1. Introduction

We consider the following system of equations:

$$\partial_t^2 u - \operatorname{div} S = f, \tag{1.1}$$

$$\partial_t \varepsilon = \operatorname{Tr}(S^* \cdot \partial_t \nabla u) + \operatorname{div} q + Q, \tag{1.2}$$

where $u = u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x))^*$ is the displacement vector of the medium, $\theta = \theta(t,x)$ is the temperature of the medium, both depending on $t \in \mathbb{R}_0^+$ and $x \in \Omega$, $\Omega \subset \mathbb{R}^3$ being a bounded domain with sufficiently smooth boundary, $\partial_t = \partial/\partial t$, $\partial_t^2 = \partial^2/\partial t^2$, $\partial_j = \partial/\partial x_j$, div stands for the divergence operator with respect to x, $f = f(t,x) = (f_1(t,x), f_2(t,x), f_3(t,x))^*$ is the body force vector, $Q = Q(t,x)$ is the intensity of the heat source, Tr is the trace operator, ε is the internal energy per unit mass, * stands for transposition, $S = \left(S^{ij}\right)$ means that S is a 3×3 matrix whose (i,j) component is S^{ij}. If $W = (w^1, \ldots, w^n)$ where $w^j = (w^{ij})_{n\times n}$ then $W = (w^{ij})_{n\times n}$ and $\operatorname{div} W = \partial_j w^j$; q is the heat flux and $q = (q_i)$ means that q is a row n-vector whose ith component is q_i and $\operatorname{div} q = \partial_j q_j$.

It is known (cf. [3], [67]) that the classical thermoelasticity theory (i.e. in which the constitutive relations are independent of the derivative $\partial_t \theta$ of the temperature) leads to a parabolic differential equation for the temperature distribution in rigid heat conductors. This implies that thermal perturbations are felt instantaneously in every part of the body (cf. [3], [67]). Although, at first sight, this outcome of the theory seems to contradict physical intuition, it can be justified by resorting to the fact that molecular motion, which plays a crucial part in transport phenomena, is very rapid except at extremely low temperatures. Hence a finite velocity of propagation for thermal perturbations is usually not observable unless experiments are performed in some neighbourhood of absolute zero such as in the case of liquid helium. In fact, thermal waves commonly known as *second sound* are detected in some metals cooled down approximately to 20 K (cf. the work [1] of Ackerman and Guyer (1968) and the works [79] of Taylor *et al.* (1969) and [47] of Jackson and Walker (1971).

Below, we consider the theory of thermoelasticity in which we removed an infinite velocity of propagation for thermal disturbance in rigid conductors described by a parabolic equation. This means that we would like to obtain hyperbolic thermoelasticity theory.

One approach to remedy this apparent flaw (an infinite velocity of propagation for thermal disturbances in rigid conductors described by a parabolic equation) is to include the *temperature rate* among the constitutive variables, which results in the presence of the second-order time derivative of the temperature field in the energy balance. However, the Clausius–Duhem inequality, in the form employed up to now, eliminates the temperature-rate-dependence from all the constitutive functions except for the constitutive function of the heat flux. Hence, in order to obtain a well posed theory for temperature-rate-dependent thermoelastic solids we have to resort to an entropy principle in its full generality presented in [78]. Such a theory of thermoelasticity was proposed by Müller in [66] where the entropy flux is postulated to be a constitutive function.

A similar idea was presented by Green and Lindsay (cf. [40]), who advocated rather special constitutive relations for the entropy supply in rigid conductors, which are simple generalizations of the conventional forms. Suhubi [78] extended these results to thermoelasticity theory and obtained a hyperbolic system of equations describing temperature--rate-dependent thermoelastic solids. Using Suhubi's approach, we now define the thermoelastic solids as a class of simple thermomechanical materials in which the response functions depend only on ∇u, θ, $\partial_t\theta$, $\nabla\theta$, where $\nabla u = (\partial_1 u, \partial_2 u, \partial_3 u)^*$, $\partial_t\theta = \partial\theta/\partial t$, $\nabla\theta = (\partial_1\theta, \partial_2\theta, \partial_3\theta)^*$. So, we assume the following constitutive relations for the internal energy ε, the stress tensor S and the heat flux q:

$$\varepsilon = \widehat{\varepsilon}(\nabla u, \theta, \partial_t\theta, \nabla\theta), \tag{1.3}$$

$$S = \widehat{S}(\nabla u, \theta, \partial_t\theta), \tag{1.4}$$

$$q = \widehat{q}(\nabla u, \partial_t\theta, \nabla\theta). \tag{1.5}$$

Taking into account the relations (1.3)–(1.5), we can rewrite the system (1.1)–(1.2) as follows:

$$\begin{aligned} \partial_t^2 u_i - c_{i\alpha j\beta}(\nabla u, \theta, \partial_t\theta)\frac{\partial^2 u_j}{\partial x_\alpha \partial x_\beta} + m_{i\alpha}(\nabla u, \theta, \partial_t\theta)\frac{\partial\theta}{\partial x_\alpha} \\ + M_{i\alpha}(\nabla u, \theta, \partial_t\theta)\frac{\partial^2\theta}{\partial t\partial x_\alpha} = f, \end{aligned} \tag{1.6}$$

$$\begin{aligned} \partial_t^2\theta + a(\theta, \partial_t\theta, \nabla\theta, \nabla u)\partial_t\theta - k_{\alpha\beta}(\theta, \partial_t\theta, \nabla\theta, \nabla u)\frac{\partial^2\theta}{\partial x_\alpha\partial x_\beta} \\ - \overline{b}_{ij\alpha}(\nabla u, \theta, \partial_t\theta, \nabla\theta)\frac{\partial^2 u_j}{\partial x_\alpha \partial x_i} + \overline{c}_{i\alpha}(\theta, \partial_t\theta, \nabla\theta, \nabla u)\frac{\partial^2 u_j}{\partial x_\beta\partial t} \\ + \overline{d}_i(\theta, \partial_t\theta, \nabla\theta, \nabla u)\frac{\partial^2\theta}{\partial t\partial x_i} = \overline{g}/\frac{\partial\widehat{\varepsilon}}{\partial(\partial_t\theta)} \end{aligned} \tag{1.7}$$

where

$$c_{i\alpha j\beta} = \frac{\partial S^{i\alpha}}{\partial(\partial_\beta u_j)}, \tag{1.8}$$

$$m_{i\alpha} = \frac{\partial S^{i\alpha}}{\partial\theta}, \qquad M_{i\alpha} = \frac{\partial S^{i\alpha}}{\partial(\partial_t\theta)} = \frac{\partial S^{i\alpha}}{\partial(\partial_t\theta)}, \tag{1.9}$$

$$a(\theta, \partial_t\theta, \nabla\theta, \nabla u) = \frac{\partial\varepsilon}{\partial\theta}\bigg/\frac{\partial\varepsilon}{\partial(\partial_t\theta)}, \tag{1.10}$$

$$k_{\alpha\beta}(\theta, \partial_t\theta, \nabla\theta, \nabla u) = \frac{a_{\alpha\beta}(\nabla u, \partial_t\theta, \nabla\theta)}{\partial\widehat{\varepsilon}/\partial(\partial_t\theta)}, \tag{1.11}$$

$$\overline{b}_{ij\alpha}(\nabla u, \theta, \partial_t\theta, \nabla\theta) = \frac{b_{ij\alpha}(\nabla u, \partial_t\theta, \nabla\theta)}{\partial\widehat{\varepsilon}/\partial(\partial_t\theta)}, \tag{1.12}$$

$$c_{i\alpha}(\nabla u, \theta, \partial_t\theta, \nabla\theta) = \frac{S^{i\alpha}(\nabla u, \partial_t\theta, \nabla\theta) - \partial\widehat{\varepsilon}/\partial(\partial_\alpha u_i)}{\partial\widehat{\varepsilon}/\partial(\partial_t\theta)}, \tag{1.13}$$

$$\overline{d}_i(\nabla u, \theta, \partial_t\theta, \nabla\theta) = \frac{\overline{d}_i(\nabla u, \partial_t\theta, \nabla\theta) - \partial\widehat{\varepsilon}/\partial(\partial_i\theta)}{\partial\widehat{\varepsilon}/\partial(\partial_t\theta)}, \tag{1.14}$$

$$b_{ij\alpha} = \frac{\partial q_i}{\partial_\beta(\partial u_j \partial x_\alpha)}, \quad d_i = \frac{\partial q_i}{\partial_t\theta}, \quad a_{\alpha\beta} = \frac{\partial q_i}{\partial(\partial\theta/\partial x_\beta)}. \tag{1.15}$$

REMARK 1.1. Since $\partial\varepsilon/\partial(\partial_t\theta) > 0$ (cf. [78]), (1.7) is a hyperbolic equation in θ which predicts a finite velocity of propagation for thermal perturbations. So, the system (1.6)–(1.7) is the nonlinear hyperbolic system of thermoelasticity theory.

We will pose the initial conditions

$$\begin{aligned} u(0,x) &= u^0(x), \quad (\partial_t u)(0,x) = u^1(x), \\ \theta(0,x) &= \theta^0(x), \quad (\partial_t\theta)(0,x) = \theta^1(x), \end{aligned} \tag{1.16}$$

with given data u^0, θ^0 and u^1, θ^1, and Dirichlet type boundary conditions (physically—rigidly clamped, constant temperature)

$$u(t,\cdot)|_{\partial\Omega} = 0, \quad \theta(t,\cdot)|_{\partial\Omega} = 0. \tag{1.17}$$

The linear hyperbolic system (1.6)–(1.7) with constant coefficients was investigated in [74] using the Cagniard–de Hoop method.

In the paper [14] the theorem about existence, uniqueness and regularity of the weak solution to the first initial-boundary value problem for the linear hyperbolic system was proved by applying the Faedo–Galerkin method in Sobolev spaces.

In [20] the global (in time) existence theorem was proved for the solution of the initial value problem for the nonlinear system (1.1)–(1.2) using the L^p-L^q time decay estimate for the solution of the associated linearized problem, an energy estimate, and methods of Sobolev spaces. The aim of this paper is to prove a local existence theorem for the solution of the initial-boundary value problem (1.6)–(1.7) in the class of smooth functions with respect to time and taking values in suitable Sobolev spaces with respect to the spatial variables.

The corresponding existence theorem is proved by using the semigroup theory for the linearized problem associated with the nonlinear one. Using the energy method we prove an energy estimate for the solution of the initial-boundary value problem to the linearized system (1.6)–(1.7). Applying the Banach fixed point theorem, we prove that the solution of the nonlinear initial-boundary value problem (1.6)–(1.7) exists and is unique.

The paper is organized as follows. In Section 2 some notations and formulae are presented. Section 3 presents the existence theorem for the solution to the initial-boundary value problem. In Section 4 we prove an energy estimate for the linearized system of hyperbolic thermoelasticity. In Section 5 the proof of the main theorem is presented. Sections 6–8 are devoted to some applications of the above method to nonlinear microelasticity theory.

Sections 9–11 present the application of this method to nonlinear thermodiffusion in a solid body. In Section 12 some general remarks are given.

2. Basic notation and formulae

We first introduce some function spaces. Let G be an open bounded set in the Euclidean space E^r with regular boundary ∂G. $L^p(G)$ is the space of (equivalence classes of) measurable functions u such that

$$\|u\|_{L^p(G)} = \Big(\int_G |u(x)|^p\,dx\Big)^{1/p} < \infty, \qquad 1 \le p < \infty, \tag{2.1}$$

$$\|u\|_{L^\infty(G)} = \operatorname*{ess\,sup}_{x\in G} |u(x)|, \qquad p = \infty. \tag{2.2}$$

Taken with the norm (2.1) or (2.2), $L^p(G)$ is a Banach space; if $p = 2$, then $L^2(G)$ is a Hilbert space, with scalar product

$$(u,v)_{L^2(G)} = \int_G u(x)v(x)\,dx. \tag{2.3}$$

The Sobolev space $W_p^m(G)$, $1 \le p \le \infty$, consists of functions u belonging to $L^p(G)$ with weak derivatives $\partial^\alpha u$, $|\alpha| \le m$, belonging to $L^p(G)$:

$$W_p^m(G) = \{u \in L^p(G) : \partial^\alpha u \in L^p(G) \text{ for } |\alpha| \le m\}. \tag{2.4}$$

With the norm

$$\|u\|_{W_p^m(G)} = \Big(\sum_{|\alpha|\le m} \|\partial^\alpha u\|^p_{L^p(G)}\Big)^{1/p}, \tag{2.5}$$

it is a Banach space.

The case $p = 2$ is fundamental. To simplify the writing, we put

$$W_2^m(G) = H^m(G);$$

with the scalar product

$$(u,v)_{H^m(G)} = \sum_{|\alpha|\le m} (\partial^\alpha u, \partial^\alpha v)_{L^2(G)}, \tag{2.6}$$

this is a Hilbert space. The norm in this space is given by

$$\|u\|_m = \Big(\sum_{|\alpha|\le m} (\partial^\alpha u, \partial^\alpha v)_{L^2(G)}\Big)^{1/2}. \tag{2.7}$$

Let $C_0^\infty(G)$ denote the space of compactly supported infinitely differentiable real-valued functions defined on G. By $H_0^m(G)$ we denote the Hilbert space obtained as the completion of $C_0^\infty(G)$ in the norm (2.7). $H_0^m(G)$ is a subspace of $H^m(G)$.

Theorem 2.1 (Sobolev imbedding theorem). *If G is a bounded domain with smooth boundary ∂G and $u \in H^m(G)$ where $m > n/2$ and $k \ge 0$ is an integer such that $m > n/2 + k$, then $u \in C^k(G)$ and*

$$\sup_{x\in G} |D^\alpha u| \le \|u\|_m, \qquad |\alpha| \le k. \tag{2.8}$$

Theorem 2.2 (The Poincaré inequality). *If $u \in H_0^m(G)$, then*

$$\|u\|_m^2 \le C \sum_{|\alpha|\le m} \int_G |\partial^\alpha u|^2\,dx \qquad \forall u \in H_0^m(G), \tag{2.9}$$

where $C = C(G,m)$.

THEOREM 2.3 (Gronwall's inequality). *Let* $u, v \in C([a, b])$, $u \geq 0$. *If*

$$v(t) \leq C + \int_a^t v(s)u(s)\, ds, \qquad a \leq t \leq b,\ C \geq 0, \tag{2.10}$$

then

$$v(t) \leq C \exp \int_a^t u(s)\, ds, \qquad a \leq t \leq b. \tag{2.11}$$

THEOREM 2.4 (Gårding's inequality). *Let A be a strongly elliptic operator of order $2m$. Then there exist constants $\alpha_0 > 0$, $\lambda_0 > 0$ such that*

$$(-1)^m \operatorname{Re}(Au, u) \geq \alpha_0 \|u\|_m^2 - \lambda_0 \|u\|_2^2 \quad \textit{for } u \in C_0^\infty(G). \tag{2.12}$$

In particular, we shall use the notations

$$\partial_j = \frac{\partial}{\partial x_j} \qquad (j = 1, 2, 3), \tag{2.13}$$

$$\partial_x^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3} \qquad (|\alpha| = \alpha_1 + \alpha_2 + \alpha_3). \tag{2.14}$$

For any integer $N \geq 0$, we write

$$\begin{aligned} D^N u &= (\partial_t^j \partial_x^\alpha u;\ j + |\alpha| = N), & \overline{D}^N u &= (\partial_t^j \partial_x^\alpha u;\ j + |\alpha| \leq N), \\ D_x^N u &= (\partial_x^\alpha u;\ |\alpha| = N), & \overline{D}_x^N u &= (\partial_x^\alpha u;\ |\alpha| \leq N). \end{aligned} \tag{2.15}$$

If $f = (f_1, \ldots, f_n)$ then $f \in X$ for a space with norm $\|\cdot\|_X$ means that each component $f_1, \ldots, f_n$ of f is in X and

$$\|f\|_X = \|f_1\|_X + \ldots + \|f_n\|_X. \tag{2.16}$$

For any $0 \leq m < \infty$ and $T > 0$, we also use the notation

$$|u|_{m,T} = \sup_{0 \leq t \leq T} \|u(t)\|_m \tag{2.17}$$

where $\|\cdot\|_0$ denotes $\|\cdot\|_{L^2(G)}$.

Below, we will present the existence theorems for an Abstract Linear Evolution System basing on semigroup theory.

Results concerning abstract linear evolution systems will be used in the proof of our theorems (cf. Sections 3–11). So, we shall discuss a slightly modified theory of Kato [54] concerning the following abstract linear evolution system:

$$\partial_t U + A(t)U = F(t), \qquad 0 < t \leq T, \tag{2.18}$$

with the initial value

$$U(0) = U^0 \tag{2.19}$$

where $T > 0$ is a fixed constant.

We begin with a simple existence theorem for (2.18), (2.19). Let X_0, Y_1 be a pair of real Banach spaces with the norms (without confusion with notations given above) denoted by $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively. A triple $(A; X_0, Y_1)$, consisting of a family $A = (A(t);\ t \in [0, T])$, is called a *CD-system* (following Kato [54]) if the following conditions are satisfied:

(i) $A = (A(t); t \in [0,T])$ is a stable family of (negative) generators of C_0-semigroups on X_0, with stability constants M, β.

(ii) The domain $D(A(t)) = Y_1$ of $A(t)$ is independent of t.

(iii) $A(t) \in \mathrm{Lip}([0,t], L(Y_1; X_0))$ or equivalently $\partial_t A \in L^\infty([0,t], L(Y_1; X_0))$.

We have the following two lemmas, which follow from Theorems 1.2 and 4.1 of [54].

Lemma 2.1. *Let $(A; X_0, Y_1)$ be a CD-system. Let $U^0 \in Y_1$ and $F \in \mathrm{Lip}([0,T], X_0)$. Then there is a unique solution U of* (2.18), (2.19) *satisfying*

$$U \in C^0([0,T], Y_1) \cap C^1([0,T], X_0), \qquad U(0) = U^0. \tag{2.20}$$

Lemma 2.2. *For the solution U given by Lemma* 2.1 *we have*

$$\sup_{0 \le t \le T} (\|U\|_1 + \|\partial_t U\|_0)(t) \le K\Big(\|U^0\|_1 + \|F(0)\|_0 + \int_0^T \|\partial_t F(\tau)\|_0 \, d\tau\Big) \tag{2.21}$$

where $K > 0$ is a constant independent of U^0 and F.

We now give an existence theorem slightly different from Lemma 2.1.

Theorem 2.5. *Suppose that X_0, Y_1 are real, separable Hilbert spaces. Let $(A; X_0, Y_1)$ be a CD-system. Let $U^0 \in Y_1$, $F \in C^0([0,T], X_0)$ and $F_t \in L^1([0,T], X_0)$. Then problem* (2.18), (2.19) *has a unique solution U with*

$$U \in C^0([0,T], Y_1) \cap C^1([0,T], X_0), \qquad U(0) = U^0. \tag{2.22}$$

Proof. We choose a sequence $\delta_n \to 0$ and define

$$F_n(t) = \int_0^T \phi_{\delta_n}(t-\tau) F(\tau) \, d\tau \tag{2.23}$$

where ϕ_{δ_n} is the Friedrichs mollifier (cf. [20]). Consider the following approximate problem for (2.18), (2.19):

$$\begin{cases} \partial_t U_n + A(t) U_n = F_n(t), \\ U_n(0) = U^0. \end{cases} \tag{2.24}$$

Note that $F_n \in \mathrm{Lip}([0,T], X_0)$ for each $n > 0$. By Lemma 2.1, (2.24) admits a unique solution U_n satisfying

$$U_n \in C^0([0,T], Y_1) \cap C^1([0,T], X_0), \qquad U_n(0) = U^0. \tag{2.25}$$

By (2.21) the following estimate is valid:

$$\sup_{0 \le t \le T} (\|U_n(t)\|_1 + \|\partial_t U_n\|_0) \le \text{const} \qquad (\text{independent of } n) \tag{2.26}$$

provided that n is sufficiently large. Hence we can take a subsequence of $\{U_n\}$, still denoted by $\{U_n\}$, such that

$$\begin{cases} U_n \to U \in L^\infty([0,T], Y_1) \text{ in the weak}^* \text{ topology of } L^\infty([0,T], X_1), \\ \partial_t U_n \to \partial_t U \in L^\infty([0,T], Y_0) \text{ in the weak}^* \text{ topology of } L^\infty([0,T], X_0). \end{cases} \tag{2.27}$$

By passing to the limit in (2.24) and by (2.27) we see that U is a solution of (2.18)–(2.19). To show that U also satisfies (2.22) we use (2.24) and (2.21) to get

$$(2.28)\qquad \sup_{0\le t\le T}(\|U_n-U_m\|_1+\|\partial_t(U_n-U_m)\|_0)(t) \le K\Big(\|F_n-F_m(0)\|_0+\int_0^T\|\partial_t(F_n-F_m)(\tau)\|_0\,d\tau\Big)\to 0 \quad \text{as } n,m\to\infty.$$

Therefore $\{U_n\}$ is a Cauchy sequence in $C^0([0,T],Y_1)$ and $C^1([0,T],X_0)$. In view of (2.27) we conclude that U satisfies (2.22). The uniqueness follows from the a priori estimate (2.21). This completes the proof.

In what follows we shall investigate the higher order differentiability of the solution given by Theorem 2.5. We introduce a double scale of real Banach spaces X_j, Y_j of the following structure:

$$(2.29)\qquad \begin{cases} X_0\supset X_1\supset\ldots\supset X_{s-1},\\ Y_0=Y_0\supset Y_1\supset\ldots\supset Y_{s-1}\supset Y_s, \qquad s\ge 1.\end{cases}$$

Here it is assumed that Y_1 is a closed subspace of X_1 and $Y_j=Y_1\cap X_j$ for $1\le j\le s-1$, $s\ge 2$. We denote by $\|\cdot\|_j$ the norm in X_j (and also in Y_j). We consider the family $A=(A(t))$ together with a double scale of the form (2.29), and introduce the following assumptions:

(L_1) (Stability) $(A;X_0,Y_1)$ is a CD-system with stability constants M,β.

(L_2) (Smoothness)

$$\partial_t^r A\in \mathrm{Lip}([0,T],L(Y_{j+r+1};X_j)),\qquad 0\le j\le s-r-1,$$

up to $r=s-1$. This implies that

$$(2.30)\qquad \partial_t^{r+1}A\in \mathrm{Lip}([0,T],L(Y_{j+r+1};X_j))$$

for the same range of r,j,s.

(L_3) (Ellipticity) For a.e. $t\in[0,T]$ and $0\le j\le s-1$, $\phi\in Y_1$ and $A(t)\phi\in X_j$ together imply $\phi\in Y_{j+1}$, with

$$(2.31)\qquad \|\phi\|_{j+1}\le K(\|A(t)\phi\|_j+\|\phi\|_0)$$

where $K>0$ is a constant.

We list some consequences (Propositions 2.1–2.4 below) of these assumptions, which are given in [54].

Proposition 2.1. *Let $\lambda>\beta$. Then $A(t)+\lambda$ is an isomorphism of Y_{j-1} onto X_j for all $t\in[0,T]$, $1\le j\le s$. The resolvent $R(t)=(A(t)+\lambda)^{-1}$ is an isomorphism of X_{j-1} onto Y_j, and*

$$(2.32)\qquad \partial_t^r R\in \mathrm{Lip}([0,T],L(X_{j+r+1};Y_j))\quad \text{for } 0\le j\le s-r,\ 0\le j\le s-1.$$

Proposition 2.2. *If $s>2$, set $C(t)=(\partial_t A)(t)R(t,\lambda)$. Then*

$$(2.33)\qquad C\in L^\infty([0,T],L(X_j;X_j))\qquad \text{for } 0\le j\le s-1,$$

$$(2.34)\qquad \partial_t^r C\in \mathrm{Lip}([0,T],L(X_{j+r+1};X_j))\quad \text{for } 0\le j+r\le s-2.$$

Proposition 2.3. *Let $s\ge 2$ and set $A_1(t)=A(t)-C(t)$. Then the family A_1 satisfies (L_1) to (L_3) for the subscale of height $s-1$, possibly with modified constants M,β,K.*

For the inhomogeneous term $F(t)$ in (2.18) we shall assume

$$(L_4) \qquad \partial_t^k F \in C^0([0,T], X_{s-1-k}), \quad k = 0,1,\dots,s-1, \quad \partial_t^s F \in L^1([0,T], X_0).$$

REMARK 2.1. The condition (L_4) on the inhomogeneous term is weaker than that required in [54]. From now on, we assume the conditions (L_1)–(L_4). Theorem 2.5 shows that (2.18)–(2.19) has a unique solution $U \in C^0([0,T], Y_1) \cap C^1([0,Y], X_0)$ with $U(0) = U^0 \in Y_1$. In order to obtain the desired regularity we have to assume that U^0 and F satisfy certain natural compatibility conditions of higher order. To formulate them precisely, we first give the following proposition, which may be obtained by the same argument as for Proposition 3.1 of [54] and so its proof will be omitted here.

PROPOSITION 2.4. *Let $U \in C^0([0,T], Y_s)$ be a solution of* (2.18), (2.19). *Then $\partial_t^k U \in C^0([0,T], Y_{s-k})$, $0 \le k \le s-1$, and*

$$\partial_t^r U(t) = \partial_t^{r-1} F(t) - \sum_{k=0}^{r-1} \binom{r-1}{k} (\partial_t^{r-1-k} A)(t) \partial_t^k U(t), \qquad r = 0,1,\dots,s. \tag{2.35}$$

Proposition 2.4 implies, in particular, that if we compute $U^1, U^2, \dots, U^s$ successively from

$$U^r = \partial_t^{r-1} F(t) - \sum_{k=0}^{r-1} \binom{r-1}{k} (\partial_t^k A)(0) U^{r-1-k}, \qquad 1 \le r \le s, \tag{2.36}$$

then we have the *compatibility condition*

$$U^r \in Y_{s-r}, \qquad r = 0,1,\dots,s. \tag{2.37}$$

We call (2.37) the compatibility condition of order $s-1$ with respect to A and F. We are now able to state the basic regularity theorem of this section.

THEOREM 2.6. *Let X_0 and Y_1 be real, separable Hilbert spaces. Assume the conditions (L_1)–(L_4). If $U^0 \in Y_s$, then the solution given by Theorem* 2.5 *belongs to $C^0([0,T], Y_s)$ (hence $\partial_t^k U \in C^0([0,T], Y_{s-k})$, $k = 0,1,\dots,s-1$, by Proposition* 2.4*) if and only if U^0 and F satisfy the compatibility condition* (2.37) *with respect to the family A and F. In this case the initial data satisfy*

$$U^r = \partial_t^k U(0), \qquad r = 0,1,\dots,s. \tag{2.38}$$

Proof. Since the compatibility condition is necessary for the theorem to hold by Proposition 2.4, it suffices to prove its sufficiency. We employ an idea due to Kato [54] to prove the sufficiency by induction on s. Since the case of $s = 1$ has been established by Theorem 2.5, we assume that $s \ge 2$ and that the sufficiency has been proved with s replaced by $s-1$, and proceed to the proof for the given s.

To this end we first solve the new equation

$$V_t + A_1 V = \lambda F - F_t + CF =: F_1, \qquad 0 < t < T, \tag{2.39}$$

with the initial value

$$V(0) = (A+\lambda)U^0 - F(0) \equiv V^0, \tag{2.40}$$

where $A_1 = A - C$, $c = (\partial_t \lambda)(t) R(t)$ and $R(t) \equiv R(t,\lambda) = (A+\lambda)^{-1}$ $(\lambda > \beta)$. It has been shown in Propositions 2.1–2.3 that A_1 satisfies the conditions (L_1)–(L_3), with s

replaced by $s-1$. By Proposition 2.2 we find after a calculation that the right hand side of (2.39) also satisfies condition (L_4) with s replaced by $s-1$. Regarding the compatibility condition, we have

PROPOSITION 2.5. *U^0 satisfies the compatibility condition of order $s-1$ with respect to A and F if and only if V^0 satisfies the condition of order $s-2$ with respect to A_1 and F_1.*

Proof. The sequence V^r $(0 \le r \le s-1)$ can be computed recursively from (2.35), in which $F = F_1$, and A and U^0 are replaced by A_1 and V^0, respectively. Furthermore, we have

$$V^r = \lambda U^r - U^{r+1}, \qquad 0 \le r \le s-1,$$

which may be shown by induction on r; the computation is somewhat tedious but straightforward and will be omitted here. Hence V^0 satisfies the compatibility condition of order $s-2$ with respect to A_1 and F_1 if and only if U^0 satisfies the condition of order $s-1$ with respect to A and F.

We are now able to complete the proof of Theorem 2.6. If the compatibility condition for U^0 holds for the family A and F, Proposition 2.5 implies that the same is true for V^0, A_1 and F_1. It follows from the induction hypothesis that (2.39)–(2.40) has a unique solution

$$\partial_t^k V \in C^0([0,T], X_{s-1-k}), \qquad k = 0, 1, \ldots, s-1. \tag{2.41}$$

Now set

$$U = R(t,\lambda)(V+F). \tag{2.42}$$

It follows from Proposition 2.1, condition (L_4) and (2.41) that $U \in C^0([0,T], Y_s)$. With the help of (2.39) and $\partial_t R = -RC$ one obtains

$$\begin{aligned} U_t &= R[V_t + F_t - C(V+F)] = R(\lambda F - AV) \\ &= R[\lambda(V+F) - (A+\lambda)V] = \lambda U - V = -AU + F, \end{aligned}$$

which shows that U is a solution of (2.18), (2.19). Obviously, $U(0) = R(0)(V^0 + F(0)) = U^0$, so U also satisfies the initial condition. Therefore U is identical with the solution guaranteed by Theorem 2.5 since $U \in C^0([0,T], Y_s)$ as shown above; this completes the induction by Proposition 2.4 and proves Theorem 2.6.

Now, we consider the regularity for the elliptic system.

We shall investigate the regularity of the elliptic system (2.43), (2.44) below. The notations appearing here are the same as above:

$$Lu := C_{i\alpha j\beta}(x) \frac{\partial^2 u_j}{\partial x_\alpha \partial x_\beta} = f_i(x), \qquad x \in \Omega, \tag{2.43}$$

$$u_i|_{\partial\Omega} = 0, \qquad i = 1,2,3, \tag{2.44}$$

where $u = (u_1, u_2, u_3)^*$. Assume that

$$C_{i\alpha j\beta} \in C^0(\Omega) \cap L^\infty(\Omega), \qquad D_x^1 C_{i\alpha j\beta} \in H^{s-1}(\Omega), \tag{2.45}$$

$$C_{i\alpha j\beta}(x) = C_{i\beta j\alpha}(x) \quad \text{for } x \in \Omega, \tag{2.46}$$

where $s \geq [3/2] + 2 = 3$ is an arbitrary but fixed integer, and there is a positive constant $\nu > 0$ such that

$$C_{i\alpha j\beta}(x)\xi_i\xi_j\eta_\alpha\eta_\beta \geq \nu|\xi|^2|\eta|^2 \quad \forall \xi, \eta \in \mathbb{R}^3. \tag{2.47}$$

We have

THEOREM 2.7. *Let* (2.45)–(2.47) *hold. Then for all* $k = 0, 1, \ldots, s$, *if* $f = (f_1, f_2, f_3)^* \in H^k(\Omega)$, *then the solution* $u = (u_1, u_2, u_3)^*$ *of* (2.43), (2.44) *is in* $H^{k+2}(\Omega)$ *and satisfies*

$$\|u\|_{k+2} \leq C_s(\|Lu\|_k + \|u\|) \tag{2.48}$$

where C_s *is a positive constant which depends continuously on* $\|C_{i\alpha j\beta}\|_{L^\infty}$ *and* $\|D_x^1 C_{i\alpha j\beta}\|_{s-1}$.

Proof. If Ω is bounded, Theorem 2.7 was proved in [65]. (The symmetry of a_{ij} from [65] can be assumed.) For unbounded Ω if the coefficients of L have continuous bounded derivatives up to order s, then the theorem is also valid (cf. [73]). With the help of this result for unbounded Ω, following a procedure similar to that in Theorem 4 of [65] we get the assertion.

REMARK 2.2. Theorem 2.7 was also obtained by Kawashima and Matsumura in [56]. If (2.44) is replaced by the Neumann boundary condition, then a similar result holds (see [72]).

REMARK 2.3. It can be easily shown that under the conditions of Theorem 2.7 the following holds:

$$\|u\|_1^2 \leq C_s\left\{\left(C_{i\alpha j\beta}\frac{\partial u_j}{\partial x_\beta}, \frac{\partial u_i}{\partial x_a}\right) + \|u\|^2\right\} \quad \text{for } u \in H_0^1(\Omega). \tag{2.49}$$

For the elliptic equation

$$\begin{cases} Lv := a_{\alpha\beta}(x)\dfrac{\partial^2 v}{\partial x_\alpha \partial x_\beta} = g, & x \in \Omega, \\ v|_{\partial\Omega} = 0 \end{cases} \tag{2.50}$$

we have a similar result. Suppose

$$\begin{cases} a_{\alpha\beta} \in C^0(\Omega) \cap L^\infty(\Omega), \quad D_x^1 a_{\alpha\beta} \in H^{s-1}(\Omega), \quad a_{\alpha\beta} = a_{\beta\alpha}, \\ a_{\alpha\beta}(x)\xi_\alpha\xi_\beta \geq \nu|\xi|^2 \end{cases} \tag{2.51}$$

for some positive constant ν and all $\xi \in \mathbb{R}^3$. Here $s \geq 3$ is an arbitrary but fixed integer. We have (the proof is simpler and will be omitted here)

THEOREM 2.8. *Let* (2.51) *hold. Then for* $k = 0, 1, \ldots, s$, *if* $g \in H^k(\Omega)$, *then the solution* v *of* (2.50) *is in* $H^{k+2}(\Omega)$ *and satisfies*

$$\|v\|_{k+2} \leq C_s(\|Lv\|_k + \|v\|).$$

Here the positive constant C_s *depends continuously on* $\|a_{\alpha\beta}\|_{L^\infty}$ *and* $\|D_x^1 a_{\alpha\beta}\|_{s-1}$.

We prove the following two theorems, which are used in Section 3 and Sections 4–11 respectively. Define

$$(\eta)_\delta(t, x) = \int_0^T \phi_\delta(t - \tau)\eta(\tau, x)\, d\tau, \quad t \in [0, T],\ x \in \Omega.$$

Here ϕ_δ is the Friedrichs mollifier. We have

THEOREM 2.9. *Let* $a \in C^0([0,T], L^2(\Omega))$, $\partial_t a \in L^\infty([0,T], L^\infty(\Omega))$ *and* $v \in C^0([0,T], L^2(\Omega))$. *Set* $v(t,x) = (\partial/\partial t)[(av)_\delta - a(v)_\delta](t,x)$. *Then*

$$\int_\varepsilon^{T-\varepsilon} \|\phi_\delta(\tau,\cdot)\|^2\,d\tau \to 0 \qquad \text{as } \delta \to 0$$

for all small $\varepsilon > 0$.

Proof. Clearly,

$$\begin{aligned}\phi_\delta(t,x) &= \frac{\partial}{\partial t}\int_0^T \phi_\delta(t-\tau)(a(\tau,x)-a(t,x))v(\tau,x)\,d\tau \\ &= -\int_0^T \frac{\partial}{\partial \tau}\{\phi_\delta(t-\tau)(a(\tau,x)-a(t,x))\}v(\tau,x)\,d\tau \\ &\quad + \int_0^T \phi_\delta(t-\tau)[a'(\tau,x)-a'(\tau,x)]v(\tau,x)\,d\tau.\end{aligned}$$

Let $0 < \delta < \varepsilon$; keeping in mind that $\phi_\delta(t,x) = \phi_\delta(T-t,x)$ for $t \in [\varepsilon, T-\varepsilon]$, we infer that

$$\begin{aligned}\phi_\delta(t,x) = \int_0^T &\frac{\partial}{\partial t}\{\phi_\delta(t-\tau)(a(\tau,x)-a(t,x))\}(v(\tau,x)-v(t,x))\,d\tau \\ &+ \int_0^T \phi_\delta(t-\tau)[a'(\tau,x)-a'(\tau,x)]v(\tau,x)\,d\tau, \qquad t \in [\varepsilon, T-\varepsilon].\end{aligned}$$

Hence by the Schwarz inequality,

$$\begin{aligned}&\int_\varepsilon^{T-\varepsilon} \|\phi_\delta(\tau)\|^2\,d\tau \\ &\quad \le C\Big\{\sup_{\substack{t,\tau\in[0,T]\\|t-\tau|\le\delta}} \|v(t)-v(\tau)\|^2 + \iint_{\substack{t,\tau\in[0,T]\\|t-\tau|\le\delta}} \|a'(\tau)-a'(t)\|_{L^\infty}^2\,dt\,d\tau \int_0^T \|v(\tau)\|^2\,d\tau\Big\} \to 0\end{aligned}$$

as $\delta \to 0$. This proves the theorem.

The following lemma can be shown by an argument similar to the one used in Lemma 3 of Appendix in [73] and we omit its proof here.

LEMMA 2.3. *Let* a_j, $j=1,\dots,m$, *be nonnegative integers and* $\beta_j, j=1,\dots,m$, *be dimensional multi-indices. Put* $r = \sum_{j=1}^m |\beta_j| a_j \ge [3/2]+1 = 2$. *If* $D^r u_j(t,\cdot) \in L^\infty([0,T], L^2(\Omega))$, $j = 1,\dots,m$, *then*

$$\|\{(D^{a_1}u_1)^{\beta_1}\dots(D^{a_m}u_m)^{\beta_m}\}(t)\| \le C\prod_{j=1}^m \|u_j(t)\|_r^{|\beta_j|} \qquad \text{for } t \in [0,T]$$

where $C = C(m,r,\Omega)$.

With the help of the Leibniz formula and Lemma 2.3 we can show (the proof is omitted)

Theorem 2.10. *Assume* $D^r u_j \in L^\infty([0,T], L^2(\Omega))$ *(for some* $r \geq 1$*) and* $u_j \in L^\infty([0,T], L^2(\Omega))$ *for* $j = 1, \ldots, m$, $f(u_1, \ldots, u_m)$ *has continuous derivatives up to order* r. *Then*

$$\|D^r f(u_1, \ldots, u_m)\| \leq C \sum_{i=1}^{r} \|D^r u(t)\|^i \quad \text{for } t \in [0,T]$$

where $u = (u_1, \ldots, u_m)$, $C = \sup_{0 \leq t \leq T} C_1(\|u(t)\|_{L^\infty})$ *is a positive constant and* $C_1 : [0,\infty) \to (0,\infty)$ *is a continuous function.*

3. The main theorem

In this section we formulate the theorem about existence and uniqueness (*local in time*) of the solution to the initial-boundary value problem for the nonlinear system (3.1)–(3.2) with initial and boundary conditions (3.3)–(3.4):

$$\begin{aligned}(3.1)\quad & \partial_t^2 u_i - c_{i\alpha j\beta}(\nabla u, \theta, \partial_t \theta) \frac{\partial^2 u_j}{\partial x_\alpha \partial x_\beta} + m_{i\alpha}(\nabla u, \theta, \partial_t \theta) \frac{\partial \theta}{\partial x_\alpha} \\ & \qquad + M_{i\alpha}(\nabla u, \theta, \partial_t \theta) \frac{\partial^2 \theta}{\partial t \partial x_\beta} = f_i,\end{aligned}$$

$$\begin{aligned}(3.2)\quad & \partial_t^2 \theta + a(\theta, \partial_t \theta, \nabla \theta, \nabla u) \partial_t \theta - k_{\alpha\beta}(\theta, \partial_t \theta, \nabla \theta, \nabla u) \frac{\partial^2 \theta}{\partial x_\alpha \partial x_\beta} \\ & \qquad - \overline{b}_{ij\alpha}(\nabla u, \theta, \partial_t \theta, \nabla \theta) \frac{\partial^2 u_j}{\partial x_\alpha \partial x_\beta} + \overline{c}_{i\beta}(\theta, \partial_t \theta, \nabla \theta, \nabla u) \frac{\partial^2 u_j}{\partial t \partial x_\beta} \\ & \qquad + \overline{d}_\alpha(\theta, \partial_t \theta, \nabla \theta, \nabla u) \frac{\partial^2 \theta}{\partial t \partial x_\alpha} = \overline{Q}\end{aligned}$$

with initial conditions

$$(3.3)\qquad \begin{aligned} u(0,x) &= u^0(x), \quad (\partial_t u)(0,x) = u^1(x), \\ \theta(0,x) &= \theta^0(x), \quad (\partial_t \theta)(0,x) = \theta^1(x), \end{aligned}$$

and boundary conditions

$$(3.4)\qquad u(t,\cdot)|_{\partial\Omega} = 0, \quad \theta(t,\cdot)|_{\partial\Omega} = 0.$$

Theorem 3.1 (Local-in-time existence). *Let the following assumptions be satisfied*:

1° $s \geq \lfloor 3/2 \rfloor + 4 = 5$ *is an arbitrary but fixed integer.*

2° $\partial_t^k f_i, \partial_t^k Q \in C^0([0,T], H^{s-2-k}(\Omega))$, $k = 1, \ldots, s-2$, $\partial_t^{s-1} f_i, \partial_t^{s-1} Q \in L^0([0,T], L^2(\Omega))$.

3° *There is a constant* $\kappa_0 > 0$ *such that*

$$(3.5)\qquad (P_{\alpha\beta} \zeta | \zeta) \xi_\alpha \zeta_\beta \geq \kappa_0 |\xi|^2 |\zeta|$$

for $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4) \in \mathbb{R}^4$ *where*

$$\begin{aligned} P_{\alpha\beta} &= [p_{i\alpha j\beta}]_{i,j=1,\ldots,4}, \\ p_{i\alpha j\beta} &= (1 - \delta_{i4})(1 - \delta_{j4}) c_{i\alpha j\beta} + \delta_{i4} \delta_{j4} k_{\alpha\beta} + \delta_{i4}(1 - \delta_{j4}) b_{\beta j\alpha}, \end{aligned}$$

$$p_{i\alpha j\beta} = p_{i\beta j\alpha}, \qquad c_{i\alpha j\beta} \in C^{s-1}(\mathbb{R}^9 \times \mathbb{R} \times \mathbb{R}),$$
$$M_{i\alpha}, m_{i\alpha} \in C^{s-1}(\mathbb{R}^9 \times \mathbb{R} \times \mathbb{R}), \qquad k_{\alpha\beta} \in C^{s-1}(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^9), \tag{3.6}$$
$$b_{\beta j\alpha}, b_{j\beta}, d_\alpha \in C^{s-1}(\mathbb{R}^9 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3), \qquad a \in C^{s-1}(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3).$$

4° *The initial data* $u^0, \theta^0, u^1, \theta^1$ *satisfy*

$$u^0, \theta^0 \in H^s(\Omega) \cap H_0^1(\Omega), \tag{3.7}$$

$$u^1, \theta^1 \in H^{s-1}(\Omega) \cap H_0^1(\Omega), \tag{3.8}$$

and the compatibility conditions

$$u^k, \theta^k \in H^{s-k}(\Omega) \cap H_0^1(\Omega) \qquad (2 \le k \le s-1), \tag{3.9}$$

$$u^s, \theta^s \in L^2(\Omega), \tag{3.10}$$

where $u^k = \partial^k u(0)/\partial t^k$ *and* $\theta^k = \partial^k \theta(0)/\partial t^k$ *are calculated formally* (*and recursively*) *in terms of* $u^0, \theta^0, u^1, \theta^1$ *using system* (1.1)–(1.4), *i.e.*

$$\begin{aligned} u_i^k = \Bigg(&\partial_t^{k-2} f_i^* + \sum_{m=0}^{k-2} \binom{k-2}{m} \partial_t^m c_{i\alpha j\beta} \partial_\alpha \partial_\beta u_j^{k-2-m} \\ &+ \sum_{m=0}^{k-2} \binom{k-2}{m} \partial_t^m M_{i\alpha} \partial_\alpha \theta^{k-1-m} \Bigg)(x), \\ \theta^k = \Bigg(&\partial_t^{k-2} Q^* + \sum_{m=0}^{k-2} \binom{k-2}{m} \partial_t^m k_{\alpha\beta} \partial_\alpha \partial_\beta \theta^{k-2-m} \\ &+ \sum_{m=0}^{k-2} \binom{k-2}{m} \partial_t^m k_{\beta j\alpha} \partial_\alpha \partial_\beta u_j^{k-2-m} \\ &+ \sum_{m=0}^{k-2} \binom{k-2}{m} \partial_t^m d_\alpha \partial_\alpha \theta^{k-2-m} + \sum_{m=0}^{k-2} \binom{k-2}{m} \partial_t^m b_{j\beta} \partial_\beta u_j^{k-1-m} \Bigg)(x) \end{aligned} \tag{3.11}$$

where $f_i^* = f_i - m_{i\alpha}\partial_\alpha \theta$.

Then for sufficiently small $T > 0$ *there exists a unique solution* (u, θ) *to the initial-boundary value problem* (3.1)–(3.4) *with the following properties*:

$$u \in \bigcap_{k=0}^{s-1} C^k([0,T], H^{s-k}(\Omega) \cap H_0^1(\Omega)), \qquad \partial_t^s u \in C^0([0,T], L^2(\Omega)), \tag{3.12}$$

$$\theta \in \bigcap_{k=0}^{s-1} C^k([0,T], H^{s-k}(\Omega) \cap H_0^1(\Omega)), \qquad \partial_t^s \theta \in C^0([0,T], L^2(\Omega)). \tag{3.13}$$

The proof of Theorem 3.1 is divided into three steps.

1. Proof for the linear hyperbolic system obtained by linearization of (3.1)–(3.4).

2. Proof of an energy estimate for the linear system.

3. Proof of existence and uniqueness of solution of the initial-boundary value problem for the nonlinear system (3.1)–(3.4) by applying a fixed point theorem.

4. Energy estimate

4.1. Linearized system of hyperbolic thermoelasticity. In this section, we investigate the initial-boundary value problem for a linear hyperbolic system which arises by linearization of (1.1)–(1.3). So, we shall investigate the solvability of the following problem:

$$\partial_t^2 u_i - \overline{c}_{i\alpha j\beta}(x,t)\partial_\alpha\partial_\beta u_j + \overline{M}_{i\alpha}(x,t)\partial_\alpha\partial_t\theta = \overline{f}_i(x,t), \tag{4.1}$$

$$\begin{aligned}\partial_t^2\theta - \overline{k}_{\alpha\beta}(x,t)\partial_\alpha\partial_\beta\theta - \overline{b}_{\beta j\alpha}(x,t)\partial_t\partial_\beta u_j \\ + \overline{b}_{j\beta}(x,t)\partial_t\partial_\beta u_j + \overline{d}_i(x,t)\partial_t\partial_i\theta = \overline{Q}(t,x),\\ (t,x)\in[0,T]\times\overline{\Omega},\qquad i=1,2,3,\end{aligned} \tag{4.2}$$

with initial conditions

$$\begin{aligned} u_i(0,x) &= u_i^0(x), &\quad (\partial_t u_i)(0,x) &= u_i^1(x),\\ \theta(0,x) &= \theta^0(x), &\quad (\partial_t\theta)(0,x) &= \theta^1(x),\end{aligned} \tag{4.3}$$

and boundary conditions

$$u_i(t,x)|_{\partial\Omega} = 0, \qquad \theta(t,x)|_{\partial\Omega} = 0. \tag{4.4}$$

4.2. Energy estimate for the linear hyperbolic system. We start with a result on the existence of solution for (4.1)–(4.3). The Faedo–Galerkin method may be used to prove an existence-uniqueness theorem. We also apply the methods of semigroup theory (cf. [54]).

THEOREM 4.1 (Existence, uniqueness and regularity for (4.1)–(4.3)). *Let the following assumptions be satisfied*:

1° $s \ge \lfloor 3/2\rfloor + 4 = 5$ *is an arbitrary but fixed integer.*

2°
$$\begin{aligned}
&\overline{c}_{i\alpha j\beta}\in C^0([0,T]\times\Omega)\cap L^\infty([0,T],L^\infty(\Omega)),\\
&D_x\overline{c}_{i\alpha j\beta}\in L^\infty([0,T],H^{s-2}(\Omega)),\\
&\partial_t^k\overline{c}_{i\alpha j\beta}\in L^\infty([0,T],H^{s-1-k}(\Omega))\quad \textit{for } k=1,\dots,s-1,\\
&\overline{k}_{\alpha\beta}\in C^0([0,T]\cap\overline{\Omega})\cap L^\infty([0,T],L^\infty(\Omega)),\\
&D_x\overline{k}_{\alpha\beta}\in L^\infty([0,T],H^{s-2}(\Omega)),\\
&\partial_t^k\overline{k}_{\alpha\beta}\in L^\infty([0,T],H^{s-1-k}(\Omega))\quad \textit{for } k=1,\dots,s-1,\\
&\overline{b}_{j\beta}\in C^0([0,T]\times\Omega),\qquad D_x\overline{b}_{j\beta}\in L^\infty([0,T],H^{s-2}(\Omega)),\\
&\partial_t^k\overline{b}_{j\beta}\in L^\infty([0,T],H^{s-1-k}(\Omega)),\\
&\overline{M}_{i\alpha}\in C^0([0,T]\times\Omega),\qquad D_x\overline{M}_{i\alpha}\in L^\infty([0,T],H^{s-2}(\Omega)),\\
&\partial_t^k\overline{M}_{i\alpha}\in L^\infty([0,T],H^{s-1-k}(\Omega)),\qquad k\le s-2,\\
&\partial_t^{s-1}\overline{M}_{i\alpha},\partial_t^{s-1}\overline{b}_{j\beta}\in L^\infty([0,T],L^\infty(\Omega)),\\
&|\overline{M}_{i\alpha}|_{s-1,T},|\overline{d}_\alpha|_{s-1,T},|\overline{b}_{j\beta}|_{s-1,T}\le c,
\end{aligned}$$

where c is a small constant.

3° $\overline{c}_{i\alpha j\beta} = \overline{c}_{j\beta i\alpha}$.

4° $\overline{k}_{\alpha\beta}=\overline{k}_{\beta\alpha}$ *and there exists a constant* $\gamma_0>0$ *such that*

$$\|W\|_1^2\le\gamma_0(\overline{c}_{i\alpha j\beta}\partial_\beta W_j,\partial_\alpha W_i)+\|W\|_0^2$$

for all $W\in H_0^1(\Omega)$, $t\in[0,T]$, *where*

$$W=(u_1,u_2,u_3,\theta)^*,\qquad \overline{c}_{\alpha\beta}=[\overline{c}_{i\alpha j\beta}]_{i,j=1,2,3},$$
$$\overline{c}_{i\alpha j\beta}=(1-\delta_{i4})(1-\delta_{j4})\overline{c}_{i\alpha j\beta}+\delta_{i4}\delta_{j4}\overline{k}_{\alpha\beta}+\delta_{i4}(1-\delta_{j4})b_{\beta j\alpha},$$

δ_{ij} *being the Kronecker delta.*

5° *For almost every* $t\in[0,T]$ *the condition* $\overline{c}_{\alpha\beta}\partial_\alpha\partial_\beta W\in H^k(\Omega)$ *together with* $W\in H_0^1(\Omega)$ *implies*

$$W\in H^{k+2}(\Omega)$$

and

$$\|W\|_{k+2}^2\le\gamma_1(\|\overline{c}_{i\alpha j\beta}\partial_\alpha\partial_\beta W_j\|_k^2+\|W\|_0^2),$$

where $\gamma_1>0$ *is a constant.*

6°
$$\partial_t^k\overline{f},\partial_t^k\overline{Q}\in C^0([0,T],H^{s-2-k}(\Omega)),\qquad k=0,1,\dots,s-2,$$
$$\partial_t^{s-1}\overline{f},\partial_t^{s-1}\overline{Q}\in L^2([0,T],L^2(\Omega)).$$

7°
$$u^k\equiv\partial_k^t u|_{t=0},\theta^k\equiv\partial_k^t\theta_{t=0}\in H^{s-k}(\Omega)\cap H_0^1(\Omega),\qquad 0\le k\le s-1,$$
$$u^s\in L^2(\Omega),\qquad \theta^s\in L^2(\Omega)\in H^{s-k}(\Omega)\cap H_0^1(\Omega).$$

Then there is a unique solution (u,θ) *of problem* (4.1)–(4.4) *with the properties*:

$$\begin{aligned}&\partial_t^k u,\partial_t^k\theta\in C^0([0,T],H^{s-k}(\Omega)\cap H_0^1(\Omega)),\qquad 0\le k\le s-1,\\&\partial_t^s\theta,\partial_t^s u\in C^0([0,T],L^2(\Omega)).\end{aligned}\tag{4.5}$$

Proof. We apply Kato's approach (cf. [54], [23], [74]). We can convert problem (4.1)–(4.4) to an equivalent (evolution) problem of the form

$$\partial_t V+AV=F,\tag{4.6}$$
$$V(0,x)=V^0(x),\tag{4.7}$$

where

$$V=(u_1,u_2,u_3,\theta,\partial_t u_1,\partial_t u_2,\partial_t u_3,\partial_t\theta)^*,\tag{4.8}$$
$$V^0=(u_1^0,u_2^0,u_3^0,\theta^0,u_1^1,u_2^1,u_3^1,\theta^1)^*,\tag{4.9}$$
$$F=(0,0,0,0,\overline{f}_1,\overline{f}_2,\overline{f}_3,\overline{\theta})^*,\tag{4.10}$$
$$A=\begin{bmatrix}-I_{4\times4} & -I_{4\times4}\\ [-\widetilde{c}_{i\alpha j\beta}\partial_\alpha\partial_\beta]_{i,j=1,2,3,4} & [\overline{g}_{i\alpha j}\partial_\alpha]_{i,j=1,2,3,4}\end{bmatrix}_{8\times8},\tag{4.11}$$
$$\begin{aligned}&\widetilde{c}_{\alpha\beta}=[\widetilde{c}_{i\alpha j\beta}]_{i,j=1,\dots,4},\\&\widetilde{c}_{i\alpha j\beta}=(1-\delta_{i4})(1-\delta_{j4})\overline{c}_{i\alpha j\beta}+\delta_{j4}\delta_{i4}\overline{k}_{\alpha\beta}+\delta_{i4}(1-\delta_{j4})\overline{b}_{\beta j\alpha},\end{aligned}\tag{4.12}$$
$$\begin{aligned}&\widetilde{g}_\alpha=[\widetilde{g}_{i\alpha j}]_{i,j=1,\dots,4},\\&\widetilde{g}_{i\alpha j}=(1-\delta_{i4})\delta_{j4}\overline{M}_\alpha+(1-\delta_{j4})\delta_{j4}\overline{b}_{j\beta}+\delta_{j4}\delta_{i4}d_\alpha.\end{aligned}\tag{4.13}$$

The operator

$$A : D(A) \to X_0 \tag{4.14}$$

defined by (4.11) has domain

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega). \tag{4.15}$$

In the space

$$X = H_0^1(\Omega) \times L^2(\Omega) \tag{4.16}$$

we introduce the time-dependent inner product

$$\langle U, V\rangle = (\widetilde{c}_{i\alpha j\beta}\partial_\beta w_j, \partial_\alpha w_i^*)_0 + (w, w^*)_0 + (v, v^*)_0 \tag{4.17}$$

where

$$(\widetilde{c}_{i\alpha j\beta}\partial_\beta w_j, \partial_\alpha w_i^*)_0 = (\widetilde{c}_{\alpha\beta}\partial_\beta w, \partial_\alpha w^*)_0, \tag{4.18}$$

$$V = (w, v)^*, \qquad V^* = (w^*, v^*)^* \in X, \tag{4.19}$$

$$w = (u_1, u_2, u_3, \theta)^*, \qquad v = (\partial_t u_1, \partial_t u_2, \partial_t u_3, \partial_t \theta)^*. \tag{4.20}$$

Note that the norm $|||\cdot|||$ corresponding to $\langle\cdot,\cdot\rangle$ is equivalent to the usual norm in $H_0^1(\Omega)\times L^2(\Omega)$. We show that the triple $(A; X, D(X))$ forms a CD-system. First, we notice that

$$AV = (-v, \widetilde{c}_{\alpha\beta}\partial_\alpha\partial_\beta w + \overline{g}_\alpha\partial_\alpha v)^*. \tag{4.21}$$

So, we have

$$\langle AV, V\rangle = (-\widetilde{c}_{\alpha\beta}\partial_\beta v, \partial_\alpha w)_0 + (-v, w)_0 + (-\overline{c}_{\alpha\beta}\partial_\alpha\partial_\beta w, v)_0 + (\overline{g}_\alpha\partial_\alpha w, v). \tag{4.22}$$

After some calculation, we get

$$\langle AV, V\rangle = (\partial_\beta\widetilde{c}_{\alpha\beta}\partial_\alpha w, v)_0 + (-v, w)_0 - \tfrac{1}{2}(\partial_\alpha\widetilde{g}_\alpha w, v). \tag{4.23}$$

Taking into account the assumption of Theorem 4.1, we obtain

$$\langle AV, V\rangle \ge -c_1|||V|||^2, \qquad c_1 > 0. \tag{4.24}$$

Hence

$$\begin{aligned} |||(\lambda I + A)V|||^2 &= \langle(\lambda I + A)V, (\lambda I + A)V\rangle \\ &= \lambda^2|||V|||^2 + |||A|||^2 + 2\lambda\langle AV, V\rangle \\ &\ge \lambda^2|||V|||^2 + 2\lambda\langle AV, V\rangle \ge (\lambda^2 - 2\lambda c_1)|||V|||^2. \end{aligned} \tag{4.25}$$

From (4.24) and (4.25) we get

$$|||(\lambda I + A)V|||^2 \ge (\lambda - 2c_1)^2|||V|||^2 \quad \text{for } \lambda > 2c_1 > 0. \tag{4.26}$$

It follows that the operator $(\lambda I + A)^{-1}$ exists. Now, we have

$$\langle(\lambda I + A)V, V\rangle = \lambda\langle V, V\rangle + \langle AV, V\rangle \ge \lambda|||V|||^2 - c_1|||V|||^2 = (\lambda - c_1)|||V|||^2 \ge 0.$$

So, in view of the Lax theorem the operator $\lambda I + A$ is invertible on X_0. Now, we have

$$|||(\lambda I + A)^{-1}||| = \sup_{\|V^*\|=1} |||(\lambda I + A)^{-1}V^*|||. \tag{4.27}$$

Putting $V^* = (\lambda I + A)V$, we get

$$|||(\lambda I + A)^{-1}||| = \sup_{\|V^*\|=1} |||V||| \le \frac{1}{\lambda - 2c_1}. \tag{4.28}$$

From (4.25) we get

$$|||V||| \le \frac{|||(\lambda I + A)V|||}{\lambda - 2c_1}. \tag{4.29}$$

Since the operator A is closed, and in view of (4.29) and the considerations in Section 2 and the Hille–Yosida theorem, it follows that the first condition of CD is satisfied.

Because $D(A)$ is independent of t, taking into account the assumption of Theorem 4.1 we have

$$\partial_t A \in L^\infty(I, L(D(A); X)).$$

Moreover $(A; X, D(A))$ is a CD-system in the sense of Kato (cf. [54]). Let

$$X_0 = Y_0 = H_0^1(\Omega) \times L^2(\Omega), \tag{4.30}$$

$$X_j = H^{j+1}(\Omega) \cap H_0^1(\Omega) \times H^j(\Omega) \quad \text{for } j \ge 1, \tag{4.31}$$

$$Y_j = H^{j+1}(\Omega) \cap H_0^1(\Omega) \times H^j(\Omega) \cap H_0^1(\Omega), \tag{4.32}$$

$$|||V|||_j = \|w\|_{j+1} + \|v\|_j \quad \text{for } V \in X_j. \tag{4.33}$$

For the spaces defined by (4.30)–(4.32) the following conditions are satisfied:

- the triple $(A; X, D(A))$ is a CD-system,
- $\partial_t^{r+1} A \in L^\infty(I, L(Y_{j+r+1}; X_j))$, $0 \le j \le s-1$ (this follows from the conditions $r \le s-1$ of Theorem 4.1),
- $|||V|||_{j+1} \le c(|||A|||_j + |||V|||_0)$, $j = 1, \ldots, s-1$ (under the assumption that $|\widetilde{g}_{i\alpha j}|_{s-1,T} \le c$ for c sufficiently small),
- $\partial_t^k F \in C^0(I, X_{s-1-k})$, $k = 0, 1, \ldots, s-2$, $\partial_t^{s-1} F \in L^1(I, X_0)$ (this follows from the assumption of Theorem 4.1).

Taking this into account and basing on Theorems 2.11 and 2.12 we get the existence, uniqueness and regularity of the solution to problem (4.1)–(4.4). This ends the proof of Theorem 4.1.

In the second step, we formulate an energy estimate for (4.1)–(4.4).

THEOREM 4.2 (Energy estimate for (4.1)–(4.4)). *If the assumptions of Theorem 4.1 are satisfied then the solution of (4.1)–(4.3) guaranteed by Theorem 4.1 satisfies the inequality*

$$|\overline{D}^s (u, \theta)^*|_{0,T}^2 \le K_0 K_1 e^{K_2 \sqrt{T}(1 + 1/\sqrt{T} + T)} \tag{4.34}$$

with positive constants K_0, K_1, K_2 where

$$K_0 = \sum_{k=0}^{s} \|(u^k, \theta^k)^*\|_{s-k}^2 + (1+T)|\overline{D}^{s-2}(\bar{f}, \overline{\theta})^*|_{s-k,T}^2 + \int_0^T \|\partial_t^{s-1}(\bar{f}, \overline{\theta})^*\|_0^2 \, dt, \tag{4.35}$$

$$K_1 = K_1(B_1, \gamma_0, \gamma_1) > 0, \tag{4.36}$$

$$K_2 = K_2(B_2, \gamma_0, \gamma_1) > 0, \tag{4.37}$$

$$B_1 = \|\overline{c}_{i\alpha j\beta}(0)\|_0 + \|\overline{k}_{\alpha\beta}(0)\|_0 + \|b_{\alpha j\beta}(0)\|_0, \tag{4.38}$$

$$B_2 = |\partial_\beta \overline{b}_{j\beta}|_{s-2,T} + |\partial_\alpha \overline{c}_{i\alpha j\beta}|_{s-2,T} + |\partial_\alpha \overline{k}_{\alpha\beta}|_{s-2,T} + |\partial_\alpha b_{\alpha j\beta}|_{s-2,T} + \sum_{k=0}^{s-1}(|\partial_t^k \overline{c}_{i\alpha j\beta}|_{s-k-1,T} + |\partial_t^k \overline{k}_{\alpha\beta}|_{s-k-1,T} + |\partial_t^k \overline{b}_{j\beta}|_{s-k-1,T} + |\partial_t^k b_{\alpha j\beta}|_{s-k-1,T} + |\partial_t^k \overline{M}_{i\alpha}|_{s-k-1,T}) \tag{4.39}$$

(*the constants* γ_0, γ_1 *are given in the assumption of Theorem* 4.1).

Proof. Using the notations (4.12), (4.13) and (4.20) we can write the system (4.1)–(4.2) as follows:

$$\partial_t^2 w_i - \widetilde{c}_{i\alpha j\beta}\partial_\alpha\partial_\beta w_j + \widetilde{g}_{i\alpha j}\partial_\alpha\partial_\beta w_j = \overline{f}_i, \tag{4.40}$$

where $i, j = 1, 2, 3, 4$ and

$$\overline{f} = (\overline{f}_1, \overline{f}_2, \overline{f}_3, \overline{Q})^*. \tag{4.41}$$

Differentiating (4.40) with respect to time $n-1$ times ($1 \le n \le s-1$) we get

$$\partial_t^{n+1} w_i - \overline{c}_{i\alpha j\beta}\partial_t^{n-1}\partial_\alpha\partial_\beta w_j = h_i^{n-1} \tag{4.42}$$

where

$$h_i^{n-1} = \partial_t^{n-1}\overline{f}_i - \partial_t^{n-1}(\overline{g}_{i\alpha j}\partial_\alpha\partial_t w_j) + \sum_{k=0}^{n-1}\binom{n-1}{k}\partial_t^k \overline{c}_{i\alpha j\beta}\partial_t^{n-1-k}\partial_\alpha\partial_\beta w_j. \tag{4.43}$$

Multiplying (4.42) by $\partial_t^n W_i$ and integrating with respect to $(t, x) \in (0, T) \times \mathbb{R}$ we get

$$\|\partial_t^n w\|_0^2 + \|\partial_t^{n-1} w\|_1^2 \le C(\beta_1, \gamma_0)(\|w^n\|_0^2 + \|w^{n-1}\|_1^2 + \|\partial_t^{n-1} w\|_0^2) + C(\beta_1, \gamma_0)\int_0^t (\|\partial_\tau^n w(\tau)\|_0^2 + \|\partial_\tau^{n-1} w(\tau)\|_1^2)\, d\tau + C(\gamma_0)\int_0^t \|h^{n-1}(\tau)\|_0^2\, d\tau \tag{4.44}$$

where

$$w = (u, \theta)^*, \quad w^n = (u^n, \theta^n)^*. \tag{4.45}$$

Taking into account that

$$\int_0^t \|h^{n-1}(\tau)\|_0^2\, d\tau \le (1+T)|\partial_t^{n-1} f|_{0,T}^2 + C(B_2)\int_0^t \|\overline{D}^{s-1} W(\tau)\|_0^2\, d\tau \tag{4.46}$$

we get (for $n \le s-1$)

$$\|\partial_t^n w\|_0^2 + \|\partial_t^{n-1} w\|_1^2 \le C(\beta_1, \gamma_0)(\|w^n\|_0^2 + \|w^{n-1}\|_1^2 + \|\partial_t^{n-1} w\|_0^2) + (1+T)|\partial_t^{n-1} f|_{0,T}^2 + C(\beta_1, \gamma_0)\int_0^t \|\overline{D}^{s-1} w(\tau)\|_0^2\, d\tau. \tag{4.47}$$

Summing the inequalities (4.47) for $n = 1, \ldots, s-1$, we get

$$\sum_{n=0}^{s-2}\|\partial_t^n w\|_1 + \|\partial_t^{s-1} w\|_0 \le C(\beta_1, \gamma_0)K_0 + C(\beta_2, \gamma_0)\int_0^t \|\overline{D}^{s-1} w(\tau)\|_0^2\, d\tau. \tag{4.48}$$

In order to estimate $\partial_t^s w$ we use the Friedrichs mollifier (cf. [22]). Applying the mollifier J_δ (cf. [22]) to both sides of (4.42) under the assumption that $n = s-1$, $0 < \delta < \varepsilon < T$, for $t \in [\varepsilon, T-\varepsilon]$, we get

$$(\partial_t^{n+1} w_i)_\delta - (\overline{c}_{i\alpha j\beta}\partial_t^{s-2}\partial_\alpha\partial_\beta w_j)_\delta = (h_i^{s-2})_\delta + (R_i)_\delta \tag{4.49}$$

where

$$(R_i)_\delta = (\overline{c}_{i\alpha j\beta}\partial_t^{s-2}\partial_\alpha\partial_\beta w_j)_\delta - \overline{c}_{i\alpha j\beta}(\partial_t^{s-2}\partial_\alpha\partial_\beta w_j). \tag{4.50}$$

Differentiating (4.49) with respect to t, integrating over $(\varepsilon, t)\times\Omega$, $\varepsilon \le t \le T-\varepsilon$, using the properties of the Friedrichs mollifier $(\partial_t^s w)_\delta|_{\partial\Omega} = 0$ and $(\partial_t w)_\delta = \partial_t (w)_\delta$, after integrating by parts we have

$$\begin{aligned}\|(\partial_t^s w)_\delta(t)\|_0^2 + \|(\partial_t^{n-1} w)_\delta(t)\|_1^2 &\le C(B_1,\gamma_0)(\|(\overline{D}^s w)_\delta(t)\|_0^2 + \|(\partial_t^{s-1} w)_\delta(t)\|^2) \\ &\quad + C(B_1,\gamma_0)\left(1+\frac{1}{\sqrt{T}}\right)\int_\varepsilon^T \|(\overline{D}^s w)_\delta(\tau)\|\,d\tau \\ &\quad + \sqrt{T}\int_\varepsilon^T \|\partial_t(h^{s-2})_\delta(\tau)\|_0^2\,d\tau + \int_\varepsilon^T \|\partial_\tau R_\delta(\tau)\|^2\,d\tau.\end{aligned} \tag{4.51}$$

Letting $\varepsilon \to 0$ we get $\delta \to 0$.

So, using the assumption of Theorem 4.1 we have

$$\int_\varepsilon^t \|\partial_t h^{s-2}(\tau)\|_0^2\,d\tau \le \int_\varepsilon^t \|\partial_t^{s-1}\overline{f}(\tau)\|_0^2\,d\tau + C(B_2,\gamma_0)\int_\varepsilon^t \|\overline{D}^s w(\tau)\|_0^2\,d\tau. \tag{4.52}$$

So, we get

$$\begin{aligned}\|\partial_t^s w\|_0^2 + \|\partial_t^{s-1} w\|_1^2 &\le C(B_1,\gamma_0)K_0 \\ &\quad + C(B_2,\gamma_0)\left(\sqrt{T}+\frac{1}{\sqrt{T}}+\frac{1}{T}\right)\int_\varepsilon^T \|\overline{D}^s w(\tau)\|_0^2\,d\tau,\end{aligned} \tag{4.53}$$

$$\|\partial_t^{s-2} w\|_2^2 \le \gamma_0(\|\partial_t^s w(t)\|_0^2 + \|h^{s-2}(t)\|_0^2 + \|\partial_t^{s-2} w(t)\|_0^2), \tag{4.54}$$

$$\|h^{s-2}(t)\|_0^2 \le C(B_1,\gamma_0)K_0 + C(B_2)\sqrt{T}\int_0^t \|\overline{D}^s w(\tau)\|_0^2\,d\tau. \tag{4.55}$$

So, we have

$$\|\partial_t^{s-2} w\|_2^2 \le C(B_1,\gamma_0,\gamma_1)K_0 + C(B_2,\gamma_0,\gamma_1)\left(\frac{1}{T}+\sqrt{T}+\frac{1}{\sqrt{T}}\right)\int_0^t \|\overline{D}^s w(\tau)\|_0^2\,d\tau. \tag{4.56}$$

Putting $n = s-2$ in (4.42), using the assumption of Theorem 4.1 and acting as above we get an estimate for $\|\partial_t^{s-3} w(t)\|_3^2$. Acting in the same way for $k = 2,\dots,s$ we can notice that $\|\partial_t^k w\|_{s-k}^2$ $(4 \le k \le s)$ are bounded by the right hand side of (4.56). Finally, we have

$$\begin{aligned}|\overline{D}^s w|_{0,T}^2 &\le C(B_1,\gamma_0,\gamma_1)K_0 \\ &\quad + C(B_2,\gamma_0,\gamma_1)\left(\frac{1}{T}+\sqrt{T}+\frac{1}{\sqrt{T}}\right)\int_0^T \|\overline{D}^s w(\tau)\|_0^2\,d\tau.\end{aligned} \tag{4.57}$$

Applying the Gronwall inequality to (4.57), we get

$$|\overline{D}^s w|_{0,T}^2 \le K_1 K_0 e^{(1+1/\sqrt{T}+T)\sqrt{T}}.$$

This completes the proof of Theorem 4.2.

5. Proof of Theorem 3.1

The proof of Theorem 3.1 is based on the Banach fixed point theorem. We denote by $Z(N,T)$ the set of functions (u,θ) which satisfy

$$\partial_t^k u_i, \partial_t^k \theta \in L^\infty([0,T], H^{s-k}(\Omega)), \quad 0 \le k \le s \tag{5.1}$$

($s \ge \lfloor 3/2 \rfloor + 4 = 5$ being an arbitrary but fixed integer), with boundary and initial conditions of the form

$$u_i|_{\partial\Omega} = 0, \quad \theta|_{\partial\Omega} = 0, \tag{5.2}$$

$$(\partial_t^k u_i)(0,x) = u_i^k, \quad (\partial_t^k \theta)(0,x) = \theta^k, \quad 0 \le k \le s-2,\ i = 1,2,3, \tag{5.3}$$

and the inequality

$$|\overline{D}^s u|_{0,T}^2 + |\overline{D}^s \theta|_{0,T}^2 \le N^2 \tag{5.4}$$

for N large enough.

Proof of Theorem 3.1. Let

$$(\overline{u}, \overline{\theta}) \in Z(N,T). \tag{5.5}$$

We consider system (4.1)–(4.2) with

$$\overline{c}_{i\alpha j\beta} = c_{i\alpha j\beta}(\nabla \overline{u}, \overline{\theta}, \partial_t \overline{\theta}), \tag{5.6}$$

$$\overline{M}_{i\alpha} = M_{i\alpha}(\nabla \overline{u}, \overline{\theta}, \partial_t \overline{\theta}), \tag{5.7}$$

$$\overline{k}_{\alpha\beta} = k_{\alpha\beta}(\nabla \overline{u}, \nabla\overline{\theta}, \overline{\theta}, \partial_t \overline{\theta}), \tag{5.8}$$

$$\overline{b}_{j\beta} = b_{j\beta}(\nabla \overline{u}, \nabla\overline{\theta}, \overline{\theta}, \partial_t \overline{\theta}), \tag{5.9}$$

$$\overline{f}_i = f_i(t,x) - m_{i\alpha}(\nabla \overline{u}, \overline{\theta}, \partial_t \overline{\theta})\partial_\alpha \overline{\theta}, \tag{5.10}$$

$$\overline{Q}_i = Q_i(t,x) - a(\nabla \overline{u}, \nabla\overline{\theta}, \overline{\theta}, \partial_t \overline{\theta})\partial_t \overline{\theta}. \tag{5.11}$$

We rewrite this system in the form

$$\begin{aligned}\partial_t^2 u_i - c_{i\alpha j\beta}(\nabla \overline{u}, \overline{\theta}, \partial_t \overline{\theta})\partial_\alpha\partial_\beta u_j + M_{i\alpha}(\nabla \overline{u}, , \overline{\theta}, \partial_t \overline{\theta})\partial_\alpha \partial_t \theta \\ = f_i(t,x) - m_{i\alpha}(\nabla \overline{u}, \overline{\theta}, \partial_t \overline{\theta})\partial_\alpha \overline{\theta},\end{aligned} \tag{5.12}$$

$$\begin{aligned}\partial_t^2 \theta - k_{\alpha\beta}(\nabla \overline{u}, \nabla\overline{\theta}, \overline{\theta}, \partial_t \overline{\theta})\partial_\alpha\partial_\beta \theta + b_{j\beta}(\nabla \overline{u}, \nabla\overline{\theta}, \overline{\theta}, \partial_t \overline{\theta})\partial_\beta \partial_t u_j \\ + b_{j\beta}(\nabla \overline{u}, \nabla\overline{\theta}, \overline{\theta}, \partial_t \overline{\theta})\partial_\alpha \partial_t u_j + d_\alpha(\nabla \overline{u}, \nabla\overline{\theta}, \overline{\theta}, \partial_t \overline{\theta})\partial_\alpha \partial_\beta u_j \\ = Q(t,x) - a(\nabla \overline{u}, \nabla\overline{\theta}, \overline{\theta}, \partial_t \overline{\theta})\partial_\alpha \overline{\theta}\end{aligned} \tag{5.13}$$

with boundary and initial conditions (4.3)–(4.4).

The functions u and θ appearing in (5.12) and (5.13) are the solution of system (5.12)–(5.13) with conditions (4.3)–(4.4). Taking into account the class of functions (4.3)–(4.4) we can apply Theorems 4.1 and 4.2. It follows that for every $(\overline{u}, \overline{\theta}) \in Z(N,T)$ there exists a unique solution (u,θ) to problem (5.12)–(5.13) with initial-boundary conditions (1.16)–(1.17). This means there exists a mapping

$$\sigma : Z(N,T) \ni (\overline{u}, \overline{\theta}) \mapsto \sigma(\overline{u}, \overline{\theta}) = (u,\theta). \tag{5.14}$$

STATEMENT I. *σ maps the set $Z(N,T)$ into itself for N large and T small enough.*

First, we introduce the notation

$$
\begin{aligned}
(5.15)\qquad E_0 = {} & \sum_{k=0}^{s}(\|u^k\|_{s-k}^2 + \|\theta^k\|_{s-k}^2) \\
& + \sum_{k=0}^{s-2} |\partial_t^k (f,Q)|^2_{s-2-k,T} + \int_0^T \|\partial_t^{s-1}(f,Q)\|^2_{s-2-k}\, dt.
\end{aligned}
$$

Using the properties of the elements of the set $Z(N,T)$ and Theorem 2.10 and applying the Sobolev inequality, we get the following estimate for the function $\overline{f}$ given by (5.10):

$$
\begin{aligned}
(5.16)\qquad \int_0^T \|\partial_t^{s-1}\overline{f}_i\|_0^2\, dt &= \int_0^T \|\partial_t^{s-1}(f_i - m_{i\alpha}(\nabla\overline{u},\overline{\theta},\partial_t\overline{\theta})\partial_\alpha\theta)\|_0^2\, dt \\
&\le \int_0^T \Big[\Big(\sum_{k=0}^{s} \|D^s(\nabla\overline{u},\overline{\theta},\partial_t\overline{\theta})\|_0^k\Big)^2 + \|\partial_t^{s-1}f_i\|_0^2\Big]\, dt \\
&\le C(N)(1+T) + C(E),
\end{aligned}
$$

$$
\begin{aligned}
(5.17)\qquad \sum_{k=0}^{s-2} |\partial_t^k \overline{f}|^2_{0,T} &\le \sum_{k=0}^{s-2} |\partial_t^k f_i - \partial_t^k m_{i\alpha}(\nabla\overline{u},\overline{\theta},\partial_t\overline{\theta})\partial_\alpha\overline{\theta}|^2_{s-2-k,T} \\
&\le (T+T^2)C(N) + C(E_0),
\end{aligned}
$$

and similarly

$$
\begin{aligned}
(5.18)\qquad \int_0^T \|\partial_t^{s-1}\overline{Q}\|_0^2\, dt &= \int_0^T \|\partial_t^{s-1}Q - \partial_t^{s-1}(a(\nabla\overline{u},\overline{\theta},\partial_t\overline{\theta})\partial_\alpha\theta)\|_0^2\, dt \\
&\le C(E_0) + C(N) + (1+T),
\end{aligned}
$$

$$
\begin{aligned}
(5.19)\qquad \sum_{k=0}^{s-2} |\partial_t^k \overline{Q}|_{0,T} &\le \sum_{k=0}^{s-2} |\partial_t^k Q_i - a(\nabla\overline{u},\overline{\theta},\partial_t\overline{\theta})\partial_\alpha\overline{\theta}|^2_{s-2-k,T} \\
&\le (T+T^2)C(N) + C(E_0).
\end{aligned}
$$

Putting (5.16)–(5.19) into the energy estimate (4.34) of Theorem 4.2, we get

$$(5.20)\qquad |\overline{D}^s(u,\theta)|^2_{0,T} \le C(E_0,\gamma_0,\gamma_1)(1 + C(N)(T^{1/2}+T+T^2))e^{\sqrt{T}(1+1/\sqrt{T}+T)}C(N).$$

Let N be large enough that

$$(5.21)\qquad 2C(E_0,\gamma_0,\gamma_1) < N^2.$$

Since $\eta(T)$ is a continuous function and $\eta(0)=1$, there exists $T>0$ such that

$$(5.22)\qquad \eta(T) := (1 + C(N)(T^{1/2}+T+T^2))e^{\sqrt{T}(1+1/\sqrt{T}+T)C(N)} \le 2.$$

So, in view of this fact, we get from (5.20) the inequality

$$(5.23)\qquad |\overline{D}^s(u,\theta)|^2_{0,T} \le N^2.$$

From (5.23) it follows that

$$(5.24)\qquad (u,\theta) \in Z(N,T).$$

Statement II. *The mapping $\sigma : Z(N,T) \to Z(N,T)$ is a contraction for T small enough.*

Let W denote the complete metric space given by

$$W = \{(\overline{u}, \overline{\theta}) : \overline{D}^1(\overline{u}, \overline{\theta}) \in L^\infty([0,T], L^2(\Omega))\} \tag{5.25}$$

with metric

$$\varrho((\overline{u}, \overline{\theta}), (u, \theta)) = |\overline{D}^1(\overline{u} - u)|^2_{0,T} + |\overline{D}^1(\overline{\theta} - \theta)|^2_{0,T}. \tag{5.26}$$

It is easy to see that $Z(N,T)$ is a closed subset of W. Let $(\overline{u}, \overline{\theta}), (\overline{u}^*, \overline{\theta}^*) \in Z(N,T)$. Then

$$\sigma(\overline{u}, \overline{\theta}) = (u, \theta) \in Z(N,T), \qquad \sigma(\overline{u}^*, \overline{\theta}^*) = (u^*, \theta^*) \in Z(N,T), \tag{5.27}$$

where $(\overline{u}, \overline{\theta})$, $(\overline{u}^*, \overline{\theta}^*)$ are the solutions of problem (4.1)–(4.4) where the coefficients (5.6)–(5.9) and the right hand sides of (5.10)–(5.11) depend on $(\overline{u}, \overline{\theta})$ and $(\overline{u}^*, \overline{\theta}^*)$ respectively, i.e.

$$\begin{aligned} &\partial_t^2 u_i - c_{i\alpha j\beta}(\overline{\theta}, \nabla\overline{u}, \partial_t\overline{\theta})\partial_\alpha\partial_\beta u_j + M_{i\alpha}(\overline{\theta}, \nabla\overline{u}, \partial_t\theta)\partial_\alpha\partial_t\theta \\ &\qquad = f_i(t,x) - m_{i\alpha}(\overline{\theta}, \nabla\overline{u}, \partial_t\theta)\partial_\alpha\overline{\theta}, \end{aligned} \tag{5.28}$$

$$\begin{aligned} &\partial_t^2\theta - k_{\alpha\beta}(\overline{\theta}, \partial_t\theta, \nabla\overline{\theta}, \nabla\overline{u})\partial_\alpha\partial_\beta\theta + b_{j\beta}(\overline{\theta}, \partial_t\theta, \nabla\overline{\theta}, \nabla\overline{u})\partial_\beta\partial_t u_j \\ &\qquad - b_{\alpha j\beta}(\overline{\theta}, \partial_t\theta, \nabla\overline{u})\partial_\alpha\partial_\beta u_j + d_\alpha(\overline{\theta}, \partial_t\theta, \nabla\overline{\theta}, \nabla\overline{u})\partial_t\partial_\alpha\theta \\ &\qquad = Q(t,x) - a(\overline{\theta}, \partial_t\theta, \nabla\overline{\theta}, \nabla\overline{u})\partial_\alpha\overline{\theta}, \end{aligned} \tag{5.29}$$

$$\begin{aligned} &\partial_t^2 u_i^* - c_{i\alpha j\beta}(\overline{\theta}^*, \nabla\overline{u}^*, \partial_t\overline{\theta}^*)\partial_\alpha\partial_\beta u_j^* + M_{i\alpha}(\overline{\theta}^*, \nabla\overline{u}^*, \partial_t\overline{\theta}^*)\partial_\alpha\partial_t\theta^* \\ &\qquad = f_i(t,x) - m_{i\alpha}(\overline{\theta}^*, \nabla\overline{u}^*, \partial_t\overline{\theta}^*)\partial_\alpha\overline{\theta}^*, \end{aligned} \tag{5.30}$$

$$\begin{aligned} &\partial_t^2\theta^* - k_{\alpha\beta}(\overline{\theta}^*, \partial_t\overline{\theta}^*, \nabla\overline{\theta}^*, \nabla\overline{u}^*)\partial_\alpha\partial_\beta\theta^* + b_{j\beta}(\overline{\theta}^*, \partial_t\overline{\theta}^*, \nabla\overline{\theta}^*, \nabla\overline{u}^*)\partial_\beta\partial_t u_j^* \\ &\qquad - b_{\beta j\alpha}(\overline{\theta}^*, \partial_t\overline{\theta}^*, \nabla\overline{\theta}^*, \nabla\overline{u}^*)\partial_\alpha\partial_\beta u_j^* + d_\alpha(\overline{\theta}^*, \partial_t\overline{\theta}^*, \nabla\overline{\theta}^*, \nabla\overline{u}^*)\partial_t\partial_\alpha\theta^* \\ &\qquad = Q(t,x) - a(\overline{\theta}^*, \partial_t\overline{\theta}^*, \nabla\overline{\theta}^*, \nabla\overline{u}^*)\partial_\alpha\overline{\theta}^*. \end{aligned} \tag{5.31}$$

Subtracting (5.28), (5.30) and (5.29), (5.31) respectively, we get

$$\begin{aligned} &\partial_t^2(u_i - u_i^*) - c_{i\alpha j\beta}(\overline{\theta}, \nabla\overline{u}, \partial_t\overline{\theta})\partial_\alpha\partial_\beta(u_j - u_j^*) \\ &\qquad + (M_{i\alpha}(\overline{\theta}, \nabla\overline{u}, \partial_t\overline{\theta})\partial_\alpha\partial_t(\theta - \theta^*) + m_{i\alpha}(\overline{\theta}, \nabla\overline{u}, \partial_t\overline{\theta}))\partial_\alpha(\overline{\theta} - \overline{\theta}^*) \\ &\qquad = (c_{i\alpha j\beta}(\overline{\theta}, \nabla\overline{u}, \partial_t\overline{\theta}) - c_{i\alpha j\beta}(\overline{\theta}^*, \nabla\overline{u}^*, \partial_t\overline{\theta}^*))\partial_\alpha\partial_\beta u_j^* \\ &\qquad + (M_{i\alpha}(\overline{\theta}, \nabla\overline{u}, \partial_t\overline{\theta}) - M_{i\alpha}(\overline{\theta}^*, \nabla\overline{u}^*, \partial_t\overline{\theta}^*))\partial_\alpha\partial_t\theta^* \\ &\qquad - m_{i\alpha}(\overline{\theta}^*, \nabla\overline{u}^*, \partial_t\overline{\theta}^*)\partial_\alpha\theta^*, \end{aligned} \tag{5.32}$$

$$\begin{aligned} &\partial_t^2(\theta - \theta^*) - k_{\alpha\beta}(\overline{\theta}, \nabla\overline{\theta}, \nabla\overline{u}, \partial_t\overline{\theta})\partial_\alpha\partial_\beta(\theta - \theta^*) \\ &\qquad + b_{j\beta}(\overline{\theta}, \nabla\overline{\theta}, \nabla\overline{u}, \partial_t\overline{\theta})\partial_\beta\partial_t(u - u_j^*) - b_{\beta j\alpha}(\overline{\theta}, \nabla\overline{\theta}, \nabla\overline{u}, \partial_t\overline{\theta})\partial_\alpha\partial_\beta(u - u_j^*) \\ &\qquad + d_\alpha(\overline{\theta}, \nabla\overline{\theta}, \nabla\overline{u}, \partial_t\overline{\theta})\partial_t\partial_\alpha(\overline{\theta} - \overline{\theta}^*) + a(\overline{\theta}, \nabla\overline{\theta}, \nabla\overline{u}, \partial_t\overline{\theta})\partial_\alpha(\overline{\theta} - \overline{\theta}^*) \\ &\quad = (k_{\alpha\beta}(\overline{\theta}, \nabla\overline{\theta}, \nabla\overline{u}, \partial_t\overline{\theta}) - k_{\alpha\beta}(\overline{\theta}^*, \nabla\overline{\theta}^*, \nabla\overline{u}^*, \partial_t\overline{\theta}^*))\partial_\alpha\partial_\beta\theta^* \\ &\qquad + (b_{j\beta}(\overline{\theta}, \nabla\overline{\theta}, \nabla\overline{u}, \partial_t\overline{\theta}) - b_{j\beta}(\overline{\theta}^*, \nabla\overline{\theta}^*, \nabla\overline{u}^*, \partial_t\overline{\theta}^*))\partial_\beta\partial_t u_j^* \\ &\qquad + (b_{\beta j\alpha}(\overline{\theta}, \nabla\overline{u}, \partial_t\overline{\theta}) - b_{\beta j\alpha}(\overline{\theta}^*, \nabla\overline{u}^*, \partial_t\overline{\theta}^*))\partial_\alpha\partial_\beta u_j^* \\ &\qquad + (d_\alpha(\overline{\theta}, \nabla\overline{\theta}, \nabla\overline{u}, \partial_t\overline{\theta}) - d_\alpha(\overline{\theta}^*, \nabla\overline{\theta}^*, \nabla\overline{u}^*, \partial_t\overline{\theta}^*))\partial_t\partial_\alpha u_j^* \\ &\qquad + (a(\overline{\theta}, \nabla\overline{\theta}, \nabla\overline{u}, \partial_t\overline{\theta}) - a(\overline{\theta}^*, \nabla\overline{\theta}^*, \nabla\overline{u}^*, \partial_t\overline{\theta}^*))\partial_t\overline{\theta}^*. \end{aligned} \tag{5.33}$$

Multiplying (5.32), (5.33) by $\partial_t(u-u^*)$ and $\partial_t(\theta-\theta^*)$ respectively, and integrating by parts over $[0,T]\times\Omega$, performing partial integration with respect to x, taking into account that

$$
\begin{aligned}
&(u_i-u_i^*)|_{\partial\Omega}=0, \quad \partial_t^k(u_i-u_i^*)(0,x)=0, \quad k=0,1,\\
&(\theta_i-\theta_i^*)|_{\partial\Omega}=0, \quad \partial_t^k(\theta_i-\theta_i^*)(0,x)=0, \quad k=0,1,
\end{aligned} \tag{5.34}
$$

and using the fact that

$$
|\overline{D}^2(\overline{u},\overline{\theta},\overline{u}^*,\overline{\theta}^*,u,\theta,u^*,\theta^*)|_{0,T}<C(N) \tag{5.35}
$$

and the mean value theorem, we get

$$
\begin{aligned}
\|\overline{D}^1(u-u^*)\|_0^2+\|\overline{D}^1(\theta-\theta^*)\|_0^2
&\le C(N)\left(1+\frac{1}{\sqrt{T}}\right)\int_0^T(\|\overline{D}^1(\overline{u}-\overline{u}^*)\|_0^2+\|\overline{D}^1(\overline{\theta}-\overline{\theta}^*)\|_0^2)\,dt\\
&\quad+T(1+T)^2C(N)(|\overline{D}^1(\overline{u}-\overline{u}^*)|_0^2+|\overline{D}^1(\overline{\theta}-\overline{\theta}^*)|_0^2).
\end{aligned} \tag{5.36}
$$

Applying Gronwall's inequality to (5.36) we get

$$
\begin{aligned}
|\overline{D}^1(u-u^*)|_{0,T}^2+|\overline{D}^1(\theta-\theta^*)|_{0,T}^2
\le C(N)T(1+T)^2(|\overline{D}^1(\overline{u}-\overline{u}^*)|_{0,T}^2+|\overline{D}^1(\overline{\theta}-\overline{\theta}^*)|_{0,T}^2)e^{(1+1/\sqrt{T})C(N)T}.
\end{aligned} \tag{5.37}
$$

Introducing the notation

$$
\lambda=C(N)T(1+T)^2e^{(1+1/\sqrt{T})C(N)T} \tag{5.38}
$$

we get

$$
|\overline{D}^1(u-u^*)|_{0,T}^2+|\overline{D}^1(\theta-\theta^*)|_{0,T}^2\le\lambda(|\overline{D}^1(\overline{u}-\overline{u}^*)|_{0,T}^2+|\overline{D}^1(\overline{\theta}-\overline{\theta}^*)|_{0,T}^2). \tag{5.39}
$$

From (5.38) it follows that choosing T small enough, we get $\lambda<1$. Therefore the mapping σ is a contraction. So, in view of the Banach fixed point theorem, σ has a unique fixed point $(u,\theta)\in Z(N,T)$.

This implies that problem (1.1)–(1.4) has a unique solution on $0\le t\le T$.

6. Applications to nonlinear microelasticity theory. Formulation of the main theorem

Below, we show how the approach presented in Sections 2–5 works in nonlinear microelasticity theory. So, we consider the nonlinear hyperbolic system of six partial differential equations of second order describing a microelastic medium in the three-dimensional space (cf. [28]):

$$
\partial_t^2u_i-c_{i\alpha j\beta}(\nabla u,\nabla\varphi)\partial_\alpha\partial_\beta u_j+\alpha_{ij}(\nabla u,\nabla\varphi)\varepsilon_{jlk}\partial_l\varphi_k=f_i, \tag{6.1}
$$

$$
\begin{aligned}
\partial_t^2\varphi_i-d_{i\alpha j\beta}(\nabla u,\nabla\varphi)\partial_\alpha\partial_\beta\varphi_j+\alpha_{ij}(\nabla u,\nabla\varphi)\varphi_j&\\
-\alpha_{ij}(\nabla u,\nabla\varphi)\varepsilon_{jlk}\partial_lu_k=Y_i,&\quad i=1,2,3,
\end{aligned} \tag{6.2}
$$

where $u=u(t,x)=(u_1(t,x),u_2(t,x),u_3(t,x))^*$ is the displacement vector of the medium, $\varphi=\varphi(t,x)=(\varphi_1(t,x),\varphi_2(t,x),\varphi_3(t,x))^*$ is the microrotation vector, depending on

$t \in \mathbb{R}_0^+$ and $x \in \Omega$, $\Omega \subset \mathbb{R}^3$ being a bounded domain with $\partial\Omega$ smooth enough; $\nabla u = (\partial_1 u, \partial_2 u, \partial_3 u)$, $\nabla\varphi = (\partial_1\varphi, \partial_2\varphi, \partial_3\varphi)$ are the spatial gradients of the functions u, φ respectively; $c_{i\alpha j\beta}(\cdot)$, $d_{i\alpha j\beta}(\cdot), \alpha_{ij}(\cdot), \overline{\alpha}_{ij}(\cdot)$ are the nonlinear coefficients depending on the gradients of the unknown functions; $f = f(t,x) = (f_1(t,x), f_2(t,x), f_3(t,x))^*$ is the body force vector, $Y = Y(t,x) = (Y_1(t,x), Y_2(t,x), Y_3(t,x))^*$ is the body couple vector; the symbol ε_{ijk} is defined as follows:

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{when the permutation } (i,j,k) \text{ is even,} \\ -1 & \text{when the permutation } (i,j,k) \text{ is odd,} \end{cases} \qquad i,j,k = 1,2,3,$$

with the following initial conditions:

$$u(0,x) = u^0(x)\ , \qquad (\partial_t u)(0,x) = u^1(x), \tag{6.3}$$

$$\varphi(0,x) = \varphi^0(x), \qquad (\partial_t\varphi)(0,x) = \varphi^1(x), \tag{6.4}$$

where $u^0, \varphi^0, u^1, \varphi^1$ are given data, and with the boundary conditions:

$$u(t,\cdot)|_{\partial\Omega} = 0, \qquad \varphi(t,\cdot)|_{\partial\Omega} = 0. \tag{6.5}$$

REMARK 6.1. Putting into (6.1)–(6.2)

$$\begin{aligned} c_{i\alpha j\beta} &= \mu\delta_{\alpha\beta}(\lambda+\mu)\delta_{ij}, \qquad \alpha_{ij}(\cdot) = 2\alpha, \qquad \overline{\alpha}_{ij}(\cdot) = 4\alpha, \\ d_{i\alpha j\beta} &= (\gamma+\varepsilon)\delta_{\alpha\beta} + (\delta+\gamma+\varepsilon)\delta_{ij}, \end{aligned}$$

where $\delta_{\alpha\beta}$ denotes the Kronecker symbol $(\alpha,\beta = 1,2,3)$, we obtain from (6.1)–(6.2) the linear hyperbolic system with constant coefficients describing the microelastic medium.

The initial-boundary value problem for the linear system of microelasticity theorem was investigated by W. Nowacki [68] using integral transformations.

Now, we formulate the main theorem about local (in time) existence of the solution of the initial-boundary value problem for the nonlinear system (6.1)–(6.2).

THEOREM 6.1 (Local-in-time existence). *Let the following conditions be satisfied*:

1° $s \geq \lfloor 3/2 \rfloor + 4 = 5$ *is an arbitrary but fixed integer.*

2° $\partial_t^k f_i, \partial_t^k Y_i \in C^0([0,T], H^{s-2-k}(\Omega))$, $k = 1,\dots,s-2$, $\partial_t^{s-1} f_i, \partial_t^{s-1} Y_i \in L^2([0,T], L^2(\Omega))$.

3° *There are two constants* γ_1, γ_2 *such that*

$$(c_{\alpha\beta}\xi_\alpha\xi_\beta\eta,\eta) \geq \gamma_1|\xi|^2|\eta|^2, \qquad (d_{\alpha\beta}\xi_\alpha\xi_\beta\eta,\eta) \geq \gamma_2|\xi|^2|\eta|^2$$

for $\xi = (\xi_1,\xi_2,\xi_3), \eta = (\eta_1,\eta_2,\eta_3) \in \mathbb{R}^3$ *where*

$$\begin{gathered} c_{\alpha\beta} = [c_{i\alpha j\beta}], \qquad d_{\alpha\beta} = [d_{i\alpha j\beta}], \\ \alpha_{ij}, \overline{\alpha}_{ij} \in C^{s-1}(\mathbb{R}^{18}) \qquad (i,j = 1,2,3), \\ c_{i\alpha j\beta} = c_{j\beta i\alpha}, \qquad d_{i\alpha j\beta} = d_{j\beta i\alpha}. \end{gathered}$$

4° *The initial data* $u^0, \varphi^0, u^1, \varphi^1$ *satisfy*

$$u^0, \varphi^0 \in H^s(\Omega) \cap H_0^1(\Omega), \qquad u^1, \varphi^1 \in H^{s-1}(\Omega) \cap H_0^1(\Omega)$$

and the compatibility conditions

$$\begin{gathered} u^k \in H^{s-k}(\Omega) \cap H_0^1(\Omega) \quad (2 \leq k \leq s-1), \qquad u^s \in L^2(\Omega), \\ \varphi^k \in H^{s-k}(\Omega) \cap H_0^1(\Omega) \quad (2 \leq k \leq s-1), \qquad \varphi^s \in L^2(\Omega), \end{gathered}$$

where $u^k = \partial^k u(0,x)/\partial t^k$, $\varphi^k = \partial^k \varphi(0,x)/\partial t^k$ *are calculated formally* (*and recursively*) *in terms of* $u^0, u^1, \varphi^0, \varphi^1$ *using system* (6.1)–(6.2).

Then for sufficiently small $T > 0$ *there exists a unique solution* (u,φ) *to the initial-boundary value problem* (6.1)–(6.4) *with the following properties*:

$$u \in \bigcap_{k=0}^{s-1} C^k([0,T], H^{s-k}(\Omega) \cap H_0^1(\Omega)), \qquad \varphi \in \bigcap_{k=0}^{s-1} C^k([0,T], H^{s-k}(\Omega) \cap H_0^1(\Omega)),$$
$$\partial_t^s u \in C^0([0,T], L^2(\Omega)), \qquad \partial_t^s \varphi \in C^0([0,T], L^2(\Omega)).$$

The proof of Theorem 6.1 is divided into three steps:

1° Proof for the linear system obtained by linearization of (6.1)–(6.4) in the case of two linear hyperbolic systems.

2° Proof of an energy estimate for the linear system.

3° Proof of existence and uniqueness of solution of the initial-boundary value problem for the nonlinear system (6.1)–(6.4) by applying a fixed point theorem.

7. Energy estimate for the linearized microelasticity system

7.1. Linearized system of microelasticity theory. In this subsection, we investigate two initial-boundary value problems for two linear hyperbolic systems. These systems arise from the linearized system (6.1)–(6.4).

So, we shall investigate the solvability of the following problems.

1° The initial-boundary value problem for the linear hyperbolic system

$$\partial_t^2 u_i - \overline{c}_{i\alpha j\beta}(x,t)\frac{\partial^2 u_j}{\partial x_\alpha \partial x_\beta} = \overline{h}_i(x,t) \qquad ((t,x) \in [0,T] \times \overline{\Omega},\ i = 1,2,3) \tag{7.1}$$

with initial conditions

$$u_i(0,x) = u_i^0(x), \qquad (\partial_t u_i)(0,x) = u_i^1(x) \tag{7.2}$$

and boundary conditions

$$u_i(t,\cdot)|_{\partial\Omega} = 0 \qquad (t \in [0,T]). \tag{7.3}$$

2° The initial-boundary value problem for the linear system

$$\partial_t^2 \varphi_i - \overline{d}_{i\alpha j\beta}(x,t)\frac{\partial^2 \varphi_j}{\partial x_\alpha \partial x_\beta} = \overline{k}_i(x,t) \qquad ((t,x) \in [0,T] \times \overline{\Omega},\ i = 1,2,3) \tag{7.4}$$

with initial conditions

$$\varphi_i(0,x) = \varphi_i^0(x), \qquad (\partial_t \varphi_i)(0,x) = \varphi_i^1(x) \tag{7.5}$$

and boundary conditions

$$\varphi_i(t,\cdot)|_{\partial\Omega} = 0 \qquad (t \in [0,T]). \tag{7.6}$$

7.2. Energy estimate for the linear system of microelasticity theory. We start with the existence of solution to the initial-boundary value problem (7.1)–(7.4).

Theorem 7.1 (Existence and regularity for (7.1)–(7.3)). *Let the following assumptions be satisfied*:

1° $s \geq \lfloor 3/2 \rfloor + 4 = 5$ *is an arbitrary but fixed integer.*

2°
$$\overline{c}_{i\alpha j\beta} \in C^0([0,T] \times \overline{\Omega}) \cap L^\infty([0,T], L^\infty(\Omega)),$$
$$D_x \overline{c}_{i\alpha j\beta} \in L^\infty([0,T], H^{s-2}(\Omega)),$$
$$\partial_t^k \overline{c}_{i\alpha j\beta} \in L^\infty([0,T], H^{s-1-k}(\Omega)) \quad (k = 1, \ldots, s-1).$$

3° $\overline{c}_{i\alpha j\beta} = \overline{c}_{j\beta i\alpha}$ *for* $t \in [0,T] \times \Omega$ *and if* $u \in H_0^1(\Omega)$, *then*
$$\|u\|_1^2 \leq \gamma_0 \left\{ \left(\overline{c}_{i\alpha j\beta}(t) \frac{\partial u_j}{\partial x_\beta}, \frac{\partial u_i}{\partial x_\alpha} \right) + \|u\|_0^2 \right\}$$
for $t \in [0,T]$, *where* $\gamma_0 > 0$ *is some constant.*

4°
$$\overline{c}_{i\alpha j\beta} \frac{\partial^2 u_j}{\partial x_\alpha \partial x_\beta} \in H^k(\Omega)$$
for $t \in [0,T]$, *and if* $u \in H_0^1(\Omega)$, *then* $u \in H^{k+2}(\Omega)$ *and*
$$\|u\|_{k+2} \leq \gamma_1 \left(\left\| \overline{c}_{i\alpha j\beta}(t) \frac{\partial^2 u_j}{\partial x_\alpha \partial x_\beta} \right\|_k^2 + \|u\|_0^2 \right)$$
$(0 \leq k \leq s-2)$ *for* $t \in [0,T]$, *where* $\gamma_1 > 0$ *is some constant.*

5° $\partial_t^k \overline{h} \in C^0([0,T], H^{s-2-k}(\Omega)) \quad (0 \leq k \leq s-2), \qquad \partial_t^{s-1} \overline{h} \in L^2([0,T], L^2(\Omega)).$

Then there exists a unique solution $u = (u_1, u_2, u_3)^*$ *of problem* (7.1)–(7.4) *with the properties*

$$\begin{aligned} &\partial_t^s u \in C^0([0,T], L^2(\Omega)), \\ &\partial_t^k u \in C^0([0,T], H^{s-k}(\Omega) \cap H_0^1(\Omega)) \quad (0 \leq k \leq s-1). \end{aligned} \tag{7.11}$$

Sketch of proof. The assertion follows from semigroup theory (cf. Section 2) and the proof of Theorem 4.1.

We can convert problem (4.1)–(4.4) into an equivalent (evolution) problem of the form

$$\partial_t V + AV = F, \tag{7.12}$$
$$V(0,x) = V(x), \tag{7.13}$$

where

$$V = (u_1, u_2, u_3, \partial_t u_1, \partial_t u_2, \partial_t u_3)^*, \tag{7.14}$$
$$V(0) = V^0 = (u_1^0, u_2^0, u_3^0, u_1^1, u_2^1, u_3^1)^*, \tag{7.15}$$
$$F = (0, \overline{h}), \tag{7.16}$$
$$A(t) = \begin{pmatrix} 0 & -I \\ -\overline{c}_{i\alpha j\beta} \dfrac{\partial^2}{\partial x_\alpha x \partial_\beta} & 0 \end{pmatrix}, \tag{7.17}$$

the operator

$$A : D(A) \to X_0 \tag{7.18}$$

being defined by (7.17) with

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega), \qquad X_0 = H_0^2(\Omega) \times L^2(\Omega). \tag{7.19}$$

Using the same considerations as in the proof of Theorem 4.1, we show that $(A, X_0, D(A))$ is a CD-system and that $A(t)$ satisfies also the other conditions which allow one to prove the required regularity (cf. (4.30)–(4.33)). Now, we formulate an energy estimate for problem (7.1)–(7.3).

THEOREM 7.2 (Energy estimate for (7.1)–(7.3)). *If the assumptions of Theorem* 7.1 *are satisfied*, *then the solution of problem* (7.1)–(7.3) *guaranteed by Theorem* 7.1 *satisfies the inequality*

$$|\overline{D}^s u|_{0,T}^2 \le K_0 K_1 e^{K_2 \zeta(T)} \tag{7.20}$$

with positive constants K_0, K_1, K_2, *where*

$$K_0 = \sum_{k=0}^{s} \|u^k\|_{s-k}^2 + (1+T)|\overline{D}^{s-2}\overline{h}|_{0,T}^2 + T^{1/2} \int_0^T |\partial_t^{s-1}\overline{h}(t)|^2 \, dt,$$

and $K_1 = K_1(L_0, \gamma_0, \gamma_1)$ *and* $K_2 = K_2(L, \gamma_0, \gamma_1)$ *depend continuously on their arguments*, *where*

$$\begin{aligned} L_0 &= \|\overline{c}_{i\alpha j\beta}(0)\|_{L^\infty} + \|D_x \overline{c}_{i\alpha j\beta}(0)\|_{s-3}, \\ L &= \sup_{0 \le t \le T} \|\overline{c}_{i\alpha j\beta}(t)\|_\infty + |D_x \overline{c}_{i\alpha j\beta}(0)|_{s-2,T} + \sum_{k=0}^{s-1} |\partial_t^k \overline{c}_{i\alpha j\beta}|_{s-k-1,T}, \\ \zeta(T) &= T^{1/2}(1 + T^{1/2} + T + T^{3/2}). \end{aligned} \tag{7.21}$$

Sketch of proof. Differentiating (4.1) $n-1$ times $(1 \le n \le s-1)$ formally with respect to t, multiplying by $\partial_t^n u_i$ and then integrating over $(0,t) \times \Omega$, using integration by parts with respect to x, the Schwarz inequality, Friedrich's mollifier (in order to estimate $\partial_t^s u(t,x)$); cf. the proof of Theorem 4.2), and the assumption of the theorem, we get

$$\|\overline{D}^s u(t)\|_2^2 = C(L, \gamma_0, \gamma_1) K_0 + C(L, \gamma_0, \gamma_1)(1 + T^{1/2} + T + T^{3/2}) \int_0^t \|\overline{D}^s u(\tau)\|_2^2 \, d\tau. \tag{7.22}$$

Applying Gronwall's inequality to (7.22), we immediately get the energy estimate (7.20).

As the second step, we start with the existence theorem for the initial-boundary value problem (7.4)–(7.6).

THEOREM 7.3 (Existence, uniqueness and regularity for (7.4)–(7.6)). *Let the following assumptions be satisfied*:

1° $s \ge \lfloor 3/2 \rfloor + 4 = 5$ *is an arbitrary but fixed integer.*

2°
$$\begin{aligned} \overline{d}_{i\alpha j\beta} &\in C^0([0,T] \times \overline{\Omega}) \cap L^\infty([0,T], L^\infty(\Omega)), \\ D_x \overline{d}_{i\alpha j\beta} &\in L^\infty([0,T], H^{s-2}(\Omega)) \qquad (k = 1, \ldots, s-1), \\ \partial_t^k \overline{d}_{i\alpha j\beta} &\in L^\infty([0,T], H^{s-1-k}(\Omega)). \end{aligned}$$

3° $\overline{d}_{i\alpha j\beta} = \overline{d}_{j\beta i\alpha}$ *for* $(t,x) \in [0,T] \times \overline{\Omega}$ *and if* $\varphi \in H_0^1(\Omega)$, *then*

$$\|\varphi\|_1^2 \le \gamma_0' \left\{ \left(\overline{d}_{i\alpha j\beta}(t) \frac{\partial \varphi_j}{\partial x_\beta}, \frac{\partial \varphi_i}{\partial x_\alpha} \right) + \|\varphi\|_0^2 \right\}$$

for $t \in [0,T]$, *where* $\gamma_0' > 0$ *is some constant.*

4°
$$\overline{d}_{i\alpha j\beta} \frac{\partial^2 \varphi_j}{\partial x_\alpha \partial x_\beta} \in H^k(\Omega)$$

for $t \in [0,T]$, *and if* $\varphi \in H_0^1(\Omega)$, *then* $\varphi \in H^{k+2}(\Omega)$ *and*

$$\|\varphi\|_{k+2} \le \gamma_1' \left(\left\| \overline{d}_{i\alpha j\beta}(t) \frac{\partial^2 \varphi}{\partial x_\alpha \partial x_\beta} \right\|_k^2 + \|\varphi\|_0^2 \right)$$

$(0 \le k \le s-2)$ *for* $t \in [0,T]$, *where* $\gamma_1' > 0$ *is some constant.*

5°
$$\begin{aligned} &\partial_t^{s-1} \overline{k} \in L^2([0,T], L^2(\Omega)) \quad (s > 5), \\ &\partial_t^k \overline{k} \in C^0([0,T], H^{s-2-k}(\Omega)) \quad (0 \le k \le s-2). \end{aligned}$$

Then there exists a unique solution $\varphi = (\varphi_1, \varphi_2, \varphi_3)^*$ *of problem* (7.4)–(7.6) *with the properties*

$$\begin{aligned} &\partial_t^s \varphi \in C^0([0,T], L^2(\Omega)), \\ &\partial_t^k \varphi \in C^0([0,T], H^2(\Omega) \cap H_0^1(\Omega)) \quad (0 \le k \le s-1). \end{aligned} \tag{7.23}$$

Sketch of proof. Introducing the vector $V = (\varphi_1, \varphi_2, \varphi_3, \partial_t \varphi_1, \partial_t \varphi_2, \partial_t \varphi_3)^*$ we can convert problem (7.4)–(7.6) into an equivalent (evolution) problem of the form

$$\partial_t V + AV = G, \tag{7.24}$$

$$V(0,x) = V^0(x), \tag{7.25}$$

where

$$V(0) = V^0 = (\varphi_1^0, \varphi_2^0, \varphi_3^0, \varphi_1^1, \varphi_2^1, \varphi_3^1)^*, \qquad G = (0, \overline{k})^*,$$

$$A(t) = \begin{pmatrix} 0 & -I \\ -\overline{d}_{i\alpha j\beta}\, \partial_\alpha \partial_\beta & 0 \end{pmatrix}.$$

Using similar considerations to those in the proofs of Theorems 7.1 and 4.1 we obtain the assertion of Theorem 7.3. Using the same approach as in the proof of Theorem 7.2, we can also obtain the following energy estimate for the solution of problem (7.4)–(7.6).

THEOREM 7.4 (Energy estimate for (7.4)–(7.6)). *If the assumptions of Theorem* 7.3 *are satisfied*, *then the solution of problem* (7.4)–(7.6) *guaranteed by Theorem* 7.3 *satisfies the inequality*

$$|\overline{D}^s \varphi|_{0,T}^2 \le M_0 M_1 e^{M_2 \eta(T)} \tag{7.26}$$

with positive constants M_0, M_1, M_2, *where*

$$M_0 = \sum_{k=0}^{s} \|\varphi^k\|_{s-k}^2 + (1+T)|\overline{D}^{s-2} \overline{h}|_{0,T}^2 + T^{1/2} \int_0^T |\partial_t^{s-1} \overline{h}(t)|^2\, dt$$

and $M_1 = M_1(P_0, \gamma_0', \gamma_1')$ *and* $M_2 = M_2(P, \gamma_0', \gamma_1')$ *depend continuously on their arguments*; *here*

$$P_0 = \|\overline{d}_{i\alpha j\beta}(0)\|_{L^\infty} + \|D_x \overline{d}_{i\alpha j\beta}(0)\|_{s-3},$$

$$P = \sup_{0\le t\le T} \|\overline{d}_{i\alpha j\beta}(t)\|_{L^\infty} + |D_x \overline{d}_{i\alpha j\beta}(0)|_{s-2,T} + \sum_{k=1}^{s-1} |\partial_t^k \overline{d}_{i\alpha j\beta}|_{s-k-1,T}$$

and

$$\eta(T) = T^{1/2}(1 + T^{1/2} + T + T^{3/2}). \tag{7.27}$$

Proof. It runs in the same way as that of Theorem 7.1.

8. Proof of Theorem 6.1

Let $\boldsymbol{W}(N,T)$ be the set of functions (u,φ) satisfying

$$\partial_t^k \varphi_i, \partial_t^k u_j \in L^\infty([0,T], H^{s-k}(\Omega)), \quad 0 \le k \le s,\ j = 1,2,3, \tag{8.1}$$

$s \ge \lfloor 3/2 \rfloor + 4 = 5$ being an arbitrary but fixed integer, with boundary and initial conditions of the form

$$u_j|_{\partial\Omega} = 0, \quad \varphi_j|_{\partial\Omega} = 0, \tag{8.2}$$

$$\partial_t^k u_j(0,x) = u_j^k(x), \quad 0 \le k \le s-1, \tag{8.3}$$

$$\partial_t^k \varphi_j(0,x) = \varphi_j^k(x), \quad 0 \le k \le s-1, \tag{8.4}$$

and the inequality

$$|\overline{D}^s u|_{0,T}^2 + |\overline{D}^s \varphi|_{0,T}^2 \le N^2 \tag{8.5}$$

for N large enough.

A mapping σ_1: $\boldsymbol{W}(N,T) \to \boldsymbol{W}(N,T)$ is defined as follows:

$$\sigma_1 : \boldsymbol{W}(N,T) \ni (\overline{u}, \overline{\varphi}) \mapsto \sigma_1(\overline{u}, \overline{\varphi}) = (u, \varphi) \tag{8.6}$$

where u is the solution of (7.1)–(7.3) according to Theorem 7.1 with

$$\overline{c}_{i\alpha j\beta} = c_{i\alpha j\beta}(\nabla\overline{u}, \nabla\overline{\varphi}), \tag{8.7}$$

$$h_i = \alpha_{ij}(\nabla\overline{u}, \nabla\overline{\varphi})\varepsilon_{jlk}\frac{\partial\overline{\varphi}_k}{\partial x_l} + f_i, \tag{8.8}$$

$$\overline{u}^0 = u^0, \quad \overline{u}^1 = u^1, \tag{8.9}$$

and φ is the solution of (7.4)–(7.6) according to Theorem 7.3 with

$$\overline{d}_{i\alpha j\beta} = d_{i\alpha j\beta}(\nabla\overline{u}, \nabla\overline{\varphi}), \tag{8.10}$$

$$\overline{k}_i = -\alpha_{ij}(\nabla\overline{u}, \nabla\overline{\varphi}) + \alpha_{ij}(\nabla\overline{u}, \nabla\overline{\varphi})\varepsilon_{jlk}\frac{\partial\overline{u}_k}{\partial x_l} + \varphi_i, \tag{8.11}$$

$$\overline{\varphi}^0 = \varphi^0, \quad \overline{\varphi}^1 = \varphi^1. \tag{8.12}$$

Then σ_1 maps $\boldsymbol{W}(N,T)$ into itself provided N is sufficiently large and T is sufficiently small. To prove this, we use the energy estimates (7.22)–(7.26) and the same arguments as in the proof of Theorem 3.1.

For this, we introduce the notation

$$\varepsilon_0 = \sum_{k=0}^{s} \|u^k\|_{s-k}^2 + \sum_{k=0}^{s} \|\varphi^k\|_{s-k}^2 + \sum_{k=0}^{s-2} |\partial_t^k(\overline{h},\overline{k})|_{s-2-k,T}^2 + \int_0^T \|\partial_t^{s-1}(\overline{h},\overline{k})(\tau)\|_0^2\, d\tau. \tag{8.13}$$

Taking into account the properties of elements of $\boldsymbol{W}(N,T)$, applying the Sobolev inequality and the mean value theorem, we get, for the function $\overline{h}$ defined by (8.8),

$$\begin{aligned}\|\partial_t^{s-1}\overline{h}\|_0^2 &\le 2\Big(C\sum_{i=1}^{s-2}\|\overline{D}^{s-1}(\nabla\overline{u},\nabla\overline{\varphi})\|_0^i\Big)^2 + 2C\|\partial_t^{s-1}\overline{h}\|_0^2 \\ &\le C(N) + C\|\partial_t^{s-1}\overline{h}\|_0^2\end{aligned} \tag{8.14}$$

and

$$\int_0^T \|\partial_t^{s-1}\overline{h}\|\, dt \le C(N)(1+T) + CE_0. \tag{8.15}$$

Acting similarly and using the fact that

$$\gamma(t) = \gamma(0) + \int_0^t \partial_t\gamma(\tau)\, d\tau$$

we get

$$\sum_{k=0}^{s-2}\bigg\{|\partial_t^k\overline{h}|_{s-2-k,T}^2 + \bigg|\partial_t^k\alpha_{ij}(\nabla\overline{u},\nabla\overline{\varphi})\varepsilon_{jkh}\frac{\partial\overline{u}_k}{\partial x_i}\bigg|_{s-2-k,T}^2\bigg\} \le C(E_0) + C(N)T(1+T). \tag{8.16}$$

Using the same estimate, we get

$$\int_0^T \|\partial_t^{s-1}\overline{k}\|\, dt \le C(N)(1+T) + C(E_0) \tag{8.17}$$

and

$$\begin{aligned}\sum_{k=0}^{s-2}\bigg\{|\partial_t^k\overline{k}|_{s-2-k,T}^2 + |\partial_t^k\alpha_{ij}(\nabla\overline{u},\nabla\overline{\varphi})\overline{\varphi}_j|_{s-2-k,T}^2 \\ + \bigg|\partial_t^k\alpha_{ij}(\nabla\overline{u},\nabla\overline{\varphi})\varepsilon_{jhk}\frac{\partial\overline{u}_h}{\partial x_i}\bigg|_{s-2-k,T}^2\bigg\} \\ \le C(E_0) + C(N)T(1+T).\end{aligned} \tag{8.18}$$

Putting (8.13) and (8.14) into the energy estimate (7.20), putting (7.17) into the energy estimate (7.26) and adding the resulting inequalities, we get

$$\begin{aligned}|\overline{D}^s u|_{0,T}^2 + |\overline{D}^s\varphi|_{0,T}^2 &\le C'(E_0,\gamma_0,\gamma_1,\gamma_0',\gamma_1') \\ &\quad\cdot\Big(1 + C(N)T^{1/2}\sum_{i=0}^{6} T^{i/2}\Big) \\ &\quad\cdot e^{C(N)T^{1/2}\sum_{i=0}^{5} T^{i/2}}.\end{aligned} \tag{8.19}$$

Let N be so large that

$$2C'(E_0,\gamma_0,\gamma_1,\gamma_0',\gamma_1') < N^2. \tag{8.20}$$

There exists $T>0$ small enough that

$$\eta(T) = \Big(1 + C(N)T^{1/2}\sum_{i=0}^{6} T^{i/2}\Big)e^{C(N)T^{1/2}\sum_{i=0}^{5}T^{i/2}} < 2 \tag{8.21}$$

(since $\eta(0)=1$ and $\eta(T)$ is a continuous function). So, we get

$$|\overline{D}^s u|_{0,T}^2 + |\overline{D}^s\varphi|_{0,T}^2 \le N^2. \tag{8.22}$$

This means that $(u,\varphi)\in \boldsymbol{W}(N,T)$. Finally, we notice that $\boldsymbol{W}(N,T)$ is a closed subspace of the complete metric space defined by

$$Y = \{(\overline{u},\overline{\varphi}) : \overline{D}^1\overline{u}, \overline{D}^1\overline{\varphi} \in L^\infty([0,T],L^2(\Omega))\} \tag{8.23}$$

with metric δ given by

$$\delta((\overline{u},\overline{\varphi}),(u,\varphi)) = |\overline{D}^1(\overline{u}-u)|_{0,T}^2 + |\overline{D}^1(\overline{\varphi}-\varphi)|_{0,T}^2. \tag{8.24}$$

Below, we prove that the mapping σ_1 is a contraction for T small enough, with respect to the metric δ given by (8.24). Using (8.7)–(8.12) we can see that $u-u^*, \varphi-\varphi^*$ satisfy the system

$$\begin{aligned}
&\partial_t(u_i-u_i^*) - c_{i\alpha j\beta}(\nabla\overline{u},\nabla\overline{\varphi})\frac{\partial^2(u_j-u_j^*)}{\partial x_\alpha\partial x_\beta} \\
&\quad = (c_{i\alpha j\beta}(\nabla\overline{u}^*,\nabla\overline{\varphi}^*) - c_{i\alpha j\beta}(\nabla\overline{u},\nabla\overline{\varphi}))\frac{\partial^2 u_j^*}{\partial x_\alpha\partial x_\beta} \\
&\qquad + \Big(\alpha_{ij}(\nabla\overline{u},\nabla\overline{\varphi}) - \alpha_{ij}(\nabla\overline{u}^*,\nabla\overline{\varphi}^*)\varepsilon_{ihk}\frac{\partial\overline{\varphi}_k}{\partial x_i} + \alpha_{ij}(\nabla\overline{u}^*,\nabla\overline{\varphi}^*)\varepsilon_{ihk}\Big)\Big(\frac{\partial\overline{\varphi}_k}{\partial x_i} - \frac{\partial\overline{\varphi}_k^*}{\partial x_i}\Big),
\end{aligned} \tag{8.25}$$

$$\begin{aligned}
&\partial_t^2(\varphi_i-\varphi_i^*) - d_{i\alpha j\beta}(\nabla\overline{u},\nabla\overline{\varphi})\frac{\partial^2(\varphi_j-\varphi_j^*)}{\partial x_\alpha\partial x_\beta} \\
&\quad = (d_{i\alpha j\beta}(\nabla\overline{u}^*,\nabla\overline{\varphi}^*) - d_{i\alpha j\beta}(\nabla\overline{u},\nabla\overline{\varphi}))\frac{\partial^2\varphi_j^*}{\partial x_\alpha\partial x_\beta} \\
&\qquad + (\overline{\alpha}_{ij}(\nabla\overline{u},\nabla\overline{\varphi}) - \overline{\alpha}_{ij}(\nabla\overline{u}^*,\nabla\overline{\varphi}^*))\overline{\varphi}_j + \overline{\alpha}_{ij}(\nabla\overline{u}^*,\nabla\overline{\varphi}^*)(\varphi_i-\varphi_i^*) \\
&\qquad + (\alpha_{ij}(\nabla\overline{u},\nabla\overline{\varphi}) - \alpha_{ij}(\nabla\overline{u}^*,\nabla\overline{\varphi}^*))\varepsilon_{jlk}\frac{\partial\overline{\varphi}_k}{\partial x_i} \\
&\qquad + \alpha_{ij}(\nabla\overline{u}^*,\nabla\overline{\varphi}^*)\varepsilon_{jlk}\Big(\frac{\partial\overline{\varphi}_k}{\partial x_i} - \frac{\partial\overline{\varphi}_k^*}{\partial x_i}\Big).
\end{aligned} \tag{8.26}$$

Multiplying (8.15), (8.16) by $\partial_t(u-u^*)$, $\partial_t(\varphi-\varphi^*)$ respectively, and integrating over $[0,T]\times\Omega$, performing partial integration with respect to x, taking into account that

$$\begin{aligned}
(u_i-u_i^*)|_{\partial\Omega} &= 0, \quad \partial_t^k(u_i-u_i^*)(0,x) = 0, \quad k=0,1, \\
(\varphi_i-\varphi_i^*)|_{\partial\Omega} &= 0, \quad \partial_t^k(\varphi_i-\varphi_i^*)(0,x) = 0, \quad k=0,1,
\end{aligned} \tag{8.27}$$

and using a similar approach to that in the proof of Theorem 3.1, we get

$$\|\overline{D}^1(u-u^*)\|_0^2+\|\overline{D}^1(\varphi-\varphi^*)\|_0^2 \le C(N)\Big(1+\frac{1}{\sqrt{T}}\Big)\int_0^T[\|\overline{D}^1(u-u^*)\|_0^2+\|\overline{D}^1(\varphi-\varphi^*)\|_0^2]\,dt + T^{1/2}(1+T)^2C(N)[\|\overline{D}^1(\overline{u}-\overline{u}^*)\|_{0,T}^2+\|\overline{D}^1(\overline{\varphi}-\overline{\varphi}^*)\|_{0,T}^2]. \tag{8.28}$$

Applying Gronwall's inequality to (8.28) we get

$$|\overline{D}^1(u-u^*)|_{0,T}^2+|\overline{D}^1(\varphi-\varphi^*)|_{0,T}^2 \le \varepsilon(|\overline{D}^1(\overline{u}-\overline{u}^*)|_{0,T}^2+|\overline{D}^1(\overline{\varphi}-\overline{\varphi}^*)|_{0,T}^2) \tag{8.29}$$

where

$$\varepsilon = C(N)T^{1/2}(1+T)^2e^{C(N)(T+T^{1/2})}. \tag{8.30}$$

From (8.30) it follows that choosing T small enough, we get $\varepsilon<1$. Therefore the mapping σ_1 is a contraction. So, in view of the Banach fixed point theorem, σ has a unique fixed point $(u,\varphi)\in \boldsymbol{W}(N,T)$. This implies that problem (6.1)–(6.2) with conditions (6.3)–(6.4) has a unique solution on $0\le t\le T$. This completes the proof of Theorem 6.1.

9. Application to nonlinear thermodifusion in a solid body

In this section we extend our approach to the nonlinear hyperbolic-parabolic system of equations describing the behaviour of a thermodiffusion medium in continuum mechanics.

More precisely we consider the initial-boundary value problem for the nonlinear hyperbolic-parabolic system of equations describing the process of thermodiffusion in a solid body (cf. [29], [51]):

$$\begin{aligned}
&\partial_t^2u_i - C_{i\alpha j\beta}(\nabla u,\theta_1,\theta_2)\frac{\partial^2u_j}{\partial x_\alpha\partial x_\beta} + \overline{c}^1_{i\alpha}(\nabla u,\theta_1,\theta_2)\frac{\partial\theta_1}{\partial x_\alpha}+\overline{c}^2_{i\alpha}(\nabla u,\theta_1,\theta_2)\frac{\partial\theta_2}{\partial x_\alpha}=f_i,\\
&c(\nabla u,\theta_1,\theta_2)\partial_t\theta_1 - a^1_{\alpha\beta}(\nabla u,\theta_1,\theta_2,\nabla\theta_1,\nabla\theta_2)\frac{\partial^2\theta_1}{\partial x_\alpha\partial x_\beta} + \overline{c}^1_{i\alpha}(\nabla u,\theta_1,\theta_2)\frac{\partial^2u_i}{\partial x_\alpha\partial t}+d(\nabla u,\theta_1,\theta_2,\nabla\theta_1,\nabla\theta_2)\frac{\partial\theta_2}{\partial t}=Q_1,\\
&n(\nabla u,\theta_1,\theta_2)\partial_t\theta_2 - a^2_{\alpha\beta}(\nabla u,\theta_1,\theta_2,\nabla\theta_1,\nabla\theta_2)\frac{\partial^2\theta_2}{\partial x_\alpha\partial x_\beta} + \overline{c}^2_{i\alpha}(\nabla u,\theta_1,\theta_2)\frac{\partial^2u_i}{\partial x_\alpha\partial t}+d(\nabla u,\theta_1,\theta_2,\nabla\theta_1,\nabla\theta_2)\frac{\partial\theta_1}{\partial t}=Q_2,
\end{aligned} \tag{9.1}$$

where $u=u(t,x)=(u_1(t,x),u_2(t,x),u_3(t,x))^*$ is the displacement vector of the medium, $\theta_1=\theta_1(t,x)$ denotes the temperature of the medium, $\theta_2=\theta_2(t,x)$ denotes the chemical potential depending on $t\in\mathbb{R}_0^+$ and $x\in\Omega$, $\Omega\subset\mathbb{R}^3$ being a bounded domain with $\partial\Omega$ smooth; $\nabla u=(\partial_1u,\partial_2u,\partial_3u)$, $\nabla\theta_1=(\partial_1\theta_1,\partial_2\theta_1,\partial_3\theta_1)$, $\nabla\theta_2=(\partial_1\theta_2,\partial_2\theta_2,\partial_3\theta_2)$; $C_{i\alpha j\beta}(\cdot),\overline{c}^1_{i\alpha}(\cdot)$, $\overline{c}^2_{i\alpha}(\cdot),c(\cdot),a^1_{\alpha\beta}(\cdot),d^1(\cdot),n(\cdot),a^2_{\alpha\beta}(\cdot)$ are nonlinear coefficients depending on the unknown functions and their gradients, smooth enough; $f=f(t,x)=(f_1(t,x),$ $f_2(t,x),f_3(t,x))^*$ denotes the body force vector; $Q_1=Q_2(t,x)$, $Q_2=Q_2(t,x)$ are the intensity of the heat source and the intensity of the source of diffusing mass; * denotes transposition; the initial conditions are

$$(9.2)\qquad \begin{aligned} u(0,x) &= u^0(x), & (\partial_t u)(0,x) &= u^1(x),\\ \theta_1(0,x) &= \theta_1^0(x), & \theta_2(0,x) &= \theta_2^0(x), \end{aligned}$$

with given data u^0, u^1, θ_1^0, θ_2^0, and boundary conditions (of Dirichlet type) are

$$(9.3)\qquad u(t,\cdot)|_{\partial\Omega} = 0, \quad \theta_1(t,\cdot)|_{\partial\Omega} = 0, \quad \theta_2(t,\cdot)|_{\partial\Omega} = 0,$$

Putting in the system (9.1)

$$(9.4)\qquad \begin{aligned} &c_{i\alpha j\beta}(\cdot) = \mu\delta_{\alpha\beta} + (\lambda+\mu)\delta_{ij}, \quad \overline{c}_{i\alpha}^1 = \gamma_1\delta_{i\alpha}, \quad \overline{c}_{i\alpha}^2 = \gamma_2\delta_{i\alpha},\\ &c(\cdot) = c, \quad \overline{a}_{\alpha\beta}^1 = \delta_{\alpha\beta}, \quad a_{\alpha\beta}^2 = D\delta_{\alpha\beta}, \quad d^1(\cdot) = d, \quad n(\cdot) = n, \end{aligned}$$

we obtain the linear hyperbolic-parabolic system describing thermodiffusion in a solid body. The linear hyperbolic-parabolic system of thermodiffusion in a solid body was investigated in [38, 39, 40, 69] using the method of integral transformations. In [37] some theorems about existence and uniqueness of solution for initial-boundary value problems were proved using the Faedo–Galerkin method in suitable Sobolev spaces. The aim of this section is to prove a local existence theorem for the nonlinear problem (9.1)–(9.7) in the class of smooth functions under the assumptions given below. Before starting the main theorem we rewrite system (9.1) in the form (under the assumption $cn - d^2 > 0$)

$$(9.5)\qquad \begin{aligned} &\partial_t^2 u_i - C_{i\alpha j\beta}(\nabla u, \theta_1, \theta_2)\frac{\partial^2 u_j}{\partial x_\alpha \partial x_\beta}\\ &\qquad + \overline{c}_{i\alpha}^1(\nabla u, \theta_1, \theta_2)\frac{\partial \theta_1}{\partial x_\alpha} + \overline{c}_{i\alpha}^2(\nabla u, \theta_1, \theta_2)\frac{\partial \theta_2}{\partial x_\alpha} = f_i(t,x),\\ &\partial_t\theta_1 - \widetilde{a}_{\alpha\beta}^{11}(\nabla u, \theta_1, \theta_2, \nabla\theta_1, \nabla\theta_2)\frac{\partial^2\theta_1}{\partial x_\alpha\partial x_\beta} - \widetilde{a}_{\alpha\beta}^{12}(\nabla u, \theta_1, \theta_2, \nabla\theta_1, \nabla\theta_2)\frac{\partial^2\theta_2}{\partial x_\alpha\partial x_\beta}\\ &\qquad = \mathbb{C}_{i\alpha}^1(\nabla u, \theta_1, \theta_2\nabla\theta_1, \nabla\theta_2)\frac{\partial^2 u_i}{\partial x_\alpha\partial t} + g_1(\nabla u, \theta_1, \theta_2, \nabla\theta_1, \nabla\theta_2, t, x),\\ &\partial_t\theta_2 - \overline{a}_{\alpha\beta}^{21}(\nabla u, \theta_1, \theta_2, \nabla\theta_1, \nabla\theta_2)\frac{\partial^2\theta_1}{\partial x_\alpha\partial x_\beta} - \overline{a}_{\alpha\beta}^{22}(\nabla u, \theta_1, \theta_2, \nabla\theta_1, \nabla\theta_2)\frac{\partial^2\theta_2}{\partial x_\alpha\partial x_\beta}\\ &\qquad = \mathbb{C}_{i\alpha}^2(\nabla u, \theta_1, \theta_2, \nabla\theta_1, \nabla\theta_2)\frac{\partial^2 u_i}{\partial x_\alpha\partial t} + g_2(\nabla u, \theta_1, \theta_2, \nabla\theta_1, \nabla\theta_2, t, x), \end{aligned}$$

where

$$(9.6)\qquad \begin{aligned} &\overline{a}_{\alpha\beta}^{11} = \frac{n}{\delta}a_{\alpha\beta}^1, \quad \overline{a}_{\alpha\beta}^{12} = -\frac{d}{\delta}a_{\alpha\beta}^1,\\ &\overline{a}_{\alpha\beta}^{21} = -\frac{d}{\delta}a_{\alpha\beta}^2, \quad \overline{a}_{\alpha\beta}^{22} = \frac{c}{cn-d^2}a_{\alpha\beta}^2, \quad \delta = cn - d^2,\\ &\mathbb{C}_{i\alpha}^1(\nabla u, \theta_1, \theta_2, \nabla\theta_1, \nabla\theta_2) = \frac{dC_{i\alpha}^2 - nc_{i\alpha}^1}{\delta},\\ &\mathbb{C}_{i\alpha}^2(\nabla u, \theta_1, \theta_2, \nabla\theta_1, \nabla\theta_2) = \frac{dC_{i\alpha}^1 - cC_{i\alpha}^2}{\delta},\\ &g_1(\nabla u, \theta_1, \theta_2, \nabla\theta_1, \nabla\theta_2, t, x) = \frac{Q_1 n - dQ_2}{\delta},\\ &g_2(\nabla u, \theta_1, \theta_2, \nabla\theta_1, \nabla\theta_2, t, x) = \frac{dQ_2 - dQ_1}{\delta}. \end{aligned}$$

Now, we formulate the main theorem:

THEOREM 9.1 (Local-in-time existence). *Let the following conditions be satisfied*:

1° $s \geq \lfloor 3/2 \rfloor + 4 = 5$ *is an arbitrary but fixed integer.*

2° $\partial_t^k f_i, \partial_t^k Q_1, \partial_t^k Q_2 \in C^0([0,T], H^{s-2-k}(\Omega))$, $k = 0, 1, \ldots, s-2$, $\partial_t^{s-1} Q_1$, $\partial_k^{s-1} Q_2 \in L^2([0,T], L^2(\Omega))$.

3° *There are two constants* γ_1, γ_2 *such that*

$$(c_{\alpha\beta}\xi_\alpha\xi_\beta\eta, \eta) > \gamma_1|\xi|^2|\eta|^2, \qquad (\overline{a}_{\alpha\beta}\xi_\alpha\xi_\beta\overline{\eta}, \overline{\eta}) \geq \gamma_2|\xi|^2|\overline{\eta}|^2$$

for $\overline{\eta} = (\overline{\eta}_1, \overline{\eta}_2) \in \mathbb{R}^2$, $\xi = (\xi_1, \xi_2, \xi_3)$, $\eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3$, *where*

$$\begin{aligned}
&c_{\alpha\beta} = [c_{i\alpha j\beta}], \qquad i = 1,2,3,\ j = 1,2,3,\\
&\overline{a}_{\alpha\beta} = [\overline{a}^{ij}_{\alpha\beta}], \qquad i = 1,2,\ j = 1,2,\\
&a^{ij}_{\alpha\beta}, d \in C^{s-1}(\mathbb{R}^{17}),\\
&c_{i\alpha j\beta}, c, n, \widetilde{c}^1_{i\alpha}, \widetilde{c}^2_{i\alpha} \in C^{s-1}(\mathbb{R}^{11}),\\
&c_{i\alpha j\beta} = c_{j\beta i\alpha}, \qquad \overline{a}^{ij}_{\alpha\beta} = \overline{a}^{ji}_{\beta\alpha}.
\end{aligned}$$

4° *The initial data satisfy*: $\theta_1^0, \theta_2^0, u^0 \in H^0(\Omega) \cap H_0^1(\Omega)$, $u^1 \in H^{s-1}(\Omega) \cap H_0^1(\Omega)$ *for* $x \in \overline{\Omega}$ *and the compatibility conditions*

$$\begin{aligned}
u^k &\in H^{s-k}(\Omega) \cap H_0^1(\Omega), \quad 2 \leq k \leq s-1, \qquad u^s \in L^2(\Omega),\\
\theta_1^k &\in H^{s-k}(\Omega) \cap H_0^1(\Omega), \quad 1 \leq k \leq s-2, \quad \theta_1^{s-1} \in L^2(\Omega),\\
\theta_2^k &\in H^{s-k}(\Omega) \cap H_0^1(\Omega), \quad 1 \leq k \leq s-2, \quad \theta_2^{s-1} \in L^2(\Omega),
\end{aligned}$$

where $u^k = \partial^k u(0,\cdot)/\partial t^k$, $\theta_1^k = \partial^k \theta_1(0,\cdot)/\partial t^k$, $\theta_2^k = \partial^k \theta_2(0,\cdot)/\partial t^k$ *and they are calculated formally* (*and recursively*) *in terms of* $u^0, u^1, \theta_0^1, \theta_0^2$ *using* (9.8).

Then for sufficiently small $T > 0$ *there exists a unique solution* (u, θ_1, θ_2) *of the initial value problem* (9.1)–(9.2) *with the following properties*:

$$\begin{aligned}
u &\in \bigcap_{k=1}^{s-1} C^k([0,T], H^{s-k}(\Omega) \cap H_0^1(\Omega)),\\
\partial_t^s u &\in C^0([0,T], L^2(\Omega)),\\
\theta_1 &\in \bigcap_{k=1}^{s-2} C^k([0,T], H^{s-k}(\Omega) \cap H_0^1(\Omega)),\\
\partial_t^{s-1}\theta_1 &\in C^0([0,T], L^2(\Omega)), \qquad \partial_t^{s-1}\nabla\theta_1 \in L^2([0,T], L^2(\Omega)),\\
\theta_2 &\in \bigcap_{k=1}^{s-2} C^k([0,T], H^{s-k}(\Omega) \cap H_0^1(\Omega)),\\
\partial_t^{s-1}\theta_2 &\in C^0([0,T], L^2(\Omega)), \qquad \partial_t^{s-1}\nabla\theta_2 \in L^2([0,T], L^2(\Omega)).
\end{aligned} \tag{9.8}$$

The proof of Theorem 9.1 is divided into three steps:

1° Proof for the linear system of equations obtained by linearization of system (9.1)–(9.3) in the cases of
   (a) one linear hyperbolic system,
   (b) one linear parabolic system.

2° Proof of an energy estimate for these systems.

3° Proof of existence and uniqueness of solution of the initial-boundary value problem to the nonlinear system (9.1)–(9.2) by applying a fixed point theorem.

10. Energy estimate for the linearized system of thermodiffusion in a solid body

10.1. Linearized system of thermodiffusion in a solid body. In this subsection we shall investigate two initial-boundary value problems for one linear hyperbolic system and one linear parabolic system. These systems arise from the linearized system (9.1)–(9.2). So we shall investigate the solvability of the following problem:

1° The initial-boundary value problem for the linear hyperbolic system

$$\partial_t^2 u_i - \overline{c}_{i\alpha j\beta}(t,x)\frac{\partial^2 u_j}{\partial x_\alpha \partial x_\beta} = \overline{f}_i(t,x), \quad (t,x)\in[0,T]\times\Omega,\ i=1,2,3, \tag{10.1}$$

with initial conditions

$$u_i(0,x) = u_i^0(x), \quad (\partial_t u_i)(0,x) = u_i^1(x), \tag{10.2}$$

and boundary conditions

$$u_i(t,\cdot)|_{\partial\Omega} = 0, \quad t\in[0,T], \tag{10.3}$$

2° The initial-boundary value problem for the linear parabolic system

$$\partial_t\theta_1 - a_{\alpha\beta}^{11}(t,x)\frac{\partial^2\theta_1}{\partial x_\alpha\partial x_\beta} - a_{\alpha\beta}^{12}(t,x)\frac{\partial^2\theta_2}{\partial x_\alpha\partial x_\beta} = \overline{g}_1(t,x), \tag{10.4}$$

$$\partial_t\theta_2 - a_{\alpha\beta}^{21}(t,x)\frac{\partial^2\theta_1}{\partial x_\alpha\partial x_\beta} - a_{\alpha\beta}^{22}(t,x)\frac{\partial^2\theta_2}{\partial x_\alpha\partial x_\beta} = \overline{g}_2(t,x) \tag{10.5}$$

with initial conditions

$$\theta_1(0,x) = \theta_1^0(x), \quad \theta_2(0,x) = \theta_2^0(x), \tag{10.6}$$

and boundary conditions

$$\theta_1(t,\cdot)|_{\partial\Omega} = 0, \quad \theta_2(t,\cdot)|_{\partial\Omega} = 0, \quad t\in[0,T]. \tag{10.7}$$

10.2. Energy estimate for the linear system of thermodiffusion in a solid body

10.2.1. *Energy estimate for the linear hyperbolic system.* We start with results on the existence of solution for problem (10.1)–(10.3).

THEOREM 10.1 (Existence, uniqueness and regularity for problem (10.1)–(10.3)). *Let the following assumptions be satisfied*:

1° $s > \lfloor 3/2\rfloor + 4 = 5$ *is an arbitrary but fixed integer.*

2° $\overline{c}_{i\alpha j\beta} \in C^0([0,T]\times\overline{\Omega})\cap L^\infty([0,T],L^\infty(\Omega))$, $\quad D_x\overline{c}_{i\alpha j\beta}\in L^\infty([0,T],H^{s-2}(\Omega))$, $\partial_t^k\overline{c}_{i\alpha j\beta}\in L^\infty([0,T],H^{s-1-k}(\Omega))$ *for* $k=1,\dots,s-1$.

3° $\overline{c}_{i\alpha j\beta}=\overline{c}_{j\beta i\alpha}$ *and there exists a constant* $\gamma_0>0$ *such that*

$$\|u\|_1^2\le\gamma_0\left(\overline{c}_{i\alpha j\beta}(t)\frac{\partial u_j}{\partial x_\beta},\frac{\partial u_i}{\partial x_\alpha}\right)+\|u\|_0^2$$

for all $u\in H_0^1(\Omega)$, $t\in[0,T]$.

4° *For almost all* $t\in[0,T]$, *the condition*

$$\overline{c}_{i\alpha j\beta}\frac{\partial^2 u_j}{\partial x_\alpha\partial x_\beta}\in H^k(\Omega)$$

together with $u\in H_0^1(\Omega)$ *implies*

$$u\in H^{k+2}(\Omega)\quad\text{and}\quad\|u\|_{k+2}^2\le\gamma_1\left(\left\|\overline{c}_{i\alpha j\beta}(t)\frac{\partial^2 u_j}{\partial x_\alpha\partial x_\beta}\right\|_k^2+\|u\|_0^2\right),$$

where $\gamma_1>0$ *is some constant.*

Then there exists a unique solution $u=(u_1,u_2,u_3)^*$ *of problem* (10.1)–(10.3) *with the properties*

$$\partial_t^k u\in C^0([0,T],H^{s-k}(\Omega)\cap H_0^1(\Omega)),\qquad 0\le k\le s-1,$$

and

$$\partial_t^s u\in C^0([0,T],L^2(\Omega)).\tag{10.8}$$

The proof runs in the same way as that of Theorem 4.1.

Now, we formulate an energy estimate for problem (10.1)–(10.4).

THEOREM 10.2 (Energy estimate for problem (10.1)–(10.4)). *If the assumptions of Theorem* 10.1 *are satisfied*, *then the solution of problem* (10.1)–(10.3) *guaranteed by Theorem* 10.1 *satisfies the inequality*

$$|\overline{D}^s u|_{0,T}^2\le\overline{K}_1\overline{K}_0e^{\overline{K}_2\xi(T)}\tag{10.9}$$

with positive constants $\overline{K}_1,\overline{K}_0,\overline{K}_2$, *where*

$$\overline{K}_0=\sum_{k=0}^s\|u^k\|_{s-k}^2+(1+T)|\overline{D}^{s-2}\overline{f}|_{0,T}^2+T^{1/2}\int_0^T\|\partial_t^{s-1}\overline{f}(t)\|^2\,dt,$$
$$\overline{K}_1=\overline{K}_1(L_0,\gamma_0,\gamma_1)>0,\qquad\overline{K}_2=\overline{K}_2(L,\gamma_0,\gamma_1)>0$$

depend continuously on their arguments, *where*

$$\begin{aligned}L_0&=\|\overline{c}_{i\alpha j\beta}(0)\|_{L^\infty}+\|D_x\overline{c}_{i\alpha j\beta}(0)\|_{s-3},\\ L&=\sup_{0\le t\le T}\|\overline{c}_{i\alpha j\beta}(t)\|_{L^\infty}+|D_x\overline{c}_{i\alpha j\beta}|_{s-2,T}+\sum_{k=1}^{s-1}|\partial_t^k\overline{c}_{i\alpha j\beta}|_{s-1-k,T},\end{aligned}\tag{10.10}$$

and

$$\xi(T)=T^{1/2}(1+T^{1/2}+T+T^{3/2}),\tag{10.11}$$

The proof runs in the same way as that of Theorem 7.2.

10.2.2. *Energy estimate for the linear parabolic system.* In the second step we consider solvability of the initial-boundary value problem for the linear parabolic system (10.4)–(10.5) with conditions (10.6)–(10.7).

First we introduce the vector $V = (\theta_1, \theta_2)^*$ and convert the initial-boundary value problem (10.1)–(10.7) to the form

$$\partial_t V - a_{\alpha\beta}(t,x)\frac{\partial^2 V}{\partial x_\alpha \partial x_\beta} = \overline{G}(t,x), \tag{10.12}$$

with

$$V(0,x) = V^0(x), \qquad V(t,\cdot)|_{\partial\Omega} = 0, \tag{10.13}$$

where

$$a_{\alpha\beta}(t,x) = \begin{pmatrix} a^{11}_{\alpha\beta}(t,x) & a^{12}_{\alpha\beta}(t,x) \\ a^{21}_{\alpha\beta}(t,x) & a^{22}_{\alpha\beta}(t,x) \end{pmatrix}, \qquad \overline{G}(t,x) = (\overline{g}_1(t,x), \overline{g}_2(t,x))^*. \tag{10.14}$$

In order to formulate an energy estimate for problem (10.10)–(10.11) we present two theorems, whose proofs can be found in [30].

THEOREM 10.3. *Let the following conditions be satisfied*:

$$\begin{aligned}
&\overline{D}^1 a^{ij}_{\alpha\beta}(t,x) \in C^0([0,T]\times\overline{\Omega}) \cap L^\infty([0,T], L^\infty(\Omega)),\\
&\partial_t \nabla a^{ij}_{\alpha\beta}(t,x) \in L^\infty([0,T], L^\infty(\Omega)), \qquad \overline{G} \in C^0([0,T], L^2(\Omega)),\\
&\partial_t \overline{G} \in L^2([0,T], H^{-1}(\Omega)), \qquad V^0 \in H^1_0(\Omega),\\
&V^1 := a_{\alpha\beta}(0)\frac{\partial^2 V^0}{\partial x_\alpha \partial x_\beta} + \overline{G}(0) \in L^2(\Omega),
\end{aligned}$$

and

$$a^{ij}_{\alpha\beta}(t,x) = a^{ji}_{\beta\alpha}(t,x) \quad \text{for } (t,x) \in [0,T]\times\overline{\Omega}, \tag{10.15}$$

$$(a_{\alpha\beta}\xi_\alpha\xi_\beta\eta, \eta) \ge \gamma_3 |\xi|^2 |\eta|^2 \quad \text{for } \xi \in \mathbb{R}^3,\ \eta \in \mathbb{R}^2, \tag{10.16}$$

and for some constant $\gamma_3 > 0$. Then there exists a unique solution $V = (\theta_1, \theta_2)^*$ to problem (10.10)–(10.11) *with the properties*

$$\begin{aligned}
\theta_1 &\in C^0([0,T], H^2(\Omega) \cap H^1_0(\Omega)),\\
\partial_t \theta_1 &\in C^0([0,T], L^2(\Omega)),\\
\partial_t \nabla\theta_1 &\in L^2([0,T], L^2(\Omega)),\\
\theta_2 &\in C^0([0,T], H^2(\Omega) \cap H^1_0(\Omega)),\\
\partial_t \theta_2 &\in C^0([0,T], L^2(\Omega)),\\
\partial_t \nabla\theta_2 &\in L^2([0,T], L^2(\Omega)).
\end{aligned} \tag{10.17}$$

Now, we formulate a regularity theorem for solutions of problem (10.10)–(10.11).

THEOREM 10.4. *Let the following conditions be satisfied*:

1° $s \ge \lfloor 3/2 \rfloor + 4 = 5$ *is an arbitrary but fixed integer.*

2°
$$\begin{aligned}
&a^{ij}_{\alpha\beta} \in C^0([0,T]\times\overline{\Omega}) \cap L^\infty([0,T], L^\infty(\Omega)),\\
&D_x a^{ij}_{\alpha\beta} \in L^\infty([0,T], H^{s-2}(\Omega)),\\
&\partial_t^k a^{ij}_{\alpha\beta} \in L^\infty([0,T], H^{s-1-k}(\Omega)), \qquad 1 \le k \le s-2,\\
&\partial_t^{s-1} a^{ij}_{\alpha\beta} \in L^2([0,T], L^2(\Omega)).
\end{aligned}$$

3° *For all* $\theta_1, \theta_2 \in H_0^1(\Omega)$ *and* $t \in [0,T]$, *the inequality*

$$\|\theta_1\|_1^2 + \|\theta_2\|_1^2 \le \gamma_4 \Big\{ \Big(a_{\alpha\beta}^{ij} \frac{\partial \theta_i}{\partial x_\beta}, \frac{\partial \theta_j}{\partial x_\alpha} \Big) + \|\theta_1\|^2 + \|\theta_2\|^2 \Big\}$$

is satisfied for some constant $\gamma_4 > 0$.

4° *For* $t \in [0,T]$,

$$a_{\alpha\beta}^{ij}(t) \frac{\partial^2 \theta_i}{\partial x_\alpha \partial x_\beta} \in H^k(\Omega) \qquad \text{with } \theta_1, \theta_2 \in H_0^1(\Omega),$$

implies that $\theta_1, \theta_2 \in H^{k+2}(\Omega)$, *and*

$$\|V\|_{k+2} \le \gamma_3 \bigg(\Big\| a_{\alpha\beta}(t) \frac{\partial^2 V}{\partial x_\alpha \partial x_\beta} \Big\|_k + \|V\|_0 \bigg),$$

where $V = (\theta_1, \theta_2)^*$, $0 \le k \le s-2$ *and* γ_3 *is some constant.*

5°
$$\begin{aligned} \partial_t^k \overline{g}_1 &\in C^0([0,T], H^{s-2-k}(\Omega)), \qquad 0 \le k \le s-2, \\ \partial_t^{s-1} \overline{g}_1 &\in L^2([0,T], H^{-1}(\Omega)), \\ \partial_t^k \overline{g}_2 &\in C^0([0,T], H^{s-2-k}(\Omega)), \qquad 0 < k \le s-2, \\ \partial_t^{s-1} \overline{g}_2 &\in L^2([0,T], H^{-1}(\Omega)), \end{aligned}$$

Then there exists a unique solution $V = (\theta_1, \theta_2)^*$ *of problem* (10.10)–(10.11) *with the properties*

$$\begin{aligned} \partial_t^k \theta_1 &\in C^0([0,T], H^{s-2-k}(\Omega) \cap H_0^1(\Omega)), \qquad 0 \le k \le s-2, \\ \partial_t^{s-1} \theta_1 &\in C^0([0,T], L^2(\Omega)), \qquad \partial_t^{s-1} \nabla \theta_1 \in L^2([0,T], L^2(\Omega)), \\ \partial_t^k \theta_2 &\in C^0([0,T], H^{s-2-k}(\Omega) \cap H_0^1(\Omega)), \qquad 0 \le k \le s-2, \\ \partial_t^{s-1} \theta_2 &\in C^0([0,T], L^2(\Omega)), \qquad \partial_t^{s-1} \nabla \theta_2 \in L^2([0,T], L^2(\Omega)). \end{aligned} \tag{10.18}$$

Next we present an energy estimate for the solution of problem (10.10)–(10.11).

THEOREM 10.5 (Energy estimate for the parabolic system (10.10)–(10.11)). *Let the conditions of Theorem* 10.4 *be satisfied. Then the solution to problem* (10.10)–(10.11) *satisfies the inequality*

$$\begin{aligned} \sum_{k=0}^{s-2} |\partial_t^k \theta_1|_{s-k,T}^2 + \sum_{k=0}^{s-2} |\partial_t^k \theta_2|_{s-k,T}^2 + |\partial_t^{s-1} \theta_1|_{0,T}^2 + |\partial_t^{s-1} \theta_2|_{0,T}^2 \\ + \int_0^T [\|\partial_\tau^{s-1} \nabla \theta_1(\tau)\|^2 + \|\partial_\tau^{s-1} \nabla \theta_2(\tau)\|^2]\, d\tau \le K_3 M_0 e^{K_4 \eta(T)}, \end{aligned} \tag{10.19}$$

where

$$\begin{aligned} M_0 = (1+T) \Big\{ \sum_{k=0}^{s-2} (\|\theta_1^k\|_{s-k}^2) + \|\theta_2^k\|_{s-k}^2 + \|\theta_1^{s-1}\|^2 + \|\theta_2^{s-1}\|^2 + |\overline{D}^{s-2} \overline{g}_1|_{0,T}^2 \\ + |\overline{D}^{s-2} \overline{g}_2|_{0,T}^2 + \int_0^T [\|\partial_t^{s-1} \overline{g}_1(\tau)\|_{H^{-1}}^2 + \|\partial_t^{s-1} \overline{g}_2(\tau)\|_{H^{-1}}^2]\, d\tau \Big\}, \end{aligned} \tag{10.20}$$

$K_3 = K_3(P_0, \gamma_2, \gamma_3) > 0$, $K_4 = K_4(P, \gamma_2, \gamma_3) > 0$, γ_2, γ_3 *are given in the assumption of Theorem* 10.9,

$$
\begin{aligned}
P = & \sup_{0\le t\le T} \sum_{i,j=1}^{2} \|a^{ij}_{\alpha\beta}(t)\|_{L^\infty} + \sum_{i,j=1}^{2} \|D_\alpha a^{ij}_{\alpha\beta}\|_{s-2,T} \\
& + \sum_{k=1}^{s-2} \sum_{i,j=1}^{2} |\partial_t^k a^{ij}_{\alpha\beta}|_{s-1-k,T} + \int_0^T \sum_{i,j=1}^{2} \|\partial_t^{s-1} a^{ij}_{\alpha\beta}(\tau)\|^2 \, d\tau, \\
P_0 = & \sum_{i,j=1}^{2} \|a^{ij}_{\alpha\beta}(0)\|_{L^\infty} + \sum_{i,j=1}^{2} \|D_x a^{ij}_{\alpha\beta}(0)\|_{s-3},
\end{aligned}
\tag{10.21}
$$

and $\eta(T) = T(1+T)$.

Proof. This theorem can be found in [29].

11. Proof of Theorem 9.1

The proof of Theorem 9.1 is based on the Banach fixed point theorem. For this reason we denote by $\boldsymbol{V}(N,T)$ the set of functions u satisfying the conditions

$$
\begin{aligned}
&\partial_t^k u \in L^\infty([0,T], H^{s-k}(\Omega)), \quad 0 \le k \le s, \\
&\partial_t^k \theta_1 \in L^\infty([0,T], H^{s-k}(\Omega)), \quad 0 \le k \le s-2, \\
&\partial_t^{s-1} \theta_1 \in L^\infty([0,T], L^2(\Omega)), \quad \partial_t^{s-1}\nabla\theta_1 \in L^2([0,T], L^2(\Omega)), \\
&\partial_t^k \theta_2 \in L^\infty([0,T], H^{s-k}(\Omega)), \quad 0 \le k \le s-2, \\
&\partial_t^{s-1} \theta_2 \in L^\infty([0,T], L^2(\Omega)), \quad \partial_t^{s-1}\nabla\theta_2 \in L^2([0,T], L^2(\Omega)),
\end{aligned}
\tag{11.1}
$$

with boundary and initial conditions of the form

$$
\begin{aligned}
&u_i|_{\partial\Omega} = 0, \quad \theta_1|_{\partial\Omega} = 0, \quad \theta_2|_{\partial\Omega} = 0, \\
&\partial_t^k u(0,x) = u^k(x), \quad 0 \le k \le s-1, \quad \partial_t^k \theta_1(0,x) = \theta_1^k(x), \quad 0 \le k \le s-2, \\
&\partial_t^k \theta_2(0,x) = \theta_2^k(x), \ 0 \le k \le s-2,
\end{aligned}
\tag{11.2}
$$

and the inequality

$$
\begin{aligned}
|\overline{D}^s u|^2_{0,T} &+ \sum_{k=0}^{s-2} |\partial_t^k \theta_1|^2_{s-k,T} + |\partial_t^{s-1}\theta_1|^2_{0,T} + \sum_{k=0}^{s-2} |\partial_t^k \theta_2|^2_{s-k,T} + |\partial_t^{s-1}\theta_2|^2_{0,T} \\
&+ \int_0^T \|\partial_\tau^{s-1}\nabla\theta_1(\tau)\|^2 \, d\tau + \int_0^T \|\partial_\tau^{s-1}\nabla\theta_2(\tau)\|^2 \, d\tau \le N^2
\end{aligned}
\tag{11.3}
$$

for N large enough.

Proof of Theorem 9.1. Let

$$(\overline{u}, \overline{\theta}_1, \overline{\theta}_2) \in \boldsymbol{V}(N,T).$$

We consider

1° the system

$$\partial_t^2 u_i - \overline{c}_{i\alpha j\beta} \frac{\partial^2 u_j}{\partial x_\alpha \partial x_\beta} = \overline{f}_i \quad \text{for } i = 1,2,3$$

with

$$
\begin{aligned}
&\overline{c}_{i\alpha j\beta} := c_{i\alpha j\beta}(\nabla\overline{u},\overline{\theta}_1,\overline{\theta}_2),\\
&\overline{f}_i := \widetilde{c}^{1}_{i\alpha}(\nabla\overline{u},\overline{\theta}_1,\overline{\theta}_2)\frac{\partial\overline{\theta}_1}{\partial x_\alpha} + \widetilde{c}^{2}_{i\alpha}(\nabla\overline{u},\overline{\theta}_1,\overline{\theta}_2)\frac{\partial\overline{\theta}_2}{\partial x_\alpha} + f_i(t,x),
\end{aligned}
\tag{11.4}
$$

2° the system

$$
\partial_t\theta_1 - a^{11}_{\alpha\beta}\frac{\partial^2\theta_1}{\partial x_\alpha\partial x_\beta} - a^{12}_{\alpha\beta}\frac{\partial^2\theta_2}{\partial x_\alpha\partial x_\beta} = \overline{g}_1,\qquad \partial_t\theta_2 - a^{21}_{\alpha\beta}\frac{\partial^2\theta_1}{\partial x_\alpha\partial x_\beta} - a^{22}_{\alpha\beta}\frac{\partial^2\theta_2}{\partial x_\alpha\partial x_\beta} = \overline{g}_2
$$

with

$$
\begin{aligned}
&a^{11}_{\alpha\beta} := \overline{a}^{11}_{\alpha\beta}(\nabla\overline{u},\overline{\theta}_1,\overline{\theta}_2,\nabla\overline{\theta}_1,\nabla\overline{\theta}_2),\qquad a^{12}_{\alpha\beta} := \overline{a}^{12}_{\alpha\beta}(\nabla\overline{u},\overline{\theta}_1,\overline{\theta}_2,\nabla\overline{\theta}_1,\nabla\overline{\theta}_2),\\
&a^{21}_{\alpha\beta} := \overline{a}^{21}_{\alpha\beta}(\nabla\overline{u},\overline{\theta}_1,\overline{\theta}_2,\nabla\overline{\theta}_1,\nabla\overline{\theta}_2),\qquad a^{22}_{\alpha\beta} := \overline{a}^{22}_{\alpha\beta}(\nabla\overline{u},\overline{\theta}_1,\overline{\theta}_2,\nabla\overline{\theta}_1,\nabla\overline{\theta}_2),\\
&\overline{g}_1 = \mathbb{C}^1_{i\alpha}(\nabla\overline{u},\overline{\theta}_1,\overline{\theta}_2,\nabla\overline{\theta}_1,\nabla\overline{\theta}_2)\frac{\partial^2 u_i}{\partial x_\alpha\partial t} + g_1(\nabla\overline{u},\overline{\theta}_1,\overline{\theta}_2,\nabla\overline{\theta}_1,\nabla\overline{\theta}_2,t,x),\\
&g_1 = \frac{Q_1 n - dQ_2}{\delta},\qquad g_2 = \frac{cQ_2 - dQ_1}{\delta},\\
&\overline{g}_2 = \mathbb{C}^2_{i\alpha}(\nabla\overline{u},\overline{\theta}_1,\overline{\theta}_2,\nabla\overline{\theta}_1,\nabla\overline{\theta}_2)\frac{\partial^2 u_i}{\partial x_\alpha\partial t} + g_2(\nabla\overline{u},\overline{\theta}_1,\overline{\theta}_2,\nabla\overline{\theta}_1,\nabla\overline{\theta}_2,t,x),
\end{aligned}
\tag{11.5}
$$

where u is the solution of (10.1), (10.3), (11.4).

By σ_2 we denote the mapping which maps $(\overline{u},\overline{\theta}_1,\overline{\theta}_2)$ to the solution (u,θ_1,θ_2) of problem (10.1)–(10.3), (5.4), (10.4)–(10.8), (11.5), i.e.

$$
\sigma_2 : \boldsymbol{V}(N,T) \ni (\overline{u},\overline{\theta}_1,\overline{\theta}_2) \mapsto \sigma_2(\overline{u},\overline{\theta}_1,\overline{\theta}_2) = (u,\theta_1,\theta_2).
\tag{11.6}
$$

The following statements are true.

Statement I. *σ_2 maps $\boldsymbol{V}(N,T)$ into itself for N large and T small enough.*

In the proof of this statement we use the energy estimate for the linearized hyperbolic system of equations (cf. Theorems 10.1, 10.2) and the energy estimate for the linearized parabolic system of equations (cf. Theorem 10.4, 10.5).

Statement II. *The mapping $\sigma_2 : \boldsymbol{V}(N,T) \to \boldsymbol{V}(N,T)$ is a contraction for T small enough.*

For this let W_1 denote the complete metric space given by

$$
\begin{aligned}
W_1 := \{(\overline{u},\overline{\theta}_1,\overline{\theta}_2) : \overline{D}^1\overline{u},\overline{\theta}_1,\overline{\theta}_2 \in L^\infty([0,T],L^2(\Omega)),&\\
\nabla\overline{\theta}_1 \in L^2([0,T],L^2(\Omega)),\ \nabla\overline{\theta}_2 \in L^2([0,T],L^2(\Omega))\}&,
\end{aligned}
\tag{11.7}
$$

with metric given by

$$
\begin{aligned}
\varphi^2((\overline{u},\overline{\theta}_1,\overline{\theta}_2),(u,\theta_1,\theta_2)) := |\overline{D}^1(\overline{u}-u)|^2_{0,T} + |\overline{\theta}_1-\theta_1|^2_{0,T} + |\overline{\theta}_2-\theta_2|^2_{0,T}&\\
+ \int_0^T \|\nabla(\overline{\theta}_1-\theta_1)(\tau)\|_0^2\,d\tau + \int_0^T \|\nabla(\overline{\theta}_2-\theta_2)(\tau)\|_0^2\,d\tau&.
\end{aligned}
\tag{11.8}
$$

Then $\boldsymbol{V}(N,T)$ is a closed subset of W_1.

Let $(\overline{u},\overline{\theta}_1,\overline{\theta}_2)$ and $(\overline{u}^*,\overline{\theta}_1^*,\overline{\theta}_2^*)\in \boldsymbol{V}(N,T)$. Then

$$\sigma_2(\overline{u},\overline{\theta}_1,\overline{\theta}_2)=(u,\theta_1,\theta_2)\in \boldsymbol{V}(N,T),\qquad \sigma_2(\overline{u}^*,\overline{\theta}_1^*,\overline{\theta}_2^*)=(u^*,\theta_1^*,\theta_2^*)\in \boldsymbol{V}(N,T),$$

where (u,θ_1,θ_2), $(u^*,\theta_1^*,\theta_2^*)$ are the solutions of problems (10.1)–(10.3), (11.4) and (10.4)–(10.7), (11.5) respectively, where the coefficients and the right hand side depend on $(\overline{u},\overline{\theta}_1,\overline{\theta}_2)$ and $(\overline{u}^*,\overline{\theta}_1^*,\overline{\theta}_2^*)$. Subtracting the resulting systems of equations and using some calculations, we get

$$\begin{aligned}
&\partial_t^2(u_i-u_i^*)-c_{i\alpha j\beta}(\nabla\overline{u},\overline{\theta}_1,\overline{\theta}_2)\frac{\partial^2(u_j-u_j^*)}{\partial x_\alpha\partial x_\beta}\\
&\quad=(c_{i\alpha j\beta}(\nabla\overline{u}^*,\overline{\theta}_1^*,\overline{\theta}_2^*)-c_{i\alpha j\beta}(\nabla\overline{u},\overline{\theta}_1,\overline{\theta}_2))\frac{\partial^2 u_j^*}{\partial x_\alpha\partial x_\beta}\\
&\qquad+(\widetilde{c}^1_{i\alpha}(\nabla\overline{u},\overline{\theta}_1,\overline{\theta}_2)-\widetilde{c}^1_{i\alpha}(\nabla\overline{u}^*,\overline{\theta}_1^*,\overline{\theta}_2^*))\frac{\partial\overline{\theta}_1}{\partial x_\alpha}+\widetilde{c}^1_{i\alpha}(\nabla\overline{u}^*,\overline{\theta}_1^*,\overline{\theta}_2^*)\left(\frac{\partial\overline{\theta}_1}{\partial x_\alpha}-\frac{\partial\overline{\theta}_1^*}{\partial x_\alpha}\right)\\
&\qquad+(\widetilde{c}^2_{i\alpha}(\nabla\overline{u},\overline{\theta}_1,\overline{\theta}_2)-\widetilde{c}^2_{i\alpha}(\nabla\overline{u}^*,\overline{\theta}_1^*,\overline{\theta}_2^*))\frac{\partial\overline{\theta}_2}{\partial x_\alpha}+\widetilde{c}^2_{i\alpha}(\nabla\overline{u}^*,\overline{\theta}_1^*,\overline{\theta}_2^*)\left(\frac{\partial\overline{\theta}_2}{\partial x_\alpha}-\frac{\partial\overline{\theta}_2^*}{\partial x_\alpha}\right)
\end{aligned}$$

and

$$\begin{aligned}
(11.9)\quad&\partial_t(\theta_i-\theta_i^*)-\overline{a}^{ij}_{\alpha\beta}(\nabla\overline{u},\overline{\theta}_1,\overline{\theta}_2,\nabla\overline{\theta}_1,\nabla\overline{\theta}_2)\frac{\partial^2(\theta_j-\theta_j^*)}{\partial x_\alpha\partial x_\beta}\\
&\quad=(\overline{a}^{ij}_{\alpha\beta}(\nabla\overline{u}^*,\overline{\theta}_1^*,\overline{\theta}_2^*,\nabla\overline{\theta}_1^*,\nabla\overline{\theta}_2^*)-\overline{a}^{ij}_{\alpha\beta}(\nabla\overline{u},\overline{\theta}_1,\overline{\theta}_2,\nabla\overline{\theta}_1,\nabla\overline{\theta}_2))\frac{\partial^2\theta_j^*}{\partial x_\alpha\partial x_\beta}\\
&\qquad+\mathbb{C}^i_{j\alpha}(\nabla\overline{u},\overline{\theta}_1,\overline{\theta}_2,\nabla\overline{\theta}_1,\nabla\overline{\theta}_2)\frac{\partial^2(u_j-u_j^*)}{\partial x_\alpha\partial t}\\
&\qquad+(\mathbb{C}^i_{j\alpha}(\nabla\overline{u},\overline{\theta}_1,\overline{\theta}_2,\nabla\overline{\theta}_1,\nabla\overline{\theta}_2)-\mathbb{C}^i_{j\alpha}(\nabla\overline{u}^*,\overline{\theta}_1^*,\overline{\theta}_2^*,\nabla\overline{\theta}_1^*,\nabla\overline{\theta}_2^*))\frac{\partial^2 u_j^*}{\partial x_\alpha\partial t}\\
&\qquad+(g_i(\nabla\overline{u},\theta_1,\overline{\theta}_2,\nabla\overline{\theta}_1,\nabla\overline{\theta}_2,x,t)-g_i(\nabla\overline{u}^*,\overline{\theta}_1^*,\overline{\theta}_2^*,\nabla\overline{\theta}_1^*,\nabla\overline{\theta}_2^*,x,t)),\qquad i=1,2.
\end{aligned}$$

Using the fact that

$$\sup_{0\le t\le T}\|\overline{D}^2(\overline{u},\overline{\theta}_1,\overline{\theta}_2,\overline{u}^*,\overline{\theta}_1^*,\overline{\theta}_2^*,u,\theta_1,\theta_2,u^*,\theta_1^*,\theta_2^*\|\le CN,$$

and $(u_i-u_i^*)|_{\partial\Omega}=0$, $\partial_t^k(u_i-u^*)(0,x)=0$, $k=0,1$, and the mean value theorem

$$\begin{aligned}
C(\nabla\overline{u},\overline{\theta})-C(\nabla\overline{u}^*,\overline{\theta}^*)&=C(\nabla\overline{u}^*+(\nabla\overline{u}-\nabla\overline{u}^*),\overline{\theta}^*+(\overline{\theta}-\overline{\theta}^*))-C(\nabla\overline{u}^*,\overline{\theta}^*)\\
&=\nabla_\xi C(\xi,\zeta)(\nabla\overline{u}-\nabla\overline{u}^*)+\nabla_\zeta C(\xi,\zeta)\bullet(\overline{\theta}-\overline{\theta}^*),
\end{aligned}$$

and the Schwarz inequality, after some calculations, we get

$$\begin{aligned}
(11.10)\quad\|\overline{D}^1(u-u^*)\|_0^2\le C(N)\biggl\{&\left(1+\frac{1}{T^{1/2}}\right)\int_0^T\|\overline{D}_1(u-u^*)\|_0^2\,d\tau\\
&+T^{1/2}(1+T)\Bigl[|\overline{D}_1(\overline{u}-\overline{u}^*)|_{0,T}^2+|\overline{\theta}_1-\overline{\theta}_1^*|_{0,T}^2+|\overline{\theta}_2-\overline{\theta}_2^*|_{0,T}^2\\
&+\int_0^T\|\nabla(\overline{\theta}_1-\overline{\theta}_1^*)\|_0^2\,d\tau+\int_0^T\|\nabla(\overline{\theta}_2-\overline{\theta}_2^*)\|_0^2\,d\tau\Bigr]\biggr\}
\end{aligned}$$

and

$$
\begin{aligned}
(11.11)\qquad & \|\theta_1-\theta_1^*\|_0^2+\int_0^t\|\nabla(\theta_1-\theta_1^*)\|_0^2\,d\tau+\|\theta_2-\theta_2^*\|_0^2+\int_0^t\|\nabla(\theta_2-\theta_2^*)\|_0^2\,d\tau\\
&\le C(N)\biggl\{\biggl(1+\frac{1}{T^{1/2}}\biggr)\int_0^t[\|\theta_1-\theta_1^*\|_0^2+\|\theta_2-\theta_2^*\|_0^2+\|\overline{D}^1(u-u^*)\|_0^2]\,d\tau\\
&\quad+T^{1/2}(1+T)\Bigl[|\overline{D}^1(\overline{u}-\overline{u}^*)|_{0,T}^2+|\overline{\theta}_1-\overline{\theta}_1^*|_{0,T}^2+|\overline{\theta}_2-\overline{\theta}_2^*|_{0,T}^2\\
&\quad+\int_0^t[\|\nabla(\overline{\theta}_1-\overline{\theta}_1^*)\|_0^2+\|\nabla(\overline{\theta}_2-\overline{\theta}_2^*)\|_0^2]\,d\tau\Bigr]\biggr\}
\end{aligned}
$$

we deduce from (11.10)–(11.11) that

$$
\begin{aligned}
(11.12)\qquad & |\overline{D}^1(u-u^*)|_{0,T}^2+|\theta_1-\theta_1^*|_{0,T}^2+|\theta_2-\theta_2^*|_{0,T}^2\\
&\quad+\int_0^T\|\nabla(\theta_1-\theta_1^*)\|_0^2\,d\tau+\int_0^T\|\nabla(\theta_2-\theta_2^*)\|_0^2\,d\tau\\
&\le C(N)\biggl\{\biggl(1+\frac{1}{T^{1/2}}\biggr)\int_0^T[\|\overline{D}^1(u-u^*)\|_0^2+\|\theta_1-\theta_1^*\|_0^2+\|\theta_2-\theta_2^*\|_0^2]\,d\tau\\
&\quad+T^{1/2}(1+T)\Bigl[|\overline{D}^1(\overline{u}-\overline{u}^*)|_{0,T}^2+|\overline{\theta}_1-\overline{\theta}_1^*|_{0,T}^2+|\overline{\theta}_2-\overline{\theta}_2^*|_{0,T}^2\\
&\quad+\int_0^T\|\nabla(\overline{\theta}_1-\overline{\theta}_1^*)\|_0^2\,d\tau+\int_0^T\|\nabla(\overline{\theta}_2-\overline{\theta}_2^*)\|_0^2\,d\tau\Bigr]\\
&\quad+\biggl(1+\frac{1}{T^{1/2}}\biggr)\int_0^T\int_0^T(\|\nabla(\theta_1-\theta_1^*)\|_0^2+\|\nabla(\theta_2-\theta_2^*)\|_0^2)\,d\tau\,ds\biggr\}.
\end{aligned}
$$

Applying to (11.12) the Gronwall inequality (cf. (2.11)) we get

$$
\begin{aligned}
(11.13)\qquad & |\overline{D}^1(u-u^*)|_{0,T}^2+|\theta_1-\theta_1^*|_{0,T}^2+|\theta_2-\theta_2^*|_{0,T}^2\\
&\quad+\int_0^T\|\nabla(\theta_1-\theta_1^*)\|_0^2\,d\tau+\int_0^T\|\nabla(\theta_2-\theta_2^*)\|_0^2\,d\tau\\
&\le\delta\Bigl(|\overline{D}^1(\overline{u}-\overline{u}^*)|_{0,T}^2+|\overline{\theta}_1-\overline{\theta}_1^*|_{0,T}^2+|\overline{\theta}_2-\overline{\theta}_2^*|_{0,T}^2\\
&\quad+\int_0^T\|\nabla(\overline{\theta}_1-\overline{\theta}_1^*)\|_0^2\,d\tau+\int_0^T\|\nabla(\overline{\theta}_2-\overline{\theta}_2^*)\|_0^2\,d\tau\Bigr).
\end{aligned}
$$

Choosing T small enough, we get $\delta<1$. Therefore the mapping σ_2 is a contraction. So, in view of the Banach fixed theorem the contraction mapping σ_2 has a unique fixed point $(u,\theta_1,\theta_2)\in\boldsymbol{V}(N,T)$. This implies that problem (9.1) with conditions (9.2)–(9.3) has a unique solution on $0\le t\le T$. This completes the proof of Theorem 9.1.

12. General remarks

Many physical phenomena arising in mathematical physics are described not only by quasilinear or linear hyperbolic systems (as in the case of nonlinear hyperbolic thermoelasticity theory and nonlinear microelasticity theory), but by quasilinear or nonlinear coupled hyperbolic-parabolic systems of composite type or by parabolic nonlinear coupled systems as well. Such is the case of:

1. classical thermoelasticity theory, which is described by a nonlinear hyperbolic-parabolic coupled system consisting of four nonlinear partial differential equations;
2. nonlinear thermodiffusion, which is described by a nonlinear hyperbolic-parabolic coupled system (cf. [51], [29]):
   - consisting of five nonlinear partial differential equations describing thermodiffusion in a solid body (cf. Section 9 and [51], [29]),
   - consisting of eight coupled nonlinear partial differential equations describing thermodiffusion in a micropolar medium (cf. [23]);
3. nonlinear diffusion, which is described by a nonlinear coupled parabolic system of equations (cf. [30]).

We can extend the method presented above to prove (local-in-time) existence of solution of the initial-boundary value problem for the nonlinear hyperbolic system of equations and the nonlinear hyperbolic-parabolic system describing the medium in continuum mechanics. Such is the case of a hyperbolic system of partial differential equations describing the so-called nonsimple thermoelastic materials (cf. [32]) and the case of thermodiffusion in a micropolar medium (cf. [23]).

The strategy of the proof is to consider linear hyperbolic and linear parabolic systems associated with the nonlinear ones, and to apply a fixed point principle. The three major steps are the following:

I. Investigate the linear hyperbolic system using the approach of Kato via semigroup theory.

II. Investigate the linear parabolic system using the Faedo–Galerkin method (cf. [18]) or apply Kato's approach.

III. Show that the solution of the initial-boundary problem can be obtained as the unique fixed point of a contraction mapping in a suitable function space (cf. Sections 5, 8, 11).

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