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Abstract

Let G be a locally compact group. We shall study the Banach algebras which are the group algebra $L^1(G)$ and the measure algebra $M(G)$ on G, concentrating on their second dual algebras. As a preliminary we shall study the second dual $C_0(\Omega)''$ of the C^*-algebra $C_0(\Omega)$ for a locally compact space Ω, recognizing this space as $C(\widetilde{\Omega})$, where $\widetilde{\Omega}$ is the hyper-Stonean envelope of Ω.

We shall study the C^*-algebra $B^b(\Omega)$ of bounded Borel functions on Ω, and we shall determine the exact cardinality of a variety of subsets of $\widetilde{\Omega}$ that are associated with $B^b(\Omega)$.

We shall identify the second duals of the measure algebra $(M(G), \star)$ and the group algebra $(L^1(G), \star)$ as the Banach algebras $(M(\widetilde{G}), \Box)$ and $(M(\Phi), \Box)$, respectively, where $\Box$ denotes the first Arens product and $\widetilde{G}$ and Φ are certain compact spaces, and we shall then describe many of the properties of these two algebras. In particular, we shall show that the hyper-Stonean envelope $\widetilde{G}$ determines the locally compact group G. We shall also show that $(\widetilde{G}, \Box)$ is a semigroup if and only if G is discrete, and we shall discuss in considerable detail the product of point masses in $M(\widetilde{G})$. Some important special cases will be considered.

We shall show that the spectrum of the C^*-algebra $L^\infty(G)$ is determining for the left topological centre of $L^1(G)''$, and we shall discuss the topological centre of the algebra $(M(G)'', \Box)$.

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1. Introduction

Our aim in this memoir is to study the Banach algebras which are the second dual algebras $(M(G)'', \Box)$ and $(L^1(G)'', \Box)$ of the measure algebra $(M(G), \star)$ and the group algebra $(L^1(G), \star)$, respectively, of a locally compact group G. Here $\Box$ denotes the (first) Arens product on the second dual space A'' of a Banach algebra A. We are particularly interested in the case where the group G is not discrete; the discrete case was studied in our earlier memoir [17]. Thus we must discuss in some depth a compact space $\widetilde{G}$ which we call the *hyper-Stonean envelope* of a locally compact group G, and also the subspace Φ of $\widetilde{G}$, where Φ is the character space, or spectrum, of the C^*-algebra $L^\infty(G)$. The space $\widetilde{G}$ is analogous to the semigroup $(\beta S, \Box)$ which is the Stone–Čech compactification of a semigroup S (see [17]), and we shall discuss the 'semigroup-like' properties of $(\widetilde{G}, \Box)$; however, we shall prove that $(\widetilde{G}, \Box)$ is actually only a semigroup in the special case where G is discrete.

As a preliminary to our discussion of $\widetilde{G}$ we shall develop the theory of the hyper-Stonean envelope $\widetilde{\Omega}$ of a locally compact space Ω; in our approach, $\widetilde{\Omega}$ is the character space of the second dual $C_0(\Omega)''$ of $C_0(\Omega)$. Many of these results are known, and indeed they go back to the seminal paper of Dixmier [24] of 1951. However we cast the material in a different context, and prove some new results that we shall require later.

The present chapter contains a review of some notation that we shall use and background material involving Banach spaces, Banach algebras, and their second duals. A summary of our results and some acknowledgements are given at the end of this chapter.

Basic notation. We shall use the following notation.

The rational, real, and complex fields are $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. We denote the set of integers by $\mathbb{Z}$, and set $\mathbb{Z}^+ = \{n \in \mathbb{Z} : n \geq 0\}$ and $\mathbb{N} = \{n \in \mathbb{Z} : n > 0\}$; for $n \in \mathbb{N}$, we set $\mathbb{Z}_n^+ = \{0, \dots, n\}$ and $\mathbb{N}_n = \{1, \dots, n\}$. Further,

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} \quad \text{and} \quad \mathbb{I} = [0,1] \subset \mathbb{R}.$$

However, for $p \in \mathbb{N}$, we set

$$\mathbb{Z}_p = \{0, 1, \dots, p-1\};$$

this set is a group with respect to addition modulo p. Further, we set

$$D_p = \mathbb{Z}_p^{\aleph_0} = \{\varepsilon = (\varepsilon_j : j \in \mathbb{N}) : \varepsilon_j \in \mathbb{Z}_p \ (j \in \mathbb{N})\}.$$

The set D_p is a group with respect to the coordinatewise operations.

The cardinality of a set S is denoted by $|S|$; the first infinite cardinal is $\aleph_0$; the first uncountable cardinal is $\aleph_1$; the cardinality of the continuum is denoted by $\mathfrak{c}$, so that

$\mathfrak{c} = 2^{\aleph_0}$, and the *continuum hypothesis* (CH) is the assertion that $\aleph_1 = \mathfrak{c}$; the *generalized continuum hypothesis* (GCH) implies that $2^{\mathfrak{c}} = 2^{\aleph_1} = \aleph_2$ and that $2^{2^{\mathfrak{c}}} = \aleph_3$.

The characteristic function of a subset S of a set is denoted by χ_S; the function constantly equal to 1 on a set S is also denoted by 1_S or 1. The symmetric difference of two subsets S and T of a given set is denoted by $S \bigtriangleup T$.

Let E be a linear space (always taken to be over the complex field $\mathbb{C}$), and let S be a subset of E. The *convex hull* of S is $\langle S \rangle$, and the *linear span* of S is $\operatorname{lin} S$. The set of *extreme points* of a convex subset S of E is denoted by $\operatorname{ex} S$.

Algebras and modules. Let A be an algebra, which is always taken to be linear and associative. The following notation is as in [13].

The identity of A (if it exists) is e_A; the algebra formed by adjoining an identity to a non-unital algebra A is denoted by $A^{\#}$, and $A^{\#} = A$ when A has an identity. The *centre* of A is $\mathfrak{Z}(A)$. For a subset S of A, we set

$$S^{[2]} = \{ab : a, b \in S\} \quad \text{and} \quad S^2 = \operatorname{lin} S^{[2]}.$$

A *character* on A is a homomorphism from A onto the field $\mathbb{C}$; the character space of A is the collection of characters on A, and it is denoted by Φ_A. For $a \in A$, we define

$$L_a : b \mapsto ab, \qquad R_a : b \mapsto ba, \qquad A \to A.$$

Suppose that B is a subalgebra of A and that I is an ideal in A. Then the product in $B \times I$ is given by

$$(b_1, x_1)(b_2, x_2) = (b_1 b_2, b_1 x_2 + x_1 b_2 + x_1 x_2) \qquad (b_1, b_2 \in B,\ x_1, x_2 \in I);$$

in this case A is a *semidirect product* of B and I, written $A = B \ltimes I$.

Let E be an A-bimodule, so that E is a linear space and there are bilinear maps

$$(a, x) \mapsto a \cdot x, \qquad (a, x) \mapsto x \cdot a, \qquad A \times E \to E,$$

such that $a \cdot (b \cdot x) = ab \cdot x$, $(x \cdot b) \cdot a = x \cdot ba$, and $a \cdot (x \cdot b) = (a \cdot x) \cdot b$ for $a, b \in A$ and $x \in E$. In this case, set

$$A \cdot E = \{a \cdot x : a \in A,\ x \in E\}, \quad AE = \operatorname{lin} A \cdot E,$$

and similarly for $E \cdot A$ and EA. Suppose that A has an identity e_A. Then the bimodule E is *unital* if $e_A \cdot x = x \cdot e_A = x$ $(x \in E)$. In general, an A-bimodule E is *neo-unital* if $A \cdot E = E \cdot A = A$.

For details on bimodules, see [13, §1.4].

Banach spaces. Throughout our terminology and notations for Banach spaces and algebras will be in accord with those in [13], where further details may be found. We recall some notation.

Let E be a Banach space. The closed unit ball in E is $E_{[1]}$. The dual space and second dual space of E are denoted by E' and E'', respectively; we write $\langle x,\, \lambda \rangle$ for the action of $\lambda \in E'$ on $x \in E$ and $\langle \mathrm{M},\, \lambda \rangle$ for the action of $\mathrm{M} \in E''$ on $\lambda \in E'$, etc.; the weak-$*$ topology on E' is $\sigma(E', E)$, so that $(E'_{[1]}, \sigma(E', E))$ is always compact; we set

$$\langle \kappa_E(x),\, \lambda \rangle = \langle x,\, \lambda \rangle \qquad (x \in E,\ \lambda \in E'),$$

so defining the canonical embedding $\kappa_E : E \to E''$, and we set

$$\langle \mathrm{M},\, \kappa_{E'}(\lambda)\rangle = \langle \mathrm{M},\, \lambda\rangle \quad (\lambda \in E',\, \mathrm{M} \in E''),$$

so defining the canonical embedding $\kappa_{E'} : E' \to E'''$. Of course, $\kappa_E(E_{[1]})$ is $\sigma(E'', E')$-dense in $E''_{[1]}$ and

$$\langle \kappa_E(x),\, \kappa_{E'}(\lambda)\rangle = \langle x,\, \lambda\rangle \quad (x \in E,\, \lambda \in E').$$

We shall often identify E with $\kappa_E(E)$, and regard it as a $\|\cdot\|$-closed subspace of E''.

We first recall some standard results of functional analysis that will be used more than once.

PROPOSITION 1.1. *Let E be a non-zero Banach space.*

- (i) *The space $(E'_{[1]}, \sigma(E', E))$ is metrizable if and only if $(E, \|\cdot\|)$ is separable.*
- (ii) *The following are equivalent conditions on an element $\mathrm{M} \in E''$:*
  - (a) $\mathrm{M} \in E$;
  - (b) M *is continuous on $(E', \sigma(E', E))$;*
  - (c) M *is continuous on $(E'_{[1]}, \sigma(E', E))$.*
- (iii) *Suppose that $|E| = \kappa$. Then $|E'| \leq 2^\kappa$.*

Proof. For (i) and (ii), see [26, Theorems V.5.1, V.5.6], for example. For (iii), we have $|E'| \leq |\mathbb{C}^E| = \mathfrak{c}^\kappa = 2^\kappa$. ■

The elements M of E'' which satisfy the equivalent conditions of clause (ii) above are the *normal* elements of E''.

Let E and F be normed spaces. Then we write $\mathcal{B}(E, F)$ for the space of bounded linear operators from E to F; this space is taken with the operator norm. A map $T : E \to F$ is a *linear homeomorphism* if T is a bijection and if $T \in \mathcal{B}(E, F)$ and $T^{-1} \in \mathcal{B}(F, E)$. The spaces E and F are *linearly homeomorphic* if there is a linear homeomorphism from E to F, and E and F are *isometrically isomorphic* if there is a linear isometry from E onto F; in the latter case, we write $E \cong F$.

Let X be a linear subspace of a Banach space E. Then

$$X^\circ = \{\lambda \in E' : \lambda \,|\, X = 0\},$$

so that X' is isometrically isomorphic to E'/X°.

Banach algebras. Let A be a Banach algebra. We recall that all characters on A are continuous, and that Φ_A is a locally compact subspace of the unit ball $(A'_{[1]}, \sigma(A', A))$ of A'. In the case where A has an identity e_A, we have

$$\Phi_A \subset \{\lambda \in A' : \langle e_A,\, \lambda\rangle = \|\lambda\| = 1\},$$

and Φ_A is compact.

A *bounded approximate identity* in A is a bounded net (e_α) in A such that

$$\lim_\alpha a e_\alpha = \lim_\alpha e_\alpha a = a \qquad (a \in A).$$

The theory of Banach A-bimodules is given in [13]. Indeed, a *Banach A-bimodule* is an A-bimodule E which is a Banach space and such that

$$\max\{\|a \cdot x\|, \|x \cdot a\|\} \leq \|a\| \, \|x\| \quad (a \in A,\, x \in E).$$

For example, A is a Banach A-bimodule over itself. Let E be a Banach A-bimodule. Then the dual space E' is also a Banach A-bimodule for the operations defined by

$$\langle x,\, a \cdot \lambda\rangle = \langle x \cdot a, \lambda\rangle, \quad \langle x,\, \lambda \cdot a\rangle = \langle a \cdot x, \lambda\rangle \quad (a \in A,\, x \in E,\, \lambda \in E').$$

In particular, A' is the *dual module* of A, and $\overline{\mathrm{lin}}(A \cdot A')$ is a closed submodule of A'. Further, the second dual A'' is a Banach A-bimodule. A Banach A-bimodule E is *essential* if

$$\overline{AE} = \overline{EA} = E.$$

We shall use the following result, which is a version of Cohen's factorization theorem [13, Corollary 2.9.31].

PROPOSITION 1.2. *Let A be a Banach algebra with a bounded approximate identity, and let E be an essential Banach A-bimodule. Then E is neo-unital. In particular, $A = A^{[2]}$, and $A \cdot A' \cdot A$ is a closed submodule of A'.* ■

A Banach algebra A is said to be a *dual* Banach algebra if there is a closed A-submodule E of A' such that $E' = A$ as a Banach space; in this case, E is a *predual* of A. It is easy to see that a Banach space E is a predual of A in this sense if and only if $E' = A$ and multiplication in A is separately $\sigma(A, E)$-continuous. For example, each von Neumann algebra is a dual Banach algebra [102, Examples 4.4.2(c)]. For further details, see [16, Chapter 2] and [102, §4.4]; for recent accounts of dual Banach algebras, see [19, 20].

We shall refer briefly to the very extensive theory of *amenable* Banach algebras; for the general theory of these algebras, see [13, 59, 102], and for characterizations involving the algebras that we shall be concerned with, see [17].

Arens products and topological centres. Let A be a Banach algebra. Then there are two natural products on the second dual A'' of A; they are called the *Arens products*, and are denoted by $\Box$ and $\Diamond$, respectively. They were introduced by Arens [2], and studied in [10]; for further discussions of these products, see [13, 16, 17], for example.

We recall briefly the definitions. As above, A' and A'' are Banach A-bimodules. For $\lambda \in A'$ and $\mathrm{M} \in A''$, define $\lambda \cdot \mathrm{M} \in A'$ and $\mathrm{M} \cdot \lambda \in A'$ by

$$\langle a,\, \lambda \cdot \mathrm{M}\rangle = \langle \mathrm{M},\, a \cdot \lambda\rangle, \qquad \langle a,\, \mathrm{M} \cdot \lambda\rangle = \langle \mathrm{M},\, \lambda \cdot a\rangle \quad (a \in A).$$

For $\mathrm{M}, \mathrm{N} \in A''$, define

$$\langle \mathrm{M} \Box \mathrm{N},\, \lambda\rangle = \langle \mathrm{M},\, \mathrm{N} \cdot \lambda\rangle, \qquad \langle \mathrm{M} \Diamond \mathrm{N},\, \lambda\rangle = \langle \mathrm{N},\, \lambda \cdot \mathrm{M}\rangle \quad (\lambda \in A').$$

THEOREM 1.3. *Let A be a Banach algebra. Then $(A'', \Box)$ and $(A'', \Diamond)$ are Banach algebras containing A as a closed subalgebra.* ■

The Arens products $\Box$ and $\Diamond$ are determined by the following formulae, where all limits are taken in the weak-$*$ topology $\sigma(A'', A')$ of A''. Let $\mathrm{M}, \mathrm{N} \in A''$, and take nets

(a_α) and (b_β) in A such that $\mathrm{M} = \lim_\alpha a_\alpha$ and $\mathrm{N} = \lim_\beta b_\beta$. Then

$$\mathrm{M} \,\Box\, \mathrm{N} = \lim_\alpha \lim_\beta a_\alpha b_\beta, \qquad \mathrm{M} \diamond \mathrm{N} = \lim_\beta \lim_\alpha a_\alpha b_\beta. \tag{1.1}$$

The two maps $\mathrm{M} \mapsto \mathrm{M} \,\Box\, \mathrm{N}$ and $\mathrm{M} \mapsto \mathrm{N} \diamond \mathrm{M}$ are weak-$*$ continuous on A'' for each $\mathrm{N} \in A''$.

We shall use the following equation. Let A be a Banach algebra, let $a \in A$, and let $\varphi \in \Phi_A$. Then clearly $a \cdot \varphi = \langle a,\, \varphi\rangle\varphi$. Thus, taking weak-$*$ limits, we see that

$$\mathrm{M} \cdot \varphi = \langle \mathrm{M},\, \varphi\rangle\varphi \qquad (\mathrm{M} \in A'',\ \varphi \in \Phi_A). \tag{1.2}$$

Proposition 1.4.

(i) *Let A and B be Banach algebras, and suppose that $\theta : A \to B$ is a continuous homomorphism. Then the map $\theta'' : (A'', \Box) \to (B'', \Box)$ is a continuous homomorphism with range contained in the $\sigma(B'', B')$-closure of $\theta(A)$.*

(ii) *Let A be a Banach algebra, and let E be a Banach A-bimodule. Then E'' is a Banach $(A'', \Box)$-module in a natural way.*

(iii) *Let A be a Banach algebra, let E and F be Banach A-bimodules, and then take $T : E \to F$ to be a continuous A-bimodule homomorphism. Then $T'' : E'' \to F''$ is a continuous $(A'', \Box)$-bimodule homomorphism.*

Proof. These are contained in [13, §2.6], or follow directly from results there; in particular, see Theorem 2.6.15 and equation (2.6.26) of [13]. ■

Let A be a dual Banach algebra with predual E, where E regarded as a subset of A', so that $E^\circ = \{\mathrm{M} \in A'' : \mathrm{M} \,|\, E = 0\}$. Then

$$(A'', \Box) = A \ltimes E^\circ \tag{1.3}$$

as a semidirect product [16, Theorem 2.15].

Definition 1.5. Let A be a Banach algebra. Then the *left* and *right topological centres* of A'' are

$$\mathfrak{Z}_t^{(\ell)}(A'') = \{\mathrm{M} \in A'' : \mathrm{M} \,\Box\, \mathrm{N} = \mathrm{M} \diamond \mathrm{N}\ (\mathrm{N} \in A'')\},$$
$$\mathfrak{Z}_t^{(r)}(A'') = \{\mathrm{M} \in A'' : \mathrm{N} \,\Box\, \mathrm{M} = \mathrm{N} \diamond \mathrm{M}\ (\mathrm{N} \in A'')\},$$

respectively. The *topological centre* is $\mathfrak{Z}_t(A'') = \mathfrak{Z}_t^{(\ell)}(A'') \cap \mathfrak{Z}_t^{(r)}(A'')$.

We also recall that

$$\mathfrak{Z}_t^{(\ell)}(A'') = \{\mathrm{M} \in A'' : L_\mathrm{M} : \mathrm{N} \mapsto \mathrm{M} \,\Box\, \mathrm{N} \text{ is weak-}* \text{ continuous on } A''\},$$
$$\mathfrak{Z}_t^{(r)}(A'') = \{\mathrm{M} \in A'' : R_\mathrm{M} : \mathrm{N} \mapsto \mathrm{N} \diamond \mathrm{M} \text{ is weak-}* \text{ continuous on } A''\}.$$

In the case where A is commutative, we have

$$\mathrm{M} \diamond \mathrm{N} = \mathrm{N} \,\Box\, \mathrm{M} \qquad (\mathrm{M}, \mathrm{N} \in A''),$$

and so $\mathfrak{Z}_t^{(\ell)}(A'')$ and $\mathfrak{Z}_t^{(r)}(A'')$ are each just the (algebraic) centre $\mathfrak{Z}(A'')$ of the algebra $(A'', \Box)$.

Proposition 1.6. *Let A be a Banach algebra. Then $A \subset \mathfrak{Z}_t(A'') = \mathfrak{Z}_t^{(\ell)}(A'') \cap \mathfrak{Z}_t^{(r)}(A'')$.* ■

The following definitions were given in [16]. Further, many examples showing the possibilities that can occur were given in [16, Chapter 4].

Definition 1.7. Let A be a Banach algebra. Then A is *Arens regular* if

$$\mathfrak{Z}_t^{(\ell)}(A'') = \mathfrak{Z}_t^{(r)}A'') = A'',$$

left strongly Arens irregular if

$$\mathfrak{Z}_t^{(\ell)}(A'') = A,$$

right strongly Arens irregular if

$$\mathfrak{Z}_t^{(r)}(A'') = A,$$

and *strongly Arens irregular* if it is both left and right strongly Arens irregular.

A closed subalgebra and a quotient algebra of an Arens regular Banach algebra are themselves Arens regular.

Definition 1.8. Let A be a left strongly Arens irregular Banach algebra. Then a subset V of A'' is *determining for the left topological centre* of A'' if $\mathrm{M} \in A$ whenever

$$\mathrm{M} \Box \mathrm{N} = \mathrm{M} \Diamond \mathrm{N} \quad (\mathrm{N} \in V).$$

Thus A'' is determining for the left topological centre whenever A is left strongly Arens irregular, and possibly smaller subsets of A'' have this property.

The above definition was first given in [17, Definition 12.4]; care is required because this term has been used in a slightly different sense elsewhere.

Let S be a semigroup, and let $\ell^1(S)$ be the corresponding semigroup algebra. In [17], it is shown that, in the case where S belongs to an interesting class of semigroups which is strictly larger than the class of cancellative semigroups, certain subsets V of βS of cardinality 2 are determining for the left topological centre of $\ell^1(S)''$; for strong versions of this and other related results, see [7] and [31]. There are some related results for subsemigroups of the real line in [14, Chapter 9]. We shall address similar questions in Chapter 9.

Introverted subspaces. We recall the definition of introverted subspaces of the dual module A' of a Banach algebra A. Our definition is slightly more general than the one in [16, Definition 5.1] in that now we do not require X to be closed in A'.

Definition 1.9. Let A be a Banach algebra, and let X be a left (respectively, right) A-submodule of A'. Then X is *left-introverted* (respectively, *right-introverted*) if $\mathrm{M} \cdot \lambda \in X$ (respectively, $\lambda \cdot \mathrm{M} \in X$) whenever $\lambda \in X$ and $\mathrm{M} \in A''$; a sub-bimodule X of A' is *introverted* if it is both left- and right-introverted.

Let X be a faithful, left-introverted subspace of A'. Then $\overline{X}$ is also a left-introverted subspace of A', and X° is a weak-$*$ closed ideal in $(A'', \Box)$: this is proved in [16, Theorem 5.4(ii)], but was actually given earlier in [83, Theorem 3.2]. Thus A''/X° is a quotient Banach algebra; the product in this algebra is again denoted by $\Box$. Since $X' = A''/X^\circ$ as a Banach space, we may regard $(X', \Box)$ as a Banach algebra; the formula for the product in X' is

$$\langle \mathrm{M} \Box \mathrm{N},\, \lambda \rangle = \langle \mathrm{M},\, \mathrm{N} \cdot \lambda \rangle \quad (\lambda \in X).$$

Definition 1.10. Let A be a Banach algebra. For $\lambda \in A'$, set

$$K(\lambda) = \{a \cdot \lambda : a \in A_{[1]}\}. \tag{1.4}$$

The element λ is [*weakly*] *almost periodic* if the map

$$a \mapsto a \cdot \lambda, \quad A \to A',$$

is [weakly] compact.

Thus $K(\lambda)$ is a convex subset of A'. We take $\overline{K(\lambda)}$ to be the closure of $K(\lambda)$ in $(A', \|\cdot\|)$; by Mazur's theorem, $\overline{K(\lambda)}$ is also equal to the closure of $K(\lambda)$ in $(A', \sigma(A', A''))$. It is always true that the closure of $K(\lambda)$ in $(A', \sigma(A', A))$ is

$$\overline{K(\lambda)}^{\sigma(A',A)} = \{\mathrm{M} \cdot \lambda : \mathrm{M} \in A''_{[1]}\},$$

and of course $\overline{K(\lambda)} \subset \overline{K(\lambda)}^{\sigma(A',A)}$. Thus λ is almost periodic if and only if $\overline{K(\lambda)}$ is compact in $(A', \|\cdot\|)$, and weakly almost periodic if and only if $\overline{K(\lambda)}$ is compact in $(A', \sigma(A', A''))$.

Definition 1.11. Let A be a Banach algebra. Then the Banach spaces of almost periodic and weakly almost periodic functionals on A are denoted by

$$AP(A) \quad \text{and} \quad WAP(A),$$

respectively.

Thus $AP(A) \subset WAP(A)$, and it is easily seen that both $AP(A)$ and $WAP(A)$ are Banach A-submodules of A'. By [93] (see [16, Proposition 3.11]), $\lambda \in WAP(A)$ if and only if

$$\langle \mathrm{M} \,\Box\, \mathrm{N}, \lambda\rangle = \langle \mathrm{M} \diamond \mathrm{N}, \lambda\rangle \quad (\mathrm{M}, \mathrm{N} \in A''),$$

and so $\lambda \in WAP(A)$ if and only if

$$\lim_m \lim_n \langle a_m b_n,\, \lambda\rangle = \lim_n \lim_m \langle a_m b_n,\, \lambda\rangle$$

whenever (a_m) and (b_n) are bounded sequences in A and both iterated limits exist.

The following result, from [93], is also contained in [16, Theorem 3.14, Proposition 5.7].

Proposition 1.12. *Let A be a Banach algebra. Then A is Arens regular if and only if $WAP(A) = A'$.* ■

We consider the relation between the space $WAP(A)$ and the two sets $A' \cdot A$ and $A \cdot A'$.

First, as in [16, Example 4.9(i)], let A be a non-zero Banach algebra with $A^2 = \{0\}$. Then A is Arens regular, and so $WAP(A) = A'$, but $A' \cdot A = A \cdot A' = \{0\}$, and so $\overline{A'A} \subsetneq WAP(A)$. Second, let $A = \ell^1(G)$ for an infinite group G, as described below. Then we shall see that $WAP(A) = WAP(G)$, the space of weakly almost periodic functions on G, whereas, in this case, $A' \cdot A = A' = \ell^\infty(G)$, and so $WAP(A) \subsetneq A' \cdot A$.

Now suppose that A has a bounded approximate identity. Clause (i) of the following result is contained in [16, Propositions 2.20 and 3.12], following [71, Proposition 3.3]; clause (ii) is part of [80, Theorem 3.6].

Proposition 1.13. *Let A be a Banach algebra with a bounded approximate identity. Then:*

(i) *$AP(A)$ and $WAP(A)$ are neo-unital Banach A-bimodules, with*

$$AP(A) \subset WAP(A) \subset (A' \cdot A) \cup (A \cdot A');$$

(ii) *$WAP(A) = A' \cdot A$ if and only if $A \cdot A'' \subset \mathfrak{Z}_t^{(\ell)}(A'')$.* ■

For a further discussion of $AP(A)$ and $WAP(A)$, see [16, 25, 83].

We shall also use the following propositions. The first is exactly [75, Lemma 1.2]; clause (ii) was given earlier in [83, Theorem 3.1].

PROPOSITION 1.14. *Let A be a Banach algebra, and let X be a left A-submodule of A'. Then X is left-introverted if and only if*

$$\overline{K(\lambda)}^{\sigma(A',A)} \subset X$$

for each $\lambda \in X$. Further, suppose that X is an A-submodule of A'. Then:

(i) *X is introverted whenever X is weak-$*$ closed;*
(ii) *X is introverted whenever $X \subset WAP(A)$.* ■

In particular, in the case where A is Arens regular, each $\|\cdot\|$-closed, A-submodule of A' is introverted, and so $(X', \Box)$ is a Banach algebra.

PROPOSITION 1.15. *Let A be a Banach algebra, and let X be a left-introverted subspace of A'. Then the following are equivalent conditions on X:*

(a) *$X \subset AP(A)$;*
(b) *the product*

$$(\mathrm{M}, \mathrm{N}) \mapsto \mathrm{M} \Box \mathrm{N}, \quad X'_{[1]} \times X'_{[1]} \to X'_{[1]},$$

is jointly continuous with respect to the weak-$*$ topology $\sigma(X', X)$ on X'.

Proof. (a)⇒(b). Let (M_α) and (N_β) be nets in $X'_{[1]}$ converging in the weak-$*$ topology to M and N in $X'_{[1]}$, respectively. By taking norm-preserving extensions, we may suppose that all these elements belong to $A''_{[1]}$.

Let $\lambda \in X$, so that $\lambda \in AP(A)$ by (a), and hence the set $K(\lambda)$ is relatively compact in the Banach space $(A', \|\cdot\|)$. The identity map

$$(\overline{K(\lambda)}, \|\cdot\|) \to (\overline{K(\lambda)}, \sigma(A', A))$$

is a continuous map from a compact space onto a Hausdorff space, and so the topologies $\sigma(A', A)$ and $\|\cdot\|$ agree on $\overline{K(\lambda)}$ and

$$\overline{K(\lambda)} = \{\mathrm{M} \cdot \lambda : \mathrm{M} \in X'_{[1]}\}.$$

The net $(\mathrm{N}_\beta \cdot \lambda)$ converges to $\mathrm{N} \cdot \lambda$ in $(A', \sigma(A', A))$, and so $(\mathrm{N}_\beta \cdot \lambda)$ converges to $\mathrm{N} \cdot \lambda$ in $(A', \|\cdot\|)$. Hence

$$\begin{aligned} &|\langle \mathrm{M}_\alpha \Box \mathrm{N}_\beta, \lambda\rangle - \langle \mathrm{M} \Box \mathrm{N}, \lambda\rangle| \\ &\qquad \le |\langle \mathrm{M}_\alpha \Box \mathrm{N}_\beta, \lambda\rangle - \langle \mathrm{M}_\alpha \Box \mathrm{N}, \lambda\rangle| + |\langle \mathrm{M}_\alpha \Box \mathrm{N}, \lambda\rangle - \langle \mathrm{M} \Box \mathrm{N}, \lambda\rangle| \\ &\qquad \le \|\mathrm{N}_\beta \cdot \lambda - \mathrm{N} \cdot \lambda\| + |\langle \mathrm{M}_\alpha, \mathrm{N} \cdot \lambda\rangle - \langle \mathrm{M}, \mathrm{N} \cdot \lambda\rangle|, \end{aligned}$$

and so

$$\lim_{(\alpha,\beta)} \langle \mathrm{M}_\alpha \Box \mathrm{N}_\beta, \lambda\rangle = \langle \mathrm{M} \Box \mathrm{N}, \lambda\rangle,$$

where the limit is taken over the product directed set. This holds for each $\lambda \in X$, and so (b) follows.

(b)⇒(a). Let $\lambda \in X_{[1]}$, and consider the map

$$\rho_\lambda : \mathrm{M} \to \mathrm{M} \cdot \lambda, \quad (X'_{[1]}, \sigma(X', X)) \mapsto (X_{[1]}, \sigma(X, X')).$$

We *claim* that ρ_λ is continuous. Indeed, let (M_α) converge to M in $(X'_{[1]}, \sigma(X', X))$, and take $\mathrm{N} \in X'$. Then $\langle \mathrm{N}, \rho_\lambda(\mathrm{M}_\alpha)\rangle = \langle \mathrm{N} \,\Box\, \mathrm{M}_\alpha, \lambda\rangle \to \langle \mathrm{N} \,\Box\, \mathrm{M}, \lambda\rangle = \langle \mathrm{N}, \rho_\lambda(\mathrm{M})\rangle$, giving the claim. (At this point, we are using only the separate continuity of the product.) It follows that $\rho_\lambda(X'_{[1]})$, the weak-$*$ closure of $K(\lambda)$, is weakly compact in the space X, and hence in A'.

Let (M_α) be a net in $X'_{[1]}$. Then $(\mathrm{M}_\alpha \cdot \lambda)$ is a net in $\overline{K(\lambda)}$; by passing to a subnet, we may suppose that $\mathrm{M}_\alpha \to \mathrm{M}$ in $(X', \sigma(X', X))$ for some $\mathrm{M} \in X'_{[1]}$ and that $\mathrm{M}_\alpha \cdot \lambda \to \mathrm{M} \cdot \lambda$ in $(A, \sigma(A, A'))$.

We next *claim* that $\mathrm{M}_\alpha \cdot \lambda \to \mathrm{M} \cdot \lambda$ in $(A, \|\cdot\|)$. Assume towards a contradiction that this is not the case. Then, by passing to a subnet, we may suppose that there exists $\varepsilon > 0$ such that $\|\mathrm{M}_\alpha \cdot \lambda - \mathrm{M} \cdot \lambda\| > \varepsilon$ for each α. For each α, choose $\mathrm{N}_\alpha \in X'_{[1]}$ such that

$$|\langle \mathrm{M}_\alpha \cdot \lambda - \mathrm{M} \cdot \lambda,\, \mathrm{N}_\alpha\rangle| > \varepsilon.$$

Again by passing to a subnet, if necessary, we may suppose that the net (N_α) converges to N in $(X', \sigma(X', X))$. Now we have

$$\begin{aligned} \varepsilon &< |\langle \mathrm{M}_\alpha \cdot \lambda - \mathrm{M} \cdot \lambda,\, \mathrm{N}_\alpha\rangle| \\ &\le |\langle \mathrm{N}_\alpha \,\Box\, \mathrm{M}_\alpha, \lambda\rangle - \langle \mathrm{N} \,\Box\, \mathrm{M}, \lambda\rangle| + |\langle \mathrm{N} \,\Box\, \mathrm{M}, \lambda\rangle - \langle \mathrm{N}_\alpha \,\Box\, \mathrm{M}, \lambda\rangle|. \end{aligned}$$

But the limit of both terms on the right-hand side is 0 by (b), and so we obtain the required contradiction. Thus the claim holds.

The claim implies that $\overline{K(\lambda)}$ is compact in $(A, \|\cdot\|)$, and hence that $\lambda \in AP(A)$, giving (a). ∎

Let I be a closed ideal in a Banach algebra A, with the embedding $\iota : I \to A$. Then $\iota' : A' \to I'$ is a continuous surjection which is an A-bimodule homomorphism. Let X be a $\|\cdot\|$-closed A-submodule of A'. Then $Y := \overline{\iota'(X)}$ is a Banach A-submodule of I'. We use the above notation in the following proposition.

PROPOSITION 1.16. *Suppose that X is introverted in A'. Then Y is introverted in I', and there is a continuous A-bimodule monomorphism*

$$\tau : Y' = I''/Y^\circ \to X' = A''/X^\circ.$$

Further, $\tau : (Y', \Box) \to (X', \Box)$ is a continuous embedding identifying Y' as a closed ideal in X'.

Suppose that $\iota' : X \to Y$ is an injection. Then $\tau : Y' \to X'$ is a surjection, and so Y is introverted in I' if and only if X is introverted in A'.

Proof. To show that Y is left-introverted in I', we apply Proposition 1.14.

Let $\lambda \in \iota'(X)$, and let K_λ be the closure of $\{a \cdot \lambda : a \in I_{[1]}\}$ in the topology $\sigma(I', I)$. Let (a_α) be a net in $I_{[1]}$ such that $a_\alpha \cdot \lambda \to \mu$ in $(I', \sigma(I', I))$. Then there exist $\widetilde{\lambda} \in X$ and $\widetilde{\mu} \in A'$ such that $\iota'(\widetilde{\lambda}) = \lambda$ and $\iota'(\widetilde{\mu}) = \mu$. By passing to a subnet, we may suppose that $a_\alpha \cdot \widetilde{\lambda} \to \widetilde{\mu}$ in $(A', \sigma(A', A))$. Since $\widetilde{\lambda} \in X$ and X is left-introverted in A', it follows from Proposition 1.14 that $\widetilde{\mu} \in X$, and so $\mu \in \iota'(X)$. Thus $K_\lambda \subset Y$, and so $\iota'(X)$ is left-introverted in I', again by Proposition 1.14. Hence Y is left-introverted in I'.

Similarly, Y is right-introverted in I', and so Y is introverted in I'.

The existence of the specified map τ is clear. By Proposition 1.4, the map

$$\iota'' : (I'', \Box) \to (A'', \Box)$$

is a continuous injection, and it follows easily that $\tau : (Y', \Box) \to (X', \Box)$ is a continuous embedding.

Certainly $(I'', \Box)$ is a closed ideal in $(A'', \Box)$, and so $(Y', \Box)$ is a closed ideal in $(X', \Box)$. It is also clear that X is introverted in A' whenever Y is introverted in I' in the case where $\tau : Y' \to X'$ is a surjection. ■

We recall the standard result that every C^*-algebra A is Arens regular, and that its second dual $(A'', \Box)$ is also a C^*-algebra; for an identification of $(A'', \Box)$ using universal representations, see [13, Theorem 3.2.36]. In the present work, we wish to avoid using the representation theory of C^*-algebras, and to give direct proofs.

We have obtained the following result, using Proposition 1.14(ii).

PROPOSITION 1.17. *Let A be a C^*-algebra, and let X be a Banach A-submodule of A'. Then X is introverted, X° is a weak-$*$ closed ideal in the C^*-algebra $(A'', \Box)$, and $(X', \Box)$ is a C^*-algebra.* ■

Lau algebras. It will be seen that the main examples that we shall consider later are examples of 'Lau algebras'; we introduce these algebras here in an abstract manner.

DEFINITION 1.18. A *Lau algebra* is a pair (A, M), where:

(i) A is a Banach algebra and M is a C^*-algebra which is isometrically isomorphic to A' as a Banach space;
(ii) the identity of M is a character on A.

In this case, M is a von Neumann algebra; every von Neumann algebra has an identity. It is a standard fact [112, Corollary III.3.9] that there is a unique (as a Banach space, up to isometric isomorphism) predual M_* of each von Neumann algebra M; thus $A = M_*$ as a Banach space. Further, the product in M is separately continuous when M has the $\sigma(M, A)$-topology (see [104, Theorem 1.7.8]). Thus M is a dual Banach algebra, and A is a Banach M-submodule of A''. For $\mu \in M$, define continuous linear operators $L_\mu, R_\mu : A \to A$ by

$$\langle L_\mu a,\, \nu\rangle = \langle a,\, \mu \cdot \nu\rangle, \qquad \langle R_\mu a,\, \nu\rangle = \langle a,\, \nu \cdot \mu\rangle \qquad (a \in A,\, \nu \in M). \tag{1.5}$$

(In fact, for each $a \in A$, the elements $L_\mu a$ and $R_\mu a$ are defined as members of A'', but they are continuous on $(M, \sigma(M, A))$ because the product in M is separately $\sigma(M, A)$-continuous, and so they belong to A by Proposition 1.1(ii).)

We shall usually write A' for M and regard A itself as a Lau algebra; we shall denote the identity of A' by e. The class of Lau algebras was introduced in [70], where they were called 'F-algebras'; they were renamed as 'Lau algebras' in [92].

Examples of Lau algebras include the group algebra and the measure algebra of a locally compact group G (see Chapter 6), the Fourier algebra $A(G)$ and the Fourier–Stieltjes algebra $B(G)$ of a locally compact group G (see [30]), the measure algebra $M(S)$ of a locally compact semitopological semigroup S (see [39]), the convolution measure algebras studied by Taylor [114], the 'L-algebras' considered by McKilligan and White [83], the predual of a Hopf–von Neumann algebra [111], and the algebras $L^1(K)$ (in the case where K has a 'left Haar measure') and $M(K)$ of a locally compact hypergroup K [27, 100, 107] or of semi-convos [57].

Definition 1.19. Let A be a Lau algebra. A closed subspace X of A' is a *left-* (respectively, *right-*) *introverted* C^*-subalgebra of A' if:

(i) X is a C^*-subalgebra of A';
(ii) X is a left-introverted (respectively, right-introverted) A-submodule of A'.

The space X is an *introverted* C^**-subalgebra* if it is both left- and right-introverted.

In particular, A' itself is an introverted C^*-subalgebra.

Let A be a Lau algebra, so that A' is a C^*-algebra. We denote by $\mathcal{P}(A)$ the cone of elements of A which act as positive linear functionals on A'. The set of elements $p \in \mathcal{P}(A)$ with $\langle p,\, e\rangle = 1$ is denoted by $\mathcal{P}_1(A)$. It is shown in [70] and [92] that $\mathcal{P}_1(A)$ is a subsemigroup of $(A, \cdot)$. Note that $(A'', \Box)$ is also a Lau algebra.

Definition 1.20. Let A be a Lau algebra. Let X be a left-introverted C^*-subalgebra of A'. A *topological left-invariant mean* on X is an element $m \in \mathcal{P}_1(A'')$ such that

$$\langle m,\, x\cdot p\rangle = \langle m,\, x\rangle \quad (x \in X,\, p \in \mathcal{P}_1(A)).$$

Let X be an introverted C^*-subalgebra of A'. Then a *topological invariant mean* on X is an element $m \in \mathcal{P}_1(A'')$ such that

$$\langle m,\, x\cdot p\rangle = \langle m,\, p\cdot x\rangle = \langle m,\, x\rangle \quad (x \in X,\, p \in \mathcal{P}_1(A)).$$

The algebra A is *left-amenable* if, for each Banach A-bimodule E such that

$$a\cdot x = \langle a,\, e\rangle x \quad (a \in A,\, x \in E),$$

every bounded derivation from A into E' is inner.

The following result is [70, Theorem 4.1] and [92, Propsition 3.5].

Proposition 1.21. *Let A be a Lau algebra. Then A is left-amenable if and only if A' has a topological left-invariant mean.* ■

There is a similar definition of a right-amenable Lau algebra A and a topological right-invariant mean. Now suppose that M_1 and M_2 are topological left- and right-invariant means, respectively. Then $\mathrm{M}_1 \Box \mathrm{M}_2$ is both a topological left- and right-invariant mean on A'.

Let S be a semigroup. Then the semigroup algebra $\ell^1(S)$ has been intensively studied recently; see [17, 18], for example. Clearly $\ell^1(S)$ is a Lau algebra (where $\ell^1(S)' = \ell^\infty(S)$, with the pointwise product). It is shown in [70, Corollary 4.2] that $\ell^1(S)$ is left-amenable if and only if S is left-amenable as a semigroup. However $\ell^1(S)$ need not be amenable even when S is abelian. For example, a necessary condition for this is that $S^2 = S$: this follows from [13, Theorem 2.8.63] because $\ell^1(S)$ is not essential whenever $S^2 \subsetneq S$, in the terminology of the reference. In particular, $(\ell^1(\mathbb{N}), \star)$ is not weakly amenable.

For further studies of Lau algebras, see [71, 81, 84, 92].

Summary. In Chapter 2, we shall give further background involving topological spaces, continuous functions, and measures. In particular, we shall define in Definition 2.6 the class of hyper-Stonean spaces, and we shall characterize these spaces in Theorem 2.9 and Proposition 2.17.

In Chapter 3, we shall first discuss the second dual algebra of the commutative C^*-algebra $C_0(\Omega)$, which is the algebra of all continuous functions that vanish at infinity on a locally compact space Ω. This second dual space has the form $C(\widetilde{\Omega})$ for a certain hyper-Stonean space $\widetilde{\Omega}$, called the hyper-Stonean envelope of Ω in Definition 3.2. The second dual space of $M(\Omega)$, the Banach space of all complex-valued, regular Borel measures on Ω, is identified with $M(\widetilde{\Omega})$. We shall also discuss $B^b(\Omega)$, the C^*-algebra of all bounded Borel functions on Ω; we shall regard $B^b(\Omega)$ as a C^*-subalgebra of $C(\widetilde{\Omega})$.

Let Ω be a non-empty, locally compact space. In Chapter 4, we shall discuss subspaces of $M(\Omega)$ which are modules over the algebra $C_0(\Omega)$. We shall also discuss further the hyper-Stonean space $\widetilde{\Omega}$, and explain that we cannot, in general, recover Ω from $\widetilde{\Omega}$.

A particularly important case for us is that in which Ω is an uncountable, compact, and metrizable space (such as $\Omega = \mathbb{I}$). Indeed, it will be shown in Theorem 4.16 that there is a unique hyper-Stonean space X which is the hyper-Stonean envelope of each such space; we shall give a topological characterization of this space X. We shall calculate the cardinalities of various subsets of X in this case. We shall also discuss the character space Φ_b of $B^b(\Omega)$, and we shall calculate the cardinalities of various subsets of $\widetilde{\Omega}$ which are defined in terms of the algebra $B^b(\Omega)$.

In Chapter 5, we shall recall the definitions and some basic properties of the measure algebra $M(G)$ and the group algebra $L^1(G)$ of a locally compact group G, and develop the properties of the hyper-Stonean envelope $\widetilde{G}$ of G. We shall also consider some introverted subspaces of dual spaces; these will include $LUC(G)$ and the spaces $AP(G)$ and $WAP(G)$ of almost periodic and weakly almost periodic functions on G; we shall discuss the relation of these spaces to the more mysterious C^*-algebras $AP(M(G))$ and $WAP(M(G))$. We shall also discuss Taylor's structure semigroup of a locally compact abelian group and the more abstract notion of the structure semigroup of the Lau algebras that were introduced in Chapter 1.

We shall continue in Chapter 6 with the proofs of some formulae that will be required later for products in the Banach algebra $(M(\widetilde{G}), \Box)$. Our proofs will frequently use the fact that points of $\widetilde{G}$ can be identified with certain ultrafilters.

The main theorem of Chapter 7 is Theorem 7.9, which shows that we can recover a locally compact group G from knowledge of the hyper-Stonean envelope $\widetilde{G}$; this answers a question raised in [34]. The special case where G is compact was resolved earlier by Ghahramani and McClure in [35].

Let G be a locally compact group. In Chapter 8, we shall investigate whether or not $(\widetilde{G}, \Box)$ is a semigroup. Indeed, we shall prove in Theorem 8.16 that $(\widetilde{G}, \Box)$ is a semigroup only in the special case where G is discrete. In the case where G is not discrete, we shall study in considerable detail the products of two point masses in $(M(\widetilde{G}), \Box)$, showing that this product must be a point mass in certain cases and that there are always two points in $\widetilde{G}$ such that their product is a continuous measure. In many groups G, including the circle group $(\mathbb{T}, \cdot)$, the space $\widetilde{G}$ contains two point masses whose product is neither discrete nor continuous. As important special groups we shall consider $\mathbb{T}$ and the groups D_p.

In the final chapter, Chapter 9, we shall consider the topological centres of $L^1(G)''$ and $M(G)''$ in the case where G is a non-discrete, locally compact group, concentrating on the case where G is compact. We shall essentially show in Corollary 9.5 that the spectrum Φ of $L^\infty(G)$ is determining for the left topological centre of $L^1(G)''$; this gives a strong form of the known result that $L^1(G)$ is always strongly Arens irregular. We do not know which subsets of Φ are determining for the left topological centre of $L^1(G)''$.

In Chapter 9, we shall also attack the question of whether or not the measure algebra $M(G)$ is always strongly Arens irregular; this question was raised by Lau in [72] and Ghahramani and Lau in [34]. Unfortunately we are not able to resolve this point, but we do give some partial results. [Added in proof: an announcement in May, 2009, by V. Losert, M. Neufang, J. Pachl, and J. Steprāns states that $M(G)$ is strongly Arens irregular for each locally compact group; see [82].]

Our memoir concludes with a list of problems that we believe to be both open and interesting.

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2. Locally compact spaces, continuous functions, and measures

Locally compact spaces. Let (X, τ) be a topological space. The interior, closure, and frontier (or boundary) of a subset S of X are denoted by $\operatorname{int} S$, by $\overline{S}$ or $\overline{S}^{\tau}$, and by ∂S or $\partial_X S$, respectively; the family of open neighbourhoods of a point $x \in X$ is denoted by $\mathcal{N}_x$. A G_δ-set is a countable intersection of open sets. The space X with the discrete topology is denoted by X_d. A subset S of X is *meagre* if $S = \bigcup S_n$, where $\operatorname{int} \overline{S_n} = \emptyset$ $(n \in \mathbb{N})$. A Hausdorff topological space X is *extremely disconnected* if the closure of every open set is itself open; this is equivalent to requiring that $\overline{U} \cap \overline{V} = \emptyset$ for every pair $\{U, V\}$ of open sets with $U \cap V = \emptyset$. A topological space is *second countable* if its topology has a countable base. A locally compact, second countable topological space is metrizable. The *weight*, $w(X)$, of X is the minimum cardinal of a base for the topology. Clearly $|X| \leq 2^{w(X)}$ whenever X is Hausdorff.

Let (Ω, τ) be a non-empty, locally compact space. (Our convention is that each locally compact space is Hausdorff, and that a hypothesised compact space is non-empty.) The one-point compactification of Ω is $\Omega_\infty = \Omega \cup \{\infty\}$ (and $\Omega_\infty = \Omega$ when Ω is compact). Further, $\beta\Omega$ is the Stone–Čech compactification of Ω and $\Omega^* = \beta\Omega \setminus \Omega$ is the growth of Ω [17, 37, 52, 117]. In particular, $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$. Compact, extremely disconnected topological spaces are also called *Stonean* spaces. In particular, each non-empty, open subset of a Stonean space contains a non-empty, clopen subset.

For example, a compact space X is Stonean if and only if it is a retract of a space βD for some discrete space D. We shall use Gleason's theorem [38] (see [3, Theorems 7.4, 7.14], [106, Theorem 25.5.1], or [117, §10.51]) that a compact space X is extremely disconnected if and only if it is *projective*, in the sense that it is projective in the category of compact spaces. We shall also use the following standard fact: for each dense subspace U of a Stonean space Ω, each bounded, continuous function on U can be extended to a continuous function on Ω, and so $\beta U = \Omega$ (see [24] and [112, Corollary III.1.8]).

For substantial accounts of Stone–Čech and other compactifications of topological spaces and semigroups, see [52, 117].

A topological space is an *F-space* if, for each real-valued, continuous function f on X, the sets $\{x \in X : f(x) > 0\}$ and $\{x \in X : f(x) < 0\}$ have disjoint closures. Thus every extremely disconnected space is an F-space. For characterizations of F-spaces, see [13, Proposition 4.2.18(ii)] and [37, §14.25]. By [37, 14N(5)], every infinite, compact F-space contains a homeomorphic copy of $\beta\mathbb{N}$.

For the following basic result, see [52, Theorem 3.58] and [117, Proposition 3.21].

PROPOSITION 2.1. *Let D be an infinite, discrete space with $|D| = \kappa$. Then $|\beta D| = 2^{2^\kappa}$. In particular, $|\Omega| \geq |\beta\mathbb{N}| = |\mathbb{N}^*| = 2^{\mathfrak{c}}$ for each infinite Stonean space Ω. Further, we have $w(\beta\mathbb{N}) = w(\mathbb{N}^*) = \mathfrak{c}$.* ∎

Let X be a topological space. Then $\mathfrak{I}_X$ denotes the family of subsets of X which are both compact and open, so that $\mathfrak{I}_X$ is a family of subsets of X which is closed under finite unions and intersections; in the case where Ω is a compact space, $\mathfrak{I}_\Omega$ is the family of clopen sets. A compact space Ω satisfies CCC, the *countable chain condition*, if each pairwise disjoint family of non-empty, open subsets in $\mathfrak{I}_\Omega$ is countable.

We now recall the definition of certain specific compact topological spaces that will be used later.

Let $p \in \mathbb{N}$ with $p \geq 2$. We recall that $\mathbb{Z}_p = \{0, \dots, p-1\}$, taken with the discrete topology. Let κ be an infinite cardinal. Then the *Cantor cube of weight* κ is the product space $\mathbb{Z}_p^\kappa$ (with the product topology). The space $\mathbb{Z}_p^\kappa$ is compact, totally disconnected, and perfect. In particular, we set

$$D_p = \mathbb{Z}_p^{\aleph_0},$$

so that D_p is a metrizable space with $|D_p| = \mathfrak{c}$. Every compact, totally disconnected, perfect, metrizable space is homeomorphic to D_2. For each $k \in \mathbb{N}$, take $\tau_1, \dots, \tau_k < \kappa$ with $\tau_1 < \tau_2 < \cdots < \tau_k$, and then set $F = \{\tau_1, \dots, \tau_k\}$, so that $|F| = k$. Now take $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{Z}_p^k$, and define

$$U_{F,\alpha} = \{(\varepsilon_\tau) \in \mathbb{Z}_p^\kappa : \varepsilon_{\tau_i} = \alpha_i \ (i \in \mathbb{N}_k)\}, \tag{2.1}$$

so that $U_{F,\alpha}$ is a clopen subset of $\mathbb{Z}_p^\kappa$; the sets $U_{F,\alpha}$ are called the *basic clopen subsets* of $\mathbb{Z}_p^\kappa$. These sets form a base of cardinality κ for the topology of $\mathbb{Z}_p^\kappa$, and so $w(\mathbb{Z}_p^\kappa) = \kappa$; also each clopen set is a finite, pairwise disjoint union of these basic clopen sets. Thus we have

$$|\mathbb{Z}_p^\kappa| = 2^\kappa, \qquad |\mathfrak{I}_{\mathbb{Z}_p^\kappa}| = w(\mathbb{Z}_p^\kappa) = \kappa. \tag{2.2}$$

Each $x \in \mathbb{I}$ has a ternary expansion as

$$x = \sum_{n=1}^\infty \frac{\varepsilon_n(x)}{3^n},$$

where $\varepsilon_n(x) \in \mathbb{Z}_3^+$. (We agree to resolve ambiguity by requiring that no expansion is equal to 2 eventually; since the points with an ambiguous expansion form a countable set, and we shall be considering continuous measures on $\mathbb{I}$ when this expansion is relevant, the ambiguous points will, in any case, have measure 0.) The space D_2 is homeomorphic to the Cantor subset K of $\mathbb{R}$ by the map

$$(\varepsilon_n) \mapsto 2\sum_{n=1}^\infty \frac{\varepsilon_n}{3^n}, \qquad D_2 \to K. \tag{2.3}$$

Borel sets. The σ-algebra generated by a family $\mathcal{S}_0$ of subsets of a set S is denoted by $\sigma(\mathcal{S}_0)$; it can be represented as

$$\bigcup\{\mathcal{S}_\alpha : \alpha < \omega_1\},$$

where $\mathcal{S}_1$ consists of the complements of the sets in $\mathcal{S}_0$ and $\mathcal{S}_\alpha$ consists of all countable unions of sets in $\bigcup\{\mathcal{S}_\beta : \beta < \alpha\}$ for odd ordinals $\alpha > 1$ and of all countable intersections of sets in this family for even ordinals $\alpha > 0$. Hence $|\sigma(\mathcal{S}_0)| \leq 2^{|\mathcal{S}_0|}$; in the case where $|\mathcal{S}_0| \geq \mathfrak{c}$, we have $|\sigma(\mathcal{S}_0)| = |\mathcal{S}_0|$.

Let (X, τ) be a Hausdorff topological space. Then $\mathfrak{B}_X$ is the family of all Borel subsets of X, so that $\mathfrak{B}_X$ is the σ-algebra generated by τ. Certainly $\mathfrak{I}_X \subset \mathfrak{B}_X$. We record the following well-known facts about the σ-algebra $\mathfrak{B}_X$.

Let X_2 be a subspace (with the relative topology) of a Hausdorff space X_1. Then, by [11, Lemma 7.2.2], we have

$$\mathfrak{B}_{X_2} = \{B \cap X_2 : B \in \mathfrak{B}_{X_1}\}.$$

Let X_1 and X_2 be Hausdorff topological spaces. A map $\eta : X_1 \to X_2$ is a *Borel map* if

$$\eta^{-1}(U) \in \mathfrak{B}_{X_1} \quad (U \in \mathfrak{B}_{X_2}).$$

Let B_1 and B_2 be Borel subsets of X_1 and X_2, respectively. Then a *Borel isomorphism* from B_1 to B_2 is a bijection $\eta : B_1 \to B_2$ such that both η and η^{-1} are Borel maps; B_1 and B_2 are *Borel isomorphic* if there exists such a Borel isomorphism. By [11, Lemma 7.2.1], each continuous map $\eta : X_1 \to X_2$ is a Borel map.

PROPOSITION 2.2.

(i) *Let Ω be an uncountable, compact, metrizable space. Then, for each uncountable set $B \in \mathfrak{B}_\Omega$, we have $w(\Omega) = \aleph_0$, and $|\Omega| = |B| = \mathfrak{c}$.*

(ii) *Let B_1 and B_2 be Borel subsets of two compact, metrizable spaces with $|B_1| = |B_2|$. Then B_1 and B_2 are Borel isomorphic.*

(iii) *Let B be an uncountable Borel subset of a compact, metrizable space. Then B contains $\mathfrak{c}$ pairwise disjoint sets, each homeomorphic to D_2. In particular, B contains an uncountable, compact space.*

(iv) *Let Ω be an uncountable, compact, metrizable space. Then $|\mathfrak{B}_\Omega| = \mathfrak{c}$.*

Proof. (i) & (ii) Each compact, metrizable space is complete [29, Theorem 4.3.28] and separable [29, Theorem 4.1.18], and so is a Polish space. A metrizable space has a countable base (i.e., is second countable) if and only if it is separable [29, Theorem 4.1.16]. Clauses (i) and (ii) now follow from [11, Theorem 8.3.6].

(iii) By [11, Corollary 8.2.14], B contains a subset that is homeomorphic to the set D_2, and so it suffices to prove the result for the space D_2 itself. Clearly there is a continuous bijection $\theta : D_2 \to D_2 \times D_2$. For each $\alpha \in D_2$, we set $F_\alpha = \theta^{-1}(\{\alpha\} \times D_2)$, so that F_α is a compact subset of D_2 homeomorphic to D_2. The family $\{F_\alpha : \alpha \in D_2\}$ is pairwise disjoint, and so has the required properties.

(iv) By (iii), $|\mathfrak{B}_\Omega| \geq \mathfrak{c}$. By (i), $w(\Omega) = \aleph_0$. Since each open set is a countable union of basic open sets, $\mathfrak{B}_\Omega$ is the σ-algebra generated by the basic open sets, and hence $|\mathfrak{B}_\Omega| \leq 2^{\aleph_0} = \mathfrak{c}$. Hence $|\mathfrak{B}_\Omega| = \mathfrak{c}$. ■

Clause (ii), above, is a form of the *Borel isomorphism theorem*. For example, it follows from (ii) that D_2 and $\mathbb{T}$ are Borel isomorphic.

Continuous functions. Let Ω be a non-empty, locally compact space. Then $C^b(\Omega)$ denotes the space of bounded, continuous, complex-valued functions on Ω, and $C_0(\Omega)$ denotes the subspace of all functions in $C^b(\Omega)$ which vanish at infinity, so that $C^b(\Omega)$ and $C_0(\Omega)$ are commutative C^*-algebras for the pointwise product of functions and the uniform norm $|\cdot|_\Omega$ on Ω; the latter norm is defined by

$$|f|_\Omega = \sup\{|f(x)| : x \in \Omega\} \quad (f \in C^b(\Omega))$$

(see [13, 17] for details).

Of course, $C^b(\Omega)$ is isometrically isomorphic as a C^*-algebra to $C(\beta\Omega)$ (see [37]), and we shall identify these spaces. In particular,

$$\ell^\infty(\Omega) \cong C(\beta\Omega_d).$$

We shall often set $E = C_0(\Omega)$. The space of real-valued functions in E is $E_{\mathbb{R}} = C_0(\Omega)_{\mathbb{R}}$. We shall use the natural ordering on $E_{\mathbb{R}}$: for $\lambda \in E_{\mathbb{R}}$, we have

$$\lambda \geq 0 \quad \text{if} \quad \lambda(x) \geq 0 \quad (x \in \Omega);$$

the positive cone of E is denoted by $E^+ = C_0(\Omega)^+$. Then $(E_{\mathbb{R}}, \leq)$ is a Banach lattice in a standard sense. Further, E itself is a (complex) Banach lattice. We recall that a Banach lattice such as $(E_{\mathbb{R}}, \leq)$ is *Dedekind complete* if every subset which is bounded above has a supremum.

For early discussions of the Banach space $C_0(\Omega)$, see [3, 106]; for background on Banach lattices, with particular reference to the Banach lattice $C_0(\Omega)_{\mathbb{R}}$, see [60, §3.4] and [65]. The Banach algebra $C_0(\Omega)$ is discussed at some length in [13, §4.2].

PROPOSITION 2.3. *Let Ω be a compact space.*

(i) *The space $(C(\Omega), |\cdot|_\Omega)$ is separable if and only if Ω is metrizable.*
(ii) *Suppose that Ω is metrizable and infinite. Then $|C(\Omega)| = \mathfrak{c}$.*

Proof. (i) is [1, Theorem 4.1.3], for example. For (ii), the space Ω is separable, and so $|C(\Omega)| = \mathfrak{c}^{\aleph_0} = \mathfrak{c}$. ■

In the case where Ω is compact, 1_Ω is the identity of $C(\Omega)$; in the general locally compact case, $C_0(\Omega)$ has a bounded approximate identity. The idempotents of $C_0(\Omega)$ are the characteristic functions of the sets in $\mathfrak{I}_\Omega$, and we regard $\mathfrak{I}_\Omega$ as a subset of $C_0(\Omega)$.

Let Ω be a non-empty, locally compact space. The *evaluation functional* on $E = C_0(\Omega)$ at a point $x \in \Omega$ is

$$\varepsilon_x : \lambda \mapsto \lambda(x), \quad E \to \mathbb{C},$$

so that $\varepsilon_x \in E'$; we also identify ε_∞ with the zero linear functional on E. We can regard Ω_∞ as a subset of E' by identifying $x \in \Omega_\infty$ with $\varepsilon_x \in E'$.

We now recall some well-known and standard facts about continuous mappings between compact spaces and algebras of continuous functions.

Let Ω_1 and Ω_2 be two compact spaces. First, let $\eta : \Omega_1 \to \Omega_2$ be a continuous map, and define

$$\eta^\circ : \lambda \mapsto \lambda \circ \eta, \quad C(\Omega_2) \to C(\Omega_1). \tag{2.4}$$

Then η° is a continuous $*$-homomorphism with $\|\eta^\circ\| = 1$. Further, η° is an injection/a surjection if and only if η is a surjection/an injection, respectively.

Conversely, let $\theta : C(\Omega_2) \to C(\Omega_1)$ be a $*$-homomorphism. Then θ is continuous with $\|\theta\| = 1$, and there exists a continuous map $\eta : \Omega_1 \to \Omega_2$ with $\theta = \eta^\circ$; indeed, we have $\eta = \theta' \,|\, \Omega_1$.

We shall use the standard Banach–Stone theorem; see [1, Theorem 4.1.5] and [12, Theorem VI.2.1], for example. For clause (ii) below, see also [60, Corollary 3.4.8].

Theorem 2.4. *Let Ω_1 and Ω_2 be two compact spaces.*

(i) *Suppose that $T : C(\Omega_1) \to C(\Omega_2)$ is an isometric linear isomorphism. Then there are a homeomorphism $\eta : \Omega_2 \to \Omega_1$ and $\theta \in C(\Omega_2)$ such that $|\theta(y)| = 1$ $(y \in \Omega_2)$ and*

$$(T\lambda)(y) = \theta(y)(\lambda \circ \eta)(y) \quad (y \in \Omega_2,\ \lambda \in C(\Omega_1)).$$

(ii) *Suppose that $T : C(\Omega_1) \to C(\Omega_2)$ is an isometric linear isomorphism such that $T(1) = 1$. Then T is an isomorphism of C^*-algebras.*

(iii) *The commutative C^*-algebras $C(\Omega_1)$ and $C(\Omega_2)$ are isomorphic as C^*-algebras if and only if Ω_1 and Ω_2 are homeomorphic as topological spaces.* ■

Let Ω be a compact space, and let A be a uniformly closed subalgebra of $C(\Omega)$ such that A contains the constant function 1_Ω and such that, for each $x \in \Omega$, there exists $\lambda \in A$ with $\lambda(x) \neq 0$. We say that A *separates the points of* Ω if, for each $x, y \in \Omega$ with $x \neq y$, there exists $\lambda \in A$ with $\lambda(x) \neq \lambda(y)$. For $x, y \in \Omega$, set $x \sim_A y$ or $x \sim y$ if $\lambda(x) = \lambda(y)$ $(\lambda \in A)$, so that $\sim_A$ is an equivalence relation on Ω, and set

$$[x] = \{y \in \Omega : y \sim_A x\} \quad (x \in \Omega).$$

Then $\{[x] : x \in \Omega\}$ is a partition of Ω into closed subsets; we may identify the character space of A with the compact space $\Omega/\!\sim_A$ which is the quotient space of Ω by the relation $\sim_A$, and then identify A with $C(\Omega/\!\sim_A)$.

Let F be a closed subspace of Ω. Then we remark that

$$[F] := \bigcup\{[x] : x \in \Omega\}$$

is closed in Ω. For let (x_α) be a net in $[F]$ such that $x_\alpha \to x_0$ in Ω. For each α, there exists $y_\alpha \in F$ with $y_\alpha \sim_A x_\alpha$. By passing to a subnet, we may suppose that (y_α) converges to y_0 in F. Clearly $x_0 \sim_A y_0$, and so $x_0 \in [F]$. Thus $[F]$ is closed.

There are many statements that are equivalent to the fact that a compact space is extremely disconnected; we collect some of these in the following theorem.

Theorem 2.5. *Let Ω be a compact space. Then the following statements about Ω are equivalent:*

(a) *Ω is extremely disconnected, and so Ω is Stonean;*

(b) *Ω is projective in the category of compact spaces;*

(c) *the Banach lattice $C(\Omega)_{\mathbb{R}}$ is Dedekind complete;*

(d) *$C(\Omega)$ is injective in the category of commutative C^*-algebras and continuous $*$-homomorphisms;*

(e) *$C(\Omega)$ is injective in the category of Banach spaces and contractive linear maps.*

Proof. The equivalence of (a) and (b) is Gleason's theorem [38], and the equivalence with (d) is also in [38]. The equivalence of (a) and (c) is given in [13, Proposition 4.2.29] and [112, Proposition III.1.7]), and the equivalence of (a), (c), and (e) is given in [1, §4.3].

For a short and attractive direct exposition of all these equivalences, see [43, Theorem 2.4]. ∎

DEFINITION 2.6. Let Ω be a compact space. Then Ω is *hyper-Stonean* if $C(\Omega)$ is isometrically isomorphic to the dual space of another Banach space.

Thus Ω is hyper-Stonean if $C(\Omega)$ is a von Neumann algebra [13, Definition 3.2.35]. A Banach space F such that $F' = C(\Omega)$ is a *predual* of $C(\Omega)$. In this case, the predual of $C(\Omega)$ is unique and is denoted by $C(\Omega)_*$; this space defines the canonical weak-$*$ topology $\sigma(C(\Omega), C(\Omega)_*)$ on $C(\Omega)$. We shall identify this predual shortly.

By [13, Proposition 4.2.29(ii)], a hyper-Stonean space is Stonean. The seminal work on hyper-Stonean spaces is the classical paper of Dixmier [24].

For example, $C(\beta\mathbb{N}) = \ell^\infty$ is isometrically the dual of ℓ^1, and so $\beta\mathbb{N}$ is a hyper-Stonean space. Note that the closed subspace $\mathbb{N}^*$ of $\beta\mathbb{N}$ is not extremely disconnected [37, Exercise 6W], and so the compact space $\mathbb{N}^*$ is not Stonean.

Measures. Let Ω be a non-empty, locally compact space. We shall consider 'measures' on Ω; these are the complex-valued, regular Borel measures defined on the σ-algebra $\mathfrak{B}_\Omega$, and they form the Banach space $M(\Omega)$ in a standard way, so that

$$\|\mu\| = |\mu|(\Omega) \quad (\mu \in M(\Omega)).$$

The sets of real-valued and positive measures in $M(\Omega)$ are denoted by $M(\Omega)_\mathbb{R}$ and $M(\Omega)^+$, respectively. A measure μ in $M(\Omega)^+$ with $\|\mu\| = 1$ is a *probability measure*; the collection of probability measures on Ω is denoted by $P(\Omega)$, so that $P(\Omega)$ is the *state space* of the C^*-algebra $C_0(\Omega)$.

The *support* of a measure μ on Ω is denoted by $\operatorname{supp}\mu$.

Let Ω be a non-empty, locally compact space, and let $\mu, \nu \in M(\Omega)$. Then we write $\mu \ll \nu$ if μ is absolutely continuous with respect to $|\nu|$, and $\mu \perp \nu$ if μ and ν are mutually singular. We recall that $\mu \perp \nu$ if and only if

$$\|\mu + \nu\| = \|\mu - \nu\| = \|\mu\| + \|\nu\|. \tag{2.5}$$

The dual space of $E = C_0(\Omega)$ is E', and this space is identified with $M(\Omega)$; the duality is

$$\langle \lambda,\, \mu\rangle = \int_\Omega \lambda \,\mathrm{d}\mu \quad (\lambda \in C_0(\Omega),\, \mu \in M(\Omega)).$$

Certainly $M(\Omega)$ is a Banach E-module. The dual module action $\lambda \cdot \mu$ of $\lambda \in E$ on $\mu \in E'$ is just the usual product $\lambda\mu$; in particular, when Ω is compact, $1_\Omega \cdot \mu = \mu$. The space $M(\Omega)_\mathbb{R}$ is again a Banach lattice in an obvious way; it is the dual lattice to $(E_\mathbb{R}, \leq)$. Again we regard $M(\Omega)$ as a (complex) Banach lattice.

The subspaces of $M(\Omega)$ consisting of the *discrete* and *continuous* measures are $M_d(\Omega)$ and $M_c(\Omega)$, respectively. Let $\mu \in M(\Omega)$. Then the discrete and continuous parts of μ are denoted by μ_d and μ_c, respectively; we have $\mu = \mu_d + \mu_c$ with $\|\mu\| = \|\mu_d\| + \|\mu_c\|$, and thus we have a decomposition of Banach spaces

$$E' = M(\Omega) = M_d(\Omega) \oplus_1 M_c(\Omega).$$

We shall identify $M_d(\Omega)$ with $\ell^1(\Omega)$. In the case where Ω is an uncountable, compact, metrizable space, $M_c(\Omega) \neq \{0\}$. (In fact, we have $M(\Omega) = M_d(\Omega)$ if and only if the topological space Ω is scattered, in the sense that each non-empty subset A of Ω contains a point that is isolated in A [76].)

We have

$$E'' = M(\Omega)' \cong C(\beta\Omega_d) \oplus_1 M_c(\Omega)'.$$

In particular, there is an embedding

$$j_d : \ell^\infty(\Omega) \to C(\beta\Omega_d) = M(\Omega_d)'. \tag{2.6}$$

Particular measures in $P(\Omega) \cap M_d(\Omega)$ are the point masses δ_x, defined for $x \in \Omega$; we shall sometimes regard Ω as a subset of $P(\Omega)$ by identifying $x \in \Omega$ with δ_x. In the above identification of E' with $M(\Omega)$, we are identifying ε_x with δ_x for each $x \in \Omega$. It is easy to see that the extreme points of the unit ball $M(\Omega)_{[1]}$ are those measures of the form $\zeta\delta_x$, where $\zeta \in \mathbb{T}$ and $x \in \Omega$, and so we can identify Ω with $\operatorname{ex} P(\Omega)$.

Let Ω be a non-empty, locally compact space, and let $\mu \in M(\Omega)$ and $B \in \mathfrak{B}_\Omega$. Then

$$(\mu\,|\,B)(C) = \mu(B \cap C) \qquad (C \in \mathfrak{B}_\Omega),$$

so that $\mu\,|\,B \in M(\Omega)$; if $\mu \in M(\Omega)^+$ and $\mu(B) \neq 0$, then we set

$$\mu_B = \frac{\mu\,|\,B}{\mu(B)}, \tag{2.7}$$

so that $\mu_B \in M(\Omega)_{[1]}$.

We shall require the following well-known lemma.

Lemma 2.7. *Let Ω be a locally compact space, let Q be a countable, dense subset of Ω, and let $\mu \in M_c(\Omega)^+$. Then Ω contains a dense G_δ-subset D such that $Q \subset D$ and $\mu(D) = 0$.*

Proof. Set $Q = \{x_n : n \in \mathbb{N}\}$. Since μ is continuous, it follows that, for each $k, n \in \mathbb{N}$, there is an open neighbourhood $U_{k,n}$ of x_n such that $\mu(U_{k,n}) < 1/2^n k$. Set

$$U_k = \bigcup\{U_{k,n} : n \in \mathbb{N}\} \qquad (k \in \mathbb{N}).$$

Then each U_k is an open subset of Ω with $\mu(U_k) < 1/k$. The set $D := \bigcap U_k$ is a G_δ-subset of Ω; it is dense because it contains $\{x_n : n \in \mathbb{N}\}$, and clearly $\mu(D) = 0$. ∎

The following concept originates in [24]; see also [3] and [112, Definition III.1.10], for example.

Definition 2.8. Let Ω be a non-empty, locally compact space. A measure $\mu \in M(\Omega)$ is *normal* if μ is order-continuous, in the sense that $\langle f_\alpha,\, \mu\rangle \to 0$ for each decreasing net $(f_\alpha : \alpha \in A)$ in $(C(\Omega)_\mathbb{R}, \leq)$ such that the infimum (in $C(\Omega)_\mathbb{R}$) of the family $\{f_\alpha : \alpha \in A\}$ is 0.

The set of normal measures on Ω is denoted by $N(\Omega)$; it is easy to see that a measure $\mu \in M(\Omega)$ is normal if and only if $|\mu|$ is normal [3, Lemma 8.3] and that $N(\Omega)$ is a closed linear subspace of $M(\Omega)$ [3, Theorem 8.8]. In the case where Ω is Stonean, the support of a normal measure is a clopen subspace of Ω [3, Theorem 8.6].

We now record the following theorem, taken from [3, Theorem 8.19], [24], and [112, Definition III.1.14 and Theorem III.1.18]; it shows that several different definitions of 'hyper-Stonean' in the literature are equivalent.

THEOREM 2.9. *Let Ω be a Stonean space. Then the following are equivalent:*

(a) *Ω is hyper-Stonean;*
(b) *for each $\lambda \in C(\Omega)^+$ with $\lambda \neq 0$, there exists $\mu \in N(\Omega)^+$ with $\langle \lambda, \mu \rangle \neq 0$;*
(c) *the union of the supports of the normal measures is dense in Ω;*
(d) *there is a locally compact space Γ and a positive measure ν on Γ such that $C(\Omega)$ is C^*-isomorphic to $L^\infty(\Gamma, \nu)$.* ■

It is clear from the above that a clopen subspace of a hyper-Stonean space is hyper-Stonean.

We now characterize normal measures on Ω.

DEFINITION 2.10. Let Ω be a non-empty, locally compact space. Then $\mathcal{K}_\Omega$ denotes the family of compact subsets K of Ω for which $\operatorname{int} K = \emptyset$.

The next result was essentially proved by Dixmier in the seminal paper [24, Proposition 1, §2]. The equivalence of (a) and (b) is [112, Proposition III.1.11].

THEOREM 2.11. *Let Ω be a Stonean space, and let $\mu \in M(\Omega)^+$. Then the following conditions on μ are equivalent:*

(a) *μ is a normal measure on Ω;*
(b) *$\mu(K) = 0 \ (K \in \mathcal{K}_\Omega)$.*

In the case where Ω is hyper-Stonean, the conditions are also equivalent to:

(c) *$\mu \in C(\Omega)_*$.* ■

Thus the unique predual $C(\Omega)_*$ of $C(\Omega)$ is $N(\Omega)$. It follows that a measure $\mu \in M_{\mathbb{R}}(\Omega)$ is order-continuous on $C_{\mathbb{R}}(\Omega)$ if and only if it is weak-$*$ continuous.

Note that our theorem implies that the restriction of a measure in $N(\Omega)^+$ to a Borel subset of Ω also belongs to $N(\Omega)^+$.

In fact, a more general result is well-known. Indeed, by [60, Definition 7.1.11], a state μ on a von Neumann algebra R is *normal* if it is order-continuous, in the sense that $\mu(a_\alpha) \to \mu(a)$ for each increasing net (a_α) in R with least upper bound a; by [60, Theorem 7.1.12], a state on R is normal if and only if it is weak-operator continuous on $R_{[1]}$ (and several other equivalences are given in this reference); by [60, Theorem 7.4.2], the weak-$*$ topology on $R_{[1]}$ coincides with the weak-operator topology on $R_{[1]}$, and the predual R_* of R is just the space of normal states. Thus clauses (a) and (c) in the above theorem are equivalent in a wider context.

DEFINITION 2.12. Let $\mathcal{F}$ be a family of positive measures on a non-empty, locally compact space Ω. Then $\mathcal{F}$ is *singular* if any two distinct measures in $\mathcal{F}$ are mutually singular.

The collection of such singular families on Ω is ordered by inclusion. It is clear from Zorn's lemma that the collection has a maximal member that contains any specific singular family; this is a *maximal singular family*. We may suppose that such a maximal singular family contains all the measures that are point masses and that all other members are continuous measures, so that, in the case where Ω is discrete, a maximal singular family consists just of the point masses. We shall also refer to a *maximal singular family of continuous measures* in an obvious sense.

We shall see in Proposition 4.10, below, that any two such maximal singular families of continuous measures have the same cardinality.

PROPOSITION 2.13. *Let Ω be an uncountable, compact, metrizable space. Then*

$$|M(\Omega)| = \mathfrak{c}.$$

Further, there is a maximal singular family of measures in $M(\Omega)^+$ that consists of exactly $\mathfrak{c}$ point masses and $\mathfrak{c}$ continuous measures.

Proof. By Proposition 2.2(i), the topology of Ω has a countable base, say $\mathcal{B}$; we may suppose that this base is closed under finite unions. Each open set in Ω is a countable, increasing union of members of $\mathcal{B}$, and so each $\mu \in M(\Omega)$ is determined by its values on $\mathcal{B}$. Hence $|M(\Omega)| \leq \mathfrak{c}$.

Let $\{F_\alpha : \alpha \in D_2\}$ be a family of pairwise disjoint subsets of Ω, with each set F_α homeomorphic to D_2; such a family is constructed in Proposition 2.2(iii). For each α, there is a continuous measure μ_α with $\operatorname{supp} \mu_\alpha = F_\alpha$. Let $\mathcal{F}_0$ be the family consisting of all the point masses and all the measures μ_α, so that $\mathcal{F}_0$ is a singular family of measures, and let $\mathcal{F}$ be a maximal singular family containing $\mathcal{F}_0$. By Proposition 2.2(i), $|\Omega| = \mathfrak{c}$, and so $\mathcal{F}$ contains exactly $\mathfrak{c}$ point masses. Since $\mathcal{F}$ contains each measure μ_α, $\mathcal{F}$ contains at least $\mathfrak{c}$ continuous measures, and so $|M(\Omega)| \geq \mathfrak{c}$. Since $|M(\Omega)| \leq \mathfrak{c}$, the family $\mathcal{F}$ contains at most $\mathfrak{c}$ continuous measures. ■

Again let Ω be a non-empty, locally compact space, and let μ be a fixed continuous positive measure on Ω (so that it is not necessarily the case that $\mu \in M(\Omega)$ because we allow the possibility that $\mu(\Omega) = \infty$). Then $M_{ac}(\Omega, \mu)$ and $M_s(\Omega, \mu)$ denote the subspaces of $M(\Omega)$ consisting of measures which are *absolutely continuous* and *singular* (and non-discrete) with respect to μ, respectively, and we have an ℓ^1-Banach space decomposition

$$M(\Omega) = \ell^1(\Omega) \oplus_1 M_{ac}(\Omega, \mu) \oplus_1 M_s(\Omega, \mu).$$

In the case where the measure μ is σ-finite or is the left Haar measure on a locally compact group, we may identify $M_{ac}(\Omega, \mu)$ with $L^1(\Omega, \mu)$ via the Radon–Nikodým theorem, and so, in the case where μ is continuous, we have

$$M(\Omega) = \ell^1(\Omega) \oplus_1 L^1(\Omega, \mu) \oplus_1 M_s(\Omega, \mu). \tag{2.8}$$

Let $\mu \in M(\Omega)$. Then, in the above cases, the dual of the Banach space $L^1(\Omega, \mu)$ is the space $L^\infty(\Omega, \mu)$. This space is a commutative, unital C^*-algebra with respect to the pointwise operations, and thus its character space is a compact space.

DEFINITION 2.14. Let Ω be a non-empty, locally compact space, and let μ be a positive measure on Ω. Then the character space of $L^\infty(\Omega, \mu)$ is denoted by Φ_μ.

Thus $L^\infty(\Omega,\mu)$ is isometrically $*$-isomorphic to $C(\Phi_\mu)$; the map that implements this isomorphism is the Gel'fand transform

$$\mathcal{G}_\mu : L^\infty(\Omega,\mu) \to C(\Phi_\mu).$$

The space Φ_μ is hyper-Stonean. Clearly, the second dual $L^1(\Omega,\mu)''$ of $L^1(\Omega,\mu)$ is the dual space $C(\Phi_\mu)' = M(\Phi_\mu)$.

Let $\mathcal{F} = \{\nu_i : i \in I\}$ be a maximal singular family of positive measures on Ω. In the case where $\nu_i \in M(\Omega)$, we may suppose that $\|\nu_i\| = 1$ for each $i \in I$; the character space of $L^\infty(\Omega,\nu_i)$ is denoted by Φ_i. Clearly, each measure $\nu \in M(\Omega)$ can be written in the form

$$\nu = \sum_{i\in I} f_i\nu_i,$$

where $f_i \in L^1(\Omega,\nu_i)$ $(i \in I)$ and $\|\nu\| = \sum_{i\in I}\|f_i\|_1$, and so

$$M(\Omega) = \bigoplus\nolimits_1 \{L^1(\Omega,\nu_i) : i \in I\}.$$

Thus

$$M(\Omega)' = \bigoplus\nolimits_\infty \{L^\infty(\Omega,\nu_i) : i \in I\} = \bigoplus\nolimits_\infty \{C(\Phi_i) : i \in I\}. \tag{2.9}$$

Boolean algebras. We recall some basic facts about Boolean algebras. For background, see [33].

Let B be a Boolean algebra. Then B is *complete* if every non-empty subset S of B has a supremum, denoted by $\bigvee S$, and an infimum, denoted by $\bigwedge S$. For example, the family of all clopen subsets of a topological space X is a Boolean algebra with respect to the Boolean operations $\cup$ and $\cap$; this Boolean algebra is complete if and only if X is extremely disconnected.

Let B be a Boolean algebra. An *ultrafilter* p on B is a subset of B which is maximal with respect to the property that $b_1 \wedge \cdots \wedge b_n \neq 0$ whenever $b_1,\dots,b_n \in p$. The family of ultrafilters on B is the *Stone space* of B, denoted by $S(B)$; a topology on $S(B)$ is defined by taking the sets

$$\{p \in S(B) : b \in p\}$$

for $b \in B$ as a base of the open sets of $S(B)$. In this way $S(B)$ is a totally disconnected compact space; it is extremely disconnected if and only if B is complete as a Boolean algebra, and in this case it is a Stonean space. Conversely, let Ω be a totally disconnected compact space. Then Ω is the Stone space of the Boolean algebra $\mathfrak{I}_\Omega$.

Let Ω be a non-empty, locally compact space, and let μ be a positive measure on Ω. Then $\mathfrak{N}_\mu$ is the family of sets $B \in \mathfrak{B}_\Omega$ with $\mu(B) = 0$, and we define

$$\mathfrak{B}_\mu = \mathfrak{B}_\Omega/\mathfrak{N}_\mu;$$

clearly, $\mathfrak{B}_\mu$ is a complete Boolean algebra, and so its Stone space $S(\mathfrak{B}_\mu)$ is extremely disconnected.

Let $B \in \mathfrak{B}_\Omega$. Then χ_B (or, more precisely, the equivalence class $[\chi_B]$) is an idempotent in $L^\infty(\Omega,\mu)$, and so $\mathcal{G}_\mu(\chi_B)$ is an idempotent in $C(\Phi_\mu)$; we set

$$K_B \cap \Phi_\mu = \{\varphi \in \Phi_\mu : \mathcal{G}_\mu(\chi_B)(\varphi) = 1\}, \tag{2.10}$$

so that

$$\{K_B \cap \Phi_\mu : B \in \mathfrak{B}_\Omega\} = \mathfrak{I}_{\Phi_\mu}.$$

In particular, suppose that $B \in \mathfrak{B}_\Omega$ and $\mu(B) = 0$. Then $K_B \cap \Phi_\mu = \emptyset$.

Clearly, $S(\mathfrak{B}_\mu)$ is homeomorphic to the space Φ_μ. Indeed, first let p be an ultrafilter in $\mathfrak{B}_\mu$. Then

$$\bigcap\{K_B \cap \Phi_\mu : B \in p\}$$

is a singleton in Φ_μ, and so we can regard p as a point of Φ_μ. Conversely, each element $\varphi \in \Phi_\mu$ defines the ultrafilter in $\mathfrak{B}_\Omega$ which is the equivalence class corresponding to the family

$$\{B \in \mathfrak{B}_\Omega : \varphi(\chi_B) = 1\}.$$

This family is directed by reverse inclusion, and so defines a net; we write '$\lim_{B\to\varphi}$' for convergence along this net. Thus we see that the corresponding net

$$\left\{\mu_B = \frac{\mu\,|\,B}{\mu(B)} : B \to \varphi\right\}$$

in $L^\infty(\Omega,\mu)_{[1]}$ converges weak-$*$ to δ_φ in $M(\Phi_\mu)$; this net is called the *canonical net* that converges to δ_φ. Specifically, for each $\lambda \in L^\infty(\Omega,\mu)$, we have

$$\lim_{B\to\varphi}\langle\lambda,\,\mu_B\rangle = \lim_{B\to\varphi}\frac{1}{\mu(B)}\int_B \lambda\,\mathrm{d}\mu = \mathcal{G}_\mu(\lambda)(\varphi). \tag{2.11}$$

It is clear that, for each $x \in \Omega_\infty$ such that $\mu(U) > 0$ for each $U \in \mathcal{N}_x$, there exists $\varphi \in \Phi_\mu$ such that $\mathcal{N}_x \subset \varphi$. In particular, for each $x \in \operatorname{supp}\mu$, there exists $\varphi \in \Phi_\mu$ with $\mathcal{N}_x \subset \varphi$. It is also clear that, for each $\varphi \in \Phi_\mu$, there exists a unique point $x \in \operatorname{supp}\mu \cup \{\infty\}$ with $\mathcal{N}_x \subset \varphi$. Thus we can define a map

$$\pi_\mu : \varphi \mapsto x, \quad \Phi_\mu \to \Omega_\infty, \tag{2.12}$$

and so $|\Phi_\mu| \geq |\operatorname{supp}\mu|$. We see from the definition of the topology on the Stone space Φ_μ that π_μ is continuous.

PROPOSITION 2.15. *Let Ω be a non-empty, locally compact space, and let μ be a positive measure on Ω such that* $\operatorname{supp}\mu = \Omega$ *and $|\mathfrak{B}_\mu| = \kappa$ for an infinite cardinal κ.*

(i) *Each non-empty, clopen subset of Φ_μ has the form $K_B \cap \Phi_\mu$ for some $B \in \mathfrak{B}_\Omega \setminus \mathfrak{N}_\mu$, and the family*

$$\{K_B \cap \Phi_\mu : B \in \mathfrak{B}_\Omega \setminus \mathfrak{N}_\mu\}$$

is a base for the topology of Φ_μ.

(ii) *$w(\Phi_\mu) = \kappa$ and $|\Omega| \leq |\Phi_\mu| = 2^\kappa$.*

(iii) *Φ_μ satisfies* CCC.

(iv) *Φ_μ has no isolated points if and only if μ is continuous.*

Proof. (i) & (ii) These are clear from our earlier remarks.

(iii) Let $\{U_i : i \in I\}$ be a pairwise disjoint family of non-empty, open subsets of Φ_μ. For each $i \in I$, choose a non-empty, clopen set $K_i \subset U_i$. Then there exists $B_i \in \mathfrak{B}_\Omega \setminus \mathfrak{N}_\mu$ with $K_{B_i} \cap \Phi_\mu = K_i$. The family $\{B_i : i \in I\} \subset \mathfrak{B}_\Omega$ is pairwise disjoint, and $\mu(B_i) > 0$ $(i \in I)$. Thus I is countable.

(iv) Suppose that μ is not continuous, so that there exists $x \in \Omega$ with $\mu(\{x\}) > 0$. Then $\varphi := \{B \in \mathfrak{B}_\Omega : x \in B\}$ is an ultrafilter in $S(\mathfrak{B}_\mu)$, and clearly φ is an isolated point of Φ_μ.

Suppose that φ is an isolated point of Φ_μ. Then there exists $B \in \mathfrak{B}_\Omega$ with $\mu(B) > 0$ such that $\{\psi \in \Phi_\mu : B \in \psi\} = \{\varphi\}$. Since μ is regular, we may suppose that B is compact. Thus there is a unique point $x \in B$ such that $\mu(U \cap B) = \mu(B)$ $(U \in \mathcal{N}_x)$. Clearly $\mu(\{x\}) = \mu(B) > 0$, and so μ is not continuous. ∎

EXAMPLE 2.16. Take $p \geq 2$, and let $\Omega = \mathbb{Z}_p^\kappa$ be the Cantor cube of weight κ described above, where κ is an infinite cardinal. Let m_p be the measure that gives the value $1/p$ to each point of $\mathbb{Z}_p$, and let m be the corresponding product measure on $\mathbb{Z}_p^\kappa$. Then $m \in M_c(\Omega)^+$, $\|m\| = 1$, and $\operatorname{supp} m = \Omega$.

By (2.2), $|\Omega| = 2^\kappa$ and $w(\Omega) = |\mathfrak{I}_\Omega| = \kappa$, and so $|\mathfrak{B}_m| \geq \kappa$.

Now suppose that $\kappa \geq \mathfrak{c}$. Since $|\mathfrak{I}_\Omega| \geq \mathfrak{c}$, we have $|\sigma(\mathfrak{I}_\Omega)| = \kappa$. Let $B \in \mathfrak{B}_\Omega$. The space Ω is totally disconnected and m is regular, and so, for each $\varepsilon > 0$, there exists $C_\varepsilon \in \mathfrak{I}_\Omega$ such that $m(B \bigtriangleup C_\varepsilon) < \varepsilon$. It follows that there exists $C \in \sigma(\mathfrak{I}_\Omega)$ with $m(B \bigtriangleup C) = 0$, and hence $m(B) = m(C)$. Thus $|\mathfrak{B}_m| \leq \kappa$, and so $|\mathfrak{B}_m| = \kappa$.

By Proposition 2.15(ii), $w(\Phi_m) = \kappa$ and so $|\Phi_m| = 2^\kappa$. (That $|\Phi_m| \leq 2^\kappa$ also follows because each character on $L^\infty(\Omega, \mu)$ is determined by its values on the characteristic functions of Borel sets of Ω.) ∎

The following result characterizes the sets Φ_μ of Definition 2.14.

PROPOSITION 2.17. *Let X be a hyper-Stonean space. Then the following conditions on X are equivalent:*

(a) *X satisfies* CCC*;*
(b) *X is homeomorphic to a space Φ_μ for some positive measure μ on a non-empty, locally compact space;*
(c) *there exists $\mu \in N(X)$ with $\|\mu\| = 1$ and* $\operatorname{supp} \mu = X$ *such that X is homeomorphic to Φ_μ.*

Proof. Trivially (c)⇒(b); we have shown that (b) ⇒ (a).

(a)⇒(c). We have remarked that each normal measure on X has clopen support.

Take $\mathcal{N}$ to be a family of normal measures in $P(X)$ such that $\mathcal{N}$ is maximal subject to the condition that the supports of the measures in the family are pairwise disjoint; certainly such a maximal family exists. By hypothesis, $\mathcal{N}$ is countable, and so can be enumerated as (μ_n). Define $\mu = \sum_{n=1}^\infty \mu_n/2^n$. Then μ is a normal measure with $\|\mu\| = 1$ and $\operatorname{supp} \mu = X$.

We shall show that Φ_μ is homeomorphic to X. To see this, let θ be the canonical map from $\mathfrak{B}_X$ onto $\mathfrak{B}_\mu$, so that θ is a Boolean epimorphism. We *claim* that

$$\theta \,|\, \mathfrak{I}_X : \mathfrak{I}_X \to \mathfrak{B}_\mu$$

is an isomorphism of Boolean algebras. Clearly $\theta \,|\, \mathfrak{I}_X$ is a Boolean monomorphism. Now take $B \in \mathfrak{B}_X$. Then there is a sequence (K_n) of compact subsets of B with $\mu(B) = \mu(U)$, where $U = \bigcup\{K_n : n \in \mathbb{N}\}$, an open set in X. We have $\overline{U} \in \mathfrak{I}_X$. For each $n \in \mathbb{N}$, we have $\mu(K_n) = \mu(\overline{\operatorname{int} K_n})$, and so we may suppose that $K_n = \overline{\operatorname{int} K_n}$, and thus that $K_n \in \mathfrak{I}_X$.

Further, $\overline{U} \setminus U \in \mathcal{K}_X$, and so $\mu(\overline{U} \setminus U) = 0$ by Theorem 2.11. Thus $\mu(\overline{U}) = \mu(B)$. This shows that $\theta \,|\, \mathfrak{I}_X$ is a surjection onto $\mathfrak{B}_\mu$.

We have shown that $\mathfrak{I}_X$ and $\mathfrak{B}_\mu$ are isomorphic Boolean algebras, and so their respective Stone spaces are homeomorphic; these Stone spaces are X and Φ_μ, respectively. ■

Let Ω be a non-empty, locally compact space, and let μ be a positive measure on Ω. Then the pair $(\mathfrak{B}_\mu, \mu)$ is termed a *measure algebra* by Halmos in [45, p. 167]; however, we shall call it a *measure Boolean algebra* to avoid possible confusion with a later usage of the term 'measure algebra'. The special case in which $\Omega = \mathbb{I}$ and μ is the Lebesgue measure on $\mathbb{I}$ is called the *measure Boolean algebra of the unit interval*; the corresponding space Φ_μ is called the *hyper-Stonean space of the unit interval* in [32, A7H].

Definition 2.18. The hyper-Stonean space of the unit interval is denoted by $\mathbb{H}$.

Thus $\mathbb{H}$ is the character space of the C^*-algebra $L^\infty(\mathbb{I}, m)$.

Let Ω_1 and Ω_1 be two non-empty, locally compact spaces, and let μ_1 and μ_2 be positive measures on Ω_1 and Ω_2, respectively. An *isomorphism* between the measure Boolean algebras $(\mathfrak{B}_{\mu_1}, \mu_1)$ and $(\mathfrak{B}_{\mu_2}, \mu_2)$ is a map $\eta : \mathfrak{B}_{\mu_1} \to \mathfrak{B}_{\mu_2}$ which is a Boolean algebra isomorphism and $\mu_2(\eta(B)) = \mu_1(B)$ for each $B \in \mathfrak{B}_{\Omega_1}$; the two measure Boolean algebras $(\mathfrak{B}_{\mu_1}, \mu_1)$ and $(\mathfrak{B}_{\mu_2}, \mu_2)$ are *isomorphic* if there is such an isomorphism between them. In this latter case, the spaces $L^1(\Omega_1, \mu_1)$ and $L^1(\Omega_2, \mu_2)$ are isomorphic as Banach lattices.

A measure ring $(\mathfrak{B}_\mu, \mu)$ is *separable* in the sense of [45, p. 168] if the space $(\mathfrak{B}_\Omega, \rho)$ is a separable metric space for the metric ρ, defined by setting

$$\rho(B, C) = \mu(B \bigtriangleup C) = \|\chi_B - \chi_C\|_1 \quad (B, C \in \mathfrak{B}_\Omega).$$

This is the case if and only if the Banach space $(L^1(\Omega, \mu), \|\cdot\|_1)$ is separable. The measure Boolean algebra of a compact, metrizable space Ω is separable because $w(\Omega) = \aleph_0$.

We shall require a famous isomorphism theorem; a proof involving just measures is given in [45, §41, Theorem C], and a proof involving von Neumann algebras is given in [112, Theorem III.1.22].

Theorem 2.19. *Let Ω be a non-empty, locally compact space, and let $\mu \in M_c(\Omega)^+$ be such that $\|\mu\| = 1$ and such that the Banach space $(L^1(\Omega, \mu), \|\cdot\|_1)$ is separable. Then the measure Boolean algebra $(\mathfrak{B}_\mu, \mu)$ is isomorphic to the measure Boolean algebra of the unit interval, and the Banach spaces $L^1(\Omega, \mu)$ and $L^1(\mathbb{I}, m)$ are isometrically isomorphic as Banach lattices.*

In particular, in the case where Ω is uncountable, locally compact, and second countable, there exists a measure $\mu \in M_c(\Omega)^+$ such that the spaces $L^1(\Omega, \mu)$ and $L^1(\mathbb{I}, m)$ are isometrically isomorphic as Banach lattices. ■

Corollary 2.20. *Let Ω_1 and Ω_2 be two locally compact and second countable spaces, and suppose that $\mu_1 \in M_c(\Omega_1)^+$ and $\mu_2 \in M_c(\Omega_2)^+$, with $\mu_1, \mu_2 \neq 0$. Then the compact spaces Φ_{μ_1} and Φ_{μ_2} are homeomorphic.* ■

Corollary 2.21. *Let Ω be an uncountable, compact, metrizable space. Let $\mu \in M_c(\Omega)^+$ with $\mu \neq 0$. Then $|\Phi_\mu| = 2^{\mathfrak{c}}$ and $w(\Phi_\mu) = \mathfrak{c}$. In particular,*

$$|\mathbb{H}| = 2^{\mathfrak{c}} \quad \textit{and} \quad w(\mathbb{H}) = \mathfrak{c}.$$

Proof. By Proposition 2.2(iv), $|\mathfrak{B}_\Omega| = \mathfrak{c}$, and so $|\mathfrak{B}_\mu| \leq \mathfrak{c}$. By Proposition 2.15(ii), we have $|\Phi_\mu| \leq 2^{\mathfrak{c}}$ and $w(\Phi_\mu) \leq \mathfrak{c}$.

Since Φ_μ is Stonean and infinite, it contains a copy of $\beta\mathbb{N}$, and so $w(\Phi_\mu) \geq w(\beta\mathbb{N}) = \mathfrak{c}$ by Proposition 2.1. Thus $|\Phi_\mu| = 2^{\mathfrak{c}}$ and $w(\Phi_\mu) = \mathfrak{c}$.

We give a direct proof of the fact that $|\Phi_\mu| \geq 2^{\mathfrak{c}}$. By Corollary 2.20, it suffices to suppose that $\Omega = \mathbb{I}$ and that μ is Lebesgue measure on $\mathbb{I}$. For $n \in \mathbb{N}$, set $F_n = [t_{2n+1}, t_{2n}]$, where (t_n) is a sequence in $\mathbb{I}$ such that $t_n \searrow 0$. For each $S \subset \mathbb{N}$, set $B_S = \bigcup\{F_n : n \in S\}$, and, for each $p \in \mathbb{N}^*$, set

$$C_p = \bigcap\{K_{B_S} : S \in p\}.$$

Then C_p is a non-empty, closed subset of Φ_μ, and $C_p \cap C_q = \emptyset$ whenever p and q are distinct points of $\mathbb{N}^*$. By Proposition 2.1, $|\mathbb{N}^*| = 2^{\mathfrak{c}}$, and so it follows that $|\Phi_\mu| \geq 2^{\mathfrak{c}}$. ■

Thus, with GCH, we have $|\mathbb{H}| = \aleph_2$ and $w(\mathbb{H}) = \aleph_1$.

Corollary 2.22. *The space $\mathbb{H}$ is a topological space X with the following properties:*

- (i) *X is a hyper-Stonean space;*
- (ii) *X has no isolated points;*
- (iii) *X satisfies* CCC*;*
- (iv) *the space $(C(X)_{[1]}, \sigma(C(X), N(X)))$ is metrizable.*

Conversely, each topological space X satisfying (i)–(iv) *is homeomorphic to $\mathbb{H}$.*

Further, $|X| = 2^{\mathfrak{c}}$ and $w(X) = \mathfrak{c}$ for each such space X.

Proof. We have seen that $\mathbb{H}$ satisfies clauses (i)–(iii). The space $\mathbb{H}$ satisfies (iv) because the Banach space $F := (L^1(\mathbb{I}, m), \|\cdot\|_1)$ is separable (where m is Lebesgue measure on $\mathbb{I}$), and so $(F'_{[1]}, \sigma(F', F))$ is metrizable by Proposition 1.1(i); here, $F' = L^\infty(\mathbb{I}, m) \cong C(\mathbb{H})$.

Conversely, suppose that X is a topological space satisfying clauses (i)–(iv). By Proposition 2.17, X is homeomorphic to a space Φ_μ for some $\mu \in N(X)$ with $\|\mu\| = 1$ and $\operatorname{supp}\mu = X$. By (ii) and Proposition 2.15, μ is a continuous measure. By (iv) and Proposition 1.1(i), $(L^1(\Omega, \mu), \|\cdot\|_1)$ is separable, and so, by Theorem 2.19, $L^1(\Omega, \mu)$ is isomorphic as a Banach lattice to $L^1(\mathbb{I}, m)$. Thus $C(X)$ and $C(\mathbb{H})$ are isomorphic as Banach lattices, and hence as C^*-algebras, whence X and $\mathbb{H}$ are homeomorphic. ■

We note that clause (iv) of the above characterization of $\mathbb{H}$ is necessary: there is a compact space Ω and $\mu \in M_c(\Omega)^+$ such that $X = \Phi_\mu$ is a hyper-Stonean space with no isolated points, X satisfies CCC, $|X| = 2^{\mathfrak{c}}$ and $w(X) = \mathfrak{c}$, but (iv) fails. Indeed, set $\Omega = \mathbb{Z}_2^{\mathfrak{c}}$, the Cantor cube of weight $\mathfrak{c}$, let m be the corresponding product measure described above, and set $X = \Phi_m$. Since m is continuous, Φ_m has no isolated points and Φ_m satisfies CCC. As in Example 2.16, we have

$$w(\Phi_m) = |\mathfrak{B}_m| = \mathfrak{c} \quad \text{and} \quad |\Phi_m| = 2^{\mathfrak{c}}.$$

However, for each $\tau < \mathfrak{c}$, set $B_\tau = \{\varepsilon \in \Omega : \varepsilon_\tau = 1\}$, so that $B_\tau \in \mathfrak{I}_\Omega$. Clearly we have $m(B_{\tau_1} \bigtriangleup B_{\tau_2}) = 1/2$ whenever $\tau_1, \tau_2 < \mathfrak{c}$ with $\tau_1 \neq \tau_2$, and so the measure Boolean algebra $(\mathfrak{B}_m, m)$ is not separable; equivalently, the space $(C(X)_{[1]}, \sigma(C(X), N(X)))$ is not metrizable.

We can give a condition that is apparently weaker than clause (iv) of the above corollary, but is actually equivalent to it.

PROPOSITION 2.23. *Let X be a topological space that satisfies clauses* (i)–(iii) *above. Then X satisfies* (iv) *if and only if each subspace of $C(X)_{[1]}$ which is discrete in the weak-$*$ topology is countable.*

Proof. Suppose that X satisfies (iv). Then certainly each weak-$*$ discrete subset of $C(X)_{[1]}$ is countable.

For the converse, suppose that each weak-$*$ discrete subset of $C(X)_{[1]}$ is countable.

By Proposition 2.17, there are a compact space Ω and a positive measure $\mu \in M(\Omega)^+$ such that $X = \Phi_\mu$. Assume towards a contradiction that there is an uncountable family (B_α) in $\mathfrak{B}_\Omega \setminus \mathfrak{N}_\mu$ and $\delta > 0$ such that $\rho(B_\alpha, B_\beta) > \delta$ whenever $\alpha \neq \beta$. The characteristic function of B_α is χ_α.

We *claim* that, for each α, it is not the case that χ_α is in the $\|\cdot\|_1$-closed convex hull of $\{\chi_\beta : \beta \neq \alpha\}$. Indeed, let $n \in \mathbb{N}$ and $t_1, \dots, t_n \in \mathbb{I}$ with $\sum_{i=1}^n t_i = 1$, and set $\lambda = \sum_{i=1}^n t_i \chi_{\beta_i}$, where $\beta_i \neq \alpha$ $(i \in \mathbb{N}_n)$. We have

$$\chi_\alpha - \lambda = \sum_{i=1}^n t_i(\chi_\alpha - \chi_{\beta_i}) = \sum_{i=1}^n t_i(\chi_{\alpha\setminus\beta_i} - \chi_{\beta_i\setminus\alpha}),$$

and so

$$\|\chi_\alpha - \lambda\|_1 = \sum_{i=1}^n t_i(\mu(B_\alpha \setminus B_{\beta_i}) + \mu(B_{\beta_i} \setminus B_\alpha)) = \sum_{i=1}^n t_i\|\chi_\alpha - \chi_{\beta_i}\|_1 > \delta,$$

where $\|\cdot\|_1$ is the norm in $L^1(\Omega, \mu)$. The claim follows.

Now regard the family (χ_α) as a subspace of $L^\infty(\Omega, \mu) = C(\Phi_\mu)$. For each α, there is a linear functional M on $L^\infty(\Omega, \mu)$ such that M is continuous with respect to the seminorm $\|\cdot\|_1$ on $L^\infty(\Omega, \mu)$ and such that

$$\langle \chi_\alpha, \mathrm{M}\rangle < \inf\{\langle \chi_\beta, \mathrm{M}\rangle : \beta \neq \alpha\}.$$

The linear functional $\lambda \mapsto \langle \lambda, \mu\rangle$ is order-continuous on $C(\Phi_\mu)_\mathbb{R}$, and so M is order-continuous on $C(\Phi_\mu)_\mathbb{R}$. Thus M is a normal measure on Φ_μ; by Theorem 2.11, M is weak-$*$ continuous on $C(\Phi_\mu)$, and so χ_α does not belong to the weak-$*$ closure of $\{\chi_\beta : \beta \neq \alpha\}$.

It follows that (χ_α) is an uncountable weak-$*$ discrete subset of $C(X)_{[1]}$. This is a contradiction, and so the measure Boolean algebra $(\mathfrak{B}_\mu, \mu)$ is separable. Thus X satisfies (iv). ∎

3. Specific second dual algebras

In this chapter, we shall begin our study of the second duals of the Banach algebras $C_0(\Omega)$ and $M(\Omega)$ for a locally compact space Ω. We shall also introduce $B^b(\Omega)$, the C^*-algebra of bounded Borel functions on Ω.

Second duals of algebras of continuous functions. Let Ω be a non-empty, locally compact space, and again set $E = C_0(\Omega)$. Since E is a commutative C^*-algebra, E is Arens regular, and E'' is also a commutative C^*-algebra, with just one Arens product, which we denote by juxtaposition. Thus

$$\langle \lambda,\, \Lambda \cdot \mu\rangle = \langle \Lambda, \mu \cdot \lambda\rangle \quad (\lambda \in E,\, \mu \in E',\, \Lambda \in E'')$$

and

$$\langle \Lambda_1\Lambda_2,\, \mu\rangle = \langle \Lambda_1,\, \Lambda_2 \cdot \mu\rangle \quad (\Lambda_1, \Lambda_2 \in E'',\, \mu \in E').$$

Since E has a bounded approximate identity, E'' (with this Arens product) has an identity, and so E'' is isometrically isomorphic to $C(\widetilde{\Omega})$ for a certain compact space $\widetilde{\Omega}$. As in [112, III.2.3], $C(\widetilde{\Omega})$ is the *enveloping von Neumann algebra* of E.

Definition 3.1. Let Ω be a non-empty, locally compact space. Then the character space of the commutative C^*-algebra $C_0(\Omega)''$ is denoted by $\widetilde{\Omega}$.

The general proof that a C^*-algebra is Arens regular and that its second dual is also a C^*-algebra involves a considerable theory of C^*-algebras; we note that a direct proof that $C_0(\Omega)''$ is isometrically isomorphic to $C(\widetilde{\Omega})$ for a compact space $\widetilde{\Omega}$ is given in [105, §4].

We regard $C_0(\Omega)$ as a closed subalgebra of $C(\widetilde{\Omega})$ via the map κ_E; when Ω is not compact, we identify $C(\Omega_\infty)$ with the closed subalgebra

$$\{z1 + \lambda : z \in \mathbb{C},\, \lambda \in C_0(\Omega)\}$$

of $C(\widetilde{\Omega})$. Clearly $(E'')_{\mathbb{R}}$ is a Banach lattice and $\kappa_E : E_{\mathbb{R}} \to (E'')_{\mathbb{R}}$ is isotonic.

The topology on the space $\widetilde{\Omega}$ is called σ, so that σ is the weak-$*$ topology $\sigma(E''', E'')$ restricted to $\widetilde{\Omega}$. Since E'' is certainly a dual space, $(\widetilde{\Omega}, \sigma)$ is hyper-Stonean.

Definition 3.2. Let Ω be a non-empty, locally compact space. Then the corresponding space $\widetilde{\Omega}$ is the *hyper-Stonean envelope* of Ω.

The term 'hyper-Stonean cover' is used for our 'hyper-Stonean envelope' in [125], where some references to earlier works are given. In [125], there is a characterization of $\widetilde{\Omega}$ in terms of certain 'Kelley ideals'.

Let $\varphi \in \widetilde{\Omega}$. Then $\varepsilon_\varphi \in E'''$, and $\varepsilon_\varphi \,|\, C_0(\Omega)$ is a character on E or 0, say $\pi(\varepsilon_\varphi) = \varepsilon_{\pi(\varphi)}$ for a point $\pi(\varphi) \in \Omega_\infty$. The map

$$\pi : (\widetilde{\Omega}, \sigma) \to (\Omega_\infty, \tau)$$

is a continuous surjection.

We remark that a *cover* of a compact space Ω is a pair (X, f), where X is a compact space and $f : X \to \Omega$ is a continuous surjection. Thus $(\widetilde{\Omega}, \pi)$ is a cover of Ω. The cover (X, f) is said to be *essential* [43, Definition 2.10] if, for each compact space Y and each continuous function $h : Y \to X$ with $f(h(Y)) = \Omega$, necessarily $h(Y) = X$, and the cover (X, f) is *projective* if it is essential and X is a projective (equivalently, extremely disconnected) space. As in [43, Theorem 2.16], we see that each closed subset X of $\widetilde{\Omega}$ that is minimal with respect to the property that $\pi(X) = \Omega$ is a projective cover of Ω; such a cover is unique up to a homeomorphism that commutes with the covering map π. In this case, $C(X)$ is the so-called *injective envelope* of $C(\Omega)$.

Definition 3.3. Let Ω be a non-empty, locally compact space, and let $x \in \Omega_\infty$. Then

$$\Omega_{\{x\}} = \pi^{-1}(\{x\})$$

is the *fibre* of Ω at x.

Each fibre $\Omega_{\{x\}}$ is a closed subspace of $(\widetilde{\Omega}, \sigma)$, and clearly we have

$$\widetilde{\Omega} = \bigcup \{\Omega_{\{x\}} : x \in \Omega_\infty\}.$$

We shall see in Example 3.16 below that a fibre $\Omega_{\{x\}}$ is not necessarily open.

Let $\Lambda \in E''$ and $\mu \in E'$. Then we *claim* that

$$\operatorname{supp}(\Lambda \cdot \mu) \subset \operatorname{supp} \mu. \tag{3.1}$$

Indeed, let $\lambda \in C_0(\Omega)$ with $\operatorname{supp} \lambda \subset \Omega \setminus \operatorname{supp} \mu$. Then clearly $\lambda\mu = 0$, and so we have $\langle \lambda,\, \Lambda \cdot \mu \rangle = \langle \Lambda,\, \lambda\mu \rangle = 0$. Thus the claim follows.

There is a natural embedding ι of Ω into $\widetilde{\Omega}$. Indeed, let $x \in \Omega$. Then

$$\varepsilon_x'' : \Lambda \mapsto \langle \Lambda,\, \varepsilon_x \rangle, \qquad E'' \to \mathbb{C},$$

is a character on E'' extending ε_x; the second dual ε_x'' is given by a point of $\widetilde{\Omega}$, say by $\iota(x)$. Clearly $\iota : \Omega \to \widetilde{\Omega}$ is an injection and $\pi \circ \iota$ is the identity on Ω. The map $\iota^{-1} : \iota(\Omega) \to \Omega$ is continuous, and so $\tau \subset \sigma \,|\, \Omega$. We now identify x with $\iota(x)$, and regard Ω as a subset (but not a topological subspace) of $\widetilde{\Omega}$. For $x \in \Omega$, we identify ε_x with $\delta_x \in M(\Omega)$. For a subset U of Ω, we denote by $\overline{U}$ the closure of U in $(\widetilde{\Omega}, \sigma)$, and we set $U^* = \overline{U} \setminus U$. In particular, $\overline{\Omega}$ is the closure of Ω in $(\widetilde{\Omega}, \sigma)$.

Let $x \in \Omega$. Then the map

$$\Lambda_x : \mu \mapsto \mu(\{x\}), \qquad M(\Omega) \to \mathbb{C},$$

belongs to $M(\Omega)' = E'' = C(\widetilde{\Omega})$. For $y \in \Omega$, we have

$$\varepsilon_y''(\Lambda_x) = \langle \Lambda_x,\, \varepsilon_y \rangle = \langle \Lambda_x,\, \delta_y \rangle,$$

and so $\Lambda \,|\, \Omega = \chi_{\{x\}}$. This shows that (Ω, σ) is a discrete space. We shall see below that Ω is open in the hyper-Stonean envelope $(\widetilde{\Omega}, \sigma)$.

Let $x \in \Omega$. In the case where x is isolated in Ω, set $Y = \Omega\backslash\{x\}$. Then $E = C_0(Y)\oplus\mathbb{C}\delta_x$, and so $E'' = C(\widetilde{Y}) \oplus \mathbb{C}\delta_x$. Clearly $\pi(\widetilde{Y}) = Y$, and so $\Omega_{\{x\}} = \{x\}$.

PROPOSITION 3.4. *Let Ω be a non-empty, locally compact space. Then $\kappa_E(C_0(\Omega))$ consists of the functions $\Lambda \in C(\widetilde{\Omega}, \sigma)$ such that $\Lambda\,|\,\Omega_{\{x\}}$ is constant for each $x \in \Omega_\infty$ and $\Lambda\,|\,\Omega_{\{\infty\}} = 0$.*

Proof. Take $\lambda \in E = C_0(\Omega)$. For each $x \in \Omega_\infty$, we see that $\kappa_E(\lambda)\,|\,\Omega_{\{x\}}$ takes the constant value $\lambda(x)$ and that $\kappa_E(\lambda)\,|\,\Omega_{\{\infty\}} = 0$.

Now suppose that $\Lambda \in C(\widetilde{\Omega})$ and that Λ is constant on each set $\Omega_{\{x\}}$. We *claim* that $\lambda := \Lambda\,|\,\Omega \in C(\Omega, \tau)$. For let (x_α) be a net in Ω with limit $x_0 \in \Omega$ with respect to the topology τ. Since $(\widetilde{\Omega}, \sigma)$ is compact, we may suppose by passing to a subnet that there exists $\varphi_0 \in \widetilde{\Omega}$ such that $x_\alpha \to \varphi_0$ in $(\widetilde{\Omega}, \sigma)$. Since $\pi : (\widetilde{\Omega}, \sigma) \to (\Omega, \tau)$ is continuous, $x_\alpha \to \pi(\varphi_0)$ in (Ω, τ), and so $\pi(\varphi_0) = x_0$. Thus $\lambda(x_\alpha) = \Lambda(x_\alpha) \to \Lambda(\varphi_0) = \lambda(x_0)$. It follows that $\lambda \in C(\Omega)$. By the same argument, $\lambda \in C_0(\Omega)$ in the case where $\Lambda\,|\,\Omega_{\{\infty\}} = 0$. Clearly $\kappa_E(\lambda) = \Lambda$, and so the result follows. ■

COROLLARY 3.5. *Let Ω be a non-empty, locally compact space. Then $\pi^{-1}(\Omega)$ is a dense, open subspace of $\widetilde{\Omega}$.* ■

We shall see in Example 3.16 below that, in general, there is no continuous projection of $C(\widetilde{\Omega})$ onto $C(\Omega)$.

The following result is a slightly more general version of [56, Lemma 2.3]. We say that an element $\lambda \in L^\infty(\Omega)$ is *continuous at* $x \in \Omega$ if the equivalence class of λ contains a function which is continuous at x.

PROPOSITION 3.6. *Let Ω be a non-empty, locally compact space, and take μ to be a positive measure on Ω. Suppose that there exists $V \in \mathcal{N}_x$ such that $\mu(U) > 0$ for each non-empty, open subset U of Ω with $U \subset V$. Let $\lambda \in L^\infty(\Omega, \mu)$, and suppose that $\mathcal{G}_\mu(\lambda)$ is constant on $\Phi_\mu \cap \Omega_{\{x\}}$. Then λ is continuous at x.*

Proof. We note that the set $\Phi_\mu \cap \Omega_{\{x\}}$ is not empty because it contains each weak-$*$ accumulation point of the net $\{\mu_B : B \in \mathcal{N}_x\}$.

We may suppose that λ is real-valued. Assume towards a contradiction that λ is not continuous at x. Then there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ such that, setting

$$A = \{x \in V : \lambda(x) < \alpha\}, \qquad B = \{x \in V : \lambda(x) > \beta\},$$

we have $A \cap B = \emptyset$ and both A and B meet each neighbourhood of x in a non-empty, open set; by hypothesis, each such intersection has strictly positive μ-measure, and so $\{A\} \cup \mathcal{N}_x$ and $\{B\} \cup \mathcal{N}_x$ are contained in ultrafilters $\varphi, \psi \in \Phi_\mu \cap \Omega_{\{x\}}$, respectively, with $\varphi \neq \psi$. We have $\mathcal{G}_\mu(\lambda)(\varphi) \leq \alpha$ and $\mathcal{G}_\mu(\lambda)(\psi) \geq \beta$, a contradiction of the fact that $\mathcal{G}_\mu(\lambda)$ is constant on $\Phi_\mu \cap \Omega_{\{x\}}$.

Thus λ is continuous at x. ■

Second duals of spaces of measures. Let Ω be a non-empty, locally compact space, and again set $E = C_0(\Omega)$. The dual space of $E'' = C(\widetilde{\Omega})$ is $E''' = M(\widetilde{\Omega})$. We denote by $\kappa = \kappa_{E'}$ the canonical mapping of E' into E''', and sometimes identify $\mu \in M(\Omega)$ with

$\kappa(\mu) \in M(\widetilde{\Omega})$. Thus we have

$$\langle \Lambda,\, \mu\rangle = \int_{\widetilde{\Omega}} \Lambda \,\mathrm{d}\mu \quad (\Lambda \in C(\widetilde{\Omega}),\, \mu \in M(\Omega)). \tag{3.2}$$

There is a continuous projection $\pi : E''' \to E'$ which is the dual of the injection $\kappa_E : E \to E''$, and which is defined by

$$\langle \lambda,\, \pi(\mathrm{M})\rangle = \langle \kappa_E(\lambda),\, \mathrm{M}\rangle \quad (\lambda \in E,\, \mathrm{M} \in E'''),$$

and so we also have a map

$$\pi = \kappa_E' : M(\widetilde{\Omega}) \to M(\Omega). \tag{3.3}$$

The map $\pi \,|\, \widetilde{\Omega} : \widetilde{\Omega} \to \Omega$ coincides with the previously-defined map π. Further,

$$M(\widetilde{\Omega}) = M(\Omega) \oplus_1 E^\circ,$$

where

$$E^\circ = \{\mathrm{M} \in M(\widetilde{\Omega}) : \mathrm{M} \,|\, \kappa_E(E) = 0\} = \ker \pi.$$

For a compact subset K of Ω, we write $K \prec \lambda$ whenever $\lambda \in C_0(\Omega)$ with $\lambda(\Omega) \subset \mathbb{I}$ and $\lambda \,|\, K = 1$. In the case where $\mathrm{M} \in M(\widetilde{\Omega})^+$ and K is a compact subset of Ω, we have

$$\begin{aligned}
\pi(\mathrm{M})(K) &= \inf \left\{ \int_\Omega \lambda \,\mathrm{d}(\pi(\mathrm{M})) : K \prec \lambda \right\} \\
&= \inf \left\{ \int_{\widetilde{\Omega}} \kappa_E(\lambda) \,\mathrm{dM} : K \prec \lambda \right\} \\
&= \inf \left\{ \int_{\widetilde{\Omega}} \Lambda \,\mathrm{dM} : \pi^{-1}(K) \prec \Lambda \right\} = \mathrm{M}(\pi^{-1}(K)),
\end{aligned}$$

and so

$$\pi(\mathrm{M})(B) = \mathrm{M}(\pi^{-1}(B)) \quad (\mathrm{M} \in M(\widetilde{\Omega}),\, B \in \mathfrak{B}_\Omega). \tag{3.4}$$

It follows that E° is the weak-$*$ closed linear span of measures of the form $\delta_\varphi - \delta_\psi$, where φ, ψ are two points of $\widetilde{\Omega}$ in the same fibre. It also follows that

$$\|\mathrm{M}\| = \mathrm{M}(\widetilde{\Omega}) = \pi(\mathrm{M})(\Omega) = \|\pi(\mathrm{M})\| \quad (\mathrm{M} \in \mathrm{M}(\widetilde{\Omega})^+). \tag{3.5}$$

We shall use the following theorem.

Theorem 3.7. *Let Ω_1 and Ω_2 be two compact spaces, and suppose that there is a Banach lattice isomorphism $T : M(\Omega_1) \to M(\Omega_2)$. Then the dual map*

$$T' : C(\widetilde{\Omega}_2) \to C(\widetilde{\Omega}_1)$$

is a Banach lattice isomorphism and a unital $*$-isomorphism, and $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ are homeomorphic.

Proof. Certainly $T' : C(\widetilde{\Omega}_2) \to C(\widetilde{\Omega}_1)$ is a Banach lattice isomorphism such that T' maps the identity function on $\widetilde{\Omega}_2$ to the identity function on $\widetilde{\Omega}_1$. By Theorem 2.4(ii), T' is a unital $*$-isomorphism. The map $T'' \,|\, \widetilde{\Omega}_2 : \widetilde{\Omega}_2 \to \widetilde{\Omega}_1$ is a homeomorphism. ∎

Let Ω be a non-empty, locally compact space. Then of course the predual $C(\widetilde{\Omega})_*$ of $C(\widetilde{\Omega})$ is $\kappa(M(\Omega))$, and so we may extend Theorem 2.11 to obtain the following characterization of $\kappa(M(\Omega))$.

Theorem 3.8. *Let Ω be a non-empty, locally compact space, and let* $\mathrm{M} \in M(\widetilde{\Omega})^+$. *Then the following conditions on* M *are equivalent:*

(a) $\mathrm{M} \in \kappa(M(\Omega))$;
(b) M *is weak-$*$ continuous as a linear functional on* $C(\widetilde{\Omega})$;
(c) M *is a normal measure;*
(d) $\mathrm{M}(K) = 0$ $(K \in \mathcal{K}_{\widetilde{\Omega}})$. ■

Bounded Borel functions. We now define a further important C^*-algebra.

Definition 3.9. Let Ω be a non-empty, locally compact space. Then $B^b(\Omega)$ denotes the space of bounded Borel functions on Ω.

Clearly $(B^b(\Omega), |\cdot|_\Omega)$ is a unital C^*-subalgebra of $(\ell^\infty(\Omega), |\cdot|_\Omega)$ with $C^b(\Omega) \subset B^b(\Omega)$. It is also clear that the space

$$\mathrm{lin}\{\chi_B : B \in \mathfrak{B}_\Omega\}$$

is a $|\cdot|_\Omega$-dense linear subspace of $B^b(\Omega)$.

Indeed, $B^b(\Omega)$ is a well-known Banach algebra. This algebra is closely related to the algebra of Baire functions, which can be defined by a transfinite recursion through the Baire classes. The Baire functions of order 0 are the functions in $C^b(\Omega)$. Given a definition of the Baire class of order β for each $\beta < \alpha$, the class of order α is the space of bounded functions on Ω which are pointwise limits of sequences of functions in the union of the earlier classes; the construction terminates at $\alpha = \omega_1$. The *Baire functions* on Ω are the members of this final class. Each Baire class is itself a Banach algebra which is a closed subalgebra of $B^b(\Omega)$. In the case where the space Ω is (locally compact and) second countable, the algebra of Baire functions is equal to $B^b(\Omega)$ [50, (11.46)]; in particular, this is true for $\Omega = \mathbb{R}$ with the usual topology.

Definition 3.10. The character space of the unital C^*-algebra $(B^b(\Omega), |\cdot|_\Omega)$ is denoted by Φ_b.

Proposition 3.11. *Let Ω be an infinite, compact metrizable space. Then* $|B^b(\Omega)| = \mathfrak{c}$ *and* $|\Phi_b| = 2^{\mathfrak{c}}$.

Proof. By Proposition 2.3(ii), $|C(\Omega)| = \mathfrak{c}$. Thus each Baire class of order less than ω_1 has cardinality $\mathfrak{c}$, and so the algebra of Baire functions on Ω has cardinality $\mathfrak{c}$. Since the latter algebra is equal to $B^b(\Omega)$, we have $|B^b(\Omega)| = \mathfrak{c}$.

We have $|\Phi_b| \leq |B^b(\Omega)'| = 2^{\mathfrak{c}}$. Let D be a countable subset of Ω. Then $\ell^\infty(D)$ is a closed C^*-subalgebra of $B^b(\Omega)$ and the character space of $\ell^\infty(D)$ is βD, which, by Proposition 2.1, has cardinality $2^{\mathfrak{c}}$. Thus $|\Phi_b| \geq 2^{\mathfrak{c}}$. Hence $|\Phi_b| = 2^{\mathfrak{c}}$. ■

Definition 3.12. Let Ω be a non-empty, locally compact space. For $\lambda \in B^b(\Omega)$, define $\kappa_E(\lambda)$ on $E' = M(\Omega)$ by

$$\langle \kappa_E(\lambda), \mu \rangle = \int_\Omega \lambda \,\mathrm{d}\mu \quad (\mu \in M(\Omega)). \tag{3.6}$$

We see immediately that $\kappa_E(\lambda) \in M(\Omega)' = C(\widetilde{\Omega})$ and that we have $\kappa_E(\lambda) \mid \Omega = \lambda$ for $\lambda \in B^b(\Omega)$.

Let $\lambda \in B^b(\Omega)$ and $\mu \in M(\Omega)$. Then $\kappa_E(\lambda) \cdot \mu$ is the measure $\lambda\mu$.

Now take $\lambda_1, \lambda_2 \in B^b(\Omega)$ and $\mu \in M(\Omega)$. Then

$$\begin{aligned}\langle \kappa_E(\lambda_1)\kappa_E(\lambda_2),\, \mu\rangle &= \langle \kappa_E(\lambda_1),\, \kappa_E(\lambda_2)\cdot\mu\rangle = \langle \kappa_E(\lambda_1),\, \lambda_2\mu\rangle \\ &= \int_\Omega \lambda_1\lambda_2 \,\mathrm{d}\mu = \langle \kappa_E(\lambda_1\lambda_2),\, \mu\rangle,\end{aligned}$$

and so $\kappa_E(\lambda_1\lambda_2) = \kappa_E(\lambda_1)\kappa_E(\lambda_2)$. It follows from Corollary 3.5 that $\kappa_E(1_\Omega) = 1_{\widetilde{\Omega}}$, and so the map

$$\kappa_E : B^b(\Omega) \to C(\widetilde{\Omega})$$

is a unital, isometric $*$-isomorphism identifying $B^b(\Omega)$ as a closed, self-adjoint subalgebra of $C(\widetilde{\Omega})$ containing the identity function, and it extends the previously defined map κ_E.

The algebra $\kappa_E(B^b(\Omega))$ is a uniformly closed C^*-subalgebra of $C(\widetilde{\Omega})$. In the case where there is a non-Borel set in Ω, it cannot be that $\kappa_E(B^b(\Omega))$ separates the points of $\widetilde{\Omega}$. For, if this were so, we would have $\kappa_E(B^b(\Omega)) = C(\widetilde{\Omega})$ by the Stone–Weierstrass theorem. However $B^b(\Omega)_{\mathbb{R}}$ is not a complete lattice (the family of characteristic functions of finite subsets of a non-Borel subset of Ω, ordered by inclusion, is an increasing net in $B^b(\Omega)$ that does not have a supremum), but $C(\widetilde{\Omega})_{\mathbb{R}}$ is a complete lattice.

The character space Φ_b is the compact space $\widetilde{\Omega}/\sim$, where

$$\varphi \sim \psi \quad \text{if} \quad \kappa_E(\lambda)(\varphi) = \kappa_E(\lambda)(\psi) \quad (\lambda \in B^b(\Omega)).$$

Since $\mathrm{lin}\{\chi_B : B \in \mathfrak{B}_\Omega\}$ is dense in $B^b(\Omega)$, it follows that

$$\varphi \sim \psi \quad \text{if and only if} \quad \kappa_E(\chi_B)(\varphi) = \kappa_E(\chi_B)(\psi) \quad (B \in \mathfrak{B}_\Omega).$$

DEFINITION 3.13. Let Ω be a non-empty, locally compact space, and take $\varphi, \psi \in \widetilde{\Omega}$. Then φ and ψ are *Borel equivalent* if $\varphi \sim \psi$.

The equivalence class under the relation $\sim$ that contains φ is denoted by $[\varphi]$. Clearly we have $[\varphi] \subset \Omega_{\{x\}}$ for $\varphi \in \widetilde{\Omega}$, where $x = \pi(\varphi)$. Since $C(\Phi_b)_{\mathbb{R}}$ is not a complete lattice, Φ_b is not a Stonean space. We shall make further remarks about the equivalence classes $[\varphi]$ and the space Φ_b in Chapter 4.

For each $B \in \mathfrak{B}_\Omega$, the function $\kappa_E(\chi_B)$ is an idempotent in $C(\widetilde{\Omega})$, and so $\kappa_E(\chi_B)$ is the characteristic function of a clopen subset, say K_B, of $\widetilde{\Omega}$.

DEFINITION 3.14. Let Ω be a non-empty, locally compact space, and let $B \in \mathfrak{B}_\Omega$. Then

$$K_B = \{\varphi \in \widetilde{\Omega} : \kappa_E(\chi_B)(\varphi) = 1\}.$$

Thus

$$\kappa_E(\chi_B) = \chi_{K_B} \quad (B \in \mathfrak{B}_\Omega). \tag{3.7}$$

Clearly $\kappa_E(\chi_B)\,|\,\Omega = \chi_B$, and so $K_B \cap \Omega = B$, whence $\overline{B} \subset K_B$. Let $B, C \in \mathfrak{B}_\Omega$. Then

$$\chi_{B\cap C} = \chi_B \cdot \chi_C \quad \text{and} \quad \chi_{B\cup C} = \chi_B + \chi_C - \chi_B \cdot \chi_C,$$

and so

$$K_B \cap K_C = K_{B\cap C} \quad \text{and} \quad K_B \cup K_C = K_{B\cup C}.$$

In particular, if $B \cap C = \emptyset$, then $\overline{B} \cap \overline{C} = \emptyset$. Suppose that $B \in \mathfrak{B}_\Omega$ and that $x \in \Omega$. Then $\langle \kappa_E(\chi_B), \delta_x\rangle$ is 1 or 0 according as $x \in B$ or $x \notin B$. Thus the map $B \mapsto K_B$ is an injection. We shall use the following immediate proposition later.

PROPOSITION 3.15. *Let Ω be a non-empty, locally compact space, and let $\varphi, \psi \in \widetilde{\Omega}$. Then $\varphi \sim \psi$ if and only if*

$$\varphi \in K_B \ \Leftrightarrow\ \psi \in K_B$$

for each $B \in \mathfrak{B}_\Omega$. ■

Note that the family $\{K_B : B \in \mathfrak{B}_\Omega\}$ is not a base for the topology of $\widetilde{\Omega}$.

Let $B \in \mathfrak{B}_\Omega$ and $\mu \in M(\Omega)$. Then $\kappa(\mu) \in M(\widetilde{\Omega})$, and

$$\kappa(\mu)(K_B) = \langle \chi_{K_B},\, \kappa(\mu)\rangle = \langle \chi_B,\, \mu\rangle = \mu(B). \tag{3.8}$$

Let $\{B_n : n \in \mathbb{N}\}$ be a family in $\mathfrak{B}_\Omega$, and set $B = \bigcup_{n\in\mathbb{N}} B_n$, so that $B \in \mathfrak{B}_\Omega$. The following example shows that, in general, it is not true that $K_B = \bigcup_{n\in\mathbb{N}} K_{B_n}$.

EXAMPLE 3.16. In the special case where $\Omega = S$ is a discrete space, we have $E = c_0(S)$ and $\widetilde{\Omega} = \beta S$, the Stone–Čech compactification of S, and hence $E'' = C(\beta S)$. Further,

$$B^b(S) = \ell^\infty(S) = E''.$$

The above map π takes S^* to the point ∞ of Ω_∞. The fibre $S_{\{\infty\}}$ is not open in βS.

Let $S = \mathbb{N}$. Then we see that, for each $n \in \mathbb{N}$, we have $K_{\{n\}} = \{n\}$ and $K_{\mathbb{N}} = \beta\mathbb{N}$, whereas $\bigcup_{n\in\mathbb{N}} K_{\{n\}} = \mathbb{N}$. Note that, by Phillips' Lemma [1, §2.5], there is no continuous projection of $C(\beta\mathbb{N}) = \ell^\infty$ onto c_0. ■

PROPOSITION 3.17. *Let Ω be a non-empty, locally compact space, and let $\{B_n : n \in \mathbb{N}\}$ be a family in $\mathfrak{B}_\Omega$. Set $B = \bigcup_{n\in\mathbb{N}} B_n$. Then*

$$K_B \setminus \bigcup\{K_{B_n} : n \in \mathbb{N}\} \in \mathcal{K}_\Omega, \tag{3.9}$$

and so $K_B = \overline{\bigcup\{K_{B_n} : n \in \mathbb{N}\}}$.

Proof. Set $K = K_B \setminus \bigcup_{n\in\mathbb{N}} K_{B_n}$.

Each set K_{B_n} is clopen, and so K is a closed subset of $\widetilde{\Omega}$. Hence K is compact in $\widetilde{\Omega}$.

To show that $\operatorname{int} K = \emptyset$, we may suppose that $B_n \subset B_{n+1}$ ($n \in \mathbb{N}$). For each measure $\mu \in M(\Omega)^+$, we have $\mu(B_n) \to \mu(B)$ by the monotone convergence theorem, and so, by (3.8), $\kappa(\mu)(K_{B_n}) \to \kappa(\mu)(K_B)$ as $n \to \infty$.

Assume towards a contradiction that $\operatorname{int} K \neq \emptyset$. Since the space $\widetilde{\Omega}$ is extremely disconnected, there is a non-empty, clopen subset W of $\widetilde{\Omega}$ with $W \subset K$; we have $\chi_W \in E''$. It follows that $W \subset K_B \setminus K_{B_n}$ ($n \in \mathbb{N}$), and so, for each $\mu \in M(\Omega)^+$, we have

$$0 \le \kappa(\mu)(W) \le \kappa(\mu)(K_B \setminus K_{B_n}) = \kappa(\mu)(K_B) - \kappa(\mu)(K_{B_n}) \to 0$$

as $n \to \infty$. Thus

$$\langle \chi_W,\, \mu\rangle = \langle \chi_W,\, \kappa(\mu)\rangle = \kappa(\mu)(W) = 0.$$

This holds for each $\mu \in M(\Omega)^+$, and hence for each $\mu \in M(\Omega)$, and so $\chi_W = 0$ in $E'' = C(\widetilde{\Omega})$. Hence $W = \emptyset$, and this is the required contradiction. ■

Let $x \in \Omega$. What is the set $K_{\{x\}}$? It is easy to see that

$$\{x\} \subset K_{\{x\}} \subset \Omega_{\{x\}}.$$

We *claim* that $K_{\{x\}} \cap \overline{\Omega} = \{x\}$. For this, we may suppose that x is not isolated in Ω, for otherwise the claim follows trivially. Now take $\varphi \in \overline{\Omega} \setminus \{x\}$. There is a net $(x_\alpha : \alpha \in A)$ in

Ω with $x_\alpha \to \varphi$ in $(\widetilde{\Omega}, \sigma)$. The set $\{\alpha \in A : x_\alpha = x\}$ cannot be cofinal in the directed set A (or otherwise $\varphi = x$), and so we may suppose that $(x_\alpha) \subset \Omega \setminus \{x\}$. Since $\kappa_E(\chi_{\{x\}}) \in C(\widetilde{\Omega})$ and $\kappa_E(\chi_{\{x\}})(x_\alpha) = 0$ for each α, we have $\kappa_E(\chi_{\{x\}})(\varphi) = 0$. Thus $\varphi \notin K_{\{x\}}$. This shows that $K_{\{x\}} \cap \overline{\Omega} = \{x\}$, as claimed. We shall see later that $K_{\{x\}} = \{x\}$.

PROPOSITION 3.18. *Let Ω be a non-empty, locally compact space. Then*

$$\pi(K_B) = \overline{B}^{\tau} \quad \text{and} \quad K_B \supset \pi^{-1}(\operatorname{int} B)$$

for each $B \in \mathfrak{B}$.

Proof. Clearly $B \subset \pi(K_B)$, and so $\overline{B}^{\tau} \subset \pi(K_B)$.

For the converse, suppose that $x \in \Omega \setminus \overline{B}^{\tau}$. Then there exists $\lambda \in C_0(\Omega)_{\mathbb{R}}$ with $\lambda \mid B = 1$ and $\lambda(x) = 0$. We have $\chi_B \leq \lambda$ in $E_{\mathbb{R}}$, and so $\kappa_E(\chi_B) \leq \kappa_E(\lambda)$ in $(E'')_{\mathbb{R}}$. The function $\kappa_E(\lambda)$ takes the constant value 0 on the fibre $\Omega_{\{x\}}$, and so $K_B \cap \Omega_{\{x\}} = \emptyset$. Thus $x \notin \pi(K_B)$. This shows that $\pi(K_B) = \overline{B}^{\tau}$.

Set $U = \operatorname{int} B$, and take $x \in U$. Then there exists $\lambda \in C_0(\Omega)_{\mathbb{R}}$ such that $\lambda(x) = 1$ and $\lambda \leq \chi_U$ in $E_{\mathbb{R}}$, and so $\kappa_E(\lambda) \leq \kappa_E(\chi_U)$ in $(E'')_{\mathbb{R}}$. The function $\kappa_E(\lambda)$ takes the constant value 1 on the fibre $\Omega_{\{x\}}$, and so $K_B \supset \Omega_{\{x\}}$. ■

A bounded linear operator. Let Ω_1 and Ω_2 be two compact spaces, and suppose that $\eta : \Omega_1 \to \Omega_2$ is a continuous map. Then we have defined a continuous $*$-homomorphism

$$\theta = \eta^\circ : C(\Omega_2) \to C(\Omega_1).$$

We now have the dual map

$$\theta' : M(\Omega_1) \to M(\Omega_2);$$

the map θ' is a homomorphism of Banach lattices, and it is an isometric isomorphism whenever η is a homeomorphism. More generally, let Ω_1 and Ω_2 be two non-empty, locally compact spaces, and let $\eta : \Omega_1 \to \Omega_2$ be a continuous map. Then (cf. (2.4)) the continuous $*$-homomorphism

$$\eta^\circ : \lambda \mapsto \lambda \circ \eta, \qquad C_0(\Omega_2) \to C^b(\Omega_1), \tag{3.10}$$

may not have its range contained in $C_0(\Omega_1)$. However, suppose that $\eta : \Omega_1 \to \Omega_2$ is a Borel map, so that $\lambda \circ \eta \in B^b(\Omega_1)$, and, for each $\mu \in M(\Omega_1)$, set

$$\nu(\lambda) = \int_{\Omega_1} (\lambda \circ \eta) \, \mathrm{d}\mu \quad (\lambda \in C_0(\Omega_2)).$$

It is clear that ν is a bounded linear functional on $C_0(\Omega_2)$, and so we may regard ν as a measure on Ω_2; we set

$$\overline{\eta}\,(\mu) = \nu \quad (\mu \in M(\Omega_1)),$$

so that $\overline{\eta} : M(\Omega_1) \to M(\Omega_2)$ is a bounded linear operator with $\|\overline{\eta}\| = 1$ such that

$$\int_{\Omega_2} \lambda \, \mathrm{d}\overline{\eta}(\mu) = \int_{\Omega_1} (\lambda \circ \eta) \, \mathrm{d}\mu \quad (\lambda \in C_0(\Omega_2),\, \mu \in M(\Omega_1)). \tag{3.11}$$

It follows that

$$\overline{\eta}\,(\mu)(B) = \mu(\eta^{-1}(B)) \quad (B \in \mathfrak{B}_{\Omega_2},\, \mu \in M(\Omega_1)). \tag{3.12}$$

In particular, $\overline{\eta}(\delta_x) = \delta_{\eta(x)}$ $(x \in \Omega_1)$, so that $\overline{\eta} | \Omega_1 = \eta$.

Suppose that $\mu_1, \mu_2 \in M(\Omega_1)^+$ with $\mu_1 \ll \mu_2$. Then $\overline{\eta}\,(\mu_1), \overline{\eta}\,(\mu_2) \in M(\Omega_2)^+$, and it is clear from (3.12) that we have $\overline{\eta}\,(\mu_1) \ll \overline{\eta}\,(\mu_2)$. It follows that

$$\overline{\eta}\,(L^1(\Omega_1, \mu)) \subset L^1(\Omega_2, \overline{\eta}\,(\mu)). \tag{3.13}$$

We shall make further remarks about the maps $\overline{\eta}$ and $\overline{\eta}\,'$ in the next chapter.

Conversely, suppose that $T : M(\Omega_1) \to M(\Omega_2)$ is an isometric Banach lattice isomorphism. Then $T \,|\, \operatorname{ex} P(\Omega_1)$ is a bijection from Ω_1 to Ω_2, and so $|\Omega_1| = |\Omega_2|$.

Proposition 3.19. *Let Ω_1 and Ω_2 be two non-empty, locally compact spaces, and let $\eta : \Omega_1 \to \Omega_2$ be a Borel map. Suppose that η is an injection. Then*

$$\|\overline{\eta}\,(\mu)\| = \|\mu\| \quad (\mu \in M(\Omega_1)).$$

In particular, $\overline{\eta} : M(\Omega_1) \to M(\Omega_2)$ is an injection.

Proof. Take $\mu \in M(\Omega_1)$ with $\|\mu\| = 1$, say $\mu = \mu_1 - \mu_2 + \mathrm{i}(\mu_3 - \mu_4)$, where $\mu_j \in M(\Omega_1)^+$ for $j = 1, 2, 3, 4$. Set $\nu_j = \overline{\eta}\,(\mu_j) \in M(\Omega_2)^+$ for $j = 1, 2, 3, 4$, and set

$$\nu = \overline{\eta}\,(\mu) = \nu_1 - \nu_2 + \mathrm{i}(\nu_3 - \nu_4).$$

Take $\varepsilon > 0$. For $j = 1, 2, 3, 4$, there exist Borel sets B_j in Ω_2 such that $\nu_j(B) \geq 0$ for each Borel subset B of B_j and

$$\sum_{j=1}^{4} \nu_j(B_j) > \|\nu\| - \varepsilon.$$

Set $C_j = \eta^{-1}(B_j)$, a Borel set in Ω_1, so that $\mu_j(C_j) = \nu_j(B_j)$.

Since η is an injection, the sets C_1, C_2, C_3, C_4 are pairwise disjoint, and so

$$\|\mu\| \geq \sum_{j=1}^{4} \mu_j(C_j) = \sum_{j=1}^{4} \nu_j(B_j) > \|\nu\| - \varepsilon.$$

This holds for each $\varepsilon > 0$, and so $\|\mu\| = \|\nu\|$. ■

Corollary 3.20. *Let Ω_1 and Ω_2 be two uncountable, compact, metrizable spaces. Then the spaces $M(\Omega_1)$ and $M(\Omega_2)$ are isometrically isomorphic as Banach spaces and lattices.*

Proof. By Proposition 2.2(ii), there is a map $\eta : \Omega_1 \to \Omega_2$ which is a Borel isomorphism. As above, we define $\overline{\eta} : M(\Omega_1) \to M(\Omega_2)$. By Proposition 3.19, $\overline{\eta}$ is an isometric isomorphism of Banach spaces. Clearly $\overline{\eta}$ preserves the lattice operations. ■

However $\overline{\eta}$ is not necessarily a surjection even when η is a continuous surjection: for a counter-example, let $\Omega_1 = \mathbb{I}_d$ and $\Omega_2 = \mathbb{I}$, and take η to be the identity map. We shall give an example for which $\overline{\eta}$ is a surjection in Proposition 5.2(i). In the case where Ω_1 is compact and η is a surjection, $\overline{\eta}$ is obviously a surjection.

4. The topological structure of $\widetilde{\Omega}$

Submodules of $M(\Omega)$ and clopen subspaces of $\widetilde{\Omega}$. Let Ω be a non-empty, locally compact space, and again set $E = C_0(\Omega)$. We are identifying $M(\Omega)$ as the dual module E' of E.

Let X be a Banach E-submodule of $M(\Omega)$, and let $j_X : X \to M(\Omega)$ denote the injection. By Proposition 1.17, X° is a weak-$*$ closed ideal in $C(\widetilde{\Omega})$, and so the hull of X° is a closed subset, say L, of $\widetilde{\Omega}$. The ideal X° has a bounded approximate identity, say (Λ_α), in $X^\circ_{[1]}$; since $C(\widetilde{\Omega})_{[1]}$ is weak-$*$ compact and X° is weak-$*$ closed, (Λ_α) has a limit, say Λ, in $X^\circ_{[1]}$. Certainly $\Lambda(\varphi) = 1$ $(\varphi \in L)$, and so $\Lambda = \chi_L$. This shows that L is a clopen subset of $\widetilde{\Omega}$. Set $\widetilde{\Omega}_X = \widetilde{\Omega} \setminus L$, so that $\widetilde{\Omega}_X$ is also a clopen subset of $\widetilde{\Omega}$. Clearly we can identify X' with the commutative C^*-algebra $C(\widetilde{\Omega}_X)$, and so $\widetilde{\Omega}_X$ is the character space of X'. In this way, j'_X is just the restriction map from $C(\widetilde{\Omega})$ to $C(\widetilde{\Omega}_X)$; in particular, $j'_X(1_{\widetilde{\Omega}})$ is the characteristic function of $\widetilde{\Omega}_X$.

Conversely, let L be a clopen subset of $\widetilde{\Omega}$, so that $\chi_L \in C(\widetilde{\Omega})$, and define

$$X_L = \{\chi_L \cdot \mu : \mu \in M(\Omega)\}.$$

Then X_L is a $\|\cdot\|$-closed E-submodule of $M(\Omega)$, and clearly $\widetilde{\Omega}_{X_L} = L$. We have established the following result; it is essentially a special case of [112, Theorem III.2.7]. The collections of $\|\cdot\|$-closed submodules of $M(\Omega)$ and of clopen subsets of $\widetilde{\Omega}$ are both ordered by inclusion.

Proposition 4.1. *Let Ω be a non-empty, locally compact space. Then the above correspondence is an isotonic bijection between the collections of $\|\cdot\|$-closed submodules of $M(\Omega)$ and of clopen subsets of the hyper-Stonean envelope $\widetilde{\Omega}$.*

Further, for each Banach submodule X of $M(\Omega)$, there is a unique Banach submodule Y of $M(\Omega)$ such that $M(\Omega) = X \oplus Y$. ∎

Corollary 4.2. *Let Ω be a non-empty, locally compact space, and let $\varphi \in \widetilde{\Omega}$. Then φ is an isolated point of $\widetilde{\Omega}$ if and only if $\varphi \in \Omega$.*

Proof. Let $x \in \Omega$. Then $X = \mathbb{C}\delta_x$ is a one-dimensional submodule of $M(\Omega)$, and so $\widetilde{\Omega}_X$ is a singleton. Since $\varepsilon_x \in X'$, we have $\widetilde{\Omega}_X = \{x\}$, and so x is an isolated point in $\widetilde{\Omega}$.

Conversely, suppose that φ is an isolated point in $\widetilde{\Omega}$, and let X be the submodule of $M(\Omega)$ corresponding to the clopen subset $\{\varphi\}$ of $\widetilde{\Omega}$. Then the space $X' = C(\{\varphi\})$ is one-dimensional, and so X is one-dimensional. Let $\mu \in X \setminus \{0\}$. Assume towards a contradiction that $\operatorname{supp}\mu$ contains two distinct points x and y, and take $\lambda \in C(\Omega)$ with $\lambda(x) = 1$ and $\lambda(y) = 0$. Then $\lambda\mu \in X$, but $\lambda\mu \notin \mathbb{C}\mu$, a contradiction. Thus $\operatorname{supp}\mu = \{x\}$ for some $x \in \Omega$, and hence $\mu = \delta_x$ and $\varphi = x$. ∎

Recovery of Ω from $\widetilde{\Omega}$. Let Ω be a non-empty, locally compact space. Corollary 4.2 shows that we can recover the set Ω from the hyper-Stonean envelope $\widetilde{\Omega}$; indeed, Ω was identified with the set of isolated points of $\widetilde{\Omega}$. Thus, if Ω_1 and Ω_2 are locally compact spaces such that $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ are homeomorphic, then necessarily we have $|\Omega_1| = |\Omega_2|$. However, we shall now show that we cannot recover the topology τ on a compact space Ω from its hyper-Stonean envelope; indeed, we cannot even recover $C(\Omega)$ as a Banach space, even when we restrict ourselves to countable, compact spaces.

For example, set $\Omega = \mathbb{N}$. Then $\widetilde{\Omega}$ is the space $\beta\mathbb{N}$, and so $\widetilde{\Omega}$ is homeomorphic to $\beta\mathbb{N}$.

Now let (Ω, τ) be any countable, locally compact space, and take $x \in \Omega$. Since $\iota(x)$ is an isolated point in $\widetilde{\Omega}$, we may say that $\delta_x \in C(\widetilde{\Omega})$; further $\delta_x \cdot \delta_y = 0$ whenever $x, y \in \Omega$ with $x \neq y$, and $\delta_x \cdot \delta_x = \delta_x$ whenever $x \in \Omega$. Set $L = \operatorname{lin}\{\delta_x : x \in \Omega\} \subset C(\widetilde{\Omega})$. Then the product of two elements of L is determined independently of the topology τ. We *claim* that L is weak-$*$ dense in $C(\widetilde{\Omega})$. Indeed, assume towards a contradiction that L is not weak-$*$ dense in $C(\widetilde{\Omega})$. Then there exists a non-zero, weak-$*$ continuous element $\mu \in M(\widetilde{\Omega})$ such that $\mu \mid L = 0$. By Proposition 1.1(ii), it follows that $\mu \in M(\Omega)$. But $M(\Omega) = \ell^1(\Omega)$ because Ω is countable, and $\langle \delta_x, \mu \rangle = 0$ $(x \in \Omega)$, whence $\mu = 0$, the required contradiction. Hence L is weak-$*$ dense in $C(\widetilde{\Omega})$, and so the structure of $C(\widetilde{\Omega})$ is determined as a Banach algebra independently of the topology τ. We have established the following result.

THEOREM 4.3. *Let (Ω, τ) be a countable, locally compact space. Then $\widetilde{\Omega}$ is homeomorphic to $\beta\mathbb{N}$ with its usual topology.* ■

It is certainly not the case that any two countable, compact spaces are homeomorphic. For example, consider the compact spaces $\omega + 1$, $2 \cdot \omega + 1$, and $\omega^\omega + 1$, where ω is the first infinite ordinal, and the spaces are taken with the order topology; these three spaces are countable and compact, but no two of them are mutually homeomorphic. In particular, there are three distinct topologies on each infinite, countable set rendering it a compact space. (In fact, there are at least $\aleph_1$ such topologies.) The two Banach spaces $C(\omega+1)$ and $C(2\cdot\omega+1)$ are linearly homeomorphic, but the Banach spaces $C(\omega+1)$ and $C(\omega^\omega+1)$ are not linearly homeomorphic. For these remarks on Banach spaces, see [106, Notes 2.5.14], for example.

Partitions of $\widetilde{\Omega}$. Let Ω be a non-empty, locally compact space.

We denote by

$$j_d : M_d(\Omega) \to M(\Omega) \quad \text{and} \quad j_c : M_c(\Omega) \to M(\Omega)$$

the natural injections. Clearly $M_d(\Omega)$ and $M_c(\Omega)$ are both closed E-submodules of $M(\Omega)$, and j_d and j_c are E-module homomorphisms. We recall that $M_d(\Omega)$ is $\sigma(E', E)$-dense in $M(\Omega)$. Thus, by Propositions 1.17 and 4.1, $C(\widetilde{\Omega})$ is the direct sum of two closed ideals,

$$j_d'(C(\widetilde{\Omega})) = M_d(\Omega)' = \ell^\infty(\Omega) \quad \text{and} \quad j_c'(C(\widetilde{\Omega})) = M_c(\Omega)'.$$

The character spaces of these two ideals are denoted by $\widetilde{\Omega}_d$ and $\widetilde{\Omega}_c$, respectively, so that $\{\widetilde{\Omega}_d, \widetilde{\Omega}_c\}$ is partition of $\widetilde{\Omega}$ into clopen sets. We have $\Omega \subset \widetilde{\Omega}_d$. Take $\Lambda \in C(\widetilde{\Omega})$ with $\Lambda \mid \Omega = 0$. Then $\Lambda \mid M_d(\Omega) = 0$, and so $\varphi(\Lambda) = 0$ $(\varphi \in \widetilde{\Omega}_d)$. Thus Ω is dense in $\widetilde{\Omega}_d$, and hence $\overline{\Omega} = \widetilde{\Omega}_d$. Clearly $\widetilde{\Omega}_d$ can be identified with $\beta\Omega_d$, the Stone–Čech compactification of Ω with the discrete topology. We now say that

$$\{\overline{\Omega}, \widetilde{\Omega}_c\} = \{\beta\Omega_d, \widetilde{\Omega}_c\}$$

is a partition of $\widetilde{\Omega}$ into clopen sets.

Let $\lambda \in B^b(\Omega)$ and $\mu \in M(\Omega)$. It is clear that

$$\langle \kappa_E(\lambda)\,|\,\widetilde{\Omega}_d,\, \mu\rangle = \langle \kappa_E(\lambda),\, \mu_d\rangle, \qquad \langle \kappa_E(\lambda)\,|\,\widetilde{\Omega}_c,\, \mu\rangle = \langle \kappa_E(\lambda),\, \mu_c\rangle.$$

Thus $\kappa_E(\lambda)\,|\,\widetilde{\Omega}_d = j_d(\lambda)$; we caution that $\kappa_E(\lambda) = j_d(\lambda)$ for each $\lambda \in B^b(\Omega)$ only if Ω is discrete.

For each $x \in \Omega$, we have $\langle \kappa_E(\chi_{\{x\}}),\, \mu_c\rangle = 0$, and so $\kappa_E(\chi_{\{x\}}) = \kappa_E(\chi_{\{x\}})\,|\,\widetilde{\Omega}_d$ and $K_{\{x\}} \subset \overline{\Omega}$. Hence we can now see that $K_{\{x\}} = \{x\}$; this again shows that the set $\{x\}$ is open in $(\widetilde{\Omega}, \sigma)$ for each $x \in \Omega$, and so Ω is open in $\widetilde{\Omega}$. In particular, $\kappa_E(\chi_{\{x\}}) = \chi_{\{x\}}$, and so the equivalence class $[x]$ is just the singleton $\{x\}$.

Let μ be a continuous positive measure on Ω such that μ is either σ-finite or the left Haar measure on a locally compact group. Then, as in (2.8),

$$M(\Omega) = \ell^1(\Omega) \oplus_1 L^1(\Omega, \mu) \oplus_1 M_s(\Omega, \mu).$$

Each of the three spaces $M_d(\Omega, \mu)$, $M_{ac}(\Omega, \mu)$, and $M_s(\Omega, \mu)$ is a closed, complemented E-submodule of $M(\Omega)$, and so is an introverted space; we obtain a further partition of $\widetilde{\Omega}$ into three corresponding clopen subsets. In this case, we have

$$M(\Omega)' = \ell^\infty(\Omega) \oplus_1 L^\infty(\Omega, \mu) \oplus_1 M_s(\Omega, \mu)'.$$

The character space of the C^*-algebra $L^\infty(\Omega, \mu)$ has already been called Φ_μ; the character space of $M_s(\Omega, \mu)'$ is denoted by $\Phi_{s,\mu}$, and so we have a partition

$$\{\beta\Omega_d, \Phi_\mu, \Phi_{s,\mu}\}$$

of $\widetilde{\Omega}$ into clopen subsets; thus

$$C(\widetilde{\Omega}) = C(\beta\Omega_d) \oplus_\infty C(\Phi_\mu) \oplus_\infty C(\Phi_{s,\mu}).$$

Let Ω be a non-empty, locally compact space, and let μ be a positive measure on Ω. We recall that the map $\pi_\mu : \Phi_\mu \to \operatorname{supp}\mu \cup \{\infty\}$ was defined in equation (2.12).

Proposition 4.4. *Let Ω be a non-empty, locally compact space, and let μ be a positive measure on Ω. Then:*

(i) $\pi|\Phi_\mu = \pi_\mu$;
(ii) $\operatorname{supp}\mu \subset \pi(\Phi_\mu) \subset (\operatorname{supp}\mu) \cup \{\infty\}$.

Proof. (i) Take $\varphi \in \Phi_\mu$, and set $\pi(\varphi) = x \in \Omega_\infty$. For each $U \in \mathcal{N}_x$, there is an element $\lambda \in C_0(\Omega)^\#$ and $V \in \mathcal{N}_x$ with $\lambda(y) = 1$ $(y \in V)$ and $0 \le \lambda \le \chi_U$. It follows that

$$\varphi(\kappa_E(\chi_U)) \ge \varphi(\kappa_E(\lambda)) = \mathcal{G}_\mu(\lambda)(\varphi) = \lim_{B \to \varphi} \frac{1}{\mu(B)} \int_B \lambda \, \mathrm{d}\mu$$

by (2.11). In the above limit, we may suppose that $B \subset V$, and so $\varphi(\kappa_E(\chi_U)) \ge 1 = \varepsilon_x(\lambda)$. This shows that $U \in \varphi$, and hence we have $\mathcal{N}_x \subset \varphi$. By the definition of x, we have $\pi_\mu(\varphi) = x$, and so $\pi_\mu(\varphi) = \pi(\varphi)$.

(ii) We know that $\pi(\Phi_\mu) \subset (\operatorname{supp}\mu) \cup \{\infty\}$.

Now set $U = \Omega \setminus \operatorname{supp} \mu$; we may suppose that $U \neq \emptyset$. By Proposition 3.18, we have $K_U \supset \pi^{-1}(U)$. Also $\mu(U) = 0$, and so $K_U \cap \Phi_\mu = \emptyset$. Thus $\pi^{-1}(U) \cap \Phi_\mu = \emptyset$, and so $\pi(\Phi_\mu) \cap U = \emptyset$. This shows that $\operatorname{supp} \mu \subset \pi(\Phi_\mu)$. ■

Again let μ be a positive measure on Ω. Take $\Lambda \in C(\widetilde{\Omega}) = M(\Omega)'$, and set

$$\Lambda_\mu = \Lambda \,|\, L^1(\Omega, \mu) \in L^\infty(\Omega, \mu).$$

Then, following our identifications, we have

$$\mathcal{G}_\mu(\Lambda_\mu) = \Lambda \,|\, \Phi_\mu. \tag{4.1}$$

It follows that the notation $K_B \cap \Phi_\mu$ for $B \in \mathfrak{B}_\Omega$ is consistent with that used earlier in (2.10).

We identify $L^1(\Omega, \mu)''$ with $M(\Phi_\mu)$. It follows from Theorem 3.8 that the canonical image of $L^1(\Omega, \mu)$ in $M(\Phi_\mu)$ is given by

$$L^1(\Omega, \mu) = \{\mathrm{M} \in M(\Phi_\mu) : \mathrm{M}(K) = 0 \ (K \in \mathcal{K}_{\Phi_\mu})\}. \tag{4.2}$$

Let $\mu, \nu \in M(\Omega)^+$. Then it is clear that $\Phi_\mu \subset \Phi_\nu$ if and only if $\mu \ll \nu$, that $\Phi_\mu \cap \Phi_\nu = \emptyset$ if and only if $\mu \perp \nu$, that $\Phi_{\mu+\nu} = \Phi_\mu \cup \Phi_\nu$, and that $\Phi_{\mu\wedge\nu} = \Phi_\mu \cap \Phi_\nu$. These remarks are also contained in [40, §4]. Let (μ_n) be a sequence of measures in $M(\Omega)^+$, and set $\mu = \sum_{n=1}^\infty \mu_n/2^n$. Then $\mu \in M(\Omega)^+$, and

$$\Phi_\mu = \overline{\bigcup \{\Phi_{\mu_n} : n \in \mathbb{N}\}}. \tag{4.3}$$

As in Definition 3.9, we have an embedding $\kappa_E : B^b(\Omega) \to C(\widetilde{\Omega})$. Let μ be a positive measure on Ω. Then we have a restriction map

$$\rho_\mu : C(\widetilde{\Omega}) \to C(\Phi_\mu).$$

On the other hand, there is a quotient map

$$q_\mu : B^b(\Omega) \to L^\infty(\Omega, \mu),$$

formed by identifying $\lambda \in B^b(\Omega)$ with its equivalence class in $L^\infty(\Omega, \mu)$. (In fact, every equivalence class in $L^\infty(\Omega, \mu)$ contains a representative in the second Baire class; see [65, (4.1.3)].) We have

$$\langle q_\mu(\lambda),\, f\rangle = \int_\Omega f\lambda \,\mathrm{d}\mu = \langle \kappa_E(\lambda), f\mu\rangle \qquad (f \in L^1(\Omega, \mu)).$$

Hence, by (4.1), we have

$$\mathcal{G}_\mu(q_\mu(\lambda)) = \rho_\mu(\kappa_E(\lambda)) \qquad (\lambda \in B^b(\Omega)),$$

whence $\mathcal{G}_\mu \circ q_\mu = \rho_\mu \circ \kappa_E$; this shows that the diagram

$$\begin{tikzcd}
B^b(\Omega) \arrow[r, tail, "\kappa_E"] \arrow[d, "q_\mu"'] & C(\widetilde{\Omega}) \arrow[d, "\rho_\mu"] \\
L^\infty(\Omega, \mu) \arrow[r, tail, two heads, "\mathcal{G}_\mu"] & C(\Phi_\mu)
\end{tikzcd}$$

is commutative, and that $\kappa_E(B^b(\Omega)) \,|\, \Phi_\mu = C(\Phi_\mu)$.

Definition 4.5. Let Ω be a non-empty, locally compact space. Then

$$U_\Omega = \bigcup\{\Phi_\mu : \mu \in M(\Omega)^+\}.$$

Clearly a point $\varphi \in U_\Omega$ belongs to Φ_μ if and only if $\varphi(\lambda) = 0$ for each $\lambda \in B^b(\Omega)$ such that

$$\int_\Omega |\lambda| \,\mathrm{d}\mu = 0.$$

In the case where Ω is discrete, the corresponding set U_Ω is the set of ultrafilters on Ω that contain a countable set; for example, $U_{\mathbb{N}} = \beta\mathbb{N}$.

Proposition 4.6. *Let Ω be a non-empty, locally compact space. Then U_Ω is a dense, open subset of $\widetilde{\Omega}$ and $\beta U_\Omega = \widetilde{\Omega}$. Further, the space $\kappa_E(B^b(\Omega))$ separates the points of U_Ω.*

Proof. Clearly U_Ω is an open subset of $\widetilde{\Omega}$.

To show that U_Ω is dense in $\widetilde{\Omega}$, let $\Lambda \in C(\widetilde{\Omega})$ be such that $\Lambda \,|\, U_\Omega = 0$. Then, for each $\mu \in M(\Omega)^+$, we see that $\Lambda \,|\, \Phi_\mu$, regarded as a linear functional on $L^1(\Omega, \mu)$, is zero, and so $\Lambda = 0$. This implies that U_Ω is dense in $\widetilde{\Omega}$. Thus $\beta U_\Omega = \widetilde{\Omega}$.

Now take $\varphi, \psi \in U_\Omega$ with $\varphi \neq \psi$. Since $\Phi_{\mu+\nu} = \Phi_\mu \cup \Phi_\nu$ $(\mu, \nu \in M(\Omega)^+)$, we may suppose that there exists $\mu \in M(\Omega)^+$ such that $\varphi, \psi \in \Phi_\mu$. Further, since the map $\rho_\mu \circ \kappa_E : B^b(\Omega) \to C(\Phi_\mu)$ is an epimorphism, $\kappa_E(B^b(\Omega))$ separates φ and ψ. ∎

Corollary 4.7. *Let Ω be a non-empty, locally compact space, let $x \in \Omega$, and let $N \in \mathcal{N}_x$. Then each $\psi \in \pi^{-1}(N)$ is in the weak-$*$ closure of the set*

$$\{\mu_C : \mu \in M(\Omega)^+,\, C \in \mathfrak{B}_\mu,\, C \subset N\}.$$

Proof. Let $\psi \in \Omega_{\{x\}}$. By Proposition 4.6, it suffices to suppose that $\psi \in \pi^{-1}(N) \cap U_\Omega$, and hence that $\psi \in \Phi_\mu$ for some $\mu \in M(\Omega)^+$. Thus the result now follows from equation (2.11). ∎

Suppose that Ω is not scattered. Then it is not true that the family $\{K_B \cap U_\Omega : B \in \mathfrak{B}\}$ forms a base for the topology of U_Ω. For take $\mu \in M(\Omega)^+$ such that $\Phi_\mu \cap \Omega = \emptyset$. It cannot be that Φ_μ contains a set of the form $K_B \cap U_\Omega$ because $K_B \cap U_\Omega \cap \Omega = K_B \cap \Omega = B$ for any non-empty $B \in \mathfrak{B}$.

Let $\mathcal{F} = \{\nu_i : i \in I\}$ be a maximal singular family of positive measures on Ω, as in Chapter 2. The corresponding clopen subsets of $\widetilde{\Omega}$ are then called Φ_i. It follows from (2.9) that

$$C(\widetilde{\Omega}) = M(\Omega)' = \bigoplus\nolimits_\infty \{C(\Phi_i) : i \in I\}.$$

Proposition 4.8. *Let Ω be a non-empty, locally compact space, and let $\mathcal{F} = \{\nu_i : i \in I\}$ be a maximal singular family of positive measures on Ω. Then the family $\{\Phi_i : i \in I\}$ is pairwise disjoint, and $U_{\mathcal{F}} := \bigcup\{\Omega_i : i \in I\}$ is a dense, open subset of $\widetilde{\Omega}$ with $\beta U_{\mathcal{F}} = \widetilde{\Omega}$.*

Proof. This is essentially the same as the proof of Proposition 4.6. ∎

In the case where Ω is discrete, so that $\mathcal{F}$ is the collection of point masses, we have $U_{\mathcal{F}} = \Omega \subset \beta\Omega$.

Proposition 4.9. *Let Ω be a non-empty, locally compact space.*

- (i) *Let $\varphi \in \widetilde{\Omega}$. Then $\varphi \in U_\Omega$ if and only if φ has a basis of clopen neighbourhoods such that each set in the basis satisfies* CCC *on clopen subsets.*
- (ii) *Let L be a clopen subset of $\widetilde{\Omega}$ that satisfies* CCC *on clopen subsets. Then there is a measure $\mu \in M(\Omega)^+$ such that $L = \Phi_\mu$.*

Proof. Let $\mathcal{F} = \{\nu_i : i \in I\}$ be a maximal singular family of measures in $M(\Omega)^+$, and let Φ_i be as above for $i \in I$; we may suppose that $\|\nu_i\| = 1$ $(i \in I)$.

(i) Suppose that $\varphi \in \Phi_\mu$, where $\mu \in M(\Omega)^+$. Then φ has a neighbourhood basis of clopen sets, and each set in this basis satisfies CCC on clopen subsets by Proposition 2.15(iii).

Suppose that $\varphi \notin U_\Omega$, and let V be a clopen neighbourhood of φ. By equation (4.3), the set $\{i \in I : V \cap \Phi_i \neq \emptyset\}$ is not countable, and so V does not satisfy CCC on clopen subsets.

(ii) Clearly $\{\Phi_i \cap L : i \in I\}$ is a pairwise disjoint family of clopen subsets, and so, by hypothesis, there is a countable subset J of I such that $\Phi_i \cap L \neq \emptyset$ if and only if $i \in J$. Set

$$V = \bigcup\{\Phi_i \cap L : i \in J\} \quad \text{and} \quad F = \overline{V}.$$

Then V is open in L, and F is a clopen subset of L because L is a Stonean space. The set $L \setminus F$ is a clopen subset of $\widetilde{\Omega}$ such that $(L \setminus F) \cap \Phi_i = \emptyset$ $(i \in I)$. By Proposition 4.8, $\bigcup\{\Phi_i : i \in I\}$ is dense in $\widetilde{\Omega}$, and so $L \setminus F = \emptyset$. By (4.3), there exists $\mu \in M(\Omega)^+$ such that $L = \Phi_\mu$. ■

Proposition 4.10. *Let Ω be a non-empty, locally compact space, and let $\mathcal{F}_c$ and $\mathcal{G}_c$ be two maximal singular families of positive, continuous measures on Ω. Then $|\mathcal{F}_c| = |\mathcal{G}_c|$.*

Proof. Suppose that $\mathcal{F}_c = \{\mu_i : i \in I\}$ and $\mathcal{G}_c = \{\nu_j : j \in J\}$, where $\mu_i, \nu_j \in M_c(\Omega)^+$. We *claim* that $|I| = |J|$.

We may suppose that I and J are infinite.

Assume towards a contradiction that $|I| < |J|$. For each $i \in I$, consider the set

$$H_i = \{j \in J : \Phi_{\nu_j} \cap \Phi_{\mu_i} \neq \emptyset\}.$$

By Proposition 2.15(iii), Φ_{μ_i} satisfies CCC, and so it follows that $|H_i| \leq \aleph_0$. Also we have $\bigcup\{H_i : i \in I\} = J$ because $\mathcal{F}_c$ is a maximal family. Thus $|J| \leq \aleph_0 \cdot |I| = |I|$, a contradiction.

We conclude that $|I| = |J|$. ■

A homomorphism. Let Ω_1 and Ω_2 be two non-empty, locally compact spaces, and then take $\eta : \Omega_1 \to \Omega_2$ to be a continuous map; we have defined in (3.11) the bounded linear operator $\overline{\eta}'' : M(\Omega_1) \to M(\Omega_2)$ by the formula

$$\int_{\Omega_2} \lambda \, \mathrm{d}\overline{\eta}(\mu) = \int_{\Omega_1} (\lambda \circ \eta) \, \mathrm{d}\mu \quad (\lambda \in C_0(\Omega_2),\, \mu \in M(\Omega_1)). \tag{4.4}$$

We now have bounded linear operators

$$\overline{\eta}' : C(\widetilde{\Omega}_2) \to C(\widetilde{\Omega}_1) \quad \text{and} \quad \overline{\eta}'' : M(\widetilde{\Omega}_1) \to M(\widetilde{\Omega}_2).$$

In the case where Ω_1 and Ω_2 are compact spaces, we have $\overline{\eta}' = \theta''$, where $\theta = \eta^\circ$, and so $\overline{\eta}' : C(\widetilde{\Omega}_2) \to C(\widetilde{\Omega}_1)$ is a continuous $*$-homomorphism. It does not seem to be immediate that $\overline{\eta}\,'$ is a homomorphism in the general case; we shall now prove this. We are grateful to Colin Graham for an active discussion on this result.

Equation (4.4) holds for $\lambda \in C_0(\Omega_2)$; we first note that it also holds for $\lambda \in B^b(\Omega_2)$. Note that $\lambda \circ \eta \in B^b(\Omega_1)$, regarded as a subset of $C(\widetilde{\Omega}_1)$, whenever $\lambda \in B^b(\Omega_2)$, and so $\langle \lambda \circ \eta, \mu\rangle$ and $(\lambda \circ \eta)\cdot\mu$ are defined.

Let $\mu \in M(\Omega_1)$, and set $\nu = \overline{\eta}(\mu)$. Consider $\lambda \in B^b(\Omega_2)$. There is a sequence (λ_k) in $C_0(\Omega_2)$ such that $|\lambda_k|_{\Omega_2} \leq |\lambda|_{\Omega_2}$ $(k \in \mathbb{N})$ and such that $\lambda_k \to \lambda$ (p.p. ν) on Ω_2. Thus

$$|\lambda_k \circ \eta|_{\Omega_1} \leq |\lambda|_{\Omega_2} \quad (k \in \mathbb{N})$$

and $\lambda_k \circ \eta \to \lambda \circ \eta$ (p.p. μ) on Ω_1. (If the first convergence fails on the set B, where $\nu(B) = 0$, then the second convergence holds off the set $\eta^{-1}(B)$, and $\mu(\eta^{-1}(B)) = 0$ by (3.12).) Equation (4.4) holds whenever λ is replaced by λ_k; it follows from the dominated convergence theorem that (4.4) holds for our $\lambda \in B^b(\Omega_2)$.

Theorem 4.11. *Let Ω_1 and Ω_2 be two non-empty, locally compact spaces, and consider a continuous map $\eta : \Omega_1 \to \Omega_2$. Then the map $\overline{\eta}\,' : C(\widetilde{\Omega}_2) \to C(\widetilde{\Omega}_1)$ is a continuous $*$-homomorphism. Further, $\overline{\eta}\,'(C(\Phi_{\overline{\eta}(\mu)})) \subset C(\Phi_\mu)$ for each $\mu \in M(\Omega_1)^+$.*

Proof. Take $\mu \in M(\Omega_1)$. Clearly

$$\langle \overline{\eta}\,'(\lambda),\, \mu\rangle = \langle \lambda,\, \overline{\eta}\,(\mu)\rangle = \langle \lambda \circ \eta,\, \mu\rangle \quad (\mu \in M(\Omega_1),\, \lambda \in C_0(\Omega_2))$$

by the above remark, and so $\overline{\eta}\,'(\lambda) = \lambda \circ \eta \in C(\widetilde{\Omega}_1)$ for all $\lambda \in C_0(\Omega_2)$.

Let $\lambda \in B^b(\Omega_2)$ and $\mu \in M(\Omega_1)$. We first *claim* that

$$\overline{\eta}\,(\overline{\eta}\,'(\lambda)\cdot\mu) = \overline{\eta}\,((\lambda \circ \eta)\cdot\mu) = \lambda\cdot\overline{\eta}\,(\mu). \tag{4.5}$$

Indeed, for all $\lambda_1 \in C_0(\Omega_2)$, we have

$$\begin{aligned}\langle \lambda_1,\, \overline{\eta}\,((\lambda \circ \eta)\cdot\mu)\rangle &= \int_{\Omega_2} (\lambda_1 \circ \eta)(\lambda \circ \eta)\,\mathrm{d}\mu = \int_{\Omega_2} (\lambda_1\lambda \circ \eta)\,\mathrm{d}\mu\\ &= \langle \lambda_1\lambda,\, \overline{\eta}\,(\mu)\rangle = \langle \lambda_1,\, \lambda\cdot\overline{\eta}\,(\mu)\rangle,\end{aligned}$$

giving (4.5).

We recall that we write Λ_ν for $\Lambda | L^1(\Omega_2, \nu)$ when $\Lambda \in C(\widetilde{\Omega}_2)$, so that Λ_ν is regarded as an element of $B^b(\Omega_2)$; we shall write $\Lambda_{1,\nu}$ for $(\Lambda_1)_\nu$, etc.

Now take $\mu \in M(\Omega_1)^+$, and set $\nu = \overline{\eta}(\mu)$. Let $\Lambda_1, \Lambda_2 \in C(\widetilde{\Omega}_2)$. Then we have

$$\begin{aligned}\langle \overline{\eta}\,'(\Lambda_1\Lambda_2),\, \mu\rangle &= \langle \Lambda_1\Lambda_2,\, \nu\rangle = \langle (\Lambda_1\Lambda_2)_\nu,\, \nu\rangle = \langle \Lambda_{1,\nu}\Lambda_{2,\nu},\, \nu\rangle\\ &= \langle \Lambda_{1,\nu},\, \Lambda_{2,\nu}\cdot\nu\rangle = \langle \Lambda_{1,\nu},\, \overline{\eta}\,(\overline{\eta}\,'(\Lambda_{2,\nu})\cdot\mu)\rangle\end{aligned}$$

by (4.5). Also, we have

$$\langle \overline{\eta}\,'(\Lambda_1)\overline{\eta}\,'(\Lambda_2),\, \mu\rangle = \langle \overline{\eta}\,'(\Lambda_1),\, \overline{\eta}\,'(\Lambda_2)\cdot\mu\rangle = \langle \Lambda_1,\, \overline{\eta}\,(\overline{\eta}\,'(\Lambda_2)\cdot\mu)\rangle.$$

Since $L^1(\Omega_1, \mu)$ is an introverted subspace of $M(\Omega_1)$, we know that $\overline{\eta}\,'(\Lambda_2)\cdot\mu \in L^1(\Omega_1, \mu)$; it now follows from (3.13) that $\overline{\eta}\,(\overline{\eta}\,'(\Lambda_2)\cdot\mu)$ belongs to $L^1(\Omega_2, \nu)$, and so

$$\langle \Lambda_1,\, \overline{\eta}\,(\overline{\eta}\,'(\Lambda_2)\cdot\mu)\rangle = \langle \Lambda_{1,\nu},\, \overline{\eta}\,(\overline{\eta}\,'(\Lambda_2)\cdot\mu)\rangle.$$

Since $\overline{\eta}\,'(\Lambda_1)\overline{\eta}\,'(\Lambda_2) = \overline{\eta}\,'(\Lambda_2)\overline{\eta}\,'(\Lambda_1)$, we obtain

$$\langle \overline{\eta}\,'(\Lambda_1)\overline{\eta}\,'(\Lambda_2),\, \mu\rangle = \langle \Lambda_{1,\nu},\, \overline{\eta}\,(\overline{\eta}\,'(\Lambda_{2,\nu})\cdot\mu)\rangle.$$

Thus $\langle \overline{\eta}\,'(\Lambda_1\Lambda_2),\, \mu\rangle = \langle \overline{\eta}\,'(\Lambda_1)\overline{\eta}\,'(\Lambda_2),\, \mu\rangle$. The above equality holds for all $\mu \in M(\Omega_1)$, and so we conclude that

$$\overline{\eta}\,'(\Lambda_1\Lambda_2) = \overline{\eta}\,'(\Lambda_1)\overline{\eta}\,'(\Lambda_2) \quad (\Lambda_1, \Lambda_2 \in C(\widetilde{\Omega}_2)),$$

and hence that $\overline{\eta}\,'$ is a homomorphism; clearly it is $*$-homomorphism.

It is clear from (3.13) that $\overline{\eta}\,'(C(\Phi_{\overline{\eta}(\mu)})) \subset C(\Phi_\mu)$ for each $\mu \in M(\Omega_1)^+$. ∎

COROLLARY 4.12. *Let Ω_1 and Ω_2 be two non-empty, locally compact spaces, and let $\eta : \Omega_1 \to \Omega_2$ be a continuous map. Then $\overline{\eta}\,''(\widetilde{\Omega}_1) \subset \widetilde{\Omega}_2$ and the map*

$$\widetilde{\eta} := \overline{\eta}\,'' \,|\, \widetilde{\Omega}_1 : \widetilde{\Omega}_1 \to \widetilde{\Omega}_2 \tag{4.6}$$

is a continuous map with $\overline{\eta}\,' = (\widetilde{\eta}\,)^\circ$ such that $\widetilde{\eta}$ extends η and such that $\widetilde{\eta}(\Phi_\mu) \subset \Phi_{\overline{\eta}(\mu)}$ for each $\mu \in M(\Omega_1)^+$.

Further:

(i) *the map $\widetilde{\eta}$ is injective whenever $\eta : \Omega_1 \to \Omega_2$ is injective, and in this case we have $\overline{\eta}\,''(M(\widetilde{\Omega}_1)) \subset M(\widetilde{\eta}\,(\widetilde{\Omega}_1))$ and $\overline{\eta}\,''(M_c(\widetilde{\Omega}_1)) \subset M_c(\widetilde{\eta}\,(\widetilde{\Omega}_1))$;*
(ii) *the map $\widetilde{\eta}$ is surjective whenever $\overline{\eta} : M(\Omega_1) \to M(\Omega_2)$ is surjective, and in this case $(\overline{\eta}\,'')^{-1}(M_c(\widetilde{\Omega}_2)) \subset M_c(\widetilde{\eta}\,(\widetilde{\Omega}_1))$ and $\widetilde{\eta}(\Phi_\mu) = \Phi_{\overline{\eta}(\mu)}$ for each $\mu \in M(\Omega_1)^+$.*

Proof. It is immediate from the theorem that $\widetilde{\eta} : \widetilde{\Omega}_1 \to \widetilde{\Omega}_2$ has the specified properties. Further, we see that the map $\widetilde{\eta}$ is injective/surjective if and only if $\overline{\eta}\,''$ is injective/surjective if and only if $\overline{\eta}$ is injective/surjective.

(i) By Proposition 3.19, $\overline{\eta}$ is injective whenever η is injective, and this implies that $\overline{\eta}''$ is injective, and hence $\widetilde{\eta}$ is injective.

(ii) Since $\overline{\eta} : M(\Omega_1) \to M(\Omega_2)$ is surjective, the C^*-homomorphism

$$\overline{\eta}\,' : C(\widetilde{\Omega}_2) \to C(\widetilde{\Omega}_1)$$

is injective, and so we may regard $C(\widetilde{\Omega}_2)$ as a closed C^*-subalgebra of $C(\widetilde{\Omega}_1)$. Thus points of $\widetilde{\Omega}_2$ correspond to closed subsets of $\widetilde{\Omega}_1$, and so each such point is the image of a point in $\widetilde{\Omega}_1$. ∎

The space Φ_b. Let Ω be an infinite, locally compact space.

The character space Φ_b of $B^b(\Omega)$ is the Stone space of the Boolean algebra $\mathfrak{B}_\Omega$, and so is totally disconnected. In fact $\mathfrak{B}_\Omega$ is σ-complete, and so Φ_b is *basically disconnected*, in the sense that every cozero set in Φ_b has an open closure [37, Exercise 1H]. It follows from Proposition 2.1 that $|\Phi_b| \geq 2^{\mathfrak{c}}$.

Let $B(\Omega)$ be the quotient of $B^b(\Omega)$ by the closed linear subspace consisting of the functions which are zero outside a meagre subspace of Ω. Then $B(\Omega)$ is a commutative C^*-algebra, and so has the form $C(T)$ for a certain Stonean space T, formed by identifying points of the character space of $B^b(\Omega)$; since $B(\Omega)$ is a complete Boolean algebra, T is extremely disconnected [112, Theorem III.1.25]. We remark that $B(\Omega)$ is called the *Dixmier algebra* of Ω. It is proved in [24] that every Stonean space arises as the character space of such an algebra; in the case where Ω is compact, the character space of $B(\Omega)$ is homeomorphic to the projective cover of Ω, and so $B(\Omega)$ is (isometrically isomorphic to) the injective envelope of $C(\Omega)$.

Since $B^b(\Omega)$ is a C^*-subalgebra of $\ell^\infty(\Omega)$, we can identify Φ_b as a quotient of $\beta\Omega_d$. For $\varphi \in \widetilde{\Omega}$, let $[\varphi]$ be the closed subset of $\widetilde{\Omega}$ defined above. The following obvious remark will be strengthened later.

Proposition 4.13. *Let Ω be a non-empty, locally compact space, and let $\varphi \in \widetilde{\Omega}$. Then $[\varphi] \cap \beta\Omega_d$ is a non-empty, closed subset of $\beta\Omega_d$, and these sets partition $\beta\Omega_d$. Indeed, for $\varphi = x \in \Omega$, we have $[x] = \{x\}$, and for $\varphi \notin \Omega$, the set $[\varphi] \cap \beta\Omega_d$ is a non-empty, closed subset of $\Omega_d^* = \beta\Omega_d \setminus \Omega$.* ∎

It follows that Ω is dense in Φ_b. By Proposition 4.6, the sets $[\varphi]$ and $[\psi]$ are disjoint whenever $\varphi, \psi \in U_\Omega$ with $\varphi \neq \psi$. Thus we have described a continuous surjection

$$\eta : \varphi \mapsto [\varphi] \cap \beta\Omega_d, \qquad \widetilde{\Omega} \to \Phi_b; \tag{4.7}$$

the map $\eta \,|\, U_\Omega$ is an injection of U_Ω onto a dense subset of Φ_b.

The restriction map $\eta \,|\, \beta\Omega_d : \beta\Omega_d \to \Phi_b$ is also a continuous surjection.

Proposition 4.14. *Let Ω be a non-empty, locally compact space.*

- (i) *There is a C^*-monomorphism $\kappa_E : \ell^\infty(\Omega) \to C(\widetilde{\Omega})$ that extends the above embedding $\kappa_E : B^b(\Omega) \to C(\widetilde{\Omega})$.*
- (ii) *There is a retraction from $\widetilde{\Omega}$ onto $\beta\Omega_d$.*

Proof. (i) Since $\widetilde{\Omega}$ is Stonean, it follows from Theorem 2.5 that $C(\widetilde{\Omega})$ is injective in the category of commutative C^*-algebras and continuous $*$-homomorphisms, and so there is a C^*-homomorphism

$$\theta : \ell^\infty(\Omega) \to C(\widetilde{\Omega})$$

that extends $\kappa_E : B^b(\Omega) \to C(\widetilde{\Omega})$.

Let $I = \ker\theta$, a closed ideal in $C(\beta\Omega_d)$. There is a closed subspace F of $\beta\Omega_d$ such that $I = \{\lambda \in C(\beta\Omega_d) : \lambda | F = 0\}$. It cannot be that there exists $x \in \Omega \setminus F$, for otherwise $\theta(\delta_x) = \kappa_E(\delta_x) = 0$, which is not the case. Thus $\Omega \subset F$, and so $F = \beta\Omega_d$ and $I = \{0\}$, showing that θ is a monomorphism.

(ii) The map $\theta' : \widetilde{\Omega} \to \beta\Omega_d$ is a continuous map. Let $x \in \Omega$, and set $y = \theta'(x) \in \beta\Omega_d$. Then

$$\varepsilon_y(\delta_x) = (\varepsilon_x \circ \theta)(\delta_x) = (\varepsilon_x \circ \kappa_E)(\delta_x) = 1,$$

and so $y = x$. Thus θ' is the identity map on Ω, and hence is the identity map on $\beta\Omega_d$. This shows that $\theta' : \widetilde{\Omega} \to \beta\Omega_d$ is a retraction. ∎

We note that the map $\kappa_E : \ell^\infty(\Omega) \to C(\widetilde{\Omega})$ is not a unique extension of the map $\kappa_E : B^b(\Omega) \to C(\widetilde{\Omega})$, although $\kappa_E(\lambda) | \widetilde{\Omega}_d$ is uniquely specified for each $\lambda \in \ell^\infty(\Omega)$.

The image $\kappa_E(\ell^\infty(\Omega))$ is a closed subalgebra of $C(\widetilde{\Omega})$, and so it separates at least as many pairs of points of $\widetilde{\Omega}$ as $B^b(\Omega)$ does. For example, $\kappa_E(\ell^\infty(\Omega))$ separates all pairs of points in the space $\beta\Omega_d$. We wonder whether, given two points $\varphi, \psi \in \widetilde{\Omega}$, there is such an embedding κ_E such that $\kappa_E(\ell^\infty(\Omega))$ separates φ and ψ.

Metrizable spaces. We now consider an uncountable, compact, metrizable space Ω, and summarize our results in this setting.

Note that each uncountable, second countable, locally compact space (such as $\mathbb{R}$) has a one-point compactification that is metrizable, and so the results of this section apply to such spaces, with very slight changes of wording.

PROPOSITION 4.15. *Let Ω_1 and Ω_2 be two uncountable, compact, metrizable spaces. Then the Banach spaces $M(\Omega_1)$ and $M(\Omega_2)$ are isometrically isomorphic.*

Proof. This follows easily from Proposition 2.2, which states that Ω_1 and Ω_2 are Borel isomorphic. ∎

Let Ω be an uncountable, compact, metrizable space. Then there is a maximal singular family $\mathcal{F}_c = \{\mu_i : i \in I\}$ of continuous measures in $M_c(\Omega)^+$ such that $|\mathcal{F}_c| = \mathfrak{c}$; such a family is exhibited in Proposition 2.13. Then $\widetilde{\Omega}$ contains the following clopen subsets: $\beta\Omega_d$ and the sets Φ_i for $i \in I$, and all these sets are pairwise disjoint. It follows from Proposition 2.15(iii) that the sets Φ_i all satisfy CCC on clopen subsets, and from Proposition 4.8 that $\beta U = \widetilde{\Omega}$, where $U = \Omega \cup \bigcup\{\Phi_i : i \in I\}$ is a dense, open subset of $\widetilde{\Omega}$.

THEOREM 4.16. *Let Ω be an uncountable, compact, metrizable space. Then the hyper-Stonean envelope $X = \widetilde{\Omega}$ has the following properties:*

- (i) *X is a hyper-Stonean space;*
- (ii) *the set S of isolated points of X has cardinality $\mathfrak{c}$, the closure Y of S in X is a clopen subspace of X, and Y is homeomorphic to βS_d;*
- (iii) *$X \backslash Y$ contains a pairwise disjoint family $\mathcal{F}$ of $\mathfrak{c}$ clopen subspaces, each homeomorphic to $\mathbb{H}$;*
- (iv) *the union $U_{\mathcal{F}}$ of the sets of $\mathcal{F}$ is dense in $X \setminus Y$ and is such that $\beta U_{\mathcal{F}} = X \setminus Y$.*

Further, any two spaces X_1 and X_2 satisfying the clauses (i)–(iv) *are mutually homeomorphic.*

Proof. We have shown that $X = \widetilde{\Omega}$ satisfies clauses (i)–(iv).

Let X_1 and X_2 be two spaces satisfying clauses (i)–(iv). The sets of isolated points of X_1 and X_2 are S_1 and S_2, respectively. Since $|S_1| = |S_2|$, there is a bijection from S_1 to S_2, and this extends to a homeomorphism from βS_1 to βS_2, and so the respective closures Y_1 and Y_2 of S_1 and S_2 in X_1 and X_2 are clopen subsets of X_1 and X_2, respectively, such that Y_1 and Y_2 are homeomorphic.

Let the families specified in (iii) corresponding to X_1 and X_2 be $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, listed as $(H_{1,\tau} : \tau < \mathfrak{c})$ and $(H_{2,\tau} : \tau < \mathfrak{c})$. For each $\tau < \mathfrak{c}$, there is a homeomorphism from $H_{1,\tau}$ onto $H_{2,\tau}$, and hence there is a homeomorphism from $U_{\mathcal{F}_1}$ onto $U_{\mathcal{F}_2}$. Since $\beta U_{\mathcal{F}_i} = X_i \setminus Y_i$ for $i = 1, 2$, this homeomorphism extends to a homeomorphism of $X_1 \setminus Y_1$ onto $X_2 \setminus Y_2$. ∎

Thus there is a unique space X that is the hyper-Stonean envelope of all uncountable, compact, metrizable spaces. We shall obtain some further properties of this space involving the calculations of some cardinalities.

THEOREM 4.17. *Let Ω be an uncountable, compact, metrizable space, and set $X = \widetilde{\Omega}$. Then:*

(i) $|C(X)| = 2^{\mathfrak{c}}$ *and* $|X| = 2^{2^{\mathfrak{c}}}$*;*
(ii) $|U_\Omega| = 2^{\mathfrak{c}}$ *and* $w(U_\Omega) = \mathfrak{c}$*;*
(iii) $|\widetilde{\Omega}_c \setminus U_\Omega| = 2^{2^{\mathfrak{c}}}$.

Proof. (i) Certainly, we have $|X| \geq |\beta\Omega_d|$. By Proposition 2.2(i), we have $|\Omega| = \mathfrak{c}$, and so $|\beta\Omega_d| = 2^{2^{\mathfrak{c}}}$ by Proposition 2.1. By Proposition 2.13, $|M(\Omega)| = \mathfrak{c}$, and so, by Proposition 1.1(iii), we have $|C(X)| \leq 2^{\mathfrak{c}}$ and $|X| \leq |C(X)'| \leq 2^{2^{\mathfrak{c}}}$. Finally, $|C(X)| \geq |\ell^\infty(\Omega_d)| = 2^{\mathfrak{c}}$. We obtain (i) by combining the above inequalities.

(ii) For each $\mu \in M_c(\Omega)^+$ such that $\mu \neq 0$, we have $|\Phi_\mu| = 2^{\mathfrak{c}}$ and $w(\Phi_\mu) = \mathfrak{c}$ by Corollary 2.21. For each $\mu \in M_d(\Omega)^+$, we also have $|\Phi_\mu| \leq |\beta\mathbb{N}| = 2^{\mathfrak{c}}$ and hence $w(\Phi_\mu) \leq w(\beta\mathbb{N}) = \mathfrak{c}$. For general $\mu \in M(\Omega)^+$, we have $\Phi_\mu = \Phi_{\mu_c} \cup \Phi_{\mu_d}$, and so $|\Phi_\mu| \leq 2^{\mathfrak{c}}$ and $w(\Phi_\mu) \leq w(\beta\mathbb{N}) \leq \mathfrak{c}$. By Proposition 2.13, we have $|M(\Omega)| = \mathfrak{c}$, and so it follows that $|U_\Omega| = 2^{\mathfrak{c}}$ and $w(\Phi_\mu) = \mathfrak{c}$.

(iii) Consider a maximal singular family $\mathcal{F}_c$ of continuous measures, as in Proposition 4.8, so that $\{\Phi_i : i \in I\}$ is a pairwise disjoint family, now of cardinality $\mathfrak{c}$.

Let A be the algebra of all functions on $U_{\mathcal{F}} := \bigcup\{\Omega_i : i \in I\}$ that are constant on each set Φ_i. Each function in A has a continuous extension to $\widetilde{\Omega}_c$, and so we may regard A as a closed subalgebra of $C(\widetilde{\Omega}_c)$. The character space Φ_A of A is a quotient of $\widetilde{\Omega}_c$. However it is clear that we can identify Φ_A with βI. By Proposition 2.1, $|\beta I| = 2^{2^{\mathfrak{c}}}$, and so $|\widetilde{\Omega}_c| \geq 2^{2^{\mathfrak{c}}}$. Since $|X| = 2^{2^{\mathfrak{c}}}$, we have $|\widetilde{\Omega}_c| = 2^{2^{\mathfrak{c}}}$. Since $|U_\Omega| = 2^{\mathfrak{c}}$, we have $|\widetilde{\Omega}_c \setminus U_\Omega| = 2^{2^{\mathfrak{c}}}$. ■

Thus, with GCH, we have $|X| = \aleph_3$, but $|U_\Omega| = \aleph_2$.

We know that the set $U_\Omega = \bigcup\{\Phi_\mu : \mu \in M(\Omega)^+\}$ is a proper subset of $\widetilde{\Omega}$. However, for each $\mu \in M(\Omega)^+$, set

$$[\Phi_\mu] := \bigcup\{[\varphi] : \varphi \in \Phi_\mu\}.$$

By an earlier remark on p. 22, $[\Phi_\mu]$ is a closed subset of $\widetilde{\Omega}$. It seemed possible that the subset $\bigcup\{[\Phi_\mu] : \mu \in M(\Omega)^+\}$ would be equal to the whole of $\widetilde{\Omega}$. However Theorems 4.19 and 4.24 below show that this is far from the case whenever Ω is an uncountable, compact, metrizable space.

We shall also need the following definitions from [52, Definitions 3.13 and 3.60].

Let D be a set, and let κ be an infinite cardinal. Then a *κ-uniform ultrafilter* on D is an ultrafilter $\mathcal{U}$ on D such that each set in $\mathcal{U}$ has cardinality at least κ. Let $\mathcal{A}$ be a family of subsets of D. Then $\mathcal{A}$ has the *κ-uniform finite intersection property* if each finite subfamily of $\mathcal{A}$ has an intersection of cardinality at least κ. Theorem 3.62 of [52] is the following.

THEOREM 4.18. *Let D be an infinite set of cardinality κ, and let $\mathcal{A}$ be a family of at most κ subsets of D such that $\mathcal{A}$ has the κ-uniform finite intersection property. Then there are at least 2^{2^κ} κ-uniform ultrafilters on D that contain $\mathcal{A}$.* ■

THEOREM 4.19. *Let Ω be an uncountable, compact, metrizable space. Then*

$$|\beta\Omega_d \setminus [U_\Omega]| = 2^{2^{\mathfrak{c}}}.$$

Proof. First, choose a countable, dense subset of Ω, say

$$Q = \{q_m : m \in \mathbb{N}\}.$$

Consider the family of G_δ-subsets B of Ω such that $B \supset Q$; each such B is a Borel set. It follows from the Baire category theorem that B is uncountable, and so $|B| = \mathfrak{c}$ by Proposition 2.2(i). The family $\mathcal{F}$ of all such sets B is a filter of Borel subsets of Ω and also $|\mathcal{F}| = \mathfrak{c}$, and so, by Theorem 4.18, there are $2^{2^{\mathfrak{c}}}$ $\mathfrak{c}$-uniform ultrafilters $\mathcal{U}$ on Ω with $\mathcal{F} \subset \mathcal{U}$. We identify these ultrafilters with points ψ of $\beta\Omega_d$.

Let ψ be such an ultrafilter. We *claim* that, for each $\mu \in M(\Omega)^+$, there exists $B \in \mathfrak{B}_\Omega$ with $B \in \psi$ and such that $\mu(B) = 0$.

First, suppose that $\mu \in M_d(\Omega)^+$, and set

$$C = \operatorname{supp} \mu \quad \text{and} \quad B = \Omega \setminus C.$$

Since C is countable and ψ is a $\mathfrak{c}$-uniform ultrafilter, it is not true that $C \in \psi$. Thus B is a Borel set, $B \in \psi$, and $\mu(B) = 0$.

Second, suppose that $\mu \in M_c(\Omega)^+$. By Lemma 2.7, there is a G_δ-subset B of Ω containing Q, and so again $B \in \mathcal{F} \subset \psi$ with $\mu(B) = 0$.

Now let $\mu \in M(\Omega)^+$. There exist $\mu_1 \in M_d(\Omega)^+$ and $\mu_2 \in M_c(\Omega)^+$ with $\mu = \mu_1 + \mu_2$. Take subsets $B_1, B_2 \in \mathfrak{B}_\Omega$ such that $B_1, B_2 \in \psi$ and $\mu_1(B_1) = \mu_2(B_2) = 0$, and set $B = B_1 \cap B_2$, so that $B \in \mathfrak{B}_\Omega$ with $B \in \psi$ and $\mu(B) = 0$.

For each $\varphi \in \Phi_\mu$, we have $\kappa_E(\chi_B)(\varphi) = 0$, whereas $\kappa_E(\chi_B)(\psi) = 1$ because $B \in \psi$. This shows that $\psi \notin [\Phi_\mu]$.

Thus $|\beta\Omega_d \setminus [U_\Omega]| = 2^{2^{\mathfrak{c}}}$. ■

We now seek to make some calculations of the cardinality of the sets $[\varphi]$ for $\varphi \in \widetilde{\Omega}$. We shall first associate with each such φ a certain filter of Borel sets.

Definition 4.20. Let Ω be a non-empty, locally compact space, and take $\varphi \in \widetilde{\Omega}$. Then

$$\mathcal{G}_\varphi = \{B \in \mathfrak{B}_\Omega : \varphi \in K_B\}.$$

Clearly $\mathcal{G}_\varphi$ is a subset of $\mathfrak{B}_\Omega$ that is closed under finite intersections. In the case where Ω is compact and metrizable, $|\mathcal{G}_\varphi| \leq \mathfrak{c}$.

Recall from Proposition 2.2(i) that, for each $B \in \mathfrak{B}_\Omega$, either B is countable or $|B| = \mathfrak{c}$.

We begin with a preliminary lemma and corollary.

Let Ω be an uncountable, compact, metrizable space. As above, we take

$$\mathcal{F}_c = \{\mu_i \in M_c(\Omega)^+ : i \in I\}$$

to be a maximal singular family of continuous measures in $M_c(\Omega)^+$, so that, by Proposition 2.13, $|\mathcal{F}_c| = \mathfrak{c}$. For each $B \in \mathfrak{B}_\Omega$, we set

$$J_B = \{i \in I : K_B \cap \Phi_i \neq \emptyset\}.$$

Lemma 4.21. *Let Ω be an uncountable, compact, metrizable space, and let $B \in \mathfrak{B}_\Omega$ with B uncountable. Then:*

(i) $|J_B| = \mathfrak{c}$;

(ii) $K_B \cap (\widetilde{\Omega}_c \setminus U_\Omega) \neq \emptyset$.

Proof. (i) By Proposition 2.2(iii), the set B contains an uncountable, compact subset, say C. We *claim* that the family

$$\{\mu_i | C : i \in J_B\}$$

is a maximal singular family of continuous measures in $M_c(C)^+$. Indeed, all pairs of distinct elements of this family are mutually singular. Suppose that $\nu \in M_c(C)^+$ is such that $\nu \perp (\mu_i|C)$ for each $i \in J_B$. Then $\nu \perp \mu_i$ for each $i \in I$, and so $\nu = 0$. This gives the claim.

By Proposition 4.10, $|J_B| = \mathfrak{c}$.

(ii) Assume towards a contradiction that $K_B \cap \widetilde{\Omega}_c \subset U_\Omega$. Then

$$K_B \subset \bigcup \{\Phi_\mu : \mu \in M_c(\Omega)^+\}.$$

Since K_B is compact, since each Φ_μ is open, and since $\{\Phi_\mu : \mu \in M_c(\Omega)^+\}$ is closed under finite unions, there exists $\mu \in M_c(\Omega)^+$ such that $K_B \subset \Phi_\mu$. By (i), $\{i \in I : \Phi_\mu \cap \Phi_i \neq \emptyset\}$ is uncountable. But this contradicts the fact that Φ_μ satisfies CCC. Thus $K_B \cap \widetilde{\Omega}_c \not\subset U_\Omega$. ∎

Corollary 4.22. *Let Ω be an uncountable, compact, metrizable space, and take*

$$\varphi \in \widetilde{\Omega}_c \cup (\beta\Omega_d \setminus U_\Omega).$$

Then there exists $\psi \in \widetilde{\Omega}_c \setminus U_\Omega$ such that $\psi \sim \varphi$.

Proof. Since $\varphi \in \widetilde{\Omega}_c \cup (\beta\Omega_d \setminus U_\Omega)$, each $B \in \mathcal{G}_\varphi$ is uncountable. The set $K_B \cap (\widetilde{\Omega}_c \setminus U_\Omega)$ is closed in the compact space $\widetilde{\Omega}_c \cap U_\Omega$, and so, by Lemma 4.21, this set is not empty. Thus

$$\bigcap \{K_B \cap (\widetilde{\Omega}_c \setminus U_\Omega) : B \in \mathcal{G}_\varphi\} \neq \emptyset;$$

choose ψ in the set on the left. Then $\psi \in \widetilde{\Omega}_c \setminus U_\Omega$ and $\psi \in K_B$ whenever $\varphi \in K_B$, and so $\psi \sim \varphi$. ∎

Theorem 4.23. *Let Ω be an uncountable, compact, metrizable space, and let $\varphi \in \widetilde{\Omega}$.*

- (i) *Suppose that there exists $B \in \mathcal{G}_\varphi$ such that B is countable. Then $[\varphi] = \{\varphi\}$, and so $|[\varphi]| = 1$.*
- (ii) *Suppose that each $B \in \mathcal{G}_\varphi$ is uncountable. Then*

$$|[\varphi] \cap \beta\Omega_d| = 2^{2^{\mathfrak{c}}}.$$

- (iii) *Suppose that $\varphi \in \widetilde{\Omega}_c$. Then*

$$|[\varphi] \cap \widetilde{\Omega}_c| = 2^{2^{\mathfrak{c}}}.$$

Proof. (i) Suppose that $\psi \in [\varphi]$. Since $\chi_B \in B^b(\Omega)$ and $\varphi \in K_{\chi_D} = \beta D \subset \beta\Omega_d$, necessarily $\psi \in \beta D$. Since $\ell^\infty(B) \subset B^b(\Omega)$ and the functions in $\ell^\infty(B)$ separate the points of βB, it follows that $\psi = \varphi$.

(ii) We first note that $|\mathcal{G}_\varphi| \leq \mathfrak{c}$ and that each member of $\mathcal{G}_\varphi$ has cardinality $\mathfrak{c}$. Since $\mathcal{G}_\varphi$ is closed under finite intersections, it is clear that $|\mathcal{G}_\varphi|$ has the $\mathfrak{c}$-uniform finite intersection property. By Theorem 4.18, we have

$$|\{\psi \in \beta\Omega_d : \psi \supset \mathcal{G}_\varphi\}| = 2^{2^{\mathfrak{c}}}.$$

However, for each $\psi \supset \mathcal{G}_\varphi$ and each $B \in \mathcal{G}_\varphi$, we have $\psi \in K_B$, and so $\psi \sim \varphi$. It follows that $|[\varphi] \cap \beta\Omega_d| = 2^{2^{\mathfrak{c}}}$.

(iii) First, we consider the case where $\varphi \in \widetilde{\Omega}_c \setminus U_\Omega$. Again consider the above family $\mathcal{F}_c$, so that $\{\Phi_i : i \in I\}$ is a pairwise disjoint family of cardinality $\mathfrak{c}$ of subsets of $\widetilde{\Omega}$.

For each $B \in \mathcal{G}_\varphi$, define J_B as above. By Lemma 4.21(i), $|J_B| = \mathfrak{c}$. Certainly

$$|\{J_B : B \in \mathcal{G}_\varphi\}| \le |\mathfrak{B}_\Omega| = \mathfrak{c}$$

by Proposition 2.2(iv). Thus, by Theorem 4.18, there are $2^{2^{\mathfrak{c}}}$ ultrafilters $\mathcal{U}$ on I each containing $\{J_B : B \in \mathcal{G}_\varphi\}$.

For each such ultrafilter $\mathcal{U}$ and each $B \in \mathcal{G}_\varphi$, define

$$C(\mathcal{U}, B) = \bigcap_{U \in \mathcal{U}} \Big\{ \overline{\bigcup_{i \in U} K_B \cap \Phi_i} \Big\} \quad \text{and} \quad C(\mathcal{U}) = \bigcap \{C(\mathcal{U}, B) : B \in \mathcal{G}_\varphi\}.$$

Since each set $\overline{\bigcup_{i \in U} K_B \cap \Phi_i}$ is a non-empty, closed subset of the compact space $\widetilde{\Omega}_c$, it follows that $C(\mathcal{U}) \neq \emptyset$ for each such $\mathcal{U}$. Suppose that $\mathcal{U}_1$ and $\mathcal{U}_2$ are distinct ultrafilters on I containing $\{J_B : B \in \mathcal{G}_\varphi\}$ and that $B_1, B_2 \in \mathcal{G}_\varphi$. Then $C(\mathcal{U}_1, B_1) \cap C(\mathcal{U}_2, B_2) = \emptyset$, and so $C(\mathcal{U}_1) \cap C(\mathcal{U}_2) = \emptyset$. Thus there are $2^{2^{\mathfrak{c}}}$ sets of the form $C(\mathcal{U})$ and the family of these sets is pairwise disjoint.

Let $\mathcal{U}$ be an ultrafilter on I containing $\{J_B : B \in \mathcal{G}_\varphi\}$, and let $\psi \in C(\mathcal{U})$. For each $B \in \mathcal{G}_\varphi$, we have $\psi \in C(\mathcal{U}, B) \subset K_B$, and so $\psi \sim \varphi$.

We have shown that $|[\varphi] \cap \widetilde{\Omega}_c| = 2^{2^{\mathfrak{c}}}$ for this element φ.

Second, we consider the case where $\varphi \in \widetilde{\Omega}_c \cap U_\Omega$. By Corollary 4.22, there exists $\psi \in \widetilde{\Omega}_c \setminus U_\Omega$ such that $\psi \sim \varphi$. Thus, we have

$$|[\varphi] \cap \widetilde{\Omega}_c| = |[\psi] \cap \widetilde{\Omega}_c| = 2^{2^{\mathfrak{c}}},$$

as required. ■

Theorem 4.24. *Let Ω be an uncountable, compact, metrizable space. Then*

$$|[U_\Omega]| = |[U_\Omega] \cap \widetilde{\Omega}_c| = |\widetilde{\Omega}_c \setminus [U_\Omega]| = 2^{2^{\mathfrak{c}}}.$$

Proof. Take $\varphi \in \widetilde{\Omega}_c \cap U_\Omega$. By Theorem 4.23(ii), $|[\varphi]| = 2^{2^{\mathfrak{c}}}$. Since $[\varphi] \subset [U_\Omega]$, we have $|[U_\Omega]| = 2^{2^{\mathfrak{c}}}$. Similarly, the fact that $|[U_\Omega] \cap \widetilde{\Omega}_c| = 2^{2^{\mathfrak{c}}}$ follows from Theorem 4.23(iii).

By Theorem 4.19, there exists $\varphi \in \beta\Omega_d \setminus [U_\Omega]$. By Corollary 4.22, there exists an element $\psi \in \widetilde{\Omega}_c \setminus U_\Omega$ such that $\psi \sim \varphi$. Since $\varphi \notin [U_\Omega]$, we have $[\psi] \cap U_\Omega = \emptyset$. By Theorem 4.20, $|[\psi]| = 2^{2^{\mathfrak{c}}}$. Thus $|\widetilde{\Omega}_c \setminus [U_\Omega]| = 2^{2^{\mathfrak{c}}}$. ■

Thus, with GCH, we have $|\widetilde{\Omega}| = |[U_\Omega]| = \aleph_3$, but $|U_\Omega| = \aleph_2$.

Of course, it is not the case that any two uncountable, compact, metrizable spaces Ω_1 and Ω_2 are homeomorphic. However, by Milyutin's theorem [1, Theorem 4.4.8], $C(\Omega_1)$ and $C(\Omega_2)$ are isomorphic as Banach spaces. Thus it seems possible that $C(\Omega_1)$ and $C(\Omega_2)$ are isomorphic as Banach spaces whenever the hyper-Stonean envelope of each of Ω_1 and Ω_2 is the above space X. However, this is not the case, as the following example shows.

Example 4.25. There is a compact, uncountable, non-metrizable space Ω such that the hyper-Stonean envelope $\widetilde{\Omega}$ is homeomorphic to $\widetilde{\mathbb{I}}$.

Let $\Omega = \mathbb{I} \times \{0, 1\}$ as a set, and identify $\mathbb{I}$ with the subset $\mathbb{I} \times \{0\}$ of Ω. Let Ω be ordered lexicographically, and then assign the interval topology to Ω, so that a base of open sets for the topology on Ω is formed by sets of the form

$$U = ((a, i), (b, j)),$$

where $a, b \in \mathbb{I}$ and $i, j \in \{0, 1\}$ and where either $a < b$ or $a = b$, $i = 0$, and $j = 1$; the relative topology from Ω on $\mathbb{I}$ coincides with the Sorgenfrey topology [29, Example 1.2.2], which is generated by intervals of the form $(a, b]$. The space Ω is compact, but it is not metrizable because the Sorgenfrey topology on $\mathbb{I}$ is not metrizable.

Clearly $\mathbb{I}$ and Ω have the same cardinality, so the spaces $M_d(\mathbb{I})$ and $M_d(\Omega)$ of discrete measures can be identified. Hence the topological spaces $\beta\mathbb{I}_d$ and $\beta\Omega_d$ are homeomorphic.

We *claim* that it is also true that the spaces $M_c(\mathbb{I})$ and $M_c(\Omega)$ of continuous measures can be identified. To see this, first consider an open interval U in Ω of the above form, and set $V = (a, b) \times \{0, 1\} \subset \Omega$ (with $V = \emptyset$ when $a \geq b$). We note that $V \supset U$ and that $|V \setminus U| \leq 2$, so that the symmetric difference $U \bigtriangleup V$ is always finite. Now consider the family $\mathcal{F}$ of subsets E of Ω which have the property that $E \bigtriangleup (B \times \{0, 1\})$ is countable for some Borel subset B of $\mathbb{I}$. The family $\mathcal{F}$ is a σ-algebra, and $\mathcal{F}$ contains all open intervals in Ω. It is easy to see that each open subset of Ω is a countable union of open intervals, and so $\mathcal{F}$ contains all open sets in Ω. Hence $\mathcal{F}$ contains all Borel subsets of Ω, so that, in fact, $\mathcal{F} = \mathfrak{B}_\Omega$. Let $\mu \in M_c(\Omega)$, and define $T\mu \in M_c(\mathbb{I})$ by

$$(T\mu)(B) = \mu(B \times \{0, 1\}) \quad (B \in \mathfrak{B}_{\mathbb{I}}),$$

so that $T : M_c(\Omega) \to M_c(\mathbb{I})$ is a linear isometry. For each $\nu \in M_c(\mathbb{I})$, define

$$\mu(E) = \nu(B) \quad (E \in \mathfrak{B}_\Omega),$$

where $B \in \mathfrak{B}_{\mathbb{I}}$ is such that $E \Delta (B \times \{0, 1\})$ is countable. Then $\mu(E)$ is well-defined, $\mu \in M_c(\Omega)$, and $T\mu = \nu$. Thus T is a surjection. It follows that the spaces Φ_μ, which is a clopen subspace of $\widetilde{\Omega}$, and $\Phi_{T\mu}$, which is a clopen subspace of $\widetilde{\mathbb{I}}$, are homeomorphic.

A maximal singular family of positive measures on Ω consists of $\mathfrak{c}$ discrete measures and $\mathfrak{c}$ continuous measures, and so it follows from our basic construction that $\widetilde{\Omega}$ and $\widetilde{\mathbb{I}}$ are homeomorphic.

It cannot be that $C(\Omega)$ is linearly homeomorphic to $C(\mathbb{I})$, or else $C(\Omega)$ would be separable and Ω would be metrizable by a remark on page 21. ■

The above example gives rise to an interesting phenomenon, which we now describe.

Example 4.26. Our Example 4.25 leads to examples of two compact, uncountable spaces, Ω_1 and Ω_2, with Ω_1 metrizable and Ω_2 non-metrizable, such that the two Banach spaces defined to be $E_1 := C(\Omega_1)$ and $E_2 := C(\Omega_2)$ have the property that E_1' and E_2' are isometrically isomorphic, but are such that E_1 is separable, but E_2 is non-separable.

Indeed, we take Ω_1 to be the closed unit interval $\mathbb{I}$ and Ω_2 to be the space constructed in Example 4.25. Then $C(\Omega_1)'$ and $C(\Omega_2)'$ are each isometrically isomorphic to $N(\widetilde{\mathbb{I}})$. For a non-empty, compact space Y, the Banach space $C(Y)$ is separable if and only if the space Y is metrizable. Thus $C(\Omega_1)$ is separable, but $C(\Omega_2)$ is not separable.

A stronger example is given in [99, Proposition 5.5]: there is a non-separable compact space K such that $C(K)'$ is isometrically isomorphic to $C(\mathbb{I})'$. ■

$C(X)$ as a bidual space. Let X be a hyper-Stonean space. It is natural to ask when X is the hyper-Stonean envelope of some compact space Ω. Our **conjecture** is the following.

Suppose that $C(X)$ is isometrically the second dual space of a Banach space. Then there is a locally compact space Ω such that $X = \widetilde{\Omega}$.

We note that it does not follow from the fact that F is a Banach space such that $F'' = C(X)$ for a compact space X that F has the form $C_0(\Omega)$ for some locally compact space Ω. For example, it is shown in [4] that there is a Banach space F such that F' is isometrically linearly isomorphic to ℓ^1, so that $F'' \cong C(\beta\mathbb{N})$, but such that F is not isomorphic to any complemented subspace of a space of the form $C(K)$; the space F is not isomorphic to any Banach lattice. However this does not give a counter-example to our conjecture. For further study of preduals of $\ell^1(\mathbb{Z})$, see [23].

The following result proves a special case of this conjecture.

PROPOSITION 4.27. *Let X be a hyper-Stonean space. Suppose that there is a Banach lattice F such that F'' is isometrically isomorphic to $C(X)$ as a Banach lattice. Then there is a compact space Ω such that $X = \widetilde{\Omega}$.*

Proof. The dual of F is the Banach lattice $N(X)$ of normal measures on X, and this is an L-space. By [106, Theorem 27.1.1], F is an M-space. By a theorem of Kakutani ([61], [106, §13.3]), an M-space is equal to $C_0(\Omega)$ as a Banach lattice for some locally compact space Ω. Since F'' is Banach lattice isomorphic to $C(X)$, there is an isometric isomorphism from $C(\widetilde{\Omega})$ onto $C(X)$. By the Banach–Stone Theorem 2.4(i), X is homeomorphic to $\widetilde{\Omega}$. ∎

A further special case of the conjecture, that in which $C(X)$ is isometrically the second dual space of a *separable* Banach space, has been resolved by Lacey in a striking manner: indeed, the two cases that we are considering are the only two cases.

First, let X be an infinite compact space for which $C(X)$ is isometrically the second dual space of a Banach space. Then the space $N(X)$ of normal measures on X is itself the dual of a Banach space, say $N(X) = F'$. Since $N(X)$ has the form $L^1(\mu)$ for a measure μ, this says that 'F is a L_1-predual space', in the terminology of [67, §22]. We denote by $\operatorname{ex} X$ the set of extreme points of the closed unit ball $N(X)_{[1]}$. It is easy to see that points of $\operatorname{ex} X$ are exactly the point masses at the isolated points of X, and so we can identify $\operatorname{ex} X$ with this set of isolated points. It follows from the Krein–Milman theorem that $\operatorname{ex} X$ is infinite. (In the case where $C(X) = C_0(\Omega)''$ for a locally compact space Ω, we can, by Corollary 4.2, identify Ω as a set with the isolated points of X, and hence with $\operatorname{ex} N(X)_{[1]}$.)

EXAMPLE 4.28. The compact space $X := \widetilde{\mathbb{I}} \setminus \beta\mathbb{I}_d$ has no isolated points, and so X is a hyper-Stonean space such that $C(X)$ is not the second dual of any Banach space. ∎

The following theorem is an immediate consequence of a theorem of Lacey [67, §22, Theorem 5]; it was first proved in [66], and a slightly stronger theorem of Hess is proved by a shorter proof in [46]. We are indebted to Frederick Dashiell and Thomas Schlumprecht for a discussion of the literature on this question.

THEOREM 4.29. *Let X be an infinite compact space for which $C(X)$ is isometrically the second dual space of a separable Banach space. Then $\operatorname{ex} X$ is infinite. Further, there are only two possibilities for the space X (up to homeomorphism): either*

(i) $\operatorname{ex} X$ *is countable, $X = \beta\mathbb{N}$, and $C(X) = c_0''$; or*
(ii) $\operatorname{ex} X$ *is uncountable, $X = \widetilde{\mathbb{I}}$, and $C(X) = C(\mathbb{I})''$.* ∎

The analogous question in the isomorphic (not isometric) theory of Banach spaces was resolved in a similar way by Stegall [110]; for related work, see [44].

A historical remark. Let Ω be a compact space. Then in fact the hyper-Stonean envelope $\widetilde{\Omega}$ was already constructed in the PhD thesis of the third author, written more than 50 years ago [89] (see also [90, 91])! Let L be an Archimedean vector lattice, and choose a family (e_i) in L^+ that is maximal with respect to the property that $e_i \wedge e_j = 0$ whenever $i \neq j$. For each i, there is a space U_i of 'ultrafilters' such that

$$\Big\{x \in L : |x| = \bigvee \{|x| \wedge ne_i : n \in \mathbb{N}\}\Big\}$$

can be represented by a space of continuous functions on U_i with values in $\mathbb{R} \cup \{-\infty, \infty\}$, each function taking values in $\mathbb{R}$ save on a nowhere dense subset of U_i. The space U_i is Stonean for each i if and only if L is complete. Form the disjoint union U of the sets U_i, giving U the topology such that each U_i is clopen in U, and set $X = \beta U$. Then there is a representation of L as a space of functions on X. In the special case where $L = M(\Omega)$, we obtain a representation of this form, with $X = \widetilde{\Omega}$ such that a measure $\mu \in M(\Omega)_{\mathbb{R}}$ is represented by a continuous function $\widehat{\mu} : X \to \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$. Further, for each $\lambda \in B^b(\Omega)$ and $\mu \in M(\Omega)$, we have

$$\int_\Omega \lambda \,\mathrm{d}\mu = \kappa_E(\lambda) \cdot \widehat{\mu}.$$

Essentially the same representation of $C(\widetilde{\Omega}) = M(\Omega)'$ as ours is given by Gordon in [40, §6] and by Wong in [123], extending a theorem of Šreĭder [109]. We now recover these results from our remarks above.

Let Ω be a non-empty, locally compact space, and form the hyper-Stonean envelope $\widetilde{\Omega}$. We adopt the above notation involving $\mathcal{F}$; further, we write $\mathcal{G}_i$ for the Gel'fand transform $\mathcal{G}_{\nu_i}$ for each $i \in I$. We take $\Lambda \in C(\widetilde{\Omega}) = M(\Omega)'$. For each $i \in I$, we set $\Lambda_i = \Lambda \,|\, L^1(\Omega, \nu_i)$, so that, by (4.1), we have $\mathcal{G}_i(\Lambda_i) = \Lambda \,|\, \Phi_i$. The family $(\Lambda_i : i \in I)$, which represents Λ, is a *generalized function* in the sense of [123].

We now consider the famous memoir [114] of J. L. Taylor. In [114, §2.4], the compact spaces $\widetilde{\Omega}$ and Φ_μ (for $\mu \in M(\Omega)^+$) are termed the 'standard domains' of the L-spaces $M(\Omega)$ and $L^1(\Omega, \mu)$. The canonical embedding $\kappa : M(\Omega) \to M(\widetilde{\Omega})$ is the 'standard representation' of $M(\Omega)$; for each $\mu \in M(\Omega)^+$, the map

$$f \mapsto \kappa(f)|\Phi_\mu, \qquad L^1(\Omega, \mu) \to M(\Phi_\mu),$$

is the 'standard representation' of $L^1(\Omega, \mu)$.

The second dual space of $C_0(\Omega)$ has been widely studied. For example, see [106, §27.2]. An early paper of Kaplan is [63], which mainly studies $E_{\mathbb{R}} = C_0(\Omega)_{\mathbb{R}}$, $(E')_{\mathbb{R}}$, and $(E'')_{\mathbb{R}}$ as Banach lattices. The study is continued in [64] and further papers of Kaplan; for a comprehensive account of this work, see [65].

Now here are some remarks of Gordon from [40, §5]. Our space U_Ω is called Y in [40, §5]. The family of all subsets of U_Ω of the form Φ_μ forms a basis of open sets for a topology, called the δ-topology in [40]; clearly this topology agrees with the relative topology from $(\widetilde{\Omega}, \sigma)$ on each Φ_μ. A subset K of U_Ω has the form Φ_μ for some $\mu \in M(\Omega)^+$ if and only if K is open and compact in the δ-topology of U_Ω.

5. Locally compact groups

Topological semigroups . Before beginning this chapter, we wish to recall quickly some basic facts about topological semigroups that we shall use.

Let S be a semigroup, with the product of two elements denoted by juxtaposition. For $t \in S$, we set

$$L_t : s \mapsto ts, \qquad R_t : s \mapsto st, \qquad S \to S.$$

For subsets A and B of S, set $AB = \{st : s \in A,\, t \in B\}$. A non-empty subset I of S is a *left ideal* in S if $SI \subset I$ and a *right ideal* if $IS \subset I$; I is an *ideal* if it is both a left and a right ideal in S. A *minimum ideal* in S is an ideal which is minimum in the family of all ideals in S when this family is ordered by inclusion. A minimum ideal of S is unique if it exists; it is denoted by $K(S)$, and is often called the *kernel* of S.

A semigroup S which is also a topological space is a *right topological semigroup* if the map R_t is continuous on S for each $t \in S$, and a *semitopological semigroup* (respectively, a *topological semigroup*) if the product map

$$(s,t) \mapsto st, \qquad S \times S \to S,$$

is separately continuous (respectively, continuous). A group G is a *topological group* if it is a topological semigroup and the map

$$s \mapsto s^{-1}, \qquad S \to S,$$

is continuous. In the case where S is (locally) compact as a topological space, we say that S is a (*locally*) *compact, right topological semigroup* or a (*locally*) *compact topological semigroup*, or a (*locally*) *compact group*, respectively. For an extensive account of topological semigroups, see [52]; see also [5] and [17, Definition 3.24].

For example, let T be a semigroup. Then, for each $s \in T$, the map L_s has an extension to a continuous map $L_s : \beta T \to \beta T$. For each $u \in \beta T$, define $s \,\Box\, u = L_s(u)$. Next, the map $R_u : s \mapsto s \,\Box\, u,\ T \to \beta T$, has an extension to a continuous map $R_u : \beta T \to \beta T$ for $u \in \beta T$. Define

$$u \,\Box\, v = R_v(u) \qquad (u, v \in \beta T).$$

Then $S = (\beta T, \Box)$ is a compact, right topological semigroup.

There is a major structure theorem for compact, right topological semigroups (and for more general semigroups); see [17, Theorem 3.25] and [52]. We state the (small) part of this theorem that we shall use.

Theorem 5.1. *Let S be a compact, right topological semigroup. Then the minimum ideal $K(S)$ exists. Further, the families of minimal left ideals and of minimal right ideals of S form partitions of $K(S)$; in particular, $L \cap K(S) \neq \emptyset$ for each left ideal L of S.* ■

The measure algebra of a locally compact group. Our next step is to take G to be a locally compact group, with left Haar measure denoted by m or m_G. We apply the theory of earlier chapters, with G replacing Ω. The topology on G is again denoted by τ; the identity of G is e or e_G, and we again set $E = C_0(G)$.

For example, we have introduced the Cantor cube $\mathbb{Z}_p^\kappa$ of weight κ; here $p \geq 2$ and κ is an infinite cardinal. The space $\mathbb{Z}_p^\kappa$ is a totally disconnected, perfect compact space. The set $\mathbb{Z}_p$ is a finite group with respect to addition modulo p, and $\mathbb{Z}_p^\kappa$ is a group with respect to the coordinatewise operations, denoted by $+$. Clearly $(\mathbb{Z}_p^\kappa, +)$ is a compact group. In Example 2.16, we described a measure m on Z_p^κ; this is easily seen to be the Haar measure on $\mathbb{Z}_p^\kappa$.

We now define the group algebra $(L^1(G), \star)$ and the measure algebra $M = (M(G), \star)$ of a locally compact group G; for details, see [48], [49], and [13, §3.3]. Indeed, for measures $\mu, \nu \in M(G)$, we set

$$(\mu \star \nu)(B) = \int_G \mu(Bs^{-1})\,\mathrm{d}\nu(s) \qquad (B \in \mathfrak{B}_G),$$

so that $\mu \star \nu \in M(G)$; the measure $\mu \star \nu$ is also defined as an element of $C_0(G)'$ by the formula

$$\langle \lambda, \mu \star \nu \rangle = \int_G \int_G \lambda(st)\,\mathrm{d}\mu(s)\,\mathrm{d}\nu(t) \qquad (\lambda \in C_0(G)).$$

Then $(M(G), \star, \|\cdot\|)$ is a Banach algebra, called the *measure algebra* of G. This algebra has an identity δ_{e_G}; the algebra is commutative if and only if G is abelian.

Let $\mu, \nu \in M(G)^+$. Then $\mu \star \nu \in M(G)^+$, and $\|\mu \star \nu\| = \|\mu\|\|\nu\|$.

For $f, g \in L^1(G)$, identified with the measures $f\,\mathrm{d}m$ and $g\,\mathrm{d}m$, respectively, we have

$$(f \star g)(t) = \int_G f(s) g(s^{-1}t)\,\mathrm{d}m(s) \qquad (t \in G).$$

The measure algebra $(M(G), \star)$ is always semisimple [13, Theorem 3.3.36]. The subspaces $M_c(G)$ and $L^1(G)$, identified with $M_{ac}(G)$, are closed ideals in $M(G)$, and $\ell^1(G)$ is a closed subalgebra of $M(G)$, so that

$$M(G) = \ell^1(G) \ltimes M_c(G) = \ell^1(G) \oplus_1 L^1(G) \oplus_1 M_s(G).$$

In the case where G is compact, $m_G \in M(G)^+$; in this case, we normalize m_G so that $m_G(G) = 1$.

The group algebra $L^1(G)$ will often be denoted just by A; by Wendel's theorem, the multiplier algebra of A is the measure algebra $M = (M(G), \star)$ [13, Theorem 3.3.40], and we regard A as a closed ideal in M. The point masses in M have the form δ_s for $s \in G$. The Banach algebra A has a bounded approximate identity, for example, the net $\{\chi_U/m(U) : u \in \mathcal{U}\}$, where $\mathcal{U}$ is the family of compact neighbourhoods of e_G, directed by reverse inclusion, is a bounded approximate identity.

For a function f on G, we set $\check{f}(s) = f(s^{-1})$ $(s \in G)$. The module operations in the space $L^\infty(G) = L^1(G)' = A'$ are given by

$$f \cdot \lambda = (\check{f}/\Delta) \star \lambda, \qquad \lambda \cdot f = \lambda \star \check{f} \quad (f \in A,\, \lambda \in A'),$$

where Δ is the modular function of G.

Let H be a closed subgroup of G, so that H is also a locally compact group, with left Haar measure m_H. We regard m_H as a measure on G by setting

$$m_H(B) = m_H(B \cap H) \quad (B \in \mathfrak{B}_G).$$

Let G be a locally compact group. The map

$$\mu \mapsto \mu(G) = \langle \mu,\, 1 \rangle, \quad M(G) \to \mathbb{C},$$

is a character on $M(G)$, called the *augmentation character*. (This may be the only character on $M(G)$.) In the case where G is compact, we clearly have $m_G \in M(G)^+$ and

$$\mu \star m_G = m_G \star \mu = \langle \mu,\, 1 \rangle m_G \quad (\mu \in M(G)). \tag{5.1}$$

In particular, $m_G \star m_G = m_G$. Further, each $\nu \in M(G)$ such that

$$\mu \star \nu = \nu \star \mu = \langle \mu,\, 1 \rangle \nu \quad (\mu \in M(G))$$

is a left-invariant measure on G, and so has the form ζm_G for some $\zeta \in \mathbb{C}$ (using the argument in [13, Proposition 3.3.53]).

Let G be a locally compact group, and let N be a closed, normal subgroup of G. Then $H := G/N$ is a locally compact group for the quotient topology, and the quotient map $\eta : G \to H$ is a continuous, open map which is a group epimorphism; see [48, §5] and [95, §3.1]. The induced map

$$\overline{\eta} : M(G) \to M(H)$$

was defined in equation (3.12), and $\widetilde{\eta} = \overline{\eta}'' \mid G : G \to H$ was defined in (4.6). Here we write Φ_G and Φ_H for the character spaces of $L^\infty(G, m_G)$ and $L^\infty(H, m_H)$, respectively.

Proposition 5.2.

(i) *Let G and H be locally compact groups, and let $\eta : G \to H$ be a continuous, open epimorphism. Then the induced map $\overline{\eta} : (M(G), \star) \to (M(H), \star)$ is a continuous epimorphism.*

(ii) *Let G and H be compact groups, and let $\eta : G \to H$ be a continuous epimorphism. Then*

$$\overline{\eta}(m_G) = m_H, \quad \widetilde{\eta}(\widetilde{G}) = \widetilde{H}, \quad \text{and} \quad \widetilde{\eta}(\Phi_G) = \Phi_H.$$

Proof. (i) Let $N = \eta^{-1}(\{e_H\})$, the kernel of η. By [48, (5.27)], we have $H = G/N$ as a locally compact group. It follows from (3.11) that $\overline{\eta}$ is exactly the map described in [95, (8.2.12)]. Thus the result is [95, Proposition 8.2.8].

(ii) By (5.1) and (i),

$$\delta_x \star \overline{\eta}(m_G) = \overline{\eta}(m_G) \star \delta_x = \overline{\eta}(m_G) \quad (x \in H),$$

and so $\overline{\eta}(m_G) = m_H$. By Corollary 4.12(ii), $\widetilde{\eta}(\widetilde{G}) = \widetilde{H}$ and hence $\widetilde{\eta}(\Phi_{m_G}) = \Phi_{\overline{\eta}(m_G)}$. Thus $\widetilde{\eta}(\Phi_G) = \Phi_H$. ■

We state the following closely related result; it is immediate from Proposition 3.19. A similar result is given as [55, Proposition 2.1(i)], where it is stated for abelian groups. For a general theory of the embeddings of group algebras, culminating in Cohen's idempotent theorem, see [101, Chapter 4].

Proposition 5.3. *Let G and H be locally compact groups, and let $\eta : G \to H$ be a Borel monomorphism. Then the induced map $\overline{\eta} : (M(G), \star) \to (M(H), \star)$ is an isometric injection.* ■

The hyper-Stonean envelope of G. Let G be a locally compact group. Then the hyper-Stonean envelope of the space G is denoted by $\widetilde{G}$. As before, the canonical projection is $\pi : \widetilde{G} \to G_\infty$, the dual space of $M(G)$ is $C_0(G)'' = C(\widetilde{G})$, and the second dual space is $M(\widetilde{G}) = M(G)''$. Here $M(G)'$ is a commutative C^*-algebra, and its identity, the constant function 1, when regarded as a functional on $M(G)$, is just the augmentation character. Thus $M(G)$ is a Lau algebra in the sense of Chapter 1. We have noted that the dual space $L^\infty(G)$ of the group algebra $L^1(G)$ is a C^*-algebra, and again the constant function 1, when regarded as a functional on $L^1(G)$ is just the augmentation character restricted to $L^1(G)$, and so $L^1(G)$ is also a Lau algebra. Here $\Box$ and $\Diamond$ are the Arens products from page 8.

Definition 5.4. Let G be a locally compact group. Then $(M(\widetilde{G}), \Box)$ and $(M(\widetilde{G}), \Diamond)$ are the unital Banach algebras formed by identifying $M(\widetilde{G})$ with the Banach algebras $(M(G)'', \Box)$ and $(M(G)'', \Diamond)$.

The space $E = C_0(G)$ is a $\|\cdot\|$-closed subspace of $M(G)' = C(\widetilde{G})$. For $\mu \in M(G)$ and $\lambda \in C_0(G)$, we have

$$(\lambda \cdot \mu)(t) = \int_G \lambda(ts)\,\mathrm{d}\mu(s), \qquad (\mu \cdot \lambda)(t) = \int_G \lambda(st)\,\mathrm{d}\mu(s) \qquad (t \in G),$$

and so $C_0(G)$ is a submodule of $M(G)'$. Thus $M(G)$ is a dual Banach algebra [102, Exercise 4.4.1], and hence

$$M(\widetilde{G}) = M(G)'' = M(G) \ltimes E^\circ, \tag{5.2}$$

where we are identifying $M(G)$ with $\kappa(M(G))$. In particular, the map

$$\pi = \kappa_E' : (M(\widetilde{G}), \Box) \to (M(G), \star) \tag{5.3}$$

is a continuous epimorphism, as in [56, Theorem 3.3].

Take $\mathrm{M}, \mathrm{N} \in M(\widetilde{G})^+$. Then $\mathrm{M} \Box \mathrm{N} \in M(\widetilde{G})^+$, and

$$\begin{aligned}\|\mathrm{M} \Box \mathrm{N}\| &= (\mathrm{M} \Box \mathrm{N})(\widetilde{G}) = \pi(\mathrm{M} \Box \mathrm{N})(G) = (\pi(\mathrm{M}) \star \pi(\mathrm{N}))(G)\\ &= \pi(\mathrm{M})(G)\pi(\mathrm{N})(G) = \mathrm{M}(\widetilde{G})\mathrm{N}(\widetilde{G}) = \|\mathrm{M}\|\,\|\mathrm{N}\|\end{aligned}$$

by (3.5). In particular, let $\varphi, \psi \in \widetilde{G}$. Then $\delta_\varphi \mathbin{\scriptstyle\Box} \delta_\psi \in M(\widetilde{G})^+$, and $\|\delta_\varphi \mathbin{\scriptstyle\Box} \delta_\psi\| = 1$, where we write $\delta_\varphi \mathbin{\scriptstyle\Box} \delta_\psi$ for $\delta_\varphi \Box \delta_\psi$.

Proposition 5.5. *Let G be a locally compact group. Then the following conditions on $\mathrm{M} \in M(\widetilde{G})$ are equivalent:*

(a) M *is invertible in* $(M(\widetilde{G}), \Box)$ *with* $\|\mathrm{M}\| = \|\mathrm{M}^{-1}\| = 1$;
(b) *there exists* $s \in G$ *and* $\zeta \in \mathbb{T}$ *such that* $\mathrm{M} = \zeta\delta_s$.

Proof. This is [34, Theorem 3.5]. ■

Let B be a Borel subset of G. Then we have defined the subset K_B of $\widetilde{G}$ in Chapter 1. It is clear that

$$\delta_s \,\Box\, \chi_{K_B} = \chi_{K_{Bs^{-1}}} \quad (s \in G).$$

For example, let G be a discrete group. Then $M(G) = \ell^1(G)$ and $\widetilde{G}$ is identified with βG. For a general locally compact group G, we have $\beta G_d = \overline{G}$ and $(\beta G_d, \Box)$ is a compact, right topological semigroup which is a subsemigroup of $(M(\beta G_d), \Box)$.

Let G be a compact group. Then it follows immediately from (5.1) by taking weak-$*$ limits that

$$\mathrm{M} \,\Box\, m_G = m_G \,\Box\, \mathrm{M} = \langle \mathrm{M}, 1\rangle m_G \quad (\mathrm{M} \in M(\widetilde{G})). \tag{5.4}$$

Proposition 5.6. *Let* G *be a compact group. Suppose that* $\mathrm{N} \in M(\widetilde{G})$ *satisfies the equations*

$$\mathrm{M} \,\Box\, \mathrm{N} = \mathrm{N} \,\Box\, \mathrm{M} = \langle \mathrm{M}, 1\rangle \mathrm{N} \quad (\mathrm{M} \in M(\widetilde{G})) \tag{5.5}$$

and

$$\mathrm{N} \,\Box\, \mathrm{N} = \mathrm{N}. \tag{5.6}$$

Then $\mathrm{N} = m_G$ *or* $\mathrm{N} = 0$.

Proof. Since N satisfies (5.5), necessarily

$$m_G \,\Box\, \mathrm{N} = \mathrm{N} \,\Box\, m_G = \mathrm{N}.$$

By (5.4), we have $\mathrm{N} \,\Box\, m_G = \langle \mathrm{N}, 1\rangle m_G$, and so $\mathrm{N} = \zeta m_G$, where $\zeta = \langle \mathrm{N}, 1\rangle$. By (5.6), $\zeta^2 = \zeta$, and so $\zeta = 0$ or $\zeta = 1$, giving the result. ■

We now apply the theory of Chapters 3 and 4, with G for Ω and m as left Haar measure on G.

Proposition 5.7. *Let* G *be a locally compact group, let* X *be a Banach* $C_0(G)$*-submodule of* $M(G)$*, and denote the character space of the commutative* C^**-algebra* X' *by* Φ_X*. Then* Φ_X *is a clopen subset of* $\widetilde{G}$.

Suppose further that X *is a subalgebra* (*respectively, ideal*) *of the Banach algebra* $(M(G), \star)$*. Then*

$$(X'', \Box) = (M(\Phi_X), \Box)$$

is a closed subalgebra (*respectively, ideal*) *of* $(M(\widetilde{G}), \Box)$.

Proof. Since X is a Banach E-submodule of $M(G)$, it follows from Proposition 1.17 that $(X', \Box)$ is a commutative C^*-algebra. By Proposition 4.1, Φ_X is a clopen subset of $\widetilde{G}$ and we can identify X' with $C(\Phi_X)$. Hence we can identify X'' as a Banach space with $M(\Phi_X)$.

If X is a subalgebra or ideal of $(M(G), \star)$, the Banach algebra $(X'', \Box)$ is a closed subalgebra or ideal, respectively, of $(M(\widetilde{G}), \Box)$. ■

Definition 5.8. Let G be a locally compact group. Then Φ, $\widetilde{G}_c$, and $\widetilde{G}_d$ are the character spaces of $L^\infty(G)$, $M_c(G)'$, and $\ell^\infty(G)$, respectively.

Thus Φ, $\widetilde{G}_c$, and $\widetilde{G}_d = \overline{G}$ are clopen subsets of $\widetilde{G}$.

Corollary 5.9. *Let G be a locally compact group. Then*

$$(L^\infty(G)', \Box) = (M(\Phi), \Box) \quad \textit{and} \quad (M_c(G)'', \Box) = (M(\widetilde{G}_c), \Box)$$

are closed ideals and $(\ell^\infty(G)', \Box) = (M(\overline{G}), \Box)$ is a closed subalgebra of $(M(\widetilde{G}), \Box)$. ■

We note that, in the special case where G is compact, $L^1(G)$ is an ideal in $(M(\Phi), \Box)$ [118] and hence in $(M(\widetilde{G}), \Box)$ [35, Lemma 4].

Introverted subspaces. Let G be a locally compact group. Since $L^1(G)$ and $M(G)$ are Lau algebras, we have definitions of introverted C^*-subalgebras X of $L^\infty(G)$ and of $M(G)'$, and also of topologically invariant means on X. For example, a closed subspace X of $L^\infty(G)$ is an introverted C^*-subalgebra if X is a C^*-subalgebra of $L^\infty(G)$ and X is an introverted $L^1(G)$-submodule of $L^\infty(G)$. A topologically invariant mean on X is an element $m \in L^\infty(G)'$ such that $\|m\| = \langle 1, m\rangle = 1$ and

$$\langle m, \lambda \cdot \mu\rangle = \langle m, \mu \cdot \lambda\rangle = \langle m, \lambda\rangle \quad (\lambda \in X,\ \mu \in \mathcal{P}(L^1(G))).$$

The following result is given in [71, Corollary 4.3]; it also uses Johnson's famous theorem [59], [13, Theorem 5.6.42] on the amenability of $L^1(G)$.

Theorem 5.10. *Let G be a locally compact group. Then the following are equivalent:*

(a) *G is amenable;*
(b) *$L^1(G)$ is amenable;*
(c) *$L^1(G)$ is left-amenable;*
(d) *$M(G)$ is left-amenable.* ■

We also record the following theorem of Dales, Ghahramani, and Helemskii [15].

Theorem 5.11. *Let G be a locally compact group. Then $M(G)$ is amenable if and only if G is discrete and amenable.* ■

In the present case, an introverted C^*-subalgebra X of $L^\infty(G)$ or of $M(G)'$ is commutative. The character space of such a commutative C^*-algebra X is denoted by Φ_X; it is formed by identifying points of Φ, the character space of $L^\infty(G)$. As in Chapter 1, $(X', \Box)$ is a Banach algebra; it is identified with $(M(\Phi_X), \Box)$. The quotient map

$$R_X : \Lambda \mapsto \Lambda | C_0(G), \quad (X', \Box) \to (M(G), \star),$$

is a continuous epimorphism.

In the case where $X \subset C^b(G)$, define $\theta\mu \in X'$ for $\mu \in M(G)$ by

$$\langle \theta\mu, \lambda\rangle = \int_G \lambda(s)\,\mathrm{d}\mu(s) \quad (\lambda \in X).$$

Then $\theta : (M(G), \star) \to (M(\Phi_X), \Box)$ is an isometric embedding and

$$M(\Phi_X) = \theta(M(G)) \ltimes C_0(G)^\circ;$$

we regard $M(G)$ as a closed subalgebra of $M(\Phi_X)$. We also regard Φ_X as a compact subset of $M(\Phi_X)$, and so we see that $(\Phi_X, \Box)$ is a compact, right topological semigroup

[52, Theorem 21.43]. Thus we have quotient maps

$$q_G : (M(\widetilde{G}), \Box) \to (M(\Phi_X), \Box) \quad \text{and} \quad q_G : \widetilde{G} \to \Phi_X; \tag{5.7}$$

both of the maps $q_G : \widetilde{G} \to \Phi_X$ and $q_G : \Phi \to \Phi_X$ are continuous epimorphisms. There is a natural embedding of G in Φ_X, and so we can regard G as a dense, open subspace of Φ_X.

For a discussion of the above objects, see [5, §4.4], [17, Chapter 5], [52, Chapter 21], and [75].

The space $LUC(G)$. Let G be a locally compact group. Then $LUC(G)$ denotes the closed subspace of $C^b(G)$ consisting of the left uniformly continuous functions on G: these are the functions $\lambda \in C^b(G)$ such that the map

$$t \mapsto \lambda \cdot t, \quad G \to C^b(G),$$

is continuous, where $(\lambda \cdot t)(s) = \lambda(ts)$ $(s, t \in G)$. We set

$$Z = LUC(G),$$

so that $1 \in Z \subset C^b(G)$. The canonical embedding $\kappa_E : Z \to C(\widetilde{G})$ identifies Z as a unital C^*-subalgebra of $C(\widetilde{G})$. [In [48], Z is the space of right uniformly continuous functions on G.]

Let $\lambda \in L^\infty(G)$. Then λ is in the equivalence class of a function in $LUC(G)$ if and only if the map

$$t \mapsto \lambda \cdot t, \quad G \to L^\infty(G),$$

is continuous, and so the space Z is a left-introverted C^*-subalgebra of $L^\infty(G) = C(\Phi)$; for these results, see [123, Lemma 6.2] and [16, Proposition 7.15]. Since the map $\Lambda \mapsto \Lambda|\Phi$ from $\kappa_E(Z)$ onto $\kappa_E(Z)|\Phi$ is an injection, the space Z is clearly also a left-introverted C^*-subalgebra of $M(G)' = C(\widetilde{G})$. We note that Z is also a left-introverted C^*-subalgebra of $\ell^\infty(G) = L^1(G_d)$, and the two respective products on Z coincide [16, Proposition 7.20].

The character space Φ_Z of Z is formed by identifying points of $\widetilde{G}$ that are not separated by $\kappa_E(Z)$. (For $x \in G$, the equivalence classes in $\widetilde{G}$ are just the fibres $G_{\{x\}}$.) The space Φ_Z is denoted by $\gamma_u(G) = \mathcal{LUC}(G)$ in [52, Example 21.5.6] and by $G^{\mathcal{L}C}$ in [5].

We shall use the following theorem; it is [52, Exercise 21.5.3].

THEOREM 5.12. *Let G be a locally compact group, and let A and B be subsets of G such that $A \cap UB = \emptyset$ for some $U \in \mathcal{N}_{e_G}$. Then $\overline{A} \cap \overline{B} = \emptyset$ in Φ_Z.* ■

The spaces $AP(G)$ and $WAP(G)$. Let G be a locally compact group. For $\lambda \in L^\infty(G)$, set

$$LO(\lambda) = \{\lambda \cdot \delta_t : t \in G\}, \quad RO(\lambda) = \{\delta_t \cdot \lambda : t \in G\},$$

so that $LO(\lambda)$ and $RO(\lambda)$ are the *left-orbit* and *right-orbit* of λ, respectively. Then λ is *almost periodic* if the set $LO(\lambda)$ (equivalently, $RO(\lambda)$) is relatively compact in the $\|\cdot\|$-topology on $L^\infty(G)$ and *weakly almost periodic* if the set $LO(\lambda)$ (equivalently, $RO(\lambda)$) is relatively compact in the weak topology on $L^\infty(G)$; the spaces of almost periodic and weakly almost periodic functions on G are denoted by $AP(G)$ and $WAP(G)$, respectively. For the equivalence of the 'left' and 'right' versions of these definitions, see [5, pp. 130, 139].

The spaces $AP(G)$ and $WAP(G)$ are introverted C^*-subalgebras of $L^\infty(G)$. Further, by [5, p. 138], $C_0(G) \subset AP(G)$ if and only if G is compact.

Recall that $AP(A)$ and $WAP(A)$ for a Banach algebra A were defined in Definition 1.11. We have

$$AP(G) = AP(L^1(G)) \quad \text{and} \quad WAP(G) = WAP(L^1(G)).$$

The first proof of this is due to Wong, in the sense that it is an immediate consequence of [122, Lemma 6.3]; see also [69, Corollary 4.2(b)] and [28, 115]. It also follows from [122, Lemma 6.3] that $WAP(G) \subset LUC(G)$, and so

$$AP(G) \subset WAP(G) \subset LUC(G) \subset C^b(G) \subset L^\infty(G)$$

and $C_0(G) \subset WAP(G)$.

It follows from Proposition 1.14 that $AP(M(G))$ and $WAP(M(G))$ are introverted subspaces of $M(G)'$, and hence of $L^\infty(G)$. However it was not clear that $AP(M(G))$ and $WAP(M(G))$ are C^*-algebras; in fact, this has been proved recently by Daws in a striking paper [21]. For further related work, see [22] and [103].

Theorem 5.13. *Let G be a locally compact abelian group. Then both $AP(M(G))$ and $WAP(M(G))$ are introverted C^*-subalgebras of $M(G)'$.* ■

We also have the following result, which is surely well-known.

Theorem 5.14. *Let G be a locally compact group. Then:*

(i) *$WAP(G) \subset WAP(M(G))$;*
(ii) *$AP(G) \subset AP(M(G))$;*
(iii) *the space $WAP(G)$ is an introverted C^*-subalgebra of $M(G)'$.*

Proof. Set $M = M(G)$ and $\sigma = \sigma(M', M'')$.

(i) Let $\lambda_0 \in WAP(G)$, so that $\lambda_0 \in C^b(G) \subset M'$. The set $RO(\lambda_0)$ is relatively compact in $\sigma(C(\Phi), M(\Phi))$, and hence in σ. Let K be the σ-closed convex hull of $RO(\lambda_0)$. By the Krein–Šmulian theorem [13, Theorem A.3.29], K is compact in $(C(\widetilde{G}), \sigma)$.

Let p be the product topology on $\mathbb{C}^G$. Then (K, p) is Hausdorff and (K, σ) is compact. But $p \leq \sigma$, and so p and σ agree on K.

Let $\mu \in M(G)_{[1]}$. We regard μ as an element of $C^b(G)'$, and then take a norm-preserving extension of μ to be an element of the Banach space $\ell^\infty(G) = C(\beta G_d)$. The unit ball $\ell^1(G)_{[1]}$ is weak-$*$ dense in $C(\beta G_d)_{[1]}$, and so there is a net (μ_α) in $M_d(G)_{[1]}$ such that $\langle \mu_\alpha,\, \lambda \rangle \to \langle \mu_\alpha,\, \lambda \rangle$ for each $\lambda \in C^b(G)$. Hence

$$(\mu_\alpha \cdot \lambda)(t) = \langle \mu_\alpha,\, \lambda \cdot \delta_t \rangle \to \langle \mu,\, \lambda \cdot \delta_t \rangle = (\mu \cdot \lambda)(t) \quad (t \in G).$$

This shows that $\mu_\alpha \cdot \lambda \to \mu \cdot \lambda$ in (K, p), and hence in (K, σ). Since $RO(\lambda_0) \subset K$ and (K, σ) is compact, it follows that $\{\mu \cdot \lambda : \mu \in M(G)_{[1]}\} \subset K$. This implies that $K(\lambda)$, as defined in (1.4), is compact in (M', σ), showing that $\lambda_0 \in WAP(M(G))$.

(ii) This is similar.

(iii) The argument in (i) shows that $\mu \cdot \lambda \in WAP(G)$ whenever $\lambda \in WAP(G)$ and $\mu \in M(G)$, and so $WAP(G)$ is a Banach left $M(G)$-submodule of the dual module $M(G)'$. Similarly, $WAP(G)$ is a Banach right $M(G)$-submodule of $M(G)'$, and clearly $WAP(G)$

is a Banach $M(G)$-sub-bimodule of $M(G)'$. By (i), we have

$$WAP(G) \subset WAP(M(G)),$$

and so, by Proposition 1.14(ii), $WAP(G)$ is introverted in $M(G)'$. Certainly $WAP(G)$ is a C^*-subalgebra of $M(G)'$ with $C_0(G) \subset WAP(G)$, and so $WAP(G)$ is an introverted C^*-subalgebra of $M(G)'$. ■

Proposition 5.15. *Let G be a locally compact group, and take $\lambda \in \ell^\infty(G)$. Then we have $\kappa_E(\lambda)\,|\,\widetilde{\Omega}_d \in WAP(M(G))$ if and only if $\lambda \in WAP(G_d)$.*

Proof. For $\lambda \in \ell^\infty(G)$, set $T\lambda = \kappa_E(\lambda)|\widetilde{G}_d$ (so that $T\lambda$ is uniquely defined in $C(\widetilde{G})$).

Let $\lambda \in \ell^\infty(G)$. For $\mu, \nu \in M(G)$, we have

$$\langle \nu,\ \mu\cdot T\lambda\rangle = \langle \mu\star\nu,\ j_d'(\kappa_E(\lambda))\rangle = \langle (\mu\star\nu)_d,\ j_d'(\kappa_E(\lambda))\rangle = \langle \mu_d\star\nu_d,\ j_d'(\kappa_E(\lambda))\rangle = \langle \nu_d,\ \mu_d\cdot\lambda\rangle,$$

where the last two dualities are ℓ^1-ℓ^∞-dualities. Thus $\overline{K(T\lambda)}$ in $C(\widetilde{G})$ is equal to $\overline{K(\lambda)}$ in $\ell^\infty(G)$, and so these two sets are weakly compact in the appropriate space if and only if the other has the same property.

The result follows. ■

Since $WAP(M(G))$ contains $WAP(G_d)$, which includes $C_0(G_d)$ as a subspace, we see that $WAP(G) = WAP(M(G))$ if and only if G is discrete. It seems that the spaces $AP(M(G))$ and $WAP(M(G))$ are not well understood in the case where G is not discrete.

Let G be a locally compact group. It is interesting to ask when $WAP(M(G))$ has a topological invariant mean; we have the following partial result.

Proposition 5.16. *Let G be a locally compact group. Suppose that G is discrete or amenable. Then $WAP(M(G))$ has a topological invariant mean.*

Proof. This is immediate in the case where G is discrete.

Now suppose that G is amenable. Then, by Proposition 5.10, $M(G)$ is a left-amenable Banach algebra, and so, by Proposition 1.21, $M(G)'$ has a topological left-invariant mean. Similarly, $M(G)'$ has a topological right-invariant mean, and hence a $M(G)'$ has a topological invariant mean. The restriction of this mean to $WAP(M(G))$ is a topological invariant mean on $WAP(M(G))$. ■

The structure semigroup of G. The structure semigroup of a locally compact abelian group G was introduced by J. L. Taylor in [113] and discussed in some detail by Taylor in [114, Chapters 3, 4]; the work is also described in the text [41, §5.1] of Graham and McGehee. This structure semigroup has been used by Brown [6] and by Chow and White [9]; an important early paper of Hewitt and Kakutani is [47].

We shall present what appears to be a somewhat more direct and abstract approach to the definition and the results. The definition is also applicable to non-abelian groups, but the semigroup may be trivial in the non-abelian case.

Definition 5.17. Let G be a locally compact group. The character space of the Banach algebra $M(G)$ is $\Phi_{M(G)} = \Phi_M$.

Let $\Lambda \in \Phi_M$. Then Λ is an element of $M(G)' = C(\widetilde{G})$ with $|\Lambda|_{\widetilde{G}} = 1$, and so Φ_M is a subset of $C(\widetilde{G})_{[1]}$; in particular, Φ_M inherits a product from $C(\widetilde{G})_{[1]}$. A key fact is that

Φ_M is closed under complex conjugation and this product, so that Φ_M is a $*$-semigroup. This follows from results in [109]; an explicit, simple proof is given in [94]; a result that applies when the group G is replaced by an arbitrary locally compact abelian semigroup with separately continuous product is given in [9] and [96, Theorem (4.1)].

The constant function 1 on $\widetilde{G}$, regarded as a continuous linear functional on $M(G)$, is exactly the augmentation character on $M(G)$, and so we may say that $1 \in \Phi_M$.

Suppose that the locally compact group G is abelian. Then the set of elements Λ of $C(\widetilde{G})_{[1]}$ with the property that $|\Lambda(\varphi)| = 1$ $(\varphi \in \widetilde{G})$ is just the canonical image of $\Gamma := \widehat{G}$, the dual group of G [114, Corollary p. 36]. Let $\varphi, \psi \in \widetilde{G}$ with $\varphi \neq \psi$. There exists $\mathrm{M} \in C_{\mathbb{R}}(\widetilde{G})$ with $\mathrm{M}(\varphi) = 0$ and $\mathrm{M}(\psi) = 1$, and then $\exp(\mathrm{iM}) \in \Gamma$ and also $\exp(\mathrm{iM})(\varphi) \neq \exp(\mathrm{iM})(\psi)$. Thus Γ separates the points of $\widetilde{G}$. By the Stone–Weierstrass theorem, $\operatorname{lin}\Gamma$ is $|\cdot|_{\widetilde{G}}$-dense in $C(\widetilde{G})$.

Definition 5.18. Let G be a locally compact group. Then X_G is the $|\cdot|_{\widetilde{G}}$-closure of $\operatorname{lin}\Phi_M$ in $C(\widetilde{G})$.

Thus X_G is a unital C^*-subalgebra of $C(\widetilde{G})$.

Let $\mu \in M(G)$ and $\gamma \in \Phi_M$. Then

$$(\mu \cdot \gamma)(\nu) = \gamma(\mu \star \nu) = \gamma(\mu)\gamma(\nu) \quad (\nu \in M(G)),$$

and so $\mu \cdot \gamma = \gamma(\mu)\gamma$. It follows that X_G is an $M(G)$-submodule of $M(G)'$. In fact, each element of $\operatorname{lin}\Phi_M$ has finite-dimensional range as an operator on $M(G)'$, and so

$$X_G \subset AP(G) \subset AP(M(G)) \subset WAP(M(G)) \subset M(G)' = C(\widetilde{G}).$$

In the case where the group G is compact, we have

$$AP(G) = WAP(G) = C(G);$$

see [101].

Proposition 5.19. *Let G be a locally compact group. Then X_G is an introverted subspace of $M(G)'$, and $(X_G', \Box) = (X_G', \Diamond)$ is a Banach algebra.*

Proof. This follows immediately from Proposition 1.14(ii). ■

The following definition was first given by Taylor; see [113, 114].

Definition 5.20. Let G be a non-discrete, locally compact group. Then the character space of X_G is denoted by $S(G)$ and called the *structure semigroup* of $M(G)$.

Suppose that G is a (discrete) abelian group. Then $S(G)$ is the *Bohr compactification* of G; the space $(S(G), \Box)$ is a compact group.

The justification for the term 'semigroup' will come in Proposition 5.21 below. Set $X = X_G$. We see that $S(G) \subset X'_{[1]}$; as usual, $S(G)$ is given the relative $\sigma(X', X)$-topology. The quotient map

$$q_G : (M(\widetilde{G}), \Box) \to (X', \Box)$$

is a continuous homomorphism that induces a continuous homeomorphism of $\widetilde{G}$ onto $S(G)$. The space $S(G)$ inherits the multiplication $\Box$ from $(X'_{[1]}, \Box)$.

The following result is a theorem of Rennison [96, Theorem (5.2)]; we shall obtain it in a more general context in the next section.

PROPOSITION 5.21. *Let G be a locally compact abelian group. Then $(S(G), \Box)$ is a compact, abelian topological semigroup.* ■

In [96, Theorem (5.4)], the semi-characters on $S(G)$ are identified with the Gel'fand transforms of elements of $\Phi_{M(G)}$, and in [96, Theorem (6.5)] it is shown that the semigroup $(S(G), \Box)$ is exactly the structure semigroup of $M(G)$ which was defined by Taylor [114] in a more complicated manner. In [83, Theorem 5.2], McKilligan and White consider the situation where $M(G)$ is replaced by a general 'L-algebra' $\mathfrak{A}$; $\mathfrak{A}'$ is again a commutative C^*-algebra, and X_G is replaced by a general introverted subspace X of $\mathfrak{A}'$ such that $1 \in X \subset WAP(\mathfrak{A})$; they give a necessary and sufficient condition for the character space of X to be a subsemigroup of $X'_{[1]}$ with respect to the relative Arens product $\Box$.

For further study of the structure semigroup, see [114], [41, §5.1], and §4, Chapitre IV, of the substantial text [53].

The structure semigroup for Lau algebras. The notion of a Lau algebra was recalled in Chapter 1.

Let A be a commutative Lau algebra, with character space Φ_A, so that $\Phi_A \subset A'$, where $(A', \cdot)$ is a C^*-algebra (not necessarily commutative); the identity of A' is e. Recall from equation (1.5) the definitions of $L_\mu a, R_\mu a \in A$ for $a \in A$ and $\mu \in A'$.

Further suppose that T is a subset of A' such that T is a subsemigroup of $(A', \cdot)$, and let T have the relative $\sigma(A', A)$-topology from A'. Then T is a semitopological semigroup because the product in A' is separately $\sigma(A', A)$-continuous. For each $a \in A$, define

$$\widehat{a} : \varphi \mapsto \varphi(a), \qquad T \to \mathbb{C},$$

and set

$$B(T) = \{\widehat{a} : a \in A\}.$$

Then $B(T)$ is a subalgebra of $C^b(T)$. Clearly the map $a \mapsto \widehat{a},\ A \to B(T)$, is a homomorphism, and it is an injection if and only if $\operatorname{lin} T$ is $\sigma(A', A)$-dense in A'.

Now let $\lambda \in A'$ and $f \in B(T)$. Define

$$(\ell_\varphi f)(\psi) = f(\varphi \cdot \psi) \qquad (\varphi, \psi \in T),$$

so that

$$\widehat{L_\varphi a}(\psi) = \langle a,\, \varphi \cdot \psi\rangle = (\ell_\varphi f)(\psi) \qquad (\psi \in T),$$

and hence $\widehat{L_\varphi a} = \ell_\varphi f$ whenever $f = \widehat{a}$. We now suppose throughout that $\operatorname{lin} T$ is $\sigma(A', A)$-dense in A', so that, for each $f \in B(T)$, there is a unique $a \in A$ with $f = \widehat{a}$. In the case where A has an identity u, we have $L_\varphi u = u$ $(\varphi \in T)$. Define

$$\lambda_\ell(f)(\varphi) = \langle L_\varphi a,\, \lambda\rangle = \langle a,\, \varphi \cdot \lambda\rangle = \langle R_\lambda a,\, \varphi\rangle \qquad (\varphi \in T),$$

so that $\lambda_\ell(f) \in C^b(T)$. Indeed, $\lambda_\ell f = \widehat{R_\lambda a} \in B(T)$.

Let $a, b \in A$ and $\varphi, \psi \in T$, so that $\varphi \cdot \psi \in T \subset \Phi_A \cup \{0\}$. Then

$$\begin{aligned}\langle R_\varphi(ab),\, \psi\rangle &= \langle ab,\, \varphi \cdot \psi\rangle = \langle a,\, \varphi \cdot \psi\rangle\langle b,\, \varphi \cdot \psi\rangle \\ &= \langle R_\varphi a,\, \psi\rangle\langle R_\varphi b,\, \psi\rangle = \langle (R_\varphi a)(R_\varphi b),\, \psi\rangle,\end{aligned}$$

and so

$$R_\varphi(ab) = (R_\varphi a)(R_\varphi b) \in A. \tag{5.8}$$

Proposition 5.22. *Let A be a commutative Lau algebra, and let T be a subsemigroup of $\Phi_A \cup \{0\}$ such that $e \in T$ and $\operatorname{lin} T$ is $\sigma(A', A)$-dense in A'. Let $\lambda \in A'$. Then L_λ is an automorphism on $A = B(T)$ if and only if $\lambda \in \Phi_A \cup \{0\}$.*

Proof. Suppose that $\lambda \in \Phi_A \cup \{0\}$. We have noted that L_λ is a bounded linear operator on A. Let $a, b \in A$. For each $\varphi \in T$, we have

$$\begin{aligned}\langle L_\lambda(ab),\, \varphi\rangle &= \langle R_\varphi(ab),\, \lambda\rangle && \text{by (1.5)}\\ &= \langle (R_\varphi a)(R_\varphi b),\, \lambda\rangle && \text{by (5.8)}\\ &= \langle R_\varphi a,\, \lambda\rangle\langle R_\varphi b,\, \lambda\rangle && \text{because } \lambda \in \Phi_A \cup \{0\}\\ &= \langle L_\lambda a,\, \varphi\rangle\langle L_\lambda b,\, \varphi\rangle &&\\ &= \langle (L_\lambda a)(L_\lambda b),\, \varphi\rangle && \text{because } \varphi \in \Phi_A \cup \{0\}.\end{aligned}$$

Thus $L_\lambda(ab) = (L_\lambda a)(L_\lambda b)$, and so L_λ is an automorphism on A.

Conversely, suppose that L_λ is an automorphism on A. We have

$$\langle ab,\, \lambda\rangle = \langle L_\lambda(ab),\, e\rangle = \langle L_\lambda a,\, e\rangle\langle L_\lambda b,\, e\rangle = \langle a,\, \lambda\rangle\langle b,\, \lambda\rangle,$$

and so $\lambda \in \Phi_A \cup \{0\}$. ∎

A subsemigroup of $(A', \cdot)$ is a *$*$-semigroup* if it is closed under the involution on A'.

Proposition 5.23. *Let A be a commutative Lau algebra. Then the following are equivalent:*

(a) *A is semisimple and $\Phi_A \cup \{0\}$ is a $*$-semigroup with respect to the product and involution on A';*

(b) *there is a $*$-subsemigroup T of A' with $e \in T \subset \Phi_A \cup \{0\}$ such that $\operatorname{lin} T$ is $\sigma(A', A)$-dense in A'.*

Proof. (a)⇒(b) Set $T = \Phi_A \cup \{0\}$. Clearly $e \in T$ and T is a $*$-subsemigroup T of A'. Assume towards a contradiction that there exists $\lambda \in A'$ with λ not in the $\sigma(A', A)$-closure of $\operatorname{lin} T$. Then there exists $a \in A$ such that $\langle a,\, \lambda\rangle = 1$ and $\langle a,\, \varphi\rangle = 0$ $(\varphi \in \Phi_A)$, so that $\widehat{a} = 0$. Since A is semisimple, $a = 0$, a contradiction. Thus $\operatorname{lin} T$ is $\sigma(A', A)$-dense in A'.

(b)⇒(a) Suppose that $a \in A$ with $\widehat{a} = 0$. Since $T \subset \Phi_A \cup \{0\}$ and $\operatorname{lin} T$ is $\sigma(A', A)$-dense in A', it follows that $\langle a,\, \lambda\rangle = 0$ $(\lambda \in A')$, and so $a = 0$. Thus A is semisimple.

Let $\varphi, \psi \in \Phi_A \cup \{0\}$. For each $a, b \in A$, we have

$$\begin{aligned}\langle ab,\, \varphi \cdot \psi\rangle = \langle L_\varphi(ab),\, \psi\rangle &= \langle (L_\varphi a)(L_\varphi b),\, \psi\rangle \quad \text{by Proposition 5.22}\\ &= \langle L_\varphi a,\, \psi\rangle\langle L_\varphi b,\, \psi\rangle = \langle a,\, \varphi \cdot \psi\rangle\langle b,\, \varphi \cdot \psi\rangle,\end{aligned}$$

and so $\varphi \cdot \psi \in \Phi_A \cup \{0\}$. It follows that $\Phi_A \cup \{0\}$ is a semigroup in A'; clearly $\Phi_A \cup \{0\}$ is a $*$-semigroup. ∎

An example given in [74, Corollary 3.8] exhibits a commutative, semisimple Lau algebra such that $\Phi_A \cup \{0\}$ is a $*$-semigroup, but Φ_A itself is not a semigroup.

Definition 5.24. Let A be a commutative Lau algebra such that $(A', \cdot)$ is commutative. Then X_A is the $\|\cdot\|$-closure of $\operatorname{lin} \Phi_A$ in A'.

Thus X_A is a commutative, unital C^*-subalgebra of A'. As before, X_A is an introverted subspace of A', and so $(X'_A, \Box)$ is a Banach algebra for the product $\Box$ inherited from $(A', \Box)$. Also as before, we have $X_A \subset AP(A) \subset WAP(A)$.

In general, $AP(A)$ and $WAP(A)$ need not be subalgebras of the C^*-algebra A'. For example, let K be a hypergroup with a left Haar measure (see [57, 100]). Then the hypergroup algebra $A = L^1(K)$ is a Lau algebra. Since A has a bounded approximate identity [57, 100], it follows from Proposition 1.2 that $A \cdot A' \cdot A = A \cdot A'$ is a closed linear subspace of A', and hence $WAP(A) \subset A \cdot A' \subset C^b(K)$ [107, Lemma 2]. Let $AP(K)$ and $WAP(K)$ denote the spaces of elements $\lambda \in C^b(K)$ such that $\{\ell_x\lambda : x \in K\}$ is relatively compact in the norm and weak topologies, respectively, of $C^b(K)$. Then, by [108, Remark 2.4(i)], we have $AP(K) = AP(A)$ and $WAP(K) = WAP(A)$. However there is an example in [68] of a hypergroup K such that $AP(K)$ is not a subalgebra of $C^b(K)$, and in [121] there is an example of a hypergroup K such that neither $AP(K)$ nor $WAP(K)$ is an algebra. Thus it follows that, in general, neither $AP(A)$ nor $WAP(A)$ is a subalgebra of A'.

As we have remarked, Daws [21] has proved that $AP(M(G))$ and $WAP(M(G))$ are C^*-algebras when G is a locally compact group. It remains an interesting open question whether $AP(A)$ and $WAP(A)$ are necessarily C^*-subalgebras of A' when A' is a Hopf–von Neumann algebra, not necessarily commutative. This problem has been studied by Chou when A is the Fourier algebra $A(G)$ of a locally compact group [8]; see also [103].

DEFINITION 5.25. Let A be a commutative Lau algebra such that $(A', \cdot)$ is commutative. Then the character space of X_A is denoted by $S(A)$; it is the *structure semigroup* of A.

Thus the definition of $S(A)$ generalizes that of $S(G)$ in the case where G is a locally compact abelian group, in which case $A = M(G)$ is a commutative Lau algebra and $A' = C(\widetilde{G})$ is a commutative von Neumann algebra.

THEOREM 5.26. *Let A be a commutative Lau algebra such that $(A', \cdot)$ is commutative. Suppose that there is a $*$-subsemigroup T of A' with $e \in T \subset \Phi_A \cup \{0\}$ such that* $\operatorname{lin} T$ *is $\sigma(A', A)$-dense in A'. Then $(S(A), \Box)$ is a compact, abelian topological semigroup.*

Proof. The space $S(A)$ is compact because X_A is a unital, commutative C^*-algebra.

By (1.2), $\mathrm{M} \cdot \varphi = \langle \mathrm{M}, \varphi\rangle\varphi$ ($\mathrm{M} \in A''$, $\varphi \in \Phi_A$). Thus

$$\langle s \Box t,\, \varphi\rangle = \langle s,\, t \cdot \varphi\rangle = \langle s, \varphi\rangle\langle t,\, \varphi\rangle \quad (s, t \in S(A),\ \varphi \in \Phi_A).$$

Let $\varphi, \psi \in \Phi_M$. Then $\varphi \cdot \psi \in \Phi_A \cup \{0\}$ by Proposition 5.23, and so

$$\langle s \Box t,\, \varphi \cdot \psi\rangle = \langle s, \varphi \cdot \psi\rangle\langle t,\, \varphi \cdot \psi\rangle = \langle s, \varphi\rangle\langle s, \psi\rangle\langle t, \varphi\rangle\langle t, \psi\rangle = \langle s \Box t, \varphi\rangle\langle s \Box t, \psi\rangle.$$

Thus $s \Box t \in S(A)$ because $\operatorname{lin} \Phi_A$ is dense in X_A, and so $(S(G), \Box)$ is a semigroup.

That $(S(A), \Box)$ is a compact, topological semigroup follows from Proposition 1.15. ∎

COROLLARY 5.27. *Let G be a locally compact abelian group. Then $(S(G), \Box)$ is a compact, abelian topological semigroup.*

Proof. Let $A = M(G)$ and $T = \Gamma$, the dual group of G. Then A and T satisfy the conditions in the above theorem. ∎

Submodules of $M(G)''$. Let G be a locally compact group. A closed subspace F of $M(\widetilde{G})$ is *translation-invariant* if

$$s \cdot \Lambda \cdot t \in F \quad (s, t \in G,\, \Lambda \in F).$$

Again set $E = C_0(G)$ and $A = L^1(G, m_G)$. Let

$$D = \ell^1(G)$$

denote the subspace of M consisting of the discrete measures, so that D is a closed subalgebra of M, and let $M_s = M_s(G)$ denote the space of (non-discrete) singular measures on G, so that M_s is a closed linear subspace of M. It was first shown by Hewitt and Zuckerman [51] that, for every non-discrete, locally compact abelian group, there is a probability measure $\mu \in M_s(G)$ such that $\mu \star \mu \in L^1(G)$, and so $M_s(G)$ is not a subalgebra of $(M(G), \star)$. As in (2.8), we have

$$M = A \oplus_1 D \oplus_1 M_s = D \ltimes M_c$$

as a direct ℓ^1-sum of Banach spaces; each of A, D, M_c, and M_s is a translation-invariant E-submodule of M. It follows that

$$M'' = A'' \oplus D'' \oplus M_s'' = D'' \ltimes M_c'',$$

where each of A'', D'', and M_s'' is a translation-invariant E-submodule of M''. Further, A'' and M_c'' are closed ideals in $(M'', \Box)$ and D'' is a closed subalgebra $(M'', \Box)$. We note that the weak-$*$ topologies on the spaces A'', D'', and M_s'' are just the appropriate relative weak-$*$ topology from M''. The canonical embedding is now

$$\kappa = \kappa_M : M \to M'' = M(\widetilde{G}).$$

Set $\mathfrak{A} = A \oplus D$. Then $\mathfrak{A}$ is a closed subalgebra of $(M, \star)$ and

$$\mathfrak{A} = D \ltimes A = \ell^1(G) \ltimes L^1(G) \quad \text{and} \quad \mathfrak{A}'' = D'' \ltimes A''.$$

For details of the remarks concerning the Banach algebra $M(G)$, see [13, §3.3] and [48, (19.20) and (19.26)].

As we have stated, $L^1(G, m_G)$ is a closed ideal of $M(G)$. Now take $\mu \in M(\Omega)^+$. In general, $L^1(G, \mu)$ is not a subalgebra of $M(G)$, but there may be singular measures μ for which this is true; for example, this is the case if $\mu = m_H$ is the left Haar measure on a closed, non-open subgroup H of G.

Recall that the character space of $L^\infty(G) = L^\infty(G, m_G) = A'$ is denoted just by Φ. Of course, Φ is a clopen subset of $\widetilde{G}$, and $\pi(\Phi) = G$. Thus we may suppose that the family $\{\Omega_i : i \in I\}$ of subsets of $\widetilde{G}$ described in Proposition 4.8 contains the singletons $\{x\}$ for $x \in G$ and the compact space Φ. The space Φ is the topic of the paper [78], where it is called the *spectrum* of $L^\infty(G)$. For $x \in G$, we set

$$\Phi_{\{x\}} = \widetilde{G}_{\{x\}} \cap \Phi = \{\varphi \in \Phi : \pi(\varphi) = x\}.$$

Let $\mu \in M(G)^+$ and $x \in G$, and set $\nu = \delta_x \star \mu$. Then it is clear that

$$\Phi_\nu = \delta_x \star \Phi_\mu := \{\delta_x \star \delta_\psi : \psi \in \Phi_\mu\}.$$

PROPOSITION 5.28. *Let G be a locally compact group, and suppose that $\mu \in M_s(G)^+$. Then $\Phi_\mu \cap (\delta_x \star \Phi_\mu) = \emptyset$ for almost all $x \in (G, m_G)$.*

Proof. By [41, Corollary 8.3.3], $\{x \in G : \delta_x \star \mu \not\perp \mu\}$ is a Borel set, say B, such that $m_G(B) = 0$. (The result in [41] is stated for abelian groups, but the proof of this result applies also to general, non-abelian groups.) Thus, for $x \in G \setminus B$, we have $\delta_x \star \mu \perp \mu$, and so $\Phi_\mu \cap (\delta_x \star \Phi_\mu) = \emptyset$. ■

In the case where G is compact, infinite, and metrizable, the space Φ is homeomorphic to $\mathbb{H}$, the hyper-Stonean space of the unit interval.

We have noted in Corollary 5.9 that $(M(\Phi), \Box)$ is a closed ideal in $(M(\widetilde{G}), \Box)$. In particular, for each $\varphi \in \Phi$, we have the map

$$L_\varphi : \mathrm{M} \mapsto \delta_\varphi \Box \mathrm{M}, \qquad M(\widetilde{G}) \to M(\Phi). \tag{5.9}$$

The compact spaces corresponding to A, D, M_s, and M are denoted by Φ, $\Phi_D = \beta G_d$, Φ_s, and $\widetilde{G}$, respectively. (In fact they are the character spaces of the C^*-algebras A', $D' = \ell^\infty(G)$, M_s', and M', respectively.) We have shown that

$$\{\Phi, \beta G_d, \Phi_s\}$$

is a partition of $\widetilde{G}$ into clopen subsets and that $\overline{G} = \beta G_d$. It follows from our remarks that

$$M(G)'' = M(\widetilde{G}) = M(\Phi) \oplus_1 M(\beta G_d) \oplus_1 M(\Phi_s)$$

as a Banach space, that $M(\Phi)$ is a closed ideal in $(M(\widetilde{G}), \Box)$, and that $M(\beta G_d)$ and $\mathfrak{A}''$ are closed subalgebras in $(M(\widetilde{G}), \Box)$.

Proposition 5.29. *Let G be a locally compact group. Then $\mathfrak{A} = \ell^1(G) \ltimes L^1(G)$ is strongly Arens irregular.*

Proof. This follows from [17, Proposition 2.25]. ■

6. Formulae for products

In this chapter, we shall establish some formulae for products in the algebra $(M(\widetilde{G}), \Box)$ that we shall require. Our method is based on the use of ultrafilters.

Let G be a locally compact group. We shall use the following notation. First, take a positive measure $\mu \in M(G)^+$ and $B \in \mathfrak{B}_G$ with $\mu(B) \neq 0$. We recall that we are setting $\mu_B = (\mu | B)/\mu(B)$. Fix $L \in \mathfrak{B}_G$. Then we now make the following definition of a function $\lambda_{\mu,B}$:

$$\lambda_{\mu,B}(t) := \frac{\mu(B \cap Lt^{-1})}{\mu(B)} = \mu_B(Lt^{-1}) \quad (t \in G). \tag{6.1}$$

Thus

$$\lambda_{\mu,B}(t) = \int_B \chi_{Lt^{-1}}(s)\,\mathrm{d}\mu_B(s) \quad (t \in G). \tag{6.2}$$

Then $\lambda_{\mu,B}$ is a function on G that belongs to $B^b(G)$.

We also recall that $U_G = \bigcup\{\Phi_\mu : \mu \in M(G)^+\}$, as in Definition 4.5.

Proposition 6.1. *Let G be a locally compact group.*

(i) *Let $\mu, \nu \in M(G)$. Then*

$$\langle \kappa_E(\lambda),\, \mu \star \nu\rangle = \int_G \int_G \lambda(st)\,\mathrm{d}\mu(s)\,\mathrm{d}\nu(t) \quad (\lambda \in B^b(G)). \tag{6.3}$$

(ii) *Let $\varphi, \psi \in \widetilde{G}$ with $\varphi \in \Phi_\mu$ and $\psi \in \Phi_\nu$, where $\mu, \nu \in M(G)^+$. Then*

$$\langle \kappa_E(\lambda),\, \delta_\varphi \Box \delta_\psi\rangle = \lim_{B\to\varphi} \lim_{C\to\psi} \int_B \int_C \lambda(st)\,\mathrm{d}\mu_B(s)\,\mathrm{d}\nu_C(t) \tag{6.4}$$

for each $\lambda \in B^b(G)$.

(iii) *Let $\varphi \in \Phi_\mu$, where $\mu \in M(G)^+$, and let $L \in \mathfrak{B}_G$. Suppose that $\psi \in \overline{G}$ has the form $\psi = \lim_\alpha s_\alpha$, where (s_α) is a net in G. Then*

$$\langle \chi_{K_L},\, \delta_\varphi \Box \delta_\psi\rangle = \lim_{B\to\varphi} \lim_\alpha (\mu_B \star \delta_{s_\alpha})(L). \tag{6.5}$$

(iv) *For each $\varphi \in U_G$, each $\nu \in M(G)^+$, each $L \in \mathfrak{B}_G$, and each $B \in \mathfrak{B}_G$ with $\nu(B) > 0$, we have*

$$(\nu_B \Box \delta_\varphi)(K_L) = \langle \kappa_E(\lambda_{\nu,B}),\, \delta_\varphi\rangle. \tag{6.6}$$

Proof. (i) Let $\lambda \in B^b(\Omega)$. By (3.6), we have

$$\langle \kappa_E(\lambda),\, \mu \star \nu\rangle = \int_G \lambda\,\mathrm{d}(\mu \star \nu).$$

By a standard theorem [48, Theorem (19.10)],

$$\int_G \lambda \,\mathrm{d}(\mu \star \nu) = \int_G \int_G \lambda(st)\,\mathrm{d}\mu(s)\,\mathrm{d}\nu(t);$$

this theorem applies because $\lambda \in L^1(G, |\mu| \star |\nu|)$. The result follows.

(ii) For each $\Lambda \in C(\widetilde{G})$, we have

$$\langle \Lambda,\, \delta_\varphi \mathbin{\Box} \delta_\psi\rangle = \lim_{B\to\varphi}\lim_{C\to\psi}\langle \Lambda,\, \mu_B \star \mu_C\rangle$$

by (1.1). In particular,

$$\langle \kappa_E(\lambda),\, \delta_\varphi \mathbin{\Box} \delta_\psi\rangle = \lim_{B\to\varphi}\lim_{C\to\psi}\langle \kappa_E(\lambda),\, \mu_B \star \mu_C\rangle \qquad (\lambda \in B^b(G)).$$

The result now follows from (i).

(iii) It follows from (i) that

$$\langle \kappa_E(\lambda),\, \delta_\varphi \mathbin{\Box} \delta_\psi\rangle = \lim_{B\to\varphi}\lim_{\alpha}\int_B \lambda(ss_\alpha)\,\mathrm{d}\mu_B(s) \qquad (\lambda \in B^b(G)).$$

Apply this with $\lambda = \chi_L \in B^b(G)$, so that $\kappa_E(\lambda) = \chi_{K_L}$ in $C(\widetilde{G})$. We also have

$$\int_L \chi_L(ss_\alpha)\,\mathrm{d}\mu_B(s) = \int_B \chi_{Ls_\alpha^{-1}}(s)\,\mathrm{d}\mu_B(s) = \lambda_{\mu,B}(s_\alpha)$$

by (6.1), and so (6.5) follows.

(iv) Take $\mu \in M(G)^+$ and $C \in \mathfrak{B}_G$ with $\mu(C) > 0$. Then we have

$$\begin{aligned}(\nu_B \star \mu_C)(L) &= \int_G\int_G \chi_L(st)\chi_B(s)\chi_C(t)\,\mathrm{d}\nu_B(s)\,\mathrm{d}\mu_C(t)\\ &= \int_G \lambda_{\nu,B}(t)\chi_C(t)\,\mathrm{d}\mu_C(t) = \langle \kappa_E(\lambda_{\nu,B}),\, \mu_C\rangle.\end{aligned}$$

By Corollary 4.7, we can take the limits $\lim_{C\to\psi}$ to see that (6.6) holds. ■

The following result extends a theorem of Işik, Pym, and Ülger [56, Theorem 3.2] (with a different proof); see also Corollary 6.4. We recall that $\pi : (M(\widetilde{G}), \Box) \to (M(G), \star)$, defined in (3.3) is a continuous epimorphism; cf. (5.3); we shall often write $\varphi \mathbin{\Box} \psi$ and $\varphi \diamond \psi$ for $\delta_\varphi \mathbin{\Box} \delta_\psi$ and $\delta_\varphi \mathbin{\Diamond} \delta_\psi$, respectively.

Proposition 6.2. *Let G be a locally compact group. Then:*

(i) $\varphi \mathbin{\Box} \psi = \varphi \in \Phi \subset \widetilde{G}$ *and* $\psi \diamond \varphi = \varphi \in \Phi \subset \widetilde{G}$ *for each* $\varphi \in \Phi$ *and* $\psi \in \widetilde{G}_{\{e\}}$*;*
(ii) *in the case where the group G is compact,*

$$\mathrm{M} \mathbin{\Box} \mathrm{N} = \mathrm{M} \mathbin{\Box} \pi(\mathrm{N}), \qquad \mathrm{N} \mathbin{\Diamond} \mathrm{M} = \pi(\mathrm{N}) \mathbin{\Diamond} \mathrm{M} \qquad (\mathrm{M} \in M(\Phi),\, \mathrm{N} \in M(\widetilde{G})).$$

Proof. (i) First, we fix $\psi \in \widetilde{G}_{\{e\}}$ and a set $B \in \mathfrak{B}_G$ such that $0 < m(B) < \infty$, where $m = m_G$.

For each $\varepsilon > 0$ and each $A \in \mathfrak{B}_G$ with $m(A) < \infty$, there exists $N \in \mathcal{N}_e$ such that

$$m(At^{-1} \setminus A) < \varepsilon m(B) \qquad (t \in N).$$

For each $\mu \in M(G)^+$ and $C \in \mathfrak{B}_G$ with $\mu(C) > 0$, we have

$$\begin{aligned}\int_B \int_C \chi_A(st)\,\mathrm{d}m_B(s)\,\mathrm{d}\mu_C(t) &= \int_B \int_C \chi_{At^{-1}}(s)\,\mathrm{d}m_B(s)\,\mathrm{d}\mu_C(t)\\ &\leq \int_B \chi_A(s)\,\mathrm{d}m_B(s) + \int_C \frac{m(At^{-1}\setminus A)}{m(B)}\,\mathrm{d}\mu_C(t).\end{aligned}$$

Thus, in the case where $C \subset N$, it follows from (6.3) that

$$\langle \chi_A,\, m_B \star \mu_C\rangle \leq \langle \chi_A,\, m_B\rangle + \varepsilon. \tag{6.7}$$

By Corollary 4.7, we can take the limits $\lim_{C\to\psi}$ to see that

$$\langle \kappa_E(\chi_A), m_B \Box \delta_\psi\rangle \leq \langle \kappa_E(\chi_A), m_B\rangle + \varepsilon.$$

This holds for each $\varepsilon > 0$, and so

$$\langle \kappa_E(\chi_A), m_B \Box \delta_\psi\rangle \leq \langle \kappa_E(\chi_A), m_B\rangle.$$

However, this inequality also holds if A be replaced by $G \setminus A$, and so

$$\langle \kappa_E(\chi_A), m_B \Box \delta_\psi\rangle = \langle \kappa_E(\chi_A), m_B\rangle.$$

It follows that

$$\langle \kappa_E(\lambda), m_B \Box \delta_\psi\rangle = \langle \kappa_E(\lambda), m_B\rangle \quad (\lambda \in B^b(\Omega)).$$

Since $m_B \in M(\Phi)$ and $\kappa_E(B^b(G))\,|\,\Phi = C(\Phi)$, we have $m_B \Box \delta_\psi = m_B$. Finally, we take the limits $\lim_{B\to\varphi}$ to see that $\varphi \Box \psi = \varphi \in \Phi \subset \widetilde{G}$.

Similarly, $\delta_\psi \diamond \delta_\varphi = \delta_\varphi$.

(ii) We return to the above formula $m_B \Box \delta_\psi = m_B$, which holds for each $\psi \in \widetilde{G}_{\{e\}}$ and $B \in \mathfrak{B}_G$ with $m(B) > 0$.

Now suppose that $\psi \in \widetilde{G}$. Since G is compact, there exists $s \in G$ with $\pi(\psi) = s$. Then $\psi \Box s^{-1} \in \widetilde{G}_{\{e\}}$, and so $m_B \Box \delta_\psi \Box s^{-1} = m_B$, whence

$$m_B \Box \psi = m_B \star s = m_B \star \pi(\psi).$$

This formula extends to give $m_B \Box \mathrm{N} = m_B \Box \pi(\mathrm{N})$ for each N which is a linear combination of point masses in $M(\widetilde{G})$, and then, by taking weak-$*$ limits, for each $\mathrm{N} \in M(\widetilde{G})$.

We now take limits $\lim_{B\to\varphi}$ to establish that $\varphi \Box \mathrm{N} = \varphi \Box \pi(\mathrm{N})$ for each $\varphi \in \Phi$ and $\mathrm{N} \in M(\widetilde{G})$, and then take linear combinations of point masses in Φ and further weak-$*$ limits to see that

$$\mathrm{M} \Box \mathrm{N} = \mathrm{M} \Box \pi(\mathrm{N}) \quad (\mathrm{M} \in M(\Phi),\, \mathrm{N} \in M(\widetilde{G}));$$

this last step is valid because the map R_{N} is weak-$*$ continuous on $(M(\widetilde{G}), \Box)$.

Similarly, $\mathrm{N} \diamond \mathrm{M} = \pi(\mathrm{N}) \diamond \mathrm{M}$ $(\mathrm{M} \in M(\Phi), \mathrm{N} \in M(\widetilde{G}))$. ■

It follows in particular that $\varphi \Box \psi = \varphi$ $(\varphi, \psi \in \Phi_{\{e\}})$, and so $(\Phi_{\{e\}}, \Box)$ is a left-zero semigroup, as in [56, Theorem 3.2].

Corollary 6.3. *Let G be a locally compact group, let $\mathrm{M} \in M(\Phi)$, and let $\psi \in \widetilde{G}_{\{e\}}$. Then $\mathrm{M} \Box \delta_\psi = \delta_\psi \diamond \mathrm{M} = \mathrm{M}$.* ■

The above result in the case where $\psi \in \Phi_{\{e\}}$ says that the element δ_ψ is a *mixed identity* for $M(\Phi) = L^1(G)''$ in the sense of [13, Definition 2.6.21].

Corollary 6.4. *Let G be a locally compact group, and let $\varphi \in \widetilde{G}$. Then the following are equivalent:*

(a) $\varphi \in \widetilde{G}_{\{e\}}$;
(b) $\psi \,\Box\, \varphi = \psi \ (\psi \in \Phi)$;
(c) $\psi_0 \,\Box\, \varphi = \psi_0$ *for some* $\psi_0 \in \Phi_{\{e\}}$.

Proof. That (a)⇒(b) is part of Corollary 6.3, and (b)⇒(c) is trivial. Suppose that (c) holds. Then, by (5.3), $\pi(\psi_0) \star \pi(\varphi) = \pi(\psi_0)$ in $M(G)$. But $\pi(\psi_0) = \delta_e$, and so $\pi(\varphi) = \delta_e$, giving (a). ∎

The following result, which characterizes $M(\Phi)$ as a subset of $M(\widetilde{G})$, will be important later.

Theorem 6.5. *Let G be a locally compact group, and suppose that* $\mathrm{M} \in M(\widetilde{G})$. *Then the following conditions on* M *are equivalent:*

(a) $\mathrm{M} \in M(\Phi)$;
(b) $\mathrm{M} \,\Box\, \delta_\varphi = \mathrm{M}$ *for all* $\varphi \in \widetilde{G}_{\{e\}}$;
(c) *there exists* $\varphi \in \Phi_{\{e\}}$ *such that* $\mathrm{M} \,\Box\, \delta_\varphi = \mathrm{M}$.

Proof. That (a)⇒(b) is part of Corollary 6.3, and the proof of (b)⇒(c) is trivial. Since $M(\Phi)$ is an ideal in $(M(\widetilde{G}), \Box)$, we have (c)⇒(a). ∎

Definition 6.6. Let G be a locally compact group. For an element $\mu \in M(G)^+$, set

$$\mathfrak{A}_\mu = \{A \in \mathfrak{B}_G : \mu(\partial A) = 0\}.$$

Lemma 6.7. *Let G be a locally compact group, let $\mu \in M(G)^+$, let $A \in \mathfrak{A}_\mu$, and let $\varepsilon > 0$.*

(i) *There exists $N \in \mathcal{N}_e$ with $\mu(At^{-1} \setminus A) < \varepsilon$ and $\mu(t^{-1}A \setminus A) < \varepsilon$ for each $t \in N$.*
(ii) *Let $B \in \mathfrak{B}_G$ with $\mu(B) > 0$ and $\nu \in M(G)^+ \setminus \{0\}$. Then there exists $N \in \mathcal{N}_e$ such that*

$$|\langle \chi_A,\, \mu_B \star \nu_C\rangle - \langle \chi_A,\, \mu_B\rangle| < \varepsilon \tag{6.8}$$

and

$$|\langle \chi_A,\, \nu_C \star \mu_B\rangle - \langle \chi_A,\, \mu_B\rangle| < \varepsilon \tag{6.9}$$

whenever $C \in \mathfrak{B}_G$ with $C \subset N$ and $\nu(C) > 0$.

Proof. (i) Since $\mu(\partial A) = 0$, there is an open set U with $\partial A \subset U$ and $\mu(U) < \varepsilon$. Set $V = U \cup \operatorname{int} A$, so that V is an open set in G. We have $V \supset \partial A \cup \operatorname{int} A = \overline{A}$, and so there is a symmetric set $N \in \mathcal{N}_e$ such that $AN \cup NA \subset V$. In this case

$$(AN \cup NA) \setminus A \subset V \setminus A \subset V \setminus \operatorname{int} A \subset U,$$

and so $\mu((AN \cup NA) \setminus A) < \varepsilon$. The result follows.

(ii) Essentially as in the proof of (6.7), but using the estimate on μ from clause (i), we see that $\langle \chi_A,\, \mu_B \star \nu_C\rangle \le \langle \chi_A,\, \mu_B\rangle + \varepsilon$ whenever $C \in \mathfrak{B}_G$ with $C \subset N$ and $\nu(C) > 0$. Again this leads to (6.8). Similarly, we see that (6.9) holds. ∎

Lemma 6.8. *Let G be a locally compact group, let $\mu \in M(G)^+$, and let $A \in \mathfrak{A}_\mu$. Then*

$$\langle \chi_{K_A},\, \delta_\varphi \,\Box\, \delta_\psi\rangle = \langle \chi_{K_A},\, \delta_\psi \diamond \delta_\varphi\rangle = \langle \chi_{K_A},\, \delta_\varphi\rangle \tag{6.10}$$

for each $\varphi \in \Phi_\mu$ and $\psi \in \widetilde{G}_{\{e\}}$.

Proof. We consider (6.8) and (6.9), and first take the limit $\lim_{C\to\psi}$ and then the limit $\lim_{B\to\varphi}$ to see that

$$|\langle \chi_{K_A},\, \delta_\varphi \,\Box\, \delta_\psi\rangle - \langle \chi_{K_A},\, \delta_\varphi\rangle| \leq \varepsilon \quad \text{and} \quad |\langle \chi_{K_A},\, \delta_\psi \diamond \delta_\varphi\rangle - \langle \chi_{K_A},\, \delta_\varphi\rangle| \leq \varepsilon.$$

However these two inequalities hold for each $\varepsilon > 0$, and so the result follows. ∎

THEOREM 6.9. *Let G be a locally compact group, and let $A \in \mathfrak{B}_G$.*

(i) *Let* $\mathrm{M} \in M(K_A \setminus K_{\partial A})$ *and* $\psi \in \widetilde{G}_{\{e\}}$. *Then*

$$\langle \chi_{K_A},\, \mathrm{M} \,\Box\, \delta_\psi\rangle = \langle \chi_{K_A},\, \mathrm{M}\rangle. \tag{6.11}$$

(ii) *Let* $\mathrm{M} \in M(\widetilde{G}_{\{e\}})$ *with* $\langle \mathrm{M},\, 1\rangle = 1$, *and let* $\varphi \in \widetilde{G}_{\{e\}} \setminus K_{\partial A}$. *Then*

$$\langle \chi_{K_A},\, \mathrm{M} \diamond \delta_\varphi\rangle = \langle \chi_{K_A},\, \delta_\varphi\rangle. \tag{6.12}$$

Proof. (i) First take $\mathrm{M} = \delta_\varphi$, where $\varphi \in K_A \setminus K_{\partial A}$.

Let (φ_α) be a net in U_G with $\lim_\alpha \varphi_\alpha = \varphi$. Since $\varphi \notin K_{\partial A}$, we may suppose that $\varphi_\alpha \notin K_{\partial A}$, and hence that $\partial A \notin \varphi_\alpha$ and $G \setminus \partial A \in \varphi_\alpha$, for each α. Fix α, and choose $\mu_\alpha \in M(\Omega)^+$ such that $\varphi_\alpha \in \Phi_{\mu_\alpha}$; we may suppose that $\mu_\alpha(\partial A) = 0$ because $\varphi_\alpha \in \Phi_{\nu_\alpha}$, where $\nu_\alpha = \mu_\alpha \,|\, (G \setminus \partial A)$, and we can replace μ_α by ν_α, if necessary.

For each α, we apply Lemma 6.8, with φ replaced by φ_α, to see that

$$\langle \chi_{K_A},\, \delta_{\varphi_\alpha} \,\Box\, \delta_\psi\rangle = \langle \chi_{K_A},\, \delta_{\varphi_\alpha}\rangle.$$

By taking limits in α, it follows that

$$\langle \chi_{K_A},\, \delta_\varphi \,\Box\, \delta_\psi\rangle = \langle \chi_{K_A},\, \delta_\varphi\rangle.$$

Thus the result holds in this special case.

Now, by taking linear combinations of point masses of the form δ_φ and then weak-$*$ limits, we see that equation (6.11) holds for each $\mathrm{M} \in M(K_A \setminus K_{\partial A})$.

(ii) For each $\psi \in \widetilde{G}_{\{e\}}$ and each $\mu \in M(G)^+$ such that $A \in \mathfrak{A}_\mu$, it follows from (6.9) and Corollary 4.7 that

$$|\langle \chi_{K_A},\, \delta_\psi \diamond \mu_B\rangle - \langle \chi_{K_A},\, \mu_B\rangle| \leq \varepsilon.$$

This inequality holds for each $\varepsilon > 0$, and so

$$\langle \chi_{K_A},\, \delta_\psi \diamond \mu_B\rangle = \langle \chi_{K_A},\, \mu_B\rangle.$$

Since $\mathrm{M} \in M(\widetilde{G}_{\{e\}})$ and $\langle \mathrm{M},\, 1\rangle = 1$, we see that M is the weak-$*$ limit of linear combinations of measures of the form $\sum_j \alpha_j \delta_{\psi_j}$ such that each $\psi_j \in \widetilde{G}_{\{e\}}$ and $\sum_j \alpha_j = 1$. It follows that

$$\langle \chi_{K_A},\, \mathrm{M} \diamond \mu_B\rangle = \lim \sum_j \alpha_j \langle \chi_{K_A},\, \delta_{\psi_j} \diamond \mu_B\rangle = \langle \chi_{K_A},\, \mu_B\rangle.$$

Now let (φ_α) be a net in U_G with $\lim_\alpha \varphi_\alpha = \varphi$. Since $\varphi \notin K_{\partial A}$, we may suppose that $G \setminus \partial A \in \varphi_\alpha$ for each α. Fix α, and choose $\mu_\alpha \in M(\Omega)^+$ such that $\varphi_\alpha \in \Phi_{\mu_\alpha}$; we may suppose that $\mu_\alpha(\partial A) = 0$. Thus δ_φ is in the closure of the set $\{\mu_C : A \in \mathfrak{A}_\mu,\, C \in \mathfrak{B}_\mu\}$, and so (6.12) follows. ∎

Let $\varphi, \psi \in \widetilde{G}$. We recall that $\varphi \sim \psi$ if $\kappa_E(\lambda)(\varphi) = \kappa_E(\lambda)(\psi)$ for each $\lambda \in B^b(\Omega))$. We now slightly extend this notation.

Definition 6.10. Let G be a locally compact group, and take elements $\mathrm{M}, \mathrm{N} \in M(\widetilde{G})$. Then

$$\mathrm{M} \sim \mathrm{N} \quad \text{if} \quad \langle \kappa_E(\lambda),\, \mathrm{M}\rangle = \langle \kappa_E(\lambda),\, \mathrm{N}\rangle \quad (\lambda \in B^b(\Omega)).$$

We say that M and N are *Borel equivalent* if $\mathrm{M} \sim \mathrm{N}$.

Theorem 6.11. *Let G be a locally compact group, and let $\varphi, \psi \in \widetilde{G}$ be such that $\varphi \sim \psi$. Then $\mathrm{M} \,\Box\, \varphi \sim \mathrm{M} \,\Box\, \psi$ and $\varphi \diamond \mathrm{M} \sim \psi \diamond \mathrm{M}$ for each $\mathrm{M} \in M(\widetilde{G})$.*

Proof. First suppose that $\varphi, \psi \in U_G$. For each $\nu \in M(G)^+$, each $L \in \mathfrak{B}_G$, and each $B \in \mathfrak{B}_G$, we have

$$(\nu_B \,\Box\, \delta_\varphi)(K_L) = \langle \kappa_E(\lambda_{\nu,B}),\, \delta_\varphi\rangle \quad \text{and} \quad (\nu_B \,\Box\, \delta_\psi)(K_L) = \langle \kappa_E(\lambda_{\nu,B}),\, \delta_\psi\rangle$$

by (6.6). By taking suitable limits, we see that these equations also hold when $\varphi, \psi \in \widetilde{G}$.

Since $\varphi \sim \psi$, we see that $\langle \kappa_E(\lambda_{\nu,B}),\, \delta_\varphi\rangle = \langle \kappa_E(\lambda_{\nu,B}),\, \delta_\psi\rangle$, and so

$$(\nu_B \,\Box\, \delta_\varphi)(K_L) = (\nu_B \,\Box\, \delta_\psi)(K_L).$$

Again by taking limits over a canonical net, we see that

$$(\delta_\theta \,\Box\, \delta_\varphi)(K_L) = (\delta_\theta \,\Box\, \delta_\psi)(K_L) \quad (\theta \in \widetilde{G}).$$

Finally, taking linear combinations of the point masses δ_θ and further weak-$*$ limits, we see that

$$(\mathrm{M} \,\Box\, \delta_\varphi)(K_L) = (\mathrm{M} \,\Box\, \delta_\psi)(K_L) \quad (\mathrm{M} \in M(\widetilde{G})).$$

Thus

$$\langle \kappa_E(\chi_L),\, \mathrm{M}\rangle = \langle \kappa_E(\chi_L),\, \mathrm{N}\rangle.$$

The above equation holds for each $L \in \mathfrak{B}_G$, and this is sufficient to imply that $\mathrm{M} \,\Box\, \varphi \sim \mathrm{M} \,\Box\, \psi$.

Similarly $\varphi \diamond \mathrm{M} \sim \psi \diamond \mathrm{M}$ for each $\mathrm{M} \in M(\widetilde{G})$. ■

In Example 8.20, below, we shall see that there exist $\varphi, \psi \in \widetilde{G}$ with $\varphi \sim \psi$ and $\theta \in \widetilde{G}$ such that $\varphi \,\Box\, \theta \nsim \psi \,\Box\, \theta$.

7. The recovery of G from $\widetilde{G}$

Introduction. Let G and H be locally compact groups, and consider the compact spaces $\widetilde{G}$ and $\widetilde{H}$ and the Banach spaces $M(\widetilde{G})$ and $M(\widetilde{H})$. Then we have seen in Chapter 3 that we cannot recover the locally compact spaces G and H from the information that we are given. Indeed, by Theorem 4.3, the space $\widetilde{\Omega}$ is homeomorphic to $\beta\mathbb{N}$ whenever (Ω, τ) is a countable, locally compact space, and, by Theorem 4.16, there is a unique (up to homeomorphism) hyper-Stonean envelope for all uncountable, compact, metrizable spaces.

We now ask whether the fact that Banach algebras $(M(\widetilde{G}), \Box)$ and $(M(\widetilde{H}), \Box)$ are the 'same' entails that $G \sim H$, in the sense that there is a homeomorphic group isomorphism from G onto H.

We first note that we must interpret the word 'same' in the previous paragraph to mean that there is an *isometric* isomorphism from $(M(\widetilde{G}), \Box)$ onto $(M(\widetilde{H}), \Box)$. For let G be the dihedral group of order eight and let H be the quaternion group. Then $(M(\widetilde{G}), \Box) = (\ell^1(G), \star)$ is isomorphic to $(M(\widetilde{H}), \Box) = (\ell^1(H), \star)$, but it is not true that $G \sim H$ [88, §1.9.1].

The character spaces of the C^*-algebras $L^\infty(G)$ and $L^\infty(H)$ are denoted by Φ_G and Φ_H, respectively, in this chapter.

History. We recall some brief history of these questions. Let G and H be locally compact groups. The first result is Wendel's theorem ([119], [88, §1.9.13]), which we state explicitly.

THEOREM 7.1. *Let G and H be locally compact groups. Then there is an isometric isomorphism from $L^1(G)$ onto $L^1(H)$ if and only if $G \sim H$.* ■

In fact, by a theorem of Kalton and Wood [62], we have $G \sim H$ whenever there is an isomorphism from $L^1(G)$ onto $L^1(H)$ with norm less than $\sqrt{2}$.

It was proved by Johnson [58] that $G \sim H$ if and only if there is an isometric isomorphism from $M(G)$ onto $M(H)$; see also Rigelhof [98].

It was further proved in [77] that $G \sim H$ whenever there is an isometric isomorphism θ from $(LUC(G)', \Box)$ onto $(LUC(H)', \Box)$; in this case, θ maps $M(G)$ onto $M(H)$ and $L^1(G)$ onto $L^1(H)$ [36]. We state a related result from [36, Theorem 3.1(c)] that we shall require. (Earlier partial results are listed in [36].)

THEOREM 7.2. *Let G and H be locally compact groups, and let*

$$\theta : (L^1(G)'', \Box) \to (L^1(H)'', \Box)$$

be an isometric isomorphism. Then θ maps $L^1(G)$ onto $L^1(H)$, and so $G \sim H$. ■

The question whether $G \sim H$ when there is an isometric isomorphism from $(M(\widetilde{G}), \Box)$ onto $(M(\widetilde{H}), \Box)$ was specifically raised by Ghahramani and Lau in [34, Problem 2, p. 184]. Our aim in the present chapter is to resolve this question affirmatively. We think that some of the results obtained en route to this are of independent interest.

The result in the case where G and H are both abelian and have non-measurable cardinal was given by Neufang in [87, Corollary 3.7]. [Added in proof: By [82], the condition on cardinality is not required.]

An isomorphism. We shall first note that the groups G and H are isomorphic when $(M(\widetilde{G}), \Box)$ and $(M(\widetilde{H}), \Box)$ are isometrically isomorphic as Banach algebras; the difficulty is to show that this isomorphism from G to H is also a homeomorphism. The following result is [34, Corollary 3.6].

Proposition 7.3. *Let G and H be locally compact groups, and let*

$$\theta : (M(\widetilde{G}), \Box) \to (M(\widetilde{H}), \Box)$$

be an isometric isomorphism. Then, for each $\varphi \in \widetilde{G}$, there exists $\theta(\varphi) \in \widetilde{H}$ and $\zeta_\varphi \in \mathbb{T}$ such that $\theta(\delta_\varphi) = \zeta_\varphi \delta_{\theta(\varphi)}$. Further, for each $s \in G$, we have $\theta(s) \in H$, and $\theta : G \to H$ is an isomorphism.

Proof. For each $\varphi \in \widetilde{G}$, the element δ_φ is an extreme point of $M(\widetilde{G})_{[1]}$, and so, since θ is isometric, $\theta(\delta_\varphi)$ is an extreme point of $M(\widetilde{H})_{[1]}$. Hence $\theta(\delta_\varphi)$ has the form $\zeta_\varphi \delta_{\theta(\varphi)}$ for some $\theta(\varphi) \in \widetilde{H}$ and $\zeta_\varphi \in \mathbb{T}$. We thus obtain a map

$$\theta : \varphi \mapsto \theta(\varphi), \quad \widetilde{G} \to \widetilde{H};$$

since θ is a bijection, we see that this new map is also a bijection.

Take $s \in G$. Then δ_s has inverse $\delta_{s^{-1}}$ in $(M(\widetilde{G}), \Box)$, and so $\theta(\delta_s)$ has inverse $\theta(\delta_{s^{-1}})$ in $(M(\widetilde{H}), \Box)$; further, $\|\theta(\delta_s)\| = \|\theta(\delta_{s^{-1}})\| = 1$. By Proposition 5.5, $\theta(\delta_s) \in H$. It follows that $\theta(G) = H$ and hence that $\theta : G \to H$ is an isomorphism (as is the map $s \mapsto \zeta_s$, $G \to \mathbb{T}$). ∎

The case of compact groups. The following partial answer to our question was first proved by Ghahramani and McClure in [35]; our proof is similar to, but perhaps a little shorter than, their proof.

Theorem 7.4. *Let G and H be compact groups. Suppose that there is an isometric isomorphism from $(M(\widetilde{G}), \Box)$ onto $(M(\widetilde{H}), \Box)$. Then $G \sim H$.*

Proof. The normalized Haar measures on G and H are m_G and m_H, respectively.

First, let $\theta : (M(\widetilde{G}), \Box) \to (M(\widetilde{H}), \Box)$ be an isomorphism, and set

$$\mathrm{N} = \theta(m_G) \in M(\widetilde{H}).$$

It follows from (5.4) that N satisfies (5.5) and (5.6) (with respect to the group H), and so, by Proposition 5.6, $\mathrm{N} = m_H$ or $\mathrm{N} = 0$. Since θ is an injection, $\mathrm{N} \neq 0$. We conclude that $\theta(m_G) = m_H$. We now identify elements of $L^1(G)$ and $L^1(H)$ with the corresponding elements in $M(\widetilde{G})$ and $M(\widetilde{H})$, respectively, so that we can say that $\theta(1_G) = 1_H$.

It follows from (2.5) that a linear isometry θ from $M(\widetilde{G})$ to $M(\widetilde{H})$ has the property that $\theta(\mathrm{M}_1) \perp \theta(\mathrm{M}_2)$ in $M(\widetilde{H})$ whenever $\mathrm{M}_1 \perp \mathrm{M}_2$ in $M(\widetilde{G})$.

Now let $\theta : (M(\widetilde{G}), \Box) \to (M(\widetilde{H}), \Box)$ be an isometric isomorphism. Take $B \in \mathfrak{B}_G$, and set $C = G \setminus B$. Then $1_G = \chi_B + \chi_C$, with $\chi_B \perp \chi_C$, and so $1_H = \theta(\chi_B) + \theta(\chi_C)$, with $\theta(\chi_B) \perp \theta(\chi_C)$. Hence $\theta(\chi_B)$ and $\theta(\chi_C)$, as elements of $M(\widetilde{H})$, must be the restrictions of $\kappa(m_H) = 1_H = \chi_{\Phi_H}$ to two disjoint Borel subsets of $\widetilde{H}$. Thus $\theta(\chi_B)$ and $\theta(\chi_C)$ are positive, normal measures on $\widetilde{H}$, and so we may regard them as elements of $M(H)$. We now see that $\theta(\chi_B)$ and $\theta(\chi_C)$ are the restrictions of 1_H to two disjoint Borel subsets of H. In particular, $\theta(\chi_B) \in L^1(H)$.

It follows that $\theta(f) \in L^1(H)$ for each $f \in L^1(G)$.

Since $\theta : (M(\widetilde{G}), \Box) \to (M(\widetilde{H}), \Box)$ is an isometric isomorphism, it follows that the map $\theta : (L^1(G), \star) \to (L^1(H), \star)$ is an isometric isomorphism. By Wendel's theorem, Theorem 7.1, $G \sim H$. ∎

We can easily extend Theorem 7.4 slightly at this stage at the cost of borrowing a result of Neufang from our Chapter 9.

PROPOSITION 7.5. *Let G be a compact group, and let H be a locally compact group with non-measurable cardinal. Suppose that there is an isometric isomorphism from the Banach algebra $(M(\widetilde{G}), \Box)$ onto $(M(\widetilde{H}), \Box)$. Then $G \sim H$.*

Proof. We shall obtain a contradiction from the assumption that H is non-compact; by Theorem 7.4, this is sufficient for the result.

We have $m_G \in M(G) \subset M(\widetilde{G})$; set $\mathrm{M} = \theta(m_G) \in M(\widetilde{H})$, so that $\|\mathrm{M}\| = 1$. We see that $\mathrm{M} \Box \mathrm{N} = \mathrm{N} \Box \mathrm{M}$ $(\mathrm{N} \in M(\widetilde{H}))$, and so, by Proposition 1.7, $\mathrm{M} \in \mathfrak{Z}_t^{(\ell)}(M(\widetilde{H}))$. Since H is non-compact with non-measurable cardinal, it follows from a theorem of Neufang which is our Theorem 9.6 that $\mathrm{M} \in M(H)$, say $\mathrm{M} = \mu$.

Take $t \in H$. By Proposition 7.3, there exist $s \in G$ and $\zeta \in \mathbb{T}$ with $\zeta\theta(\delta_s) = \delta_t$. Since $m_G \star \delta_s = m_G$, we have

$$\mu \star \delta_t = \zeta\mu. \tag{7.1}$$

Let K be a compact subset of H. Since H is not compact, there is a sequence (t_n) in H such that the sets Kt_n for $n \in \mathbb{N}$ are pairwise disjoint. It follows from (7.1) that $|\mu|(Kt_k) = |\mu|(K)$ for each $k \in \mathbb{N}$, and so

$$n|\mu|(K) = \sum_{k=1}^{n} |\mu|(Kt_n) = |\mu|\Big(\bigcup\{Kt_k : k \in \mathbb{N}_n\}\Big) \le |\mu|(H) = \|\mu\| \quad (n \in \mathbb{N}).$$

Thus $|\mu|(K) = 0$. This holds for each compact subset K of H, and so $|\mu| = 0$, a contradiction of the fact that $\|\mu\| = 1$.

Thus H is compact, as required. ∎

Suppose that G and H are locally compact, abelian groups with non-measurable cardinal, and that there is an isometric isomorphism from $(M(\widetilde{G}), \Box)$ onto $(M(\widetilde{H}), \Box)$. Then it also follows from Neufang's theorem, as remarked in [34], that $G \sim H$. [Added in proof: By [82], the condition on cardinality in Proposition 7.5 and this remark is not required.]

The general case. We now turn to the general case, in which it may be that neither G nor H is compact.

Let H be a locally compact group. Recall from Chapter 5 that $Z = LUC(H)$ is the left-introverted C^*-subalgebra of $M(H)' = C(\widetilde{H})$ consisting of the left uniformly continuous functions on H, so that $(Z', \Box)$ is a Banach algebra and $(\Phi_Z, \Box)$ is a compact, right topological semigroup containing H as a dense open subspace. As in (5.7), we have a continuous surjection $q_H : \widetilde{H} \to \Phi_Z$.

Let G and H be locally compact groups, and let $\theta : (M(\widetilde{G}), \Box) \to (M(\widetilde{H}), \Box)$ be an isometric isomorphism. We adopt the notation given in Proposition 7.3, and again set $Z = LUC(H)$. Take $\varphi \in \widetilde{G}_{\{e_G\}}$. Then $\theta(\varphi) \in \widetilde{H}$ and $q_H(\theta(\varphi)) \in \Phi_Z$; we define

$$u = q_H(\theta(\varphi)).$$

Let $\mathrm{M} \in M(\Phi_G)$. Then it follows from Theorem 6.5 that $\mathrm{M} \mathbin{\square} \delta_\varphi = \mathrm{M}$, and so

$$q_H(\theta(\mathrm{M})) \mathbin{\square} u = q_H(\theta(\mathrm{M})) \quad (\mathrm{M} \in M(\Phi_G)). \tag{7.2}$$

The following result is crucial for our proof.

Lemma 7.6. *We have $u = e_H$, and $\theta(\varphi) \in \widetilde{H}_{\{e_H\}}$.*

Proof. This result is trivial if H is compact, and so we may suppose that H is not compact.

We shall first consider the special case in which $\varphi \in \Phi_G$; our immediate aim is to prove that $u \in H$ in this case.

We assume towards a contradiction that $u \in \Phi_Z \setminus H$. Let κ be the smallest cardinal such that u is in the closure in Φ_Z of the union of κ compact subsets of H (so that $\kappa \geq \omega$), and choose a sequence $(K_\alpha : \alpha < \kappa)$ of compact subsets of H such that

$$u \in \overline{\bigcup\{K_\alpha : \alpha < \kappa\}}.$$

We also choose a symmetric, compact neighbourhood U of e_H.

Clearly there is a strictly increasing sequence $(C_\alpha : \alpha < \kappa)$ of symmetric subsets of H such that the following properties hold:

(i) $U \subset C_0$ and $K_\alpha \subset C_\alpha$ $(\alpha < \kappa)$;
(ii) the set C_α is compact when $\alpha < \omega$, and C_α is the union of at most $|\alpha|$ compact subsets of H when $\omega \leq \alpha < \kappa$;
(iii) $C_\alpha^2 \subset C_{\alpha+1}$ $(\alpha < \kappa)$;
(iv) $C_\alpha \supsetneq \bigcup\{C_\beta : \beta < \alpha\}$ $(\alpha < \kappa)$.

It follows from (ii) and the fact that κ is the smallest cardinal with certain properties that $u \notin \overline{C}_\alpha$ for any $\alpha < \kappa$.

We now set $H_0 = \bigcup\{C_\alpha : \alpha < \kappa\}$. Since each C_α is symmetric and $C_\alpha C_\beta \subset C_{(\alpha\vee\beta)+1}$, we see that H_0 is a subgroup of H. The closure of H_0 in Φ_Z is denoted by $\overline{H}_0$, so that $u \in \overline{H}_0$ and $\overline{H}_0$ is a subsemigroup of $(\Phi_Z, \Box)$. It follows that $\overline{H}_0$ is itself a compact, right topological semigroup. (In fact, we can identify $\overline{H}_0$ with the character space of $LUC(H_0)$.)

By Corollary 5.9, $M(\Phi_G)$ is a closed ideal in $(M(\widetilde{G}), \Box)$, and so $\theta(M(\Phi_G))$ is a closed ideal in $(M(\widetilde{H}), \Box)$. We define

$$J = q_H(\theta(M(\Phi_G))) \subset \Phi_Z .$$

We *claim* that $J \cap \overline{H}_0$ is an ideal in the semigroup $(\overline{H}_0, \Box)$. First note that $u \in J$ because $\varphi \in \Phi_G$, and so $J \cap \overline{H}_0 \neq \emptyset$. Now take $x \in J \cap \overline{H}_0$ and $y \in \overline{H}_0$, say $x = q_H(\theta(\mathrm{M}))$ for some $\mathrm{M} \in M(\Phi_G)$ and $y = q_H(\psi)$ for some $\psi \in \widetilde{H}$. By Proposition 7.3, there exists $\varphi \in \widetilde{G}$ and $\zeta \in \mathbb{T}$ such that $\psi = \zeta\theta(\varphi)$. But now

$$x \,\Box\, y = q_H(\theta(\mathrm{M})) \,\Box\, q_H(\zeta\delta_{\theta(\varphi)}) = q_H(\zeta\theta(\mathrm{M}) \,\Box\, \delta_{\theta(\varphi)}) \in J,$$

and also $x \,\Box\, y \in \overline{H}_0$, and so $x \,\Box\, y \in J \cap \overline{H}_0$. Similarly, $y \,\Box\, x \in J \cap \overline{H}_0$, and so $J \cap \overline{H}_0$ is an ideal in $\overline{H}_0$, as claimed.

Since $(\overline{H}_0, \Box)$ is a compact, right topological semigroup, it follows from Theorem 5.1 that $\overline{H}_0$ has a minimum ideal, $K(\overline{H}_0)$. Clearly $K(\overline{H}_0) \subset J \cap \overline{H}_0$. Now (7.2) yields

$$x \,\Box\, u = x \quad (x \in K(\overline{H}_0) \subset J). \tag{7.3}$$

For each $s \in H_0$, define

$$f(s) = \min\{\alpha < \kappa : s \in C_\alpha\}.$$

Suppose that $s, t \in H_0$ are such that $f(s) < f(t)$. Then $f(st) \in \{f(t), f(t)+1\}$ whenever $f(t)$ is a limit ordinal or 0 and $f(st) \in \{f(t)-1, f(t), f(t)+1\}$ otherwise.

Each ordinal α has the form $\alpha = \lambda(\alpha) + n(\alpha)$, where $\lambda(\alpha)$ is a limit ordinal or 0, and $n(\alpha) \in \mathbb{N}$.

For $k \in \mathbb{Z}_8$, we define

$$D_k = \{s \in H_0 : n(f(s)) \equiv k \pmod 8\},$$

so that $\{D_1, \dots, D_8\}$ is a partition of H_0 and $\overline{D}_1 \cup \cdots \cup \overline{D}_8 = \overline{H}_0$. For each $k \in \mathbb{Z}_8$ and each $\alpha < \kappa$, the set $D_k \setminus C_\alpha$ is infinite by (iv) above. Thus, for each $k \in \mathbb{Z}_8$, the family

$$\{\overline{(D_k \setminus C_\alpha)} : \alpha < \kappa\}$$

of closed subsets of the compact space $\overline{H}_0$ has the finite intersection property, and so we may choose

$$y_k \in \bigcap\{\overline{(D_k \setminus C_\alpha)} : \alpha < \kappa\}.$$

For $k \in \mathbb{Z}_8$, we further define

$$F_k = \{st \in H_0 : t \in D_k,\ s \in H_0 \text{ with } f(s) < f(t)\}.$$

Then, for each $k \in \mathbb{Z}_8$, we have

$$F_k \subset D_{k-1} \cup D_k \cup D_{k+1} \quad \text{and} \quad UF_k \subset D_{k-2} \cup D_{k-1} \cup D_k \cup D_{k+1} \cup D_{k+2},$$

where the subscripts are calculated in $\mathbb{Z}_8$. It follows that $F_k \cap UF_\ell = \emptyset$ whenever $k = \ell + 4$, and so, by Theorem 5.12, we have

$$\overline{F}_k \cap \overline{F}_\ell = \emptyset \quad \text{whenever} \quad \ell = k + 4 \text{ in } \mathbb{Z}_8. \tag{7.4}$$

For each $x \in \overline{H}_0$ and $k \in \mathbb{Z}_8$, we can write

$$x \,\Box\, y_k = \lim_{s \to x} \lim_{t \to y_k} \{st : s \in H_0,\ t \in D_k \setminus C_{f(s)}\}.$$

Since $f(t) > f(s)$ for $t \in H \setminus C_{f(s)}$ and since R_{y_k} is continuous on $\overline{H}_0$, it follows that

$$\overline{H}_0 \,\Box\, y_k \subset \overline{F}_k .$$

By Theorem 5.1, the left ideal $\overline{H}_0 \,\Box\, y_k$ of $\overline{H}_0$ has a non-empty intersection with the minimum ideal $K(\overline{H}_0)$ of $\overline{H}_0$, and so there exists an element $x_k \in \overline{F}_k \cap K(\overline{H}_0)$.

There exists $k_0 \in \mathbb{Z}_8$ such that $u \in \overline{D}_{k_0}$. For each $x \in K(\overline{H}_0)$, we can write

$$x \,\Box\, u = \lim_{s\to x}\,\lim_{t\to u}\{st : s \in H_0,\, t \in D_k \setminus C_{f(s)}\}\,;$$

this holds because $u \notin \overline{C}_\alpha$ for any $\alpha < \kappa$. It follows that

$$x = x \,\Box\, u \in \overline{F}_{k_0} \qquad (x \in K(\overline{H}_0)), \tag{7.5}$$

where we are using (7.3).

We take $\ell_0 = k_0 + 4$ (in $\mathbb{Z}_8$), so that $x_{\ell_0} \in \overline{F}_{\ell_0} \cap K(\overline{H}_0)$. But $x_{\ell_0} \in \overline{F}_{k_0}$ by (7.5). This is a contradiction of (7.4).

We conclude that $u \in H$ in the special case in which $\varphi \in \Phi_G$.

Now consider the more general case in which $\varphi \in \widetilde{G}_{\{e_G\}}$. We choose $\psi \in (\Phi_G)_{\{e_G\}}$, so that $\psi \,\Box\, \varphi = \psi$ in $\widetilde{G}$. Set $v = q_H(\theta(\psi)) \in \Phi_Z$. By the special case that we have just proved, $v \in H$. But $v \,\Box\, u = v$ in $(\Phi_Z, \Box)$, and so, acting on the left with v^{-1}, we see that $u = e_H$, as required. It follows that $\theta(\varphi) \in \widetilde{H}_{\{e_H\}}$. ∎

Let G and H be locally compact groups, as in the theorem, but now suppose further that H is σ-compact and non-compact. Then the above proof can be considerably simplified. Indeed, in this case, the sequence $(C_n : n < \omega)$ is any strictly increasing sequence of compact subspaces of H such that $U \subset C_0$ and $\bigcup_{n<\omega} C_n = H$, and we can take $H_0 = H$. Thus the argument used in the above proof shows the following; it would be interesting to know if the result is still true when H is not necessarily σ-compact.

Proposition 7.7. *Let H be a locally compact group which is σ-compact and non-compact. Then, for each $u \in \Phi_Z \setminus H$, there is a left ideal L of $(\Phi_Z, \Box)$ such that $(\Phi_Z \,\Box\, u) \cap L = \emptyset$.* ∎

We obtain the following consequence of the above lemma.

Proposition 7.8. *Let G and H be locally compact groups, and let*

$$\theta : (M(\widetilde{G}), \Box) \to (M(\widetilde{H}), \Box)$$

be an isometric isomorphism. Then θ induces a bijection $\theta : \widetilde{G}_{\{e_G\}} \to \widetilde{H}_{\{e_H\}}$ and an isometric isomorphism $\theta : (M(\Phi_G), \Box) \to (M(\Phi_H), \Box)$.

Proof. It is clear from Lemma 7.6 that $\theta : \widetilde{G}_{\{e_G\}} \to \widetilde{H}_{\{e_H\}}$ is a bijection.

Take $\mathrm{M} \in M(\Phi_G)$, and set $\mathrm{N} = \theta(\mathrm{M}) \in M(\widetilde{H})$. Choose an element $\varphi \in (\Phi_H)_{\{e_H\}}$. Then there exists $\psi \in \widetilde{G}_{\{e_G\}}$ such that $\theta(\psi) = \varphi$. By Theorem 6.5, (a)⇒(b), $\mathrm{M} \,\Box\, \delta_\psi = \mathrm{M}$ in $(M(\widetilde{G}), \Box)$, and so we see that $\mathrm{N} \,\Box\, \delta_\varphi = \mathrm{N}$ in $(M(\widetilde{H}), \Box)$. By Theorem 6.5, (c)⇒(a), we have $\mathrm{N} \in M(\Phi_H)$. Thus $\theta(M(\Phi_G)) \subset M(\Phi_H)$. We conclude that θ is an isometric isomorphism from $(M(\Phi_G), \Box)$ onto $(M(\Phi_H), \Box)$. ∎

Theorem 7.9. *Let G and H be locally compact groups, and suppose that there is an isometric isomorphism from $(M(\widetilde{G}), \Box)$ onto $(M(\widetilde{H}), \Box)$. Then $G \sim H$.*

Proof. By Proposition 7.8, there is an isometric isomorphism from $(M(\Phi_G), \Box)$ onto $(M(\Phi_H), \Box)$. By Theorem 7.2, $G \sim H$. ∎

8. The compact space $\widetilde{G}$

Introduction. Let G be a locally compact group. We now enquire whether or not $(\widetilde{G}, \Box)$ is a semigroup. Specifically, we take $\varphi, \psi \in \widetilde{G}$, so that $\delta_\varphi \Box \delta_\psi$ is a measure on $\widetilde{G}$; we say that $\varphi \Box \psi \in \widetilde{G}$ if $\delta_\varphi \Box \delta_\psi$ is a point mass in $\widetilde{G}$; in the contrary case, we say that $\varphi \Box \psi \notin \widetilde{G}$.

Definition 8.1. Let G be a locally compact group. Then a subset S of $\widetilde{G}$ *is a semigroup* if $\varphi \Box \psi \in S$ whenever $\varphi, \psi \in S$.

In particular, we shall consider whether or not $\widetilde{G}$ itself is a semigroup. More generally, let S and T be subsets of $\widetilde{G}$. We shall consider, first, whether or not $\varphi \Box \psi \in \widetilde{G}$ for each $\varphi \in S$ and $\psi \in T$, and, second, if so, the subset of $\widetilde{G}$ to which $\varphi \Box \psi$ belongs. Indeed, we shall say that

$$S \Box T \subset U$$

if $\varphi \Box \psi$ is point mass in U for each $\varphi \in S$ and $\psi \in T$.

For example, recall that, in the case where the group G is discrete, so that $\widetilde{G} = \beta G$, it is certainly the case that $(\beta G, \Box)$ is a semigroup; indeed, it is a compact, right topological semigroup. This semigroup has been extensively discussed [17, 52]. Thus, for a general locally compact group G, the subset $\overline{G} = \beta G_d$ of $(\widetilde{G}, \Box)$ is always a semigroup. Also, the following result follows easily from Proposition 6.2(ii); recall that Φ is the character space of $L^\infty(G)$.

Proposition 8.2. *Let G be a compact group. Then $\Phi \Box \widetilde{G} \subset \Phi$, and, in particular, $(\Phi, \Box)$ is a semigroup.* ■

Indeed, we have noted in Corollary 6.3 that $(\Phi_{\{e\}}, \Box)$ is a left-zero semigroup.

The above proposition does not extend to all locally compact groups G. Indeed, it is shown in [78, Corollary 4.4] that $(\Phi, \Box)$ is a semigroup if and only if G is either compact or discrete; we state this result in the following form.

Proposition 8.3. *Let G be a non-discrete, locally compact group that is not compact. Then $(\widetilde{G}, \Box)$ is not a semigroup.* ■

One of our aims is to prove that $(\widetilde{G}, \Box)$ is not a semigroup for each non-discrete, locally compact group. The above proposition shows that it would be sufficient to restrict considerations to infinite, compact groups G. However we shall prove the result for general non-discrete, locally compact groups without appealing to Proposition 8.3; one reason for this is that we shall find different elements $\varphi, \psi \in \widetilde{G}$ such that $\varphi \Box \psi \notin \widetilde{G}$ from those that arise in Proposition 8.3.

Here is another obvious remark. As before, we set $E = C_0(G)$.

Proposition 8.4. *Let G be a locally compact group. Then*

$$\overline{G}\,\square\,\widetilde{G}\subset\widetilde{G}\quad\text{and}\quad\widetilde{G}\,\square\,G\subset\widetilde{G}.$$

Proof. Let $s\in G$. First take $\lambda\in E$. Then we recall that $\lambda\cdot s\in E$ is defined by the equation $(\lambda\cdot s)(t)=\lambda(st)$ $(t\in G)$. For $\lambda_1,\lambda_2\in E$, we have $\lambda_1\lambda_2\cdot s=(\lambda_1\cdot s)(\lambda_2\cdot s)$, and so

$$\langle\lambda_1\lambda_2,\,s\star\varphi\rangle=\langle\lambda_1,\,s\star\varphi\rangle\langle\lambda_2,\,s\star\varphi\rangle\qquad(\varphi\in\widetilde{G}).$$

Now take $\Lambda_1,\Lambda_2\in C(\widetilde{G})$. Taking weak-$*$ limits, we see that

$$\langle\Lambda_1\Lambda_2,\,s\star\varphi\rangle=\langle\Lambda_1,\,s\star\varphi\rangle\langle\Lambda_2,\,s\star\varphi\rangle\qquad(\varphi\in\widetilde{G}),$$

and so $\Lambda_1\Lambda_2\cdot s=(\Lambda_1\cdot s)(\Lambda_2\cdot s)$. Thus

$$\langle\Lambda_1\Lambda_2,\,s\cdot\varphi\rangle=\langle\Lambda_1\Lambda_2\cdot s,\,\varphi\rangle=\langle\Lambda_1\cdot s,\,\varphi\rangle\langle\Lambda_2\cdot s,\,\varphi\rangle\qquad(\Lambda_1,\Lambda_2\in C(\widetilde{G}),\ \varphi\in\widetilde{G}),$$

and so $s\cdot\varphi\in\widetilde{G}$. Similarly, $\varphi\cdot s\in\widetilde{G}$. We have shown that $G\,\square\,\widetilde{G}\subset\widetilde{G}$ and $\widetilde{G}\,\square\,G\subset\widetilde{G}$.

Since multiplication on the right is continuous on $(M(\widetilde{G}),\square)$, it also follows that $\overline{G}\,\square\,\widetilde{G}\subset\widetilde{G}$. ∎

However multiplication on the left is not continuous on $(M(\widetilde{G}),\square)$, and so we cannot say that $\widetilde{G}\,\square\,\overline{G}\subset\widetilde{G}$. Indeed, this is not true in general, as we shall see below.

In fact, since D', A', and M_s' are translation-invariant subspaces of the space M', we see that $\overline{G}\,\square\,\overline{G}\subset\overline{G}$, that $\overline{G}\,\square\,\Phi\subset\Phi$, and that $\overline{G}\,\square\,\Phi_s\subset\Phi_s$.

Proposition 8.5. *Let G be a compact group, and let*

$$\mathfrak{A}=\ell^1(G)\ltimes L^1(G).$$

Then $(\mathfrak{A}'',\square)=(M(\beta G_d\cup\Phi),\square)$, and $(\beta G_d\cup\Phi,\square)$ is a subsemigroup of $(M(\beta G_d\cup\Phi),\square)$.

Proof. Set $S=\beta G_d\cup\Phi$. By the standard result, $(\beta G_d,\square)$ is a semigroup, and so we have $\beta G_d\,\square\,\beta G_d\subset\beta G_d$; by Proposition 8.2, $\Phi\,\square\,S\subset\Phi$; by Proposition 8.4, $\beta G_d\,\square\,\Phi\subset\Phi$. Thus $S\,\square\,S\subset S$. ∎

Relation between groups. As a preliminary to our main investigations, we consider the relation between the statements that $(\widetilde{G},\square)$ and $(\widetilde{H},\square)$ are semigroups when G and H are related groups.

Let G and H be locally compact groups, and let $\eta:G\to H$ be a continuous map that is also a group homomorphism; as in Chapter 2, we can define a continuous linear operator $\overline{\eta}:M(G)\to M(H)$ with $\|\overline{\eta}\|=1$ and $\overline{\eta}(\delta_s)=\delta_{\eta(s)}$ $(s\in G)$. Let $\mu,\nu\in M(G)$. For each $\lambda\in C_0(H)$, we have

$$\begin{aligned}\langle\lambda,\,\overline{\eta}(\mu\star\nu)\rangle&=\int_G\int_G(\lambda\circ\eta)(st)\,\mathrm{d}\mu(s)\,\mathrm{d}\nu(t)\\&=\int_G\int_G\lambda(\eta(s)\eta(t))\,\mathrm{d}\mu(s)\,\mathrm{d}\nu(t)\\&=\int_H\int_H\lambda(uv)\,\mathrm{d}\overline{\eta}(\mu)(u)\,\mathrm{d}\overline{\eta}(\nu)(v)=\langle\lambda,\,\overline{\eta}(\mu)\star\overline{\eta}(\nu)\rangle.\end{aligned}$$

It follows that $\overline{\eta}(\mu \star \nu) = \overline{\eta}(\mu) \star \overline{\eta}(\nu)$, and so the map $\overline{\eta} : (M(G), \star) \to (M(H), \star)$ is a continuous homomorphism. Hence

$$\overline{\eta}'' : (M(\widetilde{G}), \Box\,) \to (M(\widetilde{H}), \Box)$$

is a continuous homomorphism with $\|\overline{\eta}''\| = 1$. Further, as in equation (4.6) of Corollary 4.12, we can define a continuous map $\widetilde{\eta} : \widetilde{G} \to \widetilde{H}$.

Proposition 8.6. *Let G and H be locally compact groups, and let $\eta : G \to H$ be a continuous homomorphism.*

(i) *Suppose that η is an injection and that $(\widetilde{G}, \Box)$ is not a semigroup. Then $(\widetilde{H}, \Box)$ is not a semigroup.*
(ii) *Suppose that η is an open surjection and that $(\widetilde{H}, \Box)$ is not a semigroup. Then $(\widetilde{G}, \Box)$ is not a semigroup.*

Proof. (i) By Corollary 4.12(i), $\widetilde{\eta} : \widetilde{G} \to \widetilde{H}$ is an injection.

There exist $\varphi, \psi \in \widetilde{G}$ such that $\delta_\varphi \mathbin{\Box} \delta_\psi \in M(\widetilde{G}) \setminus \widetilde{G}$. We have $\widetilde{\eta}(\varphi), \widetilde{\eta}(\psi) \in \widetilde{H}$, and

$$\delta_{\widetilde{\eta}(\varphi)} \mathbin{\Box} \delta_{\widetilde{\eta}(\psi)} = \overline{\eta}''(\delta_\varphi \mathbin{\Box} \delta_\psi) \in M(\widetilde{H}).$$

Further, $\|\delta_\varphi \mathbin{\Box} \delta_\psi\| = 1$, and so $\overline{\eta}''(\delta_\varphi \mathbin{\Box} \delta_\psi) \neq 0$ because $\overline{\eta}''$ is an injection.

Assume towards a contradiction that $\overline{\eta}''(\varphi \mathbin{\Box} \psi) \in \widetilde{H}$. Then $\overline{\eta}''(\varphi \mathbin{\Box} \psi) \in \widetilde{\eta}\,(\widetilde{G})$, for otherwise $\widetilde{\eta}\,(\varphi \mathbin{\Box} \psi) = 0$, and so $\varphi \mathbin{\Box} \psi \in \widetilde{G}$ because $\widetilde{\eta}$ is an injection, a contradiction.

Thus it follows that $\overline{\eta}''(\varphi \mathbin{\Box} \psi) \notin \widetilde{H}$, and so $(\widetilde{H}, \Box)$ is not a semigroup.

(ii) Since η is an open surjection, Proposition 5.2(i) shows that $\overline{\eta} : M(G) \to M(H)$ is a surjection. By Corollary 4.12(ii), the map $\widetilde{\eta} : \widetilde{G} \to \widetilde{H}$ is a surjection. It follows that there exist $\varphi, \psi \in \widetilde{G}$ such that $\widetilde{\eta}(\varphi) \mathbin{\Box} \widetilde{\eta}(\psi) \notin \widetilde{H}$.

Assume towards a contradiction that $\varphi \mathbin{\Box} \psi \in \widetilde{G}$. Then $\widetilde{\eta}(\varphi \mathbin{\Box} \psi) = \widetilde{\eta}(\varphi) \mathbin{\Box} \widetilde{\eta}(\psi) \in \widetilde{H}$, a contradiction. This shows that $(\widetilde{G}, \Box)$ is not a semigroup. ∎

Specific compact groups. We shall now show that the inclusion $\widetilde{G} \mathbin{\Box} \overline{G} \subset \widetilde{G}$ often fails; we shall establish a strong form of this result in the special cases where G is the circle group $\mathbb{T}$ or a compact, totally disconnected group, and then generalize the result to arbitrary locally compact groups.

We shall first introduce some preliminary notation.

We shall identify $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$, and use numbers θ in the interval $[0,1)$ to represent the point $\exp(2\pi \mathrm{i}\theta)$ in $\mathbb{T}$. Haar measure $m_{\mathbb{T}}$ on $\mathbb{T}$ gives the measure m on $[0,1)$. The fibre in $\widetilde{\mathbb{T}}$ above the identity element of the group $\mathbb{T}$ is $\widetilde{\mathbb{T}}_{\{0\}}$. As before, D_p is the compact group $\mathbb{Z}_p^{\aleph_0}$. We regard D_2 as a closed subset of D_p for each $p \geq 2$; however note that D_2 is not a subgroup of D_p whenever $p \geq 3$. We also regard D_2 as a subset of $\mathbb{T}$, as follows.

Definition 8.7. For $\varepsilon = (\varepsilon_j) \in D_2$, define

$$\zeta(\varepsilon) = \sum_{j=1}^{\infty} \frac{\varepsilon_j}{3^j} \in \mathbb{T},$$

and set $L = \zeta(D_2)$.

Thus L is the set of numbers whose ternary expansion contains only 0's and 1's; it is a Cantor set with $\{0, 1/2\} \subset L \subset [0, 1/2] \subset \mathbb{T}$. The map $\zeta : D_2 \to L$ is a homeomorphism.

Let $\mu \in M(\mathbb{I})$ be defined by

$$\mu(B) = m_{D_2}(B \cap L) \quad (B \in \mathfrak{B}_{\mathbb{I}}), \tag{8.1}$$

where now m_{D_2} is Haar measure on D_2, identified with L, so that μ is a fixed, positive singular measure on $\mathbb{I}$ with $\operatorname{supp} \mu = L$.

For $X \in \mathfrak{B}_L$, $r \in \mathbb{N}$, and $j \in \{0, 1\}$, set

$$\pi_{r,j}(X) = \{x \in X : \varepsilon_r(x) = j\}.$$

Then we note that $(\pi_{r,0}(X) - 3^{-r}) \cap L = \emptyset$, so that $\mu(\pi_{r,0}(X) - 3^{-r}) = 0$, and then that $\mu(\pi_{r,1}(X) - 3^{-r}) = \mu(\pi_{r,1}(X))$ because the map $x \mapsto x - 3^{-r}$ applied to $\pi_{r,1}(X)$ just corresponds to a translation in D_2.

Let $X, Y \in \mathfrak{B}_L$ and $r \in \mathbb{N}$. Then we have

$$(X - 3^{-r}) \bigtriangleup (Y - 3^{-r}) = (X \bigtriangleup Y) - 3^{-r},$$

and so

$$\mu((X - 3^{-r}) \bigtriangleup (Y - 3^{-r})) = \mu(\pi_{r,1}(X \bigtriangleup Y) - 3^{-r}) = \mu(\pi_{r,1}(X \bigtriangleup Y)).$$

It follows that

$$\mu((X - 3^{-r}) \bigtriangleup (Y - 3^{-r})) \leq \mu(X \bigtriangleup Y). \tag{8.2}$$

We shall also consider the groups of p-adic integers, where p is a prime (and $p \geq 2$); this group is described in [48, §10]. Following [48], we denote the group by Δ_p, and regard an element of Δ_p as a sequence in $\mathbb{Z}_p^{\aleph_0}$; for $r \in \mathbb{Z}^+$, the element $(\delta_{r,n} : n \in \mathbb{Z}^+)$ is denoted by u_r. We note that Δ_p is monothetic, with generator u_1.

Theorem 8.8. *Let G be either the circle group $\mathbb{T}$ or $(\mathbb{R}, +)$ or the compact group D_p or the group Δ_p of p-adic integers, where p is a prime. Then there exist $\mu \in M_s(G)^+$ and $\psi \in G_d^*$ such that:*

(i) $\varphi \,\Box\, \psi \notin \widetilde{G}$ *for each* $\varphi \in \Phi_\mu$;
(ii) $|(\mathrm{M} \,\Box\, \delta_\psi)(\Phi_\mu)| \leq 1/2$ *for each* $\mathrm{M} \in M(\Phi_\mu)_{[1]}$.

Proof. (i) We give the proof first in the case where $G = \mathbb{T}$ or $G = \mathbb{R}$.

We have defined $\mu \in M_s(\mathbb{I})^+$ in (8.1), and we can regard μ as an element of $M_s(G)^+$.

For $r \in \mathbb{N}$, the element $3^{-r} \in L$. Now take $x = \sum_{j=1}^\infty \varepsilon_j / 3^j \in L$. For each $r \geq 2$, we have $x + 3^{-r} \in L$ if and only if $\varepsilon_r = 0$.

Fix $\varepsilon > 0$. Then we *claim* that, for each $B \in \mathfrak{B}_L$ with $\mu(B) > 0$, there exists $r_\varepsilon \in \mathbb{N}$ such that

$$\left| \mu(B \cap (L - 3^{-r})) - \frac{1}{2}\mu(B) \right| < \varepsilon \mu(B) \quad (r > r_\varepsilon). \tag{8.3}$$

To see this, first suppose that $B \subset L$ is a basic clopen subset of the form $U_{F,\alpha}$, as in (2.1) above. Then

$$B \cap (L - 3^{-r}) = \{\varepsilon \in U_{F,\alpha} : \varepsilon_r = 0\},$$

and so, for each $r > \max F$, we have

$$\mu(B \cap (L - 3^{-r})) = \frac{1}{2^{k+1}} = \frac{1}{2}\mu(B),$$

giving (8.3) in this case. Since each clopen subset of L is a finite union of pairwise disjoint, basic clopen sets, our claim holds for each non-empty, clopen set $B \in \mathfrak{B}_L$.

For an arbitrary $B \in \mathfrak{B}_L$ with $\mu(B) > 0$, there is an open and closed subset V of L such that $\mu(B \bigtriangleup V) < \varepsilon\mu(B)$. It follows from (8.2) that

$$\mu((B - 3^{-r}) \bigtriangleup (V - 3^{-r})) < \varepsilon\mu(B),$$

and so our claim holds for this set B. This establishes the general claim.

It follows from (8.3) that

$$\lim_{r\to\infty} (\mu_B \star \delta_{3^{-r}})(L) = \lim_{r\to\infty} \frac{\mu(B \cap (L - 3^{-r}))}{\mu(B)} = \frac{1}{2}$$

for each $B \in \mathfrak{B}_G$ with $\mu(B) > 0$.

Now take $\varphi \in \Phi_\mu$ and $\psi \in G_d^*$ to be any accumulation point of the set $\{3^{-r} : r \in \mathbb{N}\}$. By equation (6.5) in Proposition 6.1(iii), we have $\langle \chi_{K_L},\, \delta_\varphi \mathbin{\scriptstyle\square} \delta_\psi \rangle = 1/2$, and so $\delta_\varphi \mathbin{\scriptstyle\square} \delta_\psi$ is not a point mass, whence $\varphi \mathbin{\scriptstyle\square} \psi \notin \widetilde{G}$.

The case where G is either D_p or Δ_p for some $p \geq 3$ is essentially the same: we embed D_2 in Δ_p, as before, and note that the sum of two elements of D_2 in Δ_p is just the same as their sum in D_p.

Suppose that $G = D_2$, and again set $L = \zeta(D_2) \subset \mathbb{I}$ and take μ to be Haar measure on D_2, as in (8.1), with μ transferred to L. For each $n \in \mathbb{N}$, set

$$A_n = \{(\varepsilon_r) \in L + u_{2n+1} : \varepsilon_{2n} = 0\} \quad \text{and} \quad s_n = u_{2n} + u_{2n+1},$$

and set $A = \bigcup\{A_n : n \in \mathbb{N}\}$. Then we see that

$$L \cap (A - s_n) = \{(\varepsilon_r) \in L : \varepsilon_{2n} = 1\} \quad (n \in \mathbb{N}).$$

For each clopen subset B of L, we have

$$\mu(B \cap (A - s_n)) = \frac{1}{2}\mu(B)$$

for each sufficiently large $n \in \mathbb{N}$. Now let $\varphi \in \Phi_\mu$, and take ψ to be any accumulation point of the set $\{s_n : n \in \mathbb{N}\}$. It follows essentially as before that $\langle \chi_{K_L},\, \delta_\varphi \mathbin{\scriptstyle\square} \delta_\psi \rangle = 1/2$, and so $\delta_\varphi \mathbin{\scriptstyle\square} \delta_\psi$ is not a point mass.

The final case in which $G = \Delta_2$ is essentially the same.

(ii) Clearly $\Phi_\mu \subset K_L$, and so

$$0 \leq (\delta_\varphi \mathbin{\scriptstyle\square} \delta_\psi)(\Phi_\mu) \leq \frac{1}{2} \quad (\varphi \in \Phi_\mu).$$

Since $M(\Phi_\mu)_{[1]}$ is the weak-$*$ closure of the convex hull of the measures δ_φ for $\varphi \in \Phi_\mu$, we have $|(\mathrm{M} \mathbin{\scriptstyle\square} \delta_\psi)(\Phi_\mu)\rangle| \leq 1/2$ for each $\mathrm{M} \in M(\Phi_\mu)_{[1]}$, as required. ■

Corollary 8.9. *Let* $G = \mathbb{T}$. *Then* $\Phi_s \,\square\, \overline{G} \not\subset \widetilde{G}$, *and* $(\widetilde{G}, \square)$ *is not a semigroup.* ■

We shall now show that $\Phi_s \,\square\, \Phi \not\subset \widetilde{G}$ and that $\Phi_s \,\square\, \Phi_s \not\subset \widetilde{G}$; we shall first work with the key group $\mathbb{T}$. We again require some preliminary notation. We recall that we are writing $\mathbb{Z}_n$ for the set $\{0, 1, 2, \dots, n-1\}$.

We fix a sequence (r_n) in $\mathbb{N}$ such that $4 \leq r_n < r_{n+1}$ $(n \in \mathbb{N})$ and

$$\sum_{n=1}^{\infty} \frac{1}{r_n} < \infty. \tag{8.4}$$

Next we define a new sequence $(d_n : n \in \mathbb{Z}^+)$ by requiring that $d_0 = 1$ and that $d_n = r_n d_{n-1}$ $(n \in \mathbb{N})$. Each $x \in [0,1)$ has a expression in the form

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n(x)}{d_n}, \qquad \text{where} \qquad \varepsilon_n(x) \in \mathbb{Z}_{r_n} \ (n \in \mathbb{N}),$$

where we note that

$$\sum_{n=1}^{\infty} \frac{r_n - 1}{d_n} = 1.$$

(The expression for x is unique provided that we exclude the case where $\varepsilon_n(x) = r_n - 1$ eventually; this ambiguity involves only countably many points of $[0,1)$.)

Let $x, y \in [0,1)$ and $n \in \mathbb{N}$. Then we see that

$$\varepsilon_n(x+y) = \varepsilon_n(x) + \varepsilon_n(y)$$

provided that $\varepsilon_n(x) + \varepsilon_n(y) < r_n - 1$, for, in this case, there is no 'carrying of decimals'.

We now define three subsets L_0, L_1, L_2 of $[0,1)$.

The set L_0 consists of those elements $x \in [0,1)$ such that

$$\varepsilon_n(x) \in \{0,1\} \quad (n \in \mathbb{N}).$$

Thus L_0 is a Borel subset of $[0,1)$ with $m(L_0) = 0$. We can identify L_0 as a topological space with a dense subset of D_2, and we again denote by μ the positive measure on $[0,1)$ that corresponds to the Haar measure on D_2, as in (8.1), so that $\mu(L_0) = 1$ and μ is singular with respect to m. We fix φ to be any element of Φ_μ, so that $\varphi \in \Phi_s$.

The set L_1 consists of those elements $x \in [0,1)$ such that

$$\varepsilon_n(x) \notin \{2, r_n - 1\} \quad (n \in \mathbb{N}).$$

Thus L_1 is a compact subset of $[0,1)$ with

$$m(L_1) = \prod_{n=1}^{\infty} (1 - \frac{2}{r_n}),$$

and so $m(L_1) > 0$ by (8.4).

The set L_2 consists of those elements $x \in [0,1)$ such that

$$\varepsilon_n(x) \notin \{2, r_n - 2, r_n - 1\} \quad (n \in \mathbb{N})$$

and $\varepsilon_n(x) = 1$ for exactly one value of $n \in \mathbb{N}$, say for $n = n_x$. Thus $L_0 \subset L_2 \subset L_1$ and L_2 is a countable union of Borel subsets of $[0,1)$, and hence is a Borel subset of $\mathbb{T}$. We observe that $m(L_2 \cap U) > 0$ for each neighbourhood U of 0 in $[0,1)$, and so there is a point $\psi \in \Phi \cap \widetilde{\mathbb{T}}_{\{0\}}$ such that L_2 belongs to the ultrafilter ψ.

The key step in our construction is contained in the following lemma, which uses the above notation.

Lemma 8.10. *We have* $(\delta_\varphi \,\Box\, \delta_\psi)(K_{L_1}) = 1/2$.

Proof. We first consider a basic clopen subset B of L_0 of the form

$$B = \{x \in L_0 : \varepsilon_i(x) = u_i \quad (i \in \mathbb{N}_k)\},$$

where $k \in \mathbb{N}$ and $u_1, \dots, u_k \in \mathbb{Z}_2$. We then choose a Borel subset C of L_2 such that, for each $t \in C$, we have $\varepsilon_i(t) = 0$ $(i \in \mathbb{N}_k)$.

We first fix $t \in C$, and consider $\mu((L_1-t)\cap B)$. Indeed, take $x \in L_1$, and set $\alpha_i = \varepsilon_i(x)$ and $\beta_i = \varepsilon_i(t)$ for $i \in \mathbb{N}$. We *claim* that $x-t \in B$ if and only if the following two conditions hold:

(1) $\alpha_i = u_i$ $(i \in \mathbb{N}_k)$; (2) $\alpha_i - \beta_i \in \{0,1\}$ $(i \in \mathbb{N})$.

To see this, first suppose that (1) and (2) hold, and set

$$y = \sum_{i=1}^{\infty} \frac{\alpha_i - \beta_i}{d_i}.$$

Then $y \in B$ because $\beta_i = 0$ $(i \in \mathbb{N}_k)$, and $x = y + t$ because

$$\alpha_i = (\alpha_i - \beta_i) + \beta_i < r_i - 1 \quad (i \in \mathbb{N}).$$

Thus $x - t \in B$. Conversely, suppose that $y := x - t \in B$. Then (1) holds. Since $t \in L_1$, we have $\beta_i + \varepsilon_i(y) \le r_i - 1$ $(i \in \mathbb{N})$, and so $\alpha_i = \beta_i + \varepsilon_i(y)$ $(i \in \mathbb{N})$, and this implies that $\alpha_i - \beta_i \in \{0,1\}$ $(i \in \mathbb{N})$, giving (2). This establishes the claim.

Next set $n = n_t$, so that $n > k$ and $\beta_n = 1$. Suppose that $x - t \in B$. Then $\alpha_n \in \{1,2\}$ by (2). But we know that $\alpha_n \neq 2$ because $x \in L_1$, and so $\alpha_n = 1$. For each $i \in \mathbb{N}$ with $i > k$ and with $i \neq n$, we can choose $x \in L_1$ with $\varepsilon_i(x) = \beta_i$, and we can also choose $y \in L_1$ with $\varepsilon_i(y) = 1 + \beta_i$; we can make these choices independently of any of the other coordinates of x or y, respectively. Thus we see that

$$(L_1 - t) \cap B = \{z \in B : \varepsilon_n(z) = 0\}.$$

This implies that

$$\mu((L_1 - t) \cap B) = \frac{1}{2}\mu(B) \quad (t \in C). \tag{8.5}$$

Since each clopen subset is the union of a finite, pairwise disjoint family of basic open sets, equation (8.5) easily extends to arbitrary clopen subsets B of L_0.

Essentially as before, we see that, given $\varepsilon > 0$, there is a neighbourhood U of 0 in $[0,1)$ such that

$$\left|\mu((L_1 - t) \cap B) - \frac{1}{2}\mu(B)\right| < \varepsilon\mu(B) \quad (t \in C)$$

for each $C \in \psi$ such that $C \subset L_2 \cap U$.

Recall that

$$(\mu_B \star m_C)(L_1) = \frac{1}{\mu(B)m(C)} \int_C \mu((L_1 - t) \cap B)\,\mathrm{d}m(t)$$

whenever B, C are Borel sets with $\mu(B), m(C) > 0$. Thus

$$\left|(\mu_B \star m_C)(L_1) - \frac{1}{2}\right| < \varepsilon$$

for each $C \in \psi$ such that $C \subset L_2 \cap U$.

We again take limits along the ultrafilters, first letting $C \to \psi$, and then letting $B \to \varphi$, to see that $(\delta_\varphi \,\Box\, \delta_\psi)(K_{L_1}) = 1/2$, as required. ■

We now give an analogous result for compact, totally disconnected groups.

We first describe a class of *sequential pro-finite groups*; our groups are certain projective limits of finite groups. Indeed, each such group has the following form. For each

$n \in \mathbb{N}$, let G_n be a finite group of cardinality $|G_n|$, with identity denoted by e_n. Suppose that there are group homomorphisms $\theta_{n,m} : G_n \to G_m$, defined whenever $m, n \in \mathbb{N}$ and $m \leq n$, such that $\theta_{m,m}$ is the identity on G_m for each $m \in \mathbb{N}$, and such that $\theta_{p,n} \circ \theta_{n,m} = \theta_{p,m}$ whenever $m \leq n \leq p$. Then the group G is the projective limit of this system. Thus, as a group,

$$G = \Big\{ (x_n) \in \prod_{n=1}^{\infty} G_n : \theta_{n,m}(x_n) = x_m \ (m \leq n) \Big\},$$

and G has the relative product topology from $\prod_{n=1}^{\infty} G_n$. These groups are examples of *pro-finite groups*; general pro-finite groups replace the set $\mathbb{N}$ by more general directed sets. These groups are discussed in [88, §12.3] and [120]. A pro-finite group G is sequential if and only if e_G is a countable intersection of open, normal subgroups [120, Proposition 4.1.3]. Let G be a compact, totally disconnected group. Then it follows from [48, §8] that G has a quotient that is a sequential pro-finite group.

Let G be an infinite, sequential pro-finite group, with the above representation. The Haar measure on G is denoted by m. We set

$$K_n = \ker \theta_{n,n-1} \quad (n \geq 2),$$

By relabelling the groups, we may suppose that

$$|G_{n+1}| > 2^{2n} |G_n| \quad (n \in \mathbb{N}),$$

so that $|K_n| > 2^{2(n-1)}$ $(n \geq 2)$.

We begin by defining a continuous homomorphism from the group D_2 into G. Indeed, we shall first define by induction on $n \in \mathbb{N}$ an element $\zeta(\varepsilon)$ in the group G_n for each $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{Z}_2^n$ with $\varepsilon_1 = 0$. In the case where $n = 1$ and ε is the singleton 0, we set $\zeta(\varepsilon) = e_1 \in G_1$. Now suppose that $n \geq 2$, and assume that $\zeta(\varepsilon) = a_\varepsilon$ has been defined in G_m for each $\varepsilon \in \mathbb{Z}_2^m$ whenever $m < n$. For each $\varepsilon \in \mathbb{Z}_2^{n-1}$, choose distinct elements $a_\varepsilon{}^\frown 0$ and $a_\varepsilon{}^\frown 1$ in G_n such that

$$\theta_{n,n-1}(a_{\varepsilon^\frown 0}) = \theta_{n,n-1}(a_{\varepsilon^\frown 1}) = a_\varepsilon .$$

Further, in the case where $\varepsilon = (0, \dots, 0) \in \mathbb{Z}_2^n$, we insist that $a_{\varepsilon^\frown 0} = e_n$; this is compatible with the previous instruction. This completes the inductive definition. Next, for $\varepsilon \in D_2$, we define $\zeta(\varepsilon)$ to be the unique sequence a_ε in G such that

$$(a_\varepsilon) {\restriction} n = \zeta(\varepsilon {\restriction} n) \quad (n \in \mathbb{N}).$$

It is clear that $\zeta : D_2 \to G$ is a well-defined, continuous embedding.

The set L_0 is defined to be the image $\zeta(D_2)$ of D_2 in G, so that L_0 is a compact subset of G. The measure on L_0 that corresponds to Haar measure on D_2 is again denoted by μ, so that $\mu(L_0) = 1$ and μ is singular with respect to m. We fix φ to be any element of Φ_μ, so that $\varphi \in \Phi_s$.

For each $n \in \mathbb{N}$, we define

$$A_n = \{ a_\varepsilon : \varepsilon \in \mathbb{Z}_2^n, \ \varepsilon_1 = 0 \}, \qquad B_n = \{ a_\varepsilon : \varepsilon \in \mathbb{Z}_2^n, \ \varepsilon_1 = \varepsilon_n = 0 \},$$

so that $B_n \subset A_n \subset L_0$, and then, for $n \geq 2$, choose $c_n \in K_n \setminus A_n^{-1} A_n$; the latter is possible because $|A_n| = 2^{n-1}$ and $|K_n| > 2^{2(n-1)}$ for each $n \geq 2$. We note in particular that $c_n \neq e_n$.

We next define

$$L_1 = \{(x_n) \in G : x_n \notin B_n c_n \ (n \geq 2)\}.$$

Clearly L_1 is a Borel subset of G.

Further, for each $m \geq 2$, we define

$$L_{2,m} = \{x = (x_n) \in G : x_m = c_m\}.$$

We observe that, for each $(x_n) \in L_{2,m}$ and each $r \in \mathbb{N}$ with $r \neq m$, necessarily $x_r \neq c_r$; this holds because $\theta_{m,r}(x_m) = e_r$ $(r < m)$ and $\theta_{r,m}(c_r) = e_m \neq c_m$ $(r > m)$. Thus m is the unique element $n \in \mathbb{N}$ such that $x_n = c_n$, say $m = n_x$.

Finally, we define

$$L_2 = \bigcup\{L_{2,m} : m \in \mathbb{N}\},$$

so that L_2 is a Borel subset of G. We observe that $m(L_2 \cap U) > 0$ for each neighbourhood U of 0 in G, and so there is a point $\psi \in \Phi_{\{e\}}$ such that L_2 belongs to the ultrafilter ψ.

The following lemma is essentially the same as Lemma 8.10.

LEMMA 8.11. *We have* $(\delta_\varphi \,\Box\, \delta_\psi)(K_{L_1}) = 1/2$.

Proof. Let B be a basic clopen subset of L_0 consisting of the elements $(x_n) \in L_0$ such that $x_i = u_i$ $(i \in \mathbb{N}_k)$ for some $k \in \mathbb{N}$ and $u_1, \dots, u_k \in \mathbb{Z}_2$, and let C be the subset of L_2 consisting of the elements $(x_n) \in L_2$ with $x_i = e_i$ $(i \in \mathbb{N}_k)$.

Fix $t = (t_n) \in C$, and let $x \in L_1$. We *claim* that $xt^{-1} \in B$ if and only if the following two conditions hold: (1) $x_i = u_i$ $(i \in \mathbb{N}_k)$; (2) $x_i c_i^{-1} \in A_i \setminus B_i$ $(i \in \mathbb{N})$. This is a slight variation of the earlier argument.

Thus we see that

$$L_1 t^{-1} \cap B = \{(b_n) \in B : b_{n_t} \notin B_{n_t}\},$$

which implies that

$$\mu(L_1 t^{-1} \cap B) = \frac{1}{2}\mu(B) \qquad (t \in C);$$

again this equation easily extends to arbitrary clopen subsets B of L_0.

The remainder of the proof is as before. ∎

We have established the following theorem.

THEOREM 8.12. *Let G be $\mathbb{T}$ or a sequential pro-finite group. Then there exist $\mu \in M_s(G)^+$ and $\psi \in \Phi$ such that $\varphi \,\Box\, \psi \notin \widetilde{G}$ for each $\varphi \in \Phi_\mu$.* ∎

An inspection of the above proofs shows that the only property of the measure m that was used is that $m(L_2 \cap U) > 0$ for each neighbourhood U of 0 in $[0, 1)$; there are many singular measures $\nu \in M_s(G)^+$ such that $\nu(L_2 \cap U) > 0$ for each such neighbourhood U. Thus we also obtain the following theorem.

THEOREM 8.13. *Let G be $\mathbb{T}$ or a sequential pro-finite group. Then there exist $\mu \in M_s(G)^+$ and $\psi \in \Phi_s$ such that $\varphi \,\Box\, \psi \notin \widetilde{G}$ for each $\varphi \in \Phi_\mu$.* ∎

We shall discuss below a version of the above results for more general groups.

The following table summarizes the inclusions that we have established at least for the compact groups specified in the above theorems. Let R, S, and T be subsets of $\widetilde{G}$, with R in the left-hand column and S in the top row. The conclusion '$\subset T$' implies that, for each $\varphi \in R$ and $\psi \in S$, it follows that $\varphi \mathbin{\Box} \psi \in \widetilde{G}$, and, further, that $\varphi \mathbin{\Box} \psi \in T$. The conclusion '$\not\subset \widetilde{G}$' implies that there exist $\varphi \in R$ and $\psi \in S$ such that $\varphi \mathbin{\Box} \psi \notin \widetilde{G}$.

$\Box$	βG_d	Φ	Φ_s
βG_d	$\subset \beta G_d$	$\subset \Phi$	$\subset \Phi_s$
Φ	$\subset \Phi$	$\subset \Phi$	$\subset \Phi$
Φ_s	$\not\subset \widetilde{G}$	$\not\subset \widetilde{G}$	$\not\subset \widetilde{G}$

General groups. Let $\mathcal{C}$ be the class of all non-discrete, locally compact groups G such that $(\widetilde{G}, \Box)$ is not a semigroup. Our aim is to show that $\mathcal{C}$ is the class of all non-discrete, locally compact groups. (We recall that it is already known for all non-discrete, locally compact groups which are not compact that $(\widetilde{G}, \Box)$ is not a semigroup, but we shall not use this result.)

We first reduce to the case of non-discrete, locally compact abelian groups. The following result may be well known; we are indebted to George Willis for some of the references in the proof.

THEOREM 8.14. *Every non-discrete, locally compact group has a closed subgroup which is a non-discrete, locally compact abelian subgroup.*

Proof. Let G be a non-discrete, locally compact group, and let the component of the identity of G be G_0.

Suppose first that $G_0 = \{e_G\}$, the identity of G. By [48, Theorems (7.3) and (7.7)], G is totally disconnected and contains an infinite, compact subgroup. By a very deep theorem of Zelmanov [126], each infinite compact group contains an infinite (and hence non-discrete), compact abelian subgroup.

Next suppose that $G_0 \neq \{e_G\}$. Then G_0 has a compact normal subgroup, say K, such that G_0/K is a Lie group [85, §4.6].

If K is infinite, then again K contains an infinite, compact abelian subgroup.

If K is finite, then G_0 itself is a Lie group, and so G_0 contains a 1-parameter subgroup (isomorphic to $\mathbb{R}$ or $\mathbb{T}$) [85, §4.2].

Thus in each case G contains a closed subgroup which is a non-discrete, locally compact abelian subgroup. ■

We now call in aid a structure theorem for non-discrete, locally compact abelian groups; the theorem is implied by [41, Theorem 6.8.4], which is called a 'standard theorem'. For a prime number p, the group Δ_p is the group of p-adic integers, as is explained on [41, p. 191], and $D_p = (Z_p)^\infty$ in the notation of [41].

THEOREM 8.15. *Let $\mathcal{B}$ be the class of all locally compact abelian groups G such that:*

(i) $\mathbb{R}, \mathbb{T} \in \mathcal{B}$;

(ii) $\Delta_p, D_p \in \mathcal{B}$ *for all prime numbers* p;

(iii) $G \in \mathcal{B}$ *whenever* G *is a locally compact abelian group such that* G *contains as a subgroup a member of* $\mathcal{B}$;

(iv) $G \in \mathcal{B}$ *whenever* G *is a locally compact abelian group such that* G *has a quotient that is a member of* $\mathcal{B}$.

Then $\mathcal{B}$ *contains all non-discrete, locally compact abelian groups.* ■

Thus we can conclude with the following theorem.

THEOREM 8.16. *Let G be a non-discrete, locally compact group. Then $(\widetilde{G}, \Box)$ is not a semigroup.*

Proof. By Theorem 8.14 and Proposition 8.6, it suffices to prove that $(\widetilde{G}, \Box)$ is not a semigroup for each non-discrete, locally compact abelian group G.

Let $\mathcal{B}$ be the class of all non-discrete, locally compact abelian groups G such that $(\widetilde{G}, \Box)$ is not a semigroup. Then we see that the class $\mathcal{B}$ satisfies all the clauses of Theorem 8.15; indeed we have shown in Theorem 8.8 that $\mathcal{B}$ satisfies clauses (i) and (ii), and in Proposition 8.6 that $\mathcal{B}$ satisfies clauses (iii) and (iv) of Theorem 8.15. Thus $\mathcal{B}$ is the class of all non-discrete, locally compact abelian groups.

This completes the proof of the theorem. ■

A similar extension of Theorem 8.12 can be given.

THEOREM 8.17. *Let G be a compact group. Suppose that there is a continuous epimorphism from G onto either $\mathbb{T}$ or a sequential pro-finite group. Then there exist $\varphi \in \widetilde{G}$ and $\psi \in \Phi$ such that $\varphi \,\Box\, \psi \notin \widetilde{G}$.*

Proof. Let H be either $\mathbb{T}$ or a sequential pro-finite group. Then, by Theorem 8.12, there exist elements $\varphi_1 \in \widetilde{H}$ and $\psi_1 \in \Phi_H$ such that $\varphi_1 \,\Box\, \psi_1 \notin \widetilde{H}$. By Proposition 5.2(ii), $\widetilde{\eta}(\widetilde{G}) = \widetilde{H}$, and $\widetilde{\eta}(\Phi_G) = \Phi_H$, and so there exist elements $\varphi \in \widetilde{G}$ and $\psi \in \Phi_G$ such that $\widetilde{\eta}(\varphi) = \varphi_1$ and $\widetilde{\eta}(\psi) = \psi_1$. Since $\widetilde{\eta}(\delta_\varphi \,\Box\, \delta_\psi) = \delta_{\varphi_1} \,\Box\, \delta_{\psi_1}$, we have $\varphi \,\Box\, \psi \notin \widetilde{G}$. ■

In particular, the above theorem applies to each group of the form $G \times H$, where G is an infinite, compact, totally disconnected group, and to each non-trivial, connected, solvable group.

Further calculations on products. Within the proof of Theorem 8.8, we showed that there are a measure $\mu \in M_s(\mathbb{T})^+$ and elements $\psi \in \mathbb{T}_d^*$ such that, for each $\varphi \in \Phi_\mu$, the measure $\delta_\varphi \,\Box\, \delta_\psi \in M(\widetilde{\mathbb{T}})$ satisfies $\langle \chi_{K_L}, \delta_\varphi \,\Box\, \delta_\psi \rangle = 1/2$, and hence is not a point mass. We now gain further information about measures similar to $\delta_\varphi \,\Box\, \delta_\psi$. (We restrict attention to the group $\mathbb{T}$; similar remarks apply to other groups.)

In the next theorem, μ and ψ are fixed as above; the measure $\mu \in M_s(\mathbb{T})^+$ was defined in (8.1).

THEOREM 8.18. *For each $\varphi \in \Phi_\mu$, there is a non-zero, continuous measure* $\mathrm{M} \in M_c(\widetilde{\mathbb{T}})^+$ *such that*

$$\delta_\varphi \,\square\, \delta_\psi = \frac{1}{2}\delta_\varphi + \mathrm{M}.$$

In particular, the measure $\delta_\varphi \,\square\, \delta_\psi$ is neither continuous nor discrete.

Proof. This proof comes in two parts, which together establish the theorem, and in fact give slightly more information. The space L was specified in Definition 8.7.

(1) We shall show first that the restriction of $\delta_\varphi \,\square\, \delta_\psi$ to the set K_L is $\delta_\varphi/2$.

Recall from (2.1) that the basic clopen subsets of L have the form

$$U_{F,\alpha} = \{(\varepsilon_n) \in L : \varepsilon_{n_i} = \alpha_i \ (i \in \mathbb{N}_k)\}$$

for fixed $F = \{n_1, \dots, n_k\}$ and $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{Z}_2^k$, and that each clopen set is a finite union of pairwise disjoint, basic, clopen subsets of L.

We first make the following *claim*. Let U and V be clopen subsets of L with $U \subset V$. Then there exists $r_0 \in \mathbb{N}$ such that

$$\mu(U \cap (V - 3^{-r})) = \frac{1}{2}\mu(U) \quad (r > r_0). \tag{8.6}$$

First suppose that U and V are basic clopen subsets, say $U = U_{G,\beta}$ and $V = U_{F,\alpha}$, where $F \subset G$, $\alpha \in \mathbb{Z}_2^{|F|}$, $\beta \in \mathbb{Z}_2^{|G|}$, and $\beta \,|\, F = \alpha$, so that it is indeed true that $U \subset V$. Take $r > \max G$, and define γ on $G \cup \{r\}$ by requiring that $\gamma \,|\, G = \beta$ and $\gamma_r = 0$. Then $U \cap (V - 3^{-r}) = U_{G \cup \{r\},\gamma}$. Thus

$$\mu(U \cap (V - 3^{-r})) = \mu(U_{G\cup\{r\},\gamma}) = \left(\frac{1}{2}\right)^{|G|+1} = \frac{1}{2}\mu(U). \tag{8.7}$$

For the general case, take clopen sets $U, V \subset L$ with $U \subset V$. Then there exist a finite subset F of $\mathbb{N}$ and elements $\alpha^1, \dots, \alpha^m, \beta^1, \dots, \beta^n$ in $\mathbb{Z}_2^{|F|}$ such that $\alpha^1, \dots, \alpha^m$ are distinct and $\beta^1, \dots, \beta^n$ are distinct and

$$U = \bigcup_{i=1}^{m} U(F, \alpha^i) \quad \text{and} \quad V = \bigcup_{j=1}^{n} U(F, \beta^j)$$

(and each union is composed of pairwise disjoint sets). By (8.7), there exists $r_0 \in \mathbb{N}$ such that $r_0 > \max F$ and

$$\mu(U_{\alpha^i} \cap U_{\beta^j} \cap (U_{\beta^j} - 3^{-r})) = \frac{1}{2}\mu(U_{\alpha^i} \cap U_{\beta^j}) \quad (r > r_0,\ i \in \mathbb{N}_m,\ j \in \mathbb{N}_n),$$

where we are writing U_{α^i} for $U(F, \alpha^i)$, etc. Now

$$V \cap (V - 3^{-r}) = \bigcup_{j=1}^{n} U_{\beta^j} \cap (U_{\beta^j} - 3^{-r}) \quad (r > r_0)$$

because $U_{\beta^i} \cap (U_{\beta^j} - 3^{-r}) = \emptyset$ whenever $r > r_0$ and $i, j \in \mathbb{N}_n$ with $i \neq j$, and so

$$\mu(U \cap V \cap (V - 3^{-r})) = \frac{1}{2}\mu(U \cap V) \quad (r > r_0).$$

Since $U \cap V = U$, our first claim (8.6) holds.

Our second *claim* is the following. Let $B, C \in \mathfrak{B}_L$ with $B \subset C$ and $\mu(B) > 0$. Then, for each $\varepsilon > 0$, there exists $r_\varepsilon \in \mathbb{N}$ such that

$$\left|\mu(B \cap (C - 3^{-r})) - \frac{1}{2}\mu(B)\right| < \varepsilon\mu(B) \quad (r > r_\varepsilon). \tag{8.8}$$

To see that this holds, first take clopen subsets U and V in L such that

$$\mu(B \bigtriangleup U) < \frac{1}{2}\varepsilon\mu(B) \quad \text{and} \quad \mu(C \bigtriangleup V) < \frac{1}{2}\varepsilon\mu(B).$$

Set $W = U \cap V$. Then

$$\mu(B \bigtriangleup W) < \varepsilon\mu(B)$$

because $B \bigtriangleup W \subset (B \bigtriangleup U) \cup (C \setminus V) \subset (B \bigtriangleup U) \cup (C \bigtriangleup V)$, and so

$$|\mu(W) - \mu(B)| < \varepsilon\mu(B). \tag{8.9}$$

It follows from our first claim that there exists $r_\varepsilon \in \mathbb{N}$ such that

$$\mu(W \cap (V - 3^{-r})) = \frac{1}{2}\mu(W) \quad (r > r_\varepsilon). \tag{8.10}$$

Now fix $r > r_\varepsilon$. As in the proof of Theorem 8.8, we have

$$\mu((C - 3^{-r}) \bigtriangleup (V - 3^{-r})) < \frac{1}{2}\varepsilon\mu(B) \quad (r > r_\varepsilon). \tag{8.11}$$

It follows from (8.9) and (8.10) that

$$\left|\mu(B \cap (V - 3^{-r})) - \frac{1}{2}\mu(B)\right| < \frac{1}{2}\varepsilon\mu(B) \quad (r > r_\varepsilon),$$

and then from (8.11) it follows that our second claim, (8.8), holds.

Now suppose that $C \in \varphi$. For each $B \in \varphi$ such that $B \subset C$ and $\mu(B) > 0$ and for each $\varepsilon > 0$, we have seen that there exists $r_\varepsilon \in \mathbb{N}$ such that

$$\left|(\mu_B \star \delta_{3^{-r}})(C) - \frac{1}{2}\right| = \left|\frac{\mu(B \cap (C - 3^{-r}))}{\mu(B)} - \frac{1}{2}\right| < \varepsilon \quad (r > r_\varepsilon).$$

We take limits as a subnet of the point masses $\delta_{3^{-r}}$ converges to δ_ψ, and then take limits $\lim_{B \to \varphi}$; by (6.5), we have

$$\langle \chi_{K_C}, \delta_\varphi \Box \delta_\psi \rangle = \frac{1}{2}.$$

Thus $(\delta_\varphi \Box \delta_\psi)(K_C) = 1/2$. We already know that $(\delta_\varphi \Box \delta_\psi)(K_L) = 1/2$, and so

$$(\delta_\varphi \Box \delta_\psi)(K_{L \setminus C}) = 0 \quad (C \in \varphi).$$

Thus the restriction of $\delta_\varphi \Box \delta_\psi$ to the set K_L is $\delta_\varphi/2$, and so part (1) is proved.

(2) We shall show now that the restriction of $\delta_\varphi \Box \delta_\psi$ to the set $\widetilde{\mathbb{T}} \setminus K_L = K_{\mathbb{T} \setminus L}$ is a continuous measure (it is positive and has mass 1/2).

First, recall that each $x \in \mathbb{I}$ has a ternary expansion of the form $x = \sum_{n=1}^{\infty} \varepsilon_n(x)/3^n$, where $\varepsilon_n(x) \in \mathbb{Z}_3^+$. For each $n \in \mathbb{N}$, set

$$C_n = \{x \in \mathbb{I} : \varepsilon_n(x) = 2,\, \varepsilon_r(x) \in \{0, 1\}\ (r \in \mathbb{N} \setminus \{n\})\},$$

so that each C_n is a closed subset of $\mathbb{I}$ with $C_n \cap L = \emptyset$ and the sets C_n are pairwise disjoint, and then set

$$C = \bigcup_{n=1}^{\infty} C_n,$$

so that $C \in \mathfrak{B}_{\mathbb{T}}$ and $C \cap L = \emptyset$.

We first *claim* that $\operatorname{supp}(\delta_\varphi \mathbin{\Box} \delta_\psi) \subset K_{L\cup C}$. Indeed, suppose that $x \in \mathbb{T} \setminus (L \cup C)$. Then it is easily checked that $x - 3^{-r} \in \mathbb{T} \setminus L$ for each $r \in \mathbb{N}$, and so

$$(\mu_B \star \delta_{3^{-r}})(L \cup C) = 0$$

for each $B \in \mathfrak{B}_L$ with $\mu(B) > 0$. Thus

$$(\delta_\varphi \mathbin{\Box} \delta_\psi)(\widetilde{\mathbb{T}} \setminus K_{L\cup C}) = \lim_{B\to\varphi} \lim_{r\to\infty} (\mu_B \star \delta_{3^{-r}})(L \cup C) = 0,$$

giving the claim.

For each $A \in \mathfrak{B}_C$, we have

$$(\delta_\varphi \mathbin{\Box} \delta_\psi)(K_A) = \langle \chi_{K_A}, \delta_\varphi \mathbin{\Box} \delta_\psi \rangle = \lim_{B\to\varphi} \lim_{n\to\infty} \mu_B((A \cap C_n) - 3^{-n}).$$

Fix $k \in \mathbb{N}$, and enumerate the set $\mathbb{Z}_2^k$ as $\{\alpha^1, \dots, \alpha^{2^k}\}$. For each $i \in \mathbb{N}_{2^k}$ and each $n \in \mathbb{N}$, define

$$A_{i,n} = \{x \in C_n : \varepsilon_{m+n} = \alpha^i_m \ (m \in \mathbb{N}_k)\},$$

so that $A_{i,n}$ is a closed subset of C_n and $\{A_{1,n}, \dots, A_{2^k,n}\}$ is a partition of C_n. Now, for each $i \in \mathbb{N}_{2^k}$, define

$$A_i = \bigcup_{n=1}^{\infty} A_{i,n},$$

so that each $A_i \in \mathfrak{B}_C$ and $\{A_1, \dots, A_{2^k}\}$ is a partition of C. We shall show that

$$(\delta_\varphi \mathbin{\Box} \delta_\psi)(K_{A_i}) = \frac{1}{2^{k+1}} \quad (i \in \mathbb{N}_{2^k}); \tag{8.12}$$

from this we see that each singleton in $K_{\mathbb{T}\setminus L}$ has mass at most $1/2^k$ with respect to the measure $\delta_\varphi \mathbin{\Box} \delta_\psi$. Since this is true for each $k \in \mathbb{N}$, it will follow that $(\delta_\varphi \mathbin{\Box} \delta_\psi) | K_{\mathbb{T}\setminus L}$ is a continuous measure, as required.

Fix $i \in \mathbb{N}_{2^k}$. We first observe that, for every basic open subset U of L, there exists $r_0 \in \mathbb{N}$ such that

$$\mu(U \cap ((A_i \cap C_r) - 3^{-r})) = \frac{1}{2^{k+1}}\mu(U) \quad (r > r_0).$$

This statement extends to all clopen subsets U of L because each such set is the union of a pairwise disjoint family of basic open sets. Now take $\varepsilon > 0$. For each $B \in \mathfrak{B}_L$ with $\mu(B) > 0$, there is a clopen subset U of L with $\mu(B \mathbin{\triangle} U) < \varepsilon\mu(B)$, and then, as before,

$$|\mu(U \cap ((A_i \cap C_r) - 3^{-r})) - \mu(B \cap ((A_i \cap C_r) - 3^{-r}))| < \varepsilon\mu(B)$$

for each $r \in \mathbb{N}$. Thus

$$\frac{1}{2^{k+1}}(1 - \varepsilon) < \mu_B((A_i \cap C_r) - 3^{-r}) < \frac{1}{2^{k+1}}(1 + \varepsilon) \quad (r > r_0).$$

By taking limits in the usual way, we see that (8.12) follows.

This completes the proof. ■

In comparison, we note that, for each $\varphi \in \mathbb{T}$ and $\psi \in \overline{\mathbb{T}}$, the measure $\delta_\varphi \diamond \delta_\psi = \delta_\psi \mathbin{\Box} \delta_\varphi$ is a point mass, and so $\varphi \diamond \psi \in \widetilde{\mathbb{T}}$; this is a consequence of Proposition 8.4.

Corollary 8.19. *Let μ be as above. Then $\delta_\varphi \notin \mathfrak{Z}_t^{(\ell)}(M(\widetilde{\mathbb{T}}))$ for each $\varphi \in \Phi_\mu$.* ■

A stronger result than the above will be proved in Theorem 9.8.

Example 8.20. We give an example to show that there is a compact group G and elements $\varphi, \psi, \theta \in \widetilde{G}$ with $\varphi \sim \psi$, but such that

$$\varphi \square \theta \nsim \psi \square \theta;$$

this contrasts with Theorem 6.11.

We take $G = \mathbb{T}$. As in Theorem 8.18, there exist $\varphi \in \Phi$, $\theta \in \widetilde{\mathbb{T}}$, and $L \in \mathfrak{B}_{\mathbb{T}}$ such that $(\delta_\varphi \square \delta_\theta)(K_L) = 1/2$. By Proposition 4.13, there exists $\psi \in \beta\mathbb{T}_d$ such that $\psi \sim \varphi$. Now $\psi \square \theta \in \widetilde{\mathbb{T}}$, and so $\delta_\psi \square \delta_\theta$ is point mass. Thus $(\delta_\psi \square \delta_\theta)(K_L) \in \{0, 1\}$. Hence $\delta_\psi \square \delta_\theta \nsim \delta_\varphi \square \delta_\theta$. ■

We shall now show that the product of two point masses in $M(\widetilde{\mathbb{T}})$ might be a continuous measure on $\widetilde{\mathbb{T}}$.

Let G be a locally compact group. For $n \in \mathbb{N}$ and $\mathrm{M} \in M(\widetilde{G})$, we write $\mathrm{M}^{\square n}$ for the n^{th} power of M in the algebra $(M(\widetilde{G}), \square)$. For ψ in the semigroup $(\overline{G}, \square)$, the n^{th} power of ψ in the semigroup is $\psi^{\square n}$, so that $\delta_\psi^{\square n}$ is the point mass at $\psi^{\square n}$. The set $\{\psi^{\square n} : n \in \mathbb{N}\}$ of points in $\overline{G}$ has an accumulation point, say ξ, and then δ_ξ is a weak-$*$ accumulation point of the set $\{\delta_\psi^{\square n} : n \in \mathbb{N}\}$ in $M(\widetilde{G})_{[1]}$.

We let L, μ, φ, and ψ have the same meaning as above.

Theorem 8.21. *Let $\mu \in M_s(\mathbb{T})^+$ be as specified. Then there is an element $\xi \in \overline{\mathbb{T}}$ such that, for each $\varphi \in \Phi_\mu$, the measure $\delta_\varphi \square \delta_\xi$ belongs to $M_c(\widetilde{\mathbb{T}})^+$.*

Proof. The element $\xi \in \overline{\mathbb{T}}$ is taken to be any accumulation point of the set $\{\psi^{\square n} : n \in \mathbb{N}\}$, which was specified above.

Let $\varphi \in \Phi_\mu$. The proof that $\delta_\varphi \square \delta_\xi$ is a continuous measure on $\widetilde{\mathbb{T}}$ is similar to that of Theorem 8.18; it comes in three parts.

(1) We shall show first that the restriction of $\delta_\varphi \square \delta_\xi$ to the set K_L is 0.

We *claim* the following. Let $B, C \in \mathfrak{B}_L$ with $B \subset C$ and $\mu(B) > 0$. Then, for each $\varepsilon > 0$, there exists $r_\varepsilon \in \mathbb{N}$ such that

$$\left| \mu(B \cap (C - (3^{-r_1} + \cdots + 3^{-r_n}))) - \frac{1}{2^n}\mu(B) \right| < \varepsilon \mu(B) \tag{8.13}$$

whenever $r_n > \cdots > r_2 > r_1 > r_\varepsilon$. This is proved by a slight variation of the proof of the corresponding claim in Theorem 8.18.

Let $B \in \varphi$ with $B \subset L$ and $\mu(B) > 0$, and take $\varepsilon > 0$. For each $n \in \mathbb{N}$, we have seen that there exists $r_\varepsilon \in \mathbb{N}$ such that

$$\left| (\mu_B \star \delta_{3^{-r_1}} \star \cdots \star \delta_{3^{-r_n}})(L) - \frac{1}{2^n} \right| < \varepsilon$$

whenever $r_n > \cdots > r_1 > r_\varepsilon$. We take limits successively over $r_n, \dots, r_1$ as subnets of the point masses $\delta_{3^{-r}}$ converge to δ_ψ to see that

$$\left| (\mu_B \square \delta_\psi)^{\square n} \left(K_L - \frac{1}{2^n} \right) \right| \leq \varepsilon,$$

using (3.8). We next take limits over a subnet of $(\psi^{\square n})$ to see that

$$\langle \chi_{K_L}, \mu_B \square \delta_\xi \rangle = (\mu_B \square \delta_\xi)(K_L) \leq \varepsilon.$$

Finally, we take limits $\lim_{B \to \varphi}$ to see that $(\delta_\varphi \square \delta_\xi)(K_L) \leq \varepsilon$. Since this holds for each $\varepsilon > 0$, we have $(\delta_\varphi \square \delta_\xi)(K_L) = 0$, as required.

(2) There is a Borel subset C of $\mathbb{T}$ such that $(\delta_\varphi \mathbin{\Box} \delta_\psi) \,|\, K_{\mathbb{T}\setminus(L\cup C)} = 0$.

Let $\mathcal{F}$ denote the family of non-empty, finite subsets of the set $\{r_n : n \in \mathbb{N}\}$ that was specified above. For each $F \in \mathcal{F}$, we take m_F to be the maximum of F in $\mathbb{N}$, we set

$$x_F = \sum \{3^{-r} : r \in F\} \in \mathbb{T},$$

and we define

$$C_F := \{x \in \mathbb{T} : \varepsilon_r(x) = 2 \text{ if and only if } r \in F\} \subset \mathbb{T},$$

so that C_F is a closed subset of $\mathbb{T}$ and the sets C_F are pairwise disjoint. Now set

$$X = \{x_F : F \in \mathcal{F}\} \quad \text{and} \quad C = \bigcup \{C_F : F \in \mathcal{F}\}.$$

Since $\mathcal{F}$ is countable, $C \in \mathfrak{B}_{\mathbb{T}}$. Further, for each $t \in \mathbb{T}$ and $x \in X$ such that $t - x \in L$, we see easily that $\varepsilon_r(t-x) = \varepsilon_r(t)$ for each $r \in \mathbb{N} \setminus F$, and so $t \in L \cup C$. It follows that, for each $B \in \mathfrak{B}_{\mathbb{T}}$ with $\mu(B) > 0$ and each $x \in X$, we have

$$(\mu_B \star \delta_x)(\mathbb{T} \setminus (L \cup C)) = 0.$$

Thus $(\delta_\varphi \mathbin{\Box} \delta_\xi) \,|\, K_{\mathbb{T}\setminus(L\cup C)} = 0$, establishing (2).

(3) The restriction of $\delta_\varphi \mathbin{\Box} \delta_\psi$ to K_C is continuous.

We fix $k \in \mathbb{N}$. Let B be a non-empty, clopen subset of L. Then there exists $m \geq 2$ such that B is specified by the first m coordinates in the ternary expansion of a point of $\mathbb{T}$.

Let $\alpha \in \mathbb{Z}_3^k$. For each $F \in \mathcal{F}$, define

$$A_{\alpha,F} = \{t \in C_F : \varepsilon_{m_F+m}(t) = \alpha_m \ (m \in \mathbb{N}_k)\},$$

so that $A_{\alpha,F}$ is a closed subset of C_F and $\{A_{\alpha,F} : \alpha \in \mathbb{Z}_3^k\}$ is a partition of C_F. Next define

$$A_\alpha = \bigcup \{A_{\alpha,F} : F \in \mathcal{F}\},$$

so that each $A_\alpha \in \mathfrak{B}_{\mathbb{T}}$ and $\{A_\alpha : \alpha \in \mathbb{Z}_3^k\}$ is a partition of C into 3^k subsets.

Now suppose that $G \in \mathcal{F}$ and that G is such that $\min G > r_m \geq m$. Let $F \in \mathcal{F}$. For each $t \in C_F$, the point $t - x_G$ can only be in L if $F \subset G$. This shows that $L \cap (C_F - x_G) = \emptyset$ whenever $F \in \mathcal{F}$ with $F \not\subset G$. Now suppose that $F \subset G$, so that

$$r_n \geq \min F \geq \min G > r_m,$$

where $n \in \mathbb{N}$ is such that $m_F = r_n$. Thus $n > m$. The set

$$\{3^{-r_1} + \cdots + 3^{-r_{n+1}} : r_{n+1} - r_n > k\}$$

belongs to $\psi^{\Box n}$, and so we may suppose that $r_{n+1} - r_n > k$. This implies that

$$G \cap \{r_n + 1, \ldots, r_n + k\} = \emptyset, \tag{8.14}$$

and so, for each $\alpha \in \mathbb{Z}_3^k$, we have

$$L \cap ((A_\alpha \cap C_F) - x_G) \subset A_{\alpha,F}.$$

In addition, for each $t \in C_F$, the element $t - x_G$ is in L only if we have $\varepsilon_r(t) = 1$ $(r \in G \setminus F)$, and then $\varepsilon_r(t - x_G) = 1$ $(r \in F)$ and $\varepsilon_r(t - x_G) = 0$ $(r \in G \setminus F)$. It follows that, for each

$t \in A_\alpha \cap C_F$, the element $t - x_G$ is in L only if $\varepsilon_r(t)$ takes specified values on G and on the set $\{r_n + 1, \dots, r_n + k\}$; now (8.14) implies that

$$\mu(B \cap ((A_\alpha \cap C_F) - x_G) \leq 2^{-k} 2^{-|G|} \mu(B).$$

Since the number of subsets F of G is $2^{|G|}$, it follows that

$$\mu(B \cap (A_\alpha - x_G)) = \mu(B \cap ((A_\alpha \cap C) - x_G)) \leq 2^{-k} \mu(B).$$

We have shown that, for each $k \in \mathbb{N}$, for each non-empty, clopen set $B \in \mathfrak{B}_L$, for each $x \in X$, and each $\alpha \in \mathbb{Z}_3^k$, we have $(\mu_B \star \delta_x)(A_\alpha) \leq 2^k$. As in the proof of Theorem 8.18, we now see that, for each $k \in \mathbb{N}$, for each $B \in \mathfrak{B}_{\mathbb{T}}$ with $\mu(B) > 0$, each $\varepsilon > 0$, and each $\alpha \in \mathbb{Z}_3^k$, there exists $m \in \mathbb{N}$ such that

$$(\mu_B \star \delta_x)(A_\alpha) \leq 2^{-k} + \varepsilon$$

whenever $x = x_G$ for some $G \in \mathcal{F}$ for which $\min G > m$. As in part (1), we can take limits as $x \to \xi$ through a suitable net, and then take the limit $\lim_{B \to \varphi}$ to see that, for each A of the form A_α, we have

$$(\delta_\varphi \,\square\, \delta_\xi)(K_A) \leq 2^{-k} + \varepsilon$$

for each $k \in \mathbb{N}$ and each $\varepsilon > 0$. Since $\{A_\alpha : \alpha \in \mathbb{Z}_3^k\}$ is a partition of C, it follows that each point in C has measure at most $2^{-k} + \varepsilon$. This is true for each $k \in \mathbb{N}$ and $\varepsilon > 0$, and so each point in C has measure 0. Thus $(\delta_\varphi \,\square\, \delta_\xi) \,|\, K_C$ is a continuous measure.

It follows from (1), (2), and (3) that $\delta_\varphi \,\square\, \delta_\xi$ is a continuous measure on $\widetilde{\mathbb{T}}$. ■

We now obtain information about groups other than $\mathbb{T}$. The same proof implies that any locally compact group G which contains a copy of $\mathbb{T}$ as a subgroup or which can be mapped onto $\mathbb{T}$ by a continuous, open epimorphism has the property that $\widetilde{G}$ contains two point masses whose box product is a continuous measure on $\widetilde{G}$.

The arguments given above also apply to the groups $\mathbb{R}$, Δ_p, and D_p for each prime p; with the aid of the details given in Proposition 8.6, we can prove the following theorem by essentially the arguments used to establish Theorem 8.16.

THEOREM 8.22. *Let G be a non-discrete, locally compact group. Then there exist $\varphi, \xi \in \widetilde{G}$ such that $\delta_\varphi \,\square\, \delta_\xi$ is a continuous measure on $\widetilde{G}$.* ■

We do not know whether or not the product of two point masses can be a finite sum of point masses, without being a point mass.

We conclude this chapter with a weaker result than Theorem 8.16; however in this case the proof is considerably shorter.

It may be that there is a topology τ on G such that $\tau_G \leq \tau \leq d$, such that $\tau \neq \tau_G$ and $\tau \neq d$, where d is the discrete topology, and such that (G, τ) is a locally compact group. In this case, we denote the character space of the commutative C^*-algebra $L^\infty(G, \tau)$ by $\Phi(\tau)$. Such a phenomenon does not happen when $G = \mathbb{T}$, for example; see [97]. However, in the case where $G = G_1 \times G_2$, where G_1 and G_2 are compact, infinite groups, the topology formed by taking the product of the given topology on G_1 and the discrete topology on G_2 has the specified properties. This question is related to that of the 'spine' of the algebra $M(G)$; see [54, 55].

PROPOSITION 8.23. *Let (G, τ_G) be a locally compact group, and suppose that τ is a topology on G such that $\tau \supset \tau_G$ and (G, τ) is a locally compact group. Then:*

(i) *$L^1(G, \tau)$ embeds isometrically in $M(G)$ as a closed subalgebra;*

(ii) *there is a continuous $C_0(G)$-module epimorphism*

$$P : M(G) \to L^1(G, \tau)$$

which is the identity on $L^1(G, \tau)$;

(iii) *the map $P'' : (M(\widetilde{G}), \Box) \to (M(\Omega(\tau)), \Box)$ is a continuous E''-module epimorphism which is the identity on $M(\Omega(\tau))$.*

Proof. (i) This is immediate from Proposition 3.19.

(ii) Let m_τ denote left Haar measure on (G, τ). We denote by $\mathcal{C}$ the family of τ-compact subsets K of G such that $m_\tau(K) > 0$. For $K \in \mathcal{C}$, set

$$V_K = \{\mu \in M(G) : |\mu|(K) = 0\}.$$

Next, set

$$M_{\mathcal{C},\tau} = \bigcap \{V_K : K \in \mathcal{C}\}.$$

Let $K \in \mathcal{C}$ and $t \in G$. Then the set Kt^{-1} is τ-compact and

$$m_\tau(Kt^{-1}) = m_\tau(K)\Delta(t^{-1}) > 0,$$

where Δ is the modular function on G, and so $Kt^{-1} \in \mathcal{C}$. Similarly, $t^{-1}K \in \mathcal{C}$.

We *claim* that $M_{\mathcal{C},\tau}$ is a closed ideal in $M(G)$. Indeed, take $\mu \in M_{\mathcal{C},\tau}$ and $\nu \in M(G)$; we shall show that $\mu \star \nu, \nu \star \mu \in M_{\mathcal{C},\tau}$. Clearly we may suppose that $\mu, \nu \geq 0$. Then, for each $K \in \mathcal{C}$, we have

$$(\mu \star \nu)(K) = \int_G \mu(Kt^{-1}) \, \mathrm{d}\nu(t) = 0$$

because $\mu(Kt^{-1}) = 0$, and so $\mu \star \nu \in M_{\mathcal{C},\tau}$. Similarly, $\nu \star \mu \in M_{\mathcal{C},\tau}$.

Let $\mu \in M(G)$. By the Lebesgue decomposition theorem, there exist $\mu_a \in L^1(G, \tau)$ and $\mu_s \in M(G)$ with $\mu_s \perp m_\tau$ such that $\mu = \mu_a + \mu_s$. Clearly $\mu_s \in M_{\mathcal{C},\tau}$, and so we have $M(G) = L^1(G, \tau) \ltimes M_{\mathcal{C},\tau}$; this implies that (ii) holds.

(iii) This follows from (ii) and Proposition 1.4(iii). ∎

The following result is a special case of Theorem 8.16.

THEOREM 8.24. *Let (G, τ_G) be a locally compact group, and suppose that τ is a non-discrete topology on G such that $\tau \supsetneq \tau_G$ and (G, τ) is a locally compact group. Then $(\widetilde{G}, \Box)$ is not a semigroup.*

Proof. The topological space (G, τ) is neither compact nor discrete, and so, by [78], $(\Phi(\tau), \Box)$ is not a semigroup. It follows from Proposition 8.23 that $(\widetilde{G}, \Box)$ is not a semigroup. ∎

COROLLARY 8.25. *Let G_1 and G_2 be infinite, compact groups, and set $G = G_1 \times G_2$. Then $(\widetilde{G}, \Box)$ is not a semigroup.* ∎

9. Topological centres

In this chapter we shall seek to determine the topological centres of the Banach algebras $(L^1(G)'', \Box)$ and $(M(G)'', \Box)$, and also which subsets of the spaces $L^1(G)''$ and $M(G)''$ are determining for the left topological centres, where G is a locally compact group.

The character space of $L^\infty(G)$. Let G be a locally compact group, and again set $A = L^1(G)$. We have denoted by Φ the character space of $L^\infty(G)$. It was first proved by Young in [124] that A is not Arens regular, the case where G is abelian having been settled by Civin and Yood in [10]; see also [115, 116] and [13, Theorems 2.9.39, 3.3.28]. It was proved by Işık, Pym, and Ülger in [56] that, in the case where G is compact, $(\Phi, \Box)$ is a semigroup and that A is strongly Arens irregular. It also follows from [56, Theorem 3.4] that each element of the semigroup $(\Phi, \Box)$ is right cancellable. The main result was eventually established when Lau and Losert proved in [73] that A is strongly Arens irregular for each locally compact group G. Finally Neufang [86] gave a shorter proof of a stronger (see below) version of the result.

We shall now prove that certain subsets of A'' are determining for the left topological centre of A''; after giving the statement of our result in Corollary 9.5, we shall compare our result with earlier theorems.

We shall use the following proposition.

The character space Φ_Z of the C^*-algebra $Z = LUC(G)$ was described in Chapter 5; as before, we regard G as a subset of Φ_Z. Recall that we have a continuous surjection $q_G : \Phi \to \Phi_Z$. For a subset T of G, we temporarily denote by T^* the growth of T in Φ_Z, so that $T^* = \overline{T} \setminus G \subset \Phi_Z$.

Proposition 9.1. *Let G be a locally compact, non-compact group, and set $Z = LUC(G)$. Take $U \in \mathcal{N}_e$. Then there exist an infinite cardinal κ, a sequence $(t_\alpha : \alpha < \kappa)$ in G such that the family*

$$\{Ut_\alpha : \alpha < \kappa\}$$

of subsets of G is pairwise disjoint, and elements $a, b \in T^*$ (where $T = \{t_\alpha : \alpha < \kappa\}$) with the following property: each $\mathrm{M} \in M(\Phi_Z)$ such that $L_{\mathrm{M}} \mid \overline{T} : \overline{T} \to M(\Phi_Z)$ is continuous at both a and b belongs to $M(G)$.

Proof. This is a result that is shown within the proof of [17, Theorem 12.22] (but is not stated explicitly there). ∎

We fix the objects constructed in the above proposition.

Since the family $\{Ut_\alpha : \alpha < \kappa\}$ is pairwise disjoint, we can identify $\overline{T}$ with βT. For each $\alpha < \kappa$, choose $\varphi_\alpha \in \Phi$ such that $q_G(\varphi_\alpha) = \pi(\varphi_\alpha) = t_\alpha$, and define

$$\rho : t_\alpha \mapsto \varphi_\alpha, \qquad T \to \Phi.$$

Then ρ has a continuous extension $\rho : \overline{T} \to \Phi$; we set $K = \rho(\overline{T}) \subset \Phi$. Clearly, $(q_G \,|\, K) \circ \rho$ is the identity map on $\overline{T}$. Now we choose elements $\varphi_a, \varphi_b \in K$ such that $q_G(\varphi_a) = a$ and $q_G(\varphi_b) = b$.

Proposition 9.2. *Let G be a locally compact, non-compact group, and let $\mathrm{M} \in M(\Phi)$ be such that $L_{\mathrm{M}} : \Phi \mapsto M(\Phi)$ is continuous at the two points φ_a and φ_b. Then $\pi(\mathrm{M}) \in M(G)$.*

Proof. Let (s_i) be a net in T such that $s_i \to a$ in Φ_Z. Then we have $\rho(s_i) \to \varphi_a$ in Φ, and so $\mathrm{M} \,\Box\, \rho(s_i) \to \mathrm{M} \,\Box\, \varphi_a$ in $M(\Phi)$ because the map L_{M} is continuous at a. Hence $\pi(\mathrm{M}) \,\Box\, s_i \to \pi(\mathrm{M}) \,\Box\, a$ in Φ_Z. This shows that $L_{\pi(\mathrm{M})}$ is continuous at a. Similarly, $L_{\pi(\mathrm{M})}$ is continuous at b. It follows from Proposition 9.1 that $\pi(\mathrm{M}) \in M(G)$. ■

Proposition 9.3. *Let G be a locally compact group. Let $\nu \in M(G)$ be such that*

$$\lambda \cdot \nu \in C(G)$$

for each $\lambda \in L^\infty(G)$. Then $\nu \in L^1(G)$.

Proof. This is a slight modification of [49, Theorem (35.13)], which gives the result in the case where G is compact. ■

We continue to set $A = L^1(G)$, $A'' = M(\Phi)$, and $Z = LUC(G)$.

Theorem 9.4. *Let G be a locally compact group. Let $\mathrm{M} \in M(\Phi)$ be such that*

$$\mathrm{M} \,\Box\, \delta_\varphi = \mathrm{M} \diamond \delta_\varphi \qquad (\varphi \in \Phi_{\{e\}}),$$

and, in the case where G is not compact, $\mathrm{M} \,\Box\, \delta_\varphi = \mathrm{M} \diamond \delta_\varphi$ for $\varphi \in \{\varphi_a, \varphi_b\}$. Then $\mathrm{M} \in L^1(G)$.

Proof. In the case where G is not compact, we have $\pi(\mathrm{M}) \in M(G)$ by Proposition 9.2. In the case where G is compact, we have $Z = C(G)$, and $\pi(\mathrm{M}) \in Z' = M(G)$.

Take $\lambda \in A' = L^\infty(G)$. For each $g \in A$, we have

$$\langle \pi(\mathrm{M}) \cdot \lambda, g \rangle = \langle \lambda,\, g \star \pi(\mathrm{M}) \rangle = \langle \lambda \cdot g,\, \pi(\mathrm{M}) \rangle = \langle \lambda \cdot g,\, \mathrm{M} \rangle$$

because $\lambda \cdot g \in Z$. However $\langle \lambda \cdot g,\, \mathrm{M} \rangle = \langle \mathrm{M} \cdot \lambda,\, g \rangle$ by definition, and so $\pi(\mathrm{M}) \cdot \lambda = \mathrm{M} \cdot \lambda$ in A'.

Let $\varphi \in \Phi_{\{e\}}$. Since δ_φ is a mixed identity for $M(\Phi)$, we have $\mathrm{M} \,\Box\, \delta_\varphi = \delta_\varphi \diamond \mathrm{M} = \mathrm{M}$. Since $\mathrm{M} \,\Box\, \delta_\varphi = \mathrm{M} \diamond \delta_\varphi$, we have $\mathrm{M} \diamond \delta_\varphi = \mathrm{M}$. Thus, for each $\lambda \in A'$, we have

$$\langle \lambda \cdot \pi(\mathrm{M}),\, \delta_\varphi \rangle = \langle \lambda \cdot \mathrm{M},\, \delta_\varphi \rangle = \langle \lambda,\, \mathrm{M} \diamond \delta_\varphi \rangle = \langle \lambda,\, \mathrm{M} \rangle.$$

This shows that the function $\lambda \cdot \pi(\mathrm{M})$ is constant on the fibre $\Phi_{\{e\}}$. By Proposition 3.6, $\lambda \cdot \pi(\mathrm{M})$ is continuous at e. Similarly, $\lambda \cdot \pi(\mathrm{M})$ is continuous at each point of G. By Proposition 9.3, $\pi(\mathrm{M}) \in L^1(G)$, say $\pi(\mathrm{M}) = f \in L^1(G)$. It follows that $\mathrm{M} \cdot \lambda = f \cdot \lambda$, and so

$$\langle \lambda,\, \mathrm{M} \rangle = \langle \lambda,\, \delta_\varphi \,\Box\, \mathrm{M} \rangle = \langle \mathrm{M} \cdot \lambda,\, \delta_\varphi \rangle = \langle f \cdot \lambda,\, \delta_\varphi \rangle = \langle \lambda,\, f \rangle.$$

This holds for each $\lambda \in A'$, and so $\mathrm{M} = f \in L^1(G)$. ■

Corollary 9.5.

(i) *Let G be a compact group. Then $\Phi_{\{e\}}$ is determining for the left topological centre of $L^1(G)''$.*

(ii) *Let G be a locally compact, non-compact group. Then there exist $\varphi_a, \varphi_b \in \Phi$ such that*

$$\Phi_{\{e\}} \cup \{\varphi_a, \varphi_b\}$$

is determining for the left topological centre of $L^1(G)''$. ■

We now compare our result to some earlier theorems.

First suppose that G is compact. Then the proof in [56] that, in this case, $L^1(G)$ is strongly Arens irregular actually shows that the family of right identities in $(M(\Phi), \Box)$ is determining for the left topological centre of $L^1(G)''$. In fact, by Corollary 6.3, the element δ_φ is a right identity in the algebra $(M(\Phi), \Box)$ for each $\varphi \in \Phi_{\{e\}}$, and so our result is slightly stronger.

Second, suppose that G is a locally compact, non-compact group. Then a set which is determining for the left topological centre of $L^1(G)''$ is specified in [86, Theorem 1.1]: one can choose any subset S of Φ such that $q_G(S) = \Phi_Z$. Such a set S is neither smaller nor larger than our set $\Phi_{\{e\}} \cup \{\varphi_a, \varphi_b\}$. A further paper of Filali and Salmi [31] establishes in an attractive way that $L^1(G)$ is strongly Arens irregular, and unifies this result with several related results.

After the above was written, we received (in May, 2009) the very impressive paper [7] of Budak, Işik, and Pym that proves a much stronger result in the non-compact case in their Theorem 1.2(iii), namely that, for a locally compact, non-compact group G, there are just two points $\varphi_a, \varphi_b \in \Phi$ with the property that $\{\varphi_a, \varphi_b\}$ is determining for the left topological centre of $(L^1(G)'', \Box)$. This result does not apply to compact groups, such as $\mathbb{T}$.

Let G be a compact group (such as $\mathbb{T}$). Could it be that a smaller set than $\Phi_{\{e\}}$ is sufficient to determine the topological centre of $L^1(G)$? In fact, at least in the case where G has a basis of $\mathfrak{c}$ open sets, there are at most $\mathfrak{c}$ clopen subsets of the fibre $\Phi_{\{e\}}$. Choose a point in the fibre for each such set, thus obtaining a dense subset of the fibre. The continuity argument in Proposition 3.6 still works by using just these points, so we only need $\mathfrak{c}$ points in the fibre for the above result, whereas the fibre has cardinality at least $2^{\mathfrak{c}}$. The main **question** is: *Is there always a finite or countable set S of points in $\Phi_{\{e\}}$ such that S is determining for the left topological centre of $L^1(G)$?* We are not able to decide this.

The topological centre of the measure algebra. We now turn to the topological centre question for $M(G)$.

The question whether or not $M(G)$ is strongly Arens irregular was raised by Lau in [72, Problem 11, p. 89] and Ghahramani and Lau in [34, Problem 1, p. 184]. The question was solved in the case where G is non-compact and with non-measurable cardinal by Neufang in [87, Theorem 3.5]. In fact the following theorem is proved (but not explicitly stated in our form) in [87, Theorem 3.5].

THEOREM 9.6 (Neufang). *Let G be a locally compact, non-compact group with non-measurable cardinal. Suppose that* $\mathrm{M} \in M(\widetilde{G})$ *is such that* $\mathrm{M} \,\Box\, \delta_\varphi = \mathrm{M} \diamond \delta_\varphi$ *for each* $\varphi \in \beta G_d$. *Then* $\mathrm{M} \in M(G)$. *In particular, $M(G)$ is strongly Arens irregular.* ■

Thus we can concentrate on the case where G is a compact group; our investigations have focused to no avail on the special case in which G is the unit circle $\mathbb{T}$. We shall obtain a partial result.

We shall require the following preliminary result.

PROPOSITION 9.7. *Let G be a compact, infinite, metrizable group. Then there exist an element* $\mu \in M(G)^+$ *and four sets* A_1, A_2, A_3, A_4 *in* $\mathfrak{A}_\mu$ *with* $\mu(A \cap N) > 0$ *for each* $N \in \mathcal{N}_e$ *and* $A \in \{A_1, A_2, A_3, A_4\}$, *such that*

$$\bigcup\{K_{A_j} \setminus K_{\partial A_j} : j = 1,2,3,4\} \supset \widetilde{G}_{\{e\}} \setminus \{e\}, \tag{9.1}$$

and such that

$$K_{A_1} \cap K_{A_3} = K_{A_2} \cap K_{A_4} = \{e\}. \tag{9.2}$$

Proof. Choose $\mu \in M_c(G)^+$ with the property that $\mu(N) > 0$ for each $N \in \mathcal{N}_e$ (for example, Haar measure m has this property).

The metric on G is denoted by d; for each $r \in \mathbb{R}^+$, we set

$$S_r = \{s \in G : d(s,e) = r\} \quad \text{and} \quad B_r = \{s \in G : d(s,e) < r\},$$

so that S_r and B_r are the sphere and open ball, respectively, in G of radius r around e.

Since $\{r \in \mathbb{R}^+ : \mu(S_r) > 0\}$ is a countable set, there is a sequence (r_n) in $\mathbb{R}^+$ with $r_n \searrow 0$ such that $\mu(S_{r_n}) = 0$ and $\mu(B_{r_{n+1}}) < \mu(B_{r_n})$ for each $n \in \mathbb{N}$. We note that

$$\overline{\bigcup\{S_{r_{2n}} : n \in \mathbb{N}\}} \cap \overline{\bigcup\{S_{r_{2n-1}} : n \in \mathbb{N}\}} = \{e\}. \tag{9.3}$$

For $n \in \mathbb{N}$, set $U_n = B_{r_n} \setminus B_{r_{n+1}}$, so that each U_n belongs to $\mathfrak{A}_\mu$ and $\mu(U_n) > 0$, and then set

$$A_j = \bigcup\{U_{4n+j} : n \in \mathbb{Z}^+\} \quad (j = 1,2,3,4).$$

so that each A_j belongs to $\mathfrak{A}_\mu$ and is such that $\mu(A_j \cap N) > 0$ for each $N \in \mathcal{N}_e$. It follows from (9.3) that $\bigcap\{\partial A_j : j = 1,2,3,4\} = \{e\}$, and so we have

$$\bigcup_{j=1,2,3,4} (A_j \setminus \partial A_j) = \bigcup_{n \in \mathbb{N}} U_n \setminus \bigcap_{j=1,2,3,4} \partial A_j = B_{r_1} \setminus \{e\},$$

which gives (9.1). Clearly $\overline{A_1} \cap \overline{A_3} = \overline{A_2} \cap \overline{A_4} = \{e\}$, and this gives (9.2). ■

THEOREM 9.8. *Let G be a compact, infinite, metrizable group. Then there exist four points* $\psi_1, \psi_2, \psi_3, \psi_4 \in \widetilde{G}_{\{e\}}$ *with the property that the only measures* $\mathrm{M} \in M(\widetilde{G}_{\{e\}})^+$ *such that*

$$\mathrm{M} \,\Box\, \delta_{\psi_j} = \mathrm{M} \diamond \delta_{\psi_j} \quad (j = 1,2,3,4) \tag{9.4}$$

have the form $\mathrm{M} = \zeta\delta_e$ *for some* $\zeta \in \mathbb{C}$.

Proof. We shall actually suppose further that $\mathrm{M} \in M(\widetilde{G}_{\{e\}})^+$ is such that $\mathrm{M}(\{e\}) = 0$, and shall show that $\mathrm{M} = 0$; this is sufficient for the result.

Let $\mu \in M(G)^+$, and take the four sets A_1, A_2, A_3, A_4 to be as specified in Proposition 9.7. For $j = 1, 2, 3, 4$, we have $\mu(A_j \cap N) > 0$ for each $N \in \mathcal{N}_e$, and so there exists $\psi_j \in \widetilde{G}_{\{e\}} \cap \Phi_\mu$ such that $A_j \in \psi_j$.

By (9.1), it suffices to prove that $\mathrm{M}(K_A \setminus K_{\partial A}) = 0$ for each $A \in \{A_1, A_2, A_3, A_4\}$; we fix such a set A, and replace the measure M by the restriction $\mathrm{M} \,|\, (K_A \setminus K_{\partial A})$. By (9.2), there exists $B \in \{A_1, A_2, A_3, A_4\}$ with $K_A \cap K_B = \emptyset$; the element of $\{\psi_1, \psi_2, \psi_3, \psi_4\}$ corresponding to B is ψ.

To obtain a contradiction, we may suppose that we have $\mathrm{M}(K_A) \neq 0$, and hence that $\langle \mathrm{M},\, 1\rangle > 0$; by replacing M by $\mathrm{M}/\langle \mathrm{M},\, 1\rangle$, we may suppose that $\langle \mathrm{M},\, 1\rangle = 1$. It follows from (6.11) in Theorem 6.9 that

$$\langle \chi_{K_A},\, \mathrm{M} \,\Box\, \delta_\psi\rangle = \langle \chi_{K_A},\, \mathrm{M}\rangle = \mathrm{M}(K_A).$$

Since $\psi \in \widetilde{G}_{\{e\}} \setminus K_A$ and $\langle \mathrm{M},\, 1\rangle = 1$, it follows from (6.12) that

$$\langle \chi_{K_A},\, \mathrm{M} \diamond \delta_\psi\rangle = \langle \chi_{K_A},\, \delta_\psi\rangle,$$

and so $\langle \chi_{K_A},\, \mathrm{M} \diamond \delta_\psi\rangle = 0$ because $\psi \notin K_A$.

Since $\mathrm{M} \diamond \delta_\psi = \mathrm{M} \,\Box\, \delta_\psi$, we have $\mathrm{M}(K_A) = 0$. ∎

We note that, in the special case where the group G is totally disconnected, two points ψ_1, ψ_2 suffice for the above argument to apply.

We now consider the case where G might not be metrizable.

THEOREM 9.9. *Let G be a compact, infinite group. Then the only measure $\mathrm{M} \in M(\widetilde{G}_{\{e\}})^+$ such that $\mathrm{M} \,\Box\, \delta_\psi = \mathrm{M} \diamond \delta_\psi$ for each $\psi \in \widetilde{G}_{\{e\}}$ has the form $\mathrm{M} = \zeta\delta_e$ for some $\zeta \in \mathbb{C}$.*

Proof. For each $U \in \mathcal{N}_e$, there is a closed, normal subgroup N of G such that $N \subset U$ and $H := G/N$ is a compact, infinite, metrizable group [48, Theorem (8.7)]. The quotient map is $\eta : G \to H$, and there is an induced continuous homomorphism $\overline{\eta} : M(G) \to M(H)$. We have $\overline{\eta}(\mathrm{M}) \,\Box\, \delta_\psi = \overline{\eta}(\mathrm{M}) \diamond \delta_\psi$ for each $\psi \in \widetilde{H}_{\{e\}}$, and so, by Theorem 9.8, $\overline{\eta}(\mathrm{M}) \in \mathbb{C}\delta_{e_H}$. It follows that $\operatorname{supp} \mathrm{M} \subset U$.

However this holds for each $U \in \mathcal{N}_e$, and so $\operatorname{supp} \mathrm{M} = \{e_G\}$, as required. ∎

Clearly the above results are unsatisfactory, in that they leave open the question that motivated our work.

In fact, the question of the strong Arens irregularity of $M(G)$ has been resolved by V. Losert, M. Neufang, J. Pachl, and J. Steprāns with their exciting proof [82] of the following result.

THEOREM 9.10. *Let G be a locally compact group. Then $M(G)$ is strongly Arens irregular.* ∎

10. Open problems

We list here some problems that we believe are open.

1. Let X be a compact space such that $C(X)$ is isometrically isomorphic to the second dual space of a Banach space. Is it necessarily true that there is a locally compact space Ω such that $X = \widetilde{\Omega}$? Which hyper-Stonean spaces X are such that $C(X) = F''$ for some Banach space F? For some partial results, see Proposition 4.27 and Theorem 4.29.
2. Let A be a commutative Lau algebra such that A' is a commutative von Neumann algebra. We have
$$X_A \subset AP(A) \subset WAP(A) \subset A'.$$
When are $AP(A)$ and $WAP(A)$ C^*-subalgebras of A'? When does $X_A = AP(A)$? In particular, let G be a locally compact group, so that
$$X_G \subset AP(G) \subset AP(M(G)) \subset WAP(M(G)) \subset M(G)' = C(\widetilde{G}).$$
Now $AP(M(G))$ and $WAP(M(G))$ are C^*-subalgebras of the space $M(G)'$ [21]. When is it true that $AP(G) = AP(M(G))$? Does this imply that G is discrete? It is shown in [103] that the method of Daws in [21] does not extend directly to all such cases.
3. Let G be a locally compact group. Do $WAP(M(G))$ or $AP(M(G))$ always have a topological invariant mean. If so, is it unique?
4. Suppose that G and H are locally compact groups and that $(WAP(M(G))', \Box)$ and $(WAP(M(H))', \Box)$ are isometrically isomorphic. Are G and H then isomorphic?
5. Let G be a locally compact group. Can we find two points φ and ψ in $\widetilde{G}$ such that $\delta_\varphi \,\Box\, \delta_\psi$ is not a point mass, but such that it is a finite sum of point masses in $M(\widetilde{G})$?
6. Let G be a compact group. We have shown in Corollary 9.5(i) that $\Phi_{\{e\}}$ is determining for the left topological centre of $(L^1(G)'', \Box)$. Is there a finite or countable subset V of $\Phi_{\{e\}}$ such that V is so determining?
7. Let G be a compact group. Is $\widetilde{G}$ determining for the left topological centre of $M(G)''$? If so, is there a 'small' subset of $\widetilde{G}$ that is so determining?
8. Let G be a locally compact, non-compact group. Is there a 'small' subset of $\widetilde{G}$ that is determining for the left topological centre of $M(G)''$?

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