Summary

The plan of the paper is as follows. In Chapter 0 we set up notation and terminology.

In Chapter 1 we investigate the properties of consistence, strongness and semimodularity, each of which may be viewed as generalization of modularity. Section 1.1 deals with some conditions characterizing consistence in lower continuous strongly coatomic lattices. Here we prove that a finite lattice is the lattice of closed sets of a closure space with the Steinitz exchange property if and only if it is a consistent lattice. Section 1.2 extends Faigle's concept of strongness from lattices of finite length to arbitrary lattices. Any atomistic lattice is strong whereas the converse does not hold in general. It is shown (Theorem 1.15) that a lower continuous strongly atomic lattice in which each atom has a complement is strong precisely when it is atomistic. For the class of strongly coatomic lower continuous lattices we prove that in semimodular lattices, the concepts of strongness and of consistence are equivalent (Theorem 1.26). Section 1.3 combines semimodularity with the strongness property. In Section 1.4 we characterize atomistic lattices. These characterizations are given in terms of concepts related to pure elements and neat elements.

Chapter 2 considers join decompositions in lattices. In Section 2.1, some sufficient conditions are given under which every element of a lattice has a join decomposition (Proposition 2.2). The goal of Section 2.2 is to characterize modularity of lattices in terms of the Kurosh–Ore Replacement Properties (Theorem 2.13). Finally, in Section 2.3, we study lattices with unique irredundant join decompositions.

In Chapter 3 Problem IV.15 of Grätzer [1978] is solved. In Section 3.1 we introduce the notion of a c-join in lattices, where c is a distributive element of the lattice. Sections 3.2 and 3.3 present some properties of c-joins and c-decomposition functions. An important role in our investigations is played by the B_c-condition defined in Section 3.4. Section 3.5 is devoted to the study of finite c-decompositions of elements in a modular lattice. We give here a generalization of some results of papers of Močulskiĭ [1955, 1961] and Walendziak [1991b]. We find (Theorem 3.25) a common generalization of the Kurosh–Ore Theorem and the Schmidt–Ore Theorem for arbitrary modular lattices, solving a problem of G. Grätzer. In Sections 3.6 and 3.7 we consider infinite c-decompositions. For investigations of such representations, property (B_c) does not yield anything. Therefore, we shall use property (B_c^*) defined in Section 3.6. As a main result of Chapter 3 we give the c-Decomposition Theorem (3.40) which implies (in particular) Crawley's Theorem for direct decompositions (Corollary 3.44) and a generalization of the Kurosh–Ore Theorem to infinite join decompositions (Corollary 3.46).

The decomposition theory of Chapters 2 and 3 enables us to devolop a structure theory for algebras. In Chapter 4 we consider weak direct representations of a universal

algebra. The existence of such representations is studied (Theorems 4.6 and 4.17). Here some applications to algebras whose congruences permute (groups, rings, modules, quasigroups, relatively complemented lattices, etc.) and to congruence distributive algebras (lattices, modular median algebras and trellises) are indicated.

Chapter 5 contains a common generalization of full subdirect products and of weak direct products. This is an $\langle \mathcal{L}, \varphi \rangle$-representation of a subalgebra A of a direct product B of algebras, where $\mathcal{L}$ is an ideal in the power set of the index set of the direct product and φ is a binary relation on B, and A is a subdirect product satisfying certain conditions involving $\mathcal{L}$ and φ. In Section 5.3 we prepare for further investigations, first by introducing the notions of φ-product and φ-isotopy for congruences, and then by proving a few lemmas about these notions. In Section 5.4, $\langle \mathcal{L}, \varphi \rangle$-representations of algebras are associated with systems of their congruence relations (Theorem 5.25). Section 5.5 gives sufficient conditions for an algebra to be isomorphic to an $\langle \mathcal{L}, \varphi \rangle$-product with simple factors (Theorem 5.34) and with directly indecomposable factors (Theorem 5.41). The Third Existence Theorem 5.45 concerns restricted full subdirect products. These results imply some theorems on subdirect, full subdirect and weak direct representations.

Finally Section 5.6 contains uniqueness theorems. The first uniqueness result (Theorem 5.48) concerns restricted full subdirect representations of algebras with distributive congruence lattices. Here we generalize the results of Draškovičová [1987] for a congruence distributive algebra A with the property that the set of all decomposition congruences of A is closed under arbitrary joins. Another uniqueness result in this section is the Unique Factorization Theorem 5.54. In particular, Theorem 5.54 implies Theorem 3 of Walendziak [1993c] and has as corollaries uniqueness results for irredundant restricted subdirect representations (Proposition 5.56) and for restricted full subdirect representations (Proposition 5.57). We note that Proposition 5.56 yields that any two irredundant subdirect representations of a congruence distributive algebra with subdirectly irreducible factors are isomorphic (Corollary 5.59). Our application of Proposition 5.57 to weak direct products is Corollary 5.64. In particular we obtain Birkhoff's Theorem which asserts that every congruence permutable algebra with congruence lattice of finite length and a one-element subalgebra is uniquely factorable. In Chapter 5 we also give other applications to algebras whose congruences permute and to congruence distributive algebras.

0. Basic notions

Let L be a lattice. Lattice join, meet, inclusion and proper inclusion are denoted respectively by $\vee$, $\wedge$, $\leq$ and $<$. If L contains a least or a greatest element, these elements will be denoted by 0 or 1, respectively. The dual of L is the lattice L^∂ with the same underlying set, but with $a \leq b$ in L^∂ if and only if $b \leq a$ in L. We say that a and b in L are *comparable* if either $a \leq b$ or $b \leq a$, otherwise a and b are *incomparable*. By $[a, b]$ ($a \leq b$; $a, b \in L$) we denote an *interval*, that is, the set of all $c \in L$ for which $a \leq c \leq b$. Two intervals of the form $[a \wedge b, b]$ and $[a, a \vee b]$ are said to be *transposed*. We say that b *covers* a if $a < b$ and $[a, b] = \{a, b\}$; in this case we write $a \prec b$ or $b \succ a$ and also say that b is an *upper cover* of a (or: a is a *lower cover* of b). Let us write $a \preceq b$ if $a \prec b$ or $a = b$.

An element $p \in L$ is called an *atom* (resp. a *coatom*) if $0 \prec p$ (resp. $p \prec 1$). We denote by $\mathrm{A}(L)$ the set of all atoms of L. The lattice L is called *atomic* if L has a least element and the interval $[0, a]$ contains an atom for each $a > 0$; and *weakly atomic* if for any $a, b \in L$ with $a > b$, there exist $u, v \in L$ such that $b \leq v \prec u \leq a$. If a lattice L (perhaps with no least element) has the property that the interval $[a, b]$ contains an atom whenever $b > a$ in L, we say that L is *strongly atomic*. Each strongly atomic lattice is weakly atomic, and each strongly atomic lattice having a least element is atomic.

A lattice is *coatomic* (resp. *strongly coatomic*) if its dual is atomic (resp. strongly atomic). A lattice is called *atomistic* if every nonzero element is a join of atoms. L is *coatomistic* if L^∂ is atomistic.

An element u of a lattice L is *join irreducible* if $u = a \vee b$ implies $u = a$ or $u = b$. An element $m \in L$ is *meet irreducible* if $m = a \wedge b$ implies $m = a$ or $m = b$. By $\mathrm{V}(L)$ (resp. $\Lambda(L)$) we denote the set of all join irreducible (resp. meet irreducible) elements of L. In a strongly coatomic lattice the unique lower cover of a nonzero join irreducible element u is denoted by u^*.

Let L be a complete lattice. An element $u \in L$ is called *completely join irreducible* if for all $T \subseteq L$, $u = \bigvee T$ implies $u \in L$. *Completely meet irreducible* elements are defined dually. Let $\mathrm{J}(L)$ (resp. $\mathrm{M}(L)$) be the set of all completely join irreducible (resp. completely meet irreducible) elements of L. Clearly every completely join irreducible element is join irreducible. For complete strongly coatomic lattices the two concepts coincide.

Let T be a subset of a lattice L. We say T has the *weak isomorphism property* if for each $t \in T$ and each $a \in L$,

$$[a, a \vee t] \cong [a \wedge t, a]$$

(that is, the intervals $[a, a \vee t]$ and $[a \wedge t, a]$ are isomorphic). A lattice L is called *modular* if for all $a, b, c \in L$, $c \leq b$ implies $(c \vee a) \wedge b = c \vee (a \wedge b)$. We know (see e.g. Grätzer [1978], p. 162) that transposed intervals of a modular lattice are isomorphic. This yields, in particular, that every subset of a modular lattice satisfies the so-called *neighborhood condition*

$$\text{(N)} \qquad a \wedge b \prec b \;\Rightarrow\; b \prec a \vee b,$$

and the *dual neighborhood condition*

$$\text{(N}^*\text{)} \qquad b \prec a \vee b \;\Rightarrow\; a \wedge b \prec a.$$

A lattice is called *upper semimodular* (briefly: *semimodular*) if it satisfies the neighborhood condition (N); it is called *lower semimodular* if it satisfies the dual neighborhood condition (N*). A lattice L is said to satisfy the *upper covering condition* iff $a \preceq b$ implies $a \vee c \preceq b \vee c$ for all $a, b, c \in L$. The *lower covering condition* is the dual. It is well known that a lattice is semimodular (resp. lower semimodular) iff it satisfies the upper covering condition (resp. lower covering condition).

A lattice L with least element 0 and greatest element 1 is said to be *complemented* if for each $a \in L$ there exists a $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$. The element b is said to be a *complement* of a. If every interval of a lattice L is a complemented lattice, then L is *relatively complemented*.

Let $E \subseteq L$. If each element of L is a join of elements of E, then we call L an E-*lattice*. Then L is an A-*lattice* if L is atomistic. L is a V-*lattice* (resp. J-*lattice*) if for every $a \in L$ there is a subset T of $\mathrm{V}(L)$ (resp. $\mathrm{J}(L)$) such that $a = \bigvee T$. An AC-*lattice* is an A-lattice with the covering property:

$$\text{(C)} \qquad (b \in L,\ p \in \mathrm{A}(L) \text{ and } b \wedge p = 0) \ \Rightarrow\ b \prec b \vee p.$$

A lattice L with the property that each of its nonempty subsets contains a maximal element is said to satisfy the *ascending chain condition* (ACC). If each nonempty subset of L has a minimal element, then L satisfies the *descending chain condition* (DCC). Obviously, if a lattice satisfies DCC, then it is strongly atomic. The ACC also has an important generalization. An element c of a complete lattice L is called *compact* if $S \subseteq L$ and $c \le \bigvee S$ imply $c \le \bigvee S'$ for some finite subset S' of S. Let $\mathrm{K}(L)$ be the set of all compact elements of L. L is an *algebraic* or a *compactly generated* lattice if L is complete and each of its elements is a join of compact elements. It is easy to see that if L satisfies ACC, then L is algebraic (every element of a complete lattice L is compact iff L satisfies ACC; see Crawley–Dilworth [1973], p. 14).

Let L be a complete lattice. An element $q \in L$ is called *precompact* if $S \subseteq L$ and $q = \bigvee S$ imply $q = \bigvee S'$ for some finite subset S' of S. Let $\mathrm{Q}(L)$ denote the set of all precompact elements of L. If L is a Q-lattice, then L is said to be *prealgebraic*. It is obvious that $\mathrm{K}(L) \subseteq \mathrm{Q}(L)$, and each algebraic lattice is prealgebraic.

Compact generation has a useful generalization. Define a lattice L to be *upper continuous* if L is complete and, for every $a \in L$ and every chain C in L,

$$a \wedge \bigvee C = \bigvee \{a \wedge c : c \in C\}.$$

The lattice L is *lower continuous* if its dual lattice is upper continuous, and it is *continuous* if it is both upper and lower continuous. It can be shown that every algebraic lattice is upper continuous and weakly atomic. Crawley–Dilworth [1973] (see Theorem 2.4) show that if a is an element of an upper continuous lattice L, $S \subseteq L$, and if $\mathrm{F}(S)$ is the set of all finite subsets of S, then

$$\text{(UC)} \qquad a \wedge \bigvee S = \bigvee \{a \wedge \bigvee S' : S' \in \mathrm{F}(S)\}.$$

Every lower continuous lattice L has the dual property to (UC), namely:

$$\text{(LC)} \qquad a \vee \bigwedge S = \bigwedge \{a \vee \bigwedge S' : S' \in \mathrm{F}(S)\},$$

for all $a \in L$ and $S \subseteq L$. A complete lattice L is called *Brouwerian* if for each $a \in L$ and each $T \subseteq L$,

$$a \wedge \bigvee T = \bigvee \{a \wedge t : t \in T\}.$$

L is *dually Brouwerian* if L^{∂} is Brouwerian. Clearly, every dually Brouwerian lattice is lower continuous.

In a complete strongly coatomic lattice L we put

$$a_+ = \bigwedge \{b \in L : b \prec a\},$$

for $a \in L$, $a \neq 0$, that is, a_+ is the meet of all lower covers of a. We say that a complete strongly coatomic lattice L is *lower locally modular* (resp. *lower locally distributive*) if for each $a \in L$, $a \neq 0$, the interval $[a_+, a]$ is a modular (resp. distributive) sublattice.

In a dual way, we define (*upper*) *locally modular* lattices and (*upper*) *locally distributive* lattices. We note that the concepts of local distributivity and local modularity go back to Dilworth [1940, 1941].

We denote by $\mathbf{K}$ the class of all lower continuous strongly coatomic lattices.

1. Consistent, strong, and atomistic lattices

1.1. Consistent lattices. Kung [1985] introduced the notion of a consistent lattice. A lattice L is said to be *consistent* iff $a \in L$ and $u \in \mathrm{V}(L)$ imply that $a \vee u \in \mathrm{V}([a, a \vee u])$. Geometric lattices (i.e., complete, semimodular, atomistic lattices in which all atoms are compact) and modular lattices are consistent (see Kung [1985]). The pentagon lattice is an example of a consistent lattice which is neither modular nor geometric.

We will need the following

LEMMA 1.1. *If a, b are elements of a lattice $L \in \mathbf{K}$ and $b \nleq a$, then there exists $u \in \mathrm{V}(L)$ such that $u \leq b$ and $u \nleq a$.*

Proof. Since L is strongly coatomic and $a \wedge b < b$, there exists a $p \in L$ such that $a \wedge b \leq p \prec b$. Let

$$T = \{x \in L : x \leq b \text{ and } x \nleq p\}.$$

Then T is nonempty, since $b \in T$. Let C be a chain in T. The lower continuity yields

$$p \vee \bigwedge C = \bigwedge \{p \vee c : c \in C\} = b.$$

Thus $\bigwedge C \in T$, and T contains a minimal element u by the dual of Zorn's Lemma. Clearly, $u \in \mathrm{V}(L)$, $u \leq b$ and $u \nleq a$. ∎

PROPOSITION 1.2 (see Walendziak [1994d], Theorem 1). *A lattice $L \in \mathbf{K}$ is consistent iff L satisfies the dual of property* $(*)$ (see Crawley–Dilworth [1973], p. 53), *namely*:

(+) *For all $a, b \in L$, if the interval $[a \wedge b, b]$ has exactly one coatom, then the interval $[a, a \vee b]$ has exactly one coatom.*

Proof. Suppose that the lattice L is consistent. Let $a, b \in L$ and let p be a unique element such that $a \wedge b \leq p \prec b$. By Lemma 1.1 there is a join irreducible element u such that $u \vee p = b$. We set $t = a \vee u$. It is obvious that $t \wedge b \in [a \wedge b, b]$ and $u \leq t \wedge b \nleq p$. Since L is strongly coatomic, and p is a single coatom in $[a \wedge b, b]$ we conclude that $t \wedge b = b$. Then $b \leq t$, and therefore $t = a \vee b$. Consistence implies that $a \vee u \in \mathrm{V}([a, 1])$, i.e., $a \vee b$ $(= a \vee u)$ is a join irreducible element of the sublattice $[a, 1]$. Hence the interval $[a, a \vee b]$ has exactly one coatom.

Conversely, assume that L satisfies (+). Let $u \in \mathrm{V}(L)$ and $a \in L$. If a and u are comparable, then obviously $a \vee u \in \mathrm{V}([a, 1])$. Suppose that a, u are incomparable. Since $u \in \mathrm{V}(L)$, the sublattice $[a \wedge u, u]$ has exactly one coatom. By (+), $[a, a \vee u]$ has exactly one coatom. Hence $a \vee u \in \mathrm{V}([a, 1])$, and therefore L is consistent. ∎

By the dual of Lemma 3 of Walendziak [1990b] we have

LEMMA 1.3. *Let L be a lower locally modular lattice of* $\mathbf{K}$. *If* $b, p, q \in L$, *and if* $p, q \prec b \vee (p \wedge q)$ *and* $p \wedge b = q \wedge b$, *then* $p = q$.

Now we prove the following

PROPOSITION 1.4. *Every lower locally modular lattice belonging to* $\mathbf{K}$ *is lower semimodular and consistent.*

Proof. Let L be a lattice satisfying the assumptions of the proposition. From the dual of Theorem 3.7 of Crawley–Dilworth [1973] it follows that L is lower semimodular (in the proof of that theorem, just the upper continuity of L was used). We verify that L is also consistent. It is sufficient to show that L satisfies (+). Suppose on the contrary that there exist $a, b \in L$ such that the interval $[a \wedge b, b]$ has exactly one coatom and $[a, a \vee b]$ contains two distinct coatoms p and q. Obviously we have

$$p, q \prec a \vee b = b \vee (p \wedge q). \tag{1}$$

By lower semimodularity, $p \wedge b \preceq b$ and $q \wedge b \preceq b$. Since $[a \wedge b, b]$ has exactly one coatom,

$$p \wedge b = q \wedge b. \tag{2}$$

From (1) and (2) we conclude by Lemma 1.3 that $p = q$. This contradiction shows that L satisfies (+). Then from Proposition 1.2 it follows that L is consistent. ■

A finite lattice L is called *meet-distributive* if $[a_+, a]$ is a boolean interval of L for all $a \in L$.

As an immediate consequence of Proposition 1.4 we obtain

COROLLARY 1.5 (cf. Reuter [1989], Lemma 1). *A meet-distributive lattice is consistent.*

REMARK 1.6. Since every lattice of finite length is lower continuous and strongly coatomic, from Proposition 1.4 we have Proposition 20.2 of Stern [1991b].

REMARK 1.7. The converse of Proposition 1.4 is not true. The lattice of Figure 1 is a consistent lattice which is not lower locally modular.

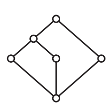

Fig. 1

Combining the dual of the Theorem in Walendziak [1990b] (p. 554) and Proposition 1.2 we obtain

COROLLARY 1.8. *If L is a lower semimodular*, *lower continuous lattice satisfying the ascending chain condition*, *then the following statements are equivalent*:

(i) *L is lower locally modular.*
(ii) *L is consistent.*

We now describe the relationship between consistent lattices and a class of Steinitz spaces.

Let S be a finite set. By $\mathrm{P}(S)$ we denote the set all subsets of S. A function $\mathrm{Cl} : \mathrm{P}(S) \to \mathrm{P}(S)$ is called a *closure operator* on S if it has the properties (for all $A, B \subseteq S$):

1) $\mathrm{Cl}A) = \mathrm{Cl}^2(A)$.
2) $A \subseteq \mathrm{Cl}(A)$.
3) If $A \subseteq B$, then $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(B)$.

A *closure space* $\mathbb{S}$ is a pair (S, Cl) where S is a finite set and Cl is a closure operator on S. $\mathrm{Cl}(A)$ is more commonly denoted by $\overline{A}$. A subset $A \subseteq S$ is *closed*, or a *flat*, if $A = \overline{A}$. The lattice of closed sets of $\mathbb{S}$ is given by

$$\mathrm{L}(\mathbb{S}) = \{A \subseteq S : A = \overline{A}\}.$$

The analogy with vector spaces leads to the following notions. An element $p \in S$ is a *point* of the closure space $\mathbb{S}$ if p is a nonzero join irreducible element of $\mathrm{L}(\mathbb{S})$. We call the set $B \subseteq A$ of points a *basis* for A if

$$\overline{B} = \overline{A} \quad \text{and} \quad \overline{B - b} \neq \overline{A} \quad \text{for all } b \in B.$$

We say that a closure space $\mathbb{S}$ has the *Steinitz exchange property* if for all $A \subseteq S$ and bases B_1, B_2 for A, if $b_1 \in B_1$, then there is $b_2 \in B_2$ such that $(B_1 - b_1) \cup \{b_2\}$ is a basis for A. A closure space $\mathbb{S}$ with the Steinitz exchange property will be called a *Steinitz space*. $\mathbb{S}$ is called a *matroid* if it has the Steinitz–MacLane exchange property:

$$p \in \overline{A \cup q} - \overline{A} \;\Rightarrow\; q \in \overline{A \cup p} \quad (A \subseteq S,\ p, q \in S).$$

Finally, we say that a closure space $\mathbb{S}$ is a *convex geometry* if $\mathbb{S}$ has the so-called anti-exchange property ($A \subseteq S$, $p, q \in S$):

$$(p, q \in \overline{A \cup q} - \overline{A} \text{ and } p \neq q) \;\Rightarrow\; q \notin \overline{A \cup p}.$$

It is well known that a finite lattice L is isomorphic to the lattice of flats of a matroid iff L is geometric. A result due to Edelman [1980] (see Theorem 3.3) states that a lattice is meet-distributive iff it is isomorphic to the lattice of all closed sets of a convex geometry.

We first prove the following

PROPOSITION 1.9. *Let $\mathbb{S}$ be a closure space. Then $\mathbb{S}$ is a Steinitz space iff $\mathrm{L}(\mathbb{S})$ is a consistent lattice.*

Proof. Let $\mathbb{S}$ be a Steinitz space and set $L = \mathrm{L}(\mathbb{S})$. To prove that L is consistent, let $A \in L$ and $U \in \mathrm{V(L)}$. Suppose that $A \vee U = B \vee C$, where $B, C \geq A$. Let $\{a_1, \ldots, a_m\}$, $\{b_1, \ldots, b_k\}$ and $\{c_1, \ldots, c_n\}$ be bases for A, B and C, respectively. Since U is join irreducible in L, we conclude that $U = \overline{p}$ for some point p. Without loss of generality we can assume that $\{a_1, \ldots, a_{m'}, p\}$ and $\{b_1, \ldots, b_{k'}, c_1, \ldots, c_{n'}\}$ ($m' \leq m$, $k' \leq k$, $n' \leq n$) are bases for $A \vee U$. Since $\mathbb{S}$ has the Steinitz exchange property, for p there is $b \in \{b_1, \ldots, b_{k'}, c_1, \ldots, c_{n'}\}$ such that $\{a_1, \ldots, a_{m'}, b\}$ is a basis for $A \vee U$. Assume that $b = b_1$. Then

$$B \leq A \vee U = \overline{a}_1 \vee \ldots \vee \overline{a}_{m'} \vee \overline{b}_1 \leq A \vee B = B,$$

i.e., $A \vee U = B$. Thus $A \vee U \in \mathrm{V}([A, A \vee U])$.

Suppose $L = \mathrm{L}(\mathbb{S})$ is a consistent lattice. We claim that $\mathbb{S}$ has the Steinitz exchange property. Indeed, let $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_n\}$ be bases for $A \subseteq S$. Then

$$\overline{A} = \overline{a}_1 \vee \ldots \vee \overline{a}_m = \overline{b}_1 \vee \ldots \vee \overline{b}_n.$$

Let $i \in \{1, \ldots, m\}$. Set $B = \overline{a}_1 \vee \ldots \vee \overline{a}_{i-1} \vee \overline{a}_{i+1} \vee \ldots \vee \overline{a}_m$. By consistence,

$$\overline{A} = (B \vee \overline{b}_1) \vee \ldots \vee (B \vee \overline{b}_n) = B \vee \overline{a}_i \in \mathrm{V}([B, \overline{A}\,]).$$

Consequently, $\overline{A} = B \vee \overline{b}_j$ for some $j \in \{1, \ldots, n\}$. Hence $\{a_1, \ldots, a_{i-1}, b_j, a_{i+1}, \ldots, a_m\}$ is a basis for A, and therefore, $\mathbb{S}$ is a Steinitz space. ■

Remark 1.10. Since every geometric lattice is consistent, any matroid is a Steinitz space. Similarly, every meet-distributive lattice is consistent, and hence each convex geometry is a Steinitz space.

Example 1.11. Let Q_8 be a quaternion group (of order 8). We denote by $\mathrm{Sg}(A)$ the subgroup of Q_8 generated by $A \subseteq Q_8$. It is obvious that (Q_8, Sg) is a closure space. Let $L = \mathrm{Sub}(Q_8)$ be the lattice of all subgroups of Q_8. Then L is the lattice of flats of the space (Q_8, Sg) and it is shown in Figure 2. It is a modular lattice but it is neither geometric nor meet-distributive. Hence (Q_8, Sg) is a Steinitz space but it is neither a matroid nor a convex geometry.

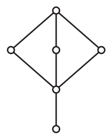

Fig. 2

Now we prove

Theorem 1.12 (Walendziak [1997], Theorem). *Let L be a finite lattice. Then L is isomorphic to the lattice of all closed sets of a Steinitz space iff L is consistent.*

Proof. If $\mathbb{S}$ is a Steinitz space and $L = \mathrm{L}(\mathbb{S})$, then L is a consistent lattice, by Proposition 1.9.

Conversely, let L be a finite consistent lattice. Define $S = \mathrm{V}(L)$ and

$$\overline{A} = \{p \in S : p \leq \bigvee A\}$$

for $A \subseteq S$. It is easy to see that $^-$ is a closure operator on S. Then $\mathbb{S} = (S, ^-)$ is a closure space. Observe that $A = \{p \in S : p \leq a\}$ $(a \in L)$ is a closed set of $\mathbb{S}$. Indeed, every element of L is a join of join irreducible elements, thus $a = \bigvee A$, showing that $A = \overline{A}$. Set $L' = \mathrm{L}(\mathbb{S})$, and define the map

$$f : a \in L \mapsto \{p \in S : p \leq a\} \in L'.$$

Let $A \in L'$. Then $\{p \in S : p \leq \bigvee A\} = A$, and therefore, $f(\bigvee A) = A$. Thus f is onto L'. Since $a = \bigvee \{p \in S : p \leq a\} = \bigvee f(a)$ for each $a \in L$, f is one-to-one. Obviously,

$$\{p \in S : p \leq a \wedge b\} = \{p \in S : p \leq a\} \cap \{p \in S : p \leq b\},$$

and so $f(a \wedge b) = f(a) \wedge f(b)$. The formula $f(a \vee b) = f(a) \cup f(b)$ is equivalent to

$$\{p \in S : p \leq a \vee b\} = \overline{B}, \tag{3}$$

where $B = \{p \in S : p \leq a \text{ or } p \leq b\}$. By the definition of $^-$,

$$\overline{B} = \{p \in S : p \leq \bigvee B\}.$$

Since every element of L is a join of join irreducible elements, we have $\bigvee B = a \vee b$. Therefore, (3) holds, and hence $f(a \vee b) = f(a) \cup f(b)$. Thus f is an isomorphism between L and L'. Therefore the lattice L' is consistent. By Proposition 1.9, $\mathbb{S}$ is a Steinitz space. Consequently, L is isomorphic to the lattice $\mathrm{L}(\mathbb{S})$ for some Steinitz space $\mathbb{S}$. ■

1.2. Strong lattices. Now we introduce the concept of a strong lattice. For lattices of finite length the definition of strongness is given by Stern [1989] by the property

(St) $\qquad (u \in \mathrm{V}(L) - \{0\},\ a \in L \text{ and } u \leq a \vee u^*) \ \Rightarrow\ u \leq a.$

We extend the notion of strongness from lattices of finite length to arbitrary lattices. Namely, we introduce the following

Definition 1.13. We say that a lattice L is *strong* if the following condition is satisfied:

(S) $\qquad (u \in \mathrm{V}(L) - \{0\},\ a, b \in L \text{ and } b < u \leq a \vee b) \ \Rightarrow\ u \leq a.$

This concept is also dealt with in Walendziak [1994b]. It is easy to see that in strongly coatomic lattices (in particular: in lattices of finite length) properties (St) and (S) are equivalent. We remark that any atomistic lattice (in particular: each geometric lattice) is strong. (Indeed, each join irreducible element of an atomistic lattice is an atom.)

Now we observe that any modular lattice is strong. Let L be a modular lattice and let $u \in \mathrm{V}(L)$, $a, b \in L$ with $b < u \leq a \vee b$. By the modular law, $u = (a \wedge u) \vee b$. Since u is join irreducible this implies $u = a \wedge u$, that is, $u \leq a$, which means that L is strong.

Also, it is not difficult to give examples of lattices which are strong but neither modular nor atomistic (see Section 23 of Stern [1991b]).

Theorem 1.14. *Let L be an atomic V-lattice. If each atom of L has a complement, then L is strong iff L is atomistic.*

Proof. Assume that L is strong. Let a be a nonzero element of L. Since L is a V-lattice,

$$a = \bigvee\{u : u \in U \subseteq \mathrm{V}(L)\}.$$

Suppose that a join irreducible element $u \in U$ is not an atom. Since L is atomic, there exists an atom $p \in L$ such that $p < u$. Let p' be a complement of p. This means that $1 = p \vee p'$ and $0 = p \wedge p'$. Then $p < u \leq p \vee p'$ and strongness implies $u \leq p'$. Thus $p < p'$, which contradicts $p \wedge p' = 0$. It follows that L is atomistic. The converse is clear. ■

Now we prove the following

Theorem 1.15 (Walendziak [1994e], Theorem 2). *A lower continuous strongly atomic lattice in which each atom has a complement is atomistic iff it is strong.*

Proof. Observe that if a lattice L is lower continuous and strongly atomic, then L is a V-lattice. Indeed, let $a \in L$ and $b = \bigvee\{u \in \mathrm{V}(L) : u \leq a\}$. Assume that $b < a$. Since L

is strongly atomic there exists an element $p \in L$ such that $b \prec p \leq a$. Consider the set $T = \{t \in L : b \vee t = p\}$. It is nonempty, since $p \in T$. Let C be a chain in T. The lower continuity yields

$$b \vee \bigwedge C = \bigwedge \{b \vee c : c \in C\} = p.$$

Thus $\bigwedge C \in T$ and by the dual of Zorn's Lemma T contains a minimal element v. Clearly, $v \in \mathrm{V}(L)$ and $v \leq a$. Consequently, $v \leq b$, and hence $p = b \vee v = b$, a contradiction. Thus $a = \bigvee \{u \in \mathrm{V}(L) : u \leq a\}$, which shows that L is a V-lattice. Now the assertion follows from Theorem 1.14. ■

We recall that a lattice L satisfies the *descending chain condition* (DCC) if each nonempty subset of L contains a minimal element. It is obvious that any lattice satisfying the DCC is lower continuous and strongly atomic. Therefore we obtain the following

COROLLARY 1.16. *Suppose that a lattice L satisfies the DCC and each atom of L has a complement. Then L is atomistic iff it is strong.*

REMARK 1.17. Since every lattice of finite length satisfies the DCC, this corollary implies the theorem of Stern [1989].

We know (see Crawley–Dilworth [1973], Theorem 4.1) that every upper continuous, semimodular, atomistic lattice is relatively complemented. This together with Corollary 1.16 yields

COROLLARY 1.18. *Let L be a semimodular, upper continuous, strong lattice with DCC. If each atom of L has a complement, then L is relatively complemented.*

REMARK 1.19. Corollary 1.18 generalizes the corollary of Stern [1989].

PROPOSITION 1.20. *A lower semimodular strongly coatomic lattice is strong.*

Proof. Let L be a lower semimodular strongly coatomic lattice and assume that L is not strong. Then there exists a join irreducible element $u \in \mathrm{V}(L)$ such that

$$u \leq a \vee u^* \quad \text{but} \quad u \nleq a$$

for some $a \in L$. Clearly $a < a \vee u^*$, and since L is strongly coatomic, there is $b \in L$ such that $a \leq b \prec a \vee u^*$. It is easy to see that $u \nleq b$, and hence $b < b \vee u \leq a \vee u^*$. Consequently, $b \vee u = a \vee u^*$, and therefore $b \prec b \vee u$. By lower semimodularity it follows that $b \wedge u \prec u$, that is, $b \wedge u = u^*$. This means $u^* \leq b$ and thus

$$b \leq b \vee u = a \vee u^* \leq b,$$

contradicting $u \nleq b$. ■

REMARK 1.21. The preceding proposition generalizes Lemma 2 of Stern [1991a], since any lattice of finite length is strongly coatomic.

EXAMPLE 1.22. Let L be the lattice diagrammed in Figure 3.

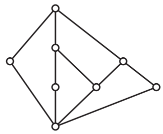

Fig. 3

Then L is strong but not lower semimodular. This example implies that the converse of Proposition 1.20 is not true.

PROPOSITION 1.23 (Walendziak [1999], Proposition 1). *A lattice L is strong iff it does not contain a pentagon isomorphic to the lattice in Figure 4 (where $u \in \mathrm{V}(L)$).*

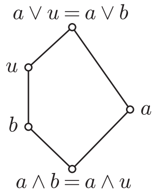

Fig. 4

Proof. Assume that L is not strong. Then there are $a, c \in L$, $u \in \mathrm{V}(L)$ such that $c < u \leq a \vee c$ and $u \nleq a$. Let $b = c \vee (a \wedge u)$. Since u is join irreducible, $b < u$. We have

$$a \wedge b \leq a \wedge u \leq a \wedge [c \vee (a \wedge u)] = a \wedge b,$$

and hence $a \wedge b = a \wedge u$. Now we observe that $a \wedge b < b$. Namely, $a \wedge b = b$ yields $b \leq a$ and thus $u \leq a \vee b = a$ contradicting our assumption $u \nleq a$. It is easy to see that $a \wedge b < a$ and $a < a \vee b \leq a \vee u$. On the other hand, $a \vee u \leq a \vee b$. Therefore, $a \vee b = a \vee u$, and thus L contains a pentagon isomorphic to the lattice of Figure 4. The converse is trivial. ∎

As a preparation for the next result we need the following

LEMMA 1.24. *Let L be a strong lattice and $c, d \in L$ with $c \prec d$. If $u \in \mathrm{V}(L)$ and $b \in L$ are such that $b < u \leq d$ but $u \nleq c$, then $b \leq c$.*

Proof. Suppose that $b \nleq c$. We have $c \leq b \vee c \leq d$ and $c \prec d$. Then $u \leq d = b \vee c$ and strongness implies $u \leq c$, a contradiction. ∎

PROPOSITION 1.25. *Let L be a strongly coatomic lattice. If L is semimodular and strong, then L is consistent.*

Proof. Let L satisfy the above assumptions, and suppose that L is not consistent. This means that there exist $a \in L$ and $u \in \mathrm{V}(L)$ with $a \vee u \notin \mathrm{V}([a, 1])$. Thus there are two distinct elements $c_1, c_2 \in [a, a \vee u]$ which are covered by $a \vee u$. Since $u \nleq c_1, c_2$, by Lemma 1.24 we get $u^* \leq c_i$ for $i = 1, 2$. Thus $u^* \leq u \wedge (c_1 \wedge c_2) \leq u$ and obviously $u \nleq c_1 \wedge c_2$. Hence $u \wedge (c_1 \wedge c_2) = u^* \prec u$. By semimodularity we conclude that

$$c_1 \wedge c_2 \prec (c_1 \wedge c_2) \vee u = a \vee u.$$

This is a contradiction since $c_1 \wedge c_2 < c_1 \prec a \vee u$ by construction. ∎

The main result of the present section is

THEOREM 1.26 (Walendziak [1994b], Theorem 1). *A semimodular lattice $L \in \mathbf{K}$ is consistent iff it is strong.*

Proof. Let L be a semimodular lower continuous strongly coatomic lattice. Assume first that L is consistent but not strong. Let a join irreducible element $u \in \mathrm{V}(L)$ be such that $u \leq a \vee u^*$ and $u \nleq a$ for some $a \in L$. Thus the set

$$T = \{x \in L : u \leq x \vee u^* \text{ and } u \nleq x\}$$

is not empty. Let C be a chain in T. Lower continuity implies

$$u^* \vee \bigwedge C = \bigwedge\{c \vee u^* : c \in C\} \geq u.$$

Clearly, $u \nleq \bigwedge C$. Therefore $\bigwedge C \in T$, and T contains a minimal element b, by the dual of Zorn's Lemma. Since L is strongly coatomic we may choose $p \in L$ with $p \prec b$. Observe that

$$p \vee u^* < p \vee u. \tag{4}$$

Indeed, if $p \vee u^* = p \vee u$, then $p \in T$, contradicting the minimality of b. Now we observe that $b \leq p \vee u^*$ is not possible, since $b \leq p \vee u^*$ would imply $b \vee u^* \leq p \vee u^* < p \vee u \leq b \vee u^*$, a contradiction. Since $p \prec b$ and $b \nleq p \vee u^*$ we get

$$b \wedge (p \vee u^*) = p \prec b.$$

Hence, by semimodularity we conclude that

$$p \vee u^* \prec b \vee p \vee u^* = b \vee u^* = b \vee u.$$

Thus we have

$$p \vee u^* < p \vee u \leq b \vee u \quad \text{and} \quad p \vee u^* < b \vee u.$$

Consequently,

$$p \vee u = b \vee u, \tag{5}$$

and therefore $p \vee u = (p \vee u^*) \vee b$. Consistence implies that $p \vee u$ is a join irreducible element of the sublattice $[p, 1]$. This together with (4) and (5) yields $b \vee u = p \vee u = b$, which contradicts the fact that $u \nleq b$. It follows that L must be strong.

The converse follows from Proposition 1.25. ∎

REMARK 1.27. The preceding theorem generalizes Theorem 27.1 of Stern [1991b] (see also Faigle [1980], p. 33, and Reuter [1989], p. 125). Example 1.22 shows that the assumption of semimodularity cannot be dropped in Theorem 1.26. Indeed, the lattice of Figure 3 is an example of a nonsemimodular lattice which is strong but not consistent.

1.3. Strongly semimodular lattices

DEFINITION 1.28 (Faigle [1980]). A lattice is called *strongly semimodular* if it is both strong and semimodular.

All geometric lattices are strongly semimodular. Each modular lattice is obviously strongly semimodular.

As a preparation we need the following

LEMMA 1.29. *Let $L \in \mathbf{K}$. If $p \prec q$ $(p, q \in L)$, then there exists a join irreducible element $u \in \mathrm{V}(L)$ such that $p \vee u = q$ and $p \wedge u = u^*$.*

Proof. The set $T = \{t \in L : p \vee t = q\}$ is nonempty, since $q \in T$. Let C be a chain in T. Lower continuity yields

$$p \vee \bigwedge C = \bigwedge \{p \vee c : c \in C\} = q.$$

Thus $\bigwedge C \in T$, and T contains a minimal element u, by the dual of Zorn's Lemma. Clearly, $u \in \mathrm{V}(L)$, $p \vee u = q$ and from $u \nleq p$ it follows that $p \wedge u \leq u^*$. Observe that $u^* \leq p$. Indeed, if $u^* \nleq p$, then $p \vee u^* = q$, that is, $u^* \in T$ and $u^* < u$, contradicting the minimality of u. Thus we have $u^* \leq p \wedge u$. Hence $p \wedge u = u^*$. ■

REMARK 1.30. For lattices of finite length this lemma was proved in Stern [1982] (see also Stern [1991b], p. 25).

Our main result of this section is

THEOREM 1.31 (Walendziak [1996a,c]). *Let $L \in \mathbf{K}$. Then*:

(i) *L is semimodular iff L has the exchange property*:

(EP) *For all $u, v \in \mathrm{V}(L)$ and arbitrary $b \in L$, $v \leq b \vee u$ and $v \nleq b \vee u^*$ imply $u \leq b \vee v \vee u^*$.*

(ii) *L is strongly semimodular iff L has the property*:

($\overline{\mathrm{EP}}$) *For all $u, v \in \mathrm{V}(L)$ and arbitrary $b \in L$, $v \leq b \vee u$ and $v \nleq b \vee u^*$ imply $u \leq b \vee v$.*

Proof. (i) Suppose that L is a semimodular lattice. Let $u, v \in \mathrm{V}(L)$ and $b \in L$ be such that $v \leq b \vee u$ and $v \nleq b \vee u^*$. Observe that $u \nleq b \vee u^*$. Indeed, if $u \leq b \vee u^*$, then $v \leq b \vee u = b \vee u^*$, a contradiction. Thus we have

$$u \wedge (b \vee u^*) = u^* \prec u.$$

Hence, by semimodularity, we conclude that

$$b \vee u^* \prec (b \vee u^*) \vee u = b \vee u.$$

From this and from $v \nleq b \vee u^*$ we get $v \vee b \vee u^* = b \vee u$. Consequently, $u \leq b \vee v \vee u^*$.

Let L satisfy (EP). Let $a, b \in L$ be elements for which $a \wedge b \prec a$. Without loss of generality we may assume that a, b are incomparable. By Lemma 1.29, there exists a join irreducible element $u \in \mathrm{V}(L)$ such that $(a \wedge b) \vee u = a$ and $a \wedge b \wedge u = u^*$. We shall prove that $b \prec b \vee u$. To obtain a contradiction, suppose that $b < q < b \vee u$ for some $q \in L$. Since L is strongly coatomic, there is $p \in L$ with $b \leq p \prec q$. By Lemma 1.29 we get the existence of a join irreducible element $v \in \mathrm{V}(L)$ with $p \vee v = q$. It follows that $v \leq b \vee u$ and $v \nleq b = b \vee u^*$. Applying (EP) we obtain $u \leq b \vee v \vee u^* = b \vee v$. Then $b \vee u \leq b \vee v \leq q$. This contradiction shows that $b \prec b \vee u$. We have

$$a \vee b = (a \wedge b) \vee u \vee b = b \vee u.$$

Consequently, $b \prec a \vee b$, which shows that L is semimodular.

(ii) Let L be a strongly semimodular lattice, and let $u, v \in \mathrm{V}(L)$ and $b \in L$ be such that $v \leq b \vee u$ and $v \nleq b \vee u^*$. Applying (EP) we obtain $u \leq b \vee v \vee u^*$, and strongness implies $u \leq b \vee v$. Hence ($\overline{\mathrm{EP}}$) holds.

Now let L satisfy $(\overline{\mathrm{EP}})$. By (i), L is semimodular. We show that L is also strong. Suppose, on the contrary, that there exists $u \in \mathrm{V}(L)$ such that property (St) is not satisfied. Then the set

$$T = \{t \in L : u \leq t \vee u^* \text{ and } u \nleq t\}$$

is not empty. Let C be a chain in L. Lower continuity yields

$$u^* \vee \bigwedge C = \bigwedge \{u^* \vee c : c \in C\} \geq u.$$

It is obvious that $u \nleq \bigwedge C$. Thus $\bigwedge C \in T$, and T contains a minimal element a, by the dual of Zorn's Lemma. Since $a \neq 0$ and L is strongly coatomic, we may choose $p \in L$ with $p \prec a$. By Lemma 1.29 there exists a join irreducible element v such that $p \vee v = a$ and $p \wedge v = v^*$. We shall prove that $u \nleq p \vee u^*$. Assume, on the contrary, that $u \leq p \vee u^*$. By the choice of p we have $u \leq p \leq a$, a contradiction. Then $u \nleq p \vee u^*$ and hence $u \nleq p \vee u^* \vee v^*$, since $v^* \leq p$. Obviously, $u \leq a \vee u^* = p \vee u^* \vee v$. Therefore, using the exchange property $(\overline{\mathrm{EP}})$ we get $v \leq p \vee u$. Observe now that $v \nleq p \vee u^*$ since otherwise

$$p \vee u^* = p \vee v \vee u^* = a \vee u^* \geq u.$$

Hence applying $(\overline{\mathrm{EP}})$ we conclude that $u \leq p \vee v = a$, which contradicts $u \nleq a$. It follows that L must be strong. Thus L is strongly semimodular. ∎

Remark 1.32. Since every lattice of finite length is lower continuous and strongly coatomic, Theorem 1.31 gives the Theorem of Stern [1990b] and Theorem 1 of Faigle–Richter–Stern [1984] (see also Theorem 26.5 of Stern [1991b]).

We know that in atomistic lattices each join irreducible element is an atom. Then, as the consequence of Theorem 1.31 we get the following result which is a generalization of the Corollary of Stern [1990b].

Corollary 1.33. *For every atomistic lattice $L \in \mathbf{K}$ the following conditions are equivalent*:

(i) *L is semimodular.*

(ii) *L has the Steinitz–MacLane exchange property, that is, for all atoms $p, q \in L$ and for arbitrary $b \in L$, the relations $p \leq b \vee q$ and $p \nleq b$ imply $q \leq b \vee p$.*

1.4. Characterizations of atomistic lattices. We characterize atomistic lattices in terms of concepts related to pure elements and neat elements. We first recall the notion of pure elements in lattices. This notion was introduced independently by Head [1966] and Kertész [1968].

An element a of a complete lattice L is *pure* in L if for each $c \in \mathrm{K}([a, 1])$ there exists $b \in L$ such that $c = a \dot{\vee} b$ (i.e., $c = a \vee b$ and $a \wedge b = 0$).

We now introduce the notion of a neat element in lattices following Delany [1968]. An element $a \in L$ (L is a lattice with 0) is called *neat* if $a \prec b$ ($b \in L$) implies the existence of $c \in L$ such that $b = a \dot{\vee} c$.

Applications of these concepts in group theory can be found in Delany [1968], Head [1966] and Honda [1956]. For our aims, we sharpen these concepts by introducing weakly pure and strongly neat elements. (Note that Stern [1984] uses the term "strongly neat" in another sense.)

Definition 1.34 (Walendziak [2000b], Definition 1). An element $a \in L$ is called *weakly pure* if for every $v \in \mathrm{J}(L)$ there exists $b \in L$ such that $a \vee v = a \dot{\vee} b$.

Lemma 1.35. *In an upper continuous lattice every pure element is weakly pure.*

Proof. Let L be an upper continuous lattice and let $a \in L$. Suppose that a is pure and let $v \in \mathrm{J}(L)$. Then v is compact in L (see Crawley [1962], Lemma 3), and therefore $c = a \vee v$ is compact in $[a, 1]$. Since a is pure, there exists $b \in L$ such that $c = a \dot{\vee} b$. This means that a is weakly pure. ∎

The converse of Lemma 1.35 is not true. In Figure 5 we give an example of a lattice having a weakly pure element, a, which is not pure.

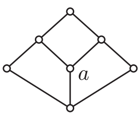

Fig. 5

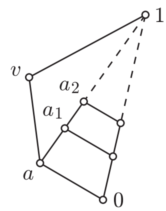

Fig. 6

The following example shows that in Lemma 1.35 the assumption that L is upper continuous cannot be dropped. Let L be the lattice diagrammed in Figure 6. Observe that the element a is pure but not weakly pure. It is sufficient to show that the element v is completely join irreducible but not compact. Obviously, $v \in \mathrm{J}(L)$ and $v \leq 1 = \bigvee\{a_i : i = 1, 2, \ldots\}$. Since $v \not\leq \bigvee\{a_i : i \in I\}$ for every finite subset I of $\{1, 2, \ldots\}$, it follows that v is not compact.

Definition 1.36 (Walendziak [2000b], Definition 2). Let L be a lattice with 0. An element $a \in L$ is called *strongly neat* if $a \prec b$ $(b \in L)$ implies the existence of an atom p such that $b = a \vee p$.

By definition, it is clear that in a lattice with 0 a strongly neat element is a fortiori neat. It is obvious that in every lower semimodular lattice L any element $a \in L$ is strongly neat if and only if it is neat. In Figure 7 we give an example of a lattice having a neat element, a, which is not strongly neat.

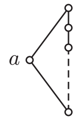

Fig. 7

Theorem 1.37 (Walendziak [2000b], Theorem 1). *Let L be a* J*-lattice. Then the following four conditions are equivalent*:

(i) *L is atomistic.*
(ii) *Every element of L is weakly pure.*
(iii) *Every element of L is neat.*
(iv) *Every element of L is strongly neat.*

Proof. (i)⇒(ii). Let L be an A-lattice, $a \in L$ and let v be a completely join irreducible element of L. Since L is atomistic, $\mathrm{J}(L) = \mathrm{A}(L)$, and therefore v is an atom of L. If $v \leq a$, then $a = a \dot\vee 0$. If $v \nleq a$, then $a \vee v = a \dot\vee v$. Thus a is a weakly pure element of L.

(ii)⇒(iii). Let $a \in L$ and let b be an upper cover of a, i.e. $a \prec b$. Since L is a J-lattice, the relation $a \prec b$ implies the existence of a completely join irreducible element v such that $v \nleq a$ and $v \leq b$, and therefore $b = a \vee v$. By (ii), the element a is weakly pure and so by definition there exists $c \in L$ with $b = a \dot\vee c$. Consequently, a is neat.

(iii)⇒(iv). Let $a, b \in L$ with $a \prec b$. Let $v \in \mathrm{J}(L)$ be such that $b = a \vee v$. We set

$$u = \bigvee\{x \in L : x < v\}$$

(u exists, because L is complete). Since v is completely join irreducible we conclude that $u \prec v$. By (iii) the element u is neat. Hence there exists $c \in L$ for which $v = u \dot\vee c$. Then $u = 0$, because v is join irreducible. From this we deduce that v is an atom of L. Consequently, a is strongly neat.

(iv)⇒(i). It is easy to see that every $v \in \mathrm{J}(L)$ is an atom of L. This means that $\mathrm{J}(L) = \mathrm{A}(L)$. Since L is a J-lattice, it is also an atomistic lattice. ∎

Theorem 1.38. *For a complete weakly atomic lattice L the following are equivalent*:

(i) *L is atomistic.*
(ii) *Every element of L is strongly neat.*

Proof. If L is an atomistic lattice and $a \prec b$ $(a, b \in L)$, then there exists an atom p such that $p \leq b$ and $p \nleq a$. Therefore $b = a \vee p$. It follows that the arbitrarily chosen element $a \in L$ is strongly neat.

Let (ii) hold, and let $a \in L - \{0\}$. Since L is weakly atomic, there exist $x, y \in L$ such that $0 \leq x \prec y \leq a$. By (ii) the element x is strongly neat and so by definition there exists an atom $p \in \mathrm{A}(L)$ with $y = x \vee p$. Since $p \leq a$, the set $P = \{r \in \mathrm{A}(L) : r \leq a\}$ is nonvoid. Suppose that $b = \bigvee P < a$. By weak atomicity of L there exist $u, v \in L$ such that $b \leq u \prec v \leq a$. Since u is strongly neat, we get the existence of an atom q such that $v = u \vee q$. Then $q \in \mathrm{A}(L)$ and $q \leq a$, and therefore, $q \leq b$. Hence $v = u \vee q \leq u \vee b = u$, a contradiction. Thus

$$a = \bigvee\{r \in \mathrm{A}(L) : r \leq a\},$$

i.e., every element ($\neq 0$) of L is the join of the atoms contained in it. ∎

Since every algebraic lattice is weakly atomic, Theorem 1.38 yields

Corollary 1.39. *An algebraic lattice L is atomistic if and only if every element of L is strongly neat.*

The next proposition is a generalization of Theorem 2 from Kertész–Stern [1974].

Proposition 1.40. *Every element of an* AC-*lattice is pure.*

Proof. Let L be an AC-lattice, $a \in L$ and let c be a compact element in $[a, 1]$. Since L is atomistic, $c = \bigvee\{p_i : i \in I\}$, where p_i $(i \in I)$ are atoms. Hence $c = \bigvee\{a \vee p_i : i \in I\}$, and we have $c = a \vee p_1 \vee \ldots \vee p_n$, because $c \in \mathrm{K}([a, 1])$. Without loss of generality we can assume that

$$p_i \nleq a \quad \text{and} \quad p_i \nleq a \vee p_1 \vee \ldots \vee p_{i-1} \quad \text{for } i = 2, \ldots, n.$$

We prove by induction that

$$a \wedge (p_1 \vee \ldots \vee p_k) = 0 \quad \text{for all } 1 \leq k \leq n. \tag{6}$$

This is true for $k = 1$. Let $i \in \{2, \ldots, n\}$. We set

$$b = p_1 \vee \ldots \vee p_{i-1},$$

and suppose that $a \wedge b = 0$. Since $p_i \nleq b$ we see that $p_i \wedge b = 0$. Property (C) yields

$$b \prec b \vee p_i. \tag{7}$$

As $p_i \nleq a \vee b$ we have $b \vee p_i \nleq a \vee b$ and therefore,

$$b \leq (a \vee b) \wedge (b \vee p_i) < b \vee p_i.$$

From (7) we conclude that

$$(a \vee b) \wedge (b \vee p_i) = b.$$

Hence

$$a \wedge (b \vee p_i) \leq (a \vee b) \wedge (b \vee p_i) = b.$$

Consequently, $a \wedge (b \vee p_i) \leq a \wedge b = 0$, that is,

$$a \wedge (p_1 \vee \ldots \vee p_{i-1} \vee p_i) = 0,$$

completing the proof of (6). Thus

$$c = a \vee p_1 \vee \ldots \vee p_n \quad \text{and} \quad a \wedge (p_1 \vee \ldots \vee p_n) = 0,$$

which means that a is pure in L. ∎

THEOREM 1.41 (Walendziak [2000b], Theorem 3). *Let L be an upper continuous* J-*lattice satisfying* (C). *Then the following statements are equivalent*:

(i) *L is atomistic.*
(ii) *Every element of L is pure.*
(iii) *Every element of L is weakly pure.*
(iv) *Every element of L is neat.*
(v) *Every element of L is strongly neat.*

Proof. The implication (i)⇒(ii) follows from Proposition 1.40. (ii) implies (iii) by Lemma 1.35. The equivalence of conditions (i), (iii), (iv) and (v) follows from Theorem 1.37. ∎

For lattices of finite length, Theorem 1.41 gives

COROLLARY 1.42. *Let L be a lattice of finite length with* (C). *Then all statements of Theorem* 1.41 *are equivalent.*

THEOREM 1.43. *Let L be an algebraic lattice satisfying the covering property* (C) *and the following condition*:

$$(\square) \qquad (a \prec a \vee b \text{ and } a \vee b \in \mathrm{K}(L)) \;\Rightarrow\; a \wedge b \prec b.$$

Then the following four statements are equivalent:

(i) *L is atomistic.*
(ii) *Every element of L is pure.*
(iii) *Every element of L is neat.*
(iv) *Every element of L is strongly neat.*

Proof. The implication (i)$\Rightarrow$(ii) follows from Proposition 1.40. It is clear that (ii) implies (iii).

Now suppose that L satisfies (iii). We show first that L is atomic. Let $0 \neq b \in \mathrm{K}(L)$, and set

$$T = \{t \in T : t < b\}.$$

Obviously, $T \neq \emptyset$. Let C be a chain in T. Assume that $b = \bigvee C$. Since b is compact, there is a finite subset C' of C such that $b = \bigvee C'$. As C is a chain we have $b = c_0$ for some $c_0 \in C$. This contradiction shows that $\bigvee C < b$, and therefore $\bigvee C \in T$. By Zorn's Lemma, T contains a maximal element a. It is easy to see that $a \prec b$. By condition (iii) the element a is neat. Hence there exists $c \in L$ for which $b = a \dot{\vee} c$. Property ($\square$) gives

$$0 = a \wedge c \prec c \leq b.$$

This means that every interval $[0, b]$ ($b \in \mathrm{K}(L)$), $b \neq 0$) contains an atom. Since L is algebraic, we deduce that L is atomic. In atomic lattices every neat element is strongly neat. Therefore, from (iii) we obtain (iv).

Finally, (iv) implies (i) by Theorem 1.38. ∎

COROLLARY 1.44. *Let L be an algebraic lattice with* (C) *and* ($\square$). *If one of the conditions* (i)–(iv) *of the preceding theorem is satisfied*, *then L is modular.*

Proof. Let L be an algebraic AC-lattice (i.e., a matroid lattice) satisfying ($\square$). Let $\mathrm{Fin}(L)$ denote the set of all finite elements of L (i.e., $a \in \mathrm{Fin}(L)$ iff a is the join of a finite number of atoms). It is obvious that $\mathrm{Fin}(L) \subseteq \mathrm{K}(L)$. From ($\square$) we conclude that $\mathrm{Fin}(L)$ is a lower semimodular sublattice of L. It follows from Theorem 9.5 of Maeda–Maeda [1970] that $\mathrm{Fin}(L)$ is a modular lattice. Now Theorem 14.1 from Maeda–Maeda [1970] implies that L is modular. ∎

THEOREM 1.45 (Walendziak [2000b], Theorem 5). *A lattice L is atomistic if and only if L is prealgebraic and satisfies the following condition*:

$$(*) \qquad \text{If } b \prec q \ (q \in \mathrm{Q}(L)), \text{ then there is } p \in \mathrm{A}(L) \text{ with } q = b \vee p.$$

Proof. Let L be a Q-lattice with property ($*$). To show that L is atomistic, it is sufficient to prove that each precompact element is a join of atoms. To see this consider a precompact element $q \in \mathrm{Q}(L)$. Suppose that

$$(8) \qquad a = \bigvee\{p \in \mathrm{A}(L) : p \leq q\} < q.$$

We put

$$T = \{x \in L : a \leq x < q\}.$$

Then T is nonvoid, since $a \in T$. Let C be a chain in T. Then $\bigvee C \in T$, because $q \in \mathrm{Q}(L)$. Therefore T contains a maximal element b by Zorn's Lemma. The maximality of b shows that $a \leq b \prec q$. Applying $(*)$ we get the existence of an atom $p_0 \in \mathrm{A}(L)$ with $q = b \vee p_0$. Obviously, $p_0 \leq a \leq b$, and consequently $q = b$, which is impossible. Hence our assumption (8) was false, i.e., q is the join of the atoms contained in it. Thus L is an atomistic lattice.

The converse is immediate. ■

By Theorem 1.45 we obtain

Corollary 1.46. *A prealgebraic lattice L is atomistic if and only if L satisfies* $(*)$.

2. Join decompositions in lattices

2.1. J-lattices. If an element $a \in L$ has a representation $a = \bigvee T$ (resp. $a = \bigwedge T$) with $T \subseteq \mathrm{J}(L)$ (resp. $T \subseteq \mathrm{M}(L)$), then we say that a has a *join decomposition* (resp. *meet decomposition*). A join decomposition $a = \bigvee T$ is *irredundant* if $a > \bigvee(T - \{t\})$ for each $t \in T$. L is a J-*lattice* if each element of L has a join decomposition. Crawley–Dilworth [1973] (p. 39) mentioned that if L is a lattice with the ascending chain condition, then every element of L has an irredundant finite meet decomposition. Therefore, every element of L has an irredundant finite join decomposition if L satisfies the descending chain condition.

Most of the investigations in this section will concern lower continuous lattices with the *hereditary property* (HJ), defined in Richter [1991] as follows:

(HJ) $$(a \in L \text{ and } u \in \mathrm{J}(L)) \;\Rightarrow\; a \vee u \in \mathrm{J}([a, 1]).$$

It is obvious that every modular lattice has this property. We remark that for complete strongly coatomic lattices the property of being consistent and property (HJ) are equivalent. In arbitrary lattices, this equivalence does not hold. For instance, the lattice of Figure 8 is consistent but it does not have the hereditary property (HJ).

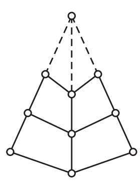

Fig. 8

First, we shall prove the following simple but useful lemma.

Lemma 2.1. *Let L be a lower continuous lattice and let $u, v \in L$. If u is covered by v, then each minimal element of the set $P = \{p \in L : v = u \vee p\}$ is completely join irreducible.*

Proof. P is nonempty, since $v \in P$. Let C be a chain in P. By lower continuity, $u \vee \bigwedge C = \bigwedge\{u \vee c : c \in C\} = v$. Then $\bigwedge C \in P$ and P contains a minimal element q by the dual of Zorn's Lemma. Now we prove that q is completely join irreducible in L. Indeed, let $q = \bigvee T$ and $t < q$ for all $t \in T$. From the minimality of q and the fact that $u \prec v$ we infer that $u \vee t = u$ for every $t \in T$. Consequently, $q = \bigvee T \leq u$, and hence $v = u \vee q = u$. This contradiction shows that $q \in \mathrm{J}(L)$. ■

The following result is a generalization of the classical existence theorem (cf. Crawley–Dilworth [1973], Theorem 6.1).

Proposition 2.2 (Walendziak [1993d], Theorem 1). *If a lower continuous lattice L is weakly atomic, then it is a* J*-lattice.*

Proof. Let a be an arbitrary element of L, and set

$$b = \bigvee\{x \in \mathrm{J}(L) : x \leq a\}.$$

Suppose now $b < a$. Since L is weakly atomic, there exist $u, v \in [a, b]$ such that $u \prec v$. Let P be the set of all $p \in L$ with $v = u \vee p$, and let q be a minimal element of P. From Lemma 2.1 it follows that $q \in \mathrm{J}(L)$. By the definition of b we have $q \leq b$. Hence $v = u \vee q \leq u \vee b = u$, a contradiction. Thus $a = \bigvee\{x \in \mathrm{J}(L) : x \leq a\}$ is a join decomposition of a. ■

Proposition 2.2 implies

Corollary 2.3 (see Draškovičová [1974], Theorem 4). *Every weakly atomic dually Brouwerian lattice is a* J*-lattice.*

Crawley [1962] (Lemma 3) showed that in an upper continuous lattice, every completely join irreducible element is compact. We know that any algebraic lattice is weakly atomic (Crawley–Dilworth [1973], Theorem 2.2). From the last two facts and Proposition 2.2 we get

Corollary 2.4 (cf. Geissinger–Graves [1972], Corollary 2). *For a continuous lattice L, the following statements are equivalent*:

(i) *L is weakly atomic.*
(ii) *L is a* J*-lattice.*
(iii) *L is algebraic.*
(iv) *L is dually algebraic.*
(v) *Every element of L has a meet decomposition.*

Now we prove

Proposition 2.5. *A* J*-lattice with hereditary property* (HJ) *is weakly atomic.*

Proof. Let L be a J-lattice satisfying (HJ). Let $a, b \in L$, $b < a$ and let $a = \bigvee T$ be a join decomposition. Since $b < a$ there is $t_0 \in T$ such that $t_0 \not\leq b$. We set

$$v = t_0 \vee b \quad \text{and} \quad u = \bigvee\{x \in L : b \leq x < v\}$$

(u exists, since $b < v$ and L is complete). From (HJ) it follows that v is completely join irreducible in $[b, v]$, and hence $u < v$. Now, by the definition of u we obtain $u \prec v$. ■

As a consequence of Propositions 2.2 and 2.5 we get the following

THEOREM 2.6. *Let L be a lower continuous lattice satisfying* (HJ). *Every element of L has a join decomposition iff L is weakly atomic.*

We say that a complete lattice L has *irredundant join decompositions* if each element of L has at least one irredundant join decomposition.

We close this section with the following result.

PROPOSITION 2.7 (Richter [1982a], Theorem 10). *Every lattice belonging to* $\mathbf{K}$ *has irredundant join decompositions.*

2.2. The Kurosh–Ore replacement property. The most important result on join decompositions of an element of a modular lattice is the Kurosh–Ore Theorem.

THEOREM 2.8 (Kurosh [1935], Ore [1936]). *Let L be a modular lattice and let $a \in L$. If $a = x_1 \vee \ldots \vee x_n$ and $a = y_1 \vee \ldots \vee y_m$ are irredundant join decompositions of a, then for every x_i there is a y_j such that*

$$a = x_1 \vee \ldots \vee x_{i-1} \vee y_j \vee x_{i+1} \vee \ldots \vee x_n$$

and $n = m$.

The following definition is suggested by the Kurosh–Ore Theorem.

A complete lattice L has the *Kurosh–Ore Replacement Property* for join decompositions ($\bigvee$-KORP, for short) if each element of L has at least one irredundant join decomposition, and whenever $a = \bigvee T = \bigvee R$ are two irredundant join decompositions, for each $t \in T$ there exists $r \in R$ such that $a = \bigvee (T - \{t\}) \vee r$ is also an irredundant join decomposition.

The $\bigwedge$-KORP is defined dually. The concept of consistency relates to the $\bigvee$-KORP. Indeed, we have the following result.

PROPOSITION 2.9 (Richter [1982a]). *A lattice $L \in \mathbf{K}$ has the $\bigvee$-KORP iff it is consistent.*

Combining Theorem 1.26 and Proposition 2.9 we get

COROLLARY 2.10 (Walendziak [1994b], Theorem 2). *For every semimodular lattice belonging to $\mathbf{K}$, the following conditions are equivalent*:

(i) *L has the $\bigvee$-KORP.*
(ii) *L is consistent.*
(iii) *L is strong.*

REMARK 2.11. The preceding result is a generalization of Theorem 4 of Reuter [1989].

PROPOSITION 2.12. *Let L be an upper continuous, strongly atomic lattice. If L is locally modular, then L has the $\bigwedge$-KORP.*

Proof. By the dual of Proposition 1.4, L is dually consistent. The dual of Proposition 2.9 shows that L has the $\bigwedge$-KORP. ■

THEOREM 2.13 (Walendziak [1999], Theorem 1). *Let L be a lattice such that both L and its dual L^∂ are algebraic and strongly atomic (i.e., $L, L^\partial \in \mathbf{K}$). If L is semimodular or lower semimodular, then L has both the $\bigwedge$-KORP and the $\bigvee$-KORP iff L is modular.*

Proof. Without loss of generality we can assume that L is semimodular. Let L have both the $\bigwedge$-KORP and the $\bigvee$-KORP. We know that if an algebraic, strongly atomic lattice is both semimodular and lower semimodular, then it is modular (see Crawley–Dilworth [1973], Theorem 3.6). Therefore, we only need to show that L is lower semimodular. Then we prove that L satisfies (N*).

Assume that $x \prec x \vee y$. We conclude from Proposition 2.9 that L is dually consistent. By the dual of Proposition 1.2, the interval $[x \wedge y, y]$ has exactly one atom, say p. We now prove that $p = y$. On the contrary, suppose that $p < y$. Since every element of L has at least one irredundant join decomposition, we conclude that there is $u \in \mathrm{J}(L)$ such that $u \leq y$ and $u \not\leq p$. From Corollary 2.10 it follows that L is strong. We have

$$x \leq x \vee u^* \leq x \vee y \quad \text{and} \quad x \prec x \vee y.$$

Observe that $x = x \vee u^*$. Indeed, if $x \vee u^* = x \vee y$, then $u \leq x \vee u^*$ and strongness implies $u \leq x$, a contradiction. Therefore, $u^* \leq x$. Hence, $u \wedge x \wedge y = u^* \prec u$, and by semimodularity we deduce that $x \wedge y \prec u \vee (x \wedge y) \leq y$. Then $p = u \vee (x \wedge y)$, and this contradicts the fact that $u \not\leq p$. Thus $x \wedge y \prec p = y$, that is, (N*) holds in L, and, in consequence, L is modular.

The converse is clear by the Kurosh–Ore Theorem (see Theorem 2.8). ■

Remark 2.14. The preceding theorem generalizes Theorem 6 of Stern [1996], since any lattice satisfying the descending chain condition is strongly atomic.

Theorem 2.15 (Walendziak [1999], Theorem 3). *If L is a lattice such that L and L^∂ belong to* $\mathbf{K}$, *then L is strong and locally modular if and only if L is modular.*

Proof. If L is locally modular, then L is also semimodular (by the dual of Proposition 1.4). From Proposition 2.12 and Corollary 2.10 we conclude that L has both the $\bigwedge$-KORP and the $\bigvee$-KORP. Therefore, by Theorem 2.13, L is modular.

The converse is obvious. ■

Finally we recall that a complete lattice L has the *Kurosh–Ore property* for join decompositions ($\bigvee$-KOP, for short) if every element of L has an irredundant finite join decomposition and for each $a \in L$, the number of join irreducible elements in any irredundant finite join decomposition of a is unique. In a dual way one defines the $\bigwedge$-KOP. It is obvious that the KORP implies the corresponding KOP, whereas the converse does not hold in general. Consider, for instance, the lattice of Figure 9.

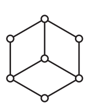

Fig. 9

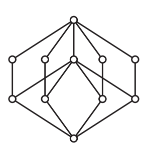

Fig. 10

This lattice is denoted by S_7 and will be called the *hexagon*. The lattice S_7 has the $\bigvee$-KOP whereas the $\bigvee$-KORP does not hold. In semimodular algebraic lattices satisfying the DCC, the $\bigwedge$-KORP is equivalent to the $\bigwedge$-KOP (see Crawley–Dilworth [1973], Theorems 7.6 and 7.7). Hence in Theorem 6 of Stern [1996] we may replace the $\bigwedge$-KORP by the $\bigwedge$-KOP, but here it is not possible to replace the $\bigvee$-KORP by the $\bigvee$-KOP, that is, the question of Stern [1996] has a negative answer. Indeed, let L be the lattice diagrammed in Figure 10. Then L is locally modular, and therefore it has the $\bigwedge$-KORP (and, evidently, the $\bigwedge$-KOP). This lattice also has the $\bigvee$-KOP, whereas the $\bigvee$-KORP does not hold.

2.3. Lattices with unique irredundant join decompositions. Throughout this section L will denote a lower continuous strongly coatomic lattice. For $a \in L$, set

$$P_a = \{p \in L : p \prec a\}.$$

Then $a_+ = \bigwedge P_a$. By Proposition 2.7 we deduce that L has irredundant join decompositions. If every element of L has exactly one irredundant join decomposition, then we say L has *unique irredundant join decompositions*.

We begin with the following four lemmas.

LEMMA 2.16. *Suppose that the lattice L has the following property*:

(**) *For every $a \in L$ and for every $u, w \in \mathrm{J}(L)$, if $u \vee a = w \vee a$ and $u \vee w \nleq a$, then $u = w$.*

Then L is lower semimodular.

Proof. Let $a, b \in L$ be elements for which $a \prec a \vee b$. Without loss of generality we may suppose that a, b are incomparable. We show that then $a \wedge b \prec b$, which means that L is lower semimodular. Assume that there exists $c \in L$ such that $a \wedge b < c < b$. By Lemma 1.1 there are completely join irreducible elements $u \leq b$ and $w \leq c$ such that $u \nleq c$ and $w \nleq a \wedge b$. Consequently, $u \vee a = w \vee a$ and $u \vee w \nleq a$. From (**) it follows that $u = w$. This contradiction shows that $a \wedge b \prec b$. ∎

LEMMA 2.17. *Let L be a lower locally distributive lattice and let $a \in L$. If $p \in P_a$ and $u, w \in \mathrm{J}(L) \cap [0, a]$, then*

$$p \vee (u \wedge w) = (p \vee u) \wedge (p \vee w). \tag{1}$$

Proof. Assume that the assumptions of Lemma 2.17 hold but

$$p \vee (u \wedge w) < (p \vee u) \wedge (p \vee w).$$

Therefore, $p \vee (u \wedge w) = p$ and $p \vee u = p \vee w = a$. Then $u \wedge w \leq p$, $u \nleq p$ and $w \nleq p$. Set $b = u \vee w$; then $w < b$. By Proposition 1.4, L is lower semimodular. As $b \nleq p$ this implies that $p \wedge b \prec b$. Since L is strongly coatomic, there exists $q \in L$ such that $w \leq q \prec b$. It is obvious that $s = u \vee b_+ \nleq q$ and $s \leq b$. Lower semimodularity now implies that $b_+ \leq s \wedge q \prec s \leq b$. By the definition of b_+ it is clear that

$$b_+ = \bigwedge \{r \wedge s : b_+ \leq r \prec b \text{ and } s \nleq r\}.$$

By lower semimodularity, $r \wedge s \prec s$ whenever $s \nleq r \prec b$. Therefore, b_+ is a meet of lower covers of s. But $s = u \vee b_+ \in \mathrm{J}([b_+, 1])$, since the lattice L is consistent (see Proposition 1.4). Consequently,

$$s \wedge q = b_+. \tag{2}$$

Since $u \not\leq p$ and $w \not\leq p$ we have $s \not\leq p \wedge b$ and $p \wedge b \neq q$. This together with the fact that $p \wedge b \prec b$ yields

$$(p \wedge b) \vee s = b \quad \text{and} \quad (p \wedge b) \vee q = b.$$

By the distributivity of $[b_+, b]$ we infer

$$(p \wedge b) \vee (s \wedge q) = [(p \wedge b) \vee s] \wedge [(p \wedge b) \vee q] = b.$$

On the other hand, by (2),

$$(p \wedge b) \vee (s \wedge q) = (p \wedge b) \vee b_+ = p \wedge b < b,$$

a contradiction. Thus (1) holds. ■

LEMMA 2.18. *Let L ($\in \mathbf{K}$) be a lattice having the property* $(**)$, *and let $a \in L$. Then $p \vee \bigwedge(P_a - \{p\}) = a$ for each $p \in P_a$.*

Proof. Let $p \in P_a$. Assume that there is a finite subset R of $P_a - \{p\}$ with minimal number of elements such that $p \geq \bigwedge R$. Let $r \in R$ and set $s = \bigwedge(R - \{r\})$. Obviously, $s \not\leq p$. By Lemma 1.1 there are join irreducible elements u and w such that $u \leq r$, $w \leq s$ and $u \vee p = w \vee p = a$. From $(**)$ it follows that $u = w$. Consequently, $u \leq r$ and $u \leq s$. Hence $u \leq r \wedge s = \bigwedge R \leq p$, a contradiction. Then for every finite subset X of $P_a - \{p\}$, $\bigwedge X \not\leq p$. Therefore, by (LC) we have $p \vee \bigwedge(P_a - \{p\}) = \bigwedge\{p \vee \bigwedge X : X$ is a finite subset of $P_a - \{p\}\} = a$. ■

LEMMA 2.19. *If L satisfies* $(**)$, *then for each $a \in L$ the sublattice $[a_+, a]$ is distributive.*

Proof. First we prove that $[a_+, a]$ is a coatomistic lattice. Let b ($\neq a$) be an arbitrary element of $[a_+, a]$ and let $d = \bigwedge\{p \in P_a : b \leq p\}$. Suppose that $b < d$. Since L is strongly coatomic, there exists $c \in L$ such that $b \leq c \prec d$. By Lemma 2.16, L is lower semimodular. Therefore, if $p \in P_a$ and $d \not\leq p$, then $p \wedge d \prec d$. Observe that $c \neq p \wedge d$ for every $p \in P_a$. Indeed, if $c = p_0 \wedge d$ for some $p_0 \in P_a$, then $p_0 \geq p_0 \wedge d = c \geq b$ and hence $p_0 \geq d$. Consequently, $c = d$, contrary to the fact that $c \prec d$. Therefore,

$$c \geq b \geq a_+ \geq \bigwedge\{p \wedge d : p \in P_a\} \geq \bigwedge(P_d - \{c\}),$$

and hence $c \vee \bigwedge(P_d - \{c\}) = c$. On the other hand, by Lemma 2.18 we get $c \vee \bigwedge(P_d - \{c\}) = d$. This contradiction shows that $b = d$. Thus every element of $[a_+, a]$ is a meet of lower covers of a. Hence $[a_+, a]$ is coatomistic. Since $[a_+, a]$ is also lower continuous and lower semimodular, by Theorem 4.1 of Crawley–Dilworth [1973] we conclude that $[a_+, a]$ is complemented. We show that

(3) $$\text{if } a_+ = \bigwedge P \text{ where } P \subseteq P_a, \text{ then } P = P_a.$$

Indeed, if $P \neq P_a$, then there is an element $q \in P_a - P$, and we have

$$a > q = q \vee \bigwedge P \geq q \vee \bigwedge(P_a - \{q\}) = a \quad \text{(by Lemma 2.18)},$$

a contradiction. Now we prove that $[a_+, a]$ is a uniquely complemented lattice. Let $x \in [a_+, a]$ and suppose that there exist $x_1, x_2 \in [a_+, a]$ such that

(4) $$x \wedge x_1 = x \wedge x_2 = a_+$$

and

(5) $$x \vee x_1 = x \vee x_2 = a.$$

Since the lattice $[a_+, a]$ is coatomistic, there are subsets R, S and T of P_a such that $x = \bigwedge R$, $x_1 = \bigwedge S$, and $x_2 = \bigwedge T$. By (4), $a_+ = \bigwedge(R \cup S) = \bigwedge(R \cup T)$ and from (3) it follows that $R \cup S = R \cup T = P_a$. By (5) we have $R \cap S = R \cap T = \emptyset$. Consequently, $R = S$ and hence $x_1 = x_2$. Thus $[a_+, a]$ is uniquely complemented. Then, by Theorem 4.5 of Crawley–Dilworth [1973], $[a_+, a]$ is a distributive lattice. ■

In this section, the major result is

THEOREM 2.20. *If L is a lower continuous strongly coatomic lattice, then the following statements are equivalent*:

(i) *L has unique irredundant join decompositions.*
(ii) *L satisfies $(**)$.*
(iii) *L is lower locally distributive.*

Proof. (i)⇒(ii). Assume that L has unique irredundant join decompositions but it does not satisfy $(**)$. Then there are $a \in L$ and distinct $u, v \in \mathrm{J}(L)$ such that $u \vee a = w \vee a$ and $u \vee w \not\leq a$. By lower continuity there exist $c, d \leq a$ which are minimal with respect to $u \vee c = b$ and $w \vee d = b$, respectively. Let $c = \bigvee R$ and $d = \bigvee T$ be irredundant join decompositions. Then $b = u \vee \bigvee R = w \vee \bigvee T$ are two irredundant join decompositions of b. They are also distinct, since $u \neq w$ and $u \notin T$. This contradiction proves that L has the property $(**)$.

(ii)⇒(iii). By Lemma 2.19.

(iii)⇒(i). Now suppose that L is lower locally distributive. Let $a \in L$ and let $a = \bigvee R = \bigvee T$ be two irredundant join decompositions. Pick $r \in R$ and set $s = \bigvee(R - \{r\})$. Obviously, $s < a$. Then, as L is strongly coatomic there exists $p \in L$ with $s \leq p \prec a$. Clearly, there is $t \in T$ such that $t \not\leq p$. Consequently, $p \vee r = p \vee t = a$. By Lemma 2.17,

$$p \vee (r \wedge t) = (p \vee r) \wedge (p \vee t).$$

Hence $r \wedge t \not\leq p$. The lower semimodularity of L implies that $p \wedge r \prec r$ and $p \wedge t \prec t$. Suppose that $r \neq t$. Then either $r \wedge t < r$ or $r \wedge t < r$. If $r \wedge t < r$, then there exists $q \in [r \wedge t, r]$ such that $q \prec r$. Since $r \in \mathrm{J}(L)$, r has exactly one lower cover and hence $p \wedge r = q \geq r \wedge t$. But this is impossible since $r \wedge t \not\leq p$. Similarly, if $r \wedge t < t$, then $p \wedge t \geq r \wedge t$. Therefore, $p \geq r \wedge t$, a contradiction. Thus $r = t$ and we infer that $R = T$. Consequently, L has unique irredundant join decompositions. ■

REMARK 2.21. Since every dual algebraic lattice is lower continuous, this theorem implies the dual of Theorem 7.4 of Crawley–Dilworth [1973]. Theorem 2.20 also implies Theorem 6 of Walendziak [1993d] and the dual of the Theorem of Walendziak [1995].

3. c-Decompositions in modular lattices

3.1. Preliminaries. In this chapter L always denotes a complete modular lattice. If $a \in L$, then we say that a is a *direct join* of the elements a_i $(i \in I)$, and we write

$$a = \dot{\bigvee}\{a_i : i \in I\},$$

if $a = \bigvee\{a_i : i \in I\}$ and for each $i \in I$, $a_i \wedge \bigvee\{a_j : j \in I - \{i\}\} = 0$. The direct join of finitely many elements $a_1, \ldots, a_n$ is also written $a_1 \dot{\vee} \ldots \dot{\vee} a_n$. An element $a \in L$ is called *directly join irreducible* (or *directly indecomposable*) if $0 < a$ and if $a = b \dot{\vee} c$ implies $b = 0$ or $c = 0$.

The first lattice theoretic theorem on direct decompositions was given by Ore [1936]. Ore's Theorem may be stated as follows:

THEOREM 3.1. *Let L be a modular lattice of finite length and consider two direct decompositions*

$$1 = a_1 \dot{\vee} \ldots \dot{\vee} a_m = b_1 \dot{\vee} \ldots \dot{\vee} b_n \quad (a_i, b_j \in L)$$

of the unit element of L into directly join irreducible summands a_i, b_j. Then $m = n$ and there is a permutation λ of the set $I = \{1, \ldots, n\}$ such that

$$1 = a_1 \dot{\vee} \ldots \dot{\vee} a_{i-1} \dot{\vee} b_{\lambda(i)} \dot{\vee} a_{i+1} \dot{\vee} \ldots \dot{\vee} a_n$$

for all $i \in I$.

We obtain this theorem as a corollary from Theorem 3.25 in Section 3.5. Many interesting results on direct decompositions in modular lattices can be found in Kurosh [1943, 1946], Baer [1947, 1948], and Hostinsky [1951]. A number of papers are devoted to this topic, for example, Graev [1947], Livšic [1951] and Jakubík [1955]. Richter [1982b] gave a necessary and sufficient condition for an element in an algebraic modular lattice to be a direct join of completely join irreducible elements. Direct decompositions are also considered in Crawley [1962], Močulskiĭ [1955, 1961, 1962, 1968], and Walendziak [1979, 1980, 1991b].

In this chapter we give a common generalization to both the Theorem of Kurosh–Ore (Theorem 2.8) and the Theorem of Ore (Theorem 3.1). Before giving this generalization we still need a few notions.

Let c be a distributive element of L. Then c satisfies the following condition:

(D) $\qquad$ For all $x, y \in L$, $\quad c \vee (x \wedge y) = (c \vee x) \wedge (c \vee y)$.

Since L is modular, (D) is equivalent to the following property:

(Sn) $\qquad$ $x \wedge (c \vee y) = (x \wedge c) \vee (x \wedge y)$ $\quad$ for all $x, y \in L$.

(See e.g. Grätzer [1978], p. 145.) We denote by $\mathrm{D}(L)$ the set of all distributive elements of L.

DEFINITION 3.2. Let T be a subset of L and $a \in L$. If $a = \bigvee T$ and for each $t \in T$,

$$t \wedge \bigvee(T - \{t\}) \leq c,$$

then we say that a is the *c-join* of T, and we write $a = \sum_c T$. We will write simply $a = \sum T$ when no confusion can arise. The c-join of finitely many elements $t_1, \ldots, t_n$ is also written $t_1 +_c \ldots +_c t_n$ (or briefly, $t_1 + \ldots + t_n$). A representation of an element as a c-join of elements of the lattice L is said to be a *c-decomposition* of the element.

Observe that joins and direct joins are special cases of c-joins. Indeed,

$$a = \textstyle\sum_1 T \quad \text{iff} \quad a = \bigvee T, \quad \text{and} \quad a = \sum_0 T \quad \text{iff} \quad a = \dot{\bigvee} T.$$

Let $a \in L$. An element b ($\neq a$) is called a *c-summand* of a if $a = b +_c x$ for some element x ($\neq a$). We denote by $\mathrm{S}(c, L)$ the set of all c-summands of the unit element of L. An element $a \in L$ is called *c-irreducible* if for any $x, y \in L$, $a = x +_c y$ implies $a = x$ or $a = y$.

It is easy to see that $a \in L$ is 1-irreducible iff it is join irreducible, and a is 0-irreducible iff a is directly join irreducible.

Let $a \in \mathrm{S}(c, L)$. An element $b \in L$ is called a *c-complement* of a if $1 = a +_c b$. If an element $a \in L$ has a c-decomposition

$$a = \textstyle\sum_c \{a_i : i \in I\} \tag{1}$$

we define $\overline{a}_{j,k,\ldots,n} = \bigvee\{a_i : i \in I - \{j, k, \ldots, n\}\}$ for each subset $\{j, k, \ldots, n\}$ of I. Denote by α_i the function of L defined by the formula

$$x\alpha_i = a_i \wedge (x \vee \overline{a}_i).$$

The maps α_i, $i \in I$, are called the *decomposition functions* related to (1); any α_i is called the *decomposition function with respect to the c-summand a_i* of the c-decomposition (1).

Let $a \in \mathrm{S}(c, L)$. Define the set $\mathrm{DF}(c, a)$ of maps of L by $\alpha \in \mathrm{DF}(c, a)$ iff there exists a c-complement b of a such that $x\alpha = a \wedge (x \vee b)$ for every $x \in L$.

Let $\mathrm{DF}(c, L)$ denote the smallest set satisfying (i) and (ii):

(i) If $\alpha \in \mathrm{DF}(c, a)$ for some $a \in \mathrm{S}(c, L)$, then $\alpha \in \mathrm{DF}(c, L)$.
(ii) If $\varphi, \psi \in \mathrm{DF}(c, L)$, then $\varphi\psi \in \mathrm{DF}(c, L)$.

($\varphi\psi$ is the map of L defined by $x(\varphi\psi) = (x\varphi)\psi$, $x \in L$.) The elements of the set $\mathrm{DF}(c, L)$ are called the *c-decomposition functions* of L.

Let $\alpha, \beta \in \mathrm{DF}(c, L)$. We say that $\langle \alpha, \beta \rangle$ is a *pair of complementary c-decomposition functions* of L if there exist $a, b \in L$ such that $1 = a +_c b$ and α, β are decomposition functions with respect to a and b, respectively.

For an element $a \in L$ we denote by $\mathrm{F}(c, a)$ the set of all functions $\varphi \in \mathrm{DF}(c, L)$ such that $a\varphi = a$ and from $x \leq a$, $x\varphi \leq c$ it follows that $x \leq c$.

In Sections 3.2 and 3.3 we will present some of the most important properties of c-joins and c-decomposition functions. The material of Chapter 3 is taken from Walendziak [1986, 1989, 1990a].

3.2. Properties of c-joins and c-decomposition functions. The most important form of modularity is the following:

(M) If $x_i, y_i \in L$ ($i = 1, \ldots, n$) such that $x_i \leq y_{i'}$ for all $i \neq i'$, then

$$(x_1 \vee \ldots \vee x_n) \wedge y_1 \wedge \ldots \wedge y_n = (x_1 \wedge y_1) \vee \ldots \vee (x_n \wedge y_n).$$

Let $c \in \mathrm{D}(L)$. We recall that if a is a c-join of T ($T \subseteq L$), we also write $a = \sum T$ instead of $a = \sum_c T$.

I. *Let I be a finite set of indices and K_j, $j = 1, \ldots, n$, be nonempty subsets of I with $\bigcup\{K_j : j = 1, \ldots, n\} = I$ and $K_{j_1} \cap K_{j_2} = \emptyset$ for $j_1 \neq j_2$. If $a = \sum\{a_i : i \in I\}$ and $b_j = \bigvee\{a_i : i \in K_j\}$, then*

$$a = b_1 + \ldots + b_n.$$

Proof. Obviously, $a = b_1 \vee \ldots \vee b_n$. Moreover,

$$\begin{aligned} b_j \wedge \bigvee\{b_m : m \neq j\} &= \bigvee\{a_i : i \in K_j\} \wedge \bigvee\{a_i : i \in I - K_j\} \\ &\leq \bigvee\{a_i : i \in K_j\} \wedge \bigwedge\{\bigvee\{a_m : m \in I - \{i\}\} : i \in K_j\} \\ &= \bigvee\{a_i \wedge \bigvee\{a_m : m \in I - \{i\}\} : i \in K_j\} \\ &\qquad \text{(observe } a_i \leq \bigvee\{a_m : m \in I - \{i'\}\} \text{ for } i \neq i', \text{ and apply (M)).} \\ &\leq c. \end{aligned}$$

Therefore, $a = b_1 + \ldots + b_n$. ∎

II. *Consider an index set I and index sets J_i for each $i \in I$. If $a = \sum\{a_i : i \in I\}$ and if $a_i = \sum\{a_{ij} : j \in J_i\}$ for $i \in I$, then*

$$a = \sum\{a_{ij} : i \in I,\ j \in J_i\}.$$

Proof. Indeed,

$$\begin{aligned} & a_{ij} \wedge (\bigvee\{a_m : m \neq i\} \vee \bigvee\{a_{in} : n \in J_i - \{j\}\}) \\ &\quad = a_{ij} \wedge a_i \wedge (\bigvee\{a_m : m \neq i\} \vee \bigvee\{a_{in} : n \in J_i - \{j\}\}) \\ &\quad = a_{ij} \wedge [(a_i \wedge \bigvee\{a_m : m \neq i\}) \vee \bigvee\{a_{in} : n \in J_i - \{j\}\}] \quad \text{(by modularity)} \\ &\quad \leq a_{ij} \wedge (c \vee \bigvee\{a_{in} : n \in J_i - \{j\}\}) \\ &\quad = (a_{ij} \wedge c) \vee [a_{in} \wedge \bigvee\{a_{in} : n \in J_i - \{j\}\}] \quad \text{(by (Sn))} \\ &\quad \leq c. \end{aligned}$$ ∎

III. *Let $a = \sum\{a_i : i \in I\}$, and let α_i $(i \in I)$ be the decomposition functions related to this c-decomposition of a. Let $x \in L$. If I_1 is a finite subset of I such that $x \leq \bigvee\{a_i : i \in I_1\}$, then $x \leq \bigvee\{x\alpha_i : i \in I_1\}$.*

Proof. Compute:

$$\begin{aligned} \bigvee\{x\alpha_i : i \in I_1\} &= \bigvee\{a_i \wedge (x \vee \overline{a}_i) : i \in I_1\} \\ &= \bigvee\{a_i : i \in I_1\} \wedge \bigwedge\{x \vee \overline{a}_i : i \in I_1\} \\ &\qquad \text{(observe } a_i \leq \overline{a}_i \text{ for each } i \neq j \text{ and apply (M))} \\ &\geq x. \end{aligned}$$ ∎

Let

$$1 = a + b, \tag{2}$$

and let α, β be the decomposition functions with respect to a and b, respectively.

IV. *For every $x \in L$, $x \leq x\alpha \vee x\beta$.*

Proof. Follows from Property III. ∎

V. *Let $x \in L$. If $x\alpha \leq c$ and $x \wedge b \leq c$, then $x \leq c$.*

Proof. Indeed, by modularity,

$$x \leq x \vee b = (a \vee b) \wedge (x \vee b) = (a \wedge (x \vee b)) \vee b = x\alpha \vee b \leq b \vee c.$$

Since c is distributive, we have $c = c \vee (x \wedge b) = (c \vee x) \wedge (c \vee b) = c \vee x$. Hence, $x \leq c$. ∎

VI. *Let $x \in L$. Then $x \leq a$ implies $x \leq x\alpha \leq x \vee c$.* ■

VII. *Let $\varphi \in \mathrm{DF}(c, L)$ and let $T \subseteq L$. Then $(\bigvee T)\varphi = \bigvee\{t\varphi : t \in T\}$.*

Proof. In view of the definition of $\mathrm{DF}(c, L)$ it is sufficient to prove that the statement holds for $\varphi = \alpha$. Let $t \in T$. By Property IV, $t \leq t\alpha \vee t\beta$. Then $\bigvee T \leq \bigvee\{t\alpha : t \in T\} \vee b$, and hence $(\bigvee T)\alpha \leq a \wedge (\bigvee\{t\alpha : t \in T\} \vee b)$. Since $\bigvee\{t\alpha : t \in T\} \leq a$, by modularity,

$$a \wedge (\bigvee\{t\alpha : t \in T\} \vee b) = \bigvee\{t\alpha : t \in T\} \vee (a \wedge b) = \bigvee\{t\alpha : t \in T\}.$$

Therefore, $(\bigvee T)\alpha \leq \bigvee\{t\alpha : t \in T\}$. On the other hand, $t\alpha \leq (\bigvee T)\alpha$, and hence $\bigvee\{t\alpha : t \in T\} \leq (\bigvee T)\alpha$. ■

VIII. *If $\varphi \in \mathrm{DF}(c, L)$, then $c\varphi \leq c$.*

Proof. We first observe that $c\alpha \leq c$. Indeed, applying (Sn) we obtain

$$c\alpha = a \wedge (c \vee b) = (a \wedge c) \vee (a \wedge b) \leq c.$$

Now, by the definition of $\mathrm{DF}(c, L)$ we get the assertion. ■

IX. *Let $\varphi \in \mathrm{DF}(c, L)$. Then for any $x \in L$ with $x \leq 1\varphi$ there exists $y \in L$ satisfying $x \leq y\varphi \leq x \vee c$.*

Proof. For $\varphi = \alpha$ the statement follows from Property VI. Now assume the statement to hold for ψ and let $\varphi = \psi\alpha$. Let $x \leq (1\psi)\alpha$ and set $z = (x \vee b) \wedge 1\psi$. By modularity,

$$[1\psi \wedge (x \vee b)] = (1\psi \vee b) \wedge (x \vee b),$$

i.e., $z \vee b = (1\psi \vee b) \wedge (x \vee b)$. Now compute:

$$z\alpha = a \wedge (z \vee b) = (x \vee b) \wedge (1\psi)\alpha = x \vee (b \wedge 1\psi\alpha).$$

From this we obtain $x \leq z\alpha \leq x \vee c$. We have $z \leq 1\psi$, and by the induction hypothesis there is a $y \in L$ such that $z \leq y\psi \leq z \vee c$. Applying Properties VII and VIII we get

$$x \leq z\alpha \leq y\psi\alpha \leq z\alpha \vee c\alpha \leq x \vee c.$$

Then $x \leq y\varphi \leq x \vee c$. ■

Let $\varphi \in \mathrm{DF}(c, L)$. We denote by $\mathrm{k}(\varphi)$ the join of all $x \in L$ such that $x\varphi \leq c$, i.e., $\mathrm{k}(\varphi) = \bigvee\{x \in L : x\varphi \leq c\}$.

By Property VII, we have

$$\mathrm{k}(\varphi)\varphi \leq c. \tag{3}$$

Note that

$$\mathrm{k}(\varphi^n) \leq \mathrm{k}(\varphi^{n+1}) \quad \text{for all } n = 1, 2, \ldots \tag{4}$$

Indeed, $\mathrm{k}(\varphi^n)\varphi^{n+1} = (\mathrm{k}(\varphi^n)\varphi^n)\varphi \leq c\varphi \leq c$ (by Property VIII), and by the definition of $\mathrm{k}(\varphi^{n+1})$ we get (4).

LEMMA 3.3. *Let n be a natural number. If* $\mathrm{k}(\varphi^n) = \mathrm{k}(\varphi^{n+1})$, *then* $1\varphi^n \wedge \mathrm{k}(\varphi^n) \leq c$.

Proof. We prove by induction on i that $\mathrm{k}(\varphi^{n+i}) = \mathrm{k}(\varphi^n)$. This is true for $i = 1$. We suppose that $\mathrm{k}(\varphi^{n+i}) = \mathrm{k}(\varphi^n)$. By (3), we conclude that $[\mathrm{k}(\varphi^{n+i+1})\varphi]\varphi^{n+i} \leq c$. Therefore, $\mathrm{k}(\varphi^{n+i+1})\varphi \leq \mathrm{k}(\varphi^{n+i}) = \mathrm{k}(\varphi^n)$, and hence $\mathrm{k}(\varphi^{n+i+1})\varphi^{n+1} \leq \mathrm{k}(\varphi^n)\varphi^n \leq c$. Thus $\mathrm{k}(\varphi^{n+i+1})\varphi \leq \mathrm{k}(\varphi^{n+1})$. Moreover, by (4),

$$\mathrm{k}(\varphi^{n+1}) \leq \mathrm{k}(\varphi^{n+i+1}),$$

and we deduce that $\mathrm{k}(\varphi^{n+i+1}) = \mathrm{k}(\varphi^{n+1}) = \mathrm{k}(\varphi^n)$. Thus, by induction, we obtain $\mathrm{k}(\varphi^{n+i}) = \mathrm{k}(\varphi^n)$ for all $i = 1, 2, \ldots$ In particular,

$$\mathrm{k}(\varphi^{2n}) = \mathrm{k}(\varphi^n). \tag{5}$$

We put $x = 1\varphi^n \wedge \mathrm{k}(\varphi^n)$. By Property IX it follows that there exists $y \in L$ such that $x \leq y\varphi^n \leq x \vee c$. Applying Properties VII and VIII and inequality (3) we have

$$y\varphi^{2n} \leq (x \vee c)\varphi^n = x\varphi^n \vee c\varphi^n \leq \mathrm{k}(\varphi^n)\varphi^n \vee c\varphi^n \leq c.$$

Hence $y \leq \mathrm{k}(\varphi^{2n})$ and using equality (5) we get $y \leq \mathrm{k}(\varphi^n)$. Therefore, $x \leq y\varphi^n \leq \mathrm{k}(\varphi^n)\varphi^n \leq c$. ■

LEMMA 3.4. *Let* $x_1, x_2 \in L$, $x_1 \geq \mathrm{k}(\varphi)$ *and* $x_2 \geq \mathrm{k}(\varphi)$. *If* $x_1\varphi = x_2\varphi$, *then* $x_1 = x_2$.

Proof. We use induction on the length of φ. Let $\varphi = \alpha$, and suppose that $x_1, x_2 \geq \mathrm{k}(\alpha)$ and $x_1\alpha = x_2\alpha$. Then $[a \wedge (x_1 \vee b)] \vee b = [a \wedge (x_2 \vee b)] \vee b$, and by modularity, we obtain

$$x_1 \vee b = x_2 \vee b. \tag{6}$$

Since $b\alpha = a \wedge b \leq c$, we have $b \leq \mathrm{k}(\alpha)$. Therefore, $x_1 \geq b$ and $x_2 \geq b$. Hence in view of (6) we get $x_1 = x_2$. Thus, for $\varphi = \alpha$, the proof of Lemma 3.4 is complete.

Now assume the statement holds for ψ and let $\varphi = \psi\alpha$. Let $x_1\varphi = x_2\varphi$ ($x_1, x_2 \geq \mathrm{k}(\varphi)$), that is, $a \wedge (x_1\psi \vee b) = a \wedge (x_2\psi \vee b)$. Consequently, $x_1\psi \vee b = x_2\psi \vee b$, and hence

$$1\psi \wedge (x_1\psi \vee b) = 1\psi \wedge (x_2\psi \vee b).$$

Since $x_1\psi \leq 1\psi$ and $x_2\psi \leq 1\psi$, by modularity, we obtain

$$x_1\psi \vee (1\psi \wedge b) = x_2\psi \vee (1\psi \wedge b). \tag{7}$$

We set $x = 1\psi \wedge b$. Then $x \leq 1\psi$, and by Property IX there exists $y \in L$ such that $x \leq y\psi \leq x \vee c$. Therefore, we have

$$y\varphi = y\psi\alpha \leq (x \vee c)\alpha \leq x\alpha \vee c \leq b\alpha \vee c = c.$$

Hence $y\varphi \leq c$, that is, $y \leq \mathrm{k}(\varphi)$. Thus $1\psi \wedge b \leq \mathrm{k}(\varphi)\psi \leq x_1\psi$, and similarly, $1\psi \wedge b \leq x_2\psi$. Hence in view of (7) we obtain $x_1\psi = x_2\psi$. Obviously, we have $\mathrm{k}(\psi) \leq \mathrm{k}(\varphi)$, and therefore $\mathrm{k}(\psi) \leq x_1$, $\mathrm{k}(\psi) \leq x_2$. Applying the induction hypothesis we get $x_1 = x_2$. ■

LEMMA 3.5. *For every* $\varphi \in \mathrm{DF}(c, L)$ *the following conditions are equivalent*:

(i) *There exists a natural number n such that* $1\varphi^n = 1\varphi^{n+1}$ *and* $\mathrm{k}(\varphi^n) = \mathrm{k}(\varphi^{n+1})$.

(ii) *There exists a natural number n such that* $\varphi \in \mathrm{F}(c, 1\varphi^n)$.

Proof. (i)⇒(ii). Since $1\varphi^n = 1\varphi^{n+1}$, we have $(1\varphi^n)\varphi = 1\varphi^n$. Let $x \leq 1\varphi^n$ and $x\varphi \leq c$. Then $x\varphi^n \leq c$, and hence $x \leq \mathrm{k}(\varphi^n)$. Consequently, $x \leq 1\varphi^n \wedge \mathrm{k}(\varphi^n)$, and by Lemma 3.3, $x \leq c$. Therefore, $\varphi \in \mathrm{F}(c, 1\varphi^n)$.

(ii)⇒(i). Suppose that $\varphi \in \mathrm{F}(c, 1\varphi^n)$. This clearly forces

$$1\varphi^n = 1\varphi^{n+1}.$$

Set $x = \mathrm{k}(\varphi^{n+1})\varphi^n$. Then $x \le 1\varphi^n$ and $x\varphi \le c$. Since $\varphi \in \mathrm{F}(c, 1\varphi^n)$, we have $x \le c$. This means that $\mathrm{k}(\varphi^{n+1}) \le \mathrm{k}(\varphi^n)$. We conclude from (4) that $\mathrm{k}(\varphi^n) = \mathrm{k}(\varphi^{n+1})$. ∎

3.3. Distinguished c-decomposition functions. We say that a c-decomposition function φ of L is *distinguished* if $\varphi = \alpha\delta\alpha\varepsilon\alpha$, where $\alpha \in \mathrm{DF}(c, a)$ for some $a \in \mathrm{S}(c, L)$ and $\langle \delta, \varepsilon \rangle$ is a pair of complementary c-decomposition functions of L.

Suppose the unit element of the lattice L has two c-decompositions: (2) and

(8) $$1 = d + e.$$

Let $\langle \alpha, \beta \rangle$ and $\langle \delta, \varepsilon \rangle$ be the pairs of decomposition functions related to the c-decompositions (2) and (8), respectively. Then, for instance, $\alpha\delta\alpha\varepsilon\alpha$, $\beta\delta\beta\varepsilon\beta$ and $\varepsilon\alpha\varepsilon\beta\varepsilon$ are distinguished c-decomposition functions of L.

We first observe that for every $x \in L$,

(9) $$x\delta\alpha\varepsilon = x\delta\beta\varepsilon.$$

Indeed, in view of modularity,

$$\begin{aligned} x\delta\alpha\varepsilon &= e \wedge ([a \wedge (x\delta \vee b)] \vee d) = e \wedge (x\delta \vee [a \wedge (x\delta \vee b)] \vee d) \\ &= e \wedge ([(x\delta \vee a) \wedge (x\delta \vee b)] \vee d) = e \wedge (x\delta \vee [b \wedge (x\delta \vee a)] \vee d) = x\delta\beta\varepsilon. \end{aligned}$$

Similarly,

(10) $$x\alpha\delta\beta = x\alpha\varepsilon\beta, \quad x\beta\varepsilon\alpha = x\beta\delta\alpha, \quad x\varepsilon\beta\delta = x\varepsilon\alpha\delta.$$

Lemma 3.6. *For $x \in L$, $x\alpha\delta\alpha\varepsilon\alpha = x\alpha\varepsilon\alpha\delta\alpha$.*

Proof. Applying (9) and (10) we get

$$x\alpha\delta\alpha\varepsilon\alpha = x\alpha\delta\beta\varepsilon\alpha = x\alpha\varepsilon\beta\varepsilon\alpha = x\alpha\varepsilon\beta\delta\alpha = x\alpha\varepsilon\alpha\delta\alpha. \ ∎$$

We put $\eta = \alpha\delta\alpha\varepsilon\alpha$, $\sigma = \alpha\delta\alpha$, and $\chi = \alpha\varepsilon\alpha$.

Lemma 3.7. *If m is a natural number, then*

$$\mathrm{k}(\sigma^m) \vee \mathrm{k}(\chi^m) \le \mathrm{k}(\eta^m).$$

Proof. By Lemma 3.6 and Property VIII we have

$$\mathrm{k}(\sigma^m)\eta^m = \mathrm{k}(\sigma^m)(\sigma\chi)^m = \mathrm{k}(\sigma^m)\sigma^m\chi^m \le c\chi^m \le c,$$

and hence $\mathrm{k}(\sigma^m) \le \mathrm{k}(\eta^m)$. Similarly, $\mathrm{k}(\chi^m) \le \mathrm{k}(\eta^m)$. ∎

Lemma 3.8. *If $x \le a$ and if m is a natural number, then*

(11) $$x \le \bigvee\{x\sigma^{m-i}\chi^i : i = 1, \ldots, m\}.$$

Proof. We use induction on m. Since $x \le x\alpha$ and $x\alpha \le x\alpha\delta \vee x\alpha\varepsilon$ (by Property IV), we get

(12) $$x \le x\sigma \vee x\chi,$$

that is, (11) holds for $m = 1$. Assume now the assertion to be true for $m-1$. Then (11) can be deduced as follows:

$$\begin{aligned}x &\le \bigvee\{x\sigma^{m-1-i}\chi^i : i = 0,1,\ldots,m-1\}\\ &\le \bigvee\{(x\sigma \vee x\chi)\sigma^{m-1-i}\chi^i : i = 0,1,\ldots,m-1\}\\ &= \bigvee\{x\sigma^{m-i}\chi^i : i = 0,\ldots,m-1\} \vee \bigvee\{x\sigma^{m-1-i}\chi^{i+1} : i = 0,\ldots,m-1\}\\ &= \bigvee\{x\sigma^{m-i}\chi^i : i = 0,1,\ldots,m\}. \ \blacksquare\end{aligned}$$

LEMMA 3.9. *Let m be a natural number. Then*

$$\mathrm{k}(\eta^m)\alpha \le (a \wedge \mathrm{k}(\sigma^m)) \vee (a \wedge \mathrm{k}(\chi^m)).$$

Proof. First we prove the inequality for $m = 1$. From (12) it follows that

$$\mathrm{k}(\eta)\alpha \le \mathrm{k}(\eta)\sigma \vee \mathrm{k}(\eta)\chi.$$

Since $\mathrm{k}(\eta)\sigma \le a$ and $(\mathrm{k}(\eta)\sigma)\chi = \mathrm{k}(\eta)\eta \le c$, we obtain

$$\mathrm{k}(\eta)\sigma \le a \wedge \mathrm{k}(\chi).$$

Similarly, $\mathrm{k}(\eta)\chi \le a \wedge \mathrm{k}(\sigma)$. Then

$$\mathrm{k}(\eta)\alpha \le (a \wedge \mathrm{k}(\sigma)) \vee (a \wedge \mathrm{k}(\chi)).$$

Now assume the statement to hold for $m-1$. By Lemma 3.8,

$$\mathrm{k}(\eta^m)\alpha \le \bigvee\{\mathrm{k}(\eta^m)\sigma^{m-i}\chi^i : i = 0,1,\ldots,m\}. \tag{13}$$

Let $i \in \{1,\ldots,m-1\}$. Then $\mathrm{k}(\eta^m)\sigma^{m-i}\chi^i \le \mathrm{k}(\eta^{m-1})$, and consequently, applying the induction hypothesis and (4) we obtain

$$\begin{aligned}\mathrm{k}(\eta^m)\sigma^{m-i}\chi^i &= \mathrm{k}(\eta^m)\sigma^{m-i}\chi^i\alpha \le \mathrm{k}(\eta^{m-1})\alpha\\ &\le (a \wedge \mathrm{k}(\sigma^{m-1})) \vee (a \wedge \mathrm{k}(\chi^{m-1})) \le (a \wedge \mathrm{k}(\sigma^m)) \vee (a \wedge \mathrm{k}(\chi^m)).\end{aligned}$$

It is easy to see that $\mathrm{k}(\eta^m)\sigma^m \le a \wedge \mathrm{k}(\chi^m)$ and $\mathrm{k}(\eta^m)\chi^m \le a \wedge \mathrm{k}(\sigma^m)$. Therefore, from (13) we conclude that

$$\mathrm{k}(\eta^m)\alpha \le (a \wedge \mathrm{k}(\sigma^m)) \vee (a \wedge \mathrm{k}(\chi^m)). \ \blacksquare$$

LEMMA 3.10. *For any natural numbers m and n we have*

$$a \wedge \mathrm{k}(\sigma^m) \wedge \mathrm{k}(\chi^n) \le c.$$

Proof. We proceed by induction on n. We set

$$x = a \wedge \mathrm{k}(\sigma^m) \wedge \mathrm{k}(\chi).$$

Using Lemmas 3.8 and 3.6 we get

$$x \le \bigvee\{x\sigma^{m-i}\chi^i : i = 0,\ldots,m\} = x\sigma^m \vee \bigvee\{x\chi^i\sigma^{m-i} : i = 1,\ldots,m\}.$$

From this, applying inequality (3) and Property VIII we deduce that $x \le c$. Suppose that $a \wedge \mathrm{k}(\sigma^m) \wedge (\chi^{n-1}) \le c$ and put $y = a \wedge \mathrm{k}(\sigma^m) \wedge \mathrm{k}(\chi^n)$. By Lemma 3.8,

$$y \le y\sigma^m \vee \bigvee\{y\sigma^{m-i}\chi^i : i = 1,\ldots,m\}. \tag{14}$$

Obviously, $\mathrm{k}(\sigma^m)\chi \le \mathrm{k}(\sigma^m)$ and $\mathrm{k}(\chi^n)\chi \le \mathrm{k}(\chi^{n-1})$. Then

$$y\chi \le a \wedge \mathrm{k}(\sigma^m) \wedge \mathrm{k}(\chi^{n-1}) \le c.$$

From this and (14) it follows that $y \le c$. $\blacksquare$

From Lemmas 3.7, 3.9 and 3.10 we have

LEMMA 3.11. *For any natural number* m, *we have*

$$a \wedge \mathrm{k}(\eta^m) = (a \wedge \mathrm{k}(\sigma^m)) + (a \wedge \mathrm{k}(\chi^m)).$$

3.4. B_c**-lattices.** In Walendziak [1986] (p. 350) we gave the following

DEFINITION 3.12. Let $a \in \mathrm{S}(c, L)$. We say that a satisfies the B_c-*condition* in the lattice L if for every $\alpha \in \mathrm{DF}(c, a)$ and for every pair $\langle\delta, \varepsilon\rangle$ of complementary c-decomposition functions of L, either $\alpha\delta\alpha \in \mathrm{F}(c, a)$ or $\alpha\varepsilon\alpha \in \mathrm{F}(c, a)$. If every c-irreducible c-summand of the unit element of L satisfies the B_c-condition, then we call L a B_c-*lattice*.

PROPOSITION 3.13. *Let a be a c-irreducible c-summand of* 1, $\alpha \in \mathrm{DF}(c, a)$ *and let* $\langle\delta, \varepsilon\rangle$ *be a pair of complementary c-decomposition functions of L. Put $\eta = \alpha\delta\alpha\varepsilon\alpha$. If condition* (i) (*or equivalently* (ii)) *of Lemma* 3.5 *holds for $\varphi = \eta$, then a satisfies the* B_c*-condition* (*in L*).

Proof. Suppose that $\mathrm{k}(\eta^m) = \mathrm{k}(\eta^{m+1})$ and $1\eta^{m+1} = 1\eta^m$ for some $m \in \mathbb{N}$. By Property VII,

$$(1\eta^m \vee \mathrm{k}(\eta^m))\eta^m = 1\eta^{2m} \vee \mathrm{k}(\eta^m)\eta^m = 1\eta^m.$$

From Lemma 3.4 we obtain $1 = 1\eta^m \vee \mathrm{k}(\eta^m)$. Hence, by modularity,

$$a = 1\eta^m \vee (a \wedge \mathrm{k}(\eta^m)).$$

According to Lemma 3.3 we have

$$a = 1\eta^m + (a \wedge \mathrm{k}(\eta^m)).$$

From Lemma 3.11 and Property II we conclude that

$$a = 1\eta^m + (a \wedge \mathrm{k}(\eta^m)) + (a \wedge \mathrm{k}(\chi^m)),$$

where $\sigma = \alpha\delta\alpha$, and $\chi = \alpha\varepsilon\alpha$. We shall consider three cases.

CASE 1: $a = 1\eta^m$. We deduce from Lemma 3.6 that $a = 1\chi^m\sigma^m$, hence $a \le a\sigma \le a$, and finally $a\sigma = a$. Suppose now that $x \le a$ and $x\sigma \le c$. By Property VIII, $x\eta^m \le c$, and therefore $x \le \mathrm{k}(\eta^m)$. Then $x \le 1\eta^m \wedge \mathrm{k}(\eta^m)$, and hence in view of Lemma 3.3 we obtain $x \le c$. Thus $\alpha\delta\alpha = \sigma \in \mathrm{F}(c, a)$.

CASE 2: $a \le \mathrm{k}(\sigma^m)$. By Lemma 3.8, $a \le a\sigma^m \vee a\chi$. But $a\sigma^m \le c$ since $a \le \mathrm{k}(\sigma^m)$. Therefore $a = a\sigma^m + a\chi$. The element a is c-irreducible, and so $a = a\sigma^m$ or $a = a\chi$. If $a = a\sigma^m$, then $\sigma \in \mathrm{F}(c, a)$ by the proof of Case 1. Assume that $a = a\chi$. Let $x \le a$ and $x\chi \le c$. From Lemmas 3.8 and 3.6 it follows that

$$x \le x\sigma^m \vee \bigvee\{x\chi^i\sigma^{m-i} : i = 1, \ldots, m\}.$$

Hence $x \le x\sigma^m \vee c$. But $x\sigma^m \le a\sigma^m \le \mathrm{k}(\sigma^m)\sigma^m \le c$. Then $x \le c$. Thus $\alpha\varepsilon\alpha = \chi \in \mathrm{F}(c, a)$.

CASE 3: $a \le \mathrm{k}(\chi^m)$. In this case, the proof is similar.

Now, we conclude from Definition 3.12 that a satisfies the B_c-condition. ∎

An immediate consequence of Proposition 3.13 is

THEOREM 3.14. *Suppose that every distinguished c-decomposition function φ of L satisfies condition* (i) (*or equivalently* (ii)) *of Lemma* 3.5. *Then L is a* B_c*-lattice.*

PROPOSITION 3.15. *Let a be a c-irreducible c-summand of* 1. *If for every $\alpha \in \mathrm{DF}(c,a)$ and for every pair $\langle\delta,\varepsilon\rangle$ of complementary c-decomposition functions of L the sublattice $[0, 1\alpha\delta\alpha\varepsilon\alpha]$ is of finite length, then a satisfies the* B_c*-condition.*

Proof. Let $\eta = \alpha\delta\alpha\varepsilon\alpha$. It is obvious that

$$1\eta \wedge \mathrm{k}(\eta) \leq 1\eta \wedge \mathrm{k}(\eta^2) \leq \ldots \leq 1\eta \wedge \mathrm{k}(\eta^i) \leq \ldots \leq 1\eta$$

and

$$1\eta \geq 1\eta^2 \geq \ldots \geq 1\eta^i \geq \ldots$$

Since $[0, 1\eta]$ is of finite length, there is a natural number m such that $1\eta^m = 1\eta^{m+1}$ and $1\eta \wedge \mathrm{k}(\eta^m) = 1\eta \wedge \mathrm{k}(\eta^{m+1})$. Then

$$(1\eta^m)\eta = 1\eta^m.$$

Let $x \leq 1\eta^m$ and $x\eta \leq c$. We have $x \leq 1\eta^m$, and by Property IX we deduce that there exists $y \in L$ such that $x \leq y\eta^{m+1} \leq x \vee c$. Hence $y\eta^{m+2} \leq x\eta \vee c\eta \leq c$, and therefore, $y\eta \leq 1\eta \wedge \mathrm{k}(\eta^{m+1}) = 1\eta \wedge \mathrm{k}(\eta^m) \leq \mathrm{k}(\eta^m)$. Consequently,

$$x \leq y\eta^{m+1} = (y\eta)\eta^m \leq \mathrm{k}(\eta^m)\eta^m \leq c.$$

Thus $\eta \in \mathrm{F}(c, 1\eta^m)$. From Proposition 3.13 it follows that a satisfies the B_c-condition. ∎

Proposition 3.15 gives

PROPOSITION 3.16. *Let a be a c-irreducible c-summand of* 1 *such that the sublattice $[0,a]$ is of finite length. Then a satisfies the* B_c*-condition.*

Hence we have

PROPOSITION 3.17. *Every modular lattice of finite length is a* B_c*-lattice, where c is a distributive element of this lattice.*

PROPOSITION 3.18. *Every complete modular lattice is a* B_1*-lattice.*

Proof. Let L be a complete modular lattice. Let $\alpha \in \mathrm{S}(1, L)$ and suppose that a is join irreducible. It is sufficient to show that a satisfies the B_1-condition. Let b be an element of L such that $1 = a \vee b$, i.e.,

$$1 = a +_1 b.$$

Let α, β be the decomposition functions related to this 1-decomposition of 1, and let $\langle\delta,\varepsilon\rangle$ be a pair of complementary 1-decomposition functions of L. From Property IV we have $a \leq a\delta \vee a\varepsilon$. Hence $(b \vee a\delta) \vee (b \vee a\varepsilon) \geq b \vee a = 1$, that is,

$$1 = (b \vee a\delta) \vee (b \vee a\varepsilon).$$

By the weak isomorphism property (see Chapter 0) the lattices $[b, 1]$ and $[a \wedge b, a]$ are isomorphic. But a is join irreducible in L, and therefore in $[a \wedge b, a]$, thus 1 is join irreducible in $[b, 1]$. Hence $1 = b \vee a\delta$ or $1 = b \vee a\varepsilon$.

If $1 = b \vee a\delta$, then $a\alpha\delta\alpha = a\delta\alpha = a \wedge (a\delta \vee b) = a$, and therefore $\alpha\delta\alpha \in \mathrm{F}(1, a)$. Similarly, if $1 = b \vee a\varepsilon$, then $\alpha\varepsilon\alpha \in \mathrm{F}(1, a)$. Thus a satisfies the B_1-condition. ∎

Let G be a group. By $\mathrm{L}(G)$ we denote the lattice of all normal subgroups of G. We say that G is of *finite length for normal subgroups* if the lattice $\mathrm{L}(G)$ is of finite length.

If $G = A_1 \times \ldots \times A_n$ is the direct product (direct sum) of groups $A_1, \ldots, A_n$, then $G = A_1 \dot{\vee} \ldots \dot{\vee} A_n$ in $L(G)$.

Lemma 3.19. *Let G be a group, and let*

$$G = A \dot{\vee} B = D \dot{\vee} E. \tag{15}$$

Let $\langle \alpha, \beta \rangle$ and $\langle \delta, \varepsilon \rangle$ be the pairs of decomposition functions related to (15). *Then $G\alpha\delta\alpha\varepsilon\alpha \subseteq \mathrm{Z}(G)$, where $\mathrm{Z}(G)$ denotes the center of G.*

Proof. By Lemma 3.6,

$$G\alpha\delta\alpha\varepsilon\alpha = G\alpha\delta\beta\varepsilon\alpha.$$

We observe that an arbitrary element of A is permutable with every element of $B'\varepsilon$, where $B' = G\alpha\delta\beta$. Indeed, let $x \in A$ and $y \in B'\varepsilon = E \cap B' \cdot D$. Clearly, $y = b' \cdot d$, where $b' \in B'$ and $d \in D$. We know that every element of G can be written uniquely as a product of an element of D and another element of E. Let $x = d_1 \cdot e$, where $d_1 \in D$, $e \in E$. We compute:

$$\begin{aligned} d_1 \cdot (e \cdot y) &= x \cdot b' \cdot d = b' \cdot x \cdot d = y \cdot d^{-1} \cdot x \cdot d \\ &= y \cdot d^{-1} \cdot d_1 \cdot e \cdot d = (d^{-1} \cdot d_1 \cdot d) \cdot (y \cdot e). \end{aligned}$$

So, by the uniqueness of the decomposition, we conclude that $e \cdot y = y \cdot e$. Then

$$x \cdot y = d_1 \cdot e \cdot y = d_1 \cdot y \cdot e = y \cdot d_1 \cdot e = y \cdot x.$$

Now, it is easy to see that an arbitrary element of A is permutable with every element of $G\alpha\delta\beta\varepsilon\alpha$. Therefore, if $g = a \cdot b$ ($a \in A$, $b \in B$) and if $h \in A\alpha\delta\beta\varepsilon\alpha$, then

$$g \cdot h = a \cdot b \cdot h = a \cdot h \cdot b = h \cdot a \cdot b = h \cdot g.$$

Thus, $G\alpha\delta\alpha\varepsilon\alpha \subseteq \mathrm{Z}(G)$. ■

Proposition 3.20. *Let G be a group. If the center $Z(G)$ of G is of finite length for normal subgroups, then* $\mathrm{L}(G)$ *is a* $\mathrm{B}_{\mathcal{E}}$*-lattice* ($\mathcal{E}$ *is a trivial subgroup of* G).

Proof. Follows from Lemma 3.19 and Proposition 3.15. ■

We now give an example of a complete modular lattice which is not a B_0-lattice.

Example 3.21. Let $\mathbb{Z}$ denote the additive group of integers, and let G be the direct product of two copies of $\mathbb{Z}$. Then $\mathrm{L}(G)$ is a complete modular lattice. Set

$$\begin{aligned} A &= \{(m,0) : m \in \mathbb{Z}\}, & B &= \{(0,m) : m \in \mathbb{Z}\}, \\ D &= \{(m,2m) : m \in \mathbb{Z}\}, & E &= \{(m,3m) : m \in \mathbb{Z}\}. \end{aligned}$$

It is obvious that $A, B, D, E \in \mathrm{L}(G)$. We see at once that two direct decompositions (15) hold. Observe that A is directly join irreducible in $\mathrm{L}(G)$. Indeed, let

$$A = A_1 \dot{\vee} A_2 \quad (A_1, A_2 \neq \{(0,0)\}).$$

Clearly, $A_1 = \{(ma_1, 0) : m \in \mathbb{Z}\}$ for some $a_1 \in \mathbb{Z} - \{0\}$ and $A_2 = \{(ma_2, 0) : m \in \mathbb{Z}\}$ for some $a_2 \in \mathbb{Z} - \{0\}$. We have $(a_1 a_2, 0) \in A_1 \cap A_2 = \{(0,0)\}$, a contradiction. Let $\langle \alpha, \beta \rangle$ and $\langle \delta, \varepsilon \rangle$ be the pairs of decomposition functions related to the direct decompositions (15) of G. We want to verify that A does not satisfy the B_0-condition. It is sufficient to show that $A\alpha\delta\alpha \neq A$ and $A\alpha\varepsilon\alpha \neq A$. Compute:

$$\begin{aligned} A\alpha\delta\alpha &= A\delta\alpha = (D \cap (A \vee E))\alpha = (D \cap \{(m, 3n) : m, n \in \mathbb{Z}\})\alpha \\ &= A \cap (\{(3m, 6m) : m \in \mathbb{Z}\} \vee B) = A \cap \{(3m, n) : m, n \in \mathbb{Z}\} \\ &= \{(3m, 0) : m \in \mathbb{Z}\} \neq A. \end{aligned}$$

Similarly, $A\alpha\varepsilon\alpha = \{(2m, 0) : m \in \mathbb{Z}\} \neq A$. Therefore $\mathrm{L}(G)$ is not a B_0-lattice.

Let (2) and (8) be two c-decompositions of the unit element of L, and let $\langle \alpha, \beta \rangle$ and $\langle \delta, \varepsilon \rangle$ be the corresponding pairs of decomposition functions. Now, we will prove the following

Lemma 3.22. *The following conditions are equivalent*:

(i) $\alpha\delta\alpha \in \mathrm{F}(c, a)$.
(ii) $1 = d \wedge (a + e) + b$.
(iii) $1 = b \wedge (a + e) + d$.

Proof. (i)$\Rightarrow$(ii). Let $\alpha\delta\alpha \in \mathrm{F}(c, a)$. Then $a\alpha\delta\alpha = a$. Hence $a \wedge (a\delta \vee b) = a$. Thus $[a \wedge (a\delta \vee b)] \vee b = 1$. Since $a\delta \vee b \geq b$, by modularity, $[a \wedge (a\delta \vee b)] \vee b = (a \vee b) \wedge (a\delta \vee b)$, and so

$$1 = a\delta \vee b.$$

We will prove that $x = a\delta \wedge b \leq c$. We have $x \leq 1\alpha\delta$. By Property IX, there exists $y \in L$ such that $x \leq y\alpha\delta \leq x \vee c$. Hence using Properties VII and VIII we obtain

$$(y\alpha)\alpha\delta\alpha = y\alpha\delta\alpha \leq x\alpha \vee c\alpha = (a \wedge b) \vee c\alpha \leq c.$$

Hence we infer that $y\alpha \leq c$. Then $x \leq y\alpha\delta \leq c\delta \leq c$, and therefore

$$1 = a\delta + b.$$

We now prove that $a \wedge e \leq c$. Applying Properties VII and VIII we have

$$(a \wedge e)\alpha\delta\alpha \leq (e \vee c)\delta\alpha = e\delta\alpha \vee c\delta\alpha \leq c.$$

Then $a \wedge e \leq c$, by the definition of $\mathrm{F}(c, a)$.

(ii)$\Rightarrow$(iii). From (ii) we deduce that

$$a + e = d \wedge (a + e) + b \wedge (a + e). \tag{16}$$

This gives (iii).

(iii)$\Rightarrow$(i). Let (iii) hold. By modularity, we obtain (16), and hence (ii) is satisfied. Now, it is easy to see that

$$a\alpha\delta\alpha = a.$$

Suppose that $x \leq a$ and $x\alpha\delta\alpha \leq c$. By Property VI, we get $x \leq x\alpha$. Consequently, $x\delta\alpha \leq c$. Moreover, $x\delta \wedge b \leq a\delta \wedge b \leq c$, and therefore $x\delta \leq c$ by Property V. We have $x \wedge e \leq c$, because $x \wedge e \leq a \wedge e \leq c$. Then using Property V we obtain $x \leq c$. Thus, $\alpha\delta\alpha \in \mathrm{F}(c, a)$. ∎

Lemma 3.23. *Let* (2) *and*

$$1 = d_1 + \ldots + d_n \tag{17}$$

be two c-decompositions of 1. *Let* α, β *and* δ_i, $i = 1, \ldots, n$, *be the related decomposition functions. If* a *satisfies the* B_c*-condition, then there exists* $i \in \{1, \ldots, n\}$ *such that* $\alpha\delta_i\alpha \in \mathrm{F}(c, a)$.

Proof. Suppose that $b < 1$ (if $b = 1$, then obviously $\alpha\delta_i\alpha \in \mathrm{F}(c, a)$ for each $i \in \{1, \ldots, n\}$). Let $n = 2$. Then the assertion follows from Definition 3.12. Assume it holds for $n - 1$ and let $\alpha\delta_i\alpha \notin \mathrm{F}(c, a)$. Set $d = d_2 + \ldots + d_n$. Clearly,

$$1 = d_1 + d. \tag{18}$$

We denote by δ_1, δ the decomposition functions related to (18). Since a satisfies the B_c-condition and $\alpha\delta_1\alpha \notin \mathrm{F}(c, a)$, it follows that $\alpha\delta\alpha \in \mathrm{F}(c, a)$. From Lemma 3.22 we conclude that $1 = b \wedge (a + d_1) + d$. Therefore

$$1 = d_2 + d', \tag{19}$$

where $d' = d_1\beta + d_3 + \ldots + d_n$. Let δ_2', δ' be the decomposition functions related to (19). We shall consider two cases.

Case 1: $\alpha\delta_2'\alpha \in \mathrm{F}(c, a)$. Lemma 3.22 now implies $1 = d_2 \wedge (a + d') + b$. Since

$$a + d_1\beta = a + b \wedge (a + d_1) = (a + b) \wedge (a + d_1) = a + d_1$$

we obtain

$$a + d' = a + d_1 + d_3 + \ldots + d_n.$$

Then $1 = d_2 \wedge (a + d_1 + d_3 + \ldots + d_n) + b$, and, by Lemma 3.22, $\alpha\delta_2\alpha \in \mathrm{F}(c, a)$.

Case 2: $\alpha\delta'\alpha \in \mathrm{F}(c, a)$. Applying Lemma 3.22 to the c-decompositions (2) and (19) we deduce that $1 = b \wedge (a + d_2) + d'$. Thus

$$1 = d_3 + \ldots + d_n + b', \tag{20}$$

where $b' = d_1\beta + d_2\beta$. We denote by δ_i', β', $i = 3, \ldots, n$, the decomposition functions related to (20). Observe that $\alpha\beta'\alpha \notin \mathrm{F}(c, a)$. Indeed, suppose on the contrary that $a = a\alpha\beta'\alpha$. Then $a = a \wedge (a\alpha\beta' \vee b) = a \wedge b$ (since $a\alpha\beta' \le b$). Hence $1 = a \vee b = b < 1$, a contradiction. By the induction hypothesis, there exists i, $3 \le i \le n$, such that $\alpha\delta_i'\alpha \in \mathrm{F}(c, a)$. Let, for example $i = n$. Applying Lemma 3.22 to (20), we conclude that

$$1 = d_n \wedge (a + b' + d_3 + \ldots + d_{n-1}) + b.$$

We have $a + d_1\beta = a + d_1$, and similarly, $a + d_2\beta = a + d_2$. Then $a + b' = a + d_1 + d_2$, and hence

$$1 = d_n \wedge (a + d_1 + d_2 + \ldots + d_{n-1}) + b.$$

Therefore, by Lemma 3.22, $\alpha\delta_n\alpha \in \mathrm{F}(c, a)$. ■

In the proof of this lemma we applied Properties I and II several times.

3.5. Finite c-decompositions. Let L be a complete modular lattice. Recall that a subset T of L is called *join irredundant* if $\bigvee T > \bigvee (T - \{t\})$ for each $t \in T$. If an element $a \in L$ is a c-join of T, where T is a join irredundant subset of L, then we write

$$a = \dot{\sum}_c T \tag{21}$$

and we say that (21) is an *irredundant c-decomposition* of a. A join irredundant subset T is called *c-independent* if for each $t \in T$, $t \wedge \bigvee(T - \{t\}) \le c$. It is obvious that

$$a = \dot{\sum}_c T \quad \text{iff} \quad a = \bigvee T \text{ and } T \text{ is } c\text{-independent}.$$

For simplicity of notation, we sometimes write $\dot{\sum} T$ instead of $\dot{\sum}_c T$. If $T = \{t_1, \ldots, t_n\}$, then we can write (21) in the form $a = t_1 \dot{+}_c \ldots \dot{+}_c t_n$ (or briefly, $a = t_1 + \ldots + t_n$).

As a preparation for the next result we need the following

LEMMA 3.24. *If*

$$1 = a \dot{+} b = d \dot{+} e, \tag{22}$$

where d is c-irreducible, and if α, β and δ, ε are the decomposition functions with respect to the c-summands a, b and d, e of the c-decompositions (22), *then $\alpha\delta\alpha \in \mathrm{F}(c, a)$ iff $1 = d \dot{+} b = a \dot{+} e$.*

Proof. Necessity. From Lemma 3.22 we conclude that $1 = a\delta + b$. Then $d = d \wedge (a\delta \vee b)$, and hence, by modularity, $d = a\delta \vee (d \wedge b)$. Moreover, $a\delta \wedge (d \wedge b) \le a\delta \wedge b \le c$, and therefore $d = a\delta + d \wedge b$. We have $d \ne d \wedge b$, since otherwise $1 = a\delta \vee b \le d \vee b = b$, a contradiction. Since d is c-irreducible, we obtain $d = a\delta$. Consequently, $1 = d + b$ and hence $1 = d \dot{+} b$. We now show that $1 = a \dot{+} e$. We have

$$1 = d + e = a\delta + e = d \wedge (a \vee e) + e = (a \vee e) \wedge (d \vee e) = a \vee e.$$

Furthermore, by Lemma 3.22, $a \wedge e \le c$. Therefore, $1 = a \dot{+} e$.

Sufficiency. Let $1 = d \dot{+} b = a \dot{+} e$. Hence, $1 = d + b = d \wedge (a + e) + b$, and by Lemma 3.22, we deduce that $\alpha\delta\alpha \in \mathrm{F}(c, a)$. ∎

The next theorem is the principal result of this section.

THEOREM 3.25. *Let L be a complete modular lattice, and let c be a distributive element of L. If the unit element of L has two irredundant finite c-decompositions*

$$1 = a_1 \dot{+} a_2 \dot{+} \ldots \dot{+} a_m \tag{23}$$

and

$$1 = b_1 \dot{+} b_2 \dot{+} \ldots \dot{+} b_n \tag{24}$$

into c-irreducible elements a_i, b_j satisfying the B_c-condition, then $m = n$ and for every a_i there exists b_j such that we have the c-decomposition

$$1 = a_1 \dot{+} \ldots \dot{+} a_{i-1} \dot{+} b_j \dot{+} a_{i+1} \dot{+} \ldots \dot{+} a_n. \tag{25}$$

Proof. Let α_i $(i = 1, \ldots, m)$, β_j $(j = 1, \ldots, n)$ be the decomposition functions related to (23) and (24), respectively. By Property I,

$$1 = a_1 \dot{+} \overline{a}_1. \tag{26}$$

Applying Lemma 3.23 to (26) and (24), we conclude that there exists $j \in \{1, \ldots, n\}$ such that $\alpha_1\beta_j\alpha_1 \in \mathrm{F}(c, a_1)$. Let for example $j = 1$. Then, by Lemma 3.24,

$$1 = b_1 \dot{+} \overline{a}_1 = a_1 \dot{+} \overline{b}_1. \tag{27}$$

Observe that the set $\{b_1, a_2, \ldots, a_m\}$ is join irredundant. Indeed, if for instance $a_2 \le b_1 \vee \overline{a}_{1,2}$ (where $\overline{a}_{1,2} = a_3 \vee \ldots \vee a_m$), then $1 = b_1 + \overline{a}_{1,2}$. By Lemmas 3.23 and 3.24, there exists an $i \in \{1, \dots, m\}$ such that $1 = a_i + \overline{a}_{1,2}$. This means that the set $\{a_1, a_2, \dots, a_m\}$ is not join irredundant, contrary to our assumptions. Therefore the set $\{b_1, a_2, \dots, a_m\}$ is join irredundant, and with (27) we have

$$1 = b_1 \dot{+} a_2 \dot{+} \dots \dot{+} a_m,$$

proving the first statement.

Repeating this we eventually obtain $1 = b_{j_1} \dot{+} \dots \dot{+} b_{j_m}$, and so $\{j_1, \dots, j_m\} = \{1, \dots, n\}$. This shows that $m = n$. ■

Definition 3.26. Let

$$1 = a_1 \dot{+}_c \dots \dot{+}_c a_m = b_1 \dot{+}_c \dots \dot{+}_c b_n.$$

We say that these c-decompositions are *exchange isomorphic* if $m = n$ and there is a permutation λ of the set $I = \{1, \dots, m\}$ such that

$$1 = a_1 \dot{+}_c \dots \dot{+}_c a_{i-1} \dot{+}_c b_{\lambda(i)} \dot{+}_c a_{i+1} \dot{+}_c \dots \dot{+}_c a_m,$$

for all $i \in I$.

Corollary 3.27. *Let $c \in \mathrm{D}(L)$. If L is a B_c-lattice, then any two irredundant finite c-decompositions of* 1 *with c-irreducible summands are exchange isomorphic.*

Remark 3.28. The case $c = 0$ yields the Theorem of Ore (cf. Theorem 3.1) since, by Proposition 3.17, every modular lattice of finite length is a B_0-lattice. For $c = 1$ we get the Kurosh–Ore Theorem. (Indeed, Proposition 3.18 shows that every complete modular lattice is a B_1-lattice.)

Proposition 3.20 and Corollary 3.27 together yield

Corollary 3.29. *Let G be a group such that the center $Z(G)$ is of finite length for subgroups. If*

$$G = G_1 \times \dots \times G_m = H_1 \times \dots \times H_n,$$

where G_i and H_j $(i = 1, \dots, m;\ j = 1, \dots, n)$ are directly indecomposable, then $m = n$ and, after renumbering, $G_i \cong H_i$ for $1 \le i \le n$.

Combining Theorem 3.14 with Corollary 3.27 we get

Corollary 3.30. *If for every distinguished c-decomposition function φ of L, condition* (i) (*or equivalently* (ii)) *of Lemma* 3.5 *is satisfied, then any two irredundant finite c-decompositions of* 1 *with c-irreducible summands are exchange isomorphic.*

Remark 3.31. For $c = 0$, from Corollary 3.30 we obtain Theorem 5 (for direct decompositions with directly join irreducible summands) of Walendziak [1991b]. From this corollary we also get Theorems 11 and 12 of Močulskiĭ [1955].

By Theorem 3.25 and Proposition 3.16 we have

Corollary 3.32. *Let two irredundant c-decompositions* (23) *and* (24) *of* 1 *be given. If each $[0, a_i]$ and each $[0, b_j]$ is of finite length and a_i, b_j are c-irreducible, then $m = n$ and for every a_i there is b_j such that we have the c-decomposition* (25).

Hence we obtain

Corollary 3.33 (Schmidt [1970]; see also Walendziak [1986]). *Let L be a modular lattice of finite length. If* (23) *and* (24) *are two irredundant c-decompositions of* 1 *with c-irreducible summands*, *then $m = n$ and for every a_i there exists b_j such that we have the c-decomposition* (25).

3.6. Property (B_c^*)**. Preliminary lemmas.** Recall that $\mathrm{K}(L)$ denotes the set of all compact elements of L.

Definition 3.34 (see Walendziak [1989], Definition 2). Let $a \in \mathrm{S}(c, L)$, $\alpha \in \mathrm{DF}(c, a)$, and let

$$1 = \textstyle\sum_c \{d_i : i \in I\} \tag{28}$$

be an arbitrary c-decomposition of 1. We denote by δ_i, $i \in I$, the decomposition functions related to (28). If $a \in \mathrm{K}(L)$ and if there exists $i \in I$ such that $\alpha\delta_i\alpha \in \mathrm{F}(c, a)$, then we say that a satisfies the B_c^*-*condition* (in L).

It is easy to see that if $a \in \mathrm{S}(c, L)$ satisfies the B_c^*-condition, then a also satisfies the B_c-condition.

We first prove

Lemma 3.35. *Let $a \in \mathrm{K}(L)$ and let a be a c-irreducible c-summand of* 1*. If $[0, a]$ is of finite length*, *then a satisfies the* B_c^**-condition.*

Proof. Let $\alpha \in \mathrm{DF}(c, a)$ and let b be a c-complement of a such that $x\alpha = a \wedge (x \vee b)$ for all $x \in L$. Let (28) be an arbitrary c-decomposition of 1, and denote by δ_i $(i \in I)$ the related decomposition functions. Since a is compact, there is a finite subset $I_1 \subseteq I$ such that $a \le \bigvee\{d_i : i \in I_1\}$. By Property III, $a \le \bigvee\{a\delta_i : i \in I_1\}$. We put $s = c \wedge d$. Set $d = \bigvee\{a\delta_i : i \in I_1\}$. Then $a \le d$. Observe that s is a distributive element in $[0, d]$. Let $x, y \in [0, d]$. Compute:

$$\begin{aligned}(c \wedge d) \vee (x \wedge y) &= [c \vee (x \wedge y)] \wedge d = [(c \vee x) \wedge (c \vee y)] \wedge d \\ &= (c \vee x) \wedge d \wedge (c \vee y) \wedge d = [(c \wedge d) \vee x] \wedge [(c \wedge d) \vee y].\end{aligned}$$

Then $s \in \mathrm{D}([0, d])$. Obviously

$$d = \textstyle\sum_s \{a\delta_i : i \in I_1\}. \tag{29}$$

Since $1 = a \vee b$, we have $d = d \wedge (a \vee b)$ and hence, by modularity, we obtain $d = a \vee (b \wedge d)$. Clearly, $a \wedge b \wedge d \le c \wedge d = s$. Therefore,

$$d = a +_s (b \wedge d). \tag{30}$$

Let δ_i', $i \in I$, and α', β' be the decomposition functions related to (29) and (30), respectively. It is easily seen that a is s-irreducible in $[0, d]$. Since $[0, a]$ is of finite length, from Proposition 3.16 we conclude that a satisfies the B_s-condition in $[0, d]$. Applying Lemma 3.23 to the s-decompositions (30) and (29) we deduce that there exists $i \in I_1$ such that $\alpha'\delta_i'\alpha' \in \mathrm{F}(s, a)$. Then $a\alpha'\delta_i'\alpha' = a$, and therefore $a \wedge [a\delta_i' \vee (b \wedge d)] = a$. It follows that $a \le a\delta_i' \vee (b \wedge d)$. From this, since $a\delta_i' \le a\delta_i$, we have $a \le a\delta_i \vee b$. Consequently, $1 = a \vee b \le a\delta_i' \vee b$, and hence $1 = a\delta_i \vee b = a\alpha\delta_i \vee b$. Then

$$a = a\alpha\delta_i\alpha. \tag{31}$$

Suppose now that $x \leq a$ and $x\alpha\delta_i\alpha \leq c$. Since $x\alpha\delta_i'\alpha' \leq x\alpha\delta_i\alpha$, we obtain $x\alpha'\delta_i'\alpha' \leq c$. Moreover, $x\alpha'\delta_i'\alpha' \leq a \leq d$. Thus $x\alpha'\delta_i'\alpha' \leq c \wedge d = s$. Therefore, since $x \leq a$ and $\alpha'\delta_i'\alpha' \in \mathrm{F}(s,a)$, we get $x \leq s$. Hence $x \leq c$. From this and (31) we conclude that $\alpha\delta_i\alpha \in \mathrm{F}(c,a)$. ■

Lemma 3.36. *Let a be a 1-summand of the unit element of L. If a is join irreducible and compact, then a satisfies the B_1^*-condition.*

Proof. Let $\alpha \in \mathrm{DF}(1,a)$, and let b be a complement of a such that $x\alpha = a \wedge (x \vee b)$ for every $x \in L$. Let

$$1 = \bigvee\{d_i : i \in I\},$$

and denote by δ_i $(i \in I)$ the related decomposition functions. Since a is compact, there is a finite subset $I_1 \subseteq I$ such that $a \leq \bigvee\{d_i : i \in I_1\}$. By Property III, we have

$$a \leq \bigvee\{a\delta_i : i \in I_1\}.$$

Hence $1 = a \vee b \leq \bigvee\{a\delta_i \vee b : i \in I_1\}$, that is,

$$1 = \bigvee\{a\delta_i \vee b : i \in I_1\}. \tag{32}$$

By the weak isomorphism property, the lattices $[b,1]$ and $[a \wedge b, b]$ are isomorphic. But a is join irreducible in L, and therefore, in $[a \wedge b, b]$. Thus $1 \in \mathrm{J}([b,1])$. Then from (32) we conclude that there exists $i \in I_1$ such that $1 = a\delta_i \vee b$. Hence

$$a\alpha\delta_i\alpha = a\delta_i\alpha = a \wedge (a\delta_i \vee b) = a,$$

and consequently, $\alpha\delta_i\alpha \in \mathrm{F}(1,a)$. Therefore, a satisfies the B_1^*-condition. ■

Lemma 3.37. *Let*

$$1 = a \dot{+} b = d \dot{+} e. \tag{33}$$

If the elements b and e are comparable, d is c-irreducible and a satisfies the B_c-condition, then

$$1 = d \dot{+} b = a \dot{+} e. \tag{34}$$

Proof. Let $\langle\alpha,\beta\rangle$ and $\langle\delta,\varepsilon\rangle$ be the pairs of decomposition functions related to the c-decompositions (33). Suppose that $b \leq e$. If $e \leq b$, then the proof is similar. Observe that $\alpha\varepsilon\alpha \notin \mathrm{F}(c,a)$. Indeed, suppose on the contrary that $a = a\alpha\varepsilon\alpha$. Then

$$a = a \wedge (a\alpha\varepsilon \vee b) \leq a \wedge e \quad \text{(since } b \leq e\text{)}.$$

Hence $a \leq e$, a contradiction. Since a satisfies the B_c-condition and $\alpha\varepsilon\alpha \notin \mathrm{F}(c,a)$, we deduce that $\alpha\delta\alpha \in \mathrm{F}(c,a)$. Therefore, by Lemma 3.24, we obtain (34). ■

Lemma 3.38. *Suppose the unit element of L has two irredundant c-decompo-sitions:*

$$1 = \dot{\sum}\{a_i : i \in I\} \tag{35}$$

and

$$1 = \dot{\sum}\{b_j : j \in J\} \tag{36}$$

with c-irreducible summands. If each b_j $(j \in J)$ satisfies the B_c^*-condition, then for every finite subset $J' = \{j_1,\dots,j_k\} \subseteq J$ there exists a finite subset $I' = \{i_1,\dots,i_k\} \subseteq I$ such *that*

$$1 = a_{i_n} \dot{+} \dot{\sum}\{b_j : j \neq j_n\} \tag{37}$$
$$= b_{j_n} \dot{+} b_{j_{n-1}} \dot{+} \ldots \dot{+} b_{j_1} \dot{+} \dot{\sum}\{a_i : i \in I - \{i_1, \ldots, i_n\}\}$$

for all $n = 1, \ldots, k$.

Proof. We argue by induction on the number of elements in J'. We show the statement for $J' = \{j_1\}$. We have two c-decompositions of 1: (35) and

$$1 = b_{j_1} \dot{+} \overline{b}_{j_1}. \tag{38}$$

Let α_i, $i \in I$, and $\beta_{j_1}, \overline{\beta}_{j_1}$ be the decomposition functions related to (35) and (38), respectively. By Definition 3.34, there exists $i_1 \in I$ such that $\beta_{j_1}\alpha_{i_1}\beta_{j_1} \in \mathrm{F}(c, b_{j_1})$. We consider two c-decompositions:

$$1 = b_{j_1} \dot{+} \overline{b}_{j_1} = a_{i_1} \dot{+} \overline{a}_{i_1}.$$

From Lemma 3.24 we obtain

$$1 = a_{i_1} \dot{+} \overline{b}_{j_1} = b_{j_1} \dot{+} \overline{a}_{i_1}. \tag{39}$$

Now we prove that the set $T = \{a_{i_1}\} \cup \{b_j : j \in J - \{j_1\}\}$ is join irredundant. Assume on the contrary that there exists $j_2 \in J - \{j_1\}$ such that $1 = a_{i_1} \dot{+} \overline{b}_{j_1,j_2}$. Then

$$1 = b_{j_1} \dot{+} \overline{b}_{j_1} = a_{i_1} \dot{+} \overline{b}_{j_1,j_2}. \tag{40}$$

Since b_{j_1} satisfies the B_c^*-condition, it also satisfies the B_c-condition. Applying Lemma 3.37 to the c-decompositions (40) we deduce that $1 = b_{j_1} \dot{+} \overline{b}_{j_1,j_2} = \overline{b}_{j_2}$. This means that the set $\{b_j : j \in J\}$ is not join irredundant, contrary to our assumptions. Therefore, the set T is join irredundant, and similarly, the set $\{b_{j_1}\} \cup \{a_i : i \in J - \{j_1\}\}$ is join irredundant. Then, by Property II, from (39) we obtain

$$1 = a_{i_1} \dot{+} \dot{\sum}\{b_j : j \neq j_1\} = b_{j_1} \dot{+} \dot{\sum}\{a_i : i \neq i_1\}.$$

Thus, Lemma 3.38 is true for $J' = \{j_1\}$.

Let us assume this statement for every $(k-1)$-element subset of J and set $J' = \{j_1, \ldots, j_k\}$. By the induction hypothesis for the subset $\{j_1, \ldots, j_{k-1}\}$ of J' there exists $\{i_1, \ldots, i_{k-1}\} \subseteq I$ such that (37) holds for each $n = 1, \ldots, k-1$. In particular,

$$1 = b_{j_{k-1}} \dot{+} \ldots \dot{+} b_{j_1} \dot{+} \dot{\sum}\{a_i : i \in I - \{i_1, \ldots, i_{k-1}\}\}. \tag{41}$$

We consider the c-decompositions (41) and (36). By the first part of the proof, there is $i_k \in I - \{i_1, \ldots, i_{k-1}\}$ such that (37) holds for $n = k$. Thus, there exists $I' = \{i_1, \ldots, i_k\} \subseteq I$ such that (37) holds for all $n = 1, \ldots, k$, and the proof is complete. ∎

LEMMA 3.39. *Suppose the element* 1 *of* L *has an irredundant* c*-decomposition* (36) *into* c*-irreducible summands satisfying the* B_c^**-condition. If* 1 *also has an irredundant* c*-decomposition*

$$1 = \dot{\sum}\{a_i : i \in I'\} \dot{+} \dot{\sum}\{b_j : j \in J'\} \tag{42}$$

such that J' *is a proper subset of* J, *and* a_i *is* c*-irreducible and compact for each* $i \in I'$, *then there are two countable or finite* (*with an equal number of elements*) *subsets* $I_0 =$

$\{i_1, \ldots, i_n, \ldots\} \subseteq I'$ *and* $J_0 = \{j_1, \ldots, j_n, \ldots\} \subseteq J'' = J - J'$ *such that*

$$(43) \qquad \begin{aligned} 1 &= a_{i_n} \dot{+} \dot{\sum}\{b_j : j \neq j_n\} \\ &= b_{j_n} \dot{+} \ldots \dot{+} b_{j_1} \dot{+} \dot{\sum}\{b_j : j \in J'\} \dot{+} \dot{\sum}\{a_i : i \in I' - \{i_1, \ldots, i_n\}\} \end{aligned}$$

for all $n = 1, 2, \ldots,$ *and*

$$(44) \qquad \bigvee\{a_i : i \in I_0\} \leq \bigvee\{b_j : j \in J' \cup J_0\}.$$

Proof. Let $j_1 \in J''$. By Lemma 3.38, there is an $i_1 \in I'$ such that

$$1 = a_{i_1} \dot{+} \dot{\sum}\{b_j : j \in J - \{j_1\}\}$$

and

$$(45) \qquad 1 = b_{j_1} \dot{+} \dot{\sum}\{b_j : j \in J'\} \dot{+} \dot{\sum}\{a_i : i \in I' - \{i_1\}\}.$$

The element a_{i_1} is compact and hence there is a finite subset $\{j_2, \ldots, j_k\} \subseteq J'' - \{j_1\}$ such that

$$a_{i_1} \leq \bigvee\{b_j : j \in J' \cup J_1\}, \quad \text{where} \quad J_1 = \{j_1, j_2, \ldots, j_k\}.$$

Applying Lemma 3.38 to the c-decompositions (45) and (36) we conclude that there exist distinct indices $i_2, \ldots, i_k \in I' - \{i_1\}$ such that (43) holds for each $n = 2, \ldots, k$. In particular,

$$(46) \qquad 1 = b_{j_k} \dot{+} \ldots \dot{+} b_{j_1} \dot{+} \dot{\sum}\{b_j : j \in J'\} \dot{+} \dot{\sum}\{a_i : i \in I' - I_1\},$$

where $I_1 = \{i_1, \ldots, i_k\}$. Again $a_{i_2} \vee \ldots \vee a_{i_k}$ is compact, and there exists a finite subset $\{j_{k+1}, \ldots, j_m\} \subseteq J'' - J_1$ such that

$$a_{i_2} \vee \ldots \vee a_{i_k} \leq \bigvee\{b_j : j \in J' \cup J_2\},$$

where $J_2 = J_1 \cup \{j_{k+1}, \ldots, j_m\}$. Now we apply Lemma 3.38 to the c-decompositions (46) and (36), and to the elements $b_{j_{k+1}}, \ldots, b_{j_m}$. As before, we get the existence of distinct elements $i_{k+1}, \ldots, i_m \in I' - I_1$ such that (43) holds for each $n = k+1, \ldots, m$. By continuing this process, we obtain two subsets $I_0 = \{i_1, \ldots, i_n, \ldots\}$ and $J_0 = \{j_1, \ldots, j_n, \ldots\}$ such that (43) and (44) hold. ∎

3.7. Infinite c-decompositions. Now, we suppose that a distributive element c of L has the following property:

(△) For each $a \in L$ and for each $S \subseteq L$, if $a \wedge \bigvee S' \leq c$ for every finite subset S' of S, then $a \wedge \bigvee S \leq c$.

The main result of Chapter 3 is the following theorem.

The c-Decomposition Theorem 3.40 (see Walendziak [1989], Theorem 1). *Let L be a complete modular lattice and let c be a distributive element of L with property* (△). *If the unit element of L has two irredundant c-decompositions* (35) *and* (36) *into c-irreducible elements satisfying the* B_c^**-condition, then there is a bijection $\lambda : I \to J$ such that, for every $i \in I$,*

$$(47) \qquad 1 = a_i \dot{+} \dot{\sum}\{b_j : j \neq \lambda(i)\}.$$

Proof. Let W be the set of all ordered triples $(M, \leq^M, f_M)$ where $M \subseteq I$, $\leq^M$ is a well-ordering of M, f_M is a one-to-one mapping of M to J and for each $m \in M$,

$$\begin{aligned}(48)\qquad 1 &= a_m \dot{+} \dot{\sum}\{b_j : j \neq f_M(m)\}\\ &= \dot{\sum}\{b_{f_M(i)} : i \in (m]\} \dot{+} \dot{\sum}\{a_i : i \in I - (m]\},\end{aligned}$$

where $(m] = \{i \in M : i \leq^M m\}$, and

$$(49)\qquad \bigvee\{a_i : i \in M\} \leq \bigvee\{b_{f_M(i)} : i \in M\}.$$

Then W is nonempty since it contains the triple consisting of the empty set, the empty relation, and the empty mapping (here, we are considering relations and functions as sets of ordered pairs). Define a partial order $\leq_W$ in W by $(M, \leq^M, f_M) \leq_W (M', \leq^{M'}, f_{M'})$ if either $M = M'$ or $M = \{i \in M' : i \leq^{M'} m\}$ for some $m \in M'$, $\leq^{M'}$ restricted to M coincides with $\leq^M$, and the restriction of $f_{M'}$ to M coincides with f_M.

Let $(M_k, \leq^{M_k}, f_{M_k})$ $(k \in K)$ be a chain in W. Set

$$\overline{M} = \bigcup\{M_k : k \in K\}, \quad \leq^{\overline{M}} = \bigcup\{\leq^{M_k} : k \in K\}, \quad f_{\overline{M}} = \bigcup\{f_{M_k} : k \in K\}.$$

It is obvious that $(\overline{M}, \leq^{\overline{M}}, f_{\overline{M}}) \in W$ and that $(\overline{M}, \leq^{\overline{M}}, f_{\overline{M}})$ is an upper bound of the chain $(M_k, \leq^{M_k}, f_{M_k})$ $(k \in K)$. Therefore, by Zorn's Lemma, W contains a maximal element $(N, \leq^N, f_N)$.

We consider the set

$$S = \{b_{f_N(i)} : i \in N\} \cup \{a_i : i \in I - N\}.$$

By (49) we have

$$\bigvee\{a_i : i \in N\} \leq \bigvee\{b_{f_N(i)} : i \in N\}.$$

Hence,

$$1 = \bigvee\{a_i : i \in N\} \vee \bigvee\{a_i : i \in I - N\} \leq \bigvee\{b_{f_N(i)} : i \in N\} \vee \bigvee\{a_i : i \in I - N\}.$$

Thus, $1 = \bigvee S$. By (48), all finite subsets of S are c-independent.

Now, we prove that the set S is join irredundant. Suppose on the contrary that $s_0 \leq \bigvee(S - \{s_0\})$ for some $s_0 \in S$. But s_0 is compact and hence $s_0 \leq \bigvee(T - \{s_0\})$, where T is a finite subset of S containing s_0. Thus T is join redundant, contrary to the c-independence of T.

Let $s \in S$ and let S' be a finite subset of $S - \{s\}$. Since $S' \cup \{s\}$ is c-independent, we get $s \wedge \bigvee S' \leq c$. Then, by property $(\triangle)$, we conclude that $s \wedge \bigvee(S - \{s\}) \leq c$. Therefore, S is c-independent. Thus, $1 = \dot{\sum} S$, and hence, if we set $I' = I - N$ and $J' = f_N(N)$, then we obtain the c-decomposition (42).

Now we prove that $N = I$. Suppose on the contrary that $N \neq I$, that is, $I' \neq \emptyset$. Consequently, $J' \neq J$. Applying Lemma 3.39 to the c-decompositions (42) and (36) we get two subsets $I_0 = \{i_1, \ldots, i_n, \ldots\} \subseteq I'$ and $J_0 = \{j_1, \ldots, j_n, \ldots\} \subseteq J - J'$ such that (43) and (44) hold.

Set $P = N \cup I_0$. Define the well-ordering $\leq^P$ of P by the following rules: if $i, i' \in N$, then $i \leq^P i'$ iff $i \leq^N i'$, and for every $i \in N$,

$$i <^P i_1 <^P i_2 <^P \ldots <^P i_n <^P \ldots$$

Define the mapping f_P by $f_P(i) = f_N(i)$ for every $i \in N$, and

$$f_P(i_n) = j_n \quad \text{for } n = 1, 2, \ldots$$

By (43) and (44), the triple $(P, \leq^P, f_P)$ belongs to W. It is obvious that $(P, \leq^P, f_P)$ is greater than $(N, \leq^N, f_N)$. This contradiction forces the equality $N = I$. Then $I' = \emptyset$ and from (42) we have $J' = J$. Therefore $\lambda = f_N$ is a one-to-one mapping of I onto J such that, for each $i \in I$, we have the c-decomposition (47). ∎

REMARK 3.41. In a modular lattice of finite length every c-irreducible c-summand of 1 is compact and, by Lemma 3.35, it satisfies the B_c^*-condition. Thus from Theorem 3.40 we get Corollary 3.33.

Theorem 3.40 and Lemma 3.35 yield

COROLLARY 3.42. *Let L be a complete modular lattice and let $c \in \mathrm{D}(L)$ have property $(\triangle)$. Let*

$$1 = \dot{\sum}_c T = \dot{\sum}_c R,$$

where all c-summands are c-irreducible, $T \cup R \subseteq \mathrm{K}(L)$, and for every $a \in R \cup T$ the interval $[0, a]$ is of finite length. Then there is a bijection $\lambda : T \to R$ such that, for each $t \in T$,

$$1 = t \dotplus_c \dot{\sum}_c \{r : r \neq \lambda(t)\}.$$

COROLLARY 3.43 (Walendziak [1990a], Theorem 3). *Let L be an upper continuous modular lattice and let $c \in \mathrm{D}(L)$. Let*

$$1 = \dot{\sum}_c \{a_i : i \in I\} = \dot{\sum}_c \{b_j : j \in J\}$$

be two irredundant c-decompositions of 1 with all summands c-irreducible. If the intervals $[0, a_i]$ and $[0, b_j]$ $(i \in I, j \in J)$ are of finite length, then there exists a one-to-one mapping λ of I onto J such that, for each $i \in I$, the c-decomposition (47) holds.

Proof. We first observe that if L is an upper continuous lattice, then every element $c \in L$ has property $(\triangle)$. Let $a \in L$ and S be a subset of L. Suppose that $a \wedge \bigvee S' \leq c$ for every finite subset S' of S. By (UC),

$$a \wedge \bigvee S = \bigvee \{a \wedge \bigvee S' : S' \in \mathrm{F}(S)\}.$$

Therefore, $a \wedge \bigvee S \leq c$, because $a \wedge \bigvee S' \leq c$ for every $S' \in \mathrm{F}(S)$. Thus c has property $(\triangle)$.

Crawley [1962] proved that if a is an element of an upper continuous lattice such that $[0, a]$ is of finite length, then a is compact. Therefore the elements a_i $(i \in I)$ and b_j $(j \in J)$ are compact. Moreover, by Lemma 3.35, they satisfy the B_c^*-condition. Now the assertion follows from Theorem 3.40. ∎

From Corollary 3.43, in the case $c = 0$, we get

COROLLARY 3.44 (Crawley [1962]). *Let L be an upper continuous modular lattice and let $a \in L$. Suppose that*

$$a = \dot{\bigvee} \{a_i : i \in I\} = \dot{\bigvee} \{b_j : j \in J\}$$

are two direct decompositions of a, where a_i $(i \in I)$ and b_j $(j \in J)$ are directly join irreducible. If each $[0, a_i]$ and each $[0, b_j]$ is of finite length, then there exists a bijection

$\lambda : I \to J$ such that, for each $i \in I$,

$$1 = a_i \dot\vee \dot\bigvee\{b_j : j \neq \lambda(i)\}.$$

Corollary 3.45. *Let L be a complete modular lattice and let $a \in L$. Assume that*

$$a = \bigvee T = \bigvee R,$$

where T and R are join irredundant sets of join irreducible compact elements of L. Then there is a bijection $\lambda : T \to R$ such that, for each $t \in T$, $a = t \vee \bigvee\{r : r \neq \lambda(t)\}$.

Proof. It is obvious that the element $c = 1$ satisfies ($\triangle$). By Lemma 3.36 each $r \in R \cup T$ satisfies the B_1^*-condition. Now from Theorem 3.40 our corollary follows. ∎

We recall from Section 2.1 that if an element $a \in L$ has a representation $a = \bigvee T$ with $T \subseteq \mathrm{J}(L)$, then we say that a has a join decomposition. Richter [1982a] (see Theorem 7) proved that if $a = \bigvee T = \bigvee R$ are two join decompositions of a in a complete lattice satisfying the hereditary property (HJ), then any element $t \in T$ can be replaced by an $r \in R$. But if these join decompositions are irredundant there are no statements about the cardinality of T and R except in the finite case. For upper continuous modular lattice we are able to generalize the Theorem of Kurosh–Ore to infinite join decompositions and to make a statement about the cardinality of T and R. We remark that for algebraic strong semimodular J-lattices this is due to Richter [1991] (see Theorem 24). Now we prove the following

Corollary 3.46. *If L is an upper continuous modular lattice and if $a = \bigvee T = \bigvee R$ are two irredundant join decompositions of a, then T and R have the same cardinality, and any element t of T can be replaced by an $r \in R$.*

Proof. In the proof of Lemma 3 of Crawley [1962] it was shown that every completely join irreducible element of an upper continuous lattice is compact. Therefore $T \cup R \subseteq \mathrm{K}(L)$. Moreover, every completely join irreducible element of a complete lattice is join irreducible. Hence Corollary 3.45 implies the assertion. ∎

4. Weak direct products of algebras

4.1. Definition and preliminaries. The material of this chapter is taken from Walendziak [2000a]. If $(A_i : i \in I)$ is a system of similar algebras, then $\prod(A_i : i \in I)$, or $\prod A_i$, denotes the direct product of algebras A_i, $i \in I$. For $x, y \in \prod A_i$ we define

$$\mathrm{I}(x, y) = \{i \in I : x(i) \neq y(i)\}.$$

Definition 4.1 (cf. Grätzer [1979], p. 139). A subalgebra A of $\prod A_i$ is called a *weak direct product* of A_i, $i \in I$, if the following two conditions are satisfied:

(A1) If $x, y \in A$, then the set $\mathrm{I}(x, y)$ is finite.

(A2) If $x \in A$, $y \in \prod A_i$ and if $\mathrm{I}(x, y)$ is finite, then $y \in A$.

We write $A = \prod^W (A_i : i \in I)$, or $A = \prod^W A_i$, to denote that A is a weak direct product of A_i, $i \in I$. If $(A_i : i \in I)$ is a system of groups, rings or modules, then

$$\prod^W (A_i : i \in I) = \bigoplus (A_i : i \in I),$$

where $\bigoplus$ denotes the direct sum. If the set I is finite, then the concepts of the weak direct product and direct product coincide.

Let $\mathrm{Con}(A)$ denote the set of all congruence relations on an algebra A. Then $\mathrm{Con}(A)$ forms an algebraic lattice with 0_A and 1_A, the smallest and the largest congruence relations, respectively (occasionally, they are denoted simply by 0 and 1).

The *relational product* of two congruences α and β is the relation

$$\alpha \circ \beta = \{(a, b) : (a, c) \in \alpha \text{ and } (c, b) \in \beta \text{ for some element } c\}.$$

An algebra A is called *directly indecomposable* if it is nontrivial and is not isomorphic to a direct product of two nontrivial algebras. A is called *subdirectly irreducible* if $|A| > 1$ and 0_A is a completely meet irreducible element of $\mathrm{Con}(A)$. We say that an algebra A is *simple* if it has exactly two congruences, 0_A and 1_A; A has *permuting congruences*, or *is congruence permutable* if for any $\alpha, \beta \in \mathrm{Con}(A)$, $\alpha \circ \beta = \beta \circ \alpha$; and A is called *congruence modular* (*distributive*) if $\mathrm{Con}(A)$ is modular (distributive).

Proposition 4.2 (Hu [1969], Lemma 11). *Let A be a congruence permutable algebra, and let $\theta_i \in \mathrm{Con}(A)$, $i \in I$. Then*

$$A \cong \prod^W (A/\theta_i : i \in I) \quad \textit{if and only if} \quad 0_A = \bigcap\{\theta_i : i \in I\} \textit{ and } 1_A = \bigvee\{\overline{\theta}_i : i \in I\},$$

where $\overline{\theta}_i = \bigcap\{\theta_j : j \in I - \{i\}\}$.

A congruence $\alpha \in \mathrm{Con}(A)$ is called a *decomposition congruence* if there is a $\beta \in \mathrm{Con}(A)$ such that $\alpha \cap \beta = 0_A$ and $\alpha \circ \beta = 1_A$. Let $\mathrm{DCon}(A)$ denote the set of all decomposition congruences of A. We call a sublattice of a complete lattice $\bigvee$-*closed* whenever it is closed under arbitrary joins.

Lemma 4.3. *If* $\mathrm{DCon}(A)$ *is a* $\bigvee$*-closed modular sublattice of* $\mathrm{Con}(A)$, *then* $\mathrm{DCon}(A)$ *is atomistic.*

Proof. Let $\alpha \in \mathrm{DCon}(A)$, $\alpha \neq 0_A$ and let α' be a complement of α. Choose $a, b \in A$ such that $(a, b) \notin \alpha'$ and consider the set

$$\Delta = \{\beta \in \mathrm{D\,Con}(A) : (a, b) \notin \beta \text{ and } \alpha' \leq \beta\}.$$

Then Δ is nonempty, since $\alpha' \in \Delta$. Let Γ be a chain in Δ. It is easy to see that $\bigvee\Gamma \in \Delta$. Therefore, Δ contains a maximal element δ by Zorn's Lemma. Since $\mathrm{DCon}(A)$ is complemented and modular, it is relatively complemented. Let δ' be a relative complement of δ in $[\alpha', 1_A]$. Then $\delta \cap \delta' = \alpha'$ and $\delta \vee \delta' = 1_A$. From the maximality of δ we infer that $\delta \prec 1_A$. By modularity, $\alpha' \prec \delta'$ and hence $0_A = \alpha \cap \alpha' \prec \alpha \cap \delta' \leq \alpha$. Therefore, $\mathrm{DCon}(A)$ is atomic. Now, by Theorem 4.3 of Crawley–Dilworth [1973], $\mathrm{DCon}(A)$ is atomistic. ∎

Lemma 4.4. *Let A be an algebra such that* $\mathrm{DCon}(A)$ *is a sublattice of* $\mathrm{Con}(A)$. *If θ is a coatom of* $\mathrm{DCon}(A)$, *then A/θ is directly indecomposable.*

Proof. Suppose on the contrary that there exist two congruences α, β such that $\theta < \alpha$, $\beta < 1_A$, $\alpha \circ \beta = 1_A$ and $\alpha \cap \beta = \theta$. Let θ' be a congruence satisfying $0_A = \theta \cap \theta'$ and $1_A = \theta \circ \theta'$. Obviously,

$$\alpha \cap (\beta \cap \theta') = 0_A. \tag{1}$$

Observe that

$$\alpha \circ (\beta \cap \theta') = 1_A. \tag{2}$$

Indeed, let $x, y \in A$, and choose $z, t \in A$ such that $(x, z) \in \alpha$, $(z, y) \in \beta$, $(z, t) \in \theta$, and $(t, y) \in \theta'$. Then $(x, t) \in \alpha$ and $(t, y) \in \beta \cap \theta'$. Therefore, $(x, y) \in \alpha \circ (\beta \cap \theta')$, and hence (2) holds. From (1) and (2) we conclude that $\alpha \in \mathrm{DCon}(A)$, contradicting the fact that θ is a coatom of $\mathrm{DCon}(A)$. Consequently, A/θ is directly indecomposable. ∎

LEMMA 4.5. *Let A be an algebra and let Γ be a $\bigvee$-closed sublattice of* $\mathrm{Con}(A)$. *If Γ is an atomistic semimodular lattice, then there exist coatoms $\theta_i \in \Gamma$ $(i \in I)$ such that $0_A = \bigcap\{\theta_i : i \in I\}$ and $1_A = \bigvee\{\overline{\theta}_i : i \in I\}$.*

Proof. Let Ω be the set of all atoms of Γ. Since the lattice Γ is atomistic, $1_A = \bigvee \Omega$. Let $\{\alpha_i : i \in I\}$ be a maximal subset of Ω such that

$$\alpha_i \cap \bigvee\{\alpha_j : j \in I - \{i\}\} = 0_A$$

for all $i \in I$ (apply Zorn's Lemma). From Theorem 6.5 of Crawley–Dilworth [1973] we deduce that

$$1_A = \bigvee\{\alpha_i : i \in I\}. \tag{3}$$

Set $\theta_i = \bigvee\{\alpha_j : j \neq i\}$ for $i \in I$. By semimodularity, θ_i $(i \in I)$ is a coatom of Γ. We put

$$\gamma = \bigcap\{\theta_i : i \in I\}.$$

Let P be the set of all subsets $J \subseteq I$ such that

$$\gamma \cap \bigvee\{\alpha_j : j \in J\} = 0_A.$$

P is nonempty since $\emptyset \in P$. Let J_k $(k \in K)$ be a chain in P. By upper continuity we conclude that $\bigcup\{J_k : k \in K\} \in P$. Therefore, by Zorn's Lemma, P contains a maximal element M. Now we prove that $M = I$. Suppose on the contrary that $M \neq I$. Let $i_0 \in I - M$. Obviously,

$$\gamma \leq \theta_{i_0} \quad \text{and} \quad \delta = \bigvee\{\alpha_i : i \in M\} \leq \theta_{i_0}.$$

Since $0_A \prec \alpha_{i_0}$, by semimodularity we get

$$\delta \prec \alpha_{i_0} \vee \bigvee\{\alpha_i : i \in M\} = \bigvee\{\alpha_i : i \in N\},$$

where $N = M \cup \{i_0\}$. Then $\theta_{i_0} \cap \bigvee\{\alpha_i : i \in N\} = \delta$. Hence $\gamma \cap \bigvee\{\alpha_i : i \in N\} = \gamma \cap \delta = 0_A$. Consequently, $N \in P$. This contradiction forces $M = I$. Thus $\gamma = \gamma \cap \bigvee\{\alpha_i : i \in I\} = 0_A$, i.e.,

$$\bigcap\{\theta_i : i \in I\} = 0_A. \tag{4}$$

Since $\theta_j \geq \alpha_i$ for $j \neq i$, we obtain $\overline{\theta}_i = \bigcap\{\theta_j : j \neq i\} \geq \alpha_i$. From (3) we have

$$1_A = \bigvee\{\overline{\theta}_i : i \in I\},$$

and by (4), the proof is complete. ∎

4.2. Some existence theorems. The first major result is

THEOREM 4.6. *Let A be an algebra such that* $\mathrm{DCon}(A)$ *is an atomistic semimodular* $\bigvee$*-closed sublattice of* $\mathrm{Con}(A)$. *Suppose that for every atom α of* $\mathrm{DCon}(A)$, *if β is a complement of α in* $\mathrm{DCon}(A)$, *then $1_A = \alpha \circ \beta$. Then there are coatoms θ_i $(i \in I)$ of* $\mathrm{DCon}(A)$ *such that A is isomorphic to a weak direct product of the* (*directly indecomposable*) *algebras A/θ_i, $i \in I$.*

Proof. By the proof of Lemma 4.5 there exist atoms α_i $(i \in I)$ of $\mathrm{DCon}(A)$ such that

$$1_A = \bigvee\{\alpha_i : i \in I\} \quad \text{and} \quad \alpha_i \cap \bigvee\{\alpha_j : j \in I - \{i\}\} = 0_A$$

for all $i \in I$. Set $\theta_i = \bigvee\{\alpha_j : j \neq i\}$. By semimodularity, θ_i $(i \in I)$ is a coatom of $\mathrm{DCon}(A)$, and it is a complement of α_i. Therefore, by assumption,

$$1_A = \theta_i \circ \alpha_i \tag{5}$$

for all $i \in I$. Since $\theta_j \geq \alpha_i$ for $j \neq i$, we obtain $\overline{\theta}_i = \bigcap\{\theta_j : j \neq i\} \geq \alpha_i$. Then

$$1_A = \bigvee\{\overline{\theta}_i : i \in I\}, \tag{6}$$

and by (5),

$$1_A = \theta_i \circ \overline{\theta}_i. \tag{7}$$

From the proof of Lemma 4.5 it follows that

$$\bigcap\{\theta_i : i \in I\} = 0_A. \tag{8}$$

We denote by f the function from A to $B = \prod(A/\theta_i : i \in I)$ defined by

$$f(x) = (x/\theta_i : i \in I) \qquad (x \in A).$$

From (8) we conclude that f is an embedding. Let $x, y \in A$. We show that $R = \{i \in I : x/\theta_i \neq y/\theta_i\}$ is finite. By (6), $(x, y) \in \bigvee\{\overline{\theta}_i : i \in I\}$. So there are $i_1, \ldots, i_n \in I$ such that $(x, y) \in \overline{\theta}_{i_1} \vee \ldots \vee \overline{\theta}_{i_n}$. Observe that

$$R \subseteq \{i_1, \ldots, i_n\}. \tag{9}$$

Indeed, let $x/\theta_i \neq y/\theta_i$, for some $i \in I$, and suppose on the contrary that $i \notin \{i_1, \ldots, i_n\}$. Therefore, $\overline{\theta}_{i_1} \vee \ldots \vee \overline{\theta}_{i_n} \leq \theta_i$, and hence $(x, y) \in \theta_i$, i.e., $x/\theta_i = y/\theta_i$, a contradiction. From (9) we deduce that R is finite. Now we prove that

$$\text{if } x \in A,\ y \in B \text{ and } |\{i \in I : x/\theta_i \neq y/\theta_i\}| < \aleph_0, \text{ then } y \in f(A). \tag{10}$$

Suppose that the set $S = \{i \in I : x/\theta_i \neq y/\theta_i\}$ contains only one element i_1. Let $t \in A$ satisfy $t/\theta_{i_1} = y(i_1)$. Since $1_A = \theta_{i_1} \circ \overline{\theta}_{i_1}$ there is $z \in A$ such that $(t, z) \in \theta_{i_1}$ and $(z, x) \in \overline{\theta}_{i_1}$. It is easy to see that $y = f(z)$. Consequently, $y \in f(A)$. From this we see by induction that (10) holds. Hence $f(A)$ is a weak direct product of the algebras A/θ_i, $i \in I$. That the algebras A/θ_i are directly indecomposable follows from Lemma 4.4. ∎

COROLLARY 4.7. *Let A be an algebra such that* $\mathrm{DCon}(A)$ *is a* $\bigvee$*-closed sublattice of* $\mathrm{Con}(A)$ *and suppose that for any $\alpha, \beta \in$* $\mathrm{DCon}(A)$, *α and β permute. Then A is isomorphic to a weak direct product of directly indecomposable algebras.*

Proof. Since $\alpha \vee \beta = \alpha \circ \beta$ for all $\alpha, \beta \in \mathrm{DCon}(A)$, we conclude that $\mathrm{DCon}(A)$ is a modular lattice. By Lemma 4.3, $\mathrm{DCon}(A)$ is atomistic. Applying Theorem 4.6 we see that there exist coatoms θ_i $(i \in I)$ of $\mathrm{DCon}(A)$ such that $A \cong \prod^W(A/\theta_i : i \in I)$. From Lemma 4.4 we see that the algebras A/θ_i are directly indecomposable. ■

Corollary 4.8 (Hashimoto [1957], Theorem 4.5). *If an algebra A has permuting congruences and* $\mathrm{DCon}(A)$ *is a* $\bigvee$*-closed sublattice of* $\mathrm{Con}(A)$, *then there exists a system* $(A_i : i \in I)$ *of directly indecomposable algebras such that* $A \cong \prod^W(A_i : i \in I)$.

Corollary 4.9 (Hashimoto [1957], Theorem 5.1). *Let A be any algebra whose congruences permute and whose congruence lattice is complemented. Then A is isomorphic to a weak direct product of simple algebras.*

Proof. Note that in Crawley–Dilworth [1973] (see Theorem 4.3) it is shown that every algebraic complemented modular lattice is atomistic. Therefore, $\mathrm{DCon}(A) = \mathrm{Con}(A)$ is atomistic. By Theorem 4.6, there are coatoms θ_i $(i \in I)$ of $\mathrm{Con}(A)$ such that $A \cong \prod^W(A/\theta_i : i \in I)$. It is obvious that A/θ_i $(i \in I)$ are simple algebras. ■

Congruence permutable algebras include groups, rings, modules, quasigroups, Heyting algebras and relatively complemented lattices. In the case of groups, Corollary 4.9 implies the following statement.

Corollary 4.10 (Hashimoto [1957], p. 104). *Let G be an Ω-group. If for every normal Ω-subgroup H of G there is a normal Ω-subgroup K such that $G = H \oplus K$, then G is a direct sum of simple Ω-groups.*

Remark 4.11. Corollary 4.9 includes the result of Blair [1953] on the decomposition of rings into simple rings.

For modular algebraic lattice L the following statements are equivalent:

(i) L is complemented.

(ii) The join of the atoms of L is 1.

(See Lemma 4.83 of McKenzie–McNulty–Taylor [1987].) Therefore, Corollary 4.9 yields

Corollary 4.12. *A module M which is the sum*

$$M = \sum\{V : V \leq M \text{ and } V \text{ is a simple module}\}$$

is a direct sum of simple submodules.

For vector spaces we obtain the following

Corollary 4.13. *Every vector space V is a direct sum of one-dimensional subspaces of V.*

For lattices, we have

Corollary 4.14 (Dilworth [1950], Theorem 4.4). *A relatively complemented lattice L satisfying the ascending chain condition is isomorphic to a direct product of finitely many simple relatively complemented lattices.*

Proof. It is well known that L is congruence permutable. By Theorem 10.8 of Crawley–Dilworth [1973], $\mathrm{Con}(L)$ is complemented. Hence there exists a system $(L_i : i \in I)$ of simple lattices such that $L \cong \prod^W (L_i : i \in I)$. According to Proposition 4.2 we may assume that each L_i is L/θ_i, with

$$0_L = \bigcap\{\theta_i : i \in I\} \quad \text{and} \quad 1_L = \bigvee\{\overline{\theta}_i : i \in I\}$$

where $\overline{\theta}_i = \bigcap\{\theta_j : j \in I - \{i\}\}$. Observe that

$$\overline{\theta}_{i_1} < \overline{\theta}_{i_1} \vee \overline{\theta}_{i_2} < \overline{\theta}_{i_1} \vee \ldots \vee \overline{\theta}_{i_n} < \ldots < 1_L = \bigvee\{\overline{\theta}_i : i \in I\}$$

for $i_j \in I$ $(j = 1, 2, \ldots)$. Indeed, if $\overline{\theta}_{i_0} \leq \bigvee\{\overline{\theta}_j : j \neq i_0\}$ for some $i_0 \in I$, then $\overline{\theta}_{i_0} \leq \theta_{i_0}$, and hence $\theta_{i_0} = 1_L$, a contradiction.

But $\mathrm{Con}(L)$ satisfies the ascending chain condition (see Theorem 4.3 of Dilworth [1950]), and therefore, I is finite. Consequently, L is isomorphic to a direct product of simple lattices L_i, which clearly must be relatively complemented. ∎

PROPOSITION 4.15. *Let A be an algebra such that* $\mathrm{DCon}(A)$ *is a modular $\bigvee$-closed sublattice of* $\mathrm{Con}(A)$ *and suppose that every atom of* $\mathrm{DCon}(A)$ *has a unique complement. Then A is isomorphic to a weak direct product of directly indecomposable algebras.*

Proof. By Lemma 4.3, $\mathrm{DCon}(A)$ is an atomistic lattice. Let α be an atom of $\mathrm{DCon}(A)$, and let β be a complement of α in $\mathrm{DCon}(A)$. Then $1 = \alpha \circ \beta$, because α has a unique complement. Applying Theorem 4.6 we obtain the assertion. ∎

As a consequence of Proposition 4.15 we get the following

COROLLARY 4.16 (Hashimoto [1957], p. 106). *If A is a congruence distributive algebra such that* $\mathrm{DCon}(A)$ *is a $\bigvee$-closed sublattice of* $\mathrm{Con}(A)$, *then A can be decomposed into a weak direct product of directly indecomposable factors.*

THEOREM 4.17. *Let A be any algebra such that 1_A is a join of join irreducible elements of* $\mathrm{Con}(A)$ *and suppose that every decomposition congruence on A is neutral* (*i.e., it is standard and codistributive*) *in* $\mathrm{Con}(A)$. *Then A is isomorphic to a weak direct product of directly indecomposable algebras.*

Proof. We first prove that $\alpha \vee \beta = \alpha \circ \beta$ for $\alpha \in \mathrm{DCon}(A)$ and $\beta \in \mathrm{Con}(A)$. Let α' be a congruence satisfying $0_A = \alpha \cap \alpha'$ and $1_A = \alpha \circ \alpha'$. Assume that $(x, y) \in \alpha \vee \beta$. Obviously, there is a $z \in A$ such that $(x, z) \in \alpha$ and $(z, y) \in \alpha'$. Consequently, $(z, y) \in \alpha' \cap (\alpha \vee \beta)$. Since α is a neutral element in $\mathrm{Con}(A)$, we have

$$\alpha' \cap (\alpha \vee \beta) = (\alpha' \cap \alpha) \vee (\alpha' \cap \beta) = \alpha' \cap \beta.$$

Then $(z, y) \in \beta$, and hence $(x, y) \in \alpha \circ \beta$. Thus $\alpha \vee \beta = \alpha \circ \beta$. Therefore, every element of $\mathrm{DCon}(A)$ is permutable with any congruence on A. Now it is easy to see that $\mathrm{DCon}(A)$ is a distributive sublattice of $\mathrm{Con}(A)$. It is sufficient to show that $\mathrm{DCon}(A)$ is $\bigvee$-closed.

Let $\Gamma = \{\alpha_i : i \in I\} \subseteq \mathrm{DCon}(A)$, and let α'_i denote the congruence satisfying $0_A = \alpha_i \cap \alpha'_i$ and $1_A = \alpha_i \circ \alpha'_i$. We prove that $\alpha = \bigvee\Gamma \in \mathrm{DCon}(A)$. Write

$$\Psi = \{\beta \in \mathrm{Con}(A) : \beta \nleq \alpha \text{ and } \beta \text{ is join irreducible}\}.$$

and put $\alpha' = \bigvee\Psi$. Let β be a join irreducible element of $\mathrm{Con}(A)$. By the definition of α, if $\beta \nleq \alpha$, then $\beta \leq \alpha'$. Therefore, $\beta \leq \alpha \vee \alpha'$. Since 1_A is a join of join irreducible elements of $\mathrm{Con}(A)$, we conclude that

$$1_A = \alpha \vee \alpha'.$$

We claim that

$$\text{if } \beta \in \Psi, \text{ then } \beta \cap \alpha_i = 0_A \text{ for any } i \in I.$$

Indeed, $\beta = \beta \cap (\alpha_i \vee \alpha_i')$ and we have

$$\beta = (\beta \cap \alpha_i) \vee (\beta \cap \alpha_i').$$

Since $\beta \nleq \alpha_i$ and β is join irreducible, we get $\beta = \beta \cap \alpha_i'$. Hence $\beta \leq \alpha_i'$ and consequently, $\beta \cap \alpha_i \leq \alpha_i \cap \alpha_i' = 0_A$. Thus $\beta \cap \alpha_i = 0_A$. Now, by (UC),

$$\alpha_i \cap \alpha' = \alpha_i \cap \bigvee \Psi = \bigvee \{\alpha_i \cap \bigvee \Phi : \Phi \in \mathrm{F}(\Psi)\}.$$

But $\alpha_i \cap \bigvee \Phi = \bigvee \{\alpha_i \cap \beta : \beta \in \Phi\} = 0_A$, since α_i is neutral and $\alpha_i \cap \beta = 0_A$. Therefore, $\alpha_i \cap \alpha' = 0_A$.

Compute:

$$\begin{aligned} \alpha \cap \alpha' &= \alpha' \cap \bigvee \{\alpha_i : i \in I\} \\ &= \bigvee \{\alpha' \cap \bigvee \{\alpha_j : j \in J\} : J \in \mathrm{F}(I)\} \quad \text{(use (UC))} \\ &= \bigvee \{\alpha' \cap \alpha_i : i \in I\} \quad \text{(since } \alpha_i \ (i \in I) \text{ are neutral)} \\ &= 0_A. \end{aligned}$$

Thus $1_A = \alpha \vee \alpha'$ and $\alpha \cap \alpha' = 0_A$. But α' permutes with all congruences α_i, $i \in I$, and hence α' permutes with $\bigvee \Gamma = \alpha$ (see Lemma 3.1 of Dilworth [1950]). Consequently, $1_A = \alpha \circ \alpha'$ and $\alpha \cap \alpha' = 0_A$, i.e., $\alpha \in \mathrm{DCon}(A)$. ∎

COROLLARY 4.18. *If A is an algebra such that* Con(A) *is a distributive lower continuous lattice*, *then A is isomorphic to a weak direct product of directly indecomposable algebras.*

Proof. By lower continuity of Con(A), 1_A is a join of join irreducible elements of Con(A). Therefore, the conclusion follows from Theorem 4.17. ∎

Since every complete Boolean algebra is lower continuous, we deduce from Corollary 4.18 the following

COROLLARY 4.19. *Let A be any algebra whose congruence lattice is a Boolean algebra. Then A can be decomposed into a weak direct product of directly indecomposable factors.*

Recall that a complete lattice L is called *completely distributive* if for arbitrary sets I, J_i $(i \in I)$ the identity

$$\bigwedge \{\bigvee \{a_{ij} : j \in J_i\} : i \in I\} = \bigvee \{\bigwedge \{a_{ip(i)} : i \in I\} : p \in \prod (J_i : i \in I)\}$$

(or the dual one) holds in L.

It is well known that if Con(A) is completely distributive, then any $\alpha \in \mathrm{Con}(A)$ is a join of join irreducible congruences on A. Hence, if Con(A) is a completely distributive lattice, then the assumptions of Theorem 4.17 are satisfied. Therefore we have

COROLLARY 4.20 (Draškovičová [1987], Theorem 1.7). *If* Con(A) *is a completely distributive lattice*, *then the algebra A is isomorphic to a weak direct product of directly indecomposable algebras.*

COROLLARY 4.21. *A relatively complemented lattice L satisfying the descending chain condition is isomorphic to a direct product of finitely many directly indecomposable relatively complemented lattices.*

Proof. By Theorem 4.3 of Dilworth [1950], $\mathrm{Con}(L)$ satisfies the descending chain condition (i.e., each nonempty subset of $\mathrm{Con}(L)$ has a minimal element). Hence $\mathrm{Con}(L)$ is lower continuous. From Corollary 4.18 we conclude that $L \cong \prod^W (L_i : i \in I)$, where L_i $(i \in I)$ are directly indecomposable lattices. According to Lemma 4.5 we may assume that each L_i is L/θ_i, with

$$0_L = \bigcap\{\theta_i : i \in I\} \quad \text{and} \quad 1_L = \bigvee\{\overline{\theta}_i : i \in I\}.$$

It is easy to see that

$$1_L > \bigvee\{\overline{\theta}_i : i \neq i_1\} > \ldots > \bigvee\{\overline{\theta}_i : i \neq i_1, \ldots, i_n\} > \ldots$$

But $\mathrm{Con}(L)$ satisfies the descending chain condition, and therefore, I is finite. Thus, L is isomorphic to a direct product of lattices L_i, which clearly must be relatively complemented. ■

Let L be a lattice. We say that L is *discrete* if all bounded chains in L are finite (see Jakubík [1971]). L is called *weakly discrete* if there exists a maximal finite chain between any comparable elements (Draškovičová [1987]). Each discrete lattice is weakly discrete.

Observe that, if L is weakly discrete, then 1_L is a join of join irreducible congruences on L. Obviously

$$1_L = \bigvee\{\mathrm{Cg}^L(a,b) : a, b \in L,\ a < b\},$$

where $\mathrm{Cg}^L(a,b)$ is the congruence relation on L generated by $\{(a,b)\}$. Let

$$a = a_0 \prec a_1 \prec \ldots \prec a_{n-1} \prec a_n = b.$$

Then

$$\mathrm{Cg}^L(a,b) = \bigvee\{\mathrm{Cg}^L(a_i, a_{i+1}) : i = 0, \ldots, n-1\}.$$

It is clear that $\mathrm{Cg}^L(a_i, a_{i+1})$, $i = 0, \ldots, n-1$, are join irreducible congruences. Consequently, 1_L is a join of join irreducible congruences on L. Therefore, Theorem 4.17 yields

Corollary 4.22 (Draškovičová [1987] and Jakubík [1971]). *If a lattice L is weakly discrete or if L is discrete, then L is isomorphic to a weak direct product of directly indecomposable lattices.*

5. $\langle \mathcal{L}, \varphi \rangle$-representations of algebras

5.1. Introduction. Let $(A_i : i \in I)$ be a system of similar algebras. Recall that $\prod(A_i : i \in I)$ or $\prod A_i$ denotes the direct product of algebras. If $A = A_i$ for all $i \in I$, we write A^I and call it a *direct power* of A. In case $I = \{1, 2\}$, we write $A_1 \times A_2$.

Let $A \subseteq \prod A_i$ be a subdirect product. Then A is called a *full subdirect product* of the A_i, $i \in I$, if the condition (A2) of Definition 4.1 is satisfied.

Obviously, any weak direct product of the algebras A_i $(i \in I)$ is a full subdirect product of them. If I is finite, then the concepts of the weak direct product, full subdirect product and direct product coincide.

Let I be a nonvoid set. Let $\mathcal{P} = \mathrm{P}(I)$ and $\mathcal{F} = \mathrm{F}(I)$ denote the sets of all subsets and of all finite subsets of I, respectively. We denote by $\mathbf{P}(I)$ the Boolean algebra $\langle \mathrm{P}(I), \cap, \cup, ', \emptyset, I\rangle$. The notation $\mathcal{L} \trianglelefteq \mathbf{P}(I)$ means that $\mathcal{L}$ is an ideal of $\mathbf{P}(I)$.

Walendziak [1994a] introduced the following concept:

Definition 5.1. Let A_i $(i \in I)$ be similar algebras and let $\mathcal{L} \trianglelefteq \mathbf{P}(I)$. We say that a subalgebra A of the direct product $\prod(A_i : i \in I)$ is an *$\mathcal{L}$-restricted full subdirect product* of the algebras A_i, $i \in I$, and write

$$A = \prod\nolimits^{\mathcal{L}}(A_i : i \in I)$$

iff the following two conditions hold:

(B1) A is a full subdirect product of A_i, $i \in I$.

(B2) For all $x, y \in A$, $\mathrm{I}(x, y) \in \mathcal{L}$.

This notion is a common generalization of weak direct products ($\mathcal{L} = \mathrm{F}(I)$) and full subdirect products ($\mathcal{L} = \mathrm{P}(I)$).

Let $A \subseteq \prod(A_i : i \in I)$ be a subdirect product and let $\mathcal{L}$ be an ideal of $\mathbf{P}(I)$. Then A is called an *$\mathcal{L}$-restricted subdirect product* (see Hashimoto [1957], p. 92) if it satisfies (B2). If, in addition, A has the property that for every $x \in A$ and every $y \in \prod A_i$, $\mathrm{I}(x, y) \in \mathcal{L}$ implies $y \in A$, then we say that A is an *$\mathcal{L}$-restricted direct product* (see Grätzer [1979], p. 140 or Walendziak [1991a] , p. 219). These notions are generalized in

Definition 5.2 (Walendziak [1998]). Let A be a subdirect product of algebras A_i, $i \in I$, and let $\mathcal{L}$, $\mathcal{L}'$ be ideals of $\mathbf{P}(I)$. We say that A is an *$\langle\mathcal{L}, \mathcal{L}'\rangle$-product* of A_i, and we write

$$A = \prod\nolimits_{\mathcal{L}}^{\mathcal{L}'}(A_i : i \in I), \quad \text{or} \quad A = \prod\nolimits_{\mathcal{L}}^{\mathcal{L}'} A_i,$$

if A satisfies (B2) and the following condition:

(B3) $(x \in A,\ y \in \prod A_i, \text{ and } \mathrm{I}(x, y) \in \mathcal{L}') \Rightarrow y \in A.$

Obviously, $A = \prod_{\mathcal{L}}^{\mathcal{L}} A_i$ if A is an $\mathcal{L}$-restricted direct product of algebras A_i, $i \in I$. In particular, if $\mathcal{L} = \mathcal{L}' = \mathcal{P}$ we obtain the direct product. If $\mathcal{L}' = \{\emptyset\}$ in Definition 5.2, we get the concept of an $\mathcal{L}$-restricted subdirect product. We note that if $\mathcal{L} = \mathcal{P}$, then an $\mathcal{L}$-restricted subdirect product is a subdirect product. It is easily seen that $\prod_{\mathcal{L}} A_i$ is an $\mathcal{L}$-restricted full subdirect product of the A_i, $i \in I$. Finally, a full subdirect product is a $\langle\mathcal{P}, \mathcal{F}\rangle$-product.

Definition 5.3 (Walendziak [1992], Definition 1). Let A_i $(i \in I)$ be algebras of the same type, $B = \prod(A_i : i \in I)$, and let $\psi \subseteq B \times B$. We say that a subdirect product A of A_i $(i \in I)$ is a *ψ-product* of these algebras if the following condition holds:

(C1) For every $(x_i : i \in I) \in A^I$, if $(x_i, x_j) \in \psi$ for each $i, j \in I$, then $(x_i(i) : i \in I) \in A$.

We note that the concept of ψ-product could be explained as some form of "convexity" (see Walendziak [1993b], p. 320). Observe that subdirect and direct products of algebras are special cases of ψ-products. Indeed, let A be a subalgebra of the direct product B of similar algebras A_i $(i \in I)$. It is obvious that A is a subdirect product if and only if A is a 0_B-product of algebras A_i, $i \in I$.

Now we claim that

$$A \text{ is a } 1_B\text{-product of the } A_i \ (i \in I) \ \Leftrightarrow \ A = B.$$

Clearly, B is a 1_B-product. Conversely, let $x \in B$ and $x(i) = a_i \in A_i$ for all $i \in I$. Since A is a subdirect product, there is a system $(x_i : i \in I) \in A^I$ such that $x_i(i) = a_i$ for all $i \in I$. From Definition 5.3 it follows that $(x_i(i) : i \in I) \in A$ and hence $x \in A$.

ψ-products are studied in Walendziak [1993a,b]. In this chapter, generalizing restrictd subdirect, full subdirect, and weak direct products under the name of $\langle \mathcal{L}, \psi \rangle$-products, some classical theorems on direct, subdirect, weak direct and full subdirect representations are deduced from our more general new results.

5.2. $\langle \mathcal{L}, \psi \rangle$-products of algebras. Now we introduce the following concept:

DEFINITION 5.4 (Walendziak [1993c], Definition 1). Let $(A_i : i \in I)$ be a system of similar algebras, $\mathcal{L}$ be an ideal of $\mathbf{P}(I)$, and let ψ be a binary relation on $B = \prod(A_i : i \in I)$. A subalgebra A of B is called an *$\langle \mathcal{L}, \psi \rangle$-product* of the algebras A_i, $i \in I$, if it is a subdirect product (of these algebras) satisfying conditions (B2) and

(C2) If $i \in I$ and $(x, y) \in A^2 \cap \psi$, then $w_i(x, y) \in A$,

where the element $z = w_i(x, y)$ is defined by $z(i) = x(i)$ and $z(j) = y(j)$ for $j \neq i$.

We write $A = \prod_\psi^\mathcal{L}(A_i : i \in I)$, or $A = \prod_\psi^\mathcal{L} A_i$, to denote that A is an $\langle \mathcal{L}, \psi \rangle$-product of A_i, $i \in I$. If $\psi = 1_B$, we write $\prod^\mathcal{L}(A_i : i \in I)$ for $\prod_\psi^\mathcal{L}(A_i : i \in I)$. If $C = A_i$ for all $i \in I$ we call $\prod_\psi^\mathcal{L}(A_i : i \in I)$ an *$\langle \mathcal{L}, \psi \rangle$-power* of C with exponent I.

EXAMPLE 5.5. Let I be an index set and let $G = Z_2^I$, where Z_2 is the two-element group. For $x \in G$, we define the *support* of x, denoted by $\operatorname{supp}(x)$, as $\operatorname{supp}(x) = \{i \in I : x(i) \neq 0\}$. Let I' be a subset of I, and set

$$\mathcal{L} = \{X \cup Y : X \text{ is a finite subset of } I' \text{ and } Y \subseteq I - I'\},$$
$$\psi = \{(x, y) \in G^2 : x(i) = y(i) \text{ for all } i \in I - I'\}.$$

Define

$$H_1 = \{x \in G : x(i) = x(j) \text{ for all } i, j \in I - I'\},$$
$$H_2 = \{x \in G : I' \cap \operatorname{supp}(x) \text{ is finite}\},$$
$$H_3 = \{x \in G : \operatorname{supp}(x) \text{ is finite}\}, \text{ and}$$
$$H_4 = \{x \in G : \operatorname{supp}(x) \text{ is finite or } I - \operatorname{supp}(x) \text{ is finite}\}.$$

It is easy to see that H_1 is a $\langle \mathcal{P}, \psi \rangle$-power of Z_2 with exponent I, and H_2 is an $\mathcal{L}$-resticted full subdirect power. Moreover, $H_1 \cap H_2$ is an $\langle \mathcal{L}, \psi \rangle$-power of Z_2, and H_3 is a weak direct power (that is, $H_3 = \bigoplus(A_i : i \in I)$, where $A_i = Z_2$ for all $i \in I$). Finally, H_4 is a full subdirect power of Z_2, but it is not a weak direct power.

EXAMPLE 5.6. Let I be a set and $(R_i : i \in I)$ be a system of rings. For $x \in \prod R_i$, let $\operatorname{supp}(x) = \{i \in I : x(i) \neq 0\}$. For an infinite cardinal number $\mathfrak{m}$, the *$\mathfrak{m}$-product* of the R_i, $i \in I$, is defined to be the subring

$$\prod\nolimits^\mathfrak{m}(R_i : i \in I) = \{x \in \prod R_i : |\operatorname{supp}(x)| < \mathfrak{m}\}.$$

(See, for example, Dauns [1987].) Let

$$\mathcal{L} = \{J \subseteq I : |J| < \mathfrak{m}\}.$$

Observe that $A = \prod^{\mathfrak{m}} R_i$ is an $\mathcal{L}$-restricted full subdirect product of the R_i. Clearly, $A \subseteq \prod R_i$ is a full subdirect product. Let $x, y \in A$. Since

$$\mathrm{I}(x, y) \subseteq \mathrm{supp}(x) \cup \mathrm{supp}(y),$$

we conclude that $\mathrm{I}(x, y) \in \mathcal{L}$, and therefore, A satisfies (B2). Then $A = \prod^{\mathcal{L}} R_i$.

Example 5.7. Let $(M_i : i \in I)$ be a system of left R-modules, and let $\mathcal{D}$ be a dual ideal of $\mathbf{P}(I)$ such that $I - \{i\} \in \mathcal{D}$ for all $i \in I$. We define the *$\mathcal{D}$-product* of the M_i's to be

$$\textstyle\prod_{\mathcal{D}}(M_i : i \in I) = \{x \in \prod M_i : \{i \in I : x(i) = 0\} \in \mathcal{D}\}.$$

(This notion is due to Loustaunau [1990].) It is easily seen that $\prod_{\mathcal{D}} M_i$ is an $\mathcal{L}$-restricted full subdirect product of modules M_i, where $\mathcal{L} = \{I - J : J \in \mathcal{D}\}$.

Example 5.8. Let L_i, $i \in I$, be lattices with zero, and let $\mathcal{L}$ be an ideal of $\mathbf{P}(I)$ containing all finite subsets of I. We set $L = \prod(L_i : i \in I)$ and define a binary relation θ on L as follows:

$$x\theta y \;\Leftrightarrow\; \mathrm{I}(x, y) \in \mathcal{L}.$$

Since θ is a congrunce relation of L, we can form the lattice L/θ called a *reduced product* of L_i, $i \in I$ (see Grätzer [1979], Section 22, or Grätzer [1978], Chapter V). Let $f : L \to L/\theta$ be the natural epimorphism. The f-inverse image of the zero of L/θ (that is, the set $\{x \in L : f(x) = 0/\theta\}$) is an $\mathcal{L}$-restricted full subdirect product of L_i, $i \in I$.

Proposition 5.9. *Let A, A_i $(i \in I)$ be similar algebras, $B = \prod A_i$, and let ψ be an equivalence relation over B. If A is a ψ-product of the algebras A_i, $i \in I$, then A is a $\langle \mathcal{P}, \psi\rangle$-product of these algebras.*

Proof. Take $i_0 \in I$. Let $x, y \in A$ with $(x, y) \in \psi$, and let $z \in B$ be defined as follows: $z(i_0) = x(i_0)$ and $z(i) = y(i)$ for all $i \in I - \{i_0\}$. We put $x_{i_0} = x$ and $x_i = y$ if $i \neq i_0$. By (C1), $(x_i(i) : i \in I) \in A$. But $(x_i(i) : i \in I) = z$, and therefore $z \in A$. Then (C2) holds, and thus $A = \prod_{\psi}^{\mathcal{P}}(A_i : i \in I)$. ∎

Remark 5.10. The converse of Proposition 5.9 is not true in general: the group H_4 (see Example 5.5) is a $\langle \mathcal{P}, 1_B\rangle$-power of Z_2, but it is not a direct power.

Proposition 5.11. *Let A be a subalgebra of $B = \prod A_i$ and let $\mathcal{L}$ be an ideal of $\mathbf{P}(I)$. Then*:

(i) $A = \prod_{0_B}^{\mathcal{P}}(A_i : i \in I)$ *iff A is a subdirect product of the algebras A_i, $i \in I$.*
(ii) $A = \prod_{0_B}^{\mathcal{L}}(A_i : i \in I)$ *iff A is an $\mathcal{L}$-restricted subdirect product of the A_i, $i \in I$.*
(iii) $A = \prod^{\mathcal{L}}(A_i : i \in I)$ *iff A is an $\mathcal{L}$-restricted full subdirect product of the A_i, $i \in I$.*
(iv) $A = \prod^{\mathcal{P}} A_i$ *iff A is a full subdirect product of A_i.*
(v) $A = \prod^{\mathcal{F}} A_i$ *iff A is a weak direct product.*

Proof. The statements (i)–(iv) are obvious. To prove (v), assume first that A is an $\langle \mathcal{F}, 1_B\rangle$-product of the A_i, $i \in I$. It is clear that (A1) holds. Observe that (A2) is also satisfied. Let $x \in A$, $y \in B$ and suppose that the set $\mathrm{I}(x, y)$ contains only one element i_1. Since A is a subdirect product of A_i $(i \in I)$, there is $t \in T$ such that $t(i_1) = y(i_1)$. Take $z = w_{i_1}(t, x)$. By Definition 5.4, $z \in A$. Since $y = z$, we have $y \in A$. From this we see by induction that (A2) holds. Then A is a weak direct product of algebras A_i, $i \in I$. Conversely, assume that A satisfies conditions (A1) and (A2). It is easy to see that conditions (C1) and (C2) hold with $\mathcal{L} = \mathcal{F}$ and $\psi = 1_B$. Therefore, $A = \prod^{\mathcal{F}}(A_i : i \in I)$. ∎

5.3. φ-product of congruences and φ-isotopy. Let $\{\theta_i : i \in I\}$ be a set of congruences of an algebra A. For any set $M \subseteq I$ we define

$$\theta(M) = \bigcap\{\theta_j : j \in I - M\}.$$

We shall use the notation $\overline{\theta}_i$ for $\theta(\{i\})$, $i \in I$. Let φ be a binary relation on A, and let $\mathcal{L}$ be an ideal of $\mathbf{P}(I)$. For $\alpha \in \mathrm{Con}(A)$, we write

$$\alpha = \prod\nolimits_{\varphi}^{\mathcal{L}}\{\theta_i : i \in I\}, \quad \text{or} \quad \alpha = \prod\nolimits_{\varphi}^{\mathcal{L}} \theta_i,$$

if the following conditions are satisfied:

(D0) $\alpha = \bigcap\{\theta_i : i \in I\}$.
(D1) $1_A = \bigvee\{\theta(M) : M \in \mathcal{L}\}$.
(D2) For all $i \in I$, $\varphi \subseteq \theta_i \circ \overline{\theta}_i$.

If $\mathcal{L} = \mathrm{P}(I)$ we write

$$\alpha = \prod\nolimits_{\varphi}\{\theta_i : i \in I\} \tag{1}$$

instead of $\alpha = \prod_{\varphi}^{\mathcal{L}}\{\theta_i : i \in I\}$, and we say that α is the *φ-product* of the congruences θ_i $(i \in I)$. In this case, if $I = \{1, \ldots, n\}$, we write $\alpha = \theta_1 \times_{\varphi} \ldots \times_{\varphi} \theta_n$. For abbreviation, we let $\prod^{\mathcal{L}}\{\theta_i : i \in I\}$ stand for $\prod_{1_A}^{\mathcal{L}}\{\theta_i : i \in I\}$. If the set $\{\theta_i : i \in I\}$ is meet irredundant, then we say that (1) is an *irredundant φ-product decomposition* of α.

It is easy to see that if $\mathcal{L} = \mathrm{P}(I)$, then the condition (D1) holds. Therefore,

$$\alpha = \prod\nolimits_{\varphi}\{\theta_i : i \in I\} \;\Leftrightarrow\; \alpha = \bigcap \theta_i \text{ and } \varphi \subseteq \theta_i \circ \overline{\theta}_i \text{ for each } i. \tag{2}$$

Hence,

$$\alpha = \theta_1 \times_{\varphi} \theta_2 \;\Leftrightarrow\; \alpha = \theta_1 \cap \theta_2 \text{ and } \varphi \subseteq (\theta_1 \circ \theta_2) \cap (\theta_2 \circ \theta_1).$$

From (2) we also have

PROPOSITION 5.12. *We have*:

(i) $\alpha = \prod_{0_A}\{\theta_i : i \in I\}$ *iff* $\alpha = \bigcap\{\theta_i : i \in I\}$.
(ii) $\alpha = \prod\{\theta_i : i \in I\}$ *iff* $\alpha = \bigcap\{\theta_i : i \in I\}$ *and* $1_A = \theta_i \circ \overline{\theta}_i$ *for each* i.

PROPOSITION 5.13. *Let A be a congruence permutable algebra*, $\alpha \in \mathrm{Con}(A)$, *and let θ_i $(i \in I)$ be congruences of A such that $\alpha = \bigcap\{\theta_i : i \in I\}$. Then*

$$\alpha = \prod\nolimits^{\mathcal{F}}\{\theta_i : i \in I\} \;\Leftrightarrow\; 1_A = \bigvee\{\overline{\theta}_i : i \in I\}.$$

Proof. Let $\alpha = \prod^{\mathcal{F}} \theta_i$. Then $1_A = \theta_i \circ \overline{\theta}_i$ for all i, and

$$1_A = \bigvee\{\theta(M) : M \in \mathcal{F}\}. \tag{3}$$

Observe that

(4) $$\text{For every } \emptyset \neq M \in \mathcal{F}, \quad \theta(M) \leq \bigvee\{\overline{\theta}_i : i \in M\}.$$

We apply induction on $|M|$. The case $|M| = 1$ is trivial. Assume that the inequality holds for all $M \subseteq I$ with $|M| < n$. Let $M = \{1, \ldots, n\} \subseteq I$ and $x, y \in \theta(M)$. Since $1_A = \theta_n \circ \overline{\theta}_n$, there is a $z \in A$ such that $(x, z) \in \theta_n$ and $(z, y) \in \overline{\theta}_n$. Therefore, $(x, z) \in \theta(\{1, \ldots, n-1\})$, and by the induction hypothesis, $(x, z) \in \overline{\theta}_1 \vee \ldots \vee \overline{\theta}_{n-1}$. Then $(x, y) \in \bigvee\{\overline{\theta}_i : i \in M\}$, and consequently, we obtain (4). From this and (3) we conclude that $1_A = \bigvee\{\overline{\theta}_i : i \in I\}$.

For the converse, let $1_A = \bigvee\{\overline{\theta}_i : i \in I\}$. Hence we get (3). Let $i \in I$. Obviously, $\bigvee\{\overline{\theta}_j : j \neq i\} \leq \theta_i$, and therefore, $1_A = \theta_i \vee \overline{\theta}_i$. Then $1_A = \theta_i \circ \overline{\theta}_i$, since the congruences of A permute. Thus $\alpha = \prod^{\mathcal{F}} \theta_i$. ■

Definition 5.14. Let φ be a binary relation on an algebra A. An element $\alpha \in \mathrm{Con}(A)$ is called *φ-indecomposable* if $\alpha \neq 1$ and if $\alpha = \theta_1 \times_\varphi \theta_2$, then $\alpha = \theta_1$ or $\alpha = \theta_2$.

A trivial verification shows that the following proposition holds.

Proposition 5.15. *Let $\alpha \in \mathrm{Con}(A)$ with $\alpha \neq 1$. Then*:

(i) *α is 0-indecomposable iff α is a meet irreducible element of* $\mathrm{Con}(A)$.

(ii) *α is 1-indecomposable iff for any θ_1, $\theta_2 \in \mathrm{Con}(A)$, if $\alpha = \theta_1 \times_1 \theta_2$, then $\theta_1 = 1$ or $\theta_2 = 1$* (i.e., α is indecomposable in the sense of McKenzie–McNulty–Taylor [1987], p. 269).

Lemma 5.16. *Let A be an algebra and $\alpha \in \mathrm{Con}(A)$. Then A/α is directly indecomposable iff α is 1-indecomposable.*

Proof. By Lemma 2 of McKenzie–McNulty–Taylor [1987] (p. 269) we deduce that A/α is directly indecomposable iff α is indecomposable. Now using Proposition 5.15(ii) we get the conclusion. ■

Definition 5.17. Let $\varphi \in \mathrm{Con}(A)$. We say that the congruences of an algebra A *φ-permute* if for any congruences α and β on A, $\alpha \cap \varphi$ and $\beta \cap \varphi$ permute.

It is obvious that for every algebra A the congruences of A 0_A-permute and that 1_A-permuting is the same thing as permuting.

Lemma 5.18. *Let φ be a codistributive element of* $\mathrm{Con}(A)$. *Suppose that the congruences of A φ-permute and denote by L the dual lattice of* $\mathrm{Con}(A)$. *Let α, θ_i $(i \in I)$ be congruences on A. Then*

$$\alpha = \prod\nolimits_\varphi\{\theta_i : i \in I\} \iff \alpha = \sum\nolimits_\varphi\{\theta_i : i \in I\} \text{ in } L.$$

Proof. The congruence φ is distributive in L. Assume that $\sum_\varphi\{\theta_i : i \in I\}$ (see Section 3.1). Then

$$\alpha = \bigvee\{\theta_i : i \in I\} \quad \text{and} \quad \theta_i \cap \bigvee\{\overline{\theta}_j : j \neq i\} \subseteq \varphi \text{ for each } i \in I.$$

In other words, $\alpha = \bigcap\{\theta_i : i \in I\}$ and $\varphi \leq \theta_i \vee \bigcap\{\overline{\theta}_j : j \neq i\}$ in $\mathrm{Con}(A)$ for all $i \in I$. Therefore, $\varphi = \varphi \cap (\theta_i \vee \overline{\theta}_i)$, and since φ is codistributive in $\mathrm{Con}(A)$ we obtain $\varphi = (\varphi \cap \theta_i) \vee (\varphi \cap \overline{\theta}_i)$. From the fact that the congruences of A φ-permute we conclude that

$$\varphi = (\varphi \cap \theta_i) \circ (\varphi \cap \overline{\theta}_i),$$

and hence $\varphi \subseteq \theta_i \circ \overline{\theta}_i$ for each $i \in I$. Thus $\alpha = \sum_{\varphi}\{\theta_i : i \in I\}$. The converse is obvious. ∎

LEMMA 5.19. *Let φ be a codistributive element of* $\mathrm{Con}(A)$, *and suppose that the congruences of A φ-permute. Then for $\alpha \in \mathrm{Con}(A)$, α is φ-indecomposable iff α is φ-irreducible in the dual lattice of* $\mathrm{Con}(A)$.

Proof. This follows immediately from Lemma 5.18. ∎

DEFINITION 5.20. Let A be an algebra and let φ be a congruence relation on A.

(i) The congruences α and β on A are said to be *φ-isotopic* (in symbols, $\alpha \approx_{\varphi} \beta$) if $0 = \alpha \times_{\varphi} \gamma = \beta \times_{\varphi} \gamma$ for some $\gamma \in \mathrm{Con}(A)$ with $\gamma \neq 0$.

(ii) We call algebras B and C *φ-isotopic*, written $B \approx_{\varphi} C$, if there exist φ-isotopic congruences α and β on A such that $B \cong A/\alpha$ and $C \cong A/\beta$. To shorten notation, we let $B \approx C$ stand for $B \approx_1 C$.

LEMMA 5.21 (see Walendziak [1993c], Lemma 7). *Let A be a congruence distributive algebra, and let $\alpha, \beta \in \mathrm{Con}(A)$. If α and β are meet irreducible and 0_A-isotopic, then $\alpha = \beta$.*

LEMMA 5.22. *Let A, B and C be algebras and let A have a one-element subalgebra. If $B \approx C$, then $B \cong C$.*

Proof. Let α and β be 1_A-isotopic congruences on A such that $B \cong A/\alpha$ and $C \cong A/\beta$. By the proof of Lemma 6 of Walendziak [1993c] we conclude that $A/\alpha \cong A/\beta$. Therefore, $B \cong C$. ∎

5.4. $\langle \mathcal{L}, \varphi \rangle$-representations of algebras—a characterization theorem

DEFINITION 5.23. Let A_i $(i \in I)$ and A be similar algebras. Let φ be a binary relation on A, and let $\mathcal{L}$ be an ideal of the Boolean algebra $\mathbf{P}(I)$. If $f : A \to \prod A_i$ is an embedding such that $f(A) = \prod_{\psi}^{\mathcal{L}}(A_i : i \in I)$ where $\psi = f(\varphi)$, then we say that $\langle (A_i : i \in I), f \rangle$ is an *$\langle \mathcal{L}, \varphi \rangle$-representation* of A. In this case, we also say that A is isomorphic to an $\langle \mathcal{L}, \psi \rangle$-product of algebras A_i $(i \in I)$, and write

$$A \cong \textstyle\prod_{\psi}^{\mathcal{L}}(A_i : i \in I).$$

For each index $i \in I$, we denote by f_i the ith *f-projection function* from A onto A_i, that is,

$$f_i(x) = (f(x))(i) \quad (x \in A).$$

It is easy to see that if $f : A \to \prod A_i$ is an embedding, then $\langle (A_i : i \in I), f \rangle$ is an $\langle \mathcal{L}, \varphi \rangle$-representation of A iff the following two conditions hold:

(C1′) For all $x, y \in A$, $\mathrm{I}(f(x), f(y)) \in \mathcal{L}$.

(C2′) If $i \in I$ and $(x, y) \in \varphi$, then $w_i(f(x), f(y)) \in f(A)$.

An $\langle \mathcal{L}, \varphi \rangle$-representation $\langle (A_i : i \in I), f \rangle$ of A is called

(i) *subdirect* if $\mathcal{L} = \mathcal{P}$ and $\varphi = 0_A$,
(ii) *$\mathcal{L}$-restricted subdirect* if $\varphi = 0_A$,
(iii) *finitely restricted subdirect* if $\mathcal{L} = \mathcal{F}$ and $\varphi = 0_A$,
(iv) *full subdirect* if $\mathcal{L} = \mathcal{P}$ and $\varphi = 1_A$,
(v) *$\mathcal{L}$-restricted full subdirect* if $\varphi = 1_A$,
(vi) *weak direct* if $\mathcal{L} = \mathcal{F}$ and $\varphi = 1_A$.

For a system $(\theta_i : i \in I)$ of congruences on A, we denote by f_θ the function from A to $\prod(A/\theta_i : i \in I)$ defined by

$$f_\theta(x) = (x/\theta_i : i \in I) \quad (x \in A).$$

If f is a function from A to B, then the *kernel* of f, written $\ker(f)$, is defined to be the binary relation $\{(a, b) \in A^2 : f(a) = f(b)\}$.

A trivial verification shows that the following holds.

Proposition 5.24. *Let $\langle (A_i : i \in I),\ f \rangle$ be an $\langle \mathcal{L}, \varphi \rangle$-representation of A and let* $\ker(f_i) = \theta_i$ $(i \in I)$. *Then $\langle (A/\theta_i : i \in I), f_\theta \rangle$ is also an $\langle \mathcal{L}, \varphi \rangle$-representation of A.*

The next result is a characterization theorem for $\langle \mathcal{L}, \varphi \rangle$-representations.

Theorem 5.25 (Walendziak [1993c], Theorem 1). *Let A be an algebra*, $\varphi \subseteq A^2$, *and let $(\theta_i : i \in I)$ be a system of congruences on A. Let $\mathcal{L}$ be an ideal of* $\mathbf{P}(A)$. *Then $\langle (A/\theta_i : i \in I), f_\theta \rangle$ is an $\langle \mathcal{L}, \varphi \rangle$-representation of A iff* $0_A = \prod_{\varphi}^{\mathcal{L}} \{\theta_i : i \in I\}$.

Proof. Necessity. Since f_θ is one-to-one, (D0) holds for $\alpha = 0_A$. To prove (D1), let $x, y \in A$. Clearly,

$$M = \{i \in I : x/\theta_i \neq y/\theta_i\} = \{i \in I : f_\theta(x)(i) \neq f_\theta(y)(i)\} \in \mathcal{L},$$

and hence $(x, y) \in \theta(M)$. Then $(x, y) \in \bigvee\{\theta(M) : M \in \mathcal{L}\}$, and therefore, (D1) is satisfied. Moreover, (D2) follows from (C2′). Indeed, fix an $i \in I$ and let $(x, y) \in \varphi$. Set $x' = f_\theta(x)$ and $y' = f_\theta(y)$. By (C2′), $z' = w_i(x', y') \in A' = f_\theta(A)$. Let $z = f^{-1}(z')$. It is easy to see that $(x, z) \in \theta_i$ and $(z, y) \in \overline{\theta}_i$. Consequently, $\varphi \subseteq \theta_i \circ \overline{\theta}_i$.

Sufficiency. It is obvious that f_θ is an embedding and that A' is a subdirect product of the algebras $A_i = A/\theta_i$, $i \in I$. Let $x, y \in A$. Now we prove that

$$M = \{i \in I : x/\theta_i \neq y/\theta_i\} \in \mathcal{L}. \tag{5}$$

By (D1), $(x, y) \in \bigvee\{\theta(M) : M \in \mathcal{L}\}$. So there are finitely many sets $M_1, \ldots, M_n \in \mathcal{L}$ such that $(x, y) \in \theta(M_1) \vee \ldots \vee \theta(M_n)$. Observe that

$$M \subseteq M_1 \cup \ldots \cup M_n. \tag{6}$$

Indeed, let $x/\theta_i \neq y/\theta_i$ for some $i \in I$, and suppose on the contrary that $i \notin M_1 \cup \ldots \cup M_n$. Therefore, $\theta(M_1) \vee \ldots \vee \theta(M_n) \leq \theta_i$, and hence $(x, y) \in \theta_i$, that is, $x/\theta_i = y/\theta_i$, a contradiction. From (6), by the definition of ideal we deduce that $M \in \mathcal{L}$. Thus (5) is satisfied. Now let $i \in I$, and let $x', y' \in A'$ be such that $(x', y') \in \psi = f_\theta(\varphi)$. Take $x, y \in A$ with $x' = f_\theta(x)$ and $y' = f_\theta(y)$. Obviously, $(x, y) \in \varphi$ and by (D2) there exists $z \in A$ such that $(x, z) \in \theta_i$ and $(z, y) \in \overline{\theta}_i$. Hence $w_i(x', y') = f_\theta(z) \in A'$. Therefore, $A' = \prod_{\psi}^{\mathcal{L}}(A/\theta_i : i \in I)$, which was to be proved. ∎

The following well known fact is a consequence of Theorem 5.25 and Proposition 5.12(i).

COROLLARY 5.26. *$\langle(A/\theta_i : i \in I),\ f_\theta\rangle$ is a subdirect representation of A iff $0 = \bigcap\{\theta_i : i \in I\}$.*

Let $\mathcal{L}$ be an ideal of $\mathbf{P}(I)$. Using Theorem 5.25 we obtain

COROLLARY 5.27. *$\langle(A/\theta_i : i \in I),\ f_\theta\rangle$ is an $\mathcal{L}$-restricted subdirect representation of A iff $0 = \bigcap\{\theta_i : i \in I\}$ and $1 = \bigvee\{\theta(M) : M \in \mathcal{L}\}$.*

By Theorem 5.25 and Proposition 5.12(ii) we have

COROLLARY 5.28. *$\langle(A/\theta_i : i \in I),\ f_\theta\rangle$ is a full subdirect representation of A iff $0 = \bigcap\{\theta_i : i \in I\}$ and $1 = \theta_i \circ \overline{\theta}_i$ for all $i \in I$.*

Combining Theorem 5.25 and Proposition 5.13 we get

COROLLARY 5.29 (Hu [1969], Lemma 11). *Let A be an algebra whose congruences permute. Then $\langle(A/\theta_i : i \in I), f_\theta\rangle$ is a weak direct representation of A iff $0 = \bigcap\{\theta_i : i \in I\}$ and $1 = \bigvee\{\overline{\theta}_i : i \in I\}$.*

It is easy to verify the following

COROLLARY 5.30 (Walendziak [1991a], Corollary 8). *If the congruences of an algebra A permute and I is a finite set, then $\langle(A/\theta_i : i \in I), f_\theta\rangle$ is a direct representation of A iff $0 = \bigcap \theta_i$ and $1 = \theta_i \vee \overline{\theta}_i$ for all $i \in I$.*

Now we define the notion of an irredundant $\langle\mathcal{L}, \varphi\rangle$-representation.

DEFINITION 5.31. Let $\langle(A_i : i \in I), f\rangle$ be an $\langle\mathcal{L}, \varphi\rangle$-representation of an algebra A. For each $i \in I$, define the mapping $\overline{f}_i$ of A to $\prod(A_j : j \neq i)$ by

$$\overline{f}_i(x)(j) = f_j(x) \quad \text{for all } j \neq i.$$

If none of the mappings $\overline{f}_i$ $(i \in I)$ is an embedding we say that the $\langle\mathcal{L}, \varphi\rangle$-representation $\langle(A_i : i \in I), f\rangle$ is *irredundant*.

THEOREM 5.32. *An $\langle\mathcal{L}, \varphi\rangle$-representation $\langle(A_i : i \in I), f\rangle$ of A is irredundant iff the set $\{\ker(f_i) : i \in I\}$ is meet irredundant.*

Proof. We put $\theta_i = \ker(f_i)$ for $i \in I$, and assume the $\langle\mathcal{L}, \varphi\rangle$-representation $\langle(A_i : i \in I), f\rangle$ of A to be irredundant. Suppose on the contrary that $0 = \bigcap\{\theta_j : j \neq i\}$ for some $i \in I$. For each $x \in A$, let $\overline{f}_i(x)$ be the restriction of $f(x)$ to $I - \{i\}$. Obviously $\overline{f}_i$ is an embedding, contrary to our assumption.

Conversely, let $\{\theta_i : i \in I\}$ be meet irredundant. Assume that $\overline{f}_i$ is one-to-one for some $i \in I$, and let $(x, y) \in \bigcap\{\theta_j : j \neq i\}$. Then $f_j(x) = f_j(y)$ for each $j \neq i$. Consequently, $\overline{f}_i(x) = \overline{f}_i(y)$, and hence $x = y$. Therefore, $0 = \bigcap\{\theta_j : j \neq i\}$, and thus $\{\theta_i : i \in I\}$ is meet irredundant, a contradiction. ■

It is easy to see that the following proposition holds.

PROPOSITION 5.33. *Let $\langle(A_i : i \in I), f\rangle$ be an $\langle\mathcal{L}, 1_A\rangle$-representation of A. If $|A_i| > 1$ for each $i \in I$, then the representation is irredundant.*

5.5. The existence of irredundant $\langle\mathcal{L},\varphi\rangle$-representations. First we present the following result:

The First Existence Theorem 5.34. *Let φ be a codistributive element of* $\mathrm{Con}(A)$*. Suppose that the congruences of A φ-permute and* $\mathrm{Con}(A)$ *is semimodular and atomistic. Then there exists a system $(A_i : i \in I)$ of simple algebras and an embedding $f : A \to \prod A_i$ such that $\langle (A_i : i \in I), f\rangle$ is an irredundant $\langle\mathcal{L},\varphi\rangle$-representation of A, where $\mathcal{L}$ is an ideal of $\mathbf{P}(I)$ containing all finite subsets of I.*

Proof. By Lemma 4.5, there exist coatoms θ_i $(i \in I)$ of $\mathrm{Con}(A)$ such that $0_A = \bigcap\{\theta_i : i \in I\}$ and $1_A = \bigvee\{\overline{\theta}_i : i \in I\}$. Let $\mathcal{L}$ be an ideal of $\mathbf{P}(I)$ with $\mathcal{F} \subseteq \mathcal{L}$. We have

$$1_A = \bigvee\{\overline{\theta}_i : i \in I\} = \bigvee\{\theta(\{i\}) : i \in I\} \le \bigvee\{\theta(M) : M \in \mathcal{L}\},$$

and therefore condition (D1) holds.

Let $i \in I$. Since $\overline{\theta}_j \le \theta_i$ for all $j \ne i$, we conclude that $1_A = \theta_i \vee \overline{\theta}_i$. Now it is easy to see that the set $\{\theta_i : i \in I\}$ is meet irredundant. Moreover, since φ is codistributive and the congruences of A φ-permute, we get

$$\varphi = \varphi \cap (\theta_i \vee \overline{\theta}_i) = (\varphi\cap\theta_i) \vee (\varphi \cap \overline{\theta}_i) = (\varphi\cap\theta_i)\circ(\varphi\cap\overline{\theta}_i).$$

Hence $\varphi \subseteq \theta_i \circ \overline{\theta}_i$, that is, condition (D2) is satisfied. Thus

$$0_A = \textstyle\prod_{\varphi}^{\mathcal{L}}\{\theta_i : i \in I\}.$$

By Theorem 5.25, $\langle (A/\theta_i : i \in I), f_\theta\rangle$ is an $\langle\mathcal{L},\varphi\rangle$-representation of A. This representation is irredundant, because $\{\theta_i : i \in I\}$ is meet irredundant. Since θ_i is a coatom of $\mathrm{Con}(A)$, we conclude that A/θ_i is simple. ∎

Corollary 5.35. *Let A be any algebra with* $\mathrm{Con}(A)$ *semimodular and atomistic. Then A is isomorphic to an irredundant subdirect product of simple algebras.*

Let S be a semilattice. By Corollary 2 of Hall [1971] we know that $\mathrm{Con}(S)$ is a semimodular lattice. If each interval of S is a finite chain (i.e., S is a locally finite tree), then $\mathrm{Con}(S)$ is also atomistic (see Auinger [1990]). It is well known that a semilattice S is simple iff $|S| = 2$. Thus, we have

Corollary 5.36. *Every locally finite tree is isomorphic to an irredundant subdirect product of two-element semilattices.*

Note that in Crawley–Dilworth [1973] (see Theorem 4.2) it is shown that every algebraic complemented modular lattice is atomistic. Therefore Theorem 5.34 implies

Corollary 5.37 (Hashimoto [1957], Theorem 5.1). *If the congruence lattice of an algebra A is complemented and modular, then there is an irredundant finitely restricted subdirect representation of A with simple factors.*

Remark 5.38. Theorem 5.34 also gives the following result of Tanaka [1952]: If A is an algebra with a Boolean congruence lattice, then A is a subdirect product of simple algebras.

We know (see Crawley–Dilworth [1973], Theorem 10.7) that if a lattice L has the projectivity property and if L is weakly discrete, then $\mathrm{Con}(L)$ is a Boolean algebra. Thus, from Corollary 5.37 we obtain

Corollary 5.39. *If a weakly discrete lattice L has the projectivity property, then L is isomorphic to an irredundant finitely restricted subdirect product of simple lattices.*

It is well known that every algebra whose congruences permute has modular congruence lattice. Therefore we get

Corollary 5.40. *Let A be any algebra whose congruences permute and whose congruence lattice is complemented. Then there exists a full subdirect representation of A with simple factors.*

The Second Existence Theorem 5.41. *Let φ be a codistributive element of* $\mathrm{Con}(A)$. *Suppose that for all $\alpha \in \mathrm{DCon}(A)$ and $\beta \in \mathrm{Con}(A)$, $\alpha \cap \varphi$ and $\beta \cap \varphi$ permute. If* $\mathrm{DCon}(A)$ *is a modular $\bigvee$-closed sublattice of* $\mathrm{Con}(A)$, *then there is a system $(A_i : i \in I)$ of directly indecomposable algebras and an embedding $f : A \to \prod A_i$ such that $\langle (A_i : i \in I), f\rangle$ is an irredundant $\langle \mathcal{L}, \varphi\rangle$-representation of A, where $\mathcal{L}$ is an ideal of $\mathbf{P}(I)$ with the property that $\mathcal{F} \subseteq \mathcal{L}$.*

Proof. By Lemma 4.3, $\mathrm{DCon}(A)$ is an atomistic lattice. Applying Lemma 4.5 we deduce that there are coatoms θ_i $(i \in I)$ of $\mathrm{DCon}(A)$ such that

$$0_A = \bigcap\{\theta_i : i \in I\} \quad \text{and} \quad 1_A = \bigvee\{\overline{\theta}_i : i \in I\}.$$

Let $\mathcal{L}$ be an ideal of $\mathbf{P}(I)$ containing $\mathcal{F}$. By the proof of Theorem 5.34 we see that $1 = \bigvee\{\theta(M) : M \in \mathcal{L}\}$ and $\varphi \subseteq \theta_i \circ \overline{\theta}_i$ for all $i \in I$. Therefore,

$$0 = \textstyle\prod_{\varphi}^{\mathcal{L}}\{\theta_i : i \in I\}.$$

According to Theorem 5.25, we conclude that $\langle (A/\theta_i : i \in I), f_\theta\rangle$ is an $\langle \mathcal{L}, \varphi\rangle$-representation of A. This representation is irredundant, because the set $\{\theta_i : i \in I\}$ is meet irredundant. Since θ_i is a coatom of $\mathrm{DCon}(A)$, from Lemma 4.4 it follows that every A/θ_i is directly indecomposable. ■

As a consequence of Theorem 5.41 we get the following

Corollary 5.42 (Hashimoto [1957], Theorem 4.2). *Let A be an algebra such that* $\mathrm{DCon}(A)$ *is a modular $\bigvee$-closed sublattice of* $\mathrm{Con}(A)$. *Then A is isomorphic to an irredundant finitely restricted subdirect product with directly indecomposable factors.*

In the case of $\varphi = 0_A$, Theorem 5.41 implies

Corollary 5.43 (Hashimoto [1957], Theorem 4.5). *Let A be any algebra whose congruences permute and whose decomposition congruences form a $\bigvee$-closed sublattice of* $\mathrm{Con}(A)$. *Then there is a weak direct representation of A with directly indecomposable factors.*

From Theorem 6.2 of Hashimoto [1957] we have

Lemma 5.44. *Let A be an algebra with* $\mathrm{Con}(A)$ *distributive. Then* $\mathrm{DCon}(A)$ *is a Boolean sublattice of* $\mathrm{Con}(A)$ *and every element of* $\mathrm{DCon}(A)$ *is permutable with any congruence on A.*

Now we are able to give our existence theorem for restricted full subdirect representations.

THE THIRD EXISTENCE THEOREM 5.45 (Walendziak [1994b], Theorem 4). *Let A be a congruence distributive algebra. If* $\mathrm{DCon}(A)$ *is $\bigvee$-closed in* $\mathrm{Con}(A)$, *then there exists a family A_i $(i \in I)$ of directly indecomposable algebras such that A is isomorphic to an $\mathcal{L}$-restricted full subdirect product of A_i $(i \in I)$, where $\mathcal{L}$ is an ideal of $\mathbf{P}(I)$ containing all finite subsets of I.*

Proof. By Lemma 5.44, every $\alpha \in \mathrm{DCon}(A)$ is permutable with any $\beta \in \mathrm{Con}(A)$. Consequently, if $\varphi = 0_A$, then the hypotheses of Theorem 5.41 are satisfied. Therefore, Theorem 5.41 clearly forces the assertion. ∎

For other existence theorems we refer the reader to Walendziak [1996b].

5.6. Uniqueness theorems. For the next result we need the following

LEMMA 5.46. *Let I, J be two sets of indices and $\mathcal{L}_1, \mathcal{L}_2$ be ideals of the Boolean algebras $\mathbf{P}(I)$, $\mathbf{P}(J)$, respectively. Let A be an algebra with* $\mathrm{Con}(A)$ *distributive. If*

$$0 = \prod\nolimits^{\mathcal{L}_1}\{\alpha_i : i \in I\} = \prod\nolimits^{\mathcal{L}_2}\{\beta_j : j \in J\}$$

for congruences α_i, β_j on A, then there exist congruences γ_{ij} $(i \in I, j \in J)$ such that, for all i and j,

$$\alpha_i = \prod\nolimits^{\mathcal{L}_2}\{\gamma_{ij} : j \in J\} \quad \textit{and} \quad \beta_j = \prod\nolimits^{\mathcal{L}_1}\{\gamma_{ij} : i \in I\}.$$

Proof. For $i \in I$ and $j \in J$ we put $\gamma_{ij} = \alpha_i \vee \beta_j$. Fix $i \in I$. First we show that

$$\alpha_i = \bigcap\{\gamma_{ij} : j \in J\}. \tag{7}$$

By distributivity of $\mathrm{Con}(A)$, for any j we have

$$\overline{\alpha}_i \cap \gamma_{ij} = \overline{\alpha}_i \cap (\alpha_i \vee \beta_j) = \overline{\alpha}_i \cap \beta_j \leq \beta_j.$$

Hence

$$\overline{\alpha}_i \cap \bigcap\{\gamma_{ij} : j \in J\} = \bigcap\{\overline{\alpha}_i \cap \gamma_{ij} : j \in J\} \leq \bigcap\{\beta_j : j \in J\} = 0.$$

Therefore, using distributivity we get

$$\bigcap\{\gamma_{ij} : j \in J\} = \bigcap\{\gamma_{ij} : j \in J\} \cap (\alpha_i \vee \overline{\alpha}_i) = \alpha_i \cap \bigcap\{\gamma_{ij} : j \in J\} = \alpha_i.$$

Thus (7) is satisfied. For $M \in \mathcal{L}_2$ we set $\gamma(M) = \bigcap\{\gamma_{ij} : j \notin M\}$. Now we prove that

$$1 = \bigvee\{\gamma(M) : M \in \mathcal{L}_2\}. \tag{8}$$

Let $x, y \in A$. Then $(x, y) \in \bigvee\{\beta(M) : M \in \mathcal{L}_2\}$. Hence, we can choose a finite number of sets $M_1, \ldots, M_n \in \mathcal{L}_2$ such that $(x, y) \in \beta(M_1) \vee \ldots \vee \beta(M_n)$. We set $M = \{j \in J : (x, y) \notin \gamma_{ij}\}$. Observe that $M \subseteq M_1 \cup \ldots \cup M_n$. Indeed, suppose $j \in M$ and $j \notin M_1 \cup \ldots \cup M_n$. It is obvious that $\beta(M_k) \leq \beta_j$ for each $k = 1, \ldots, n$. Therefore, $\beta(M_1) \vee \ldots \vee \beta(M_n) \leq \beta_j \leq \gamma_{ij}$, which gives us a contradiction. Consequently, $M \subseteq M_1 \cup \ldots \cup M_n$, and hence $M \in \mathcal{L}_2$. Thus $(x, y) \in \gamma(M)$, and we conclude that (8) holds.

For each $j \in J$, write $\overline{\gamma}_{ij}$ for $\bigcap\{\gamma_{ik} : k \in J - \{j\}\}$. Clearly, $\gamma_{ij} \geq \beta_j$ and $\overline{\gamma}_{ij} \geq \overline{\beta}_j$. Since $1 = \beta_j \circ \overline{\beta}_j$ we have

$$1 = \gamma_{ij} \circ \overline{\gamma}_{ij} \tag{9}$$

for all $j \in J$. From (7), (8) and (9) it follows that $\alpha_i = \prod^{\mathcal{L}_2}\{\gamma_{ij} : j \in J\}$. The proof that $\beta_j = \prod^{\mathcal{L}_1}\{\gamma_{ij} : i \in I\}$ is similar. ■

PROPOSITION 5.47. *Under the assumptions of Lemma* 5.46, *if*

$$A \cong \prod\nolimits^{\mathcal{L}_1}(A_i : i \in I) \quad \textit{and} \quad A \cong \prod\nolimits^{\mathcal{L}_2}(B_j : j \in J),$$

then there exist algebras C_{ij} $(i \in I,\ j \in J)$ *such that*, *for all* i *and* j,

$$A_i = \prod\nolimits^{\mathcal{L}_2}(C_{ij} : j \in J) \quad \textit{and} \quad B_j = \prod\nolimits^{\mathcal{L}_1}(C_{ij} : i \in I).$$

Proof. Let $\langle (A_i : i \in I), g\rangle$ be an $\mathcal{L}_1$-restricted full subdirect representation of the algebras A_i and $\langle (B_j : j \in J), h\rangle$ be an $\mathcal{L}_2$-restricted full subdirect representation of the algebras B_j. We set $\alpha_i = \ker(g_i)$ and $\beta_j = \ker(h_j)$ $(i \in I,\ j \in J)$, where g_i is the ith g-projection function and h_j is the jth h-projection function. From Proposition 5.24 and Theorem 5.25 we conclude that

$$0 = \prod\nolimits^{\mathcal{L}_1}\{\alpha_i : i \in I\} = \prod\nolimits^{\mathcal{L}_2}\{\beta_j : j \in J\}.$$

For $i \in I$ and $j \in J$ we set $\gamma_{ij} = \alpha_i \vee \beta_j$. From Lemma 5.46 it follows that

$$\alpha_i = \prod\nolimits^{\mathcal{L}_2}\{\gamma_{ij} : j \in J\} \quad \text{and} \quad \beta_j = \prod\nolimits^{\mathcal{L}_1}\{\gamma_{ij} : i \in I\}.$$

By the proof of Theorem 5.25 we conclude that

$$A/\alpha_i = \prod\nolimits^{\mathcal{L}_2}(A/\gamma_{ij} : j \in J) \quad \text{and} \quad A/\beta_j = \prod\nolimits^{\mathcal{L}_1}(A/\gamma_{ij} : i \in I).$$

Therefore, $A_i = \prod^{\mathcal{L}_2}(C_{ij} : j \in J)$ and $B_j = \prod^{\mathcal{L}_1}(C_{ij} : i \in I)$, where $C_{ij} = A/\gamma_{ij}$. ■

Now we prove the following uniqueness theorem for restricted full subdirect representations of algebras:

THEOREM 5.48 (Walendziak [1994a], Theorem 3). *Let A be a congruence distributive algebra. Let I, J be two sets of indices and $\mathcal{L}_1$, $\mathcal{L}_2$ be ideals of* $\mathbf{P}(I)$, $\mathbf{P}(J)$, *respectively. If A has an $\mathcal{L}_1$-restricted full subdirect representation $\langle (A_i : i \in I), g\rangle$ and also has an $\mathcal{L}_2$-restricted full subdirect representation $\langle (B_j : j \in J), h\rangle$, where the algebras A_i, B_j $(i \in I,\ j \in J)$ are directly indecomposable*, *then there is a bijection $\lambda : I \to J$ for which the following conditions hold*:

(i) *For each $i \in I$, there exists an isomorphism $t_i : A_i \to B_{\lambda(i)}$ such that $t_i \circ g_i = h_{\lambda(i)}$.*
(ii) $\lambda(\mathrm{I}(g(x), g(y))) = \mathrm{J}(h(x), h(y))$ *for all $x, y \in A$.*

Proof. Let α_i $(i \in I)$ and β_j $(j \in J)$ be the kernels of g_i and h_j, respectively. For each $i \in I$ and each $j \in J$ set

$$\gamma_{ij} = \alpha_i \vee \beta_j \quad \text{and} \quad C_{ij} = A/\gamma_{ij}.$$

By Proposition 5.47, $A_i = \prod^{\mathcal{L}_2}(C_{ij} : j \in J)$ and $B_j = \prod^{\mathcal{L}_1}(C_{ij} : i \in I)$. Since A_i is directly indecomposable, there exists a $\lambda(i) = j \in J$ such that $A_i \cong C_{ij}$. We have

$$A/\alpha_i \cong A_i \cong C_{ij} = A/(\alpha_i \vee \beta_j).$$

Then $\alpha_i = \alpha_i \vee \beta_j$, and hence $\alpha_i \geq \beta_j$. Since B_j is directly indecomposable there is a $\sigma(j) = i' \in I$ such that $B_j \cong C_{i'j}$. Now we infer similarly that $\beta_j \geq \alpha_{i'}$. Consequently, $\alpha_i \geq \beta_j \geq \alpha_{i'}$. Observe that $i = i'$. Indeed, if $i \neq i'$, then $\overline{\alpha}_i \leq \alpha_{i'} \leq \alpha_i$, and hence $\alpha_i = 1_A$, contrary to the fact that A_i is directly indecomposable. Therefore, $(\sigma \circ \lambda)(i) = i$

for all $i \in I$, and similarly, $(\lambda \circ \sigma)(j) = j$ for all $j \in J$. Thus σ is a two-sided inverse of λ, and this proves that λ is a bijection.

If $\lambda(i) = j$, then $A_i \cong C_{ij} \cong B_j$ and it is clear that the mapping t_i defined on A_i by $t_i(g_i(x)) = h_j(x)$ is an isomorphism of A_i with B_j.

To prove (ii), let $x, y \in A$. We have

$$\begin{aligned} i \in \mathrm{I}(g(x), g(y)) \;&\Leftrightarrow\; g_i(x) \neq g_i(y) \;\Leftrightarrow\; (t_i \circ g_i)(x) \neq (t_i \circ g_i)(y) \\ &\Leftrightarrow\; h_{\lambda(i)}(x) \neq h_{\lambda(i)}(y) \;\Leftrightarrow\; \lambda(i) \in \mathrm{J}(h(x), h(y)). \end{aligned}$$

Therefore, (ii) is satisfied. ∎

We know that any weak direct product of algebras A_i is a full subdirect product of these algebras. Generally, a full subdirect product of A_i, $i \in I$, is not a weak direct product of A_i, $i \in I$ (e.g., the group H_4 of Example 5.5).

Now we get

THEOREM 5.49 (Walendziak [1994a], Theorem 5). *Let A be a congruence distributive algebra such that* $\mathrm{DCon}(A)$ *is a $\bigvee$-closed sublattice in* $\mathrm{Con}(A)$. *If A is a full subdirect product of directly indecomposable algebras A_i, $i \in I$, then A is a weak direct product of these algebras.*

Proof. Let $A = \prod^{\mathcal{P}}(A_i : i \in I)$, where A_i, $i \in I$, are directly indecomposable algebras. By Theorem 5.44 (for $\mathcal{L} = \mathcal{F}$) there exists a system $(B_j : j \in J)$ of directly indecomposable algebras and an embedding $f : A \to \prod B_j$ such that $\langle (B_j : j \in J), h \rangle$ is a weak direct representation of A. Theorem 5.48 yields a bijection $\lambda : I \to J$ such that $\lambda(\mathrm{I}(x, y)) = \mathrm{J}(h(x), h(y))$ for all $x, y \in A$. Since the set $\mathrm{J}(h(x), h(y))$ is finite, so is $\mathrm{I}(x, y)$. Therefore, A is a weak direct product of the algebras A_i, $i \in I$. ∎

The following lemma can be deduced from the proof of Lemma 1.4 of Draškovičová [1987].

LEMMA 5.50. *If A is an algebra whose congruence lattice is completely distributive*, *then* $\mathrm{DCon}(A)$ *is a $\bigvee$-closed sublattice of* $\mathrm{Con}(A)$.

REMARK 5.51. By Lemma 5.50, Theorem 5.49 implies Theorem 2.1 of Jakubík [1971].

By Theorems 5.45 and 5.48 we obtain

PROPOSITION 5.52. *Let A be an algebra. If A satisfies the hypotheses of Theorem* 5.49, *then A can be decomposed uniquely* (*up to isomorphism*) *into a weak direct product* (*a full subdirect product*) *of directly indecomposable algebras.*

REMARK 5.53. Combining Lemma 5.50 with Proposition 5.52 yields Theorems 1.6 and 1.7 of Draškovičová [1987].

By the proof of Corollary 3.42, if L is a lower continuous lattice, then every element $c \in L$ has the following property:

($\triangledown$) For each $a \in L$ and each $S \subseteq L$, if $c \leq a \vee \bigwedge S'$ for every finite subset S' of S, then $c \leq a \vee \bigwedge S$.

We remark that $(\bigtriangledown)$ is the dual of property $(\triangle)$ defined in Section 3.7. Recall from Section 5.3 that if $\varphi \in \mathrm{Con}(A)$, then for algebras B and C, $B \approx_\varphi C$ if there are φ-isotopic congruences β and γ on A such that $B \cong A/\beta$ and $C \cong A/\gamma$.

Our principal uniqueness result is

THE UNIQUE FACTORIZATION THEOREM 5.54. *Let A be a congruence modular algebra, and let φ be a distributive element of* $\mathrm{Con}(A)$ *having* $(\bigtriangledown)$. *Suppose that the congruences on A φ-permute. Let α_i $(i \in I)$ and β_j $(j \in J)$ be φ-indecomposable congruences on A satisfying the* B^*_φ*-condition in the dual lattice of* $\mathrm{Con}(A)$, *and let $\mathcal{L}_1$, $\mathcal{L}_2$ be ideals of the Boolean algebras $\mathbf{P}(I)$, $\mathbf{P}(J)$, respectively. If $\langle (A_i : i \in I), g\rangle$ is an irredundant $\langle \mathcal{L}_1, \varphi\rangle$-representation of A with* $\ker(g_i) = \alpha_i$, *and $\langle (B_j : j \in J), h\rangle$ is an irredundant $\langle \mathcal{L}_2, \varphi\rangle$-representation of A with* $\ker(h_j) = \beta_j$, *then there is a bijection $\lambda : I \to J$ such that $A_i \approx_\varphi B_{\lambda(i)}$ for all $i \in I$.*

Proof. By Proposition 5.24 and Theorem 5.25,

$$0 = \textstyle\prod_\varphi^{\mathcal{L}_1}\{\alpha_i : i \in I\} = \prod_\varphi^{\mathcal{L}_2}\{\beta_j : j \in J\}.$$

Hence

$$0 = \textstyle\prod_\varphi \alpha_i = \prod_\varphi \beta_j. \tag{10}$$

The sets $\{\alpha_i : i \in I\}$ and $\{\beta_j : j \in J\}$ are meet irredundant (see Proposition 5.32). Let L be the dual of $\mathrm{Con}(A)$. The congruence φ is distributive in L (since $\mathrm{Con}(A)$ is modular) and φ has property $(\triangle)$ (in L). From (10) and from the fact that $\{\alpha_i : i \in I\}$ and $\{\beta_j : j \in J\}$ are join irredundant subsets of L we see by Lemma 5.18 that

$$1 = \textstyle\dot\sum_\varphi\{\alpha_i : i \in I\} = \dot\sum_\varphi\{\beta_j : j \in J\} \tag{11}$$

and by Lemma 5.19 we know that each α_i and β_j is φ-irreducible. Applying Theorem 3.40 to the two φ-decompositions (11) we deduce that there exists a bijection $\lambda : I \to J$ such that, for each $i \in I$,

$$1 = \alpha_i \dotplus_\varphi \textstyle\dot\sum_\varphi\{\beta_j : j \neq \lambda(i)\}.$$

Hence and from (11) we deduce, by Property I of Chapter 3, that

$$1 = \alpha_i \dotplus_\varphi \bigvee\{\beta_j : j \neq \lambda(i)\} = \beta_{\lambda(i)} \dotplus_\varphi \bigvee\{\beta_j : j \neq \lambda(i)\}$$

and using Lemma 5.18 we have

$$0 = \alpha_i \times_\varphi \bigcap\{\beta_j : j \neq \lambda(i)\} = \beta_{\lambda(i)} \times_\varphi \bigcap\{\beta_j : j \neq \lambda(i)\}$$

in $\mathrm{Con}(A)$. Therefore, for all $i \in I$,

$$\alpha_i \approx_\varphi \beta_{\lambda(i)}. \tag{12}$$

Since $A_i \cong A/\alpha_i$ and $B_j \cong A/\beta_j$, from (12) it follows that $A_i \approx_\varphi B_{\lambda(i)}$. ■

For the next result we need the following

LEMMA 5.55. *Let L be a complete distributive lattice, and let $a \in \mathrm{S}(1, L)$. If a is completely join irreducible, then a is compact.*

Proof. Let $T \subseteq L$ and $a \leq \bigvee T$. Let $b \in L$ be such that $1 = a \vee b$. By distributivity of L we have

$$t \leq t \vee b = (a \vee b) \wedge (t \vee b) = (a \wedge t) \vee b \quad \text{for each } t \in T.$$

Hence, $\bigvee T \leq \bigvee\{a \wedge t : t \in T\} \vee b$. Therefore,

$$a = a \wedge [\bigvee\{a \wedge t : t \in T\} \vee b] = \bigvee\{a \wedge t : t \in T\} \vee (a \wedge b).$$

Because a is completely join irreducible and $a \nleq b$, there is $t_0 \in T$ such that $a = a \wedge t_0$. Thus a is compact. ■

PROPOSITION 5.56. *Assume that A is an algebra whose congruence lattice is distributive. Let I, J be two sets of indices and $\mathcal{L}_1$, $\mathcal{L}_2$ be ideals of the Boolean algebras $\mathbf{P}(I)$, $\mathbf{P}(J)$, respectively. If $\langle (A_i : i \in I), g\rangle$ is an irredundant $\mathcal{L}_1$-restricted subdirect representation of A and $\langle (B_j : j \in J), h\rangle$ is an irredundant $\mathcal{L}_2$-restricted subdirect representation of A, and if the factors A_i, B_j are subdirectly irreducible, then there exists a bijection $\lambda : I \to J$ with $A_i \cong B_{\lambda(i)}$ for all $i \in I$.*

Proof. It is obvious that $\varphi = 0$ satisfies $(\triangledown)$. We put $\alpha_i = \ker(g_i)$ and $\beta_j = \ker(h_j)$. Since $A_i \cong A/\alpha_i$ and $B_j \cong A/\beta_j$ are subdirectly irreducible, the congruences α_i and β_j are completely meet irreducible. By Proposition 5.15(i), α_i and β_j are 0-indecomposable. From Lemmas 5.55 and 3.36 it follows that each α_i and β_j satisfies the B_1^*-condition in the dual lattice of $\mathrm{Con}(A)$. By the proof of Theorem 5.54, there is a bijection $\lambda : I \to J$ such that α_i and $\beta_{\lambda(i)}$ are 0-isotopic for all $i \in I$. From this together with Lemma 5.21 we have $\alpha_i = \beta_{\lambda(i)}$ for each i. Then

$$A_i \cong A/\alpha_i = A/\beta_{\lambda(i)} \cong B_{\lambda(i)}. \ ■$$

PROPOSITION 5.57. *Let A be any algebra whose congruences permute and whose congruence lattice is lower continuous. Let I, J be two sets of indices and $\mathcal{L}_1$, $\mathcal{L}_2$ be ideals of* $\mathbf{P}(I)$, $\mathbf{P}(J)$, *respectively. If*

$$A \cong \prod\nolimits^{\mathcal{L}_1}(A_i : i \in I) \quad \textit{and} \quad A \cong \prod\nolimits^{\mathcal{L}_2}(B_j : j \in J),$$

where the algebras A_i, B_j $(i \in I,\ j \in J)$ are directly indecomposable and the lattices $\mathrm{Con}(A_i)$ *and* $\mathrm{Con}(B_j)$ *are of finite length, then $|I| = |J|$. Moreover, if in addition A has a one-element subalgebra, then there is a bijection $\lambda : I \to J$ such that $A_i \cong B_{\lambda(i)}$ for each $i \in I$.*

Proof. As every algebra whose congruences permute has a modular congruence lattice, $\mathrm{Con}(A)$ is modular. Obviously, $\varphi = 1$ is a distributive element of $\mathrm{Con}(A)$ with property $(\triangledown)$, because $\mathrm{Con}(A)$ is lower continuous. (See the proof of Corollary 3.43.) Let $g : A \to \prod A_i$ and $h : A \to \prod B_j$ be embeddings such that

$$g(A) = \prod\nolimits^{\mathcal{L}_1}(A_i : i \in I) \quad \text{and} \quad h(A) = \prod\nolimits^{\mathcal{L}_2}(B_j : j \in J).$$

Set $\alpha_i = \ker(g_i)$ and $\beta_j = \ker(h_j)$. The algebras $A_i \cong A/\alpha_i$ and $B_j \cong A/\beta_j$ are directly indecomposable, and therefore, the congruences α_i and β_j are 1-indecomposable, by Lemma 5.16. From the Correspondence Theorem 4.12 of McKenzie–McNulty–Taylor [1987] we have

$$[\alpha_i, 1] \cong \mathrm{Con}(A_i) \quad \text{and} \quad [\beta_j, 1] \cong \mathrm{Con}(B_j).$$

Thus, the intervals $[\alpha_i, 1]$ and $[\beta_j, 1]$ are of finite length. Let L denote the dual lattice of $\mathrm{Con}(A)$. In L, the intervals $[0, \alpha_i]$ and $[0, \beta_j]$ are of finite length. Crawley [1962] (see Lemma 3) has shown that if a is an element of an upper continuous lattice such that $[0, a]$ is of finite length, then a is compact. Consequently, α_i and β_j are compact in L. By Lemma 5.19, they are directly join irreducible (0-irreducible) in L. Lemma 3.35 shows that they satisfy the B_0^*-condition. Thus, the assumptions of Theorem 5.54 are satisfied. Therefore, there exists a bijection $\lambda : I \to J$ such that $A_i \approx B_{\lambda(i)}$ for each $i \in I$.

The final assertion follows from Lemma 5.22. ∎

Since everly dual algebraic lattice is lower continuous, Proposition 5.57 generalizes a result of Walendziak [1994c] (see Theorem 3).

By Proposition 5.56 we obtain

Corollary 5.58. *Let A be any algebra and suppose that* $\mathrm{Con}(A)$ *is distributive. If $\langle (A_i : i \in I), g\rangle$ and $\langle (B_j : j \in J), h\rangle$ are two irredundant finitely restricted subdirect representations of A with subdirectly irreducible factors, then there is a bijection $\lambda : I \to J$ such that $A_i \approx B_{\lambda(i)}$ for each $i \in I$.*

We call two subdirect (direct) representations $\langle (A_i : i \in I), g\rangle$ and $\langle (B_j : j \in J), h\rangle$ *isomorphic* if there exists a bijection $\lambda : I \to J$ such that $A_i \approx B_{\lambda(i)}$ for each $i \in I$.

Proposition 5.56 also gives the following

Corollary 5.59. *Let A be an algebra whose congruence lattice is distributive. Then any two irredundant subdirect representations of A with subdirectly irreducible factors are isomorphic.*

Remark 5.60. We know that lattices are congruence distributive. Therefore, Corollary 5.59 implies Theorem 11.5 of Crawley–Dilworth [1973].

Example 5.61 (Skala [1971]). A *weakly associative lattice*, or a *trellis*, is an algebra with two binary operations, $+$ and $\cdot$, that satisfies the identities

$$x \cdot y = y \cdot x, \quad x \cdot (y + x) = x, \quad (x \cdot z + y \cdot z) + z = z$$

and their duals. Weakly associative lattices are congruence distributive.

Example 5.62 (Draškovičová [1987]). We call a set A with one ternary operation (xyz) a *modular median algebra* if the following identities are satisfied in A:

$$(xyy) = y, \tag{13}$$

$$((xyz)tz) = (xz(tzy)). \tag{14}$$

Any modular median algebra is congruence distributive (see Remark 3.11 of Draškovičová [1987]).

Corollary 5.59 yields the following

Corollary 5.63. *Let A be a modular median algebra (lattice, Heyting algebra, trellis). Then any two irredundant subdirect representations of A with subdirectly irreducible factors are isomorphic.*

Using Proposition 5.57 we obtain

Corollary 5.64. *Let A be any algebra whose congruences permute and whose congruence lattice is lower continuous. Suppose that A has a one-element subalgebra. If $\langle (A_i : i \in I), g\rangle$ and $\langle (B_j : j \in J), h\rangle$ are two weak direct representations (full subdirect representations) of A such that the factors A_i, B_j are directly indecomposable and the lattices* $\mathrm{Con}(A_i)$ *and* $\mathrm{Con}(B_j)$ *are of finite length, then a bijection $\lambda : I \to J$ exists for each $i \in I$.*

Let M be a module over a ring R. Then M is called *noetherian* (resp. *artinian*) if every nonempty set of submodules has a maximal (resp. minimal) element. We say that M is *of finite length* if M is noetherian and artinian.

Corollary 5.65. *Let M be an artinian module, and let*

$$M = M_1 \oplus \ldots \oplus M_n = N_1 \oplus \ldots \oplus N_m,$$

where each M_i and each N_j is directly indecomposable and noetherian. Then $m = n$ and, after renumbering, $M_i \cong N_i$ for $1 \le i \le n$.

Proof. Since M is artinian, $\mathrm{Con}(M)$ satisfies the descending chain condition, and hence $\mathrm{Con}(M)$ is lower continuous. It is obvious that the lattices $\mathrm{Con}(M_i)$ and $\mathrm{Con}(N_j)$ are of finite length. Now the assertion follows from Corollary 5.64. ∎

It is obvious that every lattice of finite length is lower continuous. Therefore, from Corollary 5.64 we get at once

Corollary 5.66 (Birkhoff [1967], p. 169). *Let A be a congruence permutable algebra with a one-element subalgebra, and let* $\mathrm{Con}(A)$ *be of finite length. Then any two finite direct representations of A with directly indecomposable factors are isomorphic.*

By Corollary 5.66 we obtain

Corollary 5.67 (see Kasch [1982], Corollary 7.3.6). *Let M be an R-module of finite length, and let*

$$M = M_1 \oplus \ldots \oplus M_n = N_1 \oplus \ldots \oplus N_m.$$

If all M_i and N_j are directly indecomposable, then $m = n$ and, after renumbering, $M_i \cong N_i$ for $1 \le i \le n$.

Remark 5.68. It is easy to see that the Krull–Schmidt Theorem (see e.g. Kurosh [1967], Section 47), which asserts that every group whose normal subgroup lattice is of finite length can be decomposed uniquely (up to isomorphism) into a direct product of directly indecomposable groups, is a consequence of Corollary 5.66. We also note that the assertion of Corollary 5.66 holds, for example, if A is a quasigroup (or a ring) with congruence lattice of finite length.

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List of symbols and notations

Sets

$\emptyset$	empty set
$\in$	element inclusion
$\subseteq$	inclusion
$\cap, \bigcap$	intersection
$\cup, \bigcup$	union
$A - B$	set-theoretic difference
$A \cup a$	a shorthand for $A \cup \{a\}$
$A - a$	a shorthand for $A - \{a\}$
(a, b)	ordered pair
$\lvert X \rvert$	cardinality of the set X
$\mathrm{P}(I)$, $\mathcal{P}$	set of all subsets of I
$\mathrm{F}(I)$, $\mathcal{F}$	set of all finite subsets of I
$A \times B$	cartesian product of A and B
$A_1 \times \ldots \times A_n$	cartesian product of $A_1, \ldots, A_n$
A^2	$A \times A$
$f : A \to B$	f is a function from A into B
$f(a), af$	value of f at a
Cl	closure operator
$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integers
$\cong$	isomorphism
$\Rightarrow$	implication
$\Leftrightarrow$	logical equivalence
iff	if and only if
$\blacksquare$	end of proof

Lattices

L	lattice
L^∂	lattice dual to L
0	least element
1	greatest element
$\vee, \bigvee$	join
$\wedge, \bigwedge$	meet
$\leq$	partial ordering relation
$[x, y]$	interval
$x \prec y$	x is a lower cover of y

$y \prec x$	x is an upper cover of y
$y \preceq x$	x is an upper cover of y or x equals y
S_7	hexagon
a_+	meet of all lower covers of a ($\neq 0$)
u^*	uniquely determined lower cover of $u \in \mathrm{V}(L)$
$\mathrm{V}(L)$	set of all join irreducibles of L
$\mathrm{J}(L)$	set of all completely join irreducibles of L
$\mathrm{A}(L)$	set of all atoms of L
$\Lambda(L)$	set of all meet irreducibles of L
$\mathrm{M}(L)$	set of all completely meet irreducibles of L
$\mathrm{D}(L)$	set of all distributive elements of L
$\mathrm{K}(L)$	set of all compact elements of L
$\mathrm{Q}(L)$	set of all precompact elements of L
$\mathbf{K}$	class of all lower continuous strongly coatomic lattices
$\dot{\vee}, \dot{\bigvee}$	direct join
$+_c, \sum_c$	c-join
$\dot{+}_c, \dot{\sum}_c$	irredundant c-join
$\mathrm{S}(c, L)$	set of all c-summands of L
$\mathrm{DF}(c, L)$	set of all c-decomposition functions of L
$\mathrm{k}(\varphi)$	join of all $x \in L$ such that $x\varphi \leq c$
$\mathrm{L}(G)$	lattice of all normal subgroups of G
$\mathbf{P}(I)$	Boolean algebra $\langle \mathrm{P}(I), \cap, \cup, ', \emptyset, I \rangle$
$\mathcal{L} \trianglelefteq \mathbf{P}(I)$	$\mathcal{L}$ is an ideal of $\mathbf{P}(I)$
(C)	covering property
(N)	neighborhood condition
(N*)	dual neighborhood condition
(ACC)	ascending chain condition
(DCC)	descending chain condition
(HJ)	hereditary property
KOP	Kurosh–Ore property
$\bigvee$-KORP	Kurosh–Ore replacement property for join decompositions
$\bigwedge$-KORP	Kurosh–Ore replacement property for meet decompositions

Algebras

$\mathrm{Con}(A)$	set of all congruence relations on A
$\mathrm{DCon}(A)$	set of all decomposition congruences of A
$\mathrm{Cg}^A(X)$	congruence relation on A generated by X
0_A	identity congruence on A
1_A	universal congruence on A
A/α	factor algebra
a/α	congruence class of a modulo α
$\alpha \circ \beta$	relational product of α and β
$\mathrm{I}(x, y)$	$\{i \in I : x(i) \neq y(i)\}$
$\prod(A_i : i \in I)$	direct product of algebras A_i ($i \in I$)
$\ker(f)$	kernel of f
p_i	ith projection function

f_i	ith f-projection function
$\prod_{\psi}^{\mathcal{L}}(A_i : i \in I)$	$\langle \mathcal{L}, \psi \rangle$-product of algebras A_i
$\prod^{\mathcal{L}}(A_i : i \in I)$	$\mathcal{L}$-restricted full subdirect product of A_i
$\prod_{\mathcal{L}}^{\mathcal{L}'}(A_i : i \in I)$	$\langle \mathcal{L}, \mathcal{L}' \rangle$-product of algebras
$\prod_{\mathcal{D}}(M_i : i \in I)$	$\mathcal{D}$-product of modules M_i
$\prod^{\mathfrak{m}} R_i$	$\mathfrak{m}$-product of rings R_i
$\bigoplus G_i$	direct sum of groups G_i
$\mathrm{supp}(x)$	support of x
$\mathrm{Z}(G)$	center of a group G
$\alpha \times_{\varphi} \beta$	φ-product of congruences α and β
$\prod_{\varphi}\{\theta_i : i \in I\}$	φ-product of congruences θ_i
$A \approx_{\varphi} B$	algebras A and B are φ-isotopic
$A \approx B$	algebras A and B are 1_A-isotopic

Index of terms