I. Ulam's stability problem and twisted sums: two faces of a coin

Ulam's stability problem for additive maps and the theory of extensions of Banach spaces have remained so far unaware of each other. To support this assertion it is enough to peruse the survey articles [36, 52] or the recent book [51] and observe that no single reference is made to papers in Banach space theory; and, conversely, a monograph on extensions of Banach spaces such as [20] does not say a word about Ulam's problem and its relatives. Or e.g. that the problem posed by Rassias in 1990 [83] had been essentially solved by Ribe in 1979 [89].

In fact, both worlds constitute two faces of the same coin. In this spirit, the purpose of this paper is to progress towards a global understanding of the problem, and we shall do so by developing a theory of vector-valued nearly additive mappings on semigroups. While a unified theory seems to be out of reach by now, the part of the theory involving quasi-normed groups and maps with values in Banach spaces seems to be ripe with these pages. We hasten to alert the reader that this paper can by no means be considered a survey, although it contains occasional historical comments; two excellent survey papers are [36, 52].

1. Background on Ulam's problem. More than half a century ago, S. M. Ulam (see [101, 102, 103] or [104]) posed the problem of finding conditions under which a "nearly additive" mapping must be "near" to an additive mapping. More precisely, Ulam asked:

PROBLEM 1.1. *Let G_1 be a group and let G_2 be a metric group with metric $d(\cdot,\cdot)$. Given $\delta > 0$, does there exist $\varepsilon > 0$ such that to each mapping $F : G_1 \to G_2$ satisfying $d(F(x+y), F(x)+F(y)) \le \varepsilon$ for all $x, y \in G_1$ there corresponds a homomorphism $A : G_1 \to G_2$ with $d(F(x), A(x)) \le \delta$ for all $x \in G_1$?*

In this paper, we shall mainly think about maps with values in Banach (or quasi-Banach) spaces. Thus, we speak of additive maps instead of homomorphisms. Of course, the problem for representations (i.e., with G_2 either the unitary group of a Hilbert space or the group of automorphisms of a Banach space) or characters (i.e., with $G_2 = \mathbb{T}$, the circle group) also received attention (explicitly in [68, 48, 45, 22, 23, 94, 12] and also implicitly in [56, 57, 54]), but we restrict ourselves to the case in which the target group is a Banach space with the usual metric induced by the norm.

To motivate the problem and to fix ideas, throughout this chapter we consider maps acting between real Banach spaces. An *additive mapping* $A : Z \to Y$ is one satisfying

$$A(x+y) = A(x) + A(y)$$

for all $x, y \in Z$. Consider now a "small" perturbation of A, say $F(x) = A(x) + B(x)$ with $\|B(x)\| \le \varepsilon$ for all $x \in Z$. Then the perturbed map F is "approximately additive" in the sense that $F(x+y)$ is not too far from $F(x) + F(y)$. In fact,

$$\|F(x+y) - F(x) - F(y)\| = \|B(x+y) - B(x) - B(y)\| \le 3\varepsilon.$$

This is the obvious way of constructing a "nearly additive" map. The question of Ulam is whether that is the only way. Needless to say, there are several possibilities, as we shall see, for understanding a *small* perturbation or a *nearly additive* map.

The first partial answer to Ulam's problem was given by D. H. Hyers [50] as follows.

THEOREM 1.2. *Let $F : Z \to Y$ be a mapping acting between Banach spaces and satisfying $\|F(x+y) - F(x) - F(y)\| \le \varepsilon$ for some $\varepsilon \ge 0$ and every $x, y \in Z$. Then there exists a unique additive map $A : Z \to Y$ such that, for all $x \in Z$, $\|F(x) - A(x)\| \le \varepsilon$.*

Proof. The hypothesis about F and a straightforward induction argument yield

$$\|F(2^n x) - 2^n F(x)\| \le (2^n - 1)\varepsilon.$$

Thus for $n, k \in \mathbb{N}$ one has

$$\|F(2^{n+k}x) - 2^k F(2^n x)\| \le 2^k \varepsilon.$$

Dividing by 2^{n+k} one obtains the estimate

$$\left\| \frac{F(2^{n+k}x)}{2^{k+n}} - \frac{F(2^n x)}{2^n} \right\| \le \frac{\varepsilon}{2^n},$$

which shows that for every $x \in Z$, the sequence $F(2^n x)/2^n$ converges in Y. Define

$$A(x) = \lim_{n\to\infty} \frac{F(2^n x)}{2^n}.$$

Clearly, $\|F(x) - A(x)\| \le \varepsilon$ for all x. Let us verify that A is additive. Fix $x, y \in Z$. Then

$$\|A(x+y) - A(x) - A(y)\| = \lim_{n\to\infty} \left\| \frac{F(2^n(x+y))}{2^n} - \frac{F(2^n x^n)}{2^n} - \frac{F(2^n y)}{2^n} \right\| \le \lim_{n\to\infty} \frac{\varepsilon}{2^n} = 0,$$

as desired. The uniqueness of the approximating map is clear. ■

Nowadays this result is commonly considered as one that originated the Hyers–Ulam stability theory. Nevertheless, as pointed out by the referee, one should remember that already in 1925 Pólya and Szegő in their famous book "Problems and Theorems in Analysis I" proved a similar result for real sequences:

THEOREM 1.3 [79, Chapter 3, Problem 99]. *For any real sequence (a_n) satisfying the inequality $|a_{n+m} - a_n - a_m| < 1$ for all $n, m \in \mathbb{N}$, there exists a finite limit $\omega = \lim_{n\to\infty} a_n/n$ and one has $|a_n - \omega n| < 1$ for all $n \in \mathbb{N}$.*

In the case where $Y = \mathbb{R}$ Hyers' theorem becomes an immediate consequence of this result. Be that as it may, later, Rassias [81] considered mappings with unbounded Cauchy differences and proved the next Theorem 1.4 for $0 \le p < 1$ thus generalizing Hyers' result. Rassias [81] observed that the same proof also worked for $p < 0$ and asked about the possibility of extending the result for $p \ge 1$; the affirmative answer for $p > 1$ was provided by Gajda [37]. Thus one has the following starting result. The case $p = 0$ is Theorem 1.2.

THEOREM 1.4 (Hyers, Rassias, Gajda). *Let p be a fixed real number different from* 1. *Suppose $F : Z \to Y$ is a mapping satisfying*

$$\|F(x+y) - F(x) - F(y)\| \le \varepsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in Z$. Then there exists a unique additive map $A : Z \to Y$ such that

$$\|F(x) - A(x)\| \le \frac{2\varepsilon}{|2^p - 2|}\|x\|^p$$

for all $x \in Z$.

Proof. The case $-\infty < p < 1$ is as follows. By induction on n one sees that

$$\left\|\frac{F(2^n x)}{2^n} - F(x)\right\| \le \varepsilon\|x\|^p \sum_{i=0}^{n-1} 2^{(p-1)i} \le \frac{2\varepsilon}{2 - 2^p}\|x\|^p$$

for every n and x. On the other hand, for $n, k \in \mathbb{N}$ and $x \in Z$ one has

$$\left\|\frac{F(2^{n+k}x)}{2^{k+n}} - \frac{F(2^n x)}{2^n}\right\| \le 2^{n(p-1)}\frac{2\varepsilon}{2 - 2^p}\|x\|^p,$$

so that $A(x) = \lim_{n\to\infty} F(2^n x)/2^n$ defines a map from Z to Y with $\|F(x) - A(x)\| \le (2\varepsilon/(2 - 2^p))\|x\|^p$. Finally, since

$$\|F(2^n(x+y)) - F(2^n x) - F(2^n y)\| \le 2^{np}\varepsilon(\|x\|^p + \|y\|^p),$$

dividing by 2^n and letting $n \to \infty$ we obtain the additivity of A. This ends the proof in case $p < 1$.

The proof for $p > 1$ is similar. In this case we use the behavior of F near zero. The approximating additive map is now given by

$$A(x) = \lim_{n\to\infty} 2^n F\left(\frac{x}{2^n}\right).$$

In any case the uniqueness of the additive map is obvious. ∎

For more general results in this direction the reader can consult Rassias [81, 84], Gajda [37], Isac and Rassias [53], Rassias and Šemrl [86], Găvruţa [40], Jung [59] and the surveys of Hyers and Rassias [52] and Forti [36]. It should be noted that most of these results are particular cases of a general result of Forti [33] about stability of functional equations of the form $g(F(x,y)) = G(f(x), f(y))$. See [51] for an account.

Unfortunately, Theorem 1.4 gives no information about homogeneous maps; recall that a mapping $F : E \to F$ acting between vector spaces is *homogeneous* provided $F(\lambda x) = \lambda F(x)$ for all $x \in E$ and all scalars $\lambda \in \mathbb{R}$. In fact, it is easily seen that a homogeneous map (or even a 2-*homogeneous map*, i.e., such that $F(2x) = 2F(x)$) which satisfies the hypothesis of Theorem 1.4 is already additive. So, in a sense, to prove Theorem 1.4 one only needs to find a 2-homogeneous map near F and then the commutativity of the domain automatically gives additivity.

The remainder case $p = 1$ is exceptional. The following is more than a mere example.

EXAMPLE 1.5 (mainly Ribe [89]). *Let $F : \mathbb{R} \to \mathbb{R}$ be given by $F(t) = t\log_2|t|$, assuming $0\log_2 0 = 0$. Then $|F(s+t) - F(s) - F(t)| \le |s| + |t|$ for all $s, t \in \mathbb{R}$ but there is no additive function $A : \mathbb{R} \to \mathbb{R}$ satisfying $|F(t) - A(t)| \le C|t|$.*

Proof. To verify the first part, let us consider first the case in which s and t have the same sign. One has

$$\begin{aligned}|F(s+t)-F(s)-F(t)| &= |(s+t)\log_2(s+t)-s\log_2 s-t\log_2 t| \\ &= |s\log_2(s+t)+t\log_2(s+t)-s\log_2 s-t\log_2 t| \\ &= \left|s\log_2\left(\frac{s}{s+t}\right)+t\log_2\left(\frac{t}{s+t}\right)\right| \\ &\leq (|s|+|t|)\left|\frac{s}{s+t}\log_2\left(\frac{s}{s+t}\right)+\frac{t}{s+t}\log_2\left(\frac{t}{s+t}\right)\right|.\end{aligned}$$

Since the maximum of $|t\log_2 t|$ for $0 \leq t \leq 1$ is $1/2$ (attained at $t = 1/2$) we obtain

$$|F(s+t)-F(s)-F(t)| \leq |s|+|t|.$$

Now, if s and t have distinct signs, we may and do assume that $s > 0,\ t < 0$ and $s+t > 0$. Taking into account that F is an odd map, we have

$$|F(s+t)-F(s)-F(t)| = |F(s)-F(-t)-F(s+t)| \leq |-t|+|s+t| \leq |s|+|t|.$$

To end the proof, observe that $\lim_{t\to\infty} F(t)/t = \infty$, which makes impossible an estimate $|F(t)-A(t)| \leq C|t|$ for any continuous additive (i.e., linear) mapping A, let alone discontinuous additive A. ∎

This example also appears in Johnson's paper [57]. Other examples were found by Gajda [38] and Rassias and Šemrl [86] and will be presented in Section 3.

At this point, it is convenient to introduce some notation and terminology. Thus, a mapping $F : Z \to Y$ is called *quasi-additive* if it satisfies

$$\|F(x+y)-F(x)-F(y)\| \leq K(\|x\|+\|y\|)$$

for some K and all $x, y \in Z$. A *quasi-linear* map is a quasi-additive homogeneous map. The smallest possible constant K in the preceding inequality is called the *quasi-additivity constant* of F and denoted by $Q(F)$. Given two maps F and G acting between the same spaces, we consider the (possibly infinite) distance

$$\operatorname{dist}(F,G) = \inf\{C : \|F(x)-G(x)\| \leq C\|x\| \text{ for all } x\},$$

where the infimum of the empty set is treated as $+\infty$. Observe that for homogeneous maps one has

$$\operatorname{dist}(F,G) = \sup_{\|x\|\leq 1} \|F(x)-G(x)\|.$$

By a *bounded* map B we mean one satisfying an estimate $\|B(x)\| \leq C\|x\|$ for some constant C and all x (that is, bounded maps are maps at finite distance from the zero map). A mapping F is *asymptotically additive* (respectively, *asymptotically linear*) if it is at finite distance from some additive (respectively, linear) map. This means that it can be written as $F = A+B$, with A additive and B bounded. Thus Example 1.5 shows that not every quasi-additive map is asymptotically additive.

The problem of the lack of stability for $p = 1$ led Ger [41, 42] to consider mappings satisfying either

$$\text{(P)} \qquad |F(x+y)-F(x)-F(y)| \leq \varepsilon(\|x\|+\|y\|-\|x+y\|)$$

or

$$(G) \qquad |F(x+y) - F(x) - F(y)| \le \varepsilon \|x\|,$$

and undertake the first serious attempt to formalize a theory for nearly additive maps (see also Johnson's final comments in [57]). Maps satisfying the first condition will be called here *pseudo-additive* in accordance with the pseudo-linearity introduced by Lima and Yost [71] for the study of properties of semi-L-summands in Banach space theory (another token of the extant fracture between Banach space theory and functional equations theory; see Subsection 2.4 in this chapter). For maps satisfying the second condition we have coined the term *Ger-additive*. We shall show that the corresponding homogeneous notion, i.e. Ger-linear maps, allows a nice formulation of some open problems in Banach space theory.

Returning to the history of Ulam's problem and its derivations, Ger shows in [41] that a map $F : Z \to \mathbb{R}^n$ satisfying (P) admits an additive map A at distance at most $n\varepsilon$. The strange bound "n" is due to the method of proof, which uses a sandwich theorem of Kranz [70] (see also Gajda and Kominek [39], or Section 2 in Chapter II). Ger gives some sufficient conditions for the uniqueness of A and displays an example to show that the additive approximating map is not necessarily unique.

Ger himself is aware of the fact that the poor estimate (useless, in principle, to extend the results to infinite-dimensional Banach spaces) is due to the method of proof. In the subsequent paper [42] he uses invariant means to show (among other results, and in a more general setting; see Section 4 in Chapter IV) that pseudo-additive or Ger-additive mappings $F : Z \to \mathbb{R}$ are asymptotically additive. Ger also realizes that the target space can be replaced by, say, a reflexive Banach space.

This point deserves a comment. Two remarkable achievements in the theory were given by Székelyhidi in the papers [97, 99]: in the first one he introduces invariant means to get a proof of Hyers's Theorem 1.2, a new technique that shaked the basement of the theory; in the second paper he observed that, thanks to the use of invariant means, whenever there exists an affirmative answer to Ulam's problem for real-valued nearly additive functions in the sense of Hyers, there is also an affirmative solution for Hyers-additive functions with values in a reflexive Banach space (actually, a semi-reflexive locally convex space). Forti [35] and Gajda [37] extend this result replacing "semi-reflexive" by "sequentially complete"... returning to the original divide-by-2^n proof of Hyers! Let us observe that such extensions are no longer valid for other types of nearly additive mappings, as we shall see later. The reader is cordially invited to peruse [8] for a comprehensive treatment of the vector-valued situation.

2. Background on twisted sums. Returning to the exceptional case $p = 1$, the existence of a non-trivial quasi-additive (in fact, quasi-linear) map $\ell_1 \to \mathbb{R}$ was obtained independently and almost simultaneously by Ribe [89], Kalton [60] and Roberts [90]. In Banach space theory quasi-linear maps have been introduced by Kalton [60]; see also [62] in connection with the so-called twisted sums of Banach or quasi-Banach spaces, i.e., spaces X containing a given subspace Y in such a way that the quotient space X/Y is a given space Z. Recall from [64] that a *quasi-norm* on a (real or complex) vector space X

is a non-negative real-valued function on X satisfying:

(1) $\|x\| = 0$ if and only if $x = 0$;
(2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $x \in X$ and $\lambda \in \mathbb{R}$;
(3) $\|x + y\| \leq \Delta(\|x\| + \|y\|)$ for some fixed $\Delta \geq 1$ and all $x, y \in X$.

A quasi-normed space is a vector space X together with a specified quasi-norm. On such a space one has a (linear) topology defined as the smallest linear topology for which the set

$$B = \{x \in X : \|x\| \leq 1\}$$

(the unit ball of X) is a neighborhood of 0. A warning is in order here: a quasi-norm need not be continuous with respect to itself! Of course, it is continuous at 0, but not necessarily at every point. Fortunately, the Aoki–Rolewicz theorem shows that every quasi-norm admits an equivalent quasi-norm that is continuous with respect to the former. Topologized in that way, X becomes a locally bounded space (i.e., it has a bounded neighborhood of 0); and, conversely, every locally bounded topology on a vector space is induced by a quasi-norm. A *quasi-Banach* space is a complete quasi-normed space.

At this point, some elements borrowed from homological algebra greatly simplify the exposition.

2.1. *Exact sequences of quasi-Banach spaces.* This section and Chapter VII are the only places in which we shall adopt the language of categorical algebra: a small annoyance for some readers, rewarded with precise statements of the problems.

We denote by **Q** the category of quasi-Banach spaces, in which objects are quasi-Banach spaces and arrows are continuous linear maps. The full subcategory in which the objects are Banach spaces is denoted by **B**. A *short exact sequence* in **Q** or **B** is a diagram $0 \to Y \to X \to Z \to 0$ in the category with the property that the kernel of each arrow coincides with the image of the preceding one. The open mapping theorem [64] guarantees that Y is a subspace of X such that the corresponding quotient X/Y is Z. Two exact sequences $0 \to Y \to X \to Z \to 0$ and $0 \to Y \to X_1 \to Z \to 0$ are said to be *equivalent* if there exists an arrow $T : X \to X_1$ making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \to & Y & \to & X & \to & Z & \to & 0 \\ & & \| & & \downarrow T & & \| & & \\ 0 & \to & Y & \to & X_1 & \to & Z & \to & 0 \end{array}$$

The three-lemma [49] and the open mapping theorem imply that T must be an isomorphism. An exact sequence is said to *split* if it is equivalent to the trivial sequence $0 \to Y \to Y \oplus Z \to Z \to 0$; equivalently, Y is complemented in X (the injection $Y \to X$ admits a left inverse) or, equivalently, the quotient map $X \to X/Y$ admits a right inverse.

2.2. *Quasi-linear maps.* The by now classical theory of Kalton and Peck [60, 62] (see also the monograph [20]) describes twisted sums of quasi-Banach spaces in terms of quasi-linear maps.

A quasi-linear map $F : Z \to Y$ gives rise to a twisted sum of Y and Z, denoted by $Y \oplus_F Z$, endowing the product space $Y \times Z$ with the quasi-norm $\|(y, z)\|_F = \|y - F(z)\|$

$+ \|z\|$. To verify that $\|\cdot\|_F$ is a quasi-norm, note that

$$\begin{aligned}\|(y+u,z+v)\|_F &= \|y+u-F(z+v)\|_Y + \|z+v\|_Z\\ &\le \|y-F(z)+u-F(v)+F(z)+F(v)-F(z+v)\|_Y + \|z+v\|_Z\\ &\le \Delta_Y(\|y-F(z)+u-F(v)\|_Y + \|F(z)+F(v)-F(z+v)\|_Y) + \Delta_Z(\|z\|_Z + \|v\|_Z)\\ &\le \Delta_Y(\Delta_Y(\|y-F(z)\|_Y + \|u-F(v)\|_Y) + Q(F)(\|z\|_Z+\|v\|_Z)) + \Delta_Z(\|z\|_Z+\|v\|_Z)\\ &\le C(\|(y,z)\|_F + \|(u,v)\|_F).\end{aligned}$$

Clearly, the map $Y \to Y \oplus_F Z$ sending y to $(y,0)$ is an into isometry while the map $Y \oplus_F Z \to Z$ sending (y,z) to z is surjective and continuous. In this way Y can be thought of as a subspace of $Y \oplus_F Z$ such that the corresponding quotient space is Z. Conversely, given an exact sequence $0 \to Y \to X \to Z \to 0$, if one takes a bounded homogeneous selection B for the quotient map (which always exists, by the open mapping theorem) and then a (not necessarily bounded) linear selection L, their difference $F = B - L$ takes values in Y instead of X and is a quasi-linear map $Z \to Y$ since

$$\|F(z+v) - F(z) - F(v)\|_Y = \|B(z+v) - B(z) - B(v)\|_X \le C(\|z\|_Z + \|v\|_Z).$$

The two processes are inverse to each other in a strong sense: equivalent sequences correspond to "equivalent" quasi-linear maps (here, two quasi-linear maps acting between the same spaces are said to be equivalent if their difference is asymptotically linear). In particular, trivial exact sequences correspond to trivial quasi-linear maps. This fundamental result of Kalton [60] establishes the basic connection between the theory of twisted sums and Ulam's problem; we state it again:

THEOREM 1.6. *A twisted sum $Y \oplus_F Z$ defined by a quasi-linear map $F : Z \to Y$ is trivial if and only if there there exists a linear map $L : Z \to Y$ such that* $\operatorname{dist}(F, L) < \infty$.

In other words, a twisted sum is trivial (a direct sum) if and only if the corresponding quasi-linear map is asymptotically linear. Notice that a homogeneous map is asymptotically additive if and only if it is asymptotically linear (see Lemma 1.8 below). Hence, the existence of non-trivial twisted sums, such as $0 \to c_0 \to \ell_\infty \to \ell_\infty/c_0 \to 0$, immediately provides the existence of quasi-linear non-asymptotically linear maps, hence a negative answer to the case $p = 1$ in Ulam's problem. A generalization of Theorem 1.6 will be proved in Chapter V (Theorem 5.12).

REMARK 1.7. Some comments about the rôle of homogeneity are in order. Consider now additive maps $\mathbb{R} \to \mathbb{R}$. Linear maps are of course additive; however, additive but non-linear functions can be obtained by considering a Hamel basis $\{t_\alpha : \alpha \in \Gamma\}$ of the vector space $\mathbb{R}$ over $\mathbb{Q}$ and assigning arbitrary values t'_α to the elements of the basis. Then, since every real number t can be represented in a unique way as a finite linear combination of elements of the basis with rational coefficients $t = \sum_\alpha q_\alpha t_\alpha$, one can define an additive map $A : \mathbb{R} \to \mathbb{R}$ as $A(t) = \sum_\alpha q_\alpha t'_\alpha$ (actually, this was the original purpose of Hamel to introduce what are now called "Hamel" bases; see [47]). The resulting map is not linear unless $t'_\alpha = ct_\alpha$ for some fixed real number c and all $\alpha \in \Gamma$. Moreover, it is well known that additive maps are linear if and only if one of the following conditions holds (the list is by no means exhaustive): bounded on some bounded set, integrable, continuous at some point, measurable, ... (See [27] or [1]. Extensions of these results to other types of nearly additive maps have been considered in [25].) Thus, every "reasonable" map which is asymptotically additive must be asymptotically linear:

LEMMA 1.8. *Let $F : Z \to Y$ be a mapping acting between quasi-normed spaces with* $\mathrm{dist}(F, A) < \infty$ *for some additive $A : Z \to Y$. Suppose that, for every fixed $z \in Z$, the map $t \in \mathbb{R} \mapsto F(tx) \in Y$ is continuous at $t = 0$. Then A is linear. In particular, an asymptotically additive map which is continuous at zero along lines is asymptotically linear.*

Proof. The hypotheses imply that A itself is continuous along lines, hence linear. ∎

It remains, however, to find out what happens with maps $Z \to \mathbb{R}$, i.e., if there exists a non-trivial sequence $0 \to \mathbb{R} \to X \to Z \to 0$ with Z a Banach space. If the answer is affirmative then observe that X cannot be isomorphic to a Banach space (i.e., it cannot be locally convex); hence one is asking for a negative solution to the three-space problem for local convexity (see Subsection 3.4 in this chapter and Proposition 5.9; or else, [20]). Ribe's example [89] of a twisted sum of $\mathbb{R}$ and ℓ_1 that is not a direct sum (that is, an exact sequence $0 \to \mathbb{R} \to X \to \ell_1 \to 0$ that is not trivial) gives an affirmative answer to the problem; so, there exist quasi-linear mappings $F : \ell_1 \to \mathbb{R}$ that are not asymptotically linear. Ribe's map (see Subsection 3.4 of this chapter for a detailed exposition) is defined (except for some technicalities involving that we only define the map over the finite sequences of ℓ_1, and putting $0 \log 0 = 0$) as

$$F(x) = \sum_n x_n \log |x_n| - \Big(\sum_n x_n\Big) \log \Big(\sum_n |x_n|\Big).$$

In fact, it is very difficult to define explicitly quasi-linear maps on infinite-dimensional Banach spaces. Fortunately, quasi-linear maps extend from dense subspaces:

LEMMA 1.9 (Kalton and Peck [62]; see also [20]). *A quasi-linear map $F_0 : Z_0 \to Y$ defined on a dense subspace Z_0 of a quasi-Banach space Z can be extended to a quasi-linear map $F : Z \to Y$. Moreover, F is asymptotically linear if and only if F_0 is.*

2.3. *Locally convex twisted sums and zero-linear maps.* The local convexity of twisted sums of Banach spaces can be characterized in terms of the quasi-linear maps defining them. Following [13, 18, 20], let us say that a homogeneous mapping $F : Z \to Y$ acting between normed spaces is *zero-linear* if there is a constant K such that

$$\Big\| \sum_{i=1}^{n} F(x_i) \Big\| \le K \Big(\sum_{i=1}^{n} \|x_i\| \Big)$$

whenever $\{x_i\}_{i=1}^n$ is a finite subset of Z such that $\sum_{i=1}^n x_i = 0$. The smallest constant K satisfying the preceding inequality will be called the *zero-linearity constant* of F and denoted by $Z(F)$. Observe that zero-linearity is equivalent to satisfying an estimate

$$\Big\| F\Big(\sum_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} F(x_i) \Big\| \le K \Big(\sum_{i=1}^{n} \|x_i\| \Big)$$

for a possibly different constant K.

PROPOSITION 1.10 ([20]). *A twisted sum of Banach spaces $Y \oplus_F Z$ is locally convex (being thus isomorphic to a Banach space) if and only if F is zero-linear.*

The proof can be found in Proposition 5.9. For this reason, one of the five types of nearly additive maps considered in Chapter II are the zero-additive maps. Observe that, for instance, Ribe's map is not zero-linear: $F(e_i) = 0$ while $F(\sum_{i=1}^n e_i) = -n \log n$; hence an estimate $\|F(\sum_{i=1}^n e_i) - \sum_{i=1}^n F(e_i)\| \le K(\sum_{i=1}^n \|e_i\|)$ would imply that $n \log n \le Kn$, which is absurd.

We are going to show that, on Banach spaces, real-valued zero-additive maps actually coincide with asymptotically additive maps. Clearly, the Hahn–Banach theorem in combination with Theorem 1.6 and the preceding proposition says that every zero-linear map $F : Z \to \mathbb{R}$ admits a linear map at finite distance. However, this indirect argument gives no information about the distance to the approximating linear map. We present now a direct proof with sharp estimates for the distance to linear maps.

PROPOSITION 1.11. *Let $F : Z \to \mathbb{R}$ be a zero linear mapping. There is a linear functional $A : Z \to \mathbb{R}$ such that* $\operatorname{dist}(F, A) \le Z(F)$.

Proof. The proof goes as the classical proof of the Hahn–Banach theorem via Zorn's lemma. The main problem is that an induction hypothesis such as: "there is a linear functional A defined on a subspace U of Z such that

$$|F(z) - A(z)| \le Z(F)\|z\| \tag{1}$$

for all $z \in U$" is not strong enough to ensure that A can be extended onto a larger subspace, say $W = [w] \oplus U$, in such a way that (1) still holds for $z \in W$.

Our strategy is to use as induction hypothesis that, given a zero-linear map F, there is a linear mapping $A : U \to \mathbb{R}$ satisfying

$$\Big|\sum_i F(z_i) - A\Big(\sum_i z_i\Big)\Big| \le Z(F)\Big(\sum \|z_i\|\Big) \tag{2}$$

for every finite set $\{z_i\} \subset Z$ such that $\sum z_i \in U$. Plainly, this implies, when $U = Z$, that $\operatorname{dist}(F, A) \le Z(F)$.

Assume that a linear mapping A has been defined on a subspace $U \subset Z$ in such a way that (2) holds when $z = \sum z_i \in U$. Fixing $w \notin U$, we want to see that it is possible to define $A(w) = a \in \mathbb{R}$ in such a way that (2) holds for $z = \sum z_i \in [w] \oplus U$. Since U is a linear subspace, one can assume that $z = w - u$ with $u \in U$. In this case it suffices to prove that there exists a number a satisfying

$$\Big|a - A(u) - \sum_i F(z_i)\Big| \le Z(F)\Big(\sum_i \|z_i\|\Big)$$

when $w - u = \sum z_i$ and $u \in U$. This is equivalent to

$$A(u) + \sum_i F(z_i) - Z(F)\Big(\sum_i \|z_i\|\Big) \le a \le A(u) + \sum_i F(z_i) + Z(F)\Big(\sum_i \|z_i\|\Big).$$

So, the question is whether

$$Au + \sum_i F(z_i) - Z(F)\Big(\sum_i \|z_i\|\Big) \le Av + \sum_j F(s_j) + Z(F)\Big(\sum_j \|s_j\|\Big),$$

whenever $w-u=\sum z_i$, $w-v=\sum s_j$, $u,v\in U$ and $z_i,s_j\in Z$. The preceding inequality can be written as

$$A(u)-A(v)+\sum_i F(z_i)-\sum_j F(s_j)\le Z(F)\Big(\sum_i \|z_i\|+\sum_j \|s_j\|\Big),$$

and follows from the induction hypothesis, which yields

$$\Big|A(u-v)-\Big(\sum_i F(-z_i)+\sum_j F(s_j)\Big)\Big|\le Z(F)\Big(\sum_i \|-z_i\|+\sum_j \|s_j\|\Big),$$

since $u-v=-(w-u)+(w-u)=\sum_i -z_i+\sum_j s_j$ and $u-v\in U$.

Observe that only the homogeneity of F has been needed so far. In fact, the induction step holds independently of the "value" (or meaning) of $Z(F)$. The zero-additivity of F only appears as the condition one needs to make this inductive procedure start: when $U=\{0\}$ the inequality of the proof states that whenever (z_i) is a finite collection of points of Z with $\sum z_i=0$ then

$$\Big|\sum F(z_i)\Big|\le Z(F)\Big(\sum \|z_i\|\Big).$$

Now, the rest of the proof is a typical application of Zorn's lemma. (Of course, there is no need for Zorn if Z is finite-dimensional.) ■

The preceding result (and its proof) should be compared to [93] (see also [51, pp. 25–30]).

2.4. *Pseudo-linear maps and semi-L-summands.* Proposition 1.10 characterizes those quasi-linear maps F for which the induced quasi-norm $\|(\cdot,\cdot)\|_F$ is equivalent to a norm. As David Yost remarked to us, the question of when $\|(\cdot,\cdot)\|_F$ is itself a norm can be easily answered in terms of F:

PROPOSITION 1.12 (Lima and Yost [71]). *Let F be a homogeneous mapping acting between Banach spaces. Then $\|(\cdot,\cdot)\|_F$ is a norm if and only if F is pseudo-linear with constant* 1.

The proof can be found in Proposition 5.11. However, making $\|\cdot\|_F$ a norm has unexpected side-effects. Recall that a *Chebyshev subspace* of a Banach space is one whose metric projection is single-valued (i.e., for each x in the larger space there is a unique point $\pi(x)$ in the subspace minimizing $\|x-\pi(x)\|$). A subspace Y of a Banach space X is called *absolutely Chebyshev* if it is Chebyshev and there is some function $\Psi:\mathbb{R}^2\to\mathbb{R}$ such that, for all $x\in X$, $\|x\|=\Psi(\|\pi(x)\|,\|x-\pi(x)\|)$. In the case $\Psi(s,t)=s+t$ the subspace Y is said to be a *semi-L-summand* in X. It can be proved that every absolutely Chebyshev subspace in a Banach space becomes a semi-L-summand after a suitable renorming [71] (thus, from the isomorphic point of view, every absolutely Chebyshev subspace is a semi-L-summand). We list without proofs some elementary facts concerning metric projections onto semi-L-summands.

LEMMA 1.13. *Let Y be a semi-L-summand in X and let $\pi:X\to Y$ be the metric projection. Then:*

- *π is homogeneous*;
- *$\pi(x+y)=\pi(x)+y$ for all $x\in X$, $y\in Y$*;

- $\|\pi(x+y)-\pi(x)-\pi(y)\|_Y \leq \|x+Y\|_{X/Y}+\|y+Y\|_{X/Y}-\|x+y+Y\|_{X/Y}$.

We obtain (see also [20, Appendix 1.9]):

PROPOSITION 1.14. *The exact sequence* $0 \to Y \to X \to Z \to 0$ *of Banach spaces is defined by a pseudo-linear map* $F : Z \to Y$ *if and only if* Y *is a semi-L-summand of* X.

Proof. If Y is a semi-L-summand then the properties of the metric projection appearing in Lemma 1.13 hold, and we shall show that, if we pick a linear selection $L : Z \to X$ for the quotient map, the map

$$F(x+Y) = x - \pi(x) - L(x+Y)$$

is (well defined and) pseudo-linear:

$$\begin{aligned}
&\|F(x+z+Y)-F(x+Y)-F(z+Y)\| = \|\pi(x+z)-\pi(x)-\pi(z)\|\\
&\quad = \|\pi(x+z-\pi(x)-\pi(z))\|\\
&\quad = \|x+z-\pi(x)-\pi(z)\| - \|(x+z-\pi(x)-\pi(z))-(\pi(x+z)-\pi(x)-\pi(z))\|\\
&\quad \leq \|x-\pi(x)\|+\|z-\pi(z)\|-\|x+z-\pi(x+z)\|\\
&\quad = \|x+Y\|+\|z+Y\|-\|x+z+Y\|.
\end{aligned}$$

Conversely, if a quasi-linear map $F : Z \to Y$ is actually pseudo-linear then $\|\cdot\|_F$ is a norm on X and Y is a semi-L-summand since $\pi(y,z) = y - F(z)$, which obviously is the metric projection, satisfies

$$\|(y,z)\|_F = \|y-F(z)\|+\|z\| = \|\pi(y,z)\|+\|(y,z)-\pi(y,z)\|. \ \blacksquare$$

An obvious question, to which some partial answer will be given in Chapter IV, is:

PROBLEM 1.15. *Is every absolutely Chebyshev subspace a complemented subspace?*

It should now be clear that this problem is equivalent to the following one:

PROBLEM 1.16. *Is every pseudo-linear map at finite distance from some linear map?*

3. Examples of nearly additive maps. In this section we shall exhibit examples of nearly additive (non-asymptotically additive) maps. First of all let us observe that for most purposes quasi-additive maps on quasi-Banach spaces can be assumed to be odd.

LEMMA 1.17. *Let* F *be a quasi-additive map acting between Banach spaces and let*

$$F_{\mathrm{o}}(x) = \frac{F(x)-F(-x)}{2}$$

be its odd part. Then F_{o} *is quasi-additive*, *with* $Q(F_{\mathrm{o}}) \leq Q(F)$ *and* $\operatorname{dist}(F,F_{\mathrm{o}}) \leq Q(F)$.

Proof. That $Q(F_{\mathrm{o}}) \leq Q(F)$ is obvious. To see that $\operatorname{dist}(F,F_{\mathrm{o}}) \leq Q(F)$, consider the even part of F:

$$F_{\mathrm{e}}(X) = \frac{F(x)+F(-x)}{2}.$$

Since $F = F_{\mathrm{o}} + F_{\mathrm{e}}$, the proof will be complete if we show that $\|F_{\mathrm{e}}(x)\| \leq Q(F)\|x\|$. From

$$\|F(x+y)-F(x)-F(y)\| \leq Q(F)(\|x\|+\|y\|),$$

by taking $x = 0$ and $y = 0$, it follows that $F(0) = 0$. Hence

$$2\|F_{\mathrm{e}}(x)\| = 2\|F(0) - F(x) - F(-x)\| \le Q(F)(\|x\| + \|-x\|) \le 2Q(F)\|x\|. \blacksquare$$

When the corresponding definitions are given, it is equally easy to show that this holds for pseudo-additive and Ger-additive maps.

3.1. *The Kalton–Peck maps.* Kalton and Peck consider in [62] quasi-additive odd maps f such that $\lim_{t\to 0} f(tx) = 0$. Perhaps the simplest examples of such maps are provided by functions $\mathbb{R} \to \mathbb{R}$ having the form

$$f_\theta(t) = t\theta(\log_2 |t|)$$

(if $t \neq 0$; and $f_\theta(0) = 0$), where $\theta : \mathbb{R} \to \mathbb{R}$ is a Lipschitz map. Observe that the map in Example 1.5 corresponds to the choice of θ as the identity function on $\mathbb{R}$.

Let $\mathrm{Lip}(\theta)$ denote the Lipschitz constant of the map θ.

Lemma 1.18. *The function f_θ is quasi-additive with $Q(f_\theta) \le \mathrm{Lip}(\theta)$.*

Proof. The proof follows the lines of that of Example 1.5. If $s, t > 0$ then

$$\begin{aligned} |f_\theta(s+t) - f_\theta(s) - f_\theta(t)| &= |(s+t)\theta(\log_2(s+t)) - s\theta(\log_2 s) - t\theta(\log_2 t)| \\ &\le |t\theta(\log_2(s+t)) - t\theta(\log_2 t)| + |s\theta(\log_2(s+t)) - s\theta(\log_2 s)| \\ &= \mathrm{Lip}(\theta) \cdot \left| s \log_2\left(\frac{s}{s+t}\right) + t\log_2\left(\frac{t}{s+t}\right)\right| \\ &\le \mathrm{Lip}(\theta) \cdot (|s| + |t|)\left|\frac{s}{s+t}\log_2\left(\frac{s}{s+t}\right) + \frac{t}{s+t}\log_2\left(\frac{t}{s+t}\right)\right| \\ &\le \mathrm{Lip}(\theta) \cdot (|s| + |t|). \end{aligned}$$

Now, if s and t have different signs, we may assume that $s > 0$, $t < 0$ and $s + t > 0$. Taking into account that f_θ is odd, we have

$$\begin{aligned} |f_\theta(s+t) - f_\theta(s) - f_\theta(t)| = |f_\theta(s) - f_\theta(-t) - f_\theta(s+t)| &\le \mathrm{Lip}(\theta)(|-t| + |s+t|) \\ &\le \mathrm{Lip}(\theta)(|s| + |t|). \end{aligned}$$

Finally, the odd character of f_θ shows that it is quasi-additive with $Q(f_\theta) \le \mathrm{Lip}(\theta)$. ■

The result can be completed as follows.

Proposition 1.19. *The map f_θ is asymptotically additive if and only if θ is uniformly bounded.*

Proof. It is obvious that if θ is uniformly bounded then $\mathrm{dist}(f_\theta, 0) < \infty$.

Conversely, if f_θ is asymptotically additive, then by Lemma 1.8 it is asymptotically linear and, therefore, $\mathrm{dist}(f_\theta, 0) < \infty$. Hence there is a constant C such that

$$|t\theta(\log_2 |t|)| \le C|t|,$$

that is,

$$\sup_{-\infty<s<\infty} |\theta(s)| = \sup_{-\infty<t<\infty} |\theta(\log_2 |t|)| \le C. \blacksquare$$

Moreover, Kalton and Peck ([62, Theorem 3.7]) establish that those are essentially all such maps. The proof can be seen as a clever returning to Hyers' proof:

PROPOSITION 1.20. *Let $F:\mathbb{R}\to\mathbb{R}$ be a quasi-additive function which is continuous at zero. Then there exists a Lipschitz function $\theta:\mathbb{R}\to\mathbb{R}$ such that* $\operatorname{dist}(F,f_\theta)<\infty$.

Proof. By Lemma 1.17, we may assume that F is odd. It obviously suffices to construct a Lipschitz map θ with

$$\sup_{0<t<\infty}\left|\frac{F(t)}{t}-\theta(\log_2|t|)\right|<\infty.$$

Let Q denote the quasi-additivity constant of F. Then

$$|F(2^{n+1})-2F(2^n)|\le 2^{n+1}Q,$$

hence

$$\left|\frac{F(2^{n+1})}{2^{n+1}}-\frac{F(2^n)}{2^n}\right|\le Q.$$

The Lipschitz function θ is chosen with $\operatorname{Lip}(\theta)\le Q$ and such that, for all $n\in\mathbb{Z}$,

$$\theta(n)=2^{-n}f(2^n).$$

The function f_θ satisfies $f_\theta(2^n)=f(2^n)$ for all integers n and is quasi-additive, with constant at most Q. We close the proof with a couple of observations:

LEMMA 1.21. *Every quasi-additive odd map $\mathbb{R}\to\mathbb{R}$ which is continuous at zero is bounded on compact sets.*

Proof. Consider, for instance, the interval $[0,1]$; if F is not bounded on it there would be some sequence t_n in $[0,1]$ for which $|F(t_n)|>n+1$. There is no loss of generality in assuming that t_n converges (in any case, a subsequence will do) to some $t\in[0,1]$. Since

$$|F(t)-F(t-t_n)-F(t_n)|\le Q(F)(|t-t_n|+|t_n|)\le 2Q(F)$$

it follows that

$$|F(t_n)|\le 2Q(F)+|F(t)|+|F(t-t_n)|.$$

Taking now limits as $n\to\infty$, since $\lim F(t-t_n)=F(0)=0$ one finds that eventually

$$n\le 2Q(F)+|F(t)|,$$

which is absurd. ∎

The following estimate is due to Kalton [60]. The proof is a straightforward induction that we leave to the reader.

LEMMA 1.22. *Let F be a quasi-additive function on the line. Then, for all $s_1,\dots,s_m$, one has*

$$\left|F\Big(\sum_{n=1}^m s_n\Big)-\sum_{n=1}^m F(s_n)\right|\le Q\sum_{n=1}^m n|s_n|,$$

where Q is the quasi-additivity constant of F. ∎

LEMMA 1.23. *Suppose $F:\mathbb{R}\to\mathbb{R}$ is a quasi-additive odd map which is continuous at zero. If $F(2^n)=0$ for all integers n, then F is bounded (in the sense of being at finite distance from the zero map). More precisely, if C is a (uniform) bound for F in $[0,1]$ and Q its quasi-additivity constant, then*

$$|F(t)|\le 2(Q+C)|t|.$$

Proof. Since F is odd, it is enough to work with positive t. We proceed inductively on the interval $[2^{n-1}, 2^n)$, $n \in \mathbb{N}$, where t falls. If $t \in [1/2, 1)$ the result is clear. Let $t \in [1, 2)$. Then

$$|F(t)| = |F(t) - F(1) - F(t-1) + F(t-1)| \leq Qt + C \leq (Q+C)t.$$

Let now $t \in [2, 4)$. One has to consider two cases, depending on whether $2 \leq t < 3$ or $3 \leq t < 4$. In the first case,

$$|F(t)| = |F(t) - F(2) - F(t-2) + F(t-2)| \leq Qt + C \leq (Q+C)t;$$

in the second case (recall that F is odd),

$$|F(t)| = |F(t) - F(4) - F(t-4) + F(t-4)| \leq Qt + C \leq (Q+C)t.$$

Now, the induction argument. Consider $t \in [2^{n-1}, 2^n]$. If $t < 2^{n-2} + 2^{n-1}$, then

$$\begin{aligned}|F(t)| &= |F(t) - F(2^{n-1}) - F(t - 2^{n-1}) + F(t - 2^{n-1})| \leq Qt + |F(t - 2^{n-1})| \\ &\leq Qt + Q(t - 2^{n-1}) + C = Q(2t - 2^{n-1}) + C \leq Q2^n + C \\ &\leq (Q+C)2^n \leq (Q+C)2t.\end{aligned}$$

If $2^{n-2} + 2^{n-1} \leq t$ then (recalling again that F is odd)

$$\begin{aligned}|F(t)| &= |F(t) - F(2^n) - F(t - 2^n) + F(t - 2^n)| \leq Qt + |F(t - 2^n)| \\ &\leq Qt + Q(2^n - t) + C = Q2^n + C \leq (Q+C)2^n \leq 2(Q+C)t.\end{aligned}$$

We now examine the situation for $t \in [0, 1/2]$. Any $t \in [2^{-n}, 2^{-n+1}]$ can be written as $t = 2^{-n}\sum_{j=1}^{\infty} \delta_j 2^{-j}$, where δ_j is 0 or 1. Set $t_m = 2^{-n}\sum_{j=1}^{m} \delta_j 2^{-j}$; since $|F(t) - F(t - t_m) - F(t_m)| \leq Q|t|$ one then has

$$|F(t)| \leq |F(t - t_m)| + |F(t_m)| + Q|t|,$$

and since F is continuous at 0, $F(t - t_m) \to 0$ as $m \to \infty$. Hence,

$$|F(t)| \leq Q|t| + \limsup_{m\to\infty} |F(t_m)|.$$

Finally, we can use Lemma 1.22 to estimate $F(t_m)$:

$$|F(t_m)| = \left|F\Big(2^{-n}\sum_{j=1}^{m} \delta_j 2^{-j}\Big)\right| \leq Q\sum_{j=1}^{m} j\delta_j 2^{-n-j} \leq KQ2^{-n} \leq 8Qt$$

since $\sum_{n=1}^{\infty} n2^{-n} = 2$. ■

The proof of Proposition 1.20 is now complete. ■

3.2. *The Rassias–Šemrl map.* This was considered to be the first example (see [86, Theorem 2]) of a continuous quasi-additive but non-asymptotically additive map $\mathbb{R} \to \mathbb{R}$ (although, as we already mentioned, Ribe's example came first). It is defined as

$$f(x) = x\log_2(1 + |x|).$$

This function is odd, continuous and convex on $\mathbb{R}^+$. We show that $Q(f) = 1$. Since f is convex, for positive x, y one has

$$|f(x+y) - f(x) - f(y)| \leq f(x+y) - 2f\left(\frac{x+y}{2}\right)$$

and the estimate holds. For negative x, y it also holds since f is odd. Thus, there remains the case in which x and y have different signs. There is no loss of generality to assume that $x > 0$, $y < 0$ and $|y| < |x|$. By the convexity of f one has

$$|f(x+y) - f(x) - f(y)| = -f(x+y) + f(x) + f(y) = f(x) - f(x+y) - f(-y);$$

since both $x + y$ and $-y$ are positive, the proof is complete.

According to Proposition 1.20 there is a Kalton–Peck map f_θ such that $\operatorname{dist}(f, f_\theta) < \infty$. In fact, the Rassias–Šemrl map *is* a Kalton–Peck map. To see this, define

$$\theta(t) = \log_2(1 + 2^t).$$

Then

$$\theta(\log_2 |x|) = \log_2(1 + |x|),$$

so that $f = f_\theta$. On the other hand,

$$\frac{d\theta}{dt}(t) = \frac{2^t}{1 + 2^t},$$

which yields $\operatorname{Lip}(\theta) = 1$. In view of Lemma 1.18 this also shows that $Q(f) \leq 1$.

3.3. *Gajda's map.* In [38] (see also [51, pp. 24–25], the following example of a uniformly bounded quasi-additive but non-asymptotically additive map $f : \mathbb{R} \to \mathbb{R}$ is presented. The idea is that if f is continuous at zero and satisfies

$$\lim_{x \to 0} \frac{f(x)}{x} = \infty$$

then it cannot be asymptotically additive, since the additive map approximating f has to be linear (a possibility that the previous condition prevents). It remains to get such an f that is quasi-additive, though. Fix $Q > 0$, and define

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^n}$$

where ϕ is the function given by

$$\phi(x) = \begin{cases} -Q/6 & \text{if } x \leq -1, \\ Qx/6 & \text{if } -1 < x < 1, \\ Q/6 & \text{if } x \geq -1. \end{cases}$$

Gajda proves that f is quasi-additive with constant Q.

Also, uniformly bounded quasi-additive non-asymptotically additive maps can be produced as Kalton–Peck maps f_θ for suitable choices of θ. Simply take

$$\theta(s) = \begin{cases} s & \text{if } x \leq 0, \\ 0 & \text{if } x \geq 0. \end{cases}$$

3.4. *Ribe's map.* Requiring the maps to be homogeneous, one cannot expect to have non-trivial examples $\mathbb{R} \to \mathbb{R}$. Or $\mathbb{R}^n$ for that matter. Passing to infinite-dimensional Banach spaces, however, let us show the existence of quasi-linear (i.e., quasi-additive homogeneous) maps $\ell_1 \to \mathbb{R}$ that are not asymptotically linear. Kalton and Peck show in [62] that a quasi-additive odd map f that is continuous at 0 along lines can be rendered quasi-linear by the process

$$F(x) = \|x\| f\left(\frac{x}{\|x\|}\right).$$

Thus, it is enough to describe a suitable quasi-additive map that is not asymptotically additive. To do that, observe that a quasi-additive map $f : \mathbb{R} \to \mathbb{R}$ yields a quasi-additive map defined on the dense subspace of ℓ_1 spanned by the finitely supported sequences as follows: $g(x) = \sum f(x_n)$. Moreover, such a map can be extended to the whole ℓ_1 by Lemma 1.9. So, taking as starting point the map $f(t) = x \log_2 |t|$ one arrives at the map defined on the subspace of finite sequences of ℓ_1 by

$$F(x) = \sum_i x_i \log_2 |x_i| - \Big(\sum_i x_i\Big) \log_2 \Big|\sum_i x_i\Big|.$$

That F is quasi-linear with $Q(F) = 2$ is not difficult to prove and has been essentially done in Example 1.5. This map is not asymptotically linear since an estimate $\|F(x) - A(x)\| \le C\|x\|$ and $F(e_n) = 0$ would imply that A is bounded by, say, M on e_n. Hence $|A(\sum_{i=1}^n e_i)| \le Mn$. On the other hand, $F(\sum_{i=1}^n e_i) = n \log_2 n$.

3.5. *K-spaces.* Ribe's map $F : \ell_1 \to \mathbb{R}$ yields a twisted sum of $\mathbb{R}$ and ℓ_1 that cannot be locally convex: otherwise, by the Hahn–Banach theorem, the copy of $\mathbb{R}$ would be complemented, the sequence would split and F would be asymptotically linear.

There exist Banach spaces Z such that every quasi-linear map $Z \to Y$ is zero-linear (i.e., every twisted sum of no matter which Banach space and Z is locally convex). In particular, every twisted sum of $\mathbb{R}$ and Z is trivial (i.e., every quasi-linear map $Z \to \mathbb{R}$ is asymptotically linear). A theorem of Dierolf [28] states that the two statements are actually equivalent: every twisted sum with a Banach space is locally convex if and only if every twisted sum with the line is. This last formulation, however, has the advantage that it can also be applied to quasi-Banach spaces. One has:

Definition 1.24. A quasi-Banach space Z is said to be a *K-space* if every exact sequence $0 \to \mathbb{R} \to X \to Z \to 0$ splits.

This notion was introduced by Kalton and Peck [63]. Let us recall again that a quasi-Banach space is a K-space if and only if every quasi-linear map $Z \to \mathbb{R}$ is asymptotically additive.

The main examples of K-spaces are, amongst Banach spaces, the B-convex spaces [60] (recall that B-convexity means "having non-trivial type") and those of type $\mathcal{L}_\infty$ (cf. [66]); and, in general, the $\mathcal{L}_p$-spaces for $0 < p \le \infty$, $p \ne 1$ (see [60]). Thus every exact sequence $0 \to \mathbb{R} \to X \to Z \to 0$ in which Z is either a B-convex or an $\mathcal{L}_p$-space with $p \ne 1$ splits. Since, on the other hand, there exist non-trivial exact sequences of reflexive Banach spaces, it is clear that the results of Gajda, Székelyhidi and Ger we mentioned in Section 1 for Hyers-additive and pseudo-additive maps no longer work for zero-additive maps. Let us display an example.

3.6. *The Kalton–Peck–Ribe maps.* In some sense, these are the "coordinate-by-coordinate" version of Ribe's map obtained by Kalton and Peck in [62] as follows. Let X be a quasi-Banach space having an unconditional basis $(e_n)_{n=1}^\infty$ so that, for every $x \in X$, one has

$x = \sum_{n=1}^{\infty} x(n)e_n$. We assume that X does not contain a copy of c_0. First consider the map defined on the finitely supported sequences of X by

$$f(x)(n) = x(n)(-\log|x(n)|),$$

and then make it homogeneous:

$$F(x) = \|x\| f\left(\frac{x}{\|x\|}\right).$$

Of course, it is also possible to replace the invisible identity by a Lipschitz function; that is, given a Lipschitz function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ we set

$$f_\psi(x)(n) = x(n)\psi(-\log|x(n)|)$$

and then homogenize the function to

$$F_\psi(x) = \|x\| f_\psi\left(\frac{x}{\|x\|}\right).$$

It is a highly non-trivial task now to show that two such functions F_ψ and F_θ are equivalent if and only if $\psi - \theta$ is uniformly bounded. Hence, F_ψ is trivial if and only if ψ is uniformly bounded. See the original paper [62] or the monograph [20] for a full treatment.

II. Nearly additive mappings on controlled groups

In this chapter we introduce "nearly additive" maps on arbitrary groups and semigroups, taking values in Banach or quasi-Banach spaces.

1. Controlled groups and asymptotically additive maps. In what follows, G and S stand, respectively, for groups and semigroups acting as domains of mappings. Despite the use of additive notation for the operations these are *not* assumed to be commutative. We shall consider semigroups equipped with a certain "control" functional $\varrho : S \to \mathbb{R}^+$ which provides a rudimentary "topological" structure. We remark that by a control functional we mean any non-negative function on the (semi)group under consideration. If some additional property is required, we shall state it explicitly. We write (S, ϱ) if it is necessary to explicitly mention both the semigroup and the control functional.

The most interesting type of control functional are the quasi-norms. A *quasi-norm* is a non-negative real-valued map on a group G such that $\varrho(x) = 0$ if and only if $x = 0$, the function is symmetric in the sense that $\varrho(x) = \varrho(-x)$ for all $x \in G$, and the following condition holds: $\varrho(x + y) \le \Delta(\varrho(x) + \varrho(y))$ for some constant Δ and all $x, y \in G$. If this condition holds for $\Delta = 1$, then ϱ is called a *norm*. A (quasi-) normed group is a group together with a specified (quasi-) norm.

Let S be a semigroup with control functional ϱ and let Y be a quasi-Banach space (or even a *commutative* quasi-normed group) with quasi-norm $\|\cdot\|$. Given two maps $F, G : S \to Y$, one can consider the following possibly infinite "distance":

$$\operatorname{dist}(F, G) = \inf\{K : \|F(x) - G(x)\| \le K\varrho(x) \text{ for all } x \in S\}.$$

As before, the infimum of an empty set is treated as $+\infty$. By a *bounded* map we mean one at finite distance from the zero map. Following tradition we sometimes write $\|F\|$ instead of $\operatorname{dist}(F,0)$. The group of all bounded maps $B : S \to Y$ is denoted by $\mathcal{B}(S,Y)$. It is a quasi-normed group under the quasi-norm $\|B\|_{\mathcal{B}(S,Y)} = \operatorname{dist}(B,0)$. Observe that $\mathcal{B}(S,Y)$ is a quasi-Banach or Banach space if (and only if) Y is.

A map $A : S \to Y$ will be called *additive* if

$$A(x+y) = A(x) + A(y)$$

for all $x, y \in S$. The set of all additive maps from S to Y is a group (a linear space if Y is) that we denote by $\operatorname{Hom}(S,Y)$. The set

$$\operatorname{Hom}_{\mathcal{B}}(S,Y) = \operatorname{Hom}(S,Y) \cap \mathcal{B}(S,Y)$$

is a closed subgroup of $\mathcal{B}(S,Y)$ and therefore a quasi-normed group. When $Y = \mathbb{R}$ we write S^* or $(S,\varrho)^*$ instead of $\operatorname{Hom}_{\mathcal{B}}(S,Y)$ and we speak of the *dual space* of S. Notice that S^* is always a Banach space.

As before, we say that a map $F : S \to Y$ is *asymptotically additive* if $\operatorname{dist}(F,A)$ is finite for some additive map A or, in other words, if it is at finite distance from $\operatorname{Hom}(S,Y)$. These (trivial) maps are just perturbations of additive maps by bounded maps. The group of all asymptotically additive maps between S and Y is denoted by $\mathcal{T}(S,Y)$ or $\mathcal{T}((S,\varrho),Y)$. Notice that

$$\mathcal{T}(S,Y) = \operatorname{Hom}(S,Y) + \mathcal{B}(S,Y).$$

Remark 2.1. The only places in which mappings with values in general (commutative) quasi-normed groups are considered are Chapters V and VII.

2. Five definitions of nearly additive maps. Let F be a mapping defined on a controlled semigroup (S,ϱ) and taking values in a (quasi-) Banach space or a commutative quasi-normed group Y. There are several possibilities open to decide in which sense F behaves like an additive map. They usually take the form

$$\|F(x+y) - F(x) - F(y)\| \le K\Lambda(x,y)$$

for some constant K and some function Λ related to the control functional of S. We pass to give five definitions of nearly additive mapping that articulate the theory. It is true that, at first sight, they are much of a muchness; but we expect to have been able to emphasize the striking differences between the theories they originate. See also Chapter V.

Hyers additivity. It is defined by the choice

$$\Lambda(x,y) = 1.$$

The name comes from [50] where Hyers proved that a nearly additive mapping (in this sense) between Banach spaces is asymptotically additive with respect to $\varrho = 1$. See [36, 52] for a sound background, and the comments made in the introductory chapter for an overall panorama. The space of Hyers-additive maps acting between (G,ϱ) and $(Y,\|\cdot\|)$ is denoted by $\mathcal{H}(G,Y)$.

Quasi-additivity. It is defined by the choice

$$\Lambda(x,y) = \varrho(x) + \varrho(y).$$

The smallest constant K for which the inequality holds is referred to as the *quasi-additivity constant* of F and denoted by $Q(F)$. This notion constitutes the bridge towards the theory of extensions of (quasi-) Banach spaces in which, as we have seen, homogeneous quasi-additive maps are called quasi-linear. The space of quasi-additive maps acting between (G,ϱ) and $(Y,\|\cdot\|)$ is denoted by $\mathcal{Q}(G,Y)$.

Pseudo-additivity. It is defined by the choice

$$\Lambda(x,y) = \varrho(x) + \varrho(y) - \varrho(x+y)$$

and inspired by pseudo-linearity of Lima and Yost [71]. We denote by $P(F)$ the pseudo-additivity constant of F. As we have seen, in Banach space theory pseudo-linear maps generate semi-L-summands. Observe the rather restrictive character of this notion: the mere existence of pseudo-additive mappings forces ϱ to be subadditive. The space of pseudo-additive maps acting between (G,ϱ) and $(Y,\|\cdot\|)$ is denoted by $\mathcal{P}(G,Y)$.

Ger-additivity. It is defined by the choice

$$\Lambda(x,y) = \varrho(x)$$

or, equivalently, $\Lambda(x,y) = \min\{\varrho(x),\varrho(y)\}$. For F odd this is equivalent to the original definition $\Lambda(x,y) = \varrho(x+y)$. See also [25] and the remarks below.

The meaning of $G(F)$ should be clear. The space of Ger-additive maps acting between (G,ϱ) and $(Y,\|\cdot\|)$ is denoted by $\mathcal{G}(G,Y)$. There are close connections between Ger-additive maps and the existence of Lipschitz selectors for the metric projection in Banach spaces. See Chapter V, Section 6.

Zero-additivity. It appears when considering relations between an arbitrary (but finite) number of variables. Following [18, 20, 13, 9], we say that a map F is *zero-additive* if there is a constant K such that

$$\Big\|\sum_{i=1}^{n} F(x_i) - \sum_{j=1}^{m} F(y_j)\Big\| \le K\Big(\sum_{i=1}^{n} \varrho(x_i) + \sum_{j=1}^{m} \varrho(y_j)\Big)$$

for each n and m whenever x_i and y_j are such that $\sum_{i=1}^{n} x_i = \sum_{j=1}^{m} y_j$.

The smallest constant for which the inequality holds is denoted by $Z(F)$ and referred to as the *zero-additivity constant* of F. The space of all zero-additive maps acting between (G,ϱ) and $(Y,\|\cdot\|)$ is denoted by $\mathcal{Z}(G,Y)$. We have already mentioned that these mappings appear in connection with locally convex extensions of Banach spaces; see [18, 20, 13] and also Section 3 in Chapter V.

Remarks 2.2. • Although it may seem at first glance that Hyers additivity is the most restrictive of those notions, what is more true is that this notion stands apart from the others. Observe that Hyers's condition reduces to pseudo-additivity with respect to the control functional $\varrho = 1$. Thus, the constant functions are Hyers-additive although they are usually not quasi-additive.

• The interested reader could pay some attention to the paper [25], in which the general definition of Λ-additive map is introduced, and *regular* Λ-additive maps $\mathbb{R} \to \mathbb{R}$ are considered. That approach makes one realize that there are two (different) things that could be called Ger-additive maps: when, as in the text, one makes the choice $\Lambda(x,y) = \varrho(x)$, or else, when one makes the choice $\Lambda(x,y) = \varrho(x+y)$. They are not equivalent.

• It is quite clear that Lemma 1.17 is true for our five types of nearly additive maps, as it is true for any class $\mathcal{A}$ of quasi-additive maps satisfying two conditions: (a) if $F \in \mathcal{A}$ then the composition of F with the automorphism $x \mapsto -x$ is in $\mathcal{A}$; (b) $\mathcal{A}$ is a real vector space. These two conditions ensure that the odd and even parts of a map in $\mathcal{A}$ are in $\mathcal{A}$.

For maps taking values in Banach spaces, the following implications hold.

LEMMA 2.3. *Let S be a controlled semigroup and Y be a Banach space. Then:*

(a) $\mathcal{P}(S,Y) \subset \mathcal{Q}(S,Y)$ *and* $Q(F) \le P(F)$ *for every* $F \in \mathcal{P}(S,Y)$.
(b) $\mathcal{P}(S,Y) \subset \mathcal{Z}(S,Y)$ *and* $Z(F) \le P(F)$ *for every* $F \in \mathcal{P}(S,Y)$.
(c) $\mathcal{G}(S,Y) \subset \mathcal{Q}(S,Y)$ *and* $Q(F) \le G(F)$ *for every* $F \in \mathcal{G}(S,Y)$.
(d) $\mathcal{G}(S,Y) \subset \mathcal{Z}(S,Y)$ *and* $Z(F) \le G(F)$ *for every* $F \in \mathcal{G}(S,Y)$.

If G is a group endowed with a symmetric control functional one moreover has

(e) $\mathcal{P}(S,Y) \subset \mathcal{G}(S,Y)$ *with* $G(F) \le 2P(F)$ *for every* $F \in \mathcal{P}(S,Y)$.

If, in addition, G is a quasi-normed group then

(f) $\mathcal{Z}(S,Y) \subset \mathcal{Q}(S,Y)$ *with* $Q(F) \le (1+\Delta)Z(F)$, *where Δ is the constant appearing in the definition of the quasi-norm ϱ.*

Proof. The implications (a), (c) and (f) are obvious. That pseudo-additivity implies zero-additivity *even in a not necessarily commutative semigroup* deserves some comments. Let us show that a pseudo-additive mapping $F : (S,\varrho) \to Y$ with $P(F) = 1$ satisfies the inequality

$$\Big\|F\Big(\sum_{i=1}^n x_i\Big) - \sum_{i=1}^n F(x_i)\Big\| \le \sum_{i=1}^n \varrho(x_i) - \varrho\Big(\sum_{i=1}^n x_i\Big)$$

for each n and all $x_i \in S$. We proceed by induction on n. For $n=1$ and $n=2$ it is obvious and the definition of pseudo-additivity, respectively. We assume it is true for $n=k$ and proceed for $k+1$:

$$\begin{aligned}
&\Big\|F\Big(\sum_{i=1}^{k+1} x_i\Big) - \sum_{i=1}^{k+1} F(x_i)\Big\| \\
&\qquad = \Big\|F\Big(\sum_{i=1}^{k+1} x_i\Big) - F\Big(\sum_{i=1}^{k} x_i\Big) + F\Big(\sum_{i=1}^{k} x_i\Big) - \sum_{i=1}^{k+1} F(x_i)\Big\| \\
&\qquad \le \Big\|F\Big(\sum_{i=1}^{k+1} x_i\Big) - F\Big(\sum_{i=1}^{k} x_i\Big) - F(x_{k+1})\Big\| + \Big\|F\Big(\sum_{i=1}^{k} x_i\Big) - \sum_{i=1}^{k} F(x_i)\Big\|
\end{aligned}$$

$$\leq \varrho\Big(\sum_{i=1}^{k} x_i\Big) + \varrho(x_{k+1}) - \varrho\Big(\sum_{i=1}^{k+1} x_i\Big) + \sum_{i=1}^{k} \varrho(x_i) - \varrho\Big(\sum_{i=1}^{k} x_i\Big)$$
$$= \sum_{i=1}^{k+1} \varrho(x_i) - \varrho\Big(\sum_{i=1}^{k+1} x_i\Big).$$

This shows the containment (b).

That Ger-additivity implies zero-additivity is quite easy: in fact, a straightforward induction argument shows that Ger-additive maps satisfy the estimate

$$\Big\|F\Big(\sum_{i=1}^{n} x_i\Big) - \sum_{i=1}^{n} F(x_i)\Big\| \leq G(F)\Big(\sum_{i=1}^{n-1} \varrho(x_i)\Big).$$

This proves (d).

To see (e), note that if F is pseudo-additive with respect to ϱ, then ϱ must be subadditive. Hence, for every $x, y \in G$, one has

$$\varrho(y) - \varrho(x+y) \leq \varrho(-x).$$

Thus, if ϱ is symmetric, then every pseudo-additive map is Ger-additive with $G(F) \leq 2P(F)$. ∎

3. Forti's example. In general, no "approximate additivity" condition guarantees that a quasi-additive map $F : (G, \varrho) \to \mathbb{R}$ defined on an arbitrary controlled group is asymptotically additive. Thus, there is no room for a "simple" theory. The following crucial example due to Forti makes this apparent. In what follows, $\mathbb{F}_2$ denotes the free group generated by the symbols a and b, the operation being juxtaposition.

EXAMPLE 2.4 (Forti [34, 35]). *There is a map $F : \mathbb{F}_2 \to \mathbb{R}$ with $F(xy) - F(x) - F(y) \in \{-1, 0, 1\}$ for every $x, y \in \mathbb{F}_2$ and such that $F - A$ is (uniformly) unbounded on $\mathbb{F}_2$ for every $A \in \mathrm{Hom}(\mathbb{F}_2, \mathbb{R})$.*

Proof. Let $x \in \mathbb{F}_2$ be written in the "reduced" form, that is, x does not contain pairs of the form aa^{-1}, $a^{-1}a$, bb^{-1} or $b^{-1}b$, and it is written without exponents different from 1 and -1. Let $r(x)$ be the number of pairs of the form ab in x and let $s(x)$ be the number of pairs of the form $b^{-1}a^{-1}$ in x. Now, put $F(x) = r(x) - s(x)$. Clearly, one has $F(xy) - F(x) - F(y) \in \{-1, 0, 1\}$ for all $x, y \in \mathbb{F}_2$.

To end, let $S \subset \mathbb{F}_2$ be the semigroup generated by the commutator $aba^{-1}b^{-1}$. Then $S \subset \ker A$ for all $A \in \mathrm{Hom}(\mathbb{F}_2, \mathbb{R})$, while F is unbounded on S. Hence there in no additive A which (uniformly) approximates F on $\mathbb{F}_2$. ∎

Now, let $\delta : \mathbb{F}_2 \to \mathbb{R}^+$ be the discrete norm given by

$$\delta(x) = \begin{cases} 0 & \text{for } x = \emptyset \text{ (the ``empty'' word)}, \\ 1 & \text{for } x \neq \emptyset. \end{cases}$$

Since $F(\emptyset) = 0$, it is clear that $F : (\mathbb{F}_2, \delta) \to \mathbb{R}$ is pseudo-additive, Ger-additive, zero-additive and quasi-additive with constant 1 yet not asymptotically additive.

REMARK 2.5. This example also shows that Hyers' Theorem 1.2 does not hold for Z an arbitrary group. Notice that, for each x, the sequence $F(2^n x)/2^n$ still converges, but the resulting map is not additive. The reason for that disaster is that $2^n(x+y)$ may be different from $2^n x + 2^n y$ in absence of commutativity.

For related examples, see [55, 30, 31, 32].

4. Four stability properties. In view of the preceding example, we consider the following properties. Let S be a controlled semigroup and Y a quasi-Banach or Banach space.

DEFINITION 2.6. We say that the pair $[S, Y]$ has *property* (Z) (respectively, (Q), (G) or (P)) if every zero-additive (respectively, quasi-additive, Ger-additive or pseudo-additive) map $F : S \to Y$ is asymptotically additive.

In the spirit of Problem 1.1, one could require for every $\delta > 0$ the existence of some $\varepsilon > 0$ such that, for each $F : S \to Y$ with $Z(F) \le \varepsilon$ (respectively, $Q(F) \le \varepsilon$, $G(F) \le \varepsilon$ or $P(F) \le \varepsilon$) there is $A \in \mathrm{Hom}(S, Y)$ satisfying $\mathrm{dist}(F, A) \le \delta$. Since the numbers $Z(F)$, $Q(F)$, $G(F)$, $P(Z)$ and $\mathrm{dist}(F, \mathrm{Hom}(S, Y))$ depend homogeneously on F, that requirement would lead to some of the following properties.

DEFINITION 2.7. Let (S, ϱ) be a controlled semigroup, Y a (quasi-) Banach space and M a constant. We define the following properties of the pair $[(S, \varrho), Y]$:

(MP) For each $F \in \mathcal{P}((S, \varrho), Y)$ there is $A \in \mathrm{Hom}(S, Y)$ such that $\mathrm{dist}(F, A) \le MP(F)$.
(MG) For each $F \in \mathcal{G}((S, \varrho), Y)$ there is $A \in \mathrm{Hom}(S, Y)$ such that $\mathrm{dist}(F, A) \le MG(F)$.
(MZ) For each $F \in \mathcal{Z}((S, \varrho), Y)$ there is $A \in \mathrm{Hom}(S, Y)$ such that $\mathrm{dist}(F, A) \le MZ(F)$.
(MQ) For each $F \in \mathcal{Q}((S, \varrho), Y)$ there is $A \in \mathrm{Hom}(S, Y)$ such that $\mathrm{dist}(F, A) \le MQ(F)$.

Our immediate objective will be to show that, in fact, a pair $[G, Y]$ has property (Z) (respectively, (Q), (G) or (P)) if and only if it has, for some constant M, property (MZ) (respectively, (MQ), (MG) or (MP)). For obvious reasons we understand this as a "uniform boundedness principle". Let us prove the principle without further delay.

III. Uniform boundedness principles

The search for uniform boundedness principles for nearly additive maps is paved by the somewhat astonishing result of Forti [35] that we report now.

THEOREM 3.1 (Forti [35]). *Let S be a semigroup and Y a Banach space. Let $F : S \to Y$ be a mapping satisfying $\|F(x+y) - F(x) - F(y)\| \le \varepsilon$ for all $x, y \in S$. Suppose A is an additive mapping such that $F - A$ is uniformly bounded on S. Then, in fact, one has $\|F(x) - A(x)\| \le \varepsilon$ for all $x \in S$.*

We recognize here a uniform boundedness principle, namely: if every Hyers-additive mapping F is at finite distance from additive maps, then the distance depends only on the Hyers additivity constant of F.

In this chapter we prove the following generalization of Forti's result (see [10]).

THEOREM 3.2. *Let (G, ϱ) be a controlled group and Y a Banach space. Suppose that every zero-additive map $G \to Y$ is asymptotically additive. Then the pair $[(G, \varrho), Y]$ has property $(K\mathrm{Z})$ for some $K \geq 0$.*

Although we shall concentrate on zero-additive maps, only minor changes are needed to prove the corresponding results for other types of maps (and also for Y a quasi-Banach space). Let us also remark that the scalar case $Y = \mathbb{R}$ is not simpler than the general one. The somewhat involved proof combines ideas of Giudici (see [36], pp. 149–150) with the following general results about decomposition of groups that will be used extensively in what follows.

1. Decomposition of groups. Let $(G, +)$ be a group. A *commutator* is an element $x \in G$ which can be written as $x = a + b - a - b$. The *commutator subgroup* G_1 of G is the subgroup spanned by the commutators. It is easily seen that G_1 is a normal subgroup of G and also that $A(G) = G/G_1$ is abelian, so that one has the "abelianizing" sequence

$$0 \to G_1 \to G \to A(G) \to 0$$

which decomposes G into the commutator G_1 and the "abelian part" $A(G)$. The quotient map $\pi_1 : G \to A(G)$ has the following universal property: every group homomorphism ψ from G into a commutative group H factorizes through π_1 in the sense that there exists a unique homomorphism $\psi_1 : A(G) \to H$ for which $\psi = \psi_1 \circ \pi_1$.

Now consider the torsion subgroup of $A(G)$, that is,

$$T = \{x \in A(G) : nx = 0 \text{ for some } n \in \mathbb{N}\}.$$

Thus, we have another exact sequence

$$0 \to T \to A(G) \to A(G)/T \to 0$$

in which $A(G)/T$ is torsion-free (although not generally free). So, we have a diagram $G \xrightarrow{\pi_1} A(G) \xleftarrow{i} T$ which can be completed to give the so-called *pull-back diagram*

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 & & A(G)/T & = & A(G)/T & & \\
 & & \uparrow & & \uparrow & & \\
0 \to G_1 \to & & G & \to & A(G) & \to 0 \\
\| & & \uparrow & & \uparrow & & \\
0 \to G_1 \to & & \mathrm{PB} & \to & T & \to 0 \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
\end{array}$$

where $\mathrm{PB} = \pi_1^{-1}(i(T))$ is a normal subgroup of G containing both the commutator subgroup and the torsion elements of G, and which we denote by G_0 (actually, the sequence $0 \to G_1 \to G_0 \to T \to 0$ shows that G_0 can be regarded as a "mixture" of G_1 and T).

We are, however, interested in the middle vertical sequence

$$0 \to G_0 \to G \to G/G_0 \to 0$$

which decomposes G into the "bad part" G_0 and the "good part" G/G_0. In fact, we prove now that G/G_0 is the greatest quotient of G which embeds in a vector space. This is the content of the next lemmata.

LEMMA 3.3. *G/G_0 is a subgroup of a vector space over $\mathbb{Q}$.*

Proof. Since G/G_0 is both abelian and torsion-free, it obviously suffices to show that every abelian torsion-free group M is a subgroup of a vector space over $\mathbb{Q}$.

Consider M as a $\mathbb{Z}$-module in the obvious way and let $\varphi : M \to M \otimes_{\mathbb{Z}} \mathbb{Q}$ be the homomorphism given by the usual change of base procedure (that is, $\varphi(x) = x \otimes 1$). It is easily seen that φ is injective if (and only if) M is torsion-free (see [4]). So the obvious observation that $M \otimes_{\mathbb{Z}} \mathbb{Q}$ is a vector space over $\mathbb{Q}$ ends the proof. ∎

LEMMA 3.4. *Every homomorphism from G into a torsion-free abelian group factorizes through the natural quotient map $\pi_0 : G \to G/G_0$. In particular, such homomorphisms must vanish on G_0.*

Proof. Suppose $\psi : G \to H$ is a homomorphism, where H is abelian. Then ψ factorizes through the quotient map $\pi_1 : G \to A(G)$, so that $\psi = \psi_1 \circ \pi_1$ for some homomorphism $\psi_1 : A(G) \to H$. If, besides this, H is torsion-free, then $\psi_1 = \psi_2 \circ \pi_2$, where $\psi_2 : A(G)/T \to H$ is a homomorphism and $\pi_2 : A(T) \to A(T)/T$ is the natural quotient map. Since $A(G)/T$ is isomorphic to G/G_0 the result follows. ∎

An obvious consequence is the following.

COROLLARY 3.5. *For every Banach space Y one has $G_0 = \bigcap_{A \in \mathrm{Hom}(G,Y)} \ker A$.*

So, if we want to approximate a given map $F : (G, \varrho) \to Y$ by some additive mapping, then F must be bounded on G_0. Keep this in mind: sometimes it will be sufficient.

2. Proof of Theorem 3.2. The main step in the proof of Theorem 3.2 is the following result, which has its own intrinsic interest. The result seems to be new even for $\varrho = 1$ and shows that the space of Hyers-additive maps (modulo homomorphisms) is always a Banach space under the norm $\|F\|_{\mathcal{H}} = \sup\{\|F(x+y) - F(x) - F(y)\| : x, y \in G\}$. (See the papers [31–33] for a description of some related spaces.)

THEOREM 3.6. *Let (G, ϱ) be a controlled group. Then there exists a function $\varrho^* : G \to \mathbb{R}$ (depending only on ϱ) with the following property: for every Banach space Y and every zero-additive map $F : G \to Y$ there is an additive map $A : G \to Y$ such that $\|F(x) - A(x)\| \le Z(F)\varrho^*(x)$ for all $x \in G$. Moreover, when Y is fixed, A depends linearly on F.*

A crucial property of G_0 is isolated in Lemma 3.8. Lemma 3.7 will simplify the proof.

LEMMA 3.7. *Let F be a zero-additive map from an arbitrary controlled group (G, ϱ) into a Banach space. Then*

$$\|F(x) + F(-x)\| \le Z(F)(\varrho(x) + \varrho(-x)).$$

Proof. For each $n \in \mathbb{N}$ one has

$$\|nF(x) + nF(-x) - F(0)\| \leq Z(F)(n\varrho(x) + n\varrho(-x) + \varrho(0)),$$

which gives

$$\left\| F(x) + F(-x) - \frac{1}{n} F(0) \right\| \leq Z(F)\left(\varrho(x) + \varrho(-x) + \frac{1}{n}\varrho(0) \right).$$

The result follows on taking limits as $n \to \infty$. ■

LEMMA 3.8. *Let (G, ϱ) be a controlled group. There is a functional $\eta : G_0 \to \mathbb{R}$ such that, for every zero-additive map F from G into a Banach space, one has $\|F(x)\| \leq Z(F)\eta(x)$ for all $x \in G_0$.*

Proof. For each $x \in G_0$ there is $n(x) \geq 0$ such that $n(x)x \in G_1$. Since G_1 is generated by the set of commutators $\{z + y - z - y : z, y \in G\}$, there is an integer $m(x)$ and elements $z_i(x)$, $y_i(x)$, $1 \leq i \leq m(x)$, such that

$$\sum_{i=1}^{n(x)} x = \sum_{i=1}^{m(x)} (z_i(x) + y_i(x) - z_i(x) - y_i(x))$$

Hence

$$\begin{aligned} \|n(x)F(x)\| &\leq \Big\| n(x)F(x) - \sum_{i=1}^{m(x)} (F(z_i(x)) + F(y_i(x)) + F(-z_i(x)) + F(-y_i(x))) \Big\| \\ &\quad + \sum_{i=1}^{m(x)} \|(F(z_i(x)) + F(y_i(x)) + F(-z_i(x)) + F(-y_i(x)))\| \\ &\leq Z(F)\Big(n(x)\varrho(x) + 2\sum_{i=1}^{m(x)} \varrho(z_i(x)) + \varrho(y_i(x)) + \varrho(-z_i(x)) + \varrho(-y_i(x)) \Big). \end{aligned}$$

Thus, the choice

$$\eta(x) = \varrho(x) + \frac{2}{n(x)} \sum_{i=1}^{m(x)} (\varrho(z_i(x)) + \varrho(y_i(x)) + \varrho(-z_i(x)) + \varrho(-y_i(x)))$$

does what was announced. ■

Proof of Theorem 3.6. Let $\{e_i\}$ be a fixed basis for $G/G_0 \otimes_{\mathbb{Z}} \mathbb{Q}$ over $\mathbb{Q}$. Without loss of generality we may assume that e_i belongs to G/G_0 for all i. Let $\pi : G \to G/G_0$ be the quotient map. For each i, choose $g_i \in G$ such that $\pi(g_i) = e_i$. Now, let Y be a Banach space and let $F : (G, \varrho) \to Y$ be a zero-additive map. We define a $\mathbb{Q}$-linear map $L : G/G_0 \otimes_{\mathbb{Z}} \mathbb{Q} \to Y$ by putting $L(e_i) = F(g_i)$. Finally, let $A : G \to Y$ be given by $A(x) = L(\pi(x))$.

Clearly, A is additive and depends linearly on F. We shall estimate $\|F(x) - A(x)\|$ as a function of $Z(F)$ and $x \in G$. Since $\{e_i\}$ is a basis for $G/G_0 \otimes_{\mathbb{Z}} \mathbb{Q}$ there are rational numbers q_i such that $\pi(x) = \sum_i q_i e_i$. Write $q_i = m_i/n_i$ as an irreducible fraction with $n_i > 0$. Let moreover $N = \prod_i n_i$ and $M_i = m_i \prod_{j \neq i} n_j$. Clearly,

$$\pi(Nx) = N\pi(x) = \sum_i M_i e_i.$$

Let $y = \sum_i M_i g_i$. Obviously, $Nx - y$ lies in G_0, so that $ANx = A(y)$. One has

$$\Big\| \sum_{j=1}^{N} F(x) - \sum_{j=1}^{N} A(x) \Big\| = \Big\| \sum_{j=1}^{N} F(x) - A(y) \Big\| = \Big\| \sum_{j=1}^{N} F(x) - \sum_i M_i F(g_i) \Big\|$$

$$\leq Z(F) \cdot \eta\Big(Nx - \sum_i M_i g_i\Big) + \Big\| \sum_{j=1}^{N} F(x) - \sum_i M_i F(g_i) - F\Big(\sum_{j=1}^{N} x - \sum_i M_i g_i\Big) \Big\|$$

$$\leq Z(F) \Big\{ \eta\Big(Nx - \sum_i M_i g_i\Big) + \varrho\Big(Nx - \sum_i M_i g_i\Big) + N\varrho(x) + \sum_i M_i \varrho(g_i) \Big\}.$$

Therefore, the choice

$$\varrho^*(x) = \frac{1}{N} \Big\{ \eta\Big(Nx - \sum_i M_i g_i\Big) + \varrho\Big(Nx - \sum_i M_i g_i\Big) + N\varrho(x) + \sum_i M_i \varrho(g_i) \Big\}$$

is as expected. ∎

2.1. *Completeness of the space of all zero-additive maps.* For the proof of Theorem 3.2 we still need to develop some ideas. Let (G, ϱ) be a controlled group and Y a Banach space. Let $\mathscr{Z}(G, Y)$ be the (real) vector space of all zero-additive maps from G to Y. Consider the zero-additivity constant $Z(\cdot)$ as a seminorm on $\mathscr{Z}(G, Y)$. It is clear that the kernel of $Z(\cdot)$ is $\mathrm{Hom}(G, Y)$, so that $Z(\cdot)$ defines a norm on the quotient space $\mathscr{Z}(G, Y)/\mathrm{Hom}(G, Y)$.

Proposition 3.9. *The space $Z(G, Y)/\mathrm{Hom}(G, Y)$ endowed with $Z(\cdot)$ is a Banach space.*

Proof. Let $([G_n])_n$ be a $Z(\cdot)$-Cauchy sequence in $\mathscr{Z}(G, Y)/\mathrm{Hom}(G, Y)$. We want to see that there are representatives F_n of $[G_n]$ such that $(F_n)_n$ is pointwise convergent on G: if A_n is the additive map associated with G_n as in the proof of Theorem 3.6 then we set $F_n = G_n - A_n$. If ϱ^* is chosen as in Theorem 3.6, one has

$$\|F_n(x) - F_m(x)\| \leq Z(G_n - G_m)\varrho^*(x)$$

so that F_n is pointwise convergent on G. We set

$$F(x) = \lim_{n\to\infty} F_n(x).$$

It is not hard to see that F is zero-additive on G and that $[G_n] = [F_n]$ converges to $[F]$ with respect to $Z(\cdot)$. ∎

The space obtained from the space of trivial maps by taking quotient with respect to additive maps carries the natural quotient quasi-norm; we show that it is also complete:

Lemma 3.10. *Let (G, ϱ) be a controlled group, Y a Banach space and $\mathcal{B}(G, Y)$ the linear space of all bounded maps $G \to Y$. Then $(\mathcal{B}(G, Y) + \mathrm{Hom}(G, Y))/\mathrm{Hom}(G, Y)$ endowed with the norm*

$$\|[F]\| = \mathrm{dist}(F, \mathrm{Hom}(G, Y))$$

is a Banach space.

Proof. Simply observe that $(\mathcal{B}(G, Y) + \mathrm{Hom}(G, Y))/\mathrm{Hom}(G, Y)$ endowed with the metric $\mathrm{dist}(\cdot, \mathrm{Hom}(G, Y))$ is naturally isometric to $\mathcal{B}(G, Y)/(\mathcal{B}(G, Y) \cap \mathrm{Hom}(G, Y))$, where $\mathcal{B}(G, Y)$ is equipped with the natural norm $\|F\| = \mathrm{dist}(F, 0)$, which is a Banach space. ∎

2.2. *The end of the proof.* After all this machinery, we are ready to end the

Proof of Theorem 3.2. First observe that since for any controlled group G the space $\mathcal{Z}(G,Y)$ contains $\mathcal{B}(G,Y)+\mathrm{Hom}(G,Y)$, the hypothesis "zero-additive maps $G\to Y$ are at finite distance from additive ones" means that both spaces coincide. Thus both $Z(\cdot)$ and $\mathrm{dist}(\cdot,\mathrm{Hom}(G,Y))$ are defined on $\mathcal{Z}(G,Y)/\mathrm{Hom}(G,Y)$, making it complete by Theorem 3.6 and Lemma 3.10. Since $Z(\cdot)$ is dominated by $\mathrm{dist}(\cdot,\mathrm{Hom}(G,Y))$, the open mapping theorem implies that $Z(\cdot)$ and $\mathrm{dist}(\cdot,\mathrm{Hom}(G,Y))$ are K-equivalent on $\mathcal{Z}(G,Y)/\mathrm{Hom}(G,Y)$ for some $K\ge 0$, which is nothing but a restatement of property (KZ). ∎

The proofs of Theorems 3.6 and 3.2 can be adapted for other types of nearly additive maps:

THEOREM 3.11. *Let (G,ϱ) be a controlled group and Y a Banach space. Suppose that every Ger-additive* (*respectively*, *pseudo-additive*, *quasi-additive*) *map from G to Y is asymptotically additive. Then the pair $[(G,\varrho),Y]$ has property* (KG) (*respectively*, (KP), (KQ)) *for some $K\ge 0$.*

We also have

PROPOSITION 3.12. *Let (G,ϱ) be a controlled group and Y a Banach space.*

- *The space $\mathcal{G}(G,Y)/\mathrm{Hom}(G,Y)$ endowed with $G(\cdot)$ is a Banach space.*
- *The space $\mathcal{P}(G,Y)/\mathrm{Hom}(G,Y)$ endowed with $P(\cdot)$ is a Banach space.*
- *The space $\mathcal{Q}(G,Y)/\mathrm{Hom}(G,Y)$ endowed with $Q(\cdot)$ is a Banach space.*

IV. Asymptotically additive maps

This chapter contains some positive answers to Ulam's problem for various types of nearly additive mappings. In view of Forti's Example 2.4 quoted in Chapter II some restriction on the domain group is necessary to get stability. In this regard, we follow the lines along which Hyers' Theorem 1.2 has been generalized to weakly commutative and amenable groups. For the sake of completeness, these groups are introduced in Section 1. Section 2 concentrates on separation of subadditive and superadditive functionals by additive functions. That this topic is the third face of our strange coin will be proved in Section 3 where we study the stability of real-valued functions. Section 4 deals with vector-valued maps.

1. Amenable and weakly commutative groups and semigroups. Let S be a semigroup and let $\ell_\infty(S)$ be the real Banach space of all uniformly bounded functions $S\to\mathbb{R}$. Linear functionals on $\ell_\infty(S)$ can be viewed as (signed, finitely additive) measures on the power set of S, so we write $\int_S f\,d\mu$ (instead of $\mu(f)$) for the value of $\mu\in\ell_\infty(S)^*$ at $f\in\ell_\infty(S)$.

A *mean* on S is a linear functional $\mu:\ell_\infty(S)\to\mathbb{R}$ which is positive (that is, $\int_S f\,d\mu\ge 0$ for $f\ge 0$) and normalized ($\int_S 1_S\,d\mu=1$). These conditions imply that μ is bounded and, in fact, that $\|\mu\| = 1$. A mean is said to be *left invariant* on S if for every $y \in S$ and $f \in \ell_\infty(S)$ one has

$$\int_S f(x+y)\,d\mu(x) = \int_S f(x)\,d\mu(x)$$

and is said to be *right invariant* if

$$\int_S f(y+x)\,d\mu(x) = \int_S f(x)\,d\mu(x).$$

A semigroup is called *left* (respectively *right*) *amenable* if it admits at least one left (respectively right) invariant mean. A group is left amenable if and only if it is right amenable. The following classes of groups and semigroups are amenable: commutative groups and semigroups; finite groups; solvable groups; directed limits of amenable groups; normal subgroups and quotient groups of amenable groups. If G has an amenable normal subgroup H with G/H amenable, then G is itself amenable [26]. We refer the reader to Greenleaf's booklet [44] for background on amenability.

Our interest in amenability stems from the fact that some partial affirmative answers to Ulam's problem are available for amenable groups.

We consider now a quite different generalization of commutativity. A semigroup S is said to be *weakly commutative* if given $x, y \in S$ there is $n \in \mathbb{N}$ so that

$$2^n(x+y) = 2^n x + 2^n y.$$

The main tool for dealing with weakly commutative semigroups is the following simple

LEMMA 4.1. *If x and y are fixed elements of a weakly commutative semigroup, then there exists an infinite sequence $n(k)$ such that*

$$2^{n(k)}(x+y) = 2^{n(k)}x + 2^{n(k)}y.$$

Proof. We construct the sequence $n(k)$ by induction on k. The first term is given by the weak commutativity of S. Suppose $n(k)$ has been choosen in such a way that $2^{n(k)}(x+y) = 2^{n(k)}x + 2^{n(k)}y$. Put $u = 2^{n(k)}$ and $v = 2^{n(k)}$ and let $m \geq 1$ be such that $2^m(u+v) = 2^m u + 2^m v$. We have

$$\begin{aligned} 2^{m+n(k)}(x+y) &= 2^m(2^{n(k)}(x+y)) = 2^m(2^{n(k)}x + 2^{n(k)}y) = 2^m(u+v) \\ &= 2^m u + 2^m v = 2^m(2^{n(k)}x) + 2^m(2^{n(k)}y) = 2^{m+n(k)}x + 2^{m+n(k)}y. \end{aligned}$$

So, we can take $n(k+1) = m + n(k)$. ∎

2. Sandwich theorems. This section has a preparatory character and is mainly taken from [39]. We discuss the following two related problems.

PROBLEM 4.2. *Suppose $\alpha, \beta : G \to \mathbb{R}$ are maps with α superadditive (that is, $\alpha(x+y) \geq \alpha(x) + \alpha(y)$ for all $x, y \in G$), β subadditive, and $\alpha(x) \leq \beta(x)$ for all $x \in G$. Does there exist an additive map $A : G \to \mathbb{R}$ separating α from β, that is, satisfying*

$$\alpha(x) \leq A(x) \leq \beta(x)$$

for every $x \in G$?

Problem 4.3. *Let $\beta : G \to \mathbb{R}$ be a subadditive mapping. Does there exist an additive map $A : G \to \mathbb{R}$ dominated by β, that is, with $A(x) \le \beta(x)$ for every $x \in G$?*

After the equivalence Theorems 4.14 and 4.15 (to be proved in the next section), the close connections between this kind of problems and Ulam's problem for real-valued maps will become apparent.

First of all, let us show that Problem 4.3 is a restriction of Problem 4.2. Suppose β is subadditive on G. Then the mapping $\alpha : G \to \mathbb{R}$ given by $\alpha(x) = -\beta(-x)$ is superadditive, with $\alpha \le \beta$. (Thus, subadditive functionals always dominate some superadditive functional.) Moreover, it is easily seen that, if $A : G \to \mathbb{R}$ is additive, then A is dominated by β if and only if it separates α from β.

The following partial affirmative answer to the first problem is due to Gajda and Kominek [39] and generalizes previous results of Mazur and Orlicz [76], Sikorski [95], Pták [80], Kaufman [67] and Kranz [70].

Theorem 4.4 ([39]). *Suppose G is a weakly commutative group and that $\alpha, \beta : G \to \mathbb{R}$ are such that α is superadditive, β is subadditive and $\alpha \le \beta$. Then there exists an additive A separating α from β.*

For the proof, we need the following simple results.

Lemma 4.5 ([39]). *Let S be a semigroup and let $\gamma : S \to \mathbb{R}$ be subadditive or superadditive. If γ is 2-homogeneous (that is, $\gamma(2x) = 2\gamma(x)$ for all $x \in S$), then it is $\mathbb{N}$-homogeneous: $\gamma(nx) = n\gamma(x)$ for all $x \in S$ and $n \in \mathbb{N}$.*

Proof. We only consider the case in which γ is subadditive. If γ is 2-homogeneous, then a simple induction shows that $\gamma(2^k x) = 2^k\gamma(x)$ for all $x \in S$ and $k \in \mathbb{N}$.

Now, let $x \in S$ and let n be a positive integer which is different from 2^k for any k. Then either $n = 1$ and the result is trivial or $n = 2^k + r$ with $k \in \mathbb{N}$ and $1 \le r \le 2^k - 1$. Hence

$$2^{k+1}\gamma(x) \le \gamma(2^{k+1}x) = \gamma((nx) + (2^k - r)(x)) \le \gamma(nx) + (2^k - r)\gamma(x).$$

Thus, $n\gamma(x) \le \gamma(nx)$, whereas the converse inequality follows from the subadditivity of γ. ∎

Lemma 4.6 ([39]). *Let S be a semigroup and let $\gamma : S \to \mathbb{R}$ be subadditive or superadditive. If γ is 2-homogeneous, then $\gamma(x + y) = \gamma(y + x)$ for all $x, y \in S$.*

Proof. We only consider the subadditive case. By the previous lemma, γ is $\mathbb{N}$-homogeneous. Hence,

$$\gamma(x+y) = \frac{\gamma(k(x+y))}{k} = \frac{\gamma(x + (k-1)(y+x) + y)}{k} \le \frac{\gamma(x) + \gamma(y)}{k} + \frac{k-1}{k}\gamma(y+x).$$

Letting $k \to \infty$, we get $\gamma(x+y) \le \gamma(y+x)$ and, by symmetry, we have $\gamma(x+y) = \gamma(y+x)$, as desired. ∎

The following lemma allows us to replace the original functionals of Theorem 4.4 we want to separate by new ones with some additional properties.

LEMMA 4.7 ([39]). *Let S be a weakly commutative semigroup, and let $\alpha, \beta : S \to \mathbb{R}$ be respectively superadditive and subadditive, with $\alpha(x) \leq \beta(x)$ for all $x \in S$. Then there exist $\widetilde{\alpha}, \widetilde{\beta} : S \to \mathbb{R}$ such that*:

(a) $\alpha(x) \leq \widetilde{\alpha}(x) \leq \widetilde{\beta}(x) \leq \beta(x)$ *for all* $x \in S$.
(b) $\widetilde{\alpha}$ *is superadditive and* $\widetilde{\beta}$ *subadditive.*
(c) $\widetilde{\alpha}$ *and* $\widetilde{\beta}$ *are* $\mathbb{N}$*-homogeneous.*
(d) *For every* $x, y \in S$, *one has* $\widetilde{\alpha}(x+y) = \widetilde{\alpha}(y+x)$ *and* $\widetilde{\beta}(x+y) = \widetilde{\beta}(y+x)$.

Proof. First of all note that, for $x \in S$ and $n \in \mathbb{N}$, one has

$$\alpha(x) \leq \frac{\alpha(2^n x)}{2^n} \leq \frac{\alpha(2^{n+1}x)}{2^{n+1}} \leq \frac{\beta(2^{n+1}x)}{2^{n+1}} \leq \frac{\beta(2^n x)}{2^n} \leq \beta(x).$$

So, we can define $\widetilde{\alpha}, \widetilde{\beta} : S \to \mathbb{R}$ by

$$\widetilde{\alpha}(x) = \lim_{n\to\infty} \frac{\alpha(2^n x)}{2^n}, \qquad \widetilde{\beta}(x) = \lim_{n\to\infty} \frac{\beta(2^n x)}{2^n}.$$

Obviously, (a) holds. Let us prove (b). To verify the subadditivity of $\widetilde{\alpha}$, let $x, y \in S$. Take a sequence $n(k)$ so that

$$2^{n(k)}(x+y) = 2^{n(k)}x + 2^{n(k)}y.$$

One has

$$\begin{aligned}\widetilde{\alpha}(x)(x+y) &= \lim_{n\to\infty} \frac{\alpha(2^n(x+y))}{2^n} = \lim_{k\to\infty} \frac{\alpha(2^{n(k)}(x+y)}{2^{n(k)}} = \lim_{k\to\infty} \frac{\alpha(2^{n(k)}x + 2^{n(k)}y)}{2^{n(k)}} \\ &\leq \lim_{k\to\infty} \frac{\alpha(2^{n(k)}x)}{2^{n(k)}} + \lim_{k\to\infty} \frac{\alpha(2^{n(k)}y)}{2^{n(k)}} = \widetilde{\alpha}(x) + \widetilde{\alpha}(y),\end{aligned}$$

as required. The superadditivity of $\widetilde{\beta}$ can be proved analogously. Finally, (c) and (d) follow from the obvious 2-homogeneity of $\widetilde{\alpha}$ and $\widetilde{\beta}$ and the previous lemmata. ∎

Proof of Theorem 4.4. Consider the family $\mathcal{F}$ of all pairs (ϕ, ψ), where ϕ is superadditive on G, ψ is subadditive and $\phi \leq \psi$. We introduce a partial ordering in $\mathcal{F}$ by

$$(\phi_1, \psi_1) \leq (\phi_2, \psi_2) \iff \phi_1 \leq \phi_2 \leq \psi_2 \leq \psi_1.$$

It is obvious that every linearly ordered subfamily of $\mathcal{F}$ has an upper bound in $\mathcal{F}$. Since $\mathcal{F}$ is not empty (it contains (α, β)), the Kuratowski–Zorn lemma yields a maximal pair (a, b) with $(\alpha, \beta) \leq (a, b)$. Let $\widetilde{a}, \widetilde{b}$ be the functionals obtained by applying Lemma 4.7 to a and b. By the maximality of the pair (a, b), we see that $a = \widetilde{a}$ and $b = \widetilde{b}$ and, in particular, we have

$$a(x+y) = a(y+x), \qquad b(x+y) = b(y+x)$$

for all $x, y \in G$.

On the other hand, $a(0) = b(0) = 0$ (for if not, then taking into account that superadditive functionals take non-positive values at 0 and that subadditive functionals take non-negative values at 0, one could obtain a new pair exceeding (a, b) by redefining the values at 0).

After these preparations, we prove that $a = b$ on the whole of G. In this case, it is clear that $A = a = b$ is an additive functional separating α from β.

For the indirect proof, suppose there is $c \in G$ and a number r such that

$$a(c) < r < b(c).$$

We claim that either

(1) $mr + b(s) \geq a(mc + s)$ for every $m = 0, 1, \ldots$ and all $s \in G$, or
(2) $b(nc + t) \geq nr + G(t)$ for every $n = 0, 1, \ldots$ and all $t \in G$

(with the convention that $0x = 0$ for all $x \in G$). Suppose that neither (1) nor (2) holds. Then there exist $m, n \in \mathbb{Z}$ and $s, t \in G$ such that

$$mr + b(s) < a(mc + s), \qquad b(nc + t) < nr + a(t),$$

whence

$$nb(s) + mb(nc + t) < ma(t) + na(mc + s).$$

Since a is superadditive and b is subadditive (and taking into account that $a(x + y) = a(y + x)$ and $b(x + y) = b(y + x)$ for all x, y) we would obtain

$$\begin{aligned} b(ns + mt + nmc) &\leq nb(s) + b(mt + nmc) = nb(s) + b((m-1)t + n(m-1)c + nc + t) \\ &= nb(s) + b((m-1)t + n(m-1)c) + b(nc + t) \\ &\leq \ldots \leq nb(s) + mb(nc + t) < ma(t) + na(mc + s) \\ &\leq a(mt) + (n-2)a(mc + s) + a(mc + s) + a(s + mc) \\ &\leq a(mt) + (n-2)a(mc + s) + a(mc + 2s + mc) \\ &= a(mt) + (n-2)a(mc + s) + a(2mc + 2s) \\ &\leq \ldots \leq a(mt) + a(nmc + ns) \leq a(nmc + ns + mt) \\ &= a(ns + mt + nmc), \end{aligned}$$

which is impossible, since b dominates a.

Suppose (1) holds. For each $x \in G$, put

$$b_0(x) = \inf\{mr + b(s) : x = mc + s,\ s \in G,\ m = 0, 1, \ldots\}.$$

It follows that $a(x) \leq b_0(x) \leq b(x)$ for all $x \in G$ and $b_0(c) \leq r < b(c)$. Next, let m and n be arbitrarily chosen. Clearly,

$$\begin{aligned} (m+n)r + b(-(m+n)c + x + y) &= (m+n)r + b(y - (m+n)c + x) \\ &\leq (m+n)r + b(y - nc) + b(-mc + x) \\ &= mr + b(-mc + x) + nr + b(-nc + y). \end{aligned}$$

Hence, by the very definition of b_0, we get

$$b_0(x + y) \leq b_0(x) + b_0(y),$$

that is, b_0 is subadditive and $(a, b) \leq (a, b_0)$. Since $b_0(c) < b(c)$, this contradicts the maximality of the pair (a, b) and ends the proof in case (1).

Assuming (2), we define

$$a_0(x) = \sup\{nr + a(t) : x = nc + t,\, t \in G,\, m = 0, 1, \ldots\}$$

for $x \in G$. Then, similarly to case (1), one can show that a_0 is superadditive and also that $a(x) \le a_0(x) \le b(x)$ and $a(c) < r < a_0(c)$, again contradicting the maximality of (a, b). ■

We do not know if Theorem 4.4 is true for amenable groups. Nevertheless, a subadditive functional on an amenable group always dominates some additive map:

THEOREM 4.8. *Let $\beta : G \to \mathbb{R}$ be a subadditive functional. If G is amenable, then there exists an additive map A such that $A \le \beta$.*

Proof. Let $\beta : G \to \mathbb{R}$ be subadditive. Then, for all $x, y \in G$, one has

$$-\beta(-x) \le \beta(x+y) - \beta(y) \le \beta(x),$$

so that the mapping $y \in G \mapsto \beta(x+y) - \beta(y)$ is bounded (in the usual sense) on G for each x. Let dy denote a (left) invariant mean for G and define $A : G \to \mathbb{R}$ as

$$A(x) = \int_G (\beta(x+y) - \beta(y))\, dy.$$

Obviously, A is dominated by β. We claim that A is additive. Indeed, take $x, z \in G$. Using linearity and invariance of the mean, we get

$$\begin{aligned} A(x+z) &= \int_G (\beta(x+z+y) - \beta(y))\, dy = \int_G (\beta(x+z+y) - \beta(y))\, dy \\ &= \int_G (\beta(x+z+y) - \beta(z+y) + \beta(z+y) - \beta(y))\, dy \\ &= \int_G (\beta(x+z+y) - \beta(z+y))\, dy + \int_G (\beta(z+y) - \beta(y))\, dy \\ &= \int_G (\beta(x+y) - \beta(y))\, dy + \int_G (\beta(z+y) - \beta(y))\, dy = A(x) + A(z). \ ■ \end{aligned}$$

Theorem 4.8 (and also 4.4) is a "global" result which explains a common property of all subadditive functionals defined on an amenable group. One may wonder under which conditions a (particular) given subadditive functional must dominate an additive one. A simple criterion is the following (a similar result holds for the analogous question about separation).

COROLLARY 4.9. *Let β be a subadditive functional on an arbitrary group G. Then there exists an additive map dominated by β if and only if β is non-negative on the commutator subgroup of G.*

Proof. Since additive functionals vanish on the commutator subgroup, the necessity of the condition is clear. Let us prove its sufficiency. Suppose β is a subadditive functional on G. Denote by G_1 the commutator subgroup of G and by π the natural homomorphism $G \to G/G_1$. Consider the functional $\gamma : G/G_1 \to [-\infty, \infty)$ defined as

$$\gamma(\pi(x)) = \inf\{\beta(y) : y \in \pi(x)\}.$$

We show that γ is subadditive. Indeed, let $x, y \in G$. Take sequences (x_n), (y_n) so that

$x_n \in \pi(x)$, $y_n \in \pi(y)$ and

$$\gamma(\pi(x)) = \lim_{n\to\infty} \beta(x_n), \qquad \gamma(\pi(y)) = \lim_{n\to\infty} \beta(y_n).$$

Since $x_n + y_n$ belongs to $\pi(x) + \pi(y)$, we have

$$\begin{aligned}\gamma(\pi(x) + \pi(y)) &\le \inf_n \beta(x_n + y_n) \le \inf_n \{\beta(x_n) + \beta(y_n)\} \\ &\le \lim_{n\to\infty} \beta(x_n) + \lim_{n\to\infty} \beta(y_n) = \gamma(\pi(x) + \gamma(\pi(y)).\end{aligned}$$

Next, assume that β is non-negative on G_1. We prove that γ takes only finite values. Suppose on the contrary there is $x \in G$ with $\gamma(\pi(x)) = -\infty$. Then

$$\gamma(0) \le \gamma(\pi(x)) + \gamma(-\pi(x)) = -\infty,$$

and therefore

$$\gamma(0) = \inf\{\beta(y) : y \in G_1\} = -\infty,$$

which is a contradiction. Hence γ is everywhere finite and subadditive on G/G_1. But G/G_1 is abelian, so according to Theorem 4.4 (or Theorem 4.8) there exists an additive functional $a : G/G_1 \to \mathbb{R}$ dominated by γ. Taking $A = a \circ \pi$, we obtain an additive map on G with

$$A(x) = a(\pi(x)) \le \gamma(\pi(x)) \le \beta(x)$$

for all $x \in G$. ■

We do not know whether the conclusion of Theorem 4.8 implies the separation property. It seems very likely that amenable groups must have the separation property, but we only have the following partial result.

Proposition 4.10 ([39]). *Let S be a (left or right) amenable semigroup and let α and β be respectively superadditive and subadditive, with $\alpha \le \beta$. If moreover either $\alpha(x + y) = \alpha(y + x)$ or $\beta(x + y) = \beta(y + x)$ holds true for every $x, y \in S$, then α and β can be separated by an additive mapping.*

Proof. We may assume that S is left amenable and that $\beta(x + y) = \beta(y + x)$ for every $x, y \in S$. Notice that

$$\beta(x + y) - \alpha(y) \ge \alpha(x + y) - \alpha(y) \ge \alpha(x), \qquad x, y \in S.$$

Hence one can define a functional h on S by

$$h(x) = \inf\{\beta(x + y) - \alpha(y) : y \in S\}.$$

The subadditivity of β yields

$$\begin{aligned}h(x + y) &= \inf\{\beta(x + y + z) - \alpha(z) : z \in S\} \le \inf\{\beta(x) + \beta(y + z) - \alpha(x) : z \in S\} \\ &= \beta(x) + \inf\{\beta(y + z) - \alpha(x) : z \in S\} = \beta(x) + h(y).\end{aligned}$$

On the other hand,

$$\begin{aligned}h(x + y) &= \inf\{\beta(x + y + z) - \alpha(z) : z \in S\} \\ &\ge \inf\{\beta(y + x + z) - \alpha(z + x) + \alpha(x) : z \in S\} \\ &= \alpha(x) + \inf\{\beta(y + x + z) - \alpha(z + x) : z \in S\} \\ &\ge \alpha(x) + \inf\{\beta(y + w) - \alpha(w) : w \in S\} = \alpha(x) + h(y).\end{aligned}$$

Putting these estimates together, we infer that, for every x, y,

$$\alpha(x) \leq h(x+y) - h(y) \leq \beta(x).$$

So, we obtain the desired separating (additive) functional by taking

$$A(x) = \int_S (h(x+y) - h(y))\, dy,$$

where dy represents a left invariant mean for S. ■

Theorem 4.8 is, quite plainly, a "Hahn–Banach" type result (see [5, théorème 2]). The following nice result, also due to Gajda and Kominek, generalizes another well known theorem by Banach about supporting functionals: for every point x_0 of a Banach space there is a norm-one linear functional f such that $f(x_0)$ equals $\|x_0\|$.

LEMMA 4.11 ([39]). *Let $\beta : G \to \mathbb{R}$ be a 2-homogeneous subadditive functional. If the group G is either weakly commutative or amenable then, for every $x_0 \in G$, there exists an additive mapping $A : G \to \mathbb{R}$ such that $A(x_0) = \beta(x_0)$ with $A(x) \leq \beta(x)$ for all $x \in G$.*

Proof. Let $\beta : G \to \mathbb{R}$ be a 2-homogeneous (hence $\mathbb{N}$-homogeneous) subadditive functional. One has

$$\beta(x_0) + \beta(nx_0) = \beta((n+1)x_0), \qquad \beta(-x + (n+1)x_0) \leq \beta(-x + nx_0) + \beta(x_0).$$

Hence

$$\beta(nx_0) - \beta(-x + nx_0) \leq \beta((n+1)x_0) - \beta(-x + (n+1)x_0),$$

so that the sequence $(\beta(nx_0) - \beta(-x + nx_0))_n$ is non-decreasing and bounded above by $\beta(x)$ for every fixed $x \in S$. Put

$$\alpha(x) = \lim_{n\to\infty} (\beta(nx_0) - \beta(-x + nx_0)).$$

Clearly, $\alpha(x) \leq \beta(x)$ for all $x \in G$ and $\alpha(x_0) = \beta(x_0)$. Thus the proof will be complete if we show that α and β can be separated by an additive functional. In view of Theorem 4.4 (if G is weakly commutative) and Proposition 4.10 (for G amenable) it suffices to verify that α is superadditive. Indeed, let $x, y \in G$. By Lemma 4.6 one has

$$\begin{aligned}
\alpha(x) + \alpha(y) &= \lim_{n\to\infty} (\beta(nx_0) - \beta(-x + nx_0)) + \lim_{n\to\infty} (\beta(nx_0) - \beta(-y + nx_0)) \\
&= \lim_{n\to\infty} (\beta(2nx_0) - \beta(-x + nx_0) - \beta(-y + nx_0)) \\
&\leq \lim_{n\to\infty} (\beta(2nx_0) - \beta(-x + nx_0 - y)) = \lim_{n\to\infty} (\beta(2nx_0) - \beta(-y - x + nx_0)) \\
&= \alpha(x+y). \ ■
\end{aligned}$$

2.1. *Embedding quasi-normed groups into Banach spaces.* In some aspects, amenable and weakly commutative normed groups behave like Banach spaces. As an application of the results of this section, we now show that, under rather mild assumptions, an amenable or weakly commutative normed group (G, ϱ) must be an additive subgroup of a Banach space (ϱ being the restriction of the norm).

PROPOSITION 4.12. *Let (G, ϱ) be a normed group. Assume that ϱ is 2-homogeneous. If G is either weakly commutative or amenable, then there exists a Banach space X and a* *group homomorphism $\delta : G \to X$ such that $\varrho(x) = \|\delta(x)\|$ for all $x \in G$. If in addition ϱ is non-degenerate, then δ is an into isomorphism.*

Proof. Let $(G, \varrho)^*$ be the Banach space of all real-valued bounded homomorphisms on (G, ϱ) equipped with the norm

$$\|A\| = \inf\{K : |A(x)| \le K\varrho(x)\}$$

and let X be the dual space of $(G, \varrho)^*$ with the dual norm. Define $\delta : G \to X$ by $\delta(x)(A) = A(x)$. Obviously, δ is a group homomorphism. Moreover,

$$\|\delta\| = \sup\{|A(x)| : \|A\| \le 1\} \le \varrho(x).$$

It remains to prove that $\|\delta(x)\|_X \ge \varrho(x)$ for each $x \in G$. Fix $x \in G$. By the lemma just proved, the hypothesis on G together with the subadditivity and the 2-homogeneity of ϱ implies the existence of an additive map $A : G \to \mathbb{R}$ satisfying

$$A(x) = \varrho(x), \quad A(y) \le \varrho(y) \quad \text{for all } y \in G.$$

Since ϱ is symmetric and non-negative one has $|A(y)| \le \varrho(y)$ for all $y \in G$, so that $\|A\| = 1$. It follows that

$$\|\delta(x)\|_X \ge |\delta(x)(A)| = |A(x)| = \varrho(x),$$

which proves the first statement. The second is now obvious. ∎

COROLLARY 4.13. *Let G be an arbitrary group endowed with a 2-homogeneous norm ϱ. There exists a group homomorphism δ from G into a Banach space X such that $\|\delta(x)\|_X = \varrho(x)$ for every $x \in G$ if and only if* $\ker\varrho$ *is a normal subgroup of G containing the commutator subgroup of G.*

3. The scalar case. In this section we only consider real-valued maps. Vector-valued maps will be considered in the next section. We begin with the following somewhat surprising result.

THEOREM 4.14 ([9]). *Let S be a semigroup and let $M \ge 1$ be a fixed number. The following statements are equivalent:*

(a) *For every ϱ, the pair $[(S, \varrho), \mathbb{R}]$ has property $(M\mathrm{Z})$.*

(b) *For every subadditive ϱ, the pair $[(S, \varrho), \mathbb{R}]$ has property $(M\mathrm{P})$.*

Moreover, for $M = 1$, (a) *and* (b) *are equivalent to*

(c) *For every $\alpha, \beta : S \to \mathbb{R}$ such that α is superadditive, β is subadditive and $\alpha \le \beta$, there exists an additive A separating α from β.*

Proof. That (a) implies (b) for any M is clear since pseudo-additivity implies zero-additivity (see Lemma 2.3(b)). We now prove that (c) implies (a) for $M = 1$. We need some notation. For a real-valued mapping a on a semigroup S, define

$$a^*(x) = \sup\Big\{\sum_{i=1}^{n} a(x_i) : x = \sum_{i=1}^{n} x_i\Big\}, \quad a_*(x) = \inf\Big\{\sum_{i=1}^{n} a(x_i) : x = \sum_{i=1}^{n} x_i\Big\}.$$

Obviously, $a_* \le a \le a^*$. Moreover, a^* is superadditive and a_* is subadditive.

Suppose that S has property (c) and let $F : S \to \mathbb{R}$ be zero-additive with respect to ϱ. We may assume $Z(F) = 1$. We claim that $(F - \varrho)^*(x) \le (F + \varrho)_*(x)$ for all $x \in S$ (which implies that both functions take only finite values). Indeed, let $\sum_{i=1}^n x_i = \sum_{j=1}^m y_j$. One has to verify that

$$\sum_{i=1}^n F(x_i) - \sum_{i=1}^n \varrho(x_i) \le \sum_{j=1}^m \varrho(y_j) + \sum_{j=1}^m F(y_j)$$

or, in other words, that

$$\sum_{i=1}^n F(x_i) - \sum_{j=1}^m F(y_j) \le \sum_{i=1}^n \varrho(x_i) + \sum_{j=1}^m \varrho(y_j),$$

which immediately follows from zero-additivity. The hypothesis on S implies the existence of an additive A separating $(F - \varrho)^*$ from $(F + \varrho)_*$. Hence

$$F(x) - \varrho(x) \le A(x) \le F(x) + \varrho(x)$$

for all $x \in S$ or, which is the same, $\operatorname{dist}(F, A) \le 1$, and thus, S satisfies the condition (a) for $M = 1$.

It remains to show that the property (b) for $M = 1$ implies the separation property (c). We need a way to translate a problem about separation into an equivalent problem about approximation of pseudo-additive mappings by additive mappings. So, let α and β be respectively superadditive and subadditive mappings such that $\alpha \le \beta$. Define

$$F = \frac{\beta + \alpha}{2}, \qquad \varrho = \frac{\beta - \alpha}{2},$$

Obviously, ϱ is non-negative and subadditive. The pseudo-additivity of F with respect to ϱ follows from the hypotheses on α and β by routine computations. Finally, observe that $\operatorname{dist}(F, A) \le 1$ implies that A separates α from β. Hence property (b) for $M = 1$ implies (c).

To complete the proof we show that (b) implies (a) for any $M \ge 1$. Assume that S has property (b) for some $M \ge 1$ and let F be zero-additive with respect to a given ϱ. We may assume that $Z(F) = 1$. Put

$$\alpha = (F - \varrho)^*, \qquad \beta = (F + \varrho)_*.$$

As above, α is superadditive, β is subadditive and $\alpha \le \beta$. Hence taking

$$G = \frac{\beta + \alpha}{2} = \frac{(F - \varrho)^* + (F + \varrho)_*}{2}, \qquad \sigma = \frac{\beta - \alpha}{2} = \frac{(F + \varrho)_* - (F - \varrho)^*}{2},$$

one sees that G is pseudo-additive with respect to σ. The hypothesis gives an additive A such that $\operatorname{dist}(G, A) \le M$. This can be written as

$$\frac{(1 - M)(F + \varrho)_* + (1 + M)(F - \varrho)^*}{2} \le \frac{(1 + M)(F + \varrho)_* - (1 - M)(F - \varrho)^*}{2}.$$

Since $M \ge 1$, taking into account that $(F + \varrho)_* \le F + \varrho$ and $(F - \varrho)^* \ge F - \varrho$ one deduces that also

$$\frac{(1 - M)(F + \varrho) + (1 + M)(F - \varrho)}{2} \le A \le \frac{(1 + M)(F + \varrho) - (1 - M)(F - \varrho)}{2}.$$

Rearranging the parentheses, one obtains

$$F - M\varrho \le A \le F + M\varrho. \ \blacksquare$$

It should be noted that (a), (b) (and (c)) are statements about S and not about a particular pair (S,ϱ).

It will be convenient to have a version of Theorem 4.14 for symmetric control functionals (that is, with $\varrho(-x) = \varrho(x)$ for all x) when dealing with quasi-normed groups.

Theorem 4.15. *For a group G the following statements are equivalent:*

(a) *For every symmetric ϱ the pair $[(G,\varrho),\mathbb{R}]$ has the property* (1Z) *for odd maps.*

(b) *For every symmetric ϱ the pair $[(G,\varrho),\mathbb{R}]$ has the property* (1P) *for odd maps.*

(c) *For every subadditive* (*but not necessarily symmetric*) *mapping $\beta : G \to \mathbb{R}$ there exists an additive map $A : G \to \mathbb{R}$ such that $A(x) \le \beta(x)$ for all $x \in G$.*

Moreover, each of these conditions implies the following:

(d) *For every symmetric ϱ the pair $[(G,\varrho),\mathbb{R}]$ has property* (1G) *for odd maps.*

Proof. That (a) implies (b) and (d) is trivial. We show that (b) implies (c). Let $\beta : G \to \mathbb{R}$ be subadditive. Define F and ϱ by

$$F(x) = \frac{\beta(x) - \beta(-x)}{2}, \qquad \varrho(x) = \frac{\beta(x) + \beta(-x)}{2}.$$

Clearly, F is odd and ϱ even. Moreover, it is easily seen that F is pseudo-additive with respect to ϱ, with constant 1. The hypothesis about G implies the existence of an additive map $A : G \to \mathbb{R}$ satisfying $|A(x) - F(x)| \le \varrho(x)$ for all $x \in G$. This can be written as

$$-\beta(x) - \beta(-x) \le 2(A(x) - \beta(x) + \beta(-x)) \le \beta(x) + \beta(-x),$$

which yields $A(x) \le \beta(x)$, as desired.

Finally, we prove the implication (c)$\Rightarrow$(a). Let F be zero-additive with respect to ϱ, with F odd and ϱ even. Put

$$\beta(x) = \inf\Big\{\sum_{i=1}^{n} F(x_i) + \sum_{i=1}^{n} Z(F)\varrho(x_i) : x = \sum_{i=1}^{n} x_i\Big\}.$$

(Note that β is well defined and, in fact, one has $\beta(x) \ge F(x) - Z(F)\varrho(x)$.) Clearly, β is subadditive. Hence, it dominates some additive map $A : G \to \mathbb{R}$. Therefore,

$$A(x) \le \beta(x) \le F(x) + Z(F)\varrho(x)$$

for every $x \in G$. In particular, we have $A(x) - F(x) \le Z(F)\varrho(x)$. Since ϱ is even and both F and A are odd maps, one also has

$$|A(x) - F(x)| \le Z(F)\varrho(x). \ \blacksquare$$

Theorem 4.16 ([9]). *If G is a weakly commutative group*, *then for every ϱ the pair $[(G,\varrho),\mathbb{R}]$ has property* (1Z) (*hence* (1P) *and* (1G)).

Proof. This is an obvious consequence of Theorems 4.14 and 4.4. ■

We do not know if this is true for amenable groups. Fortunately, the following "symmetric" version of Theorem 4.16 holds.

THEOREM 4.17. *Let $\varrho : G \to \mathbb{R}$ be a symmetric control functional on an amenable group. Then the pair* $[(G, \varrho), \mathbb{R}]$ *has property* (1Z) (*hence* (1P) *and* (1G)) *for odd maps.*

Proof. This obviously follows from Theorems 4.15 and 4.8. ■

This result shows, in combination with Forti's Example 2.4, that $\mathbb{F}_2$ is not an amenable group.

COROLLARY 4.18. *Let $\varrho : G \to \mathbb{R}$ be a symmetric control functional on an amenable group. Then the pair* $[(G, \varrho), \mathbb{R}]$ *has property* (2Z) (*hence* (2P) *and* (2G)).

Proof. This follows from the inequality of Lemma 3.7. ■

REMARK 4.19. No semigroup has properties (a), (b) or (d) of Theorem 4.15 for any $M < 1$: given a semigroup S, consider the constant mapping $F = 1$ which is pseudo-additive, Ger-additive and zero-additive with respect to $\varrho = 1$. Since an additive map $A : S \to \mathbb{R}$ should satisfy $A(0) = 0$, we have $|F(0) - A(0)| = 1$.

Further results on the asymptotically additive character of Λ-additive maps can be found in [25] and [7].

4. Vector-valued maps. In this section we give some partial affirmative answers to Ulam's problem for vector-valued maps. Non-trivial exact sequences of Banach spaces do exist, hence zero-additive maps may be (and generally are) far from additive ones. However, as the following result shows, Ger-additive maps behave very well on amenable semigroups when the range is a Banach space complemented in its bidual.

THEOREM 4.20 (Mainly Ger [41]; see also [11]). *Let S be an amenable semigroup and Y a Banach space which is complemented in its second dual by a projection of norm K. Then, for every ϱ, the pair* $[(S, \varrho), Y]$ *has property* (KG). *If, besides this, S is a group and ϱ is symmetric, then* $[(S, \varrho), Y]$ *has property* (2KP).

Proof. Suppose $F : S \to Y$ is a Ger-additive map. We may, and do, assume that $G(F) = 1$. First of all, observe that for each fixed $x \in S$ the map $y \in S \mapsto F(x + y) - F(y) \in Y$ is bounded by $\varrho(x) + \|F(x)\|$. Fix a (say left) invariant mean dy for S and define $B(x) \in Y^{**}$ by the "Pettis integral"

$$B(x)(y^*) = \int_S y^*(F(x + y) - F(y))\, dy.$$

Clearly, $B(x)$ belongs to Y^{**} for all $x \in S$ and, in fact, one has $\|B(x) - F(x)\|_{Y^{**}} \le \varrho(x)$ for all $x \in S$. That B is additive can be proved as in Theorem 4.8, by using the invariance and linearity of dy.

Finally, if P is a bounded linear projection from Y^{**} onto Y with $\|P\| \le K$, we see that $A = P \circ B$ is an additive mapping $S \to Y$ with $\operatorname{dist}(F, A) \le K$.

The second assertion is obvious. ■

One may wonder about the necessity of some extra hypothesis on Y in the theorem and about the behavior of invariant means in the proof. The first question posed below will be solved at the end of Chapter V. The second question has been considered in the paper [8], which the interested reader is invited to peek.

PROBLEM 4.21. *Is it true that every Ger-additive map from an amenable group G into a Banach space is asymptotically additive? What if G is itself a Banach space?*

It follows from Theorem 3.2 that an affirmative answer to this problem would imply the existence of a universal constant C such that all pairs $[(G,\varrho),Y]$ with G amenable have property (CG). Unfortunately, the answer to this problem, which will be proved at the end of Chapter V, is no.

The following result is, in a sense, a vector-valued version of Theorem 4.4.

THEOREM 4.22. *Let (G,ϱ) be a weakly commutative group endowed with a symmetric functional and let Y be a Banach space complemented in its second dual by a projection of norm K. Then $[(G,\varrho),Y]$ has property* ($3K$P)

Beginning of the proof of Theorem 4.22. Let $F : G \to Y$ be pseudo-additive with respect to ϱ. In this part of the proof we assume neither that Y is complemented in its bidual nor that ϱ is symmetric, but only that G is weakly commutative. Moreover, without loss of generality, we may suppose $P(F) = 1$.

First of all, we replace F and ϱ by new maps having certain additional properties. Define

$$\sigma(x) = \lim_{n\to\infty} \frac{\varrho(2^n x)}{2^n} \qquad (x \in G).$$

This definition makes sense because, for every $x \in G$, the sequence $\varrho(2^n x)/2^n$ is non-negative and decreasing (this follows from the subadditivity of ϱ). Clearly, $\sigma(2x) = 2\sigma(x)$ for all $x \in G$. Moreover, σ is subadditive (see Lemma 4.7) and $\sigma \le \varrho$. Now, put

$$H(x) = \lim_{n\to\infty} \frac{F(2^n x)}{2^n} \qquad (x \in G).$$

Again, the limit defining H exists for every $x \in G$: fix $x \in G$ and let $n, m \in \mathbb{N}$. It is easily seen that

$$|F(2^{n+m}x) - 2^m F(2^n x)| \le 2^m \varrho(2^n x) - \varrho(2^{n+m}x).$$

Thus, letting $n \to \infty$, we have

$$\left| \frac{F(2^{n+m}x)}{2^{n+m}} - \frac{F(2^n x)}{2^n} \right| \le \frac{\varrho(2^n x)}{2^n} - \frac{\varrho(2^{n+m}x)}{2^{n+m}} \to \sigma(x) - \sigma(x) = 0.$$

Next, we prove that H is pseudo-additive with respect to σ with constant at most 1. Let $x, y \in G$. Take a sequence $n(k)$ so that

$$2^{n(k)}(x+y) = 2^{n(k)}x + 2^{n(k)}y.$$

Then, for every k, one has

$$\|F(2^{n(k)}(x+y)) - F(2^{n(k)}x) - F(2^{n(k)}y)\| \le \varrho(2^{n(k)}x) + \varrho(2^{n(k)}y) - \varrho(2^{n(k)}(x+y)).$$

Hence,

$$\begin{aligned}\|H(x+y) - H(x) - H(y)\| &= \lim_{k\to\infty} \left\| \frac{F(2^{n(k)}(x+y))}{2^{n(k)}} - \frac{F(2^{n(k)}x)}{2^{n(k)}} - \frac{F(2^{n(k)}y)}{2^{n(k)}} \right\| \\ &\le \lim_{k\to\infty} \frac{\varrho(2^{n(k)}x)}{2^{n(k)}} + \frac{\varrho(2^{n(k)}y)}{2^{n(k)}} - \frac{\varrho(2^{n(k)}(x+y))}{2^{n(k)}} \\ &= \sigma(x) + \sigma(y) - \sigma(x+y).\end{aligned}$$

The main step of the proof appears now.

LEMMA 4.23. *The maps H and σ factorize through the quotient $\pi : G \to G/G_0$.*

Proof. First, we show that both σ and H vanish on G_0. For σ this is an obvious consequence of Lemma 4.11, which asserts that our σ can be written as a pointwise supremum of additive functionals, and additive functionals must vanish on G_0. On the other hand, if $y^* \in Y^*$ with $\|y^*\| \le 1$, then $\sigma + y^* \circ H$ is 2-homogeneous (since σ and H are) and subadditive and hence it vanishes on G_0. Since y^* is arbitrary and σ vanishes on G_0, so does H. Let $x \in G$ and $h \in G_0$. Then

$$\sigma(x+h) \le \sigma(x) + \sigma(h) = \sigma(x) \le \sigma(x+h-h) \le \sigma(x+h) + \sigma(-h) = \sigma(x+h),$$

so that $\sigma(x+h) = \sigma(x)$. This clearly implies the existence of a map $\widetilde{\sigma} : G/G_0 \to \mathbb{R}$ such that $\sigma = \widetilde{\sigma} \circ \pi$.

Now, take $x \in G$ and $h \in G_0$. Then

$$\|H(x+h) - H(x)\| = \|H(x+h) - H(x) - H(h)\| \le \sigma(x) + \sigma(h) - \sigma(x+h) = 0.$$

Thus $H = \widetilde{H} \circ \pi$ for some $\widetilde{H} : G/G_0 \to Y$. Clearly, one has

$$\|\widetilde{H}(x+y) - \widetilde{H}(x) - \widetilde{H}(y)\| \le \widetilde{\sigma}(x) + \widetilde{\sigma}(y) - \widetilde{\sigma}(x+y)$$

for all $x, y \in G/G_0$ and $\widetilde{H}$ is pseudo-additive with respect to $\widetilde{\sigma}$. ■

End of the proof of Theorem 4.22. Let $\widetilde{H}, \widetilde{\sigma}$ be as above. Observe that $\widetilde{\sigma}$ is symmetric if ϱ is. Since G/G_0 is commutative and commutative groups are always amenable, we can apply the "besides this" part of Theorem 4.20 to get an additive map $B : G/G_0 \to Y$ such that $\|B(\pi(x)) - \widetilde{H}(\pi(x))\| \le 2K\widetilde{\sigma}(\pi(x))$ for every $x \in G$. Taking $A = \pi \circ B$ one obtains

$$\|A(x) - H(x)\| \le 2K\sigma(x) \qquad (x \in G).$$

Finally, let us estimate $\operatorname{dist}(F, A)$ on (G, ϱ). From

$$|F(2^n x) - 2^n F(x)| \le 2^n \varrho(x) - \varrho(2^n x),$$

it follows that $\operatorname{dist}(F, H) \le 1$ on (G, ϱ). Since $\operatorname{dist}(A, H) \le 2K$ on (G, σ), and taking into account that ϱ dominates σ, we see that $\operatorname{dist}(F, A) \le 3K$ on (G, ϱ). Thus $[(G, \varrho), Y]$ has $(3K\mathrm{P})$, and the proof is complete. ■

An interesting consequence of the first part of the proof of Theorem 4.22 is the following.

PROPOSITION 4.24. *Suppose there exists a pseudo-additive mapping $f : (G, \varrho) \to Y$ that is not asymptotically additive, where G is some weakly commutative group and ϱ is some symmetric control functional. Then there is a pseudo-linear map from a Banach space Z to Y which is far from any linear* (*or additive*) *mapping $Z \to Y$.*

Proof. If such an f exists, then we can construct another counterexample (which we also denote by f) as in Lemma 4.23. Thus we may assume that the domain group G is commutative and torsion-free (hence it embeds via $G \to G \otimes_{\mathbb{Z}} \mathbb{Q}$ into a linear space over the rationals), that both f and σ are 2-homogeneous and also that σ is symmetric and non-degenerate. Since the even part of f is bounded, we may assume even that f is odd. It follows that σ and f are, respectively, $\mathbb{N}$-homogeneous and $\mathbb{Z}$-homogeneous. Put $Z = G \to G \otimes_{\mathbb{Z}} \mathbb{Q}$ and define $\Sigma : Z \to \mathbb{R}$ as

$$\Sigma\Big(\sum_i g_i \otimes q_i\Big) = \frac{\sigma(n(\sum_i g_i \otimes q_i))}{n},$$

where n is the least positive integer for which $n(\sum_i g_i \otimes q_i)$ belongs to (the image of) G. It is easily verified that Σ is a non-degenerate, positively $\mathbb{Q}$-homogeneous, subadditive (= sublinear) functional which extends σ to Z. Actually, these properties determine Σ completely. The same procedure provides us with a $\mathbb{Q}$-homogeneous extension F for f. Just looking at the definitions, one sees that F is pseudo-linear (with constant 1, say) with respect to Σ, but the distance from F to the set of additive maps $Z \to Y$ is infinite.

Now, using the theory developed in the first chapter, we construct the "twisted sum" $Y \oplus_F Z$ which is the linear space (over $\mathbb{Q}$) $Y \times Z$ endowed with the "norm"

$$\|(y,z)\|_F = \|y - F(z)\|_Y + \Sigma(z).$$

Clearly, $\|(\cdot,\cdot)\|_F$ is positively $\mathbb{Q}$-homogeneous, symmetric and subadditive. Obviously, Y embeds isometrically as a (rational) subspace of $Y \oplus_F Z$ via $y \in Y \mapsto (y,0) \in Y \oplus_F Z$. Moreover, there is no bounded additive projection of $Y \oplus_F Z$ onto Y, since F is not asymptotically additive.

Consider the "metric" projection $m : Y \oplus_F Z \to Y$ given by

$$m(y,z) = y - F(z).$$

(This map finds, for each $(y,z) \in Y \oplus_F Z$, the nearest point in Y.) The pseudo-linearity of F with respect to the norm of Z implies that, for every $x \in Y \oplus_F Z$, one has

$$\|x\|_F = \|m(x)\|_F + \|x - m(x)\|_F,$$

and also that m is a Lipschitz map.

Let X denote the completion of $Y \oplus_F Z$ as a metric space. Clearly, it is a real Banach space whose norm $\|\cdot\|_X$ is obtained by extending Σ to the whole of X. Finally, observe that Y was complete, so it becomes a closed subspace of X. Let $\widetilde{m} : X \to Y$ be the extension of m by (uniform) continuity. It is easily seen that $\widetilde{m}$ is now an $\mathbb{R}$-homogeneous semiprojection onto Y satisfying again

$$\|x\|_X = \|\widetilde{m}(x)\|_Y + \|x - \widetilde{m}(x)\|_X$$

for all $x \in X$. Therefore Y is a semi-L-summand in X, corresponding to some pseudo-linear mapping $\widetilde{F} : X/Y \to Y$. Since Y is uncomplemented in X (otherwise Y would be boundedly complemented in $Y \oplus_F Z$), this map $\widetilde{F}$ is not asymptotically additive. ■

V. Nearly additive mappings and exact sequences

We now arrive at one of the central themes of the paper. We are going to establish the connections between nearly additive maps and exact sequences of quasi-normed groups and bounded group homomorphisms. Then we shall consider different types of nearly additive maps and shall see how the different properties of the maps reflect different properties of the exact sequences they define.

1. Quasi-normed groups as topological groups. In this section we collect some specific features of quasi-normed groups in order to clarify the exposition.

As we already said, the most interesting control functionals are the quasi-norms. These are functionals $\varrho : G \to \mathbb{R}^+$ satisfying

- $\varrho(x) = 0$ if and only if $x = 0$.
- $\varrho(-x) = \varrho(x)$ for all x.
- $\varrho(x+y) \le \Delta(\varrho(x) + \varrho(y))$ for some constant Δ independent of $x, y \in G$.

A quasi-normed group is a group together with a specified quasi-norm. Such a group admits a unique structure of topological group for which the sets

$$\{x \in G : \varrho(x) \le \varepsilon\} \qquad (\varepsilon > 0)$$

form a base of neighborhoods at zero.

A group homomorphism $\psi : (G_1, \varrho_1) \to (G_2, \varrho_2)$ is called *bounded* if there exists some constant K such that $\varrho_2(\psi(x)) \le K\varrho_1(x)$ for all $x \in G_1$. Observe that all bounded group homomorphisms between quasi-normed groups are continuous (observe also that continuous homomorphisms between quasi-normed groups need not be bounded). Quasi-normed groups and bounded homomorphisms constitute a category that we denote by **G**. Note that **G** includes **Q** and **B**.

Two quasi-norms ϱ_1, ϱ_2 defined on the same group are called *equivalent* if $\mathrm{Id} : (G, \varrho_1) \to (G, \varrho_2)$ is an isomorphism in **G**. This means that

$$k\varrho_1(x) \le \varrho_2(x) \le K\varrho_1(x)$$

for some $k, K > 0$ and all $x \in G$.

Simple examples show that a quasi-norm ϱ is not necessarily continuous with respect to the topology it itself induces. For instance, the quasi-norm defined on $\mathbb{R}$ by

$$\varrho(x) = \begin{cases} |x| & \text{if } x \in \mathbb{Q}, \\ 2|x| & \text{otherwise}, \end{cases}$$

has constant $\Delta = 2$ and induces the usual topology on $\mathbb{R}$, but it is discontinuous at all points save 0. Also, the "balls"

$$\{x : \varrho(x) \le \varepsilon\} = [-\varepsilon/2, \varepsilon/2] \cup ([-\varepsilon, \varepsilon] \cap \mathbb{Q})$$

are not closed (they are only neighborhoods of 0) and, similarly, the sets $\{x : \varrho(x) < \varepsilon\}$ are not open.

An interesting class of continuous quasi-norms are the so-called *p-norms* $(0 < p \le 1)$. These are quasi-norms satisfying the inequality

$$\varrho(x+y)^p \le \varrho(x)^p + \varrho(y)^p.$$

The following generalization of the Aoki–Rolewicz theorem due to Peetre and Sparr [77] asserts that every quasi-norm on a commutative group is equivalent to a (continuous) p-norm. Precisely,

THEOREM 5.1. *Let ϱ be a quasi-norm on a commutative group G. Then there is a p-norm σ such that $\sigma(x) \le \varrho(x) \le 2^{1/p}\sigma(x)$ for all x, where p is obtained from $\Delta_\varrho = 2^{1/p-1}$.*

Proof. Let Δ be the quasi-norm constant of ϱ. Clearly,

$$\varrho(x+y) \le 2\Delta \max\{\varrho(x), \varrho(y)\} \quad (x, y \in G).$$

Write $\Delta = 2^{1/p-1}$, that is, $2\Delta = 2^{1/p}$. One then has

$$\varrho(x+y)^p \le 2 \max\{\varrho(x)^p, \varrho(y)^p\} \quad (x, y \in G).$$

Let $\alpha : G \to \mathbb{R}^+$ be given by $\alpha(x) = \varrho(x)^p$. We have

$$\alpha(x+y) \le 2 \max\{\alpha(x), \alpha(y)\}. \tag{1}$$

Now, define

$$\beta(x) = \inf\Big\{ \sum_{i=1}^{n} \alpha(x_i) : x = \sum_{i=1}^{n} x_i \Big\}.$$

On the one hand, it is clear that $\beta(x) \le \alpha(x)$ and that β is subadditive; on the other hand, if $x = \sum_{i=1}^n x_i$ then a simple induction on the inequality (1) yields

$$\alpha\Big(\sum_{i=1}^{n} x_i\Big) \le \sup_{1\le i\le n} 2^{k_i}\alpha(x_i)$$

whenever $\sum_{i=1}^n 2^{-k_i} \le 1$. So, choose k_i so that

$$2^{-k_i} \le \frac{\alpha(x_i)}{\sum_{j=1}^n \alpha(x_j)} \le 2^{1-k_i}.$$

Since $\sum_{i=1}^n 2^{-k_i} \le 1$, we have

$$\alpha\Big(\sum_{i=1}^{n} x_i\Big) \le \sup_{1\le i\le n} 2^{k_i}\alpha(x_i) \le 2\sum_{i=1}^{n} \alpha(x_i).$$

It follows that $\beta \le \alpha \le 2\beta$, that is,

$$\beta(x)^{1/p} \le \varrho(x) \le 2^{1/p}\beta(x)^{1/p}.$$

Thus, we complete the proof by taking $\sigma(x) = \beta(x)^{1/p}$. ■

PROBLEM 5.2. *Can the hypothesis of commutativity be removed in the Aoki–Rolewicz theorem?*

The following criterion will be useful.

LEMMA 5.3. *Let ϱ be a quasi-norm on an arbitrary group G. Then ϱ is equivalent to a p-norm $(0 < p \le 1)$ if and only if there is K such that*

$$\varrho\Big(\sum_{i=1}^{n} x_i\Big)^p \le K \sum_{i=1}^{n} \varrho(x_i)^p$$

for each n and all x_i.

Proof. The necessity of the condition is obvious. As for the sufficiency, observe that the p-convex envelope of ϱ given by

$$\sigma(x) = \inf\Big\{ \Big(\sum_i \varrho(x_i)^p\Big)^{1/p} : x = \sum_i x_i \Big\}$$

is a p-norm satisfying $\sigma(x) \le \varrho(x) \le K^{1/p}\sigma(x)$ for all $x \in G$. ■

Suppose (x_n) is a sequence convergent to x in the quasi-normed group (G, ϱ). Since ϱ need not be continuous one cannot guarantee that $\varrho(x) = \lim_{n\to\infty} \varrho(x_n)$. We do have the following result.

Lemma 5.4. *Let (x_n) be a sequence convergent to x in (G, ϱ). Then*

$$\Delta^{-1} \limsup_{n\to\infty} \varrho(x_n) \le \varrho(x) \le \Delta \liminf_{n\to\infty} \varrho(x_n),$$

where Δ is the quasi-norm constant of ϱ.

Proof. The first inequality follows from

$$\varrho(x_n) \le \Delta(\varrho(x_n - x) + \varrho(x)),$$

while the second one follows from

$$\varrho(x) \le \Delta(\varrho(x - x_n) + \varrho(x_n)). \ \blacksquare$$

Corollary 5.5. *Let G_0 be a dense subgroup of the quasi-normed group G and let Y be a complete and commutative quasi-normed group. Suppose $A_0 : G_0 \to Y$ is an additive mapping satisfying $\|A_0(x)\| \le K\varrho(x)$ for $x \in G_0$. Then there exists a unique continuous additive $A : G \to Y$ extending A_0. Moreover, $\|A(x)\|_Y \le K\Delta_Y\Delta_G\varrho(x)$ holds for all $x \in G$, where Δ_Y and Δ_G are the quasi-norm constants of Y and G.*

Proof. The first part is obvious by taking

$$A(x) = \lim_y A(y)$$

for $y \in G_0$ converging to x in (G, ϱ). Now, fix $x \in G$ and take a sequence (x_n) in G_0 converging to x. Then $A_0(x_n)$ converges to $A(x)$ in Y. Applying Lemma 5.4 twice, we get

$$\|A(x)\|_Y \le \Delta_Y \liminf_{n\to\infty} \|A_0(x_n)\| \le K\Delta_Y \limsup_{n\to\infty} \varrho(x_n) \le K\Delta_Y\Delta_G\varrho(x),$$

as desired. $\blacksquare$

2. Extensions of quasi-normed groups. Let G be a quasi-normed group and let $S \subset G$ be a closed normal subgroup. The quotient group G/S is then quasi-normed by

$$\varrho_{G/S}(\pi(x)) = \inf\{\varrho(z) : \pi(x) = \pi(z)\},$$

where $\pi : G \to G/S$ is the canonical quotient map. This suggests the following notion of extension which is the natural one in $\mathbf{G}$.

Definition 5.6. A short exact sequence $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0$ of quasi-normed groups is said to be an *extension* (of Y by Z in $\mathbf{G}$) if both $j : Y \to j(Y)$ and the natural map $X/j(Y) \to Z$ are isomorphisms in $\mathbf{G}$.

Two extensions $0 \to Y \to X \to Z \to 0$ and $0 \to Y \to X_1 \to Z \to 0$ are said to be *equivalent* if there exists an isomorphism $T : X \to X_1$ in $\mathbf{G}$ making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \to & Y & \to & X & \to & Z & \to & 0 \\ & & \| & & \downarrow T & & \| & & \\ 0 & \to & Y & \to & X_1 & \to & Z & \to & 0 \end{array}$$

An extension $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0$ is said to *split* in $\mathbf{G}$ if it is equivalent to the trivial extension $0 \to Y \to Y \oplus Z \to Z \to 0$. Here, the quasi-norm of $Y \oplus Z$ is given by $\|(y,z)\| = \|y\|_Y + \|z\|_Z$. This means that j admits a retraction in $\mathbf{G}$, or equivalently, that q admits a section in $\mathbf{G}$. An extension $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0$ is called *singular* if it splits when regarded algebraically (that is, j admits a, not necessarily bounded or even continuous, left inverse in $\mathrm{Hom}(X,Y)$ or, equivalently, q has a right inverse in $\mathrm{Hom}(Z,X)$).

3. Nearly additive maps and twisted quasi-norms. In a number of constructions it will be convenient to consider only odd mappings. When $F : G \to Y$ is a quasi-additive mapping on a quasi-normed group and Y is a quasi-Banach space there is no serious loss of generality in assuming F to be odd because of the following lemma. The proof is almost the same as that of Lemma 1.17 and will be omitted.

LEMMA 5.7. *Let $F:(G,\varrho)\to Y$ be a quasi-additive map and let $F_{\mathrm{o}}(x)=(F(x)-F(-x)/2$ be its odd part. Then F_{o} is quasi-additive*, *with $Q(F_{\mathrm{o}}) \le \Delta_Y Q(F)$ and $\mathrm{dist}(F,F_{\mathrm{o}}) \le \Delta_Y Q(F)$.*

Given an odd mapping $F : (G,\varrho) \to Y$ (where Y is an abelian quasi-normed group) it is possible to construct a "twisted" symmetric control functional on the product group $Y \times G$ by putting

$$\|(y,x)\|_F = \|y - F(x)\|_Y + \varrho(x).$$

PROPOSITION 5.8. *The functional $\|(\cdot,\cdot)\|_F$ is a quasi-norm on the group $Y \times G$ if and only if F is quasi-additive.*

Proof. Assume that F is quasi-additive with constant K. Then

$$\begin{aligned}
&\|(u,x) + (v,y)\|_F \\
&= \|(u+v, x+y)\|_F = \|u+v-F(x+y)\|_Y + \varrho(x+y) \\
&\le \Delta_Y\{\|u+v-F(x)-F(y)\|_Y + \|F(x)+F(y)-F(x+y)\|\} + \Delta_\varrho\{\varrho(x)+\varrho(y)\} \\
&\le \Delta_Y^2\{\|u-F(x)\|_Y + \|v-F(y)\|_Y\} + (K\Delta_Y + \Delta_\varrho)(\varrho(x)+\varrho(y)) \\
&\le \max\{\Delta_Y^2, K\Delta_Y + \Delta_\varrho\} \cdot (\|(u,x)\|_F + \|(v,y)\|_F),
\end{aligned}$$

so that $\|\cdot\|_F$ satisfies the Δ-condition. For the converse, assume that

$$\|(u,x) + (v,y)\|_F \le \Delta\{\|(u,x)\|_F + \|(v,y)\|_F\}.$$

Taking $u = F(x)$ and $v = F(y)$, one obtains

$$\|F(x)+F(y)-F(x+y)\| + \varrho(x+y) \le \Delta(\varrho(x)+\varrho(y))$$

and the result follows. ∎

Now we consider some stronger properties of $\|\cdot\|_F$.

PROPOSITION 5.9. *Let $F : (G,\varrho) \to Y$ be an odd map. Suppose that ϱ is* (*equivalent to*) *a p-norm $(0 < p \le 1)$. Then $\|\cdot\|_F$ is equivalent to a p-norm on $Y \times G$ if and only if* *there is K such that*

$$\Big\|\sum_{i=1}^{n}F(x_i)-\sum_{j=1}^{m}F(y_j)\Big\|_Y^p\le K\Big(\sum_{i=1}^{n}\varrho(x_i)^p+\sum_{j=1}^{m}\varrho(y_j)^p\Big)$$

for each n and m whenever x_i and y_j are such that $\sum_{i=1}^n x_i=\sum_{j=1}^m y_j$. In particular, if ϱ is a norm, then $\|\cdot\|_F$ is equivalent to a norm if and only if F is zero-additive.

Proof. We consider the case $p=1$, the other being analogous. Assume that F is zero-additive with constant K. It is straightforward that

$$\Big\|\sum_{i=1}^{n}(y_i,z_i)\Big\|_F\le 2K\sum_{i=1}^{n}\|(y_i,x_i)\|_F,$$

and therefore $\|\cdot\|_F$ is equivalent to a norm. Conversely, assume that $\|\cdot\|_F$ is equivalent to a norm $\|\cdot\|$. We may assume that $\|x\|\le\|x\|_F\le K\|x\|$ for some K and all $x\in Y\times G$. If $\sum_{i=1}^n z_i=0$, then

$$\Big\|\sum_{i=1}^{n}F(z_i)\Big\|_Y=\Big\|\sum_{i=1}^{n}(F(z_i),z_i)\Big\|_F\le K\sum_{i=1}^{n}\|(F(z_i),z_i)\|_F\le K\sum_{i=1}^{n}\varrho(x_i);$$

it follows that F is zero-additive with constant at most K. ∎

Corollary 5.10. *Suppose G is commutative and $F:(G,\varrho)\to Y$ quasi-additive. Then there exist $0<p\le 1$ and $K\ge 0$ such that*

$$\Big\|\sum_{i=1}^{n}F(x_i)-\sum_{j=1}^{m}F(y_j)\Big\|_Y^p\le K\Big(\sum_{i=1}^{n}\varrho(x_i)^p+\sum_{j=1}^{m}\varrho(y_j)^p\Big)$$

for each n and m whenever x_i and y_j are such that $\sum_{i=1}^n x_i=\sum_{j=1}^m y_j$.

Proposition 5.11. *Suppose that ϱ is a norm. Then $\|\cdot\|_F$ is a norm if and only if F is pseudo-additive with constant 1.*

Proof. This also appears in [20, Appendix 1.9]. If F is pseudo-additive with constant 1, then $\|\cdot\|_F$ is quite clearly a norm. Conversely, if $\|\cdot\|_F$ is a norm then

$$\begin{aligned}\|F(x+y)-F(x)-F(y)\|+\varrho(x+y)&=\|(F(x)+F(y),x+y)\|_F\\&\le\|(F(x),x)\|_F+\|(F(y),y)\|_F=\varrho(x)+\varrho(y),\end{aligned}$$

and F is pseudo-additive with constant 1. ∎

4. Singular extensions of quasi-normed groups. Let us write $Y\oplus_F G$ for the quasi-normed group $(Y\times G,\|\cdot\|_F)$. Observe that $Y\oplus_F G$ contains a normal subgroup "isometric" to Y (namely, $\{(y,0):y\in Y\}$) such that the corresponding quotient is isometric to (G,ϱ), so that there is an extension in $\mathbf{G}$:

$$0\to Y\xrightarrow{i}Y\oplus_F G\xrightarrow{\pi}(G,\varrho)\to 0$$

The following result asserts that while these sequences are always singular their splitting in the category of quasi-normed groups and bounded homomorphisms is connected to the problem of finding a (not necessarily bounded, or continuous) group homomorphism $A:G\to Y$ approximated by F.

THEOREM 5.12. *Let* $0 \to Y \xrightarrow{i} Y \oplus_F G \xrightarrow{\pi} (G,\varrho) \to 0$ *be a sequence induced by a quasi-additive odd map* $F:(G,\varrho) \to Y$. *The following are equivalent:*

(a) *There is a bounded additive section for* π.
(b) *There is a bounded additive retraction for* i.
(c) *There is an additive map* $A: G \to Y$ *at finite distance from* F.

Proof. (a)⇒(c). Assume that $S:(G,\varrho) \to Y \oplus_F G$ is a homomorphism such that $\pi \circ S = \mathrm{Id}_G$ and $\|S(x)\|_F \le M\varrho(x)$. Observe that S must have the form $S(x) = (A(x), x)$ for some $A \in \mathrm{Hom}(G,Y)$. Moreover, one has

$$\|F(x) - A(x)\|_Y + \varrho(x) = \|(A(x),x)\|_F \le M\varrho(x),$$

which yields (c).

(c)⇒(b). If (c) holds, let K be the distance between F and A and define $R: Y \oplus_F G \to Y$ by $R(u,x) = u - A(x)$. Clearly,

$$\begin{aligned}\|R(u,x)\|_Y = \|u - A(x)\|_Y &\le \Delta_Y(\|u - F(x)\|_Y + \|F(x) - A(x)\|_Y)\\ &\le \Delta_Y(\|u - F(x)\|_Y + K\varrho(x)) \le \max\{1,K\}\Delta_Y\|(u,x)\|_F.\end{aligned}$$

The implication (b)⇒(a) is left to the reader as an easy exercise. ∎

We now identify singular extensions in **G** as those extensions that can be obtained by quasi-additive maps. Observe that, for instance, every short exact sequence of quasi-Banach spaces is a singular extension.

PROPOSITION 5.13. *Let* Z *and* Y *be quasi-normed abelian groups. Every singular extension* $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0$ *is equivalent to an extension* $0 \to Y \to Y \oplus_F Z \to Z \to 0$ *induced by a quasi-additive map* $F: Z \to Y$.

Proof. Since $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0$ is an extension, we may (and do) assume that Y is a normal subgroup of X with X/Y isomorphic to Z as quasi-normed groups. This obviously implies the existence of a bounded odd selection $B: Z \to X$ for the quotient map $X \to Z$.

On the other hand, if the given extension is singular, there is $L \in \mathrm{Hom}(Z,X)$ such that $q \circ L = \mathrm{Id}_Z$. Set $F(z) = B(z) - L(z)$ for all $z \in Z$; then $q \circ F = 0$ and F takes values in Y instead of X. We see that $F: Z \to Y$ is quasi-additive. Take $x,y \in Z$. Then

$$\begin{aligned}\|F(x+y) - F(x) - F(y)\|_Y &= \|F(x+y) - F(x) - F(y) - L(x+y) + L(x) + L(y)\|_Y\\ &= \|B(x+y) - B(x) - B(y)\|_Y\\ &\le \Delta_X\|B(x+y)\|_X + \Delta_X^2(\|B(x)\|_X + \|B(y)\|_X)\\ &\le \|B\|(\Delta_X\Delta_Z + \Delta_X^2)(\|x\|_Z + \|y\|_Z).\end{aligned}$$

It remains to exhibit an isomorphism $T: Y \oplus_F Z \to X$ in **G** making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \to & Y & \xrightarrow{i} & Y \oplus_F Z & \xrightarrow{\pi} & Z & \to & 0\\ & & \| & & \downarrow T & & \| & & \\ 0 & \to & Y & \xrightarrow{j} & X & \xrightarrow{q} & Z & \to & 0\end{array}$$

Define $T(y,z) = y - L(z)$. It is easily seen that T is a homomorphism (this requires commutativity in X, which is guaranteed by the singular character of the extension X). It is also clear that $T \circ i = j$ and $q \circ T = \pi$. We show that T is bounded:

$$\begin{aligned}\|T(y,z)\|_X &= \|y - F(z) + F(z) - L(z)\|_X \le \Delta_X(\|y - F(z)\|_Y + \|F(z) - L(z)\|_X)\\ &\le \max\{\Delta_X, \|B\|\}\|(y,z)\|_F.\end{aligned}$$

To end, observe that $T^{-1}(x) = (x - L(q(x)), q(x))$. Thus, the following computation shows that T^{-1} is bounded and completes the proof:

$$\begin{aligned}\|T^{-1}(x)\|_F &= \|(x - L(q(x)), q(x))\|_F = \|x - L(q(x)) - F(q(x))\|_X + \|q(x)\|_Z\\ &= \|x - B(q(x))\|_X + \|q(x)\|_Z \le \mathrm{const} \cdot \|x\|_X. \ \blacksquare\end{aligned}$$

5. The behavior of nearly additive maps on dense subgroups. At first sight, nearly additive maps have nothing to do with continuity. The possibility of thinking about such maps as "objects that generate short exact sequences of quasi-normed groups" provides us a link between quasi-additive maps and topology. An illustrative example is the following theorem.

THEOREM 5.14. *Let (G,ϱ) be a quasi-normed group, G_0 a dense subgroup and Y a quasi-Banach space. Let further $F : (G,\varrho) \to Y$ be a quasi-additive mapping. Assume that there is $A_0 \in \mathrm{Hom}(G_0, Y)$ such that $\|F(x) - A_0(x)\|_Y \le M\varrho(x)$ for some M and all $x \in G_0$. Then there exists a unique $A \in \mathrm{Hom}(G,Y)$ extending A_0 and satisfying $\|F(x) - A(x)\|_Y \le K\varrho(x)$ for some K and all $x \in G$.*

Proof. We give the proof for odd mappings. Taking into account Lemma 5.7 it is not hard to adapt the proof for arbitrary mappings.

Let $F : (G,\varrho) \to Y$ be quasi-additive and odd. Before embarking on the proof, let us observe that $Y \times G_0$ is a dense subgroup of $Y \oplus_F G$. Indeed, let $(y,x) \in Y \times G$ and $\varepsilon > 0$. Choose $x \in G_0$ such that $\varrho(\xi - x) \le \varepsilon$ and let $\psi = y - F(\xi - x)$. Then

$$\|(y,x) - (\psi,\xi)\|_F = \|(y - \psi) - F(x - \xi)\|_Y + \varrho(x - \xi) \le \varepsilon,$$

as desired. Assume that there is $A_0 \in \mathrm{Hom}(G_0, Y)$ such that

$$\|F(\xi) - A_0(\xi)\| \le M\varrho(\xi)$$

for all $\xi \in G_0$. Putting $R_0(y,x) = y - A_0(x)$, one obtains a homomorphism $Y \times G_0 \to Y$ such that $R_0(y,0) = y$ for all $y \in Y$ and satisfying

$$\|R_0(\psi,\xi)\| \le \max\{1, M\}\|(\psi,\xi)\|_F$$

for all $(\psi,\xi) \in Y \times G_0$. An obvious "homogeneity" argument shows that, in fact,

$$\|R_0(\psi,\xi)\| \le M\|(\psi,\xi)\|_F \qquad ((\psi,\xi) \in Y \times G_0).$$

According to Corollary 5.5, taking

$$R(y,x) = \lim_{\|(\psi - y, \xi - x)\|_F \to 0,\ (\psi,\xi) \in Y \times G_0} R_0(\psi,\xi),$$

we obtain a homomorphism $R : Y \times G \to Y$ extending R_0, satisfying again $R(y,0) = y$ and

$$\|R(y,x)\| \le M\Delta_F\|(y,x)\|_F$$

for all y and x, where Δ_F denotes the quasi-norm constant of $\|\cdot\|_F$. It follows from the proof of Proposition 5.8 that $\Delta_F \le Q(F) + \Delta_\varrho$.

Finally, observe that R must have the form $R(y,x) = y - A(x)$ for some $A \in \mathrm{Hom}(G,Y)$ extending A_0 and satisfying

$$\|F(x) - A(x)\| \le M(Q(F) + \Delta_\varrho)\varrho(x)$$

for every $x \in G$. The uniqueness of A is clear. ∎

REMARK 5.15. Again, a "homogeneity" argument shows that one actually has

$$\mathrm{dist}(F,A) \le (1+\Delta_\varrho)\max\{Q(F), \mathrm{dist}(F|_{G_0}, A_0)\}.$$

For an arbitrary map F, Lemma 5.7 yields

$$\mathrm{dist}(F,A) \le 2(1+\Delta_\varrho)\max\{Q(F), \mathrm{dist}(F|_{G_0}, A_0)\}.$$

COROLLARY 5.16. *Let (G,ϱ) be a quasi-normed group and let G_0 be a dense subgroup. Let further $F : G_0 \to Y$ be asymptotically additive. Then every quasi-additive map $G \to Y$ extending F is asymptotically additive.*

COROLLARY 5.17. *Let (G,ϱ) be a normed group and let G_0 be a dense subgroup. Let further $F : G_0 \to Y$ be zero-additive. Then every quasi-additive map $G \to Y$ extending F is zero-additive.*

6. Ger-additive maps and exact sequences. In this section we shall show that not every Ger-additive map between Banach spaces is asymptotically additive. This will be done by exhibiting a Banach space X having an uncomplemented subspace Y admitting a Lipschitz semiprojection. The precise statement of the equivalence between the two assertions is the following:

PROPOSITION 5.18. *Let $0 \to Y \to X \to Z \to 0$ be an exact sequence of quasi-Banach spaces. The following are equivalent:*

(a) *The quotient map $X \to Z$ admits a Lipschitz selection.*

(b) *The inclusion $Y \to X$ admits a Lipschitz semiprojection* (*i.e., there is a Lipschitz map $\pi : X \to Y$ such that $\pi(x + \pi(y)) = \pi(x) + \pi y$ for all $x, y \in X$*).

(c) *Some quasi-linear map defining the sequence is Ger-additive.*

To obtain its proof, let us consider the problem of finding the nearest point in Y for a given point $(y,x) \in Y \oplus_F G$. Clearly, the best approximation is given by the "metric projection" $(y,x) \in Y \oplus_F G \mapsto m(y,x) = y - F(x) \in Y$, corresponding to the "optimal selection" $x \in G \mapsto S(x) = (F(x), x) \in Y \oplus_F G$ for the quotient map. Both maps are bounded, but generally discontinuous. One has

PROPOSITION 5.19. *For a quasi-additive odd map F, the following are equivalent:*

(a) *The "metric projection" $m : Y \oplus_F G \to Y$ is a Lipschitz continuous map.*

(b) *The "optimal selection" $S : G \to Y \oplus_F G$ is Lipschitz continuous.*

(c) *F is Ger-additive.*

Proof. (c)⇔(b). Observe that $\|S(x+z) - S(x)\|_F = \|F(x+z) - F(x) - F(z)\|_Y + \varrho(z)$, hence one has $\|S(x+z) - S(x)\|_F \le K\varrho(z)$ for all $x, z \in G$ if and only if F is Ger-additive.

(a)⇔(c). Observe that the condition $\|m(y+u,x+z)-m(y,x)\|_Y \le K\|(u,z)\|_F$ can be written as $\|u-F(x+z)-F(x)\|_Y \le K\{\|u-F(z)\|_Y+\varrho(z)\}$. Taking $u=F(z)$ one deduces that (a) implies (c). And conversely, if F is Ger-additive, then

$$\begin{aligned}\|u-F(x+z)-F(x)\|_Y &\le \Delta_Y(\|u-F(z)\|_Y+\|F(z)-F(x+z)-F(x)\|_Y)\\ &\le \max\{1,G(F)\}\cdot\Delta_Y\cdot(\|u-F(z)\|_Y+\varrho(z)). \ \blacksquare\end{aligned}$$

From the above the proof of Proposition 5.18 is straightforward.

Moreover, the preceding argument yields the following.

Corollary 5.20. *Let F be a Ger-additive map. If F is at finite distance from an additive map A then the difference $B=F-A$ is a Lipschitz map* (*instead of a mere bounded map*).

From Proposition 5.18 it is clear that the existence of a Lipschitz semiprojection onto an uncomplemented subspace is equivalent to the existence of a Ger-linear map between Banach spaces that is not asymptotically linear. We already know that Ger-linear maps are asymptotically linear when the target space is complemented in its bidual (see Theorem 4.20). The following example of what we shall call a Johnson–Lindenstrauss space in Chapter VII provides a counterexample in the general case.

Example 5.21. *An uncomplemented subspace admitting a Lipschitz semiprojection. A Ger-additive map $F: c_0(\Gamma)\to c_0$ which fails to be asymptotically additive.*

Proof. The construction has been taken from [58], while the argument that it admits a Lipschitz semiprojection comes from [2]. Consider an uncountable family Γ of infinite subsets of $\mathbb{N}$ with the property that for two different γ and μ in Γ the intersection $\gamma\cap\mu$ is finite. Let JL_∞ be the closed linear span of c_0 and the characteristic functions of the sets in Γ in ℓ_∞. Via the natural embedding one has the exact sequence

$$0\to c_0 \xrightarrow{i} \mathrm{JL}_\infty \xrightarrow{q} c_0(\Gamma)\to 0.$$

The sequence clearly does not split because $c_0(\Gamma)$ is not a subspace of ℓ_∞.

In view of Proposition 5.18, it will suffice to show that the quotient map q admits a Lipschitz selection.

Let $c_{00}(\Gamma)$ be the linear subspace spanned by the "basis" $(e_\gamma)_{\gamma\in\Gamma}$. Clearly, it is a dense subspace of $c_0(\Gamma)$. For $x\in c_{00}(\Gamma)$, let

$$x=\sum_{n=1}^N a_n e_{\gamma_n}-\sum_{m=1}^M b_m e_{\mu_m}$$

with $a_1\ge a_2\ge\ldots,\ b_1\ge b_2\ge\ldots$ be the unique representation of x as the difference of two disjointly supported positive elements x^+ and x^-. Put

$$\gamma_n^*=\gamma_n\setminus\bigcup_{i=1}^{n-1}\gamma_i,\qquad \mu_m^*=\gamma_m\setminus\bigcup_{j=1}^{m-1}\gamma_j\quad (1\le n\le N,\ 1\le m\le M)$$

and set

$$\psi(x)=\sum_{n=1}^N a_n 1_{\gamma_n^*}-\sum_{m=1}^M b_m 1_{\mu_m^*}.$$

Clearly, $q \circ \psi$ is the identity on $c_{00}(\Gamma)$. That ψ is a Lipschitz map easily follows from the equality

$$\psi(x)(k) = \operatorname{dist}(x^+, Z_k) - \operatorname{dist}(x^-, Z_k) \quad (k \in \mathbb{N}),$$

in which $Z_k = \operatorname{span}\{e_\gamma : k \notin \gamma\}$.

Finally, since $c_{00}(\Gamma)$ is dense in $c_0(\Gamma)$ we can extend ψ to a Lipschitz mapping $\widetilde{\psi} : c_{00}(\Gamma) \to \mathrm{JL}_\infty$, which is quite clearly a Lipschitz selection for the quotient map. ■

REMARK 5.22. Taking into account the Corson–Klee lemma (uniformly continuous mappings between Banach spaces are Lipschitzian for large distances [24, Proposition 4.3]), and the obvious fact that every invariant mean on a normed space vanishes on boundedly supported functions, one can extract from Proposition 5.18 and the proof of Theorem 4.20 the following: Let Y be a subspace of a Banach space X; assume that there is a (not necessarily homogeneous) uniformly continuous projection $\pi : X \to Y$ (i.e., a mapping satisfying $\pi(y) = y$ for all $y \in Y$). If Y is complemented in its second dual then it is a complemented subspace of X. This result, which is essentially known (see [72] and [78, pp. 61–62]), can be regarded in some sense as a result on stability of linear projections on Banach spaces.

A related open question is:

PROBLEM 5.23. *Given a sequence* $0 \to Y \to X \to Z \to 0$ *of quasi-Banach spaces, how to detect if it comes defined by some Ger-linear map?*

VI. Duality theory and localization techniques

This chapter aims to translate the standard techniques of Banach space theory (duality, localization) to quasi-normed groups. We shall show that this can be done with success for what concerns nearly additive mappings.

We start with duality theory for quasi-normed groups. The duality we are going to consider is not Pontryagin duality as topological groups (that is, taking as dual the group of characters). We shall rather consider the Banach space $(G, \varrho)^* = \operatorname{Hom}_{\mathcal{B}}(G, \mathbb{R})$ as a dual object for the quasi-normed group (G, ϱ) (in a similar sense to making the Banach space $C(K)$ of continuous functions over a compact space K a dual object for K). Recall from Chapter II that the map

$$f \in (G, \varrho)^* \mapsto \|f\| = \inf\{K > 0 : |f(x)| \le K\varrho(x) \text{ for all } x \in G\}$$

defines a complete norm on $(G, \varrho)^*$ (even when ϱ is only a quasi-norm). Recall also that the canonical homomorphism $\delta : (G, \varrho) \to (G, \varrho)^{**}$ defined by $\delta(x)(f) = f(x)$ is continuous, with $\|\delta(x)\| \le \delta(x)$; it is also injective when $(G, \varrho)^*$ separates the points of G.

1. The containing Banach space of a quasi-normed group. The *containing Banach space* of a quasi-normed group (G, ϱ) is the closed subspace $\operatorname{co}(G, \varrho) \subset (G, \varrho)^{**}$ spanned by $\delta(G)$, with the (restriction of the) norm of $(G, \varrho)^{**}$. The Banach space $\operatorname{co}(G, \varrho)$ is defined by the following universal property: every bounded homomorphism from (G, ϱ) into a Banach space factorizes through $\delta : (G, \varrho) \to \mathrm{co}(G, \varrho)$, with equal norm. In particular, the Banach spaces $(G, \varrho)^*$ and $\mathrm{co}(G, \varrho)^*$ are naturally isometric.

Our aim is to prove that zero-additive maps on (G, ϱ) with values in a Banach space "approximately" factorize through $\mathrm{co}(G, \varrho)$. Let us recall again that the absence of a Hahn–Banach theorem forces us to state as hypotheses facts that the Banach space structure gives for free. Thus, although, as proved in Chapter IV, every $\mathbb{R}$-valued zero-additive map in a linear space is asymptotically additive, this is not necessarily so for zero-additive maps on arbitrary quasi-normed groups (see Example 2.4). Hence, to some extent, the following notion of Z-group is the "group version" of being a K-space.

Definition 6.1. We say that a quasi-normed group (G, ϱ) is a *Z-group* if every real-valued zero-additive mapping on (G, ϱ) is asymptotically additive.

Of course, it is also possible to define Q-, G-, P- and H-groups. However, observe that Q-groups must be very scarce, while the theory for G-, P- and H-groups is contained in that of Z-groups. By Theorems 4.17 and 4.16, amenable and weakly commutative groups are Z-groups. Moreover, as we proved in Chapter III, Theorem 3.2, Z-groups (respectively, G- and P-groups) coincide with groups (G, ϱ) for which the pair $\{(G, \varrho), \mathbb{R}\}$ has, for some $M \geq 0$, the property $(M\mathrm{Z})$ (respectively, $(M\mathrm{G})$ or $(M\mathrm{P})$).

Lemma 6.2. *A quasi-normed group (G, ϱ) is a Z-group if and only if, given an exact sequence $0 \to Y \to Y \oplus_F G \to (G, \varrho) \to 0$ defined by a zero-additive odd map $F : (G, \varrho) \to Y$, the dual sequence $0 \to (G, \varrho)^* \to (Y \oplus_F G)^* \to Y^* \to 0$ is exact.*

Proof. ($\Rightarrow$) The only non-trivial point is to verify that the restriction map $(Y \oplus_F G)^* \to Y^*$ is onto. Let $y^* \in Y^*$. Then $y^* \circ F$ is zero-additive on (G, ϱ) and there exists $A \in \mathrm{Hom}(G, \mathbb{R})$ at finite distance from $y^* \circ F$. It is easily seen that the mapping $(y, s) \mapsto y^*(y) - A(s)$ is a bounded additive extension of y^* to $Y \oplus_F G$.

($\Leftarrow$) Let $F : (G, \varrho) \to \mathbb{R}$ be a zero-additive map. Since the sequence $0 \to (G, \varrho)^* \to (\mathbb{R} \oplus_F G)^* \to \mathbb{R}^* \to 0$ is exact the identity of $\mathbb{R}$ can be lifted to a bounded additive map R on $\mathbb{R} \oplus_F G$. Clearly, R must have the form $R(t, x) = t - A(x)$, where A is a real-valued additive map on G at finite distance from F. ■

Theorem 6.3. *Let (G, ϱ) be a Z-group and Y Banach space complemented in its bidual. Let $F : \mathrm{co}(G, \varrho) \to Y$ be zero-linear. Then F is asymptotically linear if (and only if) $F \circ \delta$ is asymptotically additive.*

Proof. Let $F : \mathrm{co}(G, \varrho) \to Y$ be zero-linear. Clearly, $F \circ \delta : (G, \varrho) \to Y$ is zero-additive, $\delta : (G, \varrho) \to \mathrm{co}(G, \varrho)$ being the canonical map. One has the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & Y & \to & Y \oplus_F \mathrm{co}(G, \varrho) & \to & \mathrm{co}(G, \varrho) & \to & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle\delta} & & \\ 0 & \to & Y & \to & Y \oplus_F G & \to & (G, \varrho) & \to & 0 \end{array}$$

where all arrows represent bounded group homomorphisms. Since $F \circ \delta$ is asymptotically additive the lower row splits. By dualization one obtains the following diagram of Banach spaces and operators:

$$\begin{array}{ccccccccc} 0 \leftarrow Y^* \leftarrow & (Y \oplus_F \mathrm{co}(G,\varrho))^* & \leftarrow & (\mathrm{co}(G,\varrho))^* \leftarrow 0 \\ \| & \downarrow & & \| \\ 0 \leftarrow Y^* \leftarrow & (Y \oplus_F G)^* & \leftarrow & (G,\varrho)^* \leftarrow 0 \end{array}$$

Since the lower row splits, so does the upper row. Hence, its dual sequence

$$0 \to Y^{**} \to (Y \oplus_F \mathrm{co}(G))^{**} \to (G,\varrho)^{**} \to 0$$

also splits. Since Y is complemented in its bidual the starting sequence

$$0 \to Y \to Y \oplus_F \mathrm{co}(G,\varrho) \to \mathrm{co}(G,\varrho) \to 0$$

splits (see [13]). Therefore, F is asymptotically linear. ■

COROLLARY 6.4. *If every zero-additive map $(G,\varrho) \to Y$ is asymptotically additive then every locally convex twisted sum of Y and $\mathrm{co}(G,\varrho)$ is trivial.*

The converse implication also holds under the assumption that Y is complemented in its bidual. The proof is a consequence of the following technical result, worth the awkward computations it contains.

THEOREM 6.5. *Let (G,ϱ) be a Z-group and Y a Banach space. Let $j : Y \to Y^{**}$ denote the natural inclusion map. Given a zero-additive map $F : (G,\varrho) \to Y$ there is a zero-linear map $H : \mathrm{co}(G,\varrho) \to Y^{**}$ such that the difference $j \circ F - H \circ \delta$ can be decomposed as the sum of an additive map $G \to Y^{**}$ and a bounded map $G \to Y^{**}$.*

Proof. The proof uses the technique developed in [13] for the construction of the adjoint zero-linear map. Let $F : (G,\varrho) \to Y$ be zero-additive. Consider the sequence

$$0 \to Y \to Y \oplus_F G \to (G,\varrho) \to 0.$$

By Lemma 6.2, the dual sequence of Banach spaces

$$0 \to (G,\varrho)^* \to (Y \oplus_F G)^* \to Y^* \to 0$$

is exact. The open mapping theorem implies that for every $y^* \in Y^*$ there is $\Phi(y^*) \in \mathrm{Hom}(G,\mathbb{R})$ such that $\|y^* \circ F - \Phi(y^*)\| \le C\|y^*\|$, where C is a constant depending on F but not on y^*. Obviously, one can assume that $\Phi : Y^* \to \mathrm{Hom}(G,\mathbb{R})$ is homogeneous. Fix a Hamel basis $\{f_i\}$ for Y^* over $\mathbb{R}$ and define a linear map $L_\Phi : Y^* \to \mathrm{Hom}(G,\mathbb{R})$ by $L_\Phi(\sum_i \lambda_i f_i) = \sum_i \lambda_i \Phi(f_i)$. We claim that $F^* = L_\Phi - \Phi$ is a zero-linear map from Y^* into $(G,\varrho)^*$. To see this, let $y^* = \sum_i \lambda_i f_i$. Then, for every x in G one has

$$\begin{aligned} |F^*(y^*)(x)| &= |L_\Phi(y^*)(x) - \Phi(y^*)(x)| = \Big|\sum_i \lambda_i \Phi(f_i)(x) - \Phi\Big(\sum_i \lambda_i f_i\Big)(x)\Big| \\ &= \Big|\sum_i \lambda_i \Phi(f_i)(x) - \sum_i \lambda_i f_i F(x) + \sum_i \lambda_i f_i F(x) - \Phi\Big(\sum_i \lambda_i f_i\Big)(x)\Big| \\ &\le 2C\Big(\sum_i |\lambda_i| \cdot \|f_i\|\Big)\varrho(x), \end{aligned}$$

hence $F^*(y^*) \in (G,\varrho)^*$. Moreover, given $g_i, h_j \in Y^*$ such that $\sum_{i=1}^n g_i = \sum_{j=1}^m h_j$, one has

$$\begin{aligned}\Big\|\sum_{i=1}^n F^*(g_i) - \sum_{j=1}^m F^*(h_j)\Big\| &= \operatorname{dist}\Big(\sum_{i=1}^n \Phi^*(g_i), \sum_{j=1}^m \Phi^*(h_j)\Big)\\ &= \operatorname{dist}\Big(\sum_{i=1}^n \Phi^*(g_i) - \sum_{i=1}^n g_i\circ F, \sum_{j=1}^m \Phi^*(h_j) - \sum_{j=1}^m h_j \circ F\Big)\\ &\le C\Big(\sum_{i=1}^n \|g_i\| + \sum_{j=1}^m \|h_j\|\Big),\end{aligned}$$

so that F^* is zero-linear (actually, this map defines the sequence $0 \to (G,\varrho)^* \to (Y\oplus_F G)^* \to Y^* \to 0$).

With the same procedure we can construct the bitranspose zero-linear map $F^{**} : (G,\varrho)^{**} \to Y^{**}$ by considering, for each $x^{**} \in (G,\varrho)^{**}$, a functional $\Psi(x^{**}) \in \operatorname{Hom}(Y^*,\mathbb{R})$ such that $\|x^{**}F^* - \Psi(x^{**})\| \le K\|x^{**}\|$ and putting $F^{**}(x^{**}) = L_\Psi(x^{**}) - \Psi(x^{**})$. We want to see that $F^{**}\circ\delta - F$ is asymptotically additive as a map $(G,\varrho) \to Y^{**}$. Take $x\in G$ and $y^* \in (G,\varrho)^*$. Then

$$\begin{aligned}&F^{**}(\delta(x))y^* - y^*F(x)\\ &\quad= L_\Psi(\delta(x))y^* - \Psi(\delta(x))y^* - y^*F(x)\\ &\quad= L_\Psi(\delta(x))y^* - \Psi(\delta(x))y^* + \delta(x)F^*y^* - \delta(x)F^*y^* - y^*F(x)\\ &\quad= L_\Psi(\delta(x))y^* - \Psi(\delta(x))y^* + \delta(x)F^*y^* - L_\Phi(y^*)x + \Phi(y^*)x - y^*F(x)\\ &\quad= A(x)(y^*) + B(x)(y^*),\end{aligned}$$

where

$$B(x)(y^*) = -\Psi(\delta(x))y^* + \delta(x)F^*y^* + \Phi(y^*)x - y^*F(x)$$

is bounded since

$$|B(x)(y^*)| \le (K+C)\|y^*\|\varrho(x),$$

while

$$A(x)(y^*) = L_\Psi(\delta x)y^* - L_\Phi(y^*)x$$

is additive and takes values in Y^{**}, instead of $\operatorname{Hom}(Y^*,\mathbb{R})$, B is bounded and $F^{**}\circ\delta(x) - F(x)$ belongs to Y^{**}. ■

COROLLARY 6.6. *If, moreover, Y is complemented in Y^{**} by a projection π then $\pi\circ H : \operatorname{co}(G,\varrho) \to Y$ gives a zero-linear map such that $\pi\circ H - F$ is a trivial map $G \to Y$. In particular, if every locally convex twisted sum of Y and $\operatorname{co}(G,\varrho)$ is trivial then every zero-additive map $(G,\varrho) \to Y$ is asymptotically additive.*

COROLLARY 6.7. *Let (G,ϱ) be a quasi-normed group admitting a dense subgroup that is either weakly commutative or amenable and let Y be a Banach space complemented in the bidual. Then $[(G,\varrho),Y]$ has property* (Z) *if and only if* $[\operatorname{co}(G,\varrho),Y]$ *does.*

2. Projective and almost projective groups. It is well known that the Banach spaces $\ell_1(I)$ are projective: every exact sequence of Banach spaces $0 \to Y \to X \to$

$\ell_1(I) \to 0$ splits; equivalently, every zero-linear map $\ell_1(I) \to Y$ admits a linear map at finite distance. The following stronger fact holds.

THEOREM 6.8. *For every Banach space Y and every index set I the pair $[\ell_1(I), Y]$ has property* (8Z).

Proof. Let $F : \ell_1(I) \to Y$ be a zero-additive mapping. Consider the following dense subgroup of $\ell_1(I)$:

$$G_0 = \{x \in \ell_1(I) : x(i) \in \mathbb{Q} \text{ for all } i \text{ and } x(i) = 0 \text{ for almost all } i\}.$$

It is not hard to see that the formula $A(\sum_i q_i e_i) = \sum_i q_i F(e_i)$ defines an additive mapping $A : G_0 \to Y$ such that

$$\|F(x) - A(x)\| = \Big\| F\Big(\sum_i q_i e_i\Big) - \sum_i q_i F(e_i)\Big\| \le 4Z(F)$$

Hence $[G_0, Y]$ has property $(4Z)$, and thus $[\ell_1(I), Y]$ (using the results in Section 5 of Chapter V) has property (8Z). ∎

$\mathcal{L}_1$-spaces are not "as projective" as $\ell_1(I)$-spaces; after all, only exact sequences (of Banach spaces) $0 \to Y \to X \to L_1(\mu) \to 0$ in which Y is complemented in the bidual split [72]; that is, the couples $(L_1(\mu), Y)$ in which Y is complemented in the bidual have some property $(K\mathrm{Z})$ for homogeneous maps. This property, called "almost projectivity" in [14], actually characterizes $\mathcal{L}_1$-spaces in the following sense:

PROPOSITION 6.9. *A Banach space X is an $\mathcal{L}_1$-space if and only if every exact sequence $0 \to Y \to E \to X \to 0$ in which Y is reflexive splits.*

DEFINITION 6.10. We say that a quasi-normed group (G, ϱ) is *almost projective* if every zero-additive map from (G, ϱ) into a Banach space complemented in its bidual is asymptotically additive.

THEOREM 6.11. *If (G, ϱ) is an almost projective quasi-normed group, then* $\mathrm{co}(G, \varrho)$ *is an almost projective Banach space.*

Proof. This follows from Theorem 6.3 on taking into account that $\mathbb{R}$ is complemented in its bidual. ∎

The previous results are also significant when translated to the domain of quasi-Banach spaces. In Banach spaces, the hypothesis of the following theorem means that Z is an $\mathcal{L}_1$-space (see [14]). Thus, we are asking for quasi-Banach spaces whose Banach envelope is an $\mathcal{L}_1$-space.

COROLLARY 6.12. *Let Z be a quasi-Banach space such that every twisted sum of a reflexive Banach space with Z is trivial. Then* $\mathrm{co}(Z)$ *is an almost projective Banach space.*

This assertion is a weaker form of Theorem 6.11 for homogeneous maps. The converse holds under the additional hypothesis of being a K-space:

COROLLARY 6.13. *Let Z be a quasi-Banach K-space. If* $\mathrm{co}(Z)$ *is an almost projective Banach space, then every twisted sum of a Banach space complemented in its bidual and Z is trivial.*

Again, this is a weaker form of Theorem 6.5 for homogeneous maps.

These results assert something sensible about K-spaces with trivial dual since 0 is an almost projective Banach space. Precisely, they say that if Z is a quasi-Banach K-space with trivial dual and Y is a Banach space complemented in the bidual, then every twisted sum of Y with Z splits. In [64, pp. 46–47] it is mentioned that "the containing Banach space of a non-locally convex space exhibits a certain degree of ℓ_1-like behavior"; this consists in that when Z^* separates points, ℓ_1 is finitely represented in $\mathrm{co}(Z)$. Corollary 6.13 displays another aspect of this ℓ_1-like behavior for the opposite case, when $Z^* = 0$. The two cases are represented by the spaces ℓ_p and L_p for $0 < p < 1$. In the first case, in which the dual separates points, $\mathrm{co}(\ell_p) = \ell_1$; in the second case, in which $L_p^* = 0$ and thus $\mathrm{co}(L_p) = 0$, Corollary 6.13 asserts that every twisted sum of a Banach space complemented in its bidual and L_p is trivial.

3. Injective spaces. The following result characterizes Banach spaces capable of playing the rôle of the scalar field in the results we proved in Section 3 of Chapter IV. A Banach space is said to be *injective* if it is complemented in every larger space that contains it. Typical injective spaces are the spaces $\ell_\infty(I)$, while all injective spaces are complemented subspaces of some $\ell_\infty(I)$.

THEOREM 6.14. *For a Banach space Y the following are equivalent:*

(a) *Y is an injective space.*

(b) *There exists $K \geq 0$ such that $[(S,\varrho), Y]$ has property $(KM\mathrm{Z})$ whenever $[(S,\varrho),\mathbb{R}]$ has property $(M\mathrm{Z})$.*

Proof. (b)⇒(a). Assume that (b) holds and let $Y \to \ell_\infty(I)$ be an isometric embedding. Let $F : \ell_\infty(I)/Y \to Y$ be a zero-linear mapping defining the sequence

$$0 \to Y \to \ell_\infty(I) \to \ell_\infty(I)/Y \to 0.$$

Since $\ell_\infty(I)/Y$ is commutative, the pair $[\ell_\infty(I)/Y, \mathbb{R}]$ has property (1Z) and the hypothesis implies that $[\ell_\infty(I)/Y, Y]$ has property $(K\mathrm{Z})$ for some K, so that there is an additive $A : \ell_\infty(I)/Y \to Y$ such that $\mathrm{dist}(F, A) \leq K$. By Theorem 5.12 there is a (necessarily homogeneous) additive bounded projection from $\ell_\infty(I)$ onto Y. Thus Y is complemented in $\ell_\infty(I)$ and, therefore, injective.

(a)⇒(b). For the converse, suppose Y injective and let the pair $[(S,\varrho),\mathbb{R}]$ have the property $(M\mathrm{Z})$. Let $F : (S,\varrho) \to Y$ be a zero-additive map. Fix an isometric linear embedding $\Phi : Y \to \ell_\infty(I)$ and consider the map $\Phi \circ F : (S,\varrho) \to \ell_\infty(I)$. Writing $\Phi\circ F = (F_\alpha)_{\alpha\in I}$, we can find for every $\alpha \in I$ an $A_\alpha \in \mathrm{Hom}(S,\mathbb{R})$ such that $\mathrm{dist}(F_\alpha, A_\alpha) \leq MZ(F_\alpha)$ and thus $\mathrm{dist}(\Phi \circ F, A) \leq MZ(\Phi \circ F)$, where $A = (A_\alpha)_\alpha$. Finally, let π be a bounded projection from $\ell_\infty(I)$ onto Y. Then $F = \pi \circ \Phi \circ F$ and

$$\mathrm{dist}(F, \pi\circ A) = \mathrm{dist}(\pi \circ \Phi F, \pi \circ A) \leq \|\pi\|\mathrm{dist}(\Phi\circ F, A) \leq M\|\pi\|Z(F),$$

and thus $[(S,\varrho), Y]$ has property $(\|\pi\|M\mathrm{Z})$. ■

4. Sobczyk's theorem Hyers's way. Sobczyk's theorem [96, 17] asserts that a copy of the space c_0 inside a separable Banach space is always complemented; after Zippin's theorem [106] this weaker form of injectivity in fact characterizes c_0. Separability being a 3-space property [20], the statement is equivalent to saying that every zero-linear map from a separable Banach space into c_0 must be asymptotically linear. We think it is worthwhile to present an uncanny interpretation from the point of view of nearly additive maps on semigroups.

THEOREM 6.15. *Let S be a countable semigroup such that $[(S,\varrho),\mathbb{R}]$ has property* $(M\mathrm{Z})$ *for some M. For every index set I the pair $[(S,\varrho),c_0(I)]$ has property* $(2M\mathrm{Z})$.

Proof. Let $F:(S,\varrho)\to c_0(I)$ be zero-additive. If one writes $F=(F_\alpha)_{a\in I}$, then every F_α is again zero-additive with $Z(F_\alpha)\le Z(F)$, hence for each α, there exists $A_\alpha\in \mathrm{Hom}(S,\mathbb{R})$ such that $\mathrm{dist}(F_\alpha,A_\alpha)\le MZ(F)$. This implies that the map $x\in S\mapsto A(x)=(A_\alpha(x))$ takes values in $\ell_\infty(I)$ being, in fact, at distance at most $MZ(F)$ from F. We set

$$\Im=\left\{B\in \mathrm{Hom}(S,\mathbb{R}):\sup_{x\in S}\frac{B(x)}{1+\|A(x)\|_\infty}<\infty\right\}$$

endowed with the distance

$$d(B,C)=\sum_{n=1}^{\infty}\frac{|B(x_n)-C(x_n)|}{2^n(1+\|A(x_n)\|_\infty)},$$

where $\{x_n\}$ is an enumeration of the elements of S. Bounded sets are relatively compact in $\Im$, and so is the closure of $\{A_\beta\}_{\beta\in I}$. For each $\alpha\in I$, let B_α be a point in $\Im$ minimizing the distance from A_α to the set of accumulation points of $\{A_\beta\}_{\beta\in I}$ in $\Im$. It is easily seen that, for each $x\in S$, one has $(A-B)(x)=((A_\alpha-B_\alpha)(x))_\alpha\in c_0(I)$. Moreover, and this is the key point, the B_α's are uniformly bounded on S. Indeed, let $\beta\in I$. Then there is a sequence $B_{\alpha(n)}$ converging to B_β in the metric of $\Im$, which obviously implies that $B_{\alpha(n)}$ converges to B_β pointwise on S. Thus, fixing $x\in S$, one has

$$\begin{aligned}|B_\beta(x)|&=\lim_{n\to\infty}|A_{\alpha(n)}(x)|\le\limsup_{n\to\infty}(|A_{\alpha(n)}(x)-F_{\alpha(n)}(x)|+|F_{\alpha(n)}(x)|)\\&\le \mathrm{dist}(F_\alpha,A_\alpha)\varrho(x)+\lim_{n\to\infty}|F_{\alpha(n)}(x)|\le MZ(F)\varrho(x).\end{aligned}$$

Therefore,

$$\|F(x)-(A-B)(x)\|_{c_0}\le\|F(x)-A(x)\|_{\ell_\infty}+\|B(x)\|_{\ell_\infty}\le 2MZ(F)\varrho(x),$$

so that the pair $[(S,\varrho),c_0(I)]$ has property $(2M\mathrm{Z})$. ∎

COROLLARY 6.16. *Let G be a countable Z-group. Then, for every index set I, the pair $[S,c_0(I)]$ has property* (Z).

Thus, taking into account the extension results in Section 5 of Chapter V) one has.

THEOREM 6.17 (Sobczyk's theorem). *Let (G,ϱ) be a quasi-normed group admitting a countable dense subgroup that is either amenable or weakly commutative. Then, for any index set I, the pair $[(G,\varrho),c_0(I)]$ has property* (Z).

This approach not only re-proves known results; once it has been shown that there is a proof that does not involve the Banach space structure at all, the following non-locally convex extension is possible.

THEOREM 6.18. *Every twisted sum of $c_0(I)$ with a separable quasi-Banach K-space splits.*

5. Localization techniques. The purpose of this section is to show that, despite the poorer structure involved, it is possible to derive information about asymptotic stability on a semigroup when some information about the subgroups is available. The quasi-Banach analogue, considered afterwards, is that of $\mathcal{L}_p$-spaces, $0 < p \leq \infty$.

THEOREM 6.19. *Let (S, ϱ) be a controlled semigroup and let Y be a Banach space complemented in its second dual by a projection π. Assume that S is a directed union of a system of semigroups $\{S_\alpha\}$ such that $[(S_\alpha, \varrho), Y]$ has property $(M\mathrm{Z})$ for all α. Then $[(S, \varrho), Y]$ has property $(\|\pi\| M\mathrm{Z})$.*

Proof. Let $F : (S, \varrho) \to Y$ be a zero-additive map. Consider the net $\{S_\alpha\}$ ordered by inclusion, and, for each α, let F_α be the restriction of F to S_α. Clearly, $Z(F_\alpha) \leq Z(F)$ for all α. The hypothesis yields additive maps $A_\alpha : S_\alpha \to Y$ such that $\operatorname{dist}(A_\alpha, F_\alpha) \leq MZ(F)$. Let $\mathcal{U}$ be an ultrafilter refining the Fréchet filter on $\{S_\alpha\}$ and define a mapping $B : S \to Y^{**}$ as follows:

$$B(x) = \text{weak*-}\lim_{\mathcal{U}} A_\alpha(x)$$

(observe that for each $x \in S$, $A_\alpha(x)$ is eventually well defined). The definition of $B(x)$ makes sense because for each $x \in S$, one has

$$\|A_\alpha(x) - F_\alpha(x)\| \leq MZ(F)\varrho(x)$$

and thus $\{A_\alpha(x)\}$ is bounded in Y (hence in Y^{**}) by $MZ(F)\varrho(x) + \|F(x)\|$. Moreover, it is clear that $B \in \operatorname{Hom}(S, Y^{**})$ and also that

$$\|B(x) - F(x)\| \leq \lim_{\mathcal{U}} \|A_\alpha(x) - F(x)\| = \lim_{\mathcal{U}} \|A_\alpha(x) - F_\alpha(x)\| \leq MZ(F)\varrho(x).$$

Now, $A = \pi \circ B$ is an additive map at distance at most $\|\pi\| MZ(F)$ from $\pi \circ F = F$. ∎

Of course, "zero-additive" can be replaced by "quasi-additive", "Ger-additive" and "pseudo-additive" with no changes in the proof.

COROLLARY 6.20. *Let Z be an $\mathcal{L}_p$-space, $0 < p \leq \infty$, and let Y be a Banach space complemented in its second dual by a projection π. Assume that $[\ell_p^n, Y]$ has property $(M\mathrm{Q})$ for all n. Then $[Z, Y]$ has property $(\|\pi\| M\mathrm{Q})$.*

The result can be dualized by imposing the "local conditions" on the quasi-Banach target space. In that case only the values $1 \leq p \leq \infty$ can be considered since the result yielding the existence of projections on an $\mathcal{L}_p$-space no longer works on quasi-Banach spaces.

THEOREM 6.21. *Let (S, ϱ) be a controlled group and $1 \leq p \leq \infty$. The following statements are equivalent:*

(a) *For any $\mathcal{L}_p$-space Y complemented in its bidual the pair $[(S, \varrho), Y]$ has property $(M\mathrm{Z})$.*

(b) *The pair $[(S, \varrho), \ell_p(\mathbb{N})]$ has property $(M\mathrm{Z})$ for some M.*

(c) *There is $C \geq 0$ such that $[(S, \varrho), \ell_p^n]$ has property $(C\mathrm{Z})$ for all n.*

Proof. Since the implications (a)⇒(b) and (b)⇒(c) are obvious, the point is to prove that (c) implies (a). This proof is in a sense a remake of Theorem 6.19; but we present it anyway for the sake of completeness. Let $F : (S, \varrho) \to Y$ be a zero-additive mapping, where Y is an $\mathcal{L}_p$-space for some $1 \le p \le \infty$. By [74] there is $\lambda > 0$, a net $\{Y_\alpha\}$ of finite-dimensional subspaces of Y such that $Y = \bigcup_\alpha Y_\alpha$ and, for each α, an isomorphism $T_\alpha : Y_\alpha \to \ell_p(\dim Y_\alpha)$ with $\|T_\alpha\| \cdot \|T_\alpha^{-1}\| \le \lambda$ and a projection $P_\alpha : Y \to Y_\alpha$ such that $\|P_\alpha\| \le \lambda$. For each α, consider the mapping $T_\alpha P_\alpha F : (S, \varrho) \to \ell_p(\dim Y_\alpha)$. Clearly, $Z(T_\alpha P_\alpha F) \le \|T_\alpha \circ P_\alpha\| Z(F) = \lambda^2 Z(F)$ and the hypothesis yields an additive $A_\alpha : S \to \ell_p(\dim Y_\alpha)$ such that $Z(T_\alpha P_\alpha F, A_\alpha) \le C\lambda^2 Z(F)$, which implies that the distance between $P_\alpha F$ and $B_\alpha = T_\alpha^{-1} \circ A_\alpha$ is at most $C\lambda^3 Z(F)$. The remainder of the proof goes as before, by defining

$$A(x) = \pi(\text{weak*-}\lim_{\mathcal{U}} B_\alpha(x))$$

and checking that it has all the desired properties. ∎

Of course, "zero-additive" can be replaced by "quasi-additive", "Ger-additive" and "pseudo additive" with no changes in the proof. On the other hand, the sequence $0 \to c_0 \to \ell_\infty \to \ell_\infty/c_0 \to 0$ together with Sobczyk's theorem shows that the hypothesis "Y complemented in its second dual" appearing in Theorems 6.19 and 6.21 is not superfluous.

VII. Homology sequences, and applications

This chapter introduces elements from homological algebra to derive new results about the asymptotic stability of nearly additive mappings. To keep our promise of not using the categorical language, we shall state the homology sequences in terms of nearly additive maps (although all results in the chapter can be read as results about singular extensions of quasi-normed abelian groups, according to Proposition 5.13). This new topic was explored in [15], where the knowledgeable reader, interested in the appropriate translation of the results here presented to the standard algebraic setting, can find rather complete information.

In this chapter all groups are assumed to be commutative. In what follows, Ξ denotes one of the five classes of nearly additive mappings. When the quasi-normed groups involved are quasi-normed spaces, Ξ can also denote one of the five classes of nearly linear mappings. In that case $\mathcal{T}$ (the corresponding space of "trivial" maps) stands for the space of asymptotically linear maps. Given two Ξ-additive maps F and G, we say that F is a *version* of G (or vice versa) if $F \equiv G$ modulo $\mathcal{T}$. With a slight abuse of notation, we shall denote the group of equivalence clases of Ξ-additive or linear maps from Z to Y by $\Xi(Z,Y)/\mathcal{T}(Z,Y)$, although this is an abbreviation for

$$\frac{\Xi(Z,Y)}{\mathcal{T}(Z,Y) \cap \Xi(Z,Y)}.$$

THEOREM 7.1. *Let $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0$ be an exact sequence of quasi-normed groups defined by a Ξ-additive map F_0. Let E be a quasi-normed group. There exists an exact sequence of groups*

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathcal{B}}(Z,E) &\xrightarrow{q^*} \mathrm{Hom}_{\mathcal{B}}(X,E) \xrightarrow{j^*} \mathrm{Hom}_{\mathcal{B}}(Y,E) \\ &\xrightarrow{F_0^*} \Xi(Z,E)/\mathcal{T}(Z,E) \xrightarrow{q^*} \Xi(X,E)/\mathcal{T}(X,E) \xrightarrow{j^*} \Xi(Y,E)/\mathcal{T}(Y,E). \end{aligned}$$

Proof. To clarify the nature of the sequence, let us recall that q^*, j^* and F_0^* mean composition with q, j and F_0 respectively. Precisely: $q^*(T) = T \circ q$; $j^*(T) = T \circ j$; $F_0^*(T) = [T \circ F_0]$; $q^*([F]) = [F \circ q]$; $j^*([F]) = [F \circ j]$. That the sequence is exact at $\mathrm{Hom}_{\mathcal{B}}(X, E)$ is easily checked. The exactness at $\Xi(X,E)/\mathcal{T}(X,E)$, $\Xi(Z,E)/\mathcal{T}(Z,E)$ and $\mathrm{Hom}_{\mathcal{B}}(Y,E)$ is the contents of the following three lemmata.

LEMMA 7.2. *Let $0 \to Y \to X \to Z \to 0$ be an exact sequence of quasi-normed groups defined by a Ξ-additive map F_0. Let E be a quasi-normed group and let $F : X \to E$ be a Ξ-additive map. If $F|_Y \equiv 0$ then there exists a Ξ-additive map $G : Z \to E$ so that F is a version of $G \circ q$.*

Proof. Let $F_0 = B_0 - L_0$ be a Ξ-additive map defining the starting sequence. Let $F = B_1 - L_1$ so that $B_1|_Y = B$ and $L_1|_Y = L$ are E-valued. Consider E-valued extensions B_2 and L_2 of B and L respectively. The map $(B_1 - B_2) - (L_1 - L_2)$ is E-valued. We define

$$G = (B_1 - B_2) \circ B_0 - (L_1 - L_2) \circ L_0.$$

The map $G : Z \to E$ is well defined. We show that $G \circ q$ is a version of F:

$$\begin{aligned}
&(B_1 - B_2)B_0q - (L_1 - L_2)L_0q - (B_1 - L_1) \\
&\qquad = (B_1B_0q - L_1L_0q) - (B_2B_0q - L_2L_0q) - (B_1 - L_1) \\
&\qquad = B_1(B_0q - \mathrm{Id}_X) - L_1(L_0q - \mathrm{Id}_X) - (B_2B_0q - L_2L_0q) \\
&\qquad = B_2(B_0q - \mathrm{Id}_X) - L_2(L_0q - \mathrm{Id}_X) - (B_2B_0q - L_2L_0q) \\
&\qquad = L_2 - B_2. \ \blacksquare
\end{aligned}$$

LEMMA 7.3. *Let $0 \to Y \to X \to Z \to 0$ be an exact sequence of quasi-normed groups defined by a Ξ-additive map F_0. Let E be a quasi-normed group and let $F : Z \to E$ be a Ξ-additive map such that $F \circ q$ is trivial. Then there exists a bounded additive map $T : Y \to E$ so that $T \circ F_0 \equiv F$.*

Proof. If $Fq = B - L$ then $B|_Y = L|_Y$ is a bounded additive map we shall call T. Now,

$$\begin{aligned}
TF_0q + Fq &= TF_0q - (L - B) = L(B_0q - L_0q) - (L - B) \\
&= L(B_0q - \mathrm{Id}_X) - LL_0q + B = B(B_0q - \mathrm{Id}_X) - LL_0q + B \\
&= BB_0q - LL_0q. \ \blacksquare
\end{aligned}$$

LEMMA 7.4. *Let $0 \to Y \to X \to Z \to 0$ be an exact sequence of quasi-normed groups defined by a Ξ-additive map F_0. Let E be a quasi-normed group and let $T : Y \to E$ be a bounded additive map such that $TF_0 \equiv 0$. There exists a bounded additive map $\tau : X \to E$ such that $\tau j \equiv T$.*

Proof. It is easy to verify that $F_0q \equiv 0$ and that if we write $F_0q = b + l$ then $b|_Y = -l|_Y = \mathrm{Id}_Y$. On the other hand, $TF_0 = B + L$; hence, $Bq + Lq = TF_0q = Tb + Tl$, which means that $\tau = Tb - Bq = Lq - Tl$ is a bounded additive map $X \to E$. That $\tau|_Y \equiv T$ is a consequence of $\tau j = (Tb - Bq)j = Tbj = T$. $\blacksquare$

The "dual" results also hold:

THEOREM 7.5. *Let $0 \to Y \to X \to Z \to 0$ be an exact sequence of quasi-normed groups defined by a Ξ-additive map F_0. Let E be a quasi-normed group. There exists an exact* *sequence of groups*

$$0 \to \mathrm{Hom}_{\mathcal{B}}(E,Y) \xrightarrow{j_*} \mathrm{Hom}_{\mathcal{B}}(E,X) \xrightarrow{q_*} \mathrm{Hom}_{\mathcal{B}}(E,Z)$$
$$\xrightarrow{(F_0)_*} \varXi(E,Y)/\mathcal{T}(E,Y) \xrightarrow{j_*} \varXi(E,X)/\mathcal{T}(E,X) \xrightarrow{q_*} \varXi(E,Z)/\mathcal{T}(E,Z).$$

Proof. The sequence is given by: $q_*(S) = qS$; $j_*(S) = jS$; $(F_0)_*(T) = [F_0T]$; $q_*([F]) = [qF]$; $j_*([F]) = [jF]$. That the sequence is exact at $\mathrm{Hom}_{\mathcal{B}}(E,X)$ is obvious. The exactness at $\varXi(E,X)/\mathcal{T}(E,X)$ is precisely the content of the first lemma below, the exactness at $\varXi(E,Y)/\mathcal{T}(E,Y)$ is proved in the second lemma, while the exactness at $\mathrm{Hom}_{\mathcal{B}}(E,Z)$ is proved in the third lemma.

LEMMA 7.6. *Let $0 \to Y \to X \to Z \to 0$ be an exact sequence of quasi-normed groups defined by a $\varXi$-additive map. Let $F : E \to X$ be a $\varXi$-additive map. If $qF \equiv 0$ then there exists an Y-valued version of F.*

Proof. Let $F_0 = B_0 - L_0$ be a $\varXi$-additive map defining the starting sequence. Since $qF = B - L$ consider
$$G = F - (B_0B - L_0L).$$
Since $B_0B - L_0L : E \to X$, G is a version of F. Moreover, G is Y-valued since
$$q(B_0B - L_0L) = q(B_0B - B_0L + B_0L - L_0L) = qB_0qF - qF_0L = qF. \ \blacksquare$$

LEMMA 7.7. *Let $0 \to Y \to X \to Z \to 0$ be an exact sequence of quasi-normed groups defined by a $\varXi$-additive map. Let E be a quasi-normed group. If $F : E \to Y$ is a $\varXi$-additive map such that $iF \equiv 0$ then there exists a bounded additive map $T : E \to Z$ so that $F_0T \equiv F$.*

Proof. Since $iF = B - L$, we see that $qB = qL = A$ is bounded and additive simultaneously. Now,
$$\begin{aligned} F_0qB + F &= F_0qB - (L - B) = (B_0q - L_0q)B + B - L \\ &= B_0qB + (\mathrm{Id}_X - L_0q)B - L = B_0qB + (\mathrm{Id}_X - L_0q)L - L = B_0qB - L_0qL. \ \blacksquare \end{aligned}$$

LEMMA 7.8. *Let $0 \to Y \to X \to Z \to 0$ be an exact sequence of quasi-normed groups defined by a $\varXi$-additive map F_0. Let A be a quasi-normed group. Let $T : A \to Z$ be a bounded additive map. If $F_0T \equiv 0$ then there exists a bounded additive lifting $\tau : A \to X$ for T.*

Proof. It is again easy to verify that $jF_0 \equiv 0$ and that if we write $jF_0 = b + l$ then $qb = -ql = \mathrm{Id}_Z$. On the other hand, $F_0T = B + L$; hence, $jB + jL = jF_0T = bT + lT$, which means that $\tau = bT - jB = jL - lT$ is a bounded additive map $A \to X$. That $q\tau = T$ is a consequence of $q\tau = q(bT - jB) = qbT = T$. $\blacksquare$

1. "Three-space" problems. Now it is time to use the techniques of the previous section to obtain concrete results. Recall that a property P is said to be a 3-*space property* (for a given class of extensions) if every extension (in the given class) of two groups with P has P. (We shall maintain the name 3-space although applied to groups.) Also, given two properties P and Q of Banach spaces we say that a X is a *P-by-Q space* if it is a twisted sum of a space having P and a space having Q. This name has been taken from group theory [46]. It has been transplanted to Banach space theory in [21]. It is clear that P is a 3-space property (for Banach spaces) if and only if P-by-P implies P.

Observe that "abelian" is not a 3-space property on groups: the symmetric group S_3 of permutations of three elements contains a normal cyclic subgroup of order 3 (that generated by the cycle (123)) so that the quotient space is (necessarily) the cyclic group of order 2. Moreover, the operation -by- is associative on quasi-Banach spaces (i.e. (P-by-Q)-by-R and P-by-(Q-by-R) are the same property); however, -by- is not associative on groups: if P_n is the property of "being cyclic of order n" then the alternating group A_4 is (P_2-by-P_2)-by-P_3, but not P_2-by-(P_2-by-P_3).

We use the following notation: given a class Ξ of nearly additive (or nearly linear) maps and two classes $\mathcal{A}$, $\mathcal{B}$ of quasi-normed groups (or quasi-Banach spaces), we write

$$\Xi(\mathcal{A},\mathcal{B})\equiv 0$$

if every Ξ-additive (linear) map acting between a space in $\mathcal{A}$ and a space in $\mathcal{B}$ is trivial.

Our first application of homology sequences is:

Lemma 7.9. *Let $0\to Y\to X\to Z\to 0$ be an exact sequence defined by a Ξ-additive* (*linear*) *map. If $\Xi(Y,E)\equiv 0$ and $\Xi(Z,E)\equiv 0$, then $\Xi(X,E)\equiv 0$.*

And, of course, the dual version:

Lemma 7.10. *Let $0\to Y\to X\to Z\to 0$ be an exact sequence defined by a Ξ-additive* (*linear*) *map. If $\Xi(A,Y)\equiv 0$ and $\Xi(A,Z)\equiv 0$, then $\Xi(A,X)\equiv 0$.*

One therefore has

Theorem 7.11. *The properties $\Xi(\mathcal{A},\cdot)\equiv 0$ and $\Xi(\cdot,\mathcal{B})\equiv 0$ are 3-space properties for singular extensions induced by Ξ-additive* (*or linear*) *maps.*

Concrete situations to which this general result can be applied are described now.

2. K-spaces. Recall that a quasi-normed space Z is called a K-space if every quasi-linear map $F:Z\to\mathbb{R}$ is asymptotically linear. One has:

Corollary 7.12. *To be a K-space is a 3-space property for quasi-Banach spaces.*

Proof. This is Theorem 7.11 for $\mathcal{B}=\mathbb{R}$ and with Ξ the class of quasi-linear maps. ■

3. $\mathcal{L}_1$-spaces. Let $\mathcal{CB}$ denote the class of Banach spaces complemented in their biduals. The fact that every exact sequence $0\to Y\to X\to Z\to 0$ of Banach spaces with Y in $\mathcal{CB}$ and Z an $\mathcal{L}_1$-space splits was first proved by Lindenstrauss [73]. This result actually characterizes the $\mathcal{L}_1$-spaces. We prove that:

Proposition 7.13. *A Banach space X is an $\mathcal{L}_1$-space if and only if $\mathcal{Z}(X,\mathcal{CB})\equiv 0$.*

Proof. The "only if" has already been proved in Chapter VI; we prove the "if" part. We show that X^* is injective. Let $G:Z\to X^*$ be a zero-linear map. By hypothesis, $G^*|_X:X\to Z^*$ is trivial, and so is $(G^*|_X)^*:Z^{**}\to X^*$, as well as $(G^*|_X)^*|_Z:Z\to X^*$. It is now a simple matter to verify that $(G^*|_X)^*|_Z$ and G are equivalent. ■

However, it is not true that $\mathcal{Z}(\mathcal{L}_1, V) \equiv 0$ implies that V is complemented in the bidual. We present an example.

PROPOSITION 7.14. *Let Z be a space in $\mathcal{CB}$ with the Radon–Nikodym property. If V is such that $V^{**}/V = Z$, then $\mathcal{Z}(\mathcal{L}_1, V) \equiv 0$.*

Proof. Let $q : \ell_1(\Gamma) \to V^{**}$ be a quotient map. It is easy to see that $\ell_1(\Gamma)/q^{-1}(V) = V^{**}/V$. The Radon–Nikodym property of V^{**}/V yields that $q^{-1}(V)$ is an ultrasummand [65]. Moreover, the exact sequence

$$0 \to q^{-1}(V) \to \ell_1(\Gamma) \oplus V \to V^{**} \to 0$$

with inclusion $x \mapsto (x, qx)$ and quotient map $(a, b) \mapsto qa - b$ shows that $\mathcal{Z}(\mathcal{L}_1, \ell_1(I) \oplus V) \equiv 0$, and consequently $\mathcal{Z}(\mathcal{L}_1, V) \equiv 0$. ■

Observe that V is not necessarily in $\mathcal{CB}$. For instance, if JT denotes the James-Tree space and B denotes the standard predual of JT then there is a non-trivial exact sequence $0 \to B \to \mathrm{JT}^* \to \ell_2(\Gamma) \to 0$ (see [20]); thus, B is not in $\mathcal{CB}$ while $\mathcal{Z}(\mathcal{L}_1, B) \equiv 0$.

A different type of examples are provided in the next section.

4. Johnson–Lindenstrauss spaces. Basic information about the class of weakly compactly generated Banach spaces (in short, WCG) can be found in the monograph [20]. By a *Johnson–Lindenstrauss space* we mean a non-WCG twisted sum of two WCG spaces. In [58] specific examples were constructed of non-trivial exact sequences $0 \to c_0 \to \mathrm{JL}_\infty \to c_0(\Gamma) \to 0$ and $0 \to c_0 \to \mathrm{JL}_p \to \ell_p(\Gamma) \to 0$ for $1 < p < \infty$. A version of Sobczyk's theorem (asserting that c_0 is complemented in WCG spaces) implies that JL_p are not WCG spaces for $1 \le p \le \infty$.

LEMMA 7.15. *The Johnson–Lindenstrauss spaces* JL_p, $1 < p < \infty$, *are not $\mathcal{CB}$ spaces.*

Proof. Observe the diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 \to & & c_0 & \to & \mathrm{JL}_p & \to \ell_p(\Gamma) \to 0 \\
 & & \downarrow & & \downarrow & \| \\
0 \to & & \ell_\infty & \to & \mathrm{JL}_p^{**} & \to \ell_p(\Gamma) \to 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \ell_\infty/c_0 & = & \mathrm{JL}_p^{**}/\,\mathrm{JL}_p & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

in which the second row is the bitranspose of the first one. Were JL_p complemented in JL_p^{**} then ℓ_∞/c_0 would be a subspace of $\ell_\infty \oplus \ell_p(\Gamma)$, which is not the case. ■

Denote by $\mathcal{A}^{\mathrm{s}}$ the class of separable spaces in $\mathcal{A}$. Let us recall again that a Banach space X is an $\mathcal{L}_1$-space if and only if $\mathcal{Z}(X, \mathcal{CB}) \equiv 0$. In spite of not being in $\mathcal{CB}$,

$$\mathcal{Z}(\mathcal{L}_1^{\mathrm{s}}, \mathrm{JL}_p) \equiv 0$$

since $\mathcal{Z}(c_0(I), \mathcal{L}_1^{\mathrm{s}}) \equiv 0$ is given by Sobczyk's theorem, while $\mathcal{Z}(\ell_p(\Gamma), \mathcal{L}_1) \equiv 0$ is a consequence of $\ell_p(\Gamma)$ being a $\mathcal{CB}$ space.

And, in full generality,

Corollary 7.16. *$\mathcal{Z}(\mathcal{L}_1^s, c_0(I)$-by-$\mathcal{CB}) \equiv 0$.*

If $\mathcal{S}$ denotes the class of separable spaces, Sobczyk's theorem asserts that

$$\mathcal{Z}(\mathcal{S}, c_0(I)) \equiv 0,$$

while Zippin's theorem [106] asserts that if Y is a separable Banach space such that $\mathcal{Z}(\mathcal{S}, Y) \equiv 0$ then Y is isomorphic to c_0. The injective spaces show that the separability assumption is essential. The Johnson–Lindenstrauss space JL_∞ yields a non-injective space such that $\mathcal{Z}(\mathcal{S}, \mathrm{JL}_\infty) \equiv 0$ although it is not isomorphic to $c_0(I)$.

5. Kalton–Pełczyński spaces. From the results in [65], we say that a Banach space X is a *Kalton–Pełczyński space* if $\mathcal{Z}(X, \ell_2) \equiv 0$. Of course, $\mathcal{L}_1$-spaces are Kalton–Pełczyński spaces; but there are more: for instance, an uncomplemented copy of ℓ_1, say Y, inside a bigger $\mathcal{L}_1$-space X produces a KP-space X/Y which is not an $\mathcal{L}_1$-space.

Theorem 7.17. *To be a Kalton–Pełczyński space is a 3-space property for Banach spaces.*

We can present two proofs for this result. One follows from the definition of Kalton–Pełczyński spaces and Theorem 7.11. There is, however, a more general approach.

Lemma 7.18. *Let P be a property stable by products. Then the property $K(P)$ defined as "being a quotient of some $\ell_1(I)$ by a subspace having P" is a 3-space property.*

Proof. Let $0 \to Y \to X \to Z \to 0$ be an exact sequence in which both Y and Z have the property $K(P)$. Observe the diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
0 \to & & Y & \to X \to & Z & \to 0 \\
 & & \uparrow & & \uparrow & & \\
 & & \ell_1(I) & & \ell_1(J) & & \\
 & & \uparrow & & \uparrow & & \\
 & & K_Y & & K_Z & & \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
\end{array}$$

in which both K_Y and K_Z have property P. It is not difficult to see the existence of an exact sequence $0 \to K_X \to \ell_1(\Gamma) \to X \to 0$ that can be inserted in the middle column making all rows exact and the full diagram commutative. From this, one easily derives that $K_X = K_Y \oplus K_Z$, and X has property $K(P)$. ∎

Lemma 7.19 ([65]). *A Banach space X is a Kalton–Pełczyński space if and only if it is a $K(P)$-space for $P =$ "satisfying Grothendieck's theorem" (every operator into a Hilbert space is 1-summing).*

Because of this lemma and since it is clear that satisfying Grothendieck's theorem is stable by products, it follows that being a Kalton–Pełczyński space is a 3-space property. We cannot help mentioning that satisfying the equation $L(X, \ell_2) = \Pi_1(X, \ell_2)$ is not only stable by products but also a 3-space property (see [20]).

6. The road beyond. What has been displayed so far is not all there is, and what there is does not cover what one would like to know. For instance, what happens with non-singular extensions of topological groups? A simple example such as $0 \to 2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$ is not covered by this theory. In this chapter we have presented a glimpse of homological elements, but many other basic questions remain untouched. For instance, the elements of homological algebra rely on the definition of equivalent exact sequences. The translation to quasi-Banach space theory is easy thanks to two very special features: all extensions are singular (since, as vector spaces, all extensions are trivial) and the open mapping theorem holds (which translates an exact sequence into a *topologically* exact sequence). But in topological groups no open mapping theorem exists. Hence, even if the first difficulty (what occurs with non-singular extensions) can be surmounted, there appears a second one equally formidable. At this point it might be interesting to look at paper [19]: there, a very nice result of Dierolf and Schwanengel [29] is revisited to show that the 3-lemma (hence, the notion of equivalent sequences) still works well for topologically exact sequences of topological groups (even whithout the open mapping theorem). This allows one to transport the basic elements of homological algebra (pull-back and push-out constructions) to *abelian* topological groups. What lies beyond abelian groups, we do not know.

References

[1] J. Aczél, *On applications and theory of functional equations*, in: Elemente der Mathematik vom höheren Standpunkt, Birkhäuser, Basel, 1969.

[2] I. Aharoni and J. Lindenstrauss, *Uniform equivalence between Banach spaces*, Bull. Amer. Math. Soc. 84 (1978) 281–283.

[3] J. R. Alexander, C. E. Blair and L. A. Rubel, *Approximate versions of Cauchy's functional equation*, Illinois J. Math. 39 (1995), 278–287.

[4] M. F. Atiyah and I. G. Macdonald, *Introducción al Álgebra conmutativa*, Reverté, Barcelona, 1980.

[5] S. Banach, *Sur les fonctionnelles linéaires* (*II*), Studia Math. 1 (1929), 223–239.

[6] —, *Théorie des opérations linéaires*, Chelsea, 1978.

[7] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, Vol. 1, Amer. Math. Soc. Colloq. Publ. 48, Amer. Math. Soc., 2000.

[8] H. Bustos-Domecq, *Vector-valued invariant means*, *revisited*, J. Math. Anal. Appl., to appear.

[9] F. Cabello Sánchez, *Some remarks stemming from Ulam's problem about nearly additive mappings*, Aequationes Math. 56 (1998), 233–242.

[10] —, *Uniform boundedness theorems for nearly additive mappings*, Abh. Math. Sem. Univ. Hamburg 69 (1999), 167–173.

[11] —, *Stability of additive maps on large subsets*, Proc. Amer. Math. Soc. 128 (2000), 1071–1077.

[12] F. Cabello Sánchez, *Pseudo-characters and almost multiplicative functionals*, J. Math. Anal. Appl. 248 (2000), 275–289.

[13] F. Cabello Sánchez and J. M. F. Castillo, *Duality and twisted sums of Banach spaces*, J. Funct. Anal. 175 (2000), 1–16.

[14] —, —, *Uniform boundedness and twisted sums of Banach spaces*, preprint.

[15] —, —, *The long homology sequence for quasi-Banach spaces, with applications*, Positivity, to appear.

[16] F. Cabello Sánchez, J. M. F. Castillo, N. J. Kalton and D. Yost, *Twisted sums with $C(K)$-spaces*, Trans. Amer. Math. Soc., to appear.

[17] F. Cabello Sánchez, J. M. F. Castillo and D. Yost, *Sobczyk's theorems from A to B*, Extracta Math. 15 (2000), 391–420.

[18] J. M. F. Castillo, *Snarked sums of Banach spaces*, ibid. 12 (1997), 117–128.

[19] —, *On the "three-space" problem for locally quasi-convex topological groups*, Arch. Math. (Basel) 74 (2000), 253–262.

[20] J. M. F. Castillo and M. González, *Three Space Problems in Banach Space Theory*, Lecture Notes in Math. 1667, Springer, 1997.

[21] J. M. F. Castillo, M. González, A. M. Plichko and D. Yost, *Twisted properties of Banach spaces*, Math. Scand., to appear.

[22] D. Cenzer, *The stability problem for transformations of the circle*, Proc. Roy. Soc. Edinburgh Sect. A 84 (1979), 279–281.

[23] —, *The stability problem: new results and counterexamples*, Lett. Math. Phys. 10 (1985), 155–160.

[24] H. Corson and V. Klee, *Topological classification of convex sets*, in: Convexity, Proc. Sympos. Pure Math. 7, Amer. Math. Soc., 1963, 37–51.

[25] J. Crespo, *Un estudio acerca de la regularidad de las funciones aproximadamente aditivas*, memoria licenciatura, Uex, 2001.

[26] M. M. Day, *Amenable semigroups*, Illinois J. Math. 1 (1957), 509–544.

[27] J. Dhombres, *Some aspects of functional equations*, lecture notes, Department of Mathematics, Chulalongkorn Univ., 1979.

[28] S. Dierolf, *Über Vererbbarkeitseigenschaften in topologischen Vektorräumen*, Dissertation, Ludwig Maximilians Universität, 1973, Munich.

[29] S. Dierolf and U. Schwanengel, *Examples of locally compact non-compact minimal topological groups*, Pacific J. Math. 82 (1979), 349–355.

[30] V. A. Faiziev, *Spaces of pseudocharacters on free products of semigroups*, Mat. Zametki 52 (1992), no. 6, 119–130 (in Russian); English transl.: Math. Notes 52 (1992), 1255–1264.

[31] —, *Pseudocharacters on semidirect products of semigroups*, Mat. Zametki 53 (1993), no. 2, 132–139 (in Russian); English transl.: Math. Notes 53 (1993), 208–213.

[32] —, *Pseudocharacters on free semigroups invariant with respect to their automorphism groups*, Proc. Amer. Math. Soc. 126 (1998), 333–342.

[33] G. L. Forti, *An existence and stability theorem for a class of functional equations*, Stochastica 4 (1980), 23–30.

[34] —, Remark 11 in: The 22nd International Symposium on Functional Equations, Aequationes Math. 29 (1985), 90–91.

[35] —, *The stability of homomorphisms and amenability, with applications to functional equations*, Abh. Math. Sem. Univ. Hamburg 57 (1987), 215–226.

[36] —, *Hyers–Ulam stability of functional equations in several variables*, Aequationes Math. 50 (1995), 143–190.

[37] Z. Gajda, *On stability of the Cauchy equation on semigroups*, ibid. 36 (1988), 76–79.

[38] —, *On stability of additive mappings*, Internat. J. Math. Math. Sci. 14 (1991), 431–434.

[39] Z. Gajda and Z. Kominek, *On separation theorems for subadditive and superadditive functionals*, Studia Math. 100 (1991), 25–38.

[40] P. Găvruţa, *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. 184 (1994), 431–436.

[41] R. Ger, *On functional inequalities stemming from stability questions*, in: General Inequalities 6, W. Walter (ed.), Internat. Ser. Numer. Math. 103, Birkhäuser, Basel, 1992, 227–240.

[42] —, *The singular case in the stability behaviour of linear mappings*, in: Selected Topics in Functional Equations and Iteration Theory, Proc. Austrian–Polish Seminar (Graz, 1991), Grazer Math. Ber. 316 (1992), 59–70.

[43] —, *Delta-exponential mappings in Banach algebras*, in: General Inequalities 7, C. Bandle *et al.* (eds.), Internat. Ser. Numer. Math. 123, Birkhäuser, Basel, 1997, 285–296.

[44] F. P. Greenleaf, *Invariant Means on Topological Groups*, Van Nostrand Math. Stud. 16, Van Nostrand, New York, 1969.

[45] K. Grove, H. Karcher and E. A. Roh, *Jacobi fields and Finsler metrics on compact Lie groups with an application to differentiable pinching problems*, Math. Ann. 211 (1974), 7–21.

[46] P. Hall, *Finiteness conditions for soluble groups*, Proc. London Math. Soc. 4 (1954), 419–436.

[47] G. Hamel, *Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung*: $f(x+y) = f(x) + f(y)$, Math. Ann. 60 (1905), 459–462.

[48] P. de la Harpe et M. Karoubi, *Représentations approchées d'un groupe dans une algèbre de Banach*, Manuscripta Math. 22 (1977), 293–310.

[49] P. J. Hilton and U. Stammbach, *A Course in Homological Algebra*, Grad. Texts in Math. 4, Springer, 1970.

[50] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. 271 (1941), 222–224.

[51] D. H. Hyers, G. Isac and T. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[52] D. H. Hyers and T. M. Rassias, *Approximate homomorphisms*, Aequationes Math. 44 (1992), 125–153.

[53] G. Isac and T. M. Rassias, *On the Hyers–Ulam stability of ψ-additive mappings*, J. Approx. Theory 72 (1993), 131–137.

[54] K. Jarosz, *Perturbations of Banach Algebras*, Lecture Notes in Math. 1120, Springer, Berlin, 1985.

[55] B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. 127 (1972).

[56] —, *Approximately multiplicative functionals*, J. London Math. Soc. 34 (1986), 489–510.

[57] —, *Approximately multiplicative maps between Banach algebras*, ibid. 37 (1988), 294–316.

[58] W. B. Johnson and J. Lindenstrauss, *Some remarks on weakly compactly generated Banach spaces*, Israel J. Math. 17 (1974), 219–230.

[59] S.-M. Jung, *On the Hyers–Ulam–Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. 204 (1996), 221–226.

[60] N. J. Kalton, *The three space problem for locally bounded F-spaces*, Compositio Math. 37 (1978), 243–276.

[61] —, *Convexity, type and the three space problem*, Studia Math. 69 (1981), 247–287.

[62] N. J. Kalton and N. T. Peck, *Twisted sums of sequence spaces and the three space problem*, Trans. Amer. Math. Soc. 255 (1979), 1–30.

[63] —, —, *Quotients of $L_p(0,1)$ for $0 \le p < 1$*, Studia Math. 64 (1979), 65–75.

[64] N. J. Kalton, N. T. Peck and J. W. Roberts, *An F-space Sampler*, London Math. Soc. Lecture Note Ser. 89, Cambridge Univ. Press, 1984.

[65] N. J. Kalton and A. Pełczyński, *Kernels of surjections from $\mathcal{L}_1$-spaces with an application to Sidon sets*, Math. Ann. 309 (1997), 135–158.

[66] N. J. Kalton and J. W. Roberts, *Uniformly exhaustive submeasures and nearly additive set functions*, Trans. Amer. Math. Soc. 278 (1983), 803–816.

[67] R. Kaufman, *Interpolation of additive functionals*, Studia Math. 27 (1966), 269–272.

[68] D. Kazhdan, *On ε-representations*, Israel J. Math. 43 (1982), 315–323.

[69] H. König, *On the abstract Hahn–Banach theorem due to Rodé*, Aequationes Math. 34 (1987), 89–95.

[70] P. Kranz, *Additive functionals on abelian semigroups*, Comment. Math. Prace Mat. 16 (1972), 239–246.

[71] Å. Lima and D. Yost, *Absolutely Chebyshev subspaces*, in: Workshop/Miniconference on Functional Analysis/Optimization (Canberra, 1988), S. Fitzpatrick and J. Giles (eds.), Proc. Centre Math. Anal. Austral. Nat. Univ., 20, Austral. Nat. Univ., 1988, 116–127.

[72] J. Lindenstrauss, *On non-linear projections in Banach spaces*, Michigan Math. J. 11 (1964), 268–287.

[73] —, *On a certain subspace of l_1*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 12 (1964), 539–542.

[74] J. Lindenstrauss and H. P. Rosenthal, *The $\mathcal{L}_p$-spaces*, Israel J. Math. 7 (1969), 325–349.

[75] K. D. Magill (reviewer), review of the paper: Z. Gajda, *Invariant means and representations of semigroups in the theory of functional equations*, Prace Nauk. Uniw. Śl. Katow. 1273 (1992); MR 94b:39034.

[76] S. Mazur et W. Orlicz, *Sur les espaces métriques linéaires*, Studia Math. 13 (1953), 137–179.

[77] J. Peetre and G. Sparr, *Interpolation of normed abelian groups*, Ann. Mat. Pura Appl. 92 (1972), 217–262.

[78] A. Pełczyński, *Linear extensions, linear averagings, and their applications to linear topological classification of spaces of continuous functions*, Dissertationes Math. (Rozprawy Mat.) 58 (1968).

[79] G. Pólya and G. Szegő, *Problems and Theorems in Analysis I*, Springer, 1976.

[80] V. Pták, *On a theorem of Mazur and Orlicz*, Studia Math. 15 (1956), 365–366.

[81] T. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. 72 (1978), 297–300.

[82] —, *On a modified Hyers–Ulam sequence*, J. Math. Anal. Appl. 158 (1991), 106–113.

[83] —, Problem 16 (2), in: The Twenty Seventh International Symposium on Functional Equations, Aequationes Math. 39 (1990), 308–309.

[84] —, *On the stability of functional equations originated by a problem of Ulam*, preprint, 1999.

[85] T. M. Rassias and P. Šemrl, *On the behaviour of mappings which do not satisfy the Hyers–Ulam stability*, Proc. Amer. Math. Soc. 114 (1992), 989–993.

[86] —, —, *On the Hyers–Ulam stability of linear mappings*, J. Math. Anal. Appl. 173 (1993), 325–338.

[87] T. M. Rassias and J. Tabor, *What is left of Hyers–Ulam stability?*, J. Natural Geom. 1 (1992), 65–69.

[88] J. Rätz, *On approximately additive mappings*, in: General Inequalities 2, E. F. Beckenbach (ed.), Internat. Ser. Numer. Math. 47, Birkhäuser, Basel, 1980, 233–251.

[89] M. Ribe, *Examples for the nonlocally convex three space problem*, Proc. Amer. Math. Soc. 73 (1979), 351–355.

[90] J. W. Roberts, *A nonlocally convex F-space with the Hahn–Banach approximation property*, in: Banach Spaces of Analytic Functions, Lecture Notes in Math. 604, Springer, Berlin, 1977, 76–81.

[91] G. Rodé, *Eine abstrakte Version des Satzes von Hahn–Banach*, Arch Math. (Basel) 31 (1978), 474–481.

[92] S. Rolewicz, *Metric Linear Spaces*, Monograf. Mat. 56, PWN, Warszawa, 1972, and Reidel, Dordrecht, 1984.

[93] P. Šemrl, *The stability of approximately additive functions*, in: Stability of Mappings of Hyers–Ulam type, T. M. Rassias and J. Tabor (eds.), Hadronic Press, Palm Harbor, FL, 1994, 135–140; MR95i:39029.

[94] A. I. Shtern, *Quasirepresentations and pseudorepresentations*, Funktsional. Anal. i Prilozhen. 25 (1991), no. 2, 70–73 (in Russian); English transl.: Functional Anal. Appl. 25 (1991), 140–143.

[95] R. Sikorski, *On a theorem of Mazur and Orlicz*, Studia Math. 13 (1953), 180–182.

[96] A. Sobczyk, *Projection of the space m on its subspace c_0*, Bull. Amer. Math. Soc. 47 (1941), 938–947.

[97] L. Székelyhidi, *Note on a stability theorem*, Canad. Math. Bull. 25 (1982), 500–501.

[98] —, Remark 17, in: Report on the 22nd International Symposium on Functional Equations, Aequationes Math. 29 (1985), 95–96.

[99] —, *Note on Hyers' theorem*, C. R. Math. Rep. Acad. Sci. Canada 8 (1986), 127–129.

[100] —, *An abstract superstability theorem*, Abh. Math. Sem. Univ. Hamburg 59 (1989), 81–83.

[101] S. M. Ulam, *A Collection of Mathematical Problems*, Wiley-Interscience, 1960.

[102] —, *Problems in Modern Mathematics*, Wiley, 1964.

[103] —, *Sets*, *Numbers and Universes*: *Selected Works*, Mathematicians of Our Times 9, MIT Press, 1974.

[104] —, *An anecdotal history of the Scottish Book*, in: The Scottish Book, R. D. Mauldin (ed.), Birkhäuser, 1981.

[105] D. Yost, *On the Johnson–Lindenstrauss space*, Extracta Math. 12 (1997), 185–192.

[106] M. Zippin, *The separable extension problem*, Israel J. Math. 26 (1977), 372–387.