Contents

Abstract

We overview the literature concerning bilinear operations on vector-valued distributions in general, and more specifically the convolution of vector-valued functions or distributions. We compare and evaluate the different approaches to this problem of L. Schwartz on the one hand and of Y. Hirata and R. Shirarishi on the other. Moreover we discuss applications of the general existence and uniqueness results to different branches of mathematical analysis like partial differential equations or harmonic analysis.

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1. Introduction

Whereas L. Schwartz' theory of distributions in the form (of parts) of his monograph [47] is world-wide generally accepted, the continuation of this theory in [44], [45] and [46] remains relatively unknown. E.g., the simplified version of the theory of vector-valued distributions presented in L. Schwartz' Tata lectures [44] is quoted almost always only for the reference to the generalization of the Hille–Yosida Theorem to (quasi-)complete locally convex spaces.

Therefore the aims of our paper are:

1. to present a survey of L. Schwartz' and R. Shiraishi' theorems on the convolution of vector-valued distributions, mostly without proofs or with only sketches of proofs (quoted as "Theorems" 1–12);
2. to compare the assumptions and the proofs of the theorems (quoted as "Comparisons" 1–11);
3. to report on applications, mostly taken from the literature (quoted as "Applications" 1–22);
4. to add some new contributions to the theory of vector-valued distributions (quoted as "Propositions" 1–9);
5. to correct some errors in the literature.

In some cases, we renounce to quote the most general form of the assumptions given in [45], [46] and [50] in order to simplify the presentation (e.g., we do not use "completing sets" or "$\mathfrak{T}$-bounded mappings").

We have not found relevant applications of Theorems 7 and 8 where the assumption of a subnuclear embedding would have to be used in an essential manner.

1.1. A motivating example. By using the theory of vector-valued distributions, it is possible to reduce questions of convolvability (of distributions) to multiplicative properties (see, e.g., Application 6 or the proof of Theorem 10).

Originally however, our interest in the theory of vector-valued distributions arose from problems in "classical" mathematical physics. Let us explain this point of view by an example (which, in two space variables, also served as a starting point in [16] and in [29, p. 189, ex. 405 and *406]).

The solution $u\colon \mathbb{H}_+ \to \mathbb{R}$ to the three-dimensional wave equation with vanishing Cauchy data and Dirichlet boundary condition

$$\begin{cases} (\partial_t^2 - \Delta_3)u = 0 \quad \text{in } \mathbb{H}_+ = \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}_+, \\ u|_{t=0} = \partial_t u|_{t=0} = 0, \\ u|_{x_3=0} = k, \end{cases} \tag{P}$$

where $t \in \mathbb{R}_+$, $x' = (x_1, x_2) \in \mathbb{R}^2$, $x_3 \in \mathbb{R}_+$, $x = (x', x_3)$, with $k\colon \mathbb{R}^2 \to \mathbb{R}$ independent of t, describes the propagation of waves in a half-space generated by the excitation k. The problem (P) is a mixed initial-boundary value problem in the sense of J. Hadamard. In almost all textbooks on partial differential equations one finds conditions on the initial values ensuring the existence and regularity of solutions to the pure Cauchy problem.

In contrast, the discussion of solvability of mixed problems for hyperbolic equations is confined to some monographs, e.g., [21, 12.9 Mixed problems, pp. 162–179, 181], [40] or [48, 4.9 Mixed problems, pp. 318–337].

Let us solve the mixed problem (P) by the "continuation-reflection" method. Assume for the moment that $u \in \mathcal{E}^2(\mathbb{H}_+)$, $k \in \mathcal{E}^0(\mathbb{R}^2)$ and denote by U the continuation of u by zero with respect to t, and the continuation as an odd function with respect to x_3, i.e.,

$$U(t, x) := Y(t) \operatorname{sign}(x_3) u(t, x', |x_3|),$$

where Y denotes the Heaviside function. Furthermore,

$$K(t, x) := Y(t) k(x').$$

The distributional jump formula and the conditions in (P) yield the distributional equation

$$\partial_t^2 U - \Delta_3 U = -2Y(t) \otimes k(x') \otimes \delta'(x_3) \tag{E}$$

on $\mathbb{R}^4$.

The solution $U \in \mathcal{D}'(\mathbb{R}^4)$ to equation (E) then yields a solution u to problem (P) (in the sense of [20]) by restriction.

The uniquely determined solution U with support in the half-space $t \geq 0$ is given by

$$U = -2E_3 * (Y(t) \otimes k(x') \otimes \delta'(x_3))$$

where

$$E_3 = \frac{1}{4\pi t} \delta(t - |x|) = \frac{1}{4\pi} \partial_t \left(\frac{Y(t - |x|)}{|x|} \right)$$

is the retarded fundamental solution to the three-dimensional wave operator $\partial_t^2 - \Delta_3$.

Since

$$E_3 \in \mathcal{D}'_{[0,\infty),t}(\mathcal{E}'_x) \quad \text{and} \quad Y(t) \otimes k(x') \otimes \delta'(x_3) \in \mathcal{D}'_{[0,\infty),t} \otimes \mathcal{D}'_{x'} \otimes \mathcal{E}'_{x_3},$$

the convolution is well-defined and furnishes

$$U \in \mathcal{D}'_{[0,\infty),t}(\mathcal{D}'_x).$$

Furthermore, by shifting the differentiations, we get an explicit formula for U:

$$U = -\frac{1}{2\pi}\left(\frac{Y(t-|x|)}{|x|}\right) * (\delta(t)\otimes k(x')\otimes\delta'(x_3))$$
$$= -\frac{1}{2\pi}\partial_{x_3}\left(\frac{Y(t-|x|)}{|x|}\right) * (\delta(t)\otimes k(x')\otimes\delta(x_3)).$$

From

$$\partial_{x_3}\left(\frac{Y(t-|x|)}{|x|}\right) = -\frac{x_3}{t^2}\delta(t-|x|) - \frac{x_3}{|x|^3}Y(t-|x|),$$

we derive the representation

$$U = \frac{1}{2\pi}\left(\frac{x_3}{t^2}\delta(t-|x|) + \frac{x_3}{|x|^3}Y(t-|x|)\right) * (\delta(t)\otimes k(x')\otimes\delta(x_3)).$$

If $k \in L^1_{\mathrm{loc}}(\mathbb{R}^2_{x'})$ then U is the sum of a simple integral on $(t^2-x_3^2)^{1/2}\mathbb{S}_1$ and a double integral over $(t^2-x_3^2)^{1/2}B_1$.

The regularity of the distributional solution U in dependence on the regularity of k, e.g.

$$k \in \mathcal{E}(\mathbb{R}^2) \quad \text{or} \quad k \in \mathcal{D}_{L^2}(\mathbb{R}^2)$$

("$\mathcal{E}$-well-posedness" or "H^∞-well-posedness") can be discussed by considering the convolution of vector-valued distributions. Due to the "factors" $\delta(t)$ and $\delta(x_3)$, essentially, we have

$$U = \frac{1}{2\pi}\partial_{x_3}\left(\frac{Y(t-|x|)}{|x|}\right) *_{x'} k(x').$$

As

$$\partial_{x_3}\left(\frac{Y(t-|x|)}{|x|}\right) \in \mathcal{E}_t(\mathcal{E}'_x) = (\mathcal{E}_t \mathbin{\widehat{\otimes}} \mathcal{E}'_{x_3})(\mathcal{E}'_{x'})$$

and as the convolutions

$$*\colon \mathcal{E}'\times\mathcal{E}\to\mathcal{E} \quad \text{and} \quad *\colon \mathcal{E}'\times\mathcal{D}_{L^2}\to\mathcal{D}_{L^2}$$

are well-defined and hypocontinuous (e.g. [24, p. 407] and [47, p. 204]), we conclude by means of Theorem 7.1 in [44, p. 30] that $U \in (\mathcal{E}_t \mathbin{\widehat{\otimes}} \mathcal{E}'_{x_3})(\mathcal{E}_{x'})$ and $U \in (\mathcal{E}_t \mathbin{\widehat{\otimes}} \mathcal{E}'_{x_3})(\mathcal{D}_{L^2,x'})$, respectively. Furthermore the maps $k \mapsto U$ are continuous, i.e., the distributional solution U to (E) depends continuously on the boundary value k.

A much more general result is contained in Theorem 4 in [40, p. 102]. Classically the solution u to problem (P) could be constructed by using Green's function (see, e.g. (4.5) in [27, p. 297]) and applying the Kirchhoff formula.

1.2. Overview. There are different approaches to the problem of convolution of vector-valued distributions in the literature. These different approaches led to the following different definitions of vector-valued convolutions:

In the second part of L. Schwartz' treatment of vector-valued distributions [46], vector-valued tensor and scalar products, multiplications and convolutions are considered. Proposition 34 in [46, p. 151] yields a vector-valued convolution for nuclear normal spaces of distributions. There, a convolution map between normal spaces of distributions $\mathcal{H}$, $\mathcal{K}$

and $\mathcal{L}$ is defined as a partially continuous bilinear map the restriction of which coincides with the classical convolution

$$*\colon \mathcal{D}(\mathbb{R}^n) \times \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$$

of test functions. Since $\mathcal{H}$ and $\mathcal{K}$ are normal spaces of distributions and they contain $\mathcal{D}(\mathbb{R}^n)$ as a dense subspace, the convolution map

$$*\colon \mathcal{H} \times \mathcal{K} \to \mathcal{L}$$

is uniquely determined because it is partially continuous.

Let E and F be two separated locally convex spaces. Moreover let $\mathcal{K}$ be a nuclear space satisfying the strict approximation property and suppose its dual space $\mathcal{K}'$ is a nuclear space too. Then, denoting $\mathcal{H}(E) = \mathcal{H} \,\varepsilon\, E = \mathcal{L}_\varepsilon(\mathcal{H}'_c, E)$ and $\mathcal{K}(F)$ and $\mathcal{L}(E \widehat{\otimes}_\pi E)$ analogously, a vector-valued convolution in the sense of Proposition 34 in [46, p. 151] is a map

$$*_\pi\colon \mathcal{H}(E) \times \mathcal{K}(F) \to \mathcal{L}(E \widehat{\otimes}_\pi F)$$

satisfying the consistency property

$$(S \otimes e) *_\pi (T \otimes f) = (S * T) \otimes (e \otimes f)$$

for decomposed elements $S \in \mathcal{H}$, $T \in \mathcal{K}$, $e \in E$, and $f \in F$.

Here the appearance of the π-topology implies that the above construction can be composed with continuous bilinear maps on $E \times F$ (but not with just hypocontinuous ones).

In the more instructive presentation [44] of the theory of vector-valued distributions, L. Schwartz gave a slightly different definition of a vector-valued convolution (Theorem 14.1 in [44, p. 72]): Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be three nuclear separated locally convex spaces and

$$u\colon \mathcal{H} \times \mathcal{K} \to \mathcal{L}$$

a hypocontinuous bilinear map. Additionally let E, F and G be three Banach spaces and $\theta\colon E \times F \to G$ a continuous bilinear map. Then there is a unique hypocontinuous bilinear map

$$\overset{u}{\theta}\colon \mathcal{H}(E) \times \mathcal{K}(F) \to \mathcal{L}(G)$$

satisfying the consistency property $(S \otimes e) \overset{u}{\theta} (T \otimes f) = (SuT) \otimes \theta(e, f)$ for decomposed elements $S \in \mathcal{H}$, $T \in \mathcal{K}$, $e \in E$, and $f \in F$.

In [18, p. 558] Y. Hirata and R. Shiraishi used a different approach to the convolution of vector-valued distributions. They only considered the convolution of two distributions defined by the vector-valued tensor product and did not construct a convolution map. Starting with the convolution of kernel distributions, they defined the following ι-convolution: Let E and F be two separated locally convex spaces. Additionally let $S \in \mathcal{D}'(E)$ and $T \in \mathcal{D}'(F)$ be two vector-valued distributions. Their ι-convolution $S *_\iota T \in \mathcal{D}'(E \widehat{\otimes}_\iota F)$ is a distribution with values in the quasi-completion of the inductive tensor product (see [13, p. 73]) of E and F.

In [50] R. Shiraishi defined the θ-convolution of vector-valued distributions. Let E, F and G be three separated locally convex spaces, where G is assumed to be quasi-complete, and $\theta\colon E \times F \to G$ a partially continuous bilinear map. The θ-convolution $S *_\theta T \in \mathcal{D}'(G)$

of two vector-valued distributions $S \in \mathcal{D}'(E)$ and $T \in \mathcal{D}'(F)$ is defined analogously to the ι-convolution and the convolution of two scalar-valued distributions defined by a tensor product due to L. Schwartz in [42, exposé n° 22, p. 1] and J. Horváth in [22, (1), p. 185] and [23, p. 8]. In contrast to the ι-convolution in [18], in [50] R. Shiraishi considered θ-convolution mappings between spaces of vector-valued distributions.

In this article, we will compare these convolution mappings, the general convolution of vector-valued distributions and the more difficult convolutions where only one of the occurring bilinear maps has to be hypocontinuous and the other one is allowed to be just partially continuous ("convolution générale" in [46] and [50]).

Moreover we present applications of those convolvability results which appeared in the literature.

A preliminary version of this article is contained (as Chapter 3) in the first author's thesis [2] written under supervision of the second author.

1.3. Preliminaries and notation. For the convenience of the reader, we recall the basic concepts of the theory of topological tensor products. For the details, we refer to [45], [46] and [13].

Definition 1 (see [43, p. 91] and [45, p. 5]). Let E be a separated locally convex space and $A \subset E$ a subset of E. The set A is *quasi-closed* if it contains all points of E which adhere to a bounded subset of A. The *quasi-closed hull* of A is the intersection of all quasi-closed subsets of E containing A. An element $x \in E$ is contained in the *strict adherence* of A if it is contained in the quasi-closed hull of A. The space E has the *strict approximation property* if the identity mapping $\mathrm{id}_E\colon E \to E$ is contained in the strict adherence of $E' \otimes E$ in $\mathcal{L}_c(E, E)$, the space of linear and continuous mappings on E endowed with the topology of uniform convergence on absolutely convex compact sets. By $\widehat{E}$ we denote the quasi-completion of E. By E'_c we denote the dual space of E equipped with the topology of uniform convergence on compact absolutely convex subsets of E. A is *strictly dense* in E if E coincides with the strict adherence of A.

Remark. An example of a quasi-closed but not closed set is $\ell^1 \subset (\ell^1, \widehat{\sigma(\ell^1, c_0)})$ (see ex. 4 in [24, p. 220] and [43, p. 91]).

According to Corollary 2 in [36, p. 742] there is a Fréchet–Schwartz space with the approximation property but without the strict approximation property.

Let E, F and G be separated locally convex spaces.

Definition 2 (see [45, p. 9]). Let $\mathfrak{S}$ and $\mathfrak{T}$ be saturated families of subsets of E and F, respectively. A bilinear mapping $b\colon E \times F \to G$ is called $\mathfrak{S}$-$\mathfrak{T}$-*hypocontinuous* if its restrictions to $A \times F$ and $E \times B$ are continuous for all $A \in \mathfrak{S}$ and $B \in \mathfrak{T}$.

Note that this definition of hypocontinuity differs from the classical one (see [8, p. III.30]) as classically the elements of $\mathfrak{S}$ and $\mathfrak{T}$ are assumed to be bounded. The reason for this difference seems to be that L. Schwartz wanted continuity to be a particular case of $\mathfrak{S}$-$\mathfrak{T}$-hypocontinuity.

A bilinear map $b\colon E \times F \to G$ is called ι-, γ-, β-*continuous* if it is hypocontinuous with respect to the finite-dimensional subsets, the absolutely convex relatively compact subsets and the bounded subsets of E and F, respectively. It is called π-continuous if it is continuous, i.e., hypocontinuous with respect to all subsets of E and F. We use the following five topologies on the tensor product $E \otimes F$ starting with the finest one of them:

The ι-, γ-, β- and π-*topology* are the finest locally convex topologies on $E \otimes F$ such that the canonical mapping $E \times F \to E \otimes F \colon (e, f) \mapsto e \otimes f$ is ι-, γ-, β- and π-continuous, respectively.

The ε-*topology* on $E \otimes F$ is the topology induced by the ε-product $E \otimes F \subset E \,\varepsilon\, F$ of the spaces E and F. The ε-*product* $E \,\varepsilon\, F$ is defined as the space of all bilinear forms on $E'_c \times F'_c$ which are hypocontinuous with respect to equicontinuous subsets of E' and F'. It is endowed with the topology of uniform convergence on products of equicontinuous sets (see [45, p. 18]). A bilinear map $b \colon E \times F \to G$ is called ε-continuous if the corresponding linear map is continuous on $E \otimes_\varepsilon F$.

We denote by $E \otimes_\lambda F$ the space $E \otimes F$ endowed with the λ-topology, where $\lambda = \iota$, γ, β, π or ε.

In his thesis [13], A. Grothendieck calls the ι-topology the inductive topology (chap. I, p. 74) and the π-topology the projective topology (chap. I, p. 32). He also considers the ε-topology (chap. I, p. 89) which is often called the injective topology ([26, p. 343]). These notations seems to originate in the following properties of these topologies:

- The ι-topology commutes with inductive limits (Proposition 14 in [13, chap. I, p. 46]),
- the π-topology commutes with projective limits (2. Theorem in [26, p. 332]) and
- the ε-(tensor-)product of injective linear maps is again injective (2. Corollary in [26, p. 348]).

By $E \mathbin{\widehat{\otimes}}_\lambda F$ and $E \mathbin{\hat{\widehat{\otimes}}}_\lambda F$, we denote the quasi-completion and the completion of $E \otimes_\lambda F$, respectively for $\lambda = \iota$, γ, β, π or ε.

In the following, we denote concrete spaces of distributions or functions as in [47], [45] and [46]. For a list of the most prominent spaces of smooth functions and distributions see the Appendix.

As indicated before, the relation $\varepsilon \le \pi \le \beta \le \gamma \le \iota$ holds. Let us now discuss examples where these topologies are distinct:

$\mathcal{B}_c \otimes_\beta \mathcal{D}'_{L^1} \neq \mathcal{B}_c \otimes_\gamma \mathcal{D}'_{L^1}$ as the convolution mapping $* \colon \mathcal{B}_c \times \mathcal{D}'_{L^1} \to \mathcal{B}_c$ is γ-continuous but not β-continuous. A detailed discussion of the topology of $\mathcal{B}_c = (\mathcal{D}'_{L^1})'_c$ is found in [9].

$\mathcal{O}_C \otimes_\pi \mathcal{O}_C \neq \mathcal{O}_C \otimes_\beta \mathcal{O}_C$ as the tensor product mapping $\otimes \colon \mathcal{O}_{C,x} \times \mathcal{O}_{C,y} \to \mathcal{O}_{C,x,y}$ is β-continuous but not π-continuous (see Proposition 6 in [28, p. 8]).

$L^2 \otimes_\varepsilon L^2 \neq L^2 \otimes_\pi L^2$ as the mapping $L^2 \times L^2 \to \mathbb{C}, (f, g) \mapsto \langle f, \mathrm{vp}\frac{1}{x} * g \rangle$ is π-continuous but not ε-continuous.

If $\mathcal{H}$ is a space of distributions (see Definition 12) and E a locally convex space the space $\mathcal{H}(E)$ of vector-valued (i.e., E-valued) distributions in $\mathcal{H}$ is defined by $\mathcal{H}(E) = \mathcal{H} \varepsilon E$.

Let E be a separated locally convex space and $U \subset E$ an absolutely convex neighbourhood of zero. E_U is the space $E/(\bigcap_{\lambda \neq 0} \lambda U)$ equipped with the topology generated by the Minkowski functional of U (see [13, chap. I, p. 7] or [24, p. 208]). E_U is a normed space. By $\mathrm{can}_U \colon E \to E_U$ we denote the canonical mapping into this quotient.

Let us exemplify this definition. Choose

$$E = \mathcal{E}(\mathbb{R}) \quad \text{and} \quad U = \{f \in \mathcal{E}(\mathbb{R});\ \|f|_{[-1,1]}\|_\infty \le 1\}.$$

Then $\bigcap_{\lambda \neq 0} \lambda U = \{f \in \mathcal{E}(\mathbb{R});\ f|_{[-1,1]} = 0\}$ and

$$\mathcal{E}(\mathbb{R})_U = \{[h]; f, g \in [h] \Leftrightarrow (f-g)|_{[-1,1]} = 0\} \cong (\mathcal{E}([-1,1]), \|\cdot\|_\infty).$$

The space $\mathcal{E}(\mathbb{R})_U$, being isomorphic to a dense subspace of $(\mathcal{C}([-1,1]), \|\cdot\|_\infty)$, is a non-complete normed space.

Definition 3 ([46, p. 15]). Let E and F be separated locally convex spaces. A λ-*set* in a separated locally convex space is a bounded finite-dimensional set and a relatively compact set for $\lambda = \iota$ and $\lambda = \gamma$, respectively. It is a bounded set for $\lambda = \beta$, π or ε.

A bounded subset $\Xi \subset E \mathbin{\widehat{\otimes}}_\lambda F$ (or $E \mathbin{\widehat{\otimes}}_\lambda F$) is called σ-τ-*decomposable* if

1. Ξ is contained in the absolutely convex closed envelope of the tensor product of a σ-set of E and a τ-set of F for $\sigma = \iota$, γ or β and $\tau = \iota$, γ or β.
2. For all absolutely convex neighbourhoods of zero U in E the image $(\mathrm{can}_U \otimes \mathrm{id})(\Xi) \subset \widehat{E}_U \mathbin{\widehat{\otimes}}_\lambda F$ of Ξ is β-τ-decomposable for $\sigma = \pi$ or ε and $\tau = \iota$, γ or β.
3. For all absolutely convex neighbourhoods of zero V in F the image

$$(\mathrm{id} \otimes \mathrm{can}_V)(\Xi) \subset E \mathbin{\widehat{\otimes}}_\lambda \widehat{F}_V$$

   of Ξ is β-τ-decomposable for $\sigma = \pi$ or ε and $\tau = \iota$, γ or β.
4. Ξ is an arbitrary bounded set for $\sigma = \pi$ or ε and $\tau = \pi$ or ε.

Definition 4 (see [50, p. 178] and [50, p. 195]). Let E and F be locally convex spaces. A linear map $f\colon E \to F$ is called *quasi-continuous* (in the sense of R. Shiraishi) if for all bounded subsets $B \subset E$ the restriction $f|_B$ is continuous.

If $\mathcal{H}$ is a normal space of distributions (see Definition 12), then $\mathcal{H}$ is said to be $\dot{\mathcal{B}}$-*normal* if for all $\alpha \in \dot{\mathcal{B}}$ the multiplication $\mathcal{H} \to \mathcal{H}, S \mapsto \alpha S$ is well defined and for all bounded subsets $B \subset \dot{\mathcal{B}}$ the set of linear mappings $\{\mathcal{H} \to \mathcal{H}, S \mapsto \alpha S;\ \alpha \in B\}$ is equicontinuous.

Note that every quasi-continuous linear map is also sequentially continuous as convergent sequences together with their limit are bounded sets. Corollaire 1 in [14, p. 69] shows that for linear maps on (DF)-spaces quasi-continuity and continuity are equivalent. The same holds true for linear maps on bornological spaces.

An example of a linear map which is quasi-continuous but not continuous is given by $\mathcal{D}_F(\mathbb{R}) \to \mathbb{C}, \varphi \mapsto \langle \varphi, \sum_{k=0}^\infty \frac{\mathrm{d}^k}{\mathrm{d}x^k} \delta_k \rangle$, $\mathcal{D}_F(\mathbb{R})$ denoting the space $\mathcal{D}(\mathbb{R})$ of test functions endowed with the projective limit topology of the spectrum $(\mathcal{D}^m(\mathbb{R}^n), m \in \mathbb{N}_0)$, where $\mathcal{D}^m(\mathbb{R}^n)$ is the space of m-times continuously differentiable functions with compact support. Being a distribution of infinite order, it is discontinuous. It is quasi-continuous as for every bounded set $B \subset \mathcal{D}_F(\mathbb{R})$ there exists an m such that $\operatorname{supp} \varphi \subset [-m, m]$ for all $\varphi \in B$.

In [50, p. 174], R. Shiraishi points out that "most normal spaces of distributions $\mathcal{H}$ referred to as examples in" [45] and [46] are $\dot{\mathcal{B}}$-normal. On the other hand Theorem 2, (v) in [51, p. 16] combined with Proposition 11, (iii) in [51, p. 13] yields an example of a space which is not $\dot{\mathcal{B}}$-normal.

2. Existence and continuity of vector-valued bilinear maps

The theorems on vector-valued bilinear operations, where at least one of the defining maps has to be continuous (called “convolution élémentaire” by L. Schwartz) in [46] are based on the “Théorèmes de croisement”. We will compare the theorems in [50] on such operations with results that follow from this fundamental proposition.

THEOREM 1 (Proposition 2 in [46, p. 18], “Théorèmes de croisement”). *Let L, M, U and V be separated locally convex spaces. There is a bilinear mapping*

$$\Gamma_{\mu,\lambda}\colon (L \mathbin{\widehat{\otimes}_\lambda} U) \times (M \mathbin{\varepsilon} V) \to (L \mathbin{\widehat{\otimes}_\mu} M) \mathbin{\varepsilon} (U \mathbin{\widehat{\otimes}_\lambda} V), \qquad (\xi, \eta) \mapsto \Gamma_{\mu,\lambda}(\xi, \eta),$$

for $\lambda, \mu \le \gamma$. The mapping $\Gamma_{\mu,\lambda}$ has the following properties:

1. Consistency. *On $(L \otimes U) \times (M \otimes V)$ it coincides with the canonical mapping into $(L \otimes M) \otimes (U \otimes V)$ and, denoting by $(M \otimes V)_0$ the strict adherence of $M \otimes V$ in $M \varepsilon V$, $\Gamma_{\mu,\lambda}$ maps $(L \mathbin{\widehat{\otimes}_\lambda} U) \times (M \otimes V)_0$ into $(L \mathbin{\widehat{\otimes}_\mu} M) \mathbin{\widehat{\otimes}_\varepsilon} (U \mathbin{\widehat{\otimes}_\lambda} V)$, i.e., the diagram*

$$\begin{array}{ccc}
(L \mathbin{\widehat{\otimes}_\lambda} U) \times (M \mathbin{\varepsilon} V) & \xrightarrow{\Gamma_{\mu,\lambda}} & (L \mathbin{\widehat{\otimes}_\mu} M) \mathbin{\varepsilon} (U \mathbin{\widehat{\otimes}_\lambda} V) \\
\uparrow & & \uparrow \\
(L \mathbin{\widehat{\otimes}_\lambda} U) \times (M \otimes V)_0 & \longrightarrow & (L \mathbin{\widehat{\otimes}_\mu} M) \mathbin{\widehat{\otimes}_\varepsilon} (U \mathbin{\widehat{\otimes}_\lambda} V) \\
\uparrow & & \uparrow \\
(L \otimes U) \times (M \otimes V) & \xrightarrow{\text{can}} & L \otimes M \otimes U \otimes V
\end{array}$$

   commutes.
2. Uniqueness. *It is the unique bilinear mapping coinciding with the canonical map on $(L \otimes U) \times (M \otimes V)$ and continuous with respect to $\xi \in L \mathbin{\widehat{\otimes}_\lambda} U$ for fixed $\eta \in (M \otimes V)_0$ and continuous with respect to $\eta \in (M \otimes V)_0$ for fixed $\xi \in L \otimes U$.*
3. Compatibility. *The mapping $\Gamma_{\mu,\lambda}$ is compatible with continuous linear mappings on L, M, U and V and with refinement of the topologies λ and μ, i.e., for topologies $\mu \le \sigma$, $\lambda \le \tau$ and continuous linear maps $u\colon L \to L$, $v\colon U \to U$, $w\colon M \to M$ and $r\colon V \to V$ the diagram*

$$\begin{array}{ccc}
(L \mathbin{\widehat{\otimes}_\lambda} U) \times (M \mathbin{\varepsilon} V) & \xrightarrow{\Gamma_{\mu,\lambda}} & (L \mathbin{\widehat{\otimes}_\mu} M) \mathbin{\varepsilon} (U \mathbin{\widehat{\otimes}_\lambda} V) \\
\uparrow{\scriptstyle (u\otimes v)\times(w\otimes r)} & & \uparrow{\scriptstyle (u\otimes v)\varepsilon(w\otimes r)} \\
(L \mathbin{\widehat{\otimes}_\tau} U) \times (M \mathbin{\varepsilon} V) & \xrightarrow{\Gamma_{\sigma,\tau}} & (L \mathbin{\widehat{\otimes}_\sigma} M) \mathbin{\varepsilon} (U \mathbin{\widehat{\otimes}_\tau} V)
\end{array}$$

   commutes.

4. Continuity properties.

   (a) *If L, M, U and V are quasi-complete and ξ converges to zero in $(L \mathbin{\widehat{\otimes}_\lambda} U)$ and η stays in a $\nu = \sup\{\mu, \lambda\}$-set of $M \varepsilon V$ then $\Gamma_{\mu,\lambda}(\xi, \eta)$ converges to zero as well.*

   (b) *If η converges to zero in $M \varepsilon V$ and ξ is in a μ-λ-decomposable subset of $(L \mathbin{\widehat{\otimes}_\lambda} U)$ then $\Gamma_{\mu,\lambda}(\xi, \eta)$ converges to zero.*

   (c) *If $\lambda, \mu \le \pi$ then $\Gamma_{\mu,\lambda}$ is continuous.*

   (d) *$\Gamma_{\varepsilon,\varepsilon}$ is ε-continuous. It is the restriction of the canonical mapping*

$$(\widehat{L} \mathbin{\varepsilon} \widehat{U}) \times (M \mathbin{\varepsilon} V) \subset (\widehat{L} \mathbin{\varepsilon} \widehat{U}) \times (\widehat{M} \mathbin{\varepsilon} \widehat{V}) \to (\widehat{L} \mathbin{\varepsilon} \widehat{M}) \mathbin{\varepsilon} (\widehat{U} \mathbin{\varepsilon} \widehat{V}).$$

(e) *If L, M, U and V are quasi-complete, the mapping $\Gamma_{\mu,\varepsilon}$ is μ-continuous and $\Gamma_{\varepsilon,\lambda}$ is λ-continuous.*

The results also hold if $\frown$, $\widehat{\otimes}$ and strict adherence are replaced by $\widehat{\ }$, $\widehat{\widehat{\otimes}}$ and adherence, respectively.

Remark. Whereas the convolutions in [46] which are constructed by Theorem 1 refer to nuclear normal spaces of distributions, Theorem 1 itself is applicable also to other spaces like $\mathcal{H}(\Omega)$, c_0, ℓ^p, s, s', $\dot{\mathcal{B}}^m$ or $\mathcal{E}^m$, m finite.

Application 1. We can use Theorem 1 to prove a generalization of Proposition 4, (i) in [32, p. 373].

Let $\Omega_1 \subset \mathbb{C}^{m_1}$ and $\Omega_2 \subset \mathbb{C}^{m_2}$ open sets and $\mathcal{H}(\Omega_i)$ the space of holomorphic functions on Ω_i carrying the topology induced by $\mathcal{E}(\Omega_i)$, $i = 1, 2$. Theorem 1 yields a unique continuous bilinear map

$$\Gamma_{\varepsilon,\varepsilon}\colon (\mathcal{H}(\Omega_1) \mathbin{\widehat{\widehat{\otimes}}} \mathcal{D}'_{L^p}) \times (\mathcal{H}(\Omega_2) \mathbin{\widehat{\widehat{\otimes}}} \mathcal{D}'_{L^q}) \to \mathcal{H}(\Omega_1 \times \Omega_2) \mathbin{\widehat{\widehat{\otimes}}} (\mathcal{D}'_{L^p} \mathbin{\widehat{\widehat{\otimes}}}_\pi \mathcal{D}'_{L^q}),$$

satisfying the consistency property

$$\Gamma_{\varepsilon,\varepsilon}\,(f(z)S(x), g(w)T(y)) = f(z)\cdot g(w)\, S(x) \otimes T(y)$$

for $f \in \mathcal{H}(\Omega_1)$, $g \in \mathcal{H}(\Omega_2)$, $S \in \mathcal{D}'_{L^p}$ and $T \in \mathcal{D}'_{L^q}$.

In this case we have $L = \mathcal{H}(\Omega_1)$, $M = \mathcal{H}(\Omega_2)$, $U = \mathcal{D}'_{L^p}$, $V = \mathcal{D}'_{L^q}$, $\lambda = \varepsilon = \pi$ and $\mu = \varepsilon = \pi$.

The vector-valued convolution mapping

$$\overset{\otimes}{*}\colon (\mathcal{H}(\Omega_1) \mathbin{\widehat{\widehat{\otimes}}}_\pi \mathcal{D}'_{L^p}) \times (\mathcal{H}(\Omega_2) \mathbin{\widehat{\widehat{\otimes}}}_\pi \mathcal{D}'_{L^q}) \to \mathcal{H}(\Omega_1 \times \Omega_2) \mathbin{\widehat{\widehat{\otimes}}}_\pi \mathcal{D}'_{L^r}$$

satisfying the consistency property

$$\overset{\otimes}{*}\,((f(z)S(x), g(w)T(y))) = (f(z)\cdot g(w))(S(x) * T(y))$$

is given by

$$\overset{\otimes}{*} = (\mathrm{id}\,\varepsilon\,\tilde{*}) \circ \Gamma_{\varepsilon,\varepsilon}$$

where id denotes the identity mapping on $\mathcal{H}(\Omega_1 \times \Omega_2)$ and $\tilde{*}\colon \mathcal{D}'_{L^p} \mathbin{\widehat{\widehat{\otimes}}} \mathcal{D}'_{L^q} \to \mathcal{D}'_{L^r}$ the continuation of the linear map corresponding to the continuous convolution mapping

$$\mathcal{D}'_{L^p} \times \mathcal{D}'_{L^q} \to \mathcal{D}'_{L^r}$$

for $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$.

A classical formulation: If $F\colon \Omega_1 \to \mathcal{D}'_{L^p}$ and $G\colon \Omega_2 \to \mathcal{D}'_{L^q}$ are holomorphic distribution-valued functions then the function

$$F \overset{\otimes}{*} G\colon \Omega_1 \times \Omega_2 \to \mathcal{D}'_{L^r}, \qquad (z, w) \mapsto F(z) * G(w),$$

is a $\mathcal{D}'_{L^r}$-valued holomorphic function. Moreover the mapping

$$\overset{\otimes}{*}\colon \mathcal{H}(\Omega_1; \mathcal{D}'_{L^p}) \times \mathcal{H}(\Omega_2; \mathcal{D}'_{L^q}) \to \mathcal{H}(\Omega_1 \times \Omega_2; \mathcal{D}'_{L^r}), \qquad (F, G) \mapsto F \overset{\otimes}{*} G,$$

is continuous.

An elementary approach—without the "Théorèmes de croisement"—can be found in Proposition 6 in [1, p. 61].

Application 2 (Proposition 19 in [1, p. 61]). Let $\Omega_1 \subset \mathbb{R}^{n_1}$ and $\Omega_2 \subset \mathbb{R}^{n_2}$ be open sets and $m \in \mathbb{N}$. Combining the continuous tensor product mapping

$$\mathcal{E}^m(\Omega_1) \times \mathcal{E}^m(\Omega_2) \to \mathcal{E}^m(\Omega_1 \times \Omega_2)$$

and the hypocontinuous multiplication map

$$\mathcal{D}_{L^p} \times \mathcal{D}'_{L^q} \to \mathcal{D}'_{L^r},$$

where $r \geq 1$ and $\frac{1}{r} \leq \frac{1}{q} + \frac{1}{p}$, leads to the hypocontinuous multiplication

$$\mathcal{E}^m(\Omega_1; \mathcal{D}_{L^p}) \times \mathcal{E}^m(\Omega_2; \mathcal{D}'_{L^q}) \to \mathcal{E}^m(\Omega_1 \times \Omega_2; \mathcal{D}'_{L^r}),$$
$$(f, g) \mapsto [(x, y) \mapsto f(x) \cdot g(y)].$$

Analogously the continuous convolution

$$\mathcal{E}^m(\Omega_1; \mathcal{D}'_{L^p}) \times \mathcal{E}^m(\Omega_2; \mathcal{D}'_{L^q}) \to \mathcal{E}^m(\Omega_1 \times \Omega_2; \mathcal{D}'_{L^r}),$$
$$(f, g) \mapsto [(x, y) \mapsto f(x) * g(y)],$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0$, is the combination of the tensor product and the continuous convolution mapping $\mathcal{D}'_{L^p} \times \mathcal{D}'_{L^q} \to \mathcal{D}'_{L^r}$.

In both cases none of the results on vector-valued operations in [44], [45], [46] and [50] can be applied as the space $\mathcal{E}^m(\Omega)$ is not nuclear, but the "Théorèmes de croisement" can be used to prove its existence and hypocontinuity analogously to the considerations in Chapter 4: By Theorem 1 (Proposition 2 in [46, p. 18]), there is a hypocontinuous bilinear map

$$\Gamma_{\beta,\varepsilon} \colon (\mathcal{E}^m(\Omega_1) \widehat{\otimes}_\varepsilon E) \times (\mathcal{E}^m(\Omega_2) \,\varepsilon\, F) \to (E \widehat{\otimes}_\beta F) \,\varepsilon\, (\mathcal{E}^m(\Omega_1) \,\varepsilon\, \mathcal{E}^m(\Omega_2))$$

which coincides on $(\mathcal{E}^m(\Omega_1) \otimes E) \times (\mathcal{E}^m(\Omega_2) \otimes F)$ with the canonical mapping into the tensor product.

By [43, p. 106] and the Corollaire in [45, p. 10] the tensor product identities

$$\mathcal{E}^m(\Omega_1) \widehat{\otimes}_\varepsilon E = \mathcal{E}^m(\Omega_1; E) \quad \text{and} \quad \mathcal{E}^m(\Omega_2) \,\varepsilon\, F = \mathcal{E}^m(\Omega_2; F)$$

hold for all quasi-complete separated locally convex spaces. By Proposition 12 in [43, p. 113],

$$\mathcal{E}^m(\Omega_1) \,\varepsilon\, \mathcal{E}^m(\Omega_2) = \mathcal{E}^{m,m}(\Omega_1 \times \Omega_2)$$

where $\mathcal{E}^{m,m}(\Omega_1 \times \Omega_2)$ denotes the space of functions $f \colon \Omega_1 \times \Omega_2 \to \mathbb{C}$ such that all derivatives $\partial_x^\alpha \partial_y^\beta f(x, y)$ where $|\alpha|, |\beta| \leq m$ exist and are continuous. It is easy to check that $\mathcal{E}^{m,m}(\Omega_1 \times \Omega_2)$ is contained in $\mathcal{E}^m(\Omega_1 \times \Omega_2)$ with a finer topology. If b is the multiplication or the convolution above (or more generally a hypcontinuous bilinear map between the quasi-complete spaces E, F into the quasi-complete space G) then the preceding vector-valued operations are defined by

$$(\mathrm{id} \,\varepsilon\, \tilde{b}) \circ \Gamma_{\beta,\varepsilon} \colon \mathcal{E}^m(\Omega_1; E) \times \mathcal{E}^m(\Omega_2; F) \to \mathcal{E}^m(\Omega_1 \times \Omega_2)(G).$$

The continuity of the above convolution map can be proved by replacing $\Gamma_{\beta,\varepsilon}$ by $\Gamma_{\pi,\varepsilon}$, which is a continuous bilinear map. As in Proposition 18 in [1, p. 75], hypocontinuous maps

$$\mathcal{E}^m(\Omega; E) \times \mathcal{E}^m(\Omega; F) \to \mathcal{E}^m(\Omega; G)$$

combining the multiplication in $\mathcal{E}^m(\Omega)$ and hypocontinuous bilinear maps $b\colon E \times F \to G$ can be constructed by restricting the map $(\mathrm{id}\,\varepsilon\,\tilde{b}) \circ \Gamma_{\beta,\varepsilon}$ defined above to the diagonal in $\Omega \times \Omega$.

Further applications are given in Chapter 4 (Proposition 9) and in Chapter 3.4 (Application 16).

3. Vector-valued convolutions

In this chapter, we present the propositions on vector-valued convolutions based on the combination of the bilinear maps found in the literature. We discuss their relation to each other and to the "Théorèmes de croisement". We moreover give a result on the combination of two hypocontinuous bilinear maps which is somehow "intermediate" between the "convolution élémentaire" and the "convolution générale" of L. Schwartz.

3.1. Convolutions where at least one map is continuous ("convolution élémentaire"). The case where at least one of the bilinear maps $L \times M \to N$ and $U \times V \to W$ is continuous is the easiest case as in the "Théorèmes de croisement" the λ-topology appears twice: in the first factor of the pre-image domain $L \widehat{\otimes}_\lambda U$ and in the image domain $(L \widehat{\otimes}_\mu M)\,\varepsilon\,(U \widehat{\otimes}_\lambda V)$. As we want to have the space $L\,\varepsilon\,U$ in the pre-image domain, $\lambda = \pi$ has the advantage that for nuclear spaces U satisfying the strict approximation property we have $L \widehat{\otimes}_\pi U = L\,\varepsilon\,U$ if L and U are quasi-complete. Therefore it is sufficient to place restrictions on L and M or on U and V and to keep the other two free of any restrictions, which seems to be the goal in [46] and in [50]. For topologies other than the π-topology this is not possible.

Theorem 2 (Corollaire to Proposition 3 in [46, p. 38]). *Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be three quasi-complete distribution spaces, E, F and G three separated locally convex spaces, G quasi-complete. Moreover let $\mathcal{H}$ be a nuclear space with the strict approximation property and let $\mathcal{H}'_b = \mathcal{H}'_c$ be nuclear as well. Let*

$$u\colon \mathcal{H} \times \mathcal{K} \to \mathcal{L}, \qquad (S,T) \mapsto u(S,T),$$

be a μ-continuous bilinear map with $\mu \le \gamma$ and $\theta\colon E \times F \to G$ a continuous bilinear map. There is a (unique if $\mathcal{K}$ has the approximation property) bilinear map

$$\overset{u}{\underset{\theta}{}} : \mathcal{H}(E) \times \mathcal{K}(F) \to \mathcal{L}(G), \qquad (S,T) \mapsto \overset{u}{\underset{\theta}{}}(S,T),$$

such that $\overset{u}{\underset{\theta}{}}((S \otimes e),(T \otimes f)) = u(S,T) \otimes \theta(e,f)$ for all $S \in \mathcal{H}$, $T \in \mathcal{K}$, $e \in E$ and $f \in F$. Moreover $\overset{u}{\underset{\theta}{}}$ is hypocontinuous with respect to bounded subsets of $\mathcal{H}(E)$ and μ-sets of $\mathcal{K}(F)$. (Note that $\mathcal{H}(E) = \mathcal{H}\,\varepsilon\,E$ by definition.)

A direct application of this theorem is the following theorem on the convolution of vector-valued distributions.

Theorem 3 (Proposition 34 in [46, p. 151]). *Let $\mathcal{H}$ and $\mathcal{K}$ be normal spaces of distributions and $\mathcal{L}$ be a space of distributions. Moreover let E and F be two separated locally convex spaces. Assume $\mathcal{H}$ is a nuclear space satisfying the strict approximation property* *and admitting a nuclear dual space. Let* $*\colon \mathcal{H}\times\mathcal{K}\to\mathcal{L}$ *be a* μ*-continuous convolution mapping, where* $\mu\le\gamma$*. There is a (unique if* $\mathcal{K}$ *has the approximation property) bilinear map*

$$\overset{*}{\otimes}\colon \mathcal{H}(E)\times\mathcal{K}(F)\to\mathcal{L}(E\mathbin{\widehat{\otimes}_\pi}F),\qquad (S,T)\mapsto \overset{*}{\otimes}(S,T),$$

such that $\overset{*}{\otimes}((S\otimes e),(T\otimes f)) = S*T\otimes e\otimes f$ *for all* $S\in\mathcal{H}$*,* $T\in\mathcal{K}$*,* $e\in E$ *and* $f\in F$*. Moreover the convolution mapping* $\overset{*}{\otimes}$ *is hypocontinuous with respect to bounded subsets of* $\mathcal{H}(E)$ *and* μ*-sets of* $\mathcal{K}(F)$*.*

Application 3 (Proposition 3.7.3 in [35, p. 115] and 9 Proposition in [34, p. 147]). There is a unique hypcontinuous convolution mapping

$$\mathcal{D}'_{[a,\infty)}(\mathcal{O}'_C)\times\mathcal{D}'_{[b,\infty)}(\mathcal{S}')\to\mathcal{D}'_{[a+b)}(\mathcal{S}')$$

which is consistent for decomposed elements by Theorem 2 in [50, p. 196] (Theorem 5) or Proposition 39 in [46, p. 167] (Theorem 11). This convolution mapping is of importance in the theory of (systems of) quasihyperbolic partial differential equations considered in [34] and [35].

The most instructive theorem on the convolution of vector-valued distributions is the following:

Theorem 4 (Theorem 14.1 in [44, p. 72]). *Let* $\mathcal{H}$*,* $\mathcal{K}$*,* $\mathcal{L}$ *be three locally convex separated complete vector spaces all of which are nuclear and have nuclear dual spaces. Additionally let* $u\colon\mathcal{H}\times\mathcal{K}\to\mathcal{L}$ *be a bilinear map, hypocontinuous with respect to bounded subsets of* $\mathcal{H}$ *and* $\mathcal{K}$*. Let* E*,* F *and* G *be three Banach spaces with a continuous bilinear map* $\theta\colon E\times F\to G$*. Then there exists a unique bilinear map* $\overset{u}{\theta}\colon\mathcal{H}(E)\times\mathcal{K}(F)\to\mathcal{L}(G)$ *which satisfies*

1. $(S\otimes e)\overset{u}{\theta}(T\otimes f) = u(S,T)\otimes\theta(e,f)$ *for all* $e\in E$*,* $f\in F$*,* $S\in\mathcal{H}$ *and* $T\in\mathcal{K}$*.*
2. $(S,T)\mapsto S\overset{u}{\theta}T$ *is separately continuous in* S *and* T*.*

Moreover $\overset{u}{\theta}$ *has the following supplementary properties:*

3. $\overset{u}{\theta}$ *is hypocontinuous with respect to bounded subsets of* $\mathcal{H}(E)$ *and* $\mathcal{K}(F)$*.*
4. $S\overset{u}{\theta}T = (\mathrm{id}_\mathcal{L}\,\varepsilon\,\tilde\theta)(U_S\otimes\mathrm{id}_F)(T) = (\mathrm{id}_\mathcal{L}\,\varepsilon\,\tilde\theta)(\mathrm{id}_E\otimes U_T)(S)$ *where* $\mathrm{id}_\mathcal{L}$*,* id_E *and* id_F *are the respective identity mappings and* U_S*,* U_T *and* $\tilde\theta$ *are defined as follows.* $U_S\colon\mathcal{K}\to\mathcal{L}(E)$ *is defined by* $U_S(T)=u(S,T)$ *for any* $T\in\mathcal{K}$ *and* $U_T\colon\mathcal{H}\to\mathcal{L}(E)$ *by* $U_T(S)=u(S,T)$ *for any* $S\in\mathcal{H}$*. The map* θ *gives rise to a continuous linear map* θ' *of* $E\otimes_\pi F$ *in* G *which can be extended to a continuous linear map of* $E\mathbin{\widehat{\otimes}_\pi}F$ *in* G*, and* $\tilde\theta$ *denotes this extended map.*

Application 4. In [46, p. 154] L. Schwartz considered the following application: Let L, M and N be Banach spaces and

$$\theta\colon\mathcal{L}(L,M)\times\mathcal{L}(M,N)\to\mathcal{L}(L,N),\qquad (u,v)\mapsto v\circ u.$$

Then θ is a continuous bilinear mapping and hence Theorem 4 yields the existence of a hypocontinuous bilinear convolution map

$$\overset{*}{\theta}\colon\mathcal{D}'_+(\mathcal{L}(L,M))\times\mathcal{D}'_+(\mathcal{L}(M,N))\to\mathcal{D}'_+(\mathcal{L}(L,N)).$$

This mapping is of importance in the theory of semi-group distributions (see [30] and [52, p. 79ff]).

Application 5 (Compléments (I) in [31, pp. 250–251]). Let V be a Hilbert space and $m\colon V \times V \to \mathcal{O}'_C$ be a continuous sequilinear mapping. Setting

$$_V\langle (M\varphi)(v), u\rangle_V = {}_{\mathcal{O}_C}\langle \varphi, m(u,v)\rangle_{\mathcal{O}'_C}$$

for $\varphi \in \mathcal{O}_C$ and $u, v \in V$, we define a linear and continuous mapping

$$M\colon \mathcal{O}_C \to \mathcal{L}(V,V),$$

hence $M \in \mathcal{L}_b(\mathcal{O}_C, \mathcal{L}(V,V)) = \mathcal{O}'_C \mathbin{\widehat{\otimes}} \mathcal{L}(V,V)$. By Theorem 4 there is a hypocontinuous bilinear mapping

$$\overset{\circ}{*}\colon \mathcal{S}'(V) \times \mathcal{O}'_C(\mathcal{L}(V,V)) \to \mathcal{S}'(V), (S, M) \mapsto S \overset{\circ}{*} M,$$

which is given by the formula

$$S \overset{\circ}{*} M = \int_{\mathbb{R}^n} M(x-y)S(y)\,\mathrm{d}y$$

using the functional notation explained in Definition 9. This mapping finally allows us to "continue" the mapping m to

$$\hat{m}\colon \mathcal{S}'(V) \times V \to \mathcal{S}', \qquad (u, v) \mapsto \hat{m}(u,v) := \int_{\mathbb{R}^n} {}_V\langle u(y), Mv(x-y)\rangle_V\,\mathrm{d}y.$$

Comparison 1. Theorem 14.1 in [44, p. 72] is a special case of the Corollary to Proposition 3 in [46, p. 38], i.e., Theorem 2. Let us compare the proof of Theorem 14.1 with the one of Proposition 3 in [46, p. 37]. The construction of the bilinear map $\overset{u}{\theta}$ heavily relies on the nuclearity of $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ and the associativity of the π-tensor product. Therefore the construction in the proof of Proposition 2 in [46, p. 18] is more complicated because for non-nuclear or non-complete spaces $\mathcal{L}$ we do not necessarily have

$$\mathcal{L}\,\varepsilon\,(E \mathbin{\widehat{\otimes}_\pi} F) = \mathcal{L} \mathbin{\widehat{\otimes}_\pi} (E \mathbin{\widehat{\otimes}_\pi} F) = (\mathcal{L} \mathbin{\widehat{\otimes}_\pi} E) \mathbin{\widehat{\otimes}_\pi} F$$

and, probably more important, the equalities do not hold in general for topologies other than the π-topology (or the ε-topology). Note that the completed ι-tensor product is in general not even associative.

Moreover the proof uses Theorem 7.1 of [44, p. 30] to construct a hypocontinuous bilinear map

$$\mathcal{H}(E) \times \mathcal{K} \to \mathcal{L}(E).$$

This construction is equivalent to the one in Proposition 21 bis. of [45, p. 70], in the special case of multiplication, and the general one in the "Cas où certains des espaces sont identiques au corps des scalaires" in [46, p. 32] which is part of the "Théorèmes de croisement". In contrast, the construction in the proof of Proposition 2 in [46, p. 17] is not an application of Theorem 7.1 in [44, p. 30], but a generalisation.

If we compare the structure of the vector-valued bilinear mapping in Theorem 14.1 of [44, p. 72]

$$\mathcal{H}(E) \times (\mathcal{K} \mathbin{\widehat{\otimes}_\pi} F) \to \mathcal{L}(E \mathbin{\widehat{\otimes}_\pi} F),$$

after using the above isomorphisms, with the one in Proposition 2 of [46, p. 18] applied to the situation in Theorem 14.1

$$(\mathcal{H} \widehat{\otimes}_\pi E) \times \mathcal{K}(F) \to \mathcal{L}(E \widehat{\otimes}_\pi F)$$

the factors in the pre-image domain are "interchanged". This "permutation" is also reflected in the proof of hypocontinuity: In the proof of (iii) in [44, p. 73] it is shown that for $S \to 0$ in $\mathcal{H}(E)$ and bounded subsets $A \subset \mathcal{K}$ and $B_\alpha \subset F$ the filter $S \overset{u}{\underset{\theta}{}} T \to 0$ converges uniformly with respect to $T \in \overline{\mathrm{ac}(A \otimes B_\alpha)}$. As F is a Banach space one has $F = F_U$ for all absolutely convex neighbourhoods of zero $U \subset F$. Hence $S \overset{u}{\underset{\theta}{}} T \to 0$ converges to zero if $S \to 0$ in $\mathcal{H}(E)$ and T stays in a β-π-decomposable set. This is exactly what is shown in the proof of Proposition 2, 5° (our Theorem 1) in [46, p. 29]. Subsequently ([44, p. 75]) it is shown that the nuclearity of $\mathcal{K}'_b = \mathcal{K}'_c$ yields the β-π-decomposability of every bounded subset of $\mathcal{K}(F)$. This proof is more or less identical with the one of Proposition 1, 4° in [46, p. 16].

The proof for hypocontinuity with respect to bounded subsets of $\mathcal{H}(E)$ differs from the one in [46] as both proofs heavily use the structure of the mapping's construction which is different.

We now compare R. Shiraishi's theorems on the "elementary convolution", i.e., vector-valued convolutions where at least one of the occurring bilinear maps is continuous, in [50] with Proposition 3 of [46, p. 37] and investigate the relation of these results to L. Schwartz' "Théorèmes de croisement".

In contrast to L. Schwartz, whose theorems on vector-valued convolutions are all based on the extension of a hypocontinuous bilinear map to spaces of vector-valued distributions by the "Théorèmes de croisement", R. Shiraishi's theorems use the concept of θ-convolvability of two vector-valued distributions which is analogous to the convolution of two distributions defined by tensor products (for more details, we refer to Section 3.4). It turns out that, although these approaches are completely different, results similar to the ones of R. Shiraishi can be obtained by using the "Théorèmes de croisement", at least for μ-continuous bilinear mappings θ, $\mu \leq \gamma$. If this continuity assumption is absent the vector-valued convolution mappings constructed in [50] are in general not even partially continuous.

Theorem 5 (Theorem 2 in [50, p. 196]). *Let $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$ be three normal spaces of distributions on $\mathbb{R}^n$, $\mathcal{L}$ being assumed to be complete, and E, F, G be three separated locally convex spaces, G being assumed to be quasi-complete. Assume that $\mathcal{H}$ is nuclear, $\dot{\mathcal{B}}$-normal and $\mathcal{H} \otimes E$ is strictly dense in $\mathcal{H}(E)$. Suppose that the convolution map $*\colon \mathcal{H} \times \mathcal{K} \to \mathcal{L}$ is continuous and the bilinear map $\theta\colon E \times F \to G$ is separately continuous. Then any $S \in \mathcal{H}(E)$ and $T \in \mathcal{K}(F)$ are $*_\theta$-composable (see Definition 10) and $S \overset{*}{\underset{\theta}{}} T \in \mathcal{L}(G)$.*

1. *For fixed $S \in \mathcal{H}(E)$, the linear map $T \mapsto S \overset{*}{\underset{\theta}{}} T$ of $\mathcal{K}(F)$ into $\mathcal{L}(G)$ is quasi-continuous. Moreover if $\mathcal{L}'_c$ is barrelled, the linear map $S \mapsto S \overset{*}{\underset{\theta}{}} T$ of $\mathcal{H}(E)$ into $\mathcal{L}(G)$ is also quasi-continuous.*
2. *If θ is hypocontinuous with respect to absolutely convex compact sets of E, then, for fixed $T \in \mathcal{H}(E)$, the linear map $S \mapsto S \overset{*}{\underset{\theta}{}} T$ is uniformly continuous with respect to equicontinuous subsets of $\mathcal{L}_\varepsilon(E'_c, \mathcal{H}) = \mathcal{H} \,\varepsilon\, E$.*

3. *If θ is hypocontinuous with respect to compact absolutely convex sets of F, then, for fixed $T \in \mathcal{H}(E)$, the linear map $S \mapsto S \overset{*}{\theta} T$ is uniformly quasi-continuous with respect to equicontinuous subsets of $\mathcal{K}(F) = \mathcal{L}_\varepsilon(F'_c, \mathcal{K})$. Further, if F is quasi-complete, any compact subset of $\mathcal{K}(F)$ is such an equicontinuous subset, therefore the linear map $S \mapsto S \overset{*}{\theta} T$ is uniformly quasi-continuous with respect to compact subsets of $\mathcal{K}(F)$.*
4. *If θ is hypocontinuous with respect to bounded subsets of E and F, then so is $\overset{*}{\theta}$.*
5. *If θ is continuous, then so is $\overset{*}{\theta}$.*

If we restrict ourselves to quasi-complete $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ and to at least γ-continuous bilinear maps $\theta\colon E \times F \to G$, we obtain a similar result, where the assumption that $\mathcal{H}$ has to be $\dot{\mathcal{B}}$-normal can be dropped, using L. Schwartz' "Théorèmes de croisement":

Proposition 1. *Let $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$ be three quasi-complete spaces of distributions (or more generally three quasi-complete separated locally convex spaces) and E, F, G be three separated locally convex spaces, G being assumed to be quasi-complete. Assume that $\mathcal{H}$ is nuclear and $\mathcal{H} \otimes E$ is a strictly dense subspace of $\mathcal{H}(E) = \mathcal{H} \,\varepsilon\, E$, e.g., assume that $\mathcal{H}$ has the strict approximation property. Suppose that the convolution map $*\colon \mathcal{H} \times \mathcal{K} \to \mathcal{L}$ is continuous and the bilinear map $\theta\colon E \times F \to G$ is μ-continuous for $\mu \leq \gamma$. Then there exists a μ-continuous bilinear mapping $\overset{*}{\theta}\colon \mathcal{H}(E) \times \mathcal{K}(F) \to \mathcal{L}(G)$ which is consistent with respect to decomposed elements, i.e.,*

$$(S \otimes e) \overset{*}{\theta} (T \otimes f) = (S * T) \otimes \theta(e, f)$$

for all $S \in \mathcal{H}$, $T \in \mathcal{K}$, $e \in E$ and $f \in F$. If $\mathcal{K}$ satisfies the approximation property, or more generally if $\mathcal{K} \otimes F$ is a dense subspace of $\mathcal{K}(F)$, then $\overset{*}{\theta}$ is the uniquely determined partially continuous bilinear map consistent with respect to decomposed elements.

The proof is closely related to the proof of Proposition 3 in [46, p. 37]:

Proof. Let us temporarily assume that E and F are quasi-complete. By Proposition 2 in [46, p. 18] ("Théorèmes de croisement", our Theorem 1), there is a bilinear map

$$\Gamma_{\mu,\varepsilon}\colon (E \mathbin{\widehat{\otimes}_\varepsilon} \mathcal{H}) \times (F \,\varepsilon\, \mathcal{K}) \to (E \mathbin{\widehat{\otimes}_\mu} F) \,\varepsilon\, (\mathcal{H} \mathbin{\widehat{\otimes}_\varepsilon} \mathcal{K}).$$

As $\mathcal{H}$ is nuclear, $\mathcal{H} \mathbin{\widehat{\otimes}_\varepsilon} \mathcal{K} = \mathcal{H} \mathbin{\widehat{\otimes}_\pi} \mathcal{K}$. The equality $\mathcal{H} \mathbin{\widehat{\otimes}_\varepsilon} E = \mathcal{H}(E)$ holds as $\mathcal{H} \otimes E \subset \mathcal{H}(E)$ is strictly dense. As the ε-product is symmetrical, $\Gamma_{\mu,\varepsilon}$ is a bilinear mapping from $\mathcal{H}(E) \times \mathcal{K}(F)$ into $(\mathcal{H} \mathbin{\widehat{\otimes}_\pi} \mathcal{K}) \,\varepsilon\, (E \mathbin{\widehat{\otimes}_\mu} F)$. As the convolution is continuous, we obtain the mapping

$$(\bar{*} \otimes \mathrm{id}) \circ \Gamma_{\mu,\varepsilon}\colon \mathcal{H}(E) \times \mathcal{K}(F) \to \mathcal{L}(E \mathbin{\widehat{\otimes}_\mu} F),$$

where $\bar{*}$ denotes the mapping $\mathcal{H}(E) \to \mathcal{L}$ associated to $*$. According to Proposition 2, 5° (our Theorem 1) in [46, p. 18] the mapping $\Gamma_{\mu,\varepsilon}$ is μ-continuous and hence also $(\bar{*} \otimes \mathrm{id}) \circ \Gamma_{\mu,\varepsilon}$. As θ is μ-continuous the mapping

$$\overset{*}{\theta} := (\mathrm{id} \otimes \bar{\theta}) \circ (\bar{*} \otimes \mathrm{id}) \circ \Gamma_{\mu,\varepsilon}\colon \mathcal{H}(E) \times \mathcal{K}(F) \to \mathcal{L}(G)$$

is well-defined and μ-continuous.

If we assume that E and F are not quasi-complete, then for all $S \in \mathcal{H}(E)$ and $T \in \mathcal{K}(F)$ we have $S \in \mathcal{H}(\widehat{E})$ and $T \in \mathcal{K}(\widehat{F})$. Hence we can define the convolution $S \overset{*}{\theta} T \in \mathcal{L}(\widehat{E} \mathbin{\widehat{\otimes}_\mu} \widehat{F}) = \mathcal{L}(E \mathbin{\widehat{\otimes}_\mu} F)$. The map $(S, T) \mapsto S \overset{*}{\theta} T$ satisfies all assertions stated above.

By Theorem 1 (Proposition 2 in [46, p. 18]) the restriction of $\Gamma_{\mu,\varepsilon}$ to $(\mathcal{H}\otimes E)\times(\mathcal{K}\otimes F)$ coincides with the canonical mapping into $(\mathcal{H}\otimes\mathcal{K})\otimes(E\otimes F)$ and hence

$$(S\otimes e) \overset{*}{\underset{\theta}{}} (T\otimes f) = (S*T)\otimes\theta(e,f)$$

for all $S\in\mathcal{H}$, $T\in\mathcal{K}$, $e\in E$ and $f\in F$. If $\mathcal{K}\otimes F$ is a dense subspace of $\mathcal{K}(F)$ then $\overset{*}{\underset{\theta}{}}$ is the only partially continuous bilinear map satisfying this assertion by Proposition 7 in [8, p. III.32]. ∎

COMPARISON 2. In contrast to Theorem 2 in [50, p. 196] the only necessary assumption on $*\colon\mathcal{H}\times\mathcal{K}\to\mathcal{L}$ in Proposition 1 is that it is a continuous bilinear map. The assumption "convolution-mapping" is not needed in the proof, i.e., Proposition 1 could also be stated as:

PROPOSITION 1′. *Let $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$ be three quasi-complete separated locally convex spaces and E, F, G be three separated locally convex spaces, G being assumed to be quasi-complete. Assume that $\mathcal{H}$ is nuclear and $\mathcal{H}\otimes E$ is a strictly dense subspace of $\mathcal{H}(E)$. Suppose that the bilinear map $u\colon\mathcal{H}\times\mathcal{K}\to\mathcal{L}$ is continuous and the bilinear map $\theta\colon E\times F\to G$ is μ-continuous for $\mu\le\gamma$. Then there exists a μ-continuous bilinear mapping $\overset{u}{\underset{\theta}{}}\colon\mathcal{H}(E)\times\mathcal{K}(F)\to\mathcal{L}(G)$ which is consistent with respect to decomposed elements. If $\mathcal{K}$ has the approximation property then $\overset{u}{\underset{\theta}{}}$ is the unique partially continuous bilinear map satisfying this property.*

COMPARISON 2 (continued). Therefore the main differences between Proposition 1/1′ and Theorem 2 in [50, p. 196] are

- restriction to at least γ-continuous maps θ,
- restriction to quasi-complete spaces $\mathcal{H}$ and $\mathcal{K}$,
- generalisation to bilinear maps other than convolution and
- absence of the $\dot{\mathcal{B}}$-normality-assumption on $\mathcal{H}$.

If we compare the proof of Proposition 1 with the proof of Proposition 3 in [46, p. 37] the main difference is that the bilinear map is based on $\Gamma_{\mu,\varepsilon}$ and not on $\Gamma_{\mu,\pi}$. The reason for this difference is that the mapping

$$\Gamma_{\mu,\pi}\colon(E\,\widehat{\otimes}_\pi\,\mathcal{H})\times(F\,\varepsilon\,\mathcal{K})\to(E\,\widehat{\otimes}_\mu\,F)\,\varepsilon\,(\mathcal{H}\,\widehat{\otimes}_\pi\,\mathcal{K})$$

yields by symmetry a mapping

$$\tilde{\Gamma}_{\mu,\pi}\colon(\mathcal{H}\,\widehat{\otimes}_\pi\,E)\times(\mathcal{K}\,\varepsilon\,F)\to(\mathcal{H}\,\widehat{\otimes}_\pi\,\mathcal{K})\,\varepsilon\,(E\,\widehat{\otimes}_\mu\,F)$$

such that $\tilde{\Gamma}_{\mu,\pi}(S,T)\to0$ if $T\to0$ and S stays in a π-μ-decomposable subset of $\mathcal{H}\,\widehat{\otimes}_\pi\,E$. L. Schwartz shows in [46, p. 16] that every bounded subset of $\mathcal{H}(E)$ is μ-π-decomposable using the fact that every linear map from a nuclear space ($\mathcal{H}'_c=\mathcal{H}'_b$) into a Banach space ($\hat{E}_U$) is nuclear. This argument cannot be applied to π-μ-decomposability as in this case $\hat{\mathcal{H}}_U$ is a Banach space and E does not have a nuclear dual space in general.

APPLICATION 6 (Example (a) in [17, p. 148], Distributions in $\mathcal{O}'_C$ and $\mathcal{S}'$ are $\mathcal{S}'$-convolvable). Let $S\in\mathcal{O}'_C$ and $T\in\mathcal{S}'$. The regularization property implies

$$S(x-y)\in\mathcal{S}_y\,\widehat{\otimes}_\pi\,\mathcal{S}'_x\quad\text{and}\quad T(y-z)\in\mathcal{O}_{C,y}\,\widehat{\otimes}_\pi\,\mathcal{S}'_z.$$

By Proposition 1 there is a unique hypocontinuous bilinear mapping

$$\dot{\otimes} : (\mathcal{S}_y \mathbin{\widehat{\otimes}_\pi} \mathcal{S}'_x) \times (\mathcal{O}_{C,y} \mathbin{\widehat{\otimes}_\pi} \mathcal{S}'_z) \to \mathcal{S}_y(\mathcal{S}'_x \mathbin{\widehat{\otimes}_\pi} \mathcal{S}'_z)$$

extending the canonical mapping on the tensor products. Hence $S(x-y)T(y-z)$ is in $\mathcal{S}_y \mathbin{\widehat{\otimes}_\pi} \mathcal{S}'_{x,z}$ and therefore for all $\varphi \in \mathcal{S}_x$ and all $\psi \in \mathcal{S}_z$,

$$(\varphi * \check{S})(\psi * T) \in \mathcal{S},$$

i.e., S and T are $\mathcal{S}'$-convolvable.

Application 7 (cf. Example 1 in [50, pp. 208–209]). Let E, F and G be three quasi-complete separated locally convex spaces and $\theta\colon E \times F \to G$ a hypocontinuous bilinear map. There is a unique hypocontinuous bilinear mapping

$$\dot{\theta} : \mathcal{O}_M(E) \times \mathcal{O}_M(F) \to \mathcal{O}_M(G)$$

which is consistent for decomposed elements. By vector-valued Fourier transform, existence and uniqueness of this mapping are equivalent to that of the mapping

$$\overset{*}{\theta} : \mathcal{O}'_C(E) \times \mathcal{O}'_C(F) \to \mathcal{O}'_C(G).$$

Application 8 ((2.1.11) Prop. in [25, p. 83] and Proposition 4 in [1, p. 59]). Let $\Lambda \subset \mathbb{C}^n$ and $\Omega, \Xi \subset \mathbb{R}^d$ be open sets. The hypocontinuous convolution map

$$\begin{aligned}\dot{*} : \mathcal{H}(\Lambda; \mathcal{E}'^s(\mathbb{R}^d)) \times \mathcal{H}(\Lambda; \mathcal{D}'^t(\mathbb{R}^d)) &\to \mathcal{H}(\Lambda; \mathcal{D}'^{s+t}(\mathbb{R}^d)),\\ (f,g) &\mapsto [\lambda \mapsto f(\lambda) * g(\lambda)],\end{aligned}$$

is the uniquely determined partially continuous bilinear map which is consistent with respect to decomposed elements.

As the multiplication $\cdot\colon \mathcal{H}(\Lambda) \times \mathcal{H}(\Lambda) \to \mathcal{H}(\Lambda)$ is continuous, existence and hypocontinuity of this map can be deduced from Proposition 1. Theorem 2 in [50, p. 196] cannot be applied here as $\mathcal{H}(\Lambda)$ is not $\dot{\mathcal{B}}$-normal and the bilinear map in the pre-image domain is multiplication and not convolution. Existence and continuity of the multiplication map

$$\dot{\cdot} : \mathcal{H}(\Lambda; \mathcal{E}^m(\Omega)) \times \mathcal{H}(\Lambda; \mathcal{D}'^m(\Omega)) \to \mathcal{H}(\Lambda; \mathcal{D}'^m(\Omega))$$

(see (2.1.6) Prop. (i) in [25, p. 79] and Proposition 4 in [1, p. 59]) and of the tensor product mapping

$$\dot{\otimes} : \mathcal{H}(\Lambda; \mathcal{D}'^s(\Omega)) \times \mathcal{H}(\Lambda; \mathcal{D}'^t(\Xi)) \to \mathcal{H}(\Lambda; \mathcal{D}'^{s+t}(\Omega \times \Xi))$$

(see (2.1.9) Prop. (i) in [25, p. 81] and Proposition 4 in [1, p. 59]) can be deduced from Theorem 2 (Corollaire to Proposition 3 in [46, p. 38]) and Proposition 1′.

Application 9 ((2.1.12) in [25, p. 83] and Proposition 5 in [1, p. 60]). Let $\Lambda \subset \mathbb{C}^n$ and $\Omega \subset \mathbb{R}^d$ be an open set. The hypocontinuous convolution map

$$\begin{aligned}\dot{*} : \mathcal{H}(\Lambda; \mathcal{S}') \times \mathcal{H}(\Lambda; \mathcal{O}'_C) &\to \mathcal{H}(\Lambda; \mathcal{S}'),\\ (f,g) &\mapsto [\lambda \mapsto f(\lambda) * g(\lambda)],\end{aligned}$$

is the uniquely determined partially continuous bilinear map which is consistent with respect to decomposed elements.

As in Application 8, existence and hypocontinuity of this map can be deduced from Proposition 1′. Existence and continuity of the multiplication map

$$\overset{\cdot}{\cdot} : \mathcal{H}(\Lambda;\mathcal{S}') \times \mathcal{H}(\Lambda;\mathcal{O}_M) \to \mathcal{H}(\Lambda;\mathcal{S}')$$

can also be deduced from Proposition 1′.

For a locally convex space E, we denote by $\mathcal{D}(\mathbb{R}^n;E)$ the space of compactly supported infinitely differentiable functions with values in E, which in general differs from the ε-product of $\mathcal{D}(\mathbb{R}^n)$ and E (see [43, pp. 106, 116] or [46, p. 95]). A direct counterexample can be constructed as follows: Let $0 \neq \varphi \in \mathcal{D}(\mathbb{R}^n)$ be an arbitrary test function. Then it is easy to see that the function $\mathbb{R}^n \to \mathcal{D}(\mathbb{R}^n)$, $x \mapsto [y \mapsto \varphi(x-y)]$, is contained in $\mathcal{E}(\mathbb{R}^n;\mathcal{D}(\mathbb{R}^n))$ and by Théorème 1 in [43, p. 111] and [43, p. 106] we have $\mathcal{E}(\mathbb{R}^n;\mathcal{D}(\mathbb{R}^n)) = \mathcal{E}(\mathbb{R}^n)\varepsilon\mathcal{D}(\mathbb{R}^n)$. On the other hand, as for all $y \in \mathbb{R}^n$ there is an $x \in \mathbb{R}^n$ such that $\varphi(x-y) \neq 0$, the function $\mathbb{R}^n \to \mathcal{E}(\mathbb{R}^n)$, $y \mapsto [x \mapsto \varphi(x-y)]$, is not compactly supported. Hence $\mathcal{D}(\mathbb{R}^n;\mathcal{E}(\mathbb{R}^n)) \neq \mathcal{D}(\mathbb{R}^n)\,\varepsilon\,\mathcal{E}(\mathbb{R}^n)$.

Application 10 (cf. Theorem A in [5] and Proposition 4.5 in [11]). Given a continuous bilinear mapping $\theta\colon E_1 \times E_2 \to F$ with three quasi-complete locally convex spaces, the vector-valued convolution mapping

$$\underset{\theta}{*} : \mathcal{D}(\mathbb{R}^n;E_1) \times \mathcal{D}(\mathbb{R}^n;E_2) \to \mathcal{D}(\mathbb{R}^n;F)$$

is continuous if F has the countable neighbourhood property (see [10, 11]):

As for all compact sets $K \subset \mathbb{R}^n$ the embedding $\mathcal{D}_K(\mathbb{R}^n) \hookrightarrow \mathcal{D}(\mathbb{R}^n)$ is continuous, Proposition 1 in [45, p. 20] yields the continuity of $\mathcal{D}_K(\mathbb{R}^n;E_i) \hookrightarrow \mathcal{D}(\mathbb{R}^n)\,\varepsilon\,E_i$. Hence for $i = 1,2$ the space $\mathcal{D}(\mathbb{R}^n;E_i) = \lim_{K\to}\mathcal{D}_K(\mathbb{R}^n;E_i)$ is contained in $\mathcal{D}(\mathbb{R}^n)\,\varepsilon\,E_i$ with a finer topology. Therefore by Proposition 1, there is a unique continuous bilinear mapping

$${}_{\theta} : \mathcal{D}(\mathbb{R}^n;E_1) \times \mathcal{D}(\mathbb{R}^n;E_2) \to \mathcal{D}(\mathbb{R}^n)\,\varepsilon\,F.$$

Finally the assertion follows by Proposition 4.5 in [19, p. 65] which yields the identity $\mathcal{D}(\mathbb{R}^n)\,\varepsilon\,F = \lim_{K\to}\mathcal{D}_K(\mathbb{R}^n;F)$ as F has the countable neighbourhood property.

Note that Theorem A in [5] holds true in more general situations, e.g. for spaces defined on Lie groups.

Theorem 6 (Theorem 3 in [50, p. 199]). *Let $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$ be three normal spaces of distributions on $\mathbb{R}^N$. Let E, F, G be three separated locally convex spaces. Assume that $\mathcal{L}$, G are quasi-complete. Further assume that $\mathcal{H}$ is nuclear and $\dot{\mathcal{B}}$-normal. Suppose the convolution map $*\colon \mathcal{H} \times \mathcal{K} \to \mathcal{L}, (S,T) \mapsto S * T$ is defined and γ-continuous. Let $\theta\colon E \times F \to G$ be a continuous bilinear map.*

1. *If $\mathcal{L}$ and G are complete, or if $\mathcal{H}$ or E has the strict approximation property, then any $S \in \mathcal{H}(E)$ and $T \in \mathcal{K}(F)$ are $*_\theta$-composable (see Definition 10) and $S *_\theta T \in \mathcal{L}(G)$ where the map $\mathcal{H}(E) \to \mathcal{L}(G), S \mapsto S *_\theta T$, is continuous. If we further assume that $\mathcal{H}'_c$ is nuclear, then the bilinear map*

$$\mathcal{H}(E) \times \mathcal{K}(F) \to \mathcal{L}(G)$$

is hypocontinuous with respect to bounded subsets of $\mathcal{H}(E)$ and compact subsets of $\mathcal{K}(F)$ whenever $\mathcal{H}$, $\mathcal{K}$ are quasi-complete.

2. *If the convolution map* $*\colon \mathcal{H}\times\mathcal{K}\to\mathcal{L}$ *is* β*-continuous and if* $\mathcal{L}$ *and* G *are complete or if* $\mathcal{H}$ *or* E *has the strict approximation property and if* $\mathcal{H}$ *is quasi-complete and* $\mathcal{H}'_c$ *is nuclear, then the bilinear map*

$$\mathcal{H}(E)\times\mathcal{K}(F)\to\mathcal{L}(G),\qquad (S,T)\mapsto S *_\theta T,$$

   is β*-continuous.*

Comparison 3. The main difference with Theorems 2 and 3 (Propositions 3 and 34 in [46, p. 37 and p. 151]) is that the spaces $\mathcal{H}$ and $\mathcal{K}$ do not have to be quasi-complete. If we assume they are, we can prove the above proposition by means of Theorem 1 (Proposition 2 in [46, p. 18]):

1. If $\mathcal{H}$ or E has the strict approximation property, then $\mathcal{H}\mathbin{\widehat{\otimes}_\varepsilon} E=\mathcal{H}(E)$ and as $\mathcal{H}$ is nuclear, $\mathcal{H}\mathbin{\widehat{\otimes}_\pi} E=\mathcal{H}\mathbin{\widehat{\otimes}_\varepsilon} E=\mathcal{H}(E)$. Proposition 2 in [46, p. 17] yields a bilinear mapping
$$\Gamma_{\gamma,\pi}\colon \mathcal{H}(E)\times\mathcal{K}(F)\to(\mathcal{H}\mathbin{\widehat{\otimes}_\gamma}\mathcal{K})\,\varepsilon\,(E\mathbin{\widehat{\otimes}_\pi} F).$$
   We define $*_\theta := (\bar{*}\,\varepsilon\,\bar\theta)\circ\Gamma_{\gamma,\pi}$ where $\bar\theta\colon \mathcal{H}\mathbin{\widehat{\otimes}_\gamma}\mathcal{K}\to\mathcal{L}$ is the linear map corresponding to the convolution $*\colon\mathcal{H}\times\mathcal{K}\to\mathcal{L}$. The mapping $\mathcal{H}(E)\to\mathcal{L}(G)$, $S\mapsto S*_\theta T$, is continuous by Proposition 2, 1° in [46, p. 17]. If moreover $\mathcal{H}'_c$ is nuclear, then $*_\theta$ is hypocontinuous with respect to bounded subsets of $\mathcal{H}(E)$ and compact subsets of $\mathcal{K}(F)$ by Proposition 2, 1° and 2° in [46, p. 18] (our Theorem 1).
2. If $\mathcal{H}$ or E has the strict approximation property, as $\mathcal{H}$ is nuclear, we have $\mathcal{H}\mathbin{\widehat{\otimes}_\pi} E=\mathcal{H}\mathbin{\widehat{\otimes}_\varepsilon} E=\mathcal{H}(E)$. If $\mathcal{H}'_c$ is nuclear, by Proposition 2 in [46, p. 17] there exists a β-continuous bilinear map
$$\mathcal{H}(E)\times\mathcal{K}(F)\to\mathcal{L}(G).$$

As every nuclear space has the approximation property, we can drop the assumption that $\mathcal{H}$ or E has to have the strict approximation property, if we assume that the spaces $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$ and G are complete.

This proof is nearly identical with the proof of Proposition 3 in [46, p. 37]. If $\mathcal{H}'_c$ is nuclear and $\mathcal{H}$ has the strict approximation property, Theorem 3 in [50, p. 199] is equivalent to Theorem 2 (the Corollaire to Proposition 3 in [46, p. 39]).

In conclusion, the numerous applications in this chapter suggest that L. Schwartz' assertion

> "Les résultats précédentes ne sont pas vraiment utiles que si E et F sont des espaces de Banach; sinon, en effet, les applications bilinéaires qu'on trouve dans la pratique sont seulement hypocontinues, et non continues" ([46, p. 53])

on the operations where one of the maps has to be continuous, is too pessimistic.

3.2. Vector-valued convolutions where both mappings are hypocontinuous. Although most of the convolution mappings appearing in the theory of distributions are hypocontinuous but not continuous, both in [46] and [50] the combination of two hypocontinuous but not continuous bilinear mappings is not treated. In these articles it seems to be the goal to keep the spaces E and F free from any restrictions. Using the "Théorèmes de croisement" with $\lambda = \mu = \beta$ leads to the problem of obtaining the equality $\mathcal{H} \widehat{\otimes}_\beta E = \mathcal{H}(E)$ which is, at least for infinite-dimensional spaces, not possible in general without additional assumptions on both $\mathcal{H}$ and E. By placing some restrictions on both $\mathcal{H}$ and E, it is possible to obtain a result on vector-valued convolutions where both mappings are hypocontinuous:

PROPOSITION 2 (cf. Proposition 1 in [3, p. 6]). *Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be quasi-complete spaces of distributions (or more generally quasi-complete locally convex spaces), where $\mathcal{H}$ is nuclear and satisfies the strict approximation property. Let E, F and G be three locally convex spaces, G quasi-complete, and*

$$u\colon \mathcal{H} \times \mathcal{K} \to \mathcal{L} \quad \text{and} \quad b\colon E \times F \to G$$

be two hypocontinuous bilinear maps. If one of the assumptions

1. *$\mathcal{H}$ and E are Fréchet spaces,*
2. *$\mathcal{H}$ and E are (DF)-spaces,*

is satisfied, there is a hypocontinuous bilinear map

$$\overset{u}{\underset{b}{}} : \mathcal{H}(E) \times \mathcal{K}(F) \to \mathcal{L}(G)$$

satisfying the consistency property

$$\overset{u}{\underset{b}{}} (S \otimes e, T \otimes f) = u(S,T) \otimes b(e,f).$$

If $\mathcal{K}$ satisfies the approximation property, then $\overset{u}{\underset{b}{}}$ is the unique partially continuous bilinear map satisfying this property.

We prove this proposition by an adaptation of the proof of Proposition 3 in [46, p. 37].

Proof. Assume temporarily that E and F are quasi-complete. In Proposition 2 in [46, p. 18] (= Theorem 1) the bilinear map

$$\Gamma_{\beta,\beta}\colon (\mathcal{H} \widehat{\otimes}_\beta E) \times (\mathcal{K} \,\varepsilon\, F) \to (\mathcal{H} \widehat{\otimes}_\beta \mathcal{K}) \,\varepsilon\, (E \widehat{\otimes}_\beta F)$$

is defined. It is hypocontinuous with respect to bounded sets of $\mathcal{K}(F)$ and β-β-decomposable sets of $\mathcal{H} \widehat{\otimes}_\beta E$. On $(\mathcal{H} \otimes E) \times (\mathcal{K} \otimes F) \to \mathcal{H} \otimes \mathcal{K} \otimes E \otimes F$ it coincides with the canonical mapping.

If $\mathcal{H}$ and E are Fréchet spaces then every partially continuous bilinear map on $\mathcal{H} \times E$ is continuous by Theorem 1 in [24, p. 357], hence $\mathcal{H} \widehat{\otimes}_\beta E = \mathcal{H} \widehat{\otimes}_\pi E$.

If $\mathcal{H}$ and E are (DF)-spaces then every hypocontinuous bilinear map on $\mathcal{H} \times E$ is continuous by Théorème 2 in [14, p. 64], hence $\mathcal{H} \widehat{\otimes}_\beta E = \mathcal{H} \widehat{\otimes}_\pi E$.

As $\mathcal{H}$ is nuclear, we have $\mathcal{H} \widehat{\otimes}_\pi E = \mathcal{H} \widehat{\otimes}_\varepsilon E$. Corollaire 1 in [45, p. 47] yields $\mathcal{H} \widehat{\otimes}_\varepsilon E = \mathcal{H} \,\varepsilon\, E$ as $\mathcal{H}$ satisfies the strict approximation property. Hence $\mathcal{H} \widehat{\otimes}_\beta E = \mathcal{H}(E)$.

If $\mathcal{H}$ and E are (DF)-spaces, every bounded subset of $\mathcal{H}(E)$ is β-β-decomposable according to Proposition 1 in [46, p. 16].

If $\mathcal{H}$ and E are Fréchet spaces, Proposition 1 in [46, p. 16] shows that every bounded subset of $\mathcal{H} \widehat{\otimes} E = \mathcal{H} \widehat{\otimes}_\beta E$ is γ-β-decomposable, i.e., for all bounded sets $\Xi \subset \mathcal{H}(E)$ there exists a compact set $A \subset \mathcal{H}$ and a bounded set $B \subset E$ such that $\Xi \subset \overline{\mathrm{ac}(A \otimes B)}$. As A is compact, A is bounded and hence Ξ is β-β-decomposable.

Summing up the above results, we conclude that

$$\Gamma_{\beta,\beta}\colon \mathcal{H}(E) \times \mathcal{K}(F) \to (\mathcal{H} \mathbin{\widehat{\otimes}_\beta} \mathcal{K}) \,\varepsilon\, (E \mathbin{\widehat{\otimes}_\beta} F)$$

is hypocontinuous with respect to bounded sets of $\mathcal{H}(E)$ and $\mathcal{K}(F)$ and therefore also

$$(\tilde{u} \,\varepsilon\, \mathrm{id}) \circ \Gamma_{\beta,\beta}\colon \mathcal{H}(E) \times \mathcal{K}(F) \to \mathcal{L}(E \mathbin{\widehat{\otimes}_\beta} F),$$

where $\tilde{u}$ denotes the linear map $\mathcal{H} \widehat{\otimes} \mathcal{K} \to \mathcal{L}$ associated with u. Additionally,

$$((\tilde{u} \,\varepsilon\, \mathrm{id}) \circ \Gamma_{\beta,\beta})(S \otimes e, T \otimes f) = u(S,T) \otimes (e \otimes f),$$

as $\Gamma_{\beta,\beta}|_{(\mathcal{H}\otimes E)\times(\mathcal{K}\otimes F)} = \mathrm{can}\colon (\mathcal{H} \otimes E) \times (\mathcal{K} \otimes F) \to (\mathcal{H} \otimes \mathcal{K}) \otimes (E \otimes F)$. If $\mathcal{K}$ satisfies the approximation property then $\mathcal{K} \otimes F \subset \mathcal{K}(F)$ is a dense subspace and $(\tilde{u} \,\varepsilon\, \mathrm{id}) \circ \Gamma_{\beta,\beta}$ is the unique partially continuous bilinear mapping satisfying the above consistency property.

Now let E and F be non-quasi-complete locally convex spaces. If $S \in \mathcal{H}(E)$ and $T \in \mathcal{K}(F)$ then a fortiori $S \in \mathcal{H}(\widehat{E})$ and $T \in \mathcal{K}(\widehat{F})$. Hence $((\tilde{u} \,\varepsilon\, \mathrm{id}) \circ \Gamma_{\beta,\beta})(S,T) \in \mathcal{L}(\widehat{E} \mathbin{\widehat{\otimes}_\beta} \widehat{F})$. The identity $\widehat{E} \mathbin{\widehat{\otimes}_\beta} \widehat{F} = E \mathbin{\widehat{\otimes}_\beta} F$ finally yields

$$((\tilde{u} \,\varepsilon\, \mathrm{id}) \circ \Gamma_{\beta,\beta})(S,T) \in \mathcal{L}(E \mathbin{\widehat{\otimes}_\beta} F).$$

The map $(S,T) \mapsto \overset{u}{\underset{b}{}}(S,T) := ((\mathrm{id} \,\varepsilon\, \tilde{b}) \circ (\tilde{u} \,\varepsilon\, \mathrm{id}) \circ \Gamma_{\beta,\beta})(S,T)$ satisfies the properties stated above. ∎

Application 11. In Definition and Proposition 3 in [3, p. 9] the vector-valued multiplication mapping

$$\vdots : (\mathcal{S}'_x \mathbin{\widehat{\otimes}} \mathcal{S}'_z) \times (\mathcal{S}_x \mathbin{\widehat{\otimes}_\pi} \mathcal{O}_{C,z}) \to (\mathcal{O}'_{C,x} \mathbin{\widehat{\otimes}_\pi} \mathcal{S}'_z)$$

is used to define the W_h-transform for temperate distributions. As $\mathcal{S}'$ is a nuclear (DF)-space, existence and uniqueness of this multiplication mapping follow by Proposition 2.

Application 12. The following generalization of Proposition 1 in [18, p. 537] holds true: Let $K(x,y) \in \mathcal{D}'_{xy}(\mathbb{R}^{2n})$ such that $\mathrm{e}^{-\mathrm{i}xy} K(x,y) \in \mathcal{D}'_x \mathbin{\widehat{\otimes}_\pi} \mathcal{S}'_y$, i.e., $\mathrm{e}^{-\mathrm{i}xy} K(x,y)$ is semi-temperate ("semi-temperée") in the sense of [45, p. 123]. Then $K(x,y)$ itself has to be semi-temperate.

Proof. By Proposition 25 in [45, p. 120] there is a unique multiplication mapping

$$(\mathcal{E}_x \mathbin{\widehat{\otimes}_\pi} \mathcal{E}'_z) \times (\mathcal{D}'_x \mathbin{\widehat{\otimes}_\pi} \mathcal{S}'_y) \to \mathcal{D}'_x \mathbin{\widehat{\otimes}_\pi} (\mathcal{S}'_y \mathbin{\widehat{\otimes}_\pi} \mathcal{E}'_z).$$

As $\delta(z-x) \in \mathcal{E}'_z \mathbin{\widehat{\otimes}_\pi} \mathcal{E}_x$ and $\mathrm{e}^{-\mathrm{i}xy} K(x,y) \in \mathcal{D}'_x \mathbin{\widehat{\otimes}_\pi} \mathcal{S}'_y$ we have

$$\mathrm{e}^{-\mathrm{i}xy} K(x,y)\delta(z-x) \in \mathcal{D}'_x(\mathcal{E}'_z \mathbin{\widehat{\otimes}_\pi} \mathcal{S}'_y).$$

By Proposition 2, there is a unique vector-valued multiplication mapping

$$(\mathcal{E}'_z \mathbin{\widehat{\otimes}} \mathcal{S}'_y) \times (\mathcal{E}_z \mathbin{\widehat{\otimes}_\pi} \mathcal{O}_{M,y}) \to (\mathcal{E}'_z \mathbin{\widehat{\otimes}} \mathcal{S}'_y)$$

and finally by Proposition 21 bis in [45, p. 70],

$$(\mathcal{E}'_z \mathbin{\widehat{\otimes}} \mathcal{S}'_y) \mathbin{\widehat{\otimes}_\pi} \mathcal{D}'_x \times (\mathcal{E}_z \mathbin{\widehat{\otimes}_\pi} \mathcal{O}_{M,y}) \to (\mathcal{E}'_z \mathbin{\widehat{\otimes}} \mathcal{S}'_y) \mathbin{\widehat{\otimes}_\pi} \mathcal{D}'_x.$$

As $\mathrm{e}^{\mathrm{i}zy} \in \mathcal{E}_z \mathbin{\widehat{\otimes}_\pi} \mathcal{O}_{M,y}$, we obtain

$$K(x,y)\delta(z-x) \in \mathcal{E}'_z(\mathcal{D}'_x \mathbin{\widehat{\otimes}_\pi} \mathcal{S}'_y)$$

which yields by evaluation at $1 \in \mathcal{E}_z$ that $K(x,y) \in \mathcal{D}'_x \mathbin{\widehat{\otimes}_\pi} \mathcal{S}'_y$, i.e., $K(x,y)$ is semi-temperate itself. ∎

Remark. An immediate consequence of the above results is the following criterion of convolvability of a distribution $S \in \mathcal{D}'(\mathbb{R}^n)$ with the "imaginary Gauß kernel" $\mathrm{e}^{\mathrm{i}a|x|^2}$, given in Satz 4 in [55, p. 473]: $S \in \mathcal{S}'(\mathbb{R}^n)$.

Another application of Proposition 2 is Application 15 in Section 3.3.

In Example 5 in [50, p. 211] R. Shiraishi presents a list of continuous convolution maps between spaces of distributions. In contrast to this list, where the convolution $*\colon \mathcal{D} \times \mathcal{E} \to \mathcal{E}$ is claimed to be continuous, we have the following proposition:

Proposition 3. *The convolution mapping* $*\colon \mathcal{D} \times \mathcal{E} \to \mathcal{E}$ *is hypocontinuous but not continuous.*

Proof. The convolution $*\colon \mathcal{D} \times \mathcal{E} \to \mathcal{E}$ is hypocontinuous as $\mathcal{E}$ is contained in $\mathcal{D}'$ with a finer topology and the regularization $\mathcal{D} \times \mathcal{D}' \to \mathcal{E}$ is hypocontinuous according to Théorème XII in [47, p. 167]. Now we show that the mapping is not continuous:

If it were continuous, for all compact sets $K_1 \subset \mathbb{R}^n$ and all natural numbers $m \in \mathbb{N}_0$ there would exist a continuous seminorm p on $\mathcal{D}$ and a compact set $K_2 \subset \mathbb{R}^n$ together with a natural number $l \in \mathbb{N}_0$ and a constant $C > 0$ such that for all $\varphi \in \mathcal{D}$ and all $f \in \mathcal{E}$,

$$\sup_{x\in K_1} \sup_{|\alpha|\le m} \left| \partial^\alpha \int \varphi(x-y) f(y)\,\mathrm{d}y \right| \le C p(\varphi) \sup_{x\in K_2} \sup_{|\beta|\le l} |\partial^\beta f(x)|.$$

We choose $0 \neq f \in \mathcal{E}$ where $K_2 \cap \operatorname{supp} f = \emptyset$. Then necessarily

$$\int \partial^\alpha \varphi(x-y) f(y)\,\mathrm{d}y = 0$$

for all $x \in K_1$ and all $\varphi \in \mathcal{D}$, hence in particular for all positive φ. This means that the intersection $(K_1 + \operatorname{supp}\varphi) \cap \operatorname{supp} f$ has to be empty for all $\varphi \in \mathcal{D}$, i.e., for all compact sets $\tilde{K} \subset \mathbb{R}^n$ we have $\tilde{K} \cap \operatorname{supp} f = \emptyset$, which means that f has to be the zero function, a contradiction. ∎

Application 13. In view of Proposition 3 existence, uniqueness and hypocontinuity of the vector-valued convolution mapping

$$\overset{*}{*} : \mathcal{E}(\mathcal{D}_{L^1}) \times \mathcal{D}(\mathcal{D}'_{L^p}) \to \mathcal{E}(\mathcal{D}_{L^p})$$

(cf. Example 5 in [50, p. 211]) can best be shown using Proposition 2.

Comparison 4. In the case of two hypocontinuous bilinear mappings a construction similar to Theorem 14.1 in [44, p. 72] is not possible in general unless it is assumed that in addition to $\mathcal{H}$ and $\mathcal{L}$ both E and F are Fréchet spaces or (DF)-spaces, as it is necessary to have $\mathcal{L} \mathbin{\widehat{\otimes}_\pi} (E \mathbin{\widehat{\otimes}_\beta} F) = \mathcal{L} \mathbin{\widehat{\otimes}_\beta} (E \mathbin{\widehat{\otimes}_\beta} F)$. This assumption would lead to a continuous mapping θ (see Corollary 1 in [8, p. III.30] and Théorème 2 in [14, p. 64]). Hence such a construction is not possible in the case where both mappings are hypocontinuous but not continuous.

Comparison 5. If $u\colon \mathcal{H} \times \mathcal{K} \to \mathcal{L}$ is a convolution mapping, a scalar product or and a multiplication mapping, Proposition 2 is essentially a special case of Proposition 38 in [46, p. 159], Proposition 32 in [46, p. 127] and Proposition 20 in [46, p. 83], respectively, as we will see in Comparison 8. We nevertheless state it here and give a proof independent of these propositions, as we do not prove them here and Proposition 2 yields a practical criterion for the existence and hypocontinuity of a convolution mapping of vector-valued distributions in many applications.

3.3. Vector-valued convolutions where only one mapping needs to be hypocontinuous ("convolution générale"). In order to present the results of L. Schwartz and R. Shiraishi on vector-valued convolutions where the convolution map is μ-continuous and the bilinear map in the image domain is allowed to be just partially continuous, we need to introduce some additional concepts and notation.

If E is a separated locally convex space and $B \subset E$ is an absolutely convex bounded subset, then E_B is the normed space generated by B such that B is the unit ball of E_B (see [24, p. 207]).

REMARK. We do not use completing subsets as defined in [46, p. 198]. Whenever the use of completing subsets of a locally convex space F could imply slightly more general statements, we assume that F is a quasi-complete space, as in a quasi-complete space every closed bounded set is complete and every complete bounded absolutely convex set is completing.

DEFINITION 5 ([13, Chap. I, p. 80] and [46, p. 54]). Let E and F be separated locally convex spaces and $u\colon E \to F$ a linear map. The mapping u is called *nuclear* if it is defined by an element of $E'_A \mathbin{\widehat{\otimes}_\pi} F_B$ where A is a weakly closed, absolutely convex equicontinuous set and B is an absolutely convex bounded set such that F_B is complete.

It is called *subnuclear* if for all absolutely convex neighbourhoods of zero $U \subset E$ there exists an absolutely convex neighbourhood of zero $V \subset F$ such that $u(U) \subset V$ and the mapping

$$u_{U,V}\colon \widehat{E_U} \to \widehat{F_V}$$

is nuclear. Note that the notation E'_A and F_B is used in the sense of [24, p. 207] whereas the notation E_U and F_V is used in the sense of p. 10.

DEFINITION 6 ([46, pp. 53–55, p. 199]). Let E and F be separated locally convex spaces. A linear map $u\colon E \to F$ is called *bounded* (in the sense of Grothendieck, see [13, Chap. I, p. 8]) if there exists a neighbourhood U of zero in E such that $u(U)$ is a bounded subset of F. Bounded linear maps are continuous.

A set $H \subset \mathcal{L}(E,F)$ of linear mappings is called *equibounded* if there is a neighbourhood U of zero in E such that $\bigcup_{u\in H} u(U)$ is bounded.

A distribution $S \in \mathcal{D}'(F)$ is called *bounded* if it is bounded as a linear map $\mathcal{D} \to F$. Let $\mathcal{H}$ be a quasi-complete space of distributions. By $\mathcal{H}'(F;\beta)$ we denote the space of all bounded elements of $\mathcal{H}'_c(F)$ endowed with the topology induced by $\mathcal{H}'_c(F)$, i.e., $\mathcal{H}'(F;\beta) = \mathcal{LB}_\varepsilon(\mathcal{H}_c, F)$ (cf. [6, 7]).

REMARK (cf. [46, p. 98]). In the following cases every continuous linear map is already bounded.

1. If $\mathcal{H}$ is Fréchet space and E a (DF)-space, then every continuous linear mapping $\mathcal{H} \to E$ is bounded and every bounded (in $\mathcal{L}_s(\mathcal{H},E)$) set of continuous linear mappings is equibounded by Corollary 2 in [15, p. 168].

Moreover as $\mathcal{H}$ is a Fréchet space, the Alaoglu–Bourbaki theorem ([24, p. 201]) together with Proposition 9.8 in [24, p. 235] and Proposition 6.8 in [24, p. 218] shows that $\mathcal{H}$ carries the γ-topology ([45, p. 17]), i.e., $\mathcal{H} = (\mathcal{H}'_c)'_c$. Therefore

$$\mathcal{H}'_c(E) = \mathcal{L}_\varepsilon(\mathcal{H}, E) = \mathcal{L}_c(\mathcal{H}, E)$$

(Corollaire in [45, p. 36]) and finally by combining these results $\mathcal{H}'_c(E) = \mathcal{LB}_\varepsilon(\mathcal{H}, E)$ and every bounded set of $\mathcal{H}'_c(E)$ is already equibounded.

2. If $\mathcal{H}$ is a dual space of a distinguished Fréchet space and E is a Fréchet space then by Corollary 1 in [15, p. 167] every continuous linear mapping $\mathcal{H} \to E$ is bounded and every bounded (in $\mathcal{L}_s(\mathcal{H}, E)$) set of linear maps is already equibounded.

The space $\mathcal{H}$ carries the γ-topology, which can be seen by the same arguments as above, and therefore

$$\mathcal{H}'_c(E) = \mathcal{L}_\varepsilon(\mathcal{H}, E) = \mathcal{L}_c(\mathcal{H}, E) = \mathcal{LB}_\varepsilon(\mathcal{H}, E),$$

i.e., $\mathcal{H}'_c(E) = \mathcal{H}'_c(E; \beta)$ and every bounded set of $\mathcal{H}'_c(E)$ is already equibounded.

3. If $\mathcal{H}$ is the dual space of a Fréchet space and E is a Fréchet space then, again by Corollary 1 in [15, p. 167], every continuous linear mapping $\mathcal{H} \to E$ is bounded. Additionally every equicontinuous subset $H \subset \mathcal{L}(\mathcal{H}, E)$ is equibounded. Note that as $\mathcal{H}$ is not necessarily barrelled, it is not sufficient to assume that H is bounded for the topology of pointwise convergence as above but one really has to demand H to be equicontinuous.

Moreover, again as $\mathcal{H}$ need not be barrelled, it is unclear if $\mathcal{H}$ carries the γ-topology. Hence it is possible that $\mathcal{H}'_c(E)$ and $\mathcal{H}'(E; \beta)$ are different, as $\mathcal{L}_c(\mathcal{H}, E)$ is possibly a strict subspace of $\mathcal{H}'_c(E) = \mathcal{L}_\varepsilon((\mathcal{H}'_c)'_c, E)$.

Whereas L. Schwartz states in items 1 and 2 only sufficient conditions for the coincidence of bounded and continuous linear mappings, characterizations of pairs E, F for which the spaces $\mathcal{L}(E, F)$ and $\mathcal{LB}(E, F)$ coincide are given in [7, 54].

Let $\mathcal{H}$ and $\mathcal{K}$ be two separated locally convex spaces such that either $\mathcal{K}$ has a fundamental system of absolutely convex neighbourhoods U of zero such that $\mathcal{K}'_{U^\circ}$ has the approximation property or that $\mathcal{H}$ has a fundamental system of neighbourhoods U of zero such that $\widehat{\mathcal{H}_U}$ has the approximation property. Moreover let $\Lambda\colon \mathcal{K} \to \mathcal{H}$ be a subnuclear mapping. Let E and F be two separated locally convex spaces, with F quasi-complete. In Proposition 10 in [46, p. 57], L. Schwartz constructs a bilinear mapping

$$\mathcal{K}(E) \times \mathcal{H}'_c(F; \beta) \to E \mathbin{\widehat{\otimes}_\iota} F, \qquad (\varphi, T) \mapsto \varphi \cdot_{\iota;\Lambda} T,$$

which satisfies $(\psi \otimes e) \cdot_{\iota;\Lambda} (S \otimes f) = \langle \Lambda(\psi), S \rangle (e \otimes f)$ for $\psi \in \mathcal{K}$, $S \in \mathcal{H}'_c$, $e \in E$ and $f \in F$. This mapping can be considered as a vector-valued evaluation mapping.

Theorem 7 (Proposition 38 in [46, p. 159]). *Let $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{M}$ be quasi-complete normal spaces of distributions on $\mathbb{R}^n$, and E and F separated locally convex spaces, with F quasi-complete. Moreover let $\Lambda : \mathcal{K} \to \mathcal{H}$ be a subnuclear embedding. Assume that $\mathcal{L}$ is bornological, and either $\mathcal{K}$ has a fundamental system of absolutely convex neighbourhoods U of zero such that $\mathcal{K}'_{U^\circ}$ has the approximation property, or $\mathcal{H}$ has a fundamental system of neighbourhoods U of zero such that $\widehat{\mathcal{H}_U}$ has the approximation property.*

Let $\check{\mathcal{M}} \times \mathcal{L} \to \mathcal{K}$ be a convolution map which is hypocontinuous with respect to compact subsets of $\mathcal{L}$. By transposition we obtain a convolution $*\colon \mathcal{M} \times \mathcal{H}'_c \to \mathcal{L}'_c$, where $\check{\mathcal{M}}$ is the image of $\mathcal{M}$ under the symmetry mapping $x \mapsto -x$.

There exists a convolution mapping

$$\overset{*}{\otimes} : \mathcal{M}(E) \times \mathcal{H}'_c(F;\beta) \to \mathcal{L}'_c(E \mathbin{\widehat{\otimes}_\iota} F), \qquad (S,T) \mapsto S \overset{*}{\otimes} T,$$

such that $\langle \varphi, S \overset{*}{\otimes} T\rangle = (\check{S} * \varphi) \cdot_{\iota;\Lambda} T \in E \mathbin{\widehat{\otimes}_\iota} F$ for all $\varphi \in \mathcal{L}$ and

$$(S \otimes e) \overset{*}{\otimes} (T \otimes f) = (S * T) \otimes (e \otimes f)$$

for all $S \in \mathcal{M}$, $T \in \mathcal{H}'$, $e \in E$ and $f \in F$.

If Theorem 3 (*Theorem* 3 *in* [46, *p.* 151]) *can be applied to $\mathcal{M}$, $\mathcal{H}'_c$ and $\mathcal{L}'_c$, the image of $S \overset{*}{\otimes} T$ in $\mathcal{L}'_c(E \mathbin{\widehat{\otimes}_\pi} F)$ coincides with the convolution defined in Theorem* 3.

If S stays in a bounded subset of $\mathcal{M}(E)$ and T converges to zero in $\mathcal{H}'_c(F)$ while staying in an equibounded set, the convolution $S \overset{*}{\otimes} T$ converges to zero too in $\mathcal{L}'_c(E \mathbin{\widehat{\otimes}_\beta} F)$. If the injection Λ is nuclear it is not necessary to assume that T stays in an equibounded set.

It is possible to define $S \overset{*}{\otimes} T \in \mathcal{L}'_c(E \mathbin{\widehat{\otimes}_\beta} F)$ even if $\mathcal{L}$ is not bornological if we assume that the convolution $\check{\mathcal{M}} \times \mathcal{L} \to \mathcal{K}$ is not only hypocontinuous with respect to compact subsets of $\mathcal{L}$ but also with respect to compact subsets of $\check{\mathcal{M}}$. In both cases ($\mathcal{L}$ *bornological or the convolution hypocontinuous with respect to compact subsets of $\check{\mathcal{M}}$*), *if S converges to zero in $\check{\mathcal{M}}(E)$ and T stays in an equibounded subset of $\mathcal{H}'_c(F)$, the convolution $S \overset{*}{\otimes} T$ converges to zero in $\mathcal{L}'_c(E \mathbin{\widehat{\otimes}_\beta} F)$.*

Remark (cf. [46, p. 58]). The space $\mathcal{K}'_{U^\circ}$ is the dual space of $\widehat{\mathcal{K}_U}$. If $\mathcal{K}'_{U^\circ}$ has the approximation property then so does $\widehat{\mathcal{K}_U}$ by Proposition 36 in [13, Chap. I, p. 167]. Hence $\mathcal{K}_U$ has the approximation property and so does $\mathcal{K}$ by Lemme 19 in [13, Chap. I, p. 169]. If $\mathcal{K}$ is a nuclear space then it has all these properties, as the spaces $\mathcal{K}'_{U^\circ}$ and $\widehat{\mathcal{K}_U}$ are Hilbert spaces by Lemme 3 in [13, Chap. II, p. 37] and every Hilbert space has the approximation property by the Corollaire to Lemme 19 in [13, Chap. I, p. 170].

Therefore if we assume $\mathcal{H} = \mathcal{K}$ is a nuclear space and $\Lambda = \mathrm{id}_{\mathcal{H}}$, the assumptions on the occurring spaces are all satisfied, as the identity mapping $\mathrm{id}_{\mathcal{H}}$ is subnuclear if and only if the space $\mathcal{H}$ is nuclear by Théorème 6 in [13, Chap. II, p. 34] and [46, p. 55].

Remark. The assumption of subnuclearity is made by L. Schwartz in order to ensure the existence of the generalized evaluation mapping $\cdot_{\iota;\Lambda}$ which is used in the construction of $S \overset{*}{\otimes} T$. In the subsequent applications (Applications 14 and 15 and Proposition 2′), we do not use the concept of subnuclear mapping assuming a priori $\mathcal{H}$ is nuclear. Also in the five examples of L. Schwartz in [46, pp. 161–167], this concept is not used.

R. Shiraishi presents a closely related result which uses his θ-convolution (for his construction see Section 3.4). In this case the assumptions that $\mathcal{L}$ has to be bornological and the approximation assumptions on $\mathcal{H}$ or $\mathcal{K}$ are not necessary, but he needs $\mathcal{H}$ to be $\dot{\mathcal{B}}$-normal and $\mathcal{L}'_c$ to be quasi-complete.

Theorem 8 (Theorem 4 in [50, p. 202]). *Let $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{M}$ be normal spaces of distributions on $\mathbb{R}^n$ and E, F and G be three separated locally convex spaces where $\mathcal{L}'_c$, F and G are assumed to be quasi-complete. Moreover assume that the injection $\mathcal{K} \hookrightarrow \mathcal{H}$ is subnuclear and that $\mathcal{K}$ is $\dot{\mathcal{B}}$-normal.*

Let $*_1\colon \check{\mathcal{M}} \times \mathcal{L} \to \mathcal{K}$ *be a partially continuous convolution map and* $\theta\colon E \times F \to G$ *a partially continuous bilinear map. Then any* $S \in \mathcal{M}(E)$ *and* $T \in \mathcal{H}'_c(F;\beta)$ *are* $*_\theta$*-composable (see Definition* 10*). If the convolution mapping* $*_1$ *is hypocontinuous with respect to all compact absolutely convex sets in* $\check{\mathcal{M}}$ *or those of* $\mathcal{L}$ *then* $S \overset{*}{_\theta} T \in \mathcal{L}'_c(G)$*.*

Assume $*_1$ *is hypocontinuous with respect to compact absolutely convex subsets of* $\mathcal{L}$*.*

1. *Let* S *lie in a bounded subset of* $\mathcal{M}(E)$*. If* T *converges to zero in* $\mathcal{H}'_c(F)$ *while staying in an equibounded subset, then* $S \overset{*}{_\theta} T$ *converges to zero in* $\mathcal{L}'_c(G)$ *uniformly with respect to* S*. If the injection* $\mathcal{K} \hookrightarrow \mathcal{H}$ *is not only subnuclear but nuclear it is not necessary to assume that* T *stays in an equibounded set.*
2. *If* S *converges to zero in* $\mathcal{M}(E)$ *and* T *stays in an equibounded subset of* $\mathcal{H}'_c(F)$*, then* $S \overset{*}{_\theta} T$ *converges to zero in* $\mathcal{L}'_c(G)$*, uniformly in* T*.*

Application 14 (Exemple 2° in [46, p. 162]). Given quasi-complete separated locally convex spaces E and F, there is a vector-valued convolution mapping

$$\mathcal{O}'_C(E) \times \mathcal{S}'(F;\beta) \to \mathcal{S}'(E \mathbin{\widehat{\otimes}_\iota} F)$$

which is consistent for decomposed elements. The existence of this mapping is a consequence of Theorem 7, by taking $\mathcal{H} = \mathcal{K} = \mathcal{L} = \mathcal{S}$, $\mathcal{M} = \mathcal{O}'_C$ and $\Lambda = \mathrm{id}_{\mathcal{S}}$.

In general the vector-valued convolution on $\mathcal{O}'_C(E) \times \mathcal{O}'_C(F)$ in Application 7 is not a special case of this mapping, as the following lemma shows that, e.g., $\mathcal{O}'_C(\mathcal{O}_C)$ is not a subspace of $\mathcal{S}'(\mathcal{O}_C;\beta)$.

Lemma 1. *Let* E *be a separated locally convex space such that the embeddings* $\mathcal{S} \hookrightarrow E$ *and* $E \hookrightarrow \mathcal{E}$ *are continuous. Then the embedding* $\mathcal{S} \hookrightarrow E$ *is not bounded.*

Proof. As the composition of a bounded mapping with a continuous mapping is bounded, we just have to show that the embedding $\mathcal{S} \hookrightarrow \mathcal{E}$ is not bounded. This assertion is equivalent to

$$\forall k \in \mathbb{N}_0\ \forall j \in \mathbb{N}_0\ \forall \varepsilon > 0\ \exists K \subset \mathbb{R}^n \text{ compact } \exists m \in \mathbb{N}_0\ \forall C > 0\ \exists f \in \mathcal{S}\colon$$
$$\Big(\sup_{|\alpha|\le j} \|(1+|x|^2)^{k/2}\partial^\alpha f\|_\infty \le \varepsilon\Big) \wedge \Big(\sup_{|\alpha|\le m}\sup_{x\in K} |\partial^\alpha f(x)| > C\Big).$$

Choosing $m = j+1$ and $K = \overline{B}_1(0)$, such an f is given by

$$f(x) = \frac{C+1}{\min\{|(\partial^\beta\psi)(0)|,1\}}\,\frac{1}{\tilde{C}^{j+1}}\psi(\tilde{C}x)$$

where $\psi \in \mathcal{S}(\mathbb{R}^n)$ and β are chosen such that $|\beta| = j+1$ and $(\partial^\beta\psi)(0) \neq 0$ and where

$$\tilde{C} = \frac{(C+1)\max\{\max_{|\alpha|\le j}\|(1+|x|^2)^{k/2}\partial^\alpha\psi\|_\infty, 1\}}{\min\{|(\partial^\beta\psi)(0)|,1\}\min\{\varepsilon,1\}}. \ \blacksquare$$

We now compare these results with Theorem 3 (Proposition 34 in [46, p. 151]), Theorem 5 (Theorem 2 in [50, p. 196]) and Proposition 1/1′ and Proposition 2.

Comparison 6. The main advantage of Theorem 7 over the theorems in the preceding sections is that it is possible to combine a hypocontinuous convolution mapping in the preimage domain with a just partially continuous bilinear mapping in the image domain, as for the topology of $E \mathbin{\widehat{\otimes}_\lambda} F$, it is also possible to set $\lambda = \iota$ or $\lambda = \gamma$.

Theorem 3 (Proposition 34 in [46, p. 151]) is not a special case of Theorem 7, as for example $\mathcal{H} = \mathcal{K} = \mathcal{L} = \mathcal{D}$ and $\mathcal{M} = \mathcal{E}'$ and $\Lambda = \mathrm{id}_{\mathcal{H}}$ all assumptions in Theorem 7 are satisfied, and therefore the vector-valued convolution

$$\mathcal{E}'(E) \times \mathcal{D}'(F;\beta) \to \mathcal{D}'(E \mathbin{\widehat{\otimes}_\iota} F) \to \mathcal{D}'(E \mathbin{\widehat{\otimes}_\pi} F)$$

is well defined and hypocontinuous. But $\mathcal{D}'(F)$ and $\mathcal{D}'(F;\beta)$ do not coincide in general. For example when $F = \mathcal{D}$ these spaces are distinct, as otherwise the identity mapping $\mathrm{id}_{\mathcal{D}}$ would transform a neighbourhood of zero into a bounded subset, i.e., $\mathcal{D}$ would be normable, a contradiction.

It can be seen analogously that Theorem 5 (Theorem 2 in [50, p. 196]) and Proposition 1 are not special cases of the above propositions, using the continuous convolution mapping

$$\mathcal{E}' \times \mathcal{E}' \to \mathcal{E}'$$

and $F = \mathcal{E}$.

Also the "Théorèmes de croisement" cannot be applied to prove Theorem 7 or Theorem 8, as by using $\Gamma_{\iota,\gamma}$, if it existed, the identity $\mathcal{H} \mathbin{\widehat{\otimes}_\gamma} E = \mathcal{H}(E)$ would have to hold, which is not possible without placing restrictions on both $\mathcal{H}$ and E in general. Moreover the construction of $\Gamma_{\mu,\lambda}$ heavily uses the fact that $\mu \le \gamma$ (see [46, p. 21]).

Nevertheless there seems to be a connection of these results with the "Théorèmes de croisement" as we will see in the following comparison.

Comparison 7. Let $\mathcal{H}$ be a complete nuclear bornological separated locally convex space and E be a Montel space. Moreover assume that the space $(\mathcal{H} \mathbin{\widehat{\otimes}_\pi} E')'$ is complete.

By the Corollaire in [13, Chap. II, p. 90] we have $(\mathcal{H} \mathbin{\widehat{\otimes}_\pi} E')' = \mathcal{H}' \mathbin{\widehat{\otimes}_\iota} E$. As the dual space of $\mathcal{H} \mathbin{\widehat{\otimes}_\pi} E'$ is the space $\mathcal{B}(\mathcal{H}, E')$ of continuous bilinear forms on $\mathcal{H} \times E'$, using the algebraic isomorphism $\mathcal{B}(\mathcal{H}, E') \cong \mathcal{LB}(\mathcal{H}, (E, \sigma(E, E')))$ ([41, p. 167]), we obtain $\mathcal{H}' \mathbin{\widehat{\otimes}_\iota} E \cong \mathcal{LB}(\mathcal{H}, E)$ as vector spaces. As $\mathcal{H}'$ and E are assumed to be barrelled and every partially continuous bilinear map on a product of barrelled spaces is hypocontinuous by Theorem 2 in [24, p. 360], we finally get

$$\mathcal{H}' \mathbin{\widehat{\otimes}_\beta} E = \mathcal{H}' \mathbin{\widehat{\otimes}_\iota} E \cong \mathcal{LB}(\mathcal{H}, E) = \mathcal{H}'(E;\beta)$$

as vector spaces.

Additionally let $\mathcal{K}$ and F be quasi-complete separated locally convex spaces, with F barrelled. The "Théorèmes de croisement" (Theorem 1) yield the existence of the bilinear mapping

$$\Gamma_{\beta,\beta} \colon (\mathcal{H}' \mathbin{\widehat{\otimes}_\beta} E) \times (\mathcal{K} \mathbin{\varepsilon} F) \to (\mathcal{H}' \mathbin{\widehat{\otimes}_\beta} \mathcal{K}) \mathbin{\varepsilon} (E \mathbin{\widehat{\otimes}_\beta} F).$$

Let $* \colon \mathcal{H}' \times \mathcal{K} \to \mathcal{L}$ be a hypocontinuous convolution mapping. We obtain the mapping

$$\underset{*}{\otimes} \colon (\mathcal{H}' \mathbin{\widehat{\otimes}_\beta} E) \times (\mathcal{K} \mathbin{\varepsilon} F) \to \mathcal{L}(E \mathbin{\widehat{\otimes}_\iota} F)$$

which coincides on $(\mathcal{H}' \otimes E) \times (\mathcal{K} \otimes F)$ with the convolution mapping constructed in Theorem 7. If every subset of $\mathcal{H}' \mathbin{\widehat{\otimes}_\beta} E$ consisting of a single point is β-β-decomposable ($\star$) then $\underset{*}{\otimes}$ is partially continuous. As, under the above assumptions, the topology of $\mathcal{H}' \mathbin{\widehat{\otimes}_\beta} E$ is finer than the one of $\mathcal{H}'(E;\beta)$, both mappings are partially continuous if we endow $\mathcal{H}'(E;\beta)$ with the topology of $\mathcal{H}' \mathbin{\widehat{\otimes}_\beta} E$. Hence both mappings coincide on $(\mathcal{H}' \mathbin{\widehat{\otimes}_\beta} E) \times (\mathcal{K} \mathbin{\varepsilon} F)$.

Examples of spaces $\mathcal{H}'$ and E that satisfy the condition $(\star)$ are

- $\mathcal{H}'$ is a strict (LF)-space and E is a nuclear Fréchet space,
- $\mathcal{H}'$ is a strict (LB)-space and E is a nuclear Fréchet space, or a nuclear (DF)-space,
- $\mathcal{H}'$ and E are both Fréchet spaces or both (DF)-spaces.

This shows that in this special case the existence of the vector-valued convolution mapping stated in Theorem 7 can also be shown using the "Théorèmes de croisement". As the β-topology in general is strictly finer than the ε-topology, the "hypocontinuity" of this mapping, also in this more restricted case, is not an implication of the "Théorèmes de croisement".

Let us now compare Proposition 38 in [46, p. 159], i.e., Theorem 7 with Proposition 2. Using the Remark on page 27, we obtain the following result:

Proposition $2'$. *Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be quasi-complete normal spaces of distributions where $\mathcal{H}$ is nuclear and satisfies the strict approximation property. Let E, F and G be three locally convex spaces, with G quasi-complete, and $*\colon \mathcal{H}\times\mathcal{K}\to\mathcal{L}$ and $\theta\colon E\times F\to G$ be two hypocontinuous bilinear maps, where $*$ is the convolution defined by transposition. If one of the assumptions*

1. *$\mathcal{H}$ and E are Fréchet spaces,*
2. *$\mathcal{H}$ and E are (DF)-spaces and $\mathcal{H}$ is (infra-)barrelled,*

is satisfied, there is a hypocontinuous bilinear map $\overset{*}{\theta}\colon \mathcal{H}(E)\times\mathcal{K}(F)\to\mathcal{L}(G)$ satisfying the consistency property for decomposed elements.

Comparison 8. Therefore in all cases where the convolution $\mathcal{H}\times\mathcal{K}\to\mathcal{L}$ can be defined by transposition and $\mathcal{H}$ and E are both (DF)-spaces and $\mathcal{H}$ is (infra-)barrelled, Proposition 2 is a special case of Theorem 7 (Proposition 38 in [46, p. 159]). If $\mathcal{H}$ is not (infra-)barrelled then it is not possible to define $\mathcal{H}(E;\beta)$ in general, as it is then unclear whether $\mathcal{H}$ carries the γ-topology (cf. footnote (1) on page 54 in [46]).

An analogous result to Proposition $2'$ for multiplication and for the scalar product can be proved using Proposition 32 in [46, p. 127] and Proposition 20 in [46, p. 83], respectively. Therefore the only real advantage of Proposition 2 over Proposition $2'$ is that it can be applied to other bilinear maps than convolution or multiplication.

If we assume that $\mathcal{H}$ is $\dot{\mathcal{B}}$-normal then Proposition 2 also follows from Theorem 8 (Theorem 4 in [50, p. 202]).

Application 15 (Example 3 in [50, p. 211]). As both $\mathcal{S}$ and s are nuclear Fréchet spaces, there is a unique hypocontinuous convolution map

$$\mathcal{S}(s)\times\mathcal{S}'(s')\to\mathcal{D}'$$

which combines the hypocontinuous convolution $*\colon \mathcal{S}\times\mathcal{S}'\to\mathcal{D}'$ and the evaluation mapping $\theta=\langle -,-\rangle\colon s\times s'\to\mathbb{C}, (x,y)\mapsto\sum_{n=0}^{\infty}x(n)y(n)$ by Proposition 2, in contrast to the assertion "Then we can show that there exist vector-valued distributions $S\in\mathcal{S}(s)$ and $T\in\mathcal{S}'(s')$ which are not $*_\theta$-composable" in Example 3 in [50, p. 211].

In order to show that this convolution coincides with the θ-convolution, we have to show that there is a partially continuous θ-convolution between these spaces. Then these convolutions coincide, as both are partially continuous and satisfy the consistency property for decomposed elements.

The $*_\theta$-composability can be shown by applying Theorem 8 to $\mathcal{H} = \mathcal{K} = \mathcal{M} = \mathcal{S}$, $\mathcal{L} = \mathcal{D}$, $E = s$ and $F = s'$, which yields a hypocontinuous convolution mapping

$$\mathcal{S}(s) \times \mathcal{S}'(s';\beta) \to \mathcal{D}'.$$

As $\mathcal{S}'$ and s' are barrelled (DF)-spaces, we have $\mathcal{S}'(s';\beta) = \mathcal{S}'(s')$, as we have seen in the proof of Proposition $2'$. The flaw in the "counterexample" in Example 3 in [50, p. 211] is that, given $0 \neq \alpha \in \mathcal{D}(\mathbb{R})$, the function

$$S = \sum_{n=0}^{\infty} \tau_n \alpha \otimes e_n = [x \mapsto \{\alpha(x-n)\}_{n\in\mathbb{N}}]$$

is an element of $\mathcal{S}'(s')$ but not of $\mathcal{S}(s)$, because the seminorms on $\mathcal{S}(s)$ (cf. [43, p. 97])

$$\begin{aligned}\sup_{x\in\mathbb{R}^d}\sup_{|\gamma|\le k}\sup_{n\in\mathbb{N}} |(1+|x|^2)^{l/2}\partial^\gamma\alpha(x-n)n^j| &\ge \sup_{x\in\mathbb{R}^d}\sup_{|\gamma|\le k}\sup_{n\in\mathbb{N}} |\partial^\gamma\alpha(x-n)n^j| \\ &= \sup_{n\in\mathbb{N}}\Big\{n^j \sup_{|\gamma|\le k}\sup_{x\in\mathbb{R}^d}|\partial^\gamma\alpha(x-n)|\Big\} = \sup_{n\in\mathbb{N}}\{n^j\|\alpha\|_{W^{k,\infty}}\} = \infty\end{aligned}$$

are infinite on S for $j > 0$.

According to Proposition 7 in [24, p. 420] the convolution $\mathcal{S}' \times \mathcal{S} \to \mathcal{O}_C$ is well-defined and hypocontinuous, hence by Proposition 2 the above convolution map is a hypocontinuous map from $\mathcal{S}(s) \times \mathcal{S}'(s')$ not only into $\mathcal{D}'$ but actually into $\mathcal{O}_C$.

3.4. Convolution of vector-valued distributions defined by tensor products. The definition of the ι-convolution as well as of ι-tensor product relies on the existence of the ι-tensor product of vector-valued distributions and on the existence of vector-valued integrals.

THEOREM 9 (Part of Proposition 33 in [46, p. 145]). *Let E and F be two separated locally convex spaces and $S \in \mathcal{D}'_x(E)$ and $T \in \mathcal{D}'_y(F)$ be two vector-valued distributions. There is a unique distribution $S(x) \otimes_\iota T(y)$ such that for all $\varphi \in \mathcal{D}_x$ and all $\psi \in \mathcal{D}_y$,*

$$\langle S(x) \otimes_\iota T(y), \varphi(x)\psi(y)\rangle = \langle S, \varphi\rangle \otimes \langle T, \psi\rangle.$$

Moreover,

$$(S(x) \otimes e) \otimes_\iota (T(y) \otimes f) = (S(x) \otimes T(y)) \otimes (e \otimes f)$$

for all $S(x) \in \mathcal{D}'_x$, $T(y) \in \mathcal{D}'_y$, $e \in E$ and $f \in F$.

Sketch of proof. Identifying $S(x) \in \mathcal{D}'_x(E)$ and $T(y) \in \mathcal{D}'_y(F)$ with the linear and continuous mappings $S(x)\colon \mathcal{D}_x \to E$ and $T(y)\colon \mathcal{D}_y \to F$, the existence of the map $\otimes_\iota$ is a consequence of the defining property of tensor products:

$$\begin{array}{ccc} \mathcal{D}_x \times \mathcal{D}_y & \xrightarrow{S(x)\times T(y)} & E \times F \\ \big\downarrow \text{can} & & \big\downarrow \text{can} \\ \mathcal{D}_x \mathbin{\widehat{\otimes}_\iota} \mathcal{D}_y & \xrightarrow{S(x)\otimes_\iota T(y)} & E \mathbin{\widehat{\otimes}_\iota} F \end{array}$$

Due to the lemma below, $S(x) \otimes_\iota T(y)$ is a linear and continuous mapping

$$S(x) \otimes_\iota T(y) \colon \mathcal{D}_{x,y} \to E \mathbin{\widehat{\otimes}_\iota} F,$$

i.e., $S(x) \otimes_\iota T(y) \in \mathcal{D}'_{x,y}(E \mathbin{\widehat{\otimes}_\iota} F)$. ∎

Lemma 2 (Proposition 1 bis in [46, p. 17]). *The space $\mathcal{D}_{x,y}$ of test functions has the representation*

$$\mathcal{D}_{x,y} \cong \mathcal{D}_x \mathbin{\widehat{\otimes}_\iota} \mathcal{D}_y = \mathcal{D}_x \mathbin{\widehat{\otimes}_\iota} \mathcal{D}_y$$

and every bounded subset of $\mathcal{D}_{x,y}$ is γ-γ-decomposable.

Remark. Note that $\mathcal{D}_{x,y} \cong \mathcal{D}_x \mathbin{\widehat{\otimes}_\pi} \mathcal{D}_y$, as stated in [26, p. 500], does not hold (J. Wengenroth, personal communication). The situation is quite similar for the space $\mathcal{O}_C$ of very slowly increasing functions where

$$\mathcal{O}_{C,x,y} = \mathcal{O}_{C,x} \mathbin{\widehat{\otimes}_\iota} \mathcal{O}_{C,y} \neq \mathcal{O}_{C,x} \mathbin{\widehat{\otimes}_\pi} \mathcal{O}_{C,y}$$

(see [2, p. 78] or Remarks 1, 3. in [3]).

Definition 7 (Definition in [45, p. 129], [18, p. 545]). A distribution $T \in \mathcal{D}'(E)$ is called *summable* on $\mathbb{R}^n$ if $T \in \mathcal{D}'_{L^1}(E)$, E being a locally convex space. The integral of $T \in \mathcal{D}'_{L^1}(E)$, $\int_{\mathbb{R}^n} T(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} T$, is defined as

$$\int_{\mathbb{R}^n} T(x)\,\mathrm{d}x := (\langle 1, \cdot\rangle \,\varepsilon\, \mathrm{id}_E)(T)$$

if $\langle 1, \cdot \rangle \colon \mathcal{D}'_{L^1} \to \mathbb{C}$, is the evaluation of scalar-valued integrable distributions on the function $1 \in \mathcal{B}_c$.

A vector-valued distribution $K \in \mathcal{D}'_{x,y}(E)$ is called *partially summable with respect to y* if $K \in (\mathcal{D}'_x \mathbin{\widehat{\otimes}_\varepsilon} \mathcal{D}'_{L^1,y})(E)$.

The integral $\int_{\mathbb{R}^n} K(x,y)\,\mathrm{d}y$ is defined as

$$\int_{\mathbb{R}^n} K(x,y)\,\mathrm{d}y = (\langle 1, \cdot\rangle \,\varepsilon\, \mathrm{id}_{\mathcal{D}'_x \widehat{\otimes}_\varepsilon E})(K).$$

Application 16. The integral of a summable vector-valued distribution $T \in \mathcal{D}'_{L^1}(E)$ (E a quasi-complete locally convex space) can also be considered as the value

$$(\widetilde{\langle \cdot, \cdot \rangle} \,\varepsilon\, \mathrm{id}_E)(\Gamma_\gamma(1, T)) = (\langle 1, \cdot \rangle \,\varepsilon\, \mathrm{id}_E)(T).$$

A special case of Theorem 1 ([46, p. 32]) implies the existence of a γ-continuous bilinear mapping

$$\Gamma_\gamma \colon \mathcal{B}_c \times \mathcal{D}'_{L^1}(E) \to (\mathcal{B}_c \mathbin{\widehat{\otimes}_\gamma} \mathcal{D}'_{L^1})(E), \Gamma_\gamma = \mathrm{can}\,\varepsilon\,\mathrm{id}_E$$

(consistent for decomposed elements).

Since the evaluation mapping $\langle \cdot, \cdot \rangle \colon \mathcal{B}_c \times \mathcal{D}'_{L^1} \to \mathbb{C}$ is also γ-continuous and hence its continuation

$$\widetilde{\langle \cdot, \cdot \rangle} \colon \mathcal{B}_c \mathbin{\widehat{\otimes}_\gamma} \mathcal{D}'_{L^1} \to \mathbb{C}$$

is well-defined, we obtain by composition

$$(\widetilde{\langle \cdot, \cdot \rangle} \,\varepsilon\, \mathrm{id}_E) \circ \Gamma_\gamma \colon \mathcal{B}_c \times \mathcal{D}'_{L^1}(E) \to E$$

and by [46, p. 32],

$$(\widetilde{\langle \cdot, \cdot \rangle} \,\varepsilon\, \mathrm{id}_E) \circ \Gamma_\gamma = \langle \cdot, \cdot \rangle \,\varepsilon\, \mathrm{id}_E.$$

Definition 8 (Proposition 41 in [46, p. 174], [18, pp. 548, 556]). Let E and F be two separated locally convex spaces and $S(x) \in \mathcal{D}'_x(E)$ and $T(y) \in \mathcal{D}'_y(F)$ be two vector-valued distributions. Then $S(x)$ and $T(y)$ are called *ι-convolvable* if

$$S(x-y) \otimes_\iota T(y) \in (\mathcal{D}'_x \widehat{\otimes}_\varepsilon \mathcal{D}'_{L^1,y})(E \widehat{\otimes}_\iota F),$$

i.e., $S(x-y) \otimes_\iota T(y)$ is partially summable with respect to y. Here, $S(x-y) \otimes_\iota T(y)$ is defined as the image of $S(\xi) \otimes_\iota T(\eta)$ under the linear map $\xi = x - y$, $\eta = y$.

L. Schwartz shows that $S(x)$ and $T(y)$ are ι-convolvable if the convolution is defined by Theorem 7 (Proposition 38 in [46, p. 159]) or by Theorem 11 (Proposition 39 in [46, p. 167]) for $\mu = \iota$. Then he states the concern of symmetry of the ι-convolvability condition in Definition 8:

> "Contrairement à ce qui se passe pour la proposition 40, il y a dissymétrie entre les rôles de S et de T dans la proposition 41. Si $S(x-y) \otimes_\iota T(y)$ est partiellement sommable on y, rien ne dit qu'il en soit de même pour $S(y) \otimes_\iota T(x-y)$ et même s'il en est ainsi, rien ne dit que les intégrales en y soient les mêmes." ([46, p. 185])

This symmetry problem was solved by Y. Hirata and R. Shiraishi in the Theorem in [18, p. 558], where the equivalence of the partial summability of $S(x-y) \otimes_\iota T(y)$ and of $S(y) \otimes_\iota T(x-y)$ with respect to y is shown.

Theorem 10 (Part of the Theorem on p. 558 in [18]). *Let E and F be two separated locally convex spaces. For $S(x) \in \mathcal{D}'_x(E)$ and $T(y) \in \mathcal{D}'_y(F)$ the following conditions are equivalent:*

1. $S(x-y) \otimes_\iota T(y) \in \mathcal{D}'_x \widehat{\otimes}_\varepsilon \mathcal{D}'_{L^1,y}(E \widehat{\otimes}_\iota F)$;
2. $\varphi^\Delta(S(x) \otimes_\iota T(y)) \in \mathcal{D}'_{L^1,x,y}(E \widehat{\otimes}_\iota F)$ *for all* $\varphi \in \mathcal{D}$ *where the function* φ^Δ *is given by* $\varphi^\Delta(x,y) = \varphi(x+y)$;
3. $\delta(z-x-y) \cdot (S(x) \otimes_\iota T(y)) \in (\mathcal{D}'_z \widehat{\otimes}_\varepsilon \mathcal{D}'_{L^1,x,y})(E \widehat{\otimes}_\iota F)$.
   Note that the product of $\delta(z-x-y) \in \mathcal{E}'_z \widehat{\otimes}_\varepsilon \mathcal{E}_{x,y}$ *and* $S(x) \otimes_\iota T(y) \in \mathcal{D}'_{x,y}(E \widehat{\otimes}_\iota F)$ *is well-defined* (*by Proposition* 25 *in* [46, *p.* 120]) *and yields*
   $$\delta(z-x-y) \cdot (S(x) \otimes_\iota T(y)) \in (\mathcal{D}'_{x,y} \widehat{\otimes}_\varepsilon \mathcal{E}'_z)(E \widehat{\otimes}_\iota F);$$
4. $(S(y) \otimes_\iota T(x-y) \in \mathcal{D}'_x \widehat{\otimes}_\varepsilon \mathcal{D}'_{L^1,y}(E \widehat{\otimes}_\iota F)$.

Proof. 1⇒3: Multiplication of the distribution

$$S(x-y) \otimes_\iota T(y) \in (\mathcal{D}'_x \widehat{\otimes}_\varepsilon \mathcal{D}'_{L^1,y})(E \widehat{\otimes}_\iota F)$$

by $\delta(z-x) \in \mathcal{D}'_z \widehat{\otimes}_\varepsilon \mathcal{D}_x$ yields

$$\begin{aligned}\delta(z-x) \cdot (S(x-y) \otimes_\iota T(y)) &\in (\mathcal{D}'_z \widehat{\otimes}_\varepsilon \mathcal{E}'_x \widehat{\otimes}_\varepsilon \mathcal{D}'_{L^1,y})(E \widehat{\otimes}_\iota F) \\ &\subset \mathcal{D}'_z \widehat{\otimes}_\varepsilon (\mathcal{D}'_{L^1,x} \widehat{\otimes}_\pi \mathcal{D}'_{L^1,y})(E \widehat{\otimes}_\iota F),\end{aligned}$$

i.e., by Proposition 38 in [45, p. 135],

$$\delta(z-x) \cdot (S(x-y) \otimes_\iota T(y)) \in \mathcal{D}'_z \widehat{\otimes}_\varepsilon \mathcal{D}'_{L^1,x,y}(E \widehat{\otimes}_\iota F).$$

The linear change $\xi = x - y$, $\eta = y$ of variables implies 3.

3⇒1,4: Integration of

$$\delta(z-x-y)(S(x)\otimes_\iota T(y)\in(\mathcal{D}'_{L^1,x,y}\,\widehat{\otimes}_\varepsilon\,\mathcal{D}'_z)(E\,\widehat{\otimes}_\iota\,F)$$

with respect to x or to y furnishes statements 1 and 4 (Corollaire in [45, p. 136]).

2⇔3: The equivalence of statements 2 and 3 follows by definition taking into account the continuity of the mapping

$$\mathcal{D}_z\to\mathcal{D}'_{L^1,x,y}(E\,\widehat{\otimes}_\iota\,F),\qquad \varphi\mapsto\varphi^\Delta(S(x)\otimes_\iota T(y)),$$

which can be seen as follows:

As by Théorème XXVI in [47, p. 203] the multiplication mapping

$$\cdot\colon\mathcal{D}_{L^\infty}\times\mathcal{D}'_{L^1}\to\mathcal{D}'_{L^1}$$

is hypocontinuous, there is a unique hypocontinuous multiplication mapping

$$\cdot\colon\mathcal{D}_{L^\infty,x,y}\times\mathcal{D}'_{L^1,x,y}(E\,\widehat{\otimes}_\iota\,F)\to\mathcal{D}'_{L^1,x,y}(E\,\widehat{\otimes}_\iota\,F)$$

by Proposition 21 bis in [45, p. 70]. Moreover it is easy to see that the mapping $\mathcal{D}_z\to\mathcal{D}_{L^\infty,x,y},\varphi\mapsto\varphi^\Delta$, is continuous.

Let $K\subset\mathbb{R}^n$ be a compact set and $\psi\in\mathcal{D}(\mathbb{R}^n)$ where $\psi(x)=1$ for all $x\in K$. Then, since $S(x)$ and $T(y)$ are convolvable, $\psi^\Delta(S(x)\otimes_\iota T(y))\in\mathcal{D}'_{L^1,x,y}(E\,\widehat{\otimes}_\iota\,F)$. Hence the mapping

$$\begin{aligned}&\mathcal{D}_{K,z}\to\mathcal{D}'_{L^1,x,y}(E\,\widehat{\otimes}_\iota\,F),\\ &\quad\varphi\mapsto\varphi^\Delta\psi^\Delta(S(x)\otimes_\iota T(y))=(\varphi\psi)^\Delta(S(x)\otimes_\iota T(y))=\varphi^\Delta(S(x)\otimes_\iota T(y)),\end{aligned}$$

is continuous. Therefore, as $\mathcal{D}=\lim_{K\to}\mathcal{D}_K$, the mapping

$$\mathcal{D}_z\to\mathcal{D}'_{L^1,x,y}(E\,\widehat{\otimes}_\iota\,F),\qquad \varphi\mapsto\varphi^\Delta(S(x)\otimes_\iota T(y)),$$

is continuous.

4⇔3: The equivalence of statements 1 and 3 also shows the equivalence of 4 and 3 ■

Remark. Even in the scalar case ($E=F=\mathbb{C}$) the proof above shortens the known proofs of the equivalences (Theorem 2, in [49, p. 24], [39, pp. 194–195], Proposition 1.3.4 in [35], Proposition 3 in [33, p. 326]).

The result in Theorem 10 justifies the following definition of the ι-convolution:

Definition 9 ([18, p. 558]). If $S(x)\in\mathcal{D}'_x(E)$ and $T(y)\in\mathcal{D}'_y(F)$ fulfil the equivalent assertions in Theorem 10 then $S*_\iota T$ is defined as

$$\begin{aligned}(S*_\iota T)(x)&=\int_{\mathbb{R}^n}(S(x-y)\otimes_\iota T(y))\,\mathrm{d}y\\ &=\int_{\mathbb{R}^n}(S(y)\otimes_\iota T(x-y))\,\mathrm{d}y\\ &=\int_{\mathbb{R}^{2n}}\delta(x-\xi-\eta)(S(\xi)\otimes_\iota T(\eta))\,\mathrm{d}\xi\,\mathrm{d}\eta,\end{aligned}$$

and $S*_\iota T\in\mathcal{D}'(E\,\widehat{\otimes}_\iota\,F)$.

In [50], R. Shiraishi gave the definition of the θ-convolution of two vector-valued distributions S and T. It is a combination of the ι-convolution and a partially continuous bilinear map θ defined on the image domains of S and T.

Definition 10 ([50, pp. 181, 182]). Let E, F and G be three separated locally convex spaces, G quasi-complete and $\theta\colon E\times F\to G$ a partially continuous bilinear map. The *θ-tensor product* of $S(x)\in\mathcal{D}'_x(E)$ and $T(y)\in\mathcal{D}'_y(F)$ is defined by

$$S(x)\otimes_\theta T(y):=(\mathrm{id}\otimes\tilde\theta)(S(x)\otimes_\iota T(y))\in\mathcal{D}'_{x,y}(G)$$

as the composition of the ι-tensor product $S(x)\otimes_\iota T(y)$ and the ε-product of the identity mapping on $\mathcal{D}'_{x,y}$ and the linear mapping $\tilde\theta$ corresponding to θ.

$S(x)$ and $T(y)$ are called *θ-convolvable* if

$$S(x-y)\otimes_\theta T(y):=(\mathrm{id}\,\varepsilon\,\tilde\theta)(S(x-y)\otimes_\iota T(y))\in(\mathcal{D}'_x\mathbin{\widehat\otimes_\varepsilon}\mathcal{D}'_{L^1,y})(G).$$

Analogously to Theorem 10, R. Shiraishi shows (Proposition 5 in [50, p. 182]) that the θ-convolvability of $S(x)\in\mathcal{D}'_x(E)$ and $T(y)\in\mathcal{D}'_y(F)$ is equivalent to:

2. $\varphi^\Delta(S(x)\otimes_\theta T(y))\in\mathcal{D}'_{L^1,x,y}(G)$ for all $\varphi\in\mathcal{D}$.
3. $\delta(z-x-y)\cdot(S(x)\otimes_\theta T(y))\in(\mathcal{D}'_z\mathbin{\widehat\otimes_\varepsilon}\mathcal{D}'_{L^1,x,y})(G)$.
4. $(S(y)\otimes_\theta T(x-y)\in\mathcal{D}'_x\mathbin{\widehat\otimes_\varepsilon}\mathcal{D}'_{L^1,y}(G)$.

The θ-convolution $S*_\theta T$ is defined by

$$\begin{aligned}(S*_\theta T)(x)&=\int_{\mathbb{R}^n}(S(x-y)\otimes_\theta T(y))\,\mathrm{d}y\\&=\int_{\mathbb{R}^n}(S(y)\otimes_\theta T(x-y))\,\mathrm{d}y\\&=\int_{\mathbb{R}^{2n}}\delta(x-\xi-\eta)(S(\xi)\otimes_\theta T(\eta))\,\mathrm{d}\xi\,\mathrm{d}\eta,\end{aligned}$$

and $S*_\theta T\in\mathcal{D}'(G)$.

Comparison 9. The ι-convolution is a special case of the θ-convolution by choosing as image domain $G=E\mathbin{\widehat\otimes_\iota}F$ and by taking for θ the canonical mapping $E\times F\to E\mathbin{\widehat\otimes_\iota}F$.

Application 17 (Example 4 in [50, p. 211]). An example of two vector-valued distributions in $\mathcal{E}'(\mathcal{E})$ and $\mathcal{S}(\mathcal{S}')$ which are θ-convolvable, if we take for θ the multiplication $\theta\colon\mathcal{E}\times\mathcal{S}'\to\mathcal{D}',(\varphi,S)\mapsto\varphi\cdot S$, is given by R. Shiraishi:

$$\delta(x-y)\in\mathcal{E}'_x(\mathcal{E}_y)\quad\text{and}\quad\delta(x+y)\in\mathcal{S}_x(\mathcal{S}'_y).$$

These two distributions are θ-convolvable if

$$\delta(x-\xi-y)\otimes_\theta\delta(\xi+y)\in(\mathcal{D}'_x\mathbin{\widehat\otimes_\varepsilon}\mathcal{D}'_{L^1,\xi})(\mathcal{D}'_y)=\mathcal{D}'_{x,y}\mathbin{\widehat\otimes_\varepsilon}\mathcal{D}'_{L^1,\xi}.$$

By [45, p. 102] we have $\delta(z-\xi)\in\mathcal{B}_{c,z}\mathbin{\widehat\otimes_\varepsilon}\mathcal{D}'_{L^1,\xi}\subset\mathcal{E}_z\mathbin{\widehat\otimes_\varepsilon}\mathcal{D}'_{L^1,\xi}$ and hence $\delta(x-y-\xi)\in\mathcal{E}_{x,y}\mathbin{\widehat\otimes_\varepsilon}\mathcal{D}'_{L^1,\xi}$. Together,

$$\delta(x-\xi-y)\otimes_\theta\delta(\xi+y)=\delta(x-y-\xi)\cdot\delta(x)\in\mathcal{D}'_{x,y}\mathbin{\widehat\otimes}\mathcal{D}'_{L^1,\xi}.$$

We have

$$\delta(x-y)*_\theta\delta(x+y)=\delta(x)\otimes 1(y)\in\mathcal{M}_{\mathrm{comp},x}\otimes\mathcal{D}_{L^\infty,y}\subset\mathcal{E}'_x\otimes\mathcal{D}'_y.$$

We see that the $*_\theta$-convolution of $\delta(x-y)$ with $\delta(x+y)$ does not belong to $\mathcal{S}_x(\mathcal{D}'_y)$ in contrast to what could be expected.

But the $*_\theta$-convolution of $\delta(x-y)$ and $\delta(x+y)$ belongs to $\mathcal{O}'_{C,x}(\mathcal{D}'_y)$ because the vector-valued convolution

$$\overset{*}{.} : \mathcal{O}'_{C,x}(\mathcal{E}_y) \times \mathcal{O}'_{C,x}(\mathcal{S}'_y) \to \mathcal{O}'_{C,x}(\mathcal{D}'_y)$$

is well-defined by Theorem 6 (Theorem 3 in [50, p. 199]), or by Proposition 1.

Due to the symmetries

$$\mathcal{O}'_{C,x}(\mathcal{E}_y) = \mathcal{E}_y(\mathcal{O}'_{C,x}), \quad \mathcal{S}'_y(\mathcal{O}'_{C,x}) = \mathcal{S}'_y(\mathcal{O}'_{C,x}), \quad \mathcal{O}'_{C,x}(\mathcal{D}'_y) = \mathcal{D}'_y(\mathcal{O}'_{C,x})$$

the existence and the hypocontinuity of $\overset{*}{.}$ is also a consequence of Theorem 2 taking into account the continuity of the convolution in the space $\mathcal{O}'_C$.

R. Shiraishi's example above shows that the two distributions

$$\delta(x-y) \in \mathcal{E}'_x(\mathcal{E}_y) \quad \text{and} \quad \delta(x+y) \in \mathcal{S}_x(\mathcal{S}'_y).$$

are θ-convolvable though the convolvability cannot be shown using Theorem 7 or Theorem 8, because $\delta(x+y) \notin \mathcal{E}'_x(\mathcal{E}_y;\beta)$ and $\delta(x-y) \notin \mathcal{S}_x(\mathcal{S}'_y;\beta)$.

Application 18. Let us prove now that the conditions ensuring the holomorphy of the convolution of two distribution-valued holomorphic functions

$$S(z,x) \in \mathcal{H}(\Omega_{1,z}, \mathcal{D}'_x) \quad \text{and} \quad T(w,y) \in \mathcal{H}(\Omega_{2,w}, \mathcal{D}'_y)$$

($\Omega_1 \subset \mathbb{C}^{n_1}$, $\Omega_2 \subset \mathbb{C}^{n_2}$ two open sets) coincide with the θ-convolvability condition if the mapping

$$\theta \colon \mathcal{H}(\Omega_1) \times \mathcal{H}(\Omega_1) \to \mathcal{H}(\Omega_1 \times \Omega_2)$$

is the tensor product of holomorphic functions. By Definition 10, S and T are θ-convolvable iff

$$\varphi(x+y)(S(z,x) \otimes_\iota T(w,y)) \in \mathcal{D}'_{L^1,x,y}(\mathcal{H}(\Omega_{1,z}) \mathbin{\widehat{\otimes}_\iota} \mathcal{H}(\Omega_{2,w}))$$

for all $\varphi \in \mathcal{D}$.

The spaces $\mathcal{H}(\Omega_i)$, $i=1,2$, are nuclear Fréchet spaces, and hence

$$\mathcal{H}(\Omega_1) \mathbin{\widehat{\otimes}_\iota} \mathcal{H}(\Omega_2) = \mathcal{H}(\Omega_1) \mathbin{\widehat{\otimes}_\varepsilon} \mathcal{H}(\Omega_2).$$

By [12, p. 105], $\mathcal{H}(\Omega_1) \mathbin{\widehat{\otimes}_\varepsilon} \mathcal{H}(\Omega_2) = \mathcal{H}(\Omega_1 \times \Omega_2)$. Therefore the θ-convolvability condition reads

$$\varphi(x+y)(S(z,x) \otimes_\iota T(w,y)) \in \mathcal{D}'_{L^1,x,y}(\mathcal{H}(\Omega_{1,z} \times \Omega_{2,w}))$$

for all $\varphi \in \mathcal{D}$, i.e. the function

$$\Omega_1 \times \Omega_2 \to \mathcal{D}'_{L^1,x,y}(z,w) \mapsto \varphi(x+y)(S(z,x) \otimes_\iota T(w,y))$$

has to be holomorphic for all $\varphi \in \mathcal{D}$.

The convolvability conditions for distribution-valued holomorphic functions in [32, p. 372], [35, p. 29], [33, Problems 1 and 2] and [1, p. 65], to all of which we refer as (C), use the holomorphy of the function

$$\Omega_1 \times \Omega_2 \to \mathcal{D}'_{x,y}, \qquad (z,w) \mapsto \varphi(x+y)(S(z,x) \otimes_\iota T(w,y)),$$

in order to ensure that the function

$$\Omega_1 \times \Omega_2 \to \mathcal{D}'_{L^1,x,y}, \qquad (z,w) \mapsto \varphi(x+y)(S(z,x) \otimes_\iota T(w,y)),$$

is a holomorphic function too (cf. Remark 5 in [1, p. 66]).

The holomorphy of this last function in turn implies the conditions (C). Thus the θ-convolvability condition for holomorphic distribution-valued functions is equivalent to any one of the conditions (C) stated in the literature.

In [35, p. 30] it was shown that the holomorphy of the distribution-valued function

$$\Omega_1 \times \Omega_2 \to \mathcal{D}'_{L^1,x,y}, \qquad (z,w) \mapsto \varphi(x+y)(S(z,x) \otimes_\iota T(w,y)),$$

in general is not implied by the "algebraic" condition

$$\varphi(x+y)(S(z,x) \otimes T(w,y)) \in \mathcal{D}'_{L^1,x,y}$$

for all $\varphi \in \mathcal{D}$—in contrast to a holomorphic function $S \in \mathcal{H}(\Omega_1, \mathcal{D}'(\Omega_2))$ with values in $\mathcal{E}'(\Omega_2)$, i.e., $S(\Omega_1) \subset \mathcal{E}'(\Omega_2)$, for which we have $S \in \mathcal{H}(\Omega_1, \mathcal{E}'(\Omega_2))$ (cf. (2.1.3) in [25, p. 77]).

3.5. Extension to inductive limits: Convolutions defined by support restrictions. As in [47, p. 172] we denote by $\mathcal{D}'_{[a,\infty)} = \{T \in \mathcal{D}'(\mathbb{R}); \operatorname{supp}(T) \subset [a,\infty)\}$ the space of scalar-valued distributions with support contained in $[a,\infty)$. This space is endowed with the topology induced by $\mathcal{D}'$. More generally, if A is a closed subset of $\mathbb{R}^n$ and E is a separated locally convex space, we denote by $\mathcal{D}'_A(E)$ the space of all E-valued distributions with support contained in A. We endow $\mathcal{D}'_A(E)$ with the topology induced by $\mathcal{D}'(E)$.

According to [24, p. 383], we say that two closed subsets A, B of $\mathbb{R}^n$ satisfy condition (Σ) (in L. Schwartz' words: the mapping $A \times B \to \mathbb{R}^n$, $(x,y) \mapsto x+y$, is continuous at infinity) if the set $(A \times B) \cap K^\Delta$ is compact for every compact set $K \subset \mathbb{R}^n$. Here, $K^\Delta := \{(x,y) \in \mathbb{R}^{2n}; x+y \in K\}$. For $\varphi \in \mathcal{D}(\mathbb{R}^n)$ the function $\varphi^\Delta \in \mathcal{E}(\mathbb{R}^{2n})$ is defined by $\varphi^\Delta(x,y) = \varphi(x+y)$ as in Theorem 10(2).

In contrast to footnote (1) in [46, p. 41], we denote both the dualities between E and E' as well as between $\mathcal{H}$ and $\mathcal{H}'$, by $\langle \cdot, \cdot \rangle$.

THEOREM 11 (Proposition 39 in [46, p. 167]). *Let A and B be two closed subsets of $\mathbb{R}^n$ satisfying condition (Σ). Let E and F be two separated locally convex spaces. There is a convolution mapping*

$$\mathcal{D}'_A(E) \times \mathcal{D}'_B(F) \to \mathcal{D}'_{A+B}(E \mathbin{\widehat{\otimes}_\iota} F)$$

which is consistent with respect to decomposed elements. The convolution is defined by the formula

$$\langle \varphi, S *_\iota T \rangle = \langle 1, (S \otimes_\iota T) \cdot \varphi^\Delta \rangle,$$

for $\varphi \in \mathcal{D}(\mathbb{R}^n)$, and has the following properties.

1. Symmetry and associativity. *$S *_\iota T$ is associative and commutative with respect to the symmetry between $E \mathbin{\widehat{\otimes}_\iota} F$ and $F \mathbin{\widehat{\otimes}_\iota} E$.*
2. Continuity properties. *Denote by $S *_\mu T$ the image of $S *_\iota T$ in $E \mathbin{\widehat{\otimes}_\mu} F$ for $\mu = \gamma$, β, π or ε. Then the bilinear mapping*

$$\mathcal{D}'_A(E) \times \mathcal{D}'_B(F) \to \mathcal{D}'_{A+B}(E \mathbin{\widehat{\otimes}_\mu} F), \qquad (S,T) \mapsto S *_\mu T,$$

is μ-continuous for $\mu = \beta$, π and ε. If E and F are quasi-complete, the mapping $*_\gamma$ is γ-continuous.

3. Consistency with other convolutions. *If the convolution $S *_\pi T$ is also defined by Theorem 3 it coincides with the image in $\mathcal{D}'(E \mathbin{\widehat{\otimes}_\pi} F)$ of $S *_\iota T$ defined here if $\mathcal{H}$ or $\mathcal{K}$ has the approximation property by truncation and regularisation. If $S *_\iota T$ can also be defined by Theorem 7 then it coincides with the convolution defined here if $\mathcal{H} = \mathcal{K}$ is nuclear and has the approximation property by truncation and regularisation.*

Sketch of proof. 1. By Theorem 9 (Proposition 33 in [46, p. 145]) the mapping

$$\mathcal{D}'_x(E) \times \mathcal{D}'_y(F) \to \mathcal{D}'_{x,y}(E \mathbin{\widehat{\otimes}_\iota} F), \qquad (S(x), T(y)) \mapsto S(x) \otimes_\iota T(y),$$

is well-defined. Also the mapping

$$\mathcal{D}'_{A,x}(E) \times \mathcal{D}'_{B,y}(F) \to \mathcal{D}'_{A\times B,x,y}(E \mathbin{\widehat{\otimes}_\iota} F), \qquad (S(x), T(y)) \mapsto S(x) \otimes_\iota T(y),$$

is well-defined by Proposition 33 in [46, p. 145].

2. The multiplication mapping

$$\mathcal{D}(\mathbb{R}^n) \times \mathcal{D}'_{A\times B}(\mathbb{R}^{2n}) \to \mathcal{E}'(\mathbb{R}^{2n}), \qquad (\varphi, K) \mapsto \varphi^\Delta \cdot K,$$

is well-defined, bilinear and hypocontinuous.

3. Due to $\mathcal{E}' \subset \mathcal{D}'_{L^1}$, we get by integration the bilinear hypocontinuous mapping

$$\mathcal{D} \times \mathcal{D}'_{A\times B} \to \mathbb{C}, \qquad (\varphi, K) \mapsto \langle 1, \varphi^\Delta \cdot K\rangle.$$

Hence, by Proposition 21 bis in [45, p. 70], we derive the existence of a bilinear hypocontinuous mapping

$$\mathcal{D} \times \mathcal{D}'_{A\times B}(G) \to G$$

(G a separated, quasi-complete locally convex space) which continues the above mapping $\mathcal{D} \times \mathcal{D}'_{A\times B} \to \mathbb{C}$.

4. Combining the tensor product mapping in item 1 with the mapping in item 3, we ensure the existence of the mapping

$$\mathcal{D} \times \mathcal{D}'_{A,x}(E) \times \mathcal{D}'_{B,y}(F) \to E \mathbin{\widehat{\otimes}_\iota} F, \qquad (\varphi, S(x), T(y)) \mapsto \langle 1_{x,y}, \varphi^\Delta(S(x) \otimes_\iota T(y))\rangle,$$

which is continuous with respect to $\varphi \in \mathcal{D}$. Therefore $S(x) *_\iota T(y)$ is by definition the vector-valued distribution

$$\mathcal{D} \to E \mathbin{\widehat{\otimes}_\iota} F, \qquad \varphi \mapsto \langle 1_{x,y}, \varphi(x+y)(S(x) \otimes_\iota T(y))\rangle. \ \blacksquare$$

Comparison 10. The sketch of proof shows that the definition of $S(x) *_\iota T(y)$ coincides with the ι-convolution of Definition 8. In the proof of Proposition 39 in [46, p. 167] L. Schwartz shows, analogously to the scalar-valued case, that the linear map

$$\mathcal{D} \to E \mathbin{\widehat{\otimes}_\iota} F, \qquad \varphi \mapsto (S \otimes_\iota T) \cdot \varphi^\Delta,$$

is continuous. As the convolution mapping

$$*\colon \mathcal{D}'_A \times \mathcal{D}'_B \to \mathcal{D}'_{A+B}$$

is continuous by 5° a) Théorème in [53, p. 225], we have the following relations between Theorem 11, Proposition 1 and Theorem 5:

In the case where $\mu = \gamma$, β or π Theorem 11 is a particular case of Proposition 1, but not in the case $\mu = \iota$ as Proposition 1 is not applicable then. It is a special case of Theorem 5 in all cases where $\mu \neq \gamma$ as the γ-continuity of the convolution mapping does not follow from Theorem 5.

A special case (apart from partial quasi-continuity) of this theorem, where the sets A and B are chosen as $A = [a, \infty)$ and $B = [b, \infty)$, is the following proposition of Y. Hirata and R. Shiraishi:

THEOREM 12 (Proposition 3 in [52, p. 76]). *Let E, F and G be three separated locally convex spaces, where G is assumed to be quasi-complete. Let $\theta\colon E \times F \to G$ be a partially continuous bilinear map. Then any $S \in \mathcal{D}'_{[a,\infty)}(E)$ and $T \in \mathcal{D}'_{[b,\infty)}(F)$ are θ-composable and $S *_\theta T \in \mathcal{D}'_{[a+b,\infty)}(G)$. Moreover the θ-convolution has the following properties:*

1. Partial quasi-continuity. *The bilinear mapping*
$$\mathcal{D}'_{[a,\infty)}(E) \times \mathcal{D}'_{[b,\infty)}(F) \to \mathcal{D}'_{[a+b,\infty)}(G), \qquad (S,T) \mapsto S *_\theta T,$$
is partially quasi-continuous.
2. Hypocontinuity 1. *If θ is hypocontinuous with respect to compact absolutely convex subsets of F, then the set of linear maps $\{S \mapsto S *_\theta T;\, T \in H\}$ is equicontinuous for all equicontinuous sets $H \subset \mathcal{L}(F'_c, \mathcal{D}'_{[b,\infty)})$.*
3. Hypocontinuity 2. *If θ is hypocontinuous with respect to bounded subsets of F, then the set of linear maps $\{S \mapsto S *_\theta T;\, T \in B\}$ is equicontinuous whenever B is bounded.*
4. β-Continuity. *If θ is hypocontinuous with respect to bounded subsets, so is $*_\theta$.*
5. Continuity. *If θ is continuous, so is $*_\theta$.*

Analogously to $\bar{\mathcal{E}}'(E) = \lim_{n\to} \mathcal{D}'_{K_n}(E)$ in [45, p. 62], where $(K_n)_{n\in\mathbb{N}}$ is an exhaustion of $\mathbb{R}^n$ consisting of compact sets, we define the space of vector-valued distributions with support bounded to the left:

DEFINITION 11 ([52, p. 74]). Let E be a separated locally convex space and a a real number. We define the space
$$\bar{\mathcal{D}}'_+(E) := \lim_{a\to} \mathcal{D}'_{[a,\infty)}(E)$$
of E-valued distributions with support bounded to the left as the inductive limit of the spaces $\mathcal{D}'_{[a,\infty)}(E)$.

COMPARISON 11. Let us discuss the topological properties of the space of vector-valued distributions with support bounded to the left:

1. The ad hoc definition of $\mathcal{D}'_{[a,\infty)}(E) = \{T \in \mathcal{D}'(E);\, \mathrm{supp}(T) \subset [a,\infty)\}$ endowed with the topology induced by $\mathcal{D}'(E)$ in [52, p. 74] is equivalent to $\mathcal{D}'_{[a,\infty)}(E) = \mathcal{D}'_{[a,\infty)} \,\varepsilon\, E$ according to the definition of spaces of vector-valued distributions in [45, p. 52].
   Let $T \in \mathcal{D}'(E)$ be an E-valued distribution with support contained in $[a,\infty)$. Hence ${}^tT(E') \subset \mathcal{D}'_{[a,\infty)}$, as the support of T is the union of the supports of the scalar-valued distributions $\langle T, e'\rangle$ over all $e' \in E'$. As $\mathcal{D}'_{[a,\infty)}$ is a topological subspace of $\mathcal{D}'$, the map ${}^tT : E'_c \to \mathcal{D}'_{[a,\infty)}$ is continuous, i.e. $T \in \mathcal{D}'_{[a,\infty)}(E)$, so that
$$\{T \in \mathcal{D}'(E);\, \mathrm{supp}(T) \subset [a,\infty)\} \subset \mathcal{D}'_{[a,\infty)}(E)$$
   algebraically. The injection $i : \mathcal{D}'_{[a,\infty)} \to \mathcal{D}'$ is a strict monomorphism and hence by [44, p. 18] also the map $i \,\varepsilon\, \mathrm{id}_E : \mathcal{D}'_{[a,\infty)}(E) \to \mathcal{D}'(E)$ and every $T \in \mathcal{D}'_{[a,\infty)}(E)$ has its support contained in $[a,\infty)$. This proves the identity
$$\mathcal{D}'_{[a,\infty)}(E) = \{T \in \mathcal{D}'(E);\, \mathrm{supp}(T) \subset [a,\infty)\}$$
   as vector spaces.

2. As the topology of $\mathcal{D}'_{[a,\infty)}(E)$ is induced by $\mathcal{D}'(E)$, the above inductive limit is strict. The identity
$$\bigcup_{a\in\mathbb{R}} \mathcal{D}'_{[a,\infty)}(E) = \bigcup_{a\in\mathbb{Z}} \mathcal{D}'_{[a,\infty)}(E)$$
shows that the limit is countable. If E is a (DF)-space we have $\bar{\mathcal{D}}'_+(E) \cong \mathcal{D}'_+(E)$ algebraically and topologically, according to [52, p. 74] or the Corollaire to Proposition 6 in [13, p. 47], because a countable strict inductive limit of complete spaces is a complete space itself by [4, p. 47] and $\mathcal{D}'_+ \otimes_\pi E \subset \lim_{a\to} \mathcal{D}'_{[a,\infty)}(E) \subset \mathcal{D}'_+(E)$ algebraically and topologically, as both $\mathcal{D}'_+ \otimes_\pi E$ and $\lim_{a\to} \mathcal{D}'_{[a,\infty)}(E)$ carry the subspace topology with respect to $\mathcal{D}'_+ \widehat{\otimes}_\pi E = \mathcal{D}'_+(E)$.

Lemma 3. *Let $E = \lim_{i\to} E_i$ and $F = \lim_{i\to} F_i$ be countable strict inductive limits, G a separated locally convex space and $b_i\colon E_i \times F_i \to G$ hypocontinuous bilinear maps such that $b_i|_{E_j\times F_j} = b_j$ for all $j \le i$. Then there exists a unique hypocontinuous bilinear map $b\colon E \times F \to G$ where $b|_{E_i\times F_i} = b_i$ for all i.*

Proof. The map b, if it exists, is unique, as for all $(x, y) \in E \times F$ there is a j such that $(x, y) \in E_i \times F_i$ for all $i \ge j$ and $b(x, y) = b_i(x, y) = b_j(x, y)$. We define the map $b\colon E \times F \to G$ by $b(x, y) := b_i(x, y)$ where $i = \min\{j;\ (x, y) \in E_j \times F_j\}$. Then $b|_{E_j\times F_j} = b_j$ since $b_j(x, y) = b_i(x, y)$ for $j \ge i$. Moreover, b is partially continuous as for all $y \in F$ the map $E \to G$, $x \mapsto b(x, y)$, is continuous if and only if the maps $E_i \to G$, $x \mapsto b(x, y) = b_j(\imath_{ij}(x), y)$, are continuous ([41, 6.1, p. 54]) and by symmetry with respect to E and F.

Now we show the hypocontinuity of b with respect to bounded subsets of F. Let $B \subset F$ be a bounded subset. As F is a strict inductive limit, there is an i such that $B \subset F_j$ and bounded there for all $j \ge i$ by Proposition 3 in [15, p. 140]. Now let $x \to 0$ be a filter converging to zero in E_i. Hence it also converges to zero in E_j for $j \ge i$ and $b(x, y) = b_j(x, y) \to 0$ uniformly with respect to $y \in B$ as the maps b_j are hypocontinuous. Hence the mappings $b|_{E_i} : E_i \to \mathcal{L}_b(F, G)$ are continuous and therefore also $b : E \to \mathcal{L}_b(F, G)$ by the definition of the inductive limit topology. The hypocontinuity with respect to bounded subsets of E can be shown analogously. ∎

By continuation of convolution mappings to inductive limits, from this lemma the following vector-valued convolution mappings can be obtained:

Application 19 (Corollary in [52, p. 78]). Let E, F and G be three separated locally convex spaces, where G is assumed to be quasi-complete. Let $\theta\colon E\times F \to G$ be a partially continuous bilinear map. Then any $S \in \bar{\mathcal{D}}'_+(E)$ and $T \in \bar{\mathcal{D}}'_+(F)$ are θ-composable and $S *_\theta T \in \bar{\mathcal{D}}'_+(G)$. The bilinear mapping
$$\bar{\mathcal{D}}'_+(E) \times \bar{\mathcal{D}}'_+(F) \to \bar{\mathcal{D}}'_+(G), \qquad (S, T) \mapsto S *_\theta T,$$
is partially quasi-continuous. If θ is hypocontinuous with respect to bounded subsets of F, then the set of linear maps $\{S \mapsto S *_\theta T;\ T \in B\}$ is equicontinuous for bounded sets $B \subset F$. If θ is β- or π-continuous, so is $*_\theta$.

Application 20 (Corollaire in [46, p. 174]). Let E and F be two separated locally convex spaces. There exists a bilinear convolution map

$$\bar{\mathcal{E}}'(E) \times \mathcal{D}'(F) \to \mathcal{D}'(E \mathbin{\widehat{\otimes}_\mu} F), \qquad (S,T) \mapsto S *_\mu T,$$

for $\mu = \iota$, γ, β, π and ε. For $\mu \le \pi$ it coincides with the restriction of the convolution defined in Theorem 3 (Proposition 34 in [46, p. 155]). The mapping is μ-continuous for $\mu = \beta$ and also for $\mu = \gamma$ if E and F are quasi-complete.

For the next application, we need the following observation:

Proposition 4. *Let $E = \lim_{k\to} E_k$ be a strict inductive limit of complete separated locally convex spaces with respect to a countable index set K, and F a separated locally convex space with a neighbourhood of zero not containing any straight line or, equivalently, with a continuous seminorm which is a norm. Then every element of $E \mathbin{\varepsilon} F$ is already contained in some $E_k \mathbin{\varepsilon} F$, i.e., algebraically $E \mathbin{\varepsilon} F = \lim_{k\to}(E_k \mathbin{\varepsilon} F)$.*

Proof (cf. [45, p. 63] in the case $E = \mathcal{E}'$). Let $T \in E \mathbin{\varepsilon} F$. Up to isomorphy we have $T \in \mathcal{L}_\varepsilon(F'_c, E)$. Let U be a neighbourhood of zero in F not containing any straight lines; we assume without loss of generality that U is absolutely convex and $\sigma(F,F')$-closed. Let $x \in F$ be such that $\langle x, x'\rangle = 0$ for all $x' \in H = U^\circ$ in the polar of U. Then for all $\lambda \in \mathbb{C}$,

$$0 = |\lambda|\,|\langle x, x'\rangle| = |\langle \lambda x, x'\rangle|,$$

i.e. $H^\circ = U^{\circ\circ}$ contains the straight line $\{\lambda x;\ \lambda \in \mathbb{C}\}$. The theorem of bipolars yields $U = U^{\circ\circ}$ as U is absolutely convex and $\sigma(F,F')$-closed. Hence $x = 0$, since U does not contain any straight line. Therefore, as by Proposition 8 in [37, p. 55] we have $(F'_c)' = F$ algebraically, $H \subset F'_c$ is a total subset by Corollary 3 in [37, p. 29]. The set H is equicontinuous and so $T(H) \subset E$ is bounded. As the inductive limit $E = \lim_{k\to} E_k$ is strict, there is a $k_0 \in K$ such that $T(H) \subset E_k$ for $k \ge k_0$. The space E_k carries the subspace topology of E and hence $T(F') = T(\overline{\langle H\rangle}) \subset \overline{\langle T(H)\rangle} \subset E_k$ and $T\colon F'_c \to E_k$ is a continuous linear map, i.e. $T \in E_k \mathbin{\varepsilon} F$. ∎

Application 21. There is a unique hypocontinuous convolution mapping

$$\overset{*}{*} : \mathcal{D}'_+(\mathcal{S}') \times \bar{\mathcal{D}}'_+(\mathcal{O}'_C) \to \mathcal{D}'_+(\mathcal{S}').$$

and the spaces $\bar{\mathcal{D}}'_+(\mathcal{O}'_C)$ and $\mathcal{D}'_+(\mathcal{O}'_C)$ coincide as vector spaces but not topologically. This result can be seen as follows:

As the convolution mapping $*\colon \mathcal{S}' \times \mathcal{O}'_C \to \mathcal{S}'$ is hypocontinuous, by Theorem 12 and Lemma 3 there is a hypocontinuous convolution mapping

$$\overset{*}{*} : \bar{\mathcal{D}}'_+(\mathcal{S}') \times \bar{\mathcal{D}}'_+(\mathcal{O}'_C) \to \bar{\mathcal{D}}'_+(\mathcal{S}').$$

As $\mathcal{D}'_+$ is a countable strict inductive limit and $\mathcal{S}'$ is a (DF)-space, the Corollaire in [13, p. 47] yields the identity $\bar{\mathcal{D}}'_+(\mathcal{S}') = \mathcal{D}'_+(\mathcal{S}')$ in the sense of topological vector spaces.

Let us show that $\bar{\mathcal{D}}'_+(\mathcal{O}'_C) = \mathcal{D}'_+(\mathcal{O}'_C)$ as vector spaces but not topologically. In view of Proposition 4, we have to show that $\mathcal{O}'_C$ admits a neighbourhood of zero not containing any straight line.

The Fourier transform $\mathcal{F}\colon \mathcal{O}'_C \to \mathcal{O}_M$ is an isomorphism of topological vector spaces by Théorème XV in [47, p. 268]. The topology of $\mathcal{O}_M$ is generated by the family of seminorms

$$\Big\{q_{\varphi,\alpha}(f)=\sup_{x\in\mathbb{R}^n}|\varphi(x)(\partial^\alpha f)(x)|\Big\}_{\alpha\in\mathbb{N}^n,\,\varphi\in\mathcal{S}}$$

according to Example 15 in [24, p. 91]. The existence of a neighbourhood of 0 not containing any straight line is equivalent to the existence of a continuous seminorm which is actually a norm. As $\mathrm{e}^{-|x|^2}\in\mathcal{S}$ and $\mathrm{e}^{-|x|^2}\neq 0$ for all $x\in\mathbb{R}^n$, the seminorm $f\mapsto\|\mathrm{e}^{-|x|^2}f\|_\infty$ is actually a norm, i.e., there is a neighbourhood of 0 in $\mathcal{O}_M$ not containing any straight line and hence also in $\mathcal{O}'_C$ by the Fourier transform.

The spaces $\mathcal{D}_K$ of test functions with support in a given compact set $K\subset\mathbb{R}^n$ are closed metrizable subspaces of $\mathcal{O}_M$ ([38, p. 329]). Hence the topologies of

$$\mathcal{D}'_+(\mathcal{O}_M)=\Big(\lim_{\to a}\mathcal{D}'_{[a,\infty)}\Big)\,\hat{\otimes}_\pi\,\mathcal{O}_M\quad\text{and}\quad\bar{\mathcal{D}}'_+(\mathcal{O}_M)=\lim_{\to a}(\mathcal{D}'_{[a,\infty)}\,\hat{\otimes}_\pi\,\mathcal{O}_M)$$

do not coincide by Remark 1 in [13, chap. I, p. 47].

It is not possible to construct this convolution mapping using Theorem 7 as we have the following proposition:

PROPOSITION 5. *There exist distributions in $\mathcal{D}'_+(\mathcal{O}_M)\cong\mathcal{D}'_+(\mathcal{O}'_C)$ and $\mathcal{D}'_+(\mathcal{S}')$ which are not contained in $\mathcal{D}'_+(\mathcal{O}_M;\beta)$ and $\mathcal{D}'_+(\mathcal{S}';\beta)$, respectively.*

Proof. We first prove the result for $\mathcal{O}_M(\mathbb{R})$. For all compact sets $K\subset\mathbb{R}$ there is a $c\in\mathbb{Z}$ such that $\mathcal{D}_K$ is a topological subspace of $\mathcal{E}_{(-\infty,c]}$ and hence also of $\mathcal{D}_-$, because $\mathcal{D}_-$ is the strict inductive limit of the spaces $\mathcal{E}_{(-\infty,c]}$ by definition in [47, p. 172]. Moreover $\mathcal{D}_K$ is a topological subspace of $\mathcal{O}_M$ (see [38, p. 329]). Let $K\subset\mathbb{R}$ and $\tilde{K}\subset\mathbb{R}$ be different compact sets such that $K\subset\tilde{K}$. Let $\varphi\in\mathcal{D}$ with support contained in $\tilde{K}$ and $\varphi|_K=1$. Then the linear map $T\colon\mathcal{D}_-\to\mathcal{O}_M$, $f\mapsto f\varphi$, is continuous and its restriction $T|_{\mathcal{D}_K}$ coincides with the identity map on $\mathcal{D}_K$. If T were bounded in the sense of Grothendieck, $T|_{\mathcal{D}_K}=\mathrm{id}_{\mathcal{D}_K}$ would be bounded too, which would mean that $\mathcal{D}_K$ is normable, a contradiction. The generalization to $\mathcal{O}_M(\mathbb{R}^n)$ is immediate.

The space $\mathcal{D}_-=\lim_{c\to}\mathcal{E}_{(-\infty,c]}$ is a strict (LF)-space by [47, p. 172]. Hence, as both $\mathcal{E}_{(-\infty,c]}$ and $\mathcal{S}$ are Fréchet spaces, we have

$$\mathcal{S}\,\widehat{\otimes}_\iota\,\mathcal{D}_-=\lim_{c\to}(\mathcal{S}\,\widehat{\otimes}\,\mathcal{E}_{(-\infty,c]})\neq\mathcal{S}\,\widehat{\otimes}_\pi\,\mathcal{D}_-$$

by Proposition 14.I in [13, chap. I, p. 76] and by Remarque 1 in [13, chap. I, p. 48]. Hence there exists a discontinuous but partially continuous bilinear form $b:\mathcal{D}_-\times\mathcal{S}\to\mathbb{C}$; as $\mathcal{S}$ is a Montel space and $\mathcal{D}_-$ is ultrabornological, by [41, p. 167] this is equivalent to the existence of an unbounded continuous linear map $T:\mathcal{D}_-\to\mathcal{S}'$. ∎

APPLICATION 22 (Autre Exemple in [31, p. 252], cf. Application 5). Let V be a Hilbert space and $m\colon V\times V\to\mathcal{O}'_C$ be a continuous sesquilinear mapping. Setting

$${}_V\langle(M\varphi)(v),u\rangle_V={}_{\mathcal{O}_C}\langle\varphi,m(u,v)\rangle_{\mathcal{O}'_C}$$

for $\varphi\in\mathcal{O}_C$ and $u,v\in V$, we define a linear and continuous mapping

$$M\colon\mathcal{O}_C\to\mathcal{L}(V,V),$$

hence $M\in\mathcal{L}_b(\mathcal{O}_C,\mathcal{L}(V,V))=\mathcal{O}'_C\,\widehat{\otimes}\,\mathcal{L}(V,V)$. Now, by Theorem 4 there is a bilinear mapping

$$\overset{\circ}{*}:(\mathcal{D}'_+(\mathcal{S}'))(V)\times(\mathcal{D}'_+(\mathcal{O}'_C))(\mathcal{L}(V,V))\to(\mathcal{D}'_+(\mathcal{S}'))(V)$$

which is the combination of the evaluation mapping $\circ\colon V \times \mathcal{L}(V,V) \to V$, $(v,A) \mapsto Av$, and the convolution $*\colon \mathcal{D}'_+(\mathcal{S}') \times \mathcal{D}'_+(\mathcal{O}'_C) \to \mathcal{D}'_+(\mathcal{S}')$.

4. Necessity of nuclearity assumptions

In all propositions on vector-valued convolution mappings stated by L. Schwartz or R. Shiraishi, there are nuclearity assumptions on at least one of the occurring spaces of distributions. In this chapter we will consider the question whether these assumptions are necessary. With the methods used in [1], it can be seen that given a hypocontinuous bilinear map $b\colon E \times F \to G$ between separated locally convex spaces, there is a well-defined and hypocontinuous bilinear map

$$\dot{b} : \dot{\mathcal{B}}^m(\mathbb{R}^n; E) \times \dot{\mathcal{B}}^m(\mathbb{R}^n; F) \to \dot{\mathcal{B}}^m(\mathbb{R}^n; G)$$

satisfying a consistency condition for decomposed elements, where

$$\dot{\mathcal{B}}^m(\mathbb{R}^n; E) = \{f \in \mathcal{E}^m(\mathbb{R}^n; E);\ \partial^\alpha f \in \mathcal{C}_0(E),\ |\alpha| \leq m\},$$

m finite, denotes the space of E-valued m-times continuously differentiable functions vanishing at infinity together with their derivatives. If E and F are both neither Fréchet nor (DF)-spaces, none of the results on vector-valued multiplications or convolutions can be applied directly. The results where one of the bilinear maps has to be continuous, i.e. the propositions in Section 3.1, cannot be applied, as $\dot{\mathcal{B}}^m(\mathbb{R}^n)$ is an infinite-dimensional Banach space and hence not nuclear by Theorem 13.3 in [44, p. 69]. As we assumed E and F are neither Fréchet spaces nor (DF)-spaces, the assumptions of Proposition 2 are not fulfilled. In the following, we will see that existence and hypocontinuity of this bilinear map can be shown using the "Théorèmes de croisement". In order to do so, we need the structural results on the space $\dot{\mathcal{B}}^m(\mathbb{R}^n; E)$, where E is a separated locally convex space; Proposition 7 generalizes the corresponding result for $\mathcal{C}_0(E)$ due to A. Grothendieck in [13, chap. I, pp. 89, 90].

First let us recall the properties of approximation by truncation and regularisation.

Definition 12 ([45, p. 7]). Let $\mathcal{H}$ be a space of distributions, i.e., $\mathcal{H}$ is contained in $\mathcal{D}'$ with a finer topology.

1. $\mathcal{H}$ is called a *strictly normal space of distributions* if $\mathcal{D}$ is contained in $\mathcal{H}$ with a finer topology and is strictly dense.
2. $\mathcal{H}$ satisfies the *approximation property by truncation* if for all $\alpha \in \mathcal{D}$ the multiplication $[\alpha]\colon \mathcal{H} \to \mathcal{H}$, $T \mapsto \alpha T$, is continuous and if α converges to 1 in $\mathcal{E}$ staying bounded in $\mathcal{B}$, the mapping $[\alpha]$ converges towards the identity mapping in $\mathcal{L}_c(\mathcal{H},\mathcal{H})$.
3. $\mathcal{H}$ satisfies the *approximation property by regularisation* if for all $\rho \in \mathcal{D}$ the regularisation mapping $\{\rho\}\colon \mathcal{H} \to \mathcal{H}$, $T \mapsto \rho * T$, is continuous and whenever for $\rho \geq 0$ the support of ρ converges toward the origin while at the same time the integral $\int \rho(x)\,\mathrm{d}x$ converges to 1, then $\{\rho\}$ converges towards the identity mapping in $\mathcal{L}_c(\mathcal{H},\mathcal{H})$.

Examples of non-normal spaces of distributions are the space $\mathcal{H}(\mathbb{C}) = \mathcal{H}(\mathbb{R}^2)$ of holomorphic functions and the space $L^\infty(\Omega)$ of bounded functions. They both also fail the approximation property by truncation. The spaces $L^\infty(\mathbb{R})$ and $\mathcal{D}'_{L^\infty}(\mathbb{R})$ also fail the approximation property by regularisation as for even functions $\rho \in \mathcal{D}$, $(\operatorname{sign} x * \rho)(0) = 0$.

In Proposition 3 in [45, p. 9], L. Schwartz shows that for a normal space of distributions the approximation property by truncation together with the approximation property by regularisation yields the strict approximation property. We will now use this result to prove the strict approximation property of the space $\dot{\mathcal{B}}^m$.

Proposition 6. *The space $\dot{\mathcal{B}}^m(\mathbb{R}^n)$ of m-times continuously differentiable functions vanishing at infinity together with their derivatives is a strictly normal space of distributions and has the strict approximation property.*

Proof. The space $\mathcal{D}^m$ of m-times continuously differentiable functions with compact support is contained in $\dot{\mathcal{B}}^m$ as a dense subspace according to [43, p. 99]. As by [45, p. 12], $\mathcal{D}^m$ is a normal space of distributions, so is $\dot{\mathcal{B}}^m$. As the multiplication mapping $\dot{\mathcal{B}}^m \times \mathcal{B} \to \dot{\mathcal{B}}^m$ is continuous, for all test functions $\alpha \in \mathcal{D}$ the linear map $\dot{\mathcal{B}}^m \to \dot{\mathcal{B}}^m$, $f \mapsto \alpha f$, is continuous and the image $f \cdot B$ under the multiplication of a bounded set $B \subset \mathcal{B}$ is bounded in $\dot{\mathcal{B}}^m$ for every function $f \in \dot{\mathcal{B}}^m$. Hence by Remarques in [45, p. 8] the space $\dot{\mathcal{B}}^m$ has the approximation property by truncation, since $\dot{\mathcal{B}}^m$, being a Banach space, is barrelled. The translation map $\dot{\mathcal{B}}^m \to \dot{\mathcal{B}}^m$, $f \mapsto \tau_h f$, is an isometric linear map and hence continuous. Additionally the set of translates $\{\tau_h f; h \in K\}$, where $K \subset \mathbb{R}^n$ is a bounded set and $f \in \dot{\mathcal{B}}^m$ a fixed function, is a bounded subset of $\dot{\mathcal{B}}^m$. Hence again by Remarques in [45, p. 8], $\dot{\mathcal{B}}^m$ has moreover the approximation property by regularisation as it is complete. Therefore by Proposition 3 in [45, p. 9] it is a strictly normal space of distributions and has the strict approximation property. ∎

Proposition 7. *For a quasi-complete separated locally convex space E, the spaces $\dot{\mathcal{B}}^m(E)$ and $\dot{\mathcal{B}}^m \,\varepsilon\, E$ are isomorphic as topological vector spaces.*

Proof. Given $f \in \dot{\mathcal{B}}^m(E)$, for all elements $e' \in E'$ of the dual space of E, the function $x \mapsto \langle f(x), e'\rangle$ is an element of $\dot{\mathcal{B}}^m$ as the evaluation mapping between E and E'_c is partially continuous. Hence the linear mapping

$$T_f \colon E'_c \to \dot{\mathcal{B}}^m, \qquad e' \mapsto [x \mapsto \langle f(x), e'\rangle], \tag{4.1}$$

is well-defined. By [15, p. 29] the space $\mathcal{C}_0(\mathbb{R}^n)$ is a topological subspace of the space of continuous functions on the one-point compactification of $\mathbb{R}^n$. Hence for all $|\alpha| \le m$ the set $\partial^\alpha g(\mathbb{R}^n)$ is a compact subset of E. As E is quasi-complete, also the absolutely convex hull $C := \operatorname{ac}(\partial^\alpha g(\mathbb{R}^n))$ of $\partial^\alpha g(\mathbb{R}^n)$ is a compact set. Therefore

$$\sup_{x\in\mathbb{R}^n} |\langle(\partial^\alpha f)(x), e'\rangle| \le \sup_{e\in C} |\langle e, e'\rangle|,$$

i.e., for all $|\alpha| \le m$ the mapping

$$E'_c \to \mathcal{C}_0, \qquad e' \mapsto [x \mapsto \langle \partial^\alpha f(x), e'\rangle] = [x \mapsto \partial^\alpha \langle f(x), e'\rangle],$$

is continuous. As the topology of $\dot{\mathcal{B}}^m$ is the initial topology with respect to the differentiation mappings $\partial^\alpha \colon \dot{\mathcal{B}}^m \to \mathcal{C}_0$ for $|\alpha| \le m$ the mapping in (4.1) is continuous. This proves $T_f \in \mathcal{L}(E'_c, \dot{\mathcal{B}}^m) \cong \dot{\mathcal{B}}^m \,\varepsilon\, E$. As by Proposition 7 in [24, p. 200], the topology of E is the topology of uniform convergence on equicontinuous subsets of E', it is immediate that the mapping

$$\dot{\mathcal{B}}^m(E) \hookrightarrow \mathcal{L}_\varepsilon(E'_c, \dot{\mathcal{B}}^m) \cong \dot{\mathcal{B}}^m \,\varepsilon\, E, \qquad f \mapsto T_f,$$

is a continuous and open embedding.

As moreover $\dot{\mathcal{B}}^m \otimes E \subset \dot{\mathcal{B}}^m(E) \subset \dot{\mathcal{B}}^m \,\varepsilon\, E$ as topological subspaces, we finally get $\dot{\mathcal{B}}^m(E) = \dot{\mathcal{B}}^m \,\varepsilon\, E$ by Corollaire 1 in [45, p. 47], since $\dot{\mathcal{B}}^m$ has the strict approximation property by Proposition 6. ■

DEFINITION 13 (cf. [43, p. 113]). We denote by $\dot{\mathcal{B}}^{l,m}_{x,y}$ the space of functions

$$f\colon \mathbb{R}^{n_1+n_2} \to \mathbb{C}$$

such that the derivatives $\partial_x^\alpha \partial_y^\beta f(x,y)$ exist, are continuous and vanish at infinity for $|\alpha| \le l$ and $|\beta| \le m$. The space $\dot{\mathcal{B}}^{l,m}_{x,y}$ is equipped with the norm

$$\|f\|_{\infty,l,m} = \sup_{|\alpha|\le l,\, |\beta|\le m}\ \sup_{x,y\in\mathbb{R}^{n_1+n_2}} |\partial_x^\alpha \partial_y^\beta f(x,y)|.$$

The space $\dot{\mathcal{B}}^{l,m}_{x,y}$ can be identified with spaces of vector-valued functions:

PROPOSITION 8. *We have the isomorphisms*

$$\dot{\mathcal{B}}^l_x(\dot{\mathcal{B}}^m_y) \cong \dot{\mathcal{B}}^m_y(\dot{\mathcal{B}}^l_x) \cong \dot{\mathcal{B}}^l_x \,\varepsilon\, \dot{\mathcal{B}}^m_y \cong \dot{\mathcal{B}}^{l,m}_{x,y}$$

and $\dot{\mathcal{B}}^{m,m}_{x,y} \subset \dot{\mathcal{B}}^m_{x,y}$ *with a finer topology.*

Proof. The isomorphisms

$$\dot{\mathcal{B}}^l_x(\dot{\mathcal{B}}^m_y) \cong \dot{\mathcal{B}}^m_y(\dot{\mathcal{B}}^l_x) \cong \dot{\mathcal{B}}^l_x \,\varepsilon\, \dot{\mathcal{B}}^m_y$$

follow from Proposition 7 and the symmetry of the ε-product. As by Proposition 4.3 in [44, p. 18] the ε-product of injective continuous maps is again injective, we have

$$\dot{\mathcal{B}}^l_x(\dot{\mathcal{B}}^m_y) = \dot{\mathcal{B}}^l_x \,\varepsilon\, \dot{\mathcal{B}}^m_y \subset \mathcal{E}^l_x \,\varepsilon\, \mathcal{E}^m_y = \mathcal{E}^{l,m}_{x,y}$$

by Proposition 12 in [43, p. 113]. Given $g \in \dot{\mathcal{B}}^l_x(\dot{\mathcal{B}}^m_y)$, the function $f \in \mathcal{E}^{l,m}_{x,y}$ obtained by this isomorphism is given by $f(x,y) := g(x)(y)$. Let $|\alpha| \le l$, $|\beta| \le m$ and $|(x,y)| \to \infty$ in $\mathbb{R}^{n_1+n_2}$, hence either $|x| \to \infty$ or $|y| \to \infty$ and therefore $\partial_x^\alpha \partial_y^\beta f(x,y) \to 0$.

Conversely, let $f \in \dot{\mathcal{B}}^{l,m}_{x,y}$; again by Proposition 12 in [43, p. 113], we obtain $g \in \mathcal{E}^l_x(\mathcal{E}^m_y)$ and $g(x) \in \dot{\mathcal{B}}^m_y$ for all $x \in \mathbb{R}^{n_1}$, if we set $g(x) := f(x,\cdot)$. By Lemme 1 in [43, p. 145], g is also an m-times continuously differentiable function with values in $\dot{\mathcal{B}}^m_y$ as the space $\dot{\mathcal{B}}^m$ is complete. Moreover, as $\partial^{(\alpha,\beta)} f(x,y) \to 0$ for $|(x,y)| \to \infty$ and $|\alpha|, |\beta| \le m$, $g(x)$ converges to zero in $\dot{\mathcal{B}}^m_y$ together with its derivatives of order smaller than or equal to m for $|x| \to \infty$ which can easily be seen with a proof by contradiction. Therefore $\dot{\mathcal{B}}^l_x(\dot{\mathcal{B}}^m_y) \cong \dot{\mathcal{B}}^{l,m}_{x,y}$ algebraically.

The topological isomorphism follows from the very definition of the topology of $\dot{\mathcal{B}}^{l,m}_{x,y}$. Finally the inclusion $\dot{\mathcal{B}}^{m,m}_{x,y} \subset \dot{\mathcal{B}}^m_{x,y}$ with a finer topology follows from the inequality

$$\sup_{|(\alpha,\beta)|\le m}\ \sup_{(x,y)\in\mathbb{R}^{n_1+n_2}} |\partial_x^\alpha \partial_y^\beta f(x,y)| \le \sup_{|\alpha|,|\beta|\le m}\ \sup_{x,y\in\mathbb{R}^{n_1+n_2}} |\partial_x^\alpha \partial_y^\beta f(x,y)|. \ ■$$

This result is related to Proposition 17 in [45, p. 59], where the assertion is shown for $l = m = \infty$, and to [13, chap. I, pp. 90], where the assertion is shown for $l = m = 0$.

Finally, the following proposition shows the existence of vector-valued bilinear combinations of bilinear operations in a particular case where the space of scalar-valued distributions is not nuclear.

Proposition 9. *Let E, F and G be quasi-complete separated locally convex spaces and $\theta\colon E \times F \to G$ a hypocontinuous bilinear map. There is a unique hypocontinuous bilinear mapping*

$$\underset{\theta}{\cdot}\colon \dot{\mathcal{B}}^m(\mathbb{R}^n; E) \times \dot{\mathcal{B}}^m(\mathbb{R}^n; F) \to \dot{\mathcal{B}}^m(\mathbb{R}^n; G),$$

satisfying the consistency condition

$$\underset{\theta}{\cdot}(g \otimes e, h \otimes f) = (g \cdot h) \otimes \theta(e, f)$$

for decomposed elements. If θ is continuous, so is $\underset{\theta}{\cdot}$.

Proof. By Theorem 1 ("Théorèmes de croisement", i.e., Proposition 2 in [46, p. 18]) there is a hypocontinuous bilinear map

$$\Gamma_{\beta,\varepsilon}\colon (\dot{\mathcal{B}}^m(\mathbb{R}^n) \widehat{\otimes}_\varepsilon E) \times (\dot{\mathcal{B}}^m(\mathbb{R}^n)\, \varepsilon\, F) \to (E \widehat{\otimes}_\beta F)\, \varepsilon\, (\dot{\mathcal{B}}^m(\mathbb{R}^n) \widehat{\otimes}_\varepsilon \dot{\mathcal{B}}^m(\mathbb{R}^n))$$

which coincides on $(\dot{\mathcal{B}}^m(\mathbb{R}^n) \otimes E) \times (\dot{\mathcal{B}}^m(\mathbb{R}^n) \otimes F)$ with the canonical mapping into the tensor product. As $\dot{\mathcal{B}}^m$ satisfies the strict approximation property it follows that $\dot{\mathcal{B}}^m(\mathbb{R}^n) \widehat{\otimes}_\varepsilon E = \dot{\mathcal{B}}^m\, \varepsilon\, E$ and $\dot{\mathcal{B}}^m(\mathbb{R}^n) \widehat{\otimes}_\varepsilon \dot{\mathcal{B}}^m(\mathbb{R}^n) = \dot{\mathcal{B}}^m(\mathbb{R}^n)\, \varepsilon\, \dot{\mathcal{B}}^m(\mathbb{R}^n)$. By Proposition 7 we obtain the identity $\dot{\mathcal{B}}^m(\mathbb{R}^n) \widehat{\otimes}_\varepsilon E = \dot{\mathcal{B}}^m(\mathbb{R}^n; E)$ and the inclusion

$$(E \widehat{\otimes}_\beta F)\, \varepsilon\, (\dot{\mathcal{B}}^m(\mathbb{R}^n) \widehat{\otimes}_\varepsilon \dot{\mathcal{B}}^m(\mathbb{R}^n)) \subset \dot{\mathcal{B}}^m(\mathbb{R}^{2n})\, \varepsilon\, (E \widehat{\otimes}_\beta F) = \dot{\mathcal{B}}^m(\mathbb{R}^{2n}; E \widehat{\otimes}_\beta F).$$

Hence the bilinear map

$$\Gamma_{\beta,\varepsilon}\colon \dot{\mathcal{B}}^m(\mathbb{R}^n; E) \times \dot{\mathcal{B}}^m(\mathbb{R}^n; F) \to \dot{\mathcal{B}}^m(\mathbb{R}^{2n}; E \widehat{\otimes}_\beta F)$$

is hypocontinuous and coincides on $(\dot{\mathcal{B}}^m(\mathbb{R}^n) \otimes E) \times (\dot{\mathcal{B}}^m(\mathbb{R}^n) \otimes F)$ with the canonical mapping into the tensor product. As θ is hypocontinuous, the corresponding linear map $\tilde{\theta}\colon E \otimes_\beta F \to G$ is continuous. We also denote by $\tilde{\theta}$ its extension to the quasi-completion $E \widehat{\otimes}_\beta F$. Therefore the bilinear map

$$\underset{\theta}{\otimes} := (\mathrm{id}\, \varepsilon\, \tilde{\theta}) \circ \Gamma_{\beta,\varepsilon}\colon \dot{\mathcal{B}}^m(\mathbb{R}^n; E) \times \dot{\mathcal{B}}^m(\mathbb{R}^n; F) \to \dot{\mathcal{B}}^m(\mathbb{R}^{2n}; G)$$

is hypocontinuous and satisfies the consistency condition

$$\underset{\theta}{\otimes}(g \otimes e, h \otimes f) = (g \otimes h) \otimes \theta(e \otimes f)$$

for decomposed elements $g \otimes e \in \dot{\mathcal{B}}^m(\mathbb{R}^n) \otimes E$ and $h \otimes f \in \dot{\mathcal{B}}^m(\mathbb{R}^n) \otimes F$. As the space

$$(\dot{\mathcal{B}}^m(\mathbb{R}^n) \otimes_\varepsilon E) \times (\dot{\mathcal{B}}^m(\mathbb{R}^n) \otimes_\varepsilon F) \subset \dot{\mathcal{B}}^m(\mathbb{R}^n; E) \times \dot{\mathcal{B}}^m(\mathbb{R}^n; F)$$

is a dense subspace, $\underset{\theta}{\otimes}$ is the uniquely determined partially continuous bilinear mapping with this property. The restriction of $\underset{\theta}{\otimes}$ to the diagonal in $\mathbb{R}^{2n}$ yields the mapping $\underset{\theta}{\cdot}$. If θ is not only hypocontinuous but continuous, using $\Gamma_{\pi,\varepsilon}$ instead of $\Gamma_{\beta,\varepsilon}$ yields the assertion, i.e., $\underset{\theta}{\cdot}$ is continuous. ∎

Remark. It can be shown (cf. [2, p. 36]) that this uniquely determined hypocontinuous bilinear map is in fact the mapping

$$\dot{\mathcal{B}}^m(\mathbb{R}^n; E) \times \dot{\mathcal{B}}^m(\mathbb{R}^n; F) \to \dot{\mathcal{B}}^m(\mathbb{R}^n; G), \qquad (f, g) \mapsto [x \mapsto b(f(x), g(x))],$$

which does not follow from the proof above.

Appendix. Spaces of test functions and distributions

Let us repeat the table of the main spaces of smooth functions and distributions as presented on p. 420 of L. Schwartz' treatise on distributions [47]:

$$\begin{array}{ccccccccccccccccccc}
\mathcal{D} & \subset & \mathcal{S} & & \subset & & \mathcal{D}_{L^p} & \subset & \mathcal{D}_{L^q} & \subset & \dot{\mathcal{B}} & \subset & \mathcal{B} & \subset & \mathcal{O}_C & \subset & \mathcal{O}_M & \subset & \mathcal{E} \\
\cap & & \cap & & & & \cap & & \cap & & \cap & & \cap & & & & \cap & & \cap \\
\mathcal{E}' & \subset & \mathcal{O}'_M & \subset & \mathcal{O}'_C & \subset & \mathcal{D}'_{L^p} & \subset & \mathcal{D}'_{L^q} & \subset & \dot{\mathcal{B}}' & \subset & \mathcal{B}' & & \subset & & \mathcal{S}' & \subset & \mathcal{D}'
\end{array}$$

Here, $1 \le p \le q < \infty$ and the symbol $\subset$ means continuous injection.

The definitions are:

$$\mathcal{D} = \{\varphi \in \mathcal{C}^\infty;\ \operatorname{supp}\varphi \text{ compact}\},$$
$$\mathcal{S} = \{\varphi \in \mathcal{C}^\infty;\ \forall k \in \mathbb{N}_0\ \forall \alpha \in \mathbb{N}_0^n\colon (1+|x|^2)^{k/2}\partial^\alpha\varphi \in \mathcal{C}_0\},$$

$\mathcal{C}_0$ denoting the space of continuous functions on $\mathbb{R}^n$ vanishing at infinity,

$$\mathcal{D}_{L^p} = \{\varphi \in \mathcal{C}^\infty;\ \forall \alpha \in \mathbb{N}_0^n\colon \partial^\alpha\varphi \in L^p\}, \quad 1 \le p < \infty,$$
$$\dot{\mathcal{B}} = \{\varphi \in \mathcal{C}^\infty;\ \forall \alpha \in \mathbb{N}_0^n\colon \partial^\alpha\varphi \in \mathcal{C}_0\},$$
$$\mathcal{B} = \{\varphi \in \mathcal{C}^\infty;\ \forall \alpha \in \mathbb{N}_0^n\colon \partial^\alpha\varphi \in L^\infty\},$$
$$\mathcal{O}_C = \{\varphi \in \mathcal{C}^\infty;\ \exists k \in \mathbb{N}_0\ \forall \alpha \in \mathbb{N}_0^n\colon (1+|x|^2)^{k/2}\partial^\alpha\varphi \in \mathcal{C}_0\},$$
$$\mathcal{O}_M = \{\varphi \in \mathcal{C}^\infty;\ \forall \alpha \in \mathbb{N}_0^n\ \exists k \in \mathbb{N}_0\colon (1+|x|^2)^{k/2}\partial^\alpha\varphi \in \mathcal{C}_0\},$$
$$\mathcal{E} = \mathcal{C}^\infty.$$

$\mathcal{E}'$, $\mathcal{O}'_M$, $\mathcal{O}'_C$ and $\mathcal{S}'$ are the spaces of distributions with compact support, of very rapidly decreasing distributions, of rapidly decreasing distributions, and of temperate distributions, respectively.

The space $\dot{\mathcal{B}}'$ is defined as the closure of $\mathcal{E}'$ in $\mathcal{B}' = \mathcal{D}'_{L^\infty} = (\mathcal{D}_{L^1})'$.

The spaces $\mathcal{D}$, $\mathcal{S}$, $\mathcal{O}_C$, $\mathcal{O}_M$, $\mathcal{E}$ and their duals $\mathcal{E}'$, $\mathcal{O}'_M$, $\mathcal{O}'_C$, $\mathcal{S}'$ and $\mathcal{E}'$ are nuclear spaces. Further properties can be found in J. Horváth's table ([24, p. 442]).

The spaces $\mathcal{D}_{L^p}$, $\mathcal{D}'_{L^p}$, $1 < p < \infty$, are reflexive and complete, but neither nuclear nor Schwartz spaces.

All these spaces are normal spaces of distributions with the exception of $\mathcal{B}$ and $\mathcal{B}'$.

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