1. Introduction

Let $\mathbb{D} \subset \mathbb{C}$ denote the unit disc in the complex plane. Large parts of the structure theory for contraction operators rely on the fact that, according to von Neumann's inequality $\|p(T)\| \leq \|p\|_{\mathbb{D},\infty}$ ($p \in \mathbb{C}[z]$), each contraction T on a complex Hilbert space H possesses a contractive functional calculus Φ_T over the disc algebra $A(\mathbb{D})$. If T is assumed to be completely non-unitary, then the mapping Φ_T can even be extended to a contractive and weak* continuous $H^\infty(\mathbb{D})$-functional calculus for T. As carried out by Bercovici, Brown, Chevreau, Exner, Pearcy and others, this H^∞-functional calculus can be used to solve the invariant-subspace and the reflexivity problem for various classes of contraction operators including the completely non-unitary ones with rich spectrum in $\mathbb{D}$ (see, for instance, [3]–[7]).

Recall that an operator $T \in L(H)$ or, more generally, a commuting n-tuple $T \in L(H)^n$ is said to be *reflexive* if the WOT-closed operator algebra $\operatorname{AlgLat}(T) = \{C \in L(H) : C$ leaves invariant each closed T-invariant subspace of $H\} \supset \mathbb{C}[T]$ coincides with the WOT-closure $\mathcal{A}_T \subset L(H)$ of the polynomials in T. Hence reflexive operators are in some sense determined by the structure of their invariant-subspace lattice. If N is a normal operator, then it is an easy exercise to show that von Neumann's double commutant theorem for N can be rephrased by saying that the commuting tuple (N, N^*) is reflexive. From this point of view the reflexive operators are precisely those which satisfy a non-selfadjoint version of von Neumann's double commutant theorem. A more detailed discussion of the reflexivity concept including basic examples as well as an overview of recent results can be found in Ptak [36].

Let us say that a contraction $T \in L(H)$ belongs to the class $\mathbb{A}$ if it possesses an isometric and weak* continuous $H^\infty(\mathbb{D})$-functional calculus. Then one of the main results concerning the invariant-subspace structure of contractions can be formulated as follows:

Each contraction in the class $\mathbb{A}$ *is reflexive* (Brown and Chevreau [4]).

An important intermediate step was a result of Chevreau, Exner and Pearcy [7] saying that each contraction in the class $\mathbb{A}_{1,\aleph_0}$ is reflexive. Since, modulo the Riemann mapping theorem, every pure subnormal operator decomposes into an orthogonal sum of subnormal contractions of class $\mathbb{A}$, the above reflexivity result can be used to prove the following theorem (which has been shown by different methods earlier):

Each subnormal operator is reflexive (Olin and Thomson [32]).

In the present article we consider the corresponding questions in the case of commuting n-tuples of operators. For this purpose, we replace the unit disc $\mathbb{D}$ in $\mathbb{C}$ by a relatively compact, strictly pseudoconvex open subset $D \Subset X$ of a Stein submanifold $X \subset \mathbb{C}^n$. Instead of a contraction T on H we consider a commuting n-tuple $T \in L(H)^n$ of continuous linear operators possessing a contractive functional calculus $\Phi_T : A(D) \to L(H)$ over the algebra $A(D) = C(\overline{D}) \cap \mathcal{O}(D)$. Since the existence of such a functional calculus is equivalent to the condition that $\sigma(T) \subset \overline{D}$ and that T satisfies a von Neumann-type inequality of the form $\|f(T)\| \leq \|f\|_{\infty,\overline{D}}$ for each $f \in \mathcal{O}(\overline{D})$, these n-tuples T will be called *von Neumann n-tuples over D* in what follows. We say that a von Neumann n-tuple over D belongs to the class $\mathbb{A}$ if it possesses an isometric and weak* continuous $H^\infty(D)$-functional calculus $\Phi_T : H^\infty(D) \to L(H)$ and has in addition a ∂D-unitary dilation. By the famous unitary dilation theorem of Sz.-Nagy, the latter condition is always satisfied (with $D = \mathbb{D}$) in the case that T is a single contraction. Given a von Neumann n-tuple $T \in L(H)^n$ of class $\mathbb{A}$ over D, we say that T belongs to the class $\mathbb{A}_{1,\aleph_0}$ if the H^∞-functional calculus Φ_T satisfies the factorization property $(\mathbb{A}_{1,\aleph_0})$ (see Section 4.3). Unfortunately our generalizations of the single-operator methods do not yield a complete answer to the reflexivity problem for von Neumann n-tuples of class $\mathbb{A}$ over D. However, the following special case can be solved:

Each von Neumann n-tuple over D belonging to the class $\mathbb{A}_{1,\aleph_0}$ is reflexive.

More precisely, if $\mathcal{H}_T(\overline{D}) = \Phi_T(H^\infty(D)) \subset L(H)$ denotes the dual operator algebra generated by T over $\overline{D}$, then we show that $T \in [\mathbb{A}_{1,\aleph_0}]$ implies the super-reflexivity of $\mathcal{H}_T(\overline{D})$ (see Corollary 4.4.3). Furthermore we prove that a von Neumann n-tuple of class $\mathbb{A}$ over D belongs to the class $\mathbb{A}_{1,\aleph_0}$ if and only if T possesses a compression T_Z to a von Neumann n-tuple of class $\mathbb{A} \cap C_{\cdot 0}$ on a semi-invariant subspace $Z \subset H$ which has the ϱ-almost factorization property for some $\varrho > 0$.

Since each subnormal n-tuple with Taylor spectrum contained in D is automatically a von Neumann n-tuple over D and satisfies ϱ-almost as well as $\mathbb{A}_{1,\aleph_0}$-type factorizations, the above results allow us to draw the following conclusions (Theorems 5.3.1 and 5.4.4):

Each subnormal tuple with rich Taylor spectrum in D and each subnormal tuple possessing an isometric and weak continuous $H^\infty(D)$-functional calculus is reflexive.*

Both results were obtained by Eschmeier in the special case of the unit ball $D = \mathbb{B}_n$ in $\mathbb{C}^n$. Besides the technical effort caused by considering complex manifolds, the main difficulties that arise when we pass from the ball case to the more general situation of strictly pseudoconvex sets are due to the fact that concepts like connectedness, a smooth boundary (and hence a canonical surface area measure) and the transitivity of the automorphism group of D are no longer available in this context. To overcome these difficulties we first construct a so-called faithful Henkin measure $\mu \in M^+(\partial D)$ in such a way that the identification $H^\infty(D) \cong H^\infty(\mu)$ holds. Then using Aleksandrov's theory of μ-inner functions we are able to modify the ϱ-almost factorization techniques known from the ball case to our more general situation (see Chapter 5).

As observed by Eschmeier and Putinar (see [20, Theorem 3.4]), even on the unit ball there exist pure subnormal tuples T such that the dual algebra $\mathcal{H}_T(\overline{\mathbb{B}}_n)$ does not possess any weak* continuous characters at all, implying that, for each such T, the algebra

$\mathcal{H}_T(\overline{\mathbb{B}}_n)$ cannot in any reasonable way be isomorphic to $H^\infty(D)$ for some complex manifold (or analytic space) D. Since our methods are limited by the availability of this kind of analytic structure, other techniques may be necessary to decide the general reflexivity problem for subnormal tuples.

2. Preliminaries

This chapter provides a short overview of function theory on strictly pseudoconvex sets in Stein manifolds and its interplay with the theory of Henkin measures and operator theory. Despite the lack of precise references it is our feeling that most of the function theoretical background presented here is well known. For the sake of completeness and for the reader's convenience we have nevertheless decided to include full proofs resting on some classical results of Aleksandrov, Fornæss, Henkin and Leiterer, and Hua. Moreover, we describe a modern approach to the theory of Henkin measures and we study the basic properties of functional calculi over algebras of type $A(D)$ and $H^\infty(D)$ on a strictly pseudoconvex open subset $D \Subset X$ of a Stein submanifold $X \subset \mathbb{C}^n$.

2.1. Function theory on strictly pseudoconvex sets. Let X be a complex manifold of fixed dimension and let $D \Subset X$ be a relatively compact open subset. We select a fixed Hermitian metric on X and denote by ω the canonical volume form on X induced by the underlying Riemannian structure and the associated orientation of X. Integrating characteristic functions with respect to this volume form via the formula

$$\lambda(A) = \int_X \chi_{A\cap D} \cdot \omega \qquad (A \subset \overline{D} \text{ Borel})$$

defines a finite regular Borel measure $\lambda \geq 0$ on $\overline{D}$ with $\lambda(\partial D) = 0$ and with the property that a Borel subset $A \subset \overline{D}$ satisfying $\lambda(A) = 0$ has empty interior in X or, equivalently, in $\overline{D}$. Therefore, the inclusion map $H^\infty(D) \subset L^\infty(\lambda)$ is well defined and isometric, where we write $H^\infty(D)$ for the Banach algebra of all bounded complex-valued holomorphic functions on D, equipped with the supremum norm.

Using the Krein–Šmulian theorem and the theorem of Montel, one verifies that $H^\infty(D)$ is actually a weak* closed subspace of $L^\infty(\lambda)$ and that the point evaluations

$$\mathcal{E}_\mu : H^\infty(D) \to \mathbb{C}, \qquad f \mapsto f(\mu) \qquad (\mu \in D)$$

are weak* continuous. Here, the weak* topology refers to the natural dual pairing $\langle L^1(\lambda), L^\infty(\lambda)\rangle$. Since X is metrizable as a consequence of Whitney's embedding theorem and since λ is a compactly supported measure on X, the space $H^\infty(D)$ is the dual of the *separable* Banach space $Q(D) = L^1(\lambda)/{}^\perp H^\infty(D)$. The weak* topology with respect to this duality coincides with the relative weak* topology of $H^\infty(D)$ as a subspace of $L^\infty(\lambda)$.

Since $H^\infty(D)$ has a separable predual, a linear map $H^\infty(D) \to Y'$ into the dual of a Banach space Y is weak* continuous if and only if it is sequentially weak* continuous at the origin. In this context it is useful to know that a sequence (f_k) in $H^\infty(D)$ is a weak* zero sequence if and only if it is norm-bounded and converges to zero pointwise on D (or, equivalently, uniformly on compact subsets of D). This can be proved using the weak* continuity of the point evaluations and the theorems of Banach–Steinhaus and Montel. Combining the last two observations we deduce that the weak* topology on $H^\infty(D)$ constructed above does not depend on the special choice of a Hermitian metric on X.

From now on we specialize to the case that X is a Stein manifold and $\emptyset \neq D \Subset X$ is a relatively compact strictly pseudoconvex open subset of X in the sense that there exist an open neighborhood U of ∂D and a strictly plurisubharmonic C^2-function $\varrho : U \to \mathbb{R}$ such that

$$D \cap U = \{z \in U : \varrho(z) < 0\}.$$

Using the maximum principle for subharmonic functions, one deduces that in this case $\partial D = \partial \overline{D}$ (1). This observation leads to the following slightly modified version of Fornæss' embedding theorem which turns out to be very useful in what follows.

2.1.1. Theorem (Fornæss). *Let $D \Subset X$ be a relatively compact strictly pseudoconvex open subset of a Stein manifold X. Then there exist a Stein open neighborhood X_D of $\overline{D}$ in X, a holomorphic map $\psi : X_D \to \mathbb{C}^m$ for some $m \in \mathbb{N}$ and a bounded strictly convex domain $C \subset \mathbb{C}^m$ with C^2-boundary such that $\psi : X_D \to \psi(X_D)$ is biholomorphic onto a closed complex submanifold of $\mathbb{C}^m$ and*

$$\psi(D) = \psi(X_D) \cap C, \qquad \psi(\partial D) = \psi(X_D) \cap \partial C. \ \blacksquare$$

For a proof of this result (2), we refer the reader to [21] (Theorem 10). It should be mentioned that the set C occuring in the statement of the preceding theorem is both geometrically convex (see [21, p. 529]) and strictly pseudoconvex (Theorem 1.5.22 in [25]).

2.1.2. Corollary. *Let $D \Subset X$ be a relatively compact strictly pseudoconvex open subset of a Stein manifold X, and let X_D be as in the previous theorem.*

(a) *There is a strictly plurisubharmonic C^2-function $\varrho : V \to [-1,1]$ defined on an open neighborhood V of $\overline{D}$ with the following properties*:

$$D = \{z \in V : \varrho(z) < 0\}, \qquad \partial D = \{z \in V : \varrho(z) = 0\}, \qquad \varrho(\overline{D}) = [-1,0].$$

(b) *$\overline{D}$ is holomorphically convex in X_D and hence a Stein compactum.*

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(1) Trivially, $\partial \overline{D} \subset \partial D$ holds. Supposing that the reverse inclusion $\partial D \subset \partial \overline{D}$ fails, there must exist a point $p \in \partial D$ possessing an open neighborhood $\Omega \subset \overline{D} \cap U$, which may be assumed to be biholomorphically equivalent to the open unit ball in $\mathbb{C}^n$ ($n = \dim_{\mathbb{C}} X$). Since $\varrho|\Omega \leq 0 = \varrho(p)$, the maximum principle for real-valued (pluri-)subharmonic functions can be used to deduce that $\varrho|\Omega \equiv 0$ and hence $\Omega \cap D = \emptyset$. This clearly contradicts the fact that Ω is an open neighborhood of the boundary point $p \in \partial D$.

(2) In the original formulation of Fornæss' theorem, X is only required to be a Stein space and $\psi(X_D)$ is asserted to be a complex subvariety of $\mathbb{C}^m$. Therefore in our situation $\psi(X_D)$ is a complex subvariety of $\mathbb{C}^m$ which is, in addition, a complex manifold. This automatically implies that $\psi(X_D)$ is a complex submanifold of $\mathbb{C}^m$ (see, for example, Gunning [23, Theorem E.16]). Furthermore, the identities $\partial D = \partial \overline{D}$ and $\partial C = \partial \overline{C}$ can be used to derive the formulas for $\psi(D)$ and $\psi(\partial D)$ stated above from the ones formulated in Fornæss' paper [21] (namely, $\psi(D) \subset C$ and $\psi(X_D \setminus \overline{D}) \subset \mathbb{C}^m \setminus \overline{C}$).

(c) *For each point $w \in \partial D$ there exists a peaking function $h \in \mathcal{O}(\overline{D})$, i.e. a function $h \in \mathcal{O}(\overline{D})$ satisfying $h(w) = 1$ and $|h(z)| < 1$ for $z \in \overline{D} \setminus \{w\}$.*

Proof. Note that strict plurisubharmonicity is preserved under a biholomorphic change of variables. Hence, to prove the corollary we may assume (according to the preceding theorem) that there are a closed complex submanifold $X_D \subset \mathbb{C}^m$ and a bounded strictly convex domain $C \subset \mathbb{C}^m$ with C^2-boundary such that $D = X_D \cap C$ and $\partial D = X_D \cap \partial C$. (Here, of course, ∂D stands for the boundary of D as a subset of X_D.)

Theorem 1.5.19 in Henkin and Leiterer [25] guarantees the existence of a strictly plurisubharmonic C^2-function ϱ_0 defined in some open neighborhood $W \subset \mathbb{C}^m$ of $\overline{C}$ such that

$$C = \{z \in W : \varrho_0(z) < 0\}, \qquad \partial C = \{z \in W : \varrho_0(z) = 0\}.$$

The restriction $\varrho_1 = \varrho_0|X_D \cap W$ remains strictly plurisubharmonic and obviously satisfies the desired representation formulas for D and ∂D. Since ϱ_1 attains its minimum

$$-m = \inf_{z \in X_D \cap W} \varrho_1(z) = \min_{z \in \overline{D}} \varrho_1(z) < 0$$

at a point in D, the function $\varrho = \varrho_1/m$ has all the desired properties when restricted to an open set V in X_D, $\overline{D} \subset V \subset X_D \cap W$, which is small enough to ensure that $\varrho(V) \leq 1$. This finishes the proof of part (a).

A separation argument shows that the compact convex set $\overline{C} \subset \mathbb{C}^m$ is polynomially convex. From this we deduce that

$$\overline{D} \subset \widehat{\overline{D}}_{\mathcal{O}(X_D)} \subset X_D \cap \widehat{\overline{C}}_{\mathcal{O}(\mathbb{C}^m)} = X_D \cap \overline{C} = \overline{D},$$

where $\widehat{\cdot}_{\mathcal{O}(M)}$ stands for the $\mathcal{O}(M)$-convex hull. Thus $\overline{D}$ is in fact holomorphically convex in X_D. As a consequence, it is a Stein compactum (see Proposition VII.A.3 in Gunning–Rossi [24]), and part (b) is proved.

The last part of the assertion is well known to hold if D is replaced by the smoothly bounded (³) strictly pseudoconvex domain C (see Range [38, Corollary VI.1.14]). This clearly suffices to prove the assertion for the original set D. ∎

In what follows, the space of all regular complex Borel measures on a compact set $K \subset X$ will be denoted by $M(K)$, while $M^+(K)$ stands for the set of all positive measures in $M(K)$. Finally, we write $M_1^+(K) \subset M^+(K)$ for the set of all probability measures on K.

Let K be as above, $\mu \in M^+(K)$ and $A \subset C(K)$ a closed subspace. Recall that the triple (A, K, μ) is called *regular* (in the sense of Aleksandrov [1], [2]) if there exists a number $\tau > 0$ such that

$$\sup\Big\{ \int_K |f|^2\, d\mu : f \in A,\ |f| < \varphi \Big\} \geq \tau \int_K \varphi^2\, d\mu$$

for all positive functions $\varphi \in C(K)$, $\varphi > 0$. It is well known (see [2, Theorem 3]) that if $A \subset C(K)$ is a uniform algebra on K and if there exists a finite system $(f_j)_{j=1,\dots,N}$ of functions in A separating the points of K and satisfying the identity $\sum_{j=1}^N |f_j|^2 = 1$ on

(³) By "smoothly bounded" we always mean that the boundary is (at least) of class C^2.

the Shilov boundary ∂_A of A, then the triple (A, K, μ) is regular for every $\mu \in M^+(K)$ with $\operatorname{supp}(\mu) \subset \partial_A$.

In what follows we write $A(D)$ for the Banach algebra of all continuous complex-valued functions on $\overline{D}$ which are holomorphic on D, equipped with the supremum norm on $\overline{D}$. Some of the basic properties of the algebra $A(D)$ are collected in the following corollary.

2.1.3. Corollary. *Let $D \Subset X$ be a relatively compact strictly pseudoconvex open subset of a Stein manifold X.*

(a) *$A(D)$ is sequentially weak* dense in $H^\infty(D)$; more precisely, there exists a constant $k_D > 0$ such that, for every $f \in H^\infty(D)$, there is a sequence (f_k) in $A(D)$ with $\sup_k \|f_k\|_{\infty,\overline{D}} \le k_D \|f\|_{\infty,D}$ and $f_k \to f$ pointwise on D as $k \to \infty$. It will be shown later (see Lemma 2.2.6) that k_D can be chosen to be 1.*
(b) *There is an open neighborhood $U_X \subset X$ of $\overline{D}$ such that $\mathcal{O}(U_X)|\overline{D}$ is dense in $A(D)$ with respect to the supremum norm on $\overline{D}$.*
(c) *The character space of $A(D)$ consists precisely of the point evaluations at points of $\overline{D}$.*
(d) *The Shilov boundary of $A(D)$ is ∂D.*
(e) *For every positive measure $\mu \in M^+(\overline{D})$ supported by ∂D, the triples $(A(D), \overline{D}, \mu)$ and $(A(D)|\partial D, \partial D, \mu|\partial D)$ are regular in the sense of Aleksandrov* ([1], [2]).

Proof. An application of the embedding theorem of Fornæss (Theorem 2.1.1) allows us to assume that (possibly after shrinking) X is a closed complex submanifold of some $\mathbb{C}^m$ ($m \in \mathbb{N}$) and that there is a bounded strictly convex domain $C \subset \mathbb{C}^m$ with C^2-boundary such that $D = X \cap C$ and $\partial_X D = X \cap \partial C$. A theorem of Henkin and Leiterer (Theorem 4.11.1 in [25]) asserts that in this case the restriction operators

$$A(C) \to A(D), \quad f \mapsto f|\overline{D}, \quad \text{and} \quad H^\infty(C) \to H^\infty(D), \quad f \mapsto f|D,$$

are surjective. By Corollary VII.2.5 in Range [38], there is an open neighborhood U_C of $\overline{C}$ in $\mathbb{C}^m$ such that $\mathcal{O}(U_C)|\overline{C} \subset A(C)$ is dense with respect to the supremum norm on $\overline{C}$. After a suitable translation, we may assume that C contains the origin. A separation argument for convex sets shows that in this case $rC \supset \overline{C}$ for $r > 1$. But then it is a trivial fact that each $f \in H^\infty(C)$ can be approximated pointwise on C by the sequence of holomorphic functions $f_n(z) = f(r_n z)$, where $r_n = 1 - 1/n$ and $z \in (1/r_n) \cdot C$ for $n \ge 2$.

Therefore, both assertions (a) and (b) hold if D is replaced by the smoothly bounded strictly convex set C and X is replaced by $\mathbb{C}^m$.

A look at the commuting diagrams

$$\begin{tikzcd} H^\infty(C) \arrow[r, "\text{surj.}"] & H^\infty(D) \\ A(C) \arrow[u] \arrow[r] & A(D) \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} A(C) \arrow[r, "\text{surj.}"] & A(D) \\ \mathcal{O}(U_C) \arrow[u] \arrow[r] & \mathcal{O}(U_C \cap X) \arrow[u] \end{tikzcd}$$

where all the maps are restrictions and the upper horizontal maps are surjective, completes the proof of (a) and (b). Note that the existence of the approximation constant

$k_D > 0$ in (a) relies on the fact that the upper horizontal map on the left-hand side is open (by the open mapping principle).

To prove part (c), let $\varphi : A(D) \to \mathbb{C}$ denote a character and let $(D_n)_{n\geq 1}$ be a neighborhood basis of $\overline{D}$ consisting of Stein open sets such that $D_{n+1} \subset D_n$ $(n \geq 1)$. For each $n \in \mathbb{N}$, φ induces a character $\varphi_n : \mathcal{O}(D_n) \to A(D) \xrightarrow{\varphi} \mathbb{C}$ which is a point evaluation at a suitable point z_n of D_n by the character theorem (see [27, Theorem 57.3]). Since the holomorphic functions separate the points of the Stein manifold D_1, we conclude that there is a $z \in \bigcap_{n\geq 1} D_n = \overline{D}$ such that $z = z_n$ $(n \geq 1)$. Hence φ coincides with the point evaluation at z on each of the spaces $\mathcal{O}(D_n)|\overline{D}$ $(n \geq 1)$. To conclude the proof of part (c) it suffices to remark that, for $n \geq 1$ large enough, the latter space is dense in $A(D)$.

Part (d) follows from the maximum principle and Corollary 2.1.2(c).

Since $\overline{C}$ is bounded, we may of course assume (possibly after multiplication with a suitable constant $0 < \varepsilon < 1$) that $\overline{C}$ is contained in the open unit ball $\mathbb{B}_m \subset \mathbb{C}^m$. Applying Theorem 3 in Løw [30] to the strictly positive function

$$\varphi(z) = 1 - |z|^2 \quad (z \in \partial C)$$

yields a mapping $g \in A(C)^M$ with $|g(z)|^2 = 1 - |z|^2$ $(z \in \partial C)$. Here $M \geq 1$ denotes a suitably chosen integer. Obviously, the map

$$f : \overline{C} \to \mathbb{C}^{m+M}, \qquad z \mapsto (z, g(z)),$$

defines an element of $A(C)^{m+M}$ which is injective and maps the boundary of C into the boundary of the unit ball $\mathbb{B}_{m+M} \subset \mathbb{C}^{m+M}$. Since we know that ∂D is the Shilov boundary of $A(D)$, the criterion of Aleksandrov presented above (Theorem 3 in [2]) can be applied to show that, for every measure $\mu \in M^+(\overline{D})$ which is supported by ∂D, the triples $(A(D), \overline{D}, \mu)$ and $(A(D)|\partial D, \partial D, \mu|\partial D)$ are regular. ∎

The regularity of the triples $(A(D), \overline{D}, \mu)$ and $(A(D)|\partial D, \partial D, \mu)$ for $\mu \in M^+(\partial D)$ has striking consequences concerning the boundary values of functions in $A(D)$.

2.1.4. Corollary. *Let $D \Subset X$ be a relatively compact strictly pseudoconvex open subset of a Stein manifold X, let $\mu \in M^+(\partial D)$ denote a positive regular Borel measure on ∂D and let $\varepsilon > 0$.*

(a) *Suppose that $0 < c \leq d$ are given real numbers and that $\kappa : \partial D \to \mathbb{R}$ is a Borel measurable function with $c \leq \kappa \leq d$. Then there exists a function $g \in A(D)$ satisfying*

$$|g| \leq d \quad \text{on } \overline{D}, \qquad \mu(\{z \in \partial D : \kappa(z) \neq |g(z)|\}) < \varepsilon.$$

(b) *There are a Borel subset $Z \subset \partial D$ of measure $\mu(Z) < \varepsilon$ and a sequence $(p_k)_{k\geq 1}$ in $A(D)$ which converges pointwise to zero on D and has the following properties for all $k \geq 1$:*

$$|p_k(z)| = 1 \quad (z \in \partial D \setminus Z), \qquad |p_k(z)| < 1 \quad (z \in D).$$

Proof. (a) Using Lusin's theorem we can choose a continuous function $p : \partial D \to \mathbb{R}$, $p \leq d$, such that the set $Z_1 = \{z \in \partial D : \kappa(z) \neq p(z)\}$ is small in the sense that $\mu(Z_1) < \varepsilon/2$. The pointwise maximum $\widetilde{p} = \max(p, c)$ is continuous and satisfies $c \leq \widetilde{p} \leq d$. From the regularity of the triple $(A(D)|\partial D, \partial D, \mu)$ it follows that there exists a function $g \in A(D)$,

$|g| \le \widetilde{p}$ on ∂D, such that $\mu(Z_2) < \varepsilon/2$ where $Z_2 = \{z \in \partial D : |g(z)| \ne \widetilde{p}(z)\}$ (see [1, Theorem 37]). Our choices guarantee that $|g(z)| = \kappa(z)$ for all $z \in \partial D \setminus (Z_1 \cup Z_2)$, and by the maximum principle we have $|g| \le d$ on $\overline{D}$, as desired.

(b) Let $\psi : X_D \to \mathbb{C}^m$ and $C \subset \mathbb{C}^m$ have the properties described in Theorem 2.1.1. Fix an arbitrary point $z_0 \in C$ and a continuous function

$$\varphi : \overline{C} \to [1/2, 1] \quad \text{with} \quad \varphi|\partial C \equiv 1 \quad \text{and} \quad \varphi(z_0) = 1/2.$$

Let $\psi^*\mu \in M^+(\partial C)$ be the positive measure defined by the formula

$$\psi^*\mu(\omega) = \mu(\psi^{-1}(\omega)) \qquad (\omega \subset \partial C \text{ Borel}).$$

The regularity of $(A(C), \overline{C}, \psi^*\mu)$ allows us to choose a function $g \in A(C)$ satisfying $|g| \le \varphi \le 1$ on $\overline{C}$ and $\psi^*\mu(\{z \in \partial C : |g(z)| \ne 1\}) < \varepsilon$ (see Theorem 37 in [1]). By the maximum principle we have $|g| < 1$ on C. Otherwise, g would be a constant function of modulus 1, contradicting the fact that $|g(z_0)| \le 1/2$. In view of the relations $\psi(D) = \psi(X_D) \cap C$ and $\psi(\partial D) = \psi(X_D) \cap \partial C$ the composition $f = g \circ \psi : \overline{D} \to \mathbb{C}$ satisfies $|f| < 1$ on D and

$$\mu(\{z \in \partial D : |f(z)| \ne 1\}) = \psi^*\mu(\{z \in \partial C : |g(z)| \ne 1\}) < \varepsilon.$$

Therefore the sequence of functions defined by

$$p_k : \overline{D} \to \mathbb{C}, \quad z \mapsto (f(z))^k \qquad (k \in \mathbb{N}),$$

has all the desired properties. ■

The following theorem says that $A(D)$ is strongly tight as a uniform algebra on $\overline{D}$. For more information about tight uniform algebras, we refer the reader to Cole and Gamelin [8] and Saccone [39] (where the proof of the following theorem can be found for $X = \mathbb{C}^n$).

2.1.5. Theorem. *Let $D \Subset X$ be a relatively compact strictly pseudoconvex open subset of a Stein manifold X. Then, for each $g \in C(\overline{D})$, the Hankel-type operator*

$$S_g : A(D) \to C(\overline{D})/A(D), \qquad f \mapsto [gf],$$

is compact.

Proof. In view of the estimate $\|S_g - S_h\| \le \|g - h\|_{\infty,\overline{D}}$, valid for $g, h \in C(\overline{D})$, a Stone–Weierstraß argument reduces the assertion to the case $g \in C^1(X)$.

Let $Z^\infty_{(0,1)}(D)$ denote the Banach space of all continuous, bounded and $\overline{\partial}$-closed $(0,1)$-forms on D. There exists a compact operator $R_D : Z^\infty_{(0,1)}(D) \to C(\overline{D})$ which is a right inverse for the $\overline{\partial}$ operator (see Henkin and Leiterer [25, Theorems 4.10.4(ii) and 4.10.6 (iii)]). Defining a continuous linear operator $T_g : A(D) \to Z^\infty_{(0,1)}(D)$ by $T_g(f) = f\overline{\partial}g$ ($f \in A(D)$), it is easy to check that S_g possesses the following factorization:

$$\begin{tikzcd} A(D) \arrow[r, "S_g"] \arrow[d, "T_g"'] & C(\overline{D})/A(D) \\ Z^\infty_{(0,1)}(D) \arrow[r, "R_D"] & C(\overline{D}) \arrow[u] \end{tikzcd}$$

from which the compactness of S_g follows. ■

A famous theorem of Siu (see [41]) asserts that every Stein submanifold of a complex manifold admits a Stein open neighborhood. Now, let X be a complex manifold and let $U \subset X$ be a Stein submanifold. Since U is locally compact and carries the relative topology induced by X, it can be written as the intersection $U = A \cap W$ of a closed set $A \subset X$ and an open set $W \subset X$ (see Engelking [13, Corollary 3.3.10]). The cited theorem of Siu allows us to choose a Stein open neighborhood $V \subset X$ of U in W. Clearly, $U = A \cap V$, showing that U is a closed submanifold of V.

Hence each Stein submanifold of a complex manifold X is a closed complex submanifold of a Stein open subset of X. This observation seems to be essential for our treatment of strictly pseudoconvex sets in Stein submanifolds of $\mathbb{C}^n$.

2.1.6. Proposition. *Let $X \subset \mathbb{C}^n$ be a Stein submanifold and let $D \Subset X$ be a relatively compact strictly pseudoconvex open set.*

(a) *The closure $\overline{D}$ of D in $\mathbb{C}^n$ coincides with the closure of D in X. Moreover, $\overline{D}$ is a Stein compactum in $\mathbb{C}^n$.*
(b) *There is an open neighborhood U of $\overline{D}$ in $\mathbb{C}^n$ such that $\mathcal{O}(U)|\overline{D} \subset A(D)$ is dense with respect to the supremum norm on $\overline{D}$, and such that $\mathcal{O}(U)|D$ is sequentially weak* dense in $H^\infty(D)$.*
(c) *Given a point $\mu \in D$, there is a constant $c = c(\mu) > 0$ such that, for any $f \in H^\infty(D)$, there exist functions $f_1, \cdots, f_n \in H^\infty(D)$ satisfying $\|f_i\|_{\infty,D} \leq c\|f\|_{\infty,D}$ $(i = 1, \ldots, n)$ and*

$$f - f(\mu) = \sum_{i=1}^{n} (z_i - \mu_i) f_i.$$

Here $z_i : \mathbb{C}^n \to \mathbb{C}$ denotes the ith coordinate projection.

Proof. The set $\overline{D}^X \subset \overline{D}$ is compact in X and hence in $\mathbb{C}^n$. This shows that $\overline{D}^X$ is closed in $\mathbb{C}^n$ and therefore $\overline{D}^X = \overline{D}$. To prove that $\overline{D}$ is a Stein compactum in $\mathbb{C}^n$, let $V \subset \mathbb{C}^n$ be an open neighborhood of $\overline{D}$. Since $\overline{D}$ is a Stein compactum in X (see Corollary 2.1.2(b)), there is a Stein open subset $W \subset X$ with $\overline{D} \subset W \subset V \cap X$. As we remarked above, W is a closed complex submanifold of a Stein open subset $U \subset \mathbb{C}^n$ satisfying $\overline{D} \subset U \subset V$.

We already know (see Corollary 2.1.3) that the density relations formulated in part (b) hold if U is replaced by a suitable Stein open neighborhood U_X of $\overline{D}$ in X. Writing U_X as a closed complex submanifold of a Stein open subset $U \subset \mathbb{C}^n$ finishes the proof of part (b) if one remembers the fact that the restriction map $\mathcal{O}(U) \to \mathcal{O}(U_X)$ is surjective (see [25, Corollary 4.1.8]).

For the rest of the proof, we fix a Stein open neighborhood X_D of $\overline{D}$ in X, a Fornæss embedding $\psi : X_D \to \mathbb{C}^m$ (see Theorem 2.1.1) and a bounded strictly convex domain $C \subset \mathbb{C}^m$ with C^2-boundary, such that $\psi : X_D \to \psi(X_D)$ is biholomorphic onto a closed complex submanifold of $\mathbb{C}^m$ and the representation formulas $\psi(D) = \psi(X_D) \cap C$, $\psi(\partial D) = \psi(X_D) \cap \partial C$ hold. Assume that μ is a fixed point in D. By the open mapping principle, it suffices to prove the surjectivity of the continuous linear map

$$H^\infty(D)^n \to I_\mu, \qquad (f_i)_{i=1}^n \mapsto \sum_{i=1}^{n} (z_i - \mu_i) f_i,$$

where $I_\mu = \{g \in H^\infty(D) : g(\mu) = 0\}$. Starting with an element $f \in I_\mu$, we use the surjectivity of the restriction operator $H^\infty(C) \to H^\infty(\psi(D))$ (see Henkin and Leiterer [25, Theorem 4.11.1]) to obtain a function $F \in H^\infty(C)$ satisfying $F|\psi(D) = f \circ (\psi^{-1}|\psi(D))$. Consequently, $F(\psi(\mu)) = 0$, and the Gleason property of the smoothly bounded strictly convex domain $C \subset \mathbb{C}^m$ (cf. Grangé [22, p. 244]) allows us to choose functions $F_1, \dots, F_m \in H^\infty(C)$ solving the division problem

$$F(w) = \sum_{j=1}^{m} (w_j - \psi_j(\mu)) F_j(w) \qquad (w \in C).$$

Replacing w by $\psi(z)$ with $z \in D$ yields the relation

$$f(z) = \sum_{j=1}^{m} (\psi_j(z) - \psi_j(\mu)) F_j \circ \psi(z) \qquad (z \in D).$$

Using Siu's theorem and the fact that $X_D \subset \mathbb{C}^n$ is a Stein submanifold, we deduce that the functions $\psi_j \in \mathcal{O}(X_D)$ $(j = 1, \dots, m)$ can be extended to holomorphic functions $\Psi_j \in \mathcal{O}(U)$ on some Stein open set $U \subset \mathbb{C}^n$ containing X_D as a closed complex submanifold. By Hefer's lemma we obtain decompositions of the following type:

$$\Psi_j(z) - \Psi_j(\mu) = \sum_{i=1}^{n} (z_i - \mu_i) g_{i,j}(z) \qquad (z \in U,\ 1 \le j \le m)$$

with suitably chosen functions $g_{i,j} \in \mathcal{O}(U)$ $(1 \le i \le n,\ 1 \le j \le m)$. Therefore our calculations finally end up with the formula

$$f(z) = \sum_{j=1}^{m} \sum_{i=1}^{n} (z_i - \mu_i) g_{i,j}(z) F_j \circ \psi(z) = \sum_{i=1}^{n} (z_i - \mu_i) \sum_{j=1}^{m} g_{i,j}(z) F_j \circ \psi(z) \qquad (z \in D),$$

which completes the proof. ∎

2.2. Henkin measures. Throughout this section, let X be a Stein manifold and let $\emptyset \neq D \Subset X$ be a relatively compact strictly pseudoconvex open set.

A sequence (f_k) in $A(D)$ is said to be a *Montel sequence* if (f_k) is a weak* zero sequence in $H^\infty(D)$, i.e. if it is norm-bounded and converges to zero pointwise on D (or, equivalently, uniformly on compact subsets of D). By a *Henkin measure* (on $\overline{D}$) we mean a complex regular Borel measure $\mu \in M(\overline{D})$ satisfying

$$\int_{\overline{D}} f_k \, d\mu \to 0 \qquad \text{for every Montel sequence } (f_k) \text{ in } A(D).$$

The set of all Henkin measures on $\overline{D}$ will be denoted by $HM(\overline{D})$. It can be easily checked that $HM(\overline{D})$ is a closed subspace of $M(\overline{D})$ with respect to the total variation norm. Note that by Lebesgue's dominated convergence theorem each measure $\mu \in M(\overline{D})$ with the property that $|\mu|(\partial D) = 0$ is a Henkin measure. For this reason, the volume measure λ introduced in the preceding section is a Henkin measure on $\overline{D}$.

Cole and Range [9] studied the theory of Henkin measures on $\overline{D}$ under the additional assumption that ∂D is smooth. In what follows we consider the general case (of not necessarily smooth boundary). Our first aim is to show that $HM(\overline{D})$ is a band of measures.

This fact turns out to be a consequence of the strong tightness of the algebra $A(D)$, which implies the following important intermediate result.

2.2.1. PROPOSITION. *Let $g \in C(\overline{D})$ and let (f_k) be a Montel sequence in $A(D)$. Then there exists a Montel sequence (g_k) in $A(D)$ such that $\|f_k g - g_k\|_{\infty,\overline{D}} \to 0$ as $k \to \infty$.*

Proof. Given a subsequence (u_k) of (f_k), the compactness of the operator $S_g : A(D) \to C(\overline{D})/A(D), f \mapsto [gf]$, introduced in Theorem 2.1.5 allows us to select a subsequence $(u_{k_m})_m$ of (u_k) such that $([gu_{k_m}])_m$ converges to an element $[u] \in C(\overline{D})/A(D)$ represented by $u \in C(\overline{D})$. Using the definition of the quotient norm we obtain a sequence of functions (h_m) in $A(D)$ satisfying

$$\|gu_{k_m} - u + h_m\|_{\infty,\overline{D}} \xrightarrow{m\to\infty} 0.$$

From the fact that (u_{k_m}) is Montel we deduce that (h_m) converges to u uniformly on compact subsets of D. This proves that the function $u \in C(\overline{D})$ is analytic on D and hence $[u] = 0$.

So far we have shown that $([gf_k])$ converges to zero in $C(\overline{D})/A(D)$. Now the definition of the quotient norm obviously allows us to choose a sequence (g_k) in $A(D)$ having the desired properties. ∎

Let $\mu \in M(\overline{D})$ be an arbitrary complex measure. We use the abbreviation $L^p(\mu) = L^p(|\mu|)$, for $1 \le p \le \infty$. A measure $\nu \in M(\overline{D})$ is by definition absolutely continuous with respect to the complex measure μ ($\nu \ll \mu$) if $\nu \ll |\mu|$ in the usual sense. In this case we have $\nu = gd\mu$ with a suitable element $g \in L^1(\mu)$, as a consequence of the Radon–Nikodym theorem.

A *band of measures* $\mathcal{B}$ on $\overline{D}$ is a closed subspace $\mathcal{B} \subset M(\overline{D})$ satisfying the additional requirement that whenever $\nu \ll \mu$ with $\mu \in \mathcal{B}$ and $\nu \in M(\overline{D})$ it follows that $\nu \in \mathcal{B}$. The next theorem shows that $HM(\overline{D})$ is a band of measures.

2.2.2. THEOREM (Henkin's theorem). *Let $\mu \in HM(\overline{D})$ be a Henkin measure. Then every measure $\nu \in M(\overline{D})$ satisfying $\nu \ll \mu$ is also a Henkin measure.*

Proof. By the preceding remarks and the density of $C(\overline{D})$ in $L^1(\mu)$ it suffices to show that, given a function $g \in C(\overline{D})$ and a Montel sequence (f_k) in $A(D)$, the sequence of integrals $\int_{\overline{D}} gf_k\,d\mu$ converges to zero. An application of Proposition 2.2.1 yields a Montel sequence (g_k) in $A(D)$ satisfying $\|gf_k - g_k\|_{\infty,\overline{D}} \to 0$. Hence $\int_{\overline{D}} gf_k\,d\mu - \int_{\overline{D}} g_k\,d\mu \to 0$, but the integrals following the minus sign converge to zero since μ is a Henkin measure. ∎

The following lemma gives an impression of the size of $HM(\overline{D})$.

2.2.3. LEMMA. *Let $w \in \partial D$. Then the Dirac measure ε_w associated with w is singular to each Henkin measure $\mu \in HM(\overline{D})$. If $\dim_{\mathbb{C}}(X) \ge 1$, this implies*

$$\dim M(\overline{D})/HM(\overline{D}) = \infty.$$

Proof. Fix a peaking function $h \in A(D)$ corresponding to w (see Corollary 2.1.2). The sequence of powers $(h^k)_{k\ge1}$ is Montel and converges pointwise on $\overline{D}$ to the characteristic function $\chi_{\{w\}}$. Therefore by Lebesgue's dominated convergence theorem we have

$\mu(\{w\}) = \int_{\overline{D}} \chi_{\{w\}}\, d\mu = \lim_k \int_{\overline{D}} h^k\, d\mu = 0$ for any Henkin measure $\mu \in HM(\overline{D})$. This proves the first assertion. To prove the second one, let $w_1, \ldots, w_n \in \partial D$ be pairwise distinct elements. Since any linear combination $\sum_{i=1}^n \lambda_i \varepsilon_{w_i}$ of finite length is singular to each Henkin measure, the equivalence classes $[\varepsilon_{w_1}], \ldots, [\varepsilon_{w_n}] \in M(\overline{D})/HM(\overline{D})$ are linearly independent. To finish the proof we have to point out that ∂D cannot consist of only finitely many points. After shrinking X, if necessary, we are allowed to make the additional assumption that X is not compact (this is an easy consequence of Theorem 2.1.1 and the fact that a compact complex submanifold of $\mathbb{C}^m$ has dimension zero). Since the boundary of each component of D is contained in ∂D we may further assume that both D and X are connected. Using charts and the fact that connectedness and pathwise connectedness are the same for open subsets of manifolds one deduces that, for every $p \in X$, the set $X \setminus \{p\}$ is connected. Repeating this argument shows that, if ∂D were finite, then $(\partial D)^{\mathrm{c}} = X \setminus \partial D$ would be connected. Therefore one of the sets on the right-hand side of the decomposition

$$(\partial D)^{\mathrm{c}} = (\overline{D} \cap D^{\mathrm{c}})^{\mathrm{c}} = (\overline{D})^{\mathrm{c}} \cup D$$

would have to be empty. By hypothesis $D \neq \emptyset$, so $X = \overline{D}$ would be compact, contradicting our assumption. ■

Recall that a *dual algebra* A is by definition a Banach algebra which is isometrically isomorphic to the dual space of a Banach space A_* and which has the additional property that the multiplication in A is separately weak* continuous. If B is another dual algebra then a map $\varrho : A \to B$ is called a *dual algebra homomorphism* if ϱ is an algebra homomorphism which is contractive and weak* continuous. If ϱ is invertible and both ϱ and ϱ^{-1} are dual algebra homomorphisms then ϱ is said to be a *dual algebra isomorphism.* General duality theory shows that an algebraic isomorphism ϱ between dual algebras is a dual algebra isomorphism if and only if ϱ is isometric and weak* continuous.

Natural examples of dual algebras are given by $L^\infty(\mu)$, $\mu \in M(\overline{D})$, or $L(H)$ (which is in duality to the trace class). Since the compactum $\overline{D}$ is metrizable, we know that the predual $L^1(\mu)$ of $L^\infty(\mu)$ is separable for every choice of $\mu \in M(\overline{D})$. Obviously, each weak* closed subalgebra of a dual algebra inherits a natural dual algebra structure. This observation immediately yields the examples

$$H^\infty(\mu) = \overline{A(D)}^{w^*} \subset L^\infty(\mu),$$

which play an important role in the theory of Henkin measures, for the following reason:

2.2.4. Proposition. *Each non-zero Henkin measure $\mu \in HM(\overline{D})$ induces a homomorphism of dual algebras $r_\mu : H^\infty(D) \to H^\infty(\mu)$ of norm one extending the natural inclusion map $A(D) \hookrightarrow L^\infty(\mu)$. Moreover, r_μ is the unique weak* continuous linear map $H^\infty(D) \to H^\infty(\mu)$ with the latter extension property.*

Before going into the proof of the above proposition, let us fix the following notation: Given a normed space E and a real number $c > 0$, we set

$$(E)_c = \{x \in E : \|x\| \le c\}.$$

Proof. By Corollary 2.1.3 there exists a constant $c > 0$ such that $\overline{(A(D))_c}^{w^*} \supset (H^\infty(D))_1$. Now consider the inclusion map

$$\Phi : A(D) \to H^\infty(\mu) \subset L^\infty(\mu), \qquad f \mapsto f.$$

Given a Montel sequence (f_k) in $A(D)$ and an arbitrary function $g \in L^1(\mu)$ we infer from Theorem 2.2.2 that $\int_{\overline{D}} g f_k d\mu \to 0$ as $k \to \infty$. This proves that (f_k) tends to zero in the weak* topology of $L^\infty(\mu)$ and hence of $H^\infty(\mu)$. Having in mind the latter convergence as well as the isometric embeddings $A(D) \subset H^\infty(D) \subset L^\infty(\lambda)$ and $H^\infty(\mu) \subset L^\infty(\mu)$, the existence and the desired properties of the map r_μ are consequences of the following abstract functional-analytic lemma. ∎

2.2.5. Lemma. *Let X, Y be separable Banach spaces and let $A \subset X'$ be a linear subspace such that $\overline{(A)_c}^{w^*} \supset (X')_1$ with a suitable constant $c \geq 1$. Given a norm continuous linear map $\Phi : A \to Y'$, the following assertions hold:*

(a) *There exists a weak* continuous linear map $\widehat{\Phi} : X' \to Y'$ extending Φ if and only if, for every choice of a weak* zero sequence (f_k) in A, the sequence $(\Phi(f_k))$ forms a weak* zero sequence in Y'. In this case, $\widehat{\Phi}$ is uniquely determined by Φ and $\|\widehat{\Phi}\| \leq c\|\Phi\|$.*

(b) *Suppose that Φ possesses a weak* continuous linear extension $\widehat{\Phi}$ as described in* (a). *If X' and Y' are dual algebras, $A \subset X'$ is an algebra and Φ is an algebra homomorphism, then so is $\widehat{\Phi}$.*

(c) *If we make the additional assumptions in part* (b) *that X' and Y' are subalgebras of commutative unital C^*-algebras, that $1 \in A$ and that the map $\Phi : A \to Y'$ is a unital homomorphism, then $\widehat{\Phi}$ is a dual algebra homomorphism of norm* 1.

Proof. The "only if" part of (a) is obvious, so we assume that the convergence condition described in (a) holds for Φ. First note that the weak* topology on bounded subsets of X' and Y' is metrizable. This allows us to construct the desired extension $\widehat{\Phi} : X' \to Y'$ working with sequences instead of nets. Namely, given an element $f \in X'$ we choose a sequence (f_k) in $(A)_{c\|f\|}$ such that $f_k \xrightarrow{w^*} f$ and such that $(\Phi(f_k))$ is weak* convergent in Y'. According to the Alaoglu–Bourbaki theorem, the latter requirement can always be satisfied by passing to a suitable subsequence. The convergence condition of part (a) guarantees that the assignment

$$\widehat{\Phi}(f) = w^*\text{-}\lim_{k\to\infty} \Phi(f_k)$$

yields a well defined linear mapping $\widehat{\Phi} : X' \to Y'$. Standard arguments show that $\widehat{\Phi}$ is actually weak* continuous (see e.g. the proof of Lemma 1.1 in [15]) and norm continuous with $\|\widehat{\Phi}\| \leq c\|\Phi\|$.

Part (b) can be easily proved using the separate weak* continuity of the multiplication in X' and Y'.

Let $x \in X'$ be an arbitrary element and let $B \supset X'$ denote a unital commutative C^*-algebra containing X' isometrically. Then, as the calculation

$$\|x\| = \varrho_B(x) = \lim_{k\to\infty} \|x^k\|^{1/k} = \varrho_{X'}(x)$$

shows, the norm and the spectral radius of x as an element of X' coincide. An analogous result holds for elements of Y'. Combining this fact with the spectral inclusion $\sigma_{Y'}(\Phi(x)) \subset \sigma_{X'}(x)$, which holds since Φ is a unital homomorphism, one deduces that $\widehat{\Phi}$ has norm one. ∎

Using the fact that the induced homomorphism $r_\eta : H^\infty(D) \to H^\infty(\eta)$ of a Henkin measure $\eta \in HM(\overline{D})$ is always contractive, one can determine the approximation constant k_D of Corollary 2.1.3(a).

2.2.6. LEMMA. *The identity* $\overline{(A(D))_1}^{w^*} = (H^\infty(D))_1$ *holds. Hence, the approximation constant* k_D *in part* (a) *of Corollary* 2.1.3 *can be chosen to be* $k_D = 1$.

Proof. Assume that there exists a function $f \in H^\infty(D)$ satisfying $\|f\|_{\infty,D} \le 1$ which is not in the weak* closure of $(A(D))_1$. Using the separation theorem of Hahn–Banach we obtain an element $g \in L^1(\lambda)$ and a constant $\alpha \ge 0$ such that the measure $\nu = g d\lambda$ satisfies $|\int_D f\,d\nu| > \alpha$ and $|\int_D h\,d\nu| \le \alpha$ $(h \in (A(D))_1)$. By the Hahn–Banach extension theorem and the Riesz representation theorem, the continuous linear functional defined by $A(D) \to \mathbb{C}$, $h \mapsto \int_D h\,d\nu$, has a norm preserving extension to a continuous linear form

$$C(\overline{D}) \to \mathbb{C}, \qquad h \mapsto \int_{\overline{D}} h\,d\eta,$$

where $\eta \in M(\overline{D})$ is a measure of total variation $\|\eta\| \le \alpha$. The above choice of η guarantees that $\eta - \nu \in A(D)^\perp$ and therefore η is a Henkin measure. Since the two weak* continuous linear functionals defined by

$$H^\infty(D) \to \mathbb{C}, \quad h \mapsto \int_{\overline{D}} r_\eta(h)\,d\eta, \qquad H^\infty(D) \to \mathbb{C}, \quad h \mapsto \int_D h\,d\nu,$$

coincide on the weak* dense subset $A(D) \subset H^\infty(D)$, they are equal. Finally, using the fact that $\|r_\eta\| = 1$, we arrive at the estimate

$$\alpha < \Big|\int_D f\,d\nu\Big| = \Big|\int_{\overline{D}} r_\eta(f)\,d\eta\Big| \le \|r_\eta\| \cdot \|\eta\| \cdot \|f\|_{\infty,D} \le \alpha,$$

which is a contradiction. ∎

A Henkin measure $\mu \in HM(\overline{D})$ is called *faithful* if it has the property that the induced homomorphism $r_\mu : H^\infty(D) \to H^\infty(\mu)$ is a dual algebra isomorphism.

From the fact that $H^\infty(D) = \overline{A(D)}^{w^*} \subset L^\infty(\lambda)$ (see Corollary 2.1.3) we infer that the volume measure λ on D is a faithful Henkin measure. Provided that the boundary of D is sufficiently smooth, the surface area measure on ∂D is another natural example of a faithful Henkin measure which has the additional property of being supported only by the boundary. Using operator-theoretical techniques we will see later that, even in the case where ∂D is not smooth, there exists a faithful Henkin measure supported by ∂D.

Let $\mu \in HM(\overline{D})$. As the following lemma points out, the characteristic part of the induced homomorphism r_μ is determined by the boundary part of μ or, more precisely, by the measure $\mu_{\partial D} \in HM(\overline{D})$ defined by $\mu_{\partial D}(\omega) = \mu(\omega \cap \partial D)$ for all $\omega \in \mathcal{B}(\overline{D})$.

2.2.7. LEMMA. *For every Henkin measure $\mu \in HM(\overline{D})$, we have the identity*

$$r_\mu(f) = f \oplus r_{\mu_{\partial D}}(f)|\partial D \in L^\infty(\mu|D) \oplus L^\infty(\mu|\partial D) \qquad (f \in H^\infty(D)).$$

Here the two L^∞-spaces on the right are regarded as subspaces of $L^\infty(\mu)$ via trivial extension.

Proof. Both the left- and the right-hand side of the above formula are weak* continuous linear extensions of the natural embedding $A(D) \hookrightarrow L^\infty(\mu)$. Hence the assertion follows from the uniqueness part of Proposition 2.2.4. ∎

Now we return to the concept of a faithful Henkin measure.

2.2.8. LEMMA (Valskiĭ-type decomposition). *Let $\nu \in HM(\overline{D})$ be a faithful Henkin measure. Then each Henkin measure $\mu \in HM(\overline{D})$ has a decomposition*

$$\mu = g d\nu + \eta \qquad \textit{with } g \in L^1(\nu),\ \eta \in A(D)^\perp.$$

Proof. Choose an element $g \in L^1(\nu)$ representing the weak* continuous linear form

$$\varphi : H^\infty(\nu) \to \mathbb{C}, \qquad f \mapsto \int_{\overline{D}} r_\mu \circ r_\nu^{-1}(f)\, d\mu.$$

From the fact that the composition $r_\mu \circ r_\nu^{-1}$ acts as the identity on $A(D)$ we obtain

$$0 = \varphi(f) - \int_{\overline{D}} fg\, d\nu = \int_{\overline{D}} f\, d\mu - \int_{\overline{D}} fg\, d\nu \qquad (f \in A(D)).$$

This shows that $\mu - gd\nu \in A(D)^\perp$, as desired. ∎

The above lemma generalizes known decomposition theorems for the case that ν is the surface measure or the volume measure of a smoothly bounded strictly pseudoconvex domain D to the case that ν is an arbitrary faithful Henkin measure on a strictly pseudoconvex open set D with possibly non-smooth boundary (see e.g. Theorem 2.3 in [9]).

With an arbitrary band of measures $\mathcal{B} \subset M(\overline{D})$ one can associate a so-called *complementary band* $\mathcal{B}_{\mathrm{s}}$ defined by $\mathcal{B}_{\mathrm{s}} = \{\eta \in M(\overline{D}) : \eta \perp \nu \text{ for all } \nu \in \mathcal{B}\}$. The classical Lebesgue decomposition theorem for measures has a natural generalization in this context, namely the ℓ^1-direct sum decomposition

$$M(\overline{D}) = \mathcal{B} \oplus_1 \mathcal{B}_{\mathrm{s}}.$$

The dual space of a band $\mathcal{B}$ can be represented in the following way: The collection

$$L^\infty(\mathcal{B}) = \Big\{ (f_\mu) \in \prod_{\mu \in \mathcal{B}} L^\infty(\mu) : f_\nu = f_\mu \ (\mu\text{-a.e.})\ \forall \mu, \nu \in \mathcal{B} \text{ with } \mu \ll \nu \Big\}$$

is a commutative unital algebra over $\mathbb{C}$. Equipped with the norm

$$\|f\| = \sup_{\mu \in \mathcal{B}} \|f_\mu\|_{\infty,\mu} \qquad (f = (f_\mu) \in L^\infty(\mathcal{B})),$$

$L^\infty(\mathcal{B})$ even becomes a commutative unital C^*-algebra. The mapping

$$L^\infty(\mathcal{B}) \to \mathcal{B}', \qquad f \mapsto \varphi_f \quad \text{with} \quad \varphi_f(\mu) = \int_{\overline{D}} f_\mu\, d\mu,$$

for $\mu \in \mathcal{B}$ and $f = (f_\mu) \in L^\infty(\mathcal{B})$, is an isometric isomorphism (see [11, Theorem V.20.5]). The induced weak* topology on $L^\infty(\mathcal{B})$ can be characterized by the fact that the canonical inclusion map $(L^\infty(\mathcal{B}), \text{weak}^*) \to \prod_{\mu\in\mathcal{B}}(L^\infty(\mu), \text{weak}^*)$ is a topological monomorphism. So $L^\infty(\mathcal{B})$ is actually a dual algebra. Dualizing the direct sum decomposition $M(\overline{D}) = \mathcal{B} \oplus_1 \mathcal{B}_\mathrm{s}$ one obtains a dual algebra isomorphism

$$L^\infty(M(\overline{D})) \to L^\infty(\mathcal{B}) \oplus_\infty L^\infty(\mathcal{B}_\mathrm{s}), \qquad (f_\mu) \mapsto ((f_\mu)_{\mu\in\mathcal{B}}, (f_\mu)_{\mu\in\mathcal{B}_\mathrm{s}}).$$

For any band $\mathcal{B} \subset M(\overline{D})$, the map

$$C(\overline{D}) \xrightarrow{j_\mathcal{B}} L^\infty(\mathcal{B}), \qquad f \mapsto ([f]_\mu)_\mu,$$

is contractive. (Here $[f]_\mu$ stands for the equivalence class of f in $L^\infty(\mu)$.) Therefore, any norm dense subset of $A(D)$ gives rise to a weak* dense subset of the dual algebra

$$H^\infty(\mathcal{B}) = \overline{j_\mathcal{B}(A(D))}^{w^*} \subset L^\infty(\mathcal{B}).$$

Note that in general the map $j_\mathcal{B}$ need not be injective. However, as soon as $\mathcal{B}$ contains at least one faithful Henkin measure, $j_\mathcal{B}$ induces an isometric embedding $A(D) \xhookrightarrow{j_\mathcal{B}} H^\infty(\mathcal{B})$.

From now on, the complementary band of $HM(\overline{D})$ in $M(\overline{D})$ will be denoted by $S(\overline{D})$ and referred to as the *singular band*.

2.2.9. Lemma. *The identification $L^\infty(M(\overline{D})) = L^\infty(S(\overline{D})) \oplus_\infty L^\infty(HM(\overline{D}))$ from above induces a dual algebra isomorphism*

$$H^\infty(M(\overline{D})) = L^\infty(S(\overline{D})) \oplus_\infty H^\infty(HM(\overline{D})).$$

Proof. Obviously, $H^\infty(M(\overline{D}))$ is a weak* closed subspace of the space on the right. So by the theorem of Hahn–Banach, equality holds if every weak* continuous linear form L on $L^\infty(M(\overline{D})) = L^\infty(S(\overline{D})) \oplus L^\infty(HM(\overline{D}))$ which annihilates $H^\infty(M(\overline{D}))$ vanishes on $L^\infty(S(\overline{D})) \oplus H^\infty(HM(\overline{D}))$. As a weak* continuous linear functional on $L^\infty(M(\overline{D}))$, L is of the form

$$L(f) = \int_{\overline{D}} f_\tau \, d\tau \qquad (f \in L^\infty(S(\overline{D})) \oplus L^\infty(HM(\overline{D}))),$$

with a suitably chosen measure $\tau \in M(\overline{D})$. Our assumption implies that L annihilates $A(D)$, proving that L vanishes on $H^\infty(HM(\overline{D}))$ and that $\tau \in HM(\overline{D})$ is a Henkin measure. But then, of course, L also vanishes on $L^\infty(S(\overline{D}))$. ■

Given two measures $\mu, \nu \in M(\overline{D})$ satisfying $\mu \ll \nu$, the theorem of Radon–Nikodym ensures that there exists an element $\frac{d\mu}{d\nu} \in L^1(\nu)$ having the property that $\mu = \left(\frac{d\mu}{d\nu}\right) d\nu$. Obviously, the multiplication map $i_\mu^\nu : L^1(\mu) \to L^1(\nu)$, $f \mapsto \left(\frac{d\mu}{d\nu}\right) f$, is an isometry. Therefore, the adjoint of i_μ^ν acts as a weak* continuous contraction

$$r_\mu^\nu : L^\infty(\nu) \to L^\infty(\mu),$$

and it is an easy exercise to show that r_μ^ν maps the equivalence class $[f] \in L^\infty(\nu)$ of a bounded measurable function f to the equivalence class of the same function $[f] \in L^\infty(\mu)$. Thus, r_μ^ν is in fact a dual algebra homomorphism which induces a dual algebra homomorphism $r_\mu^\nu : H^\infty(\nu) \to H^\infty(\mu)$. Note that, in view of the uniqueness part of Proposition 2.2.4, the compatibility condition $r_\mu = r_\mu^\nu r_\nu$ is satisfied whenever ν is a Henkin measure. The following result, which is based on this elementary observation, can be thought of as an analogue of Davie's theorem for the algebra $A(D)$ (cf. [11, Theorem V.22.1]).

2.2.10. Proposition. *Let $\mathcal{B} \subset HM(\overline{D})$ be a band of measures containing a faithful Henkin measure. Then the canonical embedding $A(D) \hookrightarrow H^\infty(\mathcal{B})$ has a unique extension to a dual algebra isomorphism*

$$r : H^\infty(D) \to H^\infty(\mathcal{B}).$$

In particular, this yields a dual algebra isomorphism $H^\infty(D) \xrightarrow{\sim} H^\infty(HM(\overline{D}))$.

Proof. The uniqueness part of the assertion follows from the fact that $A(D)$ is weak* dense in $H^\infty(D)$ (see Corollary 2.1.3). Using the properties of the induced homomorphisms r_μ (see also the remarks preceding the proposition) one immediately checks that the mapping

$$r : H^\infty(D) \to H^\infty(\mathcal{B}) \subset L^\infty(\mathcal{B}), \qquad f \mapsto (r_\mu(f))_{\mu\in\mathcal{B}},$$

is a well defined dual algebra homomorphism, which acts on elements of $A(D)$ as desired. Since, by hypothesis, there exists a faithful Henkin measure ν on $\overline{D}$ belonging to $\mathcal{B}$, the mapping r is isometric. As a weak* continuous isometry, r has weak* closed range. Since the range of r is also dense in $H^\infty(\mathcal{B})$, r is in fact a dual algebra isomorphism. ∎

2.3. Extensions of the holomorphic functional calculus. Let $T = (T_1, \ldots, T_n) \in L(H)^n$ denote a commuting n-tuple of bounded linear operators on a separable complex Hilbert space H. Given a pair (A, φ_A) consisting of a locally convex algebra A and an algebra homomorphism $\varphi_A : \mathbb{C}[z_1, \ldots, z_n] \to A$ "attached" to A, we say that a map $\Phi : A \to L(H)$ is a (continuous) A-*functional calculus* (with respect to φ_A) if Φ is a (continuous) algebra homomorphism and the composition $\Phi \circ \varphi_A$ maps the constant function 1 to the identity operator on H and the ith coordinate function z_i to T_i for $i = 1, \ldots, n$. Since the map φ_A will always be clear from the context (see below), it will usually be suppressed in the notation.

Most of the time, the algebra A is an algebra of complex-valued functions (defined on a subset Ω of $\mathbb{C}^n$) containing the restrictions of polynomials. In this case φ_A is just the restriction map $\mathbb{C}[z_1, \ldots, z_n] \to A$, $p \mapsto p|\Omega$. If $\mathcal{B} \subset M(\overline{D})$ is a band of measures and $A = H^\infty(\mathcal{B})$, then we always take $\varphi_A = j_{\mathcal{B}}|\mathbb{C}[z_1, \ldots, z_n]$. If $A = H^\infty(\mu)$ for a Henkin measure $\mu \in HM(\overline{D})$, then we set $\varphi_A = r_\mu|\mathbb{C}[z_1, \ldots, z_n]$.

Every commuting n-tuple $T \in L(H)^n$ is known to possess a continuous $\mathcal{O}(\sigma(T))$-functional calculus, the so-called *holomorphic functional calculus* for T introduced by Taylor [43]. Here $\sigma(T) \subset \mathbb{C}^n$ denotes the joint spectrum of T in the sense of Taylor. If $\tau : \mathcal{O}(\sigma(T)) \to L(H)$ denotes the holomorphic functional calculus of T, then we use the standard notation $f(T) = \tau(f)$ for $f \in \mathcal{O}(\sigma(T))$. For the construction of the holomorphic functional calculus and a detailed discussion of its properties we refer the reader to the monograph [19].

From now on, let $D \Subset X$ be a relatively compact strictly pseudoconvex open subset of a Stein submanifold $X \subset \mathbb{C}^n$. In what follows, we will consider n-tuples $T \in L(H)^n$ possessing a continuous functional calculus over one of the algebras $A(D)$ or $H^\infty(\mathcal{B})$, where $\mathcal{B} \subset M(\overline{D})$ denotes an arbitrary band of measures. Our first observation is that continuous $A(D)$-calculi always extend the holomorphic functional calculus.

2.3.1. Lemma. *Suppose that $T \in L(H)^n$ is a commuting n-tuple possessing a continuous $A(D)$-functional calculus $\Phi : A(D) \to L(H)$.*

(a) *The n-tuple T satisfies $\sigma(T) \subset \overline{D}$.*
(b) *For $f \in \mathcal{O}(\overline{D})$, we have* ([4]) *$\Phi(f|\overline{D}) = f(T)$.*
(c) *The mapping Φ is uniquely determined by the formula of part* (b)*.*

Proof. Recall that the characters of $A(D)$ are precisely the point evaluations at points of $\overline{D}$ (see Corollary 2.1.3). Therefore, denoting the commutant algebra of T by $(T)' \subset L(H)$ and the n-tuple of coordinate functions by $z = (z_1, \ldots, z_n) \in A(D)^n$, we obtain the desired spectral inclusion

$$\sigma(T) \subset \sigma_{(T)'}(T) \subset \sigma_{A(D)}(z) = \overline{D}.$$

To prove the second assertion, observe that, for each Stein open set $\overline{D} \subset U \subset \mathbb{C}^n$, the composition $\mathcal{O}(U) \hookrightarrow A(D) \xrightarrow{\Phi} L(H)$ is an extension of the canonical $\mathcal{O}(\mathbb{C}^n)$-functional calculus of T to a continuous $\mathcal{O}(U)$-functional calculus. For uniqueness reasons (see Lemma 5.1.1(b) in [19]) we therefore have $\Phi(f|\overline{D}) = f(T)$ for $f \in \mathcal{O}(U)$. Now recall that $\overline{D} \subset \mathbb{C}^n$ is a Stein compactum (see Proposition 2.1.6) to complete the proof of part (b). The uniqueness assertion stated in part (c) follows from the fact that $\mathcal{O}(\overline{D})$ is a norm dense subspace of $A(D)$ (Proposition 2.1.6). ■

Given a continuous linear operator $S \in L(H)$ we use the standard notation

$$\mathrm{Lat}(S) = \{M : M \text{ is a closed linear subspace of } H \text{ satisfying } SM \subset M\}$$

to denote the invariant subspace lattice of S. Suppose that $\Phi : A \to L(H)$ is a continuous A-functional calculus for a commuting n-tuple $T \in L(H)^n$. Then a subspace $M \subset H$ is called *Φ-invariant* if $M \in \bigcap(\mathrm{Lat}(\Phi(f)) : f \in A)$.

2.3.2. Lemma. *Given a commuting n-tuple $C \in L(H)^n$ possessing a continuous $A(D)$-functional calculus $\Phi_C : A(D) \to L(H)$, the following assertions hold*:

(a) *If $M \subset H$ is a Φ_C-invariant subspace, then the restriction $T = C|M \in L(M)^n$ possesses a continuous $A(D)$-functional calculus $\Phi_T : A(D) \to L(H)$. In particular, $\sigma(T) \subset \overline{D}$ and Φ_T is of the form*

$$\Phi_T(f) = \Phi_C(f)|M \quad (f \in A(D)).$$

(b) *Suppose that $M \in \mathrm{Lat}(C)$ and that $T = C|M$ satisfies $\sigma(T) \subset \overline{D}$. Then M is Φ_C-invariant and part* (a) *applies.*
(c) *If $M \subset H$ is a reducing subspace for C, then we are in the situation of part* (b)*.*

Proof. Assuming $M \subset H$ to be Φ_C-invariant we deduce that the definition

$$\Phi_T : A(D) \to L(M), \qquad f \mapsto \Phi_C(f)|M,$$

makes sense and in fact yields a continuous $A(D)$-functional calculus for $T = C|M$. The uniqueness assertion of part (a) now follows from the preceding lemma.

([4]) Note that, although $\overline{D}$ is contained in the complex (sub-)manifold X, we use the notation $\mathcal{O}(\overline{D}) = \varinjlim(\mathcal{O}(U) : U \subset \mathbb{C}^n \text{ open and } \overline{D} \subset U)$ for the set of germs of holomorphic functions on $\overline{D}$ relative to $\mathbb{C}^n$.

To prove part (b) note that the inclusion map $i : M \hookrightarrow H$ intertwines the n-tuples T and C componentwise, i.e. $i \circ T_j = C_j \circ i$ ($j = 1, \ldots, n$). By hypothesis $\sigma(T) \cup \sigma(C) \subset \overline{D}$, and since in this case the holomorphic functional calculus preserves intertwining relations (see Lemma 2.5.8 in [19]), we deduce that

$$i \circ f(T) = f(C) \circ i \quad (f \in \mathcal{O}(\overline{D})).$$

Using the preceding lemma and the fact that $\mathcal{O}(\overline{D})$ is dense in $A(D)$ we obtain the formula

$$\langle \Phi_C(f)x, y\rangle = 0 \quad (f \in A(D),\, x \in M,\, y \in M^{\perp}),$$

which shows that M is indeed Φ_C-invariant. This proves part (b). In the case that M reduces C, the identity $\sigma(C) = \sigma(T) \cup \sigma(C|M^{\perp})$ holds, proving the last assertion. ∎

Suppose that $T = (T_1, \ldots, T_n) \in L(H)^n$ is a subnormal n-tuple, i.e. possesses an extension to a commuting n-tuple $N = (N_1, \ldots, N_n) \in L(K)^n$ of normal operators acting on a separable Hilbert space K containing H isometrically. Replacing K by a suitable intersection, we may assume that the only closed subspace of K which is reducing for N and contains H is K itself. In this case (in which N is said to be a *minimal normal extension* of T) a result of Putinar (see [37]) guarantees that the spectral inclusion $\sigma(N) \subset \sigma(T)$ holds. In particular, if $\sigma(T) \subset \overline{D}$, then we also have $\sigma(N) \subset \overline{D}$, and as an easy consequence of the spectral theorem for normal n-tuples, N possesses a contractive $A(D)$-functional calculus Φ_N. So part (b) of the preceding lemma allows us to conclude:

2.3.3. Corollary. *Each subnormal n-tuple $T \in L(H)^n$ satisfying $\sigma(T) \subset \overline{D}$ possesses a contractive $A(D)$-functional calculus.* ∎

The remainder of this section is devoted to the study of (weak*) continuous functional calculi over dual algebras of the form $H^\infty(\mathcal{B})$.

For any commuting n-tuple $T \in L(H)^n$ and any compact subset $F \subset \mathbb{C}^n$ satisfying $F \supset \sigma(T)$, the formula

$$\mathcal{H}_T(F) = \overline{\{f(T) : f \in \mathcal{O}(F)\}}^{w^*} \subset L(H)$$

defines a dual algebra of operators, called the dual algebra generated by T over F. Here, as always, $L(H)$ is assumed to carry its natural weak* topology arising from the trace duality. We should mention that if $M \in \mathrm{Lat}(T)$ is invariant for T and $\sigma(T|M) \subset F$, then the restriction to M induces a unital dual algebra homomorphism

$$\mathcal{H}_T(F) \to \mathcal{H}_{(T|M)}(F), \qquad S \mapsto S|M,$$

as can be easily checked using the compatibility of the holomorhic functional calculus with restrictions. Some basic facts concerning $H^\infty(\mathcal{B})$-functional calculi are collected in the following lemma:

2.3.4. Lemma. *Given an arbitrary band of measures $\mathcal{B} \subset M(\overline{D})$, let $j_{\mathcal{B}}$ denote the induced map $C(\overline{D}) \xrightarrow{j_{\mathcal{B}}} L^\infty(\mathcal{B})$ (see the remarks preceding Lemma 2.2.9) and let $T \in L(H)^n$ denote a commuting n-tuple.*

(a) *The space $j_{\mathcal{B}}(\mathcal{O}(\overline{D}))$ is a weak* dense subspace of $H^\infty(\mathcal{B})$.*

(b) *If $\Phi : H^\infty(\mathcal{B}) \to L(H)$ is a weak* continuous functional calculus for T, then $\sigma(T) \subset \overline{D}$, $\Phi \circ (j_{\mathcal{B}}|\mathcal{O}(\overline{D}))$ acts on $\mathcal{O}(\overline{D})$ like the holomorphic functional calculus, Φ is uniquely determined by its values on $j_{\mathcal{B}}(\mathcal{O}(\overline{D}))$ and $\Phi(H^\infty(\mathcal{B})) \subset \mathcal{H}_T(\overline{D})$ is weak* dense.*

(c) *In the case that the mapping Φ from part* (b) *is an isometry, it induces a dual algebra isomorphism $\Phi : H^\infty(\mathcal{B}) \to \mathcal{H}_T(\overline{D})$.*

(d) *If $\mathcal{B}$ is separable, then a linear map $\Phi : H^\infty(\mathcal{B}) \to L(H)$ is weak* continuous if and only if it is sequentially weak*-WOT continuous.*

Proof. Part (a) follows from Proposition 2.1.6 (b) and the remarks preceding Lemma 2.2.9. Let $\Phi : H^\infty(\mathcal{B}) \to L(H)$ be as in the statement of part (b). As a weak* continuous map, Φ is actually norm continuous. Applying Lemma 2.3.1 to the composition $A(D) \xrightarrow{j_{\mathcal{B}}} H^\infty(\mathcal{B}) \xrightarrow{\Phi} L(H)$ we deduce that $\sigma(T) \subset \overline{D}$ and that $\Phi \circ (j_{\mathcal{B}}|A(D))$ acts on $\mathcal{O}(\overline{D})$ like the holomorphic functional calculus. The remaining assertions of part (b) follow from (a) and the weak* continuity of Φ. Standard duality theory guarantees that (c) holds.

Using the theorem of Krein–Šmulian and the fact that in a dual space with separable predual the weak* topology is metrizable on bounded sets, one deduces that a linear map $\Phi : H^\infty(\mathcal{B}) \to L(H)$ is weak* continuous if and only if it is sequentially weak* continuous. Since the weak* topology and the weak operator topology coincide on norm-bounded subsets of $L(H)$ and since WOT-convergent sequences are norm-bounded (by the uniform boundedness principle), the weak* convergent sequences in $L(H)$ are precisely the WOT-converging ones. This observation finishes the proof. ∎

The proof of the following lemma is very similar to that of Lemma 2.3.2 and will therefore be omitted.

2.3.5. Lemma. *Let $\mathcal{B} \subset M(\overline{D})$ be a band of measures and let $C \in L(H)^n$ denote a commuting n-tuple possessing a weak* continuous $H^\infty(\mathcal{B})$-functional calculus $\Phi_C : H^\infty(\mathcal{B}) \to L(H)$.*

(a) *If $M \subset H$ is a Φ_C-invariant subspace, then the restriction $T = C|M \in L(M)^n$ also possesses a weak* continuous $H^\infty(\mathcal{B})$-functional calculus $\Phi_T : H^\infty(\mathcal{B}) \to L(H)$. In particular, $\sigma(T) \subset \overline{D}$ and Φ_T is of the form*

$$\Phi_T(f) = \Phi_C(f)|M \qquad (f \in H^\infty(\mathcal{B})).$$

(b) *Suppose that $M \in$* Lat(C) *and that $T = C|M$ satisfies $\sigma(T) \subset \overline{D}$. Then M is Φ_C-invariant and part* (a) *applies.*

(c) *If $M \subset H$ is a reducing subspace for C, then we are in the situation of part* (b). ∎

For the rest of the section we specialize to the band $\mathcal{B} = L^1(\lambda)$; in other words, we study $H^\infty(D)$-functional calculi. Of course, all the preceding results apply to weak* continuous $H^\infty(D)$-functional calculi. Norm continuity is sufficient for the following two assertions to hold:

2.3.6. Lemma. *Let $\Phi : H^\infty(D) \to L(H)$ be a continuous functional calculus for a commuting n-tuple $T \in L(H)^n$. Then Φ satisfies the spectral inclusion*

$$\sigma(\Phi(f)) \supset f(\sigma(T) \cap D) \qquad (f \in H^\infty(D)).$$

Proof. Fix $f \in H^\infty(D)$ and $\mu \in D$ with $f(\mu) \notin \sigma(\Phi(f))$. According to Proposition 2.1.6 the division problem

$$f - f(\mu) = \sum_{i=1}^{n} (z_i - \mu_i) f_i$$

is solvable with suitable functions $f_1, \ldots, f_n \in A(D)$. Applying the functional calculus Φ to the latter identity we deduce that

$$\Phi(f) - f(\mu) = \sum_{i=1}^{n} (T_i - \mu_i)\Phi(f_i).$$

Since the operator on the left-hand side is invertible, the point $\mu \in D$ cannot belong to $\sigma(T)$ which is a subset of the commutant spectrum of T. ∎

A subset $\sigma \subset \mathbb{C}^n$ is said to be *dominating in D* if

$$\|f\|_{\infty,D} = \sup_{z \in \sigma \cap D} |f(z)| \quad (f \in H^\infty(D)).$$

In combination with the spectral inclusion of the above lemma, we obtain the following useful sufficient condition for a contractive $H^\infty(D)$-functional calculus to be isometric.

2.3.7. Corollary. *Let $T \in L(H)^n$ be a commuting n-tuple such that the Taylor spectrum $\sigma(T)$ is dominating in D. Assume that T has a contractive functional calculus $\Phi : H^\infty(D) \to L(H)$. Then Φ has to be an isometry.*

Proof. Using the spectral inclusion derived in the preceding lemma we can apply the canonical spectral radius argument to obtain the estimate

$$\|\Phi(f)\| \geq \varrho(\Phi(f)) \geq \sup_{z \in \sigma(T) \cap D} |f(z)| = \|f\|_{\infty,D} \quad (f \in H^\infty(D)). ∎$$

For any $n \geq 1$, let $* : \mathbb{C}^n \to \mathbb{C}^n$, $z = (z_1, \ldots, z_n) \mapsto z^* = (\overline{z}_1, \ldots, \overline{z}_n)$, denote the complex conjugation. The identity $\sigma(T^*) = \sigma(T)^*$, valid for each commuting n-tuple $T \in L(H)^n$, motivates the following considerations: Given any subset $X \subset \mathbb{C}^n$, the complex conjugation induces a homeomorphism $* : X \to X^*$, which is even a diffeomorphism of class C^∞ if $X \subset \mathbb{C}^n$ is open. Given arbitrary subsets $X, Y \subset \mathbb{C}^n$ and a map $f : X \to Y$ we associate with f a mapping f_* defined by the formula

$$f_* : X^* \to Y^*, \quad f_*(z) = f(z^*)^* \quad (z \in X^*).$$

Note that $(f_*)_* = f$ and $(g \circ f)_* = g_* \circ f_*$ whenever $g : Y \to Z \subset \mathbb{C}^n$.

Now assume $X \subset \mathbb{C}^n$ to be an open set and let f denote a real- or complex-valued function on X of differentiability class C^1. Elementary calculations show that

$$\frac{\partial f_*}{\partial z_i} = \left(\frac{\partial f}{\partial z_i}\right)_*, \quad \frac{\partial f_*}{\partial \overline{z}_i} = \left(\frac{\partial f}{\partial \overline{z}_i}\right)_* \quad (i = 1, \ldots, n).$$

Therefore, the assignment $f \mapsto f_*$ induces multiplicative and conjugate linear bijections $C^k(X) \to C^k(X^*)$ $(k \geq 0)$ and $\mathcal{O}(X) \to \mathcal{O}(X^*)$. Note that, for every $f \in C(X)$ and $K \subset X$ compact, the identity $\|f\|_{\infty,K} = \|f_*\|_{\infty,K^*}$ holds, proving that the above bijections are in fact topological isomorphisms. Moreover, for any compact subset $K \subset X$ we have the identity $(\widehat{K}_{\mathcal{O}(X)})^* = (\widehat{K^*})_{\mathcal{O}(X^*)}$, which shows that X is holomorphically convex if and only if so is X^*.

Note that, since the $(\cdot)_*$-operation commutes with taking partial derivatives, a function $\varrho : X \to \mathbb{R}$ is a strictly plurisubharmonic C^2-function if and only if $\varrho_* : X^* \to \mathbb{R}$ is of the same type. This proves that $D \Subset X$ is a strictly pseudoconvex open set if and only if $D^* \Subset X^*$ is a set of the same kind.

All of the above remains true if $X \subset \mathbb{C}^n$ is taken to be a complex submanifold of dimension $1 \le k \le n$.

Now let $X \subset \mathbb{C}^n$ be a Stein submanifold and let $D \Subset X$ be a relatively compact strictly pseudoconvex open subset. Then $D^* \Subset X^*$ is of the same type, and obviously the assignment $f \mapsto f_*$ induces an isometric and conjugate linear algebra isomorphism $A(D) \to A(D^*)$ and a conjugate linear dual algebra isomorphism $H^\infty(D) \to H^\infty(D^*)$.

Suppose that $T \in L(H)^n$ is a commuting n-tuple possessing a continuous $A(D)$-functional calculus $\Phi : A(D) \to L(H)$. Then one easily verifies that the mapping Φ^* defined by

$$\Phi^* : A(D^*) \to L(H), \qquad f \mapsto \Phi(f_*)^*,$$

is a (and according to Lemma 2.3.1 even the unique) continuous $A(D^*)$-functional calculus of the adjoint n-tuple T^*. If Φ is even a (weak*) continuous $H^\infty(D)$-functional calculus, then so is

$$\Phi^* : H^\infty(D^*) \to L(H), \qquad f \mapsto \Phi(f_*)^*.$$

We have convinced ourselves in Lemma 2.3.4 that, in this context, weak* continuity is nothing else than sequential weak*-WOT continuity. If T possesses an $H^\infty(D)$-functional calculus Φ which satisfies the stronger requirement of being sequentially weak*-SOT continuous, then T is called an n-tuple of class $C_{0\cdot}$ (with respect to D). If T^* is of class $C_{0\cdot}$, or equivalently, if Φ^* is sequentially weak*-SOT continuous, then T is said to be of class $C_{\cdot 0}$ (with respect to D). Finally, the class C_{00} consists of those commuting n-tuples $T \in L(H)^n$ which are both of class $C_{0\cdot}$ and of class $C_{\cdot 0}$.

Using the metrizability of the strong operator topology on bounded subsets of $L(H)$ and our knowledge about the weak* topology of $H^\infty(D)$ one can prove the following characterization of the classes $C_{0\cdot}$ and $C_{\cdot 0}$.

2.3.8. Lemma. *Let T be a commuting n-tuple $T \in L(H)^n$ possessing a continuous $A(D)$-functional calculus $\Phi : A(D) \to L(H)$. Then T is of class $C_{0\cdot}$ (i.e. possesses a sequentially weak*-SOT continuous $H^\infty(D)$-calculus) if and only if $\Phi(f_k) \xrightarrow{SOT} 0$ for every Montel sequence (f_k) in $A(D)$. Analogously, T is of class $C_{\cdot 0}$ (i.e. T^* possesses a sequentially weak*-SOT continuous $H^\infty(D^*)$-calculus) if and only if, for every Montel sequence (f_k) in $A(D)$, $\Phi(f_k)^* \xrightarrow{SOT} 0$.* ∎

Some applications of these continuity concepts will be given in the following chapter.

3. Representations of $A(D)$ and $H^\infty(D)$

Let A be a unital Banach algebra over the complex field and let H be a separable complex Hilbert space. An algebra homomorphism $\Phi : A \to L(H)$ is said to be a *representation* of A if Φ is norm continuous and unital. In this chapter we study representations of the algebras $A(D)$ and $H^\infty(D)$, where $D \Subset X$ is always assumed to be a relatively compact strictly pseudoconvex open subset of a Stein submanifold $X \subset \mathbb{C}^n$.

A classical result of Sz.-Nagy and Foiaş says that a Hilbert-space contraction T which is neither of class $C_{0\cdot}$ nor of class $C_{\cdot 0}$ possesses a non-trivial hyperinvariant subspace or is a scalar multiple of the identity operator. Eschmeier proved in [15] that an analogous result holds for commuting n-tuples $T \in L(H)^n$ possessing a continuous $A(\mathcal{B}_n)$-functional calculus over the unit ball $\mathcal{B}_n \subset \mathbb{C}^n$. A generalization of Eschmeier's result in the context of smoothly bounded strictly pseudoconvex domains in $\mathbb{C}^n$ can be found in Pott [35]. Using a slightly modified approach we are able to free the original proofs from certain technical difficulties and, moreover, we succeed in formulating the result over an arbitrary strictly pseudoconvex set $D \subset X$. This will be done in the first section.

The following section is devoted to a decomposition theorem for commuting n-tuples $T \in L(H)^n$ possessing a contractive $A(D)$-functional calculus. Borrowing ideas from Mlak (see e.g. [31]) and Szafraniec [42], we use measure-theoretical decomposition theorems in order to obtain a decomposition result for operator-valued representations. Again, our results generalize the corresponding parts of [15] and [35]. We give an alternative approach, using band-theoretical methods inspired by the decomposition of a von Neumann operator into a singular and an absolutely continuous part, as carried out by Conway and Dudziak in [12].

In the last section of this chapter we introduce the standard model for commuting n-tuples possessing a holomorphic ∂D-unitary dilation. Here we closely follow the corresponding parts of Eschmeier [15], [18] and Pott [35].

3.1. Representations of $A(D)$ and invariant subspaces. A subspace $M \subset H$ is said to be *hyperinvariant* for a commuting n-tuple $T \in L(H)^n$ if $M \in \mathrm{Lat}(A)$ where $A \in (T)'$ runs through the commutant algebra $(T)' \subset L(H)$ of T.

3.1.1. THEOREM. *Let $T \in L(H)^n$ possess a continuous $A(D)$-functional calculus. If T is neither of class $C_{0\cdot}$ nor of class $C_{\cdot 0}$, then T has a non-trivial hyperinvariant subspace or the components of T are scalar multiples of the identity operator on H.* ∎

The proof of this theorem is based on an intermediate result which is of independent interest. Roughly speaking, it is an operator-theoretical reflection of the strong tightness of the algebra $A(D)$. During its proof we have to make use of a vector-valued generalized limit $L : \ell^\infty(H) \to H$ which can be constructed in the following way. We fix a continuous linear form $l : \ell^\infty \to \mathbb{C}$ of norm one satisfying $l((s_k)_k) = l((s_{k+1})_k)$ for each bounded sequence (s_k) of complex numbers and, in addition, $l((s_k)_k) = \lim_{k\to\infty} s_k$ for each convergent sequence (s_k) of complex numbers. Then, for any bounded sequence of vectors (x_k) in H, the continuous anti-linear form $H \to \mathbb{C}$, $y \mapsto l((\langle x_k, y\rangle)_k)$, can be represented by a unique vector $L((x_k)_k)$ in H by means of the identity

$$\langle L((x_k)_k), y\rangle = l((\langle x_k, y\rangle)_k) \qquad (y \in H).$$

One easily verifies that the map $L : \ell^\infty(H) \to H,\ (x_k)_k \mapsto L((x_k)_k)$, defined in this way is linear and continuous with norm one and has the following properties:

(a) $L((x_k)_k) = L((x_{k+1})_k)$ for each element $(x_k)_k \in \ell^\infty(H)$, and therefore $\|L((x_k)_k)\| \le \overline{\lim}_{k\to\infty}\|x_k\|$;

(b) $\langle L((x_k)_k), y\rangle = \lim_{k\to\infty}\langle x_k, y\rangle$ whenever $(x_k)_k \in \ell^\infty(H)$, $y \in H$ and the limit on the right-hand side exists (in particular: $L((x_k)_k) = x$ whenever $x_k \to x \in H$ weakly);

(c) $L((Ax_k)_k) = AL((x_k)_k)$ for each $(x_k)_k \in \ell^\infty(H)$ and each $A \in L(H)$.

We now make use of L to prove the following result.

3.1.2. Proposition. *Suppose that $T \in L(H)^n$ possesses a continuous $A(D)$-functional calculus $\Phi : A(D) \to L(H)$. If T is not of class $C_{\cdot 0}$, then there exists a non-zero bounded linear map $j : C(\partial D) \to H$ satisfying the intertwining relation*

$$j \circ M_{(f|\partial D)} = \Phi(f) \circ j \qquad (f \in A(D)).$$

Here $M_g : C(\partial D) \to C(\partial D)$, $h \mapsto gh$, denotes the multiplication operator associated with an element $g \in C(\partial D)$.

Proof. Since T is not of class $C_{\cdot 0}$, we are able to find a Montel sequence (f_k) in $A(D)$ and a vector $x \in H$ such that $\|f_k\|_{\infty,\overline{D}} \le 1$ $(k \ge 1)$ and the limit

$$\varepsilon = \lim_k \|\Phi(f_k)^* x\|$$

exists and is different from zero (see Lemma 2.3.8). Now we define a sequence of unit vectors (x_k) in H by the formula

$$x_k = \Phi(f_k)^* x/\|\Phi(f_k)^* x\| \qquad (k \ge 1).$$

Let $g \in C(\partial D)$. By Tietze's extension theorem and Proposition 2.2.1, there exists a Montel sequence (g_k) in $A(D)$ with $\|f_k g - g_k\|_{\infty,\partial D} \to 0$. Using the maximum principle and the properties of the generalized limit $L : \ell^\infty(H) \to H$ listed above one easily checks that the assignment

$$j : C(\partial D) \to H, \qquad g \mapsto L((\Phi(g_k)x_k)_k),$$

does not depend on the special choice of the Montel sequence (g_k), and that j is in fact a bounded linear map intertwining $M_{(f|\partial D)}$ on $C(\partial D)$ and $\Phi(f)$ on H, for any $f \in A(D)$. By construction we have

$$\langle \Phi(f_k)x_k, x\rangle = \langle \Phi(f_k)^* x/\|\Phi(f_k)^* x\|, \Phi(f_k)^* x\rangle = \|\Phi(f_k)^* x\| \to \varepsilon \qquad \text{as } k \to \infty.$$

Consequently,

$$\langle j(1), x\rangle = \langle L((\Phi(f_k)x_k)_k), x\rangle = \lim_{k\to\infty}\langle \Phi(f_k)x_k, x\rangle = \varepsilon > 0,$$

proving that $j \ne 0$, as desired. ∎

A bounded linear operator $j : C(\partial D) \to H$ satisfying the intertwining relation stated in the preceding proposition is said to be a $C(\partial D)$-*intertwiner* for the n-tuple T. Towards a further study of the properties of such $C(\partial D)$-intertwiners j we introduce the ideal

$$I(j) = \bigvee (I : I \subset C(\partial D) \text{ closed ideal, } I \subset \ker j),$$

which is by definition the largest ideal in $C(\partial D)$ which is contained in the kernel of j. Moreover, we define

$$s(j) = \bigcap_{f \in I(j)} Z_f \subset \partial D \quad (Z_f = \{z \in \partial D : f(z) = 0\})$$

to be the common zero set of all functions in $I(j)$. Exactly as in the case of the unit ball (see Eschmeier [15, Lemma 2.2]) one proves the following lemma.

3.1.3. Lemma. *Let $j : C(\partial D) \to H$ be a $C(\partial D)$-intertwiner for a commuting n-tuple $T \in L(H)^n$ possessing a continuous $A(D)$-functional calculus.*

(a) *For each $g \in C(\partial D)$, the composition $k = j \circ M_g$ is a $C(\partial D)$-intertwiner for T.*
(b) *Suppose that $\lambda \in s(j)$ and that $f \in C(\partial D)$ satisfies $f(\lambda) \neq 0$. Then $j \circ M_f \neq 0$.*
(c) *If $s(j) = \{\lambda\}$ for a single point λ in $\mathbb{C}^n$, then $\operatorname{ran} j \subset \bigcap_{i=1}^n \ker(\lambda_i - T_i)$ is a one-dimensional invariant subspace of T; in particular, the hyperinvariant subspace $\bigcap_{i=1}^n \ker(\lambda_i - T_i)$ of T is different from zero.* ∎

Proof of Theorem 3.1.1. By assumption, T is neither of class $C_{0\cdot}$ nor of class $C_{\cdot 0}$. An application of Proposition 3.1.2 yields a non-zero $C(\partial D)$-intertwiner $j : C(\partial D) \to H$ for T and a non-zero $C(\partial D^*)$-intertwiner $k : C(\partial D^*) \to H$ for T^*. Note that $s(j) \neq \emptyset \neq s(k)$ since j and k are different from the zero map. Moreover, by the last assertion of the preceding lemma we may assume that $s(j)$ contains more than one element. This allows us to choose elements $\lambda_2 \in s(k) \subset \partial D^*$ and $\lambda_1 \in s(j) \subset \partial D$ such that $\lambda_2^* \neq \lambda_1$. An application of Tietze's extension theorem to suitable disjoint closed neighborhoods of λ_1 and λ_2^* yields functions $\varphi_1 \in C(\partial D)$ and $\varphi_2 \in C(\partial D^*)$ having the property that $\varphi_1(\lambda_1) = 1 = \varphi_2(\lambda_2)$ and that $\omega_1 = \operatorname{supp}(\varphi_1) \subset \partial D$ and $\omega_2 = \operatorname{supp}(\varphi_2) \subset \partial D^*$ satisfy $\omega_1 \cap \omega_2^* = \emptyset$.

Regarding (via trivial extension) $\varphi_1 C(\omega_1)$ as a subspace of $C(\partial D)$ and $\varphi_2 C(\omega_2)$ as a subspace of $C(\partial D^*)$, the definitions

$$J : C(\omega_1) \to H, \quad f \mapsto j(\varphi_1 f), \qquad K : C(\omega_2) \to H, \quad g \mapsto k(\varphi_2 g),$$

make perfect sense. Using parts (a) and (b) of the preceding lemma we deduce that both J and K are non-zero bounded linear operators satisfying the intertwining relations

$$J(z_i f) = T_i J(f) \quad (f \in C(\omega_1)), \qquad K(z_i f) = T_i^* K(f) \quad (f \in C(\omega_2))$$

for $1 \leq i \leq n$. The Riesz representation theorem allows us to define a continuous linear mapping $K^0 : H \to M(\omega_2^*)$ by the formula

$$\int_{\omega_2^*} f \, d(K^0 h) = \langle h, K(f_*) \rangle \quad (f \in C(\omega_2^*)).$$

The relation $\langle h, K((z_i f)_*) \rangle = \langle T_i h, K f_* \rangle$, valid for all $h \in H$, $f \in C(\omega_2^*)$ and $i = 1, \ldots, n$, proves that

$$M'_{z_i} \circ K^0 = K^0 \circ T_i \quad (i = 1, \ldots, n),$$

where $M'_z \in L(M(\omega_2^*))^n$ denotes the adjoint of the multiplication tuple corresponding to the coordinate functions $M_z = (M_{z_1}, \ldots, M_{z_n}) \in L(C(\omega_2^*))^n$.

Given an arbitrary operator A in the commutant algebra $(T)'$ of T, the composition

$$X_A = K^0 AJ : C(\omega_1) \xrightarrow{J} H \xrightarrow{A} H \xrightarrow{K^0} M(\omega_2^*)$$

intertwines $M_z \in L(C(\omega_1))^n$ and $M'_z \in L(M(\omega_2^*))^n$ componentwise. Because of the identities $\sigma(M_z) = \omega_1$ and $\sigma(M'_z) = \omega_2^*$ this yields the intertwining relation

$$X_A f(M_z) = f(M'_z) X_A \quad (f \in \mathcal{O}(\omega_1 \cup \omega_2^*)).$$

Using the fact that ω_1 and ω_2^* are compact and disjoint, we can take f to be a function which is identically one on an open neighborhood of ω_1 and identically zero on an open neighborhood of ω_2^* to deduce that $X_A = 0$. Hence $\operatorname{ran} AJ \subset \ker K^0$ for each $A \in (T)'$. Since both K^0 and J are different from zero, we can complete the proof by the observation that

$$\bigvee(\operatorname{ran} AJ : A \in (T)') \subset \ker K^0 \neq H$$

is a non-trivial subspace of H which is hyperinvariant for T. ∎

By a *∂D-unitary n-tuple* $U \in L(H)^n$ we mean a commuting n-tuple of normal operators on H satisfying $\sigma(U) \subset \partial D$. A commuting n-tuple $T \in L(H)^n$ is said to be *completely non-∂D-unitary* if there is no reducing subspace $\{0\} \neq M \subset H$ for T such that $T|M$ is ∂D-unitary.

3.1.4. Proposition. *Let $T \in L(H)^n$ be a subnormal n-tuple satisfying $\sigma(T) \subset \overline{D}$. Suppose that T is completely non-∂D-unitary. Then T is of class $C_{\cdot 0}$.*

Proof. By Corollary 2.3.3, T possesses a contractive $A(D)$-functional calculus. Assuming that T is not of class $C_{\cdot 0}$ we infer from Proposition 3.1.2 the existence of a non-zero $C(\partial D)$-intertwiner $j : C(\partial D) \to H$ for T. Let $N \in L(K)^n$ be the minimal normal extension of T. Then the mapping j obviously induces a $C(\partial D)$-intertwiner $j : C(\partial D) \to K$ for N with $\operatorname{ran} j \subset H$. A Fuglede-type argument proves the relation $j \circ M_{\overline{z}_i} = N_i^* \circ j$ $(i = 1, \ldots, n)$ implying that

$$\operatorname{ran} j \subset \{x \in H : (N^*)^\alpha x \in H \text{ for all } \alpha \in \mathbb{N}_0^n\} = H_0.$$

The space $H_0 \subset H$ reduces N and hence T. And since $j : C(\partial D) \to H_0$, which is a $C(\partial D)$-intertwiner for $N_0 = N|H_0$, maps the spectral subspaces of M_z into the corresponding spectral subspaces of N_0, we deduce that

$$\{0\} \neq j(C(\partial D)) \subset (H_0)_{N_0}(\partial D) = H_1.$$

The space on the right-hand side reduces the normal n-tuple N_0 and the restriction $T|H_1 = N_0|H_1$ is a non-zero ∂D-unitary part of T. ∎

3.2. A decomposition of contractive $A(D)$-calculi. In this section we specialize to contractive representations of $A(D)$. A commuting n-tuple $T \in L(H)^n$ is said to be a *von Neumann n-tuple over D* if it possesses a contractive $A(D)$-functional calculus. Since $\mathcal{O}(\overline{D})$ is dense in $A(D)$, this is equivalent to demanding that $\sigma(T) \subset \overline{D}$ and, in addition, the von Neumann-type inequality $\|f(T)\| \leq \|f\|_{\infty,\overline{D}}$ holds for every $f \in \mathcal{O}(\overline{D})$. The unique contractive $A(D)$-functional calculus of a von Neumann n-tuple $T \in L(H)^n$ over D will be denoted by $\Phi_T : A(D) \to L(H)$. We say that T is *absolutely continuous* provided that Φ_T allows an extension to a weak* continuous $H^\infty(D)$-functional calculus. Such an extension is unique if it exists (see Lemma 2.3.4), and will simply again be denoted by $\Phi_T : H^\infty(D) \to L(H)$.

Given a commuting n-tuple of normal operators $N \in L(H)^n$, the von Neumann algebra generated by the identity and the components of N will be denoted by $W^*(N)$. A *scalar-valued spectral measure* for N is by definition a positive measure ν on $\sigma(N)$ having the property that ν and the projection-valued spectral measure E of N are mutually absolutely continuous (i.e. for $\omega \in \mathcal{B}(\sigma(N))$ we have $\nu(\omega) = 0 \Leftrightarrow E(\omega) = 0$). Given such a measure ν, there exists an isomorphism of von Neumann algebras $\Psi_N : L^\infty(\nu) \to W^*(N)$ mapping z_i to N_i $(i = 1, \ldots, n)$. If we have $\sigma(N) \subset \overline{D}$ (as always in what follows), then we often regard the measure $\nu \in M(\sigma(N))$ as a measure on $\overline{D}$ via trivial extension. In particular we write $\nu \in S(\overline{D})$ when we really mean that the trivial extension $\nu^{\overline{D}}(\omega) = \nu(\omega \cap \sigma(N))$ $(\omega \in \mathcal{B}(\overline{D}))$ of ν to a measure $\nu^{\overline{D}} \in M(\overline{D})$ satisfies $\nu^{\overline{D}} \in S(\overline{D})$.

3.2.1. Theorem. *Let $T \in L(H)^n$ be a von Neumann n-tuple over D. Then there exists a unique orthogonal direct sum decomposition $H = H_{\mathrm{s}} \oplus H_{\mathrm{a}}$ which reduces the operator algebra $\mathcal{H}_T(\overline{D})$ and has the following properties*:

- (a) *The restriction $T_{\mathrm{s}} = T|H_{\mathrm{s}}$ is ∂D-unitary and possesses a scalar-valued spectral measure $\nu \in S(\overline{D})$.*
- (b) *The restriction $T_{\mathrm{a}} = T|H_{\mathrm{a}}$ is an absolutely continuous von Neumann n-tuple over D.*

Moreover, *the above decomposition of H induces a decomposition of $\mathcal{H}_T(\overline{D})$ into the sum of two dual operator algebras*

$$\mathcal{H}_T(\overline{D}) = W^*(T_{\mathrm{s}}) \oplus \mathcal{H}_{T_{\mathrm{a}}}(\overline{D}).$$

The proof is divided into several steps. We first consider the uniqueness problem.

3.2.2. Lemma. *Let $T \in L(H)^n$ be a von Neumann n-tuple over D. Then there is at most one orthogonal decomposition $H = H_{\mathrm{s}} \oplus H_{\mathrm{a}}$ reducing T in such a way that the conditions* (a) *and* (b) *of the preceding theorem are satisfied.*

Proof. Suppose that there are two decompositions $H = H_{\mathrm{s}}^1 \oplus H_{\mathrm{a}}^1 = H_{\mathrm{s}}^2 \oplus H_{\mathrm{a}}^2$ with the desired properties. By hypothesis the corresponding orthogonal projections, denoted by $P_{\mathrm{s}}^1, P_{\mathrm{a}}^1, P_{\mathrm{s}}^2, P_{\mathrm{a}}^2 \in L(H)$, commute with the components of T. Therefore the closed subspace

$$Z = \overline{P_{\mathrm{s}}^1 P_{\mathrm{a}}^2 H} = \overline{P_{\mathrm{s}}^1 H_{\mathrm{a}}^2} \subset H_{\mathrm{s}}^1$$

reduces the n-tuple $T|H_{\mathrm{s}}^1$, proving that $T|Z$ is normal and possesses a scalar-valued spectral measure $\nu_Z \in S(\overline{D})$. Let $h \in Z$ denote a separating vector for $T|Z$. The scalar-valued spectral measure ν_h of $T|Z$ induced by h is mutually absolutely continuous with respect to ν_Z and thus $\nu_h \in S(\overline{D})$.

Our aim is to show that ν_h also belongs to $HM(\overline{D})$. Using the basic properties of $A(D)$-functional calculi (see Lemmas 2.3.1 and 2.3.2) we derive the formulas

$$\int_{\overline{D}} f \, d\nu_h = \langle \Phi_{(T|H_{\mathrm{s}}^1)}(f) h, h \rangle$$

and

$$\langle \Phi_{(T|H_{\mathrm{a}}^2)}(f)x,x\rangle = \langle \Phi_T(f)x,x\rangle = \langle \Phi_{(T|H_{\mathrm{s}}^1)}(f)P_{\mathrm{s}}^1x,P_{\mathrm{s}}^1x\rangle + \langle \Phi_{(T|H_{\mathrm{a}}^1)}(f)P_{\mathrm{a}}^1x,P_{\mathrm{a}}^1x\rangle,$$

valid for each $f \in A(D)$ and each $x \in H_{\mathrm{a}}^2$. Given $x \in H_{\mathrm{a}}^2$ we therefore have the estimate

$$\begin{aligned}\Big|\int_{\overline{D}} f\,d\nu_h\Big| &\le |\langle \Phi_{(T|H_{\mathrm{s}}^1)}(f)h,h\rangle - \langle \Phi_{(T|H_{\mathrm{s}}^1)}(f)P_{\mathrm{s}}^1x,P_{\mathrm{s}}^1x\rangle| + |\langle \Phi_{(T|H_{\mathrm{s}}^1)}(f)P_{\mathrm{s}}^1x,P_{\mathrm{s}}^1x\rangle|\\ &\le \|f\|_{\infty,\overline{D}}(\|h\|+\|P_{\mathrm{s}}^1x\|)\|h-P_{\mathrm{s}}^1x\|\\ &\quad + |\langle \Phi_{(T|H_{\mathrm{a}}^2)}(f)x,x\rangle - \langle \Phi_{(T|H_{\mathrm{a}}^1)}(f)P_{\mathrm{a}}^1x,P_{\mathrm{a}}^1x\rangle| \quad (f \in A(D)).\end{aligned}$$

Now let (f_k) be a Montel sequence in $A(D)$ and let $\varepsilon > 0$. Since (f_k) is norm-bounded in $A(D)$ and since $P_{\mathrm{s}}^1H_{\mathrm{a}}^2$ is dense in Z we can fix a vector $x \in H_{\mathrm{a}}^2$ in such a way that $\|h - P_{\mathrm{s}}^1x\|$ is small enough to satisfy the estimate

$$\|f_k\|_{\infty,\overline{D}}(\|h\|+\|P_{\mathrm{s}}^1x\|)\|h-P_{\mathrm{s}}^1x\| < \varepsilon/2 \quad (k \ge 1).$$

Given this $x \in H_{\mathrm{a}}^2$, the absolute continuity of $\Phi_{(T|H_{\mathrm{a}}^2)}$ and $\Phi_{(T|H_{\mathrm{a}}^1)}$ allows us to choose an index $k_0 \in \mathbb{N}$ such that

$$|\langle \Phi_{(T|H_{\mathrm{a}}^2)}(f_k)x,x\rangle - \langle \Phi_{(T|H_{\mathrm{a}}^1)}(f_k)P_{\mathrm{a}}^1x,P_{\mathrm{a}}^1x\rangle| < \varepsilon/2 \quad (k \ge k_0).$$

Therefore we finally obtain $|\int_{\overline{D}} f_k\,d\nu_h| < \varepsilon$ $(k \ge k_0)$.

Consequently, $\nu_h \in S(\overline{D}) \cap HM(\overline{D})$ is a scalar-valued spectral measure for $T|Z$ which is identically zero. This can only happen if the underlying space $\overline{P_{\mathrm{s}}^1H_{\mathrm{a}}^2} = Z$ is the zero space. Hence $H_{\mathrm{a}}^2 \subset H_{\mathrm{a}}^1$. Changing the roles of H_{a}^1 and H_{a}^2 and repeating the proof yields the reverse inclusion. ■

Let $T \in L(H)^n$ be a von Neumann n-tuple over D. To prove the existence of a decomposition $H = H_{\mathrm{s}} \oplus H_{\mathrm{a}}$ associated with T in the sense of Theorem 3.2.1 we modify the corresponding ideas of Conway and Dudziak [12]. Adjoining twice, the canonical contractive $A(D)$-functional calculus $\Phi_T : A(D) \to L(H)$ of a von Neumann n-tuple $T \in L(H)^n$ induces a weak* continuous contractive linear map $\Phi_T'' : A(D)'' \to L(H)''$. Denoting the trace class by $C^1(H)$, the adjoint of its canonical embedding $i : C^1(H) \to L(H)'$ into the bidual is a weak* continuous contraction $i' : L(H)'' \to L(H)$ satisfying $i'(A) = A$ $(A \in L(H))$. The composition

$$i' \circ \Phi_T'' : A(D)'' \to L(H)$$

therefore acts as a weak* continuous contractive linear extension of Φ_T. In view of the dualities $\langle C(\overline{D}), M(\overline{D})\rangle$ and $\langle M(\overline{D}), L^\infty(M(\overline{D}))\rangle$ general duality theory for Banach spaces yields the identification

$$A(D)'' = \overline{A(D)}^{(w^*,L^\infty(M(\overline{D})))} = H^\infty(M(\overline{D})).$$

Via this identification, $i' \circ \Phi_T''$ induces a contractive and weak* continuous functional calculus

$$\widehat{\Phi}_T : H^\infty(M(\overline{D})) \to L(H)$$

for T extending Φ_T. (The multiplicativity follows from a density argument.) The decomposition

$$H^\infty(M(\overline{D})) = L^\infty(S(\overline{D})) \oplus_\infty H^\infty(HM(\overline{D})),$$

established in Lemma 2.2.9, leads in a natural way to a decomposition of the functional calculus $\widehat{\Phi}_T$. The orthogonal projections

$$P_{\mathrm{s}} = \widehat{\Phi}_T(1 \oplus 0), \quad P_{\mathrm{a}} = \widehat{\Phi}_T(0 \oplus 1) \in L(H)$$

add up to the identity of H and possess orthogonal ranges $H_{\mathrm{s}} = P_{\mathrm{s}}H$, $H_{\mathrm{a}} = P_{\mathrm{a}}H$. Obviously, the direct sum decomposition $H = H_{\mathrm{s}} \oplus H_{\mathrm{a}}$ reduces the range of $\widehat{\Phi}_T$, and hence, for density reasons, also $\mathcal{H}_T(\overline{D})$. Moreover, $\widehat{\Phi}_T$ splits in the following sense:

$$\widehat{\Phi}_T(f \oplus g) = \widehat{\Phi}^{\mathrm{s}}_T(f) \oplus \widehat{\Phi}^{\mathrm{a}}_T(g) \quad (f \oplus g \in H^\infty(M(\overline{D}))),$$

where

$$\widehat{\Phi}^{\mathrm{s}}_T : L^\infty(S(\overline{D})) \to L(H_{\mathrm{s}}), \qquad f \mapsto \widehat{\Phi}_T(f \oplus 0)|H_{\mathrm{s}},$$
$$\widehat{\Phi}^{\mathrm{a}}_T : H^\infty(HM(\overline{D})) \to L(H_{\mathrm{a}}), \quad g \mapsto \widehat{\Phi}_T(0 \oplus g)|H_{\mathrm{a}},$$

are contractive and weak* continuous functional calculi for $T_{\mathrm{s}} = T|H_{\mathrm{s}}$ and $T_{\mathrm{a}} = T|H_{\mathrm{a}}$.

3.2.3. Lemma. *The map $\widehat{\Phi}^{\mathrm{s}}_T : L^\infty(S(\overline{D})) \to L(H_{\mathrm{s}})$ is a dual algebra homomorphism and a $*$-homomorphism.*

Proof. Only the compatibility with involutions requires an argument. We fix an arbitrary vector $x \in H_{\mathrm{s}}$. The weak* continuity of the functional $L^\infty(S(\overline{D})) \to \mathbb{C}$, $f \mapsto \langle \widehat{\Phi}^{\mathrm{s}}_T(f)x, x\rangle$, guarantees the existence of a measure $\nu_x \in S(\overline{D})$ satisfying

$$\langle \widehat{\Phi}^{\mathrm{s}}_T(f)x, x\rangle = \int_{\overline{D}} f_{\nu_x}\, d\nu_x \quad (f \in L^\infty(S(\overline{D}))).$$

From the fact that $\widehat{\Phi}^{\mathrm{s}}_T$ is contractive we deduce the formula

$$|\nu_x(\overline{D})| \le \|\nu_x\| \le \|x\|^2 = \int_{\overline{D}} d\nu_x = \nu_x(\overline{D}),$$

which immediately implies that ν_x is a positive measure. But then, for each $f \in L^\infty(S(\overline{D}))$,

$$\langle \widehat{\Phi}^{\mathrm{s}}_T(f^*)x, x\rangle = \int_{\overline{D}} f^*_{\nu_x}\, d\nu_x = \Big(\int_{\overline{D}} f_{\nu_x}\, d\nu_x\Big)^* = \langle \widehat{\Phi}^{\mathrm{s}}_T(f)^*x, x\rangle$$

and the proof is complete. ■

Let $\nu \in S(\overline{D})$ be an arbitrary element. The mapping defined by the formula

$$j_\nu : L^\infty(\nu) \to L^\infty(S(\overline{D})), \quad f \mapsto (f_\mu)_{\mu \in S(\overline{D})},$$

where $f_\mu = r^\nu_{\mu_{\mathrm{a}}}(f)$ and μ_{a} denotes the absolutely continuous part of μ appearing in the Lebesgue decomposition of μ with respect to ν, is easily seen to be a weak* continuous and isometric $*$-homomorphism.

3.2.4. Lemma. *Suppose that $H_{\mathrm{s}} \neq \{0\}$. Then there exists a positive measure $\nu \in S(\overline{D})$ having the following properties*:

(a) *The n-tuple T_{s} is ∂D-unitary with scalar-valued spectral measure ν.*
(b) *The composition $\widehat{\Phi}^{\mathrm{s}}_T \circ j_\nu : L^\infty(\nu) \to L^\infty(S(\overline{D})) \to L(H_{\mathrm{s}})$ coincides with the isomorphism of von Neumann algebras $L^\infty(\nu) \to W^*(T_{\mathrm{s}})$ associated with T_{s}.*

Proof. Choose a dense sequence of vectors (x_k) in the unit sphere of H_{s}. As carried out in detail in the previous proof, for each x_k $(k \ge 1)$, we can fix a positive measure $\nu_k \in S(\overline{D})$

satisfying

$$\langle \widehat{\Phi}^{\mathrm{s}}_T(f)x_k, x_k\rangle = \int\limits_{\overline{D}} f_{\nu_k}\, d\nu_k \qquad (f \in L^\infty(S(\overline{D}))).$$

Since $\|\nu_k\| = \|x_k\|^2 = 1$ for $k \geq 1$, we can define a non-zero measure $\nu \in S(\overline{D})$ by setting $\nu = \sum_{k=1}^\infty 2^{-k}\nu_k$.

We first want to point out that, with this choice for ν, the (obviously contractive) composition $\widehat{\Phi}^{\mathrm{s}}_T \circ j_\nu : L^\infty(\nu) \to L(H_{\mathrm{s}})$ becomes an isometry. Given $f \in L^\infty(\nu)$ and $\varepsilon > 0$, we fix a representative $f_0 : \overline{D} \to \mathbb{C}$ of f and define

$$\Delta = \{z \in \overline{D} : |f_0(z)| > \|f\|_{\infty,\nu} - \varepsilon\}.$$

One checks that $\nu(\Delta) > 0$, and hence $\nu_k(\Delta) > 0$ for some $k \geq 1$. The vector $x = \widehat{\Phi}^{\mathrm{s}}_T(j_\nu(\chi_\Delta))x_k \in H_{\mathrm{s}}$ then satisfies (note that $\nu_k \ll \nu$)

$$\|x\|^2 = \langle \widehat{\Phi}^{\mathrm{s}}_T(j_\nu(\chi_\Delta))x_k, x_k\rangle = \int\limits_{\overline{D}} (j_\nu(\chi_\Delta))_{\nu_k}\, d\nu_k = \int\limits_{\overline{D}} \chi_\Delta\, d\nu_k = \nu_k(\Delta) > 0.$$

A similar calculation shows that

$$\|\widehat{\Phi}^{\mathrm{s}}_T \circ j_\nu(f)x\|^2 = \langle \widehat{\Phi}^{\mathrm{s}}_T(j_\nu(|f|^2\chi_\Delta))x_k, x_k\rangle = \int\limits_{\Delta} |f_0|^2\, d\nu_k \geq (\|f\|_{\infty,\nu} - \varepsilon)^2\|x\|^2.$$

Note that $x \neq 0$, thus in the limit $\varepsilon \to 0$ we arrive at

$$\|f\|_{\infty,\nu} \leq \|\widehat{\Phi}^{\mathrm{s}}_T \circ j_\nu(f)\| \qquad (f \in L^\infty(\nu)),$$

proving the assertion that $\widehat{\Phi}^{\mathrm{s}}_T \circ j_\nu$ is isometric.

So far we know that the composition $\widehat{\Phi}^{\mathrm{s}}_T \circ j_\nu : L^\infty(\nu) \to L(H_{\mathrm{s}})$ induces an isomorphism of von Neumann algebras between $L^\infty(\nu)$ and the von Neumann algebra generated by 1 and $\widehat{\Phi}^{\mathrm{s}}_T \circ j_\nu(z_i)$ $(i = 1,\ldots,n)$. To finish the proof it suffices to check the identity

$$\widehat{\Phi}^{\mathrm{s}}_T \circ j_\nu(z_i) = (T_{\mathrm{s}})_i \qquad (i = 1,\ldots,n).$$

Assume, for a moment, that the latter condition is satisfied. Observe that, in this case, T_{s} possesses an involutive $L^\infty(\nu)$-functional calculus and hence is normal. Moreover, $\sigma(T_{\mathrm{s}})$ is contained in the support of ν, which is a subset of ∂D. (Note that $\nu|D$ belongs to $S(\overline{D}) \cap HM(\overline{D}) = \{0\}$.) Finally, let $\mu \in M(\overline{D})$ denote an arbitrary scalar-valued spectral measure of T_{s}. Then there is a $*$-isomorphism

$$L^\infty(\nu) \xrightarrow{\widehat{\Phi}^{\mathrm{s}}_T \circ j_\nu} W^*(T_{\mathrm{s}}) \to L^\infty(\mu)$$

acting like the identity on the polynomials $\mathbb{C}[z,\overline{z}]$. This implies that μ and ν are mutually absolutely continuous (see Kehe Zhu [44, Theorem 21.4]), and the lemma is proved modulo the above identity.

Note that, since $\nu_k \ll \nu$ $(k \geq 1)$, we have $\int_{\overline{D}} z_i\, d\nu_k = \int_{\overline{D}} (j_\nu(z_i))_{\nu_k} d\nu_k$, for $k \geq 1$ and $1 \leq i \leq n$. From this we infer that

$$\langle (T_{\mathrm{s}})_i x, x\rangle = \langle \widehat{\Phi}^{\mathrm{s}}_T(z_i)x, x\rangle = \langle \widehat{\Phi}^{\mathrm{s}}_T(j_\nu(z_i))x, x\rangle \qquad (i = 1,\ldots,n)$$

for $x = x_k$ $(k \geq 1)$, which clearly suffices to finish the proof. ∎

To handle the absolutely continuous part T_{a}, recall from Proposition 2.2.10 that there exists a dual algebra isomorphism $r : H^\infty(D) \to H^\infty(HM(\overline{D}))$ extending the canonical embedding $A(D) \hookrightarrow H^\infty(HM(\overline{D})))$. As an application we immediately obtain the following lemma.

3.2.5. LEMMA. *The composition $\widehat{\Phi}^{\mathrm{a}}_T \circ r : H^\infty(D) \to L(H_{\mathrm{a}})$ is a contractive and weak* continuous $H^\infty(D)$-functional calculus for T_{a}.* ∎

Proof of Theorem 3.2.1. It remains to settle the decomposition $\mathcal{H}_T(\overline{D}) = W^*(T_{\mathrm{s}}) \oplus \mathcal{H}_{T_{\mathrm{a}}}(\overline{D})$. Since the holomorphic functional calculus is compatible with restrictions to reducing subspaces, we immediately obtain the inclusion

$$\mathcal{H}_T(\overline{D}) \subset \mathcal{H}_{(T|H_{\mathrm{s}})}(\overline{D}) \oplus \mathcal{H}_{(T|H_{\mathrm{a}})}(\overline{D}) \subset W^*(T_{\mathrm{s}}) \oplus \mathcal{H}_{T_{\mathrm{a}}}(\overline{D}).$$

For the reverse inclusion, recall the splitting $\widehat{\Phi}_T(f \oplus g) = \widehat{\Phi}^{\mathrm{s}}_T(f) \oplus \widehat{\Phi}^{\mathrm{a}}_T(g)$ established above for $f \in L^\infty(S(\overline{D}))$ and $g \in H^\infty(HM(\overline{D}))$, and note that the left-hand side of

$$\operatorname{ran}(\widehat{\Phi}^{\mathrm{s}}_T \circ j_\nu) \oplus \operatorname{ran}(\widehat{\Phi}^{\mathrm{a}}_T \circ r) \subset \operatorname{ran}\widehat{\Phi}_T \subset \mathcal{H}_T(\overline{D})$$

is a weak* dense subset of $W^*(T_{\mathrm{s}}) \oplus \mathcal{H}_{T_{\mathrm{a}}}(\overline{D})$. ∎

3.3. Normal boundary dilations and the standard model. Let $T \in L(H)^n$ be a von Neumann n-tuple over D and let $K \supset H$ be a Hilbert space containing H as a closed subspace. A von Neumann n-tuple $N \in L(K)^n$ over D is said to be a *dilation* of T if the $A(D)$-functional calculi of T and K are joined via the relation

$$\Phi_T(f) = P_H \Phi_N(f)|H \quad (f \in A(D)),$$

where $P_H \in L(K)$ denotes the orthogonal projection from K to H. Note that the dilation requirement is equivalent to the identity $\langle \Phi_T(f)x, y\rangle = \langle \Phi_N(f)x, y\rangle$ for every $f \in A(D)$ and every choice of vectors $x, y \in H$. Observe that the adjoints $T^* \in L(H)^n$ and $N^* \in L(K)^n$ are von Neumann n-tuples over D^* via the dual representations $\Phi_{T^*} = \Phi_T^*$ and $\Phi_{N^*} = \Phi_N^*$, and that N is a dilation of T if and only if N^* is a dilation of T^*.

If the dilation N of T has the additional property of being ∂D-unitary, then N is said to be a *∂D-unitary dilation* of T. In this case, N is called a *minimal ∂D-unitary dilation* if the only reducing subspace for N containing H is K itself. Given an arbitrary ∂D-unitary dilation N of T, we can always construct a minimal ∂D-unitary dilation by restricting N to the smallest reducing subspace for N containing H.

Concerning the decomposition theorem for von Neumann n-tuples (see Theorem 3.2.1) the following observation will prove to be useful.

3.3.1. LEMMA. *Let $N \in L(K)^n$ be a von Neumann n-tuple over D which is a dilation of another von Neumann n-tuple $T \in L(H)^n$ over D. Then N_{s} is a dilation of T_{s} and N_{a} is a dilation of T_{a}. In particular, if N is a minimal ∂D-unitary dilation of T, and if T is absolutely continuous, then so is N.*

Proof. With the notations of the cited decomposition theorem we have to show that $H_{\mathrm{s}} \subset K_{\mathrm{s}}$ and $H_{\mathrm{a}} \subset K_{\mathrm{a}}$. Since $A(D)$ is weak* dense in $H^\infty(M(\overline{D}))$ (see Lemma 2.3.4) we have

$$\langle \widehat{\Phi}_T(f)x, y\rangle = \langle \widehat{\Phi}_N(f)x, y\rangle \quad (x, y \in H)$$

for any $f \in H^\infty(M(\overline{D})) = L^\infty(S(\overline{D})) \oplus H^\infty(HM(\overline{D}))$. Starting with $x \in H_{\mathrm{s}}$ we deduce that $\|x\|^2 = \langle \widehat{\Phi}_T(1 \oplus 0)x, x\rangle = \langle \widehat{\Phi}_N(1 \oplus 0)x, x\rangle = \|P_{\mathrm{s}}x\|^2$, where $P_{\mathrm{s}} \in L(K)$ denotes the orthogonal projection from K onto K_s. This proves $H_s \subset K_s$. The same arguments show that $H_a \subset K_a$. If T is absolutely continuous, then we have $H = H_a \subset K_a$ and the latter space is reducing for N. ∎

Assume that $T \in L(H)^n$ is an absolutely continuous von Neumann n-tuple over D possessing a minimal dilation to a von Neumann n-tuple $N \in L(K)^n$ over D. According to the previous lemma, N is also absolutely continuous. The fact that N is a dilation of T is expressed by the relation

$$\langle \Phi_T(f)x, y\rangle = \langle \Phi_N(f)x, y\rangle,$$

valid for any $x, y \in H$ and $f \in A(D)$. In view of the absolute continuity of T and N, the above identity even holds for arbitrary $f \in H^\infty(D)$ (see Lemma 2.3.4). Note for later use that this is equivalent to saying that

$$\Phi_T(f) = P_H\Phi_N(f)|H \qquad (f \in H^\infty(D)).$$

The existence of a ∂D-unitary dilation has remarkable consequences for the structure of a von Neumann n-tuple $T \in L(H)^n$, as the following model theorem points out.

3.3.2. Theorem. *Let $T \in L(H)^n$ be a von Neumann n-tuple over D possessing a ∂D-unitary dilation. Then T is the restriction of a von Neumann n-tuple C over D, where $C = S^* \oplus R \in L(\mathcal{S} \oplus \mathcal{R})^n$ is of the following form:*

(a) *$S^* \in L(\mathcal{S})^n$ is a von Neumann n-tuple of class $C_{0\cdot}$ over D and $R \in L(\mathcal{R})^n$ is ∂D-unitary.*
(b) *C is minimal in the sense that the only reducing subspace for C containing H is $\mathcal{S} \oplus \mathcal{R}$.*
(c) *C can be chosen to be absolutely continuous provided that T is absolutely continuous.* ∎

The proof is based on the following observation.

3.3.3. Lemma. *Let $T \in L(H)^n$ be a von Neumann n-tuple over D possessing a ∂D-unitary dilation $N \in L(K)^n$. Then*

$$K^+ = \bigvee(\Phi_N(f)H : f \in A(D))$$

is the smallest Φ_N-invariant subspace of K containing H. Moreover, H is $\Phi_{(N^+)^*}$-invariant, where $N^+ = N|K^+ \in L(K^+)^n$, and

$$\Phi_{T^*}(f) = \Phi_{(N^+)^*}(f)|H \qquad (f \in A(D^*)).$$

Proof. The minimality assertion for K^+ is obvious. Consider the closed subspace $K_0 = \bigvee\{\Phi_N(f)h - \Phi_T(f)h : f \in A(D),\ h \in H\} \subset K^+$. Whenever $f, g \in A(D)$ and $h \in H$, we have the identity

$$\Phi_N(g)(\Phi_N(f)h - \Phi_T(f)h) = \Phi_N(gf)h - \Phi_T(gf)h - (\Phi_N(g)\Phi_T(f)h - \Phi_T(g)\Phi_T(f)h),$$

proving that K_0 is Φ_N-invariant. The same is true for $H \vee K_0 \subset K^+$, thus $K^+ = H \vee K_0$ by the minimality of K^+. From the fact that N is a dilation of T one deduces that $H \perp K_0$, establishing the fact that $K^+ = H \oplus K_0$ is an orthogonal direct sum decomposition. Since both K_0 and K^+ are Φ_N-invariant, we know that K_0 is Φ_{N^+}-invariant for $N^+ = N|K^+$.

Consequently, $H = K^+ \ominus K_0$ is $\Phi_{(N^+)^*}$-invariant. Combining this with the fact that N^+ is still a dilation of T we obtain the desired formula $\Phi_{T^*}(f) = \Phi_{(N^+)^*}(f)|H$ for $f \in A(D)$. ∎

Proof of Theorem 3.3.2. We are given a von Neumann n-tuple $T \in L(H)^n$ possessing a ∂D-unitary dilation. Fix a minimal ∂D^*-unitary dilation $N \in L(K)^n$ of the adjoint tuple $T^* \in L(H)^n$. Note that, if T is absolutely continuous, then so are both T^* and, by minimality, N (see Lemma 3.3.1).

Now we consider the space K^+ (defined in the preceding lemma) with respect to the n-tuple $N \in L(K)^n$ fixed above. The spaces

$$\mathcal{R} = \bigvee (M : M \text{ is a reducing subspace for } N^+ \text{ and } N^+|M \text{ is } \partial D^*\text{-unitary}),$$
$$\mathcal{S} = K^+ \ominus \mathcal{R}$$

are reducing for N^+ such that $N^+|\mathcal{R}$ is normal and $N^+|\mathcal{S}$ is completely non-∂D^*-unitary. Since $\mathcal{R}$ is also reducing for N, it follows that $N^+|\mathcal{R}$ is even ∂D^*-unitary.

Proposition 3.1.4 asserts that $S = N^+|\mathcal{S} = N|\mathcal{S}$ is of type $C_{\cdot 0}$. Setting $R = (N^+|\mathcal{R})^*$, the preceding lemma says that

$$T = (N^+)^*|H = (S^* \oplus R)|H,$$

hence $C = (N^+)^*$ is an extension of T as described in part (a) of Theorem 3.3.2. Moreover, if T is absolutely continuous then so are N (see the remarks above) and $C = (N|K^+)^*$. To prove the asserted minimality of C, note that each subspace $M \subset K^+$ which reduces C obviously reduces N^+, and hence is Φ_N-invariant by Lemma 2.3.2. Now the minimality property of K^+ stated in the previous lemma assures that $M = K^+$ whenever M contains H. This observation finishes the proof. ∎

4. A characterization of the class $\mathbb{A}_{1,\aleph_0}$

In this chapter we examine the structure of von Neumann n-tuples $T \in L(H)^n$ over D possessing both a holomorphic ∂D-unitary dilation and an isometric and weak* continuous $H^\infty(D)$-functional calculus. As the title suggests, our main result will be a chain of equivalent conditions which are necessary and sufficient for T to belong to the class $\mathbb{A}_{1,\aleph_0}$ and which imply that the dual algebra $\mathcal{H}_T(\overline{D})$ generated by T is super-reflexive. The proof is patterned after Eschmeier [18] who settled the corresponding results in the case of the unit ball.

4.1. Factorization results on the boundary. Throughout this section, let $T \in L(H)^n$ denote a fixed absolutely continuous von Neumann n-tuple over D possessing a ∂D-unitary dilation. We choose a standard model $T = C|H$ for T in the sense of Theorem 3.3.2, where

$$C = S^* \oplus R \in L(\mathcal{S} \oplus \mathcal{R})^n$$

is the direct sum of a von Neumann n-tuple S^* of class $C_0.$ over D and an absolutely continuous ∂D-unitary part R. Our next aim is to study the boundary part R in more detail. For the rest of this section we will therefore always assume that $\mathcal{R} \neq \{0\}$. The orthogonal projection from $\mathcal{S} \oplus \mathcal{R}$ to an arbitrary closed subspace $M \subset \mathcal{S} \oplus \mathcal{R}$ will be denoted by $P_M \in L(\mathcal{S} \oplus \mathcal{R})$.

First note that a reducing subspace M for R satisfying $\overline{P_{\mathcal{R}} H} \subset M \subset \mathcal{R}$ immediately yields a reducing subspace $\mathcal{S} \oplus M \supset H$ for C. By the minimality property of C (see Theorem 3.3.2) we have $M = \mathcal{R}$. This proves that R is the minimal normal extension of $R|\overline{P_{\mathcal{R}} H}$ and allows us to choose a separating unit vector $e \in \overline{P_{\mathcal{R}} H}$ for R (see Proposition 1.9 in [16]).

Denoting the projection-valued spectral measure of R by E, we define a probability measure $\nu \in M_1^+(\overline{D})$ by the formula

$$\nu(\omega) = \langle E(\omega \cap \sigma(R))e, e\rangle \quad (\omega \subset \overline{D} \text{ Borel}).$$

Obviously, $\nu(D) = 0$ and ν is the trivial extension of a scalar-valued spectral measure for R. The associated isomorphism of von Neumann algebras will be denoted by

$$\Psi_R : L^\infty(\nu) \to W^*(R) \subset L(\mathcal{R}).$$

Lemma 2.3.2 shows that, for $f \in A(D)$, we have $\Phi_C(f) = \Phi_{S^*}(f) \oplus \Psi_R(f)$. Before we formulate the next lemma, we need to introduce some notation: Given any measure $\mu \in M^+(\overline{D})$ and $1 \le p < \infty$ we write

$$H^p(\mu) = \overline{A(D)} \subset L^p(\mu)$$

for the norm closure of the space $A(D)$ in $L^p(\mu)$.

4.1.1. Lemma. *The measure $\nu \in M_1^+(\overline{D})$ defined above is a Henkin measure, and there exists an isometric embedding $j : L^2(\nu) \to \mathcal{R}$ satisfying*

$$j \circ M_g = \Psi_R(g) \circ j \quad (g \in L^\infty(\nu)),$$

where $M_g : L^2(\nu) \to L^2(\nu)$ denotes the multiplication by the function g. In particular, the closed subspace $\mathcal{R}_0 = jL^2(\nu) \subset \mathcal{R}$ reduces R, and the closed subspace $\mathcal{R}_0^+ = jH^2(\nu) \subset \mathcal{R}_0$ satisfies $\mathcal{R}_0^+ \subset \overline{P_{\mathcal{R}} H}$ as well as $\Psi_R(f)\mathcal{R}_0^+ \subset \mathcal{R}_0^+$ $(f \in A(D))$.

Proof. Since $L^\infty(\nu) \subset L^2(\nu)$ is a dense subspace, the identity

$$\int_{\overline{D}} |f|^2 \, d\nu = \|\Psi_R(f)e\|^2 \quad (f \in L^\infty(\nu))$$

implies the existence of an isometry

$$j : L^2(\nu) \to \mathcal{R} \quad \text{with} \quad j(f) = \Psi_R(f)e \quad (f \in L^\infty(\nu)).$$

By construction, j satisfies the asserted intertwining relation, which implies that $\mathcal{R}_0$ is reducing for R and that $R|\mathcal{R}_0$ is similar to the multiplication n-tuple $M_z \in L(L^2(\nu))^n$. Since the first (and hence also the latter) n-tuple is absolutely continuous,

$$\int_{\overline{D}} f_k \, d\nu = \langle f_k \cdot 1, 1\rangle_{L^2(\nu)} \xrightarrow{k} 0$$

for each Montel sequence (f_k) in $A(D)$, proving that ν is a Henkin measure. The relation $\Psi_R(f)P_{\mathcal{R}} H = P_{\mathcal{R}} \Phi_C(f) H \subset P_{\mathcal{R}} H$, valid for all $f \in A(D)$, implies that $\mathcal{R}_0^+ = jH^2(\nu) \subset \overline{P_{\mathcal{R}} H}$. Finally, the intertwining relation established above ensures that $\mathcal{R}_0^+$ is in fact invariant under each operator of the form $\Psi_R(f)$ with $f \in A(D)$. ∎

In what follows, the set of all weak* continuous linear forms on the dual algebra $H^\infty(\nu)$ will be identified with the quotient $Q(\nu) = L^1(\nu)/{}^\perp H^\infty(\nu)$, which is a separable Banach space when equipped with the quotient norm.

For any pair of vectors $x, y \in \mathcal{R}$ we denote by $x \cdot y \in L^1(\nu)$ the Radon–Nikodym density of the measure

$$\nu_{x,y}(\omega) = \langle E(\omega \cap \sigma(R))x, y\rangle \quad (\omega \subset \overline{D} \text{ Borel})$$

with respect to the scalar-valued spectral measure ν defined above. By setting

$$x \odot y = [x \cdot y] \in Q(\nu) \quad (x, y \in \mathcal{R})$$

we obtain a weak* continuous linear form on $H^\infty(\nu)$ satisfying the estimate

$$\|x \odot y\| \le \|x \cdot y\|_{L^1(\nu)} = \|\nu_{x,y}\| \le \|x\|\,\|y\| \quad (x, y \in \mathcal{R}).$$

The last inequality can be easily checked using the Riesz representation theorem and the $L^\infty(\nu)$-functional calculus Ψ_R. Furthermore, since $\mathcal{R}_0$ is reducing for R, we deduce that $\nu_{x,y}$ vanishes identically whenever $x \in \mathcal{R}_0$ and $y \in \mathcal{R}_1 = \mathcal{R} \ominus \mathcal{R}_0$. Thus we have

$$x \cdot y = 0, \quad x \odot y = 0 \quad (x \in \mathcal{R}_0,\, y \in \mathcal{R}_1 = \mathcal{R} \ominus \mathcal{R}_0).$$

Denoting the inverse of the isometric isomorphism $j : L^2(\nu) \to \mathcal{R}_0$ defined in the previous lemma by j^{-1}, we introduce the abbreviation $\{x\} = j^{-1}(x)$ for every $x \in \mathcal{R}_0$. The intertwining relation satisfied by j immediately yields the formula

$$\{\Psi_R(f)\alpha\} = f\{\alpha\} \quad (f \in L^\infty(\nu),\, \alpha \in \mathcal{R}_0),$$

and since $\nu_{x,y}(\omega) = \langle \Psi_R(\chi_\omega) j\{x\}, j\{y\}\rangle = \langle \chi_\omega\{x\}, \{y\}\rangle_{L^2(\nu)} = \int_\omega \{x\}\overline{\{y\}}\, d\nu$ for every Borel subset $\omega \subset \overline{D}$, we conclude that

$$x \cdot y = \{x\}\overline{\{y\}} \quad (x, y \in \mathcal{R}_0).$$

As an application of the above formulas we deduce the relation

$$\Psi_R(f)\alpha \cdot \Psi_R(g)\beta = f\overline{g}\{\alpha\}\overline{\{\beta\}} \quad (f, g \in L^\infty(\nu),\, \alpha, \beta \in \mathcal{R}_0),$$

which turns out to be very useful. The following general factorization result forms the basis of the whole factorization theory to be derived in this section.

4.1.2. Lemma. *Let $\mu \in M_1^+(\overline{D})$ be an arbitrary probability measure. Then, for every choice of $L \in Q(\mu) = L^1(\mu)/{}^\perp H^\infty(\mu)$ and $\varepsilon > 0$, there exist elements $f, g \in H^2(\mu)$ satisfying $\|L - [f\overline{g}]\| < \varepsilon$ and $\|f\|, \|g\| \le \|L\|^{1/2}$.* ■

For a proof of this lemma, note that the proof of Lemma 1.4 in [16] remains true if one replaces the spaces $P^p(\mu)$ by $H^p(\mu)$ $(1 < p < \infty)$ and polynomials by functions in $A(D)$. Obvious modifications in the proof of Lemma 1.3 in [18] then lead to the above factorization result. The following proposition has been proved by Eschmeier in the case of the unit ball (see [18, Proposition 1.5]). Although the general argument scheme remains the same in the strictly pseudoconvex case, some function-theoretic details have to be adapted to the more general setting.

4.1.3. Proposition. *Let $L \in Q(\nu)$, $0 < \varrho < 1$, $\varepsilon > 0$, $a \in H$, $b \in \mathcal{R}$ be given. Then there exist vectors $x \in H$ and $c \in \mathcal{R}$ satisfying $c - b \in \mathcal{R}_0$ and a Borel subset $Z \subset \partial D$ of* *measure* $\nu(Z) < \varepsilon$ *such that*

$$\|L - P_{\mathcal{R}}(a+x) \odot c\| < \varepsilon,$$
$$\|x\| \leq 2\|L - P_{\mathcal{R}} a \odot b\|^{1/2}, \qquad \|P_{\mathcal{S}} x\| < \varepsilon, \qquad \|P_{\mathcal{R}_1} x\| < \varepsilon,$$
$$\|c\| \leq \frac{1}{\varrho}\,(\|b\| + \|L - P_{\mathcal{R}} a \odot b\|^{1/2}),$$
$$|\{P_{\mathcal{R}_0}(a+x)\}| \geq \varrho |\{P_{\mathcal{R}_0} a\}| \qquad \nu\text{-a.e. on } \partial D \setminus Z.$$

If both $a = 0$ *and* $b = 0$, *then one can achieve that* $\|x\| \leq \|L\|^{1/2}$ *and* $\|c\| \leq \|L\|^{1/2}$.

Proof. Let $\delta > 0$. For abbreviation, we set $d = \|L - P_{\mathcal{R}} a \odot b\|^{1/2}$. The proof will be divided into four steps.

(1) We construct vectors $u, v \in \mathcal{R}_0^+$ and $\widetilde{x} \in H$ having the following properties:

$$\|u\|, \|v\| \leq d, \qquad \|P_{\mathcal{S}} \widetilde{x}\| < \delta/2, \qquad \|P_{\mathcal{R}} \widetilde{x} - u\| < \delta, \qquad \|L - P_{\mathcal{R}} a \odot b - u \odot v\| < \delta.$$

For this purpose we choose—according to the previous lemma—vectors $\alpha, \beta \in \mathcal{R}_0^+$ satisfying $\|\alpha\|, \|\beta\| \leq d$ and $\|L - P_{\mathcal{R}} a \odot b - \alpha \odot \beta\| < \delta/2$. Because of $\mathcal{R}_0^+ \subset \overline{P_{\mathcal{R}} H}$ we are able to find $x' \in H$ with $\|P_{\mathcal{R}} x' - \alpha\| < \delta$. In order to define the desired vectors u, v and $\widetilde{x}$ we first choose a real number $\eta > 0$ such that

$$\int_\omega 2|\{\alpha\}\{\beta\}|\, d\nu < \frac{\delta}{2} \qquad \text{for every Borel subset } \omega \subset \partial D \text{ of measure } \nu(\omega) < \eta.$$

Then applying Corollary 2.1.4 we obtain a Borel subset $\omega \subset \partial D$ of measure $\nu(\omega) < \eta$ and a Montel sequence $(\theta_k)_{k \in \mathbb{N}}$ in $A(D)$ such that

$$\|\theta_k\|_{\infty, \overline{D}} \leq 1, \quad |\theta_k| = 1 \quad \text{on } \partial D \setminus \omega \quad (k \in \mathbb{N}).$$

The vectors defined by

$$u = \Psi_R(\theta_k)\alpha \in \mathcal{R}_0^+, \qquad v = \Psi_R(\theta_k)\beta \in \mathcal{R}_0^+, \qquad \widetilde{x} = \Phi_C(\theta_k) x' \in H$$

have all the required properties provided that $k \in \mathbb{N}$ is large enough: Obviously, the norm estimates $\|u\|, \|v\| \leq d$ hold. Because of the relation $P_{\mathcal{S}} \widetilde{x} = \Phi_{S^*}(\theta_k) P_{\mathcal{S}} x'$ and the C_0.-property of S^* we can also achieve that $\|P_{\mathcal{S}} \widetilde{x}\| < \delta/2$. Moreover, we have $\|P_{\mathcal{R}} \widetilde{x} - u\| \leq \|\Psi_R(\theta_k)\|\, \|P_{\mathcal{R}} x' - \alpha\| < \delta$, as desired. Finally, the identity

$$\Psi_R(\theta_k)\alpha \cdot \Psi_R(\theta_k)\beta = |\theta_k|^2 \{\alpha\}\overline{\{\beta\}} = \{\alpha\}\overline{\{\beta\}} \qquad (\text{on } \partial D \setminus \omega)$$

yields the norm estimate

$$\|u \odot v - \alpha \odot \beta\| \leq \|\Psi_R(\theta_k)\alpha \cdot \Psi_R(\theta_k)\beta - \{\alpha\}\overline{\{\beta\}}\|_{L^1(\nu)} \leq \int_\omega (|\theta_k|^2 + 1)|\{\alpha\}\{\beta\}|\, d\nu < \frac{\delta}{2},$$

implying $\|L - P_{\mathcal{R}} a \odot b - u \odot v\| < \delta$.

(2) We construct a Borel subset $Z \subset \partial D$ of measure $\nu(Z) < \delta$ and define vectors

$$\begin{aligned} x \in H \quad & \text{satisfying} \quad \|P_{\mathcal{S}} x\| < \delta, \quad \|P_{\mathcal{R}_1} x\| < 2\delta, \quad \|x\| \leq (1+\varrho)(d + 2\delta), \\ & \qquad\qquad\quad |\{P_{\mathcal{R}_0}(a+x)\}| \geq \varrho |\{P_{\mathcal{R}_0} a\}| \quad \nu\text{-a.e. on } \partial D \setminus Z, \\ c_0 \in \mathcal{R}_0 \quad & \text{satisfying} \quad \|c_0\| \leq \frac{1}{\varrho}\,(\|P_{\mathcal{R}_0} b\| + \|v\|), \\ & \qquad\qquad\quad \|P_{\mathcal{R}_0}(a+x) \odot c_0 - (P_{\mathcal{R}_0} a \odot P_{\mathcal{R}_0} b + P_{\mathcal{R}_0} \widetilde{x} \odot v)\| < \delta. \end{aligned}$$

Towards this end, fix a real number $\eta > 0$, $\eta < \delta$ such that, for any Borel subset $Z \subset \partial D$, we have

$$\int_Z (|\{P_{\mathcal{R}_0} a\}\{P_{\mathcal{R}_0} b\}| + |\{P_{\mathcal{R}_0} \widetilde{x}\}\{v\}|)\, d\nu < \delta \quad \text{whenever } \nu(Z) < \eta.$$

By Corollary 2.1.4 there exist a Borel subset $Z \subset \partial D$ of measure $\nu(Z) < \eta$ and a function $f \in A(D)$ of norm $\|f\|_{\infty,\overline{D}} \le 1 + \varrho$ such that, for every $z \in \partial D \setminus Z$,

$$|f(z)| = \begin{cases} 1 + \varrho & \text{if } |\{P_{\mathcal{R}_0} a\}(z)| \le |\{P_{\mathcal{R}_0} \widetilde{x}\}(z)|, \\ 1 - \varrho & \text{otherwise } (^5). \end{cases}$$

The vector $x = \Phi_C(f)\widetilde{x} \in H$ then satisfies the estimates

$$\begin{gathered} \|P_{\mathcal{S}} x\| = \|\Phi_{S^*}(f) P_{\mathcal{S}} \widetilde{x}\| \le (1+\varrho)\|P_{\mathcal{S}} \widetilde{x}\| < 2 \cdot \frac{\delta}{2} = \delta, \\ \|P_{\mathcal{R}_1} x\| = \|\Psi_R(f) P_{\mathcal{R}_1} \widetilde{x}\| \le (1+\varrho)\|P_{\mathcal{R}_1}(P_{\mathcal{R}} \widetilde{x} - u)\| < 2\delta, \\ \|x\| \le (1+\varrho)\|\widetilde{x}\| = (1+\varrho)\|u - (u - P_{\mathcal{R}} \widetilde{x}) + P_{\mathcal{S}} \widetilde{x}\| \le (1+\varrho)\left(d + \delta + \frac{\delta}{2}\right). \end{gathered}$$

For the following calculations and estimates it may be useful to have the identity $\{P_{\mathcal{R}_0} x\} = \{\Psi_R(f) P_{\mathcal{R}_0} \widetilde{x}\} = f\{P_{\mathcal{R}_0} \widetilde{x}\}$ in mind. Given $z \in \partial D \setminus Z$ such that $|\{P_{\mathcal{R}_0} a\}(z)| \le |\{P_{\mathcal{R}_0} \widetilde{x}\}(z)|$, we deduce from the very definition of f that

$$|\{P_{\mathcal{R}_0}(a + x)\}(z)| \ge |(f\{P_{\mathcal{R}_0} \widetilde{x}\})(z)| - |\{P_{\mathcal{R}_0} \widetilde{x}\}(z)| = \varrho |\{P_{\mathcal{R}_0} \widetilde{x}\}(z)|,$$

while for the remaining elements z of $\partial D \setminus Z$ we obtain the inequality

$$|\{P_{\mathcal{R}_0}(a + x)\}(z)| \ge |\{P_{\mathcal{R}_0} a\}(z)| - |(f\{P_{\mathcal{R}_0} \widetilde{x}\})(z)| \ge \varrho |\{P_{\mathcal{R}_0} a\}(z)|.$$

Combining these observations we conclude that on $\partial D \setminus Z$ we have

$$|\{P_{\mathcal{R}_0}(a + x)\}(z)| \ge \varrho \max(|\{P_{\mathcal{R}_0} \widetilde{x}\}(z)|, |\{P_{\mathcal{R}_0} a\}(z)|) \quad (z \in \partial D \setminus Z).$$

Now we set $h_0 = P_{\mathcal{R}_0} a \cdot P_{\mathcal{R}_0} b + P_{\mathcal{R}_0} \widetilde{x} \cdot v \in L^1(\nu)$ and define a Borel measurable function $h : \overline{D} \to \mathbb{C}$ by the formula

$$\overline{h}(z) = \begin{cases} \dfrac{h_0(z)}{\{P_{\mathcal{R}_0}(a + x)\}(z)} & \text{if } z \in \partial D \setminus Z \text{ and } \{P_{\mathcal{R}_0}(a + x)\}(z) \ne 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $z \in \partial D$ with $h(z) \ne 0$, we have the estimate

$$\begin{aligned} |h(z)|^2 &\le \frac{|(\{P_{\mathcal{R}_0} a\}\{P_{\mathcal{R}_0} b\})(z)|^2}{|\{P_{\mathcal{R}_0}(a + x)\}(z)|^2} + \frac{|(\{P_{\mathcal{R}_0} \widetilde{x}\}\{v\})(z)|^2}{|\{P_{\mathcal{R}_0}(a + x)\}(z)|^2} \\ &\quad + 2\frac{|(\{P_{\mathcal{R}_0} a\}\{P_{\mathcal{R}_0} b\}\{P_{\mathcal{R}_0} \widetilde{x}\}\{v\})(z)|}{|\{P_{\mathcal{R}_0}(a + x)\}(z)|^2} \\ &\le \frac{1}{\varrho^2}\,(|\{P_{\mathcal{R}_0} b\}(z)| + |\{v\}(z)|)^2, \end{aligned}$$

(5) Whenever we evaluate L^2-functions at a single point (e.g. $\{P_{\mathcal{R}_0} a\}(z)$ or $\{P_{\mathcal{R}_0} \widetilde{x}\}(z)$) we use the convention of choosing a representative of the corresponding L^2-equivalence class (which can be evaluated) and keeping this representative fixed throughout the whole proof.

proving that h lies in $L^2(\nu)$ and $\|h\| \le (\|P_{\mathcal{R}_0}b\|+\|v\|)/\varrho$. Therefore we may write $h = \{c_0\}$ for a suitable element $c_0 \in \mathcal{R}_0$ satisfying $\|c_0\| \le (\|P_{\mathcal{R}_0}b\| + \|v\|)/\varrho$. The estimate

$$\begin{aligned}\|P_{\mathcal{R}_0}(a+x) \odot c_0 - [h_0]\| &\le \|\{P_{\mathcal{R}_0}(a+x)\}\bar{h} - h_0\|_{L^1(\nu)} \\ &\le \int_Z (|\{P_{\mathcal{R}_0}a\}\{P_{\mathcal{R}_0}b\}| + |\{P_{\mathcal{R}_0}\widetilde{x}\}\{v\}|)\, d\nu < \delta\end{aligned}$$

finishes the proof of the second step.

(3) We define $c = c_0 \oplus P_{\mathcal{R}_1}b \in \mathcal{R}_0 \oplus \mathcal{R}_1 = \mathcal{R}$ guaranteeing that $c-b = c_0 - P_{\mathcal{R}_0}b \in \mathcal{R}_0$, as desired. Moreover, we can estimate the norm of c in the following way:

$$\begin{aligned}\|c\|^2 = \|c_0\|^2 + \|P_{\mathcal{R}_1}b\|^2 &\le \frac{1}{\varrho^2}\,(\|P_{\mathcal{R}_0}b\|^2 + 2\|P_{\mathcal{R}_0}b\|\|v\| + \|v\|^2 + \|P_{\mathcal{R}_1}b\|^2) \\ &\le \frac{1}{\varrho^2}(\|b\| + \|v\|)^2 \le \frac{1}{\varrho^2}\,(\|b\| + d)^2.\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\|L - P_{\mathcal{R}}(a+x) \odot c\| &= \|L - P_{\mathcal{R}_0}(a+x) \odot c_0 - P_{\mathcal{R}_1}(a+x) \odot P_{\mathcal{R}_1}b\| \\ &\le \|L - P_{\mathcal{R}}a \odot b - P_{\mathcal{R}}\widetilde{x} \odot v\| \\ &\quad + \|P_{\mathcal{R}}a \odot b + P_{\mathcal{R}}\widetilde{x} \odot v - P_{\mathcal{R}_0}(a+x) \odot c_0 - P_{\mathcal{R}_1}(a+x) \odot P_{\mathcal{R}_1}b\|.\end{aligned}$$

Abbreviating the two terms on the right-hand side of the last inequality by r and r' we observe that

$$\begin{aligned}r &\le \|L - P_{\mathcal{R}}a \odot b - u \odot v\| + \|(P_{\mathcal{R}}\widetilde{x} - u) \odot v\| \le \delta(1+d), \\ r' &\le \|P_{\mathcal{R}_0}a \odot P_{\mathcal{R}_0}b + P_{\mathcal{R}_0}\widetilde{x} \odot v - P_{\mathcal{R}_0}(a+x) \odot c_0\| + \|P_{\mathcal{R}_1}x \odot P_{\mathcal{R}_1}b\| \le \delta(1 + 2\|P_{\mathcal{R}_1}b\|).\end{aligned}$$

If we start the argument with a sufficiently small $\delta > 0$, this completes the proof of the main assertion. The extra assertion for $a = 0$ and $b = 0$ requires a deeper analysis of the above constructions which will be carried out in the next step.

(4) If $a = 0$ and $b = 0$, then we may set $f(z) \equiv 1 + \varrho$ $(z \in \overline{D})$ in the second step, and hence we have $\{P_{\mathcal{R}_0}x\} = (1+\varrho)\{P_{\mathcal{R}_0}\widetilde{x}\}$. At those points where $h(z)$ does not vanish,

$$\overline{h}(z) = \{P_{\mathcal{R}_0}\widetilde{x}\}(z)\overline{\{v\}}(z)/\{P_{\mathcal{R}_0}x\}(z) = \frac{1}{1+\varrho}\,\overline{\{v\}}(z),$$

proving the following estimate for the norm of $c = c_0$:

$$\|c\| = \|c_0\| = \|h\| \le \frac{1}{1+\varrho}\,\|v\| \le \frac{1}{1+\varrho}\,\|L\|^{1/2}.$$

The corresponding vector x from step (2) satisfies $\|P_{\mathcal{S}}x\|, \|P_{\mathcal{R}_1}x\| < 2\delta$ and

$$\|L - P_{\mathcal{R}}x \odot c\| \le \delta(2 + \|L\|^{1/2}), \qquad \|x\| \le (1+\varrho)(\|L\|^{1/2} + 2\delta).$$

After replacing the vector c by $\widetilde{c} = (1+\varrho)c$ and the vector x by

$$\widetilde{x} = \frac{1}{1+\varrho}\,\frac{\|L\|^{1/2}}{\|L\|^{1/2} + 2\delta}\,x,$$

the estimates $\|P_{\mathcal{S}}\widetilde{x}\|, \|P_{\mathcal{R}_1}\widetilde{x}\| < 2\delta$ remain true and in addition we obtain

$$\|\widetilde{c}\| \le \|L\|^{1/2}, \qquad \|\widetilde{x}\| \le \|L\|^{1/2},$$

as desired. From this and the fact that

$$\|P_{\mathcal{R}}x \odot c - P_{\mathcal{R}}\widetilde{x} \odot \widetilde{c}\| \le \left(1 - \frac{\|L\|^{1/2}}{\|L\|^{1/2} + 2\delta}\right)(\|L\|^{1/2} + 2\delta)\|L\|^{1/2}$$

we deduce that if we start the proof with a sufficiently small $\delta > 0$, then we can achieve that $\|L - P_{\mathcal{R}}\widetilde{x} \odot \widetilde{c}\| < \varepsilon$ and that all the other estimates from the assertion are valid. ∎

By a well known inductive process carried out in detail in [18] (cf. Proposition 1.6) one can generalize the previous factorization result to finite systems of functionals.

4.1.4. Proposition. *Let $N \in \mathbb{N}$, $L_1, \dots, L_N \in Q(\nu)$, $\mu_1, \dots, \mu_N > 0$, $a \in H$, $b_1, \dots, b_N \in \mathcal{R}$ be given initial data satisfying*

$$\|L_k - P_{\mathcal{R}}a \odot b_k\| < \mu_k \quad (1 \le k \le N).$$

Then, for every $\varepsilon > 0$, there are vectors $x \in H$ and $y_1, \dots, y_N \in \mathcal{R}$ such that $y_k - b_k \in \mathcal{R}_0$ $(1 \le k \le N)$ and

$$\|L_k - P_{\mathcal{R}}(a + x) \odot y_k\| < \varepsilon \quad (1 \le k \le N),$$

$$\|x\| < 2\sum_{i=1}^{N} \mu_i^{1/2}, \qquad \|P_{\mathcal{S}}x\| < \varepsilon, \qquad \|P_{\mathcal{R}_1}x\| < \varepsilon,$$

$$\|y_k\| < \|b_k\| + \mu_k^{1/2} \quad (1 \le k \le N). \ ∎$$

Again by standard arguments the approximative factorizations of the last result can be replaced by exact ones (see Corollary 1.7 in [18]).

4.1.5. Corollary. *Let $N \in \mathbb{N}$, $L_1, \dots, L_N \in Q(\nu)$, $\mu_1, \dots, \mu_N > 0$, $a \in H$, $b_1, \dots, b_N \in \mathcal{R}$ be given elements satisfying*

$$\|L_k - P_{\mathcal{R}}a \odot b_k\| < \mu_k \quad (1 \le k \le N).$$

Then, for every $\varepsilon > 0$, there are vectors $x \in H$ and $c_1, \dots, c_N \in \mathcal{R}$ such that $c_k - b_k \in \mathcal{R}_0$ $(1 \le k \le N)$ and

$$L_k = P_{\mathcal{R}}(a + x) \odot c_k \quad (1 \le k \le N),$$

$$\|x\| < 2\sum_{i=1}^{N} \mu_i^{1/2}, \qquad \|P_{\mathcal{S}}x\| < \varepsilon, \qquad \|P_{\mathcal{R}_1}x\| < \varepsilon,$$

$$\|c_k\| < \|b_k\| + \mu_k^{1/2} \quad (1 \le k \le N). \ ∎$$

4.2. Factorization results in the interior. A von Neumann n-tuple $T \in L(H)^n$ over D is said to be of class $\mathbb{A}$ if T possesses a ∂D-unitary dilation and an isometric and weak* continuous $H^\infty(D)$-functional calculus.

Note that for the latter condition to be satisfied it is sufficient that T is completely non-∂D-unitary and that $\sigma(T)$ is dominating in D. Indeed, the decomposition Theorem 3.2.1 ensures that, in this case, T is absolutely continuous and Corollary 2.3.7 then guarantees that Φ_T is isometric.

For the rest of this section, we fix a von Neumann n-tuple $T \in L(H)^n$ of class $\mathbb{A}$ over D. Let $C = S^* \oplus R \in L(\mathcal{S} \oplus \mathcal{R})^n$ denote a standard model for T as described in the statement of Theorem 3.3.2. Recall that C inherits absolute continuity from T and that the $H^\infty(D)$-calculi of T, C, S^* and R are related via the formulas

$$\Phi_T(f) = \Phi_C(f)|H, \quad \Phi_C(f) = \Phi_{S^*}(f) \oplus \Phi_R(f) \quad (f \in H^\infty(D))$$

(see Lemma 2.3.5). We define $Q(D) = L^1(\lambda)/{}^\perp H^\infty(D)$ to be the Banach space of all weak* continuous linear forms on $H^\infty(D)$. Setting $K = \mathcal{S} \oplus \mathcal{R}$ and

$$x \otimes y : H^\infty(D) \to \mathbb{C}, \quad f \mapsto \langle \Phi_C(f)x, y\rangle \quad (x, y \in K),$$

we obtain for each pair of vectors $x, y \in K$ an element $x \otimes y \in Q(D)$. The above compatibility conditions for the underlying $H^\infty(D)$-calculi imply that, for $x, y \in K$, we have $P_{\mathcal{S}}x \otimes y = x \otimes P_{\mathcal{S}}y = P_{\mathcal{S}}x \otimes P_{\mathcal{S}}y$ and $P_{\mathcal{R}}x \otimes y = x \otimes P_{\mathcal{R}}y = P_{\mathcal{R}}x \otimes P_{\mathcal{R}}y$. Similarly, given any $h \in H$ and $y \in K$ we may use the fact that $h \otimes y = h \otimes P_H y$. Furthermore, using the C_0. property of S^* and the fact that the closed unit ball of $H^\infty(D)$ is weak* compact, standard arguments (cf. Lemma 1.1 in [14]) imply that

$$P_{\mathcal{S}}x \otimes y_k \xrightarrow{k} 0 \quad \text{for every } x \in K \text{ and every weak zero sequence } (y_k) \text{ in } K.$$

4.2.1. Definition. Let $\theta \geq 0$ be a real number. We write $\mathcal{E}_\theta^r(T)$ for the set of all elements $L \in Q(D)$ for which there exist sequences (x_k), (y_k) in H satisfying the following requirements:

(a) $\|x_k\| \leq 1,\ \|y_k\| \leq 1\ (k \geq 1)$,
(b) $\overline{\lim}_{k\to\infty} \|L - x_k \otimes y_k\| \leq \theta$,
(c) $x_k \otimes z \xrightarrow{k} 0$ as $k \to \infty$ for each $z \in H$.

As pointed out in [7, Remark 3.1], one can achieve that in addition the sequence (y_k) converges to zero weakly in H.

Provided that $\mathcal{R} \neq \{0\}$ we fix a scalar-valued spectral measure $\nu \in M_1^+(\overline{D})$ as described in detail in the previous section, and we write $r_* : Q(\nu) \to Q(D)$ for the preadjoint of the dual algebra homomorphism $r = r_\nu : H^\infty(D) \to H^\infty(\nu)$ associated with the Henkin measure ν. Given $x, y \in \mathcal{R}$, the identity $\langle \Phi_C(f)x, y\rangle = \int_{\overline{D}} f\, d\nu_{x,y} = \int_{\overline{D}} f(x \cdot y)\, d\nu$ $(f \in A(D))$ proves the formula

$$r_*(x \odot y) = x \otimes y \quad (x, y \in \mathcal{R}).$$

If $\mathcal{R} = \{0\}$, then we set $H^\infty(\nu) = \{0\} = Q(\nu)$.

4.2.2. Definition. Let $0 \leq \theta < \gamma \leq 1$ be fixed real numbers. We say that T has property $F_{\theta,\gamma}^r$ if

$$\{L \in Q(D) : \|L\| \leq \gamma\} \subset \overline{\Gamma}(\mathcal{E}_\theta^r(T) \cup r_*\{L \in Q(\nu) : \|L\| \leq 1\}),$$

where $\overline{\Gamma}(\ldots)$ denotes the closed absolutely convex hull. In the case that even

$$\{L \in Q(D) : \|L\| \leq \gamma\} \subset \overline{\Gamma}(\mathcal{E}_\theta^r(T)),$$

T is said to possess property $E_{\theta,\gamma}^r$.

Exactly as in the case of the unit ball we first show how to factorize finite families of functionals using property $F_{\theta,\gamma}^r$ and the factorization results on the boundary derived in the preceding section.

4.2.3. THEOREM. *Suppose that T has property $F^r_{\theta,\gamma}$ for some $0<\theta<\gamma\le 1$. Then, given $N\in\mathbb{N}$, $L_1,\dots,L_N\in Q(D)$, $\mu_1,\dots,\mu_N>0$, $a\in H$, $w_1,\dots,w_N\in\mathcal{S}$, $b_1,\dots,b_N\in\mathcal{R}$ such that*

$$\|L_j - a\otimes(w_j+b_j)\| < \mu_j \quad (1\le j\le N),$$

there exist vectors $a'\in H$, $w'_1,\dots,w'_N\in\mathcal{S}$, $b'_1,\dots,b'_N\in\mathcal{R}$ such that

$$\|L_j - a'\otimes(w'_j+b'_j)\| < \frac{\theta}{\gamma}\,\mu_j,$$

$$\|a'-a\| < \frac{3}{\gamma^{1/2}}\sum_{i=1}^N \mu_i^{1/2}, \qquad \|w'_j - w_j\| < \Big(\frac{\mu_j}{\gamma}\Big)^{1/2}, \qquad \|b'_j\| < \|b_j\| + \Big(\frac{\mu_j}{\gamma}\Big)^{1/2}$$

for $j=1,\dots,N$.

Proof. For $j=1,\dots,N$ we define $L'_j = L_j - a\otimes(w_j+b_j)$ and $d_j=\max\{\|L'_j\|,\mu_j/2\}$ $(<\mu_j)$. We choose an $\varepsilon>0$ such that $(\theta/\gamma)d_j+4\varepsilon<(\theta/\gamma)\mu_j$ for each j. Then we set $s_j=(\theta/\gamma)d_j+\varepsilon$ $(j=1,\dots,N)$.

(1) Using the relations $\|(\gamma/d_j)L'_j\|\le\gamma$ $(j=1,\dots,N)$ and the definition of property $F^r_{\theta,\gamma}$ we deduce that

$$\Big\{\frac{\gamma}{d_j}\,L'_j : 1\le j\le N\Big\} \subset \overline{\Gamma}(\mathcal{E}^r_\theta(T)\cup r_*\{L\in Q(\nu):\|L\|\le 1\}).$$

Hence there exist natural numbers $0=k_0<k_1<\dots<k_N$ and elements

$$K_i\in\mathcal{E}^r_\theta(T), \qquad \alpha_i\in\mathbb{C} \quad (i=1,\dots,k_N), \qquad f_j\in L^1(\nu) \quad (j=1,\dots,N)$$

with

$$\Big\|L'_j - r_*([f_j]) - \sum_{i\in I_j}\alpha_i K_i\Big\| < \frac{\varepsilon}{2}, \qquad \|f_j\|_{1,\nu} + \sum_{i\in I_j}|\alpha_i| < \frac{d_j}{\gamma}$$

for $j=1,\dots,N$, where $I_j=\{k_{j-1}+1,\dots,k_j\}$. For each $K_i\in\mathcal{E}^r_\theta(T)$ there are sequences $(x^i_k)_{k\ge0}$, $(y^i_k)_{k\ge0}$ in the closed unit ball of H such that, for each multi-index $m=(m_1,\dots,m_{k_N})\in\mathbb{N}^{k_N}$ and each $j=1,\dots,N$,

$$\begin{aligned}\Big\|L'_j - r_*([f_j]) - \sum_{i\in I_j}\alpha_i x^i_{m_i}\otimes y^i_{m_i}\Big\| &< \frac{\varepsilon}{2} + \sum_{i\in I_j}|\alpha_i|\,\|K_i - x^i_{m_i}\otimes y^i_{m_i}\| \\ &\le \frac{\varepsilon}{2} + \sum_{i\in I_j}|\alpha_i|\Big(\theta + \frac{\gamma}{d_j}\,\frac{\varepsilon}{2}\Big) < \varepsilon + \frac{d_j}{\gamma}\,\theta = s_j\end{aligned}$$

and, in addition,

$$x^i_k\otimes z \xrightarrow{k\to\infty} 0 \quad (z\in H) \quad\text{and}\quad y^i_k \xrightarrow{k\to\infty} 0 \quad (\text{weakly in } H)$$

whenever $1\le i\le k_N$. For each $m=(m_1,\dots,m_{k_N})\in\mathbb{N}^{k_N}$ and $j=1,\dots,N$ we define

$$A^j(m) = a\otimes b_j + r_*([f_j]) + \sum_{i\in I_j}\alpha_i P_{\mathcal{R}}x^i_{m_i}\otimes y^i_{m_i}, \qquad Q^j(m) = a\otimes w_j + \sum_{i\in I_j}\alpha_i P_{\mathcal{S}}x^i_{m_i}\otimes y^i_{m_i}$$

and find that

$$\|L_j - Q^j(m) - A^j(m)\| < s_j \quad (1\le j\le N,\ m\in\mathbb{N}^{k_N}).$$

Finally we choose complex numbers β_i $(i = 1, \dots, k_N)$ with $\beta_i^2 = \alpha_i$ and set

$$u_m = \sum_{i=1}^{k_N} \beta_i x^i_{m_i} \in H, \qquad v_{m,j} = \sum_{i \in I_j} \overline{\beta}_i P_{\mathcal{S}} y^i_{m_i} \in \mathcal{S}.$$

(2) We prove that there exists a multi-index $m = (m_1, \dots, m_{k_N}) \in \mathbb{N}^{k_N}$ such that

$$\|u_m\|^2 < \sum_{j=1}^{N} \frac{\mu_j}{\gamma}, \quad \|v_{m,j}\|^2 < \frac{\mu_j}{\gamma}, \quad \|u_m \otimes b_j\| < \varepsilon \quad (j = 1, \dots, N),$$
$$\|Q^j(m) - (a + u_m) \otimes (w_j + v_{m,j})\| < \varepsilon.$$

To this end, let $\eta > 0$. The idea is to choose the components of $m = (m_1, \dots, m_{k_N}) \in \mathbb{N}^{k_N}$ inductively starting with a suitable index $m_1 \in \mathbb{N}$ such that

$$\|P_{\mathcal{S}} a \otimes y^1_{m_1}\| < \eta, \quad \|x^1_{m_1} \otimes w_j\| < \eta, \quad \|x^1_{m_1} \otimes b_j\| < \eta \quad (j = 1, \dots, N).$$

(Here we used the C_0.-property of S^* as well as the convergence conditions from the definition of $(x^i_k)_{k \geq 0}$.) Now suppose that $1 < k \leq k_N$ and that $m_1, \dots, m_{k-1}$ are already chosen. Then we can find a number $m_k \in \mathbb{N}$ such that

$$\|P_{\mathcal{S}} a \otimes y^k_{m_k}\| < \eta, \quad \|x^k_{m_k} \otimes w_j\| < \eta, \quad \|x^k_{m_k} \otimes b_j\| < \eta \quad (j = 1, \dots, N),$$
$$|\langle x^k_{m_k}, x^j_{m_j} \rangle| < \eta, \quad |\langle P_{\mathcal{S}} y^k_{m_k}, y^j_{m_j} \rangle| < \eta \quad (j = 1, \dots, k-1),$$
$$\|P_{\mathcal{S}} x^j_{m_j} \otimes y^k_{m_k}\| < \eta, \quad \|P_{\mathcal{S}} x^k_{m_k} \otimes y^j_{m_j}\| = \|x^k_{m_k} \otimes P_{\mathcal{S}} y^j_{m_j}\| < \eta \quad (j = 1, \dots, k-1).$$

To prove the assertion formulated at the beginning of the second step, we compute

$$\|u_m\|^2 = \langle u_m, u_m \rangle \leq \sum_{i \neq l} |\beta_i \beta_l| \cdot |\langle x^i_{m_i}, x^l_{m_l} \rangle| + \sum_{i=1}^{k_N} |\alpha_i| < c \cdot \eta + \sum_{j=1}^{N} \frac{d_j}{\gamma},$$

$$\|v_{m,j}\|^2 = \langle v_{m,j}, v_{m,j} \rangle \leq \sum_{i \neq l} |\beta_i \beta_l| \cdot |\langle P_{\mathcal{S}} y^i_{m_i}, y^l_{m_l} \rangle| + \sum_{i \in I_j} |\alpha_i| < c \cdot \eta + \frac{d_j}{\gamma},$$

$$\|u_m \otimes b_j\| \leq \sum_{i=1}^{k_N} |\beta_i| \cdot \|x^i_{m_i} \otimes b_j\| \leq \sum_{i=1}^{k_N} |b_i| \eta$$

and

$$\begin{aligned} &\|Q^j(m) - (a + u_m) \otimes (w_j + v_{m,j})\| \\ &\qquad \leq \Big\| \sum_{i \in I_j} \alpha_i P_{\mathcal{S}} x^i_{m_i} \otimes y^i_{m_i} - u_m \otimes w_j - a \otimes v_{m,j} - u_m \otimes v_{m,j} \Big\| \\ &\qquad \leq \Big\| \sum_{i \in I_j} \alpha_i P_{\mathcal{S}} x^i_{m_i} \otimes y^i_{m_i} - \sum_{i \in I_j} \beta_i P_{\mathcal{S}} x^i_{m_i} \otimes \overline{\beta}_i y^i_{m_i} \Big\| + \widetilde{c} \cdot \eta = \widetilde{c} \cdot \eta, \end{aligned}$$

where $c, \widetilde{c} > 0$ denote suitably chosen constants depending neither on η nor on m. If we start the proof of the second step with a sufficiently small $\eta > 0$, the assertion follows.

(3) Let $m \in \mathbb{N}^{k_N}$ be the multi-index constructed in the preceding part of the proof. We define

$$a_1 = a + u_m \in H, \quad w'_j = w_j + v_{m,j} \in \mathcal{S}, \quad x^i = x^i_{m_i} \in H, \quad y^i = y^i_{m_i} \in H,$$

$$h_j = f_j + \sum_{i \in I_j} \alpha_i P_{\mathcal{R}} x^i \cdot P_{\mathcal{R}} y^i \in L^1(\nu)$$

where $j = 1, \ldots, N$ and $i = 1, \ldots, k_N$. Since $\|h_j\| < d_j/\gamma$, Corollary 4.1.5 allows us to choose vectors $x \in H$ and $b_1', \ldots, b_N' \in \mathcal{R}$ satisfying

$$[h_j] + P_{\mathcal{R}} a_1 \odot b_j = P_{\mathcal{R}}(a_1 + x) \odot b_j',$$
$$\|x\| < 2 \sum_{i=1}^{N} \left(\frac{d_j}{\gamma}\right)^{1/2}, \qquad \|P_{\mathcal{S}} x\| < \frac{\varepsilon}{\|w_j'\| + 1}, \qquad \|b_j'\| < \|b_j\| + \left(\frac{d_j}{\gamma}\right)^{1/2}$$

for each $j = 1, \ldots, N$. Note that $b_j \in \mathcal{R}$ and recall the definitions of h_j and $A^j(m)$ to deduce the identity

$$\begin{aligned}(a_1 + x) \otimes b_j' &= P_{\mathcal{R}}(a_1 + x) \otimes b_j' = r_*([h_j] + P_{\mathcal{R}} a_1 \odot b_j) \\ &= r_*([f_j]) + \sum_{i \in I_j} \alpha_i P_{\mathcal{R}} x^i \otimes y^i + P_{\mathcal{R}} a_1 \otimes b_j \\ &= A^j(m) - a \otimes b_j + a_1 \otimes b_j = A^j(m) + u_m \otimes b_j \qquad (j = 1, \ldots, N).\end{aligned}$$

(4) To finish the proof we define

$$a' = a_1 + x = a + u_m + x \in H \qquad (j = 1, \ldots, N).$$

Using the results of steps (2) and (3) we derive the following estimates:

$$\|a' - a\| \le \|u_m\| + \|x\| < \frac{1}{\gamma^{1/2}} \Big(\sum_{j=1}^{N} \mu_j^{1/2} + 2 \sum_{j=1}^{N} d_j^{1/2} \Big) \le \frac{3}{\gamma^{1/2}} \sum_{j=1}^{N} \mu_j^{1/2},$$
$$\|w_j' - w_j\| = \|v_{m,j}\| < \left(\frac{\mu_j}{\gamma}\right)^{1/2} \qquad (j = 1, \ldots, N).$$

The norm estimate

$$\begin{aligned}&\|L_j - (a_1 + x) \otimes (w_j' + b_j')\| \\ &\quad = \|L_j - (a + u_m) \otimes (w_j + v_{m,j}) - x \otimes w_j' - (a_1 + x) \otimes b_j'\| \\ &\quad \le \|L_j - (a + u_m) \otimes (w_j + v_{m,j}) - (a_1 + x) \otimes b_j' + u_m \otimes b_j\| + \|u_m \otimes b_j\| + \|P_{\mathcal{S}} x\| \|w_j'\| \\ &\quad < (\|L_j - Q^j(m) - A^j(m)\| + \varepsilon) + \varepsilon + \varepsilon < s_j + 3\varepsilon < \frac{\theta}{\gamma} \mu_j \qquad (j = 1, \ldots, N)\end{aligned}$$

completes the proof. ∎

Exactly as Corollary 2.4 can be deduced from Proposition 2.3 in [18], the next result can be derived from the preceding proposition.

4.2.4. Corollary. *Assume that T has property $F^r_{\theta,\gamma}$ for some real numbers $0 \le \theta < \gamma \le 1$, and let $N \in \mathbb{N}$, $L_1, \ldots, L_N \in Q(D)$, $\mu_1, \ldots, \mu_N > 0$, $a \in H$, $w_1, \ldots, w_N \in \mathcal{S}$, $b_1, \ldots, b_N \in \mathcal{R}$ with*

$$\|L_j - a \otimes (w_j + b_j)\| < \mu_j \qquad (j = 1, \ldots, N).$$

Then one can choose vectors $a' \in H$, $w'_1, \dots, w'_N \in \mathcal{S}$, $b'_1, \dots, b'_N \in \mathcal{R}$ *satisfying*

$$L_j = a' \otimes (w'_j + b'_j), \quad \|a' - a\| < 3\alpha \sum_{i=1}^N \mu_i^{1/2}, \quad \|w'_j - w_j\| < \alpha \mu_j^{1/2}, \quad \|b'_j\| < \|b_j\| + \alpha \mu_j^{1/2}$$

for $j = 1, \dots, N$, *where* $\alpha = 1/(\gamma^{1/2} - \theta^{1/2})$. ∎

The following consequence of the above corollary will be of special interest later.

4.2.5. Corollary. *Suppose that* T *has property* $F^r_{\theta,\gamma}$ *for some* $0 \le \theta < \gamma \le 1$. *For every choice of* $a, b \in H$ *and* $L \in Q(D)$ *there are vectors* $x, y \in H$ *such that*

$$L = x \otimes y, \quad \|x - a\| \le 6\alpha \|L - a \otimes b\|^{1/2}, \quad \|y\| \le \|b\| + 6\alpha \|L - a \otimes b\|^{1/2}$$

where $\alpha = 1/(\gamma^{1/2} - \theta^{1/2})$.

Proof. We may assume that $L - a \otimes b \neq 0$. Define $\mu = 4\|L - a \otimes b\|$ and apply the previous corollary to obtain vectors $x \in H$, $y_{\mathcal{S}} \in \mathcal{S}$, $y_{\mathcal{R}} \in \mathcal{R}$ satisfying $L = x \otimes (y_{\mathcal{S}} + y_{\mathcal{R}}) = x \otimes P_H(y_{\mathcal{S}} + y_{\mathcal{R}})$ as well as the estimates $\|x - a\| < 3\alpha\mu^{1/2} = 6\alpha \|L - a \otimes b\|^{1/2}$ and

$$\|y_{\mathcal{S}}\| - \|P_{\mathcal{S}} b\| \le \|y_{\mathcal{S}} - P_{\mathcal{S}} b\| < 2\alpha \|L - a \otimes b\|^{1/2}, \quad \|y_{\mathcal{R}}\| \le \|P_{\mathcal{R}} b\| + 2\alpha \|L - a \otimes b\|^{1/2}.$$

An easy calculation shows that

$$\|P_H(y_{\mathcal{S}} + y_{\mathcal{R}})\|^2 \le \|y_{\mathcal{S}}\|^2 + \|y_{\mathcal{R}}\|^2 \le (\|b\| + 6\alpha \|L - a \otimes b\|^{1/2})^2,$$

as desired. ∎

As another consequence of Corollary 4.2.4 we obtain the following factorization result for countable families of functionals. For a proof we refer the reader once again to [18].

4.2.6. Corollary. *Suppose that* T *has property* $F^r_{\theta,\gamma}$ *for some* $0 \le \theta < \gamma \le 1$. *Then, for each* $\varepsilon > 0$, *there is a constant* $R = R(\varepsilon, \theta, \gamma) > 0$ *such that, for each sequence* $(L_k)_{k \ge 1}$ *in* $Q(D)$ *and each vector* $a \in H$, *there are elements* $x, y_k \in H$ $(k \ge 1)$ *satisfying*

$$L_k = x \otimes y_k \quad (k \ge 1), \quad \|x - a\| < \varepsilon, \quad \|y_k\| \le R k^2 \|L_k\| \quad (k \ge 1). \ ∎$$

4.3. Analytic factors and reflexivity. Let $\mathcal{A} \subset L(H)$ denote an arbitrary family of operators on H. We write $\mathrm{Lat}(\mathcal{A})$ for the set of all closed subspaces $M \subset H$ satisfying $AM \subset M$ for every $A \in \mathcal{A}$, and define

$$\mathrm{AlgLat}(\mathcal{A}) = \{C \in L(H) : \mathrm{Lat}(C) \supset \mathrm{Lat}(\mathcal{A})\},$$

which is easily seen to be a WOT-closed subalgebra of $L(H)$ containing $\mathcal{A}$ and the identity operator on H. An operator algebra $\mathcal{A} \subset L(H)$ is called *reflexive* if $\mathrm{AlgLat}(\mathcal{A}) = \mathcal{A}$. If even each WOT-closed subalgebra $\mathcal{B} \subset \mathcal{A}$ containing the identity is reflexive, then $\mathcal{A}$ is said to be *super-reflexive*.

There is a well-known argument scheme which allows one to prove that a contractive and weak* continuous representation $\Phi : H^\infty(\Omega) \to L(H)$ over a sufficiently nice domain $\Omega \subset \mathbb{C}^n$ has reflexive range $\Phi(H^\infty(\Omega))$ when Φ satisfies certain "natural" factorization conditions (see Eschmeier [16] for the case of the unit ball, and [17] for the case of the unit polydisc). Our aim is to show that a localization of Eschmeier's ideas leads to the corresponding reflexivity results on an arbitrary relatively compact open subset $\Omega \Subset X$ of an arbitrary Stein manifold X.

For the rest of this section, we fix a Stein manifold X and a relatively compact open subset $\Omega \Subset X$. Unless otherwise stated, Ω is not assumed to be connected. As always, we equip $H^\infty(\Omega)$ with its canonical dual algebra structure (see Section 2.1), and we write $Q(\Omega) = L^1(\lambda)/{}^\perp H^\infty(\Omega)$ for the Banach space of all weak* continuous linear forms on $H^\infty(\Omega)$.

Let $\Phi : H^\infty(\Omega) \to L(H)$ be an arbitrary weak* continuous representation. Recall that the dual representation $\Phi^* : H^\infty(\Omega^*) \to L(H)$ is defined by $\Phi^*(f) = \Phi(f_*)^*$ ($f \in H^\infty(\Omega^*)$). Furthermore, if $M \subset H$ is a Φ-invariant subspace, then we write $\Phi|M : H^\infty(\Omega) \to L(M)$, $f \mapsto \Phi(f)|M$, for the induced representation on M. For abbreviation, we set $\mathrm{AlgLat}(\Phi) = \mathrm{AlgLat}(\Phi(H^\infty(\Omega)))$.

Given $x, y \in H$, we define a linear functional $x \otimes y \in Q(\Omega)$ by

$$x \otimes y : H^\infty(\Omega) \to \mathbb{C}, \quad f \mapsto \langle \Phi(f)x, y\rangle.$$

In order to keep the notation simple we usually avoid an extra symbol referring to the underlying representation. If necessary, we indicate the chosen representation by a suitable subscript, e.g. "$\otimes_T$" in the case where $\Phi = \Phi_T$ for an absolutely continuous von Neumann n-tuple $T \in L(H)^n$ over D.

Let p, q be any cardinal numbers with $1 \le p, q \le \aleph_0$. The representation Φ is said to have the factorization property $(\mathbb{A}_{p,q})$ if, for each system of functionals $L_{ij} \in Q(\Omega)$ ($0 \le i < p$, $0 \le j < q$), there exist systems of vectors $(x_i)_{0\le i<p}$ and $(y_j)_{0\le j<q}$ in H solving the equations

$$L_{ij} = x_i \otimes y_j \quad (0 \le i < p,\ 0 \le j < q).$$

Instead of $(\mathbb{A}_{p,p})$ we simply write $(\mathbb{A}_p)$. We now introduce some strengthened versions of the properties $(\mathbb{A}_1)$ and $(\mathbb{A}_{1,\aleph_0})$ including norm estimates for the factors.

4.3.1. Definition. (a) The representation Φ is said to have the factorization property $(\mathbb{A}_1)^+$ if there exists a constant $R > 0$ such that, for any given vectors $a, b \in H$ and any given functional $L \in Q(\Omega)$, there are vectors $x, y \in H$ satisfying

$$L = x \otimes y, \quad \|x - a\| \le R\|L - a \otimes b\|^{1/2}, \quad \|y\| \le R(\|L - a \otimes b\|^{1/2} + \|b\|).$$

(b) The representation Φ is said to have property $(\mathbb{A}_{1,\aleph_0})^+$ if, for every $\varepsilon > 0$, every $a \in H$ and every sequence $(L_k)_{k\ge1}$ in $Q(\Omega)$, there are vectors $x \in H$, $y_k \in H$ ($k \ge 1$) and constants $R > 0$, $c \in \mathbb{N}_0$ such that

$$L_k = x \otimes y_k \quad (k \ge 1), \qquad \|x - a\| < \varepsilon, \qquad \|y_k\| \le Rk^c\|L_k\| \quad (k \ge 1).$$

For $x \in H$, the smallest Φ-invariant subspace of H containing x will be denoted by

$$H_x = H_x(\Phi) = \overline{\{\Phi(f)x : f \in H^\infty(\Omega)\}}.$$

4.3.2. Definition. A vector $x \in H$ is called an *analytic factor* of Φ if there exists a conjugate analytic function $e : \Omega \to H_x$ such that

$$x \otimes e(\lambda) = \mathcal{E}_\lambda \qquad (\lambda \in \Omega).$$

Here, as usual, $\mathcal{E}_\lambda : H^\infty(\Omega) \to \mathbb{C}$ denotes the point evaluation $f \mapsto f(\lambda)$.

It turns out that properties $(\mathbb{A}_1)^+$ and $(\mathbb{A}_{1,\aleph_0})^+$ as well as the concept of analytic factors play a crucial role in the theory of reflexivity. The following lemma is a collection of some basic facts about analytic factors.

4.3.3. Lemma. *Let $x \in H$ be an analytic factor of Φ via $e : \Omega \to H_x$.*

(a) *The function $e : \Omega \to H_x$ is uniquely determined by x.*
(b) *We have $\langle x, e(\cdot)\rangle \equiv 1$ on Ω, implying that e has no zeros.*
(c) *The set $N = \{u \in H_x : \langle u, e(\cdot)\rangle \equiv 0$ on $\Omega\}$ has empty interior in H_x.*
(d) *For every $f \in H^\infty(\Omega)$ and every $\lambda \in \Omega$, we have the identity*

$$(\Phi(f)|H_x)^* e(\lambda) = f(\lambda)^* e(\lambda);$$

in particular $\mathbb{C} \cdot e(\lambda) \in \mathrm{Lat}((\Phi|H_x)^*)$ $(\lambda \in \Omega)$.
(e) *For $f = (f_1, \dots, f_l) \in H^\infty(\Omega)^l$ $(l \geq 1)$, the Harte spectrum satisfies*

$$\sigma^{\mathrm{H}}(\Phi(f_1)|H_x, \dots, \Phi(f_l)|H_x) \supset f(\Omega).$$

(f) *The representation $\Phi|H_x$ satisfies $\|\Phi(f)|H_x\| \geq \|f\|_{\infty,\Omega}$ $(f \in H^\infty(\Omega))$.*

Proof. To prove (a) suppose that both $e : \Omega \to H_x$ and $\widetilde{e} : \Omega \to H_x$ turn x into an analytic factor. The uniqueness assertion then follows from the observation that

$$\langle \Phi(f)x, e(\lambda)\rangle = f(\lambda) = \langle \Phi(f)x, \widetilde{e}(\lambda)\rangle \quad (f \in H^\infty(\Omega),\ \lambda \in \Omega).$$

Part (b) is obvious. If the set N defined in part (c) had non-empty interior, the linear map $H_x \to \mathbb{C}^\Omega$, $u \mapsto \langle u, e(\cdot)\rangle$, would vanish on some open ball. Therefore it would vanish identically, contradicting (b). The relation

$$\langle \Phi(g)x, (\Phi(f)|H_x)^* e(\lambda)\rangle = f(\lambda)\langle \Phi(g)x, e(\lambda)\rangle,$$

valid for all $f, g \in H^\infty(\Omega)$ and $\lambda \in \Omega$, immediately yields (d). In particular,

$$0 \neq e(\lambda) \in \bigcap_{i=1}^{l} \ker(f_i(\lambda) - \Phi(f_i)|H_x)^*$$

for every finite system $(f_1, \dots, f_l) \in H^\infty(\Omega)^l$. This proves (e), since $\sigma^{\mathrm{H}}(S^*) = \sigma^{\mathrm{H}}(S)^*$ holds for each commuting n-tuple $S \in L(H_x)^l$. Finally, if we take $l = 1$, a spectral radius argument shows that $\|\Phi(f)|H_x\| \geq \varrho(\Phi(f)|H_x) \geq \|f\|_{\infty,\Omega}$ for every $f \in H^\infty(\Omega)$. ∎

4.3.4. Proposition. *Let $x \in H$ be an analytic factor of Φ via $e : \Omega \to H_x$ and let $u \in H_x$ be an element with the property that the function $\langle u, e(\cdot)\rangle$ does not vanish identically on any connected component of Ω. If $P_u \in L(H)$ denotes the orthogonal projection onto H_u and Z denotes the set of zeros of the function $\langle u, e(\cdot)\rangle$, then the map*

$$\Omega \setminus Z \to H_u, \qquad \lambda \mapsto P_u e(\lambda)/\langle e(\lambda), u\rangle,$$

possesses a unique conjugate analytic extension $e_u : \Omega \to H_u$. Moreover, u is an analytic factor of Φ via e_u.

Proof. Using part (d) of the previous lemma, one easily checks that the continuous linear map

$$i : H \to \mathcal{O}(\Omega), \qquad h \mapsto \langle h, e(\cdot)\rangle,$$

satisfies the relation

$$i(\Phi(f)h) = f \cdot i(h) \quad (f \in H^\infty(\Omega),\ h \in H_x).$$

By hypothesis, $Z \subset \Omega$ is a thin subset, and for each $h \in H_u$, the map

$$g_h : \Omega \setminus Z \to \mathbb{C}, \qquad \lambda \mapsto \frac{\langle h, e(\lambda)\rangle}{\langle u, e(\lambda)\rangle} = \frac{i(h)(\lambda)}{i(u)(\lambda)},$$

is analytic. The functional equation satisfied by i guarantees that, for $f \in H^\infty(\Omega)$, we have $i(\Phi(f)u) \in i(u) \cdot \mathcal{O}(\Omega)$. Since the right-hand side is closed (as a principal ideal in $\mathcal{O}(\Omega)$) we deduce that $i(h)/i(u) \in \mathcal{O}(\Omega)$ is analytic whenever $h \in H_u$. Therefore, for every $h \in H_u$, the analytic function $g_h \in \mathcal{O}(\Omega \setminus Z)$ defined above has a unique analytic extension $\widehat{g}_h \in \mathcal{O}(\Omega)$. Clearly, the assignment $h \mapsto g_h$ is linear, and hence so is the map

$$i_u : H_u \to \mathcal{O}(\Omega), \qquad h \mapsto \widehat{g}_h.$$

The continuity of i_u is an easy application of the closed graph theorem. For each $\lambda \in \Omega$, the composition

$$H_u \xrightarrow{i_u} \mathcal{O}(\Omega) \xrightarrow{\mathcal{E}_\lambda} \mathbb{C}, \quad h \mapsto \widehat{g}_h(\lambda),$$

is a continuous linear form on H_u, so we find a vector $e_u(\lambda) \in H_u$ such that $\langle h, e_u(\lambda)\rangle = \widehat{g}_h(\lambda)$ $(h \in H_u)$. Clearly, the map $e_u : \Omega \to H_u$ defined in this way is weakly (and hence strongly) conjugate analytic. For $\lambda \in \Omega \setminus Z$, we have the formula

$$\langle h, e_u(\lambda)\rangle = \frac{\langle h, e(\lambda)\rangle}{\langle u, e(\lambda)\rangle} = \left\langle h, \frac{P_u e(\lambda)}{\langle e(\lambda), u\rangle}\right\rangle \qquad (h \in H_u),$$

which proves the asserted extension property of e_u. It remains to check that e_u turns u into an analytic factor of Φ. Since we have

$$(u \otimes e_u(\lambda))(f) = \left\langle \Phi(f)u, \frac{P_u e(\lambda)}{\langle e(\lambda), u\rangle}\right\rangle = \frac{i(\Phi(f)u)(\lambda)}{i(u)(\lambda)} = f(\lambda)$$

for $\lambda \in \Omega \setminus Z$ and $f \in H^\infty(\Omega)$, the desired relation $u \otimes e_u(\lambda) = \mathcal{E}_\lambda$ holds on the dense subset $\Omega \setminus Z \subset \Omega$, and hence on all of Ω. ∎

Let $x \in H$ be an analytic factor of Φ and let $C \in \operatorname{AlgLat}(\Phi)$. Then $(C|H_x)^* \in \operatorname{AlgLat}((\Phi|H_x)^*)$, and Lemma 4.3.3(d) implies that, for each $\lambda \in \Omega$, there exists a unique complex number $g_{x,C}(\lambda) \in \mathbb{C}$ satisfying

$$(C|H_x)^* e(\lambda) = g_{x,C}(\lambda)^* e(\lambda).$$

The function $g_{x,C} : \Omega \to \mathbb{C}$ defined in this way has the following properties:

(a) From the very definition of $g_{x,C}$ we deduce the formula

$$\langle Cu, e(\lambda)\rangle = g_{x,C}(\lambda)\langle u, e(\lambda)\rangle \qquad (u \in H_x,\ \lambda \in \Omega),$$

which implies in particular that $g_{x,C} = \langle Cx, e(\cdot)\rangle$ is analytic.

(b) Again from the definition of $g_{x,C}$ we infer that

$$|g_{x,C}(\lambda)| \cdot \|e(\lambda)\| \le \|C\| \cdot \|e(\lambda)\| \qquad (\lambda \in \Omega),$$

and since e has no zeros, this together with (a) implies that

$$g_{x,C} \in H^\infty(\Omega) \quad \text{and} \quad \|g_{x,C}\|_{\infty,\Omega} \le \|C\|.$$

4.3.5. LEMMA. *Suppose that $x \in H$ is an analytic factor of Φ. Then the mapping $\Psi_x : \mathrm{AlgLat}(\Phi) \to H^\infty(\Omega)$, $C \mapsto g_{x,C}$, is a contractive and unital algebra homomorphism satisfying $\Psi_x \circ \Phi = I_{H^\infty(\Omega)}$.*

Proof. Using the remarks preceding the lemma we deduce that

$$g_{x,CD+\lambda E} = \langle (CD+\lambda E)x, e(\cdot)\rangle = g_{x,C}\langle Dx, e(\cdot)\rangle + \lambda g_{x,E} = g_{x,C}g_{x,D} + \lambda g_{x,E}$$

for any $\lambda \in \mathbb{C}$ and $C, D, E \in \mathrm{AlgLat}(\Phi)$. Moreover, $g_{x,\Phi(f)} = \langle \Phi(f)x, e(\cdot)\rangle = f$ for every $f \in H^\infty(\Omega)$. ∎

To prove the following proposition we make the additional assumption that Ω is connected. At the end of this section the result obtained in this way will be applied to the connected components of a not necessarily connected open set $\Omega \Subset X$.

4.3.6. PROPOSITION. *Suppose that $x \in H$ is an analytic factor of Φ and that, in addition, $\Phi|H_x$ possesses property $(\mathbb{A}_1)^+$. If Ω is connected, then, for every $C \in \mathrm{AlgLat}(\Phi)$, the function $g = \Psi_x(C)$ is the unique element $g \in H^\infty(\Omega)$ satisfying*

$$\Phi(g)|H_x = C|H_x.$$

Proof. The uniqueness assertion is an immediate consequence of the injectivity of $\Phi|H_x$ (see Lemma 4.3.3(f)). Let $e : \Omega \to H_x$ denote the unique conjugate analytic function turning x into an analytic factor of Φ, that is, $x \otimes e(\lambda) = \mathcal{E}_\lambda$ ($\lambda \in \Omega$). Since the set $N = \{u \in H_x : \langle u, e(\cdot)\rangle \equiv 0\}$ has empty interior (Lemma 4.3.3(c)), it suffices to check that

$$\langle \Phi(g)u, v\rangle = \langle Cu, v\rangle \quad (u \in H_x \setminus N,\ v \in H_x).$$

This will be done in three steps, imposing additional restrictions on u and v. First note that, according to Proposition 4.3.4, $u \in H_x \setminus N$ is an analytic factor of Φ via the unique conjugate analytic extension $e_u : \Omega \to H_u$ of the map $\Omega \setminus Z \to H_u$, $z \mapsto P_u e(z)/\langle e(z), u\rangle$, where Z denotes the zero set of the denominator. (Here we used the hypothesis that Ω is connected.)

(1) $\underline{u \otimes v = \mathcal{E}_\lambda \text{ for some } \lambda \in \Omega}$: By assumption, $u \otimes v = \mathcal{E}_\lambda = u \otimes e_u(\lambda)$, and hence the linear functionals $\langle \cdot, v\rangle|H_u$ and $\langle \cdot, e_u(\lambda)\rangle|H_u$ coincide. Since $C \in \mathrm{AlgLat}(\Phi)$ leaves H_u invariant and $Z \subset \Omega$ is thin (Ω is connected), we conclude that

$$\langle Cu, v\rangle = \langle Cu, e_u(\lambda)\rangle = \lim_{\substack{z\to\lambda\\ z\notin Z}} \left\langle Cu, \frac{P_u e(z)}{\langle e(z), u\rangle}\right\rangle = \lim_{\substack{z\to\lambda\\ z\notin Z}} \frac{\langle Cu, e(z)\rangle}{\langle u, e(z)\rangle}.$$

Using remark (a) following Proposition 4.3.4 we obtain for $g = g_{x,C} = \Psi_x(C)$ the desired relation

$$\langle Cu, v\rangle = g(\lambda) = u \otimes v(g) = \langle \Phi(g)u, v\rangle.$$

(2) $\underline{u \otimes v \in \mathrm{LH}\{\mathcal{E}_\lambda : \lambda \in \Omega\}}$: Fix a representation $u \otimes v = \sum_{i=1}^r t_i \mathcal{E}_{\lambda_i}$ with $t_i \in \mathbb{C}\setminus\{0\}$ and pairwise distinct $\lambda_i \in \Omega$ ($1 \le i \le r$). Since the Stein manifold X can be embedded as a closed complex submanifold into $\mathbb{C}^m$ with $m = 2\dim X + 1$ (see Theorem 5.3.9 in [26]), there exists an injective bounded holomorphic map $F = (F_1, \dots, F_m) \in H^\infty(\Omega)^m$. Choose polynomials $p_1, \dots, p_r \in \mathbb{C}[z_1, \dots, z_m]$ satisfying

$$p_i(F(\lambda_j)) = \delta_{ij}/t_i \quad (i, j = 1, \dots, r).$$

Because of $p_i \circ F = p_i(F_1, \dots, F_m) \in H^\infty(\Omega)$ we may define

$$v_i = (\Phi(p_i \circ F)|H_x)^* v \quad (i = 1, \dots, r).$$

Our choices guarantee that, for every $f \in H^\infty(\Omega)$ and $1 \le i \le r$, we have

$$u \otimes v_i(f) = u \otimes v(p_i \circ F \cdot f) = \sum_{j=1}^{r} t_j f(\lambda_j)\delta_{ij}/t_i = \mathcal{E}_{\lambda_i}(f),$$

implying the identity $u \otimes v = \sum_{i=1}^{r} t_i u \otimes v_i = u \otimes (\sum_{i=1}^{r} t_i^* v_i)$. Using the fact that part (1) can be applied to handle the sum in the middle, we deduce that

$$\langle \Phi(g)u, v\rangle = \sum_{i=1}^{r} t_i \langle \Phi(g)u, v_i\rangle = \sum_{i=1}^{r} t_i \langle Cu, v_i\rangle = \langle Cu, v\rangle.$$

To finish the proof we consider the general case:

(3) $\underline{u \otimes v \in \overline{\mathrm{LH}}\{\mathcal{E}_\lambda : \lambda \in \Omega\} = Q(\Omega)}$: Choose a sequence $(L_k)_{k\ge 1}$ in $\mathrm{LH}\{\mathcal{E}_\lambda : \lambda \in \Omega\}$ such that $L_k \to u \otimes v$ in $Q(\Omega)$ as $k \to \infty$. Using the fact that $\Phi|H_x$ has property $(\mathbb{A}_1)^+$, we obtain factorizations $L_k = u_k \otimes v_k$ with $u_k, v_k \in H_x$ $(k \ge 1)$ such that $u_k \to u$ as $k \to \infty$ and such that (v_k) is bounded. Since u belongs to the open set $H_x \setminus N$ and since closed balls in H_x are weakly compact, we can achieve that (possibly after passing to a suitable subsequence) $u_k \in H_x \setminus N$ $(k \ge 0)$ and that $v_k \to \widetilde{v} \in H_x$ (weakly) as $k \to \infty$. Then we have

$$u \otimes v(f) = \lim \langle \Phi(f)u_k, v_k\rangle = \langle \Phi(f)u, \widetilde{v}\rangle = u \otimes \widetilde{v}(f) \quad (f \in H^\infty(\Omega)).$$

Using this fact as well as the result of part (2) we finally obtain

$$\langle Cu, v\rangle = \langle Cu, \widetilde{v}\rangle = \lim \langle Cu_k, v_k\rangle = \lim \langle \Phi(g)u_k, v_k\rangle = u \otimes v(g) = \langle \Phi(g)u, v\rangle,$$

as desired. ∎

As an application of the preceding proposition we are able to prove the reflexivity of $\Phi(H^\infty(\Omega))$ under the assumption that a certain class of analytic factors is a dense subset of H.

4.3.7. Proposition. *Let $\Omega \Subset X$ be a relatively compact and connected open subset of a Stein manifold X and let $\Phi : H^\infty(\Omega) \to L(H)$ be a weak* continuous representation. If the set*

$$\mathcal{F}^+ = \{x \in H : x \text{ is an analytic factor of } \Phi \text{ and } \Phi|H_x \text{ satisfies } (\mathbb{A}_1)^+\}$$

is dense in H, then the operator algebra $\Phi(H^\infty(\Omega))$ is reflexive.

Proof. Fix $C \in \mathrm{AlgLat}(\Phi)$. Proposition 4.3.6 guarantees that, for every $x \in \mathcal{F}^+$, there exists a unique function $g_x \in H^\infty(\Omega)$ satisfying

$$\Phi(g_x)|H_x = C|H_x, \quad \|g_x\|_{\infty,\Omega} \le \|C\|.$$

Our aim is to show that C belongs to $\Phi(H^\infty(\Omega))$. Since $\mathcal{F}^+ \subset H$ is dense (by hypothesis), it obviously suffices to check that

$$g_x = g_y \quad \text{whenever } x, y \in \mathcal{F}^+.$$

To prove this claim, we first show that, for every $v \in H$, there exists a function $g_v \in H^\infty(\Omega)$ such that $Cv = \Phi(g_v)v$.

Starting with an arbitrary vector $v \in H$, we choose a sequence (x_k) in $\mathcal{F}^+$ converging to v. Since $\|g_{x_k}\|_{\infty,\Omega} \le \|C\|$, we may assume that (g_{x_k}) converges to a function $g_v \in H^\infty(\Omega)$ (with respect to the weak* topology). From this we deduce that

$$\langle \Phi(g_v)v, y\rangle = \lim\langle \Phi(g_{x_k})x_k, y\rangle = \lim\langle Cx_k, y\rangle = \langle Cv, y\rangle \quad (y \in H),$$

and hence $\Phi(g_v)v = Cv$, as desired.

Now, given arbitrary vectors $x, y \in \mathcal{F}^+$ we are able to choose a function $h \in H^\infty(\Omega)$ satisfying $C(x+y) = \Phi(h)(x+y)$. Obviously, this implies

$$\Phi(g_x - h)x = \Phi(h - g_y)y.$$

If it happens that $g_x = h$, then $0 = \Phi(f)\Phi(h-g_y)y = \Phi(h-g_y)\Phi(f)y$ for every $f \in H^\infty(\Omega)$. So $\Phi(h-g_y)|H_y = 0$, implying that $g_y = h = g_x$ (use Lemma 4.3.3(f)). For reasons of symmetry, we may therefore suppose in the following that both $g_x \ne h$ and $g_y \ne h$. Let $e : \Omega \to H_x$ denote the unique conjugate analytic function turning x into an analytic factor of Φ. Since $\langle \Phi(g_x - h)x, e(\cdot)\rangle = g_x - h \not\equiv 0$ and Ω is assumed to be connected, Proposition 4.3.4 guarantees that $u = \Phi(g_x - h)x \in H_x$ is an analytic factor of Φ. Furthermore, we can find a function $g_u \in H^\infty(\Omega)$ satisfying

$$\Phi(g_u)u = Cu.$$

Note that $H_u \subset H_x$ and $x \in \mathcal{F}^+$ together imply that $\Phi(g_u)u = Cu = \Phi(g_x)u$, and hence

$$\Phi(g_u)|H_u = \Phi(g_x)|H_u.$$

Since $\Phi|H_u$ is injective (see Lemma 4.3.3(f)), this implies $g_u = g_x$. Using the fact that $u = \Phi(h-g_y)y \in H_y$, the same arguments yield the identity $g_u = g_y$, and the proof is complete. ■

For the remainder of this section, let $\Omega \Subset X$ be a not necessarily connected, relatively compact open subset of a Stein manifold X. The following result guarantees that if the representation Φ has property $(\mathbb{A}_{1,\aleph_0})^+$, then it has a rich supply of analytic factors.

4.3.8. Proposition. *Suppose that Φ has property $(\mathbb{A}_{1,\aleph_0})^+$. Then*

$$\{x \in H : x \text{ is an analytic factor of } \Phi\} \subset H$$

is a dense subset of H.

Proof. Since the topology of Ω has a countable basis, every open cover of Ω has a countable subcover. Thus we are able to fix a countable holomorphic atlas for Ω consisting of charts of the form

$$\Omega \supset U_j \xrightarrow{\varphi_j} P \subset \mathbb{C}^\nu \quad (j \in \mathbb{N}_0)$$

where $\nu = \dim\Omega$ and P denotes the open unit polydisc in $\mathbb{C}^\nu$. For each multi-index $k \in \mathbb{N}_0^\nu$ and $j \in \mathbb{N}_0$ we define a linear form on $H^\infty(\Omega)$ by

$$\mathcal{E}_j^{(k)} : H^\infty(\Omega) \to \mathbb{C}, \qquad f \mapsto (1/k!)(\partial^k(f \circ \varphi_j^{-1}))(0).$$

Using the characterization of weak* zero sequences in $H^\infty(\Omega)$ presented in the second chapter one immediately checks that all these functionals are weak* continuous. For

$f \in H^\infty(\Omega)$, $j \in \mathbb{N}_0$ and $k \in \mathbb{N}_0^\nu$, the Cauchy estimates yield

$$|(\partial^k(f \circ \varphi_j^{-1}))(0)| \le k!\|f \circ \varphi_j^{-1}\|_{\infty,P} \le k!\|f\|_{\infty,\Omega},$$

and therefore $\|\mathcal{E}_j^{(k)}\| \le 1$ ($j \in \mathbb{N}_0, k \in \mathbb{N}_0^\nu$). In a first step, we prove the following:

(1) For every $a \in H$ and every $\varepsilon > 0$, there exist vectors $x \in H$ and $y_j^{(k)} \in H$ ($j \in \mathbb{N}_0$, $k \in \mathbb{N}_0^\nu$) satisfying the following requirements:

(a) $\|x - a\| < \varepsilon$.
(b) $x \otimes y_j^{(k)} = \mathcal{E}_j^{(k)}$ ($j \in \mathbb{N}_0$, $k \in \mathbb{N}_0^\nu$).
(c) For each $j \in \mathbb{N}_0$, the power series $f_j(z) = \sum_{k \in \mathbb{N}_0^\nu} y_j^{(k)} z^k$ converges on P.

To prove this, we fix an enumeration $(L_m)_{m \ge 1}$ of the set

$$\mathcal{E} = \{\mathcal{E}_j^{(k)} : j \in \mathbb{N}_0,\, k \in \mathbb{N}_0^\nu\}$$

as follows: We write $\mathcal{E}$ as the disjoint union $\mathcal{E} = \bigcup_{r=0}^\infty K_r$ of the blocks

$$K_r = \Big(\bigcup_{l=0}^{r-1} \{\mathcal{E}_l^{(k)} : |k| = r\}\Big) \cup \{\mathcal{E}_r^{(k)} : |k| \le r\} \qquad (r \ge 0).$$

Obviously, the length of the union of the first r blocks satisfies

$$\sharp(K_0 \cup \dots \cup K_r) \le (r+1)\sharp\{k \in \mathbb{N}_0^\nu : |k| \le r\} \le (r+1)^{\nu+1}.$$

Therefore we can find an enumeration $(L_m)_{m \ge 1}$ such that, for $N, i \in \mathbb{N}_0$ and $k \in \mathbb{N}_0^\nu$ with $|k| = N + i$, the functional $\mathcal{E}_N^{(k)}$ appears as L_m with $m \le (N + i + 1)^{\nu+1}$.

By hypothesis, Φ has property $(\mathbb{A}_{1,\aleph_0})^+$, which allows us to choose vectors $x \in H$, $y_i^{(k)} \in H$ ($i \in \mathbb{N}_0$, $k \in \mathbb{N}_0^\nu$) such that

$$\|x - a\| < \varepsilon, \qquad x \otimes y_i^{(k)} = \mathcal{E}_i^{(k)} \quad (i \in \mathbb{N}_0,\, k \in \mathbb{N}_0^\nu)$$

and

$$\|y_N^{(k)}\| \le R(N + i + 1)^{(\nu+1)c} \qquad (N, i \in \mathbb{N}_0,\, k \in \mathbb{N}_0^\nu,\, |k| = N + i)$$

with suitable constants $R > 0$, $c \in \mathbb{N}_0$. To conclude the proof of part (1), we have to check the convergence condition (c). To this end, fix arbitrary numbers $N, i \in \mathbb{N}_0$ and a multi-index $k \in \mathbb{N}_0^\nu$ with $|k| = N + i$. For $z \in P$ and $\varrho = \max_{j=1,\dots,\nu} |z_j| < 1$, we have the estimate

$$\|y_N^{(k)} z^k\| = \|y_N^{(k)}\| \prod_{j=1}^\nu |z_j|^{k_j} \le R(N+i+1)^{(\nu+1)c} \varrho^{N+i} \le R[(N+i+1)^{(\nu+1)c/(N+i)} \varrho]^{N+i}.$$

Since the first term between the brackets on the right-hand side tends to 1 as $i \to \infty$, and since $\varrho < 1$, the right-hand side tends to zero as $i \to \infty$. Therefore, $\sup_{k \in \mathbb{N}_0^\nu} \|y_N^{(k)} z^k\| < \infty$ ($z \in P$), proving (c).

(2) Given $a \in H$ and $\varepsilon > 0$, we fix vectors $x \in H$ and $y_j^{(k)} \in H$ ($j \in \mathbb{N}_0$, $k \in \mathbb{N}_0^\nu$) as in the statement of part (1) of the proof. Letting $P_x \in L(H)$ denote the orthogonal projection onto H_x, the assertion of the proposition follows from the following fact: The assignment

$$e : \Omega \to H_x, \qquad \lambda \mapsto P_x f_j(\overline{\varphi_j(\lambda)}) \quad (j \in \mathbb{N}_0 \text{ such that } \lambda \in U_j)$$

is a well defined conjugate analytic function satisfying $x \otimes e(\lambda) = \mathcal{E}_\lambda$ ($\lambda \in \Omega$).

First, we consider the family $e_j : U_j \to H_x$ $(j \in \mathbb{N}_0)$ of conjugate analytic functions given by

$$e_j(\lambda) = P_x f_j(\overline{\varphi_j(\lambda)}) \quad (\lambda \in U_j).$$

For each $f \in H^\infty(\Omega)$ and every $\lambda \in U_j$, we have the identity

$$\begin{aligned}(x \otimes e_j(\lambda))(f) &= (x \otimes P_x f_j(\overline{\varphi_j(\lambda)}))(f) = \sum_{k \in \mathbb{N}_0^\nu} (x \otimes y_j^{(k)})(f) \cdot (\varphi_j(\lambda))^k \\ &= \sum_{k \in \mathbb{N}_0^\nu} (1/k!)(\partial^k (f \circ \varphi_j^{-1}))(0) \cdot (\varphi_j(\lambda))^k = (f \circ \varphi_j^{-1})(\varphi_j(\lambda)) = \mathcal{E}_\lambda(f).\end{aligned}$$

In particular, if $\lambda \in U_i \cap U_j$ $(i, j \in \mathbb{N}_0)$, then

$$\langle \Phi(f)x, e_i(\lambda)\rangle = \mathcal{E}_\lambda(f) = \langle \Phi(f)x, e_j(\lambda)\rangle \quad (f \in H^\infty(\Omega)).$$

So there is a unique conjugate analytic function $e : \Omega \to H_x$ satisfying $e|U_j = e_j$ $(j \in \mathbb{N}_0)$ and $x \otimes e(\lambda) = \mathcal{E}_\lambda$ for every $\lambda \in \Omega$, as desired. ■

Before turning to the proof of the announced reflexivity result (see Theorem 4.3.9 below) recall that any isometric and weak* continuous representation $\Phi : H^\infty(\Omega) \to L(H)$ can be decomposed into "pairwise orthogonal components" corresponding to the components of the underlying open set Ω. To see this, write Ω as the disjoint union of its connected components

$$\Omega = \bigcup (\Omega_i : 1 \le i < N) \quad (N \in \mathbb{N} \cup \{\infty\})$$

and let $\chi_i \in H^\infty(\Omega)$ denote the characteristic function of Ω_i $(1 \le i < N)$. Since Φ is contractive, the operators $P_i = \Phi(\chi_i) \in L(H)$ are orthogonal projections possessing pairwise orthogonal ranges $H_i = P_i H$ which are reducing for $\Phi(H^\infty(\Omega))$. Note that, since $1 = w^*\text{-}\sum_{1 \le i < N} \chi_i$ on Ω and Φ is weak* continuous, we have $1_H = \text{WOT-}\sum_{1 \le i < N} P_i$, establishing the orthogonal direct sum decomposition

$$H = \bigoplus_{1 \le i < N} H_i.$$

Given $f \in H^\infty(\Omega_i)$, we write f_i for the trivial extension of f onto all of Ω. The formula

$$\Phi_i : H^\infty(\Omega_i) \to L(H_i), \quad f \mapsto \Phi(f_i)|H_i,$$

then obviously yields a weak* continuous and isometric representation of $H^\infty(\Omega_i)$ (note that $\Phi(f_i)|H_j = 0$ for $j \ne i$). The orthogonal decomposition of H established above induces the direct sum decomposition

$$\Phi(H^\infty(\Omega)) = \bigoplus_{1 \le i < N} \Phi_i(H^\infty(\Omega_i)),$$

where the right-hand side should be regarded as an abbreviation for the set

$$\{C \in L(H) : H_i \in \operatorname{Lat}(C) \text{ and } C|H_i \in \Phi_i(H^\infty(\Omega_i)) \text{ for all } i\}.$$

From this we conclude that if, for each $1 \le i < N$, the operator algebra $\Phi_i(H^\infty(\Omega_i)) \subset L(H_i)$ is reflexive, then so is $\Phi(H^\infty(\Omega))$.

4.3.9. THEOREM. *Let $\Omega \Subset X$ be a relatively compact open subset of a Stein manifold X and let $\Phi : H^\infty(\Omega) \to L(H)$ be a contractive and weak* continuous representation.*

If Φ has property $(\mathbb{A}_{1,\aleph_0})^+$ and if, in addition, for each analytic factor $x \in H$ of Φ the induced representation $\Phi|H_x$ has property $(\mathbb{A}_1)^+$, then the operator algebra $\Phi(H^\infty(\Omega))$ is reflexive.

Proof. First note that, according to Proposition 4.3.8 and the hypotheses of the theorem, the set

$$\mathcal{F}^+ = \{x \in H : x \text{ is an analytic factor of } \Phi \text{ and } \Phi|H_x \text{ satisfies } (\mathbb{A}_1)^+\}$$

is dense in H. In particular, Φ is isometric (see Lemma 4.3.3(f)). By the remarks preceding the theorem it therefore remains to check that, for every connected component Ω_i of Ω, the operator algebra $\Phi_i(H^\infty(\Omega_i)) \subset L(H_i)$ is reflexive (with the notations fixed above).

Towards this end, define

$$\mathcal{F}_i^+ = \{z \in H_i : z \text{ is an analytic factor of } \Phi_i \text{ and } \Phi_i|(H_i)_z \text{ satisfies } (\mathbb{A}_1)^+\},$$

where $(H_i)_z = \overline{\{\Phi_i(f)z : f \in H^\infty(\Omega_i)\}} \subset H_i$, and assume for a moment that $P_i\mathcal{F}^+ \subset \mathcal{F}_i^+$. Since $\mathcal{F}^+$ is dense in H, this would immediately imply that $P_i\mathcal{F}^+$ and hence also the larger set $\mathcal{F}_i^+$ is dense in H_i. But then Proposition 4.3.7 allows us to conclude that $\Phi_i(H^\infty(\Omega_i))$ is reflexive, as desired. To finish the proof, it therefore suffices to check that, for $1 \le i < N$, we have $P_i\mathcal{F}^+ \subset \mathcal{F}_i^+$.

For the rest of the proof, fix an index $1 \le i < N$ as well as an arbitrary vector $x \in \mathcal{F}^+$. First note that

$$\Phi(f\chi_i)x = P_i\Phi(f)x = \Phi_i(f|\Omega_i)P_ix \qquad (f \in H^\infty(\Omega)).$$

Since the vectors on the right-hand side of the latter identity run through a dense subset of $(H_i)_{P_ix}$ as f runs through $H^\infty(\Omega)$, we deduce that

$$H_x \supset \overline{P_iH_x} = (H_i)_{P_ix}.$$

In particular, if $e : \Omega \to H_x$ denotes the unique conjugate analytic function turning x into an analytic factor of Φ, then the function P_ie takes values in the space $(H_i)_{P_ix}$. Denoting the trivial extension of a function $f \in H^\infty(\Omega_i)$ to a function on all of Ω by f_i, we obtain the identity

$$\langle \Phi_i(f)P_ix, P_ie(\lambda)\rangle = \langle \Phi(f_i)x, e(\lambda)\rangle = f(\lambda) \qquad (f \in H^\infty(\Omega_i),\ \lambda \in \Omega_i),$$

from which we deduce that P_ix is an analytic factor of Φ_i via the conjugate analytic function

$$P_ie|\Omega_i : \Omega_i \to (H_i)_{P_ix}, \qquad \lambda \mapsto P_ie(\lambda).$$

Finally, we prove that $\Phi_i|(H_i)_{P_ix}$ inherits property $(\mathbb{A}_1)^+$ from $\Phi|H_x$. Recall that property $(\mathbb{A}_1)^+$ for $\Phi|H_x$ means by definition that there exists a constant $R > 0$ such that, for any $a, b \in H_x$ and $L \in Q(\Omega)$, there exist $u, v \in H_x$ satisfying

$$L = u \otimes_x v, \qquad \|u - a\| \le R\|L - a \otimes_x b\|^{1/2}, \qquad \|v\| \le R(\|L - a \otimes_x b\|^{1/2} + \|b\|),$$

where the symbol "$\otimes_x$" means that we factorize with respect to the representation $\Phi|H_x$. To prove that $\Phi_i|(H_i)_{P_ix}$ also has property $(\mathbb{A}_1)^+$, fix $a, b \in (H_i)_{P_ix}$ and $L \in Q(\Omega_i)$. First observe that L possesses a trivial extension

$$\widetilde{L} : H^\infty(\Omega) \to \mathbb{C}, \qquad f \mapsto L(f|\Omega_i),$$

to an element $\widetilde{L} \in Q(\Omega)$. Note that, since $a, b \in (H_i)_{P_i x} \subset H_i \cap H_x$, we have

$$a \otimes_x b(f) = \langle \Phi(f)a, b\rangle = \langle \Phi_i(f|\Omega_i)a, b\rangle = a \otimes_i b(f|\Omega_i) \quad (f \in H^\infty(\Omega)).$$

Here the symbol "$\otimes_i$" refers to the representation $\Phi_i|(H_i)_{P_i x}$. Therefore

$$\|\widetilde{L} - a \otimes_x b\| = \|L - a \otimes_i b\|,$$

where on the left-hand side $\|\cdot\|$ denotes the norm on the dual space of $H^\infty(\Omega)$, whereas on the right-hand side $\|\cdot\|$ denotes the norm on the dual space of $H^\infty(\Omega_i)$.

Now fix $u, v \in H_x$ solving the factorization problem $\widetilde{L} = u \otimes_x v$ and satisfying the norm estimates from the definition of property $(\mathbb{A}_1)^+$. Then both $P_i u$ and $P_i v$ belong to $(H_i)_{P_i x}$, and the identity

$$\langle \Phi_i(f)|(H_i)_{P_i x} P_i u, P_i v\rangle = \langle \Phi(f_i)u, v\rangle = \widetilde{L}(f_i) = L(f) \quad (f \in H^\infty(\Omega_i))$$

proves that $L = P_i u \otimes_i P_i v$, as intended. Since we know that the norms of the two functionals $\widetilde{L} - a \otimes_x b \in Q(\Omega)$ and $L - a \otimes_i b \in Q(\Omega_i)$ coincide and since $a \in H_i$, we obtain the desired norm estimates

$$\|P_i u - a\| \le \|u - a\| \le R\|\widetilde{L} - a \otimes_x b\|^{1/2} = R\|L - a \otimes_i b\|^{1/2},$$
$$\|P_i v\| \le \|v\| \le R(\|L - a \otimes_i b\|^{1/2} + \|b\|).$$

This finally settles the inclusion $P_i \mathcal{F}^+ \subset \mathcal{F}_i^+$ $(1 \le i < N)$ and completes the proof. ∎

We conclude the present section with a remark on Φ-semi-invariant subspaces and compressions.

Let $\Phi : A \to L(H)$ be an arbitrary continuous representation of a commutative Banach algebra A. A subspace $Z \subset H$ is called *Φ-semi-invariant* if Z is of the form $Z = M \ominus N$ with Φ-invariant subspaces $M \subset H$ and $N \subset M$. Denoting the orthogonal projection of H onto Z by $P_Z \in L(H)$, one easily verifies the identity

$$P_Z\Phi(fg)P_Z - P_Z\Phi(f)P_Z\Phi(g)P_Z = P_Z\Phi(f)(1 - P_Z)\Phi(g)P_Z = 0 \quad (f, g \in A),$$

which is responsible for the fact that the compression of Φ onto Z, i.e. the map

$$\Phi_Z : A \to L(H), \quad f \mapsto P_Z\Phi(f)|Z,$$

is multiplicative and hence also a representation of A. Obviously, if $A = H^\infty(\Omega)$ and Φ is weak* continuous, then so is Φ_Z. The following observation will be very useful:

4.3.10. Lemma. *If $x \in H$ is an analytic factor of a weak* continuous representation $\Phi : H^\infty(\Omega) \to L(H)$ via $e : \Omega \to H_x$, then the subspace*

$$Z = \bigvee\{e(\lambda) : \lambda \in \Omega\} \subset H$$

is Φ-semi-invariant, and $P_Z x$ is an analytic factor of Φ_Z. Moreover, Φ_Z is of class $C_{\cdot 0}$ (i.e. Φ_Z^* is sequentially weak-SOT continuous) and $\sigma^{\mathrm{H}}(\Phi_Z(f_1), \dots, \Phi_Z(f_m)) \supset f(\Omega)$ for every $f = (f_1, \dots, f_m) \in H^\infty(\Omega)^m$, where σ^{H} denotes the Harte spectrum.*

Proof. According to Lemma 4.3.3(d), $Z \subset H_x$ is $(\Phi|H_x)^*$-invariant, and hence $Z = H_x \ominus (H_x \ominus Z)$ is Φ-semi-invariant. Since $\langle P_Z x, e(\cdot)\rangle = 1$, an application of Proposition 4.3.4 to the vector $u = P_Z x$ yields the formula

$$\mathcal{E}_\lambda(f) = \langle \Phi(f)u, e_u(\lambda)\rangle = \langle \Phi_Z(f)u, e(\lambda)\rangle \quad (\lambda \in \Omega,\ f \in H^\infty(\Omega)),$$

where e_u is defined as in the cited lemma. This proves that $u = P_Z x$ is an analytic factor of Φ_Z via the conjugate analytic function $\Omega \to Z_u$, $\lambda \mapsto P_{Z_u} e(\lambda)$.

Note that, for $f \in H^\infty(\Omega)$, we have $\Phi_Z(f)^* = (P_Z \Phi(f)|Z)^* = P_Z(\Phi(f)|H_x)^*|Z$. Lemma 4.3.3(d) therefore implies the relation

$$\Phi_Z(f)^* e(\lambda) = f(\lambda)^* e(\lambda) \quad (\lambda \in \Omega,\ f \in H^\infty(\Omega)),$$

which can be used in an obvious way to complete the proof. ∎

4.4. A characterization of the class $\mathbb{A}_{1,\aleph_0}$ and reflexivity. Let $\mathcal{A} \subset L(H)$ be a weak* closed subalgebra. Then $Q_{\mathcal{A}} = C^1(H)/{}^\perp\mathcal{A}$ can be identified with the Banach space of all weak* continuous linear forms on $\mathcal{A}$. Given $x, y \in H$ we obtain an element $\langle x \otimes y \rangle \in Q_{\mathcal{A}}$ by setting

$$\langle x \otimes y \rangle : \mathcal{A} \to \mathbb{C}, \quad A \mapsto \langle Ax, y \rangle = \operatorname{tr}(A \circ \langle \cdot, y \rangle x).$$

Exactly as in the context of representations (see page 49) one defines the factorization properties $(\mathbb{A}_{p,q})$ for the operator algebra $\mathcal{A}$. If $\Phi : H^\infty(D) \to L(H)$ is a weak* continuous and isometric functional calculus for a commuting n-tuple $T \in L(H)^n$, then one easily checks that Φ has property $(\mathbb{A}_{p,q})$ if and only if the dual algebra $\mathcal{H}_T(\overline{D})$ generated by T over $\overline{D}$ has property $(\mathbb{A}_{p,q})$. Using a well known result concerning hereditarily reflexive algebras due to Loginov and Sulman (see Theorem 2.3 in [29]) one proves the following lemma.

4.4.1. Lemma. *Suppose that $\mathcal{A}_i \subset L(H_i)$ $(i = 1, 2)$ are reflexive operator algebras both with property $(\mathbb{A}_{p,q})$ for $1 \le p, q \le \aleph_0$. Then each weak* closed operator algebra $\mathcal{B} \subset L(H_1 \oplus H_2)$ which is contained in $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ and contains the identity operator on $H_1 \oplus H_2$ is super-reflexive and has the factorization property $(\mathbb{A}_{p,q})$. Moreover, the weak* topology and the weak operator topology coincide on $\mathcal{B}$.*

Proof. For $C \in \operatorname{AlgLat}(\mathcal{A})$ we have $C|H_i \in \operatorname{AlgLat}(\mathcal{A}_i)$ $(i = 1, 2)$. This proves the reflexivity of $\mathcal{A}$ itself. Now let $(L_{\mu\nu})$ $(0 \le \mu < p,\ 0 \le \nu < q)$ be a collection of elements in $Q_{\mathcal{A}}$. Defining

$$L^1_{\mu\nu} : \mathcal{A}_1 \to \mathbb{C}, \quad S \mapsto L_{\mu\nu}(S \oplus 0), \qquad L^2_{\mu\nu} : \mathcal{A}_2 \to \mathbb{C}, \quad T \mapsto L_{\mu\nu}(0 \oplus T),$$

we obtain families $(L^i_{\mu\nu})_{\mu\nu}$ of weak* continuous linear functionals on $\mathcal{A}_i$ $(i = 1, 2)$. By hypothesis there are vectors $(x^i_\mu)_\mu$ and $(y^i_\nu)_\nu$ in H_i solving the equations

$$L^i_{\mu\nu} = \langle x^i_\mu \otimes y^i_\nu \rangle \quad (0 \le \mu < p,\ 0 \le \nu < q,\ i = 1, 2).$$

By definition we have $\langle A_1 \oplus A_2 (x^1_\mu \oplus x^2_\mu), y^1_\nu \oplus y^2_\nu \rangle = \langle A_1 x^1_\mu, y^1_\nu \rangle + \langle A_2 x^2_\mu, y^2_\nu \rangle$ for $0 \le \mu < p$, $0 \le \nu < q$ and every choice of $A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$, proving that

$$L_{\mu\nu} = \langle (x^1_\mu \oplus x^2_\mu) \otimes (y^1_\nu \oplus y^2_\nu) \rangle \quad (0 \le \mu < p,\ 0 \le \nu < q).$$

Hence $\mathcal{A}$ has the factorization property $(\mathbb{A}_{p,q})$. From Theorem 2.3 in [29] we deduce that $\mathcal{A}$ is super-reflexive, implying that each WOT-closed subalgebra $\mathcal{B} \subset \mathcal{A}$ containing the identity is reflexive. Using the theorem of Hahn–Banach we infer that $\mathcal{B}$ inherits property $(\mathbb{A}_{p,q})$ from $\mathcal{A}$.

To finish the proof it suffices to show that the weak* topology and the weak operator topology coincide on $\mathcal{A}$. Recall that the weak* topology is always stronger than the weak operator topology. Moreover, the weak* topology is generated by the seminorms $|L(\cdot)|$ where $L \in Q_{\mathcal{A}}$. Since $\mathcal{A}$ has $(\mathbb{A}_1)$, each such seminorm $|L(\cdot)|$ can be represented in the form $|L(S)| = |\langle Sx, y\rangle|$ $(S \in \mathcal{A})$ with suitable elements $x, y \in H$ and hence is WOT-continuous. ∎

Now we intend to apply the abstract reflexivity result from the preceding section within the context of von Neumann n-tuples. Let $T \in L(H)^n$ denote an absolutely continuous von Neumann n-tuple of class $\mathbb{A}$ over D. Expressed in the notation defined above, the results obtained in Sections 4.1 and 4.2 imply that if T has property $F^r_{\theta,\gamma}$ for some $0 \le \theta < \gamma \le 1$, then the canonical functional calculus Φ_T has both properties $(\mathbb{A}_1)^+$ and $(\mathbb{A}_{1,\aleph_0})^+$. The following theorem settles a strengthened version of the reverse implication.

4.4.2. Theorem. *Let $D \Subset X$ be a strictly pseudoconvex open subset of a Stein submanifold $X \subset \mathbb{C}^n$. Suppose that a von Neumann n-tuple $T \in L(H)^n$ of class $\mathbb{A}$ over D satisfies at least one of the following conditions*:

(a) *The canonical $H^\infty(D)$-functional calculus Φ_T has property* $(\mathbb{A}_{1,\aleph_0})$.
(b) *There exists an analytic factor $x \in H$ for Φ_T.*

Then Φ_T has property $E^r_{0,1}$.

Proof. Since $\{L \in Q(D) : \|L\| \le 1\} = \overline{\Gamma}(\{\mathcal{E}_\mu : \mu \in D\})$, it suffices to check that the point evaluations $\mathcal{E}_\mu$ $(\mu \in D)$ belong to $\mathcal{E}^r_0(T)$. Towards this end, fix $\mu \in D$ and a biholomorphic chart $\varphi : U \to B \subset \mathbb{C}^\nu$ of D at μ such that $B = \varphi(U)$ is the open Euclidean unit ball in $\mathbb{C}^\nu$ $(\nu = \dim X)$ and $\varphi(\mu) = 0$. Since D is an open subset of the Stein manifold X we may assume that φ is of the form $\varphi = \psi|U$ with a holomorphic map $\psi \in \mathcal{O}(X, \mathbb{C}^\nu)$. The proof of the theorem is based on a suitable factorization of the weak* continuous linear functionals

$$\mathcal{E}^{(k)} : H^\infty(D) \to \mathbb{C}, \quad f \mapsto \partial^k(f \circ \varphi^{-1})(0) \quad (k \in \mathbb{N}_0^\nu).$$

First note that these functionals form a linearly independent family in $Q(D)$. To see this, observe that, for every polynomial $p \in \mathbb{C}[z_1, \ldots, z_\nu]$, the function $p \circ \psi \in \mathcal{O}(X)$ satisfies $\mathcal{E}^{(k)}(p \circ \psi|D) = \partial^k(p \circ \psi|U \circ \varphi^{-1})(0) = (\partial^k p)(0)$ $(k \in \mathbb{N}_0^\nu)$.

Let us now turn to the announced factorization problem. We first assume that the condition of part (b) is satisfied. Then, by the definition of an analytic factor, there exist a vector $x \in H$ as well as a conjugate analytic function $e : D \to H_x$ such that $x \otimes e(\lambda) = \mathcal{E}_\lambda$ $(\lambda \in D)$. Taking partial derivatives with respect to the chart φ we obtain the formula

$$\langle \Phi_T(f)x, \overline{\partial}^k(e \circ \varphi^{-1})(z)\rangle = \partial^k(f \circ \varphi^{-1})(z) \quad (k \in \mathbb{N}_0^\nu,\ z \in B,\ f \in H^\infty(D)),$$

which leads us to the identity

$$x \otimes \overline{\partial}^k(e \circ \varphi^{-1})(0) = \mathcal{E}^{(k)} \quad (k \in \mathbb{N}_0^\nu).$$

Hence there exist vectors $x, y_k \in H$ $(k \in \mathbb{N}_0^\nu)$ solving the equations

$$x \otimes y_k = \mathcal{E}^{(k)} \quad (k \in \mathbb{N}_0^\nu).$$

Obviously, condition (a) directly implies the solvability of this factorization problem. For abbreviation, we set $M = H_x$. Replacing y_k by $P_M y_k$, if necessary, we may of course assume that $y_k \in M$ $(k \in \mathbb{N}_0^\nu)$. Given $f, g \in H^\infty(D)$ and $k \in \mathbb{N}_0^\nu$, we have

$$\begin{aligned}\langle \Phi_T(f)x, \big(\Phi_T(g)|M\big)^* y_k\rangle &= \langle \Phi_T(gf)x, y_k\rangle = \partial^k(g\circ\varphi^{-1}\cdot f\circ\varphi^{-1})(0)\\ &= \sum_{\substack{\alpha\in\mathbb{N}_0^\nu\\ \alpha\le k}} \binom{k}{\alpha}\partial^\alpha(g\circ\varphi^{-1})(0)\partial^{k-\alpha}(f\circ\varphi^{-1})(0)\\ &= g(\mu)\langle\Phi_T(f)x, y_k\rangle + \sum_{\substack{\alpha\le k\\ |\alpha|\ne 0}}\binom{k}{\alpha} x\otimes y_\alpha(g)\langle \Phi_T(f)x, y_{k-\alpha}\rangle,\end{aligned}$$

proving that

$$(\Phi_T(g)|M - g(\mu))^* y_k = \sum_{\substack{ha\le k\\ |\alpha|\ne 0}} \binom{k}{\alpha}(x\otimes y_\alpha(g))^* y_{k-\alpha} \qquad (g\in H^\infty(D),\, k\in\mathbb{N}_0^\nu).$$

For the spaces $M_r = \mathrm{LH}\{y_j : |j|\le r\}$ $(r\ge 0)$ and $M_{-1} = \{0\}$ we therefore have the inclusion

$$(\Phi_T(g)|M - g(\mu))^* M_r \subset M_{r-1} \qquad (r\ge 0,\, g\in H^\infty(D)).$$

In particular, M_r $(r\ge 0)$ is invariant under $(\Phi_T(g)|M)^*$ for every $g\in H^\infty(D)$. Since the family $\mathcal{E}^{(k)} = x\otimes y_k$ $(k\in\mathbb{N}_0^\nu)$ is linearly independent, so is $(y_k)_k$, and we are able to choose an orthonormal system $(x_k)_{k\in\mathbb{N}_0^\nu}$ in H satisfying $M_r = \mathrm{LH}\{x_j : |j|\le r\}$. Our choices guarantee that

$$x_k\otimes x_k = \mathcal{E}_\mu \qquad (k\in\mathbb{N}_0^\nu)$$

as we will prove now. Given $k\in\mathbb{N}_0^\nu$ and $f\in H^\infty(D)$, we first apply Proposition 2.1.6 to obtain functions $g_1,\dots,g_n\in H^\infty(D)$ satisfying the identity

$$f - f(\mu) = \sum_{i=1}^n (z_i-\mu_i)g_i,$$

which allows us to compute

$$\begin{aligned} x_k\otimes x_k(f) &= x_k\otimes x_k\Big(f(\mu) + \sum_{i=1}^n (z_i-\mu_i)g_i\Big)\\ &= f(\mu) + \sum_{i=1}^n \langle x_k, (T_i|M-\mu_i)^*(\Phi_T(g_i)|M)^* x_k\rangle.\end{aligned}$$

Now it suffices to observe that all the scalar products occurring in the last term vanish since, for $1\le i\le n$, we have $(T_i|M-\mu_i)^*(\Phi_T(g_i)|M)^* x_k \subset M_{|k|-1}$.

To prove that $\mathcal{E}_\mu$ actually belongs to $\mathcal{E}_0^r(T)$ we fix a standard model $C_M = S_M^*\oplus R_M \in L(\mathcal{S}_M\oplus\mathcal{R}_M)^n$ for $(T|M)^* = C_M|M$ as described in Theorem 3.3.2. Let $P_\mathcal{S}, P_\mathcal{R}\in L(\mathcal{S}_M\oplus\mathcal{R}_M)$ denote the orthogonal projections onto $\mathcal{S}_M$ and $\mathcal{R}_M$.

We will now show by induction that $y_k\in\mathcal{S}_M$ $(k\in\mathbb{N}_0^\nu)$. For $|k| = 0$, this follows from the identity

$$((R_M)_i - \mu_i^*)P_\mathcal{R} y_0 = P_\mathcal{R}((C_M)_i - \mu_i^*)y_0 = P_\mathcal{R}(T_i|M-\mu_i)^* y_0 = 0$$

for $i = 1,\dots,n$. Indeed, $P_\mathcal{R} y_0 \ne 0$ would imply the contradiction $\mu^*\in\sigma(R_M)\subset\partial D^*$.

To conclude the induction apply the same arguments to the relation

$$((R_M)_i - \mu_i^*)P_{\mathcal{R}}y_k = P_{\mathcal{R}}(T_i|M - \mu_i)^* y_k \subset P_{\mathcal{R}}M_{|k|-1} \quad (1 \le i \le n,\ |k| \ge 1).$$

Knowing by now that $x_k \in M_{|k|} \subset \mathcal{S}_M$ $(k \in \mathbb{N}_0^\nu)$ we are able to deduce that

$$x_k \otimes z(f) = \langle x_k, \Phi_{(T|M)^*}(f_*)z\rangle = \langle x_k, \Phi_{S_M^*}(f_*)P_{\mathcal{S}}z\rangle = (P_{\mathcal{S}}z \otimes_{S_M^*} x_k(f_*))^*,$$

for each $f \in H^\infty(D)$, each $z \in M$ and each $k \in \mathbb{N}_0^\nu$. To complete the proof choose an injection $j : \mathbb{N} \to \mathbb{N}_0^\nu$ and observe that

$$x_{j(m)} \otimes x_{j(m)} = \mathcal{E}_\mu \quad (m \ge 1)$$

and that in view of the C_0.-property of S_M^* the sequence $\|x_{j(m)} \otimes z\| = \|P_{\mathcal{S}}z \otimes_{S_M^*} x_{j(m)}\|$ converges to zero for each $z \in H$. ■

Suppose that the n-tuple $T \in L(H)^n$ in the statement of the preceding theorem is of the form $T = J^{(N)} = J \oplus \cdots \oplus J \in L(K^N)^n$ for a suitable von Neumann n-tuple $J \in L(K)^n$ of class $\mathbb{A}$ over D. Then at the end of the proof we may decompose $x_{j(m)} = (x_{j(m)}^1, \ldots, x_{j(m)}^N) \in K^N$ $(m \ge 1)$ in its components relative to $K^N = H$. Since $\sum_{i=1}^N \|x_{j(m)}^i\|^2 = 1$ for each $m \ge 1$, we can choose an index $\nu \in \{1, \ldots, N\}$ in such a way that after passing to a suitable subsequence of $(x_{j(m)})$ the νth components $\omega_m = x_{j(m)}^\nu$ (lying in the closed unit ball of K) satisfy the estimate $\|\omega_m\|^2 \ge 1/N$ for every $m \ge 1$. But then an easy calculation shows that, for $m \ge 1$, we have $\|\mathcal{E}_\mu - \omega_m \otimes_J \omega_m\| \le (N-1)/N$ and that $\omega_m \otimes_J h = x_{j(m)} \otimes h^\nu \to 0$ as $m \to \infty$, where $h \in K$ denotes an arbitrary element and h^ν denotes the vector $(h\delta_{\nu i})_{1 \le i \le N}$ in K^N. Therefore we can conclude our remark with the observation that, in the above situation, the n-tuple $J \in L(K)^n$ has property $E^r_{(N-1)/N,1}$.

Now we are able to formulate the main result of this section.

4.4.3. Corollary. *Let $D \Subset X$ be a relatively compact strictly pseudoconvex open subset of a Stein submanifold $X \subset \mathbb{C}^n$ and let $T \in L(H)^n$ be a von Neumann n-tuple of class $\mathbb{A}$ over D. Then the following assertions are equivalent*:

- (a) *Φ_T has property $(\mathbb{A}_{1,\aleph_0})$.*
- (b) *T has property $E^r_{0,1}$.*
- (c) *T has property $E^r_{\theta,\gamma}$ for some $0 \le \theta < \gamma \le 1$.*
- (d) *T has property $F^r_{\theta,\gamma}$ for some $0 \le \theta < \gamma \le 1$.*
- (e) *Φ_T satisfies both $(\mathbb{A}_1)^+$ and $(\mathbb{A}_{1,\aleph_0})^+$.*
- (f) *The set of all analytic factors of Φ_T is dense in H.*
- (g) *There exists at least one analytic factor $x \in H$ of Φ_T.*
- (h) *One* (*and hence all*) *of the above assertions is* (*are*) *satisfied by $T^{(N)}$ instead of T for some* (*and then each*) *natural number $N \ge 1$.*

If any of the above conditions is satisfied, then the operator algebra $\mathcal{H}_T(\overline{D})$ is super-reflexive and possesses property $(\mathbb{A}_{1,\aleph_0})$.

Proof. By the preceding theorem, (a) implies (b). The implications (b)⇒(c)⇒(d) are obvious. Applying Corollaries 4.2.5 and 4.2.6 we deduce that (e) (and hence (a)) is a consequence of (d). Using Proposition 4.3.8 we infer that (e)⇒(f)⇒(g), and Theorem

4.4.2 finally proves that (g) implies (a). Using the remarks preceding the corollary one easily checks that the above chain of equivalences can be extended by condition (h).

To prove the reflexivity assertion suppose that $x \in H$ is an analytic factor of Φ_T. In view of condition (e) and Theorem 4.3.9 it suffices to observe that, for trivial reasons, x is an analytic factor of $\Phi_{(T|H_x)}$, and that (according to Lemma 4.3.3(f)) the absolutely continuous von Neumann n-tuple $T|H_x$ is again of class $\mathbb{A}$. Hence Theorem 4.3.9 can be applied to show that $\mathcal{H}_T(\overline{D}) = \Phi_T(H^\infty(D))$ is reflexive. Since Φ_T has property $(\mathbb{A}_{1,\aleph_0})$, the operator algebra $\mathcal{H}_T(\overline{D})$ has $(\mathbb{A}_{1,\aleph_0})$, as desired. The super-reflexivity of $\mathcal{H}_T(\overline{D})$ follows as an application of Theorem 2.3 in [29]. ■

Suppose that $T \in L(H)^n$ is a von Neumann n-tuple over D with dominating Taylor spectrum $\sigma(T)$ in D. According to Theorem 3.2.1 the n-tuple T can be written as a direct sum $T = T_{\mathrm{s}} \oplus T_{\mathrm{a}} \in L(H_{\mathrm{s}} \oplus H_{\mathrm{a}})^n$ consisting of a (singular) ∂D-unitary part T_{s} and an absolutely continuous part T_{a}. Note that since $\sigma(T_{\mathrm{s}}) \subset \partial D$ and $\sigma(T) = \sigma(T_{\mathrm{s}}) \cup \sigma(T_{\mathrm{a}})$, the spectrum of T_{a} remains dominating in D. Hence if T possesses a ∂D-unitary dilation and has dominating spectrum in D, then T_{a} is of class $\mathbb{A}$.

In the latter case, the general remarks concerning the reflexivity of direct sums formulated in Lemma 4.4.1 can be applied to obtain the following concluding result.

4.4.4. Corollary. *Suppose that $T \in L(H)^n$ is a von Neumann n-tuple over D with the property that its absolutely continuous part T_{a} is of class $\mathbb{A}$ and satisfies one of the equivalent conditions formulated in Corollary* 4.4.3. *Then the operator algebra*

$$\mathcal{H}_T(\overline{D}) = W^*(T_{\mathrm{s}}) \oplus \mathcal{H}_{T_{\mathrm{a}}}(\overline{D})$$

is super-reflexive and has property $(\mathbb{A}_{1,\aleph_0})$. In particular, the weak operator topology and the weak topology coincide on $\mathcal{H}_T(\overline{D})$, the algebras $\mathcal{A}_T = \overline{\mathbb{C}[T]}^{WOT}$ and $\mathcal{W}_T = \overline{\mathbb{C}[T]}^{w^*}$ are equal, have property $(\mathbb{A}_{1,\aleph_0})$ and are super-reflexive.*

Proof. In view of the above results it suffices to recall that $W^*(T_{\mathrm{s}})$ is reflexive (see Sarason [40]) and has property $(\mathbb{A}_{1,\aleph_0})$ (cf. Eschmeier [16]). ■

5. Subnormal n-tuples

The aim of this chapter is to point out that the reflexivity results obtained so far are general enough to imply the reflexivity of subnormal n-tuples with dominating Taylor spectrum in a strictly pseudoconvex open set D. Our considerations are based on the observation that, according to an abstract dilation theorem of Arveson, each subnormal n-tuple possesses a normal boundary dilation. This fact can also be applied to prove the existence of a faithful Henkin measure μ on $\overline{D}$ supported only by the boundary ∂D. Given such a measure we are able to apply results of Aleksandrov concerning the inner function problem to solve a factorization problem which finally leads to the desired reflexivity statement. The corresponding factorization technique can also be applied to treat the case of subnormal n-tuples possessing an isometric and weak* continuous $H^\infty(D)$-functional calculus.

5.1. On the structure of subnormal n-tuples. The following concrete dilation result is a consequence of a general dilation theorem arising in the context of operator spaces.

5.1.1. THEOREM. *Each subnormal n-tuple $T \in L(H)^n$ satisfying $\sigma(T) \subset \overline{D}$ is a von Neumann n-tuple over D which possesses a normal boundary dilation. More precisely, there exists a ∂D-unitary n-tuple $N \in L(K)^n$ defined on some (possibly larger) Hilbert space $K \supset H$ such that*

$$\Phi_T(f) = P_H \Phi_N(f)|H \quad (f \in A(D)),$$

where $P_H \in L(K)$ denotes the orthogonal projection onto H.

Proof. After replacing T by its minimal normal extension we may suppose that T is a normal n-tuple with spectrum $\sigma(T) \subset \overline{D}$. In this case T possesses an involutive (and hence completely contractive) functional calculus $\Psi : C(\overline{D}) \to L(H)$, proving in particular that T is a von Neumann n-tuple over D. For uniqueness reasons (see Lemma 2.3.1) the canonical $A(D)$-calculus Φ_T of T is nothing else than the composition $A(D) \hookrightarrow C(\overline{D}) \xrightarrow{\Psi} L(H)$. Thus Φ_T is completely contractive when $A(D)$ is regarded as an operator subspace of $C(\overline{D})$. From the fact that each isometric embedding into a commutative unital C^*-algebra induces the same operator space structure on $A(D)$ (see Theorem 3.8 in [33]) we deduce that $\Phi_T : A(D) \to L(H)$ is completely contractive when $A(D)$ is regarded as an operator subspace of $C(\partial D)$. But then an abstract dilation result due to Arveson (Corollary 6.7 in [33]) yields the existence of a Hilbert space $K \supset H$ and a unital $*$-homomorphism $\Psi_d : C(\partial D) \to L(K)$ dilating Φ_T in the sense that

$$\Phi_T(f) = P_H \Psi_d(f)|H \qquad (f \in A(D)).$$

Taking $N = (\Psi_d(z_1), \dots, \Psi_d(z_n)) \in L(K)^n$ and using once again the uniqueness property of continuous $A(D)$-calculi finishes the proof. ∎

As a consequence of the above theorem each subnormal n-tuple $T \in L(H)^n$ which satisfies $\sigma(T) \subset \overline{D}$ and possesses an isometric and weak* continuous $H^\infty(D)$-functional calculus is automatically a von Neumann n-tuple of class $\mathbb{A}$ over D. In this case, T is referred to as a subnormal n-tuple of class $\mathbb{A}$ over D.

Now let $\mu \in M_1^+(\overline{D})$ be an arbitrary probability measure on $\overline{D}$ which will later on be chosen as a scalar-valued spectral measure of the minimal normal extension of T. Recall the notations $H^\infty(\mu) = \overline{A(D)}^{w^*} \subset L^\infty(\mu)$ and $H^2(\mu) = \overline{A(D)} \subset L^2(\mu)$. Using the fact that the weak and the original closure of $A(D)$ in $L^2(\mu)$ coincide one easily deduces that $H^\infty(\mu) \cdot H^2(\mu) \subset H^2(\mu)$.

Now suppose that, in addition, μ is a Henkin measure. Then the map

$$\Phi_\mu : H^\infty(D) \to L(H^2(\mu)), \quad f \mapsto M_{r_\mu(f)},$$

where $M_g : H^2(\mu) \to H^2(\mu)$ denotes the multiplication by a function $g \in H^\infty(\mu)$, is a contractive and weak* continuous $H^\infty(D)$-functional calculus for the multiplication n-tuple $M_z = (M_{z_1}, \dots, M_{z_n}) \in L(H^2(\mu))^n$.

5.1.2. PROPOSITION. *Let $T \in L(H)^n$ be a subnormal n-tuple of class $\mathbb{A}$ over D. Then there exist a faithful Henkin probability measure $\mu \in M_1^+(\overline{D})$ and an isometric embedding*

$j : H^2(\mu) \hookrightarrow H$ satisfying

$$j \circ \Phi_\mu(f) = \Phi_T(f) \circ j \qquad (f \in H^\infty(D)).$$

Moreover, one can achieve that the support of μ is contained in the Taylor spectrum of the minimal normal extension of T.

Proof. Let E denote the projection-valued spectral measure of the minimal normal extension $N \in L(K)^n$ of T. Fix a separating unit vector $x \in H$ for N. Then the associated scalar-valued spectral measure $\mu_N = \langle E(\cdot)x, x\rangle$ is a probability measure. Since the support of E coincides with $\sigma(N)$ which is contained in $\sigma(T)$, the measure μ_N can be trivially extended to an element $\mu \in M_1^+(\overline{D})$. The spectral theorem for normal n-tuples ensures the existence of an isometric, weak* continuous and involutive functional calculus $\Psi : L^\infty(\mu) \to L(K)$ for N. According to Lemma 2.3.5(b) the weak* closed "restriction algebra"

$$\mathcal{W} = \{f \in L^\infty(\mu) : \Psi(f)H \subset H\} \subset L^\infty(\mu)$$

contains $H^\infty(\mu)$. Using the well known fact that the map $\mathcal{W} \to L(H)$, $f \mapsto \Psi(f)|H$, is isometric again (see Conway [10, Proposition 1.1]) we deduce that Ψ induces a dual algebra isomorphism

$$\gamma_T : H^\infty(\mu) \to \mathcal{H}_T(\overline{D}), \qquad f \mapsto \Psi(f)|H,$$

mapping z_i to T_i $(1 \le i \le n)$. Since, by hypothesis, T is of class $\mathbb{A}$ we have a canonical isomorphism $\Phi_T : H^\infty(D) \to \mathcal{H}_T(\overline{D})$ of dual algebras satisfying $\Phi_T(f) = \gamma_T(f)$ for every $f \in A(D)$, because of the uniqueness property of continuous $A(D)$-calculi (Lemma 2.3.1). Therefore the composition

$$r : H^\infty(D) \xrightarrow{\Phi_T} \mathcal{H}_T(\overline{D}) \xrightarrow{\gamma_T^{-1}} H^\infty(\mu)$$

is a dual algebra isomorphism extending the natural inclusion map $A(D) \hookrightarrow H^\infty(\mu)$. From this we deduce that μ is a Henkin measure with $r_\mu = r$ (see Proposition 2.2.4), proving that μ is faithful. Finally, the identity

$$\|\Phi_T(f)x\|^2 = \|\Psi(f)x\|^2 = \langle \Psi(|f|^2)x, x\rangle = \|f\|_{2,\mu}^2 \qquad (f \in A(D))$$

guarantees the existence of an isometry $j : H^2(\mu) \hookrightarrow H$ extending the map $A(D) \to H$, $f \mapsto \Phi_T(f)x$. To prove the announced intertwining relation observe that, for $f, g \in A(D)$, we have

$$j(\Phi_\mu(f)g) = \Phi_T(fg)x = \Phi_T(f)\Phi_T(g)x = \Phi_T(f)j(g).$$

Now an obvious density argument applies to complete the proof. ■

For $x, y \in H^2(\mu)$, we define

$$x \otimes y : H^\infty(D) \to \mathbb{C}, \qquad f \mapsto \langle \Phi_\mu(f)x, y\rangle = \int_{\overline{D}} r_\mu(f) x\overline{y}\, d\mu,$$

which is an element of $Q(D)$.

5.1.3. Corollary. *Let T and $j : H^2(\mu) \hookrightarrow H$ be as in the statement of the preceding proposition and let $(r_\mu)_* : Q(\mu) \to Q(D)$ denote the preadjoint of the dual algebra isomorphism r_μ.*

(a) *For any* $x, y \in H^2(\mu)$ *we have the identities* $(r_\mu)_*[x\overline{y}] = x \otimes y = jx \otimes jy$.

(b) *If in the situation of Proposition* 5.1.2 *the functional calculus* Φ_μ *has property* $(\mathbb{A}_{p,q})$ *for some cardinal numbers* $1 \le p, q \le \aleph_0$, *then so does* Φ_T.

(c) *For each* $L \in Q(D)$, *there exist* $x, y \in H$ *such that* $\|L - x \otimes y\| < \varepsilon$ *and* $\|x\|, \|y\| \le \|L\|^{1/2}$.

Proof. The fact that j preserves scalar products and intertwines the $H^\infty(D)$-calculi Φ_μ and Φ_T implies the validity of the formula

$$x \otimes y(f) = \langle \Phi_\mu(f)x, y\rangle = \langle j\Phi_\mu(f)x, jy\rangle = \langle \Phi_T(f)jx, jy\rangle = jx \otimes jy(f)$$

for any $f \in H^\infty(D)$, proving the second identity from (a). The observation that, for $f \in H^\infty(D)$, the expression $((r_\mu)_*[x\overline{y}])(f)$ is by definition equal to $\int_{\overline{D}} r_\mu(f) x\overline{y}\, d\mu$ proves the first identity.

Part (b) is an immediate consequence of (a). Moreover, using (a) the statement of part (c) can be reduced to the measure-theoretic factorization problem solved in Lemma 4.1.2. ∎

5.2. Henkin measures and ϱ-almost factorizations. Applying the structure theory developped in the previous section to the multiplication n-tuple M_z on $L^2(\lambda)$, where λ denotes the volume measure on D (see Section 2.1), we obtain the following existence result.

5.2.1. Proposition. *Let* $D \Subset X$ *be a relatively compact strictly pseudoconvex open subset of a Stein submanifold* $X \subset \mathbb{C}^n$. *Then there exist*

(a) *a Hilbert space* K *and a* ∂D-*unitary* n-*tuple* $N \in L(K)^n$ *possessing an isometric and weak* continuous* $H^\infty(D)$-*functional calculus*;

(b) *a faithful Henkin probability measure* $\mu \in M_1^+(\overline{D})$ *satisfying* $\mu(D) = 0$.

Proof. Consider the n-tuple $T = (M_{z_1}, \ldots, M_{z_n}) \in L(L^2(\lambda))^n$ of the multiplication by the coordinate functions. The map

$$\Phi_T : H^\infty(D) = H^\infty(\lambda) \to L(L^2(\lambda)), \quad g \mapsto M_g,$$

where $M_g f = gf$ ($f \in L^2(\lambda)$) is easily seen to be a contractive $H^\infty(D)$-functional calculus of class C_{00} for T. Since it is well known that $\sigma(T) = \mathrm{supp}(\lambda) = \overline{D}$, Corollary 2.3.7 applies to show that Φ_T is isometric. According to Theorem 5.1.1, T possesses a dilation to a ∂D-unitary n-tuple $N \in L(K)^n$ which may, possibly after passing to a suitable restriction, be assumed to be minimal. Then Lemma 3.3.1 asserts that N is absolutely continuous and the remarks immediately following the cited lemma yield the estimate

$$\|f\|_{\infty,D} = \|\Phi_T(f)\| \le \|\Phi_N(f)\| \le \|f\|_{\infty,D} \quad (f \in H^\infty(D)),$$

concluding the proof of part (a). Part (b) now results as an application of Proposition 5.1.2 to the ∂D-unitary n-tuple N constructed above. ∎

A faithful Henkin measure which is concentrated on the boundary ∂D as in part (b) of the above proposition can be used to deduce another characterization of class $\mathbb{A}_{1,\aleph_0}$ which is based on the following factorization property.

5.2.2. Definition. Let $\varrho > 0$ be a real number. An absolutely continuous von Neumann n-tuple $T \in L(H)^n$ over D is said to have the *ϱ-almost factorization property* if, for every $L \in Q(D)$ and every $\varepsilon > 0$, there are vectors $x, y \in H$ such that

$$\|L - x \otimes y\| < \varepsilon \quad \text{and} \quad \|x\|, \|y\| \leq \varrho \|L\|^{1/2}.$$

Note that part (c) of Corollary 5.1.3 can be rephrased by saying that each subnormal n-tuple of class $\mathbb{A}$ over D has the ϱ-almost factorization property with $\varrho = 1$.

5.2.3. Lemma. *Let E be a Banach space and let $T \in L(E)$ be a continuous linear operator whose adjoint T' is an isometry. Then T has the following lifting property: Given any $\varepsilon > 0$ and any $y \in E$, there exists a vector $x \in E$ such that $Tx = y$ and $\|x\| \leq (1+\varepsilon)\|y\|$.*

Proof. The proof relies on the fact that the map $\widehat{T}$ occurring in the canonical factorization $T : E \to E/\ker T \xrightarrow{\widehat{T}} E$ is an isometric isomorphism. To see this, note that in view of the formula $\ker T = {}^{\perp}(T'E')$ and the weak* closedness of the range $T'E'$ the adjoint of the operator $\widehat{T}$ coincides with the isometric isomorphism $T' : E' \to T'E'$. ∎

Applied to the special situation of strictly pseudoconvex sets, Aleksandrov's work on the abstract inner function problem (see [1] and [2]) immediately yields the next lemma.

5.2.4. Lemma. *Let $\mu \in M^+(\overline{D})$ be a positive Henkin measure satisfying $\mu(D) = 0$. Then there exists a weak* zero sequence $(g_k)_{k\geq 1}$ in $H^\infty(\mu)$ consisting of μ-inner functions, that is,*

$$|g_k| = 1 \qquad (\mu\text{-a.e. on } \partial D)$$

for every $k \geq 1$.

Proof. Observe that, by Corollary 2.1.3, the triple $(A(D), \overline{D}, \mu)$ is regular in the sense of Aleksandrov, and that μ vanishes at all one-point sets (Lemma 2.2.3). Therefore, according to Corollary 29 in Aleksandrov [1], the weak* closure of the set

$$I = \{g \in H^\infty(\mu) : |g| = 1 \ \mu\text{-a.e. on } \partial D\} \subset (H^\infty(\mu))_1$$

contains the closed unit ball of $A(D)$. In particular, this implies the existence of a sequence $(g_k)_{k\geq 1}$ in I converging to zero in the weak* topology. ∎

The following result is well known in the case of the unit ball (see Eschmeier [18, Proposition 3.2]). However, Eschmeier's proof is based on a trick which seems to work only under the assumption that the automorphism group acts transitively on D. Since this is far from being true for a general strictly pseudoconvex set, we have to use a different idea.

5.2.5. Proposition. *Each von Neumann n-tuple $T \in L(H)^n$ of class $\mathbb{A}$ over D which is of type $C_{\cdot 0}$ and has the ϱ-almost factorization property for some $\varrho > 0$ has property $E^r_{0,\gamma}$ with $\gamma = 1/(2\varrho^2)$.*

Proof. Suppose that we are given an element $L \in Q(D)$ of norm $\|L\| \leq \gamma$. It suffices to prove the existence of sequences $(x_k)_{k\geq 1}$, $(y_k)_{k\geq 1}$ in the closed unit ball of H such that

$$\|L - x_k \otimes y_k\| \xrightarrow{k} 0, \qquad x_k \otimes z \xrightarrow{k} 0 \quad (z \in H).$$

To realize this claim we fix a faithful Henkin probability measure $\mu \in M_1^+(\overline{D})$ satisfying $\mu(D) = 0$ (see Proposition 5.2.1). By Lemma 5.2.4 above, there is a weak* zero sequence $(g_k)_{k\geq 1}$ of μ-inner functions. Since the map $r_\mu : H^\infty(D) \to H^\infty(\mu)$ associated with μ is a dual algebra isomorphism, the sequence $\theta_k = r_\mu^{-1}(g_k)$ $(k \geq 1)$ is a weak* zero sequence in the unit ball of $H^\infty(D)$ with the property that the corresponding multiplication operators

$$M_{\theta_k} : H^\infty(D) \to H^\infty(D), \qquad f \mapsto \theta_k f,$$

are isometric (and, of course, weak* continuous) for each $k \geq 1$. Applying Lemma 5.2.3 to their preadjoints $(M_{\theta_k})_* : Q(D) \to Q(D)$ we obtain a sequence of functionals $(L_k)_{k\geq 1}$ in $Q(D)$ satisfying

$$L_k \circ M_{\theta_k} = L, \qquad \|L_k\| \leq 2\gamma \qquad (k \geq 1).$$

Now the ϱ-almost factorization property guarantees the existence of sequences $(u_k)_{k\geq 1}$ and $(y_k)_{k\geq 1}$ in H such that

$$\|u_k\|, \|y_k\| \leq \varrho(2\gamma)^{1/2} = 1, \qquad \|L_k - u_k \otimes y_k\| < 1/k \qquad (k \geq 1).$$

We show that the sequences $x_k = \Phi_T(\theta_k)u_k$ $(k \geq 1)$ and $(y_k)_{k\geq 1}$, both contained in the closed unit ball of H, have the properties described at the beginning of the proof. Towards this end, fix a sequence $(h_k)_{k\geq 1}$ of functions in $(H^\infty(D))_1$ such that $\|L - x_k \otimes y_k\| = |L(h_k) - x_k \otimes y_k(h_k)|$ $(k \geq 1)$. Then we have the estimate

$$\|L - x_k \otimes y_k\| = |L_k(\theta_k h_k) - \langle \Phi_T(\theta_k h_k)u_k, y_k\rangle| \leq \|L_k - u_k \otimes y_k\| \xrightarrow{k} 0,$$

proving the first one of the desired convergence conditions. For the second one, note that by definition $\langle \Phi_T(f)x_k, z\rangle = \langle \Phi_T(f)u_k, \Phi_T(\theta_k)^* z\rangle$ whenever $f \in H^\infty(D)$, $z \in H$ and $k \geq 1$. From this we deduce that

$$\|x_k \otimes z\| = \|u_k \otimes \Phi_T(\theta_k)^* z\| \leq \|\Phi_T(\theta_k)^* z\| \xrightarrow{k} 0 \quad (z \in H),$$

because of the $C_{\cdot 0}$-property of T. ■

Let $T \in L(H)^n$ be a von Neumann n-tuple of class $\mathbb{A}$ over D. Fix a Φ_T-semi-invariant subspace $Z \subset H$, that is, a subspace of H which can be represented in the form $Z = M \ominus N$ where $N \subset M \subset H$ are closed Φ_T-invariant subspaces. By $P_Z \in L(H)$ we denote the orthogonal projection from H onto Z. Note that the so-called compression $T_Z = (P_Z T_1|Z, \ldots, P_Z T_n|Z) \in L(Z)^n$ of T onto Z possesses the absolutely continuous contractive functional calculus $\Phi_{T_Z} = (\Phi_T)_Z : A(D) \to L(Z)$, $f \mapsto P_Z \Phi_T(f)|Z$ (see the remarks preceding Lemma 4.3.10), turning T_Z into a von Neumann n-tuple over D. Since, by hypothesis, T possesses a normal boundary dilation, so does T_Z. As will be pointed out in the next section, the following corollary allows us to give a short proof of the reflexivity of subnormal n-tuples with dominating Taylor spectrum in D.

5.2.6. Corollary. *Given a von Neumann n-tuple $T \in L(H)^n$ of class $\mathbb{A}$ over D, the following assertions are equivalent*:

(a) *Φ_T has property $(\mathbb{A}_{1,\aleph_0})$.*
(b) *There is a Φ_T-semi-invariant subspace $Z \subset H$ such that the compression T_Z of T onto Z is a von Neumann n-tuple of class $\mathbb{A}$ over D which is of type $C_{\cdot 0}$ and has the ϱ-almost factorization property (for some $\varrho > 0$).*

Proof. Note that $(\mathbb{A}_1)^+$ implies the ϱ-almost factorization property for a suitable constant $\varrho > 0$. Therefore, combining Corollary 4.4.3, Lemma 4.3.10 and Lemma 4.3.3(f) we conclude that (a) implies (b). More explicitly, if $x \in H$ is an analytic factor of Φ_T implemented by a conjugate analytic function $e : D \to H_x$, then we may set $Z = \bigvee\{e(\lambda) : \lambda \in D\}$. The reverse implication follows from Proposition 5.2.5 and Corollary 4.4.3. ∎

To treat subnormal n-tuples of class $\mathbb{A}$, we prove a more general version of the approximate factorization result formulated as Lemma 1.5 in [16]. The proof presented there relies on the corresponding one-dimensional ideas (see Chapter VII, §5, in Conway [11]). Our proof is based on the methods developed in the proof of Proposition 5.2.5 above.

Recall that we are working on a relatively compact strictly pseudoconvex open subset $D \Subset X$ of a Stein submanifold $X \subset \mathbb{C}^n$. Corollary 2.1.2 allows us to choose an open neighborhood $V \supset \overline{D}$ of $\overline{D}$ in X and a strictly plurisubharmonic C^2-function $\varrho : V \to [-1, 1]$ such that

$$D = \{z \in V : \varrho(z) < 0\}, \qquad \partial D = \{z \in V : \varrho(z) = 0\}, \qquad \varrho(\overline{D}) = [-1, 0].$$

For the following, we fix such a defining function $\varrho : V \to [-1, 1]$ and define

$$D_t = \{z \in V : \varrho(z) < t - 1\}, \quad A_t = \overline{D} \setminus D_t \quad (0 < t < 1).$$

From $\overline{D}_t \subset \overline{D} \subset V$ we deduce that $\overline{D}_t \subset \{z \in V : \varrho(z) \leq t - 1\} \subset D$ for every $0 < t < 1$, and hence D_t is a relatively compact, open subset of D. Moreover, $D_t \nearrow D$ and $A_t \searrow \partial D$ as $t \uparrow 1$.

Let $\mu \in M_1^+(\overline{D})$ be a faithful Henkin probability measure. (We do not assume that $\mu(D) = 0$.) Denote the predual of the dual algebra $H^\infty(\mu)$ by $Q(\mu) = L^1(\mu)/{}^\perp H^\infty(\mu)$ and note that, for $x, y \in L^2(\mu)$, the linear form

$$x \otimes y : H^\infty(\mu) \to \mathbb{C}, \qquad f \mapsto \int_{\overline{D}} f x \overline{y}\, d\mu,$$

defines an element $x \otimes y \in Q(\mu)$. In this context, the following approximate factorization result holds:

5.2.7. Lemma. *Let $0 < t < 1$ and let $\varepsilon > 0$, $\delta > 0$ be given real numbers. Let $g_1, \ldots, g_l$, $h_1, \ldots, h_l \in L^2(\mu)$ $(l \geq 1)$ and let $L \in Q(\mu)$ be a functional satisfying $\|L\| < \delta^2$. Then there exist vectors $x \in H^2(\mu)$, $y \in L^2(\mu|A_t)$ with the following properties*:

(a) $\|x\| < \delta$, $\|y\| < \delta$,
(b) $\|L - x \otimes y\| < \varepsilon$,
(c) $\max_{i=1,\ldots,l} \|x h_i \chi_D\|_{1,\mu} < \varepsilon$, $\max_{i=1,\ldots,l} \|y g_i \chi_D\|_{1,\mu} < \varepsilon$.

Proof. Fix a faithful Henkin probability measure $\eta \in M_1^+(\overline{D})$ satisfying $\eta(D) = 0$ (see Proposition 5.2.1). By Lemma 5.2.4, we can choose a weak* zero sequence $(g_k)_{k \geq 0}$ in $H^\infty(\eta)$ such that

$$|g_k| = 1 \quad (\eta\text{-a.e. on } \partial D).$$

Obviously, the induced multiplication operators

$$M_{g_k} : H^\infty(\eta) \to H^\infty(\eta), \qquad f \mapsto g_k f \quad (k \geq 1)$$

are isometric and weak* continuous. Since both μ and η are faithful Henkin measures, the composition

$$r = r_\mu \circ r_\eta^{-1} : H^\infty(\eta) \xrightarrow{r_\eta^{-1}} H^\infty(D) \xrightarrow{r_\mu} H^\infty(\mu)$$

is an isomorphism of dual algebras. Each of the multiplication operators

$$M_{\theta_k} : H^\infty(\mu) \to H^\infty(\mu), \qquad f \mapsto \theta_k f,$$

corresponding to the weak* zero sequence $\theta_k = r(g_k) \in H^\infty(\mu)$ $(k \geq 1)$ is a weak* continuous isometry. This is an immediate consequence of the commutativity of the diagrams

$$\begin{tikzcd} H^\infty(\eta) \arrow[r, "M_{g_k}"] \arrow[d, "r"'] & H^\infty(\eta) \arrow[d, "r"] \\ H^\infty(\mu) \arrow[r, "M_{\theta_k}"] & H^\infty(\mu) \end{tikzcd} \qquad (k \geq 1).$$

Applying Lemma 5.2.3 to the functional $L \in Q(\mu)$ and the family of weak* continuous isometries $M_{\theta_k^2} = (M_{\theta_k})^2$ $(k \geq 1)$, we obtain a sequence $(L_k)_{k\geq 1}$ in $Q(\mu)$ satisfying

$$L_k \circ M_{\theta_k^2} = L, \qquad \|L_k\| < \delta^2 \qquad (k \geq 1).$$

According to Lemma 4.1.2 there exist vectors $u_k, v_k \in H^2(\mu)$ satisfying

$$\|u_k\|, \|v_k\| < \delta \qquad \|L_k - u_k \otimes v_k\| < 1/k \qquad (k \geq 1).$$

Obviously, each of the vectors defined by

$$x_k = \theta_k u_k \in H^2(\mu), \qquad y_k = \overline{\theta}_k v_k \chi_{A_t} \in L^2(\mu|A_t) \qquad (k \geq 1)$$

has norm strictly less than δ. The estimates to be derived in what follows rely on the fact that, for $k \geq 1$,

$$\theta_k|D = r(g_k)|D = r_\eta^{-1}(g_k) \qquad \text{as elements of } L^\infty(\mu|D)$$

(see Lemma 2.2.7), and that $r_\eta^{-1}(g_k) \to 0$ as $k \to \infty$ uniformly on compact subsets of D (as a weak* zero sequence in $H^\infty(D)$). For any $f \in H^\infty(\mu)$ with $\|f\|_{\infty,\mu} \leq 1$, we have the estimate

$$\begin{aligned} |(L - x_k \otimes y_k)(f)| &= \Big| L_k(\theta_k^2 f) - \int_{\overline{D}} \theta_k u_k \theta_k \overline{v}_k \chi_{A_t} f \, d\mu \Big| \\ &\leq \|L_k - u_k \otimes v_k\| \cdot \|\theta_k^2 f\|_{\infty,\mu} + \int_{D_t} |r_\eta^{-1}(g_k)|^2 |u_k v_k| \, d\mu. \end{aligned}$$

Using the Cauchy–Schwarz inequality, we deduce that

$$\|L - x_k \otimes y_k\| \leq 1/k + \delta^2 \|r_\eta^{-1}(g_k)\|^2_{\infty,\overline{D}_t} \xrightarrow{k\to\infty} 0.$$

As an application of the dominated convergence theorem, it follows that

$$\|x_k h_i \chi_D\|_{1,\mu} = \int_D |\theta_k u_k h_i| \, d\mu \leq \delta \Big(\int_D |r_\eta^{-1}(g_k)|^2 |h_i|^2 \, d\mu \Big)^{1/2} \xrightarrow{k\to\infty} 0,$$

for $i = 1, \ldots, l$. By the same arguments, $\|y_k g_i \chi_D\|_{1,\mu} \to 0$ as $k \to \infty$ for every $i = 1, \ldots, l$. To conclude the proof, it therefore suffices to choose $k \geq 1$ large enough and to define $x = x_k \in H^2(\mu)$ and $y = y_k \in L^2(\mu|A_t)$. ∎

5.3. Subnormal n-tuples with dominating spectrum. Though looking rather artificial, the criterion established in Corollary 5.2.6 can be applied to prove the following reflexivity statement for subnormal n-tuples with dominating Taylor spectrum in D.

5.3.1. THEOREM. *Let $D \Subset X$ be a relatively compact strictly pseudoconvex open subset of a Stein submanifold $X \subset \mathbb{C}^n$. Then each subnormal n-tuple $T \in L(H)^n$ satisfying the conditions that $\sigma(T) \subset \overline{D}$ and that $\sigma(T)$ is dominating in D is reflexive. More precisely, the dual algebra $\mathcal{H}_T(\overline{D})$ is super-reflexive and has property* $(\mathbb{A}_{1,\aleph_0})$.

Proof. We start by decomposing $T = U \oplus S \in L(H_U \oplus H_S)^n$ into a ∂D-unitary part $U \in L(H_U)^n$ and a completely non-∂D-unitary part $S \in L(H_S)^n$ (see the proof of Theorem 3.3.2) which is of class $C_{\cdot 0}$ (by Proposition 3.1.4). As a consequence of the dilation Theorem 5.1.1 and the fact that $\sigma(S)$ is still dominating in D we infer that S is a subnormal n-tuple of class $\mathbb{A}$ over D and hence has the ϱ-almost factorization property (Corollary 5.1.3). But then Corollary 5.2.6 applies to show that Φ_S has property $(\mathbb{A}_{1,\aleph_0})$ and Corollary 4.4.3 asserts that the dual algebra $\mathcal{H}_S(\overline{D}) \subset L(H_S)$ is reflexive and satisfies $(\mathbb{A}_{1,\aleph_0})$. Since the von Neumann algebra $W^*(U)$ generated by the ∂D-unitary part of T shares the same properties, so does the sum $W^*(U) \oplus \mathcal{H}_S(\overline{D}) \subset L(H_U \oplus H_S)$ (see Lemma 4.4.1). Applying Lemma 2.3.5 to the reducing subspaces H_U and H_S we deduce that

$$\mathcal{H}_T(\overline{D}) \subset W^*(U) \oplus \mathcal{H}_S(\overline{D}).$$

Our considerations concerning the reflexivity and factorization properties of subdirect sums carried out in Lemma 4.4.1 complete the proof. ■

Note that, as a trivial consequence of the above theorem and Lemma 4.4.1, for each subnormal n-tuple T with dominating Taylor spectrum in D, the weak operator topology and the weak* topology coincide on $\mathcal{H}_T(\overline{D})$, implying that the operator algebras

$$\mathcal{A}_T = \overline{\{p(T) : p \in \mathbb{C}[z_1, \ldots, z_n]\}}^{WOT}, \qquad \mathcal{W}_T = \overline{\{p(T) : p \in \mathbb{C}[z_1, \ldots, z_n]\}}^{w^*}$$

coincide, are reflexive and have property $(\mathbb{A}_{1,\aleph_0})$.

5.4. Subnormal n-tuples of class $\mathbb{A}$. Let $\mu \in M_1^+(\overline{D})$ be a fixed faithful Henkin probability measure. Taking Lemma 5.2.7 as our new starting point and then proceeding as in [16], we prove that Φ_μ is of class $\mathbb{A}_{1,\aleph_0}$.

5.4.1. LEMMA. *Given $0 < t < 1$, $0 < \delta < 1/3$, $0 < \varepsilon$, $r \in \mathbb{N}$, $h_1, \ldots, h_r \in L^2(\mu)$, $L \in Q(\mu)$, $a \in L^2(\mu)$, $b \in L^2(\mu|A_t)$ such that*

$$\|L - a \otimes b\| < \delta^4,$$

there exist $x \in H^2(\mu)$, $y \in L^2(\mu|A_t)$, $Z \subset \partial D$ Borel with $\mu(Z) < \varepsilon$ and with the following properties:

$$\|L - (a + x) \otimes (b + y)\| < \varepsilon, \qquad \|x\| < 3\delta, \qquad \|y\chi_D\| < \delta^2,$$

$$\|(b + y)\chi_{\partial D}\| < \delta^2 + \frac{\|b\chi_{\partial D}\|}{1 - 2\delta}, \qquad \|b + y\| < \delta^2 + \frac{\|b\|}{1 - 2\delta},$$

$$|a + x| \geq (1 - 2\delta)|a| \quad (\mu\text{-a.e. on } \partial D \setminus Z), \qquad \|x \otimes (h_j\chi_D)\| < \varepsilon \quad (j = 1, \ldots, r).$$

Proof. The proof is divided into several steps.

(1) The construction of x: Lemma 5.2.7 allows us to choose functions $u \in H^2(\mu)$ and $v \in L^2(\mu|A_t)$ satisfying

$$\|u\| < \delta^2, \qquad \|v\| < \delta^2, \qquad \|L - a \otimes b - u \otimes v\| < \varepsilon/6,$$
$$\|u \otimes (b\chi_D)\| < \varepsilon/6, \qquad \|a \otimes (v\chi_D)\| < \varepsilon/6, \qquad \|u \otimes (h_j\chi_D)\| < \varepsilon/2 \quad (j = 1, \ldots, r).$$

Fix a constant $\eta \in (0, \varepsilon)$ such that

$$\int_Z (|uv| + (1 + 2/\delta)|ub|)\, d\mu < \varepsilon/6$$

for each Borel set $Z \subset \partial D$ with $\mu(Z) < \eta$. By Corollary 2.1.4(a) there exists a function $g \in A(D)$, $\|g\|_{\infty,\overline{D}} \le 2/\delta$, such that, on the boundary ∂D, the function $|g|$ is close to the measurable function

$$\kappa : \partial D \to \mathbb{R}, \qquad z \mapsto \begin{cases} 2/\delta & \text{if } |a(z)| \le |u(z)|/\delta, \\ 1 & \text{otherwise,} \end{cases}$$

in the sense that the Borel set

$$Z_1 = \{z \in \partial D : |g(z)| \ne \kappa(z)\} \subset \partial D$$

has measure $\mu(Z_1) < \eta/2$. Using Corollary 2.1.4 (b), we find a Montel sequence $(p_i)_{i\ge1}$ in $A(D)$ as well as a Borel set $Z_2 \subset \partial D$ of measure $\mu(Z_2) < \eta/2$ such that $|p_i| < 1$ on D and $|p_i| = 1$ on $\partial D \setminus Z_2$ $(i \ge 1)$. The dominated convergence theorem allows us to choose an index $i \ge 1$ in such a way that the function $f = p_i g \in A(D)$ satisfies the estimate

$$\|uf\chi_D\| = \Big(\int_D |up_i g|^2\, d\mu\Big)^{1/2} < \varepsilon/6 \cdot \frac{1}{\|h_j\| + \|b\| + \delta^2} \qquad (j = 1, \ldots, r).$$

Since $\|(1 + f)u\| \le \|u\| + \|p_i g\|_{\infty,\mu}\|u\| \le \delta^2(1 + 2/\delta) < 3\delta$, the function

$$x = (1 + f)u \in H^2(\mu)$$

satisfies $\|x\| < 3\delta$ as well as

$$\|x \otimes (h_j\chi_D)\| \le \|u \otimes (h_j\chi_D)\| + \|(uf) \otimes (h_j\chi_D)\| < \varepsilon/2 + \varepsilon/6 < \varepsilon,$$
$$\|x \otimes (b\chi_D)\| \le \|u \otimes b\chi_D\| + \|(uf) \otimes (b\chi_D)\| < \varepsilon/6 + \varepsilon/6 = \varepsilon/3.$$

For later reference we remark that $\|(uf) \otimes (v\chi_D)\| < \varepsilon/6$.

(2) Set $Z = Z_1 \cup Z_2$. We intend to show that

$$|a + x| \ge |u|, \qquad |a + x| \ge (1 - 2\delta)|a| \quad (\text{on } \partial D \setminus Z).$$

First observe that on $S_1 = \{z \in \partial D \setminus Z : |a(z)| \le |u(z)|/\delta\}$ we have $|g| = \kappa = 2/\delta$ and $|p_i| = 1$, and hence $|a + u| \le (1 + 1/\delta)|u|$ and $|uf| = |up_i g| = (2/\delta)|u|$. Therefore

$$|a + x| = |uf + a + u| \ge (2/\delta)|u| - (1 + 1/\delta)|u| = (1/\delta - 1)|u| \ge |u| \quad (\text{on } S_1).$$

On $S_1 \cap \{z \in \partial D : a(z) \ne 0\}$ this yields

$$\left|\frac{a + x}{a}\right| = \left|\frac{a + x}{u}\right| \cdot \left|\frac{u}{a}\right| \ge \Big(\frac{1}{\delta} - 1\Big)\delta = 1 - \delta,$$

implying that

$$|a + x| \ge (1 - \delta)|a| \quad (\text{on } S_1).$$

On the set $S_2 = \{z \in \partial D \setminus Z : |a(z)| > |u(z)|/\delta\}$ we have $|g| = 1 = |p_i|$, and therefore $|x| \le 2|u|$, implying that

$$|a + x| \ge (1/\delta - 2)|u| \ge |u| \quad (\text{on } S_2).$$

Since $|a| \le |a + x| + |x| \le |a + x| + 2|u| \le |a + x| + 2\delta|a|$ on S_2, we obtain

$$|a + x| \ge (1 - 2\delta)|a| \quad (\text{on } S_2).$$

Combining the four main estimates on the sets S_1 and S_2, we deduce that the estimates stated at the beginning of step (2) hold on the set $\partial D \setminus Z$.

(3) The construction of y: We use the set $W = (\partial D \setminus Z) \cap \{z \in \partial D : a(z) + x(z) \ne 0\}$ to define a function $w \in L^2(\mu)$ by

$$w = \begin{cases} \dfrac{\overline{u}}{\overline{a} + \overline{x}}\,(v - (1 + \overline{f})b) = \dfrac{\overline{u}v}{\overline{a} + \overline{x}} - \dfrac{\overline{x}b}{\overline{a} + \overline{x}} & \text{on } W, \\ 0 & \text{on } \overline{D} \setminus W. \end{cases}$$

The function

$$y = v\chi_D + w\chi_{\partial D} \in L^2(\mu|_{A_t})$$

then obviously satisfies $\|y\chi_D\| < \delta^2$. Moreover, we have

$$\begin{aligned}
\|y + b\|^2 &= \int_D |v + b|^2 d\mu + \int_W \left| \frac{\overline{u}}{\overline{a} + \overline{x}} v - \frac{\overline{x}}{\overline{a} + \overline{x}} b + b \right|^2 d\mu + \int_{\partial D \setminus W} |b|^2\, d\mu \\
&= \int_D |v + b|^2 d\mu + \int_W \left| \frac{\overline{u}}{\overline{a} + \overline{x}} v + \frac{\overline{a}}{\overline{a} + \overline{x}} b \right|^2 d\mu + \int_{\partial D \setminus W} |b|^2\, d\mu \\
&= \int_D |v|^2 d\mu + \int_D 2\mathrm{Re}(v\overline{b})\, d\mu + \int_D |b|^2\, d\mu + \int_{\partial D \setminus W} |b|^2\, d\mu \\
&\quad + \int_W |v|^2 \left| \frac{u}{a + x} \right|^2 d\mu + \int_W \frac{2\mathrm{Re}(v\overline{b}\overline{u}a)}{|a + x|^2} d\mu + \int_W \left| \frac{a}{a + x} \right|^2 |b|^2\, d\mu \\
&\le \int_{\overline{D}} |v|^2\, d\mu + 2 \int_{\overline{D}} |v|\, \frac{|b|}{1 - 2\delta}\, d\mu + \int_{\overline{D}} \frac{|b|^2}{(1 - 2\delta)^2}\, d\mu \\
&\le \left(\|v\| + \frac{\|b\|}{1 - 2\delta} \right)^2 < \left(\delta^2 + \frac{\|b\|}{1 - 2\delta} \right)^2,
\end{aligned}$$

as desired. An analogous calculation shows that

$$\|(y + b)\chi_{\partial D}\| < \delta^2 + \frac{\|b\chi_{\partial D}\|}{1 - 2\delta}.$$

(4) To complete the proof, it suffices to estimate the norm of

$$L - (a + x) \otimes (b + y) = L - a \otimes b - x \otimes y - a \otimes y - x \otimes b.$$

Towards this end, we write

$$\begin{aligned}
x \otimes y &= (u + uf) \otimes (v\chi_D + w\chi_{\partial D}) \\
&= u \otimes (v\chi_D + w\chi_{\partial D}) + (uf) \otimes (v\chi_D + w\chi_{\partial D}) \\
&= u \otimes v + (uf) \otimes (v\chi_D) + [u \otimes (-v\chi_{\partial D} + w\chi_{\partial D}) + (uf) \otimes (w\chi_{\partial D})].
\end{aligned}$$

Abbreviating the term between the brackets by z, we obtain a decomposition

$$L-(a+x)\otimes(b+y)=L-a\otimes b-u\otimes v-(uf)\otimes(v\chi_D)-a\otimes(v\chi_D)-x\otimes(b\chi_D)$$
$$-(z+a\otimes(w\chi_{\partial D})+x\otimes(b\chi_{\partial D})).$$

Note that, for all but the last term in the above formula, we have derived suitable estimates during steps (1)–(3) of the proof. To estimate the last term from above, observe that, for $\varphi\in H^\infty(\mu)$, we have

$$\begin{aligned}(z+a\otimes(w\chi_{\partial D})+x\otimes(b\chi_{\partial D}))(\varphi)&=\int\limits_{\partial D}\varphi(-u\overline{v}+u\overline{w}+uf\overline{w}+a\overline{w}+x\overline{b})\,d\mu\\&=\int\limits_{\partial D}\varphi((a+x)\overline{w}-u\overline{v}+x\overline{b})\,d\mu\\&=\int\limits_{Z}\varphi(-u\overline{v}+u\overline{b}+uf\overline{b})\,d\mu.\end{aligned}$$

Since $|f|\le 2/\delta$ we can conclude that

$$\|z+a\otimes(w\chi_{\partial D})+x\otimes(b\chi_{\partial D})\|\le\int\limits_{Z}(|uv|+(1+2/\delta)|ub|)\,d\mu<\varepsilon/6.$$

This finally leads to the desired estimate

$$\|L-(a+x)\otimes(b+y)\|<\varepsilon/6+\varepsilon/6+\varepsilon/6+\varepsilon/3+\varepsilon/6=\varepsilon,$$

which finishes the proof. ∎

Applying the above lemma to the setting $\|(\delta^4/\varrho)L-(\delta^2/\sqrt{\varrho})a\otimes(\delta^2/\sqrt{\varrho})b\|<\delta^4$ where $\|L-a\otimes b\|<\varrho$, one easily settles the case $m=1$ of the next result, a proof of which can be obtained by a word by word repetition of that of Lemma 2.1 in [16] and will therefore be omitted.

5.4.2. Lemma. *Given* $m\in\mathbb{N}$, $L_1,\ldots,L_m\in Q(\mu)$, $0<\alpha_1,\ldots,\alpha_m<1$, $\varrho_1,\ldots,\varrho_m>0$, $a\in L^2(\mu)$, $b_k\in L^2(\mu|A_{\alpha_k})$ $(k=1,\ldots,m)$ *with*

$$\|L_k-a\otimes b_k\|<\varrho_k\qquad(k=1,\ldots,m)$$

and given $\varepsilon>0$, $0<\delta<1/3$, *there exist functions* $x\in H^2(\mu)$, $h_k\in L^2(\mu|A_{\alpha_k})$ $(k=1,\ldots,m)$ *satisfying the following estimates for* $k=1,\ldots,m$:

$$\|L_k-(a+x)\otimes h_k\|<\varepsilon,\qquad \|x\|<\frac{3}{\delta}\sum_{i=1}^{m}\sqrt{\varrho_i},$$

$$\|(h_k-b_k)\chi_D\|<\sqrt{\varrho_k},\qquad \|h_k\chi_{\partial D}\|<\frac{\sqrt{\varrho_k}}{(1-2\delta)^{m-k}}+\frac{\|b_k\chi_{\partial D}\|}{(1-2\delta)^m}.\ ∎$$

Now a standard approximation technique carried out in detail in [16] (see Proposition 2.2) can be used to obtain a factorization result for countable families of weak* continuous linear functionals.

5.4.3. Proposition. *Given any faithful Henkin probability measure* $\mu\in M_1^+(\overline{D})$, *the representation*

$$\varPhi_\mu: H^\infty(D)\to L(H^2(\mu)),\qquad f\mapsto M_{r_\mu(f)},$$

has property $(\mathbb{A}_{1,\aleph_0})^+$. ∎

According to Proposition 4.3.8, Φ_μ possesses an analytic factor, implying that Φ_μ is isometric (Lemma 4.3.3(f)) and hence that $M_z \in L(H^2(\mu))^n$ is a von Neumann n-tuple of class $\mathbb{A}$ over D. In view of the preceding proposition, Corollary 4.4.3 can be applied to deduce that the latter multiplication n-tuple is reflexive.

5.4.4. THEOREM. *Let $D \Subset X$ be a relatively compact strictly pseudoconvex open subset of a Stein submanifold $X \subset \mathbb{C}^n$. Then each subnormal n-tuple $T \in L(H)^n$ of class $\mathbb{A}$ over D is reflexive. More precisely, the operator algebra $\mathcal{H}_T(\overline{D})$ is super-reflexive and has property $(\mathbb{A}_{1,\aleph_0})$.*

Proof. A suitable restriction of T is unitarily equivalent to a multiplication n-tuple of the form $M_z \in L(H^2(\mu))^n$ where $\mu \in M_1^+(\overline{D})$ is a faithful Henkin probability measure (Proposition 5.1.2). We have just shown that the latter n-tuple has property $(\mathbb{A}_{1,\aleph_0})$, implying that T is also of type $\mathbb{A}_{1,\aleph_0}$ (see Corollary 5.1.3). The statement of the theorem now follows from Corollary 4.4.3. ∎

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