Contents

Abstract

The spaces $L^1(m)$ of all m-integrable (resp. $L^1_w(m)$ of all scalarly m-integrable) functions for a vector measure m, taking values in a complex locally convex Hausdorff space X (briefly, lcHs), are themselves lcHs for the mean convergence topology. Additionally, $L^1_w(m)$ is always a complex vector lattice; this is not necessarily so for $L^1(m)$. To identify precisely when $L^1(m)$ is also a complex vector lattice is one of our central aims. Whenever X is sequentially complete, then this is the case. If, additionally, the inclusion $L^1(m) \subseteq L^1_w(m)$ (which always holds) is *proper*, then $L^1(m)$ and $L^1_w(m)$ contain lattice-isomorphic copies of the complex Banach lattices c_0 and ℓ^∞, respectively. On the other hand, whenever $L^1(m)$ contains an isomorphic copy of c_0, merely in the lcHs sense, then necessarily $L^1(m) \subsetneq L^1_w(m)$. Moreover, the X-valued integration operator $I_m : f \mapsto \int f\, dm$, for $f \in L^1(m)$, then fixes a copy of c_0. For X a Banach space, the validity of $L^1(m) = L^1_w(m)$ turns out to be *equivalent* to I_m being weakly completely continuous. A sufficient condition for this is the $(q, 1)$-concavity of I_m for some $1 \leq q < \infty$. This criterion is fulfilled when I_m belongs to various classical operator ideals. Unlike for $L^1_w(m)$, the space $L^1(m)$ can never contain an isomorphic copy of ℓ^∞. A rich supply of examples and counterexamples is presented. The methods involved are a hybrid of vector measure/integration theory, functional analysis, operator theory and complex vector lattices.

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1. Introduction and main results

The theory of Banach-space-valued vector measures, their associated L^1-space and the integration operator is well established, with myriad applications in classical and harmonic analysis, the geometry of Banach spaces and functional analysis; see, for example, the monographs [14], [18], [19], [31], [46], [57], and the references therein. The moment one considers, for example, a Banach space in its weak topology or applies the Banach space theory pointwise to *operator-valued* measures acting in a Banach space (for the weak or strong operator topology), then the natural framework shifts to the setting of vector measures taking values in a *locally convex Hausdorff space* (briefly, lcHs); see, for example, [31], [47], [52]. Moreover, the lcHs involved are typically no longer sequentially complete (just consider the Banach space c_0 in its weak topology, or the space of all continuous linear operators on c_0 equipped with the weak operator topology). This causes various inherent difficulties (see e.g. [41], [53]), in particular, concerning the interplay of the topological properties of the L^1-space of integrable functions and its order properties as a vector lattice of functions. Given the richness of this interplay for Banach-space-valued (or *sequentially complete* lcHs-valued) vector measures, it is desirable to better understand this interplay in general, i.e., without sequential completeness as an a priori assumption. Moreover, applications demand a theory for *complex* spaces. However, in this setting, it turns out that the L^1-space of integrable functions (over $\mathbb{C}$) may even fail to be a complex vector lattice! Nevertheless, it will be seen that many important structural results (well known for sequentially complete spaces over $\mathbb{R}$) still carry over in general. It is time to be more precise.

Let X be a lcHs, over $\mathbb{C}$. Consider an X-valued vector measure (i.e., a σ-additive set function) m defined on a measurable space (Ω, Σ). Associated with m are the complex vector space $L^1(m)$ of all (equivalence classes of) $\mathbb{C}$-valued m-integrable functions on Ω and the larger complex vector space $L^1_w(m)$ of all (equivalence classes of) $\mathbb{C}$-valued, scalarly m-integrable functions on Ω. We equip $L^1(m)$ with the mean convergence topology $\tau(m)$ (i.e., the topology of uniform convergence of indefinite integrals), which can be extended to a lcH-topology $\tau(m)_w$ on $L^1_w(m)$ so that $\tau(m)_w$, in turn, induces $\tau(m)$ on $L^1(m)$. Section 2 presents the basic concepts and results concerning the lcHs $L^1(m)$ and $L^1_w(m)$.

The main aim of this paper is to determine when $L^1(m)$ or $L^1_w(m)$ contains a lattice-isomorphic copy of the complex Banach lattice c_0 or of ℓ^∞, which is related to the inclusion $L^1(m) \subseteq L^1_w(m)$ being proper (or to $L^1(m) = L^1_w(m)$). Of course, such a lattice-isomorphic copy is also an isomorphic copy in the lcHs sense.

Results concerning lattice-isomorphic copies of $(c_0)_{\mathbb{R}} := c_0 \cap \mathbb{R}^{\mathbb{N}}$ and $(\ell^\infty)_{\mathbb{R}} := \ell^\infty \cap \mathbb{R}^{\mathbb{N}}$ in general locally solid Riesz spaces X over $\mathbb{R}$ (which are topologically complete or, at least, sequentially complete) can be found, for example, in [21], [59] and the references therein. The criteria there are typically in terms of order properties of X (e.g. Dedekind σ-completeness) and/or properties related to the topology of X (e.g., σ-Levi, σ-Lebesgue). Our aim is to focus directly on the special features of the particular complex vector spaces $L^1(m)$ and $L^1_w(m)$, which are the central spaces of importance here, rather than attempting to extend such real results to general complex Riesz spaces.

Our motivation originates from the special case when X is a Banach space; $L^1(m)$ is then a complex Banach lattice. For instance, the monograph [46, Ch. 3] treats $L^1(m)$ from the viewpoint of complex Banach lattices. According to [10, Theorem 2.2], [11, Theorem 2], if the Banach space X does not contain an isomorphic copy of c_0, then neither does $L^1(m)$; this seems to be the first result regarding the existence or not of an isomorphic copy of c_0 in $L^1(m)$. Moreover, whenever X does not contain an isomorphic copy of c_0 (equivalently, if X is weakly Σ-complete), then necessarily $L^1(m) = L^1_w(m)$ [31, Theorem II.5.1], [34, Theorem 5.1]; see also Lemmas 2.2 and 2.5(iv) below. The question of the role played by these two necessary conditions, namely that $L^1(m)$ does not contain an isomorphic copy of c_0 and that $L^1(m) = L^1_w(m)$, already has an answer (following from the well known case when the codomain space of m is a real Banach space). Namely, for a general Banach-space-valued vector measure m, the following three conditions are equivalent:

(A) $L^1(m) = L^1_w(m)$;
(B) $L^1(m)$ does not contain a *lattice*-isomorphic copy of c_0;
(C) $L^1(m)$ does not contain an isomorphic copy of c_0;

see Proposition 3.11. The equivalence of these three conditions for a vector measure with values in a *real* Fréchet space is also available (together with other equivalent conditions) [8, Proposition 3.4].

The core of this paper is the investigation of the above conditions (A)–(C) for a general lcHs-valued vector measure m, together with the question of whether or not $L^1_w(m)$ contains a lattice-isomorphic copy of c_0 or of ℓ^∞. We provide criteria which guarantee the equivalence of these three conditions. Let us emphasize again that the codomain space X of m is a *complex* lcHs and, hence, so are the lcHs $L^1(m)$ and $L^1_w(m)$. For tackling the problem of whether a lattice copy of the complex Banach lattice c_0 or ℓ^∞ is in $L^1(m)$ and $L^1_w(m)$, it is first necessary to determine whether or not $L^1(m)$ and $L^1_w(m)$ are themselves *complex vector lattices*. Accordingly, in Section 3 we define complex vector lattices and present relevant results which can be applied to $L^1(m)$ and $L^1_w(m)$. To be precise, a complex vector lattice is defined as the complexification $E_{\mathbb{C}} := E + iE$ of a real vector lattice E having the "complex modulus property", which then enables us to consider both complex conjugation and forming the modulus in $E_{\mathbb{C}}$. This class of complex vector lattices is strictly larger than that of [55] and [60]. Indeed, according to [55, Definition II.11.1], a complex vector lattice is the complexification of a real vector lattice E satisfying Axiom (OS); it is Axiom (OS) that guarantees the complex modulus property of E. On the other hand, a complex vector lattice according to [60, §91] is the complexification of a real vector lattice which is both Archimedean and uniformly complete. It turns out that a real vector lattice satisfies Axiom (OS) if and only if it is Archimedean and uniformly complete. Consequently, the class of complex vector lattices in [55] coincides with that in [60]; see Remark 3.3.

Now, let $L^1(m)_{\mathbb{R}}$ (resp. $L^1_w(m)_{\mathbb{R}}$) be the real vector subspace of $L^1(m)$ (resp. $L^1_w(m)$) consisting of all (equivalence classes of) $\mathbb{R}$-valued, m-integrable (resp. scalarly m-integrable) functions. Then $L^1(m)_{\mathbb{R}}$ is a real vector lattice with respect to the m-a.e. pointwise order. The complex vector space $L^1(m)$ is a complex vector lattice for the m-a.e. pointwise order, realized as the complexification of $L^1(m)_{\mathbb{R}}$, if and only if it is closed under complex conjugation and under forming the modulus, both being defined *pointwise* m-a.e. In this case the complex conjugation and modulus formed pointwise m-a.e. coincide with those *intrinsic* to the complex vector lattice $L^1(m)$; see Proposition 3.7(ii). Via this criterion, parts (i) and (iii) of Example 3.9 provide normed-space-valued vector measures whose associated L^1-space is a complex vector lattice in our sense, but *not* in the sense of [55] and [60]. In contrast, parts (iv) and (v) of Example 3.9 exhibit normed-space-valued vector measures whose associated L^1-spaces are not complex vector lattices at all. On the other hand, it turns out that $L^1_w(m)$ is *always* a complex vector lattice in the m-a.e. pointwise order, realized as the complexification of $L^1_w(m)_{\mathbb{R}}$; see Proposition 3.7(iv).

Let X, Y be lcHs and $T : X \to Y$ be a continuous linear operator. Then T is said to *fix a copy of* c_0 if there exist a complete subspace W of X (for the relative topology) and a bi-continuous isomorphism of c_0 onto W such that the restriction $T|_W : W \to Y$ is a bi-continuous isomorphism onto its range in Y.

It is time to formulate our three main results. The first two of them hold for a general lcHs-valued measure, independent of whether or not $L^1(m)$ is a complex vector lattice.

THEOREM 1.1. *Let $m : \Sigma \to X$ be a lcHs-valued vector measure whose associated lcHs $L^1(m)$ contains an isomorphic copy of the Banach space c_0. Then:*

(i) *The inclusion $L^1(m) \subseteq L^1_w(m)$ is proper.*
(ii) *The integration operator $I_m : L^1(m) \to X$ fixes a copy of c_0.*

Section 4 is devoted to the proof of Theorem 1.1 and some immediate consequences together with relevant examples.

A natural question suggested by Theorem 1.1 is whether or not there exists a lcHs-valued vector measure m for which $L^1(m)$ contains an isomorphic copy of ℓ^∞. The answer is negative.

THEOREM 1.2. *Let $m : \Sigma \to X$ be any lcHs-valued vector measure. Then its associated lcHs $L^1(m)$ does not contain an isomorphic copy of the Banach space ℓ^∞.*

The proof of Theorem 1.2 is presented in Section 5. Important is the fact that the L^1-space of a Banach-space-valued vector measure is always a complex Banach lattice with *order continuous* norm.

The third theorem is the following one.

THEOREM 1.3. *Let X be a sequentially complete lcHs and $m : \Sigma \to X$ be a vector measure. Then:*

(i) *The space* $L^1(m)$ *is a complex vector lattice for the* m*-a.e. pointwise order.*
(ii) *If* $L^1(m) \subsetneq L^1_w(m)$, *then* $L^1(m)$ *and* $L^1_w(m)$ *contain lattice-isomorphic copies of the complex Banach lattices* c_0 *and* ℓ^∞, *respectively.*

In Section 6 we first establish Theorem 1.3 and then analyze some of the earlier results together with various illustrative examples. To summarize Section 6 briefly, let us consider the following conditions for a general lcHs-valued vector measure m:

(a) $L^1(m) \subsetneq L^1_w(m)$;
(b) $L^1(m)$ is a complex vector lattice in the m-a.e. pointwise order and contains a *lattice*-isomorphic copy of c_0;
(c) $L^1(m)$ contains an isomorphic copy of the Banach space c_0; and
(d) $L^1_w(m)$ contains a *lattice*-isomorphic copy of ℓ^∞.

Whenever X is sequentially complete, Theorems 1.1–1.3 together guarantee the equivalence of all four of the conditions (a)–(d). Consequently, this conclusion remains valid whenever $L^1(m) = L^1(J \circ m)$, with J denoting the natural embedding of a general lcHs X into its *sequential completion* $\widetilde{X}$. A special case of this occurs when $L^1(m)$ is sequentially complete as then $L^1(m) = L^1(J \circ m)$ follows; see Lemma 2.7(ii). The just mentioned sufficient conditions for the equivalence of (a)–(d) are formally stated in Proposition 6.4. Furthermore, in Example 6.5 it is shown that the sequential completeness of X need not always imply that of $L^1(m)$ and, vice versa, that the equality $L^1(m) = L^1(J \circ m)$ does not imply the sequential completeness of X or of $L^1(m)$, in general. Actually, the identity $L^1(m) = L^1(J \circ m)$ is characterized by the *σ-monotone completeness property* of $L^1(m)$; see Lemma 6.6.

For a general lcHs-valued vector measure m, we always have

$$(b)\Rightarrow(c)\Rightarrow(d)\Rightarrow(a).$$

The reverse implications are false, in general. Indeed, we present examples corresponding to (c)$\not\Rightarrow$(b), (d)$\not\Rightarrow$(c) and (a)$\not\Rightarrow$(d); see Examples 6.8, 6.11 and 2.6(iii), respectively.

The final Section 7 is devoted to an investigation of the validity of $L^1(m) = L^1_w(m)$ for a Banach-space-valued vector measure $m : \Sigma \to X$. For instance, it is known that this is the case whenever the integration operator $I_m : L^1(m) \to X$ is either weakly compact or completely continuous. In Proposition 7.7 it is established that $L^1(m) = L^1_w(m)$ holds *precisely* when I_m is *weakly completely continuous*, a class of linear operators first considered by J. Dieudonné and A. Grothendieck. Utilizing the fact that the domain space $L^1(m)$ of I_m is a *Banach lattice*, it is shown in Proposition 7.9(i) that the $(q, 1)$-concavity of I_m, for some $1 \le q < \infty$, is sufficient (but not necessary; see Example 7.11) for I_m to be weakly completely continuous. This “concavity criterion” is satisfied by the membership of I_m in various classical operator ideals.

2. Preliminaries and basic results

Vector spaces over $\mathbb{C}$ are simply called vector spaces here. When the scalar field $\mathbb{C}$ needs to be emphasized, we speak of complex vector spaces. Vector spaces over $\mathbb{R}$ are usually called real vector spaces. Of course, vector subspaces of a real vector space are understood to be over $\mathbb{R}$ even when we do not say real vector subspaces.

Let X be a lcHs (over $\mathbb{C}$) with topological dual space X^*. The duality between X and X^* is denoted by $\langle x, x^*\rangle := x^*(x)$, for $x \in X$, $x^* \in X^*$. We denote by $\mathcal{P}(X)$ the class of all continuous seminorms on X. Given $p \in \mathcal{P}(X)$, let $U_p := \{x \in X : p(x) \le 1\}$. The *polar set* $U_p^\circ := \{x^* \in X^* : |\langle x, x^*\rangle| \le 1 \text{ for all } x \in U_p\}$ of U_p can be rewritten as

$$U_p^\circ = \{x^* \in X^* : |\langle x, x^*\rangle| \le p(x) \text{ for all } x \in X\}. \tag{2.1}$$

Let $X/p^{-1}(\{0\})$ be the quotient normed space associated with p and let X_p denote its Banach space completion with norm $\|\cdot\|_{X_p}$. The canonical map $\pi_p : X \to X_p$ is continuous and linear. Moreover, (2.1) gives

$$\pi_p^*(\mathbb{B}[X_p^*]) = U_p^\circ, \tag{2.2}$$

where π_p^* is the *adjoint map* of π_p from the dual space X_p^* (of X_p) into X^* and $\mathbb{B}[X_p^*]$ is the closed unit ball of X_p^* with respect to the dual norm.

A sequence $\{x_n\}_{n=1}^\infty$ in a lcHs X is called *summable* if the sequence $\{\sum_{n=1}^N x_n\}_{N=1}^\infty$ of its partial sums is convergent in X (with respect to its given topology). It will be clearly indicated whenever we consider different lcH-topologies on X, such as the weak topology $\sigma(X, X^*)$, for example. A sequence $\{x_n\}_{n=1}^\infty$ in X is *subseries summable* if every subsequence $\{x_{n(k)}\}_{k=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ is summable, and is *unconditionally summable* if $\{x_{\tau(n)}\}_{n=1}^\infty$ is summable for every permutation $\tau : \mathbb{N} \to \mathbb{N}$.

LEMMA 2.1. *Let X be a lcHs.*

(i) *Every subseries summable sequence in X is unconditionally summable. If X is sequentially complete, then the converse also holds.*
(ii) *A sequence in X is subseries summable in the weak topology if and only if it is subseries summable for the given topology in X.*

Proof. (i) See [27, Proposition 14.6.2 and Corollary 14.6.6].

(ii) See [37, Theorem 1]. ∎

Part (ii) of Lemma 2.1 is known as the Orlicz–Pettis Theorem, a survey of which can be found in [30], for example.

A sequence $\{x_n\}_{n=1}^\infty$ in a lcHs X is called *weakly absolutely Cauchy* if $\sum_{n=1}^\infty |\langle x_n, x^*\rangle| < \infty$ for all $x^* \in X^*$. Our terminology is essentially the same as that in [27]. Indeed, our weakly absolutely Cauchy sequences are exactly those absolutely Cauchy sequences in the weak topology according to [27, p. 305]. In Banach space theory, the fact that $\{x_n\}_{n=1}^\infty$ is weakly absolutely Cauchy is also expressed by saying that the formal series $\sum_{n=1}^\infty x_n$ is weakly unconditionally Cauchy [38, p. 390]. According to [57, p. 5], the lcHs X is said to be *weakly Σ-complete* if every weakly absolutely Cauchy sequence in X is summable in the weak topology. This concept was already considered earlier in the Banach space setting [5].

LEMMA 2.2. *Let X be a lcHs. The following conditions are equivalent:*

(i) *X is weakly Σ-complete.*
(ii) *Every weakly absolutely Cauchy sequence in X is subseries summable in the weak topology.*

(iii) *Every weakly absolutely Cauchy sequence in X is subseries summable in the given topology of X.*
(iv) *Every weakly absolutely Cauchy sequence in X is summable in the given topology of X.*

Proof. (i)⇒(ii). Use the fact that all the subsequences of a weakly absolutely Cauchy sequence are again weakly absolutely Cauchy.

(ii)⇒(iii). Apply the Orlicz–Pettis Theorem, i.e., Lemma 2.1(ii).

(iii)⇒(iv)⇒(i). These implications are clear. ∎

In [31, p. 31], a lcHs X satisfying condition (iv) above is said to have the (B-P)-*property*, so that X being weakly Σ-complete is equivalent to X having the (B-P)-property via Lemma 2.2.

Given a lcHs Y, let $\mathcal{L}(X,Y)$ denote the vector space of all continuous linear maps from X into Y. When $X = Y$, we write $\mathcal{L}(X) := \mathcal{L}(X,X)$. The *range* of an operator $T \in \mathcal{L}(X,Y)$ is denoted by $\mathcal{R}(T)$; it is always equipped with the relative topology from Y. A useful fact is that every $T \in \mathcal{L}(X,Y)$ maps each weakly absolutely Cauchy sequence in X to a weakly absolutely Cauchy sequence in Y because the adjoint T^* of T satisfies $T^*(Y^*) \subseteq X^*$ and $\langle Tx, y^*\rangle = \langle x, T^*y^*\rangle$ for every $x \in X$ and $y^* \in Y^*$.

By an isomorphism between lcHs, we mean a bi-continuous linear bijection. When there exists an isomorphism $T : c_0 \to X$ (resp. $T : \ell^\infty \to X$) onto $\mathcal{R}(T) \subseteq X$, we say that X contains an isomorphic copy of c_0 (resp. ℓ^∞). Here, the Banach spaces c_0 and ℓ^∞ are, of course, equipped with their uniform norms $\|\cdot\|_{c_0}$ and $\|\cdot\|_{\ell^\infty}$, respectively.

Lemma 2.3. *Let X be a lcHs.*

(i) *If X is weakly Σ-complete, then X does not contain an isomorphic copy of c_0.*
(ii) *Suppose that X is sequentially complete. Then X is weakly Σ-complete if and only if X does not contain an isomorphic copy of c_0.*

Proof. Part (i) is clear as c_0 is not weakly Σ-complete. For (ii), see [58, Theorem 4]. ∎

Every weakly sequentially complete lcHs is clearly weakly Σ-complete. For example, if Y is a Banach space, then the lcHs $X := Y^*_{\sigma(Y^*,Y)}$ is quasi-complete (for $\sigma(Y^*,Y)$), and hence X is weakly sequentially complete (as $\sigma(X,X^*) = \sigma(Y^*,Y)$).

Throughout this section let (Ω,Σ) denote a measurable space, in other words, Σ is a σ-algebra of subsets of a non-empty set Ω. By $\mathcal{L}^0(\Sigma)$ we denote the vector space of all $\mathbb{C}$-valued, Σ-measurable functions on Ω. Given a lcHs X, let $m : \Sigma \to X$ be a *vector measure*, that is, m is a σ-additive set function. For each $x^* \in X^*$, the complex measure $\langle m, x^*\rangle : A \mapsto \langle m(A), x^*\rangle$ on Σ induces its total variation measure $|\langle m, x^*\rangle| : \Sigma \to [0,\infty)$ [54, §6.1]. A function $f \in \mathcal{L}^0(\Sigma)$ is said to be *m-integrable* if it satisfies the following two conditions:

(I-1) $\int_\Omega |f|\, d|\langle m, x^*\rangle| < \infty$ for all $x^* \in X^*$, and
(I-2) given $A \in \Sigma$, there is a unique element $\int_A f\, dm \in X$ such that

$$\left\langle \int_A f\, dm, x^* \right\rangle = \int_A f\, d\langle m, x^*\rangle, \qquad x^* \in X^*.$$

In this case, $\int_A f\,dm$ is called the integral of f over $A \in \Sigma$ with respect to m. The resulting X-valued set function

$$m_f : A \mapsto \int_A f\,dm, \qquad A \in \Sigma, \tag{2.3}$$

is called the *indefinite integral* of f with respect to m. The set function m_f is again σ-additive thanks to the Orlicz–Pettis Theorem; see Lemma 2.1(ii). The subset $\mathcal{L}^1(m) \subseteq \mathcal{L}^0(\Sigma)$ of all m-integrable functions is a vector subspace. The vector subspace $\operatorname{sim}\Sigma \subseteq \mathcal{L}^0(\Sigma)$ of all $\mathbb{C}$-valued, Σ-simple functions is contained in $\mathcal{L}^1(m)$. This can be seen from the fact that the characteristic function χ_A of each set $A \in \Sigma$ is m-integrable with $\int_B \chi_A\,dm = m(A \cap B)$ for $B \in \Sigma$. A useful fact, for each $f \in \mathcal{L}^1(m)$ and $A \in \Sigma$, is that

$$f\chi_A \in \mathcal{L}^1(m) \quad \text{and} \quad \int_B f\chi_A\,dm = \int_{A\cap B} f\,dm \quad \text{for } B \in \Sigma. \tag{2.4}$$

In particular, if $f \in \mathcal{L}^1(m)$ is $\mathbb{R}$-*valued*, then both $f^+ := \max\{f, 0\} = f\chi_A$ and $f^- := (-f)^+ = f\chi_B$ belong to $\mathcal{L}^1(m)$, where $A := \{w \in \Omega : f(w) \geq 0\}$ and $B := \{w \in \Omega : f(w) \leq 0\}$. Hence, for such f, also $|f| \in \mathcal{L}^1(m)$.

Fix $p \in \mathcal{P}(X)$ and define $p(m)$ on $\mathcal{L}^1(m)$ by

$$p(m)(f) := \sup_{x^* \in U_p^\circ} \int_\Omega |f|\,d|\langle m, x^*\rangle|, \qquad f \in \mathcal{L}^1(m). \tag{2.5}$$

Then, for every $f \in \mathcal{L}^1(m)$, we have $p(m)(f) \geq 0$ and

$$\sup_{A\in\Sigma} p\Big(\int_A f\,dm\Big) \leq p(m)(f) \leq 4 \sup_{A\in\Sigma} p\Big(\int_A f\,dm\Big) < \infty. \tag{2.6}$$

Indeed, [34, Theorem 2.2(1)] gives $p(m)(f) = \sup_{x^*\in U_p^\circ} |\langle m_f, x^*\rangle|(\Omega)$, where the right side equals $\|m_f\|_p(\Omega)$ with $\|m_f\|_p(\cdot)$ denoting the p-semivariation of the vector measure $m_f : \Sigma \to X$ [34, Definition 1.2]. This observation together with m_f having bounded range in X establishes (2.6) [34, p. 158]. Moreover, the definition of $p(m)$ gives

$$p(m)(f) \leq p(m)(g) \quad \text{whenever} \quad f, g \in \mathcal{L}^1(m) \text{ satisfy } |f| \leq |g|. \tag{2.7}$$

Although $p(m)(f)$ in (2.5) is defined in terms of $|f|$, for every $f \in \mathcal{L}^1(m)$, we point out that the inclusion

$$\{|f| : f \in \mathcal{L}^1(m)\} \subseteq \mathcal{L}^1(m) \tag{2.8}$$

is not always valid. In other words, $\mathcal{L}^1(m)$ may *not* be closed under the pointwise modulus operation in $\mathcal{L}^0(\Sigma)$; see Section 3 for relevant results and counterexamples. Of course, as noted immediately after (2.4), if X is a *real* vector space, then (2.8) always holds. Now, since $\pi_p \in \mathcal{L}(X, X_p)$, the composition $\pi_p \circ m : \Sigma \to X_p$ is a Banach-space-valued vector measure. Moreover,

$$\mathcal{L}^1(m) \subseteq \mathcal{L}^1(\pi_p \circ m) \tag{2.9}$$

and

$$\int_A f\,d(\pi_p \circ m) = \pi_p\Big(\int_A f\,dm\Big), \qquad f \in \mathcal{L}^1(m),\ A \in \Sigma, \tag{2.10}$$

both of which are immediate from the relevant definitions. From (2.2), it follows that

$$p(m)(f) = \sup_{\xi^* \in \mathbb{B}[X_p^*]} \int_\Omega |f|\,d|\langle \pi_p \circ m, \xi^*\rangle|, \qquad f \in \mathcal{L}^1(m). \tag{2.11}$$

The lc-topology on $\mathcal{L}^1(m)$ defined by the family of seminorms $\{p(m) : p \in \mathcal{P}(X)\}$ is called the *mean convergence topology* and is the topology of uniform convergence of indefinite integrals in view of (2.6).

Define the closed subspace

$$\mathcal{N}(m) := \bigcap_{p \in \mathcal{P}(X)} p(m)^{-1}(\{0\}) \tag{2.12}$$

of $\mathcal{L}^1(m)$. By the *associated* lcHs with the mean convergence topology is meant the quotient space $L^1(m) := \mathcal{L}^1(m)/\mathcal{N}(m)$; we denote by $\tau(m)$ the corresponding quotient lc-topology on $L^1(m)$. Even though (2.8) may fail to be satisfied in general, it is always true that $|f| \in \mathcal{N}(m) \subseteq \mathcal{L}^1(m)$ whenever $f \in \mathcal{N}(m)$; see (2.5) and (2.6). Then (2.11) implies that $f \in \mathcal{L}^1(m)$ satisfies $f \in \mathcal{N}(m)$ if and only if $|f| \in \mathcal{N}(m)$. A useful fact is that

$$\mathcal{N}(m) = \bigcap_{x^* \in X^*} \mathcal{N}(\langle m, x^* \rangle), \tag{2.13}$$

which is a consequence of $X^* = \bigcup_{p \in \mathcal{P}(X)} U_p^\circ$. Each function f in $\mathcal{N}(m)$ is said to be *m-null* and its indefinite integral is the zero vector measure. A set $A \in \Sigma$ is called *m-null* if $\chi_A \in \mathcal{N}(m)$. Observe that a set $A \in \Sigma$ is m-null if and only if $m(B) = 0$ for all $B \in \Sigma$ with $B \subseteq A$. The family of all m-null sets is denoted by $\mathcal{N}_0(m)$. A property is said to hold *m-almost everywhere*, briefly m-a.e., if it holds outside an m-null set. A function $f \in \mathcal{L}^0(\Sigma)$ is called m-null if f is m-a.e. equal to 0, i.e., there is $A \in \Sigma$ satisfying both $\Omega \setminus A \in \mathcal{N}_0(m)$ and $(f\chi_A)(w) = 0$ for all $w \in \Omega$. In this case, $f \in \mathcal{N}(m) \subseteq \mathcal{L}^1(m)$. Similarly, functions $f, g \in \mathcal{L}^0(\Sigma)$ satisfy $f \geq g$ (m-a.e.) if and only if $f\chi_A \geq g\chi_A$ pointwise for some $A \in \Sigma$ with $\Omega \setminus A \in \mathcal{N}_0(m)$.

The *integration operator* $I_m : \mathcal{L}^1(m) \to X$ is defined by

$$I_m(f) := \int_\Omega f\, dm, \qquad f \in \mathcal{L}^1(m), \tag{2.14}$$

and is linear and continuous by (2.6). Since $I_m(\mathcal{N}(m)) = \{0\}$, the operator I_m induces a unique X-valued, continuous linear map on $L^1(m)$, namely

$$f + \mathcal{N}(m) \mapsto I_m f, \qquad f \in \mathcal{L}^1(m).$$

We say that a function $f \in \mathcal{L}^0(\Sigma)$ is *scalarly m-integrable* (or *weakly m-integrable*) if it satisfies (I-1). The vector subspace $\mathcal{L}^1_w(m) \subseteq \mathcal{L}^0(\Sigma)$, consisting of all the scalarly m-integrable functions on Ω, satisfies

$$\mathcal{N}(m) \subseteq \mathcal{L}^1(m) \subseteq \mathcal{L}^1_w(m) = \bigcap_{p \in \mathcal{P}(X)} \mathcal{L}^1_w(\pi_p \circ m) \subseteq \mathcal{L}^0(\Sigma). \tag{2.15}$$

The equality in (2.15) follows from $X^* = \bigcup_{p \in \mathcal{P}(X)} U_p^\circ = \bigcup_{p \in \mathcal{P}(X)} \pi_p^*(\mathbb{B}[X_p^*])$ (see (2.2)) while the rest of (2.15) is clear.

It seems that the concept of scalar m-integrability formally appeared for the first time in [34, Definition 2.5] for a normed-space-valued measure. G. F. Stefansson [56] presents a systematic study of $\mathcal{L}^1_w(m)$ and related topics in the case of Banach-space-valued measures. For Fréchet-space-valued measures m, the space $\mathcal{L}^1_w(m)$ has been investigated in [8].

Let us proceed to define a lc-topology on $\mathcal{L}^1_w(m)$. Fix $p \in \mathcal{P}(X)$ and let

$$p(m)_w(f) := \sup_{x^* \in U_p^\circ} \int_\Omega |f|\, d|\langle m, x^*\rangle|, \qquad f \in \mathcal{L}^1_w(m). \tag{2.16}$$

It turns out that $p(m)_w$ is a seminorm extending $p(m)$ from $\mathcal{L}^1(m)$ to $\mathcal{L}^1_w(m)$. To see this, it suffices to show that $p(m)_w(f) < \infty$ for each $f \in \mathcal{L}^1_w(m)$. To this end, we first show that

$$\sup_{x^* \in U_p^\circ} \left| \int_A f\, d\langle m, x^*\rangle \right| < \infty, \qquad A \in \Sigma. \tag{2.17}$$

Indeed, via [34, p. 163], there is $\xi^{**}_{A,p}$ in the bidual Banach space X_p^{**} of X_p such that $\int_A f\, d\langle \pi_p \circ m, \xi^*\rangle = \langle \xi^*, \xi^{**}_{A,p}\rangle$ for all $\xi^* \in X_p^*$. This, together with (2.2), verifies (2.17) as

$$\begin{aligned}\sup_{x^* \in U_p^\circ} \left| \int_A f\, d\langle m, x^*\rangle \right| &= \sup_{\xi^* \in \mathbb{B}[X_p^*]} \left| \int_A f\, d\langle \pi_p \circ m, \xi^*\rangle \right| \\ &= \sup_{\xi^* \in \mathbb{B}[X_p^*]} \left|\langle \xi^*, \xi^{**}_{A,p}\rangle\right| = \left\|\xi^{**}_{A,p}\right\|_{X_p^{**}} < \infty.\end{aligned}$$

Now, applying the Nikodým Boundedness Theorem [18, Theorem I.3.1] to the family of indefinite integrals $\left\{\langle m, x^*\rangle_f : x^* \in U_p^\circ\right\}$ (see (2.3) with $\langle m, x^*\rangle$ in place of m) gives $p(m)_w(f) < \infty$ because the total variation measure $|\langle m, x^*\rangle_f| : \Sigma \to [0, \infty)$ satisfies $|\langle m, x^*\rangle_f|(\Omega) = \int_\Omega |f|\, d|\langle m, x^*\rangle|$; see [54, Theorems 6.12 and 6.13]. The fact that $p(m)_w(f) < \infty$ also follows from [56, Proposition 2 and p. 227], which gives

$$\sup_{\xi^* \in \mathbb{B}[X_p^*]} \int_\Omega |f|\, d|\langle \pi_p \circ m, \xi^*\rangle| < \infty.$$

The seminorm $p(m)_w : \mathcal{L}^1_w(m) \to [0, \infty)$ also satisfies

$$p(m)_w(f) \le p(m)_w(g) \quad \text{ for all } f, g \in \mathcal{L}^1_w(m) \text{ with } |f| \le |g|. \tag{2.18}$$

We note here that $\mathcal{L}^1_w(m)$ is *always* closed under the modulus in $\mathcal{L}^0(\Sigma)$, in contrast with $\mathcal{L}^1(m)$, i.e., $|f| \in \mathcal{L}^1_w(m)$ whenever $f \in \mathcal{L}^1_w(m)$.

Remark 2.4. For a general lcHs-valued vector measure $m : \Sigma \to X$, given $f \in \mathcal{L}^1_w(m)$ and $A \in \Sigma$, the linear functional

$$x^{**}_A : x^* \mapsto \int_A f\, d\langle m, x^*\rangle, \qquad x^* \in X^*,$$

is continuous with respect to the strong dual topology $\beta(X^*, X)$ (see [27, p. 154]), i.e., x^{**}_A belongs to the bidual $X^{**} := (X^*_{\beta(X^*,X)})^*$. This is an extension of the case of a normed-space-valued measure [34, p. 163] (see also [56, Corollary 3]), which we have already used above. The case of a Fréchet-space-valued measure is also known [8, Proposition 2.3]. To verify $x^{**}_A \in X^{**}$, select $\{s_n\}_{n=1}^\infty \subseteq \operatorname{sim}\Sigma$ such that $|s_n| \le |f|$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = f$ pointwise on Ω. The subset $W := \{\int_A s_n\, dm : n \in \mathbb{N}\} \subseteq X$ is bounded because, given $p \in \mathcal{P}(X)$, we deduce from (2.6) (with $f := s_n$) and (2.18) (with $f := s_n$

and $g := f$) that

$$\sup_{n\in\mathbb{N}} p\left(\int_A s_n\,dm\right) \le \sup_{n\in\mathbb{N}} p(m)(s_n) = \sup_{n\in\mathbb{N}} p(m)_w(s_n) \le p(m)_w(f) < \infty.$$

The polar set $W^\circ := \{x^* \in X^* : |\langle x, x^*\rangle| \le 1 \text{ for all } x \in W\}$ is then a neighbourhood of 0 in $X^*_{\beta(X^*,X)}$ and

$$\begin{aligned} |x_A^{**}(x^*)| &= \left|\int_A f\,d\langle m, x^*\rangle\right| = \left|\lim_{n\to\infty}\int_A s_n\,d\langle m, x^*\rangle\right| \\ &\le \sup_{n\in\mathbb{N}}\left|\left\langle \int_A s_n\,dm, x^*\right\rangle\right| \le 1, \qquad x^* \in W^\circ, \end{aligned}$$

via the Lebesgue Dominated Convergence Theorem for a scalar measure. Thus, x_A^{**} is $\beta(X^*, X)$-continuous on X^*. □

The identity

$$\mathcal{N}(m) = \bigcap_{p\in\mathcal{P}(X)} (p(m)_w)^{-1}(\{0\}) \tag{2.19}$$

is a consequence of $X^* = \bigcup_{p\in\mathcal{P}(X)} U_p^\circ$ and the fact that if $f \in \mathcal{L}^1_w(m)$ satisfies $p(m)_w(f) = 0$ for all $p \in \mathcal{P}(X)$, then $f \in \mathcal{N}(m) \subseteq \mathcal{L}^1(m)$. Accordingly, the lcHs associated with $\mathcal{L}^1_w(m)$ is $L^1_w(m) := \mathcal{L}^1_w(m)/\mathcal{N}(m)$. Let $\tau(m)_w$ denote the resulting quotient lcH-topology on $L^1_w(m)$. The quotient vector space $\mathcal{L}^0(\Sigma)/\mathcal{N}(m)$ is denoted by $L^0(m)$. It follows from (2.15) and (2.19) that

$$L^1(m) \subseteq L^1_w(m) \subseteq L^0(m) \tag{2.20}$$

as vector subspaces of $L^0(m)$. Moreover, $\tau(m)$ on $L^1(m)$ is the relative topology from $\tau(m)_w$ on $L^1_w(m)$.

The *sequential closure* of a vector subspace F of a lcHs X is the smallest vector subspace of X which contains F and is sequentially closed. A lcHs X can always be identified with a vector subspace of its quasi-completion $\widehat{X}$ [32, pp. 296–297]. The sequential closure of X in $\widehat{X}$, denoted by $\widetilde{X}$, is called the *sequential completion* of X. Since every Cauchy sequence in $\widetilde{X}$ is Cauchy and bounded in $\widehat{X}$, this sequence has a limit in $\widehat{X}$. But, $\widetilde{X}$ is sequentially closed in $\widehat{X}$ and so this limit belongs to $\widetilde{X}$. Hence, $\widetilde{X}$ is a sequentially complete lcHs. Let

$$J : X \to \widetilde{X}$$

be the natural embedding of X into the sequentially complete lcHs $\widetilde{X}$. Each $p \in \mathcal{P}(X)$ admits a unique extension $\widetilde{p} \in \mathcal{P}(\widetilde{X})$. Conversely, every continuous seminorm on $\widetilde{X}$ is such an extension of its restriction to X. In other words

$$\mathcal{P}(\widetilde{X}) = \{\widetilde{p} : p \in \mathcal{P}(X)\}. \tag{2.21}$$

Moreover, $X^* = (\widetilde{X})^*$ and $U_p^\circ = U_{\widetilde{p}}^\circ \subseteq X^*$. In particular, if X has its weak topology $\sigma(X, X^*)$, then also $\widetilde{X}$ has its weak topology and so it is weakly sequentially complete.

Lemma 2.5. *The following statements hold for a lcHs-valued vector measure $m : \Sigma \to X$:*

(i) $\mathcal{N}(m) = \mathcal{N}(J \circ m)$ *as vector subspaces of* $\mathcal{L}^0(\Sigma)$.

(ii) $L^1_w(m) = L^1_w(J \circ m)$ *as lcHs, i.e., they are the same as vector subspaces of* $L^0(m)$ *and the topologies* $\tau(m)_w$ *and* $\tau(J \circ m)_w$ *coincide.*

(iii) *The inclusions*

$$L^1(m) \subseteq L^1(J \circ m) \subseteq L^1_w(m) \tag{2.22}$$

as vector subspaces of $L^0(m)$ *are valid. Moreover, the lcH-topology* $\tau(m)_w$ *induces* $\tau(J \circ m)$ *on* $L^1(J \circ m)$ *and* $\tau(J \circ m)$ *in turn induces* $\tau(m)$ *on* $L^1(m)$.

(iv) *If* X *is weakly* Σ*-complete, then* $L^1(m) = L^1_w(m)$ *as lcHs. In particular,* $L^1(m) = L^1(J \circ m)$ *as lcHs.*

Proof. (i) This is immediate from $X^* = (\widetilde{X})^*$ and (2.13).

(ii) The identity $\mathcal{L}^1_w(m) = \mathcal{L}^1_w(J \circ m)$ holds via $X^* = (\widetilde{X})^*$. This and part (i) lead to $L^1_w(m) = L^1_w(J \circ m)$ as vector subspaces of $L^0(m)$. Moreover, given $p \in \mathcal{P}(X)$, we have $p(m)_w = \widetilde{p}(J \circ m)_w$ because of $U_p^\circ = U_{\widetilde{p}}^\circ$ in $X^* = (\widetilde{X})^*$. So, (ii) holds.

(iii) From $X^* = (\widetilde{X})^*$ we obtain the inclusion $\mathcal{L}^1(m) \subseteq \mathcal{L}^1(J \circ m)$. This together with part (i) ensures that $L^1(m) \subseteq L^1(J \circ m)$ as vector subspaces of $L^0(m)$. That $\tau(J \circ m)$ induces $\tau(m)$ on $L^1(m)$ follows from the fact that $p(m)$ is the restriction of $\widetilde{p}(J \circ m)$ to $\mathcal{L}^1(m)$ via $U_p^\circ = U_{\widetilde{p}}^\circ$ for each $p \in \mathcal{P}(X)$.

Note that $L^1(J \circ m) \subseteq L^1_w(J \circ m)$ by (2.20), with $J \circ m$ in place of m, and that $\tau(J \circ m)_w$ induces $\tau(J \circ m)$ on $L^1(J \circ m)$. Now apply part (ii) to complete the proof of (iii).

(iv) See [31, Theorem II.5.1] together with Lemma 2.2. ∎

In the notation of Lemma 2.5 above, if $L^1(m)$ and $L^1(J \circ m)$ are equal as vector spaces, then they are equal as lcHs, which follows immediately from part (iii) there.

Throughout this paper we regard sequences with entries from $\mathbb{C}$ as $\mathbb{C}$-valued functions on $\mathbb{N}$, unless stated otherwise. Then coordinatewise multiplication of sequences can be naturally expressed as pointwise multiplication of functions defined on $\mathbb{N}$. Moreover, vector subspaces of $\mathbb{C}^\mathbb{N}$ such as c_0, ℓ^p are then function spaces on $\mathbb{N}$. Given a vector subspace Y of $\mathbb{C}^\mathbb{N}$ and $f \in \mathbb{C}^\mathbb{N}$ we write

$$f \cdot Y := \{fg : g \in Y\}.$$

Example 2.6. Let $\Omega := \mathbb{N}$ and $\Sigma := 2^\mathbb{N}$. The identity function from $\mathbb{N}$ to itself is denoted by φ, i.e., $\varphi(n) = n$ for $n \in \mathbb{N}$.

(i) Let X be the space $(1/\varphi)\operatorname{sim}\Sigma := \{s/\varphi : s \in \operatorname{sim}\Sigma\}$ equipped with the norm induced from c_0. Define $m : \Sigma \to X$ by $m(A) := \chi_A/\varphi$ for $A \in \Sigma$. Since $\widetilde{X} = c_0$, the vector measure $J \circ m : \Sigma \to c_0$ is precisely the one used in [31, Example II.5.1]. Clearly $\mathcal{N}(m) = \{0\}$. It is routine to obtain the following identities:

$$L^1(m) = \operatorname{sim}\Sigma, \qquad L^1(J \circ m) = \varphi \cdot c_0, \qquad L^1_w(m) = \varphi \cdot \ell^\infty.$$

Hence, $L^1(m) \subsetneq L^1(J \circ m) \subsetneq L^1_w(m)$, that is, both of the inclusions in (2.22) may be strict (simultaneously).

(ii) Let X be the Banach space c_0. Then the X-valued vector measure $m : A \mapsto \chi_A/\varphi$ on Σ satisfies

$$L^1(m) = L^1(J \circ m) = \varphi \cdot c_0 \quad \text{and} \quad L^1_w(m) = \varphi \cdot \ell^\infty.$$

So, $L^1(m) = L^1(J \circ m) \subsetneq L^1_w(m)$, that is, the first inclusion in (2.22) is an equality whereas the second inclusion is strict.

(iii) Let $X := (1/\varphi)\,\mathrm{sim}\,\Sigma$, equipped with the norm induced from ℓ^2, and let m be as above. Then $\widetilde{X} = \ell^2$. Moreover,

$$L^1(m) = \mathrm{sim}\,\Sigma, \qquad L^1(J \circ m) = L^1_w(m) = \varphi \cdot \ell^2,$$

and hence $L^1(m) \subsetneq L^1(J \circ m) = L^1_w(m)$, that is, the first inclusion in (2.22) is strict whereas the second inclusion is an equality. □

We shall, from now on, identify each $f \in \mathcal{L}^0(\Sigma)$ with its quotient class $f + \mathcal{N}(m) \in L^0(m)$ as in the case of scalar measure theory, except when such a distinction is required for precise arguments. When we need to emphasize that we are dealing with functions in $\mathcal{L}^0(\Sigma)$, we may speak of *individual functions*. The function spaces $\mathcal{L}^1(m)$ and $\mathcal{L}^1_w(m)$ will be identified with their quotient spaces $L^1(m)$ and $L^1_w(m)$, respectively.

We say that the vector measure m has the *Lebesgue Convergence Property*, briefly LCP, if, whenever a Σ-measurable function $f : \Omega \to \mathbb{C}$ is the m-a.e. pointwise limit of a sequence $\{f_n\}_{n=1}^\infty$ of m-integrable functions satisfying $|f_n| \le g$ (m-a.e.) for each $n \in \mathbb{N}$ and some non-negative m-integrable function g, then it follows that f is m-integrable and $\{f_n\}_{n=1}^\infty$ is $\tau(m)$-convergent to f.

LEMMA 2.7. *Let $m : \Sigma \to X$ be a lcHs-valued vector measure.*

(i) *The following conditions are equivalent for a Σ-measurable function $f : \Omega \to \mathbb{C}$:*

   (a) *f is m-integrable.*
   (b) *There exists a sequence $\{s_n\}_{n=1}^\infty \subseteq \mathrm{sim}\,\Sigma$ converging m-a.e. pointwise to f such that the sequence $\{\int_A s_n\,dm\}_{n=1}^\infty$ is convergent in X for each $A \in \Sigma$.*
   (c) *The same condition as in* (b) *with $L^1(m)$ in place of* $\mathrm{sim}\,\Sigma$.

   Moreover, if (b) *or* (c) *holds, then $\lim_{n\to\infty} \int_A s_n\,dm = \int_A f\,dm$ for every $A \in \Sigma$, and $\{s_n\}_{n=1}^\infty$ is $\tau(m)$-convergent to f in $L^1(m)$.*

(ii) *If $L^1(m)$ is sequentially complete, then $L^1(m) = L^1(J \circ m)$.*

(iii) *If X is sequentially complete, then m has the LCP.*

(iv) *If X is sequentially complete, then every $\mathbb{C}$-valued, bounded, Σ-measurable function on Ω is m-integrable.*

Proof. (i) This follows from [34, Theorem 2.4]. The current form occurs in [41, Proposition 1.2].

(ii) Let $f \in L^1(J \circ m)$. Observe first that the $(J \circ m)$-null and the m-null sets coincide by Lemma 2.5(i). Select $\{s_n\}_{n=1}^\infty \subseteq \mathrm{sim}\,\Sigma$ satisfying condition (b) in part (i). Then (i) ensures that $\{s_n\}_{n=1}^\infty$ is $\tau(J \circ m)$-convergent to f. On the other hand, since $\tau(m)$ is the topology induced by $\tau(J \circ m)$ (see Lemma 2.5(iii)), the sequence $\{s_n\}_{n=1}^\infty$ is $\tau(m)$-Cauchy, and hence admits a $\tau(m)$-limit in the sequentially complete lcHs $L^1(m)$. Then f must equal this $\tau(m)$-limit, so that $f \in L^1(m)$. Thus $L^1(J \circ m) \subseteq L^1(m)$, and hence (ii) holds via (2.22).

(iii) See [34, Theorem 2.2].

(iv) This follows from part (iii) because constant functions are m-integrable. An alternative proof is in [31, Theorem II.3.1]. ■

When X is a Banach space, Lemma 2.7(i) ensures that our m-integrability is equivalent to that in [3, Definition 2.5].

Parts (i) and (ii) of the following lemma occur in [41, Lemma 1.3] whereas part (iii) is straightforward.

LEMMA 2.8. *Given are a lcHs-valued vector measure $m : \Sigma \to X$ and a continuous linear operator T from X into a lcHs Y.*

(i) *The set function $T \circ m : \Sigma \to Y$ is a vector measure.*
(ii) *Every m-integrable function f is also $(T \circ m)$-integrable and*

$$\int_A f\, d(T \circ m) = T\Big(\int_A f\, dm\Big), \qquad A \in \Sigma.$$

(iii) *The corresponding linear map $[T]_m : L^1(m) \to L^1(T \circ m)$ which assigns to each $f \in L^1(m)$ the same function f in $L^1(T \circ m)$ is continuous.*

REMARK 2.9. The precise definition of $[T]_m$ in Lemma 2.8(iii) is

$$[T]_m(f + \mathcal{N}(m)) := Tf + \mathcal{N}(T \circ m), \qquad f \in \mathcal{L}^1(m).$$

This is well defined because (2.13), applied twice, gives

$$\mathcal{N}(m) = \bigcap_{x^* \in X^*} \mathcal{N}(\langle m, x^* \rangle) \subseteq \bigcap_{y^* \in Y^*} \mathcal{N}(\langle m, T^* y^* \rangle) = \bigcap_{y^* \in Y^*} \mathcal{N}(\langle T \circ m, y^* \rangle) = \mathcal{N}(T \circ m).$$

It is clear that $[T]_m$ is injective if and only if $\mathcal{N}(m) = \mathcal{N}(T \circ m)$. This applies, in particular, when $Y := \widetilde{X}$ and $T := J$. □

To discuss whether or not the lcHs $L^1(m)$ is complete for a lcHs-valued vector measure $m : \Sigma \to X$, we recall the concept of a *closed vector measure*. According to [31, p. 71], this means that the subset $\Sigma(m) := \{\chi_A + \mathcal{N}(m) : A \in \Sigma\} \subseteq L^1(m)$ is $\tau(m)$-complete. We identify $\Sigma(m)$ with $\{\chi_A : A \in \Sigma\}$. Let $[X]_m$ denote the sequential closure in X of the linear span of the range of m.

LEMMA 2.10. *Let $m : \Sigma \to X$ be a lcHs-valued vector measure.*

(i) *The subset $\Sigma(m)$ is always closed in the lcHs $L^1(m)$.*
(ii) *In the case when $[X]_m$ is sequentially complete, the lcHs $L^1(m)$ is complete if and only if m is a closed measure.*
(iii) *The following conditions are equivalent:*
    (a) *$L^1(m)$ is complete.*
    (b) *$L^1(m)$ is quasi-complete*
    (c) *m is a closed measure and $L^1(m)$ is sequentially complete.*
    (d) *m is a closed measure and $L^1(m) = L^1(J \circ m)$ as lcHs.*
(iv) *If X is metrizable, then m is a closed measure.*
(v) *If X is a Banach (resp. Fréchet) space, then so is $L^1(m)$.*
(vi) *If X is a normed (resp. metrizable) space, then $L^1_w(m)$ is a Banach (resp. Fréchet) space.*

Proof. (i) This fact has been used in the literature starting with [31, proof of Theorem IV.4.1], but without proof. For the sake of completeness we now present its proof.

Let $\{\chi_{A(\lambda)}\}_\lambda$ be a net in $\Sigma(m)$ having a $\tau(m)$-limit $f \in L^1(m)$. Let $B(f) := \{w \in \Omega : |\mathrm{Re}\,(f(w))| \le 1/2\} \in \Sigma$. Then $|\chi_{A(\lambda)} - \chi_{B(f)}| \le 2|\chi_{A(\lambda)} - f|$ pointwise on Ω, and hence it follows from (2.7) that

$$p(m)(\chi_{A(\lambda)} - \chi_{B(f)}) \le 2p(m)(\chi_{A(\lambda)} - f), \quad p \in \mathcal{P}(X).$$

Accordingly, the net $\{\chi_{A(\lambda)}\}_\lambda$ is $\tau(m)$-convergent to $\chi_{B(f)}$, which implies that $f = \chi_{B(f)}$ (m-a.e.). So, $\Sigma(m)$ is $\tau(m)$-closed in $L^1(m)$.

(ii) See [51, Theorem 2].

(iii) The implication (a)⇒(b) is clear. Since $\Sigma(m)$ is a bounded subset of $L^1(m)$, by part (i) we have (b)⇒(c). For (c)⇒(d) apply Lemma 2.7(ii). Now assume (d). Since m is closed if and only if $J \circ m$ is closed, part (ii) ensures that $L^1(J \circ m)$ is complete, and hence so is $L^1(m)$. So, we have established (d)⇒(a).

(iv) See [31, Theorem IV.7.1].

(v) This is a consequence of (ii) and (iv), in view of the definition of $L^1(m)$.

(vi) See [8, Theorem 2.5]. ■

Regarding part (ii) above, an earlier result [31, Theorem IV.4.1] was extended in [20, p. 139]; in [50, Proposition 1] and [51] it occurs in the current general form. Part (iii) is essentially in [51]. A generalization of (iv) is that, if m is countably determined, then m is a closed measure [42, Propositions 1.2 and 1.6]. Here m is called *countably determined* if $\mathcal{N}_0(m) = \bigcap_{n=1}^\infty \mathcal{N}_0(\langle m, x_n^*\rangle)$ for some sequence $\{x_n^*\}_{n=1}^\infty$ in X^*. More generally, m is a closed measure whenever m is absolutely continuous with respect to a localizable scalar measure [31, Theorem IV.7.3].

REMARK 2.11. Concerning Lemma 2.10(v), when X is a Banach space with norm $\|\cdot\|_X$, then the corresponding seminorm (2.5), with $p := \|\cdot\|_X$, is the *norm*

$$f \mapsto \|f\|_{L^1(m)} := \sup_{x^* \in \mathbb{B}[X^*]} \int_\Omega |f|\, d|\langle m, x^*\rangle|, \tag{2.23}$$

for which $L^1(m)$ is a Banach space. According to Lemmas 2.5(iii) and 2.10(v)&(vi), the right side of (2.23) is also the norm $\|\cdot\|_{L^1_w(m)}$ of the Banach space $L^1_w(m)$, and $L^1(m)$ is a closed subspace of $L^1_w(m)$. □

We refer to [14], [18], [46] for the theory and applications of Banach-space-valued vector measures.

3. Complex vector lattices

Our aim is to determine when $L^1(m)$, for a lcHs-valued vector measure m, is a complex vector lattice with respect to the m-a.e. pointwise order.

Let E be a *vector lattice* (also called a *Riesz space*) with order relation $\le$. In other words, E is an ordered vector space over $\mathbb{R}$ such that both

$$x \vee y := \sup\{x, y\} \quad \text{and} \quad x \wedge y := \inf\{x, y\} \tag{3.1}$$

exist in E whenever $x, y \in E$ [55, Definition II. 2.1], [61, Ch. 2, §4]. Given $x \in E$, we adopt the standard symbols:

$$x^+ := x \vee 0, \qquad x^- := (-x) \vee 0, \qquad |x| := x \vee (-x),$$

which are called the *positive part, negative part* and *modulus* of x, respectively. The *positive cone* $\{x \in E : x \geq 0\}$ of E is denoted by E^+. A vector subspace of E is called a *vector sublattice* if it is closed under the lattice operations (3.1). Each vector sublattice is, of course, a vector lattice in the order induced by E. A vector subspace F of E is a vector sublattice if and only if $x^+ \in F$ for each $x \in F$ if and only if $|x| \in F$ for each $x \in F$, which is a consequence of basic lattice identities [1, Theorem 1.1], [36, Theorem 11.8]. We say that a vector sublattice F is *order dense* in E if, given $x \in E^+ \setminus \{0\}$, there is $y \in F^+ \setminus \{0\}$ satisfying $y \leq x$ [1, Definition 1.9], [61, Definition 23.1]. A vector subspace F of E is called an *ideal* if, whenever $x \in E$ and $y \in F$ satisfy $|x| \leq |y|$, we have $x \in F$. In this case we can define the *quotient vector lattice* E/F [36, pp. 100–102]. An ideal $F \subseteq E$ is called a *σ-ideal* if, given any countable subset $H \subseteq F$ with $\sup_E H$ existing in E, we necessarily have $\sup_E H \in F$ [36, Definition 17.1(iii)], [55, p. 61]. Via standard vector lattice identities [36, Theorems 11.7 & 11.8], it is routine to verify that every ideal $F \subseteq E$ is necessarily a vector sublattice of E.

Let us formulate a known fact whose proof is straightforward.

LEMMA 3.1. *Let F be a vector sublattice of a vector lattice E. Given a subset $H \subseteq F$ which has a supremum $\sup_E H$ in E, if the element $\sup_E H$ belongs to F, then the supremum $\sup_F H$ of the set H in F exists and is precisely the element $\sup_E H$.*

The *complexification* $E_{\mathbb{C}} := E + iE$ of a vector lattice E is a complex vector space [55, p. 134], [60, Section 91], [61, Ch. 6]. Each $z \in E_{\mathbb{C}}$ corresponds to a *unique* pair $(x, y) \in E \times E$ so that $z = x + iy$; in this case x (resp. y) is called the *real* (resp. *imaginary*) *part* of z and is denoted by $\mathrm{Re}(z)$ (resp. $\mathrm{Im}(z)$). The *complex conjugate* $\overline{z} \in E_{\mathbb{C}}$ of such a z is defined as $\overline{z} := x - iy$. We say that a real vector lattice E has the *complex modulus property* if the supremum $\sup_{\theta\in[0,2\pi]} |(\cos\theta)x + (\sin\theta)y|$ exists in E for each pair $(x, y) \in E \times E$. In this case, the *modulus* $|z|$ of $z = (x + iy) \in E_{\mathbb{C}}$ is defined to be the element

$$|z| = |x + iy| := \sup_{\theta\in[0,2\pi]} |(\cos\theta)x + (\sin\theta)y| \tag{3.2}$$

of E^+. The modulus map $z \mapsto |z|$ from $E_{\mathbb{C}}$ onto E^+ satisfies the “standard” conditions of a modulus. Namely, $|z| = 0$ if and only if $z = 0$, with $|\alpha z| = |\alpha| \cdot |z|$ for all $\alpha \in \mathbb{C}$ and $z \in E_{\mathbb{C}}$ and $|z_1 + z_2| \leq |z_1| + |z_2|$ for all $z_1, z_2 \in E_{\mathbb{C}}$ [55, (3), p. 134], [60, Section 91], [61, Ch. 6]. Moreover, $|\overline{z}| = |z|$ for each $z \in E_{\mathbb{C}}$ because, with $x := \mathrm{Re}(z)$ and $y := \mathrm{Im}(z)$, we have

$$\begin{aligned} |\overline{z}| = |x - iy| &= \sup_{\theta\in[0,2\pi]} |(\cos\theta)x + (\sin\theta)(-y)| \\ &= \sup_{\theta\in[0,2\pi]} |(\cos(-\theta))x + (\sin(-\theta))y| = |x + iy| = |z|. \end{aligned}$$

It is also clear from (3.2) that if $z = x + i0$ with $x \in E^+$, then $|z| = x$. Hence, $\big||z|\big| = |z|$ for every $z \in E_{\mathbb{C}}$.

The complexification of a real vector lattice with the complex modulus property is called a *complex vector lattice*. Our class of complex vector lattices properly includes those complex lattices (or complex Riesz spaces) given in [55, Definition II.11.1], [60, p. 191] (see also [61, Ch. 6]); see Remark 3.3 below. An axiomatic way of defining complex vector lattices has been adopted in [39].

Let us discuss some sufficient conditions for E to have the complex modulus property. According to [55, Definition II.1.8], E satisfies Axiom (OS) if, whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in E^+ for which there exist $\alpha \in \ell^1$ (with $\alpha(n) \in \mathbb{R}$ for $n \in \mathbb{N}$) and $x \in E$ satisfying $x_n \le \alpha(n)x$ for each $n \in \mathbb{N}$, then the supremum $\sup_{N\in\mathbb{N}} \sum_{n=1}^{N} x_n$ exists in E. It is routine to check that an equivalent statement to Axiom (OS) occurs if it is formulated with $x \in E^+$ in place of $x \in E$. It turns out that E satisfies Axiom (OS) if and only if E is Archimedean and uniformly complete; see Remark 3.3(iii) below. A seminorm (resp. norm) q on E is called a *lattice seminorm* (resp. *lattice norm*) if $q(x) \le q(y)$ for all $x, y \in E$ with $|x| \le |y|$. The terminology *Riesz seminorm / Riesz norm* is also common [36], [60], [61]. A vector lattice equipped with a lattice norm for which it is complete is by definition a *Banach lattice*.

LEMMA 3.2. *Let E be a vector lattice.*

(i) *Given $x, y \in E$, it follows that $\sup_{\theta\in[0,2\pi]} \big|(\cos\theta)x + (\sin\theta)y\big|$ exists in E if and only if $\sup_{\theta\in[0,2\pi]}((\cos\theta)x + (\sin\theta)y)$ exists in E, in which case*

$$\sup_{\theta\in[0,2\pi]} |(\cos\theta)x + (\sin\theta)y| = \sup_{\theta\in[0,2\pi]} ((\cos\theta)x + (\sin\theta)y). \tag{3.3}$$

(ii) *The vector lattice E has the complex modulus property if and only if*

$$\sup_{\theta\in[0,2\pi]} ((\cos\theta)x + (\sin\theta)y)$$

exists in E for all pairs $x, y \in E$, in which case

$$|x + iy| = \sup_{\theta\in[0,2\pi]} ((\cos\theta)x + (\sin\theta)y). \tag{3.4}$$

(iii) *If E satisfies Axiom* (OS), *then it has the complex modulus property.*

(iv) *Each of the following conditions guarantees that E satisfies Axiom* (OS), *and hence that E has the complex modulus property:*

   (a) *E is Dedekind σ-complete.*

   (b) *E is sequentially complete with respect to a real lcH-topology generated by a family of lattice seminorms.*

   (c) *E is equipped with a lattice norm for which it is a Banach lattice.*

(v) *If E satisfies Axiom* (OS), *then so does every ideal F in E.*

(vi) *If E has the complex modulus property, then so does every ideal F in E. Moreover, given $x, y \in F$, the modulus $|x + iy|$ of $x + iy$ in $E_{\mathbb{C}}$ equals that of $x + iy$ in the complex vector lattice $F + iF$.*

Proof. (i) This is a consequence, for each $\theta \in [0, 2\pi]$, of the equalities

$$\begin{aligned}|(\cos\theta)x + (\sin\theta)y| &= ((\cos\theta)x + (\sin\theta)y) \vee (-(\cos\theta)x - (\sin\theta)y)\\ &= ((\cos\theta)x + (\sin\theta)y) \vee \big((\cos(\theta+\pi))x + (\sin(\theta+\pi))y\big).\end{aligned}$$

(ii) Apply part (i) after recalling the definition of the complex modulus property.

(iii) See [55, p. 134].

(iv) Condition (a) implies Axiom (OS) as noted in [55, p. 54]. Via [1, Theorem 5.6(iii)], condition (b) also ensures Axiom (OS). Condition (c) is a special case of (b). In each case the complex modulus property of E then follows from part (iii).

(v) The proof is a routine application of Lemma 3.1.

(vi) This is also an immediate consequence of Lemma 3.1. ∎

REMARK 3.3. (i) The complex vector lattices defined in [55, Definition II.11.1] are limited to those which are the complexification of some vector lattice satisfying Axiom (OS). Such complex vector lattices are also complex vector lattices in our sense because Axiom (OS) guarantees the complex modulus property; see Lemma 3.2(iii). The converse is not valid (see the spaces $(\operatorname{sim} 2^{\mathbb{N}})_{\mathbb{R}}$ and $\mathbb{R}^{\mathbb{N}}_{\mathrm{ec}}$ in part (iv) together with Fact 1 below), so that the class of complex vector lattices considered here is strictly larger than that in [55].

(ii) The complex vector lattices defined in [60, p. 191] are those which are the complexification of some Archimedean, uniformly complete vector lattice. It turns out that the class of complex vector lattices in [60] is exactly the same as that in [55]. This is a consequence of the following

FACT 1. A *vector lattice satisfies Axiom* (OS) *if and only if it is Archimedean and uniformly complete.*

The proof of Fact 1 is given in part (iii) below.

Every Banach lattice is Archimedean [60, p. 282], and uniformly complete [60, Theorem 100.4(ii)]. The Banach lattice $C([0,1))$ consisting of all $\mathbb{R}$-valued, continuous functions on $[0,1]$ and equipped with the uniform norm fails to be Dedekind σ-complete [36, Example 23.3(ii)]. Hence, Dedekind σ-completeness is not equivalent to Axiom (OS). On the other hand, every Dedekind σ-complete vector lattice is Archimedean [61, p. 62], and uniformly complete [61, Theorem 12.8].

(iii) Let E be a vector lattice. Recall that E is *Archimedean* if $\inf_{n\in\mathbb{N}} \frac{1}{n}u = 0$ for every $u \in E^+$ [36, p. 78]. Given $u \in E^+$, a sequence $\{x_n\}_{n=1}^\infty$ in E is said to *converge u-uniformly* to $x \in E$ if for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $|x - x_n| \le \varepsilon u$ for all $n \ge N_\varepsilon$. The definition of a *u-uniform Cauchy sequence* is similar [36, Definition 39.1]. Then E is called *uniformly complete* if, for every $u \in E^+$, all u-uniform Cauchy sequences are u-uniformly convergent in E [36, Definition 42.1].

To verify Fact 1, assume first that E satisfies Axiom (OS). Then E is Archimedean [55, p. 54]. To obtain the uniform completeness of E, let $u \in E^+ \setminus \{0\}$ and $\{x_n\}_{n=1}^\infty$ be an increasing, u-uniform Cauchy sequence in E. Set $y_n := x_n - x_1$ for $n \in \mathbb{N}$, in which case $\{y_n\}_{n=1}^\infty \subseteq E^+$ is also increasing and u-uniformly Cauchy. Select a subsequence $\{y_{n(k)}\}_{k=1}^\infty$ of $\{y_n\}_{n=1}^\infty$ such that

$$0 \le (y_{n(k+1)} - y_{n(k)}) = |y_{n(k+1)} - y_{n(k)}| \le \frac{1}{2^k}u, \qquad k \in \mathbb{N}.$$

As $\{2^{-k}\}_{k=1}^\infty \in \ell^1$, Axiom (OS) ensures that $y := \sup_{N\in\mathbb{N}} \sum_{k=1}^N y_{n(k)}$ exists in E. According to [36, Lemma 39.2], the sequence $\{y_{n(k)}\}_{k=1}^\infty$ is u-uniformly convergent to y, from which it follows that $\{x_{n(k)}\}_{k=1}^{\infty}$ is u-uniformly convergent to $y+x_1$. Therefore the original sequence $\{x_n\}_{n=1}^{\infty}$, which is u-uniformly Cauchy, is also u-uniformly convergent (to $y+x_1$).

If $\{x_n\}_{n=1}^{\infty}$ is a decreasing, u-uniform Cauchy sequence in E, then it admits a u-uniform limit because so does the increasing, u-uniform Cauchy sequence $\{-x_n\}_{n=1}^{\infty}$. So, E is uniformly complete via [36, Theorem 39.4].

Conversely, assume that E is Archimedean and uniformly complete. To prove that E satisfies Axiom (OS), let $\{x_n\}_{n=1}^{\infty} \subseteq E^+$, $x \in E^+ \setminus \{0\}$ and $\alpha \in \ell^1$ satisfy $x_n \leq \alpha(n)x$ for $n \in \mathbb{N}$. Define $y_n := \sum_{j=1}^{n} x_j$ for $n \in \mathbb{N}$. Given $\varepsilon > 0$, choose $N_\varepsilon \in \mathbb{N}$ such that $\sum_{j=k+1}^{n} \alpha(j) < \varepsilon$ whenever $n > k \geq N_\varepsilon$. Then

$$|y_n - y_k| = \sum_{j=k+1}^{n} x_j \leq \sum_{j=k+1}^{n} \alpha(j)x \leq \varepsilon x, \qquad n > k \geq N_\varepsilon.$$

So, the increasing sequence $\{y_n\}_{n=1}^{\infty}$ is x-uniformly Cauchy, and hence has an x-uniform limit y by the uniform completeness of E. Again by [36, Lemma 39.2], the supremum $\sup_{n\in\mathbb{N}} y_n$ exists in E and equals y. Consequently, E satisfies Axiom (OS).

(iv) Concerning Fact 1 above, we now exhibit three vector lattices, two of which are Archimedean but not uniformly complete with the third one being uniformly complete but not Archimedean. To this end, consider the vector lattice $\mathbb{R}^{\mathbb{N}}$ consisting of all $\mathbb{R}$-valued functions on $\mathbb{N}$ equipped with the pointwise order. Then $\mathbb{R}^{\mathbb{N}}$ and all of its vector sublattices are Archimedean [61, Theorem 9.1(iii) and Example 9.2(iii)]. Since $\mathbb{R}^{\mathbb{N}}$ is Dedekind complete, it follows from [36, p. 276] that $\mathbb{R}^{\mathbb{N}}$ is also uniformly complete. Set $e_n := \chi_{\{n\}} \in \mathbb{R}^{\mathbb{N}}$ for each $n \in \mathbb{N}$.

For our first example, the claim is that the vector sublattice $(\operatorname{sim} 2^{\mathbb{N}})_{\mathbb{R}} := \mathbb{R}^{\mathbb{N}} \cap \operatorname{sim} 2^{\mathbb{N}}$ of $\mathbb{R}^{\mathbb{N}}$ is Archimedean but not uniformly complete. That it is Archimedean has already been noted. To see that $(\operatorname{sim} 2^{\mathbb{N}})_{\mathbb{R}}$ is not uniformly complete, consider the sequence $\{f_k\}_{k=1}^{\infty}$ in $(\operatorname{sim} 2^{\mathbb{N}})_{\mathbb{R}}$ given by $f_k := \sum_{n=1}^{k} n^{-1} e_n$ for $k \in \mathbb{N}$. Given $\varepsilon > 0$, choose an $N_\varepsilon \in \mathbb{N}$ for which $1/N_\varepsilon < \varepsilon$. Then it follows that

$$|f_j - f_k| = \sum_{n=k+1}^{j} n^{-1} e_n \leq N_\varepsilon^{-1} \sum_{n=k+1}^{j} e_n \leq \varepsilon u$$

whenever $j, k \in \mathbb{N}$ satisfy $j > k \geq N_\varepsilon$, where $u := \chi_{\mathbb{N}} \in (\operatorname{sim} 2^{\mathbb{N}})_{\mathbb{R}}$. So, the sequence $\{f_k\}_{k=1}^{\infty}$ is u-uniformly Cauchy in $(\operatorname{sim} 2^{\mathbb{N}})_{\mathbb{R}}$, but it does not have a u-uniform limit in $(\operatorname{sim} 2^{\mathbb{N}})_{\mathbb{R}}$ because its u-uniform limit in the ambient vector lattice $\mathbb{R}^{\mathbb{N}}$ is the element $\sum_{n=1}^{\infty} n^{-1} e_n$ which does not belong to $(\operatorname{sim} 2^{\mathbb{N}})_{\mathbb{R}}$. Thus, $(\operatorname{sim} 2^{\mathbb{N}})_{\mathbb{R}}$ fails to be uniformly complete.

We point out that $(\operatorname{sim} 2^{\mathbb{N}})_{\mathbb{R}}$ failing to be uniformly complete can also be deduced from Fact 1 above. To see this, let $g_n := 2^{-n} e_n \in (\operatorname{sim} 2^{\mathbb{N}})_{\mathbb{R}}$ for each $n \in \mathbb{N}$, and define $\alpha \in \ell^1$ via $\alpha(k) := 2^{-k}$ for $k \in \mathbb{N}$. Clearly $0 \leq g_n \leq \alpha(n)\chi_{\mathbb{N}}$ for $n \in \mathbb{N}$. However, the supremum "$\sup_{N\in\mathbb{N}} \sum_{n=1}^{N} g_n$" does not exist in $(\operatorname{sim} 2^{\mathbb{N}})_{\mathbb{R}}$. So, $(\operatorname{sim} 2^{\mathbb{N}})_{\mathbb{R}}$ fails to satisfy Axiom (OS), and hence by Fact 1, cannot be uniformly complete (as we already know that $(\operatorname{sim} 2^{\mathbb{N}})_{\mathbb{R}}$ is Archimedean).

The second example, exhibiting the same features, is the vector sublattice $\mathbb{R}^{\mathbb{N}}_{\mathrm{ec}}$ of $\mathbb{R}^{\mathbb{N}}$ consisting of all $f \in \mathbb{R}^{\mathbb{N}}$ for which there exists $k \in \mathbb{N}$ (depending on f) such that $f(n) = f(k)$ for all $n \geq k$, i.e., f is eventually constant. Since $\mathbb{R}^{\mathbb{N}}$ is Archimedean, it follows that so is $\mathbb{R}^{\mathbb{N}}_{\mathrm{ec}}$. The functions $\{g_n\}_{n=1}^\infty$ of the previous paragraph also belong to $\mathbb{R}^{\mathbb{N}}_{\mathrm{ec}}$ and the same argument applies to show that $\mathbb{R}^{\mathbb{N}}_{\mathrm{ec}}$ fails Axiom (OS), and hence $\mathbb{R}^{\mathbb{N}}_{\mathrm{ec}}$ is not uniformly complete. Observe that if $f, g \in \mathbb{R}^{\mathbb{N}}_{\mathrm{ec}}$, then also $\sqrt{f^2+g^2} \in \mathbb{R}^{\mathbb{N}}_{\mathrm{ec}}$. Moreover, $\sqrt{f^2+g^2} = \sup_{\theta\in[0,2\pi]}((\cos\theta)f + (\sin\theta)g)$ in the order of the vector lattice $\mathbb{R}^{\mathbb{N}}_{\mathrm{ec}}$ [61, Example 13.2]. Hence, $\mathbb{R}^{\mathbb{N}}_{\mathrm{ec}}$ has the complex modulus property.

The third example alluded to is immediate from the following result.

FACT 2. *Let $F \subseteq \mathbb{R}^{\mathbb{N}}$ be any proper ideal containing $\{e_n : n \in \mathbb{N}\}$. Then the quotient vector lattice $\mathbb{R}^{\mathbb{N}}/F$ is uniformly complete but not Archimedean.*

To verify Fact 2, first recall that the quotient space $\mathbb{R}^{\mathbb{N}}/F$ is indeed a vector lattice [36, Theorem 18.9]. Moreover, according to [36, Corollary 59.4] the vector lattice $\mathbb{R}^{\mathbb{N}}/F$ is uniformly complete. To show that $\mathbb{R}^{\mathbb{N}}/F$ is not Archimedean, we adapt the argument in [36, Example 60.1(i)], which corresponds to the special case of $F := (\ell^\infty)_{\mathbb{R}}$. Since $F \neq \mathbb{R}^{\mathbb{N}}$, we may choose an element $f \in (\mathbb{R}^{\mathbb{N}})^+ \setminus F$. Now define $g \in (\mathbb{R}^{\mathbb{N}})^+$ by $g(n) := nf(n)$ for $n \in \mathbb{N}$. With $[f]$ and $[g]$ denoting the quotient classes in $\mathbb{R}^{\mathbb{N}}/F$ containing f and g, respectively, we shall verify that

$$[f] \leq k^{-1}[g], \qquad k \in \mathbb{N}. \tag{3.5}$$

If $k = 1$, then $f \leq g$ in $\mathbb{R}^{\mathbb{N}}$ gives $[f] \leq [g]$. For each $k \geq 2$, the inequality (3.5) still holds because

$$\begin{aligned}
\frac{1}{k}g - f &= \frac{1}{k}\Big(\sum_{n=1}^{k-1} g(n)e_n + \sum_{n=k}^{\infty} g(n)e_n\Big) - \Big(\sum_{n=1}^{k-1} f(n)e_n + \sum_{n=k}^{\infty} f(n)e_n\Big)\\
&= \sum_{n=1}^{k-1}\Big(\frac{n}{k}-1\Big)f(n)e_n + \sum_{n=k}^{\infty}\Big(\frac{n}{k}-1\Big)f(n)e_n\\
&\geq \sum_{n=1}^{k-1}\Big(\frac{n}{k}-1\Big)f(n)e_n,
\end{aligned}$$

with $\sum_{n=1}^{k-1}(n/k-1)f(n)e_n \in F$ by the assumptions on F. This establishes (3.5). Since $[f] \neq [0]$ in $\mathbb{R}^{\mathbb{N}}/F$, it follows from the discussion on pp. 78–79 of [36] that $\mathbb{R}^{\mathbb{N}}/F$ is not Archimedean.

(v) Let E be a vector lattice. A sequence $\{x_n\}_{n=1}^\infty \subseteq E$ is said to be *relatively uniformly convergent* to $x \in E$ if $\{x_n\}_{n=1}^\infty$ is u-uniformly convergent to x for some $u \in E^+$ [36, Theorem 16.2]. A subset W of E is called *uniformly closed* if, whenever a sequence $\{x_n\}_{n=1}^\infty$ in W is relatively uniformly convergent to some x in E, then necessarily $x \in W$ [36, p. 84].

Recall that a *Fréchet lattice* is a vector lattice with a complete lcH-topology which is generated by countably many lattice seminorms. For the particular Fréchet lattice $\mathbb{R}^{\mathbb{N}}$ of part (iv), equipped with the lcH-topology generated by the sequence of lattice seminorms $\{p_n\}_{n=1}^\infty$ with $p_n(x) := \max_{1\leq k\leq n}|x(k)|$ for $x \in \mathbb{R}^{\mathbb{N}}$, it is clear in Fact 2 above that the conditions on the ideal $F \subseteq \mathbb{R}^{\mathbb{N}}$ imply that F is *not* topologically closed in $\mathbb{R}^{\mathbb{N}}$. The following result shows that this is no coincidence.

FACT 3. *For an ideal F in a Fréchet lattice E the following assertions are equivalent*:

(i) *The quotient vector lattice E/F is Archimedean.*
(ii) *F is uniformly closed in E.*
(iii) *F is topologically closed in E.*

Proof. The equivalence (i)⇔(ii) holds for a general vector lattice and its quotient vector lattices [36, Theorem 60.2]. The equivalence (ii)⇔(iii) is a consequence of Proposition 4.2.4 in [48], once we observe that the definition of relatively uniformly convergent sequences in [48, Definition 1.5.7] is equivalent to that given above. ∎

Since a Fréchet lattice E is uniformly complete (by Lemma 3.2(iv)(b) and Fact 1), its quotient vector lattice E/F with respect to an ideal F in E is also uniformly complete [36, Corollary 59.4]. Then the equivalence (i)⇔(iii) in Fact 3, together with Fact 1, shows that an ideal F of a Fréchet lattice E is topologically closed if and only if E/F satisfies Axiom (OS).

(vi) Let us point out, for the equivalence (ii)⇔(iii) in Fact 3, that the topological completeness of E is crucial. There exist examples which satisfy condition (ii) but not condition (iii); see, for instance, [48, Example 4.2.5] and [60, Exercise 100.13].

The implication (iii)⇒(ii) in Fact 3 holds even if E is merely a metrizable locally convex vector lattice (i.e., a vector lattice with a lcH-topology generated by countably many lattice seminorms). In fact, (iii) implies that the quotient vector lattice E/F is also a lcHs [1, Theorem 4.7(iii)], and hence it is Arichmedean [1, Theorem 5.6(i)], i.e., (i) holds. Moreover, (ii) also holds via the general equivalence (i)⇔(ii); see the proof of Fact 3. □

According to [55, Definition II.11.3], a complex Banach lattice is defined as the complexification $E_{\mathbb{C}}$ of a Banach lattice E with a lattice norm $\|\cdot\|_E$, in which case $E_{\mathbb{C}}$ is complete with respect to the norm $z \mapsto \||z|\|_E$ for $z \in E_{\mathbb{C}}$. It is important to note that $E_{\mathbb{C}}$ is a complex vector lattice because of the complex modulus property of E; see Lemma 3.2(iv)(c).

The complex sequence spaces c_0 and ℓ^∞ are complex Banach lattices equipped with their respective uniform norm. Indeed, c_0 (resp. ℓ^∞) is realized as the complexification of the Banach lattice

$$(c_0)_{\mathbb{R}} := c_0 \cap \mathbb{R}^{\mathbb{N}} \quad (\text{resp. } (\ell^\infty)_{\mathbb{R}} := \ell^\infty \cap \mathbb{R}^{\mathbb{N}})$$

equipped with the uniform norm (which is a lattice norm).

Let us return to a general complex vector lattice $E_{\mathbb{C}} = E + iE$, i.e., $E_{\mathbb{C}}$ is the complexification of a vector lattice E with the complex modulus property. Relevant is the question: Given a complex vector subspace $G \subseteq E_{\mathbb{C}}$, when is G a complex vector lattice? We will say that G is *closed under complex conjugation* if, given $z \in G$, its complex conjugate $\overline{z}$ in $E_{\mathbb{C}}$ belongs to G. Similarly, G is said to be *closed under forming the modulus* if, given $z \in G$, its modulus $|z|$ in $E_{\mathbb{C}}$ (recall that $|z| \in E^+$) belongs to G. Finally,

G is called *solid* if $z \in E_{\mathbb{C}}$, $w \in G$ and $|z| \leq |w|$ imply that $z \in G$, in which case G is necessarily closed under complex conjugation (since $|z| = |\overline{z}|$ for $z \in E_{\mathbb{C}}$) and forming the modulus. To verify the latter claim, given $w \in G$ let $z := |w|$ (the modulus of w in $E_{\mathbb{C}}$). Then $z \in E_{\mathbb{C}}$ satisfies $|z| = \big||w|\big| = |w| \leq |w|$. Since G is solid, it follows that $z \in G$, i.e., $|w| \in G$.

Returning to a general complex vector subspace $G \subseteq E_{\mathbb{C}}$, define

$$G_{\mathbb{R}} := G \cap E.$$

The inclusion

$$G_{\mathbb{R}} + iG_{\mathbb{R}} \subseteq G \tag{3.6}$$

always holds because G is a vector subspace of $E_{\mathbb{C}}$. Assume further that $G_{\mathbb{R}}$ is a vector sublattice of E. Then we say that G is a complex vector lattice *in the order induced by* $E_{\mathbb{C}}$ if $G_{\mathbb{R}}$ has the complex modulus property and if G is the complexification of $G_{\mathbb{R}}$, i.e.,

$$G = G_{\mathbb{R}} + iG_{\mathbb{R}}. \tag{3.7}$$

Lemma 3.4. *Let G be a complex vector subspace of a complex vector lattice $E_{\mathbb{C}} = E + iE$ such that $G_{\mathbb{R}}$ is a vector sublattice of E.*

(i) *The complex vector subspace G is closed under complex conjugation if and only if* (3.7) *holds.*
(ii) *Assume that G is closed under complex conjugation and forming the modulus. Then G is a complex vector lattice in the order induced by $E_{\mathbb{C}}$. Moreover, given $z \in G$, the modulus of z in G equals that of z in $E_{\mathbb{C}}$.*
(iii) *If G is solid, then the same conclusion as in part* (ii) *holds.*
(iv) *Assume that $G_{\mathbb{R}}$ is order dense in E. Then G is a complex vector lattice in the order induced by $E_{\mathbb{C}}$ if and only if G is closed under complex conjugation and forming the modulus.*

Proof. (i) Suppose that G is closed under complex conjugation. Let $z \in G$, in which case $z = x + iy$ with $x, y \in E$. By assumption also $\overline{z} = x - iy \in G$. Since G is a vector space, the identities $x = \frac{1}{2}(z + \overline{z})$ and $y = \frac{1}{2i}(z - \overline{z})$ show that both x, y are in G, i.e., $x, y \in G_{\mathbb{R}} := G \cap E$. This, together with (3.6), imply (3.7).

Clearly (3.7) implies that G is closed under complex conjugation.

(ii) Let $x, y \in G_{\mathbb{R}}$, in which case $x + iy \in G$ via (3.6), and define

$$H := \{(\cos\theta)x + (\sin\theta)y : \theta \in [0, 2\pi]\} \subseteq G_{\mathbb{R}} \subseteq E. \tag{3.8}$$

Then $|x + iy| := \sup_E H \in E$ belongs to $G_{\mathbb{R}}$ (by the assumptions on G). Lemma 3.1 with $F := G_{\mathbb{R}}$ tells us that $\sup_E H$ equals the supremum of H in $G_{\mathbb{R}}$. Accordingly, $G_{\mathbb{R}}$ has the complex modulus property. This, together with part (i), ensures that G is a complex vector lattice in the order induced by $E_{\mathbb{C}}$.

The second conclusion has already been established.

(iii) This holds because the assumptions of part (ii) hold when G is solid; see the discussion prior to Lemma 3.4.

(iv) Suppose that G is a complex vector lattice in the order induced by $E_{\mathbb{C}}$. That G is closed under complex conjugation is clear from (3.7); see part (i). Next, let $x, y \in G_{\mathbb{R}}$.

Then $x + iy \in G$; see (3.6). The set $H \subseteq G_{\mathbb{R}}$ given in (3.8) admits a supremum $\sup_{G_{\mathbb{R}}} H$ in $G_{\mathbb{R}}$ (as $G_{\mathbb{R}}$ has the complex modulus property). But this supremum $\sup_{G_{\mathbb{R}}} H$ of H in $G_{\mathbb{R}}$ *equals* its supremum $\sup_E H$ in E because of the order denseness of $G_{\mathbb{R}}$ in E [1, Theorem 1.10]. So, $|x + iy| := \sup_E H = \sup_{G_{\mathbb{R}}} H \in G_{\mathbb{R}}$, and therefore G is closed under forming the modulus.

The converse implication is precisely part (ii). Note that here the order denseness of $G_{\mathbb{R}}$ in E is not required. ∎

We proceed to apply the general results just established to various function spaces associated with a vector measure. So, let X be a lcHs and m be an X-valued vector measure defined on a measurable space (Ω, Σ). To see that the vector space $\mathcal{L}^0(\Sigma)$ (see Section 2) is a complex vector lattice, observe first that the real vector space

$$\mathcal{L}^0(\Sigma)_{\mathbb{R}} := \mathcal{L}^0(\Sigma) \cap \mathbb{R}^{\Omega}$$

is a vector sublattice of the vector lattice $\mathbb{R}^{\Omega}$ in the pointwise order. It is clear that $\mathcal{L}^0(\Sigma)$ equals the complexification of $\mathcal{L}^0(\Sigma)_{\mathbb{R}}$, namely

$$\mathcal{L}^0(\Sigma) = \mathcal{L}^0(\Sigma)_{\mathbb{R}} + i\mathcal{L}^0(\Sigma)_{\mathbb{R}}. \tag{3.9}$$

A useful fact is that a subset $H \subseteq \mathbb{R}^{\Omega}$ admits a supremum in $\mathbb{R}^{\Omega}$ if and only if $\sup_{f\in H} f(w) < \infty$ for each $w \in \Omega$, in which case $\sup H \in \mathbb{R}^{\Omega}$ is the function $w \mapsto \sup_{f\in H} f(w)$ on Ω. In other words, the supremum in $\mathbb{R}^{\Omega}$ is the *pointwise supremum*. This also applies to $\mathcal{L}^0(\Sigma)_{\mathbb{R}}$ when we limit ourselves to the suprema of its countable subsets. A consequence is that $\mathcal{L}^0(\Sigma)_{\mathbb{R}}$ is Dedekind σ-complete, so that it satisfies Axiom (OS), and hence has the complex modulus property; see Lemma 3.2(ii), (iii) and the proof. So, $\mathcal{L}^0(\Sigma)$ is a complex vector lattice via (3.9). Moreover, given $f, g \in \mathcal{L}^0(\Sigma)_{\mathbb{R}}$, the modulus $|f + ig|$ of $f + ig$ in the complex vector lattice $\mathcal{L}^0(\Sigma)$ equals the pointwise modulus, i.e.,

$$|f + ig| := \sup_{\theta\in[0,2\pi]} ((\cos\theta)f + (\sin\theta)g) = \sqrt{f^2 + g^2}. \tag{3.10}$$

This is an application of Lemma 3.1 with $E := \mathbb{R}^{\Omega}$ and $F := \mathcal{L}^0(\Sigma)_{\mathbb{R}}$. Alternatively, Lemma 3.4(ii) with $E := \mathbb{R}^{\Omega}$ and $G := \mathcal{L}^0(\Sigma)$ gives both the fact that $\mathcal{L}^0(\Sigma)$ is a complex vector lattice and that (3.10) holds.

Recall from Section 2 the vector subspace $\mathcal{N}(m) \subseteq \mathcal{L}^1(m) \subseteq \mathcal{L}^0(\Sigma)$ of all m-null functions. Define

$$\mathcal{N}(m)_{\mathbb{R}} := \mathcal{N}(m) \cap \mathcal{L}^0(\Sigma)_{\mathbb{R}}$$

so that

$$\mathcal{N}(m) = \mathcal{N}(m)_{\mathbb{R}} + i\mathcal{N}(m)_{\mathbb{R}}. \tag{3.11}$$

Clearly, $\mathcal{N}(m)_{\mathbb{R}}$ is an ideal in $\mathcal{L}^0(\Sigma)_{\mathbb{R}}$; see (2.11) and (2.12). Moreover, it is also a σ-ideal. To see this let $\{f_n\}_{n\in\mathbb{N}}$ be a countable subset of $\mathcal{N}(m)_{\mathbb{R}}$ such that $\varphi := \sup_{n\in\mathbb{N}} f_n$ exists in $\mathcal{L}^0(\Sigma)_{\mathbb{R}}$. Then the increasing sequence $g_n := f_1 \vee \cdots \vee f_n \in \mathcal{N}(m)_{\mathbb{R}}$ for $n \in \mathbb{N}$, satisfies $f_n \le g_n \le \varphi$, for $n \in \mathbb{N}$. The Dedekind σ-completeness of $\mathcal{L}^0(\Sigma)_{\mathbb{R}}$ ensures that $\psi := \sup_{n\in\mathbb{N}} g_n$ exists in $\mathcal{L}^0(\Sigma)_{\mathbb{R}}$. It is routine to verify that $\varphi = \psi$. Moreover, $g_n \in \mathcal{N}(m)_{\mathbb{R}}$ for each $n \in \mathbb{N}$ and $g_n \uparrow \psi$ pointwise on Ω. The identity (2.13) and the Monotone Convergence Theorem for each scalar measure $|\langle m, x^*\rangle|$, for $x^* \in X^*$, imply that $\psi \in \mathcal{N}(m)$, i.e., $\varphi \in \mathcal{N}(m)_{\mathbb{R}}$. Accordingly, the quotient vector lattice $\mathcal{L}^0(\Sigma)_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}}$

is Dedekind σ-complete. For this fact, see [24, proof of Proposition 62G and Notes and Comments on p. 159]. So, $\mathcal{L}^0(\Sigma)_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}}$ satisfies Axiom (OS), or equivalently, it is Archimedean and uniformly complete via Fact 1 of Remark 3.3. This enables us to apply [60, pp. 198–199] to establish that the quotient vector space $L^0(m) = \mathcal{L}^0(\Sigma)/\mathcal{N}(m)$ is a complex vector lattice as follows.

Let $\Pi : \mathcal{L}^0(\Sigma) \to \mathcal{L}^0(\Sigma)/\mathcal{N}(m)$ be the quotient map and write $[h] := \Pi(h)$ for $h \in \mathcal{L}^0(\Sigma)$. Then, with the natural identification $\Pi(\mathcal{L}^0(\Sigma)_{\mathbb{R}}) = \mathcal{L}^0(\Sigma)_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}}$, we have

$$\mathcal{L}^0(\Sigma)/\mathcal{N}(m) = (\mathcal{L}^0(\Sigma)_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}}) + i(\mathcal{L}^0(\Sigma)_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}}) \tag{3.12}$$

and hence

$$[f+ig] = [f] + i[g], \qquad f, g \in \mathcal{L}^0(\Sigma)_{\mathbb{R}}, \tag{3.13}$$

which can naturally be seen from

$$(f+ig) + \mathcal{N}(m) = (f+ig) + (\mathcal{N}(m)_{\mathbb{R}} + i\mathcal{N}(m)_{\mathbb{R}}) = (f + \mathcal{N}(m)_{\mathbb{R}}) + i(g + \mathcal{N}(m)_{\mathbb{R}}).$$

So, $\mathcal{L}^0(\Sigma)/\mathcal{N}(m)$ is a complex vector lattice. In view of (3.12) we write

$$L^0(m)_{\mathbb{R}} := \mathcal{L}^0(\Sigma)_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}},$$

so that (3.12) can be rewritten simply as

$$L^0(m) = L^0(m)_{\mathbb{R}} + iL^0(m)_{\mathbb{R}}.$$

Regarding the modulus, for $h \in \mathcal{L}^0(\Sigma)$ with $f := \mathrm{Re}(h)$ and $g := \mathrm{Im}(h)$, we have the formula

$$[|h|] = \sup_{\theta \in [0,2\pi]} ((\cos\theta)[f] + \sin\theta[g]) = |[h]|. \tag{3.14}$$

Applying both of the above facts, derived from [60] and Lemma 3.4, we shall determine in Proposition 3.7 below whether or not $L^1(m) := \mathcal{L}^1(m)/\mathcal{N}(m)$ and $L^1_w(m) := \mathcal{L}^1_w(m)/\mathcal{N}(m)$ are complex vector lattices. To this end, let $\mathcal{F} \subseteq \mathcal{L}^0(\Sigma)$ be any vector subspace satisfying the two conditions that $\mathcal{F}$ contains $\mathcal{N}(m)$ and that its subset $\mathcal{F}_{\mathbb{R}} := \mathcal{F} \cap \mathcal{L}^0(\Sigma)_{\mathbb{R}}$ is a vector sublattice of $\mathcal{L}^0(\Sigma)_{\mathbb{R}}$. First, observe that the quotient vector space $\mathcal{F}/\mathcal{N}(m)$ equals $\Pi(\mathcal{F})$, and hence is a vector subspace of $L^0(m) = \mathcal{L}^0(\Sigma)/\mathcal{N}(m)$. On the other hand, $\mathcal{N}(m)_{\mathbb{R}}$ being an ideal of $\mathcal{F}_{\mathbb{R}}$ also allows us to consider the quotient vector lattice $\mathcal{F}_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}}$, which is also a vector sublattice of $\mathcal{L}^0(\Sigma)_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}}$. In particular, the order in the quotient vector lattice $\mathcal{F}_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}}$ coincides with that induced by $\mathcal{L}^0(\Sigma)_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}}$. To understand Lemma 3.5 below, recall (3.12) and observe that

$$(\mathcal{F}_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}}) + i(\mathcal{F}_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}}) \subseteq \mathcal{F}/\mathcal{N}(m) \subseteq \mathcal{L}^0(\Sigma)/\mathcal{N}(m) = L^0(m). \tag{3.15}$$

LEMMA 3.5. *Suppose that $m : \Sigma \to X$ is a non-zero, lcHs-valued vector measure defined on a measurable space (Ω, Σ). Let $\mathcal{F} \subseteq \mathcal{L}^0(\Sigma)$ be any vector subspace containing $\mathcal{N}(m)$ and such that $\mathcal{F}_{\mathbb{R}} := \mathcal{F} \cap \mathcal{L}^0(\Sigma)_{\mathbb{R}}$ is a vector sublattice of $\mathcal{L}^0(\Sigma)_{\mathbb{R}}$.*

(i) *The identity $\mathcal{F} = \mathcal{F}_{\mathbb{R}} + i\mathcal{F}_{\mathbb{R}}$ holds in $\mathcal{L}^0(\Sigma)$ if and only if the identity $\mathcal{F}/\mathcal{N}(m) = (\mathcal{F}_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}}) + i(\mathcal{F}_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}})$ holds in $L^0(m)$.*

(ii) *The vector subspace $\mathcal{F} \subseteq \mathcal{L}^0(\Sigma)$ is closed under forming the modulus if and only if the vector subspace $\mathcal{F}/\mathcal{N}(m) \subseteq L^0(m)$ is closed under forming the modulus.*

(iii) *The vector subspace $\mathcal{F} \subseteq \mathcal{L}^0(\Sigma)$ is closed under complex conjugation and forming the modulus if and only if the same holds for the vector subspace $\mathcal{F}/\mathcal{N}(m) \subseteq L^0(m)$.*

(iv) *The vector subspace* $\mathcal{F} \subseteq \mathcal{L}^0(\Sigma)$ *is solid if and only if the vector subspace* $\mathcal{F}/\mathcal{N}(m) \subseteq L^0(m)$ *is solid.*

(v) *If* $\mathcal{F} \supseteq \operatorname{sim}\Sigma$, *then* $\mathcal{F}_{\mathbb{R}}$ *and* $\mathcal{F}_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}}$ *are order dense in* $\mathcal{L}^0(\Sigma)$ *and* $L^0(m)$, *respectively.*

Proof. (i) Once we observe, given $h \in \mathcal{F} \subseteq \mathcal{L}^0(\Sigma)$ with $f := \operatorname{Re}(h)$ and $g := \operatorname{Im}(h)$, that $[\,h\,] = [\,f\,] + i[\,g\,]$ in $L^0(m)$ by (3.13), part (i) follows routinely.

(ii) By (3.14), for each $h \in \mathcal{F}$, we have

$$|[h]| = [|h|] \qquad \text{in } L^0(m), \tag{3.16}$$

which verifies the "only if" part. Conversely, assume that $\mathcal{F}/\mathcal{N}(m)$ is closed under forming the modulus. Then, given $h \in \mathcal{F}$, it follows from (3.16) that $[|h|] = |[h]| \in \mathcal{F}/\mathcal{N}(m)$. So, $|h| \in \mathcal{F}$ because $\mathcal{F} \supseteq \mathcal{N}(m)$, which establishes the "if" part.

(iii) This follows from parts (i) and (ii).

(iv) Suppose that $\mathcal{F}$ is solid. Let $h_1 \in \mathcal{L}^0(\Sigma)$ and $h_2 \in \mathcal{F}$ satisfy $|[h_1]| \le |[h_2]|$ in $L^0(m)$. Then (3.16) yields $[\,|h_1|\,] \le [\,|h_2|\,]$ in $L^0(m)$. Hence, there is $A \in \Sigma$, with $\Omega \setminus A \in \mathcal{N}_0(m)$, such that $|h_1|\chi_A \le |h_2|\chi_A \le |h_2|$. This implies that $h_1\chi_A \in \mathcal{F}$ because $h_2 \in \mathcal{F}$ with $\mathcal{F}$ solid. Thus

$$h_1 = h_1\chi_A + h_1\chi_{\Omega\setminus A} \in \mathcal{F} + \mathcal{N}(m) \subseteq \mathcal{F},$$

which yields $[h_1] \in \mathcal{F}/\mathcal{N}(m)$. So, $\mathcal{F}/\mathcal{N}(m)$ is solid.

Suppose now that $\mathcal{F}/\mathcal{N}(m)$ is solid. Let $h_1 \in \mathcal{L}^0(\Sigma)$ and $h_2 \in \mathcal{F}$ satisfy $|h_1| \le |h_2|$. Then $|\,[\,h_1\,]\,| = [\,|h_1|\,] \le [\,|h_2|\,] = |\,[\,h_2\,]\,|$ by (3.16). This and the solidness of $\mathcal{F}/\mathcal{N}(m)$ yield $[h_1] \in \mathcal{F}/\mathcal{N}(m)$, and hence $h_1 \in \mathcal{F}$ because $\mathcal{F} \supseteq \mathcal{N}(m)$.

(v) Clearly $(\operatorname{sim}\Sigma)_{\mathbb{R}} := (\operatorname{sim}\Sigma) \cap \mathcal{L}^0(\Sigma)_{\mathbb{R}}$ is order dense in $\mathcal{L}^0(\Sigma)_{\mathbb{R}}$, and hence so is $\mathcal{F}_{\mathbb{R}}$ as it contains $(\operatorname{sim}\Sigma)_{\mathbb{R}}$. To prove the order denseness of $\mathcal{F}_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}}$ in $\mathcal{L}^0(\Sigma)_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}}$, let $0 \le f \in \mathcal{L}^0(\Sigma)_{\mathbb{R}} \setminus \mathcal{N}(m)_{\mathbb{R}}$, so that $0 \le [\,f\,] \in (\mathcal{L}^0(\Sigma)_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}}) \setminus \{0\}$. Choose $0 \le s \in (\operatorname{sim}\Sigma)_{\mathbb{R}} \setminus \mathcal{N}(m)_{\mathbb{R}}$ satisfying $s \le f$. Then $0 \le [\,s\,] \le [\,f\,]$ with $[\,s\,] \ne 0$. This implies the order denseness of $\mathcal{F}_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}}$ in $\mathcal{L}^0(\Sigma)_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}}$ because $[\,s\,]$ belongs to $\mathcal{F}_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}}$. ∎

Definition 3.6. Under the same assumptions as in Lemma 3.5, we say that the quotient vector space $\mathcal{F}/\mathcal{N}(m) \subseteq L^0(m)$ is a complex vector lattice in the *m-a.e. pointwise order* if it is a complex vector lattice in the order induced by $L^0(m)$.

Proposition 3.7. *Let* $m : \Sigma \to X$ *be a non-zero, lcHs-valued vector measure defined on a measurable space* (Ω, Σ).

(i) *Both vector subspaces* $\mathcal{L}^1(m)_{\mathbb{R}} := \mathcal{L}^1(m) \cap \mathcal{L}^0(\Sigma)_{\mathbb{R}}$ *and* $\mathcal{L}^1_w(m)_{\mathbb{R}} := \mathcal{L}^1_w(m) \cap \mathcal{L}^0(\Sigma)_{\mathbb{R}}$ *of the vector lattice* $\mathcal{L}^0(\Sigma)_{\mathbb{R}}$ *are vector sublattices. Furthermore, the quotient vector lattices* $L^1(m)_{\mathbb{R}} := \mathcal{L}^1(m)_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}}$ *and* $L^1_w(m)_{\mathbb{R}} := \mathcal{L}^1_w(m)_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}}$ *are vector sublattices of the quotient vector lattice* $L^0(m)_{\mathbb{R}} := \mathcal{L}^0(\Sigma)_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}}$.

(ii) *The following conditions are equivalent:*

  (a) *The vector subspace* $\mathcal{L}^1(m) \subseteq \mathcal{L}^0(\Sigma)$ *is closed under complex conjugation and forming the modulus.*

  (b) $\mathcal{L}^1(m)$ *is a complex vector lattice in the order induced by* $\mathcal{L}^0(\Sigma)$.

(c) *The vector subspace* $L^1(m) \subseteq L^0(m)$ *is closed under complex conjugation and forming the modulus.*

(d) $L^1(m)$ *is a complex vector lattice in the* m*-a.e. pointwise order.*

(iii) *The vector subspace* $\mathcal{L}^1_w(m) \subseteq \mathcal{L}^0(\Sigma)$ *is solid, and hence is a complex vector lattice in the order induced by* $\mathcal{L}^0(\Sigma)$.

(iv) *The vector subspace* $L^1_w(m) := \mathcal{L}^1_w(m)/\mathcal{N}(m) \subseteq L^0(m)$ *is solid, and hence is a complex vector lattice in the* m*-a.e. pointwise order.*

Proof. (i) Let $f \in \mathcal{L}^1(m)_{\mathbb{R}}$ and set $A := f^{-1}([0,\infty))$. Then $f\chi_A$ is m-integrable (see (2.4)), and hence $f^+ \in \mathcal{L}^1(m)_{\mathbb{R}}$, from which it routinely follows that $\mathcal{L}^1(m)_{\mathbb{R}}$ is a vector sublattice of $\mathcal{L}^0(\Sigma)_{\mathbb{R}}$.

Next, $\mathcal{L}^1_w(m)$ is solid in $\mathcal{L}^0(\Sigma)$ by its definition, and hence $\mathcal{L}^1_w(m)_{\mathbb{R}}$ is an ideal of $\mathcal{L}^0(\Sigma)_{\mathbb{R}}$. In particular, $\mathcal{L}^1_w(m)_{\mathbb{R}}$ is a vector sublattice.

The statement regarding the quotient vector lattices $L^1(m)_{\mathbb{R}}$ and $L^1_w(m)_{\mathbb{R}}$ has already been verified, immediately prior to Lemma 3.5, with $\mathcal{F} := \mathcal{L}^1(m)_{\mathbb{R}}$ or $\mathcal{F} := \mathcal{L}^1_w(m)_{\mathbb{R}}$.

(ii) From Lemma 3.5(v) with $\mathcal{F} := \mathcal{L}^1(m) \supseteq \operatorname{sim}\Sigma$, we deduce that $\mathcal{L}^1(m)$ and $\mathcal{L}^1(m)/\mathcal{N}(m)$ are order dense in $\mathcal{L}^0(\Sigma)$ and $L^0(m)$, respectively. So, the equivalences (a)⇔(b) and (c)⇔(d) follow from Lemma 3.4(iv) with $G := \mathcal{L}^1(m)$, $E_{\mathbb{C}} := \mathcal{L}^0(\Sigma)$ and with $G := L^1(m)$, $E_{\mathbb{C}} := L^0(m)$, respectively. Next, the equivalence (a)⇔(c) is a special case of Lemma 3.5(iii) with $\mathcal{F} := \mathcal{L}^1(m)$.

(iii) and (iv). Since $\mathcal{L}^1_w(m)$ is solid in $\mathcal{L}^0(\Sigma)$, so is $L^1_w(m)$ in $L^0(m)$ via Lemma 3.5(iv) with $\mathcal{F} := \mathcal{L}^1_w(m)$. Now apply Lemma 3.4(iii) with $G := \mathcal{L}^1_w(m)$ and with $G := L^1_w(m)$ to establish parts (iii) and (iv), respectively. ■

Assume that any one of (a)–(d) in Proposition 3.7(ii) above holds. Given $h \in \mathcal{L}^1(m)$, its modulus in the complex vector lattice $\mathcal{L}^1(m)$ equals its pointwise modulus. To see this, apply both Lemma 3.4(ii) (with $G := \mathcal{L}^1(m)$ and $E_{\mathbb{C}} := \mathcal{L}^0(\Sigma)$) and (3.10) (with $f := \operatorname{Re}(h)$ and $g := \operatorname{Im}(h)$). Similarly, the modulus of $[h] \in L^1(m)$ equals $[|h|]$. The corresponding results for $\mathcal{L}^1_w(m)$ and $L^1_w(m)$ are also valid.

In view of this observation, Lemma 3.5 and Proposition 3.7, we may, for a simpler presentation, identify the quotient spaces $L^1(m) = \mathcal{L}^1(m)/\mathcal{N}(m)$ and $L^1_w(m) = \mathcal{L}^1_w(m)/\mathcal{N}(m)$ with $\mathcal{L}^1(m)$ and $\mathcal{L}^1_w(m)$, respectively, unless stated otherwise. In the same spirit, we identify $L^1(m)_{\mathbb{R}} = \mathcal{L}^1(m)_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}}$ and $L^1_w(m)_{\mathbb{R}} = \mathcal{L}^1_w(m)_{\mathbb{R}}/\mathcal{N}(m)_{\mathbb{R}}$ with $\mathcal{L}^1(m)_{\mathbb{R}}$ and $\mathcal{L}^1_w(m)_{\mathbb{R}}$, respectively.

Before discussing sufficient conditions for $L^1(m)$ to be a complex vector lattice in the m-a.e. pointwise order, observe that the following containments always hold (see (3.15) with $\mathcal{F} := \mathcal{L}^1(m)$):

$$L^1(m)_{\mathbb{R}} + iL^1(m)_{\mathbb{R}} \subseteq L^1(m) \subseteq L^0(m).$$

Remark 3.8. Let $m : \Sigma \to X$ be as in the statement of Proposition 3.7.

(i) Each of the following conditions is sufficient for $L^1(m)$ to be a complex vector lattice in the m-a.e. pointwise order:

(α) $L^1(m)$ *is solid in* $L^0(m)$.

(β) *The sequential closure* $[X]_m$ *of the linear span of* $m(\Sigma)$ *in* X *is sequentially complete.*

(γ) $L^1(m)$ is $\tau(m)$*-sequentially complete.*

Indeed, condition (α) is sufficient via Lemma 3.4(iii) with $G := L^1(m)$ and $E_{\mathbb{C}} := L^0(m)$. Conditions (β) and (γ) are also sufficient because of the implications $(\beta)\Rightarrow(\alpha)$ and $(\gamma)\Rightarrow(\alpha)$ established in Propositions 2.7 and 2.8 of [41], respectively.

We point out that (α) implies the Dedekind σ-completeness of $L^1(m)_{\mathbb{R}}$ because $L^1(m)_{\mathbb{R}}$ is then an ideal of the Dedekind σ-complete vector lattice $L^0(m)_{\mathbb{R}}$. In particular, $L^1(m)_{\mathbb{R}}$ satisfies Axiom (OS).

Conditions equivalent to (α) can be found in [41, Proposition 2.4].

We point out that conditions (β) and (γ) are unrelated, that is, (β) does not always imply (γ) (see Example 6.5(i)) and (γ) does not imply (β) in general (see Example 6.5(iv)).

(ii) Suppose that X is a Banach space. Then $L^1(m)$ is a complex Banach lattice. Indeed, first observe that $L^1(m)$ is a complex vector lattice in the m-a.e. pointwise order via (β) in part (i). Next, $L^1(m)$ is a Banach space in the norm $\|\cdot\|_{L^1(m)}$ (see Lemma 2.10(v) and (2.23)). Further, the restriction of $\|\cdot\|_{L^1(m)}$ to $L^1(m)_{\mathbb{R}}$ is a lattice norm for which $L^1(m)_{\mathbb{R}}$ is a Banach lattice. So, $L^1(m)$ is a complex Banach lattice. Similarly, $L^1_w(m)$ is also a complex Banach lattice. Indeed, by Proposition 3.7(iv) we see that $L^1_w(m)$ is a complex vector lattice and via Remark 2.11 that it is a Banach space for the norm $\|\cdot\|_{L^1_w(m)}$. Further, the restriction of $\|\cdot\|_{L^1_w(m)}$ to $L^1_w(m)_{\mathbb{R}}$ is a lattice norm for which $L^1_w(m)_{\mathbb{R}}$ is a Banach lattice. So, $L^1_w(m)$ is a complex Banach lattice.

It is worth noting that $\|\cdot\|_{L^1(m)}$ is *order continuous* in the sense that, if $\{f_\lambda\}_\lambda$ is any downwards directed net in $L^1(m)_{\mathbb{R}}$ with $\inf_\lambda f_\lambda = 0$, then $\lim_\lambda \|f_\lambda\|_{L^1(m)} = 0$, [46, Theorem 3.7(iii)]. □

Example 3.9. Let μ be Lebesgue measure on the Borel σ-algebra $\Sigma := \mathcal{B}(\Omega)$ of the interval $\Omega := (0,1]$. Suppose that $X \subseteq L^1(\mu)$ is a vector subspace satisfying $\chi_\Omega \in X$ and $fs \in X$ for all $f \in X$ and all $s \in \operatorname{sim}\Sigma$. In particular, $\operatorname{sim}\Sigma \subseteq X$. Equip X with the norm induced by $L^1(\mu)$. Then the X-valued set function $m : A \mapsto \chi_A$ on Σ is a vector measure with $L^1(m) = X$ as lcHs or, more precisely, as normed spaces [41, Corollary 3.2]. The identity function on Ω is denoted by $\mathbb{x}$.

(i) Let $X := \operatorname{sim}\Sigma$. Then $L^1(m) = \operatorname{sim}\Sigma$ is closed under complex conjugation and forming the modulus, so that $L^1(m)$ is a complex vector lattice in the m-a.e. pointwise order (see Proposition 3.7(ii)). However, $L^1(m)_{\mathbb{R}}$ does not satisfy Axiom (OS). To see this, let $A(n) := (1/(n+1), 1/n]$ and $f_n := (1/2^n)\chi_{A(n)}$, so that $0 \le f_n \le (1/2^n)\chi_\Omega$ for $n \in \mathbb{N}$, with $(1/2^n)_{n=1}^\infty \in \ell^1$. Then the supremum of $\{\sum_{n=1}^N f_n\}_{N=1}^\infty$ does not exist in $L^1(m)_{\mathbb{R}} = (\operatorname{sim}\Sigma)_{\mathbb{R}}$.

It is clear that m fails to have the LCP.

(ii) Let $X := L^\infty(\mu)$. Then $L^1(m) = L^\infty(\mu)$ is solid in $L^0(m)$. So, $L^1(m)$ is a complex vector lattice in the m-a.e. pointwise order via condition (α) in Remark 3.8(i). However, conditions (β) and (γ) there do not hold [41, Example 3.5].

As $L^1(m)$ is solid, the vector measure m has the LCP [41, Proposition 2.4].

(iii) Let $X := L^\infty(\mu) + \frac{1}{\sqrt{\mathbb{x}}}\operatorname{sim}\Sigma$. Then $L^1(m) = X$ is clearly closed under complex conjugation. The claim is that $L^1(m) = X$ is also closed under forming the modulus, which corrects the corresponding false assertion in [41, Example 3.4(i)]. To prove this, fix $f \in L^\infty(\mu)$ and $s \in \operatorname{sim}\Sigma$ and define $A := s^{-1}(\{0\})$ and $B := \{w \in \Omega : f(w) + s(w)/\sqrt{w} = 0\}$.

Clearly,

$$\left|f + \frac{1}{\sqrt{\mathbb{x}}} s\right| \chi_{A \cup B} = f \chi_{A \setminus B} \in L^\infty(\mu) \subseteq X.$$

To verify that $\left|f + \frac{1}{\sqrt{\mathbb{x}}} s\right| \in X$, observe that on $(A \cup B)^c$ we have

$$\left|f + \frac{1}{\sqrt{\mathbb{x}}} s\right| = \frac{\left|f + \frac{1}{\sqrt{\mathbb{x}}} s\right|^2 - \left(\frac{1}{\sqrt{\mathbb{x}}}|s|\right)^2}{\left|f + \frac{1}{\sqrt{\mathbb{x}}} s\right| + \frac{1}{\sqrt{\mathbb{x}}}|s|} + \frac{1}{\sqrt{\mathbb{x}}}|s| = \frac{\sqrt{\mathbb{x}}\,|f|^2 + (f\overline{s} + \overline{f}s)}{|\sqrt{\mathbb{x}}\, f + s| + |s|} + \frac{1}{\sqrt{\mathbb{x}}}|s|.$$

But, $|s| \in \operatorname{sim} \Sigma$ and so $\frac{1}{\sqrt{\mathbb{x}}}|s| \chi_{(A \cup B)^c} \in X$. So, it suffices to show that

$$\frac{\sqrt{\mathbb{x}}\,|f|^2 + (f\overline{s} + \overline{f}s)}{|\sqrt{\mathbb{x}}\, f + s| + |s|} \chi_{(A \cup B)^c} \in L^\infty(\mu) \subseteq X.$$

Now, the inequalities $\sqrt{\mathbb{x}} \leq 1$ and $|\sqrt{\mathbb{x}} + s| + |s| \geq |s|$ on Ω, and the fact that $f \in L^\infty(\mu)$, imply that on $(A \cup B)^c$ we have

$$\frac{\sqrt{\mathbb{x}}\,|f|^2 + (f\overline{s} + \overline{f}s)}{|\sqrt{\mathbb{x}}\, f + s| + |s|} \leq \frac{|f|^2}{|s|} + \frac{|f\overline{s} + \overline{f}s|}{|s|} \leq \frac{|f|^2}{|s|} + 2|f|.$$

Since $|s| \geq \min\{|s(w)| : w \notin A \cup B\} > 0$, it follows that the right side (hence also the left side) of the previous inequality does indeed belong to $L^\infty(\mu) \subseteq X$. Consequently, $L^1(m)$ is a complex vector lattice in the m-a.e. pointwise order; see Proposition 3.7(ii). Hence, by the definition of a complex vector lattice, $L^1(m)_{\mathbb{R}}$ has the complex modulus property. However, by considering the functions $0 + \frac{1}{\sqrt{\mathbb{x}}} f_n$, with f_n as in part (i), for $n \in \mathbb{N}$, the argument of part (i) shows that $L^1(m)_{\mathbb{R}}$ fails Axiom (OS).

The fact that m fails to have the LCP has been asserted in [41, Example 3.4(i)] with correct arguments.

(iv) Define $X := \operatorname{sim} \Sigma + \frac{1}{\sqrt{\mathbb{x}}} \operatorname{sim} \Sigma$. Then $L^1(m) = X$ is not closed under forming the modulus because

$$\left|\chi_\Omega + \frac{i}{\sqrt{\mathbb{x}}} \chi_\Omega\right| \notin X \tag{3.17}$$

whereas $\chi_\Omega + \frac{i}{\sqrt{\mathbb{x}}} \chi_\Omega \in X$. To prove (3.17) assume, on the contrary, that there exist $s_1, s_2 \in \operatorname{sim} \Sigma$ satisfying

$$\left|\chi_\Omega + \frac{i}{\sqrt{\mathbb{x}}} \chi_\Omega\right| = s_1 + \frac{1}{\sqrt{\mathbb{x}}} s_2. \tag{3.18}$$

We may assume that $s_1, s_2 \in (\operatorname{sim} \Sigma)_{\mathbb{R}}$. Select $n \in \mathbb{N}$, pairwise disjoint, non-μ-null sets $A(1), \ldots, A(n) \in \Sigma$ with $\bigcup_{j=1}^n A(j) = \Omega$, and scalars $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$ such that $s_1 = \sum_{j=1}^n a_j \chi_{A(j)}$ and $s_2 = \sum_{j=1}^n b_j \chi_{A(j)}$. This and (3.18) give

$$\sqrt{\mathbb{x} + \chi_\Omega} = \sqrt{\mathbb{x}}\left|\chi_\Omega + \frac{i}{\sqrt{\mathbb{x}}}\ \chi_\Omega\right| = \sum_{j=1}^n \left(a_j \sqrt{\mathbb{x}} + b_j\right) \chi_{A(j)}. \tag{3.19}$$

More precisely, we have (3.19) holding in the normed space $X \subseteq L^1(\mu)$, and hence it holds m-a.e. pointwise on Ω. Because $\Omega = \bigcup_{j=1}^n A(j)$, we may assume that

$$\sqrt{\mathbb{x} + \chi_\Omega} = a_1 \sqrt{\mathbb{x}} + b_1 \tag{3.20}$$

μ-a.e. on $A(1)$, say. In particular, (3.20) holds at *infinitely* many points in $A(1)$. Such points necessarily belong to the set

$$C := \{w \in \Omega : (1 - a_1^2)w + 1 - b_1^2 = 2a_1 b_1 \sqrt{w}\}.$$

This is a contradiction because C is either empty or a singleton set, which thereby verifies (3.17).

It is clear that $L^1(m)$ is closed under complex conjugation. That m fails to have the LCP can be deduced from

$$\left|\chi_\Omega + \frac{i}{\sqrt{\mathbb{x}}}\chi_\Omega\right| \leq \chi_\Omega + \frac{1}{\sqrt{\mathbb{x}}}\chi_\Omega \in L^1(m)$$

together with (3.17).

(v) Let $X := L^\infty(\mu) + \left(\frac{1}{\sqrt{\mathbb{x}}} + i \ln \mathbb{x}\right) \operatorname{sim} \Sigma$. Then $L^1(m) = X$ is not a complex vector lattice in the m-a.e. pointwise order because $L^1(m)$ is not closed under forming the modulus or under complex conjugation (see Proposition 3.7(ii)). However, m does have the LCP. For the details, see [41, Example 3.4(ii)]. □

Remark 3.10. Let the setting be as in Example 3.9 above. There exists a vector subspace $X \subseteq L^1(\mu)$, satisfying the requirements of Example 3.9, such that $L^1(m) = X$ is closed under forming the modulus, but not closed under complex conjugation; see [49] and [53, Theorems 2 and 4]. □

Given a lcHs-valued measure m we shall discuss, in later sections, whether or not $L^1(m)$ or $L^1_w(m)$ contains a lattice-isomorphic copy of the complex Banach lattice c_0 or ℓ^∞. For this purpose, let us clarify what we mean, more generally, by saying that $L^1(m)$ or $L^1_w(m)$ contains a lattice-isomorphic copy of a complex Banach lattice $E_\mathbb{C} = E + iE$. First, $L^1(m)$ must be a complex vector lattice in the m-a.e. pointwise order. Then a linear map $T : E_\mathbb{C} \to L^1(m)$ is called a *lattice-isomorphism* (onto its range) if T is an isomorphism (in the topological sense explained in Section 2) and if $T(E) \subseteq L^1(m)_\mathbb{R}$ with the resulting $\mathbb{R}$-linear map from E into $L^1(m)_\mathbb{R}$ being a lattice-homomorphism (i.e., preserving the lattice operations (3.1)). When such a lattice-isomorphism exists, we say that $L^1(m)$ *contains a lattice-isomorphic copy of* $E_\mathbb{C}$.

The corresponding terminology also applies to $L^1_w(m)$, in which case we do not need to assume that $L^1_w(m)$ is a complex vector lattice in the m-a.e. pointwise order because it always is; see Proposition 3.7(iv).

We end this section with a prototype for Banach spaces of a result whose extension (in subsequent sections) to the case of lcHs-valued vector measures is the main theme of this paper. For a Banach-space-valued vector measure recall from Remark 3.8(ii) that $L^1(m)$ and $L^1_w(m)$ are complex Banach lattices.

Proposition 3.11. *The following conditions are equivalent for a Banach-space-valued vector measure m:*

(i) $L^1(m) = L^1_w(m)$.
(ii) *The complex Banach lattice $L^1(m)$ does not contain a lattice-isomorphic copy of the complex Banach lattice c_0.*

(iii) *The complex Banach space $L^1(m)$ does not contain an isomorphic copy of the complex Banach space c_0.*

Proof. Condition (i) is equivalent to the weak sequential completeness of $L^1(m)$ [46, Proposition 3.38(I)]. This latter condition implies the weak Σ-completeness of $L^1(m)$, from which (iii) follows; see Lemma 2.3(i) with $X := L^1(m)$.

The implication (iii)$\Rightarrow$(ii) is clear.

Now, (ii) is equivalent to $L^1(m)_{\mathbb{R}}$ not containing a lattice-isomorphic copy of $(c_0)_{\mathbb{R}}$ [46, Lemma 3.8(i), (iv)]. But, this latter condition holds if and only if $L^1(m)_{\mathbb{R}}$ does not contain an isomorphic copy of $(c_0)_{\mathbb{R}}$ if and only if $L^1(m)_{\mathbb{R}}$ is weakly sequentially complete [2, Theorem 14.12]. This observation establishes that (ii) implies the weak sequential completeness of $L^1(m)$, and hence implies condition (i) because the weak sequential completeness of $L^1(m)$ is equivalent to that of $L^1(m)_{\mathbb{R}}$ [46, Lemma 3.35(i)]. ■

Observe in Proposition 3.11 that the implication (ii)$\Rightarrow$(iii) is proved indirectly and proceeds via the implications (ii)$\Rightarrow$(i) $\Rightarrow$(iii) together with the equivalence of (i) and the weak sequential completeness of $L^1(m)$. This is because our proof is based on the key fact [2, Theorem 14.12], in which the corresponding real case also involves the weak sequential completeness condition. It is also worthwhile to exhibit a direct proof of (ii)$\Rightarrow$(iii) for $L^1(m)$. This is presented in Remark 6.13.

4. Proof of Theorem 1.1

Throughout this section, let $m : \Sigma \to X$ be a lcHs-valued vector measure defined on a measurable space (Ω, Σ) unless stated otherwise. By X_σ we denote X equipped with its weak topology $\sigma(X, X^*)$. To avoid the trivial case, we assume that m is not the zero vector measure.

Remark 4.1. Let $i_\sigma : X \to X_\sigma$ denote the identity map. Then $i_\sigma \circ m : \Sigma \to X_\sigma$ is also a vector measure. The equality $X^* = (X_\sigma)^*$ implies that $\mathcal{L}^1(m) = \mathcal{L}^1(i_\sigma \circ m)$ as vector spaces, and that $\mathcal{N}(m) = \mathcal{N}(i_\sigma \circ m)$; see (2.13). Hence, also $L^1(m) = L^1(i_\sigma \circ m)$ as vector spaces. Similarly, $L^1_w(m) = L^1_w(i_\sigma \circ m)$ as vector spaces. □

The lcHs $L^1(m)$ has the topology $\tau(m)$; see Section 2. As indicated above, $L^1(m)$ equipped with its weak topology $\sigma(L^1(m), (L^1(m))^*)$ is denoted by $L^1(m)_\sigma$. Except for the case of a Banach space [40], there is, in general, no adequate description available of the dual space $(L^1(m))^*$. Now, for each $A \in \Sigma$, multiplication by χ_A defines the linear operator

$$M_A : L^1(m) \to L^1(m) \quad \text{via} \quad f \mapsto \chi_A f, \tag{4.1}$$

which is clearly continuous by the definition of $\tau(m)$. Moreover, M_A is also weakly continuous, i.e., $M_A \in \mathcal{L}(L^1(m)_\sigma)$ because $\mathcal{L}(L^1(m)) \subseteq \mathcal{L}(L^1(m)_\sigma)$ [32, §20, 4(5)].

Lemma 4.2. *Suppose that $\{f_k\}_{k=1}^\infty$ is a Cauchy sequence in $L^1(m)_\sigma$ converging pointwise m-a.e. to a function $f \in L^1(m)$. Then $\lim_{k\to\infty} f_k = f$ in $L^1(m)_\sigma$.*

Proof. Consider the $L^1(m)$-valued set function $[m] : A \mapsto \chi_A$ on Σ. It follows from [41, Proposition 3.1] that $[m]$ is σ-additive, that $L^1([m]) = L^1(m)$ as lcHs and that the integration operator $I_{[m]} : L^1([m]) \to L^1(m)$ is the identity.

The identity map $i_\sigma : L^1(m) \to L^1(m)_\sigma$ is linear and continuous, so that $i_\sigma \circ [m] : \Sigma \to L^1(m)_\sigma$ is also a vector measure. Moreover, $L^1(i_\sigma \circ [m]) = L^1([m])$ as vector spaces; see Remark 4.1 (with $[m]$ in place of m and $X := L^1(m)$). By J_σ we denote the natural embedding of $L^1(m)_\sigma$ into its sequential completion $(L^1(m)_\sigma)^\sim$; see Section 2. The resulting vector measure $J_\sigma \circ i_\sigma \circ [m] : \Sigma \to (L^1(m)_\sigma)^\sim$ satisfies

$$L^1(i_\sigma \circ [m]) \subseteq L^1(J_\sigma \circ i_\sigma \circ [m])$$

and $\tau(i_\sigma \circ [m])$ is the relative topology induced by $\tau(J_\sigma \circ i_\sigma \circ [m])$; see Lemma 2.5(iii) (with $i_\sigma \circ [m]$ in place of m, J_σ in place of J and $X := L^1(m)_\sigma$). Moreover, each function f_k is $(J_\sigma \circ i_\sigma \circ [m])$-integrable and, for each $A \in \Sigma$, we have

$$\begin{aligned}\int_A f_k \, d(J_\sigma \circ i_\sigma \circ [m]) &= (J_\sigma \circ i_\sigma)\Big(\int_A f_k \, d[m]\Big) = (J_\sigma \circ i_\sigma)(f_k \chi_A)\\ &= (J_\sigma \circ i_\sigma \circ M_A)(f_k), \qquad k \in \mathbb{N};\end{aligned}$$

apply Lemma 2.8 with $[m]$ in place of m, $X := L^1(m)$, $Y := (L^1(m)_\sigma)^\sim$ and $T := J_\sigma \circ i_\sigma$. Since $J_\sigma \circ i_\sigma \circ M_A$ is continuous from $L^1(m)$ into $(L^1(m)_\sigma)^\sim$, and $(L^1(m)_\sigma)^\sim$ has its weak topology, it follows that $J_\sigma \circ i_\sigma \circ M_A$ is also continuous from $L^1(m)_\sigma$ into $(L^1(m)_\sigma)^\sim$. Accordingly, the sequence $\{(J_\sigma \circ i_\sigma \circ M_A)(f_k)\}_{k=1}^\infty$ is Cauchy in $(L^1(m)_\sigma)^\sim$, and hence admits a limit there. As we know that $f \in L^1(J_\sigma \circ i_\sigma \circ [m])$, by Lemma 2.8(ii) (with $T := J_\sigma \circ i_\sigma$ and $[m]$ in place of m), we find that $\{f_k\}_{k=1}^\infty$ is $\tau(J_\sigma \circ i_\sigma \circ [m])$-convergent to f via Lemma 2.7(i) (with $J_\sigma \circ i_\sigma \circ [m]$ in place of m and $X := (L^1(m)_\sigma)^\sim$). As $f, f_k \in L^1(m) = L^1([m]) = L^1(i_\sigma \circ [m])$ for $k \in \mathbb{N}$ and since $\tau(J_\sigma \circ i_\sigma \circ [m])$ induces the topology $\tau(i_\sigma \circ [m])$ on $L^1(i_\sigma \circ [m])$, it then follows that $\{f_k\}_{k=1}^\infty$ is $\tau(i_\sigma \circ [m])$-convergent to f. Therefore

$$f = I_{i_\sigma \circ [m]}(f) = \lim_{k\to\infty} I_{i_\sigma \circ [m]}(f_k) = \lim_{k\to\infty} f_k$$

in $L^1(m)_\sigma$ because the integration operator $I_{i_\sigma \circ [m]} : L^1(i_\sigma \circ [m]) \to L^1(m)_\sigma$ is the identity and is continuous. ∎

LEMMA 4.3. *Let Y, Z and W be lcHs.*

(i) *Given a surjective linear map $R \in \mathcal{L}(Y, Z)$ and a linear map $V \in \mathcal{L}(Z, W)$, suppose that their composition $V \circ R : Y \to W$ is a surjective isomorphism. Then both R and V are also surjective isomorphisms.*

(ii) *Suppose that $R \in \mathcal{L}(Y, Z)$ and $V \in \mathcal{L}(Z, W)$ are linear maps such that their composition $V \circ R : Y \to W$ is an isomorphism onto its range. Then both $R : Y \to Z$ and the restriction $V|_{\mathcal{R}(R)} : \mathcal{R}(R) \to W$ of V to $\mathcal{R}(R) \subseteq Z$ are isomorphisms onto their respective ranges.*

Proof. (i) Let $H := V \circ R$. Clearly, V is surjective. If $z \in V^{-1}(\{0\}) \subseteq Z$, then there exists $y \in Y$ satisfying $z = Ry$ (as R is surjective). It then follows that $0 = Vz = (V \circ R)y = Hy$, and hence the injectivity of H gives $y = 0$, by which $z = 0$. Consequently, V is also injective. Since $V \circ (R \circ H^{-1})$ is the identity on W, we have $V^{-1} = R \circ H^{-1} \in \mathcal{L}(W, Z)$, and V is a surjective isomorphism.

Next, the injectivity of H implies that of R via $H = V \circ R$. Since $(H^{-1} \circ V) \circ R$ is the identity on Y, it follows that $R^{-1} = H^{-1} \circ V \in \mathcal{L}(Z, Y)$, which implies that R is also a surjective isomorphism.

(ii) First replace W by $\mathcal{R}(V \circ R)$ so that we may assume that $V \circ R$ is a surjective isomorphism. Then apply part (i) with $\mathcal{R}(R)$ in place of Z and $V|_{\mathcal{R}(R)}$ in place of V. ∎

Proof of Theorem 1.1. (i) Let $S : c_0 \to L^1(m)$ be any isomorphism onto its range. Then we can select $q \in \mathcal{P}(X)$ satisfying

$$\|\alpha\|_{c_0} \le q(m)(S(\alpha)), \qquad \alpha \in c_0. \tag{4.2}$$

With $\{e_n\}_{n=1}^\infty$ denoting the canonical basis of c_0, let $g_n := S(e_n) \in L^1(m)$ for $n \in \mathbb{N}$. The claim is that there exist a function $f \in \mathcal{L}^0(\Sigma)$, a set $B \in \Sigma$ with $q(m)(\chi_{\Omega \setminus B}) = 0$ and an increasing sequence $\{n(k)\}_{k=1}^\infty$ in $\mathbb{N}$ such that

$$\lim_{k\to\infty} \sum_{j=1}^{n(k)} (g_j \chi_B)(w) = f(w), \qquad w \in \Omega. \tag{4.3}$$

To verify this, choose a Rybakov functional $\xi_0^* \in X_q^*$ for the Banach-space-valued vector measure $m_q := \pi_q \circ m : \Sigma \to X_q$, i.e., $\mathcal{N}_0(\langle m_q, \xi_0^*\rangle) = \mathcal{N}_0(m_q)$, [18, Theorem IX.2.2], and define $x_0^* := \xi_0^* \circ \pi_q \in X^*$. Then $\langle m, x_0^*\rangle = \langle m_q, \xi_0^*\rangle$. By Lemma 2.8 (with $Y := \mathbb{C}$ and $T := x_0^* \in \mathcal{L}(X, \mathbb{C})$), consider the canonical map $[x_0^*]_m : L^1(m) \to L^1(\langle m, x_0^*\rangle)$ which assigns to each $h \in L^1(m)$ the *same function* h in $L^1(\langle m, x_0^*\rangle)$; see also Remark 2.9. In particular, $g_n = ([x_0^*]_m \circ S)(e_n)$ for all $n \in \mathbb{N}$. As $\{e_n\}_{n=1}^\infty$ is weakly absolutely Cauchy in c_0 and as $[x_0^*]_m \circ S \in \mathcal{L}(c_0, L^1(\langle m, x_0^*\rangle))$, the functions $g_n = ([x_0^*]_m \circ S)(e_n)$ for $n \in \mathbb{N}$ form a weakly absolutely Cauchy sequence in the weakly sequentially complete (hence also weakly Σ-complete) Banach space $L^1(\langle m, x_0^*\rangle)$. Therefore, $\{g_n\}_{n=1}^\infty$ is summable in the *norm* of $L^1(\langle m, x_0^*\rangle)$; see Lemma 2.2. Setting $g := \sum_{n=1}^\infty g_n$, we can select a set $B \in \Sigma$ with $\Omega \setminus B \in \mathcal{N}_0(\langle m, x_0^*\rangle)$ and an increasing sequence $\{n(k)\}_{k=1}^\infty$ in $\mathbb{N}$ such that

$$\lim_{k\to\infty} \sum_{j=1}^{n(k)} (g_j \chi_B)(w) = (g\chi_B)(w), \qquad w \in \Omega.$$

So, (4.3) holds with $f := g\chi_B$. Next, observe that

$$q(m)(\chi_{\Omega\setminus B}) = \sup_{x^* \in U_q^\circ} |\langle m, x^*\rangle|(\Omega \setminus B) = \sup_{\xi^* \in \mathbb{B}[X_q^*]} |\langle m_q, \xi^*\rangle|(\Omega \setminus B) = 0$$

because $\pi_q^*(\mathbb{B}[X_q^*]) = U_q^\circ$ (see (2.2)) and because $\Omega \setminus B \in \mathcal{N}_0(\langle m, x_0^*\rangle) = \mathcal{N}_0(m_q)$ implies that $\Omega \setminus B \in \mathcal{N}_0(\langle m_q, \xi^*\rangle)$ for all $\xi^* \in \mathbb{B}[X_q^*]$. Thus, the claim is verified. So, fix now the function f, the set B and the sequence $\{n(k)\}_{k=1}^\infty$ as in the claim.

It follows, from (2.11) (with q in place of p and $h\chi_{\Omega\setminus B}$ in place of f) and the fact that $\Omega \setminus B \in \mathcal{N}_0(\langle m_q, \xi^*\rangle)$ for all $\xi^* \in \mathbb{B}[X_q^*]$, that $q(m)(h\chi_{\Omega\setminus B}) = 0$ for all $h \in L^1(m)$. So, for each $h \in L^1(m)$, we find via the triangle inequality for $q(m)(\cdot)$ that

$$q(m)(h) = q(m)\big(h\chi_B + h\chi_{\Omega\setminus B}\big) \le q(m)\big(h\chi_B\big) \le q(m)(h).$$

That is, $q(m)(M_B(h)) = q(m)(h)$ for all $h \in L^1(m)$. In view of (4.2) we have

$$\|\alpha\|_{c_0} \le q(m)(S(\alpha)) = q(m)((M_B \circ S)(\alpha)), \qquad \alpha \in c_0.$$

Recalling that $M_B : L^1(m) \to L^1(m)$ is continuous, it follows that $M_B \circ S : c_0 \to L^1(m)$ is an isomorphism onto its range. Setting $f_k := (M_B \circ S)(\sum_{j=1}^{n(k)} e_j) \in L^1(m)$ for $k \in \mathbb{N}$, it follows that $\{f_k\}_{k=1}^\infty$ is weakly Cauchy but not weakly convergent in $L^1(m)$; this is because $\{\sum_{j=1}^{n(k)} e_j\}_{k=1}^\infty$ has this same property in c_0 and because $M_B \circ S$ is an isomorphism onto its range with respect to the weak topologies on both c_0 and $L^1(m)$. Since $f_k \to f$ pointwise on Ω (see (4.3)) we can apply Lemma 4.2 to conclude that f is *not* m-integrable.

To show that $f \in L^1_w(m)$, fix $x^* \in X^*$. Given $A \in \Sigma$, the image $\{\int_A f_k \, d\langle m, x^*\rangle\}_{k=1}^\infty \subseteq \mathbb{C}$ of the weakly Cauchy sequence $\{f_k\}_{k=1}^\infty \subseteq L^1(m)$ under $x^* \circ I_m \circ M_A \in \mathcal{L}(L^1(m), \mathbb{C}) = \mathcal{L}(L^1(m)_\sigma, \mathbb{C})$ is (weakly) Cauchy, and hence has a limit in $\mathbb{C}$. So, Lemma 2.7(i) applied to the scalar measure $\langle m, x^*\rangle$ tells us that the pointwise limit f of $\{f_k\}_{k=1}^\infty$ is $\langle m, x^*\rangle$-integrable. Since $x^* \in X^*$ is arbitrary, we conclude that $f \in L^1_w(m)$. Hence, the function f belongs to $L^1_w(m) \setminus L^1(m)$.

(ii) For a Banach-space-valued vector measure this result is known [10, pp. 43–44]. The following proof is along the lines of that in [10], suitably adapted to the more general setting. Recall that $q \in \mathcal{P}(X)$ satisfies (4.2) and that $m_q : \Sigma \to X_q$ is a Banach-space-valued vector measure. The natural map $[\pi_q]_m : L^1(m) \to L^1(m_q)$ is defined via Lemma 2.8, with $Y := X_q$ and $T := \pi_q \in \mathcal{L}(X, X_q)$. For each $\alpha \in c_0$, we deduce from (2.11) and (4.2) that

$$\|\alpha\|_{c_0} \le q(m)(S(\alpha)) = \|([\pi_q]_m \circ S)(\alpha)\|_{L^1(m_q)}. \tag{4.4}$$

It follows that the continuous linear map $([\pi_q]_m \circ S) : c_0 \to L^1(m_q)$ is an isomorphism onto its range. Recall from the proof of part (i) that the sequence $\{g_j\}_{j=1}^\infty$ is summable in the norm of $L^1(\langle m, x_0^*\rangle) = L^1(\langle m_q, \xi_0^*\rangle)$, which implies that $\lim_{j\to\infty} \|g_j\|_{L^1(\langle m_q, \xi_0^*\rangle)} = 0$. Consequently, $\{g_j\}_{j=1}^\infty$ has a subsequence which converges to 0 pointwise $\langle m_q, \xi_0^*\rangle$-a.e., and hence also m_q-a.e. because ξ_0^* is a Rybakov functional for m_q. For ease of notation, we may assume that $\{g_j\}_{j=1}^\infty$ itself is convergent to 0 pointwise m_q-a.e. According to (4.4), for each $j \in \mathbb{N}$, we have

$$\|g_j\|_{L^1(m_q)} = \|([\pi_q]) \circ S)(e_j)\|_{L^1(m_q)} \ge \|e_j\|_{c_0} = 1$$

so that the sequence $\{g_j\}_{j=1}^\infty \subseteq L^1(m_q)$ is not $\tau(m_q)$-convergent to 0 in $L^1(m_q)$. Since $g_j \to 0$ pointwise m_q-a.e., it follows from Lemma 2.7(i), with X_q in place of X and m_q in place of m, that there is a set $F \in \Sigma$ for which the sequence $\{\int_F g_j \, dm_q\}_{j=1}^\infty$ is not norm-convergent to 0 in X_q. By passing to a subsequence of $\{g_j\}_{j=1}^\infty$ if necessary, we may again assume, with $u_j := \int_F g_j \, dm_q$ for $j \in \mathbb{N}$, that

$$\inf_{j \in \mathbb{N}} \|u_j\|_{X_q} > 0. \tag{4.5}$$

For each $j \in \mathbb{N}$, the identity

$$u_j = \int_\Omega g_j \chi_F \, dm_q = (\pi_q \circ I_m \circ M_F \circ S)(e_j) \tag{4.6}$$

implies that $\{u_j\}_{j=1}^\infty$ is weakly null in X_q because it is the image of the weakly null sequence $\{e_j\}_{j=1}^\infty$ in c_0 under the operator $(\pi_q \circ I_m \circ M_F \circ S) \in \mathcal{L}(c_0, X_q)$. This fact, together with (4.5), guarantees that $\{u_j\}_{j=1}^\infty$ has a subsequence which is *basic* [38, Theorem 4.1.32]. Again, for simplicity of presentation, assume that $\{u_j\}_{j=1}^\infty$ itself is a basic sequence in X_q. Via (4.6), observe that the sequence $\{u_j\}_{j=1}^\infty$ is the image of the weakly absolutely Cauchy sequence $\{e_j\}_{j=1}^\infty$ in c_0 under the operator $(\pi_q \circ I_m \circ M_F \circ S) \in \mathcal{L}(c_0, X_q)$, and hence is itself also weakly absolutely Cauchy in X_q. This, together with (4.5), implies that the basic sequence $\{u_j\}_{j=1}^\infty$ is equivalent to the canonical basis of c_0 [38, Theorem 4.3.10]. Let $T : c_0 \to X_q$ denote the corresponding isomorphism onto its range (i.e., $T(e_j) = u_j$ for $j \in \mathbb{N}$) [38, Proposition 4.3.2].

As an equality in $\mathcal{L}(c_0, X_q)$, we have

$$T = (\pi_q \circ I_m) \circ (M_F \circ S)$$

because both operators coincide on each basis vector $e_j \in c_0$ for $j \in \mathbb{N}$; see (4.6). Setting $U := \mathcal{R}(M_F \circ S) \subseteq L^1(m)$ it follows that

(a) $(M_F \circ S) \in \mathcal{L}(c_0, L^1(m))$ is an isomorphism onto its range, and
(b) the restriction $(\pi_q \circ I_m)|_U : U \to X_q$ is an isomorphism onto its range.

This can be verified via Lemma 4.3(ii) applied to the spaces $Y := c_0$, $Z := L^1(m)$, $W := X_q$ and the operators $R := (M_F \circ S) \in \mathcal{L}(Y, Z)$ and $V := (\pi_q \circ I_m) \in \mathcal{L}(Z, W)$ because their composition $T = V \circ R : Y \to W$ is an isomorphism onto its range.

According to (a) above, U is isomorphic to c_0. The claim is that the restriction $I_m|_U : U \to X$ is an isomorphism onto its range. This again follows from Lemma 4.3(ii), now applied to the spaces $Y := U$, $Z := X$, $W := X_q$ and the operators $R := I_m|_U \in \mathcal{L}(Y, Z)$ and $V := \pi_q \in \mathcal{L}(Z, W)$, upon noting (via (b) above) that their composition $V \circ R = (\pi_q \circ I_m)|_U : Y \to Z$ is an isomorphism onto its range. So, $R = I_m|_U \in \mathcal{L}(U, X)$ is an isomorphism onto its range. Therefore, I_m fixes a copy of c_0. ∎

We point out that the argument given in the proof of Theorem 1.1(ii) shows that there exists a subsequence $\{g_{j(k)}\}_{k=1}^\infty$ of $\{g_j\}_{j=1}^\infty$ and a set $F \in \Sigma$ such that the closed subspace U of $L^1(m)$ is precisely the closed linear span of $\{g_{j(k)}\chi_F : k \in \mathbb{N}\}$ in $L^1(m)$. This corresponds to the comment given in [10, p. 44] for the case of a Banach-space-valued vector measure. The proof shows, in particular, that *every* isomorphic copy of c_0 in $L^1(m)$ generates a further isomorphic copy of c_0 in $L^1(m)$ which is fixed by I_m.

The proof of Theorem 1.1(i) is motivated by the second proof of Theorem 2.2 of [10], where X is a real Banach space; the same result appears in [11, Theorem 2] with a different proof. We formulate Theorem 2.2 of [10] in Corollary 4.4(iv) below (which will also follow immediately from Theorem 1.1).

Recall that J denotes the canonical embedding of X into its sequential completion $\widetilde{X}$; see Section 2.

Corollary 4.4. *Let $m : \Sigma \to X$ be a lcHs-valued vector measure.*

(i) *If $L^1(m) = L^1_w(m)$, then $L^1(m)$ does not contain an isomorphic copy of c_0.*
(ii) *If $L^1(J \circ m) = L^1_w(m)$, then $L^1(m)$ does not contain an isomorphic copy of c_0.*
(iii) *If $L^1(m)$ contains an isomorphic copy of c_0, then so does $L^1(\pi_q \circ m)$ for some $q \in \mathcal{P}(X)$.*

(iv) *Suppose that X is a Banach space. If X does not contain an isomorphic copy of c_0, then neither does the Banach space $L^1(m)$.*

Proof. (i) This is the contrapositive of Theorem 1.1.

(ii) The assumption $L^1(J\circ m) = L^1_w(m)$ means that $L^1(J\circ m) = L^1_w(J\circ m)$ as $L^1_w(m) = L^1_w(J\circ m)$ via Lemma 2.5(ii). So, part (i) applied to $J\circ m$ instead of m shows that $L^1(J\circ m)$ does not contain an isomorphic copy of c_0, and hence neither does $L^1(m)$ via Lemma 2.5(iii).

(iii) As in the proof of Theorem 1.1, take an isomorphism $S : c_0 \to L^1(m)$ onto its range and choose $q \in \mathcal{P}(X)$ satisfying (4.2). Now, $\pi_q \in \mathcal{L}(X, X_q)$ induces the operator $[\pi_q]_m \in \mathcal{L}(L^1(m), L^1(\pi_q \circ m))$ given by $[\pi_q]_m(f) := f$ for $f \in L^1(m)$; see Lemma 2.8(iii) with $Y := X_q$ and $T := \pi_q$. Moreover, for a given $f \in L^1(m)$, we have $q(m)(f) = \|[\pi_q]_m(f)\|_{L^1(\pi_q\circ m)}$; see (2.11) with $p := q$ and (2.23) which give

$$\|[\pi_q]_m(f)\|_{L^1(\pi_q\circ m)} = \|f\|_{L^1(\pi_q\circ m)} = \sup_{\eta\in\mathbb{B}[X_q^*]} \int_\Omega |f|\, d|\langle \pi_q\circ m, \eta\rangle|, \qquad f \in L^1(m).$$

Therefore, it follows from (4.2) that

$$\|\alpha\|_{c_0} \le q(m)(S(\alpha)) = \|([\pi_q]_m \circ S)(\alpha)\|_{L^1(\pi_q\circ m)}, \qquad \alpha \in c_0,$$

which establishes part (iii) because $[\pi_q]_m \circ S : c_0 \to L^1(\pi_q \circ m)$ is continuous.

(iv) First recall, from Lemma 2.10(v), that $L^1(m)$ is a Banach space. To verify that $L^1(m)$ does not contain an isomorphic copy of c_0 assume, on the contrary, that it does. Then Theorem 1.1 yields $L^1(m) \subsetneq L^1_w(m)$. On the other hand, since X is weakly Σ-complete (see Lemma 2.3(ii)), it follows from Lemma 2.5(iv) that $L^1(m) = L^1_w(m)$. This contradiction proves part (iv). ∎

Observe in Corollary 4.4 that part (ii) is a stronger statement than part (i), since the assumption $L^1(m) = L^1_w(m)$ in (i) implies the assumption $L^1(J\circ m) = L^1_w(m)$ in (ii). This is because of the general inclusion $L^1(m) \subseteq L^1(J\circ m) \subseteq L^1_w(m)$; see Lemma 2.5(iii). Moreover, part (ii) is *genuinely* stronger than part (i); indeed, the vector measure m in Example 2.6(iii) satisfies $L^1(m) \subsetneq L^1(J\circ m) = L^1_w(m)$.

The converse of part (iii) in Corollary 4.4 is not valid in general, as can be seen by the following example.

Example 4.5. Let the notation and setting be as in Example 2.6(iii). Then $L^1(m)$ does not contain an isomorphic copy of c_0 by Corollary 4.4(ii). On the other hand, considering the sup-norm q on $X := ((1/\varphi)\operatorname{sim}\Sigma, \|\cdot\|_{\ell^2})$ it is clear that q is continuous on X, that $X_q = c_0$ and that $(\pi_q\circ m)(A) = \chi_A/\varphi \in c_0$ for all $A \in \Sigma$. So, $L^1(\pi_q\circ m) = \varphi\cdot c_0$ by Example 2.6(ii). The corresponding integration operator is a surjective isomorphism from $L^1(\pi_q\circ m)$ onto c_0; indeed, it is the multiplication operator $f \mapsto (1/\varphi)\cdot f$ for $f \in \varphi\cdot c_0$. So, we conclude that $L^1(\pi_q\circ m)$ contains an isomorphic copy of c_0 whereas $L^1(m)$ does not. □

Since $L^1(m) = \operatorname{sim}\Sigma$ does not contain an isomorphic copy of c_0 (cf. Lemma 6.10 below), the vector measure m in Example 2.6(i) provides a counterexample to the converse of Corollary 4.4(ii).

Let us exhibit a class of vector measures m which satisfy the assumption $L^1(J\circ m) = L^1_w(m)$ of Corollary 4.4(ii).

EXAMPLE 4.6. (i) Let Y be a lcHs and let $X := Y_\sigma$, in which case $X^* = Y^*$ as vector spaces. As noted in Section 2, the sequential completion $\widetilde{X}$ of X is weakly sequentially complete. So, given a vector measure $m : \Sigma \to X$, the vector measure $J\circ m : \Sigma \to \widetilde{X}$ satisfies $L^1(J\circ m) = L^1_w(m)$ by parts (ii) and (iv) of Lemma 2.5.

Both cases, namely $L^1(m) = L^1_w(m)$ and $L^1(m) \subsetneq L^1_w(m)$, can occur, as will now be shown.

(ii) Choose $Y := c_0$ and $X := Y_\sigma$. Let $\nu : 2^{\mathbb{N}} \to Y$ be the vector measure $A \mapsto \chi_A/\varphi$ of Example 2.6(ii), where it is denoted by m. With $i_\sigma : Y \to X$ being the identity map, the vector measure $m := i_\sigma \circ \nu$ satisfies

$$L^1(m) = L^1(\nu) = \varphi\cdot c_0 \subsetneq \varphi\cdot\ell^\infty = L^1_w(\nu) = L^1_w(m).$$

Here, Remark 4.1 (with ν in place of m) gives the first and last equalities whereas the fact that $L^1(\nu) = \varphi\cdot c_0$ and $L^1_w(\nu) = \varphi\cdot\ell^\infty$ occurs in Example 2.6(ii).

(iii) Let Y be a weakly Σ-complete lcHs and $\nu : \Sigma \to Y$ be any vector measure. With i_σ denoting the identity map from Y onto $X := Y_\sigma$, it follows from Lemma 2.5(iv) and Remark 4.1 that

$$L^1(m) = L^1(\nu) = L^1_w(\nu) = L^1_w(m). \ \square$$

5. Proof of Theorem 1.2

To establish Theorem 1.2, recall that a real or complex Banach space Y has *property* (u) if every weakly Cauchy sequence $\{y_n\}_{n=1}^\infty$ in Y admits a weakly absolutely Cauchy sequence $\{z_n\}_{n=1}^\infty$ in Y such that $\{y_n - \sum_{j=1}^n z_j\}_{n=1}^\infty$ is weakly null [2, Definition 14.6]. Whereas we have defined weakly absolutely Cauchy sequences in a complex lcHs (see Section 2), there are no difficulties in also considering such sequences in a *real* lcHs, especially in a real Banach space.

LEMMA 5.1. *Let $E_{\mathbb{C}} = E + iE$ be a complex Banach lattice realized as the complexification of a real Banach lattice E.*

(i) *Let $x^*, y^* \in E^*$. Then the functional $(x^* + iy^*) : E_{\mathbb{C}} \to \mathbb{C}$ defined by*

$$\langle x+iy, x^*+iy^*\rangle := \langle x, x^*\rangle - \langle y, y^*\rangle + i(\langle x, y^*\rangle + \langle y, x^*\rangle) \tag{5.1}$$

for each $x+iy \in E_{\mathbb{C}}$ (with $x, y \in E$) is linear and continuous. Conversely, every continuous linear functional on $E_{\mathbb{C}}$ is of the form (5.1) for some $x^*, y^* \in E$. In short,

$$(E_{\mathbb{C}})^* = E^* + iE^*.$$

(ii) *Let $\{z_n\}_{n=1}^\infty$ be a sequence in $E_{\mathbb{C}}$ and set*

$$x_n := \mathrm{Re}(z_n) \in E \quad \textit{and} \quad y_n := \mathrm{Im}(z_n) \in E, \qquad n \in \mathbb{N}. \tag{5.2}$$

(a) *The sequence $\{z_n\}_{n=1}^\infty$ is weakly Cauchy in $E_{\mathbb{C}}$ if and only if both $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are weakly Cauchy in E.*

(b) *The sequence* $\{z_n\}_{n=1}^\infty$ *is weakly null in* $E_{\mathbb{C}}$ *if and only if both* $\{x_n\}_{n=1}^\infty$ *and* $\{y_n\}_{n=1}^\infty$ *are weakly null in* E.

(c) *The sequence* $\{z_n\}_{n=1}^\infty$ *is weakly absolutely Cauchy in* $E_{\mathbb{C}}$ *if and only if both* $\{x_n\}_{n=1}^\infty$ *and* $\{y_n\}_{n=1}^\infty$ *are weakly absolutely Cauchy in* E.

(iii) *If, in addition,* E *is Dedekind* σ*-complete and has order continuous norm, then* $E_{\mathbb{C}}$ *has property* (u). *Consequently,* $E_{\mathbb{C}}$ *does not contain an isomorphic copy of* ℓ^∞.

Proof. (i) See [60, pp. 323–324], [55, pp. 134–135].

(ii) This follows from part (i).

(iii) Recall that E having *order continuous norm* means that, whenever $\{x_\lambda\}_\lambda$ is a decreasing net in E^+ with $\inf x_\lambda = 0$, then $\lim_\lambda \|x_\lambda\|_E = 0$ [2, Definition 12.7]. Now, let $\{z_n\}_{n=1}^\infty$ be a weakly Cauchy sequence in $E_{\mathbb{C}}$. In the notation of (5.2), the sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are weakly Cauchy in E via part (ii)(a). Since E necessarily has property (u) [2, Theorem 14.9], we can select weakly absolutely Cauchy sequences $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ in E such that both of the sequences $\{x_n - \sum_{j=1}^n u_j\}_{n=1}^\infty$ and $\{y_n - \sum_{j=1}^n v_j\}_{n=1}^\infty$ are weakly null in E. Thus, it follows from (5.1) and part (ii)(b) that

$$\lim_{n\to\infty} \Big(x_n + iy_n - \sum_{j=1}^n (u_j + iv_j) \Big) = 0$$

weakly in $E_{\mathbb{C}}$. This, together with the fact that $\{u_n + iv_n\}_{n=1}^\infty$ is weakly absolutely Cauchy in $E_{\mathbb{C}}$ (via part (ii)(c)), guarantees property (u) of $E_{\mathbb{C}}$.

From properties (a)–(c) of part (ii) it is routine to check that $(\ell^\infty)_{\mathbb{R}}$ has property (u) whenever ℓ^∞ has property (u). But, it is known that $(\ell^\infty)_{\mathbb{R}}$ fails to have property (u) [2, Example 14.8], and so ℓ^∞ does not have property (u) either. Moreover, by a result of A. Pełczyński, every closed subspace of a Banach space (real or complex) with property (u) also has property (u) [2, Theorem 14.7]. Accordingly, $E_{\mathbb{C}}$ does not contain an isomorphic copy of ℓ^∞. ∎

Proof of Theorem 1.2. Assume, on the contrary, that there exists an isomorphism $S : \ell^\infty \to L^1(m)$ onto its range. Then there exists $q \in \mathcal{P}(X)$ such that

$$\|\beta\|_{\ell^\infty} \le q(m)(S(\beta)), \qquad \beta \in \ell^\infty. \tag{5.3}$$

Now, consider the continuous linear map

$$[\pi_q]_m : L^1(m) \to L^1(\pi_q \circ m);$$

see Lemma 2.8 with $Y := X_q$ and π_q in place of T. Then $q(m)(f) = \|[\pi_q]_m(f)\|_{L^1(m_q)}$ for $f \in L^1(m)$; see the proof of Corollary 4.4(iii). This and (5.3) yield

$$\|\beta\|_{\ell^\infty} \le \|([\pi_q]_m \circ S)(\beta)\|_{L^1(\pi_q\circ m)}, \qquad \beta \in \ell^\infty,$$

which implies that $[\pi_q]_m \circ S : \ell^\infty \to L^1(\pi_q \circ m)$ is an isomorphism onto its range (because we already know that $[\pi_q]_m \circ S$ is continuous).

On the other hand, recall from Remark 3.8(ii) that $L^1(\pi_q \circ m)$ is a complex Banach lattice realized as the complexification of the real Banach lattice $L^1(\pi_q \circ m)_{\mathbb{R}}$. Since $L^1(\pi_q \circ m)$ is solid in $L^0(\pi_q \circ m)$, Remark 3.8(i) guarantees that $L^1(\pi_q \circ m)_{\mathbb{R}}$ is Dedekind σ-complete. Also, $L^1(\pi_q \circ m)_{\mathbb{R}}$ has order continuous norm [46, Theorem 3.7(iii)]. So, the space $L^1(\pi_q \circ m)$ does not contain an isomorphic copy of ℓ^∞ (via Lemma 5.1(iii)). This contradiction establishes Theorem 1.2. ∎

6. Proof of Theorem 1.3 and further results

Our main aim in this section is to establish Theorem 1.3. In addition, further conditions involving various copies of c_0 and ℓ^∞ in $L^1(m)$ and $L^1_w(m)$, respectively, are investigated in relation to the criterion $L^1(m) \subsetneq L^1_w(m)$. Throughout this section let $m : \Sigma \to X$ be a lcHs-valued vector measure defined on a measurable space (Ω, Σ), unless stated otherwise.

Proof of Theorem 1.3. (i) That $L^1(m)$ is a complex vector lattice in the m-a.e. pointwise order has already been established (see Remark 3.8(i)(β)). Moreover, $L^1_w(m)$ is always a complex vector lattice in the m-a.e. pointwise order, whether X is sequentially complete or not; see Proposition 3.7(iv).

(ii) Recall that $L^1(m) \subsetneq L^1_w(m)$ is being assumed. Choose $f \in L^1_w(m)^+ \setminus L^1(m)$, which is possible as $L^1_w(m)$ is a complex vector lattice. For each $n \in \mathbb{N}$ and with $A(n) := f^{-1}([0,n])$, the bounded, Σ-measurable function $f_n := f\chi_{A(n)}$ is m-integrable; see Lemma 2.7(iv). As $f \notin L^1(m)$ and $\lim_{n\to\infty} f_n = f$ pointwise on Ω, it follows from Lemma 2.7(i) that there must exist $B \in \Sigma$ for which the sequence $\{\int_B f_n\,dm\}_{n=1}^\infty$ is not Cauchy in the sequentially complete lcHs X. This enables us to choose $q \in \mathcal{P}(X)$, a positive number δ, and increasing sequences $\{u(k)\}_{k=1}^\infty$ and $\{v(k)\}_{k=1}^\infty$ in $\mathbb{N}$ satisfying

$$u(1) < v(1) < u(2) < v(2) < \cdots$$

such that, for every $k \in \mathbb{N}$, we have

$$\Big\|\pi_q\Big(\int_B (f_{v(k)} - f_{u(k)})\,dm\Big)\Big\|_{X_q} = q\Big(\int_B (f_{v(k)} - f_{u(k)})\,dm\Big) \geq \delta. \tag{6.1}$$

Define, for each $k \in \mathbb{N}$, the elements

$$g_k := (f_{v(k)} - f_{u(k)})\chi_B \in L^1(m)^+ \quad \text{and} \quad \xi_k := \pi_q \circ I_m(g_k) \in X_q. \tag{6.2}$$

It is clear from (6.1) that *no* function g_k, for $k \in \mathbb{N}$, is m-null. Fix any $\alpha \in c_0$ and $N \in \mathbb{N}$. Since the functions $g_1, \ldots, g_N$ are disjointly supported (as $A(n) \uparrow \Omega$) with $g_k \leq f$ for $k \in \mathbb{N}$, it follows that

$$\Big|\sum_{k=1}^N \alpha(k) g_k\Big| = \sum_{k=1}^N |\alpha(k)| g_k \leq \Big(\max_{1\leq k\leq N} |\alpha(k)|\Big) f,$$

which implies that

$$\begin{aligned}\Big\|\sum_{k=1}^N \alpha(k)\xi_k\Big\|_{X_q} &= \Big\|(\pi_q \circ I_m)\Big(\sum_{k=1}^N \alpha(k) g_k\Big)\Big\|_{X_q} = q\Big(I_m\Big(\sum_{k=1}^N \alpha(k) g_k\Big)\Big)\\ &\leq q(m)\Big(\sum_{k=1}^N \alpha(k) g_k\Big) = q(m)_w\Big(\sum_{k=1}^N \alpha(k) g_k\Big)\\ &\leq \Big(\max_{1\leq k\leq N} |\alpha(k)|\Big) q(m)_w(f). \end{aligned} \tag{6.3}$$

Here we have applied (2.6) with $p := q$ and used the fact that $q(m)_w$, which extends $q(m)$, satisfies (2.18) with $p := q$. Now, (6.3) holds for all $\alpha \in c_0$ and $N \in \mathbb{N}$ and so $\{\xi_k\}_{k=1}^{\infty}$ is weakly absolutely Cauchy in the Banach space X_q [38, Proposition 4.3.9]. Moreover, the resulting weakly null sequence $\{\xi_k\}_{k=1}^{\infty}$ in X_q satisfies $\inf_{k\in\mathbb{N}} \|\xi_k\|_{X_q} \geq \delta > 0$ (see (6.1) and (6.2)), and therefore admits a subsequence which is a *basic sequence* in X_q [38, Theorem 4.1.32]. For ease of presentation and without loss of generality, we may assume that $\{\xi_k\}_{k=1}^{\infty}$ itself is a basic sequence. Then (6.3) together with $\inf_{k\in\mathbb{N}} \|\xi_k\|_{X_q} \geq \delta > 0$ implies that $\{\xi_k\}_{k=1}^{\infty} \subseteq X_q$ is equivalent to the canonical basis of c_0 [38, Theorem 4.3.7]. Let $S : c_0 \to X_q$ be the corresponding isomorphism onto its range, i.e., $S(e_k) := \xi_k$ for $k \in \mathbb{N}$ [38, Proposition 4.3.2].

Fix $\alpha \in c_0$. The g_k's, for $k \in \mathbb{N}$, being disjointly supported enables us to define a function $T(\alpha) : \Omega \to \mathbb{C}$ by $T(\alpha) := \sum_{k=1}^{\infty} \alpha(k) g_k$. The claim is that $T(\alpha) \in L^1(m)$ and $\lim_{N\to\infty} \sum_{k=1}^{N} \alpha(k) g_k = T(\alpha)$ with respect to $\tau(m)$. In fact, $\{\sum_{k=1}^{N} \alpha(k) g_k\}_{N=1}^{\infty} \subseteq L^1(m)$ converges pointwise on Ω to $T(\alpha)$. Moreover, given $A \in \Sigma$ and $p \in \mathcal{P}(X)$, for all $M, N \in \mathbb{N}$ with $M < N$ we have

$$p\Big(\int_A \Big(\sum_{k=M}^{N} \alpha(k) g_k \Big)\, dm \Big) \leq p(m)\Big(\sum_{k=M}^{N} \alpha(k) g_k \Big) \leq \Big(\max_{M\leq k\leq N} |\alpha(k)| \Big) p(m)_w(f), \tag{6.4}$$

which can be verified analogously to (6.3) with p in place of q. Consequently, because $\alpha \in c_0$, the sequence $\{\int_A \big(\sum_{k=1}^{N} \alpha(k) g_k \big)\, dm\}_{N=1}^{\infty}$ is Cauchy (hence, convergent) in the sequentially complete lcHs X. So, $T(\alpha) \in L^1(m)$ and $\{\sum_{k=1}^{N} \alpha(k) g_k\}_{N=1}^{\infty}$ is $\tau(m)$-convergent to $T(\alpha)$ via Lemma 2.7(i).

The so-defined linear map $T : \alpha \mapsto T(\alpha)$ is continuous from c_0 into $L^1(m)$. Indeed, via (6.4) with $M := 1$, we have

$$\begin{aligned} p(m)(T(\alpha)) &= \lim_{N\to\infty} p(m)\Big(\sum_{k=1}^{N} \alpha(k) g_k \Big) \\ &\leq \lim_{N\to\infty} \Big(\max_{1\leq k\leq N} |\alpha(k)| \Big) p(m)_w(f) = p(m)_w(f) \|\alpha\|_{c_0}. \end{aligned} \tag{6.5}$$

Now, for each $\alpha \in c_0$, observe that

$$\begin{aligned} S(\alpha) &= S\Big(\sum_{k=1}^{\infty} \alpha(k) e_k \Big) = \sum_{k=1}^{\infty} \alpha(k) S(e_k) = \sum_{k=1}^{\infty} \alpha(k) \xi_k \\ &= \sum_{k=1}^{\infty} \alpha(k) (\pi_q \circ I_m)(g_k) = (\pi_q \circ I_m)\Big(\sum_{k=1}^{\infty} \alpha(k) g_k \Big) = ((\pi_q \circ I_m) \circ T)(\alpha). \end{aligned}$$

In other words,

$$S = (\pi_q \circ I_m) \circ T \in \mathcal{L}(c_0, X_q). \tag{6.6}$$

As S is an isomorphism onto its range, both $T : c_0 \to L^1(m)$ and $(\pi_q \circ I_m)|_{\mathcal{R}(T)} : \mathcal{R}(T) \to X$ are also isomorphisms onto their respective ranges. This follows from Lemma 4.3(ii) applied to the spaces $Y := c_0$, $Z := L^1(m)$, $W := X_q$ and the operators $R := T$ and $V := \pi_q \circ I_m$, in which case (6.6) yields $S = V \circ R$.

Recalling that the functions $\{g_k : k \in \mathbb{N}\} \subseteq L^1(m)^+_{\mathbb{R}}$ are disjointly supported, it follows from the definition of T and the identity $T(e_k) = g_k$, for each $k \in \mathbb{N}$, that

$T((c_0)_{\mathbb{R}}) \subseteq L^1(m)_{\mathbb{R}}$, that $\mathcal{R}(T_{\mathbb{R}})$ is a real vector sublattice of $L^1(m)_{\mathbb{R}}$, and that the restriction $T_{\mathbb{R}} : (c_0)_{\mathbb{R}} \to L^1(m)_{\mathbb{R}}$ of T to $(c_0)_{\mathbb{R}}$ is $\mathbb{R}$-linear. Since $T_{\mathbb{R}}$ is bijective onto its range and both $T_{\mathbb{R}} : (c_0)_{\mathbb{R}} \to \mathcal{R}(T_{\mathbb{R}})$ and $T_{\mathbb{R}}^{-1} : \mathcal{R}(T_{\mathbb{R}}) \to (c_0)_{\mathbb{R}}$ are positive operators, the operators $T_{\mathbb{R}}, T_{\mathbb{R}}^{-1}$ are lattice-homomorphisms [2, Theorem 7.3]. Moreover, the identity $\mathcal{R}(T_{\mathbb{R}}) = \mathcal{R}(T) \cap L^1(m)_{\mathbb{R}}$ and the fact that T is a topological isomorphism of c_0 onto $\mathcal{R}(T)$ imply that $T_{\mathbb{R}}$ is a topological isomorphism of $(c_0)_{\mathbb{R}}$ onto $\mathcal{R}(T_{\mathbb{R}})$. In particular, $\mathcal{R}(T_{\mathbb{R}})$ has the complex modulus property (see Lemma 3.2(iv)(b)) and so $\mathcal{R}(T) = \mathcal{R}(T_{\mathbb{R}}) + i\mathcal{R}(T_{\mathbb{R}})$ is a complex vector lattice. According to the discussion after Remark 3.10 the operator T is a (complex) lattice-isomorphism of c_0 onto $\mathcal{R}(T) \subseteq L^1(m)$. This completes the proof that $L^1(m)$ contains a lattice-isomorphic copy of c_0.

We shall now construct a lattice-isomorphism $T^{(w)} : \ell^\infty \to L^1_w(m)$ onto its range by extending $T : c_0 \to L^1(m)$. Given $\beta \in \ell^\infty$, we can define the function $\sum_{k=1}^\infty \beta(k) g_k$ pointwise on Ω as the g_k's, for $k \in \mathbb{N}$, are disjointly supported. Moreover, $\sum_{k=1}^\infty \beta(k) g_k$ is dominated pointwise on Ω by $\|\beta\|_{\ell^\infty} f \in L^1_w(m)$, and hence belongs to $L^1_w(m)$ via Proposition 3.7(iv). The resulting $L^1_w(m)$-valued, linear map $T^{(w)} : \beta \mapsto \sum_{k=1}^\infty \beta(k) g_k$ on ℓ^∞ is continuous because

$$p(m)_w(T^{(w)}(\beta)) \le \|\beta\|_{\ell^\infty} p(m)_w(f), \qquad \beta \in \ell^\infty,$$

for each $p \in \mathcal{P}(X)$ via (2.18) (with $\sum_{k=1}^\infty \beta(k) g_k$ in place of f and $\|\beta\|_{\ell^\infty} f$ in place of g). The injectivity of $T^{(w)}$ is a consequence of the fact that the g_k's, for $k \in \mathbb{N}$, are disjointly supported and that no function g_k, for $k \in \mathbb{N}$, is m-null.

Next, for that particular $q \in \mathcal{P}(X)$ satisfying (6.1), we claim that

$$C\|\beta\|_{\ell^\infty} \le q(m)_w(T^{(w)}(\beta)), \qquad \beta \in \ell^\infty,$$

for some $C > 0$. To see this fix any $\beta \in \ell^\infty$. Select $C > 0$ satisfying

$$C\|\alpha\|_{c_0} \le \|S(\alpha)\|_{X_q} \le \Big\| \sum_{k=1}^\infty \alpha(k) \xi_k \Big\|_{X_q}, \qquad \alpha \in c_0, \tag{6.7}$$

which is possible as $S : c_0 \to X_q$ is an isomorphism onto its range. Then (6.3) and (6.7), both with $\alpha_N := \sum_{k=1}^N \beta(k) e_k \in c_0 \subseteq \ell^\infty$ in place of α, for $N \in \mathbb{N}$ (and observing that $S(\alpha_N) = \sum_{k=1}^N \beta(k)\xi_k$), yield

$$\begin{aligned} C\|\beta\|_{\ell^\infty} &= C \sup_{N\in\mathbb{N}} \|\alpha_N\|_{c_0} \le \sup_{N\in\mathbb{N}} \|S(\alpha_N)\|_{X_q} \\ &\le \sup_{N\in\mathbb{N}} q(m)_w\Big(\sum_{k=1}^N \beta(k) g_k\Big) = \sup_{N\in\mathbb{N}} \sup_{x^*\in U_q^\circ} \int_\Omega \Big|\sum_{k=1}^N \beta(k) g_k\Big| \, d|\langle m, x^*\rangle| \\ &= \sup_{x^*\in U_q^\circ} \sup_{N\in\mathbb{N}} \int_\Omega \Big|\sum_{k=1}^N \beta(k) g_k\Big| \, d|\langle m, x^*\rangle| = q(m)_w(T^{(w)}(\beta)). \end{aligned}$$

Here, for the last equality and each $x^* \in U_q^\circ$, we can apply the Lebesgue Dominated Convergence Theorem for scalar measures because

$$\Big|\sum_{k=1}^N \beta(k) g_k\Big| = \sum_{k=1}^N |\beta(k)| g_k \le \|\beta\|_{\ell^\infty} f, \qquad N \in \mathbb{N},$$

with $\|\beta\|_\infty f \in L^1(|\langle m, x^*\rangle|)$. Our claim is thus established. Hence, $T^{(w)} : \ell^\infty \to L^1_w(m)$ is an isomorphism onto its range.

Finally, the fact that the g_k's, for $k \in \mathbb{N}$, are disjointly supported, non-negative functions in $L^1_w(m)$ ensures that $T^{(w)}$ is a lattice-isomorphism onto its range, that is, $L^1_w(m)$ contains a lattice-isomorphic copy of ℓ^∞. The argument is an adaption of that which showed that $T : c_0 \to L^1(m)$ was a lattice-isomorphism. ■

Remark 6.1. A careful examination of the proof of Theorem 1.3 shows that it actually suffices for $[X]_m$, rather than X itself, to be sequentially complete. □

Recall from Section 2 that the natural embedding of X into its sequential completion $\widetilde{X}$ is denoted by J. Basic facts relevant to J and our vector measure $m : \Sigma \to X$ are presented in Lemma 2.5. We also recall that the identity $L^1(m) = L^1(J \circ m)$ of vector spaces implies the identity of lcHs; see the comment immediately after Lemma 2.5. So, it will suffice to speak of the identity $L^1(m) = L^1(J \circ m)$ without further explanation. The same principle applies when we speak of the identities $L^1(m) = L^1_w(m)$ and $L^1(J \circ m) = L^1_w(m)$ because such identities as vector spaces imply the identities as lcHs via Lemma 2.5(iii).

Corollary 6.2. *Let $m : \Sigma \to X$ be a lcHs-valued vector measure such that $L^1(m) \subsetneq L^1_w(m)$.*

(i) *If $L^1(m) = L^1(J \circ m)$, then $L^1(m)$ is a complex vector lattice in the m-a.e. pointwise order. Moreover, $L^1(m)$ and $L^1_w(m)$ then contain lattice-isomorphic copies of c_0 and ℓ^∞, respectively.*
(ii) *If $L^1(m)$ happens to be sequentially complete, then the same conclusion as in part* (i) *holds.*
(iii) *If X is sequentially complete, then there exists a closed vector sublattice U of $L^1(m)$ which is lattice-isomorphic to c_0 and such that the restricted integration operator $I_m : U \to X$ is a topological isomorphism onto its range.*

Proof. (i) Apply Theorem 1.3 with $\widetilde{X}$ in place of X and $J \circ m$ in place of m.

(ii) This is a special case of part (i) because $L^1(m) = L^1(J \circ m)$ via Lemma 2.7(ii).

(iii) Appealing to Remark 3.8(i)(β) we see that $L^1(m)$ is a vector lattice. In the proof of Theorem 1.3, recall the discussion immediately after (6.6) which asserts that $(\pi_q \circ I_m)|_{\mathcal{R}(T)} : \mathcal{R}(T) \to X_q$ is an isomorphism onto its range. Since $(\pi_q \circ I_m)|_{\mathcal{R}(T)} = \pi_q \circ (I_m|_{\mathcal{R}(T)})$, we can apply Lemma 4.3(ii) to the spaces $Y := \mathcal{R}(T)$, $Z := X$, $W := X_q$ and the operators $R := I_m|_{\mathcal{R}(T)}$ and $V := \pi_q$ to deduce that the restricted integration operator $I_m|_{\mathcal{R}(T)} : \mathcal{R}(T) \to X$ (which equals R) is an isomorphism onto its range. Moreover, its domain space $U := \mathcal{R}(T)$ is lattice-isomorphic to c_0 because $T : c_0 \to L^1(m)$ is a lattice-isomorphism onto $\mathcal{R}(T)$; see the first paragraph after the paragraph containing (6.6). ■

Remark 6.3. (i) A careful examination of the proof of Corollary 6.2(iii) (via Theorem 1.3(ii)) shows that *every* function $f \in L^1_w(m)^+ \setminus L^1(m)$ generates a lattice-isomorphic copy $U \subseteq L^1(m)$ of c_0, with U *depending on* f, such that $I_m : U \to X$ is a topological isomorphism onto its range.

(ii) In Corollary 6.2(iii) the restricted integration operator $I_m : U \to X$ is a topological isomorphism, but typically not a lattice-homomorphism. Indeed, in general X is only assumed to be a lcHs and not a vector lattice. But, even in the event that X is a vector lattice, I_m need *not* be a lattice-homomorphism. Consider the simplest of cases when X is the 1-dimensional real Banach lattice $\mathbb{R}$, the σ-algebra Σ is $\mathcal{B}([0,1])$ and the vector measure $m : \Sigma \to X$ is defined by $m(A) := \int_A \psi(t)\,dt$ for $A \in \Sigma$, where $\psi(t) := \pi \sin(2\pi t)$ for $t \in [0,1]$. If $\lambda(A) := \int_A |\psi(t)|\,dt$ for $A \in \Sigma$, then it is routine to check that $L^1(m) = L^1(\lambda)$ with $\|\cdot\|_{L^1(m)} = \|\cdot\|_{L^1(\lambda)}$, and that I_m is given by $f \mapsto \int_{[0,1]} f(t)\psi(t)\,dt$ for $f \in L^1(m)$. If we set $g := \chi_{[0,1/2]}$ and $h := \chi_{[1/2,1]}$, it turns out that $I_m(g \vee h) = I_m(\chi_{[0,1]}) = 0$ whereas $I_m(g) \vee I_m(h) = 1 \vee (-1) = 1$, i.e., I_m is not a lattice-homomorphism. □

Having established Theorem 1.3 and its immediate consequence Corollary 6.2, let us now formulate some relevant conditions for further investigation of a lcHs-valued vector measure $m : \Sigma \to X$:

(a) $L^1(m) \subsetneq L^1_w(m)$.
(b) $L^1(m)$ is a complex vector lattice in the m-a.e. pointwise order and contains a lattice-isomorphic copy of c_0.
(c) The lcHs $L^1(m)$ contains an isomorphic copy of c_0.
(d) $L^1_w(m)$ contains a lattice-isomorphic copy of ℓ^∞.
(e) The lcHs $L^1_w(m)$ contains an isomorphic copy of ℓ^∞.
(f) $L^1_w(m)$ contains a lattice-isomorphic copy of c_0.
(g) The lcHs $L^1_w(m)$ contains an isomorphic copy of c_0.

In conditions (d) and (f), recall from Proposition 3.7(iv) that $L^1_w(m)$ is necessarily a complex vector lattice in the m-a.e. pointwise order. This is *not* always the case for $L^1(m)$; see Example 3.9(iii), (iv). The following condition is also relevant:

(a)* $L^1(J \circ m) \subsetneq L^1_w(m)$.

This condition is precisely (a) with $J \circ m$ in place of m because $L^1_w(m) = L^1_w(J \circ m)$; see Lemma 2.5(ii).

PROPOSITION 6.4. *Let $m : \Sigma \to X$ be a lcHs-valued vector measure.*

(i) *If X is sequentially complete, then all of the conditions* (a)–(g) *are equivalent.*
(ii) *If $L^1(m) = L^1(J \circ m)$, then all of the conditions* (a)–(g) *are equivalent.*
(iii) *If $L^1(m)$ is sequentially complete, then all of the conditions* (a)–(g) *are equivalent.*
(iv) *The four conditions* (d)–(g) *are always equivalent and each one is equivalent to* (a)*.
(v) *The implications* (b)⇒(c) ⇒(d)⇒(a) *are always valid.*

Proof. (i) To establish (a)⇔(b)⇔(c) first observe that Theorem 1.3 gives (a)⇒(b). The implication (b)⇒(c) is clear. Finally, (c)⇒(a) via Theorem 1.1 (in which the sequential completeness of X is not required).

(a)⇒(d). See Theorem 1.3.

(d)⇒(e)⇒(g) and (d)⇒(f)⇒(g). These implications are obvious.

(g)⇒(a). Assume that (g) holds but (a) fails, i.e., $L^1(m) = L^1_w(m)$ contains an isomorphic copy of c_0. Then Theorem 1.1 implies that $L^1(m) \subsetneq L^1_w(m)$, i.e., (a) holds. This is a contradiction, and hence the implication (g)⇒(a) is valid.

(ii) This follows from part (i) applied to $J \circ m$ whose codomain space $\widetilde{X}$ is sequentially complete.

(iii) This is a special case of part (ii) because $L^1(m) = L^1(J \circ m)$; see Lemma 2.7(ii).

(iv) The equivalence of (d)–(g) follows from part (ii) because $L^1_w(m) = L^1_w(J \circ m)$; see Lemma 2.5(ii). Then part (i) with $J \circ m : \Sigma \to \widetilde{X}$ in place of m gives the equivalence of (a)* to any one of (d)–(g) because $L^1_w(m) = L^1_w(J \circ m)$; see Lemma 2.5(ii).

(v) Clearly (b)$\Rightarrow$(c). The implications (c)$\Rightarrow$(d)$\Rightarrow$(a) hold because Lemma 2.5(iii) yields both (c)$\Rightarrow$(g) and (a)$^*\Rightarrow$(a) and because (a)$^*\Leftrightarrow$(d)$\Leftrightarrow$(g) via part (iv). ∎

In Proposition 6.4, since the sequential completeness of either X or $L^1(m)$ implies that $L^1(m) = L^1(J \circ m)$, part (ii) is a stronger statement than (i) and (iii). There is no direct relationship between the assumptions of (i) and (iii). In other words, the sequential completeness of X does not always imply that of $L^1(m)$ (see Example 6.5(i) below) whereas the sequential completeness of $L^1(m)$ need not imply that of X (see Example 6.5(iv)). Furthermore, part (iii) of Example 6.5 exhibits a vector measure m satisfying $L^1(m) = L^1(J \circ m)$, but such that neither the codomain space X of m nor $L^1(m)$ is sequentially complete.

Example 6.5. (i) There exists a vector measure taking values in a sequentially complete lcHs such that its corresponding L^1-space is not sequentially complete. Indeed, [25, Example B in §4] provides a non-separable Hilbert space H and an $\mathcal{L}_s(H)$-valued spectral measure P defined on a measurable space (Ω, Σ) such that its range $P(\Sigma)$ is not a sequentially closed subset of $\mathcal{L}_s(H)$. Here, the definition of a *spectral measure* can be found in [46, Definition 3.16], [52, Definition III.2], for example, and $\mathcal{L}_s(H)$ denotes the vector space $\mathcal{L}(H)$ equipped with the strong operator topology [22, Definition IV.1.2]. Thanks to the completeness of H and the Banach–Steinhaus Theorem the lcHs $\mathcal{L}_s(H)$ is quasi-complete and, in particular, sequentially complete. Consequently, $P(\Sigma)$ is not a sequentially complete subset of $\mathcal{L}_s(H)$.

To prove that the lcHs $L^1(P)$ is not sequentially complete, assume the contrary. Then the closed subset $\Sigma(P) \subseteq L^1(P)$ (see Lemma 2.10(i) with $m := P$) is necessarily sequentially complete. The integration operator $I_P : L^1(P) \to \mathcal{L}_s(H)$, which is known to be an isomorphism onto its range (see [52, Theorem V.5], for example), preserves sequentially complete sets. Consequently, $P(\Sigma) = I_P(\Sigma(P))$ is sequentially complete. This is a contradiction, and hence $L^1(P)$ is not sequentially complete.

We also claim that $L^1(P) = L^1_w(P)$. Indeed, first observe that the lcHs $\mathcal{L}_s(H)$ is quasi-complete for its weak topology. This fact can be verified via the Uniform Boundedness Principle because H is quasi-complete for its weak topology [32, §23, 1(3)], and because the weak topology on $\mathcal{L}_s(H)$ is exactly the weak operator topology; see [22, Definition VI.1.3] and [32, §39, 7(2)]. In particular, $\mathcal{L}_s(H)$ is also weakly Σ-complete, and hence $L^1(P) = L^1_w(P)$ via Lemma 2.5(iv).

(ii) We now present a simple method of producing a vector measure m whose codomain space is not sequentially complete but $L^1(m) = L^1(J \circ m)$.

Take any lcHs-valued vector measure $\nu : \Sigma \to Y$, with Y sequentially complete, such that the range $\mathcal{R}(I_\nu)$ of the corresponding integration operator $I_\nu : L^1(\nu) \to Y$ is not sequentially closed in Y. Then the vector subspace $X := \mathcal{R}(I_\nu) \subseteq Y$, equipped with the induced topology from Y, is not sequentially complete; its sequential completion $\widetilde{X}$ is exactly the sequential closure of X in Y. Such an example occurs in [51, Example 1], for instance, where Y is the Banach space $c_0(\mathbb{Z})$, X is the dense subspace $\{\widehat{f} : f \in L^1([0,2\pi])\}$ of Y equipped with the norm from Y (with $\widehat{\cdot}$ denoting Fourier transform), $\nu(A) = \widehat{\chi_A}$ for each Borel set $A \subseteq [0,2\pi]$, in which case $L^1(\nu) = L^1([0,2\pi])$ with $I_\nu(f) = \widehat{f}$, for $f \in L^1(\nu)$, and $\mathcal{R}(I_\nu) = X \subsetneq Y$.

Back to the general set up with $\nu : \Sigma \to Y$ (as described above), let $m : \Sigma \to X$ denote the vector measure ν regarded as being X-valued. Then $L^1(m) = L^1(J \circ m) = L^1(\nu)$ (as lcHs), which is routine to verify.

(iii) In the notation of part (i), it follows from part (ii) with $Y := \mathcal{L}_s(H)$ and $\nu := P$ that the resulting vector measure $m : \Sigma \to X := \mathcal{R}(I_P)$ satisfies $L^1(m) = L^1(J \circ m)$. However, as noted in part (i), neither of the (isomorphic) lcHs $L^1(m)$ and X is sequentially complete.

(iv) Let us apply part (ii) to the Banach space case. We claim, given any infinite-dimensional Banach space Y, that there always exists a Y-valued vector measure ν defined on the Borel σ-algebra $\mathcal{B}([0,1])$ of $[0,1]$ such that ν satisfies the condition in part (ii). Indeed, take any vector measure $\nu : \mathcal{B}([0,1]) \to Y$ such that $I_\nu : L^1(\nu) \to Y$ is a *compact* operator and $\mathcal{R}(\nu)$ is not contained in any finite-dimensional vector subspace of Y; see Theorem 2 and its proof in [43]. Then $X := \mathcal{R}(I_\nu)$ is not closed in Y, which establishes the claim.

Now, the X-valued vector measure $m : \mathcal{B}([0,1]) \to X$ as given in part (ii) satisfies the following conditions: its codomain space X is not sequentially complete, the equalities (as lcHs) $L^1(m) = L^1(J \circ m) = L^1(\nu)$ hold and $L^1(m)$ is a Banach space. In particular, $L^1(m)$ is surely sequentially complete. ▫

In part (i) of Example 6.5 above, the fact that the codomain space $\mathcal{L}_s(H)$ of P is not metrizable is crucial. The non-metrizability follows, for example, from the fact that $\mathcal{L}_s(H)$ is quasi-complete but not complete.

Now, let us analyze the condition $L^1(m) = L^1(J \circ m)$, which is not only the assumption of Proposition 6.4(ii) but has also appeared in various other statements and examples in this paper. In Lemma 6.6 below, we characterize when the identity $L^1(m) = L^1(J \circ m)$ is valid. Recall that $L^1(m)_{\mathbb{R}}$ has the *σ-Monotone Completeness Property*, briefly σ-MCP, if every increasing $\tau(m)$-Cauchy sequence in $L^1(m)_{\mathbb{R}}$ is $\tau(m)$-convergent in $L^1(m)_{\mathbb{R}}$ [1, Definition 7.4]. Recall from Proposition 3.7(i) that $L^1(m)_{\mathbb{R}}$ is always a vector lattice (whether or not $L^1(m)$ is a complex vector lattice for the m-a.e. pointwise order).

Lemma 6.6. *Given a lcHs-valued vector measure $m : \Sigma \to X$, the identity $L^1(m) = L^1(J \circ m)$ holds if and only if $L^1(m)_{\mathbb{R}}$ has the σ-MCP.*

Proof. Suppose that $L^1(m) = L^1(J \circ m)$. According to Proposition 6.4(ii) the space $L^1(m)$ is a complex vector lattice. Take an increasing $\tau(m)$-Cauchy sequence $\{f_n\}_{n=1}^\infty \subseteq L^1(m)_{\mathbb{R}}$. To prove its $\tau(m)$-convergence (by considering $\{f_n - f_1\}_{n=1}^\infty$ if necessary), we may assume that $f_n \geq 0$ for all $n \in \mathbb{N}$. The function $F : \Omega \to [0,\infty]$ defined by the pointwise limit of $\{f_n\}_{n=1}^\infty$ on Ω is then Σ-measurable. With $B := \{w \in \Omega : F(w) < \infty\}$, we claim that $\Omega \setminus B$

is m-null. To see this, fix $x^* \in X^* \setminus \{0\}$. Then $p_{x^*}(x) := |\langle x, x^* \rangle|$, for $x \in X$, is a continuous seminorm on X and $x^* \in U^\circ_{p_{x^*}}$. Hence, (2.5) implies that $\int_\Omega |g|\, d|\langle m, x^* \rangle| \le p_{x^*}(m)(g)$ for all $g \in L^1(m)$. Since $\{f_n\}_{n=1}^\infty$ is increasing and $\tau(m)$-Cauchy, it follows that

$$\int_\Omega F\, d|\langle m, x^* \rangle| = \lim_{n\to\infty} \int_\Omega f_n\, d|\langle m, x^* \rangle| \le \sup_{n\in\mathbb{N}} p_{x^*}(m)(f_n) < \infty$$

(via the Monotone Convergence Theorem for $|\langle m, x^* \rangle|$). Consequently, F is finite-valued $|\langle m, x^* \rangle|$-a.e., so $\Omega \setminus B$ is $|\langle m, x^* \rangle|$-null. As $x^* \in X^* \setminus \{0\}$ is arbitrary, this establishes the claim.

Next we show that $f := F\chi_B \in L^1(m)$ and that $\{f_n\}_{n=1}^\infty$ is $\tau(m)$-convergent to f. First, observe that $\lim_{n\to\infty} f_n \chi_B = f$ pointwise on Ω. Given $A \in \Sigma$, the sequence $\{\int_A f_n\, dm\}_{n=1}^\infty$ is Cauchy in X (see (2.6)) and so admits a limit in $\widetilde{X}$. Hence, Lemma 2.7(i) (with $J \circ m$ in place of m) implies that $f \in L^1(J \circ m)$ and that $\{f_n\}_{n=1}^\infty$ is $\tau(J \circ m)$-convergent to f. The assumption that $L^1(m) = L^1(J \circ m)$ now gives $f \in L^1(m)$ to which $\{f_n\}_{n=1}^\infty$ is $\tau(m)$-convergent. Thus, $L^1(m)_\mathbb{R}$ has the σ-MCP.

Conversely, suppose that $L^1(m)_\mathbb{R}$ has the σ-MCP. To verify that $L^1(J\circ m)^+_\mathbb{R} \subseteq L^1(m)$, fix $f \in L^1(J \circ m)^+_\mathbb{R}$. Select an increasing sequence $\{s_n\}_{n=1}^\infty \subseteq (\operatorname{sim} \Sigma)^+_\mathbb{R}$ converging pointwise on Ω to f. Thanks to the LCP of the $\widetilde{X}$-valued vector measure $J \circ m$ (see Lemma 2.7(iii)), the sequence $\{s_n\}_{n=1}^\infty$ is $\tau(J \circ m)$-convergent to f. Consequently, the increasing sequence $\{s_n\}_{n=1}^\infty$ is $\tau(m)$-Cauchy in $L^1(m)_\mathbb{R}$, and hence admits a $\tau(m)$-limit $g \in L^1(m)_\mathbb{R}$ as $L^1(m)_\mathbb{R}$ is assumed to have the σ-MCP. Thus, $f = g \in L^1(m)_\mathbb{R}$, which establishes the inclusion $L^1(J \circ m)^+_\mathbb{R} \subseteq L^1(m)$.

Next, since $L^1(J \circ m)$ is a complex vector lattice in the $(J \circ m)$-a.e. pointwise order (see Remark 3.8(i)(β) with $J \circ m$ in place of m and $\widetilde{X}$ in place of X), it follows that

$$\begin{aligned} L^1(J \circ m) &= L^1(J \circ m)^+_\mathbb{R} - L^1(J \circ m)^+_\mathbb{R} + i(L^1(J \circ m)^+_\mathbb{R} - L^1(J \circ m)^+_\mathbb{R}) \\ &\subseteq L^1(m) \subseteq L^1(J \circ m). \end{aligned}$$

In other words, $L^1(m) = L^1(J \circ m)$. ∎

Remark 6.7. Our motivation for considering the σ-MCP is from [21]. If we have a vector measure m with values in a *real* lcHs X, then all m-integrable and all $J \circ m$-integrable functions are taken to be $\mathbb{R}$-valued, i.e., $L^1(m) = L^1(m)_\mathbb{R}$ and $L^1(J \circ m) = L^1(J \circ m)_\mathbb{R}$, and the result corresponding to Lemma 6.6 is still valid. Namely, $L^1(m)_\mathbb{R} = L^1(J \circ m)_\mathbb{R}$ if and only if $L^1(m)_\mathbb{R}$ has the σ-MCP. Of course, $J \circ m$ now takes its values in the real sequential completion of the real lcHs X. This observation and [21, Theorem 2.4] give the equivalence of (b) and (c) in the above list of conditions (a)–(g), with $(c_0)_\mathbb{R}$ in place of c_0. Consequently, we can obtain a part of Proposition 6.4(ii) in the real case. Similarly, a corresponding result for $L^1(m)_\mathbb{R}$ can be established even if we begin with a vector measure m taking values in a (complex) lcHs. □

We now turn our attention to part (v) of Proposition 6.4 for a general lcHs-valued vector measure $m : \Sigma \to X$. The reverse implications to (b)⇒(c)⇒(d)⇒(a) fail to hold, in general. Indeed, for (c)⇏(b) and (d)⇏(c), see Examples 6.8 and 6.11 below, respectively, whereas Example 2.6(iii) shows that (a)⇏(d).

EXAMPLE 6.8. We adopt Example 3.9(iv)&(v) to show that (c)$\not\Rightarrow$(b). Let Ω, Σ and φ be as in Example 2.6. By μ we denote the c_0-valued vector measure $A \mapsto \chi_A/\varphi$ on Σ. As μ is exactly the vector measure of Example 2.6(ii), where it is denoted by m, we deduce from there that $L^1(\mu) = \varphi \cdot c_0$. Moreover, $L^1(\mu)$ is a complex Banach lattice (see Remark 3.8(ii)) and $\|f\|_{L^1(\mu)} = \|f/\varphi\|_{c_0}$ for $f \in L^1(\mu)$, which can be proved directly or obtained from [46, Lemma 3.13]. With $2\mathbb{N} := \{2n : n \in \mathbb{N}\}$, let

$$\operatorname{sim} 2^{2\mathbb{N}} := \operatorname{span}\{\chi_A : A \subseteq 2\mathbb{N}\} \subseteq L^1(\mu).$$

(i) Let X be any vector subspace of $L^1(\mu)$ satisfying the following three conditions:

$$\chi_{\mathbb{N}} \in X, \tag{6.8}$$

$$f \operatorname{sim} \Sigma \subseteq X \quad \text{for all } f \in X, \tag{6.9}$$

$$(\varphi \chi_{2\mathbb{N}-1}) \cdot c_0 \subseteq X. \tag{6.10}$$

We can consider the X-valued set function $m : A \mapsto \chi_A$ on $2^{\mathbb{N}}$ because of the inclusion $\operatorname{sim} \Sigma \subseteq X$; see (6.9) with $f := \chi_{\mathbb{N}} \in X$. Equip X with the norm induced by $L^1(\mu)$. Then $m : \Sigma \to X$ is a vector measure [41, Proposition 3.1]. Furthermore, the associated integration operator $I_m : L^1(m) \to X$ is the identity map, with $L^1(m) = X$ as an isomorphism of normed spaces [41, Corollary 3.2]. We can say more: $L^1(m)$ and X even have equal norms. Indeed, given $f \in L^1(m)$, we claim that

$$\|f\|_{L^1(m)} = \sup \|I_m(fs)\|_X = \sup \|fs\|_X = \|f\|_X, \tag{6.11}$$

where the supremum is taken over the set $\{s \in \operatorname{sim} \Sigma : |s(n)| \leq 1 \text{ for all } n \in \mathbb{N}\}$. Even though m takes its values in a normed space, the first equality in (6.11) can be proved as in Lemma 3.11 of [46] (where it is only stated for Banach spaces) and the last equality in (6.11) holds because $L^1(\mu)$ induces the norm on X; again apply [46, Lemma 3.11] to the Banach-space-valued vector measure μ. The inclusion (6.10) enables us to define a linear map $T : c_0 \to L^1(m) = X$ by $T(g) := \sum_{n=1}^{\infty} \varphi(2n-1) g(n) e_{2n-1}$ for $g \in c_0$, with $\{e_n\}_{n=1}^{\infty}$ denoting the canonical basis of c_0. It follows from (6.11) that T is a linear isometry onto its range. In particular, m satisfies condition (c).

In parts (ii) and (iii) below, concrete examples of such spaces X will be presented.

(ii) The vector subspace of $L^1(\mu)$ given by

$$X := (\varphi \chi_{2\mathbb{N}-1}) \cdot c_0 + \operatorname{sim} 2^{2\mathbb{N}} + \sqrt{\varphi} \operatorname{sim} 2^{2\mathbb{N}}$$

satisfies the three conditions (6.8)–(6.10) imposed in part (i). So, part (i) implies that the X-valued vector measure $m : A \mapsto \chi_A$ on Σ satisfies $L^1(m) = X$ and condition (c). It is clear that $L^1(m)$ is closed under complex conjugation. However, $L^1(m)$ is not closed under forming the modulus because

$$\left|\chi_{2\mathbb{N}} + i\sqrt{\varphi}\, \chi_{2\mathbb{N}}\right| \notin L^1(m) = X, \tag{6.12}$$

whereas $\chi_{2\mathbb{N}} + i\sqrt{\varphi}\chi_{2\mathbb{N}} \in L^1(m)$. To verify that (6.12) is valid assume, on the contrary, that $|\chi_{2\mathbb{N}} + i\sqrt{\varphi}\chi_{2\mathbb{N}}| \in L^1(m)$. Observing that $|\chi_{2\mathbb{N}} + i\sqrt{\varphi}\chi_{2\mathbb{N}}|$ has its support in $2\mathbb{N}$, there exist $s_1, s_2 \in \operatorname{sim} 2^{2\mathbb{N}}$ such that

$$\left(\sqrt{1+\varphi}\right)\chi_{2\mathbb{N}} = \sqrt{\chi_{2\mathbb{N}} + \varphi\chi_{2\mathbb{N}}} = \left|\chi_{2\mathbb{N}} + i\sqrt{\varphi}\, \chi_{2\mathbb{N}}\right| = s_1 + \sqrt{\varphi}\, s_2,$$

that is, $\big(\sqrt{1+\varphi}\big)\chi_{2\mathbb{N}} - \sqrt{\varphi}\, s_2 = s_1$. It follows that

$$s_1 - \frac{\chi_{2\mathbb{N}}}{\sqrt{1+\varphi}+\sqrt{\varphi}} = \sqrt{\varphi}\,(\chi_{2\mathbb{N}} - s_2). \tag{6.13}$$

Since the left side of (6.13) is uniformly bounded and since $\sqrt{\varphi}$ is unbounded on $\mathbb{N}$, the function $\sqrt{\varphi}\,(\chi_{2\mathbb{N}} - s_2)$ must vanish outside some finite subset of $\mathbb{N}$, which implies that $\sqrt{\varphi}\,(\chi_{2\mathbb{N}} - s_2) \in \operatorname{sim} 2^{2\mathbb{N}}$. Consequently,

$$\frac{\chi_{2\mathbb{N}}}{\sqrt{1+\varphi}+\sqrt{\varphi}} = s_1 - \sqrt{\varphi}\,(\chi_{2\mathbb{N}} - s_2) \in \operatorname{sim} 2^{2\mathbb{N}}. \tag{6.14}$$

This contradicts the fact that the left side of (6.14) has infinite range, and hence does not belong to $\operatorname{sim} 2^{2\mathbb{N}}$. Thereby (6.12) is verified. So, $L^1(m)$ is not a complex vector lattice in the (m-a.e.) pointwise order (note that $\mathcal{N}_0(m) = \emptyset$).

That m fails the LCP can be deduced from (6.12) together with

$$\big|\chi_{2\mathbb{N}} + i\sqrt{\varphi}\,\chi_{2\mathbb{N}}\big| \le \chi_{2\mathbb{N}} + \sqrt{\varphi}\,\chi_{2\mathbb{N}} \in L^1(m).$$

(iii) Let $h := \big(\sqrt{\varphi} + i\ln\varphi\big)\chi_{2\mathbb{N}}$. Then the vector subspace

$$X := (\varphi\chi_{2\mathbb{N}-1})\cdot c_0 + \chi_{2\mathbb{N}}\cdot \ell^\infty + h \operatorname{sim} 2^{2\mathbb{N}} \subseteq L^1(\mu)$$

satisfies the three conditions (6.8)–(6.10) imposed in part (i). Again by part (i), the X-valued vector measure $m : A \mapsto \chi_A$ on Σ satisfies condition (c). Clearly $h \in X = L^1(m)$. However, neither $|h|$ nor $\overline{h}$ belongs to $L^1(m)$. So, it follows from Proposition 3.7(ii) that $L^1(m)$ is not a complex vector lattice in the m-a.e. pointwise order, and hence m fails to satisfy condition (b).

Finally, observe that every *non-negative* function belonging to $L^1(m)$ has the form $\varphi\chi_{2\mathbb{N}-1} f + \chi_{2\mathbb{N}} g$ for some non-negative elements $f \in c_0$ and $g \in \ell^\infty$. Even though $L^1(m)$ is not closed under forming the modulus, this description for elements of $L^1(m)^+$ implies that $L^1(m)$ does have the property that if F is m-integrable and satisfies $|F| \le G$ for some $G \in L^1(m)^+$, then $|F| \in L^1(m)$. In particular, if F is the m-a.e. pointwise limit of a sequence of m-integrable functions $\{F_n\}_{n=1}^\infty$ satisfying $|F_n| \le G$ for $n \in \mathbb{N}$, then also $|F| \le G$, and hence F is m-integrable. By the Dominated Convergence Theorem applied to the vector measure $m \circ J : \Sigma \to \widetilde{X}$ [46, Theorem 3.7(i)], it follows that $\{F_n\}_{n=1}^\infty$ is $\tau(m \circ J)$-convergent to F. Since the norm in the Banach space $\widetilde{X} = L^1(\mu)$ is the continuous extension of the (relative) norm in its dense subspace X, it is clear from (2.5) that $\tau(m \circ J)$ coincides with $\tau(m)$ on $L^1(m) \subseteq L^1(m \circ J)$ and so $\{F_n\}_{n=1}^\infty$ is $\tau(m)$-convergent to F. This shows that m has the LCP. ▫

Problem 6.9. Does there exist a lcHs-valued vector measure m satisfying condition (c) such that $L^1(m)$ is a complex vector lattice in the m-a.e. pointwise order, but $L^1(m)$ does not contain a *lattice*-isomorphic copy of c_0? ■

Let us undertake some preparations to show that the vector measure in Example 2.6(i) serves to establish that (a)$\not\Rightarrow$(c). By $\mathcal{F}(\mathbb{N})$ we denote the class of all non-empty, finite subsets of $\mathbb{N}$, directed by inclusion. Consider a sequence $\{y_n\}_{n=1}^\infty$ in a lcHs Y. Recall that $\{y_n\}_{n=1}^\infty$ is *unordered summable* if the net $\{\sum_{n\in\sigma} y_n\}_{\sigma\in\mathcal{F}(\mathbb{N})}$ is convergent in Y (cf. condition (c) in [15, p. 78]). Following [4, p. 86] a sequence $\{y_n\}_{n=1}^\infty$ is called *subfamily summable* if every subsequence of $\{y_n\}_{n=1}^\infty$ is unordered summable in Y.

FACT 4. *In any lcHs a series is subfamily summable if and only if it is subseries summable.*

Proof. Let Y be a lcHs. Assume that the series $\{y_n\}_{n=1}^\infty$ is subfamily summable in Y. Then an arbitrary subsequence $\{y_{n(k)}\}_{k=1}^\infty$ of $\{y_n\}_{n=1}^\infty$ is unconditionally summable (see Section 2) because unordered summability and unconditional summability for a sequence in a lcHs are equivalent [15, (1)(b), p. 79]. In particular, $\{y_{n(k)}\}_{k=1}^\infty$ is summable. This shows that $\{y_n\}_{n=1}^\infty$ is subseries summable.

Conversely, suppose that a series $\{y_n\}_{n=1}^\infty \subseteq Y$ is subseries summable. Again, take an arbitrary subsequence $\{y_{n(k)}\}_{k=1}^\infty$ of $\{y_n\}_{n=1}^\infty$. Then $\{y_{n(k)}\}_{k=1}^\infty$ is also subseries summable, and hence unconditionally summable; see Lemma 2.1(i). Again by [15, (1)(b), p. 79], the subsequence $\{y_{n(k)}\}_{k=1}^\infty$ is unordered summable. Consequently, $\{y_n\}_{n=1}^\infty$ is subfamily summable. ∎

LEMMA 6.10. *Equip the vector space* $\operatorname{sim} 2^{\mathbb{N}}$ *with the uniform norm inherited from* ℓ^∞. *There is no continuous linear injection from any infinite-dimensional Banach space into the normed space* $\operatorname{sim} 2^{\mathbb{N}}$.

Proof. Assume, on the contrary, that there exist an infinite-dimensional Banach space Y and a continuous linear injection $T : Y \to \operatorname{sim} 2^{\mathbb{N}}$. Select a linearly independent, infinite subset $\{y_n : n \in \mathbb{N}\}$ from the closed unit ball of Y. With $z_n := 2^{-n} y_n$ for $n \in \mathbb{N}$, the sequence $\{z_n\}_{n=1}^\infty$ is subseries summable because $\sum_{n=1}^\infty \|z_n\|_Y < \infty$ and because Y is complete. Then $\{T(z_n)\}_{n=1}^\infty$ is also subseries summable in $\operatorname{sim} 2^{\mathbb{N}}$ (as T is continuous and linear), and hence $\{T(z_n)\}_{n=1}^\infty$ is subfamily summable by Fact 4 above. It now follows from [4, Theorem 1] that $\operatorname{span}\{T(z_n) : n \in \mathbb{N}\}$ is finite-dimensional. This is a contradiction because T being a continuous linear injection implies that $\{T(z_n) : n \in \mathbb{N}\}$ is an infinite, linearly independent subset of $\operatorname{sim} 2^{\mathbb{N}}$. ∎

EXAMPLE 6.11. Let the notation be as in Example 2.6(i). Then $L^1(m)$ is a normed space with norm $\|f\|_{L^1(m)} := \sup_{n\in\mathbb{N}} |f(n)/\varphi(n)|$ for $f \in L^1(m) = \operatorname{sim}\Sigma$. Let $(\operatorname{sim}\Sigma)_\infty$ denote $\operatorname{sim}\Sigma$ equipped with the uniform norm (i.e., from ℓ^∞) and let j_∞ denote the natural embedding of $(\operatorname{sim}\Sigma)_\infty$ into its completion ℓ^∞. With $S : L^1(m) \to (\operatorname{sim}\Sigma)_\infty$ being the identity map, we see that the composition $j_\infty \circ S : L^1(m) \to \ell^\infty$ is a closed linear map.

Assume that condition (c) holds, that is, there is an isomorphism $T : c_0 \to L^1(m)$ onto its range. Then the composition $j_\infty \circ (S \circ T) : c_0 \to \ell^\infty$ is a closed linear map, and hence is continuous via the Closed Graph Theorem. Consequently, $S \circ T : c_0 \to (\operatorname{sim}\Sigma)_\infty$ is a continuous linear injection. This contradicts Lemma 6.10 and so condition (c) fails to hold.

On the other hand, m does satisfy condition (d) because $L^1_w(m) = (1/\varphi) \cdot \ell^\infty$. So, m is an example showing that (d)$\not\Rightarrow$(c). Moreover, we can see directly that (a)$^*\not\Rightarrow$(c) and (a)$\not\Rightarrow$(c) because $L^1(m) \subsetneq L^1(J \circ m) \subsetneq L^1_w(m)$. It is worth noting that $L^1(J \circ m) = (1/\varphi) \cdot c_0$ is lattice-isomorphic to c_0. □

In Example 6.12 below we shall construct a non-atomic vector measure satisfying condition (b) from a purely atomic vector measure with the same property. Recall that a set $A \in \Sigma$ is an *atom* for a lcHs-valued vector measure $m : \Sigma \to X$ if $m(A) \neq 0$ and if, for each $B \in \Sigma$, either $m(A \cap B) = 0$ or $m(A \setminus B) = 0$ [26, p. 7], [31, p. 32]. We call m *non-atomic* if m does not have any atoms. Analogously to the case of a scalar measure [28, p. 650], we say that m is *purely atomic* if every non-m-null set contains an atom.

EXAMPLE 6.12. Let $X := c_0$. With $\varphi : \mathbb{N} \to \mathbb{N}$ denoting the identity, let $\nu : 2^{\mathbb{N}} \to X$ be the vector measure in Example 2.6(ii), denoted there by m, i.e., $\nu(A) := \chi_A/\varphi$ for $A \in 2^{\mathbb{N}}$. Then $L^1(\nu) = \varphi \cdot c_0 \subsetneq \varphi \cdot \ell^\infty = L^1_w(\nu)$. That $L^1(\nu)$ (resp. $L^1_w(\nu)$) contains a lattice-isomorphic copy of c_0 (resp. ℓ^∞) can be seen once we observe that $\|g\|_{L^1(\nu)} = \|g/\varphi\|_{c_0}$ for $g \in L^1(\nu)$ (resp. $\|g\|_{L^1_w(\nu)} = \|g/\varphi\|_{\ell^\infty}$ for $g \in L^1_w(\nu)$). Moreover, ν is purely atomic and its atoms are all the singleton sets $\{n\}$ for $n \in \mathbb{N}$.

Define $\Omega := [0,1] \times \mathbb{N}$. For each $\psi \in \mathbb{C}^{\mathbb{N}}$, we denote by $\chi_{[0,1]} \otimes \psi$ the $\mathbb{C}$-valued function $(t,n) \mapsto \chi_{[0,1]}(t)\psi(n)$ on Ω. For a subset W of $\mathbb{C}^{\mathbb{N}}$, define $\chi_{[0,1]} \otimes W := \{\chi_{[0,1]} \otimes \psi : \psi \in W\}$. Let Σ be the product σ-algebra $\mathcal{B}([0,1]) \otimes 2^{\mathbb{N}}$, in which case

$$\Sigma = \Big\{ \bigcup_{n=1}^{\infty} (B_n \times \{n\}) : B_n \in \mathcal{B}([0,1]) \text{ for each } n \in \mathbb{N} \Big\}.$$

Lebesgue measure on $\mathcal{B}([0,1])$ is denoted by μ. The set function $m : \Sigma \to X$ defined by

$$m(A) := \int_{\mathbb{N}} \left(\int_{[0,1]} \chi_A(t,n)\, d\mu(t) \right) d\nu(n), \qquad A \in \Sigma,$$

is clearly finitely additive. Given a decreasing sequence $\{A(k)\}_{k=1}^\infty$ in Σ with $\bigcap_{k=1}^\infty A(k) = \emptyset$, the σ-additivity of μ ensures that the decreasing sequence $\{\int_{[0,1]} \chi_{A(k)}(t,\cdot)\, d\mu(t)\}_{k=1}^\infty$ of ν-integrable functions converges to 0 pointwise on $\mathbb{N}$. This enables us to apply the LCP of ν (see Lemma 2.7(iii)) to deduce that $\lim_{k\to\infty} m(A(k)) = 0$ in X. Hence, m is σ-additive. Fix $h \in X^* = \ell^1$. Given $B \in \mathcal{B}([0,1])$ and $n \in \mathbb{N}$, it follows that

$$\langle m, h\rangle(B \times \{n\}) = \langle \mu(B)\nu(\{n\}), h\rangle = (h(n)/\varphi(n))\mu(B).$$

Consequently,

$$|\langle m, h\rangle|(B \times \{n\}) = (|h(n)|/\varphi(n))\mu(B). \tag{6.15}$$

So, given $f \in \mathcal{L}^0(\Sigma)$, since Ω is the disjoint union of its subsets $\Omega(n) := [0,1] \times \{n\}$ for $n \in \mathbb{N}$, we find from (6.15) that

$$\begin{aligned} \int_\Omega |f|\, d|\langle m, h\rangle| &= \sum_{n=1}^\infty \int_{\Omega(n)} |f|\, d|\langle m, h\rangle| \\ &= \sum_{n=1}^\infty (|h(n)|/\varphi(n)) \int_{[0,1]} |f(t,n)|\, d\mu(t). \end{aligned} \tag{6.16}$$

Therefore, $f \in L^1_w(m)$ if and only if $n \mapsto (\int_{[0,1]} |f(t,n)|\, d\mu(t))/\varphi(n)$, for $n \in \mathbb{N}$, is an element of ℓ^∞, i.e.,

$$\sup_{n\in\mathbb{N}} \frac{1}{\varphi(n)} \int_{[0,1]} |f(t,n)|\, d\mu(t) < \infty, \tag{6.17}$$

in which case the left side of (6.17) equals $\|f\|_{L^1_w(m)}$. Consequently, $\chi_{[0,1]} \otimes L^1_w(\nu) \subseteq L^1_w(m)$ and $\|\chi_{[0,1]} \otimes g\|_{L^1_w(m)} = \|g\|_{L^1_w(\nu)} = \|g/\varphi\|_{\ell^\infty}$ for all $g \in L^1_w(\nu)$. So, the linear map $S : \ell^\infty \to L^1_w(m)$ defined by $S(\psi) := \chi_{[0,1]} \otimes (\varphi\psi)$, for $\psi \in \ell^\infty$, is a linear isometry onto $\mathcal{R}(S) = \chi_{[0,1]} \otimes L^1_w(\nu)$. Furthermore, S is a lattice-isomorphism onto its range, i.e., m satisfies condition (d).

Let us identify $L^1(m)$, which is strictly smaller than $L^1_w(m)$ via Proposition 6.4(i) (i.e., (d)⇒(a)). For each $f \in L^1_w(m)$ and $A \in \Sigma$ we can, similarly to (6.16), deduce that

$$\int_A f\, d\langle m, h\rangle = \sum_{n=1}^{\infty} h(n) \frac{1}{\varphi(n)} \int_{[0,1]} f(t,n)\chi_A(t,n)\, d\mu(t), \qquad h \in \ell^1.$$

Accordingly, such an f belongs to $L^1(m)$ if and only if

$$\lim_{n\to\infty} \frac{1}{\varphi(n)} \int_{[0,1]} |f(t,n)|\chi_A(t,n)\, d\mu(t) = 0, \qquad A \in \Sigma,$$

which is, in turn, equivalent to

$$\lim_{n\to\infty} \frac{1}{\varphi(n)} \int_{[0,1]} |f(t,n)|\, d\mu(t) = 0.$$

So, $S(c_0) = \chi_{[0,1]} \otimes L^1(\nu) \subseteq L^1(m)$. Let $T : c_0 \to L^1(m)$ be the restriction $S|_{c_0}$ with codomain space $L^1(m)$. Then T is a lattice-isomorphism onto $\mathcal{R}(T) = \chi_{[0,1]} \otimes L^1(\nu)$. Since $L^1(m)$ is a Banach lattice, this shows that m satisfies condition (b). Moreover, we see explicitly from

$$S(\ell^\infty \setminus c_0) = \chi_{[0,1]} \otimes (L^1_w(\nu) \setminus L^1(\nu)) \subseteq L^1_w(m) \setminus L^1(m)$$

that condition (a) is satisfied; see also (a)⇔(b) in Proposition 6.4(i).

Finally, we claim that m is non-atomic. To see this, let $A \in \Sigma$ be any non-m-null set. Then $m(A) = \sum_{n=1}^{\infty} m(\Omega(n) \cap A) \neq 0$. So, select $n \in \mathbb{N}$ with $m(\Omega(n) \cap A) \neq 0$. Then $\Omega(n) \cap A = B \times \{n\}$ for some $B \in \mathcal{B}([0,1])$. Since we have $0 \neq m(\Omega(n) \cap A) = m(B \times \{n\}) = \mu(B)\nu(\{n\})$, it follows that $\mu(B) > 0$. Writing $B = C \cup D$ for disjoint Borel sets C, D satisfying $\mu(C) > 0$ and $\mu(D) > 0$ we have $m(C \times \{n\}) = \mu(C)\nu(\{n\}) \neq 0$ and $m(D \times \{n\}) = \mu(D)\nu(\{n\}) \neq 0$. Since $C \times \{n\}$ and $D \times \{n\}$ are pairwise disjoint subsets of A we see that A is *not* an atom. That is, m is non-atomic. □

REMARK 6.13. Let $m : \Sigma \to X$ be a Banach-space-valued vector measure. As observed at the end of Section 3 the proof of the implication (ii)⇒(iii) of Proposition 3.11 is indirect. We end this section by presenting a rather interesting *direct proof* (via the contrapositive implication (iii)⇏(ii)). This is achieved by applying the classical Kadec–Pełczyński "disjointification method" to construct a lattice-isomorphic copy of c_0 in $L^1(m)$ from any given Banach space isomorphic copy of c_0 in $L^1(m)$.

More precisely, suppose that there exists a Banach space isomorphism $S_0 : c_0 \to L^1(m)$ onto its range. Then there exist positive constants a_0, b_0 satisfying

$$a_0 \|S_0(\alpha)\|_{L^1(m)} \leq \|\alpha\|_{c_0} \leq b_0 \|S_0(\alpha)\|_{L^1(m)}, \qquad \alpha \in c_0.$$

In particular, with $\{e_n\}_{n=1}^{\infty}$ denoting the canonical basis of c_0, we have

$$a_0 \leq 1/\|S_0(e_n)\|_{L^1(m)} \leq b_0, \qquad n \in \mathbb{N}.$$

This enables us to define a surjective isomorphism $T_0 : c_0 \to c_0$ by

$$T_0(\alpha) := \sum_{n=1}^{\infty} \frac{1}{\|S_0(e_n)\|_{L^1(m)}} \alpha(n)e_n, \qquad \alpha \in c_0.$$

Then the composition $S_0 \circ T_0 : c_0 \to L^1(m)$ is an *isomorphism onto its range*, i.e., onto $\mathcal{R}(S_0)$. Accordingly, with $S := S_0 \circ T_0$ and $f_n := S(e_n)$ for $n \in \mathbb{N}$, there exist constants a, b with $0 < a \le b$ such that, for every $\alpha \in c_0$, we have

$$a\Big\| \sum_{n=1}^{N} \alpha(n) f_n \Big\|_{L^1(m)} \le \Big\| \sum_{n=1}^{N} \alpha(n)e_n \Big\|_{c_0} \le b\Big\| \sum_{n=1}^{N} \alpha(n) f_n \Big\|_{L^1(m)} \tag{6.18}$$

for all $N \in \mathbb{N}$. Note also that $\|f_n\|_{L^1(m)} = 1$ for all $n \in \mathbb{N}$.

Choose $x^* \in X^*$ such that $\mu := |\langle m, x^*\rangle|$ is a probability measure on Σ with μ and m being mutually absolutely continuous [18, Theorem IX.2.2]. The natural embedding $\Phi : L^1(m) \to L^1(\mu)$ is then continuous (see (2.23)), and hence the composition $\Phi \circ S : c_0 \to L^1(\mu)$, which is also continuous, preserves weakly absolutely Cauchy sequences. So, the functions $(\Phi \circ S)(e_n) = \Phi(f_n)$, for $n \in \mathbb{N}$, form a weakly absolutely Cauchy sequence in $L^1(\mu)$. Since the weakly sequentially complete space $L^1(\mu)$ [22, Theorem IV.8.6] is necessarily weakly Σ-complete, we can apply Lemma 2.2 to deduce that $\{\Phi(f_n)\}_{n=1}^{\infty}$ is a norm-summable sequence in the Banach space $L^1(\mu)$. In particular, $\lim_{n\to\infty} \|\Phi(f_n)\|_{L^1(\mu)} = 0$.

Assume, for the moment, that there exist a subsequence $\{f_{n(k)}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ and a disjointly supported sequence $\{g_k\}_{k=1}^{\infty}$ of functions in the Banach lattice $L^1(m)$ satisfying

$$\lim_{k\to\infty} \|f_{n(k)} - g_k\|_{L^1(m)} = 0. \tag{6.19}$$

We shall proceed, leaving the verification of this fact until the end of the proof.

Without loss of generality we may suppose that

$$\|f_{n(k)} - g_k\|_{L^1(m)} \le 1/(b2^{k+1}), \qquad k \in \mathbb{N}, \tag{6.20}$$

as we can select further subsequences of $\{f_{n(k)}\}_{k=1}^{\infty}$ and of $\{g_k\}_{k=1}^{\infty}$ with this property, if necessary. Given $N \in \mathbb{N}$ and $\alpha \in c_0$, we claim that

$$\frac{2a}{3}\Big\| \sum_{k=1}^{N} \alpha(k) g_k \Big\|_{L^1(m)} \le \max_{1\le k\le N} |\alpha(k)| \le 2b\Big\| \sum_{k=1}^{N} \alpha(k) g_k \Big\|_{L^1(m)}. \tag{6.21}$$

In fact, the triangle inequality and (6.20) yield

$$\begin{aligned} a\Big\| \sum_{k=1}^{N} \alpha(k) g_k \Big\|_{L^1(m)} &\le a\Big(\Big\| \sum_{k=1}^{N} \alpha(k)(g_k - f_{n(k)}) \Big\|_{L^1(m)} + \Big\| \sum_{k=1}^{N} \alpha(k) f_{n(k)} \Big\|_{L^1(m)}\Big) \\ &\le \frac{a}{2b} \max_{1\le k\le N} |\alpha(k)| + a\Big\| \sum_{k=1}^{N} \alpha(k) f_{n(k)} \Big\|_{L^1(m)}. \end{aligned}$$

But, with $\widetilde{\alpha} := \sum_{k=1}^{\infty} \alpha(k)e_{n(k)}$ in place of α in (6.18), we have

$$a\Big\| \sum_{k=1}^{N} \alpha(k) f_{n(k)} \Big\|_{L^1(m)} = a\Big\| \sum_{r=1}^{n(N)} \widetilde{\alpha}(r) f_r \Big\|_{L^1(m)} \le \Big\| \sum_{r=1}^{n(N)} \widetilde{\alpha}(r)e_r \Big\|_{c_0} = \max_{1\le k\le N} |\alpha(k)|.$$

Since $a/(2b)+1 \le 3/2$ (recall that $a \le b$), we conclude that

$$a\Big\|\sum_{k=1}^{N}\alpha(k)g_k\Big\|_{L^1(m)} \le \frac{3}{2}\max_{1\le k\le N}|\alpha(k)|,$$

which verifies the first inequality in (6.21). On the other hand, from (6.18) and (6.20) we have

$$\begin{aligned}\max_{1\le k\le N}|\alpha(k)| &= \max_{1\le r\le n(N)}|\widetilde{\alpha}(r)| \le b\Big\|\sum_{r=1}^{n(N)}\widetilde{\alpha}(r)f_r\Big\|_{L^1(m)} = b\Big\|\sum_{k=1}^{N}\alpha(k)f_{n(k)}\Big\|_{L^1(m)}\\ &\le b\Big(\Big\|\sum_{k=1}^{N}\alpha(k)g_k\Big\|_{L^1(m)} + \Big\|\sum_{k=1}^{N}\alpha(k)(g_k-f_{n(k)})\Big\|_{L^1(m)}\Big)\\ &\le b\Big\|\sum_{k=1}^{N}\alpha(k)g_k\Big\|_{L^1(m)} + (b/2b)\max_{1\le k\le N}|\alpha(k)|,\end{aligned}$$

from which the second inequality in (6.21) follows. It then follows from (6.21) that

$$\frac{2a}{3}\Big\|\sum_{k=1}^{N}\alpha(k)|g_k|\Big\|_{L^1(m)} \le \max_{1\le k\le N}|\alpha(k)| \le 2b\Big\|\sum_{k=1}^{N}\alpha(k)|g_k|\Big\|_{L^1(m)} \tag{6.22}$$

because the disjointly supported functions g_k, for $1\le k\le N$, satisfy

$$\begin{aligned}\Big\|\sum_{k=1}^{N}\alpha(k)g_k\Big\|_{L^1(m)} &= \Big\|\Big|\sum_{k=1}^{N}\alpha(k)g_k\Big|\Big\|_{L^1(m)} = \Big\|\Big|\sum_{k=1}^{N}\alpha(k)|g_k|\Big|\Big\|_{L^1(m)}\\ &= \Big\|\sum_{k=1}^{N}\alpha(k)|g_k|\Big\|_{L^1(m)}, \qquad N\in\mathbb{N},\end{aligned}$$

as $L^1(m)$ is a complex Banach lattice for the m-a.e. pointwise order.

Given $\alpha\in c_0$, it follows from (6.22) that the sequence $\{\sum_{k=1}^{N}\alpha(k)|g_k|\}_{N=1}^{\infty}$ of partial sums is Cauchy, and hence converges in $L^1(m)$. This allows us to define a linear map $R: c_0\to L^1(m)$ by $R(\alpha) := \sum_{k=1}^{\infty}\alpha(k)|g_k|$ for $\alpha\in c_0$. By letting $N\to\infty$ in (6.22) it is clear that R is an isomorphism onto its range. Since the non-negative functions $|g_k|$, for $k\in\mathbb{N}$, are disjointly supported it is clear that R is also a lattice-isomorphism onto its range. In other words, $L^1(m)$ contains a lattice-isomorphic copy of c_0.

Finally, it remains to verify the existence of a subsequence $\{f_{n(k)}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ and of a sequence $\{g_k\}_{k=1}^{\infty}$ of disjointly supported functions in $L^1(m)$ which satisfy (6.19). We shall follow the arguments in the proof of Proposition 1.c.8 in [35], which is taken from [23, Theorem 4.1]. Actually, such arguments originated in [29, §2]. Recall the natural embedding $\Phi: L^1(m)\to L^1(\mu)$ mentioned above. No confusion will arise by writing $f=\Phi(f)$ for each $f\in L^1(m)\subseteq L^1(\mu)$. Moreover, we may assume that $\|\chi_\Omega\|_{L^1(m)}=1$; otherwise replace m by the vector measure $\kappa m:\Sigma\to X$ with $\kappa := 1/\|\chi_\Omega\|_{L^1(m)}$. Fix $\varepsilon\in(0,1)$. Define the Σ-measurable sets

$$\sigma(f,\varepsilon) := \{w\in\Omega : |f(w)|\ge\varepsilon\|f\|_{L^1(m)}\}, \qquad f\in L^1(m),$$

and the subset of $L^1(m)$ given by

$$M(\varepsilon) := \{f\in L^1(m) : \mu(\sigma(f,\varepsilon))\ge\varepsilon\}.$$

Given any $j \in \mathbb{N} \cup \{0\}$, we claim that there exists $N(\varepsilon, j) \in \mathbb{N}$ satisfying both $N(\varepsilon, j) > j$ and

$$0 < \mu(\sigma(f_{N(\varepsilon,j)}, \varepsilon)) < \varepsilon. \tag{6.23}$$

Indeed, since $\lim_{n\to\infty} \|f_n\|_{L^1(\mu)} = 0$, we can select $N(\varepsilon, j) \in \mathbb{N}$ satisfying $N(\varepsilon, j) > j$ and such that $\|f_{N(\varepsilon,j)}\|_{L^1(\mu)} < \varepsilon^2$. Recalling that $\|f_{N(\varepsilon,j)}\|_{L^1(m)} = 1$, we have

$$\varepsilon = \varepsilon \|f_{N(\varepsilon,j)}\|_{L^1(m)} \le |f_{N(\varepsilon,j)}(w)|, \quad w \in \sigma(f_{N(\varepsilon,j)}, \varepsilon),$$

and hence

$$\varepsilon\mu(\sigma(f_{N(\varepsilon,j)}, \varepsilon)) \le \|f_{N(\varepsilon,j)}\chi_{\sigma(f_{N(\varepsilon,j)},\varepsilon)}\|_{L^1(\mu)} \le \|f_{N(\varepsilon,j)}\|_{L^1(\mu)} < \varepsilon^2.$$

Accordingly, $\mu(\sigma(f_{N(\varepsilon,j)}, \varepsilon)) < \varepsilon$, which is one part of (6.23). To verify the other part of (6.23) assume, on the contrary, that $\mu(\sigma(f_{N(\varepsilon,j)}, \varepsilon)) = 0$, that is, $|f_{N(\varepsilon,j)}(w)| < \varepsilon$ for μ-a.e. $w \in \Omega$. This implies that

$$1 = \|f_{N(\varepsilon,j)}\|_{L^1(m)} \le \|\varepsilon\chi_\Omega\|_{L^1(m)} = \varepsilon,$$

which contradicts $\varepsilon \in (0, 1)$. Thus $\mu(\sigma(f_{N(\varepsilon,j)}, \varepsilon)) > 0$, and hence (6.23) is verified.

Next, define an increasing sequence $\{s(n)\}_{n=1}^\infty$ in $\mathbb{N}$ inductively via $s(1) := N(1/2, 0) \ge 1$ and $s(n) := N(2^{-n}, s(n-1))$ for $n \ge 2$. Then, given any $n \in \mathbb{N}$, it follows from (6.23) for $\varepsilon := 2^{-n}$ with $j := s(n-1)$ if $n \ge 2$ and $j := 0$ if $n = 1$ that

$$0 < \mu(\sigma(f_{s(n)}, 2^{-n})) < 2^{-n}.$$

In particular, $f_{s(n)} \notin M(2^{-n})$ for $n \in \mathbb{N}$. As $L^1(m)$ is a complex Banach lattice with order continuous norm (see Remark 3.8(ii)), by setting $z_n := f_{s(n)}$ for $n \in \mathbb{N}$, we can use the argument in the latter part of the proof of Proposition 1.c.8 in [35] to select a subsequence $\{f_{s(n_k)}\}_{k=1}^\infty$ of $\{f_{s(n)}\}_{n=1}^\infty$ and a sequence of pairwise disjoint sets $A(k) := \sigma(f_{s(n_k)}, 2^{-n_k})$ in Σ, for $k \in \mathbb{N}$, such that

$$\|f_{s(n_k)}\chi_{A(k)} - f_{s(n_k)}\|_{L^1(m)} \le 2^{1-k}, \quad k \in \mathbb{N}. \tag{6.24}$$

Then the functions $g_k := f_{s(n_k)}\chi_{A(k)} \in L^1(m)$, for $k \in \mathbb{N}$, are disjointly supported and (6.24) implies that (6.19) holds, as required. □

7. Characterization of the equality $L^1(m) = L^1_w(m)$

Throughout this section the setting is that of a Banach-space-valued vector measure $m : \Sigma \to X$, in which case both $L^1(m)$ and $L^1_w(m)$ are Banach lattices. According to Corollary 4.4(iv), if X does not contain an isomorphic copy of c_0, then neither does $L^1(m)$. Then Proposition 6.4(i)&(iii) implies (via the equivalence (a)⇔(c) of the conditions immediately prior to 6.4) that necessarily $L^1(m) = L^1_w(m)$. It is known (see [46, Proposition 3.38], for example, and the references given there) that each of the following conditions is *equivalent* to $L^1(m) = L^1_w(m)$:

- $L^1(m)$ is weakly sequentially complete.
- $L^1(m)$ does not contain an isomorphic copy of c_0.
- $L^1_w(m)$ is weakly sequentially complete.

- The Banach lattice $L^1(m)$ has the Fatou property.
- The Banach lattice $L^1_w(m)$ has order continuous norm.

All of the above equivalent conditions, characterizing the equality $L^1(m) = L^1_w(m)$, are in terms of Banach space or Banach lattice properties of $L^1(m)$ and $L^1_w(m)$. The first aim of this section is to *characterize* the equality $L^1(m) = L^1_w(m)$ in terms of the *integration operator* $I_m : L^1(m) \to X$; here Corollary 6.2 plays a crucial role. In the latter part of the section we present various sufficient conditions on I_m, mainly in terms of concavity requirements or membership of I_m in certain operator ideals, which ensure that $L^1(m) = L^1_w(m)$.

Let us begin with the following known result.

PROPOSITION 7.1. *Let m be a Banach-space-valued vector measure.*

(i) *If I_m is weakly compact, then necessarily $L^1(m) = L^1_w(m)$.*
(ii) *If I_m is completely continuous, then necessarily $L^1(m) = L^1_w(m)$.*

Recall that I_m is *weakly compact* if, whenever $\{f_n\}_{n=1}^\infty$ is a norm-bounded sequence in $L^1(m)$, then its image $\{I_m(f_n)\}_{n=1}^\infty$ has a weakly convergent subsequence [17, p. 49]. Proposition 7.1(i) occurs in [13, Corollary 2.3]; its converse is not valid in general (see Examples 7.2(i) and 7.3 below).

The integration operator I_m is *completely continuous* if it maps weakly Cauchy sequences in $L^1(m)$ to norm-convergent sequences in X [2, Theorem 19.1]. Proposition 7.1(ii) occurs (even for Fréchet-space-valued measures) in [9, Theorem 3.6]; its converse also fails to hold, in general (see Examples 7.2(ii) and 7.3 below).

It is important to note that the weak compactness of I_m need not imply its complete continuity (see Example 7.2(ii)), nor vice versa (see Examples 7.2(i) and 7.3(ii)). There also exist Banach-space-valued vector measures m for which $L^1(m) = L^1_w(m)$ but I_m is *neither* weakly compact *nor* completely continuous; see Example 7.3.

The *variation measure* $|m| : \Sigma \to [0,\infty]$ of a Banach-space-valued vector measure m is defined as for scalar measures [18, Definition I.1.4]; it is the smallest σ-additive, $[0,\infty]$-valued measure dominating m in the sense that $\|m(A)\|_X \le |m|(A)$ for every $A \in \Sigma$. Moreover, with continuous inclusions, it is always the case that

$$L^1(|m|) \subseteq L^1(m) \subseteq L^1_w(m);$$

see [46, Lemma 3.14(i) and p. 138].

EXAMPLE 7.2. Let $1 \le r \le \infty$ and consider the *Volterra* vector measure $\nu_r : \mathcal{B}([0,1]) \to L^r([0,1])$ given by

$$\nu_r(A) : t \mapsto \int_0^t \chi_A(s)\, ds, \qquad t \in [0,1],$$

for each $A \in \mathcal{B}([0,1])$. For $1 \le r < \infty$ the codomain space $L^r([0,1])$ is weakly sequentially complete, and hence also weakly Σ-complete. It then follows from Lemma 2.5(iv) (with $m := \nu_r$) that $L^1(\nu_r) = L^1_w(\nu_r)$.

(i) Suppose that $r = 1$ or ∞. Then $L^1(\nu_r) = L^1(|\nu_r|)$ and the integration operator I_{ν_r} is completely continuous but not weakly compact [46, Example 3.49(iv)]. Moreover,

$L^1(\nu_r) = L^1_w(\nu_r)$ follows from Proposition 7.1(ii), or via the weak sequential completeness of $L^1(\nu_r) = L^1(|\nu_r|)$; see the beginning of this section.

(ii) Let $1 < r < \infty$. In contrast to part (i) we know that I_{ν_r} is now weakly compact (as its codomain space $L^r([0,1])$ is reflexive), but not completely continuous [46, Proposition 3.52]. According to [46, Example 3.26] we see that ν_r has finite variation for every $1 \le r \le \infty$, but if $1 < r < \infty$, then $L^1(|\nu_r|) \subsetneq L^1(\nu_r)$. □

The following examples arise in classical harmonic analysis.

EXAMPLE 7.3. (i) Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ be the circle group equipped with normalized Haar measure. By $c_0(\mathbb{Z})$ we denote the Banach space of all functions $\psi : \mathbb{Z} \to \mathbb{C}$ for which $\lim_{|n|\to\infty} |\psi(n)| = 0$, equipped with the uniform norm. The Riemann–Lebesgue Lemma ensures that the Fourier transform $\widehat{f}$ of each $f \in L^1(\mathbb{T})$ belongs to $c_0(\mathbb{Z})$. Let $F \in \mathcal{L}(L^1(\mathbb{T}), c_0(\mathbb{Z}))$ denote the Fourier transform map, i.e., $F(f) := \widehat{f}$ for each $f \in L^1(\mathbb{T})$.

Define a vector measure $m : \mathcal{B}(\mathbb{T}) \to c_0(\mathbb{Z})$ by $m(A) = F(\chi_A) = \widehat{\chi_A}$ for $A \in \mathcal{B}(\mathbb{T})$, in which case

$$L^1(\mathbb{T}) = L^1(|m|) = L^1(m) = L^1_w(m)$$

with the integration operator $I_m = F$ [46, p. 299]. Moreover, I_m is neither weakly compact nor completely continuous [46, Proposition 7.3(iii)].

(ii) For each complex measure $\lambda : \mathcal{B}(\mathbb{T}) \to \mathbb{C}$ define the *convolution* vector measure $\nu_\lambda : \mathcal{B}(\mathbb{T}) \to L^1(\mathbb{T})$ by

$$\nu_\lambda(A) := \lambda * \chi_A : s \mapsto \int_0^{2\pi} \chi_A(s-t)\, d\lambda(t), \qquad s \in [0, 2\pi] \simeq \mathbb{T}.$$

It is known that

$$L^1(\mathbb{T}) = L^1(|\nu_\lambda|) = L^1(\nu_\lambda) = L^1_w(\nu_\lambda)$$

and that the integration operator $I_{\nu_\lambda} : L^1(\nu_\lambda) \to L^1(\mathbb{T})$ is precisely the convolution operator $f \mapsto \lambda * f$ for $f \in L^1(\mathbb{T})$ [46, Remark 7.36]. Moreover, I_{ν_λ} is compact if and only if it is weakly compact if and only if λ is absolutely continuous with respect to Haar measure. On the other hand, I_{ν_λ} is completely continuous if and only if the Fourier–Stieltjes transform $\widehat{\lambda}$ of λ (i.e., $\widehat{\lambda}(n) := \frac{1}{2\pi}\int_0^{2\pi} e^{-int}\, d\lambda(t)$ for $n \in \mathbb{Z}$) belongs to $c_0(\mathbb{Z})$ [46, Remark 7.36(ii)]. Hence, every measure λ which fails to be absolutely continuous but satisfies $\widehat{\lambda} \in c_0(\mathbb{Z})$ (see [46, p. 320] for the existence of such λ) has the property that its integration operator I_{ν_λ} is completely continuous but not weakly compact. If λ satisfies $\widehat{\lambda} \notin c_0(\mathbb{Z})$ (e.g., the Dirac measure at any point of $\mathbb{T}$ or Cantor–Lebesgue measure or certain Riesz product measures), then $L^1(\nu_\lambda) = L^1_w(\nu_\lambda)$ but I_{ν_λ} is neither weakly compact nor completely continuous. □

Recall that a continuous linear operator between Banach spaces is *strictly singular* if no restriction to any closed, infinite-dimensional subspace of the domain space is an isomorphism (onto its range). Clearly, every compact operator is strictly singular, but not conversely; see Remark 7.5 below.

PROPOSITION 7.4. *Let m be a Banach-space-valued vector measure whose integration operator I_m is strictly singular. Then necessarily $L^1(m) = L^1_w(m)$.*

Proof. Since I_m is strictly singular it cannot fix any copy of c_0. So, Corollary 6.2(iii) yields $L^1(m) = L^1_w(m)$. ∎

REMARK 7.5. There exist vector measures m such that I_m is strictly singular, but neither weakly compact nor completely continuous [45, Proposition 2.10]. The paragraph immediately prior to Example 2.14 in [45] exhibits a vector measure m such that I_m is strictly singular and weakly compact, but not compact (as $L^1(m) \neq L^1(|m|)$ [43, Theorems 1 & 4]). There also exist vector measures m satisfying $L^1(m) = L^1_w(m)$ with I_m not strictly singular. Indeed, in [46, Example 3.49(ii)] the ℓ^1-valued vector measure $m := \nu$ has the property that $I_m : L^1(m) \to \ell^1$ is a surjective linear isomorphism, and hence is surely not strictly singular. On the other hand, being isomorphic to ℓ^1, the space $L^1(m)$ is weakly sequentially complete and so $L^1(m) = L^1_w(m)$; see the beginning of this section. □

The two (rather extensive) classes of operators that occur in Proposition 7.1 are not comparable, but they have the property that I_m belonging to either class ensures that $L^1(m) = L^1_w(m)$. Proposition 7.4 exhibits the same feature. There is available a further class of operators, containing all weakly compact and all completely continuous operators, which *characterizes* the property $L^1(m) = L^1_w(m)$ via its membership of I_m; see Proposition 7.7 below.

A continuous linear operator between Banach spaces (or lcHs) is said to be *weakly completely continuous* if it maps weakly Cauchy sequences to weakly convergent sequences, a notion going back to J. Dieudonné and A. Grothendieck [6, p. 27]. Clearly, every completely continuous operator is also weakly completely continuous. The following observation shows that the same is true of weakly compact operators.

FACT 5. *Every weakly compact operator between two Banach spaces is also weakly completely continuous.*

Proof. Let $T : Y \to Z$ be a weakly compact operator between Banach spaces. Given a weakly Cauchy sequence $\{y_n\}_{n=1}^\infty$ in Y, the continuity of T from Y_σ into Z_σ implies that $\{Ty_n\}_{n=1}^\infty$ is weakly Cauchy in Z. Moreover, the weak compactness of T implies that $\{Ty_n\}_{n=1}^\infty$ lies in a weakly compact subset of Z. By the Eberlein–Šmulian Theorem [22, Theorem V.6.1], $\{Ty_n\}_{n=1}^\infty$ has a convergent subsequence in Z_σ. Since $\{Ty_n\}_{n=1}^\infty$ is itself Cauchy in Z_σ, it follows that $\{Ty_n\}_{n=1}^\infty$ is also convergent in Z_σ. Hence, T is weakly completely continuous. ∎

We also require the following useful information.

LEMMA 7.6.

(i) *The composition of a weakly completely continuous operator with any continuous linear operator* (*on the left or the right*) *is again weakly completely continuous.*
(ii) *Every Banach-space-valued continuous linear operator defined on a weakly sequentially complete Banach space is necessarily weakly completely continuous.*
(iii) *Let Y be a closed subspace of the Banach space Z and $\Phi : Y \to Z$ be the identity inclusion. Then Φ is weakly completely continuous if and only if Y is weakly sequentially complete.*

Proof. (i) This follows routinely from the definition of a weakly completely continuous operator and the fact that a continuous linear operator is also continuous for the weak topologies.

(ii) Let Y, Z be Banach spaces, with Y weakly sequentially complete, and $T \in \mathcal{L}(Y, Z)$. Fix any weakly Cauchy sequence $\{y_n\}_{n=1}^{\infty}$ in Y. Then $\{y_n\}_{n=1}^{\infty}$ has a limit vector y in Y_σ. Since $T \in \mathcal{L}(Y_\sigma, Z_\sigma)$, it follows that $\{T(y_n)\}_{n=1}^{\infty}$ converges in Z_σ. Thus, T is weakly completely continuous.

(iii) If Y is weakly sequentially complete, then Φ is weakly completely continuous by part (ii).

Conversely, suppose that Φ is weakly completely continuous. Let $\{u_n\}_{n=1}^{\infty}$ be a Cauchy sequence in Y_σ. Then $\{\Phi(u_n)\}_{n=1}^{\infty}$ converges in Z_σ, say to $z \in Z$. Since Y is norm-closed in Z it is also closed in Z_σ and so actually $z \in Y$. Given any $y^* \in Y^*$, the Hahn–Banach Theorem ensures the existence of $z^* \in Z^*$ satisfying $z^*|_Y = y^*$. From this, together with the fact that $\langle \Phi(u_n) - z, z^* \rangle \to 0$ and $\{\Phi(u_n) - z\}_{n=1}^{\infty} \subseteq Y$, we can conclude that $\lim_{n\to\infty} \langle \Phi(u_n) - z, y^* \rangle = 0$. Since $y^* \in Y^*$ is arbitrary, it follows that $u_n = \Phi(u_n) \to z$ in Y_σ. Hence, Y is weakly sequentially complete. ∎

For a Banach-space-valued vector measure $m : \Sigma \to X$ we recall from Remark 2.11 that $L^1(m)$ is a closed subspace of $L^1_w(m)$. Denote the identity inclusion of $L^1(m)$ into $L^1_w(m)$ by ρ_m. We can now formulate the main result of this section.

Proposition 7.7. *Let $m : \Sigma \to X$ be a Banach-space-valued vector measure. The following assertions are equivalent:*

(i) $L^1(m) = L^1_w(m)$.
(ii) *The identity operator* $\mathrm{id}_{L^1(m)}$ *on $L^1(m)$ is weakly completely continuous.*
(iii) *The natural inclusion $\rho_m : L^1(m) \to L^1_w(m)$ is weakly completely continuous.*
(iv) *The integration operator $I_m : L^1(m) \to X$ is weakly completely continuous.*

Proof. (i)⇔(ii). This follows from Lemma 7.6(iii) with $Y = Z := L^1(m)$ and $\Phi := \mathrm{id}_{L^1(m)}$ together with the fact (see the beginning of this section) that $L^1(m) = L^1_w(m)$ if and only if $L^1(m)$ is weakly sequentially complete.

(ii)⇒(iii). This follows from $\rho_m = \rho_m \circ \mathrm{id}_{L^1(m)}$ and Lemma 7.6(i).

(iii)⇒(i). Apply Lemma 7.6(iii) with $Y := L^1(m)$, $Z := L^1_w(m)$ and $\Phi := \rho_m$ together with the fact that $L^1(m) = L^1_w(m)$ if and only if $L^1(m)$ is weakly sequentially complete.

(i)⇒(iv). By (i) the space $L^1(m)$ is weakly sequentially complete. The desired conclusion is then immediate from Lemma 7.6(ii).

(iv)⇒(i). Assume that $L^1(m) \subsetneq L^1_w(m)$. Then I_m fixes an isomorphic copy of c_0; see Corollary 6.2(iii). That is, there exists an isomorphism (onto its range) $S \in \mathcal{L}(c_0, L^1(m))$ such that $I_m \circ S : c_0 \to X$ is an isomorphism onto its range. So, $(I_m \circ S)^{-1} \circ (I_m \circ S)$ equals the identity operator id_{c_0} on c_0. Since the standard basis vectors of c_0 form a weakly Cauchy sequence which fails to be weakly convergent, it is clear that id_{c_0} is not weakly completely continuous. In other words, the composition $(I_m \circ S)^{-1} \circ I_m \circ S$ is not weakly completely continuous. Hence, by Lemma 7.6(i) the operator I_m also fails to be weakly completely continuous. This establishes that $L^1(m) = L^1_w(m)$ whenever I_m is weakly completely continuous. ∎

REMARK 7.8. The class of weakly completely continuous operators *properly* contains the weakly compact operators and the completely continuous operators. This follows from Proposition 7.7 and Example 7.3(i) (resp. Example 7.3(ii)), where the vector measure m (resp. ν_λ with $\widehat{\lambda} \notin c_0(\mathbb{Z})$) satisfies $L^1(m) = L^1_w(m)$ (resp. $L^1(\nu_\lambda) = L^1_w(\nu_\lambda)$), but I_m (resp. I_{ν_λ}) is neither weakly compact nor completely continuous. Concerning m from Example 7.3(i), recall that $L^1(m) = L^1_w(m) = L^1(\mathbb{T})$ and so $\rho_m = \mathrm{id}_{L^1(\mathbb{T})}$ is weakly completely continuous via Proposition 7.7. However, since $L^1(\mathbb{T})$ is not reflexive, $\mathrm{id}_{L^1(m)}$ is not weakly compact. To see that $\mathrm{id}_{L^1(m)}$ is not completely continuous either, observe that the sequence $\{e^{-in(\cdot)}\}_{n=1}^\infty$ converges weakly to 0 in $L^1(\mathbb{T})$ (by the Riemann–Lebesgue Lemma), but $\{e^{-in(\cdot)}\}_{n=1}^\infty$ is not norm-convergent to 0 in $L^1(\mathbb{T})$ because $\|e^{-in(\cdot)}\|_{L^1(\mathbb{T})} = 1$ for all $n \in \mathbb{N}$. □

Checking directly whether one of the equivalences in Proposition 7.7 is satisfied may not be easy in practise. The fact that $L^1(m)$ is a complex *Banach lattice* (see Remark 3.8(ii)) provides a means to exhibit classical classes of operators with the property that I_m is weakly completely continuous whenever it belongs to one of these classes.

Let X be a Banach space and E be a complex Banach lattice (see Section 3). Given $1 \le q < \infty$, an operator $R \in \mathcal{L}(E, X)$ is said to be $(q,1)$-*concave* if there exists a constant $C_q > 0$ such that

$$\Big(\sum_{j=1}^n \|R(u_j)\|_X^q\Big)^{1/q} \le C_q \Big\|\sum_{j=1}^n |u_j|\Big\|_E$$

for all choices of finitely many elements $\{u_j\}_{j=1}^n \subseteq E$ with $n \in \mathbb{N}$ [17, p. 330].

PROPOSITION 7.9. *Let $m : \Sigma \to X$ be a Banach-space-valued vector measure.*

(i) *If the integration operator $I_m : L^1(m) \to X$ is $(q,1)$-concave for some $1 \le q < \infty$, then I_m is weakly completely continuous. In particular, $L^1(m) = L^1_w(m)$.*

(ii) *Given any $1 \le q < \infty$, the following conditions are equivalent:*

   (a) *The integration operator $I_m : L^1(m) \to X$ is $(q,1)$-concave.*
   (b) *The identity operator $\mathrm{id}_{L^1(m)} : L^1(m) \to L^1(m)$ is $(q,1)$-concave.*
   (c) *The embedding $\rho_m : L^1(m) \to L^1_w(m)$ is $(q,1)$-concave.*

Proof. (i) Let $C_q > 0$ satisfy

$$\Big(\sum_{j=1}^n \|I_m(f_j)\|_X^q\Big)^{1/q} \le C_q \Big\|\sum_{j=1}^n |f_j|\Big\|_{L^1(m)} \tag{7.1}$$

whenever $\{f_j\}_{j=1}^n \subseteq L^1(m)$, with $n \in \mathbb{N}$, is a finite set. Suppose that $L^1(m) \subsetneq L^1_w(m)$. Via Corollary 6.2(iii) there is a lattice-isomorphism $S : c_0 \to L^1(m)$ (onto its range) such that $I_m|_{\mathcal{R}(S)} : \mathcal{R}(S) \to X$ is a topological isomorphism onto its range. Consequently, $I_m \circ S : c_0 \to X$ is a topological isomorphism onto its range and so there exists a constant $M > 0$ such that

$$\|\alpha\|_{c_0} \le M \|(I_m \circ S)(\alpha)\|_X, \qquad \alpha \in c_0. \tag{7.2}$$

On the other hand, being a lattice-isomorphism, S is a positive operator, i.e., $S(c_0^+) \subseteq L^1(m)^+$ [2, Theorem 7.3]. It then follows from [46, Lemma 2.57(ii)(a)], with $q = 1$ there, that $\sum_{j=1}^{n} |S(\alpha_j)| \le S(\sum_{j=1}^{n} |\alpha_j|)$ in the order of $L^1(m)$, and hence, since $\|\cdot\|_{L^1(m)}$ is a lattice norm, that

$$\Big\| \sum_{j=1}^{n} |S(\alpha_j)| \Big\|_{L^1(m)} \le \Big\| S\Big(\sum_{j=1}^{n} |\alpha_j| \Big) \Big\|_{L^1(m)} \tag{7.3}$$

whenever $\alpha_1, \dots, \alpha_n \in c_0$ and $n \in \mathbb{N}$.

Let $\{e_j\}_{j=1}^{\infty}$ denote the canonical basis of c_0 and $\|S\|_{\mathrm{op}}$ denote the operator norm of S. Fix $n \in \mathbb{N}$. It follows from (7.1)–(7.3) that

$$\begin{aligned} n^{1/q} &= \Big(\sum_{j=1}^{n} \|e_j\|_{c_0}^q \Big)^{1/q} \le M \Big(\sum_{j=1}^{n} \|(I_m \circ S)(e_j)\|_X^q \Big)^{1/q} \\ &\le MC_q \Big\| \sum_{j=1}^{n} |S(e_j)| \Big\|_{L^1(m)} \le MC_q \Big\| S\Big(\sum_{j=1}^{n} |e_j| \Big) \Big\|_{L^1(m)} \\ &\le MC_q \|S\|_{\mathrm{op}} \Big\| \sum_{j=1}^{n} e_j \Big\|_{c_0} = MC_q \|S\|_{\mathrm{op}}. \end{aligned}$$

Since $n \in \mathbb{N}$ is arbitrary, this is impossible. So, we must have $L^1(m) = L^1_w(m)$, which, via Proposition 7.7, is equivalent to I_m being weakly completely continuous.

(ii) (a)$\Rightarrow$(b). Let $C_q > 0$ be a constant satisfying

$$\Big(\sum_{j=1}^{n} \|I_m(g_j)\|_X^q \Big)^{1/q} \le C_q \Big\| \sum_{j=1}^{n} |g_j| \Big\|_{L^1(m)} \tag{7.4}$$

whenever $\{g_j\}_{j=1}^{n} \subseteq L^1(m)$ is a finite set, with $n \in \mathbb{N}$. Fix now $n \in \mathbb{N}$ and $\{f_j\}_{j=1}^{n} \subseteq L^1(m)$. It follows from [46, Lemma 3.71] that

$$\Big(\sum_{j=1}^{n} \|f_j\|_{L^1(m)}^q \Big)^{1/q} = \sup_{s_1,\dots,s_n} \Big(\sum_{j=1}^{n} \|I_m(s_j f_j)\|_X^q \Big)^{1/q}, \tag{7.5}$$

where the supremum on the right side is taken over all choices of $s_j \in \operatorname{sim}\Sigma$ with $\sup_{w\in\Omega} |s_j(w)| \le 1$. For any such choice $s_1, \dots, s_n$ we deduce from (7.4), with $g_j := s_j f_j$ for $1 \le j \le n$, that

$$\Big(\sum_{j=1}^{n} \|I_m(s_j f_j)\|_X^q \Big)^{1/q} \le C_q \Big\| \sum_{j=1}^{n} |s_j f_j| \Big\|_{L^1(m)} \le C_q \Big\| \sum_{j=1}^{n} |f_j| \Big\|_{L^1(m)}.$$

This and (7.5) yield

$$\Big(\sum_{j=1}^{n} \|f_j\|_{L^1(m)}^q \Big)^{1/q} \le C_q \Big\| \sum_{j=1}^{n} |f_j| \Big\|_{L^1(m)}. \tag{7.6}$$

Thus, (b) holds because $n \in \mathbb{N}$ and $\{f_j\}_{j=1}^{n} \subseteq L^1(m)$ are arbitrary.

(b)$\Rightarrow$(a). By the hypothesis on $\mathrm{id}_{L^1(m)}$ there exists $C_q > 0$ such that (7.6) holds for any finite set $\{f_j\}_{j=1}^{n} \subseteq L^1(m)$, with $n \in \mathbb{N}$. Fix such a choice of $\{f_j\}_{j=1}^{n} \subseteq L^1(m)$. Since $\|I_m\|_{\mathrm{op}} = 1$ [46, p. 152], it follows from (7.6) that

$$\Big(\sum_{j=1}^{n} \|I_m(f_j)\|_X^q \Big)^{1/q} \le \Big(\sum_{j=1}^{n} \|f_j\|_X^q \Big)^{1/q} \le C_q \Big\| \sum_{j=1}^{n} |f_j| \Big\|_{L^1(m)}.$$

As $\{f_j\}_{j=1}^{n} \subseteq L^1(m)$, with $n \in \mathbb{N}$, is arbitrary, I_m is $(q, 1)$-concave.

(b)⇔(c). This equivalence is a routine consequence of the definition of $(q,1)$-concave operators and the fact that $\|\rho_m(f)\|_{L^1_w(m)} = \|f\|_{L^1(m)}$ for each $f \in L^1(m)$. ■

COROLLARY 7.10. *Let m be a Banach-space-valued vector measure.*

(i) *If the embedding $\rho_m : L^1(m) \to L^1_w(m)$ is $(q,1)$-concave for some $1 \le q < \infty$, then ρ_m is weakly completely continuous.*
(ii) *If the identity operator $\mathrm{id}_{L^1(m)}$ is $(q,1)$-concave for some $1 \le q < \infty$, then $\mathrm{id}_{L^1(m)}$ is weakly completely continuous.*

Proof. (i) From Proposition 7.9(ii) we see that I_m is $(q,1)$-concave, and hence, by Proposition 7.9(i), it follows that $L^1(m) = L^1_w(m)$. As recorded at the beginning of this section, $L^1(m)$ is then weakly sequentially complete. Now apply Lemma 7.6(ii) to conclude that ρ_m is weakly completely continuous.

(ii) A similar argument as for part (i) applies. ■

Let μ be any σ-additive, positive measure and X be any Banach space. Then *every* continuous linear operator $T : L^1(\mu) \to X$ is necessarily $(1,1)$-concave. Indeed, given any finite set $\{f_j\}_{j=1}^n \subseteq L^1(\mu)$, with $n \in \mathbb{N}$, we have

$$\sum_{j=1}^n \|T(f_j)\|_X \le \|T\|_{\mathrm{op}} \sum_{j=1}^n \|f_j\|_{L^1(\mu)} = \|T\|_{\mathrm{op}} \Big\| \sum_{j=1}^n |f_j| \Big\|_{L^1(\mu)}.$$

Since the vector measures m and ν_λ (with λ any complex measure on $\mathcal{B}(\mathbb{T})$) presented in Examples 7.3(i) and 7.3(ii), respectively, have the property that their corresponding spaces $L^1(m)$ and $L^1(\nu_\lambda)$ are of the form $L^1(\mu)$, it follows from the previous observation that the integration operators I_m and I_{ν_λ} are necessarily $(1,1)$-concave, although, as noted before, they are neither weakly compact nor completely continuous. We now exhibit a vector measure m such that I_m is weakly completely continuous but not $(q,1)$-concave for any $1 \le q < \infty$. So, Proposition 7.9 does not cover all cases.

EXAMPLE 7.11. Let $\Omega := [0,\infty)$ and $\Sigma := \mathcal{B}(\Omega)$. Denote Lebesgue measure on Σ by μ. Fix a strictly increasing sequence $\{p_k\}_{k=1}^\infty \subseteq [1,\infty)$ satisfying $\lim_{k\to\infty} p_k = \infty$. For each $k \in \mathbb{N}$ let $\Omega(k) := [(k-1),k)$, in which case the closed subspace $L^{p_k}(\Omega(k)) := \{f\chi_{\Omega(k)} : f \in L^{p_k}(\mu)\}$ of the Banach space $L^{p_k}(\mu)$ is again a Banach space for the induced norm $\|\cdot\|_{p_k}$ from $L^{p_k}(\mu)$. The ℓ^2-*direct sum* $X := (\bigoplus_{k=1}^\infty L^{p_k}(\Omega(k)))_2$ of the Banach spaces $(L^{p_k}(\Omega(k)), \|\cdot\|_{p_k})$, for $k \in \mathbb{N}$, is the vector space consisting of all $f \in L^0(\Sigma)$ satisfying

$$\Big(\sum_{k=1}^\infty \|f\chi_{\Omega(k)}\|_{p_k}^2 \Big)^{1/2} < \infty, \tag{7.7}$$

with the norm $\|f\|_X$ of f defined by the left side of (7.7), in which case X is a Banach space [17, p. xiv, Notation]. Moreover, X is also a complex Banach lattice for the μ-a.e. pointwise order and $\|\cdot\|_X$ is the corresponding (complex) lattice norm. Noting that X is an ideal in $L^0(\Sigma)$, and hence is Dedekind σ-complete [61, p. 107], and that X is also separable, it follows that X has order continuous norm [60, Theorem 117.3].

Clearly the function $g := \sum_{k=1}^\infty (1/k)\chi_{\Omega(k)}$, defined pointwise on Ω, belongs to X^+ and so $g\chi_A \in X$ for all $A \in \Sigma$ as $\|\cdot\|_X$ is a lattice norm. This enables us to define a finitely additive measure $m : \Sigma \to X$ by $m(A) := g\chi_A$ for $A \in \Sigma$. The σ-additivity of

m is a routine consequence of X having order continuous norm. The claim is that the integration operator $I_m : L^1(m) \to X$ has the form

$$I_m(f) = fg, \qquad f \in L^1(m), \tag{7.8}$$

and is a linear isometry.

To establish (7.8) first fix $f \in L^1(m)^+$. Choose an increasing sequence $\{s_j\}_{j=1}^{\infty} \subseteq \operatorname{sim} \Sigma$ converging pointwise to f with $0 \le s_j \le f$ for $j \in \mathbb{N}$. It is clear from the definition of m that

$$\int_A s_j \, dm = I_m(s_j \chi_A) = s_j g \chi_A, \qquad A \in \Sigma,\ j \in \mathbb{N}.$$

Moreover, $\lim_{j\to\infty} \|f - s_j\|_{L^1(m)} = 0$; apply Lemma 2.7(i) in the Banach space setting. Continuity of $I_m : L^1(m) \to X$ implies that $s_j g = I_m(s_j) \to I_m(f)$ in X as $j \to \infty$. Since X is an ideal in $L^0(\Sigma)$, it is a *Banach function space* over (Ω, Σ, μ), also called a *Köthe function space*, in the sense of [36, Ch. 1, §9]. Hence, there exists a pointwise μ-a.e. convergent subsequence $s_{j(n)} g \to I_m(f)$ for $n \to \infty$. This follows from the general fact that, in any Banach lattice, a norm convergent sequence has an order convergent subsequence [60, Theorem 100.6]. On the other hand, also $s_j g \to fg$ pointwise on Ω as $j \to \infty$, and hence $I_m(f) = fg$, i.e., (7.8) holds whenever $f \in L^1(m)^+$. Since every function in $L^1(m)$ can be expressed as a linear combination of four positive functions from $L^1(m)$, it follows that (7.8) holds in general.

Noting that m is a *positive vector measure*, i.e., $m(\Sigma) \subseteq X^+$, it follows from [46, Lemma 3.13], the fact that $g \ge 0$ and (7.8) that

$$\|f\|_{L^1(m)} = \|I_m(|f|)\|_X = \big\| |f| g \big\|_X = \|fg\|_X = \|I_m(f)\|_X, \qquad f \in L^1(m).$$

Hence, I_m is a linear isometry, which completes the proof of the claim.

As X is reflexive, I_m is weakly compact, and hence also weakly completely continuous; see Fact 5.

Fix $1 \le q < \infty$. Choose any $K \in \mathbb{N}$ satisfying $p_K > q$ and fix it henceforth. Select any sequence $\{A(j)\}_{j=1}^{\infty}$ of pairwise disjoint subintervals of $\Omega(K)$. For each $j \in \mathbb{N}$, define $f_j := K\mu(A(j))^{-1/p_K} \chi_{A(j)}$, in which case $gf_j = \mu(A(j))^{-1/p_K} \chi_{A(j)} \in X$. Clearly $\{f_j\}_{j=1}^{\infty} \subseteq L^1(m)^+$. Moreover, for each $j \in \mathbb{N}$, we have

$$\|I_m(f_j)\|_X = \|f_j g\|_X = \|K^{-1} f_j\|_{p_K} = 1. \tag{7.9}$$

Fix $n \in \mathbb{N}$. It follows from (7.9) that

$$\Big(\sum_{j=1}^{n} \|I_m(f_j)\|_X^q \Big)^{1/q} = n^{1/q}. \tag{7.10}$$

On the other hand, recalling that I_m is a linear isometry and the functions $\{f_j\}_{j=1}^{n} \subseteq L^1(m)^+$ are disjointly supported, we deduce from (7.9) that

$$\begin{aligned} \Big\| \sum_{j=1}^{n} |f_j| \Big\|_{L^1(m)}^{p_K} &= \Big\| I_m\Big(\sum_{j=1}^{n} |f_j| \Big) \Big\|_X^{p_K} = \Big\| \sum_{j=1}^{n} g f_j \Big\|_X^{p_K} \\ &= \Big\| \sum_{j=1}^{n} K^{-1} f_j \Big\|_{p_K}^{p_K} = \sum_{j=1}^{n} \|K^{-1} f_j\|_{p_K}^{p_K} = n. \end{aligned}$$

That is,

$$\Big\| \sum_{j=1}^{n} |f_j| \Big\|_{L^1(m)} = n^{1/p_K}. \tag{7.11}$$

Since (7.10) and (7.11) hold for all $n \in \mathbb{N}$ and $q^{-1} > p_K^{-1}$, it follows that I_m is not $(q,1)$-concave. □

Remark 7.12. Clearly every $(1,1)$-concave operator defined on a Banach lattice is $(q,1)$-concave whenever $1 \le q < \infty$. Those Banach-space-valued vector measures m whose integration operator I_m is $(1,1)$-concave have been characterized in Proposition 3.74 of [46]. Namely, I_m is $(1,1)$-concave if and only if $L^1(m) = L^1(|m|)$ as an equality of isomorphic Banach spaces. □

We now exhibit several classical *operator ideals* [17, p. 131] such that, if I_m belongs to any one of them, then I_m is $(q,1)$-concave for some suitable $1 \le q < \infty$ and so, in particular, I_m is weakly completely continuous; see Proposition 7.9(i).

Example 7.13. Consider a Banach-space-valued vector measure $m : \Sigma \to X$ and its associated integration operator $I_m : L^1(m) \to X$.

(i) Suppose that I_m is a *compact operator*. Then $L^1(m) = L^1(|m|)$ as an equality of vector spaces, [46, Proposition 3.48], and hence, also an equality as isomorphic Banach spaces (because the natural embedding from $L^1(|m|)$ into $L^1(m)$ is continuous; see the discussion prior to Example 7.2). Then Remark 7.12 tells us that I_m is $(1,1)$-concave.

(ii) Let $1 \le p \le q < \infty$ and suppose that I_m is a (q,p)-*summing operator*; see [17, p. 197] for the definition. Then I_m is also $(q,1)$-summing [17, p. 198]. Hence, I_m is $(q,1)$-concave [17, p. 330].

(iii) Let $1 \le p < \infty$. The p-*summing operators* are exactly the (p,p)-summing operators [17, p. 31 & p. 197]. So, if I_m is p-summing, then it is $(p,1)$-concave by part (ii) with $p = q$. Actually, this can be improved. Indeed, it follows essentially from [7, Theorem 2.7] and [46, Proposition 3.74] that $L^1(m) = L^1(|m|)$; more precisely, see the paragraph immediately prior to Proposition 2.4 of [44]. So, an appeal to Remark 7.12 shows that I_m is $(1,1)$-concave.

(iv) Let $1 \le p < r < \infty$. A continuous linear operator $T : Y \to Z$ between Banach spaces is called (r,p)-*mixing* if $S \circ T$ is p-summing for every Banach-space-valued r-summing operator S defined on Z, [16, p. 415]. Every (r,p)-mixing operator is necessarily (q,p)-summing with $q \in (p,\infty)$ determined by $1/q + 1/r = 1/p$, [16, p. 426, Remark 1]. In particular, if I_m is (r,p)-mixing, then I_m is (q,p)-summing for an appropriate $q \in (p,\infty)$, and hence is $(q,1)$-concave by part (ii) above.

(v) For the definition of the *cotype* of a Banach space see [17, p. 218]. Suppose that $L^1(m)$ has cotype 2. Then the identity operator $\mathrm{id}_{L^1(m)}$ is $(2,1)$-mixing [16, p. 417, 32.2(4)]. Accordingly, $\mathrm{id}_{L^1(m)}$ is $(2,1)$-concave; see part (iv) above with $p := 1$ and $r := 2$ (in which case $q = 2$). So, Proposition 7.9(ii) implies that I_m is also $(2,1)$-concave.

Next suppose that $L^1(m)$ has cotype $r > 2$. Recalling that r^* is the conjugate index of r, we see that $\mathrm{id}_{L^1(m)}$ is $((r^* - \varepsilon),1)$-mixing whenever ε satisfies $0 < \varepsilon < r^* - 1$ [16, p. 417, 32.2(5)]. So, by part (iv) above, the operator $\mathrm{id}_{L^1(m)}$ is $(q(\varepsilon),1)$-summing for

$q(\varepsilon) \in (1, \infty)$ determined by $1/q(\varepsilon) + 1/(r^* - \varepsilon) = 1$. Again Proposition 7.9(ii) implies that I_m is $(q(\varepsilon), 1)$-concave (with $q(\varepsilon) > 1$). □

Concerning Example 7.13(ii), it should be noted that there exist Banach-space-valued vector measures m for which I_m is $(2, 1)$-summing but I_m is neither weakly compact nor completely continuous. For instance, let m be the $L^1([0, 1])$-valued vector measure which assigns χ_A to each $A \in \mathcal{B}([0, 1])$. Then $L^1(m) = L^1([0, 1])$ and I_m, which is given by $I_m(f) = f$ for $f \in L^1(m)$, is a surjective linear isometry [46, Example 3.61]. It follows from Orlicz's Theorem [17, Theorem 3.12 & p. 197], that $I_m = \mathrm{id}_{L^1([0,1])}$ is $(2, 1)$-summing. On the other hand, I_m is not weakly compact or completely continuous.

In conclusion, concerning Example 7.13(v) we point out that $L^1(m)$ necessarily has cotype q whenever the codomain space X of m has cotype q with $q \in [2, \infty)$ [12, Theorem 1].

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