Introduction

Topology and order—the combination of these two branches of mathematics is a classical subject that has led to many fruitful results. There are many facets concerning the interplay of topology and order both in mathematics and theoretical computer science. On the mathematical side, important topics include the theory of *partially ordered spaces* introduced by Nachbin [44] and the investigation of *intrinsic topologies* on posets initiated by Birkhoff (cf. [4]) and Frink [20]. Given a set together with a partial order and a topology, a modest compatibility condition is to have a partially ordered space: the partial order is required to be closed in the cartesian product of the topological space with itself. That is, whenever we have convergent nets $(x_n)_n$ and $(y_n)_n$ with x_n being below y_n for all n, then this is also true for the limit points. On the other hand, a partially ordered set gives rise to several natural topologies defined in terms of the given poset structure. Typical examples are Frink's [20] *interval topology*, Scott's [48] "induced topology" (now well known as the *Scott topology*), and the *Lawson topology*. The latter is the common refinement of the Scott and the interval topology. It provides a fundamental connection between continuous lattices and compact semilattices (see Gierz *et al.* [22]).

In theoretical computer science, order and topology constitute the basis for *domain theory*. Domain theory started in the late sixties when Scott (see e.g. [48]) found the first mathematical models of the untyped λ-calculus. Thus, he provided the mathematical framework for the semantics of functional programming languages. Another concept of domain theory is the notion of *approximation*. The idea is that "infinite" or "ideal" objects of a poset are the supremum of their "finite approximations". Then every element can be approximated by the elements "essentially" below it. This leads to considering *algebraic* and *continuous domains*.

Generally, finite objects are used to approximate infinite ones in many fields of computer science dealing with the specification of non-terminating processes. Usually order or topology or even both are involved. To mention are the projection space approach (Ehrig *et al.* [17]), rank ordered sets (Bruce and Mitchell [8]), rank ordered posets (Baier and Majster-Cederbaum [2, 40]), and the theory of infinite Mazurkiewicz traces (cf. Gastin and Petit [21]).

In the present monograph we develop a general concept connecting topology and order under the aspect of approximation, especially by projections. Moreover, we establish a link to domain theory and apply our results to trace theory.

Our mathematical objects are partially ordered sets (posets) together with a special family of monotone self-maps. These maps are used both to approximate elements of the poset and to generate an intrinsic uniformity. In particular, we obtain a canonical topology that turns the poset into a partially ordered space. To a great extent our results concern monotone self-maps that are even projections.

The definition of our objects is quite natural. Firstly, we shall see that there are many close relationships between the poset on the one hand and the uniform space (topological space, respectively) on the other hand. Secondly, we provide a general mathematical framework into which several well understood structures are integrated, viz. FS-domains ([27–29]), bifinite domains ([25, 27, 47]), the formal ball model of metric spaces ([16]) as well as the "closed ball model" of ultrametric spaces ([19]), rank ordered (po)sets ([2, 8, 40]), projection spaces ([17, 23, 24]), and Mazurkiewicz traces ([12–14, 21]).

The author's motivation to study posets with an *approximating family* of mappings originated from the theory of infinite traces. *Trace monoids* (also called *free partially commutative monoids*) were introduced by Cartier and Foata [9], who investigated combinatorial problems concerning the rearrangement of words, and by Mazurkiewicz [42], who was motivated to provide a mathematical model for concurrent systems. Since then trace theory has become a very popular topic, see the recent surveys [13, 14]. In particular, the theory was extended to infinite traces (cf. Gastin and Petit [21]). The basic idea is to view a process as being built up by "atomic actions". One requires that either two actions depend on each other or they may be executed independently. Then a (possibly infinite) process is described by a *real trace*. This is a particular acyclic graph whose vertices are labelled with atomic actions and whose edges are precisely between dependent actions. Order and topology come into play by the idea of approximation. The property of a real trace to be a downwards closed subgraph (i.e. a subprocess) of another one yields a partial order called the *prefix order*. It turns out that each real trace can be approximated order-theoretically by all its finite prefixes. On the other hand, Kwiatkowska [38] defined the *prefix metric* of real traces: each real trace can also be approximated by finite traces with respect to the metric topology.

The prefix metric on traces is actually induced by a sequence of *projections*. The projections themselves are defined with the help of the *length function* of traces. A similar situation arises with regard to the *Foata normal form metric* introduced by Bonizzoni, Mauri, and Pighizzini [5]. This metric is equivalent to the prefix metric and it is also induced by projections. These projections are definable by the *height* of traces. There are other trace models (see Diekert and Gastin [12]) where ultrametrics are obtained in a similar way: again, projections are involved. Hence, a uniform view seems to be reasonable and desirable, and this is one goal of this monograph.

In fact, a first unifying approach was already given by Baier and Majster-Cederbaum [2, 40]. They considered posets with a *weight function*. This is a particular monotone map from the given poset into the set of non-negative integers with infinity. As in the concrete case of traces, the elements of the poset represent processes and the partial order is the subprocess relation. Intuitively, the weight of a process d is the maximal number of steps which is needed for an execution of d. A weight function gives rise to a sequence of projections (a *rank ordering*). The projections are used to approximate each element by its images. As for traces, they also induce an ultrametric. In [2] basic topological results are proven under the assumption of directed completeness of the partial order. Moreover, a characterization of *SFP-domains* by finitary rank orderings ([2]) provides a first link to domain theory.

SFP-domains were introduced by Plotkin [47]. They are algebraic domains with least element arising as limits of ω-chains of finite posets. Plotkin proved that the category of SFP-domains and Scott-continuous mappings is cartesian closed ([47]). Confirming a conjecture of Plotkin, Smyth [49] showed that this is in fact the largest cartesian closed category of algebraic domains with least element and countably many compact elements. *Bifinite domains* (also called *profinite* or *FB-domains*) are a natural generalization of SFP-domains. Here the conditions that there be a least element and only countably many compact elements are dropped. Still, they form a cartesian closed category (see Gunter [25], Jung [27]). By extending Smyth's result, Jung [27] determined all maximal cartesian closed categories of algebraic domains. The category of bifinite domains is one of them. A bifinite domain can be characterized as a directed complete partial order $(D, \leq)$ admitting a directed set $\mathcal{P}$ of Scott-continuous and image-finite projections such that $\sup_{p\in\mathcal{P}} p(d) = d$ for all $d \in D$ (cf. [25, 27]). Hence, a bifinite domain has a special approximating family of projections and, moreover, it is characterized by the existence of such. One should notice that it is not sufficient anymore to consider merely *sequences* of projections. Instead, *directed sets* or *nets* become necessary.

So far, we have seen that in several cases partially ordered sets are equipped with a directed, approximating family of projections. What about approximating elements with mappings that are not projections? In fact, there is a prominent class of continuous domains where elements are approximated in this manner. They are the *FS-domains* introduced by Jung [28]. In Jung's classification programme of all maximal cartesian closed categories of continuous domains ([27, 28]), they appear as such a category. Also, Jung and Sünderhauf characterized them as so-called "uniformly approximated spaces" (cf. [29]). FS-domains are continuous domains $(D, \leq)$ having a directed set $\mathcal{F}$ of special Scott-continuous mappings with $\sup_{f\in\mathcal{F}} f(d) = d$ for all $d \in D$. In general, the mappings in $\mathcal{F}$ need not be projections. This is one reason why we do not deal with projections only. Another reason is that the intrinsic topology of our structure is always zero-dimensional when induced by a family of projections. However, there are many important examples where the topology is not zero-dimensional.

Now we give a more detailed description of the contents of the present monograph. For the convenience of the reader we recall basic notation from order theory, general topology, and trace theory in Chapter 1.

Chapter 2 may be considered to form the most general part of this monograph. We give the definition of an *F-poset*, which is the mathematical structure we are interested in. An F-poset is simply a partially ordered set $(D, \leq)$ together with a directed set $\mathcal{F}$ of monotone mappings from D to itself such that each $f \in \mathcal{F}$ can be "separated" from the identity map by the square of some $g \in \mathcal{F}$, i.e. $f \leq g \circ g \leq \mathsf{id}_D$ with respect to the pointwise order. As we will see, Edalat and Heckmann's [16] formal ball model can be turned into such an F-poset.

The family $\mathcal{F}$ gives rise to a uniformity on D, which we call the *F-uniformity.* The uniform topology is the *F-topology*. We characterize all uniformities on a poset arising as F-uniformities and investigate basic properties of F-uniformity and F-topology. As we have already emphasized, the mappings in $\mathcal{F}$ will be used for approximation purposes. We therefore say that an F-poset $(D, \leq, \mathcal{F})$ is *approximating* if $\sup_{f \in \mathcal{F}} f(d) = d$ for all $d \in D$. It turns out that this order-theoretic condition is equivalent to saying that the poset is a partially ordered space in the F-topology. We shall mainly focus on approximating F-posets. Then a close relationship between the partial order and the F-topology can be established. We are interested in the question when suprema and infima of subsets $A \subseteq D$ exist and, moreover, when they are accumulation points of A with respect to the F-topology. Here continuous domains come into play. We show that each monotone net in D converges in the F-topology if and only if $(D, \leq)$ is a continuous domain such that $f(d)$ is "essentially" below d for all $f \in \mathcal{F}$ and all $d \in D$. In addition, a similar result holds concerning the convergence of monotone nets which are bounded. We give a domain-theoretic characterization of approximating F-posets that are compact in their F-topology. In this case, the F-topology coincides with the Lawson topology of the poset. Moreover, provided that they have a least element, these F-posets arise exactly from FS-domains. Part of these results can also be found in [33].

Chapter 3 introduces a special sort of F-posets involving projections. Therefore, we first collect some basic facts on projections. After that, we define a *partially ordered set with projections* (or *pop* for short) to be an F-poset $(D, \leq, \mathcal{P})$ where $\mathcal{P}$ consists of projections only. We investigate the *pop uniformity* and the *pop topology*, i.e. the F-uniformity and the F-topology of a pop, and give a list of examples. For instance, Flagg and Kopperman's [19] closed ball model carries such a pop structure. Moreover, we show how several domains of traces can be made into pop's. This enables us to apply our results in order to obtain a uniform proof for topological properties shared by all these trace models.

Coming back to the general theory, we show that approximating pop's are complete in their pop uniformity if and only if they can be represented as an inverse limit built up by the images of their projections. We resume the problem when the supremum or the infimum of a subset A exists in the closure of A. This leads to algebraic domains. Using the results of Chapter 2, we prove that each monotone net of an approximating pop $(D, \leq, \mathcal{P})$ converges in the pop topology if and only if $(D, \leq)$ is an algebraic domain with its compact elements being exactly the images of all projections of $\mathcal{P}$. Furthermore, we raise the question under which (order-theoretic) conditions an algebraic domain $(D, \leq)$ admits such a set $\mathcal{P}$. We then call $(D, \leq)$ a *P-domain.* Here posets satisfying the ascending chain condition (ACC) come into play. As we shall see, P-domains are exactly the inverse limits of posets with the ACC. Furthermore, we extend the well known "internal" description of bifinite domains (due to Plotkin [47]) to a characterization of P-domains. As for the bifinites, it employs a certain directed system of "complete" subsets of compact elements. Finally, we deduce that bifinite domains appear both as approximating pop's compact in their pop topology and as Lawson-compact P-domains. Some ideas of this chapter were first introduced in the extended abstract [32].

Chapter 4 considers homomorphisms and function spaces of posets with projections. In order to obtain a suitable notion of a homomorphism, we fix a directed index set $(I, \leq)$ and consider pop's whose projection sets are given by monotone nets indexed by $(I, \leq)$. Given two such $(I, \leq)$-*pop's*, for any $i \in I$ there is an ith projection in both of them. Hence, we are able to define a *homomorphism* to be a monotone map from one $(I, \leq)$-pop to another one that commutes with the ith projection for all $i \in I$. This is the obvious way to define homomorphisms and pursues a similar course to the notion of a projection morphism for *projection spaces* (Ehrig *et al.* [17], Große-Rhode [23, 24]) or *approximation structures* (Spreen [50]). The former occur in the area of algebraic specification. A projection space consists of a set together with a sequence of idempotent mappings ("projections") satisfying certain conditions. It dispenses with any order relation. However, the projections induce a canonical ultrametric. A projection morphism is a map that commutes with the given projections. Spreen's [50] approximation structure is a poset with a rank ordering in the sense of [2] with the additional requirements that the poset is a dI-domain and the projections are ideal-preserving stable maps. Here a projection morphism is a stable map commuting with all projections. Approximation structures are used to obtain models of the untyped λ-calculus ([50]).

Besides homomorphisms we also define *subpop's* of $(I, \leq)$-pop's and *weak homomorphisms*, which satisfy a weaker condition than homomorphisms. Then we deal with ω-*pop's*, i.e. $(\mathbb{N}_0, \leq)$-pop's. It turns out that there is a canonical partial order turning a projection space into an ω-pop. Each poset with a rank ordering is an ω-pop as well. Moreover, we describe all ω-pop's induced by a weight function. Weak homomorphisms and homomorphisms between ω-pop's can be characterized by means of a particular pseudo-ultrametric and the so-called *pseudo-weight*.

Taking up the topic of "function pop's", we equip the set of all [weak] homomorphisms between two indexed pop's with a natural pop structure. Its pop uniformity coincides with the uniformity of uniform convergence. After discussing the topological properties of such function spaces, we turn to the categorical aspect of cartesian closure. We obtain several cartesian closed categories of indexed pop's with respect to both sorts of morphisms. In case we deal with weak homomorphisms, this is straightforward because the exponential object coincides with the function space. The situation is different with regard to homomorphisms. Here the exponential object is larger: the function space embeds into it. We extend techniques developed by Herrlich and Ehrig [26]. They showed that the category of projection spaces and projection morphisms is cartesian closed. Using a similar argument, Spreen [50] proved that approximation structures also yield a cartesian closed category. Analogously to [50], we obtain $(I, \leq)$-pop's isomorphic to their own exponent by performing Scott's [48] D_∞-construction. This leads to new models of the untyped λ-calculus (cf. Barendregt [3]). The main results of Chapter 4 concerning homomorphisms also appear in the paper [35].

By an inverse limit construction Ehrig *et al.* [17] showed that there is a universal completion for projection spaces. Chapter 5 centres on the analogous problem for $(I, \leq)$-pop's. In fact, there are two different sorts of a universal completion. The first is the *pop completion.* It is related to the completion of uniform spaces. For any approximating $(I, \leq)$-pop there is another one that is complete in its pop uniformity and contains the former as a dense subpop. Moreover, it has a universal extension property concerning [weak] homomorphisms. In order to prove its existence and uniqueness, an inverse limit construction is suitable here as well. In what follows, we investigate order-theoretic properties of the pop completion and study the pop completion of function pop's.

The second completion is the *domain completion*. It is more related to the ideal completion of posets. Given any approximating $(I, \leq)$-pop, its domain completion is an approximating $(I, \leq)$-pop whose underlying poset is an algebraic domain and whose projections are Scott-continuous. The original $(I, \leq)$-pop is contained in it as a subpop (which is in general not dense). The domain completion, too, has a universal extension property with respect to [weak] homomorphisms. Here the extended maps are Scott-continuous.

A comparison of both completions shows us that the pop completion can always be embedded into the domain completion. Further, we discuss when both completions coincide. For instance, if any monotone net of the given $(I, \leq)$-pop is a Cauchy net, then its pop completion is equal to its domain completion. Especially, we have equality when we consider the closed ball model or the pop's occurring in trace theory. Note that our results extend some of Majster-Cederbaum and Baier's [40]. They carried out a comparison between the metric completion and the ideal completion of rank ordered posets induced by weight functions. A summary of our results of Chapter 5 can be found in [34].

As an application topic, the last chapter of this monograph is devoted to traces. We have already mentioned that real traces are built up by atomic actions. Given a finite set Σ of atomic actions, a (global) *dependence relation* D on Σ specifies whether two actions are dependent or not. The pair (Σ, D) is called a *dependence alphabet*. It is well known that real traces over some dependence alphabet (Σ, D) form an algebraic domain with respect to the prefix order. In fact, the domain is (isomorphic to) the ideal completion of the poset of finite traces (see Gastin and Petit [21, Section 11.3]). On the other hand, the set of real traces is the completion of the set of finite traces with respect to the prefix metric. Moreover, the metric topology is compact and coincides with the Lawson topology of the domain (Kwiatkowska [38]). The following question arises: how are Lawson topology and dependence alphabet related? We give a detailed answer by showing that the topology depends on two simple properties of the underlying dependence alphabet only. Furthermore, we obtain a topological representation of the space of real traces as a product space. Its factors are (at most) the space of all (finite or infinite) words over an alphabet of two letters and a finite power of the Aleksandrov one-point compactification of the non-negative integers. An extension of this result to countably infinite dependence alphabets has been obtained in [36] by using a tree-theoretic argument.

Finally, we turn to the domains of α-*traces* and of δ-*traces*. Collectively called *approximating traces*, these models were introduced by Diekert and Gastin [12] for the specification of non-terminating processes. Carrying a partial order (*approximation order*) and a canonical ultrametric, they have similarly favourable domain-theoretic and topological properties as real traces. Moreover, they admit a concatenation that is monotone with respect to the approximation order and continuous in the metric topology (cf. [12]). This is in contrast to real traces where the concatenation is only a partial operation which is neither monotone with respect to the prefix order nor continuous in the metric topology. We study topological properties of approximating traces by means of the underlying dependence alphabet. Provided the dependence relation is transitive, approximating traces turn out to be homeomorphic to product spaces having a similarly simple structure as the ones occurring in the characterization of the space of real traces.

1. PRELIMINARIES

This chapter fixes notation and briefly discusses some well known aspects concerning order and domain theory, general topology, and the theory of Mazurkiewicz traces. The presentation of these facts cannot be exhaustive; hence the reader is referred to the literature. For instance, basic order-theoretic notions can be found in Davey and Priestley [10]. An excellent survey on domain theory is given by Abramsky and Jung [1]. Concerning notions from general topology see e.g. Bourbaki [7], Engelking [18], Kelley [31], and Kuratowski [37]. For the definition of intrinsic topologies on posets the reader might wish to consult the compendium [22]. The detailed book by Diekert and Rozenberg [14] and the survey by Diekert and Métivier [13] may serve as references for the theory of Mazurkiewicz traces.

1.1. Basic notation from order theory

Let $(D, \leq)$ be a partially ordered set (poset). For elements $d, e \in D$ with $d \leq e$ we also write $e \geq d$. If $d \leq e$ and $d \neq e$, then we often denote this by $d < e$.

Let $A \subseteq D$ and let $d \in D$. Then d is an *upper bound* of A if $a \leq d$ for all $a \in A$. In this case we write $A \leq d$. A *bounded* subset of D is a non-empty set $A \subseteq D$ having some upper bound. A subset $A \subseteq D$ is *directed* provided that it is non-empty and for all $a, b \in A$ there is an element $c \in A$ with $\{a, b\} \leq c$. *Filtered* subsets are defined dually.

If $A \leq d$ and $A \leq e$ implies $d \leq e$ for any $e \in D$, then d is the *least upper bound* or *supremum* of A. We denote it by $\sup A$, $\sup_D A$, or $\sup_{a \in A} a$. Note that whenever we write $\sup A$ etc., then we implicitly assume the existence of the least upper bound of A. If A has a supremum which is an element of A itself, then it is the *greatest element* of A. It is denoted by $\max A$. *Lower bounds*, *greatest lower bounds* (*infima*), and *least elements* are defined dually. If A has a greatest lower bound [least element, respectively], then we denote it by $\inf A$ [$\min A$, respectively]. If D has a least element, then $(D, \leq)$ is said to be *pointed*.

Definition. Let $(D, \leq)$ be a poset.

(1) $(D, \leq)$ is a *directed complete partial order* or *dcpo* if each directed subset admits a supremum.
(2) $(D, \leq)$ is a *bounded complete partial order* or *bcpo* if each bounded subset has a supremum.

Let $(D, \leq)$ be a poset. An element $d \in D$ is called *way below* an element $e \in D$, denoted by $d \ll e$, if for all directed subsets $A \subseteq D$ with $\sup A \geq e$ there is some $a \in A$

with $a \geq d$. Let $d\downdownarrows$ denote the set of all elements of D way below d. Similarly, $d\upuparrows$ is the set of all $e \in D$ with $d \ll e$. If $d \ll d$, then d is said to be *compact*. Let $K(D)$ be the set of all compact elements of D.

DEFINITION. Let $(D, \leq)$ be a poset.

(1) A subset $B \subseteq D$ is an (*order-theoretic*) *basis* for $(D, \leq)$ if for all $d \in D$ the set $B \cap d\downdownarrows$ is directed and has d as supremum.
(2) $(D, \leq)$ is *continuous* if it has a basis.
(3) $(D, \leq)$ is *algebraic* if $K(D)$ is a basis for $(D, \leq)$.

A continuous dcpo is also called a *continuous domain*. Analogously, an *algebraic domain* is an algebraic dcpo.

Let $(D, \leq)$ be a poset. For $A \subseteq D$ let $A{\downarrow}_D := A{\downarrow} := \{d \in D \mid \exists a \in A : d \leq a\}$. A *lower set* is a subset $A \subseteq D$ with $A = A{\downarrow}$. We define $A{\uparrow}$ and *upper sets* dually. Further, we shorten $d{\downarrow} := \{d\}{\downarrow}$ and $d{\uparrow} := \{d\}{\uparrow}$ for any $d \in D$. An *ideal* is a directed, lower subset of D. Dually, a *filter* is a filtered, upper subset of D. In particular, ideals and filters are non-empty. Note that $d{\downarrow}$ is an ideal for all $d \in D$: the *principal ideal generated by* d. Similarly, $d{\uparrow}$ is the *principal filter generated by* d.

Let $\mathsf{Id}(D)$ be the set of all ideals of $(D, \leq)$. It is well known that $\mathsf{Id}(D)$ together with the inclusion forms an algebraic dcpo with $K(\mathsf{Id}(D)) = \{d{\downarrow} \mid d \in D\}$. It is called the *ideal completion* of $(D, \leq)$.

Let $(D, \leq)$ be a poset. Let $A \subseteq D$. An element $a \in A$ is called a *maximal* element of A if $a \leq b$ implies $a = b$ for all $b \in A$. We denote the set of all maximal elements of A by $\mathsf{Max}\, A$. *Minimal* elements of A are defined dually. The poset $(D, \leq)$ is said to be *well founded* provided that each non-empty subset of D contains some minimal element. Recall that $(D, \leq)$ is well founded if and only if it satisfies the *descending chain condition*, i.e. each decreasing sequence $(d_n)_{n \in \mathbb{N}}$ in D is eventually constant. In other words, there is no infinite strictly decreasing sequence of elements of D.

A partially ordered set $(D, \leq)$ is *linearly ordered* or a *chain* if $d \leq e$ or $d \geq e$ for all $d, e \in D$. A chain that is order-isomorphic to the set of natural numbers with the usual linear order is called an *ω-chain*.

For any subset M of a poset $(D, \leq)$ the restriction of $\leq$ to M is a partial order turning M itself into a poset, which we denote by $(M, \leq)$. We say that M is a (*sub*)*chain* of D if $(M, \leq)$ is a chain. Finally, a poset $(D, \leq)$ is a *tree* if each principal ideal of D is a chain. Note that we do not require a tree to have a least element.

Mappings between posets. For sets X and Y and a mapping $f : X \to Y$ the *kernel* of f is the set $\mathsf{ker}\, f := \{(x_1, x_2) \in X^2 \mid f(x_1) = f(x_2)\}$. The identity map of X is denoted by id_X. For any $M \subseteq X$ let $\mathsf{id}_{M,X} : M \to X$ denote the inclusion map.

Let $f : X \to X$. Then $\mathsf{fix}\, f := \{x \in X \mid f(x) = x\}$ is the set of all fixpoints of f. Recall that f is *idempotent* if $f \circ f = f$. This can be characterized as follows (the proof is straightforward):

1.1. LEMMA. *Let X be a set and let $f : X \to X$. Then f is idempotent if and only if $f[X] = \mathsf{fix}\, f$.* ■

If $(E, \leq)$ is a poset, then the set of all mappings from a set X to E can be turned into a poset by letting $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$ (with $f, g : X \to E$). This is the *pointwise order* of mappings.

Let $(D, \leq)$ and $(E, \leq)$ be posets and let $f : D \to E$. Then f is called *monotone* or *order-preserving* if $c \leq d$ implies $f(c) \leq f(d)$ for all $c, d \in D$. The following lemma is well known and will be used without further reference:

1.2. Lemma. *Let $(D, \leq)$ and $(E, \leq)$ be posets and let $f : D \to E$. Then f is monotone if and only if $f[A]{\downarrow} = f[A{\downarrow}]{\downarrow}$ for all subsets $A \subseteq D$.*

Proof. Clearly, $f[A]{\downarrow} \subseteq f[A{\downarrow}]{\downarrow}$ is always true and $f[A]{\downarrow} \supseteq f[A{\downarrow}]{\downarrow}$ holds whenever f is monotone. Conversely, let $c, d \in D$ with $c \leq d$ and suppose that $f(d){\downarrow} = f[d{\downarrow}]{\downarrow}$. Then $f(c) \in f[d{\downarrow}] \subseteq f[d{\downarrow}]{\downarrow} = f(d){\downarrow}$, whence $f(c) \leq f(d)$. ■

Conversely, if $f(c) \leq f(d)$ implies $c \leq d$ for all $c, d \in D$, then f is said to be *order-reflecting*. Recall that order-reflecting mappings are always injective. If f is both monotone and order-reflecting, then it is an *order embedding*. It is an *order isomorphism* if f is a surjective order embedding. For instance, the mapping $\varphi : D \to \mathsf{Id}(D)$ defined by $\varphi(d) := d{\downarrow}$ is an order embedding. If $(D, \leq)$ is an algebraic dcpo, then it is well known that the mapping $\xi : D \to \mathsf{Id}(K(D))$, $d \mapsto \{x \in K(D) \mid x \leq d\}$, is an order isomorphism.

A map $f : D \to E$ is called *Scott-continuous* if it preserves suprema of directed sets, that is, for all directed subsets $A \subseteq D$ having a supremum, the image $f[A]$ has a supremum and $f(\sup A) = \sup f[A]$. Recall that Scott-continuous mappings are in particular monotone. Clearly, order isomorphisms are Scott-continuous because they preserve all suprema and all infima.

We say that a mapping $f : D \to D$ is *below the identity* if $f \leq \mathsf{id}_D$. Moreover, f is a *projection* or a *kernel operator* if f is monotone, idempotent, and below the identity.

The following statement concerning suprema in the image of a projection is well known:

1.3. Lemma. *Let $(D, \leq)$ be a poset and let $p : D \to D$ be a projection. Let $A \subseteq p[D]$. Then $\sup_D A$ exists if and only if $\sup_{p[D]} A$ exists. In this case, $\sup_D A = p(\sup_D A) = \sup_{p[D]} A$.*

Proof. The assertion follows from Propositions 3.1.2 and 3.1.12.5 in [1] or from an easy calculation. ■

Embedding projection pairs. Let $(D, \leq)$ and $(E, \leq)$ be posets and let $f : D \to E$ and $g : E \to D$ be monotone mappings. Then (f, g) is an *embedding projection pair* or *epp* if $g \circ f = \mathsf{id}_D$ and $f \circ g \leq \mathsf{id}_E$. To simplify notation we write $(f, g) : D \to E$. Recall that f and g uniquely determine each other: $f(d) = \min g^{-1}[d{\uparrow}]$ for all $d \in D$ and $g(e) = \max f^{-1}[e{\downarrow}]$ for all $e \in E$ ([1, Prop. 3.1.10]). Furthermore, f is always Scott-continuous ([1, Prop. 3.1.12.5]), whereas g need not be. We say that an epp (f, g) is *Scott-continuous* provided that g is (and hence both f and g are) Scott-continuous. Recall that in this case we have $f[K(D)] \subseteq K(E)$.

Given any epp $(f, g) : D \to E$, the mapping $f \circ g$ is a projection of E. On the other hand, if $p : D \to D$ is a projection, then $(\mathsf{id}_{p[D],D}, p) : p[D] \to D$ is an epp.

Let $(\varGamma, \leq)$ be a directed index set and let $(D_\gamma, \leq_\gamma)_{\gamma \in \varGamma}$ be a family of posets. For all $\gamma, \mu \in \varGamma$ with $\gamma \leq \mu$ let $(f_{\gamma\mu}, g_{\gamma\mu}) : D_\gamma \to D_\mu$ be an embedding projection pair. We say that $\mathcal{S} = \{(f_{\gamma\mu}, g_{\gamma\mu}) \mid \gamma, \mu \in \varGamma,\ \gamma \leq \mu\}$ is an *inverse system (of epp's)* if $f_{\gamma\gamma} = g_{\gamma\gamma} = \mathsf{id}_{D_\gamma}$, $f_{\mu\nu} \circ f_{\gamma\mu} = f_{\gamma\nu}$, and $g_{\gamma\mu} \circ g_{\mu\nu} = g_{\gamma\nu}$ for all $\gamma \leq \mu \leq \nu$. If all epp's of the inverse system $\mathcal{S}$ are Scott-continuous, then we call $\mathcal{S}$ an *inverse system of Scott-continuous epp's*. Now let

$$D_\infty := \Big\{ (d_\gamma)_{\gamma \in \varGamma} \in \prod_{\gamma \in \varGamma} D_\gamma \,\Big|\, \forall \gamma, \mu \in \varGamma : \gamma \leq \mu \Rightarrow d_\gamma = g_{\gamma\mu}(d_\mu) \Big\}$$

and endow D_∞ with the product order, i.e. $(d_\gamma)_{\gamma \in \varGamma} \leq (e_\gamma)_{\gamma \in \varGamma}$ if and only if $d_\gamma \leq_\gamma e_\gamma$ for all $\gamma \in \varGamma$. Then we denote the *inverse limit* of $(D_\gamma, \leq_\gamma)_{\gamma \in \varGamma}$ with respect to $\mathcal{S}$ by

$$(\varprojlim_{\gamma \in \varGamma} (D_\gamma, \mathcal{S}), \leq) := (D_\infty, \leq).$$

For all $\gamma \in \varGamma$ let $f_\gamma : D_\gamma \to D_\infty$ be defined by $f_\gamma(d) := (d_\mu)_{\mu \in \varGamma}$ with $d_\mu := g_{\mu\nu}(f_{\gamma\nu}(d))$ for some $\nu \geq \gamma, \mu$. Notice that f_γ is well defined. Let $g_\gamma : D_\infty \to D_\gamma$ be the canonical projection from D_∞ onto D_γ, i.e. $g_\gamma((d_\mu)_{\mu \in \varGamma}) := d_\gamma$. Then we have the following lemma. A proof can be found e.g. in [27, p. 34 and p. 37]; cf. also [1, Theorem 3.3.7].

1.4. Lemma. *Let $\gamma \in \varGamma$. Then*

(1) $(f_\gamma, g_\gamma) : D_\gamma \to D_\infty$ *is an embedding projection pair. If all epp's in $\mathcal{S}$ are Scott-continuous, then (f_γ, g_γ) is Scott-continuous. If, in this case, $(D_\gamma, \leq_\gamma)$ is a dcpo for all $\gamma \in \varGamma$, then $(\varprojlim_{\gamma \in \varGamma} (D_\gamma, \mathcal{S}), \leq)$ is also a dcpo.*
(2) *For all $\mu \in \varGamma$ with $\mu \geq \gamma$ we have $f_\gamma = f_\mu \circ f_{\gamma\mu}$ and $g_\gamma = g_{\gamma\mu} \circ g_\mu$.*
(3) $\sup_{\gamma \in \varGamma} (f_\gamma \circ g_\gamma) = \mathsf{id}_{D_\infty}$. ∎

1.2. Topological and uniform spaces

We define topologies by means of open sets. Hence, whenever we say that τ is a topology on a set X, then τ is the family of open subsets of the topological space (X, τ). Let $\overline{A}$ denote the topological closure of a subset $A \subseteq X$. If not mentioned otherwise, no separation properties are assumed. For the convenience of the reader, we list some axioms of separation we need in what follows:

T_0: For every pair of distinct points $x, y \in X$ there exists an open set containing exactly one of these points.

T_1: For every pair of distinct points $x, y \in X$ there exists an open set $O \subseteq X$ such that $x \in O$ and $y \notin O$. Recall that this is equivalent to saying that each singleton of X is closed.

T_2 or *Hausdorff*: For every pair of distinct points $x, y \in X$ there exist open sets $O_1, O_2 \subseteq X$ such that $x \in O_1$, $y \in O_2$, and $O_1 \cap O_2 = \emptyset$.

completely regular: For all $x \in X$ and all closed subsets $F \subseteq X$ with $x \notin F$ there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \in F$.

Recall that we have the implications "$\mathrm{T}_2 \Rightarrow \mathrm{T}_1 \Rightarrow \mathrm{T}_0$" and "completely regular + $\mathrm{T}_0 \Rightarrow \mathrm{T}_2$". We note that we do not require a compact topological space to be Hausdorff.

A *zero-dimensional* space is a topological space having a basis of sets that are both open and closed. For any set X the *discrete topology* on X is the power set $\mathcal{P}(X)$ of X. We denote it by τ_{dis}. It is the finest topology on X. Clearly, (X, τ_{dis}) is zero-dimensional.

For any topological space (X, τ) let $\mathsf{isol}(X) := \{x \in X \mid \{x\} \text{ is open in } (X, \tau)\}$ be the set of all (*topologically*) *isolated* elements of X. For any ordinal number ξ the *ξth derivation* $X^{(\xi)}$ of (X, τ) is inductively defined as follows (cf. Kuratowski [37]):

$$X^{(\xi)} := X \setminus \bigcup_{\eta < \xi} \mathsf{isol}(X^{(\eta)}),$$

where $X^{(\xi)}$ carries the relative topology of X. In particular, $X^{(0)} = X$ and $X^{(1)} = X \setminus \mathsf{isol}(X)$. Clearly, $\eta \leq \xi$ implies $X^{(\eta)} \supseteq X^{(\xi)}$. The space (X, τ) is called *perfect* if $X = X^{(1)}$, i.e. $X = X^{(\xi)}$ for all ordinal numbers ξ. It is *scattered* provided that there is some ξ with $X^{(\xi)} = \emptyset$. For instance, (X, τ_{dis}) is scattered.

A *net* $(x_n)_{n \in N}$ in X is a mapping from a directed index set $(N, \leq)$ to X. It is said to *converge* to some element $x \in X$ if for all neighbourhoods U of x there is an index $n_U \in N$ such that $x_n \in U$ for all $n \geq n_U$. In this case we write $(x_n)_{n \in N} \to x$ and say that x is a *limit* (*point*) of $(x_n)_{n \in N}$. Recall that (X, τ) is Hausdorff if and only if every net in X has at most one limit. Hence, in Hausdorff spaces, limits are unique. Then we also write $\lim\,(x_n)_{n \in N} = x$ or $\lim_{n \in N} x_n = x$ if $(x_n)_{n \in N} \to x$.

Let (X, τ) and (Y, σ) be two topological spaces and let $f : X \to Y$ be a mapping. If $f : X \to Y$ is continuous, then we sometimes say that f is (τ, σ)-*continuous*. Recall that continuity of f can be characterized as follows. The map f is continuous at $x \in X$ if and only if $(x_n)_{n \in N} \to x$ implies $(f(x_n))_{n \in N} \to f(x)$ for all nets $(x_n)_{n \in N}$ in X. Moreover, f is said to be a *homeomorphism* provided that f is bijective and both f and f^{-1} are continuous. In this case, (X, τ) and (Y, σ) are *homeomorphic*.

Topologies on posets. On a poset $(D, \leq)$ several topologies can be defined; see e.g. the compendium [22] and Lawson [39]. The *upper topology* (or *weak topology*) of $(D, \leq)$ is generated by the subbasis $\{D \setminus d{\downarrow} \mid d \in D\}$. Dually, the *lower topology* (or *weak^d topology*) is generated by $\{D \setminus d{\uparrow} \mid d \in D\}$. The *interval topology* is the supremum of the upper and the lower topology. A subset $A \subseteq D$ is *Scott-closed* if it is a lower set closed under suprema of directed subsets of A (whenever these suprema exist). Complements of Scott-closed sets are *Scott-open*. The family of all Scott-open sets forms the *Scott topology* $\sigma_{(D,\leq)}$. Recall that a mapping $f : D \to E$ between two posets $(D, \leq)$ and $(E, \leq)$ is Scott-continuous if and only if it is $(\sigma_{(D,\leq)}, \sigma_{(E,\leq)})$-continuous. The *Lawson topology* $\lambda_{(D,\leq)}$ is the join of the Scott and lower topologies, i.e. it is generated by the subbasis $\{O \subseteq D \mid O \text{ Scott-open}\} \cup \{D \setminus d{\uparrow} \mid d \in D\}$. A map $f : D \to E$ is *Lawson-continuous* if it is $(\lambda_{(D,\leq)}, \lambda_{(E,\leq)})$-continuous.

It follows directly from the definitions that the upper topology is coarser than the Scott topology of $(D, \leq)$. Therefore, the interval topology is coarser than the Lawson topology. If $(D, \leq)$ is linearly ordered, then recall that the upper topology equals the

Scott topology of $(D, \leq)$; hence the interval topology coincides with the Lawson topology of $(D, \leq)$.

Uniform concepts. We define uniform spaces in the sense of A. Weil (cf. Bourbaki [7], Kelley [31]). To do this, we denote for each binary relation R on a set X the *inverse relation* by $R^{-1} := \{(y, x) \mid (x, y) \in R\}$. Given two binary relations R and S on X, the *product* of R and S is the relation $R \circ S := \{(x, z) \mid \exists y \in X : (x, y) \in S \text{ and } (y, z) \in R\}$. For each $x \in X$ let $R(x) := \{y \mid (x, y) \in R\}$.

Definition. A *uniformity* $\mathcal{U}$ on a set X is a filter of the poset $(\mathcal{P}(X \times X), \subseteq)$ satisfying the following axioms:

(U1) For all $U \in \mathcal{U}$ we have $\mathsf{id}_X \subseteq U$.
(U2) If $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$.
(U3) For each $U \in \mathcal{U}$ there exists some $V \in \mathcal{U}$ such that $V \circ V \subseteq U$.

The elements of $\mathcal{U}$ are called *entourages*. The pair $(X, \mathcal{U})$ is a *uniform space.*

Let X be a set. A *basis for a uniformity* on X is a $\subseteq$-filtered set $\mathcal{B}$ of subsets of $X \times X$ such that $\mathcal{B}{\uparrow}_{\mathcal{P}(X \times X)}$ is a uniformity on X. If $(X, \mathcal{U})$ is a uniform space and $\mathcal{B} \subseteq \mathcal{P}(X \times X)$ is filtered, then $\mathcal{B}$ is a *basis for* $\mathcal{U}$ provided that $\mathcal{U} = \mathcal{B}{\uparrow}_{\mathcal{P}(X \times X)}$.

Let $(X, \mathcal{U})$ be a uniform space. Then there is a unique topology τ on X such that, for each $x \in X$, the set $\{U(x) \mid U \in \mathcal{U}\}$ is the neighbourhood filter of x with respect to τ. It is the *topology induced by* $\mathcal{U}$ or the *uniform topology.* If $\mathcal{B}$ is a basis for $\mathcal{U}$, then the collection $\{B(x) \mid B \in \mathcal{B}\}$ is a basis for the τ-neighbourhood filter of x. Recall that (X, τ) is completely regular. Furthermore, it is Hausdorff if and only if $\bigcap \mathcal{U} = \mathsf{id}_X$.

Recall that for any subset $A \subseteq X$ of a uniform space $(X, \mathcal{U})$ the *relative uniformity* on A is given by $\mathcal{U}|_A := \{U \cap (A \times A) \mid U \in \mathcal{U}\}$. Clearly, it induces the relative topology $\tau|_A$ of the uniform topology τ.

For a set X the *discrete uniformity* is given by the basis $\{\mathsf{id}_X\}$. It is finer than all uniformities on X and is denoted by $\mathcal{U}_{\mathsf{dis}}$. Its induced topology is the discrete topology.

Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be uniform spaces and let $f : X \to Y$. Then f is said to be *uniformly continuous* or $(\mathcal{U}, \mathcal{V})$*-uniformly continuous* if for all $V \in \mathcal{V}$ there is some $U \in \mathcal{U}$ such that $(f \times f)[U] \subseteq V$. Clearly, uniformly continuous mappings are continuous with respect to the uniform topologies. Further, f is called a *uniform isomorphism* if f is bijective and both f and f^{-1} are uniformly continuous.

Let X be a set and let $(X_i, \mathcal{U}_i)_{i \in I}$ be a family of uniform spaces. For each $i \in I$ let $f_i : X \to X_i$ be a mapping. Then the *initial uniformity* of $((X_i, \mathcal{U}_i), f_i)_{i \in I}$ is the coarsest uniformity on X such that f_i is uniformly continuous for all $i \in I$. A basis for this uniformity is given by the collection of all sets of the form $\bigcap_{i \in I_0} (f_i \times f_i)^{-1}[U_i]$, where $U_i \in \mathcal{U}_i$ and $I_0 \subseteq I$ is finite. If τ_i denotes the topology induced by $\mathcal{U}_i$, then the uniform topology is the initial topology of $((X_i, \tau_i), f_i)_{i \in I}$, i.e. the coarsest topology such that f_i is continuous for all $i \in I$.

A *Cauchy net* in a uniform space $(X, \mathcal{U})$ is a net $(x_n)_{n \in N}$ in X such that for each $U \in \mathcal{U}$ there is an index $n_U \in N$ such that $(x_m, x_n) \in U$ for all $m, n \geq n_U$. A uniform space $(X, \mathcal{U})$ is said to be *complete* if each Cauchy net converges in the uniform topology.

The space is called *totally bounded* if for each $U \in \mathcal{U}$ there is a finite subset $M \subseteq X$ such that $X = \bigcup_{m \in M} U(m)$. Furthermore, $(X,\mathcal{U})$ is *compact* if X is compact in the uniform topology. Recall that $(X,\mathcal{U})$ is compact if and only if $(X,\mathcal{U})$ is complete and totally bounded. This important result will be used subsequently without being mentioned explicitly.

Uniform convergence. Let X be a set, let $(Y,\mathcal{V})$ be a uniform space, and let $F(X,Y)$ be the set of all mappings from X to Y. Let $\mathcal{A}$ be a set of subsets of X. Let $\mathcal{B}$ be a basis for $\mathcal{V}$. For all $M \subseteq X$ and all $B \in \mathcal{B}$ let

$$W(M,B) := \{(f,g) \in F(X,Y)^2 \mid \forall x \in M : (f(x),g(x)) \in B\}.$$

Then the set $\{W(\bigcup \mathcal{A}_0, B) \mid \mathcal{A}_0 \subseteq \mathcal{A} \text{ finite}, B \in \mathcal{B}\}$ is a basis for a uniformity on $F(X,Y)$. It is called the *uniformity of uniform convergence in the sets of* $\mathcal{A}$.

We consider two special cases.

(1) If $\mathcal{A} = \mathcal{P}_{\mathsf{fin}}(X)$ is the family of all finite subsets of X, then we obtain the *uniformity of pointwise convergence*. The induced topology, which is called the *topology of pointwise convergence*, coincides with the product topology on Y^X. A net in $F(X,Y)$ converging in this topology is said to *converge pointwise*.

(2) If $\mathcal{A} = \{X\}$, then the above-defined uniformity is the *uniformity of uniform convergence*. The induced topology is called the *topology of uniform convergence*. A net in $F(X,Y)$ that converges with respect to this topology is called *uniformly convergent*. Recall that for any set $\mathcal{A}'$ of subsets of X, the uniformity of uniform convergence in the sets of $\mathcal{A}'$ is coarser than the uniformity of uniform convergence. The same holds for the induced topologies.

Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be uniform spaces and let $H \subseteq F(X,Y)$. Then H is said to be *uniformly equicontinuous* if for each $V \in \mathcal{V}$ there is an entourage $U \in \mathcal{U}$ such that for all $f \in H$ we have $(f \times f)[U] \subseteq V$. Notice that then each element of H is in particular uniformly continuous.

(Ultra)metric spaces. Let X be a set and let $\varrho : X \times X \to \mathbb{R}_{\geq 0}$. Then ϱ is called a *pseudo-metric* on X if the following axioms are satisfied:

(D1) For all $x \in X$ we have $\varrho(x,x) = 0$.

(D2) For all $x,y \in X$ we have $\varrho(x,y) = \varrho(y,x)$.

(D3) (Triangle inequality) For all $x,y,z \in X$ we have $\varrho(x,z) \leq \varrho(x,y) + \varrho(y,z)$.

Furthermore, a pseudo-metric ϱ is a *pseudo-ultrametric* if

(D4) (Strong triangle inequality) For all $x,y,z \in X$ we have

$$\varrho(x,z) \leq \max\{\varrho(x,y), \varrho(y,z)\}.$$

A pseudo-metric ϱ is said to be a *metric* provided that

(D0) $\varrho(x,y) = 0$ implies $x = y$ for all $x,y \in X$.

An *ultrametric* is a pseudo-ultrametric satisfying (D0). The pair (X,ϱ) is called a *pseudo-metric*, *pseudo-ultrametric*, *metric*, or *ultrametric space*, respectively.

Let (X, ϱ) be a pseudo-metric space. Recall that the sets

$$\{(x, y) \in X^2 \mid \varrho(x, y) \leq \varepsilon\}, \qquad \varepsilon > 0,$$

form a basis for a uniformity $\mathcal{U}_\varrho$ on X. It is the *uniformity induced by* ϱ. A uniform space $(X, \mathcal{U})$ is *pseudo-metrizable* if there exists a pseudo-metric ϱ on X with $\mathcal{U} = \mathcal{U}_\varrho$. *Pseudo-ultrametrizable*, *metrizable*, and *ultrametrizable* are defined analogously. Recall that a uniform space is pseudo-metrizable if and only if it has a countable basis for its uniformity. Let τ_ϱ denote the topology induced by ϱ (i.e. induced by $\mathcal{U}_\varrho$).

Two pseudo-metrics ϱ_1 and ϱ_2 on a set X are said to be *uniformly equivalent* if $\mathcal{U}_{\varrho_1} = \mathcal{U}_{\varrho_2}$. Let (X, ϱ_X) and (Y, ϱ_Y) be pseudo-metric spaces. Let $f : X \to Y$. Then f is *metrically non-expansive* or (ϱ_X, ϱ_Y)*-non-expansive* if $\varrho_Y(f(x), f(y)) \leq \varrho_X(x, y)$ for all $x, y \in X$. The mapping f is called *isometric* or (ϱ_X, ϱ_Y)*-isometric* if $\varrho_Y(f(x), f(y)) = \varrho_X(x, y)$ for all $x, y \in X$. It is an *isometry* or (ϱ_X, ϱ_Y)*-isometry* if f is bijective and isometric.

In order to characterize the topology of traces (Chapter 6), we cite the following theorem on zero-dimensional, compact metric spaces, which is due to Pierce [46, Theorem 1.1]. I am thankful to U. Brehm, Dresden, for his hint to consider [46].

1.5. THEOREM (Pierce [46]). *Let (X, τ) and (Y, σ) be zero-dimensional, compact metrizable spaces. Let $A \subseteq X$ and $B \subseteq Y$ be closed subsets such that $X \setminus A$ and $Y \setminus B$ are scattered. Let $f : A \to B$ be a homeomorphism. Then f can be extended to a homeomorphism $\widetilde{f} : X \to Y$ if and only if $X \setminus A$ and $Y \setminus B$ are homeomorphic with respect to the relative topologies and $f[A \cap \overline{(X^{(\xi)} \setminus A)}\,] = B \cap \overline{(Y^{(\xi)} \setminus B)}$ for all ordinal numbers ξ.* ■

Let us make a few remarks with regard to the previous theorem. A topological space (X, τ) is *separable* provided that there exists a countable and dense subset of X. Recall that a pseudo-metrizable space is separable if and only if it has a countable basis for its topology. For any separable pseudo-metrizable space (X, τ) we know that each scattered subspace of X is countable and there is a countable ordinal number η with $X^{(\eta)} = X^{(\xi)}$ whenever $\eta \leq \xi$ (cf. Kuratowski [37, §23.V and §24.IV]). Since compact (pseudo-) metrizable spaces are separable, these remarks apply to the spaces in Theorem 1.5. Hence, only countable ordinals come into play.

We will apply the following corollary of Pierce's Theorem in Chapter 6:

1.6. COROLLARY. *Let (X, τ) and (Y, σ) be compact ultrametrizable spaces such that*

(1) $\mathsf{isol}(X)$ *is dense in* X *and* $\mathsf{isol}(Y)$ *is dense in* Y;
(2) $|\mathsf{isol}(X)| = |\mathsf{isol}(Y)|$;
(3) $X \setminus \mathsf{isol}(X)$ *and* $Y \setminus \mathsf{isol}(Y)$ *are homeomorphic.*

Then X and Y are homeomorphic.

Proof. Note first that as (X, τ) and (Y, σ) are ultrametrizable, the induced topological spaces are zero-dimensional. Next, let $A := X \setminus \mathsf{isol}(X)$ and let $B := Y \setminus \mathsf{isol}(Y)$. Clearly, A and B are closed, and $X \setminus A = \mathsf{isol}(X)$ and $Y \setminus B = \mathsf{isol}(Y)$ are scattered. By condition (3), A and B are homeomorphic. Moreover, the discrete spaces $X \setminus A$ and $Y \setminus B$ are also homeomorphic because of condition (2). We have $A \cap \overline{(X^{(0)} \setminus A)} = A \cap \overline{\mathsf{isol}(X)} = A$ by (1). Analogously, $B \cap \overline{(Y^{(0)} \setminus B)} = B$. Further, $A \cap \overline{(X^{(1)} \setminus A)} = A \cap \overline{((X \setminus \mathsf{isol}(X)) \setminus A)} =$

$A \cap \overline{\bar{\emptyset}} = \emptyset$ and, likewise, $B \cap \overline{(Y^{(1)} \setminus Y)} = \emptyset$. This implies that $A \cap \overline{(X^{(\xi)} \setminus A)} = \emptyset = B \cap \overline{(Y^{(\xi)} \setminus B)}$ for all ordinals $\xi \geq 1$. Theorem 1.5 tells us that X and Y are homeomorphic. ■

1.3. Dependence graphs and traces

Here we briefly recapitulate the fundamental notions of (finite and infinite) traces. For a thorough treatment we refer the reader to the exhaustive book [14] by Diekert and Rozenberg, especially to Chapter 11 on infinite traces ([21]). As to approximating traces, the reader should consult Diekert and Gastin [12].

Mazurkiewicz traces serve as a semantic model for concurrent processes. They can be seen as a generalization of words. While words can only describe sequential processes, a trace can model the concurrent execution of actions that are independent of each other. The independence of two actions is given globally, i.e. it does not depend on the actual state of the system. The process itself is a special acyclic directed graph whose vertices are labelled by the actions of a given alphabet.

After recalling basic notation concerning words, we give the definition of a dependence relation (independence relation, respectively). It determines when two actions are dependent (independent, respectively). Then we define dependence graphs as the most general concept in the theory of (infinite) traces. The notions of a real trace and a finite trace follow. We present some order-theoretic and some topological aspects of these traces. Finally, we give a very brief introduction to so-called approximating traces.

Let Σ be a finite set. Its elements are called *letters*. As usual, let $\Sigma^\star$ denote the set of all finite words over Σ. Furthermore, Σ^ω is the set of all infinite words over Σ. Let $\Sigma^\infty := \Sigma^\star \cup \Sigma^\omega$. For any $x \in \Sigma^\star$ and $y \in \Sigma^\infty$ we denote the multiplication (i.e. concatenation) of x and y by xy. An element $x \in \Sigma^\infty$ is a *prefix* of $y \in \Sigma^\infty$, denoted by $x \leq y$, if either $x = y$ or $x \in \Sigma^\star$ and $xz = y$ for some $z \in \Sigma^\infty$. For any words x and y, the *greatest common prefix* $\mathrm{gcp}(x, y)$ exists. The *prefix metric* d_{pref} on Σ^∞ is defined by $d_{\mathrm{pref}}(x, y) := 2^{-|\mathrm{gcp}(x,y)|}$ if $x \neq y$ and $d_{\mathrm{pref}}(x, x) := 0$. Thus, the longer the common prefix of two different words is, the closer they are in the prefix metric.

A *dependence relation* is a reflexive, symmetric relation on a finite set Σ. The pair (Σ, D) is called a *dependence alphabet* if $D \subseteq \Sigma \times \Sigma$ is a dependence relation. The (irreflexive and symmetric) relation $I_D := (\Sigma \times \Sigma) \setminus D$ is the *independence relation induced by* D. There is a conventional graphical representation of dependence alphabets (Σ, D) as undirected graphs without loops. The vertices are the elements of Σ. Two different elements $a, b \in \Sigma$ are connected by an edge if and only if $(a, b) \in D$. For instance, the graphical representation of $(\Sigma, D) = (\{a, b, c\}, \{(a, b), (b, a), (b, c), (c, b)\} \cup \mathsf{id}_{\{a,b,c\}})$ is a—b—c. The meaning is that a and b as well as b and c are dependent on each other, but a and c are independent.

We adopt the definition of dependence graphs from Gastin and Petit [21, Def. 11.2.1] (cf. also Diekert [11, Section 5]). A *dependence graph* $[V, E, \lambda]$ over the dependence alphabet (Σ, D) is an isomorphism class of a node-labelled graph (V, E, λ) such that (V, E) is a directed acyclic graph, V is at most countably infinite, $\lambda : V \to \Sigma$ is the labelling map, and such that the following hold:

(a) Edges between dependent vertices: $(\lambda(v), \lambda(w)) \in D \Leftrightarrow (v, w) \in \mathsf{id}_V \cup E \cup E^{-1}$ for all $v, w \in V$.
(b) The reflexive and transitive closure E^* of the edge relation E is well founded.

Notice that since (V, E) is acyclic, E^* is a partial order on V; hence (V, E^*) is a well founded poset.

For all dependence graphs $g = [V, E, \lambda]$ and all elements $a \in \Sigma$ let $|g|_a$ be the ordinal number associated with the well ordered set $\lambda^{-1}[a]$ (where $\lambda^{-1}[a]$ is endowed with the induced order of (V, E^*)). This ordinal number is called the *number of occurrences of a in g*. Then g admits a *standard representation* (V_g, E_g, λ_g) by defining V_g, E_g, and λ_g as follows. Let $V_g := \{(a, i) \mid a \in \Sigma,\ 0 \le i < |g|_a\}$. Let $v \in V$, $a := \lambda(v)$. Let i be the ordinal number associated with the well ordered set $v{\downarrow}_{E^*} \cap \lambda^{-1}[a]$. Define $f(v) := (a, i)$. This yields a bijection $f : V \to V_g$. Now let $E_g := (f \times f)[E]$ and let $\lambda_g(a, i) = a$ for all $(a, i) \in V_g$. Then f is an isomorphism from (V, E, λ) onto (V_g, E_g, λ_g), hence $g = [V_g, E_g, \lambda_g]$.

As in [21, Def. 11.2.4] we define the multiplication (or concatenation) of two dependence graphs $g_1 = [V_1, E_1, \lambda_1]$ and $g_2 = [V_2, E_2, \lambda_2]$ to be the dependence graph $g_1 g_2 := g_1 \cdot g_2 := [V, E, \lambda]$ with

$$\begin{aligned} V &= V_1 \mathbin{\dot\cup} V_2, \\ E &= E_1 \mathbin{\dot\cup} E_2 \mathbin{\dot\cup} \{(v_1, v_2) \in V_1 \times V_2 \mid (\lambda_1(v_1), \lambda_2(v_2)) \in D\}, \\ \lambda &= \lambda_1 \mathbin{\dot\cup} \lambda_2. \end{aligned}$$

Let $\mathbb{G}(\Sigma, D)$ denote the set of all dependence graphs over (Σ, D). Then $(\mathbb{G}(\Sigma, D), \cdot)$ is a monoid with $\varepsilon := [\emptyset, \emptyset, \emptyset]$ as neutral element.

The *prefix order* on $\mathbb{G}(\Sigma, D)$ is defined by $g \le h$ if there is some $z \in \mathbb{G}(\Sigma, D)$ such that $g \cdot z = h$. This is equivalent to saying that g is a downwards closed subgraph of h (cf. [21, Prop. 11.2.5]). The corresponding suffix z is unique and denoted by $g^{-1}h := z$ (see [21, Rem. 11.2.9]).

A *real trace* over the dependence alphabet (Σ, D) is a dependence graph $[V, E, \lambda]$ such that $\{u \in V \mid (u, v) \in E^*\}$ is finite for all $v \in V$, i.e. each principal ideal in (V, E^*) is finite. The set of real traces over (Σ, D) is denoted by $\mathbb{R}(\Sigma, D)$. The *alphabet* $\mathsf{alph}(t)$ of a real trace $t = [V, E, \lambda]$ is the set $\lambda[V]$. The *alphabet at infinity* of t is the set $\mathsf{alphinf}(t) := \{a \in \Sigma \mid \lambda^{-1}[a] \text{ is infinite}\}$ of all $a \in \mathsf{alph}(t)$ occurring infinitely often in t. *Finite traces* are real traces with finitely many vertices. Let $\mathbb{M}(\Sigma, D)$ denote the set of all finite traces. Recall that $\mathbb{M}(\Sigma, D)$ yields a submonoid of $\mathbb{G}(\Sigma, D)$, whereas $\mathbb{R}(\Sigma, D)$ does not. In fact, the concatenation $s \cdot t$ of two real traces s and t is a real trace if and only if $\mathsf{alphinf}(s) \times \mathsf{alph}(t) \subseteq I_D$.

We remark here that there is also an "algebraic definition" of $\mathbb{M}(\Sigma, D)$. The monoid $(\mathbb{M}(\Sigma, D), \cdot)$ is isomorphic to the quotient of the free monoid $(\Sigma^\star, \cdot)$ by the least congruence relation containing $\{(ab, ba) \mid (a, b) \in I_D\}$. In the present monograph we will not make use of this property.

Figures 1.1–1.3 illustrate some dependence graphs over the dependence alphabet a—b—c. Figure 1.1 actually contains two pictures. On the left hand side, one can find the graphical representation of a finite trace having six vertices. However, it is often not suitable to draw all arrows (edges) in our illustrations. We omit those which can be easily derived from the given ones. In case of finite traces $[V, E, \lambda]$ this means that we draw the Hasse diagram of the labelled partial order (V, E^*, λ). For instance, in Figure 1.1 the picture on the right hand side is the Hasse diagram of the labelled partial order that is induced by the trace depicted on the left. For infinite traces and their induced labelled partial orders, we cannot speak of a Hasse diagram anymore. But, anyway, since we can draw only a finite part of an infinite trace, this finite part will be "a kind of Hasse diagram".

Fig. 1.1. A finite trace and its illustration by a Hasse diagram

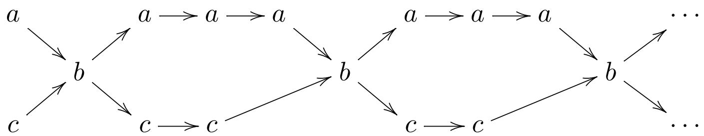


Fig. 1.2. An infinite real trace (redundant edges omitted)

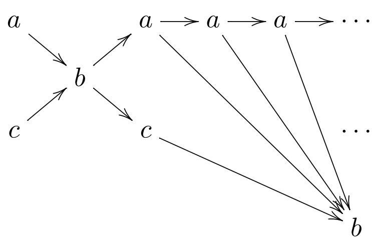


Fig. 1.3. A dependence graph that is not a real trace (redundant edges omitted)

The next theorem can be found in Gastin and Petit [21, Theorems 11.3.2 and 11.3.11 and Corollary 11.3.6].

1.7. Theorem (cf. Gastin–Petit [21]). *Let (Σ, D) be a dependence alphabet. Then*

(1) $(\mathbb{G}(\Sigma, D), \leq)$ *is a bcpo with least element ε.*
(2) $(\mathbb{R}(\Sigma, D), \leq)$ *is an algebraic dcpo with $K(\mathbb{R}(\Sigma, D)) = \mathbb{M}(\Sigma, D)$.*
(3) $(\mathbb{R}(\Sigma, D), \leq)$ *is a bcpo.*

Either let A be a bounded subset of $\mathbb{G}(\Sigma, D)$ or let A be a directed or bounded subset of $\mathbb{R}(\Sigma, D)$. Consider the traces $t = [V_t, E_t, \lambda_t]$ in A given in standard representation (V_t, E_t, λ_t). Then $\sup A = [V, E, \lambda]$ *with $V = \bigcup_{t\in A} V_t$, $E = \bigcup_{t\in A} E_t$, and $\lambda(a, i) = a$ for all $(a, i) \in V$.* ∎

For all $t \in \mathbb{M}(\Sigma, D)$ let $|t|$ be the number of vertices of the finite trace t. We set $|t| := \infty$ if $t \in \mathbb{R}(\Sigma, D) \setminus \mathbb{M}(\Sigma, D)$. Then $|t|$ is called the *length* of t.

As a generalization from words to traces, Kwiatkowska [38] defines an ultrametric on $\mathbb{R}(\Sigma, D)$ inducing the Lawson topology of $(\mathbb{R}(\Sigma, D), \leq)$. This is the *prefix metric* on traces, which is defined as follows (cf. also [21, Section 11.5.3]):

$$\ell_{\text{pref}}(s, t) := \sup\{n \in \mathbb{N}_0 \mid p \leq s \Leftrightarrow p \leq t \text{ for all } p \in \mathbb{M}(\Sigma, D) \text{ with } |p| \leq n\},$$
$$d_{\text{pref}}(s, t) := 2^{-\ell_{\text{pref}}(s,t)} \quad (\text{where } 2^{-\infty} := 0).$$

It is well known that $\mathbb{M}(\Sigma, D)$ is an open and discrete subspace of $(\mathbb{R}(\Sigma, D), \tau_{d_{\text{pref}}})$. This follows from the fact that $\ell_{\text{pref}}(s, t) \leq |t|$ and thus $d_{\text{pref}}(s, t) \geq 2^{-|t|}$ for all $s \in \mathbb{R}(\Sigma, D)$, $t \in \mathbb{M}(\Sigma, D)$, $s \neq t$. Moreover, we have the following properties:

1.8. THEOREM (Kwiatkowska [38]). *Let (Σ, D) be a dependence alphabet. Then $(\mathbb{R}(\Sigma, D), d_{\text{pref}})$ is a compact ultrametric space. The set $\mathbb{M}(\Sigma, D)$ is dense in $\mathbb{R}(\Sigma, D)$ with respect to the metric topology. The metric topology coincides with the Lawson topology of $(\mathbb{R}(\Sigma, D), \leq)$.* ∎

Consider the special dependence alphabet (Σ, Σ^2), i.e. any two elements of Σ are dependent. For each finite word $a_1 \cdots a_n \in \Sigma^\star$ let $V := \{1, \ldots, n\}$, let $E := \{(i, j) \mid 1 \leq i \leq j \leq n\}$, let $\lambda(i) := a_i$ for all $i \in V$, and let $\varphi(a_1 \cdots a_n) := [V, E, \lambda]$. Similarly, for any infinite word $y = a_1 a_2 \cdots \in \Sigma^\omega$ let $V := \mathbb{N}$, let $E := \{(i, j) \mid 1 \leq i \leq j\}$, let $\lambda(i) := a_i$ for all $i \in V$, and let $\varphi(y) := [V, E, \lambda]$. This leads to a bijection $\varphi : \Sigma^\infty \to \mathbb{R}(\Sigma, \Sigma^2)$ with $\varphi[\Sigma^\star] = \mathbb{M}(\Sigma, \Sigma^2)$. It is compatible with the multiplication $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x \in \Sigma^\star$, $y \in \Sigma^\infty$. In particular, φ is an order isomorphism with respect to the prefix orders. For all $x, y \in \Sigma^\infty$ we have $|\text{gcp}(x, y)| = \ell_{\text{pref}}(\varphi(x), \varphi(y))$; hence φ is also an isometry with respect to the prefix metrics. That is why the prefix metric on real traces is a generalization of the prefix metric on words. In light of these remarks, we shall identify $(\Sigma^\infty, \leq)$ with $(\mathbb{R}(\Sigma, \Sigma^2), \leq)$ and $(\Sigma^\infty, d_{\text{pref}})$ with $(\mathbb{R}(\Sigma, \Sigma^2), d_{\text{pref}})$. We thus write $\Sigma^\infty = \mathbb{R}(\Sigma, \Sigma^2)$ and $\Sigma^\star = \mathbb{M}(\Sigma, \Sigma^2)$.

Similarly, for an arbitrary dependence alphabet (Σ, D) and any letter $a \in \Sigma$, the finite words a^n with $n \in \mathbb{N}_0$ (a^0 is the empty word) and the infinite word a^ω correspond to real traces $[V, E, \lambda]$ with V, E, and λ being defined analogously as above. For the sake of simplicity, we denote these traces by a^n and a^ω as well. We will use this notation in Chapter 6.

Approximating traces. We recall the definition of α- and δ-traces from Diekert and Gastin [12]. Let (Σ, D) be a dependence alphabet. An *α-trace* over (Σ, D) is a pair (r, A) with $r \in \mathbb{R}(\Sigma, D)$, $A \subseteq \Sigma$, and $\mathsf{alphinf}(r) \subseteq A$. The set of all α-traces is denoted by $\mathbb{F}^\alpha(\Sigma, D)$. A *finite* α-trace is an α-trace (r, A) with $r \in \mathbb{M}(\Sigma, D)$. Note that $\mathsf{alphinf}(r) = \emptyset$ for all $r \in \mathbb{M}(\Sigma, D)$. Let $\mathbb{F}^\alpha_{\text{f}}(\Sigma, D)$ be the set of all finite α-traces over (Σ, D).

The idea ([12]) to study α-traces is to use them as a description for concurrent, possibly non-terminated processes. Given an α-trace (r, A), the first component r (the *real part*) represents that part of a process which has already been executed. The second component (the *imaginary part*) is not visible. It contains the actions that the process is able to perform in the future.

The *approximation order* $\sqsubseteq$ on $\mathbb{F}^\alpha(\Sigma, D)$ is defined as follows. Let $x = (r, A)$ and $y = (s, B)$ be α-traces. Then $x \sqsubseteq y$ provided that $r \leq s$ and $B \cup \mathsf{alph}(r^{-1}s) \subseteq A$. Intuitively, $x \sqsubseteq y$ means that y contains more information about the actual process than x. The observable information is greater $(r \leq s)$, actions in $r^{-1}s$ have to be taken from A, and the imaginary part of y is possibly smaller $(B \subseteq A)$. See [12] for more details.

We summarize some poset properties of α-traces; cf. Theorems 2 and 3 in [12].

1.9. Theorem (Diekert–Gastin [12]). *Let (Σ, D) be a dependence alphabet. Then*

(1) *$(\mathbb{F}^\alpha(\Sigma, D), \sqsubseteq)$ is an algebraic dcpo with $K(\mathbb{F}^\alpha(\Sigma, D)) = \mathbb{F}^\alpha_{\mathrm{f}}(\Sigma, D)$.*
(2) *$(\mathbb{F}^\alpha(\Sigma, D), \sqsubseteq)$ is a bcpo with least element (ε, Σ).*

For any directed or bounded subset $X \subseteq \mathbb{F}^\alpha(\Sigma, D)$ we have $\sup X = (s, B)$ with $s = \sup\{r \in \mathbb{R}(\Sigma, D) \mid \exists A \subseteq \Sigma : (r, A) \in X\}$ and $B = \bigcap\{A \subseteq \Sigma \mid \exists r \in \mathbb{R}(\Sigma, D) : (r, A) \in X\}$. ∎

For any α-trace $x = (r, A)$ let $|x| := |r|$ be the *length* of x. For all $n \in \mathbb{N}_0$ let $x[n] := \sup\{p \in \mathbb{F}^\alpha_{\mathrm{f}}(\Sigma, D) \mid p \sqsubseteq x \text{ and } |p| \leq n\}$. Diekert and Gastin [12] define an ultrametric on $\mathbb{F}^\alpha(\Sigma, D)$ as follows:

$$\ell(x, y) := \sup\{n \in \mathbb{N}_0 \mid x[n] = y[n]\}, \qquad d(x, y) := 2^{-\ell(x,y)}.$$

Proposition 9 and Theorem 5 in [12] yield

1.10. Theorem (Diekert–Gastin [12]). *Let (Σ, D) be a dependence alphabet. Then $(\mathbb{F}^\alpha(\Sigma, D), d)$ is a compact ultrametric space. The set $\mathbb{F}^\alpha_{\mathrm{f}}(\Sigma, D)$ is dense, discrete, and open in $(\mathbb{F}^\alpha(\Sigma, D), \tau_d)$. The metric topology is the Lawson topology of $(\mathbb{F}^\alpha(\Sigma, D), \sqsubseteq)$.* ∎

We remark here that one can define a multiplication $\cdot$ on $\mathbb{F}^\alpha(\Sigma, D)$ such that $(\mathbb{F}^\alpha(\Sigma, D), \cdot)$ is a monoid with neutral element $(\varepsilon, \emptyset)$ and $\cdot$ is $\sqsubseteq$-monotone and uniformly continuous with respect to the ultrametric defined above (cf. [12] for details).

Finally, we give a very brief summary on δ-traces. For a subset $A \subseteq \Sigma$ let $D(A) := \{b \in \Sigma \mid \exists a \in A : (a, b) \in D\}$ be the set of elements dependent on some letter of A. A *δ-trace* over (Σ, D) is a pair $(r, D(A))$ with $r \in \mathbb{R}(\Sigma, D)$, $A \subseteq \Sigma$, and $D(\mathsf{alphinf}(r)) \subseteq D(A)$. Again, $(r, D(A))$ is *finite* if $r \in \mathbb{M}(\Sigma, D)$. Let $\mathbb{F}^\delta(\Sigma, D)$ be the set of all δ-traces and let $\mathbb{F}^\delta_{\mathrm{f}}(\Sigma, D)$ be the set of all finite δ-traces. Let $x = (r, D(A))$ and $y = (s, D(B))$ be δ-traces. Then define $x \sqsubseteq y$ if $r \leq s$ and $D(B \cup \mathsf{alph}(r^{-1}s)) \subseteq D(A)$. Similarly to Theorem 1.9, $(\mathbb{F}^\delta(\Sigma, D), \sqsubseteq)$ is an algebraic dcpo with $K(\mathbb{F}^\delta(\Sigma, D)) = \mathbb{F}^\delta_{\mathrm{f}}(\Sigma, D)$ and a bcpo (cf. [12, Section 6]). Analogously to α-traces, Diekert and Gastin [12] define an ultrametric on $\mathbb{F}^\delta(\Sigma, D)$ with respect to which $\mathbb{F}^\delta(\Sigma, D)$ becomes a compact ultrametric space whose induced topology is the Lawson topology of $(\mathbb{F}^\delta(\Sigma, D), \sqsubseteq)$. The set of all finite δ-traces is dense, discrete, and open in $\mathbb{F}^\delta(\Sigma, D)$. The details can be found in [12].

2. CONTINUOUS DOMAINS VIA APPROXIMATING MAPPINGS

In this chapter we introduce the notion of an *F-poset*. Roughly speaking, an F-poset is a partially ordered set $(D, \leq)$ together with a directed family $\mathcal{F}$ of monotone mappings below the identity of D satisfying some additional condition. In case we have $\sup_{f \in \mathcal{F}} f(d) = d$ for all $d \in D$, we call the F-poset *approximating*. The family $\mathcal{F}$ defines a uniformity and thus a topology on D. We analyse the uniform structure and investigate the interplay of order and topology.

Section 2.1 deals with the basic properties of F-posets. After having defined F-posets, we study the uniformity induced by $\mathcal{F}$ (*F-uniformity*) and the uniform topology (*F-topology*). Several examples ranging from C*-algebras to the formal ball model of metric spaces illustrate F-posets to be a natural mathematical structure.

In Section 2.2 we relate order-theoretic properties of approximating F-posets such as directed completeness and order continuity to uniform completeness and convergence properties of nets. To do this, we investigate under which topological conditions suprema exist. The main result of this section (Theorem 2.39) states for any approximating F-poset $(D, \leq, \mathcal{F})$ that $(D, \leq)$ is a continuous dcpo with $f(d) \ll d$ for all $f \in \mathcal{F}$ and all $d \in D$ if and only if each monotone net in D converges in the F-topology. We derive a similar result (Theorem 2.43) for *local* dcpo's (i.e. each bounded directed subset has a supremum) and bounded monotone nets.

Section 2.3 is devoted to compactness. We characterize an approximating F-poset $(D, \leq, \mathcal{F})$ to be compact in its F-topology if and only if $(D, \leq)$ is a continuous dcpo such that for all $f \in \mathcal{F}$ and all $d \in D$ we have $f(d) \ll d$ and f is finitely separated from the identity of D (cf. Theorem 2.47). In this case, the F-topology coincides with the Lawson topology of $(D, \leq)$ (Corollary 2.49). Finally, we show that compact approximating F-posets with least element can be characterized as special continuous dcpo's well known in domain theory, viz. FS-domains (see Corollary 2.51).

Part of the contents of the present chapter can be found in [33]. I am grateful to A. Jung, Birmingham, who suggested extending the notion of a "pop" (see Chapter 3) to an F-poset.

2.1. Definition and basic properties of F-posets

Let $(D, \leq)$ be a poset and let $f : D \to D$ be a mapping. We define the set $B_f := \{(d, e) \in D^2 \mid f(d) \leq e,\ f(e) \leq d\}$. Clearly, B_f is a symmetric binary relation on D. Recall that $B_f(d) = \{e \in D \mid f(d) \leq e,\ f(e) \leq d\}$ for all $d \in D$. We shall mainly consider sets B_f

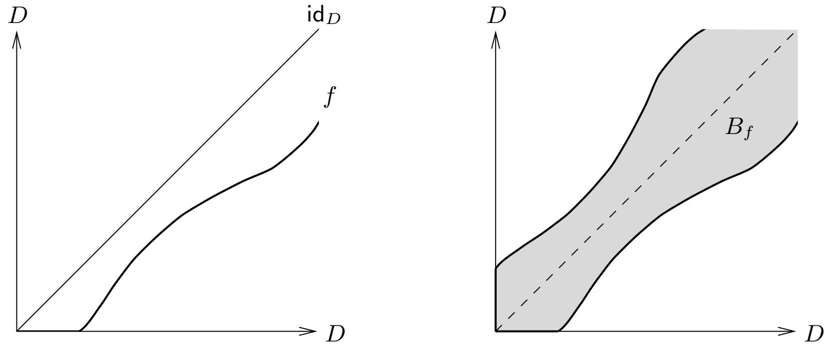


Fig. 2.1. The relation B_f for a monotone mapping $f \leq \mathsf{id}_D$

with f being a monotone mapping below the identity. Then B_f can be illustrated as in Figure 2.1.

2.1. Lemma. *Let $f : D \to D$ be a mapping and let $d \in D$. Then we have*

(1) $B_f(d) = f(d)\!\uparrow \cap f^{-1}[d\!\downarrow]$.
(2) *If f is below the identity, then $(d, f(d)) \in B_f$ and $f(d) = \min B_f(d)$.*
(3) *f is below the identity if and only if B_f is reflexive if and only if $\mathsf{ker}\, f \subseteq B_f$.*
(4) *If f is monotone and idempotent, then $B_f \subseteq \mathsf{ker}\, f$.*

Proof. (1) follows directly from the definition of $B_f(d)$.

(2) As $f(d) \leq f(d)$ and $f(f(d)) \leq f(d) \leq d$, we obtain $(d, f(d)) \in B_f$ and $f(d) \in B_f(d)$. It follows from (1) that $f(d) \leq B_f(d)$.

(3) Obviously, f is below the identity if and only if B_f is reflexive. If $f \leq \mathsf{id}_D$ and $f(d) = f(e)$, then $f(d) \leq e$ and $f(e) \leq d$, whence $(d, e) \in B_f$. Thus, $\mathsf{ker}\, f \subseteq B_f$. Conversely, if $\mathsf{ker}\, f \subseteq B_f$, then B_f is reflexive.

(4) Let $f(d) \leq e$ and $f(e) \leq d$. Then $f(d) = f(f(d)) \leq f(e) = f(f(e)) \leq f(d)$. Hence, $f(d) = f(e)$. ■

2.2. Corollary. *Let $f, g : D \to D$ be below the identity. Then $f \leq g$ if and only if $B_f \supseteq B_g$. In particular, f is uniquely determined by B_f; that is, $f = g$ if and only if $B_f = B_g$.*

Proof. First let $f \leq g$ and let $(d, e) \in B_g$. Then $f(d) \leq g(d) \leq e$ and $f(e) \leq g(e) \leq d$, i.e. $(d, e) \in B_f$. Now let $B_f \supseteq B_g$ and let $d \in D$. As $(d, g(d)) \in B_g \subseteq B_f$ (Lemma 2.1(2)), we conclude $f(d) \leq g(d)$. ■

2.3. Corollary. *Let $f : D \to D$ be a monotone mapping. Then f is a projection if and only if $B_f = \mathsf{ker}\, f$.*

Proof. If f is a projection, then $B_f = \mathsf{ker}\, f$ by Lemma 2.1(3) and (4). Conversely, let $B_f = \mathsf{ker}\, f$. By 2.1(3), f is below id_D. Let $d \in D$. As $(d, f(d)) \in B_f \subseteq \mathsf{ker}\, f$ (Lemma 2.1(2)), we have $f(d) = f(f(d))$. ■

The next definition is central for this chapter:

Definition. Let $(D, \leq)$ be a poset and let $\mathcal{F}$ be a directed family of monotone mappings below the identity with the following property: for all $f \in \mathcal{F}$ there is some $g \in \mathcal{F}$ such that $f \leq g \circ g$. Then we call the triple $(D, \leq, \mathcal{F})$ an *F-poset*. It is said to be *approximating* if $\sup_{f \in \mathcal{F}} f(d) = d$ for all $d \in D$.

Let $(D, \leq, \mathcal{F})$ be an F-poset and let $d \in D$. Notice that whenever $\sup_{g \in \mathcal{F}} g(d)$ exists, then we have $f(\sup_{g \in \mathcal{F}} g(d)) \leq f(d) \leq \sup_{g \in \mathcal{F}} g(d) \leq d$ for all $f \in \mathcal{F}$.

To give the reader a first impression, we mention the following basic example. Further examples will be presented in what follows.

2.4. Example. The reals $\mathbb{R}$ together with their usual order can be turned easily into an approximating F-poset. For all $\varepsilon > 0$ and all $x \in \mathbb{R}$ let $f_\varepsilon(x) := x - \varepsilon$ (Figure 2.2). Clearly, f_ε is a monotone mapping below the identity, $f_\varepsilon = f_{\varepsilon/2} \circ f_{\varepsilon/2}$, and $f_\delta \geq f_\varepsilon$ for all $0 < \delta \leq \varepsilon$. As $\sup_{\varepsilon > 0}(x - \varepsilon) = x$ for all $x \in \mathbb{R}$, we find $\underline{D_\mathbb{R}} := (\mathbb{R}, \leq, \{f_\varepsilon \mid \varepsilon > 0\})$ to be an approximating F-poset.

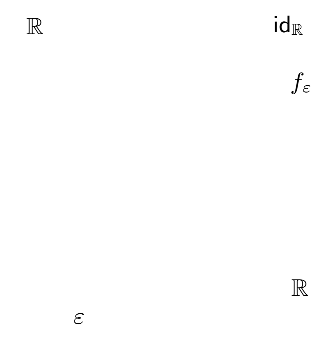


Fig. 2.2. The reals as an approximating F-poset

A uniformity for F-posets. Each F-poset $(D, \leq, \mathcal{F})$ gives rise to a canonical uniformity (and thus a topology) on D. Furthermore, we give a description of all uniformities on a poset $(D, \leq)$ generated by a family $\mathcal{F}$ such that $(D, \leq, \mathcal{F})$ is an F-poset.

2.5. Theorem. *Let $(D, \leq)$ be a poset.*

(1) *If $(D, \leq, \mathcal{F})$ is an F-poset, then $\mathcal{B} = \{B_f \mid f \in \mathcal{F}\}$ is a basis for a uniformity $\mathcal{U}_\mathcal{F}$ on D. Each element of $\mathcal{B}$ is symmetric. Moreover, the following conditions are satisfied:*

   (a) *For all $B \in \mathcal{B}$ and for all $d \in D$ there exists a least element $\min B(d)$ of $B(d)$.*
   (b) *For all $B \in \mathcal{B}$ and for all $d, e \in D$ with $d \leq e$ we have $\min B(d) \leq \min B(e)$.*
   (c) *For all $B \in \mathcal{B}$ and for all $d, e \in D$ such that $\min B(d) \leq e$ and $\min B(e) \leq d$ we have $(\min B(d), \min B(e)) \in B$.*

(2) *Let $\mathcal{U}$ be a uniformity on D and let $\mathcal{B}$ be a basis for $\mathcal{U}$ consisting of symmetric entourages such that* (a) *and* (b) *above hold. For each $B \in \mathcal{B}$ we define a mapping*

$g_B : D \to D$ *by* $g_B(d) := \min B(d)$ *for all* $d \in D$. *Then* $(D, \leq, \{g_B \mid B \in \mathcal{B}\})$ *is an F-poset.*

(3) *With the notation of* (1) *and* (2), *we have* $f = g_{B_f}$ *for all* $f \in \mathcal{F}$, *whence* $(D, \leq, \mathcal{F}) = (D, \leq, \{g_{B_f} \mid f \in \mathcal{F}\})$. *Conversely, suppose that* $\mathcal{U}$ *additionally meets condition* (c) *above. Then* $(D,\mathcal{U}) = (D,\mathcal{U}_{\{g_B \mid B \in \mathcal{B}\}})$.

Proof. (1) As $\mathcal{F}$ is directed, $\mathcal{B} = \{B_f \mid f \in \mathcal{F}\}$ is filtered by Corollary 2.2. Let $f \in \mathcal{F}$. We know that B_f is reflexive and symmetric. Choose some $g \in \mathcal{F}$ with $f \leq g \circ g$. Then it is straightforward to see that $B_g \circ B_g \subseteq B_f$. Therefore, $\mathcal{B}$ is a basis for a uniformity $\mathcal{U}_{\mathcal{F}}$ on D. By Lemma 2.1(2) we have $f(d) = \min B_f(d)$ for all $f \in \mathcal{F}$. This implies (a) and (b). To prove (c), let $f \in \mathcal{F}$ and let $d, e \in D$ with $f(d) = \min B_f(d) \leq e$ and $f(e) = \min B_f(e) \leq d$. Then $f(f(d)) \leq f(e)$ and $f(f(e)) \leq f(d)$, whence $(\min B_f(d), \min B_f(e)) = (f(d), f(e)) \in B_f$.

(2) Let $B, C \in \mathcal{B}$ with $B \supseteq C$. Let $d \in D$. Then $B(d) \supseteq C(d)$ and $\min C(d) \in B(d)$; hence $g_B(d) = \min B(d) \leq \min C(d) = g_C(d)$. Therefore, $g_B \leq g_C$. Since $\mathcal{B}$ is filtered, $\{g_B \mid B \in \mathcal{B}\}$ is directed.

Let $B \in \mathcal{B}$. Because of (b) the mapping g_B is monotone. Moreover, for all $d \in D$ we have $g_B(d) = \min B(d) \leq d$ (because $d \in B(d)$), i.e. $g_B \leq \mathsf{id}_D$. Choose some $C \in \mathcal{B}$ with $C \circ C \subseteq B$. Let $d \in D$. Then $\min C(d) \in C(d)$, $\min C(\min C(d)) \in C(\min C(d))$, and thus $\min C(\min C(d)) \in (C \circ C)(d) \subseteq B(d)$. This yields $g_B(d) = \min B(d) \leq \min C(\min C(d)) = g_C(g_C(d))$. Hence, $g_B \leq g_C \circ g_C$.

(3) For all $f \in \mathcal{F}$ and all $d \in D$ we have $g_{B_f}(d) = \min B_f(d) = f(d)$ due to Lemma 2.1(2), i.e. $g_{B_f} = f$ for all $f \in \mathcal{F}$.

Next, let $B \in \mathcal{B}$ and let $(d, e) \in B$. Then $g_B(d) = \min B(d) \leq e$ because $e \in B(d)$. Dually, we have $g_B(e) \leq d$ as B is symmetric. Hence, $(d, e) \in B_{g_B}$ and $B \subseteq B_{g_B}$. Consequently, $\mathcal{U}_{\{g_B \mid B \in \mathcal{B}\}} \subseteq \mathcal{U}$.

Again let $B \in \mathcal{B}$. Choose an entourage $C \in \mathcal{B}$ with $C \circ C \circ C \subseteq B$ and let $(d, e) \in B_{g_C}$. Since $\min C(d) = g_C(d) \leq e$ and $\min C(e) = g_C(e) \leq d$, we use (c) to obtain $(\min C(d), \min C(e)) \in C$. As $(d, \min C(d)) \in C$ and $(\min C(e), e) \in C$, we conclude that $(d, e) \in C \circ C \circ C \subseteq B$; hence $B_{g_C} \subseteq B$ and thus $\mathcal{U} \subseteq \mathcal{U}_{\{g_B \mid B \in \mathcal{B}\}}$. ∎

If $\underline{D} = (D, \leq, \mathcal{F})$ is an F-poset, then we call $\mathcal{U}_{\underline{D}} := \mathcal{U}_{\mathcal{F}}$ the *F-uniformity* of $\underline{D}$. The induced topology $\tau_{\underline{D}}$ is the *F-topology* of $\underline{D}$.

Observe that the F-uniformity is discrete if and only if $\mathsf{id}_D \in \mathcal{F}$. Nevertheless, the F-topology can be discrete although the F-uniformity is not (cf. Example 3.26 below).

2.6. Lemma. *Let* $(D, \leq, \mathcal{F})$ *be an F-poset and let* $\mathcal{F}' \subseteq \mathcal{F}$. *Then* $\mathcal{F}'$ *is cofinal in* $\mathcal{F}$ *if and only if* $(D, \leq, \mathcal{F}')$ *is an F-poset with* $\mathcal{U}_{\mathcal{F}'} = \mathcal{U}_{\mathcal{F}}$. *In this case* $\{B_{f'} \mid f' \in \mathcal{F}'\}$ *is a basis for* $\mathcal{U}_{\mathcal{F}}$.

Proof. If $\mathcal{F}' \subseteq \mathcal{F}$ is cofinal in $\mathcal{F}$, then, clearly, $(D, \leq, \mathcal{F}')$ is an F-poset. Moreover, $\{B_{f'} \mid f' \in \mathcal{F}'\}$ is a basis for $\mathcal{U}_{\mathcal{F}}$ because for given $f \in \mathcal{F}$ we find some $f' \in \mathcal{F}'$ with $f \leq f'$, whence $B_{f'} \subseteq B_f$ by Corollary 2.2. We conclude that $\mathcal{U}_{\mathcal{F}'} = \mathcal{U}_{\mathcal{F}}$. The converse can be shown similarly. ∎

Let $(D, \leq, \mathcal{F})$ be an F-poset. Let $(\langle \mathcal{F} \rangle, \circ)$ be the semigroup of all mappings from D to itself generated by $\mathcal{F}$. Furthermore, let $\mathcal{G} := \{g : D \to D \mid g \text{ monotone}, \exists f \in \mathcal{F} : g \leq f\}$.

One easily sees that $(D, \leq, \langle \mathcal{F} \rangle)$ and $(D, \leq, \mathcal{G})$ are F-posets and $\mathcal{F} \subseteq \langle \mathcal{F} \rangle \subseteq \mathcal{G}$. Since $\mathcal{F}$ is cofinal in $\mathcal{G}$ and thus in $\langle \mathcal{F} \rangle$, we deduce that $\mathcal{U}_{\mathcal{F}} = \mathcal{U}_{\langle \mathcal{F} \rangle} = \mathcal{U}_{\mathcal{G}}$ by Lemma 2.6. Hence, from a purely topological point of view, it makes no difference with which of the three mentioned F-posets we want to work.

Next, we give a simple necessary and sufficient condition for $\mathcal{U}_{\mathcal{F}}$ to be pseudo-metrizable:

2.7. PROPOSITION. *The F-uniformity of an F-poset $(D, \leq, \mathcal{F})$ is pseudo-metrizable if and only if $\mathcal{F}$ contains a cofinal ω-chain or a greatest element.*

Proof. First let $\mathcal{U}_{\mathcal{F}}$ be pseudo-metrizable. Then $\mathcal{U}_{\mathcal{F}}$ has a countable basis $\{A_n \mid n \in \mathbb{N}\}$. Let $f_1 \in \mathcal{F}$ with $B_{f_1} \subseteq A_1$. Let $n \in \mathbb{N}$ and let $f \in \mathcal{F}$ be such that $B_f \subseteq A_{n+1}$. Choose a mapping $f_{n+1} \in \mathcal{F}$ with $f_{n+1} \geq f, f_n$. Applying Corollary 2.2 we obtain $B_{f_{n+1}} \subseteq B_f \subseteq A_{n+1}$ and $B_{f_{n+1}} \subseteq B_{f_n}$. Thus, we have defined a $\subseteq$-decreasing sequence $(B_{f_n})_{n \in \mathbb{N}}$ with $B_{f_n} \subseteq A_n$ for all $n \in \mathbb{N}$. Let $N := \{f_n \mid n \in \mathbb{N}\}$. Let $f \in \mathcal{F}$. Then there is some $n \in \mathbb{N}$ with $A_n \subseteq B_f$, whence $B_{f_n} \subseteq A_n \subseteq B_f$ and $f_n \geq f$ by 2.2. This shows us that N is cofinal in $\mathcal{F}$. Now the assertion follows from the fact that $f_m \leq f_n$ for all $m \leq n$.

To prove the "if" part, apply Lemma 2.6 to see that $\mathcal{U}_{\mathcal{F}}$ has a countable basis and thus is pseudo-metrizable. ∎

Note that for countable $\mathcal{F}$ without greatest element we can find a cofinal ω-chain by induction. Hence, whenever $\mathcal{F}$ is countable, then $(D, \leq, \mathcal{F})$ is pseudo-metrizable.

With the help of the following definition, which is due to Jung [28], we are able to characterize when $(D, \mathcal{U}_{\mathcal{F}})$ is a totally bounded space:

DEFINITION. A mapping $f : D \to D$ is *finitely separated* from id_D if there is a finite set $M \subseteq D$ such that for all $d \in D$ there is some $m \in M$ with $f(d) \leq m \leq d$. Such a set M is called a *finite separating set* of f and id_D.

2.8. PROPOSITION. *Let $(D, \leq, \mathcal{F})$ be an F-poset. Then $(D, \mathcal{U}_{\mathcal{F}})$ is totally bounded if and only if each $f \in \mathcal{F}$ is finitely separated from id_D.*

Proof. Let $(D, \mathcal{U}_{\mathcal{F}})$ be totally bounded. Let $f \in \mathcal{F}$ and choose some $g \in \mathcal{F}$ such that $f \leq g \circ g$. Due to total boundedness there is a finite subset $M' \subseteq D$ with $D = \bigcup_{m' \in M'} B_g(m')$. Hence, for all $d \in D$ there is an element $m' \in M'$ such that $g(d) \leq m'$ and $g(m') \leq d$; hence $f(d) \leq g(g(d)) \leq g(m') \leq d$. Consequently, $M := g[M']$ is a finite separating set of f and id_D.

To prove the converse, let $f \in \mathcal{F}$ and let M be a finite separating set of f and id_D. Let $d \in D$. Then there is some $m \in M$ with $f(d) \leq m \leq d$ and thus $f(m) \leq f(d) \leq d$. This implies $d \in B_f(m)$. We infer that $D = \bigcup_{m \in M} B_f(m)$ and conclude that $(D, \mathcal{U}_{\mathcal{F}})$ is totally bounded. ∎

Let $(D, \leq, \mathcal{F})$ be an F-poset and let $A \subseteq D$. By extending the arguments of the previous proof we can show the following. The uniform space $(A, \mathcal{U}_{\mathcal{F}}|_A)$ is totally bounded if and only if for each $f \in \mathcal{F}$ there is a finite set $M_f \subseteq D$ such that for all $a \in A$ we find some $m \in M_f$ with $f(a) \leq m \leq a$. The "only if" part follows as above. For the "if" part let $f, g \in \mathcal{F}$ with $f \leq g \circ g$. As in the argument above we obtain $A \subseteq \bigcup_{m \in M_g} B_g(m)$ for some finite set $M_g \subseteq D$. We may assume $A \cap B_g(m) \neq \emptyset$ and choose some $a_m \in A \cap B_g(m)$ for all $m \in M_g$. Since $B_g \circ B_g \subseteq B_f$, we infer that $B_g(m) \subseteq B_f(a_m)$ for all $m \in M_g$. Therefore, $A \subseteq \bigcup_{m \in M_g} B_f(a_m)$ and $(A, \mathcal{U}_{\mathcal{F}}|_A)$ is totally bounded.

Basic properties of the F-topology. Let $\underline{D} = (D, \leq, \mathcal{F})$ be an F-poset. As the sets $B_f(d)$ with $f \in \mathcal{F}$ form a $\tau_{\underline{D}}$-neighbourhood basis of $d \in D$, we immediately obtain the following lemma concerning the convergence of nets, which we will use without citation.

2.9. Lemma. *A net $(d_n)_{n \in N}$ in D converges to some $d \in D$ with respect to the F-topology if and only if, for all $f \in \mathcal{F}$, there is an index $n_f \in N$ such that $f(d) \leq d_n$ and $f(d_n) \leq d$ for all $n \geq n_f$.* ∎

Given any F-poset $\underline{D} = (D, \leq, \mathcal{F})$, the topological space $(D, \tau_{\underline{D}})$ is completely regular. Thus, $(D, \tau_{\underline{D}})$ is Hausdorff if and only if it is T_0. Furthermore, note that $\bigcap_{f \in \mathcal{F}} B_f(d)$ is the closure of the singleton $\{d\}$ in $(D, \tau_{\underline{D}})$. We will use this fact for the following statement.

2.10. Proposition. *Let $\underline{D} = (D, \leq, \mathcal{F})$ be an F-poset.*

(1) *For all $d \in D$ the net $(f(d))_{f \in \mathcal{F}}$ converges to d. In particular, $\bigcup_{f \in \mathcal{F}} f[D]$ is dense in D.*
(2) *Let $d \in D$. If $e = \sup_{f \in \mathcal{F}} f(d)$ exists, then $e = \min \bigcap_{f \in \mathcal{F}} B_f(d) = \min \overline{\{d\}}$.*
(3) *$(D, \tau_{\underline{D}})$ is Hausdorff if and only if, for all $d, e \in D$, the following holds: if $f(d) \leq e$ and $f(e) \leq d$ for all $f \in \mathcal{F}$, then $d = e$.*
(4) *Each isolated element of $(D, \tau_{\underline{D}})$ is a fixpoint of some $f \in \mathcal{F}$. Hence, if all $f \in \mathcal{F}$ are fixpoint free, then $(D, \tau_{\underline{D}})$ has no isolated elements.*

Proof. (1) Let $f \in \mathcal{F}$. Then $f(d) \leq g(d)$ and $f(g(d)) \leq d$ for all $g \in \mathcal{F}$, $g \geq f$.

(2) As noted above, $\bigcap_{f \in \mathcal{F}} B_f(d) = \overline{\{d\}}$. Let $e := \sup_{f \in \mathcal{F}} f(d)$. Then $f(e) \leq f(d) \leq e \leq d$ and thus $e \in B_f(d)$ for all $f \in \mathcal{F}$. Let $\widetilde{e} \in \bigcap_{f \in \mathcal{F}} B_f(d)$. By Lemma 2.1(2) we have $f(d) = \min B_f(d) \leq \widetilde{e}$ for all $f \in \mathcal{F}$, whence $e \leq \widetilde{e}$.

(3) is obvious because $(D, \tau_{\underline{D}})$ is Hausdorff if and only if $\bigcap_{f \in \mathcal{F}} B_f = \mathsf{id}_D$.

(4) Let $d \in D$ and let $f \in \mathcal{F}$ with $B_f(d) = \{d\}$. Since $f(d) \in B_f(d)$ by Lemma 2.1, we infer that $f(d) = d$. ∎

Due to Proposition 2.10(1), each $d \in D$ can be approximated topologically by elements of $\bigcup_{f \in \mathcal{F}} f[D]$. If, moreover, $(D, \leq, \mathcal{F})$ is approximating, then each $d \in D$ can also be approximated order-theoretically by elements of $\bigcup_{f \in \mathcal{F}} f[D]$. The property of an F-poset to be approximating can be characterized topologically as follows:

2.11. Proposition. *Let $\underline{D} = (D, \leq, \mathcal{F})$ be an F-poset. Then the following are equivalent:*

(i) *$\underline{D}$ is approximating.*
(ii) *$(D, \tau_{\underline{D}})$ is Hausdorff and the pointwise supremum $\sup \mathcal{F}$ exists.*
(iii) *$\leq$ is closed in D^2, i.e. $(D, \leq, \tau_{\underline{D}})$ is a* partially ordered space*.*

Proof. (i)⇒(ii). Let $d, e \in D$ be such that $f(d) \leq e$ and $f(e) \leq d$ for all $f \in \mathcal{F}$. Then $d = \sup_{f \in \mathcal{F}} f(d) \leq e = \sup_{f \in \mathcal{F}} f(e) \leq d$ by (i) and thus $d = e$. Proposition 2.10(3) tells us that $(D, \tau_{\underline{D}})$ is Hausdorff.

(ii)⇒(i) follows from 2.10(2).

(i)⇒(iii). Let $d, e \in D$ and let $(d_n)_{n\in N}, (e_n)_{n\in N}$ be nets in D with $(d_n)_{n\in N} \to d$, $(e_n)_{n\in N} \to e$, and $d_n \leq e_n$ for all $n \in N$. Let $f \in \mathcal{F}$ and choose a mapping $g \in \mathcal{F}$ with $f \leq g \circ g$. There is an index $n_0 \in N$ such that $g(d) \leq d_{n_0}$ and $g(e_{n_0}) \leq e$. Therefore, $f(d) \leq g(g(d)) \leq g(d_{n_0}) \leq g(e_{n_0}) \leq e$. We deduce that $d = \sup_{f\in\mathcal{F}} f(d) \leq e$.

(iii)⇒(i). Let $d, e \in D$ be such that $f(d) \leq e$ for all $f \in \mathcal{F}$. As $(f(d))_{f\in\mathcal{F}} \to d$ (Proposition 2.10(1)), (iii) yields $d \leq e$. Consequently, $d = \sup_{f\in\mathcal{F}} f(d)$. ■

Now we relate the F-topology to the topologies defined in Section 1.2 (cf. p. 16). Note that whenever τ is a topology on a poset $(D, \leq)$ such that $\leq$ is closed in D^2 (endowed with the product topology), then principal ideals and principal filters are closed in (D, τ). Consequently, the interval topology is coarser than τ. Therefore, Proposition 2.11(i)⇒(iii) implies:

2.12. COROLLARY. *Let $\underline{D} = (D, \leq, \mathcal{F})$ be an approximating F-poset. Then the interval topology is coarser than the F-topology. In particular, the Scott topology is coarser than the F-topology if and only if the Lawson topology is coarser than the F-topology.* ■

For an important class of F-posets we shall show that the Lawson topology is equal to the F-topology; see Theorem 2.46 and Corollary 2.49 below.

2.13. COROLLARY. *Let $\underline{D} = (D, \leq, \mathcal{F})$ be an F-poset and let $h : D \to D$. If h is Scott- or Lawson-continuous, then $B_h(d)$ is Lawson-closed for all $d \in D$. If $\underline{D}$ is approximating and h is $\tau_{\underline{D}}$-continuous, then $B_h(d)$ is $\tau_{\underline{D}}$-closed for all $d \in D$.*

Proof. Recall that $B_h(d) = h(d)\!\uparrow \cap\, h^{-1}[d\!\downarrow]$ (Lemma 2.1(1)), $h(d)\!\uparrow$ is Lawson-closed, and $d\!\downarrow$ is Scott- (and Lawson-) closed. If h is Scott-continuous (Lawson-continuous), then $h^{-1}[d\!\downarrow]$ is Scott-closed (Lawson-closed), whence $B_h(d)$ is Lawson-closed. By Corollary 2.12, $h(d)\!\uparrow$ and $d\!\downarrow$ are also $\tau_{\underline{D}}$-closed provided that $\underline{D}$ is approximating. If h is $\tau_{\underline{D}}$-continuous, then $h^{-1}[d\!\downarrow]$ and thus $B_h(d)$ is $\tau_{\underline{D}}$-closed as well. ■

The mappings in $\mathcal{F}$ need not be continuous with respect to the F-topology, see Example 2.17 below. Here we characterize when a monotone map is (uniformly) continuous:

2.14. PROPOSITION. *Let $(D, \leq, \mathcal{F})$ and $(E, \leq, \mathcal{G})$ be F-posets, let $h : D \to E$ be a monotone mapping, and let $d \in D$. Then*

(1) *h is uniformly continuous with respect to the F-uniformities if and only if for all $g \in \mathcal{G}$ there is some $f \in \mathcal{F}$ with $g \circ h \leq h \circ f$.*
(2) *h is continuous at d with respect to the F-topologies if and only if for all $g \in \mathcal{G}$ there is some $f \in \mathcal{F}$ with $(g \circ h)(d) \leq (h \circ f)(d)$.*

Proof. We only prove (1); (2) is shown similarly. Note that we need monotonicity of the mapping h just for the "if" part.

First let h be uniformly continuous and let $g \in \mathcal{G}$. We find some $f \in \mathcal{F}$ with $(h \times h)[B_f] \subseteq B_g$. Let $d \in D$. Since $(d, f(d)) \in B_f$ (Lemma 2.1(2)), it follows that $(h(d), h(f(d))) \in B_g$; hence in particular $g(h(d)) \leq h(f(d))$.

Conversely, let $g \in \mathcal{G}$ and choose some $f \in \mathcal{F}$ with $g \circ h \leq h \circ f$. Let $(d, e) \in B_f$. Then $g(h(d)) \leq h(f(d)) \leq h(e)$ and $g(h(e)) \leq h(f(e)) \leq h(d)$. Thus, $(h \times h)[B_f] \subseteq B_g$. ■

Let $\underline{D} = (D, \leq, \mathcal{F})$ be an F-poset and assume that the underlying uniform space $(D, \mathcal{U}_{\underline{D}})$ (topological space $(D, \tau_{\underline{D}})$, respectively) has some property E. Then, for the sake of simplicity, we say that $\underline{D}$ *has property* E. For instance, if $(D, \mathcal{U}_{\underline{D}})$ is totally bounded, then we say that $\underline{D}$ is totally bounded. If not explicitly stated otherwise, all mentioned uniform (topological, respectively) properties refer to the F-uniformity (F-topology, respectively).

Examples. We finish this section by giving several examples of F-posets and by investigating their topology.

2.15. EXAMPLE. Let $\underline{D_{\mathbb{R}}}$ be the approximating F-poset as given in Example 2.4. Then for all $\varepsilon > 0$ we have $B_{f_\varepsilon} = \{(x, y) \in \mathbb{R}^2 \mid |x - y| \leq \varepsilon\}$; hence the F-uniformity coincides with the Euclidean uniformity and thus the F-topology is the Euclidean topology.

2.16. EXAMPLE. Consider the unit interval $[0, 1]$ with the usual linear order. Let I be a non-empty set and equip $[0, 1]^I$ with the product order. For all $n \in \mathbb{N}$, all finite $I_0 \subseteq I$ and all $(x_i)_{i\in I} \in [0, 1]^I$ let $f_{n,I_0}((x_i)_{i\in I}) := (y_i)_{i\in I}$ with $y_i = \max\{0, x_i - 1/n\}$ if $i \in I_0$ and $y_i = 0$ otherwise. Let $\mathcal{F} := \{f_{n,I_0} \mid n \in \mathbb{N},\ I_0 \subseteq I \text{ finite}\}$. It is routine to check that $([0, 1]^I, \leq, \mathcal{F})$ is an approximating F-poset. One easily sees that $B_{f_{n,I_0}} = \{((x_i)_{i\in I}, (y_i)_{i\in I}) \in ([0, 1]^I)^2 \mid |x_i - y_i| \leq 1/n \text{ for all } i \in I_0\}$; hence $\mathcal{U}_{\mathcal{F}}$ coincides with the product uniformity of the family $([0, 1], \mathcal{U}_{|\cdot|})_{i\in I}$, where $\mathcal{U}_{|\cdot|}$ is the Euclidean uniformity on $[0, 1]$. In particular, the F-topology is the product topology on $[0, 1]^I$ with $[0, 1]$ carrying the Euclidean topology. Therefore, this F-poset is compact. Proposition 2.8 tells us that the mappings $f_{n,I_0} \in \mathcal{F}$ are finitely separated from $\mathsf{id}_{[0,1]^I}$. Note that $([0, 1]^I, \leq)$ is an FS-domain in the sense of Jung [28]; cf. also Section 2.3.

Next, we consider subchains of the set $\mathcal{F}$. The following are equivalent:

(i) I is countable.
(ii) $\mathcal{F}$ has a cofinal subchain.
(iii) $\mathcal{F}$ has a subchain $\mathcal{C}$ with $\mathcal{U}_{\mathcal{C}} = \mathcal{U}_{\mathcal{F}}$.
(iv) $\mathcal{F}$ has a subchain $\mathcal{C}$ with the pointwise supremum being $\sup \mathcal{C} = \mathsf{id}_{[0,1]^I}$.

The equivalence of (ii) and (iii) follows from Lemma 2.6. The implication (ii)⇒(iv) is trivial. If I is countable, then $\mathcal{F}$ is also countable and we find a cofinal chain by induction. This proves (i)⇒(ii). It remains to show (iv)⇒(i). Let $\mathcal{C} \subseteq \mathcal{F}$ be a chain with $\sup \mathcal{C} = \mathsf{id}_{[0,1]^I}$. Clearly, $f_{n_1,I_1} \leq f_{n_2,I_2}$ if and only if $n_1 \leq n_2$ and $I_1 \subseteq I_2$. As $\sup \mathcal{C} = \mathsf{id}_{[0,1]^I}$, the union of the $\subseteq$-chain $\{I_0 \mid f_{n_0,I_0} \in \mathcal{C} \text{ for some } n_0 \in \mathbb{N}\}$ must be equal to I (if we found an $i_1 \in I$ with $i_1 \notin I_0$ whenever $f_{n_0,I_0} \in \mathcal{C}$, then the i_1th coordinate of $f_{n_0,I_0}((1)_{i\in I})$ would be equal to 0 for all $f_{n_0,I_0} \in \mathcal{C}$, a contradiction). As a consequence, I has to be countable.

More generally, note that if I is uncountable, then there is no family $\mathcal{G}$ such that $([0, 1]^I, \leq, \mathcal{G})$ is an F-poset with $\mathcal{G}$ being a chain and $\mathcal{U}_{\mathcal{G}} = \mathcal{U}_{\mathcal{F}}$. This follows from Cor. 3.2 in Nyikos and Reichel [45]. It states that a compact Hausdorff space is metrizable if and only if the (unique) uniformity inducing the topology has a linearly ordered basis.

If $|I| = 1$, then we obtain the approximating F-poset $\underline{D_{[0,1]}} := ([0, 1], \leq, \{f_n \mid n \in \mathbb{N}\})$ with $f_n(x) = \max\{0, x - 1/n\}$ $(x \in [0, 1],\ n \in \mathbb{N})$, see Figure 2.3. Its uniformity (topology) coincides with the Euclidean uniformity (Euclidean topology) on $[0, 1]$.

1 $\mathsf{id}_{[0,1]}$

f_n

0 $\frac{1}{n}$ 1

Fig. 2.3. The unit interval as an approximating F-poset

2.17. Example. Consider again the unit interval $([0,1],\leq)$. For all $n\in\mathbb{N}$ let $g_n:[0,1]\to[0,1]$ be defined as follows: $g_1(x):=0$ for all $x\in[0,1]$. For all $n\geq 2$ let $g_n(x):=0$ if $x\in[0,1/n)$, $g_n(x):=i/n$ if $x\in[(i+1)/n,(i+2)/n)$, $i\in\{0,\ldots,n-2\}$, and $g_n(x):=(n-2)/n$ if $x=1$ (cf. Figure 2.4).

1 $\mathsf{id}_{[0,1]}$

g_n

0 $\frac{1}{n}$ 1

Fig. 2.4. An approximating F-poset with discontinuous mappings

Clearly, g_n is monotone and below $\mathsf{id}_{[0,1]}$ for all $n\in\mathbb{N}$. It is easy to check that $g_n\leq g_{2n}\circ g_{2n}$ and $g_m\leq g_n$ for all $m,n\in\mathbb{N}$, $m\leq n$. Further, $\sup_{n\in\mathbb{N}} g_n=\mathsf{id}_{[0,1]}$. Therefore, $\underline{D}=([0,1],\leq,\{g_n\mid n\in\mathbb{N}\})$ is an approximating F-poset. Compare $\underline{D}$ with the approximating F-poset $\underline{D_{[0,1]}}=([0,1],\leq,\{f_n\mid n\in\mathbb{N}\})$ at the end of Example 2.16. Obviously, $g_n\leq f_n$ for all $n\in\mathbb{N}$, whence $\tau_{\underline{D}}\subseteq\tau_{\underline{D_{[0,1]}}}$ (cf. Corollary 2.2). As $\tau_{\underline{D_{[0,1]}}}$ is the Euclidean topology and therefore compact Hausdorff, we obtain $\tau_{\underline{D}}=\tau_{\underline{D_{[0,1]}}}$. Note that all g_n are discontinuous with respect to the F-topology (and not Scott-continuous, either).

2.18. Example. Let $\underline{D}=(D,\leq,\mathcal{F})$ be an F-poset such that each $f\in\mathcal{F}$ is an order isomorphism. Then we have $B_f(d)=\{e\in D\mid f(d)\leq e\leq f^{-1}(d)\}=f(d)\!\uparrow\cap f^{-1}(d)\!\downarrow$ for all $f\in\mathcal{F}$ and all $d\in D$.

An element is isolated with respect to $\tau_{\underline{D}}$ if and only if it is a fixpoint of some $f \in \mathcal{F}$; that is, $\mathsf{isol}(D) = \bigcup_{f\in\mathcal{F}} \mathsf{fix}\, f$. To see this, let $d \in D$ be a fixpoint of $f \in \mathcal{F}$. Then $B_f(d) = \{e \in D \mid d \leq e \leq d\} = \{d\}$. The converse is true by Proposition 2.10(4).

It is clear that $\underline{D}^{-1} := (D, \geq, \{f^{-1} \mid f \in \mathcal{F}\})$ is also an F-poset. The standard basis $\{B_f \mid f \in \mathcal{F}\}$ of $\mathcal{U}_{\underline{D}}$ is obviously a basis for $\mathcal{U}_{\underline{D}^{-1}}$ since $B_f = \{(d,e) \mid f^{-1}(d) \geq e \geq f(d)\}$ for all $f \in \mathcal{F}$. Thus, $\mathcal{U}_{\underline{D}} = \mathcal{U}_{\underline{D}^{-1}}$.

Recall that if $(D, \leq)$ is a chain, then an element $d \in D$ is compact if and only if it is either the least element or there is some element $c < d$ with $c{\uparrow} \cap d{\downarrow} = \{c, d\}$. The chain $(D, \leq)$ is called *dense in itself* provided that for all $c, d \in D$ with $c < d$ there is some $x \in D$ with $c < x < d$. Therefore, $K(D) = \emptyset$ if and only if $(D, \leq)$ is dense in itself and has no least element.

Now assume that $\underline{D}$ is approximating, $(D, \leq)$ is linear, and all $f \in \mathcal{F}$ are fixpoint free order isomorphisms. Then

(1) $(D, \leq)$ is dense in itself and has no least element;
(2) The F-topology coincides with the interval topology of $(D, \leq)$.

This can be checked as follows. Suppose that there is some $d \in K(D)$. Since $\sup_{f\in\mathcal{F}} f(d) = d$, we find some $f \in \mathcal{F}$ with $f(d) = d$, a contradiction. Therefore, $K(D) = \emptyset$, which proves (1). By Corollary 2.12 we know that the interval topology is coarser than $\tau_{\underline{D}}$. Let $f \in \mathcal{F}$ and let $d \in D$. We have $f(d) < d$ and thus $d < f^{-1}(d)$. Hence, $d \in (D \setminus f(d){\downarrow}) \cap (D \setminus f^{-1}(d){\uparrow}) \subseteq f(d){\uparrow} \cap f^{-1}(d){\downarrow} = B_f(d)$. This proves (2).

Notice that since $(D, \leq)$ is linearly ordered, the interval topology coincides with the Lawson topology.

2.19. EXAMPLE. Let $(G, \cdot, \leq)$ be a *linearly ordered group*, i.e. $(G, \cdot)$ is a group, $(G, \leq)$ is a linearly ordered set, and for all $x, y \in G$ the inequality $x \leq y$ implies $ax \leq ay$ and $xa \leq ya$ for all $a \in G$. Let 1 be the identity of $(G, \cdot)$. We assume $G \neq \{1\}$ in what follows. Let $P := \{a \in G \mid 1 < a\}$ and $N := \{a \in G \mid a < 1\} = \{a^{-1} \mid a \in P\}$ be the subsets of all *positive* and all *negative* elements, respectively. For each $a \in G \setminus \{1\}$ let $f_a : G \to G$ be the left translation $x \mapsto ax$. Clearly, f_a is a fixpoint free order isomorphism. If $a \in N$, then $f_a \leq \mathsf{id}_G$. For all $a, b \in G$ we have $a \leq b$ if and only if $f_a \leq f_b$. Therefore, the set $\mathcal{F} := \{f_a \mid a \in N\}$ yields a chain with respect to the pointwise order. Furthermore, the conditions (a) $\sup_{a\in N} f_a(x) = x$ for all $x \in G$, (b) $\sup N = 1$, (c) $\inf P = 1$, (d) $K(G) = \emptyset$, and (e) $(G, \leq)$ is dense in itself, are all equivalent. Clearly, (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c), and (d)$\Leftrightarrow$(e) because of Example 2.18 (recall that a linearly ordered group with more than one element has neither a least nor a largest element). The implication (a)$\Rightarrow$(e) follows from 2.18 as well, and (e)$\Rightarrow$(b) is clear.

The question arises when $\underline{G} := (G, \leq, \mathcal{F})$ is an F-poset. This can be characterized as follows:

The following are equivalent:

(i) $\underline{G}$ *is an F-poset.*
(ii) $\underline{G}$ *is an approximating F-poset.*
(iii) *For all $d \in P$ there is an element $e \in P$ with $e^2 \leq d$.*

In this case, the F-topology is precisely the interval topology of $(G, \leq)$ *(which is known to turn* $(G, \cdot)$ *into a topological group).*

Proof. (i)⇔(iii). The mapping $\varphi : a \mapsto f_a$ is an order isomorphism from $(N, \leq)$ onto $(\mathcal{F}, \leq)$ with $\varphi(a \cdot b) = f_{a \cdot b} = f_a \circ f_b = \varphi(a) \circ \varphi(b)$ for all $a, b \in N$. Therefore, (i) is equivalent to saying that for all $a \in N$ there is some $b \in N$ with $a \leq b^2$. Clearly, this statement is equivalent to (iii).

(ii)⇒(i) is trivial. To show (iii)⇒(ii), it is enough to check that $\inf P = 1$. Let $d \in G$ with $d \leq P$. Suppose that $1 < d$. By (iii) we find some $e \in P$ with $e^2 \leq d$. Since $1 < e \leq e^2 \leq d \leq P$, we obtain $e = e^2 = d$. Hence, $1 = e = d$, a contradiction.

If conditions (i)–(iii) are satisfied, then we infer from Example 2.18 that the F-topology is the interval topology of $(G, \leq)$. ∎

For the following example we need some basic facts concerning C*-algebras. We refer the reader to the standard literature, e.g. Dixmier [15], Kadison and Ringrose [30].

2.20. Example. Let A be a unital C*-algebra with identity e. For all $x \in \mathsf{A}$ let $\sigma(x)$ be the *spectrum* of x, i.e. $\sigma(x) := \{\lambda \in \mathbb{C} \mid x - \lambda e \text{ is not invertible}\}$. An element $x \in \mathsf{A}$ is called *self-adjoint* if $x = x^*$, where $*$ denotes the involution of the C*-algebra A. Let D be the real vector space of all self-adjoint elements of A. Clearly, D is closed in A with respect to the norm topology. The convex cone $D_+ := \{x \in D \mid \sigma(x) \subseteq \mathbb{R}_{\geq 0}\}$ induces a partial order $\leq$ on D via $x \leq y$ if $y - x \in D_+$.

Next, we turn $(D, \leq)$ into an F-poset by letting $f_\varepsilon(x) := x - \varepsilon e$ for all $x \in D$ and all $\varepsilon > 0$. Obviously, this yields a monotone mapping $f_\varepsilon : D \to D$. As $e \in D_+$ and $\varepsilon > 0$, we have $\varepsilon e \in D_+$, whence $f_\varepsilon \leq \mathsf{id}_D$. Clearly, $f_\varepsilon = f_{\varepsilon/2} \circ f_{\varepsilon/2}$. Since $f_\delta \geq f_\varepsilon$ for all $0 < \delta \leq \varepsilon$, we infer that $\underline{D}_{\mathsf{A}} := (D, \leq, \{f_\varepsilon \mid \varepsilon > 0\})$ is an F-poset.

Recall that D_+ is closed in the norm topology $\tau_{\|\cdot\|}$. Thus, $\leq$ is closed in $(D^2, \tau^2_{\|\cdot\|})$. As in the proof of Proposition 2.11(iii)⇒(i) we derive $\underline{D}_{\mathsf{A}}$ to be approximating. Note that in the case of $\mathsf{A} = \mathbb{C}$, we obtain $\underline{D}_{\mathbb{C}} = \underline{D}_{\mathbb{R}}$ as in Example 2.4.

2.21. Proposition. *The F-uniformity of* $\underline{D}_{\mathsf{A}}$ *coincides with the norm uniformity of the C**-algebra* A *restricted to* D*. In particular,* $\underline{D}_{\mathsf{A}}$ *is complete, and the F-topology is precisely the restriction of the norm topology to* D*.*

Proof. We have $B_{f_\varepsilon} = \{(x, y) \in D^2 \mid -\varepsilon e \leq x - y \leq \varepsilon e\}$ for all $\varepsilon > 0$. The sets $E_\varepsilon := \{(x, y) \in D^2 \mid \|x - y\| \leq \varepsilon\}$, $\varepsilon > 0$, form a basis for the norm uniformity of A restricted to D. Therefore, it is sufficient to show that $B_{f_\varepsilon} = E_\varepsilon$ for all $\varepsilon > 0$. In fact, we prove that $-\varepsilon e \leq z \leq \varepsilon e$ if and only if $\|z\| \leq \varepsilon$ ($z \in D$, $\varepsilon > 0$). The "if" part follows from the well known inequality $-\|z\| e \leq z \leq \|z\| e$. To prove the converse, we remark that by virtue of the functional calculus in C*-algebras we have $\sigma(z + \varepsilon e) = \sigma(z) + \varepsilon$ and $\sigma(-z + \varepsilon e) = -\sigma(z) + \varepsilon$. Hence, $-\varepsilon e \leq z \leq \varepsilon e$ if and only if $z + \varepsilon e, -z + \varepsilon e \in P$ if and only if $\sigma(z) \subseteq [-\varepsilon, \varepsilon]$. Recall that $\|z\| \in \sigma(z)$ or $-\|z\| \in \sigma(z)$. Therefore, $\|z\| \in [-\varepsilon, \varepsilon]$ or $-\|z\| \in [-\varepsilon, \varepsilon]$; hence $\|z\| \leq \varepsilon$. ∎

We conclude this section by equipping the *formal ball model* of Edalat and Heckmann [16] with a canonical F-poset structure:

2.22. Example. Let (X, ϱ) be a metric space and let $D := X \times \mathbb{R}_{\geq 0}$. Define a partial order on D as follows ([16]): $(x,r) \leq (y,s) \;:\Leftrightarrow\; \varrho(x,y) \leq r - s$. Recall from [16] that:

(1) $(D, \leq)$ is a continuous poset.
(2) $(D, \leq)$ is a dcpo if and only if (X, ϱ) is complete.
(3) $(D, \leq)$ has a countable basis if and only if (X, ϱ) is separable.

Now we define for all $\varepsilon > 0$ a mapping $f_\varepsilon : D \to D$ by $f_\varepsilon((x,r)) := (x, r+\varepsilon)$. One easily sees that f_ε is monotone and below the identity, and $f_\varepsilon = f_{\varepsilon/2} \circ f_{\varepsilon/2}$. Again, $0 < \delta \leq \varepsilon$ implies $f_\delta \geq f_\varepsilon$. Therefore, $\underline{D}_{\mathrm{fb}}(X, \varrho) := (D, \leq, \{f_\varepsilon \mid \varepsilon > 0\})$ is an F-poset.

Suppose that $(x, r+\varepsilon) \leq (y,s)$ for all $\varepsilon > 0$, i.e. $\varrho(x,y) \leq r + \varepsilon - s$ for all $\varepsilon > 0$. Then $\varrho(x,y) \leq r - s$ and $(x,r) \leq (y,s)$. This shows us that $\underline{D}_{\mathrm{fb}}(X, \varrho)$ is approximating.

2.23. Proposition. *The F-uniformity of $\underline{D}_{\mathrm{fb}}(X, \varrho)$ equals the product uniformity of $X \times \mathbb{R}_{\geq 0}$ and, in particular, the F-topology is the corresponding product topology.*

Proof. Let $\varepsilon > 0$. Then

$$\begin{aligned} B_{f_\varepsilon} &= \{((x,r),(y,s)) \in D^2 \mid \varrho(x,y) \leq r + \varepsilon - s,\ \varrho(x,y) \leq s + \varepsilon - r\} \\ &= \{((x,r),(y,s)) \in D^2 \mid \varrho(x,y) + s - r \leq \varepsilon,\ \varrho(x,y) + r - s \leq \varepsilon\} \\ &= \{((x,r),(y,s)) \in D^2 \mid \varrho(x,y) + |r - s| \leq \varepsilon\}. \end{aligned}$$

Hence, the assertion follows. ■

It follows immediately from the previous proposition that $\underline{D}_{\mathrm{fb}}(X, \varrho)$ is complete (separable, respectively) if and only if (X, ϱ) is complete (separable, respectively).

2.2. Continuous domains and convergence of monotone nets

In the present section we deal with uniform completeness. Therefore, we begin with a description of Cauchy nets:

2.24. Lemma. *Let $(D, \leq, \mathcal{F})$ be an F-poset and let $(d_n)_{n \in N}$ be a net in D. Then the following are equivalent:*

(i) *$(d_n)_{n \in N}$ is a Cauchy net (with respect to the F-uniformity).*
(ii) *For all $f \in \mathcal{F}$ there is an index $n_0 \in N$ with $f(d_m) \leq d_n$ and $f(d_n) \leq d_m$ for all $m, n \geq n_0$.*
(iii) *For all $f \in \mathcal{F}$ there is an index $n_0 \in N$ with $f(d_{n_0}) \leq d_n$ and $f(d_n) \leq d_{n_0}$ for all $n \geq n_0$.*

Proof. The equivalence of (i) and (ii) follows immediately from the definition of the F-uniformity. The implication (ii)⇒(iii) is trivial. To show (iii)⇒(ii), let $f \in \mathcal{F}$ and choose a mapping $g \in \mathcal{F}$ with $f \leq g \circ g$. By (iii) we find an index $n_0 \in N$ with $g(d_{n_0}) \leq d_n$ and $g(d_n) \leq d_{n_0}$ for all $n \geq n_0$. Let $m, n \in N$ such that $m, n \geq n_0$. Then $f(d_m) \leq g(g(d_m)) \leq g(d_{n_0}) \leq d_n$ and, analogously, $f(d_n) \leq d_m$. ■

Next, we show that directed completeness implies uniform completeness provided that all mappings in $\mathcal{F}$ are Scott-continuous.

2.25. PROPOSITION. *Let* $\underline{D} = (D, \leq, \mathcal{F})$ *be an F-poset with* $(D, \leq)$ *being a dcpo. Suppose further that all mappings* $f \in \mathcal{F}$ *are Scott-continuous. Then* $\underline{D}$ *is complete.*

Proof. Let $(d_n)_{n\in N}$ be a Cauchy net in D. For all $f \in \mathcal{F}$ fix $k_f \in \mathcal{F}$ and $n_f \in N$ with $f \leq k_f \circ k_f$ and $f(d_m) \leq d_n$ and $f(d_n) \leq d_m$ for all $m, n \geq n_f$. Let $A := \{f(d_n) \mid n \geq n_{k_f}, f \in \mathcal{F}\}$. We show that A is directed. Let $f(d_m), g(d_n) \in A$. Let $h \in \mathcal{F}$ with $h \geq k_f, k_g$. Let $l \in N$ with $l \geq n_{k_f}, n_{k_g}, n_{k_h}$. Then $f(d_m) \leq k_f(k_f(d_m)) \leq k_f(d_l) \leq h(d_l) \in A$. Similarly, $g(d_n) \leq h(d_l)$.

We prove that $(d_n)_{n\in N}$ converges to $\sup A$. Let $f \in \mathcal{F}$. Define $B := \{g(d_m) \mid m \geq n_{k_g}, n_f,\ g \in \mathcal{F}\}$. Clearly, $B \subseteq A$. Now B is cofinal in A because if $f(d_n) \in A$, then if we let $m \in N$ with $m \geq n_{k_{(k_f)}}, n_{k_f}, n_f$, we have $f(d_n) \leq k_f(k_f(d_n)) \leq k_f(d_m)$ because $m, n \geq n_{k_f}$ and $k_f(d_m) \in B$ since $m \geq n_{k_{(k_f)}}, n_f$. Next, for all $n \geq n_f$ we have $f[B] \leq d_n$. To see this, let $g(d_m) \in B$. Then $f(g(d_m)) \leq f(d_m) \leq d_n$ for all $n \geq n_f$. Summing up we deduce $f(\sup A) = f(\sup B) = \sup f[B] \leq d_n$ for all $n \geq n_f$. Furthermore, for all $n \geq n_{k_f}$ we have $f(d_n) \in A$, whence $f(d_n) \leq \sup A$. This yields $(d_n)_{n\in N} \to \sup A$. ∎

Note that, in general, Scott-continuity of the mappings $f \in \mathcal{F}$ is indispensable for Proposition 2.25. A counterexample is provided by Example 3.27 below. Moreover, an F-poset may be complete in its F-uniformity although the underlying poset is not a dcpo (cf. Example 2.4/2.15).

For approximating F-posets we can slightly strengthen Proposition 2.25. To do this, we need the following technical lemma. It gives us a sufficient condition for the existence of suprema and infima.

2.26. LEMMA. *Let* $(D, \leq, \mathcal{F})$ *be an approximating F-poset and let* $A \subseteq D$. *Suppose that* $\sup f[A]$ *exists and* $f(\sup g[A]) = \sup f[g[A]]$ *for all* $f, g \in \mathcal{F}$ *with* $f \leq g$. *Then* $(\sup f[A])_{f\in\mathcal{F}}$ *is a Cauchy net. If it converges with respect to the F-topology, then* A *has a supremum with* $\sup A = \lim(\sup f[A])_{f\in\mathcal{F}}$. *Similarly for the infimum.*

Proof. For all $f \in \mathcal{F}$ let $d_f := \sup f[A]$. Let $f, g \in \mathcal{F}$ with $f \leq g$. Then, for each $a \in A$, we have $f(g(a)) \leq f(a)$ and $f(f(a)) \leq g(a)$; hence $\sup f[g[A]] \leq d_f$ and $\sup f[f[A]] \leq d_g$. Therefore, $f(d_g) \leq d_f$ and $f(d_f) \leq d_g$. Consequently, $(d_f)_{f\in\mathcal{F}}$ is a Cauchy net.

Now assume that $(d_f)_{f\in\mathcal{F}}$ converges to some $d \in D$. We show that $\sup A = d$. Let $f, g \in \mathcal{F}$ such that $f \leq g \circ g$. We find some $h_0 \in \mathcal{F}$ with $g(d_h) \leq d$ for all $h \in \mathcal{F}$ with $h \geq h_0$. Let $a \in A$ and let $h \in \mathcal{F}$ be such that $h \geq g, h_0$. Then $f(a) \leq g(g(a)) \leq g(h(a)) \leq g(d_h) \leq d$. Hence, $f[A] \leq d$ for all $f \in \mathcal{F}$. Since $\mathcal{D}$ is approximating, we obtain $A \leq d$. Now let $e \in D$ with $A \leq e$. Then $f[A] \leq f(e)$ and thus $d_f \leq f(e)$ for all $f \in \mathcal{F}$. Again, as $\mathcal{D}$ is approximating, we infer $d \leq e$ by Proposition 2.11. Therefore, $\lim(\sup f[A])_{f\in\mathcal{F}} = d = \sup A$.

To prove the assertion for the infimum, only a minor modification is necessary. Let $d_f := \inf f[A]$ for all $f \in \mathcal{F}$. As for the supremum one checks that $(d_f)_{f\in\mathcal{F}}$ is a Cauchy net. Let $d \in D$ be the limit of $(d_f)_{f\in\mathcal{F}}$ and let $f \in \mathcal{F}$. Then we find some $g_0 \in \mathcal{F}$ such that $f(d) \leq d_{g_0}$. Therefore, for all $a \in A$ we have $f(d) \leq \inf g_0[A] \leq g_0(a) \leq a$. Hence, $f(d) \leq A$ for all $f \in \mathcal{F}$ and thus $d \leq A$ because $\mathcal{D}$ is approximating. Now $d = \inf A$ can be shown as for the supremum. ∎

Using Proposition 2.25 and the previous lemma, we immediately obtain:

2.27. THEOREM. *Let* $\underline{D} = (D, \leq, \mathcal{F})$ *be an approximating F-poset and let all* $f \in \mathcal{F}$ *be Scott-continuous. Then the following are equivalent*:

(i) $(D, \leq)$ *is a dcpo.*
(ii) $\underline{D}$ *is complete in its F-uniformity and for all* $f \in \mathcal{F}$ *and all directed subsets* $A \subseteq f[D]$ *the supremum of* A *in* $(D, \leq)$ *exists.* ∎

During a first course in calculus the students usually become acquainted with the following easy lemma about suprema of subsets of the reals:

Let $A \subseteq \mathbb{R}$, $x \in \mathbb{R}$. *Then* $x = \sup A$ *if and only if* $A \leq x$ *and for all* $\varepsilon > 0$ *there is some* $a \in A$ *with* $x - \varepsilon \leq a$.

Here we prove an analogous statement for approximating F-posets. For this we need a simple lemma which is known as the "squeeze rule" in calculus:

2.28. LEMMA. *Let* $(D, \leq, \mathcal{F})$ *be an F-poset, let* $(d_n)_{n\in N}$, $(e_n)_{n\in N}$, $(x_n)_{n\in N}$ *be nets in* D, *and let* $d \in D$ *with* $(d_n)_{n\in N} \to d$, $(e_n)_{n\in N} \to d$. *Further, let* $d_n \leq x_n \leq e_n$ *for all* $n \in N$. *Then* $(x_n)_{n\in N} \to d$.

Proof. Let $f \in \mathcal{F}$. We find some $n_0 \in N$ with $f(d) \leq d_n$ and $f(e_n) \leq d$ for all $n \geq n_0$. Therefore, $f(d) \leq x_n$ and $f(x_n) \leq d$ for all $n \geq n_0$. ∎

2.29. LEMMA. *Let* $(D, \leq, \mathcal{F})$ *be an approximating F-poset, let* $A \subseteq D$, *and let* $d \in D$.

(1) *The following are equivalent*:
   (i) $d = \sup A$ *and* $d \in \overline{A}$.
   (ii) $A \leq d$ *and for all* $f \in \mathcal{F}$ *there exists some* $a_f \in A$ *with* $f(d) \leq a_f$.
(2) *If* $A \leq d$ *and if* $(a_f)_{f\in\mathcal{F}}$ *is a net in* A *satisfying* $f(d) \leq a_f$ *for all* $f \in \mathcal{F}$, *then* $d = \sup A = \lim(a_f)_{f\in\mathcal{F}}$.

Proof. We show (1). The proof of (2) is evident by having a close look at the following conclusions for the implication (ii)⇒(i) in (1).

To check (i)⇒(ii), let $f \in \mathcal{F}$. As $d \in \overline{A}$, we find an element $a_f \in A \cap B_f(d)$; hence $f(d) \leq a_f$.

In order to prove (ii)⇒(i), let a_f be as in (ii) ($f \in \mathcal{F}$). Then we have $f(d) \leq a_f \leq d$ for all $f \in \mathcal{F}$. Since $(f(d))_{f\in\mathcal{F}} \to d$ (Proposition 2.10(1)), we conclude that $(a_f)_{f\in\mathcal{F}} \to d$ due to Lemma 2.28. In particular, $d \in \overline{A}$. Let $A \leq e$ for some $e \in D$. As $f(d) \leq a_f \leq e$ for all $f \in \mathcal{F}$, we obtain $d = \sup_{f\in\mathcal{F}} f(d) \leq e$. ∎

There is an analogous result for infima: let $(D, \leq, \mathcal{F})$ be an approximating F-poset, let $A \subseteq D$, and let $d \in D$. Then $d = \inf A$ and $d \in \overline{A}$ if and only if $d \leq A$ and for each $f \in \mathcal{F}$ there is an element $a_f \in A$ with $f(a_f) \leq d$. Furthermore, if $d \leq A$ and if $(a_f)_{f\in\mathcal{F}}$ is a net in A with $f(a_f) \leq d$ for all $f \in \mathcal{F}$, then $d = \inf A = \lim(f(a_f))_{f\in\mathcal{F}}$. The proof is similar to that of Lemma 2.29.

2.30. COROLLARY. *Let* $(D, \leq, \mathcal{F})$ *be an approximating F-poset and let* $A \subseteq D$ *have a supremum. Consider the following two statements*:

(a) $\sup A \in \overline{A}$.
(b) $\sup A \in A$ *or* $f(\sup A) < \sup A$ *for all* $f \in \mathcal{F}$.

Then (a) *implies* (b). *If, moreover,* $(D, \leq)$ *is linear, then* (a) *and* (b) *are equivalent.*

Proof. Let $\sup A \in \overline{A}$ and let $f \in \mathcal{F}$. By Lemma 2.29(1) we find some $a \in A$ with $f(\sup A) \leq a$. If $\sup A \notin A$, then we have $f(\sup A) \leq a < \sup A$.

Next, let $(D, \leq)$ be linear and let condition (b) be satisfied. Let $f \in \mathcal{F}$. We may assume that $\sup A \notin A$. Suppose that $f(\sup A) \not\leq a$ for all $a \in A$. Then $a < f(\sup A)$ for all $a \in A$, whence $\sup A \leq f(\sup A)$, a contradiction. Therefore, there is some $a \in A$ such that $f(\sup A) \leq a$. Now apply Lemma 2.29(1) to complete the proof. ∎

Now we characterize approximating F-posets in which suprema of directed sets A (if they exist) lie in the closure of A:

2.31. Proposition. *Let* $(D, \leq, \mathcal{F})$ *be an approximating F-poset. Then the following are equivalent*:

(i) *For all directed subsets* $A \subseteq D$ *admitting a supremum we have* $\sup A \in \overline{A}$.
(ii) $f(d) \ll d$ *for all* $f \in \mathcal{F}$ *and all* $d \in D$.

In this case, $(D, \leq)$ *is a continuous poset with basis* $\bigcup_{f \in \mathcal{F}} f[D]$. *Moreover,* $K(D) = \bigcup_{f \in \mathcal{F}} \mathsf{fix}\, f$. *In particular, each topologically isolated element is compact.*

Proof. (i)⇒(ii). Let $f \in \mathcal{F}$, let $d \in D$, and let $A \subseteq D$ be directed with $\sup A \geq d$. We infer from Lemma 2.29(1) that $f(d) \leq f(\sup A) \leq a$ for some $a \in A$.

(ii)⇒(i). Let $A \subseteq D$ be directed with supremum. Let $f \in \mathcal{F}$. As $f(\sup A) \ll \sup A$ by (ii), we find some $a_f \in A$ with $f(\sup A) \leq a_f$. Then $\sup A \in \overline{A}$ due to 2.29(1).

Now assume that (i) and (ii) are satisfied and let $d \in D$, let $f_1, f_2 \in \mathcal{F}$, and let $x_1, x_2 \in D$ with $f_1(x_1) \ll d$, $f_2(x_2) \ll d$. Since $d = \sup_{f \in \mathcal{F}} f(d)$, we find some $g_1, g_2 \in \mathcal{F}$ with $f_1(x_1) \leq g_1(d)$ and $f_2(x_2) \leq g_2(d)$. Choose a mapping $h \in \mathcal{F}$ with $h \geq g_1, g_2$. Then $f_1(x_1), f_2(x_2) \leq h(d) \in \bigcup_{f \in \mathcal{F}} f[D] \cap d{\downarrow\!\!\downarrow}$. Hence, $\bigcup_{f \in \mathcal{F}} f[D] \cap d{\downarrow\!\!\downarrow}$ is directed. As $(D, \leq, \mathcal{F})$ is approximating and $\{f(d) \mid f \in \mathcal{F}\} \subseteq \bigcup_{f \in \mathcal{F}} f[D] \cap d{\downarrow\!\!\downarrow}$, it is clear that $\sup(\bigcup_{f \in \mathcal{F}} f[D] \cap d{\downarrow\!\!\downarrow}) = \sup_{f \in \mathcal{F}} f(d) = d$.

Let $d \in D$. If $d \in K(D)$, then there is some $f \in \mathcal{F}$ with $f(d) = d$ because $\sup_{f \in \mathcal{F}} f(d) = d$. Conversely, if there is a map $f \in \mathcal{F}$ with $f(d) = d$, then $d = f(d) \ll d$. The last assertion follows from Proposition 2.10(4). ∎

In case the approximating F-poset is a dcpo and satisfies the properties of Proposition 2.31, we can switch to an "equivalent" F-poset structure where we may assume Scott-continuity of the self-maps involved. This was pointed out to me by A. Jung.

2.32. Proposition. *Let* $(D, \leq, \mathcal{F})$ *be an approximating F-poset such that* $(D, \leq)$ *is a dcpo and* $f(d) \ll d$ *for all* $f \in \mathcal{F}$ *and all* $d \in D$. *Then there is a family* $\widetilde{\mathcal{F}}$ *such that* $(D, \leq, \widetilde{\mathcal{F}})$ *is an approximating F-poset with the following properties*:

(a) $\widetilde{f}$ *is Scott-continuous and* $\widetilde{f}(d) \ll d$ *for all* $\widetilde{f} \in \widetilde{\mathcal{F}}$ *and all* $d \in D$.
(b) $\mathcal{U}_{\widetilde{\mathcal{F}}} = \mathcal{U}_{\mathcal{F}}$, *and this uniformity is complete.*

Proof. $(D, \leq)$ is continuous by Proposition 2.31. For all $f \in \mathcal{F}$ define $f^{\mathrm{c}} : D \to D$ by $f^{\mathrm{c}}(d) := \sup_{x \ll d} f(x)$ $(d \in D)$. Proposition 1.12 in Jung [27] tells us that f^{c} is the greatest Scott-continuous mapping below f (cf. also [1, Prop. 2.2.24]). If $f, g \in \mathcal{F}$, $f \leq g$, then $f^{\mathrm{c}} \leq g^{\mathrm{c}}$. Thus, $\widetilde{\mathcal{F}} := \{f^{\mathrm{c}} \mid f \in \mathcal{F}\}$ is directed. Let $f \in \mathcal{F}$ and let $d \in D$. As $f^{\mathrm{c}}(d) \leq f(d) \ll d$, the map f^{c} is below the identity and $f^{\mathrm{c}}(d) \ll d$. Now let $g \in \mathcal{F}$ be such that $f \leq g \circ g \circ g \circ g$. Since $g(d) \ll d$, we infer $g(g(d)) \leq g^{\mathrm{c}}(d)$ by definition of g^{c}. Thus, $f^{\mathrm{c}} \leq f \leq g \circ g \circ g \circ g \leq g^{\mathrm{c}} \circ g^{\mathrm{c}}$. Therefore, $(D, \leq, \widetilde{\mathcal{F}})$ is an F-poset. For all $d \in D$ we have $\sup_{f \in \mathcal{F}} f^{\mathrm{c}}(d) = \sup_{f \in \mathcal{F}} \sup_{x \ll d} f(x) = \sup_{x \ll d} \sup_{f \in \mathcal{F}} f(x) = \sup_{x \ll d} x = d$; hence $\sup \widetilde{\mathcal{F}} = \mathsf{id}_D$. Since $f^{\mathrm{c}} \leq f$ for all $f \in \mathcal{F}$, Corollary 2.2 implies that $\mathcal{U}_{\widetilde{\mathcal{F}}} \subseteq \mathcal{U}_{\mathcal{F}}$. Let $f \in \mathcal{F}$. As we have just shown, there is some $g \in \mathcal{F}$ with $f \leq g^{\mathrm{c}} \circ g^{\mathrm{c}}$, whence $f \leq g^{\mathrm{c}}$. This yields $\mathcal{U}_{\mathcal{F}} \subseteq \mathcal{U}_{\widetilde{\mathcal{F}}}$ again by Corollary 2.2. Then $\mathcal{U}_{\widetilde{\mathcal{F}}} = \mathcal{U}_{\mathcal{F}}$, and this uniformity is complete by Proposition 2.25. ■

The following lemma deals with the *existence* of suprema:

2.33. LEMMA. *Let $(D, \leq, \mathcal{F})$ be an approximating F-poset and let $A \subseteq D$. Then the following are equivalent*:

(i) *A has a supremum and* $\sup A \in \overline{A}$.
(ii) *There exists a convergent net $(a_f)_{f \in \mathcal{F}}$ such that $a_f \in A$ and $f[A] \leq a_f$ for all $f \in \mathcal{F}$.*

In this case we obtain $\sup A = \lim(a_f)_{f \in \mathcal{F}}$.

Proof. (i)⇒(ii). Due to Lemma 2.29(1) we find for all $f \in \mathcal{F}$ an element $a_f \in A$ with $f(\sup A) \leq a_f$. This yields a net $(a_f)_{f \in \mathcal{F}}$ that converges to $\sup A$ (Lemma 2.29(2)). Since f is monotone, we have $f[A] \leq f(\sup A) \leq a_f$ for all $f \in \mathcal{F}$.

(ii)⇒(i). Let $d := \lim(a_f)_{f \in \mathcal{F}}$. Let $a \in A$. By assumption, $f(a) \leq a_f$ for all $f \in \mathcal{F}$. As $(f(a))_{f \in \mathcal{F}} \to a$ (Proposition 2.10(1)), we infer from Proposition 2.11 that $a \leq d$. Thus $A \leq d$. Since $(a_f)_{f \in \mathcal{F}} \to d$, we deduce in particular that for all $f \in \mathcal{F}$ there is some $g_f \in \mathcal{F}$ with $f(d) \leq a_{g_f}$. Apply Lemma 2.29(1) to conclude $d = \sup A \in \overline{A}$. ■

Again, we can prove a similar result concerning the existence of infima. Let $(D, \leq, \mathcal{F})$ be an approximating F-poset and let $A \subseteq D$. Then A has an infimum with $\inf A \in \overline{A}$ if and only if there exists a convergent net $(f(a_f))_{f \in \mathcal{F}}$ such that $a_f \in A$ and $f(a_f) \leq A$ for all $f \in \mathcal{F}$. In this case we have $\inf A = \lim(f(a_f))_{f \in \mathcal{F}}$. The proof, which is similar to that of Lemma 2.33, is left to the interested reader.

Concerning the existence of suprema of *directed* sets, we have:

2.34. LEMMA. *Let $(D, \leq, \mathcal{F})$ be an approximating F-poset and let $A \subseteq D$ be directed. Then the following are equivalent*:

(i) *A has a supremum with* $\sup A \in \overline{A}$.
(ii) *$(a)_{a \in A}$ is convergent.*

In this case we obtain $\sup A = \lim(a)_{a \in A}$.

Proof. (i)⇒(ii). Lemma 2.29 tells us that there is a net $(a_f)_{f \in \mathcal{F}}$ with $a_f \in A$ and $f(\sup A) \leq a_f$ for all $f \in \mathcal{F}$. Let $f \in \mathcal{F}$. Then we have $f(a) \leq a \leq \sup A$ and $f(\sup A) \leq a_f \leq a$ for all $a \in A$ with $a \geq a_f$. Hence, $\sup A = \lim(a)_{a \in A}$.

(ii)⇒(i). Let $d := \lim(a)_{a\in A}$ and let $f \in \mathcal{F}$. Then, in particular, we find some $a_f \in A$ with $f(d) \leq a_f$ and $f(a') \leq d$ for all $a' \in A$ with $a' \geq a_f$. Let $a \in A$ and choose an element $a' \in A$ with $a' \geq a_f, a$. Then $f(a) \leq f(a') \leq d$. As $(D, \leq, \mathcal{F})$ is approximating, we infer $a \leq d$. Thus, $A \leq d$. Now apply 2.29(1) to obtain $d = \sup A \in \overline{A}$. ∎

Let $(D, \leq, \mathcal{F})$ be an approximating F-poset and let $A \subseteq D$ be filtered. Then $(A, \geq)$ is directed; hence we obtain a net $(a)_{a\in(A,\geq)}$. We can show that A has an infimum with $\inf A \in \overline{A}$ if and only if $(a)_{a\in(A,\geq)}$ is convergent. In this case, $\inf A = \lim(a)_{a\in(A,\geq)}$. Again, the details are left to the reader.

A net $(d_n)_{n\in N}$ in D is called *monotone* or *increasing* if $m \leq n$ implies $d_m \leq d_n$ $(m, n \in N)$. A monotone net $(d_n)_{n\in N}$ gives rise to a directed set $A := \{d_n \mid n \in N\}$. It is an easy observation that $(d_n)_{n\in N}$ converges to some d if and only if $(a)_{a\in A}$ converges to d. Thus, we obtain:

2.35. Corollary. *Let $(D, \leq, \mathcal{F})$ be an approximating F-poset and let $(d_n)_{n\in N}$ be a monotone net in D. Then the following are equivalent*:

(i) *$\{d_n \mid n \in N\}$ has a supremum with* $\sup_{n\in N} d_n \in \overline{\{d_n \mid n \in N\}}$.
(ii) *$(d_n)_{n\in N}$ is convergent.*

In this case we have $\sup_{n\in N} d_n = \lim_{n\in N} d_n$. ∎

A net is *decreasing* if for all $m, n \in N$ with $m \leq n$ we have $d_m \geq d_n$. Let $(D, \leq, \mathcal{F})$ be an approximating F-poset and let $(d_n)_{n\in N}$ be a decreasing net in D. Then $\{d_n \mid n \in N\}$ has an infimum with $\inf_{n\in N} d_n \in \overline{\{d_n \mid n \in N\}}$ if and only if $(d_n)_{n\in N}$ is convergent. In this case, $\inf_{n\in N} d_n = \lim_{n\in N} d_n$.

The previous corollary implies that any monotone net that is convergent with respect to the F-topology has a supremum. The converse statement characterizes the approximating F-posets with the properties of Proposition 2.31:

2.36. Corollary. *Let $(D, \leq, \mathcal{F})$ be an approximating F-poset. Then the following are equivalent*:

(i) *For all monotone nets $(d_n)_{n\in N}$ in D, if $(d_n)_{n\in N}$ has a supremum, then it is also convergent (and* $\sup_{n\in N} d_n = \lim_{n\in N} d_n$ *in this case).*
(ii) *$f(d) \ll d$ for all $f \in \mathcal{F}$ and all $d \in D$.*

Proof. (i)⇒(ii). Let $A \subseteq D$ be directed and assume that A has a supremum. By (i) we know that $(a)_{a\in A}$ is convergent. Lemma 2.34 tells us that $\sup A \in \overline{A}$. Now (ii) results from Proposition 2.31.

(ii)⇒(i). Let $(d_n)_{n\in N}$ be a net in D having a supremum. By (ii) and Proposition 2.31, $\sup_{n\in N} d_n \in \overline{\{d_n \mid n \in N\}}$. Then $(d_n)_{n\in N}$ converges to $\sup_{n\in N} d_n$ due to Corollary 2.35. ∎

2.37. Lemma. *Let $(D, \leq, \mathcal{F})$ be an F-poset.*

(1) *Let $A \subseteq D$ and for all $f \in \mathcal{F}$ let $a_f \in A$ with $f[A] \leq a_f$. Then $(a_f)_{f\in\mathcal{F}}$ is a Cauchy net.*
(2) *Let $A \subseteq D$ be directed. Then for all $f \in \mathcal{F}$ there exists some $a_f \in A$ with $f[A] \leq a_f$ if and only if $(a)_{a\in A}$ is a Cauchy net.*

Proof. (1) Let $f \in \mathcal{F}$. Then for all $g \in \mathcal{F}$ with $g \geq f$ we have $f(a_g) \leq a_f$ (because $a_g \in A$) and $f(a_f) \leq g(a_f) \leq a_g$ (because $a_f \in A$). Thus, $(a_f)_{f\in\mathcal{F}}$ is a Cauchy net by Lemma 2.24(iii)⇒(i).

(2) Suppose first that $(a)_{a\in A}$ is a Cauchy net and let $f \in \mathcal{F}$. Then we find an element $a_f \in A$ with $f(a) \leq a_f$ for all $a \in A$ with $a \geq a_f$. Let $b \in A$. Choose some $a \in A$ with $a \geq a_f, b$. Then $f(b) \leq f(a) \leq a_f$. We see that $f[A] \leq a_f$.

Now let $f \in \mathcal{F}$ and let $a_f \in A$ be such that $f[A] \leq a_f$. Then $f(a) \leq a_f$ and $f(a_f) \leq a_f \leq a$ for all $a \in A$ with $a \geq a_f$. Hence, $(a)_{a\in A}$ is a Cauchy net by Lemma 2.24(iii)⇒(i). ∎

A similar result holds for filtered subsets $A \subseteq D$: for all $f \in \mathcal{F}$ there exists some $a_f \in A$ with $f(a_f) \leq A$ if and only if $(a)_{a\in(A,\geq)}$ is a Cauchy net.

2.38. Corollary. *Let $(D, \leq, \mathcal{F})$ be an F-poset and assume some $f_0 \in \mathcal{F}$ is an order isomorphism. Then any monotone Cauchy net is bounded. In particular, any monotone convergent net is bounded.*

Proof. Let $(d_n)_{n\in N}$ be a monotone Cauchy net and let $A := \{d_n \mid n \in N\}$. Then A is directed and $(a)_{a\in A}$ is Cauchy as well. By Lemma 2.37(2) we find some $n_0 \in N$ with $f_0[A] \leq d_{n_0}$. Hence, $A \leq f_0^{-1}(d_{n_0})$. ∎

We use the previous results to prove the main theorem of this section. This gives us several (topological, order-theoretic) characterizations of approximating F-posets in which all monotone nets converge.

2.39. Theorem. *Let $\underline{D} = (D, \leq, \mathcal{F})$ be an approximating F-poset. Then the following are equivalent:*

(i) *$(D, \leq)$ is a dcpo and $f(d) \ll d$ for all $f \in \mathcal{F}$ and all $d \in D$.*
(ii) *Each directed subset $A \subseteq D$ has a supremum with $\sup A \in \overline{A}$.*
(iii) *Each monotone net in D is convergent with respect to the F-topology.*
(iv) *$\underline{D}$ is complete and for all directed subsets $A \subseteq D$ and all $f \in \mathcal{F}$ there exists some $a \in A$ with $f[A] \leq a$.*

If these conditions are satisfied, then we also have the following:

(1) *$(D, \leq)$ is a continuous dcpo and $\bigcup_{f\in\mathcal{F}} f[D]$ is a basis for $(D, \leq)$.*
(2) *For all directed subsets $A \subseteq D$ we have $\sup A = \lim(a)_{a\in A}$. For all monotone nets $(d_n)_{n\in N}$ in D we have $\sup_{n\in N} d_n = \lim_{n\in N} d_n$.*
(3) *Let $h : D \to D$ be a monotone mapping that is continuous with respect to the F-topology. Then h is also Scott-continuous.*

Proof. (i) and (ii) are equivalent and (1) is true due to Proposition 2.31. The equivalence of (ii) and (iii) and the validity of (2) follow from Lemma 2.34 and Corollary 2.35.

Now we prove (i),(iii)⇒(iv). Completeness of $\underline{D}$ results from (i) and Proposition 2.32. The second statement of (iv) follows from (iii) and Lemma 2.37(2).

In order to prove (iv)⇒(ii), let $A \subseteq D$ be directed. Because of (iv) we find a net $(a_f)_{f\in\mathcal{F}}$ with $a_f \in A$ and $f[A] \leq a_f$. This net is Cauchy by Lemma 2.37(1), thus convergent. In view of Lemma 2.33, A has a supremum with $\sup A \in \overline{A}$. (Alternatively, we can prove (iv)⇒(iii) as follows. Let $(d_n)_{n\in N}$ be a monotone net in D. Because of (iv) and Lemma 2.37(2) we deduce that $(d_n)_{n\in N}$ is a Cauchy net, hence convergent.)

Finally, we prove (3). Let $h : D \to D$ be monotone and continuous with respect to $\tau_{\underline{D}}$. Let $A \subseteq D$ be directed. Then $h[A]$ is directed and $\sup h[A] \le h(\sup A)$. Due to (2) we have $h(\sup A) = h(\lim(a)_{a\in A}) = \lim(h(a))_{a\in A}$. Since $h(a) \le \sup h[A]$ for all $a \in A$, we deduce $h(\sup A) = \lim(h(a))_{a\in A} \le \sup h[A]$ by Proposition 2.11. ■

2.40. Corollary. *Let $\underline{D} = (D, \le, \mathcal{F})$ be an approximating F-poset. Suppose that each directed subset has a countable cofinal chain. Then the following are equivalent*:

(i) *$(D, \le)$ is a continuous dcpo and $f(d) \ll d$ for all $f \in \mathcal{F}$ and all $d \in D$.*
(ii) *Each monotone* sequence *in D converges with respect to the F-topology.*

Proof. In view of the previous theorem it is enough to show that (ii) implies (i). Let $A \subseteq D$ be directed. Then A contains a countable cofinal subchain C by assumption, say $C = \{a_n \mid n \in \mathbb{N}\}$ with $a_n \le a_{n+1}$ for all $n \in \mathbb{N}$. By (ii) and Corollary 2.35 we infer $\sup C = \lim_{n\in\mathbb{N}} a_n$. Therefore, A has a supremum with $\sup A = \sup C \in \overline{C} \subseteq \overline{A}$. Apply Theorem 2.39 to conclude the proof. ■

2.41. Example. Let $D := \mathbb{R} \cup \{+\infty\}$ with $+\infty \notin \mathbb{R}$ and extend the linear order of the reals to a linear order of D in the usual way. For all $0 < \varepsilon \le \sqrt{2}$ consider the mapping $g_\varepsilon : D \to D$, $g_\varepsilon(d) := \min\{d - \varepsilon, 1/\varepsilon\}$, where $(+\infty) + e := +\infty$ for all $e \in D$. This is illustrated in Figure 2.5. Obviously, g_ε is monotone and below the identity. Further, one easily sees that $0 < \delta \le \varepsilon \le \sqrt{2}$ implies $g_\delta \ge g_\varepsilon$. We show that $g_\varepsilon \le g_{\varepsilon/2} \circ g_{\varepsilon/2}$ for all $0 < \varepsilon \le \sqrt{2}$. First let $d \in D$, $d - \varepsilon \le 1/\varepsilon$. Then

$$d - \frac{\varepsilon}{2} = d - \varepsilon + \frac{\varepsilon}{2} \le \frac{1}{\varepsilon} + \frac{\varepsilon}{2} = \frac{2 + \varepsilon^2}{2\varepsilon} \le \frac{2}{\varepsilon}.$$

Thus, $g_{\varepsilon/2}(g_{\varepsilon/2}(d)) = g_{\varepsilon/2}(d - \varepsilon/2) = d - \varepsilon = g_\varepsilon(d)$. Now let $d - \varepsilon \ge 1/\varepsilon$ and $d - \varepsilon/2 \le 2/\varepsilon$. Then $g_{\varepsilon/2}(g_{\varepsilon/2}(d)) = g_{\varepsilon/2}(d - \varepsilon/2) = d - \varepsilon \ge 1/\varepsilon = g_\varepsilon(d)$. Finally, let $d - \varepsilon \ge 1/\varepsilon$,

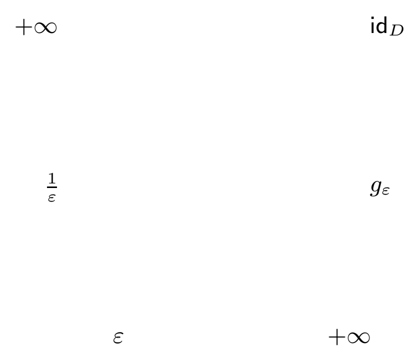


Fig. 2.5. The extended reals as an approximating F-poset

$d - \varepsilon/2 \geq 2/\varepsilon$. We deduce

$$g_{\varepsilon/2}(g_{\varepsilon/2}(d)) = g_{\varepsilon/2}\left(\frac{2}{\varepsilon}\right) = \frac{2}{\varepsilon} - \frac{\varepsilon}{2} = \frac{4-\varepsilon^2}{2\varepsilon} \geq \frac{1}{\varepsilon} = g_\varepsilon(d).$$

Therefore, $\underline{D} := (D, \leq, \{g_\varepsilon \mid 0 < \varepsilon \leq \sqrt{2}\})$ is an F-poset. It is straightforward to show that $\underline{D}$ is approximating. Clearly, $\underline{D}$ satisfies the conditions of Theorem 2.39.

Additionally, we remark here that the restriction of $\mathcal{U}_{\underline{D}}$ to $\mathbb{R}$ is strictly coarser than the Euclidean uniformity. (Use Examples 2.4, 2.15 to show this.) Nevertheless, the F-topology restricted to $\mathbb{R}$ coincides with the Euclidean topology.

We note further that the topology of the Aleksandrov one-point compactification $\mathbb{R}^*$ of $\mathbb{R}$ is strictly coarser than the F-topology of $\underline{D}$, where $+\infty$ is also considered as the point in infinity of $\mathbb{R}^*$. (Observe that $B_{g_\varepsilon}(+\infty) = [1/\varepsilon, +\infty] \subseteq D \setminus [-M, M]$ for all $M > 0$ and $0 < \varepsilon < \min\{1/M, \sqrt{2}\}$, and $D \setminus [-M, M] \not\subseteq [1/\varepsilon, +\infty]$ for all $0 < \varepsilon \leq \sqrt{2}$ and $M > 0$.)

Finally, if we restrict the topology of the two-point compactification $\mathbb{R} \cup \{-\infty, +\infty\}$ to D, then we obtain precisely the F-topology.

2.42. Example. Let (X, ϱ) be a complete metric space and consider the F-poset $\underline{D}_{\mathrm{fb}}(X, \varrho)$ of the formal ball model (Example 2.22). As $(x, r) \ll (y, s) \Leftrightarrow \varrho(x, y) < r - s$ ([16, Prop. 7]), we have $f((x, r)) \ll (x, r)$. Therefore, $\underline{D}_{\mathrm{fb}}(X, \varrho)$ satisfies the conditions of 2.39.

We know that in Theorem 2.39, condition (i) implies (1). The converse does not hold. The approximating F-poset $([0, 1], \leq, \{\mathsf{id}_{[0,1]}\})$ yields a counterexample.

The reals $(\mathbb{R}, \leq)$ do not form a dcpo, but admit suprema of bounded sets. Each bounded subset $A \subseteq \mathbb{R}$ is also directed because it is a chain, and we always have $\sup A \in \overline{A}$. Therefore, we state a "local" analogue of Theorem 2.39. Recall that a poset $(D, \leq)$ is called a *local dcpo* if each bounded, directed subset has a supremum (cf. Mislove [43]).

2.43. Theorem. *Let $\underline{D} = (D, \leq, \mathcal{F})$ be an approximating F-poset. Then the following are equivalent*:

(i) *$(D, \leq)$ is a local dcpo and $f(d) \ll d$ for all $f \in \mathcal{F}$ and all $d \in D$.*
(ii) *Each bounded, directed subset $A \subseteq D$ has a supremum with* $\sup A \in \overline{A}$.
(iii) *Each bounded, monotone net is convergent with respect to the F-topology.*
(iv) *Each bounded Cauchy net has a limit point, and for all bounded, directed sets $A \subseteq D$ and all $f \in \mathcal{F}$ there exists an element $a \in A$ with $f[A] \leq a$.*

Furthermore, if these conditions are satisfied, then we also have:

(1) *$(D, \leq)$ is a continuous local dcpo and $\bigcup_{f\in\mathcal{F}} f[D]$ is a basis for $(D, \leq)$.*
(2) *For all bounded, directed subsets $A \subseteq D$ we have* $\sup A = \lim(a)_{a\in A}$. *For all bounded, monotone nets $(d_n)_{n\in N}$ in D we have* $\sup_{n\in N} d_n = \lim_{n\in N} d_n$.
(3) *Each monotone, $\tau_{\underline{D}}$-continuous self-map of D is also Scott-continuous.*

Proof. Similar to Theorem 2.39. Only the implication (ii),(iii),(2)$\Rightarrow$(iv) requires an explanation. The second statement of (iv) follows again from (iii) and Lemma 2.37(2). Now let $(d_n)_{n\in N}$ be a bounded Cauchy net. For all $f \in \mathcal{F}$ fix $k_f \in \mathcal{F}$ and $n_f \in N$ such that $f \leq k_f \circ k_f$ and $f(d_m) \leq d_n$ and $f(d_n) \leq d_m$ for all $m, n \geq n_f$. Let $A := \{f(d_n) \mid n \geq n_{k_f},\ f \in \mathcal{F}\}$. We know from the proof of Proposition 2.25 that

A is directed. Clearly, A is bounded because $(d_n)_{n\in N}$ is bounded and all $f \in \mathcal{F}$ are below the identity. By (ii) and (2) we have $\sup A = \lim(a)_{a\in A}$. Let $f \in \mathcal{F}$. Then, in particular, we find some $a_0 \in A$ with $f(\sup A) \leq a_0$. Let $f_0 \in \mathcal{F}$ and $n_0 \geq n_{k_{f_0}}$ with $a_0 = f_0(d_{n_0})$. We deduce that $f(\sup A) \leq f_0(d_{n_0}) \leq (k_{f_0} \circ k_{f_0})(d_{n_0}) \leq k_{f_0}(d_{n_0}) \leq d_n$ for all $n \geq n_{k_{f_0}}$. Further, for all $n \geq n_{k_f}$ we have $f(d_n) \in A$, whence $f(d_n) \leq \sup A$. Consequently, $(d_n)_{n\in N}$ converges to $\sup A$. ∎

2.44. EXAMPLE. (a) Clearly, the approximating F-poset $\underline{D}_{\mathbb{R}}$ of Example 2.4 satisfies the equivalent conditions of Theorem 2.43.

(b) An approximating F-poset satisfying the conditions of 2.43 need not be complete. For instance, let $\underline{D} := (\mathbb{R}, \leq, \{g_\varepsilon \mid 0 < \varepsilon \leq \sqrt{2}\})$, where $g_\varepsilon : \mathbb{R} \to \mathbb{R}$ is defined by $g_\varepsilon(x) := \min\{x - \varepsilon, 1/\varepsilon\}$. Then $\underline{D}$ is an approximating F-poset whose uniformity is strictly weaker than the Euclidean uniformity and whose topology coincides with the Euclidean topology, cf. Example 2.41 above. It satisfies all conditions of Theorem 2.43, but it is not complete since $(n)_{n\in\mathbb{N}}$ is a (non-bounded) Cauchy net without limit.

2.45. EXAMPLE. Let A be the (commutative) unital C*-algebra of all continuous, complex-valued mappings on the unit interval $[0,1]$. Consider the approximating F-poset $\underline{D}_{\mathsf{A}}$ of Example 2.20. Its underlying poset $(D, \leq)$ consists of all continuous, real-valued mappings of $[0,1]$ together with the pointwise ordering. For all $n \in \mathbb{N}$ let $F_n : [0,1] \to \mathbb{C}$ be defined by $F_n(x) := -2^n x^n + 1$ if $x \in [0, 1/2]$ and $F_n(x) := 0$ if $x \in (1/2, 1]$. Obviously, $(F_n)_{n\in\mathbb{N}}$ is a bounded, monotone sequence in D. As $\{F_n \mid n \in \mathbb{N}\}$ has no supremum, $(D, \leq)$ cannot be a local dcpo.

Even if a directed subset of $\underline{D}$ has a supremum, it need not lie in its closure. For instance, let $G_n : [0,1] \to \mathbb{C}$, $G_n(x) := -x^n + 1$. Then $(G_n)_{n\in\mathbb{N}}$ is a monotone sequence in D with $\sup_{n\in\mathbb{N}} G_n = \underline{1}$ in $(D, \leq)$, where $\underline{1}$ is the constant map sending all elements of D to 1. But $(G_n)_{n\in\mathbb{N}}$ has no limit in D with respect to the F-topology. Otherwise, in view of Corollary 2.35, its limit would be $\underline{1}$. Then $(G_n)_{n\in\mathbb{N}} \to \underline{1}$ with respect to the norm topology (Proposition 2.21), whence in particular $(G_n(1))_{n\in\mathbb{N}} \to 1$, a contradiction.

However, if $A \subseteq D$ is directed and if $\sup A$ exists in D such that $\sup A$ coincides with the pointwise supremum of all functions in A, then $\sup A \in \overline{A}$ by Dini's Theorem.

We mention here the following: let, more generally, A be the commutative unital C*-algebra of all continuous, complex-valued mappings on a compact Hausdorff space (X, τ), and let $\underline{D}_{\mathsf{A}} = (D, \leq, \{f_\varepsilon \mid \varepsilon > 0\})$ be as in Example 2.20. Clearly, $(D, \leq)$ is a lattice. It can be shown that $(D, \leq)$ is a bcpo if and only if (X, τ) is *extremely disconnected*, i.e. each open subset of X has an open closure. We refer the interested reader to Kadison and Ringrose [30, 3.4.16, p. 223, 5.7.14, p. 373]. (For instance, the Čech–Stone compactification of $(\mathbb{N}, \tau_{\mathsf{dis}})$ is extremely disconnected; see e.g. Engelking [18, Cor. 6.2.29].)

Recall from Corollary 2.12 that the Lawson topology $\lambda_{(D,\leq)}$ is coarser than the F-topology of an approximating F-poset $(D, \leq, \mathcal{F})$ if and only if the Scott topology is. Here, we show that this situation occurs in the case when $f(d) \ll d$ for all $f \in \mathcal{F}$ and all $d \in D$. If, furthermore, all $f \in \mathcal{F}$ are finitely separated from id_D, then we even deduce $\lambda_{(D,\leq)}$ to be equal to the F-topology.

2.46. THEOREM. *Let $\underline{D} = (D, \leq, \mathcal{F})$ be an F-poset such that $f(d) \ll d$ for all $f \in \mathcal{F}$ and all $d \in D$.*

(1) *If the pointwise supremum* $\sup \mathcal{F}$ *exists, then the following are equivalent:*

  (i) *$\underline{D}$ is approximating.*

  (ii) *$(D, \leq)$ is continuous and $\lambda_{(D,\leq)} \subseteq \tau_{\underline{D}}$.*

(2) *If $(D, \mathcal{U}_{\underline{D}})$ is totally bounded, then $\tau_{\underline{D}} \subseteq \lambda_{(D,\leq)}$.*

Proof. (1) (i)⇒(ii). Note that $(D, \leq)$ is continuous and $B(D) := \bigcup_{f \in \mathcal{F}} f[D]$ is a basis for $(D, \leq)$ by Proposition 2.31. In view of Corollary 2.12 it is enough to show that $\sigma_{(D,\leq)} \subseteq \tau_{\underline{D}}$. First we prove that for all $d \in D$ the family $\{f(d)\Uparrow \mid f \in \mathcal{F}\}$ is a $\sigma_{(D,\leq)}$-neighbourhood basis of d. Clearly, $d \in f(d)\Uparrow \in \sigma_{(D,\leq)}$ for all $f \in \mathcal{F}$. Let $U \in \sigma_{(D,\leq)}$ with $d \in U$. As $U = \bigcup_{x \in B(D) \cap U} x\Uparrow$ (cf. [1, Prop. 2.3.6]), we find some $f \in \mathcal{F}$ and $e \in D$ such that $d \in f(e)\Uparrow \subseteq U$. Since $d = \sup_{f \in \mathcal{F}} f(d)$, there is a mapping $g \in \mathcal{F}$ with $f(e) \leq g(d)$. Let $h \in \mathcal{F}$ with $g \leq h \circ h$. Then $f(e) \leq h(h(d)) \ll h(d)$ and $h(d)\Uparrow \subseteq f(e)\Uparrow \subseteq U$.

Next, we show that $B_g(d) \subseteq f(d)\Uparrow$ for all $d \in D$ and all $f, g \in \mathcal{F}$ with $f \leq g \circ g$. Let $e \in B_g(d)$. Then $f(d) \leq g(g(d)) \leq g(e) \ll e$ and thus $e \in f(d)\Uparrow$. Clearly, this yields $\sigma_{(D,\leq)} \subseteq \tau_{\underline{D}}$.

(ii)⇒(i). As $(D, \leq)$ is continuous, we infer that $(D, \lambda_{(D,\leq)})$ is Hausdorff (cf. [1, Prop. 4.2.20.1]; [27, Theorem 4.7(i)]). Consequently, the finer topology $\tau_{\underline{D}}$ is Hausdorff as well. By Proposition 2.11, $\underline{D}$ is approximating.

(2) Let $d \in D$ and let $f \in \mathcal{F}$. By applying Proposition 2.8 we find a finite separating set $M \subseteq D$ of f and id_D. Let $U := f(d)\Uparrow \cap \bigcap\{D \setminus m\uparrow \mid m \in M \cap (D \setminus d\downarrow)\}$. As M is finite, we conclude $U \in \lambda_{(D,\leq)}$. By assumption we have $d \in f(d)\Uparrow$. If $m \in M \cap (D \setminus d\downarrow)$, then $d \in D \setminus m\uparrow$. Therefore, $d \in U$. We prove that $U \subseteq B_f(d)$. Let $e \in U$. Clearly, $f(d) \leq e$. Let $m \in M$ be such that $f(e) \leq m \leq e$. Suppose that $m \not\leq d$. Then $m \in M \cap (D \setminus d\downarrow)$ and thus $e \in D \setminus m\uparrow$; that is, $m \not\leq e$, a contradiction. Hence, $f(e) \leq m \leq d$. This yields $e \in B_f(d)$. ■

2.3. Compact approximating F-posets and FS-domains

This section deals with compactness of the F-topology. Its main result states that FS-domains, which have been introduced by Jung in [28], appear precisely as compact approximating F-posets with least element.

The following theorem yields a domain-theoretic characterization of approximating F-posets to be compact with respect to the F-topology. It turns out that compact approximating F-posets are exactly those approximating F-posets satisfying the conditions of Theorem 2.39 and Proposition 2.8:

2.47. THEOREM. *Let $\underline{D} = (D, \leq, \mathcal{F})$ be an approximating F-poset. Then the following are equivalent:*

(i) *$\underline{D}$ is compact.*

(ii) *$(D, \leq)$ is a (continuous) dcpo, f is finitely separated from id_D and $f(d) \ll d$ for all $f \in \mathcal{F}$ and all $d \in D$.*

Proof. (i)⇒(ii). By Proposition 2.8 all $f \in \mathcal{F}$ are finitely separated from id_D. Let $A \subseteq D$ be directed. Let $f \in \mathcal{F}$. Let $M \subseteq D$ be a finite separating set of f and id_D. For all $a \in A$ choose some $m_a \in M$ with $f(a) \leq m_a \leq a$. The set $N := \{m_a \mid a \in A\}$ is finite; let $N =: \{m_1, \ldots, m_n\}$ for some $n \in \mathbb{N}$. We find elements $a_1, \ldots, a_n \in A$ with $f(a_\nu) \leq m_\nu \leq a_\nu$ for all $\nu = 1, \ldots, n$. Let $a \in A$ be such that $a \geq a_1, \ldots, a_n$. Let $b \in A$ and let $\nu \in \{1, \ldots, n\}$ with $f(b) \leq m_\nu$. As $m_\nu \leq a_\nu \leq a$, we obtain $f(b) \leq a$. Thus, $f[A] \leq a$, and we see that $\underline{D}$ satisfies condition (iv) of Theorem 2.39. This proves (ii).

(ii)⇒(i). Apply Theorem 2.39(i)⇒(iv) and Proposition 2.8. ■

By Lemma 2 in Jung [28] a Scott-continuous mapping $f : D \to D$ that is finitely separated from id_D satisfies $f(d) \ll d$ for all $d \in D$. Hence, we obtain from the previous theorem:

2.48. Corollary. *Let $\underline{D} = (D, \leq, \mathcal{F})$ be an approximating F-poset with each $f \in \mathcal{F}$ being Scott-continuous. Then $\underline{D}$ is compact if and only if $(D, \leq)$ is a (continuous) dcpo and each $f \in \mathcal{F}$ is finitely separated from id_D.* ■

Theorem 2.47 tells us that compact approximating F-posets satisfy the conditions of Theorem 2.46(1) and (2). As a consequence, we get:

2.49. Corollary. *The F-topology of a compact approximating F-poset $(D, \leq, \mathcal{F})$ is precisely the Lawson topology of $(D, \leq)$.* ■

Next, we head for FS-domains.

2.50. Theorem. *Let $(D, \leq)$ be a poset. Then the following are equivalent:*

(i) *There exists a directed family $\mathcal{F}$ such that $(D, \leq, \mathcal{F})$ is a compact approximating F-poset.*
(ii) *There exists a directed family $\widetilde{\mathcal{F}}$ consisting of Scott-continuous mappings such that $(D, \leq, \widetilde{\mathcal{F}})$ is a compact approximating F-poset.*
(iii) *$(D, \leq)$ is a (continuous) dcpo, and there exists a directed family $\mathcal{G}$ consisting of Scott-continuous mappings finitely separated from id_D with $\sup \mathcal{G} = \mathsf{id}_D$.*

Proof. (i)⇒(ii). From Theorem 2.47 we know that $(D, \leq)$ is a (continuous) dcpo. Therefore, (ii) follows from Proposition 2.32.

(ii)⇒(iii). Again by Theorem 2.47, $(D, \leq)$ is a continuous dcpo and all $f \in \widetilde{\mathcal{F}}$ are finitely separated from the identity. Hence, set $\mathcal{G} := \widetilde{\mathcal{F}}$.

(iii)⇒(i). Define $\mathcal{F} := \{g \circ g \mid g \in \mathcal{G}\}$. Obviously, $\mathcal{F}$ is a directed family of Scott-continuous mappings below id_D and cofinal in the directed set $\{g \circ h \mid g, h \in \mathcal{G}\}$. Due to Scott-continuity we derive

$$\sup \mathcal{F} = \sup_{g,h \in \mathcal{G}} (g \circ h) = (\sup_{g \in \mathcal{G}} g) \circ (\sup_{h \in \mathcal{G}} h) = \mathsf{id}_D \circ \mathsf{id}_D = \mathsf{id}_D.$$

Analogously, $\sup_{g \in \mathcal{G}} (g \circ g \circ g \circ g) = \mathsf{id}_D$. By Jung [28, Lemma 2], we know that $g \circ g \ll \mathsf{id}_D$ for all $g \in \mathcal{G}$ because g is Scott-continuous and finitely separated from id_D. Hence, for all $g \in \mathcal{G}$ we find some $g' \in \mathcal{G}$ with $g \circ g \leq g' \circ g' \circ g' \circ g'$; that is, for all $f \in \mathcal{F}$ there is a mapping $f' \in \mathcal{F}$ with $f \leq f' \circ f'$. Consequently, $(D, \leq, \mathcal{F})$ is an approximating F-poset. As all $g \in \mathcal{G}$ are finitely separated from id_D, this is also true for $g \circ g \leq g$. Now apply Corollary 2.48 to deduce that $(D, \leq, \mathcal{F})$ is compact. ■

The next definition is due to Jung [28]:

DEFINITION. A pointed dcpo $(D, \leq)$ is an *FS-domain* if there is a directed family $\mathcal{G}$ of Scott-continuous mappings on D, each finitely separated from id_D, such that $\sup \mathcal{G} = \mathsf{id}_D$.

We just remark here that the category of FS-domains is one of the two maximal cartesian closed subcategories of the category of all pointed continuous dcpo's together with Scott-continuous mappings as morphisms (see Jung [28]; cf. also [1, Section 4]).

2.51. COROLLARY. *Let $(D, \leq)$ be a pointed poset. Then the following are equivalent*:

(i) *$(D, \leq)$ is an FS-domain.*

(ii) *There is a family $\mathcal{F}$ such that $(D, \leq, \mathcal{F})$ is a compact approximating F-poset.* ■

Note that if $(D, \leq)$ is an FS-domain, then due to Jung [28, Theorem 4(i)], the set $\mathcal{F} := \{f : D \to D \mid f \text{ Scott-continuous}, f \ll \mathsf{id}_D\}$ induces a compact approximating F-poset $(D, \leq, \mathcal{F})$ whose F-topology coincides with the Lawson topology.

Observe that the Lawson topology of an FS-domain may be the topology of a compact approximating F-poset $\underline{D} = (D, \leq, \mathcal{F})$ where all $f \in \mathcal{F}$ are *not* Scott-continuous (and thus *not* $\tau_{\underline{D}}$-continuous, cf. Theorem 2.39(3)). Such an example is given in 2.17.

3. FROM POSETS WITH PROJECTIONS TO ALGEBRAIC DOMAINS

This chapter specializes in F-posets $(D, \leq, \mathcal{P})$ in which each mapping of $\mathcal{P}$ is a projection. We call them *pop's* (*posets with projections*). The assumption of idempotence leads to topological and order-theoretic consequences. We obtain zero-dimensional spaces and ultrametrics. Order continuity of $(D, \leq)$ is replaced by algebraicity. Instead of FS-domains, bifinite domains come into play.

Section 3.1 provides us with some necessary information on projections. Though easy to prove, the presented properties of projections turn out to be quite useful.

An algebraic description of pop's is the beginning of Section 3.2. We show a pop to be representable by two families of subsets of a given set and some closure operator (Theorem 3.10). Then we turn to investigating additional properties of the F-uniformity and the F-topology of pop's, which we call *pop uniformity* and *pop topology*, respectively. Again, various examples, among them the closed ball model of ultrametric spaces and examples naturally occurring in trace theory, show us that pop's are interesting objects appearing in quite distinct areas.

Section 3.3 deals with complete approximating pop's $(D, \leq, \mathcal{P})$. They turn out to be isomorphic to an inverse limit built up by the sets $p[D]$ with $p \in \mathcal{P}$. On the other hand, inverse limits of posets induce a pop structure yielding a complete approximating pop (Theorem 3.40).

Section 3.4 consists of two parts. The first is devoted to the connection of order and topology. Resuming the results on F-posets from Section 2.2, we derive similar properties for the existence of suprema. Let $(D, \leq, \mathcal{P})$ be an approximating pop. Analogously to Theorem 2.39, we characterize $(D, \leq)$ to be an algebraic dcpo with $K(D) = \bigcup_{p \in \mathcal{P}} p[D]$ if and only if each monotone net is convergent with respect to the pop topology (Theorem 3.48). The second part of the section concerns the question under which order-theoretic conditions a poset $(D, \leq)$ admits a directed family $\mathcal{P}$ of projections such that $(D, \leq, \mathcal{P})$ has the properties just mentioned. If such a family $\mathcal{P}$ exists, then we call $(D, \leq)$ a *P-domain*. An "external" characterization (Theorem 3.51) tells us that $(D, \leq)$ is a P-domain if and only if it is order-isomorphic to an inverse limit of posets satisfying the ascending chain condition. We also obtain "internal" characterizations in the spirit of the well known description of bifinite domains by "mub-completeness" and the U^∞-operator (cf. Theorem 3.53 and Corollary 3.54).

As for F-posets we finish the present chapter with compactness in Section 3.5. We prove that compact approximating pop's appear precisely as bifinite domains (Corollary 3.57). Moreover, we characterize a P-domain to be bifinite if and only if it is compact in its Lawson topology (Corollary 3.59).

Some results of this chapter are also presented in the extended abstract [32].

3.1. Projections

As defined in Chapter 1, a *projection* on a poset is a monotone, idempotent mapping below the identity. This section deals with some of their fundamental properties.

3.1. Lemma. *Let $(D, \leq)$ be a poset and let p, $q : D \to D$ be idempotent mappings. Then we have*

(1) $p \circ q = p \Leftrightarrow \mathsf{ker}\, p \supseteq \mathsf{ker}\, q$.
(2) $q \circ p = p \Leftrightarrow p[D] \subseteq q[D]$.
(3) *If p and q are below the identity, then $p \circ q = p \Rightarrow p \leq q \Rightarrow q \circ p = p$.*
(4) *If p and q are projections, then the conditions* (i) $p \leq q$, (ii) $p \circ q = p$, (iii) $q \circ p = p$, (iv) $\mathsf{ker}\, p \supseteq \mathsf{ker}\, q$, (v) $p[D] \subseteq q[D]$ *are all equivalent.*

Proof. (1) Let $p \circ q = p$ and let $(d, e) \in \mathsf{ker}\, q$. Then $p(d) = p(q(d)) = p(q(e)) = p(e)$, i.e. $(d, e) \in \mathsf{ker}\, p$. Now let $\mathsf{ker}\, p \supseteq \mathsf{ker}\, q$, and let $d \in D$. As $(d, q(d)) \in \mathsf{ker}\, q \subseteq \mathsf{ker}\, p$, we infer $p(d) = p(q(d))$.

(2) If $q \circ p = p$, then $p[D] = q[p[D]] \subseteq q[D]$. To show the converse, recall that $p(d) \in q[D]$ is a fixpoint of q for all $d \in D$ (Lemma 1.1).

(3) Let $p \circ q = p$. Since $p \leq \mathsf{id}_D$, we have $p = p \circ q \leq q$. Now let $p \leq q$. Then $p = p \circ p \leq q \circ p \leq p$ because $q \leq \mathsf{id}_D$. Hence $p = q \circ p$.

(4) We only need to show (iii)$\Rightarrow$(ii). Let $q \circ p = p$. As p and q are monotone and below the identity, we obtain $p = p \circ p = p \circ q \circ p \leq p \circ q \leq p$, whence $p = p \circ q$. ■

3.2. Corollary. *A projection is uniquely determined by its kernel and, respectively, by its image: if p and q are projections, then $p = q \Leftrightarrow \mathsf{ker}\, p = \mathsf{ker}\, q \Leftrightarrow p[D] = q[D]$.* ■

Notice that for any projection $p : D \to D$ and any $d \in D$ we have $p(d) = \max(d{\downarrow} \cap p[D])$ (cf. p. 14).

We call a projection $p : D \to D$ *compact-valued* if $p[D] \subseteq K(D)$. Compact-valued projections will play an important rôle in view of the topological results in Section 3.4. We give the following characterization:

3.3. Lemma. *Let $p : D \to D$ be a projection on a poset $(D, \leq)$. Then the following are equivalent:*

(i) *p is compact-valued.*
(ii) *$p(d) \ll d$ for all $d \in D$.*
(iii) *$p(\sup A)$ is the greatest element of $p[A]$ for all directed subsets $A \subseteq D$ that have a supremum.*
(iv) *p is Scott-continuous and $p[A]$ has a greatest element for all directed subsets $A \subseteq D$ that have a supremum.*

Proof. (i)⇒(ii). Let $d \in D$. Since $p(d) \in K(D)$, we have $p(d) \ll p(d) \leq d$ and thus $p(d) \ll d$.

(ii)⇒(iii). Let $A \subseteq D$ be directed and suppose that $\sup A$ exists. As $p(\sup A) \ll \sup A$, we find some $a \in A$ such that $p(\sup A) \leq a$. Consequently, $p(p(\sup A)) \leq p(a) \leq p(\sup A)$ and $p(a) = p(\sup A)$. Therefore, $p(\sup A)$ is the greatest element of $p[A]$.

(iii)⇒(iv). Obvious.

(iv)⇒(i). Let $d \in D$ and let $A \subseteq D$ be directed such that $\sup A \geq p(d)$. By assumption we find an element $a \in A$ with $p(a) = \max p[A] = \sup p[A] = p(\sup A)$. Thus, we have $a \geq p(a) = p(\sup A) \geq p(d)$. This yields $p(d) \in K(D)$. ■

Keep in mind that any compact-valued projection is Scott-continuous.

3.4. Corollary. *Each Scott-continuous projection with finite range is compact-valued.*

Proof. Let p be a Scott-continuous projection on a poset $(D, \leq)$ such that $p[D]$ is finite. Let $A \subseteq D$ be directed. Then $p[A]$ is directed and finite; hence it has a greatest element. Now apply Lemma 3.3(iv)⇒(i) to conclude the proof. ■

We say that a projection $p : D \to D$ is *downwards closed* provided that $p[D]$ is a lower set.

3.5. Lemma. *Let $(D, \leq)$ be a poset and let $p : D \to D$ be a projection. Then the following are equivalent*:

(i) *p is downwards closed.*
(ii) *p preserves lower subsets.*
(iii) *p preserves ideals.*

Proof. (i)⇒(iii). Let $A \subseteq D$ be an ideal. Obviously, $p[A]$ is directed. Let $a \in A$ and let $d \in D$ with $d \leq p(a)$. As $d \leq a$, we obtain $d \in A$. By (i) we have $d \in p[D]$; hence $d = p(d) \in p[A]$.

(iii)⇒(ii). Let $A \subseteq D$ be a lower set. Then $p[A] = \bigcup_{a \in A} p[a\downarrow]$ is also a lower set because $p[a\downarrow]$ is a lower set by (iii).

(ii)⇒(i). This is trivial since D is a lower set. ■

Note that the composition of two projections need not be a projection:

3.6. Example. Let $(D, \leq)$ be as follows:

We define two projections p and q by $p(1) := 1$, $p(2) := p(3) := 2$, and $q(1) := q(2) := 1$, $q(3) := 3$. Then $q \circ p$ is a projection, whereas $p \circ q$ is not. Further, we have $q \circ p < p \circ q$.

3.7. Lemma. *Let $(D, \leq)$ be a poset and let p and q be projections on D. Then the following are equivalent*:

(i) *$p \circ q$ and $q \circ p$ are projections.*
(ii) *$p \circ q = q \circ p$.*

In this case, $p \circ q$ is the greatest projection on D below p and q.

Proof. (i)⇒(ii). By (i) we obtain $p \circ q = p \circ q \circ p \circ q \leq q \circ p = q \circ p \circ q \circ p \leq p \circ q$.

(ii)⇒(i). Clearly, $p \circ q$ is monotone and below the identity. Using (ii), we have $p \circ q \circ p \circ q = p \circ p \circ q \circ q = p \circ q$; that is, $p \circ q$ is idempotent. Similarly for $q \circ p$.

Next, suppose that (i) and (ii) are satisfied. Due to Lemma 3.1 and the equations $p \circ (p \circ q) = p \circ q = (p \circ q) \circ q$, we have $p \circ q \leq p, q$. Let $r : D \to D$ be a projection with $r \leq p, q$. Then $r = r \circ p$ and $r = r \circ q$ by Lemma 3.1. Hence, $r \circ (p \circ q) = (r \circ p) \circ q = r \circ q = r$. Again by 3.1, we obtain $r \leq p \circ q$. ■

In the remainder of this section we investigate how an Abelian semigroup of idempotent mappings of a set D induces a partial order on D with respect to which these mappings become projections. Let D be a set and let $(\mathcal{S}, \circ)$ be an Abelian semigroup consisting of idempotent mappings from D into itself. We define a binary relation $\sqsubseteq_{\mathcal{S}}$ on D as follows: for all $d, e \in D$ let

$d \sqsubseteq_{\mathcal{S}} e$ provided that $d = e$ or there exists some $p \in \mathcal{S}$ such that $d = p(e)$.

By definition, $\sqsubseteq_{\mathcal{S}}$ is reflexive. Let $c, d, e \in D$ and let $p, q \in \mathcal{S}$. To check transitivity, suppose that $c = p(d)$ and $d = q(e)$. Then $c = (p \circ q)(e)$. Hence, $\sqsubseteq_{\mathcal{S}}$ is transitive. Next, let $d = p(e)$ and $e = q(d)$. Then $d = p(q(d)) = q(p(d)) = q(p(p(e))) = q(p(e)) = e$. Thus, $\sqsubseteq_{\mathcal{S}}$ is also antisymmetric. Consequently, $\sqsubseteq_{\mathcal{S}}$ is a partial order on D. Let $p \in \mathcal{S}$ and let $d, e \in D$. Clearly, $p(d) \sqsubseteq_{\mathcal{S}} d$. Let $d \sqsubseteq_{\mathcal{S}} e$; we may assume $d = q(e)$ for some $q \in \mathcal{S}$. Then $p(d) = p(q(e)) = q(p(e))$, whence $p(d) \sqsubseteq_{\mathcal{S}} p(e)$. This shows us that all $p \in \mathcal{S}$ are projections of $(D, \sqsubseteq_{\mathcal{S}})$. We obtain:

3.8. PROPOSITION. *Let D be a set and let $(\mathcal{S}, \circ)$ be an Abelian semigroup whose elements are idempotent mappings from D into itself. Then $(D, \sqsubseteq_{\mathcal{S}})$ is a poset. Each element of $D \setminus \bigcup_{p \in \mathcal{S}} p[D]$ is maximal in $(D, \sqsubseteq_{\mathcal{S}})$. All mappings in $\mathcal{S}$ are downwards closed projections with respect to $(D, \sqsubseteq_{\mathcal{S}})$.*

Proof. We have already proven that $(D, \sqsubseteq_{\mathcal{S}})$ is a poset and that all mappings in $\mathcal{S}$ are projections. Let $p \in \mathcal{S}$ and let $d, e \in D$ with $d \sqsubseteq_{\mathcal{S}} p(e)$. Then $d = p(e)$ or there exists a projection $q \in \mathcal{S}$ with $d = q(p(e)) = p(q(e))$; hence $d \in p[D]$. Thus, $p[D]$ is a lower set. The statement on $D \setminus \bigcup_{p \in \mathcal{S}} p[D]$ is obvious. ■

In view of the previous proposition, we call $\sqsubseteq_{\mathcal{S}}$ the *projection order* of $(D, \mathcal{S})$. For mappings $f, g : D \to D$ we write $f \sqsubseteq_{\mathcal{S}} g$ if $f(d) \sqsubseteq_{\mathcal{S}} g(d)$ for all $d \in D$.

3.9. COROLLARY. *Let $(D, \leq)$ be a poset and let $(\mathcal{S}, \circ)$ be an Abelian semigroup of projections on $(D, \leq)$. Then*

(1) *$d \sqsubseteq_{\mathcal{S}} e$ implies $d \leq e$ for all $d, e \in D$, i.e. $\sqsubseteq_{\mathcal{S}}$ is a subset of $\leq$.*
(2) *$\sqsubseteq_{\mathcal{S}}$ is the least order $\leq'$ on D such that $(D, \leq')$ is a poset with each element of $\mathcal{S}$ being a projection on $(D, \leq')$.*
(3) *$p \sqsubseteq_{\mathcal{S}} q$ if and only if $p \leq q$ for all $p, q \in \mathcal{S}$, i.e. $(\mathcal{S}, \sqsubseteq_{\mathcal{S}}) = (\mathcal{S}, \leq)$.*

Proof. (1) is obvious and (2) follows from (1).

(3) All elements of $\mathcal{S}$ are projections with respect to $\sqsubseteq_{\mathcal{S}}$ (Proposition 3.8). By applying Lemma 3.1 to the posets $(D, \sqsubseteq_{\mathcal{S}})$ and $(D, \leq)$, we obtain $p \sqsubseteq_{\mathcal{S}} q$ if and only if $p \circ q = p$ if and only if $p \leq q$. ∎

We will return to this later.

3.2. F-posets with projections

The present section introduces F-posets $(D, \leq, \mathcal{F})$ where each mapping in $\mathcal{F}$ is idempotent, i.e. a projection.

Definition. Let $(D, \leq)$ be a poset and let $\mathcal{P}$ be a directed family of projections on D. We call $(D, \leq, \mathcal{P})$ a *poset with projections* or *pop*. The set $\mathcal{P}$ is the *projection set* of $(D, \leq, \mathcal{P})$.

Before we start with our topological investigations, we give an algebraic description of pop's. Recall that a *closure operator* on a poset $(D, \leq)$ is a kernel operator on $(D, \geq)$. In particular, for any set X and any $\mathcal{B} \subseteq \mathfrak{P}(X)$ a closure operator on $(\mathcal{B}, \subseteq)$ is a mapping $c : \mathcal{B} \to \mathcal{B}$ with $c(B_1) \subseteq c(B_2)$, $c(c(B)) = c(B)$, and $c(B) \supseteq B$ for all $B_1, B_2, B \in \mathcal{B}$ with $B_1 \subseteq B_2$. Now the next theorem states that any pop can be represented canonically by two families of subsets of a set X and a closure operator on $\mathfrak{P}(X)$. This was pointed out to me by B. Ganter, Dresden.

We need a definition first. Let X be a set and let $\mathcal{A} \subseteq \mathfrak{P}(X)$. Let $\mathcal{Q} \subseteq \mathfrak{P}(X)$ be directed with respect to the inclusion. Let $\mathcal{B} := \mathcal{A} \cup \{A \cap Q \mid A \in \mathcal{A},\ Q \in \mathcal{Q}\}$. Let $c : \mathcal{B} \to \mathcal{B}$ be a closure operator on $(\mathcal{B}, \subseteq)$ such that $c[\mathcal{B}] = \mathcal{A}$ (i.e. $\mathcal{A}$ is precisely the set of all "c-closed" sets of $\mathcal{B}$). Then we say that $\mathcal{R} = (X, \mathcal{A}, \mathcal{Q}, c)$ is a *pop representation system*. For instance, if $h : \mathfrak{P}(X) \to \mathfrak{P}(X)$ is a closure operator on $(\mathfrak{P}(X), \subseteq)$ such that each $A \in \mathcal{A}$ is closed with regard to h and $h(A \cap Q) \in \mathcal{A}$ for all $A \in \mathcal{A}$ and all $Q \in \mathcal{Q}$, then the restriction of h to $\mathcal{B}$ yields such a mapping c.

3.10. Theorem. (1) *Let $\underline{D} = (D, \leq, \mathcal{P})$ be a pop. Let $\mathcal{A}_{\underline{D}} := \{d{\downarrow} \mid d \in D\}$ and let $\mathcal{Q}_{\underline{D}} := \{p[D] \mid p \in \mathcal{P}\}$. Then $\mathcal{R}_{\underline{D}} := (D, \mathcal{A}_{\underline{D}}, \mathcal{Q}_{\underline{D}}, \downarrow)$ is a pop representation system.*

(2) *Let $\mathcal{R} = (X, \mathcal{A}, \mathcal{Q}, c)$ be a pop representation system. For each $Q \in \mathcal{Q}$ define a mapping $r_Q : \mathcal{A} \to \mathcal{A}$ by $r_Q(A) := c(A \cap Q)$. Then $\underline{D_{\mathcal{R}}} := (\mathcal{A}, \subseteq, \{r_Q \mid Q \in \mathcal{Q}\})$ is a pop.*

(3) *With the notation of* (1) *and* (2), *the mapping $\varphi : D \to \mathcal{A}_{\underline{D}}$ defined by $\varphi(d) := d{\downarrow}$ is an order isomorphism from $(D, \leq)$ onto $(\mathcal{A}_{\underline{D}}, \subseteq)$ with $r_{p[D]} \circ \varphi = \varphi \circ p$ for all $p \in \mathcal{P}$.*

Proof. (1) Lemma 3.1 tells us that $\mathcal{Q}_{\underline{D}}$ is directed. Obviously, $\downarrow$ is a closure operator on $\mathfrak{P}(X)$. Let $d \in D$ and let $p \in \mathcal{P}$. Recall that $p(d)$ is the greatest element of $d{\downarrow} \cap p[D]$; hence $(d{\downarrow} \cap p[D]){\downarrow} = p(d){\downarrow} \in \mathcal{A}_{\underline{D}}$. Consequently, $\mathcal{R}_{\underline{D}}$ is a pop representation system.

(2) By definition of a pop representation system, the mapping r_Q is well defined for all $Q \in \mathcal{Q}$. Let $A \in \mathcal{A}$ and let $Q \in \mathcal{Q}$. Since c is monotone, we deduce that r_Q is monotone.

As $r_Q(A) = c(A \cap Q) \subseteq c(A) = A$, we see that r_Q is below $\mathsf{id}_{\mathcal{A}}$. By monotonicity, $r_Q(r_Q(A)) \subseteq r_Q(A)$. Furthermore, $A \cap Q \subseteq c(A \cap Q)$ implies $r_Q(A) = c(A \cap Q \cap Q) \subseteq c(c(A \cap Q) \cap Q) = r_Q(r_Q(A))$. Therefore, r_Q is idempotent and thus a projection on $(\mathcal{A}, \subseteq)$. If $Q_1, Q_2 \in \mathcal{Q}$ with $Q_1 \subseteq Q_2$, then r_{Q_1} is below r_{Q_2} because c is monotone. We conclude that $\underline{D_{\mathcal{R}}}$ is a pop.

(3) We know that φ is an order isomorphism. Let $p \in \mathcal{P}$ and let $d \in D$. Then $r_{p[D]}(\varphi(d)) = r_{p[D]}(d{\downarrow}) = (d{\downarrow} \cap p[D]){\downarrow} = p(d){\downarrow} = \varphi(p(d))$. ■

Part (3) of the previous theorem states that the pop's $(D, \leq, \mathcal{P})$ and $(\mathcal{A}_{\underline{D}}, \subseteq, \{r_{p[D]} \mid p \in \mathcal{P}\})$ are "isomorphic". We will study pop homomorphisms and isomorphisms in Chapter 4.

3.11. Remark. With the notation of Theorem 3.10, we have the following:

(1) If $(D, \leq, \mathcal{P})$ is a pop with all $p \in \mathcal{P}$ being downwards closed, then $\mathcal{R}_{\underline{D}} = (D, \mathcal{A}_{\underline{D}}, \mathcal{Q}_{\underline{D}}, \mathsf{id}_{\mathcal{B}})$.
(2) If $(X, \mathcal{A}, \mathcal{Q}, \mathsf{id}_{\mathcal{B}})$ is a pop representation system, then r_Q is downwards closed for all $Q \in \mathcal{Q}$.

Proof. (1) Clearly, $d{\downarrow} \cap p[D] \subseteq (d{\downarrow} \cap p[D]){\downarrow}$. Let $b \in (d{\downarrow} \cap p[D]){\downarrow}$ and let $c \leq d$ with $p(c) = c$ and $b \leq c$. As p is downwards closed, we have $b = p(b)$ and thus $b \in d{\downarrow} \cap p[D]$. We conclude $\mathsf{id}_{\mathcal{B}}(d{\downarrow} \cap p[D]) = d{\downarrow} \cap p[D] = (d{\downarrow} \cap p[D]){\downarrow}$.

(2) Let $Q \in \mathcal{Q}$, let $A_1, A_2 \in \mathcal{A}$ with $A_1 \subseteq r_Q(A_2) = A_2 \cap Q$. Then $A_1 = A_1 \cap Q = r_Q(A_1)$. ■

Let $(D, \leq, \mathcal{P})$ be a pop. Clearly, as each $p \in \mathcal{P}$ is idempotent, $(D, \leq, \mathcal{P})$ is an F-poset. Therefore, we adopt notions like approximating F-posets, complete F-posets etc. and speak of approximating pop's, complete pop's, and so forth. Note that an F-poset $(D, \leq, \mathcal{F})$ is a pop if and only if $f \leq f \circ f$ for all $f \in \mathcal{F}$, because we always have $f \circ f \leq f$.

3.12. Remark. Let $\underline{D} = (D, \leq, \mathcal{P})$ be a pop such that the supremum of $\{p(d) \mid p \in \mathcal{P}\}$ exists for all $d \in D$. Then the pointwise supremum $\xi := \sup \mathcal{P}$ is a projection satisfying $p \circ \xi = \xi \circ p = p$ for all $p \in \mathcal{P}$. Certainly, $\underline{D}$ is approximating if and only if $\xi = \mathsf{id}_D$.

Proof. Clearly, ξ is monotone and below the identity. Let $d \in D$. We know that $p(\xi(d)) \leq p(d)$. As $p(d) \leq \xi(d)$, we deduce $p(d) = p(p(d)) \leq p(\xi(d))$. This yields $p(d) = p(\xi(d))$ for all $p \in \mathcal{P}$, whence $\xi(d) = \sup_{p \in \mathcal{P}} p(d) = \sup_{p \in \mathcal{P}} p(\xi(d)) = \xi(\xi(d))$. Therefore, ξ is a projection with $p = p \circ \xi$. From Lemma 3.1 we infer $p = \xi \circ p$. ■

Next, we formulate the analogue to Theorem 2.5 for pop's $(D, \leq, \mathcal{P})$. Recall that Corollary 2.3 tells us that $B_p = \mathsf{ker}\, p$ for all $p \in \mathcal{P}$; hence the kernels of all projections in $\mathcal{P}$ form a basis for the F-uniformity. Similarly to 2.5, we describe all uniformities on a poset induced by some directed family of projections.

3.13. Theorem. *Let $(D, \leq)$ be a poset.*

(1) *If $(D, \leq, \mathcal{P})$ is a pop, then $\{\mathsf{ker}\, p \mid p \in \mathcal{P}\}$ is a basis for the F-uniformity $\mathcal{U}_{\mathcal{P}}$ on D. For all $p \in \mathcal{P}$ and all $d \in D$ we have $B_p(d) = B_p(p(d)) = B_p(e)$ for all $e \in B_p(d)$.*

(2) *Let $\mathcal{U}$ be a uniformity on D having a basis $\mathcal{B}$ that consists of equivalence relations such that the following are satisfied:*

   (a) *For all $B \in \mathcal{B}$ and for all $d \in D$ there exists a least element $\min B(d)$ of $B(d)$.*

   (b) *For all $B \in \mathcal{B}$ and for all $d, e \in D$ with $d \leq e$ we have $\min B(d) \leq \min B(e)$.*

   For all $B \in \mathcal{B}$ define a mapping $q_B : D \to D$ by $q_B(d) := \min B(d)$. Then $(D, \leq, \{q_B \mid B \in \mathcal{B}\})$ is a pop.

(3) *With the notation of* (1) *and* (2) *we have*

   (a) $p = q_{B_p}$ *for all $p \in \mathcal{P}$, whence $(D, \leq, \mathcal{P}) = (D, \leq, \{q_{B_p} \mid p \in \mathcal{P}\})$.*

   (b) $B = \mathsf{ker}\, q_B$ *for all $B \in \mathcal{B}$, hence $(D, \mathcal{U}) = (D, \mathcal{U}_{\{q_B \mid B \in \mathcal{B}\}})$.*

Proof. (1) We already know that $B_p = \mathsf{ker}\, p$ for all $p \in \mathcal{P}$. Let $p \in \mathcal{P}$ and let $d \in D$. Lemma 2.1(2) implies that $p(d) \in B_p(d)$. It is obvious that $B_p(d) = B_p(e)$ for all $e \in B_p(d)$.

(2) In view of Theorem 2.5(2) we only need to show that each q_B is idempotent. To do this, let $B \in \mathcal{B}$ and let $d \in D$. Set $\widetilde{d} := \min B(d)$. Clearly, $(d, \widetilde{d}) \in B$. As B is an equivalence relation, we deduce that $B(d) = B(\widetilde{d})$. This implies $q_B(d) = \min B(d) = \min B(\widetilde{d}) = \min B(\min B(d)) = q_B(q_B(d))$.

(3) (a) follows from Theorem 2.5(3). In order to show (b), let $B \in \mathcal{B}$. Let $(d, e) \in B$. Then $q_B(d) = \min B(d) \leq e$ and $q_B(d) = q_B(q_B(d)) \leq q_B(e)$. Dually, as B is symmetric, we obtain $q_B(e) \leq q_B(d)$. Therefore, $(d, e) \in \mathsf{ker}\, q_B$. (See also the proof of 2.5(3).) Conversely, let $(d, e) \in \mathsf{ker}\, q_B$. Then $\min B(d) = \min B(e)$. Since $(d, \min B(d)) \in B$ and $(\min B(e), e) \in B$ (for B is symmetric), we infer $(d, e) \in B \circ B \subseteq B$. ∎

Let $\underline{D} = (D, \leq, \mathcal{P})$ be a pop. Then we call the F-uniformity $\mathcal{U}_{\underline{D}} = \mathcal{U}_{\mathcal{P}}$ the *pop uniformity* of $\underline{D}$. The induced F-topology $\tau_{\underline{D}}$ is the *pop topology* of $\underline{D}$.

3.14. Proposition. *Let $\underline{D} = (D, \leq, \mathcal{P})$ be a pop.*

(1) *The pop uniformity of $\underline{D}$ is the initial uniformity on D generated by the family $((p[D], \mathcal{U}_{\mathsf{dis}}), p)_{p \in \mathcal{P}}$.*

(2) *For all $p \in \mathcal{P}$ the restriction of the pop uniformity to $p[D]$ is the discrete uniformity.*

(3) *Each projection $p \in \mathcal{P}$ is uniformly continuous with respect to the pop uniformity.*

Proof. (1) follows from the equation $\mathsf{ker}\, p = (p \times p)^{-1}[\mathsf{id}_{p[D]}]$ for all $p \in \mathcal{P}$.

(2) Recall that the family $\{\mathsf{ker}\, q \cap p[D]^2 \mid q \in \mathcal{P}\}$ forms a basis for the relative uniformity on $p[D]$. Using Lemma 3.1, we easily infer $\mathsf{ker}\, p \cap p[D]^2 = \mathsf{id}_{p[D]}$. This yields the assertion.

(3) is a consequence of (1) and (2). ∎

The preceding proposition depicts the first major difference to arbitrary F-posets. Whilst the mappings $f \in \mathcal{F}$ of an F-poset $(D, \leq, \mathcal{F})$ need not be continuous with respect to the F-topology (Example 2.17), all projections $p \in \mathcal{P}$ of a pop $(D, \leq, \mathcal{P})$ are even uniformly continuous. Note further that Proposition 3.14 implies that the pop topology of $\underline{D}$ is the initial topology generated by the family $((p[D], \tau_{\mathsf{dis}}), p)_{p \in \mathcal{P}}$. Moreover, the pop topology restricted to $p[D]$ is the discrete topology.

We know by Proposition 2.7 that the pop uniformity is pseudo-metrizable if and only if $\mathcal{P}$ contains a countable cofinal chain. This statement can be strengthened a bit.

3.15. THEOREM. *Let* $\underline{D} = (D, \leq, \mathcal{P})$ *be a pop.*

(1) *Let* $\{p_n \mid n \in \mathbb{N}_0\}$ *be a cofinal subset of* $\mathcal{P}$ *such that* $m \leq n$ *implies* $p_m \leq p_n$. *For all* $d, e \in D$ *define*

$$\ell_{\underline{D}}(d, e) := \sup\{n \in \mathbb{N}_0 \mid p_n(d) = p_n(e)\} \in \mathbb{N}_0 \cup \{\infty\}.$$

Then, with the usual convention $2^{-\infty} := 0$, *the mapping*

$$\varrho_{\underline{D}} : D \times D \to [0, 1], \qquad (d, e) \mapsto 2^{-\ell_{\underline{D}}(d,e)},$$

is a pseudo-ultrametric on D *inducing the pop uniformity. Furthermore,* $B_{p_n}(d) = \{e \in D \mid \varrho_{\underline{D}}(d, e) \leq 2^{-n}\}$, *and* $\varrho_{\underline{D}}(d, e) = 2^{-n}$ *if and only if* $p_n(d) = p_n(e)$ *and* $p_{n+1}(d) \neq p_{n+1}(e)$.

(2) *The following are equivalent:*

  (i) *The pop uniformity of* $\underline{D}$ *is pseudo-ultrametrizable.*

  (ii) $\mathcal{P}$ *contains a cofinal* ω*-chain or a greatest element.*

Proof. (1) It is routine to check that $\varrho_{\underline{D}}$ is a pseudo-ultrametric on D. Moreover, $\ker p_n = \{(d, e) \in D^2 \mid p_n(d) = p_n(e)\} = \{(d, e) \in D^2 \mid \ell_{\underline{D}}(d, e) \geq n\} = \{(d, e) \in D^2 \mid \varrho_{\underline{D}}(d, e) \leq 2^{-n}\}$. Hence, $\varrho_{\underline{D}}$ induces the pop uniformity of $\underline{D}$ by Lemma 2.6. Finally, $\varrho_{\underline{D}}(d, e) = 2^{-n}$ if and only if $\ell_{\underline{D}}(d, e) = n$ if and only if $p_n(d) = p_n(e)$ and $p_{n+1}(d) \neq p_{n+1}(e)$.

(2) follows from (1) and Proposition 2.7. ■

Let $(D, \leq)$ be a poset and let $p : D \to D$ be a projection. Clearly, if p has finite range, then it is finitely separated from id_D because $p[D]$ is a finite separating set. We prove that the converse is also true. Let p be finitely separated from id_D and let $M \subseteq D$ be a finite separating set. Let $d \in D$ and choose some $m \in M$ with $p(d) \leq m \leq d$. Then $p(d) \leq p(m) \leq p(d)$, i.e. $p(d) = p(m)$. Consequently, $p[D] = p[M]$ is finite. Together with Proposition 2.8, this yields

3.16. PROPOSITION. *The pop uniformity of a pop* $(D, \leq, \mathcal{P})$ *is totally bounded if and only if* $p[D]$ *is finite for all* $p \in \mathcal{P}$. ■

Similarly to the remarks after Proposition 2.8, we deduce for any pop $(D, \leq, \mathcal{P})$ and any subset $A \subseteq D$ that $(A, \mathcal{U}_{\mathcal{P}}|_A)$ is totally bounded if and only if the set $p[A]$ is finite for all $p \in \mathcal{P}$.

Next, we formulate the basic properties of the pop topology. Convergence of nets can be described as follows (cf. the corresponding Lemma 2.9 for F-posets).

3.17. LEMMA. *Given a pop* $(D, \leq, \mathcal{P})$, *a net* $(d_n)_{n \in N}$ *in* D *converges to an element* $d \in D$ *if and only if, for all* $p \in \mathcal{P}$, *there is an index* $n_p \in N$ *such that* $p(d_n) = p(d)$ *for all* $n \geq n_p$. ■

Recall further that the net $(p(d))_{p \in \mathcal{P}}$ converges to d and that, in particular, $\bigcup_{p \in \mathcal{P}} p[D]$ is dense in D (Proposition 2.10(1)).

3.18. PROPOSITION. *Let* $\underline{D} = (D, \leq, \mathcal{P})$ *be a pop. Then*

(1) $\{B_p(d) \mid d \in D,\ p \in \mathcal{P}\}$ *is a basis for* $\tau_{\underline{D}}$ *consisting of clopen sets. Hence,* $(D, \tau_{\underline{D}})$ *is zero-dimensional.*
(2) $(D, \tau_{\underline{D}})$ *is Hausdorff if and only if, for all* $d, e \in D$*, the equality* $p(d) = p(e)$ *for all* $p \in \mathcal{P}$ *implies* $d = e$.
(3) $(D, \tau_{\underline{D}})$ *is Hausdorff if and only if it is totally disconnected.*
(4) *If* $(D, \tau_{\underline{D}})$ *is Hausdorff, then* $p[M]$ *is closed in* $(D, \tau_{\underline{D}})$ *for all* $M \subseteq D$.

Proof. (1) Let $d \in D$ and let $p \in \mathcal{P}$. Due to Theorem 3.13(1) we have $B_p(d) = B_p(e)$ for all $e \in B_p(d)$, whence $B_p(d)$ is open and $\{B_p(d) \mid d \in D,\ p \in \mathcal{P}\}$ is a basis for $\tau_{\underline{D}}$. Let $e \in D \setminus B_p(d)$. Then $B_p(d) \cap B_p(e) = \emptyset$. Thus, e is in the interior of $D \setminus B_p(d)$ and $B_p(d)$ is closed.

(2) results from Proposition 2.10(3).

(3) Zero-dimensional Hausdorff spaces are totally disconnected. Totally disconnected topological spaces are T_1.

(4) By Proposition 3.14(2) we know that the pop uniformity restricted to $p[M]$ is the discrete uniformity; hence $p[M]$ is complete in the relative uniformity. As $(D, \tau_{\underline{D}})$ is Hausdorff, $p[M]$ is closed. ■

Let $(D, \leq, \mathcal{P})$ be a pop. As we have seen in Example 3.6, two projections of $\mathcal{P}$ need not commute. In the following we shall *require* that any two elements of $\mathcal{P}$ commute. We call the projection set $\mathcal{P}$ *Abelian* if $p \circ q = q \circ p$ for all $p, q \in \mathcal{P}$. From Lemma 3.7 we immediately derive:

3.19. PROPOSITION. *Let* $(D, \leq, \mathcal{P})$ *be a pop. Then the projection set* $\mathcal{P}$ *is Abelian if and only if* $p \circ q$ *is a projection for all* $p, q \in \mathcal{P}$. ■

Let $(D, \leq, \mathcal{P})$ be a pop with $\mathcal{P}$ Abelian. Then $\mathcal{P}$ is uniformly equicontinuous. To see this, let $p, r \in \mathcal{P}$ and let $d, e \in D$ be such that $(d, e) \in \ker p$. Then $p(r(d)) = r(p(d)) = r(p(e)) = p(r(e))$, whence $(r(d), r(e)) \in \ker p$.

For a pop $(D, \leq, \mathcal{P})$ let $(\langle \mathcal{P} \rangle, \circ)$ denote the semigroup of mappings from D into itself generated by $\mathcal{P}$. Clearly, $\mathcal{P}$ is Abelian if and only if $(\langle \mathcal{P} \rangle, \circ)$ is an Abelian semigroup. In the case when $\mathcal{P}$ is a chain, we infer from Lemma 3.1 that $\mathcal{P}$ is Abelian and $\mathcal{P} = \langle \mathcal{P} \rangle$.

3.20. PROPOSITION. *Let* $(D, \leq, \mathcal{P})$ *be a pop with Abelian projection set* $\mathcal{P}$*. Then* $(\langle \mathcal{P} \rangle, \circ)$ *is an Abelian semigroup consisting of projections on* D*. Moreover,* $(D, \leq, \langle \mathcal{P} \rangle)$ *is a pop with an Abelian projection set. The pointwise supremum* $\sup \mathcal{P}$ *exists if and only if* $\sup \langle \mathcal{P} \rangle$ *exists, and they are equal then. The pop uniformities* $\mathcal{U}_{\mathcal{P}}$ *and* $\mathcal{U}_{\langle \mathcal{P} \rangle}$ *coincide.*

Proof. First of all, $\langle \mathcal{P} \rangle$ consists of projections only because of Proposition 3.19. The remaining assertions follow from the fact that $\mathcal{P}$ is cofinal in $\langle \mathcal{P} \rangle$ (cf. the remarks after Lemma 2.6). ■

An *inf-semilattice* is a poset $(I, \leq)$ in which any two elements have an infimum. The following simple (and well known) observation shows us that inf-semilattices appear exactly as Abelian semigroups consisting of idempotent elements only:

3.21. REMARK. (1) Let $(\mathcal{S},\circ)$ be an Abelian semigroup whose elements are all idempotent. For all $x,y\in\mathcal{S}$ define $x\preccurlyeq y$ if $x\circ y=x$. Then $(\mathcal{S},\preccurlyeq)$ is an inf-semilattice with $\inf\{x,y\}=x\circ y$ for all $x,y\in\mathcal{S}$.

(2) Let $(I,\leq)$ be an inf-semilattice and set $i\circ j:=\inf\{i,j\}$ for all $i,j\in I$. Then $(I,\circ)$ is an Abelian semigroup consisting of idempotent elements only.

Proof. (1) As all elements of $\mathcal{S}$ are idempotent, the relation $\preccurlyeq$ is reflexive. Let $x,y,z\in\mathcal{S}$. If $x\preccurlyeq y$ and $y\preccurlyeq x$, then $x\circ y=x$ and $y\circ x=y$. Since $(\mathcal{S},\circ)$ is Abelian, we infer $x=y$. Finally, let $x\preccurlyeq y$ and $y\preccurlyeq z$. Then $x\circ y=x$ and $y\circ z=y$; hence $x\circ z=x\circ y\circ z=x\circ y=x$, i.e. $x\preccurlyeq z$. Thus, $\preccurlyeq$ is a partial order. Clearly, $x\circ y\preccurlyeq x,y$. Let $z\preccurlyeq x,y$. Then $z\circ x=z$ and $z\circ y=z$, $z\circ x\circ y=z\circ y=z$, and hence $z\preccurlyeq x\circ y$.

(2) is obvious. ∎

Let $(D,\leq)$ be a poset and let $(\mathcal{S},\circ)$ be an Abelian semigroup consisting of projections on D, where $\circ$ is the composition of mappings. Then, in view of Lemma 3.1, the partial order $\preccurlyeq$ defined in 3.21(1) coincides with the pointwise order on $\mathcal{S}$.

In what follows we will consider inf-semilattices that are also directed. Basic examples are lattices and, in particular, linearly ordered sets. It turns out that directed inf-semilattices can be characterized as Abelian projection sets $\mathcal{P}$ of (approximating) pop's with $\mathcal{P}=\langle\mathcal{P}\rangle$:

3.22. PROPOSITION. (1) *Let $(D,\leq,\mathcal{P})$ be a pop with Abelian projection set $\mathcal{P}$. Then $(\langle\mathcal{P}\rangle,\leq)$ is a directed inf-semilattice with* $\inf\{q,r\}=q\circ r$ *for all $q,r\in\langle\mathcal{P}\rangle$.*

(2) *Let $(I,\leq)$ be a directed inf-semilattice. For all $i\in I$ define a mapping $p_i:I\to I$ by $p_i(j):=\inf\{i,j\}$. Then $\underline{I}:=(I,\leq,\{p_i\mid i\in I\})$ is an approximating pop with Abelian projection set. The mapping $\varphi:I\to\{p_i\mid i\in I\}$, $i\mapsto p_i$, is an order isomorphism. For all $i,j\in I$ we have $p_i\circ p_j=p_{\inf\{i,j\}}$. All projections p_i are downwards closed.*

Proof. (1) By Proposition 3.20, $\langle P\rangle$ is directed and consists of idempotent elements only. Now the assertion follows from 3.21(1) and the remark after it (or from Lemma 3.7).

(2) Clearly, p_i is a projection for all $i\in I$. As $p_i\leq p_j$ whenever $i\leq j$, we infer that $\underline{I}$ is a pop and φ is monotone. Obviously, $p_i\circ p_j=p_{\inf\{i,j\}}$ for all $i,j\in I$. In particular, $\{p_i\mid i\in I\}$ is Abelian. By definition, φ is surjective. Let $i,j\in I$ with $p_i\leq p_j$. Then $i=p_i(i)\leq p_j(i)=\inf\{i,j\}$, whence $i\leq j$. Thus, φ is order-reflecting and in particular injective. Let $k\in I$. As $p_i(k)=\inf\{i,k\}=k$ for all $i\geq k$, we obtain $k=\sup_{i\geq k}p_i(k)=\sup_{i\in I}p_i(k)$. Therefore, $\underline{I}$ is approximating. Since $p_i[I]=i{\downarrow}$, the projections p_i are downwards closed for all $i\in I$. ∎

We know that if $(D,\leq,\mathcal{P})$ is a pop with an Abelian projection set, then $(\langle\mathcal{P}\rangle,\circ)$ is an Abelian semigroup whose elements are projections (Proposition 3.20). Conversely, if D is a set and $(\mathcal{S},\circ)$ is an Abelian semigroup consisting of idempotent mappings from D into itself, then the projection order $\sqsubseteq_{\mathcal{S}}$ is a partial order on D such that each element of $\mathcal{S}$ is a (downwards closed) projection on $(D,\sqsubseteq_{\mathcal{S}})$ (Proposition 3.8). If we can ensure $\mathcal{S}$ to be directed with respect to $\sqsubseteq_{\mathcal{S}}$, then we obtain a pop.

3.23. PROPOSITION. *Let D be a set and let $(\mathcal{S}, \circ)$ be a non-empty Abelian semigroup whose elements are idempotent mappings from D into itself. Then for all $p, q \in \mathcal{S}$ there is an element $r \in \mathcal{S}$ such that $p \circ r = p$ and $q \circ r = q$ if and only if $(D, \sqsubseteq_{\mathcal{S}}, \mathcal{S})$ is a pop.*

Proof. Let $p, q \in \mathcal{S}$ and let $r \in \mathcal{S}$ with $p \circ r = p$ and $q \circ r = q$. Then $p, q \sqsubseteq_{\mathcal{S}} r$ by Lemma 3.1. Hence, $\mathcal{S}$ is directed. The converse also follows from Lemma 3.1. ∎

3.24. PROPOSITION. *Let $(D, \leq, \mathcal{P})$ be a pop with Abelian projection set such that $\mathcal{P} = \langle \mathcal{P} \rangle$. Then*

(1) *$(D, \sqsubseteq_{\mathcal{P}}, \mathcal{P})$ is a pop with all $p \in \mathcal{P}$ downwards closed.*
(2) *$\sqsubseteq_{\mathcal{P}}$ is the least order $\leq'$ on D such that $(D, \leq', \mathcal{P})$ is a pop.*
(3) *$(\mathcal{P}, \sqsubseteq_{\mathcal{P}}) = (\mathcal{P}, \leq)$.*
(4) *If $(\mathcal{P}, \leq)$ is a chain, then $(D, \sqsubseteq_{\mathcal{P}})$ is a tree.*
(5) *The pop uniformities of $(D, \leq, \mathcal{P})$ and $(D, \sqsubseteq_{\mathcal{P}}, \mathcal{P})$ coincide.*

Proof. (1) From Lemma 3.1 and Proposition 3.23 we deduce that $(D, \sqsubseteq_{\mathcal{P}}, \mathcal{P})$ is a pop. Proposition 3.8 tells us that each $p \in \mathcal{P}$ is downwards closed.

(2) and (3) follow from Corollary 3.9.

(4) Let $c, d, e \in D$ with $c, d \sqsubseteq_{\mathcal{P}} e$. Without loss of generality, let $p, q \in \mathcal{P}$ be such that $p \leq q$, $c = p(e)$, and $d = q(e)$. Then $p \circ q = p$ by Lemma 3.1 and $c = p(e) = p(q(e)) = p(d)$; hence $c \sqsubseteq_{\mathcal{P}} d$.

(5) is trivial. ∎

Let $(D, \leq, \mathcal{P})$ be a pop with Abelian $\mathcal{P}$ satisfying $\mathcal{P} = \langle \mathcal{P} \rangle$. Note that the partial orders $\leq$ and $\sqsubseteq_{\mathcal{P}}$ need not coincide. For instance, Example 3.27 below exhibits a pop whose projection set is an ω-chain, but no projection is downwards closed. But even if all projections in $\mathcal{P}$ are downwards closed, $\leq$ and $\sqsubseteq_{\mathcal{P}}$ may be different (cf. Example 3.35).

Examples. As we did for F-posets, we illustrate the manifold occurrences of pop's in different areas. We start with some basic examples and then revisit the domain-theoretic model for ultrametric spaces given by Flagg and Kopperman [19]. Moreover, we shall realize that pop's occur quite naturally in the theory of traces.

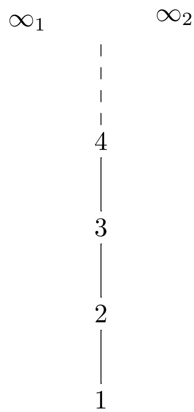


Fig. 3.1. A pop that is not Hausdorff

3.25. EXAMPLE. (a) Let $(D, \leq)$ be as in Figure 3.1. For all $n \in \mathbb{N}$ we define a projection p_n by $p_n(\infty_1) := p_n(\infty_2) := n$, $p_n(m) := \min\{m, n\}$ $(m \in \mathbb{N})$. Then $(D, \leq, \{p_n \mid n \in \mathbb{N}\})$ is a pop that is not Hausdorff (and in particular not approximating).

(b) Consider the poset $(D, \leq)$ given in Figure 3.2. We define projections p_n $(n \in \mathbb{N})$ as follows: $p_n(d) := d_n$, $p_n(e) := e_n$, $p_n(d_m) := d_{\min\{m,n\}}$, $p_n(e_m) := e_{\min\{m,n\}}$. This yields a pop $(D, \leq, \{p_n \mid n \in \mathbb{N}\})$ that is Hausdorff but not approximating (because $\{d_m \mid m \in \mathbb{N}\}$ has no supremum).

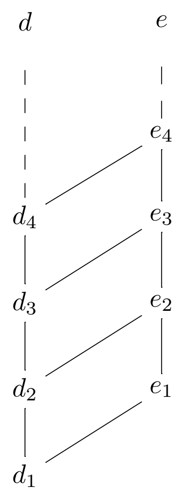


Fig. 3.2. A Hausdorff pop that is not approximating

3.26. EXAMPLE. Let I be a non-empty index set and let $D := \mathbb{N}_0^I$ be endowed with the product order. Let $n \in \mathbb{N}_0$ and let $I_0 \subseteq I$ be finite. Define a projection p_{n,I_0} by $p_{n,I_0}((k_i)_{i\in I}) := (l_i)_{i\in I}$ with $l_i = \min\{k_i, n\}$ if $i \in I_0$ and $l_i = 0$ otherwise. Clearly, $\mathcal{P} := \{p_{n,I_0} \mid n \in \mathbb{N}_0,\ I_0 \subseteq I \text{ finite}\}$ is directed and has id_D as supremum. Hence, $\underline{D} := (D, \leq, \mathcal{P})$ is an approximating pop.

Obviously, $\ker p_{n,I_0} = \{((k_i)_{i\in I}, (l_i)_{i\in I}) \in D^2 \mid \min\{k_i, n\} = \min\{l_i, n\} \text{ for all } i \in I_0\}$. Let π_i be the canonical projection from D onto the ith coordinate space. Let $(k_i)_{i\in I} \in D$ and let $I_0 \subseteq I$ be finite. Let $n \in \mathbb{N}_0$ with $n > k_i$ for all $i \in I_0$. Then

$$\begin{aligned} B_{p_{n,I_0}}((k_i)_{i\in I}) &= \{(l_i)_{i\in I} \in D \mid \min\{k_i, n\} = \min\{l_i, n\} \text{ for all } i \in I_0\} \\ &= \{(l_i)_{i\in I} \in D \mid k_i = l_i \text{ for all } i \in I_0\} = \bigcap_{i\in I_0} \pi_i^{-1}[k_i]. \end{aligned}$$

Hence, the pop topology of $\underline{D}$ is the product topology of the family $(\mathbb{N}_0, \tau_{\mathsf{dis}})_{i\in I}$.

Since each p_{n,I_0} has finite range, $\underline{D}$ is totally bounded (Proposition 3.18). Clearly, $\underline{D}$ is *not* compact.

If $|I| = 1$, then $\underline{D} = (\mathbb{N}_0, \leq, \{p_n \mid n \in \mathbb{N}_0\})$ with $p_n(m) = \min\{m, n\}$. Note that then the pop topology is discrete whereas the pop uniformity is not because $\ker p_n \supsetneq \mathsf{id}_{\mathbb{N}_0}$ for all $n \in \mathbb{N}_0$.

3.27. Example. For all $n \in \mathbb{N}_0$ and all $r \in [0,1]$ define $\mathsf{trunc}_n(r) := \lfloor r \cdot 10^n \rfloor \cdot 10^{-n}$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Thus, we get $\mathsf{trunc}_n(r)$ by truncating r after the nth decimal. Then $([0,1], \leq, \{\mathsf{trunc}_n \mid n \in \mathbb{N}_0\})$ is an approximating pop. Note that $\bigcup_{n\in\mathbb{N}_0} \mathsf{trunc}_n[[0,1]]$ is precisely the set of all terminating decimals in $[0,1]$. The projections trunc_n are not Scott-continuous: we have $\sup\{1-10^{-m} \mid m \in \mathbb{N}_0\} = 1$, whence $\mathsf{trunc}_n(\sup\{1-10^{-m} \mid m \in \mathbb{N}_0\}) = 1$; but $\sup\{\mathsf{trunc}_n(1-10^{-m}) \mid m \in \mathbb{N}_0\} = \sup\{\mathsf{trunc}_n(1-10^{-m}) \mid m \geq n\} = 1-10^{-n} \neq 1$.

As $\mathsf{trunc}_n(r) = \mathsf{trunc}_n(s)$ implies $|r-s| < 10^{-n}$, whereas the converse is not true in general, we see that the pop uniformity (pop topology, respectively) is strictly finer than the Euclidean uniformity (Euclidean topology, respectively). Furthermore, $([0,1], \leq, \{\mathsf{trunc}_n \mid n \in \mathbb{N}_0\})$ is not complete since the sequence $(1-10^{-m})_{m\in\mathbb{N}_0}$ is Cauchy, but it does not converge with respect to the pop topology.

Note that we may also consider other counting systems, i.e. bases other than 10.

3.28. Example. (a) Let X be a non-empty set, let $(D, \leq)$ be a poset with least element $\perp$, and let $F(X,D) := \{f \mid f : X \to D\}$. As usual, let $F(X,D)$ be ordered pointwise. For all $A \subseteq X$ and all $f \in F(X,D)$ we define the mapping $p_A(f) : X \to D$ by $p_A(f)(x) := f(x)$ if $x \in A$ and $p_A(f)(x) := \perp$ otherwise. Then $p_A : f \mapsto p_A(f)$ is a projection on $F(X,D)$. Let κ be an infinite cardinal. Then, clearly, $(F(X,D), \leq, \{p_A \mid |A| < \kappa\})$ is an approximating pop with an Abelian projection set. Endow D with the discrete uniformity and notice that $\mathsf{ker}\, p_A = \{(f,g) \in F(X,D)^2 \mid f|_A = g|_A\}$ for all $A \subseteq D$. Consequently, the pop uniformity of $(F(X,D), \leq, \{p_A \mid |A| < \kappa\})$ is the uniformity of uniform convergence in all sets A having cardinality less than κ.

(b) Let $\mathbb{R}_{\geq 0}[[x_1, \ldots, x_n]]$ be the set of all formal power series over the non-negative real numbers in the indeterminates $x_1, \ldots, x_n$. Let

$$f = \sum_{i_1,\ldots,i_n \in \mathbb{N}_0} a_{(i_1,\ldots,i_n)} x_1^{i_1} \cdots x_n^{i_n} \quad \text{and} \quad g = \sum_{i_1,\ldots,i_n \in \mathbb{N}_0} b_{(i_1,\ldots,i_n)} x_1^{i_1} \cdots x_n^{i_n}$$

be elements of $\mathbb{R}_{\geq 0}[[x_1, \ldots, x_n]]$. We define a partial order by letting $f \leq g$ if $a_{(i_1,\ldots,i_n)} \leq b_{(i_1,\ldots,i_n)}$ for all $(i_1, \ldots, i_n) \in \mathbb{N}_0^n$. For all $(m_1, \ldots, m_n) \in \mathbb{N}_0^n$ define

$$p_{(m_1,\ldots,m_n)}(f) := \sum_{0 \leq i_\nu \leq m_\nu,\ \nu=1,\ldots n} a_{(i_1,\ldots,i_n)} x_1^{i_1} \cdots x_n^{i_n}.$$

Clearly, $(\mathbb{R}_{\geq 0}[[x_1, \ldots, x_n]], \leq, \{p_{(m_1,\ldots,m_n)} \mid (m_1, \ldots, m_n) \in \mathbb{N}_0^n\})$ is an approximating pop. Note that $p_{(m_1,\ldots,m_n)}(f)$ is a polynomial.

Observe that $(\mathbb{R}_{\geq 0}[[x_1, \ldots, x_n]], \leq)$ can be interpreted as $(F(X,D), \leq)$ in (a) with $X = \mathbb{N}_0^n$ and $D = \mathbb{R}_{\geq 0}$. From this point of view we have $p_{(m_1,\ldots,m_n)} = p_A$ with $A = \{(i_1, \ldots, i_n) \in \mathbb{N}_0^n \mid 0 \leq i_\nu \leq m_\nu \text{ for all } \nu = 1, \ldots, n\}$. In fact, the set $\{p_{(m_1,\ldots,m_n)} \mid (m_1, \ldots, m_n) \in \mathbb{N}_0^n\}$ is cofinal in $\{p_A \mid |A| < \aleph_0\}$. By (a) we deduce that the pop uniformity is the uniformity of pointwise convergence (where $\mathbb{R}_{\geq 0}$ carries the discrete uniformity).

In the following example we derive a pop from a pop representation system.

3.29. Example. Let $(G, \cdot)$ be a group. (In fact, we may take any (universal) algebra. For the sake of simplicity, we deal with groups here.) Let $\mathcal{L}(G)$ be the family of all subgroups of G. Let $\mathcal{P}_{\mathsf{fin}}(G)$ be the set of all finite subsets of G. Given a set $M \subseteq G$, let $\langle M \rangle$ denote the smallest subgroup of G containing M. Clearly, $\langle \cdot \rangle$ is a closure operator on $(\mathcal{P}(G), \subseteq)$. Let $S \in \mathcal{L}(G)$ and let $F \in \mathcal{P}_{\mathsf{fin}}(G)$. Then certainly $\langle S \rangle = S$ and $\langle S \cap F \rangle \in \mathcal{L}(G)$, whence $\mathcal{R} = (G, \mathcal{L}(G), \mathcal{P}_{\mathsf{fin}}(G), \langle \cdot \rangle)$ is a pop representation system. As usual, we write $H_1 \leq H_2$ for any $H_1, H_2 \in \mathcal{L}(G)$ with $H_1 \subseteq H_2$. Let $\underline{D_{\mathcal{R}}} = (\mathcal{L}(G), \leq, \{r_F \mid F \in \mathcal{P}_{\mathsf{fin}}(G)\})$ be the induced pop (cf. Theorem 3.10). Recall that $(\mathcal{L}(G), \leq)$ is a complete lattice. Given any $\mathcal{M} \subseteq \mathcal{L}(G)$, we have $\sup \mathcal{M} = \langle \bigcup \mathcal{M} \rangle$ and $\inf \mathcal{M} = \bigcap \mathcal{M}$. To show that $\underline{D_{\mathcal{R}}}$ is approximating, let $S \in \mathcal{L}(G)$. Then $\sup_{F \in \mathcal{P}_{\mathsf{fin}}(G)} r_F(S) = \langle \bigcup \{ \langle S \cap F \rangle \mid F \subseteq G \text{ finite} \} \rangle = \langle \bigcup \{ \langle F' \rangle \mid F' \subseteq S \text{ finite} \} \rangle = S$.

It is well known that $K(\mathcal{L}(G)) = \{S \in \mathcal{L}(G) \mid S \text{ is finitely generated}\}$ and that $(\mathcal{L}(G), \leq)$ is algebraic. Let $S \in K(\mathcal{L}(G))$ and let $F \in \mathcal{P}_{\mathsf{fin}}(G)$ with $S = \langle F \rangle$. Then $r_F(S) = \langle F \rangle = S$. Hence, each $S \in K(\mathcal{L}(G))$ is the image of some projection r_F. Conversely, if $F \in \mathcal{P}_{\mathsf{fin}}(G)$ and $S \in K(\mathcal{L}(G))$, then $r_F(S) = \langle S \cap F \rangle$ is finitely generated. We conclude that $K(\mathcal{L}(G)) = \bigcup_{F \in \mathcal{P}_{\mathsf{fin}}(G)} r_F[\mathcal{L}(G)]$, i.e. the set of compact elements of $\mathcal{L}(G)$ coincides with the set of all images of all projections r_F. We shall investigate approximating pop's with this property in Section 3.4. From Lemma 3.3 we infer that each projection r_F is Scott-continuous. Moreover, note that for any $F \in \mathcal{P}_{\mathsf{fin}}(G)$ we have $r_F[\mathcal{L}(G)] = \{ \langle S \cap F \rangle \mid S \leq G \} = \{ \langle F' \rangle \mid F' \subseteq F \}$. This implies that $r_F[\mathcal{L}(G)]$ has at most $2^{|F|}$ elements and hence has finite range. As a consequence, $\mathcal{L}(G)$ is totally bounded with respect to the pop uniformity (Proposition 3.16). In addition, since $(\mathcal{L}(G), \leq)$ is a complete lattice and thus a dcpo and since all projections r_F are Scott-continuous, $(\mathcal{L}(G), \mathcal{U}_{\underline{D_{\mathcal{R}}}})$ is complete (Proposition 2.25). Summing up, $\underline{D_{\mathcal{R}}}$ is a compact approximating pop. By Corollary 2.49 the pop topology coincides with the Lawson topology of $(\mathcal{L}(G), \leq)$.

In view of the previous example we remark here the following. Let $(D, \leq)$ be an *algebraic Scott domain*, i.e. an algebraic dcpo which is also a bcpo. (For instance, a complete algebraic lattice is such a domain.) Then it is not hard to see that we obtain an approximating pop $\underline{D} = (D, \leq, \{p_A \mid A \in \mathcal{P}_{\mathsf{fin}}(K(D))\})$ by setting $p_A(d) := \sup(d{\downarrow} \cap A)$ for all $d \in D$, $A \in \mathcal{P}_{\mathsf{fin}}(K(D))$. Clearly, we have $K(D) = \bigcup_{A \in \mathcal{P}_{\mathsf{fin}}(K(D))} p_A[D]$ and each p_A has finite range. As pointed out before, $\underline{D}$ is compact with its pop topology being the Lawson topology of the domain.

Notice that there is a similar but more complicated construction of projections for the more general class of *bifinite domains*. This well known construction uses a certain notion of "completeness". We will come back to this later in Section 3.4.

In contrast to Edalat and Heckmann's [16] continuous poset of formal balls for arbitrary metric spaces (cf. Example 2.22), Flagg and Kopperman [19] showed that the closed balls of any *ultra*metric space together with all its singletons form an algebraic poset. As we endowed the formal ball model with an F-poset structure (2.22), we can equip the *closed ball model* of an ultrametric space with a pop structure.

Given any ultrametric space (X, ϱ), $x \in X$, and $r > 0$, let $B_r(x) := \{y \in X \mid \varrho(x, y) \leq r\}$ be the closed ball with centre x and radius r. Since $\bigcap_{r>0} B_r(x) = \{x\}$, it is natural to set $B_{2^{-\infty}}(x) := B_0(x) := \{x\}$.

3.30. Example. Let (X, ϱ) be an ultrametric space and let $D := D_{\mathrm{cb}}(X, \varrho) := \{B_{2^{-m}}(x) \mid x \in X,\ m \in \mathbb{N}_0\} \cup \{\{x\} \mid x \in X\}$. Clearly, $(D, \supseteq)$ is a poset. Recall from [19] that:

(1) $(D, \supseteq)$ is an algebraic bcpo with $K(D) = \{B_{2^{-m}}(x) \mid x \in X,\ m \in \mathbb{N}_0\}$.

(2) $(D, \supseteq)$ is a dcpo if and only if (X, ϱ) is complete.

(3) $(D, \supseteq)$ has a countable basis if and only if (X, ϱ) is separable.

For all $n \in \mathbb{N}_0$ we define p_n as follows. Let $x \in X$ and let $m \in \mathbb{N}_0$. Then set $p_n(B_{2^{-m}}(x)) := B_{2^{-\min\{m,n\}}}(x)$ and $p_n(\{x\}) := B_{2^{-n}}(x)$. We show that $\underline{D}_{\mathrm{cb}}(X, \varrho) := (D_{\mathrm{cb}}(X, \varrho), \supseteq, \{p_n \mid n \in \mathbb{N}_0\})$ is an approximating pop.

Let $n \in \mathbb{N}_0$. Let $m_1, m_2 \in \mathbb{N}_0 \cup \{\infty\}$ and let $x_1, x_2 \in X$ with $B_{2^{-m_1}}(x_1) = B_{2^{-m_2}}(x_2)$. We may assume $m_1 \leq m_2$. If $n \leq m_1$, then $B_{2^{-n}}(x_1) \supseteq B_{2^{-m_1}}(x_1) \ni x_2$, whence $B_{2^{-n}}(x_1) = B_{2^{-n}}(x_2)$ because ϱ is an ultrametric. If $m_1 \leq n \leq m_2$, then $B_{2^{-n}}(x_2) \supseteq B_{2^{-m_2}}(x_2) \ni x_1$ and thus $B_{2^{-n}}(x_2) = B_{2^{-n}}(x_1)$. If $m_2 \leq n$, then $\min\{m_1, n\} = m_1$, $\min\{m_2, n\} = m_2$, and by assumption $B_{2^{-m_1}}(x_1) = B_{2^{-m_2}}(x_2)$. This shows us that p_n is well defined. Now let $B_{2^{-m_1}}(x_1) \supseteq B_{2^{-m_2}}(x_2)$. If $m_1 \leq m_2$, then clearly $B_{2^{-\min\{m_1,n\}}}(x_1) \supseteq B_{2^{-\min\{m_2,n\}}}(x_1) = B_{2^{-\min\{m_2,n\}}}(x_2)$ because $B_{2^{-m_1}}(x_1) = B_{2^{-m_1}}(x_2)$. If $m_1 \geq m_2$, then $B_{2^{-m_1}}(x_1) = B_{2^{-m_1}}(x_2) \subseteq B_{2^{-m_2}}(x_2)$. Hence $B_{2^{-m_1}}(x_1) = B_{2^{-m_2}}(x_2)$ and thus $B_{2^{-\min\{m_1,n\}}}(x_1) = B_{2^{-\min\{m_2,n\}}}(x_2)$ by the arguments above. Therefore, p_n is monotone. Obviously, p_n is below the identity and idempotent, and $n_1 \leq n_2$ implies $p_{n_1}(d) \supseteq p_{n_2}(d)$ for all $d \in D$. Moreover, $\sup_{n \in \mathbb{N}_0} p_n(B_{2^{-m}}(x)) = \sup_{n \geq m} p_n(B_{2^{-m}}(x)) = B_{2^{-m}}(x)$ for all $m \in \mathbb{N}_0$ and $\sup_{n \in \mathbb{N}_0} p_n(\{x\}) = \bigcap_{n \in \mathbb{N}_0} B_{2^{-n}}(x) = \{x\}$ because (X, ϱ) is Hausdorff.

Next, we investigate the order structure of the closed ball model (and thereby improve Theorem 2.2 in [19]).

3.31. PROPOSITION. *The order $\supseteq$ coincides with the projection order of $(D_{\mathrm{cb}}(X, \varrho), \{p_n \mid n \in \mathbb{N}_0\})$. In particular, $(D_{\mathrm{cb}}(X, \varrho), \supseteq)$ is a tree. Further, $(D_{\mathrm{cb}}(X, \varrho), \supseteq)$ is well founded. Each directed subset of $D_{\mathrm{cb}}(X, \varrho)$ is a countable chain that is either finite or isomorphic to ω or $\omega + 1$.*

Proof. Let $x_1, x_2 \in X$, let $m_1, m_2 \in \mathbb{N}_0 \cup \{\infty\}$, and let $B_{2^{-m_1}}(x_1) \supseteq B_{2^{-m_2}}(x_2)$. If $m_1 < m_2$, then $B_{2^{-m_1}}(x_1) = B_{2^{-m_1}}(x_2) = p_{m_1}(B_{2^{-m_2}}(x_2))$. If $m_1 \geq m_2$, then $B_{2^{-m_1}}(x_1) = B_{2^{-m_1}}(x_2) \subseteq B_{2^{-m_2}}(x_2) \subseteq B_{2^{-m_1}}(x_1)$, i.e. $B_{2^{-m_1}}(x_1) = B_{2^{-m_2}}(x_2)$. Hence, $\supseteq$ is equal to $\sqsubseteq_{\{p_n \mid n \in \mathbb{N}_0\}}$ (Proposition 3.24(2)). By 3.24(4), $(D_{\mathrm{cb}}(X, \varrho), \supseteq)$ is a tree.

To prove well foundedness, it is enough to show that $(D, \supseteq)$ satisfies the descending chain condition. Thus, for all $n \in \mathbb{N}$ let $d_n \in D$ with $d_n \subseteq d_{n+1}$. Let $x_n \in X$ and let $m_n \in \mathbb{N}_0 \cup \{\infty\}$ with $d_n = B_{2^{-m_n}}(x_n)$, whence $B_{2^{-m_n}}(x_n) \subseteq B_{2^{-m_{n+1}}}(x_{n+1})$. Then $B_{2^{-m_n}}(x_n) = B_{2^{-m_n}}(x_1)$ for all $n \in \mathbb{N}$. Note that if $m_n \leq m_{n+1}$, then we obtain $B_{2^{-m_n}}(x_1) \supseteq B_{2^{-m_{n+1}}}(x_1)$ and thus $B_{2^{-m_n}}(x_1) = B_{2^{-m_{n+1}}}(x_1)$. Contraposition yields $m_n > m_{n+1}$ whenever $B_{2^{-m_n}}(x_n) \subsetneq B_{2^{-m_{n+1}}}(x_{n+1})$. Therefore, $\{d_n \mid n \in \mathbb{N}\}$ has to be finite.

Clearly, directed subsets of trees are chains. As each principal ideal is countable, each chain is countable as well. Since $(D, \supseteq)$ is well founded, every chain must be isomorphic to some countable ordinal α. The assertion follows from the fact that each principal ideal is either finite or isomorphic to $\omega + 1$. ■

We retrieve the uniform structure of (X, ϱ) when considering the maximal elements of $D = D_{\mathrm{cb}}(X, \varrho)$. Obviously, $\mathsf{Max}\, D = \{\{x\} \mid x \in X\}$. We have

3.32. PROPOSITION. *The mapping* $x \mapsto \{x\}$ *is a uniform isomorphism from* $(X, \mathcal{U}_\varrho)$ *onto* $(\mathsf{Max}\, D, \mathcal{U}_{\underline{D}_{\mathrm{cb}}(X,\varrho)}|_{\mathsf{Max}\, D})$.

Proof. Let $x, y \in X$ and let $n \in \mathbb{N}_0$. Then $\varrho(x,y) \le 2^{-n}$ if and only if $B_{2^{-n}}(x) = B_{2^{-n}}(y)$ if and only if $p_n(\{x\}) = p_n(\{y\})$. ■

Since $K(D) = \{B_{2^{-m}}(x) \mid x \in X,\ m \in \mathbb{N}_0\}$, we immediately obtain $D \setminus K(D) = \{\{x\} \mid x \text{ is not isolated in } (X, \tau_\varrho)\}$. Next, we prove that this set is precisely the set of all elements that are not isolated in $(D, \tau_{\underline{D}_{\mathrm{cb}}(X,\varrho)})$.

3.33. PROPOSITION. *An element of* $D_{\mathrm{cb}}(X, \varrho)$ *is isolated with respect to the pop topology if and only if it is compact.*

Proof. First let $d = B_{2^{-m}}(x)$ be compact ($m \in \mathbb{N}_0$, $x \in X$). Suppose that $d = \{x\}$. Let $p_m(d) = p_m(B_{2^{-\widetilde{m}}}(\widetilde{x}))$ for some $\widetilde{m} \in \mathbb{N}_0 \cup \{\infty\}$, $\widetilde{x} \in X$. Then $\{x\} = d = p_m(d) = p_m(B_{2^{-\widetilde{m}}}(\widetilde{x})) = B_{2^{-\min\{\widetilde{m},m\}}}(\widetilde{x})$, whence $\widetilde{x} = x$. If $\widetilde{m} \le m$, then $d = B_{2^{-\widetilde{m}}}(\widetilde{x})$. If $\widetilde{m} > m$, then $d = B_{2^{-m}}(\widetilde{x})$. In particular, $d = \{x\} = \{\widetilde{x}\} = B_{2^{-\widetilde{m}}}(\widetilde{x})$. Hence, $d = \{x\}$ is isolated. Now suppose that $d \ne \{x\}$. Let $m_0 \in \mathbb{N}_0$ be maximal with $d = B_{2^{-m}}(x) = B_{2^{-m_0}}(x)$. Let $p_{m_0+1}(d) = p_{m_0+1}(B_{2^{-\widetilde{m}}}(\widetilde{x}))$ for some $\widetilde{m} \in \mathbb{N}_0 \cup \{\infty\}$, $\widetilde{x} \in X$. Thus, $d = p_{m_0+1}(d) = p_{m_0+1}(B_{2^{-\widetilde{m}}}(\widetilde{x})) = B_{2^{-\min\{\widetilde{m},m_0+1\}}}(\widetilde{x}) = B_{2^{-\min\{\widetilde{m},m_0+1\}}}(x)$. By definition of m_0 we have $\widetilde{m} \le m_0$, whence $d = B_{2^{-\widetilde{m}}}(\widetilde{x})$. Therefore, d is isolated.

Conversely, let d be non-compact, i.e. $d = \{x\}$ such that x is non-isolated in (X, ϱ). Thus, $B_{2^{-n}}(x) \ne \{x\}$ and $p_n(B_{2^{-n}}(x)) = p_n(\{x\})$ for all $n \in \mathbb{N}_0$. Consequently, $\{x\}$ is not isolated in $\underline{D}_{\mathrm{cb}}(X, \varrho)$. Alternatively, by virtue of Lemma 3.3 we may apply Proposition 2.31 to deduce that each topologically isolated element is compact. ■

3.34. PROPOSITION. *Let* (X, ϱ) *be an ultrametric space. Then*

(1) $\underline{D}_{\mathrm{cb}}(X, \varrho)$ *is complete if and only if* (X, ϱ) *is complete.*
(2) $\underline{D}_{\mathrm{cb}}(X, \varrho)$ *is separable if and only if* (X, ϱ) *is separable.*
(3) $\underline{D}_{\mathrm{cb}}(X, \varrho)$ *is totally bounded* (*compact*, *respectively*) *if and only if* (X, ϱ) *is totally bounded* (*compact*, *respectively*).

Proof. Let $D = D_{\mathrm{cb}}(X, \varrho)$.

(1) First let $\underline{D}_{\mathrm{cb}}(X, \varrho)$ be complete. Proposition 3.33 implies that $\mathsf{Max}\, D$ is closed in $(D, \tau_{\underline{D}_{\mathrm{cb}}(X,\varrho)})$. As $\underline{D}_{\mathrm{cb}}(X, \varrho)$ is Hausdorff, $\mathsf{Max}\, D$ is complete with respect to the relative pop uniformity. Proposition 3.32 tells us that (X, ϱ) must be complete as well.

Conversely, let (X, ϱ) be complete. Then we know ([19, Theorem 2.5]) that $(D, \supseteq)$ is a dcpo. As each projection p_n is compact-valued, it has to be Scott-continuous (Lemma 3.3). Consequently, $\underline{D}_{\mathrm{cb}}(X, \varrho)$ is complete by Proposition 2.25.

(2) Let $\underline{D}_{\mathrm{cb}}(X, \varrho)$ be separable. As $\underline{D}_{\mathrm{cb}}(X, \varrho)$ is metrizable by Theorem 3.15, it must be second countable. Second countability is hereditary, whence $\mathsf{Max}\, D$ is second countable and thus separable with respect to the relative pop topology. Hence, the assertion results from Proposition 3.32.

Now let (X, ϱ) be separable. Then $K(D)$ is countable ([19, Cor. 2.8]). Since $K(D) = \{B_{2^{-m}}(x) \mid x \in X,\ m \in \mathbb{N}_0\} = \bigcup_{n \in \mathbb{N}_0} p_n[D]$, we deduce that $K(D)$ is dense in D with respect to the pop topology (Proposition 2.10(1)).

(3) Let $\underline{D}_{\mathrm{cb}}(X, \varrho)$ be totally bounded. Subspaces of totally bounded spaces are totally bounded, whence (X, ϱ) is totally bounded due to Proposition 3.32.

Finally, let (X, ϱ) be totally bounded. By definition of the projections each p_n has finite range. Hence, we are done by virtue of Proposition 3.16. ■

Traces as approximating pop's. In the next examples we endow real and approximating traces with a pop structure. This enables us to give easy proofs of the topological properties of traces given in Kwiatkowska [38], Bonizzoni, Mauri, and Pighizzini [5], and Diekert and Gastin [12].

3.35. Example. Let (Σ, D) be a dependence alphabet.

(a) Consider the poset $(\mathbb{R}(\Sigma, D), \leq)$ of real traces over (Σ, D). Let $t = [V, E, \lambda] \in \mathbb{R}(\Sigma, D)$ and let $n \in \mathbb{N}_0$. Let $W_n := \{v \in V \mid |v{\downarrow}_{E^*}| \leq n\}$, i.e. W_n is the set of all vertices v of t, each having at most n elements in its *past* $v{\downarrow}_{E^*}$. Let $p_n(t) := [W_n, E|_{W_n \times W_n}, \lambda|_{W_n}]$. Clearly, $p_n(t)$ is a closed and finite subgraph of t and $p_n(p_n(t)) = p_n(t)$. If $s \in \mathbb{R}(\Sigma, D)$ with $s \leq t$ and $m \in \mathbb{N}_0$ with $m \leq n$, then $p_n(s) \leq p_n(t)$ and $p_m(t) \leq p_n(t)$. Consequently, we obtain projections p_n and a pop $\underline{D}_1 = (\mathbb{R}(\Sigma, D), \leq, \{p_n \mid n \in \mathbb{N}_0\})$. It is approximating since all vertices of a real trace have finite past. Each p_n is obviously downwards closed. The projections p_n are illustrated in Figure 3.3.

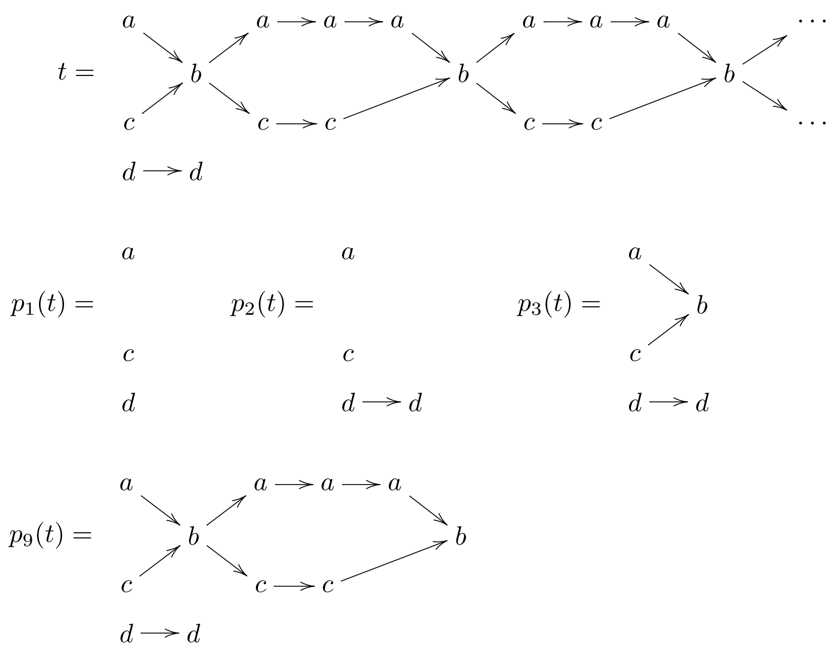


Fig. 3.3. The projections p_n on $\mathbb{R}(\Sigma, D)$ with $(\Sigma, D) = a$—b—$c \quad d$

We show that

$$p_n(t) = \sup\{s \in \mathbb{R}(\Sigma, D) \mid s \leq t,\ |s| \leq n\}.$$

If $s \leq t$ with $|s| \leq n$, then for each vertex v of s we have $|v{\downarrow}| \leq n$. Hence, v is a vertex of $p_n(t)$. As s is a closed subgraph of t, we deduce $s \le p_n(t)$. On the other hand, let $u \in \mathbb{R}(\Sigma, D)$ with $s \le u$ for all $s \le t$ with $|s| \le n$. Let v be a vertex of $p_n(t)$. Then $v{\downarrow}$ induces a closed subgraph s_v of $p_n(t) \le t$. Since $|v{\downarrow}| \le n$, we obtain $|s_v| \le n$. Consequently, $s_v \le u$. This implies that $p_n(t)$ is a closed subgraph of u, i.e. $p_n(t) \le u$. Therefore, $p_n(t) = \sup\{s \in \mathbb{R}(\Sigma, D) \mid s \le t,\ |s| \le n\}$.

As a consequence, we infer that

$$\ell_{\text{pref}}(s,t) = \sup\{n \in \mathbb{N}_0 \mid p_n(s) = p_n(t)\}$$

(cf. Proposition 4.5 below for a more general statement). By Theorem 3.15, the prefix metric thus induces the pop uniformity of $\underline{D}_1$.

Using our results on F-posets and pop's, we present a simple proof of Theorem 1.8 for $(\mathbb{R}(\Sigma, D), d_{\text{pref}})$ to be a compact metric space in which $\mathbb{M}(\Sigma, D)$ is dense and whose induced topology coincides with the Lawson topology (cf. Kwiatkowska [38]). We have $K(\mathbb{R}(\Sigma, D)) = \mathbb{M}(\Sigma, D) = \bigcup_{n \in \mathbb{N}_0} p_n[\mathbb{R}(\Sigma, D)]$ (cf. Theorem 1.7); hence p_n is Scott-continuous by Lemma 3.3 and $\mathbb{M}(\Sigma, D)$ is dense in $\mathbb{R}(\Sigma, D)$ by Proposition 2.10(1). As Σ is finite, p_n has finite range for all $n \in \mathbb{N}_0$. Hence, $(\mathbb{R}(\Sigma, D), d_{\text{pref}})$ is totally bounded by Proposition 3.16. Since $(\mathbb{R}(\Sigma, D), \le)$ is a dcpo, $(\mathbb{R}(\Sigma, D), d_{\text{pref}})$ is complete by Proposition 2.25. Therefore, $(\mathbb{R}(\Sigma, D), d_{\text{pref}})$ is compact. The induced topology is the Lawson topology because of Corollary 2.49.

(b) Compared to (a) we define a different pop structure on $(\mathbb{R}(\Sigma, D), \le)$ yielding a different ultrametric but the same uniformity and thus the same topology (namely the Lawson topology of $(\mathbb{R}(\Sigma, D), \le)$).

Let $t = [V, E, \lambda]$ be a real trace and let $v \in V$. Then the set $\{n \in \mathbb{N}_0 \mid \exists v_1, \ldots, v_n \in V : v_n = v \text{ and } (v_i, v_{i+1}) \in E \text{ for all } i = 1, \ldots, n-1\}$ is bounded. Its maximum $h(v)$ is the *height* of v. Further, for all $n \in \mathbb{N}_0$ let $V_n := \{v \in V \mid h(v) \le n\}$ and define $h_n(t) := [V_n, E|_{V_n \times V_n}, \lambda|_{V_n}]$. We call $h_n(t)$ the *nth Foata prefix* of t. By definition we have $h_n(t) \le t$. All vertices of $h_n(t)$ have height at most n. It is straightforward to check that this definition leads to a projection h_n on $(\mathbb{R}(\Sigma, D), \le)$ and that we obtain a pop $\underline{D}_2 = (\mathbb{R}(\Sigma, D), \le, \{h_n \mid n \in \mathbb{N}_0\})$. It is approximating since each vertex of any real trace has a height in the above sense. Moreover, each h_n is downwards closed. We call h_n the *nth Foata projection* of $\mathbb{R}(\Sigma, D)$. The Foata projections are visualized in Figure 3.4.

As in Theorem 3.15 we obtain an ultrametric inducing the pop uniformity. Given $s, t \in \mathbb{R}(\Sigma, D)$, we define

$$\ell_{\text{fnf}}(s,t) := \sup\{n \in \mathbb{N}_0 \mid h_n(s) = h_n(t)\}, \qquad d_{\text{fnf}}(s,t) := 2^{-\ell_{\text{fnf}}(s,t)}.$$

We call d_{fnf} the *Foata normal form metric*. We note that it is the ultrametric introduced by Bonizzoni, Mauri, and Pighizzini [5]. To see this, recall the definition of Foata normal forms: an *I_D-clique* of (Σ, D) is a subset $A \subseteq \Sigma$ such that $(a, b) \in I_D$ for all $a, b \in A$ with $a \ne b$, i.e. the elements of A are pairwise independent. Let Ω be the set of all non-empty I_D-cliques of (Σ, D). A finite word $A_1 \cdots A_n \in \Omega^\star$ (an infinite word $A_1 A_2 \cdots \in \Omega^\omega$, respectively) is called a *Foata normal form* over (Σ, D) if for all $2 \le i \le n$ (for all $2 \le i$, respectively) and for all $b \in A_i$ there is some $a \in A_{i-1}$ such that $(a, b) \in D$. Recall that there is a one-to-one correspondence between Foata normal forms and real traces (see [21, Section 11.2.3]). In fact, the Foata normal form $\mathsf{fnf}(t)$ of a trace $t \in \mathbb{M}(\Sigma, D)$

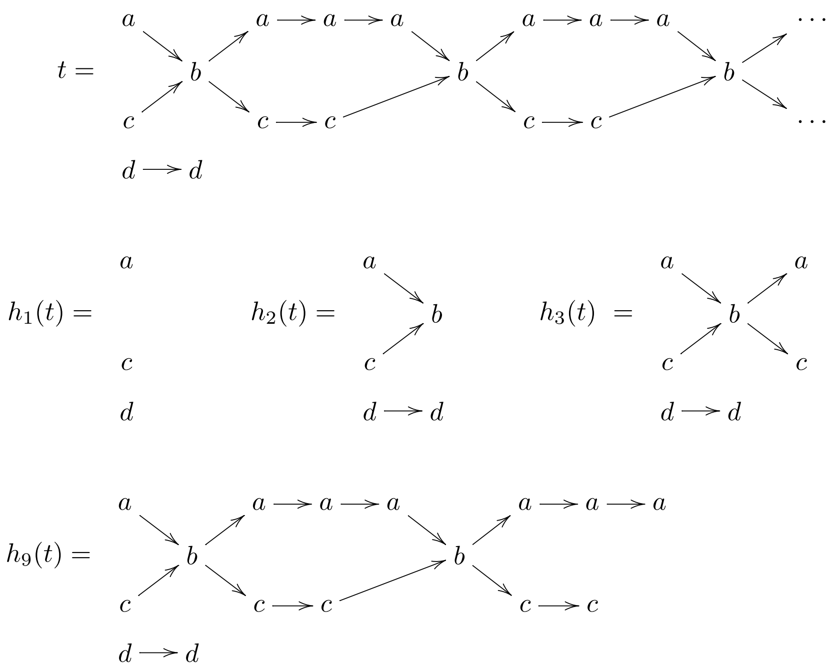


Fig. 3.4. The Foata projections h_n on $\mathbb{R}(\varSigma, D)$ with $(\varSigma, D) = a$—b—$c \quad d$

is the $\varOmega$-word $A_1 \cdots A_n$ with $A_i = \mathsf{alph}(h_{i-1}(t)^{-1} h_i(t))$ for $i = 1, \ldots, n$, where n is the maximal height of a vertex in t. The Foata normal form of an infinite real trace t equals the infinite word $A_1 A_2 \cdots$ with $A_i = \mathsf{alph}(h_{i-1}(t)^{-1} h_i(t))$ for all $i \in \mathbb{N}$. For instance, the Foata normal form of the trace t given in Figures 3.3 and 3.4 is the word

$$\{a,c,d\}\{b,d\}\{a,c\}\{a,c\}\{a\}\{b\}\{a,c\}\{a,c\}\{a\}\{b\}\cdots$$

Note that the set $\mathrm{FNF}(\varSigma, D)$ of all Foata normal forms over $(\varSigma, D)$ is a subset of $\varOmega^\infty$. Hence, it is equipped both with the prefix order and with the prefix metric d_{pref} of $\varOmega^\infty$. Clearly, $\ell_{\mathrm{fnf}}(s,t)$ is the length of the largest common prefix of the Foata normal forms of s and t; hence $d_{\mathrm{fnf}}(s,t) = d_{\mathrm{pref}}(\mathsf{fnf}(s), \mathsf{fnf}(t))$. (In [5] this equation is used as a definition.) The Foata normal form metric thus “coincides” with the prefix metric on $\mathrm{FNF}(\varSigma, D)$. More precisely, the map fnf sending each real trace t to its Foata normal form $\mathsf{fnf}(t)$ is a $(d_{\mathrm{fnf}}, d_{\mathrm{pref}})$-isometry from $\mathbb{R}(\varSigma, D)$ onto $\mathrm{FNF}(\varSigma, D)$.

Consider the projection order of $(D, \{h_n \mid n \in \mathbb{N}_0\})$. We write $\sqsubseteq_{\mathrm{fnf}}$ for $\sqsubseteq_{\{h_n \mid n \in \mathbb{N}_0\}}$. Let $s \sqsubseteq_{\mathrm{fnf}} t$ and assume that there is a number $n \in \mathbb{N}_0$ with $s = h_n(t)$. By the remarks above we have $\mathsf{fnf}(h_n(t)) \leq \mathsf{fnf}(t)$. Therefore, $\mathsf{fnf}(s) \leq \mathsf{fnf}(t)$. Conversely, let $s, t \in \mathbb{R}(\varSigma, D)$ be such that $\mathsf{fnf}(s) \leq \mathsf{fnf}(t)$. Let $s \neq t$, i.e. $\mathsf{fnf}(s) \neq \mathsf{fnf}(t)$. Hence, $\mathsf{fnf}(s)$ is a finite word $A_1 \cdots A_n$ over $\varOmega$. Since $A_1 \cdots A_n \leq \mathsf{fnf}(t)$ and $A_i = \mathsf{alph}(h_{i-1}(s)^{-1} h_i(s))$ for all $i = 1, \ldots, n$, we conclude that $A_i = \mathsf{alph}(h_{i-1}(t)^{-1} h_i(t))$ for all $i = 1, \ldots, n$ and thus $s = h_n(t)$. We infer $s \sqsubseteq_{\mathrm{fnf}} t$. As a consequence, the mapping fnf is an order isomorphism from $(\mathbb{R}(\varSigma, D), \sqsubseteq_{\mathrm{fnf}})$ onto $(\mathrm{FNF}(\varSigma, D), \leq)$.

It is well known that the prefix metric and the Foata normal form metric are uniformly equivalent ([21, Section 11.5.3]). (This can also be inferred from Corollary 2.2 and the inequalities $p_n \le h_n \le p_{n\cdot l}$ for all $n \in \mathbb{N}_0$, where $l = \max\{|A| \mid A \subseteq \Sigma$ is an I_D-clique$\}$.) Hence, the pop uniformity (pop topology) of $\underline{D}_2$ coincides with the one in (a). In particular, $(\mathbb{R}(\Sigma, D), d_{\mathrm{fnf}})$ is compact. This compactness result can also be obtained as follows: it is easy to show that $\mathrm{FNF}(\Sigma, D)$ is a closed subset of Ω^∞ with respect to the topology induced by the prefix metric. It is well known that $(\Omega^\infty, d_{\mathrm{pref}})$ is compact (this follows also from (a)). We thus find $(\mathrm{FNF}(\Sigma, D), d_{\mathrm{pref}})$ to be compact. As fnf is a $(d_{\mathrm{fnf}}, d_{\mathrm{pref}})$-isometry, $(\mathbb{R}(\Sigma, D), d_{\mathrm{fnf}})$ has to be compact as well.

3.36. EXAMPLE. Let (Σ, D) be a dependence alphabet.

(a) Consider the poset $(\mathbb{F}^\alpha(\Sigma, D), \sqsubseteq)$ of α-traces over (Σ, D). Using the Foata projections h_n from Example 3.35, we define (by abuse of language) for each $n \in \mathbb{N}_0$ a projection $h_n : \mathbb{F}^\alpha(\Sigma, D) \to \mathbb{F}^\alpha(\Sigma, D)$ by $h_n(r, A) := (h_n(r), A \cup \mathsf{alph}(h_n(r)^{-1}r))$. The reader may check without difficulty that $(\mathbb{F}^\alpha(\Sigma, D), \sqsubseteq, \{h_n \mid n \in \mathbb{N}_0\})$ is an approximating pop. (To show that it is approximating, calculate $\sup_{n\in\mathbb{N}_0} h_n(x)$ for all $x \in \mathbb{F}^\alpha(\Sigma, D)$ as in Theorem 1.9.)

Using Theorem 3.15, an ultrametric on $\mathbb{F}^\alpha(\Sigma, D)$ inducing the pop uniformity is given by the following definition:

$$\begin{aligned}\ell_\alpha((r, A), (s, B)) := \sup\{n \in \mathbb{N}_0 \mid h_n(r) = h_n(s) \text{ and}&\\ A \cup \mathsf{alph}(h_n(r)^{-1}r) = B \cup \mathsf{alph}(h_n(s)^{-1}s)\},&\\ d_\alpha((r, A), (s, B)) := 2^{-\ell_\alpha((r,A),(s,B))}.&\end{aligned}$$

We call it the *α-metric* on $\mathbb{F}^\alpha(\Sigma, D)$. We give a pop-theoretic proof of Theorem 1.10 (without the statement on $\mathbb{F}^\alpha_{\mathrm{f}}(\Sigma, D)$ to be discrete and open in $\mathbb{F}^\alpha(\Sigma, D)$) and show that the α-metric is uniformly equivalent to the distance in 1.10. To do this, we proceed as in Example 3.35(a). First of all, $K(\mathbb{F}^\alpha(\Sigma, D)) = \mathbb{F}^\alpha_{\mathrm{f}}(\Sigma, D)$ by Theorem 1.9 and, clearly, $\mathbb{F}^\alpha_{\mathrm{f}}(\Sigma, D) = \bigcup_{n\in\mathbb{N}_0} h_n[\mathbb{F}^\alpha(\Sigma, D)]$. Therefore, each h_n is compact-valued and in particular Scott-continuous (Lemma 3.3). As $(\mathbb{F}^\alpha(\Sigma, D), \sqsubseteq)$ is a dcpo (Theorem 1.9), we see that $(\mathbb{F}^\alpha(\Sigma, D), d_\alpha)$ is complete by Proposition 2.25. Since Σ is finite, each h_n has finite range, whence $(\mathbb{F}^\alpha(\Sigma, D), d_\alpha)$ is totally bounded (Proposition 3.16). Summing things up, we find $(\mathbb{F}^\alpha(\Sigma, D), d_\alpha)$ to be compact. Its induced topology is the Lawson topology of $(\mathbb{F}^\alpha(\Sigma, D), \sqsubseteq)$ by Corollary 2.49. The set of finite α-traces is dense in $\mathbb{F}^\alpha(\Sigma, D)$ in view of Proposition 2.10(1). Now let $x \in \mathbb{F}^\alpha(\Sigma, D)$ and recall that $x[n] = \sup\{p \in \mathbb{F}^\alpha(\Sigma, D) \mid p \sqsubseteq x \text{ and } |p| \le n\}$ (p. 24). Setting $p_n(x) := x[n]$, we obtain a projection p_n for all $n \in \mathbb{N}_0$ (see Proposition 4.5 below). In fact, $(\mathbb{F}^\alpha(\Sigma, D), \sqsubseteq, \{p_n \mid n \in \mathbb{N}_0\})$ is an approximating pop whose pop uniformity is induced by the ultrametric d introduced in [12] (cf. p. 24 and Theorem 3.15). Let $p = (r, A)$ and $x = (s, B)$ be α-traces with $p \sqsubseteq x$ and $|p| \le n \in \mathbb{N}_0$. Since $|r| = |p| \le n$, each vertex of r has height at most n. Therefore, $r = h_n(r)$ and $p = (r, A) = (h_n(r), A \cup \mathsf{alph}(h_n(r)^{-1}r)) = h_n(p) \sqsubseteq h_n(x)$. Consequently, $p_n(x) \sqsubseteq h_n(x)$ and thus $p_n \sqsubseteq h_n$ for all $n \in \mathbb{N}_0$. Corollary 2.2 implies that $\tau_d = \tau_{\{p_n | n\in\mathbb{N}_0\}} \subseteq \tau_{\{h_n | n\in\mathbb{N}_0\}} = \tau_{d_\alpha}$. As $(\mathbb{F}^\alpha(\Sigma, D), \tau_{d_\alpha})$ is compact Hausdorff, we deduce $\tau_d = \tau_{d_\alpha}$; hence d and d_α are uniformly equivalent.

(b) Similarly to (a), we define for each $n \in \mathbb{N}_0$ a projection h_n on the poset $(\mathbb{F}^\delta(\Sigma, D), \sqsubseteq)$ of δ-traces over (Σ, D) as follows: $h_n(r, D(A)) := (h_n(r), D(A) \cup D(\mathsf{alph}(h_n(r)^{-1}r)))$ for all $(r, D(A)) \in \mathbb{F}^\delta(\Sigma, D)$. This leads to an approximating pop $(\mathbb{F}^\delta(\Sigma, D), \sqsubseteq, \{h_n \mid n \in \mathbb{N}_0\})$. An ultrametric that induces the pop uniformity is given by

$$\begin{aligned} \ell_\delta((r, D(A)), (s, D(B))) := \sup\{n \in \mathbb{N}_0 \mid h_n(r) = h_n(s) \text{ and } \\ D(A) \cup D(\mathsf{alph}(h_n(r)^{-1}r)) = D(B) \cup D(\mathsf{alph}(h_n(s)^{-1}s))\}, \\ d_\delta((r, D(A)), (s, D(B))) := 2^{-\ell_\delta((r,D(A)),(s,D(B)))}. \end{aligned}$$

We call d_δ the *δ-metric* on $\mathbb{F}^\delta(\Sigma, D)$. Again, d_δ is uniformly equivalent to the ultrametric defined in [12, Sect. 6]. As in (a), $(\mathbb{F}^\delta(\Sigma, D), d_\delta)$ is compact and it is the metric completion of $(\mathbb{F}^\delta_{\mathrm{f}}(\Sigma, D), d_\delta)$. Its induced topology is the Lawson topology of $(\mathbb{F}^\delta(\Sigma, D), \sqsubseteq)$.

3.3. Complete approximating pop's

This section deals with approximating pop's $(D, \leq, \mathcal{P})$ whose uniformity is complete, i.e. each Cauchy net has a limit point. We characterize such pop's as inverse limits based on the sets $p[D]$ with $p \in \mathcal{P}$. Conversely, inverse limits of posets via embedding-projection pairs are complete approximating pop's with a natural pop structure.

The following analogue to Lemma 2.24 describes Cauchy nets in pop's.

3.37. Lemma. *A net $(d_n)_{n \in N}$ of a pop $(D, \leq, \mathcal{P})$ is a Cauchy net if and only if, for all $p \in \mathcal{P}$, there is an index $n_p \in N$ such that $p(d_n) = p(d_{n_p})$ for all $n \geq n_p$.* ■

Recall from Proposition 2.25 that a pop $(D, \leq, \mathcal{P})$ is complete whenever $(D, \leq)$ is a dcpo and all projections in $\mathcal{P}$ are Scott-continuous. Example 3.27 shows us that this is not true anymore if we do not assume Scott-continuity. We reformulate Theorem 2.27 for approximating pop's:

3.38. Theorem. *Let $\underline{D} = (D, \leq, \mathcal{P})$ be an approximating pop such that each $p \in \mathcal{P}$ is Scott-continuous. Then the following are equivalent:*

(i) *$(D, \leq)$ is a dcpo.*
(ii) *$\underline{D}$ is complete in its pop uniformity and $(p[D], \leq)$ is a dcpo for all $p \in \mathcal{P}$.*

Proof. This follows from Theorem 2.27 and Lemma 1.3. ■

We need the following lemma to show that the underlying poset of a complete approximating pop $(D, \leq, \mathcal{P})$ can be represented by an inverse limit built up by the sets $p[D]$ with $p \in \mathcal{P}$. Furthermore, we shall use this lemma to prove that there exists a "pop completion" (see Chapter 5).

3.39. Lemma. *Let $\underline{D} = (D, \leq, \mathcal{P})$ be a pop. Let*

$$D_\infty := \Big\{ (d_p)_{p \in \mathcal{P}} \in \prod_{p \in \mathcal{P}} p[D] \;\Big|\; \forall p, q \in \mathcal{P} : p \leq q \Rightarrow d_p = p(d_q) \Big\}$$

and define $\psi : D \to D_\infty$ by $\psi(d) := (p(d))_{p \in \mathcal{P}}$. Let D_∞ be endowed with the product order and let $\mathcal{V}$ be the uniformity on D_∞ that is induced by the product uniformity of the family $(p[D], \mathcal{U}_{\mathsf{dis}})_{p \in \mathcal{P}}$. Then

(1) *ψ is monotone and $(\mathcal{U}_{\underline{D}}, \mathcal{V})$-uniformly continuous. $\psi[D]$ is dense in $(D_\infty, \tau_{\mathcal{V}})$.*

(2) *ψ is injective if and only if $\underline{D}$ is Hausdorff. In this case, $\psi^{-1} : \psi[D] \to D$ is $(\mathcal{V}|_{\psi[D]}, \mathcal{U}_{\underline{D}})$-uniformly continuous.*

(3) *If $\underline{D}$ is approximating, then ψ^{-1} is monotone.*

(4) *If $\underline{D}$ is Hausdorff and complete, then ψ is a uniform isomorphism from $(D, \mathcal{U}_{\underline{D}})$ onto $(D_\infty, \mathcal{V})$.*

Proof. Note first that ψ is well defined: due to Lemma 3.1 we have $p(d) = p(q(d))$ for all $p, q \in \mathcal{P}$ with $p \leq q$ and all $d \in D$.

(1) Clearly, ψ is monotone. Let $\mathcal{P}_0 \subseteq \mathcal{P}$ be finite, let $N_p := \mathsf{id}_{p[D]}$ if $p \in \mathcal{P}_0$ and $N_p := p[D]^2$ if $p \in \mathcal{P} \setminus \mathcal{P}_0$. Let $N := \prod_{p\in\mathcal{P}} N_p \cap D_\infty^2$. Now choose some $q \in \mathcal{P}$ with $q \geq p$ for all $p \in \mathcal{P}_0$. Then, letting π_p be the projection of $\prod_{p\in\mathcal{P}} p[D]$ onto the pth coordinate space, for all $(d, e) \in \ker q$ and all $p \in \mathcal{P}_0$ we have $(\pi_p(\psi(d)), \pi_p(\psi(e))) = (p(d), p(e)) = (p(q(d)), p(q(e))) = (p(q(d)), p(q(d))) = (p(d), p(d)) \in N_p$. Therefore, $(\psi \times \psi)[\ker q] \subseteq N$ and we deduce that ψ is uniformly continuous.

Let $(d_p)_{p\in\mathcal{P}} \in D_\infty$. Again, let $\mathcal{P}_0 \subseteq \mathcal{P}$ be finite, let $V_p := \{d_p\}$ if $p \in \mathcal{P}_0$ and $V_p := p[D]$ if $p \in \mathcal{P} \setminus \mathcal{P}_0$. Let $V := \prod_{p\in\mathcal{P}} V_p \cap D_\infty$. Let $q \in \mathcal{P}$ with $q \geq p$ for all $p \in \mathcal{P}_0$. Then $\pi_p(\psi(d_q)) = p(d_q) = d_p$ for all $p \in \mathcal{P}_0$; hence $\psi(d_q) \in V$ and $\psi[D] \cap V \neq \emptyset$.

(2) Note that $\psi(d) = \psi(e)$ if and only if $p(d) = p(e)$ for all $p \in \mathcal{P}$. From Proposition 3.18(2) we see that ψ is injective if and only if $\underline{D}$ is Hausdorff.

Let ψ be injective. Let $p \in \mathcal{P}$, let $N_p := \mathsf{id}_{p[D]}$, and let $N_q := q[D]^2$ for all $q \in \mathcal{P} \setminus \{p\}$. Define $N := \prod_{q\in\mathcal{P}} N_q \cap \psi[D]^2$. For all $d, e \in D$ with $((q(d))_{q\in\mathcal{P}}, (q(e))_{q\in\mathcal{P}}) \in N$ we have $p(\psi^{-1}((q(d))_{q\in\mathcal{P}})) = p(d) = p(e) = p(\psi^{-1}((q(e))_{q\in\mathcal{P}}))$; that is,

$$(\psi^{-1}((q(d))_{q\in\mathcal{P}}), \psi^{-1}((q(e))_{q\in\mathcal{P}})) \in \ker p.$$

This leads to $(\psi^{-1} \times \psi^{-1})[N] \subseteq \ker p$, whence ψ^{-1} is uniformly continuous.

(3) Let $d, e \in D$ with $\psi(d) \leq \psi(e)$. Then $p(d) \leq p(e) \leq e$ for all $p \in \mathcal{P}$. As $\underline{D}$ is approximating, we deduce $d = \sup_{p\in\mathcal{P}} p(d) \leq e$.

(4) Since $\underline{D}$ is Hausdorff, ψ is a uniform isomorphism from $(D, \mathcal{U}_{\underline{D}})$ onto $(\psi[D], \mathcal{V}|_{\psi[D]})$ (see (1) and (2)). As $(D, \mathcal{U}_{\underline{D}})$ is complete, $(\psi[D], \mathcal{V}|_{\psi[D]})$ must be complete as well . As $(D_\infty, \tau_{\mathcal{V}})$ is Hausdorff, $\psi[D]$ is closed in $(D_\infty, \tau_{\mathcal{V}})$. Now (1) implies that $\psi[D] = D_\infty$. ∎

3.40. THEOREM. (1) *Let $(D, \leq, \mathcal{P})$ be a complete approximating pop. Then the family $\mathcal{S} = \{(\mathsf{id}_{p[D],q[D]}, p|_{q[D]}) \mid p, q \in \mathcal{P},\ p \leq q\}$ is an inverse system such that $(D, \leq)$ is isomorphic to $(\varprojlim_{p\in\mathcal{P}}(p[D], \mathcal{S}), \leq)$.*

(2) *Let $(D_\gamma, \leq_\gamma)_{\gamma\in\Gamma}$ be a family of posets, let $(\Gamma, \leq)$ be directed, and let $\mathcal{S} = \{(f_{\gamma\mu}, g_{\gamma\mu}) : D_\gamma \to D_\mu \mid \gamma, \mu \in \Gamma,\ \gamma \leq \mu\}$ be an inverse system. Then $(\varprojlim_{\gamma\in\Gamma}(D_\gamma, \mathcal{S}), \leq, \{f_\gamma \circ g_\gamma \mid \gamma \in \Gamma\})$ is a complete approximating pop.*

Proof. (1) Lemma 3.1 tells us that $\mathcal{S}$ is an inverse system. Due to Lemma 3.39 an order isomorphism is given by $d \mapsto (p(d))_{p\in\mathcal{P}}$ $(d \in D)$.

(2) Let $(D_\infty, \leq) := \varprojlim_{\gamma\in\Gamma}(D_\gamma, \mathcal{S})$. It is well known that $\{f_\gamma \circ g_\gamma \mid \gamma \in \Gamma\}$ is a directed set of projections on D_∞ with $\sup_{\gamma\in\Gamma}(f_\gamma \circ g_\gamma) = \mathsf{id}_{D_\infty}$ (see p. 15 for the definition of

f_γ and g_γ). Hence, $(\varprojlim_{\gamma\in\Gamma}(D_\gamma,\mathcal{S}),\leq,\{f_\gamma\circ g_\gamma \mid \gamma\in\Gamma\})$ is an approximating pop. In order to show completeness let $(c_n)_{n\in N}$ be a Cauchy net in D_∞. Then for all $\gamma\in\Gamma$ we find an index $n_\gamma\in N$ such that $f_\gamma(g_\gamma(c_n)) = f_\gamma(g_\gamma(c_{n_\gamma}))$ for all $n\geq n_\gamma$. For all $n\in N$ let $c_n =: (d_{n,\mu})_{\mu\in\Gamma}$. Next, $c := (d_{n_\gamma,\gamma})_{\gamma\in\Gamma}$ is an element of D_∞ because $d_{n_\gamma,\gamma} = g_\gamma((d_{n_\mu,\mu})_{\mu\in\Gamma}) = g_{\gamma\nu}(g_\nu((d_{n_\mu,\mu})_{\mu\in\Gamma})) = g_{\gamma\nu}(d_{n_\nu,\nu})$ for all $\nu\geq\gamma$. Finally, we show that $(c_n)_{n\in N}$ converges to c. Let $\gamma\in\Gamma$. For all $n\geq n_\gamma$ we have $f_\gamma(g_\gamma(c_n)) = f_\gamma(g_\gamma(c_{n_\gamma})) = f_\gamma(d_{n_\gamma,\gamma}) = f_\gamma(g_\gamma(c))$. ∎

3.4. Domains with compact-valued projections

3.4.1. Topological characterizations. We wish to characterize when the underlying poset of an approximating pop $\underline{D} = (D,\leq,\mathcal{P})$ is an algebraic dcpo such that its set of compact elements coincides with the set of images of all projections $p\in\mathcal{P}$. Notice that in this case we have two notions of order-theoretic approximation which coincide. For each $d\in D$ we have $d = \sup_{p\in\mathcal{P}} p(d)$ because $\underline{D}$ is approximating, $d = \sup(d{\downarrow}\cap K(D))$ because $(D,\leq)$ is algebraic, and $K(D) = \bigcup_{p\in\mathcal{P}} p[D]$. To derive such a characterization, we will use the results on F-posets obtained in Section 2.2.

It is clear that we need compact-valued projections in order to obtain $K(D) = \bigcup_{p\in\mathcal{P}} p[D]$. In fact, we have the following:

3.41. Lemma. *Let $(D,\leq,\mathcal{P})$ be an approximating pop. Then $K(D)\subseteq\bigcup_{p\in\mathcal{P}} p[D]$. In particular, $K(D) = \bigcup_{p\in\mathcal{P}} p[D]$ if and only if p is compact-valued for all $p\in\mathcal{P}$.*

Proof. Let $x\in K(D)$. As $\sup_{p\in\mathcal{P}} p(x) = x$, we conclude that there is some $p\in\mathcal{P}$ with $p(x)\geq x$ and thus $p(x) = x$. ∎

Along the lines of Section 2.2, we investigate when suprema (and infima) in approximating pop's exist.

3.42. Lemma. *Let $(D,\leq,\mathcal{P})$ be an approximating pop and let $A\subseteq D$. Suppose that $\sup p[A]$ exists such that $p(\sup q[A]) = \sup p[A]$ for all $p,q\in\mathcal{P}$ with $p\leq q$. Then $(\sup p[A])_{p\in\mathcal{P}}$ is a Cauchy net. It is convergent with respect to the pop topology if and only if A has a supremum and $p(\sup A) = \sup p[A]$ for all $p\in\mathcal{P}$. In this case we obtain $\sup A = \lim_{p\in\mathcal{P}}\sup p[A]$. Similarly for the infimum.*

Proof. The net $(\sup p[A])_{p\in\mathcal{P}}$ is Cauchy in view of Lemma 2.26 because $p\circ q = p$ for all $p,q\in\mathcal{P}$ with $p\leq q$ (Lemma 3.1). If it is convergent, then A has a supremum with $\sup A = \lim_{p\in\mathcal{P}}\sup p[A]$ again by 2.26. Since each $p\in\mathcal{P}$ is (uniformly) continuous (Proposition 3.14), we deduce $p(\sup A) = p(\lim_{q\in\mathcal{P}}\sup q[A]) = \lim_{q\in\mathcal{P}} p(\sup q[A]) = \lim_{q\in\mathcal{P}}\sup p[A] = \sup p[A]$. The "if" part is obvious because $(p(d))_{p\in\mathcal{P}}\to d$ for all $d\in D$. ∎

3.43. Lemma. *Let $(D,\leq,\mathcal{P})$ be an approximating pop and let $A\subseteq D$.*

(1) *Let $d\in D$. Then $d = \sup A$ and $d\in\overline{A}$ if and only if $p(d)$ is the greatest element of $p[A]$ for all $p\in\mathcal{P}$.*

(2) *The set A has a supremum with* $\sup A \in \overline{A}$ *if and only if* $p[A]$ *has a greatest element for all* $p \in \mathcal{P}$ *and* $(\max p[A])_{p\in\mathcal{P}}$ *is convergent in the pop topology. In this case we have* $\sup A = \lim_{p\in\mathcal{P}} \max p[A]$.

Proof. (1) Lemma 2.29 tells us that $d = \sup A \in \overline{A}$ if and only if $A \le d$ and for all $p \in \mathcal{P}$ there is an element $a_p \in A$ with $p(d) \le a_p$. In this case, we infer $p[A] \le p(d)$ and hence $p(d) = p(p(d)) \le p(a_p) \le p(d)$. Thus, $p(d) = p(a_p)$ is the greatest element of $p[A]$. To prove the converse, let $p \in \mathcal{P}$ and let $a_p \in A$ with $p(a_p) = \max p[A] = p(d)$. Then $p(d) \le a_p$. Furthermore, $p[A] \le d$ for all $p \in \mathcal{P}$ and thus $A \le d$ since $(D, \le, \mathcal{P})$ is approximating.

(2) The "only if" part follows from (1) because $\max p[A] = p(\sup A)$ for all $p \in \mathcal{P}$ and $(p(\sup A))_{p\in\mathcal{P}} \to \sup A$ by Proposition 2.10(1). To prove the converse, let $a_p \in A$ with $p(a_p) = \max p[A]$. Let d be the limit of $(\max p[A])_{p\in\mathcal{P}}$. It is easy to see that d is also the limit of the net $(a_p)_{p\in\mathcal{P}}$. Since $p[A] \le p(a_p) \le a_p$ for all $p \in \mathcal{P}$, the assertion results from Lemma 2.33. ∎

There is an analogous statement for infima: let $(D, \le, \mathcal{P})$ be an approximating pop, let $A \subseteq D$, and let $d \in D$. Then $d = \inf A$ and $d \in \overline{A}$ if and only if $p(d)$ is the least element of $p[A]$ for all $p \in \mathcal{P}$. The set A has an infimum with $\inf A \in \overline{A}$ if and only if $p[A]$ has a least element for all $p \in \mathcal{P}$ and $(\min p[A])_{p\in\mathcal{P}}$ is convergent. In this case we have $\inf A = \lim_{p\in\mathcal{P}} \min p[A]$. Again, the details are left to the reader.

We have seen that greatest (and least) elements of the sets $p[A]$ with $p \in \mathcal{P}$ play an important rôle concerning the relationship of order and topology. For directed and filtered subsets we have the following analogue to Lemma 2.37(2) and the remark after it:

3.44. Lemma. *Let* $\underline{D} = (D, \le, \mathcal{P})$ *be a pop and let* $A \subseteq D$ *be directed. Then* $(a)_{a\in A}$ *is a Cauchy net if and only if* $p[A]$ *has a greatest element for all* $p \in \mathcal{P}$. *Similarly, if* $B \subseteq D$ *is filtered, then* $(b)_{b\in(B,\ge)}$ *is a Cauchy net if and only if* $p[B]$ *has a least element for all* $p \in \mathcal{P}$. ∎

Let $(d_n)_{n\in N}$ be a monotone net in D. Then $A := \{d_n \mid n \in N\}$ is directed, and it is easy to see that $(d_n)_{n\in N}$ is Cauchy if and only if $(a)_{a\in A}$ is Cauchy. Hence, the previous lemma yields:

3.45. Corollary. *Let* $\underline{D} = (D, \le, \mathcal{P})$ *be a pop. Then the following are equivalent:*

(i) *Each monotone net in D is a Cauchy net.*
(ii) *For all directed subsets $A \subseteq D$ and for all $p \in \mathcal{P}$ the set $p[A]$ has a greatest element.* ∎

Next, we characterize when $(D, \le)$ is an algebraic poset with $K(D) = \bigcup_{p\in\mathcal{P}} p[D]$ (cf. Proposition 2.31 and Corollary 2.36 for similar statements on F-posets):

3.46. Proposition. *Let* $(D, \le, \mathcal{P})$ *be an approximating pop. Then the following are equivalent:*

(i) *$(D, \le)$ is an algebraic poset with $K(D) = \bigcup_{p\in\mathcal{P}} p[D]$.*
(ii) *p is compact-valued for all $p \in \mathcal{P}$.*

(iii) *For all directed subsets $A \subseteq D$ that admit a supremum we have* $\sup A \in \overline{A}$.
(iv) *p is Scott-continuous for all $p \in \mathcal{P}$, and all monotone nets $(d_n)_{n\in N}$ in D that have a supremum are also Cauchy nets.*
(v) *All monotone nets $(d_n)_{n\in N}$ in D that have a supremum are also convergent (and $\sup_{n\in N} d_n = \lim_{n\in N} d_n$ in this case).*

Proof. The equivalences (ii)⇔(iii)⇔(v) follow from Proposition 2.31, Corollary 2.36, and Lemma 3.3.

(ii)⇔(iv) results from Lemma 3.44 together with its succeeding remark and Lemma 3.3.

Clearly, (i) implies (ii). Now suppose (iii) holds. From Proposition 2.31 we deduce that $(D, \leq)$ is a continuous poset with basis $\bigcup_{p\in\mathcal{P}} p[D]$. From Lemma 3.41 we infer $K(D) = \bigcup_{p\in\mathcal{P}} p[D]$. In particular, $(D, \leq)$ is algebraic. ∎

3.47. Proposition. *Let $\underline{D} = (D, \leq, \mathcal{P})$ be a pop and let $(\mathsf{Id}(\bigcup_{p\in\mathcal{P}} p[D]), \subseteq)$ be the ideal completion of the images of all projections $p \in \mathcal{P}$. Define*

$$\iota : D \to \mathsf{Id}\Big(\bigcup_{p\in\mathcal{P}} p[D]\Big), \qquad d \mapsto \bigcup_{p\in\mathcal{P}} p(d)\!\downarrow.$$

Then ι is a monotone map. It is an order embedding if and only if $\underline{D}$ is approximating. In this case we call ι the ideal embedding *of $\underline{D}$.*

Proof. Clearly, as all $p \in \mathcal{P}$ are monotone, ι is monotone as well. Let $\underline{D}$ be approximating and let $d, e \in D$ with $\bigcup_{p\in\mathcal{P}} p(d)\!\downarrow \subseteq \bigcup_{p\in\mathcal{P}} p(e)\!\downarrow$. Let $p \in \mathcal{P}$. Then we find a projection $q \in \mathcal{P}$ such that $p(d) \leq q(e)$. Now choose some $r \in \mathcal{P}$ with $r \geq p, q$. Then $p(d) \leq q(e) \leq r(e)$ and $p(d) \leq p(r(e)) = p(e)$ by Lemma 3.1. Since $\underline{D}$ is approximating, $p(d) \leq p(e)$ for all $p \in \mathcal{P}$ is equivalent to $d \leq e$.

Conversely, let ι be an order embedding and let $d, e \in D$ with $p(d) \leq e$ for all $p \in \mathcal{P}$. Then $p(d)\!\downarrow = \iota(p(d)) \subseteq \iota(e) = \bigcup_{q\in\mathcal{P}} q(e)\!\downarrow$ for all $p \in \mathcal{P}$; hence $\bigcup_{p\in\mathcal{P}} p(d)\!\downarrow \subseteq \bigcup_{p\in\mathcal{P}} p(e)\!\downarrow$. As ι is an order embedding, we conclude $d \leq e$. Thus, $d = \sup_{p\in\mathcal{P}} p(d)$ and $\underline{D}$ is approximating. ∎

We will come back to this result in Section 5.2, when we consider the "domain completion" of a pop.

The following theorem is the "pop analogue" to Theorem 2.39 for F-posets:

3.48. Theorem. *Let $\underline{D} = (D, \leq, \mathcal{P})$ be an approximating pop. Then the following statements are equivalent:*

(i) *$(D, \leq)$ is an algebraic dcpo and $K(D) = \bigcup_{p\in\mathcal{P}} p[D]$.*
(ii) *$(D, \leq)$ is a dcpo and p is compact-valued for all $p \in \mathcal{P}$.*
(iii) *Each directed subset $A \subseteq D$ has a supremum with* $\sup A \in \overline{A}$.
(iv) *Each monotone net in D is convergent with respect to the pop topology.*
(v) *$\underline{D}$ is complete and $p[A]$ has a greatest element* $\max p[A]$ *for all directed subsets $A \subseteq D$ and for all $p \in \mathcal{P}$.*
(vi) *The ideal embedding of $\underline{D}$ is surjective.*

In this case we have $\sup A = \lim(a)_{a\in A} = \lim(\max p[A])_{p\in\mathcal{P}}$ *for all directed subsets $A \subseteq D$ and* $\lim_{n\in N} d_n = \sup_{n\in N} d_n$ *for all monotone nets $(d_n)_{n\in N}$ of D.*

Proof. (i), (ii), and (iii) are equivalent by Proposition 3.46. Moreover, (iii), (iv), and (v) are equivalent due to Theorem 2.39.

(i)$\Rightarrow$(vi). We know that $K(D) = \bigcup_{p\in\mathcal{P}} p[D]$. Given any $d \in D$, one easily sees that $\bigcup_{p\in\mathcal{P}} p(d)\downarrow_{K(D)} = \{x \in K(D) \mid x \leq d\}$. Since $(D, \leq)$ is algebraic, ι is an order isomorphism (cf. p. 14).

(vi)$\Rightarrow$(i). By Proposition 3.47 and (vi) the ideal embedding is an order isomorphism from $(D, \leq)$ onto the ideal completion of $\bigcup_{p\in\mathcal{P}} p[D]$. This establishes (i) because $\iota(p(d)) = p(d)\downarrow$.

The remaining assertions follow from Theorem 2.39 and Lemma 3.43. ∎

Recall that an algebraic dcpo is said to be *ω-algebraic* provided that its set of compact elements is countable.

3.49. Corollary. *Let $\underline{D} = (D, \leq, \mathcal{P})$ be a metrizable approximating pop and let $p[D]$ be countable for all $p \in \mathcal{P}$. Then the following are equivalent:*

(i) *$(D, \leq)$ is an ω-algebraic dcpo and $K(D) = \bigcup_{p\in\mathcal{P}} p[D]$.*

(ii) *Each monotone* sequence *in D is convergent.*

Proof. Due to the preceding theorem only (ii)$\Rightarrow$(i) remains to be shown. As $\underline{D}$ is metrizable, we find a countable cofinal chain $\{p_n \mid n \in \mathbb{N}_0\} \subseteq \mathcal{P}$ with $p_n \leq p_{n+1}$ for all $n \in \mathbb{N}_0$; cf. Proposition 2.7 (see also Theorem 3.15). Let $(d_m)_{m\in M}$ be a monotone net. Then $B := \{p_n(d_m) \mid m \in M,\ n \in \mathbb{N}_0\}$ is countable and directed, whence there is a cofinal chain $C = \{p_{n_k}(d_{m_k}) \mid k \in \mathbb{N}\} \subseteq B$ with $p_{n_k}(d_{m_k}) \leq p_{n_{k+1}}(d_{m_{k+1}})$ for all $k \in \mathbb{N}$. By assumption, $(p_{n_k}(d_{m_k}))_{k\in\mathbb{N}}$ is convergent. Hence, $\sup C = \lim_{k\in\mathbb{N}} p_{n_k}(d_{m_k})$ by Corollary 2.35 and thus $\sup B = \sup C$. We show that $(d_m)_{m\in M}$ converges to $\sup B$. Let $p \in \mathcal{P}$. We find some $k_0 \in \mathbb{N}$ such that $p(p_{n_{k_0}}(d_{m_{k_0}})) = p(\sup C) = p(\sup B)$, whence $p(\sup B) \leq p(d_{m_{k_0}}) \leq p(d_m)$ for all $m \geq m_{k_0}$. Let $n \in \mathbb{N}_0$ with $p_n \geq p$ and let $m \in M$. Then $p(d_m) \leq p_n(d_m) \leq \sup B$. Hence, $p(d_m) = p(\sup B)$ for all $m \geq m_{k_0}$. Therefore, $(d_m)_{m\in M}$ converges to $\sup B$. Note that $\bigcup_{p\in\mathcal{P}} p[D] = \bigcup_{n\in\mathbb{N}_0} p_n[D]$ and apply Theorem 3.48(iv)$\Rightarrow$(i) to complete the proof. ∎

3.4.2. Order-theoretic characterizations. Now we deal with the question under which (order-theoretic) conditions a directed family $\mathcal{P}$ of projections on $(D, \leq)$ exists such that Theorem 3.48(i)–(vi) is satisfied.

Definition. A dcpo $(D, \leq)$ is a *P-domain* if there exists a directed family $\mathcal{P}$ of compact-valued projections on D such that $\sup \mathcal{P} = \mathsf{id}_D$.

Hence, due to Theorem 3.48, a poset $(D, \leq)$ is a P-domain if and only if there exists a family $\mathcal{P}$ such that $(D, \leq, \mathcal{P})$ is an approximating pop satisfying the equivalent conditions of Theorem 3.48. In particular, a P-domain is always algebraic.

In what follows we give several order-theoretic characterizations of P-domains. In this connection, posets satisfying the ascending chain condition play a prominent rôle. These posets turn out to be exactly the dcpo's consisting of compact elements only. As we show, P-domains arise precisely as inverse limits of such posets. Further, we describe P-domains $(D, \leq)$ using a certain system of “complete” subsets of $K(D)$ that satisfy the ascending chain condition.

A poset $(D, \leq)$ is said to satisfy the *ascending chain condition* (ACC) or is called *Noetherian* if each monotone sequence in D is eventually constant. Notice that $(D, \leq)$ is Noetherian if and only if each non-empty subset of D has some maximal element. Further characterizations are listed in the following well known lemma (cf. e.g. Gierz *et al.* [22, Example I.1.3(4)]).

3.50. LEMMA. *Let $(D, \leq)$ be a poset. Then the following are equivalent*:

(i) *$(D, \leq)$ satisfies the ascending chain condition.*
(ii) *Each directed subset of D has a greatest element.*
(iii) *$(D, \leq)$ is a dcpo with $K(D) = D$.*
(iv) *$(D, \leq)$ is a dcpo and $f : D \to E$ is Scott-continuous for all posets $(E, \leq)$ and all monotone mappings f.*

Proof (included for the sake of convenience). (i)$\Rightarrow$(ii). Let $A \subseteq D$ be directed. Suppose that A does not have a greatest element. Then, as A is directed, it contains no maximal element, a contradiction to (i).

(ii)$\Rightarrow$(iii). Clearly, $(D, \leq)$ is a dcpo. Let $d \in D$ and let $A \subseteq D$ be directed such that $\sup A \geq d$. Since $\sup A \in A$, we conclude $d \in K(D)$.

(iii)$\Rightarrow$(iv). Let $A \subseteq D$ be directed. As $\sup A \in K(D)$, there is an element $a_0 \in A$ with $a_0 = \sup A$. Since f is monotone, $f(a_0) = f(\sup A)$ is the greatest element of $f[A]$, whence $f(a_0) = \sup f[A]$.

(iv)$\Rightarrow$(i). Suppose that there exists a strictly increasing sequence $(d_n)_{n \in \mathbb{N}}$ in $(D, \leq)$. Let $d := \sup\{d_n \mid n \in \mathbb{N}\}$. Let $E := \{0, 1\}$ with $0 < 1$ and define $f(x) := 0$ for all $x < d$ and $f(x) := 1$ otherwise ($x \in D$). Then f is a monotone mapping that is not Scott-continuous, a contradiction. ∎

We remark here that, in particular, Lemma 3.50 implies that any poset with the ascending chain condition yields a P-domain (take $\mathcal{P} = \{\mathsf{id}\}$). In fact, each P-domain can be built from Noetherian posets:

3.51. THEOREM. *Let $(D, \leq)$ be a poset. The following are equivalent*:

(i) *$(D, \leq)$ is a P-domain.*
(ii) *There exists a directed set $(\Gamma, \leq)$, a family $(D_\gamma, \leq_\gamma)_{\gamma \in \Gamma}$ of dcpo's with $K(D_\gamma) = D_\gamma$ for all $\gamma \in \Gamma$, and an inverse system $\mathcal{S}$ of Scott-continuous epp's such that $(D, \leq)$ is isomorphic to $(\varprojlim_{\gamma \in \Gamma}(D_\gamma, \mathcal{S}), \leq)$.*
(iii) *There exists a directed set $(\Gamma, \leq)$, a family $(D_\gamma, \leq_\gamma)_{\gamma \in \Gamma}$ of posets satisfying the ascending chain condition, and an inverse system $\mathcal{S}$ of epp's such that $(D, \leq)$ is isomorphic to $(\varprojlim_{\gamma \in \Gamma}(D_\gamma, \mathcal{S}), \leq)$.*

Proof. (i)$\Rightarrow$(iii). We find a directed set $\mathcal{P}$ of compact-valued projections such that $(D, \leq, \mathcal{P})$ is a complete approximating pop (Theorem 3.48). Let $p \in \mathcal{P}$ and let $A \subseteq p[D]$ be directed. Then $p[A] = A$ has a greatest element by Lemma 3.3. Lemma 3.50 tells us that $p[D]$ satisfies the ACC. Now the assertion follows from Theorem 3.40(1).

(iii)$\Rightarrow$(ii) results from Lemma 3.50. Observe that this lemma in particular implies that the given epp's are Scott-continuous.

(ii)$\Rightarrow$(i). Let $\mathcal{S} = \{(f_{\gamma\mu}, g_{\gamma\mu}) : D_\gamma \to D_\mu \mid \gamma, \mu \in \Gamma,\ \gamma \leq \mu\}$. Let (f_γ, g_γ) be the embedding projection pair as defined on p. 15. By Theorem 3.40(2), $(\varprojlim_{\gamma \in \Gamma}(D_\gamma, \mathcal{S}),$

$\leq, \{f_\gamma \circ g_\gamma \mid \gamma \in \Gamma\})$ is a (complete) approximating pop. Further, $(\varprojlim_{\gamma\in\Gamma}(D_\gamma, \mathcal{S}), \leq)$ is a dcpo and (f_γ, g_γ) is a Scott-continuous epp for all $\gamma \in \Gamma$ (Lemma 1.4(1)). In particular, $f_\gamma[D_\gamma] = f_\gamma[K(D_\gamma)] \subseteq K(\varprojlim_{\gamma\in\Gamma}(D_\gamma, \mathcal{S}))$. Hence, $f_\gamma \circ g_\gamma$ is a compact-valued projection for all $\gamma \in \Gamma$. Therefore, $(\varprojlim_{\gamma\in\Gamma}(D_\gamma, \mathcal{S}), \leq)$ and thus $(D, \leq)$ is a P-domain. ■

The reader having some knowledge in domain theory has surely noticed that P-domains generalize a class of algebraic dcpo's called "bifinite domains" (which are due to Plotkin [47]). These domains are obtained when we replace the condition "posets satisfying the ascending chain condition" by "finite posets" in Theorem 3.51(iii). Formally, we have the following definition:

Definition. A poset $(D, \leq)$ is called a *bifinite domain* if there is a directed index set $(\Gamma, \leq)$, a family $(D_\gamma, \leq_\gamma)_{\gamma\in\Gamma}$ of finite posets, and an inverse system $\mathcal{S} = \{(f_{\gamma\mu}, g_{\gamma\mu}) \mid \gamma, \mu \in \Gamma,\ \gamma \leq \mu\}$ of embedding projection pairs such that $(D, \leq)$ is order isomorphic to $(\varprojlim_{\gamma\in\Gamma}(D_\gamma, \mathcal{S}), \leq)$.

The names *profinite domains* (Gunter [25]) or *FB-domains* (Jung [27], cf. also [1]) are also in common use. By Theorem 3.51, a bifinite domain is a P-domain.

As to bifinite domains, there is a similar result to 3.51. It is due to Gunter [25, Theorem 37], cf. also Jung [27, Theorems 1.26 and 3.11]:

3.52. Theorem (Gunter [25]). *A dcpo $(D, \leq)$ is a bifinite domain if and only if there exists some directed set $\mathcal{P}$ of Scott-continuous projections with finite range such that* $\sup \mathcal{P} = \mathsf{id}_D$.

Proof. Clearly, if $(D, \leq)$ is a bifinite domain, then the assertion follows immediately from Theorem 3.40(2). To prove the converse, note that $(D, \leq, \mathcal{P})$ is an approximating pop which is complete in its pop uniformity by Proposition 2.25. Thus, Theorem 3.40(1) implies that $(D, \leq)$ is a bifinite domain. ■

Jung [27] even shows that for a bifinite domain the set of *all* its Scott-continuous projections with finite range is directed and has id_D as supremum.

Next, we derive "internal" characterizations of P-domains. For this, the following definition is crucial:

Definition. Let $(D, \leq)$ be a poset and let $B, C \subseteq D$. We say that C is *complete for* B if, whenever $B \leq d$ for some $d \in D$, there is an element $c \in C$ with $B \leq c \leq d$.

For all subsets $A \subseteq D$ of a poset $(D, \leq)$ let $\mathsf{mub}(A)$ be the set of all minimal upper bounds of A. Note that $\mathsf{mub}(A)$ may be empty. We say that $(D, \leq)$ has *property m* (or: is *mub-complete*) if, for all finite subsets $A \subseteq D$, $\mathsf{mub}(A)$ is complete for A (cf. [1, Def. 4.2.1.1], [25, p. 23], [27, p. 38]). For instance, it is well known that the set of compact elements of a bifinite domain has property m. We note that this need not be true for P-domains (see Example 3.55(a) below).

A poset $(D, \leq)$ has *property M* if it has property m and $\mathsf{mub}(A)$ is finite for all finite subsets $A \subseteq D$ (cf. [25, p. 23], [27, p. 38]). Again, the set of compact elements of a bifinite domain has property M.

For a subset $A \subseteq D$ let (cf. [25, p. 23], [27, p. 38])

$$\begin{aligned}
U(A) &:= \{x \in D \mid x \in \mathsf{mub}(M) \text{ for some finite subset } M \subseteq A\},\\
U^0(A) &:= A,\\
U^{n+1}(A) &:= U(U^n(A)) \quad \text{for all } n \in \mathbb{N}_0,\\
U^\infty(A) &:= \bigcup_{n\in\mathbb{N}_0} U^n(A).
\end{aligned}$$

Recall that if $(D, \leq)$ is a continuous dcpo, then minimal upper bounds of finite sets of compact elements are again compact (cf. [1, Prop. 2.2.18.1], [27, Prop. 1.9]). In this case we have $U^\infty(A) \subseteq K(D)$ for all $A \subseteq K(D)$.

There is a well known "internal" characterization of bifinite domains (see Plotkin [47, Theorem 5], cf. also Gunter [25, Theorem 37], Jung [27, Theorem 1.32]):

An algebraic dcpo $(D, \leq)$ is bifinite if and only if $K(D)$ has property m and $U^\infty(A)$ is finite for all finite sets $A \subseteq K(D)$.

Clearly, in this case $U^\infty(A)$ is complete for all (necessarily finite) subsets $B \subseteq U^\infty(A)$. Moreover, we obtain a compact-valued and image-finite projection $p_A : D \to D$ by setting $p_A(d) := \sup(U^\infty(A) \cap d\!\downarrow)$ for all $d \in D$ (cf. e.g. Jung [27, Prop. 1.31]). We will extend this well known construction of projections to give a description of P-domains.

As the set of compact elements of a P-domain need not satisfy property m, the following characterization (Theorem 3.53) is more abstract than the above mentioned characterization of bifinite domains. But for algebraic dcpo's $(D, \leq)$ with $K(D)$ having property m, we can formulate a simpler version of Theorem 3.53 using the U^∞-operator, (see Corollary 3.54 below).

3.53. Theorem. *Let $(D, \leq)$ be an algebraic dcpo. Then the following are equivalent*:

(i) $(D, \leq)$ *is a P-domain.*

(ii) *There is a $\subseteq$-monotone mapping $C : \mathcal{P}_{\mathsf{fin}}(K(D)) \to \mathcal{P}(K(D))$ satisfying the following conditions for all finite subsets $A \subseteq K(D)$*:

  (1) *$A \subseteq C(A)$.*

  (2) *$C(A)$ is complete for all finite subsets $B \subseteq C(A)$.*

  (3) *$C(A)$ satisfies the ascending chain condition.*

(iii) *There is a $\subseteq$-monotone mapping $C : \mathcal{P}_{\mathsf{fin}}(K(D)) \to \mathcal{P}(K(D))$ satisfying the following conditions for all finite subsets $A \subseteq K(D)$*:

  (1) *$A \subseteq C(A)$.*

  (2) *$C(A)$ is complete for all subsets $B \subseteq C(A)$.*

Proof. (i)$\Rightarrow$(iii). By definition there is a directed family $\mathcal{P}$ of projections on D with $\sup \mathcal{P} = \mathsf{id}_D$ and $p[D] \subseteq K(D)$ for all $p \in \mathcal{P}$. As follows, we choose inductively a projection $p_A \in \mathcal{P}$ for all finite subsets $A \subseteq K(D)$ with the property $p_A(a) = a$ for all $a \in A$ and $p_{A_1} \leq p_{A_2}$ for $A_1 \subseteq A_2$. For $A = \emptyset$ fix any $p_\emptyset \in \mathcal{P}$. Let $A \subseteq K(D)$ with $|A| = n \geq 1$. There exist elements $a_1, \ldots, a_n \in D$ and $p_1, \ldots, p_n \in \mathcal{P}$ such that $A = \{p_1(a_1), \ldots, p_n(a_n)\}$ because $K(D) = \bigcup_{p\in\mathcal{P}} p[D]$. As $\mathcal{P}$ is directed, we can choose a projection $p_A \in \mathcal{P}$ with $p_A \geq p_1, \ldots, p_n$ and $p_A \geq p_{A\setminus\{a\}}$ for all $a \in A$. Therefore, $p_A(a) = a$ for all $a \in A$ (cf. Lemma 3.1).

Next we set $C(A) := p_A[D]$. Clearly, $A \subseteq C(A) \subseteq K(D)$. Let $B \subseteq C(A)$ and let $d \in D$ such that $B \leq d$. Then we have $p_A(d) \in C(A)$ and $B = p_A[B] \leq p_A(d) \leq d$; that is, $C(A)$ is complete for B.

Let $A_1 \subseteq A_2$ be finite subsets of $K(D)$. Then we have $p_{A_1} \leq p_{A_2}$, and $C(A_1) \subseteq C(A_2)$ follows from Lemma 3.1.

(iii)$\Rightarrow$(ii). Let $B \subseteq C(A)$ be directed and let $d := \sup B$. As $C(A)$ is complete for B, there is some $c \in C(A)$ with $B \leq c \leq d$. Hence $d = c \in K(D)$ and d is the greatest element of B. Lemma 3.50 tells us that $C(A)$ satisfies the ACC.

(ii)$\Rightarrow$(i). Let $A \subseteq K(D)$ be finite and let $d \in D$. Condition (ii)(2), applied to the empty set, implies that $C(A) \cap d\downarrow$ is non-empty. Let $x, y \in C(A) \cap d\downarrow$. Since $C(A)$ is complete for $\{x, y\}$, there is some $z \in C(A)$ with $\{x, y\} \leq z \leq d$. Hence, $C(A) \cap d\downarrow$ is a directed subset of $C(A)$. We define $p_A(d) := \sup(C(A) \cap d\downarrow)$. This gives rise to a mapping $p_A : D \to D$. Obviously, p_A is monotone and $p_A \leq \mathsf{id}_D$. As $C(A)$ satisfies the ACC, $p_A(d)$ must be the greatest element of $C(A) \cap d\downarrow$ (Lemma 3.50). This yields $p_A(p_A(d)) = \sup(C(A) \cap p_A(d)\downarrow) = p_A(d)$. Therefore, p_A is a compact-valued projection.

For finite subsets $A_1 \subseteq A_2$ of $K(D)$ we have $C(A_1) \subseteq C(A_2)$. This leads to $p_{A_1} \leq p_{A_2}$, whence $\mathcal{P} := \{p_A \mid A \subseteq K(D) \text{ finite}\}$ is directed. We still have to show that $\sup \mathcal{P} = \mathsf{id}_D$. To see this, notice that $p_{\{x\}}(x) = \sup(C(\{x\}) \cap x\downarrow) = x$ for all $x \in K(D)$ because $x \in C(\{x\})$. Consequently, $\{p(d) \mid p \in \mathcal{P}\}$ is a cofinal subset of $K(D) \cap d\downarrow$. As $(D, \leq)$ is algebraic, we deduce $d = \sup(K(D) \cap d\downarrow) = \sup_{p \in \mathcal{P}} p(d)$. ∎

We have just proven that an operator C given as in 3.53(iii) also satisfies the conditions of 3.53(ii). An analysis of the implication (ii)$\Rightarrow$(i) in the above proof shows us that the converse is true as well: a mapping C as in 3.53(ii) satisfies the conditions of 3.53(iii).

3.54. Corollary. *Let $(D, \leq)$ be an algebraic dcpo and let $K(D)$ have property m. Then the following statements are equivalent*:

(i) *$(D, \leq)$ is a P-domain.*

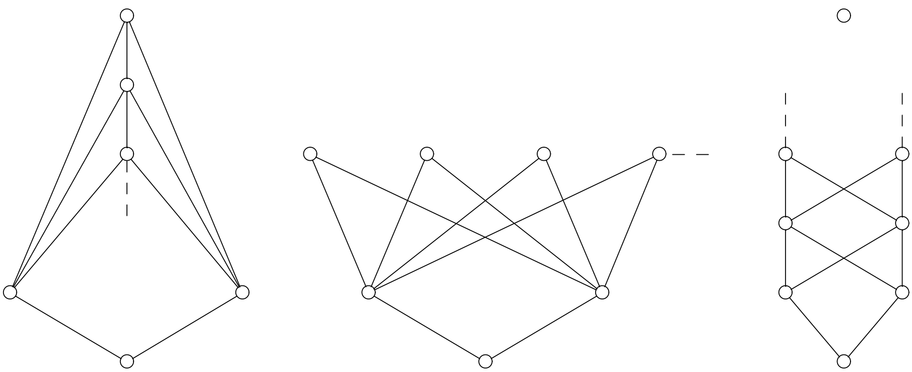
(ii) *$U^\infty(A)$ satisfies the ascending chain condition for all finite subsets $A \subseteq K(D)$.*

Proof. (i)$\Rightarrow$(ii). Let $\mathcal{P}$ be a directed set of compact-valued projections on D with $\sup \mathcal{P} = \mathsf{id}_D$. Let $A \subseteq K(D)$ be finite. As seen in the proof above, there is a projection $p \in \mathcal{P}$ such that $p(a) = a$ for all $a \in A$. By definition of the U^∞-operator it is easy to see that $p(x) = x$ for all $x \in U^\infty(A)$. Let $B \subseteq U^\infty(A)$ be directed. Then, by Theorem 3.48, $p[B]$ has a greatest element. As $p[B] = B$, we conclude that $U^\infty(A)$ satisfies the ACC (Lemma 3.50).

(ii)$\Rightarrow$(i). Let $A \subseteq K(D)$ be finite. As $K(D)$ has property m, we find $U^\infty(A)$ to be complete for all finite subsets $B \subseteq U^\infty(A)$. Clearly, if $A_1 \subseteq A_2$, then $U^\infty(A_1) \subseteq U^\infty(A_2)$. We set $C(A) := U^\infty(A)$ and complete the proof by applying Theorem 3.53(ii)$\Rightarrow$(i). ∎

Let $(D, \leq)$ be a P-domain and let $K(D)$ have property m. We have just shown that the U^∞-operator is a monotone mapping from $\mathcal{P}_{\mathsf{fin}}(K(D))$ to $\mathcal{P}(K(D))$ satisfying the conditions of Theorem 3.53. Let us note that U^∞ is even the *least* operator $C : \mathcal{P}_{\mathsf{fin}}(K(D)) \to \mathcal{P}(K(D))$ with these properties. To see this, let $A \subseteq K(D)$ be finite. We have to check that $U^\infty(A) \subseteq C(A)$. We show this by induction. Clearly, $U^0(A) = A \subseteq C(A)$. Let $x \in U^{n+1}(A)$. Then we find a finite set $M \subseteq U^n(A)$ with $x \in \mathsf{mub}(M)$. As $M \leq x$, as $M \subseteq C(A)$ by induction hypothesis, and as $C(A)$ is complete for M, there is an element $c \in C(A)$ with $M \leq c \leq x$. Consequently, $x = c \in C(A)$.

3.55. Example. The following algebraic dcpo's are well known not to be bifinite (see e.g. [1, Fig. 12], [27, Fig. 1.3, 1.4, 1.5]). The first two satisfy the ACC and hence are P-domains.



(a) (b) (c)

Fig. 3.5

(a) Figure 3.5(a) shows a P-domain $(D, \leq)$ in which $K(D) = D$ does not have property m.

(b) In Figure 3.5(b) a P-domain $(D, \leq)$ is shown in which $K(D) = D$ has property m (but not property M).

(c) The algebraic dcpo (whose set of compact elements has property M) in Figure 3.5(c) cannot be a P-domain due to Corollary 3.54.

3.5. Characterizations of bifinite domains

In this section we characterize bifinite domains as compact approximating pop's and as Lawson-compact P-domains.

The next theorem may be seen as the "pop analogue" to Theorem 2.47.

3.56. Theorem. *Let $\underline{D} = (D, \leq, \mathcal{P})$ be an approximating pop. Then the following are equivalent*:

(i) *$\underline{D}$ is compact.*

(ii) *$(D, \leq)$ is an (algebraic) dcpo and each $p \in \mathcal{P}$ is a Scott-continuous projection with finite range.*

In this case, $K(D) = \bigcup_{p \in \mathcal{P}} p[D]$. The Lawson topology of $(D, \leq)$ coincides with the pop topology.

Proof. (i)⇒(ii). Proposition 3.16 tells us that all $p \in \mathcal{P}$ have finite range. Therefore, condition (v) of Theorem 3.48 is satisfied implying that $(D, \leq)$ is an algebraic dcpo with $K(D) = \bigcup_{p \in \mathcal{P}} p[D]$. In particular, each $p \in \mathcal{P}$ is Scott-continuous (Lemma 3.3). Moreover, it follows from Corollary 2.49 that the Lawson topology and the pop topology coincide.

(ii)⇒(i). $\underline{D}$ is totally bounded because of Proposition 3.16 and complete by virtue of Proposition 2.25 (or Theorem 3.38). ■

The following corollary is similar to the characterization of FS-domains (Corollary 2.51).

3.57. COROLLARY. *Let $(D, \leq)$ be a poset. Then the following statements are equivalent*:

(i) *$(D, \leq)$ is a bifinite domain.*
(ii) *There exists a directed family $\mathcal{P}$ of projections such that $(D, \leq, \mathcal{P})$ is a compact approximating pop.* ■

Next, we show that P-domains with $K(D)$ having property M have to be bifinite already:

3.58. THEOREM. *Let $(D, \leq)$ be a poset. Then the following are equivalent*:

(i) *$(D, \leq)$ is a bifinite domain.*
(ii) *$(D, \leq)$ is a P-domain and $K(D)$ has property M.*

Proof. We need only prove (ii)⇒(i). Let $A \subseteq K(D)$ be finite. We show that $U^\infty(A)$ is finite. We do this by contradiction and assume that $U^\infty(A)$ is infinite. Then we may follow the arguments of Smyth's proof of Lemma 4 in [49], where König's Lemma is applied to show that there is a strictly increasing sequence in $U^\infty(A)$ in contradiction to Corollary 3.54. Alternatively, to prove the existence of such a sequence we can also use selection functions as done in Jung [27, proof of Lemma 2.2]. ■

Plotkin's so-called "2/3-SFP Theorem" states that an algebraic dcpo $(D, \leq)$ is Lawson-compact if and only if $K(D)$ has property M (cf. e.g. [27, Cor. 4.19]). Therefore, we immediately derive from Theorem 3.58:

3.59. COROLLARY. *A P-domain is bifinite if and only if it is compact in its Lawson topology.* ■

4. HOMOMORPHISMS AND FUNCTION SPACES OF POSETS WITH PROJECTIONS

We continue our investigation of posets with projections by studying structure preserving maps between them. These mappings induce function spaces that can be turned into pop's again. We prove that properties such as completeness and compactness of the pop uniformity are inherited by the function pop's. Moreover, in a very basic setting we study the question how to obtain both cartesian closed categories of pop's and pop's that are isomorphic to their own exponent.

In Section 4.1 we state the definition of *indexed pop's* or $(I,\leq)$*-pop's*. It is a very slight extension of the definition of pop's. Roughly speaking, an $(I,\leq)$-pop is a pop whose projection set is given by a monotone net $(p_i)_{i\in I}$ of projections. The directed index set $(I,\leq)$ is merely a fixed constant. We use it to define structure preserving maps between $(I,\leq)$-pop's. Given two $(I,\leq)$-pop's $(D,\leq,(p_i)_{i\in I})$ and $(E,\leq,(q_i)_{i\in I})$, we let a pop homomorphism be a mapping $f : D \to E$ that is compatible with the partial orders as well as with the projections. The image of each projection p_i can be viewed as the "ith level of approximation" (cf. Spreen [50]). Hence, we require f to be monotone and to preserve these levels of approximation. Besides homomorphisms, we also consider weak homomorphisms. They are monotone maps satisfying a weaker condition than homomorphisms.

Section 4.2 is devoted to a particular class of indexed pop's, namely to $(\mathbb{N}_0,\leq)$-pop's. We call them ω*-pop's*. As we shall point out, "projection spaces" in the sense of Ehrig, Große-Rhode and others [17, 23, 24], "rank ordered sets" investigated by Bruce and Mitchell [8], and "pseudo rank ordered posets" as defined by Baier and Majster-Cederbaum [2] may be regarded as ω-pop's. Moreover, we investigate length, pseudo-weight, and weight functions on posets $(D,\leq)$. Informally, if we think of D as a certain set of "processes", then $d \leq e$ can be interpreted as "d is a subprocess of e". Now a weight on D assigns to each process $d \in D$ the maximal number of steps (possibly infinitely many) which an execution of d has to perform (cf. Baier and Majster-Cederbaum [2, 40]). If d is a subprocess of e, then the weight of d is below the corresponding weight of e. A pseudo-weight function dispenses with the latter condition. We show that there is a one-to-one correspondence between pseudo-weight functions on posets and ω-pop's. Furthermore, we describe all ω-pop's coming from a weight function. Then we study weak homomorphisms and homomorphisms of ω-pop's and characterize them in terms of some canonical pseudo-ultrametric and the pseudo-weight. Finally, we apply our results to real traces. In particular, we give an order-theoretic characterization of d_{fnf}-isometries of real traces using the projection order induced by the Foata projections (Theorem 4.18).

Coming back to the general case, we discuss function spaces of $(I, \leq)$-pop's in Section 4.3. We endow several sets of mappings between $(I, \leq)$-pop's with a canonical pop structure. In particular, we consider function spaces of weak homomorphisms and homomorphisms. We prove that the uniformity of the function pop's is the uniformity of uniform convergence. This enables us to show that completeness and compactness properties transfer to function spaces. Finally, we obtain cartesian closed categories of $(I, \leq)$-pop's with respect to both weak homomorphisms and homomorphisms. Concerning weak homomorphisms, the exponential object coincides with the function space (cf. Theorem 4.29). This does not hold anymore when we deal with homomorphisms. Here, the exponential object is "larger" than the function space (see Theorem 4.35). We also consider cartesian closed categories of $(I, \leq)$-pop's whose underlying posets are dcpo's and where all pop structure preserving maps are required to be Scott-continuous (see Theorems 4.32 and 4.37). This enables us to apply Scott's D_∞-construction to get $(I, \leq)$-pop's that are pop isomorphic to their own exponent (Theorem 4.42). Thus, we obtain new models of the untyped λ-calculus.

The main results of this chapter concerning homomorphisms are presented in [35].

4.1. Homomorphisms and substructures of indexed pop's

How can we compare pop's? As is usual in mathematics, the answer should be: use homomorphisms! Thus, how can we define a suitable notion of a "pop homomorphism"? Consider, for instance, two pop's $(D, \leq, \{p_n \mid n \in \mathbb{N}_0\})$ and $(E, \leq, \{q_n \mid n \in \mathbb{N}_0\})$ whose projection sets are given by monotone sequences $(p_n)_{n\in\mathbb{N}_0}$ and $(q_n)_{n\in\mathbb{N}_0}$, respectively. Of course, a "pop homomorphism" $f : D \to E$ should be monotone. But we should also require some compatibility condition between f and the projections p_n and q_n for all $n \in \mathbb{N}_0$. For example, $q_n \circ f = f \circ p_n$ would be fine. However, there are also pop's whose projection set is not given by a sequence $(p_n)_{n\in\mathbb{N}_0}$. For this reason, we give the following definition, which slightly extends the notion of a pop.

Definition. Let $(I, \leq)$ be a directed index set, let $(D, \leq)$ be a poset, and let $(p_i)_{i\in I}$ be a monotone net of projections on D. Then we call the triple $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ a *poset with $(I, \leq)$-indexed projections*, or *$(I, \leq)$-pop* for short. The net $(p_i)_{i\in I}$ is the *projection net* of $\mathcal{D}$.

Sometimes we use the notion of an *indexed pop* to indicate that we deal with $(I, \leq)$-pop's for some directed set $(I, \leq)$.

Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop and let $\mathcal{P}_\mathcal{D} := \{p_i \mid i \in I\}$. It is obvious that then $\underline{D_\mathcal{D}} := (D, \leq, \mathcal{P}_\mathcal{D})$ is a pop, which we call the *associated pop* of $\mathcal{D}$. Note that two $(I, \leq)$-pop's may have the same associated pop. Conversely, each pop $(D, \leq, \mathcal{P})$ induces a $(\mathcal{P}, \leq)$-pop $(D, \leq, (p)_{p\in\mathcal{P}})$.

We write $\mathcal{U}_\mathcal{D}$ for the pop uniformity of $\underline{D_\mathcal{D}}$ and say that $\mathcal{U}_\mathcal{D}$ is the *pop uniformity of* $\mathcal{D}$. Similarly for the *pop topology* $\tau_\mathcal{D}$. If the associated pop $\underline{D_\mathcal{D}}$ of an $(I, \leq)$-pop $\mathcal{D}$ has some property E, then we say that $\mathcal{D}$ has property E. For instance, if $\underline{D_\mathcal{D}}$ is approximating, then we say that $\mathcal{D}$ is approximating. Furthermore, if $\mathcal{P}_{\mathcal{D}}$ is Abelian, then we call $(p_i)_{i\in I}$ an *Abelian* projection net.

We define substructures of indexed pop's as follows:

DEFINITION. Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop and let $X \subseteq D$. We say that X *induces a subpop of* $\mathcal{D}$ if $p_i[X] \subseteq X$ for all $i \in I$. Then, together with the induced order and the restricted projections, we call $\mathcal{X} = (X, \leq, (p_i|_X)_{i\in I})$ a *subpop* of $\mathcal{D}$.

Clearly, any subpop $\mathcal{X} = (X, \leq, (p_i|_X)_{i\in I})$ of an $(I, \leq)$-pop is an $(I, \leq)$-pop itself. If $\mathcal{D}$ is approximating, then $\mathcal{X}$ is also approximating.

Next, we define homomorphisms between $(I, \leq)$-pop's. As mentioned before, we require them to be compatible both with the partial orders and with the projections. We also introduce the notion of a weak homomorphism, which satisfies a weaker condition than a homomorphism.

DEFINITION. Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ and $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be $(I, \leq)$-pop's and let $f : D \to E$.

(1) We call f *non-expansive* if $q_i \circ f = q_i \circ f \circ p_i$ for all $i \in I$ (see Figure 4.1).

$$
\begin{tikzcd}
 & & & D \arrow[d, "p_i"] & \\
D \arrow[r, "f"] & E \arrow[d, "q_i"] & = & D \arrow[r, "f"] & E \arrow[d, "q_i"] \\
 & E & & & E
\end{tikzcd}
$$

Fig. 4.1. The condition $q_i \circ f = q_i \circ f \circ p_i$

(2) The mapping f *commutes with all projections* if $q_i \circ f = f \circ p_i$ for all $i \in I$ (cf. Figure 4.2).

$$
\begin{tikzcd}
D \arrow[r, "f"] \arrow[d, "p_i"'] \arrow[dr, phantom, "="] & E \arrow[d, "q_i"] \\
D \arrow[r, "f"'] & E
\end{tikzcd}
$$

Fig. 4.2. The condition $q_i \circ f = f \circ p_i$

(3) We say that f is a *weak (pop) homomorphism* if f is monotone and non-expansive.
(4) We call f a *(pop) homomorphism* provided that f is monotone and commutes with all projections.
(5) We say that f is a *(pop) embedding* if f is an order embedding and a pop homomorphism.

(6) Finally, f is a *(pop) isomorphism* if f is both an order isomorphism and a pop homomorphism. Two $(I, \leq)$-pop's are said to be *(pop) isomorphic* if there exists a pop isomorphism between them.

We remark here that for projection spaces, non-expansive mappings and mappings that commute with all projections were introduced by Ehrig *et al.* [17] (see also [23, 24]) as "projection compatible mappings" and "projection morphisms", respectively.

The property of $f : D \to E$ being non-expansive has a topological motivation. In the case of $(I, \leq) = (\mathbb{N}_0, \leq)$ we will show (Proposition 4.12 below) that non-expansive mappings f in the sense of the previous definition are precisely the (metrically non-expansive) mappings satisfying $\varrho_{\mathcal{E}}(f(d_1), f(d_2)) \leq \varrho_{\mathcal{D}}(d_1, d_2)$ for all $d_1, d_2 \in D$, where $\varrho_{\mathcal{D}}$ and $\varrho_{\mathcal{E}}$ are canonical pseudo-ultrametrics (cf. Section 4.2 below).

Intuitively, the image $p_i(d)$ can be seen as the "ith approximation" of the element $d \in D$. Hence, the property of $f : D \to E$ to commute with all projections means that f preserves all levels of approximation: the ith approximation of $f(d)$ coincides with the image of the ith approximation of d.

Clearly, if $f : D \to E$ commutes with all projections, then it is non-expansive. Hence, homomorphisms are weak homomorphisms. Each non-expansive mapping is uniformly continuous with respect to the pop uniformities. Some further relations between the above definitions are listed in the following lemma. The equivalence (3)(ii)$\Leftrightarrow$(iii) for approximation structures can be found in [50, Lemma 9].

4.1. Lemma. *Let $(D, \leq, (p_i)_{i \in I})$ and $(E, \leq, (q_i)_{i \in I})$ be $(I, \leq)$-pop's and let $f : D \to E$.*

(1) *The following are equivalent:*

  (i) *f is non-expansive.*
  (ii) *$(f \times f)[\mathsf{ker}\, p_i] \subseteq \mathsf{ker}\, q_i$ for all $i \in I$.*
  (iii) *$\mathsf{ker}\, p_i \subseteq \mathsf{ker}(q_i \circ f)$ for all $i \in I$.*

  The next statement is necessary and, if f is monotone, also sufficient for the previous ones:

  (iv) *$q_i \circ f \leq f \circ p_i$ for all $i \in I$.*

(2) *f is a weak homomorphism if and only if f is monotone and $q_i \circ f \leq f \circ p_i$ for all $i \in I$.*

(3) *The following are equivalent:*

  (i) *f commutes with all projections.*
  (ii) *f is non-expansive and $f \circ p_i = q_i \circ f \circ p_i$ for all $i \in I$.*
  (iii) *f is non-expansive and $f[p_i[D]] \subseteq q_i[E]$ for all $i \in I$.*

  In particular, f is a homomorphism if and only if f is a weak homomorphism with $f \circ p_i = q_i \circ f \circ p_i$ for all $i \in I$ if and only if f is a weak homomorphism with $f[p_i[D]] \subseteq q_i[E]$ for all $i \in I$.

(4) *Let f be bijective. If f commutes with all projections, then f and f^{-1} are non-expansive. If f is monotone and f and f^{-1} are non-expansive, then f commutes with all projections.*

(5) *f is a pop isomorphism if and only if f is bijective and both f and f^{-1} are weak homomorphisms.*

Proof. (1) Clearly, (ii) and (iii) are equivalent.

(i)$\Rightarrow$(ii). Let $i \in I$ and let $d, e \in D$ with $p_i(d) = p_i(e)$. Then $q_i(f(d)) = q_i(f(p_i(d))) = q_i(f(p_i(e))) = q_i(f(e))$.

(ii)$\Rightarrow$(i). Let $d \in D$ and let $i \in I$. As $(p_i(d), d) \in \mathsf{ker}\, p_i$, we infer $q_i(f(p_i(d))) = q_i(f(d))$.

(i)$\Rightarrow$(iv). $q_i \circ f = q_i \circ f \circ p_i \leq f \circ p_i$ for all $i \in I$.

(iv)$\Rightarrow$(i). Let f be monotone. Then for all $i \in I$ we have $q_i \circ f = q_i \circ q_i \circ f \leq q_i \circ f \circ p_i \leq q_i \circ f$, whence $q_i \circ f = q_i \circ f \circ p_i$.

(2) follows from (1) (i)$\Leftrightarrow$(iv).

(3) (i)$\Leftrightarrow$(ii) and (ii)$\Rightarrow$(iii) are clear. To prove (iii)$\Rightarrow$(ii), let $d \in D$ and let $i \in I$. We find some $e \in E$ with $f(p_i(d)) = q_i(e)$. Thus, $q_i(f(p_i(d))) = q_i(e)$ and so $f(p_i(d)) = q_i(f(p_i(d)))$. Notice that in fact we have shown $f \circ p_i = q_i \circ f \circ p_i$ to be equivalent to $f[p_i[D]] \subseteq q_i[E]$.

(4) Let f be bijective. First assume that f commutes with all projections. Then f^{-1} commutes with all projections as well. We conclude that f and f^{-1} are non-expansive. Now let f be monotone and let f and f^{-1} be non-expansive. Let $i \in I$. Then we have $q_i \circ f \leq f \circ p_i$ and $p_i \circ f^{-1} \leq f^{-1} \circ q_i$ by (1). From the latter inequality we deduce $f \circ p_i \leq q_i \circ f$ because f is monotone. This yields the assertion.

(5) results from (4). ∎

There is an obvious connection between subpop's and pop embeddings: let $\mathcal{D} = (D, \leq, (p_i)_{i \in I})$ be an $(I, \leq)$-pop. If $\mathcal{X} = (X, \leq, (p_i|_X)_{i \in I})$ is a subpop of $\mathcal{D}$, then the inclusion map $\mathsf{id}_{X,D}$ is a pop embedding. Conversely, let $\mathcal{E} = (E, \leq, (q_i)_{i \in I})$ be an $(I, \leq)$-pop and let $f : D \to E$ be a pop embedding. Then $f[D]$ induces a subpop of $\mathcal{E}$ that is pop isomorphic to $\mathcal{D}$.

A natural question is when all projections p_i of an $(I, \leq)$-pop $(D, \leq, (p_i)_{i \in I})$ are also (weak) homomorphisms. Abelian projection nets give the answer:

4.2. Lemma. *Let $\mathcal{D} = (D, \leq, (p_i)_{i \in I})$ be an $(I, \leq)$-pop. Then the following are equivalent:*

(i) *$(p_i)_{i \in I}$ is Abelian.*
(ii) *p_i is a homomorphism for all $i \in I$.*
(iii) *p_i is a weak homomorphism for all $i \in I$.*
(iv) *$p_i[D]$ induces a subpop of $\mathcal{D}$ for all $i \in I$.*

Proof. (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iv) are trivial.

(iv)$\Rightarrow$(iii). Let $i \in I$. Then $p_j(p_i(d)) \in p_i[D]$ for all $j \in I$ and all $d \in D$ because $p_i[D]$ induces a subpop. Thus, $p_j(p_i(d)) = p_i(p_j(p_i(d))) \leq p_i(p_j(d))$ for all $j \in I$, $d \in D$. Because of Lemma 4.1(2), p_i is non-expansive.

(iii)$\Rightarrow$(i). Let $i, j \in I$. Then $p_j \circ p_i \leq p_i \circ p_j \leq p_j \circ p_i$ by Lemma 4.1(2); hence $p_j \circ p_i = p_i \circ p_j$. ∎

4.3. Lemma. *Let $\mathcal{D} = (D, \leq, (p_i)_{i \in I})$ and $\mathcal{E} = (E, \leq, (q_i)_{i \in I})$ be $(I, \leq)$-pop's such that $(p_i)_{i \in I}$ and $(q_i)_{i \in I}$ are Abelian. Then any mapping $f : D \to E$ that commutes with all* *projections is monotone with respect to the projection orders, i.e. f is a homomorphism from $(D, \sqsubseteq_{\{p_i|i\in I\}}, (p_i)_{i\in I})$ to $(E, \sqsubseteq_{\{q_i|i\in I\}}, (q_i)_{i\in I})$. If, furthermore, f is injective, then f is an embedding of $(D, \sqsubseteq_{\{p_i|i\in I\}}, (p_i)_{i\in I})$ into $(E, \sqsubseteq_{\{q_i|i\in I\}}, (q_i)_{i\in I})$.*

Proof. This results from the definition of the projection order (p. 52). ■

Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop and let $\mathcal{E} = (E, \leq, (q_j)_{j\in J})$ be a $(J, \leq)$-pop. For all $i \in I$ and all $j \in J$ define $p_{ij} := p_i$ and $q_{ij} := q_j$. Obviously, $\mathcal{D}' := (D, \leq, (p_{ij})_{(i,j)\in I\times J})$ and $\mathcal{E}' := (E, \leq, (q_{ij})_{(i,j)\in I\times J})$ are $(I \times J, \leq)$-pop's with $\underline{D_{\mathcal{D}'}} = \underline{D_{\mathcal{D}}}$ and $\underline{E_{\mathcal{E}'}} = \underline{E_{\mathcal{E}}}$. Therefore, if we really want to compare an $(I, \leq)$-pop with a $(J, \leq)$-pop, we *could* do this by switching to the product index set $(I \times J, \leq)$. Nevertheless, one should be careful with this. Comparing $\mathcal{D}$ with $\mathcal{E}$ and comparing $\mathcal{D}'$ with $\mathcal{E}'$ could yield two different results. For instance, let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop such that there are indices $i_0, j_0 \in I$ with $p_{i_0} \neq p_{j_0}$. Clearly, the identity is a pop isomorphism from $\mathcal{D}$ onto itself. Now consider $\mathcal{D}' = (D, \leq, (p_{ij})_{(i,j)\in I\times I})$ with $p_{ij} = p_i$ and $\mathcal{E}' = (D, \leq, (q_{ij})_{(i,j)\in I\times I})$ with $q_{ij} = p_j$ for all $i, j \in I$. Then the identity is *not* a pop homomorphism from $\mathcal{D}'$ to $\mathcal{E}'$ because $q_{i_0 j_0} \circ \mathsf{id}_D = p_{j_0} \neq p_{i_0} = \mathsf{id}_D \circ p_{i_0 j_0}$.

4.2. ω-pop's and length functions

In this section we consider $(\mathbb{N}_0, \leq)$-pop's. We call them *ω-pop's* in what follows. Recall that the pop uniformity of an ω-pop $\mathcal{D} = (D, \leq, (p_n)_{n\in\mathbb{N}_0})$ can be given by a pseudo-ultrametric: for all $d, e \in D$ let

$$\begin{aligned} \ell_{\mathcal{D}}(d, e) &:= \sup\{n \in \mathbb{N}_0 \mid p_n(d) = p_n(e)\} \in \mathbb{N}_0 \cup \{\infty\}, \\ \varrho_{\mathcal{D}}(d, e) &:= 2^{-\ell_{\mathcal{D}}(d,e)} \qquad (\text{with } 2^{-\infty} := 0). \end{aligned}$$

Theorem 3.15(1) tells us that $\varrho_{\mathcal{D}}$ is a pseudo-ultrametric on D that induces the pop uniformity. We call $\varrho_{\mathcal{D}}$ the *canonical pseudo-ultrametric* of $\mathcal{D}$. Note that we have $\ell_{\mathcal{D}}(d, e) \geq n$ if and only if $p_n(d) = p_n(e)$ and, moreover, $\ell_{\mathcal{D}}(d, e) = n$ if and only if $p_n(d) = p_n(e)$ and $p_{n+1}(d) \neq p_{n+1}(e)$ (cf. also 3.15(1)).

4.4. Example. Let (Σ, D) be a dependence alphabet.

(a) Let $\mathcal{D}_1 = (\mathbb{R}(\Sigma, D), \leq, (p_n)_{n\in\mathbb{N}_0})$ and $\mathcal{D}_2 = (\mathbb{R}(\Sigma, D), \leq, (h_n)_{n\in\mathbb{N}_0})$, where the projections p_n and h_n are defined as in Example 3.35. Then $\varrho_{\mathcal{D}_1} = d_{\text{pref}}$ and $\varrho_{\mathcal{D}_2} = d_{\text{fnf}}$ (cf. 3.35).

(b) Let $\mathcal{D}_3 = (\mathbb{F}^\alpha(\Sigma, D), \sqsubseteq, (h_n)_{n\in\mathbb{N}_0})$ and $\mathcal{D}_4 = (\mathbb{F}^\delta(\Sigma, D), \sqsubseteq, (h_n)_{n\in\mathbb{N}_0})$ with the projections h_n as in Example 3.36. Then $\varrho_{\mathcal{D}_3} = d_\alpha$ and $\varrho_{\mathcal{D}_4} = d_\delta$.

In the following we compare ω-pop's to "projection spaces" investigated by Ehrig, Große-Rhode and others [17, 23, 24], to "rank ordered sets" defined by Bruce and Mitchell [8], and to "pseudo rank ordered posets" studied by Baier and Majster-Cederbaum [2]. It turns out that each projection space and any rank ordered set can be turned into an ω-pop using the projection order as underlying partial order. Moreover, each pseudo rank ordered poset is a special ω-pop.

ω-pop's versus projection spaces. A *projection space* is a pair (D, p) consisting of a set D and a mapping $p : \mathbb{N}_0 \times D \to D$ such that $p(n, p(m, d)) = p(\min\{m, n\}, d)$ for all $m, n \in \mathbb{N}_0$ and all $d \in D$ (cf. [23, 24]).

To turn (D, p) into an ω-pop, we have to define a suitable partial order and a monotone sequence $(p_n)_{n\in\mathbb{N}_0}$ of projections. As to the latter, for all $n \in \mathbb{N}_0$ we define $p_n : D \to D$ by $p_n(d) := p(n, d)$ for all $d \in D$. Clearly, $(\{p_n \mid n \in \mathbb{N}_0\}, \circ)$ is an Abelian semigroup of idempotent mappings. Moreover, for all $m, n \in \mathbb{N}_0$ we have $p_m \circ p_{\max\{m,n\}} = p_m$ and $p_n \circ p_{\max\{m,n\}} = p_n$. Taking the projection order of $(D, \{p_n \mid n \in \mathbb{N}_0\})$, we obtain an ω-pop $(D, \sqsubseteq_{\{p_n|n\in\mathbb{N}_0\}}, (p_n)_{n\in\mathbb{N}_0})$ by virtue of Proposition 3.23. We call it *induced* by (D, p).

Conversely, let $(D, \leq, (p_n)_{n\in\mathbb{N}_0})$ be an ω-pop. Letting $p : \mathbb{N}_0 \times D \to D$ be defined by $p(n, d) := p_n(d)$ for all $n \in \mathbb{N}_0$ and all $d \in D$, we infer that (D, p) is a projection space because of Lemma 3.1. It is the projection space *induced* by $(D, \leq, (p_n)_{n\in\mathbb{N}_0})$.

In [17] the following additional axiom is required in the definition of a projection space:

$$\forall d, e \in D : (\forall n \in \mathbb{N}_0 : p(n, d) = p(n, e)) \Rightarrow d = e.$$

If (D, p) is a projection space satisfying this condition, then its induced ω-pop is Hausdorff. This follows from Proposition 3.18(2). On the other hand, if $(D, \leq, (p_n)_{n\in\mathbb{N}_0})$ is a Hausdorff ω-pop, then the induced projection space satisfies the above condition by the same result. Note that this is equivalent to saying that the canonical pseudo-ultrametric of $(D, \leq, (p_n)_{n\in\mathbb{N}_0})$ is in fact an ultrametric.

ω-pop's versus rank ordered sets. A *rank ordered set* $(D, (p_n)_{n\in\mathbb{N}_0})$ in the sense of Bruce and Mitchell [8] consists of a set D and self-maps $p_n : D \to D$ for all $n \in \mathbb{N}_0$ such that:

(1) p_0 is a constant map.
(2) $p_m \circ p_n = p_{\min\{m,n\}}$ for all $m, n \in \mathbb{N}_0$.
(3) For all sequences $(d_n)_{n\in\mathbb{N}_0}$ in D with $d_n = p_n(d_{n+1})$ for all $n \in \mathbb{N}_0$ there exists a unique $d \in D$ such that $d_n = p_n(d)$ for all $n \in \mathbb{N}_0$.

Because of condition (2) we may view a rank ordered set as a projection space by defining p as above. Thus, with the projection order, we obtain an ω-pop $\mathcal{D} = (D, \sqsubseteq_{\{p_n|n\in\mathbb{N}_0\}}, (p_n)_{n\in\mathbb{N}_0})$. Condition (3) is a completeness property: it implies that the underlying uniform space of $\mathcal{D}$ is both Hausdorff and complete. The Hausdorff property follows from Proposition 3.18(2) and the uniqueness condition of (3) applied to sequences $(p_n(d))_{n\in\mathbb{N}_0}$ and $(p_n(e))_{n\in\mathbb{N}_0}$ with $p_n(d) = p_n(e)$ for all $n \in \mathbb{N}_0$. In order to prove completeness, let $(e_n)_{n\in\mathbb{N}_0}$ be a Cauchy sequence in D. Using Lemma 3.37, we find by induction a subsequence $(e_{n_k})_{k\in\mathbb{N}_0}$ such that $p_k(e_n) = p_k(e_{n_k})$ for all $k \in \mathbb{N}_0$ and all $n \geq n_k$. Then $p_k(e_{n_k}) = p_k(e_{n_{k+1}}) = p_k(p_{k+1}(e_{n_{k+1}}))$; hence by (3) there is a $d \in D$ with $p_k(d) = p_k(e_{n_k}) = p_k(e_n)$ for all $k \in \mathbb{N}_0$ and all $n \geq n_k$. Consequently, $(e_n)_{n\in\mathbb{N}_0}$ converges to d.

ω-pop's versus pseudo rank orderings. Roughly speaking, ω-pop's and pseudo rank ordered posets are much the same thing. Formally, Baier and Majster-Cederbaum [2] define a *pseudo rank ordering* to be a sequence $(\pi_n)_{n\in\mathbb{N}_0}$ of mappings from a pointed poset $(D,\leq)$ into itself such that the following conditions hold:

(1) π_0 sends all elements to the least element of $(D,\leq)$.
(2) π_n preserves suprema of monotone sequences (i.e. π_n is *ω-continuous*) for all $n\in\mathbb{N}_0$.
(3) $\pi_n \leq \mathsf{id}_D$ for all $n\in\mathbb{N}_0$.
(4) $\pi_m\circ\pi_n = \pi_n\circ\pi_m = \pi_m$ for all $0\leq m\leq n$.

In this case, $(D,\leq,(\pi_n)_{n\in\mathbb{N}_0})$ is said to be a *pseudo rank ordered poset.* Thus, in view of Lemma 3.1, pseudo rank ordered posets are precisely the ω-pop's $(D,\leq,(p_n)_{n\in\mathbb{N}_0})$ with $(D,\leq)$ having a least element $\perp$ such that all p_n are ω-continuous and $p_0(d)=\perp$ for all $d\in D$.

A *rank ordered poset* is a pseudo rank ordered poset $(D,\leq,(\pi_n)_{n\in\mathbb{N}_0})$ satisfying

$$\forall d,e\in D : (\forall n\in\mathbb{N}_0 : \pi_n(d)=\pi_n(e)) \Rightarrow d=e;$$

cf. the analogous condition for projection spaces on p. 87. This condition is equivalent to the Hausdorff property of the ω-pop $(D,\leq,(\pi_n)_{n\in\mathbb{N}_0})$ (see again Proposition 3.18(2)).

ω-pop's induced by weight functions. Let $(D,\leq)$ be a poset. We define a *length function* to be a mapping $|\cdot| : D\to\mathbb{N}_0\cup\{\infty\}$ such that $\sup\{x\in D \mid x\leq d,\ |x|\leq n\}$ exists for all $n\in\mathbb{N}_0$ and all $d\in D$.

For instance, if $(D,\leq)$ is a bcpo with least element, then $\{x\in D\mid x\leq d,\ |x|\leq n\}\subseteq d{\downarrow}$ has a supremum for any mapping $|\cdot| : D\to\mathbb{N}_0\cup\{\infty\}$.

4.5. Proposition. *Let $(D,\leq)$ be a poset and let $|\cdot| : D\to\mathbb{N}_0\cup\{\infty\}$ be a length function on D. For all $n\in\mathbb{N}_0$ define a mapping $q_n^{|\cdot|} : D\to D$ by*

$$q_n^{|\cdot|}(d) := \sup\{x\in D\mid x\leq d,\ |x|\leq n\}\qquad (d\in D).$$

Then $\mathcal{D}=(D,\leq,(q_n^{|\cdot|})_{n\in\mathbb{N}_0})$ is an ω-pop. Furthermore, we have

$$\ell_{\mathcal{D}}(d,e) = \sup\{n\in\mathbb{N}_0 \mid x\leq d \Leftrightarrow x\leq e \text{ for all } x\in D \text{ with } |x|\leq n\}.$$

Proof. Let $n\in\mathbb{N}_0$ and let $d\in D$. Clearly, $q_n^{|\cdot|}(d)\leq d$. We show that $\{x\in D\mid x\leq d,\ |x|\leq n\} = \{x\in D\mid x\leq q_n^{|\cdot|}(d),\ |x|\leq n\}$. The inclusion $\supseteq$ is obvious. Let $x\in D$ with $x\leq d$ and $|x|\leq n$. By definition of $q_n^{|\cdot|}(d)$ we have $x\leq q_n^{|\cdot|}(d)$. Therefore, both sets are equal; hence the suprema are the same, i.e. $q_n^{|\cdot|}(d)=q_n^{|\cdot|}(q_n^{|\cdot|}(d))$. Let $d,e\in D$ with $d\leq e$. Then $\{x\in D\mid x\leq d,\ |x|\leq n\}\subseteq\{x\in D\mid x\leq e,\ |x|\leq n\}$ and thus $q_n^{|\cdot|}(d)\leq q_n^{|\cdot|}(e)$. Next, as $\{x\in D\mid x\leq d,\ |x|\leq n\}\subseteq\{x\in D\mid x\leq d,\ |x|\leq n+1\}$, we infer $q_n^{|\cdot|}(d)\leq q_{n+1}^{|\cdot|}(d)$.

Now we prove that the sets $\{n\in\mathbb{N}_0\mid q_n^{|\cdot|}(d)=q_n^{|\cdot|}(e)\}$ and $\{n\in\mathbb{N}_0\mid x\leq d \Leftrightarrow x\leq e \text{ for all } x\in D \text{ with } |x|\leq n\}$ are the same. Let $n\in\mathbb{N}_0$ such that $q_n^{|\cdot|}(d)=q_n^{|\cdot|}(e)$. Then, as we have seen above, $\{x\in D\mid x\leq d,\ |x|\leq n\}=\{x\in D\mid x\leq q_n^{|\cdot|}(d),\ |x|\leq n\}=\{x\in D\mid x\leq q_n^{|\cdot|}(e),\ |x|\leq n\}=\{x\in D\mid x\leq e,\ |x|\leq n\}$; hence $x\leq d$ if and only if $x\leq e$ for all $|x|\leq n$. Conversely, let $n\in\mathbb{N}_0$ with $x\leq d$ if and only if $x\leq e$ for all $|x|\leq n$. Then $q_n^{|\cdot|}(d)=q_n^{|\cdot|}(e)$ by definition of $q_n^{|\cdot|}$. ∎

We call $(D, \leq, (q_n^{|\cdot|})_{n\in\mathbb{N}_0})$ the *ω-pop induced by the length function* $|\cdot|$. A length function $|\cdot|$ *induces* an ω-pop $(D, \leq, (p_n)_{n\in\mathbb{N}_0})$ if $p_n = q_n^{|\cdot|}$ for all $n \in \mathbb{N}_0$.

4.6. REMARK. Let $|\cdot| : D \to \mathbb{N}_0 \cup \{\infty\}$ be a length function on D and let $\mathcal{D} = (D, \leq, (q_n^{|\cdot|})_{n\in\mathbb{N}_0})$ be the induced ω-pop. If $d \in D$ and $n \in \mathbb{N}_0$ with $|d| \leq n$, then $q_n^{|\cdot|}(d) = d$. This follows immediately from the definition of the projections $q_n^{|\cdot|}$.

A *pseudo-weight function* is a length function $\|\cdot\| : D \to \mathbb{N}_0 \cup \{\infty\}$ with $\|q_n^{\|\cdot\|}(d)\| \leq n$ for all $d \in D$ and all $n \in \mathbb{N}_0$. In other words, a pseudo-weight function is a mapping $\|\cdot\| : D \to \mathbb{N}_0 \cup \{\infty\}$ such that for all $d \in D$ and all $n \in \mathbb{N}_0$ the set $\{x \in D \mid x \leq d,\ \|x\| \leq n\}$ has a greatest element (which is $q_n^{\|\cdot\|}(d)$ then).

4.7. REMARK. Let $\|\cdot\| : D \to \mathbb{N}_0 \cup \{\infty\}$ be a pseudo-weight function on D and let $\mathcal{D} = (D, \leq, (q_n^{\|\cdot\|})_{n\in\mathbb{N}_0})$ be the induced ω-pop. Let $d \in D$ and let $n \in \mathbb{N}_0$. Then $\|d\| \leq n$ if and only if $q_n^{\|\cdot\|}(d) = d$ (cf. Remark 4.6). In particular, $\|d\| \in \mathbb{N}_0$ if and only if $d \in \bigcup_{n\in\mathbb{N}_0} q_n^{\|\cdot\|}[D]$. Moreover, $\|d\| = \inf\{n \in \mathbb{N}_0 \mid q_n^{\|\cdot\|}(d) = d\}$ (cf. [2, Rem. 2.28]).

It turns out that each ω-pop comes from a pseudo-weight function. This is implicitly stated in [2]. On the other hand, there cannot be two different pseudo-weight functions inducing the same ω-pop:

4.8. PROPOSITION. *Let $\mathcal{D} = (D, \leq, (p_n)_{n\in\mathbb{N}_0})$ be an ω-pop.*

(1) (cf. [2, Rem. 2.28]) *Let $\|d\| := \inf\{n \in \mathbb{N}_0 \mid p_n(d) = d\}$ for all $d \in D$. Then this definition yields a pseudo-weight function $\|\cdot\|$ inducing $\mathcal{D}$.*
(2) *If $\|\cdot\|_1$ and $\|\cdot\|_2$ are pseudo-weight functions inducing $\mathcal{D}$, then $\|\cdot\|_1 = \|\cdot\|_2$.*

Proof. (1) Let $d \in D$ and let $n \in \mathbb{N}_0$. Clearly, $p_n(d) \leq d$ and $\|p_n(d)\| \leq n$ because $p_n(p_n(d)) = p_n(d)$. Hence, $p_n(d) \in \{x \in D \mid x \leq d,\ \|x\| \leq n\}$. Let $x \leq d$ with $\|x\| \leq n$. Then, as $\{m \in \mathbb{N}_0 \mid p_m(x) = x\}$ is an upper set, we infer $p_n(x) = x \leq d$, whence $x = p_n(x) \leq p_n(d)$. Thus, $p_n(d) = q_n^{\|\cdot\|}(d)$, implying the result.

(2) As $q_n^{\|\cdot\|_1} = p_n = q_n^{\|\cdot\|_2}$ for all $n \in \mathbb{N}_0$, Remark 4.7 yields $\|d\|_1 = \|d\|_2$ for all $d \in D$. ∎

Thus, due to Proposition 4.8, each ω-pop $\mathcal{D}$ has precisely one pseudo-weight function that induces $\mathcal{D}$.

4.9. PROPOSITION. *Let $\mathcal{D} = (D, \leq, (p_n)_{n\in\mathbb{N}_0})$ be an ω-pop. Let $|\cdot|$ be any length function inducing $\mathcal{D}$. Then the pseudo-weight function $\|\cdot\|$ of $\mathcal{D}$ satisfies $\|d\| \leq |d|$ for all $d \in D$. Thus, $\|\cdot\|$ is the smallest length function inducing $\mathcal{D}$.*

Proof. Proposition 4.8 tells us that $\mathcal{D}$ is induced by $\|\cdot\|$. Let $d \in D$. Then $\|d\| = n \in \mathbb{N}$ if and only if $p_n(d) = d$ and $p_{n-1}(d) \neq d$, see Remark 4.7. In this case, if we had $|d| < n$, then $d \in \{x \in D \mid x \leq d,\ |x| \leq n-1\}$, whence $d \leq p_{n-1}(d)$, a contradiction. Thus, $\|d\| = n \leq |d|$. Clearly, this is also true for $n = 0$. Let $\|d\| = \infty$, i.e. $p_n(d) \neq d$ for all $n \in \mathbb{N}_0$. Similarly, $|d| < \infty$ would yield a contradiction. ∎

As mentioned before, a "weight" on D assigns to any process $d \in D$ the maximal number of steps required for an execution of d. This happens in such a way that for all natural numbers n below the weight of d there is a greatest subprocess of d of weight at most n that coincides with d in the first n steps. Further, if d is a subprocess of e, then it is natural to assume that the weight of d is below the corresponding weight of e. This motivates the following definition (cf. [2, Def. 2.7]): a *weight function* is a monotone pseudo-weight function.

In general, an ω-pop is not induced by a weight function. We give an equivalent description of all ω-pop's coming from a weight function:

4.10. PROPOSITION. *Let $(D, \leq)$ be a poset.*

(1) *Let $\|\cdot\| : D \to \mathbb{N}_0 \cup \{\infty\}$ be a weight function. Then $(D, \leq, (q_n^{\|\cdot\|})_{n\in\mathbb{N}_0})$ is an ω-pop with all projections $q_n^{\|\cdot\|}$ downwards closed ($n \in \mathbb{N}_0$).*

(2) *Let $\mathcal{D} = (D, \leq, (p_n)_{n\in\mathbb{N}_0})$ be an ω-pop such that p_n is downwards closed for all $n \in \mathbb{N}_0$. Then the pseudo-weight function of $\mathcal{D}$ is a weight function.*

Proof. (1) Let $n \in \mathbb{N}_0$ and let $d, e \in D$ with $d \leq q_n^{\|\cdot\|}(e)$. Then $\|d\| \leq \|q_n^{\|\cdot\|}(e)\| \leq n$. Therefore, $q_n^{\|\cdot\|}(d) = d$ (see Remark 4.7).

(2) Let $\|\cdot\|$ be the canonical pseudo-weight function of $\mathcal{D}$. We need to show that $\|\cdot\|$ is monotone. Let $d, e \in D$ with $d \leq e$. Let $m \in \mathbb{N}_0$ be such that $p_m(e) = e$. Then $d \leq p_m(e)$, whence $p_m(d) = d$. Consequently, $\{m \in \mathbb{N}_0 \mid p_m(d) = d\} \supseteq \{m \in \mathbb{N}_0 \mid p_m(e) = e\}$ and so $\|d\| \leq \|e\|$. ∎

Hence, ω-pop's induced by weight functions are precisely the ω-pop's with downwards closed projections.

4.11. EXAMPLE. Let (Σ, D) be a dependence alphabet.

(a) As $(\mathbb{R}(\Sigma, D), \leq)$ is a bcpo with least element (Theorem 1.7), the map $|\cdot|$ assigning the length $|t|$ to each real trace t is a length function. Let $t = [V, E, \lambda] \in \mathbb{R}(\Sigma, D)$ and let $n \in \mathbb{N}_0$. Then we have $q_n^{|\cdot|}(t) = \sup\{s \in \mathbb{R}(\Sigma, D) \mid s \leq t,\ |s| \leq n\} = p_n(t)$, where p_n is defined as in Example 3.35(a). That is, $p_n(t) = [W_n, E|_{W_n \times W_n}, \lambda|_{W_n}]$ with $W_n := \{v \in V \mid |v{\downarrow}_{E^*}| \leq n\}$ (cf. 3.35(a)). In general, $|\cdot|$ is not a pseudo-weight function. The pseudo-weight function $\|\cdot\|$ of $(\mathbb{R}(\Sigma, D), \leq, (p_n)_{n\in\mathbb{N}_0})$ can be computed as follows: for each real trace $t = [V, E, \lambda]$ and each $n \in \mathbb{N}_0$ we have $p_n(t) = t$ if and only if $|v{\downarrow}_{E^*}| \leq n$ for each vertex $v \in V$ of t. Thus, by Proposition 4.8, $\|t\| = \sup\{|v{\downarrow}_{E^*}| \mid v \in V\}$ and $p_n = q_n^{\|\cdot\|}$ for all $n \in \mathbb{N}_0$. It is obvious that $\|\cdot\|$ is monotone, i.e. $\|\cdot\|$ is in fact a weight function. We call it the *weight* of traces.

(b) Let $t = [V, E, \lambda] \in \mathbb{R}(\Sigma, D)$ and define $h(t) := \sup\{h(v) \mid v \in V\}$. We call $h(t)$ the *height* of t. This yields a length function $h : \mathbb{R}(\Sigma, D) \to \mathbb{N}_0 \cup \{\infty\}$. For $n \in \mathbb{N}_0$ we have $q_n^h(t) = \sup\{s \in \mathbb{R}(\Sigma, D) \mid s \leq t,\ h(s) \leq n\}$. Similarly to the equation $q_n^{|\cdot|}(t) = p_n(t)$ (see (a) and Example 3.35(a)), we obtain $q_n^h(t) = h_n(t)$; hence q_n^h coincides with the Foata projection h_n and $(\mathbb{R}(\Sigma, D), \leq, (h_n)_{n\in\mathbb{N}_0})$ is induced by h. By Proposition 4.5, $\ell_{\mathrm{fnf}}(s, t) = \sup\{n \in \mathbb{N}_0 \mid p \leq s \Leftrightarrow p \leq t \text{ for all } p \in \mathbb{R}(\Sigma, D) \text{ with } h(p) \leq n\}$.

Notice that $h(q_n^h(t)) = h(h_n(t)) \leq n$ for all $t \in \mathbb{R}(\Sigma, D)$. Moreover, it is obvious that h is monotone. Therefore, h is a weight function.

Note further that $h(t) \leq \|t\| \leq |t|$ for all $t \in \mathbb{R}(\Sigma, D)$. It is not difficult to show that $h = \|\cdot\|$ if and only if D is transitive and $\|\cdot\| = |\cdot|$ if and only if $D = \Sigma^2$.

Non-expansive and non-pseudo-weight increasing mappings of ω-pop's. In the remainder of this section we shall study mappings between ω-pop's that are non-expansive or that commute with all projections. They turn out to be characterizable by properties of the canonical pseudo-ultrametric (see p. 86) and the pseudo-weight function (cf. Proposition 4.8).

We begin by showing that non-expansive mappings are exactly the maps that are metrically non-expansive with respect to the canonical pseudo-ultrametrics. For the "if" part in (1) of the following proposition, an analogous statement for projection spaces is given in [17, Fact 1.12(2)].

4.12. PROPOSITION. *Let $\mathcal{D} = (D, \leq, (p_n)_{n \in \mathbb{N}_0})$ and $\mathcal{E} = (E, \leq, (q_n)_{n \in \mathbb{N}_0})$ be ω-pop's and let $f : D \to E$.*

(1) *f is $(\varrho_\mathcal{D}, \varrho_\mathcal{E})$-non-expansive if and only if f is non-expansive.*
(2) *$\varrho_\mathcal{E}(f(d_1), f(d_2)) \geq \varrho_\mathcal{D}(d_1, d_2)$ for all $d_1, d_2 \in D$ if and only if $\mathsf{ker}\, p_n \supseteq \mathsf{ker}(q_n \circ f)$ for all $n \in \mathbb{N}_0$.*

Proof. (1) We have $\varrho_\mathcal{E}(f(d_1), f(d_2)) \leq \varrho_\mathcal{D}(d_1, d_2)$ for all $d_1, d_2 \in D$ if and only if $\ell_\mathcal{E}(f(d_1), f(d_2)) = \sup\{n \in \mathbb{N}_0 \mid q_n(f(d_1)) = q_n(f(d_2))\} \geq \sup\{n \in \mathbb{N}_0 \mid p_n(d_1) = p_n(d_2)\} = \ell_\mathcal{D}(d_1, d_2)$ for all $d_1, d_2 \in D$. The latter holds if and only if $\{n \in \mathbb{N}_0 \mid q_n(f(d_1)) = q_n(f(d_2))\} \supseteq \{n \in \mathbb{N}_0 \mid p_n(d_1) = p_n(d_2)\}$ for all $d_1, d_2 \in D$ (because these sets are lower sets of non-negative integers). Clearly, this is equivalent to $(d_1, d_2) \in \mathsf{ker}\, p_n$ implying $(d_1, d_2) \in \mathsf{ker}(q_n \circ f)$ for all $n \in \mathbb{N}_0$ and all $d_1, d_2 \in D$; that is, $\mathsf{ker}\, p_n \subseteq \mathsf{ker}(q_n \circ f)$ for all $n \in \mathbb{N}_0$. Then the assertion follows from Lemma 4.1(1)(i)⇔(iii).

(2) is proven similarly to (1). ■

4.13. COROLLARY. *Let $\mathcal{D} = (D, \leq, (p_n)_{n \in \mathbb{N}_0})$ and $\mathcal{E} = (E, \leq, (q_n)_{n \in \mathbb{N}_0})$ be ω-pop's and let $f : D \to E$.*

(1) *f is $(\varrho_\mathcal{D}, \varrho_\mathcal{E})$-isometric if and only if $\mathsf{ker}\, p_n = \mathsf{ker}(q_n \circ f)$ for all $n \in \mathbb{N}_0$.*
(2) *Let f be bijective. Then f is a $(\varrho_\mathcal{D}, \varrho_\mathcal{E})$-isometry if and only if $q_n \circ f = q_n \circ f \circ p_n$ and $p_n \circ f^{-1} = p_n \circ f^{-1} \circ q_n$ for all $n \in \mathbb{N}_0$.* ■

We know that non-expansive mappings $f : D \to E$, i.e. mappings f with $q_n \circ f = q_n \circ f \circ p_n$ for all $n \in \mathbb{N}_0$, can be described using the pseudo-ultrametrics $\varrho_\mathcal{D}$ and $\varrho_\mathcal{E}$. What about mappings with the property that $f \circ p_n = q_n \circ f \circ p_n$ for all $n \in \mathbb{N}_0$? The latter can be seen as the "second condition" for a mapping to commute with all projections (whilst "non-expansive" is the "first condition", cf. Lemma 4.1(3)). To give a characterization of these mappings, we define $f : D \to E$ to be *non-pseudo-weight increasing* provided that $\|f(d)\|_\mathcal{E} \leq \|d\|_\mathcal{D}$ for all $d \in D$, where $\|\cdot\|_\mathcal{D}$ and $\|\cdot\|_\mathcal{E}$ are the pseudo-weight functions of $\mathcal{D}$ and $\mathcal{E}$, respectively.

4.14. PROPOSITION. *Let $\mathcal{D} = (D, \leq, (p_n)_{n \in \mathbb{N}_0})$ and $\mathcal{E} = (E, \leq, (q_n)_{n \in \mathbb{N}_0})$ be ω-pop's and let $f : D \to E$. Then we have $f \circ p_n = q_n \circ f \circ p_n$ for all $n \in \mathbb{N}_0$ if and only if f is non-pseudo-weight increasing.*

Proof. To prove the "if" part, let $n \in \mathbb{N}_0$ and let $d \in D$. As $\|f(p_n(d))\|_\mathcal{E} \leq \|p_n(d)\|_\mathcal{D} \leq n$, we deduce by Remark 4.7 that $q_n(f(p_n(d))) = f(p_n(d))$. Conversely, let $d \in D$. We may assume $n := \|d\|_D \in \mathbb{N}_0$. Then, again by 4.7, $\|f(d)\|_\mathcal{E} = \|f(p_n(d))\|_\mathcal{E} = \|q_n(f(p_n(d)))\|_\mathcal{E} \leq n = \|d\|_D$. ■

We are now able to characterize mappings that commute with all projections as those which are metrically non-expansive and non-pseudo-weight increasing. In the case of bijections, we can strengthen this result. For this, we define a mapping $f : D \to E$ between two ω-pop's $\mathcal{D}$ and $\mathcal{E}$ to be *pseudo-weight preserving* if $\|f(d)\|_{\mathcal{E}} = \|d\|_{\mathcal{D}}$ for all $d \in D$.

4.15. THEOREM. *Let $\mathcal{D} = (D, \leq, (p_n)_{n\in\mathbb{N}_0})$ and $\mathcal{E} = (E, \leq, (q_n)_{n\in\mathbb{N}_0})$ be ω-pop's and let $f : D \to E$.*

(1) *The following are equivalent*:

  (i) *f commutes with all projections.*
  (ii) *f is $(\varrho_{\mathcal{D}}, \varrho_{\mathcal{E}})$-non-expansive and non-pseudo-weight increasing.*

(2) *The following are equivalent*:

  (i) *f is bijective and commutes with all projections.*
  (ii) *f is a $(\varrho_{\mathcal{D}}, \varrho_{\mathcal{E}})$-isometry that is pseudo-weight preserving.*
  (iii) *f is a pop isomorphism from $(D, \sqsubseteq_{\{p_n|n\in\mathbb{N}_0\}}, (p_n)_{n\in\mathbb{N}_0})$ onto $(E, \sqsubseteq_{\{q_n|n\in\mathbb{N}_0\}}, (q_n)_{n\in\mathbb{N}_0})$.*

(3) *The following are equivalent*:

  (i) *f is a pop isomorphism from $\mathcal{D}$ onto $\mathcal{E}$.*
  (ii) *f is an order isomorphism from $(D, \leq)$ onto $(E, \leq)$ and a $(\varrho_{\mathcal{D}}, \varrho_{\mathcal{E}})$-isometry.*
  (iii) *f is an order isomorphism from $(D, \leq)$ onto $(E, \leq)$ and pseudo-weight preserving.*

Proof. (1) follows from Propositions 4.12 and 4.14.

(2) (iii)⇒(ii) follows from part (3) and (ii)⇒(i) from part (1) of this theorem. (i)⇒(iii) is a consequence of Lemma 4.3.

(3) (i)⇒(ii) and (i)⇒(iii) are trivial. To prove (ii)⇒(i), note that f and f^{-1} are non-expansive (Proposition 4.12). By Lemma 4.1(5), f is a pop isomorphism. Finally, we show (iii)⇒(i). For all $n \in \mathbb{N}_0$ we have $f \circ p_n = q_n \circ f \circ p_n$ and $f^{-1} \circ q_n = p_n \circ f^{-1} \circ q_n$ by Proposition 4.14. From the first equality we deduce $f \circ p_n \leq q_n \circ f$, the latter implies $f^{-1} \circ q_n \leq p_n \circ f^{-1}$ and thus $q_n \circ f \leq f \circ p_n$. Hence, f commutes with all projections. ■

4.16. EXAMPLE. Let (Σ_1, D_1) and (Σ_2, D_2) be two dependence alphabets and consider the ω-pop's $(\mathbb{R}(\Sigma_1, D_1), \leq, (p_n)_{n\in\mathbb{N}_0})$ and $(\mathbb{R}(\Sigma_2, D_2), \leq, (p_n)_{n\in\mathbb{N}_0})$, where the projections p_n are those of Example 3.35(a) (cf. also 4.11(a)). Let $f : \mathbb{R}(\Sigma_1, D_1) \to \mathbb{R}(\Sigma_2, D_2)$. Then Theorem 4.15(2) tells us that f is a weight preserving $(d_{\text{pref}}, d_{\text{pref}})$-isometry if and only if f is a bijection that commutes with all projections p_n.

4.17. PROPOSITION. *Let (Σ_1, D_1) and (Σ_2, D_2) be two dependence alphabets and consider the ω-pop's $(\mathbb{R}(\Sigma_1, D_1), \leq, (p_n)_{n\in\mathbb{N}_0})$ and $(\mathbb{R}(\Sigma_2, D_2), \leq, (p_n)_{n\in\mathbb{N}_0})$. Let $f : \mathbb{R}(\Sigma_1, D_1) \to \mathbb{R}(\Sigma_2, D_2)$ be* length preserving, *i.e. $|f(t)| = |t|$ for all $t \in \mathbb{R}(\Sigma_1, D_1)$. Then f is a $(d_{\text{pref}}, d_{\text{pref}})$-isometry if and only if f is a bijection that commutes with all projections p_n. In this case f is also weight preserving.*

Proof. From the previous example we know that f is a weight preserving $(d_{\text{pref}}, d_{\text{pref}})$-isometry if (and only if) f is a bijection that commutes with all projections p_n. This proves the "if" part. Conversely, let f be a $(d_{\text{pref}}, d_{\text{pref}})$-isometry. Let $t \in \mathbb{R}(\Sigma_1, D_1)$ and let $n \in \mathbb{N}_0$. Let $t' := f(p_n(t))$. As $\ell_{\text{pref}}(p_n(t), f^{-1}(p_n(t'))) = \ell_{\text{pref}}(f(p_n(t)), p_n(t')) = \ell_{\text{pref}}(f(p_n(t)), p_n(f(p_n(t)))) \geq n$, we obtain $p_n(t) = p_n(p_n(t)) = p_n(f^{-1}(p_n(t')))$. Since f and f^{-1} are length preserving, $|p_n(t)| = |p_n(f^{-1}(p_n(t')))| \leq |f^{-1}(p_n(t'))| = |p_n(t')| \leq |t'| = |f(p_n(t))| = |p_n(t)|$. Therefore, $|p_n(t')| = |t'|$. As a consequence, $p_n(t') = t'$, i.e. $p_n(f(p_n(t))) = f(p_n(t))$. Since f is in particular non-expansive (Proposition 4.12), we conclude that f commutes with all projections. ■

When we consider the Foata projections h_n (see Examples 3.35(b) and 4.11(b)), then we obtain the following result:

4.18. THEOREM. *Let (Σ_1, D_1) and (Σ_2, D_2) be two dependence alphabets and consider the ω-pop's $(\mathbb{R}(\Sigma_1, D_1), \leq, (h_n)_{n\in\mathbb{N}_0})$ and $(\mathbb{R}(\Sigma_2, D_2), \leq, (h_n)_{n\in\mathbb{N}_0})$. Let $f : \mathbb{R}(\Sigma_1, D_1) \to \mathbb{R}(\Sigma_2, D_2)$ be a mapping. Then the following are equivalent*:

(i) *f is a $(d_{\text{fnf}}, d_{\text{fnf}})$-isometry.*
(ii) *f is a height preserving $(d_{\text{fnf}}, d_{\text{fnf}})$-isometry.*
(iii) *f is a bijection that commutes with all projections h_n.*
(iv) *f is an order isomorphism from $(\mathbb{R}(\Sigma_1, D_1), \sqsubseteq_{\text{fnf}})$ onto $(\mathbb{R}(\Sigma_2, D_2), \sqsubseteq_{\text{fnf}})$.*

Proof. (ii) and (iii) are equivalent and (iii)⇒(iv) by virtue of Theorem 4.15(2). (ii)⇒(i) is trivial.

(i)⇒(iii). Due to Theorem 4.15(1) it is enough to show that f is non-height increasing. Let $t \in \mathbb{R}(\Sigma_1, D_1)$. We may assume $n := h(t) \in \mathbb{N}_0$. Hence, each vertex of t has height at most n. Let $s \in \mathbb{R}(\Sigma_1, D_1)$ with $h_{n+1}(s) = h_{n+1}(t)$. Since $h(t) = n$, we obtain $h_{n+1}(t) = t$. Suppose that $h(s) > h(t)$. Then s has a vertex v with $h(v) = n+1$. This implies $h_{n+1}(s) \neq t$, a contradiction to $h_{n+1}(s) = h_{n+1}(t) = t$. Therefore, $h(s) \leq h(t) = n$ and thus $s = h_{n+1}(s) = h_{n+1}(t) = t$. We see that $B_{h_{n+1}}(t) = \{s \in \mathbb{R}(\Sigma_1, D_1) \mid h_{n+1}(s) = h_{n+1}(t)\} = \{t\}$. Recall that $B_{h_{n+1}}(t) = \{s \in \mathbb{R}(\Sigma_1, D_1) \mid d_{\text{fnf}}(s, t) \leq 2^{-(n+1)}\}$. As f is an isometry, we obtain $\{f(t)\} = f[B_{h_{n+1}}(t)] = B_{h_{n+1}}(f(t))$. Since $f(t), h_{n+1}(f(t)) \in B_{h_{n+1}}(f(t))$, we have $f(t) = h_{n+1}(f(t))$. Thus $h(f(t)) \leq n+1$. Suppose that $h(f(t)) = n+1$. Let v' be a vertex of $f(t)$ with $h(v') = n+1$. Hence, v' is a maximal vertex of $f(t)$. It is labelled with some $a' \in \Sigma_2$. Consider the dependence graph a' consisting of one vertex labelled with a'. We deduce for the product $f(t)a'$ that $h_{n+1}(f(t)a') = h_{n+1}(f(t))$ because $n+1 = h(f(t))$. Thus, $f(t)a' \in B_{h_{n+1}}(f(t)) \setminus \{f(t)\}$, a contradiction. Consequently, $h(f(t)) \leq n = h(t)$.

(iv)⇒(iii). First let $t \in \mathbb{M}(\Sigma_1, D_1)$ with $m := h(t)$. By definition of the Foata projections we have $h_n(t) \neq h_{n+1}(t)$ for all $n = 0, \ldots, m-1$ and $h_m(t) = t$. By definition of $\sqsubseteq_{\text{fnf}}$ we have

$$\{s \in \mathbb{R}(\Sigma_1, D_1) \mid s \sqsubseteq_{\text{fnf}} t\} = \{h_0(t) \sqsubset_{\text{fnf}} h_1(t) \sqsubset_{\text{fnf}} \cdots \sqsubset_{\text{fnf}} h_m(t)\}.$$

Analogously, for $m' := h(f(t))$ we obtain

$$\{s' \in \mathbb{R}(\Sigma_2, D_2) \mid s' \sqsubseteq_{\text{fnf}} f(t)\} = \{h_0(f(t)) \sqsubset_{\text{fnf}} h_1(f(t)) \sqsubset_{\text{fnf}} \cdots \sqsubset_{\text{fnf}} h_{m'}(f(t))\}.$$

As f is an order isomorphism with respect to $\sqsubseteq_{\text{fnf}}$, we infer

$$\begin{aligned}\{f(h_0(t)) &\sqsubset_{\text{fnf}} f(h_1(t)) \sqsubset_{\text{fnf}} \cdots \sqsubset_{\text{fnf}} f(h_m(t))\} \\ &= f[\{s \in \mathbb{R}(\Sigma_1, D_1) \mid s \sqsubseteq_{\text{fnf}} t\}] = \{s' \in \mathbb{R}(\Sigma_2, D_2) \mid s' \sqsubseteq_{\text{fnf}} f(t)\} \\ &= \{h_0(f(t)) \sqsubset_{\text{fnf}} h_1(f(t)) \sqsubset_{\text{fnf}} \cdots \sqsubset_{\text{fnf}} h_{m'}(f(t))\}.\end{aligned}$$

Consequently, $h(t) = m = m' = h(f(t))$ and $f(h_n(t)) = h_n(f(t))$ for all $n = 0, \dots, h(t)$. Since $h_n(t) = t$ and $h_n(f(t)) = f(t)$ for all $n \geq h(t) = h(f(t))$, we have $f(h_n(t)) = h_n(f(t))$ for all $n \in \mathbb{N}_0$.

Now let $t \in \mathbb{R}(\Sigma_1, D_1) \setminus \mathbb{M}(\Sigma_1, D_1)$. Clearly, infinite real traces are precisely the $\sqsubseteq_{\text{fnf}}$-maximal elements. Hence, f maps infinite real traces to infinite ones. Therefore, $h(t) = h(f(t)) = \infty$. Let $n \in \mathbb{N}_0$. As $h_n(t) \sqsubseteq_{\text{fnf}} t$, we infer $f(h_n(t)) \sqsubseteq_{\text{fnf}} f(t)$. Hence, there is some $m \in \mathbb{N}_0$ with $f(h_n(t)) = h_m(f(t))$. Since $h(t) > n$, we have $h(h_n(t)) = n$ by definition of h and h_n. We already know that $h(f(h_n(t))) = h(h_n(t)) = n$ by the above considerations on finite traces. Analogously, $h(f(t)) > m$ implies $h(h_m(f(t))) = m$. We infer that $m = n$. ∎

4.3. Function spaces of indexed pop's

This section deals with function spaces of $(I, \leq)$-pop's. We endow various sets of mappings between $(I, \leq)$-pop's with a canonical pop structure. In particular, we consider homomorphisms and weak homomorphisms. As we shall show, the induced pop uniformity coincides with the uniformity of uniform convergence. This allows us to apply results from topology to prove completeness and compactness properties of the function spaces under consideration. After that, we will see that we are in a lucky situation from some computer scientists' point of view: we will obtain several cartesian closed categories of indexed pop's. Such categories form the basis to obtain models of the λ-calculus. Indeed, by a D_∞-construction we obtain indexed pop's that are isomorphic to their own exponent.

4.3.1. Function spaces from a topological viewpoint. Let X be a set and let $\mathcal{E} = (E, \leq, (q_i)_{i \in I})$ be an $(I, \leq)$-pop. Let $F(X, E)$ be the set of all mappings from X to E and equip it with the pointwise order. For all $i \in I$ let $Q_i : F(X, E) \to F(X, E)$ be defined by $Q_i(f) := q_i \circ f$. Then, clearly, $\mathcal{F}(X, E) := (F(X, E), \leq, (Q_i)_{i \in I})$ is an $(I, \leq)$-pop.

4.19. Proposition. *The pop uniformity $\mathcal{U}_{\mathcal{F}(X,E)}$ of $\mathcal{F}(X, E)$ coincides with the uniformity of uniform convergence. In particular, a net $(f_n)_{n \in N}$ converges to some $f : X \to E$ with respect to $\tau_{\mathcal{F}(X,E)}$ if and only if it converges uniformly to f.*

Proof. A basis for the uniformity of uniform convergence is given by the sets

$$\begin{aligned}\{(f,g) \in F(X,E)^2 \mid \forall x \in X : (f(x), g(x)) \in \ker q_i\} \\ = \{(f,g) \in F(X,E)^2 \mid \forall x \in X : q_i(f(x)) = q_i(g(x))\} \\ = \{(f,g) \in F(X,E)^2 \mid \forall x \in X : Q_i(f)(x) = Q_i(g)(x)\} = \ker Q_i\end{aligned}$$

with $i \in I$ (cf. p. 18). ∎

The previous proposition turns out to be quite useful because it allows us to apply results from topology whenever we work with the pop uniformity of $\mathcal{F}(X, E)$.

For any $e \in E$ we write $\underline{e}$ for the constant map sending all elements of X to e.

4.20. Proposition. *Let $\mathcal{E} = (E, \leq, (q_i)_{i \in I})$ be an $(I, \leq)$-pop and let X be a set. Then the mapping $e \mapsto \underline{e}$ is a pop embedding of $\mathcal{E}$ into $\mathcal{F}(X, E)$. Moreover, $\mathcal{F}(X, E)$ is approximating [complete, respectively] if and only if $\mathcal{E}$ is approximating [complete, respectively].*

Proof. Let $e_1, e_2, e \in E$. Clearly, $e_1 \leq e_2$ if and only if $\underline{e_1} \leq \underline{e_2}$. For all $i \in I$ we have $Q_i(\underline{e}) = q_i \circ \underline{e} = \underline{q_i(e)}$. Therefore, $e \mapsto \underline{e}$ is a pop embedding. Hence, $\mathcal{E}$ is approximating provided that $\mathcal{F}(X, E)$ is approximating. Let $\mathcal{F}(X, E)$ be complete and let $(e_n)_{n\in N}$ be a Cauchy net in E. One easily sees that $(\underline{e_n})_{n\in N}$ is a Cauchy net in $\mathcal{F}(X, E)$, hence convergent to some mapping $f : X \to E$ in the topology of uniform convergence (Proposition 4.19). In particular, $(\underline{e_n})_{n\in N}$ converges pointwise to f. Consequently, $(e_n)_{n\in N}$ converges to $f(x)$ for any $x \in X$. (We remark here that if $\mathcal{E}$ is Hausdorff, then f has to be a constant map.)

Conversely, if $\mathcal{E}$ is approximating, then it is straightforward to check that $\mathcal{F}(X, E)$ is approximating. Let $\mathcal{E}$ be complete. As the pop uniformity of $\mathcal{F}(X, E)$ is the uniformity of uniform convergence (Proposition 4.19), we can use e.g. Theorem 1 in Bourbaki [7, Section X.1.5], to obtain completeness of $\mathcal{F}(X, E)$. ∎

Let $(D, \leq)$ be a poset, let $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be an $(I, \leq)$-pop, and set

$$M(D, E) := \{f \in F(D, E) \mid f \text{ is monotone}\},$$
$$S(D, E) := \{f \in F(D, E) \mid f \text{ is Scott-continuous}\}.$$

As all q_i are monotone ($i \in I$), the projections Q_i map $M(D, E)$ to $M(D, E)$. Hence, $M(D, E)$ induces a subpop $\mathcal{M}(D, E)$ of $\mathcal{F}(D, E)$. Provided that all q_i are Scott-continuous, each Q_i maps $S(D, E)$ to $S(D, E)$. In this case, $S(D, E)$ induces a subpop $\mathcal{S}(D, E)$ of $\mathcal{M}(D, E)$.

4.21. Proposition. *Let $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be an $(I, \leq)$-pop and let $(D, \leq)$ be a poset.*

(1) *The mapping $e \mapsto \underline{e}$ is a pop embedding of $\mathcal{E}$ into $\mathcal{M}(D, E)$. Furthermore, $\mathcal{M}(D, E)$ is approximating if and only if $\mathcal{E}$ is approximating. If $\mathcal{M}(D, E)$ is complete, then $\mathcal{E}$ is complete.*

(2) *Let $\mathcal{E}$ be approximating. Then $M(D, E)$ is closed in $F(D, E)$ with respect to the topology of pointwise convergence. In particular, $M(D, E)$ is closed in $\tau_{\mathcal{F}(D,E)}$. In addition, $\mathcal{M}(D, E)$ is complete if and only if $\mathcal{E}$ is complete.*

Proof. (1) Obviously, $\underline{e} \in M(D, E)$ for all $e \in E$, whence $e \mapsto \underline{e}$ is a pop embedding (Proposition 4.20). Therefore, if $\mathcal{M}(D, E)$ is approximating, then $\mathcal{E}$ is also approximating. A similar argument to that given in the proof of Proposition 4.20 shows us that $\mathcal{E}$ is complete if $\mathcal{M}(D, E)$ is complete. Let $\mathcal{E}$ be approximating. Since $\mathcal{F}(D, E)$ is approximating (Proposition 4.20), so is $\mathcal{M}(D, E)$.

(2) Now let $\mathcal{E}$ be approximating. Let $(f_n)_{n\in N}$ be a net in $M(D, E)$ that converges to some $f \in F(D, E)$ in the pointwise topology. Let $d_1, d_2 \in D$ with $d_1 \leq d_2$ and let $i \in I$. Then there is some $n_i \in N$ with $q_i(f(d_1)) = q_i(f_{n_i}(d_1))$ and $q_i(f(d_2)) = q_i(f_{n_i}(d_2))$. Thus, $q_i(f(d_1)) \leq q_i(f(d_2))$ for all $i \in I$. This yields $f(d_1) \leq f(d_2)$ because $\mathcal{E}$ is approximating. By Proposition 4.19, the pointwise topology is coarser than $\tau_{\mathcal{F}(D,E)}$; hence $M(D, E)$ is closed in $F(D, E)$ with respect to $\tau_{\mathcal{F}(D,E)}$. In light of Proposition 4.20, $\mathcal{M}(D, E)$ is complete if $\mathcal{E}$ is complete. ∎

4.22. Proposition. *Let $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be an $(I, \leq)$-pop and let $(D, \leq)$ be a poset.*

(1) *For all $i \in I$ the projection q_i is Scott-continuous if and only if Q_i is Scott-continuous.*

(2) *Let q_i be Scott-continuous for all $i \in I$. Then the mapping $e \mapsto \underline{e}$ is a pop embedding of $\mathcal{E}$ into $\mathcal{S}(D,E)$. Moreover, $\mathcal{S}(D,E)$ is approximating if and only if $\mathcal{E}$ is approximating. If $\mathcal{S}(D,E)$ is complete, then $\mathcal{E}$ is complete.*

(3) *Let all q_i be Scott-continuous ($i \in I$) and let $\mathcal{E}$ be Hausdorff. Then $S(D,E)$ is closed in $F(D,E)$ with respect to $\tau_{\mathcal{F}(D,E)}$. Furthermore, $\mathcal{S}(D,E)$ is complete if and only if $\mathcal{E}$ is complete.*

Proof. (1) If q_i is Scott-continuous, then Q_i is Scott-continuous as well. This follows from the fact that the composition map is Scott-continuous itself when applied to Scott-continuous mappings. Conversely, q_i is Scott-continuous if Q_i is Scott-continuous because whenever $A \subseteq E$ is directed with $e := \sup A$, then $\underline{A} := \{\underline{a} \mid a \in A\}$ is directed with $\sup \underline{A} = \underline{e}$. Thus, as $\underline{q_i(e)} = Q_i(\underline{e}) = \sup Q_i[\underline{A}] = \sup_{a\in A} \underline{q_i(a)}$, we obtain $q_i(e) = \sup q_i[A]$.

(2) Now let q_i be Scott-continuous for all $i \in I$. Clearly, $\underline{e} \in \mathcal{S}(D,E)$, i.e. $e \mapsto \underline{e}$ is a pop embedding. This tells us that $\mathcal{E}$ is approximating [complete, respectively] if $\mathcal{S}(D,E)$ is approximating [complete, respectively]. It is also clear that if $\mathcal{E}$ is approximating, then $\mathcal{S}(D,E)$ is approximating.

(3) Let q_i be Scott-continuous for all $i \in I$ and let, furthermore, $\mathcal{E}$ be Hausdorff. Let $(f_n)_{n\in N}$ be a net in $S(D,E)$ converging to a map $f : D \to E$ with respect to $\tau_{\mathcal{F}(D,E)}$. Let $A \subseteq D$ be directed such that $\sup A$ exists. Let $i \in I$ and choose some $n_i \in N$ with $Q_i(f_{n_i}) = Q_i(f)$, i.e. $q_i \circ f_{n_i} = q_i \circ f$. Then $q_i(f(\sup A)) = q_i(f_{n_i}(\sup A)) = \sup q_i[f_{n_i}[A]] = \sup q_i[f[A]] = q_i(\sup f[A])$. Thus, $q_i(f(\sup A)) = q_i(\sup f[A])$ for all $i \in I$. As $\mathcal{E}$ is Hausdorff, $f(\sup A) = \sup f[A]$ by Proposition 3.18(2). We conclude that $S(D,E)$ is closed in $F(D,E)$ with respect to the pop uniformity of $\mathcal{F}(D,E)$. In view of Proposition 4.20, $\mathcal{S}(D,E)$ is complete if $\mathcal{E}$ is complete. ■

Next, let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ and $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be $(I,\leq)$-pop's and define

$$\mathrm{NEX}(D,E) := \{f \in F(D,E) \mid f \text{ is non-expansive}\},$$
$$\mathrm{COM}(D,E) := \{f \in F(D,E) \mid f \text{ commutes with all projections}\}.$$

Assume that $(q_i)_{i\in I}$ is Abelian. Then $\mathrm{NEX}(D,E)$ induces a subpop $\mathcal{NEX}(D,E)$ of $\mathcal{F}(D,E)$ because each Q_i maps $\mathrm{NEX}(D,E)$ to itself: for all $i,j \in I$ and all $f \in \mathrm{NEX}(D,E)$ we have $q_j \circ Q_i(f) \circ p_j = q_j \circ (q_i \circ f) \circ p_j = q_i \circ (q_j \circ f \circ p_j) = q_i \circ (q_j \circ f) = q_j \circ (q_i \circ f) = q_j \circ Q_i(f)$. Furthermore, $(Q_i)_{i\in I}$ is Abelian as well. Similarly, $\mathrm{COM}(D,E)$ induces a subpop $\mathcal{COM}(D,E)$ of $\mathcal{NEX}(D,E)$ whenever $(q_i)_{i\in I}$ is Abelian.

4.23. Remark. The sets $\mathrm{NEX}(D,E)$ and $\mathrm{COM}(D,E)$ are uniformly equicontinuous. To see this, let $i \in I$. Lemma 4.1(1) tells us that $(f\times f)[\mathsf{ker}\, p_i] \subseteq \mathsf{ker}\, q_i$ for all $f \in \mathrm{NEX}(D,E)$, implying that $\mathrm{NEX}(D,E)$ is uniformly equicontinuous. Since $\mathrm{COM}(D,E) \subseteq \mathrm{NEX}(D,E)$, the assertion is also true for $\mathrm{COM}(D,E)$.

4.24. Proposition. *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ and $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be $(I,\leq)$-pop's and let $(q_i)_{i\in I}$ be Abelian.*

(1) *The mapping $e \mapsto \underline{e}$ is a pop embedding of $\mathcal{E}$ into $\mathcal{NEX}(D,E)$.*

(2) $\mathrm{NEX}(D,E)$ *is closed in $F(D,E)$ in the topology of pointwise convergence. In particular, it is closed with respect to $\tau_{\mathcal{F}(D,E)}$.*

(3) $\mathcal{NEX}(D,E)$ *is approximating* [*complete, respectively*] *if and only if* $\mathcal{E}$ *is approximating* [*complete, respectively*].

Proof. (1) Let $e \in E$ and let $i \in I$. Then $q_i \circ \underline{e} = q_i \circ \underline{e} \circ p_i$. Therefore, $\underline{e} \in \mathrm{NEX}(D,E)$. The assertion follows from Proposition 4.20.

(2) Let $(f_n)_{n \in N}$ be a net in $\mathrm{NEX}(D,E)$ converging to some $f \in F(D,E)$ in the pointwise topology. Let $i \in I$ and let $d \in D$. We find some $n_i \in N$ such that $q_i(f(d)) = q_i(f_{n_i}(d))$ and $q_i(f(p_i(d))) = q_i(f_{n_i}(p_i(d)))$. Thus, $q_i(f(d)) = q_i(f_{n_i}(d)) = q_i(f_{n_i}(p_i(d))) = q_i(f(p_i(d)))$, whence f is non-expansive.

(3) Apply (1) to see that $\mathcal{E}$ is approximating [complete, respectively] if $\mathcal{NEX}(D,E)$ is approximating [complete, respectively]. Clearly, if $\mathcal{E}$ is approximating, then $\mathcal{NEX}(D,E)$ is approximating. Use (2) and Proposition 4.20 to prove that $\mathcal{NEX}(D,E)$ is complete if $\mathcal{E}$ is complete. ∎

4.25. Theorem. *Let* $\mathcal{D} = (D, \leq, (p_i)_{i \in I})$ *and* $\mathcal{E} = (E, \leq, (q_i)_{i \in I})$ *be* $(I, \leq)$*-pop's and let* $(q_i)_{i \in I}$ *be Abelian.*

(1) *Let* $\mathcal{D}$ *be totally bounded. If* $\mathcal{E}$ *is totally bounded* [*compact, respectively*], *then* $\mathcal{NEX}(D,E)$ *is totally bounded* [*compact, respectively*].
(2) *If* $\mathcal{NEX}(D,E)$ *is totally bounded* [*compact, respectively*], *then* $\mathcal{E}$ *is totally bounded* [*compact, respectively*].

Proof. (1) As $\mathrm{NEX}(D,E)$ is uniformly equicontinuous (Remark 4.23) and $\mathcal{D}$ and $\mathcal{E}$ are totally bounded, the Ascoli Theorem tells us that $\mathrm{NEX}(D,E)$ is totally bounded with respect to the uniformity of uniform convergence; see e.g. [7, Theorem 2 in Section X.2.5]. By Proposition 4.19 the uniformity of uniform convergence coincides with the pop uniformity.

We also give an elementary proof. Due to Proposition 3.16 it suffices to show that $Q_i[\mathrm{NEX}(D,E)]$ is finite for all $i \in I$. The same result tells us that p_i and q_i have finite range for all $i \in I$ since $\mathcal{D}$ and $\mathcal{E}$ are totally bounded. Now $Q_i[\mathrm{NEX}(D,E)] = \{q_i \circ f \mid f \in \mathrm{NEX}(D,E)\} = \{q_i \circ f \circ p_i \mid f \in \mathrm{NEX}(D,E)\}$. The maps $q_i \circ f$ can therefore be interpreted as mappings from the finite set $p_i[D]$ into the finite set $q_i[E]$. Consequently, $Q_i[\mathrm{NEX}(D,E)]$ has to be finite.

If, furthermore, $\mathcal{E}$ is compact, then it is complete. Hence, $\mathcal{NEX}(D,E)$ is complete by Proposition 4.24(3) and thus compact.

(2) As Q_i has finite range for all $i \in I$ (Proposition 3.16), Proposition 4.24(1) implies that q_i has finite range for all $i \in I$. This shows us that $\mathcal{E}$ is totally bounded.

If $\mathcal{NEX}(D,E)$ is compact, then $\mathcal{E}$ is complete (and thus compact) by Proposition 4.24(3). ∎

4.26. Proposition. *Let* $\mathcal{D} = (D, \leq, (p_i)_{i \in I})$ *and* $\mathcal{E} = (E, \leq, (q_i)_{i \in I})$ *be* $(I, \leq)$*-pop's such that* $(q_i)_{i \in I}$ *is Abelian. Let* $\mathcal{E}$ *be Hausdorff.*

(1) $\mathrm{COM}(D,E)$ *is closed in* $F(D,E)$ *in the topology of pointwise convergence. In particular,* $\mathrm{COM}(D,E)$ *is closed in* $F(D,E)$ *with respect to* $\tau_{\mathcal{F}(D,E)}$.
(2) *If* $\mathcal{E}$ *is approximating* [*complete, respectively*], *then* $\mathcal{COM}(D,E)$ *is approximating* [*complete, respectively*].

Proof. (1) Let $(f_n)_{n\in N}$ be a net in $\mathrm{COM}(D,E)$ converging pointwise to a mapping $f \in F(D,E)$. Let $i \in I$ and let $d \in D$. There is an index $\widetilde{n} \in N$ such that $q_i(f(d)) = q_i(f_n(d))$ for all $n \geq \widetilde{n}$. Let $j \geq i$ and choose some $n_j \geq \widetilde{n}$ with $q_j(f(p_i(d))) = q_j(f_{n_j}(p_i(d)))$. Then, using Lemma 3.1, we infer $q_i(f(d)) = q_j(q_i(f(d))) = q_j(q_i(f_{n_j}(d))) = q_j(f_{n_j}(p_i(d))) = q_j(f(p_i(d)))$ for all $j \geq i$. As $\mathcal{E}$ is Hausdorff and $(q_j(f(p_i(d))))_{j\geq i}$ converges to $f(p_i(d))$ (cf. Proposition 2.10(1)), we conclude $q_i(f(d)) = f(p_i(d))$. Therefore, $f \in \mathrm{COM}(D,E)$.

(2) Clearly, if $\mathcal{E}$ is approximating, then $\mathcal{COM}(D,E)$ is approximating. Let $\mathcal{E}$ be complete. Then $\mathcal{F}(D,E)$ is complete by Proposition 4.20. Now completeness of $\mathcal{COM}(D,E)$ results from (1). ∎

4.27. Theorem. *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I,\leq)$-pop that is totally bounded. Let $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be a Hausdorff $(I,\leq)$-pop with Abelian projection net $(q_i)_{i\in I}$. If $\mathcal{E}$ is totally bounded [compact, respectively], then $\mathcal{COM}(D,E)$ is totally bounded [compact, respectively].*

Proof. Let $\mathcal{E}$ be totally bounded and recall that $\mathrm{COM}(D,E) \subseteq \mathrm{NEX}(D,E)$. Theorem 4.25(1) tells us that $\mathcal{NEX}(D,E)$ is totally bounded. Consequently, this is also true for $\mathcal{COM}(D,E)$. If, furthermore, $\mathcal{E}$ is complete, then $\mathcal{COM}(D,E)$ is complete by Proposition 4.26(2) and thus compact. ∎

Finally, we consider function spaces of [weak] homomorphisms. Given two $(I,\leq)$-pop's $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ and $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ such that $(q_i)_{i\in I}$ is Abelian, we define

$$[D \to E]^{\mathrm{weak}} := \{f \in F(D,E) \mid f \text{ is a weak homomorphism}\} = M(D,E) \cap \mathrm{NEX}(D,E),$$
$$[D \to E]^{\mathrm{hom}} := \{f \in F(D,E) \mid f \text{ is a homomorphism}\} = M(D,E) \cap \mathrm{COM}(D,E).$$

Provided that q_i is Scott-continuous for all $i \in I$, we set

$$\begin{aligned} [D \to E]^{\mathrm{Sweak}} &:= \{f \in F(D,E) \mid f \text{ is a Scott-continuous weak homomorphism}\} \\ &= S(D,E) \cap \mathrm{NEX}(D,E), \\ [D \to E]^{\mathrm{Shom}} &:= \{f \in F(D,E) \mid f \text{ is a Scott-continuous homomorphism}\} \\ &= S(D,E) \cap \mathrm{COM}(D,E). \end{aligned}$$

It is clear that these sets induce subpop's $[\mathcal{D} \to \mathcal{E}]^{\mathrm{weak}}$, $[\mathcal{D} \to \mathcal{E}]^{\mathrm{hom}}$, $[\mathcal{D} \to \mathcal{E}]^{\mathrm{Sweak}}$, and $[\mathcal{D} \to \mathcal{E}]^{\mathrm{Shom}}$, respectively. Note that $[\mathcal{D} \to \mathcal{E}]^{\mathrm{hom}}$ is a subpop of $[\mathcal{D} \to \mathcal{E}]^{\mathrm{weak}}$ and $[\mathcal{D} \to \mathcal{E}]^{\mathrm{Shom}}$ is a subpop of $[\mathcal{D} \to \mathcal{E}]^{\mathrm{Sweak}}$. All of them have an Abelian projection net. Using the previous results, we derive:

4.28. Theorem. *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ and $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be $(I,\leq)$-pop's with $(q_i)_{i\in I}$ Abelian.*

(1) *Let $\mathcal{E}$ be approximating. Then $[\mathcal{D} \to \mathcal{E}]^{\mathrm{weak}}$ and $[\mathcal{D} \to \mathcal{E}]^{\mathrm{hom}}$ are approximating. If $\mathcal{E}$ is complete, then $[\mathcal{D} \to \mathcal{E}]^{\mathrm{weak}}$ and $[\mathcal{D} \to \mathcal{E}]^{\mathrm{hom}}$ are also complete. If $\mathcal{D}$ and $\mathcal{E}$ are totally bounded [compact, respectively], then so are $[\mathcal{D} \to \mathcal{E}]^{\mathrm{weak}}$ and $[\mathcal{D} \to \mathcal{E}]^{\mathrm{hom}}$.*

(2) *Let $\mathcal{E}$ be approximating and q_i be Scott-continuous for all $i \in I$. Then $[\mathcal{D} \to \mathcal{E}]^{\mathrm{Sweak}}$ and $[\mathcal{D} \to \mathcal{E}]^{\mathrm{Shom}}$ are approximating. If $\mathcal{E}$ is complete, then $[\mathcal{D} \to \mathcal{E}]^{\mathrm{Sweak}}$ and*

$[\mathcal{D} \to \mathcal{E}]^{\mathrm{Shom}}$ *are also complete. If* $\mathcal{D}$ *and* $\mathcal{E}$ *are totally bounded* [*compact, respectively*], *then so are* $[\mathcal{D} \to \mathcal{E}]^{\mathrm{Sweak}}$ *and* $[\mathcal{D} \to \mathcal{E}]^{\mathrm{Shom}}$.

(3) *Let* q_i *be Scott-continuous for all* $i \in I$ *and let* $(E, \leq)$ *be a dcpo. Then* $([D \to E]^{\mathrm{Sweak}}, \leq)$ *and* $([D \to E]^{\mathrm{Shom}}, \leq)$ *are dcpo's. Moreover, the projections* $Q_i|_{[D\to E]^{\mathrm{Sweak}}}$ *and* $Q_i|_{[D\to E]^{\mathrm{Shom}}}$ *are Scott-continuous for all* $i \in I$.

Proof. (1) and (2) follow from Propositions 4.21, 4.22, 4.24, 4.26 and Theorems 4.25, 4.27.

To prove (3), it is sufficient to show that the pointwise supremum of any directed subset of $[D \to E]^{\mathrm{Sweak}}$ ($[D \to E]^{\mathrm{Shom}}$, respectively) is an element of $[D \to E]^{\mathrm{Sweak}}$ ($[D \to E]^{\mathrm{Shom}}$, respectively). Let $A \subseteq [D \to E]^{\mathrm{Sweak}}$. Clearly, the pointwise supremum $g := \sup A$ is a Scott-continuous mapping. Let $i \in I$. By recalling that the composition map (applied to Scott-continuous mappings) is Scott-continuous, we infer that $q_i \circ g = \sup\{q_i \circ f \mid f \in A\} = \sup\{q_i \circ f \circ p_i \mid f \in A\} = q_i \circ g \circ p_i$ for all $i \in I$. Similarly for $[D \to E]^{\mathrm{Shom}}$. ∎

We conclude this subsection by giving a few remarks on a different pop structure on the set $M(D, E)$. Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ and $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be $(I, \leq)$-pop's and define for all $i \in I$ a mapping $P_i : M(D, E) \to M(D, E)$ by $P_i(f) := q_i \circ f \circ p_i$. One easily checks that $(M(D, E), \leq, (P_i)_{i\in I})$ is an $(I, \leq)$-pop. We have

$$\begin{aligned}
\ker P_i &= \{(f, g) \in M(D, E)^2 \mid q_i \circ f \circ p_i = q_i \circ g \circ p_i\} \\
&= \{(f, g) \in M(D, E)^2 \mid \forall x \in p_i[D] : q_i(f(x)) = q_i(g(x))\} \\
&= \{(f, g) \in M(D, E)^2 \mid \forall x \in p_i[D] : (f(x), g(x)) \in \ker q_i\}.
\end{aligned}$$

As a consequence, the pop uniformity of $(M(D, E), \leq, (P_i)_{i\in I})$ is the uniformity of uniform convergence in the sets $p_i[D]$, $i \in I$. Thus, this uniformity is weaker than the pop uniformity of $\mathcal{M}(D, E) = (M(D, E), \leq, (Q_i|_{M(D,E)})_{i\in I})$. Note that on $[D \to E]^{\mathrm{weak}}$, it is equal to the uniformity of uniform convergence because $P_i(f) = Q_i(f)$ for all $f \in [D \to E]^{\mathrm{weak}}$ and all $i \in I$. Hence, on the function spaces $[\mathcal{D} \to \mathcal{E}]^{\mathrm{weak}}$, $[\mathcal{D} \to \mathcal{E}]^{\mathrm{hom}}$, $[\mathcal{D} \to \mathcal{E}]^{\mathrm{Sweak}}$, and $[\mathcal{D} \to \mathcal{E}]^{\mathrm{Shom}}$, both uniformities coincide.

4.3.2. Cartesian closed categories with weak homomorphisms. A category $\mathcal{C}$ is *cartesian closed* provided that the following hold:

(1) There is an object T in $\mathcal{C}$ such that for each object A in $\mathcal{C}$ there is exactly one morphism from A to T. Such a T is called a *terminal object*.
(2) For any two objects A and B in $\mathcal{C}$ there exist an object $A \times B$ in $\mathcal{C}$ and morphisms $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$ such that for any object X in $\mathcal{C}$ and morphisms $f : X \to A$, $g : X \to B$ there is a unique morphism $\langle f, g\rangle : X \to A \times B$ with $\pi_1 \circ \langle f, g\rangle = f$ and $\pi_2 \circ \langle f, g\rangle = g$. In this case, $A \times B$ is called the *product* of A and B.
(3) For any two objects A and B in $\mathcal{C}$ there exists an object B^A in $\mathcal{C}$ and a morphism $\mathrm{ev} : B^A \times A \to B$ with the following universal property: for each object X in $\mathcal{C}$ and each morphism $f : X \times A \to B$ there exists a unique morphism $\widehat{f} : X \to B^A$ such that $f = \mathrm{ev} \circ \langle \widehat{f} \circ \pi_1, \mathsf{id}_A \circ \pi_2\rangle$ (see Figure 4.3). Then B^A is said to be the *exponential object* for A and B and ev is the *evaluation morphism*.

$$
\begin{tikzcd}
X \times A \arrow[d, "\langle \widehat{f}\circ\pi_1, \mathsf{id}_A\circ\pi_2\rangle"'] \arrow[dr, "f"] & \\
B^A \times A \arrow[r, "\mathrm{ev}"'] \arrow[ur, phantom, "="] & B
\end{tikzcd}
$$

Fig. 4.3. The universal property of the exponential object

Let $(I, \leq)$ be a directed set. We define the category $\mathsf{POP}^{\text{weak}}_{(I,\leq)}$ as follows. The object class is the class of all $(I, \leq)$-pop's that have an Abelian projection net. Morphisms are all weak pop homomorphisms. With the "usual" constructions concerning the product and the exponential object (here: the function space of weak homomorphisms) we show that this category is cartesian closed. Note that projection spaces together with "projection compatible mappings" (which correspond to our non-expansive maps) form a cartesian closed category as remarked in [23, 24].

4.29. THEOREM. *The category* $\mathsf{POP}^{\text{weak}}_{(I,\leq)}$ *is cartesian closed.*

Proof. Let $(\{t\}, =)$ be a one-point poset. Then $\mathcal{T} = (\{t\}, =, (\mathsf{id}_{\{t\}})_{i\in I})$ is an object of $\mathsf{POP}^{\text{weak}}_{(I,\leq)}$. It is terminal: for each $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ in $\mathsf{POP}^{\text{weak}}_{(I,\leq)}$ the constant map sending all elements of D to t is the unique weak homomorphism from $\mathcal{D}$ to $\mathcal{T}$.

Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ and $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be in $\mathsf{POP}^{\text{weak}}_{(I,\leq)}$. Endow the cartesian product $D \times E$ with the product order and let $\mathcal{D} \times \mathcal{E} := (D \times E, \leq, (p_i \times q_i)_{i\in I})$. Clearly, $\mathcal{D} \times \mathcal{E}$ is an $(I, \leq)$-pop with an Abelian projection net, hence an object of $\mathsf{POP}^{\text{weak}}_{(I,\leq)}$. Let π_1 (π_2, respectively) be the canonical projection of $D \times E$ onto D (onto E, respectively). We know that π_1 and π_2 are monotone. For $d \in D$ and $e \in E$ we have $\pi_1((p_i \times q_i)(d, e)) = \pi_1(p_i(d), q_i(e)) = p_i(d) = p_i(\pi_1(d, e))$, whence π_1 is a homomorphism. Analogously, π_2 is a homomorphism. In particular, π_1 and π_2 are weak homomorphisms. Let $\mathcal{X} = (X, \leq, (s_i)_{i\in I})$ be in $\mathsf{POP}^{\text{weak}}_{(I,\leq)}$ and let $f : X \to D$ and $g : X \to E$ be weak homomorphisms. Define $\langle f, g\rangle(x) := (f(x), g(x))$. Clearly, this yields a weak homomorphism $\langle f, g\rangle : X \to D \times E$ with $\pi_1 \circ \langle f, g\rangle = f$ and $\pi_2 \circ \langle f, g\rangle = g$. It is obvious that $\langle f, g\rangle$ is unique with these properties.

Next, we check that the function space $[\mathcal{D} \to \mathcal{E}]^{\text{weak}}$ is the exponential object of $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ and $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ in $\mathsf{POP}^{\text{weak}}_{(I,\leq)}$. We know that $[\mathcal{D} \to \mathcal{E}]^{\text{weak}}$ is an object of $\mathsf{POP}^{\text{weak}}_{(I,\leq)}$.

Let $\mathrm{ev} : [D \to E]^{\text{weak}} \times D \to E$ be defined by $\mathrm{ev}(f, d) := f(d)$. Clearly, ev is monotone and $q_i(\mathrm{ev}(f, d)) = q_i(f(d)) = q_i(f(p_i(d))) = \mathrm{ev}(q_i \circ f, p_i(d))$ for all $f \in [D \to E]^{\text{weak}}$, $d \in D$, and all $i \in I$. Therefore, $q_i \circ \mathrm{ev} = \mathrm{ev} \circ (Q_i \times p_i)$ for all $i \in I$; hence ev is a (weak) homomorphism.

Let $\mathcal{X} = (X, \leq, (s_i)_{i\in I})$ be in $\mathsf{POP}^{\text{weak}}_{(I,\leq)}$ and let $f : X \times D \to E$ be a weak homomorphism. Define $\widehat{f} : X \to [D \to E]^{\text{weak}}$ by $\widehat{f}(x) := f_x : D \to E$ with $f_x(d) := f(x, d)$ for all $d \in D$, $x \in X$. Obviously, f_x is monotone and $q_i(f_x(d)) = q_i(f(x, d)) \leq f(s_i(x), p_i(d)) \leq$

$f(x, p_i(d)) = f_x(p_i(d))$ for all $d \in D$ and all $i \in I$, i.e. $q_i \circ f_x \leq f_x \circ p_i$ for all $i \in I$. Hence, f_x is a weak homomorphism by Lemma 4.1(2) and $\widehat{f}$ is well defined.

It is obvious that $\widehat{f}$ is monotone. Let $i \in I$, let $x \in X$, and let $d \in D$. Then $q_i(f_x(d)) = q_i(f(x,d)) \leq f(s_i(x), p_i(d)) \leq f(s_i(x), d) = f_{s_i(x)}(d)$. Hence, $Q_i(f_x) \leq f_{s_i(x)}$ for all $x \in X$, i.e. $Q_i \circ \widehat{f} \leq \widehat{f} \circ s_i$. Consequently, $\widehat{f}$ is a weak homomorphism (4.1(2)). Certainly, $\mathrm{ev} \circ \langle \widehat{f} \circ \pi_1, \mathsf{id}_D \circ \pi_2 \rangle = \mathrm{ev} \circ (\widehat{f} \times \mathsf{id}_D) = f$, and $\widehat{f}$ is unique with this property. ∎

We define the following full subcategories of $\mathsf{POP}^{\mathrm{weak}}_{(I,\leq)}$:

$\mathsf{APOP}^{\mathrm{weak}}_{(I,\leq)}$: objects are all approximating $(I, \leq)$-pop's of $\mathsf{POP}^{\mathrm{weak}}_{(I,\leq)}$,
$\mathsf{CAPOP}^{\mathrm{weak}}_{(I,\leq)}$: objects are all complete approximating $(I, \leq)$-pop's of $\mathsf{POP}^{\mathrm{weak}}_{(I,\leq)}$,
$\mathsf{CompAPOP}^{\mathrm{weak}}_{(I,\leq)}$: objects are all compact approximating $(I, \leq)$-pop's of $\mathsf{POP}^{\mathrm{weak}}_{(I,\leq)}$.

4.30. Theorem. *Let* $\mathcal{C} \in \{\mathsf{APOP}^{\mathrm{weak}}_{(I,\leq)}, \mathsf{CAPOP}^{\mathrm{weak}}_{(I,\leq)}, \mathsf{CompAPOP}^{\mathrm{weak}}_{(I,\leq)}\}$*. Then* $\mathcal{C}$ *is a cartesian closed full subcategory of* $\mathsf{POP}^{\mathrm{weak}}_{(I,\leq)}$.

Proof. In view of the proof of Theorem 4.29, it is enough to show that $\mathcal{T} = (\{t\}, =, (\mathsf{id}_{\{t\}})_{i\in I})$, $\mathcal{D} \times \mathcal{E} = (D \times E, \leq, (p_i \times q_i)_{i\in I})$, and $[\mathcal{D} \to \mathcal{E}]^{\mathrm{weak}}$ are in $\mathcal{C}$ whenever $\mathcal{D}$ and $\mathcal{E}$ are objects of $\mathcal{C}$.

Clearly, $\mathcal{T}$ is compact (whence complete) and approximating and hence an object of $\mathcal{C}$.

Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ and $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be in $\mathcal{C}$. It is clear that $\mathcal{D} \times \mathcal{E}$ is approximating if (and only if) $\mathcal{D}$ and $\mathcal{E}$ are approximating. Observe that we have $\mathsf{ker}(p_i \times q_i) = \{((d_1, e_1), (d_2, e_2)) \in (D \times E)^2 \mid p_i(d_1) = p_i(d_2) \text{ and } q_i(e_1) = q_i(e_2)\} = (\pi_1 \times \pi_1)^{-1}[\mathsf{ker}\, p_i] \cap (\pi_2 \times \pi_2)^{-1}[\mathsf{ker}\, q_i]$ for all $i \in I$. Thus, the pop uniformity of $\mathcal{D} \times \mathcal{E}$ coincides with the product uniformity of $(D, \mathcal{U}_{\mathcal{D}})$ and $(E, \mathcal{U}_{\mathcal{E}})$. As a consequence, $\mathcal{D} \times \mathcal{E}$ is complete [compact, respectively] if (and only if) $\mathcal{D}$ and $\mathcal{E}$ are complete [compact, respectively]. Therefore, $\mathcal{D} \times \mathcal{E} \in \mathcal{C}$.

Theorem 4.28(1) tells us that $[\mathcal{D} \to \mathcal{E}]^{\mathrm{weak}}$ belongs to $\mathcal{C}$ whenever $\mathcal{D}$ and $\mathcal{E}$ are objects of $\mathcal{C}$. ∎

4.31. Remark. Clearly, we have the inclusions

$$\mathsf{CompAPOP}^{\mathrm{weak}}_{(I,\leq)} \subseteq \mathsf{CAPOP}^{\mathrm{weak}}_{(I,\leq)} \subseteq \mathsf{APOP}^{\mathrm{weak}}_{(I,\leq)} \subseteq \mathsf{POP}^{\mathrm{weak}}_{(I,\leq)}.$$

Note that the object class of $\mathsf{CAPOP}^{\mathrm{weak}}_{(I,\leq)}$ "contains" all partially ordered sets in the following sense: if $(D, \leq)$ is a poset, then $(D, \leq, (\mathsf{id}_D)_{i\in I})$ is an object of $\mathsf{CAPOP}^{\mathrm{weak}}_{(I,\leq)}$ because discrete uniform spaces are complete.

Note further that if $(D, \leq, (p_i)_{i\in I})$ lies in $\mathsf{CompAPOP}^{\mathrm{weak}}_{(I,\leq)}$, then $(D, \leq)$ is a bifinite domain, cf. Corollary 3.57. Moreover, $\sup_{i\in I} p_i = \mathsf{id}_D$ and each p_i is Scott-continuous and image-finite (Theorem 3.56). But recall that the morphisms of $\mathsf{CompAPOP}^{\mathrm{weak}}_{(I,\leq)}$ are monotone, non-expansive mappings.

Next, we deal with indexed pop's whose underlying poset is a dcpo. Then, of course, we require a weak homomorphism to be Scott-continuous.

Let $(I, \leq)$ be directed. Let $\mathsf{POP}^{\mathrm{Sweak}}_{(I,\leq)}$ be the category whose objects are all $(I, \leq)$-pop's $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ with an Abelian projection net such that $(D, \leq)$ is a dcpo and p_i is Scott-continuous for all $i \in I$. Its morphisms are all weak homomorphisms that are also Scott-continuous, i.e. all Scott-continuous non-expansive mappings. Similarly to Theorem 4.29 we obtain:

4.32. THEOREM. *The category* $\mathsf{POP}^{\mathrm{Sweak}}_{(I,\leq)}$ *is cartesian closed.*

Proof. We follow the proof of Theorem 4.29 and recall that the constructions of the terminal element, the product, and the exponential object in the cartesian closed category of dcpo's with Scott-continuous mappings are also the "standard constructions" (one point dcpo, usual product, and function space of Scott-continuous maps). Then we deduce without trouble that $(\{t\}, =, (\mathsf{id}_{\{t\}})_{i\in I})$ is a terminal object of $\mathsf{POP}^{\mathrm{Sweak}}_{(I,\leq)}$, and for any $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ and $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ in $\mathsf{POP}^{\mathrm{Sweak}}_{(I,\leq)}$ the product of $\mathcal{D}$ and $\mathcal{E}$ is given by $\mathcal{D} \times \mathcal{E} = (D \times E, \leq, (p_i \times q_i)_{i\in I})$. The exponential object of $\mathcal{D}$ and $\mathcal{E}$ in $\mathsf{POP}^{\mathrm{Sweak}}_{(I,\leq)}$ is the space $[\mathcal{D} \to \mathcal{E}]^{\mathrm{Sweak}}$ of Scott-continuous weak homomorphisms. Note that the latter is an object of $\mathsf{POP}^{\mathrm{Sweak}}_{(I,\leq)}$ in view of Theorem 4.28(3). The details are left to the reader. ∎

Consider the following full subcategories of $\mathsf{POP}^{\mathrm{Sweak}}_{(I,\leq)}$:

$\mathsf{APOP}^{\mathrm{Sweak}}_{(I,\leq)}$: objects are all approximating $(I,\leq)$-pop's of $\mathsf{POP}^{\mathrm{Sweak}}_{(I,\leq)}$,
$\mathsf{CompAPOP}^{\mathrm{Sweak}}_{(I,\leq)}$: objects are all compact approximating $(I,\leq)$-pop's of $\mathsf{POP}^{\mathrm{Sweak}}_{(I,\leq)}$.

4.33. THEOREM. *The categories* $\mathsf{APOP}^{\mathrm{Sweak}}_{(I,\leq)}$ *and* $\mathsf{CompAPOP}^{\mathrm{Sweak}}_{(I,\leq)}$ *are cartesian closed full subcategories of* $\mathsf{POP}^{\mathrm{Sweak}}_{(I,\leq)}$.

Proof. Similar to the proof of Theorem 4.30. ∎

4.34. REMARK. We have the inclusions

$$\mathsf{CompAPOP}^{\mathrm{Sweak}}_{(I,\leq)} \subseteq \mathsf{APOP}^{\mathrm{Sweak}}_{(I,\leq)} \subseteq \mathsf{POP}^{\mathrm{Sweak}}_{(I,\leq)}.$$

Any dcpo $(D,\leq)$ is "contained" in $\mathsf{APOP}^{\mathrm{Sweak}}_{(I,\leq)}$ via the $(I,\leq)$-pop $(D, \leq, (\mathsf{id}_D)_{i\in I})$. Note further that due to Proposition 2.25 or Theorem 3.38 all indexed pop's in $\mathsf{POP}^{\mathrm{Sweak}}_{(I,\leq)}$ are complete with respect to their pop uniformity.

The object class of $\mathsf{CompAPOP}^{\mathrm{Sweak}}_{(I,\leq)}$ coincides with the one of $\mathsf{CompAPOP}^{\mathrm{weak}}_{(I,\leq)}$ (cf. Remark 4.31): their objects $(D, \leq, (p_i)_{i\in I})$ are bifinite domains together with a monotone net $(p_i)_{i\in I}$ of Scott-continuous projections having finite range such that $\sup_{i\in I} p_i = \mathsf{id}_D$. Again, observe that the morphisms of $\mathsf{CompAPOP}^{\mathrm{Sweak}}_{(I,\leq)}$ are both Scott-continuous and non-expansive.

4.3.3. Cartesian closed categories with homomorphisms. Recall that for every $(I,\leq)$-pop $(D, \leq, (p_i)_{i\in I})$ with an Abelian projection net $(p_i)_{i\in I}$, the composition $p_i \circ p_j$ is a projection for all $i, j \in I$ (Proposition 3.19). In what follows we shall require that these compositions appear in the net $(p_i)_{i\in I}$ already, i.e. $(\{p_i \mid i \in I\}, \circ)$ is an Abelian semigroup. Then $(\{p_i \mid i \in I\}, \leq)$ is a directed inf-semilattice and, for all $i, j \in I$, there is some $k \in I$ with $p_k = p_i \circ p_j = \inf\{p_i, p_j\}$; cf. Proposition 3.22(1). In this case, it is natural to assume $(I,\leq)$ to be a directed inf-semilattice and $p_{\inf\{i,j\}} = p_i \circ p_j$. This gives rise to the following definition:

DEFINITION. Let $(I,\leq)$ be a directed inf-semilattice and let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I,\leq)$-pop. We say that $\mathcal{D}$ is a *strict* $(I,\leq)$*-pop* if $p_{\inf\{i,j\}} = p_i \circ p_j$ for all $i, j \in I$.

Note that if $\mathcal{D}$ is a strict $(I, \leq)$-pop, then $(\{p_i \mid i \in I\}, \circ)$ is an Abelian semigroup and thus $(\{p_i \mid i \in I\}, \leq)$ is a directed inf-semilattice.

Observe that if $(I, \leq)$ is linear, then each $(I, \leq)$-pop is strict. This follows from Lemma 3.1.

Let $(I, \leq)$ be a directed inf-semilattice. We define the category $\mathsf{POP}^{\text{hom}}_{(I,\leq)}$ as follows: the object class consists of all strict $(I, \leq)$-pop's. The morphism class is the class of all homomorphisms, i.e. monotone maps commuting with all projections. The full subcategory $\mathsf{APOP}^{\text{hom}}_{(I,\leq)}$ has all approximating indexed pop's of $\mathsf{POP}^{\text{hom}}_{(I,\leq)}$ as objects.

4.35. THEOREM. *The categories* $\mathsf{POP}^{\text{hom}}_{(I,\leq)}$ *and* $\mathsf{APOP}^{\text{hom}}_{(I,\leq)}$ *are cartesian closed.*

The proof is a refinement of a proof given by Herrlich and Ehrig [26]. They showed that the category of projection spaces as objects and certain maps as morphisms (called "projection morphisms", which are similar to our mappings that commute with all projections) is cartesian closed. A related approach can be found in Spreen [50], who considers approximating ω-pop's with downwards closed projections with some additional assumptions concerning the poset structure ("dI-domains") and the homomorphisms ("stable functions").

Proof. Let $\mathcal{C} \in \{\mathsf{POP}^{\text{hom}}_{(I,\leq)}, \mathsf{APOP}^{\text{hom}}_{(I,\leq)}\}$. Clearly, $\mathcal{T} = (\{t\}, =, (\mathsf{id}_{\{t\}})_{i\in I})$ is the terminal object in $\mathcal{C}$. Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ and $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be in $\mathcal{C}$. Again, let $\mathcal{D} \times \mathcal{E} := (D \times E, \leq, (p_i \times q_i)_{i\in I})$. It is obvious that $\mathcal{D} \times \mathcal{E}$ is in $\mathcal{C}$. The canonical projections π_1 and π_2 from $D \times E$ onto D and E, respectively, are already known to be homomorphisms. The conditions on $\mathcal{D} \times \mathcal{E}$ to be the categorical product are satisfied, cf. the proof of Theorem 4.29. In particular, for any $X \in \mathcal{C}$ and homomorphisms $f : X \to D$ and $g : X \to E$ we have $\langle f, g\rangle(x) = (f(x), g(x))$ for all $x \in X$.

Unfortunately, the exponential object of $\mathcal{D}$ and $\mathcal{E}$ is not the $(I, \leq)$-pop $[\mathcal{D} \to \mathcal{E}]^{\text{hom}}$ of homomorphisms from $\mathcal{D}$ to $\mathcal{E}$ (cf. also Remark 4.36(1) below). In fact, the latter is an object of $\mathcal{C}$ that can be embedded into the exponential object (see 4.36). Thus, the exponential object is "larger" than the usual function space.

Before we continue our proof, we informally indicate how the exponential object can be derived. As in the projection space approach, the projection net $(p_i)_{i\in I}$ can be viewed as a mapping $p : I \times D \to D$ with $p(i, d) := p_i(d)$ for all $i \in I$, $d \in D$. Assume that I has no greatest element. We add a top element $\top$ and extend the partial order to $I^\top := I \dot\cup \{\top\}$ as usual. We also extend p to $I^\top \times D$ by setting $p(\top, d) := d$ for all $d \in D$. One can endow $I^\top$ with a strict $(I, \leq)$-pop structure that gives rise to an object of $\mathcal{C}$. Then p is a homomorphism from $I^\top \times D$ to D. In the same way, $(E, \leq, (q_i)_{i\in I})$ gives rise to a homomorphism $q : I^\top \times E \to E$. Now the exponential object of "$(D, \leq, p : I^\top \times D \to D)$" and "$(E, \leq, q : I^\top \times E \to E)$" is the set of all homomorphisms $f : I^\top \times D \to E$ together with a specific strict $(I, \leq)$-pop structure.

Here are the details. In fact, we have to distinguish two cases. First, if $\mathcal{C} = \mathsf{POP}^{\text{hom}}_{(I,\leq)}$, then let $I^\top := I \dot\cup \{\top\}$. In the case that $\mathcal{C} = \mathsf{APOP}^{\text{hom}}_{(I,\leq)}$, let $I^\top := I$ if I has a greatest element $\top$, and $I^\top := I \dot\cup \{\top\}$ otherwise.

Extend (if $\top \notin I$) the partial order to $I^\top$ by setting $i \leq \top$ for all $i \in I^\top$. It is clear that $(I^\top, \leq)$ is again a directed inf-semilattice. For all $i \in I$ let $t_i : I^\top \to I^\top$ be defined by $t_i(k) := \inf\{i,k\}$ for all $k \in I^\top$. Certainly, t_i is a projection and $t_{\inf\{i,j\}} = t_i \circ t_j$ for all $i, j \in I$. Thus, $\mathcal{I}^\top := (I^\top, \leq, (t_i)_{i\in I})$ is a strict $(I,\leq)$-pop. If $\mathcal{C} = \mathsf{APOP}^{\mathrm{hom}}_{(I,\leq)}$, then $\mathcal{I}^\top$ is approximating: let $k, l \in I^\top$ and let $t_i(k) \leq l$ for all $i \in I$. Obviously, if $k \in I$, then $k = t_i(k) \leq l$ for all $i \geq k$. If $k = \top$, then $i = t_i(\top) \leq l$ for all $i \in I$; hence $l = \top$. Therefore, $\sup_{i\in I} t_i(k) = k$ for all $k \in I^\top$. Thus, in either case, $\mathcal{I}^\top$ is an object of $\mathcal{C}$.

Next, we define the exponential object of $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ and $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ in $\mathcal{C}$. Let

$$E^D := \{f : I^\top \times D \to E \mid f \text{ is a homomorphism from } \mathcal{I}^\top \times \mathcal{D} \text{ to } \mathcal{E}\} = [I^\top \times D \to E]^{\mathrm{hom}}.$$

Let E^D be equipped with the pointwise order. For all $f \in E^D$ and all $i \in I$ define a mapping $R_i(f) : I^\top \times D \to E$ by $R_i(f)(k,d) := f(\inf\{i,k\}, d)$. Then we have $R_i(f) = f \circ (t_i \times \mathsf{id}_D)$. Clearly, $R_i(f)$ is monotone. For all $j \in I$ we have

$$\begin{aligned} q_j \circ R_i(f) &= q_j \circ f \circ (t_i \times \mathsf{id}_D) = f \circ (t_j \times p_j) \circ (t_i \times \mathsf{id}_D) = f \circ ((t_j \circ t_i) \times p_j) \\ &= f \circ ((t_i \circ t_j) \times p_j) = f \circ (t_i \times \mathsf{id}_D) \circ (t_j \times p_j) = R_i(f) \circ (t_j \times p_j). \end{aligned}$$

This shows us that $R_i(f) \in E^D$ for all $i \in I$ and all $f \in E^D$ and gives rise to a mapping $R_i : E^D \to E^D$. Obviously, R_i is monotone. As $f \in E^D$ is monotone and t_i is below $\mathsf{id}_{I^\top}$, we deduce $R_i(f) \leq f$; hence $R_i \leq \mathsf{id}_{E^D}$. In addition, $R_{\inf\{i,j\}}(f) = f \circ (t_{\inf\{i,j\}} \times \mathsf{id}_D) = f \circ (t_j \times \mathsf{id}_D) \circ (t_i \times \mathsf{id}_D) = R_i(f \circ (t_j \times \mathsf{id}_D)) = R_i(R_j(f))$ for all $i, j \in I$. Therefore, $\mathcal{E}^{\mathcal{D}} := (E^D, \leq, (R_i)_{i\in I})$ is a strict $(I,\leq)$-pop. If $\mathcal{C} = \mathsf{APOP}^{\mathrm{hom}}_{(I,\leq)}$, then $\mathcal{E}^{\mathcal{D}}$ is approximating: let $k \in I^\top$ and let $d \in D$. Then $\sup_{i\in I} q_i(f(k,d)) = f(k,d)$ since $\mathcal{E}$ is approximating. As $q_i(f(k,d)) = f(\inf\{i,k\}, p_i(d)) \leq f(\inf\{i,k\}, d) = R_i(f)(k,d) \leq f(k,d)$, we conclude that $\sup_{i\in I} R_i(f)(k,d) = f(k,d)$. Summing up, we see that $\mathcal{E}^{\mathcal{D}}$ is an object of $\mathcal{C}$.

We define $\mathrm{ev} : E^D \times D \to E$ by $\mathrm{ev}(f,d) := f(\top, d)$ for all $f \in E^D$, $d \in D$. Clearly, ev is monotone. Moreover, $q_i(\mathrm{ev}(f,d)) = q_i(f(\top,d)) = f(t_i(\top), p_i(d)) = R_i(f)(\top, p_i(d)) = \mathrm{ev}(R_i(f), p_i(d))$ for all $i \in I$, $f \in E^D$, $d \in D$. Hence, $q_i \circ \mathrm{ev} = \mathrm{ev} \circ (R_i \times p_i)$ for all $i \in I$. Therefore, ev is a homomorphism.

Let $\mathcal{X} = (X, \leq, (s_i)_{i\in I})$ be an object of $\mathcal{C}$ and let $f : X \times D \to E$ be a homomorphism. Define $\widehat{f} : X \to E^D$ by $\widehat{f}(x) : I^\top \times D \to E$ with

$$\widehat{f}(x)(k,d) := \begin{cases} f(s_k(x), d) & \text{if } k \in I, \\ f(x,d) & \text{if } k = \top. \end{cases}$$

First we check that $\widehat{f}$ is well defined. We have $I^\top = I$ if and only if $\mathcal{C} = \mathsf{APOP}^{\mathrm{hom}}_{(I,\leq)}$ and I has a greatest element $\top$. Then, as $\mathcal{X}$ is approximating, we conclude $s_\top(x) = x$ for all $x \in X$; hence $f(s_\top(x), d) = f(x,d)$. Again, let $\mathcal{C} \in \{\mathsf{POP}^{\mathrm{hom}}_{(I,\leq)}, \mathsf{APOP}^{\mathrm{hom}}_{(I,\leq)}\}$. If $\top \notin I$, then we set $s_\top := \mathsf{id}_X$. Let $x \in X$. It is clear that $\widehat{f}(x)$ is a monotone mapping. It also commutes with all projections since $q_i(\widehat{f}(x)(k,d)) = q_i(f(s_k(x),d)) = f(s_i(s_k(x)), p_i(d)) = f(s_{\inf\{i,k\}}(x), p_i(d)) = \widehat{f}(x)(\inf\{i,k\}, p_i(d)) = \widehat{f}(x)(t_i(k), p_i(d))$ for all $k \in I^\top$, $d \in D$, $i \in I$; that is $q_i \circ \widehat{f}(x) = \widehat{f}(x) \circ (t_i \times p_i)$ for all $i \in I$.

It is also clear that $\widehat{f}$ is monotone. Let $x \in X$, $k \in I^\top$, $d \in D$, and let $i \in I$. Then

$$R_i(\widehat{f}(x))(k,d) = \widehat{f}(x)(\inf\{i,k\}, d) = f(s_{\inf\{i,k\}}(x), d) = f(s_k(s_i(x)), d) = \widehat{f}(s_i(x))(k,d).$$

Hence, $R_i \circ \widehat{f} = \widehat{f} \circ s_i$ for all $i \in I$. Consequently, $\widehat{f}$ is a homomorphism.

For all $x \in X$ and all $d \in D$ we have $\mathrm{ev}(\widehat{f}(x), d) = \widehat{f}(x)(\top, d) = f(x, d)$, i.e.

$$\mathrm{ev} \circ \langle \widehat{f} \circ \pi_1, \mathsf{id}_D \circ \pi_2 \rangle = \mathrm{ev} \circ (\widehat{f} \times \mathsf{id}_D) = f.$$

Next, let $h : X \to E^D$ be a homomorphism with $\mathrm{ev} \circ (h \times \mathsf{id}_D) = f$. We show $h = \widehat{f}$ as in [26]: let $x \in X$, let $k \in I$, and let $d \in D$. Then $h(x)(\top, d) = \mathrm{ev}(h(x), d) = f(x, d) = \widehat{f}(x)(\top, d)$ and $h(x)(k, d) = h(x)(t_k(\top), d) = R_k(h(x))(\top, d) = h(s_k(x))(\top, d) = \mathrm{ev}(h(s_k(x)), d) = f(s_k(x), d) = \widehat{f}(x)(k, d)$. ■

4.36. Remark. Let $\mathcal{C} \in \{\mathsf{POP}^{\mathrm{hom}}_{(I,\leq)}, \mathsf{APOP}^{\mathrm{hom}}_{(I,\leq)}\}$ and let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ and $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be objects of $\mathcal{C}$. Consider the function space $[\mathcal{D} \to \mathcal{E}]^{\mathrm{hom}} = ([D \to E]^{\mathrm{hom}}, \leq, (Q_i|_{[D\to E]^{\mathrm{hom}}})_{i\in I})$ of all homomorphisms from $\mathcal{D}$ to $\mathcal{E}$. It is obvious that $[\mathcal{D} \to \mathcal{E}]^{\mathrm{hom}}$ is a strict $(I, \leq)$-pop. By Theorem 4.28(1), $[\mathcal{D} \to \mathcal{E}]^{\mathrm{hom}}$ is an object of $\mathcal{C}$. In the following we compare it with the exponential object of $\mathcal{D}$ and $\mathcal{E}$.

(1) When we define the "usual" evaluation map $e : [D \to E]^{\mathrm{hom}} \times D \to E$ by $e(f, d) := f(d)$, then it is easy to see that e is a homomorphism. If $\mathcal{X} = (X, \leq, (s_i)_{i\in I})$ is in $\mathcal{C}$, if $f : X \times D \to E$ is a homomorphism, and if we define $\widetilde{f} : X \to F(D, E)$ by $\widetilde{f}(x) : D \to E$, $\widetilde{f}(x)(d) := f(x, d)$, then we come across the problem that possibly neither $\widetilde{f}(x)$ nor $\widetilde{f}$ is a homomorphism. Clearly, $\widetilde{f}(x)$ and $\widetilde{f}$ are monotone. But $(q_i \circ \widetilde{f}(x))(d) = q_i(f(x, d)) = f(s_i(x), p_i(d)) = \widetilde{f}(s_i(x))(p_i(d))$, whence $q_i \circ \widetilde{f}(x) = \widetilde{f}(s_i(x)) \circ p_i$ instead of the desired equation "$q_i \circ \widetilde{f}(x) = \widetilde{f}(x) \circ p_i$". Similarly, one gets $(Q_i \circ \widetilde{f})(x) = (\widetilde{f} \circ s_i)(x) \circ p_i$ instead of "$Q_i \circ \widetilde{f} = \widetilde{f} \circ s_i$".

For instance, take $\mathcal{X} := \mathcal{E} := \mathcal{D}$ and suppose there is some $i \in I$ with $p_i \neq \mathsf{id}_D$. Let $d_1 \in D$ with $p_i(d_1) \neq d_1$. Let $f := \pi_1$ be the canonical projection from $D \times D$ onto the first coordinate. Then $(p_i \circ \widetilde{\pi}_1(d_1))(d_2) = p_i(\pi_1(d_1, d_2)) = p_i(d_1) \neq d_1 = \widetilde{\pi}_1(d_1)(p_i(d_2))$ for any $d_2 \in D$.

(2) We can pop embed the function space $[\mathcal{D} \to \mathcal{E}]^{\mathrm{hom}}$ into the exponential object $\mathcal{E}^{\mathcal{D}} = (E^D, \leq, (R_i)_{i\in I})$. Let $\zeta : [D \to E]^{\mathrm{hom}} \to E^D$ be defined by $\zeta(f) : I^\top \times D \to E$, with

$$\zeta(f)(k, d) := \begin{cases} f(p_k(d)) & \text{if } k \in I, \\ f(d) & \text{if } k = \top. \end{cases}$$

Then ζ is a pop embedding. In order to prove this, consider the homomorphism $e : [D \to E]^{\mathrm{hom}} \times D \to E$ from (1). The mapping $\widehat{e} : [D \to E]^{\mathrm{hom}} \to E^D$, defined by $\widehat{e}(f) : I^\top \times D \to E$ with $\widehat{e}(f)(k, d) = e(Q_k(f), d) = (Q_k(f))(d) = q_k(f(d)) = f(p_k(d))$ if $k \in I$ and $\widehat{e}(f)(\top, d) = e(f, d) = f(d)$, is a homomorphism. This is proven on p. 104. Thus, $\zeta = \widehat{e}$ is a homomorphism. If $\zeta(f) \leq \zeta(g)$, then, in particular, $f(d) = \zeta(f)(\top, d) \leq \zeta(g)(\top, d) = g(d)$ for all $d \in D$; that is, $f \leq g$. Therefore, ζ is a pop embedding.

(3) Let $\mathcal{C} = \mathsf{APOP}^{\mathrm{hom}}_{(I,\leq)}$ and consider the embedding $\zeta : [D \to E]^{\mathrm{hom}} \to E^D$ given in (2). Then $\zeta[[D \to E]^{\mathrm{hom}}]$ induces a *closed* subpop of $\mathcal{E}^{\mathcal{D}}$, i.e. the range $\zeta[[D \to E]^{\mathrm{hom}}]$ is a closed subset of E^D. This can be shown as follows. Let $(f_n)_{n\in N}$ be a net in $[D \to E]^{\mathrm{hom}}$ such that $(\zeta(f_n))_{n\in N}$ converges to some $g \in E^D$ with respect to the pop topology of $\mathcal{E}^{\mathcal{D}}$. Define $f : D \to E$ by $f(d) := \mathrm{ev}(g, d) = g(\top, d)$ for all $d \in D$. Clearly, f is monotone. Let $i \in I$. Then there exists some $n_i \in N$ such that for all $n \geq n_i$, for all $k \in I^\top$, and for all $d \in D$, we have $f_n(p_{\inf\{i,k\}}(d)) = \zeta(f_n)(\inf\{i, k\}, d) = R_i(\zeta(f_n))(k, d) = R_i(g)(k, d) = g(\inf\{i, k\}, d)$. In particular, setting $k := \top$ and switching from d to $p_i(d)$, we deduce for

all $n \geq n_i$ and all $d \in D$ that

$$\begin{aligned} q_i(f_n(d)) &= f_n(p_i(d)) = f_n(p_{\inf\{i,\top\}}(p_i(d))) = g(\inf\{i,\top\}, p_i(d)) \\ &= g(t_i(\top), p_i(d)) = q_i(g(\top, d)) = q_i(f(d)). \end{aligned}$$

Hence, $Q_i(f_n) = Q_i(f)$ for all $n \geq n_i$; that is, $(f_n)_{n \in N}$ converges to f in the pop topology of $[\mathcal{D} \to \mathcal{E}]^{\text{hom}}$. In view of Proposition 4.26(1), we obtain $f \in [D \to E]^{\text{hom}}$. As ζ is in particular continuous, we infer that $(\zeta(f_n))_{n \in N}$ converges to $\zeta(f)$. Since $\mathcal{E}^{\mathcal{D}}$ is approximating, it has to be Hausdorff (Proposition 2.11). As a consequence, $\zeta(f) = g$.

(4) Note that the pop uniformity $\mathcal{U}_{\mathcal{E}^{\mathcal{D}}}$ of the exponential object $\mathcal{E}^{\mathcal{D}}$ is finer than the uniformity of uniform convergence on E^D. To see this, recall that the latter is induced by the projections Q'_i with $Q'_i(f) = q_i \circ f$ for all $f \in E^D$, $i \in I$, cf. Proposition 4.19. We have $Q'_i(f) = q_i \circ f = f \circ (t_i \times p_i) \leq f \circ (t_i \times \mathsf{id}_D) = R_i(f)$ for all $f \in E^D$ and all $i \in I$, i.e. $Q'_i \leq R_i$ for all $i \in I$. Then $\ker Q'_i \supseteq \ker R_i$ by Lemma 3.1; hence the assertion follows.

Finally, we prove an analogous result to Theorem 4.35 by considering pop's that are also dcpo's. Let $\mathsf{POP}^{\text{Shom}}_{(I,\leq)}$ be the category whose objects are strict $(I,\leq)$-pop's $(D, \leq, (p_i)_{i \in I})$ such that $(D, \leq)$ is a dcpo and p_i is Scott-continuous for all $i \in I$. Morphisms are all Scott-continuous pop homomorphisms, i.e. all Scott-continuous mappings that commute with all projections. Let $\mathsf{APOP}^{\text{Shom}}_{(I,\leq)}$ be the full subcategory of $\mathsf{POP}^{\text{Shom}}_{(I,\leq)}$ whose objects are all approximating indexed pop's of $\mathsf{POP}^{\text{Shom}}_{(I,\leq)}$.

4.37. Theorem. $\mathsf{POP}^{\text{Shom}}_{(I,\leq)}$ *and* $\mathsf{APOP}^{\text{Shom}}_{(I,\leq)}$ *are cartesian closed.*

Proof. Let $\mathcal{C} \in \{\mathsf{POP}^{\text{Shom}}_{(I,\leq)}, \mathsf{APOP}^{\text{Shom}}_{(I,\leq)}\}$. Again, there are no problems with the terminal object and the categorical product (cf. the proof of Theorem 4.35). The only difficulty is the exponential object of $\mathcal{D} = (D, \leq, (p_i)_{i \in I})$ and $\mathcal{E} = (E, \leq, (q_i)_{i \in I})$ in $\mathcal{C}$. It is not the function space of all Scott-continuous homomorphisms, which in fact can be turned into an object of $\mathcal{C}$. Its construction is similar to the one given in the proof of Theorem 4.35. There, we introduced the set $I^\top$ and obtained an object of $\mathcal{C}$. Here, the situation is slightly more complicated because $(I^\top, \leq)$ need not be a dcpo. Instead, we consider the ideal completion $(\mathsf{Id}(I), \subseteq)$ of $(I, \leq)$. As in 4.35, there are two distinct cases. If $\mathcal{C} = \mathsf{POP}^{\text{Shom}}_{(I,\leq)}$, then let $J := \mathsf{Id}(I) \,\dot\cup\, \{\top\}$. We extend the partial order $\subseteq$ of $\mathsf{Id}(I)$ to a partial order $\leq$ of J by setting $A \leq \top$ for all $A \in J$. We also write $A \leq B$ for $A \subseteq B$, $A, B \in \mathsf{Id}(I)$. Clearly, $(J, \leq)$ is a dcpo with $\sup \mathcal{A} = \bigcup \mathcal{A}$ for all $\mathcal{A} \subseteq \mathsf{Id}(I)$, $\mathcal{A}$ directed. On the other hand, if $\mathcal{C} = \mathsf{APOP}^{\text{Shom}}_{(I,\leq)}$, then let $(J, \leq) := (\mathsf{Id}(I), \subseteq)$. In this case, $\top := I$ is the greatest element of $(J, \leq)$.

For all $i \in I$ and all $A \in \mathsf{Id}(I)$ let $u_i(A) := A \cap i{\downarrow}$ and $u_i(\top) := u_i(I) = i{\downarrow}$. Since $(I, \leq)$ is an inf-semilattice, one easily sees that $A \cap i{\downarrow}$ is non-empty and, moreover, $A \cap i{\downarrow} \in \mathsf{Id}(I)$. Therefore, we obtain a mapping $u_i : J \to J$ for all $i \in I$. Clearly, u_i is a projection and $u_{\inf\{i,j\}} = u_i \circ u_j$ for all $i, j \in I$. Thus, $\mathcal{J} := (J, \leq, (u_i)_{i \in I})$ is a strict $(I, \leq)$-pop. Let $\mathcal{A} \subseteq \mathsf{Id}(I)$ be directed. Then $u_i(\bigcup \mathcal{A}) = (\bigcup \mathcal{A}) \cap i{\downarrow} = \bigcup_{A \in \mathcal{A}} (A \cap i{\downarrow}) = \bigcup_{A \in \mathcal{A}} u_i(A)$. This implies that u_i is Scott-continuous. If $\mathcal{C} = \mathsf{APOP}^{\text{Shom}}_{(I,\leq)}$, then let $A, B \in J = \mathsf{Id}(I)$ with $A \cap i{\downarrow} = u_i(A) \subseteq B$ for all $i \in I$. We infer that $A \subseteq B$. This shows us that $\mathcal{J}$ is approximating. Summing things up, we deduce that $\mathcal{J}$ is an object of $\mathcal{C}$.

Let

$$E^D := \{f : J \times D \to E \mid f \text{ is a Scott-continuous homomorphism from } \mathcal{J} \times \mathcal{D} \text{ to } \mathcal{E}\}$$
$$= [J \times D \to E]^{\mathrm{Shom}}.$$

We define for all $i \in I$ projections $R_i : E^D \to E^D$. For any $f \in E^D$ we set $R_i(f) := f \circ (u_i \times \mathsf{id}_D)$, i.e. $R_i(f)(A, d) = f(A \cap i{\downarrow}, d)$ and $R_i(f)(\top, d) = f(i{\downarrow}, d)$ for all $A \in \mathsf{Id}(I)$, $d \in D$. As a composition of Scott-continuous mappings, $R_i(f)$ is Scott-continuous. Further, one checks $R_i(f)$ to be a homomorphism as in the proof of Theorem 4.35 (cf. p. 104). Since the composition map, applied to Scott-continuous mappings, is Scott-continuous itself, we find R_i to be Scott-continuous. Finally, R_i is a projection and $R_{\inf\{i,j\}} = R_i \circ R_j$ for all $i, j \in I$. This can be seen as on p. 104. Thus, $\mathcal{E}^{\mathcal{D}} := (E^D, \leq, (R_i)_{i \in I})$ is a strict $(I, \leq)$-pop. Theorem 4.28(3) tells us that $(E^D, \leq)$ is a dcpo. In the case of $\mathcal{C} = \mathsf{APOP}^{\mathrm{Shom}}_{(I,\leq)}$ we see that $\mathcal{E}^{\mathcal{D}}$ is approximating (cf. p. 104). Hence, $\mathcal{E}^{\mathcal{D}}$ is an object of $\mathcal{C}$.

Let $\mathrm{ev} : E^D \times D \to E$ be defined by $\mathrm{ev}(f, d) := f(\top, d)$ for all $f \in E^D$ and all $d \in D$. It is well known that ev is Scott-continuous. As on p. 104, one can show that ev commutes with all projections. Let $\mathcal{X} = (X, \leq, (s_i)_{i \in I})$ be in $\mathcal{C}$ and let $f : X \times D \to E$ be a Scott-continuous homomorphism. For any $A \in \mathsf{Id}(I)$ the set $\{s_a \mid a \in A\}$ of Scott-continuous projections is directed. Its pointwise supremum $\sup_{a \in A} s_a$ is a Scott-continuous projection again (cf. [1, Prop. 3.1.17.1], [27, Prop. 1.18(ii)]). Given $A \in \mathsf{Id}(I)$ and $i \in I$, we have

$$(*) \qquad \{s_i \circ s_a \mid a \in A\} = \{s_a \mid a \in A \cap i{\downarrow}\}.$$

To see this, let $a \in A$. Then $s_i \circ s_a = s_{\inf\{i,a\}}$ and $\inf\{i, a\} \leq a, i$; hence $\inf\{i, a\} \in A \cap i{\downarrow}$. Conversely, if $a \in A \cap i{\downarrow}$, then $s_a = s_{\inf\{i,a\}} = s_i \circ s_a$.

Define $\widehat{f} : X \to E^D$ as follows. For all $x \in X$ let $\widehat{f}(x) : J \times D \to E$ with

$$\widehat{f}(x)(A, d) := \begin{cases} f(\sup_{a \in A} s_a(x), d) & \text{if } A \in \mathsf{Id}(I), \\ f(x, d) & \text{if } A = \top. \end{cases}$$

Note that if $\mathcal{C} = \mathsf{APOP}^{\mathrm{Shom}}_{(I,\leq)}$, then $I = \top$ and $\sup_{a \in I} s_a = \mathsf{id}_X$. We show that $\widehat{f}$ is well defined. Let $\mathcal{A} \subseteq \mathsf{Id}(I)$ be directed and let $B \subseteq D$ be directed. Then

$$\begin{aligned}\widehat{f}(x)\Big(\bigcup \mathcal{A}, \sup B\Big) &= f(\sup_{a \in \bigcup \mathcal{A}} s_a(x), \sup B) = f(\sup_{A \in \mathcal{A}} \sup_{a \in A} s_a(x), \sup B) \\ &= \sup_{(A,d) \in \mathcal{A} \times B} f(\sup_{a \in A} s_a(x), d) = \sup_{(A,d) \in \mathcal{A} \times B} \widehat{f}(x)(A, d).\end{aligned}$$

We thus infer that $\widehat{f}(x)$ is Scott-continuous. Let $i \in I$, let $A \in \mathsf{Id}(I)$, and let $d \in D$. Using $(*)$ we deduce

$$\begin{aligned} q_i(\widehat{f}(x)(A, d)) &= q_i(f(\sup_{a \in A} s_a(x), d)) = f(s_i(\sup_{a \in A} s_a(x)), p_i(d)) = f(\sup_{a \in A} s_i(s_a(x)), p_i(d)) \\ &= f(\sup_{a \in A \cap i{\downarrow}} s_a(x), p_i(d)) = \widehat{f}(x)(A \cap i{\downarrow}, p_i(d)) = \widehat{f}(x)(u_i(A), p_i(d)), \\ q_i(\widehat{f}(x)(\top, d)) &= q_i(f(x, d)) = f(s_i(x), p_i(d)) = f(\sup_{a \in i{\downarrow}} s_a(x), p_i(d)) \\ &= \widehat{f}(x)(i{\downarrow}, p_i(d)) = \widehat{f}(x)(u_i(\top), p_i(d)). \end{aligned}$$

Hence, $q_i \circ \widehat{f}(x) = \widehat{f}(x) \circ (u_i \times p_i)$ for all $i \in I$, i.e. $\widehat{f}(x)$ commutes with all projections.

Let $C \subseteq X$ be directed. Let $A \in \mathsf{Id}(I)$ and let $d \in D$. Then

$$\begin{aligned}
\widehat{f}(\sup C)(A,d) &= f(\sup_{a\in A} s_a(\sup C), d) = f(\sup_{a\in A}\sup_{x\in C} s_a(x), d) = f(\sup_{x\in C}\sup_{a\in A} s_a(x), d)\\
&= \sup_{x\in C} f(\sup_{a\in A} s_a(x), d) = \sup_{x\in C} \widehat{f}(x)(A,d),\\
\widehat{f}(\sup C)(\top,d) &= f(\sup C, d) = \sup_{x\in C} f(x,d) = \sup_{x\in C}\widehat{f}(x)(\top,d).
\end{aligned}$$

Therefore, $\widehat{f}$ is Scott-continuous. Let $i \in I$, let $x \in X$, let $A \in \mathsf{Id}(I)$, and let $d \in D$. Then, again using $(*)$, we obtain

$$\begin{aligned}
R_i(\widehat{f}(x))(A,d) &= \widehat{f}(x)(A \cap i{\downarrow}, d) = f(\sup_{a\in A\cap i\downarrow} s_a(x), d)\\
&= f(\sup_{a\in A} s_a(s_i(x)), d) = \widehat{f}(s_i(x))(A,d),\\
R_i(\widehat{f}(x))(\top,d) &= \widehat{f}(x)(i{\downarrow}, d) = f(\sup_{a\in i\downarrow} s_a(x), d) = f(s_i(x), d) = \widehat{f}(s_i(x))(\top,d).
\end{aligned}$$

This yields $R_i \circ \widehat{f} = \widehat{f} \circ s_i$ for all $i \in I$. Hence, $\widehat{f}$ is a Scott-continuous homomorphism.

For all $x \in X$ and all $d \in D$ we have $\mathrm{ev}(\widehat{f}(x), d) = \widehat{f}(x)(\top, d) = f(x,d)$, whence $\mathrm{ev} \circ (\widehat{f} \times \mathsf{id}_D) = f$.

Finally, let $h : X \to E^D$ be a Scott-continuous homomorphism such that $\mathrm{ev} \circ (h \times \mathsf{id}_D) = f$. Let $x \in X$, let $A \in \mathsf{Id}(I)$, and let $d \in D$. Then $h(x)(\top,d) = \mathrm{ev}(h(x),d) = f(x,d) = \widehat{f}(x)(\top,d)$. Moreover,

$$\begin{aligned}
h(x)(A,d) &= h(x)\Big(\bigcup_{a\in A} a{\downarrow}, d\Big) = h(x)\Big(\bigcup_{a\in A} u_a(\top), d\Big) = \sup_{a\in A} h(x)(u_a(\top), d)\\
&= \sup_{a\in A} R_a(h(x))(\top,d) = \sup_{a\in A} h(s_a(x))(\top,d)\\
&= \sup_{a\in A} f(s_a(x),d) = f(\sup_{a\in A} s_a(x), d) = \widehat{f}(x)(A,d). \blacksquare
\end{aligned}$$

4.38. Remark. Let $\mathcal{C} \in \{\mathsf{POP}^{\mathrm{Shom}}_{(I,\leq)}, \mathsf{APOP}^{\mathrm{Shom}}_{(I,\leq)}\}$ and let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ and $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be objects of $\mathcal{C}$. The function space $[\mathcal{D} \to \mathcal{E}]^{\mathrm{Shom}} = ([D \to E]^{\mathrm{Shom}}, \leq, (Q_i|_{[D\to E]^{\mathrm{Shom}}})_{i\in I})$ of all Scott-continuous homomorphisms from $\mathcal{D}$ to $\mathcal{E}$ is an object of $\mathcal{C}$, cf. Theorem 4.28. As in 4.36, we can pop embed $[\mathcal{D} \to \mathcal{E}]^{\mathrm{Shom}}$ into $\mathcal{E}^{\mathcal{D}} = (E, \leq, (R_i)_{i\in I})$. Recall that $E^D = [J \times D \to E]^{\mathrm{Shom}}$, where $J = \mathsf{Id}(I)$ if $\mathcal{C} = \mathsf{APOP}^{\mathrm{Shom}}_{(I,\leq)}$ and $J = \mathsf{Id}(I) \mathbin{\dot\cup} \{\top\}$ if $\mathcal{C} = \mathsf{POP}^{\mathrm{Shom}}_{(I,\leq)}$. We define $\zeta : [D \to E]^{\mathrm{Shom}} \to E^D$ by $\zeta(f) : J \times D \to E$, where

$$\zeta(f)(A,d) := \begin{cases} f(\sup_{a\in A} p_a(d)) & \text{if } A \in \mathsf{Id}(I),\\ f(d) & \text{if } A = \top. \end{cases}$$

Then ζ is a Scott-continuous pop embedding. The proof is quite similar to the one in Remark 4.36(2) and is left to the reader.

Recall that $[\mathcal{D} \to \mathcal{E}]^{\mathrm{Shom}}$ and $\mathcal{E}^{\mathcal{D}}$ are complete (with respect to their pop uniformity). This follows from Proposition 2.25.

If $\mathcal{C} = \mathsf{APOP}^{\mathrm{Shom}}_{(I,\leq)}$, then $[\mathcal{D} \to \mathcal{E}]^{\mathrm{Shom}}$ and $\mathcal{E}^{\mathcal{D}}$ are approximating and, in particular, Hausdorff. Then $\zeta[[D \to E]^{\mathrm{Shom}}]$ induces a closed and complete approximating subpop of $\mathcal{E}^{\mathcal{D}}$.

4.3.4. $\mathcal{D}_\infty$-models for the untyped λ-calculus. The last subsection of the present chapter combines our results on the cartesian closure of the categories $\mathsf{POP}^{\mathrm{Shom}}_{(I,\le)}$ and $\mathsf{APOP}^{\mathrm{Shom}}_{(I,\le)}$ with well known domain-theoretic arguments to obtain $(I,\le)$-pop's isomorphic to their own exponent. It is Scott's celebrated D_∞-construction (cf. [48]) that we are able to perform since, as we will see, inverse limits exist and the exponent functor turns out to be continuous. Hence, this gives rise to models of the untyped λ-calculus (see e.g. Barendregt [3, Chapter 5]).

A similar approach concerning D_∞-models of Spreen's "approximation structures", which are very special ω-pop's, can be found in [50].

Let $(I,\le)$ be a directed inf-semilattice and let $\mathcal{C}$ be one of the categories $\mathsf{POP}^{\mathrm{Shom}}_{(I,\le)}$ or $\mathsf{APOP}^{\mathrm{Shom}}_{(I,\le)}$. Let $(\mathit{\Gamma},\le)$ be a directed index set and let $\mathcal{D}_\gamma = (D_\gamma,\le_\gamma,(p_i^\gamma)_{i\in I})$ be an object of $\mathcal{C}$ for all $\gamma\in\mathit{\Gamma}$. That is, $(D_\gamma,\le_\gamma,(p_i^\gamma)_{i\in I})$ is a strict $(I,\le)$-pop with $(D_\gamma,\le_\gamma)$ a dcpo and p_i^γ Scott-continuous for all $i\in I$ (see the previous section). If $\mathcal{C}=\mathsf{APOP}^{\mathrm{Shom}}_{(I,\le)}$, then $(D_\gamma,\le_\gamma,(p_i^\gamma)_{i\in I})$ is approximating.

Let $\mathcal{S}=\{(f_{\gamma\mu},g_{\gamma\mu}) : D_\gamma\to D_\mu \mid \gamma,\mu\in\mathit{\Gamma},\ \gamma\le\mu\}$ be an inverse system of Scott-continuous epp's such that both $f_{\gamma\mu}$ and $g_{\gamma\mu}$ are also homomorphisms for all $\gamma\le\mu$. Hence, $f_{\gamma\mu}$ and $g_{\gamma\mu}$ are morphisms of $\mathcal{C}$. We also say that $(f_{\gamma\mu},g_{\gamma\mu})$ is an *epp of Scott-continuous homomorphisms.*

Let $D_\infty := \{(d_\gamma)_{\gamma\in\mathit{\Gamma}}\in\prod_{\gamma\in\mathit{\Gamma}} D_\gamma \mid \forall\gamma,\mu\in\mathit{\Gamma} : \gamma\le\mu\Rightarrow d_\gamma = g_{\gamma\mu}(d_\mu)\}$ be equipped with the product order. Recall that $(D_\infty,\le)$ is a dcpo (Lemma 1.4(1)).

For $\gamma\in\mathit{\Gamma}$ let $f_\gamma : D_\gamma\to D_\infty$ be defined by $f_\gamma(d) := (d_\mu)_{\mu\in\mathit{\Gamma}}$ with $d_\mu := g_{\mu\nu}(f_{\gamma\nu}(d))$ for some $\nu\ge\gamma,\mu$ (cf. p. 15). Let $g_\gamma : D_\infty\to D_\gamma$ be defined by $g_\gamma((d_\mu)_{\mu\in\mathit{\Gamma}}) := d_\gamma$. Recall that (f_γ,g_γ) is a Scott-continuous epp by Lemma 1.4(1).

Next, we define an $(I,\le)$-pop structure on D_∞. For all $i\in I$ let $p_i : D_\infty\to D_\infty$ be defined by $p_i((d_\gamma)_{\gamma\in\mathit{\Gamma}}) := (p_i^\gamma(d_\gamma))_{\gamma\in\mathit{\Gamma}}$. Notice that p_i is well defined because $p_i^\gamma(d_\gamma) = p_i^\gamma(g_{\gamma\mu}(d_\mu)) = g_{\gamma\mu}(p_i^\mu(d_\mu))$ for all $\gamma\le\mu$. Further, one easily sees that p_i is a projection. Given $i,j\in I$ and $(d_\gamma)_{\gamma\in\mathit{\Gamma}}\in D_\infty$, we have

$$p_{\inf\{i,j\}}((d_\gamma)_{\gamma\in\mathit{\Gamma}}) = (p^\gamma_{\inf\{i,j\}}(d_\gamma))_{\gamma\in\mathit{\Gamma}} = (p_i^\gamma(p_j^\gamma(d_\gamma)))_{\gamma\in\mathit{\Gamma}} = p_i(p_j((d_\gamma)_{\gamma\in\mathit{\Gamma}})).$$

Moreover, $i\le j$ implies $p_i\le p_j$. Hence, $\mathcal{D}_\infty := (D_\infty,\le,(p_i)_{i\in I})$ is a strict $(I,\le)$-pop. If all $\mathcal{D}_\gamma$ are approximating, then $\mathcal{D}_\infty$ is also approximating.

The maps f_γ and g_γ turn out to be homomorphisms: for all $d\in D_\gamma$ and all $\mu,\nu\in\mathit{\Gamma}$ with $\nu\ge\gamma,\mu$ we have $p_i^\mu(g_\mu(f_\gamma(d))) = p_i^\mu(g_{\mu\nu}(f_{\gamma\nu}(d))) = g_{\mu\nu}(f_{\gamma\nu}(p_i^\gamma(d))) = g_\mu(f_\gamma(p_i^\gamma(d)))$. By definition of the projections p_i we obtain $p_i\circ f_\gamma = f_\gamma\circ p_i^\gamma$. On the other hand, $p_i^\gamma(g_\gamma((d_\mu)_{\mu\in\mathit{\Gamma}})) = p_i^\gamma(d_\gamma) = g_\gamma(p_i((d_\mu)_{\mu\in\mathit{\Gamma}}))$ for all $(d_\mu)_{\mu\in\mathit{\Gamma}}$, i.e. $p_i^\gamma\circ g_\gamma = g_\gamma\circ p_i$. We see that (f_γ,g_γ) is an epp of Scott-continuous homomorphisms.

Finally, for all $i\in I$ the projection p_i is Scott-continuous. To see this, let $A\subseteq D_\infty$ be directed. Then $p_i(\sup A) = (p_i^\gamma(g_\gamma(\sup A)))_{\gamma\in\mathit{\Gamma}} = (\sup p_i^\gamma[g_\gamma[A]])_{\gamma\in\mathit{\Gamma}} = (\sup g_\gamma[p_i[A]])_{\gamma\in\mathit{\Gamma}}$ $= \sup p_i[A]$. Consequently, $\mathcal{D}_\infty\in\mathcal{C}$.

We call $\mathcal{D}_\infty$ the *inverse limit* of the family $(\mathcal{D}_\gamma)_{\gamma\in\mathit{\Gamma}}$ with respect to the inverse system $\mathcal{S}$ and denote it by $\varprojlim_{\gamma\in\mathit{\Gamma}}(\mathcal{D}_\gamma,\mathcal{S})$. The epp's (f_γ,g_γ) are the *limiting epp's.*

Again, let $(\mathit{\Gamma},\le)$ be directed and let $\mathcal{D}_\gamma = (D_\gamma,\le_\gamma,(p_i^\gamma)_{i\in I})\in\mathcal{C}$ for all $\gamma\in\mathit{\Gamma}$. Let $\mathcal{S}=\{(f_{\gamma\mu},g_{\gamma\mu}) : D_\gamma\to D_\mu \mid \gamma,\mu\in\mathit{\Gamma},\ \gamma\le\mu\}$ be an inverse system of epp's of Scott-continuous homomorphisms. Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I}) \in \mathcal{C}$ and for $\gamma \in \Gamma$ let $(F_\gamma, G_\gamma) : D_\gamma \to D$ be an epp of Scott-continuous homomorphisms. Suppose that for all $\gamma \leq \mu$ we have $F_\gamma = F_\mu \circ f_{\gamma\mu}$ and $G_\gamma = g_{\gamma\mu} \circ G_\mu$. Then $(\mathcal{D}, (F_\gamma, G_\gamma)_{\gamma\in\Gamma})$ is said to *satisfy the universal property for inverse limits* (with respect to $(\mathcal{D}_\gamma)_{\gamma\in\Gamma}$ and $\mathcal{S}$) if for all $\mathcal{E} = (E, \leq, (q_i)_{i\in I}) \in \mathcal{C}$ and all epp's $(F'_\gamma, G'_\gamma) : D_\gamma \to E$ of Scott-continuous homomorphisms there is a unique epp $(F, G) : D \to E$ of Scott-continuous homomorphisms such that $F'_\gamma = F \circ F_\gamma$ and $G'_\gamma = G_\gamma \circ G$ for all $\gamma \in \Gamma$ (cf. Figure 4.4).

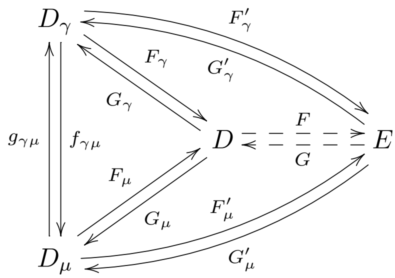


Fig. 4.4. The universal property for inverse limits

With this notation we obtain:

4.39. LEMMA. *If* $\sup_{\gamma\in\Gamma}(F_\gamma \circ G_\gamma) = \mathsf{id}_D$, *then* $(\mathcal{D}, (F_\gamma, G_\gamma)_{\gamma\in\Gamma})$ *satisfies the universal property for inverse limits. In this case we have* $F = \sup_{\gamma\in\Gamma}(F'_\gamma \circ G_\gamma)$ *and* $G = \sup_{\gamma\in\Gamma}(F_\gamma \circ G'_\gamma)$.

Proof. From domain theory it is known that $(F, G) := (\sup_{\gamma\in\Gamma}(F'_\gamma \circ G_\gamma), \sup_{\gamma\in\Gamma}(F_\gamma \circ G'_\gamma))$ is the unique Scott-continuous epp such that $F'_\gamma = F \circ F_\gamma$ and $G'_\gamma = G_\gamma \circ G$ for all $\gamma \in \Gamma$ (cf. e.g. [1, Theorem 3.3.7 and Lemma 3.3.8]). Thus, it remains to show that F and G commute with all projections. But this is clear since F_γ, G_γ, F'_γ, G'_γ are homomorphisms and all projections under consideration are Scott-continuous. ■

The converse of the previous lemma is also true. This follows from Lemma 1.4(3) and the following proposition which implies that the term "universal property for inverse limits" is justified:

4.40. PROPOSITION. *Let* $\mathcal{C} \in \{\mathsf{POP}^{\mathrm{Shom}}_{(I,\leq)}, \mathsf{APOP}^{\mathrm{Shom}}_{(I,\leq)}\}$. *Let* $\mathcal{D}_\infty = (D_\infty, \leq, (p_i)_{i\in I})$ *be the inverse limit of a family* $(\mathcal{D}_\gamma)_{\gamma\in\Gamma}$ *of* $(I, \leq)$*-pop's in* $\mathcal{C}$ *with respect to an inverse system* $\mathcal{S} = \{(f_{\gamma\mu}, g_{\gamma\mu}) : D_\gamma \to D_\mu \mid \gamma, \mu \in \Gamma,\ \gamma \leq \mu\}$ *of epp's of Scott-continuous homomorphisms. Then* $(\mathcal{D}_\infty, (f_\gamma, g_\gamma)_{\gamma\in\Gamma})$ *satisfies the universal property for inverse limits. If* $(\mathcal{D}, (F_\gamma, G_\gamma)_{\gamma\in\Gamma})$, *too, satisfies the universal property for inverse limits* (*with respect to* $(\mathcal{D}_\gamma)_{\gamma\in\Gamma}$ *and* $\mathcal{S}$ *as above*), *then there is a unique pop isomorphism* $\Phi : D_\infty \to D$ *such that* $F_\gamma = \Phi \circ f_\gamma$ *and* $G_\gamma = g_\gamma \circ \Phi^{-1}$. *We have* $\Phi = \sup_{\gamma\in\Gamma}(F_\gamma \circ g_\gamma)$ *and* $\Phi^{-1} = \sup_{\gamma\in\Gamma}(f_\gamma \circ G_\gamma)$.

Proof. The first assertion follows from Lemma 1.4(2),(3) and the first part of Lemma 4.39. The rest results from the second part of Lemma 4.39 by exploiting the universal property in the usual way. ■

Consider again $\mathcal{C} \in \{\mathsf{POP}^{\text{Shom}}_{(I,\leq)}, \mathsf{APOP}^{\text{Shom}}_{(I,\leq)}\}$ and recall that for $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ and $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ in $\mathcal{C}$ the exponential object $\mathcal{E}^{\mathcal{D}}$ is the $(I, \leq)$-pop $(E^D, \leq, (R_i)_{i\in I})$. The set E^D consists of all Scott-continuous homomorphisms from $\mathcal{J} \times \mathcal{D}$ to $\mathcal{E}$ with $\mathcal{J} = (J, \leq, (u_i)_{i\in I})$, $J = \mathsf{Id}(I) \mathbin{\dot\cup} \{\top\}$ if $\mathcal{C} = \mathsf{POP}^{\text{Shom}}_{(I,\leq)}$ and $J = \mathsf{Id}(I)$ if $\mathcal{C} = \mathsf{APOP}^{\text{Shom}}_{(I,\leq)}$ (cf. the previous section). In the latter case I is the greatest element of J and we set $\top := I$.

Let $\mathcal{C}^{\mathsf{epp}}$ be the category whose objects are precisely those of $\mathcal{C}$ and whose morphisms are epp's of Scott-continuous homomorphisms (i.e. epp's of morphisms of $\mathcal{C}$). We define a functor exp on $\mathcal{C}^{\mathsf{epp}}$ as follows: If $\mathcal{D}$ is an object of $\mathcal{C}^{\mathsf{epp}}$, then let $\mathsf{exp}(\mathcal{D}) := \mathcal{D}^{\mathcal{D}}$. If (f, g) is a morphism of $\mathcal{C}^{\mathsf{epp}}$, then define $\mathsf{exp}(f, g) := (f', g')$ where $f' : D^D \to E^E$ is defined by $f'(h) := f \circ h \circ (\mathsf{id}_J \times g)$ for all $h \in D^D$ and $g' : E^E \to D^D$ by $g'(k) := g \circ k \circ (\mathsf{id}_J \times f)$ for all $k \in E^E$. It is routine to check that f' and g' are Scott-continuous. Moreover, for $h \in D^D$ and $k \in E^E$ we have

$$g'(f'(h)) = g \circ (f \circ h \circ (\mathsf{id}_J \times g)) \circ (\mathsf{id}_J \times f) = (g \circ f) \circ h \circ (\mathsf{id}_J \times (g \circ f)) = h,$$
$$f'(g'(k)) = f \circ (g \circ k \circ (\mathsf{id}_J \times f)) \circ (\mathsf{id}_J \times g) = (f \circ g) \circ k \circ (\mathsf{id}_J \times (f \circ g)) \leq k.$$

For $i \in I$ we infer, by definition of the projections R_i, that

$$\begin{aligned} R_i(f'(h)) &= f'(h) \circ (u_i \times \mathsf{id}_D) = f \circ h \circ (\mathsf{id}_J \times g) \circ (u_i \times \mathsf{id}_D) \\ &= f \circ h \circ (u_i \times \mathsf{id}_D) \circ (\mathsf{id}_J \times g) = f \circ R_i(h) \circ (\mathsf{id}_J \times g) = f'(R_i(h)) \end{aligned}$$

and, analogously, $R_i(g'(k)) = g'(R_i(k))$. Hence, (f', g') is a morphism in $\mathcal{C}^{\mathsf{epp}}$.

Let $(\Gamma, \leq)$ be directed, let $\mathcal{D}_\gamma = (D_\gamma, \leq_\gamma, (p_i^\gamma)_{i\in I}) \in \mathcal{C}^{\mathsf{epp}}$ for all $\gamma \in \Gamma$, and let $\mathcal{S} = \{(f_{\gamma\mu}, g_{\gamma\mu}) : D_\gamma \to D_\mu \in \mathcal{C}^{\mathsf{epp}} \mid \gamma, \mu \in \Gamma,\ \gamma \leq \mu\}$ be an inverse system. By Proposition 4.40 we know that $(\varprojlim_{\gamma\in\Gamma}(\mathcal{D}_\gamma, \mathcal{S}), (f_\gamma, g_\gamma)_{\gamma\in\Gamma})$ satisfies the universal property for inverse limits with respect to $(\mathcal{D}_\gamma)_{\gamma\in\Gamma}$ and $\mathcal{S}$.

Now set

$$\mathsf{exp}(\mathcal{S}) := \{\mathsf{exp}(f_{\gamma\mu}, g_{\gamma\mu}) \mid \gamma, \mu \in \Gamma,\ \gamma \leq \mu\}.$$

Clearly, this yields an inverse system of morphisms of $\mathcal{C}^{\mathsf{epp}}$. The corresponding inverse limit is $\varprojlim_{\gamma\in\Gamma}(\mathsf{exp}(\mathcal{D}_\gamma), \mathsf{exp}(\mathcal{S}))$.

Let (F_γ, G_γ) be the respective limiting epp's. Again by Proposition 4.40, $(\varprojlim_{\gamma\in\Gamma}(\mathsf{exp}(\mathcal{D}_\gamma), \mathsf{exp}(\mathcal{S})), (F_\gamma, G_\gamma)_{\gamma\in\Gamma})$ satisfies the universal property for inverse limits with respect to $(\mathsf{exp}(\mathcal{D}_\gamma))_{\gamma\in\Gamma}$ and $\mathsf{exp}(\mathcal{S})$.

The following statement tells us that exp is *continuous*, i.e. $\mathsf{exp}(\varprojlim_{\gamma\in\Gamma}(\mathcal{D}_\gamma, \mathcal{S}))$ is pop isomorphic to $(\varprojlim_{\gamma\in\Gamma}(\mathsf{exp}(\mathcal{D}_\gamma), \mathsf{exp}(\mathcal{S})))$.

4.41. PROPOSITION. *With the notation above*, $(\mathsf{exp}(\varprojlim_{\gamma\in\Gamma}(\mathcal{D}_\gamma, \mathcal{S})), (\mathsf{exp}(f_\gamma, g_\gamma))_{\gamma\in\Gamma})$ *satisfies the universal property for inverse limits with respect to* $(\mathsf{exp}(\mathcal{D}_\gamma))_{\gamma\in\Gamma}$ *and* $\mathsf{exp}(\mathcal{S})$. *In particular, the functor* exp *is continuous.*

Proof. As usual, let $D_\infty := \{(d_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} D_\gamma \mid \forall \gamma, \mu \in \Gamma : \gamma \leq \mu \Rightarrow d_\gamma = g_{\gamma\mu}(d_\mu)\}$. For $\gamma \in \Gamma$ let $(f'_\gamma, g'_\gamma) := \mathsf{exp}(f_\gamma, g_\gamma)$. Let $h \in D_\infty^{D_\infty}$. As we have seen above, $f'_\gamma(g'_\gamma(h)) = (f_\gamma \circ g_\gamma) \circ h \circ (\mathsf{id}_J \times (f_\gamma \circ g_\gamma))$. We deduce from Lemma 1.4(3) that $\sup_{\gamma\in\Gamma} f'_\gamma(g'_\gamma(h)) = \sup_{\gamma\in\Gamma}(f_\gamma \circ g_\gamma) \circ h \circ (\mathsf{id}_J \times \sup_{\gamma\in\Gamma}(f_\gamma \circ g_\gamma)) = h$. Therefore, $\sup_{\gamma\in\Gamma}(f'_\gamma \circ g'_\gamma) = \mathsf{id}_{D_\infty^{D_\infty}}$. By Lemma 4.39 the first assertion follows. The second is a consequence of Proposition 4.40. ■

So far we have gathered all pieces for a D_∞-construction à la Scott on the level of strict $(I,\leq)$-pop's that are also dcpo's. More precisely, we state the following theorem:

4.42. THEOREM. *Let $\mathcal{C} \in \{\mathsf{POP}^{\mathrm{Shom}}_{(I,\leq)}, \mathsf{APOP}^{\mathrm{Shom}}_{(I,\leq)}\}$. Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be any object of $\mathcal{C}$ that has a least element $\perp$. Let $f_0 : D \to D^D$ be defined by $f_0(d) : J \times D \to D$,*

$$f_0(d)(A,e) := \begin{cases} \sup_{a\in A} p_a(d) & \text{if } A \in \mathsf{Id}(I), \\ d & \text{if } A = \top. \end{cases}$$

Define $g_0 : D^D \to D$ as follows:

$$g_0(h) := h(\top, \perp) = \mathrm{ev}(h, \perp).$$

Then (f_0, g_0) is an epp of Scott-continuous homomorphisms. Moreover, let $\mathcal{D}_0 := \mathcal{D}$. For all $n \in \mathbb{N}_0$ let $\mathcal{D}_{n+1} := \mathsf{exp}(\mathcal{D}_n)$ and $(f_{n+1}, g_{n+1}) := \mathsf{exp}(f_n, g_n)$. For all $m, n \in \mathbb{N}_0$ with $m \leq n$ let $f_{mn} := f_{n-1} \circ f_{n-2} \circ \cdots \circ f_{m+1} \circ f_m$ and $g_{mn} := g_m \circ g_{m+1} \circ \cdots \circ g_{n-2} \circ g_{n-1}$. Let $\mathcal{S} := \{(f_{mn}, g_{mn}) \mid m,n \in \mathbb{N}_0,\ m \leq n\}$. Then $\mathcal{D}_\infty := \varprojlim_{n\in\mathbb{N}_0}(\mathcal{D}_n, \mathcal{S})$ is pop isomorphic to $\mathsf{exp}(\mathcal{D}_\infty)$.

Proof. First of all, we verify that (f_0, g_0) is a morphism in $\mathcal{C}^{\mathsf{epp}}$. Let $d \in D$. Clearly, if $\mathcal{C} = \mathsf{APOP}^{\mathrm{Shom}}_{(I,\leq)}$, then $\top = I$ and $\sup_{a\in I} p_a(d) = d$, whence $f_0(d)$ is well defined. To show that $f_0(d)$ is Scott-continuous, let the sets $\mathcal{A} \subseteq \mathsf{Id}(I)$ and $B \subseteq D$ be directed. Then $f_0(d)(\bigcup \mathcal{A}, \sup B) = \sup_{a \in \bigcup \mathcal{A}} p_a(d) = \sup_{A\in\mathcal{A}} \sup_{a\in A} p_a(d) = \sup_{(A,e)\in\mathcal{A}\times B} f_0(d)$. Further, for all $i \in I$ and all $A \in \mathsf{Id}(I)$ we have $\{p_i \circ p_a \mid a \in A\} = \{p_a \mid a \in A \cap i{\downarrow}\}$, cf. $(*)$ on p. 107. Hence, $p_i(f_0(d)(A,e)) = p_i(\sup_{a\in A} p_a(d)) = \sup_{a\in A} p_i(p_a(d)) = \sup_{a\in A\cap i\downarrow} p_a(d) = f_0(d)(A \cap i{\downarrow}, p_i(e)) = f_0(d)(u_i(A), p_i(e))$ and $p_i(f_0(d)(\top, e)) = p_i(d) = \sup_{a\in i\downarrow} p_a(d) = f_0(d)(i{\downarrow}, p_i(e)) = f_0(d)(u_i(\top), p_i(e))$. Thus, $p_i \circ f_0(d) = f_0(d) \circ (u_i \times p_i)$. This shows us that $f_0(d)$ is a Scott-continuous homomorphism and f_0 is well defined. Obviously, f_0 is Scott-continuous. Moreover,

$$R_i(f_0(d))(A,e) = f_0(d)(A \cap i{\downarrow}, e) = \sup_{a\in A\cap i\downarrow} p_a(d) = \sup_{a\in A} p_a(p_i(d)) = f_0(p_i(d))(A,e),$$
$$R_i(f_0(d))(\top,e) = f_0(d)(i{\downarrow}, e) = \sup_{a\in i\downarrow} p_a(d) = p_i(d) = f_0(p_i(d))(\top,e).$$

That is, $R_i \circ f_0 = f_0 \circ p_i$ and f_0 is a homomorphism.

Since ev and the constant map sending all elements of D to $\perp$ are Scott-continuous homomorphisms, so is g_0.

Finally, $g_0(f_0(d)) = f_0(d)(\top, \perp) = d$ and, furthermore,

$$\begin{aligned} f_0(g_0(h))(A,e) &= f_0(h(\top,\perp))(A,e) = \sup_{a\in A} p_a(h(\top,\perp)) = \sup_{a\in A} h(u_a(\top), p_a(\perp)) \\ &= \sup_{a\in A} h(a{\downarrow}, \perp) = h(\bigcup_{a\in A} a{\downarrow}, \perp) = h(A,\perp) \leq h(A,e), \\ f_0(g_0(h))(\top,e) &= f_0(h(\top,\perp))(\top,e) = h(\top,\perp) \leq h(\top,e) \end{aligned}$$

for all $d, e \in D$, $A \in \mathsf{Id}(I)$, $h \in D^D$. As a consequence, $(f_0, g_0) \in \mathcal{C}^{\mathsf{epp}}$.

The following conclusions are standard arguments from category theory. However, we mention them here for the sake of completeness. They were first developed by Scott [48] for continuous lattices.

Since $\mathcal{D}_0 = \mathcal{D}$ and (f_0, g_0) are in $\mathcal{C}^{\mathsf{epp}}$, we may apply the functor exp consecutively to $\mathcal{D}_0$ and (f_0, g_0) to obtain $\mathcal{D}_1 = \mathsf{exp}(\mathcal{D}_0)$ and $(f_1, g_1) = \mathsf{exp}(f_0, g_0)$, to $\mathcal{D}_1$ and (f_1, g_1) to obtain $\mathcal{D}_2 = \mathsf{exp}(\mathcal{D}_1)$ and $(f_2, g_2) = \mathsf{exp}(f_1, g_1)$, and so forth. If for $m \leq n$ the maps f_{mn} and g_{mn} are defined as in the theorem, then we clearly obtain an inverse system $\mathcal{S} = \{(f_{mn}, g_{mn}) \mid m, n \in \mathbb{N}_0,\ m \leq n\}$ of morphisms of $\mathcal{C}^{\mathsf{epp}}$. Let $\mathcal{S}_{+1} := \{(f_{mn}, g_{mn}) \mid m, n \in \mathbb{N},\ m \leq n\}$ and let "$\cong$" be an abbreviation for "is pop isomorphic to". Since exp is continuous (Proposition 4.41), we infer $\varprojlim_{n\in\mathbb{N}_0}(\mathcal{D}_n, \mathcal{S}) \cong \varprojlim_{n\in\mathbb{N}_0}(\mathcal{D}_{n+1}, \mathcal{S}_{+1}) = \varprojlim_{n\in\mathbb{N}_0}(\mathsf{exp}(\mathcal{D}_n), \mathsf{exp}(\mathcal{S})) \cong \mathsf{exp}(\varprojlim_{n\in\mathbb{N}_0}(\mathcal{D}_n, \mathcal{S}))$. ■

Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I}) \in \mathcal{C}$ and let $I = \{\star\}$. Assume further that $p_\star = \mathsf{id}_D$. Then $\mathcal{D} = (D, \leq, (\mathsf{id}_D))$ can be identified with the dcpo $(D, \leq)$. Since $\mathsf{Id}(I) = \{\star\}$, we have $D^D = [\{\star\} \times D \to D]^{\mathrm{Shom}}$, $u_\star = \mathsf{id}_{\{\star\}}$, and $R_\star = \mathsf{id}_{D^D}$. We see that any Scott-continuous map from $\{\star\}\times D$ to D is a homomorphism. Consequently, $\mathsf{exp}(\mathcal{D}) = \mathcal{D}^{\mathcal{D}}$ may be identified with the dcpo of all Scott-continuous mappings from D to itself. Assume $(D, \leq)$ to have a least element $\perp$ and consider the epp (f_0, g_0) of Theorem 4.42. With the identifications made above, $f_0(d)$ is the constant map sending all elements of D to d, and g_0 maps any Scott-continuous mapping h to $h(\perp)$. Hence, in this special case the construction in Theorem 4.42 boils down to Scott's original D_∞-construction (cf. [48]).

5. COMPLETION OF POSETS WITH PROJECTIONS

Completions of mathematical structures emerge in almost all branches of mathematics. Well known are completions of metric spaces, uniform spaces, normed linear spaces, lattices, the algebraic completion of fields, the ideal completion of posets, and so forth. In [6] Bonsangue, van Breugel, and Rutten investigated the completion of generalized metric spaces. This yields a generalization both of the chain completion of (pre)ordered sets and of the metric Cauchy completion.

In a certain sense, a completion of a structure is unique and "small". It is the "smallest" structure satisfying certain nice properties such that the original structure is contained in it as a substructure.

In the present chapter we consider completions of posets with projections. Since $(I, \leq)$-pop's have both a uniformity and a partial order, we can form either the uniform completion with respect to the pop uniformity to get a Hausdorff and complete uniform space that contains the original one as a dense subspace, or the ideal completion of the poset (or a suitable subset of it) to obtain an algebraic domain. In fact, we deal with both kinds of completions and equip them with an $(I, \leq)$-pop structure again. Then we investigate how they are related.

In Section 5.1 we establish both the existence and the uniqueness of the *pop completion* (Theorem 5.4). Given any approximating $(I, \leq)$-pop $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$, there exists (up to isomorphism) a unique approximating $(I, \leq)$-pop $\mathsf{C}(\mathcal{D}) = (\mathsf{C}(D), \widehat{\leq}, (\widehat{p}_i)_{i\in I})$ with the following properties:

- $\mathsf{C}(D)$ is complete in its pop uniformity,
- $\mathcal{D}$ is a subpop of $\mathsf{C}(\mathcal{D})$ such that D is dense in $\mathsf{C}(D)$ with respect to the pop topology,
- any homomorphism f from $\mathcal{D}$ into a complete approximating $(I, \leq)$-pop $\mathcal{E}$ can be extended uniquely to a homomorphism $\overline{f}$ from $\mathsf{C}(\mathcal{D})$ to $\mathcal{E}$.

In particular, $\mathsf{C}(D)$ is the uniform completion of D with respect to the pop uniformity. The completion $\mathsf{C}(\mathcal{D})$ can be constructed as the limit of an inverse system consisting of the sets $p_i[D]$ and the projections p_i $(i \in I)$ (cf. also Lemma 3.39 and Theorem 3.40). We study the fundamental properties of the pop completion and investigate which properties of the original $(I, \leq)$-pop are transferred to its completion. In particular, the pop completion of a function pop turns out to be the function space of the respective pop completions (Theorem 5.18).

On the other hand, by taking the ideal completion of $\bigcup_{i\in I} p_i[D]$, we obtain in Section 5.2 a unique (up to isomorphism) approximating $(I, \leq)$-pop $\mathsf{J}(\mathcal{D}) = (\mathsf{J}(D), \widetilde{\leq}, (\widetilde{p}_i)_{i\in I})$

such that

- $(\mathsf{J}(D), \widetilde{\leq})$ is an algebraic dcpo and $\widetilde{p}_i$ is Scott-continuous for all $i \in I$ (well known),
- $\mathcal{D}$ is a subpop of $\mathsf{J}(\mathcal{D})$,
- any homomorphism f from $\mathcal{D}$ into an approximating $(I, \leq)$-pop $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ with $(E, \leq)$ a dcpo and q_i Scott-continuous for all $i \in I$ can be extended uniquely to a Scott-continuous homomorphism f^* from $\mathsf{J}(\mathcal{D})$ to $\mathcal{E}$

(see Theorem 5.22). We call $\mathsf{J}(\mathcal{D})$ the *domain completion* of $\mathcal{D}$.

There is a close relation between both completions. In Section 5.3 we show that $\mathsf{C}(\mathcal{D})$ can always be embedded into $\mathsf{J}(\mathcal{D})$; hence $\mathsf{C}(\mathcal{D})$ may be seen as a subpop of $\mathsf{J}(\mathcal{D})$ (Theorem 5.29). We also give several conditions leading to a canonical isomorphism between $\mathsf{C}(\mathcal{D})$ and $\mathsf{J}(\mathcal{D})$ (Theorem 5.33). For instance, if $p_i[A]$ has a greatest element for all ideals $A \subseteq D$ and all $i \in I$, then $\mathsf{C}(\mathcal{D})$ and $\mathsf{J}(\mathcal{D})$ are isomorphic. Finally, pop completion and domain completion coincide and yield a bifinite domain provided all p_i have finite range (Theorem 5.36).

Some results of the present chapter can be found in [34].

5.1. Existence and uniqueness of the pop completion

We begin with the definition of a special class of subpop's. Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop and let $\mathcal{X} = (X, \leq, (p_i|_X)_{i\in I})$ be a subpop of $\mathcal{D}$. Then $\mathcal{X}$ is a *full subpop* of $\mathcal{D}$ provided that $p_i[D] \subseteq X$ for all $i \in I$.

5.1. Example. Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop. The set $\bigcup_{i\in I} p_i[D]$ induces a full subpop of $\mathcal{D}$. Obviously, this is the least full subpop of $\mathcal{D}$. In view of Lemma 3.1, it is approximating. We denote this subpop by $\bigcup_{i\in I} p_i[\mathcal{D}]$.

Full subpop's of $\mathcal{D}$ are precisely those subpop's that are dense in D with respect to the pop topology. This is the subject of the next lemma:

5.2. Lemma. *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop. Let $\mathcal{X} = (X, \leq, (p_i|_X)_{i\in I})$ be a subpop of $\mathcal{D}$. Then $\mathcal{X}$ is a full subpop if and only if X is dense in $(D, \tau_{\mathcal{D}})$.*

Proof. If $\mathcal{X}$ is full, then X is a dense subset of D with respect to the pop topology by Proposition 2.10(1). Conversely, let X be dense in D, let $d \in D$, and let $i \in I$. Then we find some $x \in X$ with $p_i(d) = p_i(x) \in X$, whence $\mathcal{X}$ is a full subpop. ∎

The following $(I, \leq)$-pop turns out to be the proper candidate for the pop completion:

5.3. Proposition. *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop. Let*

$$D_\infty := \left\{ (d_i)_{i\in I} \in \prod_{i\in I} p_i[D] \;\middle|\; \forall i, j \in I : i \leq j \Rightarrow d_i = p_i(d_j) \right\}$$

be equipped with the product order. For all $i \in I$ define a map $r_i : D_\infty \to D_\infty$ by $r_i((d_j)_{j\in I}) := (p_j(d_i))_{j\in I}$. Then $\mathcal{D}_\infty := (D_\infty, \leq, (r_i)_{i\in I})$ is a complete approximating $(I, \leq)$-pop. If $(p_i)_{i\in I}$ is Abelian, then $(r_i)_{i\in I}$ is Abelian and $r_i((d_j)_{j\in I}) = (p_i(d_j))_{j\in I}$ for all $i \in I$ and all $(d_j)_{j\in I} \in D_\infty$. If p_i is Scott-continuous for all $i \in I$, then r_i is Scott-continuous for all $i \in I$.

Proof. Recall that $(\mathsf{id}_{p_i[D],p_j[D]}, p_i|_{p_j[D]})$ is an embedding projection pair for all $i, j \in I$ with $i \leq j$ and $\mathcal{S} = \{(\mathsf{id}_{p_i[D],p_j[D]}, p_i|_{p_j[D]}) \mid i, j \in I,\ i \leq j\}$ is an inverse system (cf. Theorem 3.40(1)). For all $i \in I$ let $f_i : p_i[D] \to D_\infty$ be defined by $f_i(d) := (p_j(d))_{j\in I}$. Let $g_i : D_\infty \to p_i[D]$ be the canonical projection from D_∞ onto $p_i[D]$. By Theorem 3.40(2), $(D_\infty, \leq, (f_i \circ g_i)_{i\in I})$ is a complete approximating $(I, \leq)$-pop. Further, $(f_i \circ g_i)((d_j)_{j\in I}) = f_i(d_i) = (p_j(d_i))_{j\in I} = r_i((d_j)_{j\in I})$ for all $i \in I$.

Now let $(p_i)_{i\in I}$ be Abelian and let $i, j \in I$. Choose some $k \in I$ with $k \geq i, j$ and recall that $d_i = p_i(d_k)$ and $d_j = p_j(d_k)$. Then $p_j(d_i) = p_j(p_i(d_k)) = p_i(p_j(d_k)) = p_i(d_j)$. As a consequence, $(r_i)_{i\in I}$ is Abelian.

Finally, assume that p_i is Scott-continuous for all $i \in I$. Then $(\mathsf{id}_{p_i[D],p_j[D]}, p_i|_{p_j[D]})$ is a Scott-continuous embedding projection pair for all $i \leq j$ by Lemma 1.3. Consequently, f_i and g_i are Scott-continuous in view of Lemma 1.4. Therefore, $r_i = f_i \circ g_i$ is also Scott-continuous. ∎

Suppose that $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ is an object of the category $\mathsf{POP}^{\text{Shom}}_{(I,\leq)}$. For $i \in I$ let $\mathcal{D}_i$ be the subpop of $\mathcal{D}$ induced by $p_i[D]$ (cf. Lemma 4.1). Then the $(I, \leq)$-pop $\mathcal{D}_\infty$ of Proposition 5.3 coincides with the inverse limit of $(\mathcal{D}_i)_{i\in I}$ with respect to the inverse system $\{(\mathsf{id}_{p_i[D],p_j[D]}, p_i|_{p_j[D]}) \mid i, j \in I,\ i \leq j\}$, which we have defined in Section 4.3.4.

Next, we formulate our existence and uniqueness theorem of the "pop completion". The approach to obtain the completion using an inverse limit construction is similar to the one given by Ehrig *et al.* [17, Theorem 1.14]. This result states the existence and uniqueness of a universal completion of (Hausdorff) projection spaces.

$$\begin{tikzcd}
\mathsf{C}(D) \arrow[dr, dashed, "\exists_1 \overline{f}"] & \\
D \arrow[u, "\psi"] \arrow[r, "f"'] \arrow[ur, phantom, "="] & E
\end{tikzcd}$$

Fig. 5.1. The universal property for the pop completion

5.4. THEOREM. *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop.*

(1) *There exist a complete approximating $(I, \leq)$-pop $\mathsf{C}(\mathcal{D}) = (\mathsf{C}(D), \widehat{\leq}, (\widehat{p}_i)_{i\in I})$ and a pop homomorphism $\psi : D \to \mathsf{C}(D)$ with the following universal property* (*Figure* 5.1): *For any complete approximating $(I, \leq)$-pop $(E, \leq, (q_i)_{i\in I})$ and any weak homomorphism $f : D \to E$ there is a unique weak homomorphism $\overline{f} : \mathsf{C}(D) \to E$ with $\overline{f} \circ \psi = f$. Moreover, if f is a homomorphism, then $\overline{f}$ is also a homomorphism.*

(2) *Let $\mathcal{D}' = (D', \leq', (p'_i)_{i\in I})$ be a complete approximating $(I, \leq)$-pop and let $\phi : D \to D'$ be a weak homomorphism such that $(\mathcal{D}', \phi)$ satisfies the universal property of* (1). *Then there exists a unique pop isomorphism $\varPhi : \mathsf{C}(D) \to D'$ with $\varPhi \circ \psi = \phi$.*

(3) *$\psi[D]$ induces a full subpop of $\mathsf{C}(\mathcal{D})$ and is dense in $(\mathsf{C}(D), \tau_{\mathsf{C}(\mathcal{D})})$.*

(4) (a) *Let $d, e \in D$. Then $\psi(d) \mathrel{\widehat{\leq}} \psi(e)$ if and only if $p_i(d) \leq p_i(e)$ for all $i \in I$.*
(b) *$\psi|_{p_i[D]}$ is an order isomorphism from $p_i[D]$ onto $\widehat{p_i}[\mathsf{C}(D)]$ for all $i \in I$.*

(5) *ψ is a pop embedding of $\mathcal{D}$ into $\mathsf{C}(\mathcal{D})$ if and only if $\mathcal{D}$ is approximating.*

(6) (a) *If $(p_i)_{i\in I}$ is Abelian, then so is $(\widehat{p_i})_{i\in I}$.*
(b) *If p_i is Scott-continuous for all $i \in I$, then ψ and $\widehat{p_i}$ are Scott-continuous for all $i \in I$.*

(7) *$\mathsf{C}(\mathcal{D})$ is pop isomorphic to $\mathcal{D}_\infty$ (cf. Proposition 5.3). More precisely, there is a unique pop isomorphism $\Psi : \mathsf{C}(D) \to \mathcal{D}_\infty$ with $(\Psi \circ \psi)(d) = (p_i(d))_{i\in I}$ for all $d \in D$.*

Proof. Let $\mathsf{C}(\mathcal{D}) := \mathcal{D}_\infty$, i.e. $(\mathsf{C}(D), \widehat{\leq}) = (D_\infty, \leq)$ and $\widehat{p_i} = r_i$ for all $i \in I$. Proposition 5.3 tells us that $\mathsf{C}(\mathcal{D})$ is a complete approximating $(I, \leq)$-pop.

Let $\mathcal{V}$ be the uniformity on D_∞ that is induced by the product uniformity of the family $(p_i[D], \mathcal{U}_{\mathsf{dis}})_{i\in I}$. Let $i \in I$ and let $g_i : D_\infty \to p_i[D]$ be the canonical projection from D_∞ onto $p_i[D]$. One easily sees that $(g_i \times g_i)^{-1}[\mathsf{id}_{p_i[D]}] = \ker \widehat{p_i}$. We conclude that $\mathcal{V} = \mathcal{U}_{\mathsf{C}(\mathcal{D})}$. Let $\psi : D \to \mathsf{C}(D)$ be defined by $\psi(d) := (p_i(d))_{i\in I}$. From Lemma 3.39 we deduce that ψ is monotone and $\psi[D]$ is dense in $\mathsf{C}(D)$. Given $i \in I$, we have $\widehat{p_i}(\psi(d)) = r_i((p_j(d))_{j\in I}) = (p_j(p_i(d)))_{j\in I} = \psi(p_i(d))$, whence ψ is a homomorphism. In particular, $\psi[D]$ induces a subpop of $\mathsf{C}(\mathcal{D})$. It is full in view of Lemma 5.2, which proves (3).

Furthermore, let $\psi(d) \mathrel{\widehat{\leq}} \psi(e)$. Then, by definition of ψ, we have $p_i(d) \leq p_i(e)$ for all $i \in I$. Conversely, if $p_i(d) \leq p_i(e)$ for all $i \in I$, then $\widehat{p_i}(\psi(d)) = \psi(p_i(d)) \mathrel{\widehat{\leq}} \psi(p_i(e)) = \widehat{p_i}(\psi(e))$ for all $i \in I$. As $\mathsf{C}(\mathcal{D})$ is approximating, we conclude that $\psi(d) \mathrel{\widehat{\leq}} \psi(e)$. This proves (4)(a). To verify (4)(b), let $i \in I$. Since ψ is a homomorphism, we have $\psi[p_i[D]] = \widehat{p_i}[\psi[D]] \subseteq \widehat{p_i}[\mathsf{C}(D)]$. On the other hand, let $\widehat{d} \in \mathsf{C}(D)$. Then $\widehat{p_i}(\widehat{d}) \in \psi[D]$ by (3), whence $\widehat{p_i}(\widehat{d}) \in \widehat{p_i}[\psi[D]] = \psi[p_i[D]]$. We obtain $\psi[p_i[D]] = \widehat{p_i}[\mathsf{C}(D)]$. As $\psi|_{p_i[D]}$ is monotone and, in addition, order-reflecting by (4)(a), it is an order isomorphism from $p_i[D]$ onto $\widehat{p_i}[\mathsf{C}(D)]$.

Let ψ be a pop embedding, let $d, e \in D$, and let $p_i(d) \leq e$ for all $i \in I$. Then $\widehat{p_i}(\psi(d)) = \psi(p_i(d)) \mathrel{\widehat{\leq}} \psi(e)$ for all $i \in I$; hence $\psi(d) \mathrel{\widehat{\leq}} \psi(e)$ since $\mathsf{C}(\mathcal{D})$ is approximating. Then $d \leq e$ and thus $\mathcal{D}$ is approximating. Conversely, if $\mathcal{D}$ is approximating, then ψ is an order embedding by Lemma 3.39 and therefore a pop embedding. This shows (5).

(6) results from Proposition 5.3 (the proof that ψ be Scott-continuous is deferred to Proposition 5.8 below).

To prove the universal property, let $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be a complete approximating $(I, \leq)$-pop and let $f : D \to E$ be a weak homomorphism. Let $(d_i)_{i\in I} \in \mathsf{C}(D)$ and let $i, j \in I$ with $j \geq i$. Then $q_i(f(d_j)) = q_i(f(p_i(d_j))) = q_i(f(d_i))$. Consequently, $(f(d_i))_{i\in I}$ is a Cauchy net in $(E, \mathcal{U}_\mathcal{E})$. As $\mathcal{E}$ is complete Hausdorff, let $\overline{f}((d_i)_{i\in I}) := \lim_{i\in I} f(d_i)$. Now let $(d_i)_{i\in I}, (\widetilde{d_i})_{i\in I} \in \mathsf{C}(D)$ with $d_i \leq \widetilde{d_i}$ for all $i \in I$. Then $f(d_i) \leq f(\widetilde{d_i})$ for all $i \in I$ and thus $\overline{f}((d_i)_{i\in I}) \leq \overline{f}((\widetilde{d_i})_{i\in I})$ because the partial order of $\mathcal{E}$ is closed by Proposition 2.11. Hence, $\overline{f}$ is monotone. Note that since $d_i \in p_i[D]$, we have $d_i = p_j(d_i)$ for all $j \geq i$. Consequently,

$$q_i(f(d_j)) = q_i(f(p_i(d_j))) = q_i(f(d_i)) = q_i(f(p_j(d_i)))$$

for all $i, j \in I$, $j \geq i$. Since $\mathcal{E}$ is Hausdorff, we conclude

$$q_i(\overline{f}((d_j)_{j\in I})) = q_i(\overline{f}((p_j(d_i))_{j\in I})) = q_i(\overline{f}(\widehat{p_i}((d_j)_{j\in I})));$$

hence $\overline{f}$ is non-expansive and thus a weak homomorphism. A similar argument shows that $\overline{f}$ is a homomorphism whenever f is a homomorphism. Furthermore, $\overline{f}(\psi(d)) = \overline{f}((p_i(d))_{i\in I}) = \lim_{i\in I} f(p_i(d)) = f(d)$ because $(p_i(d))_{i\in I} \to d$ and f is continuous. As $\mathcal{E}$ is Hausdorff, we infer from (3) that $\overline{f}$ is the unique weak homomorphism with $\overline{f}\circ\psi = f$. This proves (1).

Next, we show that $(\mathsf{C}(\mathcal{D}),\psi)$ is "unique" by exploiting the universal property in the usual way. Let $\mathcal{D}' = (D', \leq', (p_i')_{i\in I})$ be a complete approximating $(I,\leq)$-pop and let $\phi : D \to D'$ be a weak homomorphism such that $(\mathcal{D}',\phi)$ satisfies the universal property of (1). Then there is a weak homomorphism $\Psi : D' \to \mathsf{C}(D)$ with $\Psi\circ\phi = \psi$. By applying the universal property to $(\mathsf{C}(\mathcal{D}),\psi)$, we find a weak homomorphism $\Phi : \mathsf{C}(D) \to D'$ with $\Phi\circ\psi = \phi$. Now $\Psi\circ\Phi$ is a weak homomorphism from $\mathsf{C}(D)$ into itself with $(\Psi\circ\Phi)\circ\psi = \Psi\circ\phi = \psi$. Since the identity $\mathsf{id}_{\mathsf{C}(D)}$ on $\mathsf{C}(D)$ is also a weak homomorphism with $\mathsf{id}_{\mathsf{C}(D)}\circ\psi = \psi$, we obtain $\Psi\circ\Phi = \mathsf{id}_{\mathsf{C}(D)}$ due to uniqueness. Analogously, $\Phi\circ\Psi = \mathsf{id}_{D'}$. Consequently, Φ is bijective and $\Phi^{-1} = \Psi$. Thus, Φ and Φ^{-1} are weak homomorphisms. In view of Lemma 4.1(5), we deduce that Φ is a pop isomorphism.

Finally, (7) is true by definition and (2). ∎

Definition. For any $(I,\leq)$-pop $\mathcal{D}$ we call the complete approximating $(I,\leq)$-pop $\mathsf{C}(\mathcal{D})$ of Theorem 5.4 the *pop completion* of $\mathcal{D}$. The mapping $\psi : D \to \mathsf{C}(D)$ is the *canonical homomorphism*.

Theorem 5.4(1) tells us that the pop completion actually satisfies *two* universal properties, viz. one for weak homomorphisms and one for homomorphisms. With respect to both universal properties, the pop completion is unique up to pop isomorphism. This is stated in 5.4(2) for the universal property concerning the extension of weak homomorphisms. But clearly, the proof of (2) shows us that this is also true in case we deal with the extension of homomorphisms.

In addition, we just remark here that $(\mathsf{C}(D),\mathcal{U}_{\mathsf{C}(\mathcal{D})})$ is the uniform completion of the space $(D,\mathcal{U}_{\mathcal{D}})$.

5.5. Corollary. *Let $\mathcal{D} = (D,\leq,(p_i)_{i\in I})$ be an $(I,\leq)$-pop with pop completion $\mathsf{C}(\mathcal{D}) = (\mathsf{C}(D),\widehat{\leq},(\widehat{p}_i)_{i\in I})$ and canonical homomorphism $\psi : D \to \mathsf{C}(D)$. Then $\psi|_{\bigcup_{i\in I} p_i[D]}$ is a pop isomorphism from the subpop $\bigcup_{i\in I} p_i[\mathcal{D}]$ onto the subpop $\bigcup_{i\in I}\widehat{p}_i[\mathsf{C}(\mathcal{D})]$. If, furthermore, $(p_i)_{i\in I}$ is Abelian, then $\psi|_{p_i[D]}$ is a pop isomorphism from the subpop induced by $p_i[D]$ onto the subpop induced by $\widehat{p}_i[\mathsf{C}(D)]$.*

Proof. By Theorem 5.4(1) and (4), ψ is a homomorphism with $\psi[\bigcup_{i\in I} p_i[D]] = \bigcup_{i\in I}\widehat{p}_i[\mathsf{C}(D)]$. Recall that $\psi(d) \mathrel{\widehat{\leq}} \psi(e)$ if and only if $p_i(d) \leq p_i(e)$ for all $i\in I$ (5.4(4)). Thus, given $i,j,k\in I$ with $k \geq i,j$, we infer that $\psi(p_i(d)) \mathrel{\widehat{\leq}} \psi(p_j(e))$ implies $p_i(d) = p_k(p_i(d)) \leq p_k(p_j(e)) = p_j(e)$ by Lemma 3.1.

Now let $(p_i)_{i\in I}$ be Abelian. Lemma 4.2 and Theorem 5.4(6)(a) tell us that $p_i[D]$ and $\widehat{p}_i[\mathsf{C}(D)]$ induce subpop's for all $i\in I$. They are isomorphic by virtue of 5.4(4). ∎

Let $\mathcal{D} = (D,\leq,(p_i)_{i\in I})$ be an approximating $(I,\leq)$-pop. In light of Theorem 5.4(5), we identify $\mathcal{D}$ with the subpop induced by $\psi[D]$. Thus, we assume that $D\subseteq \mathsf{C}(D)$, $\widehat{\leq}|_D = \leq$, $\psi = \mathsf{id}_{D,\mathsf{C}(D)}$, and $\widehat{p}_i|_D = p_i$ for all $i\in I$. Note that for all $i\in I$ we have

$\widehat{p}_i[\mathsf{C}(D)] = p_i[D]$ by 5.4(4). Furthermore, $\mathcal{D}$ is a full subpop of $\mathsf{C}(\mathcal{D})$ and D is dense in $\mathsf{C}(D)$. Summing things up, we obtain:

5.6. Corollary. *Let $\mathcal{D}$ be an $(I, \leq)$-pop. Then the following are equivalent*:

(i) *$\mathcal{D}$ is approximating.*
(ii) *$\mathcal{D}$ is a full subpop of a complete approximating $(I, \leq)$-pop.* ∎

5.7. Corollary. *Let $\mathcal{D}$ be an approximating $(I, \leq)$-pop. Then $\mathcal{D} = C(\mathcal{D})$ (more precisely: ψ is a pop isomorphism) if and only if $\mathcal{D}$ is complete in its pop uniformity.*

Proof. This results from the fact that both D is dense in $(\mathsf{C}(D), \tau_{\mathsf{C}(\mathcal{D})})$ and $(D, \mathcal{U}_{\mathcal{D}})$ is complete; hence D is closed in $(\mathsf{C}(D), \tau_{\mathsf{C}(\mathcal{D})})$ because the latter is Hausdorff. Alternatively: the canonical homomorphism ψ is an order isomorphism due to Lemma 3.39 and thus a pop isomorphism. A further alternative is to apply Theorems 3.40(1) and 5.4(7). ∎

Let $(D, \leq, (p_i)_{i\in I})$ be an approximating $(I, \leq)$-pop. We show that whenever a subset A has a supremum or an infimum in $(D, \leq)$ that is preserved by each projection p_i, then it has a supremum or an infimum in $(\mathsf{C}(D), \widehat{\leq})$ which coincides with the one formed in $(D, \leq)$. More precisely:

5.8. Proposition. *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop and let $\mathsf{C}(\mathcal{D}) = (\mathsf{C}(D), \widehat{\leq}, (\widehat{p}_i)_{i\in I})$ be its pop completion. Let $\psi : D \to \mathsf{C}(D)$ be the canonical homomorphism. Let $A \subseteq D$ be such that* $\sup A$ *exists and* $p_i(\sup A) = \sup p_i[A]$ *for all $i \in I$. Then we have* $\psi(\sup A) = \sup \psi[A]$ *and* $\widehat{p}_i(\psi(\sup A)) = \sup \widehat{p}_i[\psi[A]]$ *for all $i \in I$. In particular, if $\mathcal{D}$ is approximating, then* $\sup_D A = \sup_{\mathsf{C}(D)} A$. *Similarly for the infimum.*

Proof. We may assume $\mathsf{C}(\mathcal{D}) = \mathcal{D}_\infty$ and $\psi(d) = (p_i(d))_{i\in I}$ for all $d \in D$, cf. Theorem 5.4(7). Clearly, $\psi[A] \leq \psi(\sup A)$. Let $(e_i)_{i\in I} \in D_\infty$ with $\psi[A] \leq (e_i)_{i\in I}$. Then $p_i(a) \leq e_i$ for all $a \in A$ and all $i \in I$, whence $p_i(\sup A) = \sup p_i[A] \leq e_i$ for all $i \in I$. Therefore, $\psi(\sup A) = \sup \psi[A]$ and $\widehat{p}_i(\psi(\sup A)) = \psi(p_i(\sup A)) = \sup \psi[p_i[A]] = \sup \widehat{p}_i[\psi[A]]$ for all $i \in I$. ∎

5.9. Proposition. *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be any $(I, \leq)$-pop with pop completion $\mathsf{C}(\mathcal{D}) = (\mathsf{C}(D), \widehat{\leq}, (\widehat{p}_i)_{i\in I})$. Let $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be a complete approximating $(I, \leq)$-pop. Let $f : D \to E$ be a weak homomorphism with unique extension $\overline{f} : \mathsf{C}(D) \to E$. If f, p_i, and q_i are Scott-continuous for all $i \in I$, then so is $\overline{f}$.*

Proof. Let $\psi : D \to \mathsf{C}(D)$ be the canonical homomorphism. Let $\widehat{A} \subseteq \mathsf{C}(D)$ be directed such that $\sup \widehat{A}$ exists. Clearly, $\overline{f}[\widehat{A}] \leq \overline{f}(\sup \widehat{A})$. Let $e \in E$ with $\overline{f}[\widehat{A}] \leq e$ and let $i \in I$. Since $\widehat{p}_i$ is Scott-continuous and $\psi_0 := \psi|_{p_i[D]}$ is an order isomorphism from $p_i[D]$ onto $\widehat{p}_i[\mathsf{C}(D)]$ (Theorem 5.4(4),(6)(b)), we use Lemma 1.3 to deduce

$$\begin{aligned} q_i(\overline{f}(\sup \widehat{A})) &= q_i(\overline{f}(\widehat{p}_i(\sup \widehat{A}))) = q_i(\overline{f}(\psi_0(\psi_0^{-1}(\widehat{p}_i(\sup \widehat{A}))))) = q_i(f(\psi_0^{-1}(\widehat{p}_i(\sup \widehat{A})))) \\ &= q_i(f(\sup_{p_i[D]} \psi_0^{-1}[\widehat{p}_i[\widehat{A}]])) = q_i(f(\sup_D \psi_0^{-1}[\widehat{p}_i[\widehat{A}]])) = \sup q_i[f[\psi_0^{-1}[\widehat{p}_i[\widehat{A}]]]] \\ &= \sup q_i[\overline{f}[\psi_0[\psi_0^{-1}[\widehat{p}_i[\widehat{A}]]]]] = \sup q_i[\overline{f}[\widehat{p}_i[\widehat{A}]]] = \sup q_i[\overline{f}[\widehat{A}]] \leq q_i(e). \end{aligned}$$

Thus, $\overline{f}(\sup \widehat{A}) \leq e$ because $\mathcal{E}$ is approximating and hence $\overline{f}(\sup \widehat{A}) = \sup \overline{f}[\widehat{A}]$. ∎

5.10. Proposition. *Let* $\mathcal{D}=(D,\leq,(p_i)_{i\in I})$ *be an* $(I,\leq)$*-pop and let* $\mathcal{X}=(X,\leq,(p_i|_X)_{i\in I})$ *be a full subpop of* $\mathcal{D}$. *Let* $\mathcal{E}=(E,\leq,(q_i)_{i\in I})$ *be a complete approximating* $(I,\leq)$*-pop and let* $f:X\to E$ *be a [weak] homomorphism. Then there exists a unique [weak] homomorphism* $\overline{f}:D\to E$ *with* $\overline{f}|_X=f$. *If, furthermore,* f, p_i, *and* q_i *are Scott-continuous for all* $i\in I$, *then so is* $\overline{f}$.

Proof. Let $d\in D$. As $\mathcal{X}$ is a full subpop, we have $p_i(d)\in X$ for all $i\in I$. Since $(p_i(d))_{i\in I}$ converges to d, we find $(p_i(d))_{i\in I}$ to be a Cauchy net in X. Due to uniform continuity of f, the net $(f(p_i(d)))_{i\in I}$ is Cauchy in E, hence convergent. Set $\overline{f}(d):=\lim_{i\in I}f(p_i(d))$.

Let $x\in X$. As f is continuous, we deduce $(f(p_i(x)))_{i\in I}\to f(x)$. Then $\overline{f}(x)=f(x)$ because $\mathcal{E}$ is Hausdorff.

Let $i\in I$ and let $d\in D$. For all $j\in I$ we have $q_i(f(p_j(d)))=q_i(f(p_i(p_j(d))))$, hence $q_i(\overline{f}(d))=q_i(f(p_i(d)))=q_i(\overline{f}(p_i(d)))$ when we pass to the limit. Thus, $\overline{f}$ is non-expansive. Similarly, if f commutes with all projections, then so does $\overline{f}$. Let $d\leq e$. Then $p_i(d)\leq p_i(e)$ and $f(p_i(d))\leq f(p_i(e))$ for all $i\in I$. Since $\mathcal{E}$ is approximating, we infer $\overline{f}(d)\leq\overline{f}(e)$ by Proposition 2.11.

As $\mathcal{E}$ is Hausdorff, $\overline{f}$ is uniquely determined.

The proof of the remaining assertion is similar to the proof of Proposition 5.9. ■

5.11. Proposition. *Let* $\mathcal{D}=(D,\leq,(p_i)_{i\in I})$ *be an* $(I,\leq)$*-pop and let* $\mathcal{X}$ *be a full subpop of* $\mathcal{D}$. *Then the pop completions* $\mathsf{C}(\mathcal{X})$ *and* $\mathsf{C}(\mathcal{D})$ *are isomorphic. In particular,* $\mathsf{C}(\mathcal{D})$ *is the pop completion of the subpop* $\bigcup_{i\in I}p_i[\mathcal{D}]$.

Proof. Let $\mathcal{X}=:(X,\leq,(p_i|_X)_{i\in I})$. Since $\mathcal{X}$ is a full subpop of $\mathcal{D}$, we obtain $p_i[X]=p_i[D]$ for all $i\in I$. Theorem 5.4(7) yields the assertion. ■

In particular, if $\mathcal{D}$ is a complete approximating $(I,\leq)$-pop with full subpop $\mathcal{X}$, then $\mathcal{D}$ is (isomorphic to) the pop completion $\mathsf{C}(\mathcal{X})$ of $\mathcal{X}$ (Corollary 5.7).

5.12. Example. Let (X,ϱ) be an ultrametric space and view the closed ball model (Example 3.30) as an ω-pop $\mathcal{D}_{\mathrm{cb}}(X,\varrho):=(D_{\mathrm{cb}}(X,\varrho),\supseteq,(p_n)_{n\in\mathbb{N}_0})$. Let $(\widehat{X},\widehat{\varrho})$ be the metric completion of (X,ϱ). Then the pop completion of the closed ball model of (X,ϱ) is the closed ball model of $(\widehat{X},\widehat{\varrho})$, that is, $\mathsf{C}(\mathcal{D}_{\mathrm{cb}}(X,\varrho))=\mathcal{D}_{\mathrm{cb}}(\widehat{X},\widehat{\varrho})$.

To see this, recall first that $\mathcal{D}_{\mathrm{cb}}(\widehat{X},\widehat{\varrho})$ is approximating and complete by Example 3.30 and Proposition 3.34. Each subset $A\subseteq X$ can be viewed as a subset of $\widehat{X}$. Let $\overline{A}^{\tau_{\widehat{\varrho}}}$ denote the closure of A in $\widehat{X}$. This yields a mapping $\zeta:d\mapsto\overline{d}^{\tau_{\widehat{\varrho}}}$ from $D_{\mathrm{cb}}(X,\varrho)$ to $D_{\mathrm{cb}}(\widehat{X},\widehat{\varrho})$ because $\overline{\{x\}}^{\tau_{\widehat{\varrho}}}=\{x\}$ and $\overline{\{y\in X\mid\varrho(x,y)\leq 2^{-m}\}}^{\tau_{\widehat{\varrho}}}=\{\widehat{y}\in\widehat{X}\mid\widehat{\varrho}(x,\widehat{y})\leq 2^{-m}\}$ for all $x\in X$ and all $m\in\mathbb{N}_0$. Clearly, ζ is a pop embedding. Due to Proposition 5.11 it suffices to show that the subpop of $\mathcal{D}_{\mathrm{cb}}(\widehat{X},\widehat{\varrho})$ induced by $\zeta[D_{\mathrm{cb}}(X,\varrho)]$ is full. But this follows from X being dense in $\widehat{X}$: for all $\widehat{x}\in\widehat{X}$ and all $n\in\mathbb{N}_0$ there is some $x\in X$ with $\widehat{\varrho}(x,\widehat{x})\leq 2^{-n}$. Since $\widehat{\varrho}$ is an ultrametric, we deduce $\{\widehat{y}\in\widehat{X}\mid\widehat{\varrho}(\widehat{x},\widehat{y})\leq 2^{-n}\}=\{\widehat{y}\in\widehat{X}\mid\widehat{\varrho}(x,\widehat{y})\leq 2^{-n}\}=\overline{\{y\in X\mid\varrho(x,y)\leq 2^{-n}\}}^{\tau_{\widehat{\varrho}}}$.

5.13. Example. Let (Σ,D) be a dependence alphabet.

(a) Consider the ω-pop's $\mathcal{D}_1=(\mathbb{R}(\Sigma,D),\leq,(p_n)_{n\in\mathbb{N}_0})$ and $\mathcal{D}_2=(\mathbb{R}(\Sigma,D),\leq,(h_n)_{n\in\mathbb{N}_0})$ (see Example 3.35). Let $\mathbb{M}(\Sigma,D)\subseteq X\subseteq\mathbb{R}(\Sigma,D)$ and let $\mathcal{X}_1=(X,\leq,(p_n|_X)_{n\in\mathbb{N}_0})$ and

$\mathcal{X}_2 = (X, \leq, (h_n|_X)_{n\in\mathbb{N}_0})$. Then $\mathsf{C}(\mathcal{X}_1) = \mathcal{D}_1$ and $\mathsf{C}(\mathcal{X}_2) = \mathcal{D}_2$ because of Proposition 5.11. (Recall that $(\mathbb{R}(\Sigma, D), \mathcal{U}_{\{p_n|n\in\mathbb{N}_0\}}) = (\mathbb{R}(\Sigma, D), \mathcal{U}_{\{h_n|n\in\mathbb{N}_0\}})$ is compact, hence complete.)

(b) Consider the ω-pop $\mathcal{D}_\alpha = (\mathbb{F}^\alpha(\Sigma, D), \sqsubseteq, \{h_n \mid n \in \mathbb{N}_0\})$ (cf. Example 3.36). Let $\mathbb{F}^\alpha_{\mathrm{f}}(\Sigma, D) \subseteq X \subseteq \mathbb{F}^\alpha(\Sigma, D)$ and let $\mathcal{X}$ be the subpop of $\mathcal{D}_\alpha$ induced by X. Then $\mathsf{C}(\mathcal{X}) = \mathcal{D}_\alpha$. A similar statement is true for δ-traces.

5.14. Proposition. *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop. Let E be a "property" that is invariant under monotone mappings (such as e.g. being bounded or directed). Suppose all subsets of D with property E have a supremum that is preserved by each p_i. Then all subsets of $\mathsf{C}(D)$ with property E have a supremum which is preserved by each $\widehat{p}_i$.*

Proof. Let $\widehat{A} \subseteq \mathsf{C}(D)$ have property E and let $i \in I$. Recall that $\psi|_{p_i[D]}$ is an order isomorphism from $p_i[D]$ onto $\widehat{p}_i[\mathsf{C}(D)]$. Let $A_i := (\psi|_{p_i[D]})^{-1}[\widehat{p}_i[\widehat{A}]]$. Then A_i has property E; hence $\sup_D A_i$ exists, $\sup_D A_i = \sup_{p_i[D]} A_i =: \sup A_i$ by Lemma 1.3, and $p_i(\sup A_j) = \sup p_i[A_j]$ for all $i, j \in I$. As $\psi|_{p_i[D]}$ is an order isomorphism from $p_i[D]$ onto $\widehat{p}_i[\mathsf{C}(D)]$, we have $\psi(\sup A_i) = \sup_{\widehat{p}_i[\mathsf{C}(D)]} \psi[A_i] = \sup_{\mathsf{C}(D)} \widehat{p}_i[\widehat{A}]$ by 1.3 and

$$\begin{aligned}\widehat{p}_i(\sup \widehat{p}_j[\widehat{A}]) &= \widehat{p}_i(\psi(\sup A_j)) = \psi(p_i(\sup A_j)) = \psi(\sup p_i[A_j]) = \sup \psi[p_i[A_j]] \\ &= \sup \widehat{p}_i[\psi[A_j]] = \sup \widehat{p}_i[\widehat{p}_j[\widehat{A}]] = \sup \widehat{p}_i[\widehat{A}]\end{aligned}$$

for all $i \leq j$. Lemma 3.42 tells us that $\sup \widehat{A}$ exists and $\widehat{p}_i(\sup \widehat{A}) = \sup \widehat{p}_i[\widehat{A}]$ for all $i \in I$. ∎

In addition, we mention here the following. Suppose the assumptions of Proposition 5.14 are satisfied. Let $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be a complete approximating $(I, \leq)$-pop and let $f : D \to E$ be a weak homomorphism such that f and each q_i preserve suprema of sets with property E. Then the unique extension $\overline{f} : \mathsf{C}(D) \to E$ preserves suprema of subsets of $\mathsf{C}(D)$ with property E as well. This follows from an analysis of the proof of Proposition 5.9.

5.15. Corollary. *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop whose underlying poset is a bcpo such that for all $i \in I$ the projection p_i preserves suprema of bounded sets. Let $\mathsf{C}(\mathcal{D}) = (\mathsf{C}(D), \widehat{\leq}, (\widehat{p}_i)_{i\in I})$ be the pop completion of $\mathcal{D}$. Then $(\mathsf{C}(D), \widehat{\leq})$ is a bcpo and $\widehat{p}_i$ preserves suprema of bounded sets for all $i \in I$.* ∎

If property E means being directed, then we can improve Proposition 5.14:

5.16. Proposition. *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop whose underlying poset is a dcpo and whose projections p_i are Scott-continuous for all $i \in I$. Let $A(D) := \{\sup_{i\in I} p_i(d) \mid d \in D\}$. Then $A(D)$ induces a full subpop $A(\mathcal{D}) = (A(D), \leq, (p_i|_{A(D)})_{i\in I})$ of $\mathcal{D}$ such that $A(\mathcal{D})$ is approximating, $(A(D), \leq)$ is a dcpo, and $p_i|_{A(D)}$ is Scott-continuous for all $i \in I$. Moreover, $A(\mathcal{D})$ is the pop completion of $\mathcal{D}$. It coincides with $\mathcal{D}$ if and only if $\mathcal{D}$ is approximating.*

Proof. Recall that the pointwise supremum $\xi := \sup_{i\in I} p_i$ is a projection with $p_i \circ \xi = \xi \circ p_i = p_i$ for all $i \in I$ (cf. Remark 3.12). Hence, $A(\mathcal{D})$ is an approximating full subpop of $\mathcal{D}$. Now $A(D) = \xi[D]$, and $(A(D), \leq)$ is therefore a dcpo with $\sup_{A(D)} C = \sup_D C$ for any directed subset $C \subseteq A(D)$ (cf. Lemma 1.3). Thus, $p_i|_{A(D)}$ is Scott-continuous for all $i \in I$. Now apply Proposition 2.25 to deduce that $A(\mathcal{D})$ is complete in its pop uniformity.

Therefore, $A(\mathcal{D}) = \mathsf{C}(A(\mathcal{D})) = \mathsf{C}(\mathcal{D})$ by Corollary 5.7 and Proposition 5.11. It is obvious that $A(\mathcal{D}) = \mathcal{D}$ if and only if $\mathcal{D}$ is approximating. ∎

Suppose that $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ is an approximating $(I, \leq)$-pop such that $(D, \leq)$ is a dcpo or even a complete lattice and each p_i preserves suprema of all (directed) subsets of D. Then, due to the previous proposition, $\mathsf{C}(\mathcal{D}) = \mathcal{D}$. This does not hold anymore if the projections p_i do not preserve suprema of directed sets, cf. Example 3.27.

5.17. PROPOSITION. *Let $\mathcal{D}$ be any $(I, \leq)$-pop with pop completion $\mathsf{C}(\mathcal{D}) = (\mathsf{C}(D), \widehat{\leq}, (\widehat{p}_i)_{i\in I})$.*

(1) *If $(D, \leq)$ is linear, then so is $(\mathsf{C}(D), \widehat{\leq})$.*
(2) *Let $i \in I$. If p_i is downwards closed, then so is $\widehat{p}_i$.*

Proof. (1) Let $\widehat{d}, \widehat{e} \in \mathsf{C}(D)$. For all $i \in I$ choose elements $d_i, e_i \in D$ such that $\psi(d_i) = \widehat{p}_i(\widehat{d})$ and $\psi(e_i) = \widehat{p}_i(\widehat{e})$ (cf. Theorem 5.4(4)). Then $(\psi(d_i))_{i\in I} \to \widehat{d}$ and $(\psi(e_i))_{i\in I} \to \widehat{e}$. Let $I_1 := \{i \in I \mid d_i \leq e_i\}$ and $I_2 := \{i \in I \mid e_i \leq d_i\}$. Then $I = I_1 \cup I_2$. As the complement of any non-cofinal subset of I is cofinal in I, we may assume that I_1 is cofinal in I. Then $\psi(d_i) \mathrel{\widehat{\leq}} \psi(e_i)$ for all $i \in I_1$, $(\psi(d_i))_{i\in I_1} \to \widehat{d}$, and $(\psi(e_i))_{i\in I_1} \to \widehat{e}$. Therefore, $\widehat{d} \mathrel{\widehat{\leq}} \widehat{e}$.

(2) Let $\widehat{c}, \widehat{d} \in \mathsf{C}(D)$ and let $i \in I$ with $\widehat{c} \mathrel{\widehat{\leq}} \widehat{p}_i(\widehat{d})$. Then $\widehat{p}_j(\widehat{c}) \mathrel{\widehat{\leq}} \widehat{p}_i(\widehat{d})$ for all $j \in I$. Due to Corollary 5.5 we find elements $c_j, d_i \in D$ such that $\psi(p_j(c_j)) = \widehat{p}_j(\widehat{c})$, $\psi(p_i(d_i)) = \widehat{p}_i(\widehat{d})$, and $p_j(c_j) \leq p_i(d_i)$. Since p_i is downwards closed, we obtain $p_j(c_j) = p_i(p_j(c_j))$ and thus $\widehat{p}_j(\widehat{c}) = \psi(p_i(p_j(c_j))) = \widehat{p}_i(\psi(p_j(c_j))) = \widehat{p}_i(\widehat{p}_j(\widehat{c}))$ for all $j \in I$. Since $\mathsf{C}(\mathcal{D})$ is Hausdorff and $\widehat{p}_i$ is continuous with respect to the pop topology, we deduce $\widehat{c} = \widehat{p}_i(\widehat{c})$. ∎

From Propositions 4.10 and 5.17(2) we infer that if $\mathcal{D}$ is an ω-pop which is induced by a weight function, then its pop completion $\mathsf{C}(\mathcal{D})$ is also induced by a weight function.

Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ and $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be $(I, \leq)$-pop's such that $\mathcal{E}$ is approximating and $(q_i)_{i\in I}$ is Abelian. Recall from Section 4.3.1 that we have approximating $(I, \leq)$-pop's $[\mathcal{D} \to \mathcal{E}]^{\text{weak}}$ (weak homomorphisms from $\mathcal{D}$ to $\mathcal{E}$) and $[\mathcal{D} \to \mathcal{E}]^{\text{hom}}$ (homomorphisms from $\mathcal{D}$ to $\mathcal{E}$). Moreover, if q_i is Scott-continuous for all $i \in I$, then we also have approximating $(I, \leq)$-pop's $[\mathcal{D} \to \mathcal{E}]^{\text{Sweak}}$ (Scott-continuous weak homomorphisms from $\mathcal{D}$ to $\mathcal{E}$) and $[\mathcal{D} \to \mathcal{E}]^{\text{Shom}}$ (Scott-continuous homomorphisms from $\mathcal{D}$ to $\mathcal{E}$). Analogously, we have approximating pop's $[\mathsf{C}(\mathcal{D}) \to \mathsf{C}(\mathcal{E})]^{\text{weak}}$, $[\mathsf{C}(\mathcal{D}) \to \mathsf{C}(\mathcal{E})]^{\text{hom}}$, $[\mathsf{C}(\mathcal{D}) \to \mathsf{C}(\mathcal{E})]^{\text{Sweak}}$, and $[\mathsf{C}(\mathcal{D}) \to \mathsf{C}(\mathcal{E})]^{\text{Shom}}$ (use Theorem 5.4(6)(b)). In the following theorem we show that $[\mathsf{C}(\mathcal{D}) \to \mathsf{C}(\mathcal{E})]^{\text{weak}}$, $[\mathsf{C}(\mathcal{D}) \to \mathsf{C}(\mathcal{E})]^{\text{hom}}$, $[\mathsf{C}(\mathcal{D}) \to \mathsf{C}(\mathcal{E})]^{\text{Sweak}}$, and $[\mathsf{C}(\mathcal{D}) \to \mathsf{C}(\mathcal{E})]^{\text{Shom}}$ are the respective pop completions of $[\mathcal{D} \to \mathcal{E}]^{\text{weak}}$, $[\mathcal{D} \to \mathcal{E}]^{\text{hom}}$, $[\mathcal{D} \to \mathcal{E}]^{\text{Sweak}}$, and $[\mathcal{D} \to \mathcal{E}]^{\text{Shom}}$. Loosely speaking, the C-operator commutes with the $[\ \to\ ]^{\cdots}$-operators. (Notice that we could turn these operators into functors of suitable categories considered in Sections 4.3.2 and 4.3.3.)

5.18. THEOREM. *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop. Let $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be an approximating $(I, \leq)$-pop with Abelian projection net. Let $\mathsf{C}(\mathcal{D}) = (\mathsf{C}(D), \widehat{\leq}, (\widehat{p}_i)_{i\in I})$ and $\mathsf{C}(\mathcal{E}) = (\mathsf{C}(E), \widehat{\leq}, (\widehat{q}_i)_{i\in I})$ be the pop completions of $\mathcal{D}$ and $\mathcal{E}$, respectively. Then*

(1) *$[\mathcal{D} \to \mathsf{C}(\mathcal{E})]^{\text{weak}}$, $[\mathsf{C}(\mathcal{D}) \to \mathsf{C}(\mathcal{E})]^{\text{weak}}$, and $\mathsf{C}([\mathcal{D} \to \mathcal{E}]^{\text{weak}})$ are pairwise pop isomorphic.*

(2) $[\mathcal{D} \to \mathsf{C}(\mathcal{E})]^{\text{hom}}$, $[\mathsf{C}(\mathcal{D}) \to \mathsf{C}(\mathcal{E})]^{\text{hom}}$, *and* $\mathsf{C}([\mathcal{D} \to \mathcal{E}]^{\text{hom}})$ *are pairwise pop isomorphic.*

(3) *If* p_i *and* q_i *are Scott-continuous for all* $i \in I$, *then an analogous result holds for the pop completions of* $[\mathcal{D} \to \mathcal{E}]^{\text{Sweak}}$ *and* $[\mathcal{D} \to \mathcal{E}]^{\text{Shom}}$, *respectively.*

Proof. We only show (1); (2) and (3) are proven similarly. (Use Theorem 5.4(6)(b) and Proposition 5.9 for (3).) As usual, we view $\mathcal{E}$ as a subpop of $\mathsf{C}(\mathcal{E})$. Then, clearly, $[\mathcal{D} \to \mathcal{E}]^{\text{weak}}$ can be seen as a subpop of $[\mathcal{D} \to \mathsf{C}(\mathcal{E})]^{\text{weak}}$. For all $i \in I$ and all $g \in [D \to \mathsf{C}(E)]^{\text{weak}}$ we have $\widehat{q}_i \circ g \in [D \to E]^{\text{weak}}$; that is, $[\mathcal{D} \to \mathcal{E}]^{\text{weak}}$ is a full subpop of $[\mathcal{D} \to \mathsf{C}(\mathcal{E})]^{\text{weak}}$. Since $[\mathcal{D} \to \mathsf{C}(\mathcal{E})]^{\text{weak}}$ is complete and approximating (cf. Theorem 4.28(1)), we infer from Corollary 5.7 and Proposition 5.11 that $[\mathcal{D} \to \mathsf{C}(\mathcal{E})]^{\text{weak}}$ is the pop completion of $[\mathcal{D} \to \mathcal{E}]^{\text{weak}}$.

Let $\psi : D \to \mathsf{C}(D)$ be the canonical homomorphism from $\mathcal{D}$ to $\mathsf{C}(\mathcal{D})$. For each $g \in [D \to \mathsf{C}(E)]^{\text{weak}}$ let $\bar{g}$ be the unique element of $[\mathsf{C}(D) \to \mathsf{C}(E)]^{\text{weak}}$ with $\bar{g} \circ \psi = g$ (Theorem 5.4(1)). Clearly, if $g_1, g_2 \in [D \to \mathsf{C}(E)]^{\text{weak}}$ with $g_1 \mathrel{\widehat{\leq}} g_2$, then $\bar{g}_1(\psi(d)) \mathrel{\widehat{\leq}} \bar{g}_2(\psi(d))$ for all $d \in D$. As $\psi[D]$ is dense in $\mathsf{C}(D)$ and $\mathsf{C}(\mathcal{E})$ is approximating, we deduce that $\bar{g}_1 \mathrel{\widehat{\leq}} \bar{g}_2$. Let $i \in I$. Then $(\widehat{q}_i \circ \bar{g}) \circ \psi = \widehat{q}_i \circ g$. By uniqueness, $\widehat{q}_i \circ \bar{g} = \overline{\widehat{q}_i \circ g}$. We conclude that the mapping $g \mapsto \bar{g}$ is a homomorphism from $[\mathcal{D} \to \mathsf{C}(\mathcal{E})]^{\text{weak}}$ to $[\mathsf{C}(\mathcal{D}) \to \mathsf{C}(\mathcal{E})]^{\text{weak}}$. If $\bar{g}_1 \mathrel{\widehat{\leq}} \bar{g}_2$ for any $g_1, g_2 \in [D \to \mathsf{C}(E)]^{\text{weak}}$, then $g_1 = \bar{g}_1 \circ \psi \mathrel{\widehat{\leq}} \bar{g}_2 \circ \psi = g_2$, hence we have an embedding. Obviously, if $h \in [\mathsf{C}(D) \to \mathsf{C}(E)]^{\text{weak}}$ and $g := h \circ \psi$, then we have $h \circ \psi = g = \bar{g} \circ \psi$ and thus $h = \bar{g}$ by uniqueness. Consequently, $[\mathcal{D} \to \mathsf{C}(\mathcal{E})]^{\text{weak}}$ and $[\mathsf{C}(\mathcal{D}) \to \mathsf{C}(\mathcal{E})]^{\text{weak}}$ are isomorphic. ∎

5.19. Remark. Let $\mathcal{D} = (D, \leq, (p_i)_{i \in I})$ and $\mathcal{E} = (E, \leq, (q_i)_{i \in I})$ be $(I, \leq)$-pop's such that $\mathcal{E}$ is approximating, $(E, \leq)$ is a dcpo, $(q_i)_{i \in I}$ is Abelian, and q_i is Scott-continuous for all $i \in I$. Then $[D \to E]^{\text{Sweak}}$ and $[D \to E]^{\text{Shom}}$ are dcpo's with respect to the pointwise order. The projection nets of $[\mathcal{D} \to \mathcal{E}]^{\text{Sweak}}$ and $[\mathcal{D} \to \mathcal{E}]^{\text{Shom}}$, respectively, consist of Scott-continuous projections (Theorem 4.28(3)). As a consequence, $[\mathcal{D} \to \mathcal{E}]^{\text{Sweak}}$ and $[\mathcal{D} \to \mathcal{E}]^{\text{Shom}}$ are complete in their pop uniformity (Proposition 2.25) and thus coincide with their respective pop completions.

5.2. Domain completion and ideal completion

In general the underlying poset of the pop completion is not a dcpo. This section deals with another completion of a given $(I, \leq)$-pop: the "domain completion". We show that each $(I, \leq)$-pop $\mathcal{D}$ admits an approximating $(I, \leq)$-pop $\mathsf{J}(\mathcal{D})$ whose partial order is a directed complete partial order and whose projection net consists of Scott-continuous projections. The completion $\mathsf{J}(\mathcal{D})$ also satisfies a universal property with regard to the extension of homomorphisms. Similarly to the pop completion we can pop embed $\mathcal{D}$ into $\mathsf{J}(\mathcal{D})$ provided that $\mathcal{D}$ is approximating.

Since the ideal completion of a suitable subset of D will be the candidate for the domain completion, we need some facts on the ideal completion of a poset. We cite the following well known statement from Markowsky and Rosen [41, Theorem 2.7] (cf. also Lawson [39, Section I], Abramsky and Jung [1, Prop. 2.2.24]):

5.20. Proposition (Markowsky–Rosen [41]). *Let* $(D, \leq)$ *be a poset and let* $(\mathsf{Id}(D), \subseteq)$ *be the ideal completion of* $(D, \leq)$. *Let* $(E, \leq)$ *be a dcpo and let* $f : D \to E$ *be a monotone mapping. Let* $\varphi_D : D \to \mathsf{Id}(D)$, $\varphi_D(d) := d\!\downarrow$, *be the canonical order embedding of* $(D, \leq)$ *into* $(\mathsf{Id}(D), \subseteq)$. *Then* $f^* : \mathsf{Id}(D) \to E$ *defined by* $f^*(A) := \sup f[A]$ *is the unique Scott-continuous mapping such that* $f^* \circ \varphi_D = f$. ■

For any $(I, \leq)$-pop $(D, \leq, (p_i)_{i \in I})$ we endow the ideal completion $(\mathsf{Id}(D), \subseteq)$ of $(D, \leq)$ with a natural $(I, \leq)$-pop structure:

5.21. Proposition. *Let* $\mathcal{D} = (D, \leq, (p_i)_{i \in I})$ *be an* $(I, \leq)$*-pop. Let* $(\mathsf{Id}(D), \subseteq)$ *be the ideal completion of* $(D, \leq)$.

(1) *For all* $i \in I$ *define a mapping* $\widetilde{p}_i : \mathsf{Id}(D) \to \mathsf{Id}(D)$ *by* $\widetilde{p}_i(A) := p_i[A]\!\downarrow$ *for all* $A \in \mathsf{Id}(D)$. *Then* $\mathsf{Id}(\mathcal{D}) := (\mathsf{Id}(D), \subseteq, (\widetilde{p}_i)_{i \in I})$ *is a complete* $(I, \leq)$*-pop with Scott-continuous projections* $\widetilde{p}_i$.
(2) *The mapping* $\varphi_D : D \to \mathsf{Id}(D)$ *defined by* $\varphi_D(d) := d\!\downarrow$ *is a pop embedding of* $\mathcal{D}$ *into* $\mathsf{Id}(\mathcal{D})$.
(3) *Let* $(E, \leq, (q_i)_{i \in I})$ *be an* $(I, \leq)$*-pop such that* $(E, \leq)$ *is a dcpo and* q_i *is Scott-continuous for all* $i \in I$. *Let* $f : D \to E$ *be a weak homomorphism and define* $f^* : \mathsf{Id}(D) \to E$ *by* $f^*(A) := \sup f[A]$. *Then* f^* *is the unique Scott-continuous weak homomorphism with* $f^* \circ \varphi_D = f$. *Moreover, if* f *is a homomorphism, then* f^* *is also a homomorphism.*
(4) $\mathsf{Id}(\mathcal{D})$ *is approximating if and only if* $D = \bigcup_{i \in I} p_i[D]$.
(5) (a) $(p_i)_{i \in I}$ *is Abelian if and only if* $(\widetilde{p}_i)_{i \in I}$ *is Abelian.*
(b) *For all* $i \in I$ *the mapping* p_i *is downwards closed if and only if* $\widetilde{p}_i$ *is downwards closed.*

Proof. (1) Let $i \in I$. Clearly, $\widetilde{p}_i$ is Scott-continuous. Let $A \in \mathsf{Id}(D)$ and let $j \in I$ with $i \leq j$. Then $p_i[p_i[A]\!\downarrow]\!\downarrow = p_i[p_i[A]]\!\downarrow = p_i[A]\!\downarrow$, whence $\widetilde{p}_i$ is idempotent. Since $p_i[A]\!\downarrow \subseteq A$, we find $\widetilde{p}_i$ to be below the identity. As $p_i[A]\!\downarrow \subseteq p_j[A]\!\downarrow$, we see that $\mathsf{Id}(\mathcal{D})$ is an $(I, \leq)$-pop. By recalling that $(\mathsf{Id}(D), \subseteq)$ is an (algebraic) dcpo, we infer that $\mathsf{Id}(\mathcal{D})$ is complete with respect to its pop uniformity (Proposition 2.25).

(2) We know that φ_D is an order embedding of $(D, \leq)$ into $(\mathsf{Id}(D), \subseteq)$. Moreover, $p_i(d)\!\downarrow = p_i[d\!\downarrow]\!\downarrow = \widetilde{p}_i(d\!\downarrow)$ for all $i \in I$ and all $d \in D$.

(3) We already know that f^* is the unique Scott-continuous mapping such that $f^* \circ \varphi_D = f$ (Proposition 5.20). Let $A \in \mathsf{Id}(D)$ and let $i \in I$. Then

$$\begin{aligned} q_i(f^*(A)) &= q_i(\sup f[A]) = \sup q_i[f[A]] = \sup q_i[f[p_i[A]]] \\ &= q_i(\sup f[p_i[A]]) = q_i(\sup f[p_i[A]\!\downarrow]) = q_i(f^*(\widetilde{p}_i(A))). \end{aligned}$$

Similarly, if f commutes with all projections, then so does f^*.

(4) Observe that $\mathsf{Id}(\mathcal{D})$ is approximating if and only if $A = \bigcup_{i \in I} \widetilde{p}_i(A)$ for all $A \in \mathsf{Id}(D)$. Consequently, if $\mathsf{Id}(\mathcal{D})$ is approximating, then $d\!\downarrow = \bigcup_{i \in I} p_i[d\!\downarrow]\!\downarrow = \bigcup_{i \in I} p_i(d)\!\downarrow$ for all $d \in D$. Thus, there is some $i \in I$ with $d \leq p_i(d)$, i.e. $d = p_i(d)$. Conversely, if $D = \bigcup_{i \in I} p_i[D]$ and $A \in \mathsf{Id}(D)$, then we certainly obtain $A = \bigcup_{i \in I} p_i[A]\!\downarrow$.

(5) (a) Let $(p_i)_{i\in I}$ be Abelian and let $i,j \in I$. Then for all $A \in \mathsf{Id}(D)$ we have $\widetilde{p}_i(\widetilde{p}_j(A)) = p_i[p_j[A]{\downarrow}]{\downarrow} = p_i[p_j[A]]{\downarrow} = p_j[p_i[A]]{\downarrow} = p_j[p_i[A]{\downarrow}]{\downarrow} = \widetilde{p}_j(\widetilde{p}_i(A)$. The converse can be shown by using principal ideals.

(b) Let $i \in I$. Let p_i be downwards closed and let $A, B \in \mathsf{Id}(D)$ with $A \subseteq \widetilde{p}_i(B) = p_i[B]{\downarrow}$. Then for each $a \in A$ we find some $b \in B$ with $a \leq p_i(b)$. As p_i is downwards closed, we deduce $a = p_i(a)$. This implies $A = p_i[A]{\downarrow} = \widetilde{p}_i(A)$.

On the other hand, if $\widetilde{p}_i$ is downwards closed and if $c, d \in D$ with $c \leq p_i(d)$, then $c{\downarrow} \subseteq p_i(d){\downarrow} = \widetilde{p}_i(d{\downarrow})$. By assumption, $c{\downarrow} = \widetilde{p}_i(c{\downarrow})$. Hence, $c \leq p_i(c)$ and therefore $c = p_i(c)$. ∎

DEFINITION. We call $\mathsf{Id}(\mathcal{D})$ the *ideal completion* of $\mathcal{D}$.

Next, we formulate the existence and uniqueness theorem of the "domain completion". Observe the analogy to Theorem 5.4.

5.22. THEOREM. *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop.*

(1) *There exist an approximating $(I, \leq)$-pop $\mathsf{J}(\mathcal{D}) = (\mathsf{J}(D), \widetilde{\leq}, (\widetilde{p}_i)_{i\in I})$ with $(\mathsf{J}(D), \widetilde{\leq})$ a dcpo and $\widetilde{p}_i$ Scott-continuous for all $i \in I$ and a pop homomorphism $\iota : D \to \mathsf{J}(D)$ with the following universal property* (Figure 5.2): *For any approximating $(I, \leq)$-pop $(E, \leq, (q_i)_{i\in I})$ with $(E, \leq)$ a dcpo and q_i Scott-continuous for all $i \in I$ and any weak homomorphism $f : D \to E$ there is a unique Scott-continuous weak homomorphism $f^* : \mathsf{J}(D) \to E$ with $f^* \circ \iota = f$. Moreover*, *if f is a homomorphism*, *then $\overline{f}$ is also a homomorphism.*

$$\begin{tikzcd}
\mathsf{J}(D) \arrow[dr, dashed, "\exists_1 f^*"] & \\
D \arrow[u, "\iota"] \arrow[r, "f"'] \arrow[ur, phantom, "="] & E
\end{tikzcd}$$

Fig. 5.2. The universal property for the domain completion

(2) *Let $\mathcal{D}' = (D', \leq', (p'_i)_{i\in I})$ be an approximating $(I, \leq)$-pop such that $(D', \leq')$ is a dcpo and p'_i is Scott-continuous for all $i \in I$. Let $\phi : D \to D'$ be a weak homomorphism such that $(\mathcal{D}', \phi)$ satisfies the universal property of* (1). *Then there exists a unique pop isomorphism $\Phi : \mathsf{J}(D) \to D'$ with $\Phi \circ \iota = \phi$.*

(3) $(\mathsf{J}(D), \widetilde{\leq})$ *is algebraic with $K(\mathsf{J}(D)) = \iota[\bigcup_{i\in I} p_i[D]] = \bigcup_{i\in I} \widetilde{p}_i[\iota[D]]$. Moreover*, *$\mathsf{J}(\mathcal{D})$ is complete in its pop uniformity.*

(4) (a) *Let $d, e \in D$. Then $\iota(d) \mathrel{\widetilde{\leq}} \iota(e)$ if and only if $p_i(d) \leq p_i(e)$ for all $i \in I$.*
(b) *$\iota|_{p_i[D]}$ is an order embedding of $p_i[D]$ into $\widetilde{p}_i[\mathsf{J}(D)]$ for all $i \in I$.*

(5) *ι is a pop embedding of $\mathcal{D}$ into $\mathsf{J}(\mathcal{D})$ if and only if $\mathcal{D}$ is approximating.*

(6) *$\mathsf{J}(\mathcal{D})$ is pop isomorphic to $\mathsf{Id}(\bigcup_{i\in I} p_i[\mathcal{D}])$. More precisely*, *there is a unique pop isomorphism $\Psi : \mathsf{J}(D) \to \mathsf{Id}(\bigcup_{i\in I} p_i[\mathcal{D}])$ with $(\Psi \circ \iota)(d) = \bigcup_{i\in I} p_i(d){\downarrow}$ for all $d \in D$.*

Proof. Let $\mathsf{J}(\mathcal{D}) := \mathsf{Id}(\bigcup_{i\in I} p_i[\mathcal{D}])$. Then $\mathsf{J}(\mathcal{D})$ is approximating by Proposition 5.21(4). Let $\iota : D \to \mathsf{J}(D)$ be defined by $\iota(d) := \bigcup_{i\in I} p_i(d){\downarrow}$. We already know by Proposition 3.47 that ι is monotone. Let $i \in I$ and let $d \in D$. Then $\widetilde{p}_i(\iota(d)) = p_i[\bigcup_{j\in I} p_j(d){\downarrow}]{\downarrow} = \bigcup_{j\geq i} p_i(p_j(d)){\downarrow} = p_i(d){\downarrow} = \iota(p_i(d))$. Therefore, ι is a homomorphism.

Let $\iota(d) \mathrel{\widetilde{\leq}} \iota(e)$ and let $i \in I$. By definition of ι we obtain $p_i(d) \in \bigcup_{j\geq i} p_j(e){\downarrow}$, whence $p_i(d) \leq p_j(e)$ for some $j \geq i$. This implies $p_i(d) \leq p_i(p_j(e)) = p_i(e)$. Conversely, if $p_i(d) \leq p_i(e)$ for all $i \in I$, then $\widetilde{p}_i(\iota(d)) = \iota(p_i(d)) \mathrel{\widetilde{\leq}} \iota(p_i(e)) = \widetilde{p}_i(\iota(e))$ for all $i \in I$. As $\mathsf{J}(\mathcal{D})$ is approximating, we obtain $\iota(d) \mathrel{\widetilde{\leq}} \iota(e)$. This proves (4).

(5) is a consequence of Proposition 3.47. (Then $\iota : D \to \mathsf{Id}(\bigcup_{i\in I} p_i[D])$ is the ideal embedding considered in Proposition 3.47 and Theorem 3.48.)

Now we prove the universal property. Let $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ and f be as in (1). Let f^* be as in Proposition 5.21(3) applied to $\mathsf{Id}(\bigcup_{i\in I} p_i[D])$, i.e. $f^*(A) := \sup f[A]$ for all $A \in \mathsf{Id}(\bigcup_{i\in I} p_i[D])$. Then f^* is a Scott-continuous [weak] homomorphism (5.21(3)). Let $i \in I$ and let $d \in D$. Scott-continuity implies

$$\begin{aligned} q_i\Big(f^*(\bigcup_{j\geq i} p_j(d){\downarrow})\Big) &= \sup_{j\geq i} q_i(f^*(p_j(d){\downarrow})) = \sup_{j\geq i} q_i(f(p_j(d))) = \sup_{j\geq i} q_i(f(p_i(p_j(d)))) \\ &= \sup_{j\geq i} q_i(f(p_i(d))) = q_i(f(p_i(d))) = q_i(f(d)). \end{aligned}$$

Hence, $q_i(f^*(\iota(d))) = q_i(f(d))$ for all $i \in I$. As $\mathcal{E}$ is approximating, we conclude that $f^*(\iota(d)) = f(d)$.

Let $g : \mathsf{J}(D) \to E$ be a Scott-continuous [weak] homomorphism with $g \circ \iota = f$. Since $\iota|_{\bigcup_{i\in I} p_i[D]}$ coincides with the canonical embedding $d \mapsto d{\downarrow}$ of $\bigcup_{i\in I} p_i[D]$ into $\mathsf{Id}(\bigcup_{i\in I} p_i[D]) = \mathsf{J}(D)$, we infer that $f^* = g$ (Proposition 5.21(3)). This proves (1).

(2) follows from the universal property as in the proof of Theorem 5.4. Furthermore, (6) holds by definition and (2).

Recall that $(\mathsf{J}(D), \widetilde{\leq})$ is algebraic with $K(\mathsf{J}(D)) = \{p_i(d){\downarrow} \mid i \in I,\ d \in D\} = \{\iota(p_i(d)) \mid i \in I,\ d \in D\} = \iota[\bigcup_{i\in I} p_i[D]] = \bigcup_{i\in I} \widetilde{p}_i[\iota[D]]$. Finally, $\mathsf{J}(\mathcal{D})$ is complete in its pop uniformity by Proposition 5.21(1); hence (3) is true. ■

Definition. We call $\mathsf{J}(\mathcal{D})$ the *domain completion* of $\mathcal{D}$. By abuse of language, we also call $\iota : D \to \mathsf{J}(D)$ the *canonical homomorphism*.

The advantage of the domain completion $\mathsf{J}(\mathcal{D})$ as opposed to the pop completion $\mathsf{C}(\mathcal{D})$ is that $\mathsf{J}(\mathcal{D})$ always yields an algebraic domain. However, $\mathsf{J}(\mathcal{D})$ has some disadvantages compared to $\mathsf{C}(\mathcal{D})$. For instance, $\iota[D]$ need not induce a full subpop of $\mathsf{J}(\mathcal{D})$ and $\iota|_{p_i[D]}$ need not be an isomorphism from $p_i[D]$ onto $\widetilde{p}_i[\mathsf{J}(D)]$. In Theorem 5.33 we will investigate pop's where these problems do not occur.

Moreover, $\mathsf{J}(\mathsf{J}(\mathcal{D}))$ need not be isomorphic to $\mathsf{J}(\mathcal{D})$ whereas $\mathsf{C}(\mathsf{C}(\mathcal{D}))$ is always isomorphic to $\mathsf{C}(\mathcal{D})$ (cf. Corollary 5.7). For example, let $\mathcal{D} = (\mathbb{N}_0, \leq, (\mathsf{id})_{i\in I})$. Then, with the abbreviation $\omega = (\mathbb{N}_0, \leq)$ and the usual ordinal number arithmetic, the underlying posets of $\mathsf{J}(\mathcal{D})$ and $\mathsf{J}(\mathsf{J}(\mathcal{D}))$ are $\omega + 1$ and $\omega + 2$, respectively.

In contrast to the pop completion, the canonical homomorphism of the domain completion need not preserve suprema. Instead, we have the following result:

5.23. Proposition. *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop with domain completion $\mathsf{J}(\mathcal{D}) = (\mathsf{J}(D), \widetilde{\leq}, (\widetilde{p}_i)_{i\in I})$. Let $\iota : D \to \mathsf{J}(D)$ be the canonical homomorphism. Then the following are equivalent*:

(i) *ι is Scott-continuous.*
(ii) *p_i is compact-valued for all $i \in I$ (cf. also the equivalent conditions in Proposition* 3.46).
(iii) *For any approximating $(I, \leq)$-pop $(E, \leq, (q_i)_{i\in I})$ with $(E, \leq)$ a dcpo and q_i Scott-continuous for all $i \in I$ and any [weak] homomorphism $f : D \to E$ we find that f is Scott-continuous.* ■

Proof. (i)⇔(ii). Let $A \subseteq D$ be directed such that A has a supremum.

Suppose first that $\iota(\sup A) = \sup \iota[A]$ and let $i \in I$ and $d \in D$ such that $\sup A \geq p_i(d)$. Since $\bigcup_{i\in I} p_i(\sup A){\downarrow} = \bigcup_{a\in A} \bigcup_{i\in I} p_i(a){\downarrow}$, we find some $j \geq i$ and an $a \in A$ with $p_i(\sup A) \leq p_j(a)$. Thus, $p_i(d) \leq p_i(\sup A) \leq p_i(p_j(a)) = p_i(a) \leq a$, and p_i is compact-valued.

In order to prove the converse, note that $\iota[A] \mathrel{\widetilde{\leq}} \iota(\sup A)$. Let $B \in \mathsf{J}(D) = \mathsf{Id}(\bigcup_{i\in I} p_i[D])$ with $\iota[A] \mathrel{\widetilde{\leq}} B$, i.e. $\bigcup_{i\in I} p_i(a){\downarrow} \subseteq B$ for all $a \in A$. From Lemma 3.3 we infer that $p_i(\sup A) \in p_i[A]$ and therefore $p_i(\sup A) \in B$ for all $i \in I$. Hence, $\iota(\sup A) \mathrel{\widetilde{\leq}} B$ and thus $\iota(\sup A) = \sup \iota[A]$.

(i)⇔(iii). If ι is Scott-continuous, then $f = f^* \circ \iota$ is Scott-continuous as a composition of Scott-continuous maps (Theorem 5.22(1)). For the converse apply (iii) to $f := \iota$. ■

5.24. Remark. Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an approximating $(I, \leq)$-pop. Corollary 5.7 tells us that the canonical homomorphism $\psi : D \to \mathsf{C}(D)$ is an isomorphism if and only if $\mathcal{D}$ is complete in its pop uniformity. The question arises when the canonical homomorphism $\iota : D \to \mathsf{J}(D)$ is surjective. The answer is given in Theorem 3.48: ι is an isomorphism if and only if $(D, \leq)$ is a dcpo with $K(D) = \bigcup_{i\in I} p_i[D]$. In this case $\mathcal{D}$ is complete in its pop uniformity, whence $\mathcal{D}$, $\mathsf{J}(\mathcal{D})$, and $\mathsf{C}(\mathcal{D})$ are pairwise isomorphic. A more detailed analysis when $\mathsf{J}(\mathcal{D})$ and $\mathsf{C}(\mathcal{D})$ are pop isomorphic is given in Section 5.3.

5.25. Remark. Analogously to Proposition 5.11 we have the following. Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop and let $\mathcal{X}$ be a full subpop of $\mathcal{D}$. Then the domain completions $\mathsf{J}(\mathcal{X})$ and $\mathsf{J}(\mathcal{D})$ are pop isomorphic. In particular, $\mathsf{J}(\mathcal{D})$ is the domain completion of the subpop $\bigcup_{i\in I} p_i[\mathcal{D}]$.

5.3. Comparison of the completions

It is natural to ask how these completions are related. This is the subject of the present section. We demonstrate how the pop completion embeds into the domain completion and give some criteria to get equality. Finally, we show when the completions yield a bifinite domain.

We need the following well known lemma to extend monotone mappings between posets to mappings between their respective ideal completions.

5.26. LEMMA. *Let $(D,\leq)$ and $(E,\leq)$ be posets and let $f : D \to E$ be monotone. Define $\widetilde{f} : \mathsf{Id}(D) \to \mathsf{Id}(E)$ by $\widetilde{f}(A) := f[A]{\downarrow}$. Then $\widetilde{f}$ is a Scott-continuous mapping. In addition, we have the following*:

(1) *f is an order embedding if and only if $\widetilde{f}$ is an order embedding.*
(2) *f is an order isomorphism if and only if $\widetilde{f}$ is an order isomorphism. In this case, $\widetilde{f}^{-1}(C) = f^{-1}[C]{\downarrow} = \widetilde{f^{-1}}(C)$ for all $C \in \mathsf{Id}(E)$.*

Proof (included for the sake of completeness). An easy calculation shows us that $\widetilde{f}$ is Scott-continuous. (Alternatively, one may apply Proposition 5.20.)

(1) Let f be an order embedding and let $A, B \in \mathsf{Id}(D)$ with $f[A]{\downarrow} \subseteq f[B]{\downarrow}$. Let $a \in A$. As $f(a) \in f[B]{\downarrow}$, we find an element $b \in B$ with $f(a) \leq f(b)$. By assumption, $a \leq b$ and thus $a \in B$, whence $A \subseteq B$.

Conversely, let $\widetilde{f}$ be an order embedding and let $c, d \in D$ with $f(c) \leq f(d)$. Then $f[c{\downarrow}]{\downarrow} = f(c){\downarrow} \subseteq f(d){\downarrow} = f[d{\downarrow}]{\downarrow}$. By assumption, $c{\downarrow} \subseteq d{\downarrow}$ and thus $c \leq d$.

(2) Let f be an order isomorphism and let $\widetilde{g} : \mathsf{Id}(E) \to \mathsf{Id}(D)$ be defined by $\widetilde{g}(C) := f^{-1}[C]{\downarrow}$. Let $A \in \mathsf{Id}(D)$ and let $C \in \mathsf{Id}(E)$. Then $\widetilde{f}(\widetilde{g}(C)) = f[f^{-1}[C]{\downarrow}]{\downarrow} = f[f^{-1}[C]]{\downarrow} = C{\downarrow} = C$. Similarly, $\widetilde{g}(\widetilde{f}(A)) = A$. Thus, (1) implies that $\widetilde{f}$ is an order isomorphism and $\widetilde{f}^{-1} = \widetilde{g}$.

To prove the converse, let $\widetilde{f}$ be an order isomorphism and let $e \in E$. Then there is some $A \in \mathsf{Id}(D)$ with $f[A]{\downarrow} = \widetilde{f}(A) = e{\downarrow}$. Therefore, we find an element $a \in A$ with $f(a) = e$. Thus, f is surjective. Again, the assertion follows from (1). ■

5.27. LEMMA. *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ and $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be $(I,\leq)$-pop's and let $f : D \to E$ be a monotone mapping. Let $\widetilde{f} : \mathsf{Id}(D) \to \mathsf{Id}(E)$ be the Scott-continuous map defined by $\widetilde{f}(A) := f[A]{\downarrow}$. Then f is a* [*weak*] *homomorphism if and only if $\widetilde{f}$ is a* [*weak*] *homomorphism.*

Proof. The "only if" part follows from Proposition 5.21(3) because $\widetilde{f} = (\varphi_E \circ f)^*$, where φ_E is the canonical order embedding of $(E,\leq)$ into $(\mathsf{Id}(E), \subseteq)$. To prove the converse, use principal ideals. ■

For any poset $(D,\leq)$ let $\varphi_D : D \to \mathsf{Id}(D)$, $d \mapsto d{\downarrow}$, be the canonical order embedding.

5.28. PROPOSITION. *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I,\leq)$-pop. Then we have the following*:

(1) *$\mathsf{J}(\mathcal{D})$ is isomorphic to the subpop of $\mathsf{Id}(\mathcal{D})$ induced by $\{\bigcup_{i\in I} \widetilde{p}_i(A) \mid A \in \mathsf{Id}(D)\}$.*
(2) *$\mathsf{Id}(\mathcal{D})$ is isomorphic to $\mathsf{J}(\mathcal{D})$ if and only if $D = \bigcup_{i\in I} p_i[D]$.*
(3) *Let $\psi : D \to \mathsf{C}(D)$ be the canonical homomorphism. Then the mapping $\widetilde{\psi} : \mathsf{Id}(D) \to \mathsf{Id}(\mathsf{C}(D))$ defined by $\widetilde{\psi}(A) := \psi[A]{\downarrow}$ is a Scott-continuous pop homomorphism. It is a pop embedding if and only if $\mathcal{D}$ is approximating. Furthermore, $\varphi_{\mathsf{C}(D)} \circ \psi = \widetilde{\psi} \circ \varphi_D$. Hence, we have commutative diagrams as in Figure* 5.3.

Proof. (1) Let $A(\mathsf{Id}(D)) := \{\bigcup_{i\in I} \widetilde{p}_i(A) \mid A \in \mathsf{Id}(D)\}$. Since the inclusion map $\mathsf{id}_{\bigcup_{i\in I} p_i[D], D}$ is a pop embedding of $\bigcup_{i\in I} p_i[\mathcal{D}]$ into $\mathcal{D}$, we deduce that the mapping $\phi : \mathsf{Id}(\bigcup_{i\in I} p_i[D]) \to \mathsf{Id}(D)$ defined by $\phi(A) := A{\downarrow}_D$ is also a pop embedding (Lemmas 5.26 and 5.27). We show that ϕ is in fact a pop isomorphism from $\mathsf{J}(\mathcal{D}) = \mathsf{Id}(\bigcup_{i\in I} p_i[\mathcal{D}])$ onto $A(\mathsf{Id}(\mathcal{D}))$. First of all, given $A \in \mathsf{Id}(\bigcup_{i\in I} p_i[D])$, we clearly have $\phi(A) = A{\downarrow}_D =$

$$
\begin{tikzcd}
\mathsf{C}(D) \arrow[r, hook, "\varphi_{\mathsf{C}(D)}"] & \mathsf{Id}(\mathsf{C}(D)) \\
D \arrow[u, "\psi"] \arrow[r, hook, "\varphi_D"'] \arrow[ur, phantom, "="] & \mathsf{Id}(D) \arrow[u, "\widetilde{\psi}"'] \\
\bigcup_{i\in I} p_i[D] \arrow[u, hook] \arrow[r, hook, "\varphi_{\bigcup_{i\in I} p_i[D]}"'] \arrow[ur, phantom, "="] & \mathsf{Id}(\bigcup_{i\in I} p_i[D]) = \mathsf{J}(D) \arrow[u, hook]
\end{tikzcd}
$$

Fig. 5.3. Canonical embeddings

$\bigcup_{i\in I} p_i[A{\downarrow}_D]{\downarrow}_D \in A(\mathsf{Id}(\mathcal{D}))$. Now let $C \in A(\mathsf{Id}(D))$. Then $C = \bigcup_{i\in I} p_i[C]{\downarrow}_D$. Let $A := C \cap \bigcup_{i\in I} p_i[D]$. Clearly, A is a lower subset of $(\bigcup_{i\in I} p_i[D], \leq)$. Given $a, b \in A$, we find some $c \in C$ with $c \geq a, b$. There are elements $i, j, k \in I$ with $a = p_i(a)$, $b = p_j(b)$, and $k \geq i, j$. Therefore, $p_k(c) \in C \cap \bigcup_{l\in I} p_l[D] = A$, $a = p_i(a) \leq p_i(c) \leq p_k(c)$, and, similarly, $b \leq p_k(c)$. Hence, A is directed and thus $A \in \mathsf{Id}(\bigcup_{i\in I} p_i[D])$. We show that $A{\downarrow}_D = C$. As $A \subseteq C$, we obtain $A{\downarrow}_D \subseteq C{\downarrow}_D = C$. Conversely, let $c \in C$. Then we find an index $i \in I$ with $c \in p_i[C]{\downarrow}$. Thus, there is some $d \in C$ such that $c \leq p_i(d) \leq d$. Since $p_i(d) \in A$, we conclude that $c \in A{\downarrow}_D$.

(2) If $\mathsf{Id}(\mathcal{D})$ is isomorphic to $\mathsf{J}(\mathcal{D})$, then $\mathsf{Id}(\mathcal{D})$ is approximating and thus $D = \bigcup_{i\in I} p_i[D]$ by Proposition 5.21(4). The converse is clear by Theorem 5.22(6).

(3) The mapping $\widetilde{\psi}$ is a Scott-continuous homomorphism by Lemma 5.27. Furthermore, $\mathcal{D}$ is approximating if and only if ψ is a pop embedding if and only if $\widetilde{\psi}$ is a pop embedding (cf. Theorem 5.4(5) and Lemma 5.26).

Let $d \in D$. Then $\varphi_{\mathsf{C}(D)}(\psi(d)) = \psi(d){\downarrow} = \psi[d{\downarrow}]{\downarrow} = \widetilde{\psi}(d{\downarrow}) = \widetilde{\psi}(\varphi_D(d))$. ∎

At first sight, the following result might surprise the reader—especially in view of the previous proposition. It states that the pop completion $\mathsf{C}(\mathcal{D})$ can be embedded into $\mathsf{J}(\mathcal{D})$ and thus into $\mathsf{Id}(\mathcal{D})$. Moreover, it turns out that the domain completion $\mathsf{J}(\mathcal{D})$ can actually be obtained as the pop completion of $\mathsf{Id}(\mathcal{D})$:

5.29. Theorem. *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop. Let $\psi : D \to \mathsf{C}(D)$ and $\iota : D \to \mathsf{J}(D)$ be the canonical homomorphisms.*

1. *The following $(I, \leq)$-pop's are pairwise pop isomorphic*: $\mathsf{J}(\mathcal{D})$, $\mathsf{J}(\mathsf{C}(\mathcal{D}))$, $\mathsf{C}(\mathsf{J}(\mathcal{D}))$, $\mathsf{C}(\mathsf{Id}(\mathcal{D}))$, *and* $\mathsf{C}(\mathsf{Id}(\mathsf{C}(\mathcal{D})))$.
2. *Let $\bar{\iota} : \mathsf{C}(D) \to \mathsf{J}(D)$ be the unique homomorphism with $\bar{\iota} \circ \psi = \iota$. Then $\bar{\iota}$ is a pop embedding of $\mathsf{C}(\mathcal{D})$ into $\mathsf{J}(\mathcal{D})$ (Figure 5.4). Assuming $\mathsf{J}(\mathcal{D}) = \mathsf{Id}(\bigcup_{i\in I} p_i[\mathcal{D}])$, we have $\bar{\iota}(\widehat{d}) = \bigcup_{i\in I} (\psi|_{p_i[D]})^{-1}(\widehat{p}_i(\widehat{d})){\downarrow}$ for all $\widehat{d} \in \mathsf{C}(D)$.*

$$
\begin{tikzcd}
 & \mathsf{Id}(D) \\
 & \mathsf{J}(D) \arrow[u, hook] \\
D \arrow[ur, "\iota"] \arrow[r, "\psi"'] \arrow[ur, phantom, shift left=3ex, "="] & \mathsf{C}(D) \arrow[u, hook, "\bar{\iota}"']
\end{tikzcd}
$$

Fig. 5.4. The pop completion embeds into the domain completion

Proof. (1) By Corollary 5.5, $\psi_0 := \psi|_{\bigcup_{i \in I} p_i[D]}$ is a pop isomorphism from $\bigcup_{i \in I} p_i[\mathcal{D}]$ onto $\bigcup_{i \in I} \widehat{p}_i[\mathsf{C}(\mathcal{D})]$. Therefore, the map $\widetilde{\psi_0} : \mathsf{J}(D) = \mathsf{Id}(\bigcup_{i \in I} p_i[D]) \to \mathsf{Id}(\bigcup_{i \in I} \widehat{p}_i[\mathsf{C}(D)]) = \mathsf{J}(\mathsf{C}(\mathcal{D}))$, $A \mapsto \psi_0[A]\!\downarrow$, is a pop isomorphism (cf. Lemmas 5.26 and 5.27).

Since $\mathsf{J}(\mathcal{D})$ is complete in its pop uniformity (Theorem 5.22(3)), the canonical homomorphism $\widehat{\psi} : \mathsf{J}(D) \to \mathsf{C}(\mathsf{J}(\mathcal{D}))$ is a pop isomorphism by Corollary 5.7.

Let $A(\mathsf{Id}(D)) := \{\bigcup_{i \in I} \widetilde{p}_i(A) \mid A \in \mathsf{Id}(D)\}$. Then we infer from Proposition 5.16 that $\mathsf{C}(\mathsf{Id}(\mathcal{D}))$ is isomorphic to the subpop induced by $A(\mathsf{Id}(D))$. Since the latter is isomorphic to $\mathsf{J}(\mathcal{D})$ by Proposition 5.28(1), we find $\mathsf{J}(\mathcal{D})$ to be isomorphic to $\mathsf{C}(\mathsf{Id}(\mathcal{D}))$. When switching from $\mathcal{D}$ to $\mathsf{C}(\mathcal{D})$ we deduce that $\mathsf{J}(\mathsf{C}(\mathcal{D}))$ is isomorphic to $\mathsf{C}(\mathsf{Id}(\mathsf{C}(\mathcal{D})))$.

(2) Consider again the pop isomorphism $\widetilde{\psi}_0 : \mathsf{J}(D) \to \mathsf{J}(\mathsf{C}(\mathcal{D}))$ from (1). Its inverse is given by $B \mapsto \psi_0^{-1}[B]\!\downarrow$ (where $B \in \mathsf{Id}(\bigcup_{i \in I} \widehat{p}_i[\mathsf{C}(D)])$). Let $\widehat{\iota} : \mathsf{C}(D) \to \mathsf{Id}(\bigcup_{i \in I} \widehat{p}_i[\mathsf{C}(D)])$ $= \mathsf{J}(\mathsf{C}(\mathcal{D}))$ be the canonical homomorphism from $\mathsf{C}(D)$ to $\mathsf{J}(\mathsf{C}(\mathcal{D}))$. With regard to Theorem 5.22(5), $\widehat{\iota}$ is a pop embedding. Thus, $\widetilde{\psi}_0^{-1} \circ \widehat{\iota}$ is a pop embedding. Given $d \in D$, we deduce

$$
\widetilde{\psi}_0^{-1}(\widehat{\iota}(\psi(d))) = \psi_0^{-1}[\bigcup_{i \in I} \widehat{p}_i(\psi(d))\!\downarrow]\!\downarrow = \bigcup_{i \in I} \psi_0^{-1}[\psi(p_i(d))]\!\downarrow = \bigcup_{i \in I} p_i(d)\!\downarrow = \iota(d).
$$

Hence, $(\widetilde{\psi}_0^{-1} \circ \widehat{\iota}) \circ \psi = \iota$. By uniqueness we infer $\bar{\iota} = \widetilde{\psi}_0^{-1} \circ \widehat{\iota}$. ∎

5.30. Example. Let (X, ϱ) be an ultrametric space and consider the closed ball model $\mathcal{D}_{\mathrm{cb}}(X, \varrho) := (D_{\mathrm{cb}}(X, \varrho), \supseteq, (p_n)_{n \in \mathbb{N}_0})$ (Examples 3.30 and 5.12). Let $(\widehat{X}, \widehat{\varrho})$ be the metric completion of (X, ϱ). Then the domain completion of the closed ball model of (X, ϱ) is pop isomorphic to the closed ball model of $(\widehat{X}, \widehat{\varrho})$. Together with Example 5.12 we thus deduce that $\mathcal{D}_{\mathrm{cb}}(\widehat{X}, \widehat{\varrho})$, $\mathsf{J}(\mathcal{D}_{\mathrm{cb}}(X, \varrho))$, and $\mathsf{C}(\mathcal{D}_{\mathrm{cb}}(X, \varrho))$ are pairwise pop isomorphic.

In order to see this, recall from Example 3.30 that $\mathcal{D}_{\mathrm{cb}}(\widehat{X}, \widehat{\varrho})$ is approximating with $(D_{\mathrm{cb}}(\widehat{X}, \widehat{\varrho}), \supseteq)$ a dcpo (even an algebraic Scott domain) such that $K(D) = \bigcup_{n \in \mathbb{N}_0} p_n[D_{\mathrm{cb}}(\widehat{X}, \widehat{\varrho})]$. Therefore, $\mathcal{D}_{\mathrm{cb}}(\widehat{X}, \widehat{\varrho})$ and $\mathsf{J}(\mathcal{D}_{\mathrm{cb}}(\widehat{X}, \widehat{\varrho}))$ are pop isomorphic by Remark 5.24. Since the former is isomorphic to $\mathsf{C}(\mathcal{D}_{\mathrm{cb}}(X, \varrho))$ (Example 5.12), we infer from Theorem 5.29(1) that the latter is isomorphic to $\mathsf{J}(\mathcal{D}_{\mathrm{cb}}(X, \varrho))$.

5.31. Example. Let (Σ, D) be a dependence alphabet.

(a) Let $\mathcal{D}_1 = (\mathbb{R}(\Sigma, D), \leq, (p_n)_{n\in\mathbb{N}_0})$ and $\mathcal{D}_2 = (\mathbb{R}(\Sigma, D), \leq, (h_n)_{n\in\mathbb{N}_0})$ (see Examples 3.35 and 5.13). Let $\mathbb{M}(\Sigma, D) \subseteq X \subseteq \mathbb{R}(\Sigma, D)$ and let $\mathcal{X}_1 = (X, \leq, (p_n|_X)_{n\in\mathbb{N}_0})$ and $\mathcal{X}_2 = (X, \leq, (h_n|_X)_{n\in\mathbb{N}_0})$. Then $\mathsf{J}(\mathcal{X}_1) = \mathcal{D}_1$ and $\mathsf{J}(\mathcal{X}_2) = \mathcal{D}_2$.

This can be shown as follows: $\mathcal{D}_1$ is approximating and $(\mathbb{R}(\Sigma, D), \leq)$ is an algebraic Scott domain with $K(\mathbb{R}(\Sigma, D)) = \bigcup_{n\in\mathbb{N}_0} p_n[\mathbb{R}(\Sigma, D)]$ (Theorem 1.7 and Example 3.35). By Remark 5.24, $\mathsf{J}(\mathcal{D}_1) = \mathcal{D}_1$. Since $\mathsf{C}(\mathcal{X}_1) = \mathcal{D}_1$ by Example 5.13, we obtain the assertion by applying Theorem 5.29(1). Analogously for $\mathcal{D}_2$.

(b) Consider the ω-pop $\mathcal{D}_\alpha = (\mathbb{F}^\alpha(\Sigma, D), \sqsubseteq, \{h_n \mid n \in \mathbb{N}_0\})$ (cf. Examples 3.36 and 5.13). Let $\mathbb{F}^\alpha_{\mathrm{f}}(\Sigma, D) \subseteq X \subseteq \mathbb{F}^\alpha(\Sigma, D)$ and let $\mathcal{X}$ be the subpop of $\mathcal{D}_\alpha$ induced by X. Then $\mathsf{J}(\mathcal{X}) = \mathcal{D}_\alpha$. A similar statement is true for δ-traces.

Consider an $(I, \leq)$-pop $(D, \leq, (p_i)_{i\in I})$ with $D = \bigcup_{i\in I} p_i[D]$. Then we know by Proposition 5.28 that $\mathsf{Id}(\mathcal{D}) = \mathsf{J}(\mathcal{D})$. Therefore, the pop embedding $\varphi_D : d \mapsto d{\downarrow}$ of $\mathcal{D}$ into $\mathsf{Id}(\mathcal{D})$ coincides with the embedding ι and thus extends to a pop embedding of $\mathsf{C}(\mathcal{D})$ into $\mathsf{Id}(\mathcal{D})$ by Theorem 5.29(2). Hence, the following corollary generalizes Theorem 3.13 in Majster-Cederbaum and Baier [40], where a similar statement is proven for pointed posets with weight functions. Recall that weight functions correspond precisely to ω-pop's with downwards closed projections (Proposition 4.10). Also, only metric completions and isometric embeddings are considered in [40], whereas we deal with pop completions and pop embeddings.

5.32. Corollary. *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop such that $D = \bigcup_{i\in I} p_i[D]$. Then the pop embedding $\varphi_D : d \mapsto d{\downarrow}$ of $\mathcal{D}$ into $\mathsf{Id}(\mathcal{D})$ extends uniquely to a pop embedding $\overline{\varphi_D}$ of $\mathsf{C}(\mathcal{D})$ into $\mathsf{Id}(\mathcal{D})$.* ■

We have already seen in the examples above that the pop completion may coincide with the domain completion. Now we investigate this phenomenon more systematically:

5.33. Theorem. *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop with pop completion $\mathsf{C}(\mathcal{D}) = (\mathsf{C}(D), \widehat{\leq}, (\widehat{p}_i)_{i\in I})$ and domain completion $\mathsf{J}(\mathcal{D}) = (\mathsf{J}(D), \widetilde{\leq}, (\widetilde{p}_i)_{i\in I})$. Let $\psi : D \to \mathsf{C}(D)$ and $\iota : D \to \mathsf{J}(D)$ be the respective canonical homomorphisms. Let $\bar{\iota} : \mathsf{C}(D) \to \mathsf{J}(D)$ be the unique homomorphism with $\bar{\iota} \circ \psi = \iota$. Then the following are equivalent:*

- (i) *Each monotone net in D is a Cauchy net.*
- (ii) *For all $A \in \mathsf{Id}(D)$ and for all $i \in I$ we find that $p_i[A]$ has a greatest element.*
- (iii) *$\iota[D]$ induces a full subpop of $\mathsf{J}(\mathcal{D})$.*
- (iv) *$\iota|_{p_i[D]}$ is an order isomorphism from $p_i[D]$ onto $\widetilde{p}_i[\mathsf{J}(D)]$ for all $i \in I$.*
- (v) *$\iota|_{\bigcup_{i\in I} p_i[D]}$ is a pop isomorphism from $\bigcup_{i\in I} p_i[\mathcal{D}]$ onto $\bigcup_{i\in I} \widetilde{p}_i[\mathsf{J}(\mathcal{D})]$.*
- (vi) *$\bar{\iota}$ is a pop isomorphism from $\mathsf{C}(\mathcal{D})$ onto $\mathsf{J}(\mathcal{D})$.*

In this case we obtain $K(\mathsf{J}(D)) = \bigcup_{i\in I} \widetilde{p}_i[\mathsf{J}(D)]$. In other words, $(\mathsf{C}(D), \widehat{\leq})$ is an algebraic dcpo with $K(\mathsf{C}(D)) = \bigcup_{i\in I} \widehat{p}_i[\mathsf{C}(D)]$ and thus a P-domain.

Proof. (i)⇔(ii) follows easily from Corollary 3.45.

(ii)⇒(vi). Theorem 5.29(2) tells us that $\bar{\iota}$ is a pop embedding. Let $\widehat{\iota} : \mathsf{C}(D) \to \mathsf{Id}(\bigcup_{i\in I} \widehat{p}_i[\mathsf{C}(D)]) = \mathsf{J}(\mathsf{C}(\mathcal{D}))$ be the canonical homomorphism from $\mathsf{C}(D)$ into $\mathsf{J}(\mathsf{C}(\mathcal{D}))$. We know that $\widehat{\iota}$ is a pop embedding and that $\bar{\iota} = \phi \circ \widehat{\iota}$ for some pop isomorphism ϕ from

$\mathsf{J}(\mathsf{C}(\mathcal{D}))$ onto $\mathsf{J}(\mathcal{D})$ (see the proof of Theorem 5.29(2)). Therefore, it suffices to show that $\widehat{\iota}$ is surjective. In order to prove this, let $i \in I$ and let $\widehat{A} \subseteq \mathsf{C}(D)$ be directed. Then $\widehat{p}_i[\widehat{A}]$ is a directed subset of $\widehat{p}_i[\mathsf{C}(D)]$. Since $\psi|_{p_i[D]}$ is an order isomorphism from $p_i[D]$ onto $\widehat{p}_i[\mathsf{C}(D)]$ (Theorem 5.4(4)), the set $B := (\psi|_{p_i[D]})^{-1}[\widehat{p}_i[\widehat{A}]]$ is a directed subset of $p_i[D]$; hence $B = p_i[B]$ and $p_i[B{\downarrow}]$ has a greatest element b by condition (ii). Therefore, $\psi(b)$ is the greatest element of $\widehat{p}_i[\widehat{A}]$. We have thus proven that $\widehat{p}_i[\widehat{A}]$ has a greatest element for all directed subset $\widehat{A} \subseteq \mathsf{C}(D)$ and all $i \in I$. Since $\mathsf{C}(\mathcal{D})$ is complete and approximating, we can apply Theorem 3.48 to deduce that $\widehat{\iota}$ is surjective. Moreover, the same result tells us that $(\mathsf{C}(D), \widehat{\leq})$ is an algebraic dcpo and $K(\mathsf{C}(D)) = \bigcup_{i\in I} \widehat{p}_i[\mathsf{C}(D)]$. This yields

$$K(\mathsf{J}(D)) = \bar{\iota}[K(\mathsf{C}(D))] = \bigcup_{i\in I} \bar{\iota}[\widehat{p}_i[\mathsf{C}(D)]] = \bigcup_{i\in I} \widetilde{p}_i[\bar{\iota}[\mathsf{C}(D)]] = \bigcup_{i\in I} \widetilde{p}_i[\mathsf{J}(D)].$$

(vi)⇒(iii). $\psi[D]$ induces a full subpop of $\mathsf{C}(\mathcal{D})$ (Theorem 5.4(3)). By (vi), $\iota[D] = \bar{\iota}[\psi[D]]$ induces a full subpop of $\mathsf{J}(\mathcal{D})$.

(iii)⇒(iv). Let $i \in I$, We already know that $\iota|_{p_i[D]}$ is an order embedding of $p_i[D]$ into $\widetilde{p}_i[\mathsf{J}(D)]$ (Theorem 5.22(4)(b)). Let $A \in \mathsf{J}(D)$. By (iii) there is an element $d \in D$ such that $\widetilde{p}_i(A) = \iota(d)$; hence $\widetilde{p}_i(A) = \widetilde{p}_i(\iota(d)) = \iota(p_i(d)) \in \iota[p_i[D]]$. Thus, $\iota|_{p_i[D]}$ is surjective.

(iv)⇒(v). Because of (iv) it is enough to show that $\iota|_{\bigcup_{i\in I} p_i[D]}$ is order reflecting. But this follows as in the proof of Corollary 5.5.

(v)⇒(ii). For the following conclusions let ${\downarrow} := {\downarrow}_{\bigcup_{i\in I} p_i[D]}$. Let $A \in \mathsf{Id}(D)$ and let $B := \{p_i(a) \mid i \in I,\ a \in A\}$. One easily sees that $B \in \mathsf{Id}(\bigcup_{i\in I} p_i[D]) = \mathsf{J}(D)$. Let $i \in I$. Using (v) we find an element $d \in p_i[D]$ such that $p_i[A]{\downarrow} = p_i[B]{\downarrow} = \widetilde{p}_i(B) = \iota(d) = d{\downarrow}$. Hence, d is the greatest element of $p_i[A]$. ■

5.34. Corollary. *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop such that $p_i[A]$ has a greatest element for all $A \in \mathsf{Id}(D)$ and all $i \in I$. Then the pop completion $\mathsf{C}(\mathcal{D})$ is isomorphic to the ideal completion $\mathsf{Id}(\mathcal{D})$ if and only if $D = \bigcup_{i\in I} p_i[D]$. In this case, $(\mathsf{C}(D), \widehat{\leq})$ is an algebraic dcpo with $K(\mathsf{C}(D)) = D$. In particular, $(\mathsf{C}(D), \widehat{\leq})$ is a P-domain.*

Proof. This follows from Theorem 5.33 and Propositions 5.28(2) and 5.21(4). ■

Note that the previous corollary generalizes Theorem 3.16 in Majster-Cederbaum and Baier [40]. This theorem states for pointed posets $(D, \leq)$ with a weight function $\|\cdot\|$ that the metric completion is isometric to the ideal completion if $D = \bigcup_{n\in\mathbb{N}_0} q_n^{\|\cdot\|}[D]$ and if $q_n^{\|\cdot\|}[A]$ is finite for all $A \in \mathsf{Id}(D)$. Here $q_n^{\|\cdot\|}$ denotes the projection defined by $q_n^{\|\cdot\|}(d) := \max\{x \in D \mid x \leq d,\ \|x\| \leq n\}$ (cf. Section 4.2). The metric under consideration is the canonical ultrametric of $(D, \leq, (q_n^{\|\cdot\|})_{n\in\mathbb{N}_0})$.

5.35. Remark. Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop. Suppose that its pop completion yields a dcpo $(\mathsf{C}(D), \widehat{\leq})$ with all projections $\widehat{p}_i$ Scott-continuous ($i \in I$). Then the canonical homomorphism $\psi : D \to \mathsf{C}(D)$ has a unique Scott-continuous extension $\psi^* : \mathsf{J}(D) \to \mathsf{C}(D)$ with $\psi^* \circ \iota = \psi$ (cf. Theorem 5.22(1)). Let $\bar{\iota} : \mathsf{C}(D) \to \mathsf{J}(D)$ be the unique homomorphism with $\bar{\iota} \circ \psi = \iota$. There is a close relation between $\bar{\iota}$ and ψ^*. We calculate $(\psi^* \circ \bar{\iota}) \circ \psi = \psi^* \circ \iota = \psi$. Then the uniqueness statement in Theorem 5.4(1) implies that $\psi^* \circ \bar{\iota} = \mathsf{id}_{\mathsf{C}(D)}$.

What about $\bar{\iota} \circ \psi^*$? We have the following equivalent statements:

(i) $\bar{\iota} \circ \psi^* = \mathsf{id}_{\mathsf{J}(D)}$.
(ii) $\bar{\iota} \circ \psi^* \leq \mathsf{id}_{\mathsf{J}(D)}$, whence $(\bar{\iota}, \psi^*)$ is an epp.
(iii) $\bar{\iota}$ is Scott-continuous.
(iv) $\bar{\iota}$ is a pop isomorphism.

In this case, $\bar{\iota}^{-1} = \psi^*$.

The implications (i)⇒(ii)⇒(iii) and (i)⇒(iv)⇒(iii) are trivial (for (ii)⇒(iii) consider the remarks on p. 14 and for (i)⇒(iv) use Theorem 5.29(2)). Only (iii)⇒(i) remains to be shown. Hence, suppose that $\bar{\iota}$ is Scott-continuous. Since $(\bar{\iota} \circ \psi^*) \circ \iota = \bar{\iota} \circ \psi = \iota$ and since $\bar{\iota} \circ \psi^*$ is Scott-continuous, we can apply the uniqueness statement of the universal property in Theorem 5.22(1) and obtain $\bar{\iota} \circ \psi^* = \mathsf{id}_{\mathsf{J}(D)}$.

Finally, we take a look at the completions of $(I, \leq)$-pop's whose projections have only finitely many images. They are closely related to bifinite domains.

5.36. Theorem. *Let $\mathcal{D} = (D, \leq, (p_i)_{i \in I})$ be an $(I, \leq)$-pop such that p_i has finite range for all $i \in I$. Then $\mathsf{C}(\mathcal{D})$ and $\mathsf{J}(\mathcal{D})$ are pop isomorphic. Moreover, $\mathsf{C}(\mathcal{D})$ is compact in its pop topology and $(\mathsf{C}(D), \widehat{\leq})$ is a bifinite domain with $K(\mathsf{C}(D)) = \bigcup_{i \in I} \widehat{p}_i[\mathsf{C}(D)]$.*

Proof. Let $\psi : D \to \mathsf{C}(D)$ be the canonical homomorphism. Since $\widehat{p}_i[\mathsf{C}(D)] = \psi[p_i[D]]$ is finite for all $i \in I$, the pop completion $\mathsf{C}(\mathcal{D})$ is totally bounded in its pop uniformity (Proposition 3.16) and thus compact. Moreover, $(\mathsf{C}(D), \widehat{\leq})$ is a bifinite domain by Proposition 5.3 and Theorem 5.4(7). As finite directed sets have a greatest element, we may apply Theorem 5.33 to deduce that $\mathsf{C}(\mathcal{D})$ and $\mathsf{J}(\mathcal{D})$ are pop isomorphic and $K(\mathsf{C}(D)) = \bigcup_{i \in I} \widehat{p}_i[\mathsf{C}(D)]$. ■

Theorem 5.36 and Corollary 5.34 yield:

5.37. Corollary. *Let $\mathcal{D} = (D, \leq, (p_i)_{i \in I})$ be an $(I, \leq)$-pop such that p_i has finite range for all $i \in I$. Then the pop completion $\mathsf{C}(\mathcal{D})$ is isomorphic to the ideal completion $\mathsf{Id}(\mathcal{D})$ if and only if $D = \bigcup_{i \in I} p_i[D]$. In this case, $(\mathsf{C}(D), \widehat{\leq})$ is a bifinite domain and $K(\mathsf{C}(D)) = D$.* ■

5.38. Example. Let (Σ, D) be a dependence alphabet.

(a) Example 5.13 tells us that $(\mathbb{R}(\Sigma, D), \leq, (p_n)_{n \in \mathbb{N}_0})$ is (isomorphic to) the pop completion of $(\mathbb{M}(\Sigma, D), \leq, (p_n|_{\mathbb{M}(\Sigma, D)})_{n \in \mathbb{N}_0})$. As $\mathbb{M}(\Sigma, D) = \bigcup_{n \in \mathbb{N}_0} p_n[\mathbb{M}(\Sigma, D)]$, the previous corollary implies that $(\mathbb{R}(\Sigma, D), \leq, (p_n)_{n \in \mathbb{N}_0})$ is also isomorphic to the ideal completion of $(\mathbb{M}(\Sigma, D), \leq, (p_n|_{\mathbb{M}(\Sigma, D)})_{n \in \mathbb{N}_0})$.

Similarly, $(\mathbb{R}(\Sigma, D), \leq, (h_n)_{n \in \mathbb{N}_0})$, the pop completion and the ideal completion of $(\mathbb{M}(\Sigma, D), \leq, (h_n|_{\mathbb{M}(\Sigma, D)})_{n \in \mathbb{N}_0})$ are pairwise pop isomorphic.

(b) The ω-pop $(\mathbb{F}^\alpha(\Sigma, D), \sqsubseteq, (h_n)_{n \in \mathbb{N}_0})$, the pop completion and the ideal completion of $(\mathbb{F}^\alpha_{\mathrm{f}}(\Sigma, D), \sqsubseteq, (h_n|_{\mathbb{F}^\alpha_{\mathrm{f}}(\Sigma, D)})_{n \in \mathbb{N}_0})$ are pairwise pop isomorphic. An analogous statement holds for δ-traces.

6. THE TOPOLOGY OF TRACES

The last chapter is devoted to investigating the Lawson topology of traces. The aim is to characterize the topology in terms of "graph-theoretic" properties of the underlying dependence alphabet.

We prove that the Lawson topology of real traces is completely determined by the number of letters independent of all other letters of the alphabet and by the existence of a pair of distinct, dependent letters (Theorem 6.4). Moreover, we show that the space of real traces is homeomorphic to a direct product that is built from the Cantor tree (provided there exist distinct, dependent letters in the underlying dependence alphabet) and a finite-dimensional grid; see Corollary 6.5 for details. We remark here that these results have arisen from many intensive and productive discussions between D. Kuske and the author. The original proof based on an explicit construction of a specific uniform isomorphism (cf. [36]). Here we present an approach using Corollary 1.6 (the corollary to Pierce's Theorem), as well as a characterization of the topologically isolated elements in the nth derivation of the space of real traces (Theorem 6.2).

Concerning approximating traces, we derive topological invariants that are actually graph-theoretic invariants of the dependence alphabet (Corollary 6.3). If the dependence relation is transitive, then we obtain a representation of the spaces of α- and δ-traces as a direct product in the same spirit as for real traces (see Theorems 6.6 and 6.8). In particular, for transitive dependence relations real and δ-traces cannot be distinguished topologically (Corollary 6.9). In order to show these results, we use Corollary 1.6 and Theorem 6.2 again.

As to our results for the topological space of real traces, a non-trivial generalization to *infinite* dependence alphabets can be found in [36].

Let (Σ, D) be a dependence alphabet. Recall from Theorem 1.8 and Example 3.35 that the Lawson topology $\lambda_{\mathbb{R}(\Sigma,D)}$ of $(\mathbb{R}(\Sigma,D),\leq)$ is induced both by the prefix metric d_{pref} and by the Foata normal form metric d_{fnf}. We know that d_{pref} is the canonical ultrametric of the ω-pop $(\mathbb{R}(\Sigma,D),\leq,(p_n)_{n\in\mathbb{N}_0})$ and d_{fnf} is the one of $(\mathbb{R}(\Sigma,D),\leq,(h_n)_{n\in\mathbb{N}_0})$ (see Example 4.4). That is, $\lambda_{\mathbb{R}(\Sigma,D)}$ coincides with the pop topologies of both ω-pop's. To obtain results on the Lawson topology, we may choose with which ω-pop we want to work. Here we prefer the latter and define

$$\mathcal{D}_{\mathbb{R}(\Sigma,D)} := (\mathbb{R}(\Sigma,D),\leq,(h_n)_{n\in\mathbb{N}_0}).$$

Analogously, Example 3.36 shows that the Lawson topology $\lambda_{\mathbb{F}^\alpha(\Sigma,D)}$ of $(\mathbb{F}^\alpha(\Sigma,D),\sqsubseteq)$ is induced by the α-metric d_α. This metric is the canonical ultrametric of $(\mathbb{F}^\alpha(\Sigma,D),\sqsubseteq,(h_n)_{n\in\mathbb{N}_0})$ (Example 4.4). Similarly for δ-traces: the Lawson topology

$\lambda_{\mathbb{F}^\delta(\Sigma,D)}$ of $(\mathbb{F}^\delta(\Sigma,D),\sqsubseteq)$ is the metric topology given by the δ-metric d_δ. The latter is the canonical ultrametric of $(\mathbb{F}^\delta(\Sigma,D),\sqsubseteq,(h_n)_{n\in\mathbb{N}_0})$. Again, to study $\lambda_{\mathbb{F}^\alpha(\Sigma,D)}$ and $\lambda_{\mathbb{F}^\delta(\Sigma,D)}$, we work with the ω-pop's

$$\mathcal{D}_{\mathbb{F}^\alpha(\Sigma,D)} := (\mathbb{F}^\alpha(\Sigma,D),\sqsubseteq,(h_n)_{n\in\mathbb{N}_0}),\qquad \mathcal{D}_{\mathbb{F}^\delta(\Sigma,D)} := (\mathbb{F}^\delta(\Sigma,D),\sqsubseteq,(h_n)_{n\in\mathbb{N}_0}).$$

We start with an investigation of direct products of these "trace pop's". Let $\mathcal{D} = (D,\leq,(p_n)_{n\in\mathbb{N}_0})$ and $\mathcal{E} = (E,\leq,(q_n)_{n\in\mathbb{N}_0})$ be ω-pop's. Recall that $\mathcal{D}\times\mathcal{E} := (D\times E,\leq,(p_n\times q_n)_{n\in\mathbb{N}_0})$ is an ω-pop, where $\leq$ is the product order on $D\times E$ and $(p_n\times q_n)(d,e) = (p_n(d),q_n(e))$ for all $d,e\in D$ and all $n\in\mathbb{N}_0$. The pop topology $\tau_{\mathcal{D}\times\mathcal{E}}$ is the product topology of $D\times E$, where D and E carry the respective pop topologies.

6.1. Proposition. *Let (Σ_1,D_1) and (Σ_2,D_2) be dependence alphabets and let $(\Sigma,D) := (\Sigma_1\mathbin{\dot\cup}\Sigma_2, D_1\mathbin{\dot\cup} D_2)$ be their disjoint union. Then the following ω-pop's are isomorphic*:

(1) $\mathcal{D}_{\mathbb{R}(\Sigma,D)}$ *and* $\mathcal{D}_{\mathbb{R}(\Sigma_1,D_1)}\times\mathcal{D}_{\mathbb{R}(\Sigma_2,D_2)}$,
(2) $\mathcal{D}_{\mathbb{F}^\alpha(\Sigma,D)}$ *and* $\mathcal{D}_{\mathbb{F}^\alpha(\Sigma_1,D_1)}\times\mathcal{D}_{\mathbb{F}^\alpha(\Sigma_2,D_2)}$,
(3) $\mathcal{D}_{\mathbb{F}^\delta(\Sigma,D)}$ *and* $\mathcal{D}_{\mathbb{F}^\delta(\Sigma_1,D_1)}\times\mathcal{D}_{\mathbb{F}^\delta(\Sigma_2,D_2)}$.

Proof. We may assume $\Sigma_1\cap\Sigma_2=\emptyset$ and $\Sigma=\Sigma_1\cup\Sigma_2$.

(1) Let $\varphi:\mathbb{R}(\Sigma_1,D_1)\times\mathbb{R}(\Sigma_2,D_2)\to\mathbb{R}(\Sigma,D)$ be defined by $\varphi(t_1,t_2):=t_1\cdot t_2$. For any $t_1\in\mathbb{R}(\Sigma_1,D_1)$, $t_2\in\mathbb{R}(\Sigma_2,D_2)$ we have $\mathsf{alph}(t_1)\times\mathsf{alph}(t_2)\subseteq I_D$ by definition of D. Therefore, $t_1\cdot t_2$ is just the disjoint union of the graphs t_1 and t_2. Consequently, φ is an order embedding and commutes with all Foata projections: $h_n(t_1)\cdot h_n(t_2) = h_n(t_1\cdot t_2)$ for all $n\in\mathbb{N}_0$. It is obvious that each $t\in\mathbb{R}(\Sigma,D)$ is the disjoint union of two (uniquely determined) traces $t_1\in\mathbb{R}(\Sigma_1,D_1)$ and $t_2\in\mathbb{R}(\Sigma_2,D_2)$; hence $t=t_1\cdot t_2$. We infer that φ is a pop isomorphism.

(2) Let $\varphi_\alpha:\mathbb{F}^\alpha(\Sigma_1,D_1)\times\mathbb{F}^\alpha(\Sigma_2,D_2)\to\mathbb{F}^\alpha(\Sigma,D)$ be defined by $\varphi_\alpha((r_1,A_1),(r_2,A_2)) := (r_1\cdot r_2, A_1\cup A_2)$. As in (1), $r_1\cdot r_2$ is the disjoint union of the graphs r_1 and r_2. Clearly, $A_1\cap A_2=\emptyset$. Let $(r_1,A_1),(s_1,B_1)\in\mathbb{F}^\alpha(\Sigma_1,D_1)$ and let $(r_2,A_2),(s_2,B_2)\in\mathbb{F}^\alpha(\Sigma_2,D_2)$. We have $r_1\leq s_1$ and $r_2\leq s_2$ if and only if $r_1r_2\leq s_1s_2$ (see above). In this case, $(r_1^{-1}s_1)\cdot(r_2^{-1}s_2) = (r_1r_2)^{-1}s_1s_2$. This yields

$$\begin{aligned}
&(r_i,A_i)\sqsubseteq(s_i,B_i)\quad\text{ for } i=1,2\\
\Leftrightarrow\;& r_i\leq s_i \text{ and } B_i\cup\mathsf{alph}(r_i^{-1}s_i)\subseteq A_i\quad\text{ for } i=1,2\\
\Leftrightarrow\;& r_1r_2\leq s_1s_2 \text{ and } (B_1\cup B_2)\cup\mathsf{alph}((r_1r_2)^{-1}s_1s_2)\subseteq A_1\cup A_2\\
\Leftrightarrow\;& (r_1r_2,A_1\cup A_2)\sqsubseteq(s_1s_2,B_1\cup B_2).
\end{aligned}$$

Consequently, φ_α is an order embedding. Again, $h_n(r_1)\cdot h_n(r_2) = h_n(r_1\cdot r_2)$. Therefore,

$$\begin{aligned}
&\varphi_\alpha(h_n(r_1,A_1),h_n(r_2,A_2))\\
&\quad=\varphi_\alpha((h_n(r_1),A_1\cup\mathsf{alph}(h_n(r_1)^{-1}r_1)),(h_n(r_2),A_2\cup\mathsf{alph}(h_n(r_2)^{-1}r_2)))\\
&\quad=(h_n(r_1)h_n(r_2),A_1\cup A_2\cup\mathsf{alph}(h_n(r_1)^{-1}r_1)\cup\mathsf{alph}(h_n(r_2)^{-1}r_2))\\
&\quad=(h_n(r_1\cdot r_2),A_1\cup A_2\cup\mathsf{alph}(h_n(r_1\cdot r_2)^{-1}r_1\cdot r_2))\\
&\quad=h_n(r_1\cdot r_2,A_1\cup A_2)=h_n(\varphi_\alpha((r_1,A_1),(r_2,A_2))).
\end{aligned}$$

We see that φ_α commutes with all projections. As in (1), φ_α is surjective because for each $(r, A) \in \mathbb{F}^\alpha(\Sigma, D)$ there exist (unique) real traces $r_1 \in \mathbb{R}(\Sigma_1, D_1)$, $r_2 \in \mathbb{R}(\Sigma_2, D_2)$ with $(r, A) = (r_1 r_2, (A \cap \Sigma_1) \cup (A \cap \Sigma_2))$. As a consequence, φ_α is a pop isomorphism.

(3) is proven similarly to (2). ∎

In what follows let $\mathbb{R}(\Sigma, D)$ be equipped with $\lambda_{\mathbb{R}(\Sigma,D)} = \tau_{d_{\text{fnf}}}$, i.e. with the pop topology of $\mathcal{D}_{\mathbb{R}(\Sigma,D)}$. Analogously, $\mathbb{F}^\alpha(\Sigma, D)$ is assumed to be endowed with $\lambda_{\mathbb{F}^\alpha(\Sigma,D)}$ and $\mathbb{F}^\delta(\Sigma, D)$ with $\lambda_{\mathbb{F}^\delta(\Sigma,D)}$. Since we deal with the projections h_n in all three spaces $\mathbb{R}(\Sigma, D)$, $\mathbb{F}^\alpha(\Sigma, D)$, and $\mathbb{F}^\delta(\Sigma, D)$, we introduce a different notation for the "balls" $B_{h_n}(x)$. Let $r \in \mathbb{R}(\Sigma, D)$, let $A \subseteq \Sigma$ be such that $\mathsf{alphinf}(r) \subseteq A$ (respectively, $D(\mathsf{alphinf}(r)) \subseteq D(A)$), and let $n \in \mathbb{N}_0$. Then we define

$$\begin{aligned}
B_{\text{fnf}}(r, n) &:= \{s \in \mathbb{R}(\Sigma, D) \mid d_{\text{fnf}}(r, s) \leq 2^{-n}\} \\
&= \{s \in \mathbb{R}(\Sigma, D) \mid h_n(r) = h_n(s)\}, \\
B_\alpha((r, A), n) &:= \{y \in \mathbb{F}^\alpha(\Sigma, D) \mid d_\alpha((r, A), y) \leq 2^{-n}\} \\
&= \{y \in \mathbb{F}^\alpha(\Sigma, D) \mid h_n(r, A) = h_n(y)\} \\
&= \{(s, B) \in \mathbb{F}^\alpha(\Sigma, D) \mid h_n(r) = h_n(s) \text{ and} \\
&\qquad\qquad A \cup \mathsf{alph}(h_n(r)^{-1} r) = B \cup \mathsf{alph}(h_n(s)^{-1} s)\}, \\
B_\delta((r, D(A)), n) &:= \{y \in \mathbb{F}^\delta(\Sigma, D) \mid d_\delta((r, D(A)), y) \leq 2^{-n}\} \\
&= \{y \in \mathbb{F}^\delta(\Sigma, D) \mid h_n(r, D(A)) = h_n(y)\} \\
&= \{(s, D(B)) \in \mathbb{F}^\delta(\Sigma, D) \mid h_n(r) = h_n(s) \text{ and} \\
&\qquad\qquad D(A) \cup D(\mathsf{alph}(h_n(r)^{-1} r)) = D(B) \cup D(\mathsf{alph}(h_n(s)^{-1} s))\}.
\end{aligned}$$

The following theorem describes the set of all topologically isolated elements in the nth derivation of $\mathbb{R}(\Sigma, D)$, $\mathbb{F}^\alpha(\Sigma, D)$, and $\mathbb{F}^\delta(\Sigma, D)$, respectively. To formulate it, we need a definition first. We call a letter $a \in \Sigma$ *isolated in* (Σ, D) if $\{a\} \times (\Sigma \setminus \{a\}) \subseteq I_D$, i.e. a is independent of all other letters of Σ. Recall from Example 3.35(b) that a subset $A \subseteq \Sigma$ is an I_D-clique provided that $(a, b) \in I_D$ for all $a, b \in A$ with $a \neq b$.

6.2. THEOREM. *Let (Σ, D) be a dependence alphabet. Then for all $n \in \mathbb{N}_0$ we have:*

(1) $\mathsf{isol}(\mathbb{R}(\Sigma, D)^{(n)}) = \{s a_1^\omega \cdots a_n^\omega \in \mathbb{R}(\Sigma, D) \mid s \in \mathbb{M}(\Sigma, D),\ a_1, \ldots, a_n$ *isolated in* $(\Sigma, D)\}$

(2) $\mathsf{isol}(\mathbb{F}^\alpha(\Sigma, D)^{(n)}) = \{(s a_1^\omega \cdots a_n^\omega, A) \in \mathbb{F}^\alpha(\Sigma, D) \mid s \in \mathbb{M}(\Sigma, D), \{a_1, \ldots, a_n\}$ *an* I_D*-clique*, $\{a_1, \ldots, a_n\} \times (A \setminus \{a_1, \ldots, a_n\}) \subseteq I_D\}$

(3) $\mathsf{isol}(\mathbb{F}^\delta(\Sigma, D)^{(n)}) = \{(s a_1^\omega \cdots a_n^\omega, D(A)) \in \mathbb{F}^\delta(\Sigma, D) \mid s \in \mathbb{M}(\Sigma, D),\ \{a_1, \ldots, a_n\}$ *an* I_D*-clique*, $\forall b \in D(a_1, \ldots, a_n) : D(b) \subseteq D(A) \Rightarrow b \in \{a_1, \ldots, a_n\}\}$.

Proof. By induction over n.

(1) For $n = 0$ we have $\mathsf{isol}(\mathbb{R}(\Sigma, D)^{(0)}) = \mathsf{isol}(\mathbb{R}(\Sigma, D)) = \mathbb{M}(\Sigma, D)$: on the one hand, $B_{\text{fnf}}(s, m) = \{s\}$ if m is greater than the height of all vertices of $s \in \mathbb{M}(\Sigma, D)$ (cf. also the remarks before Theorem 1.8). On the other hand, since $\mathbb{M}(\Sigma, D)$ is dense in $\mathbb{R}(\Sigma, D)$ (Theorem 1.8), an infinite real trace cannot be topologically isolated.

Now let $n \geq 1$ and assume that the assertion is true for all $l < n$. First let $t \in \mathsf{isol}(\mathbb{R}(\Sigma, D)^{(n)})$. Thus, we find some $\widetilde{m} \in \mathbb{N}_0$ with

$$(*) \qquad B_{\mathrm{fnf}}(t, \widetilde{m}) \cap \mathbb{R}(\Sigma, D)^{(n)} = \{t\}.$$

We claim that each letter of $\mathsf{alphinf}(t)$ is isolated in (Σ, D). For this, let $a \in \mathsf{alphinf}(t)$ and let $b \in \Sigma$ with $a \neq b$. Suppose that $(a, b) \in D$. Let $m \geq \widetilde{m}$ be such that $h_m(t)$ has a maximal vertex labelled with a. Let $t_1 := h_m(t)a^\omega$ and let $t_2 := h_m(t)b^\omega$. Clearly, $t_1, t_2 \in B_{\mathrm{fnf}}(t, m) \subseteq B_{\mathrm{fnf}}(t, \widetilde{m})$. By induction hypothesis, $t_1, t_2 \in \mathbb{R}(\Sigma, D) \setminus \bigcup_{i=0}^{n-1} \mathsf{isol}(\mathbb{R}(\Sigma, D)^{(i)}) = \mathbb{R}(\Sigma, D)^{(n)}$, yielding a contradiction to $(*)$. This proves our claim.

Let $\{a_1, \ldots, a_l\} := \mathsf{alphinf}(t)$. There is some $m \geq \widetilde{m}$ with $t = h_m(t)a_1^\omega \cdots a_l^\omega$. By hypothesis we have $l \geq n$. Suppose that $l - 1 \geq n$. Let $t_3 := h_m(t)a_1^\omega \cdots a_{l-1}^\omega a_l$. Obviously, $t_3 \neq t$ and $t_3 \in B_{\mathrm{fnf}}(t, m) \subseteq B_{\mathrm{fnf}}(t, \widetilde{m})$. But, by hypothesis, $t_3 \in \mathbb{R}(\Sigma, D) \setminus \bigcup_{i=0}^{n-1} \mathsf{isol}(\mathbb{R}(\Sigma, D)^{(i)}) = \mathbb{R}(\Sigma, D)^{(n)}$, a contradiction to $(*)$. This yields $l = n$.

Conversely, let $t = sa_1^\omega \cdots a_n^\omega$ for some $s \in \mathbb{M}(\Sigma, D)$ and some isolated letters $a_1, \ldots, a_n$. Let $\widetilde{m} \in \mathbb{N}_0$ be such that $\widetilde{m}$ is greater than the height of all vertices of s. We show that $B_{\mathrm{fnf}}(t, \widetilde{m}) \cap \mathbb{R}(\Sigma, D)^{(n)} = \{t\}$. Clearly, $t \in B_{\mathrm{fnf}}(t, \widetilde{m})$ and $t \in \mathbb{R}(\Sigma, D) \setminus \bigcup_{i=0}^{n-1} \mathsf{isol}(\mathbb{R}(\Sigma, D)^{(i)}) = \mathbb{R}(\Sigma, D)^{(n)}$ by induction hypothesis. Let $\widetilde{t} \in \mathbb{R}(\Sigma, D)^{(n)}$ with $h_{\widetilde{m}}(\widetilde{t}) = h_{\widetilde{m}}(t) = sa_1^{\widetilde{m}} \cdots a_n^{\widetilde{m}}$. Then, due to the choice of $\widetilde{m}$, we have $\mathsf{alphinf}(\widetilde{t}) \subseteq \{a_1, \ldots, a_n\}$. As $\widetilde{t} \notin \bigcup_{i=0}^{n-1} \mathsf{isol}(\mathbb{R}(\Sigma, D)^{(i)})$, we conclude $\mathsf{alphinf}(\widetilde{t}) = \{a_1, \ldots, a_n\}$ by induction hypothesis. Thus, $\widetilde{t} = sa_1^\omega \cdots a_n^\omega = t$.

(2) For $n = 0$ we have $\mathsf{isol}(\mathbb{F}^\alpha(\Sigma, D)^{(0)}) = \mathsf{isol}(\mathbb{F}^\alpha(\Sigma, D)) = \mathbb{F}_{\mathrm{f}}^\alpha(\Sigma, D)$. This results from Theorem 1.10.

Now let $n \geq 1$ and assume that the assertion is true for all $l < n$. Let $x = (r, A) \in \mathsf{isol}(\mathbb{F}^\alpha(\Sigma, D)^{(n)})$ and let $\widetilde{m} \in \mathbb{N}_0$ with

$$(**) \qquad B_\alpha(x, \widetilde{m}) \cap \mathbb{F}^\alpha(\Sigma, D)^{(n)} = \{x\}.$$

We first claim that $\mathsf{alphinf}(r)$ is an I_D-clique and $\mathsf{alphinf}(r) \times (A \setminus \mathsf{alphinf}(r)) \subseteq I_D$. Indeed, let $a \in \mathsf{alphinf}(r)$ and let $b \in A$ with $a \neq b$. Suppose that $(a, b) \in D$. Let $m \geq \widetilde{m}$ be such that $h_m(r)$ has a maximal vertex labelled with a. Consider the α-traces $x_1 := (h_m(r)a^\omega, A \cup \mathsf{alph}(h_m(r)^{-1}r))$ and $x_2 := (h_m(r)b^\omega, A \cup \mathsf{alph}(h_m(r)^{-1}r))$. Clearly, $x_1, x_2 \in B_\alpha(x, m) \subseteq B_\alpha(x, \widetilde{m})$. Further, by induction hypothesis, $x_1, x_2 \in \mathbb{F}^\alpha(\Sigma, D) \setminus \bigcup_{i=0}^{n-1} \mathsf{isol}(\mathbb{F}^\alpha(\Sigma, D)^{(i)}) = \mathbb{F}^\alpha(\Sigma, D)^{(n)}$. This contradicts $(**)$. Consequently, $(a, b) \in I_D$. We have thus proven that $\{a\} \times A \setminus \{a\} \subseteq I_D$ for all $a \in \mathsf{alphinf}(r)$. Hence, we obtain our claim.

Let $\{a_1, \ldots, a_l\} := \mathsf{alphinf}(r)$. There exists some $m \geq \widetilde{m}$ such that $\mathsf{alph}(h_m(r)^{-1}r) = \mathsf{alphinf}(r)$. Then $r = h_m(r)a_1^\omega \cdots a_l^\omega$. By hypothesis, $l \geq n$. Suppose that $l > n$. Let $x_3 := (h_m(r)a_1^\omega \cdots a_{l-1}^\omega a_l, A)$. Then $x_3 \in B_\alpha(x, m) \subseteq B_\alpha(x, \widetilde{m})$. As $l - 1 \geq n$, we deduce $x_3 \notin \bigcup_{i=0}^{n-1} \mathsf{isol}(\mathbb{F}^\alpha(\Sigma, D)^{(i)})$, whence $x_3 \in \mathbb{F}^\alpha(\Sigma, D)^{(n)}$. Again, this contradicts $(**)$. Hence, we obtain $l = n$ and $x = (h_m(r)a_1^\omega \cdots a_n^\omega, A)$.

To prove the converse inclusion, let $x = (r, A) \in \mathbb{F}^\alpha(\Sigma, D)$, let $s \in \mathbb{M}(\Sigma, D)$, let $\{a_1, \ldots, a_n\}$ be an I_D-clique, let $\{a_1, \ldots, a_n\} \times (A \setminus \{a_1, \ldots, a_n\}) \subseteq I_D$, and let $r = sa_1^\omega \cdots a_n^\omega$. Let $m \in \mathbb{N}_0$ with $r = h_m(r)a_1^\omega \cdots a_n^\omega$. We show that $B_\alpha(x, m+1) \cap \mathbb{F}^\alpha(\Sigma, D)^{(n)} = \{x\}$. Again, it is clear that $x \in B_\alpha(x, m+1)$ and $x \in \mathbb{F}^\alpha(\Sigma, D) \setminus \bigcup_{i=0}^{n-1} \mathsf{isol}(\mathbb{F}^\alpha(\Sigma, D)^{(i)}) = \mathbb{F}^\alpha(\Sigma, D)^{(n)}$ by induction hypothesis. Let $y = (t, B) \in$

$\mathbb{F}^\alpha(\Sigma, D)^{(n)}$ with $h_{m+1}(t) = h_{m+1}(r) = h_m(r)a_1 \cdots a_n$ and $B \cup \mathsf{alph}(h_{m+1}(t)^{-1}t) = A$. Suppose there is a vertex v of t of height at least $m + 2$ that is labelled with some $b \in \Sigma \setminus \{a_1, \ldots, a_n\}$. We may assume that v is of minimal height with this property. We have $b \in \mathsf{alph}(h_{m+1}(t)^{-1}t) \subseteq A$. Therefore, $(a_i, b) \in I_D$ for all $i = 1, \ldots, n$. On the other hand, since $h_{m+1}(t) = h_m(r)a_1 \cdots a_n$, each vertex v' of t with height $h(v') \geq m + 1$ and $h(v')$ strictly less than the height of v has to be labelled with some a_i because of the minimum condition on v. This implies $(a_i, b) \in D$ for some $i \in \{1, \ldots, n\}$, a contradiction. Therefore, $\mathsf{alphinf}(t) \subseteq \{a_1, \ldots, a_n\}$. Next, suppose that $\mathsf{alphinf}(t) \subsetneq \{a_1, \ldots, a_n\}$. Then $\mathsf{alphinf}(t)$ is an I_D-clique and $\mathsf{alphinf}(t) \times (B \setminus \mathsf{alphinf}(t)) \subseteq I_D$ because for all $a \in \mathsf{alphinf}(t)$ and all $b \in B \setminus \mathsf{alphinf}(t)$ we have $b \in A = \{a_1, \ldots, a_n\} \cup (A \setminus \{a_1, \ldots, a_n\})$, whence $(a, b) \in I_D$. As $l := |\mathsf{alphinf}(t)| < n$, we obtain $y = (t, B) \in \mathsf{isol}(\mathbb{F}^\alpha(\Sigma, D)^{(l)})$ by hypothesis, a contradiction to $y \in \mathbb{F}^\alpha(\Sigma, D)^{(n)}$. Therefore, $\mathsf{alphinf}(t) = \{a_1, \ldots, a_n\}$, $t = h_m(r)a_1^\omega \cdots a_n^\omega = r$, $B = B \cup \mathsf{alphinf}(t) = B \cup \mathsf{alph}(h_{m+1}(t)^{-1}t) = A$, and thus $y = x$.

(3) is shown similarly to (2). Again, for $n = 0$ we have $\mathsf{isol}(\mathbb{F}^\delta(\Sigma, D)^{(0)}) = \mathsf{isol}(\mathbb{F}^\delta(\Sigma, D)) = \mathbb{F}^\delta_{\mathrm{f}}(\Sigma, D)$ (cf. the remarks after Theorem 1.10 and [12, Section 6]).

Assume that the assertion is true for all $l < n$, where $n \geq 1$. First let $x = (r, D(A)) \in \mathsf{isol}(\mathbb{F}^\delta(\Sigma, D)^{(n)})$ and choose some $\widetilde{m} \in \mathbb{N}_0$ with

$$(***) \qquad B_\delta(x, \widetilde{m}) \cap \mathbb{F}^\delta(\Sigma, D)^{(n)} = \{x\}.$$

We claim that $\mathsf{alphinf}(r)$ is an I_D-clique and for all $b \in D(\mathsf{alphinf}(r))$ we have $b \in \mathsf{alphinf}(r)$ if $D(b) \subseteq D(A)$. To see this, let $a \in \mathsf{alphinf}(r)$ and let $b \in D(a)$ with $D(b) \subseteq D(A)$. Let $m \geq \widetilde{m}$ be such that $h_m(r)$ has a maximal vertex labelled with a. Consider the δ-traces $x_1 := (h_m(r)a^\omega, D(A) \cup D(\mathsf{alph}(h_m(r)^{-1}r)))$ and $x_2 := (h_m(r)b^\omega, D(A) \cup D(\mathsf{alph}(h_m(r)^{-1}r)))$. Clearly, $x_1, x_2 \in B_\delta(x, m) \subseteq B_\delta(x, \widetilde{m})$. If $x_1, x_2 \in \mathbb{F}^\delta(\Sigma, D)^{(n)}$, then $x_1 = x_2$ by $(***)$ and thus $a = b$. If $x_1 \notin \mathbb{F}^\delta(\Sigma, D)^{(n)}$, then $x_1 \in \bigcup_{i=0}^{n-1} \mathsf{isol}(\mathbb{F}^\delta(\Sigma, D)^{(i)})$. Hence, by induction hypothesis, $x_1 \in \mathsf{isol}(\mathbb{F}^\delta(\Sigma, D)^{(1)})$ and thus $b = a$ because $b \in D(a)$ and $D(b) \subseteq D(A)$. Similarly, if $x_2 \notin \mathbb{F}^\delta(\Sigma, D)^{(n)}$, then $x_2 \in \bigcup_{i=0}^{n-1} \mathsf{isol}(\mathbb{F}^\delta(\Sigma, D)^{(i)})$. Therefore, $x_2 \in \mathsf{isol}(\mathbb{F}^\delta(\Sigma, D)^{(1)})$ and thus $a = b$ because $a \in D(b)$ and $D(a) \subseteq D(\mathsf{alphinf}(r)) \subseteq D(A)$. In all cases, $a = b$; hence our claim is shown.

Now let $\{a_1, \ldots, a_l\} := \mathsf{alphinf}(r)$. There exists a natural number $m \geq \widetilde{m}$ such that $\mathsf{alph}(h_m(r)^{-1}r) = \mathsf{alphinf}(r)$. Then we have $r = h_m(r)a_1^\omega \cdots a_l^\omega$ and $l \geq n$ because $l < n$ would yield a contradiction to our hypothesis. As in the proof of (2), the assumption $l > n$ leads to a contradiction as well. Therefore, $l = n$ and $x = (h_m(r)a_1^\omega \cdots a_n^\omega, D(A))$.

In order to prove the other inclusion, let $x = (r, D(A)) \in \mathbb{F}^\delta(\Sigma, D)$, let $s \in \mathbb{M}(\Sigma, D)$, let $\{a_1, \ldots, a_n\}$ be an I_D-clique such that for all $b \in D(a_1, \ldots, a_n)$ with $D(b) \subseteq D(A)$ we have $b \in \{a_1, \ldots, a_n\}$, and let $r = sa_1^\omega \cdots a_n^\omega$. Let $m \in \mathbb{N}_0$ with $r = h_m(r)a_1^\omega \cdots a_n^\omega$. We prove $B_\delta(x, m+1) \cap \mathbb{F}^\delta(\Sigma, D)^{(n)} = \{x\}$. As before we obtain $x \in B_\delta(x, m+1) \cap \mathbb{F}^\delta(\Sigma, D)^{(n)}$. Let $y = (t, D(B)) \in \mathbb{F}^\delta(\Sigma, D)^{(n)}$ with $h_{m+1}(t) = h_{m+1}(r) = h_m(r)a_1 \cdots a_n$ and $D(B) \cup D(\mathsf{alph}(h_{m+1}(t)^{-1}t)) = D(A)$. As in the proof of (2), suppose there is a vertex v of t having height $h(v) \geq m + 2$ such that v is labelled with some $b \in \Sigma \setminus \{a_1, \ldots, a_n\}$. We may assume that v is of minimal height with this property. As $b \in \mathsf{alph}(h_{m+1}(t)^{-1}t)$, we have $D(b) \subseteq D(A)$ and thus $b \notin D(a_1, \ldots, a_n)$. Hence, $(a_i, b) \in I_D$ for all $i = 1, \ldots, n$. Again, this contradicts $h_{m+1}(t) = h_m(r)a_1 \cdots a_n$ and the minimum condition on v. Consequently, $\mathsf{alphinf}(t) \subseteq \{a_1, \ldots, a_n\}$. Suppose that $\mathsf{alphinf}(t) \subsetneq \{a_1, \ldots, a_n\}$. Note that $\mathsf{alphinf}(t)$ is an I_D-clique. Let $b \in D(\mathsf{alphinf}(t))$ be such that $D(b) \subseteq D(B)$. As $D(\mathsf{alphinf}(t)) \subseteq D(a_1, \ldots, a_n)$ and $D(B) \subseteq D(A)$, we conclude $b \in \{a_1, \ldots, a_n\}$. Since $\{a_1, \ldots, a_n\}$ is an I_D-clique and $b \in D(\mathsf{alphinf}(t))$, we infer $b \in \mathsf{alphinf}(t)$. But as $l := |\mathsf{alphinf}(t)| < n$, we obtain $y = (t, D(B)) \in \mathsf{isol}(\mathbb{F}^\delta(\Sigma, D)^{(l)})$ by hypothesis. This is a contradiction to $y \in \mathsf{isol}(\mathbb{F}^\delta(\Sigma, D)^{(n)})$. As a consequence, $\mathsf{alphinf}(t) = \{a_1, \ldots, a_n\}$. Hence, $t = h_m(r)a_1^\omega \cdots a_n^\omega = r$, $D(B) = D(B) \cup D(\mathsf{alphinf}(t)) = D(B) \cup D(\mathsf{alph}(h_{m+1}(t)^{-1}t)) = D(A)$, whence $y = x$. ∎

In the light of the previous theorem, we define the following “graph-theoretic” numbers for dependence alphabets (Σ, D):

$$
\begin{aligned}
i(\Sigma, D) &:= |\{a \in \Sigma \mid a \text{ is isolated in } (\Sigma, D)\}|,\\
j_\alpha(\Sigma, D) &:= \max\{|A| \mid A \subseteq \Sigma \text{ an } I_D\text{-clique}\},\\
j_\delta(\Sigma, D) &:= \max\{|A| \mid A \subseteq \Sigma \text{ an } I_D\text{-clique}, \forall b \in \Sigma\colon D(b) \subseteq D(A) \Rightarrow b \in A\}.
\end{aligned}
$$

Note that $i(\Sigma, D)$, $j_\alpha(\Sigma, D)$, $j_\delta(\Sigma, D) \in \mathbb{N}_0$ and $i(\Sigma, D) \leq j_\delta(\Sigma, D) \leq j_\alpha(\Sigma, D)$. Theorem 6.2 implies that $i(\Sigma, D)$, $j_\alpha(\Sigma, D)$, and $j_\delta(\Sigma, D)$ are topological invariants: $i(\Sigma, D)$ is the least number $n \in \mathbb{N}_0$ such that $\mathbb{R}(\Sigma, D)^{(n+1)}$ has no topologically isolated elements. Similarly for $j_\alpha(\Sigma, D)$ and $j_\delta(\Sigma, D)$. Thus, we obtain:

6.3. Corollary. *Let (Σ_1, D_1) and (Σ_2, D_2) be dependence alphabets. If*

$$
\left\{
\begin{array}{c}
\mathbb{R}(\Sigma_1, D_1) \textit{ and } \mathbb{R}(\Sigma_2, D_2)\\
\mathbb{F}^\alpha(\Sigma_1, D_1) \textit{ and } \mathbb{F}^\alpha(\Sigma_2, D_2)\\
\mathbb{F}^\delta(\Sigma_1, D_1) \textit{ and } \mathbb{F}^\delta(\Sigma_2, D_2)
\end{array}
\right\}
$$

are homeomorphic, respectively, then

$$
\left\{
\begin{array}{rcl}
i(\Sigma_1, D_1) &=& i(\Sigma_2, D_2)\\
j_\alpha(\Sigma_1, D_1) &=& j_\alpha(\Sigma_2, D_2)\\
j_\delta(\Sigma_1, D_1) &=& j_\delta(\Sigma_2, D_2)
\end{array}
\right\},
$$

respectively. ∎

We define another number dealing with a simple property of (Σ, D):

$$
m(\Sigma, D) :=
\begin{cases}
1 & \text{if there are } a, b \in \Sigma \text{ with } (a, b) \in D \text{ and } a \neq b,\\
0 & \text{otherwise, i.e. all letters of } \Sigma \text{ are isolated in } (\Sigma, D).
\end{cases}
$$

Note that $m(\Sigma, D) = 0$ if and only if $D = \mathsf{id}_\Sigma$. This implies that $\mathbb{R}(\Sigma, D)$, $\mathbb{F}^\alpha(\Sigma, D)$, and $\mathbb{F}^\delta(\Sigma, D)$ have to be countable because Σ is finite. On the other hand, $m(\Sigma, D) = 1$ if and only if we find two distinct letters in Σ that are dependent. Then $\mathbb{R}(\Sigma, D)$ and, in particular, $\mathbb{F}^\alpha(\Sigma, D)$ and $\mathbb{F}^\delta(\Sigma, D)$ are uncountable. Consequently, $m(\Sigma, D)$ is invariant under bijective maps.

Using Corollary 1.6 and the previous results of this chapter, we are able to prove our main result on the topology of real traces. It characterizes $\lambda_{\mathbb{R}(\Sigma,D)}$ in terms of the numbers $i(\Sigma, D)$ and $m(\Sigma, D)$. Thus, $\lambda_{\mathbb{R}(\Sigma,D)}$ is completely determined by two basic properties of the underlying dependence alphabet, viz. the number of isolated letters in (Σ, D) and the existence of a “non-trivial edge” in (Σ, D).

We note here that whenever we deal with direct products of topological spaces, then we assume the product to be equipped with the product topology.

6.4. THEOREM. *Let* (Σ_1, D_1) *and* (Σ_2, D_2) *be two dependence alphabets. Then the following are equivalent*:

(i) $\mathbb{R}(\Sigma_1, D_1)$ *is homeomorphic to* $\mathbb{R}(\Sigma_2, D_2)$.
(ii) $i(\Sigma_1, D_1) = i(\Sigma_2, D_2)$ *and* $m(\Sigma_1, D_1) = m(\Sigma_2, D_2)$.

Proof. (i)⇒(ii) follows from Corollary 6.3 and the remarks on $m(\Sigma, D)$ above.

(ii)⇒(i). Let $i := i(\Sigma_1, D_1) = i(\Sigma_2, D_2)$ and let $m := m(\Sigma_1, D_1) = m(\Sigma_2, D_2)$. For $\nu = 1, 2$ let $\Sigma'_\nu := \{a \in \Sigma_\nu \mid a \text{ is not isolated in } (\Sigma_\nu, D_\nu)\}$ and $D'_\nu := D_\nu|_{\Sigma'_\nu \times \Sigma'_\nu}$. Recall that for one-letter alphabets we have $\mathbb{R}(\{0\}, \{0\}^2) = \{0\}^\infty$. By applying Proposition 6.1 i times, we obtain $\mathbb{R}(\Sigma_\nu, D_\nu)$ to be homeomorphic to the product $(\{0\}^\infty)^i \times \mathbb{R}(\Sigma'_\nu, D'_\nu)$. Consequently, we may assume that $i = 0$, i.e. $\Sigma_\nu = \Sigma'_\nu$ for $\nu = 1, 2$. But then $m = 0$ is equivalent to $\Sigma_1 = \Sigma_2 = \emptyset$. Thus, without loss of generality, let $m = 1$.

Recall that $\mathbb{R}(\Sigma_\nu, D_\nu)$ is a compact, ultrametrizable space and $\mathsf{isol}(\mathbb{R}(\Sigma_\nu, D_\nu)) = \mathbb{M}(\Sigma_\nu, D_\nu)$ is dense in $\mathbb{R}(\Sigma_\nu, D_\nu)$ for $\nu = 1, 2$; cf. Theorem 1.8 or Example 3.35. As $m = 1$, the set $\mathsf{isol}(\mathbb{R}(\Sigma_\nu, D_\nu))$ is countably infinite ($\nu = 1, 2$). We see that conditions (1) and (2) of Corollary 1.6 are satisfied. To show condition (3), we claim that $\mathbb{R}(\Sigma_\nu, D_\nu) \setminus \mathsf{isol}(\mathbb{R}(\Sigma_\nu, D_\nu))$ is homeomorphic to the Cantor discontinuum. In fact, a classical theorem from general topology states that any non-empty, compact, ultrametrizable, perfect space is homeomorphic to the Cantor discontinuum. Since $\mathsf{isol}(\mathbb{R}(\Sigma_\nu, D_\nu))$ is open in $\mathbb{R}(\Sigma_\nu, D_\nu)$, we see that $\mathbb{R}(\Sigma_\nu, D_\nu) \setminus \mathsf{isol}(\mathbb{R}(\Sigma_\nu, D_\nu))$ is closed. Hence, it is a compact and ultrametrizable space. It is non-empty because $m = 1$. As $i = 0$, Theorem 6.2 tells us that $\mathbb{R}(\Sigma_\nu, D_\nu)^{(1)} = \mathbb{R}(\Sigma_\nu, D_\nu) \setminus \mathsf{isol}(\mathbb{R}(\Sigma_\nu, D_\nu))$ cannot contain any topologically isolated elements. Consequently, our claim follows. Finally, Corollary 1.6 yields the assertion. ∎

The following corollary states that $\mathbb{R}(\Sigma, D)$ is homeomorphic to a product space which is built up basically by two different components. On the one hand, we have the space $\{0\}^\infty = \{0^k \mid k \in \mathbb{N}_0\} \cup \{0^\omega\}$ together with the Lawson topology of $(\{0\}^\infty, \leq)$. Recall that this is the topology induced by the prefix metric on $\{0\}^\infty$. Obviously, this space is homeomorphic to the Aleksandrov one-point compactification of the non-negative integers equipped with the discrete topology. The second component is the space $\{0, 1\}^\infty$ of all (finite or infinite) words over $\{0, 1\}$. It is equipped with the Lawson topology with regard to the prefix order. Again, this is the topology induced by the prefix metric on $\{0, 1\}^\infty$.

6.5. COROLLARY. *Let* (Σ, D) *be a dependence alphabet. Then* $\mathbb{R}(\Sigma, D)$ *is homeomorphic to the product*

$$(\{0\}^\infty)^{i(\Sigma,D)} \times (\{0,1\}^\infty)^{m(\Sigma,D)}.$$

Proof. If $m(\Sigma, D) = 0$, then let $\Sigma_1 := \{1, 2, \ldots, i(\Sigma, D)\}$ and $D_1 := \mathsf{id}_{\Sigma_1}$. If $m(\Sigma, D) = 1$, then let $\Sigma_1 := \{1, 2, \ldots, i(\Sigma, D), i(\Sigma, D)+1, i(\Sigma, D)+2\}$ and $D_1 := \mathsf{id}_{\Sigma_1} \cup \{(i(\Sigma, D)+1, i(\Sigma, D)+2), (i(\Sigma, D)+2, i(\Sigma, D)+1)\}$. By Theorem 6.4(ii)⇒(i) we deduce that $\mathbb{R}(\Sigma, D)$ and $\mathbb{R}(\Sigma_1, D_1)$ are homeomorphic. Proposition 6.1 tells us that $\mathbb{R}(\Sigma_1, D_1)$ is homeomorphic to the product $(\{0\}^\infty)^{i(\Sigma,D)} \times (\{0,1\}^\infty)^{m(\Sigma,D)}$. ∎

As to the topology of approximating traces, we only know a partial answer. We derive an analogous result to Corollary 6.5 provided that the dependence relation is transitive, i.e. an equivalence relation. For any finite set Σ we shorten $\Sigma^\alpha := \mathbb{F}^\alpha(\Sigma, \Sigma^2)$ and $\Sigma^\delta := \mathbb{F}^\delta(\Sigma, \Sigma^2)$. In particular, $\{0\}^\alpha = \mathbb{F}^\alpha(\{0\}, \{0\}^2)$ and $\{0,1\}^\alpha = \mathbb{F}^\alpha(\{0,1\}, \{0,1\}^2)$.

6.6. THEOREM. *Let (Σ, D) be a dependence alphabet such that D is transitive. Then $j_\alpha(\Sigma, D)$ is the number of equivalence classes of D. The space $\mathbb{F}^\alpha(\Sigma, D)$ is homeomorphic to the product*

$$(\{0\}^\alpha)^{i(\Sigma,D)} \times (\{0,1\}^\alpha)^{j_\alpha(\Sigma,D)-i(\Sigma,D)}.$$

Proof. Let $i := i(\Sigma, D)$ and let $j_\alpha := j_\alpha(\Sigma, D)$. Let $\Sigma' := \{a \in \Sigma \mid a$ is not isolated in $(\Sigma, D)\}$ and let $D' := D|_{\Sigma' \times \Sigma'}$. Using Proposition 6.1 i times, we find that $\mathbb{F}^\alpha(\Sigma, D)$ is homeomorphic to $(\{0\}^\alpha)^i \times \mathbb{F}^\alpha(\Sigma', D')$. Thus, we must show that $\mathbb{F}^\alpha(\Sigma', D')$ is homeomorphic to $(\{0,1\}^\alpha)^{j_\alpha - i}$. Since D is transitive, we infer that j_α is the number of equivalence classes with respect to D. Consequently, $j_\alpha - i$ is the number of equivalence classes consisting of at least two letters. Due to Proposition 6.1 it suffices to check that Γ^α is homeomorphic to $\{0,1\}^\alpha$ for all finite sets Γ with $|\Gamma| \geq 2$. We prove this by applying Corollary 1.6 (the corollary to Pierce's Theorem) twice. Let Γ be a finite set with at least two elements. Recall that Γ^α is compact and ultrametrizable (see Theorem 1.10 or Example 3.36(a)). Let X be the first derivation of Γ^α, i.e. $X = \Gamma^\alpha \setminus \mathsf{isol}(\Gamma^\alpha)$. Hence, X is closed in Γ^α, whence it is a compact and, moreover, ultrametrizable space.

We show that $\mathsf{isol}(X)$ is dense in X. To see this, let $(r, C) \in X = \Gamma^\alpha \setminus \mathbb{F}^\alpha_f(\Gamma, \Gamma^2)$. Let $m \in \mathbb{N}_0$. We find some $\widetilde{m} \geq m$ such that $\mathsf{alph}(h_{\widetilde{m}}(r)^{-1}r) = \mathsf{alphinf}(r)$. Let $\{c_1, \ldots, c_j\} := C$. Let $s := h_{\widetilde{m}}(r)c_1 \cdots c_{j-1}$ and let $t := sc_j^\omega$. Then $(t, \{c_j\}) \in \{(sa^\omega, \{a\}) \mid s \in \Gamma^\star,$ $a \in \Gamma\} = \mathsf{isol}((\Gamma^\alpha)^{(1)}) = \mathsf{isol}(X)$ (see Theorem 6.2 and note that I_{Γ^2}-cliques cannot contain more than one letter). Furthermore, $h_{\widetilde{m}}(r) = h_{\widetilde{m}}(s)$ because r is an infinite word. Thus, $h_{\widetilde{m}}(r) = h_{\widetilde{m}}(t)$ and

$$\begin{aligned} h_{\widetilde{m}}(r, C) &= (h_{\widetilde{m}}(r), C \cup \mathsf{alph}(h_{\widetilde{m}}(r)^{-1}r)) = (h_{\widetilde{m}}(r), C) \\ &= (h_{\widetilde{m}}(t), \{c_j\} \cup \mathsf{alph}(c_1 \cdots c_{j-1}c_j^\omega)) \\ &= (h_{\widetilde{m}}(t), \{c_j\} \cup \mathsf{alph}(h_{\widetilde{m}}(t)^{-1}t)) = h_{\widetilde{m}}(t, \{c_j\}). \end{aligned}$$

This implies $h_m(r, C) = h_m(t, \{c_j\})$ and shows us that (r, C) lies in the closure of $\mathsf{isol}(X)$.

We see that $\mathsf{isol}(X) = \{(sa^\omega, \{a\}) \mid s \in \Gamma^\star,\ a \in \Gamma\}$ is countably infinite. Moreover, by Theorem 6.2 we infer $\mathsf{isol}(X^{(1)}) = \mathsf{isol}((\Gamma^\alpha)^{(2)}) = \{(sa_1^\omega a_2^\omega, A) \in \Gamma^\alpha \mid s \in \Gamma^\star,\ \{a_1, a_2\}$ an I_{Γ^2}-clique, $\{a_1, a_2\} \times (A \setminus \{a_1, a_2\}) \subseteq I_{\Gamma^2} = \emptyset\} = \emptyset$. Therefore, $X^{(1)}$ is a perfect space. Since it is also compact, ultrametrizable, and non-empty (because $|\Gamma| \geq 2$), $X^{(1)}$ is homeomorphic to the Cantor discontinuum. All these conclusions are in particular valid for the alphabet $\{0,1\}$. Letting $Y := \{0,1\}^\alpha \setminus \mathsf{isol}(\{0,1\}^\alpha)$, we conclude that $X^{(1)}$ and $Y^{(1)}$ are homeomorphic. Now conditions (1)–(3) of Corollary 1.6 are satisfied, whence $X = \Gamma^\alpha \setminus \mathsf{isol}(\Gamma^\alpha)$ and $Y = \{0,1\}^\alpha \setminus \mathsf{isol}(\{0,1\}^\alpha)$ are homeomorphic.

Note that $\mathsf{isol}(\Gamma^\alpha) = \mathbb{F}^\alpha_f(\Gamma, \Gamma^2)$ is dense in Γ^α and $\mathsf{isol}(\{0,1\}^\alpha) = \mathbb{F}^\alpha_f(\{0,1\}, \{0,1\}^2)$ is dense in $\{0,1\}^\alpha$ (Theorem 1.10, Example 3.36(a)). Further, $\mathsf{isol}(\Gamma^\alpha)$ and $\mathsf{isol}(\{0,1\}^\alpha)$ are countably infinite. By applying Corollary 1.6 to the spaces Γ^α and $\{0,1\}^\alpha$, we deduce that Γ^α and $\{0,1\}^\alpha$ are homeomorphic. ∎

6.7. REMARK. The spaces $\{0\}^\alpha$ and $\{0\}^\infty$ are homeomorphic.

To see this, note that $\{0\}^\alpha = \{(0^k, \emptyset) \mid k \in \mathbb{N}_0\} \cup \{(0^k, \{0\}) \mid k \in \mathbb{N}_0 \cup \{\omega\}\}$. Let $g : \{0\}^\alpha \to \{0\}^\infty$ be defined by $g(0^k, \{0\}) := 0^{2k}$, $g(0^k, \emptyset) := 0^{2k+1}$ $(k \in \mathbb{N}_0)$, and $g(0^\omega, \{0\}) := 0^\omega$. Clearly, g is bijective. We show that g is monotone. Let $x_1 = (r_1, A_1)$, $x_2 = (r_2, A_2) \in \{0\}^\alpha$ with $x_1 \sqsubseteq x_2$. Then $r_1 \leq r_2$ and $A_2 \subseteq A_1 \cup \mathsf{alph}(r_1^{-1}r_2)$. We may assume $r_1, r_2 \in \{0\}^\star$. There are $k, l \in \mathbb{N}_0$ with $k \leq l$ such that $r_1 = 0^k$ and $r_2 = 0^l$, whence $g(x_1) \in \{0^{2k}, 0^{2k+1}\}$ and $g(x_2) \in \{0^{2l}, 0^{2l+1}\}$. If $g(x_1) = 0^{2k}$, then $g(x_1) = 0^{2k} \leq 0^{2l} \leq g(x_2)$. If $g(x_1) = 0^{2k+1}$, then $A_1 = \emptyset$. If $A_2 = \emptyset$, then $g(x_1) = 0^{2k+1} \leq 0^{2l+1} = g(x_2)$. If $A_2 = \{0\}$, then $r_1 \neq r_2$ and thus $k \leq l - 1$. Therefore, $g(x_1) = 0^{2k+1} \leq 0^{2l-1} \leq 0^{2l} = g(x_2)$.

For all $k, n \in \mathbb{N}_0$ we have $h_n(g(0^k, \{0\})) = 0^{\min\{2k,n\}} \leq 0^{\min\{2k,2n\}} = g(0^{\min\{k,n\}}, \{0\}) = g(h_n(0^k, \{0\}))$. If $n \geq k$, then $h_n(g(0^k, \emptyset)) = 0^{\min\{2k+1,n\}} \leq 0^{2k+1} = g(h_n(0^k, \emptyset))$. If $n < k$, then $h_n(g(0^k, \emptyset)) = 0^{\min\{2k+1,n\}} \leq 0^n \leq 0^{2n} = g(h_n(0^k, \emptyset))$. By Proposition 2.14 we infer that g is continuous. Since $\{0\}^\alpha$ is compact Hausdorff, g is a homeomorphism.

6.8. THEOREM. *Let (Σ, D) be a dependence alphabet such that D is transitive. Then $j_\delta(\Sigma, D) = i(\Sigma, D)$. The space $\mathbb{F}^\delta(\Sigma, D)$ is homeomorphic to*

$$(\{0\}^\delta)^{i(\Sigma,D)} \times (\{0,1\}^\delta)^{m(\Sigma,D)}.$$

Proof. We already know that $i(\Sigma, D) \leq j_\delta(\Sigma, D)$. Let $A \subseteq \Sigma$ be an I_D-clique such that for all letters $b \in \Sigma$ with $D(b) \subseteq D(A)$ we have $b \in A$. Let $a \in A$ and let $b \in \Sigma$ with $(a, b) \in D$. As $b \in D(A)$, we infer $D(b) \subseteq D(A)$ because D is an equivalence relation. Therefore, $b \in A$. Since $(a, b) \in D$, we conclude $a = b$. This tells us that a is isolated in (Σ, D). We conclude $j_\delta(\Sigma, D) \leq i(\Sigma, D)$ and hence $j_\delta(\Sigma, D) = i(\Sigma, D)$.

As in the proof of the previous theorem we find $\mathbb{F}^\delta(\Sigma, D)$ to be homeomorphic to $(\{0\}^\delta)^{i(\Sigma,D)} \times \mathbb{F}^\delta(\Sigma', D')$, where Σ' is the set of all letters in Σ that are not isolated in (Σ, D) and D' is the restriction of D to $\Sigma' \times \Sigma'$. We may assume $m(\Sigma, D) = 1$. It thus remains to show that $\mathbb{F}^\delta(\Sigma', D')$ is homeomorphic to $\{0,1\}^\delta$. Recall that $\mathbb{F}^\delta(\Sigma', D')$ and $\{0,1\}^\delta$ are compact and ultrametrizable, $\mathsf{isol}(\mathbb{F}^\delta(\Sigma', D')) = \mathbb{F}^\delta_f(\Sigma', D')$ is dense in $\mathbb{F}^\delta(\Sigma', D')$, and $\mathsf{isol}(\{0,1\}^\delta) = \mathbb{F}^\delta_f(\{0,1\}, \{0,1\}^2)$ is dense in $\{0,1\}^\delta$ (see the remarks after Theorem 1.10 or Example 3.36(b)). In addition, $\mathsf{isol}(\mathbb{F}^\delta(\Sigma', D'))$ and $\mathsf{isol}(\{0,1\}^\delta)$ are countably infinite as $m(\Sigma', D') = m(\Sigma, D) = 1$. Thus, (1) and (2) of Corollary 1.6 are satisfied. We verify condition (3) by proving that both $\mathbb{F}^\delta(\Sigma', D') \setminus \mathsf{isol}(\mathbb{F}^\delta(\Sigma', D')) = \mathbb{F}^\delta(\Sigma', D')^{(1)}$ and $\{0,1\}^\delta \setminus \mathsf{isol}(\{0,1\}^\delta) = (\{0,1\}^\delta)^{(1)}$ are homeomorphic to the Cantor discontinuum. Clearly, $\mathbb{F}^\delta(\Sigma', D')^{(1)}$ is compact and ultrametrizable. Since $j_\delta(\Sigma', D') = i(\Sigma', D') = 0$, we deduce $\mathsf{isol}(\mathbb{F}^\delta(\Sigma', D')^{(1)}) = \emptyset$, i.e. $\mathbb{F}^\delta(\Sigma', D')^{(1)}$ is perfect. As $m(\Sigma', D') = 1$, we have $\Sigma' \neq \emptyset$ and thus $\mathbb{F}^\delta(\Sigma', D')^{(1)} \neq \emptyset$. Thus, $\mathbb{F}^\delta(\Sigma', D')^{(1)}$ is homeomorphic to the Cantor discontinuum. Similarly for $(\{0,1\}^\delta)^{(1)}$. Finally, Corollary 1.6 tells us that $\mathbb{F}^\delta(\Sigma', D')$ and $\{0,1\}^\delta$ are homeomorphic. ∎

6.9. COROLLARY. *Let (Σ, D) be a dependence alphabet such that D is transitive. Then $\mathbb{F}^\delta(\Sigma, D)$ is homeomorphic to*

$$(\{0\}^\infty)^{i(\Sigma,D)} \times (\{0,1\}^\infty)^{m(\Sigma,D)}.$$

Hence, $\mathbb{F}^\delta(\Sigma, D)$ and $\mathbb{R}(\Sigma, D)$ are homeomorphic.

Proof. We already know that the spaces $\{0,1\}^\infty$ and $\{0,1\}^\delta$ are compact and ultrametrizable, and $\mathsf{isol}((\{0,1\}^\infty)) = \{0,1\}^\star$ and $\mathsf{isol}(\{0,1\}^\delta) = \mathbb{F}_f^\delta(\{0,1\},\{0,1\}^2)$ are countably infinite. Further, $\{0,1\}^\star$ is dense in $\{0,1\}^\infty$ and $\mathbb{F}_f^\delta(\{0,1\},\{0,1\}^2)$ is dense in $\{0,1\}^\delta$. We also know that the spaces $(\{0,1\}^\infty)^{(1)} = \{0,1\}^\infty \setminus \{0,1\}^\star = \{0,1\}^\omega$ and $(\{0,1\}^\delta)^{(1)}$ are both homeomorphic to the Cantor discontinuum. Consequently, $\{0,1\}^\infty$ and $\{0,1\}^\delta$ are homeomorphic by Corollary 1.6. Note that $\{0\}^\infty$ is homeomorphic to $\{0\}^\alpha = \{0\}^\delta$ (Remark 6.7). Therefore, we are done in view of Theorem 6.8 and Corollary 6.5. ■

6.10. EXAMPLE. Let $k, l \in \mathbb{N}$. Then

(1) $(\{0,1\}^\alpha)^k$ is homeomorphic to $(\{0,1\}^\alpha)^l$ if and only if $k = l$.
(2) The spaces $(\{0,1\}^\delta)^k$ and $(\{0,1\}^\infty)^l$ are homeomorphic.

Part (1) follows from Proposition 6.1 and Corollary 6.3. Part (2) results from Proposition 6.1 and Corollary 6.9.

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