Contents

Abstract

The memoir is based on a series of six papers by the author published over the years 1995–2007. It continues the work of D. Plachky (1970, 1976). It also owes some inspiration, among others, to papers by J. Łoś and E. Marczewski (1949), D. Bierlein and W. J. A. Stich (1989), D. Bogner and R. Denk (1994), and A. Ülger (1996). Let $\mathfrak{M}$ and $\mathfrak{N}$ be algebras of subsets of a set Ω with $\mathfrak{M} \subset \mathfrak{N}$. Given a *quasi-measure* μ on $\mathfrak{M}$, i.e., $\mu \in ba_+(\mathfrak{M})$, we denote by $E(\mu)$ the convex set of all quasi-measure extensions of μ to $\mathfrak{N}$. Moreover, we denote by s, w and w^* the strong, weak and weak* topologies of the dual Banach lattice $ba(\mathfrak{N})$, respectively. Our starting point are the following two properties of $E(\mu)$ and $\operatorname{extr} E(\mu)$, which are easy consequences of known results:

(a) $(E(\mu), w^*)$ is compact;
(b) $\operatorname{extr} E(\mu)$ is closed in $(ba(\mathfrak{N}), s)$.

We study the following conditions related to (a) and (b):

(i) $(E(\mu), s)$ is compact;
(ii) $(E(\mu), w)$ is compact;
(iii) s and w coincide on $E(\mu)$;
(iv) s and w coincide on $\operatorname{extr} E(\mu)$;
(v) s and w^* coincide on $\operatorname{extr} E(\mu)$;
(vi) w and w^* coincide on $\operatorname{extr} E(\mu)$;
(vii) $\operatorname{extr} E(\mu)$ is closed in $(ba(\mathfrak{N}), w)$;
(viii) $\operatorname{extr} E(\mu)$ is closed in $(ba(\mathfrak{N}), w^*)$;
(ix) $(\operatorname{extr} E(\mu), s)$ is compact;
(x) $(\operatorname{extr} E(\mu), w)$ is compact;
(xi) $(\operatorname{extr} E(\mu), w^*)$ is compact;
(xii) $(\operatorname{extr} E(\mu), s)$ is discrete;
(xiii) $(\operatorname{extr} E(\mu), w)$ is discrete;
(xiv) $(\operatorname{extr} E(\mu), w^*)$ is discrete;
(xv) $\operatorname{extr} E(\mu)$ is dense in $(E(\mu), w)$;
(xvi) $\operatorname{extr} E(\mu)$ is dense in $(E(\mu), w^*)$.

In most cases, we find various equivalent conditions expressed in topological, affine-topological and measure-theoretic terms. To this end, we use, in particular, the *antimonogenic component* μ^{a} of μ. (This is the minimal $\nu \in ba_+(\mathfrak{M})$ such that $\nu \leq \mu$ and $E(\mu - \nu)$ is a singleton.) Here are some sample results: (viii) holds if and only if μ^{a} is atomic; both (xiii) and (xiv) are equivalent to the condition that μ^{a} have finite range; (xvi) holds if and only if μ^{a} is nonatomic. One of our main tools is an affine-topological representation of $E(\mu)$ for atomic μ as the countable Cartesian product of simplex like sets. We also study some other topological properties of $\operatorname{extr} E(\mu)$, such as zero-dimensionality and various kinds of connectedness. Some of our results involve the cardinality $\mathfrak{m}$ of $\operatorname{extr} E(\mu)$. In general, there are no restrictions on $\mathfrak{m}$ except for $\mathfrak{m} \neq 0$. However, if μ is nonatomic, then $\mathfrak{m}^{\aleph_0} = \mathfrak{m}$. The case where $\mathfrak{m} \leq \aleph_0$ is also thoroughly investigated.

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Dedicated to my mother Jadwiga
and to the memory of my father Stanisław (1913–1974)

1. Introduction

The memoir is based on a series of my papers [41], [42], [44]–[47] ([1]) published over the years 1995–2007. As those papers, it is concerned with the convex set $E(\mu)$ of all quasi-measure extensions of a given *quasi-measure* μ, i.e., a positive additive function on an algebra $\mathfrak{M}$ of subsets of a set Ω, to a larger algebra $\mathfrak{R}$ of subsets of Ω. Being contained in the dual Banach lattice $ba(\mathfrak{R})$, $E(\mu)$ is equipped with the strong, weak and weak* topologies.

It is a classical result from the 1930s, a consequence of the Hahn–Banach theorem, that $E(\mu)$ is always nonempty. Some special elements of $E(\mu)$ were studied by E. Marczewski in a series of papers (see, e.g., [55] of 1951) on stochastic and set-theoretic independence, and in a fundamental paper [53] of 1949, joint with J. Łoś.

A systematic investigation of the set $E(\mu)$ was started by D. Plachky in his Habilitationsschrift [59] of 1970 (see also [60]). In particular, he noted that $E(\mu)$ is weak* compact, and so has extreme points, due to the Krein–Milman theorem. He also established a nice and useful characterization of the extreme points of $E(\mu)$ in measure-theoretic terms (see (D) on p. 19 below). Plachky's work was continued by W. Thomson in his dissertation [66] of 1975. He was mainly concerned with extending positive functions defined on a family of sets more general than an algebra and was also inspired by a question of B. de Finetti [19, p. 79] on uniqueness of extensions.

Some other characterizations of the extreme points of $E(\mu)$ were given in my paper [39] of 1992. They were suggested by a related result of H. Gail on *measures*, i.e., σ-additive quasi-measures defined on σ-algebras of sets (see D. Bierlein and W. J. A. Stich [7]). In the special case where $\mathfrak{R}$ is generated, as an algebra, by $\mathfrak{M}$ and a finite partition of Ω, a description of the elements of $E(\mu)$ and a characterization of the extreme ones among them were given in my paper [40] of 1993. Those results were suggested by related work concerning measures (see O. Nikodym [57], D. Bierlein [6], D. Bierlein and W. J. A. Stich [7]).

The material presented in the memoir is divided into 13 consecutive sections followed by the list of references and 2 indices.

Sections 2–4 explain the notation and terminology used in the text. They are mostly standard and coincide with those of my previous papers on the subject. Sections 2–4 also contain many auxiliary results needed in the main body, Sections 5–12. The necessary references and, in some cases, complete proofs are provided, with the purpose of making the memoir reasonably self-contained.

([1]) Throughout the text the results of [41], [42], [46] and [47] are referred to in an abbreviated form, according to the following pattern: "Theorem I.1" means [41, Theorem 1]".

Section 5 is concerned with weak compactness of $E(\mu)$. It applies a classical criterion of relative weak compactness in $ba(\mathfrak{N})$, which goes back to Dunford and Schwartz [17]. The main results are Theorem 5.1, which deals with the general case, and Theorem 5.5, where $\mathfrak{M}$ and $\mathfrak{N}$ are σ-algebras and μ is a measure. The latter is related to many previous results in the literature (see [4], [21], [62], [30], [8], [44], [45]). One of the four conditions of Theorem 5.5 equivalent to the weak compactness of $E(\mu)$ reads as follows: each element of $E(\mu)$ is a measure.

In Section 6 a first step is taken towards investigating the affine-topological structure of the convex set $E(\mu)$, equipped with its three natural topologies. Namely, given pairwise disjoint $\mu_j \in ba_+(\mathfrak{M})$ with $\mu = \sum_{j=1}^{\infty} \mu_j$, a representation of $E(\mu)$ as the Cartesian product $\prod_{j=1}^{\infty} E(\mu_j)$ is obtained (see Theorem 6.1). Various specializations of this representation are formulated explicitly for use in the next sections.

Section 7 establishes, using one of those specializations, some representations of $E(\mu)$ for atomic μ. They are in the form of Cartesian products of certain simplices of quasi-measures or measures. In the latter case, those simplices are of the type $\mathcal{S}(Z)$, the convex set of Radon probability measures over a compact zero-dimensional space Z, equipped with its three natural topologies (see Theorem 7.2(b)). This result is then combined with the material of Section 14 in order to show some topological properties of $E(\mu)$ and $\operatorname{extr} E(\mu)$ in the general atomic case or under some additional assumptions on μ.

Section 8 is devoted to topological properties of the set $\operatorname{extr} E(\mu)$ for nonatomic μ. In particular, it is proved that $\operatorname{extr} E(\mu)$, equipped with the strong topology, is then pathwise connected, and is compact only if μ is *monogenic*, i.e., $E(\mu)$ is a singleton (see Theorem 8.1). Moreover, $\operatorname{extr} E(\mu)$ is weak* dense in $E(\mu)$, and $\operatorname{extr} E(\mu)$ is weakly closed only if μ is monogenic (see Theorem 8.6). It is worth-while to note that Sections 7 and 8 are mutually independent, as far as the proofs are concerned.

Section 9 is the central one in the memoir. Some results of Sections 7 and 8 are combined there in order to express topological properties of $\operatorname{extr} E(\mu)$ and affine-topological properties of $E(\mu)$ in terms of μ^{a}, the antimonogenic component of μ. In particular, it is shown that $\operatorname{extr} E(\mu)$, equipped with any of its three natural topologies, is zero-dimensional [connected] if and only if μ^{a} is atomic [nonatomic] (see Theorems 9.1 and 9.7, respectively).

Section 10 is concerned with strong compactness of $E(\mu)$. It turns out that this property is equivalent to the conjunction of weak compactness of $E(\mu)$ and atomicity of μ^{a}. Another equivalent property is the following one: $E(\mu)$, equipped with any of its three natural topologies, is affinely homeomorphic to a countable Cartesian product of finite-dimensional simplices (see Theorem 10.2).

In Section 11 it is shown that $E(\mu)$ has finitely many extreme points if and only if it is affinely isomorphic to a finite Cartesian product of finite-dimensional simplices if and only if it is finite-dimensional (see Theorem 11.2). A countable counterpart of the equivalence of the first two conditions is also established (see Theorem 11.1). Moreover, a further equivalent condition expressed in purely measure-theoretic terms is found (see Theorem 11.6).

Section 12 is devoted to the cardinality $\mathfrak{m}$ of $\operatorname{extr} E(\mu)$. In general, there are no restrictions on $\mathfrak{m}$ except for $\mathfrak{m} \neq 0$. The situation changes when we set some natural restrictions on μ alone or on the triplet $\mathfrak{M}$, $\mathfrak{R}$, μ. For example, if μ is nonatomic, then $\mathfrak{m}$ is an *ω-power*, i.e., $\mathfrak{m} = \mathfrak{m}^{\aleph_0}$ (see Theorem 12.1). Moreover, if $\mathfrak{M}$ and $\mathfrak{R}$ are σ-algebras and μ is an atomic measure, then $\mathfrak{m}$ is either finite or equals $\mathfrak{c}$ or is an ω-power $\geq 2^{\mathfrak{c}}$ (see Theorem 12.8(b)).

Section 13 discusses briefly five open problems on $E(\mu)$ and $\operatorname{extr} E(\mu)$ and calls the reader's attention to the problems formulated in other sections.

The final Section 14 gathers together some auxiliary results on the convex set $\mathcal{S}(Z)$ of Radon probability measures over a compact space Z and its extreme points. Some of those results are more or less known, but I have decided to include the simple proofs or at least sketches thereof. New seems to be, however, Theorem 14.5, which characterizes the atomic elements of $\mathcal{S}(Z)$ in topological terms. What is actually needed is the following consequence of that characterization: Z is scattered if and only if the strong and weak topologies coincide on $\mathcal{S}(Z)$ (see Corollary 14.6). Up to small details (some notation and Lemma 3.1), Section 14 can be read independently of the rest of the memoir.

Some relations between the various properties of μ, $E(\mu)$ and $\operatorname{extr} E(\mu)$, discussed in the memoir, are summarized in the diagram opposite. The symbols s, w and w^* denote the strong, weak and weak* topologies of $ba(\mathfrak{R})$, respectively. The symbols $\longrightarrow$ and $\longleftrightarrow$ have the usual meaning. The accompanying numbers refer to the corresponding results in the text.

Some of my sources of inspiration have already been indicated above. I would like to mention, pars pro toto, two more: the papers [8] by D. Bogner and R. Denk, and [68] by A. Ülger. Many results of the former paper are generalized below, especially in Section 11. The latter paper considers, in a different context, some questions on closedness, discreteness and coincidence of topologies, which suggest similar questions concerning $E(\mu)$ and $\operatorname{extr} E(\mu)$, equipped with their three natural topologies. For some answers, in our context, see, e.g., Theorems 9.1, 9.4 and 9.6, and Corollary 9.2.

The main novelty of the memoir consists in a rearrangement and systematic presentation of the material of [41], [42] and [44]–[47]. Some results that are implicit in those papers have now been formulated explicitly (see, e.g., Theorem 8.2). Among the few new results are Proposition 3.6, due to H. Weber, and Corollary 5.8. The present approach differs from the original one in two respects. First, the atomic case is treated below in a less elementary way than before, involving Radon measures. The advantage is that the representation theorems obtained are now mainly in terms of $\mathcal{S}(Z)$, where Z is a compact zero-dimensional space, a standard object in abstract analysis. Second, the nonatomic case is treated below in a more elementary way than before, avoiding any use of measures. This new approach leads to slightly stronger results and, more importantly, to a better understanding of the nature of things.

Some material of [41], [42], [44] and [47] is not incorporated into the memoir. Omitted are, first of all, the results of [41] and [42] on $E_\sigma(\mu)$, the convex subset of $E(\mu)$ consisting of measures, which is defined in the case where $\mathfrak{M}$ and $\mathfrak{R}$ are σ-algebras and μ is a measure. The reason is that those results are rather fragmentary and the set $E_\sigma(\mu)$ would deserve a separate, more thorough investigation. For a similar reason, omitted are also

$$\begin{tikzcd}
(\operatorname{extr} E(\mu), w) \text{ discrete} \arrow[r, leftrightarrow, "8.4"] & (\operatorname{extr} E(\mu), s) \text{ discrete} \arrow[r, leftrightarrow, "9.4"] & \mu^{\mathrm{a}} \text{ has finite range} \arrow[d] \\
 & \operatorname{extr} E(\mu) \text{ countable} \arrow[ur, "11.1"] & \mu^{\mathrm{a}} \text{ atomic} \\
\operatorname{extr} E(\mu) \text{ finite} \arrow[uu] \arrow[ur] \arrow[dr, "11.1"] \arrow[dd, leftrightarrow, "9.2"] & & \\
 & (E(\mu), s) \text{ compact} \arrow[r] \arrow[d, shift right] & s = w \text{ on } E(\mu) \arrow[uu, "9.6"] \\
(\operatorname{extr} E(\mu), w^*) \text{ discrete} & (E(\mu), w) \text{ compact} \arrow[u, shift right, "\mu \text{ atomic}" right, "7.7"' right] \arrow[r, leftrightarrow, "4.4(a)"] & w = w^* \text{ on } E(\mu)
\end{tikzcd}$$

almost all results of [47, Section 5] on the coincidence of strong and weak* topologies on $\operatorname{extr} E(\mu)$. Besides, the memoir does not contain any results of my recent paper [48], even though it has some connections to Section 12 (see Problem 13.3 and the comments following it).

The significance of the memoir seems to go beyond its results and problems in themselves. In fact, the material presented suggests some new directions of research in related fields. This concerns, first of all, geometric functional analysis, where a parallel study of some other classes of convex sets in general (dual) Banach spaces and Banach lattices might be of interest. Among those classes are closed unit balls, their positive parts and some extreme subsets (= faces) thereof. Sample topics for consideration include affine-topological structure, cardinality and topological properties of the sets of extreme points, coincidence of various topologies. This last topic possibly deserves attention of general topologists as well.

Next, $E(\mu)$ can be interpreted as the set of positive-operator extensions of a positive operator defined on a subspace of an appropriate vector lattice. This point of view was adopted, among other papers, in [51] and [50] and led to some interesting general results. One might try a similar approach to some of the present material.

Finally, $E(\mu)$ is the core of the cooperative game $(\Omega, \mathfrak{R}, \mu_*|\mathfrak{R})$ (see, e.g., [29], [30], [62]). Thus, one might try to carry over some of the results below to the case of more general cooperative games.

It is my pleasant duty to mention here at least some of those many who contributed, in one way or another, to the coming of the memoir into existence. First and foremost, I am indebted to my teachers at Wrocław University, Professors Edward Marczewski (1907–1976) and Czesław Ryll-Nardzewski. It was their courses of lectures, seminars and papers that kindled my interest in measure theory and functional analysis. C. Ryll-Nardzewski also supervised my master and doctor theses (part of the latter is [36]). Many results of [41], [42], [44] and [45] were first presented to the seminar on measure theory conducted by Ryll-Nardzewski. The participants of the seminar are thanked for listening to my talks and commenting on them. I am also grateful to Detlef Plachky and Wolfgang Thomsen for collaboration, which resulted, in particular, in the joint papers [50] and [51]. Special thanks are due to Dieter Bierlein, Jürgen Kindler and Hans Weber for their long-standing interest in my work and inspiring contact. I also appreciate the collaboration with Viktor Losert and Jiří Spurný on the recent joint paper [49] whose main result makes it possible to round off some theorems of the memoir.

The memoir is dedicated to my mother Jadwiga and to the memory of my father Stanisław. Not only do I owe them my very existence, but also a substantial part of my personality. Being a chemical engineer, my father was, in fact, mainly interested in mathematics, especially in solving elementary problems. From my childhood on he took care of my mathematical education and eventually encouraged me to study mathematics. He strongly believed in my abilities and was happy to see the beginnings of my mathematical career.

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2. Notation and auxiliary results from outside measure theory

The set-theoretical notation we use is mostly standard. In particular, for a set Ω we denote by 2^Ω the family of all subsets of Ω and by $|\Omega|$ the cardinality of Ω. We put

$$[\Omega]^{\aleph_0} = \{E \in 2^\Omega : |E| = \aleph_0\}.$$

In Section 12 we shall tacitly use the well-known simple proposition that

$$|[\Omega]^{\aleph_0}| = |\Omega|^{\aleph_0}$$

if Ω is infinite.

Following [15, p. 452], we say that a cardinal $\mathfrak{m} \geq 1$ is an *ω-power* if $\mathfrak{m} = \mathfrak{n}^{\aleph_0}$ for some cardinal $\mathfrak{n}$ (see [12] for relevant information). The smallest ω-power > 1 is $\mathfrak{c} = 2^{\aleph_0}$. The role of ω-powers in this memoir is limited to Section 12.

Abstract Boolean algebras play some role below even though the quasi-measures we deal with are mainly defined on an algebra of sets. In particular, we use the Stone space of a Boolean algebra A and the notion of superatomicity. The Stone space of A can be identified with the set $ult(A)$ of $\{0,1\}$-valued quasi-measures on A equipped with an appropriate topology. Recall that A is said to be *superatomic* if every subalgebra of A is atomic (see [31, Proposition 17.5] or [5, Definition 5.3.4]). This notion appears in Theorems 7.9, 9.6, 11.1 and 11.6.

Our terminology concerning general topology is standard and follows Engelking's monograph [18]. In particular, by the *density character* $\mathfrak{d}(Z)$ of a topological space Z we mean the smallest cardinal $|D|$, where D is a dense subset of Z. This cardinal function appears in Theorem 8.2 and in the passage introducing Theorem 12.1.

Only Hausdorff topologies appear in this memoir, and the minimality of compact topologies among them is tacitly used in some proofs (see [18, Corollary 3.1.14] for this standard result).

We say that a subset of a topological space is *discrete* if its relative topology is discrete. This notion appears in Theorem 7.5, Corollary 9.2, Theorem 9.4 and Proposition 14.4. A topological space Z is called *scattered* if no nonempty subset of Z is dense-in-itself (see Theorems 7.9 and 9.6, and Corollary 14.6 for the use of this notion).

Theorems 8.1 and 9.7 deal with pathwise and local pathwise connectedness. It is, therefore, worth recalling that, for Hausdorff spaces, these properties are equivalent to the formally stronger properties of arcwise and local arcwise connectedness, respectively (see [18, Problem 6.3.12] or [69, Corollary 31.6]).

We note that by the product of a family of topological spaces we always mean their Cartesian product equipped with the Tychonoff topology.

Let Z be a set equipped with two topologies τ_1 and τ_2 and let Y be a subset of Z. We write

$$\tau_1 = \tau_2 \quad \text{on } Y$$

if τ_1 and τ_2 restricted to Y coincide. For $y \in Y$ we write

$$\tau_1 = \tau_2 \quad \text{at } y \text{ on } Y$$

if the identity on Y is a homeomorphism at y with respect to the restrictions of τ_1 and τ_2 to Y. (In a special case this condition is of some importance in the geometry of Banach spaces; cf. [9, p. 50].) Clearly, $\tau_1 = \tau_2$ on Y if and only if $\tau_1 = \tau_2$ at each $y \in Y$ on Y. For the use of the last two pieces of notation see Theorems 7.4, 7.7 and 7.9, Corollary 8.7, Theorems 9.6, 10.1 and 14.5, and Corollary 14.6.

Our terminology and notation concerning general functional analysis mostly follows the Dunford–Schwartz monograph [17]. We also usually refer the reader to [17] when applying standard functional-analytic results. We now introduce some additional notation which will be often used below. We denote by s, w and w^* the strong, weak and weak* topologies in the dual space X^* of a Banach space X, respectively. The last one is also called the X topology of X^* (see [17, Section V.3]). For a subset K of X^* we shall often write

$$(K,s), \quad (K,w) \quad \text{and} \quad (K,w^*),$$

meaning that K is equippped with the corresponding relative topology.

Functional-analytic tools will be used below in the context of some special Banach lattices originating from measure theory. They are mainly of the form $ba(\mathfrak{M})$, where $\mathfrak{M}$ is an algebra of sets (see Section 3 for definition). The relevant standard vector-lattice-theoretic notation follows [3] and [63]. We also apply some simple results on vector lattices (= Riesz spaces) and positive operators on them. From among those results we now present the following two lemmas to be used in the proofs of Theorems 6.1(b) and 11.2, (ii)⇒(iii). Lemma 2.2 is a finitary version of [44, Proposition 2].

2.1. Lemma (= Lemma II.2). *Let X be a vector lattice and $x_j, y_j \in X$ for $j \in J$. If $x_j \wedge y_k = 0$ whenever $j \neq k$ and*

$$\bigvee_{j\in J} x_j = \bigvee_{j\in J} y_j,$$

then $x_j = y_j$ for all j.

Proof. In view of [3, Theorem 1.5], we have

$$x_j = x_j \wedge \bigvee_{k\in J} x_k = x_j \wedge \bigvee_{k\in J} y_k = \bigvee_{k\in J} (x_j \wedge y_k) = x_j \wedge y_j.$$

For the same reason, $y_j = x_j \wedge y_j$, whence $x_j = y_j$. ■

2.2. Lemma. *Let $x_{10}, x_{11}, \ldots, x_{n0}, x_{n1}$ be nonzero pairwise disjoint elements of a vector lattice X. Then there exist $(\eta_k^j) \in \{0,1\}^n$, $j = 1, \ldots, n+1$, such that the set*

$$\Big\{\bigvee_{k=1}^{n} x_{k\eta_k^j} : j = 1, \ldots, n+1\Big\}$$

is linearly independent.

Proof. The assertion is clear for $n = 1$. Suppose it holds for some n, and denote the corresponding elements by $y_1, \ldots, y_{n+1}$. Set

$$z_j = y_j \vee x_{n+1,0}, \qquad j = 1, \ldots, n+1.$$

Since

$$x_{n+1,0} \notin \operatorname{lin}\{y_1, \ldots, y_{n+1}\},$$

the elements $z_1, \ldots, z_{n+1}$ are also linearly independent and are all disjoint from $x_{n+1,1}$. Thus, setting $z_{n+2} = y_1 \vee x_{n+1,1}$, we complete the induction argument. ■

3. Measure-theoretic preliminaries

For standard measure-theoretic results applied below we mainly refer the reader to the book [5]. In this connection we note that our terminology differs from that of [5] at some points. In particular, we use the terms algebra [σ-algebra] of sets and quasi-measure, the corresponding terms in [5] being field [σ-field] of sets and positive finite charge.

Throughout the memoir, Ω stands for a nonempty set. The algebra and σ-algebra generated by $\mathfrak{C} \subset 2^{\Omega}$ are denoted by $\mathfrak{C}_b$ and $\mathfrak{C}_\beta$, respectively. Recall that if $\mathfrak{C}$ is finite, then there is a (finite) partition of Ω which generates $\mathfrak{C}_b$ (cf. [5, Theorem 1.1.11]).

If $\mathfrak{M}$ is an algebra of subsets of Ω and $\{E_1, \ldots, E_n\}$ is a partition of Ω, then

$$(\mathfrak{M} \cup \{E_1, \ldots, E_n\})_b = \Big\{ \bigcup_{i=1}^{n} M_i \cap E_i : M_i \in \mathfrak{M} \Big\}.$$

In the case where $\mathfrak{M}$ is a σ-algebra, so is $(\mathfrak{M} \cup \{E_1, \ldots, E_n\})_b$.

If $\mathfrak{M}$ is a σ-algebra of subsets of Ω and $\{E_1, E_2, \ldots\}$ is a partition of Ω, then

$$(\mathfrak{M} \cup \{E_1, E_2, \ldots\})_\beta = \Big\{ \bigcup_{i=1}^{\infty} M_i \cap E_i : M_i \in \mathfrak{M} \Big\}.$$

Let now $\mathfrak{M}$ be an algebra of subsets of Ω, and denote by $ba(\mathfrak{M})$ the Banach lattice of all real-valued bounded additive functions on $\mathfrak{M}$ (see [5, Section 2.2]). According to [5, Theorem 2.2.1(9)], $ba(\mathfrak{M})$ is Dedekind (= boundedly) complete. As usual, $|\varphi|$ stands for the modulus of $\varphi \in ba(\mathfrak{M})$ and $\vee$ and $\wedge$ for the maximum and minimum operations in $ba(\mathfrak{M})$, respectively. With this notation, we have $\|\varphi\| = |\varphi|(\Omega)$. Moreover, $[0, \mu]$ denotes the order interval in $ba(\mathfrak{M})$ with endpoints 0 and $\mu \in ba_+(\mathfrak{M})$. Recall that for $\mu_1, \mu_2 \in ba_+(\mathfrak{M})$ we have $\mu_1 \wedge \mu_2 = 0$ if and only if for every $\varepsilon > 0$ there exists $M \in \mathfrak{M}$ with $\mu_1(M) + \mu_2(M^c) < \varepsilon$ (see [5, Theorem 2.2.1(7)]).

With $\mu \in ba_+(\mathfrak{M})$ we associate the quotient Boolean algebra $\mathfrak{M}(\mu)$ of $\mathfrak{M}$ modulo the ideal of μ-null sets. We equip $\mathfrak{M}(\mu)$ with the usual Fréchet–Nikodym μ-metric induced by the pseudometric

$$(M_1, M_2) \mapsto \mu(M_1 \triangle M_2)$$

on $\mathfrak{M}$.

Let $\mu \in ba_+(\mathfrak{M})$. Adapting a general vector-lattice-theoretic terminology (see [3, p. 36]), we say that $\nu \in ba(\mathfrak{M})$ is a *component of* μ if

$$\nu \wedge (\mu - \nu) = 0.$$

We denote by $\mathcal{U}_\mu$ the set of all components of μ which take at most two values. As easily seen (cf. [5, Proposition 5.2.2]), for different $\nu_1, \nu_2 \in \mathcal{U}_\mu$ we have $\nu_1 \wedge \nu_2 = 0$. Therefore, $\mathcal{U}_\mu$ is countable.

We say that μ is *nonatomic* provided for every $\varepsilon > 0$ there exists an $\mathfrak{M}$-partition $\{M_1, \ldots, M_n\}$ of Ω with $\mu(M_i) < \varepsilon$ for all i (see [5, Definition 5.1.4], where the term *strongly continuous* is used). We say that μ is (*purely*) *atomic* provided $\mu \wedge \nu = 0$ for every nonatomic $\nu \in ba_+(\mathfrak{M})$. According to the Sobczyk–Hammer decomposition theorem [5, Theorem 5.2.7], μ is atomic if and only if $\mu = \sum_{\nu \in \mathcal{U}_\mu} \nu$, while μ is nonatomic if $\mathcal{U}_\mu = \{0\}$. Moreover, $\mu = \mu_1 + \mu_2$, where $\mu_1, \mu_2 \in ba_+(\mathfrak{M})$, μ_1 is atomic and μ_2 is nonatomic. We shall use this decomposition in the proofs of Theorems 9.1, 9.6, 9.7 and 12.8.

For $\mu, \nu \in ba_+(\mathfrak{M})$ we write $\mu \ll \nu$ if μ is *absolutely continuous with respect to* ν, i.e., the familiar ε–δ condition holds (see [5, Definition 6.1.1]).

As usual, we associate with $\mu \in ba_+(\mathfrak{M})$ the inner and outer quasi-measures μ_* and μ^* defined, for all $E \subset \Omega$, by the formulas

$$\mu_*(E) = \sup\{\mu(M) : E \supset M \in \mathfrak{M}\},$$
$$\mu^*(E) = \inf\{\mu(M) : E \subset M \in \mathfrak{M}\}.$$

In addition to the strong topology, we shall consider $ba(\mathfrak{M})$ with its weak and weak* topologies ([2]). The weak* topology makes the evaluation mappings $\varphi \mapsto \varphi(M)$, where $M \in \mathfrak{M}$, continuous on $ba(\mathfrak{M})$. In the sequel we shall only deal with bounded subsets of $ba(\mathfrak{M})$ and we shall tacitly use the simple assertion that on such sets the weak* topology is the weakest topology for which the evaluation mappings are continuous.

We set

$$pa(\mathfrak{M}) = \{\mu \in ba_+(\mathfrak{M}) : \mu(\Omega) = 1\},$$
$$ult(\mathfrak{M}) = \{\mu \in pa(\mathfrak{M}) : \mu \text{ is two-valued}\}.$$

We note that $pa(\mathfrak{M})$ is a nonempty convex set, which is weak* compact. The latter is, in view of Example 4.1, a special case of Proposition 4.4(a).

The following well-known lemma is due to Choquet [11, p. 245]. As noted by Plachky [60, Remark 1 on Theorem 1], it is an obvious consequence of his extremality criterion [60, Theorem 1]; see also (D) in the next section. For the reader's convenience we include a direct and simple proof below.

3.1. Lemma. *We have*

$$\operatorname{extr} pa(\mathfrak{M}) = ult(\mathfrak{M}).$$

Proof. The inclusion "$\supset$" is clear since, for every $\pi \in ult(\mathfrak{M})$ and $\varrho \in ba_+(\mathfrak{M})$ with $\varrho \le \pi$, we have $\varrho = \varrho(\Omega)\pi$.

For the converse inclusion, consider $\mu \in pa(\mathfrak{M})$ with $0 < \mu(N) < 1$ for some $N \in \mathfrak{M}$. Set

$$\mu_1(M) = \frac{\mu(M \cap N)}{\mu(N)} \quad \text{and} \quad \mu_2(M) = \frac{\mu(M \cap N^c)}{\mu(N^c)} \quad \text{for } M \in \mathfrak{M}.$$

We have $\mu_1, \mu_2 \in pa(\mathfrak{M})$ and $\mu = \mu(N)\mu_1 + \mu(N^c)\mu_2$, and so $\mu \notin \operatorname{extr} pa(\mathfrak{M})$. ∎

We proceed with an elementary lemma, which will be used in the proofs of Lemma 11.4 and Theorem 11.6. Its connection with Corollary 6.4 is explained in Remark 6.5.

3.2. Lemma. *If $\mu \in ba_+(\mathfrak{M})$ has finite range and $\mu \ne 0$, then there exists a partition $\{\Omega_1, \ldots, \Omega_p\}$ of Ω consisting of μ-atoms. Set $\mu_j(M) = \mu(M \cap \Omega_j)$, $M \in \mathfrak{M}$. Then $\mu_j \in ba_+(\mathfrak{M})$ is two-valued and*

$$\mu = \sum_{j=1}^{p} \mu_j.$$

This is essentially a special case of [5, Lemma 11.1.3].

To establish Proposition 6.9 we shall need the following lemma.

3.3. Lemma (= [44, Lemma 2]). *If $\mu \in ba_+(\mathfrak{M})$ has infinite range, then there exist nonzero $\mu_j \in ba_+(\mathfrak{M})$ with $\mu_j \wedge \mu_{j'} = 0$ whenever $j \ne j'$ and $\mu = \sum_{j=1}^{\infty} \mu_j$.*

Proof. By assumption, for every $\Omega_0 \in \mathfrak{M}$, at least one of the sets

$$\{\mu(M \cap \Omega_0) : M \in \mathfrak{M}\}, \qquad \{\mu(M \cap \Omega_0^c) : M \in \mathfrak{M}\}$$

([2]) Recall that the canonical Banach-lattice predual of $ba(\mathfrak{M})$ is the Banach lattice of uniform limits of $\mathfrak{M}$-simple functions with the uniform norm and pointwise operations and order (see [5, Section 4.7] or [17, Section IV.5]).

is infinite. Therefore, we can choose, by induction, pairwise disjoint sets $\Omega_1, \Omega_2, \ldots$ in $\mathfrak{M}$ with $\mu(\Omega_j) > 0$ for every j. Set $\mu_j(M) = \mu(M \cap \Omega_j)$ for every $M \in \mathfrak{M}$ and $j = 2, 3, \ldots$, and $\mu_1 = \mu - \sum_{j=2}^{\infty} \mu_j$. ■

The next lemma will be used in the proof of Theorem 8.1. Both parts of it have been known for a long time; see, e.g., [22, Proposition (2.5), i)⇒vi)] for part (a) and Lemma I.3 for part (b). Some more references for the latter assertion are given in [41, p. 353].

3.4. Lemma. *Let $\mu \in ba_+(\mathfrak{M})$ be nonatomic.*

(a) *There exists a countable family $\mathfrak{C} \subset \mathfrak{M}$ linearly ordered by inclusion such that $\mu(\mathfrak{C})$ is dense in $[0, \mu(\Omega)]$.*
(b) *If $\mu(\Omega) > 1$, there exist $M_j \in \mathfrak{M}$, $j = 1, 2, \ldots$, with $\mu(M_j \setminus M_k) > \frac{1}{4}$ whenever $j \neq k$.*

Proof. (a): We call a finite sequence

$$\emptyset = N_1 \subset \cdots \subset N_k = \Omega$$

in $\mathfrak{M}$ an *ε-chain*, where $\varepsilon > 0$, if $\mu(N_{i+1} \setminus N_i) < \varepsilon$ for $i = 1, \ldots, k-1$. Using the nonatomicity of μ, we can define, by induction, a $\frac{1}{n}$-chain $\mathfrak{C}_n$, $n = 1, 2, \ldots$, so that $\mathfrak{C}_1 \subset \mathfrak{C}_2 \subset \cdots$. It follows that $\mathfrak{C} = \bigcup_{n=1}^{\infty} \mathfrak{C}_n$ is as desired.

(b): By induction, we can define $\Omega_{\eta_1 \ldots \eta_j} \in \mathfrak{M}$, $\eta_i = 0$ or 1; $i = 1, \ldots, j$; $j = 1, 2, \ldots$, with the following properties:

$$\Omega_{\eta_1 \ldots \eta_j 0} \cup \Omega_{\eta_1 \ldots \eta_j 1} = \Omega_{\eta_1 \ldots \eta_j},$$
$$\Omega_{\eta_1 \ldots \eta_j 0} \cap \Omega_{\eta_1 \ldots \eta_j 1} = \emptyset \quad \text{and} \quad \mu(\Omega_{\eta_1 \ldots \eta_j}) > 2^{-j}.$$

(By definition, $\Omega_\emptyset = \Omega$.) Put

$$M_j = \bigcup_{\eta_i = 0,1} \Omega_{\eta_1 \ldots \eta_{j-1} 0}, \quad j = 1, 2, \ldots.$$

We have

$$M_j \setminus M_k = \begin{cases} \displaystyle\bigcup_{\eta_i = 0,1} \Omega_{\eta_1 \ldots \eta_{k-1} 1 \eta_{k+1} \ldots \eta_{j-1} 0} & \text{if } j > k, \\ \displaystyle\bigcup_{\eta_i = 0,1} \Omega_{\eta_1 \ldots \eta_{j-1} 0 \eta_{j+1} \ldots \eta_{k-1} 1} & \text{if } j < k, \end{cases}$$

and the assertion follows. ■

We proceed with a lemma which will be needed in the proof of Proposition 3.6.

3.5. Lemma. *Let $\mu \in ba_+(\mathfrak{M})$ be nonatomic and let $0 \leq t \leq \mu(\Omega)$. Then there exists a component ν of μ with $\nu(\Omega) = t$.*

Proof. We assume $t > 0$. Using the nonatomicity of μ, we can find $M_1 \subset M_2 \subset \cdots$ in $\mathfrak{M}$ such that

$$t - \frac{1}{j} < \mu(M_j) < t, \quad j = 1, 2, \ldots.$$

Set $\nu(M) = \lim_j \mu(M \cap M_j)$ for $M \in \mathfrak{M}$. Clearly, we have $\nu \in ba_+(\mathfrak{M})$, $\nu \leq \mu$ and $\nu(\Omega) = t$. We also have

$$\nu(M_k) = \mu(M_k), \quad k = 1, 2, \ldots.$$

Hence $\nu(\Omega \setminus M_k) = t - \mu(M_k) < \frac{1}{k}$ and $(\mu - \nu)(M_k) = 0$. Consequently, $\nu \wedge (\mu - \nu) = 0$, completing the proof. ■

The next result is a finitely additive generalization of Lemma I.5 and plays a similar role below (see the proof of Theorem 8.3). It was first established by Hans Weber in 1994 (unpublished), in answer to a question of the author, who subsequently found a proof of his own. In what follows an elementary version of Weber's proof, based only on Lemma 3.5 and the Lebesgue decomposition theorem in $ba(\mathfrak{M})$, is given.

3.6. Proposition (H. Weber). *Let $\mu_1, \ldots, \mu_n \in ba_+(\mathfrak{M})$ be nonatomic. Then there exist $\nu_1, \ldots, \nu_n \in ba_+(\mathfrak{M})$ with the following properties:*

$$\nu_i \ll \mu_i, \quad \nu_i(\Omega) = \mu_i(\Omega) \quad \textit{for } i = 1, \ldots, n,$$

$$\nu_i \wedge \nu_{i'} = 0 \quad \textit{whenever } i \neq i' \quad \textit{and} \quad \sum_{i=1}^{n} \nu_i = \sum_{i=1}^{n} \mu_i.$$

Proof. We first establish the assertion in the case where $n = 2$ and $\mu_1 \ll \mu_2$. (The property $\nu_2 \ll \mu_2$ is then a consequence of the other properties of ν_1 and ν_2.) By the Lebesgue decomposition theorem (see [5, Theorem 6.2.4]), there exist $\mu_2', \mu_2'' \in ba_+(\mathfrak{M})$ with

$$\mu_2 = \mu_2' + \mu_2'', \quad \mu_2' \wedge \mu_1 = 0 \quad \text{and} \quad \mu_2'' \ll \mu_1.$$

We then have $(\mu_1 + \mu_2'') \wedge \mu_2' = 0$. In view of Lemma 3.5, there exists a component ν_1 of $\mu_1 + \mu_2''$ with $\nu_1(\Omega) = \mu_1(\Omega)$. Setting $\nu_2 = \mu_1 + \mu_2 - \nu_1$, we are done.

In the general case, we proceed by induction. The assertion is plain for $n = 1$. Suppose it holds for some n and let $\mu_1, \ldots, \mu_{n+1} \in ba_+(\mathfrak{M})$ be nonatomic. We additionally assume that $\mu_i \wedge \mu_{i'} = 0$ whenever $1 \le i, i' \le n$ and $i \neq i'$, which is legitimate, due to the induction hypothesis. Using the Lebesgue decomposition theorem consecutively n times, we get $\lambda_i \in ba_+(\mathfrak{M})$ such that

$$\lambda_i \ll \mu_i \quad \text{and} \quad \lambda_{n+1} \wedge \mu_i = 0, \quad i = 1, \ldots, n, \quad \text{and} \quad \sum_{i=1}^{n+1} \lambda_i = \mu_{n+1}.$$

By what we have proved so far, there exist $\varkappa_i, \nu_i \in ba_+(\mathfrak{M})$ such that

$$\varkappa_i \ll \lambda_i, \quad \nu_i \ll \mu_i, \quad \varkappa_i(\Omega) = \lambda_i(\Omega), \quad \nu_i(\Omega) = \mu_i(\Omega),$$
$$\varkappa_i \wedge \nu_i = 0 \quad \text{and} \quad \varkappa_i + \nu_i = \lambda_i + \mu_i,$$

$i = 1, \ldots, n$. We then have

$$\varkappa_i \ll \mu_{n+1} \quad \text{and} \quad \nu_i \wedge \lambda_{n+1} = 0 = \nu_i \wedge \sum_{j=1}^{n} \varkappa_j, \quad i = 1, \ldots, n.$$

Moreover,

$$\sum_{i=1}^{n} \nu_i = \sum_{i=1}^{n} (\lambda_i + \mu_i - \varkappa_i) = \sum_{j=1}^{n+1} \mu_j - \sum_{i=1}^{n} \varkappa_i - \lambda_{n+1} \quad \text{and} \quad \sum_{i=1}^{n} \nu_i(\Omega) = \sum_{i=1}^{n} \mu_i(\Omega).$$

Setting

$$\nu_{n+1} = \sum_{i=1}^{n} \varkappa_i + \lambda_{n+1},$$

we complete the induction procedure. ■

If $\mathfrak{M}$ is a σ-algebra of subsets of Ω, we set

$$ca(\mathfrak{M}) = \{\varphi \in ba(\mathfrak{M}) : \varphi \text{ is } \sigma\text{-additive}\}.$$

The elements of $ca_+(\mathfrak{M})$ are called *measures*.

The following lemma is due essentially to Lembcke [33, Lemma 1]; see also [37, Lemma 4]. It will be used in the proofs of Corollary 5.9 and Theorem 11.7.

3.7. Lemma. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be σ-algebras of subsets of Ω and let $\mathfrak{E}$ be a countable partition of Ω with $\mathfrak{N} = (\mathfrak{M} \cup \mathfrak{E})_\beta$. If $\varrho \in ba_+(\mathfrak{N})$ satisfies the following two conditions:*

1° *$\varrho|\mathfrak{M}$ is a measure,*
2° *$\varrho(\Omega) = \sum_{E\in\mathfrak{E}} \varrho(E)$,*

then ϱ itself is a measure.

We now introduce some more notation which will be mainly used in Sections 7 and 14. Most of it follows essentially Semadeni's monograph [63]. Let Z be a compact space. We denote by $\mathrm{CO}(Z)$ the algebra of open-and-closed subsets of Z and by $\mathfrak{B}(Z)$ the σ-algebra of Borel subsets of Z. Let $\mathcal{M}(Z)$ stand for the Banach lattice of real-valued Radon measures on Z (see [63, Section 18]). By the Riesz representation theorem, $\mathcal{M}(Z)$ can be identified with the dual of the Banach lattice $\mathcal{C}(Z)$ of real-valued continuous functions on Z. This identification yields the weak* topology of $\mathcal{M}(Z)$.

Put

$$\mathcal{S}(Z) = \{\varphi \in \mathcal{M}_+(Z) : \varphi(Z) = 1\}.$$

Clearly, $\mathcal{S}(Z)$ is convex and weak* compact, by Alaoglu's theorem. In fact, it is a simplex with the set of extreme points closed (see [63, Section 23]; cf. also Proposition 14.1).

The next proposition is known in various versions (see, in particular, [17, Lemma IV.9.11(a)], [24, Lemma 4], Proposition IV.1(a)). We shall need it in the discussion of Example 4.1 and in the proof of Theorem 7.2(b).

3.8. Proposition. *Let Z be a compact zero-dimensional space. There exists a surjective linear mapping $T : ba(\mathrm{CO}(Z)) \to \mathcal{M}(Z)$ with the following properties:*

1° *T is an isometry and a lattice homomorphism;*
2° *T is a homeomorphism with respect to the corresponding weak* topologies;*
3° *$T(pa(\mathrm{CO}(Z))) = \mathcal{S}(Z)$.*

Without going into the details of proof, we note that the canonical predual of $ba(\mathrm{CO}(Z))$ coincides with $\mathcal{C}(Z)$ whenever Z is compact and zero-dimensional. Moreover, T is defined so that

$$T(\varphi)|\mathrm{CO}(Z) = \varphi \quad \text{for } \varphi \in ba(\mathrm{CO}(Z))$$

(see [5, Section 4.7]).

4. Auxiliary results on $E(\mu)$ and $\operatorname{extr} E(\mu)$

Throughout the rest of the memoir, $\mathfrak{M}$ and $\mathfrak{N}$ stand for algebras of subsets of Ω with $\mathfrak{M} \subset \mathfrak{N}$. Occasionally, they are assumed to be σ-algebras, which is then mentioned explicitly.

Given $\mu \in ba_+(\mathfrak{M})$, we denote by $\mathfrak{J}_\mu$ the family of all $R \in \mathfrak{R}$ such that there exists $M \in \mathfrak{M}$ with

$$R \subset M \quad \text{and} \quad \mu(M) = 0.$$

Clearly, $\mathfrak{J}_\mu$ is an ideal in $\mathfrak{R}$. Moreover, we denote by $\mathfrak{M}_\mu$ the family of all $R \in \mathfrak{R}$ such that, for every $\varepsilon > 0$, there exist $M, N \in \mathfrak{M}$ with

$$M \subset R \subset N \quad \text{and} \quad \mu(N \setminus M) < \varepsilon$$

or, equivalently, $\mu_*(R) = \mu^*(R)$. As easily seen, $\mathfrak{M}_\mu$ is a subalgebra of $\mathfrak{R}$ containing $\mathfrak{M} \cup \mathfrak{J}_\mu$ (see [36, Proposition 1], where somewhat different notation is used), and $\mathfrak{M}_\mu = (\mathfrak{J}_\mu)_b$ for $\mu \in ult(\mathfrak{M})$. In the case where $\mathfrak{R} = 2^\Omega$ it is called the *Peano–Jordan completion of* $\mathfrak{M}$ *with respect to* μ in [36]. For the use of $\mathfrak{M}_\mu$ below see Lemma 4.7 and Proposition 4.8 as well as Section 11.

Set

$$E(\mu) = \{\varrho \in ba_+(\mathfrak{R}) : \varrho|\mathfrak{M} = \mu\}.$$

Occasionally, we shall use the more comprehensive notation $E(\mu, \mathfrak{R})$ instead of $E(\mu)$.

As simple as it is, the following example is basic for our study of the sets $E(\mu)$ in the case where μ is atomic. This is due to Corollary 6.4 and Proposition 7.1.

4.1. EXAMPLE. Let $\mathfrak{M} = \{\emptyset, \Omega\}$, let $\mathfrak{R}$ be an algebra of subsets of Ω and let μ be the unique probability quasi-measure on $\mathfrak{M}$. We then have $E(\mu) = pa(\mathfrak{R})$, and so $\operatorname{extr} E(\mu) = ult(\mathfrak{R})$, by Lemma 3.1. Choosing now Ω to be a compact zero-dimensional space, we find, for $\mathfrak{R} = \mathrm{CO}(\Omega)$, that

1° $(\operatorname{extr} E(\mu), w^*)$ is homeomorphic to Ω;
2° $(\operatorname{extr} E(\mu), w)$ is discrete.

Indeed, 1° is essentially part of the Stone representation theorem and is also a consequence of Propositions 14.1 and 3.8. As for 2°, see Propositions 14.4(c) and 3.8, or Lemma II.3(a) whose proof is more direct. It follows from 1° that $|\operatorname{extr} E(\mu)|$ can be an arbitrary cardinal $\mathfrak{m} \geq 1$. This is clear if $\mathfrak{m}$ is finite. In the opposite case, we can take for Ω the one-point compactification of a discrete space of cardinality $\mathfrak{m}$ (see [31, Example 17.3]).

It is a classical result that $E(\mu)$ is nonempty for arbitrary $\mathfrak{M}$, $\mathfrak{R}$ and $\mu \in ba_+(\mathfrak{M})$. It goes back to Łomnicki and Ulam [52, pp. 255–256]; see [5, Chapter 3] for related results and references. We shall use it tacitly on some occasions. According to another classical result, due essentially to Tarski [65, lemme 4] and Ulam [67], we have

$$E(\mu) \cap ult(\mathfrak{R}) \neq \emptyset \quad \text{provided } \mu \in ult(\mathfrak{M}).$$

Let $\mu \in ba_+(\mathfrak{M})$. Clearly, for $\varrho \in E(\mu)$ and $R \in \mathfrak{R}$ we have

$$\mu_*(R) \leq \varrho(R) \leq \mu^*(R).$$

Moreover, according to Łoś and Marczewski [53, Theorem 1], given $R \in \mathfrak{R}$, there exist $\varrho_1, \varrho_2 \in E(\mu)$ with

$$\varrho_1(R) = \mu_*(R) \quad \text{and} \quad \varrho_2(R) = \mu^*(R).$$

If, in addition, $\mathfrak{R} = (\mathfrak{M} \cup \{R\})_b$, then ϱ_1 and ϱ_2 as above are unique (see [53, Section 7]). This implies that ϱ_1 and ϱ_2 are then in $\operatorname{extr} E(\mu)$. It follows, by transfinite induction, that we always have

(C) $\operatorname{extr} E(\mu) \neq \emptyset$.

This result was first noticed by Plachky [59, p. 25], who deduced it from the weak* compactness of $E(\mu)$ (see Proposition 4.4(a)) and the Krein–Milman theorem. The argument above actually yields the following two strengthenings of (C):

(C)$_*$ $\min_{\varrho \in E(\mu)} \varrho(R) = \min_{\pi \in \operatorname{extr} E(\mu)} \pi(R) = \mu_*(R)$ *for* $R \in \mathfrak{R}$;

(C)* $\max_{\varrho \in E(\mu)} \varrho(R) = \max_{\pi \in \operatorname{extr} E(\mu)} \pi(R) = \mu^*(R)$ *for* $R \in \mathfrak{R}$.

For applications of (C)$_*$ and (C)* see the proofs of Lemma 4.7, Proposition 4.9 and Theorem 5.1.

The following extremality criterion is also due to Plachky ([59, Satz 1.15] and [60, Theorem 1]); see [51, Theorem 4] for a generalization with another proof. It will be one of our main tools in the sequel.

(D) *Let* $\varrho \in E(\mu)$. *Then* $\varrho \in \operatorname{extr} E(\mu)$ *if and only if for every* $R \in \mathfrak{R}$ *and every* $\varepsilon > 0$ *there exists* $M \in \mathfrak{M}$ *with* $\varrho(R \triangle M) < \varepsilon$.

Recall that one of the consequences of (D) is Lemma 3.1 above. In fact, (D) even yields the following generalization thereof, which will be applied in the proofs of Proposition 6.6, Lemma 11.4 and Theorem 12.11.

(D)$'$ *For* $\mu \in ult(\mathfrak{M})$ *we have* $\operatorname{extr} E(\mu) = E(\mu) \cap ult(\mathfrak{R})$.

We note that (D)$'$ can also be deduced from Lemma 3.1, since $E(\mu)$ is an extreme subset of $pa(\mathfrak{R})$ whenever $\mu \in ult(\mathfrak{M})$.

The next lemma is, for $n = 2$, a reformulation of the necessity part of (D). It will be used, together with Remark 4.3, in the discussion of Example 8.8.

4.2. Lemma (= Lemma I.1). *If* $\mu \in ba_+(\mathfrak{M})$ *and* $\pi \in \operatorname{extr} E(\mu)$, *then for every* $\mathfrak{R}$-*partition* $\{R_1, \ldots, R_n\}$ *of* Ω *and every* $\varepsilon > 0$ *there exists an* $\mathfrak{M}$-*partition* $\{M_1, \ldots, M_n\}$ *of* Ω *with*

$$\sum_{i=1}^{n} \pi(R_i \triangle M_i) < \varepsilon.$$

Proof. We may assume that $n \geq 2$. Put $\varepsilon_1 = \varepsilon/(2(n-1)^2)$, and choose, by (D), $N_i \in \mathfrak{M}$ with

$$\pi(R_i \triangle N_i) < \varepsilon_1, \qquad i = 1, \ldots, n-1.$$

For $i \neq j$ we have

$$N_i \cap N_j \subset (N_i \setminus R_i) \cup (N_j \setminus R_j),$$

whence $\pi(N_i \cap N_j) < 2\varepsilon_1$. Define

$$M_1 = N_1 \quad \text{and} \quad M_{i+1} = N_{i+1} \setminus \bigcup_{j=1}^{i} N_j, \quad i = 1, \ldots, n-2.$$

It follows that $\pi(R_{i+1} \vartriangle M_{i+1}) < (2i+1)\varepsilon_1$, $i = 0, \ldots, n-2$. This implies that $\sum_{i=1}^{n-1} \pi(R_i \vartriangle M_i) < \varepsilon/2$. Define $M_n = (\bigcup_{i=1}^{n-1} M_i)^c$. Since

$$R_n \vartriangle M_n \subset \bigcup_{i=1}^{n-1} (R_i \vartriangle M_i),$$

we have $\pi(R_n \vartriangle M_n) < \varepsilon/2$. The assertion follows. ■

4.3. Remark (= Remark I.1). In the setting of Lemma 4.2 we have

$$\bigcup_{i=1}^{n} (R_i \vartriangle M_i) \cup \bigcup_{i=1}^{n} (R_i \cap M_i) = \Omega.$$

The following version of (D), which first appears in [60, Remark 2 on Theorem 1], will be applied in the proof of Proposition 12.4.

(GD) *Let $\mathfrak{E} \subset 2^\Omega$ be such that $(\mathfrak{M} \cup \mathfrak{E})_b = \mathfrak{R}$ and let $\varrho \in E(\mu)$. Then $\varrho \in \operatorname{extr} E(\mu)$ if and only if for every $E \in \mathfrak{E}$ and every $\varepsilon > 0$ there exists $M \in \mathfrak{M}$ with $\varrho(E \vartriangle M) < \varepsilon$.*

This is a consequence of (D), since the family of all $R \in \mathfrak{R}$ such that for every $\varepsilon > 0$ there exists $M \in \mathfrak{M}$ with $\varrho(E \vartriangle M) < \varepsilon$ is a subalgebra of $\mathfrak{R}$.

The next result will be used on many occasions below. It coincides with Proposition I.1. Part (a) thereof was first noticed by Plachky [59, p. 24–25].

4.4. Proposition. *Let $\mu \in ba_+(\mathfrak{M})$. Then*

(a) *$(E(\mu), w^*)$ is compact;*
(b) *$\operatorname{extr} E(\mu)$ is closed in $(ba(\mathfrak{R}), s)$.*

Proof. (a): In view of Alaoglu's theorem, it is enough to check that $E(\mu)$ is weak* closed. Now, this is seen from the representation of $E(\mu)$ in the form

$$\bigcap_{M \in \mathfrak{M}} \{\varrho \in ba(\mathfrak{R}) : \varrho(M) = \mu(M)\} \cap \bigcap_{R \in \mathfrak{R}} \{\varrho \in ba(\mathfrak{R}) : \varrho(R) \geq 0\}.$$

(b): This follows by (D), since the density condition is preserved when we pass from a sequence of quasi-measures to its strong limit. ■

The following lemma coincides with Lemma II.4, but the proof of part (d) given below is more elementary (see also [60, p. 195] for that part). It will be instrumental in the proofs of Theorems 8.1(a), (b) and 8.2.

4.5. Lemma (= Lemma II.4). *Let $\mu \in ba_+(\mathfrak{M})$ and $N \in \mathfrak{M}$. Given $\varrho_1, \varrho_2 \in E(\mu)$, define*

$$\varrho(R) = \varrho_1(R \cap N) + \varrho_2(R \cap N^c) \quad \text{for } R \in \mathfrak{R}.$$

Then

(a) *$\varrho \in E(\mu)$;*
(b) *$(\varrho - \varrho_2)(R) = (\varrho_1 - \varrho_2)(R \cap N)$ for $R \in \mathfrak{R}$;*
(c) *$\|\varrho - \varrho_2\| \leq \|\varrho_1 - \varrho_2\|$;*
(d) *if $\varrho_1, \varrho_2 \in \operatorname{extr} E(\mu)$, then $\varrho \in \operatorname{extr} E(\mu)$.*

Proof. Parts (a) and (b) are obvious, and (c) is a consequence of (b). Fix ϱ_1, ϱ_2 as in (d), and set

$$\mu'(M) = \mu(M \cap N) \quad \text{and} \quad \mu''(M) = \mu(M \cap N^c) \quad \text{for } M \in \mathfrak{M},$$
$$\varrho'(R) = \varrho(R \cap N) \quad \text{and} \quad \varrho''(R) = \varrho(R \cap N^c) \quad \text{for } R \in \mathfrak{R}.$$

Clearly, we have $\mu', \mu'' \in ba_+(\mathfrak{M})$ and $\varrho', \varrho'' \in ba_+(\mathfrak{R})$. Moreover,

$$\mu = \mu' + \mu'', \quad \varrho = \varrho' + \varrho'', \quad \varrho' \in \operatorname{extr} E(\mu') \quad \text{and} \quad \varrho'' \in \operatorname{extr} E(\mu'').$$

It follows that ϱ is in $\operatorname{extr} E(\mu)$. ∎

We continue with a result which will be applied in the proof of Theorem 8.6(c). It is essentially a rather special case of [38, Theorem 3] (cf. also [27] and [32, Theorem 2]).

4.6. Proposition (= Proposition III.1). *There exists a linear mapping*

$$T\colon ba(\mathfrak{M}) \to ba(\mathfrak{R})$$

such that for all $\varphi \in ba(\mathfrak{M})$ we have

1° $T(\varphi)|\mathfrak{M} = \varphi$;
2° *given $R \in \mathfrak{R}$ and $\varepsilon > 0$, we can find an $M \in \mathfrak{M}$ with $|T(\varphi)|(R \triangle M) < \varepsilon$.*

In particular, T is an isometry and a lattice homomorphism, and

$$T(\varphi) \in \operatorname{extr} E(\varphi) \quad \textit{provided } \varphi \geq 0.$$

Proof. Theorem 3 of [38] yields an additive mapping $T\colon ba(\mathfrak{M}) \to ba(\mathfrak{R})$ with 1° and 2°. Since

$$|T(\varphi)(R) - T(\varphi)(M)| \leq |T(\varphi)|(R \triangle M),$$

it follows from 1° and 2° that for $\varphi \geq 0$ we have

$$T(\varphi) \geq 0 \quad \text{and} \quad \|T(\varphi)\| = \|\varphi\|.$$

Consequently, T is homogeneous, and $T(\varphi) \in \operatorname{extr} E(\varphi)$ provided $\varphi \geq 0$, by (D).

In view of 1° and [5], Theorem 2.2.1(7), we get that for $\varphi_1, \varphi_2 \in ba(\mathfrak{M})$

$$\varphi_1 \wedge \varphi_2 = 0 \quad \text{implies} \quad T(\varphi_1) \wedge T(\varphi_2) = 0.$$

This shows that T is a lattice homomorphism (see [3, Theorem 7.2]). It then follows that

$$\|T(\varphi)\| = \|\,|T(\varphi)|\,\| = \|T(|\varphi|)\| = \|\varphi\|$$

for $\varphi \in ba(\mathfrak{M})$, which completes the proof. ∎

The author does not know whether the mapping T of Proposition 4.6 can be, in addition, continuous with respect to the corresponding weak* topologies.

The next simple lemma serves as a preparation for introducing a decomposition of $\mu \in ba(\mathfrak{M})$ which will be important in our study of $E(\mu)$. The equivalence of (i) and (iv) thereof is well known (see, e.g., [60, Theorem 2]). That (ii) implies (i) is a consequence of Proposition 4.4(a) and the Krein–Milman theorem, but the proof given below is purely measure-theoretic.

4.7. Lemma. *For $\mu \in ba_+(\mathfrak{M})$ the following four conditions are equivalent:*

(i) $|E(\mu)| = 1$;
(ii) $|\operatorname{extr} E(\mu)| = 1$;
(iii) *for every $\nu \in [0, \mu]$ we have $|E(\nu)| = 1$;*
(iv) $\mathfrak{M}_\mu = \mathfrak{N}$.

Proof. The implications (iv)⇒(iii) ⇒ (i)⇒(ii) are clear. Suppose (iv) fails. It then follows by $(\mathrm{C})_*$ and $(\mathrm{C})^*$ that (ii) fails as well. ■

We call $\mu \in ba_+(\mathfrak{M})$ *monogenic* (*with respect to* $\mathfrak{N}$) if the equivalent conditions of Lemma 4.7 hold. (In [59, p. 52] the term *eindeutig positiv fortsetzbar* is used.) This notion is closely related to that of a monogenic operator on a vector lattice introduced in [50]. We call $\mu \in ba_+(\mathfrak{M})$ *antimonogenic* (*with respect to* $\mathfrak{N}$) if, for every monogenic $\nu \in ba_+(\mathfrak{M})$ with $\nu \le \mu$, we have $\nu = 0$.

The following result is due essentially to Plachky [59, Satz 5.5]; see also [60, Remark 2 on Theorem 2], Lemma II.1 and [44, p. 471, (CA)].

4.8. Proposition. *For every $\mu \in ba_+(\mathfrak{M})$ there exist unique $\mu^{\mathrm{m}}, \mu^{\mathrm{a}} \in ba_+(\mathfrak{M})$ with the following properties:*

1° $\mu = \mu^{\mathrm{m}} + \mu^{\mathrm{a}}$;
2° μ^{m} *is monogenic;*
3° μ^{a} *is antimonogenic.*

Moreover, $\mathfrak{M}_{\mu^{\mathrm{a}}} = \mathfrak{M}_\mu$.

Proof. Without the final part, the lemma can be derived from [50, Theorem 1] with the help of the Riesz decomposition property [3, Theorem 3.7]. Alternatively, observe that, for $\mu_1, \mu_2 \in ba_+(\mathfrak{M})$, we have

$$\mathfrak{M}_{\mu_1+\mu_2} = \mathfrak{M}_{\mu_1} \cap \mathfrak{M}_{\mu_2}.$$

This shows that $\mu_1 + \mu_2$ is monogenic provided so are μ_1 and μ_2. In this case, $\mu_1 \vee \mu_2$ is also monogenic. Thus, the set

$$\{\nu \in [0, \mu] : \nu \text{ is monogenic}\}$$

is directed upward in the Dedekind complete lattice $ba(\mathfrak{M})$. Denote by μ^{m} its supremum, and observe that μ^{m} satisfies condition (iv) of Lemma 4.7. Setting $\mu^{\mathrm{a}} = \mu - \mu^{\mathrm{m}}$, we see that 1° and 3° also hold. Using 1° and arguing as above with $\mu_1 = \mu^{\mathrm{m}}$ and $\mu_2 = \mu^{\mathrm{a}}$, we get the final part of the assertion. ■

We call μ^{m} and μ^{a} the *monogenic* and *antimonogenic components of* μ, respectively. The latter component plays a major role below, starting with Corollary 6.2. We note that μ^{m} can also be defined by a formula involving $\mu_*|\mathfrak{N}$ (see [60, Remark 2 on Theorem 2]).

For $\mu \in ult(\mathfrak{M})$ we have $\mu = \mu^{\mathrm{m}}$ or $\mu = \mu^{\mathrm{a}}$, i.e., μ is either monogenic or antimonogenic. Therefore, the decomposition of Proposition 4.8 is straightforward for atomic μ.

The next proposition will be used in Examples 7.11 and 12.7. A part of it is a version of [45, Proposition 3]. Another part of it provides natural examples of antimonogenic quasi-measures in a classical setting.

4.9. Proposition. *Let $\mu \in ba_+(\mathfrak{M})$ and let $\mathfrak{R} = (\mathfrak{M} \cup \{E\})_b$, where $E \subset \Omega$.*

(a) *If $\mathfrak{M}(\mu)$ is metrically complete, then $|\operatorname{extr} E(\mu)| \le |\mathfrak{M}(\mu)|$.*

Suppose, in addition, that $\mu_*(E) = \mu_*(E^c) = 0$. Then

(b) *μ is antimonogenic;*
(c) *$|\operatorname{extr} E(\mu)| \ge |\mathfrak{M}(\mu)|$;*
(d) *if $\mathfrak{M}$ is a σ-algebra and μ is a measure, then $|\operatorname{extr} E(\mu)| = |\mathfrak{M}(\mu)|$.*

Proof. We first establish the following equivalence. Given $\pi_1, \pi_2 \in E(\mu)$ and $N_1, N_2 \in \mathfrak{M}$ with $\pi_i(E \triangle N_i) = 0$, $i = 1, 2$, we have

$$\pi_1 = \pi_2 \quad \text{if and only if} \quad \mu(N_1 \triangle N_2) = 0.$$

Indeed, if $\pi_1 = \pi_2$, then

$$\mu(N_1 \triangle N_2) = \pi_1(E \triangle N_2) = \pi_2(E \triangle N_2) = 0.$$

The "if" part follows from the equality

$$\pi_i((M_1 \cap E) \cup (M_2 \cap E^c)) = \mu((M_1 \cap N_i) \cup (M_2 \cap N_i^c)),$$

where $M_1, M_2 \in \mathfrak{M}$ and $i = 1, 2$.

(a): Given $\pi \in \operatorname{extr} E(\mu)$, there exists, by assumption and (D), $N_\pi \in \mathfrak{M}$ with $\pi(E \triangle N_\pi) = 0$. Thus, in view of the equivalence above, the mapping

$$\operatorname{extr} E(\mu) \ni \pi \mapsto N_\pi \in \mathfrak{M}$$

induces an injection of $\operatorname{extr} E(\mu)$ into $\mathfrak{M}(\mu)$.

(b): Let $\nu \in [0, \mu]$ be monogenic. Condition (iv) of Lemma 4.7 then shows that

$$\nu_*(E) = \nu^*(E) \quad \text{and} \quad \nu_*(E^c) = \nu^*(E^c).$$

It follows that $\nu^*(E) = \nu^*(E^c) = 0$, whence $\nu = 0$.

(c): Given $N \in \mathfrak{M}$, we have $\mu_*(E \triangle N) = 0$. Indeed, if $M \in \mathfrak{M}$ and $M \subset E \triangle N$, then $M \cap N \subset E^c$ and $M \cap N^c \subset E$, whence $\mu(M) = 0$. By $(\mathrm{C})_*$, there exists $\pi_N \in \operatorname{extr} E(\mu)$ with $\pi_N(E \triangle N) = 0$. In view of the equivalence established at the beginning of the proof, the mapping

$$\mathfrak{M} \ni N \mapsto \pi_N \in \operatorname{extr} E(\mu)$$

induces an injection of $\mathfrak{M}(\mu)$ into $\operatorname{extr} E(\mu)$.

(d): In view of a classical result, this is a consequence of (a) and (c). ∎

5. Weak compactness of $E(\mu)$

The material of this section is mainly a development of [47, Section 6]. We start by recalling that a real-valued function η on $\mathfrak{M}$ is said to be *exhaustive* or *strongly bounded* if $\eta(M_j) \to 0$ for every sequence (M_j) in $\mathfrak{M}$ with $M_j \cap M_{j'} = \emptyset$ whenever $j \neq j'$. Moreover, η is said to be *order continuous* (*at* $\emptyset$) if $\eta(M_j) \to 0$ for every decreasing sequence (M_j) in $\mathfrak{M}$ with $\bigcap_{j=1}^{\infty} M_j = \emptyset$.

The equivalence of conditions (i) and (iii) in the following theorem is also the contents of Theorem IV.5. That (ii) implies (i) is a direct consequence of Proposition 4.4(a) and Proposition III.2.

5.1. Theorem. *For $\mu \in ba_+(\mathfrak{M})$ the following three conditions are equivalent:*

(i) $(E(\mu), w)$ *is compact;*
(ii) $\operatorname{extr} E(\mu)$ *is relatively compact in* $(ba(\mathfrak{R}), w)$*;*
(iii) $\mu^*|\mathfrak{R}$ *is exhaustive.*

Proof. Observe that $E(\mu)$ is weakly closed (cf. the proof of Proposition 4.4(a)). Thus, the assertion is a consequence of (C)* and the following well-known criterion: A subset K of $ba(\mathfrak{R})$ is relatively weakly compact if and only if K is bounded and the function η_K defined by

$$\eta_K(R) = \sup\{|\varphi(R)| : \varphi \in K\}, \qquad R \in \mathfrak{R},$$

is exhaustive (see [10, Theorem], [24, Theorem 2] or [17, Theorem IV.9.12] and [13, Corollary I.5.4]). ∎

We shall give two simplest possible examples of $E(\mu)$ which are not weakly compact. In the first example μ is two-valued while in the second one μ is nonatomic.

5.2. Example (cf. Example I.1). We take up Example 4.1 and suppose $\mathfrak{R}$ is infinite. We can then find a sequence (R_j) in $\mathfrak{R}$ with $R_j \cap R_{j'} = \emptyset$ whenever $j \neq j'$. We have $\mu^*(R_j) = 1$ for all $j \in \mathbb{N}$. Therefore, $E(\mu)$ is not weakly compact, by Theorem 5.1, (i)⇒(iii). This is also seen from Proposition 14.4(d) combined with Proposition 3.8.

5.3. Example (cf. Example I.2). Let $\Omega = [0, 1)$, and set

$$\mathfrak{M} = \{[a, b) : 0 \leq a < b < 1 \text{ and } a, b \text{ are rational}\}_b.$$

Let μ be the Lebesgue measure restricted to $\mathfrak{M}$. Let, further, E_j, $j \in \mathbb{N}$, be pairwise disjoint dense subsets of Ω, and define

$$\mathfrak{R} = (\mathfrak{M} \cup \{E_1, E_2, \ldots\})_b.$$

We then have $\mu^*(E_j) = 1$ for all $j \in \mathbb{N}$, and so $E(\mu)$ is not weakly compact, by Theorem 5.1, (i)⇒(iii).

The next result is contained in Theorem I.1(a). In the special case where $\mathfrak{M}$ is a σ-algebra (and so $\mathfrak{R}$ is a σ-algebra and $E(\mu) \subset ca(\mathfrak{R})$, for the latter see Theorem 5.5, (i)⇒(iv)), it is due to Plachky [59, Satz 3.3].

5.4. Corollary. *Let $\mathfrak{R} = (\mathfrak{M} \cup \{E_1, \ldots, E_n\})_b$, where $\{E_1, \ldots, E_n\}$ is a partition of Ω, and let $\mu \in ba_+(\mathfrak{M})$. Then $(E(\mu), w)$ is compact.*

Proof. We shall verify condition (iii) of Theorem 5.1. To this end, let (R_k) be a sequence in $\mathfrak{R}$ with $R_k \cap R_{k'} = \emptyset$ whenever $k \neq k'$. We have

$$R_k = \bigcup_{i=1}^{n} M_i^k \cap E_i, \qquad \text{where } M_i^k \in \mathfrak{M},\ i = 1, \ldots, n;\ k = 1, 2, \ldots.$$

Set $N_i^1 = M_i^1$ and $N_i^k = M_i^k \setminus \bigcup_{j=1}^{k-1} M_i^j$, $i = 1, \ldots, n$; $k = 2, 3, \ldots$. With this notation, we have

$$R_k = \bigcup_{i=1}^{n} N_i^k \cap E_i \qquad \text{for all } k.$$

Since $\mu(N_i^k) \to 0$ when $k \to \infty$, it follows that $\mu^*(R_k) \to 0$, completing the argument. ∎

The following result coincides, up to condition (ii), which is new, with Theorem IV.6. Parts of that result were previously known. Namely, the equivalence of (iii) and (iv) is due, in a more general situation to Schmeidler [62, Theorem 3.2]; see also [30, Example 2]. Moreover, (v) implies (iv), by [8, Lemma 1] and Proposition 4.4(a), but the proof given below is more elementary. We also note that the equivalence of (i) and (iv) is used in establishing Corollary 10.5. For a special case of the implication (ii)⇒(iv) see Theorem 7.10(a). Condition (iv) also appears in Theorems 11.7 and 12.11.

5.5. THEOREM. *Let $\mathfrak{M}$ and $\mathfrak{R}$ be σ-algebras of subsets of Ω with $\mathfrak{M} \subset \mathfrak{R}$ and let $\mu \in ca_+(\mathfrak{M})$. Then the following five conditions are equivalent:*

(i) $(E(\mu), w)$ *is compact;*
(ii) $\operatorname{extr} E(\mu)$ *is relatively compact in* $(ba(\mathfrak{R}), w)$*;*
(iii) $\mu^*|\mathfrak{R}$ *is order continuous;*
(iv) $E(\mu) \subset ca(\mathfrak{R})$*;*
(v) $\operatorname{extr} E(\mu) \subset ca(\mathfrak{R})$.

Proof. As μ^* is isotone and σ-subadditive, condition (iii) is equivalent to condition (iii) of Theorem 5.1, by [61, Lemma 4.1] (see also [16, Theorem 5.3]). The equivalence of conditions (i)–(iii) is now seen, due to the equivalence of the corresponding conditions of Theorem 5.1.

Since $\varrho \le \mu^*|\mathfrak{R}$ for each $\varrho \in E(\mu)$, (iii) implies (iv). Clearly, (iv) implies (v).

We shall complete the proof by showing that (v) implies (iii). Fix a decreasing sequence (R_j) in $\mathfrak{R}$ with $\bigcap_{j=1}^{\infty} R_j = \emptyset$, and set

$$\mathfrak{R}_1 = (\mathfrak{M} \cup \{R_1, R_2, \ldots\})_b.$$

By [37, Theorem 1], there exists a (unique) $\pi_1 \in E(\mu, \mathfrak{R}_1)$ with $\pi_1(R_j) = \mu^*(R_j)$ for each j. Moreover, in view of [37, Remark 1], we have $\pi_1 \in \operatorname{extr} E(\mu, \mathfrak{R}_1)$. Take $\pi \in \operatorname{extr} E(\pi_1, \mathfrak{R})$ (see (C)). Then π is in $\operatorname{extr} E(\mu)$, and so, by (v), $\pi(R_j) \to 0$. Hence $\mu^*(R_j) \to 0$. ∎

5.6. REMARK (= Remark IV.4). The implication (iv)⇒(i) of Theorem 5.5 is also a consequence of [4, Satz 2.3] (see also [21, Proposition 2.13], or [70, Theorem 1.1]) and Proposition 4.4(a).

5.7. REMARK. In the setting of Theorem 5.5, the equivalent conditions (i)–(v) do not imply the strong compactness of $E(\mu)$. This is seen from Example 12.7.

5.8. COROLLARY. *Let $\mathfrak{M}$ be a σ-algebra and let $\mathfrak{R} = (\mathfrak{M} \cup \{E_1, E_2, \ldots\})_\beta$, where $\{E_1, E_2, \ldots\}$ is a partition of Ω. Then for $\mu \in ca_+(\mathfrak{M})$ the following two conditions are equivalent:*

(i) $(E(\mu), w)$ *is compact;*
(ii) $\mu^* \left(\bigcup_{i=n}^{\infty} E_i\right) \to 0$ *when* $n \to \infty$.

Proof. Clearly, (i) implies (ii), by Theorem 5.5, (i)⇒(iii). To establish the converse, we shall show that (ii) implies condition (iv) of Theorem 5.5. To this end, consider $\varrho \in E(\mu)$.

Since $\varrho \le \mu^*|\mathfrak{R}$, we infer from (ii) that

$$\varrho\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} \varrho(E_i).$$

An appeal to Lemma 3.7 completes the proof. ■

For other results involving weak compactness of $E(\mu)$ see Theorems 7.7 and 8.6(b), Corollary 12.2, Theorem IV.4, and [48, Corollary 1 and Remark 5].

6. The main product theorem

The following theorem coincides, up to a minor detail, with Theorem II.1, which has a predecessor in Lemma I.2. The surjectivity assertion of part (a) is related to [58, Theorem 4 and Corollary 5]. To obtain it, we use an argument from [39, proof of Theorem 3]. The inclusion "$\subset$" of part (b) is essentially a special case of [34, Lemma 2.1]. Most of the consequences of Theorem 6.1 established after its proof will play a major role throughout the rest of the memoir.

6.1. THEOREM. *Suppose $\mu,\mu_j \in ba_+(\mathfrak{M})$ and $\sum_{j=1}^{\infty} \mu_j = \mu$, and define*

$$T : \prod_{j=1}^{\infty} E(\mu_j) \to E(\mu)$$

by the formula $T((\varrho_j)) = \sum_{j=1}^{\infty} \varrho_j$. Then

(a) *T is an affine surjective mapping, which is continuous with respect to the corresponding strong [weak] [weak*] topologies, and*

$$T^{-1}(\operatorname{extr} E(\mu)) \subset \prod_{j=1}^{\infty} \operatorname{extr} E(\mu_j);$$

(b) *T is a homeomorphism with respect to each of these three topologies and*

$$T\Big(\prod_{j=1}^{\infty} \operatorname{extr} E(\mu_j)\Big) = \operatorname{extr} E(\mu)$$

provided $\mu_j \wedge \mu_{j'} = 0$ whenever $j \ne j'$.

Proof. (a): Clearly, T is affine. To prove the surjectivity assertion, fix ϱ in $E(\mu)$. We shall define, by induction, $\varrho_j \in E(\mu_j)$ with

$$\sum_{j=1}^{n} \varrho_j \le \varrho \quad \text{for } n = 1, 2, \ldots.$$

By Kelley's theorem ([28, Theorem 14]; see also [29, Example 13]), there exists $\varrho_1 \in E(\mu_1)$ with $\varrho_1 \le \varrho$. Suppose $\varrho_1, \ldots, \varrho_n$ with the desired properties have already been defined. Since

$$\mu_{n+1} \le \mu - \sum_{j=1}^{n} \mu_j = \Big(\varrho - \sum_{j=1}^{n} \varrho_j\Big)\Big|\mathfrak{M},$$

by the same argument, there exists $\varrho_{n+1} \in E(\mu_{n+1})$ with $\varrho_{n+1} \le \varrho - \sum_{j=1}^{n} \varrho_j$. It follows that $\sum_{j=1}^{\infty} \varrho_j \le \varrho$. As $\varrho_j \in ba_+(\mathfrak{R})$ and $\sum_{j=1}^{\infty} \varrho_j(\Omega) = \varrho(\Omega)$, we conclude that $T((\varrho_j)) = \varrho$.

To prove that T is continuous with respect to the corresponding weak* topologies, fix $\varrho_{j\alpha}, \varrho_j \in E(\mu)$, where α runs through a directed set A, with $\varrho_{j\alpha} \to \varrho_j$ weak* for all j. Given $\varepsilon > 0$ and $R \in \mathfrak{R}$, we can find j_0 with $\sum_{j=j_0+1}^{\infty} \mu_j(\Omega) < \varepsilon/4$, and then $\alpha_0 \in A$ with

$$|\varrho_{j\alpha}(R) - \varrho_j(R)| < \frac{\varepsilon}{2j_0}, \qquad j = 1, \dots, j_0;\ \alpha \ge \alpha_0.$$

It follows that

$$\Big|\sum_{j=1}^{\infty} \varrho_{j\alpha}(R) - \sum_{j=1}^{\infty} \varrho_j(R)\Big| \le \sum_{j=1}^{j_0} |\varrho_{j\alpha}(R) - \varrho_j(R)| + \sum_{j=j_0+1}^{\infty} (\varrho_{j\alpha}(R) + \varrho_j(R)) < \varepsilon$$

for all $\alpha \ge \alpha_0$.

The proof of the remaining two continuity assertions is similar. One has only to note that for $\varrho_{j\alpha}$ and ϱ_j as above we have $\|\varrho_{j\alpha}\| = \|\varrho_j\| = \mu_j(\Omega)$. The final assertion of (a) is a consequence of (D).

(b): Since $\mu_j \wedge \mu_{j'} = 0$ implies $\varrho_j \wedge \varrho_{j'} = 0$ whenever $\varrho_j \in E(\mu_j)$ and $\varrho_{j'} \in E(\mu_{j'})$ (see [5, Theorem 2.2.1(7)]), the injectivity of T is a consequence of Lemma 2.1.

That T is a homeomorphism with respect to the corresponding weak* topologies now follows by (a). Indeed, in view of Proposition 4.4(a) and Tychonoff's product theorem, the domain of T is compact.

We shall now show that T^{-1} is also continuous with respect to the corresponding strong [weak] topologies. To this end, denote by P_j the order projection on the band in $ba(\mathfrak{R})$ generated by $E(\mu_j)$ (see [3, Theorem 3.8]). We claim that $T^{-1} = (P_j|E(\mu))$. Indeed, fix $\varrho \in E(\mu)$ and let $T^{-1}(\varrho) = (\varrho_j)$. Since $P_j(\varrho_k) = 0$ whenever $j \ne k$ and $P_j(\varrho_j) = \varrho_j$, we have

$$P_j(\varrho) = P_j\Big(\sum_{k=1}^{\infty} \varrho_k\Big) = \varrho_j$$

for all j. The desired continuity of T^{-1} follows from the claim, as the P_j are strongly continuous, and so weakly continuous, by [17, Theorem V.3.15].

To complete the proof, we only need to observe that

$$\operatorname{extr} \prod_{j=1}^{\infty} E(\mu_j) = \prod_{j=1}^{\infty} \operatorname{extr} E(\mu_j). \ \blacksquare$$

The next result is a direct consequence of Theorem 6.1(a) and Proposition 4.8. It shows that when studying the affine-topological properties of $E(\mu)$ we may always assume that μ is antimonogenic.

6.2. Corollary (= [44, p. 471, (T) and (T′)]). *For $\mu \in ba_+(\mathfrak{M})$ we have*

(a) $E(\mu^{\mathrm{a}})$ *is a translate of* $E(\mu)$*;*
(b) $\operatorname{extr} E(\mu^{\mathrm{a}})$ *is a translate of* $\operatorname{extr} E(\mu)$*.*

From Theorem 6.1(b) we get the following two corollaries.

6.3. Corollary. *If $\mu_1 \in ba_+(\mathfrak{M})$ is atomic and $\mu_2 \in ba_+(\mathfrak{M})$ is nonatomic, then there exists an affine isomorphism of $E(\mu_1+\mu_2)$ onto $E(\mu_1)\times E(\mu_2)$, which is a homeomorphism with respect to the corresponding strong [weak] [weak*] topologies.*

6.4. Corollary. *If $\mu \in ba_+(\mathfrak{M})$ is atomic, then there exists an affine isomorphism of $E(\mu)$ onto $\prod_{\nu\in\mathcal{U}_\mu} E(\nu)$, which is a homeomorphism with respect to the corresponding strong [weak] [weak*] topologies.*

6.5. Remark (cf. [44, p. 471, (FD)]). We shall often use Corollary 6.4 in the special case where $\mathcal{U}_\mu$ is finite. We note that, in that case, it can be established in a much simpler way than Theorem 6.1. Indeed, it is enough to apply Lemma 3.2 and the property that, for every $S \in \mathfrak{R}$, the mapping

$$ba(\mathfrak{R}) \ni \varphi \mapsto \varphi_S \in ba(\mathfrak{R}), \quad \text{where} \quad \varphi_S(R) = \varphi(R\cap S) \quad \text{for } R \in \mathfrak{R},$$

is continuous with respect to each of the three topologies of $ba(\mathfrak{R})$ under consideration.

The following result will be used in the proof of Theorem 11.6.

6.6. Proposition. *Let $\mathfrak{N}$ be a subalgebra of $\mathfrak{R}$ with $\mathfrak{R} = (\mathfrak{M}\cup\mathfrak{N})_b$ and let $\mu \in ba_+(\mathfrak{M})$ be atomic. Then*

$$|\operatorname{extr} E(\mu)| \le |ult(\mathfrak{N})|^{|\mathcal{U}_\mu|}.$$

Proof. In view of Corollary 6.4, it suffices to show the assertion for $\mu \in ult(\mathfrak{M})$. In this case,

$$\operatorname{extr} E(\mu) \subset ult(\mathfrak{R}),$$

by (D)$'$. Fix $\pi_1, \pi_2 \in \operatorname{extr} E(\mu)$ with $\pi_1|\mathfrak{N} = \pi_2|\mathfrak{N}$. We claim that $\pi_1 = \pi_2$. Indeed, let $R \in \mathfrak{R}$, so that

$$R = \bigcup_{i=1}^{n} M_i \cap N_i \quad \text{for some } n, \text{ and } M_i \in \mathfrak{M} \text{ and } N_i \in \mathfrak{N},\ i = 1,\dots,n.$$

Suppose $\pi_1(R) = 0$. This implies that, for each i, either $\pi_1(M_i) = 0$ or $\pi_1(N_i) = 0$. Hence $\pi_2(R) = 0$, and the claim follows. Thus, the mapping $\pi \mapsto \pi|\mathfrak{N}$ of $\operatorname{extr} E(\mu)$ into $ult(\mathfrak{N})$ is injective, and we are done. ■

By Proposition 12.4, the inequality of Proposition 6.6 turns into equality provided μ is in $ult(\mathfrak{M})$ and $\mathfrak{N}$ satisfies an additional condition.

By another application of Theorem 6.1(b), we shall show that the class of convex sets $E(\mu)$, where $\mathfrak{M}$ and $\mathfrak{R}$ are arbitrary, and μ is arbitrary [atomic] [nonatomic], equipped with any of their three standard topologies, is closed under countable Cartesian products.

6.7. Proposition (cf. Proposition IV.2). *Let $\mathfrak{M}_j$ and $\mathfrak{R}_j$ be algebras of subsets of a set Ω_j with $\mathfrak{M}_j \subset \mathfrak{R}_j$ and let $\mu_j \in ba_+(\mathfrak{M}_j)$, $j = 1,2,\dots$. Then there exist algebras $\mathfrak{M}$ and $\mathfrak{R}$ of subsets of a set Ω with $\mathfrak{M} \subset \mathfrak{R}$, $\mu \in ba_+(\mathfrak{M})$ and an affine isomorphism of $E(\mu)$ onto $\prod_{j=1}^\infty E(\mu_j)$, which is a homeomorphism with respect to the corresponding strong [weak] [weak*] topologies. If μ_j are all atomic [nonatomic], then μ can also be chosen atomic [nonatomic].*

Proof. We assume, without loss of generality, that $\Omega_1, \Omega_2, \dots$ are pairwise disjoint and $\sum_{j=1}^\infty \mu_j(\Omega_j) < \infty$. Set $\Omega = \bigcup_{j=1}^\infty \Omega_j$, and let $\mathfrak{M}$ and $\mathfrak{R}$ stand for algebras of subsets of

Ω generated by $\bigcup_{j=1}^{\infty}\mathfrak{M}_j$ and $\bigcup_{j=1}^{\infty}\mathfrak{R}_j$, respectively. Define

$$\tilde{\mu}_j(M)=\mu_j(M\cap\Omega_j)\quad\text{ for } M\in\mathfrak{M} \text{ and } j=1,2,\ldots.$$

Clearly, $\tilde{\mu}_j\in ba_+(\mathfrak{M})$ and there exists a canonical affine isomorphism of $E(\mu_j)$ onto $E(\tilde{\mu}_j)$, which is a homeomorphism with respect to the corresponding strong [weak] [weak*] topologies. Set $\mu=\sum_{j=1}^{\infty}\tilde{\mu}_j$. Then $\mu\in ba_+(\mathfrak{M})$ and μ is atomic [nonatomic] if (and only if) μ_j are all atomic [nonatomic]. The assertion now follows by Theorem 6.1(b). ∎

It is a simple consequence of Proposition 6.7 that the class of convex sets $E(\mu)$ which are strongly compact [weakly compact] [satisfy the condition that $s=w$ on $E(\mu)$] is closed under countable Cartesian products.

The following result is analogous to Proposition 6.7. We omit its simple proof, which is an obvious modification of that of Proposition 6.7.

6.8. Proposition. *Let $\mathfrak{M}_j$ and $\mathfrak{R}_j$ be algebras of subsets of a set Ω_j with $\mathfrak{M}_j\subset\mathfrak{R}_j$ and let $\mu_j\in ba_+(\mathfrak{M}_j)$ have finite range, $j=1,\ldots,p$. Then there exist algebras $\mathfrak{M}$ and $\mathfrak{R}$ of subsets of a set Ω with $\mathfrak{M}\subset\mathfrak{R}$, $\mu\in ba_+(\mathfrak{M})$ having finite range and an affine isomorphism of $E(\mu)$ onto $\prod_{j=1}^{p}E(\mu_j)$, which is a homeomorphism with respect to the corresponding strong [weak] [weak*] topologies.*

The next result is an immediate consequence of Lemma 3.3, Theorem 6.1(b) and Corollary 6.2(b). It will be used in the proof of Theorem 11.1.

6.9. Proposition (= [44, Proposition 1]). *If $\mu\in ba_+(\mathfrak{M})$ and μ^{a} has infinite range, then $|\operatorname{extr}E(\mu)|\geq\mathfrak{c}$.*

It is also worth-while to note the following direct consequence of Theorem 6.1(b) and (D) although it will not be used in the sequel.

6.10. Proposition (cf. Theorem I.2(a)). *If $\mu\in ba_+(\mathfrak{M})$ is atomic, then each element of* $\operatorname{extr}E(\mu)$ *is atomic.*

7. $E(\mu)$ for atomic μ

The main result of this section, Theorem 7.2, is an affine-topological description of $E(\mu)$, equipped with its three standard topologies, for atomic μ. We start with the special case where $\mu\in ult(\mathfrak{M})$. The following proposition is essentially a combination of Lemmas III.2 and IV.2. It includes [45, Proposition 1] and is related to [44, Theorem 3(a)].

7.1. Proposition. *Let $\mu\in ult(\mathfrak{M})$ and let Z denote the Stone space of $\mathfrak{R}/\mathfrak{J}_\mu$. Then there exists a linear mapping $T:ba(\mathrm{CO}(Z))\to ba(\mathfrak{R})$ with the following properties:*

1° *T is an isometry and a lattice homomorphism;*
2° *$T(pa(\mathrm{CO}(Z)))=E(\mu)$;*
3° *$T|pa(\mathrm{CO}(Z))$ is a homeomorphism with respect to the corresponding weak [weak*] topologies;*
4° $|\operatorname{extr}E(\mu)|=|Z|$.

Proof. Set $\tilde{\mathfrak{R}} = \mathrm{CO}(Z)$, and let g stand for a Boolean isomorphism from $\mathfrak{R}/\mathfrak{J}_\mu$ onto $\tilde{\mathfrak{R}}$. Denote by h the canonical mapping from $\mathfrak{R}$ onto $\mathfrak{R}/\mathfrak{J}_\mu$. Set

$$T(\psi)(R) = \psi(g \circ h(R)) \quad \text{for } \psi \in ba(\tilde{\mathfrak{R}}) \text{ and } R \in \mathfrak{R}.$$

Clearly, $T(\psi) \in ba(\mathfrak{R})$, T is linear and $\|T(\psi)\| = \|\psi\|$ provided $\psi \geq 0$. Moreover, we have

$$T(\psi)|\mathfrak{J}_\mu = 0, \quad \text{whence} \quad T(pa(\tilde{\mathfrak{R}})) \subset E(\mu).$$

Conversely, suppose $\varrho \in E(\mu)$. Then ϱ defines a (positive additive) function $\tilde{\varrho}$ on $\mathfrak{R}/\mathfrak{J}_\mu$ with

$$\tilde{\varrho}(h(R)) = \varrho(R) \quad \text{for all } R \in \mathfrak{R}.$$

Set $\tau(P) = \tilde{\varrho}(g^{-1}(P))$ for $P \in \tilde{\mathfrak{R}}$. We have

$$\tau \in pa(\tilde{\mathfrak{R}}) \quad \text{and} \quad T(\tau) = \varrho.$$

Thus, 2° is established.

As easily seen, in view of [5, Theorem 2.2.1(7)], for $\psi_1, \psi_2 \in ba(\tilde{\mathfrak{R}})$ we have

$$\psi_1 \wedge \psi_2 = 0 \quad \text{implies} \quad T(\psi_1) \wedge T(\psi_2) = 0.$$

This shows that T is a lattice homomorphism (see [3, Theorem 7.2]). It then follows that

$$\|T(\psi)\| = \|\,|T(\psi)|\,\| = \|T(|\psi|)\| = \|\psi\|$$

for $\psi \in ba(\tilde{\mathfrak{R}})$, completing the proof of 1°.

The first part of 3° follows from 1° and [17, Theorem V.3.15]. The definition of T shows that it is continuous with respect to the topologies of pointwise convergence on $ba(\tilde{\mathfrak{R}})$ and $ba(\mathfrak{R})$. Since $pa(\tilde{\mathfrak{R}})$ and $E(\mu)$ are both weak* compact (see Proposition 4.4(a) and Example 4.1), 2° shows that the second part of 3° also holds.

Finally, 1° and 2° imply, by Lemma 3.1, that T is injective and $T(ult(\tilde{\mathfrak{R}})) = \operatorname{extr} E(\mu)$. This yields 4°. ∎

7.2. Theorem (= Theorem IV.1). *Let $\mu \in ba_+(\mathfrak{M})$ be atomic and let Z_ν denote the Stone space of $\mathfrak{R}/\mathfrak{J}_\nu$ for $\nu \in \mathcal{U}_\mu$.*

(a) *There exists an affine isomorphism of $E(\mu)$ onto*

$$\prod_{\nu \in \mathcal{U}_\mu} pa(\mathrm{CO}(Z_\nu)),$$

which is a homeomorphism with respect to the corresponding strong [weak] [weak] topologies.*

(b) *There exists an affine isomorphism of $E(\mu)$ onto*

$$\prod_{\nu \in \mathcal{U}_\mu} \mathcal{S}(Z_\nu),$$

which is a homeomorphism with respect to the corresponding strong [weak] [weak] topologies.*

Proof. Part (a) is a direct consequence of Corollary 6.4 and Proposition 7.1. Part (b) follows from (a) and Proposition 3.8. ■

7.3. Remark. Theorem 7.2(a) gives a complete affine-topological description of the sets $E(\mu)$, equipped with each of the three topologies under consideration, for atomic μ. Indeed, suppose $\mathfrak{N}_j$ is an algebra of subsets of a set Ω_j, $j = 1, 2, \ldots$. Then there exist algebras $\mathfrak{M}$ and $\mathfrak{R}$ of subsets of a set Ω, an atomic $\mu \in ba_+(\mathfrak{M})$ and an affine isomorphism of $E(\mu)$ onto $\prod_{j=1}^{\infty} pa(\mathfrak{N}_j)$, which is a homeomorphism with respect to the corresponding strong [weak] [weak*] topologies. This is seen from Proposition 6.7 and Example 4.1.

In view of Proposition 3.8, Remark 7.3 yields a similar assertion concerning Theorem 7.2(b).

For some converses of the next two results see Theorems 9.1 and 9.4, respectively.

7.4. Theorem (cf. Theorem II.2). *Let $\mu \in ba_+(\mathfrak{M})$ be atomic. Then*

(a) $s = w$ *on* $\operatorname{extr} E(\mu)$*;*
(b) $(\operatorname{extr} E(\mu), w)$ *is homeomorphic to a countable product of discrete spaces;*
(c) $(\operatorname{extr} E(\mu), w^*)$ *is compact and zero-dimensional.*

This is a consequence of Theorem 7.2(b) combined with Propositions 14.1 and 14.4(a), (c). See the proof of Theorem II.2 for a more direct argument.

7.5. Theorem (cf. Theorem II.3(a)). *Let $\mu \in ba_+(\mathfrak{M})$ have finite range. Then* $(\operatorname{extr} E(\mu), w)$ *is discrete.*

This is a consequence of Theorem 7.2(b) combined with Lemma 3.2 and Proposition 14.4(c). See the proof of Theorem II.3(a) for a more direct argument.

7.6. Remark. Theorems 7.4(b) and 7.5 give complete information on the topological space $(\operatorname{extr} E(\mu), w)$ in the respective cases. For μ with finite range this is seen from Example 4.1. For atomic μ we use, in addition, Proposition 6.7. Also, Theorem 7.4(c) gives complete information on $(\operatorname{extr} E(\mu), w^*)$ for atomic μ, in view of Example 4.1.

The following theorem is a combination of Lemma III.1 and Proposition IV.3. For related results concerning arbitrary $\mu \in ba_+(\mathfrak{M})$ see Section 10, especially Theorem 10.1. Another situation where $s = w^*$ on $\operatorname{extr} E(\mu)$ is described in Theorem IV.4.

7.7. Theorem. *Let $\mu \in ba_+(\mathfrak{M})$ be atomic. Then the following five conditions are equivalent:*

(i) $(E(\mu), s)$ *is compact;*
(ii) $(E(\mu), w)$ *is compact;*
(iii) $\mathfrak{R}/\mathfrak{J}_\nu$ *is finite for each* $\nu \in \mathcal{U}_\mu$*;*
(iv) $s = w^*$ *on* $\operatorname{extr} E(\mu)$*;*
(v) $w = w^*$ *on* $\operatorname{extr} E(\mu)$.

Moreover, under these conditions, $(E(\mu), s)$ is affinely homeomorphic to a countable product of finite-dimensional simplices.

Proof. Clearly, (i) implies (ii). In view of Theorem 7.2(b) and Proposition 14.4(d), (ii) implies (iii) and the final assertion. By Theorem 7.2(b) again, (iii) implies (i).

The implications (i)⇒(iv)⇒(v) are obvious. Suppose (v) holds. By Theorem 7.4(c), $(\operatorname{extr} E(\mu), w)$ is then compact, and so Theorem 5.1, (ii)⇒(i), yields (ii). ■

The following result is a direct consequence of Theorems 7.2(b) and 14.7. It will be applied in the proofs of Theorems 7.9 and 10.2.

7.8. Corollary. *Let μ be atomic and let $(E(\mu), w^*)$ be affinely homeomorphic to $\prod_{j=1}^{\infty}(\mathcal{S}(Z_j), w^*)$, where $Z_1, Z_2, \ldots$ are compact spaces. Then there exists an affine isomorphism of $E(\mu)$ onto $\prod_{j=1}^{\infty}\mathcal{S}(Z_j)$, which is a homeomorphism with respect to the corresponding strong [weak] [weak*] topologies.*

The next theorem is a preliminary version of Theorem 9.6. Some of its conditions are similar to the corresponding conditions of Theorem 7.7.

7.9. Theorem. *Let $\mu \in ba_+(\mathfrak{M})$ be atomic. Then the following five conditions are equivalent:*

(i) *each element of $E(\mu)$ is atomic;*
(ii) *$\mathfrak{R}/\mathfrak{J}_\nu$ is superatomic for each $\nu \in \mathcal{U}_\mu$;*
(iii) *there exist compact scattered spaces $Z_1, Z_2, \ldots$ and an affine isomorphism of $E(\mu)$ onto $\prod_{j=1}^{\infty}\mathcal{S}(Z_j)$, which is a homeomorphism with respect to the corresponding strong [weak] topologies;*
(iv) *$(E(\mu), w^*)$ is affinely homeomorphic to $\prod_{j=1}^{\infty}(\mathcal{S}(Z_j), w^*)$, where $Z_1, Z_2, \ldots$ are compact scattered spaces;*
(v) *$s = w$ on $E(\mu)$.*

Proof. We first establish the equivalence of (i) and (ii) for $\mu \in ult(\mathfrak{M})$. In view of [5, Theorem 5.3.6, (i)⇔(v)], (ii) is equivalent to the condition that the elements of $pa(\mathrm{CO}(Z))$, where Z is the Stone space of $\mathfrak{R}/\mathfrak{J}_\mu$, be all atomic. On the other hand, the elements of $E(\mu)$ can be identified with those of $pa(\mathrm{CO}(Z))$ in a canonical way (see the beginning of the proof of Proposition 7.1). The equivalence of (i) and (ii) for arbitrary atomic $\mu \in ba_+(\mathfrak{M})$ is now seen from Theorem 6.1(a).

Denote by Z_ν the Stone space of $\mathfrak{R}/\mathfrak{J}_\nu$ for $\nu \in \mathcal{U}_\mu$. In view of [31, Remark 17.2], Z_ν is scattered if and only if $\mathfrak{R}/\mathfrak{J}_\nu$ is superatomic. Thus (ii) implies (iii) and (iv), by Theorem 7.2(b). According to Corollary 14.6, (iii) implies (v). Also, (iv) implies (iii), by Theorem 7.2(b) and Corollary 7.8. Finally, (v) implies (ii), by Theorem 7.2(b) and Corollary 14.6. ■

The next result coincides with [44, Theorem 7], but the present proof is new. In the two-valued case the result is due to Bogner and Denk [8, Theorem 2, (v)⇔(i)].

7.10. Theorem. *Let $\mathfrak{M}$ and $\mathfrak{R}$ be σ-algebras of subsets of Ω with $\mathfrak{M} \subset \mathfrak{R}$ and let $\mu \in ca_+(\mathfrak{M})$. Then*

(a) *if $\operatorname{extr} E(\mu)$ is finite, then $E(\mu) \subset ca(\mathfrak{R})$;*
(b) *if μ has finite range and $E(\mu) \subset ca(\mathfrak{R})$, then $\operatorname{extr} E(\mu)$ is finite.*

Proof. (a) is a special case of Theorem 5.5, (ii)⇒(iv). The assumptions of (b) imply that $E(\mu)$ is strongly compact (see Theorem 5.5, (iv)⇒(i), and Theorem 7.7, (ii)⇒(i)).

Therefore, $\operatorname{extr} E(\mu)$ is also strongly compact, by Proposition 4.4(b). It now follows from Theorem 7.5 that $\operatorname{extr} E(\mu)$ is finite. ∎

Theorem 7.10(b) fails for general atomic measures. This is seen from the following example.

7.11. EXAMPLE (= [44, Example 2]). Let $\Omega = \mathbb{N}$ and $\mathfrak{R} = 2^{\Omega}$. Let, further, $M_j = \{2j-1, 2j\}$ for $j \in \mathbb{N}$ and $\mathfrak{M} = \{M_1, M_2, \ldots\}_\beta$. Consider $\mu \in ca_+(\mathfrak{M})$ which is uniquely determined by the condition $\mu(M_j) = 2^{-j}$ for all j. Since

$$\mathfrak{R} = (\mathfrak{M} \cup \{\{1, 3, \ldots\}\})_\beta,$$

we have $E(\mu) \subset ca(\mathfrak{R})$ (see Lemma 3.7). Nevertheless, $\operatorname{extr} E(\mu)$ has cardinality $\mathfrak{c}$. This is seen from Proposition 4.9(d) or by an application of Theorem 6.1(b) with $\mu_j \in ca_+(\mathfrak{M})$ defined by

$$\mu_j(M) = \mu(M \cap M_j) \quad \text{for } M \in \mathfrak{M} \text{ and } j \in \mathbb{N}.$$

8. Topological properties of $\operatorname{extr} E(\mu)$ for nonatomic μ

We start with a result which will be applied in the proofs of Theorems 9.1 and 9.7 (see also Example 12.7). It coincides with Theorem II.4, but the proof of (a) and (b) given below is more direct. Part (c) has a predecessor in Theorem I.3(a).

8.1. THEOREM. *Let $\mu \in ba_+(\mathfrak{M})$ be nonatomic. Then*

(a) $(\operatorname{extr} E(\mu), s)$ *is pathwise connected;*
(b) $(\operatorname{extr} E(\mu), s)$ *is locally pathwise connected;*
(c) $(\operatorname{extr} E(\mu), s)$ *is locally compact if and only if μ is monogenic.*

Proof. Given $\pi_0, \pi_1 \in \operatorname{extr} E(\mu)$ and $M \in \mathfrak{M}$, we set

$$\pi_M(R) = \pi_1(R \cap M) + \pi_0(R \cap M^c) \quad \text{for } R \in \mathfrak{R}.$$

By Lemma 4.5(c), (d), we have $\pi_M \in \operatorname{extr} E(\mu)$ and

$$\|\pi_M - \pi_0\| \le \|\pi_1 - \pi_0\| \quad \text{for } M \in \mathfrak{M}.$$

Fix $\pi_1, \pi_2 \in \operatorname{extr} E(\mu)$. We shall define a Lipschitz function f of order 2 on $[0, \mu(\Omega)]$ with values in $\operatorname{extr} E(\mu)$ having the following properties:

$$f(0) = \pi_0,\ f(\mu(\Omega)) = \pi_1,\ \text{and } \|f(t) - \pi_0\| \le \|\pi_1 - \pi_0\| \text{ for all } t \in [0, \mu(\Omega)].$$

This yields both (a) and (b). By Lemma 3.4(a), there exists $\mathfrak{C} \subset \mathfrak{M}$ linearly ordered by inclusion such that $\mu(\mathfrak{C})$ is dense in $[0, \mu(\Omega)]$. We assume that $\emptyset$, Ω are in $\mathfrak{C}$ and $\mu|\mathfrak{C}$ is injective. Set

$$g(\mu(M)) = \pi_M \quad \text{for } M \in \mathfrak{C}.$$

Given $M_1, M_2 \in \mathfrak{C}$, say $M_1 \subset M_2$, we have

$$\pi_{M_2}(R) - \pi_{M_1}(R) = \pi_1(R \cap (M_2 \setminus M_1)) - \pi_0(R \cap (M_2 \setminus M_1)),$$

and so

$$\|\pi_{M_2} - \pi_{M_1}\| \le 2\mu(M_2 \setminus M_1) = 2|\mu(M_2) - \mu(M_1)|.$$

Thus, g is a Lipschitz function of order 2 on $\mu(\mathfrak{C})$. Since $\operatorname{extr} E(\mu)$ is metrically complete (see Proposition 4.4(b)), g extends, by uniform continuity, to a function f on $[0, \mu(\Omega)]$ with the desired properties.

To establish the nontrivial implication of (c), assume that μ is not monogenic, and so $\operatorname{extr} E(\mu)$ is not a singleton. Fix $\pi_0 \in \operatorname{extr} E(\mu)$ and $r > 0$. By (a), there exists $\pi_1 \in \operatorname{extr} E(\mu)$ with $0 < \|\pi_0 - \pi_1\| < r$. Define $\varphi = \pi_0 - \pi_1$. Then $\varphi \in ba(\mathfrak{R})$ and $\varphi \neq 0$. Moreover, $|\varphi| |\mathfrak{M}$ is nonatomic since $|\varphi| \leq \pi_0 + \pi_1$. By Lemma 3.4(b), there exist $\varepsilon > 0$ and $M_j \in \mathfrak{M}$, $j = 1, 2, \ldots$, with $|\varphi|(M_j \setminus M_k) > \varepsilon$ whenever $j \neq k$. For π_{M_j} defined as at the beginning of the proof we have

$$(\pi_{M_j} - \pi_{M_k})(R \cap (M_j \setminus M_k)) = -\varphi(R \cap (M_j \setminus M_k)) \quad \text{for } R \in \mathfrak{R}.$$

so that $\|\pi_{M_j} - \pi_{M_k}\| > \varepsilon$, and we are done. (In fact, we have shown that the ball in $\operatorname{extr} E(\mu)$ with centre π and radius r is not totally bounded.) ∎

We continue with another useful property of the topological space $(\operatorname{extr} E(\mu), s)$. It is implicit in the proof of [45, Theorem 1], which coincides with Theorem 12.1 below.

8.2. Theorem. *If $\mu \in ba_+(\mathfrak{M})$ is nonatomic, then, for every nonempty relatively open subset U of $\operatorname{extr} E(\mu)$, we have*

$$\mathfrak{d}(U) = \mathfrak{d}(\operatorname{extr} E(\mu)),$$

where openness and the density character $\mathfrak{d}$ refer to the strong topology of $\operatorname{extr} E(\mu)$.

Proof. Throughout the proof, the terms "dense" and "open" mean "strongly dense" and "strongly open", respectively.

Suppose $\operatorname{extr} E(\mu)$ has cardinality ≥ 2. Fix $\pi_0 \in \operatorname{extr} E(\mu)$ and $r > 0$. Given a dense subset D of the open ball in $\operatorname{extr} E(\mu)$ with centre π_0 and radius r, we shall define a dense subset $\tilde{D}$ of $\operatorname{extr} E(\mu)$ with $|\tilde{D}| = |D|$. This yields the assertion.

In view of Theorem 8.1, D is infinite. Let $\{M_1, \ldots, M_n\}$ be an $\mathfrak{M}$-partition of Ω with $\mu(M_i) < r/2$ for all i. Denote by $\tilde{D}$ the set of all σ of the form

$$\sigma(R) = \sum_{i=1}^{n} \sigma_i(R \cap M_i) \quad \text{for } R \in \mathfrak{R}, \tag{1}$$

where $\sigma_1, \ldots, \sigma_n \in D$. Clearly, $\tilde{D} \subset E(\mu)$. Adapting the proof of Lemma 4.5(d), we infer that $\tilde{D} \subset \operatorname{extr} E(\mu)$. Since $|\tilde{D}| = |D|$, we only need to show that $\tilde{D}$ is dense in $\operatorname{extr} E(\mu)$. To this end, fix $\pi \in \operatorname{extr} E(\mu)$ and $\varepsilon > 0$. Set

$$\pi_i(R) = \pi(R \cap M_i) + \pi_0\Big(R \cap \bigcup_{j \neq i} M_j\Big) \quad \text{for } R \in \mathfrak{R},\ i = 1, \ldots, n.$$

As before, we have $\pi_i \in \operatorname{extr} E(\mu)$. Moreover,

$$\|\pi_i - \pi_0\| = |\pi - \pi_0|(M_i) \leq 2\mu(M_i) < r.$$

It follows that there exist $\sigma_i \in D$ such that

$$\|\pi_i - \sigma_i\| < \varepsilon/n, \quad i = 1, \ldots, n.$$

For σ defined by (1), we then have

$$\|\pi - \sigma\| = \sum_{i=1}^{n} |\pi - \sigma|(M_i) = \sum_{i=1}^{n} |\pi_i - \sigma_i|(M_i) \le \sum_{i=1}^{n} \|\pi_i - \sigma_i\| < \varepsilon.$$

Thus, $\tilde{D}$ is dense in $\operatorname{extr} E(\mu)$, and so the proof is complete. ■

We note that a description of $\operatorname{extr} E(\mu)$ as a metric space is given, in a special situation, in [45, Proposition 3].

The following theorem is a generalization and an improvement of Proposition I.3. It is the main tool in the proof of the forthcoming Theorem 8.6(a). Another proof of Theorem 8.3 was given by H. Weber in 1994 (unpublished).

8.3. THEOREM. *Let $\mathfrak{R} = (\mathfrak{M} \cup \mathfrak{E})_b$, where $\mathfrak{E}$ is a finite family of subsets of Ω, let $\mu \in ba_+(\mathfrak{M})$ be nonatomic, and let $\varrho \in E(\mu)$. Then there exists $\pi \in \operatorname{extr} E(\mu)$ with $\pi(E) = \varrho(E)$ for all $E \in \mathfrak{E}$ and $\pi \ll \varrho$.*

Proof. We may assume that $\mathfrak{E} = \{E_1, \ldots, E_n\}$ is a partition of Ω. Define

$$\varrho^i(M) = \varrho(M \cap E_i) \quad \text{for } M \in \mathfrak{M} \text{ and } i = 1, \ldots, n.$$

Then ϱ^i is in $ba_+(\mathfrak{M})$ and $\sum_{i=1}^{n} \varrho^i = \mu$. According to Proposition 3.6, there exist $\pi^i \in ba_+(\mathfrak{M})$ with the following properties:

$$\pi^i \ll \varrho^i, \quad \pi^i(\Omega) = \varrho^i(\Omega) \quad \text{for } i = 1, \ldots, n,$$

$$\pi^i \wedge \pi^{i'} = 0 \quad \text{whenever} \quad i \ne i' \quad \text{and} \quad \sum_{i=1}^{n} \pi^i = \mu.$$

It follows that, for $M \in \mathfrak{M}$ with $M \cap E_i = \emptyset$, we have $\pi^i(M) = 0$. Set

$$\pi\Big(\bigcup_{i=1}^{n} M_i \cap E_i\Big) = \sum_{i=1}^{n} \pi^i(M_i) \quad \text{for } M_1, \ldots, M_n \in \mathfrak{M}.$$

It is easy to check that π is well defined and is in $E(\mu)$. We also have $\pi \ll \varrho$. By [40, Theorem 3(b)], π is in $\operatorname{extr} E(\mu)$. (This can also be shown by using (D).) Finally, for $i = 1, \ldots, n$, we have

$$\pi(E_i) = \pi^i(\Omega) = \varrho^i(\Omega) = \varrho(E_i),$$

which completes the proof. ■

8.4. REMARK. Theorem 8.3, with the final assertion omitted, can be reformulated as follows. Let $\mathfrak{E} = \{E_1, \ldots, E_n\}$, and define

$$\Psi\colon E(\mu) \to \mathbb{R}^n \quad \text{by} \quad \Psi(\varrho) = (\varrho(E_1), \ldots, \varrho(E_n)) \quad \text{for } \varrho \in E(\mu).$$

Under the assumptions of Theorem 8.3, we have $\Psi(E(\mu)) = \Psi(\operatorname{extr} E(\mu))$. This shows that there is some connection between this theorem and [1], where convexity of ranges of some mappings is discussed.

8.5. REMARK. For $\mathfrak{E}$ consisting of a single set, say E, Theorem 8.3, with the final assertion omitted, can be derived in a simpler manner. In fact, there exist π_0, π_1 in $\operatorname{extr} E(\mu)$ with

$$\pi_0(E) = \mu_*(E) \quad \text{and} \quad \pi_1(E) = \mu^*(E)$$

(see pp. 18–19). We then have $\pi_0(E) \le \varrho(E) \le \pi_1(E)$. The mapping

$$\Psi\colon (\operatorname{extr} E(\mu), s) \to \mathbb{R}^1, \qquad \Psi(\pi) = \pi(E),$$

is, clearly, continuous. By Theorem 8.1(a), there exists a continuous mapping

$$f\colon [0,1] \to (\operatorname{extr} E(\mu), s) \quad \text{with} \quad f(0) = \pi_0 \text{ and } f(1) = \pi_1.$$

It follows that $\Psi \circ f(t) = \varrho(E)$ for some $t \in [0,1]$. Thus, $\pi = f(t)$ is as desired.

Parts (a) and (d) of our next result are Theorem I.3(c) and Lemma III.3, respectively, while part (c) is a special case of Theorem I.3(b). The present proofs of (a) and (c) are new. The result will be used in the proofs of Theorems 9.1 and 9.7.

8.6. THEOREM. *Let $\mu \in ba_+(\mathfrak{M})$ be nonatomic. Then*

(a) $\operatorname{extr} E(\mu)$ *is dense in* $(E(\mu), w^*)$*;*
(b) $\operatorname{extr} E(\mu)$ *is dense in* $(E(\mu), w)$ *if* $(E(\mu), w)$ *is compact;*
(c) $\operatorname{extr} E(\mu)$ *is dense in* $(E(\mu), w)$ *if* $\mathfrak{R} = (\mathfrak{M} \cup \mathfrak{E})_b$ *for some finite family* $\mathfrak{E}$ *of subsets of* Ω*;*
(d) $\operatorname{extr} E(\mu)$ *is closed in* $(E(\mu), w)$ *if and only if* μ *is monogenic.*

Proof. To establish (a), fix a finite subfamily $\mathfrak{R}_0$ of $\mathfrak{R}$ and $\varrho \in E(\mu)$. By Theorem 8.3, there exists

$$\pi_0 \in \operatorname{extr} E(\mu, (\mathfrak{M} \cup \mathfrak{R}_0)_b) \quad \text{with} \quad \pi_0(R) = \varrho(R) \text{ for } R \in \mathfrak{R}_0.$$

Taking $\pi \in \operatorname{extr} E(\pi_0, \mathfrak{R})$ (see (C)), we get $\pi \in \operatorname{extr} E(\mu)$ and $\pi(R) = \varrho(R)$ for $R \in \mathfrak{R}_0$, and so (a) holds.

Part (b) is an obvious consequence of (a), while (c) follows from (b) and Corollary 5.4.

To establish the nontrivial implication of (d), assume that μ is not monogenic. This means that $\mathfrak{M}_\mu$ is properly contained in $\mathfrak{R}$. Thus, choosing $R_0 \in \mathfrak{R} \setminus \mathfrak{M}_\mu$ and setting $\mathfrak{M}_0 = (\mathfrak{M} \cup \{R_0\})_b$, we see that μ is not monogenic with respect to $\mathfrak{M}_0$ either. Fix

$$\varrho \in E(\mu, \mathfrak{M}_0) \setminus \operatorname{extr} E(\mu, \mathfrak{M}_0)$$

and a net (π_α) in $\operatorname{extr} E(\mu, \mathfrak{M}_0)$ which converges weakly to ϱ (see (c)).

Let $T\colon ba(\mathfrak{M}_0) \to ba(\mathfrak{R})$ be given by Proposition 4.6. The net $(T(\pi_\alpha))$ is then in $\operatorname{extr} E(\mu)$. On the other hand, $T(\varrho)$ is not in $\operatorname{extr} E(\mu)$ (see (D)). Finally, $(T(\pi_\alpha))$ converges weakly to $T(\varrho)$, by [17, Theorem V.3.15], and we are done. ∎

From Proposition 4.4(b) and Theorem 8.6(d) we get the following corollary.

8.7. COROLLARY. *Let $\mu \in ba_+(\mathfrak{M})$ be nonatomic. Then $s = w$ on $E(\mu)$ if and only if μ is monogenic.*

For an application of Corollary 8.7 see the proof of Theorem 9.6.

Theorem 8.6(c) fails drastically for countable $\mathfrak{E}$. In fact, it may then happen that even the convex hull of $\operatorname{extr} E(\mu)$ is not dense in $(E(\mu), w)$. Thus, in Theorem 8.6(a) we cannot replace the weak* topology by the weak one. An example follows.

8.8. Example (= Example I.4; cf. [35, Proposition 1] and [40, Example 3]). Let $\Omega = [0,1) \times [0,1)$ and

$$\mathfrak{M} = \{[a,b) \times [0,1) : 0 \le a < b < 1 \text{ and } a,\, b \text{ are rational}\}_b,$$
$$\mathfrak{E} = \{[0,1) \times [a,b) : 0 \le a < b < 1 \text{ and } a,\, b \text{ are rational}\}.$$

Set $\mathfrak{R} = (\mathfrak{M} \cup \mathfrak{E})_b$. Let μ and ϱ be the restrictions of the two-dimensional Lebesgue measure to $\mathfrak{M}$ and $\mathfrak{R}$, respectively. We claim that $\varrho \wedge \pi = 0$ for all $\pi \in \operatorname{extr} E(\mu)$. Indeed, fix $\pi \in \operatorname{extr} E(\mu)$ and $\varepsilon > 0$. Choose $n \in \mathbb{N}$ with $2/n < \varepsilon$. Define

$$R_i = [0,1) \times \left[\frac{i-1}{n}, \frac{i}{n}\right), \qquad i = 1, \ldots, n.$$

By Lemma 4.2, there exists an $\mathfrak{M}$-partition $\{M_1, \ldots, M_n\}$ of Ω with

$$\pi\Big(\bigcup_{i=1}^{n} R_i \mathbin{\triangle} M_i\Big) < \varepsilon/2.$$

We have $M_i = P_i \times [0,1)$, where $\{P_1, \ldots, P_n\}$ is a partition of $[0,1)$. It follows that

$$\varrho(R_i \cap M_i) = \frac{1}{n}\lambda(P_i),$$

where λ denotes the one-dimensional Lebesgue measure. Hence

$$\varrho\Big(\bigcup_{i=1}^{n} R_i \cap M_i\Big) < \varepsilon/2,$$

and so

$$\varrho\Big(\bigcup_{i=1}^{n} R_i \cap M_i\Big) + \pi\Big(\bigcup_{i=1}^{n} R_i \mathbin{\triangle} M_i\Big) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this implies the claim (see Remark 4.3). Noting that

$$\{\varphi \in ba(\mathfrak{R}) : \varrho \wedge |\varphi| = 0\}$$

is a norm closed, and so weakly closed, subspace of $ba(\mathfrak{R})$, we conclude that ϱ is not in the weakly closed convex hull of $\operatorname{extr} E(\mu)$.

9. Topological properties of $E(\mu)$ and $\operatorname{extr} E(\mu)$, and the antimonogenic component of μ

In Sections 7 and 8 some theorems on topological properties of $E(\mu)$ and $\operatorname{extr} E(\mu)$, and on affine-topological properties of $E(\mu)$ have been established. Those theorems assume that μ be atomic, have finite range or be nonatomic. In this section we shall show that they can be reversed with the help of the antimonogenic component μ^{a} of μ. We start by reversing the main part of Theorem 7.4.

9.1. Theorem (cf. Theorems II.5 and III.3). *For $\mu \in ba_+(\mathfrak{M})$ the following seven conditions are equivalent:*

(i) μ^{a} *is atomic;*
(ii) $(\operatorname{extr} E(\mu), s)$ *is zero-dimensional;*
(iii) $(\operatorname{extr} E(\mu), w)$ *is zero-dimensional;*
(iv) $(\operatorname{extr} E(\mu), w^*)$ *is zero-dimensional;*
(v) $\operatorname{extr} E(\mu)$ *is closed in* $(E(\mu), w)$*;*
(vi) $\operatorname{extr} E(\mu)$ *is closed in* $(E(\mu), w^*)$*;*
(vii) $(\operatorname{extr} E(\mu), w^*)$ *is compact.*

Proof. The implications (vii)⇒(vi)⇒(v) are plain.

In the rest of the proof we assume $\mu = \mu^{\mathrm{a}}$, which we may, due to Corollary 6.2.

By Theorem 7.4, (i) implies (ii), (iii), (iv), and (vii).

Let μ_1 and μ_2 stand for the atomic and nonatomic components of μ, respectively. Each of the conditions (ii)–(iv) implies that the only nonempty connected subsets of $\operatorname{extr} E(\mu)$ with respect to the corresponding topology are singletons. It follows from Corollary 6.3 that the same is true for $\operatorname{extr} E(\mu_2)$. Combined with Theorem 8.1(a), this yields that $\operatorname{extr} E(\mu_2)$ is a singleton, and so $\mu_2 = 0$. Thus, (i) holds.

Finally, we shall derive (i) from (v). Now, (v) and Corollary 6.3 imply that $\operatorname{extr} E(\mu_2)$ is closed in $(E(\mu_2), w)$. Combined with Theorem 8.6(d), this shows that $\operatorname{extr} E(\mu_2)$ is a singleton, and so $\mu_2 = 0$. Thus, (i) holds. ∎

Condition (i) of the following corollary is discussed, in detail, in Section 11. Condition (ii) thereof is similar to conditions (ii) and (iii) of Theorem 9.4.

9.2. Corollary (= Corollary III.2). *For $\mu \in ba_+(\mathfrak{M})$ the following two conditions are equivalent:*

(i) $\operatorname{extr} E(\mu)$ *is finite;*
(ii) $(\operatorname{extr} E(\mu), w^*)$ *is discrete.*

Proof. Suppose that (ii) holds. An application of Theorem 9.1, (iv)⇒(vii), shows that $(\operatorname{extr} E(\mu), w^*)$ is compact. This together with (ii) implies (i). ∎

The next result supplements Theorem 9.1. In connection with uniqueness of $Z_1, Z_2, \ldots$ in condition (iv) below see Theorem 14.7.

9.3. Theorem (cf. Theorem IV.2). *For $\mu \in ba_+(\mathfrak{M})$ the following four conditions are equivalent:*

(i) μ^{a} *is atomic;*
(ii) $(E(\mu), s)$ *is affinely homeomorphic to* $\prod_{j=1}^{\infty}(\mathcal{S}(Z_j), s)$*, where $Z_1, Z_2, \ldots$ are compact zero-dimensional spaces;*
(iii) $(E(\mu), w)$ *is affinely homeomorphic to* $\prod_{j=1}^{\infty}(\mathcal{S}(Z_j), w)$*, where $Z_1, Z_2, \ldots$ are compact zero-dimensional spaces;*
(iv) $(E(\mu), w^*)$ *is affinely homeomorphic to* $\prod_{j=1}^{\infty}(\mathcal{S}(Z_j), w^*)$*, where $Z_1, Z_2, \ldots$ are compact zero-dimensional spaces.*

Proof. By Theorem 7.2(b) and Corollary 6.2(a), (i) implies each of the remaining conditions. Conversely, those conditions imply conditions (ii), (iii) and (iv) of Theorem 9.1, respectively (see Propositions 14.4(b), (c) and 14.1), and so that theorem yields (i). ■

We shall now reverse Theorem 7.5 as follows.

9.4. THEOREM (cf. Theorem III.4). *For $\mu \in ba_+(\mathfrak{M})$ the following three conditions are equivalent:*

(i) μ^{a} *has finite range;*
(ii) $(\operatorname{extr} E(\mu), s)$ *is discrete;*
(iii) $(\operatorname{extr} E(\mu), w)$ *is discrete.*

Proof. We assume $\mu = \mu^{\mathrm{a}}$, as we may, due to Corollary 6.2(b). Theorem 7.5 shows that (i) implies (iii). Clearly, (iii) implies (ii). Suppose (i) fails. Combining Lemma 3.3 and Theorem 6.1(b), we see that $(\operatorname{extr} E(\mu), s)$ is homeomorphic to an infinite product of spaces of cardinality ≥ 2. Hence (ii) also fails. ■

The following result is, up to condition (iv), a variant of Theorem III.4. It supplements the preceding one and is a finitary version of Theorem 9.3.

9.5. THEOREM. *For $\mu \in ba_+(\mathfrak{M})$ the following four conditions are equivalent:*

(i) μ^{a} *has finite range;*
(ii) $(E(\mu), s)$ *is affinely homeomorphic to* $\prod_{j=1}^{p}(\mathcal{S}(Z_j), s)$*, where* $Z_1, \ldots, Z_p$ *are compact zero-dimensional spaces;*
(iii) $(E(\mu), w)$ *is affinely homeomorphic to* $\prod_{j=1}^{p}(\mathcal{S}(Z_j), w)$*, where* $Z_1, \ldots, Z_p$ *are compact zero-dimensional spaces;*
(iv) $(E(\mu), w^*)$ *is affinely homeomorphic to* $\prod_{j=1}^{p}(\mathcal{S}(Z_j), w^*)$*, where* $Z_1, \ldots, Z_p$ *are compact zero-dimensional spaces.*

Proof. It follows from (i) that μ^{a} is atomic and $\mathcal{U}_{\mu^{\mathrm{a}}}$ is finite, and hence conditions (ii)–(iv) hold, by Theorem 7.2(b) and Corollary 6.2(a). Conditions (ii) and (iii) imply the corresponding conditions of Theorem 9.4, by Proposition 14.4(b), (c), and so that theorem yields (i). Finally, it follows from (iv) and Theorem 9.3, (iv)⇒(i), that μ^{a} is atomic. Invoking Theorem 7.2(b) again, combined with Theorem 14.7, we find that $\mathcal{U}_{\mu^{\mathrm{a}}}$ is finite, and so (i) holds. ■

The next result is a variant of Theorem IV.3 extended by condition (iv). It is essentially a combination of Theorem 7.9 and Corollary 8.7. It is also closely related to Theorem 10.2.

9.6. THEOREM. *For $\mu \in ba_+(\mathfrak{M})$ the following five conditions are equivalent:*

(i) *each element of* $E(\mu^{\mathrm{a}})$ *is atomic;*
(ii) μ^{a} *is atomic and* $\mathfrak{R}/\mathfrak{J}_\nu$ *is superatomic for each* $\nu \in \mathcal{U}_{\mu^{\mathrm{a}}}$*;*
(iii) *there exist compact scattered spaces* $Z_1, Z_2, \ldots$ *and an affine isomorphism of* $E(\mu)$ *onto* $\prod_{j=1}^{\infty}\mathcal{S}(Z_j)$*, which is a homeomorphism with respect to the corresponding strong [weak] topologies;*
(iv) $(E(\mu), w^*)$ *is affinely homeomorphic to* $\prod_{j=1}^{\infty}(\mathcal{S}(Z_j), w^*)$*, where* $Z_1, Z_2, \ldots$ *are compact scattered spaces;*
(v) $s = w$ *on* $E(\mu)$*.*

Proof. Suppose (i) holds. Then μ^{a} is atomic, and so (ii) follows, by the corresponding implication of Theorem 7.9. Applying Theorem 7.9 again and Corollary 6.2(a), we see that the implications (ii)⇒(iii)⇒(iv)⇒(v) also hold.

Suppose (v) holds. To derive (i), note that $s = w$ on $E(\mu^{\mathrm{a}})$, by Corollary 6.2(a). Denote by μ_1 and μ_2 the atomic and nonatomic components of μ^{a}, respectively. By Corollary 6.3, we get $s = w$ on $E(\mu_i)$, $i = 1, 2$. It now follows from Corollary 8.7 that $\mu_2 = 0$. Thus, μ^{a} is atomic, and so an application of Theorem 7.9, (v)⇒(i), yields (i). ∎

The following result shows, in particular, that the assumption of nonatomicity of μ in Theorems 8.1 and 8.6(a), (c) is, in some sense, necessary.

9.7. Theorem (= Theorem II.6). *For $\mu \in ba_+(\mathfrak{M})$ the following four conditions are equivalent:*

(i) μ^{a} *is nonatomic;*
(ii) $(\operatorname{extr} E(\mu), s)$ *is pathwise connected;*
(iii) $(\operatorname{extr} E(\mu), w^*)$ *is connected;*
(iv) $\operatorname{extr} E(\mu)$ *is dense in* $(E(\mu), w^*)$.

Proof. In view of Corollary 6.2, (i) implies (ii) and (iv), by Theorems 8.1(a) and 8.6(a), respectively. Clearly, (ii) implies (iii).

To establish the implications (iii)⇒(i) and (iv)⇒(i), denote by μ_1 and μ_2 the atomic and nonatomic components of μ^{a}, respectively. By Corollaries 6.2(a) and 6.3, there exists an affine mapping of $E(\mu)$ onto $E(\mu_1) \times E(\mu_2)$, which is a homeomorphism with respect to the corresponding weak* topologies. Thus, (iii) and (iv) imply that $\operatorname{extr} E(\mu_1)$ is weak* connected and weak* dense in $E(\mu_1)$, respectively. On the other hand, $\operatorname{extr} E(\mu_1)$ is zero-dimensional and closed in $E(\mu_1)$ with respect to the weak* topology, by Theorem 7.4(c). In both cases, μ_1 is, therefore, monogenic. Hence $\mu^{\mathrm{a}} = \mu_2$, and so (i) holds. ∎

10. Strong compactness of $E(\mu)$

The only result on strong compactness of $E(\mu)$ we have presented so far is Theorem 7.7, where μ is assumed to be atomic. We shall now dispense with this assumption and deal with arbitrary μ. The two theorems of this section establish the equivalence of strong compactness of $E(\mu)$ to eight other conditions. Those of Theorem 10.1 are of purely topological or affine-topological character. Among those of Theorem 10.2 there are three of affine-topological and one of purely measure-theoretic character.

10.1. Theorem (= Theorem III.1). *For $\mu \in ba_+(\mathfrak{M})$ the following four conditions are equivalent:*

(i) $(E(\mu), s)$ *is compact;*
(ii) $s = w^*$ *on* $E(\mu)$*;*
(iii) $(\operatorname{extr} E(\mu), s)$ *is compact;*
(iv) $(\operatorname{extr} E(\mu), w)$ *is compact.*

Proof. Clearly, (i) implies (ii). The converse implication is seen from Proposition 4.4(a).

By Proposition 4.4 (b), (i) implies (iii). Clearly, (iii) implies (iv). Finally, suppose (iv) holds. By Theorem 9.1, (v)⇒(i), μ^{a} is then atomic. It now follows from Theorem 7.7, (v)⇒(i), and Corollary 6.2 that (i) holds. ∎

The next result is Theorem III.2 supplemented by condition (vi). The equivalence of that condition to the remaining conditions below was first established by V. Losert, in answer to a question of the author (see [46, p. 469, Postscript]).

10.2. THEOREM. *For $\mu \in ba_+(\mathfrak{M})$ the following six conditions are equivalent:*

(i) *$(E(\mu), s)$ is compact;*
(ii) *$(E(\mu), w)$ is compact and μ^{a} is atomic;*
(iii) *μ^{a} is atomic and $\mathfrak{R}/\mathfrak{J}_\nu$ is finite for each $\nu \in \mathcal{U}_{\mu^{\mathrm{a}}}$;*
(iv) *$(E(\mu), s)$ is affinely homeomorphic to a countable product of finite-dimensional simplices;*
(v) *$(E(\mu), w)$ is affinely homeomorphic to a countable product of finite-dimensional simplices;*
(vi) *$(E(\mu), w^*)$ is affinely homeomorphic to a countable product of finite-dimensional simplices.*

Proof. The implication (i)⇒(ii) follows from Theorem 9.6, (v)⇒(ii).

In the rest of the proof we assume $\mu = \mu^{\mathrm{a}}$, which we may, due to Corollary 6.2(a). The implication (ii)⇒(iii) is a consequence of the corresponding implication of Theorem 7.7. In view of Theorem 7.2(a), (iii) implies (iv). The implications (iv)⇒(v)⇒(vi) and (iv)⇒(i) are clear. To complete the proof, we shall derive (iv) from (vi). Assuming (vi), we infer from Theorem 9.3, (iv)⇒(i), that μ is atomic. Therefore, Corollary 7.8 yields (iv). ∎

10.3. REMARK. Condition (iv) of Theorem 10.2 gives a complete affine-topological description of the sets $E(\mu)$ in the strongly compact case. Indeed, for every sequence $S_1, S_2, \ldots$ of finite-dimensional simplices, there exist a set Ω, algebras $\mathfrak{M}$ and $\mathfrak{R}$ of subsets of Ω with $\mathfrak{M} \subset \mathfrak{R}$ and $\mu \in ba_+(\mathfrak{M})$ such that $(E(\mu), s)$ is affinely homeomorphic to $\prod_{j=1}^{\infty} S_j$. This is seen from Remark 7.3. In fact, Ω can be chosen countable, $\mathfrak{M}$ a σ-algebra, μ a measure and $\mathfrak{R} = 2^{\Omega}$ (see Remark III.1 for details).

10.4. COROLLARY (= Corollary III.1(a)). *If $\mu \in ba_+(\mathfrak{M})$ and $(E(\mu), s)$ is compact, then either* $\operatorname{extr} E(\mu)$ *is finite or* $(\operatorname{extr} E(\mu), s)$ *is homeomorphic to the Cantor set.*

Proof. By assumption and Theorem 10.2, (i)⇒(iv), $(\operatorname{extr} E(\mu), s)$ is homeomorphic to a countable product of finite spaces. The latter space is metrizable, compact and zero-dimensional, by standard product theorems, and dense in itself provided it is infinite. Thus, the assertion follows from a well-known theorem of Brouwer (see [18, Exercise 6.2.A(c))]). ∎

We note that either of the possibilities described in Corollary 10.4 can occur; see Remark 10.3. Corollary 10.4 will be applied in the proof of Corollary 12.2.

The following corollary was first established in [46] by a different argument.

10.5. Corollary (= Corollary III.3). *Let $\mathfrak{M}$ and $\mathfrak{R}$ be σ-algebras of subsets of Ω with $\mathfrak{M} \subset \mathfrak{R}$ and let $\mu \in ca_+(\mathfrak{M})$. Then the following two conditions are equivalent:*

(i) $(E(\mu), s)$ *is compact;*
(ii) μ^{a} *is atomic and* $E(\mu) \subset ca(\mathfrak{R})$.

This is a direct consequence of Theorem 10.2, (i)⇔(ii), and Theorem 5.5, (i)⇔(iv).

For an application of Corollary 10.5 see the proof of Theorem 12.11.

11. $E(\mu)$ with finitely or countably many extreme points

Most of the material of this section is taken from [44, Sections 3 and 4]. We start with a result which includes a part of [44, Theorem 5] and is closely related to Theorems 9.5, 9.6 and 10.2.

11.1. Theorem. *For $\mu \in ba_+(\mathfrak{M})$ the following five conditions are equivalent:*

(i) $\operatorname{extr} E(\mu)$ *is finite* [*countable*];
(ii) μ^{a} *has finite range and* $\mathfrak{R}/\mathfrak{J}_\nu$ *is finite* [*countable and superatomic*] *for each* $\nu \in \mathcal{U}_{\mu^{\mathrm{a}}}$;
(iii) $(E(\mu), s)$ *is affinely homeomorphic to* $\prod_{j=1}^p (\mathcal{S}(Z_j), s)$, *where* $Z_1, \dots, Z_p$ *are finite* [*countable*] *compact spaces;*
(iv) $(E(\mu), w)$ *is affinely homeomorphic to* $\prod_{j=1}^p (\mathcal{S}(Z_j), w)$, *where* $Z_1, \dots, Z_p$ *are finite* [*countable*] *compact spaces;*
(v) $(E(\mu), w^*)$ *is affinely homeomorphic to* $\prod_{j=1}^p (\mathcal{S}(Z_j), w^*)$, *where* $Z_1, \dots, Z_p$ *are finite* [*countable*] *compact spaces.*

Proof. We shall use the following well-known result: a Boolean algebra is finite [countable and superatomic] if and only if its Stone space is finite [countable] (see [31, Theorem 5.31 and Proposition 17.10] and [63, Theorem 8.5.4, (ii)⇒(i)] for the "infinite" part).

We assume $\mu = \mu^{\mathrm{a}}$, which is allowed, due to Corollary 6.2. Suppose (i) holds. By Proposition 6.9, μ then has finite range. Using Corollary 6.4 and property 4° of Proposition 7.1, we get (ii). Theorem 7.2(b) shows that (ii) implies (iii), (iv) and (v). In view of Proposition 14.1, each of the last three conditions implies (i). ∎

The next result and Theorem 11.1 partly overlap. Nevertheless, we give an independent proof for it.

11.2. Theorem (= [44, Theorem 1]). *For $\mu \in ba_+(\mathfrak{M})$ the following three conditions are equivalent:*

(i) $\operatorname{extr} E(\mu)$ *is finite;*
(ii) $E(\mu)$ *is finite-dimensional;*
(iii) $E(\mu)$ *is affinely isomorphic to a finite product of finite-dimensional simplices.*

Proof. Suppose (i) holds. Then, in view of Proposition 4.4(a), the Krein–Milman theorem yields

$$E(\mu) = \operatorname{conv} \operatorname{extr} E(\mu),$$

and (ii) follows. Clearly, (iii) implies (i).

Suppose (ii) holds. By Proposition 7.1, (iii) then holds in the special case where $\mu \in ult(\mathfrak{M})$. To derive (iii) in the general case, we assume $\mu = \mu^{\mathrm{a}}$, as we may, due to Corollary 6.2. By Theorem 10.2, (i)⇒(ii), μ is atomic. It now follows from Theorem 6.1(a) that, for every finite $\mathcal{F} \subset \mathcal{U}_\mu$, a translate of $\sum_{\nu\in\mathcal{F}} E(\nu)$ is contained in $E(\mu)$. In view of (ii) and Lemma 2.2, we see that $E(\nu)$ is finite-dimensional for each $\nu \in \mathcal{U}_\mu$ and $\mathcal{U}_\mu$ is finite. Consequently, $E(\nu)$ is a finite-dimensional simplex, by the special case established above. Finally, applying Corollary 6.4, we obtain (iii). ■

The countable-infinite analogue of Theorem 11.2, (i)⇔(ii), fails (cf. Theorem 11.1). In fact, according to [43, Theorem 6], a further condition equivalent to those of Theorem 11.2 is the following one:

(ii)′ $\dim E(\mu) < \mathfrak{c}$.

11.3. Remark (cf. [44, Remark 1]). Conditions (iii)–(v) of Theorem 11.1 and condition (iii) of Theorem 11.2 give a complete geometric description of the sets $E(\mu)$ dealt with in those theorems. This is seen by an obvious modification of the argument given in Remark 10.3.

For the proof of our next theorem we shall need the following lemma, which is implicit in the proof of [44, Theorem 2].

11.4. Lemma. *Let* $\mu \in ba_+(\mathfrak{M})$ *and let* $\operatorname{extr} E(\mu)$ *be infinite. Then there exists* $\varrho \in E(\mu)$ *with infinite range.*

Proof. We first consider the special case where $\mu \in ult(\mathfrak{M})$. Fix different elements $\pi_1, \pi_2, \ldots$ of $\operatorname{extr} E(\mu)$, and set $\varrho = \sum_{k=1}^{\infty} 2^{-k}\pi_k$. It follows from (D)′ that $\pi_k \wedge \pi_{k'} = 0$ whenever $k \neq k'$. Therefore, we can find $R_k \in \mathfrak{R}$ with

$$\pi_1(R_k) = \cdots = \pi_{k-1}(R_k) = 0 \quad \text{and} \quad \pi_k(R_k) = 1, \qquad k = 2, 3, \ldots.$$

This implies $2^{-k} \le \varrho(R_k) \le 2^{-(k-1)}$. Hence $\varrho(\mathfrak{R})$ is infinite.

In the general case, we assume, as we may, that $\mu(\mathfrak{M})$ is finite. By Lemma 3.2, $\mathcal{U}_\mu$ is then finite, too. In view of Theorem 6.1(b), we have

$$\operatorname{extr} E(\mu) = \sum_{\nu\in\mathcal{U}_\mu} \operatorname{extr} E(\nu).$$

Consequently, $\operatorname{extr} E(\nu_0)$ is infinite for some $\nu_0 \in \mathcal{U}_\mu$. By the special case established above, we can find ϱ_{ν_0} in $E(\nu_0)$ with infinite range. Choose ϱ_ν in $E(\nu)$ arbitrarily for $\nu \in \mathcal{U}_\mu$ and $\nu \neq \nu_0$, and set $\varrho = \sum_{\nu\in\mathcal{U}_\mu} \varrho_\nu$. We have $\varrho_{\nu_0}(\mathfrak{R}) \subset \varrho(\mathfrak{R})$, which completes the proof. ■

In the case where $\mathfrak{M}$ and $\mathfrak{R}$ are σ-algebras and μ is a two-valued measure, the next result is due to Bogner and Denk [8, Theorem 2, (v)⇔(vi)⇔(vii)].

11.5. Theorem (= [44, Theorem 2]). *For* $\mu \in ba_+(\mathfrak{M})$ *the following three conditions are equivalent:*

(i) $\operatorname{extr} E(\mu)$ *is finite and* μ *has finite range;*
(ii) *each* $\varrho \in E(\mu)$ *has finite range;*
(iii) *there exists* $n \in \mathbb{N}$ *such that each* $\varrho \in E(\mu)$ *has range of cardinality* $\le n$.

Proof. Clearly, (iii) implies (ii). By Lemma 11.4, (ii) implies (i).

Suppose (i) holds, and denote by q and r the cardinalities of $\operatorname{extr} E(\mu)$ and $\mu(\mathfrak{M})$, respectively. We claim that (iii) holds with $n = r^q$. Indeed, (D) implies $\pi(\mathfrak{R}) = \mu(\mathfrak{M})$ for each $\pi \in \operatorname{extr} E(\mu)$. This yields our claim (see the proof of Theorem 11.2, (i)⇒(ii)). ■

The following theorem, which is contained in [44, Theorem 5], supplements Theorems 11.1 and 11.2. The "finite" part of it generalizes a result of Bogner and Denk [8, Theorem 2, (v)⇔(iv)]. That result is concerned with the case where $\mathfrak{M}$ and $\mathfrak{R}$ are σ-algebras and μ is a two-valued measure. In the situation where μ has finite range, still another condition equivalent to those of Theorems 11.1 and 11.6 is condition (iii) of Theorem 4 in [44].

11.6. THEOREM. *For $\mu \in ba_+(\mathfrak{M})$ the following two conditions are equivalent:*

(i) $\operatorname{extr} E(\mu)$ *is finite [countable];*
(ii) μ^{a} *has finite range and there exists a finite [countable and superatomic] subalgebra $\mathfrak{N}$ of $\mathfrak{R}$ such that $(\mathfrak{M}_\mu \cup \mathfrak{N})_b = \mathfrak{R}$.*

Proof. Since $\mathfrak{M}_\mu = \mathfrak{M}_{\mu^{\mathrm{a}}}$ and $\operatorname{extr} E(\mu)$ is a translate of $\operatorname{extr} E(\mu^{\mathrm{a}})$ for $\mu \in ba_+(\mathfrak{M})$, by Proposition 4.8 and Corollary 6.2(b), respectively, we may assume that $\mu = \mu^{\mathrm{a}}$.

(ii)⇒(i): Let $\bar{\mu}$ be the unique quasi-measure extension of μ to $\mathfrak{M}_\mu$. Clearly, we have

$$E(\mu) = E(\bar{\mu}, \mathfrak{R}) \quad \text{and} \quad |\mathcal{U}_\mu| = |\mathcal{U}_{\bar{\mu}}|.$$

Therefore, assuming (ii), we get (i) from Proposition 6.6 and the result formulated at the beginning of the proof of Theorem 11.1.

(i)⇒(ii): By Theorem 11.1, (i)⇒(ii), we deduce from (i) that μ has finite range and $\mathfrak{R}/\mathfrak{J}_\nu$ is finite [countable and superatomic] for each $\nu \in \mathcal{U}_\mu$.

We first consider the special case where $\mu \in ult(\mathfrak{M})$. Denote by h the canonical mapping from $\mathfrak{R}$ onto $\mathfrak{R}/\mathfrak{J}_\mu$. It follows from a result of von Neumann and Stone ([56, Theorem 17]; see also [23, p. 139, Corollary 2] or [44, Remark 2]) that there exists a Boolean homomorphism g from $\mathfrak{R}/\mathfrak{J}_\mu$ into $\mathfrak{R}$ such that

$$(g \circ h(R)) \vartriangle R \in \mathfrak{J}_\mu \quad \text{for all } R \in \mathfrak{R}.$$

Clearly, g is injective. Set $\mathfrak{N} = g(\mathfrak{R}/\mathfrak{J}_\mu)$. For every $R \in \mathfrak{R}$ there is then an $N \in \mathfrak{N}$ with $N \vartriangle R \in \mathfrak{J}_\mu$, and so (ii) holds.

We now consider the general case. Let $\{\Omega_1, \ldots, \Omega_p\}$ be a partition of Ω consisting of μ-atoms (see Lemma 3.2). Set

$$\mathfrak{M}_j = \{M \cap \Omega_j : M \in \mathfrak{M}\}, \qquad \mathfrak{R}_j = \{R \cap \Omega_j : R \in \mathfrak{R}\}$$

and $\tilde{\mu}_j = \mu|\mathfrak{M}_j$ for $j = 1, \ldots, p$. Then $\mathfrak{M}_j$ and $\mathfrak{R}_j$ are algebras of subsets of Ω_j with $\mathfrak{M}_j \subset \mathfrak{R}_j$, $\tilde{\mu}_j \in ba_+(\mathfrak{M}_j)$ is two-valued and

$$(\mathfrak{M}_j)_{\tilde{\mu}_j} = \{M \cap \Omega_j : M \in \mathfrak{M}_\mu\}, \qquad j = 1, \ldots, p.$$

It follows from (i) that $\operatorname{extr} E(\tilde{\mu}_j, \mathfrak{R}_j)$ is finite [countable] for all j. Using the implication (i)⇒(ii) for $p = 1$, we get a finite [countable and superatomic] subalgebra $\mathfrak{N}_j$ of

$\mathfrak{R}_j$ such that $(\mathfrak{N}_j \cup (\mathfrak{M}_j)_{\tilde{\mu}_j})_b = \mathfrak{R}_j$. Set

$$\mathfrak{N} = \Big\{\bigcup_{j=1}^{p} N_j : N_j \in \mathfrak{N}_j,\ j = 1, \ldots, p\Big\}.$$

As easily seen, $\mathfrak{N}$ satisfies (ii). ∎

Our next result complements [8, Theorem 1] and Theorem 12.11, which give, in all, seven conditions equivalent to the condition that $E(\mu) \subset ca(\mathfrak{R})$ for atomic $\mu \in ca_+(\mathfrak{M})$. In this connection see also Theorem 5.5, which is concerned with arbitrary $\mu \in ca_+(\mathfrak{M})$. We note that conditions (ii) and (iii) below are of purely measure-theoretic character, and (ii) is a σ-additive version of condition (ii) of Theorem 11.6.

11.7. Theorem (= Theorem III.5). *Let $\mathfrak{M}$ and $\mathfrak{R}$ be σ-algebras of subsets of Ω with $\mathfrak{M} \subset \mathfrak{R}$ and let $\mu \in ca_+(\mathfrak{M})$. Then the implications* (i)⇒(ii)⇒(iii) *hold for the following conditions:*

(i) *$(E(\mu), s)$ is compact;*
(ii) *there exist a partition $\{R_1, R_2, \ldots\}$ of Ω and $1 = n_0 < n_1 < n_2 < \cdots$ such that*

$$\mathfrak{R} = (\mathfrak{M}_\mu \cup \{R_1, R_2, \ldots\})_\beta \quad \textit{and} \quad \bigcup_{i=n_{j-1}}^{n_j - 1} R_i \in \mathfrak{M} \textit{ for each } j \in \mathbb{N};$$

(iii) $E(\mu) \subset ca(\mathfrak{R})$.

If μ is atomic, then conditions (i), (ii) *and* (iii) *are equivalent.*

Proof. (i)⇒(ii): By assumption and Theorem 10.2, (i)⇒(ii), μ^{a} is atomic. Take an $\mathfrak{M}$-partition $\{\Omega_1, \Omega_2, \ldots\}$ of Ω such that

$$|\mu^{\mathrm{a}}(\{M \in \mathfrak{M} : M \subset \Omega_j\})| \le 2 \quad \text{for each } j \in \mathbb{N}.$$

Set $\mu_j(M) = \mu^{\mathrm{a}}(M \cap \Omega_j)$ for all $M \in \mathfrak{M}$ and $j \in \mathbb{N}$. In view of Corollaries 6.2(a) and 6.4, $E(\mu_j)$ is strongly compact. Hence $\operatorname{extr} E(\mu_j)$ is finite for each $j \in \mathbb{N}$, by Theorem 7.5 and Proposition 4.4(b). It now follows from [8, Theorem 2], (v)⇒(iv) (cf. also Theorem 11.6, (i)⇒(ii), for a generalization), that there exist $P_1, P_2, \ldots$ in $\mathfrak{R}$ and $1 = n_0 < n_1 < n_2 < \cdots$ such that, for each $j \in \mathbb{N}$, we have

$$\{P_{n_{j-1}}, \ldots, P_{n_j - 1}\} \text{ is a partition of } \Omega, \tag{2}$$

$$\mathfrak{R} = (\mathfrak{M}_{\mu_j} \cup \{P_{n_{j-1}}, \ldots, P_{n_j - 1}\})_\beta. \tag{3}$$

Set

$$R_i = P_i \cap \Omega_j \quad \text{whenever} \quad n_{j-1} \le i \le n_j - 1.$$

Clearly, $\{R_{n_{j-1}}, \ldots, R_{n_j - 1}\}$ is an $\mathfrak{R}$-partition of Ω_j. Set

$$\mathfrak{R}' = (\mathfrak{M}_\mu \cup \{R_1, R_2, \ldots\})_\beta.$$

We complete this part of the proof by showing that $\mathfrak{R} \subset \mathfrak{R}'$. To this end, it is enough to observe that, given $R \in \mathfrak{R}$ and $j \in \mathbb{N}$, we have $R \cap \Omega_j \in \mathfrak{R}'$. Now, in view of (2) and (3), there exist $N_1, \ldots, N_{n_j - n_{j-1}}$ in $\mathfrak{M}_{\mu_j}$ with

$$R \cap \Omega_j = (N_1 \cap P_{n_{j-1}}) \cup \cdots \cup (N_{n_j - n_{j-1}} \cap P_{n_j - 1}). \tag{4}$$

We may assume that $N_k \subset \Omega_j$ for $k = 1, \ldots, n_j - n_{j-1}$. It then follows that $N_k \in \mathfrak{M}_{\mu^a} = \mathfrak{M}_\mu$ (see Proposition 4.8). Moreover, (4) implies that

$$R \cap \Omega_j = (N_1 \cap R_{n_{j-1}}) \cup \cdots \cup (N_{n_j - n_{j-1}} \cap R_{n_j - 1}),$$

and so $R \cap \Omega_j \in \mathfrak{R}'$.

(ii)⇒(iii): Let $\varrho \in E(\mu)$. Since μ is σ-additive, so is $\varrho|\mathfrak{M}_\mu$. For $R_1, R_2, \ldots$ as in (ii), the σ-additivity of μ yields

$$\sum_{i=1}^{\infty} \varrho(R_i) = \sum_{j=1}^{\infty} \varrho\Big(\bigcup_{i=n_{j-1}}^{n_j - 1} R_i\Big) = \sum_{j=1}^{\infty} \mu\Big(\bigcup_{i=n_{j-1}}^{n_j - 1} R_i\Big) = \varrho(\Omega).$$

By Lemma 3.7, it follows that $\varrho \in ca(\mathfrak{R})$.

The implication (iii)⇒(i) for atomic μ holds, by Corollary 10.5, (ii)⇒(i). ∎

11.8. Remark. The implication (i)⇒(iii) of Theorem 11.7 follows directly from Theorem 5.5, (i)⇒(iv). For another proof of the converse implication for atomic μ see [46, p. 472]. The latter implication fails for nonatomic μ, as Example 12.7 shows. The author does not know whether the implication (iii)⇒(ii) of Theorem 11.7 holds for arbitrary μ.

12. Cardinality of $\operatorname{extr} E(\mu)$

Most of the material of this section is taken from [45, Sections 3 and 4]. We start by recalling that every cardinal ≥ 1 can arise as the cardinality of $\operatorname{extr} E(\mu)$, where $\mathfrak{M} = \{\emptyset, \Omega\}$ and $\mathfrak{R}$ is suitably chosen (see Example 4.1). This is, however, not so when we set some natural restrictions on μ alone or on the triplet $\mathfrak{M}$, $\mathfrak{R}$, μ, as Theorems 12.1 and 12.8, the main results of this section, show.

In the proof of Theorem 12.1 we shall apply the following proposition from general topology: *If X is a complete metric space such that every nonempty open subset of X has weight $\mathfrak{m}$, then $|X| = \mathfrak{m}^{\aleph_0}$.* This proposition goes back to F. K. Schmidt (1932); see [64, Lemma 3.1] or [12, p. 184]. In fact, we shall need an equivalent form of it with "weight" replaced by "density character". That those cardinal functions coincide for arbitrary metric spaces is well known; see, e.g., [18, Theorem 4.1.15].

12.1. Theorem (= [45, Theorem 1]). *If $\mu \in ba_+(\mathfrak{M})$ is nonatomic, then $|\operatorname{extr} E(\mu)|$ is an ω-power.*

Proof. By Proposition 4.4(b), $\operatorname{extr} E(\mu)$ equipped with the metric inherited from $ba(\mathfrak{R})$ is complete. Consequently, the assertion follows from Theorem 8.2 and the proposition formulated above. ∎

We shall apply Theorem 12.1 in the proofs of the following corollary and Theorem 12.8.

12.2. Corollary (= Corollary III.1(b)). *If $\mu \in ba_+(\mathfrak{M})$ and $(E(\mu), w)$ is compact, then $|\operatorname{extr} E(\mu)|$ is either finite or an ω-power.*

Proof. Let μ_1 and μ_2 stand for the atomic and nonatomic components of μ. In view of Corollary 6.3, $(E(\mu_1), w)$ is also compact and

$$|\operatorname{extr} E(\mu)| = |\operatorname{extr} E(\mu_1)| \cdot |\operatorname{extr} E(\mu_2)|.$$

By Theorem 7.7, (ii)$\Rightarrow$(i), $(E(\mu_1), s)$ is compact as well. The assertion now follows from Corollary 10.4 and Theorem 12.1. ∎

12.3. REMARK (cf. Remark III.2). Corollary 12.2 provides complete information on the possible cardinality $\mathfrak{m}$ of the set $\operatorname{extr} E(\mu)$ in the case where $E(\mu)$ is weakly compact. This is clear for finite $\mathfrak{m}$ (cf. Example 4.1). For infinite $\mathfrak{m}$ see Example 12.7.

We shall illustrate Theorem 12.1 by three examples. In all those examples $\mathfrak{M}$ and $\mathfrak{R}$ are σ-algebras and μ is a nonatomic measure. Examples 12.5 and 12.6 are concrete and concern the cardinals $2^{\mathfrak{c}}$ and $2^{2^{\mathfrak{c}}}$, respectively. Example 12.7 is of general character and shows that Theorem 12.1 can be reversed in the sense that every ω-power is the cardinality of $\operatorname{extr} E(\mu)$ for a suitable product measure μ. Example 12.7 is also essential for Remark 12.3 and is relevant to Theorem 12.8. In Examples 12.5 and 12.6 we shall make use of the following proposition, which will also be applied in the proof of Theorem 12.11. We note that the inequality of this proposition can be reversed under some additional assumptions (see Proposition 6.6).

12.4. PROPOSITION (= [45, Proposition 2]). *Let $\mu \in pa(\mathfrak{M})$ and let $\mathfrak{N}$ be a subalgebra of $\mathfrak{R}$ such that $M \cap N \neq \emptyset$ for all $M \in \mathfrak{M}$ with $\mu(M) > 0$ and nonempty $N \in \mathfrak{N}$. Then for each $\nu \in ult(\mathfrak{N})$ there exists $\pi \in \operatorname{extr} E(\mu)$ with $\pi|\mathfrak{N} = \nu$. In particular,*

$$|\operatorname{extr} E(\mu)| \geq |ult(\mathfrak{N})|.$$

Proof. We may additionally assume $\mathfrak{R} = (\mathfrak{M} \cup \mathfrak{N})_b$. Indeed, given $\sigma \in \operatorname{extr} E(\mu, (\mathfrak{M} \cup \mathfrak{N})_b)$, it suffices to take $\pi \in \operatorname{extr} E(\sigma, \mathfrak{R})$ (see (C)), and observe that $\pi \in \operatorname{extr} E(\mu)$.

With this additional assumption, by a result of Marczewski ([55, Theorem I and Lemma 3]; see also [26, Proposition 2]), for every $\nu \in ult(\mathfrak{N})$, there exists a (unique) $\pi \in E(\mu)$ with $\pi(M \cap N) = \mu(M)\nu(N)$ whenever $M \in \mathfrak{M}$ and $N \in \mathfrak{N}$. By (GD), $\pi \in \operatorname{extr} E(\mu)$, which completes the proof. ∎

The following well-known result will be used in Example 12.5 and in the proofs of Theorems 12.8(a) and 12.11.

(U) *If A is an infinite Boolean σ-algebra, then $|ult(A)| \geq 2^{\mathfrak{c}}$.*

This is standard for $A = 2^{\mathbb{N}}$ (see [31, Example 9.21]). In the general case, A contains a subalgebra isomorphic to $2^{\mathbb{N}}$, and so the assertion follows by the Tarski–Ulam theorem mentioned after Example 4.1.

12.5. EXAMPLE (= [45, Example 1]). Let $\Omega = [0,1] \times [0,1]$, let

$$\mathfrak{M} = \{B \times [0,1] : B \in \mathfrak{B}([0,1])\} \quad \text{and} \quad \mathfrak{R} = \mathfrak{B}(\Omega),$$

and let μ be the restriction of the two-dimensional Lebesgue measure to $\mathfrak{M}$. Clearly, we then have $|\operatorname{extr} E(\mu)| \leq 2^{\mathfrak{c}}$. The converse inequality follows by Proposition 12.4 with

$$\mathfrak{N} = \{[0,1] \times B : B \in \mathfrak{B}([0,1])\}$$

and (U).

12.6. EXAMPLE (= [45, Example 2]). Let $\Omega = [0,1]$ and $\mathfrak{R} = 2^{\Omega}$. Moreover, let $\mathfrak{M}$ be the σ-algebra of Lebesgue measurable subsets of Ω and let μ be the Lebesgue measure on $\mathfrak{M}$.

Clearly, we then have $|\operatorname{extr} E(\mu)| \le 2^{2^{\mathfrak{c}}}$. For the converse inequality consider a partition $\{B_\omega : \omega \in \Omega\}$ of Ω such that $\mu^*(B_\omega) = 1$ for each $\omega \in \Omega$ (see [54]). Set

$$\mathfrak{N} = \Big\{ \bigcup_{\omega \in E} B_\omega : E \subset \Omega \Big\}.$$

Since the algebra $\mathfrak{N}$ is isomorphic to 2^Ω, we get $|ult(\mathfrak{N})| = 2^{2^{\mathfrak{c}}}$ by a classical theorem (see [31, Example 9.21]). It now follows from Proposition 12.4 that $|\operatorname{extr} E(\mu)| = 2^{2^{\mathfrak{c}}}$.

12.7. Example (= [45, Example 3]). Fix an ω-power $\mathfrak{m} > 1$ and a set I with $|I| = \mathfrak{m}$. Let $\Omega = \{0,1\}^I$, and denote by $\mathfrak{M}$ the standard product σ-algebra of Ω and by μ the standard product measure on $\mathfrak{M}$. Since $\big|[I]^{\aleph_0}\big| = \mathfrak{m}$, we have $|\mathfrak{M}| = \mathfrak{m}$. This implies that $|\mathfrak{M}(\mu)| = \mathfrak{m}$. Set

$$E = \{f \in \Omega : |f^{-1}(0)| \le \aleph_0\} \quad \text{and} \quad \mathfrak{N} = (\mathfrak{M} \cup \{E\})_b.$$

Clearly, $\mu_*(E) = \mu_*(E^c) = 0$. The final assertion of Proposition 4.9 now shows that $|\operatorname{extr} E(\mu)| = \mathfrak{m}$. Note that, in addition, every element of $E(\mu)$ is a measure (see Lemma 3.7). Moreover, $E(\mu)$ is weakly but not strongly compact, by Corollary 5.4 and Theorem 8.1(c), respectively.

In the proof of Theorem 12.8(a) we shall use the following special case of a fundamental result due to van Douwen ([14, Theorem 7.1]; see also [15, Theorem 11.8]): *If A is a Boolean σ-algebra, then $|ult(A)|$ is either finite or an ω-power.*

Theorem 12.8 is an improvement of [44, Theorem 6]. The proof of Theorem 12.8(a) follows that of [44, Theorem 6(b)].

12.8. Theorem (= [45, Theorem 2]). *Let $\mathfrak{M}$ and $\mathfrak{N}$ be σ-algebras of subsets of Ω with $\mathfrak{M} \subset \mathfrak{N}$ and let $\mu \in ca_+(\mathfrak{M})$.*

(a) *If μ has finite range, then $|\operatorname{extr} E(\mu)|$ is either finite or an ω-power $\ge 2^{\mathfrak{c}}$.*
(b) *If μ is atomic, then $|\operatorname{extr} E(\mu)|$ is either finite or equals $\mathfrak{c}$ or is an ω-power $\ge 2^{\mathfrak{c}}$.*
(c) *If μ is arbitrary, then $|\operatorname{extr} E(\mu)|$ is either finite or an ω-power.*

Proof. We first observe that (a) holds for $\mu \in ult(\mathfrak{M})$. Indeed, 4° of Proposition 7.1 then shows that $\operatorname{extr} E(\mu)$ and $ult(\mathfrak{N}/\mathfrak{J}_\mu)$ are equipotent. Therefore, it is enough to apply van Douwen's result formulated above and (U) to get the assertion. Parts (a) and (b) now follow, by Corollary 6.4. Using Corollary 6.3, (b) and Theorem 12.1, we get (c). ∎

In connection with Theorem 12.8(b) it is worth mentioning that it is relatively consistent with ZFC that there exists an ω-power $\mathfrak{m}$ with $\mathfrak{c} < \mathfrak{m} < 2^{\mathfrak{c}}$. Indeed, by Easton's theorem (see [25, Theorem 11.25]), it is relatively consistent with ZFC that $\mathfrak{c} = \aleph_1$ and $2^{\mathfrak{c}} = \aleph_3$. Then $\mathfrak{m} = \aleph_2$ is as desired, by Hausdorff's formula (see [25, Theorem 11.27]).

12.9. Remark. Theorem 12.8 provides complete information on the possible cardinalities of the sets $\operatorname{extr} E(\mu)$ under the respective assumptions. Indeed, according to Example 4.1, every finite cardinal ≥ 1 can occur in the setting of (a). Moreover, according to Example 12.7, every ω-power > 1 can occur in the setting of (c). That the cardinal $\mathfrak{c}$ can occur in the setting of (b) is clear from its proof (see also Example 7.11). The remaining cases are settled by the following simple example already considered by van Douwen ([14, Example 14.2]) for a similar purpose.

12.10. Example (= [45, Example 4]). Let $\mathfrak{m}$ be an ω-power $\geq 2^{\mathfrak{c}}$, let Ω be a set with $|\Omega| = \mathfrak{m}$, and let $\mathfrak{R}$ be the σ-algebra of countable and cocountable subsets of Ω. Denote by π_0 the unique element of $ult(\mathfrak{R})$ that vanishes on $[\Omega]^{\aleph_0}$. Then, if $\pi \in ult(\mathfrak{R})$ and $\pi \neq \pi_0$, there is an $R \in [\Omega]^{\aleph_0}$ with $\pi(R) = 1$, and so $\pi|2^R \in ult(2^R)$. It follows that

$$\mathfrak{m} \leq ult(\mathfrak{R}) \leq 2^{\mathfrak{c}}|[\Omega]^{\aleph_0}| = 2^{\mathfrak{c}}\mathfrak{m} = \mathfrak{m}.$$

Let $\mathfrak{M} = \{\emptyset, \Omega\}$ and let μ be the unique probability measure on $\mathfrak{M}$. Then, by the above and Lemma 3.1, we have $|\operatorname{extr} E(\mu)| = \mathfrak{m}$ (cf. Example 4.1).

Our next theorem is closely related to some results of Bogner and Denk [8]. The implication (ii)⇒(i) strengthens [8, Theorem 2, (v)⇒(i)], while the implication (i)⇒(ii) is, for $\mu \in ult(\mathfrak{M})$, a special case of [8, Theorem 2, (i)⇒(v)]. Moreover, our theorem supplements [8, Theorem 1].

12.11. Theorem (= [45, Theorem 3]). *Let $\mathfrak{M}$ and $\mathfrak{R}$ be σ-algebras of subsets of Ω with $\mathfrak{M} \subset \mathfrak{R}$ and let $\mu \in ca_+(\mathfrak{M})$ be atomic. Then the following three conditions are equivalent:*

(i) $E(\mu) \subset ca(\mathfrak{R})$;
(ii) $|\operatorname{extr} E(\mu)| \leq \mathfrak{c}$;
(iii) $|\operatorname{extr} E(\mu) \setminus ca(\mathfrak{R})| < 2^{\mathfrak{c}}$.

Proof. The implication (i)⇒(ii) is a direct consequence of Corollary 10.5, (ii)⇒(i). Clearly, (ii) implies (iii). We shall show that (iii) implies (i) for $\mu \in ult(\mathfrak{M})$. The general case then follows, by Theorem 6.1.

Suppose (i) fails for some $\mu \in ult(\mathfrak{M})$. Then, by [8, Lemma 1] and Proposition 4.4(a) (see also Theorem 5.5, (v)⇒(iv)), there exists $\pi_0 \in \operatorname{extr} E(\mu)$ which is not a measure. Since $\pi_0 \in ult(\mathfrak{R})$ (see (D)′), it follows that there exists an $\mathfrak{R}$-partition $\{R_1, R_2, \ldots\}$ of Ω with $\pi_0(R_i) = 0$ for all i. Set

$$I = \{i \in \mathbb{N} : R_i \in \mathfrak{J}_\mu\} \quad \text{and} \quad R = \bigcup_{i \in I} R_i.$$

We have $R \in \mathfrak{J}_\mu$, and so $\pi_0(R) = 0$. Therefore, $\mathbb{N} \setminus I$ is infinite, say $\mathbb{N} \setminus I = \{i_1, i_2, \ldots\}$. Set $N_1 = R_{i_1} \cup R$ and $N_k = R_{i_k}$ for $k \geq 2$. Finally, define

$$\mathfrak{N} = \Big\{\bigcup_{i \in J} N_i : J \subset \mathbb{N}\Big\}$$

and apply Proposition 12.4 and (U) to conclude that condition (iii) fails for μ. ∎

Clearly, the implications (i)⇒(iii) and (ii)⇒(iii) of Theorem 12.11 hold for arbitrary $\mu \in ca_+(\mathfrak{M})$. On the other hand, Example 12.7 shows that (i) does not imply (ii), in general. The author does not know whether the remaining implications persist for nonatomic measures.

13. Open problems

Recall that some open problems have already been mentioned in the previous sections (see the passage following Proposition 4.6, Remark 11.8, and the end of Section 12). We now list some more, with a few comments.

13.1. Problem. Characterize $\operatorname{extr} E(\mu)$, where $\mu \in ba_+(\mathfrak{M})$ is nonatomic, equipped with each of the three topologies s, w and w^*, as a topological space.

In connection with this problem see Theorems 8.1, 8.2 and 8.6. An analogous problem for atomic μ or μ with finite range is solved by Theorems 7.4 and 7.5, and Remark 7.6.

The next problem has already been mentioned in the passage introducing Proposition IV.3.

13.2. Problem. Do we have $s = w$ on $\operatorname{extr} E(\mu)$ for arbitrary $\mu \in ba_+(\mathfrak{M})$?

According to Theorem 7.4(a), $s = w$ on $\operatorname{extr} E(\mu)$ if μ is atomic. This is also the case if $(E(\mu), w)$ is compact and $\mathfrak{R} = (\mathfrak{M} \cup \mathfrak{E})_b$ with $\mathfrak{E}$ being a family of pairwise disjoint subsets of Ω (see Theorem IV.4). Some conditions equivalent to the stronger condition that $s = w$ on $E(\mu)$ are given in Theorem 9.6.

13.3. Problem. Does the inequality $|E(\mu)| \leq 2^{\aleph_0 \cdot |\operatorname{extr} E(\mu)|}$ hold for arbitrary $\mu \in ba_+(\mathfrak{M})$?

This inequality holds if μ is atomic (see [48, Theorem 3(a)]) as well as under some extra assumptions on $\mathfrak{M}$ and $\mathfrak{R}$ (see [48, Theorems 5(a) and 7]). In view of an example due to Fremlin and Plebanek [20, Theorem 3A], interpreted according to Example 4.1, the inequality in question cannot be improved, even for $\mathfrak{M} = \{\emptyset, \Omega\}$, at least in ZFC.

13.4. Problem. Let $\mathfrak{N}$ be a subalgebra of $\mathfrak{R}$ with $\mathfrak{R} = (\mathfrak{M} \cup \mathfrak{N})_b$. Suppose $\mu \in ba_+(\mathfrak{M})$ and $\mu^*|\mathfrak{N}$ is exhaustive. Is $(E(\mu), w)$ then compact?

According to Corollary 5.4, the answer is in the affirmative if $\mathfrak{N}$ is finite.

13.5. Problem. Let $\mathfrak{M}$ and $\mathfrak{N}$ be σ-algebras of subsets of Ω and let $\mathfrak{R} = (\mathfrak{M} \cup \mathfrak{N})_\beta$. Suppose $\mu \in ca_+(\mathfrak{M})$ and $\mu^*|\mathfrak{N}$ is order continuous. Is $(E(\mu), w)$ then compact?

As follows from Corollary 5.8, the answer is in the affirmative if $\mathfrak{N}$ is generated by a partition.

For another open problem see the passage following Theorem 14.7.

14. Appendix. The simplex of Radon probability measures over a compact space

Throughout this section Z stands for an arbitrary compact (Hausdorff) space. The symbols $\mathcal{B}(Z)$, $\mathcal{M}(Z)$ and $\mathcal{S}(Z)$ used below are explained in the passage following Lemma 3.7. In addition, we denote by δ_z, where $z \in Z$, the Dirac Borel measure concentrated at z. Our main present purpose is to establish Proposition 14.4 and Corollary 14.6, which have already been used in the proofs of Theorems 7.4, 7.5, 9.3 and 9.5, and 7.9, respectively. In fact, we only need them in the case where Z is zero-dimensional, but the general case is also of some interest.

The following result is standard (cf., e.g., [2, Corollary II.4.2]). For the reader's convenience we sketch a rather elementary proof below.

14.1. Proposition. *The mapping* $z \mapsto \delta_z$ *of* Z *into* $(\mathcal{M}(Z), w^*)$ *is a homeomorphic embedding with range* $\operatorname{extr}\mathcal{S}(Z)$.

Sketch of proof. Clearly, this mapping is continuous and injective, and so is a homeomorphism. Since $\mathcal{S}(Z)$ is an extreme subset of $pa(\mathcal{B}(Z))$, we have

$$\operatorname{extr}\mathcal{S}(Z) = \mathcal{S}(Z) \cap ult(\mathcal{B}(Z)),$$

by Lemma 3.1. It follows that

$$\operatorname{extr}\mathcal{S}(Z) = \{\delta_z : z \in Z\}. \ \blacksquare$$

A direct consequence of Proposition 14.1 is the following corollary.

14.2. Corollary (cf. Lemma IV.3). *Let* Z_1 *and* Z_2 *be compact spaces and let* $(\mathcal{S}(Z_1), w^*)$ *and* $(\mathcal{S}(Z_2), w^*)$ *be affinely homeomorphic. Then* Z_1 *and* Z_2 *are homeomorphic.*

The analogues of Corollary 14.2 with "w^*" replaced by "w" or "s" fail, even in the case where $s = w$ on $\mathcal{S}(Z_j)$, $j = 1, 2$ (see Remark 14.8).

Recall that $x_0 \in K$, where K is a closed bounded convex set in a Banach space X, is called *strongly exposed* if there exists $x^* \in X^*$ with the following two properties: $x^*(x_0) > x^*(x)$ for all $x \in K \setminus \{x_0\}$ and the diameter of the set $\{x \in K : x^*(x) > x^*(x_0) - \varepsilon\}$, where $\varepsilon > 0$, tends to 0 when ε tends to 0 (see [9, Definition 3.2.1]).

The following proposition is probably known.

14.3. Proposition. *The strongly exposed points of* $\mathcal{S}(Z)$ *are of the form* δ_z, $z \in Z$.

Proof. In view of Proposition 14.1, we only need to check that δ_z is a strongly exposed point of $\mathcal{S}(Z)$ for each $z \in Z$. Set

$$F_z(\varphi) = \varphi(\{z\}) \quad \text{for } \varphi \in \mathcal{M}(Z).$$

Clearly, F_z is a linear functional on $\mathcal{M}(Z)$ with $\|F_z\| = 1$. It follows that, for $\varrho \in \mathcal{S}(Z)$, we have

$$F_z(\varrho) \le 1, \quad \text{and} \quad F_z(\varrho) = 1 \text{ if and only if } \varrho = \delta_z.$$

Finally, we have, for $\varrho \in \mathcal{S}(Z)$

$$\|\varrho - \delta_z\| = |\varrho - \delta_z|(\{z\}) + |\varrho - \delta_z|(Z \setminus \{z\}) = 2(1 - \varrho(\{z\}).$$

Therefore, $F_z(\varrho) \to 1$ implies $\|\varrho - \delta_z\| \to 0$, and so F_z strongly exposes δ_z. $\blacksquare$

14.4. Proposition (cf. Lemma II.3). *We have*

(a) $s = w$ *on* $\operatorname{extr}\mathcal{S}(Z)$;
(b) $\|\delta_{z_1} - \delta_{z_2}\| = 2$ *whenever* $z_1, z_2 \in Z$ *and* $z_1 \neq z_2$;
(c) $(\operatorname{extr}\mathcal{S}(Z), w)$ *is discrete;*
(d) $(\mathcal{S}(Z), w)$ *is compact if and only if* Z *is finite.*

Proof. Part (a) is seen from Propositions 14.1 and 14.3. Part (b) follows from the formula

$$|\delta_{z_1} - \delta_{z_2}|(\{z_1\}) + |\delta_{z_1} - \delta_{z_2}|(\{z_2\}) = 2,$$

where $z_1, z_2 \in Z$ and $z_1 \neq z_2$. Part (c) is a direct consequence of (a) and (b).

Finally, suppose $(\mathcal{S}(Z), w)$ is compact. Then $w = w^*$ on $\mathcal{S}(Z)$. Consequently, $(\operatorname{extr} \mathcal{S}(Z), w)$ is compact, by Proposition 14.1. This implies, in view of (c), that Z is finite. ∎

The following theorem is closely related to Lemmas IV.1 and IV.4 and its proof uses similar ideas. In particular, the main idea of the proof of the implication (i)⇒(ii) below is an adaptation of a well-known construction due to H. Rademacher.

14.5. Theorem. *For $\varrho \in \mathcal{S}(Z)$ the following two conditions are equivalent:*

(i) *$s = w$ at ϱ on $\mathcal{S}(Z)$;*

(ii) *ϱ is atomic.*

Proof. Suppose first that (ii) holds. To derive (i), fix $\varepsilon > 0$, and then a (nonempty) finite subset Z_0 of Z with $\varrho(Z \setminus Z_0) < \varepsilon/4$. Let $\sigma \in \mathcal{S}(Z)$ be such that

$$|\sigma(\{z\}) - \varrho(\{z\})| < \frac{\varepsilon}{4|Z_0|} \quad \text{for each } z \in Z_0.$$

We claim that $\|\sigma - \varrho\| < \varepsilon$. Indeed, we have $|\sigma(Z_0) - \varrho(Z_0)| < \varepsilon/4$. Since

$$\sigma(Z \setminus Z_0) = \varrho(Z \setminus Z_0) + \varrho(Z_0) - \sigma(Z_0),$$

it follows that $\sigma(Z \setminus Z_0) < \varepsilon/2$. Consequently, we have

$$\|\sigma - \varrho\| \le \sum_{z \in Z_0} |\sigma(\{z\}) - \varrho(\{z\})| + \sigma(Z \setminus Z_0) + \varrho(Z \setminus Z_0) < \varepsilon,$$

so that the claim is established. Hence (i) holds.

We shall now prove the implication $\neg$(ii) ⇒ $\neg$(i). To this end, we first assume that $\varrho \in \mathcal{S}(Z)$ is nonatomic and define, for each finite subalgebra $\mathfrak{S}$ of $\mathcal{B}(Z)$, some element $\varrho_{\mathfrak{S}} \in ba_+(\mathcal{B}(Z))$ with the following properties:

1° $\varrho_{\mathfrak{S}}|\mathfrak{S} = \varrho|\mathfrak{S}$;

2° $\varrho_{\mathfrak{S}} \le 2\varrho$;

3° $\|\varrho - \varrho_{\mathfrak{S}}\| \ge 1$.

Fix $\mathfrak{S}$ as above, and note that

$$\mathfrak{S} = \{R_1, \ldots, R_n\}_b,$$

where $\{R_1, \ldots, R_n\}$ is an $\mathfrak{S}$-partition of Ω. In view of the nonatomicity of ϱ, we can find $R_{i_1} \in \mathcal{B}(Z)$ such that

$$R_{i1} \subset R_i \quad \text{and} \quad \varrho(R_{i1}) = \frac{1}{2}\varrho(R_i), \qquad i = 1, \ldots, n.$$

Set

$$R_{i2} = R_i \setminus R_{i_1} \quad \text{and} \quad \mathfrak{S}' = \{R_{ij} : i = 1, \ldots, n;\ j = 1, 2\}_b.$$

Take $\sigma \in ba_+(\mathfrak{S}')$ with

$$\sigma(R_{i1}) = \varrho(R_i) \quad \text{and} \quad \sigma(R_{i2}) = 0, \qquad i = 1, \ldots, n.$$

We then have $\sigma|\mathfrak{S} = \varrho|\mathfrak{S}$ and $\sigma \le 2\varrho|\mathfrak{S}'$.

Using [28, Theorem 14] or the Radon–Nikodym theorem, we can find $\varrho_{\mathfrak{S}} \in ba_+(\mathcal{B}(Z))$ satisfying 2° and

$$\varrho_{\mathfrak{S}}|\mathfrak{S}' = \sigma|\mathfrak{S}'.$$

Thus, $\varrho_{\mathfrak{S}}$ has also property 1°. To establish 3°, it is enough to observe that, for $i = 1, \ldots, n$, we have

$$|\varrho(R_{i1}) - \sigma(R_{i1})| + |\varrho(R_{i2}) - \sigma(R_{i2})| = \varrho(R_i) - \varrho(R_{i1}) + \varrho(R_{i2}) = \varrho(R_i).$$

Let $\mathcal{A}$ denote the family of all finite subalgebras of $\mathcal{B}(Z)$ upward ordered by inclusion. In view of 1° and 2°, we have $\varrho_{\mathfrak{S}} \in \mathcal{S}(Z)$ for every $\mathfrak{S} \in \mathcal{A}$. Since

$$\mathcal{B}(Z) = \bigcup_{\mathfrak{S}\in\mathcal{A}} \mathfrak{S},$$

1° shows that $\varrho_{\mathfrak{S}}(R) \to \varrho(R)$ for each $R \in \mathcal{B}(Z)$. It follows from 2° and the weak compactness of the order interval $[0, 2\varrho]$ in $ca(\mathcal{B}(Z))$ (see [3, Theorem 12.9] or the beginning of the proof of Theorem 5.1) that $\varrho_{\mathfrak{S}} \to \varrho$ weakly in $\mathcal{S}(Z)$. In view of 3°, this shows that (i) fails.

Assume now that (ii) fails. Let $\varrho' \in \mathcal{M}_+(Z)$ be nonatomic, nonzero and satisfy $\varrho' \leq \varrho$. By what we have proved so far, there exists a net $\{\varrho_\alpha\}$ in $\mathcal{M}_+(Z)$ with $\varrho_\alpha(Z) = \varrho'(Z)$ for each α which converges weakly but not strongly to ϱ'. Clearly,

$$\varrho_\alpha + (\varrho - \varrho') \in \mathcal{S}(Z)$$

and the net $\{\varrho_\alpha + (\varrho - \varrho')\}$ converges weakly but not strongly to ϱ. This completes the proof. ∎

It is a well-known theorem that a compact space is scattered if and only if every Radon measure on it is atomic (see [63, Theorem 19.7.6]). That theorem, combined with Theorem 14.5, yields the following corollary.

14.6. Corollary. *A compact space Z is scattered if and only if $s = w$ on $\mathcal{S}(Z)$.*

The following result has been applied, either directly or via Corollary 7.8, in the proofs of Theorems 7.9, 9.5 and 10.2. It is also related to Theorem 9.3.

14.7. Theorem. *Let Y_i, $i \in I$, and Z_j, $j \in J$, be compact spaces of cardinality ≥ 2. If*

$$\prod_{i\in I}(\mathcal{S}(Y_i), w^*) \quad \textit{and} \quad \prod_{j\in J}(\mathcal{S}(Z_j), w^*)$$

are affinely homeomorphic, then there exists a bijection $b\colon I \to J$ such that Y_i and $Z_{b(i)}$ are homeomorphic for each $i \in I$.

Proof. In view of [49, Theorem 1.1] and Proposition 14.1 or a special case of the former result due to V. Losert (see [49, p. 176]), there exists a bijection $b\colon I \to J$ such that $(\mathcal{S}(Y_i), w^*)$ and $(\mathcal{S}(Z_{b(i)}, w^*)$ are affinely homeomorphic. Thus, an application of Corollary 14.2 yields the assertion. ∎

As noted after Corollary 14.2, its analogues with "w^*" replaced by "w" or "s" fail. Therefore, so do the corresponding analogues of Theorem 14.7. However, the author does not know whether, in Theorem 14.7 with an analogous replacement of topologies, the factors are uniquely determined up to affine homeomorphism and order.

14.8. Remark (cf. Remark IV.1). The analogues of Corollary 14.2 with "w^*" replaced by "w" or "s" fail even in the case where the compact spaces Z_1 and Z_2 are countable infinite. (Such spaces are zero-dimensional and metrizable, see [63, Theorem 8.5.4] and

[18, Theorems 3.1.21 and 3.2.5], respectively. Moreover, according to Corollary 14.6, we have $s = w$ on $\mathcal{S}(Z_j)$.) Indeed, $\mathcal{M}(Z_j)$ and l^1 can then be identified as Banach lattices. Thus, $\mathcal{S}(Z_1)$ and $\mathcal{S}(Z_2)$ are affinely isometric. On the other hand, Z_1 and Z_2 need not be homeomorphic, of course.

Postscript. Order-theoretic properties of the sets $E(\mu)$ and $\operatorname{extr} E(\mu)$ will be studied in subsequent papers by the author (in preparation).

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Index of special symbols

Index of terms