DISSERTATIONES
MATHEMATICAE
387 (2000)

The topological classification of weak unit balls of Banach spaces

by

Taras Banakh

Abstract. For a Banach space X let $\mathcal{W}(X)$ denote the class of topological spaces homeomorphic to bounded closed subsets of (X, weak). We investigate relationships between geometric properties of a Banach space X, topological properties of its weak unit ball, and properties of the class $\mathcal{W}(X)$. In particular, we prove that two Banach spaces X, Y with Kadec norms and separable duals have homeomorphic weak unit balls if and only if $\mathcal{W}(X) = \mathcal{W}(Y)$. The weak unit ball of any infinite-dimensional Banach space with Kadec norm and separable dual is topologically homogeneous. We prove that the weak unit ball B of a Banach space is homeomorphic to the weak unit ball of c_0 if and only if B is a metrizable σZ_∞-space. Two counterexamples to some natural conjectures are presented and many open problems are formulated.

Introduction. The paper is devoted to the problem of topological classification of weak unit balls of Banach spaces. Under the *weak unit ball* of a Banach space $(X, \|\cdot\|)$ we understand the closed unit ball $B = \{x \in X : \|x\| \le 1\}$ of X endowed with the weak topology. Compared to topological classification of unit balls equipped with the norm topology, classification of weak unit balls seems to be much more complex (recall that the unit ball of any Banach space is homeomorphic to a Hilbert space, see [BP, III, §6] and [To$_2$]).

In this paper we restrict ourselves to considering Banach spaces X whose duals X^* are separable. In this case the weak unit ball B of X is metrizable, so we can apply powerful tools, recently elaborated in infinite-dimensional topology. Thus, if the converse is not stated, under a *Banach space* we shall understand an *infinite-dimensional Banach space with separable dual* (equivalently, an infinite-dimensional separable Asplund space, see [Bo, 5.4.3] or [EW]).

Our considerations concentrate around the following questions:

(A) *What can be said about the topology of the weak unit ball B of a Banach space? In particular, when is the space B topologically homogeneous? When is B homeomorphic to a model space of infinite-dimensional topology?*

2000 *Mathematics Subject Classification*: 57N20, 46B20, 46B22.

The idea to write this paper came to the author during his visit to the Mathematical Institute of the Polish Academy of Sciences in Warsaw. The author expresses his sincere gratitude to the Banach Center for the invitation and hospitality.

The research was partially supported by the grant INTAS-96-0753.

(B) *For which Banach spaces is the topology of their weak unit balls an isomorphic invariant?*

This property will be referred to as the *ball invariance property* (briefly BIP).

(C) *Under what conditions are the weak unit balls of two Banach spaces homeomorphic?*

It turns out that answers to these questions depend on

(1) the class $\mathcal{W}(X)$ of topological spaces homeomorphic to closed bounded subsets of a Banach space X endowed with the weak topology, and

(2) (for complex $\mathcal{W}(X)$) properties of the norm of X.

Let us remark that $\mathcal{W}(X)$ coincides with the class $\mathcal{F}_0(B)$ of topological spaces homeomorphic to closed subsets of the weak unit ball B of X.

The appearance of the class $\mathcal{W}(X)$ raises another question:

(D) *What connections are there between geometric properties of a Banach space X, topological properties of its weak unit ball, and properties of the class $\mathcal{W}(X)$?*

As we will show in Proposition 1.12, the class $\mathcal{W}(X)$ is not too big: for a Banach space X with separable dual, $\mathcal{W}(X)$ always lies in the Borel class $\mathcal{M}_2$ of separable metrizable absolute $F_{\sigma\delta}$-spaces, and contains the Borel class $\mathcal{M}_0$ of metrizable compacta provided X is infinite-dimensional. In the intermediate case $\mathcal{W}(X) \subset \mathcal{M}_1$ (where $\mathcal{M}_1$ is the Borel class of all Polish spaces) the questions (A)–(D) have an exhaustive answer. There is an alternative: either $\mathcal{W}(X) = \mathcal{M}_0$ and the weak unit ball B of X is homeomorphic to the Hilbert cube $Q = [-1,1]^\omega$, or $\mathcal{W}(X) = \mathcal{M}_1$ and B is homeomorphic to $s = (-1,1)^\omega$, the pseudointerior of the Hilbert cube.

Banach spaces X with $\mathcal{W}(X) \subset \mathcal{M}_1$ were extensively studied in [EW]. There it was shown that for a Banach space X with separable dual the condition $\mathcal{W}(X) \subset \mathcal{M}_1$ is equivalent to the point continuity property (briefly PCP) (see the next section for the definition). On the other hand, the condition $\mathcal{W}(X) \subset \mathcal{M}_0$ is equivalent to the reflexivity of X. This just answers the question (D) for Banach spaces with PCP. Moreover, for such spaces the questions (A)–(C) have the following answers:

(a) the weak unit balls of Banach spaces with PCP, being homeomorphic to Q or s, are topologically homogeneous;

(b) Banach spaces with PCP satisfy BIP, and

(c) two Banach spaces X, Y satisfying PCP have homeomorphic weak unit balls if and only if $\mathcal{W}(X) = \mathcal{W}(Y)$.

The situation changes for Banach spaces failing PCP: the Banach space c_0 has two equivalent norms $\|\cdot\|_\infty$ and $\|\cdot\|_{\mathrm{K}}$ such that the weak unit balls $B(c_0, \|\cdot\|_\infty)$ and $B(c_0, \|\cdot\|_{\mathrm{K}})$ are not homeomorphic. Here $\|\cdot\|_\infty$ is the standard sup-norm and $\|\cdot\|_{\mathrm{K}}$ is any Kadec norm on c_0. Thus the topology of a weak unit ball of c_0 is not an isomorphic invariant and the space c_0 fails BIP. Moreover, combining the norms $\|\cdot\|_\infty$ and $\|\cdot\|_{\mathrm{K}}$ we construct an equivalent norm $\|\cdot\|$ on c_0 such that the weak unit ball $B(c_0, \|\cdot\|)$ is not topologically homogeneous. Nonetheless, the balls $B(c_0, \|\cdot\|_\infty)$ and $B(c_0, \|\cdot\|_{\mathrm{K}})$

are topologically homogeneous. This is so because $B(c_0, \|\cdot\|_\infty)$ is a σZ_∞-space while $B(c_0, \|\cdot\|_{\mathrm{K}})$ is an ∞-Baire space. σZ_∞-spaces and ∞-Baire spaces are infinite-dimensional counterparts of first Baire category spaces and Baire spaces, respectively (see the next section for the definitions). We call a norm $\|\cdot\|$ of a Banach space X a *σZ_∞-norm* (resp. *an ∞-Baire norm*) if the weak unit ball $B(X, \|\cdot\|)$ is a σZ_∞-space (resp. an ∞-Baire space).

While equivalent ∞-Baire norms exist on each Banach space (such are Kadec norms), there are Banach spaces admitting no equivalent σZ_∞-norm. In fact, if X is a Banach space with a σZ_∞-norm $\|\cdot\|$, then the weak unit ball B of X is homeomorphic to the weak unit ball $B(c_0, \|\cdot\|_\infty)$. Moreover, B is then topologically homogeneous and $\mathcal{W}(X) = \mathcal{M}_2$ (we do not know if the condition $\mathcal{W}(X) = \mathcal{M}_2$ guarantees that X has an equivalent σZ_∞-norm). On the other hand, if X is a Banach space with an ∞-Baire norm $\|\cdot\|$, then the weak unit ball B of X is topologically homogeneous. Moreover, B is homeomorphic to $B(c_0, \|\cdot\|_{\mathrm{K}})$ if and only if $\mathcal{W}(X) = \mathcal{M}_2$ and the norm $\|\cdot\|$ of X is ∞-Baire. This follows from the Classification Theorem 1.14: for Banach spaces X, Y with separable duals and ∞-Baire norms the weak unit balls of X, Y are homeomorphic if and only if $\mathcal{W}(X) = \mathcal{W}(Y)$. This theorem answers the question (C) and raises another one: for which Banach spaces is every equivalent norm ∞-Baire? In Theorem 1.17 we prove that a Banach space X has this property if and only if X satisfies BIP if and only if every equivalent weak unit ball of X is topologically homogeneous. It should be mentioned that every equivalent weak unit ball of a Banach space X is Baire if and only if X satisfies CPCP (the convex point continuity property), see Theorem 1.21.

The above-mentioned results also raise the following questions: when does a Banach space X satisfy $\mathcal{W}(X) = \mathcal{M}_2$? When does X have an equivalent σZ_∞-norm? We will give answers to these questions for Banach spaces containing an isomorphic copy of c_0, or, more generally, for Banach spaces that fail to be strongly regular (see Theorem 1.23). If a Banach space X does not satisfy CPCP, then X has an equivalent norm such that the corresponding weak unit ball of X is a σZ_n-space for every $n \in \mathbb{N}$; moreover, in this case the class $\mathcal{W}(X)$ contains the class $\mathcal{M}_2(\text{s.c.d.c.})$ consisting of all absolute $F_{\sigma\delta}$-spaces admitting a strongly countable-dimensional metrizable compactification. In Propositions 1.34 and 1.36 we show that $\mathcal{W}(Y) = \mathcal{M}_2$ provided there is a "nice" operator $T : X \to Y$ from a Banach space with $\mathcal{W}(X) = \mathcal{M}_2$.

In the second section we present two counterexamples. The first—denoted by S_*T_∞ in [GMS$_1$]—is a strongly regular Banach space with $\mathcal{W}(S_*T_\infty) = \mathcal{M}_2$. This space contains no isomorphic copy of c_0 but has an equivalent weak unit ball homeomorphic to a weak unit ball of c_0.

The second space—denoted by B_∞ in [GMS$_2$]—is a Banach space with CPCP and $\mathcal{W}(B_\infty) \neq \mathcal{M}_\alpha$ for $\alpha = 0, 1, 2$. This example disproves two long standing natural optimistic conjectures (see [Ba$_3$] and [BDP]) and shows that the situation with topological classification of weak unit balls (even with respect to Kadec norms) is more complex than one could imagine.

How many distinct classes $\mathcal{W}(X)$, where X is a Banach space with separable dual, are there? Are there Banach spaces failing PCP whose all equivalent weak balls are topologically homogeneous? How far apart are the conditions $\mathcal{W}(X) = \mathcal{M}_2$ and PCP? Which Banach spaces do admit equivalent σZ_∞-norms? These and many other open questions are posed in the final third section.

1. Topology of weak unit balls. For a Banach space X we denote by $B(X)$ the weak unit ball of X; X^{**} is the double dual to X and $B^{**}(X)$ stands for the closed unit ball of X^{**}, equipped with the $*$-weak topology. If the Banach space X is clear from the context, we use the notations B and B^{**} in place of $B(X)$ and $B^{**}(X)$, respectively. According to the Goldstine Theorem [HHZ, 64], B is a dense convex subspace in B^{**}, so it is legal to consider the pair (B^{**}, B). If X is infinite-dimensional and X^* is separable (we consider only such Banach spaces), then the ball B^{**} is an infinite-dimensional metrizable compact convex set (see [HHZ, 62]), homeomorphic to the Hilbert cube $Q = [-1,1]^\omega$ according to the Keller Theorem [BP, p. 100]. In fact, for every norm dense subset $\{x_n^*\}_{n=1}^\infty$ of the dual unit ball of X^* the metric $d(x^{**}, y^{**}) = \sum_{n=1}^\infty 2^{-n}|x^{**}(x_n^*) - y^{**}(x_n^*)|$ generates the $*$-weak topology on B^{**}. Thus B^{**} can be considered as a convex compact subset of the locally convex linear metric space (X^{**}, d). This fact allows us to apply the powerful results of [Ba$_1$] to studying the topology of the pair (B^{**}, B). Two pairs (X, Y) and (X', Y') of topological spaces are *homeomorphic* if $h(Y) = Y'$ for some homeomorphism $h : X \to X'$.

All undefined notions from functional analysis can be found in the book [HHZ], from infinite-dimensional topology in the books [BP] or [BRZ].

Let us recall that a Banach space X is *reflexive* if $X^{**} = X$, or equivalently, if the weak unit ball of X is compact. For such Banach spaces we have

1.1. Theorem. *For an infinite-dimensional separable Banach space X the following conditions are equivalent*:

(1) *X is reflexive*;
(2) $\mathcal{W}(X) \subset \mathcal{M}_0$;
(3) $\mathcal{W}(X) = \mathcal{M}_0$;
(4) *the weak unit ball B of X is homeomorphic to the Hilbert cube Q.*

Proof. We will prove the implications (4)⇒(3)⇒(2)⇒(1)⇒(2)⇒(4).

The conditions (1) and (2) are equivalent according to [HHZ, 65]. The implication (4)⇒(3) follows from the $\mathcal{M}_0$-universality of the Hilbert cube Q (Q contains a closed topological copy of each metrizable compactum); (3)⇒(2) is trivial. Finally, if $\mathcal{W}(X) \subset \mathcal{M}_0$, then B is compact, and being a dense subset of B^{**} it must coincide with B^{**}. Since B^{**} is homeomorphic to Q, we complete the proof. ■

Next, we consider the case $\mathcal{W}(X) \subset \mathcal{M}_1$. Let us recall that a Banach space X satisfies the *point continuity property* (briefly PCP) if for every bounded weakly closed subset $A \subset X$ the identity map $(A, \text{weak}) \to (A, \text{norm})$ has a point of continuity. Equivalently, X has PCP if each bounded subset of X has a relatively weak-open subset of arbitrarily small (norm) diameter (see [EW, 3.13]). According to [EW], the weak unit ball B of a Banach space X is complete-metrizable if and only if X satisfies PCP and X^* is separable.

1.2. Theorem. *For a Banach space X with separable dual the following conditions are equivalent*:

(1) *X satisfies PCP but is not reflexive*;

(2) $\mathcal{W}(X) \subset \mathcal{M}_1$ *but* $\mathcal{W}(X) \not\subset \mathcal{M}_0$;

(3) $\mathcal{W}(X) = \mathcal{M}_1$;

(4) *the pair* (B^{**}, B) *is homeomorphic to* (Q, s).

Proof. We will prove the implications (4)⇒(3)⇒(2)⇒(1)⇒(4).

The implication (4)⇒(3) follows from the $\mathcal{M}_1$-universality of s (this means that s contains a closed topological copy of each Polish space, see [En_1, 4.3.25]); (3)⇒(2) is trivial. To prove (2)⇒(1) note that $\mathcal{W}(X) \subset \mathcal{M}_1$ implies that the weak unit ball B of X is a Polish space. Then by [EW], X satisfies PCP. If $\mathcal{W}(X) \not\subset \mathcal{M}_0$, then by Theorem 1.1, the space X is not reflexive.

Finally, to prove the implication (1)⇒(4) apply [EW] to deduce that the weak unit ball B of X is a noncompact Polish space. Since the ball B^{**} is compact and B is dense in B^{**}, and $B^{**} \not\subset \mathrm{aff}(B) = X$, the pair (B^{**}, B) is homeomorphic to (Q, s) according to [BRZ, 5.2.8]. ∎

Let us remark that among Banach spaces satisfying PCP there are Banach spaces whose double duals are separable (see [HHZ, 311]), in particular separable quasireflexive Banach spaces have this property. Also, it should be mentioned that J. Bourgain and F. Delbaen [BD] have constructed a Banach space (denoted by BD in [EW]) satisfying PCP and such that the dual BD^* is isomorphic to l_1. Hence there are Banach spaces with PCP and nonseparable double duals.

Let us recall that we defined a Banach space $(X, \|\cdot\|)$ to satisfy the *ball invariance property* (briefly BIP) if for every equivalent norm $|||\cdot|||$ on X the weak unit balls $B(X, \|\cdot\|)$ and $B(X, |||\cdot|||)$ are homeomorphic.

Theorems 1.1 and 1.2 imply

1.3. Corollary. *If an infinite-dimensional Banach space X has separable dual and satisfies PCP*, *then*

(1) *X satisfies BIP*, *and*

(2) *the weak unit ball B of X*, *being homeomorphic to Q or to s*, *is topologically homogeneous.*

Let us recall that a topological space T is *topologically homogeneous* if for any points $x, y \in T$ there exists a homeomorphism $h : T \to T$ such that $h(x) = y$.

1.4. Remark. The second statement of Corollary 1.3 is not true for Banach spaces with PCP but nonseparable duals. Indeed, if X is such a Banach space, then by PCP, the "identity embedding" $B \to X$ has a continuity point $x_0 \in B$. Then B has a countable neighborhood base at x_0 but fails to have a countable neighborhood base at the origin (this easily follows from the nonseparability of X^*). Clearly, B is not topologically homogeneous.

1.5. Question. *Is there a Banach space satisfying BIP and having nonseparable dual? Is there a Banach space with nonseparable dual and topologically homogeneous weak unit* *ball? In particular, is the weak unit ball of the Banach space $C[0,1]$ or $L_1[0,1]$ topologically homogeneous?*

Also, Corollary 1.3 does not hold for Banach spaces failing PCP.

1.6. Example (cf. [HHZ, p. 240]). *The Banach space c_0 has two equivalent norms $\|\cdot\|_\infty$ and $\|\cdot\|_{\mathrm{K}}$ such that the weak unit balls of the Banach spaces $(c_0,\|\cdot\|_\infty)$ and $(c_0,\|\cdot\|_{\mathrm{K}})$ are not homeomorphic. Thus c_0 does not satisfy BIP.*

Proof. The norm $\|\cdot\|_\infty$ is the standard sup-norm on c_0: $\|x\|_\infty = \sup_{n\in\mathbb{N}}|x_n|$, where $x=(x_n)_{n=1}^\infty \in c_0$. The weak unit ball $B(c_0,\|\cdot\|_\infty)$ is a space of the first Baire category because it can be written as $B(c_0,\|\cdot\|_\infty)=\bigcup_{n=1}^\infty B_n$, where $B_n=\{(x_i)_{i=1}^\infty \in B(c_0,\|\cdot\|_\infty) : |x_i|\le 1/2 \text{ for } i>n\}$ are closed nowhere dense subsets in $B(c_0,\|\cdot\|_\infty)$.

The norm $\|\cdot\|_{\mathrm{K}}$ is any Kadec norm on c_0 (for example, Day's locally uniformly rotund norm on c_0, see [DGZ, p. 69]). Let us recall that a norm $\|\cdot\|$ of a Banach space is called a *Kadec norm* if the weak and the norm topologies coincide on the unit sphere. It is known that every separable Banach space admits an equivalent Kadec norm; see [BP, VI, §3]. For a Kadec norm $\|\cdot\|_{\mathrm{K}}$ on c_0, the unit sphere $S\subset B(c_0,\|\cdot\|_{\mathrm{K}})$ is a dense absolute G_δ-set in $B(c_0,\|\cdot\|_{\mathrm{K}})$. This implies that the weak unit ball $B(c_0,\|\cdot\|_{\mathrm{K}})$ is a Baire space and hence is not homeomorphic to $B(c_0,\|\cdot\|_\infty)$. ∎

1.7. Example. *The Banach space c_0 has an equivalent norm $\|\cdot\|$ such that the weak unit ball $B(c_0,\|\cdot\|)$ is not topologically homogeneous.*

Proof. We shall construct this norm not on c_0 but on its isomorphic copy $c_0\oplus\mathbb{R}$. Let $\|\cdot\|_{\mathrm{K}}$ be any equivalent Kadec norm on $c_0\oplus\mathbb{R}$ such that $\|x\|_\infty \ge \|x\|_{\mathrm{K}}$ for any $x\in c_0\subset c_0\oplus\mathbb{R}$. We claim that the norm $\|\cdot\|$ on $c_0\oplus\mathbb{R}$ defined for $(x,t)\in c_0\oplus\mathbb{R}$ by

$$\|(x,t)\| = \max\{2\|x\|_\infty, \|(x,t)\|_{\mathrm{K}}\}$$

is as required. Evidently, the set

$$\begin{aligned} B &= \{(x,t)\in c_0\oplus\mathbb{R} : \|x\|_\infty \le 1/2,\ \|(x,t)\|_{\mathrm{K}}\le 1\} \\ &= \big(\tfrac12 B(c_0,\|\cdot\|_\infty)\oplus\mathbb{R}\big)\cap B(c_0\oplus\mathbb{R},\|\cdot\|_{\mathrm{K}}) \end{aligned}$$

is the closed unit ball with respect to this norm $\|\cdot\|$. We shall show that the weak unit ball $B=B(c_0\oplus\mathbb{R},\|\cdot\|)$ contains two open sets U_1, U_2 such that U_1 is of the first Baire category, while U_2 is a Baire space. Of course, this implies that B is not topologically homogeneous.

To define the set U_1 find any nonzero vector $t_1\in\mathbb{R}\subset c_0\oplus\mathbb{R}$ with $\|t_1\|_{\mathrm{K}}<1/2$ and let

$$U_1=\{(x,t)\in B : |t|<|t_1|\}.$$

Evidently, U_1 is (weakly) open in B. We claim that $U_1=\{(x,t)\in c_0\oplus\mathbb{R} : \|x\|_\infty \le 1/2,\ |t|<|t_1|\}$. This follows from the inequality $\|(x,t)\|_{\mathrm{K}}\le \|x\|_{\mathrm{K}}+\|t\|_{\mathrm{K}}<\|x\|_\infty+\|t_1\|_{\mathrm{K}}\le 1/2+1/2=1$, which holds for every $(x,t)\in c_0\oplus\mathbb{R}$ with $\|x\|_\infty\le 1/2$, $|t|<|t_1|$. Thus U_1 is homeomorphic to $B(c_0,\|\cdot\|_\infty)\times(-t_1,t_1)$. Since the weak unit ball $B(c_0,\|\cdot\|_\infty)$ is of the first Baire category, we see that U_1 is of the first Baire category as well.

To define the set U_2 find a vector $t_2\in\mathbb{R}$ with $\|t_2\|_{\mathrm{K}}=1$. Since $\|\cdot\|_{\mathrm{K}}$ is a Kadec norm, there is a weakly open neighborhood $U_2\subset B(c_0\oplus\mathbb{R},\|\cdot\|_{\mathrm{K}})$ of the point t_2 on the unit sphere of $(c_0 \oplus \mathbb{R}, \|\cdot\|_{\mathrm{K}})$ such that $U_2 \subset \{(x,t) \in c_0 \oplus \mathbb{R} : \|x\|_\infty < 1/2\}$. Since $B(c_0 \oplus \mathbb{R}, \|\cdot\|_{\mathrm{K}})$ is a Baire space, so is its open subset U_2. Now notice that $U_2 = U_2 \cap B(c_0 \oplus \mathbb{R}, \|\cdot\|)$ is an open set in the weak unit ball $B(c_0 \oplus \mathbb{R}, \|\cdot\|)$. Thus U_2 is an open Baire subspace in $B(c_0 \oplus \mathbb{R}, \|\cdot\|)$. ■

In light of Example 1.7 the following question appears.

1.8. QUESTION. *Is there a Banach space whose weak unit ball has finite homeomorphism group?*

In contrast with the weak ball from Example 1.7, the weak balls $B(c_0, \|\cdot\|_\infty)$ and $B(c_0, \|\cdot\|_{\mathrm{K}})$ are topologically homogeneous. This is so because $B(c_0, \|\cdot\|_\infty)$ is a σZ_∞-space while $B(c_0, \|\cdot\|_{\mathrm{K}})$ is an ∞-Baire space. Let us recall their definitions.

A subset A of a topological space X is called *n-dense* in X, where $n \geq 0$, if every continuous map $f : I^n \to X$ of the n-dimensional cube $I^n = [0,1]^n$ can be uniformly approximated by continuous maps into A. A subset A of a topological space X is called a *Z_n-set* if A is closed in X and its complement $X \setminus A$ is n-dense in X. A subset $A \subset X$ is called *∞-dense* (resp. a *Z_∞-set*) in X if A is n-dense (resp. a Z_n-set) in X for every $n \in \mathbb{N}$.

A topological space X is called a *σZ_n-space*, where $n \in \mathbb{N} \cup \{0, \infty\}$, if X can be expressed as a countable union $X = \bigcup_{i=1}^\infty A_i$ of Z_n-sets in X.

Let us remark that a subset A of a topological space X is dense (resp. closed and nowhere dense) in X if and only if A is 0-dense (resp. a Z_0-set) in X; X is of the first Baire category if and only if X is a σZ_0-space.

A topological space X is called an *∞-Baire space* if X contains an ∞-dense absolute G_δ-subset. Note that each ∞-Baire X space is Baire, that is, the intersection of any countable collection of open dense subsets of X is dense in X.

A norm $\|\cdot\|$ of a Banach space X is called a *σZ_∞-norm* (resp. *∞-Baire norm*) if the weak unit ball $B(X, \|\cdot\|)$ is a σZ_∞-space (resp. an ∞-Baire space).

Because each complete-metrizable space is ∞-Baire, each equivalent norm on a Banach space with PCP is ∞-Baire. Another important example of ∞-Baire norms are Kadec norms.

1.9. PROPOSITION. *Every Kadec norm on a Banach space is ∞-Baire.*

PROOF. By the definition of a Kadec norm, the weak and the norm topologies coincide on the unit sphere $S = \{x \in X : \|x\| = 1\}$ of a Banach space X whose norm is Kadec. Since in the norm topology the sphere S is complete-metrizable, we see that S is an absolute G_δ-subset in the weak unit ball B of X. It remains to verify that S is ∞-dense in B. Let $f : K \to B$ be a continuous map of a finite-dimensional cube into the weak ball B and let U be a weakly open convex neighborhood of the origin in X. It suffices to construct a continuous map $\bar{f} : K \to S$ such that $f(x) - \bar{f}(x) \in U$ for each $x \in K$. First, by the standard technique (see [CuDM, p. 841] or [Ba$_2$]), approximate f by a map $f' : K \to B$ such that $\mathrm{span}(f'(K))$ is finite-dimensional and $\|f'(x)\| < 1$, $f(x) - f'(x) \in \frac{1}{2}U$ for each $x \in K$. Next, using the fact that the set $\frac{1}{2}U$ is unbounded, find a point $y \in \frac{1}{2}U$ with $\|y\| > 2$. Finally, for every $x \in K$ let $\bar{f}(x)$ be the unique point of intersection of the segment $f'(x) + [0,1]y$ with the sphere S. It is easy to verify that the map $\bar{f} : K \to S$

is continuous. Moreover, $\bar{f}(x) - f(x) = \bar{f}(x) - f'(x) + f'(x) - f(x) \in [0,1]y + \frac{1}{2}U \subset \frac{1}{2}U + \frac{1}{2}U = U$ for each $x \in K$. ■

1.10. THEOREM. *Let X be an infinite-dimensional Banach space with separable dual. The weak unit ball B of X is topologically homogeneous provided the norm of X is a σZ_∞-norm or an ∞-Baire norm.*

PROOF. The theorem is a direct consequence of the following general fact proven in [Ba$_1$]: Let C be an infinite-dimensional symmetric convex set in a locally convex linear metric space L such that the closure $\overline{C}$ of C in L is compact and C is closed in its affine hull. Then C is topologically homogeneous provided C is a σZ_∞-space or an ∞-Baire space. ■

1.11. REMARK. The argument of Remark 1.4 shows that for every Banach space X with Kadec norm and nonseparable dual, the weak unit ball of X is not topologically homogeneous. Thus Theorem 1.10 fails beyond the class of infinite-dimensional Banach spaces with separable dual.

In fact, for σZ_∞-norms we may prove much more. But first we need the following

1.12. PROPOSITION. *If X is a Banach space with separable dual*, *then the weak unit ball B of X is an absolute $F_{\sigma\delta}$-set* (*equivalently*, $\mathcal{W}(X) \subset \mathcal{M}_2$).

PROOF. It suffices to show that B is an $F_{\sigma\delta}$-set in B^{**}. This follows from the obvious representation

$$B = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} B^{**} \cap \left(x_m + \frac{1}{n} B^{**}\right),$$

where $\{x_n\}_{n=1}^{\infty}$ is a norm dense countable subset of B. ■

Below, $\Sigma = \{(x_i)_{i\in\omega} \in Q : \sup_{i\in\omega} |x_i| < 1\}$ is the radial interior of the Hilbert cube Q.

1.13. THEOREM. *Let X be a Banach space with separable dual and σZ_∞-norm. Then the weak unit ball B of X is homeomorphic to the weak unit ball $B(c_0, \|\cdot\|_\infty)$*; *moreover*, *the pair (B^{**}, B) is homeomorphic to $(Q^\omega, \Sigma^\omega)$.*

PROOF. We will use the fact that the weak unit ball B is ∞-*convex*. The latter means that for every bounded sequence $(x_n)_{n=1}^{\infty} \subset B$ and every sequence $(t_n)_{n=1}^{\infty} \subset [0,1]$ with $\sum_{n=1}^{\infty} t_n = 1$ the point $x_\infty = \sum_{n=1}^{\infty} t_n x_n$ belongs to B. According to Corollary 10 of [Ba$_1$], an ∞-convex set C in a locally convex linear metric space is homeomorphic to Σ^ω provided C is closed in its affine hull $\mathrm{aff}(C)$, C is an absolute $F_{\sigma\delta}$-set, and C is a σZ_∞-space. Because the weak unit ball B of a Banach space with σZ_∞-norm satisfies these conditions, it is homeomorphic to Σ^ω. To show that B is homeomorphic to the weak unit ball $B(c_0, \|\cdot\|_\infty)$ it now suffices to verify that $B(c_0, \|\cdot\|_\infty)$ is homeomorphic to Σ^ω. This will be done as soon as we verify that $B(c_0, \|\cdot\|_\infty)$ is a σZ_∞-space (cf. [GM, 2.5]). For this, observe that the sets B_n from Example 1.6 are Z_∞-sets in $B(c_0, \|\cdot\|_\infty)$. Indeed, each continuous map $f : Q \to B(c_0, \|\cdot\|_\infty)$ may be approximated by a map

$f_i : Q \to B(c_0, \|\cdot\|_\infty)$ defined for $q \in Q$ by

$$\mathrm{pr}_j \circ f_i(q) = \begin{cases} \mathrm{pr}_j \circ f(q) & \text{if } j < i, \\ 1 & \text{if } j = i, \\ 0 & \text{if } j > i, \end{cases}$$

where $\mathrm{pr}_j : c_0 \to \mathbb{R}$ denotes the projection onto the jth coordinate. Evidently, for $i > n$ we have $f_i(Q) \cap B_n = \emptyset$, i.e. B_n is a Z_∞-set in $B(c_0, \|\cdot\|_\infty)$.

To prove that the pair (B^{**}, B) is homeomorphic to $(Q^\omega, \Sigma^\omega)$ we shall apply Theorem 3.1.9 of [BRZ]. This theorem implies that a pair $(\overline{T}, T)$ of topological spaces is homeomorphic to $(Q^\omega, \Sigma^\omega)$ if and only if $\overline{T}$ is homeomorphic to the Hilbert cube Q, T is homeomorphic to Σ^ω and T is homotopy dense in $\overline{T}$. The latter means that there exists a homotopy $h : \overline{T} \times [0,1] \to \overline{T}$ such that $h(\overline{T} \times (0,1]) \subset T$ and $h(x,0) = x$ for every $x \in \overline{T}$. According to [BRZ, §1.2, Ex. 12, 13] every convex subset C of a locally convex linear metric space is homotopy dense in its closure (see also [Du]). This implies that B is homotopy dense in its closure B^{**}. Since B^{**} is homeomorphic to Q and B to Σ^ω, the pair (B^{**}, B) is homeomorphic to $(Q^\omega, \Sigma^\omega)$. ■

For ∞-Baire norms the situation is more complex. For a Banach space X denote by $\mathcal{W}(X^{**}, X)$ the class of pairs homeomorphic to a pair $(K, K \cap X)$, where K is a compact subset of the space X^{**} endowed with the $*$-weak topology. Evidently, $\mathcal{W}(X^{**}, X)$ coincides with the class $\mathcal{F}_0(B^{**}, B)$ of topological copies of pairs $(K, K \cap B)$, where K is a closed subset of B^{**}. Recall that $\mathcal{F}_0(T)$ is the class of all topological copies of closed subspaces of a topological space T.

1.14. Classification Theorem. *Let X, Y be two Banach spaces with separable duals and ∞-Baire norms.*

(1) *The weak unit balls $B(X)$ and $B(Y)$ of X and Y are homeomorphic if and only if $\mathcal{W}(X) = \mathcal{W}(Y)$.*

(2) *The pairs $(B^{**}(X), B(X))$ and $(B^{**}(Y), B(Y))$ are homeomorphic if and only if $\mathcal{W}(X^{**}, X) = \mathcal{W}(Y^{**}, Y)$.*

Proof. This theorem is a direct consequence of the following general fact proven in [Ba_1]: Let C_1, C_2 be convex symmetric ∞-Baire subspaces in locally convex spaces such that the closure $\overline{C}_i$ of C_i is a metrizable compactum and C_i is closed in its affine hull $\mathrm{aff}(C_i)$ for $i = 1, 2$. Then

(1) C_1 and C_2 are homeomorphic if and only if $\mathcal{F}_0(C_1) = \mathcal{F}_0(C_2)$;

(2) the pairs $(\overline{C}_1, C_1)$ and $(\overline{C}_2, C_2)$ are homeomorphic if and only if $\mathcal{F}_0(\overline{C}_1, C_1) = \mathcal{F}_0(\overline{C}_2, C_2)$. ■

1.15. Question. *Is the Classification Theorem true for Banach spaces with nonseparable duals?*

For classes $\mathcal{K}$, $\mathcal{C}$ of topological spaces let $(\mathcal{K}, \mathcal{C})$ be the class of pairs (K, C), where $\mathcal{C} \ni C \subset K \in \mathcal{K}$.

1.16. Corollary. *For a Banach space X the following conditions are equivalent*:

(1) $\mathcal{W}(X) = \mathcal{M}_2$ *and the norm of X is ∞-Baire*;

(2) $\mathcal{W}(X^{**}, X) = (\mathcal{M}_0, \mathcal{M}_2)$ *and the norm of* X *is* ∞*-Baire*;

(3) *the weak unit ball* B *of* X *is homeomorphic to the weak ball* $B(c_0, \|\cdot\|_{\mathrm{K}})$, *where* $\|\cdot\|_{\mathrm{K}}$ *is any Kadec norm on* c_0;

(4) *the pair* (B^{**}, B) *is homeomorphic to the pair* $(B^{**}(c_0, \|\cdot\|_{\mathrm{K}}), B(c_0, \|\cdot\|_{\mathrm{K}}))$.

Proof. We shall prove the implications (4)⇒(3)⇒(1)⇒(2)⇒(4), the first of which is trivial. To verify that (3)⇒(1), observe that by Proposition 1.9, the weak unit ball $B(c_0, \|\cdot\|_{\mathrm{K}})$ is ∞-Baire and so is its topological copy B. Next, because $\mathcal{F}_0(B(c_0, \|\cdot\|_{\mathrm{K}})) = \mathcal{W}(c_0) = \mathcal{F}_0(B(c_0, \|\cdot\|_\infty)) = \mathcal{F}_0(\Sigma^\omega)$ (Theorem 1.13) and $\mathcal{F}_0(\Sigma^\omega) = \mathcal{M}_2$ (by [BM, 6.3]), we get $\mathcal{W}(X) = \mathcal{F}_0(B) = \mathcal{M}_2$.

The implication (1)⇒(2) follows from Theorem 3.1.1 of [BRZ], which implies that for every pair (K, C) with $K \in \mathcal{M}_0$ and $\mathcal{F}_0(C) = \mathcal{M}_2$ we have $\mathcal{F}_0(K, C) = (\mathcal{M}_0, \mathcal{M}_2)$.

The final implication (2)⇒(4) follows from Classification Theorem 1.14 and Theorem 1.13, according to which $\mathcal{W}(c_0^{**}, c_0) = \mathcal{F}_0(B^{**}(c_0, \|\cdot\|_\infty), B(c_0, \|\cdot\|_\infty)) = \mathcal{F}_0(Q^\omega, \Sigma^\omega) = (\mathcal{M}_0, \mathcal{M}_2)$ (for the last equality, see [BM]). ∎

In light of the last two statements, it would be interesting to detect Banach spaces whose all equivalent norms are ∞-Baire. It turns out that this property is equivalent to BIP (the ball invariance property).

1.17. Theorem. *For an infinite-dimensional Banach space* X *with separable dual the following conditions are equivalent*:

(1) X *has BIP*;

(2) *every equivalent norm* $\|\cdot\|$ *on* X *is* ∞*-Baire*;

(3) *every equivalent weak unit ball of* X *is topologically homogeneous.*

Proof. The equivalence (1)⇔(2) is a direct consequence of Classification Theorem 1.14; the implication (2)⇒(3) follows from Theorem 1.10.

To prove the implication (3)⇒(2) we will exploit the trick used in Example 1.7. But first we need to establish some elementary properties of ∞-Baire spaces. For the theory of absolute retracts we refer the reader to [Bor].

1.18. Claim. *An absolute* G_δ*-set* G *of a separable absolute neighborhood retract* T *is* ∞*-dense if and only if* G *is homotopy dense in* T.

Proof. Let us recall that a subset G of a space T is *homotopy dense in* T if there exists a homotopy $h : T \times [0, 1] \to T$ such that $h(T \times (0, 1]) \subset G$ and $h(x, 0) = x$ for every $x \in T$. Actually only the "only if" part of 1.18 requires a proof. Let G be an ∞-dense absolute G_δ-set in an absolute neighborhood retract T. According to [To$_1$] the space T embeds as an ∞-dense subset into a complete-metrizable absolute neighborhood retract $\widetilde{T}$. Then G is ∞-dense in $\widetilde{T}$ and its complement $\widetilde{T} \setminus G$ is a σZ_∞-set in $\widetilde{T}$, i.e. $\widetilde{T} \setminus G$ is a countable union of Z_∞-sets in $\widetilde{T}$. Since each σZ_∞-set in a Polish ANR has homotopy dense complement (see [BRZ, §1.4, Ex. 3]), we see that G is homotopy dense in $\widetilde{T}$, and consequently in T. ∎

1.19. Claim. *A separable absolute neighborhood retract* T *is* ∞*-Baire if and only if every point of* T *has an* ∞*-Baire neighborhood.*

Proof. If every point of T has an ∞-Baire neighborhood, then we may construct a locally finite cover $\mathcal{U}$ of T by open ∞-Baire subsets. For every $U \in \mathcal{U}$ fix an absolute G_δ-set $G_U \subset U$, ∞-dense in U. Since $\mathcal{U}$ is locally finite, the union $G = \bigcup_{U\in\mathcal{U}} G_U$ is an absolute G_δ-set. By Claim 1.18, the set $U \cap G \supset G_U$ is homotopy dense in U. Then by [BRZ, §1.2, Ex. 3], the set G is homotopy dense in T. By Claim 1.18, G is ∞-dense in T. Thus T is an ∞-Baire space. ∎

Now we are ready to prove the implication (3)⇒(2) of 1.17. Assume on the contrary that any equivalent weak ball of X is topologically homogeneous but there exists an equivalent norm $\|\cdot\|$ on X which is not ∞-Baire. Since weak balls, being convex subsets of locally convex spaces, are absolute retracts [BP, II, §3], we may apply Claim 1.19 and conclude that the weak unit ball $B(X, \|\cdot\|)$ contains a point x_0 having no ∞-Baire neighborhood. Let $f \in X^*$ be a functional with $\|f\| = 1$ and $f(x_0) = 0$. Pick a point $e \in X$ such that $\|e\| \le 2$ and $f(e) = 1$. Fix any equivalent Kadec norm $\|\cdot\|_{\mathrm{K}}$ on X such that $\|x\|_{\mathrm{K}} \le \frac{1}{2}\|x\|$ for any $x \in X$. Since the norms $\|\cdot\|$ and $\|\cdot\|_{\mathrm{K}}$ are equivalent, there exists a $\delta > 0$ such that $\delta\|x\| \le \|x\|_{\mathrm{K}}$ for every $x \in X$.

Observe that for every positive constant C the formula $T_C(x) = (x - f(x)e) + Cf(x)e$, $x \in X$, determines an isomorphism $T_C : X \to X$ (with inverse $T_C^{-1}(y) = (y - f(y)e) + \frac{1}{C}f(y)e$). Consequently, $\|x\|_C = \|T_C(x)\|_{\mathrm{K}}$ is a Kadec norm on X. Let $C = \|e\|/\|e\|_{\mathrm{K}} + 1 \ge 3$ and

$$|||x||| = \max\{\|x\|, \|x\|_C\} = \max\{\|x\|, \|x + (C-1)f(x)e\|_{\mathrm{K}}\} \quad \text{for } x \in X.$$

Evidently, $|||\cdot|||$ is an equivalent norm on X. We claim that the weak unit ball $B(X, |||\cdot|||)$ is not topologically homogeneous. To prove this, observe that for any $x \in X$ with $|f(x)| < 1/4$ we have

$$\begin{aligned}\|x\|_C &= \|x - (C-1)f(x)e\|_{\mathrm{K}}\\ &= \left\|x + \frac{\|e\|}{\|e\|_{\mathrm{K}}}f(x)e\right\|_{\mathrm{K}} \le \|x\|_{\mathrm{K}} + f(x)\|e\| \le 1/2\|x\| + 1/2.\end{aligned}$$

Consequently, for $x \in f^{-1}(-1/4, 1/4)$ the inequality $|||x||| \le 1$ is equivalent to $\|x\| \le 1$. This means that $f^{-1}(-1/4,1/4) \cap B(X, \|\cdot\|) = f^{-1}(-1/4,1/4) \cap B(X, |||\cdot|||)$ and thus the point x_0 belongs to $B(X, |||\cdot|||)$ and has no ∞-Baire neighborhood in $B(X, |||\cdot|||)$.

On the other hand, consider the point $y_0 = e/(\|e\| + \|e\|_{\mathrm{K}})$ and observe that $\|y_0\|_C = \|y_0 + (C-1)f(y_0)e\|_{\mathrm{K}} = 1$ while $\|y_0\| = \|e\|/(\|e\| + \|e\|_{\mathrm{K}}) \le 1/(1+\delta) < 1$. Let $\varepsilon = 1 - 1/(1+\delta)$ and $U = \{x \in X : |||x - y_0||| < \varepsilon/2\}$. Then for every $x \in U$ we have

$$\|x\| \le \|y_0\| + \|x - y_0\| \le \frac{1}{1+\delta} + \frac{\varepsilon}{2} = 1 - \frac{\varepsilon}{2} \le \|y_0\|_C - \|x - y_0\|_C \le \|x\|_C.$$

This implies $|||x||| = \|x\|_C$ for $x \in U$ and $U \cap B(X, |||\cdot|||) = U \cap B(X, \|\cdot\|_C)$. Since $\|\cdot\|_C$ is a Kadec norm, $V = U \cap B(X, |||\cdot|||)$ is a (weak) neighborhood of y_0 in $B(X, |||\cdot|||)$ and V is an ∞-Baire space. If $B(X, |||\cdot|||)$ were topologically homogeneous, we would find a homeomorphism h of $B(X, |||\cdot|||)$ with $h(y_0) = x_0$. Then $h(V)$ would be an ∞-Baire neighborhood of x_0 in $B(X, |||\cdot|||)$, a contradiction. ∎

1.20. Remark. Theorem 1.17 is specific to Banach spaces with separable duals and is not valid in the general case. Indeed, equivalent weak balls of nonseparable reflexive Banach spaces, being compact, are ∞-Baire spaces. Yet, they are not topologically homogeneous (see Remark 1.4).

By Corollary 1.3, every Banach space with PCP has BIP. We do not know if the converse is true. Nonetheless, we are able to show that Banach spaces with BIP satisfy CPCP, a convex analog of PCP. This follows from Theorem 1.17 and the subsequent Theorem 1.21.

Let us recall that a Banach space X satisfies the *convex point continuity property* (briefly CPCP) if for every convex bounded closed subset A in X the identity map $(A, \mathrm{weak}) \to (A, \mathrm{norm})$ has a point of continuity.

1.21. Theorem. *For a separable Banach space X the following conditions are equivalent*:

(1) *X satisfies CPCP*;

(2) *each equivalent weak unit ball of X is a Baire space*;

(3) *for each equivalent weak unit ball B of X there is $n \in \mathbb{N}$ such that B is not a σZ_n-space.*

Proof. The implication (2)$\Rightarrow$(3) is trivial.

(1)$\Rightarrow$(2). If X satisfies CPCP, then by Lemma I.0 of [GGMS] the set C of continuity points of the identity map $B(X) \to X$ is a dense G_δ-set in $B(X)$. Since C is a weak G_δ-set, C is a norm G_δ-set in X and thus C is an absolute G_δ-space. Then $B(X)$ is a Baire space as a topological space containing a dense absolute G_δ-set.

(3)$\Rightarrow$(1) Suppose the Banach space X satisfies the condition (3) of 1.21 but fails CPCP. Then by [Ha, Lemma 7] the space X admits an equivalent norm $\|\cdot\|$ for which there exists $\varepsilon > 0$ such that for every point $x \in X$ with $\|x\| < 1$ and every closed linear subspace F of finite codimension in X we have $\mathrm{diam}_{\|\cdot\|}(x + F) \cap B > \varepsilon$, where B is the weak unit ball corresponding to the norm $\|\cdot\|$. To get a contradiction, we shall prove that the weak unit ball B is a σZ_n-space for every $n \in \mathbb{N}$. For this we shall need

1.22. Claim. *For every $n \in \mathbb{N}$ there exists a constant $\delta(n) > 0$ such that for every point $a \in B$ and any nonempty relatively weak-open subsets $U_0, \ldots, U_n \subset B$ there are points $x_i \in U_i$, $0 \le i \le n$, such that $\|x - a\| > \delta(n)$ for every $x \in \mathrm{conv}\{x_0, \ldots, x_n\}$.*

Proof. Let $\alpha_0 = 2$ and $\alpha_{k+1} = 2(1 + \sum_{i=0}^{k} \alpha_i^2/\varepsilon)$ for $0 \le k \le n$. Next, go backwards and let $\lambda_n = \alpha_n/\varepsilon$ and $\lambda_k = (\alpha_k + \sum_{i=k+1}^{n} 2\lambda_i\alpha_i)/\varepsilon$ for $n > k \ge 0$. We shall show that the number $\delta(n) = (\lambda_0 + \ldots + \lambda_n)^{-1}$ satisfies our requirements.

Fix any point $a \in B$ and nonempty relatively weak-open subsets $U_0, \ldots, U_n \subset B$. For every $i \in \{0, \ldots, n\}$ pick a point $y_i \in U_i$ with $\|y_i\| < 1$. It follows from the definition of the weak topology that there exists a closed linear subspace $F_0 \subset X$ of finite codimension such that $(y_i + F_0) \cap B \subset U_i$ for every $i \in \{0, \ldots, n\}$.

By finite induction we shall construct points $x_0, \ldots, x_n \subset B$, $e_0, \ldots, e_n \in X$, functionals $e_0^*, \ldots, e_n^* \in X^*$ and closed linear subspaces $F_0 \supset F_1 \supset \ldots \supset F_n$ of finite codimension in X such that for every $k \in \{0, \ldots, n\}$ the following conditions are satisfied:

(1) $x_k \in (y_k + F_k) \cap B$;

(2) $x_k = a + e_k + \sum_{i=0}^{k-1} e_i^*(y_k - a)e_i$;

(3) $e_k^*(e_k) = 1$ and $e_k^*(e_i) = 0$ for $i < k$;

(4) $\varepsilon/2 < \|e_k\| < \alpha_k$ and $\|e_k^*\| < \alpha_k/\varepsilon$;

(5) $F_{k+1} = F_k \cap \operatorname{Ker}(e_k^*)$.

Suppose for some $k < n$ points $x_0, \dots, x_k \in B$, vectors $e_0, \dots, e_k \in X$, functionals $e_0^*, \dots, e_k^* \in X^*$ and closed linear subspaces $F_0 \supset \dots \supset F_{k+1}$ satisfying the conditions (1)–(5) have been constructed. Let

$$b_{k+1} = a + \sum_{i=0}^{k} e_i^*(y_k - a)e_i.$$

By the property of the norm of X, the diameter of the set $(y_{k+1} + F_{k+1}) \cap B$ is greater than ε. Consequently, there is a point $x_{k+1} \in (y_{k+1}+F_{k+1}) \cap B$ with $\|x_{k+1} - b_{k+1}\| > \varepsilon/2$. Let $e_{k+1} = x_{k+1} - b_{k+1}$. Clearly, conditions (1) and (2) hold. Observe that

$$\begin{aligned}\frac{\varepsilon}{2} &< \|e_{k+1}\| \le \|x_{k+1}\| + \|a\| + \sum_{i=0}^{k} |e_i^*(y_k - a)| \cdot \|e_i\| \\ &< 2 + \sum_{i=0}^{k} 2\|e_i^*\| \cdot \|e_i\| \le 2\bigg(1 + \sum_{i=0}^{k} \frac{\alpha_i^2}{\varepsilon}\bigg) = \alpha_{k+1}.\end{aligned}$$

Fix an arbitrary $m \le k$. We claim that $e_m^*(e_i) = 0$ for all $0 \le i \le k+1$ with $i \ne m$. Indeed, if $i < m$, then this follows from condition (3). If $i > m$, then $x_i - y_i \in F_i \subset F_m \subset \operatorname{Ker}(e_m^*)$ and thus

$$e_m^*(e_i) = e_m^*(x_i - b_i) = e_m^*(y_i - a) - \sum_{j=0}^{i-1} e_j^*(y_i - a)e_m^*(e_j) = \sum_{j=m+1}^{i-1} e_j^*(y_i - a)e_m^*(e_j),$$

which by induction on $i > m$ just yields $e_m^*(e_i) = 0$ for all $m < i \le k+1$.

To construct the functional e_{k+1}^* consider the linear subspace L of X spanned by the vectors e_i, $0 \le i \le k+1$. For every $x \in L$ write $x = \sum_{i=0}^{k+1} t_i(x)e_i$ for some constants $t_0(x), \dots, t_{k+1}(x) \in \mathbb{R}$. Applying the functionals e_i^*, $0 \le i \le k$, to this representation, we get $t_i(x) = e_i^*(x)$ and $t_{k+1}(x)e_{k+1} = x - \sum_{i=0}^{k} e_i^*(x)e_i$. Consequently,

$$\begin{aligned}|t_{k+1}(x)| &= \frac{1}{\|e_{k+1}\|}\bigg\|x - \sum_{i=0}^{k} e_i^*(x)e_i\bigg\| \\ &< \frac{2}{\varepsilon}\bigg(\|x\| + \sum_{i=0}^{k} \|e_i^*\| \cdot \|e_i\| \cdot \|x\|\bigg) \le \frac{2}{\varepsilon}\bigg(1 + \sum_{i=0}^{k} \frac{\alpha_i^2}{\varepsilon}\bigg)\|x\| \le \frac{\alpha_{k+1}}{\varepsilon}\|x\|.\end{aligned}$$

This yields that the map $t_{k+1} : L \to \mathbb{R}$, $t_{k+1} : x \mapsto t_{k+1}(x)$, is a well defined linear functional with norm $< \alpha_{k+1}/\varepsilon$. Applying the Hahn–Banach Theorem, extend t_{k+1} to a functional $e_{k+1}^* \in X^*$ such that $\|e_{k+1}^*\| = \|t_{k+1}\| < \alpha_{k+1}/\varepsilon$. It is clear that $e_{k+1}^*(e_{k+1}) = 1$ and $e_{k+1}^*(e_i) = 0$ for $i < k+1$. Finally, put $F_{k+2} = F_{k+1} \cap \operatorname{Ker}(e_{k+1}^*)$. Therefore, the points x_{k+1}, e_{n+1}, the functional e_{k+1}^*, and the subspace F_{k+2} satisfying conditions (1)–(5) are constructed and the induction is complete.

By the choice of the subspace F_0, we have $x_i \in y_i + F_0 \subset U_i$ for every $i \le n$. Finally, let us show that $\|x - a\| > \delta(n)$ for any $x \in \operatorname{conv}\{x_0, \dots, x_n\}$. Assume, on the contrary,

$\|x-a\| \le \delta(n)$ for some $x=\sum_{i=0}^n t_ix_i$, where $t_0,\dots,t_n \ge 0$ and $\sum_{i=0}^n t_i = 1$. Then

$$\begin{aligned} x-a &= \sum_{i=0}^n t_i(x_i-a) \\ &= \sum_{i=0}^n t_i\Big(e_i + \sum_{k=0}^{i-1} e_k^*(y_i-a)e_k\Big) = \sum_{k=0}^n t_k e_k + \sum_{i=0}^n\sum_{k=0}^{i-1} t_i e_k^*(y_i-a)e_k \\ &= \sum_{k=0}^n t_k e_k + \sum_{k=0}^n \sum_{i=k+1}^n t_i e_k^*(y_i-a)e_k = \sum_{k=0}^n \Big(t_k + \sum_{i=k+1}^n t_i e_k^*(y_i-a)\Big)e_k. \end{aligned}$$

Consequently, for every $k \le n$ we have

$$\Big|t_k + \sum_{i=k+1}^n t_i e_k^*(y_i-a)\Big| = |e_k^*(x-a)| \le \|e_k^*\|\cdot\|x-a\| < \frac{\alpha_k}{\varepsilon}\delta(n)$$

and

$$t_k < \frac{\alpha_k}{\varepsilon}\delta(n) + \sum_{i=k+1}^n t_i|e_k^*(y_i-a)| < \frac{\alpha_k}{\varepsilon}\delta(n) + \sum_{i=k+1}^n 2t_i\|e_k^*\| < \frac{\alpha_k}{\varepsilon}\delta(n) + 2\sum_{i=k+1}^n t_i\frac{\alpha_i}{\varepsilon}.$$

In particular, for $k=n$, we get

$$t_n < \frac{\alpha_n}{\varepsilon}\delta(n) = \lambda_n\delta(n),$$

and for $k=n-1$,

$$t_{n-1} < \frac{\alpha_{n-1}}{\varepsilon}\delta(n) + 2t_n\frac{\alpha_n}{\varepsilon} < \frac{\alpha_{k-1}}{\varepsilon}\delta(n) + 2\frac{\alpha_n}{\varepsilon}\lambda_n\delta(n) = \lambda_{n-1}\delta(n).$$

Continuing in this way, we get $t_k < \lambda_k\delta(n)$ for every $0 \le k \le n$. Then $1 = t_0+\dots+t_n < (\lambda_0+\dots+\lambda_n)\delta(n) = 1$, a contradiction. ∎

Now we are able to prove that the weak unit ball B is a σZ_n-space for every $n \in \mathbb{N}$. Fix any countable dense subset $A \subset B$. Clearly, $B \subset \bigcup_{a\in A}(a+\delta B)$ for every $\delta > 0$. To show that B is a σZ_n-space, it suffices to prove that the intersection $(a+\delta(n)B)\cap B$ is a σZ_n-space in B for every $n \in \mathbb{N}$ and every $a \in A$ (here $\delta(n)$ is the constant from Claim 1.22).

So, fix $a \in A$, $n \in \mathbb{N}$, a continuous map $f : I^n \to B$ of the n-dimensional cube and a weakly open convex neighborhood $U \subset X$ of the origin. We have to construct a continuous map $f' : I^n \to B$ such that $f'(I^n)\cap(a+\delta(n)B) = \emptyset$ and $f(t)-f'(t) \in U$ for every $t \in I^n$.

The uniform continuity of f implies the existence of a triangulation $\mathcal{N}$ of the cube I^n so fine that $f(\sigma)-f(\sigma) \subset \frac{1}{6}U$ for every simplex σ of $\mathcal{N}$. Let $\mathcal{N}^{(0)}$ denote the set of all vertices of the triangulation $\mathcal{N}$, $S = \{\sigma_1,\dots,\sigma_m\}$ be the set of all maximal simplices of $\mathcal{N}$ and $\sigma^{(0)}$ denote the set of vertices of a simplex $\sigma \in S$.

Put $U_0 = \frac{1}{6}U$ and $f_0 = f|\mathcal{N}^{(0)} : \mathcal{N}^{(0)} \to B$. By finite induction, for every $k \in \{1,\dots,m\}$ we shall construct a weak-open neighborhood U_k of the origin of X and a map $f_k : \mathcal{N}^{(0)} \to B$ satisfying the conditions

(1) $f_k(t) \in f_i(t) + U_i$ for every $i < k$ and $t \in \mathcal{N}^{(0)}$;
(2) $(a+\delta(n)B)\cap \operatorname{conv}(f_k(\sigma_k^{(0)})+U_k) = \emptyset$.

Suppose for some $k \le m$ and every $i < k$ the map f_i and the neighborhood U_i satisfying conditions (1), (2) have been constructed.

Since $f_{k-1}(t) \in f_i(t) + U_i$ for every $i < k$ and $t \in \mathcal{N}^{(0)}$, we may find a weak-open neighborhood W of the origin of X such that $f_{k-1}(t)+W \subset f_i(t)+U_i$ for every $i < k$ and every $t \in \mathcal{N}^{(0)}$. Consider the maximal simplex $\sigma_k \in S$. Because $\dim \sigma_k = \dim I^n = n$, we have $|\sigma_k^{(0)}| = n + 1$. Using Claim 1.22, construct a map $\widetilde{f}_k : \sigma_k^{(0)} \to B$ such that $(a + \delta(n)B) \cap \mathrm{conv}(\widetilde{f}_k(\sigma_k^{(0)})) = \emptyset$ and $\widetilde{f}_k(t) \in f_{k-1}(t) + W$ for every $t \in \sigma_k^{(0)}$. Letting $f_k(t) = f_{k-1}(t)$ for $t \in \mathcal{N}^{(0)} \setminus \sigma_k^{(0)}$ and $f_k(t) = \widetilde{f}_k(t)$ for $t \in \sigma_k^{(0)}$, extend the map $\widetilde{f}_k$ to the map $f_k : \mathcal{N}^{(0)} \to B$. By the choice of the neighborhood W, we have $f_k(t) \in f_i(t) + U_i$ for every $i < k$ and every $t \in \mathcal{N}^{(0)}$. By the Hahn–Banach Theorem, there exists an open halfspace $H \subset X$ such that $\mathrm{conv}(f_k(\sigma_k^{(0)})) \subset H$ and $H \cap (a + \delta(n)B) = \emptyset$. Find a weak-open neighborhood U_k of the origin in X such that $f_k(t) + U_k \subset H$ for every $t \in \sigma_k^{(0)}$. Then $\mathrm{conv}(f_k(\sigma_k^{(0)}) + U_k) \subset H$ and thus $(a+\delta(n)B) \cap \mathrm{conv}(f_k(\sigma_k^{(0)}) + U_k) = \emptyset$. Therefore, the map f_k and the neighborhood U_k satisfying conditions (1), (2) have been constructed.

Consider the map $f_m : \mathcal{N}^{(0)} \to B$ and let $f' : I^n \to B$ be the simplicial map of $\mathcal{N}$ extending the map f_m. We claim that $f'(I^n) \cap (a + \delta(n)B) = \emptyset$. Fix any $t \in I^n$ and find a maximal simplex $\sigma_i \in S$ containing the point t. Then $f'(t) \in \mathrm{conv}(f_m(\sigma_i^{(0)}))$. According to (1), we have $f_m(\sigma_i^{(0)}) \subset f_i(\sigma_i^{(0)}) + U_i$ and hence $f'(t) \in \mathrm{conv}(f_m(\sigma_i^{(0)})) \subset \mathrm{conv}(f_i(\sigma_i^{(0)}) + U_i)$. By (2), $f'(t) \notin a + \delta(n)B$.

It remains to show that $f(t) - f'(t) \in U$ for every $t \in I^k$. Fix any $t \in I^k$ and pick up a simplex σ of the triangulation $\mathcal{N}$ such that $t \in \sigma$. Fix any vertex $v_0 \in \sigma^{(0)}$. By (1), for every $v \in \sigma^{(0)}$ we have $f_m(v) \in f_0(v) + U_0 = f(v) + \frac{1}{6}U$ and hence

$$\begin{aligned} f_m(v) - f_m(v_0) &= (f_m(v) - f(v)) + (f(v) - f(v_0)) + (f(v_0) - f_m(v_0)) \\ &\in \tfrac{1}{6}U + \tfrac{1}{6}U + \tfrac{1}{6}U = \tfrac{1}{2}U. \end{aligned}$$

Consequently, $f_m(\sigma^{(0)}) \subset f_m(v_0) + \frac{1}{2}U$ and thus $f'(t) \in \mathrm{conv}(f_m(\sigma^{(0)})) \subset f_m(v_0) + \frac{1}{2}U$. Then

$$\begin{aligned} f'(t) - f(t) &= (f'(t) - f_m(v_0)) + (f_m(v_0) - f(v_0)) + (f(v_0) - f(t)) \\ &\in \tfrac{1}{2}U + \tfrac{1}{6}U + \tfrac{1}{6}U \subset U. \ \blacksquare \end{aligned}$$

By the preceding theorem, every Banach space X failing CPCP admits an equivalent norm $\|\cdot\|$ such that the corresponding weak unit ball B is a σZ_n-space for every $n \in \mathbb{N}$. Can the norm $\|\cdot\|$ be chosen so that the weak ball B is a σZ_∞-space? We will show that this can be done for Banach spaces which are not strongly regular. Strongly regular Banach spaces were introduced and extensively studied in [GGMS]. Below we give a definition equivalent to that given in [GGMS] (cf. [GGMS, III.6 and II.1]).

A Banach space X is called *strongly regular* if for every $\varepsilon > 0$ and every nonempty convex bounded subset $C \subset X$ there exist scalars $t_1, \dots, t_n \ge 0$ with $\sum_{i=1}^n t_i = 1$ and nonempty relatively weak-open subsets $U_1, \dots, U_n \subset C$ such that the norm diameter of $\sum_{i=1}^n t_i U_i$ is less than ε.

It is known that every Banach space with CPCP is strongly regular. On the other hand, if X contains an isomorphic copy of c_0, then it is not strongly regular.

1.23. THEOREM. *If a separable Banach space X is not strongly regular, then X has an equivalent σZ_∞-norm $\|\cdot\|$. Moreover, if the dual space X^* is separable, then the weak unit ball B with respect to this norm is homeomorphic to the weak unit ball $B(c_0,\|\cdot\|_\infty)$ of the Banach space c_0.*

PROOF. Suppose a separable Banach space $(X,\|\cdot\|)$ is not strongly regular. Then there exist $\varepsilon>0$ and a nonempty convex bounded set $C\subset X$ such that for any scalars $t_1,\dots,t_n\ge 0$ with $\sum_{i=1}^n t_i=1$ and any nonempty relatively open subsets $U_1,\dots,U_n\subset C$ we have $\operatorname{diam}(\sum_{i=1}^n t_iU_i)>\varepsilon$.

Without loss of generality, the origin of X belongs to the set C. Consider the open set $(C-C)+B^\circ$, where $B^\circ=\{x\in X:\|x\|<1\}$ is the open unit ball of X. Note that the set $C-C+B^\circ$ is bounded, convex, open, and symmetric. This implies that its gauge functional $|||x|||=\inf\{t>0:x/t\in C-C+B^\circ\}$, $x\in X$, is an equivalent norm for X.

1.24. CLAIM. *For any scalars $t_1,\dots,t_n\ge 0$ with $\sum_{i=1}^n t_i=1$ and any nonempty open subsets $V_1,\dots,V_n\subset B(X,|||\cdot|||)$ we have $\operatorname{diam}_{\|\cdot\|}\sum_{i=1}^n t_iV_i>\varepsilon$.*

PROOF. For every V_i fix a point $x_i\in V_i\cap(C-C+B^\circ)$ and write $x_i=c_i-c_i'+b_i$, where $c_i,c_i'\in C$ and $b_i\in B^\circ$. By the continuity of the addition, the point c_i has a weak-open neighborhood $U_i\subset C$ such that $U_i-c_i'+b_i\subset V_i$. Then

$$\sum_{i=1}^n t_iV_i\supset\sum_{i=1}^n t_iU_i+\sum_{i=1}^n t_i(b_i-c_i')$$

and thus $\operatorname{diam}_{\|\cdot\|}\sum_{i=1}^n t_iV_i\ge\operatorname{diam}_{\|\cdot\|}\sum_{i=1}^n t_iU_i>\varepsilon$. ■

1.25. CLAIM. *For every point $a\in X$ and nonempty weak-open subsets $U_1,\dots,U_n\subset B(X,|||\cdot|||)$ there are points $x_i\in U_i$, $1\le i\le n$, such that $\|a-\sum_{i=1}^n t_ix_i\|>\varepsilon/6$ for any scalars $t_1,\dots,t_n\ge 0$ with $\sum_{i=1}^n t_i=1$.*

PROOF. Let $R>0$ be such that $B(X,|||\cdot|||)\subset R\cdot B(X,\|\cdot\|)$. Denote by $T=\{(t_1,\dots,t_n)\in[0,1]^n:\sum_{i=1}^n t_i=1\}$ the $(n-1)$-dimensional simplex. Using its compactness, fix any finite $(\varepsilon/(6R))$-net $T_0\subset T$ with respect to the l_1-metric $d((t_i)_1^n,(t_i')_1^n)=\sum_{i=1}^n|t_i-t_i'|$ (this means that for every $t\in T$ there is $t_0\in T_0$ with $d(t,t_0)<\varepsilon/(6R)$). Let $\le$ be any linear ordering of the finite set T_0.

By finite induction for every $t=(t_1,\dots,t_n)\in T_0$ we will construct nonempty weak-open sets $U_1(t),\dots,U_n(t)$ in $B(X,|||\cdot|||)$ such that

$$(*_t)\qquad U_i(t)\subset U_i\cap\bigcap_{\tau<t}U_i(\tau),\quad 1\le i\le n,\quad \Big(a+\frac{\varepsilon}{3}B(X,|||\cdot|||)\Big)\cap\sum_{i=1}^n t_iU_i(t)=\emptyset.$$

Fix any $t=(t_1,\dots,t_n)\in T_0$ and assume that for every $\tau\in T_0$ with $\tau<t$ nonempty weak-open sets $U_i(\tau)$ satisfying $(*_\tau)$ have been constructed. Let $V_i=U_i\cap\bigcap_{\tau<t}U_i(\tau)$ for $1\le i\le n$. By Claim 1.24, $\operatorname{diam}_{\|\cdot\|}\sum_{i=1}^n t_iV_i>\varepsilon$. This implies the existence of points $x_i\in V_i$, $1\le i\le n$, such that $\|a-\sum_{i=1}^n t_ix_i\|>\varepsilon/3$. Then $\sum_{i=1}^n t_ix_i\notin a+(\varepsilon/3)B(X,\|\cdot\|)$ and we may find a weak-open neighborhood W of the point $\sum_{i=1}^n t_ix_i$ in $B(X,|||\cdot|||)$ such that $W\cap(a+(\varepsilon/3)B(X,\|\cdot\|))=\emptyset$. By the continuity of linear operations on $B(X,|||\cdot|||)$ the points $x_1,\dots,x_n$ have neighborhoods $U_1(t),\dots,U_n(t)\subset B(X,|||\cdot|||)$ such that $x_i \in U_i(t) \subset V_i$ for $1 \le i \le n$ and $\sum_{i=1}^n t_i U_i(t) \subset W$. This yields $(\sum_{i=1}^n t_i U_i(t)) \cap (a + (\varepsilon/3)B(X, \|\cdot\|)) = \emptyset$. The inductive step is complete.

Finally, for the maximal element $\tau \in T_0$ pick points $x_i \in U_i(\tau)$, $1 \le i \le n$. Since $U_i(\tau) \subset U_i(t) \cap U_i$ for $t \le \tau$, $1 \le i \le n$, we get $x_i \in U_i$, $1 \le i \le n$, and $\|a - \sum_{i=1}^n t_i x_i\| > \varepsilon/3$ for every $t = (t_1, \ldots, t_n) \in T_0$.

Then for every $t = (t_1, \ldots, t_n) \in T$, letting $t' = (t'_1, \ldots, t'_n)$ be any point of T_0 with $\sum_{i=1}^n |t_i - t'_i| < \varepsilon/(6R)$ we get

$$\Big\|a - \sum_{i=1}^n t_i x_i\Big\| \ge \Big\|a - \sum_{i=1}^n t'_i x_i\Big\| - \Big\|\sum_{i=1}^n (t'_i - t_i) x_i\Big\|$$
$$> \frac{\varepsilon}{3} - \max_{1\le i\le n} \|x_i\| \cdot \sum_{i=1}^n |t_i - t'_i| \ge \frac{\varepsilon}{3} - R\frac{\varepsilon}{6R} = \frac{\varepsilon}{6}. \ \blacksquare$$

1.26. Claim. *For every point $a \in X$ the set $B(X, \|\!|\cdot|\!\|) \cap \big(a + (\varepsilon/6)B(X, \|\cdot\|)\big)$ is a Z_∞-set in $B(X, \|\!|\cdot|\!\|)$.*

Proof. Fix a weak-open convex neighborhood U of the origin of X and a continuous map $f : I^k \to B(X, \|\!|\cdot|\!\|)$ of a finite-dimensional cube. We have to construct a continuous map $f' : I^k \to B(X, \|\!|\cdot|\!\|)$ such that $f'(I^k) \cap (a + (\varepsilon/6)B(X, \|\cdot\|)) = \emptyset$ and $f'(t) - f(t) \in U$ for every $t \in I^k$. The uniform continuity of f implies the existence of a triangulation $\mathcal{N}$ of the cube I^k so fine that $f(\sigma) - f(\sigma) \subset \frac{1}{6}U$ for every simplex σ of $\mathcal{N}$. Let $\mathcal{N}^{(0)}$ be the set of vertices of the triangulation $\mathcal{N}$. By Claim 1.25, there exists a map $f'_0 : \mathcal{N}^{(0)} \to B(X, \|\!|\cdot|\!\|)$ such that $(a + (\varepsilon/6)B(X, \|\cdot\|)) \cap \operatorname{conv}(f'_0(\mathcal{N}^{(0)}))$ and $f'_0(v) - f(v) \in \frac{1}{6}U$ for every $v \in \mathcal{N}^{(0)}$. Let $f' : I^k \to B(X, \|\!|\cdot|\!\|)$ be the simplicial map extending the map f'_0. Then $f'(I^k) \subset \operatorname{conv}(f'_0(\mathcal{N}^{(0)}))$ and hence $f'(I^k) \cap (a + (\varepsilon/6)B(X, \|\cdot\|)) = \emptyset$.

By analogy with the proof of Theorem 1.21, verify that $f(t) - f'(t) \in U$ for every $t \in I^k$. Thus $B \cap (a + (\varepsilon/6)B(X, \|\cdot\|))$ is a Z_∞-set in $B(X, \|\!|\cdot|\!\|)$. $\blacksquare$

Now we are able to prove that the weak unit ball $B(X, \|\!|\cdot|\!\|)$ is a σZ_∞-space. Let $(a_i)_{i=1}^\infty$ be a norm dense countable subset in $B(X, \|\!|\cdot|\!\|)$. Then Claim 1.26 implies that $B(X, \|\!|\cdot|\!\|) = \bigcup_{i=1}^\infty B(X, \|\!|\cdot|\!\|) \cap (a_i + (\varepsilon/6)B(X, \|\cdot\|))$ is a σZ_∞-space.

If the dual space X^* is separable, then by Theorem 1.13, the weak unit ball $B(X, \|\!|\cdot|\!\|)$ is homeomorphic to $B(c_0, \|\cdot\|_\infty)$. $\blacksquare$

Now we consider the question of how geometric properties of a Banach space X reflect in properties of the classes $\mathcal{W}(X)$ and $\mathcal{W}(X^{**}, X)$. Below, for a nonnegative integer n and a class $\mathcal{C}$ of topological spaces, $\mathcal{C}[n]$ denotes the subclass of $\mathcal{C}$ consisting of all spaces $C \in \mathcal{C}$ with $\dim C \le n$; and $\mathcal{C}$(s.c.d.c.) stands for the subclass of $\mathcal{C}$ consisting of all spaces $C \in \mathcal{C}$ having a strongly countable-dimensional metrizable compactification (recall that a topological space X is called *strongly countable-dimensional* if it can be written as a countable union of its closed finite-dimensional subspaces, see [En$_2$]). One may compare the following theorem with Theorems 1.1 and 1.2.

1.27. Theorem. *Suppose X is a Banach space with separable dual.*

(1) *X is infinite-dimensional iff $\mathcal{W}(X) \supset \mathcal{M}_0$.*
(2) *X is not reflexive iff $\mathcal{W}(X) \supset \mathcal{M}_1$ iff $\mathcal{W}(X^{**}, X) \supset (\mathcal{M}_0, \mathcal{M}_1)$.*

(3) *X fails PCP iff* $\mathcal{W}(X) \supset \mathcal{M}_2[0]$ *iff* $\mathcal{W}(X^{**}, X) \supset (\mathcal{M}_0[0], \mathcal{M}_2)$.

(4) *If X fails CPCP, then*

$$\mathcal{W}(X) \supset \mathcal{M}_2(\text{s.c.d.c.}) \quad \textit{and} \quad \mathcal{W}(\mathrm{X}^{**}, \mathrm{X}) \supset (\mathcal{M}_0(\text{s.c.d.c.}), \mathcal{M}_2).$$

(5) *If X is not strongly regular, then* $\mathcal{W}(X) = \mathcal{M}_2$ *and* $\mathcal{W}(X^{**}, X) = (\mathcal{M}_0, \mathcal{M}_2)$.

PROOF. (1) If $\mathcal{W}(X) \supset \mathcal{M}_0$, then X is infinite-dimensional. Conversely, if X is infinite-dimensional, then X contains an infinite-dimensional convex compact set. By the Keller Theorem [BP, p. 100], K is homeomorphic to the Hilbert cube Q. Since the weak and the norm topologies coincide on K, we get $\mathcal{W}(X) \supset \mathcal{F}_0(K) = \mathcal{F}_0(Q) = \mathcal{M}_0$ (the last equality follows from the $\mathcal{M}_0$-universality of Q [En$_1$, 2.3.23]).

(2) To prove the second statement it suffices to show that if X is not reflexive, then $\mathcal{W}(X^{**}, X) \supset (\mathcal{M}_0, \mathcal{M}_1)$. Since X is not reflexive, $B \neq B^{**}$ and hence we may find a sequence $(x_n)_{n=1}^\infty$ in B, weakly convergent to a point $x_0 \in B^{**} \setminus B$. By the Milman Theorem [HHZ, 74] and the Choquet Representation Theorem [HHZ, 220], the closed convex hull K of the compact set $S_0 = \{x_n : n \geq 0\}$ coincides with the set

$$\Big\{ \sum_{n=0}^\infty t_n x_n : t_n \geq 0,\ \sum_{n=0}^\infty t_n = 1 \Big\}$$

of countable convex combinations of the points x_n, $n \geq 0$.

Observe that $K \cap B = \{\sum_{n=1}^\infty t_n x_n : t_n \geq 0, \sum_{n=1}^\infty t_n = 1\}$ is the set of all countable convex combinations of the x_n's, $n \geq 1$. Then $K \setminus B = \bigcup_{m=1}^\infty K_m$, where $K_m = \{tx_0 + (1-t)y : 1/m \leq t \leq 1,\ y \in K\}$ is compact for every m. Hence $K \cap B$ is a dense G_δ-set in K. Since K is compact and convex, and $K \not\subset \operatorname{aff}(K \cap B)$, we may apply Theorem 5.2.8 of [BRZ] to conclude that the pair $(K, K \cap B)$ is homeomorphic to (Q, s). Since $\mathcal{F}_0(Q, s) = (\mathcal{M}_0, \mathcal{M}_1)$ (this follows from Toruńczyk's Theorem 4.2 in [BP, IV, §4]), we get $\mathcal{W}(X^{**}, X) = \mathcal{F}_0(B^{**}, B) \supset \mathcal{F}_0(K, K \cap B) = \mathcal{F}_0(Q, s) = (\mathcal{M}_0, \mathcal{M}_1)$.

(3) In light of Theorems 1.1 and 1.2, to prove the third statement it suffices to show that for a Banach space failing PCP we have $\mathcal{W}(X^{**}, X) = \mathcal{F}_0(B^{**}, B) \supset (\mathcal{M}_0[0], \mathcal{M}_2)$. By the Theorem of Louveau and Saint-Raymond [Ke, 28.19] this will follow as soon as we show that B is not a $G_{\delta\sigma}$-set in B^{**}. By [Ke, 22.4] there exists a pair $(K, F) \in (\mathcal{M}_0[0], \mathcal{M}_2)$ such that F is not a $G_{\delta\sigma}$-set in K. To show that B is not a $G_{\delta\sigma}$-set in B^{**} it suffices to construct a continuous map $f : K \to B^{**}$ with $f^{-1}(B) = F$ (see [Ke, 22.1]). The construction is as follows.

According to [Ku, §30.V] the set $K \backslash F$, being a $G_{\delta\sigma}$-set in K, can be written as a countable union $K \backslash F = \bigcup_{n=1}^\infty G_n$ of pairwise disjoint G_δ-subsets of K. If X fails PCP, then by [EW], $B \notin \mathcal{M}_1$ and by the Wadge Theorem [Ke, 21.14] for every $n \in \mathbb{N}$ there exists a continuous map $f_n : K \to B^{**}$ with $f_n^{-1}(B) = K \backslash G_n$. Consider the map $f = \sum_{n=1}^\infty 2^{-n} f_n : K \to B^{**}$ and observe that it is well defined and continuous. We claim that $f^{-1}(B) = F$. Indeed, if $x \in F$, then $x \notin G_n$ for all n and by the choice of the maps f_n, we get $f_n(x) \in B$, $n \in \mathbb{N}$. Then $f(x) = \sum_{n=1}^\infty 2^{-n} f_n(x) \in B$. Now assume $x \in K \backslash F$. Then there is a unique $n_0 \in \mathbb{N}$ such that $x \in G_{n_0}$ and $x \notin G_n$ for all $n \neq n_0$. By the choice of the maps f_n, we get $f_{n_0}(x) \notin X$ and $f_n(x) \in B$ if $n \neq n_0$. Then $f(x) = 2^{-n_0} f_{n_0}(x) + \sum_{n \neq n_0} 2^{-n} f_n(x)$ does not belong to B since $f_{n_0}(x) \notin X$ and $\sum_{n \neq n_0} 2^{-n} f_n(x) \in B$. Thus $f^{-1}(B) = F$, completing the proof of (3).

(4) Suppose the space X fails CPCP. Then according to Theorem 1.21, X admits an equivalent norm $\|\cdot\|$ such that the corresponding weak unit ball B is a σZ_n-space for every $n \in \mathbb{N}$. To prove (4) it suffices to show that $\mathcal{F}_0(B^{**}, B) \supset (\mathcal{M}_0(\text{s.c.d.c.}), \mathcal{M}_2)$. According to Theorem 3.2.11 of [BRZ] this will follow as soon as we construct for every pair $(K, M) \in (\mathcal{M}_0(\text{s.c.d.c.}), \mathcal{M}_2)$ a continuous map $f : K \to B^{**}$ such that $f^{-1}(B) = M$.

Fix a pair $(K, M) \in (\mathcal{M}_0(\text{s.c.d.c.}), \mathcal{M}_2)$ and write $K = \bigcup_{n=1}^{\infty} K_n$ where K_n's are finite-dimensional compacta. Clearly, for every $n \in \mathbb{N}$, the set $K_{n+1} \setminus (K_n \cup M)$ is a $G_{\delta\sigma}$-set in K. According to [Ku, §30.V] this set can be written as the union of a countable collection $\mathcal{G}_n$ of pairwise disjoint G_δ-subsets of K. Then $\mathcal{G} = \bigcup_{n \in \mathbb{N}} \mathcal{G}_n$ is a countable collection of pairwise disjoint G_δ-sets in K such that $\bigcup \mathcal{G} = K \setminus M$; moreover, the closure of each set $G \in \mathcal{G}$ lies in some K_i and thus is finite-dimensional. Let $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ be an enumeration of the collection $\mathcal{G}$.

Fix any $n \in \mathbb{N}$ and consider the pair $(\overline{G}_n, G_n)$. Since $\overline{G}_n$ is a finite-dimensional compactum and B is an absolute retract σZ_m-space for every $m \in \mathbb{N}$, we may apply Lemma 10 of [BC] and Theorem 3.1.1 of [BRZ] to find a continuous map $f_n : \overline{G}_n \to B^{**}$ such that $f_n^{-1}(B) = \overline{G}_n \setminus G_n$. By an extension theorem of [Du], the map f_n can be extended to a continuous map $\bar{f}_n : K \to B^{**}$ such that $\bar{f}_n(K \setminus \overline{G}_n) \subset B$. Then $\bar{f}_n$ has the property $\bar{f}_n^{-1}(B) = K \setminus G_n$. By analogy with the preceding case, it can be shown that the map $f : K \to B^{**}$ defined by $f(x) = \sum_{n=1}^{\infty} 2^{-n} \bar{f}_n(x)$ for $x \in K$ is as required, i.e., f is continuous and $f^{-1}(B) = M$.

(5) If the space X is not strongly regular, then by Theorem 1.23, X admits an equivalent norm $\|\cdot\|$ such that the corresponding weak unit ball B is a σZ_∞-space. Repeating the arguments of the preceding two cases and using Lemma 8.10 of [CaDM] (see also Lemma 5.4 of [DMM]) in place of Lemma 10 of [BC], we may prove that $\mathcal{W}(X^{**}, X) \supset (\mathcal{M}_0, \mathcal{M}_2)$. Another way to prove this inclusion is to apply Theorems 1.23, 1.13 and the well known equality $\mathcal{F}_0(Q^\omega, \Sigma^\omega) = (\mathcal{M}_0, \mathcal{M}_2)$ (see [BM, 6.3]). ■

1.28. Corollary. *If an infinite-dimensional Banach space X with separable dual is complemented in its double dual X^{**}, then $\mathcal{W}(X) = \mathcal{M}_\alpha$ for some $\alpha = 0, 1, 2$.*

Proof. The corollary follows from Theorems 1.1, 1.2, 1.27 and Proposition VII.4 of [GGMS], which states that for separable Banach spaces complemented in their double duals PCP is equivalent to strong regularity. ■

The first two statements of Theorem 1.27 can be extended to general Banach spaces. Let us recall that a Banach space X is *weakly sequentially complete* (resp. satisfies the *Shur property*) if every weakly Cauchy sequence in X weakly converges (resp. converges in norm). According to the classical Steinhaus Theorem (see [Wo] or [Ed, 4.21.4]) every Banach subspace of $L_1[0,1]$ is weakly sequentially complete; by the Shur Theorem [HHZ, 99], every Banach subspace of l_1 satisfies the Shur property. It should be mentioned that the Banach space l_1 satisfies PCP (see [EW, p. 346]), while $L_1[0,1]$ is not strongly regular [GGMS, §4]. Note that if a Banach space X with separable dual is weakly sequentially complete (resp. satisfies the Shur property), then X is reflexive (resp. finite-dimensional).

Below we denote by $\mathcal{M}$ the class of all separable metrizable topological spaces; $S_0 = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ stands for a convergent sequence and $S = S_0 \setminus \{0\}$.

1.29. THEOREM. *Suppose X is a Banach space.*

(1) *X is infinite-dimensional iff $\mathcal{W}(X) \supset \mathcal{M}_0$.*

(2) *X is not reflexive iff $\mathcal{W}(X) \supset \mathcal{M}_1$.*

(3) *If X does not contain an isomorphic copy of l_1, then $\mathcal{W}(X^{**}, X) \supset (\mathcal{M}_0, \mathcal{M}_1)$.*

(4) *If X is weakly sequentially complete, then $\mathcal{W}(X^{**}, X) \not\supset (\mathcal{M}_0, \mathcal{M}_1)$; moreover, $(S_0, S) \notin \mathcal{W}(X^{**}, X)$.*

(5) *If X satisfies the Shur property, then $\mathcal{W}(X) \cap \mathcal{M} \subset \mathcal{M}_1$.*

PROOF. The first statement can be proven the same way as in Theorem 1.27. If $\mathcal{W}(X) \supset \mathcal{M}_1$, then clearly X is not reflexive. Suppose, conversely, that X is a nonreflexive Banach space. Then there are two possibilities.

A) X does not contain an isomorphic copy of l_1. Since X is not reflexive, by the Eberlein–Shmul'yan Theorem [HHZ, 229] the weak unit ball B of X is not sequentially compact and thus contains a sequence (y_k) with no limit point in B. By the Rosenthal l_1 Theorem [Di, p. 201], the sequence (y_k) has a weakly Cauchy subsequence (x_n). Since B^{**} is $*$-weakly compact, the sequence (x_n) converges to some point $x_0 \in B^{**}$. Since the sequence (y_k) has no limit point in B, $x_0 \notin B$. Proceeding further as in Theorem 1.27, we prove that $(\mathcal{M}_0, \mathcal{M}_1) \subset \mathcal{W}(X^{**}, X)$ and $\mathcal{M}_1 \subset \mathcal{W}(X)$.

B) X contains an isomorphic copy of l_1. Consider the positive unit sphere $P = \{(x_i)_{i=1}^\infty \in l_1 : x_i \geq 0 \text{ for all } i \in \mathbb{N} \text{ and } \sum_{i=1}^\infty x_i = 1\}$ of l_1 and observe that P is a (weakly) closed convex set in l_1. Moreover, the weak and the norm topologies coincide on P. Since P is complete and non-locally compact, we may apply [DT] (see also [BRZ, 5.2.2]) to deduce that P is homeomorphic to s. Then $\mathcal{W}(X) \supset \mathcal{W}(l_1) \supset \mathcal{F}_0(P) = \mathcal{F}_0(s) = \mathcal{M}_1$. Thus the first three statements are proven.

The other two statements can be easily derived from the corresponding definitions. ■

1.30. QUESTION. *Is $\mathcal{W}(L_1[0,1]) \cap \mathcal{M} \subset \mathcal{M}_1$?*

Clearly, $\mathcal{W}(Y) \subset \mathcal{W}(X)$ for every Banach subspace Y of a Banach space X. What connections are there between the classes $\mathcal{W}(X)$, $\mathcal{W}(Y)$, and $\mathcal{W}(X/Y)$?

1.31. PROPOSITION. *Suppose Y is a Banach subspace of an infinite-dimensional Banach space X. If $\mathcal{W}(Y), \mathcal{W}(X/Y) \subset \mathcal{M}_1$, then $\mathcal{W}(X) = \mathcal{W}(Y \oplus (X/Y)) = \mathcal{W}(Y) \cup \mathcal{W}(X/Y)$.*

PROOF. The inclusion $\mathcal{W}(Y) \cup \mathcal{W}(X/Y) \subset \mathcal{M}_1$ implies that both Y and X/Y have separable duals. Then the dual of X is separable as well (see [HHZ, Proposition 42]) and we may apply Proposition II.2 of [GM$_1$] to conclude that $\mathcal{W}(X) \subset \mathcal{M}_1$ and $\mathcal{W}(Y \oplus (X/Y)) \subset \mathcal{M}_1$. By Theorems 1.1 and 1.2 two cases are possible: either $\mathcal{W}(X) = \mathcal{M}_0$ or $\mathcal{W}(X) = \mathcal{M}_1$. In the first case X is reflexive and thus the spaces Y, X/Y, and $Y \oplus (X/Y)$ are reflexive. By Theorem 1.1, $\mathcal{W}(X) = \mathcal{M}_0 = \mathcal{W}(Y \oplus (X/Y)) = \mathcal{W}(Y) \cup \mathcal{W}(X/Y)$.

Now consider the second case: $\mathcal{W}(X) = \mathcal{M}_1$, i.e., X is not reflexive. Since reflexivity is a three-space property [CG], either Y or X/Y is not reflexive. Applying Theorems 1.1 and 1.2, we get $\mathcal{W}(X) = \mathcal{M}_1 = \mathcal{W}(Y \oplus (X/Y)) = \mathcal{W}(Y) \cup \mathcal{W}(X/Y)$. ■

1.32. REMARK. The equality $\mathcal{W}(X) = \mathcal{W}(Y \oplus (X/Y))$ is false in general. Indeed, by Proposition 4.9 of [EW], the Banach space c_0 is a quotient of a Banach space X with

$\mathcal{W}(X) = \mathcal{M}_1$. Let $Y \subset X$ be a Banach subspace such that $c_0 = X/Y$. Then $\mathcal{W}(Y) \subset \mathcal{W}(X) = \mathcal{M}_1$ and $\mathcal{W}(X) = \mathcal{M}_1 \neq \mathcal{W}(Y \oplus (X/Y)) \supset \mathcal{W}(c_0) = \mathcal{M}_2$.

1.33. Question. *Is $\mathcal{W}(X) \subset \mathcal{W}(Y \oplus (X/Y))$ for every Banach subspace Y of a Banach space X?*

Observe that Proposition 1.31 and Theorem 1.27 imply that if Y is a Banach subspace of a Banach space X with separable dual, then $\mathcal{W}(Y) \subset \mathcal{M}_1$ and $\mathcal{W}(X) \supset \mathcal{M}_2[0]$ imply $\mathcal{W}(X/Y) \supset \mathcal{M}_2[0]$.

1.34. Proposition. *Suppose Y is a Banach subspace of a Banach space X with separable dual. If $\mathcal{W}(Y) \subset \mathcal{M}_1$ and $\mathcal{W}(X) = \mathcal{M}_2$, then $\mathcal{W}(X/Y) = \mathcal{M}_2$.*

First we recall one definition. Following [BRZ, p. 129] we define a continuous map $f : X \to Y$ between topological spaces to be an $\mathcal{M}_1$-*map* if there exist a space $M \in \mathcal{M}_1$ and a closed embedding $e : X \to Y \times M$ such that $f = \mathrm{pr} \circ e$, where $\mathrm{pr} : Y \times M \to Y$ denotes the projection. By [BRZ, §3.2, Ex. 3] (see also [Ba$_4$]), a map $f : X \to Y$ between absolute Borel spaces is an $\mathcal{M}_1$-map if and only if $f^{-1}(K) \in \mathcal{M}_1$ for every compact subset $K \subset Y$.

1.35. Lemma. *Suppose Y is a Banach subspace of a Banach space X with separable dual. If $\mathcal{W}(Y) \subset \mathcal{M}_1$, then the quotient map $P : B(X) \to B(X/Y)$ is an $\mathcal{M}_1$-map.*

Proof. Since $B(X)$ and $B(X/Y)$ are absolute Borel spaces (see Proposition 1.12) it suffices to prove that $P^{-1}(K) \in \mathcal{M}_1$ for every compact subset $K \subset B(X/Y)$. Fix a (weakly) compact set $K \subset B(X/Y)$. By the Factorization Theorem [DFJP], there exists an injective bounded linear operator $T : R \to X/Y$ from a reflexive Banach space R such that $T(B(R)) \supset K$. Since $T : B(R) \to (X/Y, \text{weak})$ is an embedding, the weak unit ball $B(R)$ of R is metrizable. Thus R^* is separable and so is the space R.

In the direct sum $X^{**} \oplus R$ consider the closed linear subspace

$$\widetilde{Z} = \{(x, y) \in X^{**} \oplus R : P^{**}(x) = T(y)\},$$

where $P^{**} : X^{**} \to (X/Y)^{**}$ is the double dual quotient operator. Let $Z = \widetilde{Z} \cap (X \oplus R)$. Denote by $P_1 : Z \to X$ and $P_2 : Z \to R$ the projections. Clearly, $P \circ P_1 = T \circ P_2$, $P_2(Z) = R$ and $Y = \mathrm{Ker}(P_2)$. Hence $R = Z/Y$. Since both R and Y have separable duals and satisfy PCP, we may apply Proposition 1.31 to conclude that $\mathcal{W}(Z) \subset \mathcal{M}_1$.

Let $K^- = B^{**}(X) \cap (P^{**})^{-1}(K)$ and consider the map $\alpha : K^- \to (\widetilde{Z}, *\text{-weak})$ defined by $\alpha(x) = (x, T^{-1} \circ P^{**}(x))$. Since the map $T : B(R) \to (X/Y, \text{weak})$ is an embedding with $T(B(R)) \supset K$, the map α is an embedding. Observe that $\alpha(K^- \cap B(X)) = \alpha(K^-) \cap Z$. Since $\mathcal{W}(Z) \subset \mathcal{M}_1$, we get $\alpha(K^-) \cap Z \in \mathcal{M}_1$, and thus $P^{-1}(K)$, being a topological copy of $\alpha(K^-) \cap Z$, belongs to the class $\mathcal{M}_1$. ■

Proof of 1.34. Proposition 1.34 follows from Lemma 1.35 and Theorem 3.2.12 of [BRZ], according to which for an $\mathcal{M}_1$-map $f : X \to Y$ between separable metrizable spaces, $\mathcal{F}_0(X) \supset \mathcal{M}_2$ implies $\mathcal{F}_0(Y) \supset \mathcal{M}_2$. ■

Can we state that $\mathcal{W}(Y) \supset \mathcal{M}_2$ provided there is a "nice" operator $T : X \to Y$ from a Banach space X with $\mathcal{W}(X) = \mathcal{M}_2$?

Let us recall that a bounded linear operator $T : X \to Y$ between Banach spaces is called a (*nice*) G_δ*-embedding* if T is injective and for every closed bounded subset $C \subset X$ its image $T(C)$ is a (weak) G_δ-set in Y; see [GM$_1$].

We define an injective bounded linear operator $f : X \to Y$ to be a *weak* G_δ*-embedding* if for every bounded weakly closed subset $C \subset X$ its image $T(C)$ is a weak G_δ-set in Y. Clearly, each nice G_δ-embedding is simultaneously a G_δ-embedding and a weak G_δ-embedding. Yet, there are G_δ-embeddings which are not weak G_δ-embeddings; see Remark 2.2.

1.36. PROPOSITION. *If $T : X \to Y$ is a weak G_δ-embedding between Banach spaces with separable duals, then $\mathcal{W}(X) = \mathcal{M}_2$ implies $\mathcal{W}(Y) = \mathcal{M}_2$.*

PROOF. Without loss of generality, $T(B(X)) \subset B(Y)$. Clearly, $T(B(X))$ is a (weak) G_δ-set in $B(Y)$. It follows from the definition of a weak G_δ-embedding that the inverse map $T^{-1} : T(B(X)) \to B(X)$ is Borel of class 1. By [BRZ, §3.2, Ex. 4] (see also [Ba$_4$]), $T : B(X) \to T(B(X))$ is an $\mathcal{M}_1$-map. Then by [BRZ, 3.2.12], $\mathcal{F}_0(B(X)) \supset \mathcal{M}_2$ implies $\mathcal{F}_0(T(B(X))) \supset \mathcal{M}_2$. Since $T(B(X))$ is a G_δ-set in $B(Y)$, we may apply Theorem 3.1.2 of [BRZ] to conclude that $\mathcal{W}(Y) = \mathcal{F}_0(B(Y)) \supset \mathcal{M}_2$. ∎

1.37. REMARK. Repeating the arguments of the preceding proof, we may show that for every weak G_δ-embedding $T : X \to Y$, $\mathcal{W}(X) \supset \mathcal{M}_2[0]$ implies $\mathcal{W}(Y) \supset \mathcal{M}_2[0]$. Combining this with Theorem 1.27, we find that for any weak G_δ-embedding $T : X \to Y$ between Banach spaces with separable duals, $\mathcal{W}(Y) \subset \mathcal{M}_1$ implies $\mathcal{W}(X) \subset \mathcal{M}_1$. For nice G_δ-embeddings this fact was proven in [GM$_1$].

2. Two counterexamples. In this section we present two counterexamples disproving certain natural optimistic conjectures. One is an example of a Banach space—denoted by S_*T_∞ in [GMS$_1$]—which is strongly regular but satisfies $\mathcal{W}(S_*T_\infty) = \mathcal{M}_2$. This space contains no isomorphic copy of c_0 but has an equivalent weak ball, homeomorphic to a weak ball of c_0. Another is an example of a Banach space—denoted by B_∞ in [GM$_1$] and [GMS$_2$]—such that B_∞ fails PCP but $\mathcal{W}(B_\infty) \neq \mathcal{M}_2$.

Both examples are function spaces on the tree $T_\infty = \bigcup_{k=0}^\infty \mathbb{N}^k$. There is a natural order on T_∞: $(n_1, \ldots, n_p) \le (m_1, \ldots, m_q)$ if $p \le q$ and $n_i = m_i$ for $i \le p$. For $t = (n_1, \ldots, n_p) \in T_\infty$ we set $|t| = p$. A *segment* of T_∞ is a set of the form $[a, b] = \{t \in T_\infty : a \le t \le b\}$, where $a, b \in T_\infty$. The origin of T_∞ is denoted by θ (that is, $\mathbb{N}^0 = \{\theta\}$).

I. *The space* S_*T_∞. We recall its construction following [GMS$_1$].

Let $X = (\sum_{n=0}^\infty \oplus l^2(\mathbb{N}^n))_{c_0}$, that is,

$$X = \Big\{x = (x_t)_{t \in T_\infty} : \|x\|^2 = \sup_n \sum_{|t|=n} |x_t|^2 < \infty \text{ and } \lim_n \sum_{|t|=n} |x_t|^2 = 0\Big\}.$$

Let K be the subset of $X^{**} = (\sum_{n=0}^\infty \oplus l^2(\mathbb{N}^n))_{l_\infty}$ consisting of all functions z on T_∞ satisfying:

- $z_\theta = 1$;
- $z_t \ge 0$ for all $t \in T_\infty$;
- $z^2_{(n_1,\ldots,n_k)} \ge \sum_{n_{k+1}=1}^\infty z^2_{(n_1,\ldots,n_{k+1})}$ for all $(n_1, \ldots, n_k) \in T_\infty$.

Let $K_0 = K \cap X$ and W be the symmetric closed convex hull of K_0 in X. The space S_*T_∞ is the interpolation space associated with W in X by the method of [DFJP]. That is,

$$S_*T_\infty = \Big\{x \in X : \|\!|x|\!\| = \Big(\sum_{i=1}^{\infty} \|x\|_n^2\Big)^{1/2} < \infty\Big\},$$

where $\|\cdot\|_n$ is the gauge functional of the set $U_n = 2^n W + 2^{-n} B(X)$ for each $n \in \mathbb{N}$.

By Theorem VI.1 of [GMS$_1$], the space S_*T_∞ fails CPCP and has separable dual; the double dual ST_∞^* of S_*T_∞ is strongly regular and the quotient ST_∞^*/S_*T_∞ is reflexive. We add to this list the following property.

2.1. THEOREM. *$\mathcal{W}(S_*T_\infty) = \mathcal{M}_2$ and $\mathcal{W}(ST_\infty^*, S_*T_\infty) = (\mathcal{M}_0, \mathcal{M}_2)$.*

PROOF. Let $j : S_*T_\infty \to X$ denote the natural "identity" map and let $L_0 = j^{-1}(K_0)$. As noted in [GMS$_1$, p. 581], L_0 is a closed bounded convex subset of S_*T_∞ and j defines a homeomorphism between L_0 and K_0 for the respective weak topologies. In light of Propositions 1.9, 1.12 and Corollary 1.16, to prove that $\mathcal{W}(ST_\infty^*, S_*T_\infty) = (\mathcal{M}_0, \mathcal{M}_2)$ and $\mathcal{W}(S_*T_\infty) = \mathcal{M}_2$ it suffices to verify that $\mathcal{W}(S_*T_\infty) \supset \mathcal{M}_2$. To show this we will apply a recent result on the topology of spaces of probability measures. Let $C = \{0,1\}^{\mathbb{N}}$ be the Cantor cube and $C_{00} = \{(x_i)_{i\in\mathbb{N}} \in C : x_i = 0 \text{ for almost all } i\}$. Let $P(C)$ denote the space of probability measures on C and $\widehat{P}(C_{00}) = \{\mu \in P(C) : \mu(C_{00}) = 1\}$ (see [BR]). By [BR, 2.7], $\widehat{P}(C_{00})$ is a σZ_∞-space homeomorphic to Σ^ω.

We now construct a homeomorphism $h : \widehat{P}(C_{00}) \to K_0$ as follows. To every point $t = (n_1, \ldots, n_k) \in T_\infty$ assign the point $v(t) = (v(t)_i)_{i=1}^{\infty}$ in C_{00}, where

$$v(t)_i = \begin{cases} 1 & \text{if } i \in \{n_1, n_1+n_2, \ldots, n_1+\ldots+n_k\}, \\ 0 & \text{otherwise}, \end{cases}$$

and the open neighborhood $V(t) = \{(x_i)_{i=1}^{\infty} \in C : x_i = v(t)_i \text{ for } i \le n_1 + \ldots + n_k\}$ of $v(t)$ in C.

Now to each measure $\mu \in P(C)$ assign the function $f(\mu)$ on T_∞ defined by $f(\mu)(t) = \sqrt{\mu(V(t))}$ for $t \in T_\infty$. It can be shown that $f(\mu) \in K$ and the map $f : P(C) \to K$ so defined is continuous with respect to the $*$-weak topology on K. Moreover, $f^{-1}(K_0) = \widehat{P}(C_{00})$ and the restriction $f|_{\widehat{P}(C_{00})} : \widehat{P}(C_{00}) \to K_0$ is a homeomorphism. Hence the space S_*T_∞ contains the closed bounded convex subset L_0 such that (L_0, weak) is a σZ_∞-space homeomorphic to Σ^ω. This yields $\mathcal{W}(S_*T_\infty) \supset \mathcal{F}_0(L_0, \text{weak}) = \mathcal{F}_0(\Sigma^\omega) = \mathcal{M}_2$ (the last equality follows from the $\mathcal{M}_2$-universality of the space Σ^ω, see [BM]). ∎

2.2. REMARK. By Theorem VI.1 of [GMS$_1$] there exists a G_δ-embedding $T : S_*T_\infty \to l^2$. Since $\mathcal{W}(l^2) \ne \mathcal{M}_2$, Theorem 2.1 and Proposition 1.36 imply that T is not a weak G_δ-embedding.

II. *The space B_∞.* We recall its construction following [GMS$_2$]. Let L be the linear space of all real functions x on T_∞ with finite support $\operatorname{supp}(x) = \{x \in T_\infty : x(t) \ne 0\}$. For every $t \in T_\infty$ let $e_t : T_\infty \to \mathbb{R}$ be the characteristic function of the set $\{t\}$. Evidently, the collection $\{e_t : t \in T_\infty\}$ forms a Hamel basis for L and each function $x \in L$ can be written as $\sum_{t\in T_\infty} x_t e_t$, where $x_t = x(t)$. On the space L consider the norm

$$\|x\| = \sup\Big(\sum_{i=1}^{n}\Big(\sum_{t\in S_i} x_t\Big)^2\Big)^{1/2},$$

the supremum taken over all families $(S_1,\dots,S_n)$ of disjoint segments of T_∞. Denote by JT_∞ the completion of the normed space $(L,\|\cdot\|)$.

Now consider the dual space JT_∞^* to JT_∞. It is easily seen that for every $t\in T_\infty$ the coordinate functional $e_t^* : x\mapsto x_t$ is continuous on L and thus e_t^* is continuously extendable over JT_∞, i.e. $e_t^*\in JT_\infty^*$. By [GMS$_2$], B_∞, the closed linear span of the set $\{e_t^* : t\in T_\infty\}$ in JT_∞^*, is a predual space to JT_∞.

According to [GMS$_2$] the space B_∞ has separable dual JT_∞, B_∞ satisfies CPCP, but fails PCP. We add to this list the following pathological property. Below we denote by $\mathcal{A}_1[1]$ the class of all at most one-dimensional σ-compact metrizable spaces; a *Peano continuum* is a metrizable connected locally connected compact space; a subset A of a topological space T is called *meager* if A is a countable union of nowhere dense subsets in T.

2.3. Theorem. *$\mathcal{M}_2[0]\subset\mathcal{W}(B_\infty)$ but $\mathcal{A}_1[1]\not\subset\mathcal{W}(B_\infty)$. Moreover, $(K,A)\notin\mathcal{W}(JT_\infty^*,B_\infty)$ for every Peano continuum K and every dense meager subset $A\subset K$.*

Proof. First we will show that the complement $JT_\infty^*\setminus B_\infty$ can be covered by countably many $*$-weakly closed subsets each of which admits a “nice” continuous map onto a zero-dimensional space. Under an *infinite branch* of T_∞ we understand any maximal chain in $(T_\infty,\le)$; a *finite branch* is a segment of the form $\{x\in T_\infty : x\le t\}$ for some $t\in T_\infty$. Denote by Γ the set of all branches of T_∞ (both finite and infinite) and consider on Γ the topology generated by the base consisting of the sets

$$\Gamma(t)=\{\gamma\in\Gamma:\gamma\ni t\},\quad \text{where } t\in T_\infty.$$

It is easy to see that each $\Gamma(t)$ is open-and-closed in Γ and Γ is a zero-dimensional metrizable space.

According to [GMS$_2$] for every functional $f\in JT_\infty^*$ and every infinite branch γ of T_∞ the limit $\lim_{t\in\gamma} f(e_t)$ exists. Moreover, $\lim_{t\in\gamma} f(e_t)\ne 0$ for some infinite branch γ of T_∞ if and only if $f\notin B_\infty$.

For a natural number $n\in\mathbb{N}$ and a point $t\in T_\infty$ consider the subset $A(n,t)\subset JT_\infty^*$ of all functionals $f\in JT_\infty^*$ for which there exists a (unique finite or infinite) branch $\gamma(t)\in\Gamma(t)$ such that for every $\tau\in\Gamma(t)$,

$$|f(e_\tau)|\begin{cases}\ge 1/n & \text{if } \tau\in\gamma(f),\\ \le 1/(2n) & \text{if } \tau\notin\gamma(f).\end{cases}$$

2.4. Claim. *The set $A(n,t)$ is $*$-weakly closed in JT_∞^*.*

Proof. Fix a functional $f_0\in JT_\infty^*$ with $f_0\notin A(n,t)$. If $|f_0(e_t)|<1/n$, then $\{f\in JT_\infty^* : |f(e_t)|<1/n\}$ is a $*$-weakly open neighborhood separating f_0 from the set $A(n,t)$. In case $|f_0(e_t)|\ge 1/n$ let $\gamma(f_0)\in\Gamma(t)$ be a maximal branch such that $|f_0(e_\tau)|\ge 1/n$ for each $\tau\in\gamma(f_0)$. We consider separately two cases.

1. The branch $\gamma(f_0)$ is finite, i.e. $\gamma(f_0) = \{\tau \in T_\infty : \tau \le \tau_0\}$ for some $\tau_0 \in \Gamma(t)$. Then by the maximality of $\gamma(f_0)$ there is a point $\tau_1 > \tau_0$ such that $\gamma(f_0) \cup \{\tau_1\}$ is a branch and $|f_0(e_{\tau_1})| < 1/n$. Since $f_0 \notin A(n,t)$ there must exist a point $\tau_2 \in \Gamma(t)$ such that $\tau_2 \notin \gamma(f_0)$ and $|f(e_{\tau_2})| > 1/(2n)$. Then the set

$$\Big\{f \in JT_\infty^* : |f(e_\tau)| > \frac{1}{2n} \text{ for } \tau \in \gamma(f_0),\ |f(e_{\tau_1})| < \frac{1}{n},\ |f(e_{\tau_2})| > \frac{1}{2n}\Big\}$$

is a $*$-weakly open neighborhood in JT_∞^* separating f_0 from the set $A(n,t)$.

2. The branch $\gamma(f_0)$ is infinite. Since $f_0 \notin A(n,t)$ there must exist a point $\tau_0 \in \Gamma(t)$ such that $\tau_0 \notin \gamma(f_0)$ and $|f(e_{\tau_0})| > 1/(2n)$. Find a point $\tau_1 \in \gamma(f_0)$ such that τ_0 and τ_1 are incomparable. Then the set

$$\Big\{f \in JT_\infty^* : |f(e_{\tau_0})| > \frac{1}{2n} \text{ and } |f(e_\tau)| > \frac{1}{2n} \text{ for } t \le \tau \le \tau_1\Big\}$$

is a $*$-weakly open neighborhood in JT_∞^* separating the point f_0 from the set $A(n,t)$. Therefore $A(n,t)$ is $*$-weakly closed in JT_∞^*. ∎

Let Γ_∞ denote the subset of Γ consisting of all infinite branches of T_∞ and let $A_\infty(n,t) = \{f \in A(n,t) : \gamma(f) \in \Gamma_\infty\}$.

2.5. Claim. *If a connected subset $C \subset A(n,t)$ meets the set $A_\infty(n,t)$, then $C \subset A_\infty(n,t)$.*

Proof. We will first show that the map $\gamma : A(n,t) \to \Gamma$ assigning to each $f \in A(n,t)$ the branch $\gamma(f) \in \Gamma$ described in the definition of $A(n,t)$ is continuous. Indeed, if $f_0 \in A(n,t)$ and $\Gamma(\tau)$, $\tau \in \gamma(f_0)$, is a neighborhood of $\gamma(f_0)$, then the set $U = \{f \in A(n,t) : |f(e_{\max(t,\tau)})| > 1/(2n)\}$ is a $*$-weakly open neighborhood of f_0 in $A(n,t)$ such that $\gamma(f) \in \Gamma(\tau)$ for every $f \in U$. Hence the map $\gamma : A(n,t) \to \Gamma$ is continuous.

Since $C \subset A(n,t)$ is connected and Γ is zero-dimensional, the image $\gamma(C)$ consists of a unique branch γ_0. Then $C \subset \gamma^{-1}(\gamma_0)$. Because $C \cap A_\infty(n,t) \ne \emptyset$, we get $\emptyset \ne \gamma(C) \cap \gamma(A_\infty(n,t)) \subset \{\gamma_0\} \cap \Gamma_\infty$, which implies $\gamma_0 \in \Gamma_\infty$ and $C \subset \gamma^{-1}(\Gamma_\infty) = A_\infty(n,t)$. ∎

2.6. Claim. $JT_\infty^* \setminus B_\infty = \bigcup_{n\in\mathbb{N},\, t\in T_\infty} A_\infty(n,t)$.

Proof. The inclusion $A_\infty(n,t) \subset JT_\infty^* \setminus B_\infty$ follows from the definition of $A_\infty(n,t)$ and the fact that $\lim_{t\in\gamma} f(e_t) \ne 0$ for some infinite branch γ implies $f \in JT_\infty^* \setminus B_\infty$.

To prove the inclusion $JT_\infty^* \setminus B_\infty \subset \bigcup_{n,t} A_\infty(n,t)$, fix any functional $f \in JT_\infty^* \setminus B_\infty$. By [GMS$_2$, 1.1], $\lim_{t\in\gamma} f(e_t) \ne 0$ for some infinite branch $\gamma \in \Gamma$. Hence we may find an $n \in \mathbb{N}$ and a $t_1 \in \gamma$ such that $|f(e_\tau)| \ge 1/n$ for all $t \in \gamma$, $t \ge t_1$. We claim that there exists a $t \in \gamma$, $t \ge t_1$, such that $|f(e_\tau)| \le 1/(2n)$ for all $\tau \ge t$, $\tau \notin \gamma$, i.e. $f \in A_\infty(n,t)$. Assuming the converse, we would find a sequence $(t_i)_{i=1}^\infty$ of pairwise incomparable points of T_∞ such that $|f(e_{t_i})| > 1/(2n)$ for all $i \in \mathbb{N}$. Fix a natural number N with $\sum_{i=1}^N 1/(i+1) > 2n\|f\|$ and define the function $x : T_\infty \to \mathbb{R}$ with finite support by

$$x_t = \begin{cases} \operatorname{sign}(f(e_{t_i})) \cdot 1/(i+1) & \text{if } t = t_i \text{ for some } 1 \le i \le N, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from the definition of the norm on JT_∞ that

$$\|x\| = \Big(\sum_{i=1}^N |x_{t_i}|^2\Big)^{1/2} = \Big(\sum_{i=1}^N \frac{1}{(i+1)^2}\Big)^{1/2}$$

$$\leq \Big(\sum_{i=1}^{N} \frac{1}{i(i+1)}\Big)^{1/2} = \Big(\sum_{i=1}^{N}\Big(\frac{1}{i} - \frac{1}{i+1}\Big)\Big)^{1/2} \leq 1.$$

On the other hand,

$$\|f\| \geq \|f\| \cdot \|x\| \geq |f(x)| = \Big|\sum_{t\in T_\infty} x_t f(e_t)\Big| = \Big|\sum_{i=1}^{N} x_{t_i} f(e_{t_i})\Big| > \frac{1}{2n}\sum_{i=1}^{N}\frac{1}{i+1} > \|f\|,$$

a contradiction.

Therefore $f \in A_\infty(n,t)$ for some $t \in T_\infty$ and $\{A_\infty(n,t)\}_{n\in\mathbb{N},t\in T_\infty}$ is a countable cover of $JT^*_\infty \setminus B_\infty$. ∎

2.7. Claim. *Suppose K is a Peano continuum and A is a dense meager subset in K. Then there is no continuous map $\alpha : K \to JT^*_\infty$ into JT^*_∞ endowed with the $*$-weak topology such that $\alpha^{-1}(B_\infty) = A$.*

Proof. Assume on the contrary that such a map α exists. Then the comeager set $K\setminus A$ is covered by the countable family $\{\alpha^{-1}(A_\infty(n,t))\}_{n\in\mathbb{N},t\in T_\infty}$. By the Baire Theorem, there is a nonempty open subset $U \subset K$ such that $U\setminus A$ lies in the closure of $\alpha^{-1}(A_\infty(n,t))$ in K for some $n \in \mathbb{N}$, $t \in T_\infty$. By the local connectedness of K, the set U can be chosen to be connected. Since the set $A(n,t) \supset A_\infty(n,t)$ is $*$-weakly closed in JT^*_∞, we get $\alpha(U) \subset \mathrm{Cl}(\alpha(U\setminus A)) \subset \mathrm{Cl}(A_\infty(n,t)) \subset A(n,t)$. Since $\alpha(U)$ is connected and $\alpha(U) \cap A_\infty(n,t) \supset \alpha(U\setminus A)$ is not empty, we may apply Claim 2.5 to conclude that $\alpha(U) \subset A_\infty(n,t) \subset JT^*_\infty\setminus B_\infty$, contrary to $\emptyset \neq \alpha(U\cap A) \subset B_\infty$. ∎

2.8. Claim. *$\mathcal{M}_0[0] \subset \mathcal{W}(B_\infty)$ but $\mathcal{A}_1[1] \not\subset \mathcal{W}(B_\infty)$.*

Proof. Since B_∞ fails PCP and has separable dual, $\mathcal{M}_2[0] \subset \mathcal{W}(B_\infty)$ according to Theorem 1.27.

Assuming $\mathcal{A}_1[1] \subset \mathcal{W}(B_\infty) = \mathcal{F}_0(B(B_\infty))$ and applying Theorem 3.1.1 of [BRZ] we would get $(\mathcal{M}_0[1], \mathcal{A}_1[1]) \subset \mathcal{F}_0(B^{**}(B_\infty), B(B_\infty))$, where $\mathcal{M}_0[1]$ is the class of all at most 1-dimensional metrizable compacta. This implies the existence of an embedding $\alpha : [0,1] \to B^{**}(B_\infty)$ with $\alpha^{-1}(B(B_\infty)) = [0,1] \cap \mathbb{Q}$, contrary to Claim 2.7. ∎

Claim 2.8 finishes the proof of Theorem 2.3. ∎

2.9. Remark. As we said, the Banach space J_*T_∞ satisfies CPCP. Theorems 2.2 and 1.27 show that this fact has a topological nature.

3. Some open questions and comments. There are two groups of questions connected with our subject. The first group concentrates around Classification Theorem 1.14. This theorem reduces the problem of topological classification of weak unit balls $B(X)$ to the classification of the classes $\mathcal{W}(X)$.

Let

$$\mathfrak{W}^{\mathrm{s}}_\infty = \{\mathcal{W}(X) : X \text{ is an infinite-dimensional Banach space with separable dual}\}.$$

Since each separable Banach space is isomorphic to a subspace of $C[0,1]$, the set $\mathfrak{W}^{\mathrm{s}}_\infty$ contains at most continuum many elements. Note that $\mathfrak{W}^{\mathrm{s}}_\infty$ is partially ordered by the natural inclusion relation.

3.1. Problem. Investigate the ordered set $\mathfrak{W}^{\mathrm{s}}_\infty$. In particular, is it infinite? Is it linearly ordered?

At the moment, all we know about $\mathfrak{W}^{\mathrm{s}}_\infty$ is:

(1) the set $\mathfrak{W}^{\mathrm{s}}_\infty$ contains at least four classes: $\mathcal{M}_0 = \mathcal{W}(l_2)$, $\mathcal{M}_1 = \mathcal{W}(J)$ (J stands for the quasireflexive James space [HHZ, 264]), $\mathcal{W}(B_\infty)$, and $\mathcal{M}_2 = \mathcal{W}(c_0)$;

(2) $\mathcal{M}_0$ and $\mathcal{M}_2$ are the smallest and the greatest elements of $\mathfrak{W}^{\mathrm{s}}_\infty$, respectively;

(3) $\mathcal{M}_1$ is a unique successor of $\mathcal{M}_0$.

In fact, the space B_∞ is one of the spaces $J_*T_{\infty,n}$, $n \ge 0$, constructed in [GM$_2$].

3.2. Question. *Is* $\mathcal{W}(J_*T_{\infty,n}) \neq \mathcal{W}(J_*T_{\infty,m})$ *for* $n \neq m$*?*

3.3. Question. *Is* $\mathcal{W}(X \oplus Y) = \max\{\mathcal{W}(X), \mathcal{W}(Y)\}$ *for infinite-dimensional Banach spaces* X *and* Y *with separable duals?*

3.4. Question. *Let* X *be an infinite-dimensional Banach space. Is* $\mathcal{W}(X \oplus X) = \mathcal{W}(X)$*? Is* $\mathcal{W}(X \oplus \mathbb{R}) = \mathcal{W}(X)$*?*

We have shown in Theorem 2.3 that $\mathcal{M}_2[0] \subset \mathcal{W}(B_\infty)$ but $\mathcal{M}_2[1] \not\subset \mathcal{W}(B_\infty)$.

3.5. Question. *Let* $n \in \mathbb{N}$*. Is there a Banach space* X *such that* $\mathcal{M}_2[n] \subset \mathcal{W}(X)$ *but* $\mathcal{M}_2[n+1] \not\subset \mathcal{W}(X)$*? Is there a Banach space* X *such that* $\mathcal{M}_2(\text{s.c.d.c.}) \subset \mathcal{W}(X)$ *but* $\mathcal{M}_2 \not\subset \mathcal{W}(X)$*?*

3.6. Question. *Are there Banach spaces* X, Y *such that* $\mathcal{W}(X) = \mathcal{W}(Y)$ *but* $\mathcal{W}(X^{**}, X) \neq \mathcal{W}(Y^{**}, Y)$*?* (*Note that* $\mathcal{W}(X^{**}, X) = \mathcal{W}(Y^{**}, Y)$ *provided* $\mathcal{W}(X) = \mathcal{W}(Y) = \mathcal{M}_\alpha$ *for some* $\alpha = 0, 1, 2$.)

The second group of problems concern relationships between the introduced properties (such as BIP, $\mathcal{W}(X) = \mathcal{M}_2$, or the existence of an equivalent σZ_∞-norm) and known geometric properties: FD (finite-dimensionality), R (reflexivity), PCP, CPCP, and SR (strong regularity).

All relationships among these properties known at the moment are shown in the diagram below. X is a Banach space with separable dual. The second line of the diagram means that *every* equivalent weak unit ball B of X has the corresponding property; the third line means that the class $\mathcal{W}(X)$ *does not* contain the corresponding class.

$$
\begin{array}{rccccccccccccc}
X \text{ has:} & (\mathrm{FD}) & \Rightarrow & (\mathrm{R}) & \Rightarrow & (\mathrm{PCP}) & \Rightarrow & (\mathrm{BIP}) & \Rightarrow & (\mathrm{CPCP}) & \Rightarrow & (\mathrm{SR}) & \Rightarrow & (c_0 \not\subset X) \\
& \Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow & & \Uparrow & & \\
B \text{ is:} & (\text{f.d.}) & \Rightarrow & (\text{compact}) & \Rightarrow & (\text{Polish}) & \Rightarrow & (\infty\text{-Baire}) & \Rightarrow & (\text{Baire}) & \Rightarrow & \multicolumn{3}{c}{(\text{not a } \sigma Z_\infty\text{-space})} \\
& \Updownarrow & & \Updownarrow & & \Updownarrow & & & & \Uparrow & & \Uparrow & & \\
\mathcal{W}(X) \not\supset: & \mathcal{M}_0 & \Rightarrow & \mathcal{M}_1 & \Rightarrow & \mathcal{M}_2[0] & \Rightarrow & \mathcal{M}_2[1] & \Rightarrow & \mathcal{M}_2(\text{s.c.d.c.}) & \Rightarrow & \mathcal{M}_2 & &
\end{array}
$$

The following questions connected with this diagram seem to be of interest.

3.7. Question. *How far apart are the properties PCP and* $\mathcal{W}(X) \neq \mathcal{M}_2$*? In particular, is there a Banach space with CPCP and* $\mathcal{W}(X) = \mathcal{M}_2$*?*

3.8. Question. *Is there a Banach space* X *with* $\mathcal{W}(X) = \mathcal{M}_2$ *admitting no equivalent* σZ_∞*-norm? Does the space* S_*T_∞ *admit an equivalent* σZ_∞*-norm?*

3.9. Question. *Is there a Banach space with BIP but without PCP? Does the space B_∞ satisfy BIP?*

It is known that the Banach space c_0 contains no infinite-dimensional conjugate subspaces.

3.10. Question. *Suppose X is a Banach space with separable dual, containing no subspace isomorphic to an infinite-dimensional dual space. Is $\mathcal{W}(X) = \mathcal{M}_2$?*

References

[Ba_1] T. Banakh, *Toward a topological classification of convex sets in infinite-dimensional Fréchet spaces*, Topology Appl., to appear.

[Ba_2] —, *Characterization of convex Z-sets in linear metric spaces*, Funktsional. Anal. i Prilozhen. 28 (1994), no. 4, 77–79 (in Russian).

[Ba_3] —, *On the weak topology of the closed unit ball of a Banach space*, Visnyk L'viv Univ. Ser. Mekh. Mat. 40 (1994), 43–47 (in Ukrainian).

[Ba_4] —, *Preserving $\mathcal{C}$-universality by perfect maps*, preprint.

[BC] T. Banakh and R. Cauty, *On universality of countable and weak products of sigma hereditarily disconnected spaces*, preprint.

[BDP] T. Banakh, T. Dobrowolski and A. Plichko, *Topological and Borel classification of operator images*, this volume, 37–52.

[BR] T. Banakh and T. Radul, *Topology of spaces of probability measures*, Mat. Sb. 188 (1997), no. 7, 23–46 (in Russian).

[BRZ] T. Banakh, T. Radul and M. Zarichnyĭ, *Absorbing Sets in Infinite-Dimensional Manifolds*, VNTL Publishers, Lviv, 1996.

[BP] C. Bessaga and A. Pełczyński, *Selected Topics in Infinite-Dimensional Topology*, Monograf. Mat. 58, PWN, Warszawa, 1975.

[BM] M. Bestvina and J. Mogilski, *Characterizing certain incomplete infinite-dimensional absolute retracts*, Michigan Math. J. 33 (1986), 291–313.

[Bor] K. Borsuk, *Theory of Retracts*, Monograf. Mat. 44, PWN, Warszawa, 1967.

[BD] J. Bourgain and F. Delbaen, *A class of special $\mathcal{L}^\infty$-spaces*, Acta Math. 145 (1980), 155–176.

[Bo] R. D. Bourgin, *Geometric Aspects of Convex Sets with the Radon–Nikodým Property*, Lecture Notes in Math. 993, Springer, 1983.

[CG] J. M. F. Castillo and M. González, *Three-Space Problems in Banach Space Theory*, Lecture Notes in Math. 1667, Springer, 1997.

[CaDM] R. Cauty, T. Dobrowolski and W. Marciszewski, *A contribution to the topological classification of the spaces $C_{\mathrm{p}}(X)$*, Fund. Math. 142 (1993), 267–301.

[CuDM] D. W. Curtis, T. Dobrowolski and J. Mogilski, *Some applications of the topological characterizations of the sigma-compact spaces l^2_f and Σ*, Trans. Amer. Math. Soc. 284 (1984), 837–846.

[DFJP] W. J. Davis, T. Figiel, W. Johnson and A. Pełczyński, *Factorizing weakly compact operators*, J. Funct. Anal. 17 (1974), 311–327.

[DGZ] R. Deville, G. Godefroy and V. Zizler, *Smoothness and Renormings in Banach Spaces*, Pitman Monogr. Surveys Pure Appl. Math. 64, Longman, 1993.

[Di] J. Diestel, *Sequences and Series in Banach Spaces*, Springer, 1984.

[DMM] T. Dobrowolski, W. Marciszewski and J. Mogilski, *On topological classification of function spaces $C_{\mathrm{p}}(X)$ of low Borel complexity*, Trans. Amer. Math. Soc. 328 (1991), 307–324.

[DT] T. Dobrowolski and H. Toruńczyk, *Separable complete ANR's admitting a group structure are Hilbert manifolds*, Topology Appl. 12 (1981), 229–235.

[Du] J. Dugundji, *An extension of Tietze's theorem*, Pacific J. Math. 1 (1951), 353–367.

[EW] G. A. Edgar and R. F. Wheeler, *Topological properties of Banach spaces*, Pacific J. Math. 115 (1984), 317–350.

[Ed] R. E. Edwards, *Functional Analysis*, Holt, Rinehart & Winston, New York, 1965.

[En_1] R. Engelking, *General Topology*, PWN, Warszawa, 1977.

[En_2] —, *Theory of Dimensions*, *Finite and Infinite*, Heldermann, Lemgo, 1995.

[GGMS] N. Ghoussoub, G. Godefroy, B. Maurey and W. Schachermayer, *Some topological and geometrical structures in Banach spaces*, Mem. Amer. Math. Soc. 378 (1987).

[GM_1] N. Ghoussoub and B. Maurey, *G_δ-embeddings in Hilbert space*, J. Funct. Anal. 61 (1985), 72–97.

[GM_2] —, —, *G_δ-embeddings in Hilbert space*, *II*, ibid. 78 (1988), 271–305.

[GMS_1] N. Ghoussoub, B. Maurey and W. Schachermayer, *Geometrical implications of certain infinite-dimensional decompositions*, Trans. Amer. Math. Soc. 317 (1990), 541–584.

[GMS_2] —, —, —, *A counterexample to a problem on points of continuity in Banach spaces*, Proc. Amer. Math. Soc. 99 (1987), 278–282.

[GM] H. Gladdines and J. van Mill, *Hyperspaces of Peano continua of Euclidean spaces*, Fund. Math. 142 (1993), 173–188.

[HHZ] P. Habala, P. Hájek and V. Zizler, *Introduction to Banach Spaces*, Matfyzpress, Praha, 1996.

[Ha] D. E. G. Hare, *A dual characterization of Banach spaces with the convex point-of-continuity property*, Canad. Math. Bull. 32 (1989), 274–280.

[Ke] A. S. Kechris, *Classical Descriptive Set Theory*, Springer, 1995.

[Ku] K. Kuratowski, *Topology*, *I*, PWN, Warszawa, 1966.

[To_1] H. Toruńczyk, *Concerning locally homotopy negligible sets and characterization of l_2-manifolds*, Fund. Math. 101 (1978), 93–110.

[To_2] —, *Characterizing Hilbert space topology*, ibid. 111 (1981), 247–262.

[Wo] P. Wojtaszczyk, *Banach Spaces for Analysts*, Cambridge Stud. Adv. Math. 25, Cambridge Univ. Press, 1991.

DISSERTATIONES
MATHEMATICAE
387 (2000)

The topological and Borel classification of operator images

by

TARAS BANAKH, TADEUSZ DOBROWOLSKI and ANATOLIĬ PLICHKO

Abstract. We investigate Borel and topological properties of operator images, i.e., spaces TX, where $T: X \to Y$ is a linear continuous operator between Fréchet spaces. In particular, we show that if TX belongs to the small Borel class $\mathcal{M}_1^2$, then the Borel type of TX fully determines the topological type of TX. By providing two nonhomeomorphic operator images of the class $\mathcal{M}_2 \backslash \mathcal{A}_2$ we show that the above result cannot be generalized to higher Borel classes.

Introduction. The paper is devoted to the study of Borel and topological properties of operator images, i.e., spaces of the form TX, where $T : X \to Y$ is a linear continuous operator between Fréchet spaces (a *Fréchet space* is a locally convex linear complete metric space). It was known that for separable X the operator image TX is an absolute Borel space. In this paper we consider the following natural

QUESTION 1. *Is the topological type of an infinite-dimensional operator image TX of a separable Fréchet space X fully determined by the Borel type of TX?*

Due to J. Saint-Raymond [SR] we know that for a separable Fréchet space X there exist some restrictions on the Borel type of TX. Namely, TX cannot be of class $\mathcal{M}_{\alpha+1} \backslash \mathcal{M}_\alpha$ for a limit ordinal α (here $\mathcal{M}_\alpha$ and $\mathcal{A}_\alpha$ are respectively multiplicative and additive Borel classes corresponding to a countable ordinal α). Moreover, if X is a separable Banach space, then $TX \notin \mathcal{M}_{\alpha+1} \backslash \bigcup_{\xi<\alpha} \mathcal{M}_\xi$ for any limit ordinal $\alpha > 0$.

We find another restriction on the Borel type of operator images: if an operator image TX of a separable Fréchet space X belongs to the ambiguous Borel class $\mathcal{M}_{\alpha+1} \cap \mathcal{A}_{\alpha+1}$ for some α, then TX belongs to the small Borel class $\mathcal{M}_\alpha^2$, i.e., TX is a difference of two sets of class $\mathcal{M}_\alpha$. Thus for an operator image TX of a separable Banach space X we have an alternative: either $TX \in \mathcal{M}_{\alpha+2} \backslash \mathcal{A}_{\alpha+2}$, or $TX \in \mathcal{A}_{\alpha+1} \backslash \mathcal{M}_{\alpha+1}$, or $TX \in \mathcal{M}_{\alpha+1}^2 \backslash (\mathcal{M}_{\alpha+1} \cup \mathcal{A}_{\alpha+1})$ for some countable ordinal α.

Now Question 1 can be specified as follows.

QUESTION 2. *How many topologically distinct operator images of separable Fréchet spaces are there in each of the classes $\mathcal{M}_\alpha \backslash \mathcal{A}_\alpha$, $\mathcal{A}_\alpha \backslash \mathcal{M}_\alpha$, $\mathcal{M}_\alpha^2 \backslash (\mathcal{M}_\alpha \cup \mathcal{A}_\alpha)$, where α is a countable ordinal?*

2000 *Mathematics Subject Classification*: Primary 57N17, 57N20, 46B20, 46A03; Secondary 46A13, 47A05, 54F65.

Answering this question we show the following:

(1) For every countable ordinal α each of the classes $\mathcal{M}^2_{\alpha+2} \setminus (\mathcal{M}_{\alpha+2} \cup \mathcal{A}_{\alpha+2})$, $\mathcal{M}_{\alpha+2} \setminus \mathcal{A}_{\alpha+2}$, $\mathcal{A}_{\alpha+1} \setminus \mathcal{M}_{\alpha+1}$ contains an operator image of a Banach space with separable dual; if α is a limit ordinal, then the class $\mathcal{M}_\alpha \setminus \mathcal{A}_\alpha$ contains an operator image of a separable Fréchet space.

(2) Up to homeomorphism each of the classes $\mathcal{M}_1 \setminus \mathcal{A}_1$, $\mathcal{A}_1 \setminus \mathcal{M}_1$, $\mathcal{M}^2_1 \setminus (\mathcal{M}_1 \cup \mathcal{A}_1)$ contains exactly one operator image. More precisely, if TX is an operator image of class $\mathcal{M}_1 \setminus \mathcal{A}_1$ (resp. $\mathcal{A}_1 \setminus \mathcal{M}_1$, $\mathcal{M}^2_1 \setminus (\mathcal{M}_1 \cup \mathcal{A}_1)$), then TX is homeomorphic to l^2 (resp. Σ, $\Sigma \times l^2$), where Σ is the linear span of the standard Hilbert cube in the separable Hilbert space l^2.

(3) The class $\mathcal{M}_2 \setminus \mathcal{A}_2$ contains two topologically distinct operator images of Banach spaces with separable duals (thus the natural temptation to extend the second statement to higher Borel classes fails).

Observe that the last statement answers Question 1 in the negative (compare this with a result of [Ca]).

1. Borel type of operator images. Let X be a metrizable space. First we recall the definition of the multiplicative and additive Borelian classes $\mathcal{M}_\alpha(X)$ and $\mathcal{A}_\alpha(X)$. Let $\mathcal{M}_0(X)$ and $\mathcal{A}_0(X)$ be the classes of closed and open subsets in X, respectively. Assuming that for a countable ordinal α the classes $\mathcal{M}_\xi(X)$, $\mathcal{A}_\xi(X)$, $\xi < \alpha$, have been defined, let $\mathcal{M}_\alpha(X)$ (resp. $\mathcal{A}_\alpha(X)$) denote the collection of subsets in X that are countable intersections (resp. countable unions) of subsets from the class $\bigcup_{\xi<\alpha} \mathcal{A}_\xi(X)$ (resp. $\bigcup_{\xi<\alpha} \mathcal{M}_\xi(X)$).

The classes $\mathcal{M}_\alpha(X) \cap \mathcal{A}_\alpha(X)$ are called *ambiguous Borel classes*. For each countable ordinal α the ambiguous class $\mathcal{M}_{\alpha+1}(X) \cap \mathcal{A}_{\alpha+1}(X)$ can be represented as a union $\bigcup_{1\le\beta<\omega_1} \mathcal{M}^\beta_\alpha(X)$, where $\mathcal{M}^\beta_\alpha(X)$ are the so-called small Borel classes (see [Ku, §37.IV]). In particular, $\mathcal{M}^1_\alpha(X) = \mathcal{M}_\alpha(X)$ and $\mathcal{M}^2_\alpha(X)$ is the class of differences $A \setminus B$, where $A, B \in \mathcal{M}_\alpha(X)$.

By $\mathcal{M}_\alpha$ (resp. $\mathcal{A}_\alpha$, $\mathcal{M}^2_\alpha$) we denote the class of separable metrizable spaces X such that for every metrizable space Y every subspace $Z \subset Y$ homeomorphic to X belongs to the class $\mathcal{M}_\alpha(Y)$ (resp. $\mathcal{A}_\alpha(Y)$, $\mathcal{M}^2_\alpha(Y)$).

Let us recall [Ku] that a function $f : X \to Y$ between separable metric spaces is *Borel of class* α if $f^{-1}(F) \in \mathcal{M}_\alpha(X)$ for every closed set $F \subset Y$ (of course, this is equivalent to saying that $f^{-1}(U) \in \mathcal{A}_\alpha(X)$ for every open set $U \subset Y$). A function $f : X \to Y$ is *of the first Baire class* provided it is a pointwise limit of a sequence of maps $X \to Y$. It is well known that any map of the first Baire class is Borel of class 1.

For a locally convex linear topological space X let X^* be the dual space to X, i.e., the space of continuous linear functionals endowed with the strong dual topology, that is, the topology of uniform convergence on bounded subsets of X; see [Ed, §8.4] or [Sch, IV, §5]. Besides the strong dual topology, the dual space X^* carries the $*$-weak topology. It is known that for a Fréchet space X its dual X^* is σ-compact with respect to the $*$-weak topology [Sch, IV, §6], its second dual X^{**} is a Fréchet space too [Sch, IV, §6] and the natural map $X \to X^{**}$ is a topological embedding.

All Fréchet and Banach spaces considered in this paper are infinite-dimensional.

The term "operator" means a "continuous linear map between Fréchet spaces". An operator $T : X \to Y$ is called *dense* (resp. *compact*) if TX is dense in Y (resp. TU is totally bounded in Y for some neighborhood U of the origin in X). It is easily seen that an operator $T : X \to Y$ between Fréchet spaces is compact if and only if the operator image TX is σ-precompact, i.e., is contained in a σ-compact subset of Y. For an operator $T : X \to Y$ we denote by $T^* : Y^* \to X^*$ and $T^{**} : X^{**} \to Y^{**}$ the dual and the second dual operators to T.

For a subset F of the dual space X^* to a Fréchet space X let $F_{(0)} = F$ and let $F_{(1)}$ be the sequential closure of F in X^* with respect to the $*$-weak topology. By transfinite induction, for an ordinal α let $F_{(\alpha)} = \bigcup_{\beta<\alpha}(F_{(\beta)})_{(1)}$.

The following characterization theorem was proved by J. Saint-Raymond (see Proposition 19, Theorems 31, 44, 47, 54 and Corollary 45 in [SR], see also [Pl_1], [Pl_2] and [Os_2]).

1.1. Theorem. *For a countable ordinal α and an injective operator $T : X \to Y$ between separable Fréchet spaces the following conditions are equivalent*:

(1) $TX \in \mathcal{M}_{\alpha+1}$;

(2) $T^{-1} : TX \to X$ *is Borel of class* α;

(3) *there is a neighborhood base $\mathcal{B}$ at $0 \in X$ such that $TB \in \mathcal{A}_\alpha(TX)$ for every $B \in \mathcal{B}$*;

(4) *there is a base of closed convex neighborhoods $\mathcal{B}$ at $0 \in X$ such that $TB \in \mathcal{M}_0(TX) \cup \bigcup_{\xi<\alpha} \mathcal{M}_\xi(TX)$ for every $B \in \mathcal{B}$*;

(5) $(T^*Y^*)_{(\alpha)} = X^*$.

If α is a limit ordinal then the conditions (1)–(5) *are equivalent to*

(6) $TX \in \mathcal{M}_\alpha(Y)$.

Note that the ordinal $\alpha = 0$ is limit! This proposition implies the following important result (cf. [SR, Theorem 38]).

1.2. Theorem. *Suppose α is a limit ordinal and $T : X \to Y$ is an injective operator between separable Fréchet spaces. Then $TX \notin \mathcal{M}_{\alpha+1}(Y) \setminus \mathcal{M}_\alpha(Y)$. Moreover, if $\alpha > 0$ and X is a Banach space, then $TX \notin \mathcal{M}_{\alpha+1} \setminus \bigcup_{\xi<\alpha} \mathcal{M}_\xi$.*

Next, we study operator images belonging to additive Borel classes.

1.3. Theorem. *Suppose α is a countable ordinal and $T : X \to Y$ is an injective operator between separable Fréchet spaces such that $TX \in \mathcal{M}_{\alpha+1}$. Then the following conditions are equivalent*:

(1) $TX \in \mathcal{A}_{\alpha+1}(Y)$;

(2) $TX \in \mathcal{M}^2_\alpha(Y)$;

(3) *there exists a neighborhood B of the origin in X such that $TF \in \mathcal{M}_\alpha(Y)$ for every closed subset $F \subset B$.*

Proof. If α is a limit ordinal, then by Theorem 1.1, $TX \in \mathcal{M}_{\alpha+1}(Y)$ implies $TX \in \mathcal{M}_\alpha(Y)$. Thus the first two condition are satisfied. Next, if F is a closed subset of X, then $TF \in \mathcal{M}_\alpha(TX)$ by Theorem 1.1. Since $TX \in \mathcal{M}_\alpha(Y)$, we get $TF \in \mathcal{M}_\alpha(Y)$.

Thus it remains to consider the case $\alpha = \beta + 1$ for some ordinal β. The implication (2)⇒(1) is trivial.

Let us verify (3)⇒(2). Using Theorem 1.1 and the condition (3), we may find a closed neighborhood B of $0 \in X$ such that $TB \in \mathcal{M}_\beta(TX)$ and $TB \in \mathcal{M}_\alpha(Y)$. Let $X_n = n \cdot TB \subset TX$ for $n \in \mathbb{N}$. Evidently, $X_n \in \mathcal{M}_\alpha(Y)$ and $X_n \in \mathcal{M}_\beta(TX)$ for each n. For each $n \in \mathbb{N}$ fix a subset $\widetilde{X}_n \in \mathcal{M}_\beta(Y)$ such that $\widetilde{X}_n \cap TX = X_n$. Then $\bigcup_{n=1}^\infty \widetilde{X}_n \in \mathcal{A}_\alpha(Y)$ and $Y \setminus \bigcup_{n=1}^\infty \widetilde{X}_n \in \mathcal{M}_\alpha(Y)$. On the other hand, since $X_n \in \mathcal{M}_\alpha(Y)$, we get $\widetilde{X}_n \setminus X_n \in \mathcal{A}_\alpha(Y)$. Consequently, $\bigcup_{n=1}^\infty(\widetilde{X}_n \setminus X_n) \in \mathcal{A}_\alpha(Y)$ and $Y \setminus \bigcup_{n=1}^\infty(\widetilde{X}_n \setminus X_n) \in \mathcal{M}_\alpha(Y)$. Since $TX = (Y \setminus \bigcup_{n=1}^\infty(\widetilde{X}_n \setminus X_n)) \setminus (Y \setminus \bigcup_{n=1}^\infty \widetilde{X}_n)$ we conclude that TX, being a difference of two $\mathcal{M}_\alpha(Y)$-sets, belongs to the small Borel class $\mathcal{M}_\alpha^2(Y)$.

Finally we verify the implication (1)⇒(3). Fix a base $U_1 \supset U_2 \supset \ldots$ of closed convex neighborhoods of the origin in the Fréchet space Y. According to Theorem 1.1, X has a base $\mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$ of closed convex neighborhoods of $0 \in X$ such that $TB_n \in \mathcal{M}_\beta(TX)$ for every $n \in \mathbb{N}$. Without loss of generality, $TB_n \subset U_n$ and $B_n \supset B_{n+1}$ for every n.

Claim A. *$TB_n \in \mathcal{M}_\alpha(Y)$ for some $n \in \mathbb{N}$.*

Proof. Assume on the contrary $TB_n \notin \mathcal{M}_\alpha(Y)$ for every $n \in \mathbb{N}$. To get a contradiction, we will show that $TX \notin \mathcal{A}_{\alpha+1}(Y)$. For this fix any subset M of the Cantor cube 2^ω with $M \in \mathcal{M}_{\alpha+1} \setminus \mathcal{A}_{\alpha+1}$. According to Theorem 2 and Remarks in [Ku, §30.V], the set $2^\omega \setminus M$ can be written as a countable union $2^\omega \setminus M = \bigcup_{n=1}^\infty M_n$ of pairwise disjoint subsets of class $\mathcal{M}_\alpha(2^\omega)$. Let $A_n = 2^\omega \setminus M_n$ for each $n \in \mathbb{N}$. Then $M = \bigcap_{n=1}^\infty A_n$ where each $A_n \in \mathcal{A}_\alpha(2^\omega)$. Notice that if $x \notin M$, then there is a unique $n \in \mathbb{N}$ with $x \notin A_n$.

Claim B. *For every $n \in \mathbb{N}$ there exists a map $f_n : 2^\omega \to U_n$ such that $f_n(A_n) \subset TB_n$ and $f_n(2^\omega \setminus A_n) \subset U_n \setminus TX$.*

Proof. To find such maps f_n we will apply the Louveau–Saint-Raymond Separation Theorem [Ke, 28.19 and 22.13]. According to this theorem, the existence of a map f_n with the required properties will follow as soon as we prove that the sets TB_n and $U_n \setminus TX$ cannot be separated by an $\mathcal{M}_\alpha$-set. The latter means that there is no set $C \in \mathcal{M}_\alpha(Y)$ with $TB_n \subset C$ and $C \cap (U_n \setminus TX) = \emptyset$. Suppose on the contrary that such a set C exists. Since $TB_n \in \mathcal{M}_\beta(TX)$, there is a set $\widetilde{C} \in \mathcal{M}_\beta(Y)$ with $TB_n = \widetilde{C} \cap TX$. Since $TB_n \subset C \cap U_n$ and $C \cap U_n \subset TX$ we get $TB_n = \widetilde{C} \cap TX \cap C \cap U_n = \widetilde{C} \cap C \cap U_n \in \mathcal{M}_\alpha(Y)$, a contradiction. ∎

Finally, consider the map $f = \sum_{n=1}^\infty 2^{-n} f_n : 2^\omega \to Y$. Using the facts that $f_n(2^\omega) \subset U_n$ and $U_1 \supset U_2 \supset \ldots$ is a base of closed convex neighborhoods of $0 \in Y$, one may show that the map f is well defined and continuous. We claim that $f^{-1}(TX) = M$. Indeed, if $x \in M = \bigcap_{n\in\mathbb{N}} A_n$, then $f_n(x) \in TB_n$ for every $n \in \mathbb{N}$. Using the fact that $B_1 \supset B_2 \supset \ldots$ is a base of closed convex neighborhoods of $0 \in X$, one can show that the series $\sum_{n=1}^\infty 2^{-n} T^{-1}(f_n(x))$ converges to some point $a \in B_1$. Then $f(x) = T(a) \in TX$ and hence $f(M) \subset TX$. Now suppose $x \in 2^\omega \setminus M$. By the choice of the sets A_n there is a unique $n_0 \in \mathbb{N}$ such that $x \notin A_{n_0}$. Then $f_{n_0}(x) \in U_{n_0} \setminus TX$ and $f_n(x) \in TB_n$ if $n \neq n_0$. Repeating the foregoing arguments, we show that the series $\sum_{n\neq n_0} 2^{-n} T^{-1}(f_n(x))$ converges to some point $a \in B_1$. Then $f(x) = T(a) + 2^{-n_0} f_{n_0}(x) \notin TX$ and hence $f(2^\omega \setminus M) \subset Y \setminus TX$.

Since $M \notin \mathcal{A}_{\alpha+1}$ and $f^{-1}(TX) = M$ we obtain $TX \notin \mathcal{A}_{\alpha+1}(Y)$, a contradiction which proves Claim A. ∎

Therefore $TB_n \in \mathcal{M}_\alpha(Y)$ for some $n \in \mathbb{N}$. If F is any closed subset of B_n, then $TF \in \mathcal{M}_\alpha(TX)$ (since T^{-1} is Borel of class α) and hence $TF \in \mathcal{M}_\alpha(TB_n)$. Since $TB_n \in \mathcal{M}_\alpha(Y)$, we get $TF \in \mathcal{M}_\alpha(Y)$. Theorem 1.3 is proved. ∎

Operators T with T^{-1} of the first Baire class appear very often in mathematical practice (see [VPP]). The following theorem gives a characterization of such operators.

1.4. Theorem. *For an injective operator $T : X \to Y$ between separable Fréchet spaces, the following conditions are equivalent*:

(1) $TX \in \mathcal{M}_2$;
(2) *the map $T^{-1} : TX \to X$ is of the first Baire class*;
(3) *the map $T^{-1} : TX \to X$ is Borel of class* 1;
(4) *there is a neighborhood base $\mathcal{B}$ at $0 \in X$ such that $TB \in \mathcal{A}_1(TX)$ for every $B \in \mathcal{B}$*;
(5) *there is a neighborhood base $\mathcal{B}$ at $0 \in X$ such that TB is closed in TX for every $B \in \mathcal{B}$*;
(6) *there is a neighborhood base $\mathcal{B}$ at $0 \in X$ such that $T(\partial B)$ is a G_δ-set in TX for every $B \in \mathcal{B}$*;
(7) $(T^*Y^*)_{(1)} = X^*$.

Proof. The equivalences (1)⇔(3)⇔(4)⇔(5)⇔(7) follow from 1.1. It is a general fact that, fo a separable Fréchet space Y, (2) is equivalent to (3) (see [Ku, §31, VIII, Theorem 7]); obviously, (6) follows from (3).

Now, we verify (6)⇒(5). For this, we will show that for every neighborhood $U \subset X$ of the origin there exists a closed neighborhood $V \subset U$ of $0 \in X$ such that TV is closed in TX. By (6), there is a neighborhood $V_0 \ni 0$ such that $V_0 - V_0 \subset U$ and $T(\partial V_0)$ is a G_δ-set in TX. Notice that $\mathrm{Int}(V_0)$ and $X \setminus \overline{V}_0$ are disjoint open sets in X whose union is $X \setminus \partial V_0$. Since $T(\partial V_0)$ is a G_δ-set in TX, $T(\mathrm{Int}(V_0)) \subset T(X \setminus \partial V_0) = \bigcup_{n\in\mathbb{N}} K_n$, where K_n's are closed sets in TX. Applying the Baire category theorem, find a convex open set $W_0 \subset \mathrm{Int}(V_0) \cap T^{-1}(K_{n_0})$ for some n_0. Then $\overline{TW_0}$, the closure of TW_0 in TX, is convex; consequently, $W = T^{-1}\overline{TW_0}$ is also convex. Notice that $\overline{TW_0} \subset K_{n_0}$. Since $K_{n_0} \cap T(\partial V_0) = \emptyset$, we obtain $W \cap \partial V_0 = \emptyset$. Since $W \cap \mathrm{Int}(V_0) \neq \emptyset$ and W is connected, we have $W \subset \mathrm{Int}(V_0)$. Then $V = W - x_0$, where $x_0 \in W_0$ is any point, is a neighborhood of $0 \in X$ such that $V \subset V_0 - V_0 \subset U$ and TV is closed in TX. Thus (5) follows. ∎

1.5. Remark. The above Baire category argument will be implicitly used several times in this article. Note that if each K_n is completely metrizable (resp. compact, relatively compact, closed in Y) then so is TV.

In the case of operator images of separable Banach spaces we provide yet another set of conditions which are equivalent to any of (1)–(7) of Theorem 1.4. They follow from the work of [Di] and [VPP]. A subset $F \subset X^*$, where X is a Banach space, is defined to be *norming* if the formula $|x| = \sup\{|x^*(x)| : x^* \in F,\ \|x^*\| \le 1\}$ defines an equivalent norm on X (equivalently, there exists a constant $C > 0$ such that $C\|x\| \le \sup\{|x^*(x)| : x^* \in F,\ \|x^*\| \le 1\}$).

1.6. Remark. For an injective operator $T : X \to Y$ of a separable Banach space X into a Fréchet space Y conditions (1)–(7) of 1.4 are equivalent to each of the following:

(1) there exists a constant $M > 0$ such that if $Tx = \lim Tx_n$ for some sequence $\{x_n\}$ and some $x \in X$ with $\|x_n\| \le 1$, then $\|x\| \le M$;

(2) there exists an equivalent norm $\|\cdot\|$ on X such that, writing B for the closed unit ball with respect to $\|\cdot\|$, TB is closed in TX;

(3) $T^*(Y^*)$ is norming in X^*; and

(4) if we canonically identify X with a subspace of X^{**}, then $X + \mathrm{Ker}(T^{**})$ is closed in X^{**}, and consequently, $X + \mathrm{Ker}(T^{**}) = X \oplus \mathrm{Ker}(T^{**})$.

1.7. Remark. Beyond the class of Banach spaces the last condition of 1.6 is not equivalent to (1)–(7) of 1.4: according to [MP], there exists an injective operator $T : X \to Y$ from a reflexive separable Fréchet space into a Banach space such that T^{-1} fails to be Borel of class 1; yet, because of the reflexivity of X, the operator T trivially satisfies condition (4) of 1.6.

Let $T : X \to Y$ be an injective operator between Fréchet spaces. Generalizing a definition of [BR] we define T to be a *G_δ-embedding* if $TB \in \mathcal{M}_1$ for every closed bounded subset $B \subset X$. Next, we define T to be a *strong G_δ-embedding* if there exists a closed neighborhood $U \subset X$ of the origin such that $TB \in \mathcal{M}_1$ for every closed subset $B \subset U$. Clearly, each strong G_δ-embedding is a G_δ-embedding. The converse is not true: the identity operator $C^\infty[0,1] \to C[0,1]$ is a G_δ-embedding (because the Fréchet space $C^\infty[0,1]$ of smooth functions on $[0,1]$ is a Montel space [Ed, 8.4.7] and thus each closed bounded subset of $C^\infty[0,1]$ is compact) but not a strong G_δ-embedding (this follows from Theorem 5.1 of [Ba_4] and Theorem 1.8 below). However, every G_δ-embedding of a Banach space is a strong G_δ-embedding.

The following theorem characterizes operator images from the class $\mathcal{M}_1^2$.

1.8. Theorem. *For an injective operator $T : X \to Y$ between separable Fréchet spaces, the following conditions are equivalent*:

(1) $TX \in \mathcal{M}_1^2$;

(2) $TX \in \mathcal{M}_2 \cap \mathcal{A}_2$;

(3) *T is a strong G_δ-embedding.*

Proof. Theorem 1.8 will follow from Theorem 1.3 as soon as we prove that $TX \in \mathcal{M}_2$ for every strong G_δ-embedding $T : X \to Y$, where X is separable. It follows from the definition of a strong G_δ-embedding that X has a base $\mathcal{B}$ of neighborhoods of the origin such that $T(\partial B) \in \mathcal{M}_1$ for every $B \in \mathcal{B}$. By Theorem 1.4(6), $TX \in \mathcal{M}_2$. ∎

1.9. Question. *Suppose $T : X \to Y$ is an injective operator between separable Banach spaces such that $TB \in \mathcal{M}_{\alpha+1}$ for every closed bounded subset $B \subset X$. Is $TX \in \mathcal{M}_{\alpha+2}$?*

Next, we investigate Borel properties of operator images of Banach spaces with separable second duals.

1.10. Theorem. *Let X be a Banach space with separable second dual X^{**} and $T : X \to Y$ be an injective operator into a Fréchet space Y such that $TX \in \mathcal{M}_{\alpha+1} \setminus \mathcal{M}_\alpha$ for some* *countable ordinal* $\alpha \geq 1$. *Then*

(1) $TX \in \mathcal{M}_\alpha^2 \setminus \mathcal{M}_\alpha$ *and*

(2) *there an injective operator* $\widetilde{T} : \widetilde{X} \to Y$ *from a separable Banach space* $\widetilde{X} \supset X$ *such that* $\widetilde{T}|_X = T$, $\widetilde{T}(\widetilde{X}) \subset T^{**}(X^{**})$ *and* $\widetilde{T}\widetilde{X} \in \mathcal{A}_\alpha(Y) \setminus \mathcal{M}_\alpha$.

Moreover, the space $\widetilde{X}$ *has separable second dual provided* X *has separable* 4*th dual* X^{****}.

Proof. Theorem 1.2 implies that the ordinal α is not limit. So $\alpha = \beta+1$ for some ordinal β. By Theorem 1.1, $TX \in \mathcal{M}_{\alpha+1}$ implies $(T^*Y^*)_{(\alpha)} = X^*$. By Theorems 31 and 35 of [SR] there exist a separable Fréchet space N_β and two injective operators $r_\beta : X \to N_\beta$ and $v_{0,\beta} : N_\beta \to Y$ such that $v_{0,\beta} \circ r_\beta = T$, $r_\beta(X)$ is dense in N_β, $v_{0,\beta}^{-1}$ is Borel of class β and $r_\beta^*(N_\beta^*) = (T^*Y^*)_{(\beta)}$. ■

Claim C. *The inverse* r_β^{-1} *is Borel of class* 1.

Proof. This follows from Theorem 1.4(7) and the equality $(r_\beta^* N_\beta^*)_{(1)} = ((T^*Y^*)_{(\beta)})_{(1)} = (T^*Y^*)_{(\alpha)} = X^*$. ■

Let B and B^{**} denote the closed unit balls of the separable Banach spaces X and X^{**}, respectively.

Claim D. $r_\beta B \in \mathcal{M}_1$.

Proof. Identifying X with a subspace in X^{**} we infer from Claim C and 1.6(4) that $X+\mathrm{Ker}(r_\beta^{**}) = L$ is closed in X^{**}. Since the space X^{**} is separable, $B^{**} \setminus L$ is a countable union of $*$-weakly compact sets. Noticing that $r_\beta^{**}(B^{**}) \setminus r_\beta(B) = r_\beta^{**}(B^{**} \setminus L)$ we see that $r_\beta^{**}(B^{**}) \setminus r_\beta(B)$ is a countable union of $*$-weakly compact subsets of N_β^{**}. Then $r_\beta(B)$ is a G_δ-set in the closed set $r_\beta^{**}(B^{**}) \subset N_\beta^{**}$. Thus $r_\beta(B) \in \mathcal{M}_1$. ■

According to the terminology of [Ku], $v_{0,\beta} : N_\beta \to v_{0,\beta}(N_\beta) \subset Y$ is a *generalized homeomorphism of class* $(0,\beta)$. Since $r_\beta(B) \in \mathcal{M}_1$, Corollary 3 of [Ku, §35.VII] implies $v_{0,\beta}(r_\beta(B)) \in \mathcal{M}_{\beta+1} = \mathcal{M}_\alpha$. Then $TX = \bigcup_{n=1}^\infty n \cdot v_{0,\beta} \circ r_\beta(B) \in \mathcal{A}_{\alpha+1}$. Applying Theorem 1.3, we get $TX \in \mathcal{M}_\alpha^2$.

Next we construct the Banach space $\widetilde{X} \supset X$ and the operator $\widetilde{T} : \widetilde{X} \to Y$.

Let $\pi : X^{**} \to X^{**}/\mathrm{Ker}(r_\beta^{**})$ denote the quotient operator and $r : X^{**}/\mathrm{Ker}(r_\beta^{**}) \to N_\beta^{**}$ be a unique injective operator with $r \circ \pi = r_\beta^{**}$. Let $\widetilde{X} = r^{-1}(N_\beta) \subset X^{**}/\mathrm{Ker}(r_\beta^{**})$. Clearly, $\pi(X) \subset \widetilde{X}$. Claim C and Theorem 1.6 yield that $X + \mathrm{Ker}(r_\beta^{**})$ is closed in X^{**} and thus $\pi|_X : X \to \widetilde{X}$ is an embedding. So we may identify X with the subspace $\pi(X)$ of $\widetilde{X}$. Clearly, $\widetilde{T}\widetilde{X} \subset T^{**}X^{**}$, and the space $\widetilde{X}$ has separable second dual provided the 4th dual X^{****} of X is separable.

Let $\widetilde{T} = v_{0,\beta} \circ r|_{\widetilde{X}} : \widetilde{X} \to Y$. Evidently, $\widetilde{T}$ extends the operator T. Since $r(X^{**}/\mathrm{Ker}(r_\beta^{**})) = r_\beta^{**}(X^{**})$ is a countable union of $*$-weakly compact subsets, $r(\widetilde{X})$ is an F_σ-set in N_β. So, we can write $r(\widetilde{X}) = \bigcup_{n=1}^\infty F_n$, where each F_n is closed and bounded in N_β. Now we distinguish two cases.

1. β is a limit ordinal. Then by Theorem 1.1, $v_{0,\beta}(N_\beta) \in \mathcal{M}_\beta(Y)$. Because $v_{0,\beta}^{-1}$ is Borel of class β, $v_{0,\beta}(F_n) \in \mathcal{M}_\beta(v_{0,\beta}(N_\beta))$ for every $n \in \mathbb{N}$. Since $v_{0,\beta}(N_\beta) \in \mathcal{M}_\beta(Y)$, we get $v_{0,\beta}(F_n) \in \mathcal{M}_\beta(Y)$ for every n. Then $\widetilde{T}\widetilde{X} = \bigcup_{n=1}^\infty v_{0,\beta}(F_n) \in \mathcal{A}_{\beta+1}(Y) = \mathcal{A}_\alpha(Y)$.

2. $\beta = \gamma + 1$ for some ordinal γ. By Theorem 38 of [SR], N_β is a Banach space. It follows from Theorems 1 and 2 of [PP, II, §1] that $(T^*Y^*)_{(\beta)} = ((T^*Y^*)_{(\gamma)})_{(1)}$ is a norm closed linear subspace in X^*. Since $r_\beta(X)$ is dense in N_β, the dual operator $r_\beta^* : N_\beta^* \to X^*$ is injective and its image $r_\beta^*(N_\beta^*) = (T^*Y^*)_{(\beta)}$ is closed in X^*. Thus $r_\beta^* : N_\beta^* \to X^*$ is an isomorphic embedding and N_β^{**} is a quotient of X^{**}. Since X^{**} is separable, the space N_β^{**} is separable as well. Because $v_{0,\beta} : N_\beta \to Y$ has inverse of Borel class β, Theorem 1.1 and the first statement of Theorem 1.10 imply $v_{0,\beta}(N_\beta) \in \mathcal{M}_\beta^2(Y)$. By Theorem 1.3, $v_{0,\beta}(F) \in \mathcal{M}_\beta(Y)$ for every closed bounded subset of N_β. Then $v_{0,\beta}(F_n) \in \mathcal{M}_\beta(Y)$ for every $n \in \mathbb{N}$ and hence $\widetilde{T}\widetilde{X} = \bigcup_{n=1}^\infty v_{0,\beta}(F_n) \in \mathcal{A}_{\beta+1}(Y) = \mathcal{A}_\alpha(Y)$.

It remains to verify that $\widetilde{T}\widetilde{X} \notin \mathcal{M}_\alpha = \mathcal{M}_{\beta+1}$. Assuming the converse, according to Theorem 1.1, $\widetilde{T}^{-1}$ would be Borel of class β. Then $\widetilde{T}(X) = TX$ would belong to the class $\mathcal{M}_\beta(\widetilde{T}\widetilde{X})$. Since $\widetilde{T}\widetilde{X} \in \mathcal{M}_\alpha$, this yields $TX \in \mathcal{M}_\alpha$, a contradiction. Theorem 1.10 is proved. ∎

2. Constructing operator images of a given Borel complexity. In this section we consider the following question: given two Banach spaces X and Y, when is it possible to construct an injective compact dense operator $T : X \to Y$ with TX of a given Borel class?

We begin with the following simple

2.1. Proposition. *Let X, Y be two separable Banach spaces and $N \subset X^*$ be a norming linear subspace. Then there exists an injective compact dense operator $T : X \to Y$ such that T^*Y^* is norming in X^* and $(T^*)^{-1}(N)$ is norming in Y^*.*

Proof. By [LT, p. 44], there is a biorthogonal sequence $\{x_n, x_n^*\}_{n\in\mathbb{N}}$ such that $\operatorname{span}(\{x_n\}_{n\in\mathbb{N}})$ is dense in X and $\operatorname{span}(\{x_n^*\}_{n\in\mathbb{N}}) \subset N$ is norming in Y^*. Similarly, for the space Y there exists a biorthogonal sequence $\{y_n, y_n^*\}_{n\in\mathbb{N}}$ such that $\operatorname{span}\{y_n\}_{n\in\mathbb{N}}$ is dense in Y and $\operatorname{span}\{y_n^*\}_{n\in\mathbb{N}} \subset Y^*$ is norming.

It is easy to see that the operator $T : X \to Y$ defined by

$$T(x) = \sum_{n=1}^\infty \frac{x_n^*(x) y_n}{2^n \|x_n^*\| \cdot \|y_n\|}$$

satisfies our requirements. ∎

Next we reduce the problem of constructing an operator image of a given Borel class in *any* Banach space to finding such an operator image in *some* Banach space.

2.2. Proposition. *Let $T : X \to Y$ be an injective compact operator between Fréchet spaces. For every separable Banach space Z there exists an injective compact dense operator $S : X \to Z$ such that $S(X)$ is homeomorphic to $T(X)$.*

Proof. First, we prove the statement assuming that Y is a Banach space. Without loss of generality, TX is dense in Y. Since T is compact, the space TX is σ-precompact, that is, TX lies in a σ-compact set in Y. Applying Proposition 2.1 with $N = X^*$, we construct an injective dense operator $P : Y \to Z$. Let $S = P \circ T : X \to Z$. Then S is injective, compact, and dense. Using the technique of [BC, §2], one may prove that the spaces TX and $P \circ T(X) = S(X)$ are homeomorphic.

Now consider the general case of a Fréchet space Y. According to the preceding case, it suffices to construct an injective compact operator $S : X \to l^2$ with $S(X)$ homeomorphic to $T(X)$.

The construction is as follows. Using the compactness of the operator T, find a convex symmetric neighborhood U of the origin in X such that the closure K of the image TU in Y is compact. Let $(f_n)_{n\in\mathbb{N}}$ be a countable set in Y^* separating points of the compactum K. Multiplying each f_n by a suitable positive constant, we may assume that $|f_n(x)| \le 2^{-n}$ for every $x \in K$, $n \in \mathbb{N}$. Consider the linear (possibly discontinuous) map $F : \mathrm{span}(K) \to l^2$ defined by $F(y) = (f_n(y))_{n\in\mathbb{N}}$. Clearly, the restriction $F|_K$ is continuous and thus $F(K)$ is a compact subset of l^2. Then the operator $S = F \circ T : X \to l^2$ is injective and compact. Using the technique of [BC, §2] one may prove that the σ-precompact spaces $T(X)$ and $S(X)$ are homeomorphic. ■

2.3. Question. *Let $T : X \to Y$ be an operator between separable Banach spaces. Is TX homeomorphic to a pre-Hilbert space* (*cf.* [We, LS9])*? It should be mentioned that by* [Ma] *there is a normed space homeomorphic to no convex set in l^2*; *by* [BRZ, 5.5.8] *this space can be chosen to be of the first Baire category.*

Now we remark that there are some restrictions on a Banach space X which forbid to construct operator images $T : X \to Y$ of high Borel complexity.

A Banach space X is called *quasireflexive* if $\dim(X^{**}/X) < \infty$.

2.4. Proposition. *Every injective operator $T : X \to Y$ from a separable quasireflexive Banach space is a G_δ-embedding with $TX \in \mathcal{M}_1^2$.*

Proof. Since X has finite codimension in X^{**}, the linear space $X + \mathrm{Ker}(T^{**})$ is closed in X^{**} and thus $TX \in \mathcal{M}_2$ according to 1.6(3) and 1.4(7). By Theorem 1.10, $TX \in \mathcal{M}_1^2$ and by Theorem 1.8, T is a G_δ-embedding. ■

2.5. Remark. The operator from Remark 1.7 shows that Proposition 2.4 is not true for quasireflexive Fréchet spaces.

On the other hand, for nonquasireflexive Banach spaces we have

2.6. Theorem. *If X is a separable nonquasireflexive Banach space*, *then for every countable ordinal α and for every Banach space Y there exists a compact injective dense operator $T : X \to Y$ with $TX \in \mathcal{M}_{\alpha+2} \setminus \mathcal{M}_{\alpha+1}$.*

Proof. By [Os$_1$], the dual space X^* of the nonquasireflexive Banach space X contains a closed linear subspace $M \subset X^*$ such that $M_{(\alpha)} \ne M_{(\alpha+1)} = X^*$ and $\|x\|_M = \sup\{|f(x)| : f \in M,\ \|f\| \le 1\} > 0$ for every $x \in X$. Then $\|\cdot\|_M$ is a norm on X. Let X_M be the completion of the normed space $(X, \|\cdot\|_M)$ and $P : X \to X_M$ be the natural "embedding" operator. It is easy to see that $M \subset P^* X_M^*$ and $N = (P^*)^{-1}(M)$ is a norming space in X_M^*.

By Proposition 2.1, there exists an injective compact dense operator $E : X_M \to Y$ such that the spaces $L = (E^*)^{-1}(N) \subset Y^*$ and $E^* Y^* \subset X_M^*$ are norming. Then $T = E \circ P : X \to Y$ is a compact injective dense operator. By the Schauder Theorem [HHZ, 130] the dual operator $T^* : Y^* \to X^*$ is compact and thus $T^* Y^* \ne X^*$. According to Theorem 1.1, to prove that $TX \in \mathcal{M}_{\alpha+2} \setminus \mathcal{M}_{\alpha+1}$ it suffices to verify that $T^* Y^*_{(\alpha)} \ne T^* Y^*_{(\alpha+1)} = X^*$.

Since $T^*Y^* = T^*Y^*_{(0)} \neq X^*$ and $M_{(\alpha)} \neq M_{(\alpha+1)} = X^*$ this will follow as soon as we prove

Claim A. $T^*Y^*_{(1)} = M_{(1)}$.

Proof. We need the following equality:

$$P^*X^*_M = M_{(1)}.$$

Denote by B_M, B^*_M and B^* the closed unit ball of the Banach spaces X_M, X^*_M and X^*, respectively. Let

$$(B^* \cap M)_\circ = \{x \in X : |x^*(x)| \leq 1 \text{ for each } x^* \in B^* \cap M\},$$
$$((B^* \cap M)_\circ)^\circ = \{x^* \in X^* : |x^*(x)| \leq 1 \text{ for each } x \in (B^* \cap M)_\circ\}.$$

Observe that $(B^* \cap M)_\circ = P^{-1}(B_M)$ and $((B^* \cap M)_\circ)^\circ = P^*(B^*_M)$. By the Bipolar Theorem [Ed, 8.1.5], $((B^* \cap M)_\circ)^\circ = B^* \cap M_{(1)}$ (see also Theorems 1 and 2 from [PP, II, §1]). Then $P^*(B^*_M) = B^* \cap M_{(1)}$, which just implies $P^*X^*_M = M_{(1)}$.

Recall that $L = (E^*)^{-1}(N)$ is norming in Y^* and thus $L_{(1)} = Y^*$ (see 1.6(3) and 1.4(7)). By the definition of the operator T, $T^*(L) \subset M$. Since the dual operator T^* is compact, it maps $*$-weakly convergent sequences into norm convergent ones. This yields that $T^*Y^* = T^*(L_{(1)})$ lies in the norm closure of M. Since M is norm closed, we get $T^*Y^* \subset M$ and thus $(T^*Y^*)_{(1)} \subset M_{(1)}$. On the other hand, since E^*Y^* is norming in X^*_M, we get $(E^*Y^*)_{(1)} = X^*_M$ and thus $T^*Y^*_{(1)} = (P^* \circ E^*(Y^*))_{(1)} \supset P^*(E^*Y^*_{(1)}) \supset P^*X^*_M = M_{(1)}$. Consequently, $T^*Y^*_{(1)} = M_{(1)}$ and Claim A, hence also Theorem 2.6, is proved. ∎

2.7. Theorem. *For every countable ordinal α and every separable Banach space Y there exists an injective compact dense operator $\widetilde{T} : \widetilde{X} \to Y$ from a Banach space $\widetilde{X}$ with separable second dual such that $\widetilde{T}\widetilde{X} \in \mathcal{A}_{\alpha+1} \setminus \mathcal{M}_{\alpha+1}$.*

Proof. First consider the case $\alpha = 0$. Let X be any separable reflexive Banach space. By Proposition 2.1, there exists an injective compact dense operator $T : X \to Y$. The space X, being reflexive, is weakly σ-compact. Then, by the compactness of T, its image TX is norm σ-compact in Y. Since T is not an embedding, $TX \in \mathcal{A}_1 \setminus \mathcal{M}_1$.

Now suppose $\alpha > 0$. Take any nonquasireflexive Banach space X with separable 4th dual X^{****} (e.g., let $X = (\sum_{n=1}^\infty \oplus J)_{l^2}$ be the l^2-sum of James quasireflexive spaces J, see [HHZ, 264]). By Theorem 2.6 there exists an injective compact dense operator $T : X \to Y$ such that $TX \in \mathcal{M}_{\alpha+2} \setminus \mathcal{M}_{\alpha+1}$. By Theorem 1.10, there is a Banach space $\widetilde{X} \supset X$ with separable second dual and an injective operator $\widetilde{T} : \widetilde{X} \to Y$ such that $\widetilde{T}|_X = T$, $\widetilde{T}\widetilde{X} \subset T^{**}X^{**} \subset Y^{**}$ and $\widetilde{T}\widetilde{X} \in \mathcal{A}_{\alpha+1} \setminus \mathcal{M}_{\alpha+1}$. Since T is compact, so is T^{**}. Then the inclusion $\widetilde{T}\widetilde{X} \subset T^{**}X^{**}$ implies that $\widetilde{T}$ is compact. Thus $\widetilde{T} : \widetilde{X} \to Y$ is an injective dense compact operator with $\widetilde{T}\widetilde{X} \in \mathcal{A}_{\alpha+1} \setminus \mathcal{M}_{\alpha+1}$. ∎

A topological space X is called *$\mathcal{C}$-universal*, where $\mathcal{C}$ is a class of topological spaces, if X contains a closed topological copy of each space $C \in \mathcal{C}$.

2.8. Theorem. *For every countable ordinal α and every separable Banach space Y there is an injective compact dense operator $T : X \to Y$ from a Banach space X with separable dual such that $TX \in \mathcal{M}_{\alpha+2} \setminus \mathcal{A}_{\alpha+2}$ and TX is an $\mathcal{M}_{\alpha+2}$-universal space.*

Proof. In Theorem 21 of [SR], J. Saint-Raymond has constructed, for every countable ordinal α, a locally compact metric countable space $K_{\alpha+1}$ and an injective operator $u_{\alpha+1} : c_0(K_{\alpha+1}) \to c_0$ from the Banach space of continuous real functions on $K_{\alpha+1}$ tending to 0 at infinity such that $u_{\alpha+1}^{-1}$ is Borel of class α. Then by Theorem 1.1, $u_{\alpha+1}(c_0(K_{\alpha+1})) \in \mathcal{M}_{\alpha+2}$. By Corollary 24 of [SR], the image $u_{\alpha+1}(c_0(K_{\alpha+1}))$ is $\mathcal{M}_{\alpha+2}$-universal. It follows from the construction of $u_{\alpha+1}$ that the operator $u_{\alpha+1}$ is compact.

Let $X = c_0(K_{\alpha+1})$. Clearly, X has separable dual isometric to l^1. By Proposition 2.2, for every separable Banach space Y there is a compact injective dense operator $T : X \to Y$ with TX homeomorphic to $u_{\alpha+1}(X)$. Clearly, $TX \in \mathcal{M}_{\alpha+2}$ and TX is $\mathcal{M}_{\alpha+2}$-universal. Since $\mathcal{M}_{\alpha+2} \not\subset \mathcal{A}_{\alpha+2}$ (see [Ke, 22.4]), $TX \notin \mathcal{A}_{\alpha+2}$ (otherwise, every space from $\mathcal{M}_{\alpha+2}$, being homeomorphic to a closed subspace in TX, would belong to the class $\mathcal{A}_{\alpha+2}$). ■

2.9. Corollary. *For every countable limit ordinal $\alpha > 0$ there is an injective operator $T : X \to Y$ between separable Fréchet spaces such that $TX \in \mathcal{M}_\alpha \setminus \mathcal{A}_\alpha$ and TX is an $\mathcal{M}_\alpha$-universal space.*

Proof. Let $\{\alpha_n\}_{n=1}^\infty$ be an increasing sequence of countable ordinals such that $\alpha_n < \alpha$ for each n and $\sup\{\alpha_n : n \in \mathbb{N}\} = \alpha$. By Theorem 2.8, for every $n \in \mathbb{N}$ there is an injective operator $T_n : X_n \to Y_n$ between separable Banach spaces such that $T_nX_n \in \mathcal{M}_{\alpha_n+2}$ is an $\mathcal{M}_{\alpha_n+2}$-universal space. Consider the Fréchet spaces $X = \prod_{n\in\mathbb{N}} X_n$ and $Y = \prod_{n\in\mathbb{N}} Y_n$ and the injective operator $T = \prod_{n\in\mathbb{N}} T_n : X \to Y$. Clearly, TX is homeomorphic to $\prod_{n\in\mathbb{N}} T_nX_n$. In a standard way [BM, §6], it can be proved that $TX \in \mathcal{M}_\alpha$ is an $\mathcal{M}_\alpha$-universal space. ■

2.10. Proposition. *For every countable ordinal α and every separable Banach space Y there exists an injective compact dense operator $T : X \to Y$ from a Banach space X with separable dual such that $TX \in \mathcal{M}^2_{\alpha+2} \setminus (\mathcal{M}_{\alpha+2} \cup \mathcal{A}_{\alpha+2})$.*

Proof. By Theorems 2.7 and 2.8, there are compact injective dense operators $T_1 : X_1 \to Y_1$, $T_2 : X_2 \to Y_2$ between Banach spaces with separable duals such that $T_1X_1 \in \mathcal{A}_{\alpha+2} \setminus \mathcal{M}_{\alpha+2}$ and $T_2X_2 \in \mathcal{M}_{\alpha+2} \setminus \mathcal{A}_{\alpha+2}$. Then $T_1 \times T_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is a compact injective dense operator from the Banach space $X = X_1 \times X_2$ with separable dual such that $T_1 \times T_2(X)$ is homeomorphic to $T_1X_1 \times T_2X_2$. Since $T_1X_1 \in \mathcal{A}_{\alpha+2} \setminus \mathcal{M}_{\alpha+2}$ and $T_2X_2 \in \mathcal{M}_{\alpha+2} \setminus \mathcal{A}_{\alpha+2}$, we get $T_1X_1 \times T_2X_2 \in \mathcal{M}^2_{\alpha+2} \setminus (\mathcal{M}_{\alpha+2} \cup \mathcal{A}_{\alpha+2})$.

By Proposition 2.2, for every separable Banach space Y there exists a compact injective dense operator $T : X \to Y$ such that TX is homeomorphic to $T_1X_1 \times T_2X_2$. Clearly, $TX \in \mathcal{M}^2_{\alpha+2} \setminus (\mathcal{M}_{\alpha+2} \cup \mathcal{A}_{\alpha+2})$. ■

2.11. Question. *Let α be a limit ordinal. Does there exist an operator image $TX \in \mathcal{M}^2_{\alpha+1} \setminus (\mathcal{A}_{\alpha+1} \cup \mathcal{M}_{\alpha+1})$? The answer is "yes" for $\alpha = 0$.*

2.12. Question. *Does there exist, for every countable ordinal α, an injective operator $T : X \to Y$ between separable Banach spaces such that $TX \in \mathcal{A}_{\alpha+1}$ is an $\mathcal{A}_{\alpha+1}$-universal space? The answer is "yes" for $\alpha = 0$.*

3. Topology of operator images. In this section we give a complete topological classification of the pairs (Y, TX), where $T : X \to Y$ is a dense operator between Fréchet spaces such that $TX \in \mathcal{M}^2_1$. It turns out that each of the classes $\mathcal{M}_1 \setminus \mathcal{A}_1$, $\mathcal{A}_1 \setminus \mathcal{M}_1$, and $\mathcal{M}_1^2 \setminus (\mathcal{M}_1 \cup \mathcal{A}_1)$ contains exactly one (up to homeomorphism) operator image. We recall that $s = (-1,1)^\omega$ is the pseudointerior of the Hilbert cube $Q = [-1,1]^\omega$ and $\Sigma = \{(x_n) \in Q : \sup_{n\in\omega} |x_n| < 1\}$ is its radial interior. Let $Y \subset X$, $Y' \subset X'$ be topological spaces. The pairs (X,Y) and (X',Y') are called *homeomorphic* if $h(Y) = Y'$ for some homeomorphism $h : X \to X'$. Further, the symbol "$\cong$" will mean "is homeomorphic to".

3.1. Theorem. *Let $T : X \to Y$ be a dense operator between infinite-dimensional Fréchet spaces. If $TX \in \mathcal{M}_1^2$ then TX is homeomorphic to one of the spaces: s, Σ or $\Sigma \times s$. More precisely, the pair (Y,TX) is homeomorphic to:*

(a) (s,s) *if* $TX \in \mathcal{M}_1 \setminus \mathcal{A}_1$;
(b) (s,Σ) *if* $TX \in \mathcal{A}_1$;
(c) $(s\times s, \Sigma\times s)$ *if* $TX \in \mathcal{A}_1(Y) \setminus \mathcal{A}_1$;
(d) $(s\times Q, \Sigma\times s)$ *if* $TX \notin \mathcal{A}_1(Y)$ *and T is compact*;
(e) $(s\times s\times Q, s\times\Sigma\times s)$ *if* $TX \notin \mathcal{A}_1(Y)$ *and T is not compact.*

We will employ the following results, where (a′) is well known, (b′) can be derived from [BP, VIII, 3.1, p. 275], and (c′)–(e′) can be found in [Ba$_1$, Proposition 6.2].

3.2. Proposition. *Let Y be a separable Fréchet space and let C be a closed infinite-dimensional convex subset of Y. Assume that $L \in \mathcal{M}_1^2$ is a dense linear subspace of Y. Then the pair (Y,L) is homeomorphic to*

(a′) (s,s) *if* $L \in \mathcal{M}_1$;
(b′) (s,Σ) *if* $L \in \mathcal{A}_1$ *and C is a compact subset of L*;
(c′) $(s\times s, \Sigma\times s)$ *if* $L \in \mathcal{A}_1(Y)$, $L \neq Y$, *and C is a non-locally compact subset of L*;
(d′) $(s\times Q, \Sigma\times s)$ *if L is contained in a σ-compact subset of Y and* $(C, C\cap L) \cong (Q,s)$;
(e′) $(s\times s\times Q, s\times\Sigma\times s)$ *if* $L\neq Y$ *and* $(C, C\cap L) \cong (s\times Q, s\times s)$.

Let us recall that a subset A of a topological space X is called a *Z-set* in X if A is closed in X and every continuous map $f : Q \to X$ of the Hilbert cube can be uniformly approximated by continuous maps into $X \setminus A$. A subset $A \subset X$ is called a *σZ-set* in X if A can be written as a countable union $A = \bigcup_{n=1}^\infty A_n$, where each A_n is a Z-set in X. A topological space X is defined to be a *σZ-space* if X is a σZ-set in X.

Z-sets can be thought of as infinite-dimensional counterparts of closed nowhere dense subsets, while σZ-spaces as counterparts of spaces of the first Baire category.

3.3. Lemma. *Let Y, C, L be as in* 3.2, *and let B be a closed infinite-dimensional convex subset of Y. Assume that $C\cap L \in \mathcal{M}_1$ is dense in C, and that $B \subset C\setminus L$. We have*:

(d″) *if C is compact, then* $(C, C\cap L) \cong (Q,s)$;
(e″) *if B is not locally compact, then* $(C, C\cap L) \cong (s\times Q, s\times s)$.

Proof. Pick a sequence $\{c_n\}_{n=1}^\infty \subset C\cap L$ which is dense in C. For every $n \geq 1$, let

$$D_n = \Big\{t_0 b + \sum_{i=1}^n t_i c_i : t_0 \geq 2^{-n},\ t_i \geq 0,\ \sum_{i=0}^n t_i = 1 \text{ and } b \in B\Big\}.$$

Write $D = \bigcup_{n=1}^\infty D_n$. We have $B \subset D_n \subset D_{n+1} \subset C\setminus L$, $n \geq 1$. Moreover, since B is closed and convex, each D_n is also closed and convex. Since $C\setminus L \in \mathcal{A}_1(C)$ and $C\cap L$ is dense in C, $C\setminus L$ is a σZ-set in C. It follows that D is also a σZ-set in C.

If C is compact, then D contains the infinite-dimensional compactum B and, by [CDM, 4.1], the pair (C, D) is homeomorphic to (Q, Σ). If B is not locally compact, then neither is C and each D_n. By [DT], C and each D_n are homeomorphic to s. Now, it follows from [Ba_1, 4.1] that (C, D) is homeomorphic to $(s \times Q, s \times \Sigma)$. In both cases, from the Maximality Theorem of Toruńczyk [BP, p. 131], we obtain $(C, C \setminus L) \cong (Q, \Sigma)$ and $(C, C \setminus L) \cong (s \times Q, s \times \Sigma)$, respectively. Now, using the fact that $(Q, Q \setminus \Sigma) \cong (Q, s)$, we easily conclude our proof. ∎

Proof of 3.1. (a) and (b). Use the standard fact that X contains a convex compactum K with $\dim K = \infty$. Let $C = TK$ and apply 3.2(a′) and (b′).

(c) Applying a Baire category argument (see 1.5), we obtain a closed convex neighborhood U of $0 \in X$ such that TU is closed in Y. Since TU spans TX and $TX \notin \mathcal{A}_1$, TU is not locally compact. Apply 3.2(c′) with $C = TU$.

(d) Assume that T is compact. Using the assumption that $TX \in \mathcal{M}_1^2$, we can write $TX = \bigcup_{n=1}^{\infty} K_n \subset \bigcup_{n=1}^{\infty} Z_n \subset Y$, where each Z_n is compact and $K_n \in \mathcal{M}_1$ is a closed subset of TX; we may additionally require that $K_n \subset Z_n$, $n \geq 1$. Again, applying a Baire category argument (see 1.5), we can find a closed convex neighborhood U of $0 \in X$ such that $\overline{TU}$ is compact and $\overline{TU} \cap TX \in \mathcal{M}_1$. We will apply 3.2(d′) with $L = TX$ and $C = \overline{TU}$.

By 3.3(d″), we only need to find a set B required therein. As in the proof of (a), we take a convex compactum $K \subset U$ with $\dim K = \infty$. Since $L \notin \mathcal{A}_1$ (otherwise, $L \in \mathcal{A}_1(Y)$), there exists $x_0 \in C \setminus L$. We set $B = \frac{1}{2}x_0 + \frac{1}{2}TK$.

(e) Suppose T is not compact. We will find sets C and B required in 3.3(e″) so that 3.2(e′) is applicable with $L = TX$. By the continuity of T, there are bases $\{U_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$, $U_1 \supset U_2 \supset \ldots$, $V_1 \supset V_2 \supset \ldots$, of closed symmetric convex neighborhoods of the origins in X and Y respectively, such that $TU_n \subset V_n$. As in the proof of (d), find a closed symmetric convex neighborhood U of $0 \in X$ such that $\overline{TU} \cap TX \in \mathcal{M}_1$. Replacing each U_n by $U \cap U_n$, we can additionally require that $\overline{TU_n} \cap TX \in \mathcal{M}_1$. We let $C = \overline{TU_1}$.

Write $D_n = \overline{TU_n}$ and $L_n = \operatorname{span}(D_n)$. We have $L_1 \supset L_2 \supset \ldots \supset TX$. If $L_k \neq L_1$ for some $k > 1$, then we also have $D_1 \setminus L_k \neq \emptyset$. Letting $B = \frac{1}{2}x_0 + \frac{1}{2}D_k$, where $x_0 \in D \setminus L_k$ is arbitrary, we get $B \subset C \setminus L_k \subset C \setminus TX$. If C (resp. B) were locally compact, then L_1 (resp L_k) would be σ-compact. This however contradicts the fact that T is not compact.

Hence, the remaining case occurs if $L_1 = L_2 = \ldots$ (this case arises if, e.g., X is a Banach space). Let τ be the locally convex metrizable topology on L_1 with the basis generated by the sets $\{D_k\}_{k=1}^{\infty}$ (use the gauge Minkowski functionals of D_k's to obtain a sequence of seminorms). Noticing that the operator $T : X \to TX \subset (L_1, \tau)$ is continuous and open, we find that TX is closed in (L_1, τ). Observe that $L_1 = \bigcup_{n=1}^{\infty} nD_1 \in \mathcal{A}_1(Y)$, and our assumptions yield $L_1 \setminus TX \neq \emptyset$; hence, $C \setminus TX \neq \emptyset$. Pick $x_0 \in \frac{1}{2}C \setminus TX$ and use the fact that the sets $\frac{1}{m}D_k$, $k, m \geq 1$, form a basis at 0 for τ to select $k, m \geq 2$ with $\left(x_0 + \frac{1}{m}D_k\right) \cap TX = \emptyset$. We finally let $B = x_0 + \frac{1}{m}D_k$. ∎

It should be mentioned that $\Sigma \cong \Omega_0$, $\Sigma \times s \cong \Omega_1$ and $\Sigma^{\omega} \cong \Omega_2$ are the first elements in the hierarchy of so-called $\mathcal{M}_\alpha$-absorbing spaces Ω_α (see [BM]). The spaces Ω_α can be realized as some special subsets of the Hilbert cube Q. Their definitions are too complex to give them here. All we need to know about these spaces is the following result proved in [BRZ, 5.3.6 and 3.1.3].

3.4. Theorem. *Let α be a countable ordinal, L be a locally convex linear metric space and $\overline{L}$ denote the completion of L with respect to any invariant metric.*

(1) *The space L is homeomorphic to Ω_α if and only if $L \in \mathcal{M}_\alpha$ is an $\mathcal{M}_\alpha$-universal σZ-space.*

(2) *The pair $(\overline{L}, L)$ is homeomorphic to $(Q \times s, \Omega_\alpha \times \Sigma)$ if and only if $L \in \mathcal{M}_\alpha$ is an $\mathcal{M}_\alpha$-universal space contained in a σ-compact subset of $\overline{L}$.*

We apply this theorem to prove

3.5. Theorem. *Let α be a countable ordinal and $T : X \to Y$ be a dense operator between Fréchet spaces.*

(1) *The operator image TX is homeomorphic to Ω_α if and only if $TX \in \mathcal{M}_\alpha$ is an $\mathcal{M}_\alpha$-universal space.*

(2) *The pair (Y, TX) is homeomorphic to $(Q \times s, \Omega_\alpha \times \Sigma)$ if and only if $TX \in \mathcal{M}_\alpha$ is an $\mathcal{M}_\alpha$-universal space and the operator T is compact.*

In an obvious way this theorem follows from Theorem 3.4 and the subsequent

3.6. Proposition. *Every noncomplete operator image is a σZ-space.*

Proof. Since TX is not complete the operator $T : X \to Y$ is not open. By the Open Mapping Principle, the image TU of some convex neighborhood U of $0 \in X$ is nowhere dense in TX. Then $\overline{TU}$, the closure of TU in TX, is a closed convex nowhere dense set in TX. Moreover, $\overline{TU}$ spans TX (because $TX = \bigcup_{n\in\mathbb{N}} n \cdot \overline{TU}$). According to [$\mathrm{Ba}_2$], the two properties of $\overline{TU}$ imply that $\overline{TU}$ as well as $n \cdot \overline{TU}$ are Z-sets in TX. Then $TX = \bigcup_{n\in\mathbb{N}} n \cdot \overline{TU}$, and it is a σZ-space. ■

By Theorem 3.5, every $\mathcal{M}_2$-universal operator image $TX \in \mathcal{M}_2$ (including those supplied by Theorem 2.8) is homeomorphic to $\Omega_2 \cong \Sigma^\omega$. Is every operator image of the class $\mathcal{M}_2 \setminus \mathcal{A}_2$ homeomorphic to Σ^ω? Here we have a negative answer.

3.7. Theorem. *For every separable Banach space Y there is an injective compact dense operator $T : X \to Y$ from a Banach space with separable dual such that $TX \in \mathcal{M}_2 \setminus \mathcal{A}_2$ but TX is not $\mathcal{M}_2$-universal. Thus, every separable Banach space contains two dense nonhomeomorphic operator images of class $\mathcal{M}_2 \setminus \mathcal{A}_2$.*

Proof. By [Ba_3, 2.2] there exists a Banach space X (denoted by B_∞ in [Ba_3], [GM], and [GMS]) with separable dual and such that the closed unit ball B of X endowed with the weak topology satisfies the following conditions: $(B, \mathrm{weak}) \in \mathcal{M}_2 \setminus \mathcal{A}_2$ and (B, weak) is not $\mathcal{M}_2$-universal.

Let $\mathcal{F} \subset B^*$ be a countable norm dense subset of the unit ball of the dual space X^*. Consider the operator $E : X^{**} \to \mathbb{R}^{\mathcal{F}}$ defined by $E(x^{**}) = (x^{**}(x^*))_{x^*\in\mathcal{F}}$. Since $\mathcal{F}$ is countable, $\mathbb{R}^{\mathcal{F}}$ is a Fréchet space. Clearly, E and $E|_X$ are compact injective operators, the image $E(B)$ of the closed unit ball B of X is closed in $E(X)$ and is homeomorphic to (B, weak). Since $(B, \mathrm{weak}) \notin \mathcal{A}_2$ and $E(B)$ is closed in $E(X)$, we get $E(X) \notin \mathcal{A}_2$. Next, by Theorem 1.4, $E(X) \in \mathcal{M}_2$. Let us show that the space $E(X)$ is not $\mathcal{M}_2$-universal.

Since the space $E(B)$, being a topological copy of (B,weak), is not $\mathcal{M}_2$-universal, there is a space $M\in\mathcal{M}_2$ which is homeomorphic to no closed subset of $E(B)$. Embed M into the Hilbert cube $Q=[0,1]^\omega$ and consider the subset

$$\Omega=((M\times\{0\})\cup(Q\times(0,1]))^\omega$$

in $(Q\times[0,1])^\omega$. Clearly, $\Omega\in\mathcal{M}_2$, Ω is a Baire space (because Ω contains a dense absolute G_δ-subset $(Q\times(0,1])^\omega$) and for every nonempty open subset $U\subset\Omega$ there is a closed embedding $e:M\to\Omega$ with $e(M)\subset U$. Assuming that the space $E(X)$ is $\mathcal{M}_2$-universal, we would find a closed embedding $i:\Omega\to E(X)$. Since $E(X)=\bigcup_{n\in\mathbb{N}} n\cdot E(B)$, by the Baire Theorem, there is $n\in\mathbb{N}$ such that the set $i^{-1}(n\cdot E(B))$ has nonempty interior in Ω. By the property of Ω, there exists a closed embedding $e:M\to\Omega$ such that $e(M)\subset i^{-1}(n\cdot E(B))$. Then $\frac{1}{n}\cdot i\circ e:M\to E(B)$ is a closed embedding, a contradiction.

By Proposition 2.2 for every separable Banach space Y there exists an injective compact dense operator $T:X\to Y$ such that TX is homeomorphic to $E(X)$. Clearly, $TX\in\mathcal{M}_2\setminus\mathcal{A}_2$ and TX is not $\mathcal{M}_2$-universal. ∎

3.8. Question. *Does the class $\mathcal{A}_2\backslash\mathcal{M}_2$ contain two topologically distinct operator images?*

References

[Ba₁] T. Banakh, *The strongly universal property in convex sets*, Mat. Stud. 10 (1998), 203–218.

[Ba₂] —, *Characterization of convex Z-sets in linear metric spaces*, Funktsional. Anal. i Prilozhen. 28 (1994), no 4, 77–79 (in Russian; English transl.: Functional Anal. Appl. 28 (1994) 285–286).

[Ba₃] —, *Topological classification of weak unit balls of Banach spaces*, this volume, 7–35.

[Ba₄] —, *Operator images homeomorphic to Σ^ω*, this volume, 53–81.

[BC] T. Banakh et R. Cauty, *Universalité forte pour les sous-ensembles totalement bornés. Applications aux espaces $C_{\mathrm p}(X)$*, Colloq. Math. 73 (1997), 25–33.

[BRZ] T. Banakh, T. Radul and M. Zarichnyĭ, *Absorbing Sets in Infinite-Dimensional Manifolds*, VNTL Publishers, Lviv, 1996.

[BP] C. Bessaga and A. Pełczyński, *Selected Topics in Infinite-Dimensional Topology*, PWN, Warszawa, 1975.

[BM] M. Bestvina and J. Mogilski, *Characterizing certain incomplete infinite-dimensional absolute retracts*, Michigan Math. J. 33 (1986), 291–313.

[BR] J. Bourgain and H. P. Rosenthal, *Applications of the theory of semi-embeddings to Banach space theory*, J. Funct. Anal. 52 (1983), 149–188.

[Ca] R. Cauty, *La classe borélienne ne détermine pas le type topologique de $C_{\mathrm p}(X)$*, Serdica Math. J. 24 (1998), 307–318.

[CDM] D. W. Curtis, T. Dobrowolski and J. Mogilski, *Some applications of the topological characterizations of the sigma-compact spaces l_2^f and Σ*, Trans. Amer. Math. Soc. 284 (1984), 837–846.

[Di] J. Dixmier, *Sur un théorème de Banach*, Duke Math. J. 15 (1948), 1057–1074.

[DT] T. Dobrowolski and H. Toruńczyk, *Separable complete ANR's admitting a group structure are Hilbert manifolds*, Topology Appl. 12 (1981), 229–235.

[Ed] G. A. Edwards, *Functional Analysis*, Holt, Rinehart & Winston, New York, 1965.

[GM] N. Ghoussoub and B. Maurey, *G_δ-embeddings in Hilbert space*, J. Funct. Anal. 61 (1985), 72–97.

[GMS] N. Ghoussoub, B. Maurey and W. Schachermayer, *A counterexample to a problem of points of continuity in Banach spaces*, Proc. Amer. Math. Soc. 99 (1987), 278–282.

[HHZ] P. Habala, P. Hájek and V. Zizler, *Introduction to Banach Spaces*, Matfyzpress, Praha, 1996.

[Ke] A. S. Kechris, *Classical Descriptive Set Theory*, Springer, 1995.

[Ku] K. Kuratowski, *Topology*, *I*, PWN, Warszawa, 1966.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, *I*, Springer, Berlin, 1977.

[Ma] W. Marciszewski, *On topological embeddings of linear metric spaces*, Math. Ann. 308 (1997), 21–30.

[MP] L. D. Menikhes and A. M. Plichko, *On the regularizability theory in topological vector spaces*, Ukraïn. Mat. Zh. 42 (1990), 777–781 (in Ukrainian).

[Os_1] M. I. Ostrovskiĭ, *w^*-derived sets of transfinite order of subspaces in a dual Banach space*, Dokl. Akad. Nauk Ukrain. SSR Ser. A 10 (1987), 9–12 (in Ukrainian).

[Os_2] —, *Weak* sequential closures in Banach space theory and their applications*, in: General Topology in Banach Spaces, A. M. Plichko and T. Banakh (eds.), NOVA Publishers, New York, to appear.

[PP] Y. I. Petunin and A. N. Plichko, *The Theory of Characteristics of Subspaces and its Applications*, Vyssha Shkola, Kyiv, 1980 (in Russian).

[Pl_1] A. N. Plichko, *Weak* sequential closures and B-measurability of maps inverse to linear continuous operators in WGG-spaces*, Sibirsk. Mat. Zh. 22 (1981), 217.

[Pl_2] —, *Decomposition of Banach space into a direct sum of separable and reflexive subspaces and Borel maps*, Serdica Math. J. 23 (1997), 335–350.

[SR] J. Saint-Raymond, *Espaces à modèle séparable*, Ann. Inst. Fourier (Grenoble) 26 (1976), no. 3, 211–256.

[Sch] H. H. Schaefer, *Topological Vector Spaces*, Macmillan, New York, 1966.

[VPP] V. A. Vinokurov, Yu. I. Petunin and A. N. Plichko, *Measurability and regularization of mappings that are inverses of continuous linear operators*, Mat. Zametki 26 (1979), 583–591 (in Russian).

[We] J. E. West, *Open problems in infinite-dimensional topology*, in: Open Problems in Topology, J. van Mill and G. M. Reed (eds.), Elsevier, Amsterdam, 1990, 523–597.

DISSERTATIONES
MATHEMATICAE
387 (2000)

Operator images homeomorphic to Σ^ω

by

Taras Banakh

Abstract. We investigate the topology of operator images, that is, spaces of the form TX, where $T : X \to Y$ is a continuous linear operator between Fréchet (= locally convex linear complete metric) spaces. Under some restrictions we confirm Dobrowolski's conjecture that there are only four topological types of separable infinite-dimensional operator images that are absolute $F_{\sigma\delta}$-sets. In particular, we show that if $T : X \to Y$ is an injective weakly compact linear operator from an infinite-dimensional separable Banach lattice X such that TX is an $F_{\sigma\delta}$-set in Y, then the operator image TX is homeomorphic to one of the spaces: s, Σ, $\Sigma \times s$, Σ^ω, where s is the pseudo-interior of the Hilbert cube and Σ is its radial interior. Moreover, we consider operator images of the classical Banach spaces c_0, l_1, L_1, $C[0,1]$. Some counterexamples to the Dobrowolski conjecture are presented as well.

Introduction. In this paper we continue investigations of the topology of operator images, started in [BDP]. We recall that an *operator image* is a space of the form TX, where $T : X \to Y$ is a continuous linear operator between Fréchet (= locally convex linear complete metric) spaces. While in [BDP] operator images of high Borel complexity were studied, in this paper we restrict ourselves to operator images belonging to the Borel class $\mathcal{M}_2$ of separable absolute $F_{\sigma\delta}$-spaces. Such operator images are typical; actually, constructing an operator image $TX \notin \mathcal{M}_2$ requires considerable effort (see [SR]). Throughout the paper the term "operator" means "continuous linear operator" between locally convex (mainly Fréchet) spaces.

In the late 80s T. Dobrowolski made a conjecture that there exist only four topological types of infinite-dimensional operator images of the Borel class $\mathcal{M}_2$. More precisely, every such operator image is homeomorphic to one of the spaces: s, Σ, $\Sigma \times s$, or Σ^ω, where $s = (-1,1)^\omega$ is the pseudo-interior of the Hilbert cube $Q = [-1,1]^\omega$, $\Sigma = \{(x_i) \in Q : \sup_{i\in\omega} |x_i| < 1\}$ is the radial interior of Q, and Σ^ω is the countable power of Σ.

Operator images homeomorphic to one of the three spaces s, Σ, $\Sigma \times s$ were characterized in [BDP, 3.1] as those belonging to the small Borel class $\mathcal{M}_1^2$ consisting of differences $A \setminus B$ of Polish (= separable complete-metrizable) spaces $B \subset A$. So the problem reduces to the following one: *is TX homeomorphic to Σ^ω provided $TX \in \mathcal{M}_2 \setminus \mathcal{M}_1^2$?*

2000 *Mathematics Subject Classification*: Primary 57N17; Secondary 46B22, 46B25, 46B42, 47A05, 54H05.

The research was partially supported by the grant INTAS-96-0753.

It turns out that the answer depends much on (1) geometric properties of the Fréchet space X, and (2) in the case of complex X on properties of the operator T. We illustrate our results by considering operator images of the classical nonreflexive Banach spaces c_0, l_1, L_1, and $C[0,1]$ (reflexive Banach spaces admit no operator image of class $\mathcal{M}_2 \setminus \mathcal{M}_1^2$).

Let $T : X \to Y$ be an injective operator between infinite-dimensional separable Fréchet spaces Y such that $TX \in \mathcal{M}_2$.

(1) If $X = c_0$, then TX is homeomorphic either to s (if T is an isomorphic embedding) or to Σ^ω.

(2) If $X = l_1$, then TX is homeomorphic to one of the spaces: s, Σ, $\Sigma \times s$ (if T is a G_δ-embedding) or Σ^ω.

(3) If $X = L_1$, then TX is homeomorphic either to s (if T is an isomorphic embedding) or to $\Sigma \times s$ (if T is a nontrivial G_δ-embedding) or to Σ^ω.

(4) If $X = C[0,1]$ and the operator T is weakly compact, then TX is homeomorphic to Σ^ω; if T is not weakly compact, then TX can be homeomorphic to any of the spaces s, $\Sigma \times s$, Σ^ω as well as to none of these spaces.

(5) If X is a Banach lattice and the operator T is weakly compact, then TX is homeomorphic to one of the spaces s, Σ, $\Sigma \times s$, Σ^ω.

(6) If X is a Banach lattice, the operator T is compact, and TX is dense in Y, then the pair (Y, TX) is homeomorphic to one of the pairs (s, Σ), $(\sigma \times Q, \Sigma \times s)$, $(s \times Q^\omega, \Sigma \times \Sigma^\omega)$.

We recall that two pairs (A, B), (A', B') of topological spaces $B \subset A$, $B' \subset A'$ are *homeomorphic* provided there exists a homeomorphism $h : A \to A'$ such that $h(B) = B'$.

A pathological non-weakly compact operator $T : C[0,1] \to Y$ disproving Dobrowolski's conjecture (that is, with $T(C[0,1])$ in $\mathcal{M}_2 \setminus \mathcal{M}_1^2$ but not homeomorphic to Σ^ω) will be constructed using another pathological Banach space B_∞ known in Banach space theory as an example of a Banach space with CPCP but without PCP (see [GMS$_1$]). The Banach space B_∞ admits an injective *compact* operator $T : B_\infty \to l^2$ such that $T(B_\infty) \in \mathcal{M}_2 \setminus \mathcal{M}_1^2$ but $T(B_\infty)$ is not homeomorphic to Σ^ω. This shows that in spite of the optimistic situation with weakly compact operators from Banach lattices, Dobrowolski's conjecture is not true even in the realm of compact operators from Banach spaces with separable duals.

Of course, the above statements are true in a more general setting. We summarize all principal results of this paper in the following

MAIN THEOREM. *Suppose $T : X \to Y$ is an injective operator between separable Fréchet spaces such that $TX \in \mathcal{M}_2$. The space TX is homeomorphic to Σ^ω in each of the following cases*:

(1) *X is not normable and T is bounded*;

(2) *T is not strictly regular*;

(3) *X is not strongly regular and T is strongly regular*;

(4) *X is not strongly regular and T is weakly compact*;

(5) *X is nowhere strongly regular and T is not an isomorphic embedding*;

(6) *every strongly regular closed bounded convex subset of X is subset-dentable and T is not a G_δ-embedding*;

(7) *X has RNP and T is not a G_δ-embedding*;

(8) *X is a Banach space complemented in its second dual X^{**} and T is not a G_δ-embedding*;

(9) *X is a Banach lattice containing no isomorphic copy of the Banach space c_0 and T is not a G_δ-embedding*;

(10) *X is a Banach lattice*, *T is weakly compact*, *and T is not a G_δ-embedding*;

(11) *the space* (X, weak) *is $\mathcal{M}_2$-universal and T is compact*;

(12) *there exists a Fréchet space Z and two injective operators $T_1 : X \to Z$ and $T_2 : Z \to Y$ such that $T = T_2 \circ T_1$ and the space T_1X is $\mathcal{M}_2$-universal*;

(13) *there exists an F_σ-embedding $E : Z \to X$ of a separable Fréchet space Z such that $T \circ E(Z)$ is homeomorphic to Σ^ω.*

Now we briefly describe the content of the paper and simultaneously explain the items of the Main Theorem.

In the first section we introduce and study strictly regular operators. These are operator counterparts of strongly regular Banach spaces introduced and studied in [GGMS]. The principal and technically most difficult result of this section is Theorem 1.5 establishing ties between strictly regular operators, G_δ-embeddings, and Fréchet spaces with the Radon–Nikodým Property.

In the second section we develop some topological tools which will be used in the subsequent sections for detecting operator images homeomorphic to Σ^ω. The main result of §2 is the characterization and factorization Theorem 2.4 (see also items (12), (13) of the Main Theorem). The characterization is given in terms of the $\mathcal{M}_2$-universality of operator images. We recall that a topological space X is called *$\mathcal{C}$-universal* provided X contains a closed topological copy of each space $C \in \mathcal{C}$. It is well known that the space Σ^ω is $\mathcal{M}_2$-universal. Thus every operator image TX homeomorphic to Σ^ω must belong to the class $\mathcal{M}_2$ and be $\mathcal{M}_2$-universal. Theorem 2.1 states that these two conditions characterize operator images homeomorphic to Σ^ω.

In the next three sections we find three quite general situations leading to operator images homeomorphic to Σ^ω. In particular, in Theorem 3.1 we show that an operator image TX is homeomorphic to Σ^ω provided $TX \in \mathcal{M}_2$, X is separable, and the operator T is not strictly regular (item (2) of the Main Theorem). Theorem 4.1 states that TX is homeomorphic to Σ^ω provided $TX \in \mathcal{M}_2$, the operator T is compact and the space (X, weak) is separable and $\mathcal{M}_2$-universal (item (11) of the Main Theorem). Theorem 4.1 will be applied in Example 7.7 to show that the strict unregularity of T does not follow from the topological equivalence of TX and Σ^ω. Finally, Theorem 5.1 asserts that TX is homeomorphic to Σ^ω provided $TX \in \mathcal{M}_2$, the Fréchet space X is not normable and the operator T is bounded. This theorem generalizes a number of results dealing with some concrete operator images of nonnormable Fréchet spaces (see [CD], [DDMM], [DiM], [DoM]).

In Main Theorem 7.1 of the final seventh section we summarize all principal results proven in the preceding sections and then apply them to operator images of separable Banach lattices, in particular the classical Banach lattices c_0, l_1, L_1, and $C[0,1]$. For the spaces c_0, l_1, and L_1 we give an exhaustive description of the topological structure of their operator images of class $\mathcal{M}_2$. Some partial results (as well as some counterexamples) are proven for operator images of the space $C[0,1]$. The results of this section should be compared with results of the sixth section devoted to constructing operator images of a given topological type.

The notations used in the paper are standard. By $\overline{A}$ we denote the closure of a subset A of a topological space X. For a subset A of a locally convex space X, $\mathrm{conv}(A)$ and $\overline{\mathrm{conv}}(A)$ denote the convex hull and the closed convex hull of the set A in X; (A, weak) denotes A equipped with the weak topology inherited from (X, weak). For a real number λ and subsets A, B of a linear space L let $A + B = \{a + b : a \in A,\ b \in B\}$ and $\lambda A = \{\lambda a : a \in A\}$. Throught the paper the term "operator" means a "continuous linear operator" between locally convex spaces.

Acknowledgments. The author expresses his sincere thanks to T. Dobrowolski and A. Plichko for valuable discussions on the subject of the paper.

1. Strictly regular operators. In this section we introduce and investigate strictly regular operators—the main tool in our subsequent study of the topology of operator images. Strictly regular operators are operator counterparts of strongly regular Banach spaces. Strongly regular spaces were introduced and studied in detail in [GGMS].

Definition. A convex subset D of a locally convex space X is defined to be *strongly regular* if for every nonempty closed bounded convex subset $C \subset D$ there exists a point $c \in C$ such that for every neighborhood $W \subset C$ of c there exist nonempty relatively weak-open subsets $U_1, \dots, U_n \subset C$ such that $\frac{1}{n}(U_1 + \ldots + U_n) \subset W$.

In case $D = X$ we get the definition of a *strongly regular* locally convex space.

In fact, our definition slightly differs from that introduced in [GGMS, p. 35] but is equivalent to it in the realm of Banach spaces (see Proposition 1.1 below).

There are two ways to adapt the above definition to operators. One way, leading to strongly regular operators, is developed in [GGMS].

Definition. An operator $T : X \to Y$ between locally convex spaces is defined to be *strongly regular* on a convex subset $D \subset X$ if for every nonempty closed bounded convex subset $C \subset D$ there exists a point $c \in C$ such that for every neighborhood $W \subset TC$ of the point $T(c)$ there exist nonempty relatively weak-open sets $U_1, \dots, U_n \subset C$ such that $\frac{1}{n}(U_1 + \ldots + U_n) \subset T^{-1}(W)$.

An operator $T : X \to Y$ is defined to be *strongly regular* provided T is strongly regular on X.

The other way leads to so-called strictly regular operators.

Definition. An operator $T : X \to Y$ between locally convex spaces is defined to be *strictly regular* on a convex subset $D \subset X$ if for every nonempty closed bounded convex subset $C \subset D$ there exists a point $c \in C$ such that for every neighborhood $W \subset C$ of c there exist nonempty open sets $U_1, \dots, U_n \subset TC$ such that $\frac{1}{n}(T^{-1}(U_1) + \ldots + T^{-1}(U_n)) \subset W$.

An operator $T : X \to Y$ is defined to be *strictly regular* provided T is strictly regular on X.

Observe that a convex subset D of a locally convex space X is strongly regular if and only if the identity operator $X \to X$ is strongly regular on D if and only if the "identity" operator $X \to (X, \text{weak})$ is strictly regular on D.

First we verify that our definition of a strongly regular operator is equivalent to the original one [GGMS, p. 40].

1.1. PROPOSITION. *Let $T : X \to Y$ be an operator between locally convex spaces. For a convex subset D of X the following conditions are equivalent*:

(1) *the operator T is strongly regular on D*;

(2) *for every closed bounded convex set $C \subset D$ there exists a point $c \in C$ such that for every neighborhood $W \subset TC$ of $T(c)$ there exist nonempty relatively weak open subsets $U_1, \dots, U_n \subset C$ and numbers $t_1, \dots, t_n \ge 0$ with $\sum_{i=1}^n t_i = 1$ such that $\sum_{i=1}^n t_i U_i \subset T^{-1}(W)$*;

(3) *for every bounded convex set $C \subset D$ and every nonempty open set $W \subset TC$ there exist nonempty relatively weak open subsets $U_1, \dots, U_n \subset C$ and numbers $t_1, \dots, t_n \ge 0$ with $\sum_{i=1}^n t_i = 1$ such that $\sum_{i=1}^n t_i U_i \subset T^{-1}(W)$*;

(4) *for every bounded convex set $C \subset D$ and every neighborhood $W \subset Y$ of the origin there exist nonempty relatively weak open subsets $U_1, \dots, U_n \subset C$ and numbers $t_1, \dots, t_n \ge 0$ with $\sum_{i=1}^n t_i = 1$ such that $\sum_{i=1}^n t_i U_i \subset x + T^{-1}(W)$ for some $x \in X$.*

PROOF. We shall prove the implications (1)⇒(2)⇒(4)⇒(3)⇒(1). In fact, the first two implications are trivial.

(4)⇒(3). Fix a bounded convex set $C \subset D$ and let $\overline{C}$ be the closure of C in D. To prove (3), it suffices to find, for every convex symmetric neighborhood $W_0 \subset Y$ of the origin and every $c \in C$, nonempty relatively weak-open subsets $U_1, \dots, U_n \subset \overline{C}$ and numbers $t_1, \dots, t_n \ge 0$ with $\sum_{i=1}^n t_i = 1$ such that $\sum_{i=1}^n t_i U_i \subset c + T^{-1}(W_0)$. For this let $V_0 = T^{-1}(W_0)$ and consider the set $S \subset \overline{C}$ consisting of all points $s \in \overline{C}$ for which there exist nonempty relatively weak-open sets $U_1, \dots, U_n \subset \overline{C}$ and numbers $t_1, \dots, t_n \ge 0$ with $\sum_{i=1}^n t_i = 1$ such that $\sum_{i=1}^n t_i U_i \subset s + \frac{1}{2}V_0$. It is easy to see that the set S is convex. It is also dense in $\overline{C}$. Indeed, assuming the converse and applying the Hahn–Banach Separation Theorem, we would find a nonempty relatively weak-open convex set $W \subset \overline{C}$ such that $W \cap S = \emptyset$. By condition (4), there are nonempty relatively weak-open sets $U_1, \dots, U_n \subset W$ and numbers $t_1, \dots, t_n \ge 0$ with $\sum_{i=1}^n t_i = 1$ such that $\sum_{i=1}^n t_i U_i \subset x + \frac{1}{4}V_0$ for some $x \in X$. Pick any $s \in \sum_{i=1}^n t_i U_i$. Since the set W is convex, $s \in W \cap \big(x + \frac{1}{4}V_0\big)$. Then $x \in s - \frac{1}{4}V_0$ and $\sum_{i=1}^n t_i U_i \subset x + \frac{1}{4}V_0 \subset s - \frac{1}{4}V_0 + \frac{1}{4}V_0 \subset s + \frac{1}{2}V_0$. Because the sets $U_1, \dots, U_n$, being weak-open in the weak-open set W in $\overline{C}$, are weak-open in $\overline{C}$, we get $s \in S \cap W$, a contradiction which proves the density of S in $\overline{C}$. Now for the point $c \in C$ pick up a point $s \in S$ with $s - c \in \frac{1}{2}V_0$. By the definition of the set S, there are nonempty relatively weak-open sets $U_1, \dots, U_n \subset \overline{C}$ and numbers $t_1, \dots, t_n \ge 0$ with $\sum_{i=1}^n t_i = 1$ such that $\sum_{i=1}^n t_i U_i \subset s + \frac{1}{2}V_0 \subset c + \frac{1}{2}V_0 + \frac{1}{2}V_0 \subset c + V_0 = c + T^{-1}(W_0)$. Hence condition (3) is proven.

(3)⇒(1). Let C be a closed bounded convex set in D, $c \in C$ and $W \subset TC$ a neighborhood of $T(c)$. Let $W_0 \subset Y$ be a convex symmetric neighborhood of the origin such that $(T(c) + W_0) \cap TC \subset W$. By (3), there exist nonempty relatively weak open convex sets $U_1,\dots,U_n\subset C$ and numbers $t_1,\dots,t_n\ge 0$ with $\sum_{i=1}^n t_i=1$ such that $\sum_{i=1}^n t_iU_i\subset c+\frac12 T^{-1}(W_0)$.

Using the boundedness of C, find $\varepsilon>0$ such that $\varepsilon(C-C)\subset\frac12 T^{-1}(W_0)$. For every $i\in\{1,\dots,n\}$ pick a rational number $r_i>0$ such that $|t_i-r_i|<\varepsilon/n$ and $\sum_{i=1}^n r_i=1$. Write $r_i=k_i/k$, where $k_i\in\mathbb N$ and $k=k_1+\ldots+k_n$. Then

$$\sum_{i=1}^n\frac{k_i}{k}U_i\subset\sum_{i=1}^n t_iU_i+\sum_{i=1}^n\Big(t_i-\frac{k_i}{k}\Big)U_i\subset c+\tfrac12 T^{-1}(W_0)+\Big(\sum_{i=1}^n|t_i-r_i|\Big)(C-C)$$
$$\subset c+\tfrac12 T^{-1}(W_0)+\varepsilon(C-C)\subset c+\tfrac12 T^{-1}(W_0)+\tfrac12 T^{-1}(W_0)\subset c+T^{-1}(W_0).$$

Finally, for every $j\in\{1,\dots,k\}$ let $V_j=U_i$, where i is chosen from the condition $k_0+\ldots+k_{i-1}<j\le k_0+\ldots+k_i$, where $k_0=0$. Then $V_1,\dots,V_k$ are nonempty relative weak-open convex sets in C such that

$$\frac1k(V_1+\ldots+V_k)=\sum_{i=1}^n\frac{k_i}{k}U_i\subset c+T^{-1}(W_0)\subset T^{-1}(W).\ \blacksquare$$

We recall that an operator $T:X\to Y$ between Fréchet spaces is called (*weakly*) *compact* provided there exists a neighborhood $U\subset X$ of the origin such that the closure $\overline{TU}$ of TU in Y is (weakly) compact in Y.

The following proposition establishes some elementary relationships between the concepts introduced.

1.2. Proposition. *Suppose $T:X\to Y$, $T':Y\to Z$ are operators between Fréchet spaces.*

(1) *If T or T' is strongly regular, then the composition $T'\circ T$ is strongly regular.*

(2) *If T is weakly compact, then T is strongly regular.*

(3) *If T is strictly regular, then T is injective.*

(4) *If $T'\circ T$ is strictly regular, then T is strictly regular.*

(5) *If T is an isomorphic embedding, and T' is strictly regular, then $T'\circ T$ is strictly regular.*

(6) *If T is simultaneously strictly regular and strongly regular, then X is strongly regular.*

(7) *If T is weakly compact strictly regular, then $T'\circ T$ is strictly regular.*

In an obvious way, Proposition 1.2 follows from the subsequent, a bit more general

1.3. Proposition. *Suppose $T:X\to Y$, $T':Y\to Z$ are operators between Fréchet spaces and D is a closed convex set in X.*

(1) *The operator $T'\circ T$ is strongly regular on D provided T is strongly regular on D or T' is strongly regular on TD.*

(2) *If the closure $\overline{TD}$ of TD in Y is weakly compact, then the set TD is strongly regular and T is strongly regular on D.*

(3) *If T is strictly regular on D, then T is injective on D.*

(4) *If $T'\circ T$ is strictly regular on D, then T is strictly regular on D.*

(5) *If $T|_D:D\to TD$ is a homeomorphism onto a closed subset TD of Y and T' is strictly regular on TD, then $T'\circ T$ is strictly regular.*

(6) *If T is simultaneously strictly regular and strongly regular on D, then D is strongly regular.*

(7) *If T is strictly regular on D and $\overline{TD}$ is weakly compact, then $T' \circ T$ is strictly regular on D.*

PROOF. The first statement trivially follows from Proposition 1.1.

(2) Suppose $\overline{TD}$ is weakly compact. First we show that $\overline{TD}$ is strongly regular. Indeed, let $C \subset \overline{TD}$ be a closed bounded convex subset. Then C is weakly compact as well. By the Krein–Milman Theorem [HHZ, 70] the set C has an extreme point $c \in C$. By the Choquet Lemma [HHZ, 73] the weak topology of C coincides with the original one at c. Consequently, for every neighborhood $W \subset C$ of c there exists a relatively weak-open neighborhood U_1 of c in C such that $U_1 \subset W$. This proves that $\overline{TD}$ is strongly regular. It follows from Proposition 1.1 that every convex subset of $\overline{TD}$ (in particular, TD) is strongly regular. Now observe that $T = \mathrm{Id} \circ T$, where $\mathrm{Id} : Y \to Y$ is the identity operator. Since TD is strongly regular, we see that Id is strongly regular on TD. Then (1) implies that $T = \mathrm{Id} \circ T$ is strongly regular on D.

The statements (3)–(5) follow immediately from the corresponding definitions.

(6) Suppose T is simultaneously strongly regular and strictly regular on D. To show that the set D is strongly regular, fix a closed bounded convex set $C \subset D$. Because the operator T is strictly regular on D, there exists a point $c \in C$ such that for every neighborhood $W \subset C$ of c there exist nonempty open sets $U_1, \dots, U_n \subset TC$ such that $T^{-1}\big(\frac{1}{n}(U_1 + \dots + U_n)\big) \subset W$. Since T is strongly regular on D, by Proposition 1.1(3), there exist nonempty relatively weak open subsets $U_{i,1}, \dots, U_{i,n_i} \subset C$ and numbers $t_{i,1}, \dots, t_{i,n_i} \ge 0$ with $\sum_{j=1}^{n_i} t_{i,j} = 1$ such that $\sum_{j=1}^{n_i} t_{i,j} U_{i,j} \subset T^{-1}(U_i)$. Then

$$\sum_{i=1}^{n} \sum_{j=1}^{n_i} \frac{t_{i,j}}{n} U_{i,j} \subset \sum_{i=1}^{n} \frac{1}{n} T^{-1}(U_i) \subset W$$

and $\sum_{i=1}^{n} \sum_{j=1}^{n_i} t_{i,j}/n = 1$. Hence, D is strongly regular according to Proposition 1.1(2).

(7) Suppose T is strictly regular on D and $\overline{TD}$ is weakly compact. To show that $T' \circ T$ is strictly regular on D, fix a closed bounded convex set $C \subset D$. By the strict regularity of T, there exists a point $c \in C$ such that for every neighborhood $W \subset C$ of c there are nonempty open sets $U_1, \dots, U_n \subset TC$ such that $T^{-1}\big(\frac{1}{n}(U_1 + \dots + U_n)\big) \subset W$.

Let $W \subset C$ be any convex neighborhood of c. By the choice of c, there exist nonempty convex open sets $U_1, \dots, U_n \in \overline{TC}$ such that

$$T^{-1}\left(\frac{1}{n}(U_1 \cap TC + \dots + U_n \cap TC)\right) \subset W.$$

Since the set $\overline{TC}$ is weakly compact, by the Krein–Milman Theorem [HHZ, 70], for every $i \in \{1, \dots, n\}$ there are extreme points $x_{i,1}, \dots, x_{i,n_i} \in \overline{TC}$ of $\overline{TC}$ and rational numbers $t_{i,1}, \dots, t_{i,n_i} > 0$ with $\sum_{j=1}^{n_i} t_{i,j} = 1$ such that $\sum_{j=1}^{n_i} t_{i,j} x_{i,j} \in U_i$. For every $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, n_i\}$ pick up an open convex neighborhood $U_{i,j} \subset \overline{TC}$ of $x_{i,j}$ such that $\sum_{j=1}^{n_i} t_{i,j} U_{i,j} \subset U_i$. According to the Choquet Lemma [HHZ, 73], we may assume each $U_{i,j}$ to be weakly open in $\overline{TC}$. Next, since the set $\overline{TC}$ is weakly compact, the map $T' : (\overline{TC}, \text{weak}) \to (T'(\overline{TC}), \text{weak})$ is a homeomorphism. Then $V_{i,j} = T'(U_{i,j}) \cap T'(TC)$

is a (weakly) open nonempty set in TC. Observe that

$$\sum_{i=1}^{n}\sum_{j=1}^{n_i} t_{i,j}(T'\circ T)^{-1}(V_{i,j})\subset W.$$

Continuing by analogy with the proof of Proposition 1.1, we find that the operator $T'\circ T$ is strictly regular on D. ■

Next, we consider relationships between strictly regular operators and G_δ-embeddings. Generalizing a definition from [BR], we define an injective operator $T:X\to Y$ between Fréchet spaces to be a *G_δ-embedding* if for every closed bounded subset $B\subset X$ the image TB is a G_δ-set in Y. It is well known that for every G_δ-embedding $T:X\to Y$ between separable Banach spaces the inverse map $T^{-1}:TX\to X$ is of the first Baire class, i.e. is a pointwise limit of a sequence of continuous maps (see e.g. [BDP, 1.8 and 1.4]). The following characterization was proven in [SR] (see also [BDP, 1.4]).

1.4. PROPOSITION. *For an injective operator $T:X\to Y$ between separable Fréchet spaces the following conditions are equivalent*:

(1) *TX is an $F_{\sigma\delta}$-set in Y*;

(2) *T^{-1} is of the first Baire class*;

(3) *T^{-1} is Borel of class* 1, *i.e. the image TU of every open set $U\subset X$ is an F_σ-set in TX*;

(4) *X has a base $\mathcal{B}$ of closed convex neighborhoods of the origin such that TB is closed in TX for every $B\in\mathcal{B}$.*

Finally, we recall the definition of a subset-dentable set (see [GGMS, p. 35]). Under a *slice* of a subset C of a locally convex space X we understand a nonempty set of the form $S=\{x\in C: f(x)>\alpha\}$ for some $\alpha\in\mathbb{R}$ and some linear continuous functional $f:X\to\mathbb{R}$. A bounded subset D of a locally convex space X is called *subset-dentable* if for every subset $C\subset D$ and every neighborhood $W\subset X$ of the origin there is a slice S of C such that $S\subset W+x$ for some $x\in X$.

A locally convex space X is said to satisfy the *Radon–Nikodým Property* (briefly RNP) if every bounded subset of X is subset-dentable.

1.5. THEOREM. *Let $T:X\to Y$ be an injective operator between separable Fréchet spaces such that T^{-1} is Borel of class* 1 *and let D be a closed convex bounded set in X.*

(1) *If TD is a G_δ-set in Y, then T is strictly regular on D.*

(2) *If D is subset-dentable and T is strictly regular on D, then TD is a G_δ-set in Y.*

PROOF. (1) Suppose TD is a G_δ-set in Y. To prove that the operator T is strictly regular on D, fix a closed bounded convex set $C\subset D$. Since T^{-1} is Borel of class 1, the image TF of any closed set $F\subset D$ is a G_δ-set in TD. This implies that TC is a G_δ-set in Y and $T^{-1}:TC\to C$ is Borel of class 1. Since TC, being a G_δ-set in the Polish space Y, is Polish, we may apply the classical Baire Theorem [Ke, 24.14] to find a continuity point $c\in TC$ of the map $T^{-1}|_{TC}$. Then for every neighborhood $W\subset C$ of the point $a=T^{-1}(c)$ there is a neighborhood $U_1\subset TD$ of c such that $T^{-1}(U_1)\subset W$. This yields that T is strictly regular on D.

(2) Suppose D is subset-dentable, T is strictly regular on D, but TD is not a G_δ-set in Y. Since both D and Y are Polish spaces and TD is not a G_δ-set in Y, we may apply Theorem I.1 of [GM] to find a closed convex symmetric neighborhood $W \subset X$ of the origin and a countable set $S \subset D$ such that the space TS has no isolated point and $x - y \notin 6W$ for any distinct $x, y \in S$. According to Proposition 1.4 we may additionally assume that TW is closed in TX.

To get a contradiction, we shall construct a subset $A \subset D$ having no small slices. Points of this set will be parametrized by elements of the tree $T_\infty = \bigcup_{n=0}^\infty \mathbb{N}^n$. For a sequence $t = (t_1, \ldots, t_n) \in T_\infty$ and an integer $i \in \mathbb{N}$, let (t, i) denote the sequence $(t_1, \ldots, t_n, i)$. Using Proposition 1.4, fix a countable base $\{W_n\}_{n=1}^\infty$ of closed convex symmetric neighborhoods of the origin in X such that TW_n is closed in TX for every $n \in \mathbb{N}$.

Claim A. *There exist two maps $a : T_\infty \to \overline{\mathrm{conv}}(S)$ and $s : T_\infty \times \mathbb{N} \to \mathbb{N}$ such that for every $t \in T_\infty$ and $n \in \mathbb{N}$,*

(a) $s(t, n+1) > s(t, n)$;

(b) $\dfrac{1}{s(t, n+1) - s(t, n)} \displaystyle\sum_{i=s(t,n)+1}^{s(t,n+1)} a(t, i) \in a(t) + W_n$;

(c) $a(t, i) - a(t, j) \notin 2W$ *if* $s(t, n-1) < j \le s(t, n) < i \le s(t, n+1)$.

First, using Claim A we complete the proof of Theorem 1.5. We claim that the set D is not subset-dentable; namely, the set $A = a(T_\infty) \subset \overline{\mathrm{conv}}(S) \subset D$ has no slice of "diameter" less than W. Assuming the converse, we would find a slice $S = \{a \in A : f(a) > \alpha\}$ of A such that $S \subset x + W$ for some $x \in X$, $\alpha \in \mathbb{R}$, and a continuous linear functional f on X. Fix any $t \in T_\infty$ with $a(t) \in S$. By the property (b), there is $n \in \mathbb{N}$ such that

$$\frac{1}{s(t, n+1) - s(t, n)} \sum_{i=s(t,n)+1}^{s(t,n+1)} a(t, i) \in S, \qquad \frac{1}{s(t, n) - s(t, n-1)} \sum_{j=s(t,n-1)+1}^{s(t,n)} a(t, j) \in S.$$

Consequently, there are j and i such that $s(t, n-1) < j \le s(t, n) < i \le s(t, n+1)$ and $f(a(t, j)) > \alpha$, $f(a(t, i)) > \alpha$. Hence, $a(t, j), a(t, i) \in S$. Since $S \subset x + W$, we get $a(t, j) - a(t, i) \in W - W = 2W$, contrary to (c).

For the proof of Claim A we will need the following statement.

Claim B. *Suppose $S_0 \subset S$ is such that the space TS_0 has no isolated point. Then there exist the following objects*:

(1) *a point* $a_0 \in \overline{\mathrm{conv}}(S_0)$;

(2) *an increasing sequence* $s : \mathbb{N} \to \mathbb{N}$;

(3) *nonempty open convex sets* $U(i) \subset T(\overline{\mathrm{conv}}(S_0))$, $i \in \mathbb{N}$, *such that for every* $n \in \mathbb{N}$ *the following two conditions are satisfied*:

(a) $T^{-1}\left(\dfrac{1}{s(n+1) - s(n)} \displaystyle\sum_{i=s(n)+1}^{s(n+1)} U(i)\right) \subset a_0 + W_n$;

(b) $x - y \notin 3W$ *for every* $x \in T^{-1}(U(i))$, $y \in T^{-1}(U(j))$, *where* $s(n-1) < j \le s(n) < i \le s(n+1)$.

Proof of Claim B. Since the operator T is strictly regular, there exists a point $a_0 \in \overline{\mathrm{conv}}(S_0)$ such that for every $n \in \mathbb{N}$ there exist nonempty open sets $V_1(n), \dots, V_{k_n}(n) \subset T(\overline{\mathrm{conv}}(S_0))$ such that

$$T^{-1}\bigg(\frac{1}{k_n}\sum_{i=1}^{k_n} V_i(n)\bigg) \subset a_0 + W_n.$$

Clearly, $\mathrm{conv}(TS_0)$ is dense in $T(\overline{\mathrm{conv}}(S_0))$. So, for every $n \in \mathbb{N}$ and $i \in \{1, \dots, k\}$ we can select a finite set $F_i(n) \subset TS_0$ such that

$$\frac{1}{|F_i(n)|}\sum_{b\in F_i(n)} b \in V_i(n).$$

Moreover, since the set TS_0 has no isolated point, the sets $F_i(n)$ can be chosen pairwise disjoint for distinct pairs (i, n). For every $b \in F_i(n)$ pick an open convex neighborhood $W(b) \subset T(\overline{\mathrm{conv}}(S_0))$ such that

$$\frac{1}{|F_i(n)|}\sum_{b\in F_i(n)} W(b) \subset V_i(n).$$

Let $s(0) = 0$ and $s(n) = s(n-1) + \sum_{i=1}^{k_n} |F_i(n)|$ for $n \in \mathbb{N}$. Next, let $b : \mathbb{N} \to \bigcup_{n=1}^{\infty}\bigcup_{i=1}^{k_n} F_i(n)$ be a bijective map such that $b(\{s(n)+1, \dots, s(n+1)\}) = \bigcup_{i=1}^{k_n} F_i(n)$ for every $n \in \mathbb{N}$. It is easy to verify that

$$T^{-1}\bigg(\frac{1}{s(n+1)-s(n)}\sum_{i=s(n)+1}^{s(n+1)} W(b(i))\bigg) \subset a_0 + W_n.$$

Using the Hahn–Banach Theorem, for every pair (i, j) with $s(n-1) < i \le s(n) < j \le s(n+1)$ find a continuous linear functional $f_i^j : Y \to \mathbb{R}$ such that

$$f_i^j(b(j) - b(i)) > 1 \ge \sup\{f_i^j(y) : y \in 6 \cdot T(W)\} \tag{1}$$

(such an f_i^j exists because $b(j) - b(i) \notin 6 \cdot T(W)$ and $6 \cdot T(W)$ is closed in TX).

Now for every $n \in \mathbb{N}$ and every $s(n) < i \le s(n+1)$ let

$$U(i) = W(b(i)) \cap \bigg(\bigcap_{s(n-1)<j\le s(n)} \Big\{y \in Y : f_j^i(y) > f_j^i(b(i)) - \frac{1}{4}\Big\}\bigg)$$
$$\cap \bigg(\bigcap_{s(n+1)<j\le s(n+2)} \Big\{y \in Y : f_i^j(y) < f_i^j(b(i)) + \frac{1}{4}\Big\}\bigg).$$

Clearly, condition (a) of Claim B is satisfied. Let us verify condition (b). Fix any $s(n-1) < i \le s(n) < j \le s(n+1)$ and $x \in T^{-1}(U(i))$, $y \in T^{-1}(U(j))$. Then $f_i^j(Ty) > f_i^j(b(j)) - 1/4$ and $f_i^j(Tx) < f_i^j(b(i)) + 1/4$, and thus

$$f_i^j(Ty - Tx) > f_i^j(b(j) - b(i)) - 1/2 > 1/2.$$

Because of (1), we get $x - y \notin 3W$. Therefore, Claim B is proven. ∎

Proof of Claim A. Claim A will be proven by induction on the stages of T_∞. Denote by θ the unique element of $\mathbb{N}^0$ and for $k \ge 0$ let $T_k = \bigcup_{n\le k} \mathbb{N}^n$. Put $C(\theta) = \overline{\mathrm{conv}}(S)$ and $U(\theta) = T(C(\theta))$. Applying Claim B, by induction on k, construct for every $t \in T_k$ the following objects:

(1) a point $a(t) \in C(t) = \mathrm{Cl}_X(T^{-1}(U(t)))$,

(2) a sequence $\{s(t,i)\}_{i=1}^\infty$ such that $s(t,i+1) > s(t,i)$ for all $i \in \mathbb{N}$,

(3) nonempty convex open sets $U(t,i) \subset T(C(t))$, $i \in \mathbb{N}$,

so that for every $n \in \mathbb{N}$ the following two conditions are satisfied:

(a) $T^{-1}\Bigl(\dfrac{1}{s(t,n+1)-s(t,n)} \displaystyle\sum_{i=s(t,n)+1}^{s(t,n+1)} U(t,i)\Bigr) \subset a(t) + W_n$,

(b) $x - y \notin 3W$ for every $x \in T^{-1}(U(t,i))$, $y \in T^{-1}(U(t,j))$, where $s(t,n-1) < j \le s(t,n) < i \le s(t,n+1)$.

Let us verify that the constructed maps $a : T_\infty \to \overline{\mathrm{conv}}(S)$ and $s : T_\infty \times \mathbb{N} \to \mathbb{N}$ satisfy the conditions (a)–(c) of Claim A. Condition (a) trivially follows from condition (2) of the inductive construction. To prove (b), observe that for every $t \in T_\infty$ and $n \in \mathbb{N}$ the set

$$\frac{1}{s(t,n+1)-s(t,n)} \sum_{i=s(t,n)+1}^{s(t,n+1)} T^{-1}(U(t,i)) \subset a(t) + W_n$$

is dense in

$$\frac{1}{s(t,n+1)-s(t,n)} \sum_{i=s(t,n)+1}^{s(t,n+1)} C(t,i).$$

Because the set $a(t) + W_n$ is closed in X, we get

$$\frac{1}{s(t,n+1)-s(t,n)} \sum_{i=s(t,n)+1}^{s(t,n+1)} a(t,i) \in a(t) + W_n,$$

i.e., condition (b) is satisfied.

Finally, to verify (c), observe that the set $2W$ lies in the interior $3W^\circ$ of $3W$ in X. Then for every $s(t,n-1) < j \le s(t,n) < i \le s(t,n+1)$ conditions (1) and (b) of Claim B imply $a(t,i) - a(t,j) \notin 3W^\circ \supset 2W$. Thus condition (c) of Claim A is also satisfied, and so Theorem 1.5 is proven. ∎

1.6. Corollary. *Let $T : X \to Y$ be an injective operator between separable Fréchet spaces such that $T^{-1} : TX \to X$ is of the first Baire class.*

(1) *If T is a G_δ-embedding, then T is strictly regular.*

(2) *If T is strictly regular and every strongly regular closed bounded convex subset in X is subset-dentable, then T is a G_δ-embedding.*

Proof. The first statement follows immediately from the first statement of Theorem 1.5. To prove the second statement, suppose that T is strictly regular and each strongly regular closed bounded convex set in X is subset-dentable, but T is not a G_δ-embedding. Then there is a closed bounded subset $B \subset X$ such that TB is not a G_δ-set in Y. It follows from Proposition 1.4 that TB is a G_δ-set in TX and an $F_{\sigma\delta}$-set in Y. Let G be a G_δ-set in Y such that $G \cap TX = TB$. Then the Borel set TB is not a G_δ-set in G and thus we may apply the classical Hurewicz Theorem [Ke, 21.18] to find a compact subset $K \subset G$ such that $K \cap TB = K \cap TX$ is not a G_δ-set in K. By [Sch, II.4.3] the closed convex hull $\overline{\mathrm{conv}}(K)$ of K in Y is compact. Consider the closed convex bounded set $D = T^{-1}(\overline{\mathrm{conv}}(K)) \cap \overline{\mathrm{conv}}(B)$ in X. Since T is strictly regular on D and the closure $\overline{TD}$ of TD in Y is compact, we may apply Proposition 1.2(6,2) to conclude that D is strongly regular. Then D is subset-dentable and by Theorem 1.5, TD is a G_δ-set in Y. On the other hand, the set $A = T^{-1}(K) \subset B$ is closed and bounded in X and thus $TA = K \cap TX$ is a G_δ-set in $TD \subset TX$. Since TD is a G_δ-set in Y, $K \cap TX$ is a G_δ-set in Y, contrary to the choice of the compactum K. ∎

In light of Corollary 1.6, it is important to detect Fréchet spaces whose every strongly regular closed bounded convex subset is subset-dentable. Besides spaces with RNP, every separable Banach space X complemented in its second dual space X^{**} has the above property (see [GGMS, VII.4]). Besides this fact and Corollary 1.6, in the proof of the subsequent statement we use a result of [Fo] according to which for every G_δ-embedding $T : X \to Y$ of a separable Banach space X into a Fréchet space Y the inverse map T^{-1} is Borel of class 1 (see also [BR, 1.9] and [BDP, 3.1]).

1.7. Corollary. *Suppose X is a Banach subspace of a separable Banach space Z, complemented in its second dual Z^{**}. An operator $T : X \to Y$ into a Fréchet space Y is a G_δ-embedding if and only if T is a strictly regular operator with the inverse T^{-1} of the first Baire class.*

It is well known that any Banach lattice X containing no isomorphic copy of the Banach space c_0 (this is denoted by $X \not\supset c_0$) is complemented in its second dual X^{**} [LT_2, 1.c.4]. In particular, both l_1 and L_1 are complemented in their second duals.

1.8. Theorem. *Suppose $T : X \to Y$ is an injective operator from a separable Banach lattice X into a Fréchet space Y such that the inverse map $T^{-1} : TX \to X$ is Borel of class* 1.

(1) *If $X \not\supset c_0$ and T is not a G_δ-embedding, then T is not strictly regular.*
(2) *If $X \supset c_0$ and T is strongly regular, then T is not strictly regular.*
(3) *If T is strongly regular and T is not a G_δ-embedding, then T is not strictly regular.*

Proof. (1) If $X \not\supset c_0$, then X is complemented in its second dual X^{**} [LT_2, 1.c.4] and by Corollary 1.7, T is not strictly regular provided T is not a G_δ-embedding.

(2) If $X \supset c_0$, then X is not strongly regular since the space c_0 is not strongly regular. Then Proposition 1.2(6) implies that the operator T is not strictly regular provided T is a strongly regular operator.

(3) This statement follows from the preceding ones. ∎

A Fréchet space X is defined to be *nowhere strongly regular* if no infinite-dimensional closed linear subspace X_0 of X is strongly regular. It is well known that the classical Banach space c_0 is not strongly regular. Moreover, since every infinite-dimensional Banach subspace of c_0 contains an isomorphic copy of c_0 (see [LT_1, 2.a.2]), the space c_0 is nowhere strongly regular. The same arguments yield that the Banach space $C(K)$ of real continuous functions on a scattered compact Hausdorff space K is nowhere strongly regular (see [LPP, 11]). We recall that a topological space X is called *scattered* if every subspace of X has an isolated point. In particular, every Polish countable space is scattered.

1.9. PROPOSITION. *Suppose $T : X \to Y$ is a strictly regular operator between Fréchet spaces.*

(1) *If X is not strongly regular, then T is not strongly regular.*

(2) *If X is nowhere strongly regular, then T is an isomorphic embedding.*

PROOF. The first statement follows immediately from Proposition 1.2(6), while the second one follows from Proposition 1.2(2,5,6), the first statement and

1.10. LEMMA. *Let $T : X \to Y$ be an injective operator between Fréchet spaces. If T is not an isomorphic embedding, then there is an infinite-dimensional closed linear subspace X_0 of X such that $T|_{X_0}$ is compact.*

PROOF. Fix an F-norm $|\cdot|$ on Y, that is, a function $|\cdot| : Y \to \mathbb{R}$ satisfying the conditions:

(i) $|y| \geq 0$ for every $y \in Y$ and $|y| = 0$ iff $y = 0$,

(ii) $|x + y| \leq |x| + |y|$,

(iii) $|ty| \leq |y|$ for every $t \in [-1, 1]$,

(iv) the metric $d(x, y) = |x - y|$ generates the topology of Y.

Use the discontinuity of T to select a sequence $\{x_n\}_{n=1}^\infty \subset X$ such that $x_n \not\to 0$ and $\sum_{n=1}^\infty |Tx_n| < \infty$. By [KPR, p. 69], there exists a subsequence $\{e_k\}_{k=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ which forms a so-called strongly regular M-basic sequence. The latter means that for each e_k there is a continuous linear functional e_k^* defined on $E = \operatorname{span}\{e_k : k \in \mathbb{N}\}$ so that $e_k^*(e_l) = \delta_l^k$, and the formula $\|x\| = \sup\{|e_k^*(x)| : k \in \mathbb{N}\}$ defines a continuous norm on E. The fact that $\sum_k |Tx_k| < \infty$ implies that TB is precompact, where B is the unit ball in $(E, \|\cdot\|)$. (If $y \in TB$ then $y = \sum_k t_k Te_k$ for some $|t_k| \leq 1$; the closure of the set consisting of such y's is compact.) Finally, we let X_0 be the closure of E. This completes the proof of Lemma 1.10 and hence of Theorem 1.9. ∎

For the Banach spaces $C(K)$ we have the following result.

1.11. THEOREM. *Let K be a compact metric space and $T : C(K) \to Y$ be a strictly regular operator into a Fréchet space Y.*

(1) *If K is countable, then T is an isomorphic embedding.*

(2) *If K is uncountable, then there exists a Banach subspace $X \subset C(K)$ such that X is isomorphic to $C[0, 1]$ and $T|_X$ is an isomorphic embedding.*

PROOF. Since for a countable K the Banach space $C(K)$ is nowhere strongly regular, the first statement follows from Proposition 1.9(2).

If K is uncountable, then the Banach space $C(K)$ is isomorphic to $C[0, 1]$ (see [Se]). So, we may assume that $K = [0, 1]$. For every $n \in \mathbb{N}$ let X_n be the subspace in $C[0, 1]$ consisting of all functions vanishing on the set $\left[0, \frac{1}{n+1}\right] \cup \left[\frac{1}{n}, 1\right]$. Evidently, each X_n is isomorphic to $C[0, 1]$. We claim that $T|_{X_n}$ is an isomorphic embedding for some n. Assuming the converse, for every $n \in \mathbb{N}$ we would find a function $f_n \in X_n$ with $\|f_n\| = 1$ and $d(T(f_n), 0) < 1/n$, where d is any invariant metric of Y. Evidently, the closed linear span Z of the set $\{f_n : n \in \mathbb{N}\}$ is isomorphic to c_0 and $T|_Z : Z \to Y$ is not an isomorphic embedding. By the preceding case, $T|_Z : Z \to Y$ is not strictly regular, a contradiction. ∎

1.12. Remark. Because every G_δ-embedding between Banach spaces is strictly regular, Theorem 1.11 generalizes Proposition 2.2 of [BR].

In connection with nowhere strongly regular Banach spaces the following question appears naturally:

1.13. Question. *Is there a nowhere strongly regular Banach space containing no isomorphic copy of* c_0*?*

2. Characterization of operator images homeomorphic to Σ^ω. In this section we describe the topological apparatus which will be used in our subsequent study of operator images. The main result here is the characterization and factorization Theorem 2.4.

We recall that a topological space X is defined to be $\mathcal{C}$-*universal*, where $\mathcal{C}$ is a class of topological spaces, if X contains a closed topological copy of each space $C \in \mathcal{C}$. We recall that $\mathcal{M}_2$ and $\mathcal{A}_2$ denote the Borel classes of separable absolute $F_{\sigma\delta}$-spaces and absolute $G_{\delta\sigma}$-spaces; $\mathcal{M}_1$ and $\mathcal{A}_1$ stand for the Borel classes of Polish spaces and metrizable σ-compact spaces, respectively.

The following fact is a partial case of Theorem 3.5 of [BDP].

2.1. Theorem. *Let $T : X \to Y$ be an operator between Fréchet spaces.*

(1) *The space TX is homeomorphic to Σ^ω if and only if $TX \in \mathcal{M}_2$ is an $\mathcal{M}_2$-universal space.*

(2) *The pair (Y, TX) is homeomorphic to $(s \times Q^\omega, \Sigma \times \Sigma^\omega)$ if and only if TX is dense in Y, TX is homeomorphic to Σ^ω, and T is compact.*

Thus for compact operators $T : X \to Y$ the problem of investigating the topology of the pair (Y, TX) reduces to studying the topology of TX. The latter reduces in turn to verifying the $\mathcal{M}_2$-universality of TX. Fortunately, for establishing the $\mathcal{M}_2$-universality of a given space there exist very powerful tools elaborated quite recently (see [BRZ, 3.2.12]). One of them is

2.2. Proposition. *A metrizable space X is $\mathcal{M}_2$-universal if for every space $M \in \mathcal{M}_2$ there is an $\mathcal{M}_1$-map $f : M \to X$.*

We recall that a map $f : M \to X$ between topological spaces is called an $\mathcal{M}_1$-*map* if there exists a Polish space P and a closed embedding $e : M \to X \times P$ such that $f = \mathrm{pr} \circ e$, where $\mathrm{pr} : X \times P \to X$ is the natural projection. The following proposition supplies us with examples of $\mathcal{M}_1$-maps (for a proof see [BRZ, §3.2, Ex. 3,4] or [Ba_2]).

2.3. Proposition. *A continuous map $f : X \to Y$ between separable metrizable spaces is an $\mathcal{M}_1$-map in each of the cases*:

(1) *f is an embedding and $f(X)$ is a G_δ-set in Y*;

(2) *f is bijective and f^{-1} is Borel of class* 1;

(3) *f is a proper map*, *i.e.*, *the preimage $f^{-1}(K)$ is compact for every compact set $K \subset Y$*;

(4) *X and Y are absolute Borel spaces and $f^{-1}(K) \in \mathcal{M}_1$ for every compact set $K \subset Y$.*

An injective operator $T : X \to Y$ between separable Fréchet spaces is called an F_σ-*embedding* if TX is an F_σ-set in Y (see [BR, p. 156]). It is known that every F_σ-embedding between separable Banach spaces is a G_δ-embedding (see [BR, 1.8] or [BDP, 1.8]). For operator images between *separable* Fréchet spaces we have a deeper result characterizing operator images homeomorphic to Σ^ω.

2.4. Theorem. *For an injective operator $T : X \to Y$ between separable Fréchet spaces with $TX \in \mathcal{M}_2$ the following conditions are equivalent*:

(1) *TX is homeomorphic to Σ^ω*;

(2) *there exists a closed subset $B \subset X$ such that the space TB is $\mathcal{M}_2$-universal*;

(3) *there exists a Fréchet space Z and two injective operators $T_1 : X \to Z$ and $T_2 : Z \to Y$ such that $T = T_2 \circ T_1$ and the operator image T_1X is $\mathcal{M}_2$-universal*;

(4) *there exists a separable Fréchet space Z and an F_σ-embedding $E : Z \to X$ such that the operator image $T \circ E(X)$ is homeomorphic to Σ^ω.*

Proof. The implications (1)⇒(2), (1)⇒(3), and (1)⇒(4) are trivial and follow from the $\mathcal{M}_2$-universality of the space Σ^ω.

(2)⇒(1). By Proposition 1.4, $T^{-1} : TX \to X$ is Borel of class 1. Thus TB is a G_δ-set in TX. Since TB is $\mathcal{M}_2$-universal, TX is $\mathcal{M}_2$-universal according to Proposition 2.3(1). Now Proposition 2.1(1) implies TX is homeomorphic to Σ^ω.

(3)⇒(1). Suppose $T_1 : X \to Z$ and $T_2 : Z \to Y$ are injective operators such that $T = T_2 \circ T_1$ and the space T_1X is $\mathcal{M}_2$-universal. Since the Fréchet space X is completemetrizable, T_1X contains a closed topological copy M of the space $\Sigma^\omega \in \mathcal{M}_2$ such that the closure K of M in X is compact (see [BRZ, 3.1.1]). Then T_2M is homeomorphic to M and $B = T_1^{-1}(M)$ is a closed set in X such that the image $TB = T_2M$, being homeomorphic to Σ^ω, is $\mathcal{M}_2$-universal. By the preceding case, TX is homeomorphic to Σ^ω.

(4)⇒(1). Standard Baire category arguments (see [BR, 1.6]) yield the existence of a closed neighborhood U of the origin in Z such that $E(U)$ is closed in X. According to the implication (2)⇒(1) it remains to verify the $\mathcal{M}_2$-universality of the space $T \circ E(U)$.

Since $T \circ E(Z)$, being homeomorphic to Σ^ω, belongs to the Borel class $\mathcal{M}_2$, Proposition 1.4 implies that $(T \circ E)^{-1}$ is Borel of class 1 and Z has a base $\mathcal{B}$ of closed neighborhoods of the origin such that $T \circ E(B)$ is closed in $T \circ E(Z)$ for every $B \in \mathcal{B}$. Fix any $B \in \mathcal{B}$ with $B \subset U$. The closedness of $T \circ E(B)$ in $T \circ E(Z) = \bigcup_{n=1}^\infty n \cdot T \circ E(B)$ and the $\mathcal{M}_2$-universality of $T \circ E(Z)$ imply the $\mathcal{M}_2$-universality of $T \circ E(B)$ (see the proof of Theorem 3.7 in [BDP] or the proof of Theorem 4.1 below). Since $T \circ E(B)$ is a G_δ-set in $T \circ E(U)$ ($(T \circ E)^{-1}$ is Borel of class 1), the space $T \circ E(U)$ is $\mathcal{M}_2$-universal by Proposition 2.3(1). ∎

2.5. Remark. In Example 7.7 we shall show that the F_σ-embedding in the fourth condition of Theorem 2.4 cannot be replaced by a G_δ-embedding.

3. Non-strictly regular operators and operator images homeomorphic to Σ^ω. The main result of this section is

3.1. Theorem. *Suppose $T: X \to Y$ is an injective operator between separable Fréchet spaces such that $TX \in \mathcal{M}_2$. If T is not strictly regular, then the space TX is homeomorphic to Σ^ω.*

In an obvious way this theorem follows from Theorem 2.4(2) and the subsequent a bit more general

3.2. Proposition. *Suppose $T: X \to Y$ is an injective operator between separable Fréchet spaces such that $TX \in \mathcal{M}_2$. If T is not strictly regular on a closed convex set $D \subset X$, then the space TD is $\mathcal{M}_2$-universal.*

Proof. Since $TX \in \mathcal{M}_2$, the separable Fréchet space X has a base $\mathcal{B}$ of closed convex neighborhoods of the origin such that TB is closed in TX for every $B \in \mathcal{B}$ (see Proposition 1.4). Suppose the operator T is not strictly regular on a closed convex set $D \subset X$. Then there exists a closed bounded convex set $C \subset D$ such that for every point $c \in C$ there exists a closed neighborhood $B(c) \in \mathcal{B}$ of the origin such that $\frac{1}{n}(U_1 + \ldots + U_n) \not\subset T(c + B(c))$ for any nonempty open sets $U_1, \ldots, U_n \subset TC$. Clearly, both sets $T(B(c))$ and $T(c + B(c))$ are closed in TX.

Claim A. *For any nonempty open sets $U_1, \ldots, U_n \subset TC$ and rational numbers $t_1, \ldots, t_n \geq 0$ with $\sum_{i=1}^n t_i = 1$ we have $\sum_{i=1}^n t_i U_i \not\subset T(c + B(c))$.*

Proof. Without loss of generality, $t_i > 0$ for all i. Write $t_i = k_i/k$, where $k_i \in \mathbb{N}$, $1 \leq i \leq n$ and $k = k_1 + \ldots + k_n$. Let $F = \{(i,j) \in \mathbb{N}^2 : 1 \leq i \leq n,\ 1 \leq j \leq k_i\}$. Evidently, $|F| = k$. For every $(i,j) \in F$ let $V_{(i,j)} = U_i$. By our hypothesis, $\frac{1}{k}\sum_{(i,j)\in F} V_{(i,j)} \not\subset T(c + B(c))$ and thus there are points $x_{(i,j)} \in V_{(i,j)}$ such that $\frac{1}{k}\sum_{(i,j)\in F} x_{(i,j)} \notin T(c + B(c))$.

Consider the set $\mathcal{F}$ of all functions $f : \{1, \ldots, n\} \to F$ such that $\mathrm{pr}_1 \circ f = \mathrm{id}$, where $\mathrm{pr}_1 : F \to \{1, \ldots, n\}$ is the projection onto the first coordinate. Evidently, $|\mathcal{F}| = k_1 \ldots k_n$. We claim that there is a function $f \in \mathcal{F}$ such that $\sum_{i=1}(k_i/k)x_{f(i)} \notin T(c + B(c))$. Assuming the converse, we would get $\sum_{i=1}^n (k_i/k)x_{f(i)} \in T(c + B(c))$ for every $f \in \mathcal{F}$. Since the set $T(c + B(c))$ is convex, this implies

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{i=1}^n \frac{k_i}{k} x_{f(i)} \in T(c + B(c)).$$

But

$$\begin{aligned}\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{i=1}^n \frac{k_i}{k} x_{f(i)} &= \frac{1}{k_1 \ldots k_n}\sum_{i=1}^n\sum_{j=1}^{k_i}\sum_{\substack{f\in\mathcal{F}\\ f(i)=j}} \frac{k_i}{k} x_{f(i)} \\ &= \frac{1}{k}\sum_{i=1}^n\sum_{j=1}^{k_i} x_{(i,j)}\Bigg(\frac{k_i}{k_1 \ldots k_n}\sum_{\substack{f\in\mathcal{F}\\ f(i)=j}} 1\Bigg) = \frac{1}{k}\sum_{i=1}^n\sum_{j=1}^{k_i} x_{(i,j)} \\ &= \frac{1}{k}\sum_{(i,j)\in F} x_{(i,j)} \notin T(c + B(c)),\end{aligned}$$

a contradiction which shows that $\sum_{i=1}^n t_i x_{f(i)} \notin T(c + B(c))$ for some $f \in \mathcal{F}$. Since $x_{f(i)} \in U_i$ for every i, we get $\sum_{i=1}^n t_i U_i \not\subset T(c + B(c))$. ∎

CLAIM B. *For any nonempty open sets $U_1, \dots, U_n \subset TC$ there are points $x_i \in U_i$, $1 \le i \le n$, such that*

$$\operatorname{conv}\{x_1, \dots, x_n\} \cap T\big(c + \tfrac{1}{2}B(c)\big) = \emptyset.$$

PROOF. Using the boundedness of the set C in X, find $\varepsilon > 0$ such that $\varepsilon(C-C) \subset \frac{1}{2}B(c)$. Pick any $m \in \mathbb{N}$ with $n^2/m < \varepsilon$. Denote by $\Delta = \{(t_1, \dots, t_n) \in [0,1]^n : \sum_{i=1}^n t_i = 1\}$ the $(n-1)$-dimensional simplex and let $\Delta_0 = \{(t_1, \dots, t_n) \in \Delta : mt_i \in \mathbb{Z} \text{ for all } i\} \subset \Delta$. Evidently, Δ_0 is a finite set in Δ such that for every $(t_1, \dots, t_n) \in \Delta$ there exists $(t_1', \dots, t_n') \in \Delta_0$ with $\sum_{i=1}^n |t_i - t_i'| < \varepsilon$. Let $\le$ be any linear ordering of the finite set Δ_0.

By finite induction, for every $t = (t_1, \dots, t_n) \in \Delta_0$ we shall construct nonempty open sets $U_1(t), \dots, U_n(t) \subset TC$ such that

$$(*_t) \qquad U_i(t) \subset U_i \cap \bigcap_{\tau < t} U_i(\tau), \quad 1 \le i \le n, \quad \text{and} \quad T(c + B(c)) \cap \sum_{i=1}^n t_i U_i(t) = \emptyset.$$

Fix any $t = (t_1, \dots, t_n) \in \Delta_0$ and assume that for every $\tau \in \Delta_0$ with $\tau < t$ nonempty open sets $U_i(\tau)$ satisfying the conditions $(*_\tau)$ have been constructed. Let $V_i = U_i \cap \bigcap_{\tau<t} U_i(\tau)$ for $1 \le i \le n$. By Claim A, $\sum_{i=1}^n t_i V_i \not\subset T(c + B(c))$. Thus there are points $x_i \in U_i$, $1 \le i \le n$, such that $\sum_{i=1}^n t_i x_i \notin T(c+B(c))$. Since the set $T(c+B(c))$ is closed in TX, there exist open sets $U_1(t), \dots, U_n(t) \subset TC$ such that $x_i \in U_i(t)$, $1 \le i \le n$, and $T(c + B(c)) \cap \sum_{i=1}^n t_i U_i(t) = \emptyset$. The inductive step is complete.

Finally, for the maximal element $\tau \in \Delta_0$ pick points $x_i \in U_i(\tau)$, $1 \le i \le n$. Since $U_i(\tau) \subset U_i(t) \subset U_i$ for all $t \le \tau$, $1 \le i \le n$, we get $x_i \in U_i$, $1 \le i \le n$, and $\sum_{i=1}^n t_i x_i \notin T(c + B(c))$ for every $t = (t_1, \dots, t_n) \in \Delta_0$.

We claim that $\sum_{i=1}^n t_i x_i \notin T\big(c + \frac{1}{2}B(c)\big)$ for every $t = (t_1, \dots, t_n) \in \Delta$. To show this, take any point $t' = (t_1', \dots, t_n') \in \Delta_0$ with $\sum_{i=1}^n |t_i - t_i'| < \varepsilon$. Observe that

$$\begin{aligned}\sum_{i=1}^n (t_i' - t_i)x_i &\in \sum_{i=1}^n (t_i' - t_i)TC \\ &\subset \Big(\sum_{i=1}^n |t_i' - t_i|\Big)T(C-C) \subset \varepsilon T(C-C) \subset T\Big(\frac{1}{2}B(c)\Big).\end{aligned}$$

Because $\sum_{i=1}^n t_i' x_i \notin T(c+B(c))$ the above relation implies $\sum_{i=1}^n t_i x_i \notin T\big(c + \frac{1}{2}B(c)\big)$. ∎

CLAIM C. *$TC \cap T\big(c + \frac{1}{2}B(c)\big)$ is a Z-set in TC.*

PROOF. Clearly, the set $TC \cap T\big(c + \frac{1}{2}B(c)\big)$ is closed in TC. To prove that it is a Z-set in TC, fix a continuous map $f : I^k \to TC$ of a finite-dimensional cube and a convex symmetric neighborhood $U \subset Y$ of the origin. We have to construct a continuous map $f' : I^k \to TC$ such that $f'(I^k) \cap T\big(c + \frac{1}{2}B(c)\big) = \emptyset$ and $f(t) - f'(t) \in U$ for every $t \in I^k$.

The uniform continuity of f implies the existence of a triangulation $\mathcal{N}$ of the cube I^k so fine that $f(\sigma) - f(\sigma) \subset \frac{1}{6}U$ for every simplex σ of $\mathcal{N}$. Let $\mathcal{N}^{(0)}$ be the set of vertices of the triangulation $\mathcal{N}$. By Claim B, there exists a map $f_0' : \mathcal{N}^{(0)} \to TC$ such that $\operatorname{conv}(f_0'(\mathcal{N}^{(0)})) \cap T\big(c + \frac{1}{2}B(c)\big) = \emptyset$ and $f_0'(v) \in f(v) + \frac{1}{6}U$ for every $v \in \mathcal{N}^{(0)}$. Let $f' : I^k \to TC$ be the simplicial map extending f_0'. Then $f'(I^k) \subset \operatorname{conv}(f_0'(\mathcal{N}^{(0)}))$ and hence $f'(I^k) \cap T(c + \frac{1}{2}B(c)) = \emptyset$.

It remains to show that $f(t)-f'(t)\in U$ for every $t\in I^k$. Fix any $t\in I^k$ and pick up a simplex σ of the triangulation $\mathcal{N}$ such that $t\in\sigma$. Let $\sigma^{(0)}$ be the set of vertices of σ. Fix any vertex $v_0\in\sigma^{(0)}$. Then for every $v\in\sigma^{(0)}$,

$$\begin{aligned} f'(v)-f'(v_0) &= (f_0'(v)-f(v))+(f(v)-f(v_0))+(f(v_0)-f_0'(v_0)) \\ &\in \tfrac{1}{6}U+\tfrac{1}{6}U+\tfrac{1}{6}U=\tfrac{1}{2}U. \end{aligned}$$

Consequently, $f'(\sigma^{(0)})\subset f'(v_0)+\frac{1}{2}U$ and thus $f'(t)\in\operatorname{conv}(f'(\sigma^{(0)}))\subset f'(v_0)+\frac{1}{2}U$. Then

$$\begin{aligned} f'(t)-f(t) &= (f'(t)-f'(v_0))+(f'(v_0)-f(v_0))+(f(v_0)-f(t)) \\ &\in \tfrac{1}{2}U+\tfrac{1}{6}U+\tfrac{1}{6}U\subset U. \ \blacksquare \end{aligned}$$

Claim D. *TC is a σZ-space.*

Proof. The space C, being metrizable and separable, is Lindelöf. Hence the cover $\{c+\frac{1}{2}B(c):c\in C\}$ of C admits a countable subcover $\{c_i+\frac{1}{2}B(c_i)\}_{i=1}^\infty$. Then TC is expressed as the countable union

$$TC=\bigcup_{i=1}^\infty TC\cap T\Big(c_i+\frac{1}{2}B(c_i)\Big)$$

of Z-sets (see Claim C). Thus TC is a σZ-space. $\blacksquare$

Claim E. *For every metrizable compact space K and every σ-compact set $A\subset K$ there exists a continuous map $f:K\to\overline{TC}$ such that $f(A)\subset TC$ and $f(K\setminus A)\subset Y\setminus TX$.*

Proof. By Proposition 1.4, T^{-1} is Borel of class 1. Consequently, TC is a G_δ-set in TX. Let G be a G_δ-set in $\overline{TC}$ such that $G\cap TX=TC$. Using Claim D and repeating the proof of Lemma 5.4 from [DMM] (see also Lemma 8.10 of [CDM]), one may construct a continuous map $f:K\to G$ such that $f^{-1}(TC)=A$. Clearly, $f(K)\subset G\subset\overline{TC}$ and $f(K\setminus A)\subset G\setminus TC\subset Y\setminus TX$.

Claim F. *The space TD is $\mathcal{M}_2$-universal.*

Proof. According to Propositions 2.2 and 2.3(3) it suffices to construct, for every space $M\in\mathcal{M}_2$, a proper map $f:M\to TD$. Embed M into a metrizable compactum K. By [Ku, §30.V], the $G_{\delta\sigma}$-set $K\setminus M$ can be written as a countable union $K\setminus M=\bigcup_{n=1}^\infty G_n$ of pairwise disjoint G_δ-sets $G_n\subset K$. By Claim E, for every $n\in\mathbb{N}$ there exists a continuous map $f_n:K\to\overline{TC}$ such that $f_n(K\setminus G_n)\subset TC$ and $f_n(G_n)\subset Y\setminus TX$. Since the set $\overline{TC}$ is bounded in Y, the map $f:K\to\overline{TC}$ defined by

$$f(x)=\sum_{n=1}^\infty 2^{-n}f_n(x)\qquad\text{for } x\in K$$

is well defined and continuous. We claim that $f^{-1}(TX)=M$. Indeed, if $x\in M$, then $x\notin G_n$ for all n and thus $f_n(x)\in TC$ for all n. Since the set C is closed, bounded and convex in the Fréchet space X, the series $\sum_{n=1}^\infty 2^{-n}T^{-1}(f_n(x))$ converges to a point of C. By the continuity of T, we get $T(x)\in TC$.

If $x \notin M$, then there is a unique $n_0 \in \mathbb{N}$ such that $x \in G_{n_0}$ and $x \notin G_n$ for $n \neq n_0$. By the choice of the map f_{n_0}, $f_{n_0}(x) \notin TX$. Repeating the preceding arguments, we get

$$\sum_{n\neq n_0} \frac{2^{-n}}{1-2^{-n_0}} f_n(x) \in TC.$$

Then

$$f(x) = 2^{-n_0} f_{n_0}(x) + (1-2^{-n_0}) \sum_{n\neq n_0} \frac{2^{-n}}{1-2^{-n_0}} f_n(x) \notin TX.$$

Since K is compact and $f^{-1}(TX) = f^{-1}(TC) = M$, the restriction $f|_M : M \to TD$ is a proper map. This completes the proof of Claim F and hence of Proposition 3.2. ■

4. Operator images of Fréchet spaces whose weak topology is $\mathcal{M}_2$-universal. By Proposition 1.2(2,6) and Theorem 3.1, if $T : X \to Y$ is a weakly compact injective operator between separable Fréchet spaces, then TX is homeomorphic to Σ^ω provided X is not strongly regular and $TX \in \mathcal{M}_2$.

It was proven in [Ba$_1$, 1.26] that for every non-strongly regular Banach space X with separable dual X^*, the space (X, weak) is $\mathcal{M}_2$-universal (here (X, weak) is X endowed with the weak topology). On the other hand, there exists a strongly regular Banach space X (denoted by S_*T_∞ in [GMS$_2$]) such that the dual X^* of X is separable but the space (X, weak) is $\mathcal{M}_2$-universal (see [Ba$_1$, 2.1]).

4.1. Theorem. *Suppose $T : X \to Y$ is an injective compact operator between separable Fréchet spaces such that $TX \in \mathcal{M}_2$. If (X, weak) is $\mathcal{M}_2$-universal, then the operator image TX is homeomorphic to Σ^ω.*

For the proof of this theorem we will need

4.2. Lemma. *Suppose $T : X \to Y$ is an injective operator between separable Fréchet spaces such that $TX \in \mathcal{M}_2$. If $D \subset X$ is a closed subset of X such that (D, weak) is $\mathcal{M}_2$-universal and the closure $\overline{TD}$ of TD in Y is compact, then TD is $\mathcal{M}_2$-universal.*

Proof. Without loss of generality, (D, weak) is homeomorphic to Σ^ω and thus is metrizable, separable and $\mathcal{M}_2$-universal. Notice that T induces a continuous map $T_{\mathrm{w}} : (D, \mathrm{weak}) \to (TD, \mathrm{weak})$ such that $i_{TD} \circ T|_D = T_{\mathrm{w}} \circ i_D$, where $i_D : D \to (D, \mathrm{weak})$ and $i_{TD} : TD \to (TD, \mathrm{weak})$ are the "identity" maps. Since $\overline{TD}$ is compact, the weak topology on TD coincides with the original one. Hence i_{TD} is a homeomorphism and the map $f = i_{TD}^{-1} \circ T_{\mathrm{w}} : (D, \mathrm{weak}) \to TD$ is bijective and continuous. We claim that f is an $\mathcal{M}_1$-map. Indeed, observe that $f^{-1} = i_D \circ T^{-1}|_{TD}$. Since T^{-1} is Borel of class 1 (see Proposition 1.4), so is $T^{-1}|_{TD}$, and by [Ku, §27.II.1], $f^{-1} = i_D \circ T^{-1}|_{TD}$ is Borel of class 1 as well. By Proposition 2.3(2), f is an $\mathcal{M}_1$-map. Since (D, weak) is $\mathcal{M}_2$-universal, the space $T(D) = f(D)$ is $\mathcal{M}_2$-universal according to Proposition 2.2. ■

Proof of Theorem 4.1. Suppose the space (X, weak) is $\mathcal{M}_2$-universal and the operator T is compact. Then there exists a closed convex neighborhood $D \subset X$ of the origin such that the closure $\overline{TD}$ of TD in Y is compact. Theorem 4.1 will follow from Lemma 4.2 and Theorem 2.4(2) as soon as we show that the space (D, weak) is $\mathcal{M}_2$-universal. Fix any space $M \in \mathcal{M}_2$. Embed M into a metrizable compactum K and consider the subspace

$$\Omega = (K \times (0,1] \cup M \times \{0\})^\omega$$

in $(K \times [0,1])^\omega$. It is easy to see that Ω is a Baire space (it contains a dense absolute G_δ-set $(K \times (0,1])^\omega$) and for every nonempty open set $U \subset \Omega$ there is a closed embedding $e : M \to \Omega$ with $e(M) \subset U$. Since $\Omega \in \mathcal{M}_2$, there exists a closed embedding $\Omega \subset (X, \text{weak})$.

Since $X = \bigcup_{n=1}^\infty n \cdot D$, by the Baire Theorem, there is a nonempty open subset $U \subset \Omega$ such that $U \subset n \cdot D$ for some n. By the property of Ω, there is a closed embedding $e : M \to \Omega$ such that $e(M) \subset U \subset n\,D$. Thus (nD, weak) contains a closed topological copy of M. Since (D, weak) is homeomorphic to (nD, weak) we see that the space (D, weak) is $\mathcal{M}_2$-universal. ■

5. Operator images of nonnormable Fréchet spaces. Recall that a Fréchet space X is called *normable* if the topology of X is generated by a norm, or equivalently, if X contains a bounded neighborhood of the origin. An operator between Fréchet spaces is called *bounded* if there exists a neighborhood U of $0 \in X$ such that TU is bounded in Y (see [Vo]). Evidently, every bounded operator is continuous. The converse is true provided X or Y is a Banach space (see also [Vo]).

5.1. Theorem. *Let $T : X \to Y$ be an injective bounded operator between Fréchet spaces. If X is not normable and $TX \in \mathcal{M}_2$ then TX is homeomorphic to Σ^ω.*

Proof. Since T is bounded, there exists an open convex symmetric neighborhood U_0 of $0 \in X$ such that TU_0 is bounded in Y. Pick a basis $\{U_n\}_{n\in\mathbb{N}}$ of open convex symmetric neighborhoods of $0 \in X$ such that

$$U_0 \supset U_1 \supset U_2 \supset \ldots \tag{1}$$

For every $n \ge 0$, let $\|\cdot\|_n$ be the gauge Minkowski functional of U_n (that is, $\|x\|_n = \inf\{t \ge 0 : x \in tU_n\},\ x \in X$). Because of (1), we have

$$\|\cdot\|_0 \le \|\cdot\|_1 \le \|\cdot\|_2 \le \ldots$$

Since T is injective and TU_0 is bounded, $\|\cdot\|_0$ and, consequently, all $\|\cdot\|_n$'s are continuous norms on X. For $n \ge 0$ let X_n be the completion of the normed space $(X, \|\cdot\|_n)$, and $T_n : X_n \to Y$ be the extension of the bounded (and, consequently, continuous) operator $T : (X, \|\cdot\|_n) \to Y$. For $m \ge n \ge 0$, let $i_n : X \to X_n$ and $i_n^m : X_m \to X_n$ be the extensions of the "identity" operators $X \to (X, \|\cdot\|_n)$ and $(X, \|\cdot\|_m) \to (X, \|\cdot\|_n)$, respectively. Notice that all those operators have dense images. The space X can be identified with the limit of the projective sequence

$$\ldots \longrightarrow X_2 \xrightarrow{i_1^2} X_1 \xrightarrow{i_0^1} X_0 \xrightarrow{T_0} Y. \tag{2}$$

Let us make two remarks.

(I) Without loss of generality, the operators i_n^m, T_n are injective. Otherwise, in place of (2), consider the projective sequence

$$\ldots \longrightarrow \widetilde{X}_2 \xrightarrow{\widetilde{i}_1^2} \widetilde{X}_1 \xrightarrow{\widetilde{i}_0^1} \widetilde{X}_0 \xrightarrow{\widetilde{T}_0} Y, \tag{3}$$

where for $m \geq n \geq 0$, $\widetilde{X}_n = X_n/\mathrm{Ker}\, T_n$, $\pi_n : X_n \to \widetilde{X}_n$ is the quotient map, $\widetilde{T}_n : \widetilde{X}_n \to Y$ is the (injective) operator determined by the condition $T_n = \widetilde{T}_n \circ \pi_n$, and $\widetilde{i}_n^m : \widetilde{X}_m \to \widetilde{X}_n$ is a (unique) operator such that $\widetilde{i}_n^m \circ \pi_m = \pi_n \circ i_n^m$. Since $\widetilde{T}_m = \widetilde{T}_n \circ \widetilde{i}_n^m$, the injectivity of $\widetilde{T}_m$ implies the injectivity of $\widetilde{i}_n^m$. Now observe that the quotient operators π_n determine a morphism (π_n) of the sequences (2) and (3), and consequently a morphism $\pi : \varprojlim X_n \to \varprojlim \widetilde{X}_n$ of their limits. One can show that the map π is continuous, injective, open, and has dense image. Since $X = \varprojlim X_n$ and $\varprojlim \widetilde{X}_n$ are Fréchet spaces, by the Open Mapping Principle the operator π is an isomorphism.

(II) Since X is not normable, without loss of generality, the norms $\|\cdot\|_n$ are pairwise inequivalent. This means that for each $n \geq 0$ the operator i_n^{n+1} is not open.

According to 2.1, to show $TX \cong \Sigma^\omega$, it is enough to verify that TX is $\mathcal{M}_2$-universal. By 2.2 and 2.3(3), $\mathcal{M}_2$-universality of TX will follow if for every $C \in \mathcal{M}_2$ we construct a proper map $p : C \to TX$. So, fix a $C \in \mathcal{M}_2$, and let K be a compactification of C. Write $C = \bigcap_{n\in\mathbb{N}} C_n$, where $C_1 \supset C_2 \supset C_3 \supset \ldots$ are σ-compact subsets of K.

For $n \geq 0$, let B_n be the closed unit ball of the Banach space X_n. Since the dense operator $i_n^{n+1} : X_{n+1} \to X_n$ is not open, $i_n^{n+1}(X_{n+1})$ lies in a σZ-set in X_n (cf. [BDP, 3.6]). Consequently, $i_n^{n+1}(X_{n+1}) \cap B_n$ is contained in a σZ-set of B_n. Since $B_n \cap i_n(X)$ is a dense convex set in $B_n \cap i_n^{n+1}(X_{n+1}) \subset B_n$, by Lemma 8.10 of [CDM], there is a continuous map $f_n : K \to B_n$ such that $f_n(C_n) \subset i_n(X)$ and $f_n(K \setminus C_n) \subset B_n \setminus i_n^{n+1}(X_{n+1})$.

Now define a map $f : K \to Y$ by

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{2^n} T_n \circ f_n(z), \quad z \in K.$$

Notice that because of (1) we have $i_n^m(B_m) \subset B_n$ for $m \leq n$. This implies that for every m the series

$$s_m(z) = \sum_{n=m}^{\infty} \frac{1}{2^n} i_m^n \circ f_n(z), \quad z \in K,$$

converges uniformly and its sum s_m is a continuous function of K into B_m. In particular, s_1 is a continuous function of K into B_1. Since $f = T_1 \circ s_1$, we see that f is a well defined continuous function.

Let us show that $f^{-1}(TX) = C$. Observe first that for every m we can write

$$f(z) = \sum_{n=1}^{m-1} \frac{1}{2^n} T_n \circ f_n(z) + T_m \circ s_m(z).$$

If $z \in C$ then $f_n(z) \in i_n(X)$ and consequently $2^{-n} T_n \circ f_n(z) \in T_n \circ i_n(X) = TX$ for every $n < m$. Since $T_m \circ s_m(z) \in T_m(B_m)$, we obtain $f(z) \in TX + T_m(B_m) \subset T_m X_m$ for every m and every $z \in C$, and hence $f(C) \subset \bigcap_{m=1}^\infty T_m X_m = TX$.

Now suppose $z \in K \setminus C$. Let $m = \min\{n : z \notin C_n\}$. Then $f_m(z) \in B_m \setminus i_m^{m+1}(X_{m+1})$ and $f_n(z) \in i_n(X)$ for $n < m$. Applying the operators T_n, we get $T_m \circ f_m(z) \in T_m X_m \setminus T_{m+1} X_{m+1}$ and $T_n \circ f_n(z) \in TX$ for $n < m$. Write

$$f(z) = \sum_{n=1}^{m-1} \frac{1}{2^n} T_n \circ f_n(z) + \frac{1}{2^m} T_m \circ f_m(z) + T_{m+1} \circ s_{m+1}(z).$$

Since $\sum_{n=1}^{m-1} 2^{-n} T_n \circ f_n(z) \in TX \subset T_{m+1}X_{m+1}$, $T_{m+1} \circ s_{m+1}(z) \in T_{m+1}X_{m+1}$, and $T_m \circ f_m(z) \notin T_{m+1}X_{m+1}$, we get $f(z) \notin T_{m+1}X_{m+1}$, and consequently, $f(z) \notin TX$. Therefore $f(K \setminus C) \subset Y \setminus TX$. This together with $f(C) \subset TX$ just yields $f^{-1}(TX) = C$.

Letting $p = f|_C$ we obtain a proper map of C into TX as required. ∎

6. Constructing operator images of a given topological type. We start from the following easy observation:

6.1. Proposition. *Suppose X, Y are separable Banach spaces and $T_0 : X_0 \to Y$ is an injective operator from a Banach subspace $X_0 \subset X$. Then there is a separable Banach space $Z \supset Y$ and an injective operator $T : X \to Z$ such that $T|_{X_0} = T_0$ and TX is homeomorphic to $T_0(X_0) \times (X/X_0)$.*

Proof. According to [HHZ, 92], Y can be identified with a Banach subspace of l_∞. The Hahn–Banach Theorem implies that the operator $T_0 : X_0 \to Y \subset l_\infty$ extends to a continuous linear operator $T_1 : X \to l_\infty$. Let $\overline{T_1 X}$ denote the closure of TX in l_∞. Clearly, $\overline{T_1 X}$ is a separable Banach space. Denote by $\pi : X \to X/X_0$ the quotient projection. Let $Z = \overline{T_1 X} \oplus X/X_0$ and $T : X \to Z$ be the operator defined by $T(x) = T_1(x) + \pi(x)$ for $x \in X$. Clearly, T is an injective operator with $T|X_0 = T_0$. We claim that the operator image TX is homeomorphic to $T_0(X_0) \times (X/X_0)$.

It follows from the Michael Selection Theorem [BP] that there exists a continuous map $s : X/X_0 \to X$ such that $\pi \circ s = \mathrm{id}$ and $s(0) = 0$. Observe that $x - s \circ \pi(x) \in X_0$ for every $x \in X$, so we may define a continuous map $h : X \to T_0(X_0) \times X/X_0$ by letting $h(x) = (T_0(x - s \circ \pi(x)), \pi(x))$. It is easy to verify that the map $H = h \circ T^{-1} : TX \to T_0(X_0) \times X/X_0$ is a homeomorphism. ∎

Next, for any of the spaces Σ, $\Sigma \times s$, Σ^ω we shall construct an injective compact operator $T : l_1 \to l_2$ such that Tl_1 is homeomorphic to the chosen space.

Recall that for a metric compactum K we denote by $C(K)$ the Banach space of real continuous functions on K. If K is a compact subset of a Banach space Y, then there is a linear operator $R : Y^* \to C(K)$ acting as $R(f) = f|_K$. Denote by $R^* : C^*(K) \to Y^{**}$ the dual operator. The compactum K will be called *functionally independent* if the map $R^* : C^*(K) \to Y^{**}$ is injective.

First, we establish that the Hilbert space l_2 contains a functionally independent copy of each metrizable compactum. Indeed, if K is a metrizable compactum, then the Banach space $C(K)$ is separable, so we can select a countable dense subset $\{f_n\}_{n=1}^\infty$ of the unit ball of $C(K)$. Define an embedding $e : K \to l_2$ by letting $e(x) = (2^{-n} f_n(x))_{n=1}^\infty$. One can easily verify that this map is continuous and injective, moreover, the operator $R^* : C^*(K) \to (l_2)^{**} = l_2$ acts by the formula $R^*(\mu) = (2^{-n}\mu(f_n))_{n=1}^\infty$, $\mu \in C^*(K)$, and is thus injective. Therefore, $e(K)$ is a functionally independent compactum in l_2.

6.2. Proposition. *Suppose K is a functionally independent compactum in l_2, $\delta : \mathbb{N} \to K$ is an injective map, and $T : l_1 \to K$ is the operator defined by $T(t) = \sum_{n=1}^\infty t_n \delta(n)$, where $t = (t_n)_{n=1}^\infty \in l_1$. Then*

(1) *T is well defined, injective, and compact;*
(2) *$T(l_1)$ is homeomorphic to Σ provided the set $\delta(\mathbb{N})$ is compact;*

(3) $T(l_1)$ *is homeomorphic to* $\Sigma \times s$ *provided* $\delta(\mathbb{N})$ *is a noncompact* G_δ*-set in* K;

(4) $T(l_1)$ *is homeomorphic to* Σ^ω *provided* $\delta(\mathbb{N})$ *is not a* G_δ*-set in* K.

PROOF. Since K is compact, the operator T is well defined and compact. Its injectivity follows from the functional independence of K.

Observe that T factors through $C^*(K)$. Namely, $T = R^* \circ E$, where the isometric embedding $E : l_1 \to C^*(K)$ is defined by

$$E(t)(f) = \sum_{n=1}^{\infty} t_n f(\delta(n)) \quad \text{for } t = (t_n)_{n=1}^{\infty} \in l_1 \text{ and } f \in C(K).$$

Denote by $M(K)$ the closed unit ball $B(C^*(K))$ of the dual Banach space $C^*(K)$ endowed with the $*$-weak topology. Since the space $M(K)$ is compact and $R^* : M(K) \to l_2$ is continuous and injective, the map $R^*|_{M(K)} : M(K) \to l_2$ is an embedding.

Now we can show that the inverse operator T^{-1} is Borel of class 1. By Proposition 1.4 this will follow as soon as we show that the image $T(B(l_1))$ of the closed unit ball of l_1 is closed in $T(l_1)$. But this is easily deduced from the compactness of $R^*(M(K))$ and the equality $T(B(l_1)) = R^*(M(K) \cap E(l_1)) = R^*(M(K)) \cap T(l_1)$.

1. Suppose the set $\delta(\mathbb{N}) \subset K$ is compact. It is well known that elements of $C^*(K)$ can be identified with countably additive signed measures on the σ-algebra of Borel subsets of K. For each signed measure $\mu \in C^*(K)$ define a measure $|\mu| \in C^*(K)$ (its variation) by letting $|\mu|(f) = \sup\{\mu(g) : 0 \le g \le f\}$ for a nonnegative $f \in C(K)$ and $|\mu|(f) = |\mu|(f_+) - |\mu|(f_-)$, where $f_+ = \max\{f, 0\}$, $f_- = \max\{-f, 0\}$, for an arbitrary $f \in C(K)$.

We claim that the set $T(B(l_1))$ is compact. Indeed, because $R^*|_{M(K)} : M(K) \to l_2$ is an embedding, $T(B(l_1))$ is homeomorphic to the subset

$$\{\mu \in M(K) : |\mu|(K \setminus \delta(\mathbb{N})) = 0\} \subset M(K). \tag{1}$$

By [Va, Theorem II.3], for every closed set $F \subset K$ and every $\alpha \in \mathbb{R}$, the set $\{\mu \in M(K) : |\mu|(K \setminus F) \le \alpha\}$ is closed in $M(K)$. This implies that $T(B(l_1))$, being homeomorphic to a closed subset of the compactum $M(K)$, is compact. Then $T(l_1) = \bigcup_{n=1}^{\infty} nT(B(l_1))$ is σ-compact and, consequently, homeomorphic to Σ according to Theorem 3.1 of [BDP].

2. Suppose $\delta(\mathbb{N})$ is a noncompact G_δ-set in K. We show that $T(B(l_1))$ is a G_δ-set in l_2. As in the preceding case, it suffices to verify that

$$G = \{\mu \in M(K) : |\mu|(K \setminus \delta(\mathbb{N})) = 0\}$$

is a G_δ-set in $M(K)$. Write $K \setminus \delta(\mathbb{N}) = \bigcup_{n=1}^{\infty} F_n$, where each set $F_n \subset F_{n+1}$ is closed in K. Since

$$G = \bigcap_{n=1}^{\infty} \left\{\mu \in M(K) : |\mu|(F_n) \le \frac{1}{n}\right\},$$

it suffices to show that each set $G_n = \{\mu \in M(K) : |\mu|(F_n) \le 1/n\}$ is a G_δ-set in $M(K)$, or equivalently, that the complement $M(K) \setminus G_n = \{\mu \in M(K) : |\mu|(F_n) > 1/n\}$ is an F_σ-set in $M(K)$. Observe that for every $r \in \mathbb{R}$ the set $\{\mu \in M(K) : \|\mu\| > r\}$ is open in $M(K)$. Note also that $|\mu|(F_n) = |\mu|(K) - |\mu|(K \setminus F_n)) = \|\mu\| - |\mu|(K \setminus F_n)$. Then

$$\begin{aligned}\{\mu \in M(K) : |\mu|(F_n) > 1/n\} \\ &= \{\mu \in M(K) : \|\mu\| - |\mu|(K \setminus F_n) > 1/n\} \\ &= \bigcup_{\substack{r,r' \in \mathbb{Q} \\ r-r'>1/n}} \{\mu \in M(K) : \|\mu\| > r\} \cap \{\mu \in M(K) : |\mu|(K \setminus F_n) \le r'\}.\end{aligned}$$

The sets $\{\mu \in M(K) : \|\mu\| > r\}$ are open and $\{\mu \in M(K) : |\mu|(K \setminus F_n) \le r'\}$ are closed in $M(K)$ (see [Va, Theorem II.3]). Consequently, $\{\mu \in M(K) : |\mu|(F_n) > 1/n\}$ is an F_σ-set, which implies that $T(B(l_1))$ is a G_δ-set in $T(l_1)$. Then $T(l_1) = \bigcup_{n=1}^\infty nT(B(l_1))$ is a countable union of completely metrizable closed subsets in $T(l_1)$. This means that $T(l_1) \in \mathcal{M}_1^2$.

Let us verify that $T(l_1)$ is not σ-compact. Observe that $T(B(l_1))$ is a noncompact convex G_δ-set in l_2, closed in the linear space $T(l_1)$. By [BRZ, 5.2.6] and [DT], the space $T(B(l_1))$ is homeomorphic to s, the pseudo-interior of the Hilbert cube. Since s is not σ-compact and $T(B(l_1))$ is closed in $T(l_1)$, we see that $T(l_1)$ is not σ-compact either. Therefore $T(l_1) \in \mathcal{M}_1^2 \setminus \mathcal{A}_1$ and by Theorem 3.1 of [BDP], $T(l_1)$ is homeomorphic to $\Sigma \times s$.

3. Suppose finally that $\delta(\mathbb{N})$ is not a G_δ-set in K. Since $\delta(\mathbb{N}) = T(E)$, where $E = \{e_n : n \in \mathbb{N}\}$ is the standard basis of l_1, the operator T is not a G_δ-embedding. Since l_1 has RNP [Bo, 4.1.3] and the map T^{-1} is Borel of class 1, Corollary 1.6 implies that the operator T is not strictly regular. By Theorem 3.1, the space $T(l_1)$ is homeomorphic to Σ^ω. ∎

6.3. Remark. It follows from the proof of the last statement of Proposition 6.2 that there exists an injective compact (and thus strongly regular) operator $T : l_1 \to l_2$ from the strongly regular space l_1 such that T is not strictly regular (compare to Proposition 1.2(6)).

6.4. Proposition. (1) *There exists an injective compact operator* $T_1 : L_1 \to l_2$ *such that* $T(L_1)$ *is homeomorphic to* Σ^ω.

(2) *There exists an injective operator* $T_2 : L_1 \to C[0,1]$ *such that* $T(L_1)$ *is homeomorphic to* $\Sigma \times s$.

Proof. (1) By Theorem 2.6 of [BDP], there exists a compact injective operator $T : L_1 \to l_2$ such that $T_1(L_1) \in \mathcal{M}_2$. Since the operator T is compact and the space L_1 is not strongly regular [GGMS], Proposition 1.2(6) implies that the operator T is not strictly regular. Then by Theorem 3.1, the space $T(L_1)$ is homeomorphic to Σ^ω.

(2) It is well known that l_1 is isometric to a subspace in L_1. By Proposition 6.2, there exists an injective operator $T_0 : l_1 \to l_2$ such that $T_0(l_1)$ is homeomorphic to Σ. Considering l_1 as a subspace of L_1 and applying Proposition 6.1, we get an injective operator $T : L_1 \to Z$ into a separable Banach space such that $T(L_1)$ is homeomorphic to $T_0(l_1) \times L_1/l_1$. Since $T_0(l_1)$ is homeomorphic to Σ, and L_1/l_1 to s (as an infinite-dimensional separable Banach space), we see that $T(L_1)$ is homeomorphic to $\Sigma \times s$. Because the Banach space $C[0,1]$ contains an isomorphic copy of each separable Banach space (see [HHZ, 97]), we may assume that $Z \subset C[0,1]$, i.e., T is an operator from L_1 into $C[0,1]$. ∎

Finally we consider the operator images of the Banach space $C[0,1]$.

6.5. PROPOSITION. (1) *For every operator $T : X \to Y$ between separable Banach spaces there is an injective operator $\widetilde{T} : C[0,1] \to C[0,1]$ such that $\widetilde{T}(C[0,1])$ is homeomorphic to $T(X) \times s$.*

(2) *There exists an injective compact operator $T_1 : C[0,1] \to l_2$ such that $T_1(C[0,1])$ is homeomorphic to Σ^ω.*

(3) *There exists an injective operator $T_2 : C[0,1] \to C[0,1]$ such that $T_2(C[0,1])$ is homeomorphic to $\Sigma \times s$.*

(4) *There exists an injective operator $T_3 : C[0,1] \to C[0,1]$ such that $T_3(C[0,1]) \in \mathcal{M}_2 \setminus \mathcal{A}_2$ but $T_3(C[0,1])$ is not homeomorphic to Σ^ω.*

PROOF. The first statement follows from Proposition 6.1 and the universality of $C[0,1]$ (it contains an isomorphic copy of each separable Banach space).

The proofs of the next two statements repeat the proof of the corresponding statements of Proposition 6.4 and use the universality of $C[0,1]$.

For the proof of the last statement, we will use the following striking example from [BDP, 3.7].

6.6. EXAMPLE. *There is an injective compact operator $T_0 : X \to l_2$ from a Banach space X with separable dual, such that $T_0(X) \in \mathcal{M}_2 \setminus \mathcal{A}_2$ and the space $T_0(X)$ is not $\mathcal{M}_2$-universal.*

It follows from the first statement of the proposition that there exists an injective operator $T : C[0,1] \to C[0,1]$ such that $T(C[0,1])$ is homeomorphic to $T_0(X) \times s$, where T_0 is the operator from Example 6.6. Since the space $T_0(X)$ is not $\mathcal{M}_2$-universal, the product $T_0(X) \times s$ is not $\mathcal{M}_2$-universal either (see [BRZ, 3.2.12]). Clearly $T_0(X) \times s \in \mathcal{M}_2 \setminus \mathcal{A}_2$. Thus $T(C[0,1]) \in \mathcal{M}_2 \setminus \mathcal{A}_2$ is not $\mathcal{M}_2$-universal and hence cannot be homeomorphic to Σ^ω. ∎

7. Main results and operator images of classical Banach spaces. We summarize the principal results proven in the preceding sections in the following

7.1. MAIN THEOREM. *Suppose $T : X \to Y$ is an injective operator between separable Fréchet spaces such that $TX \in \mathcal{M}_2$. The space TX is homeomorphic to Σ^ω in each of the following cases*:

(1) *X is not normable and T is bounded*;

(2) *T is not strictly regular*;

(3) *X is not strongly regular and T is strongly regular*;

(4) *X is not strongly regular and T is weakly compact*;

(5) *X is nowhere strongly regular and T is not an isomorphic embedding*;

(6) *every strongly regular closed bounded convex subset of X is subset-dentable and T is not a G_δ-embedding*;

(7) *X has RNP and T is not a G_δ-embedding*;

(8) *X is a Banach space complemented in its second dual X^{**} and T is not a G_δ-embedding*;

(9) *the space (X, weak) is $\mathcal{M}_2$-universal and T is compact*;

(10) *there exists a Fréchet space Z and two injective operators $T_1 : X \to Z$ and $T_2 : Z \to Y$ such that $T = T_2 \circ T_1$ and the space $T_1 X$ is $\mathcal{M}_2$-universal*;

(11) *there exists an F_σ-embedding $E : Z \to X$ of a separable Fréchet space Z such that $T \circ E(Z)$ is homeomorphic to Σ^ω.*

For injective strongly regular operators from separable Banach lattices we have the following classification result.

7.2. Theorem. *Let $T : X \to Y$ be an injective operator from an infinite-dimensional separable Banach lattice X into a Fréchet space Y such that $TX \in \mathcal{M}_2$ and TX is dense in Y.*

(1) *If T is strongly regular* (*in particular*, *weakly compact*), *then the space TX is homeomorphic to one of the spaces s, Σ, $\Sigma \times s$, Σ^ω.*

(2) *If T is compact*, *then the pair (Y, TX) is homeomorphic to one of the pairs (s, Σ), $(s \times Q, \Sigma \times s)$, $(s \times Q^\omega, \Sigma \times \Sigma^\omega)$.*

Proof. We distinguish two cases:

1° The operator T is a G_δ-embedding. Then by Theorem 3.1 of [BDP], the space TX is homeomorphic to one of the spaces: s, Σ, $\Sigma \times s$. Moreover, if the G_δ-embedding T is compact, then the pair (Y, TX) is homeomorphic either to (s, Σ) or to $(s \times Q, \Sigma \times s)$ (see [BDP, 1.8]).

2° If T is not a G_δ-embedding and T is strongly regular, then by Theorem 1.8, we find that T is not strictly regular and by Theorem 3.1, TX is homeomorphic to Σ^ω. If, additionally, T is compact, then (Y, TX) is homeomorphic to $(s \times Q^\omega, \Sigma \times \Sigma^\omega)$ according to Theorem 2.1. ∎

Now we apply the results obtained to operator images of the classical Banach spaces c_0, l_1, L_1, and $C(K)$.

7.3. Theorem. *Let $X = c_0$ or $X = C(K)$ for some compact countable space K and let $T : X \to Y$ be an injective operator into a Fréchet space Y such that $TX \in \mathcal{M}_2$ and TX is a proper dense subspace in Y.*

(1) *TX is homeomorphic to Σ^ω.*

(2) *If T is compact*, *then (Y, TX) is homeomorphic to $(s \times Q^\omega, \Sigma \times \Sigma^\omega)$.*

Proof. See Theorems 1.11, 3.1, and 2.1 (see also Main Theorem 7.1(5)). ∎

7.4. Theorem. *Let $T : l_1 \to Y$ be an injective operator into a Fréchet space Y such that $T(l_1) \in \mathcal{M}_2$ and $T(l_1)$ is a proper dense subspace in Y.*

(1) *$T(l_1)$ is homeomorphic to either Σ, $\Sigma \times s$, or Σ^ω.*

(2) *If T is compact*, *then $(Y, T(l_1))$ is homeomorphic to one of the pairs (s, Σ), $(s \times Q, \Sigma \times s)$, $(s \times Q^\omega, \Sigma \times \Sigma^\omega)$.*

Proof. It is well known that the Banach space l_1 has RNP. If T is a G_δ-embedding, then $T(l_1) \in \mathcal{M}_1^2$ (see [BDP, 1.8]) and by Theorem 3.1 of [BDP], $T(l_1)$ is homeomorphic either to Σ or to $\Sigma \times s$. Moreover, if the operator T is a compact G_δ-embedding, then the pair (Y, TX) is homeomorphic either to (s, Σ) or to $(s \times Q, \Sigma \times s)$ (see [BDP, 3.1] again).

If T is not a G_δ-embedding, then the operator T is not strictly regular according to Theorem 1.8. Applying Theorem 3.1, we deduce that $T(l_1)$ is homeomorphic to Σ^ω. If T is compact, then (Y, TX) is homeomorphic to $(s \times Q^\omega, \Sigma \times \Sigma^\omega)$ according to Theorem 2.1. ■

7.5. Theorem. *Let $T : L_1 \to Y$ be an injective operator into a Fréchet space Y such that $T(L_1) \in \mathcal{M}_2$ and $T(L_1)$ is a proper dense subspace in Y.*

(1) *$T(L_1)$ is homeomorphic either to $\Sigma \times s$ or to Σ^ω.*

(2) *If T is compact, then $(Y, T(L_1))$ is homeomorphic to $(s \times Q^\omega, \Sigma \times \Sigma^\omega)$.*

Proof. It is well known that the space L_1 is not strongly regular. If T is compact, then T is not strictly regular according to Proposition 1.2(2,6). Then by Theorem 3.1, $T(L_1)$ is homeomorphic to Σ^ω and by Theorem 2.1, the pair $(Y, T(L_1))$ is homeomorphic to $(s \times Q^\omega, \Sigma \times \Sigma^\omega)$.

If T is a noncompact G_δ-embedding, then $T(L_1)$ is homeomorphic to $\Sigma \times s$ according to Theorem 3.1 of [BDP]. Finally, if T is not a G_δ-embedding, then T is not strictly regular (see Theorem 1.8). Applying Theorem 3.1 we conclude that $T(L_1)$ is homeomorphic to Σ^ω. ■

7.6. Theorem. *Let $T : C[0,1] \to Y$ be an injective operator into a Fréchet space Y such that $T(C[0,1]) \in \mathcal{M}_2$ and $T(C[0,1])$ is a dense subspace in Y.*

(1) *If T is compact, then $(Y, T(C[0,1]))$ is homeomorphic to $(s \times Q^\omega, \Sigma \times \Sigma^\omega)$.*

(2) *If T factors through a Fréchet space containing no isomorphic copy of the Banach space $C[0,1]$, then $T(C[0,1])$ is homeomorphic to Σ^ω.*

Proof. The first statement follows from $C[0,1]$ not being strongly regular and can be proven by analogy with the corresponding statement of Theorem 7.5.

To prove the second statement, suppose Z is a Fréchet space containing no isomorphic copy of the Banach space $C[0,1]$, and $T_1 : C[0,1] \to Z$, $T_2 : Z \to Y$ are two injective operators such that $T = T_2 \circ T_1$. By Proposition 1.11(2) the operator T_1 is not strictly regular and by Proposition 1.2(4), the operator $T = T_1 \circ T_2$ is not strictly regular either. Then Theorem 3.1 implies that $T(C[0,1])$ is homeomorphic to Σ^ω. ■

Finally, we present a counterexample disproving certain natural conjectures.

7.7. Example. *There exist a strongly regular Banach space X (denoted by S_*T_∞ in [GMS$_2$]) with separable dual and two operators $T_1 : X \to l_2$ and $T_2 : l_2 \to l_2$ such that*

(1) *T_1 is a G_δ-embedding;*

(2) *T_2 is a compact F_σ-embedding;*

(3) *$T_2 \circ T_1$ is a strictly regular operator;*

(4) *the space (X, weak) is $\mathcal{M}_2$-universal;*

(5) *the space $T_2 \circ T_1(X)$ is homeomorphic to Σ^ω.*

This example shows that the F_σ-embedding in Main Theorem 7.1(11) cannot be replaced by a G_δ-embedding and that the lack of strict regularity of an operator T does not follow from the topological equivalence of TX and Σ^ω (this means that Theorem 3.1 cannot be reversed).

Proof of 7.7. By Theorem 2.1 of [Ba$_1$] and Theorem VI.1 of [GMS$_2$], the Banach space $X = S_*T_\infty$ constructed in [GMS$_1$, §VI] has the following properties:

1° the dual space to S_*T_∞ is separable;

2° the space $(S_*T_\infty, \text{weak})$ is $\mathcal{M}_2$-universal;

3° there exists a G_δ-embedding $T_1 : S_*T_\infty \to l_2$.

Let $T_2 : l_2 \to l_2$ be any compact injective operator. Since l_2 is a reflexive Banach space, T_2 is a compact F_σ-embedding. Since T_1 is a weakly compact G_δ-embedding, by Corollary 1.6 and Proposition 1.2(7), the composition $T_2 \circ T_1$ is a strictly regular operator.

Let us show that $T_2 \circ T_1(S_*T_\infty) \in \mathcal{M}_2$. By Proposition 1.4 for this it suffices to find a bounded closed convex neighborhood B of the origin in S_*T_∞ such that $T_2 \circ T_1(B)$ is closed in $T_2 \circ T_1(S_*T_\infty)$. Since T_1 is a G_δ-embedding, T^{-1} is Borel of class 1 (see [BDP, 1.8 and 1.4]). Hence there exists a closed bounded convex neighborhood $B \subset S_*T_\infty$ of the origin such that $T_1(B)$ is closed in $T_1(S_*T_\infty)$. Let D denote the closure of $T_1(B)$ in l_2. Then D is weakly compact and thus T_2D is closed in l_2. Consequently, $T_2 \circ T_1(B) = T_2(D) \cap T_1(S_*T_\infty)$ is closed in $T_2 \circ T_1(S_*T_\infty)$. Since the operator $T_2 \circ T_1$ is compact, Theorem 4.1 implies that $T_2 \circ T_1(S_*T_\infty)$ is homeomorphic to Σ^ω. ∎

References

[Ba$_1$] T. Banakh, *The topological classification of weak unit balls of Banach spaces*, this volume, 7–35.

[Ba$_2$] —, *Preserving $\mathcal{C}$-universality by perfect maps*, preprint.

[BC$_1$] T. Banakh et R. Cauty, *Universalité forte pour les sous-ensembles totalement bornés. Applications aux espaces $C_p(X)$*, Colloq. Math. 73 (1997), 25–33.

[BC$_2$] —, —, *Interplay between strongly universal spaces and pairs*, Dissertationes Math. 386 (2000).

[BDP] T. Banakh, T. Dobrowolski and A. Plichko, *The topological and Borel classification of operator images*, this volume, 37–52.

[BRZ] T. Banakh, T. Radul and M. Zarichnyĭ, *Absorbing Sets in Infinite-Dimensional Manifolds*, VNTL Publishers, Lviv, 1996.

[BP] C. Bessaga and A. Pełczyński, *Selected Topics in Infinite-Dimensional Topology*, PWN, Warszawa, 1975.

[BR] J. Bourgain and H. P. Rosenthal, *Applications of the theory of semi-embeddings to Banach space theory*, J. Funct. Anal. 52 (1983), 149–188.

[Bo] R. D. Bourgin, *Geometric Aspects of Convex Sets with the Radon–Nikodým Property*, Lecture Notes in Math. 993, Springer, 1983.

[CD] R. Cauty and T. Dobrowolski, *Applying coordinate products to the topological identification of normed spaces*, Trans. Amer. Math. Soc. 337 (1993), 625–649.

[CDM] R. Cauty, T. Dobrowolski and W. Marciszewski, *A contribution to the topological classification of the spaces $C_p(X)$*, Fund. Math. 142 (1993), 267–301.

[DDMM] J. J. Dijkstra, T. Dobrowolski, W. Marciszewski, J. van Mill and J. Mogilski, *Recent classification and characterization results in geometric topology*, Bull. Amer. Math. Soc. 22 (1990), 277–283.

[DiM] J. J. Dijkstra and J. Mogilski, *The topological product structure of systems of Lebesgue spaces*, Math. Ann. 290 (1991), 527–543.

[DMM] T. Dobrowolski, W. Marciszewski and J. Mogilski, *On topological classification of function spaces $C_p(X)$ of low Borel complexity*, Trans. Amer. Math. Soc. 328 (1991), 307–324.

[DoM] T. Dobrowolski and J. Mogilski, *Certain sequence and function spaces homeomorphic to the countable product of l^2_f*, J. London Math. Soc. 45 (1992), 566–576.

[DT] T. Dobrowolski and H. Toruńczyk, *Separable complete ANR's admitting a group structure are Hilbert manifolds*, Topology Appl. 12 (1981), 229–235.

[Fo] V. P. Fonf, *Dual subspaces and injections of Banach spaces*, Ukrain. Mat. Zh. 39 (1987), 364–369 (in Russian).

[GGMS] N. Ghoussoub, G. Godefroy, B. Maurey and W. Schachermayer, *Some topological and geometrical structures in Banach spaces*, Mem. Amer. Math. Soc. 378 (1987).

[GM] N. Ghoussoub and B. Maurey, *G_δ-embeddings in Hilbert space*, J. Funct. Anal. 61 (1985), 72–97.

[GMS$_1$] N. Ghoussoub, B. Maurey and W. Schachermayer, *A counterexample to a problem on points of continuity in Banach spaces*, Proc. Amer. Math. Soc. 99 (1987), 278–282.

[GMS$_2$] —, —, —, *Geometrical implications of certain infinite-dimensional decompositions*, Trans. Amer. Math. Soc. 317 (1990), 541–584.

[HHZ] P. Habala, P. Hájek and V. Zizler, *Introduction to Banach Spaces*, Matfyzpress, Praha, 1996.

[KPR] N. J. Kalton, N. T. Peck and J. W. Roberts, *An F-Space Sampler*, London Math. Soc. Lecture Note Ser. 89, Cambridge Univ. Press, Cambridge, 1984.

[Ke] A. S. Kechris, *Classical Descriptive Set Theory*, Springer, 1995.

[Ku] K. Kuratowski, *Topology*, *I*, PWN, Warszawa, 1966.

[LT$_1$] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, *I*, Springer, Berlin, 1977.

[LT$_2$] —, —, *Classical Banach Spaces*, *II*, Springer, Berlin, 1979.

[LPP] H. P. Lotz, N. T. Peck and H. Porta, *Semi-embeddings of Banach spaces*, Proc. Edinburgh Math. Soc. 22 (1979), 233–240.

[SR] J. Saint-Raymond, *Espaces à modèle séparable*, Ann. Inst. Fourier (Grenoble) 26 (1976), no. 3, 211–255.

[Sch] H. H. Schaefer, *Topological Vector Spaces*, Macmillan, New York, 1966.

[Se] Z. Semadeni, *Banach Spaces of Continuous Functions*, PWN, Warszawa, 1971.

[Va] V. S. Varadarajan, *Measures on topological spaces*, Mat. Sb. (N.S.) 55 (97) (1961), 35–100 (in Russian).

[Vo] D. Vogt, *Fréchetträume, zwischen denen jede stetige lineare Abbildung beschränkt ist*, J. Reine Angew. Math. 345 (1983), 182–200.