Thus, according to Gronwall's lemma,

$$\forall y \in [f(-\lambda), f(\lambda)], \quad \text{if } y \geq f(x), \quad e_\varepsilon(y) \leq \exp\bigg(\int_{f(-\lambda)}^{y} m_\lambda 2\lambda\, d\eta\bigg)\|\sigma_\varepsilon\|_{\infty,K_\lambda}.$$

We obtain the same result for $y \le f(x)$. Hence, for every $y \in [f(-\lambda), f(\lambda)]$, we get

$$\begin{aligned} e_\varepsilon(y) &\le \exp(m_\lambda 2\lambda(y - f(-\lambda)))\|\sigma_\varepsilon\|_{\infty,K_\lambda} \\ &\le \exp(m_\lambda(2\lambda)(f(\lambda) - f(-\lambda)))\|\sigma_\varepsilon\|_{\infty,K_\lambda}, \end{aligned}$$

and consequently

$$\|w_\varepsilon\|_{\infty,K_\lambda} \le \exp(m_\lambda 2\lambda(f(\lambda) - f(-\lambda)))\|\sigma_\varepsilon\|_{\infty,K_\lambda}.$$

Since $(\sigma_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$ we have $(\|\sigma_\varepsilon\|_{\infty,K_\lambda})_\varepsilon \in I_A$. Moreover $\exp(m_\lambda 2\lambda(f(\lambda) - f(-\lambda)))$ is a constant, so

$$(\|w_\varepsilon\|_{\infty,K_\lambda})_\varepsilon \in I_A.$$

This implies the 0th order estimate.

We now proceed by induction. Suppose that $(P_{K,l}(w_\varepsilon))_\varepsilon \in I_A^+$ for every $l \le n$ and let us show that $(P_{K,n+1}(w_\varepsilon))_\varepsilon \in I_A^+$. First we show hat $(P_{1,n}(w_\varepsilon))_\varepsilon \in I_A^+$ for every $n \in \mathbb{N}$. We have

$$\begin{aligned} \frac{\partial^{n+1} w_\varepsilon}{\partial x^{n+1}}(x,y) &= \frac{\partial^{n+1}\sigma_\varepsilon}{\partial x^{n+1}}(x,y) + \delta_\varepsilon(x) \\ &\quad + \int\limits_{f(x)}^{y} \frac{\partial^n}{\partial x^n}(F(x,\eta,v_\varepsilon(x,\eta)) - F(x,\eta,u_\varepsilon(x,\eta)))\,d\eta, \end{aligned}$$

with

$$\delta_\varepsilon(x) = \Big(\sum_{j=0}^{n-1} C_n^j f^{(n-j)}(x)\Big)\alpha_\varepsilon(x), \qquad (\delta_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}).$$

Hence

$$\begin{aligned} P_{K,(n+1,0)}(w_\varepsilon) &\le P_{K,(n+1,0)}(\sigma_\varepsilon) + \sup_{x\in[-\lambda,\lambda]} |\delta_\varepsilon(x)| \\ &\quad + (f(\lambda) - f(-\lambda)) \sup_{(x,y)\in K} \left|\frac{\partial^n}{\partial x^n}(F(x,y,v_\varepsilon(x,y)) - F(x,y,u_\varepsilon(x,y)))\right|. \end{aligned}$$

We have

$$\begin{aligned} \sup_{(x,y)\in K}\left|\frac{\partial^n}{\partial x^n}(F(x,\eta,v_\varepsilon(x,\eta)) - F(x,\eta,u_\varepsilon(x,\eta)))\right| &= P_{K,(n,0)}(F(\cdot,\cdot,v_\varepsilon) - F(\cdot,\cdot,u_\varepsilon)) \\ &\le P_{K,n}(F(\cdot,\cdot,v_\varepsilon) - F(\cdot,\cdot,u_\varepsilon)). \end{aligned}$$

According to the stability hypothesis, $(P_{K,(n+1,0)}(w_\varepsilon))_\varepsilon \in I_A^+$ for every $K \Subset \mathbb{R}^2$. Let us show that $(P_{2,n}(w_\varepsilon))_\varepsilon \in I_A^+$ for every $n \in \mathbb{N}$. We have

$$\begin{aligned} \frac{\partial^{n+1} w_\varepsilon}{\partial y^{n+1}}(x,y) &= \frac{\partial^{n+1}\sigma_\varepsilon}{\partial y^{n+1}}(x,y) + \mu_\varepsilon(y) \\ &\quad - \int\limits_{x}^{f^{-1}(y)} \left(\frac{\partial^n}{\partial y^n}F(\xi,y,v_\varepsilon(\xi,y)) - \frac{\partial^n}{\partial y^n}F(\xi,y,u_\varepsilon(\xi,y))\right)d\xi, \end{aligned}$$

with

$$\mu_\varepsilon(y) = \Big(\sum_{j=0}^{n-1} C_n^j (f^{-1})^{(n-j)}(y)\Big)\alpha_\varepsilon(f^{-1}(y)), \qquad (\mu_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}).$$

Hence

$$P_{K,(0,n+1)}(w_\varepsilon) \le P_{K,(0,n+1)}(\sigma_\varepsilon) + \sup_{y\in[f(-\lambda),f(\lambda)]} |\mu_\varepsilon(y)| \\ + 2\lambda \sup_{(x,y)\in K} \left| \frac{\partial^n}{\partial y^n} F(x,y,v_\varepsilon(x,y)) - \frac{\partial^n}{\partial y^n} F(x,y,u_\varepsilon(x,y)) \right|.$$

We have

$$\sup_{(x,y)\in K} \left| \frac{\partial^n}{\partial y^n} F(x,y,v_\varepsilon(x,y)) - \frac{\partial^n}{\partial y^n} F(x,y,u_\varepsilon(x,y)) \right| = P_{K,(0,n)}(F(\cdot,\cdot,v_\varepsilon) - F(\cdot,\cdot,u_\varepsilon)) \\ \le P_{K,(0,n)}(F(\cdot,\cdot,v_\varepsilon) - F(\cdot,\cdot,u_\varepsilon)).$$

According to the stability hypothesis, $(P_{K,(0,n+1)}(w_\varepsilon))_\varepsilon \in I_A$ for every $K \Subset \mathbb{R}^2$. For $\alpha+\beta = n$ and $\beta \ge 1$, we have

$$P_{K,(\alpha+1,\beta)}(w_\varepsilon) = P_{K,(\alpha,\beta-1)}(F(\cdot,\cdot,v_\varepsilon) - F(\cdot,\cdot,u_\varepsilon)) \\ \le P_{K,n-1}(F(\cdot,\cdot,v_\varepsilon) - F(\cdot,\cdot,u_\varepsilon)).$$

Finally, we have

$$P_{3,n}(w_\varepsilon) = \sup_{\alpha+\beta=n;\beta\ge 1} P_{K,(\alpha+1,\beta)}(w_\varepsilon) \le P_{K,n-1}(F(\cdot,\cdot,v_\varepsilon) - F(\cdot,\cdot,u_\varepsilon))$$

and the stability hypothesis ensures that $(P_{3,n}(w_\varepsilon))_\varepsilon \in I_A^+$. In the same way, for $\alpha+\beta = n$ and $\alpha \ge 1$, we have

$$P_{K,(\alpha,\beta+1)}(w_\varepsilon) = P_{K,(\alpha-1,\beta)}(F(\cdot,\cdot,v_\varepsilon) - F(\cdot,\cdot,u_\varepsilon)) \\ \le P_{K,n-1}(F(\cdot,\cdot,v_\varepsilon) - F(\cdot,\cdot,u_\varepsilon)).$$

So we finally have

$$P_{4,n}(w_\varepsilon) = \sup_{\alpha+\beta=n;\alpha\ge 1} P_{K,(\alpha,\beta+1)}(w_\varepsilon) \le P_{K,n-1}(F(\cdot,\cdot,v_\varepsilon) - F(\cdot,\cdot,u_\varepsilon))$$

and the stability hypothesis ensures that $(P_{4,n}(w_\varepsilon))_\varepsilon \in I_A^+$. So $(P_{K,l}(w_\varepsilon))_\varepsilon \in I_A^+$ for every $l \le n+1$. Thus $(w_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$, and consequently u is the unique solution to (P_G). ∎

4.3. Parametric singular spectrum of the solution. We study the relationship between the $\mathcal{D}'$-parametric singular spectrum of the solution u and the $\mathcal{D}'$-parametric singular spectrum of u_0.

THEOREM 38. *Put $u_0 = [u_{0,\varepsilon}]$ with $u_{0,\varepsilon}(x,y) = \chi_\varepsilon(y) - \chi_\varepsilon(f(x)) + \varphi_\varepsilon(x)$ where χ_ε indicates a primitive of $\psi_\varepsilon \circ f^{-1}$, and suppose that*

$$\forall K \Subset \mathbb{R}^2, \quad \mathcal{M}_F(K) = \sup_{(x,y)\in K; z\in\mathbb{R}} |F(x,y,z)| < \infty. \tag{4.4}$$

Then the restriction of the $\mathcal{D}'$-parametric singular spectrum of the solution u to the Cauchy problem (P_G), to the parametric singular support of u_0 is included in the restriction of the $\mathcal{D}'$-parametric singular spectrum of u_0 to the parametric singular support of u_0. In other words, over the singular support of u_0, there is no increase in the distributional singularities of u in comparison with those of u_0.

Proof. Let $(x_0, y_0) = X \in S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u_0$ and $r \in N_{\mathcal{D}',X}(u_0)$. From the definitions, it follows that $\Sigma_{\mathcal{D}',X}(u_0) \neq \emptyset$, so that $N_{\mathcal{D}',X}(u_0) \subset]0, \infty[$, which implies $r > 0$. Next let us show that $r \in N_{\mathcal{D}',X}(u)$. From the definition of $N_{\mathcal{D}',X}(u_0)$, there exists a neighborhood V_X of X such that

$$\lim_{\varepsilon \to 0} \varepsilon^r u_\varepsilon|_{V_X} \in \mathcal{D}'(V_X).$$

Let $g \in \mathcal{D}(V_X)$. So, there exists some distribution $T \in \mathcal{D}'(V_X)$ such that

$$\lim_{\varepsilon \to 0} \iint_{V_X} \varepsilon^r u_{0,\varepsilon}(x, y) g(x, y)\, dx\, dy = T(g).$$

Let us show that

$$\iint_{V_X} \varepsilon^r [u_\varepsilon(x, y) - u_{0,\varepsilon}(x, y)] g(x, y)\, dx\, dy \to 0 \qquad \text{as } \varepsilon \to 0.$$

Suppose moreover that $y \geq f(x)$. As

$$u_\varepsilon(x, y) - u_{0,\varepsilon}(x, y) = - \iint_{D(x,y,f)} F(\xi, \eta, u_\varepsilon(\xi, \eta))\, d\xi\, d\eta$$

and since (with the above notations)

$$\begin{aligned} &\Big| \iint_{V_X} \Big(\iint_{D(x,y,f)} F(\xi, \eta, u_\varepsilon(\xi, \eta))\, d\xi\, d\eta \Big) g(x, y)\, dx\, dy \Big| \\ &\qquad \leq \mathcal{M}_F(\operatorname{supp} g) \Big| \iint_{\operatorname{supp} g} \Big(\iint_{D(x,y,f)} d\xi\, d\eta \Big) g(x, y)\, dx\, dy \Big| \\ &\qquad \leq \mathcal{M}_F(\operatorname{supp} g) \Big| \iint_{\operatorname{supp} g} (A(x, y)) g(x, y)\, dx\, dy \Big| \\ &\qquad \leq 2\lambda \mathcal{M}_F(\operatorname{supp} g) \iint_{\operatorname{supp} g} |y|\, |g(x, y)|\, dx\, dy < \infty, \end{aligned}$$

we have

$$\begin{aligned} &\limsup_{\varepsilon \to 0} \Big| \iint_{V_X} \varepsilon^r [u_\varepsilon(x, y) - u_{0,\varepsilon}(x, y)] g(x, y)\, dx\, dy \Big| \\ &\qquad \leq \limsup_{\varepsilon \to 0} \varepsilon^r \Big| \iint_{V_X} \Big[\iint_{D(x,y,f)} F(\xi, \eta, u_\varepsilon(\xi, \eta))\, d\xi\, d\eta \Big] g(x, y)\, dx\, dy \Big| \\ &\qquad \leq \limsup_{\varepsilon \to 0} \varepsilon^r \Big[2\lambda (\mathcal{M}_F(\operatorname{supp} g)) \iint_{\operatorname{supp} g} |y|\, |g(x, y)|\, dx\, dy \Big] = 0, \end{aligned}$$

because $r \neq 0$. Hence

$$\lim_{\varepsilon \to 0} \iint_{V_X} \varepsilon^r u_\varepsilon(x, y) g(x, y)\, dx\, dy = \lim_{\varepsilon \to 0} \iint_{V_X} \varepsilon^r u_{0,\varepsilon}(x, y) g(x, y)\, dx\, dy = T(g).$$

It follows that

$$\lim_{\varepsilon \to 0} \varepsilon^r u_\varepsilon|_{V_X} = \lim_{\varepsilon \to 0} \varepsilon^r u_{0,\varepsilon}|_{V_X} \in \mathcal{D}'(V_X).$$

So $r \in N_{\mathcal{D}',X}(u)$, which proves the inclusion $N_{\mathcal{D}',X}(u_0) \subset N_{\mathcal{D}',X}(u)$, and consequently $\Sigma_{\mathcal{D}',X}(u) \subset \Sigma_{\mathcal{D}',X}(u_0)$. Therefore

$$S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u|_{S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u_0} \subset S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u_0|_{S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u_0}. \ \blacksquare$$

Example 39. Let

$$g \in \mathcal{D}(\mathbb{R}), \quad g \geq 0, \quad \int_{\mathbb{R}} g(x)\,dx = 1 \text{ and } f(x) = ax, \quad a > 0.$$

Let us consider the following cases:

- $\chi_\varepsilon(y) = \varepsilon^{-1} g(y\varepsilon^{-1})$ and $\varphi_\varepsilon(x) = \varepsilon^{-1} g(x\varepsilon^{-1})$, so
$$\chi_\varepsilon(f(x)) = \varepsilon^{-1} g(f(x)\varepsilon^{-1}) = \varepsilon^{-1} g(ax\varepsilon^{-1}).$$
Then
$$N_{\mathcal{D}',X}(u_0) = [1, \infty[\text{ and } S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u \subset \mathbb{R}^2 \times [0, 1[.$$
- $\chi_\varepsilon(x) = \varepsilon^{-1} g(x\varepsilon^{-1})$ and $\varphi_\varepsilon(x) = \varepsilon^{-2} g(x\varepsilon^{-1}) = \varepsilon^{-1}[\varepsilon^{-1} g(x\varepsilon^{-1})]$. Then
$$N_{\mathcal{D}',X}(u_0) = [2, \infty[\text{ and } S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u \subset \mathbb{R}^2 \times [0, 2[.$$
- $\chi_\varepsilon(x) = g(x\varepsilon^{-1})$ and $\varphi_\varepsilon(x) = g(x\varepsilon^{-1}) = \varepsilon(\varepsilon^{-1} g(x\varepsilon^{-1}))$. Then
$$N_{\mathcal{D}',X}(u_0) = [0, \infty[.$$
As $S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u \subset \mathbb{R}^2 \times \mathbb{R}_+$, we have
$$S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u \subset \mathbb{R}^2 \times \emptyset.$$

4.4. Qualitative study of the solution. Case $F = 0$. We search for a generalized solution u to the Cauchy problem (P_G) where $F = 0$, considering as data the curve γ of equation $y = f(x)$. With the above notations, considering $P_\infty(\varphi_\varepsilon, \psi_\varepsilon)$, we have

$$u_\varepsilon(x, y) = \chi_\varepsilon(y) - \chi_\varepsilon(f(x)) + \varphi_\varepsilon(x).$$

Example 40. $f(x) = ax$ $(a > 0)$, $\varphi \sim S$, $\psi = \Psi'$ and $\Psi \sim T$; $S \in \mathcal{D}'(\mathbb{R})$, $T \in \mathcal{D}'(\mathbb{R})$. Let $g \in \mathcal{D}(\mathbb{R})$ be an even function satisfying $\int_{\mathbb{R}} g(\xi)\,d\xi = 1$. Put $g_\varepsilon(x) = \varepsilon^{-1} g(x\varepsilon^{-1})$. Then $(g_\varepsilon)_\varepsilon \to \delta$ in the distributional sense. So $g = [g_\varepsilon]$ is associated to δ. Choosing

$$\varphi = [g_\varepsilon * S] \text{ and } \Psi = [g_\varepsilon * T]$$

we have the associations $\varphi \sim S$, $\Psi \sim T$, since

$$\lim_{\substack{\varepsilon \to 0 \\ \mathcal{D}'(\mathbb{R})}} (g_\varepsilon * S)_\varepsilon = S \quad \text{and} \quad \lim_{\substack{\varepsilon \to 0 \\ \mathcal{D}'(\mathbb{R})}} (g_\varepsilon * T)_\varepsilon = T.$$

The solution to $P_\infty(\varphi_\varepsilon, \psi_\varepsilon)$ is defined by

$$u_\varepsilon(x, y) = \chi_\varepsilon(y) - \chi_\varepsilon(f(x)) + \varphi_\varepsilon(x)$$

with

$$\chi_\varepsilon(y) = \int_0^y \psi_\varepsilon(f^{-1}(\eta))\,d\eta = \int_0^y \psi_\varepsilon(\eta a^{-1})\,d\eta = a \int_0^{ya^{-1}} \psi_\varepsilon(t)\,dt = a(\Psi_\varepsilon(ya^{-1}) - \Psi_\varepsilon(0))$$

where Ψ_ε is a primitive of ψ_ε. So

$$u_\varepsilon(x,y) = a\Psi_\varepsilon(ya^{-1}) - a\Psi_\varepsilon(x) + \varphi_\varepsilon(x).$$

We have here

$$u_\varepsilon(x,y) = a(g_\varepsilon * T)(ya^{-1}) - a(g_\varepsilon * T)(x) + (g_\varepsilon * S)(x).$$

Let us estimate the function $y \mapsto (g_\varepsilon * T)(ya^{-1})$ on the test function $h \in \mathcal{D}(\mathbb{R})$. By putting $H(z) = h(az)$, we can write

$$\int (g_\varepsilon * T)(ya^{-1})h(y)\,dy = a\int (g_\varepsilon * T)(z)H(z)\,dz.$$

Then define $\widetilde{T} \in \mathcal{D}'(\mathbb{R})$ by

$$\langle \widetilde{T}, h\rangle = \langle aT, [z \mapsto h(az)]\rangle = \langle aT, H\rangle.$$

Hence

$$\lim_{\varepsilon\to 0}\int (g_\varepsilon * T)(ya^{-1})h(y)dy = \lim_{\varepsilon\to 0} a\int (g_\varepsilon * T)(z)H(z)dz = \langle aT, H\rangle = \langle \widetilde{T}, h\rangle.$$

Thus

$$\lim_{\substack{\varepsilon\to 0\\ \mathcal{D}'(\mathbb{R})}} [y \mapsto (g_\varepsilon * T)(ya^{-1})] = \widetilde{T}.$$

Then we can write $[u_\varepsilon] = [w_{\varepsilon,1}] + [w_{\varepsilon,2}] + [w_{\varepsilon,3}]$, with

$$[w_{\varepsilon,1}] \sim a(1_x \otimes \widetilde{T}_y), \qquad [w_{\varepsilon,2}] \sim -a(T_x \otimes 1_y), \qquad [w_{\varepsilon,3}] \sim S_x \otimes 1_y$$

and so

$$u \sim a(1_x \otimes \widetilde{T}_y) - a(T_x \otimes 1_y) + S_x \otimes 1_y.$$

Remark 41. We can remark that

$$\langle \widetilde{\delta}, h\rangle = \langle a\delta, [z \mapsto h(az)]\rangle = ah(0) = a\langle \delta, h\rangle,$$

so that $\widetilde{\delta} = a\delta$.

Example 42. $f(x) = ax$ $(a > 0)$, $\varphi \sim \delta$, $\psi = \Psi'$, with $\Psi \sim \delta$. As $\widetilde{\delta} = a\delta$, we have

$$[u_\varepsilon] = [w_{\varepsilon,1}] + [w_{\varepsilon,2}] + [w_{\varepsilon,3}]$$

with

$$[w_{\varepsilon,1}] \sim a^2(1_x \otimes \delta_y), \qquad [w_{\varepsilon,2}] \sim -a(\delta_x \otimes 1_y), \qquad [w_{\varepsilon,3}] \sim \delta_x \otimes 1_y,$$

hence

$$u \sim a^2(1_x \otimes \delta_y) - a(\delta_x \otimes 1_y) + \delta_x \otimes 1_y.$$

Example 43. $f(x) = ax$ $(a > 0)$, $\varphi \sim \delta$, $\psi \sim \delta$. We can choose Ψ_ε such that $\Psi_\varepsilon(0) = 2^{-1}$ in such a way that

$$\lim_{\substack{\varepsilon\to 0\\ \mathcal{D}'(\mathbb{R})}} \Psi_\varepsilon = Y, \qquad \lim_{\substack{\varepsilon\to 0\\ \mathcal{D}'(\mathbb{R})}} (y \mapsto \Psi_\varepsilon(ya^{-1})) = Y.$$

Then

$$[u_\varepsilon] = [w_{\varepsilon,1}] + [w_{\varepsilon,2}] + [w_{\varepsilon,3}]$$

with

$$[w_{\varepsilon,1}] \sim a(1_x \otimes Y_y), \qquad [w_{\varepsilon,2}] \sim -a(Y_x \otimes 1_y), \qquad [w_{\varepsilon,3}] \sim \delta_x \otimes 1_y.$$

5. Generalized Goursat problem

5.1. Formulation of the problem. We search for a solution u to the Goursat problem

$$(P'_G)\begin{cases}\dfrac{\partial^2 u}{\partial x \partial y} = F(\cdot,\cdot,u),\\ u|_{(Ox)} = \varphi,\\ u|_{\gamma} = \psi,\end{cases}$$

in the algebra of generalized functions $\mathcal{A}(\mathbb{R}^2)$ defined in Section 3. The hypotheses on F and g are kept. We suppose that

- $\mathcal{A}(\mathbb{R})$ and $\mathcal{A}(\mathbb{R}^2)$ are built on the same ring of generalized constants;
- $\mathcal{A}(\mathbb{R}^2)$ is stable under F;
- For every ε, the problem

$$P'_\infty(\varphi_\varepsilon,\psi_\varepsilon)\begin{cases}\dfrac{\partial^2 u_\varepsilon}{\partial x \partial y}(x,y) = F(x,y,u_\varepsilon(x,y)),\\ u_\varepsilon(x,0)) = \varphi_\varepsilon(x),\\ u_\varepsilon(g(y),y) = \psi_\varepsilon(y),\end{cases}$$

  has a solution $u_\varepsilon \in \mathrm{C}^\infty(\mathbb{R}^2)$.

Giving a meaning to (P'_G) is first giving a meaning to

$$\frac{\partial^2 u}{\partial x \partial y} = F(\cdot,\cdot,u), \tag{5.1}$$

$$u|_{(Ox)} = \varphi \in \mathcal{A}(\mathbb{R}), \tag{5.2}$$

$$u|_{\gamma} = \psi \in \mathcal{A}(\mathbb{R}), \tag{5.3}$$

when $u \in \mathcal{A}(\mathbb{R}^2)$ and γ is the smooth submanifold of $\mathbb{R}^2$ defined by $x = g(y)$. Giving a meaning to (5.1), under the hypothesis that $\mathcal{A}(\mathbb{R}^2)$ is stable by F, amounts to saying that for a representative $(u_\varepsilon)_\varepsilon$ of u we must have, for every $(i_\varepsilon)_\varepsilon, (j_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$,

$$\left(\frac{\partial^2 (u_\varepsilon + i_\varepsilon)}{\partial x \partial y} - F(\cdot,\cdot,u_\varepsilon) + j_\varepsilon\right)_\varepsilon \in \mathcal{N}(\mathbb{R}^2).$$

As

$$\left(\frac{\partial^2 (u_\varepsilon + i_\varepsilon)}{\partial x \partial y} - \frac{\partial^2 u_\varepsilon}{\partial x \partial y}\right)_\varepsilon \in \mathcal{N}(\mathbb{R}^2), \qquad (F(\cdot,\cdot,u_\varepsilon) + j_\varepsilon - F(\cdot,\cdot,u_\varepsilon))_\varepsilon \in \mathcal{N}(\mathbb{R}^2),$$

we must verify that

$$\left(\frac{\partial^2 u_\varepsilon}{\partial x \partial y} - F(\cdot,\cdot,u_\varepsilon)\right)_\varepsilon \in \mathcal{N}(\mathbb{R}^2).$$

Giving a meaning to (5.2) and (5.3) amounts to defining $u|_{(Ox)}$ and $u|_\gamma$. As γ is a smooth submanifold of $\mathbb{R}^2$ that can be represented by a single map ($\gamma : x = g(y)$), we can identify $\mathcal{A}(\gamma)$ and $\mathcal{A}(\mathbb{R})$, and so $u|_\gamma$ and $u|_{(Ox)}$, with the elements of $\mathcal{A}(\mathbb{R})$ with representatives $(y \mapsto u_\varepsilon(g(y),y))_\varepsilon$ and $(x \mapsto u_\varepsilon(x,0))_\varepsilon$. So (5.2) is equivalent to

$$(x \mapsto ((u_\varepsilon + i_\varepsilon)(x,0) - (\varphi_\varepsilon + \alpha_\varepsilon)(x)))_\varepsilon \in \mathcal{N}(\mathbb{R}).$$

(5.3) is equivalent to

$$(y \mapsto ((u_\varepsilon + i_\varepsilon)(g(y), y) - (\psi_\varepsilon + \beta_\varepsilon)(y)))_\varepsilon \in \mathcal{N}(\mathbb{R}),$$

for all $(i_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$ and $(\alpha_\varepsilon)_\varepsilon, (\beta_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R})$. Considering

$$\begin{aligned}
&(x \mapsto ((u_\varepsilon + i_\varepsilon)(x, 0) - (u_\varepsilon(x, 0))))_\varepsilon \in \mathcal{N}(\mathbb{R}),\\
&(x \mapsto ((\varphi_\varepsilon + \alpha_\varepsilon)(x) - \varphi_\varepsilon(x)))_\varepsilon \in \mathcal{N}(\mathbb{R}),\\
&(y \mapsto ((u_\varepsilon + i_\varepsilon)(g(y), y) - u_\varepsilon(g(y), y)))_\varepsilon \in \mathcal{N}(\mathbb{R}),\\
&(x \mapsto ((\psi_\varepsilon + \beta_\varepsilon)(x) - \psi_\varepsilon(x)))_\varepsilon \in \mathcal{N}(\mathbb{R}),\\
&(y \mapsto (j_\varepsilon(y) - i_\varepsilon(g(y), y)))_\varepsilon \in \mathcal{N}(\mathbb{R})
\end{aligned}$$

this boils down to

$$\begin{aligned}
&(x \mapsto (u_\varepsilon(x, 0) - \varphi_\varepsilon(x)))_\varepsilon \in \mathcal{N}(\mathbb{R}),\\
&(y \mapsto (u_\varepsilon(g(y), y) - \psi_\varepsilon(y)))_\varepsilon \in \mathcal{N}(\mathbb{R}).
\end{aligned}$$

To sum up, (P'_G) has a meaning if, and only if,

$$\begin{cases}
\dfrac{\partial^2 u_\varepsilon}{\partial x \partial y} - F(\cdot, \cdot, u_\varepsilon) \in \mathcal{N}(\mathbb{R}^2),\\
(x \mapsto (u_\varepsilon(x, 0) - \varphi_\varepsilon(x)))_\varepsilon \in \mathcal{N}(\mathbb{R}),\\
(y \mapsto (u_\varepsilon(g(y), y) - \psi_\varepsilon(y)))_\varepsilon \in \mathcal{N}(\mathbb{R}).
\end{cases}$$

So, if for every ε, u_ε is a solution to $P'_\infty(\varphi_\varepsilon, \psi_\varepsilon)$ and $(u_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{R}^2)$ then the relations above are all the more true and $[u_\varepsilon]$ is a solution to (P'_G).

5.2. Existence and uniqueness of solutions

THEOREM 44. *Suppose that $\mathcal{A}(\mathbb{R}^2)$ is stable under F and $\mathcal{A}(\mathbb{R})$, $\mathcal{A}(\mathbb{R}^2)$ are built on the same ring $\mathcal{C} = A/I$ of generalized constants. Suppose that the data of problem (P'_G) satisfy the conditions $\varphi, \psi \in \mathcal{A}(\mathbb{R})$, $g \in \mathrm{C}^\infty(\mathbb{R})$, $\varphi = [\varphi_\varepsilon]$, $\psi = [\psi_\varepsilon]$, $\psi_\varepsilon(0) = \varphi_\varepsilon(g(0))$. Then problem (P'_G) has a unique solution in $\mathcal{A}(\mathbb{R}^2)$.*

Proof. Suppose $g(y) \le x$. Let u_ε be the solution to $P'_\infty(\varphi_\varepsilon, \psi_\varepsilon)$. According to the previous result, it is enough to prove $(u_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{R}^2)$; then $u = [u_\varepsilon]$ will be a solution to (P'_G). We will prove that

$$\forall K \Subset \mathbb{R}^2, \forall l \in \mathbb{N}, \quad (P_{K,l}(u_\varepsilon))_\varepsilon \in A_+,$$

Proceeding by induction we first show

$$\forall K \Subset \mathbb{R}^2, \quad (P_{K,0}(u_\varepsilon))_\varepsilon = (\|u_\varepsilon\|_{\infty,K})_\varepsilon \in A_+,$$

that is, the 0th order estimate is satisfied. According to Proposition 10, for every $K \Subset \mathbb{R}^2$ there exists $K_\lambda \Subset \mathbb{R}^2$ with $K \subset K_\lambda$ such that

$$\|u_\varepsilon\|_{\infty,K} \le \|u_\varepsilon\|_{\infty,K_\lambda} \le \|u_{0,\varepsilon}\|_{\infty,K_\lambda} + \frac{\Phi_{\lambda,\varepsilon}}{m_\lambda} \exp(2\lambda' m_\lambda(2\lambda)).$$

Hence $(\|u_{0,\varepsilon}\|_{\infty,K_\lambda})_\varepsilon \in A$ because $[\varphi_\varepsilon]$ and $[\psi_\varepsilon]$ are elements of $\mathcal{A}(\mathbb{R})$. The constant

$$m_\lambda = \sup_{(x,y)\in K_\lambda; t\in\mathbb{R}} \left| \frac{\partial F}{\partial z}(x, y, t) \right|$$

depends only on F, K_λ, and the constant

$$c(K_\lambda) = \frac{1}{m_\lambda}\exp(4\lambda' m_\lambda \lambda)$$

depends only on F, g, K_λ. We have

$$\Phi_{\lambda,\varepsilon} = \|F(\cdot,\cdot,0)\|_{\infty,K_\lambda} + m_\lambda \|u_{0,\varepsilon}\|_{\infty,K_\lambda}$$

so

$$\begin{aligned}\frac{\Phi_{\lambda,\varepsilon}}{m_\lambda}\exp(4\lambda' m_\lambda\lambda) &= c(K_\lambda)\Phi_{\lambda,\varepsilon}\\ &= c(K_\lambda)\|F(\cdot,\cdot,0)\|_{\infty,K_\lambda} + \exp(4\lambda' m_\lambda\lambda)\|u_{0,\varepsilon}\|_{\infty,K_\lambda}.\end{aligned}$$

Moreover, the constant

$$c_1(K_\lambda) = c(K_\lambda)\|F(\cdot,\cdot,0)\|_{\infty,K_\lambda}$$

depends only on F, K_λ, and

$$c_2(K_\lambda) = \exp(4\lambda' m_\lambda \lambda)$$

depends only on K_λ, F, g. Consequently,

$$\|u_\varepsilon\|_{\infty,K} \le \|u_\varepsilon\|_{\infty,K_\lambda} \le \|u_{0,\varepsilon}\|_{\infty,K_\lambda} + c_1(K_\lambda) + c_2(K_\lambda)\|u_{0,\varepsilon}\|_{\infty,K_\lambda},$$

so

$$\|u_\varepsilon\|_{\infty,K} \le \|u_\varepsilon\|_{\infty,K_\lambda} \le (1 + c_2(K_\lambda))\|u_{0,\varepsilon}\|_{\infty,K_\lambda} + c_1(K_\lambda).$$

We have $(\|u_{0,\varepsilon}\|_{\infty,K_\lambda})_\varepsilon \in A$, so

$$((1 + c_2(K_\lambda))\|u_{0,\varepsilon}\|_{\infty,K_\lambda})_\varepsilon \in A$$

(if $(r_\varepsilon)_\varepsilon \in A$, then $(cr_\varepsilon)_\varepsilon \in A$), and as $c_1(K_\lambda)$ is a constant $(1 \in A)$ we deduce that

$$((1 + c_2(K_\lambda))\|u_{0,\varepsilon}\|_{\infty,K_\lambda} + c_1(K_\lambda))_\varepsilon \in A.$$

A being stable we have $(\|u_\varepsilon\|_{\infty,K_\lambda})_\varepsilon \in A$ and so $(\|u_\varepsilon\|_{\infty,K})_\varepsilon \in A$. Let us show that $(P_{K,1}(u_\varepsilon))_\varepsilon \in A_+$. We have

$$\frac{\partial u_\varepsilon}{\partial x}(x,y) = \frac{\partial u_{0,\varepsilon}}{\partial x}(x,y) + \int_0^y F(x,\eta,u_\varepsilon(x,\eta))\,d\eta,$$

hence

$$\begin{aligned}P_{K,(1,0)}(u_\varepsilon) &\le \sup_K \left|\frac{\partial u_{0,\varepsilon}}{\partial x}(x,y)\right| + |y|\sup_{K_\lambda}|F(x,\eta,u_\varepsilon(x,\eta))|\\ &\le \sup_K \left|\frac{\partial u_{0,\varepsilon}}{\partial x}(x,y)\right| + \lambda\sup_{K_\lambda}|F(x,\eta,u_\varepsilon(x,\eta))|.\end{aligned}$$

As $\mathcal{A}(\mathbb{R}^2)$ is stable under F there exists C such that

$$P_{K_\lambda,(0,0)}(F(\cdot,\cdot,u_\varepsilon)) \le C P_{K_\lambda,(0,0)}(u_\varepsilon).$$

We have

$$(\|(\partial/\partial x)u_{0,\varepsilon}\|_{\infty,K})_\varepsilon = (\varphi_\varepsilon'(x))_\varepsilon \in A$$

because $[\varphi_\varepsilon]$ is an element of $\mathcal{A}(\mathbb{R})$. So

$$P_{K,(1,0)}(u_\varepsilon) \leq \left\| \frac{\partial u_{0,\varepsilon}}{\partial x} \right\|_{\infty,K} + C\lambda P_{K_\lambda,(0,0)}(u_\varepsilon).$$

As A is stable $(P_{K,(1,0)}(u_\varepsilon))_\varepsilon \in A$. We have

$$\frac{\partial u_\varepsilon}{\partial y}(x,y) = \frac{\partial u_{0,\varepsilon}}{\partial y}(x,y) + \int_{g(y)}^{x} F(\xi, y, u_\varepsilon(\xi,y))\, d\xi - g'(y)\int_0^y F(g(y),\eta,u(g(y),\eta))\, d\eta,$$

$$\begin{aligned} P_{K,(0,1)}(u_\varepsilon) &\leq \sup_K \left| \frac{\partial u_{0,\varepsilon}}{\partial y}(x,y) \right| + (x - g(y) + |y| g'(y)) \sup_{K_\lambda} |F(x,\eta,u_\varepsilon(x,\eta))| \\ &\leq \sup_K \left| \frac{\partial u_{0,\varepsilon}}{\partial y}(x,y) \right| + (g(\lambda) - g(-\lambda) + \lambda g'(y)) \sup_{K_\lambda} |F(x,\eta,u_\varepsilon(x,\eta))|. \end{aligned}$$

$\mathcal{A}(\mathbb{R}^2)$ being stable under F, there exists C such that

$$P_{K_\lambda,(0,0)}(F(\cdot,\cdot,u_\varepsilon)) \leq C P_{K_\lambda,(0,0)}(u_\varepsilon).$$

We have

$$(\|(\partial/\partial y) u_{0,\varepsilon}\|_{\infty,K})_\varepsilon \in A_+,$$

because $[\psi_\varepsilon]$ and $[\varphi_\varepsilon]$ are elements of $\mathcal{A}(\mathbb{R})$. Hence

$$P_{K,(0,1)}(u_\varepsilon) \leq \left\| \frac{\partial u_{0,\varepsilon}}{\partial y} \right\|_{\infty,K} + C(g(\lambda) - g(-\lambda) + \lambda g'(y)) P_{K_\lambda,(0,0)}(u_\varepsilon)$$

and so, as previously,

$$(\|(\partial/\partial y) u_\varepsilon\|_{\infty,K})_\varepsilon \in A_+.$$

Now we proceed by induction. Suppose that $(P_{K,l}(u_\varepsilon))_\varepsilon \in A_+$ for every $l < n$ and let us show that $(P_{K,l+1}(u_\varepsilon))_\varepsilon \in A_+$. We use the notations from Theorem 37. Let us show first that

$$(P_{1,n}(u_\varepsilon))_\varepsilon, (P_{2,n}(u_\varepsilon))_\varepsilon \in A_+$$

for every $n \in \mathbb{N}$. We have by successive derivations, for $n \geq 1$,

$$\frac{\partial^{n+1} u_\varepsilon}{\partial x^{n+1}}(x,y) = \frac{\partial^{n+1} u_{0,\varepsilon}}{\partial x^{n+1}}(x,y) + \int_0^y \frac{\partial^n}{\partial x^n} F(x,\eta,u_\varepsilon(x,\eta))\, d\eta$$

with

$$\frac{\partial^{n+1} u_{0,\varepsilon}}{\partial x^{n+1}}(x,y) = \varphi^{(n+1)}(x).$$

As we have taken $K \subset K_\lambda$, we can write

$$\sup_{(x,y)\in K} \left| \frac{\partial^{n+1} u_\varepsilon}{\partial x^{n+1}}(x,y) \right| \leq \left\| \frac{\partial^{n+1} u_{0,\varepsilon}}{\partial x^{n+1}} \right\|_{\infty,K} + \lambda \sup_{(x,y)\in K} \left| \frac{\partial^n}{\partial x^n} F(x,y,u_\varepsilon(x,y)) \right|.$$

We have

$$\sup_{(x,y)\in K} \left| \frac{\partial^n}{\partial x^n} F(x,y,u_\varepsilon(x,y)) \right| = P_{K,(n,0)}(F(\cdot,\cdot,u_\varepsilon)) \leq P_{K,n}(F(\cdot,\cdot,u_\varepsilon)),$$

moreover

$$(\|(\partial^{n+1}/\partial x^{n+1}) u_{0,\varepsilon}\|_{\infty,K})_\varepsilon \in A_+.$$

According to the stability hypothesis, a simple calculation shows that $(P_{K,n}(F(\cdot,\cdot,u_\varepsilon)))_\varepsilon \in \mathcal{A}_+$ for every $K \Subset \mathbb{R}^2$. Let us show that $(P_{2,n}(u_\varepsilon))_\varepsilon \in \mathcal{A}_+$ for every $n \in \mathbb{N}$. We have by successive derivations, for $n \geq 1$,

$$\begin{aligned}\frac{\partial^{n+1} u_\varepsilon}{\partial y^{n+1}}(x,y) = {} & \frac{\partial^{n+1} u_{0,\varepsilon}}{\partial y^{n+1}}(x,y)\\ & - \sum_{j=0}^{n-1} C_n^j g^{(n-j)}(y) \frac{\partial^j}{\partial y^j} F(g(y),y,\psi_\varepsilon(y)) - \int_x^{g(y)} \frac{\partial^n}{\partial y^n} F(\xi,y,u_\varepsilon(\xi,y))\,d\xi\\ & - \sum_{j=0}^{n-1} C_n^{j+1} g^{(n-j)}(y) \frac{\partial^j}{\partial y^j} F(g(y),y,\psi_\varepsilon(y))\\ & - g^{(n+1)}(y) \int_0^y F(g(y),\eta,u_\varepsilon(g(y),\eta))\,d\eta.\end{aligned}$$

As we have taken $K \subset K_\lambda$, we can write

$$\begin{aligned}\sup_{(x,y)\in K} \left| \frac{\partial^{n+1} u_\varepsilon}{\partial y^{n+1}}(x,y) \right| \leq {} & \left\| \frac{\partial^{n+1} u_{0,\varepsilon}}{\partial y^{n+1}} \right\|_{\infty,K} + (g(\lambda) - g(\lambda)) \sup_{(x,y)\in K} \left| \frac{\partial^n}{\partial y^n} F(x,y,u_\varepsilon(x,y)) \right|\\ & + \sup_{y\in[-\lambda,\lambda]} \sum_{j=0}^{n-1} C_{n+1}^{j+1} |g^{(n-j)}(y)| \left| \frac{\partial^j}{\partial y^j} F(g(y),y,\psi_\varepsilon(y)) \right|\\ & + \lambda g^{(n+1)}(y) \sup_{(x,y)\in K} |F(x,y,u_\varepsilon(x,y))|.\end{aligned}$$

We have

$$\sup_{(x,y)\in K} \left| \frac{\partial^n}{\partial y^n} F(x,y,u_\varepsilon(x,y)) \right| = P_{K,(0,n)}(F(\cdot,\cdot,u_\varepsilon)) \leq P_{K,n}(F(\cdot,\cdot,u_\varepsilon)),$$

and, as $\psi_\varepsilon(y) = u_\varepsilon(g(y),y)$,

$$\begin{aligned}\sup_{y\in[-\lambda,\lambda]} \left| \frac{\partial^j}{\partial y^j} F(g(y),y,\psi_\varepsilon(y)) \right| & \leq \sup_{(x,y)\in K} \left| \frac{\partial^i}{\partial y^i} F(x,y,u_\varepsilon(x,y)) \right|\\ & \leq P_{K,i}(F(\cdot,\cdot,u_\varepsilon)) \leq P_{K,n}(F(\cdot,\cdot,u_\varepsilon)),\end{aligned}$$

$$\sup_{(x,y)\in K} |F(x,y,u_\varepsilon(x,y))| \leq P_{K,1}(F(\cdot,\cdot,u_\varepsilon)).$$

According to the stability hypothesis, a simple calculation shows that, for every $K \Subset \mathbb{R}^2$ and $n \in \mathbb{N}$, $(P_{K,(0,n+1)}(u_\varepsilon))_\varepsilon \in \mathcal{A}_+$. For $\alpha + \beta = n$ and $\beta \geq 1$, we now have

$$\begin{aligned}P_{K,(\alpha+1,\beta)}(u_\varepsilon) & = \sup_{(x,y)\in K} |D^{(\alpha+1,\beta)} u_\varepsilon(x,y)| = \sup_{(x,y)\in K} |D^{(\alpha,\beta-1)} D^{(1,1)} u_\varepsilon(x,y)|\\ & = \sup_{(x,y)\in K} |D^{(\alpha,\beta-1)} F(x,y,u_\varepsilon(x,y))| = P_{K,(\alpha,\beta-1)}(F(\cdot,\cdot,u_\varepsilon))\\ & \leq P_{K,n-1}(F(\cdot,\cdot,u_\varepsilon)) \leq P_{K,n}(F(\cdot,\cdot,u_\varepsilon)).\end{aligned}$$

So we finally have

$$P_{3,n}(u_\varepsilon) = \sup_{\alpha+\beta=n;\beta\geq 1} P_{K,(\alpha+1,\beta)}(u_\varepsilon) \leq P_{K,n}(F(\cdot,\cdot,u_\varepsilon))$$

and the stability hypothesis ensures that $(P_{3,n}(u_\varepsilon))_\varepsilon \in A_+$. In the same way, for $\alpha+\beta=n$ and $\alpha \geq 1$, we have

$$\begin{aligned} P_{K,(\alpha,\beta+1)}(u_\varepsilon) &= \sup_{(x,y)\in K} |D^{(\alpha,\beta+1)} u_\varepsilon(x,y)| = \sup_{(x,y)\in K} |D^{(\alpha-1,\beta)} D^{(1,1)} u_\varepsilon(x,y)| \\ &= \sup_{(x,y)\in K} |D^{(\alpha-1,\beta)} F(x,y,u_\varepsilon(x,y))| = P_{K,(\alpha-1,\beta)}(F(\cdot,\cdot,u_\varepsilon)) \\ &\leq P_{K,n-1}(F(\cdot,\cdot,u_\varepsilon)) \leq P_{K,n}(F(\cdot,\cdot,u_\varepsilon)). \end{aligned}$$

So we finally have

$$P_{4,n}(u_\varepsilon) = \sup_{\alpha+\beta=n;\alpha\geq 1} P_{K,(\alpha,\beta+1)}(u_\varepsilon) \leq P_{K,n}(F(\cdot,\cdot,u_\varepsilon))$$

and the stability hypothesis ensures that $(P_{4,n}(u_\varepsilon))_\varepsilon \in A_+$. Finally, we clearly have $(P_{K,n+1}(u_\varepsilon))_\varepsilon \in A_+$. So $u = [u_\varepsilon]$ is a solution to (P'_G). Let us show that u is the unique solution to (P'_G). Let $v = [v_\varepsilon]$ be another solution to (P'_G). There are $(i_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$ and $(\alpha_\varepsilon)_\varepsilon$, $(\beta_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R})$ such that

$$\begin{cases} \dfrac{\partial^2 v_\varepsilon}{\partial x \partial y}(x,y) = F(x,y,v_\varepsilon(x,y)) + i_\varepsilon(x,y), \\ v_\varepsilon(x,0) = \varphi_\varepsilon(x) + \alpha_\varepsilon(x), \\ \dfrac{\partial v_\varepsilon}{\partial y}(g(y),y) = \psi_\varepsilon(y) + \beta_\varepsilon(y). \end{cases}$$

The uniqueness of the solution to (P'_G) will be a consequence of $(v_\varepsilon - u_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$. It is easy to see that

$$\Big(\iint_{D(x,y,g)} i_\varepsilon(\xi,\eta)\, d\xi\, d\eta \Big)_\varepsilon \in \mathcal{N}(\mathbb{R}^2).$$

So there is $(j_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$ such that

$$v_\varepsilon(x,y) = v_{0,\varepsilon}(x,y) + \iint_{D(x,y,g)} F(\xi,\eta,v_\varepsilon(\xi,\eta))\, d\xi\, d\eta + j_\varepsilon(x,y),$$

with

$$v_{0,\varepsilon}(x,y) = u_{0,\varepsilon}(x,y) + \theta_\varepsilon(x,y),$$

where

$$\theta_\varepsilon(x,y) = \beta_\varepsilon(y) + \alpha_\varepsilon(x) - \alpha_\varepsilon(g(y)).$$

So $(\theta_\varepsilon)_\varepsilon$ belongs to $\mathcal{N}(\mathbb{R}^2)$. Thus there is $(\sigma_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$ such that

$$v_\varepsilon(x,y) = u_{0,\varepsilon}(x,y) + \sigma_\varepsilon(x,y) + \iint_{D(x,y,g)} F(\xi,\eta,v_\varepsilon(\xi,\eta))\, d\xi\, d\eta.$$

Let us put $w_\varepsilon = v_\varepsilon - u_\varepsilon$ and show that $(w_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$. We have to prove that

$$\forall K \Subset \mathbb{R}^2, \forall n \in \mathbb{N}, \quad (P_{K,n}(w_\varepsilon))_\varepsilon \in I_A^+.$$

First we show that $(P_{K,1}(w_\varepsilon))_\varepsilon \in I_A$. We have

$$w_\varepsilon(x,y) = \iint_{D(x,y,g)} (F(\xi,\eta,v_\varepsilon(\xi,\eta)) - F(\xi,\eta,u_\varepsilon(\xi,\eta)))\, d\xi\, d\eta + \sigma_\varepsilon(x,y);$$

but

$$F(\xi,\eta,v_\varepsilon(\xi,\eta)) - F(\xi,\eta,u_\varepsilon(\xi,\eta)) \\ = (v_\varepsilon(\xi,\eta) - u_\varepsilon(\xi,\eta))\Big(\int_0^1 \frac{\partial F}{\partial z}(\xi,\eta,u_\varepsilon(\xi,\eta) + \theta(v_\varepsilon(\xi,\eta) - u_\varepsilon(\xi,\eta)))\,d\theta\Big),$$

so

$$w_\varepsilon(x,y) = \iint_{D(x,y,g)} w_\varepsilon(\xi,\eta)\Big(\int_0^1 \frac{\partial F}{\partial z}(\xi,\eta,u_\varepsilon(\xi,\eta) + \theta(w_\varepsilon(\xi,\eta)))\,d\theta\Big)\,d\xi\,d\eta + \sigma_\varepsilon(x,y).$$

Let $(x,y) \in K_\lambda$. Since $D(x,y,g) \subset K_\lambda$, if $g(y) \le x$, we have

$$\begin{aligned} |w_\varepsilon(x,y)| &\le m_\lambda \int_{g(y)}^{x}\int_0^y |w_\varepsilon(\xi,\eta)|\,d\xi\,d\eta + \|\sigma_\varepsilon\|_{\infty,K_\lambda} \\ &\le m_\lambda \int_{-g(\lambda)}^{+g(\lambda)}\int_0^y |w_\varepsilon(\xi,\eta)|\,d\xi\,d\eta + \|\sigma_\varepsilon\|_{\infty,K_\lambda}. \end{aligned}$$

Put $e_\varepsilon(y) = \sup_{\xi\in[g(-\lambda),g(\lambda)]} |w_\varepsilon(\xi,y)|$. Then

$$|w_\varepsilon(x,y)| \le m_\lambda 2\lambda' \int_0^y e_\varepsilon(\eta)\,d\eta + \|\sigma_\varepsilon\|_{\infty,k_\lambda}.$$

We deduce that, for every $y \in [0,\lambda]$, if $g(y) \le x$,

$$e_\varepsilon(y) \le m_\lambda 2\lambda' \int_0^y e_\varepsilon(\eta)\,d\eta + \|\sigma_\varepsilon\|_{\infty,K_\lambda}.$$

Thus according to Gronwall's lemma, for every $y \in [0,\lambda]$, if $g(y) \le x$,

$$e_\varepsilon(y) \le \exp\Big(\int_0^y m_\lambda 2\lambda'\,d\eta\Big)\|\sigma_\varepsilon\|_{\infty,K_\lambda}.$$

For every $y \in [0,\lambda]$, if $g(y) \le x$,

$$e_\varepsilon(y) \le \exp(m_\lambda 2\lambda' y)\|\sigma_\varepsilon\|_{\infty,K_\lambda} \le \exp(m_\lambda 2\lambda'\lambda)\|\sigma_\varepsilon\|_{\infty,K_\lambda}.$$

We obtain the same result in the other cases, hence

$$\forall y \in [-\lambda,\lambda], \qquad e_\varepsilon(y) \le \|\sigma_\varepsilon\|_{\infty,K_\lambda}\exp(m_\lambda 2\lambda'\lambda),$$

and consequently,

$$\|w_\varepsilon\|_{\infty,K_\lambda} \le \|\sigma_\varepsilon\|_{\infty,K_\lambda}\exp(m_\lambda 2\lambda'\lambda),$$

Since $(\sigma_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$ we have $(\|\sigma_\varepsilon\|_{\infty,K_\lambda})_\varepsilon \in I_A$. Moreover $\exp(m_\lambda 2\lambda'\lambda)\|\sigma_\varepsilon\|_{\infty,K_\lambda}$ is a constant, consequently $(\|w_\varepsilon\|_{\infty,K_\lambda})_\varepsilon \in I_A$. This implies the 0th order estimate. Suppose that $(P_{K,l}(w_\varepsilon))_\varepsilon \in I_A^+$ for every $l \le n$, and let us show that $(P_{K,n+1}(w_\varepsilon))_\varepsilon \in I_A^+$. First, let us show that $(P_{1,n}(w_\varepsilon))_\varepsilon \in I_A^+$ for every $n \in \mathbb{N}$. We have

$$\frac{\partial^{n+1} w_\varepsilon}{\partial x^{n+1}}(x,y) = \frac{\partial^{n+1}\sigma_\varepsilon}{\partial x^{n+1}}(x,y) + \int_0^y \frac{\partial^n}{\partial x^n}(F(x,\eta,v_\varepsilon(x,\eta)) - F(x,\eta,u_\varepsilon(x,\eta)))\,d\eta$$

so

$$P_{K,(n+1,0)}(w_\varepsilon) \leq P_{K,(n+1,0)}(\sigma_\varepsilon) + \lambda \sup_{(x,y)\in K} \left| \frac{\partial^n}{\partial x^n}(F(x,y,v_\varepsilon(x,y)) - F(x,y,u_\varepsilon(x,y))) \right|.$$

Then

$$\sup_{(x,y)\in K} \left| \frac{\partial^n}{\partial x^n}(F(x,y,v_\varepsilon(x,y)) - F(x,y,u_\varepsilon(x,y))) \right| = P_{K,(n,0)}(F(\cdot,\cdot,v_\varepsilon) - F(\cdot,\cdot,u_\varepsilon)) \leq P_{K,n}(F(\cdot,\cdot,v_\varepsilon) - F(\cdot,\cdot,u_\varepsilon)).$$

According to the stability hypothesis, $(P_{K,(n+1,0)}(w_\varepsilon))_\varepsilon \in I_A^+$ for every $K \Subset \mathbb{R}^2$. Let us show that $(P_{2,n}(w_\varepsilon))_\varepsilon \in I_A^+$ for every $n \in \mathbb{N}$. We have

$$\frac{\partial^{n+1} w_\varepsilon}{\partial y^{n+1}}(x,y) = \frac{\partial^{n+1}\sigma_\varepsilon}{\partial y^{n+1}}(x,y) - \int_x^{g(y)} \left(\frac{\partial^n}{\partial y^n} F(\xi,y,v_\varepsilon(\xi,y)) - \frac{\partial^n}{\partial y^n} F(\xi,y,u_\varepsilon(\xi,y)) \right) d\xi - g^{(n+1)}(y) \int_0^y (F(g(y),\eta,v_\varepsilon(g(y),\eta)) - F(g(y),\eta,u_\varepsilon(g(y),\eta)))\, d\eta + \mu_\varepsilon(y)$$

with

$$\mu_\varepsilon(y) = \Big(\sum_{j=0}^{n-1} C_{n+1}^{j+1} g^{(n-j)}(y)\Big)\beta_\varepsilon(y), \qquad (\mu_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}).$$

Hence

$$P_{K,(0,n+1)}(w_\varepsilon) \leq P_{K,(0,n+1)}(\sigma_\varepsilon) + \sup_{y\in[-\lambda,\lambda]} |\mu_\varepsilon(y)| + (g(\lambda) - g(-\lambda)) \sup_{(x,y)\in K} \left| \frac{\partial^n}{\partial y^n} F(x,y,v_\varepsilon(x,y)) - \frac{\partial^n}{\partial y^n} F(x,y,u_\varepsilon(x,y)) \right| + \lambda g^{(n+1)}(y) \sup_{(x,y)\in K} |F(x,y,v_\varepsilon(x,y)) - F(x,y,u_\varepsilon(x,y))|.$$

We have

$$\sup_{(x,y)\in K} \left| \frac{\partial^n}{\partial y^n} F(x,y,v_\varepsilon(x,y)) - \frac{\partial^n}{\partial y^n} F(x,y,u_\varepsilon(x,y)) \right| = P_{K,(0,n)}(F(\cdot,\cdot,v_\varepsilon) - F(\cdot,\cdot,u_\varepsilon)) \leq P_{K,(0,n)}(F(\cdot,\cdot,v_\varepsilon) - F(\cdot,\cdot,u_\varepsilon)).$$

According to the stability hypothesis, $(P_{K,(0,n+1)}(w_\varepsilon))_\varepsilon \in I_A$ for every $K \Subset \mathbb{R}^2$. For $\alpha + \beta = n$ and $\beta \geq 1$, we have

$$P_{K,(\alpha+1,\beta)}(w_\varepsilon) = P_{K,(\alpha,\beta-1)}(F(\cdot,\cdot,v_\varepsilon) - F(\cdot,\cdot,u_\varepsilon)) \leq P_{K,n-1}(F(\cdot,\cdot,v_\varepsilon) - F(\cdot,\cdot,u_\varepsilon)).$$

So we finally have

$$P_{3,n}(w_\varepsilon) = \sup_{\alpha+\beta=n,\beta\geq 1} P_{K,(\alpha+1,\beta)}(w_\varepsilon) \leq P_{K,n-1}(F(\cdot,\cdot,v_\varepsilon) - F(\cdot,\cdot,u_\varepsilon))$$

and the stability hypothesis ensures that $(P_{3,n}(w_\varepsilon))_\varepsilon \in I_A^+$. For $\alpha + \beta = n$ and $\alpha \geq 1$, we now have

$$\begin{aligned}P_{K,(\alpha,\beta+1)}(w_\varepsilon) &= P_{K,(\alpha-1,\beta)}(F(\cdot,\cdot,v_\varepsilon)-F(\cdot,\cdot,u_\varepsilon))\\ &\le P_{K,n-1}(F(\cdot,\cdot,v_\varepsilon)-F(\cdot,\cdot,u_\varepsilon)).\end{aligned}$$

So we finally have

$$P_{4,n}(w_\varepsilon) = \sup_{\alpha+\beta=n,\alpha\ge 1} P_{K,(\alpha,\beta+1)}(w_\varepsilon) \le P_{K,n-1}(F(\cdot,\cdot,v_\varepsilon)-F(\cdot,\cdot,u_\varepsilon))$$

and the hypothesis of stability ensures that $(P_{4,n}(w_\varepsilon))_\varepsilon \in I_A^+$. So for every $l \le n+1$, we have $(P_{K,l}(w_\varepsilon))_\varepsilon \in I_A^+$. Thus $(w_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$, and consequently u is the unique solution to (P'_G). ■

5.3. A degenerate Goursat problem in $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras. We search for a generalized solution u to the following Goursat problem with irregular data:

$$(P'_G)\begin{cases}\dfrac{\partial^2 u}{\partial x\partial y} = F(\cdot,\cdot,u),\\ u|_{(Ox)} = \varphi,\\ u|_{(Oy)} = \psi,\end{cases}$$

where φ and ψ are one-variable generalized functions. The notation $F(\ .\ ,\ .\ ,u)$ extends, with the above meaning, the expression $(x,y)\mapsto F(x,y,u(x,y))$ to the case where u is a generalized function of two variables x and y. (We take $g=0$.) Suppose that hypothesis (H') is satisfied, $\mathcal{A}(\mathbb{R}^2)$ is stable under F. If the data of problem (P'_G) satisfy $\varphi,\psi\in\mathcal{A}(\mathbb{R})$, $g(y)=0$, the problem has a unique solution $[u_\varepsilon]\in\mathcal{A}(\mathbb{R}^2)$ where

$$u_\varepsilon(x,y) = u_{0,\varepsilon}(x,y) + \iint\limits_{D(x,y,0)} F(\xi,\eta,u_\varepsilon(\xi,\eta))\,d\xi\,d\eta$$

and

$$u_{0,\varepsilon}(x,y) = \psi_\varepsilon(y) + \varphi_\varepsilon(x) - \varphi_\varepsilon(0).$$

THEOREM 45. *The generalized solution u to the Goursat problem (P'_G), where φ and ψ are one-variable generalized functions, is $u=[u_\varepsilon]$ such that*

$$u_\varepsilon = \lim_{n\to\infty} u_{\varepsilon,n} \quad \text{and} \quad u_{\varepsilon,n}(x,y) = u_{0,\varepsilon}(x,y) + \int_0^x\Big(\int_0^y F(\xi,\eta,u_{\varepsilon,n-1}(\xi,\eta))\,d\eta\Big)\,d\xi$$

with

$$u_{0,\varepsilon}(x,y) = \varphi_\varepsilon(x) + \psi_\varepsilon(y) - \varphi_\varepsilon(0).$$

COROLLARY 46. *With the previous notation, we have*

$$u_\varepsilon(x,y) = u_{0,\varepsilon}(x,y) + \int_0^x\Big(\int_0^y F(\xi,\eta,u_\varepsilon(\xi,\eta))\,d\eta\Big)\,d\xi.$$

5.4. Parametric singular spectrum of the solution. We study the relationship between the $\mathcal{D}'$-parametric singular spectrum of the solution u and the $\mathcal{D}'$-parametric singular spectrum of u_0.

THEOREM 47. *Put* $u_0 = [u_{0,\varepsilon}]$ *with* $u_{0,\varepsilon}(x,y) = \psi_\varepsilon(y) + \varphi_\varepsilon(x) - \varphi_\varepsilon(g(y))$ *and suppose that*

$$\forall K \Subset \mathbb{R}^2, \quad \mathcal{M}_F(K) = \sup_{(x,y)\in K; z\in\mathbb{R}} |F(x,y,z)| < \infty. \tag{5.4}$$

Then the restriction of the $\mathcal{D}'$*-parametric singular spectrum of the solution* u *to the Goursat problem* (P'_G), *to the parametric singular support of* u_0 *is included in the restriction of the* $\mathcal{D}'$*-parametric singular spectrum of* u_0 *to the parametric singular support of* u_0. *In other words, over the singular support of* u_0, *there is no increase in the distributional singularities of* u *in comparison with those of* u_0.

Proof. Let $(x_0, y_0) = X \in S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u_0$ and $r \in N_{\mathcal{D}',X}(u_0)$. From the definitions it follows that $\Sigma_{\mathcal{D}',X}(u_0) \neq \emptyset$, so that $N_{\mathcal{D}',X}(u_0) \subset \,]0,\infty[$, which implies $r > 0$. Next let us show that $r \in N_{\mathcal{D}',X}(u)$. From the definition of $N_{\mathcal{D}',X}(u_0)$, there exists a neighborhood V_X of X such that

$$\lim_{\varepsilon\to 0} \varepsilon^r u_\varepsilon|_{V_X} \in \mathcal{D}'(V_X).$$

Let $f \in \mathcal{D}(V_X)$. So, there exists some distribution $T \in \mathcal{D}'(V_X)$ such that

$$\lim_{\varepsilon\to 0} \iint_{V_X} \varepsilon^r u_{0,\varepsilon}(x,y) f(x,y)\,dx\,dy = T(f).$$

Let us show that

$$\iint_{V_X} \varepsilon^r [u_\varepsilon(x,y) - u_{0,\varepsilon}(x,y)] f(x,y)\,dx\,dy \to 0 \quad \text{as } \varepsilon \to 0.$$

Suppose that $g(y) \le x$. As

$$u_\varepsilon(x,y) - u_{0,\varepsilon}(x,y) = \iint_{D(x,y,g)} F(\xi,\eta,u_\varepsilon(\xi,\eta))\,d\xi\,d\eta$$

and since (with the above notations)

$$\begin{aligned}
\Big|\iint_{V_X}\Big(\iint_{D(x,y,g)} & F(\xi,\eta,u_\varepsilon(\xi,\eta))\,d\xi\,d\eta\Big) f(x,y)\,dx\,dy\Big| \\
&\le \mathcal{M}_F(\operatorname{supp} f)\Big|\iint_{\operatorname{supp} f}\Big(\iint_{D(x,y,g)} d\xi\,d\eta\Big) f(x,y)\,dx\,dy\Big| \\
&\le \mathcal{M}_F(\operatorname{supp} f)\Big|\iint_{\operatorname{supp} f} (A(x,y)) f(x,y)\,dx\,dy\Big| \\
&\le 2\lambda'\mathcal{M}_F(\operatorname{supp} f)\iint_{\operatorname{supp} f} |y|\,|f(x,y)|\,dx\,dy < \infty,
\end{aligned}$$

we have

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \Big|\iint_{V_X} & \varepsilon^r [u_\varepsilon(x,y) - u_{0,\varepsilon}(x,y)] f(x,y)\,dx\,dy\Big| \\
&\le \limsup_{\varepsilon\to 0} \varepsilon^r \Big|\iint_{V_X}\Big(\iint_{D(x,y,g)} F(\xi,\eta,u_\varepsilon(\xi,\eta))\,d\xi\,d\eta\Big) f(x,y)\,dx\,dy\Big| \\
&\le \limsup_{\varepsilon\to 0} \varepsilon^r \Big[2\lambda'(\mathcal{M}_F(\operatorname{supp} f))\iint_{\operatorname{supp} f} |y|\,|f(x,y)|\,dx\,dy\Big] = 0
\end{aligned}$$

with $2\lambda' = g(\lambda) - g(-\lambda)$, because $r \neq 0$. Hence

$$\lim_{\varepsilon\to 0} \iint\limits_{V_X} \varepsilon^r u_\varepsilon(x,y) f(x,y)\,dx\,dy = \lim_{\varepsilon\to 0} \iint\limits_{V_X} \varepsilon^r u_{0,\varepsilon}(x,y) f(x,y)\,dx\,dy = T(f).$$

It follows that

$$\lim_{\varepsilon\to 0} \varepsilon^r u_\varepsilon|_{V_X} = \lim_{\varepsilon\to 0} \varepsilon^r u_{0,\varepsilon}|_{V_X} \in \mathcal{D}'(V_X).$$

So $r \in N_{\mathcal{D}',X}(u)$, which proves the inclusion $N_{\mathcal{D}',X}(u_0) \subset N_{\mathcal{D}',X}(u)$, and consequently $\Sigma_{\mathcal{D}',X}(u) \subset \Sigma_{\mathcal{D}',X}(u_0)$. Therefore

$$S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u|_{S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u_0} \subset S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u_0|_{S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u_0}. \ \blacksquare$$

Example 48. Let

$$f \in \mathcal{D}(\mathbb{R}), \quad f \geq 0, \quad \int\limits_{\mathbb{R}} f(x)\,dx = 1 \quad \text{and} \quad g(y) = ya^{-1}, a > 0.$$

Let us consider the following cases:

- $\psi_\varepsilon(y) = \varepsilon^{-1} f(y\varepsilon^{-1})$ and $\varphi_\varepsilon(x) = \varepsilon^{-1} f(x\varepsilon^{-1})$; then

$$\varphi_\varepsilon(g(y)) = \varepsilon^{-1} f(g(y)\varepsilon^{-1}) = \varepsilon^{-1} f(y(a\varepsilon)^{-1}) = a(a\varepsilon)^{-1} f(y(a\varepsilon)^{-1}),$$

$$u_{0,\varepsilon}(x,y) = \psi_\varepsilon(y) + \varphi_\varepsilon(x) - \varphi_\varepsilon(g(y)) = \varepsilon^{-1} f(y\varepsilon^{-1}) + \varepsilon^{-1} f(x\varepsilon^{-1}) - a(a\varepsilon)^{-1} f(y(a\varepsilon)^{-1}).$$

  Thus

$$N_{\mathcal{D}',X}(u_0) = [1, +\infty[\quad \text{and} \quad S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u \subset \mathbb{R}^2 \times [0,1[.$$

- $\psi_\varepsilon(y) = \varepsilon^{-1} f(y\varepsilon^{-1})$ and $\varphi_\varepsilon(x) = \varepsilon^{-2} f(x\varepsilon^{-1}) = \varepsilon^{-1}[\varepsilon^{-1} f(x\varepsilon^{-1})]$; then

$$\varphi_\varepsilon(g(y)) = \varepsilon^{-2} f((g(y)\varepsilon^{-1}) = \varepsilon^{-2} f(y(a\varepsilon)^{-1}) = a(a\varepsilon)^{-2} f(y(a\varepsilon)^{-1}).$$

  Thus

$$N_{\mathcal{D}',X}(u_0) = [2, +\infty[\quad \text{and} \quad S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u \subset \mathbb{R}^2 \times [0,2[.$$

- $\psi_\varepsilon(y) = f(y\varepsilon^{-1})$ and $\varphi_\varepsilon(x) = f(x\varepsilon^{-1}) = \varepsilon(\varepsilon^{-1} f(x\varepsilon^{-1}))$; then

$$\varphi_\varepsilon(g(y)) = \varepsilon(\varepsilon^{-1} f((g(y)\varepsilon^{-1})) = \varepsilon(\varepsilon^{-1} f(y(a\varepsilon)^{-1})) = a\varepsilon((a\varepsilon)^{-1} f(y(a\varepsilon)^{-1})).$$

  Thus

$$N_{\mathcal{D}',X}(u_0) = [0, +\infty[.$$

  As $S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u \subset \mathbb{R}^2 \times \mathbb{R}_+$, we have

$$S_\varepsilon S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}} u \subset \mathbb{R}^2 \times \emptyset.$$

5.5. Qualitative study of the solution. Case $F = 0$. We search for a generalized solution u to problem (P'_G) where $F = 0$, considering as data the curve γ of equation $x = g(y)$, $\varphi = [\varphi_\varepsilon]$, $\psi = [\psi_\varepsilon]$. With the above notations, considering $P'_\infty(\varphi_\varepsilon, \psi_\varepsilon)$, we have

$$u_\varepsilon(x,y) = \psi_\varepsilon(y) + \varphi_\varepsilon(x) - \varphi_\varepsilon(g(y)).$$

Remark 49. We have $\psi(0) = \varphi(g(0))$; if $g(y) = ya^{-1}$ $(a > 0)$ then $g(0) = 0$ and consequently, $\psi(0) = \varphi(0)$.

Example 50. $g(y) = ya^{-1}$ $(a > 0)$, $\varphi \sim S$, $\psi \sim T$; $S \in \mathcal{D}'(\mathbb{R})$, $T \in \mathcal{D}'(\mathbb{R})$. Let $f \in \mathcal{D}(\mathbb{R})$ satisfy $\int_{\mathbb{R}} f(\xi)\,d\xi = 1$. Put $f_\varepsilon(x) = \varepsilon^{-1} f(x\varepsilon^{-1})$. Choosing

$$\varphi = [f_\varepsilon * S] \quad \text{and} \quad \Psi = [f_\varepsilon * T],$$

we have the associations $\varphi \sim S$, $\psi \sim T$, since

$$\lim_{\substack{\varepsilon\to 0\\ \mathcal{D}'(\mathbb{R})}} (f_\varepsilon * S)_\varepsilon = S \quad\text{and}\quad \lim_{\substack{\varepsilon\to 0\\ \mathcal{D}'(\mathbb{R})}} (f_\varepsilon * T)_\varepsilon = T.$$

We have

$$\begin{aligned} u_\varepsilon(x,y) &= \varphi_\varepsilon(x) + \psi_\varepsilon(y) - \varphi_\varepsilon(g(y)) \\ &= (f_\varepsilon * S)(x) + (f_\varepsilon * T)(y) - (f_\varepsilon * S)(ya^{-1}). \end{aligned}$$

Let $\widetilde{S} \in \mathcal{D}'(\mathbb{R})$ be such that

$$\langle \widetilde{S}, h\rangle = \langle aS, [z \mapsto h(az)]\rangle = \langle aS, H\rangle.$$

Hence

$$\lim_{\substack{\varepsilon\to 0\\ \mathcal{D}'(\mathbb{R})}} [y \mapsto (f_\varepsilon * S)(ya^{-1})] = \widetilde{S}.$$

Then we can write

$$[u_\varepsilon] = [w_{\varepsilon,1}] + [w_{\varepsilon,2}] + [w_{\varepsilon,3}]$$

with

$$[w_{\varepsilon,1}] \sim S_x \otimes 1_y, \qquad [w_{\varepsilon,2}] \sim 1_x \otimes T_y, \qquad [w_{\varepsilon,3}] \sim -(1_x \otimes \widetilde{S}_y)$$

and so

$$u \sim S_x \otimes 1_y + 1_x \otimes T_y - (1_x \otimes \widetilde{S}_y).$$

Example 51. $g(y) = ya^{-1}$ $(a > 0)$, $\varphi \sim \delta$, $\psi \sim \delta$. As $\widetilde{\delta} = a\delta$, for $S = \delta$, we have

$$[u_\varepsilon] = [w_{\varepsilon,1}] + [w_{\varepsilon,2}] + [w_{\varepsilon,3}]$$

with

$$[w_{\varepsilon,1}] \sim \delta_x \otimes 1_y, \qquad [w_{\varepsilon,2}] \sim 1_x \otimes \delta_y, \qquad [w_{\varepsilon,3}] \sim -a(1_x \otimes \delta_y).$$

Example 52. $g(y) = 0$, $\varphi \sim \delta$, $\psi \sim \delta$. Then

$$[u_\varepsilon] = [w_{\varepsilon,1}] + [w_{\varepsilon,2}] - [\varphi_\varepsilon(0)]$$

with

$$[w_{\varepsilon,1}] \sim \delta_x \otimes 1_y, \qquad [w_{\varepsilon,2}] \sim 1_x \otimes \delta_y.$$

We observe that $[\varphi_\varepsilon(0)]$ can be associated with a distribution only if $[\varphi_\varepsilon(0)] = 0$; then, in the general case, u is not associated with a distribution (V. S. Valmorin [1995a], [1995b]).

6. A characteristic Cauchy problem in $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras

The characteristic Cauchy problem

$$(P_C)\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = F(\cdot,\cdot,u), \\ u|_{(Ox)} = \varphi, \\ \left.\dfrac{\partial u}{\partial y}\right|_{(Ox)} = \psi, \end{cases}$$

has no smooth solution (not even C^2) even for smooth data. We can approach the given characteristic problem by a family of noncharacteristic problems $(P_\varepsilon)_\varepsilon$

$$(P_\varepsilon)\begin{cases} \dfrac{\partial^2 u_\varepsilon}{\partial x \partial y} = F(\cdot,\cdot,u), \\ u_\varepsilon|_{\gamma_\varepsilon} = \varphi, \\ \left.\dfrac{\partial u_\varepsilon}{\partial y}\right|_{\gamma_\varepsilon} = \psi, \end{cases}$$

by considering the line γ_ε of equation $y = \varepsilon x$. The family of solutions is a representative of a generalized function which belongs to an appropriate parametric algebra.

6.1. Case of regular data. Rewriting the solution to P_ε, we replace $f(x)$ by εx and K_λ by

$$K_\varepsilon = [-a\varepsilon^{-1}, a\varepsilon^{-1}] \times [-a, a].$$

Here we have

$$u_\varepsilon(x,y) = u_{0,\varepsilon}(x,y) - \iint\limits_{D_\varepsilon(x,y)} F(\xi,\eta,u_\varepsilon(\xi,\eta))\, d\xi\, d\eta$$

where

$$u_{0,\varepsilon}(x,y) = \varphi(x) - \varepsilon\Psi(x) + \varepsilon\Psi(y\varepsilon^{-1}),$$

Ψ is a primitive of ψ, and

$$D_\varepsilon(x,y) = \begin{cases} \{(\xi,\eta) : x \le \xi \le y\varepsilon^{-1}, \varepsilon\xi \le \eta \le y\} & \text{if } y \ge \varepsilon x, \\ \{(\xi,\eta) : y\varepsilon^{-1} \le \xi \le x, y \le \eta \le \varepsilon\xi\} & \text{if } y \le \varepsilon x. \end{cases}$$

Put

$$m_\varepsilon = \sup_{(\xi,\eta)\in K_\varepsilon; t\in\mathbb{R}} \left| \frac{\partial F}{\partial z}(\xi,\eta,t) \right|,$$

$$\Phi_\varepsilon = \sup_{K_\varepsilon} |F(x,y,0)| + m_\varepsilon \|u_{0,\varepsilon}\|_{\infty,K_\varepsilon}.$$

We make the following hypotheses:

(H1)

$$\begin{cases} \forall K \Subset \mathbb{R}^2, \forall l \in \mathbb{N}, \exists m(K,l), \max_{\alpha\in\mathbb{N}^3,|\alpha|\le l}(\sup_{(x,y)\in K; z\in\mathbb{R}} |D^\alpha F(x,y,z)|) \le m(K,l), \\ \exists (M_\varepsilon)_\varepsilon \in \mathbb{R}_*^{]0,1]}, \exists C(l) \in \mathbb{R}_+^*, m(K_\varepsilon,l) \le C(l) M_\varepsilon, \end{cases}$$

(H2)
$$\begin{cases} \exists (r_\varepsilon)_\varepsilon \in \mathbb{R}_*^{]0,1]}, \forall K_2 \Subset \mathbb{R}, \forall \alpha_2 \in \mathbb{N}, \exists D_2 \in \mathbb{R}_+^*, \exists p \in \mathbb{N}, \\ \max[\sup_{K_2} |D^{\alpha_2}\varphi(y/\varepsilon)|, \sup_{K_2} |D^{\alpha_2}\Psi(y/\varepsilon)|] \le D_2/(r_\varepsilon)^p, \end{cases}$$

(H3)
$$\begin{cases} \mathcal{C} = A/I_A \text{ is overgenerated by the following elements of } \mathbb{R}_*^{]0,1]} : \\ (\varepsilon)_\varepsilon; (r_\varepsilon)_\varepsilon; (e^{m_\varepsilon/\varepsilon})_\varepsilon; (M_\varepsilon)_\varepsilon, \end{cases}$$

(H4)
$$\begin{cases} \mathcal{A}(\mathbb{R}^2) = \mathcal{X}(\mathbb{R}^2)/\mathcal{N}(\mathbb{R}^2) \text{ is built on } \mathcal{C} \text{ with} \\ (\mathcal{E},\mathcal{P}) = (\mathrm{C}^\infty(\mathbb{R}^2), (P_{K,l})_{K\Subset\mathbb{R}^2, l\in\mathbb{N}}) \\ \text{and } \mathcal{A}(\mathbb{R}^2) \text{ is stable under } F \text{ relatively to } \mathcal{C}. \end{cases}$$

THEOREM 53. *With the previous notations and hypotheses, if u_ε is the solution to problem (P_ε), the family $(u_\varepsilon)_\varepsilon$ is a representative of a generalized function u which belongs to* *the algebra $\mathcal{A}(\mathbb{R}^2)$. Thus we can consider u as the generalized solution to the characteristic Cauchy problem (P_C).*

Proof. For $K = K_1 \times K_2 = [-a, a] \times [-a, a]$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\sup_{K_1} |D^{\alpha_1}\varphi(x)| \leq C_1(K_1, \alpha_1); \qquad \varepsilon \sup_{K_1} |D^{\alpha_1}\Psi(x)| \leq \varepsilon C_2(K_1, \alpha 1).$$

As $G(y) = \Psi \circ f_\varepsilon^{-1}(y) = \Psi(y\varepsilon^{-1})$ we have

$$\varepsilon \sup_{K_2} |D^{\alpha_2}G(y)| \leq \frac{D_2}{\varepsilon^{\alpha_2 - 1}(r_\varepsilon)^{p(\alpha_2, K_2)}},$$

so $(P_{K,\alpha}(u_{0,\varepsilon}))_\varepsilon \in A_+$. We have to show that $(P_{K,\alpha}(u_\varepsilon))_\varepsilon \in A_+$. Put

$$u_{1,\varepsilon}(x, y) = \iint\limits_{D_\varepsilon(x,y)} F(\xi, \eta, u_\varepsilon(\xi, \eta))\, d\xi\, d\eta.$$

According to the above results

$$\sup_K \Bigg| \iint\limits_{D_\varepsilon(x,y)} F(\xi, \eta, u_\varepsilon(\xi, \eta))\, d\xi\, d\eta \Bigg| \leq \frac{\Phi_\lambda}{m_\lambda} \exp[2\lambda m_\lambda(f(\lambda) - f(-\lambda))]$$

with

$$f(x) = \varepsilon x, \qquad \lambda = a\varepsilon^{-1}, \qquad m_\lambda = m_\varepsilon.$$

So

$$(f(\lambda) - f(-\lambda)) = 2a \quad \text{and} \quad 2\lambda m_\lambda(f(\lambda) - f(-\lambda)) = 4a^2\varepsilon^{-1}m_\varepsilon,$$

hence

$$\sup_{K\varepsilon} |u_{1,\varepsilon}(x, y)| \leq \frac{\Phi_\varepsilon}{m_\varepsilon} e^{(4a^2/\varepsilon)m_\varepsilon}$$

with

$$\Phi_\varepsilon = \sup_{K\varepsilon} |F(x, y, 0)| + m_\varepsilon \|u_{0,\varepsilon}\|_{\infty, K_\varepsilon} \leq C(0)M_\varepsilon + m_\varepsilon \left(\frac{3D_2}{(r_\varepsilon)^{p1}} \right)$$

where $p1 = p([-a, a], 0)$. So $(P_{K,0}(u_{1,\varepsilon}))_\varepsilon \in A_+$, hence $(P_{K,0}(u_\varepsilon))_\varepsilon \in A_+$. We have

$$\frac{\partial u_\varepsilon}{\partial x}(x, y) = \frac{\partial u_{0,\varepsilon}}{\partial x}(x, y) + \int\limits_{f(x)}^{y} F(x, \eta, u_\varepsilon(x, \eta))\, d\eta$$

and

$$\frac{\partial u_{1,\varepsilon}}{\partial x}(x, y) = \int\limits_{f(x)}^{y} F(x, \eta, u_\varepsilon(x, \eta))\, d\eta,$$

so, according to hypothesis (H1),

$$\sup_{K\varepsilon} \int\limits_{f(x)}^{y} |F(x, \eta, u_\varepsilon(x, \eta))|\, d\eta \leq 2a(m(K_\varepsilon, 0)),$$

so $(P_{K,(1,0)}(u_{1,\varepsilon}))_\varepsilon \in A_+$, hence $(P_{K,(1,0)}(u_\varepsilon))_\varepsilon \in A_+$. We have

$$\frac{\partial u_\varepsilon}{\partial y}(x, y) = \frac{\partial u_{0,\varepsilon}}{\partial y}(x, y) - \int\limits_{x}^{f^{-1}(y)} F(\xi, y, u_\varepsilon(\xi, y))\, d\xi.$$

In the same way, we get

$$\sup_{K\varepsilon}\left|\frac{\partial u_{1,\varepsilon}}{\partial y}(x,y)\right| \le \sup_{K\varepsilon}\Big(\int\limits_x^{f^{-1}(y)} |F(\xi,y,u_\varepsilon(\xi,y))|\,d\xi\Big)$$
$$\le \frac{2a}{\varepsilon}\,m(K_\varepsilon,0) \le \frac{2a}{\varepsilon}\,C(0)M_\varepsilon,$$

so $(P_{K,(0,1)}(u_{1,\varepsilon}))_\varepsilon \in A_+$, $(P_{K,(0,1)}(u_\varepsilon))_\varepsilon \in A_+$. Consequently,

$$(P_{K,1}(u_\varepsilon))_\varepsilon \in A_+.$$

Now we proceed by induction. Suppose that $(P_{K,l}(u_\varepsilon))_\varepsilon \in A_+$ for every $l \le n$, and let us show that $(P_{K,n+1}(u_\varepsilon))_\varepsilon \in A_+$. We use the notations from Theorem 37. First let us show that

$$(P_{1,n}(u_\varepsilon))_\varepsilon \in A_+, \qquad (P_{2,n}(u_\varepsilon))_\varepsilon \in A_+.$$

for every $n \in \mathbb{N}$. We have by successive derivations, for $n \ge 1$,

$$\frac{\partial^{n+1}u_{1,\varepsilon}}{\partial x^{n+1}}(x,y) = -n\varepsilon\,\frac{\partial^{n-1}}{\partial x^{n-1}}F(x,\varepsilon x,\varphi(x)) + \int\limits_{\varepsilon x}^{y} \frac{\partial^n}{\partial x^n}F(x,\eta,u_\varepsilon(x,\eta))\,d\eta.$$

Thus

$$\sup_{(x,y)\in K_\varepsilon}\left|\frac{\partial^{n+1}u_\varepsilon}{\partial x^{n+1}}(x,y)\right|$$
$$\le \sup_{x\in[-a\varepsilon^{-1},a\varepsilon^{-1}]} n\varepsilon\left|\frac{\partial^{n-1}}{\partial x^{n-1}}F(x,\varepsilon x,\varphi(x))\right| + 2a\sup_{(x,y)\in K_\varepsilon}\left|\frac{\partial^n}{\partial x^n}F(x,y,u_\varepsilon(x,y))\right|.$$

Next, from the stability property, we get

$$\sup_{(x,y)\in K_\varepsilon}\left|\frac{\partial^n}{\partial x^n}F(x,y,u_\varepsilon(x,y))\right| = P_{K_\varepsilon,(n,0)}(F(\cdot,\cdot,u_\varepsilon)) \le P_{K_\varepsilon,n}(F(\cdot,\cdot,u_\varepsilon))$$
$$\le \sum_{i=0}^{n} C_i P^i_{K_\varepsilon,n}(u_\varepsilon)$$

and

$$\sup_{x\in[-a\varepsilon^{-1},a\varepsilon^{-1}]} n\varepsilon\left|\frac{\partial^{n-1}}{\partial x^{n-1}}F(x,\varepsilon x,\varphi(x))\right| \le n\varepsilon(m(K_\varepsilon,n-1)) \le n\varepsilon C(n-1)M_\varepsilon,$$

so $(P_{K,(n+1,0)}(u_{1,\varepsilon}))_\varepsilon \in A_+$, hence

$$(P_{K,(n+1,0)}(u_\varepsilon))_\varepsilon \in A_+.$$

Let us show that $(P_{2,n}(u_\varepsilon))_\varepsilon \in A_+$ for every $n \in \mathbb{N}$. We have by successive derivations, for $n \ge 1$,

$$\frac{\partial^{n+1}u_{1,\varepsilon}}{\partial y^{n+1}}(x,y) = -n\frac{1}{\varepsilon}\,\frac{\partial^{n-1}}{\partial y^{n-1}}F\Big(\frac{y}{\varepsilon},y,\varphi\Big(\frac{y}{\varepsilon}\Big)\Big) - \int\limits_{x}^{y/\varepsilon} \frac{\partial^n}{\partial y^n}F(\xi,y,u_\varepsilon(\xi,y))\,d\xi.$$

Thus

$$\sup_{(x,y)\in K_\varepsilon}\left|\frac{\partial^{n+1}u_{1,\varepsilon}}{\partial y^{n+1}}(x,y)\right|$$
$$\leq \sup_{y\in[-a,a]} n\,\frac{1}{\varepsilon}\left|\frac{\partial^{n-1}}{\partial y^{n-1}}F(y\varepsilon^{-1},y,\varphi(y\varepsilon^{-1}))\right| + 2\lambda \sup_{(x,y)\in K_\varepsilon}\left|\frac{\partial^{n}}{\partial y^{n}}F(x,y,u_\varepsilon(x,y))\right|.$$

Next, from the stability property,

$$\begin{aligned}\sup_{(x,y)\in K_\varepsilon}\left|\frac{\partial^{n}}{\partial y^{n}}F(x,y,u_\varepsilon(x,y))\right| &= P_{K_\varepsilon,(0,n)}(F(\cdot,\cdot,u_\varepsilon))\\ &\leq P_{K_\varepsilon,n}(F(\cdot,\cdot,u_\varepsilon)) \leq \sum_{i=0}^{n} C_i P^i_{K_\varepsilon,n}(u_\varepsilon)\end{aligned}$$

and

$$\sup_{y\in[-a,a]} n\,\frac{1}{\varepsilon}\left|\frac{\partial^{n-1}}{\partial y^{n-1}}F(y\varepsilon^{-1},y,\varphi(y\varepsilon^{-1}))\right| \leq n\,\frac{1}{\varepsilon}\,(m(K_\varepsilon,n-1)) \leq n\,\frac{1}{\varepsilon}\,C(n-1)M_\varepsilon,$$

so, $(P_{K,(0,n+1)}(u_{1,\varepsilon}))_\varepsilon \in A_+$ for every $K \Subset \mathbb{R}^2$ and $n\in\mathbb{N}$, hence

$$(P_{K,(0,n+1)}(u_\varepsilon))_\varepsilon \in A_+.$$

For $\alpha+\beta=n$ and $\beta\geq 1$, we have

$$\begin{aligned}P_{K,(\alpha+1,\beta)}(u_\varepsilon) &= \sup_{(x,y)\in K}|D^{(\alpha+1,\beta)}u_\varepsilon(x,y)| = \sup_{(x,y)\in K}|D^{(\alpha,\beta-1)}D^{(1,1)}u_\varepsilon(x,y)|\\ &= \sup_{(x,y)\in K}|D^{(\alpha,\beta-1)}F(x,y,u_\varepsilon(x,y))| = P_{K,(\alpha,\beta-1)}(F(\cdot,\cdot,u_\varepsilon))\\ &\leq P_{K,n-1}(F(\cdot,\cdot,u_\varepsilon)) \leq P_{K,n}(F(\cdot,\cdot,u_\varepsilon)),\end{aligned}$$

thus

$$P_{3,n}(u_\varepsilon) = \sup_{\alpha+\beta=n;\beta\geq 1} P_{K,(\alpha+1,\beta)}(u_\varepsilon) \leq P_{K,n}(F(\cdot,\cdot,u_\varepsilon)).$$

Then the stability hypothesis ensures that $(P_{3,n}(u_\varepsilon))_\varepsilon \in A_+$. In the same way, for $\alpha+\beta=n$ and $\alpha\geq 1$, we have

$$\begin{aligned}P_{K,(\alpha,\beta+1)}(u_\varepsilon) &= \sup_{(x,y)\in K}|D^{(\alpha,\beta+1)}u_\varepsilon(x,y)| = \sup_{(x,y)\in K}|D^{(\alpha-1,\beta)}D^{(1,1)}u_\varepsilon(x,y)|\\ &= \sup_{(x,y)\in K}|D^{(\alpha-1,\beta)}F(x,y,u_\varepsilon(x,y))| = P_{K,(\alpha-1,\beta)}(F(\cdot,\cdot,u_\varepsilon))\\ &\leq P_{K,n-1}(F(\cdot,\cdot,u_\varepsilon)) \leq P_{K,n}(F(\cdot,\cdot,u_\varepsilon)).\end{aligned}$$

So we finally have

$$P_{4,n}(u_\varepsilon) = \sup_{\alpha+\beta=n;\alpha\geq 1} P_{K,(\alpha,\beta+1)}(u_\varepsilon) \leq P_{K,n}(F(\cdot,\cdot,u_\varepsilon))$$

and the stability hypothesis ensures that $(P_{4,n}(u_\varepsilon))_\varepsilon \in A_+$. In conclusion, we have

$$(P_{K,n+1}(u_\varepsilon))_\varepsilon \in A_+. \ \blacksquare$$

REMARK 54. How does this generalized function depend on the approximation of $\{y=0\}$ by $\{y=\varepsilon x\}$? The question remains open.

6.2. Case of irregular data. We can also give a meaning to the characteristic Cauchy problem (P_C) in the case where φ and ψ are themselves irregular data (for example some generalized functions) by beginning to solve

$$P_{(\varepsilon,\eta)}\begin{cases}\dfrac{\partial^2 u_{(\varepsilon,\eta)}}{\partial x\partial y}(x,y)=F(x,y,u_{(\varepsilon,\eta)}(x,y)),\\ u_{(\varepsilon,\eta)}(x,\varepsilon x)=\varphi_\eta(x),\\ \dfrac{\partial u_{(\varepsilon,\eta)}}{\partial y}(x,\varepsilon x)=\psi_\eta(x),\end{cases}$$

where $(\varphi_\eta)_\eta$ and $(\psi_\eta)_\eta$ are representatives of φ and ψ in an appropriate algebra. The parameter ε permits replacing the given problem by a noncharacteristic one, whereas the parameter η makes it regular. Ψ being a primitive of ψ, we have

$$u_{0,(\varepsilon,\eta)}(x,y)=\varphi_\eta(x)-\varepsilon\Psi_\eta(x)+\varepsilon\Psi_\eta(y\varepsilon^{-1}),$$
$$u_{(\varepsilon,\eta)}(x,y)=u_{0,(\varepsilon,\eta)}(x,y)-\iint\limits_{D_\varepsilon(x,y)}F(\xi,\theta,u_{(\varepsilon,\eta)}(\xi,\theta))\,d\xi\,d\theta.$$

Keeping hypothesis (H1) from the previous theorem, we suppose moreover that

$$\text{(H5)}\qquad\begin{cases}\exists(r_{\varepsilon,\eta})_{(\varepsilon,\eta)}\in\mathbb{R}_*^{]0,1]\times]0,1]},\ \forall K_2\Subset\mathbb{R},\forall\alpha_2\in\mathbb{N},\exists D_2\in\mathbb{R}_+^*,\exists p\in\mathbb{N},\\ \max[\sup_{K_2}|D^{\alpha_2}\varphi_\eta(y/\varepsilon)|,\sup_{K_2}|D^{\alpha_2}\Psi_\eta(y/\varepsilon)|]\le D_2/(r_{\varepsilon,\eta})^p,\end{cases}$$

$$\text{(H6)}\qquad\begin{cases}\mathcal{C}=A/I_A\text{ is overgenerated by the following elements of }\mathbb{R}_*^{]0,1]\times]0,1]}:\\ (\varepsilon)_{(\varepsilon,\eta)};(\eta)_{(\varepsilon,\eta)};(r_{\varepsilon,\eta})_{(\varepsilon,\eta)};(e^{m_\varepsilon/\varepsilon})_{(\varepsilon,\eta)};(M_\varepsilon)_{(\varepsilon,\eta)},\end{cases}$$

$$\text{(H7)}\qquad\begin{cases}\mathcal{A}(\mathbb{R}^2)=\mathcal{X}(\mathbb{R}^2)/\mathcal{N}(\mathbb{R}^2)\text{ is built on }\mathcal{C}\\ \text{with }(\mathcal{E},\mathcal{P})=(\mathrm{C}^\infty(\mathbb{R}^2),(P_{K,l})_{K\Subset\mathbb{R}^2;l\in\mathbb{N}})\\ \text{and }\mathcal{A}(\mathbb{R}^2)\text{ is stable under }F\text{ relative to }\mathcal{C}.\end{cases}$$

THEOREM 55. *With the previous notations and hypotheses, if $u_{(\varepsilon,\eta)}$ is the solution to problem $P_{(\varepsilon,\eta)}$, the family $(u_{(\varepsilon,\eta)})_{(\varepsilon,\eta)}$ is a representative of a generalized function u which belongs to the algebra $\mathcal{A}(\mathbb{R}^2)$. Thus we can consider $u=[u_{(\varepsilon,\eta)}]$ as the generalized solution to the characteristic Cauchy problem P_C.*

Proof. For $K=K_1\times K_2=[-a,a]\times[-a,a]$ and $\alpha=(\alpha_1,\alpha_2)\in\mathbb{N}^2$, there exist $C_1>0$ and $C_2>0$ such that

$$\sup_{K_1}|D^{\alpha_1}\varphi_\eta(x)|\le C_1(K_1,\alpha_1),\qquad\varepsilon\sup_{K_1}|D^{\alpha_1}\Psi_\eta(x)|\le\varepsilon C_2(K_1,\alpha 1).$$

As $G_\eta(y)=\Psi_\eta\circ f_\varepsilon^{-1}(y)=\Psi_\eta(y/\varepsilon)$, we have

$$\varepsilon\sup_{K_2}|D^{\alpha_2}G_\eta(y)|\le\frac{D_2}{\varepsilon^{\alpha_2-1}(r_{\varepsilon,\eta})^{p(\alpha_2,K_2)}},$$

so $(P_{K,\alpha}(u_{0,(\varepsilon,\eta)}))_{(\varepsilon,\eta)}\in A_+$.

We have to show that $(P_{K,n}(u_{(\varepsilon,\eta)}))_{(\varepsilon,\eta)}\in A_+$ for every integer n. Put

$$u_{1,(\varepsilon,\eta)}(x,y)=\iint\limits_{D_\varepsilon(x,y)}F(\xi,\theta,u_{(\varepsilon,\eta)}(\xi,\theta))\,d\xi\,d\theta.$$

According to the above results, we have

$$\sup_K \Big| \iint\limits_{D_\varepsilon(x,y)} F(\xi,\theta,u_{(\varepsilon,\eta)}(\xi,\theta))\,d\xi\,d\theta \Big| \le \frac{\Phi_\lambda}{m_\lambda} \exp[2\lambda m_\lambda(f(\lambda)-f(-\lambda))]$$

with $f(x)=\varepsilon x$, $\lambda = a\varepsilon^{-1}$, $m_\lambda = m_\varepsilon$. So $(f(\lambda)-f(-\lambda)) = 2a$ and

$$2\lambda m_\lambda(f(\lambda)-f(-\lambda)) = 2\,\frac{a}{\varepsilon}\,2am_\varepsilon = 4\frac{a^2}{\varepsilon}m_\varepsilon.$$

We have

$$\sup_{K\varepsilon} |u_{1,(\varepsilon,\eta)}(x,y)| \le \frac{\Phi_\varepsilon}{m_\varepsilon}\, e^{\frac{4a^2}{\varepsilon}m_\varepsilon}$$

with

$$\Phi_\varepsilon = \sup_{K\varepsilon} |F(x,y,0)| + m_\varepsilon \|u_{0,(\varepsilon,\eta)}\|_{\infty,K_\varepsilon} \le C(0)M_\varepsilon + m_\varepsilon\bigg(\frac{3D_2}{(r_{\varepsilon,\eta})^{p1}}\bigg),$$

where $p1 = p([-a,a],0)$. So $(P_{K,0}(u_{1,(\varepsilon,\eta)}))_{(\varepsilon,\eta)} \in A_+$, hence $(P_{K,0}(u_{(\varepsilon,\eta)}))_{(\varepsilon,\eta)} \in A_+$. Moreover

$$\frac{\partial u_{(\varepsilon,\eta)}}{\partial x}(x,y) = \frac{\partial u_{0,(\varepsilon,\eta)}}{\partial x}(x,y) + \int\limits_{f(x)}^{y} F(x,\theta,u_{(\varepsilon,\eta)}(x,\theta))\,d\theta.$$

We have

$$\frac{\partial u_{1,(\varepsilon,\eta)}}{\partial x}(x,y) = \int\limits_{f(x)}^{y} F(x,\theta,u_{(\varepsilon,\eta)}(x,\theta))\,d\theta,$$

so, according to hypothesis (H1),

$$\sup_{K_\varepsilon} \bigg(\int\limits_{f(x)}^{y} F(x,\theta,u_{(\varepsilon,\eta)}(x,\theta))\,d\theta \bigg) \le 2am(K_\varepsilon,0),$$

then $(P_{K,(1,0)}(u_{1,(\varepsilon,\eta)}))_{(\varepsilon,\eta)} \in A_+$, consequently,

$$(P_{K,(1,0)}(u_{(\varepsilon,\eta)}))_{(\varepsilon,\eta)} \in A_+.$$

We have

$$\frac{\partial u_{(\varepsilon,\eta)}}{\partial y}(x,y) = \frac{\partial u_{0,(\varepsilon,\eta)}}{\partial y}(x,y) - \int\limits_{x}^{f^{-1}(y)} F(\xi,y,u_{(\varepsilon,\eta)}(\xi,y))\,d\xi.$$

In the same way, we get

$$\begin{aligned}\sup_{K_\varepsilon} \bigg| \frac{\partial u_{1,(\varepsilon,\eta)}}{\partial y}(x,y) \bigg| &\le \sup_{K_\varepsilon} \int\limits_{x}^{f^{-1}(y)} |F(\xi,y,u_{(\varepsilon,\eta)}(\xi,y)|)\,d\xi \\ &\le \frac{2a}{\varepsilon} m(K_\varepsilon,0) \le \frac{2a}{\varepsilon}\,C(0)M_\varepsilon,\end{aligned}$$

so

$$(P_{K,(0,1)}(u_{1,(\varepsilon,\eta)}))_{(\varepsilon,\eta)} \in A_+, \quad (P_{K,(0,1)}(u_{(\varepsilon,\eta)}))_{(\varepsilon,\eta)} \in A_+.$$

Consequently,

$$(P_{K,1}(u_{(\varepsilon,\eta)}))_{(\varepsilon,\eta)} \in A_+.$$

We now proceed by induction. Suppose that $(P_{K,l}(u_{(\varepsilon,\eta)}))_{(\varepsilon,\eta)} \in A_+$ for every $l \leq n$, and let us show that

$$(P_{K,n+1}(u_{(\varepsilon,\eta)}))_{(\varepsilon,\eta)} \in A_+.$$

We use the notations from Theorem 37. First we show that

$$(P_{1,n}(u_{(\varepsilon,\eta)}))_{(\varepsilon,\eta)} \in A_+, \quad (P_{2,n}(u_{(\varepsilon,\eta)}))_{(\varepsilon,\eta)} \in A_+$$

for every $n \in \mathbb{N}$. We have by successive derivations, for $n \geq 1$,

$$\frac{\partial^{n+1} u_{1,(\varepsilon,\eta)}}{\partial x^{n+1}}(x,y) = -n\varepsilon \frac{\partial^{n-1}}{\partial x^{n-1}} F(x, \varepsilon x, \varphi_\eta(x)) + \int_{\varepsilon x}^{y} \frac{\partial^n}{\partial x^n} F(x, \theta, u_{(\varepsilon,\eta)}(x,\theta))\, d\theta.$$

We have

$$\begin{aligned}\sup_{(x,y)\in K_\varepsilon} \left| \frac{\partial^{n+1} u_{1,(\varepsilon,\eta)}}{\partial x^{n+1}}(x,y) \right| \leq &\sup_{x\in[-a\varepsilon^{-1}, a\varepsilon^{-1}]} n\varepsilon \left| \frac{\partial^{n-1}}{\partial x^{n-1}} F(x, \varepsilon x, \varphi_\eta(x)) \right| \\ &+ 2a \sup_{(x,y)\in K_\varepsilon} \left| \frac{\partial^n}{\partial x^n} F(x,y,u_{(\varepsilon,\eta)}(x,y)) \right|.\end{aligned}$$

Next, according to the stability property, we get

$$\begin{aligned}\sup_{(x,y)\in K_\varepsilon} \left| \frac{\partial^n}{\partial x^n} F(x,y,u_{(\varepsilon,\eta)}(x,y)) \right| &= P_{K_\varepsilon,(n,0)}(F(\cdot,\cdot,u_{(\varepsilon,\eta)})) \leq P_{K_\varepsilon,n}(F(\cdot,\cdot,u_{(\varepsilon,\eta)})) \\ &\leq \sum_{i=0}^{n} C_i P^i_{K_\varepsilon,n}(u_{(\varepsilon,\eta)}),\end{aligned}$$

and

$$\sup_{x\in[-a\varepsilon^{-1}, a\varepsilon^{-1}]} n\varepsilon \left| \frac{\partial^{n-1}}{\partial x^{n-1}} F(x, \varepsilon x, \varphi_\varepsilon(x)) \right| \leq n\varepsilon(m(K_\varepsilon, n-1)) \leq n\varepsilon C(n-1) M_\varepsilon,$$

so $(P_{K,(n+1,0)}(u_{1,(\varepsilon,\eta)}))_{(\varepsilon,\eta)} \in A_+$, hence

$$(P_{K,(n+1,0)}(u_{(\varepsilon,\eta)}))_{(\varepsilon,\eta)} \in A_+.$$

Let us show that $(P_{2,n}(u_{(\varepsilon,\eta)}))_{(\varepsilon,\eta)} \in A_+$ for every $n \in \mathbb{N}$. We have by successive derivations, for $n \geq 1$,

$$\frac{\partial^{n+1} u_{1,(\varepsilon,\eta)}}{\partial y^{n+1}}(x,y) = -n\,\frac{1}{\varepsilon}\,\frac{\partial^{n-1}}{\partial y^{n-1}} F(y\varepsilon^{-1}, y, \varphi_\eta(y\varepsilon^{-1})) - \int_{x}^{y/\varepsilon} \frac{\partial^n}{\partial y^n} F(\xi, y, u_{(\varepsilon,\eta)}(\xi,y))\, d\xi.$$

Thus

$$\begin{aligned}&\sup_{(x,y)\in K_\varepsilon} \left| \frac{\partial^{n+1} u_{1,(\varepsilon,\eta)}}{\partial y^{n+1}}(x,y) \right| \\ &\quad\leq \sup_{y\in[-a,a]} n\,\frac{1}{\varepsilon} \left| \frac{\partial^{n-1}}{\partial y^{n-1}} F(y\varepsilon^{-1}, y, \varphi_\eta(y\varepsilon^{-1})) \right| + 2\lambda \sup_{(x,y)\in K_\varepsilon} \left| \frac{\partial^n}{\partial y^n} F(x,y,u_{(\varepsilon,\eta)}(x,y)) \right|.\end{aligned}$$

Next, according to the stability property, we get

$$\sup_{(x,y)\in K_\varepsilon}\left|\frac{\partial^n}{\partial y^n}F(x,y,u_{(\varepsilon,\eta)}(x,y))\right| = P_{K_\varepsilon,(0,n)}(F(\cdot,\cdot,u_{(\varepsilon,\eta)}))$$
$$\le P_{K_\varepsilon,n}(F(\cdot,\cdot,u_{(\varepsilon,\eta)})) \le \sum_{i=0}^{n} C_i P^i_{K_\varepsilon,n}(u_{(\varepsilon,\eta)})$$

and

$$\sup_{y\in[-a,a]} n\,\frac{1}{\varepsilon}\left|\frac{\partial^{n-1}}{\partial y^{n-1}}F(y\varepsilon^{-1},y,\varphi_\eta(y\varepsilon^{-1}))\right| \le n\,\frac{1}{\varepsilon}\,m(K_\varepsilon,n-1) \le n\,\frac{1}{\varepsilon}\,C(n-1)M_\varepsilon.$$

So, for every $K \Subset \mathbb{R}^2$ and $n\in\mathbb{N}$, $(P_{K,(0,n+1)}(u_{1,(\varepsilon,\eta)}))_{(\varepsilon,\eta)} \in A_+$, hence

$$(P_{K,(0,n+1)}(u_{(\varepsilon,\eta)}))_{(\varepsilon,\eta)} \in A_+.$$

For $\alpha+\beta=n$ and $\beta\ge 1$, we have

$$\begin{aligned} P_{K,(\alpha+1,\beta)}(u_{(\varepsilon,\eta)}) &= \sup_{(x,y)\in K}|D^{(\alpha+1,\beta)}u_{(\varepsilon,\eta)}(x,y)| = \sup_{(x,y)\in K}|D^{(\alpha,\beta-1)}D^{(1,1)}u_{(\varepsilon,\eta)}(x,y)| \\ &= \sup_{(x,y)\in K}|D^{(\alpha,\beta-1)}F(x,y,u_{(\varepsilon,\eta)}(x,y))| = P_{K,(\alpha,\beta-1)}(F(\cdot,\cdot,u_{(\varepsilon,\eta)})) \\ &\le P_{K,n-1}(F(\cdot,\cdot,u_{(\varepsilon,\eta)})) \le P_{K,n}(F(\cdot,\cdot,u_{(\varepsilon,\eta)})). \end{aligned}$$

So we finally have

$$P_{3,n}(u_{(\varepsilon,\eta)}) = \sup_{\alpha+\beta=n;\beta\ge 1} P_{K,(\alpha+1,\beta)}(u_{(\varepsilon,\eta)}) \le P_{K,n}(F(\cdot,\cdot,u_{(\varepsilon,\eta)}))$$

and the stability hypothesis ensures that

$$(P_{3,n}(u_{(\varepsilon,\eta)}))_{(\varepsilon,\eta)} \in A_+.$$

In the same way, for $\alpha+\beta=n$ and $\alpha\ge 1$, we have

$$\begin{aligned} P_{K,(\alpha,\beta+1)}(u_{(\varepsilon,\eta)}) &= \sup_{(x,y)\in K}|D^{(\alpha,\beta+1)}u_{(\varepsilon,\eta)}(x,y)| = \sup_{(x,y)\in K}|D^{(\alpha-1,\beta)}D^{(1,1)}u_{(\varepsilon,\eta)}(x,y)| \\ &= \sup_{(x,y)\in K}|D^{(\alpha-1,\beta)}F(x,y,u_{(\varepsilon,\eta)}(x,y))| = P_{K,(\alpha-1,\beta)}(F(\cdot,\cdot,u_{(\varepsilon,\eta)})) \\ &\le P_{K,n-1}(F(\cdot,\cdot,u_{(\varepsilon,\eta)})) \le P_{K,n}(F(\cdot,\cdot,u_{(\varepsilon,\eta)})). \end{aligned}$$

Thus

$$P_{4,n}(u_{(\varepsilon,\eta)}) = \sup_{\alpha+\beta=n;\alpha\ge 1} P_{K,(\alpha,\beta+1)}(u_{(\varepsilon,\eta)}) \le P_{K,n}(F(\cdot,\cdot,u_{(\varepsilon,\eta)}))$$

and the stability hypothesis ensures that $(P_{4,n}(u_{(\varepsilon,\eta)}))_{(\varepsilon,\eta)} \in A_+$. In conclusion, we have $(P_{K,n+1}(u_{(\varepsilon,\eta)}))_{(\varepsilon,\eta)} \in A_+$. ∎

6.3. Qualitative study of the solution. Case $F=0$**.** We consider $\mathcal{A}(\mathbb{R})$ and $\mathcal{A}(\mathbb{R}^2)$ built on the same ring of generalized constants as before. We suppose that $\mathcal{A}(\mathbb{R}^2)$ is stable under F. For $g\in\mathcal{D}'(\mathbb{R})$, with

$$\operatorname{supp} g = [-1,1], \qquad 0\le g\le 1, \qquad g(0)=1$$

and $g^{(k)}(0)=0$ for every $k\in\mathbb{N}^*$, we consider $g_\eta(x)=\eta^{-1}g(x\eta^{-1})$ for $x\in\mathbb{R}$. Then $(g_\eta)_\eta \to \delta_x$ in the distributional sense. For $S\in\mathcal{D}'(\mathbb{R})$, $T\in\mathcal{D}'(\mathbb{R})$, choosing

$$\varphi = [g_\eta * S], \qquad \Psi = [g_\eta * T],$$

we have the associations $\varphi \sim S$, $\varPsi \sim T$, since

$$\lim_{\substack{\eta\to 0\\ \mathcal{D}'(\mathbb{R})}} (g_\eta * S)_\eta = S \quad \text{and} \quad \lim_{\substack{\eta\to 0\\ \mathcal{D}'(\mathbb{R})}} (g_\eta * T)_\eta = T.$$

Example 56. $\varphi \sim S$, $\varPsi \sim T$, $S \in \mathcal{D}'(\mathbb{R})$, $T \in \mathcal{D}'(\mathbb{R})$. We search for a generalized solution u to the following characteristic irregular Cauchy problem:

$$(P_C)\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = 0, \\ u|_{(Ox)} = S, \\ \left.\dfrac{\partial u}{\partial y}\right|_{(Ox)} = T'. \end{cases}$$

By considering the curve γ_ε of equation $y = \varepsilon x$ and by putting the data regularized by mollifiers g_η on the curve $\gamma_\varepsilon = \{y = \varepsilon x\}$, we can solve the noncharacteristic problem

$$(P_{(\varepsilon,\eta)})\begin{cases} \dfrac{\partial^2 u_{(\varepsilon,\eta)}}{\partial x \partial y}(x,y) = 0, \\ u_{(\varepsilon,\eta)}(x,\varepsilon x) = (g_\eta * S)(x), \\ \dfrac{\partial u_{(\varepsilon,\eta)}}{\partial y}(x,\varepsilon x) = (g_\eta * T')(x). \end{cases}$$

Let us determine the solution u to $(P_{(\varepsilon,\eta)})$. We have

$$\begin{aligned} u_{(\varepsilon,\eta)}(x,y) &= \varepsilon \varPsi_\eta(y\varepsilon^{-1}) - \varepsilon \varPsi_\eta(x) + \varphi_\eta(x) \\ &= \varepsilon (g_\eta * T)(y\varepsilon^{-1}) - \varepsilon (g_\eta * T)(x) + (g_\eta * S)(x). \end{aligned}$$

Hence

$$[u_{(\varepsilon,\eta)}] = [\varepsilon u_{(\varepsilon,\eta),1}] + [\varepsilon u_{(\varepsilon,\eta),2}] + [u_{(\varepsilon,\eta),3}]$$

with

$$u_{(\varepsilon,\eta),1}(x,y) = (g_\eta * T)(y\varepsilon^{-1}), \qquad [u_{(\varepsilon,\eta),2}] \sim -T_x \otimes 1_y, \qquad [u_{(\varepsilon,\eta),3}] \sim S_x \otimes 1_y.$$

Example 57. ($\varphi \sim \delta$, $\psi \sim \delta$; $F = 0$) so ($\varphi \sim S \sim \delta$, $\varPsi \sim Y \sim T$; $F = 0$). Let G a primitive of g. We have

$$\lim_{\substack{\eta\to 0\\ \mathcal{D}'(\mathbb{R})}} G_\eta = Y \quad \text{and} \quad \lim_{\substack{(\varepsilon,\eta)\to(0,0)\\ \mathcal{D}'(\mathbb{R})}} (y \mapsto G_\eta(y\varepsilon^{-1})) = Y.$$

From the above results, we obtain

$$[u_{(\varepsilon,\eta)}] = [\varepsilon u_{(\varepsilon,\eta),1}] + [\varepsilon u_{(\varepsilon,\eta),2}] + [u_{(\varepsilon,\eta),3}]$$

with

$$[u_{(\varepsilon,\eta),1}] \sim 1_x \otimes Y_y, \qquad [u_{(\varepsilon,\eta),2}] \sim -Y_x \otimes 1_y, \qquad [u_{(\varepsilon,\eta),3}] \sim \delta_x \otimes 1_y.$$

Example 58. ($\varphi \sim 0$, $\psi \sim \delta$; $F = 0$) so $\varphi \sim 0$, $\varPsi \sim Y \sim T$; $F = 0$. From the above results, we obtain

$$u_{(\varepsilon,\eta)}(x,y) = \varepsilon (g_\eta * T)(y\varepsilon^{-1}) - \varepsilon (g_\eta * T)(x).$$

Hence

$$[u_{(\varepsilon,\eta)}] = [\varepsilon u_{(\varepsilon,\eta),1}] + [\varepsilon u_{(\varepsilon,\eta),2}]$$

with

$$[u_{(\varepsilon,\eta),1}] \sim 1_x \otimes Y_y, \qquad [u_{(\varepsilon,\eta),2}] \sim -Y_x \otimes 1_y.$$

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