Introduction

We study the function spaces with dominating mixed smoothness. First spaces of this kind were defined by S. M. Nikol'skiĭ in [21] and [22]. He introduced the spaces of Sobolev type

$$S_p^{\overline{r}}W(\mathbb{R}^2) = \Big\{ f \in L_p(\mathbb{R}^2) : \|f \,|\, S_p^{\overline{r}}W(\mathbb{R}^2)\| = \|f \,|\, L_p\| + \Big\| \frac{\partial^{r_1} f}{\partial x_1^{r_1}} \,\Big|\, L_p \Big\| + \Big\| \frac{\partial^{r_2} f}{\partial x_2^{r_2}} \,\Big|\, L_p \Big\| + \Big\| \frac{\partial^{r_1+r_2} f}{\partial x_1^{r_1} \partial x_2^{r_2}} \,\Big|\, L_p \Big\| < \infty \Big\},$$

where $1 < p < \infty$, $r_i = 0, 1, 2, \dots$ $(i = 1, 2)$. The mixed derivative $\partial^{r_1+r_2} f / \partial x_1^{r_1} \partial x_2^{r_2}$ plays the dominant part here and gave the name to this class of spaces. The detailed study of such spaces was performed by many authors, for example T. I. Amanov, O. V. Besov, K. K. Golovkin, P. I. Lizorkin, S. M. Nikol'skiĭ, M. K. Potapov and H.-J. Schmeisser. We refer to [1] for a systematic treatment of this topic. As in the theory of classical Sobolev spaces, an alternative definition in terms of Fourier transform may be given (see (1.8) and (1.9)). This definition is based on a decomposition

$$f = \sum_{\overline{k} \in \mathbb{N}_0^d} (\varphi_{k_1} \otimes \dots \otimes \varphi_{k_d} \widehat{f})^\vee, \qquad \text{convergence in } S'(\mathbb{R}^d),$$

where $\{\varphi_k\}_{k \in \mathbb{N}_0}$ is a decomposition of unity on $\mathbb{R}$ known from the theory of classical Besov spaces and $\varphi_{\overline{k}} = \varphi_{k_1} \otimes \dots \otimes \varphi_{k_d}$, $\overline{k} = (k_1, \dots, k_d)$, is a tensor product.

We refer mainly to [26] as far as the Fourier-analytic approach to these spaces is concerned. In Chapter 2 of that book the classical theory of spaces with dominating mixed smoothness properties is developed. Several types of equivalent quasinorms, embedding and trace theorems and characterisation of these spaces by differences are proved there. The authors also study basic properties of crucial operators on these spaces, namely lifting and maximal operators and Fourier multipliers. We recall some facts from that book, which will be useful later on. In contrast to [26], we do not restrict the dimension of the underlying Euclidean space to $d = 2$: the results are formulated for general dimension $d \ge 2$. As mentioned in [26], this generalisation is obvious.

The second chapter is devoted to local means, and atomic, subatomic and wavelet decompositions of spaces with dominating mixed smoothness. We state the result for both Besov and Triebel–Lizorkin spaces but in some cases we give the proofs only for the Triebel–Lizorkin scale. The proofs for Besov-type spaces are omitted as they are very similar to the proofs presented here. First of all, we characterise this class of spaces by so-called local means. See Theorem 1.25 for details. This fundamental characterisation serves as a basis for all three decomposition techniques.

By an *atomic decomposition* of a function f one usually means a decomposition of the type

$$f(x) = \sum_{\nu} \sum_{m} \lambda_{\nu\, m} a_{\nu\, m}(x), \qquad \text{convergence in } S'(\mathbb{R}^d),$$

where $a_{\nu\, m}$ are some simple building blocks, called *atoms*, and $\lambda_{\nu\, m}$ are complex numbers. A function f then belongs to some function space if, and only if, the sequence of coefficients $\{\lambda_{\nu\, m}\}_{\nu,m}$ belongs to some sequence space. For the exact formulation see Theorem 2.4. Let us mention that the atoms are specified only implicitly: a function a is an atom if, and only if, it has some qualitative properties (see Definition 2.3).

By a *subatomic decomposition* we mean a decomposition of the type

$$f(x) = \sum_{\beta} \sum_{\nu} \sum_{m} \lambda^{\beta}_{\nu\, m} (\beta qu)_{\nu\, m}(x), \qquad \text{convergence in } S'(\mathbb{R}^d),$$

where $(\beta qu)_{\nu\, m}(x)$ are so-called *quarks* and $\lambda^{\beta}_{\nu\, m}$ are complex numbers. A quark is a special type of atom defined explicitly by (2.36). Hence the basic building blocks, quarks, are much more specific in this kind of decomposition. The price one has to pay for that is a more complicated connection between f and $\{\lambda^{\beta}_{\nu\, m}\}$. It is described in detail in Theorem 2.6. In this sense each of these decompositions has its advantages and disadvantages. But all of them have something in common: they establish a connection between function spaces and sequence spaces. As the sequence spaces are simpler to deal with, it turns out that this connection is very useful in many situations (embeddings, traces, entropy numbers, ...). Here we have to mention another important way to switch from function spaces to sequence spaces—the so-called φ-transform of M. Frazier and B. Jawerth. We refer to [15] and references given there for details.

The classical theory of atomic decompositions of Besov and Triebel–Lizorkin spaces was developed mainly in the works M. Frazier and B. Jawerth ([12], [13]) and H. Triebel ([33], [34]). The subatomic decomposition of these spaces is due to H. Triebel ([35], [37]). We follow their ideas and prove similar decomposition theorems for spaces with dominating mixed derivatives. This is done in Chapter 2 and is one of the main results of this work.

The last decomposition technique developed here is the wavelet decomposition. In that case a class of compactly supported wavelets is used as the building blocks (see Theorems 2.10 and 2.11 for precise formulation). The main advantage of the wavelet decomposition is the uniqueness of the series obtained. The price paid for that is the limited smoothness of the compactly supported wavelets.

In the third chapter we study the entropy numbers of embeddings of sequence spaces associated with the function spaces with dominating mixed smoothness. The notion of entropy numbers has its roots in the study of metric entropy done in the 1930's by Kolmogorov. Given a bounded linear operator T between two quasi-Banach spaces A and B ($T \in L(A,B)$), the quantity $e_k(T)$, $k \in \mathbb{N}$, denotes, roughly speaking, the smallest radius $\varepsilon > 0$ such that the image of the unit ball of A under the operator T may be covered by 2^{k-1} balls in B of radius ε. The sequence $\{e_k(T)\}_{k=1}^{\infty}$ tends to zero if, and only if, the operator T is compact. The decay of this sequence is then understood as a measure of compactness of T. The crucial property of entropy numbers was observed by Carl [6], who proved that the entropy numbers of a compact operator $T \in L(A, A)$ dominate in some sense its eigenvalues. In general, we use the method of [10] in this part.

We use the decomposition techniques to reduce this question to the sequence space level. Namely, it turns out that

$$e_k(\mathrm{id} : S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \hookrightarrow S^{\overline{r}_2}_{p_2,q_2}A(\Omega)) \approx e_k(\mathrm{id} : s^{\overline{r}_1}_{p_1,q_1}a(\Omega) \hookrightarrow s^{\overline{r}_2}_{p_2,q_2}a(\Omega)), \tag{1}$$

where the equivalence constants do not depend on $k \in \mathbb{N}$. So, in the third chapter we study mainly the entropy numbers of embeddings of sequence spaces. We restrict ourselves to the case $\overline{r}_1 = (r_1, \ldots, r_1) \in \mathbb{R}^d$ and $\overline{r}_2 = (r_2, \ldots, r_2) \in \mathbb{R}^d$. Unlike the case of the classical Besov and Triebel–Lizorkin spaces, it turns out that the estimates of entropy numbers depend on the second, fine, summability parameter q. Unfortunately, the method used here gives the optimal answer only under some restriction on the parameters involved. We prove that the embedding appearing in (1) is compact if, and only if,

$$\alpha = r_1 - r_2 - \max\left(\frac{1}{p_1} - \frac{1}{p_2}, 0\right) > 0. \tag{2}$$

But the direct method gives the estimates for (1) only for

$$\alpha > \frac{1}{\min(p_1, p_2, q_1)} - \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{\max(p_2, q_2)}.$$

We overcome this obstacle in Chapter 4 by the use of a complex interpolation method as developed by O. Mendez and M. Mitrea in [20]. Our final result may be summarised in the following way.

Under condition (2),

$$e_k(\mathrm{id} : S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \to S^{\overline{r}_2}_{p_2,q_2}A(\Omega)) \geq ck^{r_2-r_1}(\log k)^{(d-1)(r_1-r_2+1/q_2-1/q_1)_+}.$$

If $r_1 - r_2 - 1/q_1 + 1/q_2 > 0$ then

$$e_k(\mathrm{id} : S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \to S^{\overline{r}_2}_{p_2,q_2}A(\Omega)) \leq ck^{r_2-r_1}(\log k)^{(d-1)(r_1-r_2+1/q_2-1/q_1)}.$$

If $r_1 - r_2 - 1/q_1 + 1/q_2 \leq 0$ then for every $\varepsilon > 0$ there is a constant $c_\varepsilon > 0$ such that

$$e_k(\mathrm{id} : S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \to S^{\overline{r}_2}_{p_2,q_2}A(\Omega)) \leq c_\varepsilon k^{r_2-r_1}(\log k)^{\varepsilon}.$$

(See Theorem 4.11 for exact formulation.) Finally, we compare results obtained by this method with estimates on entropy numbers of embeddings of function spaces with dominating mixed smoothness obtained by Belinsky [4], Dinh Dung [8] and Temlyakov [30].

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1. Function spaces on $\mathbb{R}^d$

Our aim in this chapter is to recall the known aspects of the theory of function spaces with dominating mixed smoothness, $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ and $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$. First of all, we introduce some basic notation. Then we quote some definitions and theorems stated in [26] which are crucial in the following. In the last part we develop the so-called *local mean* characterisation of the spaces $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ and $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$.

1.1. Notation. As usual, $\mathbb{R}^d$ denotes the d-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers and $\mathbb{C}$ denotes the plane of complex numbers.

We denote points of the underlying Euclidean space by $x, y, z, \dots$. Their components are numbered from 1 to d, hence $x = (x_1, \dots, x_d)$. If $x, y \in \mathbb{R}^d$, we write $x > y$ if, and only if, $x_i > y_i$ for every $i = 1, \dots, d$. Similarly, we define the relations $x \geq y$, $x < y$, $x \leq y$. Finally, by slight abuse of notation, we write $x > \lambda$ for $x \in \mathbb{R}^d$, $\lambda \in \mathbb{R}$ if $x_i > \lambda$, $i = 1, \dots, d$. The d-dimensional vector indices will be denoted by $\bar{k}, \bar{l}, \bar{m}, \dots$ and their components are also numbered, hence $\bar{k} = (k_1, \dots, k_d)$. When $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ is a multi-index, we denote its length by $|\alpha| = \sum_{j=1}^d \alpha_j$. The derivatives $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}$ have the usual distributional meaning; moreover $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$.

Let $S(\mathbb{R}^d)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$. We denote the d-dimensional Fourier transform of a function $\varphi \in S(\mathbb{R}^d)$ by $\mathcal{F}\varphi$, $\mathcal{F}(\varphi)$ or $\widehat{\varphi}$. Its inverse is denoted by $\mathcal{F}^{-1}\varphi$, $\mathcal{F}^{-1}(\varphi)$ or $\varphi^\vee$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to the dual Schwartz space $S'(\mathbb{R}^d)$ in the usual way. Sometimes, we need to distinguish between the d-dimensional and one-dimensional Fourier transform. In that case we denote the latter by $\mathcal{F}_1$ or $^{\wedge_1}$ and its inverse by $\mathcal{F}_1^{-1}$ or $^{\vee_1}$. We point out that for functions $\varphi(x) = \varphi_1(x_1) \cdots \varphi_d(x_d) = (\varphi_1 \otimes \cdots \otimes \varphi_d)(x)$ the following formula connects $\mathcal{F}$ with $\mathcal{F}_1$:

$$(\mathcal{F}\varphi)(\xi) = (\mathcal{F}_1\varphi_1)(\xi_1) \cdots (\mathcal{F}_1\varphi_d)(\xi_d) = ((\mathcal{F}_1\varphi_1) \otimes \cdots \otimes (\mathcal{F}_1\varphi_d))(\xi), \qquad \xi \in \mathbb{R}^d. \tag{1.1}$$

Let $0 < p, q \leq \infty$. Having a sequence of complex-valued functions $\{f_{\bar{k}}\}_{\bar{k} \in \mathbb{N}_0^d}$ on $\mathbb{R}^d$, we put

$$\|f_{\bar{k}} \,|\, \ell_q(L_p)\| = \Big(\sum_{\bar{k} \in \mathbb{N}_0^d} \|f_{\bar{k}} \,|\, L_p(\mathbb{R}^d)\|^q \Big)^{1/q} = \Big(\sum_{\bar{k} \in \mathbb{N}_0^d} \Big(\int_{\mathbb{R}^d} |f_{\bar{k}}(x)\, dx|^p \Big)^{q/p} \Big)^{1/q} \tag{1.2}$$

and

$$\|f_{\bar{k}} \,|\, L_p(\ell_q)\| = \Big\| \Big(\sum_{\bar{k} \in \mathbb{N}_0^d} |f_{\bar{k}}(x)|^q \Big)^{1/q} \Big| L_p(\mathbb{R}^d) \Big\| = \Big(\int_{\mathbb{R}^d} \Big(\sum_{\bar{k} \in \mathbb{N}_0^d} |f_{\bar{k}}(x)|^q \Big)^{p/q} dx \Big)^{1/p}, \tag{1.3}$$

appropriately modified when p and/or $q = \infty$.

We write $a_+ = \max(a, 0)$ for a real number $a \in \mathbb{R}$. Furthermore, let

$$\sigma_{pq} = \left(\frac{1}{\min(p, q)} - 1 \right)_+ \quad \text{and} \quad \sigma_p = \left(\frac{1}{p} - 1 \right)_+ \tag{1.4}$$

for every $0 < p \leq \infty$ and $0 < q \leq \infty$.

All unimportant constants are denoted by c. So, the meaning of the letter c may change from one occurrence to another. By $a_k \approx b_k$ we mean that there are constants $c_1, c_2 > 0$ such that $c_1 a_k \leq b_k \leq c_2 a_k$ for every admissible k.

1.2. Definitions and basic properties. In this section we define the function spaces with dominating mixed smoothness on $\mathbb{R}^d$ and recall their basic properties as described in [26]. We quote the results for general d, although they were stated and proved only for $d = 2$ in [26]. But, as mentioned there, this generalisation is rather obvious.

1.2.1. *Definitions*

DEFINITION 1.1. Let $\Phi(\mathbb{R})$ be the collection of all systems $\{\varphi_j\}_{j=0}^{\infty} \subset S(\mathbb{R})$ such that

$$\begin{cases} \operatorname{supp} \varphi_0 \subset \{t \in \mathbb{R} : |t| \le 2\}, \\ \operatorname{supp} \varphi_j \subset \{t \in \mathbb{R} : 2^{j-1} \le |t| \le 2^{j+1}\} \quad \text{if } j = 1, 2, \dots, \end{cases} \tag{1.5}$$

for every $\alpha \in \mathbb{N}_0$ there exists a positive constant c_α such that

$$2^{j\alpha} |D^\alpha \varphi_j(t)| \le c_\alpha \quad \text{for all } j = 0, 1, 2, \dots \text{ and all } t \in \mathbb{R}, \tag{1.6}$$

and

$$\sum_{j=0}^{\infty} \varphi_j(t) = 1 \quad \text{for every } t \in \mathbb{R}. \tag{1.7}$$

For $\overline{k} = (k_1, \dots, k_d) \in \mathbb{N}_0^d$ and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ define $\varphi_{\overline{k}}(x) = \varphi_{k_1}(x_1) \cdots \varphi_{k_d}(x_d)$. Using this kind of notation, we can give a definition of the spaces $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ and $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$.

DEFINITION 1.2. Let $\overline{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$, $0 < q \le \infty$ and $\varphi = \{\varphi_j\}_{j=0}^{\infty} \in \Phi(\mathbb{R})$.

(i) Let $0 < p \le \infty$. Then $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$\|f \,|\, S^{\overline{r}}_{p,q}B(\mathbb{R}^d)\|_\varphi = \Big(\sum_{\overline{k} \in \mathbb{N}_0^d} 2^{q \overline{k} \cdot \overline{r}} \|(\varphi_{\overline{k}} \widehat{f})^\vee \,|\, L_p(\mathbb{R}^d)\|^q \Big)^{1/q} = \|2^{\overline{k} \cdot \overline{r}} (\varphi_{\overline{k}} \widehat{f})^\vee \,|\, \ell_q(L_p)\| \tag{1.8}$$

is finite.

(ii) Let $0 < p < \infty$. Then $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$\|f \,|\, S^{\overline{r}}_{p,q}F(\mathbb{R}^d)\|_\varphi = \Big\| \Big(\sum_{\overline{k} \in \mathbb{N}_0^d} |2^{\overline{k} \cdot \overline{r}} (\varphi_{\overline{k}} \widehat{f})^\vee(\cdot)|^q \Big)^{1/q} \,\Big|\, L_p(\mathbb{R}^d) \Big\| = \|2^{\overline{k} \cdot \overline{r}} (\varphi_{\overline{k}} \widehat{f})^\vee \,|\, L_p(\ell_q)\| \tag{1.9}$$

is finite.

REMARK 1.3. According to (1.7), we have

$$\sum_{k \in \mathbb{N}_0^d} \varphi_{\overline{k}}(x) = \Big(\sum_{k_1=0}^{\infty} \varphi_{k_1}(x_1) \Big) \cdots \Big(\sum_{k_d=0}^{\infty} \varphi_{k_d}(x_d) \Big) = 1 \quad \text{for all } x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

In this sense, $\{\varphi_{\overline{k}}\}_{\overline{k} \in \mathbb{N}_0^d}$ is also a decomposition of unity, in this case on $\mathbb{R}^d$.

REMARK 1.4. The symbol $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ stands, as usual, for $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ and $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ respectively.

1.2.2. *Basic inequalities.* One of the most important questions in the theory of the spaces $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ is the independence of Definition 1.2 on the system $\varphi = \{\varphi_{\overline{k}}\}_{\overline{k} \in \mathbb{N}_0^d}$. The answer is given by

THEOREM 1.5. *Let $\{\varphi_j\}_{j=0}^{\infty}, \{\psi_j\}_{j=0}^{\infty} \in \Phi(\mathbb{R})$. Let $\overline{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$ and $0 < q \le \infty$.*

(i) *Let $0 < p \le \infty$. Then $\|f \,|\, S^{\overline{r}}_{p,q}B(\mathbb{R}^d)\|_\varphi$ and $\|f \,|\, S^{\overline{r}}_{p,q}B(\mathbb{R}^d)\|_\psi$ are equivalent quasi-norms. Furthermore, $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ is a quasi-Banach space* (*Banach space if* $\min(p,q) \ge 1$) *and*

$$S(\mathbb{R}^d) \subset S^{\overline{r}}_{p,q}B(\mathbb{R}^d) \subset S'(\mathbb{R}^d).$$

(ii) *Let* $0 < p < \infty$. *Then* $\|f\,|\,S^{\overline{r}}_{p,q}F(\mathbb{R}^d)\|_\varphi$ *and* $\|f\,|\,S^{\overline{r}}_{p,q}F(\mathbb{R}^d)\|_\psi$ *are equivalent quasinorms. Furthermore*, $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ *is a quasi-Banach space* (*Banach space if* $\min(p,q) \ge 1$) *and*

$$S(\mathbb{R}^d) \subset S^{\overline{r}}_{p,q}F(\mathbb{R}^d) \subset S'(\mathbb{R}^d).$$

For the proof in the case $d = 2$, see [26, pp. 87, 93]. So, we may write $\|f\,|\,S^{\overline{r}}_{p,q}B(\mathbb{R}^d)\|$ and $\|f\,|\,S^{\overline{r}}_{p,q}F(\mathbb{R}^d)\|$ without any index φ or ψ meaning one of these equivalent quasinorms.

Remark 1.6. The reader may have noticed that we have *not* defined the spaces $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ for $p = \infty$. The reason is very similar to the case of classical Triebel–Lizorkin spaces. If one extends Definition 1.2 to the case $p = \infty$, which is actually possible, then there is no counterpart of Theorem 1.5. In particular, these spaces *do* depend on the choice of the system $\{\varphi_j\} \in \Phi(\mathbb{R})$.

We also recall the following version of the famous Nikol'skiĭ inequality which is due to B. Stöckert [29] and A. P. Uninskiĭ [39].

Theorem 1.7 (Nikol'skiĭ inequality). *Let* $0 < p \le u \le \infty$ *and* $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$. *Let* $\overline{b} = (b_1, \ldots, b_d) > 0$ *and* $Q_{\overline{b}} = [-b_1, b_1] \times \cdots \times [-b_d, b_d] \subset \mathbb{R}^d$. *Then there exists a positive constant* c, *which is independent of* $\overline{b}$, *such that*

$$\|D^\alpha f\,|\,L_u(\mathbb{R}^d)\| \le c b_1^{\alpha_1+1/p-1/u} \cdots b_d^{\alpha_d+1/p-1/u} \|f\,|\,L_p(\mathbb{R}^d)\|$$

for every $f \in S'(\mathbb{R}^d) \cap L_p(\mathbb{R}^d)$ *with* $\operatorname{supp} \widehat{f} \subset Q_{\overline{b}}$.

1.2.3. *Lifting property.* As in the case of classical Besov and Triebel–Lizorkin spaces, we can define a lifting operator.

Definition 1.8. Let $\overline{\rho} = (\rho_1, \ldots, \rho_d) \in \mathbb{R}^d$. Then we define the *lifting operator* $I_{\overline{\rho}}$ by

$$I_{\overline{\rho}} f = \mathcal{F}^{-1}(1+\xi_1^2)^{\rho_1/2} \cdots (1+\xi_d^2)^{\rho_d/2} \mathcal{F} f, \qquad f \in S'(\mathbb{R}^d). \tag{1.10}$$

Theorem 1.9. *Let* $0 < q \le \infty$, $\overline{\rho}, \overline{r} \in \mathbb{R}^d$.

(i) *Let* $0 < p \le \infty$. *Then* $I_{\overline{\rho}}$ *maps* $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ *isomorphically onto* $S^{\overline{r}-\overline{\rho}}_{p,q}B(\mathbb{R}^d)$ *and* $\|I_{\overline{\rho}} f\,|\,S^{\overline{r}-\overline{\rho}}_{p,q}B(\mathbb{R}^d)\|$ *is an equivalent quasinorm in* $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$.
(ii) *Let* $0 < p < \infty$. *Then* $I_{\overline{\rho}}$ *maps* $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ *isomorphically onto* $S^{\overline{r}-\overline{\rho}}_{p,q}F(\mathbb{R}^d)$ *and* $\|I_{\overline{\rho}} f\,|\,S^{\overline{r}-\overline{\rho}}_{p,q}F(\mathbb{R}^d)\|$ *is an equivalent quasinorm in* $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$.

The proof may again be found in [26, p. 98].

1.2.4. *Maximal operators.* Maximal operators (and their boundedness on appropriate function spaces) play a crucial role in harmonic analysis and function spaces theory. Our constructions given later are based on the Hardy–Littlewood maximal operator and the maximal operator of Peetre. Now we give the definition of the former. For the definition of the latter, see Section 1.3.1.

For every locally integrable function $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$ we define the classical Hardy–Littlewood maximal operator

$$(Mf)(x) = \sup_Q \frac{1}{|Q|} \int_Q |f(y)|\,dy, \qquad x \in \mathbb{R}^d, \tag{1.11}$$

where the supremum is taken over all cubes Q centred at x with sides parallel to coordinate axes. The symbol $|Q|$ denotes the Lebesgue mass of the cube Q. The famous Hardy–Littlewood inequality says that for every p with $1 < p \le \infty$ there is a c such that

$$\|Mf \,|\, L_p(\mathbb{R}^d)\| \le c\|f \,|\, L_p(\mathbb{R}^d)\|, \qquad f \in L_p(\mathbb{R}^d). \tag{1.12}$$

The following theorem is a vector-valued generalisation of (1.12) and is due to C. Fefferman and E. M. Stein [11].

THEOREM 1.10. *Let $1 < p < \infty$ and $1 < q \le \infty$. There exists a constant c such that*

$$\|Mf_{\overline{k}} \,|\, L_p(\ell_q)\| \le c\|f_{\overline{k}} \,|\, L_p(\ell_q)\| \tag{1.13}$$

for all sequences $\{f_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^d}$ of locally Lebesgue-integrable functions on $\mathbb{R}^d$.

To reflect the tensor structure of the decomposition of unity $\varphi = \{\varphi_{\overline{k}}\}$ used in Definition 1.2, we consider the following "directional" maximal operators. We define

$$(M_1 f)(x) = \sup_{s>0} \frac{1}{2s} \int_{x_1-s}^{x_1+s} |f(t, x_2, \ldots, x_d)|\, dt \tag{1.14}$$

and in a similar way for other variables. We denote the composition of these operators by $\overline{M} = M_d \circ \cdots \circ M_1$. The following maximal theorem is due to R. J. Bagby [2] (actually, it is a special case of a more general theorem given there).

THEOREM 1.11. *Let $1 < p < \infty$ and $1 < q \le \infty$. There exists a constant c such that*

$$\|M_i f_{\overline{k}} \,|\, L_p(\ell_q)\| \le c\|f_{\overline{k}} \,|\, L_p(\ell_q)\|, \qquad i = 1, \ldots, d, \tag{1.15}$$

for all sequences $\{f_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^d} \subset L_p(\ell_q)$ of functions on $\mathbb{R}^d$.

Iteration of this theorem shows that the estimate (1.15) also holds for the operator $\overline{M}$.

1.2.5. *Fourier multipliers.* Let $\Omega = \{\Omega_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^d}$ be the sequence of compact subsets of $\mathbb{R}^d$ defined by

$$\Omega_{\overline{k}} = \{x \in \mathbb{R}^d : |x_1| \le a_{1,k_1}, \ldots, |x_d| \le a_{d,k_d}\} \qquad \text{with } a_{1,k_1}, \ldots, a_{d,k_d} > 0.$$

THEOREM 1.12. *Let $0 < p < \infty$, $0 < q \le \infty$ and $\overline{r} = (r_1, \ldots, r_d) > 1/\min(p, q) + 1/2$. Let $\Omega = \{\Omega_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^d}$, $a_{1,k_1}, \ldots, a_{d,k_d} > 0$ be as above. Then there is a positive constant c such that*

$$\|(\varrho_{\overline{k}}\widehat{f_{\overline{k}}})^\vee \,|\, L_p(\ell_q)\| \le c\big(\sup_{\overline{k}\in\mathbb{N}_0^d} \|\varrho_{\overline{k}}(a_{1,k_1}\cdot, \ldots, a_{d,k_d}\cdot) \,|\, S^{\overline{r}}_{2,2}F(\mathbb{R}^d)\|\big) \cdot \|f_{\overline{k}} \,|\, L_p(\ell_q)\|$$

for all systems $\{f_{\overline{k}}\} \in L_p(\ell_q)$ with $\operatorname{supp} \widehat{f_{\overline{k}}} \subset \Omega_{\overline{k}}$ and all systems $\{\varrho_{\overline{k}}\} \subset S^{\overline{r}}_{2,2}F(\mathbb{R}^d)$.

REMARK 1.13. The proof may be found in [26, p. 77].

1.2.6. *Littlewood–Paley theory.* We also state a theorem of Littlewood–Paley type for spaces with dominating mixed smoothness. But first we define the Sobolev spaces with dominating mixed smoothness. This is the very direct generalisation of the definition of Nikol'skiĭ given in the Introduction.

Definition 1.14. Let $1 < p < \infty$ and $\overline{r} = (r_1, \dots, r_d) \in \mathbb{N}_0^d$. We put

$$S_p^{\overline{r}}W(\mathbb{R}^d) = \Big\{ f \in L_p(\mathbb{R}^d) : \|f \,|\, S_p^{\overline{r}}W(\mathbb{R}^d)\| = \sum_{0 \le \alpha \le \overline{r}} \|D^\alpha f \,|\, L_p(\mathbb{R}^d)\| < \infty \Big\}.$$

Clearly, we have $S_p^{\overline{0}}W(\mathbb{R}^d) = L_p(\mathbb{R}^d)$. The connection between $S_p^{\overline{r}}W(\mathbb{R}^d)$ and $S_{p,q}^{\overline{r}}F(\mathbb{R}^d)$ is then given by

Theorem 1.15. *Let $1 < p < \infty$ and $\overline{r} = (r_1, \dots, r_d) \in \mathbb{N}_0^d$. Then*

$$S_p^{\overline{r}}W(\mathbb{R}^d) = S_{p,2}^{\overline{r}}F(\mathbb{R}^d)$$

with equivalent norms.

Remark 1.16. See [26, p. 104] for details.

1.3. Local means. In this section we present the main technical tool, namely, we characterise the spaces $S_{p,q}^{\overline{r}}A(\mathbb{R}^d)$ by the so-called *local means*. In general, we follow the method presented by Rychkov [25]. Recall that the spaces $S_{p,q}^{\overline{r}}A(\mathbb{R}^d)$ were introduced in Definition 1.2 and, according to Theorem 1.5, this definition does *not* depend on the choice of the decomposition of unity $\{\varphi_j\}_{j=0}^\infty \subset \Phi(\mathbb{R})$. Hence we may fix some specific system $\{\varphi_j\}_{j=0}^\infty$ for the rest of our work.

We fix $\varphi \in S(\mathbb{R})$ with

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| \le 4/3, \\ 0 & \text{if } |x| \ge 3/2. \end{cases}$$

We put $\varphi_0 = \varphi$, $\varphi_1(x) = \varphi(x/2) - \varphi(x)$ and

$$\varphi_j(x) = \varphi_1(2^{-j+1}x), \qquad x \in \mathbb{R},\ j \in \mathbb{N}.$$

One verifies easily that (1.5)–(1.7) hold.

1.3.1. *The Peetre maximal operator.* Next we discuss the analogue of the Peetre maximal operator introduced in [23]. The construction of Peetre adapted to the case of function spaces with dominating mixed smoothness assigns to every system $\{\psi_{\overline{k}}\}_{\overline{k} \in \mathbb{N}_0^d} \subset S(\mathbb{R}^d)$, to every distribution $f \in S'(\mathbb{R}^d)$ and to every vector $\overline{a} > 0$ the following quantities:

$$\sup_{y \in \mathbb{R}^d} \frac{|(\psi_{\overline{k}}\widehat{f})^\vee(y)|}{\prod_{i=1}^d (1 + |2^{k_i}(y_i - x_i)|^{a_i})}, \qquad x \in \mathbb{R}^d,\ \overline{k} \in \mathbb{N}_0^d. \tag{1.16}$$

As $\psi_{\overline{k}} \in S(\mathbb{R}^d)$ for every $\overline{k} \in \mathbb{N}_0^d$ the product $\psi_{\overline{k}}\widehat{f}$ is well defined for every $f \in S'(\mathbb{R}^d)$ and, according to the theorem of Paley–Wiener–Schwartz (see [32] and references given there for details), $(\psi_{\overline{k}}\widehat{f})^\vee$ is an analytic function. In particular, $(\psi_{\overline{k}}\widehat{f})^\vee(y)$ makes sense pointwise.

Unfortunately, as we are also interested in non-smooth kernels (for details, see Section 2.4), we need to consider also kernels $\psi_{\overline{k}} \notin S(\mathbb{R}^d)$. We weaken the definition of the Schwartz space $S(\mathbb{R}^d)$ in a natural way and obtain the class of spaces $X^{\overline{S}}(\mathbb{R}^d)$ defined for every $\overline{S} \in \mathbb{N}_0^d$ by

$$X^{\overline{S}}(\mathbb{R}^d) = \{\varphi \in S_2^{\overline{S}}W(\mathbb{R}^d) : \|\varphi \,|\, X^{\overline{S}}(\mathbb{R}^d)\| < \infty\},$$

$$\|\varphi \,|\, X^{\overline{S}}(\mathbb{R}^d)\| = \Big(\sum_{0\le \alpha,\beta \le \overline{S}} \|x^\beta D^\alpha \varphi(x) \,|\, L_2(\mathbb{R}^d)\|^2 \Big)^{1/2}.$$

We set $\omega(x) = \prod_{i=1}^d (1+x_i^2)^{S_i/2}$ and observe that $\varphi \in X^{\overline{S}}(\mathbb{R}^d)$ if, and only if, $\omega \cdot D^\alpha \varphi \in L_2(\mathbb{R}^d)$ for every $0 \le \alpha \le \overline{S}$. This is obviously equivalent to $D^\alpha(\omega \cdot \varphi) \in L_2(\mathbb{R}^d)$ for every $0 \le \alpha \le \overline{S}$, which may be written as $\omega \cdot \varphi \in S_2^{\overline{S}}W(\mathbb{R}^d)$. Hence

$$\varphi \in X^{\overline{S}}(\mathbb{R}^d) \quad \text{if, and only if,} \quad \omega \cdot \varphi \in S_2^{\overline{S}}W(\mathbb{R}^d).$$

This allows us to characterise the dual of $X^{\overline{S}}(\mathbb{R}^d)$. We get

$$\psi \in (X^{\overline{S}}(\mathbb{R}^d))' \quad \text{if, and only if,} \quad \omega^{-1} \cdot \psi \in (S_2^{\overline{S}}W(\mathbb{R}^d))' = S_{2,2}^{-\overline{S}}F(\mathbb{R}^d).$$

As a trivial consequence of the embedding ($\overline{S} \in \mathbb{N}_0^d$)

$$X^{\overline{S}}(\mathbb{R}^d) \hookrightarrow S_2^{\overline{S}}W(\mathbb{R}^d) \hookrightarrow S_{\infty,\infty}^{\overline{S}-1/2}B(\mathbb{R}^d)$$

we get for every $\overline{K} \in \mathbb{N}_0^d$ and every $\overline{S} \ge \overline{K} + 1$,

$$X^{\overline{S}}(\mathbb{R}^d) \hookrightarrow C^{\overline{K}}(\mathbb{R}^d).$$

Having now a function $\Psi_{\overline{k}} \in X^{\overline{S}}(\mathbb{R}^d)$ and some distribution $f \in (X^{\overline{S}}(\mathbb{R}^d))'$, we write

$$(f * \Psi_{\overline{k}})(y) = \int_{\mathbb{R}^d} f(x)\Psi_{\overline{k}}(y-x)\,dx = f(\Psi_{\overline{k}}(y-\cdot)), \quad y \in \mathbb{R}^d.$$

So, given a system $\{\psi_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^d} \subset X^{\overline{S}}(\mathbb{R}^d)$ for some $\overline{S} \in \mathbb{N}_0^d$, we denote $\Psi_{\overline{k}} = \widehat{\psi}_{\overline{k}} \in X^{\overline{S}}(\mathbb{R}^d)$ and define in analogy with (1.16) for every $f \in (X^{\overline{S}}(\mathbb{R}^d))'$,

$$(\Psi_{\overline{k}}^* f)_{\overline{a}}(x) = \sup_{y\in\mathbb{R}^d} \frac{|(\Psi_{\overline{k}} * f)(y)|}{\prod_{i=1}^d (1+|2^{k_i}(y_i-x_i)|^{a_i})}, \qquad x \in \mathbb{R}^d,\ \overline{k} \in \mathbb{N}_0^d. \tag{1.17}$$

Furthermore, for $\overline{S} = \infty$, we put $X^{\overline{S}}(\mathbb{R}^d) = S(\mathbb{R}^d)$.

1.3.2. *Helpful lemmas.* We split the proof of the local-mean characteristics of Besov and Triebel–Lizorkin spaces in two parts and give now the technical lemmas. This will allow us a straightforward proof later on. The lemmas originate in [25] and we quote them only with some minor modifications, mainly forced by the tensor product structure of function spaces with dominated mixed smoothness.

We start with a lemma describing the use of the so-called moment conditions.

LEMMA 1.17. *Let $K \in \mathbb{N}_0$ and $g, h \in X^{K+2}(\mathbb{R})$. Furthermore, let $-1 \le M \le K$ be an integer and*

$$(D^\alpha \widehat{g})(0) = 0, \qquad 0 \le \alpha \le M.$$

Then for every $N \in \mathbb{N}_0$ with $0 \le N \le K$ there is a constant C_N such that

$$\sup_{z\in\mathbb{R}} |(g_b * h)(z)|(1+|z|^N) \le C_N b^{M+1}, \qquad 0 < b < 1, \tag{1.18}$$

where $g_b(t) = b^{-1}g(t/b)$.

Proof. Using the elementary properties of the Fourier transform we get

$$\mathrm{LHS}(1.18) \le c \max_{0\le\alpha\le N} \|D^\alpha[(g_b * h)^\wedge] \,|\, L_1(\mathbb{R})\|.$$

By the Leibniz formula,

$$|D^\alpha[\widehat{g}(b\,\cdot)\widehat{h}(\cdot)](\xi)| \le c \sum_{0\le\beta\le\alpha} b^\beta |(D^\beta\widehat{g})(b\xi)(D^{\alpha-\beta}\widehat{h})(\xi)|, \qquad \xi\in\mathbb{R}. \tag{1.19}$$

As $\widehat{g}\in C^{M+1}(\mathbb{R})$, we may use the Taylor formula to get

$$|(D^\beta\widehat{g})(b\xi)| \le c|b\xi|^{M-\beta+1}, \qquad 0\le\beta\le M, \tag{1.20}$$

for $|b\xi|\le 1$. But, as $D^\beta\widehat{g}\in C(\mathbb{R})$, (1.20) holds for all $b,\xi\in\mathbb{R}$. Hence, for $0\le\beta\le M$, we get

$$b^\beta|(D^\beta\widehat{g})(b\xi)(D^{\alpha-\beta}\widehat{h})(\xi)| \le c\, b^{M+1}|(D^{\alpha-\beta}\widehat{h})(\xi)|\cdot|\xi|^{(M-\beta+1)_+}, \qquad \xi\in\mathbb{R}. \tag{1.21}$$

If $M<\beta\le K$ and $0<b<1$, we have $b^\beta\le b^{M+1}$, which, together with $D^\beta\widehat{g}\in C(\mathbb{R})$, gives (1.21) for all $0\le\beta\le K$.

We put (1.21) into (1.19) and obtain (1.18). ∎

Furthermore, we shall need the following convolution inequality.

LEMMA 1.18. *Let $0<p,q\le\infty$ and $\delta>0$. Let $\{g_{\overline{k}}\}_{\overline{k}\in\mathbb{N}_0^d}$ be a sequence of nonnegative measurable functions on $\mathbb{R}^d$ and let*

$$G_{\overline{\nu}}(x) = \sum_{\overline{k}\in\mathbb{N}_0^d} 2^{-|\overline{\nu}-\overline{k}|\delta} g_{\overline{k}}(x), \qquad x\in\mathbb{R}^d,\ \overline{\nu}\in\mathbb{N}_0^d. \tag{1.22}$$

Then there is some constant $C=C(p,q,\delta)$ such that

$$\|G_{\overline{k}} \,|\, \ell_q(L_p)\| \le C\|g_{\overline{k}} \,|\, \ell_q(L_p)\|, \tag{1.23}$$

$$\|G_{\overline{k}} \,|\, L_p(\ell_q)\| \le C\|g_{\overline{k}} \,|\, L_p(\ell_q)\|. \tag{1.24}$$

Proof. STEP 1. We start with the proof of (1.23). If $p\ge 1$, by the triangle inequality we get

$$\|G_{\overline{\nu}} \,|\, L_p(\mathbb{R}^d)\| \le \sum_{\overline{k}\in\mathbb{N}_0^d} 2^{-|\overline{\nu}-\overline{k}|\delta}\|g_{\overline{k}} \,|\, L_p(\mathbb{R}^d)\|, \qquad \overline{\nu}\in\mathbb{N}_0^d.$$

When $q\le 1$, we use the embedding $\ell_q\hookrightarrow\ell_1$ to get

$$\|G_{\overline{\nu}} \,|\, \ell_q(L_p)\| \le \Big(\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{k}\in\mathbb{N}_0^d} 2^{-|\overline{\nu}-\overline{k}|\delta q}\|g_{\overline{k}} \,|\, L_p(\mathbb{R}^d)\|^q\Big)^{1/q}.$$

Interchanging the order of summation, we get (1.23) with $C=C_1=(\sum_{\overline{k}\in\mathbb{Z}^d} 2^{-|\overline{k}|\delta q})^{1/q}$. If $q>1$, we apply Young's inequality. We define

$$\begin{aligned}\lambda_{\overline{k}} &= 2^{-|\overline{k}|\delta}, \quad \overline{k}\in\mathbb{Z}^d,\\ \gamma_{\overline{k}} &= \begin{cases}\|g_{\overline{k}} \,|\, L_p(\mathbb{R}^d)\| & \text{for } \overline{k}\in\mathbb{N}_0^d,\\ 0 & \text{for } \overline{k}\in\mathbb{Z}^d\setminus\mathbb{N}_0^d.\end{cases}\end{aligned} \tag{1.25}$$

Then we get

$$\|G_{\overline{\nu}} \,|\, L_p(\mathbb{R}^d)\| \le (\lambda*\gamma)(\overline{\nu}), \qquad \overline{\nu}\in\mathbb{N}_0^d,$$

and Young's convolution inequality gives

$$\|\lambda * \gamma \,|\, \ell_q\| \le \|\lambda \,|\, \ell_1\| \cdot \|\gamma \,|\, \ell_q\|.$$

This proves (1.23) with $C = C_2 = \|\lambda \,|\, \ell_1\|$.

If $p < 1$, we use the $\ell_p \hookrightarrow \ell_1$ embedding to get

$$\int_{\mathbb{R}^d} G^p_{\overline{\nu}}(x)\, dx \le \sum_{\overline{k}\in\mathbb{N}_0^d} 2^{-|\overline{\nu}-\overline{k}|\delta p} \int_{\mathbb{R}^d} g^p_{\overline{k}}(x)\, dx$$

For $q/p \le 1$ this implies

$$\sum_{\overline{\nu}\in\mathbb{N}_0^d} \|G_{\overline{\nu}} \,|\, L_p(\mathbb{R}^d)\|^q \le \sum_{\overline{\nu}\in\mathbb{N}_0^d} \sum_{\overline{k}\in\mathbb{N}_0^d} 2^{-|\overline{\nu}-\overline{k}|\delta q} \|g_{\overline{k}} \,|\, L_p(\mathbb{R}^d)\|^q.$$

Now we again interchange the order of summation and take the $(1/q)$th power. This proves (1.23) with $C = C_1$.

Finally, if $q/p > 1$, we use again Young's inequality with λ^p and γ^p instead of λ and γ. This gives

$$\|G_{\overline{\nu}} \,|\, \ell_q(L_p)\|^p \le \|\lambda^p \,|\, \ell_1\| \cdot \|\gamma^p \,|\, \ell_{q/p}\|,$$

which proves (1.23) with $C = \|\lambda|\ell_p\|$.

Step 2. Next we turn to (1.24). This is a trivial consequence of the pointwise inequality

$$\|G_{\overline{\nu}}(x) \,|\, \ell_q\| \le C\|g_{\overline{\nu}}(x) \,|\, \ell_q\|, \qquad x \in \mathbb{R}^d, \tag{1.26}$$

with C independent of $x \in \mathbb{R}^d$.

To prove (1.26), just use the $\ell_q \hookrightarrow \ell_1$ embedding for $q \le 1$ and Young's inequality for $q > 1$. We do not give the details, which are very similar to the calculation in Step 1. ■

As we do not want to exclude the case of arbitrarily smooth functions, we use the following notation. We write that the vector $\overline{N} = \infty$ if $N_i = \infty$ for all $i = 1, \ldots, d$. The symbol $\overline{N} \in \mathbb{N}_0^d \cup \{\infty\}$ then means that either $\overline{N} = \infty$ or $\overline{N}$ is a vector of nonnegative integers.

Lemma 1.19. *Let $0 < r \le 1$, and let $\{\gamma_{\overline{\nu}}\}_{\overline{\nu}\in\mathbb{N}_0^d}$, $\{\beta_{\overline{\nu}}\}_{\overline{\nu}\in\mathbb{N}_0^d}$ be sequences taking values in $(0, \infty)$. Assume that, for some $\overline{N}^0 \in \mathbb{N}_0^d$,*

$$\gamma_{\overline{\nu}} = O(2^{\overline{\nu}\cdot\overline{N}^0}), \qquad |\overline{\nu}| \to \infty. \tag{1.27}$$

Furthermore, assume that there is $\overline{N}^1 \in \mathbb{N}_0^d \cup \{\infty\}$ with $\overline{N}^1 \ge \overline{N}^0$ such that

$$\gamma_{\overline{\nu}} \le C_{\overline{N}} \sum_{\overline{k}\in\mathbb{N}_0^d} 2^{-\overline{k}\cdot\overline{N}} \beta_{\overline{k}+\overline{\nu}} \gamma^{1-r}_{\overline{k}+\overline{\nu}}, \qquad \overline{\nu} \in \mathbb{N}_0^d,\ C_{\overline{N}} < \infty, \tag{1.28}$$

for every $0 \le \overline{N} \le \overline{N}^1$ if $\overline{N}^1$ is finite or for every $\overline{N} \in \mathbb{N}_0^d$ if $\overline{N}^1 = \infty$. Then, for the same set of $\overline{N}$,

$$\gamma^r_{\overline{\nu}} \le C_{\overline{N}} \sum_{\overline{k}\in\mathbb{N}_0^d} 2^{-\overline{k}\cdot\overline{N}r} \beta_{\overline{k}+\overline{\nu}}, \qquad \overline{\nu} \in \mathbb{N}_0^d, \tag{1.29}$$

with the same constants $C_{\overline{N}}$.

Proof. Put

$$\Gamma_{\overline{\nu},\overline{N}} = \sup_{\overline{k}\in\mathbb{N}_0^d} 2^{-\overline{k}\cdot\overline{N}} \gamma_{\overline{k}+\overline{\nu}}, \qquad \overline{\nu}, \overline{N} \in \mathbb{N}_0^d.$$

By (1.28),

$$\begin{aligned}\varGamma_{\overline{\nu},\overline{N}} &\le C_{\overline{N}} \sup_{\overline{k}\in\mathbb{N}_0^d} \sum_{\overline{l}\in\mathbb{N}_0^d} 2^{-(\overline{k}+\overline{l})\cdot\overline{N}} \beta_{\overline{l}+\overline{k}+\overline{\nu}} \gamma^{1-r}_{\overline{l}+\overline{k}+\overline{\nu}} = C_{\overline{N}} \sup_{\overline{k}\in\mathbb{N}_0^d} \sum_{\overline{l}\in\mathbb{N}_0^d+\overline{k}} 2^{-\overline{l}\cdot\overline{N}} \beta_{\overline{l}+\overline{\nu}} \gamma^{1-r}_{\overline{l}+\overline{\nu}}\\ &= C_{\overline{N}} \sum_{\overline{l}\in\mathbb{N}_0^d} 2^{-\overline{l}\cdot\overline{N}} \beta_{\overline{l}+\overline{\nu}} \gamma^{1-r}_{\overline{l}+\overline{\nu}} \le C_{\overline{N}} \varGamma^{1-r}_{\overline{\nu},\overline{N}} \sum_{\overline{l}\in\mathbb{N}_0^d} 2^{-\overline{l}\cdot\overline{N}r} \beta_{\overline{l}+\overline{\nu}}. \end{aligned} \tag{1.30}$$

When $\varGamma_{\overline{\nu},\overline{N}} < \infty$, we finish the proof by

$$\gamma^r_{\overline{\nu}} \le \varGamma^r_{\overline{\nu},\overline{N}} \le C_{\overline{N}} \sum_{\overline{l}\in\mathbb{N}_0^d} 2^{-\overline{l}\cdot\overline{N}r} \beta_{\overline{l}+\overline{\nu}}. \tag{1.31}$$

From (1.27), $\varGamma_{\overline{\nu},\overline{N}}$ is finite for all $\overline{N}^0 \le \overline{N} \le \overline{N}^1$ (or for all $\overline{N}^0 \le \overline{N}$ if $\overline{N}^1 = \infty$). As the right-hand side of (1.29) decreases when $\overline{N}$ increases in any coordinate, this proves (1.29) also for all $\overline{N} \ngeq \overline{N}^0$ with the constant $C_{\overline{N}^*}$, where $\overline{N}^*_i = \max(\overline{N}^0_i, \overline{N}_i)$. Take now any $\overline{N} \ngeq \overline{N}^0$ and apply (1.29) with $C_{\overline{N}^*}$ instead of $C_{\overline{N}}$ to get

$$\begin{aligned}\varGamma_{\overline{\nu},\overline{N}} &= \sup_{\overline{k}\in\mathbb{N}_0^d} 2^{-\overline{k}\cdot\overline{N}} \gamma_{\overline{k}+\overline{\nu}}\\ &\le \sup_{\overline{k}\in\mathbb{N}_0^d} \Big(C_{\overline{N}^*} \sum_{\overline{l}\in\mathbb{N}_0^d} 2^{-(\overline{k}+\overline{l})\cdot\overline{N}r} \beta_{\overline{l}+\overline{k}+\overline{\nu}}\Big)^{1/r} = C^{1/r}_{\overline{N}^*} \Big(\sum_{\overline{l}\in\mathbb{N}_0^d} 2^{-\overline{l}\cdot\overline{N}r} \beta_{\overline{l}+\overline{\nu}}\Big)^{1/r},\end{aligned}$$

which is finite whenever the right-hand side of (1.29) is finite (otherwise there is nothing to prove). So, even in this case, we may apply (1.30) and (1.31) and finish the proof of the lemma. ∎

1.3.3. *Comparison of different Peetre maximal operators.* In this subsection we give an inequality between different Peetre maximal operators. This inequality (together with the boundedness of the Peetre maximal operator) forms the basis for our characterisation of $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ through local means.

Because of the limited smoothness of our kernel functions (discussed in detail in Section 2.4), we cannot expect to get such an inequality for all $f \in S'(\mathbb{R}^d)$.

We start with (given) functions ψ^i_0, ψ^i_1, $i = 1, \dots, d$, defined on $\mathbb{R}$ and set

$$\begin{aligned} \psi^i_j(t) &= \psi^i_1(2^{-j+1}t), \qquad t \in \mathbb{R},\ j = 2, 3, \dots,\\ \psi_{\overline{k}}(x) &= \prod_{i=1}^d \psi^i_{k_i}(x_i), \qquad x \in \mathbb{R}^d,\ \overline{k} \in \mathbb{N}^d_0,\\ \varPsi_{\overline{k}} &= \widehat{\psi_{\overline{k}}}, \qquad \overline{k} \in \mathbb{N}^d_0. \end{aligned} \tag{1.32}$$

To (given) functions ϕ^i_0, ϕ^i_1, $i = 1, \dots, d$, we associate $\phi_{\overline{k}}$ and $\varPhi_{\overline{k}}$ in the same way. Furthermore, we suppose that $\psi_{\overline{k}}, \phi_{\overline{k}} \in X^{\overline{S}}(\mathbb{R}^d)$ for some $\overline{S} \in \mathbb{N}^d_0$.

Using this notation we may state the main result of this section.

Theorem 1.20. *Let $\overline{a}, \overline{r} \in \mathbb{R}^d$, $\overline{R} \in \mathbb{N}^d_0$, $0 < p, q \le \infty$ with $\overline{a} > 0$ and $\overline{r} < \overline{R} + 1$. If $\overline{S} > \overline{R}$ is large enough and*

$$D^l \psi^i(0) = 0, \qquad i = 1, \dots, d,\ l = 0, 1, \dots, R_i, \tag{1.33}$$

and, for every $i = 1, \ldots, d$ and some $\varepsilon > 0$,

$$|\phi_0^i(t)| > 0 \quad \textit{on } \{t \in \mathbb{R} : |t| < \varepsilon\}, \tag{1.34}$$

$$|\phi_1^i(t)| > 0 \quad \textit{on } \{t \in \mathbb{R} : \varepsilon/2 < |t| < 2\varepsilon\}, \tag{1.35}$$

then

$$\|2^{\overline{k}\cdot\overline{r}}(\Psi_{\overline{k}}^* f)_{\overline{a}} \mid \ell_q(L_p)\| \le c\|2^{\overline{k}\cdot\overline{r}}(\Phi_{\overline{k}}^* f)_{\overline{a}} \mid \ell_q(L_p)\|, \tag{1.36}$$

$$\|2^{\overline{k}\cdot\overline{r}}(\Psi_{\overline{k}}^* f)_{\overline{a}} \mid L_p(\ell_q)\| \le c\|2^{\overline{k}\cdot\overline{r}}(\Phi_{\overline{k}}^* f)_{\overline{a}} \mid L_p(\ell_q)\|, \tag{1.37}$$

for all $f \in (X^{\overline{S}}(\mathbb{R}^d))'$.

Proof. Step 1: *formal calculations.* It follows from (1.34) and (1.35) that there exist functions $\{\lambda_j^i\}_{j=0}^\infty$, $i = 1, \ldots, d$, with

$$\sum_{j=0}^\infty \lambda_j^i(t)\phi_j^i(t) = 1, \qquad t \in \mathbb{R}, \tag{1.38}$$

$$\lambda_j^i(t) = \lambda_1^i(2^{-j+1}t), \qquad t \in \mathbb{R},\ j \in \mathbb{N}, \tag{1.39}$$

$$\operatorname{supp} \lambda_0^i \subset \{t \in \mathbb{R} : |t| \le \varepsilon\}, \qquad \operatorname{supp} \lambda_j^i \subset \{t \in \mathbb{R} : 2^{j-2}\varepsilon \le |t| \le 2^j\varepsilon\}, \qquad j \in \mathbb{N}. \tag{1.40}$$

Now we define, as usual, $\lambda_{\overline{k}}(x) = \lambda_{k_1}^1(x_1)\cdots\lambda_{k_d}^d(x_d)$ for every $\overline{k} \in \mathbb{N}_0^d$. From (1.38) we obtain

$$\sum_{\overline{k}\in\mathbb{N}_0^d} \lambda_{\overline{k}}(x)\phi_{\overline{k}}(x) = 1, \qquad x \in \mathbb{R}^d.$$

Finally, we set $\Lambda_{\overline{k}} = \widehat{\lambda}_{\overline{k}}$, $\overline{k} \in \mathbb{N}_0^d$. This gives us the following identities:

$$f = \sum_{\overline{k}\in\mathbb{N}_0^d} \Lambda_{\overline{k}} * \Phi_{\overline{k}} * f, \qquad \Psi_{\overline{\nu}} * f = \sum_{\overline{k}\in\mathbb{N}_0^d} \Psi_{\overline{\nu}} * \Lambda_{\overline{k}} * \Phi_{\overline{k}} * f, \qquad \overline{\nu} \in \mathbb{N}_0^d. \tag{1.41}$$

We have

$$\begin{aligned}
|(\Psi_{\overline{\nu}} * \Lambda_{\overline{k}} * \Phi_{\overline{k}} * f)(y)| &\le \int_{\mathbb{R}^d} |(\Psi_{\overline{\nu}} * \Lambda_{\overline{k}})(z)| \cdot |(\Phi_{\overline{k}} * f)(y-z)|\,dz \\
&\le (\Phi_{\overline{k}}^* f)_{\overline{a}}(y) \int_{\mathbb{R}^d} |(\Psi_{\overline{\nu}} * \Lambda_{\overline{k}})(z)| \prod_{i=1}^d (1 + |2^{k_i} z_i|^{a_i})\,dz \\
&\equiv (\Phi_{\overline{k}}^* f)_{\overline{a}}(y) I_{\overline{\nu}\overline{k}} = (\Phi_{\overline{k}}^* f)_{\overline{a}}(y) \prod_{i=1}^d I_{\nu_i k_i},
\end{aligned} \tag{1.42}$$

where

$$I_{\nu_i k_i} = \int_{\mathbb{R}} |(\Psi_{\nu_i}^i * \Lambda_{k_i}^i)(z_i)|(1 + |2^{k_i} z_i|^{a_i})\,dz_i.$$

We claim that by Lemma 1.17,

$$I_{\nu_i k_i} \le C \begin{cases} 2^{(k_i-\nu_i)(R_i+1)} & \text{if } k_i \le \nu_i, \\ 2^{(\nu_i-k_i)(a_i+|r_i|+1)} & \text{if } k_i \ge \nu_i. \end{cases} \tag{1.43}$$

Namely, we have (for $1 \le k_i < \nu_i$) with the change of variables $2^{k_i} z_i \to z_i$,

$$\begin{aligned} I_{\nu_i k_i} &= \frac{1}{2} \int_{\mathbb{R}} |(\Psi^i_{\nu_i - k_i} * \Lambda^i_1(\cdot/2))(z_i)| (1 + |z_i|^{a_i})\, dz_i \\ &\le c \sup_{z \in \mathbb{R}} |(\Psi^i_{\nu_i - k_i} * \Lambda^i_1(\cdot/2))(z_i)| (1 + |z_i|^{a_i+2}) \le c\, 2^{(k_i - \nu_i)(R_i+1)}, \end{aligned}$$

when S_i are chosen sufficiently large.

Analogously, for $1 \le \nu_i < k_i$ with the change of variables $2^{\nu_i} z_i \to z_i$,

$$\begin{aligned} I_{\nu_i k_i} &\le 2^{(k_i - \nu_i) a_i} \int_{\mathbb{R}} |(\Psi^i_1 * \Lambda^i_{k_i - \nu_i})(z_i)| (1 + |z_i|^{a_i})\, dz_i \\ &\le c\, 2^{(\nu_i - k_i)(-a_i + M + 1)}, \end{aligned}$$

where M may be taken as large as S_i allows. Taking $M > 2a_i + |r_i|$ (which is possible for S_i large enough), we get (1.43). This covers the cases where $\nu_i, k_i \ge 1$, $\nu_i \ne k_i$. The cases $k_i = \nu_i \ge 1$, $k_i > \nu_i = 0$ and $\nu_i > k_i = 0$ can be treated separately in a similar way. The needed moment conditions are always satisfied by (1.33) or (1.40), respectively. The case $k_i = \nu_i = 0$ is covered by the constant C in (1.43).

Next, we point out that

$$\begin{aligned} (\Phi^*_{\overline{k}} f)_{\overline{a}}(y) &\le (\Phi^*_{\overline{k}} f)_{\overline{a}}(x) \prod_{i=1}^d (1 + |2^{k_i}(x_i - y_i)|^{a_i}) \\ &\le c\, (\Phi^*_{\overline{k}} f)_{\overline{a}}(x) \prod_{i=1}^d (1 + |2^{\nu_i}(x_i - y_i)|^{a_i}) \max(1, 2^{(k_i - \nu_i) a_i}). \end{aligned}$$

We put this into (1.42) and use (1.43) to get

$$\begin{aligned} \sup_{y \in \mathbb{R}^d} \frac{|(\Psi_{\overline{\nu}} * \Lambda_{\overline{k}} * \Phi_{\overline{k}} * f)(y)|}{\prod_{i=1}^d (1 + |2^{\nu_i}(x_i - y_i)|^{a_i})} &\le (\Phi^*_{\overline{k}} f)_{\overline{a}}(x) \prod_{i=1}^d I_{\nu_i k_i} \max(1, 2^{(k_i - \nu_i) a_i}) \\ &\le c\, (\Phi^*_{\overline{k}} f)_{\overline{a}}(x) \prod_{i=1}^d \begin{cases} 2^{(k_i - \nu_i)(R_i + 1)} & \text{if } k_i \le \nu_i, \\ 2^{(\nu_i - k_i)(|r_i| + 1)} & \text{if } k_i \ge \nu_i. \end{cases} \end{aligned}$$

This inequality, together with (1.41) and (1.42), gives for

$$\delta = \min\{1, R_i + 1 - r_i; i = 1, \dots, d\} > 0$$

the estimate

$$2^{\overline{\nu} \cdot \overline{r}} (\Psi^*_{\overline{\nu}} f)_{\overline{a}}(x) \le c \sum_{\overline{k} \in \mathbb{N}_0^d} 2^{-|\overline{k} - \overline{\nu}|\delta} 2^{\overline{k} \cdot \overline{r}} (\Phi^*_{\overline{k}} f)_{\overline{a}}(x), \qquad \overline{\nu} \in \mathbb{N}_0^d,\ x \in \mathbb{R}^d.$$

Lemma 1.18 now gives the desired result immediately.

STEP 2: *theoretical background.* In Step 1 we did not take care about problems caused by limited smoothness of the functions ψ^i_j, ϕ^i_j not to disturb the elegant calculation done there. Nevertheless, to complete the proof, we have to fill some gaps. We go through the proof of Step 1 once more and discuss the theoretical aspects of the calculation.

- *Functions* λ^i_j. By the choice $\lambda^i_j(t) = \varphi_j(3t/2\varepsilon)/\phi^i_j(t)$ we ensure (1.38)–(1.40). The functions φ_j, $j \in \mathbb{N}_0$, were fixed at the beginning of Section 1.3. By (1.34) and (1.35) we get $\lambda_{\overline{k}} \in X^{\overline{S}}(\mathbb{R}^d)$.

- *Identities* (1.41). First, we point out that the expression $\Lambda_{\overline{k}} * \Phi_{\overline{k}} * f$ is well defined for every $\overline{k} \in \mathbb{N}_0^d$. As the function $\lambda_{\overline{k}} = \Lambda_{\overline{k}}^{\vee}$ has compact support, we have $\Lambda_{\overline{k}} * \Phi_{\overline{k}} = (\lambda_{\overline{k}}\phi_{\overline{k}})^{\wedge} \in X^{\overline{S}}(\mathbb{R}^d)$. The same holds for $\Psi_{\overline{\nu}} * \Lambda_{\overline{k}} * \Phi_{\overline{k}}$.

  Next we prove the convergence of both sums in (1.41) for every $f \in (X^{\overline{S}}(\mathbb{R}^d))'$ and every $\overline{\nu} \in \mathbb{N}_0^d$ in $(X^{\overline{S}}(\mathbb{R}^d))'$. By duality arguments, it is enough to prove that

$$\sum_{\overline{k}\in\mathbb{N}_0^d} \psi_{\overline{\nu}}\lambda_{\overline{k}}\phi_{\overline{k}}\mu \to \psi_{\overline{\nu}}\mu, \quad \overline{\nu} \in \mathbb{N}_0^d,$$

  in $X^{\overline{S}}(\mathbb{R}^d)$ for every $\mu \in X^{\overline{S}}(\mathbb{R}^d)$. This follows from (1.38) and (1.40).

  Finally, to pass from (1.41) to (1.42), we have to ensure that (1.41) converges also pointwise. More precisely, we need to prove

$$|(\Psi_{\overline{\nu}} * f)(y)| \le \sum_{\overline{k}\in\mathbb{N}_0^d} |(\Psi_{\overline{\nu}} * \Lambda_{\overline{k}} * \Phi_{\overline{k}} * f)(y)| \tag{1.44}$$

  for all $\overline{\nu} \in \mathbb{N}_0^d$ and almost all $y \in \mathbb{R}^d$.

  Fix $\overline{\nu} \in \mathbb{N}_0^d$ and let $f_{\overline{k}}(y) = (\Psi_{\overline{\nu}} * \Lambda_{\overline{k}} * \Phi_{\overline{k}} * f)(y)$. Then we know from (1.42) that

$$|f_{\overline{k}}(y)| \le (\Phi_{\overline{k}}^* f)_{\overline{a}}(y) I_{\overline{\nu}\overline{k}}, \quad y \in \mathbb{R}^d.$$

  By (1.43) (and by Hölder's inequality for $q > 1$)

$$\sum_{\overline{k}\in\mathbb{N}_0^d} \|f_{\overline{k}} \,|\, L_p(\mathbb{R}^d)\| \le c\|2^{\overline{k}\cdot\overline{r}}(\Phi_{\overline{k}}^* f)_{\overline{a}} \,|\, \ell_q(L_p)\|.$$

  So, whenever the right-hand side of (1.36) is finite, we obtain the L_p-convergence of the series $\sum_{\overline{k}\in\mathbb{N}_0^d} |f_{\overline{k}}|$. Hence, this series converges in the Lebesgue measure as well and therefore also pointwise almost everywhere. We recommend [19] as far as various types of convergence of sequences of functions are concerned. So, whenever the right-hand side of (1.36) is finite, we get (1.44).

  When the right-hand side of (1.37) is finite, we use

$$\|2^{\overline{k}\cdot\overline{r}}(\Phi_{\overline{k}}^* f)_{\overline{a}} \,|\, \ell_{\max(p,q)}(L_p)\| \le c\|2^{\overline{k}\cdot\overline{r}}(\Phi_{\overline{k}}^* f)_{\overline{a}} \,|\, L_p(\ell_q)\|$$

  and apply the same arguments as above. ∎

REMARK 1.21. The conditions (1.33) are usually called *moment conditions* while (1.34) and (1.35) are *Tauberian conditions.*

1.3.4. *Boundedness of the Peetre maximal operator.* In this subsection we describe the boundedness of the Peetre maximal operator in the framework of weighted $L_p(\ell_q)$ and $\ell_q(L_p)$ spaces. We use the notation explained at the beginning of Section 1.3.3. In particular, we still suppose that the functions $\psi_{\overline{k}}, \overline{k} \in \mathbb{N}_0^d$, belong to the space $X^{\overline{S}}(\mathbb{R}^d)$, where the vector $\overline{S}$ will be specified later on. Our main result now is

THEOREM 1.22. *Let $\overline{a}, \overline{r} \in \mathbb{R}^d$, $0 < p, q \le \infty$. Suppose that for every $i = 1, \ldots, d$,*

$$|\psi_0^i(t)| > 0 \quad \text{on } \{t \in \mathbb{R} : |t| < \varepsilon\}, \tag{1.45}$$

$$|\psi_1^i(t)| > 0 \quad \text{on } \{t \in \mathbb{R} : \varepsilon/2 < |t| < 2\varepsilon\}. \tag{1.46}$$

(i) *If* $\overline{a} > 1/p$ *and* $\overline{S} > 0$ *is large enough then*

$$\|2^{\overline{k}\cdot\overline{r}}(\Psi^*_{\overline{k}}f)_{\overline{a}}\,|\,\ell_q(L_p)\| \le c\|2^{\overline{k}\cdot\overline{r}}(\Psi_{\overline{k}} * f)\,|\,\ell_q(L_p)\| \tag{1.47}$$

for all $f \in (X^{\overline{S}-\overline{a}-1}(\mathbb{R}^d))'$.

(ii) *If* $\overline{a} > 1/\min(p,q)$ *and* $\overline{S} > 0$ *is large enough then*

$$\|2^{\overline{k}\cdot\overline{r}}(\Psi^*_{\overline{k}}f)_{\overline{a}}\,|\,L_p(\ell_q)\| \le c\|2^{\overline{k}\cdot\overline{r}}(\Psi_{\overline{k}} * f)\,|\,L_p(\ell_q)\| \tag{1.48}$$

for all $f \in (X^{\overline{S}-\overline{a}-1}(\mathbb{R}^d))'$.

Proof. In analogy to (1.38)–(1.40) we find functions $\{\lambda^i_j\}_{j=0}^\infty, i = 1,\dots,d$, with (1.39), (1.40) and

$$\sum_{j=0}^\infty \lambda^i_j(t)\psi^i_j(t) = 1, \qquad t\in\mathbb{R}. \tag{1.49}$$

Instead of (1.41) we now get the identity

$$f = \sum_{\overline{k}\in\mathbb{N}_0^d} \Lambda_{\overline{k}} * \Psi_{\overline{k}} * f.$$

A dilation $t \mapsto 2^{-\nu_i}t$ in (1.49) leads to

$$\Psi_{\overline{\nu}} * f = \sum_{\overline{k}\in\mathbb{N}_0^d} \Lambda_{\overline{k},\overline{\nu}} * \Psi_{\overline{k},\overline{\nu}} * \Psi_{\overline{\nu}} * f, \qquad \overline{\nu}\in\mathbb{N}_0^d, \tag{1.50}$$

where

$$\Lambda_{\overline{k},\overline{\nu}}(\xi) = [\lambda_{\overline{k}}(2^{-\overline{\nu}}\cdot)]^\wedge(\xi) = 2^{|\overline{\nu}|}\Lambda_{\overline{k}}(2^{\overline{\nu}}\xi), \qquad \overline{k},\overline{\nu}\in\mathbb{N}_0^d.$$

$\Psi_{\overline{k},\overline{\nu}}$ is defined similarly. We recall that $2^{\overline{\nu}}\xi = (2^{\nu_1}\xi_1,\dots,2^{\nu_d}\xi_d)$. Hence, for $\overline{k}\ge 1$ and $\overline{\nu}\in\mathbb{N}_0^d$, we obtain $\Psi_{\overline{k},\overline{\nu}} = \Psi_{\overline{k}+\overline{\nu}}$. To simplify the notation, we point out that

$$\psi_{\overline{k}}(2^{-\overline{\nu}}x)\psi_{\overline{\nu}}(x) = \sigma_{\overline{k},\overline{\nu}}(x)\psi_{\overline{k}+\overline{\nu}}(x), \qquad \overline{k},\overline{\nu}\in\mathbb{N}_0^d,$$

where

$$\sigma_{\overline{k},\overline{\nu}}(x) = \prod_{i=1}^d \sigma^i_{k_i,\nu_i}(x_i), \qquad \sigma^i_{k_i,\nu_i}(x_i) = \begin{cases} \psi^i_{\nu_i}(x_i) & \text{if } k_i>0,\\ \psi^i_0(2^{-\nu_i}x_i) & \text{if } k_i = 0.\end{cases}$$

Hence we may rewrite (1.50) as

$$\Psi_{\overline{\nu}} * f = \sum_{\overline{k}\in\mathbb{N}_0^d} \Lambda_{\overline{k},\overline{\nu}} * \widehat{\sigma}_{\overline{k},\overline{\nu}} * \Psi_{\overline{k}+\overline{\nu}} * f, \qquad \overline{\nu}\in\mathbb{N}_0^d. \tag{1.51}$$

By Lemma 1.17,

$$|(\Lambda_{\overline{k},\overline{\nu}} * \widehat{\sigma}_{\overline{k},\overline{\nu}})(z)| \le C_{\overline{N}}2^{|\overline{\nu}|}\frac{2^{-\overline{k}\cdot\overline{N}}}{\prod_{i=1}^d(1+|2^{\nu_i}z_i|^{a_i})}$$

for $\overline{k},\overline{\nu}\in\mathbb{N}_0^d$ with any $\overline{N}\le\overline{S}-2$. The last estimate, together with (1.51), gives

$$|(\Psi_{\overline{\nu}} * f)(y)| \le C_{\overline{N}}2^{|\overline{\nu}|}\sum_{\overline{k}\in\mathbb{N}_0^d}\int_{\mathbb{R}^d}\frac{2^{-\overline{k}\cdot\overline{N}}}{\prod_{i=1}^d(1+|2^{\nu_i}(y_i-z_i)|^{a_i})}|(\Psi_{\overline{k}+\overline{\nu}} * f)(z)|\,dz \tag{1.52}$$

Fix now any $s \in (0,1]$. Divide both sides of (1.52) by $\prod_{i=1}^d (1+|2^{\nu_i}(x_i-y_i)|^{a_i})$, take the supremum over $y \in \mathbb{R}^d$ and apply the inequalities

$$(1+|2^{\nu_i}(y_i-z_i)|^{a_i})(1+|2^{\nu_i}(x_i-y_i)|^{a_i}) \geq c(1+|2^{\nu_i}(x_i-z_i)|^{a_i}),$$

$$|(\Psi_{\overline{k}+\overline{\nu}} * f)(z)| \leq |(\Psi_{\overline{k}+\overline{\nu}} * f)(z)|^s (\Psi^*_{\overline{k}+\overline{\nu}} f)_{\overline{a}}(x)^{1-s} \prod_{i=1}^d (1+|2^{k_i+\nu_i}(x_i-z_i)|^{a_i})^{1-s},$$

$$\frac{(1+|2^{k_i+\nu_i}(x_i-z_i)|^{a_i})^{1-s}}{(1+|2^{\nu_i}(x_i-z_i)|^{a_i})} \leq \frac{2^{k_i a_i}}{(1+|2^{k_i+\nu_i}(x_i-z_i)|^{a_i})^s}.$$

Finally, we get

$$(\Psi^*_{\overline{\nu}} f)_{\overline{a}}(x) \leq c_{\overline{N}} \sum_{\overline{k}\in\mathbb{N}_0^d} 2^{\overline{k}\cdot(\overline{a}-\overline{N}-1)} (\Psi^*_{\overline{k}+\overline{\nu}} f)_{\overline{a}}(x)^{1-s} \int\limits_{\mathbb{R}^d} \frac{2^{|\overline{k}+\overline{\nu}|}|(\Psi_{\overline{k}+\overline{\nu}} * f)(z)|^s}{\prod_{i=1}^d (1+|2^{k_i+\nu_i}(x_i-z_i)|^{a_i})^s}\, dz,$$

and apply Lemma 1.19 with

$$\gamma_{\overline{\nu}} = (\Psi^*_{\overline{\nu}} f)_{\overline{a}}(x), \qquad \beta_{\overline{\nu}} = \int\limits_{\mathbb{R}^d} \frac{2^{|\overline{\nu}|}|(\Psi_{\overline{\nu}} * f)(z)|^s}{\prod_{i=1}^d (1+|2^{\nu_i}(x_i-z_i)|^{a_i})^s}\, dz, \qquad \overline{\nu}\in\mathbb{N}_0^d,$$

$\overline{N}^1 = \overline{S} - \overline{a} - 1$ and $\overline{N}^0$ giving the order of the distribution f, which is finite for $\overline{S} = \infty$ and smaller than $\overline{S}$ if $\overline{S}$ is finite.

By Lemma 1.19, we obtain, for every $\overline{N} \leq \overline{S} - \overline{a} - 1$, $x \in \mathbb{R}^d$ and $\overline{\nu} \in \mathbb{N}_0^d$,

$$(\Psi^*_{\overline{\nu}} f)_{\overline{a}}(x)^s \leq C_{\overline{N}} \sum_{\overline{k}\in\mathbb{N}_0^d} 2^{-\overline{k}\cdot\overline{N}s} \int\limits_{\mathbb{R}^d} \frac{2^{|\overline{k}+\overline{\nu}|}|(\Psi_{\overline{k}+\overline{\nu}} * f)(z)|^s}{\prod_{i=1}^d (1+|2^{k_i+\nu_i}(x_i-z_i)|^{a_i})^s}\, dz. \tag{1.53}$$

We point out that (1.53) holds for $s > 1$ as well with much simpler proof. In that case, we take (1.52) with $\overline{a}+1$ instead of $\overline{a}$, divide by $\prod_{i=1}^d (1+|2^{\nu_i}(x_i-y_i)|^{a_i})$ and apply Hölder's inequality for series and integrals.

We now choose $s > 0$ with $1/a_i < s < p$ (or $1/a_i < s < \min(p,q)$, respectively) for every $i = 1, \dots, d$. Then

$$\frac{1}{\prod_{i=1}^d (1+|z_i|)^{a_i s}} \in L_1(\mathbb{R}^d),$$

and by the majorant property of the Hardy–Littlewood maximal operator $\overline{M}$ (see [28, Chapter 2]) it follows that

$$(\Psi^*_{\overline{\nu}} f)_{\overline{a}}(x)^s \leq C'_{\overline{N}} \sum_{\overline{k}\in\mathbb{N}_0^d} 2^{-\overline{k}\cdot\overline{N}s}\, \overline{M}(|\Psi_{\overline{k}+\overline{\nu}} * f|^s)(x). \tag{1.54}$$

We choose $\overline{N} > 0$ such that $\overline{N} > -\overline{r}$ and set

$$g_{\overline{k}}(x) = 2^{\overline{k}\cdot\overline{r}s}\, \overline{M}(|\Psi_{\overline{k}} * f|^s)(x).$$

Then from (1.54) we get

$$G_{\overline{\nu}}(x) = 2^{\overline{\nu}\cdot\overline{r}s} (\Psi^*_{\overline{\nu}} f)_{\overline{a}}(x)^s \leq C'_{\overline{N}} \sum_{\overline{k}\geq\overline{\nu}} 2^{s(\overline{k}-\overline{\nu})(-\overline{N}-\overline{r})} g_{\overline{k}}(x).$$

Hence, for $0<\delta<\min\{N_i+r_i \mid i=1,\dots,d\}$, we may apply Lemma 1.18 with $L_{p/s}(\ell_{q/s})$ and $\ell_{q/s}(L_{p/s})$ norm respectively. This results in

$$\|2^{\overline{k}\cdot\overline{r}s}(\Psi^*_{\overline{k}}f)_{\overline{a}}(x)\,|\,\ell_{q/s}(L_{p/s})\|\le c\|2^{\overline{k}\cdot\overline{r}s}\overline{M}(|\Psi_{\overline{k}}*f|^s)(x)\,|\,\ell_{q/s}(L_{p/s})\| \tag{1.55}$$

and

$$\|2^{\overline{k}\cdot\overline{r}s}(\Psi^*_{\overline{k}}f)_{\overline{a}}(x)\,|\,L_{p/s}(\ell_{q/s})\|\le c\|2^{\overline{k}\cdot\overline{r}s}\overline{M}(|\Psi_{\overline{k}}*f|^s)(x)\,|\,L_{p/s}(\ell_{q/s})\|. \tag{1.56}$$

In the first case, we rewrite the left-hand side of (1.55) and use the classical Hardy–Littlewood Theorem (see (1.12) for details, we recall that $s<p$) to obtain

$$\|2^{\overline{k}\cdot\overline{r}}(\Psi^*_{\overline{k}}f)_{\overline{a}}(x)\,|\,\ell_q(L_p)\|\le c\|2^{\overline{k}\cdot\overline{r}}(\Psi_{\overline{k}}*f)(x)\,|\,\ell_q(L_p)\|.$$

In the second case, we rewrite the left-hand side of (1.56) and use Theorem 1.11 (now we recall that $s<\min(p,q)$) to get

$$\|2^{\overline{k}\cdot\overline{r}}(\Psi^*_{\overline{k}}f)_{\overline{a}}(x)\,|\,L_p(\ell_q)\|\le c\|2^{\overline{k}\cdot\overline{r}}(\Psi_{\overline{k}}*f)(x)\,|\,L_p(\ell_q)\|,$$

which concludes the proof.

1.3.5. *Local means characterisation.* We summarise Sections 1.3.3 and 1.3.4 and give the usual formulation of the local means characterisation. We still use the tensor construction of functions $\psi_{\overline{k}}$ described at the beginning of Section 1.3.3. The spaces $X^{\overline{S}}(\mathbb{R}^d)$ and the Peetre maximal function $(\Psi^*_{\overline{k}}f)_{\overline{a}}$ were defined in Section 1.3.1. We still suppose that $\psi_0^i,\psi_1^i\in X^{\overline{S}}(\mathbb{R}^d)$, where the vector $\overline{S}$ will be specified later on.

THEOREM 1.23. (i) *Let* $0<p,q\le\infty$, $\overline{r},\overline{a}\in\mathbb{R}^d$, $\overline{R},\overline{S}\in\mathbb{Z}^d$ *with* $\overline{r}\le\overline{R}+1$ *and* $\overline{a}>1/p$. *If* $\overline{S}>\overline{R}$ *is large enough and*

$$D^\alpha\psi_1^i(0)=0,\quad i=1,\dots,d,\quad \alpha=0,1,\dots,R_i, \tag{1.57}$$

and

$$|\psi_0^i(t)|>0\quad \textit{on } \{t\in\mathbb{R}:|t|<\varepsilon\}, \tag{1.58}$$

$$|\psi_1^i(t)|>0\quad \textit{on } \{t\in\mathbb{R}:\varepsilon/2<|t|<2\varepsilon\} \tag{1.59}$$

for some $\varepsilon>0$, *then*

$$\|f\,|\,S^{\overline{r}}_{p,q}B(\mathbb{R}^d)\|\approx\|2^{\overline{k}\cdot\overline{r}}(\Psi_{\overline{k}}*f)\,|\,\ell_q(L_p)\|\approx\|2^{\overline{k}\cdot\overline{r}}(\Psi^*_{\overline{k}}f)_{\overline{a}}\,|\,\ell_q(L_p)\|$$

for all $f\in(X^{\overline{S}-\overline{a}-1}(\mathbb{R}^d))'$.

(ii) *Let* $0<p<\infty, 0<q\le\infty$, $\overline{r},\overline{a}\in\mathbb{R}^d$, $\overline{R},\overline{S}\in\mathbb{Z}^d$ *with* $\overline{r}\le\overline{R}+1$ *and* $\overline{a}>1/\min(p,q)$. *If* $\overline{S}>\overline{R}$ *is large enough, and* (1.57)–(1.59) *are satisfied, then*

$$\|f\,|\,S^{\overline{r}}_{p,q}F(\mathbb{R}^d)\|\approx\|2^{\overline{k}\cdot\overline{r}}(\Psi_{\overline{k}}*f)\,|\,L_p(\ell_q)\|\approx\|2^{\overline{k}\cdot\overline{r}}(\Psi^*_{\overline{k}}f)_{\overline{a}}\,|\,L_p(\ell_q)\|$$

for all $f\in(X^{\overline{S}-\overline{a}-1}(\mathbb{R}^d))'$.

REMARK 1.24. 1. Theorem 1.23 is just a reformulation of Theorems 1.20 and 1.22.
2. In the proof of Theorems 1.20 and 1.22 we followed essentially the approach described in [25]. We point out that recently very similar results were obtained in [3].
3. We may set $\overline{S}=\infty$ in Theorem 1.23. Then one obtains equivalent quasinorms on $S'(\mathbb{R}^d)$. By choosing $\overline{S}$ large, but finite, we may always ensure that the new quasinorms are equivalent at least on $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)\subset(X^{\overline{S}-\overline{a}-1}(\mathbb{R}^d))'$.

Next we reformulate Theorem 1.23 using the local means in the sense of [33].

THEOREM 1.25. *Let $0 < p, q \leq \infty$ (with $p < \infty$ in the F-case), $\overline{r} \in \mathbb{R}^d$, $\overline{S}^1, \overline{S}^2 \in \mathbb{N}_0^d$ with $\overline{S}^1 - \overline{S}^2 > 1/p + 1$ in the B-case and $\overline{S}^1 - \overline{S}^2 > 1/\min(p, q) + 1$ in the F-case. Let $\overline{R} \in \mathbb{N}_0^d$ be a vector of d nonnegative integers with $\overline{R} > \overline{r}$. Further let $k_0, k^1, \ldots, k^d$ be complex-valued functions from $X^{\overline{S}^1}(\mathbb{R})$ whose supports lie in the set $\{t \in \mathbb{R} : |t| < 1\}$ and*

$$F_1(k_0)(0) \neq 0, \quad F_1(k^i)(0) \neq 0, \quad i = 1, \ldots, d. \tag{1.60}$$

Define

$$k_0^i(t) = k_0(t). \quad k_n^i(t) = 2^n \left(\frac{d^{R_i}}{dt^{R_i}} k^i \right)(2^n t), \quad i = 1, \ldots, d, \ n \in \mathbb{N}, \ t \in \mathbb{R}.$$

As usual, denote by $k_{\overline{\nu}}(x) = k_{\nu_1}^1(x_1) \cdots k_{\nu_d}^d(x_d)$, $\overline{\nu} = (\nu_1, \ldots, \nu_d) \in \mathbb{N}_0^d$, the tensor product of these functions. The corresponding local means are defined by

$$k_{\overline{\nu}}(f)(x) = \int_{\mathbb{R}^d} k_{\overline{\nu}}(y) f(x + y)\, dy, \quad \overline{\nu} \in \mathbb{N}_0^d, \ x \in \mathbb{R}^d, \tag{1.61}$$

appropriately interpreted for any $f \in (X^{\overline{S}^1}(\mathbb{R}^d))'$. Then, if $\overline{S}^2$ is large enough,

$$\|2^{\overline{\nu} \cdot \overline{r}} k_{\overline{\nu}}(f) \,|\, L_p(\ell_q)\| \approx \|f \,|\, S_{p,q}^{\overline{r}} F(\mathbb{R}^d)\|, \quad f \in (X^{\overline{S}^2}(\mathbb{R}^d))', \tag{1.62}$$

and

$$\|2^{\overline{\nu} \cdot \overline{r}} k_{\overline{\nu}}(f) \,|\, \ell_q(L_p)\| \approx \|f \,|\, S_{p,q}^{\overline{r}} B(\mathbb{R}^d)\|, \quad f \in (X^{\overline{S}^2}(\mathbb{R}^d))'. \tag{1.63}$$

Proof. Put $\psi_0^i = F_1^{-1} k_0$ and $\psi_1^i = F_1^{-1}\big(\frac{d^{R_i}}{dt^{R_i}} k^i\big)$. Then the Tauberian conditions (1.58) and (1.59) are satisfied and (1.57) is also true. If we define $\psi_{\overline{\nu}}$, $\overline{\nu} \in \mathbb{N}_0^d$, as in (1.32), we get

$$\begin{aligned}(\psi_{\overline{\nu}} \widehat{f})^\vee(x) &= c \int_{\mathbb{R}^d} (\psi_{\overline{\nu}})^\vee(y) f(x - y)\, dy = c \int_{\mathbb{R}^d} (F \psi_{\overline{\nu}})(y) f(x + y)\, dy \\ &= c \int_{\mathbb{R}^d} \Big(\prod_{i=1}^d (F_1 \psi_{\nu_i}^i)(y_i) \Big) f(x + y)\, dy.\end{aligned} \tag{1.64}$$

Finally, if $\nu_i = 0$ we get $(F_1 \psi_0^i)(y_i) = k_0^i(y_i)$, and if $\nu_i \geq 1$ we obtain in a similar way

$$(F_1 \psi_{\nu_i}^i)(y_i) = (F_1(\psi^i(2^{-\nu_i} \cdot)))(y_i) = 2^{\nu_i} (F_1 \psi^i)(2^{\nu_i} y_i) = 2^{\nu_i} \left(\frac{d^{R_i}}{dt^{R_i}} k^i \right)(2^{\nu_i} y_i) = k_{\nu_i}^i(y_i).$$

Using this calculation and (1.64) we get

$$(\psi_{\overline{\nu}} \widehat{f})^\vee(x) = \int_{\mathbb{R}^d} k_{\overline{\nu}}(y) f(x + y)\, dy, \quad \overline{\nu} \in \mathbb{N}_0^d, \ x \in \mathbb{R}^d,$$

and the theorem follows. ∎

REMARK 1.26. We point out that $\overline{S}^1 = \overline{S}^2 = \infty$ is allowed in Theorem 1.25.

We shall need some other modifications of Theorem 1.23. But first we give some necessary notation. For $\overline{\nu} \in \mathbb{N}_0^d$ and $\overline{m} \in \mathbb{Z}^d$ we denote by $Q_{\overline{\nu}\,\overline{m}}$ the cube with centre at $2^{-\overline{\nu}} \overline{m} = (2^{-\nu_1} m_1, \ldots, 2^{-\nu_d} m_d)$ and with sides parallel to coordinate axes and of lengths $2^{-\nu_1}, \ldots, 2^{-\nu_d}$. Hence

$$Q_{\overline{\nu}\,\overline{m}} = \{x \in \mathbb{R}^d : |x_i - 2^{-\nu_i} m_i| \leq 2^{-\nu_i - 1}, \ i = 1, \ldots, d\}, \quad \overline{\nu} \in \mathbb{N}_0^d, \ \overline{m} \in \mathbb{Z}^d. \tag{1.65}$$

If $\gamma > 0$ then $\gamma Q_{\overline{\nu}\,\overline{m}}$ denotes the cube concentric with $Q_{\overline{\nu}\,\overline{m}}$ with sides also parallel to coordinate axes and of lengths $\gamma 2^{-\nu_1}, \dots, \gamma 2^{-\nu_d}$.

Defining the Peetre maximal function by (1.17), we get

$$(\Psi^*_{\overline{\nu}} f)_{\overline{a}}(x) \geq c \sup_{x-y \in \gamma Q_{\overline{\nu},0}} |(\Psi_{\overline{\nu}} * f)(y)|, \qquad \overline{\nu} \in \mathbb{N}_0^d,\ x \in \mathbb{R}^d,$$

where the constant c depends on $\overline{a}, \gamma > 0$ but neither on x nor on $\overline{\nu}$. This very simple observation together with Theorem 1.23 gives the following

Theorem 1.27. *Let $\overline{r} \in \mathbb{R}^d$ and $0 < p, q \leq \infty$ ($p < \infty$ in the F-case). Let $\overline{R} \in \mathbb{N}_0^d$ with $\overline{R} > \overline{r}$, $\overline{S}^1, \overline{S}^2 \in \mathbb{N}_0^d$ and $k_{\overline{\nu}}$ be as in Theorem 1.25. Then, for any $\gamma > 0$,*

$$\Big\|\Big(\sum_{\overline{\nu} \in \mathbb{N}_0^d} 2^{q\overline{\nu}\cdot\overline{r}} \sup_{x-y \in \gamma Q_{\overline{\nu},0}} |k_{\overline{\nu}}(f)(y)|^q\Big)^{1/q} \Big| L_p(\mathbb{R}^d)\Big\| \approx \|f \,|\, S^{\overline{r}}_{p,q}F(\mathbb{R}^d)\|, \qquad f \in (X^{\overline{S}^2}(\mathbb{R}^d))', \tag{1.66}$$

and

$$\Big(\sum_{\overline{\nu} \in \mathbb{N}_0^d} 2^{q\overline{\nu}\cdot\overline{r}} \|\sup_{x-y \in \gamma Q_{\overline{\nu},0}} |k_{\overline{\nu}}(f)(y)| \,|\, L_p(\mathbb{R}^d)\|^q\Big)^{1/q} \approx \|f \,|\, S^{\overline{r}}_{p,q}B(\mathbb{R}^d)\|, \qquad f \in (X^{\overline{S}^2}(\mathbb{R}^d))'. \tag{1.67}$$

Another modification of Theorem 1.23 is rather technical and deals with "directional" local means, namely with local means of the form ($d = 2$)

$$\int_{\mathbb{R}} k^1_{\nu_1}(y_1) f(x_1 + y_1, x_2)\, dy_1.$$

To introduce these local means in the general dimension, we define for every $A \subset \{1, \dots, d\}$,

$$k_{\overline{\nu},A}(f)(x) = \int_{\mathbb{R}^{|A|}} \Big(\prod_{i \in A} k^i_{\nu_i}(y_i)\Big) f(x_1 + y_1\chi_A(1), \dots, x_d + y_d\chi_A(d)) \Big(\prod_{i \in A} dy_i\Big). \tag{1.68}$$

This means that we restrict the integration in (1.61) to those variables y_i for which $i \in A$. The others are left unchanged.

Using this notation, we may state our next lemma.

Lemma 1.28. *Let $0 < p < \infty$, $0 < q \leq \infty$, $A \subset \{1, \dots, d\}$ and $\gamma > 0$. Let $\overline{r} \in \mathbb{R}^d$ be such that $r_i > 1/\min(p, q)$ for $i \notin A$. Let $R_i \in \mathbb{N}_0$ and $k^i_{\overline{\nu}}$ be as in Theorem 1.25 for every $i \in A$. Further let $k_{\overline{\nu},A}(f)$ be defined by (1.68). Then*

$$\Big\|\Big(\sum_{\substack{\overline{\nu} \in \mathbb{N}_0^d \\ \nu_i = 0,\, i \notin A}} 2^{q\overline{\nu}\cdot\overline{r}} \sup_{x-y \in \gamma Q_{\overline{\nu},0}} |k_{\overline{\nu},A}(f)(y)|^q\Big)^{1/q} \Big| L_p(\mathbb{R}^d)\Big\| \leq c\|f \,|\, S^{\overline{r}}_{p,q}F(\mathbb{R}^d)\| \tag{1.69}$$

for every $f \in S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$. The sum is taken over all $\overline{\nu} = (\nu_1, \dots, \nu_d) \in \mathbb{N}_0^d$ with $\nu_i = 0$ whenever $i \notin A$. The L_p-quasinorm is then taken with respect to x.

Remark 1.29. There is again a direct analogue of this lemma for the B-scale and for nonsmooth kernels. The proof follows the proof of Theorem 1.23.

2. Decomposition theorems

In this chapter we present three decomposition theorems. We give atomic, subatomic and wavelet decomposition characterisations of spaces with dominating mixed smoothness. But first of all we explain some notation used in connection with sequence spaces.

2.1. Sequence spaces. We recall that for $\overline{\nu} \in \mathbb{N}_0^d$ and $\overline{m} \in \mathbb{Z}^d$ we denote by $Q_{\overline{\nu}\,\overline{m}}$ the cube with centre at $2^{-\overline{\nu}}\overline{m} = (2^{-\nu_1}m_1, \ldots, 2^{-\nu_d}m_d)$ and with sides parallel to coordinate axes and of lengths $2^{-\nu_1}, \ldots, 2^{-\nu_d}$. By $\chi^{(p)}_{\overline{\nu}\,\overline{m}}$ we denote the p-normalised characteristic function of $Q_{\overline{\nu}\,\overline{m}}$, that is, $\chi^{(p)}_{\overline{\nu}\,\overline{m}}(x) = 2^{|\overline{\nu}|/p}\chi_{Q_{\overline{\nu}\,\overline{m}}}(x)$. Furthermore, we write $\chi_{\overline{\nu}\,\overline{m}}(x) = \chi_{Q_{\overline{\nu}\,\overline{m}}}(x)$.

Definition 2.1. If $0 < p, q \le \infty$, $\overline{r} \in \mathbb{R}^d$ and

$$\lambda = \{\lambda_{\overline{\nu}\,\overline{m}} \in \mathbb{C} : \overline{\nu} \in \mathbb{N}_0^d,\ \overline{m} \in \mathbb{Z}^d\} \tag{2.1}$$

then we define

$$s^{\overline{r}}_{p,q}b = \Big\{\lambda : \|\lambda \,|\, s^{\overline{r}}_{p,q}b\| = \Big(\sum_{\overline{\nu}\in\mathbb{N}_0^d} 2^{\overline{\nu}\cdot(\overline{r}-1/p)q}\Big(\sum_{\overline{m}\in\mathbb{Z}^d} |\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{q/p}\Big)^{1/q} < \infty\Big\} \tag{2.2}$$

and

$$s^{\overline{r}}_{p,q}f = \Big\{\lambda : \|\lambda \,|\, s^{\overline{r}}_{p,q}f\| = \Big\|\Big(\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} |2^{\overline{\nu}\cdot\overline{r}}\lambda_{\overline{\nu}\,\overline{m}}\chi_{\overline{\nu}\,\overline{m}}(\cdot)|^q\Big)^{1/q} \,\Big|\, L_p(\mathbb{R}^d)\Big\| < \infty\Big\} \tag{2.3}$$

with the usual modification for p and/or q equal to ∞.

Remark 2.2. We point out that with λ given by (2.1) and $g_{\overline{\nu}}(x) = \sum_{\overline{m}\in\mathbb{Z}^d} \lambda_{\overline{\nu}\,\overline{m}}\chi_{\overline{\nu}\,\overline{m}}(x)$, we obtain

$$\|\lambda \,|\, s^{\overline{r}}_{p,q}b\| = \|2^{\overline{\nu}\cdot\overline{r}}g_{\overline{\nu}} \,|\, \ell_q(L_p)\|, \qquad \|\lambda \,|\, s^{\overline{r}}_{p,q}f\| = \|2^{\overline{\nu}\cdot\overline{r}}g_{\overline{\nu}} \,|\, L_p(\ell_q)\|.$$

Sequence spaces of this kind were denoted by E_{dis} in [14] and may be viewed as a discrete version of $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ and $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$.

2.2. Atomic decomposition

Definition 2.3. Let $\overline{K} \in \mathbb{N}_0^d, \overline{L}+1 \in \mathbb{N}_0^d$, and $\gamma > 1$. A $\overline{K}$-times differentiable complex-valued function a is called a $[\overline{K}, \overline{L}]$-*atom centred at* $Q_{\overline{\nu}\,\overline{m}}$ if

$$\operatorname{supp} a \subset \gamma Q_{\overline{\nu}\,\overline{m}}, \tag{2.4}$$

$$|D^\alpha a(x)| \le 2^{\alpha\cdot\overline{\nu}} \quad \text{for } 0 \le \alpha \le \overline{K} \tag{2.5}$$

and

$$\int_{\mathbb{R}} x_i^j a(x)\,dx_i = 0 \quad \text{if } i = 1, \ldots, d;\ j = 0, \ldots, L_i \text{ and } \nu_i \ge 1. \tag{2.6}$$

Using this notation we may state the atomic decomposition theorem.

Theorem 2.4. *Let $0 < p, q \le \infty$ ($p < \infty$ in the F-case) and $\overline{r} \in \mathbb{R}^d$. Fix $\overline{K} \in \mathbb{N}_0^d$ and $\overline{L}+1 \in \mathbb{N}_0^d$ with*

$$K_i \ge (1 + [r_i])_+, \quad L_i \ge \max(-1, [\sigma_{pq} - r_i]), \quad i = 1, \ldots, d \tag{2.7}$$

($L_i \ge \max(-1, [\sigma_p - r_i])$ in the B-case).

(i) *If* $\lambda \in s^{\overline{r}}_{p,q}a$ *and* $\{a_{\overline{\nu}\,\overline{m}}(x)\}_{\overline{\nu}\in\mathbb{N}_0^d,\,\overline{m}\in\mathbb{Z}^d}$ *are* $[\overline{K},\overline{L}]$*-atoms centred at* $Q_{\overline{\nu}\,\overline{m}}$*, then the sum*

$$\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d}\lambda_{\overline{\nu}\,\overline{m}}a_{\overline{\nu}\,\overline{m}}(x) \tag{2.8}$$

converges in $S'(\mathbb{R}^d)$*, its limit* f *belongs to the space* $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ *and*

$$\|f\,|\,S^{\overline{r}}_{p,q}A(\mathbb{R}^d)\| \le c\|\lambda\,|\,s^{\overline{r}}_{p,q}a\|, \tag{2.9}$$

where the constant c *is universal for all admissible* λ *and* $a_{\overline{\nu}\,\overline{m}}$*.*

(ii) *For every* $f \in S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ *there is a* $\lambda \in s^{\overline{r}}_{p,q}a$ *and* $[\overline{K},\overline{L}]$*-atoms centred at* $Q_{\overline{\nu}\,\overline{m}}$ *(denoted again by* $\{a_{\overline{\nu}\,\overline{m}}(x)\}_{\overline{\nu}\in\mathbb{N}_0^d,\,\overline{m}\in\mathbb{Z}^d}$*) such that the sum* (2.8) *converges in* $S'(\mathbb{R}^d)$ *to* f *and*

$$\|\lambda\,|\,s^{\overline{r}}_{p,q}a\| \le c\|f\,|\,S^{\overline{r}}_{p,q}A(\mathbb{R}^d)\|. \tag{2.10}$$

The constant c *is again universal for every* $f \in S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$*.*

Proof. We give the proof only for the F-case. The proof for the B-scale is very similar.

STEP 1. First of all we prove the convergence of (2.8) in $S'(\mathbb{R}^d)$. Let $\varphi \in S(\mathbb{R}^d)$. We use the Taylor expansion of φ with respect to the first variable,

$$\begin{aligned}\varphi(y) = &\sum_{\alpha_1\le L_1}\frac{D^{(\alpha_1,0,\dots,0)}\varphi(2^{-\nu_1}m_1,y_2,\dots,y_d)}{\alpha_1!}\,(y_1-2^{\nu_1}m_1)^{\alpha_1} \\ &+\frac{1}{L_1!}\int_{2^{-\nu_1}m_1}^{y_1}(t_1-2^{-\nu_1}m_1)^{L_1}D^{(L_1+1,0,\dots,0)}\varphi(t_1,y_2,\dots,y_d)\,dt_1,\end{aligned} \tag{2.11}$$

and (2.6) to obtain

$$\begin{aligned}&\int_{\mathbb{R}^d}a_{\overline{\nu}\,\overline{m}}(y)\varphi(y)\,dy \\ &\quad=\int_{\mathbb{R}^d}\frac{a_{\overline{\nu}\,\overline{m}}(y)}{L_1!}\int_{2^{-\nu_1}m_1}^{y_1}(t_1-2^{-\nu_1}m_1)^{L_1}D^{(L_1+1,0,\dots,0)}\varphi(t_1,y_2,\dots,y_d)\,dt_1\,dy.\end{aligned} \tag{2.12}$$

Using an analogue of (2.11) iteratively for the remaining $d-1$ variables we see that the left-hand side of (2.12) is equal to

$$\int_{\mathbb{R}^d}\frac{a_{\overline{\nu}\,\overline{m}}(y)}{\overline{L}!}\int_{2^{-\nu_1}m_1}^{y_1}\dots\int_{2^{-\nu_d}m_d}^{y_d}\prod_{i=1}^d(t_i-2^{-\nu_i}m_i)^{L_i}D^{\overline{L}+1}\varphi(t_1,\dots,t_d)\,dt\,dy.$$

Using the support property (2.4) of $a_{\overline{\nu}\,\overline{m}}$ we may estimate the absolute value of the inner d-dimensional integral from above by ($y \in \gamma Q_{\overline{\nu}\,\overline{m}}$)

$$c\,2^{-\overline{\nu}\cdot(\overline{L}+1)}\sup_{x\in\gamma Q_{\overline{\nu}\,\overline{m}}}|(D^{\overline{L}+1}\varphi)(x)| \le c_M 2^{-\overline{\nu}\cdot(\overline{L}+1)}\langle y\rangle^{-M}\sup_{x\in\gamma Q_{\overline{\nu}\,\overline{m}}}\langle x\rangle^M|(D^{\overline{L}+1}\varphi)(x)|,$$

where M is at our disposal. Here we write $\langle x\rangle = (1+|x|^2)^{1/2}$ for $x \in \mathbb{R}^d$.

Now suppose that $p \ge 1$ and use (2.5) and Hölder's inequality to get, for M large enough,

$$\Big|\int\limits_{\mathbb{R}^d}\sum_{\overline{m}\in\mathbb{Z}^d}\lambda_{\overline{\nu}\,\overline{m}}a_{\overline{\nu}\,\overline{m}}(y)\varphi(y)\,dy\Big|$$
$$\le c\,2^{-\overline{\nu}\cdot(\overline{L}+1)}2^{-\overline{\nu}\cdot(1/p)}\sup_{x\in\mathbb{R}^d}\langle x\rangle^M|(D^{\overline{L}+1}\varphi)(x)|\int\limits_{\mathbb{R}^d}\Big(\sum_{\overline{m}\in\mathbb{Z}^d}2^{\overline{\nu}\cdot(1/p)}|\lambda_{\overline{\nu}\,\overline{m}}|\chi_{\gamma Q_{\overline{\nu}\,\overline{m}}}(y)\Big)\langle y\rangle^{-M}\,dy$$
$$\le c\,2^{-\overline{\nu}\cdot(\overline{r}+\overline{L}+1)}\cdot 2^{\overline{\nu}\cdot(\overline{r}-1/p)}\Big(\sum_{\overline{m}\in\mathbb{Z}^d}|\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{1/p}\cdot\sup_{x\in\mathbb{R}^d}\langle x\rangle^M|(D^{\overline{L}+1}\varphi)(x)|.$$

As $\lambda\in s^{\overline{r}}_{pq}f\subset s^{\overline{r}}_{p,\infty}b$ and $\overline{r}+\overline{L}+1>0$, the convergence of (2.8) in $S'(\mathbb{R}^d)$ now follows.

If $p<1$, we get a similar estimate

$$\Big|\int\limits_{\mathbb{R}^d}\sum_{\overline{m}\in\mathbb{Z}^d}\lambda_{\overline{\nu}\,\overline{m}}a_{\overline{\nu}\,\overline{m}}(y)\varphi(y)\,dy\Big|^p$$
$$\le c\,2^{-\overline{\nu}\cdot(\overline{L}+1)p}\sup_{x\in\mathbb{R}^d}|(D^{\overline{L}+1}\varphi)(x)|^p\sum_{\overline{m}\in\mathbb{Z}^d}|\lambda_{\overline{\nu}\,\overline{m}}|^p\Big|\int\limits_{\mathbb{R}^d}\chi_{\gamma Q_{\overline{\nu}\,\overline{m}}}(y)\,dy\Big|^p$$
$$\le c\,2^{-\overline{\nu}\cdot(\overline{r}+\overline{L}+1-1/p+1)p}\sup_{x\in\mathbb{R}^d}|(D^{\overline{L}+1}\varphi)(x)|^p\sum_{\overline{m}\in\mathbb{Z}^d}2^{\overline{\nu}\cdot(\overline{r}-1/p)p}|\lambda_{\overline{\nu}\,\overline{m}}|^p.$$

In this case we use the fact that $\overline{r}+\overline{L}+1-1/p+1>0$ and the embedding $s^{\overline{r}}_{p,q}f\subset s^{\overline{r}}_{p,\infty}b$.

Step 2. Next we prove (2.9). We use the equivalent quasinorms in $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ given by (1.62). Choose $\overline{R}>\overline{K}$ and define the functions $k_{\overline{l}}$ for $\overline{l}\in\mathbb{N}_0^d$ as in Theorem 1.25. Then for all $\overline{l},\overline{\nu}\in\mathbb{N}_0^d$ and all $\overline{m}\in\mathbb{Z}^d$ we have

$$2^{\overline{l}\cdot\overline{r}}k_{\overline{l}}(a_{\overline{\nu}\,\overline{m}})(x)=2^{\overline{l}\cdot\overline{r}}\int\limits_{\mathbb{R}^d}k^1_{l_1}(y_1)\cdots k^d_{l_d}(y_d)a_{\overline{\nu}\,\overline{m}}(x+y)\,dy. \tag{2.13}$$

Further calculation depends on the size of the supports of $k_{\overline{l}}$ and $a_{\overline{\nu}\,\overline{m}}$. Hence we have to distinguish between $l_i\ge\nu_i$ and $l_i<\nu_i$. This leads to 2^d cases. We describe the first one ($\overline{l}\ge\overline{\nu}$) and the last one ($\overline{l}<\overline{\nu}$) in full detail and then we discuss the "mixed" cases.

I. $\overline{l}\ge\overline{\nu}$. We suppose that $\overline{l}>0$. This only simplifies the notation, the terms with $l_i=\nu_i=0$ may be incorporated afterwards. We use the definition of $k^i_{l_i}$ and perform partial integration (K_i-times in the ith variable) to obtain

$$2^{\overline{l}\cdot\overline{r}}k_{\overline{l}}(a_{\overline{\nu}\,\overline{m}})(x)=2^{\overline{l}\cdot(\overline{r}+1)}\int\limits_{\mathbb{R}^d}\prod_{i=1}^d\Big(\frac{d^{R_i}}{dt^{R_i}}k^i\Big)(2^{l_i}y_i)a_{\overline{\nu}\,\overline{m}}(x+y)\,dy$$
$$=2^{\overline{l}\cdot\overline{r}}\int\limits_{\mathbb{R}^d}\prod_{i=1}^d\Big(\frac{d^{R_i}}{dt^{R_i}}k^i\Big)(y_i)a_{\overline{\nu}\,\overline{m}}(x_1+2^{-l_1}y_1,\dots,x_d+2^{-l_d}y_d)\,dy$$
$$=2^{\overline{l}\cdot(\overline{r}-\overline{K})}\int\limits_{\mathbb{R}^d}\prod_{i=1}^d\Big(\frac{d^{R_i-K_i}}{dt^{R_i-K_i}}k^i\Big)(y_i)(D^{\overline{K}}a_{\overline{\nu}\,\overline{m}})(x_1+2^{-l_1}y_1,\dots,x_d+2^{-l_d}y_d)\,dy.$$

Next we use the smoothness of k^i, the boundedness of their supports and the properties (2.4) and (2.5) to estimate the absolute value of this expression:

$$2^{\bar{l}\cdot\bar{r}}|k_{\bar{l}}(a_{\overline{\nu}\,\overline{m}})(x)| \le c2^{\bar{l}\cdot(\bar{r}-\overline{K})}2^{\overline{\nu}\cdot\overline{K}}\cdot$$

$$\cdot\int_{\mathbb{R}^d}\Big(\prod_{i=1}^d \chi_{\operatorname{supp} k^i}(y_i)\Big)\chi_{\gamma Q_{\overline{\nu}\,\overline{m}}}(x_1+2^{-l_1}y_1,\dots,x_d+2^{-l_d}y_d)\,dy.$$

As $\operatorname{supp} k^i \subset \{t\in\mathbb{R} : |t|\le 1\}$, $i=1,\dots,d$, it follows that

$$2^{\bar{l}\cdot\bar{r}}|k_{\bar{l}}(a_{\overline{\nu}\,\overline{m}})(x)| \le c2^{-(\overline{K}-\bar{r})(\bar{l}-\overline{\nu})}2^{\overline{\nu}\cdot(\bar{r}-1/p)}\chi^{(p)}_{\gamma Q_{\overline{\nu}\,\overline{m}}}(x). \tag{2.14}$$

II. $\bar{l}<\overline{\nu}$. The integration in (2.13) may be restricted to $\{y : |y_i|\le 2^{-l_i}\}$. We use the Taylor expansion of $k^i_{l_i}(y_i)$ with respect to the off-points $2^{-\nu_i}m_i-x_i$ up to order L_i,

$$2^{-l_i}k^i_{l_i}(y_i) = \sum_{0\le\beta_i\le L_i} c^i_{\beta_i}(x_i)(y_i-2^{-\nu_i}m_i+x_i)^{\beta_i} + 2^{l_i(L_i+1)}O(|x_i+y_i-2^{-\nu_i}m_i|^{L_i+1}), \tag{2.15}$$

and (2.6) to get

$$2^{\bar{l}\cdot\bar{r}}k_{\bar{l}}(a_{\overline{\nu}\,\overline{m}})(x) = 2^{\bar{l}\cdot(\bar{r}+1)}\int_{\{y\,:\,|y_i|\le 2^{-l_i}\}} a_{\overline{\nu}\,\overline{m}}(x+y)\prod_{i=1}^d 2^{l_i(L_i+1)}O(|x_i+y_i-2^{-\nu_i}m_i|^{L_i+1})\,dy.$$

Since $|a_{\overline{\nu}\,\overline{m}}(x+y)|\le\chi_{\gamma Q_{\overline{\nu}\,\overline{m}}}(x+y)$ we obtain

$$2^{\bar{l}\cdot\bar{r}}|k_{\bar{l}}(a_{\overline{\nu}\,\overline{m}})(x)| \le c\,2^{\bar{l}\cdot(\bar{r}+1)}2^{(\bar{l}-\overline{\nu})\cdot(\overline{L}+1)}\int_{\{y\,:\,|y_i|\le 2^{-l_i}\}}\chi_{\gamma Q_{\overline{\nu}\,\overline{m}}}(x+y)\,dy. \tag{2.16}$$

The last integral is always smaller than $c\,2^{-|\overline{\nu}|}$ and is zero if $\{y : x+y\in\gamma Q_{\overline{\nu}\,\overline{m}}\}\cap\{y : |y_i|\le 2^{-l_i}\}=\emptyset$. Hence

$$\int_{\{y\,:\,|y_i|\le 2^{-l_i}\}}\chi_{\gamma Q_{\overline{\nu}\,\overline{m}}}(x+y)\,dy \le c\,2^{-|\overline{\nu}|}\chi_{c2^{\overline{\nu}-\bar{l}}Q_{\overline{\nu}\,\overline{m}}}(x). \tag{2.17}$$

But the last expression may be estimated from above by the use of maximal operators M_i defined by (1.14),

$$2^{-|\overline{\nu}-\bar{l}|}\chi_{c2^{\overline{\nu}-\bar{l}}Q_{\overline{\nu}\,\overline{m}}}(x) \le c\,(\overline{M}\chi_{\overline{\nu}\,\overline{m}})(x). \tag{2.18}$$

Let $0<\omega<\min(1,p,q)$. Taking the $(1/\omega)$th power of (2.18) and inserting it in (2.17) we obtain

$$\int_{\{y\,:\,|y_i|\le 2^{-l_i}\}}\chi_{\gamma Q_{\overline{\nu}\,\overline{m}}}(x+y)\,dy \le c\,2^{-|\overline{\nu}|}2^{|\overline{\nu}-\bar{l}|/\omega}(\overline{M}\chi_{\overline{\nu}\,\overline{m}})^{1/\omega}(x). \tag{2.19}$$

Next we replace $\chi_{\overline{\nu}\,\overline{m}}$ by $\chi^{(p)}_{\overline{\nu}\,\overline{m}}$ in (2.19) and insert it in (2.16):

$$2^{\bar{l}\cdot\bar{r}}|k_{\bar{l}}(a_{\overline{\nu}\,\overline{m}})(x)| \le c\,2^{(\bar{l}-\overline{\nu})\cdot(\bar{r}+1+\overline{L}+1-1/\omega)}2^{\overline{\nu}\cdot(\bar{r}-1/p)}(\overline{M}\chi^{(p)\omega}_{\overline{\nu}\,\overline{m}})^{1/\omega}(x).$$

By (2.7) and (1.4) we may choose the number ω such that $\overline{\varkappa}=(\bar{r}+1+\overline{L}+1-1/\omega)>0$.

III. *Mixed terms.* We estimate for example the term with $l_1\ge\nu_1$, $l_i<\nu_i$, $i=2,\dots,d$. First we apply (2.15) for $i=2,\dots,d$ and use (2.6) to get rid of the terms with $\beta\le\overline{L}$. Then we use K_1 partial integrations in the first variable. In the expression we get we use again the support properties of the functions involved and (2.5) to obtain

$$2^{\bar{l}\cdot\bar{r}}|k_{\bar{l}}(a_{\bar{\nu}\,\bar{m}})(x)| \le 2^{\bar{\nu}\cdot\bar{r}}2^{(l_1-\nu_1)(r_1-K_1)}2^{\sum_{i=2}^{d}(l_i(r_i+1)+(l_i-\nu_i)(L_i+1)-\nu_i r_i)} \cdot \int_{A_{\bar{l}}} \chi_{\gamma Q_{\bar{\nu}\,\bar{m}}}(x_1+2^{-l_1}y_1, x_2+y_2,\dots,x_d+y_d)\,dy,$$

where $A_{\bar{l}} = \{y\in\mathbb{R}^d : |y_1|\le 1,\ |y_i|\le 2^{-l_i},\ i=2,\dots,d\}$. Due to the product structure of the integrated function we may split the last integral into a one-dimensional integral with respect to dy_1 and a $(d-1)$-dimensional integral with respect to the remaining variables. The first integral may then be estimated from above by $c\chi_{\{t\,:\,|t-2^{-\nu_1}m_1|\le 2^{-\nu_i}\}}(x_1)$. Finally, we use the maximal operators M_i, $i=2,\dots,d$, to estimate the second integral. Exactly as in the second step, it turns out that there is some vector $\bar{\varrho}>0$ such that

$$2^{\bar{l}\cdot\bar{r}}|k_{\bar{l}}(a_{\bar{\nu}\,\bar{m}})(x)| \le c\,2^{-\sum_{i=1}^{d}|l_i-\nu_i|\varrho_i}2^{\bar{\nu}\cdot(\bar{r}-1/p)}(\overline{M}\chi_{\bar{\nu}\,\bar{m}}^{(p)\omega})^{1/\omega}(x). \tag{2.20}$$

Observe that also (2.14) may be estimated from above by the right-hand side of (2.20). Hence the estimate (2.20) is valid for all $\bar{l},\bar{\nu}\in\mathbb{N}_0^d$.

Using this estimate, we get for $q\le 1$,

$$\Big|2^{\bar{l}\cdot\bar{r}}k_{\bar{l}}\Big(\sum_{\bar{\nu},\bar{m}}\lambda_{\bar{\nu}\,\bar{m}}a_{\bar{\nu}\,\bar{m}}\Big)(x)\Big|^q \le c\sum_{\bar{\nu},\bar{m}}|\lambda_{\bar{\nu}\,\bar{m}}|^q 2^{\bar{\nu}\cdot(\bar{r}-1/p)q}2^{-q\sum_{i=1}^{d}|l_i-\nu_i|\varrho_i}(\overline{M}\chi_{\bar{\nu}\,\bar{m}}^{(p)\omega})^{q/\omega}(x).$$

For $q>1$, the same estimate is justified by Hölder's inequality.

We sum over $\bar{l}$, take the $(1/q)$th power and then apply the L_p-quasinorm with respect to x. Setting $g_{\bar{\nu}\,\bar{m}} = 2^{\bar{\nu}\cdot(\bar{r}-1/p)}\lambda_{\bar{\nu}\,\bar{m}}\chi_{\bar{\nu}\,\bar{m}}^{(p)}$ we arrive at

$$\begin{aligned}
&\Big\|\Big(\sum_{\bar{l}\in\mathbb{N}_0^d}\Big|2^{\bar{l}\cdot\bar{r}}k_{\bar{l}}\Big(\sum_{\bar{\nu},\bar{m}}\lambda_{\bar{\nu}\,\bar{m}}a_{\bar{\nu}\,\bar{m}}\Big)(x)\Big|^q\Big)^{1/q}\,\Big|\,L_p(\mathbb{R}^d)\Big\| \\
&\qquad\le c\Big\|\Big(\sum_{\bar{\nu},\bar{m}}2^{\bar{\nu}\cdot(\bar{r}-1/p)q}|\lambda_{\bar{\nu}\,\bar{m}}|^q(\overline{M}\chi_{\bar{\nu}\,\bar{m}}^{(p)\omega})^{q/\omega}(x)\Big)^{1/q}\,\Big|\,L_p(\mathbb{R}^d)\Big\| \\
&\qquad= c\Big\|\Big(\sum_{\bar{\nu},\bar{m}}(\overline{M}g_{\bar{\nu}\,\bar{m}}^{\omega})^{q/\omega}(x)\Big)^{\omega/q}\,\Big|\,L_{p/\omega}(\mathbb{R}^d)\Big\|^{1/\omega}.
\end{aligned}$$

Using Theorem 1.11 and the definition of ω, we see that this expression may be estimated from above by $c\|\lambda\,|\,s_{p,q}^{\bar{r}}f\|$. On the other hand, from Theorem 1.23, this already ensures that f belongs to $S_{p,q}^{\bar{r}}F(\mathbb{R}^d)$ and proves (2.9).

Step 3. It remains to prove (ii). Assume first that

$$\overline{L}=-1,\qquad \overline{K}>\bar{r},\qquad \bar{r}>\sigma_{pq},\qquad 0<p<\infty,\qquad 0<q\le\infty. \tag{2.21}$$

Furthermore, let $\overline{N}\in\mathbb{N}_0^d$ be a vector of integers with $\overline{N}>\bar{r}$. According to the construction in [34, p. 68], we may find functions $k_0,k^1,\dots,k^d$ such that

$$k_0, k^1, \dots, k^d \in S(\mathbb{R}); \tag{2.22}$$

$$\operatorname{supp} k_0, \operatorname{supp} k^i \subset \{t \in \mathbb{R} : |t| \le 1\}, \qquad i = 1, \dots, d; \tag{2.23}$$

$$1 = F_1(k_0)(\xi) + \sum_{\nu_i=1}^{\infty} F_1(\, d^{N_i} k^i)(2^{-\nu_i}\xi), \qquad \xi \in \mathbb{R},\ i = 1, \dots, d; \tag{2.24}$$

$$F_1 k_0(0) = 1; \tag{2.25}$$

$$F_1(\, d^{N_i} k^i)(\xi) = (F_1 k_0)(\xi) - (F_1 k_0)(2\xi), \qquad \xi \in \mathbb{R},\ i = 1, \dots, d. \tag{2.26}$$

We define $k_{\overline{l}}(x)$ and $k_{\overline{l}}(f)(x)$ as in Theorem 1.25. We claim that then

$$f = \sum_{\overline{l} \in \mathbb{N}_0^d} k_{\overline{l}}(f)(x) = \lim_{P \to \infty} \sum_{\overline{l} \le P} k_{\overline{l}}(f), \qquad \text{convergence in } S'(\mathbb{R}^d). \tag{2.27}$$

To prove this, fix $\varphi \in S(\mathbb{R}^d)$. Since the Fourier transform is an isomorphic mapping from $S'(\mathbb{R}^d)$ onto itself and

$$(k_{\overline{l}}(f))^\wedge(\xi) = \Big(\prod_{i=1}^{d} F_1(k^i_{l_i})(-\xi_i)\Big)\widehat{f}(\xi),$$

it is enough to show that

$$\varphi(\xi) \sum_{\overline{l} \le P} \Big(\prod_{i=1}^{d} F_1(k^i_{l_i})(-\xi_i)\Big) \to \varphi(\xi) \qquad \text{in } S(\mathbb{R}^d). \tag{2.28}$$

The last sum may be rewritten using (2.26) as

$$\begin{aligned}
\sum_{\overline{l} \le P} \Big(\prod_{i=1}^{d} F_1(k^i_{l_i})(-\xi_i)\Big) &= \prod_{i=1}^{d} \Big((F_1 k_0)(-\xi_i) + \sum_{l_i=1}^{P} (F_1(\, d^{N_i} k^i))(-2^{-l_i}\xi_i)\Big) \\
&= \prod_{i=1}^{d} (F_1 k_0)(-2^{-P}\xi_i).
\end{aligned}$$

We denote the last expression by $1 - \Phi(2^{-P}\xi)$ and fix $M \in \mathbb{N}$. Using the fact that $\varphi \in S(\mathbb{R}^d)$ we obtain

$$\begin{aligned}
p_M(\varphi(\xi)\Phi(2^{-P}\xi)) &\le c \sup_{\substack{0 \le \overline{\alpha}, \overline{\beta} \le M \\ \xi \in \mathbb{R}^d}} 2^{-P \cdot \overline{\beta}} (D^{\overline{\alpha}}\varphi)(\xi)(D^{\overline{\beta}}\Phi)(2^{-P}\xi) \prod_{i=1}^{d} \langle \xi_i \rangle^M \\
&\le c \sup_{\substack{0 \le \overline{\beta} \le M \\ \xi \in \mathbb{R}^d}} 2^{-P \cdot \overline{\beta}} (D^{\overline{\beta}}\Phi)(2^{-P}\xi) \prod_{i=1}^{d} \langle \xi_i \rangle^{-1}
\end{aligned}$$

where the constant c does not depend on P (but depends on M). Here p_M are the functionals defining the topology on $S(\mathbb{R}^d)$, namely $p_M(\varphi) = \sup_{0 \le \overline{\alpha} \le M, x \in \mathbb{R}^d} |D^\alpha \varphi(x)| \langle x \rangle^M$.

If at least one $\beta_i > 0$, then this expression tends to zero as $P \to \infty$. If $\overline{\beta} = 0$, then we split the supremum into $\sup_{|\xi| \ge 2^P}$ and $\sup_{|\xi| < 2^P}$. The first supremum may be estimated from above by $c2^{-P}$. To estimate the second one, we notice that $|\Phi(\xi)| \le c|\xi|$ in $\{\xi : |\xi| \le 1\}$. Hence

$$c \sup_{|\xi|\le 2^P} \Phi(2^{-P}\xi) \prod_{i=1}^{d} \langle \xi_i\rangle^{-1} \le c \sup_{\xi\in\mathbb{R}^d} \frac{2^{-P}|\xi|}{\langle\xi\rangle}$$

and $p_M(\varphi(\xi)\Phi(2^{-P}\xi)) \to 0$ as $P\to\infty$. This proves (2.28) and, consequently, also (2.27). Next we find a nonnegative function ψ which satisfies

$$\psi \in S(\mathbb{R}), \quad \operatorname{supp}\psi \text{ is compact} \quad \text{and} \quad \sum_{\overline{m}\in\mathbb{Z}^d} \psi(x-\overline{m}) = 1 \text{ for } x\in\mathbb{R}^d, \tag{2.29}$$

and we define for $\overline{\nu}\in\mathbb{N}_0^d$ and $\overline{m}\in\mathbb{Z}^d$ the function $\psi_{\overline{\nu}\,\overline{m}}(x) = \psi(2^{\overline{\nu}}x-\overline{m})$. Then there is a γ such that

$$\operatorname{supp}\psi_{\overline{\nu}\,\overline{m}} \subset \gamma Q_{\overline{\nu}\,\overline{m}}, \qquad \overline{\nu}\in\mathbb{N}_0^d,\ \overline{m}\in\mathbb{Z}^d. \tag{2.30}$$

We multiply (2.27) by these decompositions of unity and obtain

$$f = \sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} \psi_{\overline{\nu}\,\overline{m}}(x) k_{\overline{\nu}}(f)(x) = \sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} \lambda_{\overline{\nu}\,\overline{m}} a_{\overline{\nu}\,\overline{m}}(x), \tag{2.31}$$

where

$$\lambda_{\overline{\nu}\,\overline{m}} = \sum_{0\le\alpha\le\overline{K}} \sup_{y\in\gamma Q_{\overline{\nu}\,\overline{m}}} |D^\alpha[k_{\overline{\nu}}(f)](y)|, \qquad a_{\overline{\nu}\,\overline{m}}(x) = \lambda_{\overline{\nu}\,\overline{m}}^{-1}\psi_{\overline{\nu}\,\overline{m}}(x) k_{\overline{\nu}}(f)(x).$$

(If some $\lambda_{\overline{\nu}\,\overline{m}} = 0$, then we take $a_{\overline{\nu}\,\overline{m}}(x) = 0$ as well.) It follows that $a_{\overline{\nu}\,\overline{m}}$ are $[\overline{K},\overline{L}]$-atoms centred at $Q_{\overline{\nu}\,\overline{m}}$. The properties (2.4) and (2.6) are satisfied trivially (recall that $\overline{L}=-1$), and the property (2.5) is fulfilled up to some constant c independent of $\overline{\nu}$, $\overline{m}$ and x. To prove that this decomposition satisfies (2.10), write

$$\|\lambda|s^{\overline{r}}_{p,q}f\| \le c \sum_{0\le\alpha\le\overline{K}} \Big\|\Big(\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} 2^{\overline{\nu}\cdot\overline{r}q} 2^{\overline{\nu}\cdot q/p} \sup_{x-y\in\gamma Q_{\overline{\nu}\,\overline{m}}} |D^\alpha[k_{\overline{\nu}}(f)(y)]|^q\Big)^{1/q} \Big| L_p\Big\| \tag{2.32}$$

and use Theorem 1.27 with $D^{\alpha_i}k_0$ and $D^{\alpha_i}k^i$ in place of k_0 and k^i. We lose the Tauberian conditions (1.60) for these new kernels but according to Theorem 1.20, they are not necessary in the proof of (2.32).

STEP 4. Now we prove the existence of the optimal decomposition for all $\overline{r}\in\mathbb{R}^d$ and $\overline{L}$ restricted by (2.7). To simplify the notation, we restrict ourselves in this step to $d=2$. So, take $f\in S^{\overline{r}}_{p,q}F(\mathbb{R}^2)$. In Definition 1.8 we may substitute $(1+x^2)^{\overline{\rho}}$ by $(1+x_1^{2\rho_1})(1+x_2^{2\rho_2})$ for $\overline{\rho}\in\mathbb{N}_0^2$ and (using twice Theorem 1.12) we obtain the counterpart of Theorem 1.9. Hence f can be decomposed as

$$f = g + \frac{\partial^{2M_1}g}{\partial x_1^{2M_1}} + \frac{\partial^{2M_2}g}{\partial x_2^{2M_2}} + \frac{\partial^{2M_1+2M_2}g}{\partial x_1^{2M_1}x_2^{2M_2}}, \tag{2.33}$$

where $\overline{M} = (M_1,M_2)\in 2\mathbb{N}_0^2$ is at our disposal and may be chosen arbitrarily large, $g\in S^{\overline{r}+2\overline{M}}_{p,q}F(\mathbb{R}^2)$ and $\|g\,|\,S^{\overline{r}+2\overline{M}}_{p,q}F(\mathbb{R}^2)\| \approx \|f\,|\,S^{\overline{r}}_{p,q}F(\mathbb{R}^2)\|$. The optimal decomposition of f will be obtained as a sum of decompositions of these four terms.

To decompose the first term, choose $\overline{M}$ such that

$$\|g\,|\,S^{\overline{K}}\mathcal{C}(\mathbb{R}^2)\| \le c\|g\,|\,S^{\overline{r}+2\overline{M}}_{p,q}F(\mathbb{R}^2)\|.$$

This is possible according to [26, Theorem 2.4.1]. Then we decompose

$$g(x)=\sum_{\overline{m}\in\mathbb{Z}^d}\psi(x-\overline{m})g(x)=\sum_{\overline{m}\in\mathbb{Z}^d}\lambda^1_{0\,\overline{m}}a^1_{0\,\overline{m}},$$

where

$$\lambda^1_{0\,\overline{m}}=c_1\sum_{0\le\alpha\le\overline{K}}\sup_{|y-\overline{m}|\le c_2}|(D^\alpha g)(y)|,$$
$$a^1_{0\,\overline{m}}=\frac{1}{\lambda^1_{0\,\overline{m}}}\,\psi(x-\overline{m})g(x),$$

for c_1,c_2 sufficiently large and for ψ satisfying (2.29) and (2.30). Then $a^1_{0\,\overline{m}}$ are $[\overline{K},\overline{L}]$-atoms centred at $Q_{0\,\overline{m}}$. Furthermore, according to Lemma 1.28, we have

$$\|\lambda^1\,|\,s^{\overline{r}}_{p,q}f\|=\Big(\sum_{\overline{m}\in\mathbb{Z}^d}|\lambda^1_{0\,\overline{m}}|^p\Big)^{1/p}\le c_1\sum_{\alpha\le\overline{K}}\|\sup_{\cdot-y\in\gamma Q_{0\,0}}|(D^\alpha g)(y)|\,|\,L_p(\mathbb{R}^d)\|$$
$$\le c\|g\,|\,S^{\overline{r}+2\overline{M}}_{p,q}F(\mathbb{R}^2)\|\le c\|f\,|\,S^{\overline{r}}_{p,q}F(\mathbb{R}^2)\|.$$

We have used Lemma 1.28 with $d=2$ and $A=\emptyset$.

As for the last term in the decomposition (2.33), we may assume that $\overline{M}$ is large enough to apply Step 3. So we may assume that we have a decomposition (2.31) for g with, say, $\lambda^4_{\overline{\nu}\,\overline{m}}$ and $a^4_{\overline{\nu}\,\overline{m}}(x)$ instead of $\lambda_{\overline{\nu}\,\overline{m}}$ and $a_{\overline{\nu}\,\overline{m}}(x)$ and $\|\lambda^4_{\overline{\nu}\,\overline{m}}\,|\,s^{\overline{r}+2\overline{M}}_{p,q}f\|\le c\|g\,|\,S^{\overline{r}+2\overline{M}}_{p,q}F(\mathbb{R}^2)\|$. As $a^4_{\overline{\nu}\,\overline{m}}(x)$ are $[\overline{K}+2\overline{M},-1]$-atoms, $2^{-2\overline{\nu}\cdot\overline{M}}D^{2(M_1,M_2)}a^4_{\overline{\nu}\,\overline{m}}(x)$ are $[\overline{K},2\overline{M}-1]$-atoms.

In the case of the second term we use the decomposition

$$g(x)=\sum_{\substack{\overline{\nu}\in\mathbb{N}^2_0\\ \nu_2=0}}\sum_{\overline{m}\in\mathbb{Z}^d}\psi_{\overline{\nu}\,\overline{m}}(x)k_{\overline{\nu},A}(g)(x)=\sum_{\substack{\overline{\nu}\in\mathbb{N}^2_0\\ \nu_2=0}}\sum_{\overline{m}\in\mathbb{Z}^d}\lambda^2_{\overline{\nu}\,\overline{m}}a^2_{\overline{\nu}\,\overline{m}}(x),$$

where $A=\{1\}$, $k_{\overline{\nu},A}(g)(x)$ are defined by (1.68),

$$\lambda^2_{\overline{\nu}\,\overline{m}}=c_1 2^{2\nu_1M_1}\sum_{\beta\le\overline{K}+(2M_1,0)}\sup_{y\in c_2Q_{\overline{\nu}\,\overline{m}}}|D^\beta(k_{\overline{\nu},A}(g))(y)|,$$
$$a^2_{\overline{\nu}\,\overline{m}}(x)=\frac{1}{\lambda^2_{\overline{\nu}\,\overline{m}}}\psi_{\overline{\nu}\,\overline{m}}(x)k_{\overline{\nu},A}(g)(x).$$

If c_1 and c_2 are large enough, then $D^{(2M_1,0)}a^2_{\overline{\nu}\,\overline{m}}(x)$ are $[\overline{K},\overline{L}]$-atoms for $L_1\le 2M_1-1$. Finally, we use Lemma 1.28 to estimate $\|\lambda^2\,|\,s^{\overline{r}}_{p,q}f\|$:

$$\|\lambda^2\,|\,s^{\overline{r}}_{p,q}f\|\le c_1\sum_{\beta\le\overline{K}+(2M_1,0)}\Big\|\Big(\sum_{\substack{\overline{\nu}\in\mathbb{N}^2_0\\ \nu_2=0}}2^{q\nu_1(2M_1+r_1)}\sup_{\cdot-y\in c_2Q_{\overline{\nu}\,0}}|D^\beta(k_{\overline{\nu},A}(g))(y)|^q\Big)^{1/q}\Big|\,L_p\Big\|$$
$$\le c\|g\,|\,S^{\overline{r}+2\overline{M}}_{p,q}F(\mathbb{R}^d)\|\le c\|f\,|\,S^{\overline{r}}_{p,q}F(\mathbb{R}^d)\|,$$

if $\overline{M}$ is chosen sufficiently large. We have used Lemma 1.28 with $D^{\beta_1}k_1$ and $D^{\beta_2}g$ instead of k_1 and f. The third term can be estimated in a similar way. The sum of these four decompositions then gives the decomposition for f.

For general d one has to use the full generality of Lemma 1.28 but the idea of the proof is still the same. ∎

2.3. Subatomic decomposition. In this section we describe the subatomic decomposition for the spaces $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$. We follow closely [35] and [37].

First of all, we shall introduce some special building blocks called quarks.

Definition 2.5. Let $\psi \in S(\mathbb{R})$ be a nonnegative function with

$$\operatorname{supp}\psi \subset \{t \in \mathbb{R} : |t| < 2^{\phi}\} \tag{2.34}$$

for some $\phi \geq 0$ and

$$\sum_{n\in\mathbb{Z}} \psi(t-n) = 1, \quad t \in \mathbb{R}. \tag{2.35}$$

We define $\Psi(x) = \psi(x_1)\cdots\psi(x_d)$ and $\Psi^{\beta}(x) = x^{\beta}\Psi(x)$ for $x = (x_1,\dots,x_d)$ and $\beta \in \mathbb{N}_0^d$. Further let $\overline{r} \in \mathbb{R}^d$ and $0 < p \leq \infty$. Then

$$(\beta qu)_{\overline{\nu}\,\overline{m}}(x) = \Psi^{\beta}(2^{\overline{\nu}}x - \overline{m}), \quad \overline{\nu} \in \mathbb{N}_0^d,\ \overline{m} \in \mathbb{Z}^d, \tag{2.36}$$

is called a β-*quark* related to $Q_{\overline{\nu}\,\overline{m}}$.

Recall that the spaces $s^{\overline{r}}_{p,q}a$ were defined by (2.2) and (2.3).

Theorem 2.6. *Let $0 < p,q \leq \infty$ (with $p < \infty$ in the F-case) and $\overline{r} \in \mathbb{R}^d$ be such that*

$$\overline{r} > \sigma_p \text{ in the } B\text{-case} \quad \text{and} \quad \overline{r} > \sigma_{pq} \text{ in the } F\text{-case.}$$

(i) *Let*

$$\lambda = \{\lambda^{\beta} : \beta \in \mathbb{N}_0^d\} \quad \text{with} \quad \lambda^{\beta} = \{\lambda^{\beta}_{\overline{\nu}\,\overline{m}} \in \mathbb{C} : \overline{\nu} \in \mathbb{N}_0^d,\ \overline{m} \in \mathbb{Z}^d\}$$

and let $\varrho > \phi$, where ϕ comes from (2.34). *If*

$$\sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}\|\lambda^{\beta} \mid s^{\overline{r}}_{p,q}a\| < \infty$$

then the series

$$\sum_{\beta\in\mathbb{N}_0^d}\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} \lambda^{\beta}_{\overline{\nu}\,\overline{m}}(\beta qu)_{\overline{\nu}\,\overline{m}}(x) \tag{2.37}$$

converges in $S'(\mathbb{R}^d)$, its limit f belongs to $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ and

$$\|f \mid S^{\overline{r}}_{p,q}A(\mathbb{R}^d)\| \leq c \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}\|\lambda^{\beta} \mid s^{\overline{r}}_{p,q}a\|. \tag{2.38}$$

$(\beta qu)_{\overline{\nu}\,\overline{m}}$ has the same meaning as in (2.36).

(ii) *Every $f \in S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ can be represented by* (2.37) *with convergence in $S'(\mathbb{R}^d)$ and*

$$\sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}\|\lambda^{\beta} \mid s^{\overline{r}}_{p,q}a\| \leq c\|f \mid S^{\overline{r}}_{p,q}A(\mathbb{R}^d)\|. \tag{2.39}$$

Proof. We give again only the proof for the F-scale. The proof for the B-scale is very similar.

Step 1. First of all, we shall discuss convergence of (2.37). It turns out that this series converges not only in $S'(\mathbb{R}^d)$ but also in some $L_u(\mathbb{R}^d), u \geq 1$.

Let $1 \leq p < \infty$. Then $\overline{r} > 0$ and we get

$$|f(x)| \leq c \sum_{\beta\in\mathbb{N}_0^d}\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} 2^{\phi|\beta|}|\lambda^{\beta}_{\overline{\nu}\,\overline{m}}|\widetilde{\chi}_{\overline{\nu}\,\overline{m}}(x), \tag{2.40}$$

where $\widetilde{\chi}_{\overline{\nu}\,\overline{m}}$ is the characteristic function of $2^{\phi+1}Q_{\overline{\nu}\,\overline{m}}$. Using Hölder's inequality twice, we get for every $\varepsilon > 0$,

$$|f(x)| \le c \sup_{\beta\in\mathbb{N}_0^d} 2^{(\phi+\varepsilon)|\beta|} \sup_{\overline{\nu}\in\mathbb{N}_0^d} 2^{|\overline{\nu}|\varepsilon} \sup_{\overline{m}\in\mathbb{Z}^d} |\lambda^\beta_{\overline{\nu}\,\overline{m}}|\widetilde{\chi}_{\overline{\nu}\,\overline{m}}(x).$$

Taking the pth power and replacing the suprema with sums we get

$$|f(x)|^p \le c \sum_{\beta\in\mathbb{N}_0^d}\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} 2^{(\phi+\varepsilon)|\beta|p}2^{|\overline{\nu}|\varepsilon p}|\lambda^\beta_{\overline{\nu}\,\overline{m}}|^p\widetilde{\chi}_{\overline{\nu}\,\overline{m}}(x). \tag{2.41}$$

Set $\widetilde{q} = \max(p,q)$ and choose ε such that $0 < 2\varepsilon < \varrho - \phi$ and $\varepsilon < \overline{r}$. Integration of (2.41) and Hölder's inequality result in

$$\begin{aligned}\|f\,|\,L_p(\mathbb{R}^d)\| &\le c \sup_{\beta\in\mathbb{N}_0^d} 2^{(\phi+2\varepsilon)|\beta|}\Big(\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} 2^{-\overline{\nu}\cdot(1/p-\varepsilon)p}|\lambda^\beta_{\overline{\nu}\,\overline{m}}|^p\Big)^{1/p}\\
&\le c \sup_{\beta\in\mathbb{N}_0^d} 2^{(\phi+2\varepsilon)|\beta|}\Big(\sum_{\overline{\nu}\in\mathbb{N}_0^d} 2^{\overline{\nu}\cdot(\overline{r}-1/p)\widetilde{q}}\Big(\sum_{\overline{m}\in\mathbb{Z}^d}|\lambda^\beta_{\overline{\nu}\,\overline{m}}|^p\Big)^{\widetilde{q}/p}\Big)^{1/\widetilde{q}}\\
&\le c \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}\|\lambda^\beta\,|\,s^{\overline{r}}_{p,\widetilde{q}}b\| \le c \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}\|\lambda^\beta\,|\,s^{\overline{r}}_{p,q}f\|. \end{aligned}\tag{2.42}$$

Therefore, for $1 \le p < \infty$, (2.37) converges in $L_p(\mathbb{R}^d)$. If $p = \infty$, we get the uniform pointwise convergence of (2.37) by similar arguments.

Let $0 < p < 1$. Then $\overline{r} > 1/p - 1$ and we get again (2.40). Integrating this estimate and using Hölder's inequality, we get for every $\varepsilon > 0$,

$$\|f\,|\,L_1(\mathbb{R}^d)\| \le c \sum_{\beta\in\mathbb{N}_0^d}\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} 2^{\phi|\beta|}2^{-|\overline{\nu}|}|\lambda^\beta_{\overline{\nu}\,\overline{m}}| \le c\sup_{\beta\in\mathbb{N}_0^d} 2^{(\phi+\varepsilon)|\beta|}\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d} 2^{-|\overline{\nu}|}|\lambda^\beta_{\overline{\nu}\,\overline{m}}|.$$

By arguments similar to (2.42) we get

$$\|f\,|\,L_1(\mathbb{R}^d)\| \le c \sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}\|\lambda^\beta\,|\,s^{\overline{r}}_{p,q}f\|$$

and (2.37) converges in $L_1(\mathbb{R}^d)$.

Step 2. We now prove that the function f defined as a limit of (2.37) belongs to $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$, and the estimate (2.38). We decompose (2.37) into

$$f = \sum_{\beta\in\mathbb{N}_0^d} f^\beta \tag{2.43}$$

with

$$f^\beta = \sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in\mathbb{Z}^d}\lambda^\beta_{\overline{\nu}\,\overline{m}}(\beta qu)_{\overline{\nu}\,\overline{m}}(x). \tag{2.44}$$

We show that $(\beta qu)_{\overline{\nu}\,\overline{m}}$ are (up to some normalising constants) $[\overline{K},-1]$-atoms centred at $Q_{\overline{\nu}\,\overline{m}}$ for every $\overline{K} \in \mathbb{N}_0^d$. The conditions (2.4) and (2.6) are satisfied trivially. To prove (2.5) we choose $0 \le \alpha \le \overline{K}$ and estimate

$$D^\alpha(\beta qu)_{\overline{\nu}\,\overline{m}}(x) = \prod_{i=1}^d 2^{\nu_i\alpha_i}D^{\alpha_i}(\psi^{\beta_i})(2^{\nu_i}x_i - m_i)$$

where $\psi^{\beta_i}(t) = t^{\beta_i}\psi(t)$. But for $0 \le \alpha_i \le K_i$ and any $t \in \operatorname{supp}\psi$ we get by the Leibniz rule

$$|D^{\alpha_i}(\psi^{\beta_i})(t)| \le c_{K_i} \sup_{\gamma_1 \le K_i} \sup_{\gamma_2 \le K_i} |D^{\gamma_1} t^{\beta_i}| \cdot |(D^{\gamma_2}\psi)(t)| \le c_{K_i,\psi} \sup_{\gamma_1 \le K_i} |D^{\gamma_1} t^{\beta_i}|.$$

The last absolute value may be estimated from above by $(1+\beta_i)^{K_i} 2^{\phi\beta_i}$. Hence we obtain

$$|D^{\alpha_i}(\psi^{\beta_i})(t)| \le c_{K_i,\psi}(1+\beta_i)^{K_i} 2^{\phi\beta_i}$$

and

$$|D^{\alpha}(\beta qu)_{\overline{\nu}\,\overline{m}}(x)| \le c_1\, 2^{\alpha\cdot\overline{\nu}}(1+\beta)^{\overline{K}} 2^{\phi|\beta|} \le c_2\, 2^{\alpha\cdot\overline{\nu}} 2^{(\phi+\varepsilon)|\beta|}$$

for every $\varepsilon > 0$. The constant c_2 is independent of β but may depend on $\overline{K}$, ψ and ε.

It follows that the functions $c_2^{-1} 2^{-(\phi+\varepsilon)|\beta|}(\beta qu)_{\overline{\nu}\,\overline{m}}(x)$ are $[\overline{K}, -1]$-atoms and (2.44) may be understood as an atomic decomposition of f^β. By Theorem 2.4 it follows that

$$\|f^\beta \,|\, S^{\overline{r}}_{p,q}F(\mathbb{R}^d)\| \le c 2^{(\phi+\varepsilon)|\beta|} \|\lambda^\beta \,|\, s^{\overline{r}}_{p,q}f\|$$

and for $\eta = \min(1,p,q)$ we get by the triangle inequality for $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$-quasinorms

$$\begin{aligned}
\|f \,|\, S^{\overline{r}}_{p,q}F(\mathbb{R}^d)\|^\eta &\le \sum_{\beta\in\mathbb{N}_0^d} \|f^\beta \,|\, S^{\overline{r}}_{p,q}F(\mathbb{R}^d)\|^\eta \\
&\le c \sum_{\beta\in\mathbb{N}_0^d} 2^{(\phi+\varepsilon)\eta|\beta|} \|\lambda^\beta \,|\, s^{\overline{r}}_{p,q}f\|^\eta \\
&\le c \sup_{\beta\in\mathbb{N}_0^d} 2^{(\phi+2\varepsilon)\eta|\beta|} \|\lambda^\beta \,|\, s^{\overline{r}}_{p,q}f\|^\eta.
\end{aligned}$$

If we choose $\varepsilon > 0$ so small that $\phi + 2\varepsilon < \varrho$ we obtain (2.38). This finishes the proof of part (i).

Step 3. By Remark 1.3 we have

$$\widehat{f}(\xi) = \sum_{\overline{\nu}\in\mathbb{N}_0^d} \varphi_{\overline{\nu}}(\xi)\widehat{f}(\xi)$$

with convergence in $S'(\mathbb{R}^d)$. Let $Q_{\overline{\nu}}$ be a cube in $\mathbb{R}^d$ centred at the origin with side lengths $2\pi 2^{\nu_1}, \ldots, 2\pi 2^{\nu_d}$. Hence $\operatorname{supp}\varphi_{\overline{\nu}} \subset Q_{\overline{\nu}}$ and we may interpret $\varphi_{\overline{\nu}}\widehat{f}$ as a periodic distribution. Using its expansion in a Fourier series we get

$$(\varphi_{\overline{\nu}}\widehat{f})(\xi) = \sum_{\overline{m}\in\mathbb{Z}^d} b_{\overline{\nu}\,\overline{m}} e^{-i(2^{-\overline{\nu}}\overline{m})\cdot\xi}, \quad \xi \in Q_{\overline{\nu}}, \tag{2.45}$$

with

$$b_{\overline{\nu}\,\overline{m}} = c\, 2^{-|\overline{\nu}|} \int_{Q_{\overline{\nu}}} e^{-i(2^{-\overline{\nu}}\overline{m})\cdot\xi} (\varphi_{\overline{\nu}}\widehat{f})(\xi)\, d\xi = c' 2^{-|\overline{\nu}|} (\varphi_{\overline{\nu}}\widehat{f})^\vee (2^{-\overline{\nu}}\overline{m}).$$

Here we have used again the notation $2^{-\overline{\nu}}\overline{m} = (2^{-\nu_1}m_1, \ldots, 2^{-\nu_d}m_d)$ for $\overline{\nu} \in \mathbb{N}_0^d$ and $\overline{m} \in \mathbb{Z}^d$.

Let now $\omega \in S(\mathbb{R}^d)$ with $\operatorname{supp}\omega \subset Q_0$ and $\omega(\xi) = 1$ if $|\xi_i| \le 2$ for all $i = 1, \ldots, d$. Then the functions $\omega_{\overline{\nu}}(\xi) = \omega(2^{-\overline{\nu}}\xi)$ satisfy

$$\operatorname{supp}\omega_{\overline{\nu}} \subset Q_{\overline{\nu}}, \qquad \omega_{\overline{\nu}}(\xi) = 1 \quad \text{if } \xi \in \operatorname{supp}\varphi_{\overline{\nu}}$$

for all $\overline{\nu} \in \mathbb{N}_0^d$. We multiply (2.45) with $\omega_{\overline{\nu}}$, extend it by zero outside $Q_{\overline{\nu}}$, and take the inverse Fourier transform:

$$(\varphi_{\overline{\nu}}\widehat{f})^\vee(x) = \sum_{\overline{m}\in\mathbb{Z}^d} b_{\overline{\nu}\,\overline{m}}\omega_{\overline{\nu}}^\vee(x - 2^{-\overline{\nu}}\overline{m}) = \sum_{\overline{m}\in\mathbb{Z}^d} 2^{|\overline{\nu}|} b_{\overline{\nu}\,\overline{m}}\omega^\vee(2^{\overline{\nu}}x - \overline{m}), \qquad x \in \mathbb{R}^d.$$

Using (2.35) and the definition of Ψ, we get

$$(\varphi_{\overline{\nu}}\widehat{f})^\vee(x) = \sum_{\overline{m}\in\mathbb{Z}^d} 2^{|\overline{\nu}|} b_{\overline{\nu}\,\overline{m}} \sum_{\overline{l}\in\mathbb{Z}^d} \Psi(2^{\overline{\nu}}x - \overline{l})\omega^\vee(2^{\overline{\nu}}x - \overline{m}).$$

Expanding the entire analytic function $\omega^\vee(2^{\overline{\nu}}\cdot - \overline{m})$ with respect to the off-point $2^{-\overline{\nu}}\overline{l}$ we arrive at

$$\begin{aligned}(\varphi_{\overline{\nu}}\widehat{f})^\vee(x) &= \sum_{\overline{m}\in\mathbb{Z}^d} 2^{|\overline{\nu}|} b_{\overline{\nu}\,\overline{m}} \sum_{\overline{l}\in\mathbb{Z}^d} \Psi(2^{\overline{\nu}}x - \overline{l}) \sum_{\beta\in\mathbb{N}_0^d} 2^{\overline{\nu}\cdot\beta}\,\frac{(D^\beta\omega^\vee)(\overline{l} - \overline{m})}{\beta!}\,(x - 2^{-\overline{\nu}}\overline{l})^\beta \\ &= \sum_{\overline{m}\in\mathbb{Z}^d} 2^{|\overline{\nu}|} b_{\overline{\nu}\,\overline{m}} \sum_{\overline{l}\in\mathbb{Z}^d} \sum_{\beta\in\mathbb{N}_0^d} \Psi^\beta(2^{\overline{\nu}}x - \overline{l})\,\frac{(D^\beta\omega^\vee)(\overline{l} - \overline{m})}{\beta!}.\end{aligned}$$

Hence

$$f = \sum_{\overline{\nu}\in\mathbb{N}_0^d} \sum_{\beta\in\mathbb{N}_0^d} \sum_{\overline{l}\in\mathbb{Z}^d} \lambda^\beta_{\overline{\nu}\,\overline{l}} \Psi^\beta(2^{\overline{\nu}}x - \overline{l}) = \sum_{\overline{\nu}\in\mathbb{N}_0^d} \sum_{\beta\in\mathbb{N}_0^d} \sum_{\overline{l}\in\mathbb{Z}^d} \lambda^\beta_{\overline{\nu}\,\overline{l}} (\beta qu)_{\overline{\nu}\,\overline{l}}(x),$$

where

$$\lambda^\beta_{\overline{\nu}\,\overline{l}} = 2^{|\overline{\nu}|} \sum_{\overline{m}\in\mathbb{Z}^d} b_{\overline{\nu}\,\overline{m}}\,\frac{(D^\beta\omega^\vee)(\overline{l} - \overline{m})}{\beta!} = c \sum_{\overline{m}\in\mathbb{Z}^d} (\varphi_{\overline{\nu}}\widehat{f})^\vee(2^{-\overline{\nu}}\overline{m})\,\frac{(D^\beta\omega^\vee)(\overline{l} - \overline{m})}{\beta!}.$$

It remains to prove (2.39). To this end we define

$$\Lambda_{\overline{\nu}\,\overline{m}} = (\varphi_{\overline{\nu}}\widehat{f})^\vee(2^{-\overline{\nu}}\overline{m})$$

and prove that

$$\sup_{\beta\in\mathbb{N}_0^d} 2^{\varrho|\beta|}\|\lambda^\beta \,|\, s^{\overline{r}}_{p,q}f\| \le c\|\Lambda \,|\, s^{\overline{r}}_{p,q}f\| \le c'\|f \,|\, S^{\overline{r}}_{p,q}F(\mathbb{R}^d)\|. \tag{2.46}$$

We start with the second inequality in (2.46). Let $x \in Q_{\overline{\nu}\,\overline{m}}$ be fixed. Then

$$|(\varphi_{\overline{\nu}}\widehat{f})^\vee(2^{-\overline{\nu}}\overline{m})| \le \sup_{x-y\in Q_{\overline{\nu},0}} |(\varphi_{\overline{\nu}}\widehat{f})^\vee(y)| \le c(\varphi^*_{\overline{\nu}}f)_{\overline{a}}(x) \tag{2.47}$$

for every $\overline{a} \in \mathbb{R}^d_+$. We multiply (2.47) by $2^{\overline{\nu}\cdot\overline{r}}$, take the qth power and sum over $\overline{m} \in \mathbb{Z}^d$ to get

$$2^{\overline{\nu}\cdot\overline{r}q} \sum_{\overline{m}\in\mathbb{Z}^d} |\Lambda_{\overline{\nu}\,\overline{m}}|^q |\chi_{\overline{\nu}\,\overline{m}}(x)|^q \le c\,2^{\overline{\nu}\cdot\overline{r}q}(\varphi^*_{\overline{\nu}}f)^q_{\overline{a}}(x), \qquad x \in \mathbb{R}^d,\ \overline{\nu} \in \mathbb{N}_0^d.$$

Taking $\overline{a} > n/\min(p,q)$, we finally get, with the help of Theorem 1.22,

$$\begin{aligned}\|\Lambda \,|\, s^{\overline{r}}_{p,q}f\| &= \Big\|\Big(\sum_{\overline{\nu}\in\mathbb{N}_0^d} \sum_{\overline{m}\in\mathbb{Z}^d} 2^{\overline{\nu}\cdot\overline{r}q} |\Lambda_{\overline{\nu}\,\overline{m}}\chi_{\overline{\nu}\,\overline{m}}(x)|^q\Big)^{1/q} \,\Big|\, L_p(\mathbb{R}^d)\Big\| \\ &\le c\Big\|\Big(\sum_{\overline{\nu}\in\mathbb{N}_0^d} 2^{\overline{\nu}\cdot\overline{r}q}(\varphi^*_{\overline{\nu}}f)^q_{\overline{a}}(x)\Big)^{1/q} \,\Big|\, L_p(\mathbb{R}^d)\Big\| \le c\|f \,|\, S^{\overline{r}}_{p,q}F(\mathbb{R}^d)\|.\end{aligned}$$

To prove the first inequality in (2.46), we mention that

$$\lambda^{\beta}_{\overline{\nu}\,\overline{l}} = \frac{1}{\beta!} \sum_{\overline{m}\in\mathbb{Z}^d} \Lambda_{\overline{\nu}\,\overline{m}} (D^{\beta}\omega^{\vee})(\overline{l}-\overline{m}) \tag{2.48}$$

and recall a result proven in [36], namely that for any given $a>0$ there are constants $c_a>0$ and $C>0$ such that

$$|D^{\beta}\omega^{\vee}(x)| \le c_a 2^{C|\beta|}(1+|x|^2)^{-a}, \qquad x\in\mathbb{R}^d,\ \beta\in\mathbb{N}_0^d. \tag{2.49}$$

Furthermore, we define

$$h^{\beta}_{\overline{\nu}}(x) = 2^{\overline{\nu}\cdot\overline{r}} \sum_{\overline{l}\in\mathbb{Z}^d} \lambda^{\beta}_{\overline{\nu}\,\overline{l}}\chi_{\overline{\nu}\,\overline{l}}(x), \tag{2.50}$$

$$H_{\overline{\nu}}(x) = 2^{\overline{\nu}\cdot\overline{r}} \sum_{\overline{l}\in\mathbb{Z}^d} \Lambda_{\overline{\nu}\,\overline{l}}\chi_{\overline{\nu}\,\overline{l}}(x), \tag{2.51}$$

and let $0<\kappa<\min(1,p,q)$. We prove (2.46) by the following chain of inequalities:

$$\begin{aligned} 2^{\varrho|\beta|}\|\lambda^{\beta}\,|\,s^{\overline{r}}_{p,q}f\| &= 2^{\varrho|\beta|}\|h^{\beta}_{\overline{\nu}}\,|\,L_p(\ell_q)\| = 2^{\varrho|\beta|}\|\,|h^{\beta}_{\overline{\nu}}|^{\kappa}\,|\,L_{p/\kappa}(\ell_{q/\kappa})\|^{1/\kappa} \\ &\le c\,2^{\varrho|\beta|}\left(\frac{2^{C|\beta|}}{\beta!}\right)^{\kappa} \|\overline{M}(|H_{\overline{\nu}}|^{\kappa})\,|\,L_{p/\kappa}(\ell_{q/\kappa})\|^{1/\kappa} \\ &\le c'\|\,|H_{\overline{\nu}}|^{\kappa}\,|\,L_{p/\kappa}(\ell_{q/\kappa})\|^{1/\kappa} = \|\Lambda\,|\,s^{\overline{r}}_{p,q}f\|. \end{aligned} \tag{2.52}$$

The equalities in (2.52) involve only the definitions of the corresponding spaces. The second inequality follows from Theorem 1.10, the choice of κ and the growth of $\beta!$ for $|\beta|\to\infty$. Hence only the first inequality in (2.52) needs to be proven.

To prove it, put (2.49) into (2.48) to obtain for every $a>0$,

$$|\lambda^{\beta}_{\overline{\nu}\,\overline{l}}| \le \frac{c_a 2^{C|\beta|}}{\beta!} \sum_{\overline{m}\in\mathbb{Z}^d} \frac{|\Lambda_{\overline{\nu}\,\overline{m}}|}{(1+|\overline{l}-\overline{m}|^2)^a}. \tag{2.53}$$

Let $x\in Q_{\overline{\nu}\,\overline{l}}$. Using the definition of $h^{\beta}_{\overline{\nu}}$ from (2.50), (2.53) and the property $\kappa<1$ we get

$$|h^{\beta}_{\overline{\nu}}(x)|^{\kappa} = 2^{\overline{\nu}\cdot\overline{r}\kappa}|\lambda^{\beta}_{\overline{\nu}\,\overline{l}}|^{\kappa} \le \frac{c_a^{\kappa}2^{C|\beta|\kappa}}{(\beta!)^{\kappa}}\,2^{\overline{\nu}\cdot\overline{r}\kappa} \sum_{\overline{m}\in\mathbb{Z}^d} \frac{|\Lambda_{\overline{\nu}\,\overline{m}}|^{\kappa}}{(1+|\overline{l}-\overline{m}|^2)^{a\kappa}}. \tag{2.54}$$

We split the summation over $\overline{m}\in\mathbb{Z}^d$ into two sums according to the size of $|\overline{l}-\overline{m}|$:

$$\sum_{\overline{m}\in\mathbb{Z}^d} \frac{|\Lambda_{\overline{\nu}\,\overline{m}}|^{\kappa}}{(1+|\overline{l}-\overline{m}|^2)^{a\kappa}} = \sum_{k=0}^{\infty} \frac{1}{(1+k^2)^{a\kappa}} \sum_{\overline{m}\,:\,|\overline{l}-\overline{m}|=k} |\Lambda_{\overline{\nu}\,\overline{m}}|^{\kappa}. \tag{2.55}$$

Finally, we estimate the last sum using the iterated maximal operator $\overline{M}$:

$$\begin{aligned} \sum_{\overline{m}\,:\,|\overline{l}-\overline{m}|=k} |\Lambda_{\overline{\nu}\,\overline{m}}|^{\kappa} &\le 2^{-\overline{\nu}\cdot\overline{r}\kappa}2^{|\overline{\nu}|} \int\limits_{y\,:\,y-x\in(k+2)Q_{\overline{\nu},0}} |H_{\overline{\nu}}(y)|^{\kappa}\,dy \\ &\le 2^{-\overline{\nu}\cdot\overline{r}\kappa}(k+2)^d\,\overline{M}(|H_{\overline{\nu}}|^{\kappa})(x). \end{aligned} \tag{2.56}$$

We combine (2.54), (2.55) and (2.56) and arrive at

$$|h^{\beta}_{\overline{\nu}}(x)|^{\kappa} \le c'_a \frac{2^{C|\beta|\kappa}}{(\beta!)^{\kappa}}\,\overline{M}(|H_{\overline{\nu}}|^{\kappa})(x)$$

for every $a > (d+1)/2\kappa$. This finishes the proof of (2.52) and, consequently, also the proof of (2.46) and hence also of part (ii) of Theorem 2.6. ∎

Next we shall deal with subatomic decompositions in the general case. Namely, we would like to prove an analogue of Theorem 2.6 without the restriction $\overline{r} > \sigma_{pq}$.

REMARK 2.7. We introduce temporarily the following notation. Let $A \subset \{1, \ldots, d\}$ and $\overline{N} = (N_1, \ldots, N_d) \in \mathbb{R}^d$. Then we define the vector $\overline{N}^A = (N_1^A, \ldots, N_d^A)$ by

$$N_i^A = \begin{cases} N_i & \text{if } i \in A, \\ 0 & \text{if } i \notin A. \end{cases}$$

Furthermore, we denote by D_i^γ the operator

$$D_i^\gamma = \frac{\partial^\gamma}{\partial x_i^\gamma}, \qquad i = 1, \ldots, n, \ \gamma \in \mathbb{N}_0,$$

and by $D_A^{\overline{L}}$ the operator

$$D_A^{\overline{L}} = \prod_{i \in A} D_i^{L_i} = D^{\overline{L}_A}, \qquad A \subset \{1, \ldots, n\}, \ \overline{L} \in \mathbb{N}_0^d.$$

THEOREM 2.8. *Let $0 < p, q \leq \infty$ ($p < \infty$ in the F-case) and $\overline{r} \in \mathbb{R}^d$. Further let $\overline{L} + 1 \in \mathbb{N}_0^d$ and $\overline{\sigma} \in \mathbb{R}^d$ satisfy*

$$L_i \geq \max(-1, [\sigma_p - r_i]), \qquad \sigma_i > \max(\sigma_p, r_i), \qquad i = 1, \ldots, d,$$

in the B-case and

$$L_i \geq \max(-1, [\sigma_{pq} - r_i]), \quad \sigma_i > \max(\sigma_{pq}, r_i), \quad i = 1, \ldots, d,$$

in the F-case.

(i) *For every $A \subset \{1, \ldots, d\}$ let*

$$\lambda^A = \{\lambda^{A,\beta} : \beta \in \mathbb{N}_0^d\} \quad \text{with} \quad \lambda^{A,\beta} = \{\lambda^{A,\beta}_{\overline{\nu}\,\overline{m}} \in \mathbb{C} : \overline{\nu} \in \mathbb{N}_0^d, \, \overline{m} \in \mathbb{Z}^d\}$$

and let $\varrho > \phi$, where ϕ is as in (2.34). If

$$\sup_{A \subset \{1,\ldots,d\}} \sup_{\beta \in \mathbb{N}_0^d} 2^{\varrho|\beta|} \|\lambda^{A,\beta} \,|\, s^{\overline{r}}_{p,q} a\| < \infty$$

then the series

$$\sum_{A \subset \{1,\ldots,d\}} \sum_{\beta \in \mathbb{N}_0^d} \sum_{\overline{\nu} \in \mathbb{N}_0^d} \sum_{\overline{m} \in \mathbb{Z}^d} \Big(\prod_{i \notin A} 2^{\nu_i(r_i - \sigma_i)} \Big) \lambda^{A,\beta}_{\overline{\nu}\,\overline{m}} [D_A^{\overline{L}+1} \Psi^\beta](2^{\overline{\nu}} x - \overline{m}) \tag{2.57}$$

converges in $S'(\mathbb{R}^d)$, its limit f belongs to $S^{\overline{r}}_{p,q} A(\mathbb{R}^d)$ and

$$\|f \,|\, S^{\overline{r}}_{p,q} A(\mathbb{R}^d)\| \leq c \sup_{A \subset \{1,\ldots,d\}} \sup_{\beta \in \mathbb{N}_0^d} 2^{\varrho|\beta|} \|\lambda^{A,\beta} \,|\, s^{\overline{r}}_{p,q} a\|. \tag{2.58}$$

(ii) *Every $f \in S^{\overline{r}}_{p,q} A(\mathbb{R}^d)$ can be represented by (2.57) with convergence in $S'(\mathbb{R}^d)$ and*

$$\sup_{A \subset \{1,\ldots,d\}} \sup_{\beta \in \mathbb{N}_0^d} 2^{\varrho|\beta|} \|\lambda^{A,\beta} \,|\, s^{\overline{r}}_{p,q} a\| \leq c \|f \,|\, S^{\overline{r}}_{p,q} A(\mathbb{R}^d)\|. \tag{2.59}$$

REMARK 2.9. Because of the notational difficulties we shall give the proof only for $d = 2$. Furthermore, we deal only with the F-scale. The proof for the B-scale is again similar and technically simpler.

Proof of Theorem 2.8 for $d = 2$. Step 1. First we discuss the convergence of (2.57). As the first sum is finite, we may discuss the convergence of the triple sum over $\beta, \overline{\nu}$ and $\overline{m}$ separately for each $A \subset \{1,2\}$. Let us do this for example for $A = \{1\}$. Then we may rewrite the terms in (2.57) as

$$2^{\nu_2(r_2-\sigma_2)}[D^{(L_1+1,0)}\Psi^\beta](2^{\overline{\nu}}x - \overline{m}) = 2^{\nu_2(r_2-\sigma_2)}2^{-\nu_1(L_1+1)}[D^{(L_1+1,0)}(\beta qu)_{\overline{\nu}\,\overline{m}}](x) \quad (2.60)$$

where $(\beta qu)_{\overline{\nu}\,\overline{m}}(x)$ are β-quarks according to Definition 2.5. As $L_1+1 > 0$ and $\sigma_2 - r_2 > 0$, we may use the same arguments as in the proof of Theorem 2.6 and obtain the same kind of convergence. In particular, the convergence of (2.57) in $S'(\mathbb{R}^d)$ is ensured.

Step 2. Let f be given by (2.57). Then we may view this decomposition as

$$f = \sum_{A \subset \{1,2\}} f^A. \quad (2.61)$$

We shall prove that, for every admissible set A,

$$\|f^A \,|\, S^{\overline{r}}_{p,q}F(\mathbb{R}^d)\| \le c \sup_{\beta \in \mathbb{N}_0^d} 2^{\varrho|\beta|}\|\lambda^{A,\beta} \,|\, s^{\overline{r}}_{p,q}f\|. \quad (2.62)$$

If $A = \emptyset$ then the decomposition of $f^\emptyset$ into the triple sum according to (2.57) can be understood as a subatomic decomposition of $f^\emptyset$ in the space $S^{\overline{\sigma}}_{p,q}F(\mathbb{R}^d)$ and from Theorem 2.6 it follows that

$$f^\emptyset \in S^{\overline{\sigma}}_{p,q}F(\mathbb{R}^d) \subset S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$$

and

$$\|f^\emptyset \,|\, S^{\overline{r}}_{p,q}F(\mathbb{R}^d)\| \le c \sup_{\beta \in \mathbb{N}_0^d} 2^{\varrho|\beta|}\|2^{\overline{\nu}\cdot(\overline{r}-\overline{\sigma})}\lambda^{\emptyset,\beta} \,|\, s^{\overline{\sigma}}_{p,q}f\| = c \sup_{\beta \in \mathbb{N}_0^d} 2^{\varrho|\beta|}\|\lambda^{\emptyset,\beta} \,|\, s^{\overline{r}}_{p,q}f\|.$$

If $A = \{1\}$ then we use (2.60) to find that $f^{\{1\}} = D^{(L_1+1,0)}g$, where

$$g \in S^{(r_1+L_1+1,\sigma_2)}_{p,q}F(\mathbb{R}^d), \qquad \|g \,|\, S^{(r_1+L_1+1,\sigma_2)}_{p,q}F(\mathbb{R}^d)\| \le c \sup_{\beta \in \mathbb{N}_0^d} 2^{\varrho|\beta|}\|\lambda^{\{1\},\beta} \,|\, s^{\overline{r}}_{p,q}f\|.$$

Hence

$$\begin{aligned}\|f^{\{1\}} \,|\, S^{(r_1,r_2)}_{p,q}F(\mathbb{R}^d)\| &\le \|f^{\{1\}} \,|\, S^{(r_1,\sigma_2)}_{p,q}F(\mathbb{R}^d)\| = \|D^{(L_1+1,0)}g \,|\, S^{(r_1,\sigma_2)}_{p,q}F(\mathbb{R}^d)\| \\ &\le \|g \,|\, S^{(r_1+L_1+1,\sigma_2)}_{p,q}F(\mathbb{R}^d)\| \le c \sup_{\beta \in \mathbb{N}_0^d} 2^{\varrho|\beta|}\|\lambda^{\{1\},\beta} \,|\, s^{\overline{r}}_{p,q}f\|. \quad (2.63)\end{aligned}$$

Using a similar technique we prove (2.62) also for $A = \{2\}$ and $A = \{1,2\}$. Now (2.61) together with (2.62) shows that (2.58) holds.

Step 3. We prove part (ii) of the theorem. By similar arguments to Step 4 of the proof of Theorem 2.4 we prove in analogy with (2.33) that for every $\overline{M} \in \mathbb{N}_0^d$ such that

$$\overline{r} + \overline{M} + 1 \ge \overline{\sigma}, \qquad \overline{M} \ge \overline{L}, \qquad \overline{M} + 1 \in 4\mathbb{N}^2,$$

there is a function $g \in S^{\overline{r}+\overline{M}+1}_{p,q}F(\mathbb{R}^d)$ with

$$f = g + \frac{\partial^{M_1+1}g}{\partial x_1^{M_1+1}} + \frac{\partial^{M_2+1}g}{\partial x_2^{M_2+1}} + \frac{\partial^{M_1+1+M_2+1}g}{\partial x_1^{M_1+1}x_2^{M_2+1}}. \quad (2.64)$$

Furthermore

$$\|g \,|\, S^{\overline{r}+\overline{M}+1}_{p,q}F(\mathbb{R}^d)\| \approx \|f \,|\, S^{\overline{r}}_{p,q}F(\mathbb{R}^d)\|. \quad (2.65)$$

Define

$$g_1 = g, \quad g_2 = D^{(M_1-L_1,0)}g, \quad g_3 = D^{(0,M_2-L_2)}g, \quad g_4 = D^{(M_1-L_1,M_2-L_2)}g.$$

Then we can rewrite (2.64) and (2.65) as

$$f = g_1 + \frac{\partial^{L_1+1} g_2}{\partial x_1^{L_1+1}} + \frac{\partial^{L_2+1} g_3}{\partial x_2^{L_2+1}} + \frac{\partial^{L_1+1+L_2+1} g_4}{\partial x_1^{L_1+1} x_2^{L_2+1}} \tag{2.66}$$

with

$$\begin{cases} g_1 \in S^{\overline{r}+\overline{M}+1}_{p,q}F(\mathbb{R}^d) \subset S^{\overline{\sigma}}_{p,q}F(\mathbb{R}^d), \\ g_2 \in S^{(r_1+L_1+1,r_2+M_2+1)}_{p,q}F(\mathbb{R}^d) \subset S^{(r_1+L_1+1,\sigma_2)}_{p,q}F(\mathbb{R}^d), \\ g_3 \in S^{(r_1+M_1+1,r_2+L_2+1)}_{p,q}F(\mathbb{R}^d) \subset S^{(\sigma_1,r_2+L_2+1)}_{p,q}F(\mathbb{R}^d), \\ g_4 \in S^{\overline{r}+\overline{L}+1}_{p,q}F(\mathbb{R}^d). \end{cases} \tag{2.67}$$

Furthermore, the norm of g_i in the corresponding space may be estimated from above by $\|f \,|\, S^{\overline{r}}_{p,q}F(\mathbb{R}^d)\|$ for all $i = 1, \ldots, 4$. We may use Theorem 2.6 for each function g_i to get four optimal decompositions and an analogue of (2.39). Putting these estimates into (2.67) and using (2.60) we get (2.59). ■

2.4. Wavelet decomposition. In this subsection we describe the wavelet decomposition for $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$. In general, we follow the ideas in [38]. First of all, we recall the following crucial theorem from wavelet theory.

THEOREM 2.10. *For any $s \in \mathbb{N}$ there are real-valued compactly supported functions*

$$\psi_0, \psi_1 \in C^s(\mathbb{R}) \tag{2.68}$$

with

$$\int_{\mathbb{R}} t^\alpha \psi_1(t)\,dt = 0, \qquad \alpha = 0, 1, \ldots, s, \tag{2.69}$$

such that

$$\{2^{j/2}\psi_{jm}(t) : j \in \mathbb{N}_0,\, m \in \mathbb{Z}\} \tag{2.70}$$

with

$$\psi_{jm}(t) = \begin{cases} \psi_0(t-m) & \text{if } j = 0,\, m \in \mathbb{Z}, \\ \sqrt{2^{-1}}\psi_1(2^{j-1}t - m) & \text{if } j \in \mathbb{N},\, m \in \mathbb{Z}, \end{cases} \tag{2.71}$$

is an orthonormal basis in $L_2(\mathbb{R})$.

We have already observed in the previous sections the importance of tensor product constructions in the theory of function spaces with dominating mixed derivative. Following this idea, we consider a tensor product version of Theorem 2.10. Let ψ_0 and ψ_1 be the functions from Theorem 2.10 satisfying (2.68) and (2.69). Let ψ_{jm} be defined by (2.71). Then we define their tensor product counterparts by

$$\Psi_{\overline{k}\,\overline{m}}(x) = \psi_{k_1m_1}(x_1) \cdots \psi_{k_dm_d}(x_d), \tag{2.72}$$

where

$$x = (x_1, \ldots, x_d) \in \mathbb{R}^d, \quad \overline{k} = (k_1, \ldots, k_d) \in \mathbb{N}_0^d, \quad \overline{m} = (m_1, \ldots, m_d) \in \mathbb{Z}^d. \tag{2.73}$$

The tensor version of Theorem 2.10 then reads:

THEOREM 2.11. *For any $s \in \mathbb{N}$ there are real-valued compactly supported functions $\psi_0, \psi_1 \in C^s(\mathbb{R})$ satisfying* (2.69) *such that*

$$\{2^{|\overline{k}|/2}\Psi_{\overline{k}\overline{m}}(x) : \overline{k} \in \mathbb{N}_0^d,\, \overline{m} \in \mathbb{Z}^d\}, \tag{2.74}$$

with $\Psi_{\overline{k}\overline{m}}$ defined by (2.72) *and* (2.71), *is an orthonormal basis in $L_2(\mathbb{R}^d)$.*

Now we have all the necessary definitions at hand and we may state our wavelet decomposition theorem. As usual $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ stands for $S^{\overline{r}}_{p,q}B(\mathbb{R}^d)$ or $S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$, and $s^{\overline{r}}_{p,q}a$ for $s^{\overline{r}}_{p,q}b$ or $s^{\overline{r}}_{p,q}f$, respectively.

THEOREM 2.12. *Let*

$$\overline{r} = (r_1, \ldots, r_d) \in \mathbb{R}^d, \qquad 0 < p \le \infty, \qquad 0 < q \le \infty$$

with $p < \infty$ in the F-case. Then there is a natural number $s(\overline{r}, p, q)$ such that the following statements hold.

(i) *Let $\lambda \in s^{\overline{r}}_{p,q}a$. Then we have*:

1. *The sum*

$$\sum_{\overline{k}\in\mathbb{N}_0^d,\, \overline{m}\in\mathbb{Z}^d} \lambda_{\overline{k}\overline{m}}\Psi_{\overline{k}\overline{m}} \tag{2.75}$$

converges in $S'(\mathbb{R}^d)$ to some distribution f.

2. *$f \in S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ and*

$$\|f \,|\, S^{\overline{r}}_{p,q}A(\mathbb{R}^d)\| \le c\|\lambda \,|\, s^{\overline{r}}_{p,q}a\|, \tag{2.76}$$

where the constant c does not depend on λ.

3. *The sum* (2.75) *converges unconditionally in $S^{\overline{r}-\varepsilon}_{p,q}A(\mathbb{R}^d)$ for any $\varepsilon > 0$.*
4. *If* $\max(p, q) < \infty$ *then* (2.75) *converges unconditionally in $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$.*

(ii) *Let $f \in S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$. Then we may define the sequence λ by*

$$\lambda_{\overline{k}\overline{m}} = 2^{|\overline{k}|}(f, \Psi_{\overline{k}\overline{m}}), \qquad \overline{k} \in \mathbb{N}_0^d,\ \overline{m} \in \mathbb{Z}^d, \tag{2.77}$$

and we have

1. *$\lambda \in s^{\overline{r}}_{p,q}a$ and*

$$\|\lambda \,|\, s^{\overline{r}}_{p,q}a\| \le c\|f \,|\, S^{\overline{r}}_{p,q}A(\mathbb{R}^d)\|, \tag{2.78}$$

where the constant c does not depend on f.

2. *The sum* (2.75) *converges in $S'(\mathbb{R}^d)$ to f.*
3. *If $\gamma \in s^{\overline{r}}_{p,q}a$ and $\sum_{\overline{k}\in\mathbb{N}_0^d,\, \overline{m}\in\mathbb{Z}^d} \gamma_{\overline{k}\overline{m}}\Psi_{\overline{k}\overline{m}}$ converges in $S'(\mathbb{R}^d)$ to f then $\gamma = \lambda$.*

Before we come to the proof of Theorem 2.12 we clarify the technical problems caused by the limited smoothness of the functions $\Psi_{\overline{k}\overline{m}}$.

2.4.1. *Duality.* As the functions $\Psi_{\overline{k}\overline{m}}$ are of bounded smoothness, they do *not* belong to $S(\mathbb{R}^d)$. According to (2.68), (2.71) and (2.72), we only have $\Psi_{\overline{k}\overline{m}} \in C^{(s,\ldots,s)}(\mathbb{R}^d)$. Hence it is impossible to view the expression $(f, \Psi_{\overline{k}\overline{m}})$ in the distributional sense for every $f \in S'(\mathbb{R}^d)$.

To give a meaning to the symbol $(f, \Psi_{\overline{k}\overline{m}})$, one has to study the dual spaces of $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ first. As far as the Fourier-analytic version of classical Besov and Triebel–Lizorkin spaces is concerned, the corresponding theory was presented in [32, Chapter 2.11]. It is not difficult to see that one may adapt these results to the spaces with dominating mixed smoothness. We do not intend to give an exhaustive theory. The only fact we need is

$$[S^{\overline{r}}_{p,p}B(\mathbb{R}^d)]' = S^{-\overline{r}+\sigma_p}_{p'p'}B(\mathbb{R}^d), \qquad \overline{r} \in \mathbb{R}^d,\ 0 < p < \infty,$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1 \quad \text{for } 1 < p < \infty, \qquad p' = \infty \quad \text{for } p \le 1.$$

The functions $D^\alpha \Psi_{\overline{k}\overline{m}}$, $0 \le \alpha \le (s,\dots,s)$, are bounded functions with compact support. Using Hölder's inequality, we see that

$$\|D^\alpha \Psi_{\overline{k}\overline{m}} \,|\, L_{\widetilde{p}}(\mathbb{R}^d)\| < \infty, \qquad 0 \le \alpha \le (s,\dots,s),\ 0 < \widetilde{p} \le \infty.$$

Using the Littlewood–Paley theory, we get

$$\Psi_{\overline{k}\overline{m}} \in S^{\overline{s}}_{\widetilde{p},2}F(\mathbb{R}^d), \qquad 1 < \widetilde{p} < \infty,$$

for $\overline{s} = (s,\dots,s)$. And, by the Sobolev embedding,

$$S^{\overline{s}}_{\widetilde{p},2}F(\mathbb{R}^d) \hookrightarrow [S^{\overline{r}-\varepsilon}_{p,p}B(\mathbb{R}^d)]' = S^{-\overline{r}+\varepsilon+\sigma_p}_{p',p'}B(\mathbb{R}^d)$$

for s large enough and every $\varepsilon > 0$. So, for $f \in S^{\overline{r}}_{p,q}A(\mathbb{R}^d) \hookrightarrow S^{\overline{r}-\varepsilon}_{p,p}B(\mathbb{R}^d)$ we may interpret $\Psi_{\overline{k}\overline{m}}$ as a bounded linear functional on a space f belongs to. And $(f, \Psi_{\overline{k}\overline{m}})$ is then the value of this functional at f.

We may also reverse these arguments. The functions $\Psi_{\overline{k}\overline{m}}$ belong to

$$S^{\overline{s}}_{\widetilde{p},2}F(\mathbb{R}^d), \qquad 1 < \widetilde{p} < \infty,$$

and $S^{\overline{s}}_{\widetilde{p},2}F(\mathbb{R}^d) \hookrightarrow S^{\overline{s}-\varepsilon}_{\widetilde{p},\widetilde{p}}B(\mathbb{R}^d)$. Hence, for s large, we get $f \in [S^{\overline{s}-\varepsilon}_{\widetilde{p},\widetilde{p}}B(\mathbb{R}^d)]'$. In this case we may interpret f as a bounded linear functional on a space $\Psi_{\overline{k}\overline{m}}$ belongs to. $(f, \Psi_{\overline{k}\overline{m}})$ is then the value of this functional at $\Psi_{\overline{k}\overline{m}}$.

Proof of Theorem 2.12(i). Let $\lambda \in s^{\overline{r}}_{p,q}f$. If

$$s > \max\{(1+[r_i])_+, [\sigma_{pq} - r_i] : i = 1,\dots,d\}$$

and $\overline{s} = (s,\dots,s) \in \mathbb{R}^d$ then $\Psi_{\overline{k}\overline{m}}$ are $[\overline{s},\overline{s}]$-atoms centred at $Q_{\overline{k}\overline{m}}$. So, for s large, all the assumptions of Theorem 2.4 are satisfied and, according to this theorem, (2.75) converges in $S'(\mathbb{R}^d)$. We denote its limit by f. The same theorem tells us that $f \in S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$ and implies even the estimate (2.76). Hence points 1 and 2 are proven. Very similar arguments apply also to the B-case.

For $\lambda \in s^{\overline{r}}_{p,q}a$ and natural number μ we define

$$\lambda^\mu = \{\lambda^\mu_{\overline{k}\overline{m}} : \overline{k} \in \mathbb{N}^d_0,\ \overline{m} \in \mathbb{Z}^d\}$$

by

$$\lambda^\mu_{\overline{k}\overline{m}} = \begin{cases} \lambda_{\overline{k}\overline{m}} & \text{if } |\overline{k}| > \mu, \\ 0 & \text{otherwise.} \end{cases}$$

If $\max(p,q) < \infty$ then

$$\lim_{\mu\to\infty} \|\lambda^\mu \,|\, s^{\overline{r}}_{p,q}a\| = 0. \tag{2.79}$$

This is clear in the b-case and one has to use Lebesgue's dominated convergence theorem in the f-case. Using (2.76), already proven, we finish the proof of 4.

In the proof of the third point, we replace (2.79) by

$$\lim_{\mu\to\infty} \|\lambda^\mu \,|\, s^{\overline{r}-\varepsilon}_{p,q}a\| = 0. \tag{2.80}$$

To see that (2.80) holds, one uses the same reasoning as in (2.79), and Hölder's inequality. This finishes the proof of (i). ■

Proof of Theorem 2.12(ii). The meaning of the expression $(f, \Psi_{\overline{k}\overline{m}})$ was already discussed in Section 2.4.1. For the rest of the proof we consider only the F-case. The proof for B-spaces is very similar.

Before we prove the first statement of the second part we do some calculation. We may rewrite the norm in $s^{\overline{r}}_{p,q}f$ as

$$\|\lambda \,|\, s^{\overline{r}}_{p,q}f\| = \|2^{\overline{k}\cdot\overline{r}} g_{\overline{k}} \,|\, L_p(\ell_q)\|, \tag{2.81}$$

where

$$g_{\overline{k}}(x) = \sum_{\overline{m}\in\mathbb{Z}^d} \lambda_{\overline{k}\overline{m}}\chi_{\overline{k}\overline{m}}(x). \tag{2.82}$$

If $x \in Q_{\overline{k}\overline{m}}$ and λ is defined by (2.77) we use (2.82) to get

$$g_{\overline{k}}(x) = \lambda_{\overline{k}\overline{m}} = 2^{|\overline{k}|} \int_{\mathbb{R}^d} \Psi_{\overline{k}\overline{m}}(y)f(y)\,dy = 2^{|\overline{k}|} \int_{\mathbb{R}^d} \psi_{k_1m_1}(y_1)\cdots\psi_{k_dm_d}(y_d)f(y)\,dy.$$

We assume that $\overline{k} \geq 1$, insert the definition (2.71) and substitute $z_i = y_i - 2^{-k_i}m_i$:

$$\begin{aligned} g_{\overline{k}}(x) &= 2^{|\overline{k}|} \int_{\mathbb{R}^d} \psi_1(2^{k_1}z_1)\cdots\psi_1(2^{k_d}z_d)f(2^{-k_1}m_1 + z_1, \ldots, 2^{-k_d}m_d + z_d)\,dz \\ &= \mathcal{K}_{\overline{k}}(f)(2^{-\overline{k}}\overline{m}). \end{aligned}$$

Here $\mathcal{K}_{\overline{k}}(f)(2^{-\overline{k}}\overline{m})$ denotes the local means

$$\mathcal{K}_{\overline{k}}(f)(y) = \int_{\mathbb{R}^d} \mathcal{K}_{\overline{k}}(z)f(y+z)\,dz, \qquad y \in \mathbb{R}^d, \tag{2.83}$$

for the kernel

$$\mathcal{K}_{\overline{k}}(z) = 2^{|\overline{k}|}\psi_1(2^{k_1}z_1)\cdots\psi_1(2^{k_d}z_d).$$

We point out that all integrals have to be interpreted in the distributional sense. If one (or more) $k_i = 0$, only notational changes are necessary. Hence, for every $x \in Q_{\overline{k}\overline{m}}$,

$$|g_{\overline{k}}(x)| \leq \sup_{y-x\in Q_{\overline{k},0}} |\mathcal{K}_{\overline{k}}(f)(y)|.$$

Applying Theorem 1.27 we see that

$$\|\lambda \,|\, s^{\overline{r}}_{p,q}f\| = \|2^{\overline{k}\cdot\overline{r}} g_{\overline{k}} \,|\, L_p(\ell_q)\| \leq c\|f \,|\, S^{\overline{r}}_{p,q}F(\mathbb{R}^d)\|.$$

This finishes the proof of 1.

To prove the second statement, we define a new function g by

$$g = \sum_{\overline{k}\in\mathbb{N}_0^d,\, \overline{m}\in\mathbb{Z}^d} \lambda_{\overline{k}\overline{m}}\Psi_{\overline{k}\overline{m}}, \tag{2.84}$$

where $\lambda_{\overline{k}\overline{m}}$ are given by (2.77). The convergence of this sum is guaranteed by $\lambda \in s^{\overline{r}}_{p,q}f$ (which we have just proved) and by part (i). It shows even that $g \in S^{\overline{r}}_{p,q}F(\mathbb{R}^d)$. We need to prove that $g = f$ or, equivalently, that

$$(g, \varphi) = (f, \varphi) \quad \text{for every } \varphi \in S(\mathbb{R}^d).$$

First we consider the expressions $(g, \Psi_{\overline{k}'\overline{m}'})$. As $\lambda \in s^{\overline{r}}_{p,q}f$, (2.84) converges in any $S^{\overline{r}-\varepsilon}_{p,2}F(\mathbb{R}^d)$, where $\varepsilon > 0$ may be chosen arbitrarily. If the number s is chosen sufficiently large then, according to Section 2.4.1, $\Psi_{\overline{k}'\overline{m}'} \in [S^{\overline{r}-\varepsilon}_{p,2}F(\mathbb{R}^d)]'$. Hence

$$\begin{aligned}(g, \Psi_{\overline{k}'\overline{m}'}) &= \lim_{\mu\to\infty} \Big(\sum_{|\overline{k}|\le\mu,\, \overline{m}\in\mathbb{Z}^d} \lambda_{\overline{k}\overline{m}} \Psi_{\overline{k}\overline{m}}, \Psi_{\overline{k}'\overline{m}'} \Big) \\ &= \lim_{\mu\to\infty} \sum_{|\overline{k}|\le\mu,\, \overline{m}\in\mathbb{Z}^d} 2^{|\overline{k}|} (f, \Psi_{\overline{k}\overline{m}})(\Psi_{\overline{k}\overline{m}}, \Psi_{\overline{k}'\overline{m}'}).\end{aligned}$$

Using orthogonality of system (2.74) we arrive at

$$(g, \Psi_{\overline{k}'\overline{m}'}) = (f, \Psi_{\overline{k}'\overline{m}'}), \qquad \overline{k}' \in \mathbb{N}_0^d,\ \overline{m}' \in \mathbb{Z}^d.$$

One may extend this argument to any finite linear combination of $\Psi_{\overline{k}'\overline{m}'}$. For a general function $\varphi \in S(\mathbb{R}^d)$ we consider its Fourier series decomposition with respect to system (2.74):

$$\varphi = \sum_{\overline{k},\overline{m}} 2^{|\overline{k}|} (\varphi, \Psi_{\overline{k}\overline{m}}) \Psi_{\overline{k}\overline{m}}. \tag{2.85}$$

As $S(\mathbb{R}^d)$ is a subset of all Fourier-analytic Besov and Triebel–Lizorkin spaces, we see that (for s large enough) (2.85) converges also in the space $[S^{\overline{r}-\varepsilon}_{p,2}F(\mathbb{R}^d)]'$. Hence we get

$$\begin{aligned}(g, \varphi) &= \lim_{\mu\to\infty} \sum_{|\overline{k}|\le\mu,\, \overline{m}\in\mathbb{Z}^d} 2^{|\overline{k}|} (\varphi, \Psi_{\overline{k}\overline{m}})(g, \Psi_{\overline{k}\overline{m}}) \\ &= \lim_{\mu\to\infty} \sum_{|\overline{k}|\le\mu,\, \overline{m}\in\mathbb{Z}^d} 2^{|\overline{k}|} (\varphi, \Psi_{\overline{k}\overline{m}})(f, \Psi_{\overline{k}\overline{m}}) = (f, \varphi).\end{aligned}$$

Hence the sum (2.75) converges to f.

The final step, namely the proof of the third statement, follows now very easily. Suppose that the assumptions are satisfied. We define the coefficients $\lambda_{\overline{k}\overline{m}}$ by (2.77) and g by (2.84). Then we get $f = g$ according to point 2. And by the same duality arguments as there we obtain

$$\gamma_{\overline{k}\overline{m}} = 2^{|\overline{k}|/2} (f, \Psi_{\overline{k}\overline{m}}) = 2^{|\overline{k}|/2} (g, \Psi_{\overline{k}\overline{m}}) = \lambda_{\overline{k}\overline{m}}, \qquad \overline{k} \in \mathbb{N}_0^d,\ \overline{m} \in \mathbb{Z}^d. \ \blacksquare$$

3. Entropy numbers—direct results

3.1. Notation and definitions. We have seen in the previous section the close connection between the function spaces $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ and the corresponding sequence spaces $s^{\overline{r}}_{p,q}a$ given by several decomposition techniques. We use these results to study the entropy numbers of embeddings of function spaces with dominating mixed smoothness on domains.

First, we define function spaces on domains by restrictions of function spaces defined on $\mathbb{R}^d$.

Definition 3.1. Let Ω be an arbitrary bounded domain in $\mathbb{R}^d$. Then $S^{\overline{r}}_{p,q}A(\Omega)$ is the restriction of $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ to Ω:

$$S^{\overline{r}}_{p,q}A(\Omega) = \{f \in D'(\Omega) : \exists g \in S^{\overline{r}}_{p,q}A(\mathbb{R}^d) \text{ with } g|\Omega = f\}, \tag{3.1}$$

$$\|f \,|\, S^{\overline{r}}_{p,q}A(\Omega)\| = \inf \|g \,|\, S^{\overline{r}}_{p,q}A(\mathbb{R}^d)\|, \tag{3.2}$$

where the infimum is taken over all $g \in S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ such that its restriction to Ω, denoted by $g|\Omega$, coincides in $D'(\Omega)$ with f.

Next, we define the sequence spaces corresponding to $S^{\overline{r}}_{p,q}A(\Omega)$. The change with respect to $s^{\overline{r}}_{p,q}a$ is rather simple. In Definition 2.2 the sum over $\overline{m} \in \mathbb{Z}^d$ represents a discrete analogue of $L_p(\mathbb{R}^d)$-norm and the sum over $\overline{\nu} \in \mathbb{N}_0^d$ the sum over all coverings of plane with dyadic cubes. So, to adapt Definition 2.2 to fit function spaces on domains, we have to restrict the sum to those $\overline{m}$ which are in some relation with Ω.

For that reason we define, for every bounded domain $\Omega \subset \mathbb{R}^d$,

$$A^{\Omega}_{\overline{\nu}} = \{\overline{m} \in \mathbb{Z}^d : Q_{\overline{\nu}\,\overline{m}} \cap \Omega \neq \emptyset\}, \qquad \overline{\nu} \in \mathbb{N}_0^d.$$

The sequence spaces associated with a bounded domain Ω are then defined by

Definition 3.2. If $0 < p \le \infty$, $0 < q \le \infty$, $\overline{r} \in \mathbb{R}^d$ and

$$\lambda = \{\lambda_{\overline{\nu}\,\overline{m}} \in \mathbb{C} : \overline{\nu} \in \mathbb{N}_0^d,\, \overline{m} \in A^{\Omega}_{\overline{\nu}}\}$$

then we define

$$s^{\overline{r},\Omega}_{p,q}b = \Big\{\lambda : \|\lambda \,|\, s^{\overline{r},\Omega}_{p,q}b\| = \Big(\sum_{\overline{\nu}\in\mathbb{N}_0^d} 2^{\overline{\nu}\cdot(\overline{r}-1/p)q}\Big(\sum_{\overline{m}\in A^{\Omega}_{\overline{\nu}}} |\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{q/p}\Big)^{1/q} < \infty\Big\} \tag{3.3}$$

and

$$s^{\overline{r},\Omega}_{p,q}f = \Big\{\lambda : \|\lambda \,|\, s^{\overline{r},\Omega}_{p,q}f\| = \Big\|\Big(\sum_{\overline{\nu}\in\mathbb{N}_0^d}\sum_{\overline{m}\in A^{\Omega}_{\overline{\nu}}} |2^{\overline{\nu}\cdot\overline{r}}\lambda_{\overline{\nu}\,\overline{m}}\chi_{\overline{\nu}\,\overline{m}(\cdot)}|^q\Big)^{1/q} \,\Big|\, L_p(\mathbb{R}^d)\Big\| < \infty\Big\}. \tag{3.4}$$

Furthermore, we define the corresponding *building blocks.*

Definition 3.3. Let $0 < p \le \infty$, $0 < q \le \infty$, $\overline{r} \in \mathbb{R}^d$ and let $\mu \in \mathbb{N}_0$ be fixed. If

$$\lambda = \{\lambda_{\overline{\nu}\,\overline{m}} \in \mathbb{C} : \overline{\nu} \in \mathbb{N}_0^d,\, |\overline{\nu}| = \mu,\, \overline{m} \in A^{\Omega}_{\overline{\nu}}\}$$

then we define

$$(s^{\overline{r},\Omega}_{p,q}b)_\mu = \Big\{\lambda : \|\lambda \,|\, (s^{\overline{r},\Omega}_{p,q}b)_\mu\| = \Big(\sum_{|\overline{\nu}|=\mu} 2^{\overline{\nu}\cdot(\overline{r}-1/p)q}\Big(\sum_{\overline{m}\in A^{\Omega}_{\overline{\nu}}} |\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{q/p}\Big)^{1/q} < \infty\Big\} \tag{3.5}$$

and

$$(s^{\overline{r},\Omega}_{p,q}f)_\mu = \Big\{\lambda : \|\lambda \,|\, (s^{\overline{r},\Omega}_{p,q}f)_\mu\| = \Big\|\Big(\sum_{|\overline{\nu}|=\mu}\sum_{\overline{m}\in A^{\Omega}_{\overline{\nu}}} |2^{\overline{\nu}\cdot\overline{r}}\lambda_{\overline{\nu}\,\overline{m}}\chi_{\overline{\nu}\,\overline{m}}(\cdot)|^q\Big)^{1/q} \,\Big|\, L_p(\mathbb{R}^d)\Big\| < \infty\Big\}. \tag{3.6}$$

Remark 3.4. 1. We point out that for the number of elements of $A^{\Omega}_{\overline{\nu}}$ we have trivially

$$\#(A^{\Omega}_{\overline{\nu}}) \approx 2^{|\overline{\nu}|}, \qquad \overline{\nu} \in \mathbb{N}_0^d, \tag{3.7}$$

where the equivalence constants depend only on Ω. The dimension of $(s^{\overline{r},\Omega}_{p,q}a)_\mu$ will be denoted by

$$D_\mu := \sum_{|\overline{\nu}|=\mu} \#(A^{\Omega}_{\overline{\nu}}), \quad \mu \in \mathbb{N}_0. \tag{3.8}$$

2. As usual, we write $s^{\overline{r},\Omega}_{p,q}a$ for $s^{\overline{r},\Omega}_{p,q}b$ or $s^{\overline{r},\Omega}_{p,q}f$ respectively. The same holds for $(s^{\overline{r},\Omega}_{p,q}a)_\mu$.

Next we define the notion of entropy numbers and recall their basic properties. We refer to [10] and references given there for details.

Definition 3.5. Let A, B be quasi-Banach spaces and let T be a bounded linear operator $T \in L(A,B)$. Let U_A and U_B denote the unit balls in A and B, respectively. Then for every $k \in \mathbb{N}$ we define the kth *dyadic entropy number* by

$$e_k(T) := \inf\Big\{\varepsilon > 0 : T(U_A) \subset \bigcup_{j=1}^{2^{k-1}} (b_j + \varepsilon U_B)\Big\}$$

for some $b_1, \dots, b_{2^{k-1}} \in B$.

Definition 3.6. Given any $p \in (0,1]$ and a quasi-Banach space B, we say that B is a *p-Banach space* if

$$\|x+y \,|\, B\|^p \le \|x \,|\, B\|^p + \|y \,|\, B\|^p \quad \text{for all } x, y \in B. \tag{3.9}$$

It can be shown that if $\|\cdot \,|\, B\|_1$ is a quasinorm on B, then there is $p \in (0,1]$ and a quasinorm $\|\cdot \,|\, B\|_2$ with (3.9) on B which is equivalent to $\|\cdot \,|\, B\|_1$. We refer again to [10] and references given there for details.

Theorem 3.7. *Let A, B, C be quasi-Banach spaces*, $S, T \in L(A,B)$, $R \in L(B,C)$. *Then*

- $\|T\| \ge e_1(T) \ge e_2(T) \ge \dots \ge 0$.
- $e_{k+l-1}(R \circ S) \le e_k(R) e_l(S)$, $k, l \in \mathbb{N}$.
- *If B is a p-Banach space, then* $e^p_{k+l-1}(S+T) \le e^p_k(S) + e^p_l(T)$.

Remark 3.8. We refer to the first property of entropy numbers from Theorem 3.7 as *monotonicity*, the second is called *submultiplicativity*, and the last one is *subadditivity*.

Although we shall not need it, we quote the fundamental result of Carl (see [6], [7] and [10] for details). It illustrates the importance of estimates of entropy numbers in the study of spectral properties of compact operators.

Theorem 3.9. *Let A be a quasi-Banach space and let $T \in L(A,A) = L(A)$ be a compact operator on A. Denote its non-zero eigenvalues* (*counted with multiplicity*) *by*

$$|\lambda_1(T)| \ge |\lambda_2(T)| \ge |\lambda_3(T)| \ge \dots > 0.$$

Then

$$|\lambda_k(T)| \le \sqrt{2}\, e_k(T).$$

In what follows we restrict ourselves to $\overline{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$ with $r_1 = \dots = r_d$.

3.2. Basic lemmas. Now we collect some basic properties of the building blocks defined by (3.5) and (3.6). We start with the following

LEMMA 3.10. 1. *Let* $0 < p_1, p_2 \le \infty$ *and* $N \in \mathbb{N}$. *Then*

$$\|id : \ell_{p_1}^N \to \ell_{p_2}^N\| = \begin{cases} 1, & p_1 \le p_2, \\ N^{1/p_2 - 1/p_1}, & p_1 \ge p_2. \end{cases} \tag{3.10}$$

2. *Let* $0 < p \le \infty$ *and* $\overline{r} = (r, \ldots, r) \in \mathbb{R}^d$. *Then*

$$(s_{p,p}^{\overline{r},\Omega} b)_\mu = (s_{p,p}^{\overline{r},\Omega} f)_\mu = 2^{\mu(r-1/p)} \ell_p^{D_\mu}, \qquad \mu \in \mathbb{N}_0, \tag{3.11}$$

$$s_{p,p}^{\overline{r},\Omega} b = s_{p,p}^{\overline{r},\Omega} f. \tag{3.12}$$

The number D_μ *is given by* (3.8).

3. *Let* $0 < p_2 \le p_1 \le \infty$, $0 < q \le \infty$ *and* $\overline{r} = (r, \ldots, r) \in \mathbb{R}^d$. *Then*

$$\|\mathrm{id} : (s_{p_1,q}^{\overline{r},\Omega} a)_\mu \to (s_{p_2,q}^{\overline{r},\Omega} a)_\mu\| \approx 1, \qquad \mu \in \mathbb{N}_0. \tag{3.13}$$

4. *Let* $0 < q_2 \le q_1 \le \infty$, $0 < p \le \infty$ *and* $\overline{r} = (r, \ldots, r) \in \mathbb{R}^d$. *Then*

$$\|\mathrm{id} : (s_{p,q_1}^{\overline{r},\Omega} a)_\mu \to (s_{p,q_2}^{\overline{r},\Omega} a)_\mu\| \approx \mu^{(d-1)(1/q_2 - 1/q_1)}, \quad \mu \in \mathbb{N}. \tag{3.14}$$

The equivalence constants in (3.13) *and* (3.14) *do not depend on* $\mu \in \mathbb{N}_0$.

Proof. The proof of 1 and 2 involves only (3.5) and (3.6). For the proof of 3 in the case $a = b$ we write

$$\begin{aligned}
\|\lambda \mid s_{p_2,q}^{\overline{r},\Omega} b\| &= \Big(\sum_{|\overline{\nu}|=\mu} 2^{\overline{\nu}\cdot(\overline{r}-1/p_2)q} \Big(\sum_{\overline{m}\in A_{\overline{\nu}}^{\Omega}} |\lambda_{\overline{\nu}\,\overline{m}}|^{p_2}\Big)^{q/p_2}\Big)^{1/q} \\
&= 2^{\mu(r-1/p_2)} \Big(\sum_{|\overline{\nu}|=\mu} \Big(\sum_{\overline{m}\in A_{\overline{\nu}}^{\Omega}} |\lambda_{\overline{\nu}\,\overline{m}}|^{p_2}\Big)^{q/p_2}\Big)^{1/q} \\
&\le c 2^{\mu(r-1/p_2)} 2^{\mu(1/p_2-1/p_1)} \Big(\sum_{|\overline{\nu}|=\mu} \Big(\sum_{\overline{m}\in A_{\overline{\nu}}^{\Omega}} |\lambda_{\overline{\nu}\,\overline{m}}|^{p_1}\Big)^{q/p_1}\Big)^{1/q} \\
&= c\|\lambda \mid s_{p_1,q}^{\overline{r},\Omega} b\|,
\end{aligned}$$

where we have used (3.10).

In the case $a = f$, by Hölder's inequality and boundedness of Ω we get

$$\begin{aligned}
\|\lambda \mid s_{p_2,q}^{\overline{r},\Omega} f\| &= \Big\|\Big(\sum_{\overline{\nu}\in\mathbb{N}_0^d} \sum_{\overline{m}\in A_{\overline{\nu}}^{\Omega}} |2^{\overline{\nu}\cdot\overline{r}} \lambda_{\overline{\nu}\,\overline{m}} \chi_{\overline{\nu}\,\overline{m}}(\cdot)|^q\Big)^{1/q} \Bigm| L_{p_2}(\mathbb{R}^d)\Big\| \\
&\le c\Big\|\Big(\sum_{\overline{\nu}\in\mathbb{N}_0^d} \sum_{\overline{m}\in A_{\overline{\nu}}^{\Omega}} |2^{\overline{\nu}\cdot\overline{r}} \lambda_{\overline{\nu}\,\overline{m}} \chi_{\overline{\nu}\,\overline{m}}(\cdot)|^q\Big)^{1/q} \Bigm| L_{p_1}(\mathbb{R}^d)\Big\| \\
&= c\|\lambda \mid s_{p_1,q}^{\overline{r},\Omega} f\|.
\end{aligned}$$

The proof of 4 involves only 1 and

$$\#\{\overline{\nu} \in \mathbb{N}_0^d : |\overline{\nu}| = \mu\} \approx \mu^{d-1}, \qquad \mu \in \mathbb{N}. \ \blacksquare$$

Next, we recall a fundamental result which is essentially due to Schütt [27] and Kühn [17].

Lemma 3.11. (i) *If* $0 < p_1 \le p_2 \le \infty$ *and* k *and* N *are natural numbers, then*

$$e_k(\mathrm{id} : \ell_{p_1}^N \to \ell_{p_2}^N) \approx \begin{cases} 1 & \text{if } 1 \le k \le \log 2N, \\ (k^{-1}\log(1+N/k))^{1/p_1-1/p_2} & \text{if } \log 2N \le k \le 2N, \\ 2^{-k/2N} N^{1/p_2-1/p_1} & \text{if } 2N \le k, \end{cases} \tag{3.15}$$

where the equivalence constants do not depend on k *and* N.
(ii) *If* $0 < p_2 < p_1 \le \infty$ *and* k *and* N *are natural numbers, then*

$$e_k(\mathrm{id} : \ell_{p_1}^N \to \ell_{p_2}^N) \approx 2^{-k/2N} N^{1/p_2-1/p_1} \tag{3.16}$$

where the implied constants again do not depend on k *and* N.

Remark 3.12. We refer to [27], [17], [10] and references given there for the proofs of this fundamental result.

Lemma 3.13. *Let*

$$\overline{r}_1 = (r_1, \dots, r_1) \in \mathbb{R}^d, \qquad \overline{r}_2 = (r_2, \dots, r_2) \in \mathbb{R}^d, \qquad 0 < p_1, p_2, q_1, q_2 \le \infty.$$

Let $k \ge 2D_\mu$. *Then*

$$e_k(\mathrm{id} : (s^{\overline{r}_1,\Omega}_{p_1,q_1} a)_\mu \to (s^{\overline{r}_2,\Omega}_{p_2,q_2} a^\dagger)_\mu) \approx 2^{-k/2D_\mu} \mu^{(d-1)(1/q_2-1/q_1)} 2^{\mu(r_2-r_1)} \tag{3.17}$$

with equivalence constants independent of k *and* μ.

Remark 3.14. The symbols a and $a^\dagger$ stand for b or f, not necessarily for the same letter. Hence the formula (3.17) represents actually *four* different equivalences and, consequently, eight inequalities are to be proven.

Proof. Set

$$\gamma_1 = \min(p_1, q_1), \qquad \gamma_2 = \min(p_2, q_2) \tag{3.18}$$

$$\delta_1 = \max(p_1, q_1), \qquad \delta_2 = \max(p_2, q_2). \tag{3.19}$$

Step 1. We use the following diagram to estimate $e_k(\mathrm{id})$ from above:

$$\begin{array}{ccc} (s^{\overline{r}_1,\Omega}_{p_1,q_1} a)_\mu & \xrightarrow{\mathrm{id}} & (s^{\overline{r}_2,\Omega}_{p_2,q_2} a^\dagger)_\mu \\ \mathrm{id}_1 \big\downarrow & & \big\uparrow \mathrm{id}_3 \\ (s^{\overline{r}_1,\Omega}_{\gamma_1,\gamma_1} a)_\mu & \xrightarrow{\mathrm{id}_2} & (s^{\overline{r}_2,\Omega}_{\delta_2,\delta_2} a^\dagger)_\mu \end{array} \tag{3.20}$$

Using the submultiplicativity of entropy numbers (see Theorem 3.7) we get

$$e_k(\mathrm{id}) \le \|\mathrm{id}_1\| \cdot \|\mathrm{id}_3\| \cdot e_k(\mathrm{id}_2). \tag{3.21}$$

To estimate $\|\mathrm{id}_1\|$ and $\|\mathrm{id}_3\|$ we use (3.13), resp. (3.14) to get

$$\|\mathrm{id}_1\| \le c\mu^{(d-1)(1/\gamma_1-1/q_1)}, \qquad \|\mathrm{id}_3\| \le c\mu^{(d-1)(1/q_2-1/\delta_2)}. \tag{3.22}$$

To estimate $e_k(\mathrm{id}_2)$ we use Lemma 3.11 and (3.11),

$$(s^{\overline{r}_1,\Omega}_{\gamma_1,\gamma_1} a)_\mu \approx 2^{\mu(r_1-1/\gamma_1)} \ell_{\gamma_1}^{D_\mu},$$

and its counterpart for $(s^{\overline{r}_2,\Omega}_{\delta_2,\delta_2} a^\dagger)_\mu$. This gives

$$e_k(\mathrm{id}_2) \le c\, 2^{\mu(-r_1+1/\gamma_1+r_2-1/\delta_2)} 2^{-k/2D_\mu} D_\mu^{1/\delta_2-1/\gamma_1}. \tag{3.23}$$

Putting (3.22) and (3.23) into (3.21) and using $D_\mu \approx \mu^{d-1}2^\mu$ we get the desired result and finish Step 1.

Step 2. We now prove the estimates from below. Let $\gamma_1, \gamma_2, \delta_1, \delta_2$ be still defined by (3.18) and (3.19), respectively. We use the diagram

$$\begin{array}{ccc} (s^{\overline{r}_1,\Omega}_{p_1,q_1}a)_\mu & \xrightarrow{\mathrm{id}} & (s^{\overline{r}_2,\Omega}_{p_2,q_2}a^\dagger)_\mu \\ \mathrm{id}_1 \Big\uparrow & & \Big\downarrow \mathrm{id}_3 \\ (s^{\overline{r}_1,\Omega}_{\delta_1,\delta_1}a)_\mu & \xrightarrow{\mathrm{id}_2} & (s^{\overline{r}_2,\Omega}_{\gamma_2,\gamma_2}a^\dagger)_\mu \end{array} \tag{3.24}$$

As $\mathrm{id}_2 = \mathrm{id}_1 \circ \mathrm{id} \circ \mathrm{id}_3$ we may use again the submultiplicativity of entropy numbers. The estimate for the entropy numbers of id_2 is given by Lemma 3.11:

$$e_k(\mathrm{id}_2) \geq c 2^{\mu(-r_1+r_2+1/\delta_1-1/\gamma_2)} 2^{-k/2D_\mu} D_\mu^{1/\gamma_2-1/\delta_1}$$

and for $\|\mathrm{id}_1\|$ and $\|\mathrm{id}_3\|$ we use estimates similar to those in Step 1:

$$\|\mathrm{id}_1\| \leq c\mu^{(d-1)(1/q_1-1/\delta_1)}, \qquad \|\mathrm{id}_3\| \leq c\mu^{(d-1)(1/\gamma_2-1/q_2)}. \tag{3.25}$$

From this the result follows immediately. ∎

Lemma 3.13 is a generalisation of Lemma 3.11 as far as the third line of (3.15) and (3.16) are concerned. So, for $k \geq 2D_\mu$, the estimate (3.17) provides four equivalences with constants independent of k and μ. In the case $k \leq 2D_\mu$ the situation is not so simple any more; we give two different estimates from above.

Lemma 3.15. *Let*

$$\overline{r}_1 = (r_1, \ldots, r_1) \in \mathbb{R}^d, \qquad \overline{r}_2 = (r_2, \ldots, r_2) \in \mathbb{R}^d, \qquad 0 < p_1, p_2, q_1, q_2 \leq \infty$$

with $p_1 \leq p_2$. *Let* $k \leq 2D_\mu$. *Then*

$$\begin{aligned} e_k(\mathrm{id} : (s^{\overline{r}_1,\Omega}_{p_1,q_1}a)_\mu \to (s^{\overline{r}_2,\Omega}_{p_2,q_2}a^\dagger)_\mu) &\leq c\mu^{(d-1)(1/\gamma_1-1/q_1+1/q_2-1/\delta_2)} 2^{\mu(-r_1+1/\gamma_1+r_2-1/\delta_2)} \\ &\quad \cdot [k^{-1}\log(\mu^{d-1}2^\mu/k+1)]^{1/\gamma_1-1/\delta_2}, \end{aligned} \tag{3.26}$$

where $\gamma_1, \gamma_2, \delta_1, \delta_2$ *are given by* (3.18) *and* (3.19). *The constant* c *is independent of* k *and* μ.

The proof of Lemma 3.15 copies exactly the first step of the proof of Lemma 3.13.

The second estimate from above follows closely the idea of Kühn, Leopold, Sickel and Skrzypczak in [18].

Lemma 3.16. *Let*

$$\overline{r}_1 = (r_1, \ldots, r_1) \in \mathbb{R}^d, \qquad \overline{r}_2 = (r_2, \ldots, r_2) \in \mathbb{R}^d, \qquad 0 < p_1, p_2, q_1, q_2 \leq \infty$$

with

$$p_1 \leq p_2, \qquad \frac{1}{p_1} - \frac{1}{p_2} > \frac{1}{q_1} - \frac{1}{q_2}.$$

Let $(d-1)\mu^{d-1}\log\mu \leq k \leq 2D_\mu = 2\sum_{|\overline{\nu}|=\mu} \# A^\Omega_{\overline{\nu}}$. *Then*

$$\begin{aligned} e_k(\mathrm{id} : (s^{\overline{r}_1,\Omega}_{p_1,q_1}b)_\mu \to (s^{\overline{r}_2,\Omega}_{p_2,q_2}b)_\mu) &\leq c 2^{\mu(-r_1+r_2+1/p_1-1/p_2)} \mu^{1/p_1-1/p_2+1/q_2-1/q_1} \\ &\quad \cdot k^{1/p_2-1/p_1}[\log(\mu^{d-1}2^\mu/k+1)]^{1/p_1-1/p_2}. \end{aligned} \tag{3.27}$$

Proof. Set $X_i = (s^{\overline{r}_i,\Omega}_{p_i,q_i} b)_\mu$, $i = 1, 2$. We shall construct an ε-net of X_2-balls covering the unit ball B_{X_1} of X_1. For that purpose we fix some ordering of the set $\{\overline{\nu} \in \mathbb{N}_0^d : |\overline{\nu}| = \mu\} = \{\overline{\nu}^1, \ldots, \overline{\nu}^{S(\mu,d)}\}$, where

$$S(\mu, d) = \#\{\overline{\nu} \in \mathbb{N}_0^d : |\overline{\nu}| = \mu\} = \binom{\mu + d - 1}{\mu}, \qquad \mu \in \mathbb{N}_0. \tag{3.28}$$

First we consider the subset of B_{X_1},

$$B = \{\lambda \in B_{X_1} : \|\lambda_{\overline{\nu}^1} \mid X_1\| \geq \|\lambda_{\overline{\nu}^2} \mid X_1\| \geq \cdots \geq \|\lambda_{\overline{\nu}^{S(\mu,d)}} \mid X_1\|\}$$

and construct an ε-net $\mathcal{N}$ in X_2 for B. Then, if Π is any permutation of the index set $\{1, \ldots, S(\mu, d)\}$ and

$$B_\Pi = \{\lambda \in B_{X_1} : \|\lambda_{\overline{\nu}^{\Pi(1)}} \mid X_1\| \geq \|\lambda_{\overline{\nu}^{\Pi(2)}} \mid X_1\| \geq \cdots \geq \|\lambda_{\overline{\nu}^{\Pi(S(\mu,d))}} \mid X_1\|\}$$

we get, by permutation of coordinates, ε-nets $\mathcal{N}_\Pi$ for B_Π, all having the same cardinality as $\mathcal{N}$, say 2^k.

Clearly, $B_{X_1} = \bigcup_\Pi B_\Pi$, where the union is over all permutations Π of $\{1, \ldots, S(\mu, d)\}$. Hence $\bigcup_\Pi \mathcal{N}_\Pi$ is an ε-net in X_2 for B_{X_1} of cardinality

$$S(\mu, d)!2^k \leq \mu^{(d-1)\mu^{d-1}} 2^k = 2^{(d-1)\mu^{d-1} \log \mu + k}.$$

It remains to construct an ε-net for B_X in X_2. For $\lambda \in B$ we have $\|\lambda_{\overline{\nu}^j} \mid X_1\| \leq j^{-1/q_1}$. If $k_1, \ldots, k_{S(\mu,d)}$ are arbitrary natural numbers, we set

$$\varepsilon_j := c j^{-1/q_1} 2^{\mu(-r_1+1/p_1+r_2-1/p_2)} [k_j^{-1} \log(2^\mu/k_j + 1)]^{1/p_1-1/p_2}$$

and, according to Lemma 3.11, we find an ε_j-net $\mathcal{N}_j$ in $2^{\mu(r_2-1/p_2)} \ell_{p_2}^{A_j}$ for $j^{-1/q_1} B_Y$, where $Y = 2^{\mu(r_1-1/p_1)} \ell_{p_1}^{A_j}$ and $A_j = \#(A^{\Omega}_{\overline{\nu}^j})$. Thus $\mathcal{N}_1 \times \cdots \times \mathcal{N}_{S(\mu,d)}$ is an ε-net in X_2 for B of cardinality $2^{k_1+\cdots+k_{S(\mu,d)}}$, where

$$\varepsilon = \Big(\sum_{j=1}^{S(\mu,d)} \varepsilon_j^{q_2}\Big)^{1/q_2}.$$

Finally, we choose k_j, $j = 1, \ldots, S(\mu, d)$. Fix $m \in \mathbb{N}$ and set $k_j = 2^m j^{-\alpha}$, where $0 < \alpha < 1$ is chosen such that $\alpha(1/p_1 - 1/p_2) > 1/q_1 - 1/q_2$ Then

$$k = \sum_{j=1}^{S(\mu,d)} k_j \approx 2^m \mu^{(d-1)(-\alpha+1)} \tag{3.29}$$

and

$$\Big(\sum_{j=1}^{S(\mu,d)} \varepsilon_j^{q_2}\Big)^{1/q_2} \approx 2^{\mu(-r_1+1/p_1+r_2-1/p_2)} 2^{m(1/p_2-1/p_1)} \cdot S(\mu, d)^{\alpha(1/p_1-1/p_2)-1/q_1+1/q_2} [\log(2^{\mu-m}\mu^\alpha + 1)]^{1/p_1-1/p_2}.$$

Substituting for 2^m from (3.29) we get

$$\Big(\sum_{j=1}^{S(\mu,d)} \varepsilon_j^{q_2}\Big)^{1/q_2} \approx 2^{\mu(-r_1+1/p_1+r_2-1/p_2)} k^{1/p_2-1/p_1} \cdot \mu^{(d-1)(1/p_1-1/p_2+1/q_2-1/q_1)} [\log(\mu^{d-1} 2^\mu/k + 1)]^{1/p_1-1/p_2},$$

which finishes the proof. ∎

3.3. Main result. In this subsection we present our main results concerning sequence spaces. Our aim is to estimate the entropy numbers of

$$\mathrm{id} : s^{\overline{r}_1,\Omega}_{p_1,q_1} a \to s^{\overline{r}_2,\Omega}_{p_2,q_2} a^\dagger. \tag{3.30}$$

First we split the identity (3.30) into a sum of identities between building blocks,

$$\mathrm{id} = \sum_{\mu=0}^{\infty} \mathrm{id}_\mu, \qquad \mathrm{id}_\mu : s^{\overline{r}_1,\Omega}_{p_1,q_1} a \to s^{\overline{r}_2,\Omega}_{p_2,q_2} a^\dagger, \tag{3.31}$$

where

$$(\mathrm{id}_\mu \, \lambda)_{\overline{\nu}\,\overline{m}} = \begin{cases} \lambda_{\overline{\nu}\,\overline{m}} & \text{if } |\overline{\nu}| = \mu, \\ 0 & \text{otherwise,} \end{cases} \tag{3.32}$$

for all $\overline{\nu} \in \mathbb{N}_0^d, \overline{m} \in A^\Omega_{\overline{\nu}}$. Next we observe that

$$e_k(\mathrm{id}_\mu) = e_k(\mathrm{id}'_\mu), \qquad k \in \mathbb{N},\ \mu \in \mathbb{N}_0, \tag{3.33}$$

where

$$\mathrm{id}'_\mu : (s^{\overline{r}_1,\Omega}_{p_1,q_1} a)_\mu \to (s^{\overline{r}_2,\Omega}_{p_2,q_2} a^\dagger)_\mu, \qquad \mu \in \mathbb{N}_0, \tag{3.34}$$

are the natural identities between our building blocks.

First, we characterise when the embedding (3.30) is compact.

THEOREM 3.17. *Let*

$$\overline{r}_1 = (r_1, \ldots, r_1) \in \mathbb{R}^d, \qquad \overline{r}_2 = (r_2, \ldots, r_2) \in \mathbb{R}^d, \qquad 0 < p_1, p_2, q_1, q_2 \le \infty. \tag{3.35}$$

Then the embedding (3.30) *is compact if and only if*

$$\alpha = r_1 - r_2 - \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > 0. \tag{3.36}$$

Proof. PART 1. In the first part we prove that (3.36) is sufficient for compactness of (3.30). First we restrict to the case

- $0 < p_1 \le p_2 \le \infty$ and $a = a^\dagger = b$.

It is an easy exercise to show that

$$\begin{aligned} \|\mathrm{id}_\mu \mid s^{\overline{r}_1,\Omega}_{p_1,q_1} b \to s^{\overline{r}_2,\Omega}_{p_2,q_2} b\| &= \|\,\mathrm{id}'_\mu \mid (s^{\overline{r}_1,\Omega}_{p_1,q_1} b)_\mu \to (s^{\overline{r}_2,\Omega}_{p_2,q_2} b)_\mu\| \\ &\le 2^{-\mu(r_1 - r_2 + 1/p_2 - 1/p_1)} S(\mu, d)^{(1/q_2 - 1/q_1)_+}, \end{aligned}$$

where $S(\mu, d)$ was defined in (3.28). So, if (3.36) is satisfied, then we may approximate the operator id by finite rank operators $P_j = \sum_{\mu=0}^{j} \mathrm{id}_\mu$.

- $0 < p_1 \le p_2 \le \infty$.

In this case we choose $\varepsilon > 0$ such that

$$r_1 - r_2 - \left(\frac{1}{p_1} - \frac{1}{p_2}\right) > 2\varepsilon$$

and use the trivial embeddings

$$s^{\overline{r}_1,\Omega}_{p_1,q_1} a \to s^{\overline{r}_1 - \varepsilon,\Omega}_{p_1,q_1} b \to s^{\overline{r}_2 + \varepsilon,\Omega}_{p_2,q_2} b \to s^{\overline{r}_2,\Omega}_{p_2,q_2} a^\dagger.$$

All these embeddings are continuous, the middle one is even compact.

• $0 < p_2 \le p_1 \le \infty$.

Now we use the embeddings
$$s^{\overline{r}_1,\Omega}_{p_1,q_1}a \to s^{\overline{r}_1,\Omega}_{p_2,q_1}a \to s^{\overline{r}_2,\Omega}_{p_2,q_2}a^\dagger.$$
We have already proven that the second embedding is compact. As the first embedding is continuous, this finishes the proof of part 1.

Part 2. If (3.36) is *not* satisfied, we construct a sequence $\{e_\mu\}_{\mu=0}^\infty$ from the unit ball of $s^{\overline{r}_1,\Omega}_{p_1,q_1}a$ such that $\|e_\mu - e_{\mu'} \mid s^{\overline{r}_2,\Omega}_{p_2,q_2}a^\dagger\| \ge c > 0$ for $\mu \ne \mu'$.

Let us start with the case $p_1 \le p_2$. For $\mu \in \mathbb{N}_0$ fixed, we choose one $\overline{\nu}_\mu \in \mathbb{N}_0^d$ with $|\overline{\nu}_\mu| = \mu$ and one $\overline{m}_\mu \in A^\Omega_{\overline{\nu}_\mu}$. Then we set
$$(e_\mu)_{\overline{\nu}\,\overline{m}} = \begin{cases} 2^{-\mu(r_1-1/p_1)} & \text{for } \overline{\nu} = \overline{\nu}_\mu,\ \overline{m} = \overline{m}_\mu, \\ 0 & \text{otherwise.} \end{cases}$$
When $p_1 > p_2$ we fix again one $\overline{\nu}_\mu \in \mathbb{N}_0^d$ with $|\overline{\nu}_\mu| = \mu$ and define $(e_\mu)_{\overline{\nu}\,\overline{m}} = 2^{-\mu r_1}$ for $\overline{\nu} = \overline{\nu}_\mu$ and $\overline{m} \in A^\Omega_{\overline{\nu}_\mu}$ and $(e_\mu)_{\overline{\nu}\,\overline{m}} = 0$ otherwise. ■

It is our main task to estimate the decay of $e_k(\mathrm{id})$ for id given by (3.30) when this sequence tends to zero, i.e. when (3.36) is satisfied. First we get estimates from below.

Theorem 3.18. *Let $\overline{r}_1, \overline{r}_2, p_1, p_2, q_1, q_2$ be given by* (3.35) *and* (3.36). *Then*
$$e_k(\mathrm{id} : s^{\overline{r}_1,\Omega}_{p_1,q_1}a \to s^{\overline{r}_2,\Omega}_{p_2,q_2}a^\dagger) \ge ck^{r_2-r_1}(\log k)^{(d-1)(r_1-r_2+1/q_2-1/q_1)_+}, \qquad k \ge 2, \tag{3.37}$$
where the constant c does not depend on k.

Proof. Step 1. For every $\mu \in \mathbb{N}$ we consider the following diagram:
$$\begin{array}{ccc} (s^{\overline{r}_1,\Omega}_{p_1,q_1}a)_\mu & \xrightarrow{\mathrm{id}'_\mu} & (s^{\overline{r}_2,\Omega}_{p_2,q_2}a^\dagger)_\mu \\ {\scriptstyle\mathrm{id}_1}\big\downarrow & & \big\uparrow{\scriptstyle\mathrm{id}_2} \\ s^{\overline{r}_1,\Omega}_{p_1,q_1}a & \xrightarrow{\mathrm{id}} & s^{\overline{r}_2,\Omega}_{p_2,q_2}a^\dagger \end{array} \tag{3.38}$$
The meaning of id and id'_μ was explained by (3.30)–(3.34). id_1 extends a given finite sequence by zeros while id_2 is the identity restricted to the μth building block. Hence
$$\mathrm{id}_1(\{\lambda_{\overline{\nu}\,\overline{m}}\} : |\overline{\nu}| = \mu,\ \overline{m} \in A^\Omega_{\overline{\nu}}) = (\{\gamma_{\overline{\nu}\,\overline{m}}\} : \gamma_{\overline{\nu}\,\overline{m}} = \lambda_{\overline{\nu}\,\overline{m}} \text{ for } |\overline{\nu}| = \mu \text{ and } \gamma_{\overline{\nu}\,\overline{m}} = 0 \text{ otherwise})$$
and
$$\mathrm{id}_2(\{\lambda_{\overline{\nu}\,\overline{m}}\} : \overline{\nu} \in \mathbb{N}_0^d,\ \overline{m} \in A^\Omega_{\overline{\nu}}) = (\{\lambda_{\overline{\nu}\,\overline{m}}\} : |\overline{\nu}| = \mu).$$
For
$$k = 2D_\mu \tag{3.39}$$
we get, by Lemma 3.13,
$$c\mu^{(1/q_2-1/q_1)}2^{\mu(r_2-r_1)} \le e_k(\mathrm{id}'_\mu) \le \|\mathrm{id}_1\| \cdot \|\mathrm{id}_2\| \cdot e_k(\mathrm{id}) = e_k(\mathrm{id}).$$
If k is given by (3.39) we get $\mu \approx \log k$ and $2^\mu \approx k/\log^{d-1} k$. Hence
$$e_k(\mathrm{id}) \ge ck^{r_2-r_1}(\log k)^{(d-1)(r_1-r_2+1/q_2-1/q_1)}.$$
By monotonicity, we extend this result to all $k \ge 2$.

Step 2. We repeat the same arguments with different building blocks. The diagram (3.38) is replaced by

$$\begin{array}{ccc} 2^{\mu(r_1-1/p_1)}\ell_{p_1}^{A_\mu} & \xrightarrow{\mathrm{id}'_\mu} & 2^{\mu(r_2-1/p_2)}\ell_{p_2}^{A_\mu} \\ \mathrm{id}_1 \big\downarrow & & \big\uparrow \mathrm{id}_2 \\ s_{p_1,q_1}^{\overline{r}_1,\Omega}a & \xrightarrow{\mathrm{id}} & s_{p_2,q_2}^{\overline{r}_2,\Omega}a^\dagger \end{array} \tag{3.40}$$

where $A_\mu = \#(A_{\overline{\nu}}^{\Omega})$ for some $\overline{\nu}$ with $|\overline{\nu}| = \mu$. Instead of Lemma 3.13 we use Lemma 3.11 to get, for $k = 2A_\mu$,

$$c\,2^{\mu(r_2-r_1)} \le e_k(\mathrm{id}'_\mu) \le \|\mathrm{id}_1\| \cdot \|\mathrm{id}_2\| \cdot e_k(\mathrm{id}) = e_k(\mathrm{id}).$$

Finally, we substitute $2^\mu \approx k$, get $e_k(\mathrm{id}) \ge ck^{r_2-r_1}$ and use monotonicity arguments to extend the result to all $k \ge 2$. ∎

Theorem 3.19. *Let $\overline{r}_1, \overline{r}_2, p_1, p_2, q_1, q_2$ be given by* (3.35) *and* (3.36). *If*

$$\alpha > V_1(p_1,q_1,p_2,q_2) := \frac{1}{\min(p_1,q_1)} - \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{\max(p_2,q_2)} \tag{3.41}$$

for $p_1 \le p_2$, and

$$\alpha > V_1(p_2,q_1,p_2,q_2) := \frac{1}{\min(p_2,q_1)} - \frac{1}{\max(p_2,q_2)} \tag{3.42}$$

for $p_1 > p_2$, then

$$e_k(\mathrm{id} : s_{p_1,q_1}^{\overline{r}_1,\Omega}a \to s_{p_2,q_2}^{\overline{r}_2,\Omega}a^\dagger) \le ck^{r_2-r_1}(\log k)^{(d-1)(r_1-r_2+1/q_2-1/q_1)}, \qquad k \ge 2, \tag{3.43}$$

where the constant c does not depend on k.

Proof. Step 1. We first restrict ourselves to the case $p_1 \le p_2$. We split id as indicated in (3.31),

$$\mathrm{id} = \sum_{\mu=0}^{J} \mathrm{id}_\mu + \sum_{\mu=J+1}^{L} \mathrm{id}_\mu + \sum_{\mu=L+1}^{\infty} \mathrm{id}_\mu,$$

where the numbers $J \le L$ will be specified later on. Furthermore, we shall later define natural numbers k_μ, $\mu = 0,\ldots,L$, and $k = \sum_{\mu=0}^{L} k_\mu$. This will yield the fundamental estimate

$$e_k^\varrho(\mathrm{id}) \le \sum_{\mu=0}^{J} e_{k_\mu}^\varrho(\mathrm{id}_\mu) + \sum_{\mu=J+1}^{L} e_{k_\mu}^\varrho(\mathrm{id}_\mu) + \sum_{\mu=L+1}^{\infty} \|\mathrm{id}_\mu\|^\varrho, \qquad \varrho = \min(1,p_2,q_2). \tag{3.44}$$

We recall that by (3.33) one may substitute $e_{k_\mu}(\mathrm{id}_\mu)$ by $e_{k_\mu}(\mathrm{id}'_\mu)$.

Step 2. Fix now $J \in \mathbb{N}$. We show how to choose the numbers L and k_μ (in dependence on J) and we estimate the three sums in (3.44).

We start with the last one. First we remark that

$$\|\mathrm{id}_\mu\| \le c\,2^{-\mu\alpha}\mu^{(d-1)(1/q_2-1/q_1)_+}, \qquad \mu \in \mathbb{N},$$

and

$$\sum_{\mu=L+1}^{\infty} \|\mathrm{id}_\mu\|^\varrho \le c \sum_{\mu=L+1}^{\infty} 2^{-\varrho\mu\alpha}\mu^{\varrho(d-1)(1/q_2-1/q_1)_+} \le c\,2^{-\varrho\alpha L}L^{\varrho(d-1)(1/q_2-1/q_1)_+}.$$

Finally, we choose $L \geq J$ large such that the last expression may be estimated from above by

$$\sum_{\mu=L+1}^{\infty} \|\mathrm{id}_\mu\|^\varrho \leq c\, J^{\varrho(d-1)(1/q_2-1/q_1)} 2^{\varrho J(r_2-r_1)}.$$

STEP 3. We estimate the first sum in (3.44). We define

$$k_\mu = 2D_\mu 2^{(J-\mu)\varepsilon} \geq 2D_\mu, \qquad \mu = 0, \ldots, J,$$

where ε is an arbitrary fixed number with $0 < \varepsilon < 1$. Then we get

$$\sum_{\mu=0}^{J} k_\mu \approx J^{d-1} 2^J. \tag{3.45}$$

By Lemma 3.13,

$$\begin{aligned} e_{k_\mu}(\mathrm{id}_\mu) &\approx 2^{-2^{(J-\mu)\varepsilon}} \mu^{(d-1)(1/q_2-1/q_1)} 2^{\mu(r_2-r_1)}, \\ \sum_{\mu=0}^{J} e_{k_\mu}^\varrho(\mathrm{id}_\mu) &\approx J^{\varrho(d-1)(1/q_2-1/q_1)} 2^{\varrho J(r_2-r_1)}. \end{aligned} \tag{3.46}$$

STEP 4. We estimate the second sum in (3.44). We set

$$k_\mu = 2D_\mu 2^{(J-\mu)\varkappa} \leq 2D_\mu, \qquad J+1 \leq \mu \leq L,$$

where $\varkappa$ is chosen such that

$$\varkappa > 1, \qquad r_1 - r_2 > \varkappa\left(\frac{1}{\gamma_1} - \frac{1}{\delta_2}\right). \tag{3.47}$$

γ_1 and δ_2 were defined by (3.18) and (3.19), respectively. Then we get

$$\sum_{\mu=J+1}^{L} k_\mu \approx J^{d-1} 2^J. \tag{3.48}$$

By Lemma 3.15 we get

$$e_{k_\mu}(\mathrm{id}_\mu) \leq c\mu^{(d-1)(1/q_2-1/q_1)} 2^{\mu(r_2-r_1)} 2^{(J-\mu)\varkappa(1/\delta_2-1/\gamma_1)} [\log(c2^{-(J-\mu)\varkappa} + 1)]^{1/\gamma_1-1/\delta_2}.$$

By (3.47) we get

$$\sum_{\mu=J+1}^{L} e_{k_\mu}^\varrho(\mathrm{id}_\mu) \approx J^{\varrho(d-1)(1/q_2-1/q_1)} 2^{\varrho J(r_2-r_1)}. \tag{3.49}$$

Finally, we put (3.45), (3.48) together with (3.46) and (3.49) into (3.44) to obtain

$$e_{c_1 J^{d-1} 2^J}(\mathrm{id}) \leq c_2 J^{(d-1)(1/q_2-1/q_1)} 2^{J(r_2-r_1)}.$$

Substituting $k = c_1 J^{d-1} 2^J$ and using monotonicity arguments, we finish the proof of the theorem for $p_1 \leq p_2$.

STEP 5. In the case $p_1 > p_2$ we use the chain of embeddings

$$s_{p_1,q_1}^{\bar{r}_1,\Omega} a \hookrightarrow s_{p_2,q_1}^{\bar{r}_1,\Omega} a \hookrightarrow s_{p_2,q_2}^{\bar{r}_2,\Omega} a^\dagger.$$

The first embedding is then continuous (as $p_1 > p_2$ and Ω is bounded), the second is covered by the previous steps. Altogether, this finishes the proof. ∎

REMARK 3.20. 1. One notices immediately a gap between (3.36) and (3.41). To eliminate this gap we use a complex interpolation method in the next chapter.
2. Lemma 3.16 allows us to reduce the gap a bit in the special case where $a = a^\dagger = b$. If we use Lemma 3.16 instead of Lemma 3.15 in Step 4 of the previous proof, we get the same result, namely (3.43), for

$$p_1 \le p_2, \quad r_1 - r_2 + \frac{1}{p_2} - \frac{1}{p_1} > 0, \qquad \frac{1}{p_1} - \frac{1}{p_2} > \frac{1}{q_1} - \frac{1}{q_2}.$$

4. Complex interpolation

In Theorem 3.18 we obtained an estimate from below for entropy numbers of the embedding

$$\mathrm{id} : s^{\overline{r}_1,\Omega}_{p_1,q_1} a \to s^{\overline{r}_2,\Omega}_{p_2,q_2} a^\dagger. \tag{4.1}$$

The corresponding estimate from above was obtained in Theorem 3.19 for

$$\alpha = r_1 - r_2 - \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > \frac{1}{\min(p_1,p_2,q_1)} - \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{\max(p_2,q_2)}. \tag{4.2}$$

So for any p_1, p_2, q_1, q_2 we have one natural bound for $r_1 - r_2$ which ensures compactness of (4.1) (see Theorem 3.17) and a second one, in general larger and given by (4.2), where the estimates from above and from below for the entropy numbers of (4.1) coincide. The main purpose of this chapter is to eliminate this gap by using a complex interpolation method. We follow closely [20].

4.1. Abstract background. In this subsection we briefly describe the complex interpolation method of [20]. We quote only the minimum needed for our purposes.

We say that two quasi-Banach spaces X_0, X_1 form an *interpolation couple* (X_0, X_1) if there is a Hausdorff topological vector space X such that X_0 and X_1 are continuously embedded in X. Given an interpolation couple (X_0, X_1), we define the space $X_0 \cap X_1$ by

$$X_0 \cap X_1 = \{x \in X : \|x \,|\, X_0 \cap X_1\| < \infty\},$$

where

$$\|x \,|\, X_0 \cap X_1\| = \max\{\|x \,|\, X_0\|, \|x \,|\, X_1\|\}.$$

Similarly, we define the space $X_0 + X_1$ by

$$X_0 + X_1 = \{x \in X : \|x \,|\, X_0 + X_1\| < \infty\},$$

where

$$\|x \,|\, X_0 + X_1\| = \inf\{\|x_0 \,|\, X_0\| + \|x_1 \,|\, X_1\| : x = x_0 + x_1,\ x_j \in X_j,\ j = 0, 1\}.$$

It is easy to verify that $X_0 \cap X_1$ and $X_0 + X_1$ are quasi-Banach spaces (see for example [5] for details).

If X is a quasi-Banach space and $\Omega \subset \mathbb{C}$ is an open subset then $f : \Omega \to X$ is called *analytic* if for each $z_0 \in \Omega$ there exists $r > 0$ such that there is a power series expansion $f(z) = \sum_{n=0}^{\infty} x_n (z - z_0)^n$, $x_n \in X$, converging uniformly for $|z - z_0| < r$.

Given an interpolation couple (X_0, X_1) of quasi-Banach spaces, we consider the class $\mathcal{F}$ of all functions f with values in $X_0 + X_1$, which are bounded and continuous on the strip

$$S = \{z \in \mathbb{C} : 0 \le \operatorname{Re} z \le 1\},$$

and analytic in the open strip

$$S_0 = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\},$$

and moreover, the functions $t \mapsto f(j + it)$ $(j = 0, 1)$ are bounded continuous functions into X_j. We endow $\mathcal{F}$ with the quasinorm

$$\|f \,|\, \mathcal{F}\| = \max\{\sup_{t\in\mathbb{R}} \|f(it) \,|\, X_0\|, \sup_{t\in\mathbb{R}} \|f(1 + it) \,|\, X_1\|\}. \tag{4.3}$$

Finally, we set

$$[X_0, X_1]_\theta := \{x \in X_0 + X_1 : x = f(\theta) \text{ for some } f \in \mathcal{F}\}, \qquad 0 < \theta < 1.$$

This space is equipped with the quasinorm

$$\|x \,|\, [X_0, X_1]_\theta\| := \inf\{\|f \,|\, \mathcal{F}\| : f \in \mathcal{F},\ f(\theta) = x\}, \qquad x \in [X_0, X_1]_\theta.$$

For the classical complex interpolation theory of Peetre, we refer again to [5] and references given there. However, it is well known that the extension of this complex interpolation method to quasi-Banach spaces is not possible due to the possible failure of the Maximum Modulus Principle in the quasi-Banach context. However, there is a significant class of quasi-Banach spaces, called *A-convex*, in which the Maximum Modulus Principle is valid (see [20] and references given there for details).

Definition 4.1. A quasi-Banach space $(X, \|\cdot \,|\, X\|)$ is called *A-convex* if there is a constant C such that for every polynomial $P : \mathbb{C} \to X$ we have

$$\|P(0) \,|\, X\| \le C \max_{|z|=1} \|P(z) \,|\, X\|.$$

The next theorem shows that in A-convex quasi-Banach spaces the Maximum Modulus Principle holds.

Theorem 4.2. *For a quasi-Banach space $(X, \|\cdot|X\|)$ the following conditions are equivalent*:

(i) *X is A-convex*,
(ii) *there exists C such that*

$$\max\{\|f(z) \,|\, X\| : z \in S_0\} \le C \max\{\|f(z) \,|\, X\| : z \in S \setminus S_0\}$$

for any function $f : S \to X$ analytic on S_0 and continuous and bounded on S.

In the special case when X_0 and X_1 are quasi-Banach lattices, it was observed by Calderón that the interpolation space $[X_0, X_1]_\theta$ coincides with the so-called *Calderón product* of X_0 and X_1, usually denoted by $X_0^{1-\theta} X_1^\theta$. We quote again necessary definitions and theorems from [20].

First, let $(\mathfrak{X}, \mathcal{S}, \mu)$ be a σ-finite measure space and let $\mathfrak{M}$ be the class of all complex-valued, μ-measurable functions on $\mathfrak{X}$. Then a quasi-Banach space $X \subset \mathfrak{M}$ is called a *quasi-Banach lattice of functions* if for every $f \in X$ and $g \in \mathfrak{M}$ with $|g(x)| \le |f(x)|$ for μ-a.e. $x \in \mathfrak{X}$ one has $g \in X$ with $\|g \,|\, X\| \le \|f \,|\, X\|$.

Furthermore, a quasi-Banach lattice of functions $(X, \|\cdot \,|\, X\|)$ is called *lattice r-convex* if

$$\Big\| \Big(\sum_{j=1}^m |f_j|^r \Big)^{1/r} \Big| X \Big\| \le \Big(\sum_{j=1}^m \|f_j \,|\, X\|^r \Big)^{1/r}$$

for any finite family $\{f_j\}_{1 \le j \le m}$ of functions from X.

The following theorem gives a very simple condition for a lattice of functions to be A-convex.

THEOREM 4.3. *Let X be a complex quasi-Banach lattice of functions. Then the following assertions are equivalent*:

(i) *X is A-convex*,

(ii) *X is lattice r-convex for some $r > 0$.*

Finally, if $(X_j, \|\cdot \,|\, X_j\|)$, $j = 0, 1$, are quasi-Banach lattices of functions and $0 < \theta < 1$ then the *Calderón product* $X_0^{1-\theta} X_1^\theta$ is the function space defined by the quasinorm

$$\|f \,|\, X_0^{1-\theta} X_1^\theta\| := \inf\{\|f_0 \,|\, X_0\|^{1-\theta} \|f_1 \,|\, X_1\|^\theta : |f| \le |f_0|^{1-\theta} |f_1|^\theta,\ f_j \in X_j,\ j = 0, 1\}.$$

The connection between complex interpolation and Calderón products is given by

THEOREM 4.4. *Let $(\mathfrak{X}, \mathcal{S}, \mu)$ be a complete separable metric space, let μ be a σ-finite Borel measure on $\mathfrak{X}$, and let X_0, X_1 be a pair of quasi-Banach lattices of functions on $(\mathfrak{X}, \mu)$. If both X_0 and X_1 are A-convex and separable, then $X_0 + X_1$ is A-convex and $[X_0, X_1]_\theta = X_0^{1-\theta} X_1^\theta$, $0 < \theta < 1$.*

As pointed out in [20] in the case of quasi-Banach sequence lattices, only one of the spaces in 4.4 must be separable.

4.2. Interpolation of $s^{\overline{r},\Omega}_{p,q}a$.

Now we apply Theorem 4.4 to interpolate the sequence spaces $s^{\overline{r},\Omega}_{p,q}a$. First, we have to prove that these spaces are A-convex. According to Theorem 4.3 it is enough to prove that they are lattice s-convex for some $s > 0$. Trivially, $s = \min(1, p, q)$ works fine in both b- and f-cases.

Hence, it is enough to compute the Calderón products

$$(s^{\overline{r}_1,\Omega}_{p_1,q_1}a)^{1-\theta} (s^{\overline{r}_2,\Omega}_{p_2,q_2}a)^\theta, \qquad 0 < \theta < 1.$$

The answer is given by

THEOREM 4.5. *Let*

$$\overline{r}_1, \overline{r}_2 \in \mathbb{R}^d, \qquad 0 < p_1, p_2, q_1, q_2 \le \infty, \qquad 0 < \theta < 1. \tag{4.4}$$

If $\overline{r}, p$ and q are given by

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \qquad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \qquad \overline{r} = (1-\theta)\overline{r}_1 + \theta \overline{r}_2, \tag{4.5}$$

then

$$(s^{\overline{r}_1,\Omega}_{p_1,q_1}a)^{1-\theta} (s^{\overline{r}_2,\Omega}_{p_2,q_2}a)^\theta = s^{\overline{r},\Omega}_{p,q}a.$$

Proof. STEP 1. First, let $\lambda \in s^{\overline{r},\Omega}_{p,q}a$ and $\lambda^j \in s^{\overline{r}_j,\Omega}_{p_j,q_j}a$, $j=1,2$, with

$$|\lambda_{\overline{\nu}\,\overline{m}}| \le |\lambda^1_{\overline{\nu}\,\overline{m}}|^{1-\theta}\cdot|\lambda^2_{\overline{\nu}\,\overline{m}}|^{\theta}, \qquad \overline{\nu}\in\mathbb{N}_0^d,\ \overline{m}\in A^{\Omega}_{\overline{\nu}}. \tag{4.6}$$

We have to show that

$$\|\lambda\,|\,s^{\overline{r},\Omega}_{p,q}a\| \le \|\lambda^1\,|\,s^{\overline{r}_1,\Omega}_{p_1,q_1}a\|^{1-\theta}\cdot\|\lambda^2\,|\,s^{\overline{r}_2,\Omega}_{p_2,q_2}a\|^{\theta}.$$

But this is a simple exercise on Hölder's inequality in both b- and f-cases.

STEP 2. Now we prove the reverse inequality for $a=b$. Given $\lambda\in s^{\overline{r},\Omega}_{p,q}b$, we will find $\lambda^j\in s^{\overline{r}_j,\Omega}_{p_j,q_j}b$, $j=1,2$, with (4.6) such that

$$\|\lambda\,|\,s^{\overline{r},\Omega}_{p,q}b\| = \|\lambda^1\,|\,s^{\overline{r}_1,\Omega}_{p_1,q_1}b\|^{1-\theta}\cdot\|\lambda^2\,|\,s^{\overline{r}_2,\Omega}_{p_2,q_2}b\|^{\theta}. \tag{4.7}$$

First we deal with the case $p_j,q_j<\infty$, $j=1,2$. We choose

$$\lambda^j_{\overline{\nu}\,\overline{m}} = c^j_{\overline{\nu}}|\lambda_{\overline{\nu}\,\overline{m}}|^{p/p_j}, \quad j=1,2,\ \overline{\nu}\in\mathbb{N}_0^d,\ \overline{m}\in A^{\Omega}_{\overline{\nu}}, \tag{4.8}$$

where

$$c^j_{\overline{\nu}} = 2^{(\overline{\nu}\cdot\overline{r})q/q_j}2^{-\overline{\nu}\cdot\overline{r}_j}\Lambda_{\overline{\nu}}^{q/q_j-p/p_j}, \qquad j=1,2,\ \overline{\nu}\in\mathbb{N}_0^d, \tag{4.9}$$

and

$$\Lambda_{\overline{\nu}} = \Big(\sum_{\overline{m}\in A^{\Omega}_{\overline{\nu}}}|\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{1/p}, \qquad \overline{\nu}\in\mathbb{N}_0^d. \tag{4.10}$$

(If $\Lambda_{\overline{\nu}}=0$ for some $\overline{\nu}\in\mathbb{N}_0^d$ we set $c_{\overline{\nu}}=0$.) By this choice we see that

$$\begin{aligned}|\lambda^1_{\overline{\nu}\,\overline{m}}|^{1-\theta}\cdot|\lambda^2_{\overline{\nu}\,\overline{m}}|^{\theta} &= 2^{\overline{\nu}\cdot\overline{r}q[\frac{1-\theta}{q_1}+\frac{\theta}{q_2}]}2^{-\overline{\nu}\cdot\overline{r}_1(1-\theta)-\overline{\nu}\cdot\overline{r}_2\theta}\Lambda_{\overline{\nu}}^{q[\frac{1-\theta}{q_1}+\frac{\theta}{q_2}]-p[\frac{1-\theta}{p_1}+\frac{\theta}{p_2}]}|\lambda_{\overline{\nu}\,\overline{m}}|^{p[\frac{1-\theta}{p_1}+\frac{\theta}{p_2}]}\\ &= |\lambda_{\overline{\nu}\,\overline{m}}|.\end{aligned}$$

This proves (4.6).

To prove (4.7) we use (4.8)–(4.10) to get

$$\|\lambda^j|s^{\overline{r}_j,\Omega}_{p_j,q_j}b\| = \Big[\sum_{\overline{\nu}\in\mathbb{N}_0^d}2^{\overline{\nu}\cdot\overline{r}_jq_j}(c^j_{\overline{\nu}})^{q_j}\Big(\sum_{\overline{m}\in A^{\Omega}_{\overline{\nu}}}|\lambda_{\overline{\nu}\,\overline{m}}|^{\frac{p}{p_j}p_j}\Big)^{q_j/p_j}\Big]^{1/q_j} = \Big[\sum_{\overline{\nu}\in\mathbb{N}_0^d}2^{\overline{\nu}\cdot\overline{r}q}\Lambda^q_{\overline{\nu}}\Big]^{1/q_j}.$$

From this (4.7) follows immediately.

If $\max(p_1,q_1,p_2,q_2)=\infty$ only notational changes are necessary.

STEP 3. For the f-case, one may modify slightly the proof for the sequence spaces $f^s_{p,q}$ given in [13, Theorem 8.2].

We start again with given $\lambda\in s^{\overline{r},\Omega}_{p,q}f$ and we need to find $\lambda^j\in s^{\overline{r}_j,\Omega}_{p_j,q_j}f$, $j=1,2$, with (4.6) such that

$$\|\lambda^1\,|\,s^{\overline{r}_1,\Omega}_{p_1,q_1}f\|^{1-\theta}\|\lambda^2\,|\,s^{\overline{r}_2,\Omega}_{p_2,q_2}f\|^{\theta} \le c\|\lambda\,|\,s^{\overline{r},\Omega}_{p,q}f\|. \tag{4.11}$$

First we deal with the case $q_j<\infty$, $j=1,2$. For every $k\in\mathbb{Z}$, let

$$A_k = \Big\{x\in\mathbb{R}^d : \Big(\sum_{\overline{\nu}\in\mathbb{N}_0^d,\,\overline{m}\in A^{\Omega}_{\overline{\nu}}}2^{\overline{\nu}\cdot\overline{r}q}|\lambda_{\overline{\nu}\,\overline{m}}|^q\chi_{\overline{\nu}\,\overline{m}}(x)\Big)^{1/q} > 2^k\Big\}$$

and

$$C_k = \{(\overline{\nu},\overline{m}) : |Q_{\overline{\nu},\overline{m}}\cap A_k| \ge |Q_{\overline{\nu}\,\overline{m}}|/2 \text{ and } |Q_{\overline{\nu},\overline{m}}\cap A_{k+1}| < |Q_{\overline{\nu}\,\overline{m}}|/2\}.$$

We note that if $(\overline{\nu},\overline{m}) \notin \bigcup_{k\in\mathbb{Z}} C_k$, then $\lambda_{\overline{\nu}\,\overline{m}} = 0$. We define the sequences λ^j, $j=1,2$, by

$$\lambda^1_{\overline{\nu}\,\overline{m}} = 2^{k\gamma} 2^{\overline{\nu}\cdot\overline{u}} |\lambda_{\overline{\nu}\,\overline{m}}|^{q/q_1}, \qquad \lambda^2_{\overline{\nu}\,\overline{m}} = 2^{k\delta} 2^{\overline{\nu}\cdot\overline{v}} |\lambda_{\overline{\nu}\,\overline{m}}|^{q/q_2},$$

where

$$\gamma = \frac{p}{p_1} - \frac{q}{q_1}, \qquad \delta = \frac{p}{p_2} - \frac{q}{q_2},$$
$$\overline{u} = q\theta\Big[\frac{\overline{r}_2}{q_1} - \frac{\overline{r}_1}{q_2}\Big], \qquad \overline{v} = q(1-\theta)\Big[\frac{\overline{r}_1}{q_2} - \frac{\overline{r}_2}{q_1}\Big]$$

if $(\overline{\nu},\overline{m}) \in C_k$, and $\lambda^1_{\overline{\nu}\,\overline{m}} = \lambda^2_{\overline{\nu}\,\overline{m}} = 0$ if $(\overline{\nu},\overline{m}) \notin \bigcup_{k\in\mathbb{Z}} C_k$. We point out that

$$(1-\theta)\gamma + \delta\theta = (1-\theta)\overline{u} + \theta\overline{v} = 0.$$

An easy calculation shows that

$$|\lambda^1_{\overline{\nu}\,\overline{m}}|^{1-\theta} \cdot |\lambda^2_{\overline{\nu}\,\overline{m}}|^{\theta} = 2^{k[(1-\theta)\gamma+\theta\delta]+\overline{\nu}\cdot[(1-\theta)\overline{u}+\theta\overline{v}]} |\lambda_{\overline{\nu}\,\overline{m}}|^{q(\frac{1-\theta}{q_1}+\frac{\theta}{q_2})} = |\lambda_{\overline{\nu}\,\overline{m}}|.$$

In the following we assume that $\gamma \ge 0$, since the other case follows by interchanging $s^{\overline{r}_1,\Omega}_{p_1q_1}f$ with $s^{\overline{r}_2,\Omega}_{p_2q_2}f$ and θ with $1-\theta$.

We prove that

$$\|\lambda^j \,|\, s^{\overline{r}_j,\Omega}_{p_jq_j}f\| \le c\|\lambda \,|\, s^{\overline{r},\Omega}_{pq}f\|^{p/p_j}, \qquad j=1,2. \tag{4.12}$$

From this, (4.11) clearly follows. To prove (4.12) for $j=1$ we write

$$\begin{aligned}
\|\lambda^1 \,|\, s^{\overline{r}_1,\Omega}_{p_1q_1}f\| &= \Big\|\Big(\sum_{k=-\infty}^{\infty} \sum_{(\overline{\nu},\overline{m})\in C_k} |2^{\overline{\nu}\cdot\overline{r}_1}\lambda^1_{\overline{\nu}\,\overline{m}}|^{q_1} \chi_{\overline{\nu}\,\overline{m}}(x)\Big)^{1/q_1} \,\Big|\, L_{p_1}\Big\| \\
&\le c\Big\|\Big(\sum_{k=-\infty}^{\infty} \sum_{(\overline{\nu},\overline{m})\in C_k} |2^{\overline{\nu}\cdot\overline{r}_1}\lambda^1_{\overline{\nu}\,\overline{m}}|^{q_1} \chi_{Q_{\overline{\nu}\,\overline{m}}\cap A_k}(x)\Big)^{1/q_1} \,\Big|\, L_{p_1}\Big\|,
\end{aligned}$$

where in the second line we use the definition of the set C_k and the boundedness of the maximal operator $\overline{M}$ as described by Theorem 1.11.

We set $D_k = \bigcup_{l=-\infty}^{k} C_l$ and continue

$$\begin{aligned}
\|\lambda^1 \,|\, s^{\overline{r}_1,\Omega}_{p_1q_1}f\| &\le c\Big\|\sum_{k=-\infty}^{\infty} \chi_{A_k\setminus A_{k+1}}(x)\Big(\sum_{(\overline{\nu},\overline{m})\in D_k} |2^{\overline{\nu}\cdot\overline{r}_1}\lambda^1_{\overline{\nu}\,\overline{m}}|^{q_1}\chi_{\overline{\nu}\,\overline{m}}(x)\Big)^{1/q_1} \,\Big|\, L_{p_1}\Big\| \\
&\le c\Big\|\sum_{k=-\infty}^{\infty} \chi_{A_k\setminus A_{k+1}}(x) 2^{k\gamma}\Big(\sum_{(\overline{\nu},\overline{m})\in D_k} 2^{\overline{\nu}\cdot\overline{r}_1 q_1} 2^{\overline{\nu}\cdot\overline{u}q_1} |\lambda_{\overline{\nu}\,\overline{m}}|^{q}\chi_{\overline{\nu}\,\overline{m}}(x)\Big)^{1/q_1} \,\Big|\, L_{p_1}\Big\| \\
&\le c\Big\|\sum_{k=-\infty}^{\infty} \chi_{A_k\setminus A_{k+1}}(x) 2^{k\gamma}\Big(\sum_{\overline{\nu}\in\mathbb{N}_0^d,\overline{m}\in A^{\Omega}_{\overline{\nu}}} 2^{\overline{\nu}\cdot\overline{r}q} |\lambda_{\overline{\nu}\,\overline{m}}|^{q}\chi_{\overline{\nu}\,\overline{m}}(x)\Big)^{1/q_1} \,\Big|\, L_{p_1}\Big\| \\
&\le c\Big\|\Big(\sum_{\overline{\nu}\in\mathbb{N}_0^d,\overline{m}\in A^{\Omega}_{\overline{\nu}}} 2^{\overline{\nu}\cdot\overline{r}q} |\lambda_{\overline{\nu}\,\overline{m}}|^{q}\chi_{\overline{\nu}\,\overline{m}}(x)\Big)^{p/qp_1} \,\Big|\, L_{p_1}\Big\| \\
&= c\|\lambda \,|\, s^{\overline{r},\Omega}_{pq}f\|^{p/p_1}.
\end{aligned}$$

The second estimate in (4.12) is similar. ∎

After these preparations we are ready to present the main result of this section. Recall that the spaces $S^{\overline{r}}_{p,q}A(\Omega)$ were defined by (3.1) and (3.2).

THEOREM 4.6. *Let $\overline{r}_j, p_j, q_j$ for $j = 1, 2$ be given by (4.4). Let $0 < \theta < 1$ and define $\overline{r}, p$ and q by (4.5). Also suppose that* $\min(q_1, q_2) < \infty$.

(i) *Then*

$$[s^{\overline{r}_1,\Omega}_{p_1,q_1}b, s^{\overline{r}_2,\Omega}_{p_2,q_2}b]_\theta = s^{\overline{r},\Omega}_{p,q}b. \tag{4.13}$$

(ii) *Furthermore, if $p_j < \infty$, $j = 1, 2$, then*

$$[s^{\overline{r}_1,\Omega}_{p_1,q_1}f, s^{\overline{r}_2,\Omega}_{p_2,q_2}f]_\theta = s^{\overline{r},\Omega}_{p,q}f. \tag{4.14}$$

Proof. This follows immediately from Theorems 4.4 and 4.5. ■

4.3. Interpolation properties of entropy numbers. Now we shall discuss the connection between the complex interpolation method developed above with entropy numbers. We use Theorem 1.3.2 from [10]. We recall that for $t > 0$, an interpolation couple (B_0, B_1) and $b \in B_0 + B_1$, Peetre's *K-functional* is given by

$$K(t, b, B_0, B_1) = \inf\{\|b_0 \,|\, B_0\| + t\|b_1 \,|\, B_1\| : b = b_0 + b_1,\ b_0 \in B_0,\ b_1 \in B_1\}.$$

THEOREM 4.7. (i) *Let A be a quasi-Banach space and let (B_0, B_1) be an interpolation couple of p-Banach spaces. Let $0 < \theta < 1$ and let B_θ be a quasi-Banach space such that $B_0 \cap B_1 \subset B_\theta \subset B_0 + B_1$ and*

$$\|b \,|\, B_\theta\| \le \|b \,|\, B_0\|^{1-\theta} \cdot \|b \,|\, B_1\|^\theta \quad \textit{for all } b \in B_0 \cap B_1.$$

Let $T \in L(A, B_0 \cap B_1)$. Then for all $k_0, k_1 \in \mathbb{N}$,

$$e_{k_0+k_1-1}(T : A \to B_\theta) \le 2^{1/p} e^{1-\theta}_{k_0}(T : A \to B_0) e^\theta_{k_1}(T : A \to B_1).$$

(ii) *Let (A_0, A_1) be an interpolation couple of quasi-Banach spaces and let B be a p-Banach space. Let $0 < \theta < 1$ and let A be a quasi-Banach space such that $A \subset A_0 + A_1$ and*

$$t^{-\theta} K(t, a, A_0, A_1) \le \|a \,|\, A\| \quad \textit{for all } a \in A \textit{ and all } t > 0.$$

Let $T : A_0 + A_1 \to B$ be linear and such that its restrictions to A_0 and A_1 are continuous. Then its restriction to A is also continuous and for all $k_0, k_1 \in \mathbb{N}$,

$$e_{k_0+k_1-1}(T : A \to B) \le 2^{1/p} e^{1-\theta}_{k_0}(T : A_0 \to B) e^\theta_{k_1}(T : A_1 \to B).$$

So, we only have to verify that the complex interpolation satisfies the assumptions of this theorem.

THEOREM 4.8. *Let B_0, B_1 be an interpolation couple of A-convex quasi-Banach spaces and let $0 < \theta < 1$. Then*

(i) $\|b \,|\, [B_0, B_1]_\theta\| \le \|b \,|\, B_0\|^{1-\theta} \cdot \|b \,|\, B_1\|^\theta$ *for all $b \in B_0 \cap B_1$.*
(ii) *Let the functionals in B_i' separate the points of B_i, $i = 0, 1$. Then*

$$t^{-\theta} K(t, b, B_0, B_1) \le \|b \,|\, [B_0, B_1]_\theta\| \quad \textit{for all } b \in [B_0, B_1]_\theta \textit{ and all } t > 0.$$

Proof. STEP 1. Fix $b \in B_0 \cap B_1$, set $M_j = \|b \,|\, B_j\|$, $j = 0, 1$, and define $g(z) = M_0^{z-1} M_1^{-z} b$. Then $\|g \,|\, \mathcal{F}\| = 1$ and

$$\|M_0^{\theta-1}M_1^{-\theta}b\,|\,[B_0,B_1]_\theta\| \le \|g(\theta)\,|\,[B_0,B_1]_\theta\| \le 1.$$

This proves (i).

STEP 2. We follow [31, 1.10.3]. There one may find a proof dealing with the classical complex interpolation method and Banach spaces. Nevertheless, the proof works also for the generalised method, as described above, and quasi-Banach sequence spaces. In particular, the Hahn–Banach theorem needed there still holds for all sequence spaces which come into play. ■

4.4. Filling the gaps. Now we use the complex interpolation and its relation to entropy numbers to close the gap mentioned at the beginning of Section 4. Namely, we are interested in those combinations of "input" parameters which satisfy

$$\begin{aligned} V_1(p_1,q_1,p_2,q_2) &:= \frac{1}{\min(p_1,p_2,q_1)} - \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{\max(p_2,q_2)} \\ &\ge r_1 - r_2 - \left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+ > 0. \end{aligned} \tag{4.15}$$

Our main result on the sequence space level is

THEOREM 4.9. *Let* $\overline{r}_j = (r_j,\ldots,r_j) \in \mathbb{R}^d$, $0 < p_j, q_j \le \infty$, $j = 1,2$, *with*

$$r_1 - r_2 - \left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+ > 0. \tag{4.16}$$

Furthermore, let $p_j < \infty$ *in the* f*-case.*

(i) *If* $r_1 - r_2 - 1/q_1 + 1/q_2 > 0$ *then*

$$e_k(\mathrm{id} : s^{\overline{r}_1,\Omega}_{p_1,q_1}a \to s^{\overline{r}_2,\Omega}_{p_2,q_2}a) \approx k^{r_2-r_1}(\log k)^{(d-1)(r_1-r_2-1/q_1+1/q_2)}, \qquad k \ge 2.$$

(ii) *If* $r_1 - r_2 - 1/q_1 + 1/q_2 \le 0$ *and* $\varepsilon > 0$ *then there are constants* c *and* C_ε *such that*

$$ck^{r_2-r_1} \le e_k(\mathrm{id} : s^{\overline{r}_1,\Omega}_{p_1,q_1}a \to s^{\overline{r}_2,\Omega}_{p_2,q_2}a) \le C_\varepsilon k^{r_2-r_1}(\log k)^{\varepsilon}, \qquad k \ge 2.$$

REMARK 4.10. Unlike Theorems 3.18 and 3.19, this theorem deals only with embeddings which stay either in the b-scale or in the f-scale. We also see that this theorem closes the gap mentioned above up to a $(\log k)^\varepsilon$ term. Furthermore, the estimate from below is covered by Theorem 3.18. In the proof we will therefore concentrate on the estimates from above.

Proof. We shall distinguish several cases. First of all, we suppose that $p_1 \le p_2$.

I. $p_1 \le q_1, q_2 \le p_2$. In this case the condition (4.15) is empty and the result is covered by Theorem 3.19.

II. $q_1 \le p_1 \le p_2 \le q_2$. We start with the subcase

IIa. $r_1 - r_2 - 1/q_1 + 1/q_2 > 0$. In this case we have

$$r_1 - r_2 - \frac{1}{p_1} + \frac{1}{p_2} > \frac{1}{q_1} - \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{q_2} = V_1(p_1,q_1,p_2,q_2)$$

and the result is again provided by Theorem 3.19.

IIb. $r_1 - r_2 - 1/q_1 + 1/q_2 \le 0$. This subcase introduces the $\log^\varepsilon$-gap. We fix $\varepsilon > 0$ and use the embedding

$$s^{\overline{r}_1,\Omega}_{p_1,q_1}a \hookrightarrow s^{\overline{r}_1,\Omega}_{p_1,q}a \hookrightarrow s^{\overline{r}_2,\Omega}_{p_2,q'}a \hookrightarrow s^{\overline{r}_2,\Omega}_{p_2,q_2}a. \tag{4.17}$$

The indices q, q' are supposed to satisfy

$$\begin{aligned} &0 < q_1 \le q \le p_1 \le p_2 \le q' \le q_2 \le \infty,\\ &\frac{1}{p_1} - \frac{1}{p_2} < \frac{1}{q} - \frac{1}{q'} < r_1 - r_2 < \frac{1}{q} - \frac{1}{q'} + \varepsilon. \end{aligned} \tag{4.18}$$

The existence of such indices follows from (4.16) and condition IIb. Hence we may apply step IIa to the middle embedding in (4.17). All the other embeddings are bounded, which gives finally

$$e_k(\mathrm{id}) \le ck^{r_2-r_1}(\log k)^\varepsilon.$$

III. $q_1 < p_1, q_2 < p_2$. We make the same splitting as in case II:

IIIa. $r_1 - r_2 - 1/q_1 + 1/q_2 > 0$. If

$$\frac{q_1}{q_2} \le \frac{p_1}{p_2}$$

we use the interpolation scheme

$$\begin{array}{ccc} & & s^{\overline{r},\Omega}_{p,q}a \\ & \nearrow & \\ s^{\overline{r}_1,\Omega}_{p_1,q_1}a & \rightarrow & s^{\overline{r}_2,\Omega}_{p_2,q_2}a \\ & \searrow & \\ & & s^{\overline{r}_1,\Omega}_{p_1,q_1}a \end{array} \tag{4.19}$$

with the corresponding equations for r, p and q:

$$r_2 = (1-\theta)r + \theta r_1, \tag{4.20}$$

$$\frac{1}{p_2} = \frac{1-\theta}{p} + \frac{\theta}{p_1}, \tag{4.21}$$

$$\frac{1}{q_2} = \frac{1-\theta}{q} + \frac{\theta}{q_1}. \tag{4.22}$$

We choose θ such that

$$0 < \frac{\frac{1}{q_2} - \frac{1}{p_2}}{\frac{1}{q_1} - \frac{1}{p_1}} \le \theta \le \frac{q_1}{q_2} \le \frac{p_1}{p_2} \le 1.$$

By this choice we ensure that the equations (4.21) and (4.22) have solutions $p, q \in (0, \infty]$ and that $p < q$. Finally, it is easy to verify that

$$r_1 - r - \frac{1}{p_1} + \frac{1}{p} > V_1(p_1, q_1, p, q) \tag{4.23}$$

and

$$r_1 - r - \frac{1}{q_1} + \frac{1}{q} > 0. \tag{4.24}$$

(One makes use of the trivial calculation

$$(1-\theta)\left(\frac{1}{q} - \frac{1}{q_1}\right) = \frac{1}{q_2} - \frac{1}{q_1} \tag{4.25}$$

which follows directly from (4.22) and its analogues for p's and r's.) This allows us to use Theorem 3.19 for the upper embedding in (4.19). Moreover, we may use the first part of Theorem 4.7. Its assumption is easy to verify (and was done in detail in the first step of the proof of Theorem 4.5). This leads to

$$e_k(\mathrm{id}) \le c(k^{r-r_1}(\log k)^{r_1-r-1/q_1+1/q})^{1-\theta} = ck^{r_2-r_1}(\log k)^{r_1-r_2-1/q_1+1/q_2}.$$

If

$$\frac{p_1}{p_2} < \frac{q_1}{q_2}$$

we use a different interpolation scheme:

$$\begin{array}{ccc} s^{\overline{r},\Omega}_{p,q}a & & \\ & \searrow & \\ s^{\overline{r}_1,\Omega}_{p_1,q_1}a & \rightarrow & s^{\overline{r}_2,\Omega}_{p_2,q_2}a \\ & \nearrow & \\ s^{\overline{r}_2,\Omega}_{p_2,q_2}a & & \end{array} \tag{4.26}$$

with the corresponding equations for r, p and q:

$$r_1 = (1-\theta)r + \theta r_2, \tag{4.27}$$

$$\frac{1}{p_1} = \frac{1-\theta}{p} + \frac{\theta}{p_2}, \tag{4.28}$$

$$\frac{1}{q_1} = \frac{1-\theta}{q} + \frac{\theta}{q_2}. \tag{4.29}$$

We choose $0 < \theta < 1$ such that

$$0 < \frac{\frac{1}{q_1} - \frac{1}{p_1}}{\frac{1}{q_2} - \frac{1}{p_2}} \le \theta \le \frac{q_2}{q_1} < \frac{p_2}{p_1} \le 1.$$

This choice ensures that there are $p, q \in (0, \infty]$ satisfying (4.28) and (4.29) and $p \le q$. Finally, it is easy to verify that

$$r - r_2 - \frac{1}{p} + \frac{1}{p_2} > V_1(p, q, p_2, q_2) \tag{4.30}$$

and

$$r - r_2 - \frac{1}{q} + \frac{1}{q_2} > 0. \tag{4.31}$$

and we may apply Theorem 3.19 to the upper embedding in (4.26). To apply the second part of Theorem 4.7, we use Theorems 4.8 and 4.6 (recall that $q_2 < \infty$ in case III). This leads to

$$e_k(\mathrm{id}) \le c(k^{r_2-r}(\log k)^{r-r_2-1/q+1/q_2})^{1-\theta} = ck^{r_2-r_1}(\log k)^{r_1-r_2-1/q_1+1/q_2}.$$

IIIb. $r_1 - r_2 - 1/q_1 + 1/q_2 \le 0$. We use the chain of embeddings (4.17) with (4.18) and

$$q_1 \le q \le p_1, \qquad q' = q_2.$$

Applying now step IIIa to the middle embedding we get the same result as in case IIb.

IV. $p_1 < q_1$, $p_2 < q_2$. We start again with the case of a positive power of the logarithm.

IVa. $r_1 - r_2 - 1/q_1 + 1/q_2 > 0$. If
$$\frac{p_1}{p_2} \le \frac{q_1}{q_2}$$
we use again the scheme (4.19) with (4.20)–(4.22) and choose $0 < \theta < 1$ such that
$$0 < \frac{\frac{1}{p_2} - \frac{1}{q_2}}{\frac{1}{p_1} - \frac{1}{q_1}} \le \theta \le \frac{p_1}{p_2} \le 1.$$
This choice ensures that equations (4.21) and (4.22) supply some $p, q \in (0, \infty]$ with $p > q$. Again, one can easily verify (4.23) and (4.24). Finally, we apply again Theorem 3.19 to the upper embedding in (4.19) and the first part of Theorem 4.7, which leads to the same result as above.

If
$$\frac{q_1}{q_2} < \frac{p_1}{p_2}$$
we use the interpolation scheme (4.26) with (4.27)–(4.29). Now we choose θ such that
$$\max\left(0, 1 - \frac{\frac{1}{p_1} - \frac{1}{q_1}}{\frac{1}{p_2} - \frac{1}{q_2}}, 1 - \frac{q_2}{q_1}\right) < 1 - \theta < \min\left(\frac{r_1 - r_2 - \frac{1}{p_1} + \frac{1}{p_2}}{\frac{1}{p_2} - \frac{1}{q_2}}, 1\right).$$
As each expression appearing in the argument on the left-hand side is smaller than both quantities on the right-hand side, this is always possible.

By this choice we ensure that (4.22) has a solution $q \in (0, \infty]$ and that $p < q$. Finally, it is easy to verify that (4.30) and (4.31) hold.

So, we may apply Theorem 3.19 to the upper embedding in (4.26). Together with Theorems 4.6 and 4.8 this leads again to
$$e_k(\mathrm{id}) \le c(k^{r_2 - r}(\log k)^{r - r_2 - 1/q + 1/q_2})^{1-\theta} = ck^{r_2 - r_1}(\log k)^{r_1 - r_2 - 1/q_1 + 1/q_2}.$$
This finishes the discussion of case IVa as long as $\min(q, q_2) < \infty$, which is equivalent to $\min(q_1, q_2) < \infty$. If $q_1 = q_2 = \infty$ then we have to modify the argument. In this case there is in general no hope to identify the interpolation space $[s^{\overline{r}_1,\Omega}_{p_1,\infty}a, s^{\overline{r}_2,\Omega}_{p_2,\infty}a]_\theta$ with the corresponding Calderón product $s^{\overline{r},\Omega}_{p,\infty}a$. But, according to [16, IV.1.11], one embedding still holds, namely
$$[s^{\overline{r}_1,\Omega}_{p_1,\infty}a, s^{\overline{r}_2,\Omega}_{p_2,\infty}a]_\theta \to s^{\overline{r},\Omega}_{p,\infty}a.$$
So we may use the following interpolation scheme:
$$\begin{array}{ccccc} & & s^{\overline{r},\Omega}_{p,\infty}a & & \\ & \nearrow & & & \\ s^{\overline{r}_1,\Omega}_{p_1,\infty}a & \to & [s^{\overline{r},\Omega}_{p,\infty}a, s^{\overline{r}_1,\Omega}_{p_1,\infty}a]_\theta & \to & s^{\overline{r}_2,\Omega}_{p_2,\infty}a \\ & \searrow & & & \\ & & s^{\overline{r}_1,\Omega}_{p_1,\infty}a & & \end{array}$$
where p and r are given by (4.21) and (4.20). Then the choice of $0 < \theta < 1$ with
$$\max(1 - p_1(r_1 - r_2), 0) < \theta < \frac{p_1}{p_2} \le 1$$
ensures that we may proceed as in Step IIIa and get the same result.

IVb. $r_1 - r_2 - 1/q_1 + 1/q_2 \le 0$. Then $q_1 \le q_2$. We use the chain of embeddings (4.17) with (4.18) and

$$q_1 = q, \qquad p_2 \le q' \le q_2.$$

Applying now step IVa to the middle embedding we get the same result as in case IIb.

In the case $p_1 > p_2$ we use the chain of embeddings

$$s^{\overline{r}_1,\Omega}_{p_1,q_1}a \hookrightarrow s^{\overline{r}_1,\Omega}_{p_2,q_1}a \hookrightarrow s^{\overline{r}_2,\Omega}_{p_2,q_2}a.$$

The first embedding is then continuous (as $p_1 > p_2$ and Ω is bounded), the second is covered by the previous steps. Altogether, this finishes the proof. ∎

4.5. Entropy numbers—conclusion. In the second chapter we have developed a strong tool connecting the function spaces $S^{\overline{r}}_{p,q}A(\mathbb{R}^d)$ with sequence spaces $s^{\overline{r}}_{p,q}a$. In the third and fourth chapters we have studied the entropy numbers of embeddings of these sequence spaces. Finally, we combine these two concepts and obtain estimates for entropy numbers of embeddings of function spaces.

We recall that the function spaces on domains were defined by (3.1) and (3.2). Our main result reads

THEOREM 4.11. *Let Ω be a bounded domain in $\mathbb{R}^d$ with $d \ge 2$. Let $0 < p_1, q_1, p_2, q_2 \le \infty$ with $p_1, p_2 < \infty$ in the F-case. Let $\overline{r}_i = (r_i, \ldots, r_i) \in \mathbb{R}^d$, $i = 1, 2$.*

(i) *The embedding*

$$\mathrm{id} : S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \to S^{\overline{r}_2}_{p_2,q_2}A^{\dagger}(\Omega) \tag{4.32}$$

is compact if and only if

$$r_1 - r_2 - \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > 0. \tag{4.33}$$

(ii) *In that case*

$$e_k(\mathrm{id} : S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \to S^{\overline{r}_2}_{p_2,q_2}A^{\dagger}(\Omega)) \ge ck^{r_2-r_1}(\log k)^{(d-1)(r_1-r_2+1/q_2-1/q_1)_+}, \quad k \ge 2, \tag{4.34}$$

with c independent of k.

(iii) *If $A = A^{\dagger} = B$ or $A = A^{\dagger} = F$ and $r_1 - r_2 - 1/q_1 + 1/q_2 > 0$ then*

$$e_k(\mathrm{id} : S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \to S^{\overline{r}_2}_{p_2,q_2}A^{\dagger}(\Omega)) \le ck^{r_2-r_1}(\log k)^{(d-1)(r_1-r_2+1/q_2-1/q_1)}, \quad k \ge 2, \tag{4.35}$$

with c independent of k.

(iv) *If $A = A^{\dagger} = B$ or $A = A^{\dagger} = F$ and $r_1 - r_2 - 1/q_1 + 1/q_2 \le 0$ then for every $\varepsilon > 0$ there is a constant $c_\varepsilon > 0$ such that*

$$e_k(\mathrm{id} : S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \to S^{\overline{r}_2}_{p_2,q_2}A^{\dagger}(\Omega)) \le c_\varepsilon k^{r_2-r_1}(\log k)^{\varepsilon}, \quad k \ge 2. \tag{4.36}$$

(v) *For general $A, A^{\dagger}$ and*

$$r_1 - r_2 - \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > V_1(\min(p_1,p_2), q_1, p_2, q_2)$$

we get finally

$$e_k(\mathrm{id} : S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \to S^{\overline{r}_2}_{p_2,q_2}A^{\dagger}(\Omega)) \le ck^{r_2-r_1}(\log k)^{(d-1)(r_1-r_2+1/q_2-1/q_1)}, \quad k \ge 2.$$

Proof. STEP 1. First we give some notation. If $f \in S^{\overline{r}_1}_{p_1,q_1}A(\Omega)$ then according to Definition 3.1 there is a function $g \in S^{\overline{r}_1}_{p_1,q_1}A(\mathbb{R}^d)$ such that

$$\|g \,|\, S^{\overline{r}_1}_{p_1,q_1}A(\mathbb{R}^d)\| \le 2\|f \,|\, S^{\overline{r}_1}_{p_1,q_1}A(\Omega)\|$$

with $g|\Omega = f$. We denote this function $g = \operatorname{ext} f$. Hence ext is a (nonlinear) bounded operator

$$\operatorname{ext} : S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \to S^{\overline{r}_1}_{p_1,q_1}A(\mathbb{R}^d).$$

On the other hand, the natural restriction of $g \in S^{\overline{r}_1}_{p_1,q_1}A(\mathbb{R}^d)$ to $D'(\Omega)$ is a bounded linear operator

$$\operatorname{tr}_\Omega : S^{\overline{r}_1}_{p_1,q_1}A(\mathbb{R}^d) \to S^{\overline{r}_1}_{p_1,q_1}A(\Omega).$$

STEP 2. To prove the first statement we introduce two diagrams which will also be of use later on. In the first one, we start with $f \in S^{\overline{r}_1}_{p_1,q_1}A(\Omega)$ and extend it to $g = \operatorname{ext} f \in S^{\overline{r}_1}_{p_1,q_1}A(\mathbb{R}^d)$. Then we apply the wavelet decomposition to g as described in 2.12. This allows us to represent g in the form

$$g = \sum_{\overline{\nu}\in\mathbb{N}_0^d,\, \overline{m}\in\mathbb{Z}^d} \lambda_{\overline{\nu}\,\overline{m}} \Psi_{\overline{\nu}\,\overline{m}}. \tag{4.37}$$

In this way, we obtain a sequence $\lambda = \{\lambda_{\overline{\nu}\,\overline{m}} : \overline{\nu} \in \mathbb{N}_0^d,\ \overline{m} \in \mathbb{Z}^d\} \in s^{\overline{r}_1}_{p_1,q_1}a$. According to Theorem 2.12, the mapping which assigns to a given function g its wavelet coefficients λ (and which will be denoted by $\mathcal{W}$) is bounded,

$$\mathcal{W} : S^{\overline{r}_1}_{p_1,q_1}A(\mathbb{R}^d) \to s^{\overline{r}_1}_{p_1,q_1}a.$$

As the distribution g need not have a bounded support, we restrict the sum in (4.37) to those $\overline{m} \in \mathbb{Z}^d$ such that $\operatorname{supp}\Psi_{\overline{\nu}\,\overline{m}} \cap \Omega \neq \emptyset$. Furthermore, we may always find a domain Ω' such that

$$\{\overline{m} \in \mathbb{Z}^d : \operatorname{supp}\Psi_{\overline{\nu}\,\overline{m}} \cap \Omega \neq \emptyset\} \subset A^{\Omega'}_{\overline{\nu}}, \qquad \overline{\nu} \in \mathbb{N}_0^d.$$

This natural restriction will be formally realised by the the operator

$$\operatorname{id}' : s^{\overline{r}_1}_{p_1,q_1}a \to s^{\overline{r}_1,\Omega'}_{p_1,q_1}a.$$

Finally, given a sequence $\lambda \in s^{\overline{r}_2,\Omega'}_{p_2,q_2}a^\dagger$, we denote by $S(\lambda)$ the distribution which arises as a wavelet sum with coefficients $\lambda_{\overline{\nu}\,\overline{m}}$,

$$S(\lambda) = \sum_{\overline{\nu}\in\mathbb{N}_0^d,\, \overline{m}\in A^{\Omega'}_{\overline{\nu}}} \lambda_{\overline{\nu}\,\overline{m}} \Psi_{\overline{\nu}\,\overline{m}}.$$

Using all this information we obtain the commutative diagram

$$\begin{array}{ccccccc}
S^{\overline{r}_1}_{p_1,q_1}A(\Omega) & \xrightarrow{\operatorname{ext}} & S^{\overline{r}_1}_{p_1,q_1}A(\mathbb{R}^d) & \xrightarrow{\mathcal{W}} & s^{\overline{r}_1}_{p_1,q_1}a & \xrightarrow{\operatorname{id}'} & s^{\overline{r}_1,\Omega'}_{p_1,q_1}a \\
\Big\downarrow{\scriptstyle \operatorname{id}_1} & & & & & & \Big\downarrow{\scriptstyle \operatorname{id}_2} \\
S^{\overline{r}_2}_{p_2,q_2}A^\dagger(\Omega) & \xleftarrow{\operatorname{tr}_\Omega} & S^{\overline{r}_2}_{p_2,q_2}A^\dagger(\mathbb{R}^d) & & \xleftarrow{\quad S \quad} & & s^{\overline{r}_2,\Omega'}_{p_2,q_2}a^\dagger
\end{array} \tag{4.38}$$

All the operators involved are bounded, under hypothesis (4.33) the embedding id_2 is even compact. This proves that the condition (4.33) is sufficient for compactness of (4.32).

To prove that this condition is also necessary, we follow the reasoning in the proof of Theorem 3.17. Suppose, that (4.33) is *not* satisfied. We shall construct a sequence $\{f_\mu\}$

bounded in $S^{\overline{r}_1}_{p_1,q_1}A(\Omega)$ such that any two different members of it have mutual distance measured in $S^{\overline{r}_2}_{p_2,q_2}A^{\dagger}(\Omega)$ greater than some constant $c > 0$.

If $p_1 \le p_2$, then for every $\mu \ge \mu'$ there are $\overline{\nu}_\mu$ and $\overline{m}_\mu$ with $|\overline{\nu}_\mu| = \mu$ and $CQ_{\overline{\nu}_\mu,\overline{m}_\mu} \subset \Omega$. We set

$$f_\mu = 2^{-\mu(r_1-1/p_1)}\Psi_{\overline{\nu}_\mu\,\overline{m}_\mu}, \qquad \mu \ge \mu'.$$

If $p_1 > p_2$, we choose for every $\mu \ge \mu''$ some $\overline{\nu}_\mu$ with $|\overline{\nu}_\mu| = \mu$ and such that $\#\{\overline{m} \in \mathbb{Z}^d : CQ_{\overline{\nu}_\mu\,\overline{m}} \subset \Omega\} \approx 2^\mu$. Then we set

$$f_\mu = 2^{-\mu r_1} \sum_{\overline{m}\,:\, CQ_{\overline{\nu}_\mu\,\overline{m}} \subset \Omega} \Psi_{\overline{\nu}_\mu\,\overline{m}}, \qquad \mu \ge \mu''.$$

Step 3. Till now we have used (4.38) only to prove the compactness of (4.32). But one may use it also for the estimates of entropy numbers of (4.32). This gives

$$e_k(\mathrm{id}_1) \le c e_k(\mathrm{id}_2), \qquad k \in \mathbb{N},$$

where the constant c covers all the bounded operators $\mathrm{ext}, \mathcal{W}, \mathrm{id}', S$ and tr_Ω. This allows us to carry over the estimate from above obtained on the sequence space level to the function space level.

Step 4. Now we prove the estimate from below, namely (4.34). To this end we consider the sets

$$B^\Omega_{\overline{\nu}} = \{\overline{m} \in \mathbb{Z}^d \;:\; CQ_{\overline{\nu}\,\overline{m}} \subset \Omega\}, \qquad \overline{\nu} \in \mathbb{N}^d_0.$$

They form a certain counterpart to $A^\Omega_{\overline{\nu}}$. There are, however, some important differences. We cannot hope for a straightforward equivalence of (3.7). Instead, there are constants μ_0, c_1 and c_2 such that for every $\mu > \mu_0$ the cardinality of the set

$$\{\overline{\nu} : |\overline{\nu}| = \mu,\ c_1 2^\mu \le \#(B^\Omega_{\overline{\nu}}) \le c_2 2^\mu\}$$

is equivalent to μ^{d-1}. This means that (3.7) does not hold for all $\nu \in \mathbb{N}^d_0$ but only for almost all $\overline{\nu}$ with $|\overline{\nu}|$ large enough.

Following the proof of Theorem 3.18 we have to choose two kinds of building blocks. In the first case, we use the sequence spaces given by the quasinorm

$$\|\lambda \,|\, (s^{\overline{r},\Omega}_{p,q}b)'_\mu\| = \Big(\sum_{|\overline{\nu}|=\mu} 2^{\overline{\nu}\cdot(\overline{r}-1/p)q}\Big(\sum_{\overline{m}\in B^\Omega_{\overline{\nu}}} |\lambda_{\overline{\nu}\,\overline{m}}|^p\Big)^{q/p}\Big)^{1/q}$$

and

$$\|\lambda \,|\, (s^{\overline{r},\Omega}_{p,q}f)'_\mu\| = \Big\|\Big(\sum_{|\overline{\nu}|=\mu}\sum_{\overline{m}\in B^\Omega_{\overline{\nu}}} |2^{\overline{\nu}\cdot\overline{r}}\lambda_{\overline{\nu}\,\overline{m}}\chi_{\overline{\nu}\,\overline{m}}(\cdot)|^q\Big)^{1/q} \,\Big|\, L_p(\mathbb{R}^d)\Big\|.$$

To estimate the entropy numbers of

$$e_k(\mathrm{id} : (s^{\overline{r}_1,\Omega}_{p_1,q_1}a)'_\mu \to (s^{\overline{r}_2,\Omega}_{p_2,q_2}a^\dagger)'_\mu)$$

for $\mu \ge \mu_0$ large enough one may use the same arguments (and get the same results) as in Lemma 3.13.

Hence for $\mu \geq \mu_0$ we use the diagram (with $k = \mu^{d-1}2^\mu$)

$$\begin{array}{ccc} (s^{\overline{r}_1,\Omega}_{p_1,q_1}a)'_\mu & \xrightarrow{\ S\ } & S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \\ {\scriptstyle \mathrm{id}_1}\big\downarrow & & {\scriptstyle \mathrm{id}_2}\big\downarrow \\ (s^{\overline{r}_2,\Omega}_{p_2,q_2}a^\dagger)'_\mu & \xleftarrow{\ \mathcal{W}\ } & S^{\overline{r}_2}_{p_2,q_2}A^\dagger(\Omega) \end{array} \tag{4.39}$$

to get

$$e_k(\mathrm{id}: S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \to S^{\overline{r}_2}_{p_2,q_2}A^\dagger(\Omega)) \geq c\, k^{r_2-r_1}(\log k)^{(d-1)(r_1-r_2+1/q_2-1/q_1)}, \qquad k \geq 2.$$

On the other hand, the diagram (and the choice $k = 2^\mu$)

$$\begin{array}{ccc} 2^{\mu(r_1-1/p_1)}\ell^{B_\mu}_{p_1} & \xrightarrow{\ S\ } & S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \\ {\scriptstyle \mathrm{id}_1}\big\downarrow & & {\scriptstyle \mathrm{id}_2}\big\downarrow \\ 2^{\mu(r_2-1/p_2)}\ell^{B_\mu}_{p_2} & \xleftarrow{\ \mathcal{W}\ } & S^{\overline{r}_2}_{p_2,q_2}A^\dagger(\Omega) \end{array} \tag{4.40}$$

gives

$$e_k(\mathrm{id}: S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \to S^{\overline{r}_2}_{p_2,q_2}A^\dagger(\Omega)) \geq ck^{r_2-r_1}, \qquad k \geq 2.$$

Here $B_\mu = \#(B^\Omega_{\overline{\nu}})$ for some $\overline{\nu}$ with $|\overline{\nu}| = \mu$ is chosen such that $B_\mu \approx \mu^{d-1}2^\mu$, $\mu \geq \mu_0$.

Step 5. The proof of (v) involves the same arguments as given in the previous steps and in Theorem 3.19. ■

Remark 4.12. Theorem 4.11 describes in detail the entropy numbers of

$$\mathrm{id}: S^{\overline{r}_1}_{p_1,q_1}A(\Omega) \to S^{\overline{r}_2}_{p_2,q_2}A^\dagger(\Omega)$$

if $A = A^\dagger$. In this case it gives (up to the $(\log k)^\varepsilon$-gap) the final answer. Let us look a bit more closely on the situation where $A = B$ and $A^\dagger = F$. The estimate from below is covered by (4.34). If $q_1 \leq p_1$ we may use the embeddings

$$S^{\overline{r}_1}_{p_1,q_1}B(\Omega) \hookrightarrow S^{\overline{r}_1}_{p_1,q_1}F(\Omega) \hookrightarrow S^{\overline{r}_2}_{p_2,q_2}F(\Omega) \tag{4.41}$$

to carry over the results obtained for $F \hookrightarrow F$ also to $B \hookrightarrow F$. If $q_2 \leq p_2$, we replace (4.41) by

$$S^{\overline{r}_1}_{p_1,q_1}B(\Omega) \hookrightarrow S^{\overline{r}_2}_{p_2,q_2}B(\Omega) \hookrightarrow S^{\overline{r}_2}_{p_2,q_2}F(\Omega). \tag{4.42}$$

But if $p_1 < q_1$ and $p_2 < q_2$ (and, for simplicity, $p_1 \leq p_2$), no trivial embedding would help. In that case we get (4.35) only for

$$r_1 - r_2 - \left(\frac{1}{p_1} - \frac{1}{p_2}\right) > \frac{1}{p_2} - \frac{1}{q_2}.$$

In the case of $A = F$ and $A^\dagger = B$ the situation is similar. We may get (4.35) whenever (4.41) is compact and $p_1 \leq q_1$ or $p_2 \leq q_2$. If $q_1 < p_1, q_2 < p_2$ and $p_1 \leq p_2$, we get the same result only for

$$r_1 - r_2 - \left(\frac{1}{p_1} - \frac{1}{p_2}\right) > \frac{1}{q_1} - \frac{1}{p_1}.$$

4.6. Comparison with known results. As the function spaces with dominating mixed smoothness have been studied systematically by many authors, there are also many important results on the estimates of the decay of entropy numbers available in the literature. Here, we compare our results supplied by decomposition techniques with those obtained by Belinsky [4], Temlyakov [30] and Dinh Dung [8].

Unfortunately, the classes of functions studied by them differ slightly from the scales $S^{\overline{r}}_{p,q}B(\Omega)$ and $S^{\overline{r}}_{p,q}F(\Omega)$. Let us sketch briefly their setting. They consider 1-periodic functions of d real variables. Hence, their domain Ω is fixed, $\Omega = [0,1)^d$. Belinsky considers four main scales of spaces with dominating mixed smoothness, $W^{\overline{r}}_p, H^{\overline{r}}_p$ on the one hand and $L_p, B^0_{\infty,1}$ on the other hand.

For $1 < p < \infty$, the space L_p of periodic functions is a direct counterpart of $S^0_{p,2}F(\Omega)$. Similarly, $B^0_{\infty,1}$ is $S^0_{\infty,1}B(\Omega)$ in our terminology. The spaces $W^{\overline{r}}_p$ defined by Belinsky by means of Weyl derivatives represent for $1 < p < \infty$ the Sobolev spaces of dominating mixed smoothness $S^{\overline{r}}_{p,2}F(\Omega)$ and, finally, the spaces $H^{\overline{r}}_p$ are sometimes called Nikol'skiĭ spaces and have their counterpart in $S^{\overline{r}}_{p,\infty}B(\Omega)$. To simplify the comparison of our results with Belinsky's, we denote the spaces $W^{\overline{r}}_p, H^{\overline{r}}_p, L_p$ and $B^0_{\infty,1}$ by $\widetilde{S}^{\overline{r}}_{p,2}F, \widetilde{S}^{\overline{r}}_{p,\infty}B, \widetilde{S}^0_{p,2}F$ and $\widetilde{S}^0_{\infty,1}B$. We now quote four results of Belinsky and compare them with their analogues obtained by our method. We set the smoothness involved to be (as in our case) $\overline{r} = (r,\dots,r) \in \mathbb{R}^d$ although the results in [4] are presented in a bit greater generality.

Theorem 4.13. (i) *Let $r > 1/p - 1/q$ and $1 < p \le q < \infty$. Then*

$$e_k(\mathrm{id} : \widetilde{S}^{\overline{r}}_{p,2}F \to \widetilde{S}^0_{q,2}F) \approx \left(\frac{\log^{d-1} k}{k}\right)^r. \tag{4.43}$$

(ii) *Let $r > 1/p - 1/q$ and $1 < p \le q < \infty$. Then*

$$e_k(\mathrm{id} : \widetilde{S}^{\overline{r}}_{p,\infty}B \to \widetilde{S}^0_{q,2}F) \approx \left(\frac{\log^{d-1} k}{k}\right)^r \log^{(d-1)/2} k. \tag{4.44}$$

(iii) *Let $r > 1/2$. Then*

$$e_k(\mathrm{id} : \widetilde{S}^{\overline{r}}_{2,2}F \to \widetilde{S}^0_{\infty,1}B) \approx \left(\frac{\log^{d-1} k}{k}\right)^r \log^{(d-1)/2} k. \tag{4.45}$$

(iv) *Let $r > 1/2$. Then*

$$e_k(\mathrm{id} : \widetilde{S}^{\overline{r}}_{2,\infty}B \to \widetilde{S}^0_{\infty,1}B) \approx \left(\frac{\log^{d-1} k}{k}\right)^r \log^{d-1} k. \tag{4.46}$$

Remark 4.14. We point out that according to Theorem 3.17, all the bounds for r in Theorem 4.13 are optimal. Due to Theorem 4.11, we achieved the same results as in (i), (iii) and (iv). The embedding appearing in (4.44) corresponds to

$$\mathrm{id} : S^{\overline{r}}_{p,\infty}B(\Omega) \to S^0_{q,2}F(\Omega)$$

in our setting. In this case, for

$$r - \left(\frac{1}{p} - \frac{1}{q}\right) > V_1(p,\infty,q,2) = \frac{1}{q} - \frac{1}{\max(q,2)}$$

by Theorem 4.11 we get

$$e_k(\mathrm{id}) \le ck^{-r}(\log k)^{(d-1)(r+1/2)}, \qquad k \ge 2.$$

So, for $q \ge 2$, our result is optimal for all possible r, but for $q < 2$ we get the optimal result only for $r > 1/p - 1/2 > 1/p - 1/q$.

In [30], Temlyakov obtained other important results on entropy numbers of embeddings of spaces with dominating mixed smoothness. Using our notation, they may be summarised as follows.

THEOREM 4.15. (i) *Let $r > 1$. Then*

$$e_k(\mathrm{id} : S^{\overline{r}}_{1,\infty}B \to S^0_{\infty,2}B) \le ck^{-r}(\log k)^{(d-1)(r+1/2)}. \tag{4.47}$$

(ii) *Let $r > 0$. Then*

$$e_k(\mathrm{id} : S^{\overline{r}}_{\infty,\infty}B \to L_1) \ge ck^{-r}(\log k)^{(d-1)(r+1/2)}. \tag{4.48}$$

(iii) *Let $r > 1$ and $1 < p, q < \infty$. Then*

$$e_k(\mathrm{id} : S^{\overline{r}}_{q,2}F \to S^0_{p,2}F) \le ck^{-r}(\log k)^{(d-1)r}. \tag{4.49}$$

(iv) *Let $r > 0$ and $1 < q < \infty$. Then*

$$e_k(\mathrm{id} : S^{\overline{r}}_{q,2}F \to L_1) \ge ck^{-r}(\log k)^{(d-1)r}. \tag{4.50}$$

REMARK 4.16. We discuss these results briefly. We point out that the bound for r is always optimal apart from case (iii). Namely, the embedding in (4.49) is compact if and only if $r > (1/q - 1/p)_+$. The inequalities (4.47) and (4.49) are completely covered by Theorem 4.11.

But (4.48) and (4.50) are of a different nature. Namely, they deal with the space $L_1(\Omega)$, which does *not* fit into our scales $S^{\overline{r}}_{p,q}A(\Omega)$. All the known decomposition techniques fail to give some decomposition of this space and, therefore, no reduction to the sequence space level is possible. The same holds for embeddings to other spaces of this kind, especially $L_\infty(\Omega)$.

Finally, we discuss the results obtained by Dinh Dung in [8].

THEOREM 4.17. *Let $1 < p_1, p_2 < \infty$, $0 < q \le \infty$ and $r > 0$. Then we have*

(i) *for either $r > 1/p_1$ and $q \ge p_1$, or $r > (1/p_1 - 1/p_2)_+$ and $q \ge \min(p_2, 2)$,*

$$e_k(\mathrm{id} : S^{\overline{r}}_{p_1,q}B \to S^0_{p_2,2}F) \approx k^{-r}(\log k)^{(d-1)(r+1/2-1/q)}, \tag{4.51}$$

(ii) *for $r > (1/p_1 - 1/p_2)_+$,*

$$e_k(\mathrm{id} : S^{\overline{r}}_{p_1,2}F \to S^0_{p_2,2}F) \approx k^{-r}(\log k)^{(d-1)r}. \tag{4.52}$$

The embedding (4.52) is (for $p_1 \le p_2$) covered by (4.43) and for general p_1 and p_2 by (4.34) and (4.35). We therefore concentrate on (4.51). In [9], Dinh Dung comments that the conditions on r and q in Theorem 4.17 ensure the positivity of the power of the logarithm in (4.51). In view of our general estimate (4.34), this should really be so. But unfortunately, the conditions given in Theorem 4.17 do *not* ensure that $r + 1/2 - 1/q > 0$. To see that, set $p_1 = p_2 < q < 2$ and $0 < r < 1/q - 1/2$. A closer inspection of the proof of Theorem 2 in [8] shows that in the case $r > (1/p_1 - 1/p_2)_+$ and $q \ge \min(p_2, 2)$ Dinh Dung proves actually a slightly weaker result, namely

$$e_k(\mathrm{id} : S^{\overline{r}}_{p_1,q}B \to S^0_{p_2,2}F) \le c\, k^{-r}(\log k)^{(d-1)(r+1/\min(p_2,2)-1/q)}, \qquad k \ge 2. \tag{4.53}$$

In this result, the power of the logarithm is always positive and, therefore, no contradiction with (4.34) occurs. We point out that our result covers and improves (4.53) as far as the set of parameters is concerned.

We start with $p_1 \le p_2$. By Remark 4.12, we get (4.51) for all $r > 1/p_1 - 1/p_2$ with $r > 1/q - 1/2$ if $q \le p_1$ or $2 \le p_2$. Moreover, for $r \le 1/q - 1/2$ we get (4.34) and an analogue of (4.36). Finally, if $r > 1/p_1 - 1/2$ we get (4.51) even if $q > p_1$ and $2 > p_2$. A similar discussion may be done for $p_1 > p_2$.

Next we present some special cases of Theorem 4.11 which have not been discussed separately yet, but which may be of some independent interest.

THEOREM 4.18. *Let* $\overline{r} = (r, \dots, r) \in \mathbb{R}^d$.

(i) *The embedding*

$$\mathrm{id} : S^{\overline{r}}_{1,1}B(\Omega) \to S^0_{\infty,\infty}B(\Omega)$$

is compact if and only if $r > 1$ *and in that case*

$$e_k(\mathrm{id}) \approx k^{-r}(\log k)^{(d-1)(r-1)}, \qquad k \ge 2.$$

(ii) *The embedding*

$$\mathrm{id} : S^{\overline{r}}_{\infty,1}B(\Omega) \to S^0_{\infty,\infty}B(\Omega)$$

is compact if and only if $r > 0$. *If* $r > 1$ *then*

$$e_k(\mathrm{id}) \approx k^{-r}(\log k)^{(d-1)(r-1)}, \qquad k \ge 2,$$

and for $0 < r \le 1$ *and every* $\varepsilon > 0$ *there are constants* c *and* c_ε *such that*

$$ck^{-r} \le e_k(\mathrm{id}) \le c_\varepsilon k^{-r}(\log k)^{\varepsilon}, \qquad k \ge 2.$$

(iii) *Let* $0 < p \le q < \infty$. *The embedding*

$$\mathrm{id} : S^{\overline{r}}_{p,2}F(\Omega) \to S^0_{q,\infty}B(\Omega)$$

is compact if and only if $r > 1/p - 1/q$. *If in this case* $r > 1/2$ *then*

$$e_k(\mathrm{id}) \approx k^{-r}(\log k)^{(d-1)(r-\frac{1}{2})}, \qquad k \ge 2,$$

and for $1/p - 1/q < r \le 1/2$ *and every* $\varepsilon > 0$ *there are constants* c *and* c_ε *such that*

$$c\,k^{-r} \le e_k(\mathrm{id}) \le c_\varepsilon k^{-r}(\log k)^{\varepsilon}, \quad k \ge 2.$$

Proof. The assertion follows from Theorem 4.11 and Remark 4.12. ∎

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