Introduction

One of the most important problems of modern analysis is to construct a functional calculus of several noncommuting operators. This problem goes back to von Neumann [55] and has its origin in mathematical foundations of quantum mechanics. Functions of noncommuting variables also appear naturally in the theory of partial differential and pseudodifferential operators and in some problems of algebra, geometry, and mathematical physics; see, e.g., [53] and references therein.

A possible way to define the value of a function f at an n-tuple $(a_1, \dots, a_n)$ of linear operators is provided by the so-called ordered representation method [53] that was introduced by Feynman and developed by Maslov [49] (see also related papers [41, 42] by Litvinov). An essential difference of this method with the single-variable case is that the assignment $f \mapsto f(a_1, \dots, a_n)$ is no longer an algebra homomorphism. Another approach to the functional calculus problem is based on the philosophy of noncommutative geometry: we may change the concept of function itself and replace the commutative algebra of functions by some noncommutative algebra.

In a coordinate-free language, a tuple of noncommuting linear operators on a Banach space E is a representation of some finitely generated associative algebra, A, that can be viewed as an "algebra of polynomial functions on a noncommutative space". Therefore a noncommutative analogue of the classical (i.e., single-variable) functional calculus problem can be formulated as follows: Is it possible to extend the given representation $A \to \mathscr{B}(E)$ to some larger algebra B containing A? Depending on their properties, such algebras B can be considered as noncommutative versions of algebras of holomorphic functions, smooth functions, continuous functions, Borel functions, etc. Note that this problem also makes sense when A is not a subalgebra of B; it is sufficient that a homomorphism $A \to B$ be fixed.

The notion of spectrum plays a key role in the functional calculus problem. To give a simple example, recall that a bounded operator T on a Banach space E has a holomorphic functional calculus on an open set $U \subset \mathbb{C}$ (i.e., the homomorphism $\mathbb{C}[t] \to \mathscr{B}(E)$, $t \mapsto T$, extends to a continuous homomorphism from the algebra $\mathscr{O}(U)$ of holomorphic functions to $\mathscr{B}(E)$) if and only if U contains the spectrum of T. It is therefore natural to look for a reasonable analogue of the notion of spectrum for several (possibly noncommuting) operators. A general approach to this problem was suggested by J. L. Taylor [75]. If $\pi\colon A \to \mathscr{B}(E)$ is a representation of an algebra A on a Banach space E, then the spectrum $\sigma(\pi, A)$ is a part of a suitably chosen set (a "structure space") Ω_A of (isomorphism classes of) locally convex A-modules, and $F \in \Omega_A$ does not belong to $\sigma(\pi, A)$ if and only if $\operatorname{Tor}_n^A(F, E) = 0$ for all $n \geq 0$. (Here Tor_n^A denotes the nth derived functor of the projective tensor product; see [26, 74] and Section 1 below.) For example, in the case $A = \mathbb{C}[t]$

one can take Ω_A to be the set of all 1-dimensional A-modules. This set is naturally parametrized by points of the complex plane, and the Taylor spectrum, $\sigma(\pi, A)$, coincides with the usual spectrum, $\sigma(T)$, of the operator $T = \pi(t)$. In the same manner, if $A = \mathbb{C}[t_1, \dots, t_n]$, then $\sigma(\pi, A)$ is a subset of $\mathbb{C}^n$. In this case, the representations of A are in bijective correspondence with the n-tuples of commuting operators, and $\sigma(\pi, A)$ is what is now called the *Taylor joint spectrum* of the n-tuple $(T_1, \dots, T_n)$, $T_i = \pi(t_i)$. In his famous papers [72]–[75], Taylor established a number of remarkable properties of the joint spectrum and constructed a multivariable version of an analytic functional calculus. For a modern treatment of this theory, see [15].

The definition of $\sigma(\pi, A)$ suggested by Taylor depends not only on the image of the representation $\pi\colon A \to \mathscr{B}(E)$ (i.e., not only on the given n-tuple of operators), but also on the algebra A. Therefore, if π can be extended to a representation ϱ of a larger algebra $B \supset A$, then one cannot expect that $\sigma(\pi, A) = \sigma(\varrho, B)$ in general. On the other hand, the equality still holds in many important cases (e.g., in the above-mentioned case $A = \mathbb{C}[t]$, $B = \mathscr{O}(U)$). Therefore it seems natural to consider only those algebras $B \supset A$ which have the property that if some representation π of A extends to a representation ϱ of B, then $\sigma(\pi, A) = \sigma(\varrho, B)$. More generally, if A is not a subalgebra of B, but a homomorphism $\theta\colon A \to B$ is given, then it is natural to require that $\sigma(\pi, A) = \theta^*(\sigma(\varrho, B))$ where $\theta^*\colon \Omega_B \to \Omega_A$ denotes the pullback along θ.

Taylor [75] introduced an appropriate class of algebra homomorphisms satisfying the above requirement and called them *localizations*. Roughly speaking, a topological algebra homomorphism $A \to B$ is a localization if it identifies the category of topological B-modules with a full subcategory of the category of topological A-modules, and if the homological relations between B-modules do not change when the modules are considered as A-modules. Since Taylor's objective was to construct a holomorphic functional calculus of several *commuting* operators, he considered mainly the case where $A = \mathbb{C}[t_1, \dots, t_n]$, the polynomial algebra endowed with the finest locally convex topology. Taylor has proved that the canonical homomorphism of $\mathbb{C}[t_1, \dots, t_n]$ to $\mathscr{O}(U)$, the Fréchet algebra of holomorphic functions on an open set $U \subset \mathbb{C}^n$, is a localization provided U is a domain of holomorphy. He has also shown that the canonical homomorphisms of $\mathbb{C}[t_1, \dots, t_n]$ to the algebra $C^\infty(V)$ of smooth functions (where $V \subset \mathbb{R}^n$ is an open set) and to the algebra $\mathscr{E}'(\mathbb{R}^n)$ of compactly supported distributions are localizations.

Thus the polynomial algebra $\mathbb{C}[t_1, \dots, t_n]$ has a rich supply of localizations. Motivated by this example, Taylor suggested a general scheme for constructing a noncommutative functional calculus, a scheme where the notion of localization plays a fundamental role. The first step of this scheme is as follows. Suppose A is a fixed finitely generated algebra (the "base algebra") endowed with the finest locally convex topology. The problem is to construct a sufficiently large family of localizations of A with values in some topological algebras having a richer structure. Having constructed such a family, one can hope to develop a reasonable spectral theory for representations of A.

As was said above, Taylor defined localizations in the topological algebra setting. In pure algebra, a notion analogous to that of localization was introduced by W. Geigle and H. Lenzing [18] under the name "homological epimorphism". This notion turned out to

be useful in the representation theory of finite-dimensional algebras (see [9]). Recently, A. Neeman and A. Ranicki [54] applied homological epimorphisms to some problems of algebraic K-theory. They use a different terminology; namely, in the case where $\theta\colon A \to B$ is a homological epimorphism, Neeman and Ranicki say that B *is stably flat over* A, while the word "localization" is used by them in a different (ring-theoretical rather than homological) sense ([1]). We adopt both the languages here and use the phrases "$\theta\colon A \to B$ is a localization" (in Taylor's sense) and "B is stably flat over A" as synonyms. The reason is that the word "localization" is used in modern mathematics in many different senses, and the terminology of [75] is not the most common one. On the other hand, it is convenient to use Taylor's terminology when it is necessary to emphasize the role of the homomorphism θ.

Taylor [75] has pointed out that a possible candidate for an algebra B which often seems to be stably flat over A is its Arens–Michael envelope $\widehat{A}$ (the completed l.m.c. envelope, in the terminology of [75]), which is defined as the completion of A with respect to the family of all submultiplicative seminorms on A. From the viewpoint of operator theory, an important property of $\widehat{A}$ (which uniquely characterizes $\widehat{A}$ within the class of Arens–Michael algebras) is that A and $\widehat{A}$ have the same set of continuous Banach space representations. If $A = \mathbb{C}[t_1, \dots, t_n]$, then $\widehat{A}$ is isomorphic to the algebra $\mathscr{O}(\mathbb{C}^n)$ of entire functions (and hence is stably flat over A). Thus the Arens–Michael envelope of a noncommutative finitely generated algebra can be viewed as an "algebra of noncommutative entire functions".

Apart from the polynomial algebra, Taylor [75, 76] has also studied localizations of the free algebra F_n on n generators. In particular, he proved that the canonical homomorphism of F_n to its Arens–Michael envelope $\widehat{F}_n$ is a localization (i.e., $\widehat{F}_n$ is stably flat over F_n). Some results on localizations of F_n were also obtained by Luminet [45].

Another important class of noncommutative algebras considered by Taylor is that of universal enveloping algebras. Let $\mathfrak{g}$ be a complex Lie algebra and $U(\mathfrak{g})$ its universal enveloping algebra. Taylor [75] proved that if $\mathfrak{g}$ is semisimple, then $\widehat{U}(\mathfrak{g})$, the Arens–Michael envelope of $U(\mathfrak{g})$, fails to be stably flat over $U(\mathfrak{g})$, in contrast to the abelian case. We generalized this result in [64] by showing that $\widehat{U}(\mathfrak{g})$ is not stably flat over $U(\mathfrak{g})$ unless $\mathfrak{g}$ is solvable. On the other hand, Dosiev [12] has recently proved that $\widehat{U}(\mathfrak{g})$ is stably flat over $U(\mathfrak{g})$ provided $\mathfrak{g}$ is metabelian (i.e., $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0$). A natural conjecture is that $\widehat{U}(\mathfrak{g})$ is stably flat over $U(\mathfrak{g})$ for each solvable (or, at least, for each nilpotent) Lie algebra $\mathfrak{g}$, but this question is still open.

In this paper we consider some "standard" locally convex algebras H that contain $U(\mathfrak{g})$ as a dense subalgebra, and we study the question of whether or not they are stably flat over $U(\mathfrak{g})$. Specifically, we concentrate on the following algebras:

- $H = \widehat{U}(\mathfrak{g})$, the Arens–Michael envelope of $U(\mathfrak{g})$;
- $H = U(\mathfrak{g})_{\mathscr{M}}$, Goodman's weighted completion of $U(\mathfrak{g})$ [19, 20];
- $H = \mathfrak{F}(\mathfrak{g})$, Rashevskii's hyperenveloping algebra [67];

([1]) Recently localizations were independently introduced by R. Meyer [51] within the framework of bornological algebras.

- $H = \mathscr{A}(G)$, Litvinov's algebra of analytic functionals on the corresponding connected, simply connected complex Lie group G [39, 40, 42].

We generalize the above-mentioned result of Dosiev and show that $\widehat{U}(\mathfrak{g})$ is stably flat over $U(\mathfrak{g})$ provided $\mathfrak{g}$ admits a positive grading. The weighted completion $U(\mathfrak{g})_{\mathscr{M}}$ is shown to be stably flat over $U(\mathfrak{g})$ for each nilpotent Lie algebra $\mathfrak{g}$ and each entire weight sequence $\mathscr{M}$ (for terminology, see [20] and Section 7 below). We also prove that Rashevskii's hyperenveloping algebra $\mathfrak{F}(\mathfrak{g})$ is stably flat over $U(\mathfrak{g})$ for every Lie algebra $\mathfrak{g}$. Finally, $\mathscr{A}(G)$ turns out to be stably flat over $U(\mathfrak{g})$ if and only if $\mathfrak{g}$ is solvable.

A common feature of the above algebras $H \supset U(\mathfrak{g})$ is that they are *well-behaved topological Hopf algebras* [1, 43]. This means that they are Hopf algebras in the tensor category of complete locally convex spaces equipped with the projective tensor product $\widehat{\otimes}$, and that their underlying locally convex spaces are either nuclear Fréchet spaces or nuclear (DF)-spaces. The category of well-behaved topological Hopf algebras has a number of remarkable properties (see [1, 43]); in particular, it is anti-equivalent to itself via the strong duality functor. To answer the question of whether or not a morphism $U(\mathfrak{g}) \to H$ in this category is a localization, we propose a general method that applies to all of the above-mentioned algebras H. This method is based on the following observation. Let $\mathfrak{g}$ be a Lie algebra, and let $V_\cdot(\mathfrak{g}) = C_\cdot(\mathfrak{g}, U(\mathfrak{g}))$ denote the Koszul resolution of the trivial $\mathfrak{g}$-module $\mathbb{C}$. The classical fact that the augmented complex $0 \leftarrow \mathbb{C} \leftarrow V_\cdot(\mathfrak{g})$ is exact is traditionally proved by introducing an appropriate filtration on this complex and then by using an induction or a spectral sequence argument (see, e.g., [6, Chap. XIII, Theorem 7.1] or [24, Chap. II, Lemme 2.2]). However, if $\mathfrak{g}$ a finite-dimensional Lie algebra over $\mathbb{C}$, it is possible to give another proof using the fact that the (topological) dual of $U(\mathfrak{g})$ is isomorphic to the Fréchet algebra $\mathbb{C}[[z_1, \ldots, z_n]]$ of formal power series. The main point is that the complex dual to $V_\cdot(\mathfrak{g})$ turns out to be isomorphic to the (formal) de Rham complex of $\mathbb{C}[[z_1, \ldots, z_n]]$. By the Poincaré lemma, the latter complex (augmented by the unit map $\mathbb{C} \to \mathbb{C}[[z_1, \ldots, z_n]]$) splits as a complex of topological vector spaces. Taking the topological dual, we conclude that $0 \leftarrow \mathbb{C} \leftarrow V_\cdot(\mathfrak{g})$ is exact.

The advantage of this proof is that it carries over to topological algebras more general than $U(\mathfrak{g})$. This suggests the following approach to the above-mentioned localization problem for $U(\mathfrak{g})$. Given a well-behaved topological Hopf algebra H and a morphism $\theta\colon U(\mathfrak{g}) \to H$, we can view H as a right $\mathfrak{g}$-module via θ. Using a version of the Cartan–Eilenberg "inverse process" (see [6, Chap. X]), we prove that θ is a localization if and only if the standard complex $C_\cdot(\mathfrak{g}, H)$ augmented by the counit map $H \to \mathbb{C}$ splits as a complex of topological vector spaces. Due to the reflexivity of the algebras involved, this happens precisely when the dual complex $0 \to \mathbb{C} \to C^\cdot(\mathfrak{g}, H')$ splits. Suppose now that H is cocommutative; then the dual algebra, H', is commutative. Under some additional conditions on H, the latter complex turns out to be isomorphic to the de Rham complex of $A = H'$. In this situation we say that A is $\mathfrak{g}$-*parallelizable*. Thus the problem of whether or not θ is a localization reduces to the question of whether or not the augmented de Rham complex $0 \to \mathbb{C} \to \Omega(A)$ splits. A sufficient condition for this to be true is that A be *contractible* in the sense of Chen [7]. Therefore in order to prove that $\theta\colon U(\mathfrak{g}) \to H$ is a localization it is sufficient to show that H' is $\mathfrak{g}$-parallelizable and contractible.

It should be noted that the above method is inspired by the following result due to Taylor [75]. Suppose that $\mathfrak{g}$ is the complexification of the Lie algebra of a real Lie group G. Then $U(\mathfrak{g})$ is canonically embedded into $\mathscr{E}'(G)$, the algebra of compactly supported distributions on G. Taylor proved that this embedding is a localization if and only if the de Rham cohomology of G vanishes. The method described above is in fact a generalization of Taylor's proof.

This paper is organized as follows. In Section 1 we recall some basic facts from topological homology (i.e., the homology theory for locally convex algebras [26]). We also discuss "continuous versions" of some concepts from pure algebra such as DG algebras, Kähler differentials and de Rham cohomology. Section 2 is devoted to a version of the Cartan–Eilenberg inverse process for topological Hopf algebras. As a byproduct, we describe the Hochschild cohomology groups of the algebras $\ell^1(G)$ (where G is a discrete group) and $\mathscr{E}'(G)$ (where G is a real Lie group) in terms of the bounded and continuous cohomology groups of G. As another application, we show that a Banach Hopf algebra with invertible antipode is amenable precisely when it is left amenable in the sense of Lau [37]. In Section 3 we discuss the notion of localization for topological algebras and introduce related concepts of weak localization and strong transversality. The latter notion is a somewhat stronger version of a transversality condition for Fréchet modules that was introduced in [70] and has proved to be extremely useful in complex analytic geometry and operator theory [32, 70, 15, 10]. Using results of the previous section, we show that for Hopf $\widehat{\otimes}$-algebras with invertible antipode the notions of localization and weak localization coincide. In Section 4 we recall some portions of Chen's algebraic homotopy theory [7] in the topological algebra framework, and apply this theory to localizations of $U(\mathfrak{g})$ within the category of well-behaved cocommutative Hopf $\widehat{\otimes}$-algebras. Given a morphism $\theta\colon U(\mathfrak{g}) \to H$ in this category such that $\operatorname{Im}\theta$ is dense in H, we show that θ is a localization provided H' is $\mathfrak{g}$-parallelizable and contractible. In Section 5 we concentrate on nilpotent Lie algebras and show that the dual of a well-behaved Hopf $\widehat{\otimes}$-algebra H containing $U(\mathfrak{g})$ is $\mathfrak{g}$-parallelizable provided H is contained in the formal power series completion $[U(\mathfrak{g})]$ of $U(\mathfrak{g})$. In Section 6 we discuss some general properties of Arens–Michael envelopes and describe the Arens–Michael envelopes of graded algebras as certain vector-valued Köthe spaces. As a corollary, we show that the dual of the Arens–Michael envelope of $U(\mathfrak{g})$ is $\mathfrak{g}$-parallelizable provided $\mathfrak{g}$ admits a positive grading. Next we introduce a notion of contractible Lie algebra. By definition, a Lie algebra $\mathfrak{g}$ is contractible if there is a smooth path in the set of all endomorphisms of $\mathfrak{g}$ connecting the zero endomorphism and the identity endomorphism of $\mathfrak{g}$. We show that if $\mathfrak{g}$ is contractible, then $\widehat{U}'(\mathfrak{g})$, the dual of the Arens–Michael envelope of $U(\mathfrak{g})$, is contractible in the sense of Chen. This result is then used to prove that the Arens–Michael envelope of $U(\mathfrak{g})$ is stably flat over $U(\mathfrak{g})$ for each positively graded $\mathfrak{g}$. As a byproduct, we show that the injective homological dimension of each nonzero $\widehat{U}(\mathfrak{g})$-$\widehat{\otimes}$-module is equal to the dimension of $\mathfrak{g}$. In Sections 7 and 8 we prove the above-mentioned results on the stable flatness of weighted completions of $U(\mathfrak{g})$, hyperenveloping algebras, and algebras of analytic functionals. Finally, in Section 9 we explain how the completions of $U(\mathfrak{g})$ considered above are related to one another, and formulate some open problems.

REMARK. A. Dosiev has kindly informed the author that he proved the stable flatness of the Arens–Michael envelope $\widehat{U}(\mathfrak{g})$ over $U(\mathfrak{g})$ under the condition that $\mathfrak{g}$ is a nilpotent Lie algebra with *normal growth* [13]. Roughly speaking, $\mathfrak{g}$ has normal growth if for each embedding of $\mathfrak{g}$ into a Banach algebra, norms of powers of elements from $[\mathfrak{g},\mathfrak{g}]$ decrease sufficiently rapidly. The class of Lie algebras with normal growth contains all metabelian Lie algebras, but it is not clear how this class is related to that of positively graded Lie algebras.

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1. Preliminaries

We shall work over the complex numbers $\mathbb{C}$. All associative algebras are assumed to be unital.

1.1. Topological algebras and modules. In this subsection we recall some basic notions from topological homology (homology theory for topological algebras). For more details, see [26], [28], and [74].

We refer to [69] and [77] for general facts on topological vector spaces. Given topological vector spaces E and F, we denote by $\mathscr{L}(E,F)$ the space of all linear continuous maps from E to F. We endow $\mathscr{L}(E,F)$ with the topology of uniform convergence on bounded subsets of E. Unless otherwise specified, $E' = \mathscr{L}(E,\mathbb{C})$ denotes the strong dual of E. The completion of E is denoted by $E^{\sim}$. If E and F are locally convex spaces (l.c.s.'s), then $E \mathbin{\widehat{\otimes}} F$ stands for their completed projective tensor product.

By a *topological algebra* we mean a topological vector space A together with the structure of an associative algebra such that the multiplication map $A \times A \to A$ is separately continuous. A complete, Hausdorff, locally convex topological algebra with jointly continuous multiplication is called a $\widehat{\otimes}$*-algebra* (see [74, 26]). If A is a $\widehat{\otimes}$-algebra, then the multiplication $A \times A \to A$ extends to a continuous linear map from the projective tensor product $A \mathbin{\widehat{\otimes}} A$ to A. In other words, a $\widehat{\otimes}$-algebra is just an algebra in the tensor category $(\mathbf{LCS}, \widehat{\otimes})$ of complete l.c.s.'s (cf. Section 2 below).

Recall that a seminorm $\|\cdot\|$ on an algebra A is called *submultiplicative* if $\|ab\| \leq \|a\|\,\|b\|$ for all $a, b \in A$. A $\widehat{\otimes}$-algebra A is called an *Arens–Michael algebra* (or a *locally m-convex algebra*) if its topology can be defined by a family of submultiplicative seminorms (see [52, 27]). In particular, any Banach algebra is an Arens–Michael algebra. By a *Fréchet algebra* we mean a metrizable (not necessarily locally m-convex) $\widehat{\otimes}$-algebra.

Each associative $\mathbb{C}$-algebra A becomes a topological algebra with respect to the finest locally convex topology. We denote the resulting topological algebra by A_{s}. If A has countable dimension as a vector space, then A_{s} is a $\widehat{\otimes}$-algebra [1]. In particular, this condition is satisfied whenever A is finitely generated.

A left $\widehat{\otimes}$*-module* over a $\widehat{\otimes}$-algebra A (a left A-$\widehat{\otimes}$-module for short) is a complete Hausdorff locally convex space X together with the structure of a left unital A-module such that the map $A \times X \to X$, $(a,x) \mapsto a \cdot x$, is jointly continuous. As above, this

means that X is a left A-module in $(\mathbf{LCS}, \widehat{\otimes})$. Given two left A-$\widehat{\otimes}$-modules X and Y, an *A-module morphism* is a continuous linear map $\varphi\colon X \to Y$ such that $\varphi(a\cdot x) = a\cdot\varphi(x)$ for all $a \in A,\ x \in X$. The vector space of all A-module morphisms from X to Y is denoted by ${}_A\mathbf{h}(X,Y)$.

Right A-$\widehat{\otimes}$-modules, A-$\widehat{\otimes}$-bimodules, and their morphisms are defined similarly. As in pure algebra, A-$\widehat{\otimes}$-bimodules can be regarded as either left or right $\widehat{\otimes}$-modules over the algebra $A^e = A\mathbin{\widehat{\otimes}}A^{\mathrm{op}}$, where A^{op} stands for the algebra opposite to A. Given two right A-$\widehat{\otimes}$-modules (respectively, A-$\widehat{\otimes}$-bimodules) X and Y, we use the notation $\mathbf{h}_A(X,Y)$ (respectively, ${}_A\mathbf{h}_A(X,Y)$) to denote the corresponding space of morphisms. The resulting module categories are denoted by A-**mod**, **mod**-A, and A-**mod**-A, respectively.

If $\theta\colon A \to B$ is a $\widehat{\otimes}$-algebra homomorphism (i.e., a unital continuous homomorphism), then each left (resp. right) B-$\widehat{\otimes}$-module X can be considered as a left (resp. right) A-$\widehat{\otimes}$-module via θ. Sometimes we will denote the resulting A-$\widehat{\otimes}$-module by ${}_\theta X$ (resp. X_θ).

If A is a commutative $\widehat{\otimes}$-algebra, then an A-$\widehat{\otimes}$-bimodule X is *symmetric* if $a\cdot x = x\cdot a$ for all $a \in A,\ x \in X$. As usual, we identify left modules, right modules, and symmetric bimodules over a commutative algebra and call them just "modules".

Let A be a $\widehat{\otimes}$-algebra and M an A-$\widehat{\otimes}$-bimodule. Recall that a continuous linear map $D\colon A \to M$ is a *derivation* if $D(ab) = Da\cdot b + a\cdot Db$ for all $a, b \in A$. Denote by $\mathrm{Der}(A,M)$ the set of all continuous derivations from A to M. We also set $\mathrm{Der}\,A = \mathrm{Der}(A,A)$. If A is commutative, we may speak about derivations of A with coefficients in left A-$\widehat{\otimes}$-modules by identifying left modules with symmetric bimodules (see above).

If X is a right A-$\widehat{\otimes}$-module and Y is a left A-$\widehat{\otimes}$-module, then their A-module tensor product $X\mathbin{\widehat{\otimes}_A}Y$ is defined to be the completion of the quotient $(X\mathbin{\widehat{\otimes}}Y)/N$, where $N \subset X\mathbin{\widehat{\otimes}}Y$ is the closed linear span of all elements of the form $x\cdot a\otimes y - x\otimes a\cdot y$ ($x \in X$, $y \in Y$, $a \in A$) ([1]). As in pure algebra, the A-module tensor product can be characterized by a certain universal property (see [26] for details).

A morphism $\sigma\colon X \to Y$ of left A-$\widehat{\otimes}$-modules is said to be an *admissible epimorphism* if there exists a continuous linear map $\tau\colon Y \to X$ such that $\sigma\tau = \mathbf{1}_Y$, i.e., if σ is a retraction when considered in the category of topological vector spaces. Similarly, a morphism $\varkappa\colon X \to Y$ is an *admissible monomorphism* if it is a coretraction in the category of topological vector spaces. Finally, a morphism $\varphi\colon X \to Y$ is *admissible* if it has a factorization $\varphi = \varkappa\sigma$ with $\varkappa$ an admissible monomorphism and σ an admissible epimorphism. Geometrically, this means that the kernel and the image of φ are complemented subspaces of X and Y, respectively, and φ is an open map of X onto its image. A chain complex $X_\bullet = (X_n, d_n)$ of left A-$\widehat{\otimes}$-modules is called *admissible* if it splits as a complex of topological vector spaces. Equivalently, $X_\bullet$ is admissible if it is exact, and all the d_n's are admissible morphisms.

REMARK 1.1. It can easily be checked that the category A-**mod** together with the class of admissible monomorphisms and epimorphisms satisfies the axioms of exact category (see

([1]) To avoid confusion, we note that this definition of $X\mathbin{\widehat{\otimes}_A}Y$ (due to Rieffel [68] and Helemskii [26]) is different from that given by Kiehl and Verdier [32] and Taylor [74] (and used also in [70] and [15]). More precisely, $X\mathbin{\widehat{\otimes}_A}Y$ is the completion of the Kiehl–Verdier–Taylor tensor product.

[66] and [31]), so that the main constructions of abstract homological algebra (derived categories, "total" derived functors, etc.) make sense in this setting. However, we shall not use such a general approach here.

An A-module $P \in A\text{-}\mathbf{mod}$ is called *projective* if for each admissible epimorphism $X \to Y$ in $A\text{-}\mathbf{mod}$ the induced map ${}_A\mathbf{h}(P,X) \to {}_A\mathbf{h}(P,Y)$ is surjective. Dually, an A-module $Q \in A\text{-}\mathbf{mod}$ is called *injective* if for each admissible monomorphism $X \to Y$ in $A\text{-}\mathbf{mod}$ the induced map ${}_A\mathbf{h}(Y,Q) \to {}_A\mathbf{h}(X,Q)$ is surjective. For each $E \in \mathbf{LCS}$ the projective tensor product $F = A \mathbin{\widehat{\otimes}} E$ has a natural structure of a left A-$\widehat{\otimes}$-module with operation defined by $a \cdot (b \otimes x) = ab \otimes x$. Such modules are called *free*. In view of the natural isomorphism ${}_A\mathbf{h}(A \mathbin{\widehat{\otimes}} E, Y) \cong \mathscr{L}(E,Y)$, $Y \in A\text{-}\mathbf{mod}$, each free module is projective. This implies that the category $A\text{-}\mathbf{mod}$ has *enough projectives*, i.e., for each $X \in A\text{-}\mathbf{mod}$ there exists an admissible epimorphism $P \to X$ with P projective. To see this, it suffices to set $P = A \mathbin{\widehat{\otimes}} X$ and to define $A \mathbin{\widehat{\otimes}} X \to X$ by $a \otimes x \mapsto a \cdot x$.

Remark 1.2. If A is a Banach algebra, then the category $A\text{-}\mathbf{mod}$ has *enough injectives* as well, i.e., each $X \in A\text{-}\mathbf{mod}$ can be embedded into an injective A-$\widehat{\otimes}$-module via an admissible monomorphism [26]. However, if A is nonnormable, then $A\text{-}\mathbf{mod}$ may fail to have nonzero injective objects [61, 62]; see also Corollary 6.21 below.

Given a left A-$\widehat{\otimes}$-module X, a *resolution* of X is a chain complex $P_\bullet = (P_n, d_n)_{n\geq 0}$ of left A-$\widehat{\otimes}$-modules together with a morphism $\varepsilon\colon P_0 \to X$ such that the augmented sequence

$$0 \leftarrow X \xleftarrow{\varepsilon} P_0 \xleftarrow{d_0} \cdots \leftarrow P_n \xleftarrow{d_n} P_{n+1} \leftarrow \cdots$$

is admissible. A *projective resolution* is a resolution in which all the P_i's are projective $\widehat{\otimes}$-modules. Since $A\text{-}\mathbf{mod}$ has enough projectives, it follows that each A-$\widehat{\otimes}$-module has a projective resolution. Therefore one can define the derived functors Ext and Tor following the general patterns of relative homological algebra (see [26]). Namely, take a projective resolution $P_\bullet$ of an A-module $X \in A\text{-}\mathbf{mod}$ and set

$$\operatorname{Ext}^n_A(X,Y) = \mathrm{H}^n({}_A\mathbf{h}(P_\bullet, Y))$$

for each $Y \in A\text{-}\mathbf{mod}$. (Here "$\mathrm{H}^n$" stands for the nth cohomology space.) Similarly,

$$\operatorname{Tor}^A_n(Y,X) = \mathrm{H}_n(Y \mathbin{\widehat{\otimes}}_A P_\bullet)$$

for each $Y \in \mathbf{mod}\text{-}A$. Of course, $\operatorname{Ext}^n_A(X,Y)$ and $\operatorname{Tor}^A_n(Y,X)$ do not depend on the particular choice of the resolution $P_\bullet$ and have the usual functorial properties (see [26] for details).

A *projective bimodule resolution* of A is a projective resolution of A in $A\text{-}\mathbf{mod}\text{-}A$.

If $M \in A\text{-}\mathbf{mod}\text{-}A$, then the *nth Hochschild cohomology* (resp. *homology*) of A with coefficients in M is defined as $\mathscr{H}^n(A,M) = \operatorname{Ext}^n_{A^e}(A,M)$ (resp. $\mathscr{H}_n(A,M) = \operatorname{Tor}^{A^e}_n(M,A)$).

A left A-$\widehat{\otimes}$-module X has *projective homological dimension* $\leq n$ if X has a projective resolution $P_\bullet$ such that $P_i = 0$ for all $i > n$. The least such integer n is denoted by $\operatorname{dh}_A X$ and is called the *projective homological dimension* of X. Equivalently,

$$\operatorname{dh}_A X = \min\{n : \operatorname{Ext}^{n+1}_A(X,Y) = 0 \ \forall Y \in A\text{-}\mathbf{mod}\}.$$

If no such n exists, one sets $\operatorname{dh}_A X = \infty$.

The *injective homological dimension* of $X \in A$-**mod** is

$$\operatorname{inj.dh}_A X = \min\{n : \operatorname{Ext}_A^{n+1}(Y, X) = 0 \ \forall Y \in A\text{-}\mathbf{mod}\}.$$

If A is a Banach algebra, then $\operatorname{inj.dh}_A X$ can also be defined as the length of the shortest injective resolution of X (cf. Remark 1.2 above).

An A-$\widehat{\otimes}$-module X is projective (resp. injective) if and only if $\operatorname{dh}_A X = 0$ (resp. $\operatorname{inj.dh}_A X = 0$).

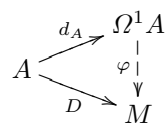
The *left global dimension* of A is

$$\operatorname{dg} A = \sup\{\operatorname{dh}_A X : X \in A\text{-}\mathbf{mod}\}.$$

Similarly, one can define homological dimension for right A-$\widehat{\otimes}$-modules and for A-$\widehat{\otimes}$-bimodules. The homological dimension of A considered as an A-$\widehat{\otimes}$-bimodule is called the *homological bidimension* of A and is denoted by $\operatorname{db} A$. For every $\widehat{\otimes}$-algebra A we have $\operatorname{dg} A \le \operatorname{db} A$.

1.2. Kähler differentials. Recall some facts about Kähler differentials and de Rham cohomology for commutative $\widehat{\otimes}$-algebras. Most of this material is well known in the purely algebraic case (see, e.g., [23] or [36]). For the $\widehat{\otimes}$-case, see [59], [48].

Let A be a commutative $\widehat{\otimes}$-algebra. A pair $(\Omega^1 A, d_A)$ consisting of an A-$\widehat{\otimes}$-module $\Omega^1 A$ and a derivation $d_A \colon A \to \Omega^1 A$ is called the *module of Kähler differentials* if for each A-$\widehat{\otimes}$-module M and for each derivation $D \colon A \to M$ there exists a unique A-$\widehat{\otimes}$-module morphism $\varphi \colon \Omega^1 A \to M$ such that the following diagram is commutative:



The derivation d_A is called the *universal derivation* of A.

Obviously, there is a natural isomorphism ${}_A\mathbf{h}(\Omega^1 A, M) \cong \operatorname{Der}(A, M)$ defined by the rule $\varphi \mapsto \varphi d_A$. In other words, $\Omega^1 A$ represents the functor $M \mapsto \operatorname{Der}(A, M)$. Hence $\Omega^1 A$ is unique up to a $\widehat{\otimes}$-module isomorphism.

The module of Kähler differentials can be constructed explicitly as follows. Let I be the kernel of the product map $A \mathbin{\widehat{\otimes}} A \to A$. Set $\Omega^1 A = (I/\overline{I^2})^\sim$ and define $d_A \colon A \to \Omega^1 A$ by $d_A a = (a \otimes 1 - 1 \otimes a) + \overline{I^2}$. Then $(\Omega^1 A, d_A)$ is the module of Kähler differentials for A (see, e.g., [59, Appendix B] or [23, §20] for the algebraic case).

If $A = C^\infty(M)$ is the Fréchet algebra of smooth functions on a manifold M, then $\Omega^1 A$ is canonically isomorphic to the module of differential 1-forms on M (cf. [59]). A similar result holds for algebras of holomorphic functions on Stein manifolds. Since we could not find this fact in the literature, we give a complete proof below.

Let $(V, \mathscr{O}_V)$ be a complex space. Consider the diagonal map $\Delta \colon V \to V \times V$, and denote by $\mathscr{I} \subset \mathscr{O}_{V \times V}$ the ideal sheaf of the subspace $\Delta(V) \subset V \times V$. Recall ([22]; cf. also [25]) that the *sheaf of 1-differentials* of V is defined as $\Omega^1_V = \Delta^*(\mathscr{I}/\mathscr{I}^2)$. There is a canonical morphism of sheaves $d \colon \mathscr{O}_V \to \Omega^1_V$ defined locally as $da = (a \otimes 1 - 1 \otimes a) + \mathscr{I}^2_{x,x}$ for each $a \in \mathscr{O}_{V,x}$, $x \in V$. If V is a complex manifold, then Ω^1_V coincides with the usual cotangent sheaf of V, the space of global sections $\Omega^1(V) = \Gamma(V, \Omega^1_V)$ is the space of

holomorphic differential 1-forms on V, and the map $d_V\colon \mathscr{O}(V) \to \Omega^1(V)$ induced by d is precisely the exterior (de Rham) derivative.

We need some facts on Stein modules [17]. Let $(V, \mathscr{O}_V)$ be a Stein space. For each coherent analytic sheaf $\mathscr{F}$ on V the space of global sections $\mathscr{F}(V) = \Gamma(V, \mathscr{F})$ has a canonical locally convex topology making it into a Fréchet $\mathscr{O}(V)$-module. Modules of this form are called *Stein modules*. Denote by $\mathrm{Coh}(V)$ the category of coherent sheaves of $\mathscr{O}_V$-modules and by $\mathrm{St}(V) \subset \mathscr{O}(V)$-**mod** the category of Stein $\mathscr{O}(V)$-modules. The functor of global sections $\Gamma\colon \mathrm{Coh}(V) \to \mathrm{St}(V)$ is exact (Cartan's Theorem B) and fully faithful [17]. Hence Γ is an equivalence of $\mathrm{Coh}(V)$ and $\mathrm{St}(V)$. If $\mathscr{F}, \mathscr{G} \in \mathrm{Coh}(V)$, and at least one of them is locally free, then there is a canonical isomorphism $\mathscr{F}(V) \mathbin{\widehat{\otimes}}_{\mathscr{O}(V)} \mathscr{G}(V) \cong (\mathscr{F} \otimes_{\mathscr{O}_V} \mathscr{G})(V)$ (see [70] or [15, 4.2.4]). If $F = \Gamma(V, \mathscr{F})$ is a Stein module and $G \subset F$ is a closed submodule, then G is also a Stein module, i.e., $G = \Gamma(V, \mathscr{G})$ for some coherent subsheaf $\mathscr{G} \subset \mathscr{F}$. In particular, each closed ideal $J \subset \mathscr{O}(V)$ has the form $J = \Gamma(V, \mathscr{J})$ for some coherent sheaf of ideals $\mathscr{J} \subset \mathscr{O}_V$. In this case, $\overline{J^2} = \Gamma(V, \mathscr{J}^2)$ (see [17]).

LEMMA 1.1. *Let $(V, \mathscr{O}_V)$ be a Stein space. Then the $\mathscr{O}(V)$-$\widehat{\otimes}$-module $\Omega^1(\mathscr{O}(V))$ of Kähler differentials is canonically isomorphic to $\Omega^1(V) = \Gamma(V, \Omega^1_V)$. Under this identification, $d_V\colon \mathscr{O}(V) \to \Omega^1(V)$ becomes a universal derivation.*

Proof. Set $V^2 = V \times V$, and let $I = \Gamma(V^2, \mathscr{I}) \subset \mathscr{O}(V^2)$ be the ideal of all functions vanishing on $\Delta(V)$. Identifying $\mathscr{O}(V^2)$ with $\mathscr{O}(V) \mathbin{\widehat{\otimes}} \mathscr{O}(V)$, we see that I becomes the kernel of the product map $\mathscr{O}(V) \mathbin{\widehat{\otimes}} \mathscr{O}(V) \to \mathscr{O}(V)$. Consider the commutative diagram

$$\begin{array}{ccccccc}
 & & \Gamma(V^2, \mathscr{I}) & \xrightarrow{\widetilde{q}} & \Gamma(V^2, \mathscr{I}/\mathscr{I}^2) & = & \Omega^1(V) \\
 & \overset{D}{\nearrow} & \| & & \uparrow{\scriptstyle j} & & \\
\mathscr{O}(V) & & & & & & \\
 & \underset{D}{\searrow} & \| & & & & \\
 & & I & \xrightarrow{q} & I/\overline{I^2} & = & \Omega^1(\mathscr{O}(V))
\end{array}$$

Here $D(a) = a \otimes 1 - 1 \otimes a$ for all $a \in \mathscr{O}(V)$, q is the quotient map, and $\widetilde{q}$ is induced by the sheaf quotient map $\mathscr{I} \to \mathscr{I}/\mathscr{I}^2$. Note also that $\Omega^1(\mathscr{O}(V)) = I/\overline{I^2}$ since I is a Fréchet space. Obviously, $d_V = \widetilde{q}D$, and $d_{\mathscr{O}(V)} = qD$. Since V is a Stein space, we see that j is an isomorphism (see the remarks preceding the statement of the lemma). The rest is clear. ∎

In some important cases the module of Kähler differentials is free and finitely generated. The next lemma gives a simple sufficient condition for this.

LEMMA 1.2. *Suppose there exist $x_1, \dots, x_n \in A$ and $\partial_1, \dots, \partial_n \in \mathrm{Der}\, A$ such that the x_i's generate a dense subalgebra of A, and that $\partial_i(x_j) = \delta_{ij}$ for each i, j. Then $\partial = (\partial_1, \dots, \partial_n)\colon A \to A^n$ is a universal derivation. In particular, $\Omega^1 A$ is isomorphic to A^n.*

Proof. Let $D\colon A \to M$ be a derivation. Denote by $(u_1, \dots, u_n)$ the standard basis in A^n (i.e., $u_i = (0, \dots, 1, \dots, 0)$ with 1 in the ith coordinate, 0 elsewhere). We have $\partial(x_i) = u_i$ for each $i = 1, \dots, n$. Define an A-$\widehat{\otimes}$-module morphism $\varphi\colon A^n \to M$ by $\varphi(u_i) = D(x_i)$ for $i = 1, \dots, n$. Then $(\varphi\partial)(x_i) = D(x_i)$ for each i. Since $x_1, \dots, x_n$ generate a dense subalgebra of A, we conclude that $\varphi\partial = D$. On the other hand, since A^n is generated (as an A-module) by $\mathrm{Im}\,\partial$, φ is a unique A-module morphism with the above property. ∎

1.3. DG $\widehat{\otimes}$-algebras and de Rham cohomology. By a *graded* $\widehat{\otimes}$*-algebra* we mean a sequence $\mathscr{A} = \{A^n\}_{n\in\mathbb{Z}_+}$ of complete l.c.s.'s together with continuous linear mappings

$$A^p \mathbin{\widehat{\otimes}} A^q \to A^{p+q}, \quad (a,b) \mapsto ab,$$

satisfying the usual associativity conditions. In particular, A^0 is a $\widehat{\otimes}$-algebra, and each A^n is an A^0-$\widehat{\otimes}$-bimodule. We will always assume that $\mathscr{A}$ is *unital*, i.e., that A^0 is unital and each A^n is a unital A^0-bimodule. A graded $\widehat{\otimes}$-algebra $\mathscr{A}$ is said to be *graded commutative* if $ab = (-1)^{pq}ba$ for each $a \in A^p$, $b \in A^q$. If $\mathscr{A}$ is graded commutative, then A^0 is commutative in the usual sense, and all the A^0-$\widehat{\otimes}$-bimodules A^n are symmetric.

Morphisms of graded $\widehat{\otimes}$-algebras are defined in an obvious way.

If $\mathscr{A}$ is a graded $\widehat{\otimes}$-algebra, then $A = \bigoplus_n A^n$ is a topological algebra with respect to the locally convex direct sum topology. If, in addition, each A^n is finite-dimensional, then the topology on A is the finest locally convex topology, so that A is a $\widehat{\otimes}$-algebra (see Subsection 1.1). In this case we will often identify $\mathscr{A}$ and A and say that $A = \bigoplus_n A^n$ *is a graded algebra.*

Let A be a commutative (ungraded) $\widehat{\otimes}$-algebra. Given an A-$\widehat{\otimes}$-module M and $n \in \mathbb{N}$, we can define the nth exterior power of M as in the purely algebraic case. Namely, consider the antisymmetrization map $\mathbf{a}_M \colon \widehat{\bigotimes}_A^n M \to \widehat{\bigotimes}_A^n M$ defined by

$$\mathbf{a}_M(x_1 \otimes \cdots \otimes x_n) = \frac{1}{n!} \sum_{\sigma\in S_n} \varepsilon(\sigma) \cdot x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)}\,. \tag{1}$$

It is easy to see that $\mathbf{a}_M$ is an A-$\widehat{\otimes}$-module morphism and that $\mathbf{a}_M^2 = \mathbf{a}_M$. Hence $\operatorname{Im}\mathbf{a}_M$ is a direct A-$\widehat{\otimes}$-module summand of $\widehat{\bigotimes}_A^n M$. We set $\widehat{\bigwedge}_A^n M = \operatorname{Im}\mathbf{a}_M$ (or, equivalently, $\widehat{\bigwedge}_A^n M = \widehat{\bigotimes}_A^n M/\operatorname{Ker}\mathbf{a}_M$), and call the resulting A-$\widehat{\otimes}$-module the *nth exterior power* of M. As usual, for each $x_1, \ldots, x_n \in M$ we denote the element $\mathbf{a}_M(x_1 \otimes \cdots \otimes x_n)$ of $\widehat{\bigwedge}_A^n M$ by $x_1 \wedge \cdots \wedge x_n$.

For each $p, q \in \mathbb{Z}_+$ we have a continuous bilinear map

$$\widehat{\bigwedge}_A^p M \times \widehat{\bigwedge}_A^q M \to \widehat{\bigwedge}_A^{p+q} M, \quad (x,y) \mapsto x \wedge y = \mathbf{a}(x \otimes y)$$

(here we set $\widehat{\bigwedge}_A^0 M = A$ and use the canonical identifications $A \mathbin{\widehat{\otimes}}_A X = X \mathbin{\widehat{\otimes}}_A A = X$). As in the purely algebraic case, the above maps make $\widehat{\bigwedge}_A M = \{\widehat{\bigwedge}_A^p M : p \in \mathbb{Z}_+\}$ into a graded commutative $\widehat{\otimes}$-algebra called the *exterior algebra* of M.

Now let $\mathscr{A}$ be a graded commutative $\widehat{\otimes}$-algebra. For each $n \in \mathbb{N}$ we have an A^0-$\widehat{\otimes}$-module morphism

$$\widehat{\bigwedge}_{A^0}^n A^1 \to A^n, \quad a_1 \wedge \cdots \wedge a_n \mapsto a_1 \cdots a_n. \tag{2}$$

The $\widehat{\otimes}$-algebra $\mathscr{A}$ is called *exterior* if (2) is an isomorphism for each $n \in \mathbb{N}$ (cf. [36]). In other words, $\mathscr{A}$ is exterior if the canonical morphism $\widehat{\bigwedge}_{A^0} A^1 \to \mathscr{A}$ is a graded $\widehat{\otimes}$-algebra isomorphism. It is easy to see that a morphism $\varphi \colon \mathscr{A} \to \mathscr{B}$ of graded exterior $\widehat{\otimes}$-algebras is an isomorphism if and only if it is an isomorphism in degrees 0 and 1.

Example 1.1. Let $(V, \mathscr{O}_V)$ be a Stein space, and let $\mathscr{F}$ be a locally free sheaf of $\mathscr{O}_V$-modules. Set $A = \mathscr{O}(V)$ and $F = \mathscr{F}(V)$. We have an obvious sheaf-theoretic version of the antisymmetrization map (1):

$$\bigotimes_{\mathscr{O}_V}^n \mathscr{F} \xrightarrow{\mathbf{a}_{\mathscr{F}}} \bigotimes_{\mathscr{O}_V}^n \mathscr{F}.$$

By definition, $\mathscr{I}m\,\mathrm{a}_{\mathscr{F}} = \bigwedge^n_{\mathscr{O}_V} \mathscr{F}$. Since the functor Γ of global sections takes tensor products over $\mathscr{O}_V$ to projective tensor products over A (see above), we deduce that the morphism $\Gamma(V, \mathrm{a}_{\mathscr{F}})$ of Stein A-modules coincides with $\mathrm{a}_F\colon \widehat{\bigotimes}^n_A F \to \widehat{\bigotimes}^n_A F$. Since Γ is exact, we conclude that $\widehat{\bigwedge}^n_A F = \operatorname{Im}\mathrm{a}_F = \Gamma(V, \mathscr{I}m\,\mathrm{a}_{\mathscr{F}}) = \Gamma(V, \bigwedge^n_{\mathscr{O}_V} \mathscr{F})$. Thus we have an isomorphism of graded $\widehat{\otimes}$-algebras $\widehat{\bigwedge}_A F \cong \Gamma(V, \bigwedge_{\mathscr{O}_V} \mathscr{F})$.

By a *differential graded $\widehat{\otimes}$-algebra* (a *DG $\widehat{\otimes}$-algebra* for short) we mean a graded $\widehat{\otimes}$-algebra $\mathscr{A}$ together with a sequence $\{d^p\colon A^p \to A^{p+1} : p \in \mathbb{Z}_+\}$ of continuous linear maps such that $d^{p+1}d^p = 0$ for all p (so that $\mathscr{A}$ becomes a cochain complex), and that $d^{p+q}(ab) = d^p(a)b + (-1)^p a d^q(b)$ for each $a \in A^p$, $b \in A^q$. In particular, d^0 is a derivation of A^0 with values in A^1. We say that a DG $\widehat{\otimes}$-algebra is graded commutative (exterior, etc.) if it has this property when considered as a graded $\widehat{\otimes}$-algebra. Morphisms of DG $\widehat{\otimes}$-algebras are morphisms of graded $\widehat{\otimes}$-algebras commuting with differentials.

Let $\Omega^1 A$ be the module of Kähler differentials of a commutative $\widehat{\otimes}$-algebra A. Then the exterior algebra $\widehat{\bigwedge}_A(\Omega^1 A)$ has a unique structure of DG $\widehat{\otimes}$-algebra such that the mapping $d^0\colon A \to \Omega^1 A$ coincides with the universal derivation d_A (cf. [3, 36]). The resulting DG $\widehat{\otimes}$-algebra is denoted by $\Omega(A)$ and is called the *algebra of differential forms* of A. The cohomology groups of $\Omega(A)$ are called the *de Rham cohomology groups* of A and are denoted by $H^n_{\mathrm{DR}}(A)$.

The algebra of differential forms has the following universal property (cf. [36]): *For each graded commutative DG $\widehat{\otimes}$-algebra $\mathscr{B}$ and each $\widehat{\otimes}$-algebra morphism $\psi\colon A \to B^0$ there exists a unique DG $\widehat{\otimes}$-algebra morphism $\varphi\colon \Omega(A) \to \mathscr{B}$ such that $\varphi^0 = \psi$.* In particular, each morphism $\psi\colon A \to B$ of $\widehat{\otimes}$-algebras uniquely extends to a morphism $\psi_*\colon \Omega(A) \to \Omega(B)$ of DG $\widehat{\otimes}$-algebras. Thus the assignment $A \mapsto \Omega(A)$ is a functor from the category of commutative $\widehat{\otimes}$-algebras to the category of graded commutative DG $\widehat{\otimes}$-algebras.

PROPOSITION 1.3. *Let V be a Stein manifold. Then the topological cohomology groups $H^n_{\mathrm{top}}(V, \mathbb{C})$ coincide with the de Rham cohomology groups $H^n_{\mathrm{DR}}(\mathscr{O}(V))$ of the Fréchet algebra $\mathscr{O}(V)$.*

Proof. For each n denote by Ω^n_V the sheaf of holomorphic differential n-forms on V. By the Poincaré lemma and Cartan's Theorem B, the de Rham complex

$$0 \to \mathbb{C} \to \mathscr{O}_V \to \Omega^1_V \to \Omega^2_V \to \cdots$$

is an acyclic resolution of the constant sheaf $\mathbb{C}$. Therefore the cohomology groups of $\Omega(V) = \Gamma(V, \Omega^\bullet_V)$ coincide with the topological cohomology groups $H^n_{\mathrm{top}}(V, \mathbb{C})$. On the other hand, the embedding $\mathscr{O}(V) \to \Omega(V)$ uniquely extends to a DG $\widehat{\otimes}$-algebra morphism $\Omega(\mathscr{O}(V)) \to \Omega(V)$ that is an isomorphism in degrees 0 and 1 (see Lemma 1.1). Since both the algebras are exterior (see Example 1.1), we conclude that $\Omega(\mathscr{O}(V)) \to \Omega(V)$ is a DG $\widehat{\otimes}$-algebra isomorphism. The rest is clear. ∎

1.4. Lie algebra actions. Throughout the paper, by a *Lie algebra* we always mean a finite-dimensional complex Lie algebra.

Let $\mathfrak{g}$ be a Lie algebra and M a right $\mathfrak{g}$-module. For each $n \in \mathbb{Z}_+$ set $C_n(\mathfrak{g}, M) = M \otimes \bigwedge^n \mathfrak{g}$. The boundary mappings $d_n \colon C_{n+1}(\mathfrak{g}, M) \to C_n(\mathfrak{g}, M)$ are defined by

$$\begin{aligned} &d_n(m \otimes X_1 \wedge \cdots \wedge X_{n+1}) \\ &\quad = \sum_{i=1}^{n+1} (-1)^{i-1} m \cdot X_i \otimes X_1 \wedge \cdots \wedge \widehat{X}_i \wedge \cdots \wedge X_{n+1} \\ &\qquad + \sum_{1 \le i < j \le n+1} (-1)^{i+j} m \otimes [X_i, X_j] \wedge X_1 \wedge \cdots \wedge \widehat{X}_i \wedge \cdots \wedge \widehat{X}_j \wedge \cdots \wedge X_{n+1}. \end{aligned}$$

(Here, as usual, the notation $\widehat{X}_i$ indicates that X_i is omitted.) The spaces $C_n(\mathfrak{g}, M)$ together with the mappings d_n form a chain complex denoted by $C_\cdot(\mathfrak{g}, M)$. The homology groups of this complex are called the *homology groups of* $\mathfrak{g}$ *with coefficients in* M and are denoted by $H_n^{\mathrm{Lie}}(\mathfrak{g}, M)$.

Now let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. We consider $U(\mathfrak{g})$ as a right $\mathfrak{g}$-module with respect to the right regular representation given by $(a, X) \mapsto aX$. The corresponding chain complex $V_\cdot(\mathfrak{g}) = C_\cdot(\mathfrak{g}, U(\mathfrak{g}))$ augmented by the counit map $\varepsilon \colon U(\mathfrak{g}) \to \mathbb{C}$ is exact (see [6, Chap. XIII]). Since all the d_n's are morphisms of left $U(\mathfrak{g})$-modules in this case, it follows that $V_\cdot(\mathfrak{g})$ is a free resolution of the trivial $\mathfrak{g}$-module $\mathbb{C}$ in the category of left $U(\mathfrak{g})$-modules (the *Koszul resolution*). If M is a right $\mathfrak{g}$-module, then $C_\cdot(\mathfrak{g}, M)$ is isomorphic to the tensor product $M \otimes_{U(\mathfrak{g})} V_\cdot(\mathfrak{g})$. Therefore $H_n^{\mathrm{Lie}}(\mathfrak{g}, M) = \mathrm{Tor}_n^{U(\mathfrak{g})}(M, \mathbb{C})$ for each $n \in \mathbb{Z}_+$.

Dually, if M is a left $\mathfrak{g}$-module, then the space $C^n(\mathfrak{g}, M)$ of *n-cochains* is defined as $\mathrm{Hom}_{\mathbb{C}}(\bigwedge^n \mathfrak{g}, M)$. Thus n-cochains are just alternating multilinear maps of n variables from $\mathfrak{g}$ with values in M. The coboundary mappings $d^n \colon C^n(\mathfrak{g}, M) \to C^{n+1}(\mathfrak{g}, M)$ are defined by

$$\begin{aligned} &d^n f(X_1 \wedge \cdots \wedge X_{n+1}) \\ &\quad = \sum_{i=1}^{n+1} (-1)^{i-1} X_i \cdot f(X_1 \wedge \cdots \wedge \widehat{X}_i \wedge \cdots \wedge X_{n+1}) \\ &\qquad + \sum_{1 \le i < j \le n+1} (-1)^{i+j} f([X_i, X_j] \wedge X_1 \wedge \cdots \wedge \widehat{X}_i \wedge \cdots \wedge \widehat{X}_j \wedge \cdots \wedge X_{n+1}). \end{aligned}$$

The spaces $C^n(\mathfrak{g}, M)$ together with the mappings d^n form a cochain complex denoted by $C^\cdot(\mathfrak{g}, M)$ (the *Chevalley–Eilenberg complex*). The cohomology groups of this complex are called the *cohomology groups of* $\mathfrak{g}$ *with coefficients in* M and are denoted by $H^n_{\mathrm{Lie}}(\mathfrak{g}, M)$. As in the case of homology groups, we have $H^n_{\mathrm{Lie}}(\mathfrak{g}, M) = \mathrm{Ext}^n_{U(\mathfrak{g})}(\mathbb{C}, M)$ for each $n \in \mathbb{Z}_+$.

Remark 1.3. Recall that each right $\mathfrak{g}$-module M can also be viewed as a left $\mathfrak{g}$-module with respect to the action $X \cdot m = -m \cdot X$ ($m \in M$, $X \in \mathfrak{g}$), and *vice versa*, so that the categories of left $\mathfrak{g}$-modules and of right $\mathfrak{g}$-modules are isomorphic. Thus one can speak about the complex $C_\cdot(\mathfrak{g}, M)$ (resp. $C^\cdot(\mathfrak{g}, M)$) in the case where M is a left (resp. right) $\mathfrak{g}$-module.

By a *right $\mathfrak{g}$-$\widehat{\otimes}$-module* we mean a complete Hausdorff l.c.s. M together with the structure of a right $\mathfrak{g}$-module such that the map $M \to M$, $m \mapsto m \cdot X$, is continuous for

each $X \in \mathfrak{g}$. If we endow $\mathfrak{g}$ with the usual topology of a finite-dimensional vector space, then the above condition means precisely that the map $M \mathbin{\widehat{\otimes}} \mathfrak{g} \to M$, $m \otimes X \mapsto m \cdot X$, is continuous. Similarly, one can speak about left $\mathfrak{g}$-$\widehat{\otimes}$-modules. If M is a right (resp. left) $\mathfrak{g}$-$\widehat{\otimes}$-module, then the strong dual, M', becomes a left (resp. right) $\mathfrak{g}$-$\widehat{\otimes}$-module via the action $\langle m, X \cdot m' \rangle = \langle m \cdot X, m' \rangle$ (resp. $\langle m' \cdot X, m \rangle = \langle m', X \cdot m \rangle$) for $m \in M$, $m' \in M'$, $X \in \mathfrak{g}$.

Remark 1.4. If we endow $U(\mathfrak{g})$ with the finest locally convex topology, then each $\mathfrak{g}$-$\widehat{\otimes}$-module M becomes a topological $U(\mathfrak{g})$-module. Note, however, that M need not be an $U(\mathfrak{g})$-$\widehat{\otimes}$-module, i.e., the action $U(\mathfrak{g}) \times M \to M$ need not be jointly continuous.

If M is a right (resp. left) $\mathfrak{g}$-$\widehat{\otimes}$-module, then the obvious identifications $M \otimes \bigwedge^n \mathfrak{g} = M \mathbin{\widehat{\otimes}} \bigwedge^n \mathfrak{g}$ and $\mathrm{Hom}_{\mathbb{C}}(\bigwedge^n \mathfrak{g}, M) = \mathscr{L}(\bigwedge^n \mathfrak{g}, M)$ enable us to consider $C_\cdot(\mathfrak{g}, M)$ (resp. $C^\cdot(\mathfrak{g}, M)$) as a complex in $\mathbf{LCS}$. If M is a right (resp. left) $\mathfrak{g}$-$\widehat{\otimes}$-module, then the complex $C^\cdot(\mathfrak{g}, M')$ (resp. $C_\cdot(\mathfrak{g}, M')$) is isomorphic to the strong dual of $C_\cdot(\mathfrak{g}, M)$ (resp. $C^\cdot(\mathfrak{g}, M)$). This readily follows from the canonical isomorphisms $\mathscr{L}(\bigwedge^n \mathfrak{g}, M) \cong (\bigwedge^n \mathfrak{g})' \mathbin{\widehat{\otimes}} M$.

Let A be a $\widehat{\otimes}$-algebra together with the structure of a left $\mathfrak{g}$-$\widehat{\otimes}$-module. Suppose that for each $X \in \mathfrak{g}$ the map $A \to A$, $a \mapsto X \cdot a$, is a derivation. In this case we say that $\mathfrak{g}$ *acts on A by derivations*. The complex $C^\cdot(\mathfrak{g}, A)$ has then a structure of DG $\widehat{\otimes}$-algebra (cf. [14]). The multiplication on $C^\cdot(\mathfrak{g}, A)$ comes from the identification of $C^\cdot(\mathfrak{g}, A)$ with the tensor product of algebras $\bigwedge \mathfrak{g}^* \mathbin{\widehat{\otimes}} A$. In particular, if A is commutative, then $C^\cdot(\mathfrak{g}, A)$ is isomorphic (as a graded $\widehat{\otimes}$-algebra) to the exterior algebra $\widehat{\bigwedge}_A C^1(\mathfrak{g}, A)$.

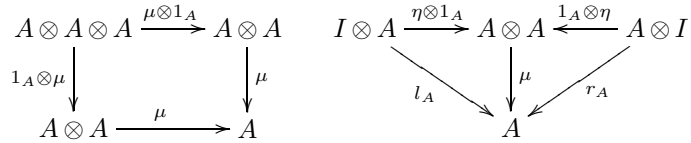
2. The inverse process for Hopf $\widehat{\otimes}$-algebras

In this section we describe a version of the Cartan–Eilenberg "inverse process" ([6, Chap. X]) adapted to the Hopf $\widehat{\otimes}$-algebra case. Originally, Cartan and Eilenberg applied the inverse process to the study of homological dimensions of group algebras and universal enveloping algebras. Subsequently some generalizations were obtained for cocommutative [33] and commutative [38] Hopf algebras. Though we believe that the algebraic versions of the results below are known, we could not find them in the literature in a form suitable for our purposes. That is why we give complete proofs.

For convenience of the reader, we recall some algebraic definitions (see [47] for details). Let $\mathcal{C}$ be a monoidal category, i.e., a category equipped with a bifunctor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a neutral object I, and natural isomorphisms

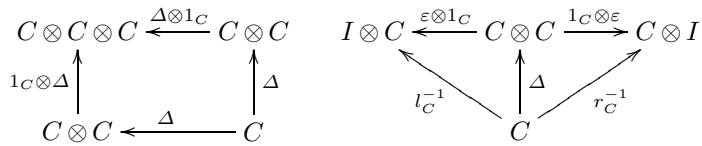
$$a_{X,Y,Z}\colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z), \qquad l_X\colon I \otimes X \to X, \qquad r_X\colon X \otimes I \to X$$

satisfying certain coherence (constraint) conditions (see, e.g., [46]). Without loss of generality (by MacLane's coherence theorem [46, Theorem 15.1]), we may assume that $\mathcal{C}$ is *strict*, so that all associativity and unit isomorphisms are identities. An *algebra* in $\mathcal{C}$ is an object A together with morphisms $\mu\colon A \otimes A \to A$ (multiplication) and $\eta\colon I \to A$ (unit) such that the diagrams



are commutative. For example, if $\mathcal{C} = \mathbf{LCS}$ is the category of complete locally convex spaces over $\mathbb{C}$, and $\otimes = \widehat{\otimes}$ is the bifunctor of the completed projective tensor product, then we obtain the definition of $\widehat{\otimes}$-algebra given in Subsection 1.1. If (A, μ_A, η_A) and (B, μ_B, η_B) are algebras in $\mathcal{C}$, then a morphism $\varphi\colon A \to B$ is an *algebra homomorphism* if $\varphi\mu_A = \mu_B(\varphi \otimes \varphi)$ and $\varphi\eta_A = \eta_B$.

Dually, a *coalgebra* in $\mathcal{C}$ is an object C together with morphisms $\Delta\colon C \to C \otimes C$ (comultiplication) and $\varepsilon\colon C \to I$ (counit) such that the diagrams



are commutative.

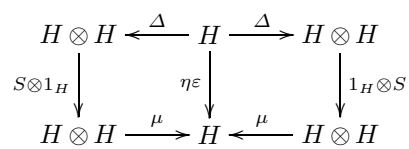
The monoidal category $\mathcal{C}$ is *braided* if it is equipped with a natural isomorphism $c_{X,Y}\colon X \otimes Y \to Y \otimes X$ satisfying the relations

$$(c_{X,Z} \otimes 1_Y)(1_X \otimes c_{Y,Z}) = c_{X \otimes Y,Z}, \qquad (1_Y \otimes c_{X,Z})(c_{X,Y} \otimes 1_Z) = c_{X,Y \otimes Z}.$$

In this case, the tensor product of any two algebras A, B in $\mathcal{C}$ is an algebra with multiplication and unit defined as the compositions

$$A \otimes B \otimes A \otimes B \xrightarrow{1_A \otimes c_{B,A} \otimes 1_B} A \otimes A \otimes B \otimes B \xrightarrow{\mu_A \otimes \mu_B} A \otimes B,$$
$$I \xrightarrow{r_I = l_I} I \otimes I \xrightarrow{\eta_A \otimes \eta_B} A \otimes B.$$

A *bialgebra* in $\mathcal{C}$ is an object H equipped with the algebra and the coalgebra structures such that $\Delta\colon H \to H \otimes H$ and $\varepsilon\colon H \to I$ are algebra homomorphisms. Finally, a *Hopf algebra* in $\mathcal{C}$ is a bialgebra H together with a morphism $S\colon H \to H$ (antipode) such that the diagram



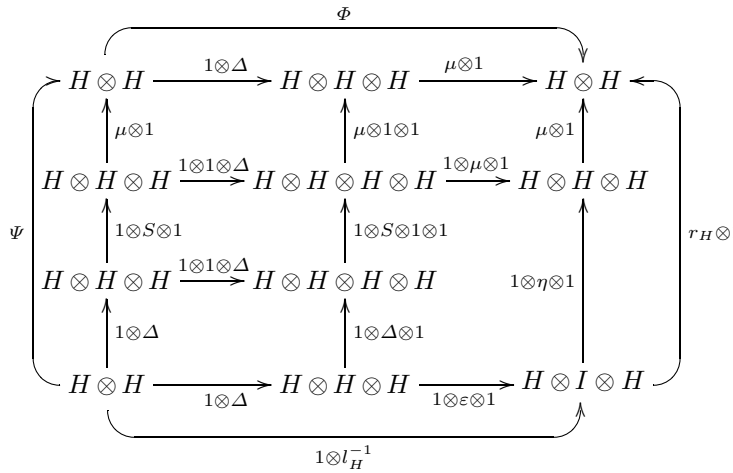
is commutative.

LEMMA 2.1. *Let H be a Hopf algebra in a braided monoidal category $\mathcal{C}$, and let $\Phi, \Psi\colon H \otimes H \to H \otimes H$ be given by*

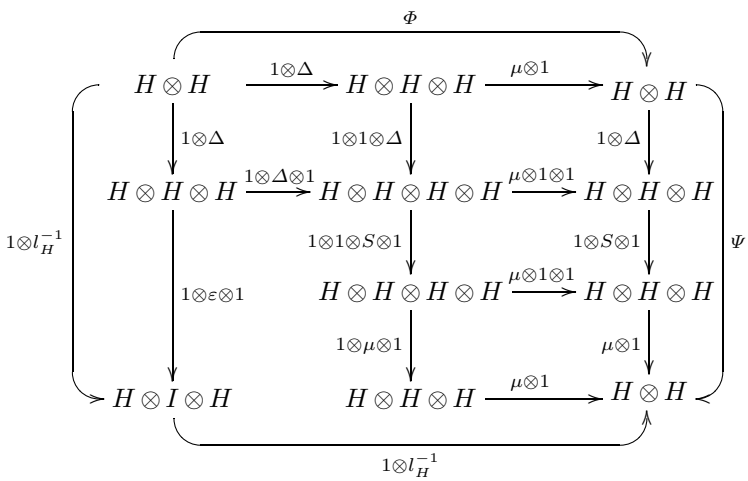
$$\Phi = (\mu \otimes 1_H)(1_H \otimes \Delta), \qquad \Psi = (\mu \otimes 1_H)(1_H \otimes S \otimes 1_H)(1_H \otimes \Delta).$$

Then $\Phi = \Psi^{-1}$.

Proof. The relation $\Phi\Psi = 1_{H\otimes H}$ follows from the commutative diagram

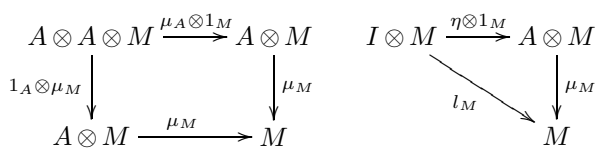


Similarly, the commutative diagram



shows that $\Psi\Phi = 1_{H\otimes H}$. ∎

Let A be an algebra in $\mathcal{C}$. Recall that a *left A-module* is an object M together with a morphism $\mu_M\colon A\otimes M\to M$ such that the diagrams



are commutative. Again, in the case $(\mathcal{C},\otimes) = (\mathbf{LCS},\widehat{\otimes})$ we obtain the definition of $\widehat{\otimes}$-module given in Subsection 1.1. If (M,μ_M) and (N,μ_N) are left A-modules, then a morphism $\varphi\colon M\to N$ in $\mathcal{C}$ is an *A-module morphism* if $\varphi\mu_M = \mu_N(1_A\otimes\varphi)$. Right A-modules and their morphisms are defined similarly.

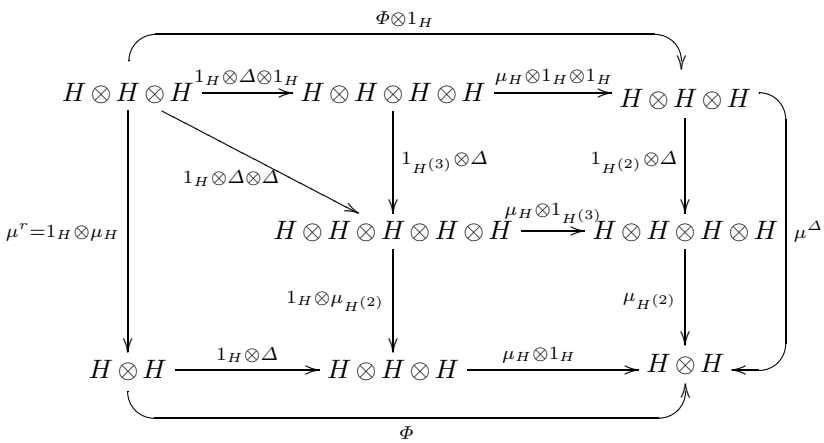
Now let $\mathcal{C}$ be a braided monoidal category, and let H be a Hopf algebra in $\mathcal{C}$. Then $H \otimes H$ has two natural structures of right H-module. The first one is given by the action of H on the right factor, and the second one arises from the algebra homomorphism $\Delta\colon H \to H \otimes H$. Thus we obtain the right H-modules $(H \otimes H, \mu^r)$ and $(H \otimes H, \mu^\Delta)$ with the actions $\mu^r, \mu^\Delta\colon (H \otimes H) \otimes H \to (H \otimes H)$ given, respectively, by

$$\mu^r\colon (H \otimes H) \otimes H \xrightarrow{\sim} H \otimes (H \otimes H) \xrightarrow{1_H \otimes \mu} H \otimes H,$$
$$\mu^\Delta\colon (H \otimes H) \otimes H \xrightarrow{1_{H\otimes H} \otimes \Delta} (H \otimes H) \otimes (H \otimes H) \xrightarrow{\mu_{H\otimes H}} H \otimes H.$$

To simplify notation, we shall often write $H^{(n)}$ to denote the n-fold tensor power $H \otimes \cdots \otimes H$.

LEMMA 2.2. *The morphism $\Phi\colon (H \otimes H, \mu^r) \to (H \otimes H, \mu^\Delta)$ defined in Lemma 2.1 is an isomorphism of right H-modules.*

Proof. By Lemma 2.1, Φ is an isomorphism in $\mathcal{C}$. To prove that Φ is a right H-module morphism, it is enough to consider the following commutative diagram:



Let H^{op} denote the algebra opposite to H, i.e., $\mu_{H^{\mathrm{op}}} = \mu_H c_{H,H}$, and let $H^e = H \otimes H^{\mathrm{op}}$. Then $S\colon H \to H^{\mathrm{op}}$ and $E = (1_H \otimes S)\Delta\colon H \to H^e$ are algebra homomorphisms [47]. Hence H^e becomes a right H-module via E. We denote this module by H^e_E.

LEMMA 2.3. *If H has invertible antipode, then the right H-modules $(H \otimes H, \mu^r)$ and H^e_E are isomorphic.*

Proof. Since S is an isomorphism in $\mathcal{C}$, it follows that $1_H \otimes S\colon H \otimes H \to H \otimes H^{\mathrm{op}} = H^e$ is an algebra isomorphism. Hence $H^e_E \cong (H \otimes H, \mu^\Delta)$ as right H-modules. Now it remains to apply Lemma 2.2. ∎

From now on, let $(\mathcal{C}, \otimes) = (\mathbf{LCS}, \widehat{\otimes})$ be the category of complete locally convex spaces over $\mathbb{C}$. By a *Hopf $\widehat{\otimes}$-algebra* we mean a Hopf algebra in **LCS** (cf. also [43, 1, 60]). Given a Hopf $\widehat{\otimes}$-algebra H, we consider $\mathbb{C}$ as a left H-module via the counit map $\varepsilon\colon H \to \mathbb{C}$.

LEMMA 2.4. *Let H be a Hopf $\widehat{\otimes}$-algebra with invertible antipode. There exists an isomorphism of left H^e-$\widehat{\otimes}$-modules*

$$\varphi\colon H_E^e \mathbin{\widehat{\otimes}_H} \mathbb{C} \to H, \qquad u \otimes 1 \mapsto \mu(u). \tag{3}$$

Proof. Consider the bilinear map

$$R\colon H^e \times \mathbb{C} \to H, \qquad (u, \lambda) \mapsto \lambda\mu(u).$$

To prove that (3) is a well-defined linear map, we have to show that

$$R\big((a \otimes b) \cdot c, \lambda\big) = R(a \otimes b, c \cdot \lambda) \tag{4}$$

for each $a, b, c \in H$ and each $\lambda \in \mathbb{C}$. To this end, note first that

$$\mu E = \mu(1_H \otimes S)\Delta = \eta\varepsilon. \tag{5}$$

Since $\mu\colon H^e \to H$ is a left H^e-module morphism, we see that

$$R((a \otimes b) \cdot c, \lambda) = \lambda\mu((a \otimes b)E(c)) = \lambda a\mu\big(E(c)\big)b = \varepsilon(c)\lambda ab. \tag{6}$$

On the other hand,

$$R(a \otimes b, c \cdot \lambda) = R(a \otimes b, \varepsilon(c)\lambda) = \varepsilon(c)\lambda ab. \tag{7}$$

Comparing (6) and (7), we obtain (4), as required. Hence φ is a well-defined, continuous linear map. Evidently, φ is also a left H^e-module morphism.

To construct the inverse of φ, consider the map

$$\psi\colon H \to H_E^e \mathbin{\widehat{\otimes}_H} \mathbb{C}, \qquad a \mapsto (a \otimes 1) \otimes 1.$$

Clearly, $\varphi\psi = 1_H$. Thus it remains to prove that $\psi\varphi = 1_{H_E^e \mathbin{\widehat{\otimes}_H} \mathbb{C}}$, which is equivalent to

$$u \otimes 1 = (\mu(u) \otimes 1) \otimes 1 \tag{8}$$

for each $u \in H^e$.

Take the map $\Phi\colon H \mathbin{\widehat{\otimes}} H \to H \mathbin{\widehat{\otimes}} H$ defined in Lemma 2.1, and set

$$\Phi' = (1_H \otimes S)\Phi\colon (H \mathbin{\widehat{\otimes}} H, \mu^r) \to H_E^e.$$

By Lemmas 2.2 and 2.3, Φ' is an isomorphism of right H-$\widehat{\otimes}$-modules. We have

$$\begin{aligned}\Phi'(a \otimes 1) &= (1_H \otimes S)(\mu \otimes 1_H)(1_H \otimes \Delta)(a \otimes 1)\\ &= (1_H \otimes S)(\mu \otimes 1_H)(a \otimes 1 \otimes 1) = (1_H \otimes S)(a \otimes 1) = a \otimes 1\end{aligned}$$

and hence

$$\Phi'(a \otimes b) = \Phi'(a \otimes 1 \cdot b) = \Phi'(a \otimes 1)E(b) = (a \otimes 1)E(b).$$

Since Φ' is bijective, it is enough to check (8) with $u = (a \otimes 1)E(b)$. Using (5) and the fact that μ is a left H^e-module morphism, we see that

$$\mu(u) = \mu((a \otimes 1)E(b)) = a\mu(E(b)) = \varepsilon(b)a.$$

Hence the right-hand side of (8) is

$$(\mu(u) \otimes 1) \otimes 1 = \varepsilon(b)(a \otimes 1) \otimes 1,$$

while the left-hand side of (8) is

$$u \otimes 1 = (a \otimes 1)E(b) \otimes 1 = (a \otimes 1) \otimes b \cdot 1 = \varepsilon(b)(a \otimes 1) \otimes 1.$$

Therefore (8) is satisfied, and so $\psi = \varphi^{-1}$. Hence φ is an isomorphism, as required. ∎

THEOREM 2.5. *Let H be a Hopf $\widehat{\otimes}$-algebra with invertible antipode, and let*

$$0 \leftarrow \mathbb{C} \leftarrow P_\bullet$$

*be a projective resolution of $\mathbb{C}$ in H-**mod**. Then the tensor product complex*

$$0 \leftarrow H \xleftarrow{\sim} H_E^e \mathbin{\widehat{\otimes}}_H \mathbb{C} \leftarrow H_E^e \mathbin{\widehat{\otimes}}_H P_\bullet \tag{9}$$

is a projective bimodule resolution of H.

Proof. By Lemma 2.3, H_E^e is a free right H-$\widehat{\otimes}$-module. Hence the augmented complex (9) is admissible. To complete the proof, it remains to apply Lemma 2.4.

Let H be a Hopf $\widehat{\otimes}$-algebra and M an H-$\widehat{\otimes}$-bimodule (i.e., a left H^e-module). We may consider M as a left H-$\widehat{\otimes}$-module via $E\colon H \to H^e$. Similarly, by considering M as a right H^e-$\widehat{\otimes}$-module, we obtain a right H-$\widehat{\otimes}$-module structure on M. The resulting left (resp. right) H-$\widehat{\otimes}$-module will be denoted by ${}_EM$ (resp. M_E).

COROLLARY 2.6. *Let H be a Hopf $\widehat{\otimes}$-algebra with invertible antipode. Then for each $M \in H$-**mod**-H there exist natural isomorphisms*

$$\mathscr{H}^n(H, M) \cong \operatorname{Ext}_H^n(\mathbb{C}, {}_EM), \qquad \mathscr{H}_n(H, M) \cong \operatorname{Tor}_n^H(M_E, \mathbb{C}).$$

Proof. Let $P_\bullet$ be a projective resolution of $\mathbb{C}$ in H-**mod**. In view of Theorem 2.5, we have

$$\mathscr{H}^n(H, M) = \mathrm{H}^n\big({}_{H^e}\mathbf{h}(H_E^e \mathbin{\widehat{\otimes}}_H P_\bullet, M)\big) \cong \mathrm{H}^n\big({}_H\mathbf{h}(P_\bullet, {}_EM)\big) = \operatorname{Ext}_H^n(\mathbb{C}, {}_EM).$$

Similarly,

$$\mathscr{H}_n(H, M) = \mathrm{H}_n(M \mathbin{\widehat{\otimes}}_{H^e} H_E^e \mathbin{\widehat{\otimes}}_H P_\bullet) \cong \mathrm{H}_n(M_E \mathbin{\widehat{\otimes}}_H P_\bullet) = \operatorname{Tor}_n^H(M_E, \mathbb{C}). \ \blacksquare$$

COROLLARY 2.7. *Let H be a Hopf $\widehat{\otimes}$-algebra with invertible antipode. Then* $\operatorname{dh}_H \mathbb{C} = \operatorname{dg} H = \operatorname{db} H$.

The following two examples are "continuous versions" of Cartan–Eilenberg's result on the Hochschild cohomology of group algebras ([6, Chap. X, §6]).

EXAMPLE 2.1. Let G be a discrete group. The Banach algebra $\ell^1(G)$ has a canonical Hopf $\widehat{\otimes}$-algebra structure uniquely determined by

$$\Delta(\delta_g) = \delta_g \otimes \delta_g, \qquad \varepsilon(f) = \sum_{g \in G} f(g), \qquad Sf(g) = f(g^{-1}).$$

(Here δ_g denotes the function which is 1 at $g \in G$ and 0 elsewhere.) Using the bar resolution of $\mathbb{C}$ in $\ell^1(G)$-**mod** (see [26]), it is easy to check that $\operatorname{Ext}_{\ell^1(G)}^n(\mathbb{C}, X)$ is isomorphic to $H_{\mathrm{b}}^n(G, X)$, the nth bounded cohomology group of G with coefficients in X ([56]; cf. also [30]). Thus we obtain the following

COROLLARY 2.8. *Let G be a discrete group and M a Banach $\ell^1(G)$-bimodule. Denote by ${}_EM$ the left G-module obtained from M by setting $gm = \delta_g \cdot m \cdot \delta_{g^{-1}}$ $(g \in G,\ m \in M)$. Then there exist canonical isomorphisms*

$$\mathscr{H}^n(\ell^1(G), M) \cong H_{\mathrm{b}}^n(G, {}_EM).$$

EXAMPLE 2.2. Let G be a real Lie group. The convolution algebra $\mathscr{E}'(G)$ of compactly supported distributions on G is a Hopf $\widehat{\otimes}$-algebra in a natural way (see, e.g., [42, 1]; cf.

also Section 8 below). Let X be a left $\mathscr{E}'(G)$-$\widehat{\otimes}$-module. As in the previous example, it can easily be checked that $\operatorname{Ext}^n_{\mathscr{E}'(G)}(\mathbb{C}, X)$ is isomorphic to $H^n_{\mathrm{c}}(G, X)$, the nth continuous (or, equivalently, differentiable) cohomology group of G with coefficients in X (cf. [24, Chap. III, Prop. 1.5]). Thus we obtain the following

COROLLARY 2.9. *Let G be a real Lie group and M an $\mathscr{E}'(G)$-$\widehat{\otimes}$-bimodule. Denote by ${}_EM$ the left G-module obtained from M by setting $gm = \delta_g \cdot m \cdot \delta_{g^{-1}}$ $(g \in G,\ m \in M)$. Then there exist canonical isomorphisms*

$$\mathscr{H}^n(\mathscr{E}'(G), M) \cong H^n_{\mathrm{c}}(G, {}_EM).$$

We end this section with an application of the above results to left amenability in the sense of Lau [37]. Recall that a Banach algebra A is said to be *amenable* [30] if $\mathscr{H}^1(A, X^*) = 0$ for each Banach A-bimodule X, i.e., if every derivation from A to X^* is inner. Suppose A is endowed with an augmentation ε_A (i.e., a continuous homomorphism $A \to \mathbb{C}$). Then A is said to be *left amenable* [37] if for each Banach A-bimodule X such that $a \cdot x = \varepsilon_A(a)x$ for all $a \in A,\ x \in X$, every derivation from A to X^* is inner.

In the next lemma, we consider $\mathbb{C}$ as a left Banach A-module via $\varepsilon_A \colon A \to \mathbb{C}$.

LEMMA 2.10. *Let A be an augmented Banach algebra. Then A is left amenable if and only if $\operatorname{Ext}^1_A(\mathbb{C}, Y^*) = 0$ for each right Banach A-module Y.*

Proof. Obviously, the A-bimodules in the definition of left amenability are precisely those of the form $X = \mathbb{C}\,\widehat{\otimes} Y$ where $Y \in \mathbf{mod}\text{-}A$. Hence $X^* \cong \mathscr{L}(\mathbb{C}, Y^*)$ (see [26, II.5.21]), and so $\mathscr{H}^1(A, X^*) \cong \operatorname{Ext}^1_A(\mathbb{C}, Y^*)$ (see [26, III.4.12]). The rest is clear. ∎

PROPOSITION 2.11. *Let H be a Banach Hopf algebra (i.e., a Hopf $\widehat{\otimes}$-algebra whose underlying locally convex space is a Banach space) with invertible antipode. Then H is left amenable if and only if H is amenable.*

Proof. The "if" part is clear. Conversely, assume H is left amenable, and let X be a Banach H-bimodule. By Corollary 2.6, we have $\mathscr{H}^1(H, X^*) \cong \operatorname{Ext}^1_H(\mathbb{C}, {}_E(X^*))$. On the other hand, it is immediate that ${}_E(X^*) = (X_E)^*$. Now the result follows from the previous lemma. ∎

3. Localizations and weak localizations

Let A be a Fréchet algebra, $X \in \mathbf{mod}\text{-}A$, and $Y \in A\text{-}\mathbf{mod}$. Then X and Y are said to be *transversal* over A (notation: $X \perp_A Y$) if $\operatorname{Tor}^A_0(X, Y)$ is Hausdorff, and $\operatorname{Tor}^A_n(X, Y) = 0$ for all $n > 0$. This notion was introduced in [70] and has proved to be extremely useful in complex analytic geometry and operator theory [32, 70, 15, 10]. We shall need a somewhat stronger condition of transversality type.

PROPOSITION 3.1. *Let A be a $\widehat{\otimes}$-algebra, $X \in \mathbf{mod}\text{-}A$, and $Y \in A\text{-}\mathbf{mod}$. The following conditions are equivalent*:

(i) *There exists a projective resolution*

$$0 \leftarrow X \leftarrow P_\bullet \tag{10}$$

of X in **mod**-A *such that the tensored complex*

$$0 \leftarrow X \mathbin{\widehat{\otimes}}_A Y \leftarrow P_\bullet \mathbin{\widehat{\otimes}}_A Y \tag{11}$$

is admissible.

(i)′ *For each projective resolution* (10) *of X in* **mod**-A *the complex* (11) *is admissible.*

(ii) *There exists a projective resolution*

$$0 \leftarrow Y \leftarrow Q_\bullet \tag{12}$$

*of Y in A-***mod** *such that the tensored complex*

$$0 \leftarrow X \mathbin{\widehat{\otimes}}_A Y \leftarrow X \mathbin{\widehat{\otimes}}_A Q_\bullet \tag{13}$$

is admissible.

(ii)′ *For each projective resolution* (12) *of Y in A-***mod** *the complex* (13) *is admissible.*

Proof. The equivalences (i)⇔(i)′ and (ii)⇔(ii)′ readily follow from the fact that any two projective resolutions of a $\widehat{\otimes}$-module are homotopy equivalent (see [26]).

Let us prove that (i)⇔(ii). Choose a projective resolution

$$0 \leftarrow A \leftarrow L_\bullet \tag{14}$$

of A in A-**mod**-A. Then the complexes

$$0 \leftarrow X \leftarrow X \mathbin{\widehat{\otimes}}_A L_\bullet, \quad 0 \leftarrow Y \leftarrow L_\bullet \mathbin{\widehat{\otimes}}_A Y$$

are projective resolutions of $X \in$ **mod**-A and $Y \in A$-**mod**, respectively. Since (i)⇔(i)′ and (ii)⇔(ii)′, we see that both (i) and (ii) are equivalent to the admissibility of the complex

$$0 \leftarrow X \mathbin{\widehat{\otimes}}_A Y \leftarrow X \mathbin{\widehat{\otimes}}_A L_\bullet \mathbin{\widehat{\otimes}}_A Y.$$

Therefore (i)⇔(ii). ∎

Definition 3.1. We say that $X \in$ **mod**-A and $Y \in A$-**mod** are *strongly transversal* over A if they satisfy the conditions of Proposition 3.1. In this case, we write $X \underline{\perp}_A Y$.

Remark 3.1. Suppose that A is a Fréchet algebra. If we require that (11) or (13) be only exact (but not necessarily admissible), then we come to the usual definition of transversality (see the beginning of this section).

Proposition 3.2. *Let $\theta\colon A \to B$ be a homomorphism of $\widehat{\otimes}$-algebras. Suppose that the map*

$$B \mathbin{\widehat{\otimes}}_A B \to B, \qquad b_1 \otimes b_2 \mapsto b_1 b_2, \tag{15}$$

is a topological isomorphism. Then the following conditions are equivalent:

(i) $B \underline{\perp}_A B$;
(ii) $B \underline{\perp}_A M$ *for each* $M \in B$-**mod**;
(iii) $M \underline{\perp}_A B$ *for each* $M \in$ **mod**-B;
(iv) $B^e \underline{\perp}_{A^e} A$.

Proof. (ii)⇒(i) and (iii)⇒(i): are clear. To prove the remaining implications, take a projective bimodule resolution (14) of A in A-**mod**-A, and note that

$$0 \leftarrow B \leftarrow L_\bullet \mathbin{\widehat{\otimes}}_A B \tag{16}$$

is a projective resolution of $B \in A$-**mod**.

(i)⇒(iv). If (i) holds, then the complex

$$0 \leftarrow B \mathbin{\widehat{\otimes}}_A B \leftarrow B \mathbin{\widehat{\otimes}}_A L_\bullet \mathbin{\widehat{\otimes}}_A B \tag{17}$$

is admissible. On the other hand, the latter complex is isomorphic to

$$0 \leftarrow B^e \mathbin{\widehat{\otimes}}_{A^e} A \leftarrow B^e \mathbin{\widehat{\otimes}}_{A^e} L_\bullet,$$

and we obtain (iv).

(iv)⇒(iii). If (iv) holds, then the complex (17) is admissible. Since $B \mathbin{\widehat{\otimes}}_A B \cong B$ is projective in B-**mod**, we see that (17) splits in B-**mod**. Hence $M \mathbin{\widehat{\otimes}}_B$ (17) is admissible. On the other hand, $M \mathbin{\widehat{\otimes}}_B$ (17) is isomorphic to $M \mathbin{\widehat{\otimes}}_A$ (16), and we obtain (iii).

The implication (iv)⟹(ii) is proved similarly. ∎

REMARK 3.2. It is easy to see that (i)⇔(iv) without the additional assumption that (15) is an isomorphism.

The following basic notion was introduced by Taylor [75]; cf. also [18] for a purely algebraic version.

DEFINITION 3.2. A homomorphism $\theta\colon A \to B$ of $\widehat{\otimes}$-algebras is a *localization* ([1]) if it satisfies the conditions of Proposition 3.2. In this case, we say (following [54]) that B is *stably flat* over A.

REMARK 3.3. Using condition (iv) of Proposition 3.2, we see that $\theta\colon A \to B$ *is a localization if and only if the functor* $B \mathbin{\widehat{\otimes}}_A (\,\cdot\,) \mathbin{\widehat{\otimes}}_A B\colon A$-**mod**-$A \to B$-**mod**-$B$ *takes some* (= *every*) *projective bimodule resolution of* A *to a projective bimodule resolution of* B. This is exactly the definition given by Taylor [75]. It can be shown that θ is a localization if and only if the (total) derived functor of $\theta^* : B$-**mod** $\to A$-**mod** is a fully faithful functor between the bounded derived categories of B-**mod** and A-**mod**, respectively. In the purely algebraic context, this was proved in [18]. For the case of topological (respectively, bornological) algebras, see [65] (respectively, [51]).

PROPOSITION 3.3. *Suppose that* $\theta\colon A \to B$ *is a localization. Then for each* $M \in B$-**mod** *the canonical map* $B \mathbin{\widehat{\otimes}}_A M \to M$, $b \otimes x \mapsto b \cdot x$, *is an isomorphism.*

Proof. Apply the functor $(\,\cdot\,) \mathbin{\widehat{\otimes}}_B M$ to (15). ∎

A useful property of localizations is that they "do not change homological relations between modules". In particular, if $A \to B$ is a localization, then $\mathscr{H}^p(B, M) = \mathscr{H}^p(A, M)$ and $\mathscr{H}_p(B, M) = \mathscr{H}_p(A, M)$ for each B-$\widehat{\otimes}$-bimodule M (see [75, Prop. 1.4 and 1.7]). The next proposition is a combination of this fact with the Cartan–Eilenberg inverse process [6, XIII.5.1].

([1]) In Taylor's paper [75], such homomorphisms are called *absolute localizations*, whereas the term "localization" is used for a somewhat wider class of homomorphisms.

PROPOSITION 3.4. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, and let $U(\mathfrak{g})$ be its universal enveloping algebra endowed with the finest locally convex topology. Suppose that $\theta\colon U(\mathfrak{g}) \to B$ is a localization. For each $M \in B$-**mod**-B denote by ${}_EM$ (resp. M_E) the left (resp., right) $\mathfrak{g}$-module obtained from M by setting $X \cdot m = \theta(X) \cdot m - m \cdot \theta(X)$ (resp., $m \cdot X = m \cdot \theta(X) - \theta(X) \cdot m$) for $X \in \mathfrak{g}$, $m \in M$. Then there exist vector space isomorphisms*

$$\mathscr{H}^p(B, M) \cong H^p_{\mathrm{Lie}}(\mathfrak{g}, {}_EM), \quad \mathscr{H}_p(B, M) \cong H_p^{\mathrm{Lie}}(\mathfrak{g}, M_E) \quad (p \in \mathbb{Z}).$$

For later reference, we note the following

PROPOSITION 3.5 ([75]). *Let $A \xrightarrow{\theta} B \xrightarrow{\lambda} C$ be $\widehat{\otimes}$-algebra homomorphisms. Suppose θ is a localization. Then λ is a localization if and only if $\lambda\theta$ is a localization.*

By an *augmented $\widehat{\otimes}$-algebra* we mean a $\widehat{\otimes}$-algebra A together with a homomorphism $\varepsilon_A\colon A \to \mathbb{C}$. Homomorphisms of augmented $\widehat{\otimes}$-algebras are defined in an obvious way. Given an augmented $\widehat{\otimes}$-algebra A, we consider $\mathbb{C}$ as an A-module via ε_A.

DEFINITION 3.3. A homomorphism $\theta\colon A \to B$ of augmented $\widehat{\otimes}$-algebras is a *weak localization* if $B \perp_A \mathbb{C}$ and if the map

$$B \mathbin{\widehat{\otimes}_A} \mathbb{C} \to \mathbb{C}, \quad b \otimes \lambda \mapsto \varepsilon(b)\lambda, \tag{18}$$

is a topological isomorphism.

Setting $M = \mathbb{C}$ in Proposition 3.3, we get the following.

PROPOSITION 3.6. *Each localization of augmented $\widehat{\otimes}$-algebras is a weak localization.*

In the case of Hopf $\widehat{\otimes}$-algebras with invertible antipodes, the converse is also true. To see this, let us first observe that if $\theta\colon U \to H$ is a homomorphism of Hopf $\widehat{\otimes}$-algebras, then the homomorphisms $E_H\theta\colon U \to H^e$ and $(\theta \otimes \theta)E_U\colon U \to H^e$ coincide. Indeed,

$$\begin{aligned}(\theta \otimes \theta)E_U &= (\theta \otimes \theta)(1_U \otimes S_U)\Delta_U = (\theta \otimes \theta S_U)\Delta_U = (\theta \otimes S_H\theta)\Delta_U \\ &= (1_H \otimes S_H)(\theta \otimes \theta)\Delta_U = (1_H \otimes S_H)\Delta_H\theta = E_H\theta.\end{aligned}$$

Hence any of the above homomorphisms can be used to make H^e into a right U-$\widehat{\otimes}$-module. It also follows from the above that the canonical isomorphisms

$$\begin{aligned} H^e_{E_H} \mathbin{\widehat{\otimes}_H} H_\theta &\to H^e, \quad x \otimes h \mapsto xE_H(h), \\ H^e_{\theta\otimes\theta} \mathbin{\widehat{\otimes}_{U^e}} U^e_{E_U} &\to H^e, \quad x \otimes w \mapsto x(\theta \otimes \theta)(w), \end{aligned} \tag{19}$$

are isomorphisms in **mod**-U.

PROPOSITION 3.7. *Let $\theta\colon U \to H$ be a homomorphism of Hopf $\widehat{\otimes}$-algebras with invertible antipodes. Then θ is a localization if and only if it is a weak localization.*

Proof. The "only if" part readily follows from Proposition 3.6. If θ is a weak localization, then the map $H \mathbin{\widehat{\otimes}_U} \mathbb{C} \to \mathbb{C}$, $h \otimes \lambda \mapsto \varepsilon(h)\lambda$, is an isomorphism. Combining this fact with Lemma 2.4 and (19), we obtain a chain of isomorphisms

$$\begin{aligned} H \mathbin{\widehat{\otimes}_U} H &\xrightarrow{\sim} H \mathbin{\widehat{\otimes}_U} U \mathbin{\widehat{\otimes}_U} H \xrightarrow{\sim} H^e_{\theta\otimes\theta} \mathbin{\widehat{\otimes}_{U^e}} U \xrightarrow{\sim} H^e_{\theta\otimes\theta} \mathbin{\widehat{\otimes}_{U^e}} U^e_{E_U} \mathbin{\widehat{\otimes}_U} \mathbb{C} \\ &\xrightarrow{\sim} H^e \mathbin{\widehat{\otimes}_U} \mathbb{C} \xrightarrow{\sim} H^e_{E_H} \mathbin{\widehat{\otimes}_H} H \mathbin{\widehat{\otimes}_U} \mathbb{C} \xrightarrow{\sim} H^e_{E_H} \mathbin{\widehat{\otimes}_H} \mathbb{C} \xrightarrow{\sim} H. \end{aligned}$$

It is easy to check that the composition of the above isomorphisms takes each $h_1 \otimes h_2 \in H \mathbin{\widehat{\otimes}}_U H$ to $h_1 h_2 \in H$. Thus we have shown that the canonical map $H \mathbin{\widehat{\otimes}}_U H \to H$ is an isomorphism.

Now let $P_\bullet$ be a projective resolution of $\mathbb{C}$ in U-**mod**, and let $\overline{P}_\bullet$ denote the augmented complex $P_\bullet \to \mathbb{C} \to 0$. By Theorem 2.5, the complex $Q_\bullet = U^e_{E_U} \mathbin{\widehat{\otimes}}_U P_\bullet$ is a projective bimodule resolution of U. In order to prove that θ is a localization, it remains to show that the augmented tensor product complex $H \mathbin{\widehat{\otimes}}_U \overline{Q}_\bullet \mathbin{\widehat{\otimes}}_U H = H^e_{\theta\otimes\theta} \mathbin{\widehat{\otimes}}_{U^e} \overline{Q}_\bullet$ is admissible.

Since θ is a weak localization, we see that $L_\bullet = H \mathbin{\widehat{\otimes}}_U P_\bullet$ is a projective resolution of $H \mathbin{\widehat{\otimes}}_U \mathbb{C} \cong \mathbb{C}$ in H-**mod**. Using again Theorem 2.5, we conclude that $H^e_{E_H} \mathbin{\widehat{\otimes}}_H L_\bullet$ is a projective bimodule resolution of H. In particular, the augmented complex $H^e_{E_H} \mathbin{\widehat{\otimes}}_H \overline{L}_\bullet$ is admissible. Now it follows from (19) that

$$H^e_{E_H} \mathbin{\widehat{\otimes}}_H \overline{L}_\bullet \cong H^e_{E_H} \mathbin{\widehat{\otimes}}_H H \mathbin{\widehat{\otimes}}_U \overline{P}_\bullet \cong H^e \mathbin{\widehat{\otimes}}_U \overline{P}_\bullet \cong H^e_{\theta\otimes\theta} \mathbin{\widehat{\otimes}}_{U^e} U^e_{E_U} \mathbin{\widehat{\otimes}}_U \overline{P}_\bullet \cong H^e_{\theta\otimes\theta} \mathbin{\widehat{\otimes}}_{U^e} \overline{Q}_\bullet.$$

Therefore $H^e_{\theta\otimes\theta} \mathbin{\widehat{\otimes}}_{U^e} \overline{Q}_\bullet$ is admissible, as required. ∎

We end this section with the following simple observation.

LEMMA 3.8. *Let $\theta\colon A \to B$ be a homomorphism of $\widehat{\otimes}$-algebras (resp. of augmented $\widehat{\otimes}$-algebras) with dense range. Then θ is a localization (resp. weak localization) if and only if $B^e \perp\!\!\!\perp_{A^e} B$ (resp. $B \perp\!\!\!\perp_A \mathbb{C}$).*

Proof. Since $\operatorname{Im}\theta$ is dense in B, the map $X \mathbin{\widehat{\otimes}}_A Y \to X \mathbin{\widehat{\otimes}}_B Y$, $x \otimes_A y \mapsto x \otimes_B y$, is a topological isomorphism for each $X \in$ **mod**-B and each $Y \in B$-**mod**. In particular, (15) (resp. (18)) is a topological isomorphism. The rest is clear. ∎

4. Localizations of $U(\mathfrak{g})$ and duality

Following [1] (cf. also [43]), we say that a Hopf $\widehat{\otimes}$-algebra is *well-behaved* if its underlying locally convex space is either a nuclear Fréchet space or a nuclear (DF)-space. Recall (see, e.g., [21]) that the strong dual of a nuclear Fréchet space is a complete nuclear (DF)-space, and *vice versa*. Moreover, if E is either a nuclear Fréchet space or a complete nuclear (DF)-space, then there is a canonical topological isomorphism $E' \mathbin{\widehat{\otimes}} E' \cong (E \mathbin{\widehat{\otimes}} E)'$. Therefore for each well-behaved Hopf $\widehat{\otimes}$-algebra H the strong dual, H', is also a well-behaved Hopf $\widehat{\otimes}$-algebra in a natural way. More precisely, the multiplication (resp. comultiplication) on H' is the dual of the comultiplication (resp. multiplication) on H, the antipode of H' is the dual of that of H, etc. Note that H is commutative (resp. cocommutative) if and only if H' is cocommutative (resp. commutative). For example, if G is a real Lie group, then the algebra $C^\infty(G)$ of smooth functions is a nuclear commutative Fréchet Hopf algebra, and its dual is the Hopf algebra $\mathscr{E}'(G)$ of compactly supported distributions. For later reference, recall that the comultiplication, the counit, and the antipode of $C^\infty(G)$ are given, respectively, by

$$(\Delta f)(x,y) = f(xy), \quad \varepsilon(f) = f(e), \quad (Sf)(x) = f(x^{-1}). \tag{20}$$

Here we identify $C^\infty(G) \mathbin{\widehat{\otimes}} C^\infty(G)$ with $C^\infty(G \times G)$ (see, e.g., [21, Chap. II, §3, no. 3]).

Another important example is $U(\mathfrak{g})$, the universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{g}$. If we endow $U(\mathfrak{g})$ with the finest locally convex topology,

then it becomes a cocommutative nuclear (DF) Hopf $\widehat{\otimes}$-algebra. Recall that the comultiplication, the counit, and the antipode of $U(\mathfrak{g})$ are uniquely determined by

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \varepsilon(X) = 0, \quad S(X) = -X \qquad (X \in \mathfrak{g}).$$

The strong dual of $U(\mathfrak{g})$ is topologically isomorphic to the Fréchet algebra of formal power series $\mathbb{C}[[z_1, \ldots, z_n]]$ with the topology of convergence of each coefficient (cf. [11, Prop. 2.7.5] and [77, Theorem 22.1]; cf. also Lemma 5.1 below).

Many other examples of well-behaved Hopf $\widehat{\otimes}$-algebras can be found in [42], [43], [44], [1], and [60].

Let $\mathfrak{g}$ be a Lie algebra. Suppose we are given a Hopf $\widehat{\otimes}$-algebra homomorphism of $U(\mathfrak{g})$ to a well-behaved Hopf $\widehat{\otimes}$-algebra H. In this section we formulate some conditions on the dual algebra, H', that are sufficient for H to be stably flat over $U(\mathfrak{g})$.

4.1. Homotopy of commutative $\widehat{\otimes}$-algebras. In this subsection we briefly discuss a continuous version of the notion of homotopy between morphisms of commutative algebras. This notion was introduced by Chen [7] in the purely algebraic case. All the definitions and the results in this subsection are straightforward adaptations of [7] to the $\widehat{\otimes}$-case.

Throughout this subsection, all $\widehat{\otimes}$-algebras are assumed to be commutative.

Definition 4.1 (cf. [7], [8]). A $\widehat{\otimes}$-algebra C is called *exact* if it has at least one nonzero augmentation $C \to \mathbb{C}$, and if there exists a derivation $\partial\colon C \to N$ to some C-$\widehat{\otimes}$-module N such that the sequence

$$0 \to \mathbb{C} \xrightarrow{\eta_C} C \xrightarrow{\partial} N \to 0$$

splits in **LCS**. A derivation ∂ with the above property is called *split exact*.

Basic examples of exact algebras are the algebra of smooth functions $C^\infty(I)$ on an interval $I \subset \mathbb{R}$, the algebra $\mathscr{O}(U)$ of holomorphic functions on a simply connected domain $U \subset \mathbb{C}$, the algebra $\mathbb{C}[[z]]$ of formal power series, the polynomial algebra $\mathbb{C}[z]$ (with the finest locally convex topology), etc. In each of the above examples, the usual derivation $d/dz\colon A \to A$ is split exact.

Definition 4.2 (cf. [7]). Two morphisms $\varphi_0, \varphi_1\colon A \to B$ of commutative $\widehat{\otimes}$-algebras are said to be *homotopic* if there exists an exact algebra C, a morphism $\Phi\colon A \to C \mathbin{\widehat{\otimes}} B$ and two augmentations $\varepsilon_0, \varepsilon_1\colon C \to \mathbb{C}$ such that

$$\varphi_i = (\varepsilon_i \otimes 1_B)\, \Phi \qquad (i = 0, 1).$$

For instance, if X and Y are smooth manifolds and $f_0, f_1\colon X \to Y$ are smooth homotopic mappings, then the induced morphisms $f_0^*, f_1^*\colon C^\infty(Y) \to C^\infty(X)$ are homotopic in the above sense. To see this, it suffices to set $C = C^\infty[0,1]$ and to reverse the arrows in the usual definition of a (smooth) homotopy between f_0 and f_1.

Two $\widehat{\otimes}$-algebras A, B are called *homotopy equivalent* if there exist morphisms $\varphi\colon A \to B$ and $\psi\colon B \to A$ such that $\psi\varphi$ is homotopic to 1_A and $\varphi\psi$ is homotopic to 1_B. A $\widehat{\otimes}$-algebra is said to be *contractible* ([1]) if it is homotopy equivalent to $\mathbb{C}$. Equivalently,

([1]) We use the word "contractible" following Chen [7]; it should be noted, however, that the notion of "contractible algebra" has a completely different meaning in the cohomology theory of locally convex algebras (see, e.g., [26]).

A is contractible iff there exist an exact algebra C, a morphism $\Phi\colon A \to C \mathbin{\widehat{\otimes}} A$, and augmentations $\varepsilon_0, \varepsilon_1\colon C \to \mathbb{C}$ and $\varepsilon_A\colon A \to \mathbb{C}$ such that $(\varepsilon_1 \otimes 1_A)\Phi = 1_A$ and $(\varepsilon_0 \otimes 1_A)\Phi = \eta_A \varepsilon_A$. For example, the algebra of smooth functions on a contractible smooth manifold is contractible. It is also easy to prove that the polynomial algebra $\mathbb{C}[z_1, \dots, z_n]$, the algebra of formal power series $\mathbb{C}[[z_1, \dots, z_n]]$, the algebra of entire functions $\mathscr{O}(\mathbb{C}^n)$ etc. are contractible.

THEOREM 4.1 ([7]). *If two morphisms $\varphi_0, \varphi_1\colon A \to B$ of commutative $\widehat{\otimes}$-algebras are homotopic, then the induced morphisms $\varphi_{0,*}, \varphi_{1,*}\colon \Omega(A) \to \Omega(B)$ are chain homotopic (as morphisms of complexes in* **LCS**).

We omit the proof, because it is an obvious modification of the proof from [7] to the $\widehat{\otimes}$-case.

COROLLARY 4.2. *If A is a contractible $\widehat{\otimes}$-algebra, then the augmented de Rham complex $0 \to \mathbb{C} \xrightarrow{\eta_A} \Omega(A)$ splits in* **LCS**.

4.2. Lie algebra actions and parallelizability

DEFINITION 4.3. Let A be a commutative $\widehat{\otimes}$-algebra, and let $\mathfrak{g}$ be a Lie algebra acting on A by derivations. We say that A is $\mathfrak{g}$-*parallelizable* if the derivation

$$d^0\colon A \to C^1(\mathfrak{g}, A), \qquad a \mapsto (X \mapsto Xa),$$

is universal, i.e., if $(C^1(\mathfrak{g}, A), d^0)$ is the module of Kähler differentials for A.

PROPOSITION 4.3. *An algebra A is $\mathfrak{g}$-parallelizable if and only if the identity map of A extends to a DG $\widehat{\otimes}$-algebra isomorphism between $C^{\cdot}(\mathfrak{g}, A)$ and $\Omega(A)$.*

Proof. The "if" part is clear. To prove the converse, recall that the universal property of $\Omega(A)$ yields a unique DG $\widehat{\otimes}$-algebra morphism $\varphi\colon \Omega(A) \to C^{\cdot}(\mathfrak{g}, A)$ such that $\varphi^0 = 1_A$. If A is $\mathfrak{g}$-parallelizable, then $\varphi^1\colon \Omega^1 A \to C^1(\mathfrak{g}, A)$ is an isomorphism. Since both $\Omega(A)$ and $C^{\cdot}(\mathfrak{g}, A)$ are exterior, we conclude that φ is an isomorphism (see Subsection 1.3). ∎

Now suppose that H is a well-behaved cocommutative Hopf $\widehat{\otimes}$-algebra, $\mathfrak{g}$ is a Lie algebra, and $\theta\colon U(\mathfrak{g}) \to H$ is a Hopf $\widehat{\otimes}$-algebra homomorphism. We consider H as a right $\mathfrak{g}$-$\widehat{\otimes}$-module via θ by setting $x \cdot X = x\theta(X)$ for each $x \in H$, $X \in \mathfrak{g}$. The strong dual space, H', is then a left $\mathfrak{g}$-$\widehat{\otimes}$-module in a natural way (see Subsection 1.4). Namely, the action of $\mathfrak{g}$ on H' is given by

$$\langle X \cdot a, x\rangle = \langle a, x\theta(X)\rangle \qquad (a \in H',\ X \in \mathfrak{g},\ x \in H). \tag{21}$$

It is easily checked that $\mathfrak{g}$ acts on H' by derivations. Indeed, for each $a, b \in H'$, $X \in \mathfrak{g}$, $x \in H$ we obtain

$$\begin{aligned} \langle X \cdot ab, x\rangle &= \langle ab, x\theta(X)\rangle = \langle a \otimes b, \Delta(x\theta(X))\rangle = \langle a \otimes b, \Delta(x)\Delta(\theta(X))\rangle \\ &= \langle a \otimes b, \Delta(x) \cdot \theta \otimes \theta(\Delta(X))\rangle = \langle a \otimes b, \Delta(x)(\theta(X) \otimes 1 + 1 \otimes \theta(X))\rangle. \end{aligned} \tag{22}$$

For each $x_1, x_2 \in H$ we have

$$\begin{aligned} \langle a \otimes b, (x_1 \otimes x_2)(\theta(X) \otimes 1)\rangle &= \langle a \otimes b, x_1\theta(X) \otimes x_2\rangle \\ &= \langle Xa, x_1\rangle\langle b, x_2\rangle = \langle Xa \otimes b, x_1 \otimes x_2\rangle. \end{aligned}$$

Therefore $\langle a \otimes b, u(\theta(X) \otimes 1)\rangle = \langle Xa \otimes b, u\rangle$ for each $u \in H \mathbin{\widehat{\otimes}} H$. Similarly, $\langle a \otimes b, u(1 \otimes \theta(X))\rangle = \langle a \otimes Xb, u\rangle$ for each $u \in H \mathbin{\widehat{\otimes}} H$. Setting $u = \Delta(x)$ and substituting in (22), we see that

$$\langle X \cdot ab, x\rangle = \langle Xa \otimes b + a \otimes Xb, \Delta(x)\rangle = \langle Xa \cdot b + a \cdot Xb, x\rangle.$$

Hence $\mathfrak{g}$ acts on H' by derivations.

In what follows, we say that the action defined by (21) is *determined by* θ. We shall sometimes refer to θ explicitly by writing $X \cdot_\theta a$ instead of $X \cdot a$ or Xa.

THEOREM 4.4. *Let $\mathfrak{g}$ be a Lie algebra, and let H be a well-behaved cocommutative Hopf $\widehat{\otimes}$-algebra. Suppose $\theta\colon U(\mathfrak{g}) \to H$ is a Hopf $\widehat{\otimes}$-algebra homomorphism with dense range. Assume that H' is $\mathfrak{g}$-parallelizable (with respect to the action determined by θ) and contractible. Then θ is a localization.*

Proof. Since H is cocommutative, we have $S^2 = 1_H$ (for a categorical proof of this classical fact, see [71, Chap. 9]). In particular, S is invertible. In view of Proposition 3.7, it suffices to show that θ is a weak localization. Set $U = U(\mathfrak{g})$, and consider the Koszul resolution

$$0 \leftarrow \mathbb{C} \xleftarrow{\varepsilon_U} V.(\mathfrak{g})$$

of the trivial $\mathfrak{g}$-module $\mathbb{C}$ (see Subsection 1.4). Clearly, the chain complexes $H \mathbin{\widehat{\otimes}_U} V.(\mathfrak{g})$ and $C.(\mathfrak{g}, H)$ are isomorphic. Due to Lemma 3.8, we need only check that the augmented complex

$$0 \leftarrow \mathbb{C} \xleftarrow{\varepsilon_H} C.(\mathfrak{g}, H)$$

splits in **LCS**. Since the above complex consists of reflexive spaces, it splits if and only if the dual complex

$$0 \to \mathbb{C} \xrightarrow{\eta_{H'}} C^{\cdot}(\mathfrak{g}, H')$$

splits. Now it remains to apply Corollary 4.2 and Proposition 4.3. ∎

5. Power series envelopes of $U(\mathfrak{g})$

Our next task is to show that the strong dual algebras of some locally convex completions of $U(\mathfrak{g})$ (for $\mathfrak{g}$ nilpotent) are indeed $\mathfrak{g}$-parallelizable. To this end, recall some facts on the "formal power series completion" $[U(\mathfrak{g})]$ of $U(\mathfrak{g})$ (see [19]).

Let $\mathfrak{g}$ be a nilpotent Lie algebra and let $I \subset U(\mathfrak{g})$ be the ideal generated by $\mathfrak{g}$. Recall that the quotient algebra $U(\mathfrak{g})/I^n$ is finite-dimensional for each n (see, e.g., [11, 2.5.1]). Endow each $U(\mathfrak{g})/I^n$ with the usual locally convex topology of a finite-dimensional vector space, and set $[U(\mathfrak{g})] = \varprojlim U(\mathfrak{g})/I^n$. Clearly, $[U(\mathfrak{g})]$ is a nuclear Fréchet–Arens–Michael algebra. We have a canonical homomorphism

$$\theta\colon U(\mathfrak{g}) \to [U(\mathfrak{g})], \quad x \mapsto x + I^n. \tag{23}$$

Since $\mathfrak{g}$ is nilpotent, it follows that $\bigcap_n I^n = \{0\}$ (see, e.g., [29, XIV.4.1]), so that (23) is injective. For notational convenience, we shall often write U instead of $U(\mathfrak{g})$ and $[U]$ instead of $[U(\mathfrak{g})]$, and we shall identify U with its canonical image in $[U]$.

It is easy to show that $[U(\mathfrak{g})]$ has a natural structure of Hopf $\widehat{\otimes}$-algebra such that (23) is a Hopf algebra homomorphism. Indeed, let $K = I \otimes U + U \otimes I \subset U \otimes U$ be the augmentation ideal of $U \otimes U$. Evidently, we have $\Delta(I) \subset K$, so $\Delta(I^n) \subset K^n$ for each n. Therefore we obtain an algebra homomorphism

$$[U] = \varprojlim U/I^n \to \varprojlim (U \otimes U)/K^n, \qquad x + I^n \mapsto \Delta(x) + K^n. \tag{24}$$

Since $U \otimes U$ is isomorphic to $U(\mathfrak{g} \times \mathfrak{g})$, and since $\mathfrak{g} \times \mathfrak{g}$ is nilpotent together with $\mathfrak{g}$, it follows that $\dim (U \otimes U)/K^n < \infty$ for each n. Hence we can endow $\varprojlim (U \otimes U)/K^n$ with a locally convex topology in the same way as we did for $[U]$. Thus (24) becomes a $\widehat{\otimes}$-algebra homomorphism.

For each n denote by $\tau_n \colon U \to U/I^n$ the quotient map, and set

$$\pi_n = \tau_n \otimes \tau_n \colon U \otimes U \to (U/I^n) \otimes (U/I^n).$$

We clearly have

$$K^n = \sum_{i+j=n} I^i \otimes I^j,$$

and so $K^{2n} \subset \operatorname{Ker} \pi_n$. Hence there exists a homomorphism

$$(U \otimes U)/K^{2n} \to (U/I^n) \mathbin{\widehat{\otimes}} (U/I^n), \qquad y + K^{2n} \mapsto \pi_n(y).$$

Taking the inverse limit and using the fact that the projective tensor product commutes with reduced inverse limits [35, 41.6], we obtain a homomorphism

$$\varprojlim (U \otimes U)/K^n \to [U] \mathbin{\widehat{\otimes}} [U].$$

(This is even an isomorphism, since $\operatorname{Ker} \pi_n \subset K^n$ for each n.) Composing with (24), we get a $\widehat{\otimes}$-algebra homomorphism

$$[\Delta] \colon [U] \to [U] \mathbin{\widehat{\otimes}} [U].$$

It is easy to check that $[\Delta]$ extends Δ in the sense that the diagram

$$\begin{array}{ccc} [U] & \xrightarrow{[\Delta]} & [U] \mathbin{\widehat{\otimes}} [U] \\ {\scriptstyle\theta}\uparrow & & \uparrow{\scriptstyle\theta\otimes\theta} \\ U & \xrightarrow{\Delta} & U \mathbin{\widehat{\otimes}} U \end{array}$$

is commutative. Since θ has dense range, the coassociativity of $[\Delta]$ readily follows from that of Δ.

Arguing as above, it is easy to construct an antipode $[S] \colon [U] \to [U]$ and a counit $[\varepsilon] \colon [U] \to \mathbb{C}$ in such a way that $[U]$ becomes a Hopf $\widehat{\otimes}$-algebra and $\theta \colon U \to [U]$ becomes a Hopf $\widehat{\otimes}$-algebra homomorphism.

A somewhat more explicit construction of $[U]$ was suggested by Goodman [19]. Fix a positive filtration $\mathscr{F}$ on $\mathfrak{g}$, i.e., a decreasing chain of subspaces

$$\mathfrak{g} = \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_\ell \supset \mathfrak{g}_{\ell+1} = 0, \qquad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}.$$

The smallest ℓ such that $\mathfrak{g}_{\ell+1} = 0$ is called the *length* of the filtration.

An example of a positive filtration is the lower central series of $\mathfrak{g}$ defined inductively by $\mathfrak{g}_{n+1} = [\mathfrak{g}, \mathfrak{g}_n]$.

Given $X \in \mathfrak{g}$, $X \neq 0$, the *$\mathscr{F}$-weight* of X is defined by $w(X) = \max\{n : X \in \mathfrak{g}_n\}$. A basis (e_i) of $\mathfrak{g}$ is called an *$\mathscr{F}$-basis* if $w(e_i) \leq w(e_{i+1})$ for all i, and $\mathfrak{g}_n = \operatorname{span}\{e_i : w(e_i) \geq n\}$ for all n. Given an $\mathscr{F}$-basis (e_i), we set $w_i = w(e_i)$ for each i. For each multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}_+^N$ ($N = \dim \mathfrak{g}$), set $|\alpha| = \sum_i \alpha_i$ and $w(\alpha) = \sum_i w_i \alpha_i$. By the Poincaré–Birkhoff–Witt theorem, the elements $e^\alpha = e_1^{\alpha_1} \cdots e_N^{\alpha_N}$ form a basis of $U(\mathfrak{g})$. For each n, set

$$J_n = \operatorname{span}\{e^\alpha : w(\alpha) \geq n\} \subset U(\mathfrak{g}). \tag{25}$$

Then we have

$$U(\mathfrak{g}) = J_0 \supset I = J_1 \supset J_2 \supset \cdots, \qquad J_i J_j \subset J_{i+j}, \tag{26}$$

so that $\{J_n\}$ is a decreasing filtration on $U(\mathfrak{g})$ satisfying $\bigcap_n J_n = \{0\}$. In particular, each J_n is an ideal of $U(\mathfrak{g})$. Goodman [19] defines $[U(\mathfrak{g})]_{\mathscr{F}}$ as the completion of $U(\mathfrak{g})$ with respect to the topology determined by the filtration $\{J_n\}$. Thus we have an algebraic isomorphism $[U(\mathfrak{g})]_{\mathscr{F}} = \varprojlim U(\mathfrak{g})/J_n$.

If we endow each $U(\mathfrak{g})/J_n$ with the usual topology of a finite-dimensional vector space, then it is easily seen that $[U(\mathfrak{g})]_{\mathscr{F}}$ is isomorphic (as a topological algebra) to the algebra $[U(\mathfrak{g})]$ introduced above. Indeed, setting $C = \max w_i$, we see that $w(\alpha) \leq C|\alpha|$ for each $\alpha \in \mathbb{Z}_+^N$, and so $J_{Cn} \subset I^n$ for each n. On the other hand, we have $I^n = J_1^n \subset J_n$ for each n. Therefore the filtrations $\{J_n\}$ and $\{I^n\}$ are equivalent, and so the algebras $[U(\mathfrak{g})]_{\mathscr{F}}$ and $[U(\mathfrak{g})]$ are isomorphic.

As a locally convex space, $[U(\mathfrak{g})]$ is isomorphic to the space of all formal power series $x = \sum_\alpha c_\alpha e^\alpha$ endowed with the topology of convergence of each coefficient (cf. [19]). More precisely, the topology on $[U(\mathfrak{g})]$ can be generated by the sequence of seminorms $\{\|\cdot\|_n : n \in \mathbb{Z}_+\}$ defined by

$$\|x\|_n = \sum_{w(\alpha) \leq n} |c_\alpha| \quad \text{for each } x = \sum_\alpha c_\alpha e^\alpha \in [U].$$

For each multi-index $\alpha \in \mathbb{Z}_+^N$ set $e_\alpha = e^\alpha/\alpha!$. Then $\Delta(e_\gamma) = \sum_{\alpha+\beta=\gamma} e_\alpha \otimes e_\beta$ (see [11, 2.7.2]), and the same relation clearly holds for $[\Delta]$.

LEMMA 5.1. *The mapping $\varkappa\colon [U]' \to \mathbb{C}[z_1, \ldots, z_N]$ defined by the rule*

$$f \mapsto \sum_\alpha f(e_\alpha) z^\alpha \tag{27}$$

is an algebra isomorphism. Moreover, $\varkappa$ is a topological isomorphism with respect to the strong topology on $[U]'$ and the finest l.c. topology on $\mathbb{C}[z_1, \ldots, z_N]$.

Proof. The continuity of f implies that there exists $n \in \mathbb{Z}_+$ such that $f(e_\alpha) = 0$ whenever $w(\alpha) \geq n$. Hence the sum on the right-hand side of (27) is finite, and $\varkappa$ is well defined. Since the e_α's generate a dense subspace of $[U]$, we see that $\varkappa$ is injective. Conversely, for every polynomial $p = \sum_\alpha \lambda_\alpha z^\alpha$ the mapping $f\colon [U] \to \mathbb{C}$, $f(\sum_\alpha c_\alpha e_\alpha) = \sum_\alpha c_\alpha \lambda_\alpha$, is a continuous linear functional on $[U]$ satisfying $\varkappa(f) = p$. Hence $\varkappa$ is bijective. A direct computation (see [11, 2.7.5]) shows that $\varkappa$ is an algebra homomorphism. Finally, since the topology on the strong dual of a countable inverse limit of finite-dimensional spaces is the finest l.c. topology (see, e.g., [77, Theorem 22.1]), we see that $\varkappa$ is a topological isomorphism. ∎

DEFINITION 5.1. Let $\mathfrak{g}$ be a nilpotent Lie algebra. By a *power series envelope* of $U(\mathfrak{g})$ we mean a Hopf $\widehat{\otimes}$-algebra H together with Hopf $\widehat{\otimes}$-algebra homomorphisms $\theta_1 \colon U(\mathfrak{g}) \to H$ and $\theta_2 \colon H \to [U(\mathfrak{g})]$ such that both θ_1 and θ_2 are injective with dense ranges, and the composition

$$U(\mathfrak{g}) \xrightarrow{\theta_1} H \xrightarrow{\theta_2} [U(\mathfrak{g})]$$

coincides with the canonical homomorphism θ defined by (23).

REMARK 5.1. Since θ is injective with dense range, the conditions "θ_1 is injective" and "θ_2 has dense range" are satisfied automatically. Note also that, since $U(\mathfrak{g})$ is cocommutative and θ_1 has dense range, H is also cocommutative. For the same reason, $S^2 = 1_H$ in H.

It is immediate from the definition that the "smallest" power series envelope of $U(\mathfrak{g})$ is $U(\mathfrak{g})$ itself, and the "largest" one is $[U(\mathfrak{g})]$.

THEOREM 5.2. *Let $\mathfrak{g}$ be a nilpotent Lie algebra, and let H be a well-behaved power series envelope of $U(\mathfrak{g})$. Then H' is $\mathfrak{g}$-parallelizable.*

Proof. Fix a positive filtration $\mathscr{F}$ on $\mathfrak{g}$, and choose an $\mathscr{F}$-basis (e_i) of $\mathfrak{g}$. Using Lemma 5.1, we may identify $[U]'$ and $\mathbb{C}[z_1, \dots, z_N]$. For each $i = 1, \dots, N$ set $x_i = \theta_2'(z_i) \in H'$. Since θ_2 is injective, it follows that $\operatorname{Im} \theta_2'$ is dense in H' with respect to the weak* topology $\sigma(H', H)$. Using the semireflexivity of H, we see that $\operatorname{Im} \theta_2'$ is dense in H' with respect to the strong topology as well. Hence $x_1, \dots, x_N$ generate a dense subalgebra of H'.

Set $A = H'$, and consider the free A-module A^N with the standard A-basis (u_i), i.e., $u_i = (0, \dots, 1, \dots, 0)$ with 1 in the ith coordinate, 0 elsewhere. Denote by $(e^i) \subset \mathfrak{g}^*$ the basis dual to (e_i) (i.e., $e^i(e_j) = \delta_{ij}$ for all i, j). Identifying the A-modules $C^1(\mathfrak{g}, A)$ and $A \otimes \mathfrak{g}^*$, we see that the elements $v_i = 1 \otimes e^i$ $(i = 1, \dots, N)$ form an A-basis of $C^1(\mathfrak{g}, A)$.

Now consider the A-module morphism $\varphi \colon A^N \to C^1(\mathfrak{g}, A)$ taking each u_i to $d^0(x_i)$. Let (φ_{ij}) be the matrix of φ with respect to the bases (u_i) and (v_i), respectively. Applying the identity $\varphi(u_j) = \sum_i \varphi_{ij} v_i$ to e_i, we see that

$$\varphi_{ij} = \varphi(u_j)(e_i) = d^0(x_j)(e_i) = e_i \cdot x_j.$$

Given $a \in H'$, denote by $\overline{a}$ the restriction of a to $U(\mathfrak{g})$ (i.e., $\overline{a} = \theta_1'(a)$). Then for each $y \in U(\mathfrak{g})$ we have

$$\langle \overline{\varphi}_{ij}, y \rangle = \langle \varphi_{ij}, \theta_1(y) \rangle = \langle e_i \cdot x_j, \theta_1(y) \rangle = \langle x_j, \theta_1(y e_i) \rangle = \langle \overline{x}_j, y e_i \rangle. \tag{28}$$

We claim that the matrix (φ_{ij}) is upper triangular with 1's on the main diagonal. Indeed, using (26), we see that $y e_i \in J_{w_i}$ for each $y \in U(\mathfrak{g})$; moreover, $y e_i \in J_{w_i+1}$ for each $y \in I = J_1$. On the other hand, it is immediate from (27) that $\overline{x}_j = z_j|_{U(\mathfrak{g})}$ vanishes on J_{w_j+1}. Hence $\langle \overline{x}_j, y e_i \rangle = 0$ for all $i > j$ and all $y \in U(\mathfrak{g})$, $\langle \overline{x}_i, y e_i \rangle = 0$ for all $y \in I$, and $\langle \overline{x}_i, e_i \rangle = 1$. Together with (28), this gives $\overline{\varphi}_{ij} = 0$ for each $i > j$, and $\overline{\varphi}_{ii} = 1$. Finally, since $\operatorname{Im} \theta_1$ is dense in H, it follows that θ_1' is injective, and so the latter relations hold with $\overline{\varphi}_{ij}$ replaced by φ_{ij}. Therefore the matrix of φ has the required form, so that φ is an isomorphism.

For each $i = 1, \dots, N$, let $p_i \colon A^N \to A$ be the projection on the ith direct summand. Evidently, $\partial_i = p_i \varphi^{-1} d^0$ is a derivation of A. It is immediate from the definition of φ that $\partial_i(x_j) = p_i(u_j) = \delta_{ij}$ for each i, j. Hence the conditions of Lemma 1.2 are satisfied, and so $\partial = (\partial_1, \dots, \partial_N) \colon A \to A^N$ is a universal derivation. Since φ is an isomorphism, we conclude that $d^0 = \varphi\partial$ is a universal derivation as well, and so A is $\mathfrak{g}$-parallelizable. ∎

Combining the above theorem with Theorem 4.4, we obtain the following.

COROLLARY 5.3. *Let $\mathfrak{g}$ be a nilpotent Lie algebra, and let H be a well-behaved power series envelope of $U(\mathfrak{g})$ such that H' is contractible. Then H is stably flat over $U(\mathfrak{g})$.*

COROLLARY 5.4. *For each nilpotent Lie algebra $\mathfrak{g}$, $[U(\mathfrak{g})]$ is stably flat over $U(\mathfrak{g})$.*

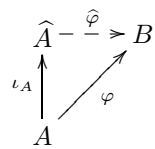
Proof. By Lemma 5.1, the algebra dual to $[U(\mathfrak{g})]$ is isomorphic to $\mathbb{C}[z_1, \dots, z_N]$ and hence is contractible. Now it remains to apply Corollary 5.3. ∎

6. Arens–Michael envelopes of universal enveloping algebras

In this section we prove that for each positively graded, finite-dimensional Lie algebra $\mathfrak{g}$ the Arens–Michael envelope of $U(\mathfrak{g})$ is stably flat over $U(\mathfrak{g})$. First we recall some facts on Arens–Michael envelopes.

6.1. Arens–Michael envelopes. Arens–Michael envelopes of topological algebras (under a different name) were introduced by Taylor ([74, Definition 5.1]). Here we follow the terminology of Helemskii's book [27].

DEFINITION 6.1 ([27, Chap. V]). Let A be a topological algebra. A pair $(\widehat{A}, \iota_A)$ consisting of an Arens–Michael algebra $\widehat{A}$ and a continuous homomorphism $\iota_A \colon A \to \widehat{A}$ is called the *Arens–Michael envelope* of A if for each Arens–Michael algebra B and for each continuous homomorphism $\varphi \colon A \to B$ there exists a unique continuous homomorphism $\widehat{\varphi} \colon \widehat{A} \to B$ making the following diagram commutative:

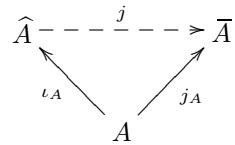


In the above situation, we say that $\widehat{\varphi}$ *extends* φ (though ι_A is not injective in general; see Remark 6.2 below).

REMARK 6.1. In the above definition, it suffices to consider only homomorphisms with values in Banach algebras. This is immediate from the fact that each Arens–Michael algebra is an inverse limit of Banach algebras (see, e.g., [27, Chap. V]).

Clearly, the Arens–Michael envelope is unique in the sense that if $(\widehat{A}, \iota_A)$ and $(\overline{A}, j_A)$ are Arens–Michael envelopes of A, then there exists a unique isomorphism $j \colon \widehat{A} \to \overline{A}$ of

topological algebras such that the following diagram is commutative:



Recall (see [74] and [27, Chap. V]) that the Arens–Michael envelope of a topological algebra A always exists and can be obtained as the completion of A with respect to the family of all continuous submultiplicative seminorms on A. This implies, in particular, that $\iota_A\colon A \to \widehat{A}$ has dense range. It is easy to see that the correspondence $A \mapsto \widehat{A}$ is a functor from the category **TA** of topological algebras to the category **AM** of Arens–Michael algebras. In what follows, we call it the *Arens–Michael functor*. Clearly, the Arens–Michael functor is the left adjoint to the forgetful functor from **AM** to **TA**.

If A is not equipped with a topology, then by an Arens–Michael envelope of A we mean the Arens–Michael envelope of the finest locally convex algebra A_{s} (see Section 1).

Here are two basic examples due to Taylor [75].

EXAMPLE 6.1. The Arens–Michael envelope of the polynomial algebra $\mathbb{C}[z_1, \dots, z_n]$ is the algebra of entire functions $\mathscr{O}(\mathbb{C}^n)$ endowed with the compact-open topology.

EXAMPLE 6.2. Let F_n be the free $\mathbb{C}$-algebra on n generators $\zeta_1, \dots, \zeta_n$. Given a k-tuple $\alpha = (\alpha_1, \dots, \alpha_k)$ of integers from $[1, n]$, we set $\zeta_\alpha = \zeta_{\alpha_1} \cdots \zeta_{\alpha_k} \in F_n$ and $|\alpha| = k$. It is convenient to agree that the identity of F_n corresponds to the tuple of length zero ($k = 0$). The algebra $\mathscr{F}_n$ of "free power series" consists of all formal expressions $a = \sum_\alpha \lambda_\alpha \zeta_\alpha$ satisfying the condition

$$\|a\|_\varrho = \sum_\alpha |\lambda_\alpha| \varrho^{|\alpha|} < \infty \qquad \text{for all } 0 < \varrho < \infty.$$

The system of seminorms $\{\|\cdot\|_\varrho : 0 < \varrho < \infty\}$ makes $\mathscr{F}_n$ into a Fréchet–Arens–Michael algebra. Evidently, F_n is a subalgebra of $\mathscr{F}_n$. Taylor [75] proved that $\mathscr{F}_n$ is the Arens–Michael envelope of F_n. Note that in the case $n = 1$ we have $F_1 = \mathbb{C}[z]$ and $\mathscr{F}_1 \cong \mathscr{O}(\mathbb{C})$.

REMARK 6.2. It should be noted that the Arens–Michael envelope can be trivial even in very simple cases. For example, let A be the Weyl algebra, i.e., the algebra with two generators p, q subject to the relation $[p, q] = 1$. It is a standard exercise from spectral theory (see, e.g., [27, Prop. 2.1.21]) to show that A has no nonzero submultiplicative seminorms. Hence $\widehat{A} = 0$.

Another example of this kind is given in [27, Chap. V].

REMARK 6.3. If $\mathfrak{g}$ is a finite-dimensional Lie algebra, then (in contrast to the previous example) the homomorphism $\iota_{U(\mathfrak{g})}\colon U(\mathfrak{g}) \to \widehat{U}(\mathfrak{g})$ is injective. This readily follows from the fact that finite-dimensional representations (and, a fortiori, Banach space representations) of $\mathfrak{g}$ separate the points of $U(\mathfrak{g})$ (see, e.g., [11, 2.5.7]).

The next proposition shows that the Arens–Michael functor commutes with quotients.

PROPOSITION 6.1. *Let A be a topological algebra and I a two-sided ideal of A. Denote by J the closure of $\iota_A(I)$ in $\widehat{A}$. Then J is a two-sided ideal of $\widehat{A}$, and the homomorphism*

$A/I \to \widehat{A}/J$ induced by $\iota_A\colon A \to \widehat{A}$ extends to a topological algebra isomorphism

$$\widehat{A/I} \cong (\widehat{A}/J)^{\sim}.$$

Proof. Since ι_A has dense range, we see that J is indeed an ideal of $\widehat{A}$, and so $(\widehat{A}/J)^{\sim}$ is an Arens–Michael algebra. Consider the homomorphism $\overline{\iota}\colon A/I \to (\widehat{A}/J)^{\sim}$ taking each $a + I \in A/I$ to $\iota_A(a) + J$. We claim that $((\widehat{A}/J)^{\sim}, \overline{\iota})$ is the Arens–Michael envelope of A/I. Indeed, each homomorphism φ from A/I to an Arens–Michael algebra C determines a homomorphism $\psi\colon A \to C$ vanishing on I. Each such homomorphism extends to a homomorphism $\widehat{\psi}\colon \widehat{A} \to C$ that vanishes on J and hence gives a homomorphism $\widehat{\varphi}\colon (\widehat{A}/J)^{\sim} \to C$. It is now elementary to check that $\widehat{\varphi}\overline{\iota} = \varphi$. The uniqueness of $\widehat{\varphi}$ is immediate from the fact that $\overline{\iota}$ has dense range. ∎

Since each separated quotient of a Fréchet space is complete, we obtain the following

Corollary 6.2. *Under the conditions of Proposition* 6.1, *assume that $\widehat{A}$ is a Fréchet algebra. Then $\widehat{A/I} \cong \widehat{A}/J$.*

Corollary 6.3. *If A is a finitely generated algebra, then $\widehat{A_{\mathrm{s}}}$ is a nuclear Fréchet algebra.*

Proof. Since A is finitely generated, it is isomorphic to a quotient of the free algebra F_n for some n. By Corollary 6.2, $\widehat{A}$ is isomorphic to a quotient of $\widehat{F}_n = \mathscr{F}_n$ (see Example 6.2). Since $\mathscr{F}_n$ is a nuclear Fréchet space [45], so is $\widehat{A}$. ∎

Another useful property of the Arens–Michael functor is that it commutes with projective tensor products.

Proposition 6.4. *Let A, B be $\widehat{\otimes}$-algebras. Then there exists a topological algebra isomorphism*

$$(A \mathbin{\widehat{\otimes}} B)\widehat{\ } \cong \widehat{A} \mathbin{\widehat{\otimes}} \widehat{B}.$$

Proof. Set $\iota = \iota_A \otimes \iota_B\colon A \mathbin{\widehat{\otimes}} B \to \widehat{A} \mathbin{\widehat{\otimes}} \widehat{B}$. Clearly, ι is a continuous homomorphism. Suppose $\varphi\colon A \mathbin{\widehat{\otimes}} B \to C$ is a homomorphism to some Arens–Michael algebra C. Then $\varphi_1\colon A \to C$, $\varphi_1(a) = \varphi(a \otimes 1)$, and $\varphi_2\colon B \to C$, $\varphi_2(b) = \varphi(1 \otimes b)$, extend to continuous homomorphisms $\widehat{\varphi}_1\colon \widehat{A} \to C$ and $\widehat{\varphi}_2\colon \widehat{B} \to C$, i.e., we have $\widehat{\varphi}_1\iota_A = \varphi_1$ and $\widehat{\varphi}_2\iota_B = \varphi_2$. Let $\widehat{\varphi}\colon \widehat{A} \mathbin{\widehat{\otimes}} \widehat{B} \to C$ be the linear continuous map associated to the bilinear map $\widehat{A} \times \widehat{B} \to C$, $(a, b) \mapsto \widehat{\varphi}_1(a)\widehat{\varphi}_2(b)$. Evidently, we have $\widehat{\varphi}\iota = \varphi$. Since ι has dense range, we conclude that $\widehat{\varphi}$ is an algebra homomorphism. For the same reason, $\widehat{\varphi}$ is a unique homomorphism extending φ. Hence $(\widehat{A} \mathbin{\widehat{\otimes}} \widehat{B}, \iota)$ is the Arens–Michael envelope of $A \mathbin{\widehat{\otimes}} B$. ∎

Proposition 6.5. *Let A be a topological algebra. Then $(A^{\mathrm{op}})\widehat{\ } \cong \widehat{A}^{\mathrm{op}}$.*

Proof. It suffices to use the 1-1 correspondence between continuous homomorphisms $A^{\mathrm{op}} \to B$ and continuous homomorphisms $A \to B^{\mathrm{op}}$. ∎

Corollary 6.6. *Let A be a $\widehat{\otimes}$-algebra. Then $(A^e)\widehat{\ } \cong (\widehat{A})^e$.*

The next proposition shows that the Arens–Michael functor can also be considered as a functor from the category $\mathbf{HTA}_{\widehat{\otimes}}$ of Hopf $\widehat{\otimes}$-algebras to the category $\mathbf{HAM}_{\widehat{\otimes}}$ of Hopf $\widehat{\otimes}$-algebras that are Arens–Michael algebras.

Proposition 6.7. *Let H be a Hopf $\widehat{\otimes}$-algebra. Then there exists a unique Hopf $\widehat{\otimes}$-algebra structure on $\widehat{H}$ such that $\iota_H\colon H \to \widehat{H}$ becomes a Hopf $\widehat{\otimes}$-algebra homomorphism. More-*

over, if L is both a Hopf $\widehat{\otimes}$-algebra and an Arens–Michael algebra, and $\varphi\colon H \to L$ is a Hopf $\widehat{\otimes}$-algebra homomorphism, then so is $\widehat{\varphi}\colon \widehat{H} \to L$.

Proof. To obtain $\Delta_{\widehat{H}}$, $\varepsilon_{\widehat{H}}$, and $S_{\widehat{H}}$, it suffices to apply the Arens–Michael functor to Δ_H, ε_H, and S_H, respectively, and to use Propositions 6.4 and 6.5. The Hopf algebra axioms (such as the coassociativity of $\widehat{\Delta}_{\widehat{H}}$ etc.) are then readily verified by applying the Arens–Michael functor to the appropriate commutative diagrams involving H.

To prove that $\widehat{\varphi}$ respects comultiplication, it is enough to show that $(\widehat{\varphi} \otimes \widehat{\varphi})\Delta_{\widehat{H}}\iota_H = \Delta_L\widehat{\varphi}\iota_H$. We have

$$(\widehat{\varphi} \otimes \widehat{\varphi})\Delta_{\widehat{H}}\iota_H = (\widehat{\varphi} \otimes \widehat{\varphi})(\iota_H \otimes \iota_H)\Delta_H = (\varphi \otimes \varphi)\Delta_H = \Delta_L\varphi = \Delta_L\widehat{\varphi}\iota_H.$$

A similar argument shows that $\widehat{\varphi}S_{\widehat{H}} = S_L\widehat{\varphi}$ and $\varepsilon_L\widehat{\varphi} = \varepsilon_{\widehat{H}}$. Hence $\widehat{\varphi}$ is a Hopf $\widehat{\otimes}$-algebra homomorphism. ∎

EXAMPLE 6.3. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. Then it follows from Proposition 6.7 and Corollary 6.3 that $\widehat{U}(\mathfrak{g})$ is a well-behaved (see the beginning of Section 4) Hopf $\widehat{\otimes}$-algebra. Denote by $\iota_{\mathfrak{g}}\colon \mathfrak{g} \to \widehat{U}(\mathfrak{g})$ the restriction of $\iota_{U(\mathfrak{g})}$ to $\mathfrak{g}$. Then it is easy to see that $\widehat{U}(\mathfrak{g})$ is characterized by the following universal property: *for each Arens–Michael algebra A and each Lie algebra homomorphism $\varphi\colon \mathfrak{g} \to A$ there exists a unique $\widehat{\otimes}$-algebra homomorphism $\psi\colon \widehat{U}(\mathfrak{g}) \to A$ such that $\psi\iota_{\mathfrak{g}} = \varphi$.* In particular, for each Lie algebra homomorphism $f\colon \mathfrak{g} \to \mathfrak{h}$ there exists a unique $\widehat{\otimes}$-algebra homomorphism $\widehat{U}(f)\colon \widehat{U}(\mathfrak{g}) \to \widehat{U}(\mathfrak{h})$ such that $\widehat{U}(f)\iota_{\mathfrak{g}} = \iota_{\mathfrak{h}}f$. Moreover, Proposition 6.7 implies that $\widehat{U}(f)$ is in fact a Hopf $\widehat{\otimes}$-algebra homomorphism (cf. [5, Chap. II, §1, no. 4]).

6.2. Arens–Michael envelopes of filtered and graded algebras. In this subsection we describe Arens–Michael envelopes of locally finite graded algebras. As a corollary, we show that the Arens–Michael envelope of the universal enveloping algebra of a nilpotent Lie algebra $\mathfrak{g}$ is a power series envelope (see Definition 5.1) provided $\mathfrak{g}$ admits a positive grading.

Recall that a *decreasing filtration* on an algebra A is a chain of linear subspaces

$$A = A_0 \supset A_1 \supset A_2 \supset \cdots \quad \text{satisfying} \quad A_iA_j \subset A_{i+j}.$$

The filtration is called *separated* if $\bigcap_n A_n = \{0\}$ and is said to be *of finite type* if $\dim A_n/A_{n+1} < \infty$ for all n. In what follows, all filtrations are assumed to have these properties.

As in Section 5, we endow each A/A_n with the usual locally convex topology of a finite-dimensional vector space, and set $[A] = \varprojlim A/A_n$.

The following proposition is immediate from the definition of $[A]$.

PROPOSITION 6.8. *For each $n \in \mathbb{Z}_+$ let V_n be a linear complement of A_{n+1} in A_n. Fix a norm on each V_n. Then, as a locally convex space, $[A]$ is isomorphic to the space of all formal series $\{a = \sum_i v_i : v_i \in V_i\}$ endowed with the family of seminorms $\{\|\cdot\|_n : n \in \mathbb{Z}_+\}$ defined by*

$$\|a\|_n = \sum_{i=0}^{n} \|v_i\| \quad \text{for each } a = \sum_i v_i.$$

Since each A/A_n is a finite-dimensional (hence Banach) algebra, we see that $[A]$ is an Arens–Michael algebra. Therefore the canonical homomorphism

$$\theta\colon A \to [A], \quad x \mapsto (x + A_n)_{n\in\mathbb{Z}_+}, \tag{29}$$

uniquely extends to a homomorphism

$$\widehat{\theta}\colon \widehat{A} \to [A], \quad \widehat{\theta}\iota_A = \theta. \tag{30}$$

PROPOSITION 6.9. *Let A be an algebra. Suppose that A admits a decreasing, separated filtration of finite type. Then the canonical homomorphism $\iota_A\colon A \to \widehat{A}$ is injective. In other words, submultiplicative seminorms separate the points of A.*

Proof. The condition $\bigcap_n A_n = \{0\}$ implies that θ is injective. Since $\theta = \widehat{\theta}\iota_A$, we conclude that ι_A is also injective. ∎

Our next task is to show that $\widehat{\theta}\colon \widehat{A} \to [A]$ is also injective provided the filtration on A comes from a grading.

Let $A = \bigoplus_{n\geq 0} A^n$ be a graded algebra (see Subsection 1.3). We assume that A is *locally finite*, i.e., that $\dim A^n < \infty$ for each n. Setting $A_n = \bigoplus_{i\geq n} A^i$, we obtain a decreasing, separated filtration of finite type on A.

The following is a direct consequence of Proposition 6.8.

PROPOSITION 6.10. *Let $A = \bigoplus_{n\geq 0} A^n$ be a locally finite graded algebra. Then, as a $\widehat{\otimes}$-algebra, $[A]$ is isomorphic to the direct product $\prod_n A^n$ endowed with the multiplication*

$$(a_i)\cdot(b_j) = (c_k), \qquad c_k = \sum_{i+j=k} a_i b_j. \tag{31}$$

In order to describe the Arens–Michael envelope of A as a certain "power series algebra", it will be convenient to use "vector-valued Köthe spaces", which are more or less straightforward generalizations of classical Köthe spaces (see, e.g., [61]).

Let $E = \{E_i : i \in \mathbb{N}\}$ be a countable family of Hausdorff locally convex spaces. For each i denote by $\mathfrak{N}(E_i)$ the set of all continuous seminorms on E_i.

DEFINITION 6.2. An *E-power set* is a family P of functions $p\colon \mathbb{N} \to \bigcup_i \mathfrak{N}(E_i)$ such that $p_i = p(i) \in \mathfrak{N}(E_i)$ for each i, and such that the following conditions are satisfied:

(1) for each $i \in \mathbb{N}$ the family of seminorms $\{p_i : p \in P\}$ generate the original topology on E_i;
(2) for each $p, q \in P$ there exists $r \in P$ such that $r_i(x) \geq \max\{p_i(x), q_i(x)\}$ for each $i \in \mathbb{N}$ and each $x \in E_i$.

DEFINITION 6.3. Given a family $E = \{E_i : i \in \mathbb{N}\}$ of Hausdorff l.c.s.'s and an E-power set P, define the *vector-valued Köthe space* $\lambda(P, E)$ by

$$\lambda(P,E) = \Big\{x = (x_i) \in \prod_i E_i : \|x\|_p = \sum_i p_i(x_i) < \infty \ \forall p \in P\Big\}.$$

REMARK 6.4. If $E_i = \mathbb{C}$ for each i, then we come to the classical notion of Köthe sequence space.

Evidently, $\lambda(P, E)$ is a Hausdorff locally convex space with respect to the family of seminorms $\{\|\cdot\|_p : p \in P\}$.

PROPOSITION 6.11. *$\lambda(P,E)$ is complete iff all the E_i's are complete.*

We omit the proof, because it is a straightforward modification of the classical fact that ℓ^1 is complete.

Now let $A = \bigoplus_{n\geq 0} A^n$ be a locally finite graded algebra. As usual, we endow each A^n with the usual topology of a finite-dimensional vector space.

DEFINITION 6.4. A *graded submultiplicative seminorm* on A is a function $p\colon \mathbb{N} \to \bigcup_n \mathfrak{N}(A^n)$ such that $p_n = p(n) \in \mathfrak{N}(A^n)$ for all $n \in \mathbb{Z}_+$, and that $p_{i+j}(ab) \leq p_i(a)p_j(b)$ for all $i,j \in \mathbb{Z}_+$ and all $a \in A^i,\ b \in A^j$.

If p is a graded submultiplicative seminorm on A, then the associated seminorm $\|\cdot\|_p\colon A \to \mathbb{R}_+$ defined by $\|a\|_p = \sum_i p_i(a_i)$ for each $a = \sum_i a_i,\ a_i \in A^i$, is submultiplicative in the usual sense. Therefore graded submultiplicative seminorms on A are in 1-1 correspondence with submultiplicative seminorms $\|\cdot\|$ on A satisfying the condition $\|a\| = \sum_i \|a_i\|$ for each $a = \sum_i a_i,\ a_i \in A^i$.

Denote by P the collection of all graded submultiplicative seminorms on A.

LEMMA 6.12. *P is an A-power set.*

Proof. To check condition (1) of Definition 6.2, it suffices to show that for each n there exists $p \in P$ such that p_n is a norm on A^n. Fix a submultiplicative norm on the finite-dimensional algebra A/A_{n+1}, denote by $\tau_{n+1}\colon A \to A/A_{n+1}$ the quotient map, and set $p_i(a) = \|\tau_{n+1}(a)\|$ for each $i \in \mathbb{Z}_+$ and each $a \in A^i$. Evidently, p is a graded submultiplicative seminorm on A. Since $\|\cdot\|$ is a norm on A/A_{n+1}, and since $A^n \cap \operatorname{Ker}\tau_{n+1} = \{0\}$, we conclude that p_n is a norm on A^n.

Given $p,q \in P$, the function $r = \max\{p,q\}$ (i.e., $r_i(a) = \max\{p_i(a), q_i(a)\}$ for each $a \in A^i$ and each $i \in \mathbb{Z}_+$) clearly belongs to P. Hence condition (2) of Definition 6.2 is also satisfied, so that P is an A-power set. ∎

THEOREM 6.13. *Let $A = \bigoplus_{n\geq 0} A^n$ be a locally finite graded algebra, and let P be the set of all graded submultiplicative seminorms on A. Denote by ι_A the canonical embedding of A into $\lambda(P,A)$ that is the identity on each A^n. Then $\lambda(P,A)$ is a subalgebra of $\prod_n A^n$, and $(\lambda(P,A),\iota_A)$ is the Arens–Michael envelope of A.*

Proof. Given $a = (a_i)$ and $b = (b_j)$ in $\lambda(P,A)$, we must show that the element $c = ab \in \prod_n A^n$ defined by (31) belongs to $\lambda(P,A)$. For each $p \in P$ we have

$$\sum_k p_k(c_k) \leq \sum_k \sum_{i+j=k} p_k(a_i b_j) \leq \sum_k \sum_{i+j=k} p_i(a_i)p_j(b_j) = \sum_i p_i(a_i) \sum_j p_j(b_j) = \|a\|_p \|b\|_p.$$

Hence $ab \in \lambda(P,A)$, and $\|ab\|_p \leq \|a\|_p\|b\|_p$. This implies, in particular, that $\lambda(P,A)$ is an Arens–Michael algebra, and ι_A is an algebra homomorphism.

Now let $\varphi\colon A \to B$ be a homomorphism to some Arens–Michael algebra B. Fix a submultiplicative seminorm $\|\cdot\|$ on B, and define $p\colon \mathbb{Z}_+ \to \bigcup_n \mathfrak{N}(A^n)$ by $p_i(a_i) = \|\varphi(a_i)\|$ for each $i \in \mathbb{Z}_+$ and each $a_i \in A^i$. Evidently, p is a graded submultiplicative seminorm on A, and $\|\varphi(a)\| \leq \|\iota_A(a)\|_p$ for each $a \in A$. This implies that φ is continuous with respect to the topology induced on A from $\lambda(P,A)$. Since A is dense in $\lambda(P,A)$, we see that there exists a unique continuous homomorphism $\widehat{\varphi}\colon \lambda(P,A) \to B$ extending φ. Hence $\lambda(P,A)$ is the Arens–Michael envelope of A. ∎

COROLLARY 6.14. *Let $A = \bigoplus_{n\geq 0} A^n$ be a locally finite graded algebra, and let $\theta\colon A \to [A]$ be the canonical homomorphism* (29). *Then the induced homomorphism $\widehat{\theta}\colon \widehat{A} \to [A]$ (see* (30)) *is injective.*

Proof. If we identify $[A]$ with $\prod_n A^n$ via Proposition 6.10 and $\widehat{A}$ with $\lambda(P, A)$ via Theorem 6.13, then $\widehat{\theta}$ becomes the natural inclusion of $\lambda(P, A)$ into $\prod_n A^n$. ∎

Now let $\mathfrak{g} = \bigoplus_{n=1}^{\ell} \mathfrak{g}^n$ be a positively graded, finite-dimensional Lie algebra. As in the case of associative algebras (see above), we may define a filtration $\mathscr{F} = \{\mathfrak{g}_n\}$ on $\mathfrak{g}$ by setting $\mathfrak{g}_n = \bigoplus_{i\geq n} \mathfrak{g}^i$. It is easy to show that the universal enveloping algebra $U(\mathfrak{g})$ has a grading $U(\mathfrak{g}) = \bigoplus_{n\geq 0} U(\mathfrak{g})^n$ such that the associated filtration on $U(\mathfrak{g})$ coincides with (25). Indeed, let $T\mathfrak{g}$ be the tensor algebra of $\mathfrak{g}$, and let L be the two-sided ideal of $T\mathfrak{g}$ generated by elements of the form $x \otimes y - y \otimes x - [x, y]$, $x, y \in \mathfrak{g}$. Then we have $U(\mathfrak{g}) \cong T\mathfrak{g}/L$. If $\mathfrak{g}$ is graded, then we can define a grading on $T\mathfrak{g}$ by

$$(T\mathfrak{g})^n = \bigoplus_{i_1+\cdots+i_k=n} \mathfrak{g}^{i_1} \otimes \cdots \otimes \mathfrak{g}^{i_k}.$$

Thus $T\mathfrak{g}$ becomes a locally finite graded algebra, and L becomes a graded ideal of $T\mathfrak{g}$. Therefore $U(\mathfrak{g}) = T\mathfrak{g}/L$ is also a locally finite graded algebra. We have

$$U(\mathfrak{g})^n = \sum_{i_1+\cdots+i_k=n} \mathfrak{g}^{i_1} \cdots \mathfrak{g}^{i_k}.$$

Choose an $\mathscr{F}$-basis (e_i) of $\mathfrak{g}$ consisting of homogeneous elements, and set

$$V^n = \operatorname{span}\{e^\alpha : w(\alpha) = n\}.$$

By the Poincaré–Birkhoff–Witt theorem, we have $U(\mathfrak{g}) = \bigoplus_n V^n$. On the other hand, it is clear that $V^n \subset U(\mathfrak{g})^n$. Since $U(\mathfrak{g}) = \bigoplus_n U(\mathfrak{g})^n$, we conclude that $V^n = U(\mathfrak{g})^n$ for all n. Hence the associated filtration of $U(\mathfrak{g})$ has the form

$$U(\mathfrak{g})_n = \bigoplus_{m\geq n} U(\mathfrak{g})^m = \operatorname{span}\{e^\alpha : w(\alpha) \geq n\} = J_n$$

(see (25)).

Now Proposition 6.7, Example 6.3, and Corollary 6.14 imply the following.

PROPOSITION 6.15. *Let $\mathfrak{g}$ be a positively graded Lie algebra. Then $\widehat{U}(\mathfrak{g})$ together with the homomorphisms $\iota_{U(\mathfrak{g})}\colon U(\mathfrak{g}) \to \widehat{U}(\mathfrak{g})$ and $\widehat{\theta}\colon \widehat{U}(\mathfrak{g}) \to [U(\mathfrak{g})]$ is a power series envelope of $U(\mathfrak{g})$.*

6.3. The contractibility of $\widehat{U}'(\mathfrak{g})$. Let $\mathfrak{g}$ be a positively graded Lie algebra. In order to prove that $\widehat{U}(\mathfrak{g})$ is stably flat over $U(\mathfrak{g})$, it now remains to show that the strong dual, $\widehat{U}'(\mathfrak{g})$, of $\widehat{U}(\mathfrak{g})$ is a contractible $\widehat{\otimes}$-algebra (see Corollary 5.3). To this end, it will be convenient to use the following Lie algebra version of contractibility.

DEFINITION 6.5. We say that a finite-dimensional Lie algebra $\mathfrak{g}$ is *contractible* if there exists a smooth mapping $h\colon [0, 1] \times \mathfrak{g} \to \mathfrak{g}$ such that

(i) for each $t \in [0, 1]$ the map $h_t\colon \mathfrak{g} \to \mathfrak{g}$, $h_t(X) = h(t, X)$, is a Lie algebra homomorphism;
(ii) $h_0 = 0$ and $h_1 = 1_{\mathfrak{g}}$.

EXAMPLE 6.4. Each positively graded Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\ell$ is contractible. To see this, it suffices to set $h_t(X) = t^n X$ for each $X \in \mathfrak{g}_n$ and each $t \in [0,1]$.

EXAMPLE 6.5. Let $\mathfrak{g}$ be the 2-dimensional Lie algebra with basis X, Y and commutation relation $[X,Y] = Y$. Take a function $f \in C^\infty(\mathbb{R})$ such that $f(t) = 0$ for each $t \leq 0$ and $f(t) = 1$ for each $t \geq 1$, and define $h_t \colon \mathfrak{g} \to \mathfrak{g}$ by $h_t(X) = f(2t)X$ and $h_t(Y) = f(2t-1)Y$. It is easy to check that h_t satisfies the conditions of Definition 6.5, and so $\mathfrak{g}$ is contractible.

REMARK 6.5. It is easy to prove that each contractible Lie algebra is solvable. Indeed, suppose that $\mathfrak{g}$ is not solvable, and consider the Levi decomposition $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{l}$ ($\mathfrak{r} = \operatorname{rad}\mathfrak{g}$, $\mathfrak{l}$ is a semisimple subalgebra, $\mathfrak{l} \neq 0$). It is clear that a semidirect summand (i.e., a retract in the category of Lie algebras) of a contractible Lie algebra is contractible. Thus it suffices to show that $\mathfrak{l}$ is not contractible. Since $\mathfrak{l}$ is a direct sum of simple algebras, we need only prove that a simple Lie algebra is not contractible. Assume towards a contradiction that $\mathfrak{g}$ is both simple and contractible, and let $h_t \colon \mathfrak{g} \to \mathfrak{g}$ be a contracting homotopy from Definition 6.5. Since $\mathfrak{g}$ is simple, each h_t is either 0 or an automorphism. Replacing, if necessary, the segment $[0,1]$ by $[t_0,1]$ where $t_0 = \max\{t : h_t = 0\}$, we may assume that h_t is an automorphism for all $t > 0$. Let $B(\cdot,\cdot)$ denote the Killing form on $\mathfrak{g}$. By Cartan's criterion, B is nondegenerate. Since h_t is an automorphism, we have $B(h_t(X), h_t(Y)) = B(X,Y)$ for all $X, Y \in \mathfrak{g}$ and all $t > 0$. Letting $t \to 0$, we obtain $B \equiv 0$, which is a contradiction.

REMARK 6.6. It should be noted that not every nilpotent Lie algebra is contractible. For example, let $\mathfrak{g}$ be the 7-dimensional Lie algebra with basis $X_1, \ldots, X_7$ and commutation relations

$$[X_1, X_i] = X_{i+1} \quad (i = 2, \ldots, 6),$$
$$[X_2, X_3] = -X_6, \quad [X_3, X_4] = X_7, \quad [X_2, X_4] = [X_2, X_5] = -X_7$$

(see [16]). Let $\{h_t : t \in [0,1]\}$ be a continuous family of endomorphisms of $\mathfrak{g}$, and let $(h_{ij}(t))$ be the matrix of h_t with respect to the basis $X_1, \ldots, X_7$. A routine calculation shows that if $h_{11}(t_0) \neq 0$ and $h_{22}(t_0) \neq 0$ at some point t_0, then $h_{ii}(t_0) = 1$ for all $i = 1, \ldots, 7$. This clearly implies that $\mathfrak{g}$ is not contractible.

Our next goal is to prove that the contractibility of $\mathfrak{g}$ implies that of $\widehat{U}'(\mathfrak{g})$. We need some facts on topological vector spaces. Most of them are standard and can be easily deduced from [69] and [21].

Let E, F, and G be locally convex spaces (l.c.s.'s). Consider the vector space $\mathfrak{B}(E \times F, G)$ of all separately continuous bilinear mappings from $E \times F$ to G. We endow this space with the topology of *bibounded convergence* (i.e., the topology of uniform convergence on direct products of bounded sets). There is a natural mapping

$$\mathscr{L}(E, \mathscr{L}(F,G)) \to \mathfrak{B}(E \times F, G) \tag{32}$$

defined by the rule $\varphi \mapsto ((x,y) \mapsto \varphi(x)(y))$. Obviously, this mapping is topologically injective (i.e., is a homeomorphism onto its image). A bilinear map $\Phi \colon E \times F \to G$ belongs to the image of the mapping (32) iff for each 0-neighborhood $U \subset G$ and each bounded set $B \subset F$ there exists a 0-neighborhood $V \subset E$ such that $\Phi(V \times B) \subset U$. Such

bilinear maps are usually called *F-hypocontinuous*. If E is barrelled, then each separately continuous map of $E \times F$ to G is F-hypocontinuous (see [69, III.5.2]), so the mapping (32) is surjective in this case. Therefore, *for each barrelled l.c.s. E and arbitrary l.c.s.'s F and G we have a topological isomorphism*

$$\mathscr{L}(E, \mathscr{L}(F, G)) \cong \mathfrak{B}(E \times F, G). \tag{33}$$

Recall also (see [21, Chapitre II, Théorème 6] or [69, IV.9.4]) that *for each complete barrelled nuclear l.c.s. E and each complete l.c.s. F there exists a natural topological isomorphism*

$$E \mathbin{\widehat{\otimes}} F \to \mathscr{L}(E', F) \tag{34}$$

defined by $x \otimes y \mapsto (x' \mapsto \langle x, x' \rangle y)$.

LEMMA 6.16. *Let E be either a nuclear Fréchet space or a complete nuclear (DF)-space, and let F be a complete nuclear barrelled l.c.s. Then for each complete l.c.s. G there exists a topological isomorphism*

$$\mathscr{L}(E, F \mathbin{\widehat{\otimes}} G) \xrightarrow{\sim} \mathscr{L}(F', E' \mathbin{\widehat{\otimes}} G) \tag{35}$$

taking each $u\colon E \to F \mathbin{\widehat{\otimes}} G$ to $v\colon F' \to E' \mathbin{\widehat{\otimes}} G$ such that

$$\langle v(y'), x \otimes z' \rangle = \langle u(x), y' \otimes z' \rangle \tag{36}$$

for each $x \in E$, $y' \in F'$, $z' \in G'$.

Proof. Applying (34) and (33), we obtain topological isomorphisms

$$\mathscr{L}(E, F \mathbin{\widehat{\otimes}} G) \cong \mathscr{L}(E, \mathscr{L}(F', G)) \cong \mathfrak{B}(E \times F', G) \cong \mathfrak{B}(F' \times E, G). \tag{37}$$

Since F is complete and nuclear, it is semireflexive [69, IV.5], and hence F' is barrelled (see [69, IV.5.5]). Further, the assumptions on E imply that E is reflexive, and E' is barrelled and nuclear [21]. Using again (33) and (34), we see that

$$\mathfrak{B}(F' \times E, G) \cong \mathscr{L}(F', \mathscr{L}(E, G)) \cong \mathscr{L}(F', \mathscr{L}(E'', G)) \cong \mathscr{L}(F', E' \mathbin{\widehat{\otimes}} G). \tag{38}$$

Combining (37) and (38), we obtain the required isomorphism (35). Relation (36) is then readily verified. ∎

Recall that for each smooth manifold M and each complete l.c.s. X there exists a topological isomorphism $C^\infty(M) \mathbin{\widehat{\otimes}} X \cong C^\infty(M, X)$ taking an elementary tensor $f \otimes x$ to the function $t \mapsto f(t)x$ (see [21, Chap. II, §3, no. 3]). Applying the previous lemma to $G = C^\infty(M)$, we obtain the following.

COROLLARY 6.17. *Let E and F be locally convex spaces satisfying the conditions of Lemma* 6.16, *and let M be a smooth manifold. Then there exists a topological isomorphism*

$$\mathscr{L}(E, C^\infty(M, F)) \cong \mathscr{L}(F', C^\infty(M, E'))$$

taking each $u\colon E \to C^\infty(M, F)$ to $v\colon F' \to C^\infty(M, E')$ such that

$$\langle v(y')(t), x \rangle = \langle y', u(x)(t) \rangle$$

for each $x \in E$, $y' \in F'$, $t \in M$.

THEOREM 6.18. *Let $\mathfrak{g}$ be a contractible, finite-dimensional Lie algebra. Then $\widehat{U}'(\mathfrak{g})$ is contractible as a commutative $\widehat{\otimes}$-algebra.*

Proof. Set $I = [0,1]$ and suppose that $h\colon I \times \mathfrak{g} \to \mathfrak{g}$ is a smooth map satisfying the conditions of Definition 6.5. Note that the space $C^\infty(I,\mathfrak{g})$ is a Lie algebra with respect to the pointwise multiplication. It is readily seen that the map

$$F\colon \mathfrak{g} \to C^\infty(I,\mathfrak{g}), \qquad F(X)(t) = h(t,X),$$

is a Lie algebra homomorphism. Using the universal property of $\widehat{U} = \widehat{U}(\mathfrak{g})$ (see Example 6.3) and the obvious fact that $C^\infty(I,\widehat{U})$ is an Arens–Michael algebra, we obtain a unique continuous homomorphism $\Psi\colon \widehat{U} \to C^\infty(I,\widehat{U})$ that fits into the commutative diagram

$$\begin{array}{ccc} \widehat{U} & \xrightarrow{\Psi} & C^\infty(I,\widehat{U}) \\ {\scriptstyle \iota_{\mathfrak{g}}}\uparrow & & \uparrow{\scriptstyle C^\infty(I,\iota_{\mathfrak{g}})} \\ \mathfrak{g} & \xrightarrow[F]{} & C^\infty(I,\mathfrak{g}) \end{array}$$

For each $t \in I$ define $\psi_t\colon \widehat{U} \to \widehat{U}$ by $\psi_t(x) = \Psi(x)(t)$. Evidently, ψ_t is an algebra homomorphism. Using the above diagram, it is readily seen that ψ_t extends h_t in the sense that $\psi_t \iota_{\mathfrak{g}} = \iota_{\mathfrak{g}} h_t$. Hence $\psi_t = \widehat{U}(h_t)$ (see Example 6.3), and so ψ_t is a Hopf $\widehat{\otimes}$-algebra homomorphism. Since $h_1 = 1_{\mathfrak{g}}$ and $h_0 = 0$, we see that $\psi_1 = 1_{\widehat{U}}$ and $\psi_0 = \eta_{\widehat{U}}\varepsilon_{\widehat{U}}$.

Now set $A = \widehat{U}'$ and let $\Phi\colon A \to C^\infty(I,A)$ be the map corresponding to Ψ under the isomorphism

$$\mathscr{L}(\widehat{U}, C^\infty(I,\widehat{U})) \cong \mathscr{L}(A, C^\infty(I,A))$$

(see Corollary 6.17). For each $a \in A$, $t \in I$, and $x \in \widehat{U}$ we have

$$\langle \Phi(a)(t), x\rangle = \langle a, \Psi(x)(t)\rangle = \langle a, \psi_t(x)\rangle = \langle a \circ \psi_t, x\rangle.$$

Hence $\Phi(a)(t) = a \circ \psi_t$. In other words, for each $t \in I$ the map $\varphi_t\colon A \to A$ defined by $\varphi_t(a) = \Phi(a)(t)$ is the dual of ψ_t. Since ψ_t is a $\widehat{\otimes}$-coalgebra homomorphism, we conclude that φ_t is a $\widehat{\otimes}$-algebra homomorphism. Hence so is Φ. Note also that $\varphi_1 = 1_A$ and $\varphi_0 = (\eta_{\widehat{U}}\varepsilon_{\widehat{U}})' = \eta_A\varepsilon_A$.

Now it is easy to check that $\Phi\colon A \to C^\infty(I,A) \cong C^\infty(I)\,\widehat{\otimes}\,A$ yields a homotopy between 1_A and $\eta_A\varepsilon_A$ (see Definition 4.2). Indeed, consider the augmentations $\varepsilon_i\colon C^\infty(I) \to \mathbb{C}$, $\varepsilon_i(f) = f(i)$ $(i = 0,1)$. Then for each $a \in A$ we have

$$\begin{aligned} ((\varepsilon_0 \otimes 1_A)\Phi)(a) &= \Phi(a)(0) = \varphi_0(a) = (\eta_A\varepsilon_A)(a), \\ ((\varepsilon_1 \otimes 1_A)\Phi)(a) &= \Phi(a)(1) = \varphi_1(a) = a. \end{aligned}$$

Hence $(\varepsilon_0 \otimes 1_A)\,\Phi = \eta_A\varepsilon_A$, and $(\varepsilon_1 \otimes 1_A)\,\Phi = 1_A$. Therefore 1_A is homotopic to $\eta_A\varepsilon_A$, i.e., A is contractible. ∎

Now, applying Corollary 5.3, Proposition 6.15, and Theorem 6.18, we obtain the following.

THEOREM 6.19. *Let $\mathfrak{g}$ be a finite-dimensional, positively graded Lie algebra. Then $\widehat{U}(\mathfrak{g})$ is stably flat over $U(\mathfrak{g})$.*

We end this section with an application of the above theorem to computing injective homological dimensions of $\widehat{U}(\mathfrak{g})$-modules. To this end, we need a formula of "Poincaré

duality" type. Let $\mathfrak{g}$ be a Lie algebra of dimension n. Recall (see, e.g., [34, 6.10]) that for each left $\mathfrak{g}$-module V there exist vector space isomorphisms

$$H^p_{\mathrm{Lie}}(\mathfrak{g}, V) \cong H^{\mathrm{Lie}}_{n-p}(\mathfrak{g}, V \otimes (\textstyle\bigwedge^n \mathfrak{g})^*) \quad (p \in \mathbb{Z}).$$

If $\mathfrak{g}$ is nilpotent, then it is easily seen that the action of $\mathfrak{g}$ on $\bigwedge^n \mathfrak{g}$ is trivial. (To see this, it suffices to take a basis (e_i) of $\mathfrak{g}$ with the property that $[e_i, e_j] \in \operatorname{span}\{e_k : k \geq \max\{i,j\}\}$ and to observe that each e_i acts on $e_1 \wedge \cdots \wedge e_n$ trivially.) Therefore the above formula takes the form

$$H^p_{\mathrm{Lie}}(\mathfrak{g}, V) \cong H^{\mathrm{Lie}}_{n-p}(\mathfrak{g}, V) \quad (p \in \mathbb{Z}).$$

Combining this with Proposition 3.4, we obtain the following.

COROLLARY 6.20. *Let $\mathfrak{g}$ be a finite-dimensional, positively graded Lie algebra, and let $n = \dim \mathfrak{g}$. Then for each $M \in \widehat{U}(\mathfrak{g})$-**mod**-$\widehat{U}(\mathfrak{g})$ there exist vector space isomorphisms*

$$\mathscr{H}^p(\widehat{U}(\mathfrak{g}), M) \cong \mathscr{H}_{n-p}(\widehat{U}(\mathfrak{g}), M) \quad (p \in \mathbb{Z}).$$

COROLLARY 6.21. *Let $\mathfrak{g}$ be a finite-dimensional, positively graded Lie algebra. Then*

(i) $\operatorname{inj.dh}_{\widehat{U}(\mathfrak{g})} M = \dim \mathfrak{g}$ *for each* $M \in \widehat{U}(\mathfrak{g})$-**mod**, $M \neq 0$;

(ii) $\operatorname{dh}_{\widehat{U}(\mathfrak{g})} M = \dim \mathfrak{g}$ *for each Banach* $M \in \widehat{U}(\mathfrak{g})$-**mod**, $M \neq 0$.

In particular, there are no nonzero injective $\widehat{U}(\mathfrak{g})$-$\widehat{\otimes}$-modules.

Proof. This is an immediate consequence of [62, Theorem 2.1], [63, Corollary 4.1.3], and Corollary 6.20. ∎

7. Weighted completions of universal enveloping algebras

In this section we describe one more class of Fréchet Hopf algebras that are stably flat completions of universal enveloping algebras. These algebras were introduced by Goodman in [19] and [20]. Each of them is a power series envelope of $U(\mathfrak{g})$ (see Definition 5.1) and consists of power series $x \in [U]$ subject to certain growth conditions.

Recall some definitions and notation from [19] and [20]. Let $\mathfrak{g}$ be a nilpotent Lie algebra, and let $N = \dim \mathfrak{g}$. Choose a positive filtration $\mathscr{F}$ on $\mathfrak{g}$, and fix an $\mathscr{F}$-basis (e_i) for $\mathfrak{g}$ (see Section 5). A sequence $\mathscr{M} = \{M_\alpha : \alpha \in \mathbb{Z}_+^N\}$ of positive numbers is an *$\mathscr{F}$-weight sequence* if $M_0 = 1$ and $M_\gamma \leq M_\alpha M_\beta$ whenever $w(\gamma) \geq w(\alpha) + w(\beta)$. Given an $\mathscr{F}$-weight sequence $\mathscr{M}$, consider the space

$$U(\mathfrak{g})_{\mathscr{M}} = \Big\{ x = \sum_\alpha c_\alpha e^\alpha \in [U] : \|x\|_r = \sum_\alpha |c_\alpha| \alpha! M_\alpha r^{w(\alpha)} < \infty \ \forall r > 0 \Big\}.$$

Clearly, $U(\mathfrak{g})_{\mathscr{M}}$ is a Fréchet space with respect to the topology defined by the family of seminorms $\{\|\cdot\|_r : r > 0\}$. Using the Grothendieck–Pietsch criterion (see, e.g., [61]), it is easy to see that $U(\mathfrak{g})_{\mathscr{M}}$ is nuclear. Goodman [19] proved that $U(\mathfrak{g})_{\mathscr{M}}$ is a subalgebra of $[U]$, and that the multiplication in $U(\mathfrak{g})_{\mathscr{M}}$ is (jointly) continuous with respect to the above topology. Note, however, that $U(\mathfrak{g})_{\mathscr{M}}$ need not be an Arens–Michael algebra.

EXAMPLE 7.1. Let $\mathfrak{g}$ be an abelian Lie algebra endowed with the trivial filtration $\mathscr{F}$ (i.e., $\mathscr{F}_1 = \mathfrak{g}$ and $\mathscr{F}_2 = 0$), and let $M_\alpha = |\alpha|^{-|\alpha|}$. Then it is easy to see that $U(\mathfrak{g})_{\mathscr{M}}$ is

isomorphic to the algebra $\mathscr{O}(\mathbb{C}^N)$ of entire functions on $\mathbb{C}^N$. Indeed, $\mathscr{O}(\mathbb{C}^N)$ is topologized by the family of seminorms $\|\cdot\|'_r$ $(r>0)$ defined by $\|f\|'_r = \sum_\alpha |c_\alpha| r^{|\alpha|}$ for each $f(z) = \sum_\alpha c_\alpha z^\alpha \in \mathscr{O}(\mathbb{C}^N)$. We clearly have $\alpha! \le |\alpha|^{|\alpha|}$ for each $\alpha \in \mathbb{Z}_+^N$. On the other hand, the Cauchy estimates applied to the entire function $z \mapsto \exp(\sum_i z_i)$ imply that $|\alpha|^{|\alpha|} \le C^{|\alpha|}\alpha!$ for some constant $C>0$. Since $w(\alpha) = |\alpha|$ in this case, we obtain $\|f\|'_{r/C} \le \|f\|_r \le \|f\|'_r$ for all polynomials f and all $r>0$. Hence the families of seminorms $\|\cdot\|_r$ and $\|\cdot\|'_r$ are equivalent, and so $U(\mathfrak{g})_{\mathscr{M}}$ and $\mathscr{O}(\mathbb{C}^N)$ are isomorphic.

EXAMPLE 7.2. Let $\mathfrak{g}$ be an abelian Lie algebra endowed with the trivial filtration, and let $M_\alpha = 1$ for all α. Then $U(\mathfrak{g})_{\mathscr{M}}$ is isomorphic to the algebra of entire functions on $\mathbb{C}^N$ of exponential order ≤ 1 and minimal type (cf. [67]).

An $\mathscr{F}$-weight sequence $\mathscr{M}$ is *entire* [20] if it satisfies the following two conditions:

$$\begin{aligned} &\sum_\alpha M_\alpha r^{w(\alpha)} < \infty \quad \text{for all } r>0, \\ &\sup_{\alpha,\beta\ne 0}\{A^{w(\alpha)/w(\beta)} M_\beta^{1/w(\beta)} M_\alpha^{-1/w(\alpha)}\} < \infty \quad \text{for some } A>0. \end{aligned} \tag{39}$$

For instance, the weight sequence of Example 7.1 is entire [20], while that of Example 7.2 is not entire.

If $\mathscr{M}$ is an entire $\mathscr{F}$-weight sequence, then the dual of $U(\mathfrak{g})_{\mathscr{M}}$ admits an explicit description as a certain function algebra [20]. Namely, let G be the connected, simply connected complex Lie group corresponding to $\mathfrak{g}$. Since $\mathfrak{g}$ is nilpotent, the exponential map $\exp\colon \mathfrak{g} \to G$ is biholomorphic. The *homogeneous norm* on G is defined by

$$|g| = \max_i |t_i|^{1/w_i} \quad \text{for each } g = \exp\Big(\sum_i t_i e_i\Big) \in G.$$

Given $z \in \mathbb{C}$, define a linear map $\delta_z\colon \mathfrak{g} \to \mathfrak{g}$ by $\delta_z(e_i) = z^{w_i} e_i$. We use the same symbol δ_z to denote the corresponding holomorphic self-map of G satisfying $\delta_z \circ \exp = \exp \circ \delta_z$. It is immediate that $\delta_1 = 1_G$, $\delta_0(g) = e$ for all $g \in G$ (here e is the identity of G), $\delta_z \delta_{z'} = \delta_{zz'}$, $\delta_z^{-1} = \delta_{z^{-1}}$ for each $z \ne 0$, and $|\delta_z g| = |z|\,|g|$ for each $z \in \mathbb{C}$, $g \in G$.

Given an entire $\mathscr{F}$-weight sequence $\mathscr{M}$, define a weight function $W_{\mathscr{M}}$ on G by

$$W_{\mathscr{M}}(g) = \sum_\alpha M_\alpha |g|^{w(\alpha)}.$$

Condition (39) implies that $W_{\mathscr{M}}$ is finite on G. For example, if $\mathfrak{g}$ is abelian and $M_\alpha = |\alpha|^{-|\alpha|}$ (see Example 7.1), then $W_{\mathscr{M}}$ satisfies the estimate

$$\exp(N|g|/C) \le W_{\mathscr{M}}(g) \le \exp(N|g|). \tag{40}$$

Given $r>0$, consider the space

$$A_{\mathscr{M},r}(G) = \Big\{ f \in \mathscr{O}(G) : N_r(f) = \sup_{g\in G} \frac{|f(g)|}{W_{\mathscr{M}}(\delta_r g)} < \infty \Big\}.$$

Evidently, $A_{\mathscr{M},r}(G)$ is a Banach space with respect to the norm N_r. Note that $W_{\mathscr{M}}(\delta_s g) \le W_{\mathscr{M}}(\delta_r g)$ whenever $0 \le s \le r$. This implies that $A_{\mathscr{M},s}(G) \subset A_{\mathscr{M},r}(G)$ for each $s \le r$, and $N_r(f) \le N_s(f)$ for each $f \in A_{\mathscr{M},s}(G)$. Therefore we may consider the locally convex space

$$A_{\mathscr{M}}(G) = \varinjlim A_{\mathscr{M},r}(G).$$

Goodman [20] proved that $A_{\mathscr{M}}(G)$ is a subalgebra of $\mathscr{O}(G)$ (under pointwise multiplication), and that the multiplication is jointly continuous with respect to the inductive limit topology on $A_{\mathscr{M}}(G)$.

For example, if $\mathfrak{g}$ is abelian and $M_\alpha = |\alpha|^{-|\alpha|}$ (Example 7.1), then it follows from (40) that $A_{\mathscr{M}}(G)$ is the algebra of entire functions on $G = \mathfrak{g}$ of exponential order ≤ 1.

Denote by $\mathscr{P}(G)$ the algebra of polynomial functions on G (i.e., of functions f such that $f \circ \exp$ is a polynomial on $\mathfrak{g}$). This is a dense subalgebra of $A_{\mathscr{M}}(G)$ (see [20]). Using the identification $\mathscr{P}(G) \otimes \mathscr{P}(G) \cong \mathscr{P}(G \times G)$, one can show that $\mathscr{P}(G)$ has a Hopf algebra structure given by (20) (cf. [19, Prop. 2.1]). The algebra $U(\mathfrak{g})$ acts on $\mathscr{P}(G)$ via left-invariant differential operators, and this leads to a canonical Hopf algebra pairing $U(\mathfrak{g}) \times \mathscr{P}(G) \to \mathbb{C}$ defined by $\langle a, f\rangle = (af)(e)$ for $a \in U(\mathfrak{g})$, $f \in \mathscr{P}(G)$ (cf. [29, XVI.3]). Goodman [20] proved that this pairing extends to a pairing $U(\mathfrak{g})_{\mathscr{M}} \times A_{\mathscr{M}}(G) \to \mathbb{C}$ and defines a topological isomorphism between $U(\mathfrak{g})_{\mathscr{M}}$ and the strong dual space of $A_{\mathscr{M}}(G)$. Since $U(\mathfrak{g})_{\mathscr{M}}$ is a nuclear Fréchet space, it follows that the multiplication on $A_{\mathscr{M}}(G)$ yields (by duality) a comultiplication $U(\mathfrak{g})_{\mathscr{M}} \to U(\mathfrak{g})_{\mathscr{M}} \mathbin{\widehat{\otimes}} U(\mathfrak{g})_{\mathscr{M}}$ that extends the comultiplication of $U(\mathfrak{g})$. Similarly, the multiplication on $U(\mathfrak{g})_{\mathscr{M}}$ yields a comultiplication $A_{\mathscr{M}}(G) \to A_{\mathscr{M}}(G) \mathbin{\widehat{\otimes}} A_{\mathscr{M}}(G)$ that extends the comultiplication of $\mathscr{P}(G)$. It is also easy to see that the antipode and the counit of $U(\mathfrak{g})$ (resp. $\mathscr{P}(G)$) extend by continuity to $U(\mathfrak{g})_{\mathscr{M}}$ (resp. $A_{\mathscr{M}}(G)$), so that $U(\mathfrak{g})_{\mathscr{M}}$ (resp. $A_{\mathscr{M}}(G)$) becomes a Hopf $\widehat{\otimes}$-algebra containing $U(\mathfrak{g})$ (resp. $\mathscr{P}(G)$) as a dense Hopf subalgebra. Thus $U(\mathfrak{g})_{\mathscr{M}}$ and $A_{\mathscr{M}}(G)$ are well-behaved Hopf $\widehat{\otimes}$-algebras dual to each other.

The above properties of $U(\mathfrak{g})_{\mathscr{M}}$ imply the following.

PROPOSITION 7.1. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with a positive filtration $\mathscr{F}$, and let $\mathscr{M}$ be an entire $\mathscr{F}$-weight sequence. Then $U(\mathfrak{g})_{\mathscr{M}}$ is a well-behaved power series envelope of $U(\mathfrak{g})$.*

PROPOSITION 7.2. *$A_{\mathscr{M}}(G)$ is contractible.*

Proof. Given a function $f\colon G \to \mathbb{C}$ and $z \in \mathbb{C}$, define $f_z\colon G \to \mathbb{C}$ by $f_z(g) = f(\delta_z g)$. Using the obvious identity $W_{\mathscr{M}}(\delta_z g) = W_{\mathscr{M}}(\delta_{|z|} g)$, we obtain

$$N_r(f_z) = \sup_g \frac{|f(\delta_z g)|}{W_{\mathscr{M}}(\delta_r g)} = \sup_h \frac{|f(h)|}{W_{\mathscr{M}}(\delta_r \delta_z^{-1} h)} = \sup_h \frac{|f(h)|}{W_{\mathscr{M}}(\delta_{r|z|^{-1}} h)} = N_{r|z|^{-1}}(f) \tag{41}$$

for each $r > 0$ and each $z \neq 0$. Therefore for each $f \in A_{\mathscr{M}}(G)$ we have $f_z \in A_{\mathscr{M}}(G)$, and the mapping $A_{\mathscr{M}}(G) \to A_{\mathscr{M}}(G)$, $f \mapsto f_z$, is continuous. Note also that $f_1 = f$ and $f_0 = f(e)1$ for each $f \in A_{\mathscr{M}}(G)$.

For each $f \in \mathscr{P}(G)$, the function $(z, g) \mapsto f_z(g)$ is clearly a polynomial on $\mathbb{C} \times G$. Therefore we have an algebra homomorphism

$$\Phi_0\colon \mathscr{P}(G) \to \mathscr{P}(\mathbb{C}, \mathscr{P}(G)) \cong \mathscr{P}(\mathbb{C} \times G), \qquad \Phi_0(f)(z) = f_z.$$

We use the same symbol Φ_0 to denote the composition of the above homomorphism with the canonical embedding $\mathscr{P}(\mathbb{C}, \mathscr{P}(G)) \hookrightarrow \mathscr{O}(\mathbb{C}, A_{\mathscr{M}}(G))$.

We claim that Φ_0 is continuous with respect to the topology on $\mathscr{P}(G)$ inherited from $A_{\mathscr{M}}(G)$ and the compact-open topology on $\mathscr{O}(\mathbb{C}, A_{\mathscr{M}}(G))$. Indeed, let $\|\cdot\|$ be a continuous

seminorm on $A_{\mathscr{M}}(G)$ and let $R > 0$. Then the rule

$$\|u\|_R = \sup\{\|u(z)\| : |z| \le R\}$$

defines a continuous seminorm on $\mathscr{O}(\mathbb{C}, A_{\mathscr{M}}(G))$. Furthermore, the compact-open topology on $\mathscr{O}(\mathbb{C}, A_{\mathscr{M}}(G))$ is generated by all seminorms of this form. Therefore to prove the continuity of Φ_0 we have to show that for each continuous seminorm $\|\cdot\|$ on $A_{\mathscr{M}}(G)$ and each $R > 0$ the seminorm $f \mapsto \|\Phi_0(f)\|_R$ is continuous on $\mathscr{P}(G)$. Since $\|\cdot\|$ is continuous on $A_{\mathscr{M}}(G)$, we see that for each $r > 0$ there exists $C > 0$ such that $\|f\| \le CN_{rR}(f)$ for all $f \in A_{\mathscr{M},rR}(G)$. Now let f be in $\mathscr{P}(G)$. Using (41) and the fact that $N_r \le N_s$ whenever $s \le r$, we obtain

$$\|\Phi_0(f)\|_R = \sup_{|z|\le R} \|f_z\| \le C \sup_{|z|\le R} N_{rR}(f_z) = C \sup_{0<|z|\le R} N_{rR|z|^{-1}}(f) = CN_r(f).$$

This means that the seminorm $f \mapsto \|\Phi_0(f)\|_R$ is continuous on $\mathscr{P}(G)$ with respect to the topology inherited from $A_{\mathscr{M}}(G)$. Therefore Φ_0 is continuous. Since $\mathscr{P}(G)$ is dense in $A_{\mathscr{M}}(G)$ (cf. [20]), we see that Φ_0 extends to a continuous homomorphism

$$\Phi\colon A_{\mathscr{M}}(G) \to \mathscr{O}(\mathbb{C}, A_{\mathscr{M}}(G)) \cong \mathscr{O}(\mathbb{C}) \mathbin{\widehat{\otimes}} A_{\mathscr{M}}(G).$$

Let $\varepsilon\colon A_{\mathscr{M}}(G) \to \mathbb{C}$, $f \mapsto f(e)$, denote the counit of $A_{\mathscr{M}}(G)$. We claim that Φ is a homotopy between $1_{A_{\mathscr{M}}(G)}$ and $\eta\varepsilon$ (see Definition 4.2). Indeed, for each $z \in \mathbb{C}$ the mappings $f \mapsto f_z$ and $f \mapsto \Phi(f)(z)$ from $A_{\mathscr{M}}(G)$ to itself are continuous, and they coincide on $\mathscr{P}(G)$. Hence $\Phi(f)(z) = f_z$ for each $f \in A_{\mathscr{M}}(G)$ and each $z \in \mathbb{C}$. In particular, $\Phi(f)(1) = f$ and $\Phi(f)(0) = f(e)1 = (\eta\varepsilon)(f)$. In other words,

$$(\varepsilon_1 \otimes 1_{A_{\mathscr{M}}(G)})\,\Phi = 1_{A_{\mathscr{M}}(G)}, \qquad (\varepsilon_0 \otimes 1_{A_{\mathscr{M}}(G)})\,\Phi = \eta\varepsilon,$$

where the augmentations $\varepsilon_k\colon \mathscr{O}(\mathbb{C}) \to \mathbb{C}$ $(k = 0, 1)$ are defined by $\varepsilon_k(f) = f(k)$. Since $\mathscr{O}(\mathbb{C})$ is an exact algebra, we conclude that $1_{A_{\mathscr{M}}(G)}$ is homotopic to $\eta\varepsilon$, and so $A_{\mathscr{M}}(G)$ is contractible. ∎

Now Proposition 7.1, Proposition 7.2, and Corollary 5.3 imply the following.

THEOREM 7.3. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with a positive filtration $\mathscr{F}$, and let $\mathscr{M}$ be an entire $\mathscr{F}$-weight sequence. Then $U(\mathfrak{g})_{\mathscr{M}}$ is stably flat over $U(\mathfrak{g})$.*

8. Algebras of analytic functionals and hyperenveloping algebras

Let $\mathfrak{g}$ be a Lie algebra, and let G denote the corresponding connected, simply connected complex Lie group. In this section we prove that the *hyperenveloping algebra* $\mathfrak{F}(\mathfrak{g})$ (see [67]) is always stably flat over $U(\mathfrak{g})$. We also show that $\mathscr{A}(G)$, the *algebra of analytic functionals* (see [42]), is stably flat over $U(\mathfrak{g})$ if and only if $\mathfrak{g}$ is solvable.

First recall some definitions. Let G be a complex Lie group. The Fréchet algebra $\mathscr{O}(G)$ of holomorphic functions on G has a canonical structure of Hopf $\widehat{\otimes}$-algebra given by (20). Since $\mathscr{O}(G)$ is nuclear, the strong dual space, $\mathscr{O}(G)'$, is a Hopf $\widehat{\otimes}$-algebra and, in addition, a nuclear (DF)-space. It is denoted by $\mathscr{A}(G)$ and is called the *algebra of analytic functionals* on G (see [42]). The product of $\alpha, \beta \in \mathscr{A}(G)$ is called the *convolution* and is

denoted by $\alpha * \beta$. By definition, we have $\langle \alpha * \beta, f\rangle = \langle \alpha \otimes \beta, \Delta f\rangle$ for each $\alpha, \beta \in \mathscr{A}(G)$ and each $f \in \mathscr{O}(G)$.

Consider the algebra $\mathscr{O}_e$ of germs of holomorphic functions at the identity $e \in G$. We endow $\mathscr{O}_e$ with its usual inductive limit topology, i.e., $\mathscr{O}_e = \varinjlim \mathscr{O}(U)$, where U runs through the collection of all neighborhoods of e. Relative to this topology, $\mathscr{O}_e$ becomes a nuclear, complete (DF)-space (see [21, Chap. II, §2, no. 3]). Moreover, the multiplication in $\mathscr{O}_e$ is jointly continuous, so that $\mathscr{O}_e$ is a $\widehat{\otimes}$-algebra.

By localizing (20) at the identity, we obtain a Hopf $\widehat{\otimes}$-algebra structure on $\mathscr{O}_e$ (cf. [44, 4.2] and [57, 3.2.3]). More exactly, take a neighborhood U of e, choose a neighborhood $V \ni e$ such that $V^2 \subset U$, and consider the map

$$\Delta_{UV} \colon \mathscr{O}(U) \to \mathscr{O}(V \times V) \cong \mathscr{O}(V) \mathbin{\widehat{\otimes}} \mathscr{O}(V), \qquad (\Delta_{UV} f)(x, y) = f(xy).$$

Composing with the restriction map $\mathscr{O}(V) \mathbin{\widehat{\otimes}} \mathscr{O}(V) \to \mathscr{O}_e \mathbin{\widehat{\otimes}} \mathscr{O}_e$ and taking the direct limit over $U \ni e$, we obtain a comultiplication $\Delta \colon \mathscr{O}_e \to \mathscr{O}_e \mathbin{\widehat{\otimes}} \mathscr{O}_e$. The counit and the antipode are defined similarly using (20). Since all Lie groups with the same Lie algebra are locally isomorphic, the Hopf algebra structure on $\mathscr{O}_e$ depends only on $\mathfrak{g}$.

By definition, the *hyperenveloping algebra* $\mathfrak{F}(\mathfrak{g})$ is the strong dual algebra of $\mathscr{O}_e$. (Note that the original definition of $\mathfrak{F}(\mathfrak{g})$ given by Rashevskii in [67] was different; we follow the approach suggested by Litvinov [42, 44].) Since $\mathscr{O}_e$ is a nuclear (DF)-space, $\mathfrak{F}(\mathfrak{g})$ is a nuclear Fréchet space.

Let $\mathfrak{m}_e$ be the ideal of $\mathscr{O}_e$ consisting of all germs vanishing at e. Consider the formal completion $\widehat{\mathscr{O}}_e = \varprojlim \mathscr{O}_e/\mathfrak{m}_e^n$. We endow each quotient $\mathscr{O}_e/\mathfrak{m}_e^n$ with the standard topology of a finite-dimensional vector space, so that $\widehat{\mathscr{O}}_e$ becomes a nuclear Fréchet algebra. Moreover, the comultiplication and the antipode of $\mathscr{O}_e$ extend to $\widehat{\mathscr{O}}_e$ (cf. [57, 3.2.3]), so that $\widehat{\mathscr{O}}_e$ has a canonical structure of Hopf $\widehat{\otimes}$-algebra.

There is a natural Hopf algebra pairing between $U(\mathfrak{g})$ and $\widehat{\mathscr{O}}_e$ defined as follows (for details, see [57, 3.2]). For each $X \in \mathfrak{g}$, let $\widetilde{X}$ denote the corresponding left-invariant vector field on G. For each open set $U \subset G$ we use the same symbol $\widetilde{X}$ to denote the corresponding derivation of $\mathscr{O}(U)$. Taking the direct limit over $U \ni e$, we see that $\widetilde{X}$ determines a derivation of $\mathscr{O}_e$ which we also denote by $\widetilde{X}$. It is easy to see that $\widetilde{X}(\mathfrak{m}_e^n) \subset \mathfrak{m}_e^{n-1}$ for each n, so that $\widetilde{X}$ extends to a derivation of $\widehat{\mathscr{O}}_e$ (again denoted by $\widetilde{X}$). The resulting map $\mathfrak{g} \to \operatorname{Der} \widehat{\mathscr{O}}_e$, $X \mapsto \widetilde{X}$, yields an algebra homomorphism $\varrho \colon U(\mathfrak{g}) \to \operatorname{End}_{\mathbb{C}} \widehat{\mathscr{O}}_e$. Thus $U(\mathfrak{g})$ acts on $\widehat{\mathscr{O}}_e$ via "formal left-invariant differential operators" (cf. Section 7). The canonical pairing between $U(\mathfrak{g})$ and $\widehat{\mathscr{O}}_e$ defined by $\langle a, \mathbf{f}\rangle = [\varrho(a)\mathbf{f}](e)$ for each $a \in U(\mathfrak{g})$, $\mathbf{f} \in \widehat{\mathscr{O}}_e$ gives an algebraic isomorphism between $\widehat{\mathscr{O}}_e$ and the algebraic dual of $U(\mathfrak{g})$ [57, 3.2.3]. If we endow $U(\mathfrak{g})$ with the finest locally convex topology, then $\widehat{\mathscr{O}}_e$ becomes the topological dual of $U(\mathfrak{g})$, and the strong dual topology on $\widehat{\mathscr{O}}_e$ coincides with the inverse limit topology introduced above (cf. the beginning of Section 4).

The restriction maps

$$\mathscr{O}(G) \to \mathscr{O}_e \to \widehat{\mathscr{O}}_e$$

are obviously Hopf $\widehat{\otimes}$-algebra homomorphisms. Taking the dual maps, we obtain Hopf $\widehat{\otimes}$-algebra homomorphisms

$$U(\mathfrak{g}) \xrightarrow{\lambda} \mathfrak{F}(\mathfrak{g}) \to \mathscr{A}(G). \tag{42}$$

Note that $\mathscr{O}_e \to \widehat{\mathscr{O}}_e$ is always injective with dense range, so that $U(\mathfrak{g}) \to \mathfrak{F}(\mathfrak{g})$ has the same property. The restriction map $\mathscr{O}(G) \to \mathscr{O}_e$ is injective provided G is connected, and has dense range provided G is a Stein group. Therefore for each connected Stein group (in particular, for each connected, simply connected complex Lie group) both the maps in (42) are injective with dense ranges (cf. [42]).

Let $\tau\colon U(\mathfrak{g}) \to \mathscr{A}(G)$ denote the composition of the above maps. It follows from the definition of the duality between $U(\mathfrak{g})$ and $\widehat{\mathscr{O}}_e$ that $\langle \tau(X), f\rangle = (\widetilde{X}f)(e)$ for all $X \in \mathfrak{g}$, $f \in \mathscr{O}(G)$. It is also easy to see that for each $X \in \mathfrak{g}$ the action of X on $\mathscr{A}(G)' = \mathscr{O}(G)$ determined by τ (see Subsection 4.2) coincides with the derivation $\widetilde{X}$. Indeed, given $x \in G$, denote by $\delta_x \in \mathscr{A}(G)$ the functional which is evaluation at x. Then for each $X \in \mathfrak{g}$, $f \in \mathscr{O}(G)$, and $x \in G$ we have

$$\begin{aligned}(X \cdot_\tau f)(x) &= \langle X \cdot_\tau f, \delta_x\rangle = \langle f, \delta_x \tau(X)\rangle \\ &= \langle \Delta f, \delta_x \otimes \tau(X)\rangle = \frac{d}{dt}\Big|_{t=0} f(x \exp tX) = (\widetilde{X}f)(x),\end{aligned}$$

i.e., $X \cdot_\tau f = \widetilde{X}f$, as required.

Similarly, for each $X \in \mathfrak{g}$ the action of X on $\mathfrak{F}(\mathfrak{g})' = \mathscr{O}_e$ determined by the canonical homomorphism $\lambda\colon U(\mathfrak{g}) \to \mathfrak{F}(\mathfrak{g})$ coincides with $\widetilde{X}$. Indeed, given $f \in \mathscr{O}_e$, denote by $\widehat{f}$ the canonical image of f in $\widehat{\mathscr{O}}_e$; then for each $X \in \mathfrak{g}$ and each $a \in U(\mathfrak{g})$ we have

$$\begin{aligned}\langle X \cdot_\lambda f, \lambda(a)\rangle &= \langle f, \lambda(aX)\rangle = \langle \widehat{f}, aX\rangle = [\varrho(aX)\widehat{f}](e) \\ &= [\varrho(a)\widetilde{X}\widehat{f}](e) = \langle (\widetilde{X}f)\widehat{\ }, a\rangle = \langle \widetilde{X}f, \lambda(a)\rangle.\end{aligned}$$

Since $\operatorname{Im}\lambda$ is dense in $\mathfrak{F}(\mathfrak{g})$, this implies $X \cdot_\lambda f = \widetilde{X}f$, as required.

PROPOSITION 8.1. *Let G be a Stein group with Lie algebra $\mathfrak{g}$. Then $\mathscr{O}(G)$ is $\mathfrak{g}$-parallelizable.*

Proof. By Lemma 1.1, we may identify the $\mathscr{O}(G)$-module $\Omega^1(\mathscr{O}(G))$ of Kähler differentials with the module $\Omega^1(G)$ of holomorphic 1-forms on G in such a way that the exterior (de Rham) derivative $d\colon \mathscr{O}(G) \to \Omega^1(G)$ becomes a universal derivation. Denote by $\mathrm{Vect}(G)$ the Lie algebra of holomorphic vector fields on G. In what follows, we identify $\mathfrak{g}$ with the Lie subalgebra of $\mathrm{Vect}(G)$ consisting of left-invariant vector fields. Each $\omega \in \Omega^1(G)$ can be viewed as an $\mathscr{O}(G)$-module morphism $\mathrm{Vect}(G) \to \mathscr{O}(G)$. Hence the rule $\omega \mapsto \omega|_{\mathfrak{g}}$ determines a linear map $\varphi\colon \Omega^1(G) \to C^1(\mathfrak{g}, \mathscr{O}(G))$ which is easily seen to be an $\mathscr{O}(G)$-module morphism. Evidently, $\varphi(df)(\widetilde{X}) = \widetilde{X}f$ for each $X \in \mathfrak{g}$, $f \in \mathscr{O}(G)$, i.e., $\varphi d = d^0$. It remains to show that φ is an isomorphism.

Given $\omega \in \mathfrak{g}^*$, denote by $\widetilde{\omega} \in \Omega^1(G)$ the corresponding left-invariant 1-form on G. Let $\psi\colon C^1(\mathfrak{g}, \mathscr{O}(G)) \to \Omega^1(G)$ be the unique $\mathscr{O}(G)$-module morphism taking each $\omega \otimes 1 \in C^1(\mathfrak{g}, \mathscr{O}(G))$ to $\widetilde{\omega}$. Recall that for each $\omega \in \mathfrak{g}^*$ the value of the left-invariant form $\widetilde{\omega}$ at a left-invariant vector field $\widetilde{X}$ is the constant function equal to $\langle \omega, X\rangle$ (see, e.g., [78, 3.12]). This means precisely that $\widetilde{\omega}|_{\mathfrak{g}} = \omega \otimes 1$, and so $\varphi\psi = 1_{C^1(\mathfrak{g}, \mathscr{O}(G))}$.

Let $\omega_1, \ldots, \omega_n$ be a basis of $\mathfrak{g}^*$. Then $\widetilde{\omega}_1(x), \ldots, \widetilde{\omega}_n(x)$ is clearly a basis of the cotangent space, T_x^*G, for each $x \in G$. Hence the forms $\widetilde{\omega}_1, \ldots, \widetilde{\omega}_n$ generate $\Omega^1(G)$ as an $\mathscr{O}(G)$-module. This implies, in particular, that ψ is surjective. Since $\varphi\psi = 1$, we see that φ and ψ are inverse to one another. This completes the proof. ∎

Combining this with Propositions 1.3 and 4.3, we obtain the following well-known fact.

COROLLARY 8.2. *Let G be a Stein group with Lie algebra $\mathfrak{g}$. Then $H^p(\mathfrak{g}, \mathscr{O}(G)) \cong H^p_{\mathrm{top}}(G, \mathbb{C})$ for each p.*

The following result is an analytic version of [75, Prop. 7.2].

THEOREM 8.3. *Let $\mathfrak{g}$ be a Lie algebra, and let G be the corresponding connected, simply connected complex Lie group. Then $\mathscr{A}(G)$ is stably flat over $U(\mathfrak{g})$ if and only if $\mathfrak{g}$ is solvable.*

Proof. Suppose that $\mathfrak{g}$ is solvable. Then G is also solvable and so is biholomorphic with $\mathbb{C}^n$ (see, e.g., [5, Chap. III, §9, no. 6]). Hence $\mathscr{O}(G) \cong \mathscr{O}(\mathbb{C}^n)$ is a contractible $\widehat{\otimes}$-algebra (cf. Subsection 4.1). On the other hand, $\mathscr{O}(G)$ is $\mathfrak{g}$-parallelizable by Proposition 8.1. It remains to apply Theorem 4.4.

Now suppose that $\mathfrak{g}$ is not solvable. Consider the Levi decomposition $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{l}$ ($\mathfrak{r} = \operatorname{rad}\mathfrak{g}$, $\mathfrak{l}$ is a semisimple subalgebra, $\mathfrak{l} \neq 0$). Let R and L be the corresponding analytic subgroups of G. Since G is simply connected, it is isomorphic to the semidirect product $R \rtimes L$ (see, e.g., [58, Chap. 6]). Since L is semisimple, we have $H^m_{\mathrm{top}}(L, \mathbb{C}) \neq 0$ for $m = \dim L$ (see, e.g., [50, Lemme 5]). Therefore $H^m_{\mathrm{top}}(G, \mathbb{C}) \neq 0$. On the other hand, $H^m_{\mathrm{top}}(G, \mathbb{C}) \cong H^m(\mathfrak{g}, \mathscr{O}(G))$ by Corollary 8.2, and so the augmented standard complex

$$0 \to \mathbb{C} \to C^{\cdot}(\mathfrak{g}, \mathscr{O}(G))$$

is not exact. Using the reflexivity argument (cf. the proof of Theorem 4.4), we see that the dual complex

$$0 \leftarrow \mathbb{C} \leftarrow C_{\cdot}(\mathfrak{g}, \mathscr{A}(G))$$

does not split in **LCS**. Therefore $U(\mathfrak{g}) \to \mathscr{A}(G)$ is not a localization, i.e., $\mathscr{A}(G)$ is not stably flat over $U(\mathfrak{g})$. ∎

We now turn to the hyperenveloping algebra $\mathfrak{F}(\mathfrak{g})$. Recall that a commutative algebra is called *local* if it has a unique maximal ideal. By a *local $\widehat{\otimes}$-algebra* we mean a commutative $\widehat{\otimes}$-algebra A which is local in the above sense and such that the maximal ideal of A is closed and has codimension 1. For example, $\mathscr{O}_e$ is a local $\widehat{\otimes}$-algebra with maximal ideal $\mathfrak{m}_e = \{f \in \mathscr{O}_e : f(e) = 0\}$ (see above), and the same is true of $\widehat{\mathscr{O}}_e$.

We need the following simple lemma.

LEMMA 8.4. *Let A be an algebra, $I \subset A$ a left ideal, and E a finite-dimensional vector space. Then for each $T \in \operatorname{Hom}_{\mathbb{C}}(E, A)$ the following conditions are equivalent*:

(i) $T \in I \cdot \operatorname{Hom}_{\mathbb{C}}(E, A)$;
(ii) $\operatorname{Im} T \subset I$.

Proof. The implication (i)⇒(ii) is clear. To prove the converse, take a basis (e_i) of E, and let (e^i) be the dual basis of E^*. Identifying $\operatorname{Hom}_{\mathbb{C}}(E, A)$ and $E^* \otimes A$, we see that $T = \sum_i e^i \otimes a_i$, where $a_i = T(e_i) \in I$. Setting $T_i = e^i \otimes 1$, we obtain $T = \sum_i a_i T_i \in I \cdot \operatorname{Hom}_{\mathbb{C}}(E, A)$, as required. ∎

LEMMA 8.5. *Let A be a local $\widehat{\otimes}$-algebra with maximal ideal $\mathfrak{m}$, and let $\mathfrak{g}$ be a Lie algebra acting on A by derivations. Suppose there exists a linear map $\chi\colon \mathfrak{g}^* \to A$ such that*

(i) $\operatorname{Im}\chi$ *generates a dense subalgebra of* A;
(ii) $X \cdot \chi(\omega) = \langle \omega, X\rangle 1 \bmod \mathfrak{m}$ *for each* $X \in \mathfrak{g}$ *and each* $\omega \in \mathfrak{g}^*$.

Then A is $\mathfrak{g}$-parallelizable.

Proof. We proceed in much the same way as in the proof of Theorem 5.2. Consider the A-module morphism $\varphi\colon A \otimes \mathfrak{g}^* \to C^1(\mathfrak{g}, A)$ uniquely determined by $1 \otimes \omega \mapsto d^0(\chi(\omega))$. Our objective is to prove that φ is an isomorphism. To this end, note that, since A is local and both $A \otimes \mathfrak{g}^*$ and $C^1(\mathfrak{g}, A)$ are free and finitely generated, we need only prove that the induced map

$$\overline{\varphi}\colon A \otimes \mathfrak{g}^*/\mathfrak{m} \cdot A \otimes \mathfrak{g}^* \to C^1(\mathfrak{g}, A)/\mathfrak{m} \cdot C^1(\mathfrak{g}, A) \tag{43}$$

is a vector space isomorphism (see [2, Chap. II, §3, no. 2]).

Since $\mathfrak{m}$ is closed and has codimension 1, there exists a continuous homomorphism $\varepsilon\colon A \to \mathbb{C}$ such that $\mathfrak{m} = \operatorname{Ker}\varepsilon$. Hence we can identify $A \otimes \mathfrak{g}^*/\mathfrak{m} \cdot A \otimes \mathfrak{g}^*$ and $\mathfrak{g}^*$ via the map

$$\alpha\colon \mathfrak{g}^* \to A \otimes \mathfrak{g}^*/\mathfrak{m} \cdot A \otimes \mathfrak{g}^*, \quad \omega \mapsto 1 \otimes \omega + \mathfrak{m} \cdot A \otimes \mathfrak{g}^*. \tag{44}$$

The inverse map is given by $a \otimes \omega + \mathfrak{m} \cdot A \otimes \mathfrak{g}^* \mapsto \varepsilon(a)\omega$.

Next consider the map

$$\beta\colon C^1(\mathfrak{g}, A)/\mathfrak{m} \cdot C^1(\mathfrak{g}, A) \to \mathfrak{g}^*, \quad T + \mathfrak{m} \cdot C^1(\mathfrak{g}, A) \mapsto \varepsilon T. \tag{45}$$

Lemma 8.4 implies that β is well defined and bijective. Indeed, the map taking each $\omega \in \mathfrak{g}^*$ to $\omega \otimes 1 + \mathfrak{m} \cdot C^1(\mathfrak{g}, A)$ is readily seen to be an inverse of β.

It is now easy to see that the map $\overline{\varphi}$ defined by (43) corresponds to the identity mapping of $\mathfrak{g}^*$ under the identifications (44) and (45). Indeed, for each $\omega \in \mathfrak{g}^*$ we have $(\overline{\varphi}\alpha)(\omega) = d^0(\chi(\omega)) + \mathfrak{m} \cdot C^1(\mathfrak{g}, A)$, and hence

$$\langle (\beta\overline{\varphi}\alpha)(\omega), X\rangle = \varepsilon(d^0(\chi(\omega))X) = \varepsilon(X \cdot \chi(\omega)) = \varepsilon(\langle \omega, X\rangle 1) = \langle \omega, X\rangle$$

for every $X \in \mathfrak{g}$. Therefore $\beta\overline{\varphi}\alpha = 1_{\mathfrak{g}^*}$, and so $\overline{\varphi}$ is an isomorphism. By the above remarks, so is φ.

Now define a derivation $d\colon A \to A \otimes \mathfrak{g}^*$ by $d = \varphi^{-1}d^0$. Note that $d(\chi(\omega)) = 1 \otimes \omega$ for each $\omega \in \mathfrak{g}^*$. Choose a basis (e_i) of $\mathfrak{g}$, and let (e^i) be the dual basis of $\mathfrak{g}^*$. We may identify $A \otimes \mathfrak{g}^*$ and A^n $(n = \dim \mathfrak{g})$ via the map

$$\psi\colon (a_1, \ldots, a_n) \in A^n \mapsto \sum_i a_i \otimes e^i \in A \otimes \mathfrak{g}^*.$$

Let $\partial = \psi^{-1}d = (\partial_1, \ldots, \partial_n)\colon A \to A^n$ be the derivation corresponding to d under the above identification. Since $d(\chi(e^j)) = 1 \otimes e^j$ for each j, it follows that $\partial_i(\chi(e^j)) = \delta_{ij}$ for each i, j. Thus the elements $x_i = \chi(e^i) \in A$ and the derivations $\partial_i \in \operatorname{Der} A$ satisfy the conditions of Lemma 1.2. Therefore ∂ is a universal derivation. Since both φ and ψ are isomorphisms, we conclude that $d^0 = \varphi d = \varphi\psi\partial$ is a universal derivation as well, i.e., A is $\mathfrak{g}$-parallelizable. ∎

THEOREM 8.6. *Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{F}(\mathfrak{g})$ is stably flat over $U(\mathfrak{g})$.*

Proof. By Theorem 4.4, it is sufficient to check that $\mathscr{O}_e$ is contractible and $\mathfrak{g}$-parallelizable.

To prove the contractibility of $\mathscr{O}_e$, it suffices to do this for the algebra $\mathscr{O}_0$ of holomorphic germs at the origin $0 \in \mathbb{C}^n$. For each $r > 0$ denote by D^n_r the polydisc in $\mathbb{C}^n$ of radius r, i.e.,

$$D^n_r = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_i| < r \ \forall i = 1, \dots, n\}.$$

Consider the homomorphism

$$\Phi_r \colon \mathscr{O}(D^n_r) \to \mathscr{O}(D^1_2) \mathbin{\widehat{\otimes}} \mathscr{O}(D^n_{r/2}) \cong \mathscr{O}(D^1_2 \times D^n_{r/2}), \qquad (\Phi_r f)(z, w) = f(zw).$$

Let $\varepsilon_k \colon \mathscr{O}(D^1_2) \to \mathbb{C}$ $(k = 0, 1)$ be given by $\varepsilon_k(f) = f(k)$. We clearly have

$$[(\varepsilon_0 \otimes 1)(\Phi_r f)](w) = f(0), \qquad [(\varepsilon_1 \otimes 1)(\Phi_r f)](w) = f(w). \tag{46}$$

Composing Φ_r with the restriction map $\mathscr{O}(D^1_2) \mathbin{\widehat{\otimes}} \mathscr{O}(D^n_{r/2}) \to \mathscr{O}(D^1_2) \mathbin{\widehat{\otimes}} \mathscr{O}_0$ and taking next the direct limit over $D_r \ni 0$, we obtain a homomorphism $\Phi \colon \mathscr{O}_0 \to \mathscr{O}(D^1_2) \mathbin{\widehat{\otimes}} \mathscr{O}_0$. Relations (46) imply that $(\varepsilon_0 \otimes 1_{\mathscr{O}_0})\Phi = \eta\varepsilon_{\mathscr{O}_0}$ and $(\varepsilon_1 \otimes 1_{\mathscr{O}_0})\Phi = 1_{\mathscr{O}_0}$, where $\varepsilon_{\mathscr{O}_0} \colon \mathscr{O}_0 \to \mathbb{C}$ is the evaluation at 0. Since $\mathscr{O}(D^1_2)$ is an exact algebra, we see that Φ is a homotopy between $1_{\mathscr{O}_0}$ and $\eta\varepsilon_{\mathscr{O}_0}$, and so $\mathscr{O}_0$ is contractible.

We now turn to the $\mathfrak{g}$-parallelizability of $\mathscr{O}_e$. Since the exponential map is biholomorphic in a neighborhood of $0 \in \mathfrak{g}$, it follows that for each $\omega \in \mathfrak{g}^*$ there exists a unique function f_ω holomorphic in a neighborhood of $e \in G$ such that $f_\omega(\exp \xi) = \omega(\xi)$ for all sufficiently small $\xi \in \mathfrak{g}$. Consider the map $\chi \colon \mathfrak{g}^* \to \mathscr{O}_e$ taking each $\omega \in \mathfrak{g}^*$ to the germ of f_ω at e.

We claim that χ satisfies the conditions of Lemma 8.5. To prove this, fix a basis (e_i) of $\mathfrak{g}$, and let $x^1, \dots, x^n$ be the corresponding canonical coordinates of the first kind on a suitable neighborhood of $e \in G$. Recall that they are defined by the rule $x^j(\exp \sum_i t^i e_i) = t^j$ for all $j = 1, \dots, n$. For each $\omega \in \mathfrak{g}^*$ we have

$$f_\omega\Big(\exp \sum_i t^i e_i\Big) = \sum_i \omega(e_i) t^i, \quad \text{i.e.,} \quad f_\omega = \sum_i \omega(e_i) x^i.$$

Therefore $\operatorname{Im}\chi$ consists of all germs of linear functions in $x^1, \dots, x^n$. Since polynomials in $x^1, \dots, x^n$ form a dense subalgebra of $\mathscr{O}_e$, we conclude that condition (i) of Lemma 8.5 is satisfied.

Finally, for each $X \in \mathfrak{g}$ and each $\omega \in \mathfrak{g}^*$ we have

$$(X \cdot \chi(\omega))(e) = (\widetilde{X} f_\omega)(e) = \frac{d}{dt}\Big|_{t=0} f_\omega(\exp tX) = \omega(X).$$

Thus we see that condition (ii) of Lemma 8.5 is also satisfied. Hence $\mathscr{O}_e$ is $\mathfrak{g}$-parallelizable. Now the result follows from Theorem 4.4. ∎

Remark 8.1. A similar argument applied to the local algebra $A = \widehat{\mathscr{O}}_e$ shows that A is contractible and $\mathfrak{g}$-parallelizable. Hence the standard cochain complex $0 \to \mathbb{C} \to C^\cdot(\mathfrak{g}, A)$ splits in **LCS**. Taking the topological dual, we recover the classical fact that the Koszul complex $0 \leftarrow \mathbb{C} \leftarrow V_\cdot(\mathfrak{g})$ is exact.

9. Relations between various completions of $U(\mathfrak{g})$

In this final section, we explain how the completions of $U(\mathfrak{g})$ considered above are related to one another, and formulate some open problems.

Let $\mathfrak{g}$ be a nilpotent Lie algebra, and let G be the corresponding connected, simply connected complex Lie group. Choose a positive filtration $\mathscr{F}$ on $\mathfrak{g}$, and let $\mathscr{M}$ be an entire $\mathscr{F}$-weight sequence (see Section 7). The algebra $A_{\mathscr{M}}(G)$ is a subalgebra of $\mathscr{O}(G)$, and it is easy to see that the inclusion map $A_{\mathscr{M}}(G) \to \mathscr{O}(G)$ is continuous. Indeed, given $r > 0$ and a compact set $K \subset G$, let $C = \sup_{g \in K} W_{\mathscr{M}}(\delta_r g)$. Then for each $f \in A_{\mathscr{M},r}(G)$ we have

$$\|f\|_K = \sup_{g \in K} |f(g)| \leq C \sup_{g \in K} \frac{|f(g)|}{W_{\mathscr{M}}(\delta_r g)} \leq C N_r(f).$$

Hence the inclusion of $A_{\mathscr{M}}(G)$ into $\mathscr{O}(G)$ is continuous.

Consider the Hopf algebra $\mathscr{P}(G)$ of polynomial functions on G. This is a Hopf $\widehat{\otimes}$-algebra with respect to the finest locally convex topology. We have a chain of canonical inclusion/restriction maps

$$\mathscr{P}(G) \to A_{\mathscr{M}}(G) \to \mathscr{O}(G) \to \mathscr{O}_e \to \widehat{\mathscr{O}}_e, \tag{47}$$

which are obviously Hopf $\widehat{\otimes}$-algebra homomorphisms. To examine the dual of this chain, choose an $\mathscr{F}$-basis $e_1, \ldots, e_N$ of $\mathfrak{g}$, and recall that there is a duality $\langle \cdot, \cdot \rangle_\varkappa$ between the formal completion $[U(\mathfrak{g})]$ of $U(\mathfrak{g})$ and the polynomial algebra $\mathbb{C}[z_1, \ldots, z_N]$ defined by $\langle a, \varphi \rangle_\varkappa = \varkappa^{-1}(\varphi)(a)$ for all $a \in [U(\mathfrak{g})]$, $\varphi \in \mathbb{C}[z_1, \ldots, z_N]$ (see Lemma 5.1). Thus the dual of the inclusion map $i\colon U(\mathfrak{g})_{\mathscr{M}} \to [U(\mathfrak{g})]$ can be viewed as a $\widehat{\otimes}$-algebra homomorphism from $\mathbb{C}[z_1, \ldots, z_N]$ to $A_{\mathscr{M}}(G)$. For each $u \in U(\mathfrak{g})_{\mathscr{M}}$ and each $\varphi \in \mathbb{C}[z_1, \ldots, z_N]$ we have

$$\langle u, i'(\varphi) \rangle = \langle i(u), \varphi \rangle_\varkappa,$$

where the brackets $\langle \cdot, \cdot \rangle$ on the left-hand side denote the duality between $U(\mathfrak{g})_{\mathscr{M}}$ and $A_{\mathscr{M}}(G)$.

We claim that $i'\colon \mathbb{C}[z_1, \ldots, z_N] \to A_{\mathscr{M}}(G)$ becomes the canonical inclusion of $\mathscr{P}(G)$ into $A_{\mathscr{M}}(G)$ if we identify $\mathbb{C}[z_1, \ldots, z_N]$ with $\mathscr{P}(G)$ using the canonical coordinates of the second kind. Indeed, for each $\alpha \in \mathbb{Z}_+^N$ and each $\psi \in \mathscr{O}(G)$ we have (in the standard multi-index notation)

$$\langle e^\alpha, \psi \rangle = [e^\alpha \psi](e) = D_z^\alpha \psi(0) \tag{48}$$

with respect to the canonical coordinates of the second kind (see [42, Lemma 7.2]). On the other hand, it follows from (27) that $\langle e^\alpha, \varphi \rangle_\varkappa = D_z^\alpha \varphi(0)$ for each $\varphi \in \mathbb{C}[z_1, \ldots, z_N] = \mathscr{P}(G)$. Together with (48), this gives $\langle e^\alpha, \varphi \rangle_\varkappa = \langle e^\alpha, \varphi \rangle$, and so $\langle e^\alpha, i'(\varphi) \rangle = \langle i(e^\alpha), \varphi \rangle_\varkappa = \langle e^\alpha, \varphi \rangle_\varkappa = \langle e^\alpha, \varphi \rangle$ for each $\alpha \in \mathbb{Z}_+^N$. This implies that $i'(\varphi) = \varphi$ for each $\varphi \in \mathscr{P}(G)$, which proves the claim.

Thus the sequence dual to (47) has the form

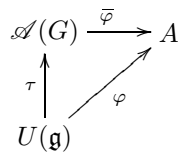
$$U(\mathfrak{g}) \to \mathfrak{F}(\mathfrak{g}) \to \mathscr{A}(G) \to U(\mathfrak{g})_{\mathscr{M}} \xrightarrow{i} [U(\mathfrak{g})]. \tag{49}$$

All the maps here are injective Hopf $\widehat{\otimes}$-algebra homomorphisms with dense ranges. Combining Theorems 8.6, 8.3, 7.3, Corollary 5.4, and Proposition 3.5, we see that all the morphisms in (49) are localizations.

PROPOSITION 9.1. *Let G be a connected, simply connected complex Lie group with Lie algebra $\mathfrak{g}$, and let $\tau\colon U(\mathfrak{g}) \to \mathscr{A}(G)$ be the canonical homomorphism (see (42)). Then there exists a unique Hopf $\widehat{\otimes}$-algebra homomorphism $j\colon \mathscr{A}(G) \to \widehat{U}(\mathfrak{g})$ such that $\iota_{U(\mathfrak{g})} = j \circ \tau$. In other words, the canonical morphism $\iota_{U(\mathfrak{g})}\colon U(\mathfrak{g}) \to \widehat{U}(\mathfrak{g})$ factors through $\mathscr{A}(G)$.*

Proof. Given a Banach algebra A and a homomorphism $\varphi\colon U(\mathfrak{g}) \to A$, we shall construct a continuous homomorphism $\overline{\varphi}\colon \mathscr{A}(G) \to A$ such that the diagram

(50)



is commutative.

For each $a \in A$ denote by $L_a\colon A \to A$ the left multiplication by a. Consider the representation

$$\pi_0\colon \mathfrak{g} \to \mathscr{L}(A), \qquad \pi_0(X) = L_{\varphi(X)}.$$

Since G is simply connected, π_0 determines a holomorphic representation $\pi\colon G \to \mathrm{GL}(A)$ such that $\exp \circ \pi_0 = \pi \circ \exp$ (see, e.g., [5, Chap. III, §6, no. 1]). For each $x \in A$ and each $y \in A'$, let $\pi_{x,y} \in \mathscr{O}(G)$ denote the corresponding matrix element of π defined by $\pi_{x,y}(g) = \langle y, \pi(g)x\rangle$. By [42, Prop. 3.5], π uniquely extends to a continuous representation $\overline{\pi}\colon \mathscr{A}(G) \to \mathscr{L}(A)$ such that $\langle y, \overline{\pi}(a')x\rangle = \langle a', \pi_{x,y}\rangle$ for all $a' \in \mathscr{A}(G)$, $x \in A$, $y \in A'$.

Consider the map $\varepsilon_1\colon \mathscr{L}(A) \to A$, $\varepsilon_1(T) = T(1)$, and define $\overline{\varphi}\colon \mathscr{A}(G) \to A$ by $\overline{\varphi} = \varepsilon_1\overline{\pi}$. We claim that $\overline{\varphi}$ makes diagram (50) commutative. Indeed, for each $X \in \mathfrak{g}$ and each $y \in A'$ we have

$$\begin{aligned}\langle y, \overline{\varphi}\tau(X)\rangle &= \langle y, \overline{\pi}(\tau(X))1\rangle = \langle \tau(X), \pi_{1,y}\rangle = (\widetilde{X}\pi_{1,y})(e)\\ &= \frac{d}{dt}\Big|_{t=0} \pi_{1,y}(\exp tX) = \frac{d}{dt}\Big|_{t=0} \langle y, \pi(\exp tX)1\rangle = \frac{d}{dt}\Big|_{t=0} \langle y, \exp \pi_0(tX)1\rangle\\ &= \frac{d}{dt}\Big|_{t=0} \langle y, \exp t\varphi(X)\rangle = \langle y, \varphi(X)\rangle,\end{aligned}$$

i.e., $\overline{\varphi}\tau = \varphi$. Hence diagram (50) is commutative. Since $\mathrm{Im}\,\tau$ is dense in $\mathscr{A}(G)$, we conclude that $\overline{\varphi}$ is an algebra homomorphism. For the same reason, $\overline{\varphi}$ is a unique linear continuous map making (50) commutative.

The above construction can easily be extended to the case where A is an Arens–Michael algebra. Indeed, we have $A = \varprojlim\{A_\nu, \sigma_\nu^\mu\}$ for some inverse system $\{A_\nu, \sigma_\nu^\mu\}$ of Banach algebras. For each ν, let $\sigma_\nu\colon A \to A_\nu$ denote the canonical map. Given a homomorphism $\varphi\colon U(\mathfrak{g}) \to A$, we can extend the homomorphism $\sigma_\nu\varphi\colon U(\mathfrak{g}) \to A_\nu$ to a homomorphism $\overline{\varphi}_\nu\colon \mathscr{A}(G) \to A_\nu$ satisfying $\overline{\varphi}_\nu\tau = \sigma_\nu\varphi$. Since such an extension is unique, we clearly have $\sigma_\nu^\mu\overline{\varphi}_\mu = \overline{\varphi}_\nu$ whenever $\mu \succ \nu$. Setting $\overline{\varphi} = \varprojlim \overline{\varphi}_\nu$, we obtain a $\widehat{\otimes}$-algebra homomorphism making (50) commutative.

Now set $A = \widehat{U}(\mathfrak{g})$ and $\varphi = \iota_{U(\mathfrak{g})}\colon U(\mathfrak{g}) \to \widehat{U}(\mathfrak{g})$. Then the above construction yields a unique $\widehat{\otimes}$-algebra homomorphism $j = \overline{\varphi}\colon \mathscr{A}(G) \to \widehat{U}(\mathfrak{g})$ satisfying $j\tau = \iota_{U(\mathfrak{g})}$. Since $\mathrm{Im}\,\tau$ is dense in $\mathscr{A}(G)$, and since $\iota_{U(\mathfrak{g})}$ is a Hopf $\widehat{\otimes}$-algebra homomorphism, we conclude that so is j. This completes the proof. ∎

The above theorem implies that for each nilpotent Lie algebra $\mathfrak{g}$ the chain of inclusions (49) can be completed as follows:

$$U(\mathfrak{g}) \longrightarrow \mathfrak{F}(\mathfrak{g}) \longrightarrow \mathscr{A}(G) \begin{array}{c} \nearrow U(\mathfrak{g})_{\mathscr{M}} \searrow \\ \searrow^{j} \quad \widehat{U}(\mathfrak{g}) \quad \nearrow_{\widehat{\theta}} \end{array} [U(\mathfrak{g})] \tag{51}$$

The following summarizes the main results of the previous sections.

THEOREM 9.2. *Suppose $\mathfrak{g}$ is a positively graded Lie algebra, G is the corresponding connected, simply connected complex Lie group, and $\mathscr{M}$ is an entire weight sequence on $\mathfrak{g}$. Then all the arrows in* (51) *are Hopf $\widehat{\otimes}$-algebra localizations.*

We end this section with some open problems.

Problem 1. Is the canonical map $U(\mathfrak{g}) \to \widehat{U}(\mathfrak{g})$ a localization for every nilpotent Lie algebra $\mathfrak{g}$?

By Proposition 3.5, we can replace $U(\mathfrak{g})$ in the above problem by either $\mathfrak{F}(\mathfrak{g})$ or $\mathscr{A}(G)$ (assuming that G is connected and simply connected).

Problem 2. Let $\mathfrak{g}$ be a nilpotent Lie algebra.

(1) Is the canonical homomorphism $\widehat{\theta}\colon \widehat{U}(\mathfrak{g}) \to [U(\mathfrak{g})]$ injective?
(2) Is the algebra $\widehat{U}'(\mathfrak{g})$ contractible?

A positive answer to Problem 2 would imply a positive solution of Problem 1 (see Corollary 5.3).

REMARK 9.1. The diagram dual to (51) has the form

$$\widehat{\mathscr{O}}_e \longleftarrow \mathscr{O}_e \longleftarrow \mathscr{O}(G) \begin{array}{c} \swarrow A_{\mathscr{M}}(G) \searrow \\ \searrow^{j'} \quad \widehat{U}'(\mathfrak{g}) \quad \swarrow_{\widehat{\theta}'} \end{array} \mathscr{P}(G)$$

Recall that all the maps here (except $\widehat{\theta}'$ and j') are the usual set-theoretic inclusions/restrictions of function algebras. Since j and $\widehat{\theta}$ have dense ranges, it follows that $\widehat{\theta}'$ and j' are injective. Hence the algebra $\widehat{U}'(\mathfrak{g})$ can be viewed as a certain algebra of holomorphic functions on G containing the polynomials. Thus question (1) of Problem 2 has a positive solution if and only if the polynomials are dense in $\widehat{U}'(\mathfrak{g})$.

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