1. Introduction

Generally speaking, almost all the continuous-time stochastic process models consist of some combination of the following:

a) diffusion,
b) deterministic motion,
c) random jumps.

We consider random dynamical systems with randomly chosen jumps acting on a given Polish space (Y, ϱ). Thus, our model is a mixture of deterministic motions and random jumps. In other words, it is an example of a non-diffusion model.

The aim of this paper is to study stochastic processes whose paths follow deterministic dynamics between random times, jump times, at which they change their position randomly. Hence, we analyse stochastic processes in which randomness appears at times $t_0 < t_1 < t_2 < \cdots$. We assume that a point $x_0 \in Y$ moves according to one of the transformations $\Pi_i : \mathbb{R}_+ \times Y \to Y$ from some set $\{\Pi_1, \ldots, \Pi_N\}$. The motion of the process is governed by the equation $X(t) = \Pi_i(t, x_0)$ until the first jump time t_1. Then we choose a transformation $q_s : Y \to Y$ from some set $\{q_1, \ldots, q_K\}$ and define $x_1 = q_s(\Pi_i(t_1, x_0))$, therefore q_s can be called a jump. The process restarts from that new point x_1 and continues as before. This gives the stochastic process $\{X(t)\}_{t \geq 0}$ with jump times $\{t_1, t_2, \ldots\}$ and post jump positions $\{x_1, x_2, \ldots\}$. The probability determining the frequency with which the maps Π_i and the jumps q_s are chosen is described by a matrix of probabilities $[p_{ij}]_{i,j=1}^N$, $p_{ij} : Y \to [0,1]$ and probability vectors $[\overline{p}_s]_{s=1}^K$, $\overline{p}_s : Y \to [0,1]$, respectively.

We are interested in the evolution of distributions of these random dynamical systems. We formulate criteria for stability and the existence of an invariant measure for such systems.

In the case of non-diffusion models, the first significant steps towards producing general models were taken by Cox [3], Gnedenko and Kovalenko [11]. The last two authors introduced a class of models called piecewise-linear Markov processes to provide a unified treatment of problems arising in queueing theory.

There is a substantial literature devoted to the problem of stability and of the existence of an invariant measure for Markov processes [37]. Different classes of Markov processes have been studied therein, for example random dynamical systems based on skew product flows [1]. Our model is not such a system. It is similar to the so-called piecewise-deterministic Markov process introduced by Davis [4]. There are some stability results for such a system based on the theory of Meyn and Tweedie [37]. However, the method of proving the existence of an invariant measure used by Meyn and Tweedie is not well adapted to general Polish spaces. In fact, it is difficult to ensure that the process under consideration satisfies all the ergodic properties on a compact set. On the other hand, the assumption of compactness is restrictive if we want to apply our model in physics and biology. Then the phase space is usually one of the function spaces and it is difficult to ensure that the ergodic properties hold on some compact set.

Our work is based on the theory of concentrating Markov operators on a Polish space (see [44]).

The system under study takes into consideration some very important and widely studied cases, namely dynamical systems generated by learning systems [2, 22, 23, 35], Poisson driven stochastic differential equations [10, 17, 34, 48, 49], iterated function systems with an infinite family of transformations [30, 50, 51], random evolutions [12, 42], randomly controlled dynamical systems [41] and irreducible Markov systems [52]. A large range of applications of such models, both in physics and biology, is worth mentioning here: the shot noise, the photo conductive detectors, the growth of the size of structural populations, the motion of relativistic particles, both fermions and bosons, and many others (see [8, 18, 24, 28]). On the other hand, it should be noted that most Markov chains appear in statistical physics and may be represented as iterated function systems (see [25]). Recently, iterated function systems have been used in studying invariant measures for the Ważewska partial differential equation which describes the process of the reproduction of red blood cells [32, 33]. Similar nonlinear first-order partial differential equations frequently appear in hydrodynamics [43]. So called irreducible Markov systems introduced by Werner (see [52]) are used for the computer modelling of various stochastic processes.

The outline of the paper is as follows. In Section 2 we set out notation and terminology. Section 3 is divided into two parts. Section 3.1 contains basic facts from the theory of Markov operators. In Section 3.2 we recall criteria for the existence of an invariant measure and for asymptotic stability on Polish spaces. These criteria are essential in the proofs of our results.

The main section of this paper is Section 4. Section 4.1 contains the description of our random dynamical systems. In Section 4.2 we consider discrete-time random dynamical systems with jumps on Polish spaces and show that a Markov operator describing the dynamics of such systems is asymptotically stable. In Section 4.3 we give sufficient conditions for asymptotic stability of a semigroup generated by the continuous-time random dynamical system in cases where the choice of jumps does not depend on a position in which it happens.

Section 5 is devoted to dimensions of measures. The lower pointwise dimension of an invariant measure for the semigroup of Markov operators generated by the continuous-time random dynamical system is estimated in Section 5.1. In Section 5.2 we give an upper bound for the concentration dimension of an invariant measure for the Markov operator describing the evolution of measures from jump to jump. Relationships between invariant measures of discrete and continuous-time random dynamical systems, and between their concentration dimensions are considered in Section 5.3. The results of Section 5 are used to evaluate the dimensions of invariant measures for dynamical systems generated by learning systems (Section 6.1) and Poisson driven stochastic differential equations (Section 6.4).

Finally, in Section 6 we apply our results to establish existence of an invariant measure and asymptotic stability of particular Markov operators. In Section 6.1 we study iterated function systems and show that the well known results proved by Barnsley and coauthors [2, 35] are a simple application of our criterion for asymptotic stability. Our next concern is the behavior of irreducible Markov systems which are an extension of iterated function systems with place dependent probabilities. Such systems on a locally compact space have been considered by Werner [52]. The irreducible Markov system is a particular example of a random dynamical system with randomly chosen jumps. However, we want to point out that the system may not satisfy the essential assumption put forward in the theorems of Section 4. This assumption can be replaced by contractiveness, which is more easily verifiable. Contractiveness has been considered in [52]. In Section 6.2 we extend Werner's result to the case of complete separable metric spaces. Section 6.3 is devoted to the mathematical theory of the cell cycle. In Section 6.4 we illustrate the usefulness of our criteria for asymptotic stability of a continuous-time random dynamical system by considering randomly connected Poisson driven stochastic differential equations.

The results of this paper are related to our papers [13–17, 19–21]. Criteria for asymptotic stability for discrete-time random dynamical systems without jumps, when Y is locally compact, are formulated in [13]. For Polish spaces these criteria are generalized in [20]. The results of Section 4.2 have been proved in [15]. In [19] we consider a continuous-time random dynamical system on Polish spaces, but also without jumps. Relationships between concentration dimensions of invariant measures of discrete and continuous time random dynamical systems are considered in [14] for the simpler case when $\{P^t\}_{t\ge 0}$ is a semigroup generated by the Poisson driven differential equation on $\mathbb{R}^d$. Poisson driven differential equations on $\mathbb{R}^d$ are studied in [14] and [17]. Some estimates of dimensions of invariant measures are formulated in [16].

2. Preliminaries

2.1. Basic notions. Let (Y,ϱ) be a Polish space, i.e. a separable, complete metric space. We denote by $B(x,r)$ the open ball with center at x and radius r. For any set $A\subset Y$, $\mathrm{cl}\,A$, $\mathrm{diam}_\varrho A$, and 1_A stand for the closure, diameter, and indicator function of A, respectively.

We denote by $\mathcal{B}(Y)$ the σ-algebra of Borel subsets of Y, by $\mathcal{M}=\mathcal{M}(Y)$ the family of all finite Borel measures on Y, and by $\mathcal{M}_s$ the space of all finite signed Borel measures on Y. We write $\mathcal{M}_1=\mathcal{M}_1(Y)$ for the family of all $\mu\in\mathcal{M}$ such that $\mu(Y)=1$. The elements of $\mathcal{M}_1$ are called *distributions*.

As usual, $B(Y)$ denotes the space of all bounded Borel measurable functions $f:Y\to\mathbb{R}$ and $C(Y)$ the subspace of all continuous functions. Both spaces are considered with the supremum norm $\|\cdot\|_0$. For $f\in B(Y)$ and $\mu\in\mathcal{M}_s$ we write

$$\langle f,\mu\rangle=\int_Y f(x)\,\mu(dx).$$

We introduce in $\mathcal{M}_s$ the *Fortet–Mourier norm* $\|\cdot\|_\varrho$ (see [6, 7, 9]) given by

$$\|\mu\|_\varrho=\sup\{|\langle f,\mu\rangle| : f\in\mathcal{F}_\varrho\}\qquad\text{for } \mu\in\mathcal{M}_s,$$

where $\mathcal{F}_\varrho$ is the set of all $f \in C(Y)$ such that $|f(x)| \le 1$ and $|f(x) - f(y)| \le \varrho(x, y)$ for $x, y \in Y$.

We say that a sequence $\{\mu_n\}_{n\ge1}$, $\mu_n \in \mathcal{M}$, *converges weakly* to a measure $\mu \in \mathcal{M}$ if

$$\lim_{n\to\infty} \langle f, \mu_n\rangle = \langle f, \mu\rangle \quad \text{for every } f \in C(Y).$$

It is well known (see [6]) that the convergence in the Fortet–Mourier norm $\|\cdot\|_\varrho$ is equivalent to the weak convergence.

We introduce the class Φ of functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:

(i) φ is continuous and $\varphi(0) = 0$;

(ii) φ is nondecreasing and concave, i.e.

$$\sum_{k=1}^{n} \alpha_k \varphi(y_k) \le \varphi\Big(\sum_{k=1}^{n} \alpha_k y_k\Big), \quad \text{where} \quad \alpha_k \ge 0, \sum_{k=1}^{n} \alpha_k = 1;$$

(iii) $\varphi(x) > 0$ for $x > 0$ and $\lim_{x\to\infty} \varphi(x) = \infty$.

We denote by Φ_0 the family of all functions satisfying (i) and (ii). A necessary and sufficient condition for a concave function φ to be subadditive on $(0, \infty)$ is that $\varphi(0+) \ge 0$. From this result we immediately obtain the triangle inequality for $\varrho_\varphi = \varphi \circ \varrho$. Thus for every $\varphi \in \Phi$ the function ϱ_φ is again a metric on Y. For notational convenience we write $\mathcal{F}_\varphi$ and $\|\cdot\|_\varphi$ instead of $\mathcal{F}_{\varrho_\varphi}$ and $\|\cdot\|_{\varrho_\varphi}$, respectively.

In our considerations an important role is played by the inequality

$$w(t) + \varphi(at) \le \varphi(t) \quad \text{for } t \ge 0, \tag{2.1.1}$$

where $w \in \Phi_0$ is a given function and $a \in [0, 1)$.

The inequality may be studied by classical methods of the theory of functional equations (see [27]). Lasota and Yorke [35] precisely discuss the cases for which (2.1.1) has a solution belonging to Φ and prove the following:

Proposition 2.1.1. *Assume that a function $w \in \Phi_0$ satisfies the Dini condition*

$$\int_0^\epsilon \frac{w(t)}{t}\, dt < \infty \quad \textit{for some } \epsilon > 0. \tag{2.1.2}$$

Let $a \in [0, 1)$. Then inequality (2.1.1) *admits a solution in Φ.*

We say that a vector $(p_1, \ldots, p_N)$, where $p_i : Y \to [0, 1]$, is *a probability vector* if

$$\sum_{i=1}^{N} p_i(x) = 1 \quad \text{for } x \in Y.$$

Analogously a matrix $[p_{ij}]_{i,j}$, where $p_{ij} : Y \to [0, 1]$ for $i, j \in \{1, \ldots, N\}$, is *a probability matrix* if

$$\sum_{j=1}^{N} p_{ij}(x) = 1 \quad \text{for } x \in Y \text{ and } i \in \{1, \ldots, N\}.$$

2.2. Dimensions of measures. For $A \subset Y$, $s > 0$ and $\delta > 0$ we define

$$\mathcal{H}^s_\delta(A) = \inf \sum_{i=1}^{\infty} (\operatorname{diam}_\varrho E_i)^s,$$

where the infimum is taken over all countable covers $\{E_i\}$ of A such that $\operatorname{diam}_\varrho E_i < \delta$. Then

$$\mathcal{H}^s(A) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(A)$$

defines the Hausdorff s-dimensional measure. The *Hausdorff dimension* of A is defined by the formula

$$\dim_H A = \sup\{s > 0 : \mathcal{H}^s(A) > 0\}.$$

(Here we assume that $\sup\emptyset = 0$.)

The *Hausdorff dimension* of $\mu \in \mathcal{M}_1$ is defined by the formula

$$\dim_H \mu = \inf\{\dim_H A : A \in \mathcal{B}(Y) \text{ and } \mu(A) = 1\}.$$

For a given $\mu \in \mathcal{M}$ we define the *lower pointwise dimension* of μ at $x \in Y$ by

$$\underline{d}\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

(here $\log 0 = -\infty$) and the *Lévy concentration function* $Q_\mu : (0,\infty) \to \mathbb{R}_+$ by (see [36])

$$Q_\mu(r) = \sup\{\mu(B(x,r)) : x \in Y\} \quad \text{for } r > 0.$$

Further, for a measure $\mu \in \mathcal{M}_1$ we define the *lower* and *upper concentration dimensions* of μ by the formulas

$$\underline{\dim}_L \mu = \liminf_{r\to 0} \frac{\log Q_\mu(r)}{\log r} \quad \text{and} \quad \overline{\dim}_L \mu = \limsup_{r\to 0} \frac{\log Q_\mu(r)}{\log r}.$$

If $\underline{\dim}_L \mu = \overline{\dim}_L \mu$ then this common value, denoted by $\dim_L \mu$, is called the *concentration dimension* (the generalized Rényi dimension) of μ (see [32, 33]).

The Hausdorff dimension and the concentration dimension are closely related to each other as is shown in the next results proved in [32]:

Proposition 2.2.1. *Let $\mu \in \mathcal{M}_1$ and $A \in \mathcal{B}(Y)$ be such that $\mu(A) > 0$. Then*

$$\dim_H A \ge \underline{\dim}_L \mu.$$

Proposition 2.2.2. *Let $A \subset Y$ be a nonempty compact set. Then*

$$\dim_H A = \sup \underline{\dim}_L \mu,$$

where the supremum is taken over all $\mu \in \mathcal{M}_1$ such that $\operatorname{supp}\mu \subset A$.

3. Properties of Markov operators

3.1. Markov operators. An operator $P : \mathcal{M} \to \mathcal{M}$ is called a *Markov operator* if

$$P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2 \quad \text{for } \lambda_1, \lambda_2 \in \mathbb{R}_+ \text{ and } \mu_1, \mu_2 \in \mathcal{M}$$

and

$$P\mu(Y) = \mu(Y) \quad \text{for } \mu \in \mathcal{M}.$$

It is easy to prove that every Markov operator can be extended to a linear operator on the space $\mathcal{M}_s$ of all signed measures.

A linear operator $U : B(Y) \to B(Y)$ is called *dual* to P if

$$\langle Uf, \mu\rangle = \langle f, P\mu\rangle \quad \text{for } f \in B(Y) \text{ and } \mu \in \mathcal{M}. \tag{3.1.1}$$

Setting $\mu = \delta_x$, the point (Dirac) measure supported at x, in (3.1.1) we obtain

$$Uf(x) = \langle f, P\delta_x\rangle \quad \text{for } f \in B(Y) \text{ and } x \in Y. \tag{3.1.2}$$

From (3.1.2) it follows immediately that U is a linear operator satisfying

$$\begin{aligned} &Uf \geq 0 \quad \text{for } f \geq 0,\ f \in B(Y),\\ &U1_Y = 1_Y,\\ &Uf_n \downarrow 0 \quad \text{for } f_n \downarrow 0,\ f_n \in B(Y). \end{aligned} \tag{3.1.3}$$

Conditions (3.1.1)–(3.1.3) allow one to reverse the roles of P and U. Namely, given U satisfying (3.1.3) we may define a Markov operator $P : \mathcal{M} \to \mathcal{M}$ by setting

$$P\mu(A) = \langle U1_A, \mu\rangle \quad \text{for } A \in \mathcal{B}(Y) \text{ and } \mu \in \mathcal{M}. \tag{3.1.4}$$

Assume now that P and U are given. If $f : Y \to \mathbb{R}_+$ is a Borel measurable function, not necessarily bounded, we may define Uf by

$$Uf(x) = \lim_{n\to\infty} Uf_n(x),$$

where $\{f_n\}_{n\geq 1}$ is an increasing sequence of bounded Borel measurable functions converging pointwise to f. From the Lebesgue monotone convergence theorem it follows that Uf satisfies (3.1.1).

A Markov operator P is called a *Markov–Feller operator* if it has a dual operator U such that

$$Uf \in C(Y) \quad \text{for } f \in C(Y).$$

A Markov operator P is called *nonexpansive* if

$$\|P\mu_1 - P\mu_2\|_\varrho \leq \|\mu_1 - \mu_2\|_\varrho \quad \text{for } \mu_1, \mu_2 \in \mathcal{M}_1.$$

Remark 3.1.1. Let P be a Markov operator and U its dual. If $U(\mathcal{F}_\varrho) \subset \mathcal{F}_\varrho$, then P is nonexpansive.

A measure μ_* is called *invariant* (or *stationary*) with respect to P if $P\mu_* = \mu_*$. A Markov operator P is called *asymptotically stable* if there exists a stationary measure $\mu_* \in \mathcal{M}_1$ such that

$$\lim_{n\to\infty} \|P^n\mu - \mu_*\|_\varrho = 0 \quad \text{for every } \mu \in \mathcal{M}_1. \tag{3.1.5}$$

Obviously a measure μ_* satisfying the above condition is unique.

When an invariant measure exists, condition (3.1.5) is equivalent to a more symmetric relation

$$\lim_{n\to\infty} \|P^n\mu_1 - P^n\mu_2\|_\varrho = 0 \quad \text{for } \mu_1, \mu_2 \in \mathcal{M}_1. \tag{3.1.6}$$

A sequence $\{\mu_n\}_{n\geq 1}$ ($\mu_n \in \mathcal{M}_1$) of distributions is called *tight* if for every $\varepsilon > 0$ there exists a compact set $K \subset Y$ such that $\mu_n(K) \geq 1 - \varepsilon$ for every $n \in \mathbb{N}$.

We say that a Markov operator $P : \mathcal{M} \to \mathcal{M}$ is *tight* if for every $\mu \in \mathcal{M}_1$ the sequence $\{P^n\mu\}_{n\geq 1}$ is tight.

A family $\{P^t\}_{t\geq 0}$ of Markov operators is called a *semigroup* if $P^{t+s} = P^t P^s$ for all $t, s \in \mathbb{R}_+$ and P^0 is the identity operator on $\mathcal{M}$.

Let $\{P^t\}_{t\geq 0}$ be given. We denote by $\{T^t\}_{t\geq 0}$ the semigroup dual to $\{P^t\}_{t\geq 0}$, i.e.

$$\langle T^t f, \mu\rangle = \langle f, P^t\mu\rangle \quad \text{for } f \in B(Y),\ \mu \in \mathcal{M}_1.$$

A measure $\mu_* \in \mathcal{M}$ is called *invariant* (or *stationary*) for the Markov semigroup $\{P^t\}_{t\geq 0}$ if $P^t\mu_* = \mu_*$ for $t \geq 0$. The Markov semigroup $\{P^t\}_{t\geq 0}$ is called *asymptotically stable* if there exists a stationary measure μ_* such that

$$\lim_{t\to\infty} \|P^t\mu - \mu_*\|_\varrho = 0 \quad \text{for } \mu \in \mathcal{M}_1.$$

3.2. Criteria for asymptotic stability. In this section we present known criteria for the existence of an invariant measure and for asymptotic stability of Markov operators on the space of Borel measures on a Polish space Y.

First results concerning the existence of invariant measures were proved for compact spaces ([23]). The classical proof goes as follows. One defines a positive invariant functional on the space of all continuous functions. By the Riesz theorem it may be represented by a measure. Since the functional is invariant, one concludes that the measure is also invariant. This scheme works smoothly when Y is a compact space. Lasota and Yorke [35] managed to extend it to the case when Y is locally compact and σ-compact. Their result on the existence of an invariant measure is similar in spirit to Komorowski's theorem [26], however, only Markov operators acting on absolutely continuous measures are considered in [26]. The approach in [35] was partially based on the idea of the lower bound function developed for Markov operators acting on L^1-space (see [31]). The authors introduced the class of so-called concentrating Markov operators and showed that every operator from this class admits an invariant measure. Furthermore, assuming that a concentrating Markov operator does not increase a distance between two measures, they showed that it must be asymptotically stable (see [35]). In order to state the result, some notation is needed.

We say that a metric $\hat\varrho$ is *equivalent* to ϱ if the classes of bounded sets and convergent sequences in the spaces $(Y, \hat\varrho)$ and (Y, ϱ) coincide. Obviously, if (Y, ϱ) is a Polish space and ϱ, $\hat\varrho$ are equivalent, then the space $(Y, \hat\varrho)$ is still a Polish space.

A Markov operator $P : \mathcal{M} \to \mathcal{M}$ is called *essentially nonexpansive* if there exists a metric $\hat\varrho$ equivalent to ϱ such that P is nonexpansive with respect to the norm $\|\cdot\|_{\hat\varrho}$, i.e.

$$\|P\mu_1 - P\mu_2\|_{\hat\varrho} \leq \|\mu_1 - \mu_2\|_{\hat\varrho} \quad \text{for } \mu_1, \mu_2 \in \mathcal{M}_1. \tag{3.2.1}$$

An operator P is called *concentrating* if for every $\varepsilon > 0$ there exist a set $A \in \mathcal{B}(Y)$ with $\operatorname{diam}_\varrho A \leq \varepsilon$ and a number $\theta > 0$ such that

$$\liminf_{n\to\infty} P^n\mu(A) > \theta \quad \text{for } \mu \in \mathcal{M}_1. \tag{3.2.2}$$

Proposition 3.2.1. *If P is an essentially nonexpansive and concentrating Markov operator then P is asymptotically stable.*

The proof can be found in [44] in the case when Y is a Polish space.

It should be noted that the definition of asymptotic stability consists of two almost independent statements: the existence of an invariant measure μ_* and the convergence condition (3.1.6). It turns out that even if the set A in condition (3.2.2) depends on the choice of initial measures, then the proof in Lasota and Yorke [35] carries over to a Polish space and leads to the following result:

Proposition 3.2.2. *Let P be a nonexpansive Markov operator. Assume that P satisfies the lower bound condition: for every $\varepsilon > 0$ there is a number $\Delta > 0$ such that for every $\mu_1, \mu_2 \in \mathcal{M}_1$ there exist $A \in \mathcal{B}(Y)$ with $\operatorname{diam}_\varrho A \le \varepsilon$ and $n_0 \in \mathbb{N}$ for which*

$$P^{n_0}\mu_i(A) \ge \Delta \quad \textit{for } i = 1, 2. \tag{3.2.3}$$

Then

$$\lim_{n\to\infty} \|P^n\mu_1 - P^n\mu_2\|_\varrho = 0 \quad \textit{for } \mu_1, \mu_2 \in \mathcal{M}_1.$$

In the setting of Polish spaces it might be difficult or even impossible to prove that a given Markov operator is concentrating. We now describe results concerning asymptotic stability of Markov operators on infinite-dimensional spaces obtained by Szarek [44] and based on the concept of tightness and the well known Prokhorov theorem. He introduced the class of globally and semi-concentrating Markov operators and gave conditions ensuring the existence of an invariant measure for nonexpansive Markov operators. It is important to emphasize that the nonexpansiveness is crucial in these considerations: Szarek [47] constructed an example which shows that it cannot be omitted.

We denote by $\mathcal{C}_\varepsilon(Y)$, $\varepsilon > 0$, ($\mathcal{C}_\varepsilon$ for abbreviation) the family of all closed sets C for which there exists a finite set $\{z_1, \dots, z_n\} \subset Y$ such that $C \subset \bigcup_{i=1}^n B(z_i, \varepsilon)$.

An operator P is called *semi-concentrating* if for every $\varepsilon > 0$ there exist $C \in \mathcal{C}_\varepsilon(Y)$ and $\theta > 0$ such that

$$\liminf_{n\to\infty} P^n\mu(C) > \theta \quad \text{for } \mu \in \mathcal{M}_1. \tag{3.2.4}$$

Remark 3.2.1. A concentrating Markov operator is semi-concentrating.

For $\mu \in \mathcal{M}_1$ we consider the limit set

$$\mathcal{L}(\mu) = \{\nu \in \mathcal{M}_1 : \text{there exists } \{n_k\} \subset \{n\} \text{ such that } \lim_{k\to\infty} \|P^{n_k}\mu - \nu\|_\varrho = 0\} \tag{3.2.5}$$

and

$$\mathcal{L}(\mathcal{M}_1) = \bigcup_{\mu\in\mathcal{M}_1} \mathcal{L}(\mu). \tag{3.2.6}$$

The following results are proved in [46]:

Proposition 3.2.3. *Let P be a nonexpansive and semi-concentrating Markov operator. Then*

(a) *P has an invariant measure;*
(b) *$\mathcal{L}(\mu) \neq \emptyset$ for arbitrary $\mu \in \mathcal{M}_1$;*
(c) *$\mathcal{L}(\mathcal{M}_1)$ is tight.*

Let $A \in \mathcal{B}(Y)$. We say that a measure $\mu \in \mathcal{M}$ is *concentrated* on A if $\mu(Y \setminus A) = 0$. We denote by $\mathcal{M}_1^A$ the set of all probability measures concentrated on A.

An operator P is called *globally concentrating* if for every $\varepsilon > 0$ and every bounded Borel set A there exist a bounded Borel set B and a number $n_0 \in \mathbb{N}$ such that

$$P^n\mu(B) \geq 1-\varepsilon \quad \text{for } n \geq n_0 \text{ and } \mu \in \mathcal{M}_1^A.$$

A continuous function $V : Y \to [0,\infty)$ is called a *Lyapunov function* if

$$\lim_{\varrho(x,z_0)\to\infty} V(x) = \infty$$

for some $z_0 \in Y$.

PROPOSITION 3.2.4. *Let P be a Markov operator and U its dual. Assume that there exists a Lyapunov function V, bounded on bounded sets, such that*

$$UV(x) \leq aV(x) + b \quad \textit{for } x \in Y,$$

where $a, b \in \mathbb{R}_+$ and $a < 1$. Then P is globally concentrating.

Moreover, for every $\varepsilon > 0$ there exists a bounded Borel set $B \subset Y$ such that

$$\liminf_{n\to\infty} P^n\mu(B) \geq 1-\varepsilon \quad \textit{for } \mu \in \mathcal{M}_1.$$

Define

$$\mathcal{E}(P) = \{\varepsilon > 0 : \inf_{\mu\in\mathcal{M}_1} \liminf_{n\to\infty} P^n\mu(A) > 0 \text{ for some } A \in \mathcal{C}_\varepsilon(Y)\}. \tag{3.2.7}$$

REMARK 3.2.2. If a Markov operator P is globally concentrating, then $\mathcal{E}(P) \neq \emptyset$.

REMARK 3.2.3. If $\inf \mathcal{E}(P) = 0$, then P is semi-concentrating.

By Proposition 3.2.2 and 3.2.3 we obtain:

THEOREM 3.2.1. *A nonexpansive, semi-concentrating Markov operator satisfying a lower bound condition* (3.2.3) *is asymptotically stable.*

4. Random dynamical systems with jumps

4.1. Introduction. Let (Y,ϱ) be a Polish space, $\mathbb{R}_+ = [0,\infty)$ and $I = \{1,\dots,N\}$, $S = \{1,\dots,K\}$, where N and K are given positive integers.

Let $\Pi_i : \mathbb{R}_+ \times Y \to Y$, $i \in I$, be a finite sequence of semidynamical systems, i.e.

$$\Pi_i(0,x) = x \quad \text{for } i \in I,\ x \in Y$$

and

$$\Pi_i(s+t,x) = \Pi_i(s,\Pi_i(t,x)) \quad \text{for } s,t \in \mathbb{R}_+,\ i \in I \text{ and } x \in Y.$$

We are given probability vectors $p_i : Y \to [0,1]$, $i \in I$, $\overline{p}_s : Y \to [0,1]$, $s \in S$, a matrix of probabilities $[p_{ij}]_{i,j\in I}$, $p_{ij} : Y \to [0,1]$, $i,j \in I$, and a family of continuous functions $q_s : Y \to Y$, $s \in S$. We denote the entire system by (Π, q, p).

Finally, let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and $\{t_n\}_{n\geq 0}$ be an increasing sequence of random variables $t_n : \Omega \to \mathbb{R}_+$ with $t_0 = 0$ and such that the increments $\Delta t_n = t_n - t_{n-1}$, $n \in \mathbb{N}$, are independent and have the same density $g(t) = \lambda e^{-\lambda t}$, $t \geq 0$.

The action of randomly chosen dynamical systems, with randomly chosen jumps, at random moments t_k corresponding to the system (Π, q, p) can be roughly described as follows.

We choose an initial point $x_0 \in Y$ and randomly select a transformation Π_i from the set $\{\Pi_1, \dots, \Pi_N\}$ in such a way that the probability of choosing Π_i is equal to $p_i(x_0)$, and we define

$$X(t) = \Pi_i(t, x_0) \quad \text{for } 0 \le t < t_1.$$

Next, at the random time t_1, at the point $\Pi_i(t_1, x_0)$ we choose a jump q_s from the set $\{q_1, \dots, q_K\}$ with probability $\overline{p}_s(\Pi_i(t_1, x_0))$. Then we define

$$x_1 = q_s(\Pi_i(t_1, x_0)).$$

After that we choose Π_{i_1} with probability $p_{ii_1}(x_1)$, define

$$X(t) = \Pi_{i_1}(t - t_1, x_1) \quad \text{for } t_1 < t < t_2$$

and at the point $\Pi_{i_1}(t_2 - t_1, x_1)$ we choose q_{s_1} with probability $\overline{p}_{s_1}(\Pi_{i_1}(t_2 - t_1, x_1))$. Then we define

$$x_2 = q_{s_1}(\Pi_{i_1}(t_2 - t_1, x_1)).$$

Finally, given x_n, $n \ge 2$, we choose Π_{i_n} in such a way that the probability of choosing Π_{i_n} is equal to $p_{i_{n-1}i_n}(x_n)$ and we define

$$X(t) = \Pi_{i_n}(t - t_n, x_n) \quad \text{for } t_n < t < t_{n+1}.$$

At the point $\Pi_{i_n}(\Delta t_{n+1}, x_n)$ we choose q_{s_n} with probability $\overline{p}_{s_n}(\Pi_{i_n}(\Delta t_{n+1}, x_n))$. Then we define

$$x_{n+1} = q_{s_n}(\Pi_{i_n}(\Delta t_{n+1}, x_n)).$$

We obtain a piecewise-deterministic trajectory for $\{X(t)\}_{t \ge 0}$ with jump times $\{t_1, t_2, \dots\}$ and post jump locations $\{x_1, x_2, \dots\}$.

We may reformulate the above considerations as follows: Let $\{\xi_n\}_{n \ge 0}$ and $\{\eta_n\}_{n \ge 1}$ be sequences of random variables, $\xi_n : \Omega \to I$ and $\eta_n : \Omega \to S$ and let $\{y_n\}_{n \ge 1}$ be auxiliary random variables, $y_n : \Omega \to Y$, such that

$$\begin{aligned} &\mathbb{P}(\xi_0 = i \mid x_0 = x) = p_i(x), \\ &\mathbb{P}(\xi_n = k \mid x_n = x \text{ and } \xi_{n-1} = i) = p_{ik}(x), \end{aligned} \tag{4.1.1}$$

and

$$\begin{aligned} &y_n = \Pi_{\xi_{n-1}}(t_n - t_{n-1}, x_{n-1}), \\ &\mathbb{P}(\eta_n = s \mid y_n = y) = \overline{p}_s(y) \end{aligned} \tag{4.1.2}$$

for $n \ge 1$, $x, y \in Y$, $k, i \in I$ and $s \in S$.

Assume that $\{\xi_n\}_{n \ge 0}$ and $\{\eta_n\}_{n \ge 0}$ are independent of $\{t_n\}_{n \ge 0}$ and that for every $n \in \mathbb{N}$ the variables $\eta_1, \dots, \eta_{n-1}$, $\xi_1, \dots, \xi_{n-1}$ are also independent.

Given an initial random variable ξ_0 the sequence of the random variables $\{x_n\}_{n \ge 0}$, $x_n : \Omega \to Y$, is given by

$$x_n = q_{\eta_n}(\Pi_{\xi_{n-1}}(t_n - t_{n-1}, x_{n-1})) \quad \text{for } n = 1, 2, \dots \tag{4.1.3}$$

and the stochastic process $\{X(t)\}_{t \ge 0}$, $X(t) : \Omega \to Y$, is given by

$$X(t) = \Pi_{\xi_{n-1}}(t - t_{n-1}, x_{n-1}) \quad \text{for } t_{n-1} \le t < t_n, \quad n = 1, 2, \dots \tag{4.1.4}$$

It is easy to see that $\{X(t)\}_{t\geq 0}$ and $\{x_n\}_{n\geq 0}$ are not Markov processes. In order to use the theory of Markov operators we must redefine the processes $\{X(t)\}_{t\geq 0}$ and $\{x_n\}_{n\geq 0}$ in such a way that the redefined processes become Markov.

For this purpose, consider the space $Y \times I$ endowed with the metric $\overline{\varrho}$ given by

$$\overline{\varrho}((x,i),(y,j)) = \varrho(x,y) + \varrho_c(i,j) \quad \text{for } x,y\in Y,\ i,j\in I, \tag{4.1.5}$$

where

$$\varrho_c(i,j) = \begin{cases} c & \text{if } i\neq j,\\ 0 & \text{if } i=j \end{cases} \tag{4.1.6}$$

and the constant c will be chosen later on. Now define a stochastic process $\{\xi(t)\}_{t\geq 0}$, $\xi(t):\Omega\to I$, by

$$\xi(t) = \xi_{n-1} \quad \text{for } t_{n-1}\leq t < t_n,\ n=1,2,\ldots$$

Then the stochastic process $\{(X(t),\xi(t))\}_{t\geq 0}$, $(X(t),\xi(t)):\Omega\to Y\times I$ has the required Markov property.

In many applications we are mostly interested in values of the process $X(t)$ at the switching points t_n. Therefore, we will also study the stochastic discrete process (post jump locations) $\{(x_n,\xi_n)\}_{n\geq 0}$, $(x_n,\xi_n):\Omega\to Y\times I$. Clearly $\{(x_n,\xi_n)\}_{n\geq 0}$ is a Markov process too.

4.2. Discrete-time random dynamical systems. Let $(Y,\|\cdot\|)$ be a separable Banach space. In this section we consider the stochastic process $\{(x_n,\xi_n)\}_{n\geq 0}$, $(x_n,\xi_n):\Omega\to Y\times I$, defined by (4.1.1)–(4.1.3) with the help of the system (Π,q,p). We are interested in the evolution of distributions corresponding to this discrete-time random dynamical system. In order to get the existence of invariant measures or asymptotic results, it is necessary to put some restrictions on the system (Π,q,p). We will need the following assumptions:

The transformations $\Pi_i:\mathbb{R}_+\times Y\to Y$, $i\in I$ and $q_s:Y\to Y$, $s\in S$, are continuous and there exists $x_*\in Y$ such that

$$\int_{\mathbb{R}_+} e^{-\lambda t}\|q_s(\Pi_j(t,x_*)) - q_s(x_*)\|\,dt < \infty \quad \text{for } j\in I,\ s\in S. \tag{4.2.1}$$

The functions $\overline{p}_s$, $s\in S$, and p_{ij}, $i,j\in I$, satisfy the following conditions:

$$\begin{aligned} &\sum_{j\in I}|p_{ij}(x)-p_{ij}(y)| \leq \psi_1(\|x-y\|) \quad \text{for } x,y\in Y,\ i\in I,\\ &\sum_{s\in S}|\overline{p}_s(x)-\overline{p}_s(y)| \leq \psi_2(\|x-y\|) \quad \text{for } x,y\in Y, \end{aligned} \tag{4.2.2}$$

where the functions $\psi_1,\psi_2\in\Phi_0$ satisfy the Dini condition (2.1.2).

We also assume that for the system (Π,q,p) there are three constants $L\geq 1$, $\alpha\in\mathbb{R}$ and $L_q>0$ such that

$$\sum_{j\in I} p_{ij}(y)\|\Pi_j(t,x)-\Pi_j(t,y)\| \leq Le^{\alpha t}\|x-y\| \quad \text{for } x,y\in Y,\ i\in I,\ t\geq 0 \tag{4.2.3}$$

and

$$\sum_{s\in S}\overline{p}_s(x)\|q_s(x)-q_s(y)\|\le L_q\|x-y\| \quad \text{for } x,y\in Y. \tag{4.2.4}$$

Further, there exists $i_0\in I$ such that

$$\inf\{p_{ii_0}(x): i\in I,\, x\in Y\}>0. \tag{4.2.5}$$

To begin our study of the stochastic process $\{(x_n,\xi_n)\}_{n\ge 0}$ consider the sequence of distributions

$$\overline{\mu}_n(A)=\mathbb{P}((x_n,\xi_n)\in A) \quad \text{for } A\in\mathcal{B}(Y\times I),\ n\ge 0.$$

It is easy to see that there exists a Markov–Feller operator $P:\mathcal{M}\to\mathcal{M}$ such that

$$\overline{\mu}_{n+1}=P\overline{\mu}_n \quad \text{for } n\ge 0.$$

The operator P is given by the formula

$$P\mu(A)=\sum_{j\in I}\sum_{s\in S}\int_{Y\times I}\int_0^\infty \lambda e^{-\lambda t}1_A(q_s(\Pi_j(t,x)),j)p_{ij}(x)\overline{p}_s(\Pi_j(t,x))\,dt\,\mu(dx,di) \tag{4.2.6}$$

and its dual operator U by

$$Uf(x,i)=\sum_{j\in I}\sum_{s\in S}\int_0^\infty \lambda e^{-\lambda t}f(q_s(\Pi_j(t,x)),j)p_{ij}(x)\overline{p}_s(\Pi_j(t,x))\,dt, \tag{4.2.7}$$

where λ is the intensity of the Poisson process which governs the increment Δt_n of the random variables $\{t_n\}_{n\ge 0}$. The operator P given by (4.2.6) is called the *transition operator* for this system.

The first result ensures the existence of an invariant distribution for the transition operator P.

THEOREM 4.2.1. *Assume that the system* (Π,p,q) *satisfies conditions* (4.2.1)–(4.2.4). *If*

$$LL_q+\alpha/\lambda<1, \tag{4.2.8}$$

then the operator P *defined by* (4.2.6) *has an invariant measure.*

The proof of Theorem 4.2.1 is based on Proposition 3.2.3. Therefore we have to show that the operator P is essentially nonexpansive and semi-concentrating. These properties are interesting in their own right and will be stated separately in the next two lemmas.

LEMMA 4.2.1. *Assume that the system* (Π,q,p) *satisfies conditions* (4.2.2)–(4.2.4) *and* (4.2.8). *Then the operator* P *given by* (4.2.6) *is essentially nonexpansive.*

Proof. Let $\psi_1,\psi_2\in\Phi_0$ be given by condition (4.2.2). Define $\overline{\psi}:\mathbb{R}_+\to\mathbb{R}$ by

$$\overline{\psi}(t)=\psi_1(t)+\psi_2\biggl(\frac{\lambda L}{\lambda-\alpha}\,t\biggr) \quad \text{for } t\ge 0.$$

It is evident that $\overline{\psi}\in\Phi_0$ and it satisfies the hypotheses of Proposition 2.1.1, thus there exists $\varphi\in\Phi$ such that

$$\overline{\psi}(t)+\varphi(at)\le\varphi(t) \quad \text{for } t\ge 0 \quad \text{with} \quad a=\frac{\lambda LL_q}{\lambda-\alpha}<1. \tag{4.2.9}$$

Since $\varphi \in \Phi$ we may choose $c \in \mathbb{R}_+$ such that $\varphi(c) > 2$. Consider the metric $\overline{\varrho}$ (see (4.1.5)) with this choice of c, i.e.

$$\overline{\varrho}((x,i),(y,j)) = \|x-y\| + \varrho_c(i,j) \quad \text{for } x,y \in Y,\ i,j \in I.$$

Fix $f \in \mathcal{F}_\varphi$. To complete the proof it is enough to show that

$$|Uf(x,i) - Uf(y,j)| \le \varphi(\overline{\varrho}((x,i),(y,j))) \quad \text{for } (x,i),(y,j) \in Y \times I, \tag{4.2.10}$$

where the operator U is given by (4.2.7). Since $\varrho_c(i,j) = c$ for $i \ne j$, $\varphi(c) > 2$, and $|f| \le 1$, condition (4.2.10) is satisfied for $i \ne j$. On the other hand, for $i = j$,

$$\begin{aligned}
&|Uf(x,i) - Uf(y,i)| \\
&\quad \le \sum_{j\in I}\sum_{s\in S}\int_0^\infty \lambda e^{-\lambda t}|f(q_s(\Pi_j(t,x)),j) - f(q_s(\Pi_j(t,y)),j)|p_{ij}(x)\overline{p}_s(\Pi_j(t,x))\,dt \\
&\qquad + \sum_{j\in I}\sum_{s\in S}\int_0^\infty \lambda e^{-\lambda t}|p_{ij}(x)\overline{p}_s(\Pi_j(t,x)) - p_{ij}(y)\overline{p}_s(\Pi_j(t,y))|\,dt \\
&\quad \le \sum_{j\in I}\sum_{s\in S}\int_0^\infty \lambda e^{-\lambda t}\varphi(\|q_s(\Pi_j(t,x)) - q_s(\Pi_j(t,y))\|)p_{ij}(x)\overline{p}_s(\Pi_j(t,x))\,dt \\
&\qquad + \sum_{j\in I}\sum_{s\in S}\int_0^\infty \lambda e^{-\lambda t}|p_{ij}(x) - p_{ij}(y)|\overline{p}_s(\Pi_j(t,x))\,dt \\
&\qquad + \sum_{j\in I}\sum_{s\in S}\int_0^\infty \lambda e^{-\lambda t}p_{ij}(y)|\overline{p}_s(\Pi_j(t,x)) - \overline{p}_s(\Pi_j(t,y))|\,dt.
\end{aligned}$$

Using consecutively (4.2.3), (4.2.4), the Jensen inequality, (4.2.2), and (4.2.9), we obtain

$$\begin{aligned}
&|Uf(x,i) - Uf(y,i)| \\
&\quad \le \varphi\Big(\sum_{j\in I}\int_0^\infty \lambda e^{-\lambda t}L_q\|\Pi_j(t,x) - \Pi_j(t,y)\|p_{ij}(x)\,dt\Big) \\
&\qquad + \sum_{j\in I}|p_{ij}(x) - p_{ij}(y)| + \sum_{j\in I}\int_0^\infty \lambda e^{-\lambda t}p_{ij}(y)\psi_2(\|\Pi_j(t,x) - \Pi_j(t,y)\|)\,dt \\
&\quad \le \varphi\Big(\int_0^\infty \lambda e^{-\lambda t}L_qLe^{\alpha t}\|x-y\|\,dt\Big) + \psi_1(\|x-y\|) + \psi_2\Big(\frac{\lambda L}{\lambda-\alpha}\|x-y\|\Big) \\
&\quad \le \varphi(a\|x-y\|) + \overline{\psi}(\|x-y\|) \le \varphi(\|x-y\|). \ \blacksquare
\end{aligned}$$

Lemma 4.2.2. *Assume that the system* (Π, q, p) *satisfies conditions* (4.2.1)–(4.2.4) *and* (4.2.8). *Then the operator* P *given by* (4.2.6) *is semi-concentrating.*

Proof. Define

$$V(x,i) = \|x\| \quad \text{for } (x,i) \in Y \times I.$$

Let us first show that there exist $a, b \in \mathbb{R}_+$, $a < 1$, such that

$$UV(x,i) \le aV(x,i) + b \quad \text{for } (x,i) \in Y \times I. \tag{4.2.11}$$

By (4.2.7) and the definition of V, we have

$$UV(x,i) \le \sum_{j\in I}\sum_{s\in S}\int_0^\infty \|q_s(\Pi_j(t,x)) - q_s(\Pi_j(t,x_*))\|\lambda e^{-\lambda t} p_{ij}(x)\overline{p}_s(\Pi_j(t,x))\,dt$$
$$+\sum_{j\in I}\sum_{s\in S}\int_0^\infty \|q_s(\Pi_j(t,x_*)) - q_s(x_*)\|\lambda e^{-\lambda t} p_{ij}(x)\overline{p}_s(\Pi_j(t,x))\,dt$$
$$+\sum_{j\in I}\sum_{s\in S}\int_0^\infty \|q_s(x_*)\|\lambda e^{-\lambda t} p_{ij}(x)\overline{p}_s(\Pi_j(t,x))\,dt,$$

where x_* is given by condition (4.2.1). Further, using (4.2.1), (4.2.3) and (4.2.4) we obtain

$$UV(x,i) \le \frac{\lambda L_q L}{\lambda-\alpha}\|x-x_*\| + \widetilde{b} \le a\|x\| + b,$$

where

$$a = \frac{\lambda L_q L}{\lambda-\alpha},$$
$$\widetilde{b} = \sum_{j\in I}\sum_{s\in S}\int_0^\infty \lambda e^{-\lambda t}\|q_s(\Pi_j(t,x_*)) - q_s(x_*)\|\,dt + \sum_{s\in S}\|q_s(x_*)\|,$$
$$b = \widetilde{b} + a\|x_*\|.$$

From (4.2.1) and the fact that the sets I and S are finite, it follows that b is finite. Since $a<1$, the proof of (4.2.11) is complete. By Proposition 3.2.4, we conclude that there exists a bounded set $A\subset Y\times I$ such that

$$\inf_{\mu\in\mathcal{M}_1}\liminf_{n\to\infty} P^n\mu(A) > 0$$

which implies that $\mathcal{E}(P)$, given by (3.2.7), is not empty.

We now claim that $\inf\mathcal{E}(P)=0$.

Suppose, contrary to our claim, that $\widetilde{\varepsilon} = \inf\mathcal{E}(P) > 0$. We consider two cases: $\alpha<0$ and $\alpha\ge 0$, where α is given by condition (4.2.3).

Case I: $\alpha<0$. We may choose $z_0\in Y$ and $r>0$ such that

$$\inf_{\mu\in\mathcal{M}_1}\liminf_{n\to\infty} P^n\mu(B(z_0,r)\times I) > 0. \tag{4.2.12}$$

Fix $t_*>0$ such that

$$\varepsilon = 4rLL_q e^{\alpha t_*} < \widetilde{\varepsilon}$$

and set

$$C_\varepsilon = \bigcup_{j\in I}\bigcup_{t\in[t_*,2t_*]}\bigcup_{s\in S}(B(q_s(\Pi_j(t,z_0)),\,\varepsilon)\times I).$$

Observe that $C_\varepsilon\in\mathcal{C}_\varepsilon$. According to (4.2.6), for arbitrary $\mu\in\mathcal{M}_1$ we have

$$\begin{aligned}(4.2.13)\qquad & P^{n+1}\mu(C_\varepsilon)\\ &= \sum_{j\in I}\sum_{s\in S}\int_{Y\times I}\int_0^\infty 1_{C_\varepsilon}(q_s(\Pi_j(t,x)),j)\lambda e^{-\lambda t}p_{ij}(x)\overline{p}_s(\Pi_j(t,x))\,dt\,P^n\mu(dx,di).\end{aligned}$$

For $x \in B(z_0, r)$ and $t > t_*$ we define

$$J(x,t) = \{j \in I : \|\Pi_j(t,x) - \Pi_j(t,z_0)\| \le 2Le^{\alpha t}\|x - z_0\|\},$$
$$S(x,t,j) = \{s \in S : \|q_s(\Pi_j(t,x)) - q_s(\Pi_j(t,z_0))\| \le 2L_q\|\Pi_j(t,x) - \Pi_j(t,z_0)\|\}.$$

Since

$$\sum_{j\in I} p_{ij}(x) = 1 \quad \text{for } i \in I \quad \text{and} \quad \sum_{s\in S} \overline{p}_s(\Pi_j(t,x)) = 1 \quad \text{for } j \in I,$$

from (4.2.3) and (4.2.4) we obtain

$$\sum_{j\in J(x,t)} p_{ij}(x) \ge \frac{1}{2} \quad \text{for } i \in I, \quad \text{and} \quad \sum_{s\in S(x,t,j)} \overline{p}_s(\Pi_j(t,x)) \ge \frac{1}{2}.$$

Let $x \in B(z_0, r)$ and $t \in [t_*, 2t_*]$. Then for every $j \in J(x,t)$ and $s \in S(x,t,j)$ we have

$$\|q_s(\Pi_j(t,x)) - q_s(\Pi_j(t,z_0))\| \le 2L_q\|\Pi_j(t,x) - \Pi_j(t,z_0)\| \le 4LL_q e^{\alpha t}\|x - z_0\| \le \varepsilon,$$

which gives $(q_s(\Pi_j(t,x)), j) \in C_\varepsilon$. Thus from (4.2.13) it follows that

$$\begin{aligned} P^{n+1}\mu(C_\varepsilon) &\ge \int\limits_{B(z_0,r)\times I} \int\limits_{t_*}^{2t_*} \sum_{j\in J(t,x)} \sum_{s\in S(x,t,j)} \lambda e^{-\lambda t} p_{ij}(x)\overline{p}_s(\Pi_j(t,x))\, dt\, P^n\mu(dx,di) \\ &\ge \frac{1}{4} e^{-\lambda t_*}(1 - e^{-\lambda t_*}) \cdot P^n\mu(B(z_0,r)\times I). \end{aligned}$$

From (4.2.12) and the last inequality, we conclude that

$$\inf_{\mu\in\mathcal{M}_1} \liminf_{n\to\infty} P^n\mu(C_\varepsilon) > 0,$$

which contradicts the fact that $\widetilde{\varepsilon} = \inf \mathcal{E}(P)$ and completes the proof in the first case.

Case II: $\alpha \ge 0$. By (4.2.8) we have $LL_q < 1$. Choose $\eta, \delta, t_* > 0$ such that

$$(1+\eta)(1+\delta)LL_q e^{\alpha t_*} < 1.$$

Finally, choose $\varepsilon_0 > \widetilde{\varepsilon}$ such that

$$\varepsilon = (1+\eta)(1+\delta)LL_q e^{\alpha t_*}\varepsilon_0 < \widetilde{\varepsilon}.$$

By the definition of $\mathcal{E}(P)$ there exists $A \in \mathcal{C}_{\varepsilon_0}$ such that

$$\beta = \inf_{\mu\in\mathcal{M}_1} \liminf_{n\to\infty} P^n\mu(A) > 0. \tag{4.2.14}$$

Without loss of generality we can assume that

$$A = \bigcup_{k=1}^{m} (B(z_k, \varepsilon_0)\times I). \tag{4.2.15}$$

We now define

$$C_\varepsilon = \bigcup_{j\in I} \bigcup_{t\in[0,t_*]} \bigcup_{s\in S} \bigcup_{k=1}^{m} (B(q_s(\Pi_j(t,z_k)),\, \varepsilon)\times I).$$

Fix $\mu \in \mathcal{M}_1$. From (4.2.14) and (4.2.15) it follows that there exists $k(n) \in \{1,\dots,m\}$ such that

$$P^n\mu(B(z_{k(n)}, \varepsilon_0)\times I) \ge \beta/m. \tag{4.2.16}$$

For $x \in B(z_{k(n)}, \varepsilon)$ and $t < t_*$ we define

$$J(x,t) = \{j \in I : \|\Pi_j(t,x) - \Pi_j(t, z_{k(n)})\| \le (1+\delta)Le^{\alpha t}\|x - z_{k(n)}\|\},$$

$$S(x,t,j) = \{s \in S : \|q_s(\Pi_j(t,x)) - q_s(\Pi_j(t, z_{k(n)}))\| \le (1+\eta)L_q\|\Pi_j(t,x) - \Pi_j(t, z_{k(n)})\|\}.$$

Analysis similar to the first case shows that

$$\sum_{j\in J(x,t)} p_{ij}(x) \ge \frac{\delta}{1+\delta} \quad \text{for } i \in I, \qquad \sum_{s\in S(x,t,j)} \overline{p}_s(\Pi_j(t,x)) \ge \frac{\eta}{1+\eta}.$$

Fix $x \in B(z_{k(n)}, \varepsilon_0)$ and $t < t_*$. Set $J_1 = J(x,t)$. Let $j \in J_1$. Then for every $s \in S_1 = S(x,t,j)$ we have

$$\begin{aligned}\|q_s(\Pi_j(t,x)) - q_s(\Pi_j(t, z_{k(n)}))\| &\le (1+\eta)L_q(1+\delta)Le^{\alpha t}\|x - z_{k(n)}\| \\ &\le (1+\eta)L_q(1+\delta)Le^{\alpha t_*}\varepsilon_0 = \varepsilon.\end{aligned}$$

Thus $(q_s(\Pi_j(t,x)), j) \in C_\varepsilon$ and

$$\begin{aligned}P^{n+1}\mu(C_\varepsilon) &\ge \int\limits_{B(z_{k(n)},\varepsilon_0)\times I} \int\limits_0^{t_*} \sum_{j\in J_1}\sum_{s\in S_1} \lambda e^{-\lambda t} p_{ij}(x)\overline{p}_s(\Pi_j(t,x))\, dt\, P^n\mu(dx, di) \\ &\ge \frac{\eta\delta}{(1+\eta)(1+\delta)}\,(1 - e^{-\lambda t_*})P^n\mu(B(z_{k(n)}, \varepsilon_0) \times I).\end{aligned}$$

Combining this with (4.2.16) gives

$$\liminf_{n\to\infty} P^n\mu(C_\varepsilon) \ge \frac{\eta\delta\beta}{(1+\eta)(1+\delta)m}\,(1 - e^{-\lambda t_*}),$$

but $\mu \in \mathcal{M}_1$ was arbitrary and $\varepsilon < \widetilde{\varepsilon}$, which is impossible. ∎

The next result gives sufficient conditions for asymptotic stability:

Theorem 4.2.2. *Under the hypotheses of Theorem* 4.2.1, *suppose that moreover condition* (4.2.5) *is satisfied and for* α *given in* (4.2.3) *one of the following holds:*

(i) $\alpha < 0$ *and there exists* $s_0 \in S$ *such that*

$$\inf_{x\in Y} \overline{p}_{s_0}(x) > 0, \tag{4.2.17}$$

(ii) $\alpha \ge 0$ *and for every* $s \in S$,

$$\inf_{x\in Y} \overline{p}_s(x) > 0. \tag{4.2.18}$$

Then the operator P *given by* (4.2.6) *is asymptotically stable.*

Proof. By Theorem 4.2.1 the operator P admits an invariant measure. By virtue of Theorem 3.2.1 it is sufficient to show that for given $\varepsilon > 0$ there exists $\theta > 0$ such that for any two measures $\mu_1, \mu_2 \in \mathcal{M}_1$, there exist a Borel measurable set $A \subset Y \times I$ with $\operatorname{diam}_{\overline{\varrho}_\varphi} A < \varepsilon$ and an integer $\tilde{n}$ such that

$$P^{\tilde{n}}\mu_k(A) \ge \theta \quad \text{for } k = 1, 2.$$

By Proposition 3.2.3 the set $\mathcal{L}(\mathcal{M}_1)$ is tight. Thus there exists a compact set $F \subset Y \times I$ such that

$$\mu(F) \ge 4/5 \quad \text{for every } \mu \in \mathcal{L}(\mathcal{M}_1).$$

We consider two cases: $\alpha < 0$ and $\alpha \ge 0$.

Case I: $\alpha < 0$. Set

$$\gamma = \inf_{x\in Y} \overline{p}_{s_0}(x) \quad \text{and} \quad \sigma = \inf_{x\in Y,\, i\in I} p_{ii_0}(x),$$

where s_0 is such that (4.2.17) holds and i_0 is given by condition (4.2.5). Obviously $\gamma > 0$ and $\sigma > 0$. Let $\varepsilon > 0$ be fixed. Choose $t_* \in \mathbb{R}_+$ such that

$$\frac{LL_q}{\sigma\gamma}\, e^{\alpha t_*} \operatorname{diam}_{\overline{\varrho}} F < \frac{\varepsilon}{2}, \tag{4.2.19}$$

where L, L_q are given by conditions (4.2.3) and (4.2.4), respectively. Define

$$F_Y = \{x \in Y : (x, i) \in F \text{ for some } i \in I\}.$$

Clearly F_Y is a compact subset of Y.

Since $q_s : Y \to Y$, $s \in S$, and $\Pi_i : \mathbb{R}_+ \times Y \to Y$, $i \in I$, are continuous, there exists $\overline{t} > t_*$ such that

$$\|q_s(\Pi_i(t, x)) - q_s(\Pi_i(t_*, x))\| < \varepsilon/8 \quad \text{for all } i \in I,\ s \in S,\ x \in F_Y,\ t \in [t_*, \overline{t}]. \tag{4.2.20}$$

Now for $x \in F_Y$ we set

$$O(x) = \{z \in F_Y : \|q_s(\Pi_i(t_*, z)) - q_s(\Pi_i(t_*, x))\| < \varepsilon/8 \text{ for } s \in S,\ i \in I\}. \tag{4.2.21}$$

Let $z_1, \ldots, z_{m_0} \in F_Y$ be such that $F \subset G$, where

$$G = \bigcup_{l=1}^{m_0} (O(z_l) \times I).$$

Note that G is an open subset of $Y \times I$. Let $\mu_1, \mu_2 \in \mathcal{M}_1$ be arbitrary. Set $\mu = (\mu_1 + \mu_2)/2$. Since $\mathcal{L}(\mu) \neq \emptyset$ (see Proposition 3.2.3), there exists a subsequence $\{n_k\}$ of $\{n\}$ and a measure $\nu \in \mathcal{L}(\mu)$ such that $P^{n_k}\mu \to \nu$ (weakly). Since $\nu(G) \geq 4/5$, the Aleksandrov theorem implies

$$\liminf_{k\to\infty} P^{n_k}\mu(G) \geq \nu(G) \geq 4/5.$$

It follows that there exists $n_0 \in \mathbb{N}$ such that

$$P^{n_0}\mu(G) = (P^{n_0}\mu_1(G) + P^{n_0}\mu_2(G))/2 \geq 3/4$$

and consequently $P^{n_0}\mu_k(G) \geq 1/2$ for $k = 1, 2$. Therefore there exist $l_1, l_2 \in \{1, \ldots, m_0\}$ and $i_1, i_2 \in I$ such that

$$P^{n_0}\mu_k(\mathcal{V}_k) \geq 1/(2m_0 N) \quad \text{for } k = 1, 2, \tag{4.2.22}$$

where

$$\mathcal{V}_k = O(z_{l_k}) \times \{i_k\}, \quad k = 1, 2.$$

From (4.2.3) and (4.2.5) it follows that

$$\|\Pi_{i_0}(t_*, z_{l_1}) - \Pi_{i_0}(t_*, z_{l_2})\| \leq \frac{L}{\sigma}\, e^{\alpha t_*} \|z_{l_1} - z_{l_2}\|. \tag{4.2.23}$$

Set

$$w_1 = \Pi_{i_0}(t_*, z_{l_1}), \quad w_2 = \Pi_{i_0}(t_*, z_{l_2}),$$

where i_0 is given by condition (4.2.5). Moreover, from condition (4.2.4) it follows that

$$\|q_{s_0}(w_1) - q_{s_0}(w_2)\| \leq \frac{L_q}{\gamma}\, \|w_1 - w_2\|. \tag{4.2.24}$$

By (4.2.19), (4.2.23) and (4.2.24) we have

$$\|q_{s_0}(w_1)-q_{s_0}(w_2)\| \le \frac{L_q}{\gamma}\,\|w_1-w_2\| \le \frac{L_qL}{\sigma\gamma}\,e^{\alpha t_*}\|z_{l_1}-z_{l_2}\| \le \frac{\varepsilon}{2}.$$

Define

$$A=(B(q_{s_0}(w_1),\varepsilon/4)\cup B(q_{s_0}(w_2),\varepsilon/4))\times\{i_0\}$$

and observe that $\operatorname{diam}_{\overline{\varrho}_\varphi} A<\varepsilon$.

For $x\in O(z_{l_1})$ and $t\in[t_*,\overline{t}]$, using (4.2.20) and (4.2.21), we have

$$\begin{aligned}\|q_{s_0}(\Pi_{i_0}(t,x))-q_{s_0}(w_1)\| &\le \|q_{s_0}(\Pi_{i_0}(t,x))-q_{s_0}(\Pi_{i_0}(t_*,x))\|\\ &\quad+\|q_{s_0}(\Pi_{i_0}(t_*,x))-q_{s_0}(\Pi_{i_0}(t_*,z_{l_1}))\| \le \varepsilon/8+\varepsilon/8=\varepsilon/4.\end{aligned}$$

This gives

$$(q_{s_0}(\Pi_{i_0}(t,x)),\,i_0)\in A \quad \text{for } x\in O(z_{l_1}),\ t\in[t_*,\overline{t}]. \tag{4.2.25}$$

Similarly,

$$(q_{s_0}(\Pi_{i_0}(t,x)),i_0)\in A \quad \text{for } x\in O(z_{l_2}),\ t\in[t_*,\overline{t}].$$

By (4.2.5), (4.2.6), (4.2.22) and (4.2.25) we have

$$\begin{aligned}P^{n_0+1}\mu_k(A)&=\sum_{j\in I}\sum_{s\in S}\int_{Y\times I}\int_0^\infty 1_A(q_s(\Pi_j(t,x)),j)\lambda e^{-\lambda t}p_{ij}(x)\overline{p}_s(\Pi_j(t,x))\,dt\,P^{n_0}\mu_k(dx,di)\\ &\ge\int_{\mathcal{V}_k}\int_{t_*}^{\overline{t}} 1_A(q_{s_0}(\Pi_{i_0}(t,x)),i_0)\lambda e^{-\lambda t}p_{ii_0}(x)\overline{p}_{s_0}(\Pi_{i_0}(t,x))\,dt\,P^{n_0}\mu_k(dx,di)\ge\theta,\end{aligned}$$

where $\theta=\gamma\sigma(e^{-\lambda t_*}-e^{-\lambda\overline{t}})/(2m_0N)$ and $k=1,2$. Since the constant θ does not depend on μ_k for $k=1,2$, the proof in the first case is complete.

Case II: $\alpha\ge0$. We introduce some further notations. Namely, for $\mathbf{s}\in S^n$, $\mathbf{i}\in I^n$ and $\boldsymbol{\tau}\in\mathbb{R}_+^n$ (i.e. $\mathbf{s}=(s_1,\dots,s_n)$, $\boldsymbol{\tau}=(\tau_1,\dots,\tau_n)$ and $\mathbf{i}=(i_1,\dots,i_n)$), we set

$$\begin{aligned}&\mathbf{q_s}=q_{s_n}\circ\cdots\circ q_{s_1},\\ &(\mathbf{q_s}\circ\Pi_{\mathbf{i}})(\boldsymbol{\tau},x)=q_{s_n}(\Pi_{i_n}(\tau_n,q_{s_{n-1}}(\Pi_{i_{n-1}}(\tau_{n-1},\dots,\Pi_{i_1}(\tau_1,x))))),\\ &\mathbf{d}\boldsymbol{\tau}=d\tau_1\cdots d\tau_n,\quad \mathbf{ds}=ds_1\cdots ds_n.\end{aligned}$$

Next, for $n\ge2$, consider the probabilities $\mathcal{P}_n:Y\times I^{n+1}\times\mathbb{R}_+^{n-1}\times S^{n-1}\to[0,1]$ and $\overline{\mathcal{P}}_n:Y\times I^n\times\mathbb{R}_+^n\times S^n\to[0,1]$ given by

$$\begin{aligned}&\mathcal{P}_n(x,i,i_1,\dots,i_{n-1},i_n,\tau_1,\dots,\tau_{n-1},s_1,\dots,s_{n-1})\\ &\qquad=p_{ii_1}(x)p_{i_1i_2}(q_{s_1}(\Pi_{i_1}(\tau_1,x)))\cdot\ldots\cdot p_{i_{n-1}i_n}((\mathbf{q_s}\circ\Pi_{\mathbf{i}})(\boldsymbol{\tau},x))\end{aligned}$$

and

$$\begin{aligned}&\overline{\mathcal{P}}_n(x,i_1,\dots,i_{n-1},i_n,\tau_1,\dots,\tau_{n-1},\tau_n,s_1,\dots,s_{n-1},s_n)\\ &\qquad=\overline{p}_{s_1}(\Pi_{i_1}(\tau_1,x))\overline{p}_{s_2}(\Pi_{i_2}(\tau_2,q_{s_2}(\Pi_{i_1}(\tau_1,x))))\cdot\ldots\cdot\overline{p}_{s_n}(\Pi_{i_n}(\tau_n,\mathbf{q_s}\circ\Pi_{\mathbf{i}}(\boldsymbol{\tau},x))),\end{aligned}$$

where $\mathbf{s}=(s_1,\dots,s_{n-1})$, $\boldsymbol{\tau}=(\tau_1,\dots,\tau_{n-1})$, $\mathbf{i}=(i_1,\dots,i_{n-1})$.

Since $\alpha\ge0$, condition (4.2.8) implies that $L_q<1$. Let $n\in\mathbb{N}$ be such that

$$L_q^n\cdot\operatorname{diam}_{\overline{\varrho}}F<\varepsilon/2. \tag{4.2.26}$$

By continuity and compactness there exists $\delta > 0$ such that

$$\|(\mathbf{q_s} \circ \Pi_{\mathbf{i}})(\boldsymbol{\tau}, x) - \mathbf{q_s}(x)\| < \varepsilon/8 \tag{4.2.27}$$

for every $\mathbf{i} \in I^n$, $\mathbf{s} \in S^n$, $\boldsymbol{\tau} \in [0,\delta]^n$ and $x \in F_Y$, where

$$F_Y = \{x \in Y : (x,i) \in F \text{ for some } i \in I\}.$$

Given $x \in Y$, define

$$O(x) = \{z \in F_Y : \|\mathbf{q_s}(x) - \mathbf{q_s}(z)\| < \varepsilon/8 \text{ for } \mathbf{s} \in S^n\}. \tag{4.2.28}$$

Clearly, $O(x)$ is an open neighborhood of x. Let $z_1, \ldots, z_{m_0} \in F_Y$ be such that $F \subset G$ where

$$G = \bigcup_{l=1}^{m_0} (O(z_l) \times I).$$

Let $\mu_1, \mu_2 \in \mathcal{M}_1$. Set $\mu = (\mu_1 + \mu_2)/2$. Since $\mathcal{L}(\mu) \neq \emptyset$ (see Proposition 3.2.3) there exists a subsequence $\{n_k\}$ of $\{n\}$ and a measure $\nu \in \mathcal{L}(\mu)$ such that $P^{n_k}\mu \to \nu$ (weakly). As in Case I there exist $n_0 \in \mathbb{N}$, $l_1, l_2 \in \{1, \ldots, m_0\}$ and $i_1, i_2 \in I$ such that

$$P^{n_0}\mu_k(\mathcal{V}_k) \geq 1/(2m_0N) \quad \text{for } k = 1,2, \tag{4.2.29}$$

where

$$\mathcal{V}_k = O(z_{l_k}) \times \{i_k\}, \quad k = 1,2.$$

Set $\tilde{z}_1 = z_{l_1}$ and $\tilde{z}_2 = z_{l_2}$. By (4.2.4) there exists $s_{0,0} \in S$ such that

$$\|q_{s_{0,0}}(\tilde{z}_1) - q_{s_{0,0}}(\tilde{z}_2)\| \leq L_q\|\tilde{z}_1 - \tilde{z}_2\|.$$

Next, for $q_{s_{0,0}}(\tilde{z}_1)$ and $q_{s_{0,0}}(\tilde{z}_2)$ we choose $s_{0,1} \in S$ such that

$$\|q_{s_{0,1}}(q_{s_{0,0}}(\tilde{z}_1)) - q_{s_{0,1}}(q_{s_{0,0}}(\tilde{z}_2))\| \leq L_q\|q_{s_{0,0}}(\tilde{z}_1) - q_{s_{0,0}}(\tilde{z}_2)\|$$

and so on. Thus there exists $\mathbf{s}_0 = (s_{0,0}, \ldots, s_{0,n-1}) \in S^n$ such that

$$\|\mathbf{q}_{\mathbf{s}_0}(\tilde{z}_1) - \mathbf{q}_{\mathbf{s}_0}(\tilde{z}_2)\| \leq L_q^n\|\tilde{z}_1 - \tilde{z}_2\|. \tag{4.2.30}$$

Define

$$A = (B(\mathbf{q}_{\mathbf{s}_0}(\tilde{z}_1), \varepsilon/4) \cup B(\mathbf{q}_{\mathbf{s}_0}(\tilde{z}_2), \varepsilon/4)) \times \{i_0\},$$

where i_0 is given by condition (4.2.5). From (4.2.26) and (4.2.30) it follows that $\operatorname{diam}_{\overline{\varrho}_\varphi} A < \varepsilon$. For $x \in O(\tilde{z}_l)$, $l = 1,2$, $\mathbf{i} \in I^n$ and $\boldsymbol{\tau} \in [0,\delta]^n$, by (4.2.27), (4.2.28), we have

$$\|(\mathbf{q}_{\mathbf{s}_0} \circ \Pi_{\mathbf{i}})(\boldsymbol{\tau}, x) - \mathbf{q}_{\mathbf{s}_0}(\tilde{z}_l)\| \leq \|(\mathbf{q}_{\mathbf{s}_0} \circ \Pi_{\mathbf{i}})(\boldsymbol{\tau}, x) - \mathbf{q}_{\mathbf{s}_0}(x)\| + \|\mathbf{q}_{\mathbf{s}_0}(x) - \mathbf{q}_{\mathbf{s}_0}(\tilde{z}_l)\| \leq \varepsilon/4.$$

This gives

$$((\mathbf{q}_{\mathbf{s}_0} \circ \Pi_{\mathbf{i}})(\boldsymbol{\tau}, x), i_0) \in A \quad \text{for } x \in O(\tilde{z}_l),\ \mathbf{i} \in I^n,\ l = 1,2 \text{ and } \boldsymbol{\tau} \in [0,\delta]^n.$$

Combining this with (4.2.5), (4.2.6), and (4.2.29), we obtain

$$P^{n_0+n}\mu_k(A) = \sum_{\mathbf{j}=(j_1,\dots,j_n)\in I^n} \int_{Y\times I}\int_{\mathbb{R}_+^n} \sum_{\mathbf{s}=(s_1,\dots,s_n)\in S^n} 1_A((\mathbf{q}_s\circ\Pi_{\mathbf{j}})(\boldsymbol{\tau},x),j_n)$$

$$\cdot\,\mathcal{P}_n(x,i,\mathbf{j},\tau_1,\dots,\tau_{n-1},s_1,\dots,s_{n-1})\cdot\overline{\mathcal{P}}_n(x,\mathbf{j},\boldsymbol{\tau},\mathbf{s})\lambda e^{-\lambda(\tau_1+\dots+\tau_n)}\,\mathbf{d}\boldsymbol{\tau}\,P^{n_0}\mu_k(dx,di)$$

$$\ge(\gamma\sigma)^n\int_{\mathcal{V}_k}\int_{[0,\delta]^n}\lambda^n e^{-\lambda(\tau_1+\dots+\tau_n)}1_A((\mathbf{q}_{\mathbf{s}_0}\circ\Pi_{\mathbf{i}_0})(\boldsymbol{\tau},x),i_0)\,\mathbf{d}\boldsymbol{\tau}\,P^{n_0}\mu_k(dx,di)$$

$$\ge(\gamma\sigma)^n\Big(\int_0^\delta\lambda e^{-\lambda\tau}\,d\tau\Big)^n P^{n_0}\mu_k(\mathcal{V}_k)\ge\frac{(\gamma\sigma)^n(1-e^{-\lambda\delta})^n}{2m_0N}\quad\text{for } k=1,2,$$

where $\mathbf{i}_0=(i_0,\dots,i_0)\in I^n$,

$$\gamma=\inf_{x\in Y,\,s\in S}\overline{p}_s(x),\qquad\sigma=\inf_{x\in Y,\,i\in I}p_{ii_0}(x),$$

and consequently the right-hand side does not depend on μ_k for $k=1,2$. ■

We conclude this section with a result describing the asymptotic behavior of distributions of the process $\{x_n\}_{n\ge0}$ on the space $(Y,\|\cdot\|)$. Let $\widetilde{\mu}$ be the distribution of the initial random vector x_0 and $\widetilde{\mu}_n$ the distribution of x_n , i.e.

$$\widetilde{\mu}_n(A)=\mathbb{P}(x_n\in A)\quad\text{for } A\in\mathcal{B}(Y)\text{ and } n\ge1. \tag{4.2.31}$$

THEOREM 4.2.3. *Under the hypotheses of Theorem* 4.2.2, *there exists a measure* $\widetilde{\mu}_0\in\mathcal{M}_1(Y)$ *such that for every* $\widetilde{\mu}$ *the sequence* $\{\widetilde{\mu}_n\}_{n\ge1}$ *given by* (4.2.31) *converges weakly to* $\widetilde{\mu}_0$. *Furthermore*, *if the initial vector* x_0 *is distributed according to* $\widetilde{\mu}_0$, *that is*,

$$\mathbb{P}(x_0\in A)=\widetilde{\mu}_0(A)\quad\textit{for } A\in\mathcal{B}(Y),$$

then $\widetilde{\mu}_n(A)=\widetilde{\mu}_0(A)$ *for* $A\in\mathcal{B}(Y)$ *and* $n\ge1$.

Proof. By Theorem 4.2.2 the operator P given by (4.2.6) is asymptotically stable. Thus there exists an invariant measure $\mu_0\in\mathcal{M}_1(Y\times I)$ such that

$$\lim_{n\to\infty}\langle\overline{f},\overline{\mu}_n\rangle=\langle\overline{f},\mu_0\rangle\quad\text{for }\overline{f}\in C(Y\times I), \tag{4.2.32}$$

where $\overline{\mu}_{n+1}=P\overline{\mu}_n$, $n=1,2,\dots$. For every function $f\in C(Y)$ we define the sequence of functions $\overline{f}_j:Y\times I\to\mathbb{R}$, $j\in I$, by the formula

$$\overline{f}_j(x,i)=\begin{cases}f(x) & \text{for } i=j,\\ 0 & \text{for } i\ne j.\end{cases}$$

It is evident that $\overline{f}_j\in C(Y\times I)$ for every $j\in I$. From (4.2.32) it follows that

$$\lim_{n\to\infty}\sum_{j\in I}\int_{Y\times I}\overline{f}_j(x,i)\,\overline{\mu}_n(dx,di)=\sum_{j\in I}\int_{Y\times I}\overline{f}_j(x,i)\,\mu_0(dx,di)$$

and consequently

$$\lim_{n\to\infty}\sum_{j\in I}\int_Y f(x)\,\overline{\mu}_n(dx\times\{j\})=\sum_{j\in I}\int_Y f(x)\,\mu_0(dx\times\{j\}).$$

Since $\overline{\mu}_n$ is the distribution of (x_n,ξ_n), from (4.2.31) we have

$$\widetilde{\mu}_n(A)=\mathbb{P}(x_n\in A)=\mathbb{P}((x_n,\xi_n)\in A\times I)=\overline{\mu}_n(A\times I)\quad\text{for } A\in\mathcal{B}(Y).$$

Taking

$$\widetilde{\mu}_0(A) = \mu_0(A \times I) \quad \text{for } A \in \mathcal{B}(Y),$$

we obtain

$$\lim_{n\to\infty} \langle f, \widetilde{\mu}_n \rangle = \langle f, \widetilde{\mu}_0 \rangle \quad \text{for } f \in C(Y),$$

as required. Furthermore,

$$\widetilde{\mu}_n(A) = \overline{\mu}_n(A \times I) = P^n\mu_0(A \times I) = \mu_0(A \times I) = \widetilde{\mu}_0(A) \quad \text{for } A \in \mathcal{B}(Y). \ \blacksquare$$

4.3. Continuous-time random dynamical systems. In this section we study the asymptotic behavior of the distributions of stochastic processes $\{(X(t), \xi(t))\}_{t\ge 0}$ and $\{X(t)\}_{t\ge 0}$, where the choice of a jump does not depend on the position in which it takes place. We additionally replace $S = \{1, \dots, K\}$ with a compact space. In [19] the authors considered continuous-time dynamical systems on Polish spaces, but without jumps.

Let $(Y, \|\cdot\|)$ be a separable Banach space and Θ be a compact metric space. Let $\{\eta_n\}_{n\ge 0}$ be a sequence of identically distributed random elements $\eta_n : \Omega \to \Theta$, $n \in \mathbb{N}$. We assume that $\{\eta_n\}_{n\ge 0}$ is independent of $\{t_n\}_{n\ge 0}$ and we denote by ν the distribution of η_n, i.e. $\nu(A) = \mathbb{P}(\eta_n^{-1}(A))$, $n \in \mathbb{N}$, $A \in \mathcal{B}(\Theta)$.

Let $q : Y \times \Theta \to Y$ be a continuous transformation such that

$$\|q(x,\cdot) - q(y,\cdot)\|_{L^1(\nu)} \le L_q\|x-y\| \quad \text{for all } x, y \in Y \tag{4.3.1}$$

with a constant $L_q \ge 0$, and

$$\tilde{c} = \int_\Theta \|q(0,\theta)\|\,\nu(d\theta) < \infty. \tag{4.3.2}$$

In the remainder of this section we require, as in Section 4.2, that $\Pi_i : \mathbb{R}_+ \times Y \to Y$, $i \in I$, are continuous semidynamical systems, $[p_i]_{i\in I}$ is a probability vector, $[p_{ij}]_{i,j\in I}$ is a probability matrix, (4.2.2) and (4.2.3) hold, the latter with the constants L, α such that

$$LL_q + \alpha/\lambda < 1, \tag{4.3.3}$$

where L_q is now given in (4.3.1) and λ is the intensity of the Poisson process which governs the increment Δt_n of the random variables $\{t_n\}_{n\ge 0}$. Finally, instead of (4.2.5) we assume that

$$\sigma = \inf\{p_{ij}(x) : i, j \in I,\ x \in Y\} > 0. \tag{4.3.4}$$

In this section we study the stochastic process $\{(X(t), \xi(t))\}_{t\ge 0}$, $(X(t), \xi(t)) : \Omega \to Y \times I$, given by

$$(X(t), \xi(t)) = (\Pi_{\xi_{n-1}}(t - t_{n-1}, x_{n-1}), \xi_{n-1}) \quad \text{for } t_{n-1} \le t < t_n,\ n = 1, 2, \dots,$$

where

$$x_n = q(\Pi_{\xi_{n-1}}(t_n - t_{n-1}, x_{n-1}), \eta_n),$$
$$\mathbb{P}(\xi_0 = i \mid x_0 = x) = p_i(x),$$
$$\mathbb{P}(\xi_n = k \mid x_n = x,\ \xi_{n-1} = i) = p_{ik}(x) \quad \text{for } n = 1, 2, \dots.$$

The semigroup $\{P^t\}_{t\geq 0}$ generated by this process is given by

$$\langle P^t\mu, f\rangle = \langle \mu, T^t f\rangle \quad \text{for } f\in C(Y\times I),\ \mu\in\mathcal{M}_1 \text{ and } t\geq 0, \tag{4.3.5}$$

where

$$T^t f(x,i) = E(f((X(t),\xi(t))_{(x,i)})) \quad \text{for } f\in C(Y\times I). \tag{4.3.6}$$

(E denotes the mathematical expectation on $(\Omega,\Sigma,\mathbb{P})$.)

THEOREM 4.3.1. *Assume that the system* (Π, q, p) *satisfies conditions* (4.3.1)–(4.3.4) *and that there exists a constant* $\beta > 0$ *such that*

$$\|\Pi_i(t,x) - x\| \leq \beta t \quad \textit{for } i\in I,\ t>0,\ x\in Y. \tag{4.3.7}$$

Then the semigroup $\{P^t\}_{t\geq 0}$ *given by* (4.3.5), (4.3.6) *is asymptotically stable.*

The proof of Theorem 4.3.1 is quite long and will be presented later on in this section. We now proceed with the following observation:

REMARK 4.3.1. Note that if Θ is equal to the finite set $S=\{1,\dots,K\}$ with the discrete topology then by setting $q_s(x) = q(x,s)$ and $\overline{p}_s(x) = \nu(\{s\})$, $x\in Y, s\in\Theta$, the system (Π,q,p) under consideration corresponds to the one in Section 4.2 with the probability vector $\overline{p}_s$ independent of x. Condition (4.2.4) is then equivalent to (4.3.1), the second inequality in (4.2.2) is trivially satisfied, and (4.2.1) follows from (4.3.1) and (4.3.7).

We next show that condition (4.3.3) in Theorem 4.3.1 is essential.

EXAMPLE 4.3.1. Let $I=\Theta=\{1\}$ and $\Pi(t,x)=x$, $q(x,1)=-x$ for $x\in Y$ and $t\geq 0$. Then $L=1$, $L_q=1$ and $\alpha=0$, so

$$LL_q + \alpha/\lambda = 1.$$

For arbitrary initial x_0 we obtain $x_n = -x_{n-1}$ for every $n\in\mathbb{N}$. By (4.1.4) we obtain

$$X(t) = x_{n-1} \quad \text{for } t_{n-1}\leq t < t_n.$$

Thus the semigroup $\{P^t\}_{t\geq 0}$ generated by this process is not asymptotically stable.

Several lemmas are needed for the proof of Theorem 4.3.1. The general idea of the proof is as follows. First, we show that the semigroup $\{P^t\}_{t\geq 0}$ is nonexpansive. Second, we prove that for some $t_*>0$ the Markov operator P^{t_*} is semi-concentrating. Finally, we show that the operator P^{t_*} satisfies a lower bound condition, which by Theorem 3.2.1 implies that the semigroup $\{P^t\}_{t\geq 0}$ is asymptotically stable.

We start with some useful notation. For $n\in\mathbb{N}$ consider the function $\overline{\Pi}_n : Y\times I^n\times\mathbb{R}_+^n\times\Theta^n\to Y$ defined by the recurrent formula

$$\begin{aligned}
&\overline{\Pi}_1(x,i,s_1,\theta_1) = q(\Pi_i(s_1,x),\theta_1);\\
&\overline{\Pi}_n(x,i,i_1,\dots,i_{n-1},s_1,\dots,s_n,\theta_1,\dots,\theta_n)\\
&\qquad = q(\Pi_{i_{n-1}}(s_n,\overline{\Pi}_{n-1}(x,i,i_1,\dots,i_{n-2},s_1,\dots,s_{n-1},\theta_1,\dots,\theta_{n-1})),\theta_n).
\end{aligned} \tag{4.3.8}$$

Next consider the transition probabilities $\mathcal{P}_n : Y \times I^{n+1} \times \mathbb{R}_+^n \times \Theta^n \to [0,1]$ given by

$$\mathcal{P}_n(x, i, i_1, \ldots, i_n, s_1, \ldots, s_n, \theta_1, \ldots, \theta_n) = p_{ii_1}(\overline{\Pi}_1(x, i, s_1, \theta_1)) \cdot \ldots \cdot p_{i_{n-1}i_n}(\overline{\Pi}_n(x, i, i_1, \ldots, i_{n-1}, s_1, \ldots, s_n, \theta_1, \ldots, \theta_n))$$

and the functions $q_n : Y \times \Theta^n \to Y$ given by

$$\begin{aligned} &q_0(x) = x, \quad q_1(x, \theta_1) = q(x, \theta_1), \\ &q_n(x, \theta_1, \ldots, \theta_{n-1}, \theta_n) = q(q_{n-1}(x, \theta_1, \ldots, \theta_{n-1}), \theta_n). \end{aligned} \tag{4.3.9}$$

REMARK 4.3.2. Observe that for every $n \in \mathbb{N}$, $s_1, \ldots, s_n \in \mathbb{R}_+, x \in Y, i, i_1, \ldots, i_{n+1} \in I$ and $\theta_1, \ldots, \theta_{n+1} \in \Theta$ we have

$$\mathcal{P}_{n+1}(x, i, i_1, \ldots, i_{n+1}, s_1, \ldots, s_{n+1}, \theta_1, \ldots, \theta_{n+1}) = \mathcal{P}_n(\overline{\Pi}_1(x, i, s_1, \theta_1), i_1, \ldots, i_{n+1}, s_2, \ldots, s_{n+1}, \theta_2, \ldots, \theta_{n+1}) p_{ii_1}(\overline{\Pi}_1(x, i, s_1, \theta_1))$$

and

$$\overline{\Pi}_{n+1}(x, i, i_1, \ldots, i_n, s_1, \ldots, s_{n+1}, \theta_1, \ldots, \theta_{n+1}) = \overline{\Pi}_n(\overline{\Pi}_1(x, i, s_1, \theta_1), i_1, \ldots, i_n, s_2, \ldots, s_{n+1}, \theta_2, \ldots, \theta_{n+1}). \tag{4.3.10}$$

Finally, given a function $f : Y \times I \to \mathbb{R}$ we consider the function $f_n : Y \times I \times \mathbb{R}_+^{n+1} \to \mathbb{R}$ defined by

$$\begin{aligned} f_n(x, i, s_1, \ldots, s_{n+1}) &= \underbrace{\int_\Theta \ldots \int_\Theta}_{n} \sum_{i_1, \ldots, i_n = 1}^{N} f(\Pi_{i_n}(s_{n+1}, \overline{\Pi}_n(x, i, i_1, \ldots, i_{n-1}, s_1, \ldots, s_n, \theta_1, \ldots, \theta_n)), i_n) \\ &\quad \cdot \mathcal{P}_n(x, i, i_1, \ldots, i_n, s_1, \ldots, s_n, \theta_1, \ldots, \theta_n) \, \nu(d\theta_1) \ldots \nu(d\theta_n). \end{aligned} \tag{4.3.11}$$

By Remark 4.3.2 we have

$$f_{n+1}(x, i, s_1, \ldots, s_{n+2}) = \int_\Theta \sum_{i_1=1}^{N} p_{ii_1}(\overline{\Pi}_1(x, i, s_1, \theta_1)) \cdot f_n(\overline{\Pi}_1(x, i, s_1, \theta_1), i_1, s_2, \ldots, s_{n+2}) \, \nu(d\theta_1). \tag{4.3.12}$$

For the convenience of the reader sometimes we will write $f_n(x, i, \mathbf{s})$ instead of $f_n(x, i, s_1, \ldots, s_{n+1})$ where $\mathbf{s} = (s_1, \ldots, s_{n+1})$ and analogously $\overline{\Pi}_n(x, \mathbf{i}, \mathbf{s}, \boldsymbol{\theta})$ instead of $\overline{\Pi}_n(x, i_1, \ldots, i_n, s_1, \ldots, s_n, \theta_1, \ldots, \theta_n)$ where $\mathbf{i} = (i_1, \ldots, i_n)$, $\mathbf{s} = (s_1, \ldots, s_n)$ and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$.

We begin with some technical estimates:

LEMMA 4.3.1. *Assume that conditions* (4.3.1), (4.3.3) *and* (4.3.4) *are satisfied. Let* $\varphi \in \Phi$ *and* $f \in \mathcal{F}_\varphi$. *Then for every* $n \in \mathbb{N}$, $i \in I$, $x, y \in Y$ *and* $s_1, \ldots, s_{n+1} \in \mathbb{R}_+$ *we have*

$$\begin{aligned} &|f_n(x, i, s_1, \ldots, s_{n+1}) - f_n(y, i, s_1, \ldots, s_{n+1})| \\ &\leq \varphi\biggl(L_q^n \frac{L^{n+1}}{\sigma} e^{\alpha(s_1 + \cdots + s_{n+1})} \|x - y\|\biggr) + \sum_{k=1}^{n} \psi_1\biggl(\frac{(L_q L)^k}{\sigma} e^{\alpha(s_1 + \cdots + s_k)} \|x - y\|\biggr). \end{aligned} \tag{4.3.13}$$

Proof. We first show that for every $n \in \mathbb{N}$, $i \in I$, $x, y \in Y$ and $s_1, \ldots, s_{n+1} \in \mathbb{R}_+$,

$$
\begin{aligned}
(4.3.14)\qquad & |f_n(x,i,s_1,\ldots,s_{n+1}) - f_n(y,i,s_1,\ldots,s_{n+1})| \\
&\quad \le \varphi((L_qL)^n e^{\alpha(s_2+\cdots+s_{n+1})}\|\Pi_i(s_1,x)-\Pi_i(s_1,y)\|) \\
&+ \sum_{k=2}^{n} \psi_1(L_q^k L^{k-1} e^{\alpha(s_2+\cdots+s_k)}\|\Pi_i(s_1,x)-\Pi_i(s_1,y)\|) + \psi_1(L_q\|\Pi_i(s_1,x)-\Pi_i(s_1,y)\|).
\end{aligned}
$$

(Here we assume that for $n<2$ the sum $\sum_{k=2}^{n}\ldots$ is equal to zero.)

The proof is by induction on n. Let $x, y \in Y, i \in I$ and $s_1, \ldots, s_{n+1} \in \mathbb{R}_+$ be fixed. Set

$$
u_1 = \Pi_i(s_1,x) \quad \text{and} \quad y_1 = \Pi_i(s_1,y).
$$

First we consider the case of $n=1$. Combining (4.2.2), (4.2.3) and (4.3.1), (4.3.8), (4.3.11), Jensen's inequality and the fact that $|f| \le 1$, we obtain

$$
\begin{aligned}
&|f_1(x,i,s_1,s_2) - f_1(y,i,s_1,s_2)| \\
&\quad \le \int_\Theta \sum_{i_1=1}^{N} |f(\Pi_{i_1}(s_2,q(u_1,\theta_1)),i_1) - f(\Pi_{i_1}(s_2,q(y_1,\theta_1)),i_1)| \cdot p_{ii_1}(q(u_1,\theta_1))\,\nu(d\theta_1) \\
&\qquad + \int_\Theta \sum_{i_1=1}^{N} |p_{ii_1}(q(u_1,\theta_1)) - p_{ii_1}(q(y_1,\theta_1))|\,\nu(d\theta_1) \\
&\quad \le \int_\Theta \sum_{i_1=1}^{N} \varphi(\|\Pi_{i_1}(s_2,q(u_1,\theta_1)) - \Pi_{i_1}(s_2,q(y_1,\theta_1))\|) p_{ii_1}(q(u_1,\theta_1))\,\nu(d\theta_1) \\
&\qquad + \int_\Theta \psi_1(\|q(u_1,\theta_1) - q(y_1,\theta_1)\|)\,\nu(d\theta_1) \\
&\quad \le \varphi(LL_q e^{\alpha s_2}\|u_1-y_1\|) + \psi_1(L_q\|u_1-y_1\|).
\end{aligned}
$$

Suppose now that inequality (4.3.14) holds for some $k \ge 1$. By virtue of (4.3.11), (4.3.12) and Remark 4.3.2 we have

$$
\begin{aligned}
&|f_{k+1}(x,i,s_1,\ldots,s_{k+2}) - f_{k+1}(y,i,s_1,\ldots,s_{k+2})| \\
&\qquad \le \int_\Theta \sum_{i_1=1}^{N} p_{ii_1}(\overline{\Pi}_1(x,i,s_1,\theta_1))|f_k(\overline{\Pi}_1(x,i,s_1,\theta_1),i_1,s_2,\ldots,s_{k+2}) \\
&\qquad\qquad - f_k(\overline{\Pi}_1(y,i,s_1,\theta_1),i_1,s_2,\ldots,s_{k+2})|\,\nu(d\theta_1) \\
&\qquad\quad + \int_\Theta \sum_{i_1=1}^{N} |p_{ii_1}(\overline{\Pi}_1(x,i,s_1,\theta_1)) - p_{ii_1}(\overline{\Pi}_1(y,i,s_1,\theta_1))|\,\nu(d\theta_1).
\end{aligned}
$$

Further, by (4.3.1), the induction hypothesis, (4.2.2), (4.2.3), and Jensen's inequality we obtain

$$
\begin{aligned}
&|f_{k+1}(x,i,s_1,\ldots,s_{k+2})-f_{k+1}(y,i,s_1,\ldots,s_{k+2})|\\
&\quad\le \int_{\Theta}\sum_{i_1=1}^{N} p_{ii_1}(q(u_1,\theta_1))\\
&\qquad\qquad\times\varphi((L_qL)^k e^{\alpha(s_3+\cdots+s_{k+2})}\|\Pi_{i_1}(s_2,q(u_1,\theta_1))-\Pi_{i_1}(s_2,q(y_1,\theta_1))\|)\,\nu(d\theta_1)\\
&\qquad+\int_{\Theta}\sum_{i_1=1}^{N} p_{ii_1}(q(u_1,\theta_1))\\
&\qquad\times\sum_{j=2}^{k}\psi_1(L_q^jL^{j-1}e^{\alpha(s_3+\cdots+s_{j+1})}\|\Pi_{i_1}(s_2,q(u_1,\theta_1))-\Pi_{i_1}(s_2,q(y_1,\theta_1))\|)\,\nu(d\theta_1)\\
&\qquad+\int_{\Theta}\sum_{i_1=1}^{N} p_{ii_1}(q(u_1,\theta_1))\psi_1(L_q\|\Pi_{i_1}(s_2,q(u_1,\theta_1))-\Pi_{i_1}(s_2,q(y_1,\theta_1))\|)\,\nu(d\theta_1)\\
&\qquad+\int_{\Theta}\psi_1(\|q(u_1,\theta_1)-q(y_1,\theta_1)\|)\,\nu(d\theta_1)\\
&\quad\le \int_{\Theta}\varphi({L_q}^kL^{k+1}e^{\alpha(s_2+\cdots+s_{k+2})}\|q(u_1,\theta_1)-q(y_1,\theta_1)\|)\,\nu(d\theta_1)\\
&\qquad+\int_{\Theta}\sum_{j=2}^{k}\psi_1(L_q^jL^je^{\alpha(s_2+\cdots+s_{j+1})}\|q(u_1,\theta_1)-q(y_1,\theta_1)\|)\,\nu(d\theta_1)\\
&\qquad+\int_{\Theta}\psi_1(L_qLe^{\alpha s_2}\|q(u_1,\theta_1)-q(y_1,\theta_1)\|)\,\nu(d\theta_1)+\psi_1(L_q\|u_1-y_1\|)\\
&\quad\le \varphi((L_qL)^{k+1}e^{\alpha(s_2+\cdots+s_{k+2})}\|u_1-y_1\|)\\
&\qquad+\sum_{j=2}^{k+1}\psi_1(L_q^jL^{j-1}e^{\alpha(s_2+\cdots+s_j)}\|u_1-y_1\|)+\psi_1(L_q\|u_1-y_1\|),
\end{aligned}
$$

which completes the proof of (4.3.14). Since

$$\|u_1-y_1\|=\|\Pi_i(s_1,x)-\Pi_i(s_1,y)\|\le\frac{L}{\sigma}\,e^{\alpha s_1}\|x-y\|,$$

we obtain (4.3.13). ∎

We can now prove the following:

LEMMA 4.3.2. *If conditions* (4.3.1), (4.3.3) *and* (4.3.4) *hold, then there exists* $t_*>0$ *such that for every* $t\ge t_*$ *the operator* P^t *given by* (4.3.5), (4.3.6) *is essentially nonexpansive.*

Proof. By virtue of (4.3.3), we can choose $t_*>0$ such that

$$r_0=\frac{L}{\sigma}\,e^{-(\lambda-\alpha-\lambda LL_q)t_*}<1.$$

Moreover, let $\overline{\psi} : \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$\overline{\psi}(t) = \begin{cases} \lambda t_* \psi_1\Big(\dfrac{LL_q}{\sigma}\, e^{-\lambda(1-LL_q)t_*} t\Big) & \text{if } \alpha < 0, \\ \lambda t_* \psi_1\Big(\dfrac{1}{\sigma}\, e^{\alpha t_*} t\Big) & \text{if } \alpha \ge 0. \end{cases}$$

Since $\overline{\psi} \in \Phi_0$ and satisfies the hypotheses of Proposition 2.1.1, there exists $\varphi \in \Phi$ such that

$$\overline{\psi}(t) + \varphi(r_0 t) \le \varphi(t) \quad \text{for } t \ge 0. \tag{4.3.15}$$

Analogously to the proof of Lemma 4.2.1, choose $c > 0$ such that $\varphi(c) \ge 2$ and consider the corresponding metric $\overline{\varrho}$.

Recall that $\|\cdot\|_\varphi$ is the Fortet–Mourier norm in $\mathcal{M}_1$ given by

$$\|\mu\|_\varphi = \sup\{|\langle f, \mu\rangle| : f \in \mathcal{F}_\varphi\},$$

where $\mathcal{F}_\varphi$ is the set of functions such that $|f| \le 1$ and

$$|f(x,i) - f(y,j)| \le \overline{\varrho}_\varphi((x,i),(y,j)) = \varphi(\overline{\varrho}((x,i),(y,j)))$$

for $x, y \in Y$, $i, j \in I$.

We will prove that P^{t_*} is nonexpansive with respect to the norm $\|\cdot\|_\varphi$. For $n \in \mathbb{N} \cup \{0\}$ we set

$$\Omega_n = \Omega_n(t_*) = \{\omega \in \Omega : t_n(\omega) \le t_* \text{ and } t_{n+1}(\omega) > t_*\}.$$

Obviously, $\mathbb{P}(\bigcup_{n=0}^\infty \Omega_n(t_*)) = 1$ and $\Omega_n(t_*) \cap \Omega_m(t_*) = \emptyset$ for $n \ne m$. Let $f : Y \times I \to \mathbb{R}$ be a bounded continuous function and let $x \in Y$ and $i \in I$ be given. We write $\mathbf{\Delta}_n = (\Delta t_1, \ldots, \Delta t_n)$, where $\Delta t_n = t_n - t_{n-1}$. A simple calculation shows that

$$\begin{aligned} &E(f((X(t_*), \xi(t_*))_{(x,i)})) \\ &\qquad = e^{-\lambda t_*} f(\Pi_i(t_*, x), i) + \sum_{n=1}^\infty \int_{\Omega_n} f_n(x, i, \mathbf{\Delta}_n(\omega), t_* - t_n(\omega))\, \mathbb{P}(d\omega). \end{aligned} \tag{4.3.16}$$

Fix an $f \in \mathcal{F}_\varphi$. Evidently $|T^{t_*} f| \le 1$, so we have to prove that

$$|T^{t_*} f(x,i) - T^{t_*} f(y,j)| \le \overline{\varrho}_\varphi((x,i),(y,j)) \quad \text{for } x, y \in Y \text{ and } i, j \in I. \tag{4.3.17}$$

Since $\varrho_c(i,j) = c$ for $i \ne j$ and $\varphi(c) \ge 2$, condition (4.3.17) is satisfied for $i \ne j$. Now, let $i = j$. We have

$$\begin{aligned} |T^{t_*} f(x,i) - T^{t_*} f(y,i)| &\le E(|f((X(t_*), \xi(t_*))_{(x,i)}) - f((X(t_*), \xi(t_*))_{(y,i)})|) \\ &\le e^{-\lambda t_*} |f(\Pi_i(t_*, x), i) - f(\Pi_i(t_*, y), i)| \\ &\quad + \sum_{n=1}^\infty \int_{\Omega_n} |f_n(x, i, \mathbf{\Delta}_n(\omega), t_* - t_n(\omega)) - f_n(y, i, \mathbf{\Delta}_n(\omega), t_* - t_n(\omega))|\, \mathbb{P}(d\omega). \end{aligned}$$

From this and (4.3.13) we obtain

$$\text{(4.3.18)}\qquad |T^{t_*}f(x,i)-T^{t_*}f(y,i)|\le e^{-\lambda t_*}\varphi(\|\Pi_i(t_*,x)-\Pi_i(t_*,y)\|)$$
$$+\sum_{n=1}^{\infty}\int_{\Omega_n}\Big[\varphi\Big(\frac{L_q^nL^{n+1}}{\sigma}\,e^{\alpha t_*}\|x-y\|\Big)+\sum_{j=1}^{n}\psi_1\Big(\frac{(L_qL)^j}{\sigma}\,e^{\alpha t_j(\omega)}\|x-y\|\Big)\Big]\,\mathbb{P}(d\omega).$$

If $\alpha<0$, then we can assume, without loss of generality, that $LL_q\ge 1$. Thus we obtain

$$|T^{t_*}f(x,i)-T^{t_*}f(y,i)|\le e^{-\lambda t_*}\varphi\Big(\frac{L}{\sigma}\,e^{\alpha t_*}\|x-y\|\Big)$$
$$+e^{-\lambda t_*}\sum_{n=1}^{\infty}\frac{(\lambda t_*)^n}{n!}\Big(\varphi\Big(\frac{L_q^nL^{n+1}}{\sigma}\,e^{\alpha t_*}\|x-y\|\Big)+\sum_{j=1}^{n}\psi_1\Big(\frac{(L_qL)^j}{\sigma}\,\|x-y\|\Big)\Big)$$
$$\le\varphi\Big(\frac{L}{\sigma}\,e^{-(\lambda-\alpha-\lambda L_qL)t_*}\|x-y\|\Big)+e^{-\lambda t_*}\sum_{n=1}^{\infty}\frac{(\lambda t_*)^n}{(n-1)!}\psi_1\Big(\frac{(L_qL)^n}{\sigma}\,\|x-y\|\Big)$$
$$\le\varphi(r_0\|x-y\|)+\overline{\psi}(\|x-y\|).$$

Suppose now that $\alpha\ge 0$. Then $LL_q<1$ and by (4.3.18) we have

$$|T^{t_*}f(x,i)-T^{t_*}f(y,i)|\le\varphi(r_0\|x-y\|)+e^{-\lambda t_*}\sum_{n=1}^{\infty}\frac{(\lambda t_*)^n}{(n-1)!}\,\psi_1\Big(\frac{e^{\alpha t_*}}{\sigma}\,\|x-y\|\Big)$$
$$\le\varphi(r_0\|x-y\|)+\overline{\psi}(\|x-y\|).$$

From the last inequality and the choice of φ it follows that

$$|T^{t_*}f(x,i)-T^{t_*}f(y,i)|\le\varphi(\|x-y\|).$$

Consequently, for every $f\in\mathcal{F}_\varphi$ and $t\ge t_*$ we have

$$|T^{t}f(x,i)-T^{t}f(y,i)|\le\varphi(\|x-y\|),$$

as required. ■

Denote by ν^n the measure on Θ^n generated by ν (i.e. $\nu^n=\nu\otimes\cdots\otimes\nu$). We need one more technical estimate:

LEMMA 4.3.3. *If conditions* (4.3.1), (4.3.2), *and* (4.3.7) *hold*, *then for every* $n\in\mathbb{N}$,

$$\text{(4.3.19)}\qquad \int_{\Theta^n}\|\overline{\Pi}_n(0,\mathbf{i},s_1,\dots,s_n,\boldsymbol{\theta})\|\,\nu^n(d\boldsymbol{\theta})\le\overline{L}_q^n\beta(s_1+\cdots+s_n)+n\tilde{c}\overline{L}_q^{n-1}$$

for $s_1,\dots,s_n\in\mathbb{R}_+,\mathbf{i}\in I^n$, *where* $\overline{L}_q=\max\{1,L_q\}$ *and* $\tilde{c}$ *is given by* (4.3.2).

Proof. For simplicity we use the notation $\mathbf{i}_k=(i,i_1,\dots,i_{k-1})\in I^k$ and $\mathbf{s}_k=(s_1,\dots,s_k)\in\mathbb{R}_+^k$. Observe that

$$\underbrace{\int_\Theta \ldots \int_\Theta}_{n} \|\overline{\Pi}_n(0,\mathbf{i}_n,\mathbf{s}_n,\theta_1,\ldots,\theta_n)\|\,\nu(d\theta_1)\ldots\nu(d\theta_n)$$
$$\le \sum_{k=0}^{n-2} \underbrace{\int_\Theta \ldots \int_\Theta}_{n} \|q_k(\overline{\Pi}_{n-k}(0,\mathbf{i}_{n-k},\mathbf{s}_{n-k},\theta_1,\ldots,\theta_{n-k}),\theta_{n-k+1},\ldots,\theta_n)$$
$$- q_{k+1}(\overline{\Pi}_{n-k-1}(0,\mathbf{i}_{n-k-1},\mathbf{s}_{n-k-1},\theta_1,\ldots,\theta_{n-k-1}),\theta_{n-k}\ldots,\theta_n)\|\,\nu(d\theta_1)\ldots\nu(d\theta_n)$$
$$+ \underbrace{\int_\Theta \ldots \int_\Theta}_{n} \|q_{n-1}(\overline{\Pi}_1(0,i,s_1,\theta_1),\theta_2,\ldots,\theta_n) - q_n(0,\theta_1,\theta_2,\ldots,\theta_n)\|\,\nu(d\theta_1)\ldots\nu(d\theta_n)$$
$$+ \underbrace{\int_\Theta \ldots \int_\Theta}_{n} \|q_n(0,\theta_1,\theta_2,\ldots,\theta_n)\|\,\nu(d\theta_1)\ldots\nu(d\theta_n).$$

By (4.3.1), (4.3.7), (4.3.8) and (4.3.9) we obtain

$$\underbrace{\int_\Theta \ldots \int_\Theta}_{n} \|q_k(\overline{\Pi}_{n-k}(0,\mathbf{i}_{n-k},\mathbf{s}_{n-k},\theta_1,\theta_2,\ldots,\theta_{n-k}),\theta_{n-k+1},\ldots,\theta_n)$$
$$- q_{k+1}(\overline{\Pi}_{n-k-1}(0,\mathbf{i}_{n-k-1},\mathbf{s}_{n-k-1},\theta_1,\ldots,\theta_{n-k-1}),\theta_{n-k}\ldots,\theta_n)\|\,\nu(d\theta_1)\ldots\nu(d\theta_n)$$
$$\le L_q^{k+1}\beta s_{n-k} \quad \text{for } k=0,1,\ldots,n-2$$

and

$$\underbrace{\int_\Theta \ldots \int_\Theta}_{n} \|q_{n-1}(\overline{\Pi}_1(0,i,s_1,\theta_1),\theta_2,\ldots,\theta_n) - q_n(0,\theta_1,\ldots,\theta_n)\|\,\nu(d\theta_1)\ldots\nu(d\theta_n) \le L_q^n\beta s_1.$$

Moreover, since $\bar{L}_q \ge 1$ we have

$$\underbrace{\int_\Theta \ldots \int_\Theta}_{n} \|q_n(0,\theta_1,\ldots,\theta_n)\|\,\nu(d\theta_1)\ldots\nu(d\theta_n) \le n\tilde{c}\bar{L}_q^{n-1},$$

which completes the proof of (4.3.19). ∎

Lemma 4.3.4. *Let the assumptions of Theorem* 4.3.1 *hold. Let* $t_* > 0$ *be such that*

$$\frac{L}{\sigma}\, e^{-(\lambda-\alpha-\lambda L_q L)t_*} < 1. \tag{4.3.20}$$

Then the Markov operator P^{t_*} *given by* (4.3.5), (4.3.6) *is globally concentrating.*

Proof. Set

$$V(x,i) = \|x\| \quad \text{for } x \in Y \text{ and } i \in I.$$

By Proposition 3.2.4 it is enough to show that there exist constants $a, b \in \mathbb{R}_+$, $a < 1$, such that

$$T^{t_*}V(x,i) \le aV(x,i) + b \quad \text{for } x \in Y,\ i \in I.$$

According to (4.3.16) we have

$$T^{t_*}V(x,i) = e^{-\lambda t_*}V(\Pi_i(t_*,x),i) + \sum_{n=1}^{\infty}\int_{\Omega_n} V_n(x,i,\mathbf{\Delta}_n(\omega),t_*-t_n(\omega))\,\mathbb{P}(d\omega),$$

where V_n is the function defined by formula (4.3.11) with f replaced by V, $\mathbf{\Delta}_n$ and Ω_n are defined in Lemma 4.3.2. Thus

$$\begin{aligned}T^{t_*}V(x,i) &\le e^{-\lambda t_*}\|\Pi_i(t_*,x)-\Pi_i(t_*,0)\| + e^{-\lambda t_*}\|\Pi_i(t_*,0)\|\\ &+\sum_{n=1}^{\infty}\int_{\Omega_n}\int_{\Theta^n}\sum_{\mathbf{i}\in I^{n-1},i_n\in I}\|\Pi_{i_n}(t_*-t_n(\omega),\overline{\Pi}_n(x,i,\mathbf{i},\mathbf{\Delta}_n(\omega),\boldsymbol{\theta}))\\ &-\Pi_{i_n}(t_*-t_n(\omega),\overline{\Pi}_n(0,i,\mathbf{i},\mathbf{\Delta}_n(\omega),\boldsymbol{\theta}))\|\cdot\mathcal{P}_n(x,i,\mathbf{i},i_n,\mathbf{\Delta}_n(\omega),\boldsymbol{\theta})\,\nu^n(d\boldsymbol{\theta})\,\mathbb{P}(d\omega)\\ &+\sum_{n=1}^{\infty}\int_{\Omega_n}\int_{\Theta^n}\sum_{\mathbf{i}\in I^{n-1},i_n\in I}\|\Pi_{i_n}(t_*-t_n(\omega),\overline{\Pi}_n(0,i,\mathbf{i},\mathbf{\Delta}_n(\omega),\boldsymbol{\theta}))\|\\ &\cdot\mathcal{P}_n(x,i,\mathbf{i},i_n,\mathbf{\Delta}_n(\omega),\boldsymbol{\theta})\,\nu^n(d\boldsymbol{\theta})\,\mathbb{P}(d\omega).\end{aligned}$$

By (4.2.3), (4.3.1)–(4.3.4), (4.3.7) and Lemma 4.3.3 we have

$$\begin{aligned}T^{t_*}V(x,i) &\le e^{-(\lambda-\alpha)t_*}\frac{L}{\sigma}\|x\| + e^{-\lambda t_*}\beta t_* + \sum_{n=1}^{\infty}\int_{\Omega_n}\frac{(LL_q)^nL}{\sigma}e^{\alpha t_*}\|x\|\,\mathbb{P}(d\omega)\\ &+\sum_{n=1}^{\infty}\int_{\Omega_n}(\beta(t_*-t_n(\omega))+\beta\bar{L}_q^n t_n(\omega)+n\tilde{c}\bar{L}_q^{n-1})\,\mathbb{P}(d\omega).\end{aligned}$$

Thus

$$T^{t_*}V(x,i) \le \frac{L}{\sigma}\,e^{-(\lambda-\alpha-\lambda L_qL)t_*}\|x\| + t_*e^{-\lambda(1-\bar{L}_q)t_*}(\beta+\lambda\tilde{c}).$$

Setting $a=\frac{L}{\sigma}\,e^{-(\lambda-\alpha-\lambda L_qL)t_*}$ and $b=t_*e^{-\lambda(1-\bar{L}_q)t_*}(\beta+\lambda\tilde{c})$ completes the proof. ∎

As a consequence of Lemma 4.3.4 and Remark 3.2.2 we have the following:

Corollary 4.3.1. *If the assumptions of Lemma* 4.3.4 *hold, then* $\mathcal{E}(P^{t_*})$ *given by* (3.2.7) *is nonempty.*

Lemma 4.3.5. *Under the hypotheses of Theorem* 4.3.1 *the operator* P^{t_*} *is semi-concentrating.*

Proof. We choose t_* such that (4.3.20) is satisfied. Set $\widetilde{P}=P^{t_*}$. By Corollary 4.3.1 we have $\mathcal{E}(\widetilde{P})\neq\emptyset$. Suppose that, contrary to our claim, $\widetilde{\varepsilon}=\inf\mathcal{E}(\widetilde{P})>0$. Let α be given by condition (4.2.3). Similarly to the proof of Theorem 4.2.2, we consider two cases: $\alpha<0$ and $\alpha\ge 0$.

Case I: $\alpha<0$. Without loss of generality we can assume that $LL_q\ge 1$. Then from (4.3.20) it follows that

$$\frac{L}{\sigma}\,e^{\alpha t_*}<1.$$

Thus we can choose $\varepsilon_0>\widetilde{\varepsilon}$ such that

$$\varepsilon=\frac{L}{\sigma}\,e^{\alpha t_*}\varepsilon_0<\widetilde{\varepsilon}.$$

By Corollary 4.3.1 there exists $\{z_1, \dots, z_m\} \subset Y$ such that

$$\inf_{\mu\in\mathcal{M}_1} \liminf_{n\to\infty} \widetilde{P}^n \mu\Big(\bigcup_{k=1}^m B(z_k,\varepsilon_0)\times I\Big) > 0. \tag{4.3.21}$$

Set

$$C_\varepsilon = \bigcup_{j=1}^N \bigcup_{k=1}^m (B(\Pi_j(t_*, z_k), \varepsilon) \times I)$$

and observe that $C_\varepsilon \in \mathcal{C}_\varepsilon$. From (4.2.3) and (4.3.4) it follows that for every $x \in B(z_k, \varepsilon_0)$, $k \in \{1, \dots, m\}$ and $j \in I$ we have

$$\|\Pi_j(t_*, x) - \Pi_j(t_*, z_k)\| \le \frac{L}{\sigma}\, e^{\alpha t_*} \|x - z_k\| = \varepsilon,$$

which gives $(\Pi_j(t_*, x), j) \in C_\varepsilon$. Obviously, from (4.3.16), we have

$$\begin{aligned}\widetilde{P}\mu(C_\varepsilon) &= \langle 1_{C_\varepsilon}, \widetilde{P}\mu\rangle = \langle T^{t_*} 1_{C_\varepsilon}, \mu\rangle = \int_{Y\times I} E(1_{C_\varepsilon}(X(t_*), \xi(t_*)))\, d\mu \\ &\ge \int_{\bigcup_{k=1}^m B(z_k,\varepsilon_0)\times I} e^{-\lambda t_*} 1_{C_\varepsilon}(\Pi_i(t_*, x), i)\, \mu(dx, di) = e^{-\lambda t_*} \mu\Big(\bigcup_{k=1}^m B(z_k, \varepsilon_0) \times I\Big)\end{aligned}$$

for $\mu \in \mathcal{M}_1$. From (4.3.21), we obtain

$$\inf_{\mu\in\mathcal{M}_1} \liminf_{n\to\infty} \widetilde{P}^n \mu(C_\varepsilon) > 0.$$

Since $\varepsilon < \widetilde{\varepsilon}$, this contradicts the fact that $\widetilde{\varepsilon} = \inf \mathcal{E}(\widetilde{P})$ and completes the proof in the first case.

Case II: $\alpha \ge 0$. Then by (4.3.3) we have $LL_q < 1$. Choose positive constants $\varepsilon_0, \eta, \delta$ and $\bar{t}_*$ such that $\varepsilon_0 > \widetilde{\varepsilon}$, $\delta < \bar{t}_*$ and

$$\varepsilon = (1+\eta) L L_q e^{\alpha \bar{t}_*}(\varepsilon_0 + 2\beta\delta) < \widetilde{\varepsilon},$$

where β is given by condition (4.3.7). Set $Q = P^{\bar{t}_*}$. By the definition of $\mathcal{E}(\widetilde{P})$ there exists $A \in \mathcal{C}_{\varepsilon_0}$ such that

$$\kappa = \inf_{\mu\in\mathcal{M}_1} \liminf_{n\to\infty} \widetilde{P}^n \mu(A) > 0.$$

Without loss of generality we can assume that

$$A = \bigcup_{i=1}^N \bigcup_{k=1}^{n_i} (B(z_{ik}, \varepsilon_0) \times \{i\}) \quad \text{for some } z_{ik} \in Y.$$

For any given $i \in I$ and $k \in \{1, \dots, n_i\}$ define

$$V_{ik} = \bigcup_{j=1}^N \bigcup_{\tau\in[0,\delta)} \bigcup_{\theta\in\Theta} B(\Pi_j(\bar{t}_* - \tau, q(\Pi_i(\tau, z_{ik}), \theta)), \varepsilon) \times \{j\}.$$

Define

$$\begin{aligned}\Theta_0 &= \Theta_0(x, \tau) \\ &= \{\theta \in \Theta : \|q(\Pi_i(\tau, x), \theta) - q(\Pi_i(\tau, z_{ik}), \theta)\| \le (1+\eta) L_q \|\Pi_i(\tau, x) - \Pi_i(\tau, z_{ik})\|\}.\end{aligned}$$

From (4.3.1) it follows that
$$\nu(\Theta_0) \ge \frac{\eta}{1+\eta}.$$
Further, from (4.2.3) it follows that for every $x \in B(z_{ik}, \varepsilon_0)$ and $\tau < \delta$ there exist $j \in I$ (depending on x and τ) and $\theta \in \Theta_0$ such that
$$\begin{aligned}&\|\Pi_j(\overline{t}_* - \tau, q(\Pi_i(\tau, x), \theta)) - \Pi_j(\overline{t}_* - \tau, q(\Pi_i(\tau, z_{ik}), \theta))\| \\ &\le Le^{\alpha(\overline{t}_* - \tau)}\|q(\Pi_i(\tau, x), \theta) - q(\Pi_i(\tau, z_{ik}), \theta)\| \le (1+\eta)LL_q e^{\alpha(\overline{t}_* - \tau)}\|\Pi_i(\tau, x) - \Pi_i(\tau, z_{ik})\|.\end{aligned}$$
Since
$$\|\Pi_i(\tau, x) - \Pi_i(\tau, z_{ik})\| \le \|\Pi_i(\tau, x) - x\| + \|x - z_{ik}\| + \|\Pi_i(\tau, z_{ik}) - z_{ik}\| \le \varepsilon_0 + 2\beta\tau,$$
we obtain
$$\|\Pi_j(\overline{t}_* - \tau, q(\Pi_i(\tau, x), \theta)) - \Pi_j(\overline{t}_* - \tau, q(\Pi_i(\tau, z_{ik}), \theta))\| \le (1+\eta)LL_q e^{\alpha(\overline{t}_* - \tau)}(\varepsilon_0 + 2\beta\tau) \le \varepsilon,$$
which implies
$$\Pi_j(\overline{t}_* - \tau, q(\Pi_i(\tau, x), \theta)) \times \{j\} \in V_{ik}.$$
Let $\mu \in \mathcal{M}_1$ be arbitrary. We have
$$Q\mu(V_{ik}) = \int\limits_{Y\times I}\int\limits_{\Omega} 1_{V_{ik}}(X(\overline{t}_*), \xi(\overline{t}_*))\, d\mathbb{P}\, d\mu.$$
Set
$$\Omega_\delta = \{\omega \in \Omega : t_1(\omega) < \delta \text{ and } \overline{t}_* < t_2(\omega)\}.$$
By the Fubini theorem we have
$$\begin{aligned}Q\mu(V_{ik}) &\ge \int\limits_{\Omega_\delta}\int\limits_{B(z_{ik},\varepsilon_0)\times\{i\}} 1_{V_{ik}}(X(\overline{t}_*), \xi(\overline{t}_*))\, d\mu\, d\mathbb{P} \\ &= \int\limits_{B(z_{ik},\varepsilon_0)\times\{i\}}\int\limits_{\Omega_\delta} f_1(x, i, \Delta t_1, \overline{t}_* - t_1)\, d\mathbb{P}\, \mu(dx, di),\end{aligned}$$
where
$$f_1(x, i, s_1, s_2) = \int\limits_{\Theta}\sum_{j=1}^{N} p_{ij}(q(\Pi_i(s_1, x), \theta))1_{V_{ik}}(\Pi_j(s_2, q(\Pi_i(s_1, x), \theta)), j)\, \nu(d\theta)$$
for $x \in Y$, $i \in I$, $s_1, s_2 \in \mathbb{R}_+$. Consequently,
$$Q\mu(V_{ik}) \ge \sigma\mu(B(z_{ik}, \varepsilon_0) \times \{i\})\mathbb{P}(\Omega_\delta).$$
By a standard calculation we obtain
$$\mathbb{P}(\Omega_\delta) = \lambda^2\int\limits_0^\delta ds_1 \int\limits_{\overline{t}_* - s_1}^{\infty} e^{-\lambda(s_1+s_2)}\, ds_2 > 0.$$

Without loss of generality we can assume that
$$t_*/\overline{t}_* = r \in \mathbb{N}.$$
Then it is evident that
$$\widetilde{P}^{n+1}\mu = Q\widetilde{P}^n Q^{r-1}\mu \quad \text{for } \mu \in \mathcal{M}_1,\ n \in \mathbb{N}.$$

Observe that for every $\mu \in \mathcal{M}_1$ and $m \in \mathbb{N}$ there exist $i_m \in I$ and $k_m \in \{1, \ldots, n_{i_m}\}$ such that

$$
\begin{aligned}
(4.3.22) \qquad \widetilde{P}^{m+1}\mu(V_{i_m k_m}) &= Q\widetilde{P}^m\hat{\mu}(V_{i_m k_m}) \geq \sigma\widetilde{P}^m\hat{\mu}(B(z_{i_m k_m}, \varepsilon_0) \times \{i_m\})\mathbb{P}(\Omega_\delta) \\
&\geq \kappa\sigma\mathbb{P}(\Omega_\delta)/(2N_0),
\end{aligned}
$$

where

$$
\hat{\mu} = Q^{r-1}\mu \quad \text{and} \quad N_0 = \sum_{i=1}^{N} n_i.
$$

Now define

$$
C = \bigcup_{i=1}^{N} \bigcup_{k=1}^{n_i} V_{ik}.
$$

Observe that there exists $\hat{\varepsilon} \in (\varepsilon, \varepsilon_0)$ such that $C \in C_{\hat{\varepsilon}}(Y \times I)$. Moreover, from (4.3.22) it follows that for arbitrary $\mu \in \mathcal{M}_1$ and $m \in \mathbb{N}$ we have

$$
\widetilde{P}^{m+1}\mu(C) \geq \kappa\sigma\mathbb{P}(\Omega_\delta)/(2N_0),
$$

which contradicts the definition of $\widetilde{\varepsilon}$ and completes the proof. ∎

Proof of Theorem 4.3.1. By Lemma 4.3.2 there exists $t_* > 0$, satisfying condition (4.3.20), such that the operator P^{t_*} is essentially nonexpansive. Moreover, for every $t \geq t_*$ the operator P^t is nonexpansive in the same norm as P^{t_*}. By Lemma 4.3.5 for every $t \geq t_*$ the operator P^t is semi-concentrating.

If $\alpha < 0$, then we may choose $t_* > 0$ such that not only (4.3.20) is satisfied but also

$$
\frac{L_q L}{\sigma}\, e^{\alpha t_*} < 1.
$$

If $\alpha \geq 0$, then we choose t_* such that (4.3.20) is satisfied. Set $\widetilde{P} = P^{t_*}$ and $\widetilde{T} = T^{t_*}$.

By Proposition 3.2.3 there exists $\mu_* \in \mathcal{M}_1$ such that $\widetilde{P}\mu_* = \mu_*$. Moreover $\mathcal{L}(\mu) \neq \emptyset$ for every $\mu \in \mathcal{M}_1$ and the set $\mathcal{L}(\mathcal{M}_1)$ is tight.

We claim that $\widetilde{P}$ is asymptotically stable. By Theorem 3.2.1 it is sufficient to show that $\widetilde{P}$ satisfies condition (3.2.3). Since $\mathcal{L}(\mathcal{M}_1)$ is tight there exists a compact set $F \subset Y \times I$ such that

$$
\mu(F) \geq 4/5 \quad \text{ for every } \mu \in \mathcal{L}(\mathcal{M}_1).
$$

Define

$$
F_Y = \{x \in Y : (x, i) \in F \text{ for some } i \in I\}.
$$

We give the proof of the lower bound condition separately in the two cases: $\alpha < 0$ and $\alpha \geq 0$.

Case I: $\alpha < 0$. Set

$$
r_0 = \frac{L_q L}{\sigma}\, e^{\alpha t_*} < 1.
$$

Let φ be the solution of (4.3.15). Fix $\varepsilon_1 > 0$. We can find $\varepsilon > 0$ such that $\varphi(\varepsilon) \leq \varepsilon_1$. Let $m \geq 2$ be such that

$$
2r_0^m \operatorname{diam}_{\overline{\varrho}} F < \varepsilon/3.
$$

Fix $i_1,\ldots,i_m\in I$, set $\mathbf{i}=(i_1,\ldots,i_{m-1})$ and $\mathbf{t}_*=\underbrace{(t_*,\ldots,t_*)}_{m-1}$. Now for $z\in F_Y$ we set

$$O(z)=\{x\in Y:\\ \|\Pi_{i_m}(t_*,\overline{\Pi}_{m-1}(q(x,\theta_1),\mathbf{i},\mathbf{t}_*,\boldsymbol{\theta}))-\Pi_{i_m}(t_*,\overline{\Pi}_{m-1}(q(z,\theta_1),\mathbf{i},\mathbf{t}_*,\boldsymbol{\theta}))\|\le\varepsilon/9\\ \text{for } \theta_1\in\Theta,\ \boldsymbol{\theta}\in\Theta^{m-1}\}.$$

Let $z_1,\ldots,z_{m_0}\in F_Y$ be such that $F\subset G$, where

$$G=\bigcup_{l=1}^{m_0}(O(z_l)\times I).$$

Note that G is an open subset of $Y\times I$. Let $\mu_1,\mu_2\in\mathcal{M}_1$ be arbitrary. Set $\mu=(\mu_1+\mu_2)/2$. Since $\mathcal{L}(\mu)\ne\emptyset$ (see Proposition 3.2.3) there exists a subsequence $\{n_k\}$ of $\{n\}$ and a measure $\nu\in\mathcal{L}(\mu)$ such that $P^{n_k}\mu\to\nu$ (weakly). Since $\nu(G)\ge 4/5$, the Aleksandrov theorem implies

$$\liminf_{k\to\infty}\widetilde{P}^{n_k}\mu(G)\ge\nu(G)\ge 4/5.$$

From this, it follows that there exists $\overline{n}\in\mathbb{N}$ such that

$$\widetilde{P}^{\overline{n}}\mu(G)=\frac{\widetilde{P}^{\overline{n}}\mu_1(G)+\widetilde{P}^{\overline{n}}\mu_2(G)}{2}\ge\frac{3}{4},$$

and consequently

$$\widetilde{P}^{\overline{n}}\mu_k(G)\ge 1/2\quad\text{for } k=1,2.$$

Therefore there exist $l_1,l_2\in\{1,\ldots,m_0\}$ and $j_1,j_2\in I$ such that

$$\widetilde{P}^{\overline{n}}\mu_k(O_k)\ge\frac{1}{2m_0N}\quad\text{for } k=1,2,$$

where $O_1=O(z_{l_1})\times\{j_1\}, O_2=O(z_{l_2})\times\{j_2\}$.

From the definition of $\overline{\Pi}_m$ and conditions (4.2.3), (4.3.1) and (4.3.4), we have

$$\int_{\Theta^m}\|\Pi_{i_m}(t_*,\overline{\Pi}_{m-1}(q(z_{l_1},\theta_1),\mathbf{i},\mathbf{t}_*,\theta_2,\ldots,\theta_m))\\ -\Pi_{i_m}(t_*,\overline{\Pi}_{m-1}(q(z_{l_2},\theta_1),\mathbf{i},\mathbf{t}_*,\theta_2,\ldots,\theta_m))\|\,\nu(d\theta_1)\cdots\nu(d\theta_m)\\ \le\left(\frac{L_qL}{\sigma}\,e^{\alpha t_*}\right)^m\|z_{l_1}-z_{l_2}\|.$$

From the last inequality it follows that $\nu^m(\Theta_0)\ge 1/2$, where

$$\begin{aligned}\Theta_0&=\Theta_0(\mathbf{i},\mathbf{t}_*,z_{l_1},z_{l_2})\\ &=\bigg\{\theta_1,\ldots,\theta_m\in\Theta:\|\Pi_{i_m}(t_*,\overline{\Pi}_{m-1}(q(z_{l_1},\theta_1),\mathbf{i},\mathbf{t}_*,\theta_2,\ldots,\theta_m))\\ &\quad-\Pi_{i_m}(t_*,\overline{\Pi}_{m-1}(q(z_{l_2},\theta_1),\mathbf{i},\mathbf{t}_*,\theta_2,\ldots,\theta_m))\|\le 2\bigg(\frac{L_qL}{\sigma}\,e^{\alpha t_*}\bigg)^m\|z_{l_1}-z_{l_2}\|\bigg\}.\end{aligned}\tag{4.3.23}$$

Since Θ^m is compact and the functions q and Π_i are continuous there are $\theta_{l_1},\ldots,\theta_{l_m}\in\Theta_0$ and a neighborhood $B(\boldsymbol{\theta}_{l_*})$ of $\boldsymbol{\theta}_{l_*}=(\theta_{l_1},\ldots,\theta_{l_m})$ such that $\nu^m(\Theta_{l_*})>0$, where

$\Theta_{l_*} = B(\boldsymbol{\theta}_{l_*}) \cap \Theta_0$ and

$$(4.3.24) \quad \|\Pi_{i_m}(t_*, \overline{\Pi}_{m-1}(q(x,\theta_1), \mathbf{i}, \mathbf{t}_*, \theta_2, \ldots, \theta_m)) - \Pi_{i_m}(t_*, \overline{\Pi}_{m-1}(q(x,\theta_{l_1}), \mathbf{i}, \mathbf{t}_*, \theta_{l_2}, \ldots, \theta_{l_m}))\| \le \varepsilon/9$$

for $x \in F_Y$ and $(\theta_1, \ldots, \theta_m) \in \Theta_{l_*}$

Moreover, from (4.3.23) we have

$$\begin{aligned}\|\Pi_{i_m}(t_*, \overline{\Pi}_{m-1}(q(z_{l_1},\theta_{l_1}), \mathbf{i}, \mathbf{t}_*, \theta_{l_2}, \ldots, \theta_{l_m}))& \\ - \Pi_{i_m}(t_*, \overline{\Pi}_{m-1}(q(z_{l_2},\theta_{l_1}), \mathbf{i}, \mathbf{t}_*, \theta_{l_2}, \ldots, \theta_{l_m}))\|& \\ \le 2\left(\frac{L_q L}{\sigma} e^{\alpha t_*}\right)^m \|z_{l_1} - z_{l_2}\| \le 2r_0^m \operatorname{diam}_{\overline{\varrho}} F \le \varepsilon/3.&\end{aligned}$$

Now define $A = A_1 \cup A_2$, where

$$A_k = B(\Pi_{i_m}(t_*, \overline{\Pi}_{m-1}(q(z_{l_k},\theta_{l_*}), \mathbf{i}, \mathbf{t}_*, \boldsymbol{\theta}_{l_*})), \varepsilon/3) \times \{i_m\} \quad \text{for } k = 1,2.$$

Then

$$\operatorname{diam}_{\overline{\varrho}_\varphi} A = \operatorname{diam}_{\varphi\circ\overline{\varrho}} A \le \varphi(\operatorname{diam}_{\overline{\varrho}} A) \le \varphi(\varepsilon) \le \varepsilon_1.$$

By continuity of Π_i, q and (4.3.10) there exists $\eta > 0$ such that

$$\begin{aligned}(4.3.25) \quad &\|\Pi_{i_m}(\delta_{m+1}, \overline{\Pi}_m(x,i,\mathbf{i},\boldsymbol{\delta},\theta_1,\ldots,\theta_m)) - \Pi_{i_m}(t_*, \overline{\Pi}_{m-1}(q(x,\theta_1),\mathbf{i},\mathbf{t}_*,\theta_2,\ldots,\theta_m))\| \\ &= \|\Pi_{i_m}(\delta_{m+1}, \overline{\Pi}_m(x,i,\mathbf{i},\boldsymbol{\delta},\theta_1,\ldots,\theta_m)) - \Pi_{i_m}(t_*, \overline{\Pi}_m(x,i,\mathbf{i},0,\mathbf{t}_*,\theta_1,\ldots,\theta_m))\| \le \varepsilon/9\end{aligned}$$

for arbitrary $(x,i) \in F, \boldsymbol{\delta} = (\delta_1, \ldots, \delta_m), \delta_1 \in (0,\eta), \delta_2, \ldots, \delta_{m+1} \in (t_* - \eta, t_* + \eta)$ and $(\theta_1, \ldots, \theta_m) \in \Theta_{l_*}$. Set

$$\Omega_* = \left\{\omega \in \Omega : \Delta t_1(\omega) \le \eta,\ \Delta t_2(\omega), \ldots, \Delta t_m(\omega) \in \left(t_* - \frac{\eta}{m-1}, t_*\right),\ t_{m+1}(\omega) > mt_*\right\}.$$

Let $n_0 = \overline{n} + m$, $\boldsymbol{\Delta}_m = (\Delta t_1, \ldots \Delta t_m)$, and $\hat{\mu}_k = \widetilde{P}^{\overline{n}}\mu_k$. We have

$$\begin{aligned}\widetilde{P}^{n_0}\mu_k(A) &= \int_{Y\times I} \widetilde{T}^m 1_A(x,i)\hat{\mu}_k(dx,di) = \int_{Y\times I} E(1_{A_k}((X(mt_*), \xi(mt_*))_{(x,i)}))\,\hat{\mu}_k(dx,di) \\ &\ge \int_{O_k}\int_{\Omega_*}\int_{\Theta_{l_*}} 1_{A_k}(\Pi_{i_m}(mt_* - t_m(\omega), \overline{\Pi}_m(x,j_k,\mathbf{i},\boldsymbol{\Delta}_m(\omega),\boldsymbol{\theta})), i_m) \\ &\quad \times \mathcal{P}_m(x,j_k,\mathbf{i},\boldsymbol{\Delta}_m(\omega),\boldsymbol{\theta})\,\nu^m(d\boldsymbol{\theta})\,\mathbb{P}(d\omega)\,\hat{\mu}_k(dx,di) \\ &\ge \sigma^m \int_{O_k}\int_{\Omega_*}\int_{\Theta_{l_*}} 1_{A_k}(\Pi_{i_m}(mt_* - t_m(\omega), \overline{\Pi}_m(x,j_k,\mathbf{i},\boldsymbol{\Delta}_m(\omega),\boldsymbol{\theta})), i_m)\,\nu^m(d\boldsymbol{\theta})\,\mathbb{P}(d\omega)\,\hat{\mu}_k(dx,di)\end{aligned}$$

for $k = 1,2$. By (4.3.24) and (4.3.25) we obtain

$$(\Pi_{i_m}(mt_* - t_m(\omega), \overline{\Pi}_m(x,j_k,\mathbf{i},\boldsymbol{\Delta}_m(\omega),\boldsymbol{\theta})), i_m) \in A_k$$

for arbitrary $\omega \in \Omega_*, (x,i) \in O_k$ and $\boldsymbol{\theta} \in \Theta_{l_*}$. Thus

$$\widetilde{P}^{n_0}\mu_k(A) \ge \frac{\sigma^m}{2m_0N}\,\nu^m(\Theta_{l_*})\mathbb{P}(\Omega_*) \quad \text{for } k = 1,2.$$

Consequently, the operator $\widetilde{P}$ satisfies the lower bound condition.

Case II: $\alpha \geq 0$. In this case condition (4.3.3) implies that $L_q < 1$. Let $m \in \mathbb{N}$ be such that

$$L_q^m \cdot \operatorname{diam}_{\overline{\varrho}} F < \varepsilon/2. \tag{4.3.26}$$

By continuity and compactness of Θ and F_Y there exists $\delta \in (0, \varepsilon(16\beta m)^{-1})$ such that

$$\|\overline{\Pi}_m(x, \mathbf{i}, \mathbf{s}, \boldsymbol{\theta}) - q_m(x, \boldsymbol{\theta})\| < \varepsilon/32 \tag{4.3.27}$$

for every $\mathbf{i} \in I^m, \mathbf{s} \in (0, \delta]^m$, $\boldsymbol{\theta} \in \Theta^m$, $x \in F_Y$. Given $\widetilde{\boldsymbol{\theta}} \in \Theta^m$ we define

$$\mathbf{V}(\widetilde{\boldsymbol{\theta}}) = \{\boldsymbol{\theta} \in \Theta^m : \|q_m(x, \boldsymbol{\theta}) - q_m(x, \widetilde{\boldsymbol{\theta}})\| < \varepsilon/32 \text{ for every } x \in F_Y\}. \tag{4.3.28}$$

Clearly, $\mathbf{V}(\widetilde{\boldsymbol{\theta}})$ is an open neighborhood of $\widetilde{\boldsymbol{\theta}}$. Since Θ^m is compact, there exists a finite family $\mathbf{V}_j = \mathbf{V}(\boldsymbol{\theta}_j)$, $j = 1, \ldots, \overline{m}$, such that $\Theta^m = \bigcup_{j=1}^{\overline{m}} \mathbf{V}_j$. Set

$$J = \{j \in \{1, \ldots, \overline{m}\} : \nu^m(\mathbf{V}_j) > 0\}$$

and

$$\vartheta = \min_{j \in J} \nu^m(\mathbf{V}_j) > 0. \tag{4.3.29}$$

Given $x \in Y$ define

$$O(x) = \{z \in Y : \|q_m(x, \boldsymbol{\theta}_j) - q_m(z, \boldsymbol{\theta}_j)\| < \varepsilon/32 \text{ for } j \in J\}. \tag{4.3.30}$$

Clearly, $O(x)$ is an open neighborhood of x. Let $z_1, \ldots, z_{m_0} \in F_Y$ be such that $F \subset G$, where G is defined by

$$G = \bigcup_{l=1}^{m_0} (O(z_l) \times I).$$

Let $\mu_1, \mu_2 \in \mathcal{M}_1$ be arbitrary. As in the first case there exist $\overline{n} = \overline{n}(\mu_1, \mu_2), l_1, l_2 \in \{1, \ldots, m_0\}$, $j_1, j_2 \in I$ such that

$$\widetilde{P}^{\overline{n}} \mu_k(O(z_{l_k}) \times \{j_k\}) \geq \frac{1}{2m_0 N}.$$

From condition (4.3.1) it follows that there exists a subset $\mathbf{\Theta}_0 = \mathbf{\Theta}_0(z_{l_1}, z_{l_2})$ of Θ^m such that $\nu^m(\mathbf{\Theta}_0) > 0$ and

$$\|q_m(z_{l_1}, \boldsymbol{\theta}) - q_m(z_{l_2}, \boldsymbol{\theta})\| \leq L_q^m \|z_{l_1} - z_{l_2}\| \quad \text{for every } \boldsymbol{\theta} \in \mathbf{\Theta}_0. \tag{4.3.31}$$

Since $\mathbf{\Theta}_0$ has a positive measure, there exists $j_0 \in J$ such that $\mathbf{\Theta}_0 \cap \mathbf{V}_0 \neq \emptyset$, where $\mathbf{V}_0 = \mathbf{V}(\boldsymbol{\theta}_{j_0})$. Choose $\boldsymbol{\theta}_0 \in \mathbf{\Theta}_0 \cap \mathbf{V}_0$, $i_m \in I$ and define

$$A_k = B(q_m(z_{l_k}, \boldsymbol{\theta}_0), \varepsilon/4) \times \{i_m\} \quad \text{for } k = 1, 2$$

and $A = A_1 \cup A_2$.

From (4.3.26) and (4.3.31) it follows that $\operatorname{diam}_{\overline{\varrho}_\varphi} A < \varepsilon$. For $\boldsymbol{\theta} \in \mathbf{V}_0$, $\mathbf{i} \in I^{m-1}$, $\mathbf{s} \in [0, \delta]^m$ and $x \in O(z_{l_k})$, $k = 1, 2$, we have, by (4.3.27)–(4.3.30),

$$\begin{aligned}
\|\overline{\Pi}_m(x, j_k, \mathbf{i}, \mathbf{s}, \boldsymbol{\theta}) - q_m(z_{l_k}, \boldsymbol{\theta}_0)\| &\leq \|\overline{\Pi}_m(x, j_k, \mathbf{i}, \mathbf{s}, \boldsymbol{\theta}) - q_m(x, \boldsymbol{\theta})\| \\
&\quad + \|q_m(x, \boldsymbol{\theta}) - q_m(x, \boldsymbol{\theta}_{j_0})\| + \|q_m(x, \boldsymbol{\theta}_{j_0}) - q_m(z_{l_k}, \boldsymbol{\theta}_{j_0})\| \\
&\quad + \|q_m(z_{l_k}, \boldsymbol{\theta}_{j_0}) - q_m(z_{l_k}, \boldsymbol{\theta}_0)\| < \varepsilon/8
\end{aligned}$$

and by (4.3.19) we obtain for every $t > 0$,

$$\begin{aligned}(4.3.32)\qquad &\|\Pi_{i_m}(t, \overline{\Pi}_m(x, j_k, \mathbf{i}, \mathbf{s}, \boldsymbol{\theta})) - q_m(z_{l_k}, \boldsymbol{\theta}_0)\| \\ &\qquad\leq \beta t + \|\overline{\Pi}_m(x, j_k, \mathbf{i}, \mathbf{s}, \boldsymbol{\theta}) - q_m(z_{l_k}, \boldsymbol{\theta}_0)\| \leq \beta t + \varepsilon/8.\end{aligned}$$

Fix $\overline{t}$ such that

$$\delta < \overline{t} < \delta + \frac{\varepsilon}{16m\beta}$$

and set

$$\Omega_* = \{\omega \in \Omega : \Delta t_i(\omega) \leq \delta \text{ for } i = 1, \ldots, m \text{ and } t_{m+1}(\omega) > m\overline{t}\}.$$

Let $n_0 = \overline{n} + m$ and $\hat{\mu}_k = \widetilde{P}^{\overline{n}}\mu_k$. Fix $i_1, \ldots, i_{m-1} \in I$, set $\mathbf{i} = (i_1, \ldots, i_{m-1})$ and $\boldsymbol{\Delta}_m = (\Delta t_1, \ldots, \Delta t_m)$. We have

$$\begin{aligned}\widetilde{P}^{n_0}\mu_k(A) &= \int_{Y\times I} \widetilde{T}^m 1_A(x,i)\,\hat{\mu}_k(dx,di) = \int_{Y\times I} E(1_{A_k}((X(m\overline{t}), \xi(m\overline{t}))_{(x,i)}))\,\hat{\mu}_k(dx,di) \\ &\geq \int_{O(z_{l_k})\times\{j_k\}} \int_{\Omega_*}\int_{\mathbf{V}_0} 1_{A_k}(\Pi_{i_m}(m\overline{t} - t_m(\omega), \overline{\Pi}_m(x,i,\mathbf{i},\boldsymbol{\Delta}_m(\omega),\boldsymbol{\theta})), i_m) \\ &\qquad \times \mathcal{P}_m(x,i,\mathbf{i},\boldsymbol{\Delta}_m(\omega),\boldsymbol{\theta})\,\nu^m(d\boldsymbol{\theta})\,\mathbb{P}(d\omega)\,\hat{\mu}_k(dx,di) \\ &\geq \sigma^m \int_{O(z_{l_k})\times\{j_k\}} \int_{\Omega_*}\int_{\mathbf{V}_0} 1_{A_k}(\Pi_{i_m}(m\overline{t} - t_m(\omega), \overline{\Pi}_m(x,i,\mathbf{i},\boldsymbol{\Delta}_m(\omega),\boldsymbol{\theta})), i_m) \\ &\qquad \times \nu^m(d\boldsymbol{\theta})\,\mathbb{P}(d\omega)\,\hat{\mu}_k(dx,di) \quad \text{for } k = 1,2.\end{aligned}$$

Keeping in mind that $\delta < \varepsilon(16\beta m)^{-1}$ by (4.3.32) we obtain

$$(\Pi_{i_m}(m\overline{t} - t_m(\omega), \overline{\Pi}_m(x, j_k, \mathbf{i}, \boldsymbol{\Delta}_m(\omega), \boldsymbol{\theta})), i_m) \in A_k$$

for arbitrary $\omega \in \Omega_*, x \in O(z_{l_k})$ and $\boldsymbol{\theta} \in \mathbf{V}_0$, $k = 1,2$. Thus

$$\widetilde{P}^{n_0}\mu_k(A) \geq \frac{\sigma^m}{2m_0N}\,\vartheta\mathbb{P}(\Omega_*) \quad \text{for } k = 1,2,$$

which completes the proof of the lower bound condition.

Now, let μ_* be the invariant distribution of P^{t_*}. Then for $t \in \mathbb{R}_+$ we have

$$P^{t_*}(P^t\mu_*) = P^t(P^{t_*}\mu_*) = P^t\mu_*.$$

Since μ_* is the unique invariant measure for the operator P^{t_*}, we have $P^t\mu_* = \mu_*$. On the other hand, using nonexpansiveness of $\{P^t\}_{t\geq t_*}$ we obtain

$$\lim_{t\to\infty}\|P^t\mu - \mu_*\|_\varphi = \lim_{t\to\infty}\|P^t\mu - P^t\mu_*\|_\varphi \leq \lim_{n\to\infty}\|(P^{t_*})^n\mu - \mu_*\|_\varphi = 0$$

for arbitrary $\mu \in \mathcal{M}_1(Y \times I)$. However, the metrics $\overline{\varrho}$ and $\varphi \circ \overline{\varrho}$ define the same space of continuous functions $C(Y \times I)$ and the weak convergence of a sequence of measures in the space $(Y \times I, \overline{\varrho})$ and $(Y \times I, \varphi \circ \overline{\varrho})$ is the same. This proves that $\{P^t\}_{t\geq 0}$ given by (4.3.5), (4.3.6) is asymptotically stable also in $(Y \times I, \overline{\varrho})$. ■

Let $\tilde{\mu}$ be the distribution of the initial random vector x_0. We denote by $Q^t\tilde{\mu}$ the distribution of $X(t)$ on the initial space $(Y, \|\cdot\|)$, i.e.

$$(4.3.33)\qquad Q^t\tilde{\mu}(A) = \mathbb{P}(X(t) \in A) \quad \text{for } A \in \mathcal{B}(Y),\ t > 0.$$

From the last theorem we may easily deduce, as in Theorem 4.2.3, a corresponding asymptotic result for the family $\{Q^t\}_{t\geq 0}$.

THEOREM 4.3.2. *Let the assumptions of Theorem* 4.3.1 *hold. Then there exists a distribution* $\widetilde{\mu}_* \in \mathcal{M}_1$ *such that for every* $\mu \in \mathcal{M}_1$ *the family* $\{Q^t\mu\}_{t\geq 0}$ *given by* (4.3.33) *is weakly convergent to* $\widetilde{\mu}_*$.

5. Dimensions of invariant measures of random dynamical systems with jumps

Dimensions of invariant sets are the most important characteristics of dynamical systems. Hausdorff dimension, introduced in 1919, is a notion of size useful for distinguishing between sets of Lebesgue measure zero. The notion has been widely investigated and used, e.g. in the theory of dynamical systems, where many interesting invariant sets are null in the sense of Lebesgue. Unfortunately, in many cases a straightforward calculation of the Hausdorff dimension is very difficult. This prompted researchers to introduce other characteristics [40], such as capacity dimension, pointwise dimension, correlation dimension, concentration dimension, etc. Using the notion of the Lévy concentration function Lasota and Myjak [32] introduced the concentration dimension (the generalized Rényi dimension) and by use of it they calculated some bounds of the concentration dimension of fractals and semifractals. It is worth noting that the concentration dimension is useful in studying the Hausdorff dimension of measures and sets [32, 40].

In this section we provide estimates for the lower pointwise dimension and the concentration dimension of invariant measures of random dynamical systems with jumps. The results of this section are related to papers [16, 38] and [45]. In [38], [45] Szarek considered the capacity and the lower pointwise dimension of invariant measures corresponding to Poisson driven stochastic differential equations. Some estimates of dimensions of invariant measures are formulated in [16].

Throughout this section we assume additionally that $\Pi_i : \mathbb{R}\times Y \to Y$, $i \in I$, are dynamical systems.

5.1. The lower pointwise dimension of an invariant measure. To estimate the lower pointwise dimension of an invariant measure for the semigroup $\{P^t\}_{t\geq 0}$ given by (4.3.5), (4.3.6) we need additional assumptions concerning the transformations $\Pi_i : \mathbb{R}\times Y \to Y$, $i \in I$.

We assume that for every $j \in I$ there exists a constant $l_j \in (0,1]$ such that

$$\|\Pi_j(t,x) - \Pi_j(t,y)\| \geq l_j\|x-y\| \quad \text{for } x,y \in Y \tag{5.1.1}$$

and for every $x \in Y$ and $j \in I$ there exist $\delta_j > 0$ and $c_{x,j} > 0$ such that

$$\|\Pi_j(t,x) - x\| \geq c_{x,j}t \quad \text{for } 0 < t < \delta_j. \tag{5.1.2}$$

As in Section 4 we require that there exist constants $L \geq 1$ and $\alpha \in \mathbb{R}$ such that

$$\sum_{j\in I} p_{ij}(y)\|\Pi_j(t,x) - \Pi_j(t,y)\| \leq Le^{\alpha t}\|x-y\| \quad \text{for } x,y \in Y,\ i\in I,\ t\geq 0. \tag{5.1.3}$$

THEOREM 5.1.1. *Let* μ_* *be an invariant measure for the semigroup* $\{P^t\}_{t\geq 0}$ *given by* (4.3.5), (4.3.6). *Assume that* $\Pi_i : \mathbb{R}\times Y \to Y$, $i \in I$, *satisfy conditions* (5.1.1)–(5.1.3). *If*

$$\sigma = \inf_{x\in Y,\, i,j\in I} p_{ij}(x) > 0, \tag{5.1.4}$$

then

$$\underline{d}\mu_*(x,i) \geq \frac{\log 3}{\log 3 + \log \frac{L}{\sigma \min_j l_j}} \quad \text{for } (x,i) \in Y\times I.$$

We start with the following technical observation:

LEMMA 5.1.1. *Assume that* μ_* *is an invariant measure for the semigroup* $\{P^t\}_{t\geq 0}$ *given by* (4.3.5), (4.3.6). *Then*

$$\mu_*(A) \geq e^{-\lambda t} \int_{Y\times I} 1_A(\Pi_i(t,x),i)\,\mu_*(dx,di) \quad \text{for } A\in \mathcal{B}(Y\times I).$$

Proof. Fix $A\in\mathcal{B}(Y\times I)$ and $t\geq 0$. We have

$$\mu_*(A) = P^t\mu_*(A) = \langle U^t 1_A, \mu_*\rangle = \int_{Y\times I} E(1_A((X(t),\xi(t))_{(x,i)}))\,\mu_*(dx,di). \tag{5.1.5}$$

Fix $(x,i)\in Y\times I$ and observe that

$$\begin{aligned} E(1_A((X(t),\xi(t))_{(x,i)})) &= \int_{\Omega} 1_A((X(t),\xi(t))_{(x,i)}(\omega))\,\mathbb{P}(d\omega) \\ &\geq \int_{\Omega_0(t)} 1_A((X(t),\xi(t))_{(x,i)}(\omega))\,\mathbb{P}(d\omega), \end{aligned}$$

where $\Omega_0(t) = \{\omega\in\Omega : t \leq t_1(\omega)\}$.

Since $(X(t),\xi(t))_{(x,i)}(\omega) = (\Pi_i(t,x),i)$ for $\omega\in\Omega_0(t)$ and $\mathbb{P}(\Omega_0(t)) = e^{-\lambda t}$, we obtain $E(1_A((X(t),\xi(t))_{(x,i)})) \geq e^{-\lambda t}1_A(\Pi_i(t,x),i)$. Combining this with (5.1.5) completes the proof. ∎

Proof of Theorem 5.1.1. We consider two cases: $\alpha\geq 0$ and $\alpha < 0$.

CASE I: $\alpha \geq 0$. Let $\overline{x}\in Y$ and $k\in I$ be fixed. Choose $\varepsilon > 0$ such that

$$r_0 = \frac{\ln(1+\varepsilon\sigma/L)}{2\alpha} \leq \delta_k \tag{5.1.6}$$

and choose $\eta > 0$ such that

$$1 - e^{-\lambda r_0} < \eta.$$

Set

$$\theta = \frac{\min_j l_j}{3(L/\sigma+\varepsilon)}, \qquad \beta = \frac{1}{(3-2\eta)(1-\eta)}, \qquad s = \frac{\log\beta}{\log\theta}.$$

Since $\sigma\min_j l_j \leq L$, we have $s < 1$.

We will show that there exists $C > 0$ such that

$$\mu_*(B((\overline{x},k),r)) \leq Cr^s \tag{5.1.7}$$

for every $r > 0$. Set

$$M = \frac{2L(L/\sigma+\varepsilon)e^{\alpha r_0}}{\sigma c_{\overline{x},k} r_0 (\min_j l_j)^s}$$

and

$$C = \max\{(\theta r_0)^{-s}, \lambda r_0 \eta^{-1}, M^{s/(1-s)}\}. \tag{5.1.8}$$

Obviously, condition (5.1.7) holds for all $r \geq r_0$. Define

$$r_* = \inf\{\bar{r} > 0 : \mu_*(B((\overline{x}, k), r)) \leq C r^s \text{ for } r > \bar{r}\}.$$

Observe that

$$r_* \leq M^{-1/(1-s)}. \tag{5.1.9}$$

We claim that $r_* = 0$. Suppose, contrary to our claim, that $r_* > 0$ and choose

$$r \in \left(\frac{r_*}{3(L/\sigma + \varepsilon)}, r_*\right]$$

such that

$$\mu_*(B((\overline{x}, k), r \min_j l_j)) > C(r \min_j l_j)^s. \tag{5.1.10}$$

Define

$$z_0 = \Pi_k(-\bar{t}, \overline{x}), \qquad z_1 = \Pi_k(\bar{t}, \overline{x})$$

where $\bar{t} = r_0(r \min_j l_j)^s$. Further, let

$$B_1 = B((\overline{x}, k), (L/\sigma + \varepsilon)r), \qquad B_2 = B((z_0, k), r), \qquad B_3 = B((z_1, k), (L/\sigma + \varepsilon)r).$$

Now, let $(y, i) \in B_2$. Then, from (5.1.3), (5.1.4), and the definition of r_0, we have

$$\|\Pi_k(\bar{t}, y) - \overline{x}\| \leq \frac{L}{\sigma} e^{\alpha \bar{t}} \|y - z_0\| \leq \frac{L}{\sigma} e^{\alpha r_0} r < \left(\frac{L}{\sigma} + \varepsilon\right) r.$$

Therefore $B_2 \subset \{(y, i) : (\Pi_i(\bar{t}, y), i) \in B_1\}$. This shows, by Lemma 5.1.1, that

$$\mu_*(B_1) \geq e^{-\lambda \bar{t}} \int_{Y \times I} 1_{B_1}(\Pi_i(\bar{t}, y), i)\, \mu_*(dy, di) \geq e^{-\lambda \bar{t}} \mu_*(B_2) \geq (1 - \eta)\mu_*(B_2). \tag{5.1.11}$$

Analogously, we obtain

$$\mu_*(B_3) \geq (1 - \eta)\mu_*(B_2). \tag{5.1.12}$$

Since $\sigma \leq L$, by (5.1.2), we have

$$\|z_1 - \overline{x}\| = \|\Pi_k(\bar{t}, \overline{x}) - \overline{x}\| \geq c_{\overline{x},k} \bar{t} \geq \frac{\sigma}{L} e^{-\alpha r_0} c_{\overline{x},k} \bar{t}.$$

From the fact that $\bar{t} < r_0$ and conditions (5.1.2), (5.1.3), it follows that

$$\|\overline{x} - z_0\| \geq \frac{\sigma}{L} e^{-\alpha \bar{t}} c_{\overline{x},k} \bar{t} \geq \frac{\sigma}{L} e^{-\alpha r_0} c_{\overline{x},k} \bar{t}. \tag{5.1.13}$$

Since

$$r < \left(\frac{\sigma c_{\overline{x},k} r_0 (\min_j l_j)^s}{2L(L/\sigma + \varepsilon) e^{\alpha r_0}}\right)^{1/(1-s)}$$

and $\bar{t} = r_0(r \min_j l_j)^s$, we obtain

$$\|z_1 - \overline{x}\| > 2(L/\sigma + \varepsilon)r, \qquad \|\overline{x} - z_0\| > 2(L/\sigma + \varepsilon)r.$$

Thus B_1, B_2, B_3 are mutually disjoint and

$$B_1 \cup B_2 \cup B_3 \subset B((\overline{x}, k), 3(L/\sigma + \varepsilon)r).$$

Set $B_4 = B((\overline{x},k), r\min_j l_j)$. We are now going to verify that

$$\mu_*(B_2) > (1-\eta)\mu_*(B_4). \tag{5.1.14}$$

Suppose that

$$\mu_*(B_2) \le (1-\eta)\mu_*(B_4). \tag{5.1.15}$$

Since $(\Pi_k(\overline{t},y),k) \notin B_4$ for $(y,k) \notin B_2$, we have

$$\begin{aligned}\mu_*(B_4) &\le e^{-\lambda\overline{t}} \int\limits_{Y\times I} 1_{B_4}(\Pi_i(\overline{t},y),i)\,\mu_*(dy,di) + 1 - e^{-\lambda\overline{t}} \le e^{-\lambda\overline{t}}\mu_*(B_2) + 1 - e^{-\lambda\overline{t}} \\ &\le \mu_*(B_2) + 1 - e^{-\lambda\overline{t}}.\end{aligned}$$

From the last inequality and (5.1.15) it follows immediately that

$$\mu_*(B_4) \le \frac{1-e^{-\lambda\overline{t}}}{\eta} \le \frac{\lambda\overline{t}}{\eta} = \frac{\lambda r_0}{\eta}\,(r\min_j l_j)^s.$$

Consequently, by the choice of C, we obtain

$$\mu_*(B_4) \le C(r\min_j l_j)^s,$$

which contradicts (5.1.10) and completes the proof of (5.1.14).

Next, from (5.1.11), (5.1.12) and (5.1.14) it follows that

$$\mu_*(B((\overline{x},k),3(L/\sigma+\varepsilon)r)) \ge (3-2\eta)\mu_*(B_2) \ge (3-2\eta)(1-\eta)\mu_*(B_4),$$

thus

$$\mu_*(B_4) \le \frac{\mu_*(B((\overline{x},k),3(L/\sigma+\varepsilon)r))}{(3-2\eta)(1-\eta)}.$$

By the last inequality and the fact that $3(L\sigma^{-1}+\varepsilon)r > r_*$ we have

$$\mu_*(B_4) \le \frac{C(3(L/\sigma+\varepsilon)r)^s}{(3-2\eta)(1-\eta)} = \frac{(3(L/\sigma+\varepsilon))^s C(r\min_j l_j)^s}{(\min_j l_j)^s(3-2\eta)(1-\eta)}.$$

Since

$$\left(\frac{3(L/\sigma+\varepsilon)}{\min_j l_j}\right)^s = (3-2\eta)(1-\eta),$$

we obtain

$$\mu_*(B_4) \le C(r\min_j l_j)^s,$$

which contradicts (5.1.10). Thus $r_* = 0$ and

$$\mu_*(B((\overline{x},k),r)) \le Cr^s \quad \text{for } r > 0.$$

From the last statement it follows that $\underline{d}\mu_*(\overline{x},k) \ge s$. Letting $\varepsilon \to 0$ and $\eta \to 0$ completes the proof in the first case.

Case II: $\alpha < 0$. The proof goes as in Case I. We only indicate the main differences.

Let $\overline{x} \in Y$ and $k \in I$ be fixed. Choose $0 < \varepsilon \le \delta_k$ and use $r_0 = \varepsilon$,

$$C = \max\left\{\frac{1}{(\theta r_0)^s}, \frac{\lambda r_0}{\eta}, \left(\frac{2L(L/\sigma+\varepsilon)}{\sigma c_{\overline{x},k} r_0(\min_j l_j)^s}\right)^{s/(1-s)}\right\}$$

instead of (5.1.6) and (5.1.8). The other constants remain the same. Then

$$r \le \left(\frac{\sigma c_{\overline{x},k} r_0 (\min_j l_j)^s}{2L(L/\sigma+\varepsilon)}\right)^{1/(1-s)}.$$

We have

$$\|z_1-\overline{x}\| = \|\Pi_k(\overline{t},\overline{x})-\overline{x}\| \ge c_{\overline{x},k}\overline{t} \ge \frac{\sigma}{L}\,c_{\overline{x},k}\overline{t},$$
$$\|\overline{x}-z_0\| \ge \frac{\sigma}{L}\,e^{-\alpha\overline{t}} c_{\overline{x},k}\overline{t} \ge \frac{\sigma}{L}\,c_{\overline{x},k}\overline{t}.$$

The rest of the proof goes as in Case I. ■

5.2. The upper bound for the concentration dimension of an invariant measure. To prove the main results of this section we need the following lemma due to Lasota and Myjak [32]:

LEMMA 5.2.1. *Let the numbers $a_i \in [0,1]$ and $b_i \in (0,1)$ for $i \in J$ be given (here J is an arbitrary set of indices). Let μ be a probability measure. Assume that for some $r_0>0$ the Lévy concentration function Q_μ satisfies the condition*

$$Q_\mu(r) \ge \sup_{i\in J} a_i Q_\mu(r/b_i) \quad \textit{for } r\in(0,r_0).$$

Then

$$\overline{\dim}_L\,\mu \le \inf_{i\in J}\frac{\log a_i}{\log b_i}.$$

THEOREM 5.2.1. *Let the assumptions of Theorem 4.2.1 hold and let μ_0 be the unique invariant measure with respect to the operator P given by (4.2.6). In addition, assume that*

$$\sigma = \inf_{x\in Y,\, i,j\in I} p_{ij}(x) > 0, \tag{5.2.1}$$

$$\gamma = \inf_{x\in Y,\, s\in S} \overline{p}_s(x) > 0, \tag{5.2.2}$$

and

$$M_0 = \frac{L_q L}{\sigma\gamma} < 1.$$

Then

$$\overline{\dim}_L\,\mu_0 \le \begin{cases} \dfrac{\log\sigma\gamma}{\log LL_q - \log\sigma\gamma} & \textit{when } \alpha\le 0,\\[2ex] \displaystyle\inf_{M\in(M_0,1)} \frac{\log(\sigma\gamma(1-M^{\lambda/\alpha}))}{\log\frac{LL_q}{\sigma\gamma M}} & \textit{when } \alpha>0.\end{cases}$$

Proof. Let $\overline{x}\in Y$, $k\in I$ and $s\in S$ be fixed. From (4.2.3), (4.2.4), (5.2.1) and (5.2.2) we have

$$\|q_s(\Pi_k(t,x))-q_s(\overline{x})\| \le \frac{L_q}{\gamma}\,\|\Pi_k(t,x)-\overline{x}\| \le \frac{LL_q}{\sigma\gamma}\,e^{\alpha t}\|x-\Pi_k(-t,\overline{x})\|.$$

Therefore

$$\Big\{x\in Y : (x,k)\in B\Big(\Big(\Pi_k(-t,\overline{x}),k\Big),\frac{r\sigma\gamma}{LL_q e^{\alpha t}}\Big)\Big\}$$
$$\subset \{x\in Y : (q_s(\Pi_k(t,x)),k)\in B((q_s(\overline{x}),k),r)\}.$$

Since μ_0 is invariant, (4.2.6) shows that

$$\mu_0(B((q_s(\overline{x}),k),r)) \geq \sigma\gamma \int_0^\infty \int_{Y\times I} 1_{B((q_s(\overline{x}),k),r)}(q_s(\Pi_k(t,x)),k)\lambda e^{-\lambda t}\,dt\,\mu_0(dx,di)$$
$$\geq \sigma\gamma \int_0^\infty \mu_0\bigg(B\bigg((\Pi_k(-t,\overline{x}),k),\frac{r\sigma\gamma}{LL_q e^{\alpha t}}\bigg)\bigg)\lambda e^{-\lambda t}\,dt.$$

This implies

$$Q_{\mu_0}(r) \geq \sigma\gamma \int_0^\infty Q_{\mu_0}\bigg(\frac{r\sigma\gamma}{LL_q e^{\alpha t}}\bigg)\lambda e^{-\lambda t}\,dt. \tag{5.2.3}$$

We now consider two cases: $\alpha \leq 0$ and $\alpha > 0$. Suppose first that $\alpha \leq 0$. Then

$$Q_{\mu_0}\bigg(\frac{r\sigma\gamma}{LL_q e^{\alpha t}}\bigg) \geq Q_{\mu_0}\bigg(\frac{r\sigma\gamma}{LL_q}\bigg) \quad \text{for } t > 0 \text{ and } r > 0.$$

Consequently,

$$Q_{\mu_0}(r) \geq \sigma\gamma Q_{\mu_0}\bigg(\frac{r\sigma\gamma}{LL_q}\bigg) \quad \text{for } r > 0.$$

From this and Lemma 5.2.1 we obtain

$$\overline{\dim}_L\,\mu_0 \leq \frac{\log\sigma\gamma}{\log LL_q - \log\sigma\gamma}.$$

Suppose now that $\alpha > 0$. Choose M such that $LL_q(\sigma\gamma)^{-1} < M < 1$. Then from (5.2.3) we obtain

$$Q_{\mu_0}(r) \geq \sigma\gamma \int_0^{\overline{t}} Q_{\mu_0}\bigg(\frac{r\sigma\gamma}{LL_q e^{\alpha t}}\bigg)\lambda e^{-\lambda t}\,dt \geq \sigma\gamma Q_{\mu_0}\bigg(\frac{Mr\sigma\gamma}{LL_q}\bigg)(1-e^{-\lambda\overline{t}}),$$

where $\overline{t} = -(\ln M)\alpha^{-1}$. From this and Lemma 5.2.1 we obtain

$$\overline{\dim}_L\,\mu_0 \leq \frac{\log(\sigma\gamma(1-M^{\lambda/\alpha}))}{\log\frac{LL_q}{\sigma\gamma M}},$$

which completes the proof, since $M \in (M_0, 1)$ was arbitrary. ■

5.3. Relationship between discrete and continuous-time random dynamical systems. Since the Markov process $\{(X(t),\xi(t))\}_{t\geq 0}$ is defined with the help of the Markov chain $\{(x_n,\xi_n)\}_{n\geq 0}$, it is natural to try to relate an invariant measure for the transition operator P, corresponding to the change of measures from jump to jump, to an invariant measure for the semigroup $\{P^t\}_{t\geq 0}$ generated by the process $\{(X(t),\xi(t))\}_{t\geq 0}$.

In this section we assume that

$$\Theta = S = \{1,\ldots,K\} \quad \text{and} \quad \overline{p}_s(x) = \nu(\{s\}) \quad \text{for } x\in Y,\ s\in S.$$

Then the results of Sections 4.2 and 4.3 hold.

We assume additionally that the first inequality in (4.2.2) is satisfied with $\psi_1(t) = \hat{\gamma}t$, $t \geq 0$, for some constant $\hat{\gamma}$.

By the definition of the process $\{(X(t),\xi(t))\}_{t\ge 0}$ and properties of Poisson processes we have

$$\mathbb{P}\{(X(h),\xi(h))_{(x,i)} = (\Pi_{\xi(t_1)}(h-t_1, q_{\eta(t_1)}(\Pi_i(t_1,x))),\xi(t_1))1_{[0,h]}(t_1) \\ + (\Pi_i(h,x),i)1_{[h,\infty)}(t_1)\} \ge 1-k_1h^2$$

for some positive constant k_1. Since $f\in C(Y\times I)$ is bounded and t_1 has the density distribution function $\lambda e^{-\lambda t}$, we obtain

$$(5.3.1)\qquad T^h f(x,i) = \sum_{j\in I}\sum_{s\in S}\int_0^h f(\Pi_j(h-t, q_s(\Pi_i(t,x))), i)p_{ij}(q_s(\Pi_i(t,x)))\overline{p}_s\lambda e^{-\lambda t}\,dt \\ + f(\Pi_i(h,x),i)e^{-\lambda h} + \varepsilon_1(h),$$

where $|\varepsilon_1(h)|\le \|f\|_0 k_1 h^2$.

In order to formulate the main result of this section we introduce two operators $H,G:\mathcal{M}_1\to\mathcal{M}_1$ by the formulas

$$H\mu(A) = \sum_{s\in S}\int_{Y\times I} 1_A(q_s(x),k)\overline{p}_s\,\mu(dx,dk),$$

$$G\mu(A) = \sum_{i\in I}\int_{Y\times I}\int_0^\infty 1_A(\Pi_i(t,x),i)p_{ki}(x)\lambda e^{-\lambda t}\,dt\,\mu(dx,dk)\qquad \text{for } A\in\mathcal{B}(Y\times I).$$

In this section we give a one-to-one correspondence between the set of P-invariant measures and the set of invariant measures for $\{P^t\}_{t\ge0}$. Similar results have been proved by Davis [4, Proposition 34.36]. They have also been studied in [13, 14, 34].

Theorem 5.3.1. *Let the assumptions of Theorem* 4.3.1 *hold. If $\mu_0\in\mathcal{M}_1$ is an invariant measure for the Markov operator P given by* (4.2.6), *then $\mu_* = G\mu_0$ is an invariant measure for the Markov semigroup $\{P^t\}_{t\ge0}$ given by* (4.3.5), (4.3.6).

On the other hand, if $\mu_*\in\mathcal{M}_1$ satisfies $P^t\mu_*=\mu_*$ for $t\ge0$, then $\mu_0=H\mu_*$ is an invariant measure for the Markov operator P.

Proof. Denote by $\{S^t\}_{t\ge0}$ the semigroup of operators corresponding to the system $\Pi_i:\mathbb{R}_+\times Y\to Y$, $i\in I$, i.e.

$$S^t f(x,i) = f(\Pi_i(t,x),i)\qquad \text{for } f\in\overline{C}_L(Y\times I), (x,i)\in Y\times I,$$

where $\overline{C}_L(Y\times I)$ denotes the closure of the space of all bounded Lipschitzean functions with the supremum norm $\|\cdot\|_0$.

We denote by $\{\widetilde{T}^t\}_{t\ge0}$ the semigroup of operators given by

$$\widetilde{T}^t f = T^t f\qquad \text{for } f\in\overline{C}_L(Y\times I).$$

Following the main ideas of the proof of Lemma 4.3.2 we obtain $\widetilde{T}^t:\overline{C}_L(Y\times I)\to\overline{C}_L(Y\times I)$. By (4.2.3), (4.3.4), and (4.3.7) it follows that $S^t:\overline{C}_L(Y\times I)\to\overline{C}_L(Y\times I)$ is a continuous semigroup.

Let A_0 be the infinitesimal generator of the semigroup $\{S^t\}_{t\ge0}$ with the domain

$$D(A_0) = \left\{f\in\overline{C}_L(Y\times I) : \lim_{t\downarrow0}\frac{1}{t}(S^t f - f) \text{ exists}\right\}.$$

Since the semigroup $\{S^t\}_{t\ge0}$ is a continuous semigroup of contractions, $D(A_0)$ is dense in $\overline{C}_L(Y\times I)$. Denote by B the infinitesimal generator for the semigroup $\{\widetilde{T}^t\}_{t\ge0}$. By (5.3.1) we have

$$Bf = A_0 f - \lambda f + \lambda QWf \quad \text{for } f\in D(B), \tag{5.3.2}$$

where $Q : C(Y\times I)\to C(Y\times I)$ and $W : C(Y\times I)\to C(Y\times I)$ are bounded linear operators given by the formulas

$$\begin{aligned} Qf(x,i) &= \sum_{s\in S} f(q_s(x),i)\overline{p}_s \quad \text{for } f\in C(Y\times I) \text{ and } (x,i)\in Y\times I,\\ Wf(x,i) &= \sum_{j\in I} f(x,j)p_{ij}(x) \quad \text{for } f\in C(Y\times I) \text{ and } (x,i)\in Y\times I. \end{aligned} \tag{5.3.3}$$

The domains $D(B)$ and $D(A_0)$ are identical.

Let us now assume that μ_0 is an invariant measure for P and let $\mu_* = G\mu_0$. Since

$$R(\lambda,A_0)f(x,i) = \int_0^\infty e^{-\lambda t}S^t f(x,i)\,dt \quad \text{for } f\in\overline{C}_L(Y\times I) \text{ and } (x,i)\in Y\times I,$$

from (4.2.7) and (5.3.3) we obtain

$$Uf = \lambda WR(\lambda,A_0)Qf \quad \text{for } f\in\overline{C}_L(Y\times I).$$

Since

$$\langle\lambda WR(\lambda,A_0)f,\mu\rangle = \langle f,G\mu\rangle \quad \text{for } f\in\overline{C}_L(Y\times I),\ \mu\in\mathcal{M}_1$$

and

$$\langle Uf,\mu\rangle = \langle f,P\mu\rangle \quad \text{for } f\in C(Y\times I),\ \mu\in\mathcal{M}_1,$$

we have

$$\begin{aligned}\langle f,\mu_*\rangle &= \langle f,G\mu_0\rangle = \langle\lambda WR(\lambda,A_0)f,\mu_0\rangle = \langle\lambda WR(\lambda,A_0)f,P\mu_0\rangle = \langle\lambda UWR(\lambda,A_0)f,\mu_0\rangle\\ &= \langle\lambda QWR(\lambda,A_0)f,G\mu_0\rangle = \langle\lambda QWR(\lambda,A_0)f,\mu_*\rangle \quad \text{for } f\in\overline{C}_L(Y\times I).\end{aligned}$$

Thus

$$\langle f,\mu_*\rangle = \langle\lambda QWR(\lambda,A_0)f,\mu_*\rangle \quad \text{for } f\in\overline{C}_L(Y\times I).$$

Substituting $f = (\lambda I - A_0)g$ gives

$$\langle(\lambda I - A_0)g,\mu_*\rangle = \langle\lambda QWg,\mu_*\rangle \quad \text{for } g\in D(A_0),$$

which according to (5.3.2) reduces to

$$\langle Bg,\mu_*\rangle = 0 \quad \text{for } g\in D(B) = D(A_0).$$

Now, since $\widetilde{T}^s h\in D(B)$ for $h\in D(B)$ and

$$\widetilde{T}^t h - h = \int_0^t B\widetilde{T}^s h\,ds \quad \text{for } h\in D(B),$$

we obtain

$$\langle\widetilde{T}^t h - h,\mu_*\rangle = 0 \quad \text{for } h\in D(B),\ t>0.$$

Since $D(B) = D(A_0)$ is dense in $\overline{C}_L(Y\times I)$ and $\widetilde{T}^t h = T^t h$ for $h\in\overline{C}_L(Y\times I)$ we have

$$\langle T^t h,\mu_*\rangle = \langle h,\mu_*\rangle \quad \text{for } h\in\overline{C}_L(Y\times I)),\ t>0.$$

The last condition is equivalent to

$$\langle h, P^t\mu_*\rangle = \langle h, \mu_*\rangle \quad \text{for } h \in \overline{C}_L(Y \times I).$$

From the Aleksandrov theorem it follows that $P^t\mu_* = \mu_*$ for $t \geq 0$, which is the desired conclusion.

Next, we show that if μ_* is an invariant measure of the semigroup $\{P^t\}_{t\geq 0}$ then $\mu_0 = H\mu_*$ is an invariant measure of the operator P. From $P^t\mu_* = \mu_*$ it follows that

$$\langle T^t g - g, \mu_*\rangle = 0 \quad \text{for } g \in C(Y \times I),\ t \geq 0.$$

Thus

$$\langle \widetilde{T}^t g - g, \mu_*\rangle = 0 \quad \text{for } g \in \overline{C}_L(Y \times I),\ t \geq 0.$$

Since B is the infinitesimal generator of the semigroup $\{\widetilde{T}^t\}_{t\geq 0}$, we obtain

$$\langle Bg, \mu_*\rangle = 0 \quad \text{for } g \in D(B).$$

According to (5.3.2) this equality may be rewritten in the form

$$\langle (\lambda I - A_0)g, \mu_*\rangle = \langle \lambda QWg, \mu_*\rangle \quad \text{for } g \in D(B) = D(A_0).$$

Substituting $g = R(\lambda, A_0)f$ gives

$$\langle f, \mu_*\rangle = \langle \lambda QWR(\lambda, A_0)f, \mu_*\rangle \quad \text{for } f \in \overline{C}_L(Y \times I),$$

which implies

$$\begin{aligned}\langle f, \mu_0\rangle &= \langle f, H\mu_*\rangle = \langle Qf, \mu_*\rangle = \langle \lambda QWR(\lambda, A_0)Qf, \mu_*\rangle \\ &= \langle \lambda WR(\lambda, A_0)Qf, \mu_0\rangle = \langle Uf, \mu_0\rangle \quad \text{for} \quad f \in \overline{C}(Y \times I).\end{aligned}$$

This, by the Aleksandrov theorem, forces $\mu_0 = P\mu_0$. ∎

We now use Theorem 5.3.1 to compare the concentration dimensions of an invariant measure for the semigroup $\{P^t\}_{t\geq 0}$ and of an invariant measure for the transition operator P describing the change of distributions from jump to jump. A similar problem for the simpler case when $\{P^t\}_{t\geq 0}$ is a semigroup generated by a Poisson driven differential equation is considered in [14] and [34].

Assume that the hypotheses of Theorem 4.3.1 are satisfied. Let $\mu_0 \in \mathcal{M}_1$ be the invariant measure for the Markov operator P given by (4.2.6) and let $\mu_* \in \mathcal{M}_1$ be the invariant measure for the Markov semigroup $\{P^t\}_{t\geq 0}$ given by (4.3.5), (4.3.6).

Define

$$L_0 = \inf_{s\in S} \inf \left\{ \frac{\|q_s(x) - q_s(y)\|}{\|x - y\|} : x \neq y \right\}. \tag{5.3.4}$$

THEOREM 5.3.2. *Assume that* $L_0 > 0$. *Then*

$$\underline{\dim}_L\, \mu_* \leq \underline{\dim}_L\, \mu_0 \quad \textit{and} \quad \overline{\dim}_L\, \mu_* \leq \overline{\dim}_L\, \mu_0. \tag{5.3.5}$$

Proof. Let $\overline{x} \in Y$ and $k \in I$ be fixed. From (5.3.4) it follows that

$$\operatorname{diam}_{\overline{\varrho}}\{(x,i) : (q_s(x), i) \in B((\overline{x}, k), r)\} \leq 2rL_0^{-1}$$

for every $s \in S$ and $0 < r < c$, where $c = \varrho_c(i,j)$ for $i \neq j$. By the definition of Q_{μ_*},

$$\mu_*(\{(x,i) : (q_s(x), i) \in B((\overline{x}, k), r)\}) \leq Q_{\mu_*}(2rL_0^{-1}). \tag{5.3.6}$$

Theorem 5.3.1 gives $\mu_0 = H\mu_*$, where

$$H\mu(A) = \sum_{s\in S} \int\limits_{Y\times I} 1_A(q_s(x), i)\overline{p}_s\, \mu(dx, di) \quad \text{for } \mu \in \mathcal{M}_1,\ A \in \mathcal{B}(Y \times I).$$

Thus

$$\mu_0(B((\overline{x}, k), r)) = \sum_{s\in S} \int\limits_{Y\times I} 1_{B((\overline{x},k),r)}(q_s(x), i)\overline{p}_s\, \mu_*(dx, di).$$

From (5.3.6) we have

$$\mu_0(B((\overline{x}, k), r)) \le Q_{\mu_*}(2rL_0^{-1}).$$

Consequently,

$$Q_{\mu_0}(r) \le Q_{\mu_*}(2rL_0^{-1}),$$

which implies (5.3.5). ∎

To obtain a lower bound for $\underline{\dim}_L\, \mu_*$ we need a more restrictive assumption concerning the transformations $\Pi_i : \mathbb{R}\times Y \to Y$, $i \in I$. Namely, we assume that there exist constants $\beta \in \mathbb{R}$ and $c_i > 0$, $i \in I$, such that

$$\|\Pi_i(t, x) - \Pi_i(t, y)\| \ge c_i e^{-\beta t}\|x - y\| \quad \text{for } t \ge 0,\ x, y \in Y \text{ and } i \in I. \tag{5.3.7}$$

Theorem 5.3.3. *Let* $\Pi_i : \mathbb{R} \times Y \to Y$, $i \in I$, *satisfy condition* (5.3.7). *If* $\lambda > \beta\,\underline{\dim}_L\, \mu_0$ *then*

$$\underline{\dim}_L\, \mu_* \ge \underline{\dim}_L\, \mu_0. \tag{5.3.8}$$

Proof. Let $\overline{x} \in Y$ and $k \in I$ be fixed. Fix $h < \underline{\dim}_L\, \mu_0$ such that $\lambda > \beta h$. From the definition of $\underline{\dim}_L\, \mu_0$ it follows that there exists $r_0 \in (0, c)$, where $c = \varrho_c(i, j)$ for $i \ne j$, such that

$$Q_{\mu_0}(r) \le r^h \quad \text{for } r \in (0, r_0). \tag{5.3.9}$$

By Theorem 5.3.1 we have $\mu_* = G\mu_0$, where

$$G\mu(A) = \sum_{j\in I} \int\limits_{Y\times I} \int\limits_0^\infty 1_A(\Pi_j(t, x), j)p_{ij}(x)\lambda e^{-\lambda t}\, dt\, \mu(dx, di) \quad \text{for } A \in \mathcal{B}(Y \times I),$$

and consequently

$$\mu_*(B((\overline{x}, k), r)) = \int\limits_0^\infty \int\limits_{Y\times I} 1_{B((\overline{x},k),r)}(\Pi_k(t, x), k)\lambda e^{-\lambda t} p_{ik}(x)\, dt\, \mu_0(dx, di),$$

by the fact that $r < r_0 < c$. Set

$$\sigma_j = \sup\{p_{ij}(x) : (x, i) \in Y \times I\}.$$

Then we obtain

$$\mu_*(B((\overline{x}, k), r)) \le \sigma_k \int\limits_0^\infty \mu_0(\{(x, i) : (\Pi_k(t, x), k) \in B((\overline{x}, k), r)\})\lambda e^{-\lambda t}\, dt.$$

By (5.3.7) we have

$$\{x : (\Pi_k(t, x), k) \in B((\overline{x}, k), r)\} \subset B(\Pi_k(-t, \overline{x}), rc_k^{-1}e^{\beta t}).$$

Thus

$$\mu_*(B((\overline{x},k),r)) \le \sigma_k N \int_0^\infty Q_{\mu_0}(rc_k^{-1}e^{\beta t})\lambda e^{-\lambda t}\,dt.$$

Let

$$r < \min\{r_0, r_0 \min_k c_k\}.$$

Consider first the case of $\beta > 0$ and define $T(r) = \beta^{-1}\ln(r_0 c_k r^{-1})$. Then

$$\mu_*(B((\overline{x},k),r)) \le \sigma_k N\Big(\int_0^{T(r)} Q_{\mu_0}(rc_k^{-1}e^{\beta t})\lambda e^{-\lambda t}\,dt + \int_{T(r)}^\infty Q_{\mu_0}(rc_k^{-1}e^{\beta t})\lambda e^{-\lambda t}\,dt\Big).$$

Since $rc_k^{-1}e^{\beta t} < r_0$ for $t \in (0, T(r))$, we can use inequality (5.3.9) to get an upper bound

$$\mu_*(B((\overline{x},k),r)) \le \left(\frac{\lambda N}{\lambda - \beta h}\cdot\frac{\sigma_k}{(c_k)^h}\right)r^h + \sigma_k N e^{-\lambda T(r)}.$$

Since $r < r_0 c_k$ and $\lambda > \beta h$, we obtain

$$\mu_*(B((\overline{x},k),r)) \le Cr^h \quad \text{for } r < \min\{r_0, r_0 \min_k c_k\}, \tag{5.3.10}$$

where

$$C = \frac{\lambda N}{\lambda - \beta h}\max_k \frac{\sigma_k}{(c_k)^h} + \frac{N}{r_0^h}\max_k \frac{\sigma_k}{(c_k)^h}.$$

Since inequality (5.3.10) is satisfied for every $(\overline{x},k) \in Y \times I$, by the definition of $Q_{\mu_*}(r)$ we obtain

$$Q_{\mu_*}(r) \le Cr^h. \tag{5.3.11}$$

When $\beta \le 0$ the calculations are even simpler and (5.3.11) holds with

$$C = \frac{\lambda N}{\lambda - \beta h}\max_k \frac{\sigma_k}{(c_k)^h}.$$

From inequality (5.3.11) it follows that

$$\underline{\dim}_L \mu_* \ge h.$$

Passing to the limit as $h \to \underline{\dim}_L \mu_0$ we obtain (5.3.8). ∎

Remark 5.3.1. Let the hypotheses of Theorem 4.3.1 hold. Assuming that $\Pi_i : \mathbb{R} \times Y \to Y$, $i \in I$, satisfy the following condition: there exist constants $c_i > 0$, $i \in I$, such that

$$\|\Pi_i(t,x) - \Pi_i(t,y)\| \ge c_i\|x - y\| \quad \text{for } t \ge 0, \quad x, y \in Y \text{ and } i \in I,$$

Theorems 5.3.2 and 5.3.3 can be restated as:

If $L_0 > 0$ then $\underline{\dim}_L \mu_* = \underline{\dim}_L \mu_0$.

6. Applications

Randomly chosen dynamical systems with randomly chosen jumps described in this paper generalize many important and widely studied random systems, for example dynamical systems generated by learning systems, by Poisson driven stochastic differential equations, iterated function systems with an infinite family of transformations, and irreducible Markov systems.

In this section we show how our results may be applied to ensure the existence of an invariant measure and asymptotic stability of corresponding Markov operators for some of these particular systems. We also use the results of Section 5 to obtain estimates for dimensions of invariant measures for dynamical systems generated by learning systems and Poisson driven stochastic differential equations.

6.1. Iterated function systems. Let $(Y, \|\cdot\|)$ be a separable Banach space. An *iterated function system* (IFS) consists of a sequence of continuous transformations

$$q_s : Y \to Y, \quad s = 1, \dots, K,$$

and a probability vector

$$\overline{p}_s : Y \to [0, 1], \quad s = 1, \dots, K.$$

Such a system is briefly denoted by $(q, \overline{p})_K = (q_1, \dots, q_K, \overline{p}_1, \dots, \overline{p}_K)$. The action of an IFS can be roughly described as follows. We choose an initial point x_0 and we randomly select from the set $S = \{1, \dots, K\}$ an integer s_0 in such a way that the probability of choosing it is $\overline{p}_{s_0}(x_0)$. If s_0 is drawn, we define $x_1 = q_{s_0}(x_0)$. Having x_1 we select s_1 in such a way that the probability of choosing it is $\overline{p}_{s_1}(x_1)$. Now we define $x_2 = q_{s_1}(x_1)$ and so on.

This system is quite often called a learning system. The system "learns" because in a new position x_n it uses a new strategy $\overline{p}(x_n)$ for choosing the next step.

In [2] Barnsley *et al.* considered the evolution of distributions due to the action of randomly chosen transformations, so called iterated function systems with place dependent probabilities, and provided sufficient conditions for the existence of an invariant measure and for stability. In [35] Lasota and Yorke generalized those results.

It is evident that IFS is a particular example of a random dynamical system with randomly chosen jumps. Consider a dynamical system of the form $I = \{1\}$ and $\Pi_1(t, x) = x$ for $x \in Y$, $t \in \mathbb{R}_+$. Moreover assume that $p_1(x) = 1$ and $p_{11}(x) = 1$ for $x \in Y$. Then we obtain an IFS $(q, \overline{p})_K$.

Denoting by $\tilde{\mu}_n$, $n \in \mathbb{N}$, the distribution of x_n, i.e., $\tilde{\mu}_n(A) = \mathbb{P}(x_n \in A)$ for $A \in \mathcal{B}(Y)$, we define $\widetilde{P}$ as the transition operator such that $\tilde{\mu}_{n+1} = \widetilde{P}\tilde{\mu}_n$ for $n \in \mathbb{N}$. The transition operator corresponding to the learning system $(q, \overline{p})_K$ is given by

$$\widetilde{P}\mu(A) = \sum_{s \in S} \int_Y 1_A(q_s(x))\overline{p}_s(x)\,\mu(dx) \quad \text{for } A \in \mathcal{B}(Y),\ \mu \in \mathcal{M}_1(Y). \tag{6.1.1}$$

From Theorems 4.2.2 and 4.2.3 we immediately obtain the following result, due to Barnsley *et al.* [2] (see also [35]):

THEOREM 6.1.1. *Let $(q, \overline{p})_K$ be an iterated function system satisfying the following conditions*:

(i) *for the probability vector $\overline{p}$, the Dini condition holds and*

$$\inf_{x \in Y} \overline{p}_s(x) > 0 \quad \textit{for } s \in S, \tag{6.1.2}$$

(ii) *the transformations* $q_s : Y \to Y$ *are continuous and satisfy* (4.2.4) *with* $L_q < 1$.

Then the operator $\widetilde{P}$ *given by* (6.1.1) *is asymptotically stable.*

Examples 6.1.1 and 6.1.2 are taken from [29] and [31]:

Example 6.1.1. The asymptotic behavior of a learning system heavily depends on the properties of the functions $\overline{p}_s$. First of all, they must be strictly positive. Consider, for example, the system $(q, \overline{p})_2$ acting on the space $Y = [0, 1]$ with the following transformations:

$$q_1(x) = 0, \quad q_2(x) = 1, \quad \overline{p}_1(x) = 1 - x, \quad \overline{p}_2(x) = x \qquad \text{for } x \in [0, 1].$$

These assumptions imply that for $x_0 = 0$ we have $x_1 = q_1(x_0) = 0$ with probability one and further by induction $x_n = 0$ with probability one for every $n \geq 0$. Analogously, if $x_0 = 1$ then also $x_n = 1$ with probability one for every $n \geq 0$. Thus in the first case $\tilde{\mu}_n(\{0\}) = 1$ and in the second $\tilde{\mu}_n(\{0\}) = 0$ for all n. This shows that the system is not asymptotically stable.

Our theorems imply the weak convergence, but the stationary measure μ_0 may be singular and in this case the convergence cannot be strong:

Example 6.1.2. Let $Y = \mathbb{R}$, $q_1(x) = x$ and $q_2(x) = 0$ for $x \in \mathbb{R}$. Evidently for every probability vector (p_1, p_2) with $p_1 < 1$ condition (4.2.4) is satisfied. Thus for every $\mu \in \mathcal{M}_1(Y)$ the sequence $\{\widetilde{P}^n \mu\}_{n \geq 1}$ given by (6.1.1) converges weakly to $\mu_0 = \delta_0$.

From the proof of Theorem 5.2.1 we immediately obtain the following result, due to Lasota and Myjak [32]:

Theorem 6.1.2. *Let* $(q, \overline{p})_K$ *be an iterated function system having an invariant measure* $\mu_0 \in \mathcal{M}_1$. *Assume that the transformations* $q_s : Y \to Y$, $s \in S$, *satisfy the Lipschitz condition*

$$\|q_s(x) - q_s(y)\| \leq L_q \|x - y\| \qquad \textit{for } x, y \in Y$$

with $L_q < 1$ *and let*

$$\gamma = \inf_{x \in Y,\, s \in S} \overline{p}_s(x) > 0.$$

Then

$$\overline{\dim}_L \, \mu_0 \leq \frac{\log \gamma}{\log L_q}.$$

6.2. Irreducible Markov systems. Werner [52] extended iterated function systems with place dependent probabilities to much more general systems, namely, graph directed constructions on locally compact spaces with an open partition.

Barnsley *et al.* [2] and Werner [52] studied the problem of the existence of an invariant measure from the probabilistic point of view. In this section we aim to show that Werner's result may be studied by employing the methods used in Section 4.2. In this way, we also extend Werner's results from a locally compact space to the more general case of a complete separable metric space.

Let (Y, ϱ) be a complete and separable metric space and let $Y_1, \ldots, Y_N$ be a partition of Y into nonempty open subsets. For each $i \in I = \{1, \ldots, N\}$, let

$$w_{i1}, \ldots, w_{iN_i} : Y_i \to Y$$

be a family of Borel measurable maps such that for each $j \in \{1,\dots,N_i\}$, $N_i \in I$, there exists $n \in I$ such that $w_{ij}(Y_i) \subset Y_n$. Furthermore, for each $i \in I$, let

$$p_{i1},\dots,p_{iN_i} : Y_i \to [0,1]$$

be a family of positive Borel measurable probability functions, that is, $p_{ij} > 0$ for all j and $\sum_{j=1}^{N_i} p_{ij}(x) = 1$ for all $x \in Y_i$.

We call I the *set of vertices*, and the subsets $Y_1,\dots,Y_N$ the *vertex sets*. Further, we call

$$E = \{(i,n_i) : i \in \{1,\dots,N\},\ n_i \in \{1,\dots,N_i\}\}$$

the *set of edges* and we write

$$p_e := p_{in} \quad \text{and} \quad w_e := w_{in} \qquad \text{for } e := (i,n) \in E.$$

For an edge $e \in E$ we denote by $\mathbf{i}(e)$ the initial vertex of e, that is, $\mathbf{i}(e) = j$ if and only if $e = (j,k)$ for some $k \in \{1,\dots,N_j\}$. The *terminal vertex* $\mathbf{t}(e)$ for $e = (j,n) \in E$ is equal to k if and only if $w_e(Y_j) \subset Y_k$.

The quadruple $G = (I,E,\mathbf{i},\mathbf{t})$ is called a *directed multigraph* or *digraph*. A sequence (finite or infinite) $(\dots,e_{-1},e_0,e_1,\dots)$ of edges is called a *path* if $\mathbf{t}(e_k) = \mathbf{i}(e_{k+1})$ for all k.

A Markov system $(Y_{\mathbf{i}(e)}, w_e, p_e)_{e\in E}$ is called *irreducible* if its directed multigraph is irreducible, that is, there is a path from any vertex to any other.

An irreducible Markov system is said to have a *period* d if the set of vertices can be partitioned into d nonempty subsets $J_1,\dots,J_d$ such that

$$\mathbf{i}(e) \in J_i \ \Rightarrow\ \mathbf{t}(e) \in J_{i+1}$$

for all $e \in E$ (with $i+1$ taken mod d), and d is the largest number with this property. An irreducible Markov system with period 1 is called *aperiodic*.

To define a Markov operator on $\mathcal{B}(Y)$ associated with the Markov system under consideration we extend p_{ij} onto the whole space Y by zero; the maps w_{ij} are extended arbitrarily.

We define the Markov operator P on $\mathcal{M}_1$ by

$$P\mu(A) = \int_Y U1_A(x)\,\mu(dx) \qquad \text{for } A \in \mathcal{B}(Y) \text{ and } \mu \in \mathcal{M}_1, \tag{6.2.1}$$

where U is the dual operator on $B(Y)$ given by

$$Uf = \sum_{e\in E} p_e f \circ w_e \qquad \text{for } f \in B(Y). \tag{6.2.2}$$

We say that a system $(Y_{\mathbf{i}(e)}, w_e, p_e)_{e\in E}$ is globally concentrating, semi-concentrating or asymptotically stable when the Markov operator P given by (6.2.1), (6.2.2) has the corresponding property.

Recall that a function $f : Y \to \mathbb{R}$ is called *Dini-continuous* if the associated *modulus of continuity* Ψ, given by

$$\Psi(\tau) = \sup\{|f(x) - f(y)| : \varrho(x,y) \le \tau,\ x,y \in Y\},$$

satisfies the Dini condition (2.1.1).

We will assume that the Markov system $(Y_{\mathbf{i}(e)}, w_e, p_e)$ is *contractive*, i.e., there exists $0 < L < 1$ such that

$$\sum_{e\in E} p_e(x)\varrho(w_e(x), w_e(y)) \le L\varrho(x,y) \quad \text{for all } x, y \in Y_i \text{ and } i \in I. \tag{6.2.3}$$

We call the constant L an *average contracting rate* of the Markov system.

The Markov system $(Y_{\mathbf{i}(e)}, w_e, p_e)_{e\in E}$ is a particular example of the following random dynamical system with randomly chosen jumps. Consider the space $\widetilde{Y} = \bigcup_{i=1}^{N}(Y_i \times \{i\})$ where $Y_i \cap Y_j = \emptyset$, $i \neq j$, $S = \{1, \dots, \max_i N_i\}$ and $\Pi_1(\tau, (x,i)) = (x,i)$ for $x \in Y_i$, $i = 1, \dots, N$ and $\tau \in \mathbb{R}_+$. Set

$$q_j(x,i) = \begin{cases} (w_{ij}(x), \mathbf{t}((i,j))) & \text{for } j = 1, \dots, N_i, \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

and

$$\overline{p}_j(x,i) = \begin{cases} p_{ij}(x) & \text{for } j = 1, \dots, N_i, \\ 0 & \text{otherwise} \end{cases}$$

for $x \in Y_i$, $i = 1, \dots, N$. In this way we obtain a system (Π, q, p).

We point out that condition (6.2.3) does not guarantee that inequality (4.2.4) is satisfied for all $x, y \in \widetilde{Y}$, hence a contractive Markov system $(Y_{\mathbf{i}(e)}, w_e, p_e)_{e\in E}$ may not satisfy the hypotheses of Theorem 4.2.2.

Example 6.2.1. Consider $Y \subset \mathbb{R}^2$ with norm $\|(x,y)\|_1 = |x| + |y|$ for $(x,y) \in \mathbb{R}^2$. Let $Y_1 = [0,1] \times [0,1]$, $Y_2 = [3/2, 2] \times [0,2]$ and $Y_3 = [0,1] \times [3/2, 2]$. Consider the maps $w_{11} : Y_1 \to Y_2$, $w_{12} : Y_1 \to Y_3$, $w_{31} : Y_3 \to Y_2$, $w_{32} : Y_3 \to Y_1$ and $w_{21} : Y_2 \to Y_1$ given by

$$\begin{aligned} &w_{11}(x,y) = \left(\tfrac{1}{2}x + \tfrac{3}{2}, 2y\right), \quad w_{12}(x,y) = \left(x, \tfrac{1}{2}y + \tfrac{3}{2}\right), \\ &w_{31}(x,y) = \left(y, -\tfrac{1}{3}x + \tfrac{7}{6}\right), \quad w_{32}(x,y) = \left(x, \tfrac{2}{3}y - \tfrac{1}{3}\right), \\ &w_{21}(x,y) = \left(\tfrac{1}{2}y, -\tfrac{2}{3}x + \tfrac{4}{3}\right), \end{aligned}$$

with the corresponding probability functions

$$p_{11} = \tfrac{1}{4}1_{Y_1}, \quad p_{12} = \tfrac{3}{4}1_{Y_1}, \quad p_{31} = \tfrac{2}{3}1_{Y_3}, \quad p_{32} = \tfrac{1}{3}1_{Y_3}, \quad p_{21} = 1_{Y_2}.$$

An easy calculation shows that they define a contractive Markov system with an average contracting rate $\frac{8}{9}$ on Y_1, Y_2, Y_3 and this system does not satisfy condition (4.2.4).

Our main results concerning irreducible Markov systems defined on Polish spaces are the following:

Theorem 6.2.1. *Let $(Y_{\mathbf{i}(e)}, w_e, p_e)_{e\in E}$ be a contractive Markov system such that each $p_e|_{Y_{\mathbf{i}(e)}}$ is Dini continuous. Then the system has an invariant measure. Moreover the set $\mathcal{L}(\mu)$ is nonempty for $\mu \in \mathcal{M}_1$ and the set $\mathcal{L}(\mathcal{M}_1)$ is tight.*

Theorem 6.2.2. *Let $(Y_{\mathbf{i}(e)}, w_e, p_e)_{e\in E}$ be an irreducible contractive Markov system such that $p_e|_{Y_{\mathbf{i}(e)}}$ is Dini continuous and there exists $\delta > 0$ such that $p_e|_{Y_{\mathbf{i}(e)}} \ge \delta$ for all $e \in E$. If, in addition, the system is aperiodic, then it is asymptotically stable.*

We have divided the proofs of Theorems 6.2.1 and 6.2.2 into a sequence of lemmas.

Lemma 6.2.1. *Suppose that $(Y_{\mathbf{i}(e)}, w_e, p_e)_{e\in E}$ is a contractive Markov system. Then $(Y_{\mathbf{i}(e)}, w_e, p_e)_{e\in E}$ is globally concentrating. Moreover for every $\varepsilon > 0$ there exists a* *bounded Borel set* $B \subset Y$ *such that*

$$\liminf_{n\to\infty} P^n\mu(B) \geq 1-\varepsilon \quad \textit{for all } \mu \in \mathcal{M}_1.$$

Proof. By Proposition 3.2.4 it is enough to show that there exists a Lyapunov function V, bounded on bounded sets, such that

$$UV(x) \leq aV(x) + b \quad \text{for } x \in Y, \tag{6.2.4}$$

where a, b are nonnegative constants and $a < 1$. Choose $y_i \in Y_i$ for $i \in I$. Set $V(x) = \varrho(x, y_1)$ for $x \in Y$. Let $0 < L < 1$ be the average contracting rate as in (6.2.3). Then

$$\begin{aligned} UV(x) &= \sum_{e\in E} p_e(x)\varrho(w_e(x), y_1) \\ &\leq \sum_{e\in E} p_e(x)(\varrho(w_e(x), w_e(y_{\mathbf{i}(e)})) + \varrho(w_e(y_{\mathbf{i}(e)}), y_1)) \\ &\leq L\varrho(x, y_{\mathbf{i}(e)}) + \sum_{e\in E} p_e(x)\varrho(w_e(y_{\mathbf{i}(e)}), y_1) \\ &\leq L\varrho(x, y_1) + L\max_{j\in I}\varrho(y_j, y_1) + \max_{e\in E,\,\in I}\varrho(w_e(y_j), y_1) \quad \text{for } x \in Y. \end{aligned}$$

Hence (6.2.4) holds with $a = L$ and

$$b = L\max_{j\in I}\varrho(y_j, y_1) + \max_{e\in E,\,\in I}\varrho(w_e(y_j), y_1). \ \blacksquare$$

From now on we will assume that p_e, $e \in E$, is Dini continuous and let Ψ_e be its modulus of continuity. Set $\Psi = \sum_{e\in E}\Psi_e$. From Proposition 2.1.1 it follows that there exists $\varphi \in \Phi_0$ satisfying

$$\Psi(\tau) + \varphi(L\tau) \leq \varphi(\tau) \quad \text{for } \tau \geq 0.$$

We denote by $\mathcal{F}_\varphi$ the family of all continuous functions $f : Y \to \mathbb{R}$ such that $|f(x)| \leq 1$ and $|f(x) - f(y)| \leq \varphi(\tilde{\varrho}(x, y))$ for all $x, y \in Y$, with

$$\tilde{\varrho}(x,y) = \begin{cases} \varrho(x,y) & \text{if } x, y \in Y_i, \text{ for } i \in I, \\ \max(c, \varrho(x,y)) & \text{otherwise}, \end{cases}$$

where $c > 0$ is such that $\varphi(c) > 2$. It is obvious that $\tilde{\varrho}$ is a metric on Y equivalent to ϱ.

LEMMA 6.2.2. *Under the hypotheses of Theorem* 6.2.1 *the operator* P *given by* (6.2.1), (6.2.2) *is essentially nonexpansive.*

Proof. Fix $f \in \mathcal{F}_\varphi$. We have

$$|Uf(x)| = \Big|\sum_e p_e(x)f(w_e(x))\Big| \leq \sum_e p_e(x) = 1 \quad \text{for } x \in Y.$$

Further, from (6.2.3) it follows that

$$\begin{aligned} |Uf(x) - Uf(y)| &= \Big|\sum_e p_e(x)f(w_e(x)) - \sum_e p_e(y)f(w_e(y))\Big| \\ &\leq \sum_e |p_e(x) - p_e(y)| + \sum_e p_e(x)|f(w_e(x)) - f(w_e(y))| \\ &\leq \Psi(\varrho(x,y)) + \sum_e p_e(x)\varphi(\varrho(w_e(x), w_e(y))) \end{aligned}$$

for $x, y \in Y_i$, $i \in I$. Since φ is concave and nondecreasing,

$$|Uf(x) - Uf(y)| \le \Psi(\varrho(x,y)) + \varphi(L\varrho(x,y)) \le \varphi(\varrho(x,y)) = \varphi(\tilde{\varrho}(x,y))$$

for $x, y \in Y_i$, $i \in I$.

If x and y are in different Y_i, then $|Uf(x) - Uf(y)| \le 2 \le \varphi(c) \le \varphi(\tilde{\varrho}(x,y))$, which completes the proof. ■

LEMMA 6.2.3. *Under the hypotheses of Theorem* 6.2.1 *the Markov system* $(Y_{\mathbf{i}(e)}, w_e, p_e)_{e\in E}$ *is semi-concentrating.*

Proof. Lemma 6.2.1 shows that there exists a bounded Borel set $B \subset Y$ such that

$$\liminf_{n\to\infty} P^n\mu(B) > 1/2 \quad \text{for all } \mu \in \mathcal{M}_1.$$

Without loss of generality we may assume that $B_i = B \cap Y_i \ne \emptyset$ for $i \in I$. Fix $\varepsilon > 0$. Choose an integer $m \in \mathbb{N}$ such that $L^m \operatorname{diam}_{\tilde{\varrho}} B < \varepsilon$. Further, let $\eta > 0$ be such that

$$(1+\eta)^m L^m \operatorname{diam}_{\tilde{\varrho}} B \le \varepsilon.$$

For any $i \in I$, fix $y_i \in B_i$ and define $C \subset Y$ by

$$C = \bigcup_{i=1}^{N} \bigcup_{e_1,\ldots,e_m \in E} B(w_{e_m} \circ \cdots \circ w_{e_1}(y_i), \varepsilon).$$

Now (6.2.3) implies that for every $y \in Y_i$ there exists $I_y \subset E^m$ such that

$$\varrho(w_{e_m} \circ \cdots \circ w_{e_1}(y_i),\, w_{e_m} \circ \cdots \circ w_{e_1}(y)) \le (1+\eta)^m L^m \varrho(y_i, y)$$

for $(e_1, \ldots, e_m) \in I_y$ and

$$\sum_{(e_1,\ldots,e_m)\in I_y} p_{e_1}(y) p_{e_2}(w_{e_1}(y)) \cdots p_{e_m}(w_{e_{m-1}} \circ \cdots \circ w_{e_1}(y)) \ge \left(\frac{\eta}{1+\eta}\right)^m.$$

Observe that for every $y \in B_i$ and $(e_1, \ldots, e_m) \in I_y$ we have $w_{e_m} \circ \cdots \circ w_{e_1}(y) \in C$. Set $\alpha = (\eta(1+\eta)^{-1})^m/2$. By induction and the definition of C for each $n \in \mathbb{N}$ we obtain

$$P^{(m+n)}\mu(C) \ge \int_B \sum_{(e_1,\ldots,e_m)\in I_y} 1_C(w_{e_m} \circ \cdots \circ w_{e_1}(y)) p_{e_1}(y)$$
$$\cdots p_{e_m}(w_{e_{m-1}} \circ \cdots \circ w_{e_1}(y)) P^n\mu(dy) \ge \left(\frac{\eta}{1+\eta}\right)^m P^n\mu(B) \quad \text{for } \mu \in \mathcal{M}_1$$

and consequently

$$\liminf_{n\to\infty}\ P^n\mu(C) \ge \alpha \quad \text{for } \mu \in \mathcal{M}_1,$$

which completes the proof. ■

Proof of Theorem 6.2.1. From Lemmas 6.2.2 and 6.2.3 it follows that P is essentially nonexpansive and semi-concentrating. A simple application of Proposition 3.2.3 finishes the proof. ■

For the proof of Theorem 6.2.2 we need to know more about properties of irreducible digraphs.

For every $j \in I$ we denote by $l(j)$ the smallest number, say k, such that there is a path $(e_1, \ldots, e_k)$ with $\mathbf{i}(e_1) = j$ and $\mathbf{t}(e_k) = j$.

A path $\mathbf{c} = (e_1, \dots, e_m)$ is called a *cycle* if $\mathbf{i}(e_1) = \mathbf{t}(e_m)$. Further, a cycle is called *simple* if it does not contain any other cycle. Let $l(\mathbf{c})$ denote the length of $\mathbf{c}$, i.e., $l(\mathbf{c}) = m$ if $\mathbf{c} = (e_1, \dots, e_m)$.

We denote by $[k_1, \dots, k_M]$ the greatest common divisor of $k_1, \dots, k_M \in \mathbb{N}$.

Finally, $\mathbf{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_m\}$ denotes the set of all simple cycles in $(Y_{\mathbf{i}(e)}, w_e, p_e)_{e\in E}$.

REMARK 6.2.1. Observe that an irreducible aperiodic Markov system $(Y_{\mathbf{i}(e)}, w_e, p_e)_{e\in E}$ satisfies

$$[l(\mathbf{c}_1), \dots, l(\mathbf{c}_M)] = 1.$$

LEMMA 6.2.4. *If an irreducible Markov system is aperiodic, then for every $k, l \in I$ there exist $m \in \mathbb{N}$ and $(e_1, \dots, e_m)$, $(\tilde{e}_1, \dots, \tilde{e}_m)$ such that $\mathbf{i}(e_1) = k$, $\mathbf{i}(\tilde{e}_1) = l$ and $\mathbf{t}(e_m) = \mathbf{t}(\tilde{e}_m)$.*

Proof. Fix $k, l \in I$. Let $(e_1^k, \dots, e_p^k)$ and $(e_1^l, \dots, e_q^l)$ be paths in $(I, E, \mathbf{i}, \mathbf{t})$ starting from k, l, respectively and containing all successive cycles from $\mathbf{C}$. Assume that $\mathbf{t}(e_k^k) = \mathbf{t}(e_q^l)$. If $p = q$, then the proof is complete. Now, assume that $p > q$. Since $[l(\mathbf{c}_1), \dots, l(\mathbf{c}_M)] = 1$, there exist integers $m_1, \dots, m_M$ such that

$$\sum_{i=1}^{M} m_i l(\mathbf{c}_i) = p - q.$$

Let $J \subset \{1, \dots, M\}$ be such that $m_j < 0$ for $j \in J$ and $m_j \geq 0$ for $j \in \{1, \dots, M\} \setminus J$. Adding to $(e_1^k, \dots, e_p^k)$ the cycle composed of the cycles $\mathbf{c}_j$ taken m_j times for $j \in J$, and similarly adding to $(e_1^l, \dots, e_q^l)$ the cycle composed of the cycles $\mathbf{c}_j$ taken m_j times for $j \in \{1, \dots, M\} \setminus J$, we finish the proof of the lemma. ■

Proof of Theorem 6.2.2. From Theorem 6.2.1 it follows that P admits an invariant probability measure. In view of Lemma 6.2.3, from Theorem 3.2.1 it follows that to finish the proof of stability it remains to show that for every $\varepsilon > 0$ there is a $\beta > 0$ such that for every $\mu_1, \mu_2 \in \mathcal{M}_1$ there exist a bounded Borel set $A \subset Y$ with $\operatorname{diam}_{\tilde{\varrho}_\varphi} A \leq \varepsilon$ and $n \in \mathbb{N}$ satisfying

$$P^n \mu_i(A) \geq \beta \quad \text{for } i = 1, 2. \tag{6.2.5}$$

Fix $\varepsilon > 0$. According to Theorem 6.2.1 there is a compact set $K_0 \subset Y$ such that

$$\tilde{\mu}(K_0) \geq 4/5 \quad \text{for all } \tilde{\mu} \in \mathcal{L}(\mathcal{M}_1).$$

By the Aleksandrov theorem there exists a sequence $\{m_n\}_{n\geq 1}$ such that for each open set G with $K_0 \subset G$,

$$\liminf_{n\to\infty} P^{m_n} \mu_i(G) > 1/2 \quad \text{for } i = 1, 2.$$

Consequently, there exist $k, l \in I$ and a subsequence $\{\tilde{m}_n\}_{n\geq 1}$ of $\{m_n\}_{n\geq 1}$ such that

$$\liminf_{n\to\infty} P^{\tilde{m}_n} \mu_1(G_1) > 1/2N \quad \text{and} \quad \liminf_{n\to\infty} P^{\tilde{m}_n} \mu_2(G_2) > \frac{1}{2N}$$

for arbitrary open neighborhoods G_1, G_2 of $\tilde{K}_k = K_0 \cap Y_k$, $\tilde{K}_l = K_0 \cap Y_l$, respectively. By Lemma 6.2.4 we choose $\tilde{m} \in \mathbb{N}$ such that for $k, l \in I$ there exist paths $(\overline{e}_1, \dots, \overline{e}_m)$, $(\tilde{e}_1, \dots, \tilde{e}_m)$ satisfying $\mathbf{i}(\overline{e}_1) = k$, $\mathbf{i}(\tilde{e}_1) = l$ and $\mathbf{t}(\overline{e}_m) = \mathbf{t}(\tilde{e}_m)$ and $m \leq \tilde{m}$. Let $F_k =$

$w_{\bar{e}_m} \circ \cdots \circ w_{\bar{e}_1}(\tilde{K}_k)$ and $F_l = w_{\tilde{e}_m} \circ \cdots \circ w_{\tilde{e}_1}(\tilde{K}_l)$. Set $F_0 = F_k \cup F_l$ and observe that $F_0 \subset Y_{\mathbf{t}(\bar{e}_m)}$. It is easily seen that

$$\liminf_{n\to\infty} P^{m_n+m}\mu_i(\tilde{G}) > \frac{1}{2N}\,\delta^m \geq \frac{1}{2N}\,\delta^{\tilde{m}} \quad \text{for } i = 1,2 \tag{6.2.6}$$

and every open neighborhood $\tilde{G}$ of F_0.

Choose an integer $n \in \mathbb{N}$ such that

$$L^n \cdot \operatorname{diam}_{\tilde{\varrho}} F_0 \leq \varepsilon/3.$$

For $x \in F_0$ and $(e_1, \ldots, e_n) \in E^n$ be such that $\mathbf{i}(e_1) = \mathbf{t}(\bar{e}_m)$ we define

$$O_{(e_1,\ldots,e_n)}(x) = \{y \in Y_{\mathbf{t}(\bar{e}_m)} : \varrho(w_{e_n} \circ \cdots \circ w_{e_1}(x), w_{e_n} \circ \cdots \circ w_{e_1}(y)) < \varepsilon/3\},$$

$$O_x = \bigcap O_{(e_1,\ldots,e_n)}(x) \quad \text{for } x \in F_0,$$

where the intersection is taken over all paths $(e_1, \ldots, e_n)$ in the digraph $(Y_{\mathbf{i}(e)}, w_e, p_e)$ starting from $\mathbf{t}(e_m)$. Since F_0 is a compact set, there exists $s_0 \geq 1$ such that

$$F_0 \subset \bigcup_{i=1}^{s_0} O_{y_i}.$$

Set $\tilde{G} = \bigcup_{i=1}^{s_0} O_{y_i}$. We claim that (6.2.5) holds with $\beta = \delta^{n+\tilde{m}}/(2Ns_0)$. Indeed, by (6.2.6) there exists $M \in \mathbb{N}$ such that

$$P^{M+m}\mu_i(\tilde{G}) > \frac{1}{2N}\,\delta^{\tilde{m}} \quad \text{for } i = 1,2,$$

therefore there exist $O_1 = O_{y_k}$ and $O_2 = O_{y_l}$ such that

$$P^{M+m}\mu_i(O_i) > \frac{1}{2Ns_0}\,\delta^{\tilde{m}},$$

thus, by (6.2.3) and the definition of O_1 and O_2, we can find a path $(e'_1, \ldots, e'_n)$ such that the set

$$A = w_{e'_n} \circ \cdots \circ w_{e'_1}(O_1) \cup w_{e'_n} \circ \cdots \circ w_{e'_1}(O_2)$$

satisfies $\operatorname{diam}_{\tilde{\varrho}_\varphi} A < \varepsilon$, which implies

$$P^{M+m+n}\mu_i(A) \geq \int_{O_i} p_{e'_1}(y) p_{e'_2}(w_{e'_1}(y)) \cdots p_{e'_n}(w_{e'_{n-1}} \circ \cdots \circ w_{e'_1}(y))$$

$$\times 1_A(w_{e'_n} \circ \cdots \circ w_{e'_1}(y)) P^{M+m}\mu_i(dy) \geq \frac{\delta^{n+\tilde{m}}}{2Ns_0} = \beta \quad \text{for } i = 1,2. \blacksquare$$

6.3. Mathematical theory of the cell cycle. We consider the infinite family of transformations $S_t : Y \to Y$ given by

$$S_t(x) = q(\Pi(t,x)) \quad \text{for } t \geq 0,\ x \in Y,$$

where $\Pi : \mathbb{R}_+ \times Y \to Y$ is a semidynamical system and $q : Y \to Y$ is a continuous function. Let $\{t_n\}_{n\geq 0}$ be a sequence of nonnegative random variables such that the increments $\Delta t_n = t_n - t_{n-1}$ are independent and have the same density distribution function $g(t) = \lambda e^{-\lambda t}$, $t \geq 0$. Set

$$x_{n+1} = S_{\Delta t_n}(x_n) \quad \text{for } n = 0, 1, \ldots. \tag{6.3.1}$$

Equation (6.3.1) defines an iterated function system with an infinite number of transformations. At each step the new transformation is selected according to the density distribution function $\lambda e^{-\lambda t}$.

Equations similar to (6.3.1) are discussed in the mathematical theory of the cell cycle [5, 29, 30, 50, 51]. For example, in [29] Lasota considered the following model:

Let $Y = \mathbb{R}^d$. The values $t_1, t_2, \ldots$ denote the birth times and x_n represents the distribution of substances of cells just before mitosis in the nth generation. Thus it is natural to assume that $q(x) = x/2$, since after mitosis each daughter cell obtains exactly half of the components of the mother cell.

Assume that Π, which describes the evolution of the amounts of real chemicals, satisfies

$$\|\Pi(t,x) - \Pi(t,y)\| \le e^{\alpha t}\|x-y\| \quad \text{for } x, y \in \mathbb{R}^d,\ t \in \mathbb{R}_+ \tag{6.3.2}$$

for some $\alpha > 0$ and there exists $x_* \in \mathbb{R}^d$ such that

$$\sup_{t\ge 0} \|\Pi(t, x_*)\| < \infty. \tag{6.3.3}$$

Further, we assume that

$$\alpha/\lambda < 1/2. \tag{6.3.4}$$

It is reasonable to think that the behavior of (6.3.1) can be described by the sequence of distributions

$$\tilde{\mu}_n(A) = \mathbb{P}(x_n \in A) \quad \text{for } n = 1, 2, \ldots,\ A \in \mathcal{B}(\mathbb{R}^d). \tag{6.3.5}$$

By Theorem 4.2.3 it follows that the sequence $\{\tilde{\mu}_n\}_{n\ge 1}$ given by (6.3.5) converges weakly to a unique $\tilde{\mu}_0$.

This fact allows one to obtain some information concerning the behavior of x_n (for example by using some ergodic theorems). Moreover, using the Aleksandrov theorem for the weak convergence, there is some biological consequence of the weak convergence of $\tilde{\mu}_n$ to $\tilde{\mu}_0$. In the space of dynamical systems (6.3.1) satisfying conditions (6.3.2)–(6.3.4) most of the systems have a singular stationary measure $\tilde{\mu}_0$. This fact may have an important biological consequence: with high probability x_n belongs to a small set. This means that the composition of substances (at birth) is not arbitrary and the cell is highly structured.

6.4. Randomly connected differential equations with Poisson-type perturbations. In this section we shall study stochastic differential equations driven by jump-type processes. They are typically of the form

$$dX(t) = a(X(t), \xi(t))\,dt + \int_{\Theta} b(X(t), \theta)\,\mathcal{N}_p(dt, d\theta) \quad \text{for } t \ge 0 \tag{6.4.1}$$

with the initial condition

$$X(0) = x_0, \tag{6.4.2}$$

where $\{X(t)\}_{t\ge 0}$ is a stochastic process with values in a separable Banach space $(Y, \|\cdot\|)$, or more explicitly

$$(6.4.3)\qquad X(t) = x_0 + \int_0^t a(X(s), \xi(s))\, ds + \int_0^t \int_\Theta b(X(s-), \theta)\, \mathcal{N}_p(ds, d\theta) \quad \text{for } t \geq 0$$

with probability one. Here $\mathcal{N}_p$ is a Poisson random counting measure, $\{\xi(t)\}_{t\geq 0}$ is a stochastic process with values in a finite set $I = \{1, \ldots, N\}$, the solution $\{X(t)\}_{t\geq 0}$ has values in Y and is right-continuous with left-hand limits, i.e. $X(t) = X(t+) = \lim_{s\to t^+} X(s)$ for all $t \geq 0$, and the left-hand limits $X(t-) = \lim_{s\to t^-} X(s)$ exist and are finite for all $t > 0$ (equalities here mean equalities with probability one).

In order to get existence and uniqueness of solutions to equation (6.4.3), it is necessary to put some restrictions on the objects a, b, ξ, and $\mathcal{N}_p$. In our study we make the following assumptions:

On a probability space $(\Omega, \Sigma, \mathbb{P})$ there is a sequence $\{t_n\}_{n\geq 0}$ of random variables such that the variables $\Delta t_n = t_n - t_{n-1}$, where $t_0 = 0$, are nonnegative, independent, and identically distributed with density $g(t) = \lambda e^{-\lambda t}$ for $t \geq 0$.

Let $\{\eta_n\}_{n\in\mathbb{N}}$ be a sequence of independent identically distributed random elements with values in a compact metric space Θ; their distribution will be denoted by ν. We assume that the sequences $\{t_n\}_{n\geq 0}$ and $\{\eta_n\}_{n\geq 0}$ are independent, which implies that the mapping $\omega \mapsto p(\omega) = (t_n(\omega), \eta_n(\omega))_{n\geq 0}$ defines a stationary Poisson point process. Then for every measurable set $Z \subset \Theta$ the random variable

$$\mathcal{N}_p((0,t] \times Z) = \#\{i : (t_i, \eta_i) \in Z\}$$

is Poisson distributed with parameter $\lambda t \nu(Z)$. $\mathcal{N}_p$ is called a Poisson random counting measure.

The coefficient $a : Y \times I \to Y$, $I = \{1, \ldots, N\}$, is Lipschitz continuous with respect to the first variable.

We define $q : Y \times \Theta \to Y$ by

$$q(x, \theta) = x + b(x, \theta) \quad \text{for } x \in Y,\ \theta \in \Theta$$

and assume that q is continuous.

Finally, suppose that $[p_{ij}]_{i,j\in I}$, $p_{ij} : Y \to [0,1]$ is a probability matrix and $[p_i]_{i\in I}$, $p_i : Y \to [0,1]$ is a probability vector.

For every $i \in I$, denote by $v_i(t) = \Pi_i(t, x)$ the solution of the unperturbed Cauchy problem

$$(6.4.4)\qquad v_i'(t) = a(v_i(t), i) \quad \text{and} \quad v_i(0) = x, \quad x \in Y.$$

Consider a sequence $\{x_n\}_{n\geq 0}$ of random variables $x_n : \Omega \to Y$ and a stochastic process $\{\xi(t)\}_{t\geq 0}$, $\xi(t) : \Omega \to I$ (describing random switching at random times t_n), such that

$$(6.4.5)\qquad \begin{aligned} &x_n = q(\Pi_{\xi(t_{n-1})}(t_n - t_{n-1}, x_{n-1}), \eta_n),\\ &\mathbb{P}(\xi(0) = k \mid x_0 = x) = p_k(x),\\ &\mathbb{P}(\xi(t_n) = s \mid x_n = y,\ \xi(t_{n-1}) = i) = p_{is}(y) \quad \text{for } n = 1, 2, \ldots,\\ &\xi(t) = \xi(t_{n-1}) \quad \text{for } t_{n-1} \leq t < t_n,\ n = 1, 2, \ldots. \end{aligned}$$

The solution of (6.4.3) is now given by

$$X(t) = \Pi_{\xi(t_{n-1})}(t - t_{n-1}, x_{n-1}) \quad \text{for } t_{n-1} \le t < t_n,\ n = 1, 2, \ldots. \tag{6.4.6}$$

For any $x \in Y$ we write $X(t)_x$ to denote the solution of problem (6.4.1), (6.4.2) with $x_0 = x$.

We are interested in the evolution of distributions of the stochastic process $\{X(t)\}_{t\ge 0}$. It is described with the help of the family $\{Q^t\}_{t\ge 0}$, given by

$$Q^t\widetilde{\mu}(A) = \mathbb{P}(X(t) \in A) = \int\limits_Y \mathbb{P}(X(t)_x \in A)\,\widetilde{\mu}(dx) \quad \text{for } A \in \mathcal{B}(Y), \tag{6.4.7}$$

where $\widetilde{\mu}$ is the distribution of the initial vector x. The stochastic process $\{(X(t), \xi(t))\}_{t\ge 0}$, $(X(t), \xi(t)) : \Omega \to Y \times I$, is a Markov process and it generates the semigroup $\{T^t\}_{t\ge 0}$ defined by

$$T^t f(x, i) = E(f((X(t), \xi(t))_{(x,i)})) \quad \text{for } f \in C(Y \times I),$$

with the corresponding semigroup of Markov operators $\{P^t\}_{t\ge 0}$, $P^t : \mathcal{M}_1 \to \mathcal{M}_1$, satisfying

$$\langle P^t\mu, f\rangle = \langle \mu, T^t f\rangle \quad \text{for } f \in B(Y \times I),\ \mu \in \mathcal{M}_1 \text{ and } t \ge 0. \tag{6.4.8}$$

As an immediate consequence of Theorems 4.3.1 and 4.3.2 we obtain the following result, which is an extension of the main theorem of [17]:

Theorem 6.4.1. *Assume that conditions* (4.3.1)–(4.3.4) *and* (4.3.7) *are satisfied. Then the semigroup* $\{P^t\}_{t\ge 0}$ *given by* (6.4.8) *is asymptotically stable and there exists a measure* $\widetilde{\mu}_* \in \mathcal{M}_1$ *such that for every* $\mu \in \mathcal{M}_1$ *the family* $\{Q^t\mu\}_{t\ge 0}$ *given by* (6.4.7) *is weakly convergent to* $\widetilde{\mu}_*$

Remark 6.4.1. In the case when the coefficient $a : \mathbb{R}^d \times I \to \mathbb{R}^d$ does not depend on the second variable, we obtain the stochastic equation considered by Traple [49], Szarek and Węgrychowicz [48].

In many applications we are mostly interested in values of the solution $X(t)$ at the switching points t_n. Setting

$$\overline{\mu}_n(A) = \mathbb{P}((X(t_n), \xi(t_n)) \in A) \quad \text{for } A \in \mathcal{B}(Y \times I),$$

we obtain $\overline{\mu}_{n+1} = P\overline{\mu}_n$, $n \in \mathbb{N}$, where P is given by

$$P\mu(A) = \sum_{j\in I} \int\limits_{\Theta} \int\limits_{Y\times I} \int\limits_{\mathbb{R}_+} \lambda e^{-\lambda t} 1_A(q(\Pi_j(t, x), \theta), j) p_{ij}(x)\, dt\, d\nu(\theta)\, d\mu(x, i) \tag{6.4.9}$$

for $A \in \mathcal{B}(Y \times I)$ and $\mu \in \mathcal{M}_1$.

We now consider the Poisson driven stochastic differential equation on a separable Banach space $(Y, \|\cdot\|)$ of the form

$$dX(t) = a(X(t))dt + b(X(t))dp \quad \text{for } t > 0 \tag{6.4.10}$$

with the initial condition

$$X(0) = x_0, \tag{6.4.11}$$

where $a, b : Y \to Y$ are Lipschitz continuous transformations, $\{p(t)\}_{t\geq 0}$ is a Poisson process and the initial condition x_0 is a random variable on Ω with values in Y, independent of $\{p(t)\}_{t\geq 0}$.

This is a particular example of equation (6.4.3) where $\Theta = I = \{1\}$, $q(x,1) = q(x) = x + b(x)$, and $\Pi_1(t,x) = \Pi(t,x)$ is the unique solution of the Cauchy problem

$$u'(t) = a(u(t)) \quad \text{for } t \geq 0, \tag{6.4.12}$$

with the initial condition

$$u(0) = x_0.$$

From Theorems 4.3.1 and 5.1.1 we obtain the following result, which is similar in spirit to the main result in [38].

THEOREM 6.4.2. *Let* Π *be the solution of the unperturbed system* (6.4.12). *Assume that there exist positive constants* α *and* L_q *such that*

$$\|x-y\| \leq \|\Pi(t,x) - \Pi(t,y)\| \leq e^{\alpha t}\|x-y\| \quad \textit{for } x, y \in Y,\ t \geq 0, \tag{6.4.13}$$

$$\|q(x) - q(y)\| \leq L_q\|x-y\| \tag{6.4.14}$$

and

$$L_q < \exp(-\alpha/\lambda). \tag{6.4.15}$$

If $a : Y \to Y$ *is bounded, then the unique invariant measure* μ_* *of the semigroup* P^t *given by* (6.4.8) *satisfies*

$$\underline{d}\mu_*(x) \geq 1 \quad \textit{for } x \in Y.$$

To obtain the upper bound for $\overline{\dim}_L\, \mu_*$ we need a more restrictive assumption concerning $q : Y \to Y$. We assume that there exist positive constants L_q and L_0 such that

$$L_0\|x-y\| \leq \|q(x) - q(y)\| \leq L_q\|x-y\| \quad \text{for } x, y \in Y. \tag{6.4.16}$$

From Theorems 5.2.1 and 5.3.2 we obtain:

THEOREM 6.4.3. *Let* Π *be the solution of* (6.4.12) *and suppose* $q : Y \to Y$ *satisfies condition* (6.4.16). *If there exists a positive constant* α *satisfying condition* (6.4.15) *such that*

$$\|\Pi(t,x) - \Pi(t,y)\| \leq e^{\alpha t}\|x-y\| \quad \textit{for } x, y \in Y,\ t \geq 0, \tag{6.4.17}$$

then

$$\overline{\dim}_L\, \mu_* \leq \overline{\dim}_L\, \mu_0 \leq \frac{\ln(1-e^{-1})}{\ln L_q + \alpha/\lambda},$$

where μ_* *and* μ_0 *are the invariant measures of the semigroup* $\{P^t\}_{t\geq 0}$ *given by* (6.4.8) *and the operator* P *given by* (6.4.9), *respectively.*

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