Introduction

One of the main results of last decades in algebraic geometry was the foundation of the Minimal Model Program, or MMP, and its proof in dimension three. Minimal Model Theory shed a new light on what is nowadays called higher dimensional geometry. In mathematics high numbers are really a matter of circumstances and here we mean greater than or equal to 3. The impact of MMP has been felt in almost all areas of algebraic geometry. In particular the philosophy and some of the main new objects like extremal rays, Fano–Mori contractions or spaces and log varieties started to play around and give fruitful answer to different problems.

The aim of the Minimal Model Program is to choose, inside a birational class of varieties, “simple” objects. The first main breakthrough of the theory is the definition of these objects: minimal models and Mori spaces. This is related to numerical properties of the intersection of the canonical class of a variety with effective cycles. After this, old objects, like the Kleiman cone of effective curves and rational curves on varieties, acquire a new significance. New ones, like Fano–Mori contractions, start to play an important role. And the tools developed to tackle these problems allow the study of formerly untouchable varieties.

Riemann surfaces were classified, in the XIXth century, according to the curvature of a holomorphic metric. Or, in other words, according to the Kodaira dimension. Surfaces needed a harder amount of work. For the first time birational modifications played an important role. The theory of (-1)-curves studied by the Italian school of Castelnuovo, Enriques and Severi, at the beginning of XXth century, allowed one to define minimal surfaces. Then the first rough classification of the latter, again by Kodaira dimension, was obtained. The Minimal Model Program is now a tool to start investigating this question in dimension 3 or higher.

In these notes we want to present our point of view on this area of research. We are not trying to give a treatment of the whole subject. Very nice books appeared recently for this purpose, and we often refer to them in the paper. We would like to present, in a sufficiently self-contained way, our contributions and interests in this field of mathematics. We will study Fano–Mori spaces both from the biregular and birational point of view. For the former we will recall and develop Kawamata’s base point free technique and some of Mori’s deformation arguments. For the latter we lean on the Sarkisov and #-Minimal Model Programs.

The content of the consecutive parts is the following. In Part 1 we collect the main

Part 2 gets the reader acquainted with the base point Free technique, BPF. For this purpose we give, or sketch, proofs of Kawamata's base point free theorem, trying to hide the technicalities.

In Part 3 we introduce the main actor of the book, Fano–Mori contractions and more generally Fano–Mori spaces. Using BPF we then describe the main properties that will allow us to study them.

Part 4 contains applications of all the above to smooth varieties. Namely we give a biregular classification of Fano–Mori spaces of dimension less than or equal to four and Mukai manifolds.

In Part 5 we present the other side of the moon, the birational world. Here a beautiful old theorem of Castelnuovo and Noether is proved in a modern language. Philosophy and applications of the Minimal Model Program for 3-folds are outlined.

These notes collect some topics we presented in three mini-courses which were held in Wykno (Pl) (1999), Recife (Br) (2000) and Ferrara (It) (2000), respectively. We discussed this subject with many people and we are grateful to them all. But we would like to distinguish Jarosław Wiśniewski and thank him deeply.

Part 1. Preliminaries

In this part we collect all definitions which are more or less standard in the algebraic geometry realm in which we live.

1.1. The Kleiman–Mori cone of a projective variety. First we fix a good category of objects (real differentiable varieties are not the good ones to extend the Riemann and Poincaré approach). Let X be a normal variety over an algebraically closed field k of dimension n, that is, an integral separated scheme which is of finite type over k. We actually also assume that $\mathrm{char}(k) = 0$; nevertheless many results at the beginning of the theory also hold in the case of positive characteristic.

We have to introduce some basic objects on X.

Let $\mathrm{Div}(X)$ be the group of Cartier divisors on X and $\mathrm{Pic}(X)$ be the group of line bundles on X. Let also $Z^1(X)$ be the group of Weil divisors and $Z_1(X)$ be the group of 1-cycles on X, i.e. the free abelian group generated, respectively, by prime divisors and reduced irreducible curves.

We will often use $\mathbb{Q}$-*Cartier divisors*, that is, linear combinations with rational coefficients of Cartier divisors. For these objects it is useful to introduce the following notations. Let $D = \sum d_i D_i \in \mathrm{Div}(X) \otimes \mathbb{Q}$ be a $\mathbb{Q}$-Cartier divisor. Then $\lfloor D \rfloor := \sum \lfloor d_i \rfloor D_i$, $\lceil D \rceil := -\lfloor -D \rfloor$ and $\langle D \rangle := D - \lfloor D \rfloor$, where $\lfloor d_i \rfloor$ is the integral part of d_i.

Then there is a pairing

$$\mathrm{Pic}(X) \times Z_1(X) \to \mathbb{Z}$$

defined, for an irreducible reduced curve $C \subset X$, by $(L, C) \to L \cdot C := \deg_C(L_{|C})$, and

Two invertible sheaves $L_1, L_2 \in Pic(X)$ are *numerically equivalent*, denoted by $L_1 \equiv L_2$, if $L_1 \cdot C = L_2 \cdot C$ for every curve $C \subset X$. Similarly, two 1-cycles C_1, C_2 are *numerically equivalent*, $C_1 \equiv C_2$, if $L \cdot C_1 = L \cdot C_2$ for every $L \in \mathrm{Pic}(X)$. Define

$$N^1X = (\mathrm{Pic}(X)/{\equiv}) \otimes \mathbb{R} \quad \text{and} \quad N_1X = (Z_1(X)/{\equiv}) \otimes \mathbb{R};$$

obviously, by definition, $N^1(X)$ and $N_1(X)$ are dual $\mathbb{R}$-vector spaces and $\equiv$ is the smallest equivalence relation for which this holds.

In particular for any divisor $H \in \mathrm{Pic}(X)$ we can view the class of H in $N^1(X)$ as a linear form on $N_1(X)$. We will use the following notation:

$$H_{\geq 0} := \{x \in N_1(X) : H \cdot x \geq 0\} \quad \text{and similarly for } > 0, \leq 0, < 0$$

and

$$H^{\perp} := \{x \in N_1(X) : H \cdot x = 0\}.$$

The fact that $\varrho := \dim_{\mathbb{R}} N^1(X)$ is finite is the Néron–Severi theorem [GH, p. 461]. The natural number ϱ is called the *Picard number* of the variety X. (Note that for a variety defined over $\mathbb{C}$ the finite dimensionality of $N_1(X)$ can be read off from the fact that $N_1(X)$ is a subspace of $H_2(X, \mathbb{R})$.)

More generally, if $f : X \to Y$ is a projective morphism and $A, B \in \mathrm{Div}(X) \otimes \mathbb{Q}$, then A is f*-numerically equivalent* to B ($A \equiv_f B$) if $A \cdot C = B \cdot C$ for any curve C contracted by f; and A is f*-linearly* equivalent to B ($A \sim_f B$) if $A - B \sim f^*M$ for some line bundle $M \in \mathrm{Pic}(Y)$. We will suppress the subscript when no confusion is likely to arise.

Note that if X is a surface then $N^1(X) = N_1(X)$; using M. Reid's words (see [Re4]): “Although very simple, this is one of the key ideas of Mori theory, and came as a surprise to anyone who knew the theory of surfaces before 1980: the quadratic intersection form of the curves on a nonsingular surface can for most purposes be replaced by the bilinear pairing between N^1 and N_1, and in this form generalizes to singular varieties and to higher dimension.”

We also notice that *algebraic equivalence* (see [GH, p. 461]) of 1-cycles implies numerical equivalence. Moreover, if X is a variety over $\mathbb{C}$ then, in terms of Hodge theory, $N^1(X) = (H^2(X, \mathbb{Z})/(\mathrm{Tors}) \cap H^{1,1}(X)) \otimes \mathbb{R}$.

We denote by $\mathrm{NE}(X) \subset N_1(X)$ the cone of effective 1-cycles, that is

$$\mathrm{NE}(X) = \Big\{ C \in N_1(X) : C = \sum r_i C_i \text{ where } r_i \in \mathbb{R},\, r_i \geq 0 \Big\},$$

where C_i are irreducible curves. Let $\overline{\mathrm{NE}(X)}$ be the closure of $\mathrm{NE}(X)$ in the real topology of $N_1(X)$. This is called the *Kleiman–Mori cone*.

We also use the following notation:

$$\overline{\mathrm{NE}(X)}_{H \geq 0} := \overline{\mathrm{NE}(X)} \cap H_{\geq 0} \quad \text{and similarly for } > 0, \leq 0, < 0.$$

One effect of taking the closure is the following trivial observation, which has many important applications: if $H \in N^1(X)$ is positive on $\overline{\mathrm{NE}(X)} \setminus 0$ then the section $(H \cdot z = 1) \cap \overline{\mathrm{NE}(X)}$ is compact. Indeed, the projectivisation of the closed cone $\overline{\mathrm{NE}(X)}$ is a closed subset of $\mathbb{P}^{\varrho - 1} = P(N_1(X))$, and therefore compact, and the section $(H \cdot z = 1)$ projects

An element $H \in N^1(X)$ is called *numerically eventually free* or *numerically effective*, for short *nef*, if $H \cdot C \geq 0$ for every curve $C \subset X$ (in other words if $H \geq 0$ on $\overline{\mathrm{NE}(X)}$).

The relation between nef and ample divisors is the content of the following Kleiman criterion that is a cornerstone of Mori theory.

Theorem 1.1.1 ([Kle]). *For $H \in \mathrm{Pic}(X)$, view the class of H in $N^1(X)$ as a linear form on $N_1(X)$. Then*

$$H \text{ is ample } \Leftrightarrow \ HC > 0 \text{ for all } C \in \overline{\mathrm{NE}(X)} \setminus \{0\}.$$

In other words the theorem says that the cone of ample divisors is the interior of the nef cone in $N^1(X)$, that is, the cone spanned by all nef divisors.

Note that it is not true that $HC > 0$ for every curve $C \subset X$ implies that H is ample (see for instance [CKM, Example 4.6.1]). The condition in the theorem is stronger.

This is only a weak form of Kleiman's criterion, since X is a priori assumed to be projective. The full strength of Kleiman's criterion gives a necessary and sufficient condition for ampleness in terms of the geometry of $\overline{\mathrm{NE}(X)}$.

Assume that X is smooth and denote by K_X the *canonical divisor* of X, that is, an element of $\mathrm{Div}(X)$ such that $\mathcal{O}_X(K_X) = \Omega_X^n$, where Ω_X is the sheaf of one-forms on X.

The first main theorem of Mori theory is the following description of the negative part, with respect to K_X, of the Kleiman–Mori cone. We recall that, by definition, a *rational curve* is an irreducible, reduced curve defined over k whose normalization is $\mathbb{P}^1$.

Theorem 1.1.2 ([Mo3], cone theorem). *Let X be a non-singular projective variety.*

(1) *There are countably many rational curves $C_i \subset X$ such that $0 < -C_iK_X \leq \dim X + 1$ and*

$$\overline{\mathrm{NE}(X)} = \overline{\mathrm{NE}(X)}_{K_X \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i].$$

(2) *For any $\varepsilon > 0$ and ample divisor H,*

$$\overline{\mathrm{NE}(X)} = \overline{\mathrm{NE}(X)}_{K_X + \varepsilon H \geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i].$$

In simple words the theorem says the following. Consider the linear form on $N_1(X)$ defined by K_X; the part of the Kleiman–Mori cone $\overline{\mathrm{NE}(X)}$ which sits in the negative semi-space defined by K_X (if not empty) is locally polyhedral and it is spanned by a countable number of *extremal rays*, $\mathbb{R}_{\geq 0}[C_i]$. Moreover each extremal ray is spanned in $N_1(X)$ by a rational curve with bounded intersection with the linear form $-K_X$, and if we move an ε away from the hyperplane $K_X = 0$ (in the negative direction) the number of extremal rays becomes finite.

There are essentially two ways of proving this theorem; the original one, which is due to Mori, is very geometric and valid in any characteristic. It is presented in [Mo3] and in many other places, for example in [KM2] and [De]. It is based on the study of deformations of a rational curve on an algebraic variety, and makes use of the theory of Hilbert schemes and of theorems like for instance 4.3.13.

Another proof was provided by Y. Kawamata ([Ka0]); it gives the cone theorem as a

THEOREM 1.1.3 ([KMM, 4.1.1], rationality theorem). *Let X be an n-dimensional variety defined over $\mathbb{C}$ which is smooth or, more generally with LT singularities (see Definition 2.2.1), for which K_X is not nef. Let L be an ample line bundle on X and define the nef value (or nef threshold) of the pair (X, L) by*

$$r = \inf\{t \in \mathbb{R} : K_X + tL \text{ is nef}\}.$$

Then the nef value is a rational number. Moreover if $a := \min\{e \in \mathbb{N} : eK_X \text{ is Cartier}\}$, and $ar := v/u$ with $(v, u) = 1$, then $v \leq a(n + 1)$.

The proof uses the base point free theorem which we will introduce in the next section. In particular it makes use of vanishing theorems and it is therefore valid only in characteristic zero.

It was noticed by M. Reid and Y. Kawamata that the rationality theorem and the base point free theorem imply immediately Mori's cone theorem, in the more general case of varieties with LT singularities.

A very nice presentation of the above theorems (Kleiman–Mori–Kawamata), together with complete proofs, in the *case of surfaces* is in [Re4, Chapter D]. This material can be presented in a few hours (3–4) to an audience with a limited knowledge of basic algebraic geometry and it can provide a good insight in the field; this is our experience at the Ferrara course.

The surface case is a perfect tutorial case in order to understand the Minimal Model Program. This was first pointed out by S. Mori who worked out a complete description of extremal rays in the case of a smooth surface (see [Mo3, Chapter 2] and also [KM2, pp. 21–23, §1.4]). Moreover he also showed how it is possible to associate to each extremal ray a morphism from the surface. When the ray is spanned by a rational curve with self-intersection -1, this is a celebrated theorem of Castelnuovo. Castelnuovo's proof is also very enlightening and it can be found in [Be, Theorem II.17], or in [Ha, Theorem V. 5.7].

1.2. Fujita Δ-genus. A classical approach to the classification of projective varieties, which dates back to the Italian school, consists of the following: (a) take a hyperplane section, (b) characterise it by induction, (c) describe the original variety by ascending the properties of the hyperplane section. To stress its classical flavor T. Fujita called it the *Apollonius method*; we will now introduce some definitions and techniques as presented in the work of T. Fujita ([Fu2]); see also Section 4.3.2.

DEFINITION 1.2.1. Let F be a variety of dimension d and let L be an ample line bundle on F. The pair (F, L) is called a *polarized variety*. We will denote by

$$\chi(F, tL) = \sum \chi_j \frac{t(t+1)\dots(t+j-1)}{j!}$$

the Hilbert polynomial of (F, L). The χ_j's are integers and $\delta(F, L) := \chi_n = L^d > 0$ is called the *degree* of (F, L), while $g(F, L) := 1 - \chi_{n-1}$ is called the *sectional genus*. The Δ-*genus* of (F, L) is defined by the formula

Definition 1.2.2. Let (F, L) be a polarized variety. Let D be a member of $|L|$ and suppose that D, as a subscheme of F, is irreducible and reduced. In such a case D is called a *rung* of (F, L). Let $r : H^0(F, L) \to H^0(D, L_D)$ be the restriction map. If r is surjective the rung is said to be *regular*.

A sequence $F = F_d \supset F_{d-1} \supset \ldots \supset F_1$ of subvarieties of F such that F_i is a rung (resp. a regular rung) of (F_{i+1}, L_{i+1}) is called a *ladder* (resp. a *regular ladder*).

Remark 1.2.3. If D is a rung then the pair (D, L_D) is a polarized variety of dimension $d-1$. The structure of (F, L) is reflected in that of (D, L_D). One can study (F, L) via (D, L_D) using induction on d. This is the main idea of the Apollonius method. In particular, $\chi(D, tL_D) = \chi(F, tL) - \chi(F, (t-1)L)$, $g(D, L_D) = g(F, L)$, $\delta(D, L_D) = \delta(F, L)$ and $\Delta(F, L) - \Delta(D, L_D) = \dim \operatorname{Coker}(r)$. If the rung is regular the two Δ-genera are the same.

In classical geometry the number $\dim \operatorname{Coker}(r)$ was called the *deficiency*.

Assume that L is very ample and let φ_L be the map associated to the elements of the complete linear system $|L|$. Then it is a classical result that $\Delta(F, L) \geq 0$ and equality holds for the so called "Varieties of Minimal Degree" [GH, p. 173]. These varieties are classified as projective spaces, hyperquadrics, scrolls over rational normal curves or generalised cones over them.

In the case of surfaces a precise statement is the following:

Proposition 1.2.4. *Let (S, L) be a pair with S a surface and L an ample line bundle on S. If $\Delta(S, L) = 0$ then the pair is one of the following*:

(1) $(\mathbb{P}^2, \mathcal{O}(e))$ *with* $e = 1, 2$,
(2) $(\mathbb{F}_r, C_0 + kf)$ *with* $k \geq r + 1$, $r \geq 0$,
(3) $(\mathbf{S}_r, \mathcal{O}_{\mathbf{S}_r}(1))$ *with* $r \geq 2$.

Here $\mathbb{F}_r$ is a Hirzebruch surface, i.e. a $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}(r) \oplus \mathcal{O})$ over the projective line $\mathbb{P}^1$ with a unique section $C_0 \subset \mathbb{F}_r$ (isomorphic to $\mathbb{P}^1$) such that $C_0^2 = -r \leq 0$ and a fiber of the projection $\mathbb{F}_r \to \mathbb{P}^1$ which we will denote by f. While $\mathbf{S}_r$ is a (normal) cone defined by contracting $C_0 \subset \mathbb{F}_r$ to a normal point; in terms of projective geometry $\mathbf{S}_r$ is a cone over $\mathbb{P}^1 \hookrightarrow \mathbb{P}^r$ embedded via the Veronese map (r-uple embedding). The restriction of the hyperplane section line bundle from $\mathbb{P}^{r+1}$ to $\mathbf{S}_r$ will be denoted by $\mathcal{O}_{\mathbf{S}_r}(1)$.

Exercise 1.2.5. Prove the classification of surfaces of minimal degree. The first step consists in showing that if a line meets a surface of minimal degree in three or more points then it lies on the surface (see for instance [GH, p. 525]).

A modern approach to this classification which extends to the case when L is merely ample is due to T. Fujita [Fu2].

Proposition 1.2.6. *Let (F, L) be a polarized variety and assume that there exists a ladder for this pair. Then $\Delta(F, L) \geq 0$ (this is actually always true, without the assumption of the existence of a ladder). If moreover the ladder is regular and for a divisor $D_1 \in |L_{|F_1}|$ the map $H^0(F_1, L_{|F_1}) \to H^0(D_1, L_{|D_1}) = \mathbb{C}^\delta$ is surjective (we will call this a complete regular ladder) then $\Delta(F, L) = 0$ and the pair (F, L) is a variety of minimal degree; in*

Proof. The proof follows immediately from the above observations plus the fact that the surjectivity of $H^0(F_1, L_1) \to H^0(D_1, L_{D_1}) = \mathbb{C}^\delta$ implies that D_1 is a rational normal curve. ∎

EXERCISE 1.2.7. Let F_1 be a curve and L a line bundle on F_1. Assume that $D \in |L|$ is an effective divisor such that $H^0(F_1, L) \to H^0(D, L_D) = \mathbb{C}^\delta$ is surjective. Prove that F_1 is a rational normal curve, when embedded by $|L|$.

Part 2. Base point free technique

In this part we introduce the base point free technique (for short BPF). This theory has been mainly developed by Kawamata, Reid, Shokurov in a series of papers (see [KMM], [Ko2] and [Ka3]). The aim of BPF is to show that an adjoint linear system, under some conditions, is free from fixed points. In the first section we will try to spare the reader a too technical approach, giving the main ideas and results, without too many definitions and details. The latter are left for the interested reader, together with examples and exercises.

2.1. Base point freeness. We start with the easy case of a curve: let C be a compact Riemann surface of genus g and let K_C be the canonical bundle of C. To give a morphism $C \to \mathbb{P}^N$ is equivalent to giving a line bundle H without base points. For this we have the well known

THEOREM 2.1.1. *If* $\deg H \geq 2g$ *then* H *has no base point.*

Proof. Let $L := H - K_C$ and let $x \in C$ be a point on C. Note that by assumption $\deg L \geq 2$ and thus

(2.1.1) $$H^1(C, K_C + L - x) = H^0(C, x - L) = 0,$$

the first equality coming from Serre duality.

Then we consider the exact sequence

$$0 \to \mathcal{O}_C(K_C + L - x) \to \mathcal{O}_C(K_C + L) \to \mathcal{O}_x(K_C + L) \to 0,$$

which comes by tensoring the structure sequence of x on C,

$$0 \to \mathcal{I}_x \to \mathcal{O}_C \to \mathcal{O}_x \to 0,$$

by the line bundle $K_C + L$.

The sequence gives rise to a long exact sequence in cohomology whose first terms are (keep in mind equation (2.1.1))

$$0 \to H^0(C, K_C + L - x) \to H^0(C, K_C + L) \to H^0(x, K_C + L) \to 0.$$

In particular we have the surjective map

$$H^0(C, H) \xrightarrow{\alpha} H^0(x, H) \to 0.$$

Furthermore x is a closed point and therefore

The surjectivity of α translates into the existence of a section of $\mathcal{O}(H)$ which is not vanishing at x. That is the pull back via α of 1. ∎

What we have done can be summarized in the following slogan, which is somehow the manifesto of the base point free technique.

Construct a section of an adjoint line bundle proving a vanishing statement, (2.1.1), *and a non-vanishing statement on a smaller dimensional variety,* (2.1.2).

What happens if we try to generalise this to higher dimensional varieties and which problems shall we encounter?

Simple observation: the point $x \in C$ is a smooth Cartier divisor, that is why with an abuse of language we wrote $H^i(C, K_C + L - x)$. This is no more true for a point on a variety X of higher dimension.

Let $x \in X$ be a point of a smooth projective variety X of dimension n. If we just want to mimic the above arguments, then in (2.1.1) we are concerned with cohomology groups of non-locally free sheaves which are difficult to interpret. Note that there is a way to make a divisor out of a point: blow it up! Do it and get a morphism $\pi : Y \to X$ with exceptional divisor E and

$$\pi^* K_X = K_Y - (n-1)E.$$

Assume now that we want to prove that x is not a base point of a divisor of the type $H := K_X + L$; we pull back the divisors on Y and we have an exact sequence, coming from the structure sequence associated to E, of the type

$$H^0(Y, \pi^*(K_X + L)) \to H^0(E, \pi^*(K_X + L)) \to H^1(Y, \pi^*(K_X + L) - E).$$

Since $H^0(Y, \pi^*(K_X+L)) = H^0(X, K_X+L)$ (Hartogs theorem) and $H^0(E, \pi^*(K_X+L)) = \mathbb{C}$, we have to prove "only" the vanishing of

$$H^1(Y, \pi^*(K_X + L) - E) = H^1(Y, K_Y + \pi^* L - nE).$$

This is of course in general not true and one has to choose carefully good assumptions on L to have a vanishing theorem of this type; let us state the best available version of it (without generalising it to a relative or to a singular situation) which is due to Kawamata–Viehweg (they worked on previous versions of Enriques, Kodaira, Ramanujan, ...)

THEOREM 2.1.2 (Vanishing theorem, see [KMM] or [EV]). *Let X be a smooth variety and let $D = \sum a_i D_i$ be a $\mathbb{Q}$-Cartier divisor satisfying the following conditions*:

(i) *D is nef and big, that is D, is nef and $D^n > 0$, where $n := \dim X$.*

(ii) *$\langle D \rangle$ (notation of Section 1.1) has support with only normal crossings (that is, each D_i is smooth and they intersect everywhere transversally).*

Then

$$H^j(X, K_X + \lceil D \rceil) = 0 \quad \text{for } j > 0.$$

Let us show how to use this vanishing theorem under a *very special hypothesis.*

Assume that L is ample (or nef and big) and that we can find a divisor $D_1 \in |L|$ such that *with* $c := n/a < 1$ *and* $\widetilde{D}_1$ *smooth. For instance assume that the only singularity of* D_1 *is an ordinary* $(n+1)$*-uple point at* x*. Then* x *is not a base point of* $K_X + L$*.*

Proof. Note that, $\pi^*(L) - \delta E := A$ is ample for every $0 < \delta \ll 1$. Then we can write

$$\pi^*(K_X + L) - E \equiv K_Y + c\widetilde{D}_1 + caE - nE + (1-c)A + (1-c)\delta E,$$

equivalently

$$\pi^*(K_X + L) - E - (1-c)\delta E - c\widetilde{D}_1 - K_Y$$

is an ample $\mathbb{Q}$-divisor on Y.

We can apply the vanishing theorem on Y and conclude that

$$H^1(Y, \pi^*(K_X + L) - E) = 0$$

since $\lceil -(1-c)\delta E - c(\widetilde{D}_1) \rceil = 0$. Thus

$$H^0(Y, \pi^*(K_X + L)) \to H^0(E, \pi^*(K_X + L)) \simeq \mathbb{C}$$

is surjective and $x \notin \mathrm{Bsl}(K_X + L)$. ■

Unfortunately it is very unlikely that our special hypothesis is satisfied. Now comes the moment to give a precise general statement and to outline its proof.

THEOREM 2.1.3 (Base point freeness, [Sh1], [Ka0] or [KMM]). *Let* X *be a variety of dimension* n, *with "good singularities"* (*i.e. smooth or LT singularities*, *see Definition* 2.2.1) *and* H *a Cartier divisor. Assume that* H *is nef and* $aH - K_X =: L$ *is ample for some* $a \in \mathbb{N}$*. Then for* $m \gg 0$ *the line bundle* mH *is generated by global sections*, *i.e. there exists an integer* m_0 *and a regular map* $\varphi : X \to W$ *given by elements in* $H^0(X, mH)$ *for any* $m \geq m_0$*.*

REMARK 2.1.4. The above theorem was proved by Y. Kawamata and V. V. Shokurov (see [Ka0] and [Sh1]) by a method which builds up from the classical methods of the Italians and which was developed in the case of surfaces by Kodaira–Ramanujan–Bombieri.

A very significant step in the understanding and spreading out of the technique was given in a beautiful paper of M. Reid (see [Re1]) which we strongly recommend to the reader.

This type of results are fundamental in algebraic geometry and they are constantly under improvement; recently important steps were achieved by Kawamata, Shokurov, Kollár, Ein-Lazarsfeld and others.

A big drawback is that the method, as it stands, is not effective, i.e. it does not give a good bound for m (in contrast to the case of curves and surfaces). Some bound can however be achieved, namely one can show that $m_0 \leq 2(n+2)!(a+n)$ (effective base point freeness: see [Ko4]). We will only outline the proof and we refer to [KMM] for many technical, and often very relevant, parts which we now state and briefly comment.

First we observe that the "perfect" assumptions we have given above are difficult to achieve in general. So more than one blow up is required and for this we need the following.

DEFINITION 2.1.5. For a pair (X, H) of a variety X and a $\mathbb{Q}$-divisor H, a *log resolution* is a proper birational morphism $f : Y \to X$ from a smooth variety Y such that the union

Theorem 2.1.6. *Let X be a variety with LT singularities, B an effective and nef $\mathbb{Q}$-divisor and L an ample divisor on X. Then there exists a log resolution $f : Y \to X$ such that*

$$K_Y = f^* K_X + \sum e_i E_i, \quad f^*(B) = B' + \sum b_i E_i, \quad f^*(L) = A + \sum p_i E_i,$$

where all relevant divisors in Y are smooth and normal crossing, all E_i are exceptional, A is an f-ample $\mathbb{Q}$-divisor, $0 \leq p_i \ll 1$ and $e_i > -1$.

The theorem follows essentially from the work of Hironaka on resolution of singularities. The statement on the e_i is the definition of LT singularities (see Definition 2.2.1) while the ampleness of A is usually called Kodaira's lemma; for a proof see [KMM, Corollary 0.3.6].

Using a log resolution instead of the blow-up we will achieve our assumption but we will very likely loose the non-vanishing part (namely $H^0(E, \pi^*(K_X + L)) = \mathbb{C}$). For this we need the next important result, due to V. V. Shokurov.

Theorem 2.1.7 (Non-vanishing theorem). *Let X be a non-singular projective variety; let N be a Cartier divisor and A a $\mathbb{Q}$-divisor on X such that:*

(i) *N is nef.*
(ii) *$\lceil A \rceil \geq 0$ and $\langle A \rangle$ has support with only normal crossings.*
(iii) *$dN + A - K_X = M$ where M is nef and big, for some positive $d \in N$.*

Then $H^0(X, mN + \lceil A \rceil) \neq \emptyset$ for all $m \gg 0$.

A proof of this theorem can be found in [KMM, 2.1.1]. It is a combination of the Riemann–Roch formula and the vanishing theorem 2.1.2.

Sketch of the proof of 2.1.3. By the non-vanishing theorem there exists an effective divisor $B \in |mH|$ for all $m \geq m_0 \gg 0$.

Noetherian argument: Let $B(\gamma)$ denote the reduced base locus of $|\gamma H|$. Clearly $B(\gamma^s) \subseteq B(\gamma^t)$ for any positive integers $s > t$. Noetherian induction implies that the sequence $B(\gamma^i)$ stabilises and we call the limit B_γ. So either B_γ is non-empty for some γ or B_γ and $B_{\gamma'}$ are empty for two relatively prime integers γ and γ'. In the latter case, take s, t such that $B(\gamma^s)$ and $B((\gamma')^t)$ are empty and use the fact that every sufficiently large integer is a linear combination of γ^s and $(\gamma')^t$ with non-negative coefficients to conclude that $|mH|$ is base point free for all $m \gg 0$.

So we must show that the assumption that some B_γ is non-empty leads to a contradiction. Let $m = \gamma^s$ be such that $B_\gamma = B(m)$ and assume that this is not empty.

With L as in the statement of the theorem and B as at the beginning of the proof, let e_i, b_i, p_i be as in Theorem 2.1.6 and define

$$c := \min \left\{ \frac{e_i + 1 - p_i}{b_i} \right\}.$$

By taking m large enough we can assume that there exists a divisor $B \in |mH|$ with arbitrarily high multiplicity along B_γ, in other words $0 < c < 1$. By changing the coefficients p_i a little we can assume that the minimum is achieved for exactly one index. Denote the

By the Bertini theorem we can assume that Z is contained in the base locus of mH, i.e. in B_γ. Then

$$K_Y + A + cB' + \sum (cb_i - e_i + p_i)E_i + f^*(m - cm)H \equiv f^*(K_X + L + mH) = f^*(m + a)H$$

and

$$\sum (cb_i - e_i + p_i)E_i = E_0 - D + \mathrm{Fr}$$

where E_0, D are effective divisors without common irreducible components and Fr is the fractional divisor with rational coefficients between 0 and 1, defined by $\mathrm{Fr} = \sum \{cb_i - e_i + p_i\}E_i$, where $\{r\}$ is the fractional part of the rational number r. Thus

$$f^*((m + a)H) + D - E_0 - Fr - cB' - K_Y \equiv A + f^*(m - cm)H$$

is ample. Write $N(m) := f^*((m + a)H) + D$ for brevity; by the vanishing theorem we then have

$$H^i(Y, N(m) - E_0) = 0 \quad \text{for } i > 0$$

(incidentally observe that also the following vanishing is true: $H^i(E_0, N(m)) = 0$ for $i > 0$).

By the first vanishing the restriction map

$$H^0(Y, N(m)) \to H^0(E_0, N(m)_{|E_0})$$

is surjective.

By the non-vanishing theorem for any $m_1 \gg m_0$ there exists a non-zero section s of $N(m_1)_{|E_0}$. By surjectivity this extends to a non-zero section of $N(m_1)$ on Y, which is not identically zero along E_0. Moreover $H^0(Y, N(m_1)) = H^0(X, (m_1 + a)H)$ since D is f-exceptional. The section s descends to a section of $(m_1 + a)H$ which does not vanish along $f(E_0) = Z \subset B_c$, which is a contradiction.

2.2. Singularities and log singularities. In the previous sections we did not introduce any technical definitions of singularities or of singular pairs. Let us do it now for the interested reader.

Definition 2.2.1. Let X be a normal variety and $D = \sum_i d_i D_i$ be an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. If $\mu : Y \to X$ is a log resolution of the pair (X, D), then we can write

$$K_Y + \mu_*^{-1} D = \mu^*(K_X + D) + F$$

with $F = \sum_j \mathrm{disc}(X, E_j, D)E_j$ for the exceptional divisors E_j. We call $e_j := \mathrm{disc}(X, E_j, D) \in \mathbb{Q}$ the *discrepancy coefficient* for E_j, and regard $-d_i$ as the discrepancy coefficient for D_i.

The variety X is said to have *terminal* (respectively *canonical*, *log terminal* (LT)) singularities if $e_j > 0$ (resp. $e_j \geq 0$, $e_j > -1$) for any j.

The pair (X, D) is said to have *log canonical* (LC) (respectively *Kawamata log terminal* (KLT)) singularities if $d_i \leq 1$ (resp. $d_i < 1$) and $e_j \geq -1$ (resp. $e_j > -1$) for any i, j of a log resolution $\mu : Y \to X$.

The *log canonical threshold* of a pair (X, D) is $\mathrm{lct}(X, D) := \sup\{t \in \mathbb{Q} : (X, tD)$ is

Definition 2.2.2 ([Ka3]). Let X be a normal variety and $D = \sum d_i D_i$ an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. A subvariety W of X is said to be a *center of log canonical singularities* for the pair (X, D) if there is a birational morphism from a normal variety $\mu : Y \to X$ and a prime divisor E on Y, not necessarily μ-exceptional, with discrepancy coefficient $e \leq -1$ and such that $\mu(E) = W$. For another such $\mu' : Y' \to X$, if the strict transform E' of E exists on Y', then we have the same discrepancy coefficient for E'. The divisor E' is considered to be equivalent to E, and the equivalence class of these prime divisors is called a *place of log canonical singularities* for (X, D). The set of all centers (respectively places) of LC singularities is denoted by $\mathrm{CLC}(X, D)$ (resp. $\mathrm{PLC}(X, D)$), the locus of all centers of LC singularities is denoted by $\mathrm{LLC}(X, D)$.

The study of these objects has been developed by Kawamata and we can summarise the main results in the following theorem.

Theorem 2.2.3 ([Ka3], [Ka4]). *Let X be a normal variety and D an effective $\mathbb{Q}$-Cartier divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. Assume that X is LT and (X, D) is LC.*

(i) *If $W_1, W_2 \in \mathrm{CLC}(X, D)$ and W is an irreducible component of $W_1 \cap W_2$, then $W \in \mathrm{CLC}(X, D)$. In particular, there exist minimal elements in $\mathrm{CLC}(X, D)$ with respect to inclusion.*

(ii) *If $W \in \mathrm{CLC}(X, D)$ is a minimal center then W is normal.*

(iii) (subadjunction formula). *Let H be an ample Cartier divisor and ε a positive rational number. If W is a minimal center for $\mathrm{CLC}(X, D)$ then there exists an effective $\mathbb{Q}$-divisor D_W on W such that $(K_X + D + \varepsilon H)_{|W} \equiv K_W + D_W$ and (W, D_W) is KLT.*

Remark 2.2.4. The first two statements are, essentially, a consequence of Shokurov's connectedness lemma, which is itself a direct consequence of the vanishing theorem 2.1.2. The subadjunction formula is quite of a different flavor and is related to semipositivity results for the relative dualising sheaf of a morphism.

In particular 2.2.3 tells us that the minimal center W is not too bad and there is some hope to be able to work on it.

Exercise 2.2.5. It is in fact not so difficult to work out all possible minimal centers $W \in \mathrm{CLC}(X, D)$, where X is a smooth surface and D any divisor (i.e. a curve). The same, a little harder, if X is a smooth 3-fold; one should keep in mind that KLT singularities are rational singularities.

Let (X, D) be a log variety and assume that (X, D) is LC and $W \in \mathrm{CLC}(X, D)$ is a minimal center. The Weil divisor D is usually called the boundary of the log pair. Then we have a log resolution $\mu : Y \to X$ with

$$K_Y = \mu^*(K_X + D) + \sum e_i E_i;$$

this time we put also the strict transform of the boundary on the right hand side. Since (X, D) is LC and $W \in \mathrm{CLC}(X, D)$, we have $e_i \geq -1$ and there is at least one $e_j = -1$ such that $\mu(E_j) = W$.

A first problem is that to apply Kawamata's BPF method we need to have a unique

Perturbation argument: Choose a generic very ample M such that $W \subset \mathrm{Supp}(M)$ and no other $Z \in \mathrm{CLC}(X, D) \setminus \{W\}$ is contained in $\mathrm{Supp}(M)$; this is always possible since W is minimal in a dimensional sense. We then perturb D to a divisor $D_1 := (1-\varepsilon_1)D+\varepsilon_2 M$, with $0 < \varepsilon_i \ll 1$ in such a way that

- (X, D_1) is LC,
- $\mathrm{CLC}(X, D_1) = W$,
- $\mu^*\varepsilon_2 M = \sum m_i E_i + P$ with P ample; this is possible by the Kodaira lemma.

After this perturbation the log resolution looks like

$$K_Y + \sum_{j=0} E_j + \Delta - A = \mu^*(K_X + D_1) - P,$$

where the E_j's are integral irreducible divisors and $\mu(E_j) = W$, A is a μ-exceptional integral divisor and $\lfloor \Delta \rfloor = 0$. It is now enough to use the ampleness of P to choose just one of the E_j. Indeed for small enough $\delta_j > 0$, $P' := P - \sum_{j=1} \delta_j E_j$ is still ample and therefore we produce the desired resolution

$$K_Y + E_0 + \Delta' - A = \mu^*(K_X + D') - P'; \tag{2.2.1}$$

where $P' + \Delta'$ is effective, fractional, and P' is ample.

If instead of an ample M we choose a nef and big divisor, we can repeat the above argument with the Kodaira lemma, but this time we cannot choose the center $\mu(E_0)$ as before, and in particular we cannot assume that at the end we are on a minimal center for (X, D).

2.2.1. *How to use singularities and the CLC locus to prove base point free-type theorems.* Assume now that X is a variety with log terminal and Gorenstein singularities and let L be an ample line bundle on X.

Let D be an effective $\mathbb{Q}$-Cartier divisor such that $D \equiv tL$ for a rational number $t < 1$. Let $W \in \mathrm{CLC}(X, D)$ be a minimal center. Perturb D using the very ample line bundle $M := mL$ for $m \gg 0$. So we can assume that there exists only one exceptional divisor in any log resolution of (X, D) with discrepancy -1 and W as center. Thus taking an embedded log resolution of the pair (X, D), $\mu : Y \to X$, we have

$$K_Y + E + F = \mu^*(K_X + D)$$

where E is a reduced divisor such that $\mu(E) = W$ and $F = \sum f_i F_i$ with $f_i < 1$. Then

$$K_Y + (1-t)\mu^* L \equiv \mu^*(K_X + L) - E - F$$

and thus

$$H^1(Y, \mu^*(K_X + L) - E + \lceil -F \rceil) = 0$$

and we obtain a surjection

$$H^0(Y, \mu^*(K_X + L) + \lceil -F \rceil) \to H^0(E, \mu^*(K_X + L) + \lceil -F \rceil).$$

The divisor $\lceil -F \rceil$ is effective and any irreducible component of $\lceil F \rceil$ is μ-exceptional; therefore $H^0(Y, \mu^*(K_X + L)) = H^0(Y, \mu^*(K_X + L) + \lceil -F \rceil)$ and we also have

Thus to find a section of $K_X + L$ not vanishing on W it is sufficient to find a non-zero section in $H^0(E, \mu^*(K_X + L) + \lceil -F \rceil)$.

The *ideal case* happens when $W = x$ is one point; in fact then $H^0(E, \mu^*(K_X + L) + \lceil -F \rceil) = \mathbb{C}$ and therefore $K + L$ is base point free at x.

2.3. Exercises and examples. The solution of the next exercise can be found in the book [BS], even under the milder hypothesis that L is ample and spanned. We propose it here because we think that the above methods are convenient to be applied to these problems and because we believe they should prove the conjecture stated in item (d) (we do not know how and therefore we adopt the trick to put it as an exercise).

Exercise 2.3.1. Let L be a very ample line bundle on a smooth projective variety X of dimension n. Prove the following:

(a) $K_X + (n+1)L$ is spanned by global sections at each point.

(b) The same is true for $K_X + nL$ unless $X = \mathbb{P}^n$ and $L = \mathcal{O}(1)$.

(c) If $n \geq 2$ the same is true for $K_X + (n-1)L$ unless $X = \mathbb{P}^n$ and $L = \mathcal{O}(1)$ or $X = \mathbb{P}^2$ and $L = \mathcal{O}(2)$ or $X = \mathbb{Q}^n$ and $L = \mathcal{O}_{\mathbb{P}^{n+1}}(1)_{\mathbb{Q}^n}$ or (X, L) is a scroll over a curve.

(d) Conjecture: If $n \geq 3$ the same is true for $K_X + (n-2)L$ as soon as it is nef and $L^n > 27$.

Hints: For (a), let $x \in X$ and take n-sections of L meeting transversally in x.

For (b) use an "induction procedure"; namely take a smooth section $D \in |L|$ passing through x (this is the Bertini theorem) and use the exact sequence

$$H^0(X, K_X + nL) \to H^0(D, K_D + L) \to 0.$$

One goes down until the dimension of D is 1, i.e. a curve, and in this case $K_D + L$ is spanned if and only if $\deg L \geq 2$. The only problem is when D is a line and therefore $X = \mathbb{P}^n$ and $L = \mathcal{O}(1)$.

For (c), as in the previous step, one can reduce the problem to the surface case; namely $X = S$ is a smooth surface and one has to prove the spannedness of $K + L$. In this case there are even stronger theorems (Reider type theorems).

Some comments to the conjecture stated in (d): by the inductive procedure it is enough to prove the statement for $n = 3$. Note that the bound $L^n > 27$ is necessary since there exists a del Pezzo 3-fold X with $-K_X = 2H$, $H^3 = 1$ and H with one base point (take $L = 3H$).

The above exercise is extremely hard when one assumes only ampleness (and not very ampleness!) of L. In fact we have:

Conjecture 2.3.2 (Fujita conjecture). Let L be an ample line bundle on a smooth projective variety of dimension n. Then $K_X + mL$ is base point free if $m \geq n+1$ and it is very ample if $m \geq n+2$.

Remark 2.3.3. Some important results toward a proof of the conjecture have been found in recent time. In particular, using an analytic approach, Demailly, Angern-Siu and Tsuji proved that if $m \geq \binom{n+1}{2}$ then $K_X + mL$ is base point free and if $m \geq \binom{n+2}{2}$ then the

The base point free part of the conjecture is true in the case $n = 1, 2, 3, 4$ by results of Reider, Ein-Lazarsfeld, Helmke, Kawamata and Fujita (see [Rei] and [Ka3]).

Part 3. Fano–Mori or extremal contractions

In this part we first define and give examples of Fano–Mori spaces. These are exactly the morphisms constructed in Part 2, and they play a central role in the Minimal Model Program. To study those objects we want to apply an inductive method as in Section 1.2. A fundamental step is therefore to ensure that we have base point free linear systems to slice the fibers. This is the content of Theorem 3.3.1, whose proof occupies the last section.

3.1. Contractions associated to a ray of the Kleiman–Mori cone. A key step in Mori theory, after the description of the structure of $\overline{\mathrm{NE}(X)}$ outlined in a previous section, is the fact that extremal rays (and in general extremal faces) give rise to morphisms of the variety. This is explained in this section.

PROPOSITION 3.1.1. *Let R be an extremal ray of the Kleiman–Mori cone $\overline{\mathrm{NE}(X)}$ such that $R \cdot K_X < 0$. Then there exists a nef Cartier divisor H_R such that $H_R \cdot z = 0$ if and only if $z \in R$.*

This proposition is proved for instance in [Ko3, III.1.4.1]. The proof makes use of the cone theorem and some easy properties of closed cones.

Then to a divisor as in the proposition we can associate a morphism via the following theorem.

THEOREM 3.1.2 (Contraction theorem). *Let X be a variety with log terminal singularities and let H be a nef Cartier divisor on X. Assume that $F := H^{\perp} \cap \overline{\mathrm{NE}(X)} \setminus \{0\}$ is contained in $\{C \in N_1(X) : K_X \cdot C < 0\}$. Then there exists a projective morphism $\varphi : X \to W$ onto a normal projective variety W which is characterised by the following properties*:

(i) *For any irreducible curve $C \subset X$, $\varphi(C)$ is a point if and only if $H \cdot C = 0$.*
(ii) *φ has connected fibers.*
(iii) *$H = \varphi^*(A)$ for some ample Cartier divisor on W.*

Proof. The proof follows immediately from Theorem 2.1.3 and Zariski's main theorem once we note that by our assumption and Kleiman's criterion for ampleness there exists a natural number a such that $aH - K_X$ is ample. ■

DEFINITION 3.1.3. A *contraction* is a surjective morphism $f : Y \to T$, with connected fibers, between normal varieties.

For a contraction $f : Y \to T$ the set

$$E = \{y \in Y : f \text{ is not an isomorphism at } y\}$$

is the *exceptional locus* of f. Let $\delta = \dim E$ where dim denotes as usual the maximum dimension of irreducible components. The contraction is called of *fiber type* if $\delta = \dim Y$, If f is birational and $\delta = \dim Y - 1$ then it is also called a *divisorial contraction*; if it is birational and $\delta \leq \dim Y - 2$ then it is called a *small contraction.*

Given a contraction $f : Y \to T$, a Cartier divisor H such that $H = \varphi^*(A)$ for some ample Cartier divisor A on T is called a *supporting divisor* for the contraction (if $H = H_R$ as in the above proposition then it is also called a supporting divisor for the ray R).

DEFINITION 3.1.4. A contraction $f : X \to W$ as in the above Theorem 3.1.2 is called *Fano–Mori* (F-M) or *extremal.* A birational contraction $f : X \to W$ is called *crepant* if $K_X = f^*K_W$.

REMARK 3.1.5. Putting together Theorem 3.1.2 and Proposition 3.1.1 we obtain the following. Given an extremal ray of the Kleiman–Mori cone $R \subset \overline{\mathrm{NE}(X)}$ such that $R \cdot K_X < 0$, there exists a projective morphism with connected fibers $\mathrm{cont}_R : X \to W$ onto a normal projective variety W, which contracts all (and only) the curves in the ray. Such a map is also called the *contraction of the extremal ray* R, or an *elementary Fano–Mori contraction.*

We stress that Theorem 3.1.2 is proved only in characteristic zero. The existence of this map in positive characteristic is an open problem.

REMARK 3.1.6. It is straightforward to prove that conversely the contraction theorem implies Theorem 2.1.3.

Note also that any supporting divisor H for a F-M contraction φ is of the type $K_X + rL$ with r a rational number and L an ample Cartier divisor. In fact let H be a Cartier divisor which is the pull back of a sufficiently ample line bundle on W. Then $mH - K_X := L$ is an ample Cartier divisor for some rational number m and thus $H = K_X + (1/m)L$.

REMARK 3.1.7. To construct a divisor as in 3.1.2, and therefore an associated morphism, one can also use the rationality theorem 1.1.3 as follows. Let X be a variety with at most log terminal singularities and let L be a Cartier divisor with nef value r. Then, by the rationality theorem, if $H' := K + rL$ there exists an integer m such that $H := mH'$ is a Cartier divisor. By definition H satisfies the assumption in 3.1.2.

The following is an important technical result whose proof may be considered an interesting exercise.

EXERCISE 3.1.8 ([KMM, Proposition 5.1.6]). Let $f : X \to W$ be a divisorial elementary Fano–Mori contraction with X smooth or with at most terminal $\mathbb{Q}$-factorial singularities. Prove that the exceptional locus of f is a unique prime divisor and W has at most terminal $\mathbb{Q}$-factorial singularities.

Hint: Assume by contradiction that there are at least two components. Show that a generic curve in one component cannot be numerically equivalent to a generic curve in the other.

3.1.1. *Local contraction.* In studying F-M contractions it makes sense to fix a fiber and understand the contraction locally, i.e. restricting to an affine neighborhood of the fixed fiber. More general complete F-M contractions can then be obtained by gluing different

For this we use the local set-up developed by Andreatta–Wiśniewski (see [AW1]), which depends on some definitions.

Definition 3.1.9. Let $f : Y \to T$ a contraction supported by $K_Y + rL$, with r rational and L ample and Cartier (i.e. a F-M contraction). Fix a fiber F of f and take an open affine $S \subset T$ such that $f(F) \in S$ and $\dim f^{-1}(s) \leq \dim F$, for $s \in S$. Let $X = f^{-1}S$ then $f : X \to S$ will be called a *local contraction around* F. If there is no need to specify fixed fibers then we will simply say that $f : X \to S$ is a local contraction. In particular $S = \mathrm{Spec}(H^0(X, \mathcal{O}_X))$.

Definition 3.1.10. Let $f : X \to S$ be a local F-M contraction around F. Let $r = \inf\{t \in \mathbb{Q} : K_Y + tH \equiv_f 0$ for some ample Cartier divisor $H \in \mathrm{Pic}(X)\}$. Assume that $K_X + rL \equiv_f \mathcal{O}_X$, that is, f is supported by $m(K_X + rL)$ for some $m \geq 1$. The Cartier divisor L will be called the *fundamental divisor* of f.

Let G be a generic non-trivial fiber of f. The *dual index* of f is

$$d(f) := \dim G - r,$$

the *character* of f is

$$\gamma(f) := \begin{cases} 1 & \text{if } \dim X > \dim S, \\ 0 & \text{if } \dim X = \dim S, \end{cases}$$

and the *difficulty* of f is

$$\Phi(f) = \dim F - r.$$

We will say that $(d(f), \gamma(f), \Phi(f))$ is the *type* of f.

3.2. Examples. A large class of examples of F-M contractions is worked out in Section 3 of the paper [AW3]; we report some of them here, referring the reader for more details to that paper. We focus on the case where X is smooth, with the purpose of showing later some classifications of F-M contractions on a smooth variety.

Example 3.2.1. Fano varieties (with the constant map $X \to \{\mathrm{pt}\}$), scrolls (i.e. $X = \mathbb{P}(E) \to Y$ where E is a vector bundle on a smooth manifold Y) and conic bundles are F-M contractions of fiber type.

Example 3.2.2. Any blow-up of a smooth smooth variety Y along a smooth subvariety Z, $X := \mathrm{Bl}_S Z \to Z$, is a birational F-M contraction.

Example 3.2.3. Blow-up a smooth surface in a 4-fold with an ordinary double point; i.e.

$$S := \{x = z = w = 0\} \subset Z := \{xy - zt + w^2\}, \qquad \varphi : X := \mathrm{Bl}_S Z \to Z.$$

A direct computation shows that X is smooth and that $\varphi^{-1}(0) = \mathbb{P}^2$.

Let L_1, L_2, L_3 be three general planes in $\mathbb{P}^3$ and let $\mathbb{P}^2$ be the base of the net $\mathcal{L} = \sum t_i L_i$. Consider the incidence variety

$$X := \{(p, L) : p \in L\} \subset \mathbb{P}^3 \times \mathbb{P}^2.$$

Then the projection $\varphi : X \to \mathbb{P}^3$ is a F-M contraction which is a $\mathbb{P}^1$-bundle generically

If we blow up a smooth surface S in X meeting the general fiber in one point we obtain a smooth conic bundle $Y \to \mathbb{P}^3$ with a 2-dimensional reducible fiber and with discriminant locus $\Delta = \varphi(S)$.

In coordinates: assume $\mathbb{P}^3 = [z_0, z_1, z_2, z_3]$, $\mathbb{P}^2 = [t_1, t_2, t_3]$, $L_i = z_i$, $i = 1, 2, 3$. Then $X = \{t_1 z_1 + t_2 z_2 + t_3 z_3 = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^3$ and let, for instance, $S = \{t_1 = z_1 = 0\}$. The special 2-dimensional fiber on Y will be $\mathbb{F}_1 \cup \mathbb{P}^2$.

On Y there are two F-M contractions, both of birational type; besides the blow-up of X along S we can contract a divisor on Y consisting of the $\mathbb{P}^2$ component of the 2-dimensional fiber and of all the components of the reducible conics not contracted to X. This is a contraction as the one described in the first part of the example (if this is not immediate now, it will be later when we give a classification of F-M contractions on smooth 4-folds).

Example 3.2.4. We now introduce a large class of examples via a standard construction; for more details see Section 3 of [AW3]. Let $\mathcal{E}$ be a vector bundle over a smooth variety F and let $\mathbf{V}(\mathcal{E}) := \mathrm{Spec}(S(\mathcal{E}))$ be the *total space of the dual* $\mathcal{E}^*$. If $S^k(\mathcal{E})$ is *generated by global sections* for some $k > 0$ let

$$\varphi : \mathbf{V}(\mathcal{E}) \to Z = \mathrm{Spec}\Big(\bigoplus_{k \geq 0} H^0(F, S^k(\mathcal{E}))\Big)$$

be the map associated to the evaluation of $S^k(\mathcal{E})$. Then φ is a contraction which gives the collapsing of the zero section of the total space $\mathbf{V}(\mathcal{E})$, $F_0 := F$, to the vertex z of the cone Z.

It is straightforward to check the following properties:

(i) The normal bundle of F_0 in $\mathbf{V}(\mathcal{E})$ is $\mathcal{E}^*$.

(ii) If $-K_Y - \det \mathcal{E}$ is ample then φ is a Fano–Mori contraction. The map φ is birational if the top Segre class of $\mathcal{E}$ is positive (if rank $\mathcal{E} = 2$ then $c_1^2 - c_2 > 0$).

(iii) $\mathbb{P}(\mathcal{O} \oplus \mathcal{E}) := \mathrm{Proj}(S(\mathcal{E} \oplus \mathcal{O}_Y))$ is the projective closure of $\mathbf{V}(\mathcal{E})$. The map φ is the restriction of the map given by the tautological bundle ξ on $\mathrm{Proj}(S(\mathcal{E} \oplus \mathcal{O}_Y))$; φ is birational if ξ is big.

(iv) (Grauert criterion) $\mathcal{E}$ is ample if and only if φ is an isomorphism outside F_0.

(v) The fiber F_0 of φ has the fiber structure (i.e. $\mathcal{I}_{F_0} = \varphi^{(-1)} m_z \mathcal{O}_X$) if and only if $\mathcal{E}$ is spanned by global sections.

Let us work out in detail the example with $F = \mathbb{P}^2$; it is possible to do the same for a 2-dimensional quadric, see [AW3], or for smooth del Pezzo surfaces. Let $\mathcal{E}$ be a rank-2 vector bundle over $\mathbb{P}^2$ such that $\mathcal{E}$ is spanned by global sections and $0 \leq c_1(\mathcal{E}) \leq 2$. These bundles were completely classified in [SW], and they are each isomorphic to one of the bundles in the following table. Performing the above construction with them we obtain eight Fano–Mori contractions with fiber $\mathbb{P}^2$; in the second column we describe these contractions. In [AW3] it is explained how to obtain these descriptions; one has to

Description of $\mathcal{E}$	Description of φ and $\mathrm{Sing}(Z)$
$\mathcal{E} = \mathcal{O} \oplus \mathcal{O}$	a scroll, Z is smooth
$\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(1)$	a smooth blow-up of a smooth curve, Z is smooth
$\mathcal{E} = T\mathbb{P}^2(-1)$	a generalised scroll, i.e. a fiber type map, general fiber isomorphic to a line and a 2-dimensional fiber, Z is smooth
$\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(2)$	the blow-up of a smooth curve C, Z is singular along C
$\mathcal{E} = \mathcal{O}(1) \oplus \mathcal{O}(1)$	a small contraction, Z is singular and the flip exists
$0 \to \mathcal{O} \to T\mathbb{P}^2(-1) \oplus \mathcal{O}(1) \to \mathcal{E} \to 0$	the blow-up of a smooth surface passing from a quadric singularity of Z
$0 \to \mathcal{O}(-1)^{\oplus 2} \to \mathcal{O}^{\oplus 4} \to \mathcal{E} \to 0$	the blow-up of a cone over a twisted cubic in a smooth Z
$0 \to \mathcal{O}(-2) \to \mathcal{O}^{\oplus 3} \to \mathcal{E} \to 0$	a conic bundle with a 2-dimensional fiber, Z is smooth

Example 3.2.5. The existence of F-M birational contractions with exceptional set of codimension greater than 1 (small contraction) was proved by P. Francia with a famous example: it is a F-M contraction on a 3-fold with terminal singularities and with exceptional locus $E \cong \mathbb{P}^1$. The example is worked out in many books, for instance in [CKM, pp. 33–34]. This is the main difficulty in the MMP over cone by S. Mori, with a tremendous work, in dimension 3 (see [Mo4]).

3.3. Relative base point freeness on Fano–Mori contractions. A F-M contraction has a supporting divisor of the type $K_X + rL$ with L an ample Cartier divisor, as noticed in 3.1.6.

This feature, which we can also call the *adjoint contraction morphism*, allows us to apply an inductive method which is typical of this theory. It is a sort of relative "Apollonius method" (see 1.2) and in [AW1] it is called a *horizontal slicing* argument (sometimes it is called simply *slicing* but we will need to distinguish it from *vertical slicing*). It can be briefly summarised as follows.

Consider a general divisor X' from the linear system $|L|$ (a hyperplane section of X if L is very ample) and assume that it is a "good" variety, i.e. has the same singularities as X, of dimension $n-1$. By adjunction, $K_{X'} = (K_X + L)_{|X'}$ and, by the vanishing theorem 2.1.2, if $r > 1$, the linear system $|m(K_{X'} + (r-1)L)|$ is just the restriction of $|m(K_X + rL)|$, so that the *adjoint contraction morphism* of X' can be related to the one of X. Moreover, fibers of the *adjoint morphism* of X' will usually be of smaller dimension and an inductive argument can be applied. The method will be further outlined in Section 4.3.2.

The *horizontal slicing* argument requires therefore the existence of a "good" divisor X' in the linear system $|L|$ (a *rung* in the language of [Fu2], see Section 1.2). The system, however, for an ample (but not very ample) L may a priori be even empty. To overcome this difficulty we use the local set-up, described in the previous section, in which the base of the *contraction morphism* will be affine. We also benefit from this situation because we may choose effective divisors which are rationally trivial.

Then the next point is to ensure that the divisor X' does not contain the whole theorem, if we ensure that the base locus of $|L|$ (L may be changed by adding a divisor trivial on fibers of φ) is empty. This is what may be called a "relative good divisor". (Now we can explain why we use the word "horizontal": we are used to thinking about the map $\varphi : X \to Z$ as going vertically; then every divisor from an ample linear system intersects every "vertical" fiber of φ of dimension ≥ 1, so it lies "horizontally".)

The above point of view was first exploited in [AW1] where the first part of the following theorem was proved. The proof used the base point free theorem method (BPF-method) of Y. Kawamata, actually a slightly improved version of it by J. Kollár (see [Ko2]) introduced in the previous section. The further refinement of the method by Y. Kawamata in [Ka3] and [Ka4] allowed an improvement of the theorem in [AW1]; this is the second part of the following theorem and it was proved in [Me3].

Theorem 3.3.1. *Let $f : X \to S$ be a local F-M space around F supported by K_X+rL and let $(d(f), \gamma(f), \Phi(f))$ be the type of f (see* 3.1.9 *and* 3.1.10*). Let also ε be a sufficiently small positive rational number. Assume one of the following two conditions is satisfied*:

- $\dim F < r+1$ *or, if f is birational*, $\dim F \leq r+1$; *equivalently the type of f is $(*, *, \Phi(f))$, with $\Phi(f) \leq 1 - \varepsilon\gamma(f)$ (see* [AW1]*),*
- *the type of f is $(d, 1, 1)$, with $d \leq 0$ or with $d = 1$ and F is reducible (see* [Me3]*).*

Then L, the fundamental divisor of the contraction, is relatively spanned, i.e. $\mathrm{Bsl}|L|$ $:= \mathrm{Supp}(\mathrm{Coker}(f^*f_*L \to L))$ *does not meet F.*

In the rest of the section we are going to prove this theorem. Let us first roughly summarise the general principles of the proof. The idea is to proceed by contradiction, we assume therefore that there is a non-empty base locus V. Then we produce a log variety non-KLT on V (with respect to a divisor in δL). Finally we use the method developed in Part 2 to produce sections of an adjoint line bundle non-vanishing along the non-KLT part of the log variety.

To apply this strategy we have a priori the main problem: namely BPF produces sections of $K_X + mL$ for $m \gg 0$, while we need sections of L itself. But in the category of local F-M contractions we have $L \equiv_f K_X + (r+1)L$. An immediate consequence is the following.

Crucial observation 3.3.2. In our set-up of F-M contractions, we will work with log pairs (X, D) such that $D \equiv_f \delta L$ and $K_X \equiv_f -rL$. In particular, by the subadjunction formula (see Theorem 2.2.3(iii)),

$$K_W + D_W \equiv_f (\delta - r + \varepsilon)L.$$

So if W is contained in a fiber and $\delta < r$ then $K_W + D_W$ is antiample.

Definition 3.3.3. A *log-Fano variety* is a KLT pair (X, Δ) such that for some positive integer m, $-m(K_X + \Delta)$ is an ample Cartier divisor. The *index* of a log-Fano variety is $i(X, \Delta) := \sup\{t \in \mathbb{Q} : -(K_X + \Delta) \equiv tH$ for some ample Cartier divisor $H\}$, and the H satisfying $-(K_X + \Delta) \equiv i(X, \Delta)H$ is called the *fundamental divisor.*

From our point of view these varieties are extremely important because we have a

PROPOSITION 3.3.4 ([Al], [Am]). *Let (X,Δ) be a log-Fano n-fold of index $i(X)$, H the fundamental divisor in X. If $i(X) > n-3$ then $h^0(X,H)>0$; moreover if $i(X)\geq n-2$ then $h^0(X,H)>1$.*

Proof. For simplicity assume that $\Delta = 0$ and $i(X)\geq n-2$; the other cases are treated similarly with some more effort. Let $p(t) := \chi(X,tH) = \sum h_j t^j$ be the Hilbert polynomial of H and $d = H^n$ (see Section 1.2). In particular

$$h_n = d/n! \quad \text{and} \quad h_{n-1} = \frac{-K_X\cdot H^{n-1}}{2(n-1)!} = \frac{i(X)d}{2(n-1)!}.$$

By the vanishing Theorem 2.1.2,

$$H^i(X,tH) = H^i(X,K_X+(tH-K_X)) = H^i(X,K_X+(i(X)+t)H) = 0$$

for $i>0$ and $t>-i(X)$. On the other hand, H is an ample divisor and therefore

$$H^0(X,tH)=0 \qquad \text{for any } t<0.$$

Combining the two we deduce that $p(t)=0$ for $-i(X)<t<0$, and $p(1)=1$. Plug this informations into $p(t)$ to get

$$\begin{aligned} p(t) &= \frac{d}{n!}(t+1)(t+2)\dots(t+n-2)\left(t^2+at+\frac{n(n-1)}{d}\right)\\ &= \frac{d}{n!}t^n + \frac{d}{n!}\left(a+\frac{(n-2)(n-1)}{2}\right)t^{n-1}+\dots \end{aligned}$$

To determine a use

$$h_{n-1} = \frac{i(X)d}{2(n-1)!}$$

so that

$$a = \frac{ni(X)-(n-2)(n-1)}{2}.$$

This yields $h^0(X,H) = p(1) > d/n + (n-1) > 1$. ■

The next lemma translates Proposition 3.3.4 into the non-vanishing theorem we need.

LEMMA 3.3.5. *Let $f: X\to S$ be a local contraction supported by K_X+rL around F. Fix a subvariety $Z\subset F$, and a $\mathbb{Q}$-divisor D, with $D\equiv_f \gamma L$. Assume that X is LT, (X,D) is LC along Z, and $W\in \mathrm{CLC}(X,D)$ is a minimal center contained in Z. Assume that one of the following conditions is satisfied:*

(i) $r-\gamma > \max\{0, \dim W - 3\}$,
(ii) $\dim W \leq 1$ *and* $r-\gamma > -1$.

Then there exists a section of $|L|$ not vanishing identically on W.

Proof. Since D is LC along W we can assume, up to perturbation, that there exists a log resolution $\mu: Y\to X$ of (X,D) with

$$K_Y - A + E + \Delta + B = \mu^*(K_X+D) - P,$$

where:

- E is an irreducible integral divisor,
- A and B are integral divisors,

Furthermore these divisors have the following properties:

- $\mu(E) = W$,
- A is μ-exceptional,
- $\lfloor \Delta \rfloor = 0$,
- $Z \cap \mu(B) = \emptyset$,
- P is $(f \circ \mu)$-ample.

Let

$$N(t) := \mu^* tL + A - \Delta - E - B - K_Y \equiv_{f \circ \mu} \mu^*(t + r - \gamma)L + P. \tag{3.3.1}$$

Then $N(t)$ is $(f \circ \mu)$-ample whenever $t + r - \gamma \geq 0$. In particular if conditions (i) or (ii) of the lemma are satisfied, by the vanishing theorem 2.1.2, we have the following surjection:

$$H^0(Y, \mu^* L + A - B) \to H^0(E, (\mu^* L + A)_{|E}).$$

Since A does not contain E and is effective, we have

$$H^0(W, L_{|W}) \hookrightarrow H^0(E, (\mu^* L + A)_{|E}).$$

In particular any section of $H^0(W, L_{|W})$ gives rise to a section in $H^0(X, L)$ not vanishing identically on W. Therefore to conclude the proof it is enough to produce a section in $H^0(W, L_{|W})$. By the subadjunction formula of Theorem 2.2.3 there exists a $\mathbb{Q}$-divisor D_W such that

$$K_W + D_W \equiv (K_X + D + \varepsilon L)_{|W} \equiv -(r - \gamma - \varepsilon)L_{|W} \tag{3.3.2}$$

for any $0 < \varepsilon \ll 1$.

In case (i) since $r - \gamma > 0$, by (3.3.2) for sufficiently small ε, (W, D_W) is a log Fano variety of index $i(W, D_W) = r - \gamma - \delta > \dim W - 3$. Therefore we can apply Proposition 3.3.4.

If $\dim W = 1$ then W is smooth. Since $r - \gamma - \varepsilon > -1$, by (3.3.2) we have

$$0 < L \cdot W \geq 2g(W) - 2$$

and thus $h^0(W, L_{|W}) > 0$ by the R-R formula. ■

We have to give the last preliminary to the proof of Theorem 3.3.1. Till now we have always worked with LC pairs. Along the proof we use pairs (X, D) which are not LC. To be able to treat this situation let us introduce the following definition and make some useful remarks.

Definition 3.3.6. *The log canonical threshold related to a scheme $V \subset X$ of a pair (X, D) is $\mathrm{lct}(X, V, D) := \inf\{t \in \mathbb{Q} : V \cap \mathrm{LLC}(X, tD) \neq \emptyset\}$. We will say that (X, D) is LC along a scheme V if $\mathrm{lct}(X, V, D) \geq 1$.*

Remark 3.3.7. Let $Z \in \mathrm{CLC}(X, \mathrm{lct}(X, V, D)D)$ be a center and assume that Z intersects V, then $(X, \mathrm{lct}(X, V, D)D)$ is LC on the generic point of Z.

If (X, D) is not LC then Theorem 2.2.3 is in general false. On the other hand the first assertion stays true, also under the weaker hypothesis that (X, D) is LC on the generic point of $W_1 \cap W_2$. In fact the discrepancy is a concept related to a valuation ν, therefore we can always substitute the variety X by an affine neighborhood of the generic point of

Proof of Theorem 3.3.1. Let $V = \mathrm{Bsl}|L| \cap F$; remember that we are in a relative situation, therefore we always need to consider objects contained in a fixed fiber to fully enjoy the geometrical consequences of the ample anticanonical class.

Our aim is to derive a contradiction producing a section of L which is not identically vanishing along V. Consider the set $\mathcal{D} = \{D\}$ of $\mathbb{Q}$-divisors D such that:

- $D \equiv_f \delta L$,
- there exists a minimal center $W_D \in \mathrm{CLC}(X, D)$ such that $W_D \subset F$ and $W_D \cap V \neq \emptyset$,
- $\dim W_D \leq r + 1 - \delta$,
- $\mathrm{lct}(X, W_D, D) = 1$.

First observe that $\mathcal{D}$ is non-empty. Consider $D_0 = f^* \sum_I l_i(g_i)$ for g_i generic functions on S vanishing at $f(F)$. Then $D_0 \equiv_f 0$ and one can choose $0 < l_i \ll 1$ such that $\mathrm{lct}(X, V, D_0) = 1$.

Claim 1. *There exists a $D \in \mathcal{D}$ such that $W_D \subset V$. Furthermore if $D \equiv_f (r+1)L$ one can choose D so that $D = D_0 + \sum_{i=1}^{r+1} H_i$ with $H_i \in |L|$ generic.*

Proof of the claim. Consider the above D_0 and let $H \in |L|$ be a generic section. Let

$$c = \inf\{t \in \mathbb{Q}^{\geq 0} : \mathrm{LLC}(D_0 + tH) \cap V \cap W_{D_0} \neq \emptyset\}.$$

Since H is a Cartier divisor vanishing on V, we have $c \leq 1$. Let $D_c = D_0 + cH$.

If $c < 1$ we assert that there exists a minimal center $W_{D_c} \in \mathrm{CLC}(X, D_c)$ with $W' \subset V$. Let us spend a few words on this. Fix a resolution $g : Y \to X$ of the singularities of X. Let $g^*H = H_Y + G$. Then by the Bertini theorem H_Y is smooth outside $\mathrm{Bsl}\,|H_Y|$. Furthermore for any g-exceptional divisor A such that $g(A) \not\subset \mathrm{Bsl}\,|L|$ we can choose an $H \in |L|$ such that $\mathrm{Supp}(H) \not\supset g(A)$. There are finitely many g-exceptional divisors in Y, therefore $g(G) \subset \mathrm{Bsl}\,|L|$. Let now $h : Z \to Y$ be a log resolution of (Y, H_Y), so that $f := g \circ h$ is a log resolution of (X, H). Let $f^*H = H_Z + \Delta$. Then $h(\Delta) \subset \mathrm{Bsl}\,|H_Y| \cup G$. Hence $f(\Delta) = g(h(\Delta)) \subset \mathrm{Bsl}\,|L|$. As a consequence $\mathrm{LLC}(X, D_c) \subset \mathrm{Bsl}\,|L| \cup \mathrm{LLC}(X, D_0)$. Furthermore for any $\varepsilon > 0$, $(X, D_c + \varepsilon H)$ is not LC along $V \cap W_{D_0}$, therefore there exists a center $W' \in \mathrm{CLC}(X, D_c)$ with $W' \cap (V \cap W_{D_0}) \neq \emptyset$ and $W' \cap F \subset V$.

To conclude consider a minimal center W_{D_c} contained in $W' \cap W_{D_0} \subset V$; keep in mind Remark 3.3.7. If $c = 1$ then both W_{D_0} and H are in $\mathrm{CLC}(X, D_1)$, and their intersection is not empty because $W_{D_0} \cap V \neq \emptyset$. Therefore by Remark 3.3.7 any irreducible component $Z \subset W_{D_0} \cap H$ is a center. Furthermore $\dim Z = \dim W_{D_0} - 1$. This means that $D_1 \in \mathcal{D}$ and $\dim W_{D_1} < \dim W_{D_0}$. Iterating this procedure we eventually produce D_{r+1} with $W_{D_{r+1}}$ a point in V. Observe that in this case $D_{r+1} = D_0 + \sum_{i=1}^{r+1} H_i$. ■

Let D be as in the claim, thus (X, D) is LC along W_D. If $r - \delta > -1$ then we can apply Lemma 3.3.5 to produce a section of L not vanishing along W_D and obtain a contradiction.

If $r - \delta = -1$ then W_D is a point in V. Moreover, according to Claim 1, in this case the divisor D is of the type

$$D = D_0 + \sum_{i=1}^{r+1} H_i$$

Let $X_j = X \cap \bigcap_{i=1}^{j} H_i$. Then X_j is LT in a neighborhood of $F_j := F \cap X_j$ for any $j \leq r+1$. The proof of this assertion is left to the reader as an exercise. (Hint: the main point to check is normality. To do it one has to use the fact that terminal singularities are smooth in codimension 2.)

By the vanishing theorem 2.1.2 we have the surjection

$$H^0(X, L) \to H^0(X_j, L_{|X_j})$$

for any $j \leq r$.

If the type of f is not $(1,1,1)$ then $f_r : X_r \to S$ is birational. In particular, by standard vanishing, $Z_r \simeq \mathbb{P}^1$ so that $L_{|Z_r}$ is spanned. The idea is to extend a section of $L_{|Z_r}$ not vanishing on Z_{r+1} to a section of $L_{|X_r}$. For details on this extension and about the case of type $(1,1,1)$ we refer to [Me3]. ■

We conclude this section with an exercise which follows easily from the main Theorem 3.3.1 and the method used in the proof of 3.3.4 (a proof can be found for instance in [Ko3, p. 245]).

Exercise 3.3.8. Let X be a Fano manifold of index $i(X)$. Then $i(X) \leq \dim X + 1$; moreover $i(X) = \dim X + 1$ if and only if $X \simeq \mathbb{P}^{\dim X}$ while $i(X) \geq \dim X$ if and only if either $X \simeq \mathbb{P}^{\dim X}$ or $X \simeq \mathbb{Q}^{\dim X}$.

Part 4. Biregular geometry

Fano–Mori contractions are fundamental tools of the Minimal Model Program; more generally they are important in problems of classification of projective varieties.

This part is devoted to the problem of describing F-M contractions. Except for a few results at the very beginning we will restrict ourselves to the smooth case, that is, we consider F-M contractions of smooth manifolds.

The singular case is very difficult and at the moment very little is known only in dimension 3 (essentially the complete classification of small extremal contractions on 3-folds with at most terminal singularities in the fundamental papers of Mori [Mo4] and of Kollár–Mori [KM1]).

We will give a complete classification of F-M contractions of smooth manifolds of dimension ≤ 4; we present this classification in a sequence of theorems in the first section.

We are interested in a local description of the contraction, in a neighborhood of a given fiber; in particular we consider a *local contraction around* F, $\varphi : X \to Z$, as defined in 3.1.9.

We present many steps of the proof of the classification; each step is important by itself and together they represent a sort of program for classifying the F-M contractions. In short they are the following:

1) Classify all possible fibers of the F-M contractions; we will succeed if their dimension is two or less.

2) When the fiber has good singularities (locally complete intersections) classify the

3) Describe a formal neighborhood of the possible fibers in X, i.e. the local contraction around F.

4) Find a commutative diagram of morphisms, preferably blow-ups and blow-downs, which includes φ and which can help understand φ (a sort of factorization of φ, for example the flip in the small contraction case).

The results contained in this part are classical for the case $n = \dim X = 2$, and they are due to the Italian school of geometry of the beginning of the previous century.

In the case $n = 3$ they were proved by S. Mori in the famous paper [Mo3], which gave rise to the so called Mori theory.

The case $n = 4$ was later considered by M. Andreatta and J. A. Wiśniewski [AW3]. [AW2] is a survey of these results on which this part is strongly based.

In Section 4.2 we present two theorems which characterise some F-M contractions of a smooth projective variety in higher dimension.

In the last section we outline the biregular classification of Fano manifolds of high index. These are the building blocks of F-M contractions and their knowledge is the starting point of any further investigation. Also in this case we will provide the known general techniques to approach the problem via adjunction methods, without any attempt to be exhaustive in the classification. In particular we will not present neither the Fano–Iskovskikh approach based on double projections, [Is], nor the Mukai vector bundle technique, [Mu], nor the Ciliberto–Lopez–Miranda deformation ideas, [CLM].

4.1. Fano–Mori contractions on a smooth n-fold with $n \leq 4$. Here we describe all F-M contractions on smooth n-folds with $n \leq 4$. The case of dimension 4 is the most elaborate. Proofs are given in the next sections.

Theorem 4.1.1. *Let X be a smooth projective surface and $R \subset \overline{\mathrm{NE}(X)}$ an extremal ray, that is, $R \cdot K_X < 0$ and R is an edge of the cone. Then the associated contraction morphism* $\mathrm{cont}_R : X \to Z$ *is one of the following*:

(1) *Z is a smooth surface and X is obtained from Z by blowing up a point*; $\varrho(Z) = \varrho(X) - 1$.

(2) *Z is a smooth curve and X is a minimal ruled surface over Z*; $\varrho(X) = 2$.

(3) *Z is a point*, $\varrho(X) = 1$ *and $-K_X$ is ample*; *in fact* $X \cong \mathbb{P}^2$.

Theorem 4.1.2. *Let X be a smooth projective* 3*-fold and $R \subset \overline{\mathrm{NE}(X)}$ an extremal ray. Then the associated contraction morphism* $\mathrm{cont}_R : X \to Z$ *is one of the following*:

(B) (Birational contractions) $\dim Z = 3$, cont_R *is a divisorial contraction and there are five types of local behavior near the exceptional divisor E*:

- B1: cont_R *is the (inverse of the) blow-up of a smooth curve in the smooth* 3*-fold Z.*
- B2: cont_R *contracts a smooth* $\mathbb{P}^2$ *with normal bundle* $\mathcal{O}(-1)$; cont_R *is the (inverse of the) blow-up of a smooth point in the smooth* 3*-fold Z.*
- B3: cont_R *contracts a smooth* 2*-dimensional quadric*, $\mathbb{F}_0$, *with normal bundle*

(locally analytically, an ordinary double point is given by the equation $x^2 + y^2 + z^2 + w^2 = 0$*).*

- B4: cont_R *contracts an irreducible singular 2-dimensional quadric,* $\mathbf{S}_2$*, with normal bundle* $\mathcal{O}(-1)$; cont_R *is the (inverse of the) blow-up of a point in* Z *which is locally analytically given by the equation* $x^2 + y^2 + z^2 + w^3 = 0$.
- B5: cont_R *contracts a smooth* $\mathbb{P}^2$ *with normal bundle* $\mathcal{O}(-2)$; cont_R *is the (inverse of the) blow-up of a point in* Z *which is locally analytically given as the quotient of* $\mathbb{C}^3$ *by the involution* $(x, y, z) \mapsto (-x, -y, -z)$.

(C) (Conic bundle) $\dim Z = 2$ *and* cont_R *is a fibration whose fibers are plane conics (general fibers are of course smooth).*

(D) (del Pezzo fibration) $\dim Z = 1$ *and* cont_R *is a fibration whose general fiber is a del Pezzo surface.*

(F) (Fano 3-folds) $\dim Z = 0$, $-K_X$ *is ample, thus* X *is a Fano 3-fold, and* $\varrho(X) = 1$.

As mentioned in the introduction to this chapter, the first theorem is due to G. Castelnuovo and F. Enriques, and the second to S. Mori. Note that actually they are true over any algebraically closed fields; the surface case follows from the fact that the Castelnuovo contraction theorem is true in any characteristic, and the 3-fold case was proved by Kollár in [Ko1], extending Mori's ideas.

The next theorem aims to give the same result for the case $n = 4$; here the situation is much more intricate and it will take some space describe it.

The result comes from many contributions, the main ones are from Y. Kawamata [Ka2] and M. Andreatta and J. A. Wiśniewski [AW3], [AW4]; in the fiber case, Y. Kachi obtained independently of [AW3] a similar classification of special 2-dimensional fibers of a conic fibration, while in the case of birational contractions contracting a divisor to a curve (part 3) Takagi obtained the same results as in Section 4 of [AW4].

THEOREM 4.1.3. *Let* X *be a smooth projective 4-fold and* $R \subset \overline{\mathrm{NE}(X)}$ *an extremal ray. Let* $\varphi := \mathrm{cont}_R : X \to Z$ *be the associated contraction morphism. Let* $F = \varphi^{-1}(z)$ *be a (geometric) fiber of* φ*; we will eventually shrink the morphism* φ *around* F *(see 3.1.9). Let* E *be the exceptional locus; in case* φ *is of fiber type we mean* $E = X$.

We divide the classification of these contractions depending on the couple of numbers $(\dim E, \dim \varphi(E))$ *which we will call the signature of the contraction; note that the pair* $(4, b)$ *will be assigned to a fiber type contraction with* $\dim Z = b$ *and the pairs* (a, b) *with* $b \geq a$ *cannot happen.*

Note also that if $\dim E = 3$ *then* E *is irreducible (see 3.1.8) and so is* $\varphi(E)$*, therefore they are both of pure dimension.*

For the notation adopted to describe some special 2-dimensional fibers see 1.2.4.

PART 0. *There is no F-M contraction of a 4-fold of signature* (a, b) *with* $a \leq 1$ *and with* $a = 2$ *and* $b = 1$.

PART 1: Small contractions, see [Ka2]. *Let* φ *be a F-M contraction of a 4-fold of signature* $(2, 0)$*. Then* $E = F \simeq \mathbb{P}^2$ *and its normal bundle is* $N_{F/X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$*. The contraction is completely determined in an analytic neighborhood by this data (see*

(*see* 3.2.4)

$$\varphi : \mathbf{V}(\mathcal{E}) \to Z = \operatorname{Spec}\Big(\bigoplus_{k\geq 0} H^0(F, S^k(\mathcal{E}))\Big)$$

where $\mathcal{E} = \mathcal{O}(1) \oplus \mathcal{O}(1)$ *and the map is associated to the evaluation of* $S^k(\mathcal{E})$*.*

In this situation the flip of φ *exists and it is obtained by blowing up* E *and then contracting the exceptional divisor in the other direction.*

PART 2: Birational; divisor to point. *Let* φ *be a F-M contraction of a* 4*-fold of signature* $(3,0)$*. Then either* E *is* $\mathbb{P}^3$, *with normal bundle* $\mathcal{O}(-a)$ *and* $1 \leq a \leq 3$, *or a* (*possibly singular*) 3*-dimensional quadric, with normal bundle* $\mathcal{O}(-a)$ *and* $1 \leq a \leq 2$, *or otherwise* $(E; -E_{|E})$ *is a del Pezzo* 3*-fold, that is,* E *has Gorenstein singularities,* $-E_{|E}$ *is ample and* $K_E = 2E_E$ (*these varieties have been classified by T. Fujita, see* [Fu2] *and* [Fu4]).

PART 3: Birational; divisor to curve. *Let* φ *be a F-M contraction of a* 4*-fold of signature* $(3,1)$*. Then*

(a) $C := \varphi(E)$ *is a smooth curve and* $\varphi : X \to Z$ *is the blow-up of* Z *along* C*.*

(b) $g := \varphi_{|E} : E \to C$ *is either a* $\mathbb{P}^2$*-bundle or a quadric bundle.*

(c1) *If* E *is a* $\mathbb{P}^2$*-bundle then the normal bundle of each fiber in* X *is either* $\mathcal{O}(-1)\oplus\mathcal{O}$ *or* $\mathcal{O}(-2)\oplus\mathcal{O}$; *in particular all fibers of* φ *are reduced and with no embedded components. In the first case* Z *is smooth and* φ *is the smooth blow-up*; *in the second* $C = \operatorname{Sing} Z$ *and* Z *is locally isomorphic to* $S_2 \times \mathbb{C}$ *where* S_2 *is the germ of singularity obtained by contracting the zero section in the total space of the bundle* $\mathcal{O}(2)$ *over* $\mathbb{P}^2$*.*

(c2) *If* E *is a quadric bundle then the general fiber is irreducible and isomorphic to a* 2*-dimensional, possibly singular, quadric. Isolated special fibers can occur and they are isomorphic either to a singular quadric or to a reduced but reducible quadric* (*i.e. union of two* $\mathbb{P}^2$ *intersecting along a line*); *in particular there are no special fibers which are isomorphic to a double plane. The normal bundle of each fiber is* $\mathcal{O}(-1) \oplus \mathcal{O}$*. Locally* Z *can be described as a hypersurface of* $\mathbb{C}^5$; *in the following table we give a list of possibilities for* $Z = V(g) \subset \mathbb{C}^5$ *according to the described combinations of general and special fibers. We choose coordinates* $(z_1, z_2, z_3, z_4, z_5)$ *such that* $C = \{z_1 = z_2 = z_3 = z_4 = 0\} \subset \mathbb{C}^5$*.*

No.	Special fiber	General fiber	g = analytic equation of Z
(1)	$\mathbb{F}_0$	$\mathbb{F}_0$	$z_1^2 + z_2^2 + z_3^2 + z_4^2$
(2)	$\mathbf{S}_2$	$\mathbb{F}_0$	$z_1^2 + z_2^2 + z_3^2 + z_5^m z_4^2,\ m \geq 1$
(3)	$\mathbf{S}_2$	$\mathbf{S}_2$	$z_1^2 + z_2^2 + z_3^2 + z_4^3$
(4)	$\mathbb{P}^2 \cup \mathbb{P}^2$	$\mathbf{S}_2$	$z_1^2 + z_2^2 + z_3^3 + z_4^3 + z_3^2 z_5^m,\ m \geq 1$
(5)	$\mathbb{P}^2 \cup \mathbb{P}^2$	$\mathbb{F}_0$	$z_1^2 + z_2^2 + z_3^3 + z_4^3 + z_3^2 z_5^m + z_3 z_4 f(z_5)$ $+z_4^2 g(z_5)$ with $z^m g(z) \neq f(z)^2/4$

PART 4: Birational; divisor to surface. *Let* φ *be a F-M contraction of a* 4*-fold of signature* $(3,2)$*. Generically the map is described by part* (1) *of Theorem* 4.2.1; *in particular* Z *as well as* $S := \varphi(E)$ *are in general smooth and* φ *is a simple blow-down of the divisor* E *to*

However there can be some special 2*-dimensional fibers* F*. If this is the case then the scheme-theoretic fiber structure over* F *is trivial, that is, the ideal* $\mathcal{I}_F$ *of* F *is equal to the inverse image of the maximal ideal of* z*, that is,* $\mathcal{I}_F = \varphi^{-1}(m_z) \cdot \mathcal{O}_X$*.*

Moreover the fiber F *and its conormal bundle* $\mathcal{I}_F/\mathcal{I}_F^2$ *as well as the singularity of* Z *and* S *at* z *can be described as follows:*

F	$N^*_{F/X}$	Sing Z	Sing S
$\mathbb{P}^2$	$T(-1) \oplus \mathcal{O}(1)/\mathcal{O}$	cone over $\mathbb{Q}^3$	smooth
$\mathbb{P}^2$	$\mathcal{O}^{\oplus 4}/\mathcal{O}(-1)^{\oplus 2}$	smooth	cone over a twisted cubic
Quadric	spinor bundle from $\mathbb{Q}^4$	smooth	non-normal

The quadric fiber can be singular, even reducible, and in the subsequent table we present a refined description of its conormal bundle. The last entry in the table provides information about the ideal of a suitable surface S*; a complete description of these ideals can be found in* [AW4].

Quadric	Conormal bundle	$\mathcal{I}(S)$ in $\mathbb{C}[[x,y,z,t]]$
$\mathbb{P}^1 \times \mathbb{P}^1$	$\mathcal{O}(1,0) \oplus \mathcal{O}(0,1)$	(xz, xt, yz, yt)
quadric cone	$0 \to \mathcal{O} \to N^* \to \mathcal{J}_{\text{line}} \to 0$	generated by 5 cubics
$\mathbb{P}^2 \cup \mathbb{P}^2$	$T_{\mathbb{P}^2}(-1) \cup (\mathcal{O} \oplus \mathcal{O}(1))$	generated by 6 quartics

Part 5: Conic bundle fibration with possibly special 2-dimensional fiber. *Let* φ *be a F-M contraction of a* 4*-fold of signature* $(4,3)$*. Then* φ *is a fibration whose general fibers are plane conics; generically the map is described by part* (2) *of Theorem* 4.2.1*. In particular* Z *is in general smooth.*

However there can be some special isolated 2*-dimensional fibers* F*; the possibilities for* F *are the following:*

- $F \simeq \mathbb{P}^2$ *and* $N^*_{F/X} \simeq \mathcal{O}^3/\mathcal{O}(-2)$ *or* $T\mathbb{P}^2(-1)$*. The scheme fiber structure* $\widetilde{F}$ *is reduced and* Z *is smooth at* $z = \varphi(F)$*.*
- F *is an irreducible quadric and* $N^*_{F/X}$ *is the pullback of* $T\mathbb{P}^2(-1)$ *via some double covering of* $\mathbb{P}^2$*. The scheme fiber structure* $\widetilde{F}$ *is reduced and* Z *is smooth at* $z = \varphi(F)$*.*
- *The following other possibilities for* F *can occur:*

$$\mathbf{S}_3, \mathbb{F}_1, \mathbb{P}^2 \cup \mathbb{P}^2, \mathbb{P}^2 \cup \mathbb{F}_0, \mathbb{P}^2 \cup_{C_0} \mathbb{F}_1, \mathbb{P}^2 \cup \mathbf{S}_2, \mathbb{P}^2 \cup \mathbb{P}^2 \cup \mathbb{P}^2, \mathbb{P}^2 \cup_f (\mathbb{F}_0) \cup_{C_0} \mathbb{P}^2,$$

where any two components intersect along a line (explicitly indicated by a subscript, when needed), and the exceptional case of $\mathbb{P}^2 \bullet \mathbb{P}^2$ *when the two components intersect at an isolated point.*

Part 6: del Pezzo and Mukai fibration and Fano 4-folds. *Let* φ *be a F-M contraction of a* 4*-fold of signature* $(4,d)$*, with* $d \le 2$*. Then* φ *is an equidimensional fibration over* Z*. If* $d = 2$ *the general fiber is a del Pezzo surface, if* $d = 1$ *then the general fiber is a Mukai variety, while if* $d = 0$*, then* $-K_X$ *is ample, thus* X *is a Fano* 4*-fold, and* $\varrho(X) = 1$*.*

Remark 4.1.4. The case $(3,0)$ is not complete, in fact it contains many non-existing cases. More precisely let E be a del Pezzo 3-fold, i.e. $-K_E = 2\mathcal{L}$ with $\mathcal{L}$ ample.

If E is smooth then one can easily construct a F-M contraction of a smooth 4-fold of signature $(3,0)$ and exceptional divisor E by taking (see 3.2.4)

$$\varphi : \mathbf{V}(\mathcal{L}) \to Z = \mathrm{Spec}\Big(\bigoplus_{k\geq 0} H^0(F, \mathcal{L}^k)\Big)$$

where $\mathcal{L} = -E_{|E}$ and the map is associated to the evaluation of $\mathcal{L}^k$.

However it is a conjecture that there is no F-M contraction of a smooth 4-fold with a non-normal exceptional divisor E; in Section 3 of [Fu4] this case is discussed deeply and a lot of limitations on E are given (see 4.3.11 and the following discussion).

If E is singular but with normal singularities then a list of possible E was given in [Be] but this list contains many redundant cases.

The case $(4,3)$ is also not complete. In particular we have examples of appropriate 2-dimensional fibers except for the cases $\mathbb{P}^2 \cup \mathbf{S}_2$, $\mathbb{P}^2 \cup \mathbb{P}^2 \cup \mathbb{P}^2$ and $\mathbb{P}^2 \cup_f (\mathbb{F}_0) \cup_{C_0} \mathbb{P}^2$; we believe these cases cannot occur.

4.2. Fano–Mori contractions on a smooth n-fold with fibers of small dimension. In this section we present two theorems which characterise some F-M contractions of a smooth projective variety in higher dimension.

The first is due to T. Ando and it deals with F-M contractions with 1-dimensional fibers.

Theorem 4.2.1 ([An]). *Let $\varphi : X \to Z$ be a (local) Fano–Mori contraction of a smooth variety X of dimension n around a fixed fiber $F = \varphi^{-1}(z)$ such that* $\dim F = 1$.

(1) *If φ is birational then F is irreducible, $F \simeq \mathbb{P}^1$, $-K_X \cdot F = 1$ and its normal bundle is $N_{F/X} = \mathcal{O}(-1) \oplus \mathcal{O}^{(n-2)}$. The target Z is smooth and φ is a blow-up of a smooth codimension 2 subvariety of Z.*

(2) *If φ is of fiber type then Z is smooth and φ is a flat conic bundle. In particular one of the following is true:*

(i) *F is a smooth $\mathbb{P}^1$ and $-K_X \cdot F = 2$, $N_{F/X} \simeq \mathcal{O}^{(n-1)}$;*

(ii) *$F = C_1 \cup C_2$ is a union of two smooth rational curves meeting at one point and $-K_X \cdot C_i = 1$, $(N_{F/X})_{|C_i} \simeq \mathcal{O}^{(n-1)}$, $N_{C_i/X} \simeq \mathcal{O}^{(n-2)} \oplus \mathcal{O}(-1)$ for $i = 1, 2$;*

(iii) *F is a smooth $\mathbb{P}^1$, $-K_X \cdot F = 1$ and the fiber structure $\widetilde{F}$ on F is of multiplicity 2 (a non-reduced conic); the normal bundle of $\widetilde{F}$ is trivial while $N_{F/X}$ is either $\mathcal{O}(1) \oplus \mathcal{O}(-1)^{(2)} \oplus \mathcal{O}^{(n-4)}$ or $\mathcal{O}(1) \oplus \mathcal{O}(-2) \oplus \mathcal{O}^{(n-3)}$ depending on whether the discriminant locus of the conic bundle is smooth at z or not.*

The above theorem was generalised to the case of a variety X with terminal Gorenstein singularities by Mori and Kollár (see [KM1, 4.9 and 4.10.1]) for $n \geq 3$.

The case of an extremal contraction of a 3-fold X with terminal non-Gorenstein singularities is much more difficult; this was discussed in the celebrated paper of Mori

The next theorem is a generalisation of the above theorem of Ando in the framework of *adjunction theory of projective varieties*, a very classical theory (see [CE]), which was revitalized and improved in modern times by A. J. Sommese and his school (see [BS]).

One of the goals of this theory is to describe varieties X polarised by an ample line bundle L by means of the Fano–Mori contraction supported by $K_X + rL$ where r is the nef value of the pair (X, L). If X is smooth and $r \geq n-2$ then this goal is achieved and we refer the reader to the book [BS] for an overview of the theory; see [And1], [And2] and [Me1] for the singular case.

The next theorem, proved in [AW1], shows that the goal is also achieved when the nef value is large with respect to the dimension of fibers of φ.

THEOREM 4.2.2 ([AW1]). *Let $\varphi : X \to Z$ be a (local) Fano–Mori contraction of a smooth variety X and let $F = \varphi^{-1}(z)$ be a fiber. Assume that φ is supported by $K_X + rL$, with L a φ-ample line bundle on X.*

(1) *If $\dim F \leq r-1$ then Z is smooth at z and φ is a projective bundle in a neighborhood of F.*

(2) *If $\dim F = r$ then, after possibly shrinking Z and restricting φ to a neighborhood of F, Z is smooth and*

- (i) *if φ is birational then φ blows a smooth divisor $E \supset X$ to a smooth codimension $r-1$ subvariety $S \supset Z$,*
- (ii) *if φ is of fiber type and $\dim Z = \dim X - r$ then φ is a quadric bundle,*
- (iii) *if it is of fiber type and $\dim Z = \dim X - r + 1$ then $r \leq \dim X/2$, $F = \mathbb{P}^r$ and the general fiber is $\mathbb{P}^{(r-1)}$.*

The basic steps of the proofs of these theorems are worked out in the next sections, together with the proofs of the results in the previous section.

4.3. The fibers of a Fano–Mori contraction. In this section we will try to give more information on the possible fibers of the F-M contractions. In particular we will classify all possible fibers of dimension less than or equal to two.

4.3.1. *Using the vanishing theorem.* We want to show how the vanishing theorem implies vanishing results on the fiber. Subsequently we show how these results, via the computation of the Hilbert polynomial of the (normalization) of the fiber, imply a bound on the dimension of the fiber.

The proof of the following proposition can be found in [Mo3, 3.20, 3.25.1], [Fu2, 11.3], [An] and [AW3, 1.2.1].

PROPOSITION 4.3.1 (Vanishing of the highest cohomology). *Let $\varphi : X \to Z$ be a local F-M contraction around F supported by $K_X + rL$ (see 3.1.9). Let F' be a subscheme of X whose support is contained in the fiber F of φ, so that $\varphi(F') = z$. If either $t > -r$ or $t = -r$ and $\dim F > \dim X - \dim Z$ then*

PROPOSITION 4.3.2. *Under the assumptions of the above proposition let also $X' \in |L|$ be the zero locus of a non-trivial section of L. Then*

$$H^{\dim(F\cap X')}(F' \cap X', tL_{|F'\cap X'}) = 0$$

if either $t > -r+1$ or $t = -r+1$ and $\dim(F \cap X') \geq \dim X - \dim Z$.

Proof. We give the proof of the second assertion, the proof of the first is similar. Note that $H^i(X, tL) = 0$ for $i > 0$ and $t > -r$ by theorem 2.1.2; moreover, $H^i(X, tL) = 0$ for $t = -r$ and $i > \dim X - \dim Z$, for the so called Grauert–Riemenschneider–Kollár vanishing theorem (see [KMM, Theorems 1.2.4 and 1.2.7]). Thus from the exact sequence

$$0 \to -L \to \mathcal{O}_X \to \mathcal{O}_{X'} \to 0$$

tensorised by tL we also have $H^i(X', tL_{X'}) = 0$ for $i > 0$ and $t > -r+1$ or $t = -r+1$ and $i > \dim X - \dim Z$.

Now let $\mathcal{I}_{F'\cap X'}$ be the ideal of $F' \cap X'$ in X' and consider the sequence

$$0 \to \mathcal{I}_{F'\cap X'} \otimes tL \to \mathcal{O}_{X'} \otimes tL \to \mathcal{O}_{F'\cap X'} \otimes tL \to 0.$$

Take the associated long exact sequence. Since $H^i(X', \mathcal{I}_{F'\cap X'} \otimes tL)_z = 0$ for $i > q := \dim F \cap X'$, the map $H^q(tL_{X'}) \to H^q(F' \cap X', tL_{F'})$ is surjective and the statement follows from what we have observed at the beginning. ■

The following result is a direct consequence of the above proposition; it was proved by T. Fujita [Fu1], following arguments of S. Mori and T. Ando.

THEOREM 4.3.3. *Let $\varphi : X \to Z$ be a local F-M contraction around F supported by $K_X + rL$. Then $\dim F \geq r-1$, and if $\dim F > \dim X - \dim Z$ then $\dim F \geq \lfloor r \rfloor$.*

Proof. Let S be a component of a fiber F of dimension s and let $g : W \to S$ its desingularisation. By Proposition 4.3.1 and the Leray spectral sequence for g, exactly as in Lemma 2.4 of [Fu1], we get

$$H^s(W, g^*(tL)) = 0$$

if $t > -r$ or $t = -r$ and $\dim F > \dim X - \dim W$.

On the other hand, since $g^*(L)$ is nef and big on W, by the Kawamata–Viehweg vanishing theorem we have $H^i(W, g^*(tL)) = 0$ for $t \geq -r$ and $0 < i < s$. Moreover, since L is ample, we also have $H^0(W, g^*(tL)) = 0$ for $t < 0$.

Consider now the Hilbert polynomial $\chi(t) := \chi(W, g^*(tL))$; it is a polynomial in t of degree equal to $\dim W = \dim S$. By what was proved above χ is zero for all integers t such that $0 > t > -r$; if $\dim F > \dim X - \dim Z$ and r is an integer then χ is zero also for $t = -r$. The inequalities follow then immediately since $\deg \chi \geq$ number of its zeros. ■

4.3.1.1. *Exercises and examples*

EXERCISE 4.3.4 (see [Fu1]). Let (X, L) be a polarized variety; $K_X + nL$ is nef except when $(X, L) = (\mathbb{P}^n, \mathcal{O}(1))$. Also, if then $n \geq 3$, then $K_X + (n-1)L$ is nef unless (X, L) is one of the following: $(\mathbb{P}^n, \mathcal{O}(1))$, X is the quadric in $\mathbb{P}^n$ and L is a hyperplane section, X

This result was proved in [Fu1] with the use of the above theorems; a completely different proof, which makes use of the deformation of rational curves and which works in any characteristic, has recently been given in [Ka-Ko].

4.3.2. *Existence of a ladder for a fiber of a F-M contraction; horizontal slicing.* Here we will develop in more detail the method outlined in the introduction of Section 3.3. With the use of Theorem 3.3.1 we will study the fibers of a F-M contraction inductively. More precisely if F is a fiber of a Fano–Mori contraction of sufficiently high dimension (i.e. with "small" difficulty) then we can construct a ladder for the pair (F, L_F) and prove that $\Delta(F, L) = 0$.

In order to do this we first start with a Bertini type theorem.

Proposition–Definition 4.3.5 ([AW1, Lemma 2.6], [Me3, Lemma 1.3]; Horizontal slicing). *Let $\varphi : X \to S$ be a local contraction around $\{F\}$ supported by $K_X + rL$. Let $H_i \in |L|$ be generic divisors and $X_k = \bigcap_{i=1}^k H_i$, a scheme-theoretic intersection*; *assume that* $\dim X_k = n - k$ (> 0) *and that* $r - k \geq 0$; *note that since X_k is a complete intersection it is $\mathbb{Q}$-Gorenstein, i.e. K_{X_k} is $\mathbb{Q}$-Cartier.*

(i) *Let $\varphi_{|X_k} = g \circ \varphi_k$ be the Stein factorisation of $\varphi_{|X_k} : X_k \to S$. Then $\varphi_k : X_k \to S_k$ is a morphism with connected fiber, around $\{F \cap \bigcap_{i=1}^k H_i\}$, supported by $K_{X_k} + (r-k)L_{|X_k}$ and S_k is affine. In particular if X_k is normal then φ_k is a local contraction.*

Assume that X has LT singularities and, if ε is a sufficiently small positive rational number, that $r \geq \varepsilon\gamma(\varphi)$ and $k \leq r + 1 - \varepsilon\gamma(\varphi)$.

(ii) *Outside* $\mathrm{Bsl}\,|L|, X_k$ *has singularities which are of the same type as the ones of X and any section of L on X_k extends to a section of L on X.*

Proof. See [Me3]. (i) is just the Stein factorisation (see [Ha, III.11.5]) and the adjunction formula, once we notice that $f_{|X_k}(X_k) = \mathrm{Spec}(H^0(H, \mathcal{O}_{X_k}))$ and that there is a morphism $S_k \to S$ induced by the ring morphism $H^0(X, \mathcal{O}_X) \to H^0(X_k, \mathcal{O}_{X_k})$.

For (ii) the first statement is just the Bertini theorem, while for the latter consider the exact sequences

$$0 \to \mathcal{O}_{X_i}(-L) \to \mathcal{O}_{X_i} \to \mathcal{O}_{X_{i+1}} \to 0, \quad 0 \to \mathcal{O}_{X_i} \to \mathcal{O}_{X_i}(L) \to \mathcal{O}_{X_{i+1}}(L) \to 0.$$

Thus to prove the assertion it is enough to prove that $H^1(X_i, \mathcal{O}_{X_i}) = 0$ for $i \leq r - \varepsilon(\dim X - \dim S)$. But this is equivalent, using inductively the first sequence tensored, to $H^1(X, -iL) = 0$ for $i \leq r - \varepsilon(\dim X - \dim S)$, which follows from the vanishing theorem 2.1.2. ∎

Theorem 4.3.6. *Let $\varphi : X \to Z$ be a local F-M contraction around F supported by $K_X + rL$ of type $(d(\varphi), \gamma(\varphi), \Phi(\varphi))$. Let S be any component of F_{red} and ε a sufficiently small positive rational number. If $\Phi(\varphi) \leq 1 - \varepsilon\gamma(\varphi)$ or the type of φ is $(d, 1, 1)$ with $d \leq 0$, then $\Delta(S, L_S) = 0$. If $\Phi(\varphi) \leq -\varepsilon\gamma(\varphi)$ or if the type of φ is $(-1, 1, 0)$, then F_{red} is irreducible and isomorphic to $\mathbb{P}^{\dim F}$.*

Proof. We present the proof from [Me3, Theorem 2.17]; this is a slight generalisation of

To prove the first assertion let S be an irreducible component of F_{red} of dimension s and let $\delta := L^s \cdot S$. We have to prove that $h^0(S, L_S) \geq \delta + r + 1$. This follows obviously if we prove that there are at least $\delta + r + 1$ independent sections of $H^0(X, L)$ not vanishing identically on S.

By Propositions 4.3.5 and 3.3.1 we reduce to the case of a contraction $\varphi : X \to T$ with 1-dimensional fiber F. Then, by assumption, we can use again 4.3.5 and 3.3.1 and go one step further with a section $H \in |L|$; $\varphi_{|H} : H \to T$ is finite and by 4.3.5 all sections of $|L_{|H}|$ extend to sections of $|L|$ proving the assertion.

Finally assume that $\Phi(\varphi) \leq -\varepsilon\gamma(\varphi)$ and assume, by contradiction, that the fiber has (at least) two irreducible components intersecting in a subvariety of dimension $t \leq r-1$. By the base point freeness of L, we can choose $t+1$ sections of L intersecting transversally in a variety with log terminal singularities and meeting the two irreducible components not in their intersection. By construction the map φ restricted to this variety has disconnected fibers, contrary to 4.3.5. Similarly one can prove that $L^r \cdot F = 1$ (we can slice to points and still have the connectedness, but then we must have only one point ...) and thus that $F = \mathbb{P}^{\dim F}$. ■

An immediate corollary in the case of 2-dimensional fiber is the following.

Corollary 4.3.7. *Let F be a 2-dimensional fiber of a F-M contraction $\varphi : X \to Z$ of a Gorenstein variety X and let F' be any component. Assume that φ is birational or that the general non-trivial fiber has dimension* 1. *Then F' is normal and the pair $(F', L_{|F'})$ has sectional* (*and thus Fujita Δ*) *genus* 0 *and therefore* (*see* 1.2.4) *it is among the following*:

(1) $(\mathbb{P}^2, \mathcal{O}(e))$ *with* $e = 1, 2$,
(2) $(\mathbb{F}_r, C_0 + kf)$ *with* $k \geq r+1$, $r \geq 0$,
(3) $(\mathbf{S}_r, \mathcal{O}_{\mathbf{S}_r}(1))$ *with* $r \geq 2$.

Moreover F is Cohen–Macaulay unless the zero locus of a general section in $|L_F|$ is disconnected.

In the next subsection we will see however that not all the possibilities can occur if the domain X is smooth (or has very good singularities). Another type of argument is needed to get rid of some cases.

To conclude the section we will mention another Bertini type theorem which has to do with the sections of the supporting divisor of the F-M contraction.

Proposition–Definition 4.3.8 (Vertical slicing, [AW1]). *Let $\varphi : X \to S$ be a local contraction supported by $K_X + rL$ with $r \geq -1 + \varepsilon\gamma(\varphi)$ and ε a sufficiently small positive rational number. Assume that X has LT singularities and let h be a general function on S. Let $X_h = \varphi^*(h)$. Then the singularities of X_h are not worse than those of X and any section of L on X_h extends to X.*

4.3.2.1. *Related topics and further results*

Exercise 4.3.9 (Lifting a contraction). Let X be a smooth complex projective variety of dimension n and L be an ample line bundle with a section $D \in |L|$ with good singularities there exists a section $s \in \Gamma(\mathcal{E})$ whose zero locus, $D = (s = 0)$, is a smooth submanifold of the expected dimension $\dim D = \dim X - r = n - r$.)

A classical and natural problem is to lift the geometric properties of D to get information on the geometry of X; a very good account of this problem can be found in [BS, Chapter 5]. In [AO1] and [AO3] the problem was considered from the point of view of Mori theory, posing the following question: assume that D is not minimal, i.e. Z has at least one extremal ray in the negative part of the Mori cone; does this ray (or the associated extremal contraction) determine a ray (or a contraction) in X, and if so, does this new ray determine the structure of X?

For instance assume that D is $\mathbb{P}^s$ or a scroll.

EXERCISE 4.3.10 (Constructing F-M contractions). Find a local Fano–Mori contraction around F supported by $K_X + rL$ of type $(d(\varphi), \gamma(\varphi), \Phi(\varphi))$ with $\Phi(\varphi) > 1 - \varepsilon\gamma(\varphi)$ and for which F is not Cohen–Macaulay or in general not normal. Can you find such an example with X smooth? (the examples (1.18) in [AW2] and (3.6) in [AW3]).

However there is the following

CONJECTURE 4.3.11. Let $\varphi : X \to Z$ be a Fano–Mori contraction of a manifold of dimension ≤ 4. Then all the fibers are normal, with the exception in (3.6) of [AW3].

Note that the conjecture, after Mori's and Andreatta–Wiśniewski's work, is open only for the case in which $\varphi : X \to Z$ is a birational Fano–Mori contraction of a manifold of dimension 4 which contracts an irreducible divisor E to a point. Moreover, by [Fu4], E, which is a del Pezzo 3-fold, is (possibly) non-normal only if $(-K_E)^3 = 7$, $\mathrm{Sing}(E) \cong \mathbb{P}^2$, the normalization of E is $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(5))$ and some other conditions are satisifed.

4.3.3. *Rational curves on the fiber of F-M contractions.* In this subsection we use another fundamental feature of a Fano–Mori contraction in order to complete the classification of possible 2-dimensional fibers: namely the existence of rational curves in its fibers.

4.3.3.1. *General facts.* We have in fact the following existence theorem due to Mori [Mo2], [Mo3] in the smooth case and extended by Kawamata to the log terminal case in [Ka0]. We recall that a *rational curve* is a curve whose normalization is $\mathbb{P}^1$. (Although we work over $\mathbb{C}$, we note that if X is smooth then the existence theorem is also true in positive characteristic; this concerns also the subsequent results obtained via deformation methods.)

THEOREM 4.3.12 (Existence of rational curves). *Let $\varphi : X \to Z$ be a Fano–Mori contraction of a variety with log terminal singularities. Then the exceptional locus of φ is covered by rational curves contracted by φ.*

In this section we study deformations of rational curves following ideas started with the paper of Mori [Mo2]. We discuss only some of the results, concentrating on the case of smooth X. We refer the reader to the book of Kollár [Ko3] for general results concerning

Theorem 4.3.13. *Let C be a (possibly reducible) connected curve such that $H^1(C, \mathcal{O}_C)=0$ and assume that C is smoothable (see* [Ko3, II.1.10] *for the definition; an example of smoothable curve is a tree of smooth rational curves, i.e.* $C = \bigcup_i R_i$ *where:* (i) *any R_i is a smooth rational curve,* (ii) *R_i intersects $\sum_{j=1}^{i-1} R_j$ in a single point which is an ordinary node of C, see* [Ko3, II.1.12]*). Suppose that $f : C \to X$ is an immersion of C into a smooth variety X. Then any component of the Hilbert scheme containing $f(C)$ has dimension at least $-K_X \cdot C + (n-3)$.*

The above result has several different versions. For example, Mori [Mo2] proved a version of it for maps of rational curves with fixed points. An important part of Mori's proof of the existence of rational curves is a technique of deforming rational curves with a fixed 0-dimensional subscheme ("bending" these curves) in order to produce rational curves of lower degree with respect to a fixed ample divisor (to "break" them). In short: if a rational curve can be deformed inside X with two points fixed then it has to break.

Mori's bend-and-break technique was used by Ionescu and Wiśniewski (see [Io, 0.4] and [Wi, 1.1]) to prove a bound on the dimension of the fiber. The reader can compare this bound with the one obtained in Theorem 4.3.3.

Theorem 4.3.14. *Let $\varphi : X \to Z$ be a Fano–Mori contraction of an extremal ray R of a smooth variety X. Let E be the exceptional locus of φ (if φ is of fiber type then $E := X$) and let S be an irreducible component of a (non-trivial) fiber F. Let $l = \min\{-K_X \cdot C$: C is a rational curve in $S\}$. Then $\dim S + \dim E \geq \dim X + l - 1$.*

Corollary 4.3.15. *Let $\varphi : X \to Z$ be a Fano–Mori contraction of a smooth variety X supported by $K_X + rL$. Let E be the exceptional locus of φ and let S be an irreducible component of a (non-trivial) fiber F. Then $\dim S + \dim E \geq \dim X + r - 1$.*

Proposition 4.3.16 ([ABW, Lemma (1.1)]). *Under the hypotheses of the above corollary, if equality holds for an irreducible component then the normalisation of S is $\mathbb{P}^s$.*

Thus one can propose the following conjecture (it was actually posed in [AW2]):

Conjecture 4.3.17. Under the hypotheses of Theorem 4.3.14, if equality holds for an irreducible component S then its normalisation is isomorphic to $\mathbb{P}^s$.

A step toward the conjecture was given by the following theorem proved in [AO2].

Theorem 4.3.18. *If the contraction $\varphi : X \to Z$ is divisorial then the conjecture is true and φ is actually a smooth blow-up (i.e. of a smooth submanifold of Z which is also smooth).*

The following result, which was a long-lasting conjecture, has been proved recently; it is not difficult to show that this proves a part of the above conjecture.

Theorem 4.3.19 ([CMS], [Keb]). *If X is a smooth projective variety of dimension n such that $K_X \cdot C \leq -n-1$ for any complete curve $C \subset X$ then $X \simeq \mathbb{P}^n$.*

In Cho–Miyaoka–Shepherd Barron's paper a more general version is stated; in particular the variety X can have normal singularities. This version should imply the above

Let us notice that the last theorem is a very nice generalisation of the famous theorem of S. Mori, i.e. the proof of the Hartshorne–Frenkel conjecture.

THEOREM 4.3.20 ([Mo2]). *If X is a smooth variety with ample tangent bundle then $X \simeq \mathbb{P}^n$.*

In a slightly different direction, also the following generalisation of Mori's theorem has recently been proved.

THEOREM 4.3.21 ([AW5]). *If X is a smooth variety which has an ample locally free subsheaf of the tangent bundle then $X \simeq \mathbb{P}^n$.*

4.3.3.2. *Rational curves on fibers of a F-M contraction of dimension* ≤ 2. Now we work out a complete classification of fibers F of dimension ≤ 2 of a F-M contraction of a smooth variety X.

LEMMA 4.3.22. *If a fiber F of a Fano–Mori contraction of a smooth n-fold X contains a component of dimension* 1 *then F is of pure dimension* 1 *and $-K_X \cdot F \leq 2$. In particular F is a line or a conic (with respect to the relative very ample line bundle $-K_X$), the last possibly reducible or non-reduced. If the contraction is birational then F is a line.*

Proof. Let F' be a 1-dimensional component of F (F' is a rational curve because of 4.3.6). Then, by 4.3.13,

$$\dim_{[F']} \operatorname{Hilb}(X) \geq -K_X \cdot F' + (n-3)$$

and therefore small deformation of F' sweeps out at least a divisor. More precisely: taking a small analytic neighborhood of $[F']$ in Hilb and the incidence variety of curves we can produce an analytic subvariety $E \subset X$ which is proper over Z such that $F \cap E = F'$ and $\dim E \geq n-1$. This implies that all components of F meeting F' are of dimension 1 and by connectedness of F we see that F is of pure dimension 1. The bound on the degree can be obtained similarly (note that because of the base point free theorem $-K_X$ is φ-very ample so that one can apply 4.3.13 to a curve consisting of two components). ■

Suppose now that $\varphi : X \to Z$ is a local Fano–Mori contraction of a smooth variety and F is an *isolated* fiber of φ of dimension ≥ 2; isolated means that all the neighboring fibers are of dimension ≤ 1. That is, because Z is affine, we can assume that all the fibers of φ except F are of dimension ≤ 1.

Note that by the base point freeness theorem $L := -K_X$ is φ-very ample (see 3.3.1; the theorem states only the relative base point freeness of L, but as noticed in [AW3, Proposition 1.3.4], after possibly shrinking the affine variety Z, the same proof yields the relative very ampleness of L).

By Lemma 4.3.22 all 1-dimensional fibers of φ are of degree 1 (lines), or ≤ 2 (conics), with respect to $-K_X$, if φ is birational or of fiber type, respectively.

Let now $C \subset F$ be a rational curve or an immersed image of a smoothable curve of genus 0. If the degree of C with respect to $-K_X$ is greater than that of 1-dimensional fibers of φ, then deformations of C in X *must remain inside* F, which, in view of 4.3.13, Lemma 4.3.23. *In the above situation*

$$\dim_{[C]} \mathrm{Hilb}(F) \geq -K_X \cdot C + (n-3).$$

Let us explain why this simple observation is useful for the understanding of the structure of the fiber F. Lemma 4.3.23 can in fact be used to rule out many redundant cases in the list 4.3.7 of possible components of F. We note that the very ampleness of $L = -K_X$ as well as the precise description of the components of the fiber in 4.3.7 allow us to choose properly the curve which satisfies the assumptions in 4.3.23.

We will give just an example which explains our argument. All possible cases are discussed in detail in [AW3, Section 4].

Suppose that S is a component of F and $S \simeq \mathbf{S}_r$ where $r \geq 3$. Then for the curve C we take the union of general $r+1$ lines passing through the vertex of $\mathbf{S}_r$ (the lines are general so that none of them is contained in any other component of F). Then Lemma 4.3.23 implies that $\mathbf{S}_r$ cannot be a component of F, for $r \geq 3$ if φ is birational and for $r \geq 4$ if φ is of fiber type.

Also this way, using 4.3.23, for a reducible fiber F we can limit the possible combinations of irreducible components of F. To show how, let us consider the following situation.

Lemma 4.3.24. *Let F be an isolated fiber of dimension ≥ 2 of a birational contraction $\varphi : X \to Z$ of a smooth n-fold X. Suppose that the exceptional locus of φ is covered by rational curves which are lines with respect to $-K_X$. If there exists a non-trivial decomposition $F = F_1 \cup F_2$ then $F_1 \cap F_2$ does not contain 0-dimensional components.*

Proof. Let $x \in F_1 \cap F_2$ be an isolated point of the intersection. Since X is smooth, $\dim_x F_1 + \dim_x F_2 \leq n$. For $i = 1, 2$ let $C_i \subset F_i$ be a line containing x. The variety parametrising deformations of C_i inside F_i with x fixed is of dimension $\leq \dim_x F_i - 1$. Indeed, take a point $y \in F_i$; then by the bend-and-break argument of Mori (see Section II.5 of [Ko3]) there is only one curve of the family passing through both x and y (i.e. through two distinct points passes only one line, with respect to any ample line bundle).

Take $C = C_1 \cup C_2$. Then

$$\dim_{[C]} \mathrm{Hilb}(F) \leq \dim_x F_1 + \dim_x F_2 - 2 \leq n - 2$$

and because $-K_X \cdot C = 2$ we arrive at a contradiction with 4.3.23. ■

Remark 4.3.25. Let us note that the above conclusion of 4.3.24 is no longer true if we do not assume that φ is birational (see [AW2, Example (2.11.2)]).

With a combination of the above arguments, all based on Lemma 4.3.23, and through a long list of cases, in Section 4 of [AW3], the following has been proved.

Proposition 4.3.26 ([AW3, Sect. 4]). *Let $\varphi : X \to Z$ be a Fano–Mori contraction of a smooth n-fold X with an isolated 2-dimensional fiber F, and let $L = -K_X$. If φ is*

$n \geq 5$	$n = 4$	$n = 3$
$(\mathbb{P}^2, \mathcal{O}(1))$	$(\mathbb{P}^2, \mathcal{O}(1))$	$(\mathbb{P}^2, \mathcal{O}(1))$
$(n \leq 6)$	$(\mathbb{F}_0, C_0 + f)$	$(\mathbb{P}^2, \mathcal{O}(2))$
	$(\mathbf{S}_2, \mathcal{O}_{\mathbf{S}_2}(1))$	$(\mathbf{S}_2, \mathcal{O}_{\mathbf{S}_2}(1))$
	$(\mathbb{P}^2 \cup \mathbb{P}^2, \mathcal{O}(1))$	$(\mathbb{F}_0, C_0 + f)$
		$(\mathbb{F}_1, C_0 + 2f)$
		$\mathbb{P}^2 \cup_{C_0} \mathbb{F}_2$, with $L_{\lvert\mathbb{P}^2} = \mathcal{O}(1), L_{\lvert\mathbb{F}_2} = C_0 + 3f$

If φ is of fiber type and L is φ-spanned then we have the following possibilities for the pair (F, L_F):

$n \geq 5$	$n = 4$, irreducible	$n = 4$, reducible	$n = 3$
$(\mathbb{P}^2, \mathcal{O}(1))$	$(\mathbb{P}^2, \mathcal{O}(1))$	$\mathbb{P}^2 \cup \mathbb{P}^2$	$(\mathbb{F}_0, C_0 + 2f)$
$(n \leq 7)$	$(\mathbb{P}^2, \mathcal{O}(2))$	$\mathbb{P}^2 \bullet \mathbb{P}^2$	$\mathbb{F}_0 \cup \mathbb{F}_1$,
$(\mathbb{F}_0, C_0 + f)$	$(\mathbf{S}_2, \mathcal{O}(1))$	$\mathbb{P}^2 \cup \mathbb{F}_0$	$L_{\mathbb{F}_0} = C_0 + f$
$(n = 5)$	$(\mathbf{S}_3, \mathcal{O}(1))$	$\mathbb{P}^2 \cup_{C_0} \mathbb{F}_1$	$L_{\mathbb{F}_1} = C_0 + 2f$
$(\mathbb{P}^2 \cup \mathbb{P}^2, \mathcal{O}(1))$	$(\mathbb{F}_1, C_0 + 2f)$	$\mathbb{P}^2 \cup \mathbf{S}_2$	
$(n = 5)$	$(\mathbb{F}_0, C_0 + f)$	$\mathbb{P}^2 \cup \mathbb{P}^2 \cup \mathbb{P}^2$	
		$\mathbb{P}^2 \cup_{C_0} \mathbb{F}_0 \cup_f \mathbb{P}^2$	

In the above list the components of reducible fibers have a common line (*in some cases we point out which line it is*) *with the unique exception of two $\mathbb{P}^2$'s which meet at a point—we denote this union by $\bullet$.* (*We suppress the description of L whenever it is clear.*)

REMARK 4.3.27. Let us say again that for almost all the above possibilities we can construct examples with appropriate isolated 2-dimensional fiber (see Section 3 of [AW3]). However, there are some exceptions for which we have been unable to construct examples and we do not expect that all of them exist. This concerns only fiber type contractions and reducible fibers: $\mathbb{P}^2 \cup \mathbb{P}^2$ for $n = 5$ and $\mathbb{P}^2 \cup \mathbf{S}_2$, $\mathbb{P}^2 \cup \mathbb{P}^2 \cup \mathbb{P}^2$ and $\mathbb{P}^2 \cup \mathbb{F}_0 \cup \mathbb{P}^2$ for $n = 4$.

4.4. The description of the normal bundle of a fiber of a F-M contraction. In order to describe a contraction locally, after having determined the fiber, one has to find the possible normal bundles of these fibers; of course when this is possible, that is, when the fiber is a local complete intersection in X.

This can be considered as a second order type problem and it is very hard compared to the determination of the fiber. If the fiber is 1-dimensional it was considered by S. Mori in the case of 3-folds and by T. Ando in general. The case of 2-dimensional fiber is one of the main achievements of [AW3].

If the fiber is a divisor in X its normal bundle is already given by the adjunction formula (since we know $K_{X|F}$); in general this gives only the first Chern class of the normal bundle.

Let us start with an easy lemma, which however gives a broader picture of what we

Lemma 4.4.1 ([AW3, Proposition (3.5)]). *Let $\varphi : X \to Z$ be a Fano–Mori or crepant contraction of a smooth variety with a fiber $F = \varphi^{-1}(z)$. Assume that F is a locally complete intersection and that the blow-up $\beta : \widehat{X} \to X$ of X along F has log terminal singularities. Denote by $\widehat{F}$ the exceptional divisor of the blow-up. Then the following properties are equivalent*:

(a) *The bundle $N^*_{F/X}$ is generated by global sections on F.*

(b) *The invertible sheaf $\mathcal{O}_{\widehat{X}}(-\widehat{F})$ is generated by global sections at any point of $\widehat{F}$.*

(c) $\varphi^{-1}m_z \cdot \mathcal{O}_X = \mathcal{I}_F$ *or, equivalently, the scheme-theoretic fiber structure of F is reduced, i.e. $\widetilde{F} = F$.*

(d) *There exists a Fano–Mori contraction $\widehat{\varphi} : \widehat{X} \to \widehat{Z} = \mathrm{Proj}_Z(\bigoplus_k m_z^k)$ onto a blow-up of Z at the maximal ideal of z, and $\varphi^*(\mathcal{O}_{\widehat{Z}}(1)) = \mathcal{O}_{\widehat{X}}(1)$.*

4.4.1. *The normal bundle of a 1-dimensional fiber.* The case in which F is a fiber of dimension 1 was mainly studied, after S. Mori, by T. Ando [An]; we will report some of his results and we will introduce an alternative proof, as done in [AW3].

Let C be an irreducible component of F. As we saw in Lemma 4.3.22, C is a rational curve and it can be either a line or a conic with respect to $-K_X$. In the latter case φ is of fiber type.

Let $\mathcal{I}$ be the ideal of $C \subset X$ (with the reduced structure) and consider the exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_X/\mathcal{I}^2 \to \mathcal{O}_X/\mathcal{I} \to 0.$$

In the associated long cohomology sequence the map of global sections $H^0(\mathcal{O}_X/\mathcal{I}^2) \to H^0(\mathcal{O}_X/\mathcal{I})$ is surjective. Moreover, by 4.3.1, we have the vanishing $H^1(\mathcal{O}_X/\mathcal{I}^2) = 0$. Therefore $H^1(\mathcal{I}/\mathcal{I}^2) = 0$, which gives a bound on $N^*_{C/X} = \mathcal{I}/\mathcal{I}^2$. Namely if $N_{C/X} = \bigoplus \mathcal{O}(a_i)$ then $a_i < 2$. On the other hand, by adjunction, $\det(N_{C/X}) = \sum a_i = \mathcal{O}(-2 - K_X \cdot C)$ and thus the list of possible values of $(a_1, \ldots, a_{n-1})$ is finite.

If φ is a good birational contraction then we even have a better bound because, similarly to the above and using 4.3.1, we actually get $H^1(N^*_{C/X} \otimes \mathcal{O}(K_X \cdot C)) = 0$. Therefore, since $K_X \cdot C = 1$, there is only one possibility, namely $N_{C/X} = \mathcal{O}(-1) \oplus \mathcal{O}^{(n-2)}$.

If φ is of fiber type then the estimate coming from this technique is not sufficient and one has to use other arguments. More precisely, one has to deal with a scheme associated to a double structure on C (see [An]).

It is also convenient to use arguments coming from deformation theory. Namely, the possibilities which can occur from the above vanishing, if $n = 3$, are the following:

$$\mathcal{O} \oplus \mathcal{O}, \quad \mathcal{O} \oplus \mathcal{O}(-1), \quad \mathcal{O}(1) \oplus \mathcal{O}(-2), \quad \mathcal{O}(1) \oplus \mathcal{O}(-1).$$

We will show that the last possibility does not occur using an argument related to the deformation technique. It can also be used to deal with 2-dimensional fibers.

Lemma 4.4.2. *The normal bundle $N_{C/X}$ cannot be $\mathcal{O}(1) \oplus \mathcal{O}(-1)$.*

Proof. Assume the contrary and let $\psi : \widehat{X} \to X$ be the blow-up of X along C. Let $E := \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(-1))$ be the exceptional divisor. Let C_0 be the curve contained in E which is the section of the ruled surface $E \to C$ corresponding to the surjective map from C_0 to C. Therefore $K_{\widehat{X}} \cdot C_0 = K_X \cdot C + E \cdot C_0 = -1$. In particular this implies that C_0 moves at least in a 1-dimensional family on $\widehat{X}$ (see 4.3.13). Since it does not move on E it means that it goes out of E. Since C_0 is contracted by $\varphi \circ \psi$ this implies that $E \cdot C_0 = 0$, a contradiction since $E \cdot C_0 = 1$. ■

4.4.2. *The normal bundle of a 2-dimensional fiber.* The case where F is a fiber of dimension 2 was studied by M. Andreatta and J. A. Wiśniewski [AW3]. We will report here the main results contained in Section 5.7 of [AW3], referring to it for proofs and more details.

Let us present in general the point of view of [AW3]. To understand higher dimensional fibers of Fano–Mori contractions we will slice them down. Thus we will need some kind of "ascending property".

Suppose that $\varphi : X \to Z$ is a Fano–Mori contraction of a smooth variety, $\mathcal{L}$ is an ample line bundle on X such that $-K_X - \mathcal{L}$ is φ-(nef & big); for instance if φ is birational one can take $\mathcal{L} = L := -K_X$. Let $F = \varphi^{-1}(z)$ be a (geometric) fiber of φ. Suppose that F is a locally complete intersection. Let $X' \in |\mathcal{L}|$ be a normal divisor which does not contain any component of F. Then the restriction of φ to X', call it φ', is a contraction, either Fano–Mori or crepant (see 4.3.5). The intersection $F' = X' \cap F$ is then a fiber of φ'. The regular sequence $(g_1, \ldots, g_r)$ of local generators of the ideal of the fiber F in X descends to a regular sequence in the local ring of X' which defines a subscheme $F \cdot X'$ supported on $F' = F \cap X'$, call it $\overline{F}'$. Note that if the divisor X' has multiplicity 1 along each of the components of F then, since a locally complete intersection has no embedded components, we get $\overline{F}' = F'$.

Lemma 4.4.3. *The scheme $\overline{F}'$ is a locally complete intersection in X' and*

$$N^*_{\overline{F}'/X'} \otimes_{\mathcal{O}_{\bar{F}'}} \mathcal{O}_{F'} \simeq (N^*_{F/X})_{|F'}.$$

If moreover X' is smooth, $\mathcal{L}$ is spanned and $\dim F' = 1$, *then*

$$H^1(F', (N^*_{F/X})_{|F'}) = 0.$$

Proof. The first part of the lemma follows from the preceding discussion so it is enough to prove the vanishing. Let $\mathcal{J}$ be the ideal of $\overline{F}'$ in X'. From 4.3.1 we know that $H^1(\overline{F}', \mathcal{O}_{X'}/\mathcal{J}^2) = 0$ and since we have an exact sequence

$$0 \to \mathcal{J}/\mathcal{J}^2 = N^*_{\overline{F}'/X'} \to \mathcal{O}_{X'}/J^2 \to \mathcal{O}_{X'}/\mathcal{J} = \mathcal{O}_{\overline{F}'} \to 0,$$

we will be done if we show $H^0(\overline{F}', \mathcal{O}_{\overline{F}'}) = \mathbb{C}$. Since $H^1(\overline{F}', \mathcal{O}_{\overline{F}'}) = 0$, this is equivalent to $\chi(\mathcal{O}_{\overline{F}'}) = 1$. The equality $H^0(\overline{F}', \mathcal{O}_{\overline{F}'}) = \mathbb{C}$ is clear if $\overline{F}'$ is reduced. But since $\mathcal{L}$ is spanned and F is a locally complete intersection, there exists a flat deformation of $\overline{F}'$ to another intersection $F \cdot X''$ which is reduced. This is what we need, because flat deformation preserves the Euler characteristic. ■

Now consider the following ascending property. Take $x \in F'$. Suppose that the ideal of F', or equivalently $N^*_{F'/X'}$, is generated by global functions from X'. That is, there exist global functions $g'_1, \ldots, g'_r \in \Gamma(X', \mathcal{O}_{X'})$ which define F' at x. Then, since $H^1(X, -\mathcal{L}) = 0$, these functions extend to $g_1, \ldots, g_r \in \Gamma(X, \mathcal{O}_X)$ which define F. Thus passing from

LEMMA 4.4.4. *If $N^*_{F'/X'}$ is spanned at a point $x \in F'$ by global functions from $\Gamma(X', \mathcal{O}_{X'})$ then $N^*_{F/X}$ is spanned at x by functions from $\Gamma(X, \mathcal{O}_X)$. If $N^*_{F'/X'}$ is spanned by global functions from $\Gamma(X', \mathcal{O}_{X'})$ everywhere on F' then $N^*_{F/X}$ is nef.*

Proof. We only have to prove the second claim of the lemma. Since $F' \subset F$ is an ample section, the set where $N^*_{F/X}$ is not generated by global sections is finite in F. Therefore the restriction $(N^*_{F/X})_{|C}$ is spanned generically for any curve $C \subset F$ and consequently it is nef. ∎

If the fiber is of dimension 2 then we have a better extension property.

LEMMA 4.4.5. *Let $\varphi : X \to Z$ be a Fano–Mori birational contraction of a smooth variety with a 2-dimensional fiber F which is a locally complete intersection. Let $L = -K_X$; it is a φ-ample line bundle which can be assumed φ-very ample* (*see* 3.3.1 *and* [AW3, *Proposition* (1.3.4)]). *Then the following conditions are equivalent*:

(a) *$N^*_{F/X}$ is generated by global sections at any point of F.*

(b) *For a generic* (*smooth*) *divisor $X' \in |L|$ the bundle $N^*_{F'/X'}$ is generated by global sections at a generic point of any component of F'.*

Proof. The implication (a)⇒(b) is clear. To prove the converse we assume the contrary. Let S denote the set of points on F where $N^*_{F/X}$ is not spanned. Because of the extension property 4.4.4 and the fact that for a 1-dimensional fiber of a F-M contraction the spannedness of the normal bundle is equivalent to spannedness at a generic point (see [AW3, Corollary 5.6.2]), the set does not contain F' and thus it is finite. Now we choose another smooth section $X'_1 \in |L|$ which meets F along a (reduced) curve F'_1 containing a point of S. (We can do this because L is φ-very ample.) The bundle $N^*_{F'_1/X'_1}$ is generated at a generic point of F'_1 so it is generated everywhere; but this, because of the extension property, implies that $N^*_{F/X}$ is generated at some point of S, a contradiction. ∎

LEMMA 4.4.6. *Let $\varphi : X \to Z$ be a Fano–Mori birational contraction of a smooth 4-fold with a 2-dimensional fiber $F = \varphi^{-1}(z)$. As usual, $L = -K_X$ is a φ-ample line bundle which may be assumed to be φ-very ample. Suppose moreover that either F is irreducible or $L^2 \cdot F \leq 2$* (*which is the case when F is an isolated 2-dimensional fiber*). *Then the scheme fiber structure $\widetilde{F}$ is reduced unless one of the following occurs*:

(a) *The fiber F is irreducible and the restriction of N_F to any smooth curve $C \in |L_{|F}|$ is isomorphic to $\mathcal{O}(-3) \oplus \mathcal{O}(1)$.*

(b) *$F = \mathbb{P}^2 \cup \mathbb{P}^2$ and the restriction of N_F to any line in one of the components is isomorphic to $\mathcal{O}(-2) \oplus \mathcal{O}(1)$.*

Proof. Consider a curve $C \in |L_{|F}|$. Since L is φ-very ample we can take a smooth $X' \in |L|$ such that $\varphi' = \varphi_{|X}$ is a crepant contraction and $C = F \cap X'$ (see 4.3.5). Then, considering the embeddings $C = F \cap X' \subset F \subset X$ and $C = F \cap X' \subset X' \subset X$, we get

$$N_{C/X} = N_{C/X'} \oplus L_C = (N_{F/X})_{|C} \oplus L_C$$

and therefore $N_{C/X'} = (N_{F/X})_{|C}$.

Now the normal bundles $N_{C/X'}$ of the crepant contraction φ' can be easily described, neither (a) nor (b) occurs then the fiber structure of the contraction φ' is reduced. Thus, using the previous lemma and the equivalence in 4.4.1, we conclude that $\widetilde{F} = F$. ∎

Now, one has to discuss the possible exceptions described in the above lemma. This is done extensively in [AW3] and the following is proved:

Theorem 4.4.7 ([AW3, Theorems 5.7.5 and 5.7.6]). *Let $\varphi : X \to Z$ be a Fano–Mori birational contraction of a smooth 4-fold with an isolated 2-dimensional fiber $F = \varphi^{-1}(z)$. Then the fiber structure $\widetilde{F}$ coincides with the geometric structure F and the conormal bundle $N^*_{F/X}$ is spanned by global sections. Moreover if $F = \mathbb{P}^2$ then $N^*_{F/X}$ is either $\mathcal{O}(1) \oplus \mathcal{O}(1)$, $T(-1) \oplus \mathcal{O}(1)/\mathcal{O}$, or $\mathcal{O}^{\oplus 4}/\mathcal{O}(-1)^{\oplus 2}$. If F is a quadric (possibly singular or even reducible) then $N^*_{F/X}$ is the spinor bundle $\mathcal{S}(1)$.*

In some respects the above results about the fiber structure of a 2-dimensional fiber are nicer than one may expect. Namely, there is no multiple fiber structure, the conormal bundle is nef and the normal bundle of the geometric isolated fiber has no section. Thus the situation is better than for 1-dimensional isolated fibers in dimensions 2 and 3: the fundamental cycle of a Du Val ADE surface singularity is non-reduced and in dimension 3 one may contract an isolated $\mathbb{P}^1$ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(-3)$. On the other hand, using a double covering construction (see [AW2, Examples (3.5)]) in dimension 5 one may contract a quadric fibration over a smooth 3-dimensional base with an isolated fiber equal to $\mathbb{P}^2$, scheme-theoretically the fiber is a double $\mathbb{P}^2$. Using the sequence of normal bundles and the deformation of lines argument, one may verify that in this case $N_F \simeq \mathcal{O}(1) \oplus \mathcal{E}^*$ where $\mathcal{E}$ is a rank 2 spanned vector bundle with $c_1 = 2$, $c_2 = 4$, so that $\dim H^1(\mathcal{E}^*) = -\chi(\mathcal{E}^*) = 3$.

Note also that if $F = \bigcup F_i$ is a divisorial fiber of a surjective map $X \to Y$, where X is smooth and $\dim Y \geq 2$ then for some $k > 0$ the line bundle $\mathcal{O}_{F_i}(-kF_i)$ has a non-trivial section and thus no multiple of $\mathcal{O}_{F_i}(F_i)$ has a section. In particular, if $\mathrm{rank}(\mathrm{Pic}(X/Y)) = 1$ then $\mathcal{O}_F(-F)$ is ample.

One can then conjecture that if F is an isolated fiber of a (Fano–Mori) contraction which is a locally complete intersection with "small" codimension then $H^0(F, N_F) = 0$.

The above result on contractions of 4-folds can be generalised to adjunction mappings of an n-fold. Namely, suppose that $\varphi : X \to Y$ is a Fano–Mori contraction of a smooth n-fold X supported by a divisor $K_X + (n-3)H$, where H is a φ-ample divisor on X. Since we are interested in the local description of φ around a non-trivial fiber $F = \varphi^{-1}(z)$, we may assume that the variety Z is affine.

Corollary 4.4.8. *Assume that φ is birational and that F is an isolated fiber of dimension $n-2$. If $n \geq 5$ then the contraction φ is small and F is an isolated non-trivial fiber. More precisely $F \simeq \mathbb{P}^{n-2}$ and $N_{F/X} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, and there exists a flip of φ (see [ABW]).*

Note that the preceding arguments, which led to the classification of the birational 4-dimensional case, depend on the isomorphism $\varphi'_*\mathcal{O}_{X'} \simeq \mathcal{O}_Z \simeq \varphi_*\mathcal{O}_X$. This fails to be true if φ is of fiber type. Namely, let $\varphi : X \to Z$ be a conic fibration, i.e. a Fano–Mori contraction such that $\dim Z = \dim X - 1$. As usual, we will assume that F is an isolated general section $X' \in |L|$ is a generically $2:1$ covering of Z. A different argument is developed for this case in [AW2], where the following theorem is proved.

THEOREM 4.4.9 ([AW3, Proposition 5.9.5 and Theorem 5.9.6]). *Let $\varphi : X \to Z$ be a conic fibration of a smooth 4-fold. Suppose that F is an irreducible isolated 2-dimensional fiber of φ which is either a projective plane or a quadric. Then the conormal bundle $N^*_{F/X}$ is nef.*

Moreover if $F \simeq \mathbb{P}^2$ then $N^*_{F/X} \simeq \mathcal{O}^3/\mathcal{O}(-2)$ or $T\mathbb{P}^2(-1)$. If F is an irreducible quadric then $N^*_{F/X}$ is the pullback of $T\mathbb{P}^2(-1)$ via some double covering of $\mathbb{P}^2$. In both cases the scheme fiber structure $\widetilde{F}$ is reduced and Z is smooth at z.

4.5. When does the normal of a fiber determine locally the contraction? We present some results which show how the normal bundle can give all the information we want on the contraction, that is, the second order approximation actually determines completely the formal neighborhood. This is of course not always the case.

In some situations the knowledge of the normal bundle $N_{F/X}$ allows to determine the singularity of Z at $z = \varphi(F)$. Recall that for a local ring $\mathcal{O}_{Z,z}$ with maximal ideal m_z one defines the graded $\mathbb{C}$-algebra $\mathrm{gr}(\mathcal{O}_{Z,z}) := \bigoplus_k m_z^k/m_z^{k+1}$. The knowledge of the ring $\mathrm{gr}(\mathcal{O}_{Z,z})$ allows one sometimes to describe the completion ring $\widehat{\mathcal{O}}_{Z,z}$. Also, we will say that a spanned vector bundle $\mathcal{E}$ on a projective variety Y is *p.n.-spanned* (p.n. stands for *projectively normal*) if for any $k > 0$ the natural morphism $S^kH^0(Y,\mathcal{E}) \to H^0(Y, S^k\mathcal{E})$ is surjective. As we noted while discussing the contraction to the vertex, projective normality allows us to compare gradings of rings “upstairs” and “downstairs”.

PROPOSITION 4.5.1 ([Mo3]). *Let $\varphi : X \to Z$ be a contraction as above. Suppose moreover that $N^*_{F/X}$ is p.n.-spanned. Then $\varphi_*(\mathcal{I}_F^k) = m_z^k$, $\varphi^{-1}(m_z^k)\cdot\mathcal{O}_X = \mathcal{I}_F^k$ and there is a natural isomorphism of graded $\mathbb{C}$-algebras*

$$\mathrm{gr}(\mathcal{O}_{Z,z}) \simeq \bigoplus_k H^0(F, S^k(N^*_{F/X})).$$

We omit the proof of the above result referring to Mori, [Mo3, p. 164], who proved it in the case when F is a divisor; the generalisation is straightforward.

The next is a version of a theorem of Mori [Mo3, 3.33], which is a generalisation of a Grauert–Hironaka–Rossi result:

PROPOSITION 4.5.2. *Suppose that F is a smooth fiber of a Fano–Mori or crepant contraction $\varphi : X \to Z$ and assume that its conormal bundle $N^* = N^*_{F/X}$ is nef. If $H^1(F, T_F \otimes S^i(N^*)) = H^1(F, N \otimes S^i(N^*)) = 0$ for $i \geq 1$ then the formal neighborhood of F in X is uniquely determined and it is the same as the formal neighborhood of the zero section in the total space of the bundle N.*

Also the following assertion is a straightforward generalisation of the celebrated Castelnuovo contraction criterion for surfaces; its proof is similar to the one of [Ha, V.5.7] (see also [AW2]).

PROPOSITION 4.5.3 (Castelnuovo criterion). *Let $\varphi : X \to Z$ be a projective morphism*

Suppose that $z \in Z$ and $F = \varphi^{-1}(z)$ is a locally complete intersection in X with conormal bundle $N^*_{F/X}$. Assume that $H^1(F, S^k(N^*_{F/X})) = 0$ for any positive integer k (note that this assumption is satisifed if φ is a Fano–Mori contraction, $N^*_{F/X}$ is nef and the blow-up of X at F has log terminal singularities). If for any $k \geq 1$ it is $S^k H^0(F, N^*_{F/X}) \simeq H^0(F, S^k(N^*_{F/X}))$ then z is a smooth point of Z and $\dim Z = \dim H^0(F, N^*_{F/X})$.

4.6. Concluding remarks on the classification of Fano–Mori contractions on a smooth n-fold with $n \leq 4$. The proofs of the theorems announced in the first two sections of this part can be given by applying the numerous results we have given up to now (very often not in a unique way). This may not be trivial, so in this section we give some possible schemes of proof.

A good starting point is to use the Ionescu–Wiśniewski inequality of 4.3.14; this gives the possibilities for the dimension of the fibers and of the exceptional locus. In particular it says that there are no small contractions (i.e. contractions whose exceptional locus has codimension ≥ 2) on a smooth 3-fold and also it proves part 0 of Theorem 4.1.3.

Description of the F-M contractions around a fiber F of dimension 1 (general case). This is given in 4.2.1; this covers almost all Theorem 4.1.1 and part of Theorems 4.1.2 and 4.1.3. The proof of 4.2.1 follows from 4.3.22 (which describes the possibilities for F), 4.4.2 and the discussion before it (which describes the possibilities for the normal bundles; in 4.4.2 only the case $n = 3$ is discussed in detail) and 4.5.3.

Note that in the surface case, if $\varphi = \mathrm{cont}_R$ is a conic bundle then φ actually gives the structure of a minimal ruled surface. In order to prove this we have to show that there are no reducible or non-reduced fibers of φ. In fact if, by contradiction, F is such a fiber then $F = \sum a_i C_i = [C]$ with $[C] \in R$. But since R is extremal this implies that $C_i \in R$ for every i. Thus $C_i^2 = 0$, since a general fiber of φ is a smooth irreducible reduced curve in the ray, and $C_i \cdot K_X < 0$. By the adjunction formula this implies that $C_i \simeq \mathbb{P}^1$ and $C_i \cdot K_X = -2$. Thus

$$-2 = (C \cdot K_X) = \sum a_i (C_i \cdot K_X) = -2 \sum a_i,$$

which gives a contradiction. Furthermore using Tsen's theorem, one can prove that X is the projectivisation of a rank 2 vector bundle on $\mathbb{P}^1$ (see [Re4, C.4.2]).

Description of the F-M contractions around a fiber F of dimension 2 (3-folds and 4-folds). In the surface case, the contraction φ contracts X to a point; that is, X has $\mathrm{Pic} = \mathbb{Z}$ and $-K_X$ is ample, i.e. X is a Fano surface of Picard number 1. Then one can prove that $X = \mathbb{P}^2$; see for instance [CKM, p. 21].

In the 3-fold case, either φ is a contraction of fiber type contracting X to a curve, with all fibers of dimension two, or φ is a birational divisorial contraction which contracts a unique prime divisor equal to F to a point: in fact, by Exercise 3.1.8, φ cannot be a contraction to a surface with some isolated 2-dimensional jumping fibers and if φ is birational then the exceptional divisor is prime.

In the first case, by adjunction, the general fiber is a smooth del Pezzo surface.

In the birational case we can apply 4.3.26, which gives all the possibilities (namely 3) we prove the uniqueness of the analytic neighborhood of the contraction around F by using 4.5.3, 4.5.1 and 4.5.2. Actually for the uniqueness in the case of a fiber isomorphic to the singular quadric an extra argument is needed (see for this [Mo3, p. 165]). Note also that all these cases exist and can be constructed via the basic example 3.2.4 except the case with a fiber isomorphic to the singular quadric; in this case if we work as in 3.2.4 we construct a singular 3-fold X. Moreover the case of the smooth quadric can be constructed via 3.2.4 but not as an elementary contraction, i.e. as a contraction of a single ray. Two good examples were given in [Mo3, 3.44.2 and 3.44.3].

We then consider a Fano–Mori elementary contraction from a 4-fold, in a neighborhood of a 2-dimensional fiber F. We start with the birational case: the fiber can be an isolated 2-dimensional fiber or can stay in a 1-dimensional family of 2-dimensional fibers (the other possibilities are ruled out by Exercise 3.1.8).

The first case is described in Part 4 of Theorem 4.1.3; to prove it we can first apply Theorem 4.3.26, which gives 4 possibilities for the fiber F, namely F can be $\mathbb{P}^2$ or a reduced quadric (smooth, singular or reducible).

Then we apply Theorem 4.4.7 which describes the possible normal bundles. If F is smooth then we can prove the uniqueness of a formal neighborhood of the fiber using 4.5.2 and construct an example using 3.2.4.

If F is a singular quadric or a reducible one (union of two $\mathbb{P}^2$ meeting along a line) then the situation is more complicated. In [AW4] one can find good examples for these situations; moreover in case F is reducible there are at least two possible analytically non-equivalent formal neighborhoods of F. That is, as one may expect, the fiber and its normal do not always determine the analytic neighborhood. The (open) question is whether in this case these two neighborhoods are the only possible ones (up to analytic equivalence).

Part 3 of Theorem 4.1.3 describes the case where F stays in a 1-dimensional family of 2-dimensional fibers. The proof is substantially different from the previous one and was given in [AW4]. Using a vertical slicing (see Proposition 4.3.8), one can prove that the general 2-dimensional fiber of this family is either $\mathbb{P}^2$ or an irreducible reduced quadric. In fact, in the notation of 4.3.8 (vertical slicing), $f_{X_h} : X_h \to f(X_h)$ is a Fano–Mori contraction from a smooth 3-fold which contracts a general fiber to a point; thus we can apply the result on 3-folds, i.e. 4.1.2. (This is actually a bit quick: in fact f_{X_h} can be non-elementary, i.e. the contraction of a face, not of a ray. In [AW4] this is in fact ruled out.) It is easy to prove that if the general fiber is $\mathbb{P}^2$ then the same holds for the special fiber. If the general fiber is a quadric then the special one is also a quadric, but very likely more singular. It turns out that there are no non-reduced quadrics, that is, double $\mathbb{P}^2$, as special fibers; the other possibilities all occur. We refer to [AW3] for further details and examples.

We finally pass to the case of fiber type Fano–Mori contractions with 2-dimensional fibers. If the 2-dimensional fiber is not isolated then the contraction is to a surface, by Exercise 3.1.8 it is equidimensional, i.e. all fibers are of dimension two, and its general fiber is a del Pezzo surface (by the adjunction formula).

If the 2-dimensional fiber F is isolated then it is one of those described in The-smooth $\mathbb{P}^2$ or a smooth quadric then the normal bundle is computed in Theorem 4.4.9. In these two cases examples can be constructed using 3.2.4; maybe one can also prove the uniqueness of a formal neighborhood of the fiber using 4.5.2 (this can be a hard computation!). Examples have been constructed for some other possible fibers but for some of them we cannot construct an example (see the remark after Theorem 4.3.26).

Description of the F-M contractions around a fiber F of dimension 3 (4-folds). If the contraction is birational then by Exercise 3.1.8, F is the unique prime divisor equal to the exceptional locus. It is immediate to see, using the adjunction formula, that F is a del Pezzo 3-fold. The problem here is to prove that F is normal and which normal non-smooth del Pezzo 3-folds can actually occur (not all of them!) (see Part 2 of Theorem 4.1.3 and the following remark).

If the contraction is of fiber type then, again by 3.1.8, all fibers are 3-dimensional and the generic one is a Mukai manifold.

4.7. Classification of Fano manifolds of high index. This section is devoted to the study of F-M contractions of a smooth manifolds with target a point, i.e. to Fano manifolds.

We already noticed that Fano varieties with high index, with respect to the dimension, are easier to understand. Namely for $i(X) \geq \dim X$ the information given by the Hilbert polynomial is already sufficient to give a complete description of all possible cases (see Exercise 3.3.8).

It is not surprising that for lower indices the world is wilder. The best known way to go further is, again, the following adjunction procedure.

Let X be a Fano manifold of index $i(X) = r$ and fundamental divisor L (see Definitions 3.3.3 and also 3.1.10). Assume that $|L|$ is not empty. Let $H \in |L|$ be a generic member. Then by the adjunction formula,

$$-K_H = -(K+L)_{|H} \sim (r-1)L_{|H}.$$

In other words whenever $r > 1$ the section H is a Fano variety of the same dual index. So that, if one is able to control the singularities of H, then it is possible to study X through H. More generally we can make the following

Definition 4.7.1. Let $f : X \to S$ be a local contraction of type (d, γ, Φ), supported by $K_X + rL$. Then we will say that f has *good divisors* if, after maybe shrinking S, the generic element $H \in |L|$ has at worst the same singularities as X and $f_{|H} : H \to S_H$ is of type $(*, *, \Phi)$.

Assume that the good divisor problem has an affirmative answer for a fixed index. Then the above observation allows one to classify all Fano manifolds of fixed dual index in an inductive way, starting from the lower dimensional ones.

This is what Fujita did (see [Fu2]) for del Pezzo manifolds, i.e. Fano manifolds with $i(X) = \dim X - 1$. His results can be summarised in the following way.

Theorem 4.7.2 ([Fu2]). *Let X be a del Pezzo manifold of dimension n. If $n = 2$ then X is either $\mathbb{Q}^2$ or $\mathbb{P}^2$ blown up in $r \leq 8$ general points. Assume $n \geq 3$. Let $d = H^n$ be*

• $d = 1$: $X_6 \subset \mathbb{P}(1^{n-1}, 2, 3)$, *i.e. a hypersurface of degree* 6 *in the weighted projective space with weights* $(1, \ldots, 1, 2, 3)$;
• $d = 2$: $X_4 \subset \mathbb{P}(1^n, 2)$;
• $d = 3$: $X_3 \subset \mathbb{P}^{n+1}$;
• $d = 4$: $X_{2,2} \subset \mathbb{P}^{n+2}$, *i.e. a complete intersection of two quadrics*;
• $d = 5$: *a linear section of* $\mathbb{G}(1, 4) \subset \mathbb{P}^9$;
• $d = 6$: X *is either* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ *or* $\mathbb{P}^2 \times \mathbb{P}^2$ *or* $\mathbb{P}_{\mathbb{P}^2}(T\mathbb{P}^2)$;
• $d = 7$: *the blow-up of* $\mathbb{P}^3$ *in one point.*

A tutorial case is the surface one, which is given through the following exercise entirely based on minimal model techniques.

EXERCISE 4.7.3. Let S be a del Pezzo surface of degree d; then the Kleiman–Mori cone is spanned by extremal rays (see Theorem 1.1.2). We also have a precise description of the contraction associated to each extremal ray (see 4.1.1).

- Assume that S has an extremal ray R whose associated contraction is birational; that is, $\varphi : S \to S'$ is the contraction of a (-1)-curve C. Prove that S' is a del Pezzo surface of degree $d + 1$.

Since going from S to S' we decrease the Picard number by one, after finitely many contractions we have a del Pezzo surface S_k with only fiber type extremal rays.

- Show that S_k is either $\mathbb{Q}^2$ or $\mathbb{P}^2$ (use 4.1.1).

We have thus established that any del Pezzo surface S is either $\mathbb{Q}^2$ or the blow up of $\mathbb{P}^2$ in a finite number of points (a blow-up of $\mathbb{Q}^2$ is in fact a blow-up of $\mathbb{P}^2$).

- Show that you cannot blow up more than 8 points with the following restrictions: no 3 are on a line and no 6 on a conic.
  Hint: Evaluate the self-intersection of K_S and the intersection of K_S with the strict transform of a line or conic.

We can now conclude:

- A surface obtained by blowing up r points of $\mathbb{P}^2$, with the above restrictions, is a del Pezzo surface.
  Hint: Study either the combinatorics of the cone of effective curves, or the linear systems of cubics with imposed conditions.

One can improve the knowledge of these surfaces by observing that whenever $d \geq 3$ then $|-K_S|$ is very ample and embeds $S \subset \mathbb{P}^d$. For $d = 2$ the complete linear system $|-K_S|$ is spanned and gives a double cover of $\mathbb{P}^2$ ramified along a quartic, while for $d = 1$ the system $|-2K_S|$ is spanned and gives a double cover of a singular quadric ramified along a sextic and the vertex.

In higher dimensions the idea of the proof is the following. First show that the good divisor problem has an affirmative answer, i.e. prove the following exercise.

EXERCISE 4.7.4. Let X be a del Pezzo manifold. Prove that X has good divisors.

Then all the information on del Pezzo surfaces can be extended.

The first two cases can be obtained with the machinery of graded rings [Mo1, §3]. We would like to stress here the following property. If a variety X, of dimension at least 4, contains a hyperplane section which is a weighted complete intersection then the variety itself is a weighted complete intersection [Mo1, Corollary 3.8].

If degree $d \geq 3$ then $|-K_X|$ is very ample. So the cases $d = 3, 4$ are immediate while for $d \geq 5$ the study is more subtle and we leave it to the interested reader (see [Fu2]).

The next case is the one of a Mukai manifold, i.e. a manifold with $i(X) = \dim X - 2$. These varieties are named after S. Mukai [Mu] who first announced their classification, assuming the existence of good divisors.

This assumption is proved in [Me2], where the base point free technique is applied to answer the good divisor problem for Mukai varieties.

The idea is simple. Let X be a Fano manifold and $|L|$ the fundamental divisor of X. Let $D \equiv \delta L$ be a $\mathbb{Q}$-divisor with $\delta < 1$. By the BPF technique there is a section of $|L|$ non-vanishing identically on $\mathrm{LLC}(X, D)$. If we combine this with the Bertini theorem we immediately see that the generic section of $|L|$ cannot have singularities worse than LC. We actually prove the following.

Theorem 4.7.5 ([Me2]). *Let X be a Mukai variety with at worst log terminal singularities. Then X has good divisors except in the following cases*:

(i) *X is a singular terminal Gorenstein 3-fold which is a "special" (see [Me2]) complete intersection of a quadric and a sextic in $\mathbb{P}(1, 1, 1, 1, 2, 3)$.*

(ii) *Let $Y \subset \mathbb{P}(1, 1, 1, 1, 1, 2)$ be a "special" complete intersection of a quadric cone and a quartic; let σ be the involution on $\mathbb{P}(1, 1, 1, 1, 1, 2)$ given by $(x_0 : x_1 : x_2 : x_3 : x_4 : x_5) \mapsto (x_0 : x_1 : x_2 : -x_3 : -x_4 : -x_5)$ and let π be the map to the quotient space. Then $X = \pi(Y)$ is a terminal non-Gorenstein 3-fold.*

In both exceptional cases the generic element of the fundamental divisor has canonical singularities and $\mathrm{Bsl}\,|-K_X|$ is a singular point. It has to be stressed that the generic 3-fold in (i) and (ii) has good divisors, but there are "special" complete intersections whose quotient has a singular point in the base locus of the anticanonical class (see [Me2, Examples 2.7, 2.8]) for details).

Proof. We prove the theorem in four steps, from the more singular to the smooth case. As usual we argue by contradiction.

Claim 2. *If X has log terminal singularities then it has good divisors.*

Proof of the claim. Proposition 3.3.4 ensures that $\dim |L| \geq 1$. Let $H \in |L|$ be a generic section and assume that H has singularities worse than LT. Let $\gamma = \mathrm{lct}(X, H)$ and $Z \in \mathrm{CLC}(X, \gamma H)$ be a minimal center.

Exercise 4.7.6. $\gamma \leq 1$ and $\operatorname{cod} Z \geq 2$.
Hint: If $\operatorname{cod} Z = 1$ then Z is a fixed component of $\mathrm{Bsl}\,|L|$, but $\dim |L| > 0$ and H is ample and therefore connected. In particular if H is reducible then there exists a codimension

This is enough to derive a contradiction. By the Bertini theorem $Z \subset \mathrm{Bsl}\,|L|$, while by Lemma 3.3.5 there exists a section of L non-vanishing on Z. ■

CLAIM 3. *If X has canonical singularities, then it has good divisors.*

Proof of the claim. By Claim 2, H has LT singularities. Let $\mu : Y \to X$ be a log resolution of (X, H), with $\mu^*H = \overline{H} + \sum r_i E_i$, where $|\overline{H}|$ is base point free, and $K_Y = \mu^* K_X + \sum a_i E_i$. Assume that H has non-canonical singularities; then, maybe after reordering the indices, we have $a_0 < r_0$. Since H is generic, $\mu(E_i) \subset \mathrm{Bsl}\,|L|$ for all i with $r_i > 0$ (see the proof of Claim 1 page 27). Let $D = H + H_1$ with $H_1 \in |L|$ a generic section. First observe that μ is a log resolution of (X, D) as well. Let $\mu^* H_1 = \overline{H}_1 + \sum r_i E_i$. Since H_1 is a Cartier divisor, the r_i are positive integers. In particular $a_0 + 1 < r_0 + r_0$, hence (X, D) is not LC. Let $\gamma = \mathrm{lct}(X, D) < 1$ and W a minimal center of $\mathrm{CLC}(X, \gamma D)$.

EXERCISE 4.7.7. Prove that $\mathrm{cod}\, W \geq 3$.
Hint: H is LT and canonical singularities are Gorenstein in codimension 2.

We can again apply Lemma 3.3.5 to derive a contradiction. ■

CLAIM 4. *If X has terminal singularities then X has good divisors unless X is as in either* (i) *or* (ii).

Proof of the claim. If X is a terminal Mukai variety of dimension ≥ 4 then by Claim 3, H has canonical singularities. Furthermore we can apply Claim 3 to $H_{|H}$, to deduce that even $H_{|H}$ has canonical singularities. Let $f : Y \to X$ be a log resolution for (X, H). Assume that $K_Y = f^*K_X + \sum a_i E_i$ and $f^*H = H_Y + \sum r_i E_i$. Then $K_{H_Y} = f^*K_H + \sum (a_i - r_i) E_i$ and, with obvious notations, $K_{H_{|HY}} = f^* K_{H_H} + \sum (a_i - 2r_i)$. We just observed that $a_i - 2r_i \geq 0$, therefore $a_i - r_i > 0$ whenever $r_i > 0$. This proves that H is terminal on the base locus of $|L|$ and we conclude by the Bertini theorem that H is terminal.

There remains the case of terminal 3-folds with $-K_X \equiv L$; this goes a bit beyond the techniques we developed and so here we only state the result (remember that terminal surface singularities are smooth points):

THEOREM 4.7.8 ([Me2]). *Let X be a terminal Mukai* 3*-fold and assume that all the divisors in the linear system $|L|$ are singular. Then X is one of the two exceptions in Theorem* 4.7.5*. They actually exist.* ■

CLAIM 5. *If X is smooth then X has good divisors.*

Proof of the claim. Assume that the generic element in $|L|$ is not smooth. Then a 3-fold section $T \subset X$ is one of the two exceptions to Claim 4. Then by the usual vanishing theorem,

$$H^0(X, L) \to H^0(T, L_{|T}) \to 0,$$

and by Theorem 4.7.8, $\mathrm{Bsl}\,|L| = \mathrm{Bsl}\,|L_{|T}|$ is just one point, say x. Let $H_i \in |L|$, for $i = 1, \ldots, n-1$, be generic elements and $D = H_1 + \ldots + H_{(n-1)}$. Then the minimal center of $\mathrm{CLC}(X, D)$ is x and (X, D) is not LC at x, since $2(n-1) > n$. We, therefore, derive a contradiction by Lemma 3.3.5. ■

With similar arguments one can handle the good divisor problem for other F-M contractions (for details and related results see [Me3]); for instance we have the following.

THEOREM 4.7.9 ([Me3]). *Let $f: X \to S$ be a local contraction of type* $(1,1,1)$. *Assume that X is smooth. Then f has good divisors.*

REMARK 4.7.10. Could this be the starting point of a relative analogue of the Fujita classification? The above theorem reduces this study to that of fibrations of surfaces. The main problem to solve is a base point free statement in a neighborhood of an irreducible non-reduced fiber. With this one could provide a structure theorem as in the absolute case, embedding these spaces in some relative (weighted) projective space. Then one should try to extend the Andreatta–Wiśniewski theory one step further. Unfortunately as far as we can say this is quite hard and requires a lot of unknown results on vector bundles on del Pezzo surfaces.

Let X be a smooth Fano n-fold of index $r = n-2$, i.e. a Mukai manifold. Let $|H|$ be the linear system of fundamental divisors. The integer

$$g = \frac{1}{2}H^n + 1$$

is called the *genus* of X (the reason will be clear after Proposition 4.7.11); by Riemann–Roch

$$\dim H^0(X, H) = n + g - 1.$$

By Theorem 4.7.5 the generic element $S \in |H|$ is smooth. As observed above this allows an inductive argument toward 3-dimensional Fano's.

PART I: If $\operatorname{rk}\operatorname{Pic}(X) = 1$ we use Iskovskikh's results [Is] on Fano 3-folds to obtain

PROPOSITION 4.7.11. *Let X be a smooth Mukai n-fold with* $\operatorname{rk}\operatorname{Pic}(X) = 1$. *Then $|H|$ is base point free and one of the following is true*:

(i) *$|H|$ is very ample and embeds X in $\mathbb{P}^{g+n-2}$. In particular $X \subset \mathbb{P}^{g+n-2}$ has a smooth curve section canonically embedded.*

(ii) *The morphism associated to sections of $|H|$ is a finite morphism of degree* 2 *either onto $\mathbb{P}^n$ (in case $g = 2$) or onto $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$ (in case $g = 3$).*

We are therefore restricted to study projective varieties with a smooth canonical curve section; the actual point of the classification is to understand all of them. (For a somewhat backward approach, see also [CLM].)

THEOREM 4.7.12 ([Mu]). *Let $X_{2g-2} \subset \mathbb{P}^{n+g-2}$ be as in point* (i) *of Proposition* 4.7.11. *If $g \leq 5$ then X is a complete intersection. Assume that $10 \geq g \geq 6$. Then we have the following picture*:

g	$n(g)$	$X^{n(g)}_{2g-2} \subset \mathbb{P}^{g+n(g)-2}$
6	6	$C(\mathbb{G}(1,4)) \cap \mathbb{Q} \subset \mathbb{P}^{10}$
7	10	$\mathrm{SO}(10,\mathbb{C})/P \subset \mathbb{P}^{15}$
8	8	$\mathbb{G}(1,5) \subset \mathbb{P}^{14}$
9	6	$\mathrm{Sp}(6,\mathbb{C})/P \subset \mathbb{P}^{13}$
10	5	$G_2/P \subset \mathbb{P}^{13}$

where $C(X)$ is the cone over X and $n(g)$ is the maximal dimension for a Mukai variety of that type (as observed before, any hyperplane section is then a Mukai variety). If $g > 10$ then $g = 12$ and $n(g) = 3$. Then some special X_{22}^3 can be seen as a smooth equivariant compactification of $\mathrm{SO}(3, \mathbb{C})/($*icosahedral group*$)$ (*see* [MU]). *In general it is possible to give a description using a net of quadrics.*

We remark that, as before, the case of genus ≤ 5 is easily obtained, while the study of the remaining cases is the heart of the proof.

PART II: If $\operatorname{rk}\operatorname{Pic}(X) > 1$ then there are at least two extremal rays on X. In the following exercise we collect all the crucial information to complete the biregular classification in this case (see [MoMu], [Mu]).

EXERCISE 4.7.13. (1) Prove that the only possible F-M birational contractions are those of a divisor to a point or to a smooth curve.

(2) Which divisors are then possible?

(3) In the case of extremal rays with associated contraction of fiber type, the base of the contraction is smooth.

Hint: Use the classification provided by Theorem 4.1.2.

Part 5. Birational geometry

One of the main goals of algebraic geometry is to achieve a birational classification of projective algebraic varieties. The Minimal Model Program, or Mori's program, is an attempt to get (part of) this classification. In the first section of this chapter we want to introduce the general philosophy of the MMP.

The developed techniques however allow one to treat other birational aspects as well. In this realm we would like to focus on two different settings: the #-minimal model and Sarkisov's program.

The former is a polarised minimal model program that enables one to study special uniruled 3-folds and it will be described in Section 5.3.

The latter is used to investigate birational morphisms between Mori spaces (see Definition 5.1.8), and its application to the projective plane is outlined in Section 5.2.

5.1. Minimal Model Program philosophy. Let us present the MMP approach toward the birational classification of algebraic varieties.

The case of smooth curves is clearly settled by the Riemann uniformisation theorem. The case of surfaces is more complicated and it was developed by Italian algebraic geometers at the beginning of the twentieth century. This big achievement was used as a model for the higher dimensional case and at the end it was included in the more general philosophy of minimal model theory as we will see in the following.

Consider a smooth projective variety X. The aim of minimal model theory is to distinguish, inside the set of varieties which are birational to X, a special "minimal" member $\widetilde{X}$ so as to reduce the study of the birational geometry of X to that of $\widetilde{X}$.

non-trivial problem and the following definition is the result of hard work of people like Mori and Reid in the late 70's.

DEFINITION 5.1.1. A variety $\widetilde{X}$ is *minimal* if

- $\widetilde{X}$ has $\mathbb{Q}$-factorial terminal singularities,
- $K_{\widetilde{X}}$ is nef.

Let us make some observations on this definition. The second condition wants to express the fact that the minimal variety is (*semi-*) *negatively curved*. We note in fact that if $\det T_X = -K_X$ admits a metric with semi-negative curvature then K_X is nef. The converse is actually an open problem (true in the case of surfaces and in general it may be considered as a conjecture).

The condition on the singularities is the real break-through of the definition. The point of view should be the following: we are in principle interested in smooth varieties but we will see that there are smooth varieties which do not admit smooth minimal models. However we can find such a model if we admit very mild singularities, the ones stated in the definition. Note also that terminal singularities are smooth in the surface case.

It happens that in the birational class of a given variety there is not a minimal model; think for instance of rational varieties. But the MMP hopes to make a list of those special varieties.

Given the definition of a minimal variety we now want to show how, starting from X, one can determine a corresponding minimal model $\widetilde{X}$. In view of 1.1.2 and 3.1.2 (or 3.1.5) the way to do it is quite natural. Namely, if K_X is not nef, then by 1.1.2 there exists an extremal ray (on which K_X is negative) and by 3.1.2 (or 3.1.5) we can construct an elementary (Fano–Mori) contraction $f : X \to X'$ which contracts all curves in this ray into a normal projective variety X'.

A naive idea at this point would be the following. If f is of fiber type, i.e. $\dim X' < \dim X$, then one hopes to recover a description of X via f. Indeed, by induction on the dimension, one should know a description of X' and of the fibers of f, which are, at least generically, Fano varieties. We will say something more on this case in the last part of the section (see 5.1.8).

If f is birational then one would substitute X with X' and proceed inductively.

The problem is of course that Theorem 3.1.2 says very little about the singularities of X' (now it starts to be clear that the choice of the singularities in the above definition is crucial). It says only that it has normal singularities.

However in the surface case the situation is optimal, namely Theorem 4.1.1 first of all says that if cont_R is birational then the image is again a smooth surface (see 4.1.1.1). Then apply recursively 4.1.1(1) to conclude that after finitely many blow-downs of (-1)-curves one reaches a smooth surface S' with either $K_{S'}$ nef or with an extremal ray of fiber type. Note that while performing the MMP we stay in the category of smooth surfaces. If cont_R is of fiber type then, again by 4.1.1, its description is very precise. We have proved the following.

THEOREM 5.1.2 (Minimal Model Program for surfaces). *Let S be a smooth surface. After* *of the following*:

(1) $K_{S'}$ *is nef, i.e.* S' *is a minimal model,*
(2) S' *is a ruled surface,*
(3) $S' \simeq \mathbb{P}^2$.

In higher dimensions the requirement on the singularity comes into play. In particular we note that cases B3, B4 and B5 in Theorem 4.1.2 lead to singular 3-folds; case B5 leads to a 2-Gorenstein singularity. However all these singularities are terminal and $\mathbb{Q}$-factorial. The fact that the cone theorem 1.1.2 and the contraction theorem 3.1.2 hold in the more general case of a variety with terminal singularities seems to give some hope.

Moreover the good property of birational contractions in the surface case ascends in higher dimensions to the fact that if an elementary F-M contraction of a smooth (or terminal $\mathbb{Q}$-factorial) variety is divisorial then the target has at worst terminal $\mathbb{Q}$-factorial singularities (see Exercise 3.1.8).

But a very serious problem is now coming up. Namely if we consider varieties with terminal singularities then they can have birational F-M contractions which are not divisorial! This was first noticed by P. Francia with a famous example (see for instance [CKM, pp. 33–34]).

Let us see why an elementary contraction $f : X \to Y$ of a variety X with $\mathbb{Q}$-Gorenstein singularities and with exceptional locus E such that $\operatorname{cod} E \geq 2$ gives problems. Let $U = X \setminus E$. Then $f_U : U \to f(U) = V$ is an isomorphism. In particular it is clear that $K_{X|U} \simeq f_U^*(K_V)$. Let M be the extension of $f_U^*(K_V)$ to X. Then the codimension assumption yields $M \simeq K_X$. On the other hand $-K_X$ is f-ample and therefore M cannot be the pullback of any $\mathbb{Q}$-Cartier divisor on Y. In other words K_Y is not $\mathbb{Q}$-Cartier!

In particular on such a Y even the definition of a minimal model does not make sense. Our naive solution came abruptly to a stop and new solutions are needed. The principal ideas are summarised in the following flip conjecture [KMM], which very roughly says that instead of contracting the exceptional locus of the "small" rays we have to make a codimension 2 surgery, called flip, that replaces the curve with another one which has a different normal sheaf.

Here is a precise statement.

Conjecture 5.1.3 (Flip conjecture). Let X be a terminal $\mathbb{Q}$-factorial variety and assume that there exists an extremal ray $\mathbb{R}^+[C] \subset \overline{\mathrm{NE}}(X)$ with associated elementary contraction $f : X \to W$; assume also that $\operatorname{cod}(\operatorname{Exc}(f)) \geq 2$. Then there exists a terminal $\mathbb{Q}$-factorial variety X^+ and a map $f^+ : X^+ \to W$ such that

(1) K_{X^+} is f^+-ample,
(2) $\operatorname{Exc}(f^+)$ has codimension at least two in X^+,
(3) the following diagram is commutative:

$$\begin{tikzcd} X \arrow[rr, dashed, "\Phi"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & W & \end{tikzcd}$$

The following theorem is the breakthrough of Mori theory for 3-folds which overcame the flip problem. The proof is very intricate and it is based on a careful classification of all possible small contractions occurring on a terminal 3-fold.

THEOREM 5.1.4 ([Mo4]). *The flip conjecture holds for* 3*-folds.*

REMARK 5.1.5. After Mori's proof of the existence of flips, different proofs of flip, even log-flip, for 3-folds were obtained mainly by Shokurov and Kollár. The best account of them is in [KU]. Very recently still a new approach of Shokurov simplified greatly the 3-fold proof and is very promising in higher dimensions.

Then, assuming the flip conjecture, one asks if this sort of inductive procedure will come to an end, namely we need a kind of termination for these birational modifications.

If f is a divisorial contraction then the Picard number drops by one so there cannot be an infinite number of those. For flips there is no such straightforward criterion and so the following termination conjecture arises.

CONJECTURE 5.1.6 (Termination conjecture). Let X be a terminal $\mathbb{Q}$-factorial variety which is not minimal. Then after finitely many flips there is an extremal ray whose exceptional locus is of codimension ≤ 1.

PROPOSITION 5.1.7 ([KMM, Theorem 5.1.15]). *The termination conjecture is true for n-folds with $n \leq 4$.*

So, assuming both the flip and termination conjectures, after finitely many birational modifications we either reach a minimal model or encounter an elementary extremal contraction of fiber type. For this we give a definition.

DEFINITION 5.1.8. A *Mori space* is a terminal $\mathbb{Q}$-factorial Fano–Mori contraction $\pi : X \to S$ such that $\dim S < \dim X$ and $\operatorname{rk}\operatorname{Pic}(X/S) = 1$.

The goal in this case is, just as for surfaces, to get a classification of Mori spaces.

In general the Mori space associated to a variety by the MMP is not uniquely determined. This problem arises when two extremal rays have non-disjoint exceptional loci. A very simple example: Let $T = E \times \mathbb{F}_1$, where E is a smooth curve of genus $g > 0$. Then there are two extremal rays, one of divisorial type and the other of fiber type. In this case the order in which the rays are contracted determines the F-M space. In one case it is a $\mathbb{P}^1$-bundle, in the other a $\mathbb{P}^2$-bundle.

Let us make some observation on these spaces. We note that the generic fiber is a variety with $\mathbb{Q}$-factorial terminal singularities with anticanonical ample bundle. Thus no multiple of the canonical bundle of X has a section, that is, the Kodaira dimension of X is $-\infty$. On the other hand we already noticed (see Section 4.3.3) that such an X is covered by rational curves. There is a deep, and still not entirely understood, relation between these two facts.

DEFINITION 5.1.9. A variety X is *uniruled* if there exists a generically finite surjective map $Y \times \mathbb{P}^1 \to X$.

PROPOSITION 5.1.10 ([KMM, Corollary 5.1.4]). *Let $\pi : X \to S$ be a Mori space. Then X*

The proposition is a consequence of the following theorem proved by means of the theory of deformation of rational curves on smooth varieties.

THEOREM 5.1.11 ([MiMo]). *Let X be a projective variety. Assume that for a general $x \in X$ there is a smooth proper curve C and a morphism $f : C \to X$ such that*

- $x \in f(C)$,
- *X is smooth along $f(C)$,*
- $\deg_C f^* K_X < 0$.

Then X is uniruled.

To conclude the above discussion let us state a minimal model conjecture.

CONJECTURE 5.1.12. Let X be a projective variety with at most terminal $\mathbb{Q}$-factorial singularities. Then there exists a minimal model X' birational to X if and only if X is not uniruled.

In dimension 3 the above conjecture is now a theorem. To prove it we still need the following observations.

PROPOSITION 5.1.13 ([Ko3]). *Let X be a smooth uniruled variety. Then there is a dense family of rational curves with negative intersection with the canonical class. In particular X has negative Kodaira dimension.*

EXERCISE 5.1.14. Let X be a 3-fold with terminal singularities covered by curves negative with respect to the canonical class. Prove that, after finitely many birational modifications, instead of reaching a minimal model one encounters an extremal ray whose exceptional locus covers the whole variety. In other words one reaches a Mori space.

This can be proved also for n-folds as soon as one assumes the flip conjecture and the termination conjecture.

Summing things up we get

THEOREM 5.1.15. *Let X be a 3-fold with terminal singularities. Then the minimal model conjecture holds.*

Let us also mention the following result.

REMARK 5.1.16. The converse of Proposition 5.1.13 is a deep and challenging problem. It is conjectured that all smooth varieties with negative Kodaira dimension are uniruled. But an affirmative answer is only known up to dimension 3, as a byproduct of MMP (see [Mi]).

As a consequence one can formulate the minimal model conjecture in this straightened form, which is also true in dimension 3.

CONJECTURE 5.1.17. Let X be a projective variety with at most terminal singularities. Then there is a minimal model X' birational to X if and only if $k(X) \neq -\infty$.

5.2. The birational geometry of the plane. Sarkisov's program concerns the study of possible birational, non-biregular, maps between Mori spaces. We do not want here to

[Co2]. However we wish to give an idea of its techniques and possible applications in the simpler set-up of surfaces; for this, using Sarkisov's dictionary, we prove the following beautiful theorem.

Theorem 5.2.1 (Noether–Castelnuovo). *The group of birational transformations of the projective plane is generated by linear transformations and the standard Cremona transformation, that is,*

$$(x_0 : x_1 : x_2) \mapsto (x_1x_2 : x_0x_2 : x_0x_1),$$

where $(x_0 : x_1 : x_2)$ are the coordinates of $\mathbb{P}^2$.

Let $\chi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map which is not an isomorphism. To study the map χ we start by factorising it into simpler birational maps, "elementary links", between Mori spaces (these maps will be either the blow-up of a point in $\mathbb{P}^2$, or an elementary transformation of a rational ruled surface, see diagram (5.2.2)). Consider $\mathcal{H} = \chi_*^{-1}\mathcal{O}(1)$, the strict transform of lines in $\mathbb{P}^2$; then $\mathcal{H}$ is without fixed components and $\mathcal{H} \subset |\mathcal{O}(n)|$ for some $n > 1$. Our point of view is to consider the general element $H \in \mathcal{H}$ as a twisted line. The factorisation we are aiming at should "untwist" H step by step so as to restore the original line, hence the starting $\mathbb{P}^2$. Observe that the fact that χ is not biregular is encoded in the base locus of $\mathcal{H}$, therefore the untwisting is clearly related to the singularities of the log pair $(\mathbb{P}^2, \mathcal{H})$, where by the pair $(\mathbb{P}^2, \mathcal{H})$ we understand the pair $(\mathbb{P}^2, H)$ were $H \in \mathcal{H}$ is a general element.

Theorem 5.2.2. *Let $\mathcal{H} \subset |\mathcal{O}(n)|$ be as above; then the pair $(\mathbb{P}^2, (3/n)\mathcal{H})$ has non-canonical singularities. In particular there is a point $x \in \mathbb{P}^2$ such that*

(5.2.1) $$\operatorname{mult}_x \mathcal{H} > n/3.$$

Proof. Take a resolution of χ

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\chi"] & & \mathbb{P}^2 \end{tikzcd}$$

and pull back the divisor $K_{\mathbb{P}^2} + (3/n)\mathcal{H}$ and $K_{\mathbb{P}^2} + (3/n)\mathcal{O}(1)$ via p and q respectively. We have

$$\begin{aligned} K_W + (3/n)\mathcal{H}_W &= p^*\mathcal{O}_{\mathbb{P}^2} + \sum_i a_i' E_i + \sum_h c_h G_h \\ &= q^*\mathcal{O}_{\mathbb{P}^2}(3(1/n - 1)) + \sum_i a_i E_i + \sum_j b_j F_j \end{aligned}$$

where E_i are p and q exceptional divisors, while F_j are q but not p exceptional divisors and G_h are p but not q exceptional divisors. Observe that since $\mathcal{O}(1)$ is base point free, the a_i's and b_j's are positive integers.

Let $l \subset \mathbb{P}^2$ be a general line in the right hand side plane. In particular q is an isomorphism on l and therefore $E_i \cdot q^*l = F_j \cdot q^*l = 0$ for all i and j.

The crucial point is that on the right hand side we have some negativity coming from the non-effective divisor $K_{\mathbb{P}^2} + (3/n)\mathcal{O}(1)$ that has to be compensated by some

More precisely, since $n > 1$, on the one hand we have

$$(K_W + (3/n)\mathcal{H}_W) \cdot q^*l = \Big(q^*\mathcal{O}_{\mathbb{P}^2}(3(1/n-1)) + \sum_i a_iE_i + \sum_j b_jF_j\Big) \cdot q^*l < 0,$$

and on the other hand,

$$0 > (K_W + (3/n)\mathcal{H}_W) \cdot q^*l = \Big(p^*\mathcal{O}_{\mathbb{P}^2} + \sum_i a_i'E_i + \sum_h c_hG_h\Big) \cdot q^*l.$$

So $c_h < 0$ for some h, that is, $(\mathbb{P}^2, (3/n)\mathcal{H})$ is not canonical.

We leave it to the reader to justify equation (5.2.1); remember that one can resolve the base locus of $\mathcal{H}$ blowing up smooth points only. ∎

The above proof can be generalised to the following set-up. Let $\pi : X \to S$ and $\varphi : Y \to W$ be two Mori spaces of dimension ≤ 3. Let $\chi : X \to Y$ be a birational non-biregular map. Choose a very ample linear system $\mathcal{H}_Y$ on Y. Let $\mathcal{H} = \chi_*^{-1}\mathcal{H}_Y$. Then by the definition of a Mori space there exists a $\mu \in \mathbb{Q}$ such that $K_X + (1/\mu)\mathcal{H} \equiv_\pi 0$.

Theorem 5.2.3 (Noether–Fano inequalities, [Co1]). *In the above notation, in particular for χ non-biregular and $K_X + (1/\mu)\mathcal{H} \equiv_\pi 0$, either $(X, (1/\mu)\mathcal{H})$ has non-canonical singularities or $K_X + (1/\mu)\mathcal{H}$ is not nef.*

We are now ready to start the factorisation of χ. For this let $x \in \mathbb{P}^2$ be a point such that $(\mathbb{P}^2, (3/n)\mathcal{H})$ is not canonical at x. Such a point exists by Theorem 5.2.2 and let $\nu : \mathbb{F}_1 \to \mathbb{P}^2$ be the blow-up of x, with exceptional divisor C_0. In the context of the Sarkisov theory it is important to look at this blow-up in the following way.

Definition 5.2.4. A *terminal extraction* is a birational morphism with connected fibers $f : Y \supset E \to X \ni x$ such that:

- X and Y are terminal varieties, Y is $\mathbb{Q}$-factorial,
- the exceptional locus is an irreducible divisor E with $f(E) \ni x$,
- $-K_Y$ is f-ample.

Exercise 5.2.5. Prove that the only terminal extraction from a smooth point of a surface is the blow-up of the maximal ideal of a point.

Hint: For a surface, terminal is equivalent to smooth. This is just restatement of Theorem 4.1.1.

Remark 5.2.6. More generally whenever a log pair $(X, (1/\mu)\mathcal{H})$ is not canonical then there exists a terminal extraction (see [Co1]).

Let us return to the proof. Observe that the natural map $\pi_1 : \mathbb{F}_1 \to \mathbb{P}^1$ is a Mori space structure. The map $\nu : \mathbb{F}_1 \to \mathbb{P}^2$, the blow-up of $\mathbb{P}^2$, is the first elementary link we define.

Let $\chi' = \chi \circ \nu : \mathbb{F}_1 \to \mathbb{P}^2$ and $\mathcal{H}' = (\chi')_*^{-1}\mathcal{O}(1)$. Let $n' = n - \mathrm{mult}_x\mathcal{H}$. Then

$$K_{\mathbb{F}_1} + (2/n')\mathcal{H}' \equiv_{\pi_1} 0.$$

We are in a position to apply Theorem 5.2.3 to the pair $(\mathbb{F}_1, (2/n')\mathcal{H}')$. First notice that $K_{\mathbb{F}_1} + (2/n')\mathcal{H}'$ is nef. In fact let $f \subset \mathbb{F}_1$ be a generic fiber of the ruled structure. Then by definition. On the other hand

$$(K_{\mathbb{F}_1} + (2/n')\mathcal{H}') \cdot C_0 = -1 + (2/n') \operatorname{mult}_x \mathcal{H} = -n + 3 \operatorname{mult}_x \mathcal{H} > 0$$

where the last inequality comes directly from (5.2.1), that is, the existence of non-canonical singularities for $(\mathbb{P}^2, (3/n)\mathcal{H})$. Thus, by Theorem 5.2.3, $K_{\mathbb{F}_1} + (2/n')\mathcal{H}'$ is not canonical and therefore the linear system $\mathcal{H}'$ admits a point $x' \in \mathbb{F}_1$ with multiplicity greater than $2/n'$.

The next step is a terminal extraction from x'. Let

$$\psi : Z \supset E \to \mathbb{F}_1 \ni x',$$

the blow-up of x'. This time Z is not a Mori space, but the strict transform of the fiber of $\mathbb{F}_1$ containing x' is now a (-1)-curve which can then be contracted by $\varphi : Z \to S$.

(5.2.2)
$$\begin{tikzcd} & Z \arrow[dl, "\psi"'] \arrow[dr, "\varphi"] & \\ \mathbb{F}_1 \arrow[rr, dashed] & & S \end{tikzcd}$$

This modification is known as an *elementary transformation* of ruled surfaces.

Exercise 5.2.7. Prove that S is either a quadric, $\mathbb{F}_0$, or $\mathbb{F}_2$.
Hint: This depends on the position of the point with respect to C_0.

Let $x_2 \subset S$ be the exceptional locus of φ^{-1} and $\mathcal{H}_2$ be the strict transform of $\mathcal{H}'$. Observe the following two facts:

(i) $(K_S + (2/n')\mathcal{H}_2) \cdot f = 0$, where, by abuse of notation, f is the strict transform of $f \subset \mathbb{F}_1$,

(ii) since $\operatorname{mult}_{x'} \mathcal{H}' > \mathcal{H}' \cdot f/2$, $(S, (2/n')\mathcal{H}_2)$ has terminal singularities at x_2.

By (i) we can apply Theorem 5.2.3 to the log pair $(S, (2/n')\mathcal{H}_2)$. Moreover by (ii) we have not introduced any new canonical singularities since the point x_2 is a terminal singularity for this pair. This is very important because it proves that after finitely many elementary transformations we reach a pair $(\mathbb{F}_k, (2/n')\mathcal{H}_r)$ with canonical singularities such that

$$K_{\mathbb{F}_k} + (2/n')\mathcal{H}_r \equiv_{\pi_k} 0.$$

Then, again by Theorem 5.2.3, the pair $(\mathbb{F}_k, (2/n')\mathcal{H}_r)$ cannot be nef.

Observe that $\mathrm{NE}(\mathbb{F}_k)$ is a 2-dimensional cone. In particular it has only two rays. One is spanned by f, a fiber of π_k. Let Z be an effective irreducible curve in the other ray. Then

(5.2.3) $$(K_{\mathbb{F}_k} + (2/n')\mathcal{H}_r) \cdot Z < 0.$$

Since $\mathcal{H}_r$ has non-fixed components, $\mathbb{F}_k$ is a del Pezzo surface and the only possibilities are therefore $k = 0, 1$.

In case $k = 1$, what is left is to simply blow down the exceptional curve $\nu : \mathbb{F}_1 \to \mathbb{P}^2$, and reach $\mathbb{P}^2$ together with a linear system $\nu_*\mathcal{H}_2 =: \widetilde{\mathcal{H}} \subset |\mathcal{O}(j)|$. Note that in this case, by (5.2.3), for some positive δ. Therefore $K_{\mathbb{P}^2} + (2/n')\widetilde{\mathcal{H}}$ is not nef. In other terms $(2/n')j < 3$, and

$$j < \frac{3(n - \operatorname{mult}_x \mathcal{H})}{2} < n.$$

Now we iterate the above argument, i.e. we restart at the beginning of the proof but with the pair $(\mathbb{P}^2, (3/j)\widetilde{\mathcal{H}})$; the above strict inequality $j < n$ tells us that after finitely many steps we untwist the map χ, i.e. we reach $\mathbb{P}^2$ with a linear system $\mathcal{H} = |\mathcal{O}(1)|$.

In case $k = 0$, observe that $\mathbb{F}_0 \simeq \mathbb{Q}^2$ is a Mori space for two different fibrations; let f_0 and f_1 be their general fibers. Moreover by (5.2.3),

$$(K_{\mathbb{F}_0} + (2/n')\mathcal{H}_r) \cdot f_1 < 0.$$

That is, there exists an

$$n_1 < n' \tag{5.2.4}$$

such that

$$(K_{\mathbb{F}_0} + (2/n_1)\mathcal{H}_r) \cdot f_0 > 0, \qquad (K_{\mathbb{F}_0} + (2/n_1)\mathcal{H}_r) \cdot f_1 = 0.$$

Again by Theorem 5.2.3, this time applied to the fibration with fiber f_1, this implies that $(\mathbb{F}_0, (2/n_1)\mathcal{H}_r)$ is not canonical and we iterate the procedure. As in the previous case the strict inequality of (5.2.4) implies termination after finitely many steps.

Thus we have factorised any birational, non-biregular, self-map of $\mathbb{P}^2$ into a sequence of "elementary links", namely elementary transformations and blow-ups of $\mathbb{P}^2$ at a point.

The next step is to interpret a standard Cremona transformation in this new language, i.e. in terms of the elementary links we have introduced above.

Exercise 5.2.8. Prove that a standard Cremona transformation is given by the following links:

$$\begin{tikzcd}
 & \mathbb{F}_1 \arrow[r, dashed] \arrow[dl] & \mathbb{F}_0 \arrow[r, dashed] & \mathbb{F}_1 \arrow[dr] & \\
\mathbb{P}^2 & & & & \mathbb{P}^2
\end{tikzcd}$$

Conversely, any map of type

$$\begin{tikzcd}
 & \mathbb{F}_1 \arrow[r, dashed] \arrow[dl] & \mathbb{F}_a \arrow[r, dashed] & \mathbb{F}_1 \arrow[dr] & \\
\mathbb{P}^2 & & & & \mathbb{P}^2
\end{tikzcd}$$

can be factorised into Cremona transformations.

Hint: A standard Cremona transformation is given by conics through 3 non-collinear points. The link above is possible only for $a = 0, 2$. Links of this kind represent birational maps given by conics with either 3 base points or 2 base points plus a tangent direction. Try to factorise the map

$$(x_0 : x_1 : x_2) \to (x_1x_2 : x_0x_2 : x_1x_2 + x_0x_2 + x_0^2).$$

Proof of Theorem 5.2.1. Let $\chi : \mathbb{P}^2 \to \mathbb{P}^2$ be a birational map and

$$\begin{tikzcd} & \mathbb{F}_1 \arrow[r, dashed, "l_0"] \arrow[dl, "\nu_1"'] & \mathbb{F}_k \arrow[r, dashed, "l_1"] & \cdots \arrow[r, dashed] & \mathbb{F}_1 \arrow[dr] & \\ \mathbb{P}^2 & & & & & \mathbb{P}^2 \end{tikzcd} \tag{5.2.5}$$

the factorisation into elementary links obtained above. Let us first make the following observation. If there is a link leading to an $\mathbb{F}_1$ then we can break the birational map by simply blowing down the (-1)-curve. That is, substitute χ with the following two pieces:

$$\begin{tikzcd} & \mathbb{F}_1 \arrow[r, dashed, "l_0"] \arrow[dl, "\nu_1"'] & \cdots \arrow[r, dashed, "l_i"] & \mathbb{F}_1 \arrow[rr, "\sim"] \arrow[dr, "\nu_2"] & & \mathbb{F}_1 \arrow[r, dashed, "l_{i+1}"] \arrow[dl, "\nu_2"'] & \cdots \arrow[r, dashed] & \mathbb{F}_1 \arrow[dr] & \\ \mathbb{P}^2 \arrow[rrrr, dashed, "\chi_1"] & & & & \mathbb{P}^2 \arrow[rrrr, dashed, "\chi_2"] & & & & \mathbb{P}^2 \end{tikzcd}$$

So we can assume that

(5.2.6) there are no links leading to $\mathbb{F}_1$ “inside” the factorisation.

Let

$$d(\chi) = \max\{k : \text{there is an } F_k \text{ in the factorisation}\}.$$

If $d(\chi) \leq 2$ we can factorise it by Exercise 5.2.8.

We now prove the theorem by induction on $d(\chi)$. Consider the left part of the factorisation (5.2.5). Since $d(\chi) > 2$, by assumption (5.2.6), l_0 is of type $\mathbb{F}_1 \to \mathbb{F}_2$ and l_1 is of type $\mathbb{F}_2 \to \mathbb{F}_3$. Then we force Cremona like diagrams in it, at the cost of introducing new singularities. Let

$$\begin{tikzcd} & \mathbb{F}_1 \arrow[r, dashed, "\alpha"] \arrow[dl, "\nu_1"'] & \mathbb{F}_0 \arrow[r, dashed, "l_0"] & \mathbb{F}_1 \arrow[dr, "\nu_2"] & & \mathbb{F}_1 \arrow[r, dashed, "\alpha^{-1}"] \arrow[dl, "\nu_2"'] & \mathbb{F}_0 \arrow[r, dashed, "l_1"] & \cdots \arrow[r, dashed] & \mathbb{F}_1 \arrow[dr] & \\ \mathbb{P}^2 & & & & \mathbb{P}^2 \arrow[rrrrr, dashed, "\chi'"] & & & & & \mathbb{P}^2 \end{tikzcd}$$

where $\alpha : \mathbb{F}_1 \to \mathbb{F}_0$ is an elementary transformation centered at a general point of $\mathbb{F}_1$, and $\mathrm{Exc}(\alpha^{-1}) = \{y_0\}$. So $\alpha_*(\mathcal{H}')$ has an ordinary singularity at y_0. Then l_0 is exactly the same modification but leads to an $\mathbb{F}_1$ and ν_2 is the blow-down of the exceptional curve of this $\mathbb{F}_1$. Observe that neither α_0 nor ν_2 are links in the Sarkisov category, in general. Nonetheless the first part can be factorised into standard Cremona transformations. Let $\chi' = \chi_1 \circ \ldots \circ \chi_k$ be a decomposition of χ into pieces satisfying (5.2.6). Then $d(\chi_i) < d(\chi)$ for all $i = 1, \ldots, k$. Therefore by the induction hypothesis also χ' can be factorised into Cremona transformations. Hence χ is factorised into Cremona transformations. ∎

5.3. #-Minimal model. We already pointed out that the Minimal Model Program allows one to attach a Mori space to a uniruled 3-fold (see 5.1.14). How can we use it to study the birational geometry of X?

The main difficulty here is that the birational modifications occurring along the MMP are difficult to follow and usually it is almost impossible to guess what is the output. We varieties using a polarising divisor; this is called a #-minimal model. Under strong assumptions on the variety studied, we are able to govern the program and understand its output.

Definition 5.3.1 ([Me4]). Let T be a terminal $\mathbb{Q}$-factorial uniruled 3-fold and $\mathcal{H}$ a movable linear system, i.e. $\dim |mH| > 0$ for $m \gg 0$, with generic element $H \in \mathcal{H}$ on T. Assume that H is nef. Then

$$\varrho = \varrho_{\mathcal{H}} = \varrho(T, \mathcal{H}) := \sup\{m \in \mathbb{Q} : H + mK_T \text{ is an effective } \mathbb{Q}\text{-divisor}\}$$

is the *threshold* of the pair $(T, \mathcal{H})$.

Since we are assuming that $\dim \mathcal{H} \geq 0$ we have $\varrho \geq 0$. A pair $(T^{\#}, \mathcal{H}^{\#})$ is called a *#-minimal model* of $(T, \mathcal{H})$ if:

(i) $T^{\#}$ has a Mori fiber space structure $\pi : T^{\#} \to W$ and $\mathcal{H}^{\#}$ is a movable Weil divisor,

(ii) there exists a birational map $\psi : T \to T^{\#}$ such that $\mathcal{H}^{\#} = \psi_* \mathcal{H}$,

(iii) if $H^{\#} \in \mathcal{H}^{\#}$ is a general member, then $\varrho(T, \mathcal{H})K_{T^{\#}} + H^{\#} \equiv_\pi \mathcal{O}_{T^{\#}}$.

To find a #-minimal model of a given pair $(T, \mathcal{H})$ let us proceed in the following way.

Let $(T_0, \mathcal{H}_0) = (T, \mathcal{H})$, where H_0 is nef by hypothesis and T_0 is uniruled; therefore to (T_0, H_0) there is naturally associated the nef value $t_0 = \sup\{m \in \mathbb{Q} : mK_{T_0} + H_0$ is nef$\}$ and a rational map $\varphi_0 : T_0 \to T_1$, which is either an extremal contraction or (if the extremal contraction is small) a flip, of an extremal ray in the face spanned by $t_0 K_{T_0} + \mathcal{H}_0$ (see Section 3.1 and in particular 3.1.7).

Consequently, on T_1 one defines a movable linear system by $\mathcal{H}_1 := \varphi_{0*}\mathcal{H}_0$. That is to say, $\varphi_{0*}H_0 \neq 0$. Note that, by construction, $t_0 K_{T_1} + H_1$ is nef, thus one inductively defines $\varphi_i : T_i \to T_{i+1}$ and $(T_{i+1}, \mathcal{H}_{i+1})$ as follows. Let $\delta = \sup\{d \in \mathbb{Q} : dK_{T_i} + (t_{i-1}K_{T_i} + H_i)$ is nef$\}$ and define $t_i := \delta + t_{i-1}$.

Exercise 5.3.2. Prove that there always exists an extremal ray $[C_i] \subset \overline{\mathrm{NE}}(T_i)$ in the face supported by $t_i K_{T_i} + H_i$.

Thus define $\varphi_i : T_i \to T_{i+1}$ to be the birational modification associated to the extremal ray $[C_i] \subset \overline{\mathrm{NE}}(T_i)$, and $\mathcal{H}_{i+1} := \varphi_{i*}\mathcal{H}_i$.

The inductive process is therefore composed of divisorial contractions and flips. Since T_0 is uniruled it does not have a minimal model (see Theorem 5.1.15). After finitely many of these birational modifications, we get a Mori fiber space.

Exercise 5.3.3. Prove that the output $(T_k, \mathcal{H}_k)$ is a #-minimal model, that is,

$$\varrho(T, \mathcal{H})K_{T_k} + H_k \equiv_\pi \mathcal{O}_{T_k}.$$

Remark 5.3.4. Note that $\mathcal{H}^{\#}$ is relatively nef. Furthermore if the rational map defined by $|mH|$ is birational then $\mathcal{H}^{\#}$ is relatively ample.

The presence of a polarisation in the #-program allows us to control the steps if we are able to impose restrictions on the threshold. Let $(T, \mathcal{H})$ be as above and assume moreover that $\varrho_{\mathcal{H}} < 1$ and that there exists a smooth surface $S \in \mathcal{H}$. Notice that the it is possible to describe in detail the #-process in a neighborhood of the surface S (see also [CF]).

PROPOSITION 5.3.5 ([Me4]). *Let $\varphi_i : T_i \to T_{i+1}$ be a birational modification in the #-program relative to $(T,\mathcal{H})$ with $\varrho_{\mathcal{H}} < 1$. Assume that $S \in \mathcal{H}_i$ is a smooth surface. Then $\varphi_i(S) = \overline{S}$ is a smooth surface and $\varphi_{i|S} : S \to \overline{S}$ is either an isomorphism or the contraction of a disjoint union of (-1)-curves.*

Sketch of proof. Since S is smooth and T_i is terminal $\mathbb{Q}$-factorial, S is not in $\mathrm{Sing}(T_i)$. In particular H_i is a Cartier divisor. We have the following cases.

- [φ_i contracts a divisor E onto a curve] Then $H_i \equiv_{\varphi_i} 0$ and $S \cap E$ is the disjoint union of (-1)-curves.
- [φ_i is a flip] S is disjoint from the flipping curve.
- [φ_i contracts a divisor E to a point] $\varphi_{i|S}$ is birational and is either an isomorphism or the contraction of a (-1)-curve. Then $(E, E_{|E}) \simeq (\mathbb{P}^2, \mathcal{O}(-1))$ and $H_{|E} \sim \mathcal{O}(1)$. ■

Using the above proposition we can control the #-minimal model and its output.

COROLLARY 5.3.6. *Let T be a terminal $\mathbb{Q}$-factorial uniruled 3-fold, $\mathcal{H}$ a movable nef linear system and $(T^{\#}, \mathcal{H}^{\#})$ a #-minimal model of $(T,\mathcal{H})$. Assume that $\varrho_{\mathcal{H}} < 1$ and $\mathcal{H}$ is base point free. Then $H^{\#} \in \mathrm{Pic}(T^{\#})$, $\mathcal{H}^{\#}$ has at most base points and $H^{\#}$ is smooth.*

Proof. By the Bertini theorem H is smooth, therefore we can apply Proposition 5.3.5 in an inductive way to reach a model $(T^{\#}, \mathcal{H}^{\#})$. ■

We need a relative version of Corollary 5.3.6, and for this we first give a definition.

DEFINITION 5.3.7 ([Me4]). Let T be a 3-fold and $\mathcal{H}$ a movable linear system with $\dim \mathcal{H} \geq 1$. Assume that $H = M + F$, where $\mathcal{M}$ is a movable linear system without fixed component and F is the fixed component. A pair $(T_1, \mathcal{H}_1)$ is called a *log minimal resolution* of the pair $(T,\mathcal{H})$ if there is a morphism $\mu : T_1 \to T$ with the following properties:

- T_1 is terminal $\mathbb{Q}$-factorial,
- $\mu_*^{-1} M = H_1$, where H_1 is a Cartier divisor, $\dim \mathrm{Bsl}(\mathcal{H}_1) \leq 0$,
- a general element $H_1 \in \mathcal{H}_1$ is a minimal resolution of a general element $M \in \mathcal{M}$.

COROLLARY 5.3.8. *For any pair $(T,\mathcal{H})$ with T an irreducible $\mathbb{Q}$-factorial 3-fold and $\mathcal{H}$ a movable linear system with $\dim \mathcal{H} \geq 1$, there exists a log minimal resolution.*

REMARK 5.3.9. Using Corollary 5.3.8 we can study any irreducible 3-fold T equipped with a movable linear system $\mathcal{H}$ with $\dim \mathcal{H} \geq 1$. Indeed we consider a log minimal resolution of $(T^{\#}, \mathcal{H})$ and then a #-minimal model of it. Note that this is well defined only up to birational equivalence.

5.4. Applications of the #-program. We now want to apply the #-theory to some concrete situations. Although the assumptions in the previous section are quite strong,

5.4.1. *3-folds with a uniruled movable system*

DEFINITION 5.4.1. Let T be a terminal $\mathbb{Q}$-factorial 3-fold and $\mathcal{H}$ a movable linear system. We say that $(T, \mathcal{H})$ is a *pair with a big uniruled system* if $H \in \mathcal{H}$ is nef and big and H is a smooth surface of negative Kodaira dimension.

EXERCISE 5.4.2. Let $(T, \mathcal{H})$ be a pair with a big uniruled system. Then T is uniruled and $\varrho(T, \mathcal{H}) < 1$.

Using #-MMP techniques we can describe in detail the possibilities that occur under these conditions.

THEOREM 5.4.3 ([Me4]). *Let $(T, \mathcal{H})$ be a pair with a big uniruled system. Then $(T^{\#}, \mathcal{H}^{\#})$ is one of the following*:

(i) *a $\mathbb{Q}$-Fano* 3-*fold of index* $1/\varrho > 1$, *with* $K_{T^{\#}} \sim -1/\varrho H^{\#}$ *and* $\Phi_{|H^{\#}|}$ *birational*; *the complete classification is given in* [CF] *and* [Sa]:

- $(\mathbb{P}(1,1,2,3), \mathcal{O}(6))$,
- $(X_6 \subset \mathbb{P}(1,1,2,3,a), X_6 \cap \{x_4 = 0\})$ *with* $3 \leq a \leq 5$,
- $(X_6 \subset \mathbb{P}(1,1,2,2,3), X_6 \cap \{x_3 = 0\})$,
- $(X_6 \subset \mathbb{P}(1,1,1,2,3), X_6 \cap \{x_0 = 0\})$,
- $(\mathbb{P}(1,1,1,2), \mathcal{O}(4))$,
- $(X_4 \subset \mathbb{P}(1,1,1,1,2), X_4 \cap \{x_0 = 0\})$,
- $(X_4 \subset \mathbb{P}(1,1,1,2,a), X_4 \cap \{x_4 = 0\})$ *with* $2 \leq a \leq 3$,
- $(\mathbb{P}^3, \mathcal{O}(a))$, *with* $a \leq 3$, $(\mathbb{Q}^3, \mathcal{O}(b))$ *with* $b \leq 2$,
- $(X_3 \subset \mathbb{P}(1,1,1,1,2), X_3 \cap \{x_4 = 0\})$, $(X_3 \subset \mathbb{P}^4, \mathcal{O}(1))$,
- $(X_{2,2} \subset \mathbb{P}^5, \mathcal{O}(1))$,
- *a linear section of the Grassmann variety parametrising lines in* $\mathbb{P}^4$, *embedded in* $\mathbb{P}^9$ *by Plücker coordinates*,
- $(\mathbb{P}(1,1,1,2), \mathcal{O}(2))$, *the cone over the Veronese surface*,

(ii) *a bundle over a smooth curve with at most finitely many fibers* $(G, H^{\#}_{|G}) \simeq (\mathbf{S}_4, \mathcal{O}(1))$, *and generic fiber* $(F, H^{\#}_{|F}) \simeq (\mathbb{P}^2, \mathcal{O}(2))$; *where* $\mathbf{S}_4$ *is the cone over the normal quartic curve and the vertex sits over a hyper-quotient singularity of type* $1/2(1,-1,1)$ *with* $f = xy - z^2 + t^k$ *for* $k \geq 1$ (*see* [YPG]),

(iii) *a quadric bundle with at most* cA_1 *singularities of type* $f = x^2 + y^2 + z^2 + t^k$ *for* $k \geq 2$, *and* $H^{\#}_{|F} \sim \mathcal{O}(1)$,

(iv) $(\mathbb{P}(E), \mathcal{O}(1))$ *where* E *is a rank* 3 *vector bundle over a smooth curve*,

(v) $(\mathbb{P}(E), \mathcal{O}(1))$ *where* E *is a rank* 2 *vector bundle over a surface of negative Kodaira dimension.*

REMARK 5.4.4. The above theorem allows one to extend the result of [CF] to 3-folds T which contain a smooth surface H of negative Kodaira dimension such that H is nef and big. Ciro Ciliberto pointed out to us that the theorem completes the research suggested by Castelnuovo [Ca, p. 187], to study linear systems of rational surfaces.

EXERCISE 5.4.5. Prove the following. Let T be a terminal 3-fold and $H \subset T$ a smooth to one of the following:

- $\mathbb{P}^3$,
- $H \times \mathbb{P}^1$,
- a terminal sextic in either $\mathbb{P}(1,1,1,2,3)$ or $\mathbb{P}(1,1,2,2,3)$,
- a terminal quartic in $\mathbb{P}(1,1,1,1,2)$,
- a terminal cubic in $\mathbb{P}^4$.

There exists a natural geometric interpretation of the conditions imposed in Theorem 5.4.3.

THEOREM 5.4.6 ([Me4]). *Let $T_d \subset \mathbb{P}^n$ be a degree d non-degenerate irreducible* 3*-fold. Suppose that $d < 2n-4$. Then any #-minimal model $(T^\#, \mathcal{H}^\#)$ of $(T_d, \mathcal{O}(1))$ is in the list of Theorem* 5.4.3.

Proof. Let $\nu : X \to T$ be a resolution of singularities and $\mathcal{H} = \nu^*\mathcal{O}(1)$. First we prove that $(K_X + H)\cdot H^2 < 0$. We argue by comparing the Castelnuovo bound on the genus of $C := H^2$ and the genus formula on the surface H. From the latter we obtain $g(C) = 1 + d/2 + (K_X + H)\cdot C/2$. For the former let $m = \lfloor \frac{d-1}{n-3} \rfloor$. Then by the Castelnuovo inequality [GH, p. 527], we have

$$g(C) \leq \frac{m(m-1)}{2}(n-3) + m(d-1-m(n-3)).$$

It is therefore enough to require that

$$1 + d/2 > \frac{m(m-1)}{2}(n-3) + m(d-1-m(n-3)); \tag{5.4.1}$$

after a small calculation one verifies that this is true whenever $d < 2n-4$. Then by the adjunction formula, $H \in \mathcal{H}$ is a smooth surface of negative Kodaira dimension and Theorem 5.4.3 applies. ∎

REMARK 5.4.7. This theorem can be interpreted as the 3-dimensional counterpart of the classical result that a non-degenerate surface $S \subset \mathbb{P}^n$ of degree $d \leq n-1$ is birational either to a rational scroll or to a projective plane [GH, p. 525]. Observe that all the listed 3-folds admit an embedding satisfying the numerical criterion.

By means of adjunction theory on terminal varieties (see [Me1]), one can prove the following higher dimensional analog of Theorem 5.4.6.

THEOREM 5.4.8 ([Me4]). *Let $X_d \subset \mathbb{P}^n$ be a non-degenerate k-fold with $k > 3$ and only $\mathbb{Q}$-factorial terminal singularities. Assume that $d < 2(n-k)-2$. Then a #-minimal model $(X^\#, H^\#)$ of $(X, \mathcal{O}(1))$ (in adjunction theory language, $(X^\#, H^\#)$ is the first reduction) is one of the following*:

(i) *a $\mathbb{Q}$-Fano n-fold of Fano index $1/\varrho > k-2$, with $K_{T^\#} \sim -(1/\varrho)H^\#$ and $\Phi_{|H^\#|}$ birational; the complete classification is given in* [Fu2] *if $X^\#$ is Gorenstein and in* [CF] *and* [Sa] *in the non-Gorenstein case*,

(ii) *a projective bundle over a curve with fibers $(F, H^\#_{|F}) \simeq (\mathbb{P}^{k-1}, \mathcal{O}(1))$, or a quadric*

(iii) $(\mathbb{P}(E), \mathcal{O}(1))$ *where E is a rank $k-1$ ample vector bundle either on $\mathbb{P}^2$ or on a ruled surface.*

REMARK 5.4.9. Note that since X_d has terminal singularities it follows that, assuming the minimal model conjecture, the above is the classification of #-models of those varieties.

5.4.2. *General elephants of $\mathbb{Q}$-Fano 3-folds.* Another direction is the study of the birational class of $\mathbb{Q}$-Fano 3-folds whose generic section has worse than canonical singularities.

CONJECTURE 5.4.10 (Reid). Let X be a $\mathbb{Q}$-Fano 3-fold and $H \in |-K_X|$ a generic section of the anticanonical divisor. Then H has at worst canonical singularities.

The motivation of this conjecture is that to classify Fano varieties, as we have learned, one uses sections of the fundamental divisor. For non-Gorenstein 3-folds with index < 1, this invariant is quite meaningless and one tries to use directly sections of $|-K_X|$. So, more than a conjecture, it is a hope that things are not too bad in this corner of the world. It has to be said that the most recent techniques to study $\mathbb{Q}$-Fano 3-folds do not rely completely on this procedure. The #-program allows one to understand the birational nature of these strange objects.

THEOREM 5.4.11 ([Me4]). *Let T be a $\mathbb{Q}$-Fano 3-fold. Assume that $\dim \phi_{|-K_T|}(T) = 3$ and the general element in $|-K_T|$ has worse than Du Val singularities. Then T is birational to a smooth Fano 3-fold $T^{\#}$ of index ≥ 2.*

The rough idea is to take a log minimal resolution of $(T, |-K_T|)$ and control the output. By the singularity requirement the generic element in $|-K_T|$ is a uniruled surface, therefore we can apply all results of previous sections. For more details see [Me4].

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