1. Introduction

Let Ω be a domain in $\mathbb{R}^N$, $N \geq 2$, and let $F = (f^1, \dots, f^N) : \Omega \to \mathbb{R}^N$ be a Schwartz distribution on Ω with values in $\mathbb{R}^N$. The divergence operator

$$\operatorname{div} : \mathcal{D}'(\Omega, \mathbb{R}^N) \to \mathcal{D}'(\Omega, \mathbb{R})$$

and its formal adjoint

$$\operatorname{curl} : \mathcal{D}'(\Omega, \mathbb{R}^N) \to \mathcal{D}'(\Omega, \mathbb{R}^{N\times N})$$

are defined, respectively, by

$$\operatorname{div} F = \frac{\partial f^1}{\partial x_1} + \dots + \frac{\partial f^N}{\partial x_N}, \qquad \operatorname{curl} F = \left[\frac{\partial f^i}{\partial x_j} - \frac{\partial f^j}{\partial x_i}\right]_{i,j=1,\dots,N}.$$

A *div-curl couple* on Ω is a pair of distributions $\Phi = [B, E]$ satisfying the conditions

$$\operatorname{div} B = 0, \qquad \operatorname{curl} E = 0.$$

For each couple $\Phi = [B, E] \in L^p(\Omega, \mathbb{R}^N) \times L^q(\Omega, \mathbb{R}^N)$, $1 < p, q < \infty$ and $pq = p + q$, it is possible to consider its *norm*

$$|\Phi(x)| = (|B(x)|^p + |E(x)|^q)^{1/2} \tag{1.1}$$

and its *Jacobian*

$$J(x, \Phi) = \langle B(x), E(x)\rangle. \tag{1.2}$$

A fundamental example is given by considering a mapping $f = (f^1, \dots, f^N) : \Omega \to \mathbb{R}^N$ in the Sobolev space $W^{1,N}(\Omega, \mathbb{R}^N)$: the vector fields $B = \nabla f^2 \times \dots \times \nabla f^N$ and $E = \nabla f^1$ are respectively in $L^{N/(N-1)}(\Omega, \mathbb{R}^N)$ and $L^N(\Omega, \mathbb{R}^N)$, and $[B, E]$ is a div-curl couple. In this case, the inner product $\langle B, E\rangle$ is exactly the Jacobian determinant of f.

A div-curl couple $\Phi \in L^p(\Omega, \mathbb{R}^N) \times L^q(\Omega, \mathbb{R}^N)$ is called a *K-quasiharmonic field* if there exists a *distortion* function $1 \leq K = K(x) < \infty$ such that

$$|\Phi(x)|^2 \leq [K(x) + K^{-1}(x)]J(x, \Phi), \qquad \text{a.e.}$$

Obviously, if $f = (f^1, \dots, f^N) \in W^{1,N}(\Omega, \mathbb{R}^N)$ is a mapping of finite distortion, that is

$$|Df(x)|^N \leq \mathcal{K}(x)J(x, f) \qquad \text{a.e.}$$

for some $1 \leq \mathcal{K}(x) < \infty$, then the couple $[B(x), E(x)] \in L^{N/(N-1)}(\Omega, \mathbb{R}^N) \times L^N(\Omega, \mathbb{R}^N)$ of the previous example yields a K-quasiharmonic field.

To every solution of an elliptic PDE there corresponds a quasiharmonic field. Consider, for example, the linear equation

$$\operatorname{div} A(x)\nabla u = 0$$

where $A(x)$ is a measurable function on Ω with values in the space of symmetric matrices in $\mathbb{R}^{N\times N}$ satisfying the ellipticity condition

$$m(x)|\xi|^2 \le \langle A(x)\xi,\xi\rangle \le M(x)|\xi|^2 \tag{1.3}$$

where $0 \le m(x) \le M(x) < \infty$. It is possible to express condition (1.3) by using just one inequality

$$|\xi|^2 + |A(x)\xi|^2 \le \mathcal{K}\langle A(x)\xi,\xi\rangle \tag{1.4}$$

for almost every $x \in \Omega$ and all $\xi \in \mathbb{R}^N$, where $\mathcal{K} = \mathcal{K}(x) \ge 1$ depends on the ellipticity bounds $m(x)$ and $M(x)$.

Inequality (1.4) can be used to formulate the ellipticity condition for the nonlinear equation

$$\operatorname{div} A(x,\nabla u) = 0,$$

namely

$$|\xi|^2 + |A(x,\xi)|^2 \le \mathcal{K}\langle A(x,\xi),\xi\rangle.$$

Since u is a solution of one of the equations above, the couple $\Phi = [A(x,\nabla u),\nabla u]$ is a K-quasiharmonic field in $L^2(\Omega,\mathbb{R}^{N\times N})$.

The relevance of quasiharmonic fields to the theory of elliptic partial differential equations is therefore evident.

More generally, we look at an elliptic complex of first order differential operators

$$\mathcal{D}'(\mathbb{R}^N,\mathbf{U}) \xrightarrow{\mathcal{P}} \mathcal{D}'(\mathbb{R}^N,\mathbf{V}) \xrightarrow{\mathcal{Q}} \mathcal{D}'(\mathbb{R}^N,\mathbf{W}) \tag{1.5}$$

where $\mathbf{U},\mathbf{V}$ and $\mathbf{W}$ are finite-dimensional inner product spaces and the symbols $\mathcal{P} = \mathcal{P}(\xi)$ and $\mathcal{Q} = \mathcal{Q}(\xi)$ are linear functions in $\xi = (\xi_1,\dots,\xi_N) \in \mathbb{R}^N$ such that

$$\operatorname{im}\mathcal{P}(\xi) = \ker\mathcal{Q}(\xi) \qquad \text{for all } \xi \ne 0.$$

Such complexes, called elliptic, can be viewed in many ways as generalizations of the complex

$$\mathcal{D}'(\mathbb{R}^N,\mathbb{R}) \xrightarrow{\nabla} \mathcal{D}'(\mathbb{R}^N,\mathbb{R}^N) \xrightarrow{\text{curl}} \mathcal{D}'(\mathbb{R}^N,\mathbb{R}^N\times\mathbb{R}^N).$$

An *elliptic couple* associated to the complex (1.5) is a pair

$$\mathcal{F} = [\mathcal{P}\alpha,\mathcal{Q}^*\beta]$$

where $\alpha \in W^{1,p}_{\text{loc}}(\Omega,\mathbf{U})$, $\beta \in W^{1,p}_{\text{loc}}(\Omega,\mathbf{W})$, and $\mathcal{Q}^*$ is the formal adjoint operator of $\mathcal{Q}$. Notice that if $\mathcal{P} = \nabla$ and $\mathcal{Q} = \text{curl}$, then the elliptic couple $\mathcal{F} = [\mathcal{P}\alpha,\mathcal{Q}^*\beta]$ is none other than a div-curl couple. Therefore, in analogy with the definitions (1.1) and (1.2), we introduce the norm

$$|\mathcal{F}| = (|\mathcal{P}\alpha|^2 + |\mathcal{Q}^*\beta|^2)^{1/2}$$

and the Jacobian

$$J(x,\mathcal{F}) = \langle\mathcal{P}\alpha,\mathcal{Q}^*\beta\rangle.$$

Moreover, an elliptic couple $\mathcal{F} = [\mathcal{P}\alpha,\mathcal{Q}^*\beta]$ is called K*-quasiharmonic*, with $1 \le K = K(x) < \infty$, if

$$|\mathcal{F}(x)|^2 \le \mathcal{K}(x)J(x,\mathcal{F})$$

where $\mathcal{K}(x) = K(x)+K^{-1}(x) \geq 2$. This more general setting allows us to give a definition of polyconvexity that can be viewed as a generalization of the classical one.

In analogy with the calculus of variations we prove that

$$\text{Polyconvexity} \;\Rightarrow\; \text{Quasiconvexity} \;\Rightarrow\; \text{Rank one convexity.}$$

Continuing this analogy one can conjecture that convexity in singular directions might imply quasiconvexity. Let us refer the interested reader to Chapter 3 for more details.

Chapter 4 is dedicated to the question of the integrability of the Jacobian determinant of some Sobolev mappings. Our main result asserts that if $f \in W^{1,N-1}(\Omega, \mathbb{R}^N)$, $N > 2$, is an orientation preserving (reversing) mapping whose cofactor matrix $|D^\sharp f|^{N/(N-1)}$ is in the space $L^P(\Omega)$, with the function P satisfying the divergence condition

$$\int_1^\infty \frac{P(t)}{t^2}\, dt = \infty,$$

then the Jacobian determinant of f is locally integrable and obeys the rule of integration by parts

$$\int_\Omega \varphi(x) J(x,f)\, dx = -\int_\Omega df^1 \wedge \ldots \wedge df^{i-1} \wedge f^i\, d\varphi \wedge df^{i+1} \wedge \ldots \wedge df^N =: \mathcal{J}_f[\varphi] \tag{1.6}$$

for all indices $i = 1, \ldots, n$ and all test functions $\varphi \in C_0^\infty(\Omega)$.

It is worth pointing out that both nonlinear elasticity and the theory of mappings of finite distortion are drawn on integral estimates of $J(x,f)$ in terms of $D^\sharp f$. In quasiconformal theory the ratio $|D^\sharp f|^N / |J(x,f)|^{N-1}$ is none other than the inner distortion function of f.

Chapter 5 is mainly dedicated to some regularity results for vector fields of bounded distortion. Starting from some inequalities for div-curl couples, under the assumption of bounded distortion, we get a family of reverse Hölder inequalities. Applications to the theory of quasiconformal mappings and PDEs are given. In particular, we recover the celebrated result of Bojarski concerning higher integrability of functions $f = (f^1, f^2) : \Omega \to \mathbb{R}^2$ of bounded distortion:

$$f \in W^{1,2-\varepsilon} \;\Rightarrow\; f \in W^{1,2+\varepsilon}.$$

The end of this chapter also contains some further regularity results for mappings having unbounded distortion in the exponential class $\mathrm{Exp}_\gamma(\Omega)$ for some $\gamma > 1$.

In Chapter 6, we discuss lower semicontinuity of integral functionals of the type

$$F(u) = \int_\Omega f(x, u, \mathcal{L}u)\, dx$$

where $u = (v, w)$ is a pair of Sobolev functions, f is a nonnegative integrand satisfying the growth condition

$$0 \leq f(x, s, \xi) \leq c(1 + |\xi|^q),$$

$q \geq p > 1$, and $\mathcal{L}u = [\mathcal{P}v, \mathcal{Q}^* w]$ with $\mathcal{P}, \mathcal{Q}$ linear differential operators forming an elliptic complex.

The conclusion is dedicated to the lower semicontinuity in the setting of functions of *bounded variation*. It is well known that if one considers an integral functional of the type

$$\int_{\Omega} f(x, u(x), Du(x))\, dx$$

with $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty]$, then the convexity in the last variable is needed in order to have lower semicontinuity in $W^{1,1}(\Omega)$. However, since the first result of this kind appeared in the celebrated paper by Serrin [55], it is well known that some regularity (or growth) assumptions on f must be present. In fact, there are counterexamples showing that the above functional is not lower semicontinuous in L^1 if f is merely a Carathéodory integrand controlled by a term like $c(1 + |Du|)$. In this paper we extend a recent result by Marcellini–Gori [44] to the BV setting by showing that no growth assumptions on f are needed, as long as we assume f nonnegative and locally Lipschitz.

2. Requisites from analysis and function spaces

This chapter is dedicated to establishing notation and an exposition of some requisites from functional analysis.

2.1. Orlicz spaces. The Orlicz spaces turn out to be instrumental in observing and formulating the phenomenon of higher integrability properties of certain nonlinear differential quantities such as the Jacobian determinant or, more generally, wedge products of differential forms.

Let Ω be an open subset of $\mathbb{R}^N$. Throughout this text we shall work with the Lebesgue measure. All functions $f : \Omega \to V$ with values in a finite-dimensional inner product space will be measurable.

By an *Orlicz function* we mean any $P : [0, \infty] \to [0, \infty]$ continuously increasing from $P(0) = 0$ to $P(\infty) = \infty$. The *Orlicz space*, denoted by $L^P(\Omega, V)$, consists of all mappings $f : \Omega \to V$ such that

$$\int_{\Omega} P(\varepsilon|f|) < \infty \qquad \text{for some } \varepsilon = \varepsilon(f) > 0.$$

This is a complete linear metric space in which the distance between f and g is defined as

$$\operatorname{dist}(f, g) = \inf\Big\{K > 0 : \int_{\Omega} P(K^{-1}|f - g|) \le K\Big\}.$$

A slight change in this formula gives us a nonlinear functional on $L^P(\Omega, V)$,

$$\|f\|_P = \inf\Big\{K > 0 : \int_{\Omega} P(K^{-1}|f|) \le 1\Big\}.$$

In general $\|\cdot\|_P$ need not be a norm, but it is a norm whenever P is convex. We refer to such P as *Young function*. In this case $L^P(\Omega, V)$ becomes a Banach space and $\|\cdot\|_P$ is called *Luxemburg norm*.

Taking $P(t) = t^p/p$, $0 < p < \infty$, we recover the Lebesgue spaces $L^p(\Omega, V)$ for which the usual notation is

$$\|f\|_p = \Big(\int_\Omega |f|^p\Big)^{1/p}.$$

In order to follow the lead of the L^p spaces, it will be necessary to put some restrictions on the Orlicz function P. In particular, we assume that P is C^∞-smooth and log-convex. The latter means that P can be represented by the integral

$$P(t) = \int_0^t \frac{\varrho(s)}{s}\,ds$$

where $\varrho \in C^\infty[0,\infty)$ is an increasing function with $\varrho(0) = 0$ and $\varrho(\infty) = \infty$. For example $\varrho(s) = s^p$ with $p > 0$. Without loss of generality we can normalize P by requiring $\varrho(1) = 1$. Note that the inverse function $\varrho^{-1} : [0,\infty) \to [0,\infty)$ also meets those conditions.

Now, given a set of such functions, say $\{\varrho_1, \dots, \varrho_k\}$, it is legitimate to define ϱ by the equation

$$\varrho^{-1}(t) = \varrho_1^{-1}(t)\dots\varrho_k^{-1}(t).$$

Then the corresponding Orlicz functions

$$P(t) = \int_0^t \frac{\varrho(s)}{s}\,ds, \qquad P_i(t) = \int_0^t \frac{\varrho_i(s)}{s}\,ds, \qquad i = 1,\dots,k,$$

satisfy *Young's inequality*

$$P(t_1,\dots,t_k) \le P_1(t_1)\dots P_k(t_k)$$

for all nonnegative numbers $t_1,\dots,t_k$. Because of this we refer to P as *Young's conjugation* of $P_1,\dots,P_k$.

The inequality above proves extremely useful in deriving the following analogue of Hölder's inequality:

$$\big\||f_1|\dots|f_k|\big\|_P \le \|f_1\|_{P_1}\dots\|f_k\|_{P_k}$$

for $f_i \in L^{P_i}(\Omega, V_i)$ with $i = 1,\dots,k$.

Now a *complementary couple* (P_1, P_2) is a pair of Orlicz functions for which P defined as above is the identity function. Many analytically pleasing functions fail to be increasing. To handle this problem we introduce the following concept.

Two functions $\Phi, \Psi \in C^\infty[0,\infty)$ are said to be *equivalent* if for every $\varepsilon > 0$ there exists a constant $K = K(\varepsilon) \ge 1$ such that

$$\Psi\Big(\frac{t}{K}\Big) \le \varepsilon\Phi(t) \le \Psi(Kt), \qquad t \ge 0.$$

Denote it briefly $\Phi \sim \Psi$.

When two equivalent functions happen to be increasing they yield the same Orlicz space.

Basic examples we can recall are the Zygmund classes, corresponding to $P(t) \sim t^p\log^\alpha(e+t)$, $p \ge 1$ and $\alpha \in \mathbb{R}$. These spaces are traditionally denoted by $L^p\log^\alpha L(\Omega, V)$.

Furthermore, for each $\lambda \in \mathbb{R}$ and $1 < p, q < \infty$, $1/p + 1/q = 1$, the pair $P(t) \sim t^p\log^{\lambda p}(e+t)$ and $Q(t) = t^q \log e^{\lambda q}(e+t)$ is a complementary couple; the complementary function to $P(t) = t\log^{\alpha}(e+t)$, $\alpha > 0$, is a function $Q(t) \sim \exp(t^{1/\alpha}) - 1$. Finally, Ψ is *stronger* than Φ, $\Psi \preccurlyeq \Phi$ for short, if for every $\varepsilon > 0$ there exists $K = K(\varepsilon) > 0$ such that

$$\Phi(t) \le \varepsilon\Psi(t) \qquad \forall t \ge 0.$$

We write $\Psi \prec \Phi$ if Ψ is stronger than Φ but they are not equivalent.

2.2. Schwartz distributions. For an arbitrary set $\Omega \subset \mathbb{R}^N$ we denote by $C_0^\infty(\Omega)$ the algebra of all infinitely differentiable functions $\phi : \mathbb{R}^N \to \mathbb{R}^N$ with compact support contained in Ω.

The *N-term multiindex* is any ordered system $\alpha = (\alpha_1, \dots, \alpha_N)$ of nonnegative integers $\alpha_1, \dots, \alpha_N$. The *length* of α is defined as $|\alpha| = \alpha_1 + \dots + \alpha_N$. The *differential of order* α is the operator $\partial^\alpha = \partial^{\alpha_1+\dots+\alpha_N}/\partial x_1^{\alpha_1}\dots\partial x_N^{\alpha_N}$ which can be applied to sufficiently smooth functions.

Let Ω be an open set of $\mathbb{R}^N$ and $\mathbf{V}$ a finite-dimensional inner product space. A *distribution* f in Ω with values in $\mathbf{V}$ is a linear form $f : C_0^\infty(\Omega) \to \mathbf{V}$ such that for every compact set $K \subset \Omega$ and any test function $\phi \in C_0^\infty(K)$ we have the estimate

$$f[\phi] \le C(K) \sum_{|\alpha|\le m} \|\partial^\alpha \phi\|_\infty.$$

In general, the integer m may depend on the compact set K. If not, we say that f has *finite order* in Ω, and the smallest such integer m is called the *order* of f in Ω. The space of all distributions will be denoted by $\mathcal{D}'(\Omega, \mathbf{V})$.

It is immediate from the definition above that the space of distributions in Ω is complete under pointwise convergence. Specifically, given a sequence $\{f_j\}$ of distributions in Ω such that $\lim_{j\to\infty} f_j[\phi]$ exists for every test function $\phi \in C_0^\infty(\Omega)$, define $f : C_0^\infty(\Omega) \to \mathbf{V}$ by $f[\phi] = \lim_{j\to\infty} f_j[\phi]$. Then $f \in \mathcal{D}'(\Omega, \mathbf{V})$, as is easy to see. We then say that

$$f = \lim_{j\to\infty} f_j \qquad \textit{in the sense of distributions.}$$

This simple notion of convergence has far reaching applications.

The reason for calling elements of $\mathcal{D}'(\Omega, \mathbf{V})$ generalized functions is that every locally integrable function $f \in L^1_{\mathrm{loc}}(\Omega, \mathbf{V})$ can be viewed as a distribution (of order zero), defined by the rule

$$\phi \mapsto \int_\Omega \phi(x) f(x)\, dx \qquad \text{for } \phi \in C_0^\infty(\Omega).$$

Hence the notation $L^1_{\mathrm{loc}}(\Omega, \mathbf{V}) \subset \mathcal{D}'(\Omega, \mathbf{V})$.

Quite often locally integrable functions are referred to as *regular distributions.* Although it is not apparent at this point, the regular distributions are dense in $\mathcal{D}'(\Omega, \mathbf{V})$. Of fundamental importance is the Dirac delta $\delta_a \in \mathcal{D}'(\Omega)$ at the point $a \in \Omega$ which assigns to each $\phi \in C_0^\infty(\Omega)$ its value at a, $\delta_a[\phi] = \phi(a)$. It has order zero but it is not a regular distribution.

Distributions of order zero, like the Dirac delta, are all represented by integration with respect to a suitable $\mathbf{V}$-valued Radon measure on Ω. This fact is usually referred to as the *Riesz representation theorem.* It asserts that each $f \in \mathcal{D}'(\Omega, \mathbf{V})$ can be written as

$$f[\phi] = \int_\Omega \phi(x)\, d\mu(x) \qquad \text{for all } \phi \in C_0^\infty(\Omega).$$

Also, let us recall that a *Radon measure* μ on Ω is such that the absolute value $|\mu|$ is a Borel measure which is finite on compact subsets. In this way we identify Radon measures with distributions of order zero. The regular distributions are the ones having no singular part with respect to the Lebesgue measure. A distribution $f \in \mathcal{D}'(\Omega, \mathbb{R})$ is said to be *positive* if $f[\phi] \geq 0$ whenever $\phi \geq 0$. Positive distributions have order zero, and therefore are represented by Borel measures.

Let $f \in C^\infty(\Omega, \mathbf{V})$. By integration by parts we have

$$\int_\Omega \phi(\partial^\alpha f) = (-1)^{|\alpha|} \int_\Omega (\partial^\alpha \phi) f \qquad \forall \phi \in C_0^\infty(\Omega).$$

This procedure can be extended to all $f \in \mathcal{D}'(\Omega, \mathbf{V})$ by setting

$$\partial^\alpha f[\phi] = (-1)^{|\alpha|} f[\partial^\alpha \phi].$$

Let us point out that the original purpose of the theory of distributions was to make it possible to differentiate locally integrable functions. From this point of view Schwartz distributions offer us the most economical extension of the space $L^1_{\mathrm{loc}}(\Omega, \mathbf{V})$ carrying out this task.

2.3. The maximal operator. The concept of the maximal function can be traced back to G. H. Hardy and J. E. Littlewood [31] and has been under study since then. This is partly due to the interest in Fourier analysis.

The objective of the present section is to describe some maximal inequalities that are crucial for the higher integrability results in PDEs and quasiconformal mappings.

Let Q_0 be a fixed cube in $\mathbb{R}^N$ with sides parallel to the axes, and, for every set $E \subset Q_0$, denote by $|E|$ the Lebesgue measure of E and by $|f|_Q$ the integral mean of $|f|$ over E:

$$|f|_E = \frac{1}{|E|} \int_E |f(x)|\,dx = \fint_E |f(x)|\,dx.$$

For $f \in L^1(Q_0)$ define the *local maximal function*

$$Mf(x) = \sup_{Q \subset Q_0} \Big\{ \fint_Q |f(y)|\,dy \Big\} \qquad \forall x \in Q_0$$

where the supremum extends over all cubes $Q \subset Q_0$ containing x with sides parallel to the coordinates axes. Note that the maximal operator is *sublinear* and *homogeneous*, that is, $M(f+g) \leq Mf + Mg$ and $M(\lambda f) = \lambda(Mf)$ for all $\lambda \geq 0$.

The Hardy–Littlewood maximal theorem plays a fundamental role in the theory of maximal functions. It ensures higher degree of integrability of some functions f as compared to that of Mf. In particular, it asserts that if $f \in L \log L(Q_0)$ then $Mf \in L^1(Q_0)$.

In [56] E. M. Stein showed that the converse of this theorem is also true; namely he proved that

$$\fint_{Q_0} |f(x)| \log\left(e + \frac{|f(x)|}{|f|_{Q_0}}\right) dx \leq 2^N \fint_{Q_0} Mf(x)\,dx$$

for all $f \in L \log L(Q_0)$, $f \neq 0$.

The following estimate strengthens the maximal theorem:

$$|\{x \in Q_0 : Mf(x) > 2t\}| \leq \frac{c(N)}{t} \int_{|f|>t} |f(x)|\,dx \tag{2.1}$$

for all $f \in L^1(Q_0)$. This is a simple consequence of the weak-type inequality

$$|\{x \in Q_0 : Mg(x) > t\}| \leq \frac{c(N)}{t} \int_{Q_0} |g(x)|\,dx \tag{2.2}$$

applied to

$$g(x) = \begin{cases} f(x) & \text{if } x \in Q_0 \cap \{|f| > t\}, \\ 0 & \text{otherwise,} \end{cases}$$

where we notice that $Mf(x) \leq t + Mg(x)$, so

$$\{x \in Q_0 : Mf(x) > 2t\} \subset \{x \in Q_0 : Mg(x) > t\}.$$

The proof of (2.2) involves Vitali's covering lemma.

Notice that an inverse estimate also holds, namely

$$|\{x \in Q_0 : Mf(x) > 2t\}| \geq \frac{c(N)}{t} \int_{|f|>t} |f(x)|\,dx. \tag{2.3}$$

The proof of (2.3) is obtained using the well known Calderón–Zygmund decomposition lemma.

2.4. Hardy spaces. Delicate cancellation properties of various nonlinear differential and integral forms cannot be discussed without introducing the Hardy spaces. It is the objective of this section to give a brief account of these spaces.

The *Hardy space* $\mathcal{H}^1(\mathbb{R}^N)$, introduced by E. Stein and G. Weiss in [57], can be characterized as follows (see [13], [8]):

$$\mathcal{H}^1(\mathbb{R}^N) = \{f \in L^1(\mathbb{R}^N) : \sup_{t \geq 0} |h_t \star f| \in L^1(\mathbb{R}^N)\}$$

where $h_t = 1/t^N h(\cdot/t)$, $h \in C_0^\infty(\mathbb{R}^N)$, $h \geq 0$, $\operatorname{supp} h \subset B(0,1)$.

Notice, of course, that $\mathcal{H}^1(\mathbb{R}^N)$ is a proper subspace of $L^1(\mathbb{R}^N)$. In particular in [7] the authors proved that $\mathcal{H}^1(\mathbb{R}^N)$ is the minimal linear subspace of $L^1(\mathbb{R}^N)$ which contains the range of the mapping

$$f \in W^{1,N}(\mathbb{R}^N, \mathbb{R}^N) \mapsto \det Df \in L^1(\mathbb{R}^N).$$

In this connection it is worth introducing also the function spaces BMO and VMO.

Definition 2.1. A function $h \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ is said to have *bounded mean oscillations* if

$$\|h\|_{\mathrm{BMO}} = \sup_{Q \subset \mathbb{R}^N} ⨍_Q |h(x) - h_Q|\,dx < \infty.$$

Observe that $\|\cdot\|_{\mathrm{BMO}}$ is a norm in the space $\mathrm{BMO}(\mathbb{R}^N)$ modulo constant functions. Clearly, bounded functions lie in $\mathrm{BMO}(\mathbb{R}^N)$, but they are not dense.

Indeed, for example, it is possible to show that the function $h(x) = \log|x|$ on the real line has bounded mean oscillation, but cannot be approximated in BMO by bounded functions.

The closure of $C_0^\infty(\mathbb{R}^N)$ in the BMO-norm is the space $\mathrm{VMO}(\mathbb{R}^N)$ introduced by Sarason [53]. It consists of functions with *vanishing mean oscillations.*

Precisely, this space is characterized by the condition:

$$\lim_{Q\subset\mathbb{R}^N} \fint_Q |h - h_Q| = 0, \qquad \text{uniformly as } |Q| + |Q|^{-1} \to \infty.$$

In words, the infinitesimal oscillations of h vanish everywhere.

Finally, there are two central facts we want to note here. The first is the result of Coifman and Rochberg [9] asserting that the logarithm of a maximal function belongs to BMO with norm bounded by a universal constant. Thus the maximal operator has a certain smoothing property.

THEOREM 2.2. *Let μ be a Radon measure on Ω, an open domain in $\mathbb{R}^N$, such that $M\mu$ is finite at some point*; *consequently at almost every point. Then*

$$\|\log(M\mu)\|_{\mathrm{BMO}(\Omega)} \le C(N).$$

The second is the duality theorem of C. Fefferman which states that $\mathrm{BMO}(\mathbb{R}^N)$ is the dual space of $\mathcal{H}^1(\mathbb{R}^N)$, and also a result of Sarason asserting that $\mathcal{H}^1(\mathbb{R}^N)$ is the dual space of $\mathrm{VMO}(\mathbb{R}^N)$. In particular, we note the following

PROPOSITION 2.3 ($\mathcal{H}^1$-BMO duality). *There is a constant $C = C(N)$ such that if $h \in \mathcal{H}^1(\mathbb{R}^N)$ and $b \in L^\infty(\mathbb{R}^N)$, then*

$$\Big| \int_{\mathbb{R}^N} h(x)b(x)\,dx \Big| \le C(N)\|h\|_{\mathcal{H}^\infty}\|b\|_{\mathrm{BMO}}.$$

3. Elliptic complexes

3.1. Introduction. The aim of this chapter is to discuss and develop my recent joint paper with A. Verde [21] in which we have continued, from a more general perspective, some themes discussed in [39] where the theory of quasiharmonic fields is formulated using singular integrals, in particular the N-dimensional Hilbert transform. This more general setting provides a better understanding of several unanswered questions in [39], especially those concerning the L^p-norm of the Hilbert transform and sharp estimates for elliptic PDEs.

We start with an exposition of some basic definitions and concepts that will be useful in what follows.

Let Ω be a domain in $\mathbb{R}^N$, $N \ge 2$. We shall consider Schwartz distributions on Ω with values in $\mathbb{R}^N$, including the Lebesgue space $L^p(\Omega,\mathbb{R}^N)$, $1 \le p < \infty$, equipped with the norm

$$\|F\|_p = \Big(\int_\Omega |F(x)|^p\,dx\Big)^{1/p}.$$

If $F \in \mathcal{D}'(\Omega,\mathbb{R}^N)$ we can speak of its differential

$$DF = [\partial f^i/\partial x_j] \in \mathcal{D}'(\Omega,\mathbb{R}^{N\times N}).$$

Then F is said to be in the *Sobolev class* $W^{1,p}(\Omega,\mathbb{R}^N)$ provided $DF\in L^p(\Omega,\mathbb{R}^{N\times N})$. Let us emphasize explicitly that in this definition we do not require F itself to be in $L^p(\Omega,\mathbb{R}^N)$.

Clearly, $W^{1,p}(\Omega,\mathbb{R}^N)$ is a Banach space equipped with the seminorm

$$\|F\|_{1,p}=\Big(\int_\Omega |DF(x)|^p\,dx\Big)^{1/p}.$$

The following two operators will be of fundamental importance to the arguments presented later: the divergence operator

$$\operatorname{div}\colon \mathcal{D}'(\Omega,\mathbb{R}^N)\to\mathcal{D}'(\Omega,\mathbb{R})$$

defined by

$$\operatorname{div} f=\frac{\partial f^1}{\partial x_1}+\ldots+\frac{\partial f^N}{\partial x_N}\qquad\text{for } f=(f^1,\ldots,f^N),$$

and its formal adjoint, denoted by $\operatorname{curl}\colon\mathcal{D}'(\Omega,\mathbb{R}^N)\to\mathcal{D}'(\Omega,\mathbb{R}^{N\times N})$, which is a matrix distribution

$$\operatorname{curl} F=\Big[\frac{\partial f^i}{\partial x_j}-\frac{\partial f^j}{\partial x_i}\Big],\qquad i,j=1,\ldots,N.$$

Note that a vector field $F=(f^1,\ldots,f^N)$ of divergence zero and curl zero (irrotational field) satisfies the generalized Cauchy–Riemann system

$$\begin{cases}\partial f^1/\partial x_1+\ldots+\partial f^N/\partial x_N=0,\\ \partial f^i/\partial x_j=\partial f^j/\partial x_i,\qquad i,j=1,\ldots,N.\end{cases}\tag{3.1}$$

Locally, such a field F is the gradient of a harmonic function, which makes it a C^∞-smooth vector field in Ω. However, distributions which are only divergence free or curl free need not be even locally integrable. The duality between div and curl can be stated as

$$\int_\Omega\langle B(x),E(x)\rangle\,dx=0$$

whenever B,E are divergence free and curl free vector fields in $L^q(\Omega,\mathbb{R}^N)$ and $L^p(\Omega,\mathbb{R}^N)$ respectively, where $1\le p,q\le\infty$ is any Hölder conjugate pair.

Definition 3.1. A *div-curl couple* on Ω consists of a pair of distributions $\Phi=[B,E]$ with $\operatorname{div}B=0$ and $\operatorname{curl}E=0$.

A div-curl couple $\Phi=[B,E]$ which satisfies the equation $B=E$ consists of two copies of a vector field satisfying the system of the Cauchy–Riemann equations (3.1), and because of this we refer to such a couple as a *harmonic field*. Denote by $\mathcal{H}^p(\Omega,\mathbb{R}^{N\times N})$, $1\le p\le\infty$, the L^p-space of div-curl couples. This space is easily seen to be a closed subspace of $L^p(\Omega,\mathbb{R}^{N\times N})$. For $\Phi=[B,E]\in\mathcal{H}^p_{\mathrm{loc}}(\Omega,\mathbb{R}^{N\times N})$ we can introduce the norm

$$|\Phi|=(|B(x)|^2+|E(x)|^2)^{1/2}$$

and the Jacobian

$$J(x,\Phi)=\langle B(x),E(x)\rangle.$$

Clearly, we have $2J(x,\Phi)\le|\Phi|^2$, where equality occurs if and only if $B=E$.

Definition 3.2. A div-curl couple $\varPhi = [B, E] \in \mathcal{H}^p_{\mathrm{loc}}(\varOmega, \mathbb{R}^{N\times N})$ is called a K-*quasiharmonic field* with *distortion* $1 \le K = K(x) < \infty$ if

$$|\varPhi(x)|^2 \le (K(x) + K^{-1}(x))J(x,\varPhi) \quad \text{a.e.} \tag{3.2}$$

The distortion function $K = K(x)$ tells us how far $\varPhi$ is from a harmonic field. Precisely, inequality (3.2) yields

$$|\varPhi^-(x)| \le \frac{K(x)-1}{K(x)+1}\,|\varPhi^+(x)|$$

where the $\pm$ components of $\varPhi$ are defined by the rules

$$\varPhi^- = \frac{1}{2}(E-B), \qquad \varPhi^+ = \frac{1}{2}(E+B).$$

Hence, harmonic fields are precisely those with the vanishing minus component, corresponding to $K(x) \equiv 1$ (no distortion).

Example 3.3. Let $f = (f^1, \ldots, f^N) : \varOmega \to \mathbb{R}^N$ be a mapping whose coordinates f^i are in $W^{1,p_i}(\varOmega)$ where $1 < p_1, \ldots, p_N < \infty$. With f we associate two vector fields $E = \nabla f^1$ and $B = \nabla f^2 \times \ldots \times \nabla f^N$. The latter stands for the cross product of $N-1$ gradient fields in $\mathbb{R}^N$.

It is well known that $\operatorname{div} B = 0$ provided $1/p_2 + \ldots + 1/p_N \le 1$. The product $\langle B, E\rangle$ is none other than the Jacobian determinant of f, that is, $\langle B, E\rangle = \det(Df(x)) = J(x, f)$.

Thus the couple $\varPhi = [B, E]$ is a quasiharmonic field if and only if

$$|\nabla f^1|^2 + |\nabla f^2 \times \ldots \times \nabla f^N|^2 \le (K(x) + K^{-1}(x))\det(Df).$$

Example 3.4. Let $f : \varOmega \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \varOmega$ the function $\xi \in \mathbb{R}^N \to f(x,\xi)$ is convex. Denote by $f^*(x,\eta) = \sup\{\langle \eta, \xi\rangle - f(x,\xi); \xi \in \mathbb{R}^N\}$ the Young conjugate of $f(x,\cdot)$. Throughout this example we assume the quadratic growth and coercivity condition which we express by a single inequality

$$|\xi|^2 + |\eta|^2 \le (K + K^{-1})[f(x,\xi) + f^*(x,\eta)]$$

where $K \ge 1$. Let $u \in W^{1,2}(\varOmega)$ be a local minimum of the variational integral

$$I[v] = \int_\varOmega f(x, \nabla v)\,dx.$$

Precisely we mean that $I[u] = \min\{I[v] : v \in u + W^{1,2}_0(\varOmega)\}$. Consider the solution $B \in L^2(\varOmega, \mathbb{R}^N)$ of the dual problem in the sense of Ekeland–Temam [12]. That is, $\operatorname{div} B = 0$ in $\varOmega$ and

$$\int_\varOmega [\langle B, \nabla u\rangle - f^*(x, B)] = \max\Big\{\int_\varOmega [\langle X, \nabla u\rangle - f^*(x, X)] : X \in L^2(\varOmega, \mathbb{R}^N), \operatorname{div} X = 0\Big\}.$$

Then the extremality relation takes the form

$$\langle B, \nabla u\rangle = f(x, \nabla u) + f^*(x, B) \quad \text{a.e. in } \varOmega.$$

Setting $E = \nabla u$ we obtain the K-quasiharmonic field $[B, E]$, which satisfies the distortion inequality

$$|B|^2 + |E|^2 \le (K + K^{-1})\langle B, E\rangle.$$

Given a vector field $F = (f^1, \dots, f^N) \in L^p(\mathbb{R}^N, \mathbb{R}^N)$, consider the Poisson equation

$$F = \Delta U = (\Delta u^1, \dots, \Delta u^N) \tag{3.3}$$

for $U = (u^1, \dots, u^N) \in \mathcal{D}'(\mathbb{R}^N, \mathbb{R}^N)$. Equation (3.3) yields the div-curl (also known as Hodge) decomposition of F:

$$F = B + E$$

where

$$B = \Delta U + \nabla \operatorname{div} U, \qquad E = \nabla \operatorname{div} U.$$

These fields are easily seen to be divergence and curl free, respectively.

Explicit calculations are possible by means of the *Riesz transform*

$$\mathbf{R} : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N, \mathbb{R}^N)$$

where

$$(\mathbf{R}f)(x) = \frac{\Gamma\left(\frac{N+1}{2}\right)}{\pi^{\frac{N+1}{2}}} \int_{\mathbb{R}^N} \frac{(x-y)f(y)}{|x-y|^{N+1}}\, dy$$

for which the following identities hold:

$$\frac{\partial^2 U}{\partial x_i \partial x_j} = -\mathbf{R}_{i,j}(F) \in L^p(\mathbb{R}^n \times \mathbb{R}^n) \quad \text{for } i, j = 1, \dots, N$$

where $\mathbf{R}_{i,j} = \mathbf{R}_i \circ \mathbf{R}_j$ are the second order Riesz transforms.

Next, F can also be written as

$$F = \nabla(\operatorname{div} U) + \operatorname{div}(\operatorname{curl} U).$$

Note that the divergence of the matrix function $\operatorname{curl} U$ is a vector field whose coordinates are obtained by simply computing the divergence of the column vectors of this matrix.

For fixed $F \in L^p(\mathbb{R}^N, \mathbb{R}^N)$, we can also define an N-dimensional version of the Hilbert transform by

$$\mathbf{S}(F) = E - B.$$

Thus $\mathbf{S}$ acts as identity on gradient fields and as minus identity on divergence free vector fields. Let us list basic properties of the operator $\mathbf{S}$:

(i) $\mathbf{S}$ is an involution, that is, $\mathbf{S} \circ \mathbf{S} = I$.

(ii) $\mathbf{S}$ is self-adjoint, that is,

$$\int_{\mathbb{R}^N} \langle \mathbf{S}F, G \rangle = \int_{\mathbb{R}^N} \langle F, \mathbf{S}G \rangle$$

for $F \in L^p(\mathbb{R}^N, \mathbb{R}^N)$ and $G \in L^q(\mathbb{R}^N, \mathbb{R}^N)$, with $1 < p, q < \infty$, $p + q = pq$.

Thus, in particular

(iii) $\mathbf{S}$ is an isometry in $L^2(\mathbb{R}^N, \mathbb{R}^N)$.

The calculation of its p-norms remains an open problem, even in the case $N = 2$. A lot of implications in the regularity theory of PDEs would follow if the exact value of $\|\mathbf{S}\|_p$ could be established.

The setting presented in [21] is the one of elliptic complexes of the first order differential operators

$$\mathcal{D}'(\mathbb{R}^N,\mathbf{U}) \xrightarrow{\mathcal{P}} \mathcal{D}'(\mathbb{R}^N,\mathbf{V}) \xrightarrow{\mathcal{Q}} \mathcal{D}'(\mathbb{R}^N,\mathbf{W})$$

where $\mathbf{U},\mathbf{V}$ and $\mathbf{W}$ are finite-dimensional inner product spaces.

Such complexes are viewed, in many ways, as generalizations of the classical exact sequence of the gradient and rotation operator

$$\mathcal{D}'(\mathbb{R}^N,\mathbb{R}) \xrightarrow{\nabla} \mathcal{D}'(\mathbb{R}^N,\mathbb{R}^N) \xrightarrow{\mathrm{curl}} \mathcal{D}'(\mathbb{R}^N,\mathbb{R}^{N\times N}).$$

3.2. Elliptic complexes. Let $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ be finite-dimensional vector spaces over the field of real numbers. We assume that they are equipped with inner products.

We consider a sequence of differential operators of first order in N independent variables with constant coefficients

$$\mathcal{D}'(\mathbb{R}^N,\mathbf{U}) \xrightarrow{\mathcal{P}} \mathcal{D}'(\mathbb{R}^N,\mathbf{V}) \xrightarrow{\mathcal{Q}} \mathcal{D}'(\mathbb{R}^N,\mathbf{W}). \tag{3.4}$$

More precisely, if $u\in\mathcal{D}'(\mathbb{R}^N,\mathbf{U})$ and $v\in\mathcal{D}'(\mathbb{R}^N,\mathbf{V})$, then

$$\mathcal{P}u=\sum_{k=1}^N A_k\frac{\partial u}{\partial x_k},\qquad \mathcal{Q}v=\sum_{k=1}^N B_k\frac{\partial v}{\partial x_k}, \tag{3.5}$$

where $A_k\in L(\mathbf{U},\mathbf{V})$ and $B_k\in L(\mathbf{V},\mathbf{W})$ for $k=1,\dots,N$. The symbols $\mathcal{P}=\mathcal{P}(\xi)$ and $\mathcal{Q}=\mathcal{Q}(\xi)$ are linear functions in $\xi=(\xi_1,\dots,\xi_N)\in\mathbb{R}^N$ valued in $L(\mathbf{U},\mathbf{V})$ and in $L(\mathbf{V},\mathbf{W})$, respectively. They are given explicitly by

$$\mathcal{P}(\xi)=\sum_{k=1}^N \xi_k A_k,\qquad \mathcal{Q}(\xi)=\sum_{k=1}^N \xi_k B_k. \tag{3.6}$$

The complex (3.4) is said to be *elliptic* if the sequence of symbols

$$\mathbf{U}\xrightarrow{\mathcal{P}(\xi)}\mathbf{V}\xrightarrow{\mathcal{Q}(\xi)}\mathbf{W} \tag{3.7}$$

is *exact*, i.e.

$$\operatorname{im}\mathcal{P}(\xi)=\ker\mathcal{Q}(\xi)\quad\text{ for all }\xi\neq 0. \tag{3.8}$$

The dual sequence consists of the formal adjoint operators

$$\mathcal{D}'(\mathbb{R}^N,\mathbf{U}) \xleftarrow{\mathcal{P}^*} \mathcal{D}'(\mathbb{R}^N,\mathbf{V}) \xleftarrow{\mathcal{Q}^*} \mathcal{D}'(\mathbb{R}^N,\mathbf{W}), \tag{3.9}$$

$$\mathcal{P}^*v=-\sum_{k=1}^N A_k^*\frac{\partial v}{\partial x_k},\qquad \mathcal{Q}^*w=-\sum_{k=1}^N B_k^*\frac{\partial w}{\partial x_k}. \tag{3.10}$$

Since inner products on $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ have been given, the dual spaces $\mathbf{U}^*$, $\mathbf{V}^*$ and $\mathbf{W}^*$ are identified with $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$, respectively. The dual complex is elliptic if the original complex is.

Given an elliptic complex we define the associated *Laplace–Beltrami operator*

$$-\varDelta=\mathcal{P}\mathcal{P}^*+\mathcal{Q}^*\mathcal{Q}:\mathcal{D}'(\mathbb{R}^N,\mathbf{V})\to\mathcal{D}'(\mathbb{R}^N,\mathbf{V}). \tag{3.11}$$

Its symbol is a quadratic form with values in $L(\mathbf{V},\mathbf{V})$,

$$\varDelta(\xi)=\Big(\sum_{j=1}^N\xi_jA_j\Big)\circ\Big(\sum_{k=1}^N\xi_kA_k^*\Big)+\Big(\sum_{j=1}^N\xi_jB_j^*\Big)\circ\Big(\sum_{k=1}^N\xi_kB_k\Big)$$

$$= \sum_{j,k=1}^{N} \xi_j \xi_k (A_j A_k^* + B_j^* B_k)$$

$$= \sum_{j,k=1}^{N} \xi_j \xi_k (A_j A_k^* + A_k A_j^* + B_j^* B_k + B_k^* B_j).$$

If we fix an arbitrary vector field $F = (f^1, \ldots, f^N) \in L^2(\mathbb{R}^N, \mathbf{V})$, we can solve the Poisson equation

$$\Delta\varphi = F \tag{3.12}$$

for φ whose second derivatives are L^2-integrable on $\mathbb{R}^N$. As a matter of fact, these derivatives can be expressed in terms of F by using singular integrals. Indeed it is possible to prove that

$$\frac{\partial^2 \varphi}{\partial x_i \partial x_j} = \int_{\mathbb{R}^N} K_{ij}(x-y) F(y)\, dy \tag{3.13}$$

where $K_{ij}(x): \mathbf{V} \to \mathbf{V}$ are Calderón–Zygmund type singular integrands. The L^p-theory yields

$$\left\| \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right\|_p \le c_p \|F\|_p \quad \text{for } 1 < p < \infty. \tag{3.14}$$

Next observe that for every vector $v \in \mathbf{V}$, we have

$$\langle \Delta(\xi) v, v\rangle = \sum_{j,k} \xi_j \xi_k \langle A_j A_k^* v, v\rangle + \sum_{j,k} \xi_j \xi_k \langle B_j^* B_k v, v\rangle = \sum_{j,k} \xi_j \xi_k [\langle A_k^* v, A_j^* v\rangle + \langle B_k v, B_j v\rangle]$$

$$= \Big|\sum_j \xi_j A_j^* v\Big|^2 + \Big|\sum_j \xi_j B_j v\Big|^2 = |\mathcal{P}^*(\xi) v|^2 + |\mathcal{Q}(\xi) v|^2 \ge 0.$$

It is important to realize that equality occurs if and only if $v = 0$. Indeed,

$$\{\mathcal{P}^*(\xi) v = 0 \text{ and } \mathcal{Q}(\xi) v = 0\} \;\Leftrightarrow\; \{v \in \ker \mathcal{Q}(\xi) \text{ and } v \in \ker \mathcal{P}^*(\xi)\}.$$

By ellipticity of the complex (3.4), $\ker \mathcal{Q}(\xi) = \operatorname{im} \mathcal{P}(\xi)$. It is well known in algebra that $\operatorname{im} \mathcal{P}(\xi)$ is orthogonal to $\ker \mathcal{P}^*(\xi)$, therefore the vector v, being orthogonal to itself, is zero. Summarizing, the operator $\Delta(\xi): \mathbf{V} \to \mathbf{V}$ is positive for $\xi \ne 0$.

In analogy with the div-curl decomposition of a vector field, the Poisson equation

$$F = \Delta\varphi \tag{3.15}$$

for $\varphi \in W^{2,p}(\mathbb{R}^N, \mathbf{V})$, $1 < p < \infty$, yields a decomposition of F,

$$F = \mathcal{P} u + \mathcal{Q}^* w \tag{3.16}$$

where $u = \mathcal{P}^* \varphi \in W^{1,p}(\mathbb{R}^N, \mathbf{U})$ and $w = \mathcal{Q}\varphi \in W^{1,p}(\mathbb{R}^N, \mathbf{W})$. In view of (3.14) we have the following estimate:

$$\|\nabla u\|_p + \|\nabla w\|_p \le c_p \|F\|_p. \tag{3.17}$$

LEMMA 3.5 (orthogonality property). *For $\alpha \in W^{1,p}(\mathbb{R}^N, \mathbf{U})$ and $\beta \in W^{1,q}(\mathbb{R}^N, \mathbf{W})$, $1/p + 1/q = 1$, the vector fields $\mathcal{P}\alpha \in L^p(\mathbb{R}^n, \mathbf{V})$ and $\mathcal{Q}^*\beta \in L^q(\mathbb{R}^N, \mathbf{V})$ are orthogonal.*

Proof. Using the equality $\operatorname{im} \mathcal{P} = \ker \mathcal{Q}$, we have

$$\int \langle \mathcal{P}\alpha, \mathcal{Q}^*\beta\rangle = \int \langle \mathcal{Q}\mathcal{P}\alpha, \beta\rangle = 0$$

if $\alpha \in W^{2,p}(\mathbb{R}^N, \mathbf{U})$ and $\beta \in W^{1,q}(\mathbb{R}^N, \mathbf{W})$, with $1/p + 1/q = 1$. Since $W^{2,p}(\mathbb{R}^N, \mathbf{U})$ is dense in $W^{1,p}(\mathbb{R}^N, \mathbf{U})$, the lemma follows by an approximation.

By this lemma, we are able to prove

THEOREM 3.6. *Each vector field $F \in L^p(\mathbb{R}^n, \mathbf{V})$, $1 < p < \infty$, admits a unique decomposition*

$$F = \mathcal{P}u + \mathcal{Q}^* w \tag{3.18}$$

with $u \in W^{1,p}(\mathbb{R}^n, \mathbf{U})$ and $w \in W^{1,p}(\mathbb{R}^n, \mathbf{W})$. In symbols,

$$L^p(\mathbb{R}^n, \mathbf{V}) = \mathcal{P}W^{1,p}(\mathbb{R}^n, \mathbf{U}) \oplus \mathcal{Q}^* W^{1,p}(\mathbb{R}^n, \mathbf{W}). \tag{3.19}$$

We also have a uniform bound for the components,

$$\|\mathcal{P}u\|_p + \|\mathcal{Q}^* w\|_p \le C_p \|F\|_p. \tag{3.20}$$

REMARK 3.7. Let us emphasize explicitly that u, w need not be unique, only their images $\mathcal{P}u$ and $\mathcal{Q}^* w$ are unique.

In case of the elliptic complex

$$\mathcal{D}'(\mathbb{R}^N, \Lambda) \xrightarrow{d} \mathcal{D}'(\mathbb{R}^n, \Lambda) \xrightarrow{d} \mathcal{D}'(\mathbb{R}^N, \Lambda)$$

formula (3.18) provides us with the familiar decomposition of a differential form as a sum of an exact and coexact form (no harmonic fields in $\mathbb{R}^N$). Because of this analogy we call (3.18) the *Hodge decomposition* associated with the given elliptic complex.

It is also possible to develop a theory of Hodge decomposition on domains $\Omega \subset \mathbb{R}^N$. But this requires some regularity of Ω if one wants to go beyond L^2-theory. The interested reader can consult [24] and the references given there.

The following inequalities allow us to improve regularity of some distributions without affecting their image under the operator $\mathcal{Q}$ or $\mathcal{P}^*$, respectively.

LEMMA 3.8. *For each distribution $F \in \mathcal{D}'(\mathbb{R}^N, \mathbf{V})$ with $\mathcal{Q}F \in L^2(\mathbb{R}^N, \mathbf{W})$, there exists $F_0 \in \ker \mathcal{Q}$ such that $F - F_0 \in W^{1,2}(\mathbb{R}^N, \mathbf{V})$ and we have a uniform bound*

$$\|F - F_0\|_{1,2} \le C \|\mathcal{Q}F\|_2.$$

We argue similarly for the dual statement.

LEMMA 3.9. *For each distribution $F \in \mathcal{D}'(\mathbb{R}^N, \mathbf{V})$ with $\mathcal{P}^* F \in L^2(\mathbb{R}^N, \mathbf{W})$, there exists F_0 such that $\mathcal{P}^* F_0 = 0$ and $F - F_0 \in W^{1,2}(\mathbb{R}^N, \mathbf{V})$ and we have a uniform bound*

$$\|F - F_0\|_{1,2} \le C \|\mathcal{P}^* F\|_2.$$

Proof of Lemma 3.8. By Hodge decomposition,

$$F = \mathcal{P}\mathcal{P}^* \varphi + \mathcal{Q}^* \mathcal{Q}\varphi.$$

Consider $F_0 = F - \mathcal{Q}^* \mathcal{Q}\varphi$. Then $\mathcal{Q}F_0 = 0$ and $F - F_0 = \mathcal{Q}^* \mathcal{Q}\varphi \in L^2(\mathbb{R}^n, \mathbf{V})$. Hence $\mathcal{Q}(F - F_0) = \mathcal{Q}F \in L^2$. Applying the Fourier transform we find that $\mathcal{Q}(\xi)\widehat{\Phi}(\xi) \in L^2$ and $\mathcal{P}^*(\xi)\widehat{\Phi}(\xi) \in L^2$, where we have set $\Phi = F - F_0$.

Let us observe the following inequality:

$$|\mathcal{Q}(\xi) y| + |\mathcal{P}^*(\xi) y| \ge c_0 |\xi| \cdot |y|$$

with a positive constant c_0. In fact, suppose that $|\xi| = 1$, $|y| = 1$ (by homogeneity). If $\mathcal{Q}(\xi)y = 0$ and $\mathcal{P}^*(\xi)y = 0$, then $y \in \ker \mathcal{Q}(\xi) \cap \ker \mathcal{P}^*(\xi)$. This implies that $y = 0$, contradicting the assumption that y was a unit vector.

Applying the above inequality to $\widehat{\Phi}(\xi)$ we have

$$c_0|\xi| \cdot |\widehat{\Phi}(\xi)| \leq |\mathcal{Q}(\xi)\widehat{\Phi}(\xi)| + |\mathcal{P}^*(\xi)\widehat{\Phi}(\xi)|.$$

This implies $|\xi|\widehat{\Phi}(\xi) \in L^2$. Hence $\Phi \in W^{1,2}(\mathbb{R}^N, \mathbf{V})$ and

$$\|\Phi\|_{1,2} \leq c(N)\|\mathcal{Q}F\|_2.$$

Let us mention that certain L^p-variants of the above inequalities are also available.

Guided by [39], we define the *Hilbert transform* $\mathbf{S}\colon L^p(\mathbb{R}^N, \mathbf{V}) \to L^p(\mathbb{R}^N, \mathbf{V})$ by the rule

$$\mathbf{S}F = \mathcal{P}u - \mathcal{Q}^*w \tag{3.21}$$

with the following properties:

(i) $\mathbf{S}$ is an involution;
(ii) $\mathbf{S}$ is self-adjoint;
(iii) $\mathbf{S}$ is an isometry in $L^2(\mathbb{R}^N, \mathbb{R}^N)$.

Let us stress again that one fundamental question of interest in the L^p-theory of PDEs concerns the sharp constant in the inequality

$$\|\mathbf{S}F\|_p \leq A_p\|F\|_p, \qquad 1 < p < \infty. \tag{3.22}$$

Several examples suggest the following conjecture of Iwaniec [34]:

$$A_p = \max\left\{p - 1, \frac{1}{p-1}\right\}. \tag{3.23}$$

The interested reader is referred to Burkholder's work [6] to find that inequality (3.22) with constant (3.23) would follow if one proves that

$$\mathcal{E}[F] = \int [A_p|\mathbf{S}F| - |F|][|\mathbf{S}F| + |F|]^{p-1} \geq 0. \tag{3.24}$$

A reason for preferring (3.24) to the inequality (3.22) is that the latter functional is convex in the so-called singular directions (see Section 4 for the definition). In light of the conjecture at (3.23) it may very well be that $\mathcal{E}$ is also quasiconvex and, consequently, inequality (3.24) would follow.

3.3. Elliptic couples and quasiharmonic fields. Following the definitions in [39] we study an extension of the notion of div-curl couples. An *elliptic couple* is the pair

$$\mathcal{F} = [\mathcal{P}\alpha, \mathcal{Q}^*\beta]$$

where $\alpha \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbf{U})$ and $\beta \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbf{W})$. Here Ω is any domain in $\mathbb{R}^N$, $N \geq 2$, and $1 < p < \infty$. The L^p-space of elliptic couples $[\mathcal{P}\alpha, \mathcal{Q}^*\beta]$ with $\alpha \in W^{1,p}(\Omega, \mathbf{U})$, and $\beta \in W^{1,p}(\Omega, \mathbf{W})$, denoted by $\mathcal{E}^p(\Omega, \mathbf{V} \times \mathbf{V})$, $1 < p < \infty$, is a closed subspace of $L^p(\Omega, \mathbf{V} \times \mathbf{V})$.

Furthermore, we introduce the norm

$$|\mathcal{F}(x)|^2 = |\mathcal{P}\alpha|^2 + |\mathcal{Q}^*\beta|^2$$

and the *Jacobian*

$$J(x,\mathcal{F}) = \langle A(x), B(x)\rangle_{\mathbf{V}} = \langle \mathcal{P}\alpha, \mathcal{Q}^*\beta\rangle$$

for $x \in \Omega$. Then the following, rather obvious, relation can be viewed as an analogue of the Hadamard inequality for determinants:

$$2J(x,\mathcal{F}) \le |\mathcal{F}(x)|^2.$$

DEFINITION 3.10. An elliptic couple $\mathcal{F} = [\mathcal{P}\alpha, \mathcal{Q}^*\beta]$ is called *K-quasiharmonic* with $1 \le K = K(x) < \infty$ if

$$|\mathcal{F}(x)|^2 \le \mathcal{K}(x) J(x,\mathcal{F})$$

where $\mathcal{K}(x) = K(x) + K^{-1}(x) \ge 2$.

This inequality yields

$$|\mathcal{F}^-(x)| \le \frac{K(x)-1}{K(x)+1}\,|\mathcal{F}^+(x)|$$

where the $\pm$ components of $\mathcal{F}$ are defined by the rules

$$\mathcal{F}^- = \tfrac{1}{2}(\mathcal{P}\alpha - \mathcal{Q}^*\beta), \qquad \mathcal{F}^+ = \tfrac{1}{2}(\mathcal{P}\alpha + \mathcal{Q}^*\beta).$$

The following result on higher integrability for the Jacobian is desired.

THEOREM 3.11. *Let $\mathcal{F} \in L^2(\Omega, \mathbf{V}\times\mathbf{V})$ be an elliptic couple. Then $J(x,\mathcal{F}) \in \mathcal{H}^1_{\mathrm{loc}}(\Omega)$.*

We only sketch the proof as it is similar to the one in [7].

Proof. Fix an arbitrary subdomain Ω' compactly contained in Ω, and fix an arbitrary $\eta \in C_0^\infty(\Omega)$ which is equal to 1 on Ω'. For each test function $\varphi \in C_0^\infty(\Omega')$ we shall estimate the integral of the Jacobian

$$\int_\Omega \varphi(x) J(x,\mathcal{F})\,dx = \int_\Omega \langle \varphi\mathcal{P}\alpha, \mathcal{Q}^*\beta\rangle = \int_{\mathbb{R}^N} \langle \varphi\mathcal{P}(\eta\alpha), \mathcal{Q}^*(\eta\beta)\rangle$$

because η equals 1 on the support of φ.

Now, we use Hodge decomposition in the entire space $\mathbb{R}^N$ to write

$$\varphi\mathcal{P}(\eta\alpha) = \mathcal{P}\alpha' + \mathcal{Q}^*\beta'.$$

Observe that the component $\mathcal{Q}^*\beta'$ can be expressed as a singular integral of $\varphi\mathcal{P}(\eta\alpha)$, say $\mathcal{Q}^*\beta' = \mathbf{B}[\varphi\mathcal{P}(\eta\alpha)]$. The singular integral operator $\mathbf{B} : L^p(\mathbb{R}^N, \mathbf{V}) \to L^p(\mathbb{R}^N, \mathbf{V})$, projection onto $\mathcal{Q}^* W^{1,p}(\mathbb{R}^N, \mathbf{W}) \subset L^p(\mathbb{R}^N, \mathbf{V})$, is bounded for all $1 < p < \infty$. It is also important to observe that $\mathbf{B}$ vanishes on the subspace $\mathcal{P}W^{1,p}(\mathbb{R}^n, \mathbf{U})$. Therefore, we can look at $\mathcal{Q}^*\beta'$ as the image of $\mathcal{P}(\eta\alpha)$ under the commutator of $\mathbf{B}$ with the multiplication by φ, namely

$$\mathcal{Q}^*\beta' = (\mathbf{B}\varphi - \varphi\mathbf{B})(\mathcal{P}(\eta\alpha)).$$

Next, we apply the celebrated commutator result of R. Coifman, R. Rochberg and G. Weiss [8], which implies that

$$\|\mathcal{Q}^*\beta'\|_2 \le C(N)\|\varphi\|_{\mathrm{BMO}}\|\mathcal{P}(\eta\alpha)\|_2.$$

Since $\mathcal{P}\alpha'$ is orthogonal to $\mathcal{Q}^*(\eta\beta)$, by Hölder's inequality we obtain

$$\int_{\mathbb{R}^n} \varphi(x) J(x,\mathcal{F})\,dx = \int_{\mathbb{R}^n} \langle \mathcal{P}\alpha', \mathcal{Q}^*(\eta\beta)\rangle + \int_{\mathbb{R}^n} \langle \mathcal{Q}^*\beta', \mathcal{Q}^*(\eta\beta)\rangle \le \|\mathcal{Q}^*\beta'\|_2 \|\mathcal{Q}^*(\eta\beta)\|_2$$

$$\leq C(N)\|\varphi\|_{\mathrm{BMO}}\|\mathcal{P}(\eta\alpha)\|_2\|\mathcal{Q}^*(\eta\beta)\|_2 \leq c(N,\eta)\|\varphi\|_{\mathrm{BMO}}\|\mathcal{F}\|_2^2.$$

In conclusion,

$$\int_\Omega \varphi(x)J(x,\mathcal{F})dx \leq C(N,\eta)\|\varphi\|_{\mathrm{BMO}}\|\mathcal{F}\|_2^2.$$

In view of the BMO-$\mathcal{H}^1$ duality it follows that $J(x,\mathcal{F}) \in \mathcal{H}^1_{\mathrm{loc}}(\Omega)$. We also have the following local bounds:

$$\|J(x,\mathcal{F})\|_{\mathcal{H}^1(\Omega')} \leq C_{\Omega'}\|\mathcal{F}\|_2^2.$$

Further, if $J(x,\mathcal{F}) \geq 0$, by Theorem of E. Stein [56] we find that the Jacobian belongs to the Zygmund class $L\log L_{\mathrm{loc}}(\Omega)$.

Just as in the theory of quasiconformal mappings, constructions of quasiharmonic fields rely on limiting processes. Therefore it is of interest to know that such fields are closed under weak convergence. The following theorem addresses this issue.

THEOREM 3.12. *Let $\mathcal{F}_\nu$ be a sequence of quasiharmonic fields converging to $\mathcal{F}$ weakly in $L^2(\Omega,\mathbf{V}\times\mathbf{V})$ and suppose that the distortion functions $\mathcal{K}_\nu$ converge to $\mathcal{K}$ weakly in $L^1(\Omega)$. Then $\mathcal{F}$ is a quasiharmonic field of distortion $\mathcal{K}$.*

For the proof we will need the following two lemmas.

LEMMA 3.13 (lower semicontinuity of the norm). *For every $\eta \in L^\infty(\Omega)$, $\eta \geq 0$ and $\mathcal{F}_\nu$ converging to $\mathcal{F}$ weakly in $L^2(\Omega,\mathbf{V}\times\mathbf{V})$,*

$$\int_\Omega \eta(x)|\mathcal{F}(x)|\,dx \leq \liminf_{\nu\to\infty}\int_\Omega \eta(x)|\mathcal{F}_\nu(x)|\,dx.$$

LEMMA 3.14 (weak continuity of the Jacobian). *Under the assumptions of Theorem* 3.12, *for every $\lambda \in L^\infty_\bullet(\Omega)$,*

$$\int_\Omega \lambda(x)J(x,\mathcal{F})\,dx = \lim_{\nu\to\infty}\int_\Omega \lambda(x)J(x,\mathcal{F}_\nu)\,dx.$$

Hereafter $L^\infty_\bullet(\Omega)$ denotes the space of bounded functions supported in a compact subset of Ω. The interested reader can find the proof of the two lemmas above in [21].

Proof of Theorem 3.12. Fix $\varepsilon > 0$ and $\delta > 0$. Then

$$\frac{|\mathcal{F}_\nu|^2}{\delta+\varepsilon|\mathcal{F}|+J(x,\mathcal{F}_\nu)} \leq \mathcal{K}_\nu(x).$$

Algebraic calculations reveal that

$$\frac{|\mathcal{F}_\nu|^2}{\delta+\varepsilon|\mathcal{F}|+J(x,\mathcal{F}_\nu)} - \frac{|\mathcal{F}|^2}{\delta+\varepsilon|\mathcal{F}|+J(x,\mathcal{F})} \geq \frac{2|\mathcal{F}|(|\mathcal{F}_\nu|-|\mathcal{F}|)}{\delta+\varepsilon|\mathcal{F}|+J(x,\mathcal{F})} - \frac{|\mathcal{F}|^2[J(x,\mathcal{F}_\nu)-J(x,\mathcal{F})]}{(\delta+\varepsilon|\mathcal{F}|+J(x,\mathcal{F}))^2}.$$

For every nonnegative test function $\varphi \in L^\infty_\bullet(\Omega)$, we can write

$$\int_\Omega \frac{\varphi|\mathcal{F}_\nu|^2}{\delta+\varepsilon|\mathcal{F}|+J(x,\mathcal{F}_\nu)}\,dx - \int_\Omega \frac{\varphi|\mathcal{F}|^2}{\delta+\varepsilon|\mathcal{F}|+J(x,\mathcal{F})}\,dx \geq \int_\Omega \frac{2\varphi|\mathcal{F}|(|\mathcal{F}_\nu|-|\mathcal{F}|)}{\delta+\varepsilon|\mathcal{F}|+J(x,\mathcal{F})}\,dx - \int_\Omega \frac{\varphi|\mathcal{F}|^2[J(x,\mathcal{F}_\nu)-J(x,\mathcal{F})]}{(\delta+\varepsilon|\mathcal{F}|+J(x,\mathcal{F}))^2}\,dx.$$

Applying the lemmas above this estimate yields

$$\int_{\Omega} \frac{\varphi|\mathcal{F}(x)|^2}{\delta+\varepsilon|\mathcal{F}(x)|+J(x,\mathcal{F})}\,dx \le \liminf_{\nu\to\infty} \int_{\Omega} \frac{\varphi|\mathcal{F}_\nu(x)|^2}{\delta+\varepsilon|\mathcal{F}(x)|+J(x,\mathcal{F}_\nu)}\,dx,$$

and from the distortion inequality

$$\le \liminf_{\nu\to\infty} \int_{\Omega} \varphi\mathcal{K}_\nu(x)dx = \int_{\Omega} \varphi(x)\mathcal{K}(x)\,dx.$$

By the monotone convergence theorem we can pass to the limit as ε goes to zero:

$$\int_{\Omega} \frac{\varphi|\mathcal{F}(x)|^2}{\delta+J(x,\mathcal{F})}\,dx \le \int_{\Omega} \varphi(x)\mathcal{K}(x)\,dx.$$

Since φ was arbitrary and nonnegative in $L^\infty_\bullet(\Omega)$, it follows that

$$\frac{|\mathcal{F}(x)|^2}{\delta+J(x,\mathcal{F})} \le \mathcal{K}(x) \quad \text{a.e.}$$

Hence

$$|\mathcal{F}(x)|^2 \le \mathcal{K}(x)[\delta+J(x,\mathcal{F})].$$

The last inequality holds for every $\delta>0$, so for $\delta=0$ as well:

$$|\mathcal{F}(x)|^2 \le \mathcal{K}(x)J(x,\mathcal{F}) \quad \text{a.e.},$$

completing the proof.

Let us conclude the present section with one more definition (see [37]). Consider a short elliptic complex

$$\mathcal{D}'(\mathbb{R}^N,\mathbf{U}) \overset{\mathcal{P}}{\to} \mathcal{D}'(\mathbb{R}^N,\mathbf{V}) \overset{\mathcal{Q}}{\to} \mathcal{D}'(\mathbb{R}^N,\mathbf{W})$$

of first order differential operators $\mathcal{P}$ and $\mathcal{Q}$ and its dual

$$\mathcal{D}'(\mathbb{R}^N,\mathbf{U}) \overset{\mathcal{P}^*}{\leftarrow} \mathcal{D}'(\mathbb{R}^N,\mathbf{V}) \overset{\mathcal{Q}^*}{\leftarrow} \mathcal{D}'(\mathbb{R}^N,\mathbf{W}).$$

The *p-harmonic couple* associated with such a sequence is a pair $\mathcal{F}=[B,E]$ with $B\in\ker\mathcal{P}^*$ and $E\in\ker\mathcal{Q}$ such that

$$\frac{|E|^p}{p}+\frac{|B|^q}{q} \le \mathcal{K}(x)\langle B,E\rangle$$

where $1<p,q<\infty$ are Hölder conjugate exponents and the distortion function $\mathcal{K}=\mathcal{K}(x)\ge 1$ satisfies

$$\int_{\Omega} e^{\gamma\mathcal{K}(x)}dx<\infty \qquad \text{for some constant } \gamma>0.$$

We say that $\mathcal{K}$ lies in the exponential class $\mathrm{Exp}_\gamma(\Omega)$. The right spaces for E and B are $L^p\log^\alpha L(\Omega,\mathbf{V})$ and $L^q\log^\alpha L(\Omega,\mathbf{V})$, respectively.

3.4. Variational integrals. This section is concerned with variational integrals defined on elliptic couples. The integrals in question take the form

$$I[\mathcal{F}] = \int_{\mathbb{R}^N} f(X,Y) \qquad \text{for } \mathcal{F}=[X,Y]\in L^p(\mathbb{R}^N,\mathbf{V}\times\mathbf{V}).$$

We assume here that the integrand $f:\mathbf{V}\times\mathbf{V}\to\mathbb{R}$ is at least continuous.

Here are three basic definitions adopted from the calculus of variations (see for example [10]). Observe that the notation $W^{1,\infty}_{\bullet}(\mathbb{R}^N, \mathbf{V})$ for the space of Lipschitz $\mathbf{V}$-valued functions with compact support in $\Omega \subset \mathbb{R}^N$ is being used in the definition below.

Definition 3.15. f is said to be *quasiconvex* if for any constant vectors $A, B \in \mathbf{V}$ we have

$$\int_{\mathbb{R}^N} [f(A + \mathcal{P}\alpha, B + \mathcal{Q}^*\beta) - f(A, B)]\, dx \geq 0$$

whenever $\alpha \in W^{1,\infty}_{\bullet}(\mathbb{R}^N, \mathbf{V})$ and $\beta \in W^{1,\infty}_{\bullet}(\mathbb{R}^N, \mathbf{W})$.

The next notion seems to be a nice extension of rank-one convexity.

Definition 3.16. We say that f is *convex in singular directions* if the real variable function

$$t \mapsto f(A + tX, B + tY)$$

is convex whenever $A, B, X, Y \in \mathbf{V}$ and X is orthogonal to Y in $\mathbf{V}$.

Finally, we give

Definition 3.17. f is said to be *polyconvex* if it can be expressed as

$$f(X, Y) = g(X, Y, \langle X, Y\rangle)$$

where $g : \mathbf{V} \times \mathbf{V} \times \mathbb{R} \to \mathbb{R}$ is convex.

In the recent years a fairly large amount of work has been done trying to understand all possible connections between these notions of convexity.

It is not difficult to see that polyconvexity implies quasiconvexity. Indeed, given $A, B \in \mathbf{V}$ and given arbitrary functions $\alpha \in W^{1,\infty}_{\bullet}(D, \mathbf{U})$ and $\beta \in W^{1,\infty}_{\bullet}(D, \mathbf{W})$, supported in a bounded domain D, we can use Jensen's inequality to obtain

$$\begin{aligned}
&\frac{1}{|D|} \int_{\mathbb{R}^N} [f(A + \mathcal{P}\alpha, B + \mathcal{Q}^*\beta) - f(A, B)]\, dx \\
&\qquad = \fint_D [g(A + \mathcal{P}\alpha, B + \mathcal{Q}^*\beta, \langle A + \mathcal{P}\alpha, B + \mathcal{Q}^*\beta\rangle) - g(A, B, \langle A, B\rangle)]\, dx \\
&\qquad \geq g\Big[\fint_D (A + \mathcal{P}\alpha, B + \mathcal{Q}^*\beta, \langle A + \mathcal{P}\alpha, B + \mathcal{Q}^*\beta\rangle)\Big] - g(A, B, \langle A, B\rangle) \\
&\qquad = g\Big(A + \fint_D \mathcal{P}\alpha, B + \fint_D \mathcal{Q}^*\beta, \langle A, B\rangle + \fint_D \langle A, \mathcal{Q}^*\beta\rangle + \fint_D \langle \mathcal{P}\alpha, B\rangle + \fint_D \langle \mathcal{P}\alpha, \mathcal{Q}^*\beta\rangle\Big) \\
&\qquad\qquad - g(A, B, \langle A, B\rangle) = 0.
\end{aligned}$$

The first four integral averages vanish, by the divergence theorem, the latter vanishes due to L^2-orthogonality of $\mathcal{P}\alpha$ and $\mathcal{Q}^*\beta$ (cf. Lemma 3.5). Thus f is quasiconvex.

It is worth pointing out that without an additional hypothesis about the elliptic complex quasiconvexity need not imply convexity in singular directions, in contrast to the classical setting.

Precisely, we have

THEOREM 3.18. *Suppose that the elliptic complex* (3.4) *satisfies the condition*

$$\bigcup_{|\xi|=1} \ker Q(\xi) = \mathbf{V}.$$

Then every quasiconvex function is convex in singular directions.

For the proof we need to show the inequality

$$f(\lambda\Phi + (1-\lambda)\Psi) \le \lambda f(\Phi) + (1-\lambda) f(\Psi)$$

whenever $0 < \lambda < 1$ and $\Phi - \Psi = [X, Y]$ with X orthogonal to Y in $\mathbf{V}$.

We can argue with the aid of the following

LEMMA 3.19. *There exist* $u \in W^{1,\infty}(\mathbb{R}^N, \mathbf{U})$, $w \in W^{1,\infty}(\mathbb{R}^N, \mathbf{W})$ *and a partition* $\mathbb{R}^N = \Omega \cup \Omega'$ *into disjoint measurable subsets such that*

$$[\mathcal{P}u, \mathcal{Q}^* w] = [(1-\lambda)\chi_\Omega - \lambda\chi_{\Omega'}](\Phi - \Psi),$$

$$\lim_{R\to\infty} \frac{|\Omega \cap B_R|}{|B_R|} = \lambda, \tag{3.25}$$

and therefore

$$\lim_{R\to\infty} \frac{|\Omega' \cap B_R|}{|B_R|} = 1 - \lambda. \tag{3.26}$$

Proof of Theorem 3.18. Consider concentric balls $B_R \subset B_{R+1}$ and a cut-off function $\eta \in C_0^\infty(B_{R+1})$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ on B_R and $|\nabla\eta(x)| \le 2$ in $\mathbb{R}^N$. The functions $\alpha = \eta u$ and $\beta = \eta w$ are Lipschitz with support in B_{R+1}, and therefore can be used as the test functions in the definition of quasiconvexity. Accordingly,

$$|B_{R+1}| f(\lambda\Phi + (1-\lambda)\Psi) \le \int_{B_{R+1}} f(\lambda\Phi + (1-\lambda)\Psi + \mathcal{F})$$

where $\mathcal{F} = [\mathcal{P}\alpha, \mathcal{Q}^*\beta]$ is an elliptic couple. We split the integral as

$$\int_{B_{R+1}} = \int_{\Omega\cap B_R} + \int_{\Omega'\cap B_R} + \int_{B_{R+1}-B_R}.$$

It is important to observe that

$$\mathcal{F} = \begin{cases} (1-\lambda)(\Phi - \Psi) & \text{on } \Omega \cap B_R, \\ -\lambda(\Phi - \Psi) & \text{on } \Omega' \cap B_R, \end{cases}$$

and $\|\mathcal{F}\|_{L^\infty(\mathbb{R}^N)} < \infty$. Hence, we obtain

$$|B_{R+1}| f(\lambda\Phi + (1-\lambda)\Psi) \le |\Omega \cap B_R| f(\Phi) + |\Omega' \cap B_R| f(\Psi) + c|B_{R+1} - B_R|$$

where c is a constant independent of R.

Finally, dividing the inequality by $|B_R|$ and letting R go to infinity, we conclude with the desired inequality

$$f(\lambda\Phi + (1-\lambda)\Psi) \le \lambda f(\Phi) + (1-\lambda) f(\Psi)$$

by the density relations (3.25) and (3.26).

4. Jacobian determinants

4.1. Introduction. Let Ω be a domain in $\mathbb{R}^N$, $N \geq 2$, and $f = (f^1, \dots, f^N) : \Omega \to \mathbb{R}^N$ a mapping of Sobolev class $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, $1 \leq p < \infty$. We denote by $Df(x) : \mathbb{R}^N \to \mathbb{R}^N$ the differential matrix and by $J = J(x, f) = \det Df(x)$ the Jacobian determinant of f. We say that f is an *orientation preserving mapping* if $J(x, f) \geq 0$ almost everywhere in Ω.

Determinants of differential matrices occur in many different contexts, such as the geometric function theory, calculus of variations, nonlinear elasticity, etc. because of their improved integrability properties. A natural question now arises: under what conditions on f is the Jacobian function locally integrable?

By Hadamard's inequality

$$|J(x, f)| \leq |Df^1(x)| \dots |Df^N(x)|$$

it follows that J is integrable as soon as $f \in W^{1,N}(\Omega, \mathbb{R}^N)$.

Stefan Müller [50] was the first to observe that under just one condition, that $J(x, f)$ does not change sign in Ω, the degree of integrability of the Jacobian of f is better than that of $|Df(x)|^N$. More precisely, Müller showed that the Jacobian of an orientation preserving mapping $f \in W^{1,N}(\Omega, \mathbb{R}^N)$ belongs to the Zygmund class $L \log L(K)$ for each compact set $K \subset \Omega$.

In its most general form, the result can be stated as follows:

$$\int_K J(x, f) \log\left(e + \frac{|J(x, f)|}{J_K}\right) dx \leq C(N, K) \int_\Omega |Df(x)|^N \, dx$$

where J_K denotes the integral mean of the Jacobian over K.

Remark 4.1. Müller's theorem is sharp for more than one reason. A counterexample by J. M. Ball and F. Murat (see [4]) shows that the condition on the sign of J cannot be removed and that the compact set K cannot be replaced by the set Ω; a counterexample by Müller himself shows that the Orlicz function $P(t) = t \log(e + t)$ cannot be replaced by a function $Q(t)$ such that $Q(t)/t \log(e + t) \to \infty$ as $t \to \infty$.

In [29], L. Greco and T. Iwaniec showed a somewhat stronger estimate by proving the local L^1-integrability of the function $J \log |Df|$.

The following theorem, obtained by T. Iwaniec and C. Sbordone and published in [38], can be viewed as dual to that of Müller:

$$\left.\begin{array}{l} |Df(x)|^N \in L \log^{-1} L(\Omega) \\ J \geq 0 \end{array}\right\} \Rightarrow J \in L^1_{\mathrm{loc}}(\Omega). \tag{4.1}$$

A precise estimate reads as follows:

$$\int_K J(x, f) \, dx \leq C(N, K) \int_\Omega \frac{|Df(x)|^N}{\log\big(e + \frac{|Df(x)|}{|Df(x)|_\Omega}\big)} \, dx$$

where K is any compact subset of Ω and $|Df|_\Omega$ denotes the integral mean of $|Df|$ over Ω. This was the first time an estimate below the natural Sobolev exponent (the dimension N) had been achieved.

Inspired by [50] and [38], H. Brezis, N. Fusco and C. Sbordone showed how to interpolate between these two results (see [5]). They proved that

$$\left.\begin{array}{l}|Df(x)|^N \in L\log^{-\alpha} L(\Omega)\\ J \geq 0\end{array}\right\} \Rightarrow J \in L\log^{1-\alpha} L_{\mathrm{loc}}(\Omega) \tag{4.2}$$

for $0 \leq \alpha \leq 1$ and gave the estimate

$$\int_K J(x,f)\log^{1-\alpha}\left(e + \frac{|J(x,f)|}{J_K}\right)dx \leq C(N,K)\int_\Omega \frac{|Df(x)|^N}{\log^\alpha\left(e + \frac{|Df(x)|^N}{|Df|_\Omega}\right)}\,dx.$$

Let Φ be a nondecreasing function on $[0,\infty]$ which is locally absolutely continuous and satisfies the following conditions:

(i) there exist constants $a > 0$ and $t_0 > 0$ such that $\Phi(t) \geq at/\log(e+t)$ for all $t \geq t_0$;

(ii) there exist constants $\alpha > 1$ and $k > 0$ such that $\Phi(\alpha t) \leq k\Phi(t)$ for all $t \geq 0$;

(iii) $\Phi'(t)/t$ is integrable in a neighborhood of zero.

Consider the function

$$\Theta(t) = t\int_0^t \frac{\Phi'(s)}{s}\,ds.$$

In [48], G. Moscariello proved that

$$\left.\begin{array}{l}|Df(x)|^N \in L^\Phi(\Omega)\\ J \geq 0\end{array}\right\} \Rightarrow J \in L^\Theta_{\mathrm{loc}}(\Omega). \tag{4.3}$$

This result is a generalization of the one by H. Brezis, N. Fusco and C. Sbordone. Indeed, if $\Phi(t) = t/\log^\alpha(e+t)$ for all $t \geq 1$, with $0 < \alpha < 1$, we see that

$$\Theta(t) \sim t\log^{1-\alpha}(e+t)$$

where $\sim$ denotes the usual equivalence notation between convex real functions. Moreover, (4.3) is stronger than (4.1). Indeed, if $\Phi(t) = t/\log(e+t)$, $t \geq 1$, then by easy calculations one can deduce that

$$\Theta(t) \sim t\log(\log(e+t))$$

and so that $L^\Theta = L\log\log L$. In other words

$$\left.\begin{array}{l}|Df(x)|^N \in L\log^{-1} L(\Omega)\\ J \geq 0\end{array}\right\} \Rightarrow J \in L\log\log L_{\mathrm{loc}}(\Omega). \tag{4.4}$$

Let $\phi_1, \ldots \phi_m : [0,\infty] \to [0,\infty]$ be log-convex functions such that:

(a) $\phi_i \succ t^{p_i}\log^{-1}(e+t)$, $i = 1,\ldots m$, for some $1 < p_1,\ldots,p_m < \infty$, $1/p_1 + \ldots + 1/p_m = 1$.

(b) There exist exponents $\alpha_i \in (1,p_i)$ with $1/\alpha_1 + \ldots + 1/\alpha_m < 1 + 1/n$, for which the functions $t \mapsto t^{-\alpha_i}\Phi_i(t)$ are increasing, $i = 1,\ldots,m$.

Next let Φ denote the log-convex function determined from the formula

$$(t\Phi')^{-1} = (t\Phi_1')^{-1}\ldots(t\Phi_m')^{-1}$$

and let Ψ be defined by

$$\Phi(t) = \Psi(t) - \int_0^t \frac{\Psi(s)}{s}\, ds. \tag{4.5}$$

Let us notice that condition (a) yields

$$\Phi(t) \succcurlyeq \frac{t}{\log(e+t)},$$

which, in turn, reveals that

$$\Psi(t) \succcurlyeq t \log\log(e+t).$$

In [30], L. Greco, T. Iwaniec and G. Moscariello proved that

$$\left.\begin{array}{l} |Df(x)|^N \in L^{\Phi}(\Omega) \\ J \geq 0 \end{array}\right\} \;\Rightarrow\; J \in L^{\Psi}_{\mathrm{loc}}(\Omega). \tag{4.6}$$

An analogous theorem in the concave case, that is, $\Psi(t) \prec t$, also holds (see [30]).

Suppose that $\Psi : [0,\infty] \to [0,\infty]$ can be represented by the following Stieltjes integral

$$\Psi(t) = \int_0^a (1-\lambda) t^{1-\lambda} dh(\lambda)$$

where $h : [0,a] \to [0,\infty]$, $0 < a < 1/(n+1)$, is an arbitrary nondecreasing function. Thus Ψ is concave and $\Psi(t) \prec t$. To each such Ψ there corresponds a log-convex function $\Phi : [0,\infty] \to [0,\infty]$ defined by a formula analogous to (4.5)

$$\Phi(t) = -\Psi(t) + \int_0^t \frac{\Psi(s)}{s}\, ds.$$

We write it in terms of h:

$$\Phi(t) = \int_0^a \lambda t^{1-\lambda} dh(\lambda).$$

In particular $\Phi(t) \preccurlyeq \Psi(t) \prec t$. Therefore, Φ and Ψ are concave and

$$\left.\begin{array}{l} |Df(x)|^N \in L^{\Phi}(\Omega) \\ J \geq 0 \end{array}\right\} \;\Rightarrow\; J \in L^{\Psi}_{\mathrm{loc}}(\Omega). \tag{4.7}$$

REMARK 4.2. It is of interest to know whether an improvement of integrability of the Jacobian truly takes place. To see this, we introduce the quotient

$$L(t) = \frac{\Psi(t)}{\Phi(t)} \geq 1$$

which measures the degree of the improvement. An easy computation shows that

$$\Psi(t) \sim t \exp\left[\int_1^t \frac{ds}{sL(s)}\right] \quad \text{when } \Psi \text{ is convex}$$

and

$$\Psi(t) \sim t \exp\left[-\int_1^t \frac{ds}{sL(s)}\right] \quad \text{when } \Psi \text{ is concave.}$$

It is clear that L cannot grow too fast. Indeed, in order to guarantee $\Psi(t) \succcurlyeq t$ and $\Psi(t) \preccurlyeq t$, respectively, we should have

$$\int_1^\infty \frac{ds}{sL(s)} = \infty. \tag{4.8}$$

Roughly speaking, every function L, continuously increasing and satisfying (4.8), represents as an improvement quotient Ψ/Φ. Of course, growth conditions imposed on Φ and Ψ yield other, rather minor, restrictions for L.

Observe that the case $t \preccurlyeq \Psi(t) \prec t \log\log t$ is not setted by the theorems above. However, further studies have filled this gap; see for example [42].

Theorem 4.3. *Let $f : \Omega \to \mathbb{R}^N$ be an orientation preserving mapping in the Sobolev class $W^{1,1}(\Omega, \mathbb{R}^N)$ with $|Df|^N \in L^\Phi(\Omega)$ where Φ is an Orlicz function satisfying the divergence condition*

$$\int_1^\infty \frac{\Phi(t)}{t^2}\, dt = \infty. \tag{4.9}$$

Then $J(x,f)$ belongs to $L^\Psi_{\mathrm{loc}}(\Omega)$ with

$$\Psi(t) = \Phi(t) + t\int_0^t \frac{\Phi(s)}{s^2}\, ds.$$

Thus, in particular, $J(x,f)$ is locally integrable.

Let us point out here that the condition (4.9) is also necessary in order to deduce the local integrability of the Jacobian. It is in this way that we consider the last result as optimal in the category of Orlicz–Sobolev spaces.

4.2. Distributional Jacobian. One of the most important concepts that occur in the theory of nonlinear differential forms and their applications to the modern theory of mappings is the distributional Jacobian. This pertains to the situations in which we impose (a priori) lesser degree of integrability of the differential with the aid of integration by parts.

Let $f = (f^1, \ldots, f^N)$ be a mapping of Sobolev class $W^{1,N}(\Omega, \mathbb{R}^N)$. According to the differential forms theory, the volume form $J(x,f)dx$ can be expressed as the wedge product of the linear forms $df^1, \ldots, df^N$

$$J(x,f)dx = df^1 \wedge \ldots \wedge df^N. \tag{4.10}$$

Now, by the Stokes theorem

$$\begin{aligned}\int_\Omega \varphi J(x,f)\, dx &= \int_\Omega \varphi(df^1 \wedge \ldots \wedge df^N) \\ &= -\int_\Omega f^i\, df^1 \wedge \ldots \wedge df^{i-1} \wedge d\varphi \wedge df^{i+1} \wedge \ldots \wedge df^N\end{aligned} \tag{4.11}$$

for each $i = 1, \ldots, N$ and for all test functions $\varphi \in C_0^\infty(\Omega)$.

The *distributional Jacobian*, denoted by $\mathcal{J}_f$, is a Schwartz distribution acting on a test function $\varphi \in C_0^\infty(\Omega)$ by the rule

$$\mathcal{J}_f[\varphi] = -\int_\Omega f^i\, df^1 \wedge \ldots \wedge df^{i-1} \wedge d\varphi \wedge df^{i+1} \wedge \ldots \wedge df^N \tag{4.12}$$

where different choices of indices $1, \ldots, N$ yield the same value of the integral.

It is clear that for the definition above, in view of the Sobolev imbedding theorem, one only needs that $|Df|^N \in L^{N/(N+1)}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, or equivalently, that $f \in W^{1,N^2/(N+1)}(\Omega, \mathbb{R}^N)$.

On the other hand, the identity (4.10) suggests investigating the partial products $df^{i_1} \wedge \ldots \wedge df^{i_l}$ corresponding to l-tuples $1 \le i_1 < \ldots < i_l \le N$. Note that

$$df^{i_1} \wedge \ldots \wedge df^{i_l} = \sum_{1 \le j_1 < \ldots < j_l \le N} \frac{\partial(f^{i_1}, \ldots, f^{i_l})}{\partial(x_{j_1}, \ldots, x_{j_l})}\, dx_{j_1} \wedge \ldots \wedge dx_{j_l}. \tag{4.13}$$

Let us denote the ordered collection (say in the lexicographical order) of all such wedge products by

$$\textstyle\bigwedge^l f = \{df^{i_l} \wedge \ldots \wedge df^{i_l} : 1 \le i_1 < \ldots < i_l \le N\} \tag{4.14}$$

and identify it with the $\binom{N}{l} \times \binom{N}{l}$ matrix of all $l \times l$-subdeterminants

$$\textstyle\bigwedge^l f = \left[\dfrac{\partial f^I}{\partial x_J}\right]_{\substack{I=(i_1,\ldots,i_l)\\ J=(j_1,\ldots,j_l)}}. \tag{4.15}$$

Thus $\bigwedge^N f = J(x, f)$ and $\bigwedge^{N-1}$ is none other than the matrix $D^\sharp f$ of cofactors of Df. We shall make use of the Hilbert–Schmidt norm in the space $\mathbb{R}^{\binom{N}{l}\times\binom{N}{l}}$ of such matrices.

$$\Big\|\textstyle\bigwedge^l f\Big\|^2 = \displaystyle\sum_{1 \le i_1 < \ldots < i_l \le N} |df^{i_1} \wedge \ldots \wedge df^{i_l}|^2 = \sum_{\substack{1 \le i_1 < \ldots < i_l \le N\\ 1 \le j_1 < \ldots < j_l \le N}} \left|\frac{\partial(f^{i_1}, \ldots, f^{i_l})}{\partial(x_{j_1}, \ldots, x_{j_l})}\right|^2. \tag{4.16}$$

It is important to realize that the l-forms at (4.14) are exact if $f \in W^{1,l}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. Precisely, we have $\| \bigwedge^l f\| \in L^1_{\mathrm{loc}}(\Omega)$ and for each $k = 1, \ldots, l$,

$$df^{i_1} \wedge \ldots \wedge df^{i_l} = d\omega_k, \qquad \omega_k = (-1)^{k-1} f^{i_k}\, df^{i_1} \wedge \ldots \wedge df^{i_{k-1}} \wedge df^{i_{k+1}} \wedge \ldots \wedge df^{i_l}.$$

By Sobolev's imbedding $f \in L^{Nl/(N-l)}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. Hence, ω_k is an $(l-1)$-form of class $L^{N/(N-1)}_{\mathrm{loc}}(\Omega)$. Various algebraic bounds for subdeterminants follow from the Hadamard-type inequality

$$\binom{N}{l}^k \Big\|\textstyle\bigwedge^k f\Big\|^{2l} \le \displaystyle\binom{N}{k}^l \Big\|\textstyle\bigwedge^l f\Big\|^{2k}, \qquad 1 \le l \le k \le N. \tag{4.17}$$

Thus, in particular

$$|J(x, f)| \le \binom{N}{l}^{-N/(2l)} \Big\|\textstyle\bigwedge^l f\Big\|^{N/l},$$

$$|D^\sharp f| \le C(N) \Big\|\textstyle\bigwedge^l f\Big\|^{(N-1)/l}, \qquad 1 \le l < N.$$

It is immediate that the distributional Jacobian can be defined whenever

$$|f|\,|D^\sharp f| \in L^1_{\mathrm{loc}}(\Omega).$$

Precisely we have

$$|\mathcal{J}_f[\varphi]| \le \int_\Omega |\nabla\varphi|\,|f|\,|D^\sharp f| < \infty. \tag{4.18}$$

We record for later use the following elementary identity:

$$\mathcal{J}_f[\varphi^N] = \int_\Omega [\varphi^N(x)\,J(x,f) - J(x,\varphi f)]\,dx \tag{4.19}$$

whenever $\varphi \in C_0^\infty(\Omega)$ and $f \in W^{1,N-1}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ are such that

$$|f|\,|D^\sharp f| \in L^1_{\mathrm{loc}}(\Omega). \tag{4.20}$$

In the remainder of this section we assume f to be in $W^{1,N^2/(N+1)}(\Omega,\mathbb{R}^N)$, or in $W^{1,N-1}(\Omega,\mathbb{R}^N)$ with $|D^\sharp f| \in L^q(\Omega)$, $q = (N^2-N)/(N^2-N-1)$. In either case condition (4.20) is fulfilled. It is generally a nontrivial question how the distributional Jacobian relates to the pointwise Jacobian $J(x,f)$. First of all, it is clear that the Jacobian has to be locally integrable. Moreover, the identity between the distributional Jacobian and the pointwise Jacobian is valid whenever $f \in W^{1,N}$. If we assume any lesser degree of integrability, the Jacobian need not be locally integrable. Even more, identity (4.19) may fail if the Jacobian is a priori integrable (see [3, 4]). In [49], S. Müller proved a conjecture of J. Ball that if $\mathcal{J}_f \in L^1$, then $\mathcal{J}_f = \det Df$. Furthermore, in [38], the validity of this identity is proved under the assumptions $|Df(x)|^N \in L\log^{-1}L(\Omega)$ and $J \ge 0$. In [27], L. Greco obtained the same identity for f an orientation preserving mapping with $|Df(x)|$ belonging to a class of functions, called Σ^N, which is strictly larger than $L^N/\log L$. The reader should notice that in Müller's result it is assumed that the distribution $\mathcal{J}_f$ is represented by a locally integrable function. This rather strong assumption is practically impossible to verify without integration by parts, a vicious circle. In this sense the result by L. Greco is more practical.

Now fix a nonnegative $\Phi \in C_0^\infty(\mathbb{B})$ with integral 1, and define $\Phi_t(x) = t^{-n}\Phi(t^{-1}x)$, $t > 0$. Given any $\mathcal{J} \in \mathcal{D}'(\Omega)$ we can speak of the convolution $\mathcal{J} * \Phi_t$, defined for $0 < t < \operatorname{dist}(x,\partial\Omega)$ by the rule

$$(\mathcal{J} * \Phi_t)(x) = \mathcal{J}[\Phi_t(\cdot - x)]. \tag{4.21}$$

This is legitimate because the function $\varphi(y) = \Phi_t(x-y)$ lies in $C_0^\infty(\Omega)$. It should be reasonably evident that

$$\mathcal{J} * \Phi_t \to \mathcal{J} \quad \text{in } \mathcal{D}'(\Omega), \text{ as } t \to 0.$$

Precisely this means

$$\mathcal{J}[\eta] = \lim_{t\to 0} \int_\Omega \eta(x)(\mathcal{J} * \Phi_t)\,dx \tag{4.22}$$

for every $\eta \in C_0^\infty(\Omega)$. The following useful approximation result holds.

Proposition 4.4. *For almost every $x \in \Omega$ we have*

$$J(x,f) = \lim_{t\to 0}(\mathcal{J}_f * \Phi_t)(x). \tag{4.23}$$

Let us end this section with the following estimate:

LEMMA 4.5. *Given $f \in W^{1,N^2/(N+1)}(\Omega, \mathbb{R}^N)$ and a test function $\varphi \in C_0^\infty(Q)$ supported in a cube $Q \subset \Omega$. Then*

$$|\mathcal{J}_f[\varphi]| \le C(N)\|\nabla\varphi\|_\infty |Q|^{\frac{N+1}{N}} \Big(\fint_Q |Df(x)|^{\frac{N^2}{N+1}}\, dx\Big)^{\frac{N+1}{N}}.$$

Proof. Let f_Q denote the L^1-mean of f over the cube Q. We have

$$\begin{aligned}
|\mathcal{J}_f[\varphi]| = |\mathcal{J}_{f-f_Q}[\varphi]| &\le \int_Q |\nabla\varphi|\, |f - f_Q|\, |Df|^{N-1} \\
&\le C(n)\, \|\nabla\varphi\|_\infty\, |Q| \Big(\fint_Q |f - f_Q|^{N^2}\Big)^{\frac{1}{N^2}} \Big(\fint_Q |Df|^{\frac{N^2}{N+1}}\Big)^{\frac{N^2-1}{N^2}} \\
&\le C(N)\|\nabla\varphi\|_\infty |Q|^{\frac{N+1}{N}} \Big(\fint_Q |Df|^{\frac{N^2}{N+1}}\Big)^{\frac{N+1}{N}}
\end{aligned}$$

by the Sobolev imbedding theorem.

4.3. Estimates of Jacobians by subdeterminants. It has become clear that in order to formulate and fully benefit from higher integrability phenomena one must study mappings in the Orlicz–Sobolev spaces $W^{1,\Phi}(\Omega, \mathbb{R}^N)$, but not too far from the natural class $W^{1,N}(\Omega, \mathbb{R}^N)$. Recall that $W^{1,\Phi}(\Omega, \mathbb{R}^N)$ consists of vector fields $f = (f^1, \dots, f^N)$ whose coordinate functions have gradient in the Orlicz space $L^\Phi(\Omega)$. It is obvious that $J(x, f)$ is integrable whenever $|Df|^N \in L^1(\Omega)$ or $|D^\sharp f|^{N/(N-1)} \in L^1(\Omega)$. We wish to investigate whether $L^1(\Omega)$ can be replaced by a slightly larger Orlicz space $L^P(\Omega)$. It involves very little loss of generality to assume that

$$L^1(\Omega) \subset L^P(\Omega). \tag{4.24}$$

This latter inclusion is guaranteed if P is concave, or simply $\sup t^{-1}P(t) < \infty$. However, the critical assumption throughout this section will be the following *divergence condition*:

$$\int_1^\infty \frac{P(s)\, ds}{s^2} = \infty, \tag{4.25}$$

which yields information about the growth of P at infinity. Examples that we have in mind are furnished by the iterated logarithms

$$P(t) = \frac{t}{\log(e + t)\log\log(e^e + t)\cdots\log\cdot\cdot\log(e^{e^{\cdot^{\cdot}}} + t)}. \tag{4.26}$$

Moreover, in order to define the distributional Jacobian it suffices to have $|D^\sharp f|$ in the space $L^q(\Omega)$, $q = (N^2 - N)/(N^2 - N - 1)$. Clearly, $1 < q < N/(N - 1)$ for $N > 2$. As a matter of fact our standing assumption (in Theorem 4.6) will be that $|D^\sharp f|^{N/(N-1)} \in L^P(\Omega)$. Practically this condition is stronger than $|D^\sharp f| \in L^q(\Omega)$, but not always. To fill this gap we really need that $P(t) \ge c \cdot t^s$, $s = (N^2 - 2N + 1)/(N^2 - N - 1)$, for large values of t. Another condition on P will be needed in the proof of Theorem 4.6, namely $[t^{(1-N)/N}P(t)]' \ge 0$. For esthetical reasons we condense all of it into one hypothesis

$$[t^{-1}P(t)]' \le 0 \le [t^{-s}P(t)]', \qquad s = \frac{N^2 - 2N + 1}{N^2 - N - 1}. \tag{4.27}$$

Such a hypothesis does not affect the behaviour of P near ∞, and therefore, we refer to it as a technical assumption.

THEOREM 4.6. *Let* $f \in W^{1,N-1}(\Omega, \mathbb{R}^N)$, $N > 2$, *be an orientation preserving* (*reserving*) *mapping such that*

$$|D^\sharp f|^{\frac{N}{N-1}} \in L^P(\Omega) \tag{4.28}$$

where P *satisfies* (4.25) *and* (4.27). *Then the Jacobian determinant of* f *is locally integrable and obeys the rule of integration by parts*

$$\int_\Omega \varphi(x) J(x,f)\,dx = -\int_\Omega df^1 \wedge \ldots \wedge df^{i-1} \wedge f^i d\varphi \wedge df^{i+1} \wedge \ldots \wedge df^N =: \mathcal{J}_f[\varphi] \tag{4.29}$$

for all indices $i = 1, \ldots, n$ *and test functions* $\varphi \in C_0^\infty(\Omega)$.

The case $P(t) = t$ has been treated in [51].

The following theorem is a refinement of some earlier results [28, 40, 36].

THEOREM 4.7. *Assume*, *in addition to the above properties of the Orlicz function*, *that the function*

$$t \mapsto t^{-\frac{N}{N+1}} P(t) \tag{4.30}$$

is increasing. Let $f = (f^1, \ldots, f^N) : \Omega \to \mathbb{R}^N$ *be an orientation preserving* (*reversing*) *map*, *with*

$$|Df|^N \in L^P(\Omega). \tag{4.31}$$

Then the Jacobian determinant is locally integrable and satisfies (4.29).

In [20] we demonstrate that both Theorems 4.7 and 4.6 are sharp, that is, they fail if the integral at (4.25) converges. Our approach relies on the effective interplay between familiar results and classical tools such as Whitney cubes, maximal functions and elementary integration theory. Even the isoperimetric inequality, of fundamental importance to us, is used here only for smooth mappings.

4.3.1. *Whitney cubes.* An *N-rectangle* $R \subset \mathbb{R}^N$ is a Cartesian product of N intervals

$$\begin{aligned} R &= (a_1, b_1] \times \ldots \times (a_n, b_n] \\ &= \{x = (x_1, \ldots, x_N) : a_\nu < x_\nu \le b_\nu \text{ for } \nu = 1, \ldots, N\}. \end{aligned} \tag{4.32}$$

One property such rectangles have is that any intersection of a finite number of rectangles is either empty or a rectangle again. A *cube* in $\mathbb{R}^N$ with side $s > 0$ is simply a rectangle R such that $b_i - a_i = s$ for $i = 1, \ldots, N$. To every integer k and a lattice point $\mathbf{j} = (j_1, \ldots, j_N) \in \mathbb{Z} \times \ldots \times \mathbb{Z}$ there corresponds a *dyadic cube*

$$Q = Q_{\mathbf{j}}^k = \{x \in \mathbb{R}^N : 2^k j_\nu < x_\nu \le 2^k + 2^k j_\nu \text{ for } \nu = 1, \ldots, N\}.$$

Dyadic cubes are very useful for constructing various disjoint covers. Any two dyadic cubes are either disjoint or one of them contains the other. This brings us to the well known Whitney decomposition.

Lemma 4.8. *Let F be a non-empty closed set in $\mathbb{R}^N$ and Ω its complement. There exists a disjoint collection $\{Q_1, Q_2, \ldots\}$ of dyadic cubes such that*

$$\Omega = \bigcup_{i=1}^{\infty} Q_i, \tag{4.33}$$

$$\operatorname{diam} Q_i \le \operatorname{dist}(Q_i, F) \le 4 \operatorname{diam} Q_i. \tag{4.34}$$

4.3.2. *Isoperimetric inequality.* Our proof of Theorem 4.6 relies on local estimates similar to those in Lemma 4.5, but with cofactors replacing the differential matrix. A device for establishing such estimates is the isoperimetric inequality. The familiar geometric form of this inequality reads

$$N^{N-1}\omega_{N-1}|U|^{N-1} \le |\partial U|^N \tag{4.35}$$

where $|U|$ stands for the volume of a region $U \subset \mathbb{R}^N$ while $|\partial U|$ is its $(N-1)$-dimensional surface area. Now, if $f : R \to U$ is a smooth diffeomorphism of a "regular" domain $R \subset \mathbb{R}^N$ onto U then $|U| = \int_R J(x,f)\,dx$, while $|\partial U|$ is dominated by $\int_{\partial R} |D^\sharp f(x)|\,dx$. In this way we obtain what is known as the integral form of the isoperimetric inequality:

$$\Big|\int_R J(x,f)\,dx\Big| \le C(N)\Big(\int_{\partial R} |D^\sharp f(x)|\,dx\Big)^{\frac{N}{N-1}} \tag{4.36}$$

with $C(N) = (N \sqrt[N-1]{\omega_{N-1}})^{-1}$. The point to make here is that (4.36) remains valid for all smooth mappings $f : R \to \mathbb{R}^N$, not necessarily diffeomorphisms.

We shall confine ourselves to the following less general but precise statement.

Lemma 4.9. *Let $f : \Omega \to \mathbb{R}^N$ be a C^∞-smooth mapping and $R \subset \Omega$ a closed N-rectangle. Then inequality* (4.36) *holds with some constant $C(N)$ depending only on the dimension.*

In the proof of Theorem 4.6 we will be dealing with Whitney's cubes, as described by Lemma 4.8, and a smooth mapping $f \in C^\infty(\Omega, \mathbb{R}^N)$. In order to estimate the integral

$$\int_\Omega J(x,f)\,dx = \sum_{i=1}^{\infty} \int_{Q_i} J(x,f)\,dx$$

in terms of the cofactors of f one would naturally try to use isoperimetric inequalities

$$\Big|\int_{Q_i} J(x,f)\,dx\Big| \le C(N)\Big(\int_{\partial Q_i} |D^\sharp f(x)|\,dx\Big)^{\frac{N}{N-1}}.$$

In general, unfortunately, we cannot control the boundary integrals by the volume integrals. The way out of difficulty is to expand slightly the cubes and choose most favourable ones, the ones with minimal boundary integral. Here is the precise construction of such cubes.

Consider concentric cubes $Q_i \subset \lambda Q_i \subset Q_i^* = (5/4)Q_i$ with the factor λ varying from 1 to 5/4. As the function $\lambda \mapsto \int_{\partial\lambda Q_i} |D^\sharp f|$ is continuous we may choose a concentric cube, denoted by $\Box_i$, such that

$$\int_{\partial \Box_i} |D^\sharp f(x)|\,dx \le \int_{\partial(\lambda Q_i)} |D^\sharp f(x)|\,dx$$

for all $1 \le \lambda \le 5/4$. Integrating with respect to the parameter λ, by Fubini's theorem, we have

$$\int_{\partial\Box_i} |D^\sharp f(x)|\, dx \le 4|Q_i|^{-\frac{1}{N}} \int_{Q_i^*\setminus Q_i} |D^\sharp f(x)|\, dx.$$

We summarize this construction in the following

LEMMA 4.10. *Given Whitney's decomposition $\Omega = \bigcup_{i=1}^{\infty} Q_i$ and $f \in C^\infty(\Omega, \mathbb{R}^N)$, there exist concentric cubes $Q_i \subset \Box_i \subset Q_i^* \subset 7NQ_i$ such that*

$$\Big(\int_{\partial\Box_i} |D^\sharp f(x)|\, dx\Big)^{\frac{N}{N-1}} \le C(n)|Q_i|\Big(\fint_{7NQ_i} |D^\sharp f(x)|\, dx\Big)^{\frac{N}{N-1}}$$

for all $i = 1, 2, \dots$

Throughout this section it will be required that a mapping $f \in W^{1,N-1}(\mathbb{R}^N, \mathbb{R}^N)$ has compact support. For abbreviation, we introduce the function

$$h = |D^\sharp f| \in L^1(\mathbb{R}^N) \tag{4.37}$$

which controls all the cofactors of $Df(x)$. Recall that the Hardy–Littlewood maximal function of h belongs to the Marcinkiewicz space:

$$Mh \in \text{weak-}L^1(\mathbb{R}^N). \tag{4.38}$$

We have the pointwise inequality for the Jacobian

$$|J(x,f)| \le |D^\sharp f(x)|^{\frac{N}{N-1}} \le [Mh(x)]^{\frac{N}{N-1}}. \tag{4.39}$$

The following estimate is crucial:

THEOREM 4.11. *For all but a countable number of parameters $t > 0$, we have*

$$\begin{aligned} \Big|\int_{Mh\le 2t} J(x,f)\, dx\Big| &\le C(N) t^{\frac{N}{N-1}} |\{x \in \mathbb{R}^N : Mh > 2t\}| \\ &\le C(N) t^{\frac{1}{N}} \int_{h>t} h(x)\, dx. \end{aligned} \tag{4.40}$$

The parameters for which this inequality holds are precisely those which satisfy the equation

$$|\{x \in \mathbb{R}^N : Mh(x) = t\}| = 0. \tag{4.41}$$

4.4. Proof of Theorem 4.7. We shall make use of the following lemma:

LEMMA 4.12. *Suppose $P : [0,\infty) \to [0,\infty)$ is continuously differentiable and satisfies*

$$\int_A^\infty \frac{P(s)ds}{s^2} = \infty \tag{4.42}$$

and

$$[t^{-1} P(t^\alpha)]' \ge 0 \tag{4.43}$$

for all $t \ge A$, where $A \ge 1$ and $\alpha > 1$ are given numbers. Let $u : X \to \mathbb{R}$ be a measurable function on a σ-finite measure space X such that

$$\int_X P(|u|^\alpha) < \infty. \tag{4.44}$$

Then

$$\inf_{t\geq A} t^{\alpha-1} \int_{|u|>t} |u| = 0. \tag{4.45}$$

Fix a nonnegative test function $\eta \in C_0^\infty(\Omega)$ equal to 1 on the support of φ. It is clear that the mapping

$$\widetilde{f} = (\varphi f^1, \eta f^2, \ldots, \eta f^N) \tag{4.46}$$

lies in the Sobolev space $W^{1,1}(\mathbb{R}^N, \mathbb{R}^N)$. As a matter of fact we have $|D\widetilde{f}|^N \in L^P(\mathbb{R}^N)$. Indeed, $|D\widetilde{f}|^N \leq C(N)|Df|^N + C(N)|f|^N$ where $|f|^N \in L^1(\Omega) \subset L^P(\Omega)$, by (4.24).

Condition (4.30) ensures that

$$\widetilde{g} = |D\widetilde{f}|^p \in L^1(\mathbb{R}^N) \qquad \text{for } p = \frac{N^2}{N+1}.$$

This justifies the use of the inequality

$$\Big| \int_{M\tilde{g}\leq 2t} J(x,\widetilde{f})\,dx \Big| \leq C(N) t^{\frac{1}{N}} \int_{\tilde{g}>t} \widetilde{g}(x)\,dx \tag{4.47}$$

for all but a countable number of the parameters $t > 0$.

Next, we apply Lemma 4.12 with $\alpha = (N+1)/N$ and $u = \widetilde{g}$ to infer that for some $A \geq 1$

$$\inf_{t\geq A} t^{\frac{1}{N}} \int_{\tilde{g}>t} \widetilde{g}(x)\,dx = 0. \tag{4.48}$$

Combining this fact with inequality (4.47) yields

$$\lim_{t\to\infty} \inf \Big| \int_{M\tilde{g}\leq 2t} J(x,\widetilde{f})\,dx \Big| = 0. \tag{4.49}$$

The rest of the proof is a simple application of the monotone convergence theorem. To this end, we split the Jacobian determinant as

$$\begin{aligned} J(x,\widetilde{f})\,dx &= d\varphi f^1 \wedge d\eta f^2 \wedge \ldots \wedge d\eta f^N = d\varphi f^1 \wedge df^2 \wedge \ldots \wedge df^N \\ &= \varphi(x) J(x,f)\,dx + f^1\, d\varphi \wedge df^2 \wedge \ldots \wedge df^N. \end{aligned}$$

Observe that

$$|f^1\, d\varphi \wedge df^2 \wedge \ldots \wedge df^N| \leq |\nabla\varphi|\,|f|\,|Df|^{N-1} \in L^1(\Omega).$$

It is at this point that we use $J(x,f) \geq 0$, precisely to ensure that the function $t \mapsto \int_{M\tilde{g}\leq 2t} \varphi(x) J(x,f)\,dx$ is increasing, and therefore, has a limit at infinity

$$\lim_{t\to\infty} \int_{M\tilde{g}\leq 2t} \varphi(x) J(x,f)\,dx = -\int_{\mathbb{R}^N} f^1\, d\varphi \wedge df^2 \wedge \ldots \wedge df^N = \mathcal{J}_f[\varphi].$$

Passing to the limit in the domain of integration we infer that $J(x,f)$ is locally integrable, and we obtain the identity

$$\int_{\Omega} \varphi(x) J(x,f)\,dx = \mathcal{J}_f[\varphi]. \tag{4.50}$$

Once we know that $J(x,f)$ is locally integrable, formula (4.50) remains valid for all test functions $\varphi \in C_0^\infty(\Omega)$, not necessarily nonnegative. We again can move the limit under the domain of integration, this time by using the Lebesgue dominated convergence theorem.

4.5. Proof of Theorem 4.6. Fix a nonnegative test function $\eta \in C_0^\infty(\Omega)$ equal to 1 on the support of φ. It is clear that the mapping

$$\widetilde{f} = (\varphi f^1, \eta f^2, \dots, \eta f^N)$$

lies in the Sobolev space $W^{1,N-1}(\mathbb{R}^N, \mathbb{R}^N)$. Indeed,

$$|D\widetilde{f}|^{N-1} \le C(N)|Df|^{N-1} + C(N)|f|^{N-1}. \tag{4.51}$$

Regarding the cofactors of $D\widetilde{f}$ we observe that

$$\begin{aligned} |D^\sharp \widetilde{f}| &\le C(N)|d(\eta f^2) \wedge \ldots \wedge d(\eta f^N)| \\ &\quad + C(N)\sum_{i=2}^{\infty} |d(\varphi f^1) \wedge df^2 \wedge \ldots \wedge df^{i-1} \wedge df^{i+1} \wedge \ldots \wedge df^N|. \end{aligned} \tag{4.52}$$

The first term takes the form

$$\begin{aligned} &(f^2 d\eta + \eta df^2) \wedge \ldots \wedge (f^N d\eta + \eta df^N) \\ &\quad = \eta^{N-1} df^2 \wedge \ldots \wedge df^N + \eta^{N-2} \sum_{i=2}^{N} df^2 \wedge \ldots \wedge df^{i-1} \wedge f^i d\eta \wedge df^{i+1} \wedge \ldots \wedge df^N \end{aligned}$$

because the other possible terms in this expansion vanish, due to the identity $d\eta \wedge d\eta = 0$. Therefore, the first term in (4.52) is dominated by

$$|\eta|^{N-1}|D^\sharp f| + |\eta|^{N-2}|d\eta|\,|f|\,|Df|^{N-2}.$$

The second term is easily seen to be bounded by

$$|\varphi|\,|D^\sharp f| + |d\varphi|\,|f|\,|Df|^{N-2}.$$

Summarizing, we obtain the inequality

$$|D^\sharp \widetilde{f}| \le C(|\varphi| + |\eta|^{N-1})|D^\sharp f| + C(|d\varphi| + |d\eta|\,|\eta|^{N-2})|f|\,|Df|^{N-2}$$

with C depending only on the dimension. On the right hand side the first term belongs to $L^1(\Omega)$ while the second lies in $L^{N/(N-1)}_{\mathrm{loc}} \subset L^1_{\mathrm{loc}}(\Omega)$. Since $\widetilde{f}$ has compact support we see that the function $\widetilde{h}(x) = |D^\sharp \widetilde{f}|$ lies in $L^1(\mathbb{R}^N)$. With these preliminaries we can apply inequality (4.40):

$$\Big| \int_{M\widetilde{h} \le 2t} J(x, \widetilde{f})\, dx \Big| \le C(n) t^{\frac{1}{N-1}} \int_{\widetilde{h} > t} \widetilde{h}(x)\, dx \tag{4.53}$$

for all but a countable number of the parameters $t > 0$. Next we observe that

$$|\widetilde{h}|^{\frac{N}{N-1}} \in L^P(\mathbb{R}^N).$$

To see this we begin with the inequality

$$|\widetilde{h}|^{\frac{N}{N-1}} \le A\,|D^\sharp f|^{\frac{N}{N-1}} + A\,|f|^{\frac{N}{N-1}}\,|Df|^{\frac{N^2-2N}{N-1}}$$

where A depends on N, φ and η. The first term belongs to $L^P(\Omega)$ by assumption at (4.28). The second term lies in $L^1_{\mathrm{loc}}(\Omega) \subset L^P_{\mathrm{loc}}(\Omega)$, by Sobolev imbedding and Hölder's inequality. Indeed, for $f \in W^{1,N-1}(\Omega, \mathbb{R}^N)$ we have $|f|^{N/(N-1)} \in L^{N^2-2N+1}_{\mathrm{loc}}(\Omega)$, while $|Df|^{(N^2-2N)/(N-1)}$ lies in $L^{(N^2-2N+1)/(N^2-2N)}(\Omega)$, the dual to $L^{N^2-2N+1}(\Omega)$. Since $\widetilde{f}$

has compact support we conclude that $|\widetilde{h}|^{N/(N-1)} \in L^P(\mathbb{R}^N)$. At this point we appeal to Lemma 4.12, with $\alpha = N/(N-1)$, to infer that for some $A \geq 1$,

$$\inf_{t\geq A} t^{\frac{1}{N-1}} \int\limits_{\tilde{h}>t} \widetilde{h}(x)\,dx = 0. \tag{4.54}$$

This combined with inequality (4.53) yields

$$\liminf_{t\to\infty} \Big| \int\limits_{M\widetilde{h}\leq 2t} J(x,\widetilde{f})\,dx \Big| = 0. \tag{4.55}$$

The rest of the proof is almost identical to that of Theorem 4.7. The only point to clarify is that the term

$$|f^1 d\varphi \wedge df^2 \wedge \ldots \wedge df^N| \leq |\nabla\varphi|\,|f|\,|D^\sharp f|$$

is integrable, by (4.18).

4.6. Examples. In this section we give quite explicit examples of Sobolev mappings with nonintegrable Jacobian and having a desired degree of regularity. They illustrate that both Theorem 4.7 and Theorem 4.6 are sharp in the Orlicz–Sobolev category. Although similar examples of radial stretchings are well known in the literature there are many interesting features still unknown.

We discuss mappings f defined on the unit ball $\mathbb{B}$ with values in $\mathbb{R}^N$ belonging to the Sobolev class $W^{1,N-1}(\mathbb{B},\mathbb{R}^N)$ of the form

$$f(x) = \lambda(|x|)x. \tag{4.56}$$

The function $t \mapsto t\lambda(t)$, for $0 \leq t \leq 1$, will be decreasing from the value ∞ at $t = 0$, to 1 at $t = 1$. Thus f will map homeomorphically the unit ball $\mathbb{B}$ onto its exterior. In particular, f will be an orientation reversing map ($J(x,f) \leq 0$) with

$$\int\limits_{\mathbb{B}} J(x,f)\,dx = -\infty.$$

Of course, if needed one may compose f with a reflection in an $(N-1)$-dimensional hyperplane to make f orientation preserving.

We may calculate the differential matrix of f and its determinant by using the familiar formulas:

$$Df(x) = \lambda(|x|)\mathbf{I} + |x|\lambda'(|x|)\frac{x\otimes x}{|x|^2}.$$

Hence

$$J(x,f) = \lambda^N(|x|) + |x|\lambda'(|x|)\lambda^{N-1}(|x|) \leq 0,$$

because $\lambda(t) + t\lambda'(t) = [t\lambda(t)]' \leq 0$. The cofactor matrix is then computed to be

$$D^\sharp f(x) = (\lambda^{N-1} + t\lambda'\lambda^{N-2})\mathbf{I} - t\lambda'\lambda^{N-2}\,\frac{x\otimes x}{|x|^2}$$

where we have denoted $\lambda(|x|)$ and $|x|$ briefly by λ and t. This formula can easily be seen by checking the identity defining the cofactor matrix

$$D^\sharp f(x)Df(x) = J(x,f)\mathbf{I}.$$

Let us disclose in advance that we shall have

$$\lambda(t) \le -t\lambda'(t) \le 2\lambda(t). \tag{4.57}$$

Consequently, the norms of the matrices in question will satisfy

$$|Df(x)| \le 3\lambda(|x|), \tag{4.58}$$

$$|D^\sharp f(x)| \le 5\lambda^{N-1}(|x|). \tag{4.59}$$

In this way the question concerning integrability of $|Df|$ and $|D^\sharp f|$ reduces to the computation of integrals for $\lambda(t)$. One integral is obvious by using polar coordinates:

$$\begin{aligned}\int_{\mathbb{B}} J(x,f)\,dx &= \omega_{N-1}\int_0^1 t^{N-1}(\lambda^N + t\lambda'\lambda^{N-1})\,dt \\ &= \frac{\omega_{N-1}}{N}\int_0^1 d[t\lambda(t)]^N = |\mathbb{B}|(1-\infty) = -\infty.\end{aligned}$$

Example 4.13. Let $\Phi : [0,\infty) \to [0,\infty)$ be an arbitrary concave function, continuously increasing from 0 to ∞, and such that

$$\int_1^\infty \frac{\Phi(s)\,ds}{s^2} = 1. \tag{4.60}$$

Define λ by the equation

$$t\lambda(t) = \left(\int_{t^{-N}}^\infty \frac{\Phi(s)\,ds}{s^2}\right)^{-\frac{1}{N^2}} \qquad \text{for } 0 < t \le 1.$$

Then the radial stretching at (4.56) lies in the Sobolev class $W^{1,N-1}(\mathbb{B},\mathbb{R}^N)$ and its cofactor matrix satisfies

$$|D^\sharp f|^{\frac{N}{N-1}} \in L^\Phi(\mathbb{B}).$$

In spite of that the Jacobian determinant of f fails to be locally integrable.

Proof. It is immediate from the definition of $\lambda(t)$ and (4.60) that the function $t \mapsto t\lambda(t)$ decreases from ∞ to 1. We also have

$$t\lambda(t) \le \left(\int_{t^{-N}}^\infty \frac{\Phi(1)\,ds}{s^2}\right)^{-\frac{1}{N^2}} = (\Phi(1))^{-\frac{1}{N^2}}t^{-\frac{1}{N}}. \tag{4.61}$$

Next we compute the logarithmic derivative of $t\lambda$:

$$\begin{aligned}\frac{(t\lambda)'}{t\lambda} &= \frac{-(\int_{t^{-N}}^\infty s^{-2}\Phi(s)\,ds)'}{N^2\int_{t^{-N}}^\infty s^{-2}\Phi(s)\,ds} = \frac{-t^{N-1}\Phi(t^{-N})}{N\int_{t^{-N}}^\infty s^{-2}\Phi(s)\,ds} \\ &\ge \frac{-t^{N-1}\Phi(t^{-N})}{N\int_{t^{-N}}^\infty s^{-2}\Phi(t^{-N})\,ds} = -\frac{1}{Nt}.\end{aligned}$$

This shows $-\frac{1}{n}\,\lambda(t) \le (t\lambda)' \le 0$, and hence (4.57) follows. Another estimate for λ follows by using concavity of Φ, namely

$$K\Phi(t^{-N}) \ge \Phi(K\,t^{-N}) \qquad \text{for every } K \ge 1.$$

We apply this to

$$K = K(t) = \Big(\int\limits_{t^{-N}}^{\infty} \frac{\Phi(s)\,ds}{s^2} \Big)^{-\frac{1}{N}} \geq 1$$

for $t \leq 1$, to obtain

$$\lambda^N(t) \leq \Phi^{-1}[K(t)\Phi(t^{-N})]. \tag{4.62}$$

Having disposed of these preliminary inequalities we are able to integrate the derivatives of f:

$$\begin{aligned} \int\limits_{\mathbb{B}} |Df(x)|^{N-1}\,dx &\leq 3^{N-1} \int\limits_{\mathbb{B}} \lambda^{N-1}(|x|)\,dx = 3^{N-1}\omega_{N-1} \int\limits_0^1 [t\lambda(t)]^{N-1}\,dt \\ &\leq C(N) \int\limits_0^1 t^{-\frac{N-1}{N}}\,dt = NC(N) < \infty \end{aligned}$$

by (4.58) and (4.61). Regarding the cofactors of Df, we make use of (4.59) and concavity of Φ to obtain

$$\int\limits_{\mathbb{B}} \Phi(|D^\sharp f|^{\frac{N}{N-1}}) \leq 5^{\frac{N}{N-1}} \int\limits_{\mathbb{B}} \Phi(\lambda^N(|x|))\,dx \leq 25\omega_{N-1} \int\limits_0^1 K(t)\Phi(t^{-N})t^{N-1}\,dt,$$

by (4.62). Finally with the aid of the substitution $\tau = t^{-N}$, we arrive at the desired estimate

$$\begin{aligned} \int\limits_{\mathbb{B}} \Phi(|D^\sharp f|^{\frac{N}{N-1}}) &\leq \frac{25\omega_{N-1}}{N} \int\limits_1^{\infty} \frac{\Phi(\tau)\,d\tau}{\tau^2(\int_\tau^\infty s^{-2}\Phi(s)\,ds)^{\frac{1}{N}}} \\ &= -\frac{25\omega_{N-1}}{N} \int\limits_1^{\infty} d\Big(\int\limits_\tau^{\infty} s^{-2}\Phi(s)\,ds \Big)^{1-\frac{1}{N}} \\ &= \frac{25\omega_{N-1}}{N-1} < \infty, \end{aligned}$$

completing the proof of Example 4.13.

4.7. Further results. Under some additional technical assumptions to Theorem 4.6 the Jacobian determinant enjoys even higher degree of integrability.

Suppose that the Orlicz function $P : [0,\infty) \to [0,\infty)$ satisfies the divergence condition (4.25) and the following technical assumptions:

$$[t^{-1-\frac{1}{N^3}}\,P(t)]' \leq 0 \leq [t^{-1+\frac{1}{N^3}}\,P(t)]' \tag{4.63}$$

for large values of t, and $t^{-2}\,P(t)$ integrable near zero. The improvement of the degree of integrability of the Jacobian will be described by the Orlicz function $\Psi : [0,\infty) \to [0,\infty)$ defined by

$$\Psi(t) := tL(t) := P(t) + t\int\limits_0^t \frac{P(s)\,ds}{s^2}.$$

THEOREM 4.14. *Let $f \in W^{1,N-1}(\Omega, \mathbb{R}^N)$ be an orientation preserving mapping such that*

$$(|D^\sharp f| + |f|\,|Df|^{N-2})^{\frac{N}{N-1}} \in L^P(\Omega). \tag{4.64}$$

Then $\det Df \in L^{\Psi}_{\mathrm{loc}}(\Omega)$; *we actually have*

$$\int_{\Omega'} \Psi(J(x,f))\,dx \le \int_{\Omega'} J(x,f) L(|D^{\sharp} f(x)|^{\frac{N}{N-1}})\,dx < \infty \tag{4.65}$$

for every compact $\Omega' \subset \Omega$.

The interested reader can find the proof in [20].

In the same paper the authors depart from the divergence condition to investigate higher integrability properties of the Jacobian in spaces weaker than $L^1_{\mathrm{loc}}(\Omega)$. If the integral $\int_0^\infty s^{-2}P(s)\,ds$ is finite then, without getting into technicalities, the Jacobian belongs to $L^{\Psi}_{\mathrm{loc}}(\Omega)$ with

$$\Psi(t) = -P(t) + t\int_t^\infty \frac{P(s)\,ds}{s^2},$$

More precisely, if we impose the *convergence condition*

$$\int_0^\infty \frac{P(s)ds}{s^2} < \infty \tag{4.66}$$

and the following technical one:

$$[t^{-1-\frac{N-2}{N^2}}P(t)]' \le 0 \le [t^{-1+\frac{N-2}{N^2}}P(t)]' \tag{4.67}$$

for large values of t, then the following theorem holds:

Theorem 4.15. *Let* $f \in W^{1,N-1}(\Omega,\mathbb{R}^N)$, $N > 2$, *be an orientation preserving mapping such that*

$$(|f|\,|Df|^{N-2} + |D^{\sharp} f|)^{\frac{N}{N-1}} \in L^P(\Omega) \tag{4.68}$$

Then $\det Df \in L^{\Psi}_{\mathrm{loc}}(\Omega)$.

In the study of Jacobians the so called grand Lebesgue spaces have emerged. Let Ω be a bounded domain in $\mathbb{R}^N$. The space $BL^p(\Omega)$ consists of the functions $h \in \bigcap_{1\le s<p} L^s(\Omega)$, $p > 1$, whose modulus of integrability

$$\mathcal{L}^p(h;\varepsilon) = \Big[\varepsilon\int_\Omega |h|^{p-\varepsilon}\Big]^{\frac{1}{p-\varepsilon}} \tag{4.69}$$

is bounded for $0 < \varepsilon \le p-1$ (see [38]). $BL^p(\Omega)$ is a Banach space equipped with the norm

$$\|h\|_{p)} = \sup_{0<\varepsilon\le p-1} \mathcal{L}^p(h;\varepsilon). \tag{4.70}$$

We say that h has *vanishing modulus of integrability* if $\mathcal{L}^p(h;\varepsilon) \to 0$ as $\varepsilon \to 0$. We write $h \in VL^p(\Omega)$. This latter space is none other than the completion of $L^p(\Omega)$ in $BL^p(\Omega)$. Some of the arguments presented here may further be extended to include the spaces of bounded or vanishing modulus of integrability (grand Lebesgue spaces).

Corollary 4.16. *Let* $f \in W^{1,N-1}(\Omega,\mathbb{R}^N)$ *be an orientation preserving map satisfying the condition*

$$|D^{\sharp} f| \in BL^{\frac{N}{N-1}}(\Omega). \tag{4.71}$$

Then $J(x,f)$ is locally integrable. Moreover, $J(x,f)$ coincides with the distributional Jacobian whenever

$$|D^\sharp f| \in VL^{\frac{N}{N-1}}(\Omega). \tag{4.72}$$

5. Mappings of finite distortion

5.1. Introduction. In this chapter we study mappings $f = (f^1, \ldots, f^N) : \Omega \to \mathbb{R}^N$ in the Sobolev class $W^{1,N}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, where Ω is a connected, open subset of $\mathbb{R}^N$, $N \geq 2$. Thus, the differential matrix $Df(x) \in \mathbb{R}^{N\times N}$ and its Jacobian determinant $J(x,f)$ are defined almost everywhere in Ω . Here $\mathbb{R}^{N\times N}$ denotes the space of all $N \times N$ matrices, equipped with the norm

$$|A| = \max\{|A\xi| : \xi \in S^{N-1}\}.$$

Throughout we assume that f is an orientation preserving mapping, that is, $J(x,f) \geq 0$.

Definition 5.1. A map $f \in W^{1,N}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is said to be *of finite distortion* if

$$|Df(x)|^N \leq K_O(x)\, J(x,f) \quad \text{a.e.} \tag{5.1}$$

for some $1 \leq K_O(x) < \infty$.

Note that Hadamard's inequality asserts that pointwise

$$J(x,f) \leq |Df(x)|^N,$$

thus the assumption $K_O(x) \geq 1$ is imposed on us. Moreover it is fundamental that the Sobolev exponent is at least the dimension of Ω so that we can integrate the Jacobian. In this case the mappings of finite distortion are actually continuous [26].

The smallest such function defined by

$$K_O(x,f) = \begin{cases} |Df(x)|^N / J(x,f) & \text{if } J(x,f) \neq 0, \\ 1 & \text{if } J(x,f) = 0, \end{cases} \tag{5.2}$$

is called the *outer distortion function* of f.

Geometrically this means that at the points where $J(x,f) > 0$ the differential $Df(x)$ maps the unit ball to an ellipsoid E and

$$K_O(x,f) = \frac{\operatorname{vol} B_O}{\operatorname{vol} E}$$

where B_O is the smallest ball circumscribed about E. In the same way, we may define the *inner distortion* of f by

$$K_I(x,f) = \frac{\operatorname{vol} E}{\operatorname{vol} B_I}$$

where B_I is the largest ball inscribed in E. We set $K_I(x,f) = 1$ at degenerate points where $Df(x) = 0$ and we call

$$K(x,f) = \max\{K_O(x,f), K_I(x,f)\}$$

the *maximal distortion*,

$$K_M(x,f) = \frac{K_O(x,f)}{K_I(x,f)}$$

the *mean distortion* and

$$H(x,f) = (K_O(x,f), K_I(x,f))^{1/N}$$

the *linear distortion.*

The linear distortion has the representation

$$H(x,f) = \frac{\max\{|Df(x)\xi| : \xi \in S^{N-1}\}}{\min\{|Df(x)\xi| : \xi \in S^{N-1}\}}$$

at points where $Df(x) \neq 0$.

All of these distortion functions coincide when $N = 2$; this is not the case when $N > 2$.

Many constructions in analysis, geometry and topology rely on limiting processes; the existence, uniqueness and compactness properties of families of mappings with finite distortion make them ideal tools for solving various problems in these areas. For instance in studying deformations of elastic bodies and the related extremals for variational integrals in certain degenerate settings, mappings of finite distortion are often the natural candidates to consider because they are closed under uniform convergence [52].

The following limit theorem holds [19]:

THEOREM 5.2. *Suppose that $f_n : \Omega \to \mathbb{R}^N$ is a sequence of mappings of finite distortion which converges weakly in $W^{1,N}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ to f and suppose that*

$$K_O(x, f_n) \le M(x) < \infty \quad \text{a.e.} \tag{5.3}$$

for $n = 1, 2, \ldots$ Then f has finite distortion and

$$K_O(x, f) \le M(x) \quad \text{a.e.} \tag{5.4}$$

This is a refinement of Reshetnyak's theorem concerning mappings f_n of bounded distortion, that is, mappings which satisfy (5.3) with $M(x) \le K$ where K is a constant. In this case, weak convergence in $W^{1,N}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ implies uniform convergence on compact sets and hence, by Reshetnyak's theorem, the limit mapping f satisfies $K_O(x,f) \le K$ instead of the pointwise bound given in (5.4).

THEOREM 5.3. *Theorem 5.2 remains valid with $K_I(x,f)$, $K_M(x,f)$ and $K(x,f)$ in place of $K_O(x,f)$.*

This is not true for the linear distortion $H(x,f)$ when $N > 2$.

While substantial progress has been made on the limit theorems, many questions still remain unanswered.

5.2. The Beltrami equation. The Beltrami equation has a long history. Gauss first studied the equation in the 1820's while investigating the problem of existence of isothermal coordinates on a given surface. The complex Beltrami equation was intensively studied by Morrey in the late 1930's, and he established the existence of homeomorphic L^2-solutions. It took another 20 years before Bers recognized that homeomorphic solutions are quasiconformal mappings.

Studying quasiconformal mappings via the Beltrami equation is a particularly valuable idea because from this point of view the mapping is the solution of an elliptic equation and as such enjoys various nice properties not obvious from the definition.

Directly from the analytic definition we see that an orientation preserving mapping of finite distortion solves the Beltrami equation

$$D^t f(x) Df(x) = J(x,f)^{\frac{2}{N}} G(x)$$

where $D^t f(x)$ stands for the transpose to $Df(x)$ and $G(x)$ is the distortion tensor of f, a symmetric positive definite matrix of determinant one. If $G(x)$ is the identity everywhere the Beltrami equation reduces to the N-dimensional Cauchy–Riemann system

$$D^t f(x) Df(x) = J(x,f)^{\frac{2}{N}} I.$$

We have the pointwise almost everywhere estimate

$$\frac{1}{K(x)}\,|\xi|^2 \le \langle G(x)\xi, \xi\rangle \le K(x)|\xi|^2$$

for vectors $\xi \in \mathbb{R}^N$ and thus the distortion function $K = K(x)$ provides ellipticity bounds for the equation. The case of K bounded gives uniform ellipticity estimates on G.

The Beltrami equation implies a number of first order differential equations analogous to the complex Cauchy–Riemann equations.

Associated with $G(x)$ is the energy integral

$$\mathcal{E}[h] = \int_{\Omega} E(x, Dh)\, dx$$

where $E(x,M) = \langle MG^{-1}(x), M\rangle^{2/N}$. Here we have used the inner product of matrices defined by

$$\langle X, Y\rangle = \operatorname{Trace} X^t Y.$$

It is the essence of the analytic theory of mappings with finite distortion that these mappings minimize the energy functional, subject to a Dirichlet boundary condition. The Euler–Lagrange equation takes the form

$$\operatorname{div} A(x, Df) = 0 \tag{5.5}$$

where $A : \Omega \times \mathbb{R}^{N\times N} \to \mathbb{R}^{N\times N}$ is given by

$$A(x,M) = \langle MG^{-1}(x), M\rangle^{\frac{N-2}{N}} MG^{-1}(x).$$

Let us stress that the equation (5.5) is of second order whereas the minimizers (mappings of finite distortion) solve the first order Beltrami equation. It is also of particular importance that each component $u = f^i$, $i = 1, \dots, N$, of a mapping of finite distortion satisfies the equation

$$\operatorname{div} A(x, \nabla u) = 0$$

called the *A-harmonic equation*. In the case that the distortion $K \equiv 1$ this reduces to

$$\operatorname{div}(|\nabla u|^{N-2}\nabla u) = 0$$

which is a special case of the p-harmonic equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \qquad p \in (1,\infty).$$

At this point the so-called div-curl fields assume particular relevance to our study.

Let us first illustrate how such fields relate to the theory of linear elliptic PDEs of the form

$$\operatorname{div} A(x)\nabla u = 0$$

where $A : \Omega \to \mathbb{R}^{N\times N}$ is a measurable function with values in symmetric matrices such that for all $\xi \in \mathbb{R}^N$

$$K^{-1}(x)|\xi|^2 \le \langle A(x)\xi, \xi\rangle \le K(x)|\xi|^2 \quad \text{a.e.} \tag{5.6}$$

Uniform ellipticity means that $1 \le K(x) \le K$ for some constant K.

Now the pair $\Phi = [B, E]$ with $E = \nabla u$ and $B = A(x)\nabla u$ is a div-curl field. Although it is not apparent at this point, condition (5.6) is equivalent to the so-called *distortion inequality* for Φ:

$$|\Phi|^2 \le [K(x) + K^{-1}(x)]J(x, \Phi) \tag{5.7}$$

where, by analogy to mappings of finite distortion in the plane, we use the notation $|\Phi|^2 = |B|^2 + |E|^2$ and $J(x, \Phi) = \langle B, E\rangle$.

An arbitrary div-curl field $\Phi \in L^p_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \times L^p_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is said to have finite distortion function $K(x)$ if (5.7) holds almost everywhere in Ω. Φ is said to be of *bounded distortion* if $1 \le K(x) \le K$. Obviously the natural integrability exponent here is $p = 2$. Thus it is interesting considering fields with exponent p different from 2. In [14], together with A. Fiorenza, we prove higher integrability results for div-curl fields of bounded distortion

$$\Phi = [B, E] \in L^{2-\varepsilon}(\Omega, \mathbb{R}^N) \times L^{2-\varepsilon}(\Omega, \mathbb{R}^N)$$

with $0 < \varepsilon < 1$.

5.3. Regularity results for vector fields of bounded distortion. This section is concerned with regularity results for vector fields of bounded distortion contained in [14] already mentioned at the end of the previous section.

The following basic estimates are established in [32] (see also [54] for the present formulation). We denote by Q_0, Q open cubes in $\mathbb{R}^N$ with sides parallel to the coordinate axes, and by $2Q$ the cube with the same centre as Q and double side-length.

THEOREM 5.4. *Let $1 < p, q < \infty$ be a Hölder conjugate pair, $1/p + 1/q = 1$, and let $1 < r, s < \infty$ be a Sobolev conjugate pair, $1/r + 1/s = 1 + 1/N$. Then there exists a constant $c_N = c_N(p, s)$ such that for each cube Q satisfying $2Q \subset Q_0 \subset \mathbb{R}^N$ we have*

$$\begin{aligned}\left| \fint_Q \frac{\langle B, E\rangle}{|B|^\varepsilon |E|^\varepsilon}\,dx\right| &\le c_N \varepsilon \Big[\fint_{2Q} |E|^{(1-\varepsilon)p}\,dx\Big]^{\frac1p}\Big[\fint_{2Q} |B|^{(1-\varepsilon)q}\,dx\Big]^{\frac1q} \\ &\quad + c_N \Big[\fint_{2Q} |E|^{(1-\varepsilon)s}\,dx\Big]^{\frac1s}\Big[\fint_{2Q} |B|^{(1-\varepsilon)r}\,dx\Big]^{\frac1r}\end{aligned} \tag{5.8}$$

whenever $0 \le 2\varepsilon \le \min\{(p-1)/p, (q-1)/q, (r-1)/r, (s-1)/s\}$ and $\operatorname{div} B = 0$, $\operatorname{curl} E = 0$.

The next proposition by Giaquinta–Modica ([23], [25]) will be useful.

PROPOSITION 5.5. *Let $g \in L^\alpha(Q_0)$, $\alpha > 1$, and $f \in L^r(Q_0)$, $r > \alpha$, be two nonnegative functions and suppose that for every cube Q such that $2Q \subset Q_0$ the following estimate* *holds*:

$$\fint_Q g^\alpha\,dx \le b\Big\{\Big(\fint_{2Q} g\,dx\Big)^\alpha + \fint_{2Q} f^\alpha\,dx\Big\} + \theta \fint_{2Q} g^\alpha\,dx \tag{5.9}$$

with $b > 1$. *There exist constants* $\theta_0 = \theta_0(\alpha, N)$, $\sigma_0 = \sigma_0(b, \theta, \alpha, r, N)$ *such that if* $\theta < \theta_0$, *then* $g \in L^{\alpha+\sigma}_{\mathrm{loc}}(Q_0)$ *for all* $0 < \sigma < \sigma_0$ *and*

$$\Big(\fint_Q g^{\alpha+\sigma}\,dx\Big)^{\frac{1}{\alpha+\sigma}} \le c\Big\{\Big(\fint_{2Q} g^\alpha\,dx\Big)^{\frac{1}{\alpha}} + \Big(\fint_{2Q} f^{\alpha+\sigma}\,dx\Big)^{\frac{1}{\alpha+\sigma}}\Big\} \tag{5.10}$$

where c *is a positive constant depending on* b, θ, α, r, N.

A variant of the result established in Proposition 5.5 can be proved. We remark that in our assumption we will consider a family of inequalities in which both the exponent of integrability of the function g and the coefficient on the right hand side depend on ε. Nevertheless, even if Proposition 5.5 cannot be applied *a priori*, in the theorem we are going to prove we get a higher integrability result for g and an estimate of the type (5.10).

THEOREM 5.6. *Let* $g \in L^{2(1-\varepsilon)}(Q_0)$ *and* $f \in L^r(Q_0)$, $0 \le \varepsilon < 1/2$, $r > 2(1-\varepsilon)$, *be nonnegative functions such that*

$$\begin{aligned}\fint_Q g^{2(1-\varepsilon)}\,dx \le{}& c_1\varepsilon \fint_{2Q} g^{2(1-\varepsilon)}\,dx \\ &+ c_2\Big\{\Big(\fint_{2Q} g^{2(1-\varepsilon)\frac{N}{N+1}}\,dx\Big)^{\frac{N+1}{N}} + \Big(\fint_{2Q} f^{2(1-\varepsilon)}\,dx\Big)\Big\}\end{aligned} \tag{5.11}$$

for every cube $Q \subset 2Q \subset Q_0$, *for some constants* $c_1 \ge 0$, $c_2 > 0$. *Then there exist* $\overline{\varepsilon} = \overline{\varepsilon}(c_1, N)$ *and* $\overline{\eta} = \overline{\eta}(c_1, c_2, r, \varepsilon, N)$ *such that if* $0 \le \varepsilon < \overline{\varepsilon}$, *then* $g \in L^{2(1-\varepsilon)+\eta}_{\mathrm{loc}}(Q_0)$ *for all* $0 \le \eta < \overline{\eta}$ *and*

$$\Big(\fint_Q g^{2(1-\varepsilon)+\eta}\,dx\Big)^{\frac{1}{2(1-\varepsilon)+\eta}} \le c\Big\{\Big(\fint_{2Q} g^{2(1-\varepsilon)}\,dx\Big)^{\frac{1}{2(1-\varepsilon)}} + \Big(\fint_{2Q} f^{2(1-\varepsilon)+\eta}\,dx\Big)^{\frac{1}{2(1-\varepsilon)+\eta}}\Big\}$$

where c *is a positive constant depending on* c_2, r, ε, N.

Proof. Since the functions $g_\varepsilon = g^{2(1-\varepsilon)N/(N+1)}$, $f_\varepsilon = f^{2(1-\varepsilon)N/(N+1)}$ satisfy the inequality

$$\fint_Q g_\varepsilon^{\frac{N+1}{N}}\,dx \le c_2\Big\{\Big(\fint_{2Q} g_\varepsilon\,dx\Big)^{\frac{N+1}{N}} + \Big(\fint_{2Q} f_\varepsilon^{\frac{N+1}{N}}\,dx\Big)\Big\} + c_1\varepsilon \fint_{2Q} g_\varepsilon^{\frac{N+1}{N}}\,dx \tag{5.12}$$

we can apply Proposition 5.5 with $\alpha = (N+1)/N$, $b = c_2$. We get $\theta_0 = \theta_0(N)$ and $\sigma_0 = \sigma_0(c_2, r, \varepsilon, N)$ such that if (5.12) holds with $c_1\varepsilon < \theta_0/2$, then $g_\varepsilon \in L^{\alpha+\sigma}_{\mathrm{loc}}(Q_0)$ for every $0 < \sigma < \sigma_0$, i.e.

$$[g^{2(1-\varepsilon)\frac{N}{N+1}}]^{\frac{N+1}{N}+\sigma} \in L^1_{\mathrm{loc}}(Q_0) \quad \forall 0 < \sigma < \sigma_0$$

and

$$\begin{aligned}\Big(\fint_Q g_\varepsilon^{\frac{N+1}{N}+\sigma}\,dx\Big)^{\frac{1}{\frac{N+1}{N}+\sigma}} \le{}& c\Big\{\Big(\fint_{2Q} g_\varepsilon^{\frac{N+1}{N}}\,dx\Big)^{\frac{N}{N+1}} \\ &+ \Big(\fint_{2Q} f_\varepsilon^{\frac{N+1}{N}+\sigma}\,dx\Big)^{\frac{N}{N+1}+\sigma}\Big\}\end{aligned} \tag{5.13}$$

with c depending on c_2, r, ε, N. Set

$$0 < \overline{\varepsilon} < \frac{\theta_0}{2c_1}, \qquad 0 < \overline{\eta} < (1-\overline{\varepsilon})\,\frac{2N\sigma_0}{N+1}.$$

If $0 \le \varepsilon < \overline{\varepsilon}$ and $0 \le \eta < \overline{\eta}$, we have

$$\varepsilon < \overline{\varepsilon} < 1 - \overline{\eta}\,\frac{N+1}{2N\sigma_0} < 1 - \eta\,\frac{N+1}{2N\sigma_0}$$

or, equivalently,

$$2(1-\varepsilon) + \eta < 2(1-\varepsilon)\frac{N}{N+1}\left[\frac{N+1}{N} + \sigma_0\right],$$

therefore we get $g \in L^{2(1-\varepsilon)+\eta}_{\mathrm{loc}}(Q_0)$ and inequality (5.13) becomes

$$\Big(\fint_Q g^{2(1-\varepsilon)+\eta}\,dx\Big)^{\frac{1}{2(1-\varepsilon)+\eta}} \le c\Big\{\Big(\fint_{2Q} g^{2(1-\varepsilon)}\,dx\Big)^{\frac{1}{2(1-\varepsilon)}} + \Big(\fint_{2Q} f^{2(1-\varepsilon)+\eta}\,dx\Big)^{\frac{1}{2(1-\varepsilon)+\eta}}\Big\}.$$

Let us observe that, upon a closer look at the proof of Theorem 5.6, one can note that the gain of integrability given by $\sigma_0 = \sigma_0(c_2, r, \varepsilon, N)$ is actually dependent only on $c_2, r/(2(1-\varepsilon)), N$. Nevertheless, if $f \equiv 0$ a.e. in Q_0, the number σ_0, and therefore also $\overline{\eta}$ and c, do not depend on ε. This remark is crucial to proving the following

Corollary 5.7. *Let $0 \le \varepsilon < 1/2$ and $g \in L^{2(1-\varepsilon)}(Q_0)$, $Q_0 \subset \mathbb{R}^N$, be such that*

$$\fint_Q g^{2(1-\varepsilon)}\,dx \le c_1\varepsilon \fint_{2Q} g^{2(1-\varepsilon)}\,dx + c_2\Big(\fint_{2Q} g^{2(1-\varepsilon)\frac{N}{N+1}}\,dx\Big)^{\frac{N+1}{N}}$$

for every cube $Q \subset 2Q \subset Q_0$. Then there exists $\overline{\varepsilon} = \overline{\varepsilon}(c_1, N)$ such that if $0 \le \varepsilon < \overline{\varepsilon}$, then $g \in L^{2+2\varepsilon}_{\mathrm{loc}}(Q_0)$ and

$$\Big(\fint_Q g^{2(1+\varepsilon)}\,dx\Big)^{\frac{1}{2(1+\varepsilon)}} \le c\Big(\fint_{2Q} g^{2(1-\varepsilon)}\,dx\Big)^{\frac{1}{2(1-\varepsilon)}} \tag{5.14}$$

where c is a positive constant depending on c_2, N.

Proof. Let us apply Theorem 5.6 with $f \equiv 0$ a.e. in Q_0. If $\varepsilon < \min(\overline{\varepsilon}, \overline{\eta}/4)$, choosing $\eta = 4\varepsilon$, from inequality (5.11) we get $g \in L^{2+2\varepsilon}_{\mathrm{loc}}(Q_0)$ and inequality (5.14) holds.

Now we are in a position to prove our higher integrability results.

Proposition 5.8. *Let $\Omega \subset \mathbb{R}^N$, $0 < \varepsilon < 1$ and $\Phi = [E, B]$ belonging to $L^{2-\varepsilon}(\Omega, \mathbb{R}^N) \times L^{2-\varepsilon}(\Omega, \mathbb{R}^N)$ such that* $\operatorname{div} B = 0$, $\operatorname{curl} E = 0$ *and*

$$|B(x)|^2 + |E(x)|^2 \le (K + K^{-1})\langle B(x), E(x)\rangle \qquad \text{a.e. in } \Omega \tag{5.15}$$

where $K \ge 1$. Then there exists $\overline{\varepsilon} = \overline{\varepsilon}(K, N)$ such that $\Phi \in L^{2+\varepsilon}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \times L^{2+\varepsilon}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ for all $0 < \varepsilon < \overline{\varepsilon}$ and

$$\Big(\fint_Q |\Phi|^{2+\varepsilon}\,dx\Big)^{\frac{1}{2+\varepsilon}} \le c\Big(\fint_{2Q} |\Phi|^{2-\varepsilon}\,dx\Big)^{\frac{1}{2-\varepsilon}} \qquad \forall Q,\ 2Q \subset \Omega$$

where c is a positive constant depending on K, N.

Proof. Fix Q a cube such that $2Q \subset \Omega$. Applying Theorem 5.4 with $p = q = 2$ and $r = s = 2N/(N+1)$, from inequality (5.8) we get

$$\fint_Q (|B|^2 + |E|^2)^{1-\varepsilon}\,dx \le c_{N,K}\varepsilon \fint_{2Q} (|B|^2 + |E|^2)^{1-\varepsilon}\,dx + c_{N,K}\Big(\fint_{2Q} (|B|^2 + |E|^2)^{(1-\varepsilon)\frac{N}{N+1}}\,dx\Big)^{\frac{N+1}{N}}$$

for ε sufficiently small. Set $g^2 = |B|^2 + |E|^2$. The last inequality implies

$$\fint_Q g^{2-2\varepsilon}\,dx \le c_{N,K}\,\varepsilon \fint_{2Q} g^{2-2\varepsilon}\,dx + c_{N,K}\Big(\fint_{2Q} g^{(2-2\varepsilon)\frac{N}{N+1}}\,dx\Big)^{\frac{N+1}{N}}.$$

By Corollary 5.7 there exists $\overline{\varepsilon} = \overline{\varepsilon}(K,N)$ such that if $0 \le \varepsilon < \overline{\varepsilon}$, then $g \in L^{2+2\varepsilon}_{\mathrm{loc}}(\Omega)$ and

$$\Big(\fint_Q g^{2+2\varepsilon}\,dx\Big)^{\frac{1}{2+2\varepsilon}} \le c\Big(\fint_{2Q} g^{2-2\varepsilon}\,dx\Big)^{\frac{1}{2-2\varepsilon}},$$

proving the assertion.

Now consider $\Phi = (E,B) \in L^{2-2\varepsilon}(\Omega,\mathbb{R}^N) \times L^{2-2\varepsilon}(\Omega,\mathbb{R}^N)$ such that

$$\operatorname{div} B = 0, \qquad \operatorname{curl} E = 0, \tag{5.16}$$

$$|B(x)|^2 + |E(x)|^2 \le (K + K^{-1})\langle B(x), E(x)\rangle + |F|^2 \tag{5.17}$$

where F is a function in $L^r(\Omega,\mathbb{R}^N)$, $r > 2 - 2\varepsilon$, for ε sufficiently small.

THEOREM 5.9. *Let $0 \le \varepsilon < 1/2$ and E, B vector fields as in* (5.16), (5.17). *Then there exist $\overline{\varepsilon} = \overline{\varepsilon}(K,N)$ and $\overline{\eta} = \overline{\eta}(K,r,\varepsilon,N)$ such that if $0 \le \varepsilon < \overline{\varepsilon}$, then $\Phi = (E,B) \in L^{2-2\varepsilon+\eta}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \times L^{2-2\varepsilon+\eta}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ for all $0 \le \eta < \overline{\eta}$ and*

$$\Big(\fint_Q |\Phi|^{2-2\varepsilon+\eta}\,dx\Big)^{\frac{1}{2-2\varepsilon+\eta}} \le c\Big\{\Big(\fint_{2Q} |\Phi|^{2-2\varepsilon}\,dx\Big)^{\frac{1}{2-2\varepsilon}} + \Big(\fint_{2Q} (|F|^2)^{\frac{2-2\varepsilon+\eta}{2}}\,dx\Big)^{\frac{1}{2-2\varepsilon+\eta}}\Big\}$$

where c is a positive constant depending on K, r, ε, N.

Proof. Fix a cube Q such that $2Q \subset \Omega$ and set

$$Q^+ = \{x \in Q \mid \langle B,E\rangle \ge 0 \text{ a.e.}\}, \qquad Q^- = \{x \in Q \mid \langle B,E\rangle \le 0 \text{ a.e.}\}.$$

Observe that by (5.17), replacing $|F|$ with f, we have

$$\begin{aligned}\int_{Q^-} \frac{-\langle B,E\rangle}{|B|^\varepsilon|E|^\varepsilon}\,dx &\le \int_{Q^-} (|B||E|)^{1-\varepsilon}\,dx \le \int_{Q^-} (|B|^2 + |E|^2)^{1-\varepsilon}\,dx \\ &\le \int_{Q^-} \Big[\Big(K + \frac{1}{K}\Big)\langle B,E\rangle + f^2\Big]^{1-\varepsilon}\,dx \le \int_{Q^-} f^{2-2\varepsilon}\,dx \le \int_Q f^{2-2\varepsilon}\,dx\end{aligned}$$

and therefore

$$\begin{aligned}\int_Q \frac{\langle B,E\rangle}{|B|^\varepsilon|E|^\varepsilon}\,dx &= \int_{Q^+} \frac{\langle B,E\rangle}{|B|^\varepsilon|E|^\varepsilon}\,dx + \int_{Q^-} \frac{\langle B,E\rangle}{|B|^\varepsilon|E|^\varepsilon}\,dx \\ &\ge \int_{Q^+} \frac{\langle B,E\rangle}{(|B|^2 + |E|^2 + f^2)^\varepsilon}\,dx - \int_Q f^{2-2\varepsilon}\,dx.\end{aligned}$$

Applying Theorem 5.4 with $p = q = 2$ and $r = s = 2N/(N+1)$, for ε sufficiently small, we get

$$\fint_Q \frac{\langle B, E\rangle}{(|B|^2+|E|^2+f^2)^{\varepsilon}}\,dx \le c_N\varepsilon \fint_{2Q} (|B|^2+|E|^2+f^2)^{1-\varepsilon}\,dx$$
$$+ c_N\Big(\fint_{2Q} (|B|^2+|E|^2+f^2)^{(1-\varepsilon)\frac{N}{N+1}}\,dx\Big)^{\frac{N+1}{N}} + \fint_{2Q} f^{2-2\varepsilon}\,dx.$$

By (5.17)

$$\langle B, E\rangle \ge c_K(|B|^2+|E|^2-f^2) = c_K(|B|^2+|E|^2+f^2) - 2c_K f^2$$

and therefore

$$\fint_Q (|B|^2+|E|^2+f^2)^{1-\varepsilon}\,dx$$
$$\le c_{N,K}\varepsilon \fint_{2Q} (|B|^2+|E|^2+f^2)^{1-\varepsilon}\,dx + c_{N,K}\Big(\fint_{2Q} (|B|^2+|E|^2+f^2)^{(1-\varepsilon)\frac{N}{N+1}}\,dx\Big)^{\frac{N+1}{N}}$$
$$+ c_K \fint_{2Q} \frac{f^2}{(|B|^2+|E|^2+f^2)^{\varepsilon}}\,dx + \fint_{2Q} f^{2-2\varepsilon}\,dx$$
$$\le c_{N,K}\varepsilon \fint_{2Q} (|B|^2+|E|^2+f^2)^{1-\varepsilon}\,dx + c_{N,K}\Big(\fint_{2Q} (|B|^2+|E|^2+f^2)^{(1-\varepsilon)\frac{N}{N+1}}\,dx\Big)^{\frac{N+1}{N}}$$
$$+ (c_K+1)\fint_{2Q} f^{2-2\varepsilon}\,dx.$$

If we set $g^2 = |B|^2+|E|^2+f^2$, the last inequality implies

$$\fint_Q g^{2-2\varepsilon}\,dx \le c_{N,K}\varepsilon \fint_{2Q} g^{2-2\varepsilon}\,dx + c_{n,K}\Big(\fint_{2Q} g^{(2-2\varepsilon)\frac{N}{N+1}}\,dx\Big)^{\frac{N+1}{N}} + (c_K+1)\fint_{2Q} f^{2-2\varepsilon}\,dx.$$

By Theorem 5.4 there exist $\overline{\varepsilon} = \overline{\varepsilon}(K,N)$ and $\overline{\eta} = \overline{\eta}(K,r,\varepsilon,N)$ such that if $0 \le \varepsilon < \overline{\varepsilon}$, then $g \in L^{2-2\varepsilon+\eta}_{\mathrm{loc}}(\Omega)$ for all $0 \le \eta < \overline{\eta}$ and

$$\Big(\fint_Q g^{2-2\varepsilon+\eta}\,dx\Big)^{\frac{1}{2-2\varepsilon+\eta}} \le c\Big\{\Big(\fint_{2Q} g^{2-2\varepsilon}\,dx\Big)^{\frac{1}{2-2\varepsilon}} + \Big(\fint_{2Q} (f^2)^{\frac{2-2\varepsilon+\eta}{2}}\,dx\Big)^{\frac{1}{2-2\varepsilon+\eta}}\Big\},$$

proving the assertion.

In [14] we give some applications to the theory of quasiconformal mappings and to the theory of regularity for very weak solutions of nonlinear elliptic equations in divergence form. In particular, the following celebrated result of Bojarski concerning higher integrability of functions $f = (f^1, f^2) : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ with bounded distortion holds:

$$f \in W^{1,2-\varepsilon}(\Omega,\mathbb{R}^2) \;\Rightarrow\; f \in W^{1,2+\varepsilon}(\Omega,\mathbb{R}^2).$$

Moreover, our method provides, for ε sufficiently small, a new proof of the regularity result

$$u \in W^{1,2-\varepsilon}(\Omega) \;\Rightarrow\; u \in W^{1,2+\varepsilon}(\Omega)$$

for very weak solutions of equations of the type

$$\operatorname{div} a(x, \nabla u) = \operatorname{div} F$$

where F is a function in $L^r(\Omega, \mathbb{R}^n)$, $r > 2-2\varepsilon$, and $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a mapping such that

$$\begin{cases} x \mapsto a(x,z) & \text{is measurable for all } z \in \mathbb{R}^N, \\ z \mapsto a(x,z) & \text{is continuous for almost every } x \in \Omega, \end{cases}$$

satisfying

$$|a(x,z)|^2 + |z|^2 \le (K + K^{-1})\langle a(x,z), z\rangle$$

for some $K \ge 1$ and x, z arbitrary vectors in $\mathbb{R}^N$.

Remark 5.10. The assumptions in Theorem 5.6 cannot be weakened. Indeed, consider f, g nonnegative functions on a cube Q_0 satisfying assumptions of the type of Theorem 5.6 with $c_1 = 0$, namely, f, g are such that $g \in L^\alpha(Q_0)$, $f \in L^{\lambda\alpha}(Q_0)$ for some $\alpha > 1$, $\lambda > 1$ and

$$\Big(\fint_Q g^\alpha\,dx\Big)^{\frac{1}{\alpha}} \le a \fint_{2Q} g\,dx + b\Big(\fint_{2Q} f^\alpha\,dx\Big)^{\frac{1}{\alpha}} \qquad \forall Q,\ 2Q \subset Q_0. \tag{5.18}$$

In this case it is known ([33]) that if λ is sufficiently close to 1, then $g \in L^{\lambda\alpha}_{\mathrm{loc}}(Q_0)$ and

$$\Big(\fint_Q g^{\lambda\alpha}\,dx\Big)^{\frac{1}{\lambda\alpha}} \le a_\lambda\Big(\fint_{2Q} g^\lambda\,dx\Big)^{\frac{1}{\lambda}} + b_\lambda\Big(\fint_{2Q} f^{\lambda\alpha}\,dx\Big)^{\frac{1}{\lambda\alpha}} \tag{5.19}$$

where a_λ and b_λ are constants depending only on N, α, a, b.

We show that even if it is still true that $g \in L^{\lambda\alpha}_{\mathrm{loc}}(Q_0)$ for any $\lambda < 1$ (sufficiently small), one cannot find any $\lambda < 1$, $a_\lambda > 0$, $b_\lambda > 0$ such that estimate (5.19) holds for any $g \in L^\alpha(Q_0)$, $f \in L^{\lambda\alpha}(Q_0)$ satisfying (5.18).

By a contradiction argument, we are able to prove that there exists $\lambda < 1$ such that *any* function $g_0 \in L^{\lambda\alpha}(Q_0)$, $g_0 > 0$, satisfies a certain reverse Hölder type inequality, which is generally false.

5.4. Further results. The aim of the present section is to illustrate some continuity properties of mappings of finite distortion. We wish to investigate them under minimal possible assumptions on the degree of integrability of the differential. It is worth pointing out that the first result in this sense is due to V. Goldstein and S. K. Vodopyanov [26]. We have already observed that they showed that mappings of finite distortion in the Sobolev class $W^{1,N}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ are actually continuous. We have repeatedly stressed that the natural Sobolev setting for mappings of finite distortion is the space $W^{1,N}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, largely due to the wish to integrate the Jacobian determinant by parts. However, matters are quite complicated if one does not know a priori that the Jacobian is locally integrable or, even if so, whether it coincides with the so-called distributional Jacobian. The first regularity results below the natural setting were recently established by K. Astala, T. Iwaniec, P. Koskela and G. Martin in [2]. Assuming that $J(x,f) \in L^1_{\mathrm{loc}}(\Omega)$ and $e^{\lambda K} \in L^1_{\mathrm{loc}}(\Omega)$ for some sufficiently large $\lambda = \lambda(N)$ they proved, in *even* dimensions, that $f \in W^{1,N}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. The standing conjecture is that one can take $\lambda(N) = 1$ as the critical exponent for the regularity conclusions; it is known that the L^N-integrability of the differential fails for any $\lambda < 1$.

In [35] the authors continue this theme of the regularity properties of mappings of finite distortion, refining and extending the earlier paper [2] to all dimensions.

Before illustrating the result proved in [35] it is worth recalling some results due to L. Migliaccio and G. Moscariello [45]. Consider the p-harmonic equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \qquad p \in (1,\infty).$$

Setting $E = \nabla u$ and $B = |\nabla u|^{p-2}\nabla u$ we obtain

$$\frac{|E|^p}{p} + \frac{|B|^q}{q} = \langle B, E\rangle.$$

Now let us consider div-curl fields $[B,E]$ coupled by the distortion inequality

$$\frac{|E|^p}{p} + \frac{|B|^q}{q} \le K(x)\langle B, E\rangle \qquad \text{a.e. in } \Omega \tag{5.20}$$

where, as usual, $1 \le K(x) \le \infty$ is a measurable function in Ω and $1 < p, q < \infty$ are conjugate Hölder exponents, $p + q = pq$. In this setting, in [45], the following higher integrability result is proved:

THEOREM 5.11. *Let $\Phi = [B,E]$ be a* div-curl *field satisfying* (5.20)*. If $K(x) \in \mathrm{Exp}_\gamma(\Omega)$ for some $\gamma > 1$, then $B \in L^p \log^\alpha L(\sigma\Omega, \mathbb{R}^N)$ and $E \in L^q \log^\alpha L(\sigma\Omega, \mathbb{R}^N)$ for any $\alpha > 0$ and $0 < \sigma < 1$. Moreover for any $\alpha > 1$,*

$$\||E|^p + |B|^q\|_{L\log^{\alpha-1/\gamma} L(\sigma\Omega)} \le c\|\langle B, E\rangle\|_{L\log^{\alpha-1} L(\Omega)}$$

where $c = c(\sigma, p, \alpha, N, \|K\|_{\mathrm{Exp}_\gamma(\Omega)})$.

Note that $\mathrm{Exp}_\gamma(\Omega)$ denotes the Orlicz space defined by the function $\Phi(t) = \exp(t^\gamma) - 1$.

The proof is obtained by using well known inequalities for nonnegative div-curl products and maximal theorems in Orlicz spaces. It is also proved that the theorem fails if $K(x)$ is assumed merely in $\mathrm{Exp}(\Omega) = \mathrm{Exp}_1(\Omega)$.

We also wish to mention an application to mappings with unbounded distortion:

PROPOSITION 5.12. *If $f \in W^{1,N}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ and satisfies the distortion inequality*

$$|Df(x)|^N \le K(x)J(x,f) \qquad \text{a.e.}$$

with $K(x) \in \mathrm{Exp}_\gamma(\Omega)$ for some $\gamma > 1$, then $|Df| \in L^N \log^\alpha L(\sigma\Omega)$ for any $\alpha \ge 0$ and $0 < \sigma < 1$.

As a consequence they also get the following continuity result:

COROLLARY 5.13. *Under the assumptions of Proposition* 5.12, *for any $\alpha > N$ and any ball $B \subset \Omega$, there exists $c = c(\alpha, B)$ such that*

$$|f(x) - f(y)| \le c(\alpha, B)(\|f\|_{L^N \log^\alpha L} + \|Df\|_{L^N \log^\alpha L})(\log(e + |x-y|^{-N}))^{1-\frac{\alpha}{N}}$$

for any $x \ne y \in B$.

The arguments above prove rather clearly that the class of mappings with exponentially integrable distortion function is optimal in many respects.

Let us conclude with one more result contained in [35].

Assume $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. Thus the differential matrix $Df(x) \in \mathbb{R}^{N\times N}$ is defined at almost every point $x \in \Omega$.

Definition 5.14. A mapping $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is said to have *finite distortion* if

(i) the Jacobian determinant is locally integrable;
(ii) there is a measurable function $K = K(x) \geq 1$, finite almost everywhere, such that
$$|Df(x)|^N \leq K(x) J(x, f).$$

Every mapping of finite distortion solves a nonlinear system of first order PDEs, the so-called Beltrami system. This in turn gives rise to a degenerate elliptic equation of the second order. We then come to the idea to approximate these second order equations by more regular ones whose solutions yield an approximation of the mapping f. Consequently, the authors of [35] proved the following result.

Theorem 5.15. *For each dimension $N \geq 2$ and $\alpha \geq 0$ there exists $\lambda_\alpha(N) \geq 1$ such that if the distortion function $K = K(x)$ of f satisfies*
$$\int_\Omega e^{\lambda K(x)}\, dx < \infty$$
for some $\lambda \geq \lambda_\alpha(N)$, then
$$\int_B |Df(x)|^N \log^\alpha\left(1 + \frac{|Df(x)|}{|Df|_B}\right) dx \leq C_\alpha(N) \int_{2B} J(x, f)\, dx$$
for any concentric balls $B \subset 2B \subset \Omega$, where $|Df|_B$ stands for the integral average of $|Df|$ over the ball B.

Let us give an idea of the proof. Consider the Beltrami equation corresponding to the mapping f, that is,
$$D^t f(x) Df(x) = J(x, f)^{2/N} G(x).$$
Associated with $G(x)$ is the energy integral
$$\mathcal{E}[f] = \int_\Omega E(x, Df)\, dx$$
where
$$E(x, Df) = \langle Df G^{-1}(x), Df \rangle^{N/2}.$$
The Euler–Lagrange equation takes the form
$$\operatorname{div} A(x, Df) = 0 \tag{5.21}$$
where
$$A(x, Df) = \langle Df G^{-1}(x), Df \rangle^{(N-2)/2} Df G^{-1}(x).$$
The energy integrand takes the form
$$E(x, Df) = N^{N/2} J(x, f)\, dx.$$
From the elementary inequality
$$\frac{1}{N} |M|^N + \frac{N-1}{N} |A(x, M)|^{N/N-1} \leq K(x) E(x, M)$$
valid for every matrix $M \in \mathbb{R}^{N \times N}$ it is possible to deduce that a mapping of distortion $K = K(x)$ gives rise to an N-harmonic couple $\Phi = [A(x, Df), Df]$ (see Chapter 3).

A suitable approximation of equation (5.21) leads to a sequence of solutions for which the following *a priori* estimate holds concerning integrability properties of p-harmonic couples. This is provided in [37].

THEOREM 5.16. *Let $h \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a mapping of finite distortion. For every integer $m \geq 0$ there exists $\lambda_p(m, N) \geq 1$ such that if the distortion function $K(x)$ satisfies*

$$\int_{\Omega} e^{\lambda K(x)}\, dx < \infty$$

with some $\lambda \geq \lambda_p(m, N)$ then

$$\|Dh\|^p_{L^p \log^m L(\Omega')} \leq C_p(\Omega') \int_{\Omega} \langle A(x, Dh), Dh\rangle\, dx \tag{5.22}$$

for every compact subset $\Omega' \subset \Omega$.

Let us emphasize explicitly that this result requires the left hand side to be finite and this can be ensured by assuming that the exponent p is close to N. More specifically, $N - 1/2 \leq p \leq N$.

It is important to observe that the estimates are preserved in passing to the limit. As a consequence, the following modulus of continuity estimate for mappings of exponentially integrable distortion holds:

COROLLARY 5.17. *For each dimension $N \geq 2$ and $s \geq 0$ there exists $\lambda_s(N) \geq 1$ such that if the distortion function $K = K(x)$ of $f : \Omega \to \mathbb{R}^N$ satisfies*

$$\int_{\Omega} e^{\lambda K(x)}\, dx < \infty$$

for some $\lambda \geq \lambda_s(N)$, then

$$|f(x) - f(y)| \leq \frac{c(N, s)}{\log^s\left(\frac{2R}{|x-y|}\right)} \Big[\int_{B(a,6R)} J(x, f)\, dx \Big]^{1/N}$$

whenever $x, y \in B(a, R) \subset B(a, 6R) \subset \Omega$.

6. Lower semicontinuity of a class of multiple integrals

6.1. Introduction. In this chapter we discuss the lower semicontinuity of an integral functional of the type

$$F(u) = \int_{\Omega} f(x, u, \mathcal{L}u)\, dx$$

where $u \in W^{1,p}(\Omega, \mathbb{R}^d)$, f is a nonnegative integrand satisfying the growth condition

$$0 \leq f(x, s, \xi) \leq c(1 + |\xi|^q) \tag{6.1}$$

$q \geq p > 1$, and $\mathcal{L}$ is a linear differential operator of first order, $\mathcal{L} : C^\infty(\Omega, \mathbb{R}^d) \to C^\infty(\Omega, \mathbb{R}^m)$.

In the special case $\mathcal{L}u = \nabla u$ and $q = p$, there is a vast literature on the lower semicontinuity properties of F (see for instance [46, 47, 1, 43, 41]).

More recently, in connection with the applications to materials exhibiting nonstandard elastic and magnetic behaviours, researchers have been interested in lower semicontinuity also when $p < q$ and $\mathcal{L}$ is a general linear operator of first order (see [15–17]). To fix ideas assume that $\mathcal{L} : C^\infty(\mathbb{R}^N, \mathbb{R}^d) \to C^\infty(\mathbb{R}^N, \mathbb{R}^m)$ is defined by

$$\mathcal{L}u = \sum_{k=1}^{N} A_k \frac{\partial u}{\partial x_k} \tag{6.2}$$

where A_k, $k = 1, \dots, N$, are given linear transformations of $\mathbb{R}^d$ into $\mathbb{R}^m$. In [22] our main result, when f depends only on ξ, is the following.

THEOREM 6.1. *Assume* $q \ge p > \max\{1, q(N-1)/N\}$. *Let* $f = f(\xi) : \mathbb{R}^m \to [0, \infty)$ *be a function satisfying* (6.1) *and* $\mathcal{L}$ *a linear differential operator of the type* (6.2). *Assume that for any* $A \in \mathbb{R}^{N\times d}$ *and any* $u \in C_0^\infty(Q, \mathbb{R}^d)$ *we have*

$$\int_Q [f(\mathcal{L}(Ax + u(x)) - f(\mathcal{L}(Ax))]\, dx \ge 0$$

where $Q = (0,1)^N$ *is the unit cube. Then for any* $u \in W^{1,p}(\Omega, \mathbb{R}^d)$ *and any sequence* $u_n \in W^{1,q}(\Omega, \mathbb{R}^d)$ *such that* $u_n \rightharpoonup u$ *weakly in* $W^{1,p}(\Omega, \mathbb{R}^d)$ *we have*

$$\int_\Omega f(\mathcal{L}u(x))\, dx \le \liminf_{n\to\infty} \int_\Omega f(\mathcal{L}u_n(x))\, dx.$$

This result, very much in the spirit of the lower semicontinuity results of Fonseca–Malý and Fonseca–Marcellini, is proved by a blow-up argument. Similar arguments are also used to extend the result to the case when f depends both on x and s.

In this framework it is natural to consider the particular case $u = (v, w)$ and $\mathcal{L}u = (\mathcal{P}v, \mathcal{Q}^*w)$ where $\mathcal{P}, \mathcal{Q}$ are linear differential operators of first order with constant coefficients forming an elliptic complex (see Chapter 3 for the definition).

It is easy to check that any functional of the type

$$G(u) = \int_\Omega g(\langle \mathcal{P}v, \mathcal{Q}^*w\rangle)\, dx, \tag{6.3}$$

where $g : \mathbb{R} \to [0, \infty)$ is convex, is quasiconvex in u. Hence Theorem 6.1 implies the lower semicontinuity of G with respect to the weak convergence in $W^{1,p}$ for all $p > (2(N-1))/N$. Functionals of type (6.3) can be viewed as a generalization of the usual polyconvex functionals. In fact if $N = 2$, taking

$$\mathcal{P}u = \nabla u, \qquad \mathcal{Q}v = \operatorname{curl} v = \frac{\partial v^2}{\partial x} - \frac{\partial v^1}{\partial y},$$

$u \in C^\infty(\mathbb{R}^2, \mathbb{R})$, $v \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$, one has an elliptic complex and $\langle \mathcal{P}u, \mathcal{Q}^*w\rangle$ is equal to the determinant of the matrix whose rows are given by ∇u and ∇w.

We shall make use of the following definition of quasiconvexity:

DEFINITION 6.2. Let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function. We say that f is *quasiconvex with respect to the operator* $\mathcal{L}$ if for almost every $x_0 \in \Omega$, for any $s_0 \in \mathbb{R}^d$ and any matrix $A \in \mathbb{R}^{N\times d}$ we have

$$\int_Q [f(x_0, s_0, \mathcal{L}(Ax + u(x))) - f(x_0, s_0, \mathcal{L}(Ax))]\, dx \ge 0 \tag{6.4}$$

for all $u \in C_0^\infty(Q, \mathbb{R}^d)$, where $Q = (0,1)^N$ is the unit cube.

Notice that by a density argument it follows that if $|f(x,s,\xi)| \le c(1+|\xi|^q)$, then (6.4) holds with $u \in W_0^{1,q}(Q,\mathbb{R}^d)$.

6.2. Main result. This section is devoted to the proof of Theorem 6.1. We consider fixed exponents $r, q \ge 1$ and $p > \max\{1, r(N-1)/N, q(N-1)/N\}$. The following lemma, proved by Fonseca–Malý [15], is very useful in what follows.

LEMMA 6.3. *Let $V \subset\subset \Omega$ and $W \subset \Omega$ be open sets, $\Omega = V \cup W$, $v \in W^{1,q}(V)$ and $w \in W^{1,q}(W)$. Let $m \in \mathbb{N}$. There exist a function $z \in W^{1,q}_{\mathrm{loc}}(\Omega)$ and open sets $V' \subset V$ and $W' \subset W$, such that $V' \cup W' = \Omega$, $z = v$ on $\Omega - W'$, $z = w$ on $\Omega - V'$,*

$$L^N(V' \cap W') \le Cm^{-1}$$

and

$$\begin{aligned}\|z\|_{L^r(V'\cap W')} &+ \|z\|_{W^{1,q}(V'\cap W')}\\ &\le Cm^{-\tau}(\|v\|_{W^{1,p}(V\cap W)} + \|w\|_{W^{1,p}(V\cap W)} + m\|w-v\|_{L^p(V\cap W)})\end{aligned}$$

where $C = C(p,q,r,V,W)$ and $\tau = \tau(N,p,q,r) > 0$.

In what follows we denote by $B_\varrho(x)$ the ball $\{y \in \mathbb{R}^N : |y-x| < \varrho\}$; if the centre of the ball is the origin we will simply write B_ϱ instead of $B_\varrho(0)$.

Proof of Theorem 6.1. The proof falls naturally into two parts.

Step 1. We prove the result in the special case that $\Omega = B_1$ and u is linear, $u(x) = Ax$ for $A \in \mathbb{R}^{N\times d}$. According to Rellich's compact imbedding theorem, we may assume that

$$\|u_n - u\|_{L^p} \le n^{-1}.$$

Let $R < 1$ and $\varrho = (R+1)/2$. We apply the lemma above to $v = u_n$, $w = u$, $V = B_\varrho$ and $W = B_1 \setminus B_\varrho$. Accordingly, we obtain $z_n \in W^{1,q}(B_1,\mathbb{R}^d)$ and open sets $V_n \subset\subset V$, $W_n \subset W$ such that $V_n \cup W_n = B_1$,

$$z_n = u_n \quad \text{on } B_1 \setminus W_n, \qquad z_n = u \quad \text{on } B_1 \setminus V_n$$

and

$$L^N(V_n \cap W_n) \le \frac{c(R)}{n}, \qquad \int_{V_n\cap W_n} |\mathcal{L}z_n|^q \le \frac{c(R,M)}{n^{\tau q}}$$

where $M = \sup \|u_n\|_{W^{1,p}}$ and $\tau > 0$ is the exponent provided by Lemma 6.3. Since $z_n - u \in W_0^{1,q}(B_1,\mathbb{R}^d)$, from the growth condition and the quasiconvexity of f, we have

$$\int_{B_1} f(\mathcal{L}u) \le \int_{B_1} f(\mathcal{L}z_n).$$

Therefore

$$\begin{aligned}\int_{B_1} f(\mathcal{L}u) - \int_{B_1} f(\mathcal{L}u_n) &\le \int_{B_1} f(\mathcal{L}z_n) - \int_{B_1} f(\mathcal{L}u_n) \le \int_{B_1\setminus V_n} f(\mathcal{L}u) + \int_{V_n\cap W_n} f(\mathcal{L}z_n)\\ &\le cL^N(B_1\setminus V_n) + \int_{V_n\cap W_n} (1+|\mathcal{L}z_n|^q)\\ &\le c(L^N(B_1\setminus B_\varrho) + n^{-1} + n^{-\tau q}) \le c(1-R+n^{-1}+n^{-\tau q}).\end{aligned}$$

The conclusion follows by letting first $n \to \infty$ and then $\varrho \to 1$.

Step 2. Let $u \in W^{1,p}(\Omega,\mathbb{R}^d)$, $u_n \in W^{1,q}(\Omega,\mathbb{R}^d)$, $u_n \rightharpoonup u$ in $W^{1,p}(\Omega,\mathbb{R}^d)$. With no loss of generality we may assume that

$$\liminf_{n\to\infty} \int_\Omega f(\mathcal{L}u_n)\,dx = \lim_{n\to\infty} \int_\Omega f(\mathcal{L}u_n)\,dx < \infty.$$

Passing to a subsequence if necessary, we obtain the existence of finite Radon nonnegative measures μ and ν such that

$$f(\mathcal{L}(u_n(x))) \to \mu \quad w^*\text{-}\mathcal{M}(\Omega), \qquad |\mathcal{L}u_n|^p \to \nu \quad w^*\text{-}\mathcal{M}(\Omega),$$

where $\mathcal{M}(\Omega)$ is the space of all Radon measures. Now our purpose is to prove that for L^N-a.e. $x_0 \in \Omega$,

$$\frac{d\mu}{dL^N}(x_0) = \lim_{\varrho\to 0^+} \frac{\mu(B_\varrho(x_0))}{\omega_N\,\varrho^N} \geq f(\mathcal{L}u(x_0)). \tag{6.5}$$

In fact if (6.5) is true, then for any $\varphi \in C_c(\Omega)$, $0 \leq \varphi \leq 1$, we have

$$\lim_{n\to\infty} \int_\Omega f(\mathcal{L}u_n) \geq \lim_{n\to\infty} \int_\Omega \varphi f(\mathcal{L}u_n) = \int_\Omega \varphi\,d\mu \geq \int_\Omega \varphi \frac{d\mu}{dL^N}\,dx \geq \int_\Omega \varphi f(\mathcal{L}u).$$

Therefore letting $\varphi \to 1$ and applying the monotone convergence theorem we may conclude that

$$\lim_{n\to\infty} \int_\Omega f(\mathcal{L}u_n) \geq \int_\Omega f(\mathcal{L}u).$$

It remains to prove (6.5). Let $x_0 \in \Omega$ be such that the limits

$$\frac{d\mu}{dL^N}(x_0) = \lim_{\varrho\to 0^+} \frac{\mu(B_\varrho(x_0))}{\omega_N\,\varrho^N}, \qquad \frac{d\nu}{dL^N}(x_0) = \lim_{\varrho\to 0^+} \frac{\nu(B_\varrho(x_0))}{\omega_N\,\varrho^N}$$

exist and are finite and

$$\lim_{\varrho\to 0^+} \frac{1}{\varrho} \fint_{B_\varrho(x_0)} |u(y) - u(x_0) - \nabla u(x_0)(y - x_0)|\,dy = 0.$$

Note that the last three conditions are satisfied by all points $x_0 \in \Omega$, except maybe on a set of L^N-measure zero. Then we select $\varrho_k \to 0^+$ such that $\mu(\partial B_{\varrho_k}(x_0)) = 0$, $\nu(\partial B_{\varrho_k}(x_0)) = 0$. Thus

$$\lim_{k\to\infty} \frac{\mu(B_{\varrho_k}(x_0))}{\omega_N\,\varrho_k^N} \geq \lim_{k\to\infty} \limsup_{n\to\infty} \fint_{B_{\varrho_k}(x_0)} f(\mathcal{L}u_n(x))\,dx = \lim_{k\to\infty} \limsup_{n\to\infty} \fint_{B_1} f(\mathcal{L}v_{n,k}(y))\,dy$$

where

$$v_{n,k} = \frac{u_n(x_0 + \varrho_k) - u(x_0)}{\varrho_k}.$$

It follows that $v_{n,k} \in W^{1,q}(B_1,\mathbb{R}^d)$,

$$\lim_{k\to\infty} \lim_{n\to\infty} \|v_{n,k} - \nabla u(x_0)x\|_{L^1(B_1)} = 0$$

and

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \|\mathcal{L}v_{n,k}\|_{L^p(B_1)} \leq \frac{d\nu}{dL^N}(x_0) < \infty.$$

Hence, we may extract a subsequence such that

$$v_{n_k,k} = v_k \rightharpoonup \nabla u(x_0)x \quad \text{weakly in } W^{1,p}(B_1,\mathbb{R}^d)$$

and

$$\frac{d\mu}{dL^N}(x_0) = \lim_{k\to\infty} \fint_{B_1} f(\mathcal{L}v_k(y))\,dy.$$

Therefore from Step 1 we get

$$\frac{d\mu}{dL^N}(x_0) = \lim_{k\to\infty} \fint_{B_1} f(\mathcal{L}v_k(y))dy \geq f(\mathcal{L}u(x_0))$$

and this concludes the proof.

It is possible to show the following extension of Theorem 6.1:

THEOREM 6.4. *Suppose that $f(x,s,\xi)$ satisfies the following conditions*:

(i) $f(x,s,\xi)$ *is quasiconvex*;
(ii) $0 \leq f(x,s,\xi) \leq c(1+|\xi|^q)$;
(iii) *for any* $(x_0,s_0) \in \Omega\times\mathbb{R}^d$ *and any* $\varepsilon>0$, *there exists* $\delta>0$ *such that if* $|x-x_0| < \delta$, $|s-s_0|<\delta$ *and* $\xi\in\mathbb{R}^{N\times d}$ *then* $f(x,s,\xi) \geq (1-\varepsilon)f(x_0,s_0,\xi)$.

Let $u_n \in W^{1,q}(\Omega,\mathbb{R}^d)$ *and* $u\in W^{1,p}(\Omega,\mathbb{R}^d)$ *be such that* $u_n \rightharpoonup u$ *in* $W^{1,p}(\Omega,\mathbb{R}^d)$. *Then*

$$\int_\Omega f(x,u,\mathcal{L}u)\,dx \leq \liminf_{n\to\infty}\int_\Omega f(x,u_n,\mathcal{L}u_n)\,dx.$$

6.3. Polyconvex case. Now let the operator $\mathcal{L}$ be defined by means of a pair of differential operators of first order in N independent variables with constant coefficients

$$C^\infty(\mathbb{R}^N,\mathbb{R}^d) \overset{\mathcal{P}}{\to} C^\infty(\mathbb{R}^N,\mathbb{R}^m) \overset{\mathcal{Q}}{\to} C^\infty(\mathbb{R}^N,\mathbb{R}^k)$$

forming an elliptic complex.

The notion of polyconvex integrands, already given in the book of Morrey [47], was deeply studied by Ball [3] providing a better understanding of several problems, especially those concerning the theory of finite elasticity.

In [22] we prove that Theorem 6.1 still holds if the function f is polyconvex according to the definition given in [21], see Chapter 3.

Note that our definition of polyconvexity agrees with the one given by Ball in dimension two, provided that we take $\mathcal{P}u = \nabla u$, $\mathcal{Q}v = \operatorname{curl} v$.

Let $f(x,y,z,\eta,\xi): \Omega\times\mathbb{R}^{d+k}\times\mathbb{R}^{2m} \to [0,\infty)$ be a Carathéodory function such that

(i) for all $x\in\Omega$, $(y,z)\in\mathbb{R}^d\times\mathbb{R}^k$ the function $(\eta,\xi)\to f(x,y,z,\eta,\xi)$ is polyconvex;
(ii) for any $(x_0,y_0,z_0)\in\Omega\times\mathbb{R}^d\times\mathbb{R}^k$ and any $\varepsilon>0$, there exists $\delta>0$ such that if $|x-x_0|<\delta$, $|(y,z)-(y_0,z_0)|<\delta$ and $\eta,\xi\in\mathbb{R}^{N\times d}$ then $f(x,y,z,\eta,\xi) \geq (1-\varepsilon)f(x_0,y_0,z_0,\eta,\xi)$.

THEOREM 6.5. *Suppose that $f(x,y,z,\eta,\xi)$ satisfies conditions* (i) *and* (ii) *and suppose* $p > 2(N-1)/N$. *Let* $\alpha_n \in W^{1,2}(\Omega,\mathbb{R}^d)$, $\beta_n\in W^{1,2}(\Omega,\mathbb{R}^k)$ *and* $\alpha\in W^{1,p}(\Omega,\mathbb{R}^d)$, $\beta\in W^{1,p}(\Omega,\mathbb{R}^k)$ *be such that* $\alpha_n \rightharpoonup \alpha$ *in* $W^{1,p}(\Omega,\mathbb{R}^d)$ *and* $\beta_n\rightharpoonup\beta$ *in* $W^{1,p}(\Omega,\mathbb{R}^k)$. *Then*

$$\int_\Omega f(x,\alpha,\beta,\mathcal{P}\alpha,\mathcal{Q}^*\beta)\,dx \leq \liminf_{n\to\infty}\int_\Omega f(x,\alpha_n,\beta_n,\mathcal{P}\alpha_n,\mathcal{Q}^*\beta_n)\,dx.$$

Proof. There exists a sequence of continuous nonnegative functions $g_j(x,y,z,\eta,\xi)$ such that each g_j is polyconvex in (η,ξ) and

$$0 \le g_j(x,y,z,\eta,\xi) \le c_j(1+|\langle \eta,\xi\rangle|), \qquad g_j(x,y,z,\eta,\xi) \le g_{j+1}(x,y,z,\eta,\xi),$$
$$f(x,y,z,\eta,\xi) = \sup_j g_j(x,y,z,\eta,\xi)$$

(see Lemma 3.2 in [18]). Observe that polyconvexity implies quasiconvexity (see Chapter 3) and that

$$g_j(x,y,z,\eta,\xi) \le c(1+|\eta|^2+|\xi|^2).$$

Therefore, Theorem 6.1 holds and we have

$$\begin{aligned}\int_\Omega g_j(x,\alpha,\beta,\mathcal{P}\alpha,\mathcal{Q}^*\beta)\,dx &\le \liminf_n \int_\Omega g_j(x,\alpha_n,\beta_n,\mathcal{P}\alpha_n,\mathcal{Q}^*\beta_n)\,dx \\ &\le \liminf_n \int_\Omega f(x,\alpha_n,\beta_n,\mathcal{P}\alpha_n,\mathcal{Q}^*\beta_n)\,dx.\end{aligned}$$

Now notice that since g_j is increasing, we get

$$\begin{aligned}\int_\Omega f(x,\alpha,\beta,\mathcal{P}\alpha,\mathcal{Q}^*\beta)\,dx &= \lim_j \int_\Omega g_j(x,\alpha,\beta,\mathcal{P}\alpha,\mathcal{Q}^*\beta)\,dx \\ &\le \liminf_n \int_\Omega f(x,\alpha_n,\beta_n,\mathcal{P}\alpha_n,\mathcal{Q}^*\beta_n)\,dx.\end{aligned}$$

This concludes the proof.

6.4. Further results. We conclude this chapter with a new result on lower semicontinuity with respect to the strong convergence in $L^1_{\mathrm{loc}}(\Omega)$ for integral functionals defined on $BV(\Omega)$, the subspace of $L^1(\Omega)$ of functions having *bounded variation.* Let us consider the functional

$$F(u,\Omega) = \int_\Omega f(x,Du(x))\,dx \tag{6.6}$$

where the integrand $f=f(x,\xi)$ satisfies the conditions:

$$\begin{cases} f \text{ is continuous in } \Omega\times\mathbb{R}^N, \\ f \text{ is nonnegative in } \Omega\times\mathbb{R}^N, \\ f(x,\xi) \text{ is convex in } \xi\in\mathbb{R}^N \text{ for every } x\in\Omega. \end{cases} \tag{6.7}$$

It is known that the functional (6.6) is not strongly lower semicontinuous if f satisfies only the above continuity and convexity properties.

In 1961 Serrin was the first to give some sufficient conditions for strong lower semicontinuity in the case $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$, see [55]. Later many authors attempted to weaken Serrin's assumption on f, also in the more general setting of $BV(\Omega)$. Nevertheless in all these results some assumptions of uniform continuity, or of uniform lower semicontinuity of $f(x,\xi)$ with respect to x have been made.

In a recent paper [44], the usual additional hypotheses have been replaced by the more general assumption of local Lipschitz continuity in the independent variable x, when $u\in W^{1,1}_{\mathrm{loc}}(\Omega)$.

Following the same idea we are able to extend this result to $u\in BV_{\mathrm{loc}}(\Omega)$, as follows [22].

Theorem 6.6. *Assume that* $f = f(x, \xi)$ *satisfies conditions* (6.7) *and that, for every compact set* $K \subset \Omega \times \mathbb{R}^N$, *there exists a constant* $L = L(K)$ *such that*

$$|f(x_1, \xi) - f(x_2, \xi)| \le L|x_1 - x_2|$$

for every $(x_1, \xi), (x_2, \xi) \in K$. *Then for every* $u_h \in W^{1,1}_{\mathrm{loc}}(\Omega)$, $u \in BV_{\mathrm{loc}}(\Omega)$ *such that* $u_h \to u$ *in* $L^1_{\mathrm{loc}}(\Omega)$, *we have*

$$\liminf_{h\to\infty} \int_\Omega f(x, \nabla u_h)\,dx \ge \int_\Omega f(x, \nabla u)\,dx + \int_\Omega f^\infty(x, D^s u)$$

where f^∞ *is the recession function of* f *and* $D^s u$ *is the singular part of the distributional derivative* Du *with respect to the Lebesgue measure.*

For the proof we will need the following two lemmas. The first one is an approximation result given by De Giorgi [11].

Lemma 6.7. *Let* $f = f(x, \xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}$ *satisfies conditions* (6.7). *Then there exists an increasing sequence of functions* $\{f_j(x, \xi)\}_{j\in\mathbb{N}}$ *that converges to* $f(x, \xi)$ *uniformly on the compact sets of* $\Omega \times \mathbb{R}^N$.

The functions f_j *can be defined as the maximum between the zero functions and a finite number of affine* (*with respect to* $\xi \in \mathbb{R}^N$) *functions*

$$a_{0,j}(x) + \sum_{i=1}^N a_j^{(i)}(x)\xi_i$$

where

$$\begin{cases} a_j^{(i)}(x) = -\int_{\mathbb{R}^N} f(x, \xi) D_i\alpha_j(\xi)\,d\xi \quad \forall i = 1, \dots, N \\ a_{0,j}(x) = \int_{\mathbb{R}^N} f(x, \xi)\{(N+1)\alpha_j(\xi) + \sum_{i=1}^N \xi_i D_i\alpha_j(\xi)\}\,d\xi \end{cases} \tag{6.8}$$

for $\alpha_j \in C_0^\infty(\mathbb{R}^N)$, $\alpha_j \ge 0$, $\int_{\mathbb{R}^N} \alpha_j(\xi)\,d\xi = 1$.

Lemma 6.8. *Let* μ *be a positive* σ*-finite Borel measure in* Ω *and let* $f_i : \Omega \to [0, \infty]$, $i \in \mathbb{N}$, *be Borel functions. Then*

$$\int_\Omega \sup_i f_i\,d\mu = \sup \sum_{i\in I} \int_{A_i} f_i\,d\mu$$

where the supremum ranges over all finite sets $I \subset \mathbb{N}$ *and all families* $\{A_i\}_{i\in I}$ *of pairwise disjoint open sets with compact closure in* Ω.

Proof of Theorem 6.6. Let $\{f_j(x, \xi)\}_{j\in\mathbb{N}}$ be the increasing sequence that converges to $f(x, \xi)$ uniformly on the compact sets of $\Omega \times \mathbb{R}^N$, as in Lemma 6.7.

For each $j \in \mathbb{N}$ the coefficients $a^{(i)}$, $i = 0, \dots, N$, in (6.8) are locally Lipschitz continuous with respect to x; in fact, for a fixed i,

$$|a_j^{(i)}(x_1) - a_j^{(i)}(x_2)| = \Big| \int_{\mathbb{R}^N} \{f(x_1, \xi) - f(x_2, \xi)\} D_i\alpha_j(\xi)\,d\xi \Big| \le m_{i,j} L(K)|x_1 - x_2|$$

for every x_1, x_2 which vary on a compact set K_O of Ω and $m_{i,j}$ given by

$$m_{i,j} = \int_{\mathbb{R}^N} |D_i\alpha_j(\xi)|\,d\xi.$$

Let further $k \in \mathbb{N}$ and let $A_0, \dots, A_k$ be pairwise disjoint open subsets of Ω. For any $j \in \{0, \dots, k\}$ and any $\phi_j \in C_0^1(A_j)$, $0 \le \phi_j \le 1$, we have

$$\begin{aligned}\int_\Omega f(x, \nabla u_h)\, dx &\ge \int_{A_j} a_{0,j}(x)\phi_j(x)\, dx + \int_{A_j} \langle a_j(x), \nabla u_h\rangle \phi_j(x)\, dx \\ &= \int_{A_j} a_{0,j}(x)\phi_j(x)\, dx + \int_{A_j} \left\langle \frac{\partial a_j}{\partial x}(x), u_h \right\rangle \phi_j(x)\, dx \\ &\quad + \int_{A_j} \langle a_j(x), u_h\rangle \nabla\phi_j(x)\, dx.\end{aligned}$$

By Lipschitz continuity of the functions $a_j^{(i)}(x)$, $i = 1, \dots, N$, we have

$$\left|\frac{\partial a_j^{(i)}}{\partial x}(x)\right| \le L.$$

Therefore, since $u_h \to u$ in $L^1_{\mathrm{loc}}(\Omega)$, we get

$$\begin{aligned}&\liminf_{h\to\infty} \int_\Omega f(x, \nabla u_h)\, dx \\ &\quad\ge \int_{A_j} a_{0,j}(x)\phi_j(x)\, dx + \int_{A_j} \left\langle \frac{\partial a_j}{\partial x}(x), u \right\rangle \phi_j(x)\, dx + \int_{A_j} \langle a_j(x), u\rangle \nabla\phi_j(x)\, dx \\ &\quad= \int_{A_j} a_{0,j}(x)\phi_j(x)\, dx + \int_{A_j} \langle a_j(x), Du\rangle \phi_j(x)\, dx \\ &\quad= \int_{A_j} [a_{0,j}(x) + \langle a_j(x), \nabla u\rangle]\phi_j(x)\, dx + \int_{A_j} \langle a_j(x), D^s u\rangle \phi_j(x).\end{aligned}$$

Taking supremum with respect to the ϕ_j above we have

$$\liminf_h \int_\Omega f(x, \nabla u_h)\, dx \ge \int_{A_j} [a_{0,j}(x) + \langle a_j(x), \nabla u\rangle]^+\, dx + \int_{A_j} \langle a_j(x), D^s u\rangle^+.$$

Since k and A_j are arbitrary, by Lemma 6.8 we conclude that

$$\liminf_h \int_\Omega f(x, \nabla u_h)\, dx \ge \int_\Omega f(x, \nabla u)\, dx + \int_\Omega f^\infty(x, D^s u).$$

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