1. Introduction

This paper can be considered as the final effort of the authors to better understand some basic properties of disjointness preserving operators (d.p.o.) on vector lattices. It is the last (but, we hope, not the least) in the series of articles [3]–[10], and it is closely related to and inspired by the work of other mathematicians [14]–[33], to cite only a few.

To explain what we mean by the "basic properties" let us recall that a (linear) operator $T : X \to Y$ between vector lattices is *disjointness preserving* if the following implication is true:

$$x_1, x_2 \in X,\ x_1 \perp x_2 \ \Rightarrow\ Tx_1 \perp Tx_2.$$

Even a superficial look at the articles mentioned above allows one to see that all of them are somehow connected with the following three problems concerning the disjointness preserving operators.

Problem A. Suppose that a disjointness preserving operator $T : X \to Y$ is *injective.* Under what additional conditions on X, Y and T is the inverse operator $T^{-1} : TX \to X$ also disjointness preserving, i.e., when

$$x_1 \perp x_2 \ \Leftrightarrow\ Tx_1 \perp Tx_2\,?$$

Problem B. Under what conditions on X, Y and on a disjointness preserving operator $T : X \to Y$ are the vector lattices X and Y *order isomorphic*?

Problem C. Under what conditions on X, Y and T is the operator T *regular*?

At this point the following question seems unavoidable by any alert reader: If all three problems above have already been studied why do we need to return to them again? Here is a brief answer. First of all we would like to point out that, as was shown by the authors, without any additional assumptions all these problems have negative solutions. On the other hand, under some very general conditions (many of which will be reproduced later) these problems do have affirmative solutions. However, for many important classes of vector lattices the situation has remained unclear so far. And the purpose of this work is to cover as much of these classes as possible, so that the above three basic problems will be solved for the most common classes of vector lattices.

It should also be pointed out that we do not claim that the above three problems exhaust the list of interesting questions about disjointness preserving operators. Plenty of work (including some by the authors) has been done on the multiplicative representation of disjointness preserving operators, their spectral properties, on polar decomposition of regular disjointness preserving operators, *et cetera.*

Problems A–C are, of course, closely related. For example, if T is a regular injection then $T^{-1} : TX \to Y$ automatically preserves disjointness, and if T is a regular bijection then X and Y are automatically order isomorphic. These implications follow from the well known criterion of regularity of disjointness preserving operators (see Theorem 2.3.2).

We do not attempt here to present the history of the work done on the above problems and refer the reader to [5]. The structure of the present paper is as follows.

In Section 2 the reader will find most of the non-standard (and part of the standard) definitions and notations, some well known and some auxiliary results used throughout the paper.

In Section 3 we completely describe (Theorem 3.1.1) the vector lattices such that any d.p.o. from them to any other vector lattice is regular. In Theorem 3.2.1 we consider a wider class $\mathcal{UI}$ of vector lattices such that for any injective d.p.o. $T : X \to Y$ the inverse operator $T^{-1} : TX \to X$ also preserves disjointness. We obtain a necessary and a sufficient condition (with a small gap between them) for the inclusion $X \in \mathcal{UI}$.

The main results of Section 4 are Theorems 4.0.1 and 4.0.2. These theorems serve as some of our principal technical tools in the next sections but they are also of independent interest and allow us to describe (Corollary 4.0.4) a large class of d-rigid and super d-rigid domains (Definition 2.3.5).

Section 5 is the central one in this paper. For a large class of domains which we call "weakly c_0-complete" and which contains, in particular, the class of all relatively uniformly complete domains, we describe conditions under which every bijective d.p.o. is a d-isomorphism and conditions under which every d-isomorphism is regular. There is a gap between our necessary and sufficient conditions due to the well known and unsolved problem: If X is a laterally σ-complete vector lattice and $d\text{-}\dim X > 1$ (see Definition 2.4.1), does there exist a *non-regular* band preserving projection $P : X \to X$?

The above-mentioned gap disappears when the domain X has either the countable sup property or the projection property. We discuss these cases in detail in Section 6. In particular, we obtain complete answers to Problems A–C for the important case when the domain X is relatively uniformly complete and the range Y has the countable sup property (Theorem 6.2.3).

Section 7 contains further discussion of the Huijsmans–de Pagter–Koldunov theorem. We use de Pagter's techniques and techniques developed in [5] to weaken the conditions imposed in the original HPK-theorem. We also prove in Theorem 7.2.6 that the conclusion of the HPK-theorem remains true when the range Y is a vector lattice with a topology defined by a countable family of lattice seminorms. Theorem 7.3.1 deals with the case when the domain X satisfies the Luxemburg condition and the range Y is relatively uniformly complete. It improves considerably our previous result in this direction—Theorem 9.3 in [5].

Finally in Section 8 we apply our results to the vector lattices of continuous functions on completely regular (Tikhonov) topological spaces.

2. Basic definitions, notations, and auxiliary results

For general information concerning vector lattices and their functional representations the reader is referred to [41], [31], and [42].

All vector lattices considered in this paper are assumed to be Archimedean and are considered over the field $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers.

2.1. Krein–Kakutani representation and related properties of vector lattices. Let X be a vector lattice and $x \in X$. Let I_x be the principal ideal generated by x in X. By the Krein–Kakutani representation theorem there is a unique (up to a homeomorphism) Hausdorff compact space K_x such that I_x is order isomorphic to a vector sublattice of $C(K_x)$ which separates points of K_x and contains constant functions (actually an order isomorphism can be chosen in such a way that x maps to the function $\mathbf{1}$).

There are many useful connections between global properties of the vector lattice X and "local" properties of the principal ideals I_x, $x \in X$ (in particular these "local" properties involve topological properties of the spaces K_x). One of the most detailed descriptions of these connections can be found in [37]. We will need some of them.

Let us recall the following definitions.

2.1.1. Definition. A vector lattice X is called *relatively uniformly complete* (briefly X is r_u-*complete* or $X \in$ (RUC)) if for any $x \in X$ the principal ideal I_x is order isomorphic to $C(K_x)$.

2.1.2. Definition. Let X be a vector lattice and let $x, u \in X$. The element u is called a *component* of x if $x - u \perp u$.

2.1.3. Definition. We will say that a vector lattice X is *weak-Freudenthal* [29], briefly $X \in$ (WF), if for any $x \in X$ and for any $u \in I_x$ the element u can be approximated by finite linear combinations of components of x in the norm of $C(K_x)$.

2.1.4. Definition. We will say that a vector lattice X has a *cofinal family of components*, briefly $X \in$ (CFC), if for any $x \in X$ and any band $U \subset X$ such that $x \not\perp U$ there is a non-zero component u of x such that $u \in U$.

2.1.5. Definition. We say that a vector lattice X has the *countable sup property*, briefly $X \in$ (CSP), if any order bounded set of pairwise disjoint non-zero elements in X is at most countable.

The proofs of the statements in the next proposition can be found for example in [37].

2.1.6. Proposition. *Let X be a vector lattice. Then*:

(1) *X has the projection property if and only if for any $x \in X$ the space K_x is extremally disconnected* (*Stonean*).
(2) *X is Dedekind complete if and only if X is r_u-complete and has the projection property.*
(3) *If for any x the space K_x is basically disconnected (quasi-Stonean) then X has the principal projection property. The converse is in general false* (*see Remark* 2.1.7).
(4) *If X is r_u-complete then it has the principal projection property if and only if for any x the space K_x is basically disconnected.*
(5) *$X \in$ (WF) if and only if for any $x \in X$ the space K_x is zero-dimensional* (*or, which for compact spaces is the same*, *totally disconnected*).

(6) $X \in (\mathrm{CFC})$ *if and only if for any* $x \in X$ *the space* K_x *has a* π*-base of clopen subsets* (*each non-empty open subset of* K_x *contains a non-empty subset clopen in* X).
(7) $X \in (\mathrm{CSP})$ *if and only if for any* $x \in X$ *the Krein–Kakutani space* K_x *satisfies the countable chain condition* (*briefly* $K_x \in (\mathrm{ccc})$), *i.e. any family of non-empty pairwise disjoint open subsets of* K_x *is at most countable.*

2.1.7. Remark. To see that the converse to Proposition 2.1.6(3) is in general false it is enough to consider a zero-dimensional compact space K which is not basically disconnected (e.g. the standard Cantor set) and to take as X the vector lattice of all finite linear combinations of characteristic functions of clopen subsets of K. The vector lattice X belongs to the class of vector lattices with a remarkable property which will be discussed in Subsection 3.1.

2.2. Vector lattices with some degree of lateral completeness. Let us first recall some standard definitions.

2.2.1. Definition.

(1) A vector lattice X is called *laterally complete* if for any family $\{x_\alpha\} \subset X$ of pairwise disjoint positive elements its supremum exists in X.
(2) A vector lattice X is called *conditionally laterally complete* if for any *order bounded* family $\{x_\alpha\} \subset X$ of pairwise disjoint positive elements its supremum exists in X.
(3) A vector lattice X is called *laterally* σ*-complete* if for any *countable* family $\{x_\alpha\} \subset X$ of pairwise disjoint positive elements its supremum exists in X.
(4) A vector lattice X is called *conditionally laterally* σ*-complete* if for any *order bounded countable* family $\{x_\alpha\} \subset X$ of pairwise disjoint positive elements its supremum exists in X.

2.2.2. Theorem (Veksler–Geĭler [40], Huijsmans–Wickstead [26], Bernau [19]).

(1) *Each conditionally laterally complete vector lattice has the projection property.*
(1′) *Each conditionally laterally* σ*-complete vector lattice has the principal projection property.*
(2) *A laterally complete band in a vector lattice is a projection band.*
(2′) *A principal laterally* σ*-complete band in a vector lattice is a projection band.*

2.2.3. Remark.

(1) A relatively uniformly complete vector lattice X is conditionally laterally complete if and only if it has the projection property [40]. But, as Example 2.2.4 shows, in general the projection property does not imply conditional lateral completeness.
(2) The condition $X^1 \in (\mathrm{CSP})$, where X^1 is the lateral completion of X, means exactly that any set of pairwise disjoint elements in X is at most countable. Clearly $X^1 \in (\mathrm{CSP})$ if $X \in (\mathrm{CSP})$ and X has a weak unit. As the example of the vector lattice c_{00} of all finite sequences shows, the converse to this statement is in general false.

(3) For vector lattices from (CSP) the notions of conditional lateral completeness and conditional lateral σ-completeness coincide.
(4) In general a laterally σ-complete vector lattice $X \in$ (CSP) need not be laterally complete but any principal band in it is laterally complete.

2.2.4. Example. Let K be an infinite extremally disconnected compact space. Let X be a subset of $C(K)$ defined in the following way: $f \in X$ if and only if for any positive real number α the set $|f|(K) \cap [\alpha, \infty)$ is finite.

The vector lattice X has the projection property because it contains the characteristic functions of all clopen subsets of K. Nevertheless we can see at once that X is not conditionally laterally complete. (It might be worth noticing that X is a c_0-complete vector sublattice, see Definition 2.2.5 below, and also a subalgebra of $C(K)$.)

In many instances when we work with disjointness preserving operators on vector lattices it is enough instead of r_u-completeness to assume only some weaker condition, a kind of "lateral r_u-completeness"—a possibility to add some series of pairwise disjoint elements. We now introduce the corresponding definitions.

2.2.5. Definition. Let X be a vector lattice. We will say that:

- $X \in (\mathrm{LC}_0)$ if for any order bounded countable family $\{u_n\}$ of pairwise disjoint elements in X and for any sequence of positive scalars ε_n such that $\varepsilon_n \to 0$ as $n \to \infty$ the element $\sum_{n=1}^{\infty} \oplus \varepsilon_n u_n$ exists in X ([1]).
- $X \in (\mathrm{LC}_1)$ if for any principal band $U = \{u\}^{dd}$ in X there is a sequence of positive scalars ε_n depending only on u and such that for any order bounded countable family $\{u_n\}$ of pairwise disjoint elements in U the element $\sum_{n=1}^{\infty} \oplus \varepsilon_n u_n$ exists in X.
- $X \in (\mathrm{LC}_2)$ if for any positive $u \in X$ there is a sequence of positive scalars ε_n depending only on u and such that for any countable family $\{u_n\}$ of pairwise disjoint elements in the interval $[0, u]$ the element $\sum_{n=1}^{\infty} \oplus \varepsilon_n u_n$ exists in X.
- $X \in (\mathrm{LC}_3)$ if for any fixed order bounded sequence u_n of non-zero, positive, pairwise disjoint elements in X there is a family of positive scalars ε_n such that

$$v_n \leq \varepsilon_n u_n \ \Rightarrow \ \sum_{n=1}^{\infty} \oplus v_n \in X.$$

- $X \in (\mathrm{LC}_4)$ if for any order bounded countable family $\{u_n\}$ of pairwise disjoint elements in X there is sequence of positive scalars ε_n such that the element $\sum_{n=1}^{\infty} \oplus \delta_n u_n$ exists in X for any sequence $\{\delta_n\}$ with $0 \leq \delta_n \leq \varepsilon_n$.

2.2.6. Proposition.

$$(\mathrm{RUC}) \subsetneq (\mathrm{LC}_0) \subsetneq (\mathrm{LC}_1) \subsetneq (\mathrm{LC}_2) \subsetneq (\mathrm{LC}_3) \subsetneq (\mathrm{LC}_4).$$

Proof. All that requires proving is that the inclusions are proper.

(1) Let X be the vector sublattice of $C[0,1]$ defined as follows. A function f from $C[0,1]$ is in X iff there is a countable family of intervals $(a_n, b_n) \subset (0,1)$ such that their

([1]) This means, as usual, that $\sup_n \varepsilon_n u_n$ exists in X.

union is dense in $(0,1)$ and f coincides with a linear function on each (a_n, b_n). Then $X \in (\mathrm{LC}_0)$ but $X \notin (\mathrm{RUC})$.

(2) Let $\mathbf{c}$ be the vector lattice of all convergent sequences and X be the linear hull in $\mathbf{c}$ of l^1 and the constant function $\mathbf{1}$. Then it is easy to see that X is a vector sublattice of $\mathbf{c}$ and that $X \in (\mathrm{LC}_1) \setminus (\mathrm{LC}_0)$.

(3) To construct a vector lattice X such that $X \in (\mathrm{LC}_2) \setminus (\mathrm{LC}_1)$ let us consider a system of positive scalars $A_n(\alpha)$ where $n \in \mathbb{N}$ and $\alpha \in (0,1)$ with the following properties:

- $\forall \alpha \in (0,1)$ $A_n(\alpha) \uparrow \infty$.
- For any sequence of positive scalars ε_n, $\varepsilon_n \downarrow 0$, we can find an α such that $A_n(\alpha)\varepsilon_n \uparrow \infty$.

For any $\alpha \in (0,1)$ we define a Lesbegue-measurable function f_α on $[0,1]$ in the following way. Let n_α be the smallest positive integer such that $[\alpha - 1/n_\alpha, \alpha + 1/n_\alpha] \subset (0,1)$. Let $f_\alpha \equiv 0$ on $[0,1] \setminus [\alpha - 1/n_\alpha, \alpha + 1/n_\alpha]$ and let $f_\alpha \equiv A_{n-n_\alpha+1}(\alpha)$ on $[\alpha - 1/n, \alpha + 1/n] \setminus [\alpha - 1/(n+1), \alpha + 1/(n+1)]$ for any $n \geq n_\alpha$.

Let X be the smallest vector sublattice of $L^0(0,1)$ (the space of all Lesbegue-measurable functions on $(0,1)$) which contains $L^\infty(0,1)$ and every function f_α, $\alpha \in (0,1)$. In other words $x \in X$ if and only if there are $g \in L^\infty(0,1)$ and $\alpha_1, \dots, \alpha_n$ such that $|x| \leq |g| + \sum_{i=1}^n |f_{\alpha_i}|$. It is easy to see that $X \in (\mathrm{LC}_2) \setminus (\mathrm{LC}_1)$.

(4) Let $\alpha \leftrightarrow \{\delta_{\alpha_n}\}$ be a one-to-one correspondence between $(0,1)$ and the set of all sequences of real numbers which decrease to 0. Let $\mathbb{D}$ be $(0,1)$ considered with the discrete topology and let $D_\alpha = \{d_{\alpha_1}, \dots, d_{\alpha_n}, \dots\}$, $\alpha \in (0,1)$, be countable pairwise disjoint subsets of D such that $\bigcup_\alpha D_\alpha = D$. Let Y be the vector sublattice of $l^\infty(D)$ given by

$$Y = \Big\{ y \in l^\infty(D) : \sum_\alpha \sum_{n=1}^\infty |y(d_{\alpha_n})|/\delta_{\alpha_n} < \infty \Big\}.$$

Let X be the linear hull of the constant function $\mathbf{1}$ and Y. Then X is a vector sublattice of $l^\infty(D)$ and it is not difficult to see that $X \in (\mathrm{LC}_3) \setminus (\mathrm{LC}_2)$.

(5) Let X be a vector sublattice of $C[0,1]$ consisting of all functions of bounded variation on $[0,1]$. Then $X \in (\mathrm{LC}_4) \setminus (\mathrm{LC}_3)$. $\blacksquare$

The next proposition will be used in Section 6.

2.2.7. Proposition. *Let X be a vector lattice with the principal projection property. Assume additionally that $X \in (\mathrm{CSP}) \cap (\mathrm{LC}_4)$. Then every principal band in X has the projection property.*

Proof. Let $U = \{u\}^{dd}$ be a principal band in X. It is enough to prove that any band V in X such that $V \subset U$ is a principal band. By Zorn's lemma there is a system $\{v_\alpha\}$ of pairwise disjoint elements such that $0 \leq v_\alpha \leq |u|$ and the system of bands $\{v_\alpha\}^{dd}$ is full in V. Because $X \in (\mathrm{CSP})$ the system $\{v_\alpha\}$ is at most countable. Because $X \in (\mathrm{LC}_4)$ there are scalars ε_n such that the element $v = \sum_n \oplus \varepsilon_n v_n$ exists in X. Clearly $V = \{v\}^{dd}$. $\blacksquare$

2.2.8. Remark. Without the assumption that $X \in (\mathrm{LC}_4)$ the statement of Proposition 2.2.7 is in general false. A counterexample is provided by the vector lattice c_{00}.

The condition (LC_3) is strong enough to guarantee the validity of our main results in Section 5. That is not the case if we assume only (LC_4). For this reason we introduce the following definition.

2.2.9. DEFINITION. We will sometimes refer to the vector lattices from (LC_0) as c_0-*complete* vector lattices and to those from (LC_3) as *weakly* c_0-*complete* vector lattices.

2.3. Disjointness preserving operators

2.3.1. DEFINITION.

(1) Let X, Y be vector lattices and $T : X \to Y$ be a linear operator. The operator T is called *disjointness preserving* (briefly *d.p.o.*) if for any $x, z \in X$ we have
$$x \perp z \;\Rightarrow\; Tx \perp Tz.$$
(2) Let Z be a vector lattice and X be a vector sublattice of Z. A linear operator $T : X \to Z$ is called *band preserving* (*b.p.o.*) if for any $x \in X$ and $z \in Z$ we have
$$x \perp z \;\Rightarrow\; Tx \perp z.$$
(3) Let X, Y be vector lattices and $T : X \to Y$ be a bijective disjointness preserving operator. The operator T is called a *d-isomorphism* if for any $x, z \in X$ we have
$$x \perp z \;\Leftrightarrow\; Tx \perp Tz,$$
in other words if the inverse operator T^{-1} also preserves disjointness.

We will need a characterization of regular disjointness preserving operators. The proof of the next result can be found in [2, Theorem 3.3] or in [25, Proposition 1.2].

2.3.2. THEOREM. *Let X, Y be vector lattices and $T : X \to Y$ be a disjointness preserving operator. The following conditions are equivalent*:

(1) *T is regular.*
(2) *T is order bounded.*
(3) *For any $u, v \in X$ such that $|u| \le |v|$ we have $|Tu| \le |Tv|$.*

2.3.3. COROLLARY. *Let X, Y be vector lattices and $T : X \to Y$ be an injective regular d.p.o. Then*
$$x \perp z \;\Leftrightarrow\; Tx \perp Tz.$$

Proof. Let $Tx \perp Tz$. Let $u = |x| \wedge |z|$. By Theorem 2.3.2, $|Tu| \le |Tx|$ and $|Tu| \le |Tz|$, whence $Tu = 0$ and because T is injective $u = 0$. ∎

If an injective operator T is a d.p.o. but the inverse $T^{-1} : TX \to X$ fails to preserve disjointness we can at least measure the degree of this failure. To this end let us recall the definition of *d-splitting number* $d(T)$ introduced in [5].

2.3.4. DEFINITION ([5, Definition 10.1]). Let $T : X \to Y$ be a disjointness preserving operator between vector lattices. We will write that $d(T) = d(T, X, Y) \le n$ for some $n \in \mathbb{N}$ if from the fact that
$$\bigwedge_{i=1}^{m} |x_i| > 0, \qquad \text{where } x_i \in X \text{ and } Tx_i \perp Tx_j \text{ for } i \ne j,$$
it follows that $m \le n$. We will write that $d(T) = n$ if $d(T) \le n$ and $d(T) \not\le n - 1$.

We now introduce two classes of vector lattices which will be of main interest to us in this paper.

2.3.5. Definition.

(1) A vector lattice X is called *d-rigid* if for any vector lattice Y and for any bijective d.p.o. $T : X \to Y$ the inverse operator T^{-1} preserves disjointness.
(2) A vector lattice X is called *super d-rigid* if any bijective operator T from X onto an arbitrary vector lattice Y is regular.

2.3.6. Remark. By Corollary 2.3.3 any super d-rigid vector lattice is d-rigid.

2.4. d-dimension and d-independence. The characterization of d-rigid and super d-rigid vector lattices obtained in this paper involves the notions of d-dimension and d-independence of elements of a vector lattice. For a more general discussion of d-dimension and the related notion of *d-bases* we refer the reader to [4] and [9].

2.4.1. Definition. We say that a vector lattice X has *d-dimension* 1, briefly $d\text{-}\dim X = 1$, if for any two elements $x, z \in X$ such that $|z| \le |x|$ the element z is a *semicomponent* of x [4, Definition 4.7], or in more detail there is a system $\{(U_\gamma, c_\gamma)\}_{\gamma\in\Gamma}$ where U_γ is a band in X and c_γ is a scalar such that

$$U_{\gamma_1} \perp U_{\gamma_2} \quad \text{if } \gamma_1 \ne \gamma_2, \qquad z \in \Big\{\bigcup_\gamma U_\gamma\Big\}^{dd}, \qquad z - c_\gamma x \perp U_\gamma \quad \forall \gamma \in \Gamma.$$

Let us recall [5, Definition 11.1] that a vector lattice X is called *essentially one-dimensional* if for any two non-disjoint elements $x, z \in X$ there are non-zero components u, v of x and z respectively and a scalar c such that $v = cu$. We can now make a trivial but useful observation.

2.4.2. Proposition. *For a vector lattice X the following two statements are equivalent*:

(1) *X is essentially one-dimensional.*
(2) *$X \in$ (CFC) and $d\text{-}\dim X = 1$.*

Vector lattices of d-dimension 1 will be of particular importance to us but we will also need the general concept of finite d-dimension:

2.4.3. Definition. Let X be a vector lattice.

(1) We will say that $d\text{-}\dim X \le n$ if for any system $x_1, \dots, x_{n+1}$ of non-zero elements in X such that $0 \le x_1 \le \dots \le x_{n+1}$ we can find a system $\{U_\alpha\}$ of bands which is full ([2]) in $U = \{x_1\}^{dd}$ and scalars $c_{1,\alpha}, \dots, c_{n+1,\alpha}$ such that $\sum_{j=1}^{n+1} |c_{j,\alpha}| > 0$ and $\sum_{j=1}^{n+1} c_{j,\alpha} x_j \perp U_\alpha$ for all α.
(2) We will say that $d\text{-}\dim X = n$ if $d\text{-}\dim X \le n$ and the statement $d\text{-}\dim X \le n-1$ is false.
(3) We will say that $d\text{-}\dim X = \infty$ if the statement $d\text{-}\dim X \le n$ is false for any $n \in \mathbb{N}$.

We will also use the notion of *d-independence* in arbitrary vector lattices [4, Definition 4.1]:

([2]) A system $\{V_\alpha\}$ of bands is called *full* in the band V if $\{\bigcup_\alpha V_\alpha\}^{dd} = V$.

2.4.4. Definition. A system $\{x_\gamma\}_{\gamma\in\Gamma}$ of elements in a vector lattice X is called *d-independent* if for every band B in X, every finite set $\{\gamma_1,\dots,\gamma_n\}\subseteq\Gamma$, and every finite set $\{c_1,\dots,c_n\}$ of non-zero scalars the following implication holds:

$$\text{if } \sum_{j=1}^{n} c_j x_{\gamma_j} \perp B, \text{ then } x_{\gamma_j}\perp B \text{ for each } j=1,\dots,n.$$

We omit the routine and simple proof of the next proposition.

2.4.5. Proposition. *Let $X\in$ (CFC). A system $\{x_\gamma\}_{\gamma\in\Gamma}$ is d-independent iff for every finite set $\{\gamma_1,\dots,\gamma_n\}\subseteq\Gamma$ every system $u_{\gamma_1},\dots,u_{\gamma_n}$ where u_{γ_i} is a non-zero component of x_{γ_i} is linearly independent.*

The following definition was introduced in [4, Definition 4.9]

2.4.6. Definition. We will say that a vector lattice X satisfies *condition* $(*)$ if for every band B in X and every $x\notin B^d$ there exists a semicomponent of x in B.

2.4.7. Proposition. *Let X be a vector lattice.*

(1) *If there are d-independent elements $x_1,\dots,x_{n+1}$ in a vector lattice X such that $|x_1|\le\ldots\le|x_{n+1}|$ then $d\text{-}\dim X>n$.*
(2) *Conversely, if X satisfies condition $(*)$ and $d\text{-}\dim X>n$ then there are d-independent elements $x_1,\dots,x_{n+1}$ in X such that $|x_1|\le\ldots\le|x_{n+1}|$.*

Proof. The first statement is obvious and the second follows from Proposition 4.10 in [4]. ∎

2.4.8. Remark.

(1) Every vector lattice with the countable sup property satisfies condition $(*)$ [4, Theorem 4.11] and in particular the statement in part (2) of Proposition 2.4.7 is true for such vector lattices.
(2) We do not know if condition $(*)$ in Proposition 2.4.7(2) is necessary but without any conditions on X the statement fails to be true. K. P. Hart [24] constructed an example of a connected F-space K such that not every function from $C(K)$ is essentially constant (see Definition 2.4.10 below). It means in particular that any two non-disjoint elements from $C(K)$ are d-dependent but $d\text{-}\dim C(K)>1$.

Proposition 2.4.7 yields immediately the following "external" characterization of vector lattices of finite d-dimension.

2.4.9. Proposition. *Let X be a vector lattice, X^1 be its lateral completion, and let $n\in\mathbb{N}$. Then*

$$d\text{-}\dim X=n \iff d\text{-}\dim X^l=n.$$

The condition $d\text{-}\dim X=1$ plays an important role in this paper; it will appear in the statements of many of our main results. For this reason we want to discuss it here in more detail.

The class of functions described in the next definition was probably first introduced in [18]. Because of the lack of commonly accepted terminology we call these functions "essentially constant".

2.4.10. Definition. Let Ω be a topological space. We will say that a function $f \in C(\Omega)$ is *essentially constant* and will write $f \in \mathrm{EC}(\Omega)$ if there are a family $\{O_\alpha\}$ of disjoint open subsets of Ω and scalars c_α such that $f \equiv c_\alpha$ on O_α and the union of the sets O_α is dense in Ω.

Let X be an r_u-complete vector lattice. It follows immediately from Definition 2.4.1 that $d\text{-}\dim X = 1$ if and only if for every $x \in X$ we have $C(K_x) = \mathrm{EC}(K_x)$.

Therefore let us recall what is known about compact spaces with the property $C(K) = \mathrm{EC}(K)$.

Clearly, a sufficient condition for $\mathrm{EC}(K) = C(K)$ is that the set of P-points [21, 4L] (in particular, of isolated points) is dense in K. The best known example of a compact space with no isolated points but a dense set of P-points is (assuming the continuum hypothesis) $\beta\mathbb{N} \setminus \mathbb{N}$ [21, 6V]. The existence of extremally disconnected compact spaces with no isolated points but a dense set of P-points is equivalent to the existence of Ulam's cardinals [35, p. 507].

The property $C(\beta\mathbb{N} \setminus \mathbb{N}) = \mathrm{EC}(\beta\mathbb{N} \setminus \mathbb{N})$ remains valid even without assuming the CH because every non-empty G_δ set in $\beta\mathbb{N} \setminus \mathbb{N}$ has non-empty interior [21, 6S].

A. Gutman [22] was probably the first to construct an extremally disconnected compact space Q without P-points and such that $C(Q) = \mathrm{EC}(Q)$. It was proved in [32, Example 2.9] that the absolute of $\beta\mathbb{N} \setminus \mathbb{N}$ has this property. Finally, the existence of an extremally disconnected compact space Q with the *countable chain property* and such that $C(Q) = \mathrm{EC}(Q)$ is equivalent to the existence of a *Suslin line* [17, Remark 1.7].

2.5. Condition $\pitchfork$

2.5.1. Definition ([5, Definition 4.3]). *Let $T : X \to Y$ be a d.p.o. We say that T satisfies condition $\pitchfork$ or that $T \in (\pitchfork)$ if for each band B in X and for any $y \in Y$ we have $Ty \perp TB \Rightarrow y \perp B$.*

There are examples of bijective d.p.o. which are not in $(\pitchfork)$ (see also Example 3.2.2) but this cannot happen if the domain X has a cofinal family of components.

2.5.2. Proposition. *Let X, Y be vector lattices, let $X \in$ (CFC), and let $T : X \to Y$ be an injective d.p.o. Then $T \in (\pitchfork)$.*

Proof. Assume to the contrary that there are a band $B \subset X$ and an $x \in X$ such that $Tx \perp TB$ but $x \not\perp B$. Because $X \in$ (CFC) the element x has a non-zero component $u \in B$. Then Tu is a non-zero component of Tx and $Tu \in TB$, a contradiction. ■

2.5.3. Remark. It was proved in [6] that a bijective d.p.o. $T : X \to Y$ satisfies $\pitchfork$ iff for any principal band B in X its image TB is a band in Y. In connection with this we want to notice that an operator from $(\pitchfork)$ is halfway between an arbitrary bijective d.p.o. and a d-isomorphism, and while a d-isomorphism induces an isomorphism of Boolean algebras of bands $\mathfrak{B}(X)$ and $\mathfrak{B}(Y)$, to an operator from $(\pitchfork)$ corresponds, in general, only an endomorphism of these algebras. Examples of bijective d.p.o. satisfying $\pitchfork$ but failing to be d-isomorphisms are quite common [5, Section 13].

Nevertheless, as was shown in [5, 6] and as will be shown in the present paper, in many important cases we can prove that a bijective d.p.o. from ($\pitchfork$) is a d-isomorphism.

The following two lemmas were proved in [6, Lemmas 3.5 and 5.3].

2.5.4. Lemma. *Let X, Y be vector lattices and let $T : X \to Y$ be a bijective disjointness preserving operator satisfying condition $\pitchfork$. Suppose that there are $a, b \in X$ such that $a \wedge b > 0$ and $|Ta| \wedge |Tb| = 0$ (that is, the inverse operator T^{-1} does not preserve disjointness). Then there are components a_1, a_2 of a and components b_1, b_2 of b such that $a = a_1 \oplus a_2$, $b = b_1 \oplus b_2$, $a \vee b = a_1 \oplus b_1$ and $a \wedge b = a_2 \oplus b_2$.*

2.5.5. Lemma. *Let $T : X \to Y$ be a disjointness preserving injection such that $T \in (\pitchfork)$ and the inverse operator T^{-1} does not preserve disjointness. Then we can find positive elements $a, b \in X$ such that*

(i) $Ta \perp Tb$,
(ii) *for each $\varepsilon > 0$ there exist linear combinations s_ε and t_ε of components of a and b, respectively, such that $|s_\varepsilon - b| \leq \varepsilon a$ and $|t_\varepsilon - a| \leq \varepsilon b$.*

The proof of Lemma 2.5.5 in [6, Lemma 5.3] shows that the following more detailed version of this lemma is true.

2.5.6. Lemma. *Let $T : X \to Y$ be a disjointness preserving injection such that $T \in (\pitchfork)$ and the inverse operator T^{-1} does not preserve disjointness. Let u, v be elements in X such that $|u| \wedge |v| > 0$ but $Tu \perp Tv$. Then we can find positive elements $a, b \in X$ such that*

(i) *a is a multiple of some component of u and b is a multiple of some component of v,*
(ii) *$Ta \perp Tb$, while $a \leq b \leq 2a$,*
(iii) *for each $\varepsilon > 0$ there exist linear combinations s_ε and t_ε of components of a and b, respectively, such that $|s_\varepsilon - b| \leq \varepsilon a$ and $|t_\varepsilon - a| \leq \varepsilon b$.*

Moreover, if we assume that $|v| \leq C|u|$ for some positive scalar C then v can be uniformly approximated by linear combinations of components of u.

2.5.7. Lemma. *Let $T : X \to Y$ be a bijective d.p.o. such that $T \in (\pitchfork)$ and T^{-1} does not preserve disjointness. Then we can find non-zero elements $a, b \in X$ such that $0 \leq a \leq b \leq 2a$, $Ta \perp Tb$, and the elements Ta and Tb are either both positive, or both negative, or one of them is positive and one negative.*

Proof. Let $c, d \in X_+$, $c \wedge d \neq 0$, and $Tc \perp Td$. Let $e_1 = (T^{-1}(Tc)_+)_+$, $e_2 = (T^{-1}(Tc)_+)_-$, $e_3 = (T^{-1}(Tc)_-)_+$, and $e_4 = (T^{-1}(Tc)_-)_-$. Substituting d for c we similarly define elements f_1, f_2, f_3, f_4. It is plain to see that $Te_i \perp Tf_j$ for $1 \leq i \leq j \leq 4$ and that there are indices i_0, j_0 such that $e_{i_0} \not\perp f_{j_0}$. Applying Lemma 2.5.6 to the elements e_{i_0} and f_{j_0} we obtain the desired result. ∎

In the case when the splitting number $d(T)$ is more than 2 we can refine the statement of Lemma 2.5.7.

2.5.8. Lemma. *Let $T : X \to Y$ be a bijective d.p.o. such that $T \in (\pitchfork)$ and $d(T) > 2$. Then there are non-zero $a, b \in X$ such that $0 \le a \le b \le 2a$, $Ta \perp Tb$, and either $Ta, Tb \ge 0$ or $Ta, Tb \le 0$.*

Proof. Because $d(T) > 2$ there are non-negative elements u_1, u_2, u_3 in X such that $u_1 \wedge u_2 \wedge u_3 \ne 0$ and the elements Tu_1, Tu_2, Tu_3 are pairwise disjoint in Y. As in the proof of Lemma 2.5.7 we can see that

$$u_i = u_{i,1} - u_{i,2} + u_{i,3} - u_{i,4}, \qquad i = 1, 2, 3,$$

where all the elements $u_{i,j}$ are non-negative, $Tu_{i,1}, Tu_{i,2} \in Y_+$, and $Tu_{i,3}, Tu_{i,4} \in Y_-$. We claim that among these 12 elements there are at least two which are not disjoint but their T-images are disjoint and of the same sign. Indeed, otherwise we would have $u_{i,j} \perp u_{m,n}$ for all the indices i, j, m, n such that $1 \le i < m \le 3$ and either $1 \le j \le n \le 2$ or $3 \le j \le n \le 4$. This immediately brings a contradiction with our assumption that $u_1 \wedge u_2 \wedge u_3 \ne 0$.

After the existence of two such elements is established it remains to apply Lemma 2.5.6 to them. ∎

We will need two more simple technical lemmas.

2.5.9. Lemma. *Let $T : X \to Y$ be a bijective d.p.o., let $T \in (\pitchfork)$, and assume that there are non-zero elements $a, b, c \in X$ such that $Ta \perp Tb$ and $Ta \perp Tc$. Then $Ta \perp T(b \wedge c)$ and $Ta \perp T(b \vee c)$.*

Proof. Let $d = b \wedge c$ and $U = \{d\}^{dd}$. Let $y = |Ta| \wedge |Td|$. The condition $T \in (\pitchfork)$ implies that $y \in TU$. Clearly there are two bands $U_1, U_2 \subset U$ such that the system $\{U_1, U_2\}$ is full in U, $d - b \perp U_1$ and $d - c \perp U_2$. Then $Td - Tb \perp TU_1$ and because $Ta \perp Tb$ we see that $y \perp TU_1$. Similarly we obtain $y \perp TU_2$. But $T \in (\pitchfork)$ and therefore the system of bands TU_1, TU_2 is full in the band TU, whence $y \perp TU$, and therefore $y = 0$.

The second statement can be verified in a similar way. ∎

2.5.10. Lemma. *Let $T : X \to Y$ be a bijective d.p.o. and let $T \in (\pitchfork)$.*

(1) *Let a, b be elements of X such that $a, b \ge 0$ and $Ta, Tb \ge 0$. Then $T(a \wedge b) \ge 0$.*
(2) *Let a, b be elements of X such that $a, b \ge 0$, $Ta \ge 0$, and $Tb \le 0$. Then $a \wedge b = c \oplus d$ where c is a component of a and d is a component of b.*

Proof. Let $U = \{a \wedge b\}^{dd}$, $U_1 = \{(a - a \wedge b\ )_+\}^{dd} \cap U$, $U_2 = \{(b - a \wedge b\ )_+\}^{dd} \cap U$, and $U_3 = U \cap U_1^d \cap U_2^d$. Then $(a \wedge b - b) \perp U_1$ and $(a \wedge b - b) \perp U_3$, whence $T(a \wedge b) - Tb \perp TU_1$ and $T(a \wedge b) - Tb \perp TU_3$. Similarly, because $a \wedge b - a \perp U_2$ and $a \wedge b - a \perp U_3$ we have $T(a \wedge b) - Ta \perp TU_2$ and $T(a \wedge b) - Ta \perp TU_3$. The system of bands U_1, U_2, U_3 is obviously full in U and because $T \in (\pitchfork)$ the system of bands TU_1, TU_2, TU_3 is full in the band TU.

(1) Because $Ta \ge 0$ and $T(a \wedge b) - Ta \perp TU_2$ we have $(T(a \wedge b))_- \perp TU_2$. Similarly, because $Tb \ge 0$ we obtain $(T(a \wedge b))_- \perp TU_1$ and $(T(a \wedge b))_- \perp TU_3$. Therefore $(T(a \wedge b))_- \perp TU$; but $(T(a \wedge b))_- \in TU$ because TU is a band in Y, whence $(T(a \wedge b))_- = 0$.

(2) First notice that because $Ta \ge 0$ and $Tb \le 0$ we have $(T(a \wedge b))_+ \perp TU_3$ and $(T(a \wedge b))_- \perp TU_3$. Therefore $T(a \wedge b) \perp TU_3$ and because $T \in (\pitchfork)$ we have $(a \wedge b) \perp U_3$, whence $U_3 = \mathbf{0}$. Now we see that $T(a \wedge b)_+ \in TU_2$ and $T(a \wedge b)_- \in TU_1$, whence

$d = T^{-1}(T(a \wedge b)_+) \in U_2$ and $c = T^{-1}(T(a \wedge b)_-) \in U_1$. It remains to notice that $a \wedge b = c + d$ and that by definition of U_1 and U_2, c is a component of a and d is a component of b. ■

The next lemma shows that operators from $(\pitchfork)$ are what we can say "laterally continuous" and will be used extensively later in this paper.

2.5.11. Lemma. *Let X, Y be vector lattices, let $T : X \to Y$ be a disjointness preserving bijection, and let $T \in (\pitchfork)$. Let $\{x_n \in X\}$ be a countable family of pairwise disjoint elements. Then:*

(1) *If the element $x = \sum_n \oplus x_n$ exists in X (3) then $Tx = \sum_n \oplus Tx_n$.*
(2) *If the element $y = \sum_n \oplus Tx_n$ exists in Y then $T^{-1}y = \sum_n \oplus x_n$.*

Proof. (1) For any n the element x_n is a component of x, whence the element Tx_n is a component of Tx. To prove that $Tx = \sum_n \oplus Tx_n$ it is enough to show that if $y \in Y^+$ and $y \perp Tx_n$ for any n then $y \perp Tx$.

Fix any $y \in Y_+$ as above and consider the element $y \wedge |Tx|$. Since T is a bijection there is $u = T^{-1}(y \wedge |Tx|) \in X$. Let us denote by B_n the band generated by x_n, and note that the pairwise disjoint bands B_n are full in the band $B = \{x\}^{dd}$. Since T satisfies $(\pitchfork)$, T sends bands to bands ([6, Proposition 3.2]) and hence TB is a band in Y. Clearly $y \wedge |Tx| \in TB$, and so $u \in B$.

Fix any index n. For each $z \in B_n$ we have $z \perp (x - x_n)$, and hence $Tz \perp T(x - x_n) = Tx \ominus Tx_n$. This plainly implies that $Tz \perp Tu = y \wedge |Tx|$ since y is disjoint from each Tx_n. That is, $TB_n \perp Tu$. By the definition of $(\pitchfork)$ we can conclude that $u \perp B_n$. This is true for each n, and therefore necessarily $u = 0$ since the bands B_n are full in B. Thus $y \wedge |Tx| = Tu = 0$. This means that y is disjoint from Tx, and the proof of (1) is finished.

(2) Fix any index n. Then $y - Tx_n \perp TB_n$ where $B_n = \{x_n\}^{dd}$, and because $T \in (\pitchfork)$ we have $T^{-1}y - x_n \perp B_n$, whence x_n is a component of $T^{-1}y$. Let $z \in X$ be such that $z \perp x_n$ for all n. Then $Tz \perp TB_n$ for all n and therefore $Tz \perp \{y\}^{dd}$. Again by the definition of $(\pitchfork)$ we have $z \perp \{T^{-1}y\}^{dd}$, whence $z \perp T^{-1}y$ and we are done. ■

2.6. The Luxemburg condition. Many results valid for the normed vector lattices do not actually require the whole power of the norm and depend on much weaker conditions that are needed just to stay away from the laterally complete vector lattices. Two such conditions, (Δ_{L}) and (Δ_{P}), were introduced by Luxemburg [30] and de Pagter [34], respectively; and they were used in [5] on several occasions. The former and some of its modifications will be the subject of this section.

We start with the definition given in [5, Definition 2.9].

2.6.1. Definition. We say that a vector lattice X satisfies:

(1) *de Pagter condition* (Δ_{P}) if for each sequence $\{x_n\} \subset X$ with pairwise disjoint non-zero elements, there exists a sequence $\{\lambda_n\}$ of scalars such that the sequence $\{\lambda_n x_n\}$ is not order bounded in X.

(3) This means, as usual, that the elements $x^+ = \sup_n x_n^+$ and $x^- = \sup_n x_n^-$ exist in X and that $x = x^+ - x^-$.

(2) *Luxemburg condition* $(\Delta_{\rm L})$ if for any non-zero $x \in X_+$ we can find pairwise disjoint components x_n of x and positive scalars λ_n such that the sequence $\{\lambda_n x_n\}$ is not order bounded in X.

Notice that without loss of generality we can assume in the above definition that the elements x, x_n and scalars λ_n are positive. Definition 2.6.1(2) (originally introduced in [30, p. 170] for Dedekind σ-complete vector lattices) is useful for atomless vector lattices only ([4]). For a vector lattice X that does not have the principal projection property this definition has a shortcoming even if X has no atoms. To explain this shortcoming, consider the vector lattice $C[0,1]$ and take any $x \in C[0,1]$ that is strictly positive. For such an x the set of components $\mathcal{C}(x)$ is trivial and, therefore, there simply does not exist a non-trivial (infinite) sequence of pairwise disjoint components of x. For this formal reason, we have to say that $C[0,1]$ does not satisfy $(\Delta_{\rm L})$. At the same time, for any $x \in C[0,1]$ that has a sequence $\{x_n\}$ of pairwise disjoint non-zero components we can certainly produce weights λ_n such that the sequence $\{\lambda_n x_n\}$ is not order bounded in $C[0,1]$. And precisely the elements with infinite sets of components are of importance. The next definition takes this into consideration and rectifies the situation.

2.6.2. Definition. A vector lattice X satisfies the *modified Luxemburg condition* $(\Delta_{\rm L}^{\rm m})$ if for each $x \in X$ whose set $\mathcal{C}(x)$ is infinite there are pairwise disjoint $x_n \in \mathcal{C}(x)$ and scalars λ_n such that the sequence $\{\lambda_n x_n\}$ is not order bounded in X.

If a vector lattice X does not have quasi-atoms, i.e., each non-zero element x in X has infinitely many components, then Definitions 2.6.1 and 2.6.2 are equivalent. In general, however, condition $(\Delta_{\rm L})$ is stronger than $(\Delta_{\rm L}^{\rm m})$, but in most situations, even when these two conditions are not equivalent, exactly the latter condition is needed.

The requirement in Definition 2.6.2 that we have to deal with the components of elements is also rather restrictive (for the same reason as explained above that some elements may not have non-trivial components); and the next definition describes a larger class of vector lattices.

Let us say that an element x of a vector lattice X is *infinite-dimensional* if the principal ideal $X(x)$ generated by x is infinite-dimensional, or equivalently, if x cannot be represented as a finite sum of atoms in X.

2.6.3. Definition. We will say that a vector lattice X satisfies the *weak Luxemburg condition* $(\Delta_{\rm L}^{\rm w})$ if for each infinite-dimensional $u \in X$ the principal ideal $X(u)$ contains a disjoint sequence which is not order bounded in X.

2.6.4. Proposition. *Let X be a vector lattice. Assume one of the following two conditions.*

(1) *X is weak Freudenthal* (*see Definition* 2.1.3).
(2) *X has the countable sup property* (*Definition* 2.1.5) *and a cofinal family of components* (*Definition* 2.1.4).

Then the conditions $X \in (\Delta_{\rm L}^{\rm m})$ and $X \in (\Delta_{\rm L}^{\rm w})$ are equivalent.

([4]) Because for any atom $a \in X$ the set $\mathcal{C}(a)$ is trivial, and hence condition $(\Delta_{\rm L})$ fails automatically for such a.

Proof. It is obvious that condition $(\Delta^{\mathrm{m}}_{\mathrm{L}})$ implies $(\Delta^{\mathrm{w}}_{\mathrm{L}})$ without any extra assumptions about the vector lattice. Only the implication $(\Delta^{\mathrm{w}}_{\mathrm{L}})\Rightarrow(\Delta^{\mathrm{m}}_{\mathrm{L}})$ is non-trivial.

(1) $X \in$ (WF). Take an arbitrary $u \in X$ with infinite set $\mathcal{C}(u)$ of components and assume, contrary to what we claim, that for any pairwise disjoint components u_n of u and for any scalars λ_n the sequence $\{\lambda_n u_n\}$ is order bounded in X. This implies immediately that if $\{v_n\}$ is a sequence whose terms are pairwise disjoint and each v_n is a linear combination of pairwise disjoint components of u, then $\{v_n\}$ is also order bounded in X. We will show that this contradicts our hypothesis that X satisfies $(\Delta^{\mathrm{w}}_{\mathrm{L}})$.

Fix an arbitrary disjoint sequence $\{z_n\}$ in $X(u)$. Because X satisfies (WF) we can find elements $v_n \in X$ such that each v_n is a finite linear combination of components of u, $v_n \in \{z_n\}^{dd}$, and

$$|v_n - z_n| < u. \tag{1}$$

Consider now the sequence $\{v_n - z_n\}$. In view of (1) this sequence is order bounded by the element u and also has pairwise disjoint terms. At the same time, since the elements v_n are pairwise disjoint and each v_n is a linear combination of pairwise disjoint components of u, as we noted above, our hypothesis that X fails condition $(\Delta^{\mathrm{m}}_{\mathrm{L}})$ implies that the sequence $\{v_n\}$ is order bounded. This clearly implies that the sequence $\{z_n\}$ is also order bounded, a contradiction.

(2) $X \in$ (CFC)$\cap$(CSP). Let $u \in X$ and let u_n be pairwise disjoint positive elements in $X(u)$ such that the sequence $\{u_n\}$ is not order bounded in X. Because $X \in$ (CFC)$\cap$(CSP) for any $n \in \mathbb{N}$ we can find pairwise disjoint positive elements $u_{k,n}$, $k \in \mathbb{N}$, and positive scalars $\lambda_{k,n}$ such that

- $u_{k,n}$ is a component of u,
- $\{u_{k,n}\}^{dd} \subset \{u\}^{dd},\ k \in \mathbb{N}$,
- the system of bands $\{u_{k,n}\}^{dd},\ k \in \mathbb{N}$, is full in the band $\{u_n\}^{dd}$,
- $(\lambda_{n,k} u_{n,k} - u_n)_- \perp \{u_{n,k}\}^{dd}$, $k, n \in \mathbb{N}$.

Then clearly the system $\{\lambda_{n,k} u_{n,k}\}$ is not order bounded in X. ∎

2.6.5. Remark. We do not know whether the statement of Proposition 2.6.4 remains true if we assume only that X has a cofinal family of components, or even that X has a cofinal family of projection bands. However, as the following example shows, without some conditions related to the existence of band projections, the statement becomes false even if the vector lattice X is r_u-complete.

2.6.6. Example. There exists an r_u-complete vector lattice X that satisfies condition $(\Delta^{\mathrm{w}}_{\mathrm{L}})$ but does not satisfy condition $(\Delta^{\mathrm{m}}_{\mathrm{L}})$.

Proof. Let $K = Q \times [0,1]$, where Q is an arbitrary σ-Stonean (= basically disconnected compact) space without isolated points.

Consider the vector lattice X of all functions on K that are continuous on K except, maybe, on a set $N \times [0,1]$, where N is a nowhere dense subset of Q depending on $x \in X$.

It is not difficult to see that the vector lattice X is r_u-complete and does not satisfy condition $(\Delta^{\mathrm{m}}_{\mathrm{L}})$. One can take the constantly one function for an element violating $(\Delta^{\mathrm{m}}_{\mathrm{L}})$. Nevertheless, clearly, $X \in (\Delta^{\mathrm{w}}_{\mathrm{L}})$. ∎

The introduced conditions $(\Delta_{\mathrm{L}}^{\mathrm{w}})$ and even $(\Delta_{\mathrm{L}}^{\mathrm{m}})$ are much weaker than the de Pagter condition (Δ_{P}). Here is an appropriate example.

2.6.7. Example. There exists an r_u-complete vector lattice X that satisfies condition $(\Delta_{\mathrm{L}}^{\mathrm{m}})$ but does not satisfy condition (Δ_{P}).

Proof. For each $n \in \mathbb{N}$ let X_n be an arbitrary atomless Banach lattice, and hence each X_n satisfies $(\Delta_{\mathrm{L}}^{\mathrm{m}})$. Consider the vector lattice X consisting of all sequences $\mathbf{x} = (x_1, \ldots, x_n, \ldots)$, where $x_n \in X_n$. The order and linear operations in X are coordinatewise. We claim that X has the required properties. It is obvious that X is r_u-complete. (Moreover, X is Dedekind complete if each X_n is, in particular in this case X satisfies $(\Delta_{\mathrm{L}}^{\mathrm{w}})$.)

Fix an arbitrary $z_n \in X_n$ and consider the sequence $\mathbf{x}_n = (0, \ldots, 0, z_n, 0, \ldots)$ in X. For any scalars λ_n the element $\mathbf{x} = (\lambda_1 z_1, \ldots, \lambda_n z_n, \ldots)$ belongs to X, and consequently the sequence $\{\lambda_n \mathbf{x}_n\}$ is order bounded in X. That is, X does not satisfy (Δ_{P}).

Finally, let us verify that the vector lattice X satisfies $(\Delta_{\mathrm{L}}^{\mathrm{m}})$. Take any non-zero $\mathbf{x} = (x_1, \ldots, x_n, \ldots) \in X$. At least one of the coordinates of $\mathbf{x}$, say x_n, is non-zero. Therefore, since X_n is atomless, we can find a disjoint sequence $\{y_k\}$ of non-trivial components of x_n. Consequently, for large enough scalars λ_k the sequence $\{\lambda_k y_k\}$ is not order bounded in X_n. This implies that X satisfies $(\Delta_{\mathrm{L}}^{\mathrm{m}})$. ∎

Our next lemma is a simple technical result that will be used later on.

2.6.8. Lemma. *Let X be a vector lattice and for some $x \in X$ let the principal ideal X_x have the principal projection property. Then each positive element $u \in B = \{x\}^{dd}$ can be represented as a supremum of a disjoint sequence in X_x.*

Proof. For each $n \in \mathbb{N}$ consider the element $(nx - u)^+$, which is obviously in X_x. Since X_x has the principal projection property, there exists the band projection P_n on the band $(nx-u)_+$. Let $u_n = P_n(u \wedge nx)$. We omit a straightforward verification that the elements $u_m - u_{m-1}$, where $u_0 = 0$, are pairwise disjoint and their supremum equals u. ∎

Now we are ready to prove an important theorem characterizing the vector lattices with condition $(\Delta_{\mathrm{L}}^{\mathrm{w}})$. A part of this theorem was stated without proof in [5, Proposition 15.1]. For Dedekind complete vector lattices this was proved in [5, Proposition 14.4]. The fact is crucial for many results proved in [5, Section 15], as well as for some of our results below.

2.6.9. Theorem. *Let $X \in (\mathrm{LC}_1)$. The following conditions are equivalent:*

(a) *X satisfies condition $(\Delta_{\mathrm{L}}^{\mathrm{w}})$.*
(b) *X does not contain any infinite-dimensional laterally σ-complete projection band.*

Proof. The implication (a)⇒(b) is trivial. Indeed, if X contains a non-trivial laterally σ-complete projection band, then certainly X cannot satisfy $(\Delta_{\mathrm{L}}^{\mathrm{w}})$. This implication does not require that $X \in (\mathrm{LC}_1)$.

To prove the converse suppose, contrary to what we claim, that $X \notin (\Delta_{\mathrm{L}}^{\mathrm{w}})$. This means that there exists an infinite-dimensional element $\overline{x} \in X_+$ such that

$(*)$ each disjoint sequence $\{x_n\}$ in $X_{\overline{x}}$ is order bounded in X.

We will establish a contradiction to (b) by showing that the principal band $B = \{\overline{x}\}^{dd}$ is a laterally σ-complete projection band.

We begin by verifying that $(*)$ implies the following property (formally much stronger than $(*)$)

$(**)$ each disjoint sequence $\{z_n\}$ in $X_{\overline{x}}$ has a supremum in X.

Consider a sequence of scalars $\varepsilon_n \searrow 0$ and a sequence $\{z_n\}$ of disjoint positive elements in $X_{\overline{x}}$. In view of $(*)$ the sequence $\{\varepsilon_n^{-1} z_n\}$ is order bounded in X by some element $\overline{z}$. Because $X \in (\mathrm{LC}_1)$ we can choose the scalars ε_n in such a way that the element $\sum_{n=1}^{\infty} \oplus z_n$ exists in X.

Property $(**)$ clearly implies that $X_{\overline{x}}$ is conditionally laterally σ-complete. In particular, $X_{\overline{x}}$ has the principal projection property and, hence, satisfies the hypotheses of Lemma 2.6.8.

Next we will show that the band $B = \{\overline{x}\}^{dd}$ is laterally σ-complete. Take any disjoint sequence $\{u_n\}$ in B_+. By Lemma 2.6.8 for each n there exists a disjoint sequence $\{u_{nk}\}_k$ in $X_{\overline{x}}$ such that $u_n = \sup_k u_{nk}$.

Consider the set $\{u_{nk} : n, k \in \mathbb{N}\}$. This is a disjoint sequence in $X_{\overline{x}}$ and hence by $(**)$ there exists $u = \sup\{u_{nk} : n, k \in \mathbb{N}\} \in X$. But obviously

$$u = \sup\{u_{nk} : n, k \in \mathbb{N}\} = \sup_n \sup_k u_{nk} = \sup_n u_n,$$

that is, B is indeed laterally σ-complete. ■

2.6.10. Corollary. *An r_u-complete vector lattice X satisfies condition $(\Delta_{\mathrm{L}}^{\mathrm{w}})$ if and only if X does not contain an infinite-dimensional universally σ-complete projection band.*

If instead of disjoint sequences we consider arbitrary disjoint sets, then the following modification of Corollary 2.6.10 is true. The proof is similar and is omitted.

2.6.11. Theorem. *Let X be an r_u-complete vector lattice. The following conditions are equivalent*:

- (a′) *For each infinite-dimensional $u \in X$ the principal ideal X_u contains a disjoint set that is not order bounded in X.*
- (b′) *X does not contain any infinite-dimensional universally complete projection band.*

It is natural to ask whether or not the assumption that $X \in (\mathrm{LC}_1)$ is essential in Theorem 2.6.9, that is, if (b) implies (a) in general vector lattices. We do not know whether the assumption $X \in (\mathrm{LC}_1)$ can be replaced by $X \in (\mathrm{LC}_3)$ or even by $X \in (\mathrm{LC}_2)$ but the next example shows that without any assumption of this kind the answer is negative, even if we require additionally that X satisfies (WF), or even that X has the projection property.

2.6.12. Example. There exists a vector lattice X that has the projection property and satisfies (b) but does not satisfy (a), that is, X does not contain any (infinite-dimensional) laterally σ-complete projection band but, nevertheless, X fails $(\Delta_{\mathrm{L}}^{\mathrm{w}})$.

Proof. Let $Y = L_\infty[0,1]$. According to [5, Corollary 13.6] there exists a vector sublattice X of the vector lattice $L_0[0,1]$ and a d-isomorphism T from X onto Y such that T is not regular. We claim that the vector lattice X provides a desired counterexample.

Since the projection property is preserved by d-isomorphisms and since, obviously, Y has the projection property, we can conclude that the vector lattice X has the projection property and, in particular, $X \in (\mathrm{WF})$.

Let us verify next that X cannot satisfy $(\Delta_{\mathrm{L}}^{\mathrm{m}})$. Indeed, otherwise by Corollary 5.3.2 ([5]) the d-isomorphism T would be regular contrary to our choice of T. Therefore, by Proposition 2.6.4, X cannot satisfy $(\Delta_{\mathrm{L}}^{\mathrm{w}})$ either.

Finally, X cannot have an infinite-dimensional laterally σ-complete band, because otherwise the vector lattice Y would, since each d-isomorphism preserves such bands. ■

3. d-universal domains

The title of this section is explained by the contents of its two subsections.

3.1. Domains on which each disjointness preserving operator is regular. The main result of this subsection is the following theorem.

3.1.1. THEOREM. *For a vector lattice X the following conditions are equivalent*:

(1) *Each disjointness preserving operator $T : X \to Y$ into an arbitrary vector lattice Y is regular.*
(2) *Each disjointness preserving operator $T : X \to X$ is regular.*
(3) *Each* injective *disjointness preserving operator $T : X \to Y$ into an arbitrary vector lattice Y is regular.*
(4) *For any $x, z \in X$ such that $|z| \le |x|$ the element z is a finite linear combination of components of x.*
(5) *For any $x \in X$ the Krein–Kakutani space K_x is zero-dimensional and the principal ideal I_x is order isomorphic to the vector lattice of all finite-valued continuous functions on K_x.*

Proof. The implications (1)⇒(2) and (1)⇒(3) are trivial.

(2)⇒(4). Assume (2) and assume to the contrary that there are $x, z \in X_+$ such that $z \le x$ but z is not a finite linear combination of components of x. Let Q be the Stone space of X; then X can be identified with a vector sublattice of $C_\infty(Q)$ in such a way that x is identified with the characteristic function of a clopen subset E of Q. Because z cannot be represented as a finite linear combination of components of x there is a point $q \in E$ such that on any open neighborhood of q the function z is not constant. Let J be the ideal in X defined as $J = \{u \in X : u \equiv 0 \text{ on some open neighborhood of } q\}$. Let $\dot{X}$ be the factor X/J. The elements $\dot{x}$ and $\dot{z}$ are linearly independent in $\dot{X}$ and therefore there is a linear functional F on $\dot{X}$ such that $F(\dot{x}) = 0$ and $F(\dot{z}) = 1$. Let G be a linear functional on X defined as $G(u) = F(\dot{u})$. Finally, let T be a linear operator from X to X defined as $Tu = G(u)x$. The operator T preserves disjointness. Indeed, if $u, v \in X$ and $u \perp v$ then at least one of the functions u and v is equal to 0 in some open neighborhood of q because Q is extremally disconnected. But the operator T is not regular because it obviously does not satisfy condition (3) in Theorem 2.3.2.

([5]) A reference "forward", but the proof of Corollary 5.3.2 does not depend on Example 2.6.12.

(3)$\Rightarrow$(4). Let T be the operator constructed in the previous step of the proof and let us define an operator $S : X \to X \oplus X$ as $Su = (Tu, u)$. Then S is an injective disjointness preserving operator but it is not regular.

The implication (4)$\Rightarrow$(1) follows immediately from Theorem 2.3.2.

Finally, the equivalence (4)$\Leftrightarrow$(5) follows from the fact that if we represent the principal ideal I_x as a vector sublattice of $C(K_x)$ then the functions from I_x separate the points of K_x and if x is represented as the function $\mathbf{1}$ then the characteristic function of any clopen subset of K_x is in I_x. ∎

We want to emphasize an important special case of Theorem 3.1.1:

3.1.2. Corollary. *Let* $X \in (\mathrm{LC}_4)$ (*see Definition* 2.2.5). *Then the following conditions are equivalent*:

(1) *Every d.p.o.* $T : X \to Y$ *into an arbitrary vector lattice* Y *is regular.*
(2) *Every d.p.o.* $T : X \to X$ *is regular.*
(3) *Every injective d.p.o. from* X *into an arbitrary vector lattice* Y *is regular.*
(4) *Every element in* X *is a finite sum of atoms.*
(5) *There is a set* Γ *such that the vector lattice* X *is order isomorphic to the vector lattice* $c_{00}(\Gamma)$ *of all functions on* Γ *which take a non-zero value only on a finite subset of* Γ. *In particular we see that* X *is a relatively uniformly complete vector lattice.*

The next question connected with Theorem 3.1.1 remains open.

3.1.3. Problem. Are conditions (1)–(5) in Theorem 3.1.1 equivalent to (2′) below?

(2′) Every *injective* d.p.o. $T : X \to X$ is regular.

Or, in other words, the problem is to describe the class of all vector lattices such that every injective d.p.o. from such a lattice *into itself* is regular.

3.1.4. Remark. The proof of Theorem 3.1.1 shows that if a vector lattice X satisfies the additional condition that for some vector lattice Z the vector lattice $X \oplus Z$ is order isomorphic to a vector sublattice of X then condition (2′) is equivalent to conditions (1)–(5).

In particular, Banach lattices like l^p, $L^p(0,1)$, or $C[0,1]$ provide examples where there exists an injective non-regular d.p.o. from the vector lattice into itself. Notice that by the Huijsmans–de Pagter–Koldunov theorem [25, 28] for any such endomorphism T on a Banach lattice we have $x \perp z \Leftrightarrow Tx \perp Tz$.

3.2. Domains on which $x \perp z \Leftrightarrow Tx \perp Tz$ for each injective disjointness preserving operator T. Here we discuss a wider class of domains, namely the class of all vector lattices X such that for an arbitrary vector lattice Y and an injective d.p.o. $T : X \to Y$ we have

$$x \perp z \;\Leftrightarrow\; Tx \perp Tz.$$

Let us denote this class of vector lattices as $\mathcal{UI}$.

3.2.1. Theorem. *Let X be a vector lattice. Then*:

(1) $X \in \mathcal{UI} \Rightarrow d\text{-}\dim X = 1$.

(2) *If we additionally assume that* $X \in$ (CFC) *then*

$$X \in \mathcal{UI} \;\Leftrightarrow\; d\text{-}\dim X = 1.$$

Proof. (1) Assume that $d\text{-}\dim X > 1$. By Proposition 2.4.9 there are $x, z \in X$, $|z| \le |x|$, and a band $B \subset X$ such that $P_B x$ and $P_B z$ are d-independent in X^u.

Without loss of generality we can assume that for any $u \in X$ and for any band $V \subset X$ such that $u \not\perp V$ there is a non-zero semicomponent (see Definition 2.4.1) v of u such that $v \in V$. Indeed, otherwise [8, Theorem 4.2] there is a *band preserving* injective operator $T : X \to X^u$ such that the inverse operator $T^{-1} : TX \to X$ does not preserve disjointness.

Let a be a non-zero semicomponent of z in B and let b be a non-zero semicomponent of x in $\{a\}^{dd}$. Then clearly $a \not\perp b$ and a and b are d-independent. The proof of Theorem 13.8 in [5] shows that there are a vector lattice Y and a disjointness preserving bijection $T : X^u \to Y$ such that $Ta \perp Tb$. The restriction $S = T|X : X \to Y$ is an injective d.p.o. and $Sa \perp Sb$ though $a \not\perp b$.

(2) The implication

$$(X \in \text{(CFC) and } d\text{-}\dim X = 1) \;\Rightarrow\; X \in \mathcal{UI}$$

is exactly the statement of Theorem 11.2 from [5]. ∎

As the following example shows, the assumption $X \in$ (CFC) in part (2) of the statement of Theorem 3.2.1 cannot be dropped.

3.2.2. Example. Let $C^\star[0,1]$ be the Banach dual of $C[0,1]$ identified as usual with the Banach space of all finite regular Borel measures on $[0,1]$. Let Y be the band of all singular (with respect to the Lebesgue measure) continuous measures in $C^\star[0,1]$. We define a linear operator $S : Y \to C[0,1]$ in the following way:

$$S\mu(t) = \mu([0,t]), \qquad \mu \in Y,\, t \in [0,1].$$

Notice that S is injective. Let $X = SY$. Then X consists of all functions of bounded variation on $[0,1]$ such that the union of the intervals where such a function is constant has Lebesgue measure 1.

It is immediate to see that X is a vector sublattice of $C[0,1]$ and that $d\text{-}\dim X = 1$. Let $T : X \to Y$ be the inverse of S. Then T is a bijective d.p.o. but its inverse S does not preserve disjointness. Moreover, we can see directly or from Theorem 4.13 in [6] that $T \notin (\pitchfork)$.

In connection with Theorem 3.2.1 and Example 3.2.2 the following problem naturally arises.

3.2.3. Problem. Describe the class $\mathcal{UI}$.

Let us mention two more specific subproblems of Problem 3.2.3:

(1) Let $X = \mathrm{EC}(0,1)$ be the vector lattice of all essentially constant continuous functions on $[0,1]$ (the union of the intervals of constancy of such a function is dense in $[0,1]$). This vector lattice is d-rigid [10, Corollary 4.4] but we do not know if it belongs to the class $\mathcal{UI}$.

(2) Let K be a connected compact Hausdorff space such that $\mathrm{EC}(K) = C(K)$. An example of such a space is provided by $\beta\mathbb{R}^+ \setminus \mathbb{R}^+$, which is a connected F-space such that any G_δ subset of it has non-empty interior (see [21, Section 14.27] and [23, p. 320]). For a different example see [17, Example 3.9]. Then d-dim $C(K) = 1$ but we do not even know whether $C(K)$ is d-rigid.

4. d-rigid vector lattices. General case

The main result of this section is the following theorem.

4.0.1. THEOREM. I. *Let X, Y be vector lattices and $T : X \to Y$ be a disjointness preserving bijection such that $T \in (\pitchfork)$ and the inverse T^{-1} does not preserve disjointness. Then there is a positive $a \in X$ for which*:

(1) *The Krein–Kakutani space $K = K_a$ is zero-dimensional.*
(2) *There is a family $\{K_\alpha\}_{\alpha\in[0,1]}$ of clopen subsets of K such that*:
   - $K_0 = \emptyset$ *and* $K_1 = K$.
   - $K_\alpha \subsetneq K_\beta$ *for* $0 \le \alpha < \beta \le 1$.
   - $A_\alpha = K_\alpha \setminus \bigcup_{\beta<\alpha} K_\beta$ *and* $B_\alpha = (\bigcap_{\beta>\alpha} K_\beta)\setminus K_\alpha$ *are nowhere dense in* K.

II. *Conversely, if there is a zero-dimensional compact space K and a family $\{K_\alpha\}_{\alpha\in[0,1]}$ of its clopen subsets with the properties listed in* I(2), *then there are vector lattices X, Y and a disjointness preserving bijection $T : X \to Y$ such that X is a vector sublattice of $C(K)$, X separates points of K and contains the constant functions, $T \in (\pitchfork)$, and T^{-1} does not preserve disjointness.*

Proof. I(1). Let $a, b \in X$ be as in the statement of Lemma 2.5.7. Without loss of generality we can assume that $Ta \ge 0$. We will identify a with the function $\mathbf{1}$ on K_a, and elements of the principal ideal I_a with the corresponding functions from $C(K_a)$. For any point $t \in K_a$ let $\mathfrak{C}(t)$ be the connected component of t. We have to prove that for any t the set $\mathfrak{C}(t)$ is a singleton. Assume to the contrary that there are $p, q \in K_a$ such that $p \ne q$ and $q \in \mathfrak{C}(p)$. Let u, v be elements of I_a such that

- $u \equiv v \equiv a$ in some open neighborhood of p,
- $\operatorname{supp} v \subset \{t : u(t) = 1\}$,
- $q \notin \operatorname{supp} u$.

First let us notice that because $Ta \perp Tb$ and $Ta - Tu \perp Tv$ we have $|Tb| \wedge |Tu| \wedge |Tv| = 0$. Let $w = (T^{-1}(|Tu| \wedge |Tv|) \wedge a) \vee (-a)$. Then $w \in I_a$ and, by Lemma 2.5.9, $Tb \perp Tw$.

We claim that $w \equiv a$ in some open neighborhood of p. Indeed, let W be a regularly open neighborhood of p such that $v \equiv a$ on W. Let $\widetilde{u}$ be the band in I_a of all functions with support in W and let $U = \{\widetilde{U}\}^{dd}$ be the corresponding band in X. Then $v - a \perp U$ and $u - a \perp U$, whence $Tu - Ta \perp TU$ and $Tv - Ta \perp TU$. Because $Ta \ge 0$ we see that $|Tu| - Ta \perp TU$ and $|Tv| - Ta \perp TU$, whence $(|Tu| \wedge |Tv|) - Ta \perp TU$. Recalling that $T \in (\pitchfork)$ we obtain $T^{-1}(|Tu| \wedge |Tv|) - a \perp U$ and it follows from the definition of w that $w - a \perp U$, whence our claim is proved.

Next let us notice that because $v \in \{u\}^{dd}$ and $T\{u\}^{dd}$ is a band in Y we have $T^{-1}(|Tu| \wedge |Tv|) \in \{u\}^{dd}$, whence $w \in \{u\}^{dd}$ and $w(q) = 0$. By Lemma 2.5.6, w can be uniformly approximated by linear combinations of components of b. But that is impossible because $\mathfrak{C}(p)$ is a connected subset of K_a, $b \equiv b(p)$ on $\mathfrak{C}(p)$, $w(p) = 1$, and $w(q) = 0$.

I(2). The set $b(K_a)$ is a closed subset of $\mathbb{R}$ and it does not have isolated points. Indeed, if t were an isolated point in $b(K_a)$ then $E = b^{-1}(t)$ would be a clopen non-empty subset of K_a and $b \equiv t$ on E. That clearly contradicts the assumption that $Ta \perp Tb$. Therefore $\operatorname{card}(b(K_a)) = \mathfrak{c}$. Let $\gamma \in b(K_a)$ and $\min b(K_a) < \gamma \leq \max b(K_a)$. Consider $u = b \wedge \alpha a$. By Lemma 2.5.4, $u = u_1 \oplus u_2$ where u_1 is a component of b. Let $E_\gamma = \operatorname{supp} u_1$; then E_γ is a clopen non-empty subset of K_a. Let $E_{\min b(K_a)} = \emptyset$. Finally, let us take $K_\alpha = E_{\psi(\alpha)}$ where ψ is a one-to-one map of $[0,1]$ onto $b(K_a)$.

II. It was proved in [8, Proof of Theorem 5.8] that $\bigcup_{\alpha\in[0,1]}\big(A_\alpha \cup B_\alpha\big) = K$ and that the function $\mathbf{f}$ defined as $\mathbf{f}(t) = \alpha$ if $t \in A_\alpha \cup B_\alpha$ is a well defined continuous function on K. Let us consider the following three vector subspaces of $C(K)$:

- X_1, the set of all finite linear combinations of components of the constant function $\mathbf{1}$;
- X_2, the set of all finite linear combinations of components of the function $\mathbf{f}$;
- X, the linear hull of X_1 and X_2.

Obviously X_1 and X_2 are vector sublattices of $C(K)$, and it is not difficult to see from the definition of $\mathbf{f}$ and the properties of the sets K_α that X is also a vector sublattice of $C(K)$. Let us define band preserving projections P and Q on X in the following way. If $x \in X$ and $x \equiv c\mathbf{1} + d\mathbf{f}$ on a clopen subset E of K then $Px \equiv c\mathbf{1}$ on E and $Qx \equiv d\mathbf{f}$ on E. The operator $P \oplus Q : X \to X_1 \oplus X_2$ is a bijective d.p.o. but its inverse does not preserve disjointness. ∎

4.0.2. Theorem. *Let X, Y be vector lattices and $T : X \to Y$ be a d-isomorphism. Assume that T is not regular. Then there is $a \in X$ such that the Krein–Kakutani compact space K_a satisfies conditions* I(1, 2) *of Theorem* 4.0.1.

Proof. Because the operator T is not regular there are elements $x, y \in X$ such that $0 \leq x \leq y$ but $(|Tx| - |Ty|)_+ \neq 0$. It follows easily from the fact that T is a d-isomorphism that $z = T^{-1}((|Tx| - |Ty|)_+) \in I_y$. We identify z with a function from $C(K_y)$. Let $t \in K_y$, $|z(t)| > 0$, and $0 < x(t) < y(t)$.

We claim that the point t has a base of clopen neighborhoods. To prove this let U, V, and W be open neighborhoods of t such that $\min(|z(s)|, x(s), y(s) - x(s)) > 0$ for $s \in \operatorname{cl} U$, $\operatorname{cl} V \subset U$, and $\operatorname{cl} W \subset V$. Let u, v be elements of I_y such that

- $\operatorname{supp} u \subset U$ and $0 \leq u \leq y$,
- $u \equiv y$ on V,
- $\operatorname{supp} v \subset V$ and $0 \leq v \leq x$,
- $v \equiv x$ on W.

Let $w = T^{-1}(|Tu| \vee |Tv|)$. Because T is a d-isomorphism we see that

- for any $k \in K_y$ either $|w(s)| = x(s)$ or $|w(s)| = y(s)$,
- $|w(s)| \equiv x(s)$ on W,
- $|w(s)| \equiv y(s)$ on $V \setminus \operatorname{supp} v$.

Recalling that $x(s) < y(s)$ on $\operatorname{cl} U$ we deduce that there is a clopen neighborhood Z of t such that $Z \subset U$ and our claim is proved.

It follows immediately from what we have just shown that without loss of generality we can assume that the compact space K_y is zero-dimensional and that $0 \le x \le y$ but $|Ty| \le |Tx|$. We identify $|Tx|$ with the constant function $\mathbf{1}$ on $K_{|Tx|}$, and $|Ty|$ with a function from $C(K_{|Tx|})$. We can see immediately that the set $E = |Ty|(K_{|Tx|})$ does not have isolated points. Let $\min E < \alpha < \max E$, $v_\alpha = \frac{1}{\alpha}|Ty| \wedge |Tx|$, and $u_\alpha = T^{-1}v_\alpha$. Then $|u_\alpha| = w_\alpha + z_\alpha$ where w_α and z_α are non-zero components of $\frac{1}{\alpha}|y|$ and $|x|$, respectively. Clearly, $\operatorname{supp} z_\alpha$ is a clopen subset of K_x and we can finish the proof exactly like the one of Theorem 4.0.1. ∎

4.0.3. Corollary. *Let K be a compact Hausdorff space and X be a vector subspace of $C(K)$ such that X separates points of K and contains the constant functions. Let Y be a vector lattice and $T : X \to Y$ be a bijective d.p.o. such that $T \in (\pitchfork)$. Assume additionally that K is either locally connected or metrizable. Then the operator T is regular and the vector lattices X and Y are order isomorphic.*

4.0.4. Corollary. *Let K be a compact Hausdorff space with a π-base of clopen subsets. Assume additionally that either*

(1) *no clopen subset of K is zero-dimensional, or*
(2) *K is metrizable.*

Let X be a vector sublattice of $C(K)$ that separates points of K. Then X is super d-rigid.

4.0.5. Remark. Comparing Theorem 2.3.2 and Corollary 2.3.3 with Theorem 5.8 and Corollary 5.9 in [8], which provide the same results in the case of arbitrary band preserving operators, one feels that all of these results should follow from some more general statement.

4.0.6. Remark. The vector lattice X described in the proof of part II of Theorem 4.0.1 has the principal projection property, and this fact gives rise to the question whether every vector lattice which is not d-rigid has a band with the principal projection property. The answer is negative: the vector lattice X described in Example 5.15 in [8] is not d-rigid and no band in it has the principal projection property. Moreover $X \in (\mathrm{LC}_4)$.

5. Weakly c_0-complete domains

5.1. The main results. Let us remind the reader that by Definition 2.2.9 weakly c_0-complete vector lattices are exactly the lattices from the class (LC_3) introduced in Definition 2.2.5.

The results of this section can be divided into three groups.

I. When is a bijective d.p.o. $T \in (\pitchfork)$ a d-isomorphism? When is the domain X d-rigid?

5.1.1. Theorem. *Let $X \in (\mathrm{LC}_3)$.*

(1) *Assume that for every conditionally laterally σ-complete projection band U in X we have d-$\dim U = 1$. Then for every vector lattice Y and for every bijective*

d.p.o. $T : X \to Y$,

$$T \in (\pitchfork) \ \Rightarrow\ T \text{ is a } d\text{-isomorphism.}$$

(2) *Assume that X contains a conditionally laterally complete projection band U such that d-$\dim U > 1$. Then there exist a vector lattice Y and a bijective d.p.o. $T : X \to Y$ such that $T \in (\pitchfork)$ but T^{-1} does not preserve disjointness.*

Recalling that on a vector lattice with a cofinal family of components every injective d.p.o. satisfies condition $\pitchfork$ we immediately get the following corollary.

5.1.2. Corollary. *Let $X \in (\mathrm{CFC}) \cap (\mathrm{LC}_3)$.*

(1) *Assume that for every conditionally laterally σ-complete projection band U in X we have d-$\dim U = 1$. Then X is d-rigid.*
(2) *Assume that X contains a conditionally laterally complete projection band U such that d-$\dim U > 1$. Then X is not d-rigid.*

5.1.3. Theorem. *Let X, Y be vector lattices and $X \in (\mathrm{LC}_3)$.*

(1) *Let $T : X \to Y$ be a bijective d.p.o. and let $T \in (\pitchfork)$. Assume that T^{-1} does not preserve disjointness and that one of the following additional conditions holds*:

(a) $d(T) > 2$,
(b) $Y \in (\mathrm{LC}_3)$.

Then X contains a laterally σ-complete band.

(2) *Assume that X contains a laterally complete band U such that d-$\dim U \geq n \geq 2$. Then there are a vector lattice $Y \in (\mathrm{LC}_3)$ and a bijective d.p.o. T such that $T \in (\pitchfork)$ and $d(T) = n$. Moreover, if d-$\dim U = \infty$ the operator T can be chosen in such a way that $d(T) = \infty$.*

II. When is a d-isomorphism regular?

5.1.4. Theorem. *Let X be a vector lattice and let $X \in (\mathrm{LC}_3)$.*

(1) *Assume that for any projection band U in X with the* principal projection property *we have d-$\dim U = 1$. Then every d-isomorphism $T : X \to Y$ where Y is an arbitrary vector lattice is regular. In particular every vector lattice d-isomorphic to X is also order isomorphic to it.*
(2) *Assume that X contains a projection band U with the* projection property *and that d-$\dim U > 1$. Then there are a vector lattice Y and a d-isomorphism $T : X \to Y$ such that T is not regular.*

5.1.5. Theorem.

(1) *Let X, Y be vector lattices from the class* (LC_3). *Assume that for every laterally σ-complete band U in X (or in Y) we have d-$\dim U = 1$. Then every d-isomorphism $T : X \to Y$ is regular and X and Y are order isomorphic.*
(2) *Let X be a vector lattice. Assume that X contains a laterally complete band U such that d-$\dim U > 1$. Then there is a d-isomorphism $T : X \to X$ such that T is not regular.*

5.1.6. Corollary. *Let X be a vector lattice such that $X \in (\mathrm{LC}_3)$ and for any laterally σ-complete band $U \subset X$, $d\text{-}\dim U = 1$. Let $P : X \to X$ be a band preserving projection. Then P is regular and therefore a band projection.*

Proof. $I + P$ is a d-isomorphism of X onto itself and it is regular by Theorem 5.1.5. ∎

From Corollary 5.1.6 and Theorem 8.5 in [5] we immediately obtain

5.1.7. Corollary. *Let $X \in (\mathrm{LC}_3)$ and let Y have a cofinal family of projection bands (i.e. for any non-zero band $U \subset X$ there is a non-zero $V \subset U$ such that V is a projection band in X). Let $T : X \to Y$ be a bijective d.p.o. Then T is a d-isomorphism.*

III. When is every bijective d.p.o. $T \in (\pitchfork)$ regular? When is the domain X super d-rigid?

Combining the results from I and II we obtain the following theorem.

5.1.8. Theorem. *Let X be a vector lattice and let $X \in (\mathrm{LC}_3)$. Assume that for every projection band U in X with the principal projection property we have $d\text{-}\dim U = 1$. Let Y be a vector lattice, $T : X \to Y$ be a bijective d.p.o., and let $T \in (\pitchfork)$. Then T is regular. If we assume additionally that $X \in (\mathrm{CFC})$ then X is super d-rigid.*

IV. Comments and remarks

In all the results stated above we see a gap between necessary and sufficient conditions for a vector lattice to be d-rigid, super d-rigid, *et cetera*. The necessary conditions involve the absence of non-trivial projection bands which are laterally complete, conditionally laterally complete, or just have the projection property. In the sufficient conditions we have to require the absence of non-trivial projection bands which are laterally σ-complete, conditionally laterally σ-complete, or have the principal projection property. This gap would be filled if we could answer the following question in the affirmative.

5.1.9. Problem. Let X be a laterally σ-complete vector lattice and let $d\text{-}\dim X > 1$. Is there a *non-regular* band preserving projection P on X? ([6])

For two important classes of vector lattices:

- vector lattices with the projection property, in particular Dedekind complete vector lattices,
- vector lattices with the countable sup property,

the gap is automatically filled in and the results become exact. We will state them explicitly in Section 6.

The statement of part (2) of Theorem 5.1.3 can be complemented in the following way. Let U be a laterally complete vector lattice. There is a band $V \subset U$ with a d-basis (see e.g. [4, Definition 4.6]) $\{x_\gamma : \gamma \in \Gamma\}$ where each x_γ is a weak unit in V. This was proved in [5, Theorem 6.4] for universally complete vector lattices but the proof remains unchanged for laterally complete ones.

([6]) Recent discussions of this question with J. van Mill and A. W. Wickstead made the second author believe that the answer to Problem 5.1.9 should be negative.

5.1.10. Proposition. *There are a vector lattice* Y *and a bijective d.p.o.* $T : U \to Y$ *such that the elements* Tx_γ, $\gamma \in \Gamma$, *are pairwise disjoint.*

5.2. Proofs of Theorems 5.1.1 and 5.1.3. We will divide the proofs into several steps-lemmas.

5.2.1. Lemma. *Let* X, Y *be vector lattices,* $X \in (\mathrm{LC}_3)$, *and* $T : X \to Y$ *be a bijective d.p.o. Assume that* $T \in (\pitchfork)$ *and that* T^{-1} *does not preserve disjointness. Then* X *contains a conditionally laterally* σ*-complete principal projection band* $U = \{a\}^{dd}$ *such that* $d\text{-}\dim U > 1$ *and the Krein–Kakutani space* K_a *is basically disconnected.*

Proof. By Theorem 4.0.1 there are non-zero elements $a, b \in X$ such that $0 \le a \le b \le 2a$, $Ta \perp Tb$, and the space K_a is zero-dimensional. Moreover, by Lemma 2.5.7 we can assume without loss of generality that either

(1) $Ta \ge 0$ and $Tb \ge 0$, or
(2) $Ta \ge 0$ and $Tb \le 0$.

Let us start with case (1). In this case we will prove that the band $U = \{a\}^{dd}$ is not only conditionally laterally σ-complete but even laterally σ-complete and therefore, by Theorem 2.2.2, a projection band in X. Let u_n, $n \in \mathbb{N}$, be a sequence of pairwise disjoint positive elements in U. We have to prove that the element $u = \sum_{n=1}^\infty \oplus u_n$ exists in U. For any $m \in \mathbb{N}$ let $u_{n,m} = u_n \wedge ma$. Then $u_{n,m} \in I_a$ and we can consider open subsets of K_a,

$$O_{n,m} = \{t \in K_a : m - 1 < u_{n,m}(t) < m\}.$$

Clearly the sets $O_{n,m}$ are pairwise disjoint (some of them might be empty). Recalling that the space K_a is zero-dimensional we see that for each $n, m \in \mathbb{N}$ we can find clopen subsets $E_{n,m,k}$, $k \in \mathbb{N}$, of the set $O_{n,m}$ such that $\bigcup_{k=1}^\infty E_{n,m,k} = O_{n,m}$. Let $\{\varepsilon_{n,m,k} : n, m, k \in \mathbb{N}\}$ be a set of positive scalars such that $\varepsilon_{n,m,k} \to 0$ as $n + m + k \to \infty$. Because the sets $E_{n,m,k}$ are zero-dimensional we can find non-negative elements $a_{n,m,k}$ and $b_{n,m,k}$ such that $a_{n,m,k}$ and $b_{n,m,k}$ are finite linear combinations of components of $a\chi_{n,m,k}$ and $b\chi_{n,m,k}$, respectively, and

$$(*) \qquad |a_{n,m,k} - u_{n,m}\chi_{E_{n,m,k}}| + |b_{n,m,k} - u_{n,m}\chi_{E_{n,m,k}}| \le \varepsilon_{n,m,k}\chi_{E_{n,m,k}}.$$

Because $X \in (\mathrm{LC}_3)$ we can choose the scalars $\varepsilon_{n,m,k}$ in such a way that the element $v = \sum_{n,m,k\in\mathbb{N}} \oplus (a_{n,m,k} - b_{n,m,k})$ exists in X. By Lemma 2.5.11,

$$Tv = \sum_{n,m,k\in\mathbb{N}} \oplus (T(a_{n,m,k}) - T(b_{n,m,k})).$$

Recalling that $Ta \perp Tb$ and that $Ta, Tb \ge 0$ we see that

$$|Tv| = \sum_{n,m,k\in\mathbb{N}} \oplus (T(a_{n,m,k}) + T(b_{n,m,k})).$$

Applying again Lemma 2.5.11 we obtain

$$T^{-1}(|Tv|) = \sum_{n,m,k\in\mathbb{N}} \oplus (a_{n,m,k} + b_{n,m,k}).$$

Therefore the element $\sum_{n,m,k\in\mathbb{N}} \oplus a_{n,m,k}$ exists in X. Taking into consideration the inequality $(*)$ we see that $u = \sum_{n=1}^\infty \oplus u_n$ exists in X.

Let us now consider case (2). We will divide the proof that X contains a non-trivial conditionally laterally σ-complete projection band into several steps.

(2a) Let $\{a_i : i \in \mathbb{N}\}$ be a countable set of pairwise disjoint components of a. We claim that the element $\sum_{n=1}^{\infty} \oplus a_i$ exists in X. Because $X \in (\mathrm{LC}_3)$ and K_a is zero-dimensional we can find non-negative elements $b_i \in X$ such that for each i, b_i is a finite linear combination of components of b and the element $c = \sum_{i=1}^{\infty} \oplus(b_i - a_i)$ exists in X. By Lemma 2.5.11, $Tc = \sum_{i=1}^{\infty} \oplus(Tb_i - Ta_i)$ and because $Ta \perp Tb$, $Tc = \sum_{i=1}^{\infty} \oplus(Tb_i \ominus Ta_i)$. Let $d = a + c$. Then

$$Td = Ta + \sum_{i=1}^{\infty} \oplus Tb_i - \sum_{i=1}^{\infty} \oplus Ta_i.$$

Because Ta_i is a component of Ta for each i and because $Ta \geq 0$ and $Tb \leq 0$ we see that $(Td)_- = \sum_{i=1}^{\infty} \oplus Tb_i$. By Lemma 2.5.11, again,

$$T^{-1}((Td)_-) = \sum_{i=1}^{\infty} \oplus b_i,$$

whence $\sum_{i=1}^{\infty} \oplus a_i$ exists in X.

(2b) In this step we want to prove that for any countable set $\{a_i : i \in \mathbb{N}\}$ of components of a and for any scalars λ_i, $i \in \mathbb{N}$, such that $0 \leq \lambda_i \leq 1$, the element $\sum_{i=1}^{\infty} \oplus \lambda_i a_i$ exists in X. As in the previous step we can find elements b_i such that b_i is a finite linear combination of components of b and the elements $\sum_{i=1}^{\infty} \oplus(b_i - a_i)$ and $\sum_{i=1}^{\infty} \oplus \lambda_i(b_i - a_i)$ exist in X. By step (2a) the element $\sum_{i=1}^{\infty} \oplus(a_i + b_i)$ exists in X, whence

$$u = \sum_{i=1}^{\infty} \oplus \left(\frac{1 + \lambda_i}{2}\, a_i + \frac{1 - \lambda_i}{2}\, b_i \right)$$

also exists in X. Let $w = T^{-1}(|Tu|)$. Recalling that $Ta \perp Tb$, $Ta \geq 0$, and $Tb \leq 0$ and applying Lemma 2.5.11 we see that

$$w = \sum_{i=1}^{\infty} \oplus \left(\frac{1 + \lambda_i}{2}\, a_i - \frac{1 - \lambda_i}{2}\, b_i \right).$$

It remains to notice that

$$z = \sum_{i=1}^{\infty} \oplus \frac{1 - \lambda_i}{2}\, (a_i - b_i)$$

exists in X and that $w - z = \sum_{i=1}^{\infty} \oplus \lambda_i a_i$.

(2c) We will now prove that if u_n, $n \in \mathbb{N}$, are pairwise disjoint elements of X and $0 \leq u_n \leq a$ then $\sum_{n=1}^{\infty} \oplus u_n$ exists in X. Indeed, we can find numbers $m(n) \in \mathbb{N}$, pairwise disjoint components $a_{n,i}$ of a, $n \in \mathbb{N}$, $1 \leq i \leq m(n)$, and scalars $\lambda_{n,i}$ such that $0 \leq \lambda_{n,i} \leq 1$ and the element

$$\sum_{n=1}^{\infty} \oplus \Big(u_n - \sum_{i=1}^{m(n)} \lambda_{n,i} a_{n,i} \Big)$$

exists in X. But $\sum_{n=1}^{\infty} \sum_{i=1}^{m(n)} \oplus \lambda_{n,i} a_{n,i}$ exists in X by step (2b), whence our claim follows.

(2d) In this step we will prove that the band $U = \{a\}^{dd}$ is conditionally laterally σ-complete. Assume that elements $u_n \in U$ are pairwise disjoint and $0 \leq u_n \leq u$ where $u \in U$. Let $v_n = u \wedge na$, $w_1 = v_1$, and $w_n = v_n \ominus v_{n-1}$ for $n \geq 2$. Then w_n are pairwise disjoint elements in I_a. Approximating w_n by finite linear combinations of components of a or b and recalling that $X \in (\mathrm{LC}_3)$ and $T \in (\pitchfork)$ we can easily construct elements $\widetilde{a}$

and $\widetilde{b}$ in X such that $u \le \widetilde{a} \le \widetilde{b} \le 2\widetilde{a}$ and $T\widetilde{a} \perp T\widetilde{b}$. Now we see that $\sum_{n=1}^{\infty} \oplus u_n$ exists in X by step (2c) applied to the ideal $I_{\widetilde{a}}$ instead of I_a.

(2e) The fact that the space K_a is basically disconnected follows immediately from step (2d). Indeed, if O is a cozero set in K_a then $O = \bigcup_{n=1}^{\infty} F_n$ where F_n are clopen disjoint subsets of K_a. The element $\sum_{n=1}^{\infty} \oplus \chi_{F_n} a$ exists in I_a, whence O is clopen in K_a.

(2f) It remains to prove that U is a projection band in X ([7]). Notice that because $T \in (\pitchfork)$ the set TU is a band in Y. Let x be a positive element in X. For any natural n let $x_n = x \wedge na$. Then because the space K_a is basically disconnected, $x_n = u_n \oplus v_n$ where u_n is a component of x and v_n is a component of na. Clearly u_n is a component of u_{n+1}. Let $w_n = u_{n+1} \ominus u_n$. Without loss of generality we can assume that $w_n \neq 0$ for every n. Let ε_n be positive scalars such that $\varepsilon_n \searrow 0$ as $n \to \infty$. Let a_n and b_n be linear combinations of components of a and b, respectively, such that

$$\{a_n\}^{dd} = \{b_n\}^{dd} \subseteq \{w_n\}^{dd}, \qquad |a_n - w_n| \le \varepsilon_n a, \qquad |b_n - w_n| \le \varepsilon_n a.$$

Because $X \in (\mathrm{LC}_3)$ we can choose the scalars ε_n in such a way that the elements $u = x + \sum_n \oplus (a_n - w_n)$ and $v = x + \sum_n \oplus (b_n - w_n)$ exist in X.

Let J be the principal ideal in Y generated by the element $y = |Tu| + |Tv|$. From the definition of u and v, from Lemma 2.5.11, and from the fact that $Ta \perp Tb$ it follows easily that $Tx \in J$. By the Krein–Kakutani theorem the ideal J is order isomorphic to a norm dense sublattice of some $C(K)$ where K is a compact Hausdorff space and the isomorphism can be chosen in such a way that the image of y is the function $\mathbf{1}$. We will identify the elements of J with the corresponding continuous functions on K.

The intersection $TU \cap J$ is a band in J; let O be the regularly (canonically) open subset of K corresponding to this band. Clearly $u - v \in U$, whence $Tu - Tv \in TU$, and therefore the functions Tu, Tv coincide on $K \setminus O$. In particular we see that if $t \in K \setminus O$ then both these functions take at this point either the value $1/2$ or $-1/2$.

We claim that $Tu \ge 0$ on O. Indeed, from the definition of u we see that there are bands $U_m \subset U$ and non-negative scalars γ_m such that $\{\bigcup_m U_m\}^{dd} = U$ and $u - \gamma_m a \perp U_m$. Then $Tu - \gamma_m Ta \perp TU_m$ and because $T \in (\pitchfork)$ we have $\{\bigcup_m TU_m\}^{dd} = TU$; it remains to recall that $Ta \ge 0$.

Similarly we conclude that $Tv \le 0$ on O and now it is plain to see that the set O is clopen in K. Therefore $Tx = y_1 \oplus y_2$ where $y_1 \in TU$ and $y_2 \perp TU$. Because $T \in (\pitchfork)$ we conclude that $x = T^{-1}y_1 \oplus T^{-1}y_2$ where $T^{-1}y_1 \in U$ and $T^{-1}y_2 \in U^d$. But x was an arbitrary element from X^+ and therefore U is a projection band in X. ■

5.2.2. Lemma. *Under the assumptions of Lemma* 5.2.1 *assume additionally that the splitting number* $d(T)$ *is at least* 3. *Then the vector lattice* X *contains a non-trivial laterally* σ*-complete band* U *and* $d\text{-}\dim U \ge 3$.

Proof. This follows immediately from Lemma 2.5.8, the first part of the proof of Lemma 5.2.1, and the obvious fact that if V is a vector lattice with a cofinal family of components, $d\text{-}\dim V \le n$, and $T : V \to W$ is an injective d.p.o., then $d(T) \le n$. ■

([7]) This does not follow automatically from step (2d). There is an example of a vector lattice X containing a non-trivial Dedekind complete band U such that no band $V \subset U$ is a projection band in X (A. I. Veksler, private communication).

5.2.3. Lemma. *Under the assumptions of Lemma* 5.2.1 *assume additionally that* $Y \in$ (LC$_3$)*. Then the vector lattice* X *contains a laterally* σ*-complete projection band.*

Proof. It follows from Lemma 5.2.1 and from the first part of its proof that without loss of generality we can assume that there are non-zero elements $a, b \in X$ such that $a \le b \le 2a$, the band $U = \{a\}^{dd}$ is a conditionally laterally σ-complete projection band in X, $Ta \ge 0$, $Tb \le 0$, and the compact space K_a is basically disconnected. Let U_n, $n \in \mathbb{N}$, be pairwise disjoint non-zero projection bands in U. Let $u_{n,k}$, $k \in \mathbb{N}$, be pairwise disjoint non-zero elements in U_n. Because $X \in$ (LC$_3$) there are elements $a_{n,k}, b_{n,k} \in X$ such that $a_{n,k}$ and $b_{n,k}$ are finite linear combinations of components of a and b, $\{a_{n,k}\}^{dd} = \{b_{n,k}\}^{dd} = \{u_{n,k}\}^{dd}$, and the elements $\sum_{n,k} \oplus (a_{n,k} - u_{n,k})$ and $\sum_{n,k} \oplus (a_{n,k} - b_{n,k})$ exist in U. By Lemma 2.5.11 the element $\sum_{n\in\mathbb{N}}^{\infty} \sum_{k\in\mathbb{N}}^{\infty} (Ta_{n,k} \oplus Tb_{n,k})$ exists in Y. Because $Y \in$ (LC$_3$) we can find positive scalars ε_n, $n \in \mathbb{N}$, such that the element

$$y = \sum_{n\in\mathbb{N}} \oplus \varepsilon_n \sum_{k\in\mathbb{N}} Ta_{n,k}$$

exists in Y. Applying again Lemma 2.5.11 we see that the element

$$T^{-1} y = \sum_{n\in\mathbb{N}} \oplus \varepsilon_n \sum_{k\in\mathbb{N}} a_{n,k}$$

exists in U. Because the band U is conditionally laterally σ-complete the element $\sum_{k\in\mathbb{N}} \oplus a_{1,k}$ exists in U_1, whence the element $\sum_{k\in\mathbb{N}} \oplus u_{1,k}$ also exists in U_1 and therefore the band U_1 is laterally σ-complete. ■

5.2.4. Lemma. *Let* X *be a vector lattice. Assume that* X *contains a conditionally laterally complete projection band* U *such that* d-$\dim U > 1$*. Then there are a vector lattice* Y *and a bijective d.p.o.* T *such that* $T \in (\pitchfork)$ *but* T^{-1} *does not preserve disjointness.*

Proof. Because the band U has the projection property and d-$\dim U > 1$ we can find two d-independent elements $a, b \in U$ such that $\{a\}^{dd} = \{b\}^{dd}$. Let $V = \{a\}^{dd}$ and let V^l be the lateral completion of V (V^l can be identified with the intersection of all laterally complete vector sublattices of V^{u}, the universal completion of V, which contain V). It follows from [4, Theorem 3.2] that there is a band preserving projection $P : V^l \to V^l$ such that $Pa = a$ and $Pb = 0$. We define a d.p.o. $S : V \to V^l \oplus V^l$ by $Sx = (Px, Px - x)$. Being conditionally laterally complete V is an ideal in V^l and therefore the arguments from the proof of Theorem 13.14 in [5] can be repeated to show that $Z = SV$ is a vector sublattice of $V^l \oplus V^l$. Let $W = V^d$, $Y = Z \oplus W$, and $T = S \oplus I_W$ where I_W is the identity operator on W. Clearly T is a bijective d.p.o. from X to Y and $T \in (\pitchfork)$, but $d(T) = 2$. ■

5.2.5. Lemma. *Let* X *be a vector lattice. Assume that* $X \in$ (LC$_3$) *and* X *contains a laterally complete band* U *such that* d-$\dim U \ge n \ge 2$*. Then there are a vector lattice* $Y \in$ (LC$_3$) *and a bijective d.p.o.* T *such that* $T \in (\pitchfork)$ *and* $d(T) = n$*. Moreover, if* d-$\dim U = \infty$ *the operator* T *can be chosen in such a way that* $d(T) = \infty$*.*

Proof. The conditions of this lemma guarantee that there are a projection band $V \subset U$ and d-independent elements $v_1, \dots, v_n \in V$ such that each v_i is a weak unit in V. By Theorem 3.2 in [4] there are band preserving projections $P_i : V \to V$ such that $P_i v_i = v_i$, $P_i v_j = 0$ if $i \ne j$, $P_i V$ is a laterally complete vector sublattice of V, and the following implication holds: $P_i x = 0, i = 1, \dots n, \Rightarrow x = 0$. Let $W = V^d$ and let $T : X \to W \oplus P_1 V \oplus \cdots \oplus P_n V$ be the operator $I_W \oplus P_1 \oplus \cdots \oplus P_n$. Then T is a bijective d.p.o., $T \in (\pitchfork)$, and $d(T) = n$.

If d-dim $U = \infty$ we can find pairwise disjoint bands $V_n \subset V$, vector lattices Y_n and bijective d.p.o. $T_n : V_n \to Y_n$ such that $d(T_n) = n$. Because U is laterally complete $V = \sum_{n=1}^{\infty} \oplus V_n$ is a projection band in X. If we take $W = V^d$ and $T = I_W \oplus \sum_{n=1}^{\infty} T_n$ then $d(T) = \infty$. ■

5.3. Proofs of Theorems 5.1.4 and 5.1.5

5.3.1. Lemma. *Let $X \in (\mathrm{LC}_3)$, Y be a vector lattice, and $T : X \to Y$ be a d-isomorphism. Assume that T is not regular. Then there is a principal band $V = \{v\}^{dd}$ in Y such that for any sequence v_n of pairwise disjoint elements in the interval $[0, v]$ and for any sequence λ_n of positive scalars there are elements $z_n \in V$ such that*

- $\{z_n\}^{dd} = \{v_n\}^{dd}$,
- $z_n \geq \lambda_n v_n$,
- *the element $z = \sum_n \oplus z_n$ exists in Y.*

Proof. By Theorem 5.1 in [5] and by Theorem 4.0.2 there are positive elements $x, x_n \in X$, positive scalars δ_n, and a positive non-zero element $v \in Y$ such that the compact space K_x is zero-dimensional, $\delta_n \downarrow 0$, $x_n \leq \delta_n x$, and $|Tx_n| \geq v$, $n \in \mathbb{N}$.

Let $V = \{v\}^{dd}$ and let λ_n, v_n be as in the statement of the lemma. Let $u_n = |T^{-1}(\lambda_n v_n)|$. Because the space K_x is zero-dimensional, for any $n \in \mathbb{N}$ we can find components $u_{n,k}$, $k \in \mathbb{N}$, of u_n and positive scalars $\mu_{n,k}$ such that $\sup_k u_{n,k} = u_n$ and $(u_{n,k} - \mu_{n,k} x)_- \perp u_{n,k}$. The last condition guarantees that there are components $x_{n,k}$ of x such that $\{x_{n,k}\}^{dd} = \{u_{n,k}\}^{dd}$. Because $X \in (\mathrm{LC}_3)$ there are positive scalars $\varepsilon_{n,k}$ such that if $0 \leq w_{n,k} \leq \varepsilon_{n,k} x$ and $w_{n,k} \in \{u_{n,k}\}^{dd}$ then the element $\sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} \oplus w_{n,k}$ exists in X.

Let us consider elements $x_{n,k}$ such that $\lambda_n x_{n,k} \leq \varepsilon_{n,k} x$ and $|Tx_{n,k}| \geq v$, $n, k \in \mathbb{N}$. Then the element $s = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \oplus u_{n,k} \wedge \lambda_n x_{n,k}$ exists in X. Recall that for a disjointness preserving operator T we have (see [33])

$$|T(a \wedge b)| \geq |Ta| \wedge |Tb|$$

for any $a, b \in X$. Let $s_n = \sum_{k=1}^{\infty} \oplus u_{n,k} \wedge \lambda_n x_{n,k}$. Then

$$\begin{aligned} |Ts_n| &= \sum_{k=1}^{\infty} \oplus |T(u_{n,k} \wedge \lambda_n x_{n,k})| \geq \sum_{k=1}^{\infty} \oplus |Tu_{n,k}| \wedge \lambda_n v = \Big(\sum_{k=1}^{\infty} \oplus |Tu_{n,k}| \Big) \wedge \lambda_n v \\ &= |Tu_n| \wedge \lambda_n v = \lambda_n v_n \wedge \lambda_n v = \lambda_n v_n. \end{aligned}$$

It remains to take $z_n = |Ts_n|$; then $z = \sum_{n=1}^{\infty} \oplus z_n = |Ts|$ exists in Y. ■

5.3.2. Corollary. *Let $X \in (\mathrm{LC}_3)$, $Y \in (\Delta^{\mathrm{L}}_{\mathrm{w}})$ and let $T : X \to Y$ be a d-isomorphism. Then the operator T is regular and the vector lattices X and Y are order isomorphic.*

5.3.3. Lemma. *Let $X \in (\mathrm{LC}_3)$ and Y be an arbitrary vector lattice. Let $T : X \to Y$ be a d-isomorphism. Assume that T is not regular. Then X contains a projection band U with the principal projection property and such that d-dim $U > 1$.*

Proof. Let $V = \{v\}^{dd}$ be a band in Y with the properties from the statement of Lemma 5.3.1. Let $w \in V$, $W = \{w\}^{dd}$, and let $y \in Y^+$. Let $z = |w| \wedge v \wedge y$. Then

$\{z\}^{dd} = W \cap \{y\}^{dd}$. We represent the principal ideal $Y(y)$ as a norm dense sublattice in $C(K_y)$ by identifying y with the function $\mathbf{1} \in C(K_y)$. For any natural n let $E_n = \{t \in K_y : 1/(n+1) \le z(t) \le 1/n\}$. Because $Y(y)$ is norm-dense in $C(K_y)$ we can find a sequence of elements z_n such that

- $z_n \le z$,
- $z_n(t) = z(t)$ for $t \in E_n$,
- $\operatorname{supp} z_n \subseteq E_{n-1} \cup E_n \cup E_{n+1}$, assuming that $E_0 = \emptyset$.

Clearly for any two natural p and q such that $p \ne q$ we have $z_{2p-1} \perp z_{2q-1}$ and $z_{2p} \perp z_{2q}$. The properties of the band V guarantee that we can find elements $w_n \in Y$ such that

- $\{w_n\}^{dd} = \{z_n\}^{dd}$,
- $w_n \ge (n+1)z_n$,
- the elements $w^{(1)} = \sum_{p=1}^{\infty} \oplus w_{2p-1}$ and $w^{(2)} = \sum_{p=1}^{\infty} \oplus w_{2p}$ exist in Y.

The element $y_W = (w^{(1)} + w^{(2)}) \wedge y$ is a component of y and clearly $y_W \in W$ and $y - y_W \in W^d$. Thus we have just proved that V is a projection band in Y with the principal projection property. Because T is a d-isomorphism $U = T^{-1}V$ is a projection band in X and U has the principal projection property.

It remains to notice that $d\text{-}\dim U > 1$ because otherwise the operator $T|U$ would be regular, which is not the case. ∎

5.3.4. Lemma. *Let X be a vector lattice. Assume that X contains a projection band U with projection property and $d\text{-}\dim U > 1$. Then there are a vector lattice Y and a d-isomorphism $T : X \to Y$ such that T is not regular.*

Proof. Without loss of generality we can assume that $U = \{a\}^{dd} = \{b\}^{dd}$ where a, b are d-independent elements in X. Let U^{u} be the universal completion of U. Then there is a band preserving projection $P : U^{\mathrm{u}} \to U^{\mathrm{u}}$ such that $Pa = a$ and $Pb = 0$. In particular the operator P is not regular. Then $S = I_U + P$, where I_U is the identity operator on U^{u}, is a non-regular d-isomorphism of U^{u} onto itself. The set SU is a vector sublattice of U^{u} because U has the projection property and therefore is component-wise closed in U^{u}. It remains to take $V = U^d$, $Y = V \oplus SU$, and $T = I_V \oplus S : X \to Y$. ∎

5.3.5. Lemma. *Let $X, Y \in (\mathrm{LC}_3)$ and $T : X \to Y$ be a d-isomorphism. Assume that T is not regular. Then X contains a laterally σ-complete band.*

Proof. Let $V = \{v\}^{dd}$ be as in Lemma 5.3.1. The proof of Lemma 5.3.1 shows that for any sequence v_n of positive pairwise disjoint elements in V we can find pairwise disjoint elements $u_n \in X$ with the properties

- the element $z = \sum_{n=1}^{\infty} \oplus z_n$ exists in X for any sequence $\{z_n\} \subset X$ such that $0 \le z_n \le u_n$,
- $w_n = |Tu_n| \ge v_n$.

Let $u = \sum_{n=1}^{\infty} \oplus u_n$. The element $w = |Tu| = \sum_n \oplus w_n$ exists in Y. Because $Y \in (\mathrm{LC}_3)$ there are positive scalars δ_n such that if y_n is a sequence of pairwise disjoint elements from the interval $[0, w]$ and $y_n \le \delta_n w_n$ then the element $\sum_n \oplus y_n$ exists in Y.

Because V has the projection property, for any n there are components $w_{n,j}$, $j = 1, \dots, m(n)$, of w_n and scalars $\lambda_{n,j}$, $0 \le \lambda_{n,j} \le 1$, such that

$$\left| v_n - \sum_{j=1}^{m(n)} \oplus \lambda_{n,j} w_{n,j} \right| \le \delta_n w.$$

Let $\widetilde{w}_n = \sum_{j=1}^{m(n)} \oplus \lambda_{n,j} w_{n,j}$. Then

$$T^{-1}\widetilde{w}_n = \sum_{j=1}^{m(n)} \oplus \lambda_{n,j} T^{-1} w_{n,j};$$

and because $T^{-1}w_{n,j}$ is a component of u_n the element $\sum_n \oplus T^{-1}\widetilde{w}_n$ exists in X. Therefore the element $\sum_n \oplus \widetilde{w}_n$ exists in Y, whence $\sum_n \oplus v_n$ exists in Y and the band V is laterally σ-complete. It remains to notice that under the conditions of this lemma the domain X and the range Y can be interchanged. ∎

5.3.6. Lemma. *Let X be a vector lattice and let U be a laterally complete band in X such that d-$\dim U > 1$. Then there is a non-regular d-isomorphism $T : X \to X$.*

Proof. As was first proved in [33] (see also [5]) there is a non-regular band preserving projection $P : U \to U$. Then $S = I + P$ is a d-isomorphism of U onto U. Recall that U is a projection band in X. Let $V = U^d$ and $T = I_V \oplus S$. Then T is as required. ∎

6. Weakly c_0-complete domains with the projection property or with the countable sup property

6.1. The general case. In the next remark we combine some simple properties of vector lattices with the countable sup property.

6.1.1. Remark.

(1) As already noticed a vector lattice X has the countable sup property if and only if for any $x \in X$ the Krein–Kakutani space K_x satisfies the countable chain condition, $K_x \in$ (ccc), i.e. any family of non-empty pairwise disjoint open subsets of K_x is at most countable.

(2) The condition $X^l \in$ (CSP), where X^l is the lateral completion of X, means exactly that any set of pairwise disjoint elements in X is at most countable. Clearly $X^l \in$ (CSP) iff $X \in$ (CSP) and X has a weak unit.

(3) Clearly, for vector lattices from (CSP) the notions of conditional lateral completeness and conditional lateral σ-completeness coincide.

(4) In general a laterally σ-complete vector lattice $X \in$ (CSP) need not be laterally complete but any principal band in it is laterally complete.

(5) In general if $X \in$ (CSP) and X has the principal projection property it might not have the projection property even if it has a weak unit. Take for example a zero-dimensional infinite compact space K and the vector lattice of all real-valued functions continuous on K and taking only a finite number of values. But if we assume additionally that $X \in (\mathrm{LC}_4)$ then any principal band in X has the projection property by Proposition 2.2.7.

From the results of Section 5 and from Remark 6.1.1 we immediately obtain the following two results.

6.1.2. THEOREM. *Let X be a weakly c_0-complete vector lattice. Assume additionally that either*

- *X has the countable sup property, or*
- *any principal band in X has the projection property.*

Then the following statements are equivalent.

(1) *Any bijective d.p.o. $T : X \to Y$ such that $T \in (⫛)$ is a d-isomorphism.*
(2) *For any conditionally laterally complete projection band U in X we have d-$\dim U = 1$.*

If we assume additionally that $X \in$ (CFC) then conditions (1) and (2) above are equivalent to

(3) *The vector lattice X is d-rigid.*

6.1.3. THEOREM. *Assume the conditions of Theorem 6.1.2. Then the following statements are equivalent:*

(1) *For any bijective d.p.o. $T : X \to Y$ such that $T \in (⫛)$ we have $d(T) \leq 2$.*
(2) *For any laterally complete band U in X, d-$\dim U \leq 2$.*

6.1.4. THEOREM. *Assume the conditions of Theorem 6.1.2. Then the following statements are equivalent.*

(1) *Any d-isomorphism $T : X \to Y$ is regular.*
(2) *For any projection band U in X with the projection property we have d-$\dim U = 1$.*

If we assume additionally that $X \in$ (CFC) then the conditions above are equivalent to

(3) *X is super d-rigid.*

6.1.5. THEOREM. *Assume the conditions of Theorem 6.1.2. The following statements are equivalent:*

(1) *Any bijective d.p.o. $T : X \to Y$ where $Y \in (\mathrm{LC}_1)$ is a d-isomorphism.*
(2) *Any d-isomorphism $T : X \to Y$ where $Y \in (\mathrm{LC}_2)$ is regular.*
(3) *For any laterally complete band U in X, d-$\dim U = 1$.*

In the case of vector lattices with the countable sup property we can say more than is stated in Theorem 6.1.5. Let us first recall that for any d.p.o. $T : X \to Y$ there is a maximal ideal $\mathcal{R}_T \subset X$ such that the restriction $T|\mathcal{R}_T$ is regular [34].

6.1.6. LEMMA. *Let $T : X \to Y$ be a bijective d.p.o. and let $T \in (⫛)$. Then the ideal $\mathcal{R}_T$ is a band in X.*

Proof. It is enough to prove that for any net x_α of positive elements in $\mathcal{R}_T$ such that $x = \sup_\alpha x_\alpha$ exists in X and for any $z \in [0, x]$ we have $|Tz| \leq |Tx|$. For any α let $V_\alpha = \{(2x_\alpha - x)_+\}^{dd}$. Then $\{V_\alpha\}$ is a full system of bands in $\{\mathcal{R}_T\}^{dd}$ and $(|Tx| - |Tz|)_- \perp TV_\alpha$ for any α. But $T \in (⫛)$, whence the system $\{TV_\alpha\}$ of bands is full in $\{T\mathcal{R}_T\}^{dd}$ and therefore $(|Tx| - |Tz|)_- = 0$. ∎

6.1.7. Remark. De Pagter proved in [34] that for any band preserving operator $T : X \to X$ the ideal $\mathcal{R}_T$ is a band in X. It would be interesting to fully describe the class of all d.p.o. for which $\mathcal{R}_T$ is a band.

6.1.8. Theorem. *Let $X \in (\mathrm{LC}_3)$ and $T : X \to Y$ be a d-isomorphism. Assume additionally that $X \in (\mathrm{CSP})$. Then*

$$X = \mathcal{R}_T \oplus \mathcal{A}_T$$

where $\mathcal{A}_T$ is the band of anti-regularity for T. If we assume additionally that $Y \in (\mathrm{LC}_2)$ then for each $z \in \mathcal{A}_T$ the band $\{z\}^{dd}$ is laterally complete.

Proof. I. Assume first that there is a positive weak unit u in X. The ideal $\mathcal{R}_T$ is a band in X by Lemma 6.1.6. Let $\mathcal{A}_T = \{\mathcal{R}_T\}^d$. If $\mathcal{A}_T = \emptyset$ there is nothing to prove. Otherwise there are [33] an $x \in [0, u]$, scalars $\varepsilon_n \searrow 0$, elements $x_n \le \varepsilon_n x$ and a non-zero $v \in Y_+$ such that $|Tx_n| \ge v$. Moreover the proof of Theorem 5.1 in [5] shows that we can take $v = (|Tz| - |Tx|)_+$ where z is any element from X such that $0 \le z \le x$ but $(|Tz| - |Tx|)_+ \ne 0$. From the proof of Lemma 5.3.1 we see that $V = \{v\}^{dd}$ is a projection band in Y and therefore $U = T^{-1}V$ is a projection band in X. Let $x_1 = P_U x$, $z_1 = P_U z$, and $v_1 = v$ where P_U is the band projection on U. Then $0 \le z_1 \le x_1$ and $|Tz_1| - |Tx_1| = v_1$.

Because X has the countable sup property we can find pairwise disjoint non-zero elements $x_n \in [0, u]$ and elements $z_n \in [0, x_n]$ such that $v_n = |Tz_n| - |Tx_n| \ge 0$ and the system of bands $U_n = T^{-1}\{v_n\}^{dd}$ is full in $\mathcal{A}_T$. Because $X \in (\mathrm{LC}_3)$ we can find scalars $\varepsilon_n \searrow 0$ such that the elements $\overline{x} = \sum_{n=1}^{\infty} \oplus\, \varepsilon_n x_n$ and $\overline{z} = \sum_{n=1}^{\infty} \oplus\, \varepsilon_n z_n$ exist in X. Then $0 \le \overline{z} \le \overline{x}$ and $\bar{v} = |T\overline{z}| - |T\overline{x}| = \sum_{n=1}^{\infty} \oplus\, \varepsilon_n v_n$. By Lemma 5.3.1, $\overline{V} = \{\bar{v}\}^{dd}$ is a projection band in Y, whence $\mathcal{A}_T = T^{-1}\overline{V}$ is a projection band in X.

II. The general case follows easily from the one already considered. Indeed, if $z \in X$ let $Z = \{z\}^{dd}$ and let $S = T|Z$. By part I, $Z = \mathcal{R}_S \oplus \mathcal{A}_S$ and obviously $\mathcal{R}_S = \mathcal{R}_T \cap Z$ and $\mathcal{A}_S = \mathcal{A}_T \cap Z$. Therefore $z = z_1 \oplus z_2$ where $z_1 \in \mathcal{R}_T$ and $z_2 \in \mathcal{A}_T$.

III. Finally, if $Y \in (\mathrm{LC}_3)$ we apply Lemma 5.3.5. ∎

6.2. Dedekind complete domains. Relatively uniformly complete domains with the countable sup property. The class of r_u-complete vector lattices is particularly important and the statements of our main results become simpler because every r_u-complete vector lattice with the projection property is Dedekind complete and every laterally complete r_u-complete vector lattice is universally complete [38, 39]. Moreover, if a Dedekind complete vector lattice has d-dimension greater than one then its d-dimension is infinite [5, Theorem 6.8].

More importantly, in this case we can prove (see Theorem 6.2.3) that if $Y^{\mathrm{u}} \in (\mathrm{CSP})$ then *any* bijective d.p.o. from X to Y is in (⫛).

The next theorem follows from our previous results.

6.2.1. Theorem. *Let X be either a Dedekind complete vector lattice or an r_u-complete vector lattice with the countable sup property.*

I. *The following conditions are equivalent*:

- *X is d-rigid.*
- *X is super d-rigid.*

- *For any Dedekind complete projection band $U \subset X$, d-dim $U = 1$.*
- *Any d-isomorphism $T : X \to Y$ is regular.*

II. *The following conditions are equivalent:*

- *For any bijective d.p.o. $T : X \to Y$ we have $d(T) \leq 2$.*
- *Any bijective d.p.o. $T : X \to Y$, where Y is an r_u-complete vector lattice, is regular and therefore a d-isomorphism.*
- *For any universally complete band $U \subset X$, d-dim $U = 1$.*

Our next result shows that Problem B has a positive solution for r_u-complete vector lattices with the countable sup property.

6.2.2. Theorem. *Let X be either a Dedekind complete vector lattice or an r_u-complete vector lattice with the countable sup property. Let Y be an r_u-complete vector lattice and let $T : X \to Y$ be a d-isomorphism. Then the vector lattices X and Y are order isomorphic.*

Proof. If X is a Dedekind complete vector lattice our statement is exactly Theorem 14.18 in [5]. If X is r_u-complete and $X \in$ (CSP) then it follows immediately from Theorem 6.1.8 that the band $\mathcal{A}_T$ is Dedekind complete and it remains to apply Theorem 14.18 from [5]. ∎

6.2.3. Theorem. *Let X be an r_u-complete vector lattice, let Y be a vector lattice such that any family of non-zero pairwise disjoint elements in Y has cardinality less than $2^{\aleph_0}$, and let $T : X \to Y$ be an injective d.p.o. Then $T \in (\pitchfork)$.*

Proof. Assume to the contrary that $T \notin (\pitchfork)$. Then there are a band $U \subseteq X$ and an element $x \in X$ such that $Tx \perp TU$ but $x \not\perp U$. Let $I = I_x$ be the principal ideal in X generated by x. We will identify I with $C(K_x)$; as usual, x will be identified with the function $\mathbf{1}$. The set $U \cap I_x$ is a band in $C(K_x)$; let O be the canonically (regularly) open subset of K_x corresponding to this band. Recall that $K_x \in$ (ccc). In what follows we repeat (up to notation) the arguments employed in [36, proof of part IV of Proposition on page 130].

Let $p, q \in O$, $p \neq q$ and let $h \in C(K)$ with $\operatorname{supp} h \subset O$, $h(p) = 0$, $h(q) = 1$, and $h(K) \subseteq [0,1]$. Let $H \subset [0,1]$ be a Cantor set; then $H = \bigcup_{\gamma \in \varGamma} H_\gamma$ where H_γ are disjoint Cantor sets and $\operatorname{card}(\varGamma) = 2^{\aleph_0}$. Let φ_γ be the Cantor function associated with H_γ. For all $\gamma \in \varGamma \backslash \varDelta$, where $\varDelta \subset \varGamma$ and $\operatorname{card}(\varDelta) < 2^{\aleph_0}$, the set $h^{-1}(H_\gamma)$ has empty interior. Therefore for any such γ the function $f_\gamma = \varphi_\gamma \circ h$ is an essentially constant non-zero function from $C(K)$ and $\varOmega(f_\gamma) \supseteq h^{-1}([0,1] \setminus H_\gamma)$ where $\varOmega(f) = \{t \in K : f$ is constant in some open neigborhood of $t\}$. Therefore if $\gamma_1, \gamma_2 \in \varGamma \setminus \varDelta$ and $\gamma_1 \neq \gamma_2$ then $\varOmega(f_{\gamma_1}) \cup \varOmega(f_{\gamma_2}) = K$.

We claim that $T(f_{\gamma_1} x) \perp T(f_{\gamma_2} x)$. To prove this consider $z \in I_x$. Then we can find elements $z_i \in I_x$, $1 \leq i \leq n$, such that $z = \sum_{i=1}^n z_i$ and for each i either $f_{\gamma_1} \equiv c_i$ on $\operatorname{supp} z_i$ or $f_{\gamma_2} \equiv c_i$ on $\operatorname{supp} z_i$ where c_i, $1 \leq i \leq n$ are some scalars. Fix i and let for definiteness $f_{\gamma_1} \equiv c_i$ on $\operatorname{supp} z_i$. Then $(f_{\gamma_1} x - c_i x) \perp z_i$ and because $Tx \perp Tz_i$ we see that $Tf_{\gamma_1} x \perp Tz_i$. Similarly, if $f_{\gamma_2} \equiv c_j$ on $\operatorname{supp} z_j$ then $Tf_{\gamma_2} x \perp Tz_j$. Therefore

$$y = |Tf_{\gamma_1} x| \wedge |Tf_{\gamma_2} x| \perp z.$$

But z is an arbitrary element of I_x, whence $y \perp TI_x$. In particular $y \perp Tf_{\gamma_k}x$, $k = 1, 2$, whence $y = 0$.

We have obtained a family of pairwise disjoint elements in Y of cardinality $2^{\aleph_0}$, in contradiction with our assumption that $Y^{\mathrm{u}} \in (\mathrm{CSP})$. ∎

7. Huijsmans–de Pagter–Koldunov theorem

The Huijsmans–de Pagter–Koldunov theorem (briefly HPK-theorem)—one of the main results in the circle of problems we are discussing—states the following.

7.0.1. Theorem ([25, 28]). *Let X be an r_u-complete vector lattice and Y be a normed vector lattice. Let $T : X \to Y$ be an injective d.p.o. Then $x \perp z \Leftrightarrow Tx \perp Tz$. Moreover, if T is a bijection then it is regular.*

In this section we will discuss two questions:

(1) To what extent can the conditions on X and Y in Theorem 7.0.1 be weakened?
(2) Under what conditions can we interchange X and Y in Theorem 7.0.1? More precisely, if we assume that Y is r_u-complete, what should be the conditions on X for the result to be true?

This section is divided into three subsections. The first one contains direct generalizations of the HPK-theorem based principally on de Pagter's technique. In the second one we consider the case when the topology on the range Y is defined by a countable family of lattice seminorms. Finally in the third one we consider the case when the domain X satisfies the weak Luxemburg condition $(\Delta_{\mathrm{L}}^{\mathrm{w}})$ and the range Y is an r_u-complete vector lattice.

7.1. The HPK-theorem. Some improvements. Here we will prove that the statements of Theorem 8 in [34] and Theorems 5.2 and 5.3 in [5] remain true if instead of considering a relatively uniformly complete domain X we require only that $X \in (\mathrm{LC}_3)$.

7.1.1. Theorem. *Let X, Y be vector lattices, $X \in (\mathrm{LC}_3)$, and $Y \in (\Delta_{\mathrm{P}})$. Let $T : X \to Y$ be a d.p.o. Then $\mathcal{R}_T$, the maximal ideal of regularity of T, is order dense in X.*

Proof. It is enough to prove that for any $x \in X$ the ideal $\mathcal{R}_T \cap I_x$ is order dense in I_x. The last statement follows from Lemma 7.1.2, which we will also use later. ∎

7.1.2. Lemma. *Let X be an order dense vector sublattice of some $C(K)$. Assume additionally that $\mathbf{1} \in X$ and $X \in (\mathrm{LC}_3)$. Let $Y \in (\Delta_{\mathrm{P}})$ and $T : X \to Y$ be a d.p.o. Let $Z = \{k \in K : x(k) = 0$ for all $x \in \mathcal{R}_T\}$. Then the set Z is at most finite.*

Proof. If Z were infinite we would be able to find disjoint regularly open sets $O_n \subset K$, $n = 1, 2, \ldots$, such that $O_n \cap Z \neq \emptyset$. Let B_n be the band in X corresponding to the set O_n. The operator $T : B_n \to Y$ cannot be regular because this would contradict the maximality of $\mathcal{R}_T$. By the McPolin–Wickstead theorem [33] for any n we can find elements $x_m^{(n)} \in B_n$ such that $\|x_m^{(n)}\|_{C(K)} \searrow 0$ as $m \to \infty$, and $|Tx_m^{(n)}| \geq y_n \in Y_+$, $y_n \neq 0$.

Let ε_n, λ_n be positive scalars. For any n choose $m(n)$ in such a way that $\|x^{(n)}_{m(n)}\| \le \varepsilon_n/\lambda_n$. Because $X \in (\mathrm{LC}_3)$ we can choose the scalars ε_n in such a way that the element $x = \sum_{n=1}^{\infty} \oplus \lambda_n x_{m(n)}$ exists in X. Then $|Tx| \ge |Tx_{m(n)}|\lambda_n \ge \lambda_n|y_n|$, in contradiction with our assumption that $Y \in (\Delta_{\mathrm{P}})$. ∎

7.1.3. Theorem. *Let $T : X \to Y$ be a d.p.o. Assume that $\mathcal{R}_T$ is order dense in X. Then either the kernel* $\ker(T)$ *of T contains a non-trivial ideal, or T is injective.*

Proof. Let u be a non-zero positive element in X such that $Tu = 0$. Because $\mathcal{R}_T$ is order dense in X we can find a positive $v \in \mathcal{R}_T$ such that $(v-u)_+ \ne 0$. Let $w = u \wedge v$ and let $z \in J = I_w \cap \{(v-u)_+\}^{dd}$. Then $(u-w) \perp z$, whence $(Tu - Tw) \perp Tz$ and because $Tu = 0$ we have $Tw \perp Tz$. But $z \in I_w$ and the restriction $T|I_w$ is regular, whence $Tz = 0$ and therefore $J \subset \ker(T)$. ∎

7.1.4. Theorem. *Let $T : X \to Y$ be an injective d.p.o. such that the ideal $\mathcal{R}_T$ is order dense in X. Then*

$$x \perp z \;\Leftrightarrow\; Tx \perp Tz.$$

Proof. Assume contrary to our claim that there are $u, v \in X_+$ such that $u \wedge v \ne 0$ but $Tu \perp Tv$. Let $w = u + v$ and let $S : I_w \to Y$ be the restriction of T to the principal ideal I_w. We identify I_w with an order dense vector sublattice of $C(K)$, where $K = K_w$ is the corresponding Krein–Kakutani space, and we identify w with the function $\mathbf{1}$. Our assumptions guarantee that the set $Z = \{k \in K : x(k) = 0 \text{ for all } x \in \mathcal{R}_S\}$ is nowhere dense in K. Let $O = \{t \in K : (u \wedge v)(t) > 0\}$. We can find non-empty open subsets U, V of O and positive elements $\widetilde{u}, \widetilde{v} \in I_w$ such that $U \subset V$, $V \cap Z = \emptyset$, $\operatorname{supp}(\widetilde{u} + \widetilde{v}) \subset V$, $\widetilde{u} \equiv u$ on U, and $\widetilde{v} \equiv v$ on U. Let z be a non-zero element of I_w such that $\operatorname{supp} z \subset U$ and $0 \le z \le u \wedge v$. Then $u - \widetilde{u} \perp z$, whence $Su - S\widetilde{u} \perp Sz$ and similarly $Sv - S\widetilde{v} \perp Sz$. But $Su \perp Sv$, whence

$$(*) \qquad |S\widetilde{u}| \wedge |S\widetilde{v}| \perp Sz.$$

On the other hand the restriction of the operator S on the principal ideal $I_{\widetilde{u}+\widetilde{v}}$ is regular and therefore by Theorem 2.3.2,

$$(**) \qquad |Sz| \le |S\widetilde{u}| \wedge |S\widetilde{v}|.$$

It follows immediately from $(*)$ and $(**)$ that $Sz = 0$, in contradiction with the injectivity of S. ∎

The operator T from Theorem 7.1.4 might be non-regular even if X and Y are Banach lattices (see Remark 3.1.4). But under an additional assumption we can prove its regularity.

7.1.5. Theorem. *Let $T : X \to Y$ be a d.p.o. such that the ideal $\mathcal{R}_T$ is order dense in X. Assume that for any full in X system of bands $\{U_\alpha\}$ the system $\{\{TU_\alpha\}^{dd}\}$ is full in $\{TX\}^{dd}$. Then the operator T is regular.*

Proof. Let $u, v \in X$ and $0 \le u \le v$. Let $I = I_v$ and K be the Krein–Kakutani space of the ideal I. The assumption that $\mathcal{R}_T$ is order dense in X implies that there is a family $\{u_\gamma, v_\gamma\}_{\gamma \in \Gamma}$ of elements of I with the following properties:

- $0 \le u_\gamma \le v_\gamma \le v$.
- $u_\gamma \equiv u$ and $v_\gamma \equiv v$ on some non-empty regularly open set $O_\gamma \subset K$.
- $\bigcup_{\gamma \in \Gamma} O_\gamma$ is dense in K.
- $I_{v_\gamma} \subset \mathcal{R}_T$.

Let B_γ be the band in X defined as $B_\gamma = \{z \in I : \operatorname{supp} z \subset O_\gamma\}^{dd}$. Then $u - u_\gamma \perp B_\gamma$ and $v - v_\gamma \perp B_\gamma$, whence $Tu - Tu_\gamma \perp TB_\gamma$ and $Tv - Tv_\gamma \perp TB_\gamma$. On the other hand, by Theorem 2.3.2, $|Tu_\gamma| \le |Tv_\gamma|$ and therefore $(|Tv| - |Tu|)_- \perp TB_\gamma$. Our assumptions imply that the system of bands $\{TB_\gamma\}^{dd}$ is full in $\{TI\}^{dd}$. Therefore $(|Tv| - |Tu|)_- = 0$ and T is regular by Theorem 2.3.2. ∎

7.1.6. Corollary. *Let $T : X \to Y$ be a bijective d.p.o. such that the ideal $\mathcal{R}_T$ is order dense in X. Then T is regular.*

7.1.7. Corollary. *Let X, Y be vector lattices, $X \in (\mathrm{LC}_3)$, and $Y \in (\Delta_{\mathrm{P}})$. Let $T : X \to Y$ be a d.p.o. Assume that $\ker(T)$ does not contain any non-trivial ideal. Then T is injective and*

$$x \perp z \iff Tx \perp Tz.$$

Moreover, if T is a bijection then T is regular.

7.2. The case when the range Y is countably normed. Let us recall the following

7.2.1. Definition. A vector lattice X is called *countably normed* if there is a countable system of lattice semi-norms p_n on X such that $p_n \le p_{n+1}$ and, for any $x \in X$,

$$\forall n \; p_n(x) = 0 \;\Rightarrow\; x = 0.$$

7.2.2. Lemma. *Let K be a compact Hausdorff space and X be an order dense vector sublattice of $C(K)$ such that $\mathbf{1} \in X$ and $X \in (\mathrm{LC}_2)$. Let Y be a countably normed vector lattice, and $T : X \to Y$ be an injective d.p.o. Assume also that there are $u, v \in X_+$ such that $u \wedge v \ne 0$ but $Tu \perp Tv$. Let $O = \{k \in K : (u \wedge v)(k) > 0\}$. Then the set O is separable and therefore $\operatorname{cl} O$ is a metrizable compact space.*

Proof. For any $n \in \mathbb{N}$ the set $J_n = \{y \in Y : p_n(y) = 0\}$ is an ideal in Y and the factor $Y_n = Y/J_n$ is a normed vector lattice with the norm p_n. For any $x \in X$ let $T_n x = \dot{T}x_n$ where $\dot{T}x_n$ is the class of Tx in the factor Y_n. Then $T_n : X \to Y_n$ is a well defined linear operator but of course it might be non-injective. Let $I_n = \mathcal{R}_{T_n}$ be the maximal ideal of regularity of T_n and let $Z_n = \{k \in K : x(k) = 0 \text{ for all } x \in I_n\}$. By Lemma 7.1.2 the set Z_n is at most finite. Moreover the proof of Theorem 7.1.4 shows that if $z \in X$, $\operatorname{supp} z \subset O$ and $\operatorname{supp} z \cap Z_n = \emptyset$ then $T_n z = 0$.

We claim that the set $Z = \bigcup_{n \in \mathbb{N}} Z_n$ is dense in O. Indeed, otherwise we can find a non-zero $z \in X$ such that for any $n \in \mathbb{N}$ we have $T_n z = 0$, whence $p_n(Tz) = 0$ for all $n \in \mathbb{N}$ and $Tz = 0$ in contradiction with the injectivity of T. ∎

7.2.3. Corollary. *Let X be a vector lattice with the principal projection property. Let Y be a countably normed vector lattice and let $T : X \to Y$ be an injective d.p.o. Then*

$$x \perp z \iff Tx \perp Tz.$$

Proof. Assume to the contrary that there are $u, v \in X_+$ such that $u \wedge v \neq 0$ but $Tu \perp Tv$. Consider the restriction of T to the main ideal X_{u+v}. Let K be the corresponding Krein–Kakutani space. The subset $\operatorname{supp}(u \wedge v)$ of K does not have isolated points, and because X_{u+v} has the principal projection property this subset cannot be metrizable, in contradiction with Lemma 7.2.2. ∎

We will need a simple lemma which is probably well known.

7.2.4. Lemma. *Let K be compact space, Z be a countable subset of K and V be a non-empty open subset of K. Then there is a non-zero function $f \in C(K)$ such that* $\operatorname{supp} f \subset V$ *and for any $z \in Z$ there is an open neighborhood $V(z)$ such that f is constant on $V(z)$.*

Proof. Let $g \in C(K)$ be a non-zero function such that $0 \leq g \leq 1$ and $\operatorname{supp} g \subset V$. Then $g(Z)$ is a countable subset of $[0,1]$ and we can find a function $\varphi \in C[0,1]$ such that $\varphi(0) = 0$, $\varphi(Z) \neq \{0\}$, and for any $z \in Z$ the function φ is constant on some open interval which contains $g(z)$. The function $\varphi \circ g$ is as required. ∎

7.2.5. Lemma. *Let K be a compact Hausdorff space, $X = C(K)$, and let Y be a countably normed vector lattice. Let $T : X \to Y$ be a disjointness preserving injection. Then $T \in (\pitchfork)$.*

Proof. If $T \notin (\pitchfork)$ then there are a regularly open set $V \subset K$ and a function $f \in C(K)$ such that $f > 0$ on V but for any $z \in C(K)$ such that $\operatorname{supp} z \subset V$ we have $Tf \perp Tv$.

For any $n \in \mathbb{N}$ let the vector lattice Y_n, the operator $T_n : X \to Y_n$ and the set $Z_n \subset K$ be defined as in the proof of Lemma 7.2.2. Recall that by Lemma 7.1.2 for any $n \in \mathbb{N}$ the set Z_n is at most finite, whence $Z = \bigcup_{n \in \mathbb{N}} Z_n$ is no more than countable. Lemma 7.2.4 guarantees that there is a non-zero $g \in C(K)$ such that $0 \leq g \leq 1$, $\operatorname{supp} g \subset V$, and for any $z \in Z$ the function g is constant on some open neighborhood of z. The function $h = gf$ is not zero. We are going to prove that $Th = 0$ in contradiction with the injectivity of T, and to do this we have to show that $p_n(Th) = 0$ for any $n \in \mathbb{N}$.

Let us fix some $n \in \mathbb{N}$. If $Z_n = \emptyset$ then the operator T_n is regular and because $h \leq f$ and $T_n h \perp T_n f$ we have $T_n h = 0$, which means exactly that $p_n(Th) = 0$.

Therefore we can assume that $Z_n = \{z_1, \dots, z_m\}$. Let V_i, $i = 1, \dots, m$, be pairwise disjoint open neighborhoods of z_i, and c_i, $i = 1, \dots, m$, be scalars such that $g \equiv c_i$ on V_i. Then we can find functions h_i, $i = 1, \dots, m$, with the following properties:

- $\operatorname{supp} h_i \subset V_i$,
- $0 \leq h_i \leq h$,
- $h_i \equiv h$ on some open neighborhood of z_i.

Let $\widetilde{h} = \sum_{j=1}^{m} h_j$. For any $j \in \{1, \dots, m\}$ we have $(c_i f - h) \perp h_i$, whence $(c_i Tf - Th) \perp Th_i$. But $Tf \perp Th$, therefore $Th \perp Th_i$, $i \in \{1, \dots, m\}$, whence $Th \perp T\widetilde{h}$, which of course implies that

$$(*) \qquad T_n h \perp T_n \widetilde{h}.$$

On the other hand, $h - \widetilde{h} \equiv 0$ on some open neighborhood of Z_n, whence (see the proof of Theorem 7.1.4)

$$(**) \qquad T_n(h - \widetilde{h}) = 0.$$

Together $(*)$ and $(**)$ imply that $T_n h = 0$ and we are done. ∎

We are ready to prove the main result of this subsection.

7.2.6. THEOREM. *Let X be an r_u-complete vector lattice and Y be a countably normed vector lattice. Let $T : X \to Y$ be a disjointness preserving bijection. Then the inverse operator $T^{-1} : Y \to X$ also preserves disjointness. Moreover the operator T is regular and the vector lattices X and Y are order isomorphic.*

Proof. By Lemma 7.2.5 the restriction of T to any principal ideal in X satisfies $\pitchfork$ and therefore $T \in (\pitchfork)$. If T^{-1} does not preserve disjointness then by Theorem 5.1.1, X contains a Dedekind σ-complete projection band U such that the operator $T^{-1} : TU \to U$ does not preserve disjointness. This contradicts Corollary 7.2.3.

We have just proved that T is a d-isomorphism. If T were not regular then by Theorem 5.1.4 and Lemma 5.3.1 the domain X would contain a Dedekind σ-complete projection band U with the following property:

- For any order bounded sequence u_n of pairwise disjoint elements in U and for any scalars λ_n the sequence $\lambda_n T u_n$ is order bounded in Y.

Clearly we can assume that U is a principal atomless band and therefore it is order isomorphic to an ideal I in $C_\infty(K)$ where K is a basically disconnected compact space without isolated points. We will identify U and I. Let e be a positive weak unit in U. For any $n \in \mathbb{N}$ let us say that a point $k \in K$ is in the set O_n if there is a clopen neighborhood V of k such that $\operatorname{supp} u \subset V \Rightarrow p_n(Tu) = 0$ for any $u \in [0, e]$. Obviously O_n is an open subset of K.

Let $F_n = K \setminus O_n$. We claim that the set F_n is at most finite. Indeed, otherwise we could find an order bounded sequence u_m of pairwise disjoint elements in U such that $p_n(Tu_m) > 0$ for any $m \in \mathbb{N}$. Let λ_m be positive scalars such that $\lambda_m p_n(Tu_m) \to \infty$ as $m \to \infty$. The sequence $\lambda_m T u_m$ is order bounded in Y, whence there is a $y \in Y$ such that $\lambda_m |Tu_m| \le |y|$ for any $m \in \mathbb{N}$. But then $p_n(y) = \infty$, a contradiction.

The set $F = \bigcup_{n=1}^{\infty} F_n$ must be dense in K. Otherwise we would find a non-zero $u \in U$ such that $p_n(Tu) = 0$ for any $n \in \mathbb{N}$; but this is impossible because K is a basically disconnected compact space without isolated points. ∎

7.3. Range-domain interchange in the HPK-theorem

7.3.1. THEOREM. *Let $T : X \to Y$ be a disjointness preserving bijection, and let the following conditions hold:*

(1) *the vector lattice X satisfies condition $(\Delta^{\mathrm{w}}_{\mathrm{L}})$,*
(2) *the vector lattice Y is r_u-complete,*
(3) *the operator T satisfies condition $\pitchfork$.*

Then the inverse operator T^{-1} is also disjointness preserving and, hence, T is a d-isomorphism. Furthermore, the operators T and T^{-1} are regular, and the vector lattices X and Y are order isomorphic.

Proof. Assume, contrary to our claim, that there are $u, v \in X$ such that $u \wedge v > 0$ and $Tu \perp Tv$. In view of Theorem 4.0.1 we can assume without loss of generality that $u \le v \le 2u$ and that the Krein–Kakutani space K_u is zero-dimensional. Moreover by

Lemma 2.5.7 we can assume that $Tu \geq 0$. Fix a decreasing sequence $\{u_n\}$ such that $u_n \to v$ in the $C(K_u)$-norm, where each u_n is a linear combination of components of u.

Since each u_n is a linear combination of non-negative components of u, the image Tu_n is a linear combination of non-negative components of Tu, and so obviously $Tu_n \perp Tv$. The condition that $\{u_n\}$ is $C(K_u)$-Cauchy in X implies easily that the sequence $\{Tu_n\}$ is $C(K_{Tu})$-Cauchy in Y and, therefore, there exists some $y \in Y$ such that $Tu_n \to y$. Clearly $y \perp Tv$.

Let $w = T^{-1}y$ and consider the pair v, w in X. As noted above, the images Tv and $y = Tw$ are disjoint. Let us verify that the elements v, w themselves cannot be disjoint. Indeed, let B be the band generated by u, that is, $B = \{u\}^{dd}$. Since T satisfies condition $\pitchfork$ the image TB is a band in Y [6, Proposition 3.2], and clearly $Tu \in TB$. Therefore, $\{Tu\}^{dd} \subseteq TB$. Recall now that y is the $C(K_{Tu})$-limit of some linear combinations of components of Tu, and so y is contained in the band $\{Tu\}^{dd}$. This implies that $w \in B$. Since v has the same width as u we conclude that $w \not\perp v$.

By Theorem 4.0.1 we can find non-zero components $\widetilde{v}$ and $\widetilde{w}$ of v and w, respectively, such that $\widetilde{v} \leq \widetilde{w} \leq 2\widetilde{v}$ and $|\widetilde{v} - \widetilde{w}| \geq c\widetilde{v}$, where c is some positive constant.

The compact space $K_{\widetilde{v}}$ cannot have isolated points (this would contradict $T\widetilde{v} \perp T\widetilde{w}$) and therefore we can find an infinite sequence $\{v_k\}$ of non-zero pairwise disjoint components of $\widetilde{v}$. Let $w_k = w \wedge 2v_k$; then for any k we have $\{v_k\}^{dd} = \{w_k\}^{dd}$ and $|v_k - w_k| \geq cv_k$.

For each k let $u_{n,k} = u_n \wedge v_k$ and note that the sequence $\{u_{n,k}\}_n$ converges in the $C(K_u)$-norm to v_k. Therefore, the sequence $\{Tu_{n,k}\}_k$ converges in the $C(K_{Tu})$-norm to some y_k, which is clearly a component of y.

For each k the element $u_{n,k}$ is a component of u_n so that $Tu_{n,k}$ is a component of Tu_n, and it is plain to see now that $T^{-1}y_k = w_k$. Let us fix some positive scalars λ_k. For any k we can find a positive integer n_k such that

$$|Tu_{n_k,k} - y_k| \leq \frac{1}{k\lambda_k}\,|y|, \qquad |u_{n_k,k} - v_k| \leq \frac{1}{2}\,|w_k - v_k|.$$

Since Y is r_u-complete, the element $y_0 = \sum_k \lambda_k(Tu_{n_k,k} - y_k)$ exists in Y. Let $x_0 = T^{-1}y_0$. Then the assumption $T \in (\pitchfork)$ implies that for each k we have

$$|x_0| \geq \lambda_k|u_{n_k,m} - w_k| \geq \frac{1}{2}\,\lambda_k|v_k - w_k| \geq \frac{c}{2}\,\lambda_m v_k.$$

Recall now that $\{v_k\}$ is an arbitrary disjoint sequence of non-zero components of $\widetilde{v}$, and therefore the last inequality shows that $X \notin (\Delta^{\mathrm{m}}_{\mathrm{L}})$. But the space $K_{\widetilde{v}}$ is zero-dimensional and Proposition 2.6.4 implies that $X \notin (\Delta^{\mathrm{w}}_{\mathrm{L}})$, in contradiction with our assumption.

As soon as we know that T is a d-isomorphism, Corollary 5.3.2 implies that T is regular. Finally, it remains to notice that by Theorem 4.12 in [5] the regularity of a d-isomorphism T implies that T^{-1} is also regular and that X is order isomorphic to Y. ∎

7.3.2. Remark. Example 3.2.2 shows that the assumption $T \in (\pitchfork)$ in Theorem 7.3.1 cannot be dropped. Indeed, in that example X is a normed lattice, Y is a Banach lattice, and $T : X \to Y$ is a bijective d.p.o. which is not a d-isomorphism.

8. Applications to spaces of continuous functions

In this section we apply our previous results to the important case when either the domain X, or the range Y, or both are vector lattices of continuous functions on a Tikhonov (completely regular) topological space.

We refer the reader to [14, 20] and to [6, Section 4] for a more complete discussion of related problems and more extensive bibliography.

For the reader's convenience let us notice that according to the terminology introduced in [14] a bijective d.p.o. $T : C(\Omega_1) \to C(\Omega_2)$, where Ω_1 and Ω_2 are Tikhonov spaces, is called a *separating map* and if T is a d-isomorphism it is called a *biseparating map.*

The following important result was proved in [15].

8.0.1. THEOREM ([15]). *Let Ω_1, Ω_2 be Tikhonov spaces and $T : C(\Omega_1) \to C(\Omega_2)$ be a d-isomorphism. Then there are a homeomorphism φ of $v\Omega_1$ onto $v\Omega_2$, where $v\Omega_1$ and $v\Omega_2$ are realcompactifications* ([21, Section 8.4]) *of Ω_1 and Ω_2, respectively, and a non-vanishing function $w \in C(\Omega_2)$ such that*

$$Tf = w(f \circ \varphi), \qquad f \in C(\Omega_1).$$

A generalization of Theorem 8.0.1 to the case of Φ-algebras was obtained in [20]. (see also [14] and [27]). Let us also remind the reader that a representation of a regular d.p.o. as a weighted composition on absolutes (or Stone spaces) of X and Y is always possible [1, 2].

We will need the following result.

8.0.2. THEOREM ([20, Theorem 5.5]). *Let Ω be a Tikhonov space. Then for every Dedekind σ-complete band U in $C(\Omega)$ we have $d\text{-}\dim U = 1$.*

Our next two results follow immediately from Corollary 5.1.2 and Theorems 5.1.4, 5.1.5, and 8.0.2.

8.0.3. THEOREM. *Let Ω be a Tikhonov space with a π-base of clopen subsets. Then*:

(1) *For any r_u-complete vector lattice Y* ([8]) *and for any bijective d.p.o. $T : C(\Omega) \to Y$ the operator T is a regular d-isomorphism.*

(2) *If Y is an* arbitrary *vector lattice and $T : C(\Omega) \to Y$ is a bijective d.p.o. then $d(T) \le 2$ (see Definition 2.3.4).*

(3) *If we additionally assume that any clopen basically disconnected subset E in Ω is "Specker" (i.e. every continuous function on E is essentially constant, Definition 2.4.10) then $C(\Omega)$ is super d-rigid, i.e. for any vector lattice Y and any bijective d.p.o. $T : C(\Omega) \to Y$ the operator T is a regular d-isomorphism.*

8.0.4. THEOREM. *Let Ω be a Tikhonov space, Y be an r_u-complete vector lattice, and $T : C(\Omega) \to Y$ be a d-isomorphism. Then T is regular and Y is order isomorphic to $C(\Omega)$.*

The next theorem follows from Theorem 6.2.3.

8.0.5. THEOREM. *Let Ω_1 and Ω_2 be Tikhonov spaces, and assume that any family of pairwise disjoint open subsets of Ω_2 has cardinality less than $2^{\aleph_0}$. Then any bijective d.p.o. $T : X \to Y$ is a regular d-isomorphism.*

([8]) In particular for $Y = C(\Gamma)$ where Γ is a Tikhonov space.

Finally, from Theorem 7.2.6 we obtain

8.0.6. THEOREM. *Let Ω_1 and Ω_2 be Tikhonov spaces, and assume that the space Ω_2 is σ-compact. Then every bijective d.p.o. $T : X \to Y$ is a regular d-isomorphism.*

8.0.7. PROBLEM. Can we drop either the condition that Ω has a π-base of clopen subsets in Theorem 8.0.3 or the condition on Ω_2 in Theorem 8.0.5?

The only case known to us when the answer to Problem 8.0.7 is positive is the one when $\Omega_1 = [0, 1]$ (see [10]).

We want to mention a special case of Problem 8.0.7 which, we think, might be crucial to solving the general problem.

8.0.8. PROBLEM. Let $\Omega_1 = [0, 1] \times [0, 1]$ and Ω_2 be an arbitrary Tikhonov space. Is any bijective d.p.o. $T : C(\Omega_1) \to C(\Omega_2)$ a d-isomorphism?

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