1. Introduction

Let $A$ and $B$ be subsets of $\mathbb{R}^n$. Their sum $A + B$, called the Minkowski sum of $A$ and $B$, is defined by

$$A + B = \{a + b \mid a \in A, b \in B\}.$$ 

Being one of the most fundamental operations on sets in spaces with the addition operation, Minkowski sum has been used, both implicitly and explicitly, in virtually all branches of mathematics. However, there have not been many investigations of the properties of Minkowski sum itself. One notable exception is the pioneering works by H. Brunn, H. Minkowski, and others on the so-called Brunn–Minkowski theory, which compares the volumes of the Minkowski sum and its summands [1, 16, 19].

Meanwhile, mainly due to the convenience of describing various geometric relations, the Minkowski sum has been adopted and used extensively in engineering and computer science. A few examples are mechanical engineering (collision-free path planning [13]), image processing and mathematical morphology [6, 20], computer graphics (metamorphosis [5]), geometric modeling (offset and sweep curve/surface generation [15, 21], computation of CSG operations [18]), and computational geometry [7].

The common problem persistent in all such applications is the efficient computation of Minkowski sums [8, 9, 11]. But the need for dealing with complex geometric objects encountered in real-world applications makes this goal seem far from satisfactory. Thus there naturally arises the need for fundamental geometric and topological analysis of Minkowski sum, which should be more detailed than just comparing volumes.

In this paper, we will investigate some global topological properties of Minkowski sum in relation to the geometric structures of its summands. Although Minkowski sum has a simple definition, it may lead to a lot of complicated phenomena. In general, the Minkowski sum operation does not preserve the topological properties of the sets in the Euclidean space. To give an idea, we first show some examples: see Figures 1, 2 and 3. Note that all the summands in these figures are homeomorphic to the unit disk. But in Figure 1, the result of the Minkowski sum is not simply connected. In Figure 2, the Minkowski sum is not simply connected, and its boundary is not homeomorphic to the unit circle. Worse still, the Minkowski sum has infinitely many “holes” in Figure 3.

These examples show that even when the summands are topologically simple, their Minkowski sum can become quite complex in the topological sense. Especially, Minkowski sum does not preserve even the simplest topological property of the sets in $\mathbb{R}^2$, that is, that of being homeomorphic to the unit disk.

Thus the following natural problem arises:
Problem 1. Find a class of sets in $\mathbb{R}^2$ which are homeomorphic to the unit disk, such that the Minkowski sums of sets in that class are always homeomorphic to the unit disk.

An immediate answer to this problem is the class of all convex sets which are homeomorphic to the unit disk, since it can be shown easily that the Minkowski sum of convex sets is also convex. But a serious drawback of the convexity is that it is too strong: there are too many useful sets which are not convex. So another important problem is:

Problem 2. Find a class of sets in $\mathbb{R}^2$ which contains all convex sets homeomorphic to the unit disk, and is maximal among all the classes satisfying the condition in Problem 1.

If we consider two bounded sets $A$ and $B$ in the plane as rigid, mutually impenetrable objects, then the complement of the Minkowski sum $A + B$ in $\mathbb{R}^2$ represents the set of all possible relative positions of the translates of $A$ and $-B$. One such configuration can be continuously moved into another by translation without mutual penetrating if and only if the two configurations are in the same connected component of the complement of $A + B$ in $\mathbb{R}^2$. So, the Minkowski sum $A + B$ is simply connected if and only if any two relative
positions of the translates of $A$ and $-B$ can be continuously moved into each other by translation without mutual penetrating, or, in other words, any relative positions can be continuously pulled over to separate $A$ and $-B$ indefinitely.

We will show that there exists an important class of planar domains that we call \emph{semi-convex}, which satisfy the conditions both in Problems 1 and 2. Intuitively speaking, a planar domain is semi-convex if the normal vector field along the boundary does not turn concavely by more than the angle $\pi$. We mention that our definition of semi-convexity differs from that introduced in [14]. It is also significantly more general than the usual notion of star-shapedness, and, to the author’s knowledge, it is the first among the many variations of convexity which has an optimal property with respect to the Minkowski sum.

In general, the boundary curves of a Minkowski sum are results of the operation called \emph{convolution} on the boundary curves of the summands. The convolution can be considered as a basic building block in analyzing the Minkowski sum of the shapes represented by boundary curves. But there has been few precise mathematical studies on the convolution of curves in the literature. Also, we will observe in Section 2 that the convolution can behave wildly unless we restrict the class of the curves to be convolved, which is a fact not often noted in both theory and practice. So in Section 2, we carefully analyze the mathematical properties of the convolution of curves, and classify the curve classes according to their differential regularity with particular regard to convolution.

Often in practice, the curve pieces used to describe shape boundaries come from specific fixed classes such as the class of rational curves or various classes of splines (e.g., the NURBS curves). However, most of these important curve classes are not closed under convolution, which makes it impossible to represent the Minkowski sum boundary in a uniform manner (i.e., with the curve pieces in the same curve class used to represent the summands), and thus causes serious problems in practice. Meanwhile, it also turns out that the curve classes $C^{k,l}$ and $C_c^{k,l}$ introduced in Section 2 are not closed under convolution. These facts imply that the usual conditions on the boundary curves such as rationality or differentiability are not preserved under Minkowski sum. In particular, it is not clear whether the notion of semi-convexity is closed under Minkowski sum, unless we restrict the boundary curves to be in special curve classes. Thus it is a necessary and important problem to find a condition on classes of curves which guarantees closedness under convolution.

In Section 2, we introduce special curve classes, called \emph{Minkowski classes}, which are closed under convolution. An important example of a Minkowski class, denoted by $\mathcal{W}$, is given in Section 3, for which we use Łojasiewicz’s structure theorem for real-analytic varieties [12]. It is shown that $\mathcal{W}$ contains practically all the curves used in engineering applications. This in particular means that it is not too restrictive to consider the Minkowski sum only in the category of $\mathcal{M}$-domains for a Minkowski class $\mathcal{M}$. Here, an \emph{$\mathcal{M}$-domain} means a subset in $\mathbb{R}^2$ whose boundary consists of finitely many curves in $\mathcal{M}$.

Note that we consider a fairly general class of domains, including ones with corners on their boundaries. In fact, this is also necessary, since such domains can arise naturally as a result of the Minkowski sum operation on quite nice domains. To handle them, we introduce two concepts: \emph{sector} in Section 4, and \emph{virtual boundary} in Section 5. A sector is a
local germ of a domain near a boundary point, whether cornered or not. So, by examining
the effect of Minkowski sum on sectors, we can understand the essential and local behavior
of Minkowski sum. By integrating these results, we obtain the global result in Section 6
that the set of all $\mathcal{M}$-domains is closed under Minkowski sum for any Minkowski class
$\mathcal{M}$, which is a basis for the further closedness result for semi-convexity.

The notion of virtual boundary is a generalization of that of the usual boundary
in a way that incorporates corners in a uniform manner. It is defined to be in one-to-
one continuous correspondence with the outer normal vectors on the boundary including
those at the corners. Together with the analysis of sectors, the notion of virtual boundary
enables a uniform and easy treatment of cornered domains, thus reducing the globally
complex problem of Minkowski sum to the analysis of a few local genotypes of the sectors.

The notion of semi-convexity, which generalizes that of convexity, will be formally
introduced in Section 7. Let $\mathcal{M}$ be a Minkowski class. It is proved that the Minkowski
sum of any two semi-convex $\mathcal{M}$-domains is homeomorphic to the unit disk, which answers
Problem 1 above within the category of $\mathcal{M}$-domains. In Section 8, we prove that for any
$\mathcal{M}$-domain which is homeomorphic to the unit disk but is not semi-convex, there exists
a semi-convex $\mathcal{M}$-domain such that their Minkowski sum is not homeomorphic to the
unit disk. This answers Problem 2 above within the category of $\mathcal{M}$-domains. In fact,
it is shown that the set of all semi-convex $\mathcal{M}$-domains is uniquely maximal among all
the classes of $\mathcal{M}$-domains which satisfy the condition in Problem 1 and contain all the
$\mathcal{M}$-domains called flag domains. Finally, we prove in Section 9 that the set of all semi-
convex $\mathcal{M}$-domains is closed under Minkowski sum. In proving these results, we will use
the Gauss–Bonnet Theorem, translated into the language of virtual boundary, as one of
the main tools. In Section 10, we summarize the results in this paper, and discuss some
further research directions.

Since semi-convexity is geometric in nature, the properties of semi-convex domains
proved in this paper reveal a new relationship between the geometric and topological
properties of Minkowski sum. Also, since semi-convexity can easily be checked algorithm-
ically, it is expected to be utilized in various application areas using Minkowski sum.

## 2. Curves

The boundaries of reasonable domains consist of curves. So, to analyse domains, we
first analyse curves. In this section, we define various special curve classes according to
their regularity, and study their properties with respect to the operation of convolution.
In particular, the Minkowski classes are introduced, which are defined essentially to be
closed under convolution. We also set up some conventions and notations which will be
used throughout this paper.

Let $\mathbf{v} = (v_1, v_2)$, $\mathbf{w} = (w_1, w_2)$ be in $\mathbb{R}^2$. We write $\mathbf{v} \parallel \mathbf{w}$ if either at least one of $\mathbf{v}$
and $\mathbf{w}$ is $0 = (0, 0)$, or $\mathbf{v} = k \mathbf{w}$ for some $k \in \mathbb{R}$. Let $p \in \mathbb{R}^2$ and $r > 0$. By $B_r(p)$, we
always denote the closed ball in $\mathbb{R}^2$, centered at $p$ and with radius $r$. The open ball will
be denoted by $B^*_r(p)$. The unit circle in $\mathbb{R}^2$ will be denoted by $S^1$. Thus, $S^1 = \{ \mathbf{v} \in \mathbb{R}^2 \mid |\mathbf{v}| = 1 \} = \partial B_1(0)$. 
2.1. Convolution

**Definition 2.1 (C^{k;l} curve).** Let $k, l = 1, 2, \ldots, \infty, \omega$ (\(\omega\) for real-analytic), and $k \geq l$. Let $n = 1, 2, \ldots$. A curve $\gamma : (a, b) \to \mathbb{R}^n$ is called a $C^k$ curve if there exists a reparametrization $\tilde{\gamma} : (\tilde{a}, \tilde{b}) \to \mathbb{R}^2$ of $\gamma$ such that $\tilde{\gamma}' \neq 0$ on $(\tilde{a}, \tilde{b})$, and $\tilde{\gamma}$ is $C^k$. A curve $\gamma : [a, b] \to \mathbb{R}^n$ is called a $C^{k;l}$ curve if the restriction of $\gamma$ to $(a, b)$ is a $C^k$ curve, and there exists an extension $\tilde{\gamma} : (a - \varepsilon, b + \varepsilon) \to \mathbb{R}^n$ of $\gamma$ for some $\varepsilon > 0$, such that $\tilde{\gamma}$ is a $C^l$ curve.

Here, it is important to note that $\tilde{\gamma}' \neq 0$. Without this condition, a curve $\gamma$ may not be a $C^k$ curve, even if it is $k$-times differentiable.

**Definition 2.2 (the class $C^{k;l}$).** Let $k, l = 1, 2, \ldots, \infty, \omega$ and $k \geq l$. Then we denote by $C^{k;l}$ the class of all $C^{k;l}$ curves in $\mathbb{R}^2$ defined on closed intervals, which have no self-intersections. An element in $C^{k;l}$ will be called a $C^{k;l}$-curve.

Note that closed loops are excluded in this definition. The inclusion relations between the classes $C^{k;l}$ in Figure 4 are immediate from the definition.

\[
\begin{array}{cccc}
C^{1:1} & C^{1:2} & C^{1:\omega} & C^{1:0} \\
C^{2:1} & C^{2:2} & C^{2:\omega} & C^{2:0} \\
\vdots & \vdots & \vdots & \vdots \\
C^{\infty:1} & C^{\infty:2} & C^{\infty:\omega} & C^{\infty:0} \\
\end{array}
\]

**Fig. 4.** Inclusion relations for $C^{k;l}$ and $C^{k;l}_e$

**Remark 2.1.** Given a $C^{k;l}$-curve $\gamma : [a, b] \to \mathbb{R}^2$, we usually assume that it is defined on some slightly larger open interval $(a - \varepsilon, b + \varepsilon)$, and $\gamma$ is $k$-times differentiable on $(a, b)$, $l$-times differentiable on $(a - \varepsilon, b + \varepsilon)$, and $\gamma' \neq 0$ on $(a - \varepsilon, b + \varepsilon)$.

Let $\gamma : [a, b] \to \mathbb{R}^2$ be a $C^{1:1}$-curve, and let $\tilde{\gamma} : (a - \varepsilon, b + \varepsilon) \to \mathbb{R}^2$ be a $C^1$ extension of $\gamma$. It is easy to see that the limit

\[
\nu[\gamma](t) = \lim_{\tau \to t} \frac{\tilde{\gamma}(\tau) - \tilde{\gamma}(t)}{|\tilde{\gamma}(\tau) - \tilde{\gamma}(t)|}
\]

exists in $S^1$ for every $t \in [a, b]$, and $\nu[\gamma] : [a, b] \to S^1$ is continuous. We will denote $\nu[\gamma]$ by $\nu[\gamma]$. Note that these are independent of the choice of $\tilde{\gamma}$. Let $\mu : \mathbb{R} \to S^1$ be the covering map defined by $\mu(t) = (\cos t, \sin t)$ for $t \in \mathbb{R}$. Now there exists a continuous function $\theta : [a, b] \to \mathbb{R}$ such that $\nu[\gamma](t) = \mu(\theta(t))$ for every $t \in [a, b]$. We call $\theta$ an angle function of $\gamma$. Note that if $\tilde{\theta}$ is another angle function of $\gamma$, then, for some integer $n$, we have $\tilde{\theta}(t) = \theta(t) + 2n\pi$ for every $t \in [a, b]$. So the following is well defined:

**Definition 2.3 (convex curve).** Let $\gamma : [a, b] \to \mathbb{R}^2$ be a $C^{1:1}$-curve, and let $\theta : [a, b] \to \mathbb{R}$ be an angle function of $\gamma$. Then $\gamma$ is called **convex** if $\theta$ is either strictly increasing or strictly decreasing, provided it is not constant. The **signature** of $\gamma$, $\sigma(\gamma)$, is defined to be $+ \ (\text{resp., } -)$ if $\theta$ is strictly increasing (resp., strictly decreasing), and $0$ if $\theta$ is constant.
For \( k, l = 1, 2, \ldots, \infty, \omega \) with \( k \geq l \), we denote by \( C_c^{k:l} \) the class of all convex curves in \( C^{k:l} \). An element of \( C_c^{k:l} \) will be called a \( C_c^{k:l} \)-curve.

From the above definition, the inclusion relations between the classes \( C_c^{k:l} \) in Figure 4 are obvious.

**Definition** 2.4 (*-admissible curves). Two \( C_c^{1:1} \)-curves \( \gamma_1, \gamma_2 \) are said to be *-admissible to each other if \( v[\gamma_1] \parallel v[\gamma_2] \) and \( \sigma(\gamma_1) = \sigma(\gamma_2) \neq 0 \).

Note that the *-admissibility is a transitive relation. Let \( \gamma_i : [a_i, b_i] \to \mathbb{R}^2, i = 1, \ldots, n \), be \( C_c^{1:1} \)-curves which are *-admissible to each other. Let \( \tilde{\theta}_i : [a_i, b_i] \to \mathbb{R} \) be an angle function of \( \gamma_i \) for each \( i \). For each \( i \), define \( \tilde{\theta}_i : [a_i, b_i] \to \mathbb{R} \) by \( \tilde{\theta}_i(t) = \theta_i(t) + \pi \) if \( v[\gamma_i] = v[\gamma_1] \), and \( \theta_i(t) = \tilde{\theta}_i(t) + \pi \) if \( v[\gamma_i] = -v[\gamma_1] \). Then, with no loss of generality, we can assume \( \tilde{\theta}_i(a_i) = \theta_1(a_1) \) for each \( i \). Define \( \alpha = \min \{ \theta_1(b_1), \ldots, \theta_n(b_n) \} \) if \( \sigma(\gamma_1) = + \), and \( \alpha = \max \{ \theta_1(b_1), \ldots, \theta_n(b_n) \} \) if \( \sigma(\gamma_1) = - \). Let \( h : [0, 1] \to \mathbb{R} \) be the linear function with \( h(0) = \theta_1(a_1) \) and \( h(1) = \alpha \). Now we define \( \gamma = \gamma_1 \ast \cdots \ast \gamma_n : [0, 1] \to \mathbb{R}^2 \), the convolution of \( \gamma_1, \ldots, \gamma_n \), by

\[
\gamma(t) = \gamma_1(\theta_1^{-1}(h(t))) + \cdots + \gamma_n(\theta_n^{-1}(h(t)))
\]

for \( t \in [0, 1] \). Note that \( v[\gamma_1](\theta_1^{-1}(h(t))) \parallel \cdots \parallel v[\gamma_n](\theta_n^{-1}(h(t))) \) for every \( t \).

From the definition, it is clear that the result of convolution does not depend on the order of operations. It is also easy to see that convolutions are continuous curves. But in general, a convolution of \( C_c^{1:1} \)-curves can exhibit quite anomalous behavior, and it cannot be expected to be even a \( C_c^{1:1} \)-curve. This can happen even when the terms belong to \( C_c^{\omega:\omega} \), as can be seen from the following example:

**Example 2.1.** For some small \( \delta > 0 \), let \( \gamma_+, \gamma_- : [0, \delta] \to \mathbb{R}^2 \) be given by \( \gamma_{\pm}(t) = (t, f_{\pm}(t)) \) for \( t \in [0, \delta] \), where \( f_{\pm} : [0, \delta] \to \mathbb{R} \) are defined by

\[
f_{\pm}(t) = \int_0^t \frac{1}{\xi^2} \exp \left( -\frac{1}{\xi} \right) \left[ 4 \pm \left\{ 1 + \sqrt{2} \sin \left( \frac{1}{\xi} - \frac{\pi}{4} \right) \right\} \right] d\xi
\]

for \( t \in [0, \delta] \). Note that

\[
0 < \frac{1}{\xi^2} \exp \left( -\frac{1}{\xi} \right) \left[ 4 \pm \left\{ 1 + \sqrt{2} \sin \left( \frac{1}{\xi} - \frac{\pi}{4} \right) \right\} \right] \leq (5 + \sqrt{2}) \frac{1}{\xi^2} \exp \left( -\frac{1}{\xi} \right)
\]

for every \( \xi > 0 \). So we have

\[
0 < f_{\pm}(t) \leq (5 + \sqrt{2}) \int_0^t \frac{1}{\xi^2} \exp \left( -\frac{1}{\xi} \right) d\xi = (5 + \sqrt{2}) \exp \left( -\frac{1}{t} \right)
\]

for every \( t \in (0, \delta] \). This shows that \( f_+ \) and \( f_- \) are well defined. It is easy to see that \( f_{\pm} \) are real-analytic on \( (0, \delta] \), and \( \lim_{t \to 0^+} f_{\pm}^{(k)} = 0 \) for every \( k < \infty \). Hence \( \gamma_{\pm} \in C^{\omega:\omega} \). Note that

\[
f''_{\pm}(t) = \frac{1}{t^4} \exp \left( -\frac{1}{t} \right) \left[ (1 - 2t) \left\{ 4 \pm \left\{ 1 + \sqrt{2} \sin \left( \frac{1}{t} - \frac{\pi}{4} \right) \right\} \right\} \right] \mp \sqrt{2} \cos \left( \frac{1}{t} - \frac{\pi}{4} \right)
\]
for \( t > 0 \). So it follows that \( f''(t) > 0 \) for \( t \in (0, \delta) \) if we choose sufficiently small \( \delta > 0 \). This shows that \( \gamma_+, \gamma_- \in C^{\omega,\infty}_c \), and \( v[\gamma_1] = v[\gamma_2] = (1, 0), \sigma(\gamma_1) = \sigma(\gamma_2) = + \). Let \( f = f_+ - f_- \). Then

\[
    f(t) = 2 \int_{0}^{t} \frac{1}{\xi^2} \exp \left( -\frac{1}{\xi} \right) \left( 1 + \sqrt{2} \sin \left( \frac{1}{\xi} - \frac{\pi}{4} \right) \right) d\xi = 2 \exp \left( -\frac{1}{t} \right) \left( 1 + \sin \frac{1}{t} \right)
\]

for \( t \in (0, \delta) \). Let \( t_n = (3\pi/2 + 2\pi N + 2n\pi)^{-1} \) for \( n = 1, 2, \ldots, \) where \((3\pi/2 + 2\pi N + 2\pi)^{-1} \leq \delta < (3\pi/2 + 2\pi N)^{-1} \). Let \( S = \{(s, t) \in [0, \delta] \times [0, \delta] \mid \gamma_+(s) = \gamma_-(t) \} \). It is easy to see that \( S = \{(t_n, t_n) \mid n = 1, 2, \ldots \}\cup\{(0, 0)\} \). Note also that \( f'(t_n) = f'_+(t_n) - f'_-(t_n) = 0 \) for all \( n \). Now \( \gamma_+ \) and \( -\gamma_- \) are in \( C^{\omega,\infty}_c \), and \(*\)-admissible to each other. Let \( \gamma = \gamma_1 * (-\gamma_2) : [0, 1] \to \mathbb{R}^2 \). Then, from the above argument, it is easy to see that there exist sequences \( a_n, b_n \searrow 0 \) with \( a_{n+1} < b_n < a_n \) such that \( \gamma(a_n) = 0 \) and \( \gamma(b_n) \neq 0 \) for every \( n \). Clearly, this cannot happen for a \( C^{1,1} \)-curve. Thus we conclude that \( \gamma \notin C^{1,1} \).

The following lemma shows that the convolution behaves as expected if we know beforehand that it has only a mild regularity, i.e., \( C^{1,1} \).

**Lemma 2.1.** Let \( \gamma_i : [a_i, b_i] \to \mathbb{R}^2, i = 1, 2, \) be two \( C^{1,1}_c \)-curves which are \(*\)-admissible to each other. Let \( \gamma = \gamma_1 * \gamma_2 \). Suppose \( \gamma \in C^{1,1} \). Then, for any \( t, t_1, t_2 \) such that \( \gamma_1(t_1), \gamma_2(t_2) \) are summed to the convolution \( \gamma_1 * \gamma_2 \), we have

\[
    v[\gamma](t) \parallel v[\gamma_1](t_1) \parallel v[\gamma_2](t_2).
\]

In consequence, \( \gamma \) is in \( C^{1,1} \) and is \(*\)-admissible to \( \gamma_1 \) and \( \gamma_2 \).

**Proof.** Let \( v = v[\gamma_1](t_1) = \pm v[\gamma_2](t_2) \). First, note that

\[
    \frac{\gamma(t) - \gamma(t_1)}{\gamma(t) - \gamma(t_1)} = \frac{1}{v_1 + k v_2} \cdot v_1 + \frac{1}{v_2 + \frac{1}{k} v_1} \cdot v_2,
\]

where

\[
    v_1 = \frac{\gamma_1(t_1) - \gamma(t_1)}{\gamma(t) - \gamma(t_1)} , \quad v_2 = \frac{\gamma_2(t_2) - \gamma(t_2)}{\gamma(t) - \gamma(t_2)} , \quad k = \frac{\gamma_2(t_2) - \gamma(t_2)}{\gamma(t) - \gamma(t_1)} , \quad \gamma(t) = \gamma_1(t_1) + \gamma_2(t_2).
\]

Let \( v = \lim_{t \to t} v_1 = \pm \lim_{t \to t} v_2 \). Then

\[
    v[\gamma](t) = \lim_{t \to t} \left( \frac{1}{v_1 + k v_2} \pm \frac{1}{v_2 + \frac{1}{k} v_1} \right) \cdot v = \lim_{t \to t} \frac{1 \pm k}{v_1 + k v_2} \cdot v.
\]

Since we know that \( v[\gamma](t) \in S^1 \), it follows that \( v[\gamma](t) = v \) or \(-v \). Now the rest of the proof follows easily. ■

### 2.2. Minkowski class

**Definition 2.5.** (Minkowski class) A subclass \( \mathcal{M} \) of \( C^{1,1}_c \) is called a **Minkowski class** if the following two conditions are satisfied:

1. \( \mathcal{M} \) is closed under **restriction**, i.e., if \( \gamma : [a, b] \to \mathbb{R}^2 \) is in \( \mathcal{M} \), then \( \gamma|_{[c, d]} \) is also in \( \mathcal{M} \) for any \( [c, d] \subset [a, b] \).

2. \( \mathcal{M} \) is closed under **initial convolution**, i.e., for any two \(*\)-admissible \( \mathcal{M} \)-curves \( \gamma_1 : [a_1, b_1] \to \mathbb{R}^2 \) and \( \gamma_2 : [a_2, b_2] \to \mathbb{R}^2 \), the convolution \( \gamma_1|[a_1, a_1+\varepsilon] * \gamma_2|[a_2, a_2+\varepsilon] \) is either an \( \mathcal{M} \)-curve or constant for some \( \varepsilon > 0 \).
As an example, let $\mathcal{LA}$ be the set of all line segments and circular arcs in $\mathbb{R}^2$. It can be easily checked that $\mathcal{LA}$ is a Minkowski class. In Section 3, we will present a non-trivial Minkowski class $\mathcal{W}$, which is significantly larger than $\mathcal{LA}$.

Let $\gamma_i : [a_i, b_i] \to \mathbb{R}^2$, $i = 1, 2$, be two continuous curves. We say that $\gamma_1, \gamma_2$ have an intersection at $(s, t)$ if $\gamma_1(s) = \gamma_2(t)$. We say that $\gamma_1, \gamma_2$ have an isolated intersection at $(s, t)$ if $\gamma_1(s) = \gamma_2(t)$ and $\gamma_1(s') \neq \gamma_2(t')$ for every $(s', t') \in (s - \varepsilon, s + \varepsilon) \times (t - \varepsilon, t + \varepsilon) \setminus \{(s, t)\}$ for some $\varepsilon > 0$.

The next lemma shows an important property of Minkowski classes:

**Lemma 2.2.** Any two $\gamma_1, \gamma_2$ in a Minkowski class $\mathcal{M}$ cannot have infinitely many isolated intersections.

**Proof.** With no loss of generality, assume $a_1 = a_2 = 0$, where $\gamma_i : [a_i, b_i] \to \mathbb{R}^2$ for $i = 1, 2$. Suppose $\gamma_1$ and $\gamma_2$ have infinitely many isolated intersections. Since $[a_1, b_1] \times [a_2, b_2]$ is compact, there exists an accumulation point of the isolated intersections, which we can assume to be $\gamma_1(0) = \gamma_2(0)$. Also, we can assume $\gamma_1(0) = \gamma_2(0) = 0$ and $\nu(\gamma_1) = (1, 0)$.

Since $\gamma_1(0) = \gamma_2(0)$ is an accumulation point of the isolated intersections, we can also assume that $\nu(\gamma_2) = \nu(\gamma_1) = (1, 0)$ and $\sigma(\gamma_1) = \sigma(\gamma_2) = +$. Thus, for $i = 1, 2$, we can write $\gamma_i(t) = (t, f_i(t))$ for small $t \geq 0$, where $f_i$ is a $C^1$ function such that $f_i(0) = f'_i(0) = 0$, and $f'_i$ is strictly increasing. Now there exists a sequence $t_n \searrow 0$ such that $\gamma_1$ and $\gamma_2$ have an isolated intersection at $(t_n, t_n)$ for every $n$. If $f'_1(t_n) = f'_2(t_n)$ except for at most finitely many $n$’s, then the convolution $\gamma = \gamma_1 * (-\gamma_2)$ would not be in $C_{1,1}^{1,1}$, which can be seen from the argument in Example 2.1. So we can assume $f'_1(t_n) \neq f'_2(t_n)$ for every $n$. We can also assume that $f_1(t) \neq f_2(t)$ if $t \neq t_n$ for any $n$. In this case, it is easy to see that $\gamma(t_n)$’s are in the regions $D_1$ and $D_3$ alternating with $n$, where $D_1 = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$ and $D_3 = \{(x, y) \in \mathbb{R}^2 \mid x < 0, y < 0\}$. But this is impossible, since $\gamma$ should be in $\mathcal{M}$, and thus in $C_{1,1}^{1,1}$.

**Remark 2.2.** Example 2.1 shows that two $C_1^{\omega;\infty}$-curves can have infinitely many isolated intersections, which implies that $C_{1,1}^{k;l}$ is not a Minkowski class for $k, l = 1, 2, \ldots, \infty, \omega$, $k \geq l$, except for $C_{1,1}^{\omega;\omega}$. Later, we will also see that $C_{1,1}^{\omega;\omega}$ is not a Minkowski class.

Let $\gamma_i : [a_i, b_i] \to \mathbb{R}^2$, $i = 1, 2$, be two one-to-one continuous curves. We write $\gamma_1 \approx \gamma_2$ if there exist $a_i < c_i \leq b_i$ for $i = 1, 2$ and a homeomorphism $h : [a_1, c_1] \to [a_2, c_2]$ such that $h(a_1) = a_2$ and $\gamma_1(t) = \gamma_2(h(t))$ for every $t \in [a_1, c_1]$. We write $\gamma_1 \sim \gamma_2$ if $\gamma_1$ can be moved to a curve $\tilde{\gamma}_1$ by a rigid motion in the plane so that $\tilde{\gamma}_1 \approx \gamma_2$. Note that both the relations $\approx$ and $\sim$ are symmetric and transitive.

Let $\gamma_i : [a_i, b_i] \to \mathbb{R}^2$, $i = 1, 2$, be two $C_1^{1,1}$-curves. Note that, with appropriate rigid motions in the plane, we can always move $\gamma_1$ and $\gamma_2$ to obtain curves $\tilde{\gamma}_1, \tilde{\gamma}_2$ respectively so that $\tilde{\gamma}_1(a_1) = \tilde{\gamma}_2(a_2) = 0$, $\nu(\tilde{\gamma}_1) = \nu(\tilde{\gamma}_2) = (1, 0)$ and $\sigma(\tilde{\gamma}_1), \sigma(\tilde{\gamma}_2) \geq 0$. We write $\gamma_1 \triangleright \gamma_2$ (resp., $\gamma_1 \triangleleft \gamma_2$) if there exist continuous functions $f_1, f_2 : [0, \varepsilon) \to \mathbb{R}$, for some $\varepsilon > 0$, such that the graph of $f_i$ is contained in the image of $\tilde{\gamma}_i$ for $i = 1, 2$, and $f_1(x) > f_2(x)$ (resp., $f_1(x) < f_2(x)$) for every $x \in (0, \varepsilon]$.

Let $\mathcal{M}$ be a Minkowski class. As an important consequence of Definition 2.5 and Lemma 2.2, note that, given any $\gamma_1, \gamma_2$ in $\mathcal{M}$, there are only three possibilities: either $\gamma_1 \triangleright \gamma_2$, or $\gamma_1 \triangleleft \gamma_2$, or $\gamma_1 \sim \gamma_2$. Suppose $\gamma_1$ and $\gamma_2$ are $*$-admissible to each other. Then
the convolution $\gamma = \gamma_1 \ast \gamma_2$ is \textit{initially constant} (that is, constant for some interval from the start) if and only if $\gamma_1 \sim \gamma_2$ and $v[\gamma_1] = -v[\gamma_2]$. For the remaining cases, $\gamma$ is initially in $\mathcal{M}$, and the next lemma shows the relation between $\gamma$ and $\gamma_1, \gamma_2$ with regard to the above relations $\succ, \prec$ and $\sim$. See Figure 5 for the illustration of these results.

**Lemma 2.3** (convolution in Minkowski class). Let $\mathcal{M}$ be a Minkowski class, and let $\gamma_i : [a_i, b_i] \to \mathbb{R}^2$, $i = 1, 2$, be two $\mathcal{M}$-curves which are $\ast$-admissible to each other. Let $\gamma$ be an initial piece of the convolution $\gamma_1 \ast \gamma_2$, which is either in $\mathcal{M}$, or is a constant.

1. Suppose $v[\gamma_1] = v[\gamma_2]$. Then $\gamma$ is always in $\mathcal{M}$, $v[\gamma] = v[\gamma_1] = v[\gamma_2]$, $\sigma(\gamma) = \sigma(\gamma_1) = \sigma(\gamma_2)$, and $\gamma \prec \gamma_1$, $\gamma \prec \gamma_2$.

2. Suppose $v[\gamma_1] = -v[\gamma_2]$. Then $\gamma$ is constant if and only if $\gamma_1 \sim \gamma_2$. If $\gamma_1 \succ \gamma_2$ (resp., $\gamma_1 \prec \gamma_2$), then $\gamma \in \mathcal{M}$, $v[\gamma] = v[\gamma_2]$ (resp., $v[\gamma] = v[\gamma_1]$), $\sigma(\gamma) = \sigma(\gamma_1) = \sigma(\gamma_2)$, and $\gamma \succ \gamma_2$ (resp., $\gamma \prec \gamma_1$).

**Proof.** With no loss of generality, assume that $a_1 = a_2 = 0$, $\gamma_1(0) = \gamma_2(0) = 0$, $v[\gamma_1] = (1, 0)$, and $\sigma(\gamma_1) = \sigma(\gamma_2) = +$. There are two possibilities for $v[\gamma_2]$: $(1, 0)$ and $(-1, 0)$. We can assume $\gamma_1(t) = (t, f_1(t))$, $\gamma_2(t) = (\pm t, \pm f_2(t))$ ($\pm$ depending on the direction of $v[\gamma_2]$) for small $t \geq 0$, where $f_i$ is a $C^1$ function such that $f_i(0) = f'(0) = 0$ and $f'_i$ is strictly increasing for $i = 1, 2$. Since either $\gamma_1 \succ \gamma_2$, $\gamma_1 \sim \gamma_2$, or $\gamma_1 \prec \gamma_2$, we can assume that either $f_1(t) > f_2(t)$, $f_1(t) = f_2(t)$, or $f_1(t) < f_2(t)$ for every small $t > 0$.

Consider first the case when $v[\gamma_2] = (1, 0)$. By Lemma 2.1, it is clear that $\gamma \in \mathcal{M}$, $v[\gamma] = (1, 0)$ and $\sigma(\gamma) = +$. So we can write $\gamma(t) = (t, f(t))$ for small $t \geq 0$, where $f$ is a $C^1$ function such that $f(0) = f'(0) = 0$ and $f'$ is strictly increasing for small $t$. Since $\gamma$ is in $\mathcal{M}$, we can see that, for $i = 1, 2$, $f(t)$ is either greater than, equal to, or less than $f_i(t)$ for every small $t > 0$.

Now, for any small $t > 0$, we can take small $t_1, t_2 > 0$ such that $t = t_1 + t_2$, $f'_1(t_1) = f'(t_2)$, and $f(t) = f_1(t_1) + f_2(t_2)$. By Lemma 2.1, $f'(t) = f'_1(t_1) = f'(t_2)$. Since $t > t_1, t_2$ and $f'_1, f'_2$ are strictly increasing, we have $f'_1(t), f'_2(t) > f'(t)$. Thus $f_i(t) > f(t)$, $i = 1, 2$, for every small $t > 0$, which implies that $\gamma \prec \gamma_1$ and $\gamma \prec \gamma_2$. This shows (1).

![Fig. 5. Convolutions of $\mathcal{M}$-curves](image)

Now consider the case when $v[\gamma_2] = (-1, 0)$. Obviously, $\gamma$ is constant if and only if $\gamma_1 \sim \gamma_2$. So assume $\gamma_1 \not\sim \gamma_2$. Then either $\gamma_1 \succ \gamma_2$ or $\gamma_1 \prec \gamma_2$. Suppose $\gamma_1 \prec \gamma_2$. By Lemma 2.1, either $v[\gamma] = (1, 0)$ or $v[\gamma] = (-1, 0)$. If $v[\gamma] = (-1, 0)$, then we must have $f'_1(t) > f'_2(t)$ for small $t > 0$, since $f'_1, f'_2$ are strictly increasing. It follows that $f_1(t) > f_2(t)$ for sufficiently small $t > 0$, which contradicts the assumption that $\gamma_1 \prec \gamma_2$. So $v[\gamma] = (1, 0)$. Since $\gamma \in \mathcal{M}$ and $\sigma(\gamma) = +$, we can assume $\gamma(t) = (t, f(t))$ for small $t \geq 0$, where $f$ is a
for any small $t > 0$, we can take small $t_1, t_2 > 0$ such that $t = t_1 - t_2$, $f'_1(t_1) = f'(t_2)$, and $f(t) = f_1(t_1) - f_2(t_2)$. By Lemma 2.1, $f'(t) = f'_1(t_1) = f'(t_2)$. Since $t < t_1$ and $f'_1$ is strictly increasing, we have $f'_1(t) < f'(t)$, and thus $f_1(t) < f(t)$ for small $t > 0$. This implies that $\gamma \bowtie \gamma_1$. By a symmetric argument, we can also show that $v[\gamma] = v[\gamma_2]$ and $\gamma \bowtie \gamma_2$, when $\gamma_1 \bowtie \gamma_2$. Thus we have shown (2).

\section{The class $\mathcal{W}$}

In this section, we present an important example of a Minkowski class, called $\mathcal{W}$, which is large enough to contain practically all the important curves such as the NURBS curves. We will need the following proposition which is part of Łojasiewicz’s Structure Theorem for real-analytic varieties ([10], [12]).

**Proposition 1** (S. Łojasiewicz). Let $\Phi : U \rightarrow \mathbb{R}$ be a real-analytic function on an open set $U \ni 0$ in $\mathbb{R}^n$, $n \geq 1$, and let $Z = \{(x_1, \ldots, x_n) \in U \mid \Phi(x_1, \ldots, x_n) = 0\}$. Then there exist $T \in SO(n, \mathbb{R})$ and an open set $N \ni 0$ such that the set $Z \cap N$ can be decomposed as

$$Z \cap N = V^0 \cup \ldots \cup V^{n-1},$$

where each $V^k$ can be decomposed again as

$$V^k = \bigcup_{i=1}^{p_k} R^k_i$$

for some $0 \leq p_k < \infty$. Here, each $R^0_i$ is a point, and for each $R^k_i$ with $k \geq 1$, there exist a connected open set $U^k_i \subset \mathbb{R}^k$ and real-analytic functions $\xi^k_{i,k+1}, \ldots, \xi^k_{i,n}$ on $U^k_i$ such that

$$R^k_i = T \cdot \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid (x_1, \ldots, x_k) \in U^k_i, x_j = \xi^k_{i,j}(x_1, \ldots, x_k) \text{ for } j = k+1, \ldots, n \}.$$

In fact, what we really need is the following consequence of the above proposition.

**Corollary 1.** Let $\Phi : U \rightarrow \mathbb{R}$ be a real-analytic function on an open set $U \ni 0$ in $\mathbb{R}^n$, $n \geq 1$, and let $Z = \{(x_1, \ldots, x_n) \in U \mid \Phi(x_1, \ldots, x_n) = 0\}$. Then there exists an open neighborhood $N$ of $0$ in $\mathbb{R}^2$ such that the set $Z \cap N$ has a finite number of connected components.

By using the above result, we first see how convolution behaves in the class $C^{\omega, \omega}_c$.

**Lemma 3.1.** Let $\gamma_i : [a_i, b_i] \rightarrow \mathbb{R}^2$, $i = 1, \ldots, n$, be $C^{\omega, \omega}_c$-curves which are $*$-admissible to each other. Then, for some $\varepsilon_1, \ldots, \varepsilon_n > 0$, $\gamma = \gamma_1|_{[a_1, a_1+\varepsilon_1]} \ast \ldots \ast \gamma_n|_{[a_n, a_n+\varepsilon_n]}$ is either constant, or is a $C^{\omega, 1}_c$-curve which is $*$-admissible to each $\gamma_i$.

**Proof.** We assume $a_1 = \ldots = a_n = 0$, $\gamma_1(0) = \ldots = \gamma_n(0) = 0$, $\sigma(\gamma_1) = \ldots = \sigma(\gamma_n) = +$, and $v[\gamma_1] = (1, 0)$. For each $i$, let $\theta_i$ be the angle function of $\gamma_i$ such that $\theta_i(0) = 0$ or $\pi$, and define $\theta_i : [0, b_i] \rightarrow \mathbb{R}$ by $\theta_i = \theta_i$ if $v[\gamma_i] = (1, 0)$, and $\theta_i = \theta_i - \pi$ if $v[\gamma_i] = (-1, 0)$. Then $\theta_i$ is strictly increasing and $\theta_i(0) = 0$ for every $i$. Take small $0 < \varepsilon_i \leq b_i$ for each $i$ such that $\theta_i(\varepsilon_1) = \ldots = \theta_n(\varepsilon_n)$. Let $\alpha = \theta_1(\varepsilon_1)$. Since $\gamma_i$’s are in $C^{\omega, \omega}_c$, we view each $\gamma_i$
as defined and real-analytic on \((-\delta, \varepsilon_i]\) for some \(\delta > 0\). We can also assume that each \(\gamma_i\) is unit-speed.

Let \(U = (-\delta, \varepsilon_1) \times \ldots \times (-\delta, \varepsilon_n) \subset \mathbb{R}^n\). Then \(F : U \to \mathbb{R}\) and \(G : U \to \mathbb{R}^2\) defined by

\[
F(x_1, \ldots, x_n) = \sum_{j \neq k} \gamma'_j(x_j) \times \gamma'_k(x_k)^2, \quad G(x_1, \ldots, x_n) = \sum_{i=1}^{n} \left( \prod_{p \neq i} \kappa_p(x_p) \right) \gamma'_i(x_i)
\]

are real-analytic on \(U\). Here, for each \(i\), \(\kappa_i : (-\delta, \varepsilon_i] \to \mathbb{R}\) is the curvature function of \(\gamma_i\), i.e., \(\kappa_i(x_i) = \gamma''_i(x_i) \times \gamma''_i(x_i)\). Let \(Z_F\) be the zero set of \(F\) in \(U\). Let \(Q = [0, \varepsilon_1) \times \ldots \times [0, \varepsilon_n)\). Then it is easy to see that \(Z_F \cap Q = \{ \zeta(t) \mid t \in [0, \alpha) \}\), where the one-to-one map \(\zeta : [0, \alpha] \to \mathbb{R}^n\) is defined by \(\zeta(t) = (\theta_1^{-1}(t), \ldots, \theta_n^{-1}(t))\).

Note that \(\kappa_i(x_i) = \theta_i'(x_i)\) for \(x_i \in [0, \varepsilon_i]\) for each \(i\). So \(\theta_i^{-1}\) is real-analytic on \([0, \alpha]\) for each \(i\), and hence \(\zeta\) is real-analytic on \([0, \alpha]\). Note that \(\gamma(t) = \gamma_1(\theta_1^{-1}(t)) + \ldots + \gamma_n(\theta_n^{-1}(t))\) for \(t \in [0, \alpha]\). So \(\gamma\) is also real-analytic on \((0, \alpha]\).

Now

\[
\gamma'(t) = \sum_{i=1}^{n} \gamma'_i(\theta_i^{-1}(t)) \frac{1}{\kappa_i(\theta_i^{-1}(t))} = \frac{1}{\prod_{i=1}^{n} \kappa_i(\theta_i^{-1}(t))} G(\zeta(t)).
\]

Note that \(|G \circ \zeta|^2\) is a real-analytic function on \((0, \alpha]\). If \(|G \circ \zeta|^2 \equiv 0\) on \((0, \alpha]\), then \(\gamma\) is constant. Suppose \(|G \circ \zeta|^2 \not\equiv 0\) on \((0, \alpha]\). Let \(S = \{ t \in [0, \alpha] \mid |G \circ \zeta(t)| = 0 \}\). Suppose \(S\) has infinitely many elements. Since \(|G \circ \zeta|^2\) is real-analytic on \((0, \alpha]\), there exists a sequence \(t_k \searrow 0\) in \((0, \alpha]\) such that \(S = \{ t_k \mid k = 1, 2, \ldots \}\). Define the real-analytic function \(\Phi\) by \(\Phi = F + |G|^2\) on \(U\). Let \(Z_{\Phi}\) be the zero set of \(\Phi\) in \(U\). By Corollary 1, there exists an open connected neighborhood \(N\) of \(0\) in \(U\) such that \(Z_{\Phi} \cap N\) has a finite number of connected components. Let \(x_k = \zeta(t_k)\) for \(k = 1, 2, \ldots\). Since \(t_k \searrow 0\) and \(\zeta(0) = 0\), infinitely many \(x_k\)'s are in \(N\). Denote them again by \(x_k, k = 1, 2, \ldots\). Then \(Z_{\Phi} \cap N \cap Q = \{ x_k \mid k = 1, 2, \ldots \} \cup \{ 0 \}\). This means that \(Z_{\Phi} \cap N\) has infinitely many isolated points, which contradicts Corollary 1. Thus \(S\) is finite. Now we can take \(\varepsilon_i\)'s small enough again such that \(\gamma'(t)\) never vanishes on \((0, \alpha]\). So \(\gamma\) on \((0, \alpha]\) is a \(C^\omega\) curve. Note that \(\gamma'(t) \parallel \gamma''(\theta_i^{-1}(t))\) for every \(t \in (0, \alpha]\) and \(i = 1, \ldots, n\). So \(\gamma\) is convex, \(C^1\) on \([0, \alpha]\), \(\nu[\gamma](1, 0)\), and \(\sigma(\gamma) = +\). We can take \(\varepsilon_i\)'s smaller still so that \(\gamma\) is one-to-one. Thus we have proved that \(\gamma\) is a \(C^\omega_{c^1}\)-curve \(\ast\)-admissible to each \(\gamma_i\).

We have seen that convolutions of any \(C^\omega_{c^\omega}\)-curves belong to \(C^\omega_{c^1}\). In fact, this is the best we can get. A convolution of \(C^\omega_{c^\omega}\)-curves may not even be a \(C^\omega_{c^2}\)-curve, which can be seen from the following example:

**Example 3.1.** Let

\[
\gamma_1(t) = \left( t, \frac{1}{2} t^2 \right), \quad t \in [0, 1], \quad \gamma_2(\theta) = (- \sin \theta, \cos \theta), \quad \theta \in [0, \pi/4].
\]

Then \(\gamma_1, \gamma_2 \in C^\omega_{c^\omega}\). It is easy to see that, with some reparametrization,

\[
\gamma(\theta) = \left( \tan \theta - \sin \theta, \frac{1}{2} \tan^2 \theta + \cos \theta \right), \quad \theta \in [0, \pi/4],
\]
where $\gamma = \gamma_1 * \gamma_2$. From this, we can show that
\[
\lim_{\theta \to 0} \frac{|\gamma'(\theta) \times \gamma''(\theta)|}{|\gamma'(\theta)|^3} = \infty.
\]
So the curvature of $\gamma$ blows up at $\theta = 0$, which is impossible for a $C^{\omega,2}_c$-curve. Thus $\gamma \notin C^{\omega,2}_c$.

Note that Example 3.1 shows that the class $C^{\omega,1}_c$ is not a Minkowski class.

Now we define the curve class $W$, which is an example of a Minkowski class.

**Definition 3.1 (the class $W$).** $W$ is the set of all straight line segments and all $C^{\omega,1}_c$-curves which are of the form $\gamma_1 * \ldots * \gamma_n$ for some $\gamma_1, \ldots, \gamma_n$ in $C^{\omega,\omega}_c$, $n \geq 1$.

As an easy consequence of Lemma 3.1, we have the following fact:

**Theorem 3.1.** $W$ is a Minkowski class.

**Proof.** First, it is obvious that $W$ satisfies condition (1) in Definition 2.5. Let $\gamma_1, \gamma_2 \in W$ be $*$-admissible to each other. Then $\gamma_1 = \alpha_1 * \ldots * \alpha_m$ and $\gamma_2 = \beta_1 * \ldots * \beta_n$ for some $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \in C^{\omega,\omega}_c$. By the definition of convolution, $\gamma_1 = \alpha_1 * \ldots * \alpha_m$ for some $1 \leq m < \infty$. Now from Lemma 3.1, condition (2) in Definition 2.5 is satisfied. ■

Note that $W$ is the smallest Minkowski class containing $C^{\omega,\omega}_c$. Now we explore the relations of $W$ with other curve classes. Note first that $C^{\omega,\omega}_c \subseteq W \subseteq C^{\omega,1}_c$ by definition. Example 2.1 and Lemma 2.2 show that $W \neq C^{\omega,1}_c$. Example 3.1 shows $W \neq C^{\omega,\omega}_c$. So $C^{\omega,\omega}_c \subseteq W \subseteq C^{\omega,1}_c$. Examples 2.1 and 3.1 also show respectively that $C^{\omega,\omega}_c \subseteq W$ and $W \subseteq C^{\omega,1}_c$. Moreover, Example 3.2 below shows that $W \cap (C^{\omega,n}_c \setminus C^{\omega,n+1}_c) \neq \emptyset$ for every $1 \leq n < \infty$. Combining all these, Figure 6 shows the inclusion relations between $W$ and other curve classes.

**Example 3.2.** Let $n \geq 1$ be an integer. For some small $0 < T < 1$, let $\gamma_1(t) = (t, f(x))$, $\gamma_2(t) = (-t, -g(t))$ for $t \in [0, T]$, where
\[
f(t) = \frac{t}{2n + 1} t^{2n+1}, \quad g(t) = \frac{1}{2n + 1} t^{2n+1}.
\]
Clearly, $\gamma_1, \gamma_2 \in C^{\omega: \omega}_c$. Putting $f'(t) = g'(s)$, we have $s = t - t^2$. So, with reparametrization, we have

$$\gamma(t) = \gamma_1(t) + \gamma_2(s) = \left(t^2 \int_0^t (\tau - \tau^2)^{2n} d\tau - \frac{1}{2n + 1} (t - t^2)^{2n+1}\right),$$

where $\gamma = \gamma_1 * \gamma_2$. Let

$$F(t) = \left(t^2 \int_0^t (\tau - \tau^2)^{2n} d\tau - \frac{1}{2n + 1} (t - t^2)^{2n+1}\right).$$

By Lemma 3.1, we know that $\gamma$ is in $C^{\omega: 1}_c$. Note that, for $1 \leq k < \infty$, $\gamma$ is in $C^{\omega: k}_c$ for $k = 1, 2, \ldots$ if and only if the limit $\lim_{t \searrow 0} d^k F/du^k$ exists, where $u = t^2$. Now

$$F = \left(t^2 \int_0^t (\tau - \tau^2)^{2n} d\tau - \frac{1}{2n + 1} (\sqrt{u} - u)^{2n+1}\right).$$

So

$$\frac{dF}{du} = (\sqrt{u} - u)^{2n} - (\sqrt{u} - u)^{2n} \left(\frac{1}{2\sqrt{u}} - 1\right) = (t - t^2)^{2n} - (t - t^2)^{2n} \left(\frac{1}{2t} - 1\right)$$

$$= -\frac{1}{2} t^{2n-1} + \text{higher order terms in } t.$$

Note that $dt^m/du = \frac{1}{2} m t^{m-2}$ for every integer $m$. So, for each $k = 1, 2, \ldots$, we have

$$\frac{d^k F}{du^k} = a_k t^{2n+1-2k} + \text{higher order terms in } t,$$

for some $a_k \neq 0$. It follows that $\lim_{t \searrow 0} d^n F/du^n = 0$ and $\lim_{t \searrow 0} d^{n+1} F/du^{n+1} = -\infty$. This shows that $\gamma \in C^{\omega: n}_c \setminus C^{\omega: n+1}_c$.

4. Sectors and domains

We will now define the exact meaning of the word domain used in this paper. With our definition, the domains can be of fairly general shape. For example, ones consisting only of curve segments, which cannot be regarded as domains in the conventional sense, are also included. Our analysis of domains and their Minkowski sums will be based on the global integration of various local results. The sector introduced below is a basic local object we will use.

Let $C$ be a class of curves in $C^{1: 1}$. We say that $C$ is closed under restriction if, for every $\gamma : [a, b] \rightarrow \mathbb{R}^2$ in $C$, $\gamma|_{[c, d]}$ is also in $C$ for every $[c, d] \subset [a, b]$. We will only consider the curve classes which are closed under restriction. Note that $C^{k:l}, C^{k:l}_e$ for $k, l = 1, 2, \ldots, \infty, \omega, k \geq l$ and every Minkowski class satisfy this condition.

DEFINITION 4.1 (sector). Let $C$ be a class of curves in $C^{1: 1}$ which is closed under restriction. A closed set $S$ in $\mathbb{R}^2$ is called a $C$-sector with center $p \in \mathbb{R}^2$ and radius $r > 0$ if $S$ is bounded by three continuous curves $\alpha, \beta, \gamma$ which satisfy the following conditions:

1. $\alpha : [a_1, a_2] \rightarrow B_r(p)$ and $\beta : [b_1, b_2] \rightarrow B_r(p)$ are $C$-curves such that $\alpha(a_1) = \beta(b_1) = p$. 

Minkowski sum 17
(2) The functions $g_\alpha : [a_1, a_2] \rightarrow [0, r]$ and $g_\beta : [b_1, b_2] \rightarrow [0, r]$ defined by $g_\alpha(t) = |\alpha(t) - p|$ and $g_\beta(t) = |\beta(t) - p|$ are homeomorphisms.

(3) Either $\alpha([a_1, a_2]) = \beta([b_1, b_2])$, or $\alpha$ and $\beta$ have no intersections except at $p$.

(4) $\gamma$ traverses $\partial B_r(p)$ from $\alpha(a_2)$ to $\beta(b_2)$ in the counter-clockwise direction.

Here, if $\alpha(a_2) = \beta(b_2)$ (or equivalently, if $\alpha([a_1, a_2]) = \beta([b_1, b_2])$), then $\gamma$ is constant just at the point $\alpha(a_2)$, and $S$ is just the set of all points on the curve $\alpha$ (or equivalently, $\beta$). The two curves $\beta$ and $\alpha$ are called the start curve and the end curve of $S$ respectively. The cone $C(S)$ of $S$ is defined as

$$C(S) = \{ v \in S^1 \mid \exists \gamma \in C^{1,1} : [0, 1] \rightarrow S \text{ such that } \gamma(0) = p, \gamma'(0) = v \}.$$ 

$S$ is called sharp (resp., dull, flat) if the center angle of $C(S)$ is less than $\pi$ (resp., greater than $\pi$, equal to $\pi$). If $\alpha([a_1, a_2]) = \beta([b_1, b_2])$, then we call $S$ degenerate; otherwise it is non-degenerate.

Let $S_1$ and $S_2$ be two $C^{1,1}$-sectors with center $p$ and radius $r$. Then $S_1$ and $S_2$ are said to be non-overlapping if $S_1 \cap S_2 = \{ p \}$. We list some elementary properties of sectors, which follow immediately from Definition 4.1 and Lemma 2.2.

**Lemma 4.1.** (1) Let $C$ be a class of curves in $C^{1,1}$ which is closed under restriction, and let $S$ be a $C$-sector with center $p$ and radius $r$. Then $B_{r'}(p) \cap S$ is a $C$-sector with center $p$ and radius $r'$ for every $0 < r' \leq r$.

(2) Let $\mathcal{M}$ be a Minkowski class, and let $S_1$ and $S_2$ be two $\mathcal{M}$-sectors with center $p$ and radius $r$. Then there exists $0 < r' \leq r$ such that, for every $0 < g \leq r'$, the set $B_{g}(p) \cap (S_1 \cup S_2)$ is either $B_{g}(p)$, or an $\mathcal{M}$-sector with center $p$ and radius $g$, or a union of two non-overlapping $\mathcal{M}$-sectors with center $p$ and radius $g$.

**Proof.** (1) is obvious, and (2) is immediate from Lemma 2.2.

Now we define the domains:

**Definition 4.2 (domain).** Let $C$ be a class of curves in $C^{1,1}$. A subset $\Omega$ of $\mathbb{R}^2$ is called a $C$-domain if it satisfies the following conditions:

(1) $\Omega$ is connected and compact.

(2) $\partial \Omega$ is a union of a finite number of $C$-curves, no two of which meet at infinitely many points.
Note that, in view of this definition, the Minkowski sum in Figure 3 is not a $C^{\omega:\infty}$-domain, though its boundary consists of finitely many $C^{\omega:\infty}$-curves. In fact, it is not even a $C^{1:1}$-domain. But the domains in our definition can be of fairly general shape such as the one in Figure 8.

![Fig. 8. Example of a “domain” with general shape](image)

**Remark 4.1.** If $C$ is $C^{\omega:\omega}$, $C^{\omega:\omega}_C$, or a Minkowski class ($W$, for example), then condition (2) in Definition 4.2 can be omitted.

Now we start to use the local object sector to describe global properties.

**Lemma 4.2 (local condition for domain).** Let $\Omega$ be a connected compact set in $\mathbb{R}^2$, and let $C$ be a class of curves in $C^{1:1}$ which is closed under restriction. Then the following two conditions are equivalent:

1. $\Omega$ is a $C$-domain.
2. For every point $p \in \partial \Omega$, there exists $r > 0$ such that $B_r(p) \cap \Omega$ is a union of a finite number of mutually non-overlapping $C$-sectors with center $p$ and radius $r$.

**Proof.** Suppose $\Omega$ is a $C$-domain. Let $p \in \partial \Omega$. Since $C \subset C^{1:1}$, it is easy to see from Definition 4.2(2) that there exist $r > 0$ and $C$-curves $\gamma_i : [0, a_i] \to B_r(p)$ for $i = 1, \ldots, n$ for some $1 \leq n < \infty$ such that $B_r(p) \cap \partial \Omega = \bigcup_{i=1}^{n} \gamma_i([0, a_i])$ and the function $g_i : [0, a_i] \to [0, r]$ defined by $g_i(t) = |\gamma_i(t) - p|$ is a homeomorphism with $g_i(0) = 0$ for each $i$. Again by Definition 4.2(2), we can assume $\gamma_i$ and $\gamma_j$ do not meet except at $p$ for every $1 \leq i \neq j \leq n$. Now it is clear that $B_r(p) \cap \Omega$ is a union of a finite number of mutually non-overlapping $C$-sectors with center $p$ and radius $r$. Thus (1) implies (2).

Conversely, suppose (2). Then, for every $p \in \partial \Omega$, we can choose $r(p) > 0$ such that $B_{r(p)}(p) \cap \Omega$ is a finite union of mutually non-overlapping $C$-sectors with center $p$ and radius $r$, and $B_{r(p)}(p) \cap \partial \Omega$ is a union of a finite number of $C$-curves, each pair of which have no intersections except at $p$. Note that $\{B_{r(p)}^\circ(p) \cap \partial \Omega \mid p \in \partial \Omega\}$ is an open cover of the compact set $\partial \Omega$. So there exist a finite number of points $p_1, \ldots, p_n \in \partial \Omega$ such that $\partial \Omega = \bigcup_{i=1}^{n} B_{r(p_i)}^\circ(p_i) \cap \partial \Omega$. Thus $\partial \Omega = \bigcup_{i=1}^{n} B_{r(p_i)}(p_i) \cap \partial \Omega$ is a union of a finite number of $C$-curves. From the definition of sector, no pair of these $C$-curves has infinitely many isolated intersections. Thus $\Omega$ is a $C$-domain. ■
As can be seen from Definition 4.2, the domains can have quite general shapes. We give a special name for domains with some relatively good geometry.

**Definition 4.3** (regular domain). A $C^{1,1}$-domain is called *regular* if each connected component of $\partial \Omega$ is homeomorphic to $S^1$, and is not itself a connected component of $\Omega$.

So, the snowman in Figure 8 is not a regular domain. Also, the Minkowski sum in Figure 2 is a $C^{\omega,\omega}$-domain, but not a regular $C^{\omega,\omega}$-domain. Note that, for any $\mathcal{C} \subset C^{1,1}$, the number of connected components of $\partial \Omega$ is finite for a $C$-domain $\Omega$.

**Lemma 4.3** (local condition for regular domain). Let $\Omega$ be a connected compact set in $\mathbb{R}^2$, and let $\mathcal{C}$ be a class of curves in $C^{1,1}$ which is closed under restriction. Then the following two conditions are equivalent:

1. $\Omega$ is a regular $\mathcal{C}$-domain.
2. For every point $p$ in $\partial \Omega$, there exists $r > 0$ such that $B_r(p) \cap \Omega$ is a non-degenerate $\mathcal{C}$-sector with center $p$ and radius $r$.

**Proof.** Suppose $\Omega$ is a regular $\mathcal{C}$-domain. Let $p \in \partial \Omega$. Since $\mathcal{C} \subset C^{1,1}$, there exists $r > 0$ such that $B_r(p) \cap \partial \Omega$ is a union of two $\mathcal{C}$-curves $\gamma_i : [0, a_i] \to B_r(p)$, $i = 1, 2$, such that $\gamma_1(0) = \gamma_2(0) = p$, $\gamma_1$ and $\gamma_2$ do not meet except at $p$, and the function $g_i : [0, a_i] \to [0, r]$ defined by $g_i(t) = |\gamma_i(t) - p|$ is a homeomorphism for $i = 1, 2$. Note that $B_r(p) \cap \Omega \neq B_r(p)$, since $p \in \partial \Omega$. So $B_r(p) \cap \Omega$ is either a non-degenerate $\mathcal{C}$-sector with center $p$ and radius $r$, or $B_r(p) \cap \Omega = \gamma_1([0, a_1]) \cup \gamma_2([0, a_2])$. Suppose the latter. Then the connected component of $\partial \Omega$ which contains $B_r(p) \cap \Omega$ is itself a connected component of $\Omega$. So we conclude that $B_r(p) \cap \Omega$ is a non-degenerate $\mathcal{C}$-sector with center $p$ and radius $r$. Thus (1) implies (2).

Conversely, suppose (2). Then $\partial \Omega$ is locally homeomorphic to $\mathbb{R}$ at every point in $\partial \Omega$, and $\partial \Omega$ is a disjoint union of a finite number of 1-dimensional (topological) manifolds embedded in $\mathbb{R}^2$. Since $\partial \Omega$ is bounded, each of these manifolds is homeomorphic to $S^1$. So $\partial \Omega$ is a disjoint union of a finite number of sets homeomorphic to $S^1$. Note that each of these sets consists of a finite number of $\mathcal{C}$-curves, since $S^1$ is compact. From the assumption, none of the connected components of $\partial \Omega$ is itself a connected component of $\Omega$. Thus (2) implies (1). \hfill \blacksquare

**Remark 4.2.** A subset $\Omega$ of $\mathbb{R}^2$ is a regular $C^{\omega,\omega}$-domain if and only if it satisfies the standing assumptions for domains in [2] and [3]. Note that a domain is a $C^{\omega,\omega}$-domain if and only if it is a $C_c^{\omega,\omega}$-domain, since a $C^{\omega,\omega}$-curve can be cut into a finite number of $C_c^{\omega,\omega}$-curves.

Finally, we introduce the following terminology.

**Definition 4.4** (sharp corner, dull corner and flat point). Let $\Omega$ be a regular $C^{1,1}$-domain. Then a point $p \in \partial \Omega$ is called a *sharp corner* (resp., *dull corner, flat point*) if there exists $r > 0$ such that $B_r(p) \cap \Omega$ is a sharp sector (resp., dull sector, flat sector) with center $p$ and radius $r$.

Note that the above properties are of a local nature of $\Omega$ around $p$, and thus are independent of the choice of $r$. 
5. Virtual boundary

In this section, we introduce the concept of virtual boundary for regular domains. This will enable us to treat the regular domains in a more uniform manner, whether they have corners or not.

Let $\Omega$ be a regular $C^{1,1}$-domain. By definition, each connected component of $\partial \Omega$ is homeomorphic to $S^1$. Among them, exactly one is the outer boundary, and the remaining ones are inner boundaries. To each of these components, we give the standard orientation, i.e., counter-clockwise orientation for the outer boundary, and clockwise orientation for the inner boundaries. Let $C$ be a connected component of $\partial \Omega$. Fix an orientation-preserving covering map $h : \mathbb{R} \to C$. For any continuous curve $\gamma : [a, b] \to C$, there exists a lifting of $\gamma$ to $\mathbb{R}$ with respect to $h$, i.e., a continuous function $\tilde{\gamma} : [a, b] \to \mathbb{R}$ such that $\gamma(t) = h(\tilde{\gamma}(t))$ for $t \in [a, b]$. We define

$$O_\Omega(\gamma) = \begin{cases} + & \text{if } \tilde{\gamma}(b) - \tilde{\gamma}(a) > 0, \\ 0 & \text{if } \tilde{\gamma}(b) - \tilde{\gamma}(a) = 0, \\ - & \text{if } \tilde{\gamma}(b) - \tilde{\gamma}(a) < 0. \end{cases}$$

Note that this definition is independent of the choice of $h$. We say that $\gamma$ is in the standard orientation on $\Omega$ if $O_\Omega(\gamma) = +$.

**Definition 5.1** (normal cone). Let $\Omega$ be a regular $C^{1,1}$-domain, and let $p \in \partial \Omega$. Let $\gamma_+, \gamma_- : [0, \varepsilon] \to \partial \Omega$ be one-to-one $C^{1,1}$-curves such that $\gamma_+(0) = \gamma_-(0) = p$ and $O_\Omega(\gamma_\pm) = \pm$. Then the normal cone of $\Omega$ at $p$, denoted by $NC_\Omega(p)$, is defined as follows:

1. If $p$ is a sharp corner, then $NC_\Omega(p) = \{ n \in S^1 \mid n \cdot v[\gamma_+] \leq 0 \text{ and } n \cdot v[\gamma_-] \leq 0 \}$.
2. If $p$ is a dull corner, then $NC_\Omega(p) = \{ n \in S^1 \mid n \cdot v[\gamma_+] \geq 0 \text{ and } n \cdot v[\gamma_-] \geq 0 \}$.
3. If $p$ is a flat point, then $NC_\Omega(p)$ consists of the (unit) vector obtained by rotating $v[\gamma_+]$ clockwise through $90^\circ$.

We set $v_\Omega^+(p) = v[\gamma_+]$ and $v_\Omega^-(p) = -v[\gamma_-]$. Note that these are independent of the choice of $\gamma_\pm$. Also, $v_\Omega^+(p) = v_\Omega^-(p)$ if and only if $p$ is a flat point of $\Omega$. In this case, we write $v_\Omega(p) = v_\Omega^+(p) = v_\Omega^-(p)$. We denote by $n_\Omega^+(p)$ (resp., $n_\Omega^-(p)$, $n_\Omega(p)$) the vector obtained by rotating $v_\Omega^+(p)$ (resp., $v_\Omega^-(p)$, $v_\Omega(p)$) clockwise through $90^\circ$. Note that $n_\Omega^+(p)$ and $n_\Omega^-(p)$ are the two ends of $NC_\Omega(p)$.

**Definition 5.2** (virtual boundary). Let $\Omega$ be a regular $C^{1,1}$-domain. Then the virtual boundary of $\Omega$, denoted by $\partial^v \Omega$, is defined to be

$$\partial^v \Omega = \{(p, \mathbf{n}) \in \partial \Omega \times S^1 \mid \mathbf{n} \in NC_\Omega(p)\}.$$
The definition is independent of the choice of the homotopy of \( \hat{H} \). We will say that \( \phi \) is in the standard orientation on \( \Omega \) if \( O_{\Omega}(\phi) \) is +.

Let \( \phi : [a, b] \to \mathbb{R}^2 \times S^1 \), \( \phi(t) = (\gamma(t), n(t)) \), be a continuous map. Then there exists a continuous function \( \theta : [a, b] \to R \) such that \( n(t) = \mu(\theta(t)) \), where \( \mu(s) = (\cos s, \sin s) \) for \( s \in R \). We call \( \theta \) an angle function of \( \phi \). We define \( \Theta(\phi) \), the total angle of \( \phi \), by

\[
\Theta(\phi) = \theta(b) - \theta(a).
\]

Note that the total angle is independent of the choice of angle functions.

We use the following notations throughout this paper: Let \( X \) be a topological space. Let \( \gamma : [a, b] \to X \) be a continuous curve. Then the curve \( \gamma : [a, b] \to X \) is defined by

\[
\gamma(t) = \gamma(a + b - t)
\]
for \( t \in [a, b] \). Let \( \gamma_i : [a_i, b_i] \to X \), \( i = 1, 2 \), be two continuous curves with \( \gamma_1(b_1) = \gamma_2(a_2) \). Then the curve \( \gamma = \gamma_1 \cdot \gamma_2 : [a, b] \to X \) is defined by

\[
\gamma(t) = \begin{cases} 
\gamma_1(t) & \text{if } t \in [a_1, b_1], \\
\gamma_2(t - b_1 + a_2) & \text{if } t \in [b_1, b_1 + b_2 - a_2].
\end{cases}
\]

We denote by \( \text{Ind}_\nu(p) \) the index of \( p \in \mathbb{R}^2 \) with respect to a continuous closed curve \( \gamma : [a, b] \to \mathbb{R}^2 \setminus \{p\} \) \( (\gamma(a) = \gamma(b)) \). It is well known that the index of a point takes integer values and remains the same if we vary the curve homotopically.

The following lemmas are easy consequences of the above definitions.

**Lemma 5.1.** Let \( \Omega \) be a regular \( C^{1,1} \)-domain, and let \( \phi : [a, b] \to \partial^u \Omega \), \( \phi_i : [a_i, b_i] \to \partial^u \Omega \), \( i = 1, 2 \), be continuous maps such that \( \phi_1(b_1) = \phi_2(a_2) \). Then:

1. \( \Theta(\phi) = -\Theta(\phi) \).
2. \( \Theta(\phi_1 \cdot \phi_2) = \Theta(\phi_1) + \Theta(\phi_2) \).
3. Suppose \( \phi_0 : [a, b] \to \partial^u \Omega \) is a continuous map which is homotopic to \( \phi \) in \( \partial^u \Omega \) relative to \( \phi(a) \) and \( \phi(b) \), i.e., there exists a continuous map \( H : [a, b] \times [0, 1] \to \partial^u \Omega \) such that \( H(t, 0) = \phi_0(t) \), \( H(t, 1) = \phi(t) \) for \( t \in [a, b] \), and \( H(a, s) = \phi(a) \), \( H(b, s) = \phi(b) \) for \( s \in [0, 1] \). Then \( \Theta(\phi_0) = \Theta(\phi) \).

**Proof.** (1), (2) are obvious from the definitions. For (3), let \( H(t, s) = (\gamma(t, s), n(t, s)) \) for \( (t, s) \in [a, b] \times [0, 1] \). From the assumption, there exists a continuous map \( \theta : [a, b] \times [0, 1] \to R \) such that \( (\mu \circ \theta)(t, s) = n(t, s) \), where \( \mu : R \to S^1 \) is defined by \( \mu(t) = (\cos t, \sin t) \) for \( t \in R \). Thus \( \Theta(\phi_0) = \theta(b, 0) - \theta(a, 0) = \theta(b, 1) - \theta(a, 1) = \Theta(\phi) \), since \( n(a, s) = n(a, 0) \) and \( n(b, s) = n(b, 0) \) for every \( s \in [0, 1] \).

**Lemma 5.2.** Let \( \Omega \) be a regular \( C^{1,1} \)-domain, and let \( p \in \text{int} \Omega \). Let \( \phi : [a, b] \to \partial^u \Omega \), \( \phi(t) = (\gamma(t), n(t)) \), be a continuous map such that \( \phi(a) = \phi(b) \) and \( O_{\Omega}(\phi) = + \). Let \( C \) be the connected component of \( \partial \Omega \) such that \( \gamma([a, b]) \subset C \).
(1) If $\phi|_{[a,b]}$ is one-to-one, then
\[
\Theta(\phi) = \begin{cases} 
2\pi & \text{if } C \text{ is the outer boundary of } \partial\Omega, \\
-2\pi & \text{if } C \text{ is an inner boundary of } \partial\Omega.
\end{cases}
\]

(2)
\[
\text{Ind}_\gamma(p) = \begin{cases} 
\frac{1}{2\pi}\Theta(\phi) & \text{if } C \text{ is the outer boundary,} \\
0 & \text{if } C \text{ is an inner boundary.}
\end{cases}
\]

Proof. (1) This is an easy consequence of the Gauss–Bonnet theorem (see [17, Theorem 8.4]).

(2) It is obvious that $\text{Ind}_\gamma(p) = 0$ if $C$ is an inner boundary. Suppose $C$ is the outer boundary, and let $\hat{C}$ be the connected component of $\partial v$ corresponding to $C$. Let $\phi_0 : [0,1] \to \hat{C}$, $\phi_0(t) = (\gamma(t), \mathbf{n}_0(t))$, be a continuous map such that $\phi_0(0) = \phi_0(1) = \phi(a) = \phi(b)$, $\phi_0|_{[0,1]}$ is one-to-one, and $O_{\Omega}(\phi_0) = +$. By (1), we have $\Theta(\phi_0) = 2\pi$. It is easy to see that $\phi$ is homotopic to $\phi_0 \cdot \cdots \cdot \phi_0$ if $\text{Ind}_\gamma(p) > 0$, to $\phi_0 \cdot \cdots \cdot \phi_0$ if $\text{Ind}_\gamma(p) < 0$, and to the constant map $\phi(a) (= \phi(b))$ if $\text{Ind}_\gamma(p) = 0$. Now the assertion follows from Lemma 5.1.

Let $A$ be a subset of $\mathbb{R}^2$ and $p \in \mathbb{R}$. Then we define
\[
A + p = \{q + p \mid q \in A\} \quad \text{and} \quad -A = \{-q \mid q \in A\}.
\]

For later reference, we collect the following elementary facts which can be easily deduced from the definitions.

Lemma 5.3. Let $\Omega$ be a regular $C^{1,1}$-domain, and let $q \in \mathbb{R}^2$. Let $p \in \partial \Omega$, and let $\phi : [a,b] \to \partial^v \Omega$, $\phi(t) = (\gamma(t), \mathbf{n}(t))$, be a continuous map. Then:

1. $\mathbf{n}^\pm_{\Omega+q}(p + q) = \mathbf{n}^\pm_{\Omega}(p)$ and $\mathbf{v}^\pm_{\Omega+q}(p + q) = \mathbf{v}^\pm_{\Omega}(p)$.
2. $\mathbf{n}^\pm_{\Omega}(-p) = -\mathbf{n}^\pm_{\Omega}(p)$ and $\mathbf{v}^\pm_{\Omega}(-p) = -\mathbf{v}^\pm_{\Omega}(p)$.
3. $\Theta(\phi + q) = \Theta(\phi)$, where $\phi + q : [a,b] \to \partial^v (\Omega + q)$ is defined by $(\phi + q)(t) = (\gamma(t) + q, \mathbf{n}(t))$ for $t \in [a,b]$.
4. $\Theta(\phi) = \Theta(\phi)$, where $-\phi : [a,b] \to \partial^v (-\Omega)$ is defined by $(-\phi)(t) = (-\gamma(t), -\mathbf{n}(t))$ for $t \in [a,b]$.

Now we define the angle of convexity of a regular domain. This will be used in defining the semi-convexity of domains in Section 7.

Definition 5.3 (angle of convexity). Let $\Omega$ be a regular $C^{1,1}$-domain. The angle of convexity of $\Omega$, denoted by $\Theta(\Omega)$, is defined by
\[
\Theta(\Omega) = \inf\{\Theta(\phi) : \phi \in S\},
\]
where $S$ is the set of all continuous maps from a closed interval to $\partial^v \Omega$ such that $O_{\Omega}(\phi) = +$.

Finally, we introduce the notion of contact position which is important for analyzing Minkowski sum.
Definition 5.4 (contact position). Two regular $C^{1:1}$-domains $\Omega_1$ and $\Omega_2$ are said to be in contact position to each other if they meet at their boundaries only, i.e., $\Omega_1 \cap \Omega_2 = \partial \Omega_1 \cap \partial \Omega_2 \neq \emptyset$.

Let $\Omega_1$ and $\Omega_2$ be two simply connected regular $C^{1:1}$-domains which are in contact position to each other. Let $U$ be the unbounded component of $\mathbb{R}^2 \setminus (\Omega_1 \cup \Omega_2)$. Suppose $p_1 \neq p_2$ are two points in $\partial \Omega_1 \cap \partial \Omega_2$. For $i = 1, 2$, let $\phi_i : [0, 1] \to \partial^n \Omega_i$, $\phi_i(t) = (\gamma_i(t), \mathbf{n}_i(t))$, be one-to-one continuous maps such that $\phi_1(0) = (p_1, \mathbf{n}_{\Omega_1}(p_1))$, $\phi_1(1) = (p_2, \mathbf{n}_{\Omega_1}(p_2))$, $\phi_2(0) = (p_2, \mathbf{n}_{\Omega_2}(p_2))$, $\phi_2(1) = (p_1, \mathbf{n}_{\Omega_2}(p_1))$, and $O_{\Omega_1}(\phi_1) = O_{\Omega_2}(\phi_2) = +$. Note that, by interchanging $p_1$ and $p_2$ if necessary, we can assume $(\gamma_i([0, 1] \setminus \{p_1, p_2\}) \cap U = \emptyset$ for $i = 1, 2$. Let $\alpha_1$ (resp., $\alpha_2$) be the non-negative angle of counter-clockwise rotation from $-\mathbf{v}_{\Omega_2}(p_1)$ to $\mathbf{v}_{\Omega_1}^+(p_1)$ (resp., from $-\mathbf{v}_{\Omega_1}(p_2)$ to $\mathbf{v}_{\Omega_2}^+(p_2)$). See Figure 9.

![Fig. 9. Contact position](image)

With the above notations, we have the following lemma:

Lemma 5.4. Let $\Omega_1$ and $\Omega_2$ be simply connected regular $C^{1:1}$-domains which are in contact position to each other. Suppose $p_1 \neq p_2 \in \partial \Omega_1 \cap \partial \Omega_2$ and $\alpha_i$, $\phi_i$ for $i = 1, 2$ are as above. Then:

1. $\Theta(\phi_1) + \Theta(\phi_2) + \alpha_1 + \alpha_2 = 0$.
2. If $\Theta(\Omega_1), \Theta(\Omega_2) \geq -\Theta$ for some $\Theta \geq 0$, then $-\Theta \leq \Theta(\phi_i) \leq \Theta$ for $i = 1, 2$.
3. There exists a continuous map $H : [0, 1] \times [0, 1] \to V$ such that $H(t, 0) = \gamma_1(t)$, $H(t, 1) = \gamma_2(t)$ for $t \in [0, 1]$, and $H(0, s) = p_1$, $H(1, s) = p_2$ for $s \in [0, 1]$, where $V$ is the region in $\mathbb{R}^2 \setminus (\Omega_1 \cup \Omega_2)$ bounded by $\gamma_1$ and $\gamma_2$.

Proof. (1) This is an easy consequence of Lemma 5.2.

(2) By (1), we have $\Theta(\phi_1) = -\Theta(\phi_2) - \alpha_1 - \alpha_2$. Since $\Theta(\Omega_1), \Theta(\Omega_2) \geq -\Theta$, we have $\Theta(\phi_1), \Theta(\phi_2) \geq -\Theta$. Note that $\alpha_1, \alpha_2 \geq 0$ by definition. Thus $\Theta(\phi_1) \leq \Theta$. We can also see that $\Theta(\phi_2) \leq \Theta$ in the same way.

(3) Obvious. See Figure 9. ■
6. Minkowski sum of domains

Now we consider the Minkowski sum of domains. For reasonable results, we restrict our
analysis to $\mathcal{M}$-domains, where $\mathcal{M}$ is a Minkowski class. After introducing the preliminary
facts in Section 6.1, we analyze the behavior of the Minkowski sum of $\mathcal{M}$-sectors in
Sections 6.2 and 6.3. Finally, by using these results, we show in Section 6.4 that the set
of all $\mathcal{M}$-domains is closed under Minkowski sum for any Minkowski class $\mathcal{M}$.

6.1. Preliminaries. Let $A$ and $B$ be two subsets of $\mathbb{R}^2$. We define
\[ A + B = \{ p + q \mid p \in A, q \in B \}, \]
and call it the Minkowski sum of $A$ and $B$. The map $M_{A,B} : A \times B \to A + B$ defined by
$M_{A,B}(p,q) = p + q$ for $p \in A$, $q \in B$ is called the Minkowski map associated to $A$ and $B$.
Note that $M_{A,B}$ is continuous for any $A, B \subseteq \mathbb{R}^2$. The following are easy consequences
of the definition.

**Lemma 6.1.** Let $A, B \subseteq \mathbb{R}^2$. Suppose $A = \bigcup_{i \in I} A_i$ and $B = \bigcup_{j \in J} B_j$. Then
\[ A + B = \bigcup_{i \in I, j \in J} (A_i + B_j). \]
**Proof.** $\supseteq$ is trivial. Suppose $p \in A + B$. Then there exist $p_1 \in A$ and $p_2 \in B$ such that
$p = p_1 + p_2$. So there exist $i \in I$ and $j \in J$ such that $p_1 \in A_i$ and $p_2 \in B_j$. This shows $\subseteq$. ■

**Lemma 6.2.** Let $A, B \subseteq \mathbb{R}^2$, and let $p \in \partial(A + B)$. Then, for any $p_1 \in A$ and $p_2 \in B$ such that $p = p_1 + p_2$, we have $p_1 \in \partial A$ and $p_2 \in \partial B$. Equivalently, we have
\[ M_{A,B}^{-1}(\partial(A + B)) \subseteq \partial A \times \partial B. \]
**Proof.** Suppose $p_1 \in \text{int } A$. Then we can take a small ball $B_r(p_1)$ around $p_1$ such that
$B_r(p_1) \subset A$. Clearly, $B_r(p) = B_r(p_1) + p_2 \subset A + B$, and this implies that $p \in \text{int}(A + B)$. This contradicts the assumption, and we conclude $p_1 \in \partial A$. In the same way, we can show that $p_2 \in \partial B$. ■

**Lemma 6.3.** Let $\Omega_1, \Omega_2 \subset \mathbb{R}^2$, and let $\Omega = \Omega_1 + \Omega_2$. Let $p \in \mathbb{R}^2$. Then:

1. $p \in \Omega$ if and only if $\Omega_1 \cap (-\Omega_2 + p) \neq \emptyset$.
2. If $\text{int } \Omega_1 \cap (-\Omega_2 + p) \neq \emptyset$, then $p \in \text{int } \Omega$.
3. Suppose $\Omega_1$, $\Omega_2$ are regular $C^{1,1}$-domains. If $p \in \partial \Omega$, then $\Omega_1$ and $-\Omega_2 + p$ are in
contact position to each other.

**Proof.** Suppose $p \notin \Omega$. Then there exist $p_1 \in \Omega_1$ and $p_2 \in \Omega_2$ such that $p_1 + p_2 = p$.
So $\Omega_1 \ni p_1 = -p_2 + p \in -\Omega_2 + p$, which means that $\Omega_1 \cap (-\Omega_2 + p) \neq \emptyset$. Conversely,
suppose $\Omega_1 \cap (-\Omega_2 + p) \neq \emptyset$. Then there exists $p_2 \in \Omega_2$ such that
$p_1 = -p_2 + p$. Thus $p = p_1 + p_2 \notin \Omega$. This shows (1).

Suppose $p_1 \in \text{int } \Omega_1 \cap (-\Omega_2 + p)$. Let $p_2 = -p_1 + p$. Then $p_2 \in \Omega_2$, and $p = p_1 + p_2 \in \Omega$.
Since $p_1 \in \text{int } \Omega_1$, we have $p \notin \partial \Omega$ by Lemma 6.2. Thus $p \notin \text{int } \Omega$. This shows (2).

Suppose $p \in \partial \Omega$. By (1), $\Omega_1 \cap (-\Omega_2 + p) \neq \emptyset$. Let $p_1 \in \Omega_1 \cap (-\Omega_2 + p)$, and let
$p_2 = -p_1 + p$. Then we have $p_1 \in \Omega_1$, $p_2 \in \Omega_2$ and $p_1 + p_2 = p$. By Lemma 6.2, $p_1 \in \partial \Omega_1$
and $p_2 \in \partial \Omega_2$, and so $p_1 = -p_2 + p \in \partial(-\Omega_2 + p)$. Thus $p_1 \in \partial \Omega_1 \cap \partial(-\Omega_2 + p)$. Since $p_1$
is taken arbitrarily, it follows that $\Omega_1$ and $-\Omega_2 + p$ are in contact position to each other. This shows (3). ■

**Remark 6.1.** The converse of (3) in Lemma 6.3 is false: it is possible that $\Omega_1$ and $-\Omega_2 + p$ are in contact position to each other, but still $p \notin \partial \Omega$.

**Definition 6.1 (admissible sectors).** Two $C^{1,1}$-sectors $S_1$ and $S_2$ with respective centers $p_1$, $p_2$ and radius $r$ are said to be admissible to each other if they satisfy the following conditions:

1. $\text{int}(S_1 - p_1) \cap (- (S_2 - p_2)) = \emptyset$ and $\text{int}(S_2 - p_2) \cap (- (S_1 - p_1)) = \emptyset$.
2. For $i = 1, 2$, let $\gamma_i$ be the end curve or start curve of $S_i$. If the two curves $\gamma_1 - p_1$ and $-(\gamma_2 - p_2)$ (or equivalently, $-(\gamma_1 - p_1)$ and $\gamma_2 - p_2$) meet at a point in $\mathbb{R}^2$ other than 0, then $\gamma_1$, $\gamma_2$ have the same image.

It is easy to see that if $S_1$ and $S_2$ are admissible to each other, then so are $B_{r'}(p_1) \cap S_1$ and $B_{r'}(p_2) \cap S_2$ for every $0 < r' \leq r$.

**Lemma 6.4.** Let $\mathcal{M}$ be a Minkowski class, and let $\Omega_1$ and $\Omega_2$ be two $\mathcal{M}$-domains. Let $p_1 \in \partial \Omega_1$ and $p_2 \in \partial \Omega_2$. Suppose $p = p_1 + p_2 \in \partial \Omega$, where $\Omega = \Omega_1 + \Omega_2$. Then for every sufficiently small $r > 0$, we have:

1. For $i = 1, 2$, $B_r(p_i) \cap \Omega_i = \bigcup_{k=1}^{n_i} S_i^k$, where $S_i^k$ is an $\mathcal{M}$-sector with center $p_i$ and radius $r$ for $k = 1, \ldots, n_i$, and $S_i^k$'s are mutually non-overlapping.
2. $S_1^k$ and $S_2^l$ are admissible to each other for every $k = 1, \ldots, n_1$ and $l = 1, \ldots, n_2$.

**Proof.** (1) follows from Lemma 4.2. For (2), fix $S_1^k$ and $S_2^l$. Let $\alpha_1$, $\beta_1$ be the end curve and start curve of $S_1^k - p_1$ respectively, and let $\alpha_2$, $\beta_2$ be the end curve and start curve of $-(S_2^l - p_2)$ respectively. Note that $S_1^k - p_1$ and $-(S_2^l - p_2)$ are $\mathcal{M}$-sectors with center 0 and radius $r$. Since $\mathcal{M}$ is a Minkowski class, we can assume that any two of $\alpha_1$, $\beta_1$, $\alpha_2$, $\beta_2$ either have the same image, or do not meet except at 0. So, if $S_1^k$ and $S_2^l$ are not admissible, then we would have either $\text{int} S_1^k \cap (-S_2^l + p) \neq \emptyset$ or $\text{int} S_2^l \cap (-S_1^k + p) \neq \emptyset$. Then by Lemma 6.3(2), $p \in \text{int} \Omega$, which is a contradiction. ■

Let $S$ be a finite union of mutually non-overlapping $C^{1,1}$-sectors $S_1, \ldots, S_n$ with center $p$ and radius $r > 0$. Then we define $C(S) = \bigcup_{k=1}^{n} C(S_k)$.

**Lemma 6.5.** Let $\Omega_1$ and $\Omega_2$ be two $C^{1,1}$-domains, and let $\Omega = \Omega_1 + \Omega_2$. Let $p_1 \in \partial \Omega_1$, $p_2 \in \partial \Omega_2$, and choose $r > 0$ such that $S_i = B_r(p_i) \cap \Omega_i$ is a finite union of mutually non-overlapping $C^{1,1}$-sectors with center $p_i$ and radius $r$ for $i = 1, 2$. Suppose $(p_1, p_2) \in M_{\Omega_1, \Omega_2}^{-1}(\partial \Omega)$ and $S_1$ is a flat $C^{1,1}$-sector with center $p_1$ and radius $r$. Then $C(S_2) \subset C(S_1)$.

**Proof.** We can assume $C(S_1) = \{(x, y) \in S^1 \mid y \leq 0\}$. Suppose $C(S_2) \not\subset C(S_1)$. Then there exists a $C^{1,1}$-curve $\gamma : [0, \varepsilon] \to S_2$ such that $\gamma(0) = p_2$ and $\nu[\gamma] \not\subset C(S_1)$. So $\tilde{\gamma}(0) = p_1$, $\tilde{\gamma}([0, \varepsilon]) \subset -S_2 + p$, and $\nu[\tilde{\gamma}] \subset \{(x, y) \in S^1 \mid y < 0\}$, where $p = p_1 + p_2$ and the $C^{1,1}$-curve $\tilde{\gamma} : [0, \varepsilon] \to \mathbb{R}^2$ is defined by $\tilde{\gamma}(t) = -\gamma(t) + p$ for $t \in [0, \varepsilon]$. It follows that $\text{int} S_1 \cap (-S_2 + p) \neq \emptyset$. So by Lemma 6.3(2), $p \in \text{int}(S_1 + S_2) \subset \text{int} \Omega$, which is a contradiction. ■

**Lemma 6.6.** Let $C$ be a subclass of $C^{1,1}$ which is closed under restriction, and let $\Omega_1$ and $\Omega_2$ be two $C$-domains. Let $p \in \partial \Omega$, where $\Omega = \Omega_1 + \Omega_2$. Then for any $\varepsilon > 0$, there exist
0 < r_1, \ldots, r_n < \varepsilon and (p_1^1, p_1^2), \ldots, (p_n^1, p_n^2) in M_{\Omega_1, \Omega_2}^{-1}(p) for some 1 \leq n < \infty such that each \( B_{r_k}(p_i^k) \cap \Omega_i \) is a finite union of mutually non-overlapping \( C \)-sectors with center \( p_i^k \) and radius \( r_k \), and \( M_{\Omega_1, \Omega_2}^{-1}(p) \subset U \), where

\[
U = \bigcup_{k=1}^{n} (B_{r_k}^0(p_1^k) \cap \Omega_1) \times (B_{r_k}^0(p_2^k) \cap \Omega_2).
\]

**Proof.** By Lemma 6.2, \( M_{\Omega_1, \Omega_2}^{-1}(p) \subset \partial \Omega_1 \times \partial \Omega_2 \). So, by Lemma 4.2, we can choose \( 0 < r(p_1, p_2) < \varepsilon \) for each \( (p_1, p_2) \in M_{\Omega_1, \Omega_2}^{-1}(p) \) such that \( B_{r(p_1, p_2)}(p_i) \cap \Omega_i \) is a finite union of \( C \)-sectors with center \( p_i \) and radius \( r(p_1, p_2) \) for \( i = 1, 2 \). Note that

\[
\{(B_{r(p_1, p_2)}^0(p_1) \cap \Omega_1) \times (B_{r(p_1, p_2)}^0(p_2) \cap \Omega_2) \mid (p_1, p_2) \in M_{\Omega_1, \Omega_2}^{-1}(p)\}
\]

is an open cover of the compact set \( M_{\Omega_1, \Omega_2}^{-1}(p) \) in \( \Omega_1 \times \Omega_2 \). Thus there exists a finite subcover \( \{(B_{r_k}^0(p_1^k) \cap \Omega_1) \times (B_{r_k}^0(p_2^k) \cap \Omega_2) \mid 1 \leq k \leq n\} \), which completes the proof. \( \blacksquare \)

### 6.2. Minkowski sum of admissible sectors

Let \( \gamma : [a, b] \to \mathbb{R}^2 \) be a continuous curve. We define \( \hat{\gamma} : [a, b] \to \mathbb{R}^2 \) by

\[
\hat{\gamma}(t) = \gamma(a + b - t) + \gamma(a) - \gamma(b).
\]

Note that if we translate the image of \( \gamma \) so that \( \gamma(b) \) is moved to \( \gamma(a) \), then we get the image of \( \hat{\gamma} \). Note also that \( \hat{\gamma}(a) = \gamma(a) \). See Figure 10.

![Fig. 10. \( \gamma \) and \( \hat{\gamma} \)](image)

**Lemma 6.7.** Let \( M \) be a Minkowski class, and let \( S_1 \) and \( S_2 \) be two admissible \( M \)-sectors with center 0 and radius \( R > 0 \). For some sufficiently small \( 0 < r \leq R \), let \( S'_i = B_r(0) \cap S_i \) for \( i = 1, 2 \), and let \( S = S'_1 + S'_2 \). Let \( \alpha_i \) and \( \beta_i \) be the end curve and start curve of \( S'_i \) respectively for \( i = 1, 2 \). Then, for every sufficiently small \( \varepsilon > 0 \), the set \( B_\varepsilon(0) \cap \partial S \) is contained in the union of the images of the following curves:

1. \( \alpha_1, \beta_1, \alpha_2, \beta_2 \).
2. \( \alpha_1 \ast \alpha_2, \alpha_1 \ast \beta_2, \beta_1 \ast \alpha_2, \beta_1 \ast \beta_2 \) (if defined).
3. \( \hat{\alpha}_1, \hat{\beta}_2 \) (if \( \alpha_1, -\beta_2 \) have the same image), and \( \hat{\beta}_1, \hat{\alpha}_2 \) (if \( \beta_1, -\alpha_2 \) have the same image).

**Proof.** By abuse of notation, we denote the image of a curve \( \gamma \) also by \( \gamma \). Set \( M = M_{S'_1, S'_2} \). Note that \( M^{-1}(\partial S) \subset \partial S'_1 \times \partial S'_2 \) by Lemma 6.2. Let \( A_i = \partial S'_i \setminus (\alpha_i \cup \beta_i) \) for \( i = 1, 2 \). Then \( \partial S'_1 \times \partial S'_2 = (A_1 \times A_2) \cup (A_1 \times (\alpha_2 \cup \beta_2)) \cup ((\alpha_1 \cup \beta_1) \times A_2) \cup ((\alpha_1 \cup \beta_1) \times (\alpha_2 \cup \beta_2)) \). Suppose \( M((p_1, p_2)) \in \partial S \) for some \( (p_1, p_2) \in A_1 \times A_2 \). Let \( p = p_1 + p_2 \). Lemma 6.5 yields \( p_1 = p_2 \). So \( |p| = |2p_1| = 2r \). This shows that \( M(A_1 \times A_2) \cap (B_\varepsilon(0) \cap \partial S) = \emptyset \) for sufficiently small \( \varepsilon > 0 \).
Suppose that \( \{ (p_1^n, p_2^n) \} \) is a sequence in \((A_1 \times (\alpha_2 \cup \beta_2)) \cup ((\alpha_1 \cup \beta_1) \times A_2) \) such that 
\[ M((p_1^n, p_2^n)) = p_1^n + p_2^n \to 0 \] as \( n \to \infty \). We can assume that \( (p_1^n, p_2^n) \in A_1 \times (\alpha_2 \cup \beta_2) \) for every \( n \). Suppose \( \alpha_i(0) = \beta_i(0) = 0 \) and \( |\beta_i(a_i)| = |\beta_i(b_i)| = r \) for \( i = 1, 2 \). Since 
\[ p_1^n \in A_1, \quad p_2^n \in \alpha_2 \cup \beta_2, \quad \text{and} \quad S'_1, S'_2 \] are admissible, it is easy to see that there exists a subsequence \( \{ p_{n_k} \} \) such that either \( p_{n_k}^{p_1} \to \alpha_1(a_1) \) or \( p_{n_k}^{p_1} \to \beta_1(b_1) \) as \( k \to \infty \). Denote this subsequence again by \( \{ p^n \} \); we can assume that \( p_1^n \to \alpha_1(a_1) \) as \( n \to \infty \). Since \( S'_1 \) and \( S'_2 \) are admissible to each other, it follows that \( \beta_2(b_2) = -\alpha_1(a_1) \) and \( p_2^n \to \beta_2(b_2) \). So we must have \( \alpha_1 \approx -\beta_2 \). Since we have assumed \( r \) to be sufficiently small, we can also assume that \( \beta_2 \) and \( \partial B_r(0) \) meet transversally at \( \beta_2(b_2) \). So, from Lemma 6.5, \( M((p_1^n, p_2^n)) \notin \partial S \) for every sufficiently large \( n \). Thus \( M((A_1 \times (\alpha_2 \cup \beta_2)) \cup ((\alpha_1 \cup \beta_1) \times A_2)) \cap (B_\varepsilon(0) \cap \partial S) = \emptyset \) for sufficiently small \( \varepsilon > 0 \).

It follows that \( B_\varepsilon(0) \cap \partial S \subset M((\alpha_1 \cup \beta_1) \times (\alpha_2 \cup \beta_2)) \) for sufficiently small \( \varepsilon > 0 \). Set \( \alpha_i^n = \alpha_i((0, a_i)) \) and \( \beta_i^n = \beta_i((0, b_i)) \) for \( i = 1, 2 \). We divide \((\alpha_1 \cup \beta_1) \times (\alpha_2 \cup \beta_2)\) into the four parts \( \alpha_1^n \times \alpha_2^n, \beta_1^n \times \beta_2^n, \alpha_1^n \times \beta_2^n, \beta_1^n \times \alpha_2^n \), and the twelve parts \( \alpha_1 \times \{0\}, \{0\} \times \alpha_2, \{0\} \times \beta_2, \alpha_1 \times \{\alpha_2(a_2)\}, \alpha_1 \times \{\beta_2(b_2)\}, \beta_1 \times \{\alpha_2(a_2)\}, \beta_1 \times \{\beta_2(b_2)\}, \{\alpha_1(a_1)\} \times \alpha_2, \{\beta_1(b_1)\} \times \alpha_2, \{\alpha_1(a_1)\} \times \beta_2, \{\beta_1(b_1)\} \times \beta_2 \). Since \( r \) is assumed to be small, Lemma 6.5 shows that the intersections of \( \partial S \) and the images of the first four parts under \( M \) are contained in the union of \( \alpha_1 \times \alpha_2, \beta_1 \times \beta_2, \alpha_1 \times \beta_2, \alpha_2 \times \beta_1 \). The images of the last twelve parts under \( M \) are \( \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2, \alpha_2 + \alpha_1, \beta_2 + \beta_1, \beta_2 + \alpha_1, \beta_1 + \alpha_2, \beta_2 + \alpha_1, \beta_1 + \alpha_2, \beta_2 + \beta_1, \beta_2 + \beta_1 \) respectively. It is easy to see that if 
\[ 0 \in \alpha_1 + \beta_2(b_2), \quad \text{then} \quad \beta_2(b_2) = -\alpha_1(a_1) \quad \text{and} \quad \alpha_1 + \beta_2(b_2) = -\alpha_1(a_1) \], since \( \beta_2(b_2) \in \partial B_r(0) \) and \( \alpha_1 \cap \partial B_r(0) = \alpha_1(a_1) \). Also, if \( 0 \in \alpha_1 + \alpha_2(a_2) \), then \( \alpha_2(a_2) = -\alpha_1(a_1) \), which implies \( \beta_2(b_2) = -\alpha_1(a_1) \) and \( \alpha_1 + \alpha_2(a_2) = -\alpha_1(a_1) \) since \( S'_1 \) and \( S'_2 \) are admissible to each other. Applying the same argument to the eight curves \( \alpha_1 + \alpha_2, \alpha_1 + \beta_2, \beta_1 + \alpha_2, \beta_1 + \beta_2, \alpha_2 + \alpha_1, \beta_2 + \beta_1, \beta_2 + \alpha_1, \beta_1 + \alpha_2 \), we can see that, among these curves, the ones containing \( 0 \) are \( \hat{\alpha}_1, \hat{\beta}_2 \) (if \( \alpha_1, -\beta_2 \) have the same image), and \( \hat{\beta}_1, \hat{\alpha}_2 \) (if \( \beta_1, -\alpha_2 \) have the same image). Now combining the above arguments, we obtain the desired result. 

From the above result, we are now able to derive the following theorem:

**Theorem 6.1** (Minkowski sum of admissible sectors). Let \( \mathcal{M} \) be a Minkowski class, and let \( S_1, S_2 \) be admissible \( \mathcal{M} \)-sectors with respective centers \( p_1, p_2 \) and radius \( R > 0 \). Let \( S'_i = B_r(p_i) \cap S_i \) for \( i = 1, 2 \) for some sufficiently small \( 0 < r < R \), and let \( S = S'_1 + S'_2 \). Then, for every sufficiently small \( \varepsilon > 0 \), either \( B_\varepsilon(p) \cap S = B_\varepsilon(p) \), or \( B_\varepsilon(p) \cap S \) is a finite union of mutually non-overlapping \( \mathcal{M} \)-sectors with center \( p \) and radius \( \varepsilon \), where \( p = p_1 + p_2 \).

**Proof.** Note that \( B_\varepsilon(p) \cap S = [B_\varepsilon(0) \cap \{(S'_1 - p_1) + (S'_2 - p_2)\}] + p \) for every \( r > 0 \) and \( \varepsilon > 0 \). So we can assume that \( p_1 = p_2 = 0 \). By Lemma 6.7, we can take a finite number of \( \mathcal{M} \)-curves \( \gamma_1, \ldots, \gamma_n : [0, \varepsilon] \to \mathbb{R}^2 \) for some \( n \geq 1 \) such that \( \gamma_1(0) = \ldots = \gamma_n(0) = 0 \) and \( B_\varepsilon(0) \cap \partial S \subset \bigcup_{k=1}^n \gamma_k([0, \varepsilon]) \subset S \) for every sufficiently small \( \varepsilon > 0 \). Since \( \mathcal{M} \) is a Minkowski class, we can assume that, for every sufficiently small \( \varepsilon > 0 \), \( \gamma_k([0, \varepsilon]) \cap \partial B_\varepsilon(0) \) is a singleton for each \( k \), and \( \gamma_i([0, \varepsilon]) \cap \gamma_j([0, \varepsilon]) = \{0\} \) for every \( i \neq j \). Since \( \partial S \) is compact
and \( q \) is small, we can assume that either \( B_\theta(0) \cap \partial S = \emptyset \), or there exists \( 0 < m \leq n \) such that \( B_\theta(0) \cap \partial S = \bigcup_{k=1}^{m} \gamma_k([0, \varepsilon]) \).

### 6.3. Minkowski sum of admissible non-degenerate sectors

When both \( S_1 \) and \( S_2 \) are non-degenerate, we have more refined results, which will provide a local building block for dealing with semi-convexity later.

**Lemma 6.8.** Let \( \mathcal{M} \) be a Minkowski class, and let \( S_1, S_2 \) be non-degenerate \( \mathcal{M} \)-sectors with center 0 and radius \( r > 0 \) which are admissible to each other. Suppose there exist \( r_1, \ldots, r_n > 0 \) and \((p_1^1, p_2^1), \ldots, (p_1^n, p_2^n) \in M_{S_1, S_2}^{-1}(0)\) such that \( B_{r_k}((p_1^k) \cap S_i \) is an \( \mathcal{M} \)-sector with center \( p_1^k \) and radius \( r_k \) for each \( i \) and \( k \), and \( M_{S_1, S_2}^{-1}(0) \subset U \), where \( U = \bigcup_{k=1}^{n} (B_{r_k}((p_1^k) \cap S_1) \times (B_{r_k}((p_2^k) \cap S_2) \). Then \( M_{S_1, S_2}(U \setminus M_{S_1, S_2}^{-1}(0)) \) is connected.

**Proof.** Set \( M = M_{S_1, S_2} \), and denote the image of a curve \( \gamma \) also by \( \gamma \). Let \( U_k = (B_{r_k}((p_1^k) \cap S_1) \times (B_{r_k}((p_2^k) \cap S_2) \) for \( k = 1, \ldots, n \). Note that \( M(U \setminus M^{-1}(0)) = \bigcup_{k=1}^{n} M(U_k \setminus M^{-1}(0)) \) since \( M \) is continuous. We first show that \( U_k \setminus M^{-1}(0) \) is connected for each \( k \). Let \((q_1^k, q_2^k) \) be an \( (q_1^k, q_2^k) \in U_k \setminus M^{-1}(0) \). It is easy to see that \( B_{r_k}((p_1^k) \cap S_1 \) is a non-degenerate \( \mathcal{M} \)-sector with center \( p_1^k \) and radius \( r_k \). So we can take a continuous curve \( \gamma_1 : [0, 1] \to B_{r_k}((p_1^k) \cap S_1 \) such that \( \gamma_1(0) = q_1^k, \gamma_1(1) = q_2^k, \) and \( \gamma_1([0, 1]) \subset \{ \gamma(0) = q_2^k = \gamma_2(1) = q_2^2 \}. \) Define \( \gamma : [0, 1] \to U_k \) by \( \gamma(t) = (\gamma_1(t), \gamma_2(t)) \). Since \( \{p_1 \mid (p_1, p_2) \in M^{-1}(0) \) for some \( p_2 \in \Omega_2 \) if \( \{p_1 \mid (p_1, p_2) \in M^{-1}(0) \) for some \( p_2 \in \Omega_2 \) is empty, \( \gamma([0, 1]) \subset U_k \setminus M^{-1}(0) \), and this shows \( U_k \setminus M^{-1}(0) \) is connected.

Now, since \( S_1, S_2 \) are admissible to each other, we can assume that \( M^{-1}(0) \) is one of \( \{ (0, 0) \}, \{ (\alpha_1(t), \beta_1(t)) \mid 0 \leq t \leq \varepsilon \}, \{ (\beta_1(t), \alpha_2(t)) \mid 0 \leq t \leq \varepsilon \}, \{ (\alpha_1(t), \beta_2(t)) \mid 0 \leq t \leq \varepsilon \}, \{ (\beta_1(t), \alpha_2(t)) \mid 0 \leq t \leq \varepsilon \} \). So we can assume that \( (U_k \setminus M^{-1}(0)) \cap (U_{k+1} \setminus M^{-1}(0)) \neq \emptyset \) for \( k = 1, \ldots, n - 1 \), since \( M^{-1}(0) \subset U \). Hence the set \( U \setminus M^{-1}(0) = \bigcup_{k=1}^{n} M(U_k \setminus M^{-1}(0)) \) is connected. Thus \( M(U \setminus M^{-1}(0)) \) is connected, since \( M \) is continuous.

**Theorem 6.2 (Minkowski sum of admissible non-degenerate sectors).** Let \( \mathcal{M} \) be a Minkowski class, and let \( S_1, S_2 \) be non-degenerate \( \mathcal{M} \)-sectors with respective centers \( p_1, p_2 \) and radius \( R > 0 \), which are admissible to each other. Let \( S = S_1' + S_2' \), where \( S_1' = B_{r_1}(p_1) \cap S_1 \) for \( i = 1, 2 \) for some sufficiently small \( 0 < r < R \). Let \( p = p_1 + p_2 \). Suppose \( B_{\phi_\theta}(p) \cap S \neq B_{\phi_\theta}(p) \) for every \( \theta > 0 \). Then, for every sufficiently small \( \theta > 0 \), we have:

1. \( B_{\phi_\theta}(p) \cap S \) is a non-degenerate \( \mathcal{M} \)-sector with center \( p \) and radius \( \theta \).

2. Let \( \alpha \) be the end curve and start curve of \( B_{\phi_\theta}(p) \cap S \). Suppose that the image of \( \alpha \) (resp., \( \beta \)) contained in one of the images of \( \alpha_1 + p_2, \beta_1 + p_2, \alpha_2 + p_1, \beta_2 + p_1, \alpha_1 \ast \alpha_2, \beta_1 \ast \beta_2, \alpha_1 \ast \beta_2, \beta_1 \ast \alpha_2 \), where \( \alpha_i = \beta_i \) are the end curve and start curve of \( S_i \) respectively for \( i = 1, 2 \). Then there exists a continuous map \( \phi_i^\alpha : [0, \varepsilon] \to \partial^c S_i', \phi_i^\beta : [0, \varepsilon] \to \partial^c S_i', \phi_i^\alpha(t) = (\gamma_i^\alpha(t), \mathbf{n}_i^\alpha(t)) \) (resp., \( \phi_i^\beta : [0, \varepsilon] \to \partial^c S_i', \phi_i^\beta(t) = (\gamma_i^\beta(t), \mathbf{n}_i^\beta(t)) \), for \( i = 1, 2 \), with the following properties:

   1. \( \gamma_i^\alpha(0) = p_1 \) and \( \gamma_i^\alpha(0) = p_2 \) (resp., \( \gamma_i^\beta(0) = p_1 \) and \( \gamma_i^\beta(0) = p_2 \)).
(2) $\alpha(t) = \gamma_1^0(t) + \gamma_2^0(t)$ (resp., $\beta(t) = \gamma_1^0(t) + \gamma_2^0(t)$) for every $t \in [0, \varepsilon]$.

(3) $n_{B_{\varrho}(p)}^+(\alpha(t)) = n_{S_1}^+(t)$ (resp., $n_{B_{\varrho}(p)\cap S}(\beta(t)) = n_{S_1}^+(t)$) for every $t \in [0, \varepsilon]$.

(4) For $i = 1, 2$, $\phi_i^0$ and $\gamma_i^0$ (resp., $\phi_i^0$ and $\gamma_i^0$) are either one-to-one or constant, and if one of $O_{S_1}(\gamma_1^0)$ and $O_{S_2}(\gamma_2^0)$ (resp., $O_{S_1}(\gamma_1^0)$ and $O_{S_2}(\gamma_2^0)$) is $-$ (resp., $+$), then the other is $+$ (resp., $-$).

3. Suppose that the image of $\alpha$ (resp., $\beta$) is not contained in any of the images of $\alpha_1 + p_2, \beta_1 + p_2, \alpha_2 + p_1, \beta_2 + p_1, \alpha_1 \ast \alpha_2, \beta_1 \ast \beta_2, \alpha_1 \ast \beta_2, \beta_1 \ast \alpha_2$. Then $\alpha \setminus \{p\} \subset \interior(S_1 + S_2)$ (resp., $\beta \setminus \{p\} \subset \interior(S_1 + S_2)$).

Proof. We denote $M_{S_1, S_2}$ by $M$. We can assume $p_1 = p_2 = 0$. By Theorem 6.1, $B_\varrho(0) \cap S$ is either $B_\varrho(0)$ or a finite union of mutually non-overlapping $M$-sectors with center 0 and radius $\varrho$, for sufficiently small $\varrho > 0$. Note that $S_1'$ and $S_2'$ are non-degenerate. So by Lemma 6.6, there exist $0 < r_1, \ldots, r_n < \varrho / 2$ and $(p_1^0, p_2^0), \ldots, (p_n^0, p_2^n) \in M^{-1}(0)$ such that $B_{r_k}(p_k^0) \cap S_1'$ is a non-degenerate $M$-sector with center $p_k^0$ and radius $r_k$ for $i = 1, 2$, $k = 1, \ldots, n$, and $M^{-1}(0) \subset U$, where $U = \bigcup_{k=1}^n (B_{r_k}(p_k^0) \cap S_1') \times (B_{r_k}(p_k^0) \cap S_2')$. Note that $M((S_1' \times S_2') \setminus U)$ is compact and does not contain 0. So there exists $0 < \varepsilon < \varrho$ such that $B_{\varepsilon}(0) \cap M((S_1' \times S_2') \setminus U) = \emptyset$. It follows that $B_{\varepsilon}(0) \cap S = B_{\varepsilon}(0) \cap M(U)$. Since $r_k < \varrho / 2$ for $k = 1, \ldots, n$, it is clear that $M(U \setminus M^{-1}(0)) \subset B_{\varepsilon}(0) \cap (S \setminus \{0\})$. By Lemma 6.8, $M(U \setminus M^{-1}(0))$ is connected, since both $S_1'$ and $S_2'$ are non-degenerate. So $M(U \setminus \{0\})$ is contained in one connected component of $B_{\varepsilon}(0) \cap (S \setminus \{0\})$. Since $B_{\varepsilon}(0) \cap S = B_{\varepsilon}(0) \cap M(U)$, it follows that $B_{\varepsilon}(0) \cap (S \setminus \{0\})$ has exactly one connected component. This implies that $B_{\varepsilon}(0) \cap S$ is an $M$-sector with center 0 and radius $\varrho$, since we assumed that $B_{\varepsilon}(0) \cap S \neq B_\varrho(0)$. Since $0 \in S_1', S_2'$, we have $B_{\varepsilon}(0) \cap S_1', B_{\varepsilon}(0) \cap S_2' \subset B_{\varepsilon}(0) \cap S$. So we conclude that $B_{\varepsilon}(0) \cap S$ is a non-degenerate $M$-sector with center 0 and radius $\varrho$, since $S_1'$ and $S_2'$ are non-degenerate. Thus we have shown 1.

Suppose the image of $\alpha$ (resp., $\beta$) is contained in one of the images of the curves $\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_1 \ast \alpha_2, \beta_1 \ast \beta_2, \alpha_1 \ast \beta_2, \beta_1 \ast \alpha_2$. We can assume that $\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_1 \ast \alpha_2, \beta_1 \ast \beta_2, \beta_1 \ast \alpha_2$ and $\alpha$, $\beta$ are parametrized as follows: $\gamma(t) = 0$ for any $\gamma$ among the above curves, and, for any $\ast$-admissible $\gamma_1, \gamma_2$ among the above curves, $\nu[\gamma_1](t) \parallel \nu[\gamma_2](t)$ for every feasible $t$. Now, depending on in which of the images of the curves $\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_1 \ast \alpha_2, \beta_1 \ast \beta_2, \beta_1 \ast \alpha_2$ the image of $\alpha$ (resp., $\beta$) is contained, we construct $\phi_i^0 : [0, \varepsilon] \to \partial^u S_1'$ (resp., $\phi_i^0 : [0, \varepsilon] \to \partial^u S_1'$) for $i = 1, 2$ as follows:

<table>
<thead>
<tr>
<th>$\alpha$ (resp., $\beta$)</th>
<th>$\phi_1^0(t)$ (resp., $\phi_1^0(t)$)</th>
<th>$\phi_2^0(t)$ (resp., $\phi_2^0(t)$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1$</td>
<td>$(\alpha_1(t), n_{S_1}^+(\alpha_1(t)))$</td>
<td>$(0, n_{S_1}^+(\alpha_1(t)))$</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>$(\beta_1(t), n_{S_1}^+(\beta_1(t)))$</td>
<td>$(0, n_{S_1}^+(\beta_1(t)))$</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>$(0, n_{S_2}^+(\alpha_2(t)))$</td>
<td>$(\alpha_2(t), n_{S_2}^+(\alpha_2(t)))$</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>$(0, n_{S_2}^+(\beta_2(t)))$</td>
<td>$(\beta_2(t), n_{S_2}^+(\beta_2(t)))$</td>
</tr>
<tr>
<td>$\alpha_1 \ast \alpha_2$</td>
<td>$(\alpha_1(t), n_{S_1}^+(\alpha_1(t)))$</td>
<td>$(\alpha_2(t), n_{S_2}^+(\alpha_2(t)))$</td>
</tr>
<tr>
<td>$\beta_1 \ast \beta_2$</td>
<td>$(\beta_1(t), n_{S_1}^+(\beta_1(t)))$</td>
<td>$(\beta_2(t), n_{S_2}^+(\beta_2(t)))$</td>
</tr>
<tr>
<td>$\alpha_1 \ast \beta_2$</td>
<td>$(\alpha_1(t), n_{S_1}^+(\alpha_1(t)))$</td>
<td>$(\beta_2(t), n_{S_2}^+(\beta_2(t)))$</td>
</tr>
<tr>
<td>$\beta_1 \ast \alpha_2$</td>
<td>$(\beta_1(t), n_{S_1}^+(\beta_1(t)))$</td>
<td>$(\alpha_2(t), n_{S_2}^+(\alpha_2(t)))$</td>
</tr>
</tbody>
</table>
From the above table, it is easy to check that \( \phi_i^\alpha \) and \( \phi_i^\beta \) (resp., \( \phi_i^\beta \) and \( \phi_i^\beta \)) satisfy (1) and (2) of 2. It is also clear that \( \phi_i^\alpha \) and \( \gamma_i^\alpha \) (resp., \( \phi_i^\beta \) and \( \gamma_i^\beta \)) are either one-to-one or constant for \( i = 1, 2 \). Note that \( B_\phi(0) \cap S, S_1', S_2' \) are non-degenerate \( M \)-sectors, and \( B_\phi(0) \cap S_i' \subset B_\phi(0) \cap S \) for \( i = 1, 2 \). Suppose the image of \( \alpha \) is contained in the image of \( \beta \). Since \( B_\phi(0) \cap S_i' \subset B_\phi(0) \cap S \), it follows that \( \alpha_1 \) and \( \beta_1 \) have the same image. But this is impossible, since \( S_i' \) is non-degenerate. So the image of \( \alpha \) cannot be contained in the image of \( \beta \). In the same way, the image of \( \alpha \) cannot be contained in the image of \( \beta_2 \), and the image of \( \beta \) cannot be contained in the images of \( \alpha_1 \) or \( \alpha_2 \). Suppose the image of \( \alpha \) is contained in the image of \( \beta_1 * \beta_2 \). By Lemma 2.3, \( v[\beta_1 * \beta_2] = v[\beta_1] \) or \( v[\beta_2] \). With no loss of generality, suppose \( v[\beta_1 * \beta_2] = v[\beta_1] = (1, 0) \). Clearly, \( v[\alpha] = (1, 0) \). Take non-zero points \( q_1, q_2, q \) in the images of \( \beta_1, \beta_2, \beta \) respectively such that \( q = q_1 + q_2 \). Note that these points can be taken arbitrarily close to 0. So there exists a small \( \delta > 0 \) such that \( \{q_1 + u \cdot (0, -1) \mid 0 \leq u \leq \delta \} \subset S_1' \) and \( q_2 + \{q_1 + u \cdot (0, -1) \mid 0 \leq u \leq \delta \} = \{q + u \cdot (0, -1) \mid 0 \leq u \leq \delta \} \subset B_\phi(0) \cap S \). But this contradicts the assumption that \( \alpha \) is the end curve of \( B_\phi(0) \cap S \). So the image of \( \alpha \) cannot be contained in the image of \( \beta_1 * \beta_2 \). In the same way, the image of \( \beta \) cannot be contained in the image of \( \alpha_1 * \alpha_2 \). Now, the above table shows that if one of \( O_{S_1}(\gamma_1^\alpha) \) and \( O_{S_2}(\gamma_2^\alpha) \) (resp., \( O_{S_1}(\gamma_1^\beta) \) and \( O_{S_2}(\gamma_2^\beta) \)) is \( \in \) (resp., \( \notin \)), then the other is \( \notin \) (resp., \( \in \)). This proves (4) of 2.

Suppose the image of \( \alpha \) is contained in \( \alpha_1 * \alpha_2 \). Then, for every \( t \), either \( n_{S_1}(\alpha_1(t)) = n_{S_2}^+(\alpha_2(t)) \) or \( n_{S_2}(\alpha_1(t)) = -n_{S_2}(\alpha_2(t)) \), since \( \alpha_1 \) and \( \alpha_2 \) are \( \ast \)-admissible to each other. Suppose the latter is true. Since \( S_1 \) and \( S_2 \) are non-degenerate, we can take \( t_0 \) such that, for every sufficiently small \( \delta > 0 \), \( \alpha_1(t_0) - \delta \cdot n_{S_1}(\alpha_1(t)) \in S_1 \) and \( \alpha_2(t_0) - \delta \cdot n_{S_2}(\alpha_2(t)) \in S_2 \). This implies \( \alpha(t_0) \pm \delta \cdot n_{S_2}(\alpha(t_0)) \in S \), contrary to \( \alpha \subset \partial S \). So \( n_{S_1}(\alpha_1(t)) = n_{S_2}^+(\alpha_2(t)) \) for every \( t \), and hence, \( n_{S_2}(\alpha(t)) = n_{S_1}(\alpha_1(t)) = n_{S_2}(\alpha_2(t)) \) for every \( t \). We can show that (3) of 2 is true for the remaining cases in a similar way.

Now we show 3. Suppose the image of \( \alpha \) (resp., \( \beta \)) is not contained in any of the images of \( \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_1 * \alpha_2, \beta_1 * \beta_2, \alpha_1 * \beta_2, \beta_1 * \alpha_2 \). Then by Lemma 6.7, the image of \( \alpha \) (resp., \( \beta \)) is contained in one of the images of \( \hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2 \). We first show that the image of \( \alpha \) cannot be contained in the images of \( \hat{\alpha}_1 \) or \( \hat{\alpha}_2 \), and the image of \( \beta \) cannot be contained in the images of \( \hat{\beta}_1 \) or \( \hat{\beta}_2 \). Suppose the image of \( \alpha \) is contained in the image of \( \hat{\alpha}_1 \). By Lemma 6.7, \( \alpha_1 - \beta_2 \) have the same image. We can assume that \( v[\alpha_1] = (1, 0) \). Since \( \alpha \) is small, there exists \( f : [-r', 0] \to \mathbb{R} \) whose graph is the image of \( \alpha_1 \). Note that \( |(-r', f(-r'))| = r \). The graph of \( g : [0, r'] \to \mathbb{R} \) defined by \( g(x) = f(x - r') - f(-r') \) is the image of \( \alpha_1 \). Since \( S_1' \) is non-degenerate, there exist \( \varepsilon, \delta > 0 \) such that \( \{(-\varepsilon, y) \mid f(-\varepsilon) - \delta \leq y \leq f(-\varepsilon)\} \subset S_1' \). Since \( \alpha_1, -\beta_2 \) have the same image, we have \( (r', -f(-r')) \subset S_2' \). Note that we can take \( \delta \) and as small as desired. So \( \{(-\varepsilon + r', y) \mid g(-\varepsilon + r') - \delta \leq y \leq g(-\varepsilon + r')\} = (r', -f(-r')) + \{(-\varepsilon, y) \mid f(-\varepsilon) - \delta \leq y \leq f(-\varepsilon)\} \subset B_\phi(0) \cap S \). This means that \( \hat{\alpha}_1 \cap B_\phi(0) \) cannot be the end curve of \( B_\phi(0) \cap S \), contrary to the assumption. Thus the image of \( \alpha \) cannot be contained in the image of \( \hat{\alpha}_1 \). In the same way, the image of \( \alpha \) cannot be contained in the image of \( \hat{\alpha}_2 \), and the image of \( \beta \) cannot be contained in the images of \( \hat{\beta}_1 \) or \( \hat{\beta}_2 \).

Suppose the image of \( \alpha \) is contained in the image of \( \hat{\beta}_1 \). By Lemma 6.7, \( \beta_1 - \alpha_2 \) have the same image. Suppose \( \sigma(\beta_1) = 0 \). Then \( \beta_1, \alpha_2 \) have the same image. So the image of \( \alpha \) is contained in the image of \( \alpha_2 \), contrary to the assumption. Suppose \( \sigma(\beta_1) = +. Then
the image of \( \tilde{\beta}_1 \) intersects \( \text{int} S_2' \), since \( S_2' \) is non-degenerate, \( \alpha_2 = -\beta_1 \), and \( r \) is assumed to be small. So \( B_{\tilde{\varphi}}(0) \cap \tilde{\beta}_1 \) cannot be the end curve of \( B_{\tilde{\varphi}}(0) \cap S \), which is a contradiction. Thus we must have \( \sigma(\beta_1) = -\) and \( \sigma(\tilde{\beta}_1) = + \). We can assume that there exists \( \tilde{r} > r \) such that \( T = B_{\tilde{\varphi}}(0) \cap \{(B_\varphi(0) \cap S_1) + (B_\varphi(0) \cap S_2)\} \) is a non-degenerate \( \mathcal{M} \)-sector with center 0 and radius \( \varrho \), and \( \tilde{\alpha}, \tilde{\beta}_1 \) have the same image, where \( \tilde{\alpha} \) is the end curve of \( T \) and \( \tilde{\beta}_1 \) is the start curve of \( B_{\varphi}(0) \cap S_1 \). Since \( \sigma(\tilde{\beta}_1) = + \), \( \beta_1 = B_{\varphi}(0) \cap \tilde{\beta}_1 \), and \( r \) is small, it is easy to see that \( \alpha \setminus \{0\} \subset \text{int} T \). Thus \( \alpha \setminus \{0\} \) is in the interior of \( S_1 + S_2 \). In the same way, we can show that \( \alpha \setminus \{0\} \subset \text{int}(S_1 + S_2) \) if \( \alpha, \tilde{\beta}_2 \) have the same image, and \( \beta \setminus \{0\} \subset \text{int}(S_1 + S_2) \) if \( \beta, \tilde{\alpha}_1 \) have the same image or \( \beta, \tilde{\alpha}_2 \) have the same image. This shows 3. \( \blacksquare \)

6.4. Closedness of Minkowski sum. Using the results of Section 6.2, we now analyze the Minkowski sum from a more global point of view, i.e., the Minkowski sum of general domains. It turns out that, for any Minkowski class \( \mathcal{M} \), the Minkowski sum of \( \mathcal{M} \)-domains is also an \( \mathcal{M} \)-domain, and thus the set of all \( \mathcal{M} \)-domains is closed under Minkowski sum. Note that this is not true for an arbitrary curve class \( \mathcal{C} \) which is closed under restriction. See Figure 3 for an example.

First, we prove a lemma which will also be used later in Section 7:

**Lemma 6.9.** Let \( \mathcal{M} \) be a Minkowski class, and let \( \Omega_1 \) and \( \Omega_2 \) be two \( \mathcal{M} \)-domains. Let \( \Omega = \Omega_1 + \Omega_2 \). Then, for every point \( p \in \partial \Omega \) and for every \( r > 0 \), there exist a finite number of pairs \((p_{1,k}, p_{2,k})\), \((p_{1,n}, p_{2,n})\) in \( M_{\Omega_1, \Omega_2}(p) \), \( 0 < r_1, \ldots, r_n < r \), such that, for every sufficiently small \( \varrho > 0 \), the following are satisfied:

1. \( S_{i,k} = B_{r_k}(p_{i,k}) \cap \Omega_i \) is a finite union of mutually non-overlapping \( \mathcal{M} \)-sectors with center \( p_{i,k} \) and radius \( r_k \) for every \( i = 1, 2 \) and \( k = 1, \ldots, n \).

2. \( B_{\varphi}(p) \cap (S_{1,k} + S_{2,k}) \) is a finite union of mutually non-overlapping \( \mathcal{M} \)-sectors with center \( p \) and radius \( \varrho \) for \( k = 1, \ldots, n \).

3. \( B_{\varphi}(p) \cap \Omega \) is a finite union of mutually non-overlapping \( \mathcal{M} \)-sectors with center \( p \) and radius \( \varrho \), and \( B_{\varphi}(p) \cap \Omega = \bigcup_{k=1}^{n} \{B_{\varphi}(p) \cap (S_{1,k} + S_{2,k})\} \).

**Proof.** Suppose \( p \in \partial \Omega \) and \( r > 0 \). By Lemma 6.6, there exist finitely many pairs \((p_{1,i}, p_{2,i})\), \((p_{1,n}, p_{2,n})\) in \( M_{\Omega_1, \Omega_2}(p) \) and \( 0 < r_1, \ldots, r_n < r \) such that \( S_{i,k} = B_{r_k}(p_{i,k}) \cap \Omega_i \) is a finite union of mutually non-overlapping \( \mathcal{M} \)-sectors with center \( p_{i,k} \) and radius \( r_k \) for \( i = 1, 2 \), \( k = 1, \ldots, n \), and \( M_{\Omega_1, \Omega_2}(p) \subset U \), where \( U = \bigcup_{k=1}^{n} \bigcup_{j=1}^{k} \bigcup_{j'=1}^{k'} (B_{r_k}(p_{1,i}) \cap \Omega_1) \times (B_{r_k}(p_{2,i}) \cap \Omega_2) \). Thus (1) is satisfied.

Let \( S_{i,k} = \bigcup_{j=1}^{k} S_{i,k,j} \), where \( S_{i,k,j} \)’s are mutually non-overlapping \( \mathcal{M} \)-sectors with center \( p_{i,k} \) and radius \( r_k \). Note that \( r_k \)'s can be taken to be arbitrarily small. So by Lemma 6.4, we can assume that \( \{S_{1,k,j} \} \) and \( \{S_{2,k,j'} \} \) are admissible to each other for every \( k = 1, \ldots, n \) and \( 1 \leq j \leq n_k' \), \( 1 \leq j' \leq n_k \). By Theorem 6.1, the set \( B_{\varphi}(p) \cap (S_{1,k,j} + S_{2,k,j'}) \) is either \( B_{\varphi}(p) \), or a finite union of mutually non-overlapping \( \mathcal{M} \)-sectors with center \( p \) and radius \( \varrho \) for sufficiently small \( \varrho > 0 \). So by Lemmas 6.1 and 4.1(2), the set \( B_{\varphi}(p) \cap (S_{1,k} + S_{2,k}) \) is either \( B_{\varphi}(p) \), or a finite union of mutually non-overlapping \( \mathcal{M} \)-sectors with center \( p \) and
radius \( \varrho \) for sufficiently small \( \varrho > 0 \). It follows that \( B_\varrho(p) \cap (S_1^k + S_2^k) \) is a finite union of mutually non-overlapping \( \mathcal{M} \)-sectors with center \( p \) and radius \( \varrho \), since \( p \in \partial \Omega \) and \( S_1^k + S_2^k \subset \Omega \). Thus (2) is satisfied.

Lemma 4.1(2) shows that \( \bigcup_{k=1}^n \{ B_\varrho(p) \cap (S_1^k + S_2^k) \} \) is a finite union of mutually non-overlapping \( \mathcal{M} \)-sectors with center \( p \) and radius \( \varrho \) for sufficiently small \( \varrho > 0 \), since \( p \in \partial \Omega \) and \( \bigcup_{k=1}^n (S_1^k + S_2^k) \subset \Omega \). Note that the set \( M_{\Omega_1,\Omega_2}((\Omega_1 \times \Omega_2) \setminus U) \) in \( \Omega \) is compact, and does not contain \( p \), so, for sufficiently small \( \varrho > 0 \), we have \( B_\varrho(p) \cap M_{\Omega_1,\Omega_2}((\Omega_1 \times \Omega_2) \setminus U) = \emptyset \). This implies that \( B_\varrho(p) \cap \Omega = B_\varrho(p) \cap M_{\Omega_1,\Omega_2}(U) \). Thus (3) is satisfied, since \( M_{\Omega_1,\Omega_2}(U) = \bigcup_{k=1}^n (S_1^k + S_2^k) \).

It is now easy to prove the following result:

**Theorem 6.3 (closedness under Minkowski sum).** Let \( \mathcal{M} \) be a Minkowski class, and let \( \Omega_1 \) and \( \Omega_2 \) be two \( \mathcal{M} \)-domains. Then their Minkowski sum \( \Omega = \Omega_1 + \Omega_2 \) is an \( \mathcal{M} \)-domain.

**Proof.** First, note that \( \Omega \) is compact and connected, since it is the image of the compact connected set \( \Omega_1 \times \Omega_2 \) under the continuous Minkowski map \( M_{\Omega_1,\Omega_2} \). By Lemma 6.9, there exist \( r > 0 \) such that \( B_r(p) \cap \Omega \) is a finite union of mutually non-overlapping \( \mathcal{M} \)-sectors with center \( p \) and radius \( r \) for every \( p \in \partial \Omega \). Thus \( \Omega \) is an \( \mathcal{M} \)-domain by Lemma 4.2.

### 7. Minkowski sum of semi-convex domains

Let us first define the *semi-convexity*:

**Definition 7.1 (semi-convex domain).** A regular \( C^{1,1} \)-domain \( \Omega \) is called **semi-convex** if \( \Theta(\Omega) \geq -\pi \).

**Remark 7.1.** In fact, if \( \Theta(\Omega) > -2\pi \) for a regular \( C^{1,1} \)-domain \( \Omega \), then \( \Omega \) must be simply connected. So a semi-convex domain is automatically simply connected. It is also easy to see that a regular \( C^{1,1} \)-domain \( \Omega \) is **convex** if and only if \( \Theta(\Omega) = 0 \).

The domains in Figures 11 and 12 are examples of regular \( C^{1,1} \)-domains which are semi-convex and not semi-convex respectively.

![Fig. 11. Examples of semi-convex domains](image)

In this section, we will show that the Minkowski sum of two semi-convex \( \mathcal{M} \)-domains is homeomorphic to the unit disk in \( \mathbb{R}^2 \) for any Minkowski class \( \mathcal{M} \). This answers Problem 1 posed in Section 1 within the category of \( \mathcal{M} \)-domains.
Let $\mathcal{M}$ be a Minkowski class, and let $\Omega_1$ and $\Omega_2$ be two semi-convex $\mathcal{M}$-domains. Let $\Omega = \Omega_1 + \Omega_2$ be their Minkowski sum. The proof is divided into two major steps: First, we show that $\Omega$ is regular in Section 7.1, and then we show that $\Omega$ is simply connected in Section 7.2. The result will finally follow, since a domain is homeomorphic to the unit disk if and only if it is regular and simply connected.

7.1. Regularity

**Lemma 7.1.** Let $\mathcal{M}$ be a Minkowski class, and let $\Omega_1$, $\Omega_2$ be regular $\mathcal{M}$-domains. Let $\Omega = \Omega_1 + \Omega_2$, and let $p \in \partial \Omega$. Suppose $B_r(p) \cap \Omega = \bigcup_{k=1}^{n} S^k$, where $S^k$’s are mutually non-overlapping $\mathcal{M}$-sectors with center $p$ and radius $r$. Then there exist $q > 0$ and $(p_1^1, p_2^1), \ldots, (p_1^n, p_2^n)$ in $M_{\Omega_1, \Omega_2}(p)$ such that $S_i^k = B_q(p_i^k) \cap \Omega_i$ is a non-degenerate $\mathcal{M}$-sector with center $p_i^k$ and radius $q$, and $S_i^k - p_i^k \subset (S_i^k + S_i^k) - p \subset S^k - p$ for each $i = 1, 2$ and $k = 1, \ldots, n$.

**Proof.** By Lemma 6.9, there exist $r_1, \ldots, r_m > 0$, $(q_1^1, q_2^1), \ldots, (q_m^m, q_2^m)$ in $M_{\Omega_1, \Omega_2}(p)$, and $0 < q < \min \{r/2, r_1, \ldots, r_m\}$ such that $T_i^j = B_{r_j}(q_i^j) \cap \Omega_i$ is a finite union of mutually non-overlapping $\mathcal{M}$-sectors with center $q_i^j$ and radius $r_j$ for $i = 1, 2$ and $j = 1, \ldots, m$, $B_{2q}(p) \cap (T_1^j + T_2^j)$ is a finite union of mutually non-overlapping $\mathcal{M}$-sectors with center $p$ and radius $2q$ for each $j$, and $B_{2q}(p) \cap \Omega = \bigcup_{j=1}^{m} B_{2q}(p) \cap (T_1^j + T_2^j)$. Since $\Omega_1$ and $\Omega_2$ are regular, each $T_i^j$ is a non-degenerate $\mathcal{M}$-sector. Since $r_j$’s can be taken arbitrarily small, we can assume that $T_1^j$ and $T_2^j$ are admissible to each other for each $j$. So by Theorem 6.2, $B_{2q}(p) \cap (T_1^j + T_2^j)$ is a non-degenerate $\mathcal{M}$-sector with center $p$ and radius $2q$, and $B_{2q}(p) \cap \Omega = \bigcup_{k=1}^{n} B_{2q}(p) \cap S^k$. So it is easy to see that there exists $1 \leq j_k \leq m$ such that $B_{2q}(p) \cap (T_1^{j_k} + T_2^{j_k}) \subset B_{2q}(p) \cap S^k$ for each $k$. Let $p_i^k = q_i^{j_k}$, and let $S_i^k = B_{q_i^k}(p_i^k) \cap \Omega_i = B_{q_i^k}(q_i^{j_k}) \cap T_i^{j_k}$ for $i = 1, 2$ and $k = 1, \ldots, n$. Then $S_i^k = B_{q_i^k}(q_i^{j_k}) \cap T_i^{j_k}$, and $S_i^k - p_i^k \subset (S_i^k + S_i^k) - p$ for each $i = 1, 2$ and $k = 1, \ldots, n$.

**Lemma 7.2.** Let $\mathcal{M}$ be a Minkowski class. Let $\gamma_i : [0, a_i] \to \mathbb{R}^2$, $i = 1, 2$, be two $\mathcal{M}$-curves such that $\gamma_1(0) = \gamma_2(0) = 0$ and their images $S_1 = \gamma_1([0, a_1])$, $S_2 = \gamma_2([0, a_2])$ are degenerate $\mathcal{M}$-sectors with center $0$ and radius $r > 0$. Suppose $S_1$, $S_2$ are admissible to each other, and $S_1 \neq -S_2$. Take $0 < r' \leq r$ such that either $B_{r'}(0) \cap S_1 = B_{r'}(0) \cap S_2,$
or $B_{r'}(0) \cap S_1, B_{r'}(0) \cap S_2$ do not meet except at 0. Let $S$ be the $\mathcal{M}$-sector with center 0 and radius $r'$ which is uniquely determined by the following conditions:

1. $S$ is bounded by $B_{r'}(0) \cap S_1, B_{r'}(0) \cap S_2$ and an arc in $\partial B_{r'}(0)$.
2. $S$ is a sharp sector if $v[\gamma_1] \neq -v[\gamma_2]$.
3. When $v[\gamma_1] = -v[\gamma_2]$, the start curve of $S$ is $B_{r'}(0) \cap S_1$ (resp., $B_{r'}(0) \cap S_2$), and the end curve of $S$ is $B_{r'}(0) \cap S_2$ (resp., $B_{r'}(0) \cap S_1$) if $\gamma_1 \prec \gamma_2$ (resp., $\gamma_1 \succ \gamma_2$).

Then $B_{\theta}(0) \cap S \subset S_1 + S_2$ for every sufficiently small $\theta > 0$.

**Proof.** We can assume $v[\gamma_1] = (\cos \theta, \sin \theta), v[\gamma_2] = (\cos (\pi - \theta), \sin (\pi - \theta))$ for some $0 \leq \theta \leq \pi/2$. In case $\theta = 0$, we can also assume that $\sigma(\gamma_1) = +$, $\sigma(\gamma_2) = 0$ or $+$, and $\gamma_1 \succ \gamma_2$. Then $(0, \varrho) \in S_1 + S_2$ for every sufficiently small $\varrho > 0$ when $\theta \neq \pi/2$. Note that $(0, \varrho) \in S$ for sufficiently small $\varrho > 0$ when $\theta \neq \pi/2$. In case $\theta = \pi/2$, it is also easy to see that there exists a point in $S_1 + S_2$ for every sufficiently small $\varrho > 0$. By Lemma 6.7 and Theorem 6.2(3), there exist $0 < r'' < r'$ and $\varrho > 0$ such that the set $B_{\theta}(0)\cap S_1 + (B_{r'}(0) \cap S_2)$ is in the union of the images of $\gamma_1, \gamma_2$ and $\gamma_1 \gamma_2$ (if defined). Hence, $B_{\theta}(0) \cap S \subset S_1 + S_2$ for every sufficiently small $\varrho > 0$.

**Theorem 7.1** (regularity of Minkowski sum of semi-convex domains). Let $\mathcal{M}$ be a Minkowski class, and let $\Omega_1$ and $\Omega_2$ be semi-convex $\mathcal{M}$-domains. Then their Minkowski sum $\Omega = \Omega_1 + \Omega_2$ is a regular $\mathcal{M}$-domain.

**Proof.** By Theorem 6.3, $\Omega$ is an $\mathcal{M}$-domain. Suppose $\Omega$ is not regular. Then there exists a point $p \in \partial \Omega$ and $r > 0$ such that $B_r(p) \cap \Omega = \bigcup_{k=1}^n S_k$, where $S_k$'s are mutually non-overlapping $\mathcal{M}$-sectors with center $p$ and radius $r$ and $n \geq 2$. By Lemma 7.1, there exist $(p_1, p_2), (p_2, p_2)$ in $M^{1}_{\Omega_1, \Omega_2}(p)$ and $\varrho > 0$ such that, for each $k = 1, 2$ and $i = 1, 2$, $S_k = B_{\varrho}(p_k) \cap \Omega_i$ is a non-degenerate $\mathcal{M}$-sector with center $p_k$ and radius $\varrho$, and $S_k - p_k \subset (S_k + S_k) - p < S_k - p$. Let $\Omega_2 = -\Omega_2 + p$. Then by Lemma 6.3(3), $\Omega_1$ and $\Omega_2$ are in contact position with each other, and meet at $p_1$ and $p_2$. Since $S_1$ and $S_2$ are non-overlapping, it is easy to see that $p_1 \neq p_2$. For $i = 1, 2$, let $\phi_i : [0, 1] \rightarrow \partial^r \Omega_i$, $\phi_i(t) = (\gamma_i(t), n_i(t))$, be a one-to-one continuous map such that $\phi_1(0) = (p_1, n_{\Omega_1}(p_1)), \phi_1(1) = (p_2, n_{\Omega_2}(p_2)), \phi_2(0) = (p_2, n_{\Omega_2}(p_2)), \phi_2(1) = (p_2, n_{\Omega_2}(p_2))$. By interchanging $(p_1, p_2)$ and $(p_2, p_1)$ if necessary, we can assume that $\partial^r \Omega_1(\phi_1) = \partial^r \Omega_2(\phi_2) = +$, and $(\gamma_1([0, 1]) \setminus \{p_1, p_2\}) \cap U = \emptyset$, where $U$ is the unbounded component of $\mathbb{R}^2 \setminus (\Omega_1 \cup \Omega_2)$. Define $\phi_2 : [0, 1] \rightarrow \partial^r \Omega_2, \tilde{\phi}_2(t) = (\gamma_2(t), \tilde{n}_2(t)), \tilde{\phi}_2(t) = (\gamma_2(t), \tilde{n}_2(t))$, by $\tilde{\phi}_2(t) = (-\gamma_2(t) + p, -n_2(t))$ for $t \in [0, 1]$. Then $\tilde{\phi}_2([0, 1]) \setminus \{p_1, p_2\}) \cap U = \emptyset$. Since $\Omega_1$ and $\Omega_2$ are semi-convex, we have $-\pi \leq \partial^r \Omega_1, \partial^r \Omega_2 \leq \pi$ by Lemma 5.4(2). So $-\pi \leq \partial^r \Omega_i \leq \pi$ for $i = 1, 2$, since $\partial^r \Omega_2(\tilde{\phi}_2) = \partial^r \Omega_2(\phi_2$) by Lemma 5.3.

We will show that, in fact, $-\pi < \partial^r \Omega_i \leq \pi$ for $i = 1, 2$. Suppose $\partial^r \Omega_1 = -\pi$. We can assume $n_1(0) = (-1, 0)$. Let $\alpha_1^k$ and $\beta_1^k$ be the end curve and start curve of the $\mathcal{M}$-sector $S_k$ respectively. Then $v[\alpha_1^k] = v[\beta_1^k] = (0, -1)$. Suppose $\sigma(\alpha_1^k) = +$. Then there exists $t_0 \in (0, 1)$ such that $\partial^r \Omega_1|_{[0, t_0]} > 0$. So by Lemma 5.1, we have $\partial^r \Omega_1|_{[0, t_0]} = \partial^r \Omega_1|_{[0, t_0]} > 0$. This is impossible since $\Omega_1$ is semi-convex. Thus $\sigma(\alpha_1^k) = 0$ or $-\pi$. In the same way, $\sigma(\beta_1^k) = 0$ or $+$. Since $S_1 - p_1 \subset S_1 - p, S_1 - p_2 \subset S_1 - p$, and $S_1 - p_1 \subset S_1 - p, S_1 - p_2 \subset S_1 - p$, and $S_1 - p$
and $S^2 - p$ are non-overlapping, it follows that $v[β^1] = v[α^2] = v[β^1] = v[α^2] = (0, -1)$, and either $σ(β^1) = -$ or $σ(α^2) = +$, where $α^k$ and $β^k$ are the end curve and start curve of $S^k$ for $k = 1, 2$. Let $α_1$ be the non-negative angle of counter-clockwise rotation from $-v_{Ω_2}(p_1^1)$ to $v_{Ω_1}(p_1^1) = v[α_1]$, and let $α_2$ be the non-negative angle of counter-clockwise rotation from $-v_{Ω_2}(p_1^2)$ to $v_{Ω_1}(p_1^2) = -v[α_2^2]$. Suppose $α_1 < π$.

Then by Lemma 7.2, the Minkowski sum of $β_1^1$ and $β_2^2$, which is contained in $S^1$, must intersect $S^2$. But this is impossible, since $S^1$ and $S^2$ are non-overlapping. So $α_1 ≥ π$. In the same way, $α_2 ≥ π$. By Lemma 5.4(1), we have $Θ(φ_1) + Θ(φ_2) + α_1 + α_2 = 0$, and hence $Θ(φ_1) + Θ(φ_2) + α_1 + α_2 = 0$. Since $-π ≤ Θ(φ_2) ≤ π$ and $α_1 + α_2 ≥ 2π$, we must have $α_1 = α_2 = π$ and $Θ(φ_2) = -π$. So $v[β_2^2] = v[α_2^2] = (0, -1)$. Remember that either $σ(β^1) = -$ or $σ(α^2) = +$. Suppose $σ(β^1) = -$. Then $σ(β_2^2)$ is also $-$. So there exists $t_0 ∈ (0, 1)$ such that $Θ(φ_2|_{t_0, 1}) > 0$. By Lemma 5.1, we have $Θ(φ_2|[0, t_0]) = Θ(φ_2) - Θ(φ_2|_{t_0, 1}) < -π$, which is impossible since $Ω_2$ is semi-convex. In the same way, we get a contradiction if $σ(α^2) = -. Thus we conclude that $Θ(φ_1) ≠ -π$. By a symmetric argument, $Θ(φ_2) ≠ -π$. It follows from Lemma 5.4(1) that $Θ(φ_i) ≠ π$ for $i = 1, 2$. Thus $-π < Θ(φ_i) < π$ for $i = 1, 2$.

![Fig. 13. C(S1) and C(S2)](image)

Now let $θ_1 (≤ 0)$ be the angle of clockwise rotation from $v_{Ω_1}(p_1^1)$ to $-v_{Ω_1}(p_1^2)$, and $θ_2 (≤ 0)$ be the angle of clockwise rotation from $v_{Ω_2}(p_2^1)$ to $-v_{Ω_2}(p_2^2)$. Note that $θ_i = Θ(φ_i) - π + 2πn_i$ for some $n_i ∈ Z$ for $i = 1, 2$. Since $S^1$ and $S^2$ are non-overlapping, we have $-2π ≤ θ_i ≤ 0$ (see Figure 13). So $n_i = 0$ for $i = 1, 2$, since $-π < Θ(φ_i) < π$. Thus $θ_i = Θ(φ_i) - π$ for $i = 1, 2$. Let $α_i'$ be the angle of rotation in $C(S^1)$ from $-v_{Ω_2}(p_1^2)$ to $v_{Ω_1}(p_1^1)$, and $α_2'$ be the angle of rotation in $C(S^2)$ from $-v_{Ω_1}(p_2^2)$ to $v_{Ω_2}(p_2^1)$. We understand $α_i'$ to be positive if the rotation is counter-clockwise, and negative if the rotation is clockwise. Note that $-v_{Ω_2}(p_1^1) = v_{Ω_2}(p_2^2)$, and $-v_{Ω_1}(p_2^1) = v_{Ω_2}(p_2^1)$. So clearly $α_i' = α_i - π$ for $i = 1, 2$ by Lemma 7.2.

From the definitions, it is obvious that $θ_1 + θ_2 + α_1' + α_2' = -2π$ (see Figure 13). So from the above relations between $θ_i$’s and $Θ(φ_i)$’s, and $α_i'$’s and $α_i$’s, it follows that $Θ(φ_1) + Θ(φ_2) + α_1 + α_2 = 2π$, which contradicts Lemma 5.4(1).

7.2. Simple connectedness. In this section, we show that the Minkowski sum of two semi-convex $M$-domains is simply connected for any Minkowski class $M$. 


Let $M$ be a Minkowski class, and let $\Omega$ be a simply connected regular $M$-domain. For each $q \in \Omega$, we fix a homotopy $H_{\Omega,q} : \Omega \times [0,1] \to \Omega$ such that $H_{\Omega,q}(p,0) = p$ and $H_{\Omega,q}(p,1) = q$ for every $p \in \Omega$. For each $q \in \mathbb{R}^2$, we define $I_q : \mathbb{R}^2 \to \mathbb{R}^2$ by $I_q(p) = -p+q$ for $p \in \mathbb{R}^2$. Note that $I_q \circ I_q$ is the identity map.

**Lemma 7.3.** Let $M$ be a Minkowski class, and let $\Omega_1$ and $\Omega_2$ be two semi-convex $M$-domains with $0 \in \Omega_1, \Omega_2$. Let $\Omega = \Omega_1 + \Omega_2$, and let $p \in \partial \Omega$. Then there exist one-to-one continuous maps $\phi^+ : [0,1] \to \partial^n \Omega$, $\phi^-(t) = (\gamma^+(t), \mathbf{n}^+(t))$, and continuous maps $\phi_i^+, \phi_i^- : [0,1] \to \partial^n \Omega_i$, $\phi_i^+(t) = (\gamma_i^+(t), \mathbf{n}_i^+(t))$ for $i = 1, 2$, which satisfy the following conditions:

1. $\phi^+(0) = (p, n_i^+(p))$, $O_\Omega(\phi^+) = \pm$, and $\gamma^+(t)$ is a flat point for every $t \in (0,1]$.
2. Each of $\phi_i^+$’s and $\gamma_i^+$’s is either one-to-one or constant, and if one of $O_{\Omega_i}(\gamma_i^+)$ and $O_{\Omega_2}(\gamma_i^+)$ is $\mp$, then the other is $\pm$.
3. $\gamma^+(t) = \gamma_1^+(t) + \gamma_2^+(t)$ and $\mathbf{n}^+(t) = \mathbf{n}_1^+(t) = \mathbf{n}_2^+(t)$ for $t \in [0,1]$.
4. $\Theta(\phi^+) = \Theta(\phi_1^+) = \Theta(\phi_2^+)$ and, for $i = 1, 2$, $\gamma_i^+$ is homotopic to $\gamma^+$ in $\mathbb{R}^2 \setminus \text{int } \Omega_i$ via the homotopy $H_i^+ : [0,1] \times [0,1] \to \mathbb{R}^2 \setminus \text{int } \Omega_i$, defined by

$$H_i^+(t,s) = \begin{cases} I_{\gamma_i^+(t)}(H_{\Omega_1,0}(\gamma_i^+(t),s)), & \text{if } t \in [0,1] \times [0,1]. \
\end{cases}$$

**Proof.** By Theorem 7.1, $\Omega$ is a regular $M$-domain. Let $p \in \partial \Omega$. By Lemma 4.3, there exists $r > 0$ such that $B_r(p) \cap \Omega$ is a non-degenerate $M$-sector with center $p$ and radius $r$. By Lemma 6.9, there exist $0 < r_1, \ldots , r_n < r$, $0 < \rho < r$, and $(p_1^1, p_2^1), \ldots , (p_n^1, p_2^1)$ in $M_{\Omega_1, \Omega_2}(p)$ such that $S_i^k = B_{r_k}(p_i^k) \cap \Omega_i$ is a finite union of mutually non-overlapping $M$-sectors with center $p_i^k$ and radius $r_k$ for $i = 1, 2$ and $k = 1, \ldots , n$, and $S^k = B_{\rho}(p) \cap (S_1^k + S_2^k)$ is a finite union of mutually non-overlapping sectors with center $p$ and radius $\rho$, and $S = B_{\rho}(p) \cap \Omega = \bigcup_{k=1}^n S_i^k$. Since $r$ can be taken arbitrarily small and $\Omega_1, \Omega_2$ are regular, we can assume that $B_{r_k}(p_i^k) \cap \Omega_i$ is a non-degenerate $M$-sector with center $p_i^k$ and radius $r_k$ for each $i$ and $k$. By Theorem 6.2, we can also assume each $S_i^k$ is a non-degenerate $M$-sector with center $p$ and radius $\rho$, since $r_k$’s can be taken arbitrarily small. Note that $S$ is a non-degenerate $M$-sector with center $p$ and radius $\rho$. Let $\gamma^+, \gamma^- : [0,1] \to S$ be the end curve and start curve of $S$ respectively. Since $S = \bigcup S_i^k$, there exist $1 \leq k^+, k^- \leq n$ such that $\gamma^+$ and $\gamma^−$ are the end curve of $S_{k^+}^k$ and the start curve of $S_{k^-}^k$ respectively. Since $\gamma^+$ and $\gamma^−$ are in the boundary of $\Omega$, they are in the boundary of $S_{k^+}^k + S_{k^+}^k$ and $S_{k^-}^k + S_{k^-}^k$ respectively. So by Theorem 6.2, there exist $0 < \varepsilon < 1$ and continuous maps $\phi_i^+ : [0,\varepsilon] \to \partial^n S_i^k$, $\phi_i^+(t) = (\gamma_i^+(t), \mathbf{n}_i^+(t))$, and $\phi_i^- : [0,\varepsilon] \to \partial^n S_i^k$, $\phi_i^-(t) = (\gamma_i^-(t), \mathbf{n}_i^-(t))$, for $i = 1, 2$, such that $\phi_i^+(0) = p_i^1$, $\gamma_i^+(t) = \gamma_i^+(t) + \gamma_2^+(t)$ and $\mathbf{n}_i^+(t)$ for $t \in [0,\varepsilon]$, each $\phi_i^+$ and each $\gamma_i^+$ is either one-to-one or constant, and if one of $O_{S_{k^+}^k}(\gamma_i^+)$ and $O_{S_{k^+}^k}(\gamma_i^−)$ is $\mp$, then the other is $\pm$.

Define $\phi^+, \phi^- : [0,\varepsilon] \to \partial^n S$, $\phi^+(t) = (\gamma^+(t), \mathbf{n}^+(t))$, by $\phi^+(t) = (\gamma^+(t), \mathbf{n}_1^+(t))$ for $t \in [0,\varepsilon]$. Note that $\phi_1^+, \phi_2^+$ and $\phi_2^-$ are in $\partial^n \Omega_1, \partial^n \Omega_2$ and $\partial^n \Omega$ respectively. Thus, by reparametrizing them on the interval $[0,1]$, (1)–(3) are checked easily.

Now we show (4). First, $\Theta(\phi^+) = \Theta(\phi_1^+) = \Theta(\phi_2^+)$. Since $\mathbf{n}^+(t) = \mathbf{n}_1^+(t) = \mathbf{n}_2^+(t)$ for every $t \in [0,1]$. Note that, for $i = 1, 2$, $H_i^+(t,0) = \gamma_i^+(t)$ and $H_i^+(t,1) = \gamma^+(t)$ for every
Let us introduce the following useful notations: Let $F_1 : [a_1, b_1] \times [c, d] \to \mathbb{R}^2$ and $F_2 : [a_2, b_2] \times [c, d] \to \mathbb{R}^2$ be two homotopies such that $F_1(b_1, s) = F_2(a_2, s)$ for every $s \in [c, d]$. Then we define $F_1 \cdot F_2 : [a_1, b_1 + b_2 - a_2] \times [c, d] \to \mathbb{R}^2$ by
\[
(F_1 \cdot F_2)(t, s) = \begin{cases} 
F_1(t, s) & \text{if } (t, s) \in [a_1, b_1] \times [c, d], \\
F_2(t - b_1 + a_2, s) & \text{if } (t, s) \in [b_1, b_1 + b_2 - a_2] \times [c, d].
\end{cases}
\]

Let $G_1 : [a, b] \times [c_1, d_1] \to \mathbb{R}^2$ and $G_2 : [a, b] \times [c_2, d_2] \to \mathbb{R}^2$ be two homotopies such that $G_1(t, d_1) = G_2(t, c_2)$ for every $t \in [a, b]$. Then we define $\underbrace{G_2}_{G_1} : [a, b] \times [c_1, d_1 + d_2 - c_2] \to \mathbb{R}^2$ by
\[
\begin{pmatrix} G_2 \\ G_1 \end{pmatrix}(t, s) = \begin{cases} 
G_1(t, s) & \text{if } (t, s) \in [a, b] \times [c_1, d_1], \\
G_2(t, s - d_1 + c_2) & \text{if } (t, s) \in [a, b] \times [d_1, d_1 + d_2 - c_2].
\end{cases}
\]

It is clear that $F_1 \cdot F_2$ and $\underbrace{G_2}_{G_1}$ are well defined and continuous. See Figure 14.

Let $F_{ij} : [a_i, b_i] \times [c_j, d_j] \to \mathbb{R}^2$ be a homotopy for $i, j = 1, 2$. Suppose that $F_{ij}(b_1, s) = F_{2j}(a_2, s)$ for every $s \in [c_j, d_j]$ and $j = 1, 2$, and $F_{i1}(t, d_1) = F_{i2}(t, c_2)$ for every $t \in [a_i, b_i]$ and $i = 1, 2$. Then we define $\underbrace{F_{12}}_{F_{21}} \cdot F_{22} : [a_1, b_1 + b_2 - a_2] \times [c_1, d_1 + d_2 - c_2] \to \mathbb{R}^2$ by
\[
\begin{pmatrix} F_{12} \cdot F_{22} \\ F_{11} \cdot F_{21} \end{pmatrix} = \begin{pmatrix} F_{12} \\ F_{11} \end{pmatrix} \cdot \begin{pmatrix} F_{22} \\ F_{21} \end{pmatrix} = (F_{12} \cdot F_{22}) \cdot (F_{11} \cdot F_{21}).
\]

See Figure 15.
For any \( m, n \geq 1 \), we define in an obvious way the appropriate homotopy, when given the homotopies \( F_{ij}, i = 1, \ldots, m, j = 1, \ldots, n \) with the continuity conditions on the common boundaries.

Now, let \( M \) be a Minkowski class, and let \( \Omega_1 \) and \( \Omega_2 \) be semi-convex \( M \)-domains with \( 0 \in \Omega_1, \Omega_2 \). Let \( \Omega = \Omega_1 + \Omega_2 \). Suppose \( \phi^k : [0, 1] \to \partial^v \Omega, \phi^k(t) = (\gamma^k(t), n^k(t)) \), and \( \phi^k_i : [0, 1] \to \partial^v \Omega_i, \phi^k_i(t) = (\gamma^k_i(t), n^k_i(t)) \), are continuous maps for \( k, i = 1, 2 \), which satisfy the following conditions:

1. \( \gamma^k \) is one-to-one and \( O_{\Omega}(\gamma^k) = + \), and \( \gamma^1(1) = \gamma^2(0) \).
2. Each of \( \phi^k \)'s and \( \gamma^k_i \)'s is either one-to-one or constant, and if one of \( O_{\Omega_i}(\gamma^k_i) \), \( O_{\Omega_2}(\gamma^k_2) \) is -, then the other is +.
3. \( \gamma^k_i(t) = \gamma^k_i(1) + \gamma^k_i(t) \) and \( n^k_i(t) = n^k_i(1) = n^k_i(0) \) for every \( t \in [0, 1] \).
4. \( \Theta(\phi^k) = \Theta(\phi^k_1) = \Theta(\phi^k_2) \), and for \( i, k = 1, 2 \), \( \gamma^k_i \) is homotopic to \( \gamma^k \) in \( \mathbb{R}^2 \setminus \text{int} \Omega_i \) via the homotopy \( H^k_i : [0, 1] \times [0, 1] \to \mathbb{R}^2 \setminus \text{int} \Omega_i \), where

\[
H^k_i(t, s) = I_{\gamma^k_i(t)}(H_{\Omega_i, 0}(\gamma^k_i(t), s)) \quad \text{and} \quad H^k_2(t, s) = I_{\gamma^k(t)}(H_{\Omega_2, 0}(\gamma^k(t), s)),
\]

for \( (t, s) \in [0, 1] \times [0, 1] \) and \( k = 1, 2 \).

Let \( p = \gamma^1(1) = \gamma^2(0) \). From the assumptions on \( \phi^k \)'s, it is obvious that \( \phi^1(1) = (p, n_\Omega(p)) \) and \( \phi^2(0) = (p, n_\Omega(p)) \). Let \( \psi : [0, 1] \to p \times NC_{\Omega}(p) \subset \partial^v \Omega, \psi(t) = (\eta(t), m(t)) \), be a continuous map which is either one-to-one or constant and \( m(0) = n_\Omega(p), m(1) = n_\Omega(p) \). Note that \( \psi(0) = \phi^1(1), \psi(1) = \phi^2(0), \) and \( \eta(t) = p \) for \( t \in [0, 1] \). Note also that \( \psi, m \) are one-to-one if \( p \) is a corner point, and constant if \( p \) is a flat point.

Let \( p^1_i = \gamma^1_i(1) \) and \( p^2_i = \gamma^2_i(0) \) for \( i = 1, 2 \). Note that \( p = p^1_1 + p^2_1 = p^1_2 + p^2_2 \), i.e., \( (p^1_i, p^2_i) \) and \( (p^2_i, p^2_i) \) are in \( M^{\perp}_{\Omega_i, \Omega_2}(p) \). Let \( \psi_i : [0, 1] \to \partial^v \Omega_i, \psi_i(t) = (\eta_i(t), m_i(t)) \), be a continuous map which is either one-to-one or constant, and \( \psi_i(0) = (p^1_i, n^1_i(1)) = \phi^1_i(1), \psi_i(1) = (p^2_i, n^2_i(0)) = \phi^2_i(0). \) Let \( \tilde{\Omega} = -\Omega_i + p \). Note that \( \Omega_1 \) and \( \Omega_2 \) are in contact position to each other, and \( p^1_1, p^2_1 \in \Omega_1 \cap \tilde{\Omega}_1 \). Also, \( \Omega_2 \) and \( \tilde{\Omega}_2 \) are in contact position to each other and \( p^2_2, p^2_2 \in \Omega_2 \cap \Omega_1 \). We assume that \( (\eta_i([0, 1]) \setminus \{p^1_i, p^2_i\}) \cap \tilde{U}_i = \emptyset, \) where \( U_1 \) is the unbounded component of \( \mathbb{R}^2 \setminus (\Omega_1 \cup \tilde{\Omega}_1) \), and \( U_2 \) is the unbounded component of \( \mathbb{R}^2 \setminus (\Omega_2 \cup \tilde{\Omega}_1) \).

Note that \( p^1_1 = p^2_1 \) if and only if \( p^2_1 = p^2_2 \). Suppose first \( p^1_1 = p^2_1 \). Then clearly \( \eta_1, \eta_2 \) are constant, and \( \psi_1, \psi_2 \) are either one-to-one or constant. Let \( p^1_1 = p^2_1 = p^2_1 \) and

\[
\begin{array}{cccc}
F_{12} & & & F_{22} \\
& & & \\
F_{11} & & & F_{21}
\end{array}
\]

Fig. 15. The homotopy \( \phi_{12}, \phi_{22}, \phi_{11}, \phi_{21} \).
Take $r > 0$ such that $S_i = B_r(p_i) \cap \Omega_i$ is a non-degenerate $\mathcal{M}$-sector with center $p_i$ and radius $r$ for $i = 1, 2$, and $S = B_{2r}(p) \cap \Omega$ is a non-degenerate $\mathcal{M}$-sector with center $p$ and radius $2r$. We can assume that $S_1, S_2$ are admissible to each other by Lemma 6.4. Note that $S_i - p_i \subset (S_1 + S_2) - p \subset S - p$. Since $m(0) = m_1(0) = m_2(0)$ and $m(1) = m_1(1) = m_2(1)$, we have $\Theta(\psi_i) = \Theta(\psi) + 2n_i \pi$ for some $n_i \in \mathbb{Z}$. Note that $-\pi \leq \Theta(\psi), \Theta(\psi_1), \Theta(\psi_2) \leq \pi$, since $m, m_1, m_2$ rotate in $NC_{\Omega}(p), NC_{\Omega_1}(p_1), NC_{\Omega_2}(p_2)$ respectively. So, if $-\pi < \Theta(\psi) < \pi$, we get $\Theta(\psi) = \Theta(\psi_1) = \Theta(\psi_2)$.

Suppose $\Theta(\psi) = \pi$. Then $S$ becomes a sharp sector, and $C(S)$ contains only one element. We can assume that $C(S) = \{(0, -1)\}$. Since $S_i - p_i \subset S - p$ for $i = 1, 2$, $S_1, S_2$ are also sharp sectors, and $C(S_1) = C(S_2) = \{(0, -1)\}$. So we must have $\Theta(\psi_1) = \Theta(\psi_2) = \pi$. Thus $\Theta(\psi) = \Theta(\psi_1) = \Theta(\psi_2)$. Suppose $\Theta(\psi) = -\pi$. Let $\alpha$ and $\beta$ be the end curve and start curve of $S$ respectively, and let $\alpha$ and $\beta$ be the end curve and start curve of $S_i$ respectively. In this case, $S$ becomes a dull sector, and $v[\alpha] = v[\beta]$. With no loss of generality, assume $v[\alpha] = v[\beta] = (0, 1)$. Note that $\Theta(\psi_1), \Theta(\psi_2)$ are $\pi$ or $-\pi$. Since $\Theta(\psi_1), \Theta(\psi_2) \neq 0$, $S_1$ and $S_2$ cannot be flat sectors. Since $S_1, S_2$ are admissible to each other, they cannot be dull sectors simultaneously. Suppose both $S_1$ and $S_2$ are sharp sectors. Then it is easy to see that $C(S_i) = \{(0, 1)\}$ for $i = 1, 2$. So, from Lemma 7.2, we can see that at least one of $\alpha$ and $\beta$ is not contained in $S_1 + S_2$, which contradicts the assumption that $\gamma^1 = \gamma_1^1 + \gamma_2^1$ and $\gamma^2 = \gamma_1^2 + \gamma_2^2$. So $S_1, S_2$ cannot be sharp sectors simultaneously. It follows that one of $S_1$ and $S_2$, say $S_1$, is a sharp sector and the other is a dull sector. Then it is easy to see that $v[\alpha_1] = v[\beta_1] = (0, -1), v[\alpha_2] = v[\beta_2] = (0, 1)$, and so $\Theta(\psi_1) = \Theta(\psi_2) = -\pi$. Thus we conclude that $\Theta(\psi) = \Theta(\psi_1) = \Theta(\psi_2)$ if $p_1^1 = p_2^1$ (or equivalently, $p_1^2 = p_2^2$).

Suppose now $p_1^1 \neq p_2^1$. Then it is easy to see that one of $O_{\Omega_1}(\eta_1), O_{\Omega_2}(\eta_2)$ is $+$ and the other is $-$. Moreover, $O_{\Omega_i}(\eta_k[a, b])$ cannot be $-O_{\Omega_i}(\eta_i)$ for any $[a, b] \in [0, 1]$, for $i = 1, 2$. We will also show that $\Theta(\psi) = \Theta(\psi_1) = \Theta(\psi_2)$ in this case. First, it is easy to see that $\Theta(\psi_1) = \Theta(\psi_2)$ since $m_1(0) = m_2(0), m_1(1) = m_2(1)$, and $\Omega_1, \Omega_2$ are in contact position to each other. Since $m(0) = m_1(0)$ and $m(1) = m_1(1)$, we have $\Theta(\psi_1) = \Theta(\psi) + 2n\pi$ for some $n \in \mathbb{Z}$. We have seen that one of $O_{\Omega_1}(\eta_1), O_{\Omega_2}(\eta_2)$ is $+$ and the other is $-$. So, one of $O_{\Omega_1}(\psi_1), O_{\Omega_2}(\psi_2)$, say $O_{\Omega_1}(\psi_1)$, is $+$ and the other is $-$. Since $\Omega_1, \Omega_2$ are semi-convex, we have $\Theta(\psi_1) \geq -\pi$ and $\Theta(\psi_2) \leq \pi$. So $-\pi \leq \Theta(\psi_1) = \Theta(\psi_2) \leq \pi$. Thus, $\Theta(\psi) = \Theta(\psi_1) = \Theta(\psi_2)$ if $-\pi < \Theta(\psi) < \pi$.

It remains to consider the cases when $\Theta(\psi) = \pi$ or $-\pi$. Take $r > 0$ such that $S_i^k = B_r(p_i^k) \cap \Omega_i$ is a non-degenerate $\mathcal{M}$-sector with center $p_i^k$ and radius $r$ for $i, k = 1, 2$, and $S = B_{2r}(p) \cap \Omega$ is a non-degenerate $\mathcal{M}$-sector with center $p$ and radius $2r$. We can assume $S_i^k$ and $S_2^k$ are admissible to each other. Note that $S_i^k - p_i^k \subset (S_i^k + S_2^k) - p \subset S - p$. Let $\alpha$ and $\beta$ be the end curve and start curve of $S$ respectively, and let $\alpha^k_i$ and $\beta^k_i$ be the end curve and start curve of $S_i^k$ respectively.

Suppose $\Theta(\psi) = \pi$. Then $S$ is a sharp sector, and $C(S)$ contains only one element, which we assume to be $(0, -1)$. Since $S_i^k - p_i^k \subset S - p$, we have $C(S_i^k) = \{(0, -1)\}$. So $v[\alpha^k_i] = v[\beta^k_i] = (0, -1)$. We can assume that $O_{\Omega_1}(\psi_1) = +$ and $O_{\Omega_2}(\psi_2) = -$. Suppose $\Theta(\psi_1) = \Theta(\psi_2) = -\pi$. Note that $m(0) = (1, 0)$ and $m(1) = (-1, 0)$. So $-\pi = \Theta(\psi_1) = \pi + \Theta(\psi_1^i) + \pi$, where $\psi_i^i : [0, 1] \to \partial^n \Omega_i$ is a one-to-one continuous map such that $\psi_i^i(0) = (p_1^i, n_{\Omega_i}^i(p_1^i)), \psi_i^i(1) = (p_2^i, n_{\Omega_i}^i(p_2^i))$, and $O_{\Omega_i}(\psi_i^i) = +$. Now $\Theta(\psi_i^i) = -3\pi$,
which is a contradiction since $\Omega_1$ is semi-convex. So we must have $\Theta(\psi_1) = \Theta(\psi_2) = \pi$, and hence $\Theta(\psi) = \Theta(\psi_1) = \Theta(\psi_2)$ if $\Theta(\psi) = \pi$.

Suppose $\Theta(\psi) = -\pi$. Then $S$ is a dull sector, and $\mathbf{v}[\alpha] = \mathbf{v}[\beta]$, which we assume to be $(0,1)$. Let $S' = B_{2r}(p) \setminus S$. Suppose $\Theta(\psi_1) = \Theta(\psi_2) = \pi$. We can assume that $O_{\Omega_1}(\psi_1) = +$ and $O_{\Omega_2}(\psi_2) = -$. Note that $\mathbf{m}(0) = (-1,0)$, $\mathbf{m}(1) = (1,0)$, and $S^i_k - p^i_k \subset S^i_k + S^i_k - p \subset S - p$. Let $\mathbf{v}[\beta_1] = (cos \theta, sin \theta)$. If $\pi/2 < \theta < 3\pi/2$, then $\mathbf{m}(0) = (-1,0)$ cannot be in $NC_{\Omega_1}(p_1^i)$. If $-\pi/2 \leq \theta < \pi/2$, then there exists $t_0 \in (0,1)$ such that $\Theta(\psi_2[t_0, t_1]) < 0$. So $\Theta(\psi_2[t_0, t_1]) = \Theta(\psi_2) - \Theta(\psi_2[t_0, t_1]) > \pi$, which is impossible since $\Omega_2$ is semi-convex and $O_{\Omega_2}(\psi_2) = -$. Thus we must have $\mathbf{v}[\beta_2] = (0,1)$. Note also that $\sigma(\beta_2^i) \neq -$ for the same reason. In the same way, we can see that $\mathbf{v}[\alpha_2^i] = (0,1)$ and $\sigma(\alpha_2^i) \neq +$. Let $\mathbf{v}[\alpha_1^i] = (cos \theta_1, sin \theta_1)$ and $\mathbf{v}[\beta_2^i] = (cos \theta_2, sin \theta_2)$. Suppose $\pi/2 < \theta_1 < 3\pi/2$. Then we should have $\mathbf{v}[\beta_2^i] = (0,1)$ in order for $\mathbf{m}(0) = (-1,0)$ to be in $NC_{\Omega_1}(p_1^i)$. Since $\mathbf{v}[\beta_2^i] = (0,1)$, it follows from Lemma 7.2 that $B_{\epsilon}(p) \subset S^1_k + S^2_k \subset S$ for sufficiently small $\epsilon > 0$, which is impossible. Suppose $\theta_1 = \pi/2$. Then, in order for $\mathbf{m}(0)$ to be in $NC_{\Omega_1}(p_1^i)$, we must have $\mathbf{v}[\beta_1^i] = (0,1)$ again, and $C(S^i_1) = \{(0,1)\} \in \partial B_1(0)$. If $C(S^i_1) = \partial B_1(0)$, then we would also have the same contradiction $B_{\epsilon}(p) \subset S^1_k + S^2_k$ by Lemma 7.2. So $C(S^i_1) = \{(0,1)\}$. Let $W_1$ be the sharp sector with center 0 and radius $2r$, whose start curve and end curve are $\beta - p$ and $\{(x,0) \mid 0 \leq x \leq 2r\}$ respectively. Let $W_2$ be the sharp sector with center 0 and radius $2r$, whose start curve and end curve are $\{(x,0) \mid -2r \leq x \leq 0\}$ and $\alpha - p$ respectively. Note that $\alpha_1^i - p^i_1, \beta_1^i - p^i_1 \subset W_1$ and $\beta_2^i - p^i_2 \subset W_2$ for some $i,j = 1,2$. If $i \neq j$, then $B_{\epsilon}(p) \cap S' \subset W_1 + W_2 \subset S$ by Lemma 7.2, which is a contradiction. So $i = j$. Suppose $i = j = 2$. Since $S^2_k - p^2_2 \subset S - p$, we also have $\alpha_2^i - p^2_2 \subset W_2$ and $\mathbf{v}[\alpha_2^i] = (0,1)$. Now from Lemma 2.3, we can see that $\beta - p$ cannot be any of $\alpha_1^i - p^i_1, \beta_1^i - p^i_1, \alpha_2^i - p^2_2, \beta_2^i - p^2_2$, or their convolutions. From Lemma 6.7 and Theorem 6.2(3), we see that this contradicts the assumption that $\gamma^i = \gamma_1^i + \gamma_2^i$. Suppose $i = j = 1$. Since $\sigma(\beta_2^i) \neq -$, we must have $\sigma(\beta) \neq -$ and $\sigma(\alpha) = +$. So, $\alpha_2^i - p^2_2, \beta_2^i - p^2_2 \subset W_1$ and $\mathbf{v}[\beta_2^i] = (0,1)$, since $\mathbf{v}[\alpha_2^i] = (0,1)$, $\sigma(\alpha_2^i) \neq +$ and $S^2_2 - p^2_2 \subset S - p$. Since $\gamma^2 = \gamma_1^2 + \gamma_2^2$, $\alpha - p$ should be one of $\alpha_1^2 - p^2_1, \beta_1^2 - p^2_1, \alpha_2^2 - p^2_2, \beta_2^2 - p^2_2$, or their convolutions by Lemma 6.7 and Theorem 6.2(3). There are only two cases to make this possible: Either $\alpha - p$ is one of $\alpha_1^i - p^i_1, \beta_1^i - p^i_1, \sigma(\beta_2^i) = +$ and $\alpha = \beta_2^i + \gamma$ for $\gamma = \alpha_1^2$ or $\beta_2^2$. But it is easy to see from Lemma 7.2 that, in both cases, $S^2_1 + S^2_2$ would intersect $B_{\epsilon}(p) \cap S'$ for sufficiently small $\epsilon > 0$, which is a contradiction. Thus $\mathbf{v}[\alpha_1^i] \neq (0,1)$. So we have $-\pi/2 \leq \theta_1 < \pi/2$. Similarly, $\pi/2 < \theta_2 \leq 3\pi/2$.

Suppose $\theta_1 = -\pi/2$, i.e., $\mathbf{v}[\alpha_1^i] = (0, -1)$. Let $\alpha_1^i$ be the non-negative angle of counter-clockwise rotation from $-\mathbf{v}[\beta_2^i]$ to $\mathbf{v}[\alpha_1^i]$ in $V$, where $V$ is the region bounded by $\eta_1$ and $-\eta_2 + p$. Let $\alpha_2^i$ be the non-negative angle of counter-clockwise rotation from $\mathbf{v}[\beta_2^i]$ to $-\mathbf{v}[\alpha_2^i]$ in $V$. Suppose either $\sigma(\alpha_1^i) = -$ or $\sigma(\alpha_2^i)$. Then $\alpha_1^i = 2\pi$, since $S^1_k, S^2_1$ are admissible to each other. For $i = 1,2$, we can choose $[a_i, b_i] \subset [0,1]$ such that $\psi_1(\alpha_1) = (p^i_1, n^i_1(p^i_1), \psi_1(b_i) = (p^i_1, n^i_1(p^i_1))$, and $\psi_2(\alpha_2) = (p^2_2, n^2_2(p^2_2), \psi_2(b_2) = (p^2_2, n^2_2(p^2_2))$. Note that $a_1 = a_2 = 0$, since $\mathbf{v}[\alpha_1^i] = -\mathbf{v}[\beta_2^i] = (0, -1)$. By Lemma 5.4(1), $\Theta(\psi_1) - \Theta(\psi_2) = 2\pi + \alpha_2 = 0$. Since $\Omega_1, \Omega_2$ are semi-convex, we must have $\Theta(\psi_1) - \Theta(\psi_2) = \pi$ and $\alpha_2 = 0$. Since $\alpha_2 = 0$ and $\mathbf{m}(1) = \mathbf{m}1(1) = \mathbf{m}2(1) = (1,0)$, it follows
that $\Theta(\psi_1) = -\pi$, which contradicts the assumption. Thus if $v[\alpha_1^1] = (0, -1)$ then we must have $\sigma(\alpha_1^1) \neq -$ and either $\alpha_1^1 \triangleright \beta_2^1$ or $\alpha_1^1 \sim \beta_2^1$. Similarly, if $v[\beta_2^2] = (0, -1)$ then $\sigma(\beta_2^2) \neq +$ and either $\beta_2^2 \triangleright \alpha_2^2$ or $\beta_2^2 \sim \alpha_2^2$. Now it is easy to see from Lemma 7.2 that $(S_1^1+S_2^1)\cup(S_2^2+S_2^2) \subset S$. Thus we must have $\Theta(\psi_1) = \Theta(\psi_2) = -\pi$.

Summarizing the above arguments, we conclude that $\Theta(\psi) = \Theta(\psi_1) = \Theta(\psi_2)$ in any case.

For $i = 1, 2$, define $\tilde{E}_i : [0, 1] \rightarrow \mathbb{R}^2$ by $\tilde{E}_i(t) = -\eta_i(t) + p$. It is easy to see that $\tilde{E}_i$ is in $\partial\Omega_i$ and $\tilde{E}_i(0) = p_1^1, \tilde{E}_i(1) = p_2^2, \tilde{E}_2(0) = p_1^1, \tilde{E}_2(1) = p_2^2$. Let $V_1$ be the region enclosed by $\eta_1$ and $\tilde{E}_2$, and $V_2$ be the region enclosed by $\eta_2$ and $\tilde{E}_1$. By Lemma 5.4(3), there exists a homotopy $A_i : [0, 1] \times [0, 1] \rightarrow V_i$ such that $A_1(t, 0) = \eta_1(t), A_1(t, 1) = \tilde{E}_2(t), A_1(0, s) = p_1^1, A_1(1, s) = p_2^2$, and $A_2(t, 0) = \eta_2(t), A_2(t, 1) = \tilde{E}_1(t), A_2(0, s) = p_1^2, A_2(1, s) = p_2^2$ for every $(t, s) \in [0, 1] \times [0, 1]$. For $i = 1, 2$, let $B_i : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^2$ be the homotopies defined by

$$B_1(t, s) = I_p(H_{\Omega_2,0}(\eta_2(t), s)), \quad B_2(t, s) = I_p(H_{\Omega_1,0}(\eta_1(t), s)),$$

for $(t, s) \in [0, 1] \times [0, 1]$. Then $B_1(t, 0) = \tilde{E}_2(t), B_2(t, 0) = \eta_1(t),$ and $B_1(t, 1) = B_2(t, 1) = p$ for $t \in [0, 1]$. It is also easy to see that $B_i([0, 1] \times [0, 1]) \subset \mathbb{R}^2 \setminus \text{int} \Omega_i$. For $i, k = 1, 2$ we define $E_i^k : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^2$ by

$$E_i^k(t, s) = \gamma_i^k(t)$$

for $(t, s) \in [0, 1] \times [0, 1]$.

Now we can see that the homotopy $G_i = \eta_1^1 \cdot \eta_i \cdot \eta_2^2$ is well defined, where $H_i$’s are defined as in Lemma 7.3, and $G_i([0, 3] \times [0, 2]) \subset \mathbb{R}^2 \setminus \text{int} \Omega_i$ for $i = 1, 2$. See Figure 16. Note that $\gamma_1^1 \cdot \eta_1 \cdot \gamma_2^2$ is homotopic to $\gamma_1^1 \cdot \eta_2 \cdot \gamma_2^2$ in $\mathbb{R}^2 \setminus \text{int} \Omega_i$ via $G_i$.

![Fig. 16. The homotopy $G_i = \eta_1^1 \cdot \eta_i \cdot \eta_2^2$](image)

Let $\tilde{\phi} : [0, 1] \rightarrow \partial^v \Omega, \tilde{\phi}(t) = (\tilde{\gamma}(t), \tilde{n}(t))$, be a locally one-to-one, continuous map such that $O_{\Omega}(\tilde{\phi}) = +$ and $\tilde{\gamma}(0), \tilde{\gamma}(1)$ are flat points. Since $[0, 1]$ is compact, Lemma 7.3 shows that there exist $\tilde{\gamma}(0) = p^0, \ldots, p^n = \tilde{\gamma}(1) \in \partial \Omega$ and continuous maps $\phi^k : [0, 1] \rightarrow \partial^v \Omega,$
\( \phi^k(t) = (\gamma^k(t), n^k(t)) \), and \( \phi^k_i : [0, 1] \rightarrow \partial^n \Omega_i, \phi^k_i(t) = (\gamma^k_i(t), n^k_i(t)) \), for \( i = 1, 2 \) and 
\( k = 1, \ldots, n \), such that:

1. \( \gamma^k(0) = p^{k-1}, \gamma^k(1) = p^k \), \( \gamma^k \) is one-to-one, and \( \Omega_i(\gamma^{k}) = + \).
2. Each of \( \phi^k_i \)'s and \( \gamma^k_i \)'s is either one-to-one or constant, and, if one of \( \Omega_i(\gamma^{k}_1) \), \( \Omega_{2}(:, \gamma_{2}^{k}) \) is -, then the other is +.
3. \( \gamma^k(t) = \gamma^k_1(t) + \gamma^k_2(t) \) and \( n^k(t) = n^k_1(t) = n^k_2(t) \) for \( t \in [0, 1] \).
4. \( \Theta(\phi^k_i) = \Theta(\phi^k_1) = \Theta(\phi^k_2) \), and, for \( i = 1, 2 \), \( \gamma_i^k \) is homotopic to \( \gamma^k \) in \( \mathbb{R}^2 \setminus \text{int} \Omega_i \) via the homotopy \( H^k_i : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^2 \setminus \text{int} \Omega_i \), where

\[
H^k_i(t, s) = I_{\gamma^k_i}(t)(H_{\Omega_i,0}(\gamma^k(t), s)) \quad \text{and} \quad H^k_2(t, s) = I_{\gamma^k_i}(t)(H_{\Omega_i,0}(\gamma^k_1(t), s)),
\]

for \( (t, s) \in [0, 1] \times [0, 1] \).

5. There exists a continuous, onto, non-decreasing function \( \tilde{h} : [0, 1] \rightarrow [0, n] \) such that \( \tilde{h}(t) = (\gamma^1 \cdot \ldots \cdot \gamma^n)(\tilde{h}(t)) \) for \( t \in [0, 1] \).

From the above arguments, there exist continuous maps \( \psi^k : [0, 1] \rightarrow \partial^n \Omega, \psi^k(t) = (\eta^k(t), m^k(t)) \), and \( \psi^k_i : [0, 1] \rightarrow \partial^n \Omega_i, \psi^k_i(t) = (\eta^k_i(t), m^k_i(t)) \), and a homotopy \( A^k_i : [0, 1] \times [0, 1] \rightarrow V^k_i \) for \( i = 1, 2 \) and \( k = 1, \ldots, n - 1 \) (\( k = 1, \ldots, n \) if \( \tilde{h}(0) = \tilde{h}(1) \)), where \( V^k_i \subset \mathbb{R}^2 \setminus (\Omega_i \cup (-\Omega_{3-i} + p^k)) \) is the region bounded by \( \eta^k_i \) and \( \tilde{\eta}^k_{3-i} = -\eta^k_{3-i} + p^k \), such that \( \Theta(\psi^k) = \Theta(\psi^k_1) = \Theta(\psi^k_2) \), and \( \gamma^k_i \cdot \eta^k_i \cdot \gamma^{k+1} \) is homotopic to \( \gamma^k \cdot \eta^k \cdot \gamma^{k+1} \) in \( \mathbb{R}^2 \setminus \text{int} \Omega_i \) via

\[
e^k_i \cdot \eta^k_i \cdot \gamma^{k+1}_i \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad k = 1, \ldots, n - 1 \quad (k = 1, \ldots, n \text{ if } \tilde{h}(0) = \tilde{h}(1)).
\]

Here, we let \( \phi^{n+1} = \phi^1, \gamma^{n+1} = \gamma^1 \), and \( \phi^{n+1}_i = \phi^1_i, \gamma^{n+1}_i = \gamma^1_i, H^k_{i+1} = H^k_i, E^k_{i+1} = E^k_i \) for \( i = 1, 2 \).

For \( i = 1, 2 \) and \( k = 1, \ldots, n - 1 \) (or \( n \)), define \( E^k_i(t, s) = \gamma^k_i(t) \) for \( (t, s) \in [0, 1] \times [0, 1] \), and

\[
B^k_i(t, s) = I_{\mu^k_i}(H_{\Omega_i,0}(\eta^k_i(t), s)), \quad B^k_2(t, s) = I_{\mu^k_i}(H_{\Omega_i,0}(\eta^k_1(t), s)),
\]

for \( (t, s) \in [0, 1] \times [0, 1] \).

For \( i = 1, 2 \), let

\[
\phi = \phi^1 \cdot \phi^2 \cdot \ldots \cdot \phi^{n-1} \cdot \phi^n, \\
\phi_i = \phi^1_i \cdot \phi^2_i \cdot \ldots \cdot \phi^{n-1}_i \cdot \phi^n_i, \\
\gamma = \gamma^1 \cdot \eta^1 \cdot \gamma^2 \cdot \ldots \cdot \eta^{n-1} \cdot \gamma^n, \\
\gamma_i = \gamma^1_i \cdot \eta^1_i \cdot \gamma^2_i \cdot \ldots \cdot \eta^{n-1}_i \cdot \gamma^n_i, \\
P_i = E^1_i \cdot A^1_i \cdot E^2_i \cdot \ldots \cdot A^{n-1}_i \cdot E^n_i, \\
Q_i = H^1_i \cdot B^1_i \cdot H^2_i \cdot \ldots \cdot B^{n-1}_i \cdot H^n_i.
\]

When \( \tilde{\phi}(0) = \tilde{\phi}(1) \), we let

\[
\phi = \phi^1 \cdot \phi^2 \cdot \ldots \cdot \phi^{n-1} \cdot \phi^n, \\
\phi_i = \phi^1_i \cdot \phi^2_i \cdot \ldots \cdot \phi^{n-1}_i \cdot \phi^n_i, \\
\gamma = \gamma^1 \cdot \eta^1 \cdot \gamma^2 \cdot \ldots \cdot \eta^{n-1} \cdot \gamma^n \cdot \eta^n, \\
\gamma_i = \gamma^1_i \cdot \eta^1_i \cdot \gamma^2_i \cdot \ldots \cdot \eta^{n-1}_i \cdot \gamma^n_i \cdot \eta^n_i,
\]
Note that \( Q_i(t,s) = I_{\gamma(t)}(H_{\Omega_{2n-i}}(0)(I_{\gamma(t)}(\tilde{\gamma}_i(t)), s)) \) for \((t, s) \in [0, 2n - 1] \times [0, 1] \) (for \((t, s) \in [0, 2n] \times [0, 1] \) if \( \tilde{\phi}(0) = \tilde{\phi}(1) \), for \( i = 1, 2 \). Let \( H_i = \frac{Q_i}{p_i} \). See Figure 17. Now it is easy to see that \( \gamma_i \) is homotopic to \( \tilde{\gamma}_i \) in \( \mathbb{R}^2 \) via \( P_i \), and \( \tilde{\gamma}_i \) is homotopic to \( \gamma \) in \( \mathbb{R}^2 \) via \( Q_i \). So \( \gamma_i \) is homotopic to \( \gamma \) in \( \mathbb{R}^2 \) via \( H_i \). Furthermore, if \( \tilde{\phi}(0) = \tilde{\phi}(1) \), then \( H_i(0, s) = H_i(2n, s) \) for \( s \in [0, 2] \). It is also easy to see that \( \Theta(\phi) = \Theta(\phi_1) = \Theta(\phi_2) \).

Finally, we obtain the following theorem by using the above arguments:

**Theorem 7.2** (simple connectedness of Minkowski sum of semi-convex domains). Let \( \mathcal{M} \) be a Minkowski class, and let \( \Omega_1 \) and \( \Omega_2 \) be semi-convex \( \mathcal{M} \)-domains. Then their Minkowski sum \( \Omega = \Omega_1 + \Omega_2 \) is a simply connected regular \( \mathcal{M} \)-domain.
Proof. From Theorem 7.1, $\Omega$ is a regular $\mathcal{M}$-domain. We can assume $0 \in \text{int} \Omega_1, \text{int} \Omega_2$. Clearly, this implies $0 \in \text{int} \Omega$. Suppose $\Omega$ is not simply connected. Then there exists an inner boundary $C$ of $\Omega$. Let $\tilde{C}$ be the connected component of $\partial^v \Omega$ corresponding to $C$, and let $\tilde{\phi} : [0, 1] \to \tilde{C}$, $\tilde{\phi}(t) = (\tilde{\gamma}(t), \tilde{n}(t))$, be a continuous map such that $\tilde{\phi}(0) = \tilde{\phi}(1)$, $\tilde{\gamma}(0) = \tilde{\gamma}(1)$ is a flat point, $\tilde{\phi}|_{[0, 1]}$ is one-to-one, and $O_{\Omega}(\tilde{\phi}) = +$. (That is, $\tilde{\phi}$ traverses $\tilde{C}$ exactly once in the standard orientation.) Then $\Theta(\tilde{\phi}) = -2\pi$ by Lemma 5.2(1).

Now take $\phi, \phi_i$, and $H_i$ for $i = 1, 2$ as in the above arguments. We have $\Theta(\tilde{\phi}) = \Theta(\phi) = \Theta(\phi_1) = \Theta(\phi_2)$, and $\gamma_i$ is homotopic to $\gamma$ in $\mathbb{R}^2 \setminus \text{int} \Omega_i$ via $H_i$ for $i = 1, 2$. Also, $\gamma$ and $\tilde{\gamma}$ are homotopic in $\partial \Omega$. So $\text{Ind}_{\gamma}(0) = \text{Ind}_{\gamma}(0) = \text{Ind}_{\gamma_i}(0) = \text{Ind}_{\gamma_2}(0)$, since $0 \in \text{int} \Omega, \text{int} \Omega_1, \text{int} \Omega_2$. Since $C$ is an inner boundary of $\Omega$ and $0 \in \text{int} \Omega$, we have $\text{Ind}_{\gamma}(0) = 0$ by Lemma 5.2(2). So $\text{Ind}_{\gamma_i}(0) = 0$ for $i = 1, 2$. It follows that $\Theta(\phi_1) = \Theta(\phi_2) = 0$ again by Lemma 5.2(2), since $\Omega_1$ and $\Omega_2$ are simply connected and hence have no inner boundaries. So $\Theta(\phi) = 0$, which is a contradiction. ■

8. Maximality of semi-convexity

Let $\mathcal{C}$ be a subclass of $C_c^{1,1}$ which is closed under restriction. In this section, we show that for any regular $\mathcal{C}$-domain which is not semi-convex, there exists a semi-convex $\mathcal{C}$-domain so that their Minkowski sum is not simply connected. Combined with Theorem 7.2, this answers Problem 2 posed in Section 1 within the category of $\mathcal{M}$-domains for any Minkowski class $\mathcal{M}$. In fact, it is shown that we can choose this domain among a special kind of semi-convex $\mathcal{C}$-domains, which we call flag domains. Note that $\mathcal{C}$ need not be a Minkowski class.

First, we observe the following easy fact:

**Lemma 8.1.** Let $\Omega$ be a regular $C_c^{1,1}$-domain which is not semi-convex. Then there exists a one-to-one continuous map $\phi : [-\varepsilon, 1 + \varepsilon] \to \partial^v \Omega$, $\phi(t) = (\gamma(t), n(t))$, for some $\varepsilon > 0$, which satisfies the following conditions:

1. $O_{\Omega}(\phi) = +$ and $\Theta(\phi|_{[0, 1]}) = -\pi$.
2. $-\pi < \Theta(\phi|_{[s, t]}) < \pi$ for every proper subinterval $[s, t]$ of $[0, 1]$.
3. Let $\theta : [-\varepsilon, 1 + \varepsilon] \to \mathbb{R}$ be an angle function of $\phi$. Then $\theta$ is strictly decreasing on $[-\varepsilon, \varepsilon]$ and $[1 - \varepsilon, 1 + \varepsilon]$.

**Proof.** Since $\Omega$ is not semi-convex, there exists a one-to-one continuous map $\tilde{\phi} : [a, b] \to \partial^v \Omega$ such that $O_{\Omega}(\tilde{\phi}) = +$ and $\Theta(\phi) < -\pi$. Let $\tilde{\theta} : [a, b] \to \mathbb{R}$ be an angle function of $\tilde{\phi}$. Since $\Omega$ is a $C_c^{1,1}$-domain, we can divide $[a, b]$ into a finite number of subintervals on which $\tilde{\theta}$ is either strictly increasing or strictly decreasing or constant. It follows that the number of critical values of $\tilde{\theta}$ is finite. So we can take $a < a' < b' < b$ such that $\Theta(\tilde{\theta}|_{[a', b']}) = -\pi$ and $\tilde{\theta}$ is strictly monotone near every $t \in [a, b]$ such that $\tilde{\theta}(t) = \tilde{\theta}(a')$ or $\tilde{\theta}(b')$. Now there exist $a' < b' < b'' < b'$ such that $\tilde{\theta}(a'') = \tilde{\theta}(a'), \tilde{\theta}(b'') = \tilde{\theta}(b')$, $\tilde{\theta}$ is strictly decreasing near $a''$ and $b''$, and $\tilde{\theta}(b'') < \tilde{\theta}(t) < \tilde{\theta}(a'')$ for every $t \in (a'', b'')$. So we can take a strictly increasing continuous function $h : [-\varepsilon, 1 + \varepsilon] \to [a, b]$ for some $\varepsilon > 0$ such that $h(0) = a''$, $h(1) = b''$ and $\tilde{\theta}$ is strictly decreasing on $h([-\varepsilon, \varepsilon])$ and $h([1 - \varepsilon, 1 + \varepsilon])$. 


Taking $\phi(t) = \tilde{\phi}(h(t))$ for $t \in [-\varepsilon, 1 + \varepsilon]$, we can easily check that $\phi$ satisfies conditions (1)–(3). \[ \square \]

For any $p \in \mathbb{R}^2$, we denote the $x$-coordinate of $p$ by $p_x$, and the $y$-coordinate of $p$ by $p_y$.

**Theorem 8.1 (maximality of semi-convexity).** Let $\mathcal{C} \subset C^1_\varepsilon$ be closed under restriction, and let $\Omega_1$ be a regular $\mathcal{C}$-domain which is not semi-convex. Then there exists a semi-convex $\mathcal{C}$-domain $\Omega_2$ such that $\Omega = \Omega_1 + \Omega_2$ is not simply connected.

**Proof.** By Lemma 8.1, we can take a one-to-one continuous map $\phi : [-\varepsilon', 1 + \varepsilon'] \to \partial^n \Omega_1$, $\phi(t) = (\gamma(t), \mathbf{n}(t))$, for some $\varepsilon' > 0$, such that $\partial_{\Omega_1}(\phi) = +$, $\Theta(\phi|_{[0,1]}) = -\pi$, $-\pi < \Theta(\phi|_{[s,t]}) < \pi$ for every proper subinterval $[s, t]$ of $[0,1]$, and $\theta$ is strictly decreasing on $[-\varepsilon', \varepsilon']$ and $[1-\varepsilon', 1+\varepsilon']$, where $\theta$ is an angle function of $\phi$. We can assume $\mathbf{n}(0) = (-1,0)$. Let $a = \gamma(1)_x$ and $b = \gamma(0)_x$. Let $\mathbf{v}(t)$ be the unit vector obtained by rotating $\mathbf{n}(t)$ counter-clockwise through $90^\circ$ for $t \in [-\varepsilon', 1 + \varepsilon']$. Suppose $\gamma(t_0)_x \geq b$ for some $t_0 \in (0,1)$ such that $\gamma(t_0) \neq \gamma(0)$. Then there exists $t_1 \in (0, t_0)$ such that $\mathbf{v}(t_1)_x \geq 0$. So $\mathbf{n}(t_1)_y \leq 0$, and hence $\Theta(\phi|_{[0,t_1]}) \geq 0$, since $-\pi < \Theta(\phi|_{[0,t_1]}) < \pi$. It follows that $\Theta(\phi|_{[t_1,1]}) = \Theta(\phi|_{[0,1]}) - \Theta(\phi|_{[0,t_1]}) \leq -\pi$, which is impossible. Thus $\gamma(t)_x < b$ for every $t \in (0,1)$ such that $\gamma(t) \neq \gamma(0)$. Anlogously, $\gamma(t)_x > a$ for every $t \in (0,1)$ such that $\gamma(t) \neq \gamma(1)$. Thus $a < \gamma(t)_x < b$ for every $t \in (0,1)$ such that $\gamma(t) \neq \gamma(0), \gamma(1)$. Suppose $\gamma(t_3)_x = \gamma(3)_x$ for some $0 < t_3 < t_2 < 1$ such that $\gamma(t_3) \neq \gamma(t_2)$. Then there exists $t_4 \in (t_1, t_2)$ such that $\mathbf{v}(t_4)_x = 0$. So $\mathbf{n}(t_3)_y = 0$, which implies $\mathbf{n}(t_3) = (1,0)$ or $(-1,0)$. It follows that either $|\Theta(\phi|_{[0,t_4]})| \geq \pi$ or $|\Theta(\phi|_{[t_4,1]})| \geq \pi$. But this contradicts the assumption that $-\pi < \Theta(\phi|_{[0,t_4]}), \Theta(\phi|_{[t_4,1]}) < \pi$. Thus $\gamma(t_1) \neq \gamma(t_2)_x$ for every $t_1, t_2 \in (0,1)$ such that $\gamma(t_1) \neq \gamma(t_2)$. From these observations, it is clear that there exists a continuous function $f : [a, b] \to \mathbb{R}$ whose graph is $\gamma([0,1])$.

Let $\gamma^+ : [0,1] \to \partial \Omega_1$ be a one-to-one continuous curve such that $\gamma^+(0) = \gamma(0)$ and $\partial_{\Omega_1}(\gamma^+) = +$. Note that if $\gamma^+((0, \varepsilon'')) \in \{(x,y) \in \mathbb{R}^2 \mid a < x < b, y > f(x)\}$ for every small $\varepsilon'' > 0$, then $\theta$ cannot be strictly decreasing on $[1-\varepsilon', 1+\varepsilon']$. So we can take a continuous function $g : [a, a+\varepsilon] \to \mathbb{R}$, for some small $\varepsilon > 0$, such that the graph of $g$ is contained in $\partial \Omega_1$, $g(a) = f(a)$, and $f(x) < g(x)$ for every $x \in (a, a+\varepsilon)$. In the same way, we can take a continuous function $h : [b-\varepsilon, b] \to \mathbb{R}$ such that the graph of $h$ is contained in $\partial \Omega_1$, $h(b) = f(b)$, and $f(x) < h(x)$ for every $x \in [b-\varepsilon, b]$. See Figure 18.

We can assume $a + \varepsilon < 0 < b - \varepsilon$ and $f(0) = 0$. Let $F$, $G$ and $H$ be the graphs of $f$, $g$ and $h$ respectively. Let $A = \partial \Omega_1 \setminus (F \cup G \cup H)$. Since $F$ and $A$ are compact and $F \cap A = \emptyset$, we can take $\delta > 0$ such that

$$2\delta < \min \{d(F,A), g(a+\varepsilon) - f(a+\varepsilon), h(b-\varepsilon) - f(b-\varepsilon)\}.$$ 

Let $F_0, F_2$ be the graphs of $f + \delta, f + 2\delta$ respectively. Since $2\delta < g(a+\varepsilon) - f(a+\varepsilon)$ and $2\delta < h(b-\varepsilon) - f(b-\varepsilon)$, $F_2$ must meet both $G$ and $H$. Let $a_1 = \max \{p_x : p \in F_0 \cap G\}$ and $b_1 = \min \{p_x : p \in F_2 \cap H\}$. Then the set

$$\Omega_2' = \{(x,y) \in \mathbb{R}^2 \mid a_1 \leq x \leq b_1, f(x) + \delta \leq y \leq f(x) + 2\delta\}$$

is a simply connected regular $\mathcal{C}$-domain, and $\Omega_1$ and $\Omega_2'$ are in contact position to each other. It is also easy to see that $\Omega_2'$ is semi-convex.
Let $\Omega_2 = -\Omega'_2 + (0, \delta)$ and $\Omega = \Omega_1 + \Omega_2$. Clearly, $\Omega_2$ is also semi-convex. Define $\beta_- : [a, a_1] \to \mathbb{R}^2$ and $\beta_+ : [b_1, b] \to \mathbb{R}^2$ by

$$
\beta_-(t) = (t, g(t)) - (a_1, g(a_1)) + (0, \delta), \quad t \in [a, a_1],
$$
$$
\beta_+(t) = (t, h(t)) - (b_1, h(b_1)) + (0, \delta), \quad t \in [b_1, b].
$$

If $\beta_-$ does not meet $F$, then let $l_-$ be the line segment that starts from $\beta_-(a)$ and goes in the direction of $(0, -1)$ until it meets $F$. Also, if $\beta_+$ does not meet $F$, then let $l_+$ be the line segment that starts from $\beta_+(b)$ and goes in the direction of $(0, -1)$ until it meets $F$. Note that $a + \varepsilon < 0 < b - \varepsilon$. Let $D$ be the simply connected regular $C$-domain which is enclosed by the curves $F$, $\beta_-$, $\beta_+$ (and $l_-$, $l_+$ if needed), and let $\beta : [0, 1] \to \partial D$ be a closed curve which traverses $\partial D$ once in the standard orientation of $D$. Now note that $(-\Omega_2 + p) \cap \Omega_1 \neq \emptyset$ for every $p \in \partial D$. So $\beta(t) \in \Omega$ for every $t \in [0, 1]$ by Lemma 6.3(1). On the other hand, note that $-\Omega_2 + (0, \delta/2) = \Omega_2 - (0, \delta/2)$ has no intersections with $\Omega_1$. So $(0, \delta/2) \notin \Omega$ again by Lemma 6.3(1). Since $(0, \delta/2) \in \text{int} D$, we have $\text{Ind}_{\beta}((0, \delta/2)) = 1$. Now suppose $\Omega$ is simply connected. Then $\text{Ind}_{\beta}(p) = 0$ for every $p \notin \Omega$ and every closed curve $\beta$ in $\Omega$. So we have $\text{Ind}_{\beta}((0, \delta/2)) = 0$. This is a contradiction.

**Remark 8.1.** Theorem 8.1 does not guarantee that for every regular non-semi-convex domain, there exists a convex domain such that their Minkowski sum is not simply connected. In fact, this is false: Let $\Omega$ be the domain depicted in Figure 12 (left). The Minkowski sum of $\Omega$ and any convex domain is simply connected. This can be easily seen from the fact that there should be a “trapping region” in order for a Minkowski sum to be non-simply connected.

Note that the domain $\Omega_2$ in the proof of Theorem 9.2 is of a special shape, which is not always shared by every semi-convex domain. Since these domains play an important role in Section 9, we give a name to them:

**Definition 8.1 (flag domain).** A simply connected regular $C^{1,1}$-domain $\Omega$ is called a flag domain if there exists a piecewise $C^1$ function $f : [a, b] \to \mathbb{R}$ such that:
(1) $-\infty < f'(x+) < f'(x-) < \infty$ for every $x \in [a, b]$.

(2) For some rigid motion in $\mathbb{R}^2$,
$$
\Omega = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, f(x) \leq y \leq f(x) + d\}
$$
for some $d > 0$.

See Figure 20 for an example of a flag domain. It is easy to see that a flag domain is semi-convex, but not vice versa. Note that the domain $\Omega_2$ in Theorem 9.2 is a flag domain. Thus we have the following statement which is stronger than Theorem 9.2:

**Theorem 8.2.** Let $\mathcal{C} \subseteq C^{1,1}_\mathcal{C}$ be closed under restriction, and let $\Omega_1$ be a regular $\mathcal{C}$-domain which is not semi-convex. Then there exists a flag $\mathcal{C}$-domain $\Omega_2$ such that the Minkowski sum $\Omega = \Omega_1 + \Omega_2$ is not simply connected.

9. Closedness of semi-convexity

In this section, we show that the Minkowski sum of two semi-convex $\mathcal{M}$-domains is again a semi-convex $\mathcal{M}$-domain for any Minkowski class $\mathcal{M}$. Thus the set of all semi-convex $\mathcal{M}$-domains is closed under Minkowski sum.

We start with some basic observations:

**Lemma 9.1.** Let $\Omega_1$ and $\Omega_2$ be two simply connected regular $C^{1,1}$-domains such that $\Omega_1 \subseteq \Omega_2$, and let $p \in \partial \Omega_1 \cap \partial \Omega_2$, $q_i \in \partial \Omega_i$ for $i = 1, 2$ with $q_2 \neq p$. For $i = 1, 2$, let $\gamma_i : [0, 1] \to \partial \Omega_i$ be continuous maps such that $\gamma_i(0) = p$, $\gamma_i(1) = q_i$, and let $\beta : [0, 1] \to \Omega_2 \setminus \text{int} \Omega_1$ be a continuous map such that $\beta(0) = q_1$, $\beta(1) = q_2$, and either $\beta$ is constant or $\beta((0, 1]) \subseteq \Omega_2 \setminus \Omega_1$. Suppose there exists a homotopy $H : [0, 1] \times [0, 1] \to \mathbb{R}^2 \setminus \text{int} \Omega_1$ such that $H(t, 0) = \gamma_1(t)$, $H(t, 1) = \gamma_2(t)$ for $t \in [0, 1]$, and $H(0, s) = p$, $H(1, s) = \beta(s)$ for $s \in [0, 1]$. Then $O_{\Omega_1}(\gamma_1) \cdot O_{\Omega_2}(\gamma_2) \neq -.\!

**Proof.** Let $\tilde{\gamma}_2 : [0, 1] \to \partial \Omega_2$ be a one-to-one continuous map such that $\tilde{\gamma}_2(0) = p$, $\tilde{\gamma}_2(1) = q_2$, and $O_{\Omega_2}(\tilde{\gamma}_2) = +$. Let $\tilde{\gamma}_1 : [0, 1] \to \partial \Omega_1$ be a continuous map such that $\tilde{\gamma}_1(0) = p$, $\tilde{\gamma}_1(1) = q_1$, $O_{\Omega_1}(\tilde{\gamma}_1) \neq -$, $\tilde{\gamma}_1|_{\{0, 1\}}$ is either one-to-one or constant. Clearly, we can find a homotopy $\tilde{H} : [0, 1] \times [0, 1] \to \mathbb{R}^2 \setminus \text{int} \Omega_1$ such that $\tilde{H}(t, 0) = \tilde{\gamma}_1(t)$, $\tilde{H}(t, 1) = \tilde{\gamma}_2(t)$ for every $t \in [0, 1]$, and $\tilde{H}(0, s) = p$, $\tilde{H}(1, s) = \beta(s)$ for every $s \in [0, 1]$. For $i = 1, 2$, let $\nu_i : [0, 2] \to \mathbb{R}$ be the continuous function such that $\nu_i(0) = 0$ and $\mu_i(\nu_i(t)) = (\tilde{\gamma}_i \cdot \tilde{\gamma}_i)(t)$ for $t \in [0, 2]$, where $\mu_i : \mathbb{R} \to \partial \Omega_i$ are covering maps in the standard orientation of $\partial \Omega_i$ with period 1 such that $\mu_i(0) = p$. See Section 5 for the definition of $\gamma$ for a curve $\gamma$.

Clearly, $0 \leq \nu_1(1) \leq 1$ and $0 < \nu_2(1) < 1$. Note also that the two closed curves $\tilde{\gamma}_1 \cdot \tilde{\gamma}_1$ and $\tilde{\gamma}_2 \cdot \tilde{\gamma}_2$ are homotopic in $\mathbb{R}^2 \setminus \text{int} \Omega_1$. So $\text{Ind}_{\tilde{\gamma}_1}(\tilde{\gamma}_1)(0) = \text{Ind}_{\tilde{\gamma}_2}(\tilde{\gamma}_2)(0)$, where we have assumed $0 \in \text{int} \Omega_1$. Note that $\text{Ind}_{\tilde{\gamma}_i}(\tilde{\gamma}_i)(0) = \nu_i(2)$ for $i = 1, 2$. So $\nu_1(2) = \nu_2(2) \in \mathbb{Z}$. Thus the assertion follows, since $O_{\Omega_i}(\gamma_i)$ is the sign of $\nu_i(1) - \nu_i(2)$.

**Lemma 9.2.** Let $\Omega$ be a simply connected regular $C^{1,1}$-domain. Let $(p_1, n_1)$ and $(p_2, n_2)$ be two points in $\partial^e \Omega$ such that $n_1 = -n_2$. Suppose
$$
\Omega \cap \{(p_1 + t \cdot n_1 \mid t > 0) \cup (p_2 + t \cdot n_2 \mid t > 0)\} = \emptyset.
$$
Then $\Theta(\phi) = \pi$ for any one-to-one continuous map $\phi : [0, 1] \to \partial^v \Omega$, $\phi(t) = (\gamma(t), n(t))$, such that $\phi(0) = (p_1, n_1), \phi(1) = (p_2, n_2)$ and $O_{\Omega}(\phi) = +$.

Proof. We can assume that $n_1 = (1, 0)$. Since $\Omega$ is bounded, there exist $a_1 < a_2$ and $b_1 < b_2$ such that $\Omega \subset \{(x, y) \in \mathbb{R}^2 \mid a_1 < x < a_2, b_1 < y < b_2\}$. See Figure 19.

![Fig. 19. Figure for Lemma 9.2](image)

Let $l_1 = \{(x, y) \in \mathbb{R}^2 \mid x = a_1, (p_2)_y \leq y \leq b_2\}$ and $l_2 = \{(x, y) \in \mathbb{R}^2 \mid x = a_2, (p_1)_y \leq y \leq b_2\}$. Let $m_1 = \{(x, y) \in \mathbb{R}^2 \mid a_1 \leq x \leq (p_2)_x, y = (p_2)_y\}$ and $m_2 = \{(x, y) \in \mathbb{R}^2 \mid (p_1)_x \leq x \leq a_2, y = (p_1)_y\}$. Let $l = \{(x, y) \in \mathbb{R}^2 \mid a_1 \leq x \leq a_2, y = b_2\}$. By the assumptions, the curve $\gamma$ and the line segments $m_2, l_2, l, m_1$ constitute the boundary of a simply connected regular $C^{1,1}$-domain, which we call $\Omega'$. Let $\psi : [0, 1] \to \partial^v \Omega'$ be a one-to-one continuous map such that $\psi(0) = (p_1, (0, -1)), \psi(1) = (p_2, (0, -1))$ and $O_{\Omega'}(\psi) = +$. It is easy to see that $\Theta(\psi) = 2\pi$. By Lemma 5.2(1), we have $\Theta(\psi) + \pi/2 - \Theta(\phi) + \pi/2 = 2\pi$. Thus $\Theta(\phi) = \pi$. ■

**Lemma 9.3.** Let $\Omega_1$ be a flag $C^{1,1}$-domain and $\Omega_2$ a semi-convex $C^{1,1}$-domain. Suppose that $\Omega_1$ and $\Omega_2$ are in contact position to each other, and that $V$ is a bounded connected component of $\mathbb{R}^2 \setminus (\Omega_1 \cup \Omega_2)$. Then for any $p_1 \in \partial V \setminus \partial \Omega_2$, there exist $p_2 \in \partial V \setminus \partial \Omega_1$ and a continuous curve $\beta : [0, 1] \to \overline{V}$ such that $\beta(0) = p_1, \beta(1) = p_2, \beta((0, 1)) \subset V$, and $(\Omega_1 + \beta(u) - \beta(0)) \cap \Omega_2 \neq \emptyset$ for every $u \in [0, 1]$.

Proof. We can assume that

$$\Omega_1 = \{(x, y) \mid f(x) \leq y \leq f(x) + d, |x| \leq 1\},$$

for some piecewise $C^1$ function $f : [-1, 1] \to \mathbb{R}$. Let $F$ and $F_d$ be the graphs of $f$ and $f + d$ respectively, and let $l_-, l_+$ be the line segments (without end points) joining $(-1, f(-1))$ to $(-1, f(-1) + d)$ and $(1, f(1))$ to $(1, f(1) + d)$ respectively. See Figure 20. Note that $\partial \Omega_1 = F \cup F_d \cup l_- \cup l_+$. If $p_1 \in F$ (resp., $p_1 \in F_d$), then take $p_2 \in \partial V \setminus \partial \Omega_1$ such that $(p_2)_x = (p_1)_x$ and $(p_2)_y = \max \{p_y : p \in \partial V \setminus \partial \Omega_1, p_x = (p_1)_x, p_y < (p_1)_y\}$ (resp., $(p_2)_y = \min \{p_y : p \in \partial V \setminus \partial \Omega_1, p_x = (p_1)_x, p_y > (p_1)_y\}$). If $p_1 \in l_-$ (resp., $p_1 \in l_+$), we take $p_2 \in \partial V \setminus \partial \Omega_1$ such that $(p_2)_y = (p_1)_y$ and $(p_2)_x = \max \{p_x : p \in \partial V \setminus \partial \Omega_1, p_y = (p_1)_y, p_x < (p_1)_x\}$ (resp., $(p_2)_x = \min \{p_x : p \in \partial V \setminus \partial \Omega_1, p_y = (p_1)_y, p_x > (p_1)_x\}$). In
any case, we define $\beta(u) = (1 - u)p_1 + up_2$ for $u \in [0, 1]$. It is clear that $\beta(0) = p_1$, $\beta(1) = p_2$, $\beta([0,1]) \subset V$, and $\beta((0,1)) \subset V$.

Now we only have to show that $(\Omega_1 + \beta(u) - \beta(0)) \cap \Omega_2 \neq \emptyset$ for every $u \in [0, 1]$. For $i = 1, 2$, let $\phi_i : [0,1] \to \partial \Omega_i$, $\phi_i(t) = (\gamma_i(t), n_i(t))$, be a one-to-one continuous map such that $O_{\Omega_i}(\phi_i) = +$, $\gamma_i([0,1]) = \partial V \cap \partial \Omega_i$, $\phi_i(0) = (\gamma_i(0), n_{\Omega_i}^+(\gamma_i(0)))$, $\phi_i(1) = (\gamma_i(1), n_{\Omega_i}^-(\gamma_i(1)))$. Note that $\Omega_1$ is semi-convex, since a flag domain is semi-convex. So by Lemma 5.4(2), we have $-\pi \leq \Theta(\phi_i) \leq \pi$ for $i = 1, 2$. From this, it is easy to see that at least one of $F$, $F_d$, $l_-$, $l_+$ has no intersections with $\gamma_1([0,1])$, and if $\gamma_1([0,1])$ intersects one of $l_+$, $l_-$, then it does not intersect the other. Thus, by symmetry, it is sufficient to consider the following four cases when $\gamma_1([0,1])$ intersects only (1) $F$, (2) $l_-$, (3) $F$ and $l_-$, (4) $F$, $l_-$ and $F_d$. See Figure 21.

First consider case (1). Let $U = \{(x,y) \in \mathbb{R}^2 \mid \gamma_1(0)x \leq x \leq \gamma_1(1)x, \ y \leq f(x)\}$. Suppose there does not exist $t_1'$ nor $t_2'$ in $[0,1]$ such that $\gamma_2([0,t_1']) \subset U$, $\gamma_2(t_1')x =
γ1(0)x, and γ2([t'2, 1]) ⊂ U, γ2(t'2)x = γ1(1)x. Then there exist 0 < t1 < t2 < 1 and ε > 0 such that n2(t1) = −n2(t2) = (−1, 0), Θ(φ1||t1−ε,t1]) , Θ(φ1||t2,t2+ε]) < 0, and \( V \cap \{ \gamma2(ti) - u \cdot n2(ti) | u > 0 \} = \emptyset \) for i = 1, 2. By applying Lemma 9.2 to V, we have \( \Theta(φ2||t1,t2]) = −\pi \). So \( \Theta(φ2||t1−ε,t2+ε]) < −\pi \), which is impossible since \( Ω2 \) is semi-convex. Thus at least one of \( t'_1, t'_2 \) above should exist. Then we can see easily that \( (Ω1 + β(u) − β(0)) \cap Ω2 \neq \emptyset \) for every \( u \in [0, 1] \).

Case (2) can be treated with the same argument as in (1). Consider case (3). Note that the case when \( γ1(1) = (−1, f(−1)) \) can be treated by the same method as for case (2). So we assume \( γ1(1) ≠ (−1, f(−1)) \). Suppose \( p1 ∈ F \). Let \( U = \{(x, y) ∈ \mathbb{R}^2 | −1 ≤ x ≤ γ1(1)x, y ≤ f(x)\} \). Suppose there does not exist \( t'_1 \in [0, 1] \) such that \( γ2([0, t'_1]) \subset U, γ2(t'1)x = −1 \). Then there exist \( 0 < t1 < t2 < 1 \) and ε > 0 such that \( n2(t1) = −n2(t2) = (−1, 0), Θ(φ2||t1−ε,t1]) , Θ(φ2||t2,t2+ε]) < 0, and \( V \cap \{ γ2(ti) - u \cdot n2(ti) | u > 0 \} = \emptyset \) for i = 1, 2. By Lemma 9.2, we have \( Θ(φ2||t1,t2]) = −\pi \). So \( Θ(φ2||t1−ε,t2+ε]) < −\pi \), which is impossible since \( Ω2 \) is semi-convex. Thus there exists \( t'_1 \in [0, 1] \) as above, and it follows easily that \( (Ω1 + β(u) − β(0)) \cap Ω2 \neq \emptyset \) for every \( u \in [0, 1] \).

Suppose \( p1 ∈ L \). Let \( U1 \) be the (closed) region bounded by \( γ1(0)+u \cdot (−1, 0) | u ≥ 0 \}, \{ γ1(1) + u \cdot (0, −1) | u ≥ 0 \} and γ1, which does not contain Ω1. Let \( U2 = \{(x, y) ∈ \mathbb{R}^2 | x ≤ γ1(1)x, y ≤ f(x)\} \). Suppose there does not exist \( t'_1, t'_2 \in [0, 1] \) such that \( γ2([0, t'_1]) ⊂ U1, γ2(t'_1)y ≥ γ1(1)y \) for every \( t \in [0, t'_1] \), \( γ2(t'_1)y = γ1(0)y \), and \( γ2([t'_2, 1]) ⊂ U2, γ2(t'_2)y = f(−1) \). Then there exist \( 0 < t1 < t2 < 1 \) and ε > 0 such that \( n2(t1) = −n2(t2) = (0, 1), Θ(φ2||t1−ε,t1]) , Θ(φ2||t2,t2+ε]) < 0, and \( V \cap \{ γ2(ti) - u \cdot n2(ti) | u > 0 \} = \emptyset \) for i = 1, 2. By Lemma 9.2, we have \( Θ(φ2||t1,t2]) = −\pi \). It follows that \( Θ(φ2||t1−ε,t2+ε]) < −\pi \), which is impossible since \( Ω2 \) is semi-convex. So at least one of \( t'_1, t'_2 \) exists. Now it is easy to see that \( (Ω1 + β(u) − β(0)) \cap Ω2 \neq \emptyset \) for every \( u \in [0, 1] \).

Finally, consider case (4). Note that the cases when \( γ1(0) = (−1, f(−1) + d) \) or \( γ1(1) = (−1, f(−1)) \) can be treated with the same methods as for cases (2) and (3). So assume \( γ1(0) ≠ (−1, f(−1)) \) and \( γ1(1) ≠ (−1, f(−1)) \). By using the same argument as for case (3) when \( p1 ∈ F \), we can see that \( (Ω1 + β(u) − β(0)) \cap Ω2 \neq \emptyset \) for every \( u \in [0, 1] \) if \( p ∈ F ∪ Fd \). Suppose \( p ∈ L \). Let \( U3 \) be the (closed) region bounded by \( γ1(1)+u \cdot (0, −1) | u ≥ 0 \}, \{ (−1, f(−1) + d) + u \cdot (−1, 0) | u ≥ 0 \} and γ1([0, 1]) \cap (F ∪ L), which does not contain Ω1. Let \( U4 \) be the (closed) region bounded by \( γ1(0)+u \cdot (0, 1) | u ≥ 0 \}, \{ (−1, f(−1)) + u \cdot (−1, 0) | u ≥ 0 \} and γ1([0, 1]) \cap (Fd ∪ L), which does not contain Ω1. Suppose there do not exist \( t'_1, t'_2 \in [0, 1] \) such that \( γ2([0, t'_1]) ⊂ U3, γ2(t'_1)y ≥ γ1(1)y \) for every \( t \in [0, t'_1] \), \( γ2(t'_1)y = f(−1) + d \), and \( γ2([t'_2, 1]) ⊂ U4, γ2(t'_2)y ≤ γ1(0)y \) for every \( t \in [t'_2, 1] \), \( γ2(t'_2)y = f(−1) \). Then there exist \( 0 < t1 < t2 < 1 \) and ε > 0 such that \( n2(t1) = −n2(t2) = (0, 1), Θ(φ2||t1−ε,t1]) , Θ(φ2||t2,t2+ε]) < 0, and \( V \cap \{ γ2(ti) - u \cdot n2(ti) | u > 0 \} = \emptyset \) for i = 1, 2. By Lemma 9.2, we have \( Θ(φ2||t1,t2]) = −\pi \), and so \( Θ(φ2||t1−ε,t2+ε]) < −\pi \). But this is impossible since \( Ω2 \) is semi-convex. So at least one of \( t'_1, t'_2 \) exists. Now clearly \( (Ω1 + β(u) − β(0)) \cap Ω2 \neq \emptyset \) for every \( u \in [0, 1] \).

**Theorem 9.1 (semi-convex + flag ⇒ semi-convex).** For any Minkowski class \( M \), the Minkowski sum of a semi-convex \( M \)-domain and a flag \( M \)-domain is a semi-convex \( M \)-domain.
Proof. Let $\mathcal{M}$ be a Minkowski class. Let $\Omega_1$ be a flag $\mathcal{M}$-domain, and let $\Omega_2$ be a semi-convex $\mathcal{M}$-domain. We can assume that $0 \in \Omega_1, \Omega_2$. Since a flag domain is semi-convex, the Minkowski sum $\Omega = \Omega_1 + \Omega_2$ is a simply connected regular $\mathcal{M}$-domain by Theorem 7.2. Suppose $\Omega$ is not semi-convex. Then we can take a one-to-one continuous map $\tilde{\phi} : [0,1] \to \partial^\infty \Omega, \tilde{\phi}(t) = (\tilde{\gamma}(t), \tilde{n}(t))$, such that $O_2(\tilde{\phi}) = +$ and $\Theta(\tilde{\phi}) < -\pi$. We can assume that $\tilde{\gamma}(0)$ and $\tilde{\gamma}(1)$ are flat points. Now we can take the maps $\phi, \phi_1, \phi^k, \phi^k, \psi^k, \psi^k$ associated to $\tilde{\phi}$ as in Section 7.2. We also use all the related notations therein.

Let $\mu : \mathbb{R} \to \partial \Omega$ and $\mu_i : \mathbb{R} \to \partial \Omega_i$ for $i = 1, 2$ be covering maps in the standard orientations of $\partial \Omega$ and $\partial \Omega_i$ respectively with period 1, such that $\mu(0) = \gamma(0)$ and $\mu_i(0) = \gamma_i(0)$. Then there exist continuous functions $\nu, \nu_1, \nu_2 : [0,2n-1] \to \mathbb{R}$ such that $\nu(0) = \nu_1(0) = \nu_2(0) = 0$ and $\gamma = \mu \circ \nu, \gamma_i = \mu_i \circ \nu_i$. Note that such $\nu$ and $\nu_i$'s are unique, and $O_2(\gamma|[a,b])$ and $O_{\Omega_i}(\gamma_i|[a,b])$ are the signs of $\nu(b) - \nu(a)$ and $\nu_i(b) - \nu_i(a)$ respectively, for any $[a,b] \subset [0,2n-1]$.

Note that $\Theta(\tilde{\phi}) = \Theta(\phi) = \Theta(\phi_1) = \Theta(\phi_2)$. Since $\Omega_1$ is semi-convex and $\Theta(\phi_1) < -\pi$, we have $O_{\Omega_1}(\phi_1) = -$. It follows that $O_{\Omega_1}(\gamma_1) = -$, since $\Theta(\phi_1) < -\pi$. So $\nu_1(2n-1) < 0$.

Note that $\nu_1$ is either non-decreasing or non-increasing on the interval $[k-1,k]$ for $k = 1, \ldots, 2n-1$. So there exist $0 = a_0 \leq b_0 < a_1 < \ldots < b_{m-1} < a_m \leq b_m = 2n-1$ such that $\nu_1$ is non-increasing on $[a_j, b_j]$ for $j = 0, \ldots, m$, and $\nu_1(a_{j+1}) - \nu_1(b_j) = 0$ for $j = 0, \ldots, m-1$. We can also assume that $\nu_2$ is non-decreasing on $[a_j, b_j]$ for $j = 0, \ldots, m$.

Suppose $\gamma(c) \neq \gamma_1(c) + \gamma_2(c)$ for some $c = a_0, b_0, \ldots, a_m, b_m$. Note that $\Omega_2$ and $-\Omega_1 + \gamma(c)$ are in contact position to each other by Lemma 6.3(3). Since $\gamma(c) \neq \gamma_1(c) + \gamma_2(c)$, it follows that, for some $k, \gamma_1(c)$ is on $\eta_k^f$ for $i = 1, 2$. Let $V$ be the connected component of the set $\mathbb{R}^2 \setminus (\Omega_2 \cup (-\Omega_1 + \gamma(c)))$ such that $-\gamma_1(c) + \gamma(c) \in \overline{V}$. Note that $V$ is bounded by $\eta_k^f$ and $\tilde{\eta}^f$. By applying Lemma 9.3 to $-\Omega_1 + \gamma_1(c)$ and $\Omega_2$, we have a continuous curve $\beta_c : [0,1] \to \overline{V}$ such that $\beta_c(0) = -\gamma_1(c) + \gamma(c), \beta_c(1) \in \partial V \setminus \partial(-\Omega_1 + \gamma(c)), \beta_1((0,1)) \subset V$, and $(-\Omega_1 + \gamma(c) + \beta_1(u) - \beta_0(0)) \cap \Omega_2 \neq \emptyset$ for every $u \in [0,1]$. Now from the constructions in Section 7, it is easy to see that we can take $\gamma_2$ (more exactly, $\eta_k^f$'s) and $A_2^f$'s such that:

(1) if $\gamma(c) = \gamma_1(c) + \gamma_2(c)$, then $P_2(c, s) = \gamma_2(c)$ for $s \in [0,1]$,
(2) if $\gamma(c) \neq \gamma_1(c) + \gamma_2(c)$, then $P_2(c, s) = \beta_c(s)$ for $s \in [0,1],$

for each $c = a_0, b_0, \ldots, a_m, b_m$.

Now we show $\nu_2(a_{j+1}) - \nu_2(b_j) \geq 0$ for $j = 0, \ldots, m-1$. Fix $j$, and let $b = b_j, a_{j+1} = a$. Note that $b < a$, $O_2(\gamma|[a,b]) = +$, and either $\gamma(b) = \gamma_1(b) + \gamma_2(b)$ or $\gamma(a) = \gamma_1(a) + \gamma_2(a)$.

Suppose $\gamma(b) = \gamma_1(b) + \gamma_2(b)$. Let $\tilde{\Omega}_1 = \Omega_1 - \gamma_1(b)$, and let $\tilde{\Omega} = \Omega - \gamma_1(b)$. Then $\tilde{\Omega} = \tilde{\Omega}_1 + \Omega_2$, and $\Omega_2 \subset \tilde{\Omega}$ since $0 \in \tilde{\Omega}_1$. Define $\hat{\gamma}_1(t) = \gamma_1(t) - \gamma_1(b)$ and $\hat{\gamma}(t) = \gamma(t) - \gamma_1(b)$ for $t \in [0,2n-1]$. Then, clearly, $O_{\tilde{\Omega}_1}(\hat{\gamma}_1|[a,b]) = O_{\tilde{\Omega}}(\gamma|[a,b]) = +$, $O_{\tilde{\Omega}_1}(\hat{\gamma}_1|[a,b]) = 0$, and $\gamma_2(b) = \hat{\gamma}(b)$. Define $Q_2(t, s) = I_{\hat{\gamma}(t)}(H_{\Omega_1,0}(I_{\gamma}(P_2(t, 1)), s))$ for $(t, s) \in [0,2n-1] \times [0,1]$ and $H_2 = \frac{Q_2}{r_2}$. Then $H_2$ is well defined and continuous, $\gamma_2|[a,b]$ is homotopic to $\hat{\gamma}|[a,b]$ in $\mathbb{R}^2 \setminus \text{int} \Omega_2$ via $H_2|[a,b] \times [0,2]$, and $H_2(b, s) = \gamma_2(b)$ for $s \in [0,2]$. 

Suppose $\gamma(a) = \gamma_1(a) + \gamma_2(a)$. Then also $\tilde{H}_2(a, s) = \gamma_2(a)$ for $s \in [0, 2]$. So by Lemma 9.1, we have $O_{\Omega_2}(\gamma_2 |_{[b, a]}) \neq -$, which implies $\nu_2(a) - \nu_2(b) \geq 0$. Suppose $\gamma(a) \neq \gamma_1(a) + \gamma_2(a)$. Then $P_2(a, s) = \tilde{\beta}_a(s)$ and $\tilde{Q}_2(a, s) = \tilde{\beta}_a(1) = -\gamma_1(a) + \gamma(a)$ for $s \in [0, 1]$. So $\tilde{H}_2(a, (0, 2]) \subseteq \mathbb{R}^2 \setminus \Omega_2$. Since $\nu_1(b) = \nu_1(a)$ (hence $\gamma_1(b) = \gamma_1(a)$) and $\beta_a(0) = -\gamma_1(a) + \gamma(a)$, we have $-\Omega_1 + \gamma(a) + \beta_a(u) - \beta_a(0) = -\tilde{\Omega}_1 + \beta_a(u)$ for $u \in [0, 1]$. So $(-\tilde{\Omega}_1 + \beta_a(u)) \cap \Omega_2 \neq \emptyset$ for $u \in [0, 1]$, and hence $\beta_a([0, 1]) \subseteq \delta$ by Lemma 6.3(1). So $\tilde{H}_2(a, (0, 2]) \subseteq \delta \setminus \Omega_2$. Thus by applying Lemma 9.1 again, we have $O_{\Omega_2}(\gamma_2 |_{[b, a]}) \neq -$, which implies that $\nu_2(a) - \nu_2(b) \geq 0$. In the same way, $\nu_2(a) - \nu_2(b) \geq 0$, when $\gamma(b) \neq \gamma_1(b) + \gamma_2(b)$ and $\gamma(a) = \gamma_1(a) + \gamma_2(a)$. Thus $\nu_2(a_{j+1}) - \nu_2(b_j) \geq 0$ for $j = 0, \ldots, m - 1$.

Now since $\nu_2$ is non-decreasing on $[a_j, b_j]$ for $j = 0, \ldots, m$, we have $\nu_2(2n - 1) \geq 0$, and hence $O_{\Omega_2}(\gamma_2) \neq -$. Note that $O_{\Omega_2}(\gamma_2) \neq 0$, since $\Theta(\phi_2) < -\pi$. So $O_{\Omega_2}(\gamma_2) = +$. But this is impossible, since $\Omega_2$ is semi-convex and $\Theta(\phi_2) < -\pi$.

Finally, we prove the main theorem of this section:

**Theorem 9.2 (semi-convex + semi-convex ⇒ semi-convex).** For any Minkowski class $\mathcal{M}$, the Minkowski sum of two semi-convex $\mathcal{M}$-domains is also a semi-convex $\mathcal{M}$-domain.

**Proof.** Let $\Omega_1$ and $\Omega_2$ be two semi-convex $\mathcal{M}$-domains, and let $\Omega = \Omega_1 + \Omega_2$ be their Minkowski sum. We know from Theorem 7.2 that $\Omega$ is a simply connected regular $\mathcal{M}$-domain. Suppose $\Omega$ is not semi-convex. Then, by Theorem 8.2, there exists a flag $\mathcal{M}$-domain $\Omega_3$ such that $\Omega + \Omega_3$ is not simply connected. By Theorem 9.1, $\Omega_2 + \Omega_3$ is a semi-convex $\mathcal{M}$-domain. So $\Omega + \Omega_3 = \Omega_1 + (\Omega_2 + \Omega_3)$ is simply connected by Theorem 7.2. This is a contradiction. ■

**10. Conclusion**

Here we briefly summarize the important results in this paper, and mention some further research directions. Let $\mathcal{M}$ be a Minkowski class. We denote the major classes of domains in this paper as follows:

- $\mathcal{M}$ = The set of all $\mathcal{M}$-domains.
- $\mathcal{D}$ = The set of all $\mathcal{M}$-domains homeomorphic to the unit disk.
- $\mathcal{S}$ = The set of all semi-convex $\mathcal{M}$-domains.
- $\mathcal{F}$ = The set of all flag $\mathcal{M}$-domains.
- $\mathcal{C}$ = The set of all convex $\mathcal{M}$-domains homeomorphic to the unit disk.

The inclusion relations between them are shown in Figure 22.

The inclusions $\mathcal{C} \subset \mathcal{S} \subset \mathcal{D} \subset \mathcal{M}$ and $\mathcal{F} \subset \mathcal{S}$ are all proper. By Theorem 6.3, $\mathcal{M}$ is closed under Minkowski sum. Let $\mathcal{D}$ be the class of all subsets $\mathcal{X}$ of $\mathcal{D}$ such that $A + B \in \mathcal{D}$ for every $A, B \in \mathcal{X}$. By Theorem 7.2, $\mathcal{S}$ is in $\mathcal{D}$, and is maximal in $\mathcal{D}$ with respect to the inclusion by Theorem 8.1. In fact, $\mathcal{S}$ is the unique maximal element in $\mathcal{D}_\mathcal{F}$ by Theorem 8.2, where $\mathcal{D}_\mathcal{F} = \{ \mathcal{X} \in \mathcal{D} | \mathcal{F} \subset \mathcal{X} \}$. Finally, $\mathcal{S}$ is closed under Minkowski sum by Theorem 9.2.

Now let us mention some further research directions in the subject of semi-convexity. First, note that the semi-convexity is amenable to the algorithmic setting in that only the rotation of normal vectors needs to be checked. Also, it is a natural generalization
of the usual convexity. Usually, the computation of the Minkowski sum of general shapes can essentially be divided into a few steps:

1. Decompose the shapes into unions of simpler shapes, which are usually convex.
2. Select the simple parts which can contribute to the boundary of the Minkowski sum.
3. Do Minkowski sum operations on these selected parts.
4. Integrate the results to form the Minkowski sum boundary, and hence the Minkowski sum itself.

The most important reason for using convex shapes in Step 1 is that they are closed under Minkowski sum. But in general, the number of the decomposed parts will be large since the convexity is very restrictive, and this results in the slow-down of the algorithms. So if we can use semi-convex shapes instead of convex ones, it would be possible to compute the Minkowski sum in a significantly more efficient way.

An immediate further research direction is to generalize the semi-convexity to 3 or higher dimensions, which would be most needed in various applications. Also, note that the current definition of semi-convexity requires some differentiability of the boundary, i.e., $C^{1,1}$. Compared to the fact that convexity has no such a priori requirements, this may be considered as a severe restriction. So an important next step would be to remove the regularity requirements from the definition of semi-convexity, which will be dealt with in [4] along with relationships of semi-convexity with other notions such as visibility.

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References


