Contents

Abstract

Let X be a space. A space Y is called an *extension* of X if Y contains X as a dense subspace. For an extension Y of X the subspace $Y \setminus X$ of Y is called the *remainder* of Y. Two extensions of X are said to be *equivalent* if there is a homeomorphism between them which fixes X pointwise. For two (equivalence classes of) extensions Y and Y' of X let $Y \leq Y'$ if there is a continuous mapping of Y' into Y which fixes X pointwise. Let $\mathcal{P}$ be a topological property. An extension Y of X is called a *$\mathcal{P}$-extension* of X if it has $\mathcal{P}$. If $\mathcal{P}$ is compactness then $\mathcal{P}$-extensions are called *compactifications*.

The aim of this article is to introduce and study classes of extensions (which we call *compactification-like $\mathcal{P}$-extensions*, where $\mathcal{P}$ is a topological property subject to some mild requirements) which resemble the classes of compactifications of locally compact spaces. We formally define compactification-like $\mathcal{P}$-extensions and derive some of their basic properties, and in the case when the remainders are countable, we characterize spaces having such extensions. We then consider the classes of compactification-like $\mathcal{P}$-extensions as partially ordered sets. This consideration leads to some interesting results which characterize compactification-like $\mathcal{P}$-extensions of a space among all its Tychonoff $\mathcal{P}$-extensions with compact remainder. Furthermore, we study the relations between the order-structure of classes of compactification-like $\mathcal{P}$-extensions of a Tychonoff space X and the topology of a certain subspace of its outgrowth $\beta X \setminus X$. We conclude with some applications, including an answer to an old question of S. Mrówka and J. H. Tsai: For what pairs of topological properties $\mathcal{P}$ and $\mathcal{Q}$ is it true that every locally-$\mathcal{P}$ space with $\mathcal{Q}$ has a one-point extension with both $\mathcal{P}$ and $\mathcal{Q}$? An open question is raised.

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1. Introduction

Let X be a space. A space Y is called an *extension* of X if Y contains X as a dense subspace. If Y is an extension of X then the subspace $Y \setminus X$ of Y is called the *remainder* of Y. Two extensions of X are said to be *equivalent* if there exists a homeomorphism between them which fixes X pointwise. This defines an equivalence relation on the class of all extensions of X. The equivalence classes will be identified with individuals when this causes no confusion. For two (equivalence classes of) extensions Y and Y' of X we let $Y \leq Y'$ if there exists a continuous mapping of Y' into Y which fixes X pointwise. The relation $\leq$ defines a partial order on the set of all (equivalence classes of) extensions of X (see Section 4.1 of [29] for more details). Let $\mathcal{P}$ be a topological property. An extension Y of X is called a *$\mathcal{P}$-extension* of X if it has $\mathcal{P}$. If $\mathcal{P}$ is compactness then $\mathcal{P}$-extensions are called *compactifications*. The aim of this article is to introduce and study classes of extensions (which we call *compactification-like $\mathcal{P}$-extensions* where $\mathcal{P}$ is a topological property) which look like the classes of compactifications of locally compact spaces. These are, for a Tychonoff space X:

- The class of *minimal $\mathcal{P}$-extensions* of X, consisting of those Tychonoff $\mathcal{P}$-extensions Y of X with compact remainder such that Y is minimal (with respect to the subspace relation $\subseteq$) among all Tychonoff $\mathcal{P}$-extensions of X with compact remainder. (In other words, one cannot construct any other Tychonoff $\mathcal{P}$-extension of X with compact remainder by deleting points from the space Y.)
- The class of *optimal $\mathcal{P}$-extensions* of X, consisting of those Tychonoff $\mathcal{P}$-extensions Y of X with compact remainder such that the topology of Y is maximal (with respect to the inclusion relation $\subseteq$) among all topologies on Y which turn Y into a Tychonoff $\mathcal{P}$-extension of X with compact remainder and Y is minimal (with respect to the subspace relation $\subseteq$) among all Tychonoff $\mathcal{P}$-extensions of X with compact remainder. (In other words, one cannot construct any other Tychonoff $\mathcal{P}$-extension of X with compact remainder either by adding sets to the topology of Y or deleting points from Y.)

Here the topological property $\mathcal{P}$ is subject to some mild restrictions and will include most of the important covering properties (such as compactness, the Lindelöf property, countable compactness, paracompactness and metacompactness) as special cases.

This article is organized as follows:

In Chapter 2 we give the formal definitions of compactification-like $\mathcal{P}$-extensions and we derive some of their basic properties.

In Chapter 3 we consider the case when the extensions have countable remainders and characterize those Tychonoff spaces which have a compactification-like $\mathcal{P}$-extension with countable remainder.

In Chapter 4 we consider the classes of compactification-like $\mathcal{P}$-extensions of a Tychonoff space X as partially ordered sets. Besides the standard partial order $\leq$ we consider two other partial orders $\leq_{\text{inj}}$ and $\leq_{\text{surj}}$. These considerations lead to some interesting results which characterize compactification-like $\mathcal{P}$-extensions of X among all Tychonoff $\mathcal{P}$-extensions of X with compact remainder. Furthermore, we study the relationships between the order-structure of classes of compactification-like $\mathcal{P}$-extensions of X (partially ordered with $\leq$) and the topology of a certain subspaces of its outgrowth $\beta X \setminus X$. We conclude this chapter with a result which characterizes the largest (with respect to $\leq$) compactification-like $\mathcal{P}$-extension of X. This largest element, which we explicitly introduce as a subspace of the Stone–Čech compactification βX of X, turns out to be even the largest among all Tychonoff $\mathcal{P}$-extensions of X with compact remainder.

In Chapter 5 we give some applications of our study. These applications include the relations between compactification-like $\mathcal{P}$-extensions and tight $\mathcal{P}$-extensions with compact remainder (a *tight $\mathcal{P}$-extension* of a space X is a Tychonoff $\mathcal{P}$-extension of X which does not contain properly any other $\mathcal{P}$-extension of X as a subspace) and an answer to an old question of S. Mrówka and J. H. Tsai in [28]: For what pairs of topological properties $\mathcal{P}$ and $\mathcal{Q}$ is it true that every (Tychonoff) locally-$\mathcal{P}$ (non-$\mathcal{P}$) space with $\mathcal{Q}$ has a one-point (Tychonoff) extension with both $\mathcal{P}$ and $\mathcal{Q}$?

We conclude with an open question which naturally arises in connection with our study.

We now review some of the terminology, notation and well known results which will be used in the sequel. Our definitions mainly come from the standard text [5] (thus in particular, compact spaces are Hausdorff, perfect mappings are continuous with Hausdorff domains, etc.). Other useful sources are [8], [29] and [40].

The letters $\mathbb{R}$, $\mathbb{I}$ and $\mathbb{N}$ denote the real line, the closed unit interval and the set of all positive integers, respectively. By ω and Ω we denote the first infinite ordinal and the first uncountable ordinal, respectively, and by $\aleph_0$ and $\aleph_1$ we denote their cardinalities. The cardinality of a set A is denoted by $\text{card}(A)$. For a subset A of a space X we let $\text{cl}_X A$, $\text{int}_X A$ and $\text{bd}_X A$ denote the closure, the interior and the boundary of A in X, respectively. A subset of a space is said to be *clopen* if it is simultaneously closed and open. A *zero-set* of a space X is a set of the form $Z(f) = f^{-1}(0)$ for some continuous $f : X \to \mathbb{I}$. Any set of the form $X \setminus Z$, where Z is a zero-set of a space X, is called a *cozero-set* of X. We denote the set of all zero-sets of X by $\mathscr{Z}(X)$ and the set of all cozero-sets of X by $\text{Coz}(X)$. For a Tychonoff space X the *Stone–Čech compactification* of X is the largest (with respect to the partial order $\leq$) compactification of X and is denoted by βX. The Stone–Čech compactification of a Tychonoff X is characterized among the compactifications of X by either of the following properties:

- Every continuous mapping from X to a compact space is continuously extendible over βX.
- Every continuous mapping from X to $\mathbb{I}$ is continuously extendible over βX.

- For every $Z, S \in \mathscr{Z}(X)$ such that $Z \cap S = \emptyset$ we have
$$\operatorname{cl}_{\beta X} Z \cap \operatorname{cl}_{\beta X} S = \emptyset.$$
- For every $Z, S \in \mathscr{Z}(X)$ we have
$$\operatorname{cl}_{\beta X}(Z \cap S) = \operatorname{cl}_{\beta X} Z \cap \operatorname{cl}_{\beta X} S.$$

A continuous mapping $f : X \to Y$ is called *perfect* if X is a Hausdorff space, f is closed (not necessarily surjective) and continuous and any fiber $f^{-1}(y)$, where $y \in Y$, is a compact subset of X. A topological property $\mathcal{P}$ is said to be *invariant under perfect mappings* (*inverse invariant under perfect mappings*, respectively) if for any perfect surjective mapping $f : X \to Y$ the space Y (X, respectively) has $\mathcal{P}$ provided that X (Y, respectively) has $\mathcal{P}$. A topological property $\mathcal{P}$ is called *perfect* if it is both invariant and inverse invariant under perfect mappings. A topological property $\mathcal{P}$ is said to be *hereditary with respect to closed subsets* (*hereditary with respect to open subsets*, respectively) if any closed (open, respectively) subset of a space with $\mathcal{P}$ also has $\mathcal{P}$. A topological property $\mathcal{P}$ is called *finitely additive* if whenever $X = X_1 \oplus \cdots \oplus X_n$ and each X_i has $\mathcal{P}$ then X also has $\mathcal{P}$. Let $\mathcal{P}$ be a topological property. A space X is called *locally-$\mathcal{P}$* if each $x \in X$ has an open neighborhood U in X whose closure $\operatorname{cl}_X U$ has $\mathcal{P}$. Note that if X is a regular (Hausdorff) space and $\mathcal{P}$ is closed hereditary, then X is locally-$\mathcal{P}$ if and only if each point x of X has a local base consisting of open neighborhoods U of x such that $\operatorname{cl}_X U$ has $\mathcal{P}$.

2. Compactification-like $\mathcal{P}$-extensions

In this chapter we give definitions and derive some basic results which will be used throughout.

Definition 2.1. Let X be a space, let $\mathcal{P}$ be a topological property and let Y be a Tychonoff $\mathcal{P}$-extension of X with compact remainder.

The extension Y of X is called *minimal* if Y is minimal (with respect to the subspace relation $\subseteq$) among all Tychonoff $\mathcal{P}$-extensions of X with compact remainder, that is, Y does not contain properly any other Tychonoff $\mathcal{P}$-extension of X with compact remainder. In other words, one cannot obtain any other Tychonoff $\mathcal{P}$-extension of X with compact remainder by deleting points from the space Y.

The extension Y of X is called *optimal* if the topology of Y is maximal (with respect to the inclusion relation $\subseteq$) among all topologies on Y which turn Y into a Tychonoff $\mathcal{P}$-extension of X with compact remainder, and Y is minimal (with respect to the subspace relation $\subseteq$) among all Tychonoff $\mathcal{P}$-extensions of X with compact remainder. In other words, one cannot obtain any other Tychonoff $\mathcal{P}$-extension of X with compact remainder either by adding sets to the topology of Y or deleting points from Y.

We refer to either minimal $\mathcal{P}$-extensions or optimal $\mathcal{P}$-extensions as *compactification-like $\mathcal{P}$-extensions*.

Notation 2.2. Let X be a space and let $\mathcal{P}$ be a topological property. Denote by $\mathscr{E}(X)$ the set of all Tychonoff extensions of X with compact remainder and denote by either $\mathscr{E}^{\mathcal{P}}(X)$

or $\mathscr{E}_{\mathcal{P}}(X)$ the set of all elements of $\mathscr{E}(X)$ which have $\mathcal{P}$. Also, let $\mathscr{M}_{\mathcal{P}}(X)$ and $\mathscr{O}_{\mathcal{P}}(X)$ denote the set of all minimal $\mathcal{P}$-extensions of X and the set of all optimal $\mathcal{P}$-extensions of X, respectively, and if $\mathcal{Q}$ is a topological property, let

$$\mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X) = \mathscr{E}^{\mathcal{Q}}(X) \cap \mathscr{E}_{\mathcal{P}}(X), \quad \mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X) = \mathscr{E}^{\mathcal{Q}}(X) \cap \mathscr{M}_{\mathcal{P}}(X), \quad \mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X) = \mathscr{E}^{\mathcal{Q}}(X) \cap \mathscr{O}_{\mathcal{P}}(X).$$

Note that by the definitions

$$\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X) \subseteq \mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X).$$

Remark. Topological properties $\mathcal{P}$ considered in this article are assumed to be non-empty, that is, it is assumed that there exists at least one space with $\mathcal{P}$. This in particular implies that for a clopen hereditary topological property $\mathcal{P}$ the empty set has $\mathcal{P}$, or, if a space is non-$\mathcal{P}$ then it is non-empty as well.

The following subspace of βX will play a crucial role in our study.

Definition 2.3. For a Tychonoff space X and a topological property $\mathcal{P}$ define

$$\lambda_{\mathcal{P}} X = \bigcup \big\{ \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} Z : Z \in \mathscr{Z}(X) \text{ has } \mathcal{P} \big\}.$$

Note that any topological property which is hereditary with respect to clopen subsets and inverse invariant under perfect mappings is hereditary with respect to closed subsets of Hausdorff spaces (see Theorem 3.7.29 of [5]). This simple fact will be used in a number of places throughout.

Lemma 2.4. *Let X be a Tychonoff space and let $\mathcal{P}$ be a clopen hereditary finitely additive perfect topological property. Then for any subset A of X if $\operatorname{cl}_{\beta X} A \subseteq \lambda_{\mathcal{P}} X$ then $\operatorname{cl}_X A$ has $\mathcal{P}$.*

Proof. By the compactness of $\operatorname{cl}_{\beta X} A$ and the definition of $\lambda_{\mathcal{P}} X$ we have

$$\operatorname{cl}_{\beta X} A \subseteq \bigcup_{i=1}^{n} \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} Z_i \subseteq \operatorname{cl}_{\beta X} Z$$

where each $Z_1, \ldots, Z_n \in \mathscr{Z}(X)$ has $\mathcal{P}$ and $Z = Z_1 \cup \cdots \cup Z_n$. Since $\mathcal{P}$ is finitely additive and invariant under perfect mappings and Z is the finite union of its closed subspaces Z_i, each having $\mathcal{P}$, it follows that Z has $\mathcal{P}$ (see Theorem 3.7.22 of [5]). Now since $\operatorname{cl}_X A \subseteq Z$ the set $\operatorname{cl}_X A$ has $\mathcal{P}$, as it is closed in Z. ∎

Lemma 2.5. *Let $\mathcal{P}$ be a topological property which is clopen hereditary and inverse invariant under perfect mappings and let $f : X \to Y$ be a perfect mapping. Then if Y is locally-$\mathcal{P}$ and Hausdorff then X is locally-$\mathcal{P}$.*

Proof. First note that $f[X]$ is locally-$\mathcal{P}$. To show this let $y \in f[X]$. Since Y is locally-$\mathcal{P}$ there exists an open neighborhood V of y in Y such that $\operatorname{cl}_Y V$ has $\mathcal{P}$. Now $V \cap f[X]$ is an open neighborhood of y in $f[X]$, the image $f[X]$ is closed in Y (as f is perfect and thus closed) and

$$\operatorname{cl}_{f[X]}\big(V \cap f[X]\big) = \operatorname{cl}_Y\big(V \cap f[X]\big) \cap f[X] \subseteq \operatorname{cl}_Y V.$$

Therefore $\operatorname{cl}_{f[X]}(V \cap f[X])$ has $\mathcal{P}$, as it is closed in $\operatorname{cl}_Y V$. Since $f : X \to f[X]$ is perfect and surjective we may assume in the statement of the lemma that f is moreover surjective.

Let $x \in X$. There exists an open neighborhood W of $f(x)$ in Y such that $\mathrm{cl}_Y W$ has $\mathcal{P}$. Since

$$f|f^{-1}[\mathrm{cl}_Y W] : f^{-1}[\mathrm{cl}_Y W] \to \mathrm{cl}_Y W$$

is perfect and surjective and $\mathcal{P}$ is inverse invariant under perfect mappings, $f^{-1}[\mathrm{cl}_Y W]$ has $\mathcal{P}$. Now $f^{-1}[W]$ is an open neighborhood of x in X, and since $\mathrm{cl}_X f^{-1}[W] \subseteq f^{-1}[\mathrm{cl}_Y W]$ and the latter has $\mathcal{P}$, its closed subset $\mathrm{cl}_X f^{-1}[W]$ also has $\mathcal{P}$. ∎

A topological property $\mathcal{P}$ is said to *satisfy Mrówka's condition* (W) if it satisfies the following: If X is a Tychonoff space in which there exists a point p with an open base $\mathscr{B}$ for X at p such that $X \setminus B$ has $\mathcal{P}$ for any $B \in \mathscr{B}$, then X has $\mathcal{P}$ (see [25]). If $\mathcal{P}$ is a topological property which is closed hereditary and productive then Mrówka's condition (W) is equivalent to the following condition: If a Tychonoff space X is the union of a compact space and a space with $\mathcal{P}$ then X has $\mathcal{P}$ (see [17]). In [25] S. Mrówka showed that if $\mathcal{P}$ is a topological property which is closed hereditary, finitely additive with respect to closed subsets (that is, whenever a space is the union of a finite number of its closed subsets each having $\mathcal{P}$, then it has $\mathcal{P}$) and invariant under continuous mappings then any Tychonoff locally-$\mathcal{P}$ space can be embedded as an open subspace in a Tychonoff space with $\mathcal{P}$ if and only if Mrówka's condition (W) holds.

In this article we will be dealing with certain classes of topological properties. For convenience, we make the following definition.

Definition 2.6. Let $\mathcal{P}$ be a topological property. Then $\mathcal{P}$ is said to be a *compactness-like topological property* if $\mathcal{P}$ is a clopen hereditary finitely additive perfect topological property which satisfies Mrówka's condition (W). If $\mathcal{Q}$ also is a topological property, then we say that $\mathcal{P}$ *and* $\mathcal{Q}$ *is a pair of compactness-like topological properties* (here the order of $\mathcal{P}$ and $\mathcal{Q}$ is important) if $\mathcal{P}$ is a compactness-like topological property and $\mathcal{Q}$ is a clopen hereditary topological property which is inverse invariant under perfect mappings and satisfies Mrówka's condition (W). (Examples of pairs of compactness-like topological properties are given in Example 2.16.)

Lemma 2.7. *Let $\mathcal{P}$ be a topological property which is inverse invariant under perfect mappings and satisfies Mrówka's condition* (W). *If X is a Tychonoff space in which there exists a compact subset A with an open base $\mathscr{B}$ for X at A such that $X \setminus B$ has $\mathcal{P}$ for any $B \in \mathscr{B}$, then X has $\mathcal{P}$.*

Proof. If $A = \emptyset$ then the conclusion holds trivially, as in this case $\emptyset \in \mathscr{B}$. Suppose that A is non-empty. Let T be the space obtained from X by contracting the set A to a point p and let $q : X \to T$ denote the corresponding quotient mapping. Note that since A is compact, T is Tychonoff. Now $\{q[B] : B \in \mathscr{B}\}$ is an open base for T at p such that $T \setminus q[B] = X \setminus B$ has $\mathcal{P}$ for any $B \in \mathscr{B}$. Since $\mathcal{P}$ satisfies Mrówka's condition (W), the space T, and thus its inverse image X under the perfect surjective mapping q, has $\mathcal{P}$. ∎

Note that if A is a dense subset of a space X and U is an open subset of X then $\mathrm{cl}_X U = \mathrm{cl}_X(U \cap A)$ and thus $U \subseteq \mathrm{int}_X \mathrm{cl}_X(U \cap A)$. In particular, if X is a Tychonoff

space, $f : \beta X \to \mathbb{I}$ is continuous and $r \in (0,1)$ then

$$f^{-1}\big[[0,r)\big] \subseteq \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X}\big(f^{-1}\big[[0,r)\big] \cap X\big).$$

We use such simple observations frequently in the future.

A Hausdorff space is called *zero-dimensional* if the set of all its clopen subsets constitutes an open base for it. A Tychonoff space is called *strongly zero-dimensional* if its Stone–Čech compactification is zero-dimensional. For a regular space X let (EX, k_X) denote the absolute of X (see Theorem 6.6(e) of [29] or Problem 6.3.20 of [5]). The space EX is (extremely disconnected and) zero-dimensional (thus strongly zero-dimensional; see Theorem 6.4 of [29]) and $k_X : EX \to X$ is a perfect (irreducible) surjective mapping.

The following lemma is quite fundamental in our study. We state and prove it in its general form for future reference.

Lemma 2.8. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X and Y be Tychonoff spaces such that Y has $\mathcal{Q}$, let $f : X \to Y$ be a perfect surjective mapping, let $T \in \mathscr{E}(Y)$, let αT be a compactification of T and let $\phi : \beta X \to \alpha T$ be the continuous extension of f. The following are equivalent:*

(1) $T \in \mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(Y)$.
(2) *X is locally-$\mathcal{P}$ and $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[T \setminus Y]$.*

Proof. (1) *implies* (2). Since $\mathcal{P}$ is hereditary with respect to closed subsets of Hausdorff spaces, the space Y, having a $\mathcal{P}$-extension with compact remainder, is locally-$\mathcal{P}$ and therefore so is X by Lemma 2.5. Next, we show that $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[T \setminus Y]$. Suppose to the contrary that there exists an $x \in \beta X \setminus \lambda_{\mathcal{P}} X$ such that $x \notin \phi^{-1}[T \setminus Y]$. Let $g : \beta X \to \mathbb{I}$ be continuous with $g(x) = 0$ and $g[\phi^{-1}[T \setminus Y]] \subseteq \{1\}$ and let $Z = g^{-1}[[0, 1/2]]$. We verify that $Z \cap X \in \mathscr{Z}(X)$ has $\mathcal{P}$. Since $Z \cap \phi^{-1}[T \setminus Y] = \emptyset$ we have

$$\phi[Z] \subseteq \phi\big[\beta X \setminus \phi^{-1}[T \setminus Y]\big] = \phi\big[\phi^{-1}\big[\alpha T \setminus (T \setminus Y)\big]\big] \subseteq \alpha T \setminus (T \setminus Y)$$

and thus $S = \phi[Z] \cap T \subseteq Y$. Therefore S has $\mathcal{P}$, as it is closed in T, because Z is compact. Since $f | f^{-1}[S] : f^{-1}[S] \to S$ is perfect and surjective and $\mathcal{P}$ is inverse invariant under perfect mappings, $f^{-1}[S]$ has $\mathcal{P}$. Thus

$$Z \cap X \subseteq f^{-1}\big[f[Z \cap X]\big] \subseteq f^{-1}\big[\phi[Z \cap X] \cap Y\big] \subseteq f^{-1}\big[\phi[Z] \cap T\big] = f^{-1}[S],$$

which implies that $Z \cap X$ has $\mathcal{P}$, as it is closed in $f^{-1}[S]$. Now

$$x \in g^{-1}\big[[0, 1/2)\big] \subseteq \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X}\big(g^{-1}\big[[0, 1/2]\big] \cap X\big) = \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X}(Z \cap X) \subseteq \lambda_{\mathcal{P}} X,$$

which is a contradiction, as $x \notin \lambda_{\mathcal{P}} X$.

(2) *implies* (1). Suppose moreover that X is strongly zero-dimensional. Let

$$\mathscr{B} = \big\{T \setminus f\big[\operatorname{cl}_X f^{-1}\big[T \setminus \phi[U]\big]\big] : U \text{ is clopen in } \beta X \text{ and } \phi^{-1}[T \setminus Y] \subseteq U\big\}.$$

We verify that $\mathscr{B}$ is an open base for T at $T \setminus Y$ such that $T \setminus B$ has both $\mathcal{P}$ and $\mathcal{Q}$ for any $B \in \mathscr{B}$. By Lemma 2.7 this will imply that T has $\mathcal{P}$ and $\mathcal{Q}$. Let U be a clopen subset of βX such that $\phi^{-1}[T \setminus Y] \subseteq U$. Consider

$$B = T \setminus f\big[\operatorname{cl}_X f^{-1}\big[T \setminus \phi[U]\big]\big] \in \mathscr{B}.$$

Since ϕ is surjective (as its image contains $Y = f[X] = \phi[X]$ and Y is dense in T) we have $T \setminus Y = \phi[\phi^{-1}[T \setminus Y]] \subseteq \phi[U]$ and thus $T \setminus \phi[U] \subseteq Y$. Since

$$f^{-1}\big[T \setminus \phi[U]\big] \subseteq \phi^{-1}\big[T \setminus \phi[U]\big] \subseteq \phi^{-1}\big[\alpha T \setminus \phi[U]\big] = \beta X \setminus \phi^{-1}\big[\phi[U]\big] \subseteq \beta X \setminus U$$

we have $\mathrm{cl}_X f^{-1}[T \setminus \phi[U]] \subseteq \beta X \setminus U$, which yields

$$\begin{aligned} T \setminus B &= f\big[\mathrm{cl}_X f^{-1}\big[T \setminus \phi[U]\big]\big] = \phi\big[\mathrm{cl}_X f^{-1}\big[T \setminus \phi[U]\big]\big] \\ &\subseteq \phi[\beta X \setminus U] \subseteq \phi\big[\beta X \setminus \phi^{-1}[T \setminus Y]\big] = \phi\big[\phi^{-1}\big[\alpha T \setminus (T \setminus Y)\big]\big] \subseteq \alpha T \setminus (T \setminus Y). \end{aligned}$$

Therefore since U is clopen in βX the set $\phi[\beta X \setminus U]$ is compact and thus $\mathrm{cl}_{\alpha T}(T \setminus B) \subseteq \phi[\beta X \setminus U]$. By the above this implies that

$$\mathrm{cl}_T(T \setminus B) \cap (T \setminus Y) = \mathrm{cl}_{\alpha T}(T \setminus B) \cap (T \setminus Y) = \emptyset$$

and therefore since f is closed,

$$\mathrm{cl}_T(T \setminus B) = \mathrm{cl}_Y(T \setminus B) = T \setminus B.$$

This shows that each $B \in \mathscr{B}$ is open in T. Obviously, each $B \in \mathscr{B}$ contains $T \setminus Y$. Next, we show that each open neighborhood W of $T \setminus Y$ in αT contains an element of $\mathscr{B}$. Since X is strongly zero-dimensional, βX is zero-dimensional. Now since $\phi^{-1}[W]$ is an open neighborhood of the compact set $\phi^{-1}[T \setminus Y]$ in βX there exists a clopen subset U of βX such that $\phi^{-1}[T \setminus Y] \subseteq U \subseteq \phi^{-1}[W]$ (see Theorem 6.2.4 of [5]). Note that

$$\begin{aligned} B = T \setminus f\big[\mathrm{cl}_X f^{-1}\big[T \setminus \phi[U]\big]\big] &\subseteq T \setminus f\big[f^{-1}\big[T \setminus \phi[U]\big]\big] \\ &= T \setminus \big(T \setminus \phi[U]\big) \subseteq \phi[U] \subseteq \phi\big[\phi^{-1}[W]\big] \subseteq W \end{aligned}$$

and that $B \in \mathscr{B}$. This shows that $\mathscr{B}$ is an open base for T at $T \setminus Y$. Now let $B \in \mathscr{B}$. Then $T \setminus B = f[\mathrm{cl}_X f^{-1}[T \setminus \phi[U]]]$ for some clopen subset U of βX containing $\phi^{-1}[T \setminus Y]$. Since f is closed, $T \setminus B$ is closed in Y, and since by our assumption Y has $\mathcal{Q}$ and $\mathcal{Q}$ is hereditary with respect to closed subsets of Hausdorff spaces, $T \setminus B$ has $\mathcal{Q}$. Also, as argued above $f^{-1}[T \setminus \phi[U]] \subseteq \beta X \setminus U$, which implies that

$$\mathrm{cl}_{\beta X} f^{-1}\big[T \setminus \phi[U]\big] \subseteq \beta X \setminus U \subseteq \beta X \setminus \phi^{-1}[T \setminus Y] \subseteq \lambda_{\mathcal{P}} X.$$

By Lemma 2.4 the set $C = \mathrm{cl}_X f^{-1}[T \setminus \phi[U]]$ has $\mathcal{P}$. Now $f|C : C \to f[C]$ is perfect (as C is closed in X) and surjective, and $\mathcal{P}$ is invariant under perfect mappings, thus $T \setminus B = f[C]$ has $\mathcal{P}$. This shows that (1) holds in this case.

We now turn to the general case in which X is an arbitrary Tychonoff space. Let (EX, k) denote the absolute of X. By our assumption X is locally-$\mathcal{P}$, and since $k : EX \to X$ is perfect, by Lemma 2.5 the space EX is locally-$\mathcal{P}$. Let $K : \beta EX \to \beta X$ be the continuous extension of k. Then $\phi K : \beta EX \to \alpha T$ continuously extends fk and therefore by the above, to show that $T \in \mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(Y)$ we only need to verify that

$$\beta EX \setminus \lambda_{\mathcal{P}} EX \subseteq (\phi K)^{-1}[T \setminus Y].$$

But by our assumption $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[T \setminus Y]$. Thus as we will see it suffices to show that $K^{-1}[\lambda_{\mathcal{P}} X] \subseteq \lambda_{\mathcal{P}} EX$. Let $t \in K^{-1}[\lambda_{\mathcal{P}} X]$. Let U be an open neighborhood of $K(t)$ in βX such that $\mathrm{cl}_{\beta X} U \subseteq \lambda_{\mathcal{P}} X$. Let $h : \beta X \to \mathbb{I}$ be continuous with $h(K(t)) = 0$ and $h[\beta X \setminus U] \subseteq \{1\}$. Let

$$Z = h^{-1}\big[[0, 1/2]\big] \cap X \in \mathscr{Z}(X).$$

Then since

$$\mathrm{cl}_{\beta X} Z \subseteq \mathrm{cl}_{\beta X}(U \cap X) = \mathrm{cl}_{\beta X} U \subseteq \lambda_{\mathcal{P}} X,$$

by Lemma 2.4 the set Z has $\mathcal{P}$ and therefore its inverse image $k^{-1}[Z] \in \mathscr{Z}(EX)$ under the perfect surjective mapping $k|k^{-1}[Z] : k^{-1}[Z] \to Z$ has $\mathcal{P}$. By the definition of $\lambda_{\mathcal{P}} EX$ we have $\mathrm{int}_{\beta EX} \mathrm{cl}_{\beta EX} k^{-1}[Z] \subseteq \lambda_{\mathcal{P}} EX$. By Theorem 3.7.16 of [5] (or Theorem 1.8(i) of [29]) and since $K|EX = k$ is perfect, $K[\beta EX \setminus EX] \subseteq \beta X \setminus X$. But K is surjective, as its image contains $X = k[EX] = K[EX]$, and thus $K[\beta EX \setminus EX] = \beta X \setminus X$. We have

$$\begin{aligned} k^{-1}[Z] &= k^{-1}\big[h^{-1}\big[[0,1/2]\big] \cap X\big] = K^{-1}\big[h^{-1}\big[[0,1/2]\big] \cap X\big] \\ &= K^{-1}\big[h^{-1}\big[[0,1/2]\big]\big] \cap K^{-1}[X] = K^{-1}\big[h^{-1}\big[[0,1/2]\big]\big] \cap EX \end{aligned}$$

and therefore

$$\begin{aligned} t \in K^{-1}\big[h^{-1}\big[[0,1/2)\big]\big] &\subseteq \mathrm{cl}_{\beta EX} K^{-1}\big[h^{-1}\big[[0,1/2)\big]\big] \\ &= \mathrm{cl}_{\beta EX}\big(K^{-1}\big[h^{-1}\big[[0,1/2)\big]\big] \cap EX\big) \\ &\subseteq \mathrm{cl}_{\beta EX}\big(K^{-1}\big[h^{-1}\big[[0,1/2]\big]\big] \cap EX\big) = \mathrm{cl}_{\beta EX} k^{-1}[Z], \end{aligned}$$

which yields $t \in \mathrm{int}_{\beta EX} \mathrm{cl}_{\beta EX} k^{-1}[Z]$ and thus $t \in \lambda_{\mathcal{P}} EX$. This shows that $K^{-1}[\lambda_{\mathcal{P}} X] \subseteq \lambda_{\mathcal{P}} EX$. Now

$$\begin{aligned} \beta EX \setminus \lambda_{\mathcal{P}} EX &\subseteq \beta EX \setminus K^{-1}[\lambda_{\mathcal{P}} X] \\ &= K^{-1}[\beta X \setminus \lambda_{\mathcal{P}} X] \subseteq K^{-1}\big[\phi^{-1}[T \setminus Y]\big] = (\phi K)^{-1}[T \setminus Y], \end{aligned}$$

which shows (1). ∎

The list of topological properties $\mathcal{P}$ and $\mathcal{Q}$ satisfying the assumption of Lemma 2.8 is quite long and includes most of the important covering properties (see Example 2.16).

Remark. Lemma 2.8 (and thus its consequences) remains valid if one omits $\mathcal{Q}$ from its statement. This is because one can replace $\mathcal{Q}$ by regularity (note that by Theorem 3.7.23 of [5] regularity is inverse invariant under perfect mappings and satisfies Mrówka's condition (W) vacuously) and observes that for this specific choice of $\mathcal{Q}$ the phrases "Tychonoff space with $\mathcal{Q}$" and "$\mathcal{P}$ and $\mathcal{Q}$ is a pair of compactness-like topological properties" can be replaced by "Tychonoff space" and "$\mathcal{P}$ is a compactness-like topological property", respectively.

Remark. Lemma 2.8 is stronger than what we normally need, as we generally apply it in the special case when $Y = X$, $f = \mathrm{id}_X$ and $\alpha T = \beta T$. Lemma 2.8 is quite fundamental in our study and it is interesting to know whether the requirement "$\mathcal{P}$ satisfies Mrówka's condition (W)" (implicit in the definition of the compactness-like topological property $\mathcal{P}$) can be omitted from its statement. In Example 2.16 we give an example of a Tychonoff space X, a topological property $\mathcal{P}$ which does not satisfy Mrówka's condition (W) and a Tychonoff extension of X with compact remainder, for which the conclusion of (the special case of) Lemma 2.8 does not hold.

We will make frequent use of the following well known result sometimes without explicitly referring to it. The proof is included here for the sake of completeness.

Lemma 2.9. *Let X be a Tychonoff space and let Y be a Tychonoff extension of X with the compact remainder $Y \setminus X = \{p_i : i \in I\}$ where p_i's are bijectively indexed. Let*

$\phi : \beta X \to \beta Y$ be the continuous extension of id_X*. Let T be the space obtained from βX by contracting any fiber $\phi^{-1}(p_i)$ where $i \in I$ to a point a_i. Then $T = \beta Y$ (identifying each a_i with p_i) and $\phi = q$ where $q : \beta X \to T$ is the quotient mapping.*

Proof. We first show that T is a compactification of Y. To show that T is Hausdorff let $s, t \in T$ be distinct elements. Consider the following cases:

Case 1. Suppose that $s, t \in T \setminus \{a_i : i \in I\}$. Then $s, t \in \beta X \setminus \phi^{-1}[Y \setminus X]$ and thus there exist disjoint open neighborhoods U and V of s and t in βX, respectively, each disjoint from $\phi^{-1}[Y \setminus X]$. The sets $q[U]$ and $q[V]$ are disjoint open neighborhoods of s and t in T, respectively.

Case 2. Suppose that $s = a_i$ for some $i \in I$ and $t \in T \setminus \{a_i : i \in I\}$. Then $\phi^{-1}[Y \setminus X]$ is a compact subset of βX not containing t and thus there exist disjoint open subsets U and V of βX such that $\phi^{-1}[Y \setminus X] \subseteq U$ and $t \in V$. Now $q[U]$ and $q[V]$ are disjoint open neighborhoods of s and t in T, respectively. The case when $s \in T \setminus \{a_i : i \in I\}$ and $t = a_j$ for some $j \in I$ is analogous.

Case 3. Suppose that $s = a_i$ and $t = a_j$ for some $i, j \in I$. Let U_i and U_j be disjoint open neighborhoods of p_i and p_j in βY, respectively. Then since $q^{-1}[q[\phi^{-1}[U_k]]] = \phi^{-1}[U_k]$ where $k = i, j$ are open subsets of βX and $\phi^{-1}(p_k) \subseteq \phi^{-1}[U_k]$ the sets $q[\phi^{-1}[U_k]]$ where $k = i, j$ are disjoint open neighborhoods of s and t in T, respectively.

This shows that T is Hausdorff and therefore it is compact, as it is a continuous image of βX. Note that Y is a subspace of T. To show this, first note that since βY is also a compactification of X, as X is dense in Y and thus in βY, and $\phi | X = \mathrm{id}_X$, by Theorem 3.5.7 of [5] we have $\phi[\beta X \setminus X] = \beta Y \setminus X$. Now if W is open in βY, since $q^{-1}[q[\phi^{-1}[W]]] = \phi^{-1}[W]$ is open in βX the set $q[\phi^{-1}[W]]$ is open in T, and therefore $W \cap Y = q[\phi^{-1}[W]] \cap Y$ is open in Y as a subspace of T. For the converse, note that if W is an open subset of T then

$$W \cap Y = \big(\beta Y \setminus \phi\big[\beta X \setminus q^{-1}[W]\big]\big) \cap Y$$

and therefore (since $\phi[\beta X \setminus q^{-1}[W]]$ is compact and thus closed in βY) the set $W \cap Y$ is open in Y in its original topology. Clearly, Y is dense in T and therefore T is a compactification of Y. To show that $T = \beta Y$ it suffices to verify that any continuous $f : Y \to \mathbb{I}$ can be continuously extended over T. Indeed, consider the continuous mapping

$$g = fq : S = X \cup \phi^{-1}[Y \setminus X] \to \mathbb{I}.$$

Note that since $X \subseteq S \subseteq \beta X$ we have $\beta S = \beta X$ (see Corollary 3.6.9 of [5]). Let $g_\beta : \beta X \to \mathbb{I}$ be the continuous extension of g. Define $F : T \to \mathbb{I}$ such that $F(x) = g_\beta(x)$ for any $x \in \beta X \setminus \phi^{-1}[Y \setminus X]$ and $F(p_i) = f(p_i)$ for any $i \in I$. Then $F | Y = f$ and since $Fq = g_\beta$ is continuous, F is continuous. This shows that $T = \beta Y$. Note that this also implies that $\phi = q$, as $\phi, q : \beta X \to \beta Y$ are continuous and $\phi | X = \mathrm{id}_X = q | X$. ∎

The following simple observation will be of frequent use in the future, sometimes with no explicit reference.

Lemma 2.10. *Let X be a Tychonoff space and let $\mathcal{P}$ be a clopen hereditary topological property which is inverse invariant under perfect mappings. Then $X \subseteq \lambda_{\mathcal{P}} X$ if and only if X is locally-$\mathcal{P}$.*

Proof. Suppose that X is locally-$\mathcal{P}$. Let $x \in X$ and let U be an open neighborhood of x in X whose closure $\mathrm{cl}_X U$ has $\mathcal{P}$. Let $f : X \to \mathbb{I}$ be continuous with $f(x) = 0$ and $f[X \setminus U] \subseteq \{1\}$ and let $f_\beta : \beta X \to \mathbb{I}$ be the continuous extension of f. Let $Z = f^{-1}[[0, 1/2]] \in \mathscr{Z}(X)$. Then $Z \subseteq U$ and thus Z has $\mathcal{P}$, as it is closed in $\mathrm{cl}_X U$. Now

$$x \in f_\beta^{-1}\big[[0, 1/2)\big] \subseteq \mathrm{int}_{\beta X}\, \mathrm{cl}_{\beta X}\, f^{-1}\big[[0, 1/2]\big] = \mathrm{int}_{\beta X}\, \mathrm{cl}_{\beta X}\, Z \subseteq \lambda_{\mathcal{P}} X$$

and therefore $X \subseteq \lambda_{\mathcal{P}} X$. For the converse suppose that $X \subseteq \lambda_{\mathcal{P}} X$. Let $x \in X$. Then $x \in \lambda_{\mathcal{P}} X$ and therefore $x \in \mathrm{int}_{\beta X}\, \mathrm{cl}_{\beta X}\, S$ for some $S \in \mathscr{Z}(X)$ which has $\mathcal{P}$. Let $V = (\mathrm{int}_{\beta X}\, \mathrm{cl}_{\beta X}\, S) \cap X$. Then V is an open neighborhood of x in X. Since $V \subseteq S$ the set $\mathrm{cl}_X V$ has $\mathcal{P}$, as it is closed in S. Thus X is locally-$\mathcal{P}$. ∎

Our next theorem gives characterizations of the elements of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. Compare with its dual result on $\mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(X)$ (Theorem 2.15).

THEOREM 2.11. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff space with $\mathcal{Q}$ and let $Y \in \mathscr{E}_{\mathcal{P}}^{\mathcal{Q}}(X)$. The following are equivalent:*

(1) $Y \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$.
(2) *For any $p \in Y \setminus X$ the set $\phi^{-1}(p) \setminus \lambda_{\mathcal{P}} X$ is non-empty where $\phi : \beta X \to \beta Y$ is the continuous extension of* id_X.
(3) *For any open subset V of Y such that $\mathrm{cl}_X(V \cap X)$ has $\mathcal{P}$ we have $V \cap (Y \setminus X) = \emptyset$.*
(4) *For any $T \in \mathscr{E}_{\mathcal{P}}^{\mathcal{Q}}(X)$ and any continuous $f : T \to Y$ such that $f|X = \mathrm{id}_X$, the mapping f is surjective.*
(5) *For any $T \in \mathscr{E}_{\mathcal{P}}^{\mathcal{Q}}(X)$ such that $Y \leq T$ there exists a continuous surjective $f : T \to Y$ such that $f|X = \mathrm{id}_X$.*

Proof. Let $\phi : \beta X \to \beta Y$ be the continuous extension of id_X.

(1) *implies* (2). Consider the subspace

$$T = X \cup \big\{p \in Y \setminus X : \phi^{-1}(p) \setminus \lambda_{\mathcal{P}} X \neq \emptyset\big\}$$

of Y. We show that $T \setminus X = \phi[\beta X \setminus \lambda_{\mathcal{P}} X]$. First note that by Lemma 2.8 we have $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X]$ and that X is locally-$\mathcal{P}$. Now if $t \in \beta X \setminus \lambda_{\mathcal{P}} X$ then $\phi(t) = p \in Y \setminus X$ and thus $\phi^{-1}(p) \setminus \lambda_{\mathcal{P}} X$ is non-empty, as it contains t. Therefore $\phi(t) = p \in T \setminus X$. This shows that $\phi[\beta X \setminus \lambda_{\mathcal{P}} X] \subseteq T \setminus X$. To show the reverse inclusion note that if $p \in T \setminus X$ then there exists some $t \in \phi^{-1}(p) \setminus \lambda_{\mathcal{P}} X \subseteq \beta X \setminus \lambda_{\mathcal{P}} X$ and thus $p = \phi(t) \in \phi[\beta X \setminus \lambda_{\mathcal{P}} X]$. This shows that $T \in \mathscr{E}(X)$. Now since

$$\phi^{-1}[T \setminus X] = \phi^{-1}\big[\phi[\beta X \setminus \lambda_{\mathcal{P}} X]\big] \supseteq \beta X \setminus \lambda_{\mathcal{P}} X$$

by Lemma 2.8 it follows that $T \in \mathscr{E}_{\mathcal{P}}(X)$. By the minimality of Y we have $T = Y$ and in particular $T \setminus X = Y \setminus X$.

(2) *implies* (1). Let $T \in \mathscr{E}_{\mathcal{P}}(X)$ be such that $T \subseteq Y$. By (the remark following) Lemma 2.8 we have $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[T \setminus X]$. Now if there exists some $p \in Y \setminus T$ then

$$\phi^{-1}(p) \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}(p) \cap \phi^{-1}[T \setminus X] = \emptyset,$$

which contradicts (2). Thus $T = Y$. This shows the minimality of Y.

(2) *implies* (3). To show (3) let V be an open subset of Y such that $V \cap (Y \setminus X)$ is non-empty. We need to show that $\mathrm{cl}_X(V \cap X)$ is non-$\mathcal{P}$. Let $V = W \cap Y$ where W is an

open subset of βY. Let $p \in V \cap (Y \setminus X)$. Let $g : \beta Y \to \mathbb{I}$ be continuous with $g(p) = 0$ and $g[\beta Y \setminus W] \subseteq \{1\}$ and let

$$Z = (g\phi)^{-1}\big[[0, 1/2]\big] \cap X \in \mathscr{Z}(X).$$

Note that

$$Z = (g\phi)^{-1}\big[[0, 1/2]\big] \cap X = \phi^{-1}\big[g^{-1}\big[[0, 1/2]\big]\big] \cap X = g^{-1}\big[[0, 1/2]\big] \cap X \subseteq W \cap X = V \cap X.$$

Thus if $\mathrm{cl}_X(V \cap X)$ has $\mathcal{P}$ then its closed subset Z also has $\mathcal{P}$. Now

$$\begin{aligned}\phi^{-1}(p) \subseteq \phi^{-1}\big[g^{-1}\big[[0, 1/2)\big]\big] &= (g\phi)^{-1}\big[[0, 1/2)\big] \\ &\subseteq \mathrm{int}_{\beta X}\,\mathrm{cl}_{\beta X}\big((g\phi)^{-1}\big[[0, 1/2]\big] \cap X\big) = \mathrm{int}_{\beta X}\,\mathrm{cl}_{\beta X}\, Z \subseteq \lambda_{\mathcal{P}} X,\end{aligned}$$

contradicting (2). Therefore $\mathrm{cl}_X(V \cap X)$ is non-$\mathcal{P}$.

(3) *implies* (2). Suppose to the contrary that $\phi^{-1}(p) \setminus \lambda_{\mathcal{P}} X = \emptyset$ for some $p \in Y \setminus X$. Then $p \notin \phi[\beta X \setminus \lambda_{\mathcal{P}} X]$. Let W be an open neighborhood of p in βY such that $\mathrm{cl}_{\beta Y} W \cap \phi[\beta X \setminus \lambda_{\mathcal{P}} X] = \emptyset$. We have

$$\phi^{-1}[\mathrm{cl}_{\beta Y} W] \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[\mathrm{cl}_{\beta Y} W] \cap \phi^{-1}\big[\phi[\beta X \setminus \lambda_{\mathcal{P}} X]\big] = \phi^{-1}\big[\mathrm{cl}_{\beta Y} W \cap \phi[\beta X \setminus \lambda_{\mathcal{P}} X]\big] = \emptyset$$

and thus

$$\mathrm{cl}_{\beta X}(W \cap X) = \mathrm{cl}_{\beta X}\big(\phi^{-1}[W] \cap X\big) = \mathrm{cl}_{\beta X}\,\phi^{-1}[W] \subseteq \phi^{-1}[\mathrm{cl}_{\beta Y} W] \subseteq \lambda_{\mathcal{P}} X.$$

Lemma 2.4 implies that $\mathrm{cl}_X(W \cap X)$ has $\mathcal{P}$. Now $V = W \cap Y$ is an open neighborhood of p in Y such that $\mathrm{cl}_X(V \cap X) = \mathrm{cl}_X(W \cap X)$ has $\mathcal{P}$, contradicting (3).

(2) *implies* (4). Let $T \in \mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$ and let $f : T \to Y$ be continuous with $f|X = \mathrm{id}_X$. Let $f_\beta : \beta T \to \beta Y$ and $\psi : \beta X \to \beta T$ be the continuous extensions of f and id_X, respectively. Then since $f_\beta \psi | X = \phi | X$ we have $f_\beta \psi = \phi$. Lemma 2.8 implies that $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \psi^{-1}[T \setminus X]$. Also, for any $p \in Y \setminus X$, since $\phi^{-1}(p) \setminus \lambda_{\mathcal{P}} X$ is non-empty, $p \in \phi[\phi^{-1}(p) \setminus \lambda_{\mathcal{P}} X]$. Thus

$$\begin{aligned}Y \setminus X &\subseteq \bigcup\big\{\phi\big[\phi^{-1}(p) \setminus \lambda_{\mathcal{P}} X\big] : p \in Y \setminus X\big\} \subseteq \phi[\beta X \setminus \lambda_{\mathcal{P}} X] \\ &= f_\beta\big[\psi[\beta X \setminus \lambda_{\mathcal{P}} X]\big] \subseteq f_\beta\big[\psi\big[\psi^{-1}[T \setminus X]\big]\big] \subseteq f_\beta[T \setminus X] = f[T \setminus X] \subseteq f[T].\end{aligned}$$

Since $f|X = \mathrm{id}_X$ this shows that $Y \subseteq f[T]$, that is, f is surjective. That (4) implies (5) is trivial.

(5) *implies* (2). Consider the subspace $T = X \cup \phi[\beta X \setminus \lambda_{\mathcal{P}} X]$ of βY. By Lemma 2.8 we have $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X]$ and X is locally-$\mathcal{P}$. Thus

$$T = X \cup \phi[\beta X \setminus \lambda_{\mathcal{P}} X] \subseteq X \cup \phi\big[\phi^{-1}[Y \setminus X]\big] \subseteq X \cup (Y \setminus X) = Y.$$

By Lemma 2.10 we have $X \subseteq \lambda_{\mathcal{P}} X$. Now $T \setminus X = \phi[\beta X \setminus \lambda_{\mathcal{P}} X]$ is compact, and since

$$\phi^{-1}[T \setminus X] = \phi^{-1}\big[\phi[\beta X \setminus \lambda_{\mathcal{P}} X]\big] \supseteq \beta X \setminus \lambda_{\mathcal{P}} X,$$

by Lemma 2.8 it follows that $T \in \mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$. It is clear that $Y \leq T$, as $T \subseteq Y$. By (5) there exists a continuous surjective $f : T \to Y$ such that $f|X = \mathrm{id}_X$. But $f|X = \mathrm{id}_T|X$, which yields $f = \mathrm{id}_T$ and therefore $Y = f[T] = T$. Now it is clear that for any

$$p \in Y \setminus X = T \setminus X = \phi[\beta X \setminus \lambda_{\mathcal{P}} X]$$

the set $\phi^{-1}(p) \setminus \lambda_{\mathcal{P}} X$ is non-empty. ∎

REMARK. Theorem 2.11 fails if one omits the requirement "$\mathcal{P}$ satisfies Mrówka's condition (W)" (implicit in the definition of the compactness-like topological property $\mathcal{P}$) from its statement (see Example 2.16).

In the next theorem we give characterizations of the elements of $\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$. We need to prove a few lemmas first.

NOTATION 2.12. Let X be a Tychonoff space and let Y be a Tychonoff extension of X. Let $\phi : \beta X \to \beta Y$ be the (unique) continuous mapping which extends id_X. Denote

$$\mathscr{F}_X(Y) = \big\{\phi^{-1}(p) : p \in Y \setminus X\big\}.$$

We may write $\mathscr{F}(Y)$ instead of $\mathscr{F}_X(Y)$ when no confusion arises.

In [20] the author associated to each compactification αX of a Tychonoff space X a set (called the *β-family* of αX)

$$\mathscr{F}_\alpha = \big\{f_\alpha^{-1}(p) : p \in \alpha X \setminus X\big\}$$

where $f_\alpha : \beta X \to \alpha X$ is the continuous extension of id_X. It is then shown that for any compactifications $\alpha_1 X$ and $\alpha_2 X$ of a Tychonoff space X we have $\alpha_1 X \leq \alpha_2 X$ if and only if each set in $\mathscr{F}_{\alpha_2}$ is a subset of a set in $\mathscr{F}_{\alpha_1}$. This provides the motivation for the statement of the next lemma.

LEMMA 2.13. *Let X be a Tychonoff space and let $Y_1, Y_2 \in \mathscr{E}(X)$. The following are equivalent:*

(1) $Y_1 \leq Y_2$.
(2) *Any element of $\mathscr{F}(Y_2)$ is contained in an element of $\mathscr{F}(Y_1)$.*

Proof. Let $\phi_i : \beta X \to \beta Y_i$ for $i = 1, 2$ be the continuous extension of id_X.

(1) *implies* (2). By definition there exists a continuous $f : Y_2 \to Y_1$ such that $f|X = \mathrm{id}_X$. Let $f_\beta : \beta Y_2 \to \beta Y_1$ be the continuous extension of f. The continuous mappings $f_\beta\phi_2, \phi_1 : \beta X \to \beta Y_1$ coincide with id_X on X and thus are identical. Also, since X is dense in βY_i, as it is dense in Y_i for $i = 1, 2$ the space βY_i is a compactification of X. Therefore since $f_\beta|X = \mathrm{id}_X$, by Theorem 3.5.7 of [5] we have $f_\beta[\beta Y_2 \setminus X] = \beta Y_1 \setminus X$. Now let $F_2 \in \mathscr{F}(Y_2)$. Then $F_2 = \phi_2^{-1}(p)$ for some $p \in Y_2 \setminus X$. By the above $f_\beta(p) \in \beta Y_1 \setminus X$ and thus $f(p) \in Y_1 \setminus X$, as $f_\beta(p) = f(p)$. Let $F_1 = \phi_1^{-1}(f(p)) \in \mathscr{F}(Y_1)$. Then

$$F_2 = \phi_2^{-1}(p) \subseteq \phi_2^{-1}\big[f_\beta^{-1}\big(f_\beta(p)\big)\big] = (f_\beta\phi_2)^{-1}\big(f_\beta(p)\big) = \phi_1^{-1}\big(f_\beta(p)\big) = \phi_1^{-1}\big(f(p)\big) = F_1.$$

(2) *implies* (1). We define $f : Y_2 \to Y_1$ as follows. If $t \in Y_2 \setminus X$ then $\phi_2^{-1}(t) \in \mathscr{F}(Y_2)$ and thus by our assumption $\phi_2^{-1}(t) \subseteq \phi_1^{-1}(s)$ for some (unique, as ϕ_2 is surjective) $s \in Y_1 \setminus X$. Define $f(t) = s$ in this case. If $t \in X$ define $f(t) = t$. We show that f is continuous; this will show that $Y_1 \leq Y_2$. By Lemma 2.9 the space βY_2 is the quotient space of βX obtained by contracting each $\phi_2^{-1}(p)$ where $p \in Y_2 \setminus X$ to a point and ϕ_2 is the quotient mapping. Thus in particular Y_2 is the quotient space of $X \cup \phi_2^{-1}[Y_2 \setminus X]$ with the quotient mapping

$$\phi_2|\big(X \cup \phi_2^{-1}[Y_2 \setminus X]\big) : X \cup \phi_2^{-1}[Y_2 \setminus X] \to Y_2.$$

Therefore to show that f is continuous it suffices to show that $f\phi_2|(X \cup \phi_2^{-1}[Y_2 \setminus X])$ is continuous. We show this by verifying that $f\phi_2(t) = \phi_1(t)$ for any $t \in X \cup \phi_2^{-1}[Y_2 \setminus X]$.

This obviously holds if $t \in X$. If $t \in \phi_2^{-1}[Y_2 \setminus X]$ then $\phi_2(t) \in Y_2 \setminus X$. Let $s \in Y_1 \setminus X$ be such that $\phi_2^{-1}(\phi_2(t)) \subseteq \phi_1^{-1}(s)$. Then $f\phi_2(t) = s$. But since $t \in \phi_2^{-1}(\phi_2(t))$ we have $t \in \phi_1^{-1}(s)$ and thus $\phi_1(t) = s$. Therefore $f\phi_2(t) = \phi_1(t)$ also in this case. ∎

LEMMA 2.14. *Let X be a Tychonoff space and let $\mathcal{P}$ be a clopen hereditary topological property which is inverse invariant under perfect mappings. Suppose that $Z \subseteq C$ where $Z \in \mathscr{Z}(X)$, $C \in \operatorname{Coz}(X)$ and $\operatorname{cl}_X C$ has $\mathcal{P}$. Then $\operatorname{cl}_{\beta X} Z \subseteq \lambda_{\mathcal{P}} X$.*

Proof. The zero-sets Z and $X \setminus C$ of X, being disjoint, are completely separated in X. Let $f : X \to \mathbb{I}$ be continuous with $f[Z] \subseteq \{0\}$ and $f[X \setminus C] \subseteq \{1\}$ and let $f_\beta : \beta X \to \mathbb{I}$ be the continuous extension of f. Let $S = f^{-1}[[0, 1/2]] \in \mathscr{Z}(X)$. Then $S \subseteq C$ and therefore S has $\mathcal{P}$, as it is closed in $\operatorname{cl}_X C$. We have

$$\operatorname{cl}_{\beta X} Z \subseteq Z(f_\beta) \subseteq f_\beta^{-1}\big[[0, 1/2)\big] \subseteq \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} f^{-1}\big[[0, 1/2]\big] = \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} S \subseteq \lambda_{\mathcal{P}} X. \ ∎$$

In the following theorem we characterize the elements of $\mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(X)$.

THEOREM 2.15. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff space with $\mathcal{Q}$ and let $Y \in \mathscr{E}_{\mathcal{P}}^{\mathcal{Q}}(X)$. The following are equivalent:*

(1) $Y \in \mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(X)$.
(2) $\phi^{-1}[Y \setminus X] = \beta X \setminus \lambda_{\mathcal{P}} X$ *where* $\phi : \beta X \to \beta Y$ *is the continuous extension of* id_X.
(3) *For any* $Z \in \mathscr{Z}(X)$ *such that* $Z \subseteq C$ *for some* $C \in \operatorname{Coz}(X)$ *such that* $\operatorname{cl}_X C$ *has* $\mathcal{P}$ *we have* $\operatorname{cl}_Y Z \cap (Y \setminus X) = \emptyset$.
(4) *For any* $T \in \mathscr{E}_{\mathcal{P}}^{\mathcal{Q}}(X)$ *and any continuous injective* $f : T \to Y$ *such that* $f|X = \operatorname{id}_X$, *the mapping f is a homeomorphism.*
(5) *For any* $T \in \mathscr{E}_{\mathcal{P}}^{\mathcal{Q}}(X)$ *if* $Y \leq T$ *then* $T \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$.

Proof. Let $\phi : \beta X \to \beta Y$ be the continuous extension of id_X.

(1) *implies* (2). By Theorem 2.11 the set $\phi^{-1}(p) \setminus \lambda_{\mathcal{P}} X$ is non-empty for any $p \in Y \setminus X$. Let S be the space obtained from βX by contracting each $\phi^{-1}(p) \setminus \lambda_{\mathcal{P}} X$ where $p \in Y \setminus X$ to a point s_p with the quotient mapping $q : \beta X \to S$. Note that S is a continuous image of βX. Therefore to prove that S is compact it suffices to show that it is Hausdorff. Suppose that $s, z \in S$ are distinct. Consider the following cases:

CASE 1. Suppose that $s, z \in \lambda_{\mathcal{P}} X$. Since s and z can be separated in $\lambda_{\mathcal{P}} X$ by disjoint open subsets and $\lambda_{\mathcal{P}} X$ is open in βX they can also be separated in S.

CASE 2. Suppose that $s \in \lambda_{\mathcal{P}} X$ and $z \in q[\beta X \setminus \lambda_{\mathcal{P}} X]$. Let U and V be disjoint open neighborhoods of s and $\beta X \setminus \lambda_{\mathcal{P}} X$ in βX, respectively. Then $q[U]$ and $q[V]$ are disjoint open subsets of S separating s and z.

CASE 3. Suppose that $s, z \in q[\beta X \setminus \lambda_{\mathcal{P}} X]$. Let

$$s = q\big[\phi^{-1}(p) \setminus \lambda_{\mathcal{P}} X\big] \quad \text{and} \quad z = q\big[\phi^{-1}(y) \setminus \lambda_{\mathcal{P}} X\big]$$

for some $p, y \in Y \setminus X$. Let U and V be disjoint open neighborhoods of p and y in βY, respectively. Then $q[\phi^{-1}[U]]$ and $q[\phi^{-1}[V]]$ are disjoint open subsets of S separating s and z.

Define $f : S \to \beta Y$ by $f(x) = p$ if $x \in q[\phi^{-1}(p)]$ for some $p \in Y \setminus X$, and $f(x) = x$ otherwise. Note that this makes sense by the construction of βY and the representation of ϕ given in Lemma 2.9. By the definition of f we have $fq = \phi$ and therefore f is

continuous. Consider the subspace $T = X \cup q[\beta X \setminus \lambda_{\mathcal{P}} X]$ of S. Note that by Lemma 2.10 we have $X \subseteq \lambda_{\mathcal{P}} X$. Thus T contains X as a dense subspace. It is also clear that X, and therefore T, is dense in S. Since $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq q^{-1}[T \setminus X]$ and X is locally-$\mathcal{P}$, as $\mathscr{E}_{\mathcal{P}}(X)$ is non-empty, by Lemma 2.8 we have $T \in \mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$. Now $f|T : T \to Y$ is a continuous bijective mapping such that $f|X = \mathrm{id}_X$. Thus by the maximality of the topology of Y we have $T = Y$ (identifying each s_p with p where $p \in Y \setminus X$). Now S is a compactification of Y and therefore there exists a continuous $g : \beta Y \to S$ such that $g|Y = \mathrm{id}_Y$. Since $fg|Y = \mathrm{id}_Y$ it follows that $fg = \mathrm{id}_{\beta Y}$. On the other hand $gf|Y = \mathrm{id}_Y$, which yields $gf = \mathrm{id}_S$ and thus $g = f^{-1}$. Now as noted before $fq = \phi$, and therefore for any $p \in Y \setminus X$ we have

$$\phi^{-1}(p) = (fq)^{-1}(p) = q^{-1}\big[f^{-1}(p)\big] = q^{-1}\big(g(p)\big) = q^{-1}(p) = \phi^{-1}(p) \setminus \lambda_{\mathcal{P}} X.$$

Thus $\phi^{-1}(p) \subseteq \beta X \setminus \lambda_{\mathcal{P}} X$ for any $p \in Y \setminus X$ and therefore $\phi^{-1}[Y \setminus X] \subseteq \beta X \setminus \lambda_{\mathcal{P}} X$. But by Lemma 2.8 we have $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X]$, which shows equality.

(2) *implies* (4). Let $T \in \mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$ and let $f : T \to Y$ be a continuous injective mapping which fixes X pointwise. Let $f_\beta : \beta T \to \beta Y$ and $\psi : \beta X \to \beta T$ be the continuous extensions of f and id_X, respectively. Since $f_\beta \psi | X = \mathrm{id}_X = \phi | X$ we have $f_\beta \psi = \phi$. Also $f[T \setminus X] \subseteq Y \setminus X$. To show this, suppose to the contrary that $f(t) \in X$ for some $t \in T \setminus X$. Let U and V be disjoint open neighborhoods of $f(t)$ and t in T, respectively. Since $Y \setminus X$ is compact, X is open in Y and thus $U \cap X$, being open in X, is an open neighborhood of $f(t)$ in Y. Let W be an open neighborhood of t in T such that $f[W] \subseteq U \cap X$. Since $W \cap V$ is open in T and non-empty, as $t \in W \cap V$ and X is dense in T, the set $W \cap V \cap X$ is non-empty. But if $x \in W \cap V \cap X$ then $x = f(x) \in U$, which is a contradiction, as $U \cap V = \emptyset$.

Claim 1. *If $t \in T \setminus X$ and $y = f(t)$ then $\psi^{-1}(t) \subseteq \phi^{-1}(y)$.*

Proof of the claim. We have $y = f(t) = f_\beta(t)$ and thus $t \in f_\beta^{-1}(y)$. Therefore

$$\psi^{-1}(t) \subseteq \psi^{-1}\big[f_\beta^{-1}(y)\big] = (f_\beta \psi)^{-1}(y) = \phi^{-1}(y).$$

Claim 2. *If $t \in T \setminus X$ and $y = f(t)$ then $\psi^{-1}(t) = \phi^{-1}(y)$.*

Proof of the claim. By Claim 1, $\psi^{-1}(t) \subseteq \phi^{-1}(y)$. Let $z \in \phi^{-1}(y)$. By Lemma 2.8 we have $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \psi^{-1}[T \setminus X]$. Thus since

$$\phi^{-1}(y) \subseteq \phi^{-1}[Y \setminus X] = \beta X \setminus \lambda_{\mathcal{P}} X$$

we have $z \in \psi^{-1}[T \setminus X]$. Let $\psi(z) = t' \in T \setminus X$ and let $y' = f(t') \in Y \setminus X$. By the first claim $\psi^{-1}(t') \subseteq \phi^{-1}(y')$ and therefore $z \in \phi^{-1}(y')$. Thus $\phi^{-1}(y) \cap \phi^{-1}(y')$ is non-empty and $f(t) = y = y' = f(t')$. But f is injective and therefore $t = t'$, which yields $z \in \psi^{-1}(t') = \psi^{-1}(t)$. This shows that $\phi^{-1}(y) \subseteq \psi^{-1}(t)$, which together with the above proves the claim.

Claim 3. $\{\psi^{-1}(t) : t \in T \setminus X\} = \{\phi^{-1}(y) : y \in Y \setminus X\}$.

Proof of the claim. By Claim 2 it suffices to show that for any $y \in Y \setminus X$ we have $\phi^{-1}(y) = \psi^{-1}(t)$ for some $t \in T \setminus X$. Let $y \in Y \setminus X$ and $z \in \beta X$ be such that $\phi(z) = y$.

By Lemma 2.8 we have $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \psi^{-1}[T \setminus X]$ and thus, since

$$\phi^{-1}(y) \subseteq \phi^{-1}[Y \setminus X] = \beta X \setminus \lambda_{\mathcal{P}} X$$

it follows that $z \in \psi^{-1}[T \setminus X]$. Let $t = \psi(z) \in T \setminus X$. Then $z \in \phi^{-1}(y) \cap \psi^{-1}(t)$. If $y' = f(t)$ then by Claim 2, $\psi^{-1}(t) = \phi^{-1}(y')$. Thus $\phi^{-1}(y) \cap \phi^{-1}(y')$ is non-empty and therefore $y = y'$. Thus $\phi^{-1}(y) = \phi^{-1}(y') = \psi^{-1}(t)$, which proves the claim.

By Lemma 2.9 the spaces βY and βT are respectively obtained from βX by contracting the sets $\phi^{-1}(y)$ where $y \in Y \setminus X$ and $\psi^{-1}(t)$ where $t \in T \setminus X$ to points, and ϕ and ψ are the corresponding quotient mappings. Thus by Claim 3, $\phi = \psi$ and therefore

$$Y = X \cup \phi\Big[\bigcup \big\{\phi^{-1}(y) : y \in Y \setminus X\big\}\Big] = X \cup \psi\Big[\bigcup \big\{\psi^{-1}(t) : t \in T \setminus X\big\}\Big] = T.$$

This shows Y and T are equivalent extensions of X. Let $g : Y \to T$ be a homeomorphism such that $g|X = \mathrm{id}_X$. Then the continuous mapping $fg : Y \to Y$ coincides with id_Y on the dense subset X of Y. This (since Y is Hausdorff) implies that $fg = \mathrm{id}_Y$ and thus $f = g^{-1}$ is a homeomorphism.

(4) *implies* (1). Let Y' be the set Y equipped with a topology which is finer than the topology of Y and turns it into an element of $\mathscr{E}_{\mathcal{P}}(X) = \mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$. Since $f : Y' \to Y$ defined by $f(y') = y'$ for any $y' \in Y'$ is continuous and injective, by our assumption it is a homeomorphism. This shows that the topology of Y is maximal among the topologies which turn Y into an element of $\mathscr{E}_{\mathcal{P}}(X)$. Next, suppose that $T \in \mathscr{E}_{\mathcal{P}}(X)$ is such that $T \subseteq Y$. Since the inclusion mapping $f : T \to Y$ defined by $f(t) = t$ for any $t \in T$ is continuous and injective, by our assumption it is a homeomorphism. But this implies that $T = Y$, which proves the minimality of Y among the elements of $\mathscr{E}_{\mathcal{P}}(X)$.

(2) *implies* (3). Let $Z \in \mathscr{Z}(X)$ be such that $Z \subseteq C$ for some $C \in \mathrm{Coz}(X)$ such that $\mathrm{cl}_X C$ has $\mathcal{P}$. By Lemma 2.14 we have $\mathrm{cl}_{\beta X} Z \subseteq \lambda_{\mathcal{P}} X$. Let U be an open neighborhood of $\beta X \setminus \lambda_{\mathcal{P}} X$ in βX which misses $\mathrm{cl}_{\beta X} Z$. Now (by the construction of βY and the representation of ϕ given in Lemma 2.9) the set $\phi[U]$ is an open neighborhood of $p \in Y \setminus X$ in βY which misses Z. Therefore

$$\mathrm{cl}_Y Z \cap (Y \setminus X) = \mathrm{cl}_{\beta Y} Z \cap (Y \setminus X) = \emptyset.$$

(3) *implies* (2). By Lemma 2.8 we have $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X]$. To show the reverse inclusion suppose to the contrary that $t \in \lambda_{\mathcal{P}} X$ for some $t \in \phi^{-1}[Y \setminus X]$. Note that $\lambda_{\mathcal{P}} X$ is open in βX. Let U be an open neighborhood of t in βX such that $\mathrm{cl}_{\beta X} U \subseteq \lambda_{\mathcal{P}} X$. Let $f : \beta X \to \mathbb{I}$ be continuous with $f(t) = 0$ and $f[\beta X \setminus U] \subseteq \{1\}$. Define

$$Z = f^{-1}\big[[0, 1/3]\big] \cap X \quad \text{and} \quad C = f^{-1}\big[[0, 1/2)\big] \cap X.$$

Then $Z \in \mathscr{Z}(X)$, $C \in \mathrm{Coz}(X)$ and $Z \subseteq C$. Also, since $\mathrm{cl}_{\beta X}(U \cap X) = \mathrm{cl}_{\beta X} U \subseteq \lambda_{\mathcal{P}} X$, by Lemma 2.4 the set $\mathrm{cl}_X(U \cap X)$ has $\mathcal{P}$. Therefore, since

$$C = f^{-1}\big[[0, 1/2)\big] \cap X \subseteq U \cap X,$$

the set $\mathrm{cl}_X C$ has $\mathcal{P}$, as it is closed in $\mathrm{cl}_X(U \cap X)$. By our assumption this implies that $\mathrm{cl}_Y Z \cap (Y \setminus X) = \emptyset$. But $\phi(t) \in Y \setminus X$ and thus $\phi(t) \notin \mathrm{cl}_{\beta Y} Z$. Let V be an open neighborhood of $\phi(t)$ in βY such that $V \cap Z = \emptyset$. Now since

$$\phi^{-1}[V] \cap \phi^{-1}[Z] = \phi^{-1}[V \cap Z] = \emptyset$$

it follows that

$$\phi^{-1}[V] \cap \mathrm{cl}_{\beta X} Z \subseteq \phi^{-1}[V] \cap \mathrm{cl}_{\beta X}\, \phi^{-1}[Z] = \emptyset.$$

Thus $t \notin \mathrm{cl}_{\beta X} Z$, which is a contradiction, as

$$t \in f^{-1}\big[[0,1/3)\big] \subseteq \mathrm{cl}_{\beta X}\big(f^{-1}\big[[0,1/3]\big] \cap X\big) = \mathrm{cl}_{\beta X}\, Z.$$

(2) *implies* (5). Let $T \in \mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$ be such that $Y \leq T$. Let $f : T \to Y$ continuously extend id_X. Arguing as in (2)⇒(4) we have $f[T \setminus X] \subseteq Y \setminus X$. Let $f_\beta : \beta T \to \beta Y$ and $\psi : \beta X \to \beta T$ be the continuous extensions of f and id_X, respectively. Then $\phi = f_\beta \psi$, as they coincide with id_X on X. To show that $T \in \mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$, by Theorem 2.11, it suffices to verify that $\psi^{-1}(p) \setminus \lambda_{\mathcal{P}} X$ is non-empty for any $p \in T \setminus X$. Let $p \in T \setminus X$. Then

$$\psi^{-1}(p) \subseteq \psi^{-1}\big[f_\beta^{-1}\big(f_\beta(p)\big)\big] = (f_\beta\psi)^{-1}\big(f_\beta(p)\big) = \phi^{-1}\big(f_\beta(p)\big) = \phi^{-1}\big(f(p)\big) \subseteq \phi^{-1}[Y \setminus X].$$

Now since $\phi^{-1}[Y \setminus X] = \beta X \setminus \lambda_{\mathcal{P}} X$ and ψ is surjective, $\psi^{-1}(p) \setminus \lambda_{\mathcal{P}} X = \psi^{-1}(p)$ is non-empty.

(5) *implies* (2). Note that (5) in particular implies that $Y \in \mathscr{M}_{\mathcal{P}}(X)$. Thus by Theorem 2.11 the set $\phi^{-1}(p) \setminus \lambda_{\mathcal{P}} X$ is non-empty for any $p \in Y \setminus X$. Since by Lemma 2.8 we have $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X]$ and X is locally-$\mathcal{P}$, to show (2) it suffices to verify that $\phi^{-1}[Y \setminus X] \subseteq \beta X \setminus \lambda_{\mathcal{P}} X$. Suppose to the contrary that $\phi^{-1}(p') \cap \lambda_{\mathcal{P}} X$ is non-empty for some $p' \in Y \setminus X$. Let $t' \in \phi^{-1}(p') \cap \lambda_{\mathcal{P}} X$. Let Z be the quotient space of βX obtained by contracting each (non-empty) subset $\phi^{-1}(p) \setminus \lambda_{\mathcal{P}} X$ where $p \in Y \setminus X$ to a point z_p with the quotient mapping $q : \beta X \to Z$. Then as in (1)⇒(2) one can verify that Z is compact. Consider the subspace

$$T = q\big[X \cup (\beta X \setminus \lambda_{\mathcal{P}} X) \cup \{t'\}\big]$$

of Z. Then T is a Tychonoff extension of X with the compact remainder

$$T \setminus X = q\big[(\beta X \setminus \lambda_{\mathcal{P}} X) \cup \{t'\}\big].$$

Note that T is dense in Z and therefore Z is a compactification of T. Let $f : \beta T \to Z$ and $\psi : \beta X \to \beta T$ be the continuous extensions of id_T and id_X, respectively. Since $f\psi : \beta X \to Z$ agrees with q on X we have $f\psi = q$. By Lemma 2.8 and since $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq q^{-1}[T \setminus X]$ (and X is locally-$\mathcal{P}$) it follows that $T \in \mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$. We verify that $Y \leq T$; our assumption will then imply that $T \in \mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$, from which we will derive a contradiction. By Lemma 2.13 to show that $Y \leq T$ it suffices to verify that each $\psi^{-1}(t)$ where $t \in T \setminus X$ is contained in $\phi^{-1}(p)$ for some $p \in Y \setminus X$. Let $t \in T \setminus X$. Note that by Theorem 3.5.7 of [5] (and since $f|T = \mathrm{id}_T$ and Z is a compactification of T) we have $f[\beta T \setminus T] = Z \setminus T$ and thus $f^{-1}(t) = \{t\}$. Therefore

$$\psi^{-1}(t) = \psi^{-1}\big[f^{-1}(t)\big] = (f\psi)^{-1}(t) = q^{-1}(t)$$

and thus by the definition of Z it follows that $\psi^{-1}(t) \subseteq \phi^{-1}(p)$ for some $p \in Y \setminus X$. This shows that $Y \leq T$. Consider the subspace $T' = T \setminus \{t'\}$ of T. Then T' is a Tychonoff extension of X with the compact remainder $T' \setminus X = q[\beta X \setminus \lambda_{\mathcal{P}} X]$. By Lemma 2.8 and since $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq q^{-1}[T' \setminus X]$ (and X is locally-$\mathcal{P}$) it follows that $T' \in \mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$. But this contradicts the minimality of T, as T' is properly contained in T. ∎

REMARK. Theorem 2.15 fails if one omits the requirement "$\mathcal{P}$ satisfies Mrówka's condition (W)" (implicit in the definition of the compactness-like topological property $\mathcal{P}$) from its statement (see Example 2.16 below).

In the following we provide examples of pairs $\mathcal{P}$ and $\mathcal{Q}$ of compactness-like topological properties. The topological properties $\mathcal{P}$ and $\mathcal{Q}$ assumed in the statement of Lemma 2.8 (and thus in the statements of all its corollaries which constitute the main results of this article) are required to be a pair of compactness-like topological properties.

EXAMPLE 2.16. Let α, θ, κ and μ be infinite cardinals and let X be a Hausdorff space. For a collection $\mathscr{A}$ of subsets of X and an $x \in X$ let

$$O(x, \mathscr{A}) = \operatorname{card}\big(\{A \in \mathscr{A} : x \in A\}\big).$$

For more details on the following definitions see [1], [33] and [38]. The space X is called (2) *μ-Lindelöf* ((3) *$[\theta, \kappa]$-compact*, respectively) if every open cover of X (of cardinality $\le \kappa$, respectively) has a subcover of cardinality $\le \mu$ ($< \theta$, respectively). The space X is called (4) *paracompact*, (5) *metacompact*, (7) *subparacompact*, (11) *para-Lindelöf*, (12) *meta-Lindelöf*, (14) *screenable*, (15) *σ-metacompact*, (9) *σ-para-Lindelöf* if every open cover of X has a (4)$'$ locally finite open, (5)$'$ point-finite open, (7)$'$ σ-locally finite closed, (11)$'$ locally countable open, (12)$'$ point-countable open, (14)$'$ σ-disjoint open, (15)$'$ σ-point-finite open, (9)$'$ σ-locally countable open refinement. The space X is called (16) *weakly θ-refinable*, (8) *θ-refinable* (or *submetacompact*), (17) *weakly $\delta\theta$-refinable*, (13) *$\delta\theta$-refinable* (or *submeta-Lindelöf*) if every open cover of X has an open refinement $\mathscr{V} = \bigcup\{\mathscr{V}_n : n \in \mathbb{N}\}$ such that for any $x \in X$ there exists some $n \in \mathbb{N}$ with (16)$'$ $0 < O(x, \mathscr{V}_n) < \aleph_0$, (8)$'$ $0 < O(x, \mathscr{V}_n) < \aleph_0$ and each $\mathscr{V}_n$ covers X, (17)$'$ $0 < O(x, \mathscr{V}_n) \le \aleph_0$, (13)$'$ $0 < O(x, \mathscr{V}_n) \le \aleph_0$ and each $\mathscr{V}_n$ covers X. The space X is called (10) *α-bounded* if any subset of X of cardinality $\le \alpha$ has compact closure in X. Moreover, let (1) be compactness and (6) be countable paracompactness.

Let $\mathcal{P}$ = regularity + (i) where $i = 1, \dots, 10$ and $\mathcal{Q}$ = regularity + (i) where $i = 1, \dots, 17$. Then $\mathcal{P}$ and $\mathcal{Q}$ is a pair of compactness-like topological properties. That $\mathcal{Q}$ is hereditary with respect to clopen subsets follows from Theorem 7.1 of [1]. Also, by (a modification of) Theorem 3.7.24 and Exercise 5.2.G of [5] and Theorem 5.9 of [1] it follows that $\mathcal{Q}$ is inverse invariant under perfect mappings. (For the case of α-boundedness note that for a perfect surjective $f : X \to Y$, when Y is α-bounded, if $A \subseteq X$ has cardinality $\le \alpha$ then $\operatorname{card}(f[A]) \le \alpha$ and thus $\operatorname{cl}_Y f[A]$ is compact. But since

$$A \subseteq f^{-1}\big[f[A]\big] \subseteq f^{-1}\big[\operatorname{cl}_Y f[A]\big]$$

and the latter is compact (as f is perfect), its closed subset $\operatorname{cl}_X A$ is also compact, that is, X is α-bounded.) Next, we verify that $\mathcal{Q}$ satisfies Mrówka's condition (W). We prove this for the cases when $\mathcal{Q}$ is paracompactness and subparacompactness. The remaining cases can be proved analogously.

Let $\mathcal{Q}$ be paracompactness (subparacompactness, respectively). Let X be a Tychonoff space, let $p \in X$ and let $\mathscr{B}$ be an open base for X at p such that $X \setminus B$ is paracompact (subparacompact, respectively) for any $B \in \mathscr{B}$. Let $\mathscr{U}$ be an open cover of X. Let $p \in B \subseteq \operatorname{cl}_X B \subseteq U$ where $U \in \mathscr{U}$ and $B \in \mathscr{B}$. Then $\mathscr{V} = \{V \setminus B : V \in \mathscr{U}\}$ is an open

cover of $X \setminus B$. Thus there exists a locally finite open (in $X \setminus B$) refinement $\mathscr{W}$ of $\mathscr{V}$ (a σ-locally finite closed (in $X \setminus B$) refinement $\mathscr{W}$ of $\mathscr{V}$, respectively). Now if

$$\mathscr{A} = \{X \setminus \mathrm{cl}_X B : W \in \mathscr{W}\} \cup \{U\}$$

($\mathscr{A} = \mathscr{W} \cup \{\mathrm{cl}_X B\}$, respectively) then $\mathscr{A}$ is a locally finite open refinement of $\mathscr{U}$ ($\mathscr{A}$ is a σ-locally finite closed refinement of $\mathscr{U}$, respectively). Thus X is paracompact (subparacompact, respectively).

Note that α-boundedness satisfies Mrówka's condition (W) by Theorem 3.1 of [16]. That $\mathcal{P}$ is finitely additive and invariant under perfect mappings follow from Theorems 5.1, 5.5, 7.3 and 7.4 of [1] and Exercises 5.2.B and 5.2.G of [5]. Theorem 3.1 of [16] provides a few more examples of topological properties satisfying (W). Among them we mention realcompactness and Dieudonné completeness which are hereditary with respect to clopen subsets and inverse invariant under perfect mappings (with Tychonoff domains); see Theorems 3.11.4 and 3.11.14 and Problem 8.5.13 of [5]. That Dieudonné completeness is inverse invariant under perfect mappings is well known; it can also be proved by using the fact that a Tychonoff space X is Dieudonné complete if and only if for any $p \in \beta X \setminus X$ there exists a paracompact subset T of βX such that $X \subseteq T \subseteq \beta X \setminus \{p\}$ (see Problem 8.5.13 of [5]) and that paracompactness is inverse invariant under perfect mappings.

In addition to the above topological properties the list of topological properties satisfying (W) includes: screenability, N-compactness [27], almost realcompactness [7] and zero-dimensionality (see [16] and [17] for details).

In the following we give an example of a topological property $\mathcal{P}$ which does not satisfy (W). At the same time we show that the requirement "$\mathcal{P}$ satisfies Mrówka's condition (W)" (implicit in the definition of the compactness-like topological property $\mathcal{P}$) cannot be omitted from the statements of Lemma 2.8 and Theorems 2.11 and 2.15 upon them the rest of this article rely.

Example 2.17. Let X be a locally compact paracompact non-σ-compact space. Then X can be represented as

$$X = \bigoplus_{i \in I} X_i \tag{2.1}$$

for some index set I, where each X_i is σ-compact and non-compact (see Theorem 5.1.27 and Exercise 3.8.C of [5]). Assume the representation given in (2.1). Let $\mathcal{P}$ be σ-compactness. Obviously, $\mathcal{P}$ is clopen hereditary, finitely additive and perfect. We show that $\mathcal{P}$ does not satisfy Mrówka's condition (W). Note that with the above notation

$$\lambda_{\mathcal{P}} X = \bigcup \Big\{ \mathrm{cl}_{\beta X} \Big(\bigcup_{i \in J} X_i \Big) : J \subseteq I \text{ is countable} \Big\}.$$

Also, note that since X is non-σ-compact, $\beta X \setminus \lambda_{\mathcal{P}} X$ is non-empty. Contract the compact subset $\beta X \setminus \lambda_{\mathcal{P}} X$ of βX to a point p to obtain a space T and denote by $q : \beta X \to T$ the quotient mapping. Note that T is compact (as it is a Hausdorff continuous image of βX) and contains X as a dense subspace. Consider the subspace $Y = X \cup \{p\}$ of T. We show that for any open neighborhood V of p in Y the set $Y \setminus V$ is σ-compact while Y itself is

not σ-compact. Let V be an open neighborhood of p in Y. Let V' be an open subset of T such that $V' \cap Y = V$. Then since $p \in V'$ we have

$$\beta X \setminus \lambda_{\mathcal{P}} X = q^{-1}(p) \subseteq q^{-1}[V']$$

and thus $\beta X \setminus q^{-1}[V'] \subseteq \lambda_{\mathcal{P}} X$. Therefore by compactness

$$\beta X \setminus q^{-1}[V'] \subseteq \mathrm{cl}_{\beta X}\Big(\bigcup_{i\in J_1} X_i\Big) \cup \cdots \cup \mathrm{cl}_{\beta X}\Big(\bigcup_{i\in J_m} X_i\Big) = \mathrm{cl}_{\beta X}\Big(\bigcup_{i\in J} X_i\Big)$$

where $m \in \mathbb{N}$, each $J_1, \ldots, J_m \subseteq I$ is countable and $J = J_1 \cup \cdots \cup J_m$. Now

$$Y \setminus V = \big(\beta X \setminus q^{-1}[V']\big) \cap X \subseteq \bigcup_{i\in J} X_i$$

being closed in the latter (σ-compact) set is σ-compact. To show that Y is not σ-compact suppose the contrary and let $Y = \bigcup_{n=1}^{\infty} K_n$ where K_n is compact for any $n \in \mathbb{N}$. Let $p \in K_j$ where $j \in \mathbb{N}$. Then

$$\beta X \setminus \lambda_{\mathcal{P}} X = q^{-1}(p) \subseteq q^{-1}[K_j]$$

and thus $K_n = q^{-1}[K_n] \subseteq \lambda_{\mathcal{P}} X$ for any $j \neq n \in \mathbb{N}$. Arguing as above for any $j \neq n \in \mathbb{N}$ we have

$$K_n \subseteq \mathrm{cl}_{\beta X}\Big(\bigcup_{i\in H_n} X_i\Big)$$

where $H_n \subseteq I$ is countable, but (since $K_n \subseteq X$) this implies that $K_n \subseteq \bigcup_{i\in H_n} X_i$. Let $H = \bigcup_{j\neq n=1}^{\infty} H_n$. Then

$$\bigcup_{j\neq n=1}^{\infty} K_n \subseteq \bigcup_{i\in H} X_i. \tag{2.2}$$

Choose some $u \in I \setminus H$. (This is possible, as H is countable and I is uncountable, because by our assumption X is non-σ-compact and X_i's are σ-compact.) Since by our assumption X_u is non-compact, $\mathrm{cl}_{\beta X} X_u \setminus X_u$ is non-empty. Let $t \in \mathrm{cl}_{\beta X} X_u \setminus X_u \subseteq \lambda_{\mathcal{P}} X$. Then $t \in T \setminus Y$. We show that $t \in \mathrm{cl}_T K_j$, contradicting the compactness of K_j. Let W be an open neighborhood of $t = q(t)$ in T. Then $q^{-1}[W]$ is an open neighborhood of t in βX and therefore $X_u \cap q^{-1}[W]$ is non-empty. Let $x \in X_u \cap q^{-1}[W]$. Note that

$$\begin{aligned} X \cup (\beta X \setminus \lambda_{\mathcal{P}} X) = q^{-1}[Y] &= q^{-1}\Big[\bigcup_{n=1}^{\infty} K_n\Big] \\ &= q^{-1}\Big[\bigcup_{j\neq n=1}^{\infty} K_n\Big] \cup q^{-1}[K_j] = \bigcup_{j\neq n=1}^{\infty} K_n \cup q^{-1}[K_j]. \end{aligned}$$

By (2.2) and since $X_u \cap \bigcup_{i\in H} X_i = \emptyset$ we have $x \notin \bigcup_{j\neq n=1}^{\infty} K_n$ and thus by the above $x \in q^{-1}[K_j]$. Therefore $x = q(x) \in W \cap K_j$ and thus $W \cap K_j$ is non-empty. This shows that $t \in \mathrm{cl}_T K_j$. Therefore Y is not σ-compact. Thus for this specific choice of $\mathcal{P}$ Mrówka's condition (W) fails. By Lemma 2.9 we have $T = \beta Y$ and if $\phi : \beta X \to \beta Y$ is the continuous extension of id_X then $\phi = q$. Also, by the above Y does not have $\mathcal{P}$ while $\phi^{-1}[Y \setminus X] = \beta X \setminus \lambda_{\mathcal{P}} X$. Therefore in the statements of Lemma 2.8 and Theorems 2.11 and 2.15 the requirement "$\mathcal{P}$ satisfies Mrówka's condition (W)" cannot be omitted.

3. Compactification-like $\mathcal{P}$-extensions with countable remainder

It is a well known result of P. Alexandroff that every locally compact non-compact space has a compactification with one-point remainder, called the *one-point compactification* or the *Alexandroff compactification* of X. One can consider $\mathcal{P}$-extensions with one-point remainder (see [9], [12], [13], [14] and [15] for some recent results) or more generally, $\mathcal{P}$-extensions with countable remainder for various topological properties $\mathcal{P}$. Below, after some definitions we state some known results which motivated our study in this chapter.

Definition 3.1. Let $n \in \mathbb{N}$. An extension with n-point (countable, respectively) remainder is called an *n-point* (a *countable-point*, respectively) *extension*. Similar definitions apply for compactifications.

Countable-point compactifications are also called *$\aleph_0$-point compactifications* or *countable compactifications*. Throughout this article countable means countable and infinite.

Notation 3.2. For a Tychonoff space X the set of all compactifications of X is denoted by $\mathscr{K}(X)$.

In [18] K. D. Magill Jr. gave the following characterization of those spaces which have an n-point compactification and thus generalized the well known result of P. Alexandroff.

Theorem 3.3 (Magill [18]). *Let X be a locally compact space and let $n \in \mathbb{N}$. The following are equivalent:*

(1) *$\mathscr{K}(X)$ contains an element with n-point remainder.*
(2) *$X = K \cup U_1 \cup \cdots \cup U_n$, where $K, U_1, \ldots, U_n$ are pairwise disjoint, each $U_1, \ldots, U_n$ is open in X and $K \cup U_i$ is non-compact for any $i = 1, \ldots, n$.*

Also, in a separate article [19], Magill characterized those spaces having a countable-point compactification with compact remainder. Recall that a space X is called *totally disconnected* if all (connected) components in X are one-point sets.

Theorem 3.4 (Magill [19]). *Let X be a locally compact space. The following are equivalent:*

(1) *$\mathscr{K}(X)$ contains an element with countable remainder.*
(2) *$\mathscr{K}(X)$ contains an element with n-point remainder for any $n \in \mathbb{N}$.*
(3) *$\mathscr{K}(X)$ contains an element with infinite totally disconnected remainder.*
(4) *$\beta X \setminus X$ has an infinite number of (connected) components.*

Magill's studies were continued by various authors (see e.g. [2], [39] and [11] among others). In [11] T. Kimura generalized Magill's result of [18] (Theorem 3.3) and gave the following characterization of spaces having a countable-point compactification with compact remainder.

Theorem 3.5 (Kimura [11]). *Let X be a locally compact space. The following are equivalent:*

(1) *$\mathscr{K}(X)$ contains an element with countable remainder.*
(2) *There exists a bijectively indexed collection $\{U_n : n \in \mathbb{N}\}$ of pairwise disjoint open subsets of X with compact boundary and non-compact closure.*

In [23] J. R. McCartney generalized Magill's result still further. To state McCartney's result, we need some preliminaries.

Let A be an infinite compact countable space. As in [22] we define the successive derived sets $A^{(\zeta)}$ of A for any $\zeta < \Omega$ by $A^{(0)} = A$, $A^{(\zeta+1)} = (A^{(\zeta)})'$ and

$$A^{(\zeta)} = \bigcap \{A^{(\eta)} : \eta < \zeta\}$$

whenever ζ is a limit ordinal. Then $A^{(\zeta)}$'s form a decreasing sequence of compact subsets of A. Note that if for some $\zeta < \Omega$ the set $A^{(\zeta)}$ is infinite then $A^{(\zeta)} \setminus A^{(\zeta+1)}$ is also infinite, as otherwise, since

$$A^{(\zeta)} = (A^{(\zeta)} \setminus A^{(\zeta+1)}) \cup A^{(\zeta+1)},$$

the set $A^{(\zeta+1)}$ will be a compact non-empty space without isolated points and therefore $A^{(\zeta+1)} \subseteq A$ will be uncountable. Since

$$\bigcup \{A^{(\zeta)} \setminus A^{(\zeta+1)} : \zeta < \Omega\} \subseteq A$$

and A is countable there exists some $\lambda < \Omega$ such that $A^{(\lambda)} \setminus A^{(\lambda+1)} = \emptyset$. Suppose that λ is the least with this property. Then by the above $A^{(\lambda)}$ is finite and thus it is empty. Note that λ is not a limit ordinal, as otherwise, by definition $A^{(\lambda)}$ is non-empty, as it is the intersection of a collection of compact non-empty subsets with the finite intersection property. Let $\lambda = \sigma + 1$. Then $A^{(\sigma)}$ is non-empty and since $(A^{(\sigma)})' = A^{(\lambda)} = \emptyset$, it is finite. We say that an infinite compact countable space A is *of type* (σ, n), where $\sigma < \Omega$ and $n \in \mathbb{N}$, if $\operatorname{card}(A^{(\sigma)}) = n$. From the above discussion it is clear that the type of A exists and is uniquely determined.

THEOREM 3.6 (McCartney [23]). *Let X be a locally compact space and let $0 < \sigma < \Omega$. The following are equivalent:*

(1) *$\mathscr{K}(X)$ contains an element with countable remainder of type $(\sigma, 1)$.*
(2) *There exists a family $\{\mathscr{U}_\zeta : \zeta < \sigma\}$ of infinite collections of pairwise disjoint open subsets of X with compact boundary satisfying the following:*
   (a) *For any $\zeta < \sigma$, $U \in \mathscr{U}_\zeta$ and finite $\mathscr{W} \subseteq \bigcup\{\mathscr{U}_\eta : \eta < \zeta\}$ the set $\operatorname{cl}_X U \setminus \bigcup \mathscr{W}$ is non-compact.*
   (b) *For any distinct $\zeta, \eta < \sigma$, $U \in \mathscr{U}_\zeta$ and $V \in \mathscr{U}_\eta$ there exists a finite $\mathscr{W} \subseteq \bigcup\{\mathscr{U}_\xi : \xi < \zeta\}$ such that either*

$$\operatorname{cl}_X U \setminus \Big(V \cup \bigcup \mathscr{W}\Big) \text{ or } (\operatorname{cl}_X U \cap \operatorname{cl}_X V) \setminus \bigcup \mathscr{W}$$

   is compact.
   (c) *For any $\zeta < \eta < \sigma$ and $U \in \mathscr{U}_\eta$ there exists an infinite $\mathscr{V} \subseteq \mathscr{U}_\zeta$ such that for any $V \in \mathscr{V}$ there exists a finite $\mathscr{W} \subseteq \bigcup\{\mathscr{U}_\xi : \xi < \zeta\}$ such that $\operatorname{cl}_X V \setminus (U \cup \bigcup \mathscr{W})$ is compact.*

It is worth mentioning that the general problem of characterizing spaces with a countable-point compactification remains open. (See [10], also [3], for a characterization of metrizable spaces having such compactifications.)

Our aim in this chapter is to generalize the above results to compactification-like $\mathcal{P}$-extensions. Note that part (2) of the lemma below generalizes Magill's theorem in [19]

(Theorem 3.4) provided that one replaces $\mathcal{P}$ and $\mathcal{Q}$, respectively, by compactness and regularity, and note that for these specific choices of $\mathcal{P}$ and $\mathcal{Q}$ and a locally compact space X we have $\lambda_{\mathcal{P}}X = X$ and the two notions "Y is a minimal $\mathcal{P}$-extension of X with $\mathcal{Q}$" and "Y is a compactification of X" coincide.

LEMMA 3.7. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff space with $\mathcal{Q}$.*

(1) *Let $n \in \mathbb{N}$. The following are equivalent:*
   (a) *$\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ contains an element with n-point remainder.*
   (b) *$\mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(X)$ contains an element with n-point remainder.*
   (c) *X is locally-$\mathcal{P}$ and $\beta X \setminus \lambda_{\mathcal{P}}X$ contains n pairwise disjoint non-empty clopen subsets.*
   (d) *X is locally-$\mathcal{P}$ and $\beta X \setminus \lambda_{\mathcal{P}}X$ has at least n (connected) components.*

(2) *The following are equivalent:*
   (a) *$\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ contains an element with countable remainder.*
   (b) *$\mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(X)$ contains an element with countable remainder.*
   (c) *X is locally-$\mathcal{P}$ and $\beta X \setminus \lambda_{\mathcal{P}}X$ contains an infinite number of pairwise disjoint non-empty clopen subsets.*
   (d) *X is locally-$\mathcal{P}$ and $\beta X \setminus \lambda_{\mathcal{P}}X$ has an infinite number of (connected) components.*

(3) *Let $0 < \sigma < \Omega$ and let $n \in \mathbb{N}$. The following are equivalent:*
   (a) *$\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ contains an element with countable remainder of type (σ, n).*
   (b) *$\mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(X)$ contains an element with countable remainder of type (σ, n).*
   (c) *X is locally-$\mathcal{P}$ and there exists a family $\{\mathscr{H}_\zeta : \zeta \leq \sigma\}$ of collections of pairwise disjoint non-empty clopen subset of $\beta X \setminus \lambda_{\mathcal{P}}X$ satisfying the following:*
      (i) *For any $\zeta < \sigma$, $\operatorname{card}(\mathscr{H}_\zeta) = \aleph_0$ and $\operatorname{card}(\mathscr{H}_\sigma) = n$.*
      (ii) *For any $\zeta \leq \sigma$ and $H \in \mathscr{H}_\zeta$ we have*
      $$H \setminus \bigcup\{G \in \mathscr{H}_\eta : \eta < \zeta\} \neq \emptyset.$$
      (iii) *For any $\zeta < \eta \leq \sigma$, $H \in \mathscr{H}_\zeta$ and $G \in \mathscr{H}_\eta$ either*
      $$H \subseteq G \cup \bigcup\{F \in \mathscr{H}_\xi : \xi < \zeta\} \quad \text{or} \quad H \cap G \subseteq \bigcup\{F \in \mathscr{H}_\xi : \xi < \zeta\}.$$
      (iv) *For any $\zeta < \eta \leq \sigma$ and $H \in \mathscr{H}_\eta$ the set*
      $$\Big\{F \in \mathscr{H}_\zeta : F \subseteq H \cup \bigcup\{G \in \mathscr{H}_\xi : \xi < \zeta\}\Big\}$$
      is infinite.

Proof. (1). It is clear that (1.c) implies (1.d) and that (1.b) implies (1.a).

(1.a) *implies* (1.c). Consider some $Y \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ with an n-point remainder $Y \setminus X = \{p_1, \ldots, p_n\}$. Let $\phi : \beta X \to \beta Y$ be the continuous extension of id_X. Let $V_1, \ldots, V_n$ be pairwise disjoint open neighborhoods of $p_1, \ldots, p_n$ in βY, respectively. By Lemma 2.8 we have $\beta X \setminus \lambda_{\mathcal{P}}X \subseteq \phi^{-1}[Y \setminus X]$ and X is locally-$\mathcal{P}$. Let $i = 1, \ldots, n$. Then
$$\phi\big[\phi^{-1}[V_i] \setminus \lambda_{\mathcal{P}}X\big] \subseteq \phi[\beta X \setminus \lambda_{\mathcal{P}}X] \subseteq \phi\big[\phi^{-1}[Y \setminus X]\big] \subseteq Y \setminus X$$

and therefore

$$\phi\big[\phi^{-1}[V_i] \setminus \lambda_{\mathcal{P}} X\big] \cap V_i \subseteq (Y \setminus X) \cap V_i = \{p_i\}.$$

This gives

$$\begin{aligned}\phi^{-1}[V_i] \setminus \lambda_{\mathcal{P}} X &\subseteq \phi^{-1}\big[\phi\big[\phi^{-1}[V_i] \setminus \lambda_{\mathcal{P}} X\big]\big] \cap \phi^{-1}[V_i] \\ &= \phi^{-1}\big[\phi\big[\phi^{-1}[V_i] \setminus \lambda_{\mathcal{P}} X\big] \cap V_i\big] \subseteq \phi^{-1}(p_i),\end{aligned}$$

which implies that

$$\phi^{-1}[V_i] \setminus \lambda_{\mathcal{P}} X = \phi^{-1}(p_i) \setminus \lambda_{\mathcal{P}} X. \tag{3.1}$$

Let H_i denote the set in (3.1). Then H_i is clopen in $\beta X \setminus \lambda_{\mathcal{P}} X$ and it is non-empty, as otherwise

$$\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X] \setminus \phi^{-1}(p_i) = \phi^{-1}\big[(Y \setminus X) \setminus \{p_i\}\big] = \phi^{-1}\big[\big(Y \setminus \{p_i\}\big) \setminus X\big],$$

which by Lemma 2.8 implies that the subspace $Y \setminus \{p_i\}$ of Y has $\mathcal{P}$, contradicting the minimality of Y. That H_i's are pairwise disjoint follows from their definitions.

(1.d) *implies* (1.b). Let $C_1, \dots, C_n$ be n distinct components of $\beta X \setminus \lambda_{\mathcal{P}} X$. Then arguing as in the proof of Theorem 2.1 of [19] and since in compact spaces components and quasi-components coincide, each component is the intersection of all clopen subsets containing it (see Theorem 6.1.23 of [5]). Since

$$C_1 \subseteq (\beta X \setminus \lambda_{\mathcal{P}} X) \setminus \bigcup_{k=2}^{n} C_k$$

with the latter an open subset of $\beta X \setminus \lambda_{\mathcal{P}} X$, by the compactness of $\beta X \setminus \lambda_{\mathcal{P}} X$, there exists a clopen subset H_1 of $\beta X \setminus \lambda_{\mathcal{P}} X$ such that

$$C_1 \subseteq H_1 \subseteq (\beta X \setminus \lambda_{\mathcal{P}} X) \setminus \bigcup_{k=2}^{n} C_k.$$

Suppose inductively that for some $j = 1, \dots, n-1$ the pairwise disjoint clopen subsets $H_1, \dots, H_j$ of $\beta X \setminus \lambda_{\mathcal{P}} X$ are defined in such a way that

$$C_i \subseteq H_i \subseteq (\beta X \setminus \lambda_{\mathcal{P}} X) \setminus \Big(\bigcup_{k=1}^{i-1} H_k \cup \bigcup_{k=i+1}^{n} C_k \Big)$$

for any $i = 1, \dots, j$. Note that $C_{j+1} \cap H_i = \emptyset$ when $i = 1, \dots, j$. Thus

$$C_{j+1} \subseteq (\beta X \setminus \lambda_{\mathcal{P}} X) \setminus \Big(\bigcup_{k=1}^{j} H_k \cup \bigcup_{k=j+2}^{n} C_k \Big).$$

Let H_{j+1} be a clopen subset of $\beta X \setminus \lambda_{\mathcal{P}} X$ such that

$$C_{j+1} \subseteq H_{j+1} \subseteq (\beta X \setminus \lambda_{\mathcal{P}} X) \setminus \Big(\bigcup_{k=1}^{j} H_k \cup \bigcup_{k=j+2}^{n} C_k \Big).$$

By Lemma 2.10 we have $X \subseteq \lambda_{\mathcal{P}} X$. Let T be the quotient space of βX obtained by contracting the compact subsets

$$H_1, \dots, H_{n-1}, (\beta X \setminus \lambda_{\mathcal{P}} X) \setminus \bigcup_{k=1}^{n-1} H_k$$

of $\beta X \setminus X$ to points $p_1, \ldots, p_n$, respectively. Then T is Tychonoff and contains X as a dense subspace. Consider the subspace $Y = X \cup \{p_1, \ldots, p_n\}$ of T. By Lemma 2.8 we have $Y \in \mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$. Since T is a compactification of Y there exists a continuous mapping $\gamma : \beta Y \to T$ such that $\gamma|Y = \mathrm{id}_Y$. Let $\psi : \beta X \to \beta Y$ be the continuous extension of id_X. Then since $\gamma\psi|X = \mathrm{id}_X = q|X$ we have $\gamma\psi = q$. By Theorem 3.5.7 of [5] we have $\gamma[\beta Y \setminus Y] = T \setminus Y$. Thus

$$\psi^{-1}(p_i) = \psi^{-1}\big[\gamma^{-1}(p_i)\big] = (\gamma\psi)^{-1}(p_i) = q^{-1}(p_i)$$

for any $i = 1, \ldots, n$. Therefore

$$\psi^{-1}[Y \setminus X] = q^{-1}[Y \setminus X] = \beta X \setminus \lambda_{\mathcal{P}} X,$$

which by Theorem 2.15 implies that $Y \in \mathscr{O}_{\mathcal{P}}(X)$.

(2). It is clear that (2.b) implies (2.a).

(2.a) *implies* (2.d). Consider some $Y \in \mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ with countable remainder. Let $\phi : \beta X \to \beta Y$ be the continuous extension of id_X. By Lemma 2.8 we have $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X]$ and X is locally-$\mathcal{P}$. Let $n \in \mathbb{N}$ and consider some distinct elements $p_1, \ldots, p_n \in Y \setminus X$. Define a continuous $f : \beta Y \to [1, n]$ such that $f(p_i) = i$ for any $i = 1, \ldots, n$. Since $f[Y \setminus X]$ is countable we can find some real numbers $r_1, \ldots, r_{n+1}$ such that

$$r_1 < 1 < r_2 < 2 < \cdots < r_n < n < r_{n+1} \quad \text{and} \quad r_i \notin f[Y \setminus X] \text{ for any } i = 1, \ldots, n+1.$$

Let $i = 1, \ldots, n$. Define $B_i = f^{-1}[(r_i, r_{i+1})]$. Then

$$\bigcup_{i=1}^{n} \phi^{-1}[B_i] = \phi^{-1}\Big[\bigcup_{i=1}^{n} B_i\Big] \supseteq \phi^{-1}[Y \setminus X] \supseteq \beta X \setminus \lambda_{\mathcal{P}} X.$$

Now $\phi^{-1}[B_i] \setminus \lambda_{\mathcal{P}} X$ is non-empty, as B_i is non-empty and by Theorem 2.11 the set $\phi^{-1}(p) \setminus \lambda_{\mathcal{P}} X$ is non-empty for any $p \in Y \setminus X$. Thus $\phi^{-1}[B_i] \setminus \lambda_{\mathcal{P}} X$, where $i = 1, \ldots, n$, are n pairwise disjoint non-empty open (and therefore clopen, as their union is $\beta X \setminus \lambda_{\mathcal{P}} X$) subsets of $\beta X \setminus \lambda_{\mathcal{P}} X$, which implies that $\beta X \setminus \lambda_{\mathcal{P}} X$ has at least n components. Since n is arbitrary the result follows.

(2.d) *implies* (2.c). Let C and D be distinct components of $\beta X \setminus \lambda_{\mathcal{P}} X$. Let U be open in $\beta X \setminus \lambda_{\mathcal{P}} X$ and such that $C \subseteq U$ and $D \cap U = \emptyset$. Then arguing as in (1.d)⇒(1.b) and since U is an open subset of $\beta X \setminus \lambda_{\mathcal{P}} X$ containing C there exists a clopen subset V of $\beta X \setminus \lambda_{\mathcal{P}} X$ such that $C \subseteq V \subseteq U$. Define H_1 to be either of the (non-empty) sets V or $(\beta X \setminus \lambda_{\mathcal{P}} X) \setminus V$ which misses an infinite number of components of $\beta X \setminus \lambda_{\mathcal{P}} X$. Now inductively suppose that $H_1, \ldots, H_n$ are defined such that H_i's are pairwise disjoint non-empty clopen subsets of $\beta X \setminus \lambda_{\mathcal{P}} X$ and $H_1 \cup \cdots \cup H_n$ misses an infinite number of components of $\beta X \setminus \lambda_{\mathcal{P}} X$. Let E and F be distinct components of $\beta X \setminus \lambda_{\mathcal{P}} X$ missing $H_1 \cup \cdots \cup H_n$ and let W be open in $\beta X \setminus \lambda_{\mathcal{P}} X$ and such that $E \subseteq W$ and $F \cap W = \emptyset$. Then since $W \setminus (H_1 \cup \cdots \cup H_n)$ is an open neighborhood of E in $\beta X \setminus \lambda_{\mathcal{P}} X$, arguing as above there exists a non-empty clopen subset H_{n+1} of $\beta X \setminus \lambda_{\mathcal{P}} X$ such that $H_{n+1} \cap H_i = \emptyset$ for any $i = 1, \ldots, n$, and it misses an infinite number of components of $\beta X \setminus \lambda_{\mathcal{P}} X$ contained in $(\beta X \setminus \lambda_{\mathcal{P}} X) \setminus (H_1 \cup \cdots \cup H_n)$. The sequence $H_1, H_2, \ldots$ consists of pairwise disjoint non-empty clopen subsets of $\beta X \setminus \lambda_{\mathcal{P}} X$.

(2.c) *implies* (2.b). Suppose that X is locally-$\mathcal{P}$ and there exists a bijectively indexed sequence $H_1, H_2, \ldots$ of pairwise disjoint non-empty clopen subsets of $\beta X \setminus \lambda_{\mathcal{P}} X$. By

Lemma 2.10 we have $X \subseteq \lambda_{\mathcal{P}} X$. Let T be the space obtained from βX by contracting the sets

$$(\beta X \setminus \lambda_{\mathcal{P}} X) \setminus \bigcup_{k=2}^{\infty} H_k, H_2, H_3, \dots$$

into points $p_1, p_2, \dots$, respectively, with the quotient mapping $q : \beta X \to T$. Then T is compact, since as we show it is Hausdorff (and a continuous image of βX). Suppose that $x, y \in T$ are distinct elements. We consider the following three cases:

Case 1. Suppose that $x, y \in \lambda_{\mathcal{P}} X$. Since x and y can be separated by disjoint open subsets in $\lambda_{\mathcal{P}} X$ and $\lambda_{\mathcal{P}} X$ is open in βX they can also be separated by disjoint open subsets in T.

Case 2. Suppose that $x \in \lambda_{\mathcal{P}} X$ and $y = p_j$ for some $j \in \mathbb{N}$. Let P and Q be disjoint open neighborhoods of x and $\beta X \setminus \lambda_{\mathcal{P}} X$ in βX, respectively. Then $q[P]$ and $q[Q]$ are disjoint open subsets of T separating x and y. The case when $x = p_i$ for some $i \in \mathbb{N}$ and $y \in \lambda_{\mathcal{P}} X$ is analogous.

Case 3. Suppose that $x = p_i$ and $y = p_j$ for some $i, j \in \mathbb{N}$. Then either $i \geq 2$ or $j \geq 2$, say $j \geq 2$. Let P and Q be disjoint open neighborhoods of $(\beta X \setminus \lambda_{\mathcal{P}} X) \setminus H_j$ and H_j in βX, respectively. Then $q[P]$ and $q[Q]$ are disjoint open subsets of T separating x and y.

Note that T contains X as a dense subspace. Consider the subspace $Y = X \cup \{p_1, p_2, \dots\}$ of T. Then Y is a countable-point extension of X with the compact remainder $Y \setminus X = q[\beta X \setminus \lambda_{\mathcal{P}} X]$. Now since T is a compactification of Y and $q^{-1}[Y \setminus X] = \beta X \setminus \lambda_{\mathcal{P}} X$, by Lemma 2.8 we have $Y \in \mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$. To complete the proof we only need to verify that Y is optimal. But this follows by an argument similar to the one in (1.d)⇒(1.b).

(3). It is clear that (3.b) implies (3.a).

(3.a) *implies* (3.c). Consider some $Y \in \mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ with countable remainder of type (σ, n). For any $\zeta \leq \sigma$ let

$$(Y \setminus X)^{(\zeta)} \setminus (Y \setminus X)^{(\zeta+1)} = \{p_i^\zeta : i \in J_\zeta\}$$

where p_i^ζ's are bijectively indexed. Note that if $\zeta < \sigma$ then $\operatorname{card}(J_\zeta) = \aleph_0$, as otherwise $(Y \setminus X)^{(\zeta)}$ is finite and thus, since

$$(Y \setminus X)^{(\sigma)} \subseteq (Y \setminus X)^{(\zeta+1)} = \big((Y \setminus X)^{(\zeta)}\big)',$$

it follows that $(Y \setminus X)^{(\sigma)} = \emptyset$, contradicting $\operatorname{card}((Y \setminus X)^{(\sigma)}) = n > 0$. Also

$$\operatorname{card}(J_\sigma) = \operatorname{card}\big((Y \setminus X)^{(\sigma)} \setminus (Y \setminus X)^{(\sigma+1)}\big) = \operatorname{card}\big((Y \setminus X)^{(\sigma)}\big) = n.$$

Let $J_\zeta = \mathbb{N}$ for any $\zeta < \sigma$ and $J_\sigma = \{1, \dots, n\}$. For any $\zeta \leq \sigma$ and $i \in J_\zeta$ there exists an open neighborhood V_i^ζ of p_i^ζ in βY such that $V_i^\zeta \cap (Y \setminus X)^{(\zeta)} = \{p_i^\zeta\}$. Indeed, we prove the following.

Claim. *For any $\zeta \leq \sigma$ there exists a collection $\{W_i^\zeta : i \in J_\zeta\}$ of open subsets of βY such that $W_i^\zeta \cap (Y \setminus X)^{(\zeta)} = \{p_i^\zeta\}$ for any $i \in J_\zeta$ and $W_i^\zeta \cap W_j^\zeta = \emptyset$ for any distinct $i, j \in J_\zeta$.*

Proof of the claim. Let $\zeta < \sigma$. We inductively define W_i^ζ's for any $i \in J_\zeta$. Let W_1^ζ be an open neighborhood of p_1^ζ in βY such that $\operatorname{cl}_{\beta Y} W_1^\zeta \subseteq V_1^\zeta$. For an $m \in \mathbb{N}$ suppose that

the open subsets $W_1^\zeta, \dots, W_m^\zeta$ of βY are defined such that $W_i^\zeta \cap W_j^\zeta = \emptyset$ for any distinct $i, j = 1, \dots, m$ and that

$$p_i^\zeta \in W_i^\zeta \subseteq \mathrm{cl}_{\beta Y} W_i^\zeta \subseteq V_i^\zeta \setminus \mathrm{cl}_{\beta Y} \Big(\bigcup_{j=1}^{i-1} W_j^\zeta \Big)$$

for any $i = 1, \dots, m$. Since

$$W_i^\zeta \cap (Y \setminus X)^{(\zeta)} \subseteq V_i^\zeta \cap (Y \setminus X)^{(\zeta)} = \{p_i^\zeta\}$$

this implies that $W_i^\zeta \cap (Y \setminus X)^{(\zeta)} = \{p_i^\zeta\}$ for any $i = 1, \dots, m$; also note that $p_{m+1}^\zeta \notin \mathrm{cl}_{\beta Y} W_i^\zeta$. Thus $V_{m+1}^\zeta \setminus \mathrm{cl}_{\beta Y}(W_1^\zeta \cup \dots \cup W_m^\zeta)$ is an open neighborhood of p_{m+1}^ζ in βY. Define W_{m+1}^ζ to be an open neighborhood of p_{m+1}^ζ in βY such that

$$\mathrm{cl}_{\beta Y} W_{m+1}^\zeta \subseteq V_{m+1}^\zeta \setminus \mathrm{cl}_{\beta Y} \Big(\bigcup_{j=1}^{m} W_j^\zeta \Big).$$

The case when $\zeta = \sigma$ is analogous.

Let $\zeta \leq \sigma$ and $i \in J_\zeta$. Let $f_i^\zeta : \beta Y \to \mathbb{I}$ be continuous with $f_i^\zeta(p_i^\zeta) = 0$ and $f_i^\zeta[\beta Y \setminus W_i^\zeta] \subseteq \{1\}$ and let $r_i^\zeta \in (0,1) \setminus f_i^\zeta[Y \setminus X]$, which exists, as $Y \setminus X$ is countable. Denote

$$C_i^\zeta = (f_i^\zeta)^{-1}\big[[0, r_i^\zeta)\big], \quad D_i^\zeta = (f_i^\zeta)^{-1}\big[[0, r_i^\zeta]\big] \quad \text{and} \quad H_i^\zeta = \phi^{-1}[C_i^\zeta] \setminus \lambda_{\mathcal{P}} X$$

where $\phi : \beta X \to \beta Y$ is the continuous extension of id_X. For any $\zeta \leq \sigma$ let

$$\mathscr{H}_\zeta = \{H_i^\zeta : i \in J_\zeta\}.$$

We verify that the collection $\{\mathscr{H}_\zeta : \zeta \leq \sigma\}$ has the desired properties. First note that for any $\zeta \leq \sigma$ and any distinct $i, j \in J_\zeta$ we have

$$H_i^\zeta \cap H_j^\zeta \subseteq \phi^{-1}[C_i^\zeta] \cap \phi^{-1}[C_j^\zeta] \subseteq \phi^{-1}[W_i^\zeta] \cap \phi^{-1}[W_j^\zeta] = \phi^{-1}[W_i^\zeta \cap W_j^\zeta] = \emptyset.$$

By definition H_i^ζ's are open in $\beta X \setminus \lambda_{\mathcal{P}} X$. We show that H_i^ζ's are closed in $\beta X \setminus \lambda_{\mathcal{P}} X$ and non-empty. Let $\zeta \leq \sigma$ and $i \in J_\zeta$. Note that by the choice of r_i^ζ we have

$$(f_i^\zeta)^{-1}(r_i^\zeta) \cap (Y \setminus X) = \emptyset$$

and thus

$$C_i^\zeta \cap (Y \setminus X) = D_i^\zeta \cap (Y \setminus X). \tag{3.2}$$

Therefore

$$\phi^{-1}[C_i^\zeta] \cap \phi^{-1}[Y \setminus X] = \phi^{-1}\big[C_i^\zeta \cap (Y \setminus X)\big] = \phi^{-1}\big[D_i^\zeta \cap (Y \setminus X)\big] = \phi^{-1}[D_i^\zeta] \cap \phi^{-1}[Y \setminus X].$$

By Lemma 2.8 we have $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X]$ (and X is locally-$\mathcal{P}$), which by the above yields

$$H_i^\zeta = \phi^{-1}[C_i^\zeta] \setminus \lambda_{\mathcal{P}} X = \phi^{-1}[D_i^\zeta] \setminus \lambda_{\mathcal{P}} X. \tag{3.3}$$

This shows that H_i^ζ is closed in $\beta X \setminus \lambda_{\mathcal{P}} X$. Next, suppose to the contrary that $H_i^\zeta = \emptyset$. Consider the subspace

$$Y' = X \cup \big((Y \setminus X) \setminus C_i^\zeta\big)$$

of Y. Then Y' is an extension of X and $Y' \setminus X$ is compact, as it is closed in $Y \setminus X$. Also

$$\phi^{-1}[Y' \setminus X] = \phi^{-1}\big[(Y \setminus X) \setminus C_i^\zeta\big] = \phi^{-1}[Y \setminus X] \setminus \phi^{-1}[C_i^\zeta] \supseteq \beta X \setminus \lambda_{\mathcal{P}} X.$$

Thus by Lemma 2.8 we have $Y' \in \mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$. Note that Y' is properly contained in Y, as $p^\zeta_i \notin Y'$, because $f^\zeta_i(p^\zeta_i) = 0$ and thus $p^\zeta_i \in C^\zeta_i$. But this contradicts the minimality of Y. This shows that each $\mathscr{H}_\zeta$, where $\zeta \leq \sigma$, is a collection of pairwise disjoint non-empty clopen subsets of $\beta X \setminus \lambda_{\mathcal{P}} X$. Also, (3.c.i) holds, as $\operatorname{card}(\mathscr{H}_\zeta) = \operatorname{card}(J_\zeta)$ for any $\zeta \leq \sigma$.

We now verify (3.c.ii). Let $\zeta \leq \sigma$ and $H \in \mathscr{H}_\zeta$. Suppose to the contrary that

$$H \setminus \bigcup \{G \in \mathscr{H}_\eta : \eta < \zeta\} = \emptyset.$$

Let $H = H^\zeta_i$ for some $i \in J_\zeta$. Since H^ζ_i is compact, as it is closed in $\beta X \setminus \lambda_{\mathcal{P}} X$, there exists some $H^{\zeta_j}_{i_j} \in \mathscr{H}_{\zeta_j}$ with $i_j \in J_{\zeta_j}$ and $\zeta_j < \zeta$, where $j = 1, \ldots, k$ and $k \in \mathbb{N}$, such that $H^\zeta_i \subseteq H^{\zeta_1}_{i_1} \cup \cdots \cup H^{\zeta_k}_{i_k}$. Consider the subspace

$$Y' = X \cup \Big((Y \setminus X) \setminus \Big(C^\zeta_i \setminus \bigcup_{j=1}^{k} D^{\zeta_j}_{i_j}\Big)\Big)$$

of Y. Using (3.3) we have

$$\phi^{-1}\Big[C^\zeta_i \setminus \bigcup_{j=1}^{k} D^{\zeta_j}_{i_j}\Big] \setminus \lambda_{\mathcal{P}} X = \Big(\phi^{-1}[C^\zeta_i] \setminus \bigcup_{j=1}^{k} \phi^{-1}[D^{\zeta_j}_{i_j}]\Big) \setminus \lambda_{\mathcal{P}} X = H^\zeta_i \setminus \bigcup_{j=1}^{k} H^{\zeta_j}_{i_j} = \emptyset$$

and thus

$$\phi^{-1}[Y' \setminus X] = \phi^{-1}\Big[(Y \setminus X) \setminus \Big(C^\zeta_i \setminus \bigcup_{j=1}^{k} D^{\zeta_j}_{i_j}\Big)\Big] = \phi^{-1}[Y \setminus X] \setminus \phi^{-1}\Big[C^\zeta_i \setminus \bigcup_{j=1}^{k} D^{\zeta_j}_{i_j}\Big] \supseteq \beta X \setminus \lambda_{\mathcal{P}} X.$$

Note that $Y' \setminus X$ is compact, as it is closed in $Y \setminus X$, and therefore by Lemma 2.8 we have $Y' \in \mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$. We show that Y' is properly contained in Y. This contradicts the minimality of Y and proves (3.c.ii). Indeed, we verify that $p^\zeta_i \notin Y'$. By the definition of f^ζ_i we have $p^\zeta_i \in C^\zeta_i$. Also, for any $j = 1, \ldots, k$ we have $p^\zeta_i \notin D^{\zeta_j}_{i_j}$, as otherwise since $D^{\zeta_j}_{i_j} \subseteq W^{\zeta_j}_{i_j}$ it follows that $p^\zeta_i \in W^{\zeta_j}_{i_j}$. But since $\zeta_j < \zeta$ and thus $\zeta_j + 1 \leq \zeta$ we have

$$p^\zeta_i \in (Y \setminus X)^{(\zeta)} \subseteq (Y \setminus X)^{(\zeta_j + 1)} = \big((Y \setminus X)^{(\zeta_j)}\big)'$$

and therefore $W^{\zeta_j}_{i_j}$, being an open neighborhood of p^ζ_i in βY, has an infinite intersection with $(Y \setminus X)^{(\zeta_j)}$, contradicting the definition of $W^{\zeta_j}_{i_j}$.

Next, we show (3.c.iii). Suppose that $\zeta < \eta \leq \sigma$, $H \in \mathscr{H}_\zeta$ and $G \in \mathscr{H}_\eta$. Let $H = H^\zeta_i$ and $G = H^\eta_j$ for some $i \in J_\zeta$ and $j \in J_\eta$. We have the following cases:

Case 1. Suppose that $p^\zeta_i \in C^\eta_j$. First note that each $p \in Y \setminus X$ is of the form p^ξ_k for some $\xi \leq \sigma$ and $k \in J_\xi$. To show this let $\alpha < \Omega$ be the least ordinal such that $p \notin (Y \setminus X)^{(\alpha)}$. Such an α exists, as $(Y \setminus X)^{(\sigma+1)} = \emptyset$, and it is necessarily not a limit ordinal, as otherwise

$$p \in \bigcap \big\{(Y \setminus X)^{(\xi)} : \xi < \alpha\big\} = (Y \setminus X)^{(\alpha)}.$$

Let ξ be such that $\alpha = \xi + 1$. Then $p \in (Y \setminus X)^{(\xi)} \setminus (Y \setminus X)^{(\xi+1)}$ and thus $p = p^\xi_k$ for some $k \in J_\xi$. Since $D^\zeta_i \subseteq W^\zeta_i$, by an argument similar to the one in (3.c.ii), for any $p^\xi_k \in D^\zeta_i$ where $k \in J_\xi$ we have $\xi \leq \zeta$ and if $\xi = \zeta$ then $p^\xi_k = p^\zeta_i$, as by the

definition of W_i^ζ we have $W_i^\zeta \cap (Y \setminus X)^{(\zeta)} = \{p_i^\zeta\}$. Therefore in this case

$$(D_i^\zeta \setminus C_j^\eta) \cap (Y \setminus X) \subseteq \{p_k^\xi : \xi < \zeta \text{ and } k \in J_\xi\} \subseteq \bigcup\{C_k^\xi : \xi < \zeta \text{ and } k \in J_\xi\},$$

which (since $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X]$) implies that

$$\begin{aligned} H_i^\zeta \setminus H_j^\eta &= \big(\phi^{-1}[D_i^\zeta] \setminus \phi^{-1}[C_j^\eta]\big) \setminus \lambda_{\mathcal{P}} X \\ &\subseteq \big(\big(\phi^{-1}[D_i^\zeta] \setminus \phi^{-1}[C_j^\eta]\big) \cap \phi^{-1}[Y \setminus X]\big) \setminus \lambda_{\mathcal{P}} X \\ &\subseteq \bigcup\big\{\phi^{-1}[C_k^\xi] \setminus \lambda_{\mathcal{P}} X : \xi < \zeta \text{ and } k \in J_\xi\big\} = \bigcup\{H_k^\xi : \xi < \zeta \text{ and } k \in J_\xi\}. \end{aligned}$$

Thus

$$H_i^\zeta \subseteq H_j^\eta \cup \bigcup\{F \in \mathscr{H}_\xi : \xi < \zeta\}$$

and (3.c.iii) holds in this case.

Case 2. Suppose that $p_i^\zeta \notin C_j^\eta$. Arguing as in Case 1 we have

$$D_i^\zeta \cap C_j^\eta \cap (Y \setminus X) \subseteq \{p_k^\xi : \xi < \zeta \text{ and } k \in J_\xi\} \subseteq \bigcup\{C_k^\xi : \xi < \zeta \text{ and } k \in J_\xi\}.$$

Therefore

$$\begin{aligned} H_i^\zeta \cap H_j^\eta &= \big(\phi^{-1}[D_i^\zeta] \cap \phi^{-1}[C_j^\eta]\big) \setminus \lambda_{\mathcal{P}} X \\ &= \big(\phi^{-1}[D_i^\zeta] \cap \phi^{-1}[C_j^\eta] \cap \phi^{-1}[Y \setminus X]\big) \setminus \lambda_{\mathcal{P}} X \\ &\subseteq \bigcup\big\{\phi^{-1}[C_k^\xi] \setminus \lambda_{\mathcal{P}} X : \xi < \zeta \text{ and } k \in J_\xi\big\} = \bigcup\{H_k^\xi : \xi < \zeta \text{ and } k \in J_\xi\}. \end{aligned}$$

Thus (3.c.iii) holds in this case as well.

Finally, we verify (3.c.iv). Suppose that $\zeta < \eta \leq \sigma$ and $H \in \mathscr{H}_\eta$. Let $H = H_j^\eta$ for some $j \in J_\eta$. We first verify that

$$C_j^\eta \cap \big((Y \setminus X)^{(\zeta)} \setminus (Y \setminus X)^{(\zeta+1)}\big)$$

is infinite. Suppose it is finite. Since

$$p_j^\eta \in (Y \setminus X)^{(\eta)} \subseteq (Y \setminus X)^{(\zeta+1)} = \big((Y \setminus X)^{(\zeta)}\big)'$$

and C_j^η is an open neighborhood of p_j^η in βY the set

$$C_j^\eta \cap (Y \setminus X)^{(\zeta)} = \big(C_j^\eta \cap \big((Y \setminus X)^{(\zeta)} \setminus (Y \setminus X)^{(\zeta+1)}\big)\big) \cup \big(C_j^\eta \cap (Y \setminus X)^{(\zeta+1)}\big)$$

is infinite. But then since by (3.2) we have

$$C_j^\eta \cap (Y \setminus X)^{(\zeta+1)} = D_j^\eta \cap (Y \setminus X)^{(\zeta+1)}$$

the latter set is an infinite compact space without isolated points and therefore uncountable. This contradiction shows that $p_i^\zeta \in C_j^\eta$ for an infinite number of $i \in J_\zeta$. But if $p_i^\zeta \in C_j^\eta$ for some $i \in J_\zeta$, arguing as in Case 1 of (3.c.iii) we have

$$H_i^\zeta \subseteq H_j^\eta \cup \bigcup\{G \in \mathscr{H}_\xi : \xi < \zeta\}.$$

This shows (3.c.iv).

(3.c) *implies* (3.b). Consider a family $\{\mathscr{H}_\zeta : \zeta \leq \sigma\}$ of collections of pairwise disjoint non-empty clopen subsets of $\beta X \setminus \lambda_{\mathcal{P}} X$ satisfying (3.c.i)–(3.c.iv). For any $\zeta \leq \sigma$ let $\mathscr{H}_\zeta = \{H_i^\zeta : i \in J_\zeta\}$ where H_i^ζ's are bijectively indexed. Note that if $(\zeta, i) \neq (\eta, j)$, where

$\zeta, \eta \leq \sigma$, $i \in J_\zeta$ and $j \in J_\eta$, then $H_i^\zeta \neq H_j^\eta$. This is clear if $\zeta = \eta$, and if $\zeta < \eta$ then it follows from (3.c.ii), as

$$\emptyset \neq H_i^\eta \setminus \bigcup \{G \in \mathscr{H}_\xi : \xi < \eta\} \subseteq H_i^\eta \setminus H_j^\zeta.$$

Similarly for $\eta < \zeta$. Before we proceed with the main proof we prove the following generalized version of (3.c.iv).

Claim. *For any $\zeta, \eta_1, \ldots, \eta_k < \eta \leq \sigma$, where k is a non-negative integer ($\eta_1, \ldots, \eta_k$ may not be distinct) $H \in \mathscr{H}_\eta$ and $H_i \in \mathscr{H}_{\eta_i}$ for any $i = 1, \ldots, k$, the set*

$$\Big\{F \in \mathscr{H}_\zeta : F \subseteq \Big(H \setminus \bigcup_{i=1}^{k} H_i\Big) \cup \bigcup \{G \in \mathscr{H}_\xi : \xi < \zeta\}\Big\} \tag{3.4}$$

is infinite.

Proof of the claim. If $k = 0$ then the claim is simply (3.c.iv). Assume that $k > 0$. We use transfinite induction on η. Suppose that $\eta = 1$, $k \in \mathbb{N}$, $\zeta, \eta_1, \ldots, \eta_k < \eta$, $H \in \mathscr{H}_\eta$ and $H_i \in \mathscr{H}_{\eta_i}$ for any $i = 1, \ldots, k$. Then $\zeta, \eta_1, \ldots, \eta_k = 0$. By (3.c.iv) the set $\mathscr{F} = \{F \in \mathscr{H}_\zeta : F \subseteq H\}$ is infinite. Now since the elements of $\mathscr{H}_\zeta$ are pairwise disjoint, each $F \in \mathscr{F} \setminus \{H_1, \ldots, H_k\}$ misses H_i for any $i = 1, \ldots, k$, and thus $F \subseteq H \setminus (H_1 \cup \cdots \cup H_k)$. Therefore (3.4) holds for $\eta = 1$. Now inductively suppose that (3.4) holds for any $\xi < \eta$. Let $\zeta, \eta_1, \ldots, \eta_k < \eta \leq \sigma$, where $k \in \mathbb{N}$, $H \in \mathscr{H}_\eta$ and $H_i \in \mathscr{H}_{\eta_i}$ for any $i = 1, \ldots, k$. We may assume that $\eta_1, \ldots, \eta_{l-1} < \eta_l = \cdots = \eta_k$ for some $l \in \mathbb{N}$ with $l \leq k$. Let

$$\mathscr{K} = \Big\{K \in \mathscr{H}_\zeta : K \subseteq \Big(H \setminus \bigcup_{i=1}^{k} H_i\Big) \cup \bigcup \{G \in \mathscr{H}_\xi : \xi < \zeta\}\Big\}.$$

We consider the following cases:

Case 1. Suppose that $\eta_k < \zeta$. Since $\zeta < \eta$, by (3.c.iv) the set

$$\mathscr{F} = \Big\{F \in \mathscr{H}_\zeta : F \subseteq H \cup \bigcup \{G \in \mathscr{H}_\xi : \xi < \zeta\}\Big\}$$

is infinite. Now for any $F \in \mathscr{F}$ we have

$$F \setminus \bigcup \{G \in \mathscr{H}_\xi : \xi < \zeta\} \subseteq H \setminus \bigcup \{G \in \mathscr{H}_\xi : \xi < \zeta\} \subseteq H \setminus \bigcup_{i=1}^{k} H_i$$

and thus $F \in \mathscr{K}$. Therefore in this case $\mathscr{F} \subseteq \mathscr{K}$ and thus $\mathscr{K}$ is infinite.

Case 2. Suppose that $\eta_k = \zeta$. By (3.c.iv) the set

$$\mathscr{F} = \Big\{F \in \mathscr{H}_\zeta : F \subseteq H \cup \bigcup \{G \in \mathscr{H}_\xi : \xi < \zeta\}\Big\}$$

is infinite. Let $F \in \mathscr{F} \setminus \{H_1, \ldots, H_k\}$. Then each $H_i \in \mathscr{H}_{\eta_i} = \mathscr{H}_\zeta$ where $i = l, \ldots, k$ is disjoint from $F \in \mathscr{H}_\zeta$. On the other hand, since $\eta_i < \zeta$ for any $i = 1, \ldots, l-1$ we have

$$F \setminus \{G \in \mathscr{H}_\xi : \xi < \zeta\} \subseteq H \setminus \bigcup_{i=1}^{l-1} H_i,$$

which combined with the above gives $F \in \mathscr{K}$. Therefore $\mathscr{F} \setminus \{H_l, \ldots, H_k\} \subseteq \mathscr{K}$ and thus $\mathscr{K}$ is infinite.

Case 3. Suppose that $\eta_k > \zeta$. Since $\eta_k < \eta$, by (3.c.iv) the set

$$\mathscr{L} = \Big\{L \in \mathscr{H}_{\eta_k} : L \subseteq H \cup \bigcup\{G \in \mathscr{H}_\xi : \xi < \eta_k\}\Big\}$$

is infinite. Choose some $L \in \mathscr{L} \setminus \{H_l, \ldots, H_k\}$. Then

$$L \subseteq H \cup \bigcup\{G \in \mathscr{H}_\xi : \xi < \eta_k\}$$

and thus by compactness $L \subseteq H \cup G_1 \cup \cdots \cup G_m$, where $G_i \in \mathscr{H}_{\xi_i}$, $\xi_i < \eta_k$ for any $i = 1, \ldots, m$ and $m \in \mathbb{N}$. Now since

$$\zeta, \xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_{l-1} < \eta_k < \eta$$

by our induction assumption the set

$$\mathscr{F} = \Big\{F \in \mathscr{H}_\zeta : F \subseteq \Big(L \setminus \Big(\bigcup_{i=1}^{m} G_i \cup \bigcup_{i=1}^{l-1} H_i\Big)\Big) \cup \bigcup\{G \in \mathscr{H}_\xi : \xi < \zeta\}\Big\}$$

is infinite. If $F \in \mathscr{F}$ then

$$F \setminus \bigcup\{G \in \mathscr{H}_\xi : \xi < \zeta\} \subseteq L \setminus \Big(\bigcup_{i=1}^{m} G_i \cup \bigcup_{i=1}^{l-1} H_i\Big) = \Big(L \setminus \bigcup_{i=1}^{m} G_i\Big) \setminus \bigcup_{i=1}^{l-1} H_i \subseteq H \setminus \bigcup_{i=1}^{l-1} H_i,$$

which together with the fact that $L \in \mathscr{H}_{\eta_k}$ is disjoint from $H_i \in \mathscr{H}_{\eta_i} = \mathscr{H}_{\eta_k}$ for any $i = l, \ldots, k$, gives

$$F \setminus \bigcup\{G \in \mathscr{H}_\xi : \xi < \zeta\} \subseteq H \setminus \bigcup_{i=1}^{k} H_i.$$

This shows that $F \in \mathscr{K}$. Therefore $\mathscr{F} \subseteq \mathscr{K}$ and thus $\mathscr{K}$ is infinite in this case as well.

This proves the claim.

We now return to the main proof. Fix some $k \in J_\sigma$. For any $\zeta \leq \sigma$ and $i \in J_\zeta$ define

$$P_i^\zeta = H_i^\zeta \setminus \bigcup\{H \in \mathscr{H}_\xi : \xi < \zeta\}$$

if $(\zeta, i) \neq (\sigma, k)$, and

$$P_i^\zeta = (\beta X \setminus \lambda_{\mathcal{P}} X) \setminus \bigcup\{H \in \mathscr{H}_\xi : \xi \leq \sigma \text{ and } H \neq H_k^\sigma\}$$

if $(\zeta, i) = (\sigma, k)$.

Claim. *The collection $\{P_i^\zeta : \zeta \leq \sigma$ and $i \in J_\zeta\}$ is bijectively indexed and partitions $\beta X \setminus \lambda_{\mathcal{P}} X$ into pairwise disjoint non-empty subsets.*

Proof of the claim. We first show that

$$\bigcup\{P_i^\zeta : \zeta \leq \sigma \text{ and } i \in J_\zeta\} = \beta X \setminus \lambda_{\mathcal{P}} X. \tag{3.5}$$

Let $x \in \beta X \setminus \lambda_{\mathcal{P}} X$. If $x \notin H_i^\zeta$ for any $\zeta \leq \sigma$ and $i \in J_\zeta$, then clearly $x \in P_k^\sigma$. In the other case there exists some $\zeta \leq \sigma$ such that $x \in H_i^\zeta$ for some $i \in J_\zeta$. Let ζ be the least with this property. If $(\zeta, i) \neq (\sigma, k)$, then by definition it is clear that $x \in P_i^\zeta$. If $(\zeta, i) = (\sigma, k)$ then again $x \in P_i^\zeta$, as the elements of $\mathscr{H}_\zeta$ are pairwise disjoint and $x \notin H$ for any $H \in \mathscr{H}_\eta$ with $\eta < \zeta$. This shows (3.5). Next, we show that $P_i^\zeta \cap P_j^\eta = \emptyset$ whenever $\zeta, \eta \leq \sigma$, $i \in J_\zeta$, $j \in J_\eta$ and $(\zeta, i) \neq (\eta, j)$. First suppose that $(\zeta, i), (\eta, j) \neq (\sigma, k)$. If $\zeta = \eta$ then clearly

$P_i^\zeta \cap P_j^\eta \subseteq H_i^\zeta \cap H_j^\eta = \emptyset$. If $\zeta < \eta$ then $P_i^\zeta \cap P_j^\eta \subseteq H_i^\zeta \cap P_j^\eta = \emptyset$. Similarly for $\eta < \zeta$. Next, suppose that $(\zeta, i) = (\sigma, k)$. Then $P_i^\zeta \cap P_j^\eta \subseteq P_i^\zeta \cap H_j^\eta = \emptyset$. Similarly for $(\eta, j) = (\sigma, k)$. Finally, we verify that P_i^ζ's are non-empty. Let $\zeta \leq \sigma$ and $i \in J_\zeta$. If $(\zeta, i) \neq (\sigma, k)$ then P_i^ζ is non-empty by (3.c.ii). If $(\zeta, i) = (\sigma, k)$, then again using (3.c.ii) we have

$$\emptyset \neq H_i^\zeta \setminus \bigcup\{H \in \mathscr{H}_\xi : \xi < \zeta\} \subseteq P_i^\zeta.$$

The fact that P_i^ζ's are bijectively indexed is now immediate.

Now let T be the space obtained from βX by contracting each P_i^ζ where $\zeta \leq \sigma$ and $i \in J_\zeta$ to a point p_i^ζ and denote by $q : \beta X \to T$ the corresponding quotient mapping. By Lemma 2.10 we have $X \subseteq \lambda_{\mathcal{P}} X$. Consider the subspace

$$Y = X \cup \{p_i^\zeta : \zeta \leq \sigma \text{ and } i \in J_\zeta\}$$

of T. In the remainder of the proof we show that $Y \in \mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(X)$ and that the remainder of Y is of type (σ, n). We first show that T is Hausdorff. Let $s, t \in T$ be distinct. Consider the following cases:

Case 1. Suppose that $s, t \in T \setminus \{p_i^\zeta : \zeta \leq \sigma \text{ and } i \in J_\zeta\}$. Then $s, t \in \lambda_{\mathcal{P}} X$. Now since $\lambda_{\mathcal{P}} X$ is open in βX and s and t can be separated in $\lambda_{\mathcal{P}} X$ by disjoint open subsets they can also be separated by disjoint open subsets in T.

Case 2. Suppose that $s \in T \setminus \{p_i^\zeta : \zeta \leq \sigma \text{ and } i \in J_\zeta\}$ and $t = p_j^\eta$ for some $\eta \leq \sigma$ and $j \in J_\eta$. Then $s \in \lambda_{\mathcal{P}} X$. Now if U and V are disjoint open subsets of βX containing s and $\beta X \setminus \lambda_{\mathcal{P}} X$, respectively, then $q[U]$ and $q[V]$ are disjoint open neighborhoods of s and t in T, respectively.

Case 3. Suppose that $s = p_i^\zeta$ and $t = p_j^\eta$ for some $\zeta, \eta \leq \sigma$, $i \in J_\zeta$ and $j \in J_\eta$. Without any loss of generality we may assume that $\zeta \leq \eta$ and $(\zeta, i) \neq (\sigma, k)$. Since H_i^ζ is clopen in $\beta X \setminus \lambda_{\mathcal{P}} X$ the sets H_i^ζ and $(\beta X \setminus \lambda_{\mathcal{P}} X) \setminus H_i^\zeta$ are compact open subsets of $\beta X \setminus \lambda_{\mathcal{P}} X$. Let U and V be disjoint open subsets of βX such that

$$H_i^\zeta = U \cap (\beta X \setminus \lambda_{\mathcal{P}} X) \quad \text{and} \quad (\beta X \setminus \lambda_{\mathcal{P}} X) \setminus H_i^\zeta = V \cap (\beta X \setminus \lambda_{\mathcal{P}} X).$$

Claim. *For any $\xi \leq \sigma$ and $l \in J_\xi$ if $P_l^\xi \cap H_i^\zeta$ is non-empty then $P_l^\xi \subseteq H_i^\zeta$.*

Proof of the claim. Suppose that $P_l^\xi \cap H_i^\zeta$ is non-empty for some $\xi \leq \sigma$ and $l \in J_\xi$. This implies that $(\xi, l) \neq (\sigma, k)$, as we are assuming that $(\zeta, i) \neq (\sigma, k)$, and thus by definition $P_k^\sigma \cap H_i^\zeta = \emptyset$. Therefore

$$P_l^\xi = H_l^\xi \setminus \bigcup\{H \in \mathscr{H}_\alpha : \alpha < \xi\}.$$

Note that $\zeta < \xi$ implies that $P_l^\xi \cap H_i^\zeta = \emptyset$ and thus $\xi \leq \zeta$. If $\zeta = \xi$, then since $P_l^\xi \cap H_i^\zeta \subseteq H_l^\xi \cap H_i^\zeta$, the last set is non-empty and therefore $P_l^\xi \subseteq H_l^\xi = H_i^\zeta$. If $\zeta > \xi$, then by (3.c.iii) we have either

$$H_l^\xi \subseteq H_i^\zeta \cup \bigcup\{H \in \mathscr{H}_\alpha : \alpha < \xi\} \quad \text{or} \quad H_l^\xi \cap H_i^\zeta \subseteq \bigcup\{H \in \mathscr{H}_\alpha : \alpha < \xi\}.$$

The latter case leads to a contradiction, as $P_l^\xi \cap H_i^\zeta \subseteq H_l^\xi \cap H_i^\zeta$ and

$$P_l^\xi \cap \bigcup\{H \in \mathscr{H}_\alpha : \alpha < \xi\} = \emptyset.$$

The former case gives $P_l^\xi \subseteq H_i^\zeta$, which proves the claim.

From the claim it follows that $q^{-1}[q[U]] = U$ and $q^{-1}[q[V]] = V$. Thus $q[U]$ and $q[V]$ are open subsets of T and they are disjoint. It is also clear that $p_i^\zeta \in q[U]$, as $P_i^\zeta \subseteq H_i^\zeta \subseteq U$. It remains to show that $p_j^\eta \in q[V]$. Note that by our assumption $\zeta \leq \eta$. To show that $P_j^\eta \cap H_i^\zeta = \emptyset$ we consider the following cases:

Case 3.a. Suppose that $\zeta = \eta$ and $(\eta, j) \neq (\sigma, k)$. Then since $p_i^\zeta \neq p_j^\eta$ we have $i \neq j$ and therefore $P_j^\eta \cap H_i^\zeta \subseteq H_j^\eta \cap H_i^\zeta = \emptyset$.

Case 3.b. Suppose that $\zeta = \eta$ and $(\eta, j) = (\sigma, k)$. By our assumption $(\zeta, i) \neq (\sigma, k)$ or equivalently $H_i^\zeta \neq H_k^\sigma$. Therefore $P_j^\eta \cap H_i^\zeta = \emptyset$ by the definition of P_j^η.

Case 3.c. Suppose that $\zeta < \eta$ and $(\eta, j) \neq (\sigma, k)$. By the definition of P_j^η we have $P_j^\eta \subseteq H_j^\eta \setminus H_i^\zeta$ and thus $P_j^\eta \cap H_i^\zeta = \emptyset$.

Case 3.d. Suppose that $\zeta < \eta$ and $(\eta, j) = (\sigma, k)$. Then $P_j^\eta \cap H_i^\zeta = \emptyset$ by the definition of P_j^η.

Thus in each case $P_j^\eta \cap H_i^\zeta = \emptyset$. Therefore $P_j^\eta \subseteq (\beta X \setminus \lambda_{\mathcal{P}} X) \setminus H_i^\zeta \subseteq V$ and thus $p_j^\eta \in q[V]$.

This shows that T is Hausdorff and therefore compact, being a continuous image of βX. It is easy to see that T contains X as a dense subspace, and thus since $X \subseteq Y \subseteq T$, it follows that T is a compactification of Y and that Y is a Tychonoff extension of X. Also $Y \setminus X = q[\beta X \setminus \lambda_{\mathcal{P}} X]$ is compact. From these by Lemma 2.8 we have $Y \in \mathscr{E}_{\mathcal{P}}^{\mathcal{Q}}(X)$. Now by Theorem 2.15 and an argument similar to the one in (1.d)⇒(1.b) it follows that $Y \in \mathscr{O}_{\mathcal{P}}(X)$. It thus remains to show that $Y \setminus X$ is of type (σ, n), that is, $\mathrm{card}((Y \setminus X)^{(\sigma)}) = n$. Indeed, we prove the following.

Claim. *For any $\zeta \leq \sigma$ we have*

$$(Y \setminus X)^{(\zeta)} = \{p_j^\eta : \zeta \leq \eta \leq \sigma \text{ and } j \in J_\eta\}. \tag{3.6}$$

Proof of the claim. The proof is by transfinite induction on ζ. Note that (3.6) clearly holds when $\zeta = 0$, as by definition $(Y \setminus X)^{(0)} = Y \setminus X$. Suppose that $0 < \alpha \leq \sigma$ and that (3.6) holds for any $\zeta < \alpha$. We show that (3.6) holds for α as well. Consider the following cases:

Case 1. Suppose that α is a successor ordinal. Let $\alpha = \gamma + 1$. Then by our induction assumption

$$(Y \setminus X)^{(\gamma)} = \{p_j^\eta : \gamma \leq \eta \leq \sigma \text{ and } j \in J_\eta\}. \tag{3.7}$$

Let $p_j^\eta \in (Y \setminus X)^{(\alpha)}$ where $\eta \leq \sigma$ and $j \in J_\eta$. Since $(Y \setminus X)^{(\alpha)} \subseteq (Y \setminus X)^{(\gamma)}$, by (3.7) we have $\gamma \leq \eta$. We show that $\gamma \neq \eta$. Suppose the contrary. Clearly $\eta < \sigma$, as $\eta = \sigma$ implies that $\sigma = \gamma < \alpha$. Let U be an open subset of βX such that $H_j^\eta = U \cap (\beta X \setminus \lambda_{\mathcal{P}} X)$. Then as in the proof of the previous claim, $P_l^\xi \subseteq H_j^\eta$ for any $\xi \leq \sigma$ and $l \in J_\xi$ such that $P_l^\xi \cap H_j^\eta$ is non-empty. Thus $q^{-1}[q[U]] = U$ and therefore $q[U]$ is open in T. Since $(\eta, j) \neq (\sigma, k)$, by definition $P_j^\eta \subseteq H_j^\eta \subseteq U$. Thus $q[U]$ is an open neighborhood of p_j^η in T and therefore $q[U] \cap (Y \setminus X)^{(\gamma)}$ is infinite. Choose some $s \in q[U] \cap (Y \setminus X)^{(\gamma)}$ such that $s \neq p_j^\eta, p_k^\sigma$. Then by (3.7) we have

$s = p_l^\xi$ for some $\gamma \leq \xi \leq \sigma$ and $l \in J_\xi$. Since $p_l^\xi \in q[U]$ the set $P_l^\xi \cap H_j^\eta$ is non-empty and thus $P_l^\xi \subseteq H_j^\eta$. Consider the following cases:

Case 1.a. Suppose that $\xi > \gamma$. Then since we are assuming that $\gamma = \eta$, by the definition of P_l^ξ we have $P_l^\xi = P_l^\xi \cap H_j^\eta = \emptyset$, which is a contradiction.

Case 1.b. Suppose that $\xi = \gamma$. Then $\eta = \gamma = \xi$ and therefore, since by the choice of s we have $p_l^\xi \neq p_j^\eta$, it follows that $P_l^\xi = P_l^\xi \cap H_j^\eta \subseteq H_l^\xi \cap H_j^\eta = \emptyset$, which is again a contradiction.

Thus in each case we are led to a contradiction, which shows that $\gamma < \eta$ or $\alpha = \gamma + 1 \leq \eta$. Therefore

$$(Y \setminus X)^{(\alpha)} \subseteq \{p_j^\eta : \alpha \leq \eta \leq \sigma \text{ and } j \in J_\eta\}. \tag{3.8}$$

Next, we show that the reverse inclusion holds in (3.8). Consider an element p_j^η where $\alpha \leq \eta \leq \sigma$ and $j \in J_\eta$. Let V be an open neighborhood of p_j^η in T. We show that $V \cap (Y \setminus X)^{(\gamma)}$ is infinite, which proves that

$$p_j^\eta \in \big((Y \setminus X)^{(\gamma)}\big)' = (Y \setminus X)^{(\gamma+1)} = (Y \setminus X)^{(\alpha)}.$$

First note that

$$H_j^\eta \subseteq q^{-1}[V] \cup \bigcup\{H \in \mathscr{H}_\xi : \xi < \eta\}. \tag{3.9}$$

This readily follows from the definition of P_j^η in the case when $(\eta, j) \neq (\sigma, k)$. If otherwise $(\eta, j) = (\sigma, k)$, note that by the definition of P_j^η we have

$$H_j^\eta \setminus \bigcup\{H \in \mathscr{H}_\xi : \xi < \eta\} \subseteq P_j^\eta \subseteq q^{-1}[V]$$

From (3.9) and by compactness there exist $\xi_i < \eta$ and $k_i \in J_{\xi_i}$ where $i = 1, \ldots, m$ and $m \in \mathbb{N}$ such that

$$H_j^\eta \subseteq q^{-1}[V] \cup \bigcup_{i=1}^m H_{k_i}^{\xi_i}.$$

By the first claim and since $\gamma, \xi_1, \ldots, \xi_m < \eta$ the set

$$\mathscr{F} = \Big\{F \in \mathscr{H}_\gamma : F \subseteq \Big(H_j^\eta \setminus \bigcup_{i=1}^m H_{k_i}^{\xi_i}\Big) \cup \bigcup\{G \in \mathscr{H}_\xi : \xi < \gamma\}\Big\}$$

is infinite. Now for any $H_l^\gamma \in \mathscr{F}$ where $l \in J_\gamma$, since $(\gamma, l) \neq (\sigma, k)$, as $\gamma < \alpha \leq \sigma$, we have

$$P_l^\gamma = H_l^\gamma \setminus \bigcup\{G \in \mathscr{H}_\xi : \xi < \gamma\} \subseteq H_j^\eta \setminus \bigcup_{i=1}^m H_{k_i}^{\xi_i} \subseteq q^{-1}[V]$$

and thus $p_l^\gamma \in V$. Therefore $V \cap (Y \setminus X)^{(\gamma)}$ is infinite. This shows that

$$p_j^\eta \in \big((Y \setminus X)^{(\gamma)}\big)' = (Y \setminus X)^{(\alpha)},$$

which proves the reverse inclusion in (3.8).

Case 2. Suppose that α is a limit ordinal. We have

$$\begin{aligned}(Y \setminus X)^{(\alpha)} &= \bigcap_{\gamma<\alpha} (Y \setminus X)^{(\gamma)} \\ &= \bigcap_{\gamma<\alpha} \{p_j^\eta : \gamma \leq \eta \leq \sigma \text{ and } j \in J_\eta\} = \{p_j^\eta : \alpha \leq \eta \leq \sigma \text{ and } j \in J_\eta\}.\end{aligned}$$

This completes the inductive proof of the claim.

In particular, we have shown that $\operatorname{card}((Y \setminus X)^{(\sigma)}) = \operatorname{card}(J_\sigma) = n$. Thus $Y \setminus X$ is of type (σ, n). ∎

REMARK. Note that in Lemma 3.7 above part (3) implies (2), but since the proof for (3) is quite long, a separate proof is given for (2).

The characterization given in Lemma 3.7 is external (to X). Our next theorem is dual to Lemma 3.7 and gives an internal characterization of those spaces which have a compactification-like $\mathcal{P}$-extension with finite remainder, countable remainder and countable remainder of type (σ, n). But we first need a few more lemmas.

LEMMA 3.8. *Let X be a Tychonoff space and let $\mathcal{P}$ be a clopen hereditary finitely additive perfect topological property. Then for any subset A of X if $\operatorname{cl}_{\beta X} A \subseteq \lambda_{\mathcal{P}} X$ then $\operatorname{cl}_X A \subseteq Z \subseteq C$ for some $Z \in \mathscr{Z}(X)$ and $C \in \operatorname{Coz}(X)$ such that $\operatorname{cl}_X C$ has $\mathcal{P}$.*

Proof. The sets $\operatorname{cl}_{\beta X} A$ and $\beta X \setminus \lambda_{\mathcal{P}} X$ are disjoint closed subsets of βX and thus they are completely separated in βX. Let $f : \beta X \to \mathbb{I}$ be continuous with $f[\operatorname{cl}_{\beta X} A] \subseteq \{0\}$ and $f[\beta X \setminus \lambda_{\mathcal{P}} X] \subseteq \{1\}$. Let

$$Z = f^{-1}\big[[0, 1/3]\big] \cap X \in \mathscr{Z}(X) \quad \text{and} \quad C = f^{-1}\big[[0, 1/2)\big] \cap X \in \operatorname{Coz}(X).$$

Then $\operatorname{cl}_X A \subseteq Z \subseteq C$ and since

$$\operatorname{cl}_{\beta X} C = \operatorname{cl}_{\beta X}\big(f^{-1}\big[[0, 1/2)\big] \cap X\big) = \operatorname{cl}_{\beta X} f^{-1}\big[[0, 1/2)\big] \subseteq f^{-1}\big[[0, 1/2]\big] \subseteq \lambda_{\mathcal{P}} X$$

by Lemma 2.4 the set $\operatorname{cl}_X C$ has $\mathcal{P}$. ∎

LEMMA 3.9. *Let X be a Tychonoff space and let A be an infinite compact countable subset of X. Then there exists a bijectively indexed collection $\{V_n : n \in \mathbb{N}\}$ of pairwise disjoint open subsets of X such that $V_n \cap A$ is compact and non-empty for any $n \in \mathbb{N}$.*

Proof. We inductively define a sequence $V_1, V_2, \ldots$ of pairwise disjoint open subsets of X such that $V_n \cap A$ is compact and non-empty, $A \setminus \operatorname{cl}_X(V_1 \cup \cdots \cup V_n)$ is infinite and $V_n \cap A = \operatorname{cl}_X V_n \cap A$ for any $n \in \mathbb{N}$. Let $a, b \in A$ be distinct and let $f : X \to \mathbb{I}$ be continuous with $f(a) = 0$ and $f(b) = 1$. Since $f[A]$ is countable there exists some $r \in (0,1) \setminus f[A]$. Either $f^{-1}[[0, r)] \cap A$ or $f^{-1}[(r, 1]] \cap A$, say the latter, is infinite. Let $V_1 = f^{-1}[[0, r)]$. Since $V_1 \cap A = f^{-1}[[0, r]] \cap A$ is closed in A, it is compact, and thus since

$$f^{-1}\big[(r, 1]\big] \subseteq X \setminus f^{-1}\big[[0, r]\big] \subseteq X \setminus \operatorname{cl}_X V_1$$

the set $A \setminus \operatorname{cl}_X V_1$ is infinite and

$$V_1 \cap A = f^{-1}\big[[0, r]\big] \cap A = \operatorname{cl}_X V_1 \cap A.$$

Suppose that for some $n \in \mathbb{N}$ the pairwise disjoint open subsets $V_1, \ldots, V_n$ of X are defined such that $A \setminus \operatorname{cl}_X(V_1 \cup \cdots \cup V_n)$ is infinite, $V_i \cap A$ is compact and non-empty and $V_i \cap A = \operatorname{cl}_X V_i \cap A$ for $i = 1, \ldots, n$. Choose some distinct $c, d \in A \setminus \operatorname{cl}_X(V_1 \cup \cdots \cup V_n)$ and let $g : X \to \mathbb{I}$ be continuous with $g(c) = 0$ and $g(d) = 1$. Choose some $s \in (0,1) \setminus g[A]$. Then at least one of

$$\Big(g^{-1}\big[[0, s)\big] \setminus \operatorname{cl}_X\Big(\bigcup_{i=1}^{n} V_i\Big)\Big) \cap A \quad \text{and} \quad \Big(g^{-1}\big[(s, 1]\big] \setminus \operatorname{cl}_X\Big(\bigcup_{i=1}^{n} V_i\Big)\Big) \cap A$$

say the latter, is infinite. Define

$$V_{n+1} = g^{-1}\big[[0,s)\big] \setminus \mathrm{cl}_X\Big(\bigcup_{i=1}^{n} V_i\Big).$$

Then $V_1, \dots, V_{n+1}$ are pairwise disjoint and since $\mathrm{cl}_X V_{n+1} \subseteq g^{-1}[[0,s]]$ we have

$$\Big(g^{-1}\big[(s,1]\big] \setminus \mathrm{cl}_X\Big(\bigcup_{i=1}^{n} V_i\Big)\Big) \cap A \subseteq A \setminus \mathrm{cl}_X\Big(\bigcup_{i=1}^{n+1} V_i\Big).$$

Therefore $A \setminus \mathrm{cl}_X(V_1 \cup \dots \cup V_{n+1})$ is infinite. By the choice of V_i's we have

$$A \setminus \bigcup_{i=1}^{n} V_i = A \setminus \bigcup_{i=1}^{n} \mathrm{cl}_X V_i.$$

Therefore

$$\begin{aligned}
\mathrm{cl}_X V_{n+1} \cap A &\subseteq \Big(g^{-1}\big[[0,s]\big] \setminus \bigcup_{i=1}^{n} V_i\Big) \cap A = \Big(A \setminus \bigcup_{i=1}^{n} V_i\Big) \cap g^{-1}\big[[0,s]\big] \\
&= \Big(A \setminus \bigcup_{i=1}^{n} \mathrm{cl}_X V_i\Big) \cap g^{-1}\big[[0,s]\big] = \Big(A \setminus \bigcup_{i=1}^{n} \mathrm{cl}_X V_i\Big) \cap g^{-1}\big[[0,s)\big] \\
&= \Big(g^{-1}\big[[0,s)\big] \setminus \bigcup_{i=1}^{n} \mathrm{cl}_X V_i\Big) \cap A = V_{n+1} \cap A,
\end{aligned}$$

which implies that $V_{n+1} \cap A = \mathrm{cl}_X V_{n+1} \cap A$ is compact, as it is closed in the compact set A. This completes the inductive step. ∎

Let X be a Tychonoff space and let αX be a compactification of X. For an open subset U of X, the *extension of U to αX* is defined to be

$$\mathrm{Ex}_{\alpha X} U = \alpha X \setminus \mathrm{cl}_{\alpha X}(X \setminus U).$$

If γX denotes the Freudenthal compactification of a rim-compact space X (a space which has a base consisting of open subsets with compact boundary) then for any open subset U of X we have $\mathrm{cl}_{\gamma X} U \setminus X = \mathrm{Ex}_{\gamma X} U \setminus X$ (see [34], as mentioned in [11]). Using this, in [11] the author defined an appropriate upper semicontinuous decomposition of γX and then proved that a locally compact space X has a countable-point compactification if and only if it has a pairwise disjoint sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets each with compact boundary and non-compact closure. Here we only deal with extensions in βX. Also, we use the simplified notation $\mathrm{Ex}_X U$ instead of $\mathrm{Ex}_{\beta X} U$. The following lemma is well known (see Lemma 7.1.13 of [5] or Lemma 3.1 of [37]).

Lemma 3.10. *Let X be a Tychonoff space and let U and V be open subsets of X. Then*

(1) $X \cap \mathrm{Ex}_X U = U$ *and thus* $\mathrm{cl}_{\beta X} \mathrm{Ex}_X U = \mathrm{cl}_{\beta X} U$.
(2) $\mathrm{Ex}_X(U \cap V) = \mathrm{Ex}_X U \cap \mathrm{Ex}_X V$.

The following lemma was proved by E. G. Skljarenko [32] and rediscovered by E. K. van Douwen [37].

LEMMA 3.11 (Skljarenko [32] and van Douwen [37]). *Let X be a Tychonoff space and let U be an open subset of X. Then*

$$\mathrm{bd}_{\beta X}\,\mathrm{Ex}_X\,U = \mathrm{cl}_{\beta X}\,\mathrm{bd}_X\,U.$$

LEMMA 3.12. *Let X be a Tychonoff space and let $\mathcal{P}$ be a clopen hereditary topological property which is inverse invariant under perfect mappings. Let U be an open subset of X such that $\mathrm{bd}_X\,U \subseteq Z \subseteq C$ where $Z \in \mathscr{Z}(X)$, $C \in \mathrm{Coz}(X)$ and $\mathrm{cl}_X\,C$ has $\mathcal{P}$. Then*

$$\mathrm{cl}_{\beta X}\,U \setminus \lambda_{\mathcal{P}}X = \mathrm{Ex}_X\,U \setminus \lambda_{\mathcal{P}}X.$$

Proof. By Lemma 2.14 we have $\mathrm{cl}_{\beta X}\,Z \subseteq \lambda_{\mathcal{P}}X$. The result then follows, as by Lemmas 3.10 and 3.11 we have

$$\mathrm{cl}_{\beta X}\,U = \mathrm{cl}_{\beta X}\,\mathrm{Ex}_X\,U = \mathrm{Ex}_X\,U \cup \mathrm{bd}_{\beta X}\,\mathrm{Ex}_X\,U = \mathrm{Ex}_X\,U \cup \mathrm{cl}_{\beta X}\,\mathrm{bd}_X\,U$$

and $\mathrm{cl}_{\beta X}\,\mathrm{bd}_X\,U \subseteq \mathrm{cl}_{\beta X}\,Z$. ∎

In [23] J. R. McCartney characterized those spaces which have a compactification with compact countable remainder of type (σ, n) (Theorem 3.6). Indeed, in the proof, for a given space X which satisfies the properties of Theorem 3.6, the author formed a new set Y by adjoining a set of points to X and then constructed a topology on Y that turned it into a compactification of X with the desired properties. The proof given in Theorem 3.13(3) below can be applied to give an alternative proof to this theorem of McCartney in [23] (Theorem 3.6). Also, note that parts (1.d) and (2.d) below generalize and give alternative proofs for the theorems of K. D. Magill Jr. and T. Kimura in [18] and [11], respectively (Theorems 3.3 and 3.5, respectively). One should simply replace $\mathcal{P}$ and $\mathcal{Q}$, respectively, by compactness and regularity and note that, for any compact subset A of a locally compact space X there exists a continuous $f : X \to \mathbb{I}$ such that $f[A] \subseteq \{0\}$, and that $f^{-1}[[0, r]]$ is compact for any $r \in (0, 1)$.

THEOREM 3.13. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff space with $\mathcal{Q}$.*

(1) *Let $n \in \mathbb{N}$. The following are equivalent:*

   (a) *$\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ contains an element with n-point remainder.*
   (b) *$\mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(X)$ contains an element with n-point remainder.*
   (c) *X is locally-$\mathcal{P}$ and $X = K \cup U_1 \cup \cdots \cup U_n$ where $K, U_1, \ldots, U_n$ are pairwise disjoint, each $U_1, \ldots, U_n$ is open in X with non-$\mathcal{P}$ closure and $\mathrm{bd}_X\,K \subseteq Z \subseteq C$ for some $Z \in \mathscr{Z}(X)$ and $C \in \mathrm{Coz}(X)$ such that $\mathrm{cl}_X\,C$ has $\mathcal{P}$.*
   (d) *X is locally-$\mathcal{P}$ and $X = U \cup Z_1 \cup \cdots \cup Z_n$ where $U, Z_1, \ldots, Z_n$ are pairwise disjoint, $\mathrm{cl}_X\,U$ has $\mathcal{P}$ and each $Z_1, \ldots, Z_n \in \mathscr{Z}(X)$ is non-$\mathcal{P}$.*

(2) *The following are equivalent:*

   (a) *$\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ contains an element with countable remainder.*
   (b) *$\mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(X)$ contains an element with countable remainder.*
   (c) *X is locally-$\mathcal{P}$ and there exists a bijectively indexed collection $\{U_n : n \in \mathbb{N}\}$ of pairwise disjoint open subsets of X, each with non-$\mathcal{P}$ closure and such that for*

any $n \in \mathbb{N}$ there exist some $Z_n \in \mathscr{Z}(X)$ and $C_n \in \operatorname{Coz}(X)$ such that $\operatorname{cl}_X C_n$ has $\mathcal{P}$ and $\operatorname{bd}_X U_n \subseteq Z_n \subseteq C_n$.

(d) *X is locally-$\mathcal{P}$ and there exists a bijectively indexed collection $\{Z_n : n \in \mathbb{N}\}$ of non-$\mathcal{P}$ zero-sets of X such that $X = Z_1 \supseteq Z_2 \supseteq \cdots$ and such that for any $n \in \mathbb{N}$ there exist non-$\mathcal{P}$ sets $S_n \in \mathscr{Z}(X)$ and $K_n \subseteq X$ such that $Z_n \setminus Z_{n+1} = S_n \cup K_n$, where $K_n \subseteq T_n \subseteq C_n$ for some $T_n \in \mathscr{Z}(X)$ and $C_n \in \operatorname{Coz}(X)$ such that $\operatorname{cl}_X C_n$ has $\mathcal{P}$.*

(3) *Let $0 < \sigma < \Omega$ and let $n \in \mathbb{N}$. The following are equivalent:*

(a) *$\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ contains an element with countable remainder of type (σ, n).*

(b) *$\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$ contains an element with countable remainder of type (σ, n).*

(c) *X is locally-$\mathcal{P}$ and there exists a family $\{\mathscr{U}_\zeta : \zeta \leq \sigma\}$ of collections of pairwise disjoint non-empty open subset of X satisfying the following:*

(i) *For any $\zeta < \sigma$, $\operatorname{card}(\mathscr{U}_\zeta) = \aleph_0$ and $\operatorname{card}(\mathscr{U}_\sigma) = n$.*

(ii) *For any $\zeta \leq \sigma$ and $U \in \mathscr{U}_\zeta$ there exist some $Z \in \mathscr{Z}(X)$ and $C \in \operatorname{Coz}(X)$ such that $\operatorname{cl}_X C$ has $\mathcal{P}$ and $\operatorname{bd}_X U \subseteq Z \subseteq C$.*

(iii) *For any $\zeta \leq \sigma$, $U \in \mathscr{U}_\zeta$ and finite $\mathscr{V} \subseteq \bigcup\{\mathscr{U}_\eta : \eta < \zeta\}$ the set $\operatorname{cl}_X U \setminus \bigcup \mathscr{V}$ is non-$\mathcal{P}$.*

(iv) *For any $\zeta < \eta \leq \sigma$, $U \in \mathscr{U}_\zeta$ and $V \in \mathscr{U}_\eta$ there exist some $Z \in \mathscr{Z}(X)$ such that Z has $\mathcal{P}$ and a finite $\mathscr{V} \subseteq \bigcup\{\mathscr{U}_\xi : \xi < \zeta\}$ such that either*

$$\operatorname{cl}_X U \setminus \Big(V \cup \bigcup \mathscr{V}\Big) \subseteq Z \text{ or } (\operatorname{cl}_X U \cap \operatorname{cl}_X V) \setminus \bigcup \mathscr{V} \subseteq Z.$$

(v) *For any $\zeta < \eta \leq \sigma$ and $U \in \mathscr{U}_\eta$ there exists an infinite $\mathscr{V} \subseteq \mathscr{U}_\zeta$ such that for any $V \in \mathscr{V}$ there exist some $Z \in \mathscr{Z}(X)$ which has $\mathcal{P}$ and a finite $\mathscr{W} \subseteq \bigcup\{\mathscr{U}_\xi : \xi < \zeta\}$ such that $\operatorname{cl}_X V \setminus (U \cup \bigcup \mathscr{W}) \subseteq Z$.*

Proof. (1). By Lemma 3.7 it follows that (1.a) and (1.b) are equivalent.

(1.a) *implies* (1.c). Consider some $Y \in \mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ with an n-point remainder $Y \setminus X = \{p_1, \ldots, p_n\}$. Let $V_1, \ldots, V_n$ be pairwise disjoint open neighborhoods of $p_1, \ldots, p_n$ in βY, respectively. Let $\phi : \beta X \to \beta Y$ denote the continuous extension of id_X. Let $i = 1, \ldots, n$. Let $f_i : \beta X \to \mathbb{I}$ be continuous with

$$f_i\big[\phi^{-1}(p_i)\big] \subseteq \{0\} \quad \text{and} \quad f_i\big[\beta X \setminus \phi^{-1}[V_i]\big] \subseteq \{1\}.$$

Let

$$U_i = f_i^{-1}\big[[0, 1/2)\big] \cap X \quad \text{and} \quad K = X \setminus \bigcup_{i=1}^{n} U_i.$$

Then $X = K \cup U_1 \cup \cdots \cup U_n$ and since $U_i \subseteq \phi^{-1}[V_i]$ (and V_i's are pairwise disjoint) the sets $K, U_1, \ldots, U_n$ are pairwise disjoint. To show that $\operatorname{cl}_X U_i$ has $\mathcal{P}$ suppose the contrary. Let

$$S = f_i^{-1}\big[[0, 1/3]\big] \cap X \in \mathscr{Z}(X).$$

Then S has $\mathcal{P}$, as it is closed in $\operatorname{cl}_X U_i$. Therefore

$$\phi^{-1}(p_i) \subseteq f_i^{-1}\big[[0, 1/3)\big] \subseteq \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X}\big(f_i^{-1}\big[[0, 1/3]\big] \cap X\big) = \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} S \subseteq \lambda_{\mathcal{P}} X.$$

By Lemma 2.8 the space X is locally-$\mathcal{P}$ and $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X]$. Now again by Lemma 2.8 and since by the above

$$\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X] \setminus \phi^{-1}(p_i) = \phi^{-1}\big[(Y \setminus X) \setminus \{p_i\}\big] = \phi^{-1}\big[\big(Y \setminus \{p_i\}\big) \setminus X\big],$$

the extension $Y \setminus \{p_i\}$ of X has $\mathcal{P}$. This contradicts the minimality of Y. Let

$$Z = X \cap \bigcup_{i=1}^{n} f_i^{-1}(1/2) \in \mathscr{Z}(X) \quad \text{and} \quad C = X \cap \bigcup_{i=1}^{n} f_i^{-1}\big[(1/3, 2/3)\big] \in \operatorname{Coz}(X).$$

Arguing as in the proof of Lemma 3.7 ((1.a)⇒(1.c)) we have

$$\phi^{-1}(p_i) \setminus \lambda_{\mathcal{P}} X = \phi^{-1}[V_i] \setminus \lambda_{\mathcal{P}} X.$$

Therefore by the definition of f_i we have

$$\operatorname{cl}_{\beta X} C \subseteq \bigcup_{i=1}^{n} f_i^{-1}\big[[1/3, 2/3]\big] \subseteq \bigcup_{i=1}^{n} \big(\phi^{-1}[V_i] \setminus \phi^{-1}(p_i)\big) \subseteq \lambda_{\mathcal{P}} X,$$

which by Lemma 2.4 implies that $\operatorname{cl}_X C$ has $\mathcal{P}$. Finally

$$\begin{aligned} \operatorname{bd}_X K &= \operatorname{cl}_X K \cap \operatorname{cl}_X (X \setminus K) \subseteq X \cap \bigcap_{j=1}^{n} f_j^{-1}\big[[1/2, 1]\big] \cap \bigcup_{i=1}^{n} f_i^{-1}\big[[0, 1/2]\big] \\ &= X \cap \bigcup_{i=1}^{n} \bigcap_{j=1}^{n} f_j^{-1}\big[[1/2, 1]\big] \cap f_i^{-1}\big[[0, 1/2]\big] \subseteq X \cap \bigcup_{i=1}^{n} f_i^{-1}(1/2), \end{aligned}$$

which implies that $\operatorname{bd}_X K \subseteq Z \subseteq C$.

(1.c) *implies* (1.d). First note that since $U_1, \dots, U_n$ are pairwise disjoint and open, for any distinct $i, j = 1, \dots, n$ we have $\operatorname{bd}_X U_i \cap U_j = \emptyset$. Let $i = 1, \dots, n$. Then

$$\operatorname{bd}_X U_i \subseteq (X \setminus U_i) \cap \bigcap_{i \neq j = 1}^{n} (X \setminus U_j) = X \setminus \bigcup_{j=1}^{n} U_j = K,$$

which combined with $\operatorname{bd}_X U_i \subseteq \operatorname{cl}_X U_i \subseteq \operatorname{cl}_X (X \setminus K)$ gives $\operatorname{bd}_X U_i \subseteq \operatorname{bd}_X K$. By Lemma 3.12 this implies that $\operatorname{cl}_{\beta X} U_i \setminus \lambda_{\mathcal{P}} X = \operatorname{Ex}_X U_i \setminus \lambda_{\mathcal{P}} X$; let H_i denote the last set. By Lemma 2.4 the set H_i is non-empty, as by our assumption $\operatorname{cl}_X U_i$ is non-$\mathcal{P}$. Let $f_i : \beta X \to \mathbb{I}$ be continuous with

$$f_i[H_i] \subseteq \{0\} \quad \text{and} \quad f_i[\beta X \setminus \operatorname{Ex}_X U_i] \subseteq \{1\}$$

for $i = 1, \dots, n-1$ and

$$f_n\Big[(\beta X \setminus \lambda_{\mathcal{P}} X) \setminus \bigcup_{i=1}^{n-1} H_i\Big] \subseteq \{0\} \quad \text{and} \quad f_n\Big[\bigcup_{i=1}^{n-1} \operatorname{cl}_{\beta X} U_i\Big] \subseteq \{1\}.$$

Let

$$Z_i = f_i^{-1}\big[[0, 1/2]\big] \cap X \in \mathscr{Z}(X) \quad \text{and} \quad U = X \setminus \bigcup_{i=1}^{n} Z_i.$$

By the definition of f_i we have $Z_i \subseteq \operatorname{Ex}_X U_i$ for any $i = 1, \dots, n-1$. Now since $\operatorname{Ex}_X U_i$'s are pairwise disjoint (as U_i's are; see Lemma 3.10) Z_i's are pairwise disjoint when $i =$

$1,\dots,n-1$, and therefore when $i=1,\dots,n$, as

$$Z_n \cap \bigcup_{i=1}^{n-1} \operatorname{cl}_{\beta X} U_i = \emptyset$$

and $Z_i \subseteq \operatorname{cl}_{\beta X} U_i$ for any $i=1,\dots,n-1$, as $\operatorname{Ex}_X U_i \subseteq \operatorname{cl}_{\beta X} U_i$ (see Lemma 3.10). Since

$$U = X \setminus \bigcup_{i=1}^{n} f_i^{-1}\big[[0,1/2]\big] \subseteq \beta X \setminus \bigcup_{i=1}^{n} f_i^{-1}\big[[0,1/2)\big] \subseteq \lambda_{\mathcal{P}} X$$

we have $\operatorname{cl}_{\beta X} U \subseteq \lambda_{\mathcal{P}} X$, and thus by Lemma 2.4 it follows that $\operatorname{cl}_X U$ has $\mathcal{P}$. To complete the proof we need to verify that Z_i is non-$\mathcal{P}$. But this follows easily, as otherwise

$$H_i \subseteq f_i^{-1}\big[[0,1/2)\big] \subseteq \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X}\big(f_i^{-1}\big[[0,1/2]\big] \cap X\big) = \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} Z_i \subseteq \lambda_{\mathcal{P}} X,$$

which contradicts the fact H_i is non-empty.

(1.d) *implies* (1.a). For any $i=1,\dots,n$ let $f_i : X \to \mathbb{I}$ be continuous with

$$f_i[Z_i] \subseteq \{0\} \quad \text{and} \quad f_i\Big[\bigcup_{k=1}^{i-1} Z_k \cup \bigcup_{k=i+1}^{n} Z_k\Big] \subseteq \{1\}$$

and let $F_i : \beta X \to \mathbb{I}$ be the continuous extension of f_i.

Claim. *Let $r \in (0,1)$ and let $i,j=1,\dots,n$ be distinct. Then*

$$\operatorname{cl}_{\beta X} f_i^{-1}\big[[0,r]\big] \cap \operatorname{cl}_{\beta X} f_j^{-1}\big[[0,r]\big] \subseteq \lambda_{\mathcal{P}} X.$$

Proof of the claim. Let $r<s<1$. By the definition of f_i's and since $Z_i \cap Z_j = \emptyset$ we have

$$S = f_i^{-1}\big[[0,s]\big] \cap f_j^{-1}\big[[0,s]\big] \subseteq (Z_i \cup U) \cap (Z_j \cup U) \subseteq U.$$

If $k=i,j$ then

$$\operatorname{cl}_{\beta X} f_k^{-1}\big[[0,r]\big] \subseteq F_k^{-1}\big[[0,r]\big] \subseteq F_k^{-1}\big[[0,s)\big] \subseteq \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} f_k^{-1}\big[[0,s]\big]$$

and therefore

$$\begin{aligned}\operatorname{cl}_{\beta X} f_i^{-1}\big[[0,r]\big] \cap \operatorname{cl}_{\beta X} f_j^{-1}\big[[0,r]\big] &\subseteq \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} f_i^{-1}\big[[0,s]\big] \cap \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} f_j^{-1}\big[[0,s]\big] \\ &= \operatorname{int}_{\beta X}\big(\operatorname{cl}_{\beta X} f_i^{-1}\big[[0,s]\big] \cap \operatorname{cl}_{\beta X} f_j^{-1}\big[[0,s]\big]\big) \\ &= \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X}\big(f_i^{-1}\big[[0,s]\big] \cap f_j^{-1}\big[[0,s]\big]\big) = \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} S.\end{aligned}$$

Note that $S \in \mathscr{Z}(X)$ has $\mathcal{P}$, as it is closed in $\operatorname{cl}_X U$, and thus $\operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} S \subseteq \lambda_{\mathcal{P}} X$.

Claim. *Let $r \in (0,1)$. Then*

$$\operatorname{cl}_{\beta X}\Big(\bigcap_{i=1}^{n} f_i^{-1}\big[[r,1]\big]\Big) \subseteq \lambda_{\mathcal{P}} X.$$

Proof of the claim. Let $0<t<r$ and let

$$T = \bigcap_{i=1}^{n} f_i^{-1}\big[[t,1]\big] \in \mathscr{Z}(X).$$

Then

$$\begin{aligned}\operatorname{cl}_{\beta X}\Big(\bigcap_{i=1}^{n} f_i^{-1}\big[[r,1]\big]\Big) &\subseteq \bigcap_{i=1}^{n} F_i^{-1}\big[[r,1]\big] \subseteq \bigcap_{i=1}^{n} F_i^{-1}\big[(t,1]\big] \subseteq \bigcap_{i=1}^{n} \operatorname{int}_{\beta X}\operatorname{cl}_{\beta X} f_i^{-1}\big[[t,1]\big]\\ &= \operatorname{int}_{\beta X}\Big(\bigcap_{i=1}^{n}\operatorname{cl}_{\beta X} f_i^{-1}\big[[t,1]\big]\Big) = \operatorname{int}_{\beta X}\operatorname{cl}_{\beta X}\Big(\bigcap_{i=1}^{n} f_i^{-1}\big[[t,1]\big]\Big)\\ &= \operatorname{int}_{\beta X}\operatorname{cl}_{\beta X} T.\end{aligned}$$

Now since

$$T = \bigcap_{i=1}^{n} f_i^{-1}\big[[t,1]\big] \subseteq \bigcap_{i=1}^{n}(X\setminus Z_i) \subseteq X\setminus\bigcup_{i=1}^{n} Z_i = U$$

and $\operatorname{cl}_X U$ has $\mathcal{P}$, its closed subset T has $\mathcal{P}$, and therefore $\operatorname{int}_{\beta X}\operatorname{cl}_{\beta X} T \subseteq \lambda_{\mathcal{P}} X$.

Now let $r \in (0,1)$ be fixed. Note that

$$\beta X\setminus\lambda_{\mathcal{P}}X = \Big(\operatorname{cl}_{\beta X}\Big(\bigcup_{i=1}^{n} f_i^{-1}\big[[0,r]\big]\Big)\cup\operatorname{cl}_{\beta X}\Big(\bigcap_{i=1}^{n} f_i^{-1}\big[[r,1]\big]\Big)\Big)\setminus\lambda_{\mathcal{P}}X$$

and thus by the above claim

$$\beta X\setminus\lambda_{\mathcal{P}}X = \operatorname{cl}_{\beta X}\Big(\bigcup_{i=1}^{n} f_i^{-1}\big[[0,r]\big]\Big)\setminus\lambda_{\mathcal{P}}X = \bigcup_{i=1}^{n}\big(\operatorname{cl}_{\beta X} f_i^{-1}\big[[0,r]\big]\setminus\lambda_{\mathcal{P}}X\big).$$

By the first claim it now follows that $\beta X\setminus\lambda_{\mathcal{P}}X$ is the union of n of its pairwise disjoint closed (and thus clopen) subsets which are also non-empty, as $Z_i \subseteq f_i^{-1}[[0,r]]$ for any $i = 1,\dots,n$, and therefore since Z_i is non-$\mathcal{P}$, using Lemma 2.4 we have

$$\emptyset \neq \operatorname{cl}_{\beta X} Z_i\setminus\lambda_{\mathcal{P}}X \subseteq \operatorname{cl}_{\beta X} f_i^{-1}\big[[0,r]\big]\setminus\lambda_{\mathcal{P}}X.$$

(2) (2.a) *implies* (2.c). Consider some $Y \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ with countable remainder. By Lemma 3.9 there exists a bijectively indexed collection $\{V_n : n\in\mathbb{N}\}$ of pairwise disjoint open subsets of βY such that $B_n = V_n\cap(Y\setminus X)$ is compact and non-empty for any $n\in\mathbb{N}$. Let $n\in\mathbb{N}$. Let $f_n : \beta X\to\mathbb{I}$ be continuous with

$$f_n\big[\phi^{-1}[B_n]\big] \subseteq \{0\} \quad\text{and}\quad f_n\big[\beta X\setminus\phi^{-1}[V_n]\big] \subseteq \{1\}$$

where $\phi : \beta X\to\beta Y$ is the continuous extension of id_X and define $U_n = f_n^{-1}[[0,1/2)]\cap X$. Then U_n's are pairwise disjoint open subsets of X. Also, $\operatorname{cl}_X U_n$ is non-$\mathcal{P}$, as otherwise, arguing as in (1.a)⇒(1.c) we have $f_n^{-1}[[0,1/3)] \subseteq \lambda_{\mathcal{P}}X$. Consider the subspace $Y' = Y\setminus B_n$ of Y. By Lemma 2.8 we have $\beta X\setminus\lambda_{\mathcal{P}}X \subseteq \phi^{-1}[Y\setminus X]$ and X is locally-$\mathcal{P}$. Since $\phi^{-1}[B_n] \subseteq f_n^{-1}[[0,1/3)]$ we have

$$\phi^{-1}[Y'\setminus X] = \phi^{-1}\big[(Y\setminus B_n)\setminus X\big] = \phi^{-1}\big[(Y\setminus X)\setminus B_n\big] = \phi^{-1}[Y\setminus X]\setminus\phi^{-1}[B_n] \supseteq \beta X\setminus\lambda_{\mathcal{P}}X.$$

Now since $Y'\setminus X = (Y\setminus X)\setminus V_n$ is compact, Lemma 2.8 implies that Y' has $\mathcal{P}$, contradicting the minimality of Y. Note that

$$\operatorname{bd}_X U_n = \operatorname{cl}_X U_n\cap\operatorname{cl}_X(X\setminus U_n) \subseteq f_n^{-1}\big[[0,1/2]\big]\cap f_n^{-1}\big[[1/2,1]\big]\cap X = f_n^{-1}(1/2)\cap X.$$

Therefore if

$$Z_n = f_n^{-1}(1/2)\cap X \in \mathscr{Z}(X) \quad\text{and}\quad C_n = f_n^{-1}\big[(1/3,2/3)\big]\cap X \in \operatorname{Coz}(X)$$

then $\mathrm{bd}_X U_n \subseteq Z_n \subseteq C_n$. Since $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X]$, arguing as in the proof of Lemma 3.7 ((1.a)⇒(1.c)), we have

$$\phi^{-1}[B_n] \setminus \lambda_{\mathcal{P}} X = \phi^{-1}[V_n] \setminus \lambda_{\mathcal{P}} X$$

and thus by the definition of f_n it follows that

$$\mathrm{cl}_{\beta X} C_n \subseteq f_n^{-1}\big[[1/3, 2/3]\big] \subseteq \phi^{-1}[V_n] \setminus \phi^{-1}[B_n] \subseteq \lambda_{\mathcal{P}} X.$$

By Lemma 2.4 this implies that $\mathrm{cl}_X C_n$ has $\mathcal{P}$.

(2.c) *implies* (2.a). Let $\{U_n : n \in \mathbb{N}\}$ satisfy the assumption of the theorem. Let $n \in \mathbb{N}$. By Lemma 3.12 we have $\mathrm{cl}_{\beta X} U_n \setminus \lambda_{\mathcal{P}} X = \mathrm{Ex}_X U_n \setminus \lambda_{\mathcal{P}} X$; let H_n denote the last set. Note that H_n is clopen in $\beta X \setminus \lambda_{\mathcal{P}} X$ and that H_n's are pairwise disjoint, as U_n's are (see Lemma 3.10). Also, H_n is non-empty, as otherwise $\mathrm{cl}_{\beta X} U_n \subseteq \lambda_{\mathcal{P}} X$, which by Lemma 2.4 implies that $\mathrm{cl}_X U_n$ has $\mathcal{P}$, contradicting our assumption.

(2.a) *implies* (2.d). By Lemma 3.7 the space X is locally-$\mathcal{P}$ and there exists a bijectively indexed sequence $H_1, H_2, \dots$ of pairwise disjoint non-empty clopen subsets of $\beta X \setminus \lambda_{\mathcal{P}} X$. Let $n \in \mathbb{N}$. Let $f_n : \beta X \to \mathbb{I}$ be continuous with

$$f_n\big[(\beta X \setminus \lambda_{\mathcal{P}} X) \setminus H_n\big] \subseteq \{0\} \quad \text{and} \quad f_n[H_n] \subseteq \{1\}$$

and let

$$Z_n = \bigcap_{i=1}^{n-1} f_i^{-1}\big[[0, 1/2]\big] \cap X \in \mathscr{Z}(\beta X)$$

with the empty intersection interpreted as βX. Clearly $X = Z_1 \supseteq Z_2 \supseteq \cdots$. Note that

$$H_n \subseteq (\beta X \setminus \lambda_{\mathcal{P}} X) \setminus \bigcup_{i=1}^{n-1} H_i \subseteq \bigcap_{i=1}^{n-1} \big((\beta X \setminus \lambda_{\mathcal{P}} X) \setminus H_i\big) \subseteq \bigcap_{i=1}^{n-1} f_i^{-1}\big[[0, 1/2)\big]. \tag{3.10}$$

To show that Z_n is non-$\mathcal{P}$, suppose the contrary. Then

$$\begin{aligned}
\bigcap_{i=1}^{n-1} f_i^{-1}\big[[0, 1/2)\big] &\subseteq \bigcap_{i=1}^{n-1} \mathrm{int}_{\beta X}\, \mathrm{cl}_{\beta X}\big(f_i^{-1}\big[[0, 1/2]\big] \cap X\big) \\
&= \mathrm{int}_{\beta X}\Big(\bigcap_{i=1}^{n-1} \mathrm{cl}_{\beta X}\big(f_i^{-1}\big[[0, 1/2]\big] \cap X\big)\Big) \\
&= \mathrm{int}_{\beta X}\, \mathrm{cl}_{\beta X}\Big(\bigcap_{i=1}^{n-1} f_i^{-1}\big[[0, 1/2]\big] \cap X\Big) = \mathrm{int}_{\beta X}\, \mathrm{cl}_{\beta X}\, Z_n \subseteq \lambda_{\mathcal{P}} X
\end{aligned}$$

and therefore $H_n \subseteq \lambda_{\mathcal{P}} X$, which is a contradiction, as $H_n \subseteq \beta X \setminus \lambda_{\mathcal{P}} X$ is non-empty. Now

$$\begin{aligned}
Z_n \setminus Z_{n+1} &= \Big(\bigcap_{i=1}^{n-1} f_i^{-1}\big[[0, 1/2]\big] \cap X\Big) \setminus \Big(\bigcap_{i=1}^{n} f_i^{-1}\big[[0, 1/2]\big] \cap X\Big) \\
&= \big(\beta X \setminus f_n^{-1}\big[[0, 1/2]\big]\big) \cap \bigcap_{i=1}^{n-1} f_i^{-1}\big[[0, 1/2]\big] \cap X
\end{aligned}$$

$$= f_n^{-1}\big[(1/2,1]\big] \cap \bigcap_{i=1}^{n-1} f_i^{-1}\big[[0,1/2]\big] \cap X$$

$$= \big(f_n^{-1}\big[(1/2,2/3)\big] \cup f_n^{-1}\big[[2/3,1]\big]\big) \cap \bigcap_{i=1}^{n-1} f_i^{-1}\big[[0,1/2]\big] \cap X = K_n \cup S_n$$

where

$$K_n = f_n^{-1}\big[(1/3,2/3)\big] \cap \bigcap_{i=1}^{n-1} f_i^{-1}\big[[0,1/2]\big] \cap X,$$

$$S_n = f_n^{-1}\big[[2/3,1]\big] \cap \bigcap_{i=1}^{n-1} f_i^{-1}\big[[0,1/2]\big] \cap X.$$

Then $S_n \in \mathscr{Z}(X)$ and $K_n \subseteq T_n \subseteq C_n$, where

$$T_n = f_n^{-1}\big[[1/2,2/3]\big] \cap X \in \mathscr{Z}(X) \quad \text{and} \quad C_n = f_n^{-1}\big[(1/3,3/4)\big] \cap X \in \operatorname{Coz}(X).$$

Since

$$\operatorname{cl}_{\beta X} C_n = \operatorname{cl}_{\beta X}\big(f_n^{-1}\big[(1/3,3/4)\big] \cap X\big) \subseteq f_n^{-1}\big[[1/3,3/4]\big] \subseteq \lambda_{\mathcal{P}} X$$

by Lemma 2.4 the set $\operatorname{cl}_X C_n$ has $\mathcal{P}$. If S_n has $\mathcal{P}$, then using (3.10) and arguing as above,

$$H_n \subseteq f_n^{-1}\big[(2/3,1]\big] \cap \bigcap_{i=1}^{n-1} f_i^{-1}\big[[0,1/2)\big] \subseteq \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} S_n \subseteq \lambda_{\mathcal{P}} X$$

which as we argued above is a contradiction. Thus S_n is non-$\mathcal{P}$. Finally, note that as argued above $H_n \subseteq \operatorname{cl}_{\beta X} S_n$, and therefore S_n is non-empty. This implies that $Z_n \setminus Z_{n+1}$ is non-empty and thus Z_n's are bijectively indexed.

(2.d) *implies* (2.a). Let $n \in \mathbb{N}$. Then

$$X = Z_1 = \bigcup_{i=1}^{n-1} (Z_i \setminus Z_{i+1}) \cup Z_n = \bigcup_{i=1}^{n-1} (S_i \cup K_i) \cup Z_n$$

and thus

$$\beta X = \bigcup_{i=1}^{n-1} (\operatorname{cl}_{\beta X} S_i \cup \operatorname{cl}_{\beta X} K_i) \cup \operatorname{cl}_{\beta X} Z_n.$$

By Lemma 2.14 we have $\operatorname{cl}_{\beta X} K_i \subseteq \operatorname{cl}_{\beta X} T_i \subseteq \lambda_{\mathcal{P}} X$ for any $i \in \mathbb{N}$. Therefore

$$\beta X \setminus \lambda_{\mathcal{P}} X = \bigcup_{i=1}^{n-1} (\operatorname{cl}_{\beta X} S_i \setminus \lambda_{\mathcal{P}} X) \cup (\operatorname{cl}_{\beta X} Z_n \setminus \lambda_{\mathcal{P}} X).$$

Since $S_1, \dots, S_{n-1}, Z_n \in \mathscr{Z}(X)$ are pairwise disjoint, their closures in βX also are pairwise disjoint. Thus by the above $\beta X \setminus \lambda_{\mathcal{P}} X$ is the union of n of its pairwise disjoint non-empty (as $S_1, \dots, S_{n-1}, Z_n$ are non-$\mathcal{P}$; see Lemma 2.4) closed (and therefore clopen) subsets. Lemma 3.7 now completes the proof.

(3). The equivalence of (3.a) and (3.b) follows from Lemma 3.7.

(3.a) *implies* (3.c). By Lemma 3.7 the space X is locally-$\mathcal{P}$ and there exists a family $\{\mathscr{H}_\zeta : \zeta \leq \sigma\}$ of collections of pairwise disjoint non-empty clopen subsets of $\beta X \setminus \lambda_{\mathcal{P}} X$ satisfying conditions (3.c.i)–(3.c.iv) of Lemma 3.7. For any $\zeta \leq \sigma$ let $\mathscr{H}_\zeta = \{H_i^\zeta : i \in J_\zeta\}$

be bijectively indexed. Let $J_\zeta = \mathbb{N}$ for any $\zeta < \sigma$ and $J_\sigma = \{1, \dots, n\}$. Also, for any $\zeta \le \sigma$ and $i \in J_\zeta$ let A_i^ζ be an open subset of βX such that $H_i^\zeta = A_i^\zeta \setminus \lambda_{\mathcal{P}} X$.

Claim. *For any $\zeta \le \sigma$ there exists a collection $\{W_i^\zeta : i \in J_\zeta\}$ of pairwise disjoint open subsets of βX such that $W_i^\zeta \setminus \lambda_{\mathcal{P}} X = H_i^\zeta$ for any $i \in J_\zeta$.*

Proof of the claim. Let $\zeta < \sigma$. We inductively define W_i^ζ's for $i \in J_\zeta$. Let W_1^ζ be an open subset of βX such that $H_1^\zeta \subseteq W_1^\zeta \subseteq \operatorname{cl}_{\beta X} W_1^\zeta \subseteq A_1^\zeta$. For an $m \in \mathbb{N}$, suppose inductively that the open subsets $W_1^\zeta, \dots, W_m^\zeta$ of βX are defined in such a way that

$$H_i^\zeta \subseteq W_i^\zeta \subseteq \operatorname{cl}_{\beta X} W_i^\zeta \subseteq A_i^\zeta \setminus \operatorname{cl}_{\beta X}\Big(\bigcup_{j=1}^{i-1} W_j^\zeta\Big)$$

for any $i = 1, \dots, m$. Note that $H_{m+1}^\zeta \cap \operatorname{cl}_{\beta X} W_i^\zeta = \emptyset$ for any $i = 1, \dots, m$. Let W_{m+1}^ζ be an open subset of βX such that

$$H_{m+1}^\zeta \subseteq W_{m+1}^\zeta \subseteq \operatorname{cl}_{\beta X} W_{m+1}^\zeta \subseteq A_{m+1}^\zeta \setminus \operatorname{cl}_{\beta X}\Big(\bigcup_{j=1}^{m} W_j^\zeta\Big).$$

Similarly for the case when $\zeta = \sigma$.

Let $\zeta \le \sigma$ and $i \in J_\zeta$. Since W_i^ζ is an open neighborhoods of the compact (and thus closed) subset H_i^ζ of βX, there exists a continuous $f_i^\zeta : \beta X \to \mathbb{I}$ with

$$f_i^\zeta[H_i^\zeta] \subseteq \{0\} \quad \text{and} \quad f_i^\zeta[\beta X \setminus W_i^\zeta] \subseteq \{1\}.$$

Define

$$U_i^\zeta = (f_i^\zeta)^{-1}\big[[0, 1/2)\big] \cap X.$$

Let $\mathscr{U}_\zeta = \{U_i^\zeta : i \in J_\zeta\}$ for any $\zeta \le \sigma$. We verify that the family $\{\mathscr{U}_\zeta : \zeta \le \sigma\}$ satisfies (3.c.i)–(3.c.v). Let $\zeta \le \sigma$. Let $i \in J_\zeta$. Since

$$H_i^\zeta \subseteq (f_i^\zeta)^{-1}\big[[0, 1/2)\big] \subseteq \operatorname{cl}_{\beta X}(f_i^\zeta)^{-1}\big[[0, 1/2)\big] = \operatorname{cl}_{\beta X}\big((f_i^\zeta)^{-1}\big[[0, 1/2)\big] \cap X\big) = \operatorname{cl}_{\beta X} U_i^\zeta$$

and H_i^ζ is non-empty, U_i^ζ is non-empty. Also, for any distinct $i, j \in J_\zeta$ we have $U_i^\zeta \cap U_j^\zeta \subseteq W_i^\zeta \cap W_j^\zeta = \emptyset$. Thus the collection $\mathscr{U}_\zeta = \{U_i^\zeta : i \in J_\zeta\}$ is bijectively indexed and consists of pairwise disjoint non-empty open subsets of X. Note that (3.c.i) holds trivially.

To show (3.c.ii) note that for any $\zeta \le \sigma$ and $i \in J_\zeta$ we have

$$\operatorname{bd}_X U_i^\zeta = \operatorname{cl}_X U_i^\zeta \cap \operatorname{cl}_X(X \setminus U_i^\zeta) \subseteq (f_i^\zeta)^{-1}\big[[0, 1/2]\big] \cap (f_i^\zeta)^{-1}\big[[1/2, 1]\big] \cap X \subseteq (f_i^\zeta)^{-1}(1/2).$$

Thus

$$\operatorname{cl}_{\beta X} \operatorname{bd}_X U_i^\zeta \setminus \lambda_{\mathcal{P}} X \subseteq (f_i^\zeta)^{-1}(1/2) \setminus \lambda_{\mathcal{P}} X \subseteq (W_i^\zeta \setminus H_i^\zeta) \setminus \lambda_{\mathcal{P}} X = \emptyset,$$

which by Lemma 3.8 shows (3.c.ii).

Next, we verify (3.c.iii). Let $\zeta \le \sigma$, $U \in \mathscr{U}_\zeta$ and $\mathscr{V} \subseteq \bigcup\{\mathscr{U}_\eta : \eta < \zeta\}$ be finite. Suppose to the contrary that $\operatorname{cl}_X U \setminus \bigcup \mathscr{V}$ has $\mathcal{P}$. Let $U = U_i^\zeta$ for some $i \in J_\zeta$ and $\mathscr{V} = \{U_{j_1}^{\eta_1}, \dots, U_{j_m}^{\eta_m}\}$, where m is a non-negative integer, and $\eta_k < \zeta$ and $j_k \in J_{\eta_k}$ for any $k = 1, \dots, m$. Let

$$S = \Big((f_i^\zeta)^{-1}\big[[0, 1/3]\big] \setminus \bigcup_{k=1}^{m} (f_{j_k}^{\eta_k})^{-1}\big[[0, 1/2)\big]\Big) \cap X \in \mathscr{Z}(X).$$

By the above $(f_i^\zeta)^{-1}[[0,1/3]] \cap X \subseteq \mathrm{cl}_X U_i^\zeta$. Thus $S \subseteq \mathrm{cl}_X U \setminus \bigcup \mathscr{V}$ and then S, being closed in the last set, has $\mathcal{P}$. This implies that

$$H_i^\zeta \setminus \bigcup_{k=1}^m W_{j_k}^{\eta_k} \subseteq (f_i^\zeta)^{-1}\big[[0,1/3)\big] \setminus \bigcup_{k=1}^m (f_{j_k}^{\eta_k})^{-1}\big[[0,1/2]\big] \subseteq \mathrm{int}_{\beta X}\,\mathrm{cl}_{\beta X}\, S \subseteq \lambda_{\mathcal{P}} X.$$

Therefore

$$H_i^\zeta \setminus \bigcup\{G \in \mathscr{H}_\eta : \eta < \zeta\} \subseteq H_i^\zeta \setminus \bigcup_{k=1}^m H_{j_k}^{\eta_k} = \Big(H_i^\zeta \setminus \bigcup_{k=1}^m W_{j_k}^{\eta_k}\Big) \cap (\beta X \setminus \lambda_{\mathcal{P}} X) = \emptyset,$$

which contradicts (3.c.ii) of Lemma 3.7. Thus (3.c.iii) holds.

To show (3.c.iv) let $\zeta < \eta \leq \sigma$, $U \in \mathscr{U}_\zeta$ and $V \in \mathscr{U}_\eta$. Then $U = U_i^\zeta$ and $V = U_j^\eta$ for some $i \in J_\zeta$ and $j \in J_\eta$. By (3.c.iii) of Lemma 3.7 we have either

$$H_i^\zeta \subseteq H_j^\eta \cup \bigcup\{F \in \mathscr{H}_\xi : \xi < \zeta\} \tag{3.11}$$

or

$$H_i^\zeta \cap H_j^\eta \subseteq \bigcup\{F \in \mathscr{H}_\xi : \xi < \zeta\}. \tag{3.12}$$

We consider the following cases. For simplicity of notation let $r = 1/2$. Note that by the definition of f_i^ζ for any $\zeta \leq \sigma$ and $i \in J_\zeta$ we have

$$H_i^\zeta \subseteq (f_i^\zeta)^{-1}\big[[0,r)\big] \setminus \lambda_{\mathcal{P}} X \subseteq (f_i^\zeta)^{-1}\big[[0,r]\big] \setminus \lambda_{\mathcal{P}} X \subseteq W_i^\zeta \setminus \lambda_{\mathcal{P}} X = H_i^\zeta$$

and therefore

$$(f_i^\zeta)^{-1}\big[[0,r]\big] \setminus \lambda_{\mathcal{P}} X = (f_i^\zeta)^{-1}\big[[0,r)\big] \setminus \lambda_{\mathcal{P}} X = H_i^\zeta.$$

Case 1. If (3.11) holds then by compactness $H_i^\zeta \subseteq H_j^\eta \cup H_{k_1}^{\xi_1} \cup \cdots \cup H_{k_m}^{\xi_m}$, where m is a non-negative integer, $\xi_l < \zeta$ and $k_l \in J_{\xi_l}$ for any $l = 1, \ldots, m$. We have

$$\mathrm{cl}_{\beta X}\Big(\mathrm{cl}_X U_i^\zeta \setminus \Big(U_j^\eta \cup \bigcup_{l=1}^m U_{k_l}^{\xi_l}\Big)\Big) \subseteq (f_i^\zeta)^{-1}\big[[0,r]\big] \setminus \Big((f_j^\eta)^{-1}\big[[0,r)\big] \cup \bigcup_{l=1}^m (f_{k_l}^{\xi_l})^{-1}\big[[0,r)\big]\Big)$$

and thus

$$\mathrm{cl}_{\beta X}\Big(\mathrm{cl}_X U_i^\zeta \setminus \Big(U_j^\eta \cup \bigcup_{l=1}^m U_{k_l}^{\xi_l}\Big)\Big) \setminus \lambda_{\mathcal{P}} X \subseteq H_i^\zeta \setminus \Big(H_j^\eta \cup \bigcup_{l=1}^m H_{k_l}^{\xi_l}\Big) = \emptyset.$$

Now by Lemma 3.8 it follows that

$$\mathrm{cl}_X U_i^\zeta \setminus \Big(U_j^\eta \cup \bigcup_{l=1}^m U_{k_l}^{\xi_l}\Big) \subseteq Z \subseteq C$$

for some $Z \in \mathscr{Z}(X)$ and $C \in \mathrm{Coz}(X)$, where $\mathrm{cl}_X C$, and therefore Z, has $\mathcal{P}$, as it is closed in $\mathrm{cl}_X C$. Thus (3.c.iv) holds in this case.

Case 2. If (3.12) holds then by compactness $H_i^\zeta \cap H_j^\eta \subseteq H_{k_1}^{\xi_1} \cup \cdots \cup H_{k_m}^{\xi_m}$, where m is a non-negative integer, $\xi_l < \zeta$ and $k_l \in J_{\xi_l}$ for any $l = 1, \ldots, m$. We have

$$\mathrm{cl}_{\beta X}\Big((\mathrm{cl}_X U_i^\zeta \cap \mathrm{cl}_X U_j^\eta) \setminus \bigcup_{l=1}^m U_{k_l}^{\xi_l}\Big) \subseteq \big((f_i^\zeta)^{-1}\big[[0,r]\big] \cap (f_j^\eta)^{-1}\big[[0,r]\big]\big) \setminus \bigcup_{l=1}^m (f_{k_l}^{\xi_l})^{-1}\big[[0,r)\big]$$

and thus

$$\mathrm{cl}_{\beta X}\Big((\mathrm{cl}_X U_i^\zeta \cap \mathrm{cl}_X U_j^\eta) \setminus \bigcup_{l=1}^m U_{k_l}^{\xi_l}\Big) \setminus \lambda_{\mathcal{P}} X \subseteq (H_i^\zeta \cap H_j^\eta) \setminus \bigcup_{l=1}^m H_{k_l}^{\xi_l} = \emptyset.$$

Now as in the previous case (3.c.iv) follows.

Finally, we verify (3.c.v). Let $\zeta < \eta \leq \sigma$ and $U \in \mathscr{U}_\eta$. Then $U = U_j^\eta$ for some $j \in J_\eta$. By (3.c.iv) of Lemma 3.7 there exists an infinite $J \subseteq J_\zeta$ such that

$$H_i^\zeta \subseteq H_j^\eta \cup \bigcup \{G \in \mathscr{H}_\xi : \xi < \zeta\}$$

for any $i \in J$. Now an argument similar to the one above shows that

$$\mathrm{cl}_X U_i^\zeta \setminus \Big(U_j^\eta \cup \bigcup_{l=1}^m U_{k_l}^{\xi_l} \Big) \subseteq Z$$

for some $Z \in \mathscr{Z}(X)$ which has $\mathcal{P}$, some non-negative integer m, some $\xi_l < \zeta$ and some $k_l \in J_{\xi_l}$ where $l = 1, \ldots, m$. Thus (3.c.v) holds.

(3.c) *implies* (3.a). To prove (3.a) we verify condition (3.c) of Lemma 3.7. Suppose that X is locally-$\mathcal{P}$ and there exists a family $\{\mathscr{U}_\zeta : \zeta \leq \sigma\}$ of collections of pairwise disjoint non-empty open subsets of X satisfying (3.c.i)–(3.c.v). For any $\zeta \leq \sigma$ let $\mathscr{U}_\zeta = \{U_i^\zeta : i \in J_\zeta\}$ be bijectively indexed. Then $\mathrm{card}(J_\zeta) = \aleph_0$ if $\zeta < \sigma$, and $\mathrm{card}(J_\sigma) = n$. Let $\zeta \leq \sigma$ and $i \in J_\zeta$. Define

$$H_i^\zeta = \mathrm{cl}_{\beta X} U_i^\zeta \setminus \lambda_\mathcal{P} X.$$

By Lemma 3.12 and (3.c.ii) we have $H_i^\zeta = \mathrm{Ex}_X U_i^\zeta \setminus \lambda_\mathcal{P} X$, which shows that H_i^ζ is clopen in $\beta X \setminus \lambda_\mathcal{P} X$. Also, H_i^ζ is non-empty, as otherwise $\mathrm{cl}_{\beta X} U_i^\zeta \subseteq \lambda_\mathcal{P} X$, which by Lemma 2.4 implies that $\mathrm{cl}_X U_i^\zeta$ has $\mathcal{P}$, contradicting (3.c.iii). Since $H_i^\zeta \subseteq \mathrm{Ex}_X U_i^\zeta$ and U_i^ζ's are pairwise disjoint, by Lemma 3.10 the sets H_i^ζ are also pairwise disjoint. For any $\zeta \leq \sigma$ let $\mathscr{H}_\zeta = \{H_i^\zeta : i \in J_\zeta\}$, which is bijectively indexed, as H_i^ζ's are non-empty, and for any distinct $i, j \in J_\zeta$ we have $H_i^\zeta \cap H_j^\zeta = \emptyset$. We verify that the family $\{\mathscr{H}_\zeta : \zeta \leq \sigma\}$ has the desired properties. Condition (3.c.i) of Lemma 3.7 holds trivially.

To prove condition (3.c.ii) of Lemma 3.7 let $H \in \mathscr{H}_\zeta$ for some $\zeta \leq \sigma$, and suppose to the contrary that

$$H \setminus \bigcup \{G \in \mathscr{H}_\eta : \eta < \zeta\} = \emptyset.$$

Then $H = H_i^\zeta$ for some $i \in J_\zeta$. By compactness $H_i^\zeta \subseteq H_{k_1}^{\eta_1} \cup \cdots \cup H_{k_m}^{\eta_m}$, where $m \in \mathbb{N}$, $\eta_l < \zeta$ and $k_l \in J_{\eta_l}$ for any $l = 1, \ldots, m$. We have

$$\mathrm{cl}_{\beta X} \Big(\mathrm{cl}_X U_i^\zeta \setminus \bigcup_{l=1}^m U_{k_l}^{\eta_l} \Big) \subseteq \mathrm{cl}_{\beta X} U_i^\zeta \setminus \bigcup_{l=1}^m \mathrm{Ex}_X U_{k_l}^{\eta_l}$$

and therefore

$$\mathrm{cl}_{\beta X} \Big(\mathrm{cl}_X U_i^\zeta \setminus \bigcup_{l=1}^m U_{k_l}^{\eta_l} \Big) \setminus \lambda_\mathcal{P} X \subseteq H_i^\zeta \setminus \bigcup_{l=1}^m H_{k_l}^{\eta_l} = \emptyset.$$

Lemma 2.4 now implies that $\mathrm{cl}_X U_i^\zeta \setminus (U_{k_1}^{\eta_1} \cup \cdots \cup U_{k_m}^{\eta_m})$ has $\mathcal{P}$. But this contradicts (3.c.iii).

Next, we show condition (3.c.iii) of Lemma 3.7. Suppose that $\zeta < \eta \leq \sigma$, $H \in \mathscr{H}_\zeta$ and $G \in \mathscr{H}_\eta$. Let $H = H_i^\zeta$ and $G = H_j^\eta$ where $i \in J_\zeta$ and $j \in J_\eta$. By (3.c.iv) there exist a finite $\mathscr{V} \subseteq \bigcup \{\mathscr{U}_\xi : \xi < \zeta\}$ and a $Z \in \mathscr{Z}(X)$ such that Z has $\mathcal{P}$, and either

$$\mathrm{cl}_X U_i^\zeta \setminus \Big(U_j^\eta \cup \bigcup \mathscr{V} \Big) \subseteq Z \tag{3.13}$$

or

$$(\mathrm{cl}_X U_i^\zeta \cap \mathrm{cl}_X U_j^\eta) \setminus \bigcup \mathscr{V} \subseteq Z. \tag{3.14}$$

Let $\mathscr{V} = \{U_{k_1}^{\xi_1}, \ldots, U_{k_m}^{\xi_m}\}$, where m is a non-negative integer, $\xi_l < \zeta$ and $k_l \in J_{\xi_l}$ for any $l = 1, \ldots, m$. We consider the following cases:

Case 1. If (3.13) holds then

$$\mathrm{Ex}_X U_i^\zeta \subseteq \mathrm{cl}_{\beta X} \mathrm{Ex}_X U_i^\zeta = \mathrm{cl}_{\beta X} U_i^\zeta \subseteq \mathrm{cl}_{\beta X}\Big(U_j^\eta \cup \bigcup_{l=1}^m U_{k_l}^{\xi_l} \cup Z\Big)$$

and thus

$$\mathrm{Ex}_X U_i^\zeta \setminus \Big(\mathrm{cl}_{\beta X} U_j^\eta \cup \bigcup_{l=1}^m \mathrm{cl}_{\beta X} U_{k_l}^{\xi_l}\Big) \subseteq \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} Z \subseteq \lambda_{\mathcal{P}} X.$$

From this it follows that

$$H \subseteq G \cup \bigcup_{l=1}^m H_{k_l}^{\xi_l} \subseteq G \cup \bigcup \{F \in \mathscr{H}_\xi : \xi < \zeta\}.$$

Case 2. If (3.14) holds then using Lemma 3.10 we have

$$\begin{aligned}\mathrm{Ex}_X U_i^\zeta \cap \mathrm{Ex}_X U_j^\eta &\subseteq \mathrm{cl}_{\beta X}(\mathrm{Ex}_X U_i^\zeta \cap \mathrm{Ex}_X U_j^\eta)\\ &= \mathrm{cl}_{\beta X} \mathrm{Ex}_X(U_i^\zeta \cap U_j^\eta) = \mathrm{cl}_{\beta X}(U_i^\zeta \cap U_j^\eta) \subseteq \mathrm{cl}_{\beta X}\Big(\bigcup_{l=1}^m U_{k_l}^{\xi_l} \cup Z\Big)\end{aligned}$$

and thus

$$(\mathrm{Ex}_X U_i^\zeta \cap \mathrm{Ex}_X U_j^\eta) \setminus \bigcup_{l=1}^m \mathrm{cl}_{\beta X} U_{k_l}^{\xi_l} \subseteq \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} Z \subseteq \lambda_{\mathcal{P}} X,$$

which yields

$$H \cap G \subseteq \bigcup_{l=1}^m H_{k_l}^{\xi_l} \subseteq \bigcup \{F \in \mathscr{H}_\xi : \xi < \zeta\}.$$

This shows condition (3.c.iii) of Lemma 3.7 in either case.

Finally, to show (3.c.iv) of Lemma 3.7 suppose that $\zeta < \eta \leq \sigma$ and $H \in \mathscr{H}_\eta$. Let $H = H_j^\eta$ for some $j \in J_\eta$. By (3.c.v) there exists an infinite $J \subseteq J_\zeta$ such that for any $i \in J$ there exist a $Z \in \mathscr{Z}(X)$ such that Z has $\mathcal{P}$ and a finite $\mathscr{W} \subseteq \bigcup \{\mathscr{U}_\xi : \xi < \zeta\}$ such that $\mathrm{cl}_X U_i^\zeta \setminus (U_j^\eta \cup \bigcup \mathscr{W}) \subseteq Z$. Arguing as above we obtain

$$H_i^\zeta \subseteq H \cup \bigcup \{F \in \mathscr{H}_\xi : \xi < \zeta\}.$$

Thus

$$\Big\{F \in \mathscr{H}_\zeta : F \subseteq H \cup \bigcup \{G \in \mathscr{H}_\xi : \xi < \zeta\}\Big\}$$

is infinite. ∎

The following generalizes a theorem of K. D. Magill Jr. in [19] (Theorem 3.4).

Corollary 3.14. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff space with $\mathcal{Q}$. The following are equivalent:*

(1) *$\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ contains an element with n-point remainder (equivalently, $\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$ contains an element with n-point remainder) for any $n \in \mathbb{N}$.*
(2) *$\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ contains an element with countable remainder (equivalently, $\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$ contains an element with countable remainder).*

Proof. This follows from Lemma 3.7 and the observation that $\beta X \setminus \lambda_{\mathcal{P}} X$ has an infinite number of components if and only if it has at least n components for any $n \in \mathbb{N}$. ∎

THEOREM 3.15. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff space.*

(1) *Let $n \in \mathbb{N}$. If X has a perfect image Y with $\mathcal{Q}$ such that $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(Y)$ ($\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(Y)$, respectively) contains an element with n-point remainder, then so does X.*
(2) *If X has a perfect image Y with $\mathcal{Q}$ such that $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(Y)$ ($\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(Y)$, respectively) contains an element with countable remainder, then so does X.*
(3) *Let $0 < \sigma < \Omega$ and let $n \in \mathbb{N}$. If X has a perfect image Y with $\mathcal{Q}$ such that $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(Y)$ ($\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(Y)$, respectively) contains an element with countable remainder of type (σ, n), then so does X.*

Proof. We prove the theorem in the case of minimal extensions. From this and Theorem 3.13 the result will then follow in the case of optimal extensions as well.

(1). Suppose that $f : X \to Y$ is a perfect surjective mapping such that Y has $\mathcal{Q}$ (thus X also has $\mathcal{Q}$, as $\mathcal{Q}$ is inverse invariant under perfect mappings) and that $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(Y)$ contains an element with n-point remainder where $n \in \mathbb{N}$. Note that Y (having a Tychonoff extension) is Tychonoff and thus by Theorem 3.13 the space Y is locally-$\mathcal{P}$ and $Y = K \cup U_1 \cup \cdots \cup U_n$, where $K, U_1, \ldots, U_n$ are pairwise disjoint, each $U_1, \ldots, U_n$ is open in Y with non-$\mathcal{P}$ closure and $\operatorname{bd}_Y K \subseteq Z \subseteq C$ for some $Z \in \mathscr{Z}(Y)$ and $C \in \operatorname{Coz}(Y)$ such that $\operatorname{cl}_Y C$ has $\mathcal{P}$. Then

$$X = f^{-1}[Y] = f^{-1}[K] \cup \bigcup_{i=1}^{n} f^{-1}[U_i]$$

and $f^{-1}[K], f^{-1}[U_1], \ldots, f^{-1}[U_n]$ are pairwise disjoint. We show that the closure of each open subset $f^{-1}[U_1], \ldots, f^{-1}[U_n]$ of X is non-$\mathcal{P}$. Suppose to the contrary that $\operatorname{cl}_X f^{-1}[U_i]$ has $\mathcal{P}$ for some $i = 1, \ldots, n$. Now since

$$f|\operatorname{cl}_X f^{-1}[U_i] : \operatorname{cl}_X f^{-1}[U_i] \to f\big[\operatorname{cl}_X f^{-1}[U_i]\big]$$

is a perfect surjective mapping and $\mathcal{P}$ is invariant under perfect mappings, $f[\operatorname{cl}_X f^{-1}[U_i]]$ has $\mathcal{P}$. Since f is surjective we have

$$U_i = f\big[f^{-1}[U_i]\big] \subseteq f\big[\operatorname{cl}_X f^{-1}[U_i]\big]$$

and since f is closed, it follows that $\operatorname{cl}_Y U_i \subseteq f[\operatorname{cl}_X f^{-1}[U_i]]$ and thus $\operatorname{cl}_Y U_i$, being closed in the latter, has $\mathcal{P}$. But this is a contradiction. Also,

$$\begin{aligned}\operatorname{bd}_X f^{-1}[K] &= \operatorname{cl}_X f^{-1}[K] \cap \operatorname{cl}_X\big(X \setminus f^{-1}[K]\big) = \operatorname{cl}_X f^{-1}[K] \cap \operatorname{cl}_X f^{-1}[Y \setminus K] \\ &\subseteq f^{-1}[\operatorname{cl}_Y K] \cap f^{-1}\big[\operatorname{cl}_Y(Y \setminus K)\big] \\ &= f^{-1}\big[\operatorname{cl}_Y K \cap \operatorname{cl}_Y(Y \setminus K)\big] = f^{-1}[\operatorname{bd}_Y K] \subseteq f^{-1}[Z] \subseteq f^{-1}[C]\end{aligned}$$

and $f^{-1}[Z] \in \mathscr{Z}(X)$ and $f^{-1}[C] \in \mathrm{Coz}(X)$. Note that since the mapping

$$f|f^{-1}[\mathrm{cl}_Y C] : f^{-1}[\mathrm{cl}_Y C] \to \mathrm{cl}_Y C$$

is perfect and surjective (since f is surjective), the set $\mathrm{cl}_Y C$ has $\mathcal{P}$, and (since $\mathcal{P}$ is inverse invariant under perfect mappings) the set $f^{-1}[\mathrm{cl}_Y C]$, and thus its closed subset $\mathrm{cl}_X f^{-1}[C]$, has $\mathcal{P}$. Finally, Lemma 2.5 implies that X is locally-$\mathcal{P}$. Thus the result follows from Theorem 3.13.

(2). This is analogous to part (1) using the characterization given in Theorem 3.13.

(3). Suppose that $f : X \to Y$ is a perfect surjective mapping such that Y has $\mathcal{Q}$ and that $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(Y)$ contains an element with countable remainder of type (σ, n). Note that as in part (1) it follows that X has $\mathcal{Q}$ and Y is Tychonoff. By Theorem 3.13 the space Y is locally-$\mathcal{P}$ and there exists a family $\{\mathscr{U}_\zeta : \zeta \leq \sigma\}$ of collections of pairwise disjoint non-empty open subsets of Y satisfying conditions (3.c.i)–(3.c.v) of that theorem. For any $\zeta \leq \sigma$ let $\mathscr{A}_\zeta = \{f^{-1}[U] : U \in \mathscr{U}_\zeta\}$. Then each $\mathscr{A}_\zeta$, where $\zeta \leq \sigma$, consists of pairwise disjoint non-empty (since f is surjective) open subsets of X. We verify that $\{\mathscr{A}_\zeta : \zeta \leq \sigma\}$ satisfies conditions (3.c.i)–(3.c.v) of Theorem 3.13. Condition (3.c.i) holds trivially: since f is surjective, each $\mathscr{A}_\zeta$ with $\zeta \leq \sigma$ is bijectively indexed and thus $\mathrm{card}(\mathscr{A}_\zeta) = \mathrm{card}(\mathscr{U}_\zeta)$ for any $\zeta \leq \sigma$. Condition (3.c.ii) follows by an argument similar to part (1) and the fact that $\{\mathscr{U}_\zeta : \zeta \leq \sigma\}$ satisfies a similar condition. To show condition (3.c.iii) suppose to the contrary that for some $\zeta \leq \sigma$, $U \in \mathscr{U}_\zeta$ and finite $\mathscr{V} \subseteq \bigcup\{\mathscr{U}_\eta : \eta < \zeta\}$ the set $\mathrm{cl}_X f^{-1}[U] \setminus f^{-1}[\bigcup \mathscr{V}]$ has $\mathcal{P}$. Since f is closed and surjective we have

$$\begin{aligned}\mathrm{cl}_Y U \setminus \bigcup \mathscr{V} &\subseteq \mathrm{cl}_Y f\big[f^{-1}[U]\big] \setminus \bigcup \mathscr{V} \\ &\subseteq f\big[\mathrm{cl}_X f^{-1}[U]\big] \setminus f\Big[f^{-1}\Big[\bigcup \mathscr{V}\Big]\Big] \subseteq f\Big[\mathrm{cl}_X f^{-1}[U] \setminus f^{-1}\Big[\bigcup \mathscr{V}\Big]\Big].\end{aligned}$$

But the latter has $\mathcal{P}$, as $\mathcal{P}$ is invariant under perfect mappings, thus its closed subset $\mathrm{cl}_Y U \setminus \bigcup \mathscr{V}$ has $\mathcal{P}$, which is a contradiction. To show condition (3.c.iv) suppose that $\zeta < \eta \leq \sigma$, $U \in \mathscr{U}_\zeta$ and $V \in \mathscr{U}_\eta$. Since $\{\mathscr{U}_\zeta : \zeta \leq \sigma\}$ satisfies a similar condition, there exist a $Z \in \mathscr{Z}(Y)$ which has $\mathcal{P}$, and a finite $\mathscr{V} \subseteq \bigcup\{\mathscr{U}_\xi : \xi < \zeta\}$ such that either

$$\mathrm{cl}_Y U \setminus \Big(V \cup \bigcup \mathscr{V}\Big) \subseteq Z \quad \text{or} \quad (\mathrm{cl}_Y U \cap \mathrm{cl}_Y V) \setminus \bigcup \mathscr{V} \subseteq Z.$$

Since $f|f^{-1}[Z] : f^{-1}[Z] \to Z$ is perfect and surjective (since f is surjective) and $\mathcal{P}$ is inverse invariant under perfect mappings, $f^{-1}[Z] \in \mathscr{Z}(X)$ has $\mathcal{P}$. In the first case

$$\mathrm{cl}_X f^{-1}[U] \setminus \Big(f^{-1}[V] \cup f^{-1}\Big[\bigcup \mathscr{V}\Big]\Big) \subseteq f^{-1}[\mathrm{cl}_Y U] \setminus \Big(f^{-1}[V] \cup f^{-1}\Big[\bigcup \mathscr{V}\Big]\Big) \subseteq f^{-1}[Z]$$

and in the second case

$$\begin{aligned}\big(\mathrm{cl}_X f^{-1}[U] \cap \mathrm{cl}_X f^{-1}[V]\big) \setminus f^{-1}\Big[\bigcup \mathscr{V}\Big] \\ \subseteq \big(f^{-1}[\mathrm{cl}_Y U] \cap f^{-1}[\mathrm{cl}_Y V]\big) \setminus f^{-1}\Big[\bigcup \mathscr{V}\Big] \subseteq f^{-1}[Z].\end{aligned}$$

The proof for condition (3.c.v) is analogous. Note that by Lemma 2.5 the space X is locally-$\mathcal{P}$. The result now follows. ∎

4. Compactification-like $\mathcal{P}$-extensions as partially ordered sets

In this chapter we consider classes of compactification-like $\mathcal{P}$-extensions of a Tychonoff space X as partially ordered sets. We define two partial orders $\leq_{\mathrm{inj}}$ and $\leq_{\mathrm{surj}}$ (besides $\leq$ itself) on the set of all extensions of X. These partial orders behave nicely when restricted to classes of compactification-like $\mathcal{P}$-extensions of X and their introduction leads to some interesting results which characterize compactification-like $\mathcal{P}$-extensions of X among all Tychonoff $\mathcal{P}$-extensions of X with compact remainder. We continue with the study of relationships between the order-structure of classes of compactification-like $\mathcal{P}$-extensions of X (partially ordered with $\leq$) and the topology of the subspace $\beta X \setminus \lambda_{\mathcal{P}} X$ of its outgrowth $\beta X \setminus X$. This generalizes a well known result of K. D. Magill Jr. in [20] which relates the order-structure of the set of all compactifications of a locally compact space X and the topology of $\beta X \setminus X$. We conclude this chapter with a result which characterizes the largest (with respect to $\leq$) compactification-like $\mathcal{P}$-extension of X. This largest element (which we explicitly introduce as a subspace of the Stone–Čech compactification βX of X) turns out to be also the largest among all Tychonoff $\mathcal{P}$-extensions of X with compact remainder.

We start with the following definition.

Definition 4.1. Let X be a space and let Y and Y' be extensions of X. We write $Y \leq_{\mathrm{inj}} Y'$ if there exists a continuous injective $f : Y' \to Y$ such that $f|X = \mathrm{id}_X$.

The relation $\leq_{\mathrm{inj}}$ defines a partial order on the set of all extensions of a space X. The following lemma (see also Lemma 4.6) is a counterpart of Lemma 2.13.

Lemma 4.2. *Let X be a Tychonoff space and let $Y_1, Y_2 \in \mathscr{E}_{\mathcal{P}}(X)$ be such that $Y_1 \leq Y_2$. The following are equivalent:*

(1) $Y_1 \leq_{\mathrm{inj}} Y_2$.
(2) *Any element of $\mathscr{F}(Y_1)$ contains at most one element of $\mathscr{F}(Y_2)$.*

Proof. Let $\phi_i : \beta X \to \beta Y_i$ where $i = 1, 2$, be the continuous extension of id_X. Since $Y_1 \leq Y_2$ there exists a continuous $f : Y_2 \to Y_1$ such that $f|X = \mathrm{id}_X$. Let $f_\beta : \beta Y_2 \to \beta Y_1$ be the continuous extension of f. As shown in the proof of Lemma 2.13 we have $f_\beta \phi_2 = \phi_1$ and $f[Y_2 \setminus X] \subseteq Y_1 \setminus X$.

(1) *implies* (2). Suppose that $f : Y_2 \to Y_1$ introduced above is moreover injective. Let $p \in Y_1 \setminus X$ and let $p_i \in Y_2 \setminus X$, where $i = 1, 2$, be such that $\phi_2^{-1}(p_i) \subseteq \phi_1^{-1}(p)$. Choose some $s_i \in \phi_2^{-1}(p_i)$ for any $i = 1, 2$ (such s_i's exist, as ϕ_2 is surjective). Then

$$f(p_1) = f_\beta(p_1) = f_\beta\big(\phi_2(s_1)\big) = \phi_1(s_1) = p = \phi_1(s_2) = f_\beta\big(\phi_2(s_2)\big) = f_\beta(p_2) = f(p_2),$$

which implies that $p_1 = p_2$. Thus $\phi_2^{-1}(p_1) = \phi_2^{-1}(p_2)$.

(2) *implies* (1). We show that the mapping $f : Y_2 \to Y_1$ introduced above is injective. Let $p_i \in Y_2 \setminus X$, where $i = 1, 2$, be such that $f(p_1) = f(p_2)$ and let $p \in Y_1 \setminus X$ denote their common value. Note that

$$\phi_2^{-1}(p_i) \subseteq \phi_2^{-1}\big[f^{-1}(p)\big] \subseteq \phi_2^{-1}\big[f_\beta^{-1}(p)\big] = (f_\beta \phi_2)^{-1}(p) = \phi_1^{-1}(p)$$

for any $i = 1, 2$, which by (2) implies that $\phi_2^{-1}(p_1) = \phi_2^{-1}(p_2)$ and therefore $p_1 = p_2$. ∎

NOTATION 4.3. Let R be a relation on a set X and let $Y \subseteq X$. Denote

$$R|Y = \big\{(y,x) \in R : y \in Y\big\}.$$

In the next result we give an order-theoretic characterization of $\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$. (Compare with its dual result Theorem 4.8 on $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$.) Recall that a subset A of a partially ordered set $(X, \leq)$ is said to be *cofinal* if for any $x \in X$ there exists some $a \in A$ with $x \leq a$.

THEOREM 4.4. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff space with $\mathcal{Q}$. Then*

(1) $\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X) = \{Y : Y$ *is maximal in* $(\mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X), \leq_{\rm inj})\}$.
(2) $\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$ *is the smallest cofinal subset of* $(\mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X), \leq_{\rm inj})$.
(3) $\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$ *is the unique cofinal subset of* $(\mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X), \leq_{\rm inj})$ *on which the two relations* $\leq_{\rm inj}$ *and* $=$ *coincide.*
(4) $\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$ *is the largest subset* $\mathscr{E}$ *of* $\mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$ *such that*

$$(\leq_{\rm inj}|\mathscr{E}) \subseteq =. \tag{4.1}$$

Proof. (2). To show that $\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$ is cofinal in $\mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$ with respect to $\leq_{\rm inj}$ let $Y \in \mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$. Let $\phi : \beta X \to \beta Y$ be the continuous extension of id_X. By Lemma 2.8 the space X is locally-$\mathcal{P}$ and $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X]$. Also, by Lemma 2.10 we have $X \subseteq \lambda_{\mathcal{P}} X$. Let

$$P = \big\{p \in Y \setminus X : \phi^{-1}(p) \setminus \lambda_{\mathcal{P}} X \neq \emptyset\big\}.$$

Form the quotient space T of βX by contracting each subset $\phi^{-1}(p) \setminus \lambda_{\mathcal{P}} X$ where $p \in P$ to a point t_p and denote by $q : \beta X \to T$ the quotient mapping. Arguing as in the proof of Theorem 2.15 ((1)⇒(2)) shows that T is compact. Consider the subspace $Z = X \cup q[\beta X \setminus \lambda_{\mathcal{P}} X]$ of T. Then Z is a Tychonoff extension of X with the compact remainder $Z \setminus X = q[\beta X \setminus \lambda_{\mathcal{P}} X]$. Note that T is a compactification of Z. Let $\psi : \beta X \to \beta Z$ and $f : \beta Z \to T$ be the continuous extensions of id_X and id_Z, respectively. Since the continuous mapping $f\psi : \beta X \to T$ coincides with q on X we have $f\psi = q$. By Lemma 2.8 and Theorem 2.15 to show that $Z \in \mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$ it suffices to show that $\psi^{-1}[Z \setminus X] = \beta X \setminus \lambda_{\mathcal{P}} X$. But this follows, as by Theorem 3.5.7 of [5] (and since βZ and T are compactifications of Z and f is continuous with $f|Z = \mathrm{id}_Z$) we have $f[\beta Z \setminus Z] = T \setminus Z$ and therefore

$$\psi^{-1}[Z \setminus X] = \psi^{-1}\big[f^{-1}[Z \setminus X]\big] = (f\psi)^{-1}[Z \setminus X] = q^{-1}[Z \setminus X] = \beta X \setminus \lambda_{\mathcal{P}} X.$$

Note that for any $p \in P$ (and again, since $f[\beta Z \setminus Z] = T \setminus Z$) we have

$$\psi^{-1}(t_p) = \psi^{-1}\big[f^{-1}(t_p)\big] = (f\psi)^{-1}(t_p) = q^{-1}(t_p) \subseteq \phi^{-1}(p) \setminus \lambda_{\mathcal{P}} X.$$

Define $g : Z \to Y$ by $g(t_p) = p$ when $p \in P$ and $g(x) = x$ when $x \in X$. By the proof of Lemma 2.13 ((2)⇒(1)) (note that $\psi^{-1}(t_p) \subseteq \phi^{-1}(p)$ for any $p \in P$) the mapping g is continuous, and by its definition, it is moreover injective. Thus $Y \leq_{\rm inj} Z$. This shows the cofinality of $\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$ in $\mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$ with respect to $\leq_{\rm inj}$.

To complete the proof we need to show that $\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$ is contained in every subset $\mathscr{S}$ of $\mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$ cofinal with respect to $\leq_{\rm inj}$. Indeed, let $Z \in \mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$. Then $Z \leq_{\rm inj} S$ for some $S \in \mathscr{S}$. We show that S and Z are equivalent extensions of X. To show this, by Lemma 2.13 it suffices to verify that $\mathscr{F}(S) = \mathscr{F}(Z)$. Let $\psi : \beta X \to \beta Z$ and $\varphi : \beta X \to \beta S$ be the continuous extensions of id_X. By Theorem 2.15 we have $\psi^{-1}[Z \setminus X] = \beta X \setminus \lambda_{\mathcal{P}} X$.

Now since $Z \leq S$ (as $Z \leq_{\mathrm{inj}} S$) by Lemma 2.13, for any $s \in S \setminus X$ we have $\varphi^{-1}(s) \subseteq \psi^{-1}(z)$ for some $z \in Z \setminus X$. Therefore $\varphi^{-1}[S \setminus X] \subseteq \beta X \setminus \lambda_{\mathcal{P}} X$ and thus since by Lemma 2.8 we have $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \varphi^{-1}[S \setminus X]$ it follows that $\varphi^{-1}[S \setminus X] = \beta X \setminus \lambda_{\mathcal{P}} X$. Now let $s' \in S \setminus X$. Then by Lemma 2.13 (and since $Z \leq S$) we have $\varphi^{-1}(s') \subseteq \psi^{-1}(z')$ for some $z' \in Z \setminus X$. Suppose that $\varphi^{-1}(s') \neq \psi^{-1}(z')$. There exists some $s'' \in S \setminus X$ such that $s'' \neq s'$ and $\varphi^{-1}(s'') \cap \psi^{-1}(z')$ is non-empty. Thus $\varphi^{-1}(s'') \subseteq \psi^{-1}(z')$. But by Lemma 4.2 this implies that $\varphi^{-1}(s'') = \varphi^{-1}(s')$, which is a contradiction, as $s'' \neq s'$ (and φ is surjective). This shows that $\varphi^{-1}(s') = \psi^{-1}(z')$. Therefore $\mathscr{F}(S) \subseteq \mathscr{F}(Z)$. To show the reverse inclusion note that for any $z \in Z \setminus X$, since $\psi^{-1}(z) \subseteq \beta X \setminus \lambda_{\mathcal{P}} X$ the set $\psi^{-1}(z) \cap \varphi^{-1}(s)$ is non-empty for some $s \in S \setminus X$, and thus $\psi^{-1}(z) = \varphi^{-1}(s)$, as $\mathscr{F}(S) \subseteq \mathscr{F}(Z)$ (and the elements of $\mathscr{F}(Z)$ are pairwise disjoint). Therefore $\mathscr{F}(Z) \subseteq \mathscr{F}(S)$, which shows the equality in the latter. By Lemma 2.13 we have $S \leq Z$ and $Z \leq S$, which implies that Z and S are equivalent. Thus $Z \in \mathscr{S}$. This shows that $\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X) \subseteq \mathscr{S}$.

(1). By Theorem 2.15 any element of $\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$ is maximal in $\mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$ with respect to $\leq_{\mathrm{inj}}$. The converse follows from part (2), because if $Y \in \mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$ is maximal with respect to $\leq_{\mathrm{inj}}$ then $Y \leq_{\mathrm{inj}} T$ for some $T \in \mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$, which yields $Y = T$ and thus $Y \in \mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$.

(3). Note that by (2) the set $\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$ is cofinal in $\mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$ with respect to $\leq_{\mathrm{inj}}$. Also, by (1) the relations $\leq_{\mathrm{inj}}$ and $=$ coincide on $\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$. Now let $\mathscr{E}$ be a subset of $\mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$ cofinal with respect to $\leq_{\mathrm{inj}}$ and such that $\leq_{\mathrm{inj}}$ and $=$ coincide on $\mathscr{E}$. Let $S \in \mathscr{E}$. By the cofinality of $\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$ (with respect to $\leq_{\mathrm{inj}}$) we have $S \leq_{\mathrm{inj}} T$ for some $T \in \mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$, and by the cofinality of $\mathscr{E}$ we have $T \leq_{\mathrm{inj}} Z$ for some $Z \in \mathscr{E}$. Then $S \leq_{\mathrm{inj}} Z$ and thus (since $S, Z \in \mathscr{E}$) we have $S = Z$. Therefore $S = T$, which implies that $S \in \mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$. This shows that $\mathscr{E} \subseteq \mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$. Note that by (2) we have also $\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X) \subseteq \mathscr{E}$, which proves equality.

(4). By part (1) the set $\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$ satisfies (4.1). Now let $\mathscr{E}$ be a subset of $\mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$ which satisfies (4.1). Let $S \in \mathscr{E}$. By part (2) the set $\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$ is cofinal in $\mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$ with respect to $\leq_{\mathrm{inj}}$. Therefore there exists some $T \in \mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$ such that $S \leq_{\mathrm{inj}} T$. By (4.1) we have $S = T$, which implies that $S \in \mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$. Thus $\mathscr{E} \subseteq \mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$. ∎

Definition 4.5. Let X be a space and let Y and Y' be extensions of X. We let $Y \leq_{\mathrm{surj}} Y'$ if there exists a continuous surjective $f : Y' \to Y$ such that $f|X = \mathrm{id}_X$.

The relation $\leq_{\mathrm{surj}}$ defines a partial order on the set of all extensions of a space X.

Lemma 4.6. *Let X be a Tychonoff space and let $Y_1, Y_2 \in \mathscr{E}_{\mathcal{P}}(X)$ be such that $Y_1 \leq Y_2$. The following are equivalent:*

(1) $Y_1 \leq_{\mathrm{surj}} Y_2$.
(2) *Any element of $\mathscr{F}(Y_1)$ contains at least one element of $\mathscr{F}(Y_2)$.*

Proof. Let $\phi_i : \beta X \to \beta Y_i$, where $i = 1, 2$, be the continuous extension of id_X. Since $Y_1 \leq Y_2$ there exists a continuous $f : Y_2 \to Y_1$ such that $f|X = \mathrm{id}_X$. Let $f_\beta : \beta Y_2 \to \beta Y_1$ be the continuous extension of f. As shown in the proof of Lemma 2.12 of [14] we have $f_\beta \phi_2 = \phi_1$ and $f[Y_2 \setminus X] \subseteq Y_1 \setminus X$.

(1) *implies* (2). Suppose that $f : Y_2 \to Y_1$ introduced above is moreover surjective. Let $p_1 \in Y_1 \setminus X$ and let $p_2 \in Y_2 \setminus X$ be such that $f(p_2) = p_1$. Then

$$\phi_2^{-1}(p_2) \subseteq \phi_2^{-1}\big[f^{-1}(p_1)\big] \subseteq \phi_2^{-1}\big[f_\beta^{-1}(p_1)\big] = (f_\beta \phi_2)^{-1}(p_1) = \phi_1^{-1}(p_1).$$

(2) *implies* (1). We show that $f : Y_2 \to Y_1$ is surjective. Let $p_1 \in Y_1 \setminus X$. Let $p_2 \in Y_2 \setminus X$ be such that $\phi_2^{-1}(p_2) \subseteq \phi_1^{-1}(p_1)$. Choose an $s \in \phi_2^{-1}(p_2)$ (ϕ_2 is surjective). Then since

$$s \in \phi_1^{-1}(p_1) = (f_\beta \phi_2)^{-1}(p_1) = \phi_2^{-1}\big[f_\beta^{-1}(p_1)\big]$$

we have $p_2 = \phi_2(s) \in f_\beta^{-1}(p_1)$, which implies that $f(p_2) = f_\beta(p_2) = p_1$. ∎

LEMMA 4.7. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff space with $\mathcal{Q}$. Let $Y \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ and let $T \in \mathscr{E}_{\mathcal{P}}^{\mathcal{Q}}(X)$ be such that $T \leq_{\rm surj} Y$. Then $T \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$.*

Proof. Let $F \in \mathscr{F}(T)$. By Lemma 4.6 there exists $G \in \mathscr{F}(Y)$ such that $G \subseteq F$. By Theorem 2.11 the set $G \setminus \lambda_{\mathcal{P}} X$ is non-empty and thus $F \setminus \lambda_{\mathcal{P}} X$ is non-empty. By Theorem 2.11 the result follows. ∎

In the next result we give an order-theoretic characterization of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$.

THEOREM 4.8. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff space with $\mathcal{Q}$. Then*

(1) *$\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ is the largest cofinal subset of $(\mathscr{E}_{\mathcal{P}}^{\mathcal{Q}}(X), \leq)$ on which the relations $\leq$ and $\leq_{\rm surj}$ coincide.*
(2) *$\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ is the largest subset of $(\mathscr{E}_{\mathcal{P}}^{\mathcal{Q}}(X), \leq_{\rm surj})$ in which $\mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(X)$ is cofinal.*
(3) *$\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ is the largest subset $\mathscr{E}$ of $\mathscr{E}_{\mathcal{P}}^{\mathcal{Q}}(X)$ such that*

$$(\leq | \mathscr{E}) \subseteq \leq_{\rm surj}. \tag{4.2}$$

(4) *$\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ is the smallest cofinal subset $\mathscr{E}$ of $(\mathscr{E}_{\mathcal{P}}^{\mathcal{Q}}(X), \leq)$ such that*

$$\big((\mathscr{E}_{\mathcal{P}}^{\mathcal{Q}}(X) \times \mathscr{E}) \cap \leq_{\rm surj}\big) \subseteq \mathscr{E} \times \mathscr{E}_{\mathcal{P}}^{\mathcal{Q}}(X). \tag{4.3}$$

Proof. (1). First we show that $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ is cofinal in $\mathscr{E}_{\mathcal{P}}^{\mathcal{Q}}(X)$ with respect to $\leq$. Let $Y \in \mathscr{E}_{\mathcal{P}}^{\mathcal{Q}}(X)$. Let $\phi : \beta X \to \beta Y$ be the continuous extension of id_X. Consider the subspace

$$T = X \cup \big\{p \in Y \setminus X : \phi^{-1}(p) \setminus \lambda_{\mathcal{P}} X \neq \emptyset\big\}$$

of Y. We show that $T \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ and $Y \leq T$. Obviously, T is a Tychonoff extension of X. By Lemma 2.8 we have $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X]$ (and X is locally-$\mathcal{P}$) and thus $T \setminus X = \phi[\beta X \setminus \lambda_{\mathcal{P}} X]$ is compact. Also, since βY is a compactification of T and by the definition of T we have $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[T \setminus X]$, again by Lemma 2.8 it follows that T has both $\mathcal{P}$ and $\mathcal{Q}$. Let $\psi : \beta X \to \beta T$ and $f : \beta T \to \beta Y$ be the continuous extensions of id_X and id_T, respectively. The continuous mappings $f\psi$ and ϕ agree on X, and therefore they are identical. Since βY is a compactification of T (and $f|T = \mathrm{id}_T$), by Theorem 3.5.7 of [5] we have $f[\beta T \setminus T] = \beta Y \setminus T$. Thus

$$\psi^{-1}(p) = \psi^{-1}\big[f^{-1}(p)\big] = (f\psi)^{-1}(p) = \phi^{-1}(p)$$

for any $p \in T \setminus X$. Lemma 2.13 then yields $Y \leq T$. By the definition of T we have

$$\psi^{-1}(p) \setminus \lambda_{\mathcal{P}} X = \phi^{-1}(p) \setminus \lambda_{\mathcal{P}} X \neq \emptyset$$

for any $p \in T \setminus X$, which by Theorem 2.11 implies that $T \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$.

By Theorem 2.11 the relations $\leq$ and $\leq_{\rm surj}$ coincide on $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. Now let $\mathscr{E}$ be a subset of $\mathscr{E}_{\mathcal{P}}^{\mathcal{Q}}(X)$ cofinal with respect to $\leq$ and such that $\leq$ and $\leq_{\rm surj}$ coincide on $\mathscr{E}$. Let $S \in \mathscr{E}$. By Theorem 4.4(2) the set $\mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(X)$ is cofinal in $\mathscr{E}_{\mathcal{P}}^{\mathcal{Q}}(X)$ with respect to $\leq$.

Therefore there exists $T \in \mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$ with $S \leq T$. By the cofinality of $\mathscr{E}$ with respect to $\leq$ there exists $Z \in \mathscr{E}$ with $T \leq Z$. Then $S \leq Z$ and thus (since $S, Z \in \mathscr{E}$) by our assumption $S \leq_{\mathrm{surj}} Z$. But by Theorem 2.15 (since $T \leq Z$) we have $Z \in \mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$, which by Lemma 4.7 yields $S \in \mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$. Thus $\mathscr{E} \subseteq \mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$.

(2). By the definitions we have $\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X) \subseteq \mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$. Also, if $Y \in \mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ then by Theorem 4.4(2) we have $Y \leq T$ for some $T \in \mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$ and thus $Y \leq_{\mathrm{surj}} T$ by Theorem 2.11. This shows the cofinality of $\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$ in $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$. Now let $\mathscr{E}$ be a subset of $\mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$ in which $\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$ is cofinal with respect to $\leq_{\mathrm{surj}}$. Let $S \in \mathscr{E}$. By cofinality there exists $Z \in \mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$ with $S \leq_{\mathrm{surj}} Z$. By Lemma 4.7 we have $S \in \mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$. Thus $\mathscr{E} \subseteq \mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$.

(3). By Theorem 2.11(1.e) the set $\mathscr{E} = \mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ satisfies (4.2). Now let $\mathscr{E}$ be a subset of $\mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$ which satisfies (4.2). Let $S \in \mathscr{E}$. By (1) the set $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ is cofinal in $\mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$ with respect to $\leq$. Therefore there exists $Y \in \mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ with $S \leq Y$. Thus $S \leq_{\mathrm{surj}} Y$ by (4.2). By Lemma 4.7 it follows that $S \in \mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$. Therefore $\mathscr{E} \subseteq \mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$.

(4). By (1) the set $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ is cofinal in $\mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$ with respect to $\leq$. Also, by Lemma 4.7 the set $\mathscr{E} = \mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ satisfies (4.3). Now let $\mathscr{E}$ be a subset of $\mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$ cofinal with respect to $\leq$ and satisfying (4.3). Let $Y \in \mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$. By the cofinality of $\mathscr{E}$ we have $Y \leq S$ for some $S \in \mathscr{E}$. By Theorem 2.11(1.e) we have $Y \leq_{\mathrm{surj}} S$ and thus by (4.3) it follows that $Y \in \mathscr{E}$. Therefore $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X) \subseteq \mathscr{E}$. ∎

Recall that a partially ordered set $(L, \leq)$ is called a *lattice* if together with any pair of elements $a, b \in L$ it contains their least upper bound $a \vee b$ and their greatest lower bound $a \wedge b$. Our next purpose is to generalize the following result of K. D. Magill Jr. of [20], which relates the order-structure of the lattice of compactifications of a locally compact space X to the topology of the outgrowth $\beta X \setminus X$. (The theorem has been generalized in various directions; see [24] for a different proof; see [31] for generalizations to non-locally compact spaces; see [42] and [4] for a zero-dimensional version, and [30] for extension to mappings.) Our results here will relate the order-structure of classes of compactification-like $\mathcal{P}$-extensions of a Tychonoff space X to the topology of the subspace $\beta X \setminus \lambda_{\mathcal{P}} X$ of βX.

THEOREM 4.9 (Magill [20]). *Let X and Y be locally compact non-compact spaces. The following are equivalent:*

(1) $(\mathscr{K}(X), \leq)$ *and* $(\mathscr{K}(Y), \leq)$ *are order-isomorphic.*
(2) $\beta X \setminus X$ *and* $\beta Y \setminus Y$ *are homeomorphic.*

REMARK. The above theorem fails if the spaces under consideration are not locally compact (see [35]).

The following simple observation will be used quite often in the future (sometimes without explicit reference).

LEMMA 4.10. *Let X be a Tychonoff locally-$\mathcal{P}$ space where $\mathcal{P}$ is a clopen hereditary finitely additive perfect topological property. Then X is non-$\mathcal{P}$ if and only if $\lambda_{\mathcal{P}} X$ is non-compact if and only if $\lambda_{\mathcal{P}} X \neq \beta X$.*

Proof. If X has $\mathcal{P}$ then by the definition of $\lambda_{\mathcal{P}} X$ (and since obviously $X \in \mathscr{Z}(X)$) we have $\beta X = \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} X \subseteq \lambda_{\mathcal{P}} X$. Thus $\lambda_{\mathcal{P}} X = \beta X$ is compact. Note that if $\lambda_{\mathcal{P}} X$ is compact,

then since $X \subseteq \lambda_{\mathcal{P}} X$ (as X is locally-$\mathcal{P}$; see Lemma 2.10) we have $\mathrm{cl}_{\beta X} X \subseteq \lambda_{\mathcal{P}} X$. Therefore by Lemma 2.4 the space X has $\mathcal{P}$. ∎

Recall that if $(A, \leq)$ and $(B, \leq)$ are partially ordered sets, a mapping $f : A \to B$ is said to be an *order-homomorphism* if for any $c, d \in A$ we have $f(c) \leq f(d)$ whenever $c \leq d$. An order-homomorphism $f : A \to B$ is called an *order-isomorphism* if it is bijective and $f^{-1} : B \to A$ is also an order-homomorphism. Two partially ordered sets $(A, \leq)$ and $(B, \leq)$ are said to be *order-isomorphic* (denoted by $(A, \leq) \cong (B, \leq)$) if there exists an order-isomorphism between them.

Lemma 4.11. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff locally-$\mathcal{P}$ non-$\mathcal{P}$ space with $\mathcal{Q}$. Then*

$$\big(\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X), \leq\big) \cong \big(\mathscr{K}(\lambda_{\mathcal{P}} X), \leq\big).$$

Proof. Let $Y \in \mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$. Let $\phi : \beta X \to \beta Y$ be the continuous extension of id_X. Recall that βY is the quotient space of βX obtained by contracting each $\phi^{-1}(p)$, where $p \in Y \setminus X$, to a point, with ϕ as the corresponding quotient mapping (see Lemma 2.9). By Theorem 2.15 we have $\phi^{-1}[Y \setminus X] = \beta X \setminus \lambda_{\mathcal{P}} X$ and thus we may assume that $\lambda_{\mathcal{P}} X \subseteq \beta Y$. Also, X is dense in βY, as X is dense in Y and by Lemma 2.10 we have $X \subseteq \lambda_{\mathcal{P}} X$. Therefore $\lambda_{\mathcal{P}} X$ is dense in βY and thus βY is a compactification of $\lambda_{\mathcal{P}} X$. Define

$$\Theta : \big(\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X), \leq\big) \to \big(\mathscr{K}(\lambda_{\mathcal{P}} X), \leq\big)$$

by

$$\Theta(Y) = \beta Y$$

for any $Y \in \mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$. By the above Θ is well defined. We verify that Θ is an order-isomorphism.

Claim. *Θ is an order-homomorphism.*

Proof of the claim. Let $Y_1 \leq Y_2$ where $Y_1, Y_2 \in \mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$. By definition there exists a continuous $f : Y_2 \to Y_1$ such that $f|X = \mathrm{id}_X$. Let $f_\beta : \beta Y_2 \to \beta Y_1$ be the continuous extension of f. By the above $\beta Y_i \in \mathscr{K}(\lambda_{\mathcal{P}} X)$ for any $i = 1, 2$. Then $f_\beta | \lambda_{\mathcal{P}} X = \mathrm{id}_{\lambda_{\mathcal{P}} X}$, as they both coincide with id_X on X and thus by definition $\Theta(Y_1) = \beta Y_1 \leq \beta Y_2 = \Theta(Y_2)$.

Claim. *Θ is surjective.*

Proof of the claim. Let $T \in \mathscr{K}(\lambda_{\mathcal{P}} X)$. Consider the subspace $Y = X \cup (T \setminus \lambda_{\mathcal{P}} X)$ of T. We verify that $Y \in \mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$ and that $\Theta(Y) = T$. Note that X is dense in T and therefore X is dense in Y, as X is dense in $\lambda_{\mathcal{P}} X$ and $\lambda_{\mathcal{P}} X$ is dense in T. By definition $\lambda_{\mathcal{P}} X$ is an open subset of βX and thus it is locally compact. Also, $X \subseteq \lambda_{\mathcal{P}} X$ and therefore $Y \setminus X = T \setminus \lambda_{\mathcal{P}} X$ is compact. This shows that $Y \in \mathscr{E}(X)$. Also, $\beta \lambda_{\mathcal{P}} X = \beta X$, as $X \subseteq \lambda_{\mathcal{P}} X \subseteq \beta X$. Let $g : \beta X \to T$ be the continuous extension of $\mathrm{id}_{\lambda_{\mathcal{P}} X}$. By Theorem 3.5.7 of [5] we have $g[\beta X \setminus \lambda_{\mathcal{P}} X] = T \setminus \lambda_{\mathcal{P}} X$. Thus

$$\beta X \setminus \lambda_{\mathcal{P}} X \subseteq g^{-1}\big[g[\beta X \setminus \lambda_{\mathcal{P}} X]\big] = g^{-1}[T \setminus \lambda_{\mathcal{P}} X] = g^{-1}[Y \setminus X].$$

Since X is locally-$\mathcal{P}$, by Lemma 2.8 we have $Y \in \mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$. To show that Y is optimal let $Z \in \mathscr{Z}(X)$ be such that $Z \subseteq C$ for some $C \in \mathrm{Coz}(X)$ such that $\mathrm{cl}_X C$ has $\mathcal{P}$. By Lemma 2.14 we have $\mathrm{cl}_{\beta X} Z \subseteq \lambda_{\mathcal{P}} X$. Therefore since $Z = g[Z] \subseteq g[\mathrm{cl}_{\beta X} Z]$ and the latter

is compact, $\mathrm{cl}_T Z \subseteq g[\mathrm{cl}_{\beta X} Z]$. Since $g[\mathrm{cl}_{\beta X} Z] \subseteq g[\lambda_{\mathcal{P}} X] = \lambda_{\mathcal{P}} X$ we have $\mathrm{cl}_T Z \subseteq \lambda_{\mathcal{P}} X$ and thus

$$\mathrm{cl}_Y Z \cap (Y \setminus X) \subseteq \mathrm{cl}_T Z \cap (T \setminus \lambda_{\mathcal{P}} X) = \emptyset.$$

Theorem 2.15 now implies that $Y \in \mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$. Let $\phi : \beta X \to \beta Y$ be the continuous extension of id_X. By Theorem 2.15 we have $\phi^{-1}[Y \setminus X] = \beta X \setminus \lambda_{\mathcal{P}} X$, which implies that $\phi|\lambda_{\mathcal{P}} X = \mathrm{id}_{\lambda_{\mathcal{P}} X}$. (Recall the construction of βY and the representation of ϕ given in Lemma 2.9.) Let $h : \beta Y \to T$ be the continuous extension of id_Y. The continuous mapping $h\phi : \beta X \to T$ is such that $h\phi|X = \mathrm{id}_X = g|X$ and therefore $h\phi = g$. Thus (and since $\phi|\lambda_{\mathcal{P}} X = \mathrm{id}_{\lambda_{\mathcal{P}} X}$) we have $h|\lambda_{\mathcal{P}} X = g|\lambda_{\mathcal{P}} X = \mathrm{id}_{\lambda_{\mathcal{P}} X}$ and therefore, since $h|Y = \mathrm{id}_Y$ and $Y \cup \lambda_{\mathcal{P}} X = \beta Y$ it follows that $h = \mathrm{id}_{\beta Y}$. In particular, $\mathrm{id}_{\beta Y} = h : \beta Y \to T$ is continuous and it is surjective (as its image contains X and X is dense in T) and thus, since βY is compact, it is a homeomorphism. Therefore $T = \beta Y = \Theta(Y)$.

Claim. *For any $Y_1, Y_2 \in \mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$ if $\Theta(Y_1) \leq \Theta(Y_2)$ then $Y_1 \leq Y_2$.*

Proof of the claim. Let $\Theta(Y_1) \leq \Theta(Y_2)$ for some $Y_1, Y_2 \in \mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X)$. Since $\beta Y_1 \leq \beta Y_2$, by definition there exists a continuous $l : \beta Y_2 \to \beta Y_1$ such that $l|\lambda_{\mathcal{P}} X = \mathrm{id}_{\lambda_{\mathcal{P}} X}$. By Theorem 3.5.7 of [5] we have $l[\beta Y_2 \setminus \lambda_{\mathcal{P}} X] = \beta Y_1 \setminus \lambda_{\mathcal{P}} X$. Note that $Y_i \setminus X = \beta Y_i \setminus \lambda_{\mathcal{P}} X$ for any $i = 1, 2$. To see this, observe that if $\phi_i : \beta X \to \beta Y_i$ where $i = 1, 2$, denotes the continuous extension of id_X then βY_i is the quotient space of βX obtained by contracting the fibers $\phi_i^{-1}(p)$ where $p \in Y_i \setminus X$ to points with the quotient mapping ϕ_i, and by Theorem 2.15 we have $\phi_i^{-1}[Y_i \setminus X] = \beta X \setminus \lambda_{\mathcal{P}} X$. Therefore $l[Y_2 \setminus X] = Y_1 \setminus X$. Thus $l|Y_2 : Y_2 \to Y_1$ and obviously it continuously extends id_X. Therefore by definition $Y_1 \leq Y_2$.

The third claim implies that Θ is injective and that Θ^{-1} is an order-homomorphism. This shows that Θ is an order-isomorphism. ∎

Recall that a partially ordered set $(L, \leq)$ is called a *complete upper semilattice* (*complete lower semilattice*, respectively) if for any non-empty subset A of L the least upper bound $\bigvee A$ (the greatest lower bound $\bigwedge A$, respectively) exists in L. A partially ordered set $(L, \leq)$ is called a *complete lattice* if it is both a complete upper semilattice and a complete lower semilattice. It is well known that for any Tychonoff X the set $\mathscr{K}(X)$ of its all compactifications, partially ordered with $\leq$, is a complete upper semilattice, and it is a complete lattice if and only if X is locally compact (see Propositions 4.2(a) and 4.3(e) of [29]). The following corollary of Lemma 4.11 is now immediate.

Corollary 4.12. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff locally-$\mathcal{P}$ non-$\mathcal{P}$ space with $\mathcal{Q}$. Then $(\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X), \leq)$ is a complete lattice.*

The following theorem relates the order-structure of the set of optimal $\mathcal{P}$-extensions of a Tychonoff locally-$\mathcal{P}$ space X and the topology of the subspace $\beta X \setminus \lambda_{\mathcal{P}} X$ of βX. This generalizes K. D. Magill Jr.'s theorem [20, Theorem 5.1] provided that one replaces $\mathcal{P}$ and $\mathcal{Q}$, respectively, by compactness and regularity, and notes that for these specific choices of $\mathcal{P}$ and $\mathcal{Q}$ and a locally compact space X we have $\lambda_{\mathcal{P}} X = X$ and $\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X) = \mathscr{K}(X)$.

Theorem 4.13. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X and Y be Tychonoff locally-$\mathcal{P}$ non-$\mathcal{P}$ spaces with $\mathcal{Q}$. The following are equivalent:*

(1) $(\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(X), \leq)$ *and* $(\mathscr{O}^{\mathcal{Q}}_{\mathcal{P}}(Y), \leq)$ *are order-isomorphic.*
(2) $\beta X \setminus \lambda_{\mathcal{P}} X$ *and* $\beta Y \setminus \lambda_{\mathcal{P}} Y$ *are homeomorphic.*

Proof. Note that $\lambda_{\mathcal{P}} X$ is locally compact, as it is open in βX, and by Lemma 4.10 (since X is non-$\mathcal{P}$) it is non-compact. Also, since X is locally-$\mathcal{P}$, by Lemma 2.10 we have $X \subseteq \lambda_{\mathcal{P}} X$ and thus (since $\lambda_{\mathcal{P}} X \subseteq \beta X$) we have $\beta\lambda_{\mathcal{P}} X = \beta X$. Similar statements hold for Y. By Theorem 5.1 the partially ordered sets $\mathscr{K}(\lambda_{\mathcal{P}} X)$ and $\mathscr{K}(\lambda_{\mathcal{P}} Y)$ are order-isomorphic if and only if $\beta\lambda_{\mathcal{P}} X \setminus \lambda_{\mathcal{P}} X$ $(= \beta X \setminus \lambda_{\mathcal{P}} X)$ and $\beta\lambda_{\mathcal{P}} Y \setminus \lambda_{\mathcal{P}} Y$ $(= \beta Y \setminus \lambda_{\mathcal{P}} Y)$ are homeomorphic. Now Lemma 4.11 shows the equivalence of (1) and (2). ∎

Our next purpose is to state and prove a result for minimal $\mathcal{P}$-extensions which is analogous to (1)⇒(2) in Theorem 4.13. As we will see, there is no counterpart for (2)⇒(1) in Theorem 4.13 in the case of minimal $\mathcal{P}$-extensions. This will be shown by means of an example. (This is the first place in this article where the duality between minimal $\mathcal{P}$-extensions and optimal $\mathcal{P}$-extensions disappears.) The example, however, is long and quite technical, and requires several lemmas. The reader who is not interested in the construction of the example may skip Lemmas 4.21, 4.22, 4.26, 4.28, 4.29, 4.30, 4.31, 4.33 and 4.34 and replace Lemma 4.33 by Lemma 4.32 in the proof of Theorem 4.36 ((1)⇒(2)).

The following lemma is a counterpart of Lemma 4 in [20].

Lemma 4.14. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff locally-$\mathcal{P}$ non-$\mathcal{P}$ space with $\mathcal{Q}$. For an $n \in \mathbb{N}$, let $K_1, \ldots, K_n$ be n pairwise disjoint compact subsets of $\beta X \setminus X$ such that $K_i \setminus \lambda_{\mathcal{P}} X$ is non-empty for any $i = 1, \ldots, n$. Then there exists a unique Y in $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ such that*

$$\mathscr{F}(Y) = \Big\{\{p\} : p \in (\beta X \setminus \lambda_{\mathcal{P}} X) \setminus \bigcup_{i=1}^{n} K_i\Big\} \cup \{K_1, \ldots, K_n\}. \tag{4.4}$$

Proof. Let T be the space obtained from βX by contracting the sets $K_1, \ldots, K_n$ to points $p_1, \ldots, p_n$, respectively, and denote by $q : \beta X \to T$ the quotient mapping. Since K_i's are compact, T is Hausdorff and thus compact, being a continuous image of βX. Consider the subspace

$$Y = q\Big[X \cup (\beta X \setminus \lambda_{\mathcal{P}} X) \cup \bigcup_{i=1}^{n} K_i\Big]$$

of T. Then Y is a Tychonoff extension of X with the compact remainder

$$Y \setminus X = q\Big[(\beta X \setminus \lambda_{\mathcal{P}} X) \cup \bigcup_{i=1}^{n} K_i\Big].$$

Note that T is a compactification of Y and thus $Y \in \mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$ by Lemma 2.8. Also, by Lemma 2.9 if $\phi : \beta X \to \beta Y$ continuously extends id_X, then βY coincides with the quotient space of βX obtained by contracting each fiber $\phi^{-1}(p)$ where $p \in Y \setminus X$ to a point, that is, $\beta Y = T$. This shows (4.4). The fact that $Y \in \mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ follows from Theorem 2.11.

For the uniqueness part, let $Y' \in \mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ be such that $\mathscr{F}(Y') = \mathscr{F}(Y)$. Let $\psi : \beta X \to \beta Y'$ be the continuous extension of id_X. By Lemma 2.9 we have $\beta Y' = T$ and

$\psi = q$. Thus

$$Y' = q\Big[X \cup (\beta X \setminus \lambda_{\mathcal{P}} X) \cup \bigcup_{i=1}^{n} K_i\Big] = Y. \ \blacksquare$$

NOTATION 4.15. Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff locally-$\mathcal{P}$ non-$\mathcal{P}$ space with $\mathcal{Q}$. Let $n \in \mathbb{N}$ and let $K_1, \dots, K_n$ be n pairwise disjoint compact subsets of $\beta X \setminus X$ such that $K_i \setminus \lambda_{\mathcal{P}} X$ is non-empty for any $i = 1, \dots, n$. Denote by $e_X(K_1, \dots, K_n)$ the unique element of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ such that

$$\mathscr{F}\big(e_X(K_1, \dots, K_n)\big) = \Big\{\{p\} : p \in (\beta X \setminus \lambda_{\mathcal{P}} X) \setminus \bigcup_{i=1}^{n} K_i\Big\} \cup \{K_1, \dots, K_n\}.$$

The next lemma is a counterpart of Lemma 6 in [20].

LEMMA 4.16. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff locally-$\mathcal{P}$ non-$\mathcal{P}$ space with $\mathcal{Q}$. Let K_i, where $i = 1, 2$, be a compact subset of $\beta X \setminus X$ such that $K_i \setminus \lambda_{\mathcal{P}} X$ is non-empty. Then*

(1) $e_X(K_1) \wedge e_X(K_2) = e_X(K_1, K_2)$ *if* $K_1 \cap K_2 = \emptyset$.
(2) $e_X(K_1) \wedge e_X(K_2) = e_X(K_1 \cup K_2)$ *if* $K_1 \cap K_2 \neq \emptyset$.

Here $\wedge$ is the operation in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$.

Proof. This follows from Lemma 2.13. In (2) note that if $Y \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ is such that $Y \leq e_X(K_i)$ for any $i = 1, 2$, then by Lemma 2.13 we have $K_i \subseteq F_i$ for some $F_i \in \mathscr{F}(Y)$. But by our assumption $K_1 \cap K_2$ is non-empty and thus $F_1 \cap F_2$ is non-empty, which implies that $F_1 = F_2$. Therefore $K_1 \cup K_2 \subseteq F_1$ and thus again by Lemma 2.13 it follows that $Y \leq e_X(K_1 \cup K_2)$. $\blacksquare$

Let $(X, \leq)$ be a partially ordered set with the largest element u. An element $a \in X$ is called an *anti-atom* in X if $a \neq u$ and there exists no $x \in X$ with $a < x < u$.

The following lemma is a counterpart of Lemma 9 in [20].

LEMMA 4.17. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff locally-$\mathcal{P}$ non-$\mathcal{P}$ space with $\mathcal{Q}$. The following are equivalent:*

(1) *Y is an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$.*
(2) *$Y = e_X(\{a, b\})$ for some distinct $a, b \in \beta X \setminus X$ such that either $a \notin \lambda_{\mathcal{P}} X$ or $b \notin \lambda_{\mathcal{P}} X$.*

Proof. Note that $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ has the largest element $\zeta_{\mathcal{P}} X = X \cup (\beta X \setminus \lambda_{\mathcal{P}} X)$. To show this first note that by Lemma 2.10 we have $X \subseteq \lambda_{\mathcal{P}} X$. Thus $\zeta_{\mathcal{P}} X$ is a Tychonoff extension of X which (since $\lambda_{\mathcal{P}} X$ is open in βX) has a compact remainder. Since $X \subseteq \zeta_{\mathcal{P}} X \subseteq \beta X$ we have $\beta \zeta_{\mathcal{P}} X = \beta X$ (see Corollary 3.6.9 of [5]). It follows from Lemma 2.8 (with $Y = X$, $f = \mathrm{id}_X$, $T = \zeta_{\mathcal{P}} X$, $\alpha T = \beta X$ and $\phi = \mathrm{id}_{\beta X}$) that $\zeta_{\mathcal{P}} X$ has both $\mathcal{P}$ and $\mathcal{Q}$ and from Theorem 2.11 that $\zeta_{\mathcal{P}} X \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. That $\zeta_{\mathcal{P}} X$ is the largest element of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ now follows from Theorem 2.11 and Lemma 2.13.

That (2) implies (1) is trivial.

(1) *implies* (2). Suppose that Y is an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. We show that except for a 2-element set the rest of the sets in $\mathscr{F}(Y)$ are singletons; the uniqueness part of Lemma 4.14 will then imply (2). Suppose to the contrary that there exist distinct $F_1, F_2 \in \mathscr{F}(Y)$ such that $\mathrm{card}(F_i) \geq 2$ for any $i = 1, 2$. By Theorem 2.11 the set $F_i \setminus \lambda_{\mathcal{P}} X$

is non-empty for any $i = 1, 2$; choose distinct $a_i, b_i \in F_i$ such that $a_i \notin \lambda_{\mathcal{P}} X$. Then $e_X(\{a_i, b_i\})$, where $i = 1, 2$, are distinct elements of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ and $Y \leq e_X(\{a_i, b_i\})$, which contradicts (1). Thus there is at most one set in $\mathscr{F}(Y)$ which is not a singleton, and since $Y \neq \zeta_{\mathcal{P}} X$, there is at least one such set. Let $F \in \mathscr{F}(Y)$ be such that $\operatorname{card}(F) \geq 2$. Suppose that $\operatorname{card}(F) > 2$. By Theorem 2.11 the set $F \setminus \lambda_{\mathcal{P}} X$ is non-empty. Let $a \in F \setminus \lambda_{\mathcal{P}} X$ and let $b, c \in F$ be distinct elements distinct from a. Then $e_X(\{a, b\})$ and $e_X(\{a, c\})$ are distinct elements of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ and $Y \leq e_X(\{a, b\})$ and $Y \leq e_X(\{a, c\})$. This contradiction proves that $\operatorname{card}(F) = 2$. ∎

Definition 4.18. Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff locally-$\mathcal{P}$ non-$\mathcal{P}$ space with $\mathcal{Q}$. An anti-atom $Y = e_X(\{a, b\})$ of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ is said to be *of type* (I) if $\{a, b\} \cap \lambda_{\mathcal{P}} X$ is non-empty; otherwise, Y is said to be *of type* (II).

The purpose of the next two lemmas is to give an order-theoretic characterization of anti-atoms of type (I) (and thus anti-atoms of type (II) as well) in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$.

Lemma 4.19. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff locally-$\mathcal{P}$ non-$\mathcal{P}$ space with $\mathcal{Q}$ such that* $\operatorname{card}(\lambda_{\mathcal{P}} X \setminus X) \geq 2$. *Then*

(1) $\operatorname{card}(\beta X \setminus \lambda_{\mathcal{P}} X) = 1$ *if and only if for any pair of distinct anti-atoms Y and Y' in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ we have*

$$\operatorname{card}\big(\{T : T \text{ is an anti-atom in } \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X) \text{ and } Y \wedge Y' \leq T\}\big) = 2.$$

(2) $\operatorname{card}(\beta X \setminus \lambda_{\mathcal{P}} X) = 2$ *if and only if there exists an anti-atom Y in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ such that for any anti-atom Y' in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ with $Y' \neq Y$ we have*

$$\operatorname{card}\big(\{T : T \text{ is an anti-atom in } \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X) \text{ and } Y \wedge Y' \leq T\}\big) = 3.$$

(3) $\operatorname{card}(\beta X \setminus \lambda_{\mathcal{P}} X) \geq 3$ *if and only if there exist some anti-atoms Y, Y' and Y'' in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ such that*

$$\operatorname{card}\big(\{T : T \text{ is an anti-atom in } \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X) \text{ and } Y \wedge Y' \wedge Y'' \leq T\}\big) = 6.$$

Here $\wedge$ is the operation in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$.

Proof. Since X is locally-$\mathcal{P}$, by Lemma 2.10 we have $X \subseteq \lambda_{\mathcal{P}} X$, and since X is moreover non-$\mathcal{P}$, by Lemma 4.10 the set $\beta X \setminus \lambda_{\mathcal{P}} X$ is non-empty.

(1). Suppose that $\operatorname{card}(\beta X \setminus \lambda_{\mathcal{P}} X) = 1$. Let $\beta X \setminus \lambda_{\mathcal{P}} X = \{a\}$.

Let Y and Y' be distinct anti-atoms in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. Then by Lemma 4.17 we have

$$Y = e_X\big(\{a, b\}\big) \quad \text{and} \quad Y' = e_X\big(\{a, c\}\big)$$

for some $b, c \in \lambda_{\mathcal{P}} X \setminus X$. By Lemmas 2.13 and 4.14 the elements b and c are distinct. By Lemma 4.16 we have

$$Y \wedge Y' = e_X\big(\{a, b\}\big) \wedge e_X\big(\{a, c\}\big) = e_X\big(\{a, b, c\}\big).$$

Using Lemmas 2.13, 4.14 and 4.17 it now follows that there are only two anti-atoms T in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ with $Y \wedge Y' \leq T$, namely,

$$e_X\big(\{a, b\}\big) \quad \text{and} \quad e_X\big(\{a, c\}\big).$$

To show the converse, suppose that $\operatorname{card}(\beta X \setminus \lambda_{\mathcal{P}} X) \neq 1$. Choose distinct $a, b \in \beta X \setminus \lambda_{\mathcal{P}} X$ and $c \in \lambda_{\mathcal{P}} X \setminus X$. By Lemma 4.17 the elements

$$Y = e_X\big(\{a,b\}\big), \quad Y' = e_X\big(\{a,c\}\big) \quad \text{and} \quad Y'' = e_X\big(\{b,c\}\big)$$

are anti-atoms in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ and by Lemmas 2.13 and 4.14 they are distinct. By Lemma 4.16 we have

$$Y \wedge Y' = e_X\big(\{a,b\}\big) \wedge e_X\big(\{a,c\}\big) = e_X\big(\{a,b,c\}\big)$$

and if T is either Y, Y' or Y'' then by Lemmas 2.13, 4.14 and 4.17 we have $Y \wedge Y' \leq T$.

(2). Suppose that $\operatorname{card}(\beta X \setminus \lambda_{\mathcal{P}} X) = 2$. Let $\beta X \setminus \lambda_{\mathcal{P}} X = \{a, b\}$.

Let $Y = e_X(\{a,b\})$. Then by Lemma 4.17 the element Y is an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. Now let Y' be an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ with $Y' \neq Y$. By Lemma 4.17 we have $Y' = e_X(\{c,d\})$ for some distinct $c, d \in \beta X \setminus X$ with either $c \notin \lambda_{\mathcal{P}} X$ or $d \notin \lambda_{\mathcal{P}} X$. Without any loss of generality we may assume that $c \notin \lambda_{\mathcal{P}} X$ and $c = a$. By Lemma 4.16 we have

$$Y \wedge Y' = e_X\big(\{a,b\}\big) \wedge e_X\big(\{a,d\}\big) = e_X\big(\{a,b,c\}\big).$$

Now using Lemmas 2.13, 4.14 and 4.17 if T is either

$$e_X\big(\{a,b\}\big), \quad e_X\big(\{a,d\}\big) \quad \text{or} \quad e_X\big(\{b,d\}\big) \tag{4.5}$$

then T is an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ and $Y \wedge Y' \leq T$, and conversely any anti-atom T in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ with $Y \wedge Y' \leq T$ is of the above form. By Lemmas 2.13 and 4.14 the elements in (4.5) are distinct.

To show the converse, suppose that $\operatorname{card}(\beta X \setminus \lambda_{\mathcal{P}} X) \neq 2$. Either $\operatorname{card}(\beta X \setminus \lambda_{\mathcal{P}} X) = 1$ or $\operatorname{card}(\beta X \setminus \lambda_{\mathcal{P}} X) \geq 3$. Consider the following cases:

Case 1. Suppose that $\operatorname{card}(\beta X \setminus \lambda_{\mathcal{P}} X) = 1$. Let $\beta X \setminus \lambda_{\mathcal{P}} X = \{a\}$. Let Y be an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. By Lemma 4.17 we have $Y = e_X(\{a,b\})$ for some $b \in \lambda_{\mathcal{P}} X \setminus X$. Let $c \in \lambda_{\mathcal{P}} X \setminus X$ be distinct from b. (Such a c exists, as we are assuming that $\operatorname{card}(\lambda_{\mathcal{P}} X \setminus X) \geq 2$.) Let $Y' = e_X(\{a,c\})$. Then by Lemma 4.17 the element Y' is an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ and by Lemmas 2.13 and 4.14 we have $Y' \neq Y$. By Lemma 4.16 we have

$$Y \wedge Y' = e_X\big(\{a,b\}\big) \wedge e_X\big(\{a,c\}\big) = e_X\big(\{a,b,c\}\big).$$

Now using Lemmas 2.13, 4.14 and 4.17 it follows that the anti-atoms T in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ with $Y \wedge Y' \leq T$ are exactly $e_X(\{a,b\})$ and $e_X(\{a,c\})$.

Case 2. Suppose that $\operatorname{card}(\beta X \setminus \lambda_{\mathcal{P}} X) \geq 3$. Let Y be an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. By Lemma 4.17 we have $Y = e_X(\{a,b\})$ for some distinct $a, b \in \beta X \setminus X$ with either $a \notin \lambda_{\mathcal{P}} X$ or $b \notin \lambda_{\mathcal{P}} X$. Choose $c \in \beta X \setminus \lambda_{\mathcal{P}} X$ and $d \in \lambda_{\mathcal{P}} X \setminus X$ such that neither $c \notin \{a,b\}$ nor $d \notin \{a,b\}$. (Again, we are using the fact that $\operatorname{card}(\lambda_{\mathcal{P}} X \setminus X) \geq 2$.) Let $Y' = e_X(\{c,d\})$. Then by Lemma 4.17 the element Y' is an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ and by Lemmas 2.13 and 4.14 we have $Y' \neq Y$. By Lemma 4.16 we have

$$Y \wedge Y' = e_X\big(\{a,b\}\big) \wedge e_X\big(\{c,d\}\big) = e_X\big(\{a,b\},\{c,d\}\big).$$

Now as above the anti-atoms T in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ with $Y \wedge Y' \leq T$ are exactly $e_X(\{a,b\})$ and $e_X(\{c,d\})$.

Thus in either case for a given anti-atom Y in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ we can find an anti-atom Y' in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ distinct from Y with at most two anti-atoms T in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ with $Y \wedge Y' \leq T$.

(3). Suppose that $\operatorname{card}(\beta X \setminus \lambda_{\mathcal{P}} X) \geq 3$.

Choose distinct $a, b, c \in \beta X \setminus \lambda_{\mathcal{P}} X$ and $d \in \lambda_{\mathcal{P}} X \setminus X$. By Lemma 4.17 the elements

$$Y = e_X\big(\{a, b\}\big), \quad Y' = e_X\big(\{b, c\}\big) \quad \text{and} \quad Y'' = e_X\big(\{c, d\}\big)$$

are anti-atoms in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. By Lemma 4.16 we have

$$\begin{aligned} Y \wedge Y' \wedge Y'' &= e_X\big(\{a, b\}\big) \wedge e_X\big(\{b, c\}\big) \wedge e_X\big(\{c, d\}\big) \\ &= e_X\big(\{a, b, c\}\big) \wedge e_X\big(\{c, d\}\big) = e_X\big(\{a, b, c, d\}\big) \end{aligned}$$

and therefore, using Lemmas 2.13, 4.14 and 4.17 it follows that there are six anti-atoms T in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ with $Y \wedge Y' \wedge Y'' \leq T$, namely,

$$e_X\big(\{a, b\}\big), e_X\big(\{a, c\}\big), e_X\big(\{a, d\}\big), e_X\big(\{b, c\}\big), e_X\big(\{b, d\}\big) \text{ and } e_X\big(\{c, d\}\big).$$

To show the converse, suppose that $\operatorname{card}(\beta X \setminus \lambda_{\mathcal{P}} X) \leq 2$. Consider the following cases:

Case 1. Suppose that $\operatorname{card}(\beta X \setminus \lambda_{\mathcal{P}} X) = 1$. Let $\beta X \setminus \lambda_{\mathcal{P}} X = \{a\}$. Let Y, Y' and Y'' be anti-atoms in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. By Lemma 4.17 we have

$$Y = e_X\big(\{a, b\}\big), Y' = e_X\big(\{a, c\}\big) \quad \text{and} \quad Y'' = e_X\big(\{a, d\}\big)$$

for some $b, c, d \in \lambda_{\mathcal{P}} X \setminus X$. Using Lemma 4.16 we have

$$\begin{aligned} Y \wedge Y' \wedge Y'' &= e_X\big(\{a, b\}\big) \wedge e_X\big(\{a, c\}\big) \wedge e_X\big(\{a, d\}\big) \\ &= e_X\big(\{a, b, c\}\big) \wedge e_X\big(\{a, d\}\big) = e_X\big(\{a, b, c, d\}\big) \end{aligned}$$

and therefore, using Lemmas 2.13, 4.14 and 4.17 it follows that the only anti-atoms T in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ with $Y \wedge Y' \wedge Y'' \leq T$ are $e_X(\{a, b\}), e_X(\{a, c\})$ and $e_X(\{a, d\})$.

Case 2. Suppose that $\operatorname{card}(\beta X \setminus \lambda_{\mathcal{P}} X) = 2$. Let $\beta X \setminus \lambda_{\mathcal{P}} X = \{a, b\}$. Let Y, Y' and Y'' be anti-atoms in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. Consider the following cases:

Case 2.a. Suppose that $e_X(\{a, b\}) \notin \{Y, Y', Y''\}$. Consider the following cases:

Case 2.a.i. Suppose that

$$Y = e_X\big(\{c, d\}\big), \quad Y' = e_X\big(\{c, e\}\big) \quad \text{and} \quad Y'' = e_X\big(\{c, f\}\big)$$

where c is either a or b and $d, e, f \in \lambda_{\mathcal{P}} X \setminus X$. By Lemma 4.16 we have

$$\begin{aligned} Y \wedge Y' \wedge Y'' &= e_X\big(\{c, d\}\big) \wedge e_X\big(\{c, e\}\big) \wedge e_X\big(\{c, f\}\big) \\ &= e_X\big(\{c, d, e\}\big) \wedge e_X\big(\{c, f\}\big) = e_X\big(\{c, d, e, f\}\big) \end{aligned}$$

and therefore, again using Lemmas 2.13, 4.14 and 4.17 it follows that the only anti-atoms T in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ with $Y \wedge Y' \wedge Y'' \leq T$ are $e_X(\{c, d\})$, $e_X(\{c, e\})$ and $e_X(\{c, f\})$.

Case 2.a.ii. Suppose that either

$$Y = e_X\big(\{a, d\}\big), \quad Y' = e_X\big(\{a, e\}\big) \quad \text{and} \quad Y'' = e_X\big(\{b, f\}\big)$$

or

$$Y = e_X\big(\{b, d\}\big), \quad Y' = e_X\big(\{b, e\}\big) \quad \text{and} \quad Y'' = e_X\big(\{a, f\}\big)$$

where $d, e, f \in \lambda_{\mathcal{P}} X \setminus X$. Without any loss of generality we may assume that the first of the above cases occurs. Suppose that $f \notin \{d, e\}$. Then by Lemma 4.16 we have

$$\begin{aligned} Y \wedge Y' \wedge Y'' &= e_X\big(\{a, d\}\big) \wedge e_X\big(\{a, e\}\big) \wedge e_X\big(\{b, f\}\big) \\ &= e_X\big(\{a, d, e\}\big) \wedge e_X\big(\{b, f\}\big) = e_X\big(\{a, d, e\}, \{b, f\}\big) \end{aligned}$$

and therefore as above it then follows that the anti-atoms T in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ with $Y \wedge Y' \wedge Y'' \leq T$ are exactly $e_X(\{a, d\})$, $e_X(\{a, e\})$ and $e_X(\{b, f\})$. Now suppose that $f \in \{d, e\}$, say $f = d$. Then by Lemma 4.16 we have

$$\begin{aligned} Y \wedge Y' \wedge Y'' &= e_X\big(\{a, d\}\big) \wedge e_X\big(\{a, e\}\big) \wedge e_X\big(\{b, f\}\big) \\ &= e_X\big(\{a, d, e\}\big) \wedge e_X\big(\{b, d\}\big) = e_X\big(\{a, b, d, e\}\big) \end{aligned}$$

and therefore as above it follows that the anti-atoms T in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ such that $Y \wedge Y' \wedge Y'' \leq T$ are exactly $e_X(\{a, b\})$, $e_X(\{a, d\})$, $e_X(\{a, e\})$, $e_X(\{b, d\})$ and $e_X(\{b, e\})$.

Case 2.b. Suppose that $e_X(\{a, b\}) \in \{Y, Y', Y''\}$, say $Y = e_X(\{a, b\})$. Consider the following cases:

Case 2.b.i. Suppose that $Y' = e_X(\{c, d\})$ and $Y'' = e_X(\{c, e\})$ where c is either a or b, say $c = a$, and $d, e \in \lambda_{\mathcal{P}} X \setminus X$. Then by Lemma 4.16 we have

$$\begin{aligned} Y \wedge Y' \wedge Y'' &= e_X\big(\{a, b\}\big) \wedge e_X\big(\{a, d\}\big) \wedge e_X\big(\{a, e\}\big) \\ &= e_X\big(\{a, b, d\}\big) \wedge e_X\big(\{a, e\}\big) = e_X\big(\{a, b, d, e\}\big) \end{aligned}$$

and therefore using Lemmas 2.13, 4.14 and 4.17 it follows that the anti-atoms T in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ such that $Y \wedge Y' \wedge Y'' \leq T$ are exactly $e_X(\{a, b\})$, $e_X(\{a, d\})$, $e_X(\{a, e\})$, $e_X(\{b, d\})$ and $e_X(\{b, e\})$.

Case 2.b.ii. Suppose that $Y' = e_X(\{a, d\})$ and $Y'' = e_X(\{b, e\})$ where $d, e \in \lambda_{\mathcal{P}} X \setminus X$. Then by Lemma 4.16 we have

$$\begin{aligned} Y \wedge Y' \wedge Y'' &= e_X\big(\{a, b\}\big) \wedge e_X\big(\{a, d\}\big) \wedge e_X\big(\{b, e\}\big) \\ &= e_X\big(\{a, b, d\}\big) \wedge e_X\big(\{b, e\}\big) = e_X\big(\{a, b, d, e\}\big) \end{aligned}$$

and therefore as above it follows that the anti-atoms T in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ such that $Y \wedge Y' \wedge Y'' \leq T$ are exactly $e_X(\{a, b\})$, $e_X(\{a, d\})$, $e_X(\{a, e\})$, $e_X(\{b, d\})$ and $e_X(\{b, e\})$.

Thus in either case for any given anti-atoms Y, Y' and Y'' of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ there are at most five anti-atoms T in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ with $Y \wedge Y' \wedge Y'' \leq T$. ∎

Lemma 4.20. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff locally-$\mathcal{P}$ non-$\mathcal{P}$ space with $\mathcal{Q}$ such that* $\operatorname{card}(\lambda_{\mathcal{P}} X \setminus X) \geq 2$.

(1) *Suppose that* $\operatorname{card}(\beta X \setminus \lambda_{\mathcal{P}} X) = 1$. *Then any anti-atom of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ is of type* (I).
(2) *Suppose that* $\operatorname{card}(\beta X \setminus \lambda_{\mathcal{P}} X) = 2$. *Then an anti-atom Y of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ is of type* (I) *if and only if there exists an anti-atom Y' of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ such that*

$$\operatorname{card}\big(\{T : T \text{ is an anti-atom in } \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X) \text{ and } Y \wedge Y' \leq T\}\big) = 2.$$

(3) *Suppose that* $\operatorname{card}(\beta X \setminus \lambda_{\mathcal{P}} X) \geq 3$. *Then an anti-atom* Y *of* $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ *is of type* (I) *if and only if there exists an anti-atom* Y' *of* $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ *with* $Y' \neq Y$ *such that for any anti-atom* Y'' *of* $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ *we have*

$$\operatorname{card}\big(\{T : T \text{ is an anti-atom in } \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X) \text{ and } Y \wedge Y' \wedge Y'' \leq T\}\big) \leq 5.$$

Here $\wedge$ *is the operation in* $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$.

Proof. By Lemmas 2.10 and 4.10 we have $X \subseteq \lambda_{\mathcal{P}} X$ and that $\beta X \setminus \lambda_{\mathcal{P}} X$ is non-empty.

(1). This is obvious.

(2). Suppose that $\operatorname{card}(\beta X \setminus \lambda_{\mathcal{P}} X) = 2$. Let $\beta X \setminus \lambda_{\mathcal{P}} X = \{a, b\}$. Suppose that Y is an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ of type (I). Then $Y = e_X(\{c, e\})$ where c is either a or b, say $c = a$, and $e \in \lambda_{\mathcal{P}} X \setminus X$. Choose some $d \in \lambda_{\mathcal{P}} X \setminus X$ such that $d \neq e$. (Such a d exists, as we are assuming that $\operatorname{card}(\lambda_{\mathcal{P}} X \setminus X) \geq 2$.) Then by Lemma 4.17 the element $Y' = e_X(\{b, d\})$ is an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. By Lemma 4.16 we have

$$Y \wedge Y' = e_X\big(\{a, e\}\big) \wedge e_X\big(\{b, d\}\big) = e_X\big(\{a, e\}, \{b, d\}\big)$$

and thus the anti-atoms T in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ with $Y \wedge Y' \leq T$ are exactly $e_X(\{a, e\})$ and $e_X(\{b, d\})$ which by Lemmas 2.13 and 4.14 are distinct.

To show the converse, suppose that an anti-atom Y of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ is not of type (I). Then necessarily $Y = e_X(\{a, b\})$. Let Y' be an anti-atom of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. If $Y' = Y$ then the only anti-atom T of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ with $Y = Y \wedge Y' \leq T$ is Y itself. Suppose that $Y' \neq Y$. Using Lemmas 2.13, 4.14 and 4.17 we have $Y' = e_X(\{c, e\})$, where c is either a or b, say $c = a$, and $e \in \lambda_{\mathcal{P}} X \setminus X$. By Lemma 4.16 we have

$$Y \wedge Y' = e_X\big(\{a, b\}\big) \wedge e_X\big(\{a, e\}\big) = e_X\big(\{a, b, e\}\big)$$

and therefore, again using Lemmas 2.13, 4.14 and 4.17 there are exactly three anti-atoms T of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ with $Y \wedge Y' \leq T$, namely, $e_X(\{a, b\})$, $e_X(\{a, e\})$ and $e_X(\{b, e\})$. Thus in either case the number of anti-atoms T in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ with $Y \wedge Y' \leq T$ is not 2.

(3). Suppose that $\operatorname{card}(\beta X \setminus \lambda_{\mathcal{P}} X) \geq 3$. Suppose that Y is an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ of type (I). By Lemma 4.17 we have $Y = e_X(\{a, b\})$ for some distinct $a, b \in \beta X \setminus X$ such that either $a \notin \lambda_{\mathcal{P}} X$ or $b \notin \lambda_{\mathcal{P}} X$. Choose some $c \in \lambda_{\mathcal{P}} X \setminus X$ distinct from b (this is possible as we are assuming that $\operatorname{card}(\lambda_{\mathcal{P}} X \setminus X) \geq 2$) and let $Y' = e_X(\{a, c\})$. Then Y' is an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ by Lemma 4.17, and $Y' \neq Y$ by Lemmas 2.13 and 4.14. Let Y'' be an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. By Lemma 4.17 we have $Y'' = e_X(\{d, e\})$ where $d, e \in \beta X \setminus X$ are distinct and either $d \notin \lambda_{\mathcal{P}} X$ or $e \notin \lambda_{\mathcal{P}} X$. Consider the following cases:

Case 1. Suppose that $\{a, b, c\} \cap \{d, e\} = \emptyset$. By Lemma 4.16 we have

$$\begin{aligned} Y \wedge Y' \wedge Y'' &= e_X\big(\{a, b\}\big) \wedge e_X\big(\{a, c\}\big) \wedge e_X\big(\{d, e\}\big) \\ &= e_X\big(\{a, b, c\}\big) \wedge e_X\big(\{d, e\}\big) = e_X\big(\{a, b, c\}, \{d, e\}\big) \end{aligned}$$

and therefore using Lemmas 2.13, 4.14 and 4.17 the anti-atoms T in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ such that $Y \wedge Y' \wedge Y'' \leq T$ are exactly $e_X(\{a, b\})$, $e_X(\{a, c\})$ and $e_X(\{d, e\})$.

Case 2. Suppose that $\{a,b,c\}\cap\{d,e\}\neq\emptyset$. By Lemma 4.16 we have

$$\begin{aligned} Y\wedge Y'\wedge Y'' &= e_X\big(\{a,b\}\big)\wedge e_X\big(\{a,c\}\big)\wedge e_X\big(\{d,e\}\big)\\ &= e_X\big(\{a,b,c\}\big)\wedge e_X\big(\{d,e\}\big) = e_X\big(\{a,b,c,d,e\}\big). \end{aligned}$$

Consider the following cases:

Case 2.a. Suppose that $a\in\{d,e\}$, say $a=d$. Consider the following cases:

Case 2.a.i. Suppose that $\{d,e\}\cap\lambda_{\mathcal{P}}X=\emptyset$. Now using Lemmas 2.13, 4.14 and 4.17 the anti-atoms T in $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ with $Y\wedge Y'\wedge Y''\leq T$ are exactly $e_X(\{a,b\})$, $e_X(\{a,c\})$, $e_X(\{a,e\})$, $e_X(\{b,e\})$ and $e_X(\{c,e\})$.

Case 2.a.ii. Suppose that $\{d,e\}\cap\lambda_{\mathcal{P}}X\neq\emptyset$. Then necessarily $e\in\lambda_{\mathcal{P}}X$ and therefore as above the anti-atoms T in $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ such that $Y\wedge Y'\wedge Y''\leq T$ are exactly $e_X(\{a,b\})$, $e_X(\{a,c\})$ and $e_X(\{a,e\})$.

Case 2.b. Suppose that $a\notin\{d,e\}$. Then $\{b,c\}\cap\{d,e\}$ is non-empty. Without any loss of generality we may assume that $c=d$. This implies that $e\notin\lambda_{\mathcal{P}}X$ and therefore again using Lemmas 2.13, 4.14 and 4.17 the anti-atoms T in $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ with $Y\wedge Y'\wedge Y''\leq T$ are exactly $e_X(\{a,b\})$, $e_X(\{a,c\})$, $e_X(\{a,e\})$, $e_X(\{b,e\})$ and $e_X(\{c,e\})$.

Thus for this choice of Y', for any anti-atom Y'' of $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ the number of anti-atoms T of $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ with $Y\wedge Y'\wedge Y''\leq T$ is at most 5.

To show the converse, suppose that an anti-atom Y of $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ is not of type (I). Then $Y=e_X(\{a,b\})$ for some distinct $a,b\in\beta X\setminus\lambda_{\mathcal{P}}X$. Let Y' be an anti-atom in $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ distinct from Y. By Lemma 4.17 we have $Y'=e_X(\{c,d\})$ for some distinct $c,d\in\beta X\setminus X$ with either $c\notin\lambda_{\mathcal{P}}X$ or $d\notin\lambda_{\mathcal{P}}X$. Consider the following cases:

Case 1. Suppose that $\{a,b\}\cap\{c,d\}=\emptyset$. Let $Y''=e_X(\{b,c\})$. By Lemma 4.17 the element Y'' is an anti-atom in $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$. By Lemma 4.16 we have

$$\begin{aligned} Y\wedge Y'\wedge Y'' &= e_X\big(\{a,b\}\big)\wedge e_X\big(\{c,d\}\big)\wedge e_X\big(\{b,c\}\big)\\ &= e_X\big(\{a,b\}\big)\wedge e_X\big(\{b,c,d\}\big) = e_X\big(\{a,b,c,d\}\big) \end{aligned}$$

and therefore using Lemmas 2.13, 4.14 and 4.17 there are exactly six anti-atoms T in $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ such that $Y\wedge Y'\wedge Y''\leq T$, namely, $e_X(\{a,b\})$, $e_X(\{a,c\})$, $e_X(\{a,d\})$, $e_X(\{b,c\})$, $e_X(\{b,d\})$ and $e_X(\{c,d\})$.

Case 2. Suppose that $\{a,b\}\cap\{c,d\}\neq\emptyset$. Without any loss of generality we may assume that $b=c$. Consider the following cases:

Case 2.a. Suppose that $d\notin\lambda_{\mathcal{P}}X$. Choose some $e\in\lambda_{\mathcal{P}}X\setminus X$. (This is possible as we are assuming that $\operatorname{card}(\lambda_{\mathcal{P}}X\setminus X)\geq 2$.)

Case 2.b. Suppose that $d\in\lambda_{\mathcal{P}}X$. Choose some $e\in\beta X\setminus\lambda_{\mathcal{P}}X$ distinct from both a and b. (This is possible as we are assuming that $\operatorname{card}(\beta X\setminus\lambda_{\mathcal{P}}X)\geq 3$.)

Now let $Y''=e_X(\{a,e\})$. By Lemma 4.17 the element Y'' is an anti-atom in $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$. By Lemma 4.16 we have

$$\begin{aligned} Y\wedge Y'\wedge Y'' &= e_X\big(\{a,b\}\big)\wedge e_X\big(\{b,d\}\big)\wedge e_X\big(\{a,e\}\big)\\ &= e_X\big(\{a,b,d\}\big)\wedge e_X\big(\{a,e\}\big) = e_X\big(\{a,b,d,e\}\big) \end{aligned}$$

and therefore as above there are exactly six anti-atoms T in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ such that $Y \wedge Y' \wedge Y'' \leq T$, namely, $e_X(\{a,b\})$, $e_X(\{a,d\})$, $e_X(\{a,e\})$, $e_X(\{b,d\})$, $e_X(\{b,e\})$ and $e_X(\{d,e\})$.

Thus in either case for a given anti-atom Y' of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ distinct from Y there is an anti-atom Y'' in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ with exactly six anti-atoms T in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ such that $Y \wedge Y' \wedge Y'' \leq T$. ∎

The following lemma together with Lemmas 4.19 and 4.20 above gives an order-theoretic characterization of one-point extensions in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$.

LEMMA 4.21. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff locally-$\mathcal{P}$ non-$\mathcal{P}$ space with $\mathcal{Q}$ and let $Y \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. The following are equivalent:*

(1) *Y is a one-point extension of X.*
(2) *$Y \leq T$ for any anti-atom T of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ of type* (II).

Proof. Let $\phi : \beta X \to \beta Y$ be the continuous extension of id_X. By Lemma 2.8 we have $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X]$. Also, Lemmas 2.10 and 4.10 imply that $X \subseteq \lambda_{\mathcal{P}} X$ and that $\beta X \setminus \lambda_{\mathcal{P}} X$ is non-empty.

(1) *implies* (2). Note that $\mathscr{F}(Y) = \{\phi^{-1}[Y \setminus X]\}$. Let T be an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ of type (II). Then $Y = e_X(\{a,b\})$ for some distinct $a, b \in \beta X \setminus \lambda_{\mathcal{P}} X$. Since $\{a,b\} \subseteq \phi^{-1}[Y \setminus X]$ it follows from Lemmas 2.13 and 4.14 that $Y \leq T$.

(2) *implies* (1). Note that $Y \setminus X$ is non-empty, as $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X]$ and $\beta X \setminus \lambda_{\mathcal{P}} X$ is non-empty. Suppose that $\mathrm{card}(Y \setminus X) \geq 2$. Let $F, G \in \mathscr{F}(Y)$ be distinct. By Theorem 2.11 both $F \setminus \lambda_{\mathcal{P}} X$ and $G \setminus \lambda_{\mathcal{P}} X$ are non-empty. Let $a \in F \setminus \lambda_{\mathcal{P}} X$ and $b \in G \setminus \lambda_{\mathcal{P}} X$. Then $T = e_X(\{a,b\})$ is an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ of type (II), while $Y \not\leq T$. This shows that $Y \setminus X$ is a one-point set. ∎

The following lemma is well known. It is included here for the sake of completeness.

LEMMA 4.22. *Let X be a Tychonoff space. Then for any compact non-empty subset C of $\beta X \setminus X$ there exists a unique one-point Tychonoff extension Y of X with $C = \phi^{-1}[Y \setminus X]$, where $\phi : \beta X \to \beta Y$ is the continuous extension of* id_X.

Proof. Let Z be the quotient space of βX obtained by contracting C to a point p with the quotient mapping $q : \beta X \to Z$. Note that Z is compact, being a Hausdorff continuous image of βX. Consider the subspace $Y = X \cup \{p\}$ of Z. Then Y is a one-point Tychonoff extension of X. We show that $Z = \beta Y$ and $q = \phi$ where $\phi : \beta X \to \beta Y$ is the continuous extension of id_X. Note that Z is a compactification of Y, as it contains Y as a dense subspace. Thus to show that $Z = \beta Y$ it suffices to verify that any continuous $h : Y \to \mathbb{I}$ is continuously extendable over Z. Indeed, let $G : \beta X \to \mathbb{I}$ be the continuous extension of $hq|(X \cup C) : X \cup C \to \mathbb{I}$. (Note that $\beta(X \cup C) = \beta X$, as $X \subseteq X \cup C \subseteq \beta X$; see Corollary 3.6.9 of [5].) Define $H : T \to \mathbb{I}$ such that $H|(\beta X \cup C) = G|(\beta X \cup C)$ and $H(p) = h(p)$. Then $H|Y = h$, and since $Hq = G$ is continuous, H is continuous. This shows that $Z = \beta Y$. That $q = \phi$ follows easily, as they are both continuous and coincide with id_X on the dense subset X of βX. We have

$$C = q^{-1}(p) = q^{-1}[Y \setminus X] = \phi^{-1}[Y \setminus X].$$

For the uniqueness part, note that if T is also a one-point Tychonoff extension of X such that $C = \psi^{-1}[T \setminus X]$, where $\psi : \beta X \to \beta T$ is the continuous extension of id_X, then $\mathscr{F}(T) = \{C\} = \mathscr{F}(Y)$ and thus $T = Y$ by Lemma 2.13. ∎

NOTATION 4.23. For a Tychonoff space X and a compact non-empty subset C of $\beta X \setminus X$ denote by $e_C X$ the unique one-point Tychonoff extension Y of X with $C = \phi^{-1}[Y \setminus X]$, where $\phi : \beta X \to \beta Y$ is the continuous extension of id_X.

NOTATION 4.24. Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff locally-$\mathcal{P}$ non-$\mathcal{P}$ space with $\mathcal{Q}$. Denote

$$M_{\mathcal{P}}^X = e_C X$$

where $C = \beta X \setminus \lambda_{\mathcal{P}} X$.

REMARK. In Notation 4.24 the set $C = \beta X \setminus \lambda_{\mathcal{P}} X$ is a compact subset of $\beta X \setminus X$ and it is non-empty; see Lemma 4.10. Therefore the above definition of $e_C X$ makes sense.

In the following we associate to any element Y in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ a certain one-point extension Y_U in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. This will be used in Lemma 4.31 when we order-theoretically characterize the locally compact elements of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$.

NOTATION 4.25. Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff locally-$\mathcal{P}$ non-$\mathcal{P}$ space with $\mathcal{Q}$ and let $Y \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. Denote

$$Y_U = e_C X$$

where $C = \phi^{-1}[Y \setminus X]$ and $\phi : \beta X \to \beta Y$ is the continuous extension of id_X.

REMARK. The definition in Notation 4.25 makes sense, as C is a compact subset of $\beta X \setminus X$ and since by Lemma 2.8 we have $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X]$, it is non-empty (as $\beta X \setminus \lambda_{\mathcal{P}} X$ is non-empty; see Lemma 4.10).

The following lemma together with Lemmas 4.19, 4.20 and 4.21 gives an order-theoretic characterization of the element Y_U we already associated to any Y in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$.

LEMMA 4.26. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff locally-$\mathcal{P}$ non-$\mathcal{P}$ space with $\mathcal{Q}$ and let $Y \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. Then Y_U is the largest $T \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ satisfying the following:*

(1) *T is a one-point extension of X.*
(2) *$T \leq Y'$ for any anti-atom Y' of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ of type* (I) *such that $Y \leq Y'$.*

Proof. Let $\phi : \beta X \to \beta Y$ and $\phi_U : \beta X \to \beta Y_U$ denote the continuous extensions of id_X. By Lemma 4.22 we have $\phi^{-1}[Y \setminus X] = \phi_U^{-1}[Y_U \setminus X]$ and thus $Y_U \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ by Lemma 2.8 and Theorem 2.11, as $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X]$ and by Lemma 4.10 the set $\beta X \setminus \lambda_{\mathcal{P}} X$ is non-empty. Obviously, Y_U satisfies (1). To show that Y_U satisfies (2), let Y' be an anti-atom of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ of type (I) such that $Y \leq Y'$. Let $Y' = e_X(\{a, b\})$. Then by Lemmas 2.13 and 4.14 we have $\{a, b\} \subseteq F$ for some $F \in \mathscr{F}(Y)$. Since $F \subseteq \phi^{-1}[Y \setminus X]$ we have $\{a, b\} \subseteq \phi_U^{-1}[Y_U \setminus X]$ and therefore, again by Lemmas 2.13 and 4.14 it follows that $Y_U \leq Y'$. Now we show that Y_U is the largest element of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ satisfying (1)–(2). Let (1)–(2) hold for some $T \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. Let $\psi : \beta X \to \beta T$ be the continuous extension of id_X. To show that $T \leq Y_U$, by Lemma 2.13, it suffices to show that $\phi_U^{-1}[Y_U \setminus X] \subseteq \psi^{-1}[T \setminus X]$.

Let $c \in \phi_U^{-1}[Y_U \setminus X]$. Suppose that $c \in \beta X \setminus \lambda_{\mathcal{P}} X$. Then obviously $c \in \psi^{-1}[T \setminus X]$, as $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \psi^{-1}[T \setminus X]$ by Lemma 2.8. Now suppose that $c \in \lambda_{\mathcal{P}} X$. Since $c \in \phi^{-1}[Y \setminus X]$ there exists $G \in \mathscr{F}(Y)$ such that $c \in G$. By Theorem 2.11 the set $G \setminus \lambda_{\mathcal{P}} X$ is non-empty. Let $d \in G \setminus \lambda_{\mathcal{P}} X$. Then $Y' = e_X(\{c, d\})$ is an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ of type (I) which by Lemmas 2.13 and 4.14 is such that $Y \leq Y'$. Thus by our assumption $T \leq Y'$. Therefore again using Lemmas 2.13 and 4.14 it follows that $\{c, d\} \subseteq \psi^{-1}[T \setminus X]$. Thus $c \in \psi^{-1}[T \setminus X]$ in this case as well. Hence $\phi_U^{-1}[Y_U \setminus X] \subseteq \psi^{-1}[T \setminus X]$, which completes the proof. ∎

In the following we define, and then order-theoretically characterize, certain elements of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. This will have an application in the order-theoretic characterization of locally compact elements of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ given in Lemma 4.31.

Definition 4.27. Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff locally-$\mathcal{P}$ non-$\mathcal{P}$ space with $\mathcal{Q}$. An element $Y \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ is called *almost optimal* provided that $\lambda_{\mathcal{P}} X \cap \phi^{-1}[Y \setminus X]$ is compact, where $\phi : \beta X \to \beta Y$ is the continuous extension of id_X.

Lemma 4.28. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff locally-$\mathcal{P}$ non-$\mathcal{P}$ space with $\mathcal{Q}$. Let $\{Y_i : i \in I\} \subseteq \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ be a non-empty collection of one-point extensions of X. Then*

(1) *The least upper bound $\bigvee_{i \in I} Y_i$ exists in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$.*
(2) *If $Y = \bigvee_{i \in I} Y_i$ then Y is a one-point extension of X and*

$$\phi^{-1}[Y \setminus X] = \bigcap_{i \in I} \phi_i^{-1}[Y_i \setminus X]$$

where $\phi : \beta X \to \beta Y$ and $\phi_i : \beta X \to \beta Y_i$ for any $i \in I$ denote the continuous extensions of id_X.

Here $\vee$ is the operation in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$.

Proof. Let $\phi_i : \beta X \to \beta Y_i$ for any $i \in I$ be the continuous extension of id_X. Let

$$C = \bigcap_{i \in I} \phi_i^{-1}[Y_i \setminus X].$$

Then C is compact, as it is closed in βX, and obviously $C \subseteq \beta X \setminus X$, as $\phi_i | X = \mathrm{id}_X$ for any $i \in I$ (and I is non-empty). By Lemma 2.8 we have $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi_i^{-1}[Y_i \setminus X]$ for any $i \in I$. Therefore $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq C$, which implies that C is non-empty, as by Lemma 4.10 the set $\beta X \setminus \lambda_{\mathcal{P}} X$ is non-empty. Let $Y = e_C X$. Then Y is a one-point Tychonoff extension of X and $Y \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ by Lemma 2.8 and Theorem 2.11, since if $\phi : \beta X \to \beta Y$ denotes the continuous extensions of id_X, then using Lemma 4.22 we have $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq C = \phi^{-1}[Y \setminus X]$. By Lemma 2.13 it is obvious that $Y_i \leq Y$ for any $i \in I$, as $\phi^{-1}[Y \setminus X] = C \subseteq \phi_i^{-1}[Y_i \setminus X]$. We only need to show that $Y \leq Y'$ for any $Y' \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ which satisfies $Y_i \leq Y'$ for any $i \in I$. Indeed, let $F \in \mathscr{F}(Y')$. By Lemma 2.13 we have $F \subseteq \phi_i^{-1}[Y_i \setminus X]$ for any $i \in I$ and thus

$$F \subseteq \bigcap_{i \in I} \phi_i^{-1}[Y_i \setminus X] = C = \phi^{-1}[Y \setminus X].$$

Therefore $Y \leq Y'$ again by Lemma 2.13. ∎

The following lemma together with Lemmas 4.19, 4.20 and 4.21 gives an order-theoretic characterization of almost optimal elements of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. Recall that a Tychonoff space X is locally compact if and only if it is open in βX (see Theorem 3.5.8 of [5]). We use this well known fact in the proof of the following lemma.

LEMMA 4.29. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a locally compact locally-$\mathcal{P}$ non-$\mathcal{P}$ space with $\mathcal{Q}$ and let $Y \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. The following are equivalent:*

(1) *Y is almost optimal.*
(2) *For any collection $\{Y_i : i \in I\} \subseteq \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ of one-point extensions of X such that $Y_U \vee \bigvee_{i\in I} Y_i = M_{\mathcal{P}}^X$ we have $Y_U \vee \bigvee_{j=1}^k Y_{i_j} = M_{\mathcal{P}}^X$ for some $k \in \mathbb{N}$ and some $i_1, \dots, i_k \in I$.*

Here $\vee$ is the operation in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$.

Proof. Let $\phi : \beta X \to \beta Y$ and $\phi_U : \beta X \to \beta Y_U$ denote the continuous extensions of id_X. By Lemma 4.22 we have $\phi^{-1}[Y \setminus X] = \phi_U^{-1}[Y_U \setminus X]$.

(1) *implies* (2). Suppose that $\lambda_{\mathcal{P}} X \cap \phi^{-1}[Y \setminus X]$ is compact. Let $\{Y_i : i \in I\} \subseteq \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ be a collection of one-point extensions of X with $Y_U \vee \bigvee_{i\in I} Y_i = M_{\mathcal{P}}^X$. By Lemmas 4.22 and 4.28 we have

$$\phi_U^{-1}[Y_U \setminus X] \cap \bigcap_{i\in I} \phi_i^{-1}[Y_i \setminus X] = \beta X \setminus \lambda_{\mathcal{P}} X$$

where $\phi_i : \beta X \to \beta Y_i$ for any $i \in I$ denotes the continuous extension of id_X. Now

$$\lambda_{\mathcal{P}} X \cap \phi^{-1}[Y \setminus X] \cap \bigcap_{i\in I} \phi_i^{-1}[Y_i \setminus X] = \lambda_{\mathcal{P}} X \cap \phi_U^{-1}[Y_U \setminus X] \cap \bigcap_{i\in I} \phi_i^{-1}[Y_i \setminus X] = \emptyset$$

and therefore by the compactness of $\lambda_{\mathcal{P}} X \cap \phi^{-1}[Y \setminus X]$ it follows that

$$\lambda_{\mathcal{P}} X \cap \phi^{-1}[Y \setminus X] \cap \bigcap_{j=1}^{k} \phi_{i_j}^{-1}[Y_{i_j} \setminus X] = \emptyset$$

for some $k \in \mathbb{N}$ and some $i_1, \dots, i_k \in I$. This implies that

$$\phi^{-1}[Y \setminus X] \cap \bigcap_{j=1}^{k} \phi_{i_j}^{-1}[Y_{i_j} \setminus X] \subseteq \beta X \setminus \lambda_{\mathcal{P}} X.$$

But by Lemma 2.8 we have $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X]$ and $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi_i^{-1}[Y_i \setminus X]$ for any $i \in I$. Thus from the above

$$\phi_U^{-1}[Y_U \setminus X] \cap \bigcap_{j=1}^{k} \phi_{i_j}^{-1}[Y_{i_j} \setminus X] = \phi^{-1}[Y \setminus X] \cap \bigcap_{j=1}^{k} \phi_{i_j}^{-1}[Y_{i_j} \setminus X] = \beta X \setminus \lambda_{\mathcal{P}} X.$$

Lemma 4.28 now implies that $Y_U \vee \bigvee_{j=1}^k Y_{i_j} = M_{\mathcal{P}}^X$.

(2) *implies* (1). First note that $X \subseteq \lambda_{\mathcal{P}} X$ (see Lemma 2.10) and $\beta X \setminus \lambda_{\mathcal{P}} X$ is non-empty (see Lemma 4.10). To show (1) we have to verify that $\lambda_{\mathcal{P}} X \cap \phi^{-1}[Y \setminus X]$ is compact. Note that $\lambda_{\mathcal{P}} X \cap \phi^{-1}[Y \setminus X] \subseteq \lambda_{\mathcal{P}} X \setminus X$, as obviously $\phi^{-1}[Y \setminus X] \subseteq \beta X \setminus X$, because $\phi | X = \mathrm{id}_X$. Let $\{U_i : i \in I\}$ be an open cover of $\lambda_{\mathcal{P}} X \cap \phi^{-1}[Y \setminus X]$ in $\lambda_{\mathcal{P}} X \setminus X$. Note that each U_i, where $i \in I$, is open in $\beta X \setminus X$, as U_i is open in $\lambda_{\mathcal{P}} X \setminus X$ and the latter

is open in $\beta X \setminus X$, because $\lambda_{\mathcal{P}} X$ is open in βX. Let $C_i = (\beta X \setminus X) \setminus U_i$ where $i \in I$. Then C_i is closed in $\beta X \setminus X$ and thus compact, as $\beta X \setminus X$ is compact (because X is locally compact), and it is non-empty, as it contains $\beta X \setminus \lambda_{\mathcal{P}} X$. Let $Y_i = e_{C_i} X$ and let $\phi_i : \beta X \to \beta Y_i$ denote the continuous extension of id_X. By Lemma 4.22 we have

$$\phi_i^{-1}[Y_i \setminus X] = C_i = (\beta X \setminus X) \setminus U_i. \tag{4.6}$$

We have

$$\begin{aligned}\phi^{-1}[Y \setminus X] \cap \bigcap_{i\in I} \phi_i^{-1}[Y_i \setminus X] &= \phi^{-1}[Y \setminus X] \cap \bigcap_{i\in I} \big((\beta X \setminus X) \setminus U_i\big) \\ &= \phi^{-1}[Y \setminus X] \cap \Big((\beta X \setminus X) \setminus \bigcup_{i\in I} U_i\Big) \\ &\subseteq \phi^{-1}[Y \setminus X] \cap \big((\beta X \setminus X) \setminus \big(\lambda_{\mathcal{P}} X \cap \phi^{-1}[Y \setminus X]\big)\big) \\ &\subseteq \beta X \setminus \lambda_{\mathcal{P}} X\end{aligned}$$

and therefore

$$\phi_U^{-1}[Y_U \setminus X] \cap \bigcap_{i\in I} \phi_i^{-1}[Y_i \setminus X] = \phi^{-1}[Y \setminus X] \cap \bigcap_{i\in I} \phi_i^{-1}[Y_i \setminus X] = \beta X \setminus \lambda_{\mathcal{P}} X$$

as by Lemma 2.8 we have $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X]$ and $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi_i^{-1}[Y_i \setminus X]$ for any $i \in I$. Lemma 4.28 now implies that $Y_U \vee \bigvee_{i\in I} Y_i = M_{\mathcal{P}}^X$, which yields $Y_U \vee \bigvee_{j=1}^k Y_{i_j} = M_{\mathcal{P}}^X$ for some $k \in \mathbb{N}$ and some $i_1, \ldots, i_k \in I$. Again, using Lemma 4.28 we have

$$\phi^{-1}[Y \setminus X] \cap \bigcap_{j=1}^k \phi_{i_j}^{-1}[Y_{i_j} \setminus X] = \phi_U^{-1}[Y_U \setminus X] \cap \bigcap_{j=1}^k \phi_{i_j}^{-1}[Y_{i_j} \setminus X] = \beta X \setminus \lambda_{\mathcal{P}} X$$

and thus by (4.6) it follows that

$$\begin{aligned}\phi^{-1}[Y \setminus X] \cap \Big((\beta X \setminus X) \setminus \bigcup_{j=1}^k U_{i_j}\Big) &= \phi^{-1}[Y \setminus X] \cap \bigcap_{j=1}^k \big((\beta X \setminus X) \setminus U_{i_j}\big) \\ &= \phi^{-1}[Y \setminus X] \cap \bigcap_{j=1}^k \phi_{i_j}^{-1}[Y_{i_j} \setminus X] = \beta X \setminus \lambda_{\mathcal{P}} X.\end{aligned}$$

Therefore

$$\lambda_{\mathcal{P}} X \cap \phi^{-1}[Y \setminus X] \cap \Big((\beta X \setminus X) \setminus \bigcup_{j=1}^k U_{i_j}\Big) = \emptyset,$$

which implies that

$$\lambda_{\mathcal{P}} X \cap \phi^{-1}[Y \setminus X] \subseteq \bigcup_{j=1}^k U_{i_j}.$$

That is $\lambda_{\mathcal{P}} X \cap \phi^{-1}[Y \setminus X]$ is compact. ∎

Lemma 4.30. *Let X be a locally compact space and let $Y \in \mathscr{E}(X)$. The following are equivalent:*

(1) *Y is locally compact.*

(2) *$\phi^{-1}[Y \setminus X]$ is open in $\beta X \setminus X$, where $\phi : \beta X \to \beta Y$ is the continuous extension of* id_X.

Proof. Recall that by Lemma 2.9 the space βY is the quotient space of βX obtained by contracting each $\phi^{-1}(p)$ where $p \in Y \setminus X$ to a point, with the continuous extension $\phi : \beta X \to \beta Y$ of id_X as the quotient mapping.

(1) *implies* (2). Note that Y is open in βY and thus $\phi^{-1}[Y]$ is open in βX. Therefore

$$\phi^{-1}[Y \setminus X] = \phi^{-1}[Y] \cap (\beta X \setminus X)$$

is open in $\beta X \setminus X$.

(2) *implies* (1). Let U be an open subset of βX such that $\phi^{-1}[Y \setminus X] = U \cap (\beta X \setminus X)$. Since X is locally compact, it is open in βX. Thus $U \cup X$ is open in βX and therefore $Y = \phi[U \cup X]$ is open in βY. This shows that Y is locally compact. ∎

The following lemma together with Lemmas 4.19, 4.20, 4.21, 4.26 and 4.29 gives an order-theoretic characterization of locally compact elements of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$.

Lemma 4.31. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a locally compact locally-$\mathcal{P}$ non-$\mathcal{P}$ space with $\mathcal{Q}$ and let $Y \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. The following are equivalent:*

(1) *Y is locally compact.*
(2) *There exists an almost optimal one-point extension $Y' \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ such that for any anti-atom T of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ of type* (I)*, either $Y_U \leq T$ or $Y' \leq T$, but not both.*

Proof. Let $\phi : \beta X \to \beta Y$ and $\phi_U : \beta X \to \beta Y_U$ denote the continuous extensions of id_X. By Lemmas 2.8 and 4.22 we have $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X]$ and $\phi^{-1}[Y \setminus X] = \phi_U^{-1}[Y_U \setminus X]$. Also, by Lemmas 2.10 and 4.10 we have $X \subseteq \lambda_{\mathcal{P}} X$ and $\beta X \setminus \lambda_{\mathcal{P}} X$ is non-empty.

(1) *implies* (2). By Lemma 4.30 the set $\phi^{-1}[Y \setminus X]$ is open in $\beta X \setminus X$. Let

$$C = \big((\beta X \setminus X) \setminus \phi^{-1}[Y \setminus X]\big) \cup (\beta X \setminus \lambda_{\mathcal{P}} X).$$

Then $C \subseteq \beta X \setminus X$ is compact, being the union of two compact subspaces, and it is obviously non-empty, as it contains $\beta X \setminus \lambda_{\mathcal{P}} X$. Let $Y' = e_C X$. Then Y' is a Tychonoff one-point extension of X and by Lemma 4.22, if $\psi : \beta X \to \beta Y'$ denotes the continuous extension of id_X then $\psi^{-1}[Y' \setminus X] = C$. Therefore by Lemma 2.8 and Theorem 2.11 we have $Y' \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$, as $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq C$. Also, Y' is almost optimal, as

$$\lambda_{\mathcal{P}} X \cap \psi^{-1}[Y' \setminus X] = \lambda_{\mathcal{P}} X \cap C = (\beta X \setminus X) \setminus \phi^{-1}[Y \setminus X]$$

is compact. (Note that $(\beta X \setminus X) \setminus \phi^{-1}[Y \setminus X] \subseteq \lambda_{\mathcal{P}} X$, as $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X]$.) Now consider an anti-atom T of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ of type (I). Then $T = e_X(\{a, b\})$ for some $a \in \beta X \setminus \lambda_{\mathcal{P}} X$ and $b \in \lambda_{\mathcal{P}} X \setminus X$. Consider the following cases:

Case 1. Suppose that $b \in \phi^{-1}[Y \setminus X]$. Then $a \in \phi^{-1}[Y \setminus X]$, as $a \in \beta X \setminus \lambda_{\mathcal{P}} X$ and thus $\{a, b\} \subseteq \phi_U^{-1}[Y_U \setminus X]$. Lemmas 2.13 and 4.14 now imply that $Y_U \leq T$.

Case 2. Suppose that $b \notin \phi^{-1}[Y \setminus X]$. Then necessarily $b \in C = \psi^{-1}[Y' \setminus X]$. But also $a \in \psi^{-1}[Y' \setminus X]$, as $a \in \beta X \setminus \lambda_{\mathcal{P}} X$ and $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \psi^{-1}[Y' \setminus X]$. Thus $\{a, b\} \subseteq \psi^{-1}[Y' \setminus X]$. Again Lemmas 2.13 and 4.14 imply that $Y' \leq T$.

Obviously, $Y_U \leq T$ and $Y' \leq T$ cannot hold simultaneously, as then $\{a,b\} \subseteq \phi_U^{-1}[Y_U \setminus X]$ and $\{a,b\} \subseteq \psi^{-1}[Y' \setminus X]$. But $b \in \psi^{-1}[Y' \setminus X] = C$ implies, by the choice of b, that $b \in (\beta X \setminus X) \setminus \phi^{-1}[Y \setminus X]$, which is not possible, as $b \in \phi^{-1}[Y \setminus X]$.

(2) *implies* (1). Let $\psi : \beta X \to \beta Y'$ denote the continuous extension of id_X. By Lemma 4.30 to prove that Y is locally compact it suffices to show that $\phi^{-1}[Y \setminus X]$ is open in $\beta X \setminus X$. We show this by verifying that

$$(\beta X \setminus X) \setminus \phi^{-1}[Y \setminus X] = \lambda_{\mathcal{P}} X \cap \psi^{-1}[Y' \setminus X]. \tag{4.7}$$

Choose some $a \in \beta X \setminus \lambda_{\mathcal{P}} X$. Let $b \in (\beta X \setminus X) \setminus \phi^{-1}[Y \setminus X]$ and suppose to the contrary that $b \notin \psi^{-1}[Y' \setminus X]$. Let $T = e_X(\{a,b\})$. Then T is an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ of type (I), and by Lemmas 2.13 and 4.14 neither $Y_U \leq T$ nor $Y' \leq T$, as neither $\{a,b\} \subseteq \phi^{-1}[Y \setminus X]$ nor $\{a,b\} \subseteq \psi^{-1}[Y' \setminus X]$. This is a contradiction. Therefore

$$(\beta X \setminus X) \setminus \phi^{-1}[Y \setminus X] \subseteq \lambda_{\mathcal{P}} X \cap \psi^{-1}[Y' \setminus X]. \tag{4.8}$$

To show the reverse inclusion in (4.8), let $c \in \lambda_{\mathcal{P}} X \cap \psi^{-1}[Y' \setminus X]$. Suppose to the contrary that $c \notin (\beta X \setminus X) \setminus \phi^{-1}[Y \setminus X]$, or equivalently that $c \in \phi^{-1}[Y \setminus X]$, as $c \in \beta X \setminus X$, because $c \in \psi^{-1}[Y' \setminus X]$ and $\psi^{-1}[Y' \setminus X] \subseteq \beta X \setminus X$, since $\psi | X = \mathrm{id}_X$. Let $T' = e_X(\{a,c\})$. Then T' is an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ of type (I) and by Lemmas 2.13 and 4.14 both $Y_U \leq T'$ and $Y' \leq T'$, as both $\{a,c\} \subseteq \phi^{-1}[Y \setminus X]$ and $\{a,c\} \subseteq \psi^{-1}[Y' \setminus X]$, because by Lemma 2.8 we have $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X]$ and $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \psi^{-1}[Y' \setminus X]$ and $a \in \beta X \setminus \lambda_{\mathcal{P}} X$. This contradicts our assumption and proves (4.7). Now since $\lambda_{\mathcal{P}} X \cap \psi^{-1}[Y' \setminus X]$ is compact, as Y' is almost optimal, $(\beta X \setminus X) \setminus \phi^{-1}[Y \setminus X]$ is compact and thus closed in $\beta X \setminus X$. Equivalently, $\phi^{-1}[Y \setminus X]$ is open in $\beta X \setminus X$, as $\phi^{-1}[Y \setminus X] \subseteq \beta X \setminus X$, because $\phi | X = \mathrm{id}_X$. ∎

The following lemma together with Lemmas 4.19 and 4.20 gives an order-theoretic characterization of optimal elements of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$.

LEMMA 4.32. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff locally-$\mathcal{P}$ non-$\mathcal{P}$ space with $\mathcal{Q}$ and let $Y \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. The following are equivalent:*

(1) $Y \in \mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(X)$.
(2) *There exists no anti-atom T in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ of type* (I) *with $Y \leq T$.*

Proof. Let $\phi : \beta X \to \beta Y$ denote the continuous extension of id_X.

(1) *implies* (2). Let T be an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ such that $Y \leq T$. Let $T = e_X(\{a,b\})$ where $a,b \in \beta X \setminus X$ are distinct such that either $a \notin \lambda_{\mathcal{P}} X$ or $b \notin \lambda_{\mathcal{P}} X$. By Lemmas 2.13 and 4.14 we have $\{a,b\} \subseteq F$ for some $F \in \mathscr{F}(Y)$. But $F \subseteq \phi^{-1}[Y \setminus X]$ and $\phi^{-1}[Y \setminus X] = \beta X \setminus \lambda_{\mathcal{P}} X$ by Theorem 2.15. Thus $\{a,b\} \subseteq \beta X \setminus \lambda_{\mathcal{P}} X$, which shows that T is of type (II).

(2) *implies* (1). By Theorem 2.15 to show (1) it suffices to show that $\phi^{-1}[Y \setminus X] = \beta X \setminus \lambda_{\mathcal{P}} X$. Suppose otherwise. By Lemma 2.8 we have $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X]$. Thus $\phi^{-1}[Y \setminus X] \nsubseteq \beta X \setminus \lambda_{\mathcal{P}} X$. Choose $b \in \phi^{-1}[Y \setminus X]$ such that $b \notin \beta X \setminus \lambda_{\mathcal{P}} X$. Then $b \in \phi^{-1}(p)$ for some $p \in Y \setminus X$. By Theorem 2.11 the set $\phi^{-1}(p) \setminus \lambda_{\mathcal{P}} X$ is non-empty. Choose an $a \in \phi^{-1}(p) \setminus \lambda_{\mathcal{P}} X$. Note that $a,b \in \beta X \setminus X$, as $a,b \in \phi^{-1}[Y \setminus X]$ and $\phi | X = \mathrm{id}_X$. Consider the anti-atom $T = e_X(\{a,b\})$ of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. Then T is of type (I), and since $\{a,b\} \subseteq \phi^{-1}(p)$, by Lemmas 2.13 and 4.14 we have $Y \leq T$. This is a contradiction. ∎

The following is an immediate corollary of our previous lemmas.

LEMMA 4.33. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X and Y be Tychonoff locally-$\mathcal{P}$ non-$\mathcal{P}$ spaces with $\mathcal{Q}$ such that* $\operatorname{card}(\lambda_{\mathcal{P}}X \setminus X) \geq 2$ *and* $\operatorname{card}(\lambda_{\mathcal{P}}Y \setminus Y) \geq 2$. *Let*

$$\Theta : \big(\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X), \leq\big) \to \big(\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(Y), \leq\big)$$

be an order-isomorphism. Let $T \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. Then

(1) *If T is an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ (an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ of type* (I)*, an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ of type* (II)*, respectively), then so is $\Theta(T)$.*
(2) *If T is optimal, then so is $\Theta(T)$.*
(3) *If T is a one-point extension, then so is $\Theta(T)$.*

Suppose that X and Y are moreover locally compact. Then

(4) *If T is almost optimal, then so is $\Theta(T)$.*
(5) *If T is locally compact, then so is $\Theta(T)$.*

Proof. This follows from the previous lemmas, as Lemmas 4.19 and 4.20 imply (1), part (1) and Lemma 4.32 imply (2), part (1) and Lemma 4.21 imply (3), part (3) and Lemma 4.29 imply (4), and finally parts (1), (3), (4) and Lemma 4.31 imply (5), noting that by Lemma 4.26 (and parts (1) and (3)) we have $\Theta(S_U) = (\Theta(S))_U$ for any $S \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. ∎

LEMMA 4.34. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff locally-$\mathcal{P}$ non-$\mathcal{P}$ space with $\mathcal{Q}$. Let $T = e_X(\{a, b\})$ be an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ of type* (I) *where $a \notin \lambda_{\mathcal{P}}X$ and $b \in \lambda_{\mathcal{P}}X$. Let $T' = e_X(\{c, d\})$ be an anti-atom in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ of type* (I) *such that $T \neq T'$. The following are equivalent:*

(1) $b \notin \{c, d\}$.
(2) $\operatorname{card}(\{T'' : T''$ *is an anti-atom in* $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ *and* $T \wedge T' \leq T''\}) = 2$.

Proof. (1) *implies* (2). Consider the following cases:

CASE 1. Suppose that $a \in \{c, d\}$, say $a = c$. By Lemma 4.16 we have

$$T \wedge T' = e_X\big(\{a, b\}\big) \wedge e_X\big(\{a, d\}\big) = e_X\big(\{a, b, d\}\big).$$

Now using Lemmas 2.13, 4.14 and 4.17 there are only two anti-atoms T'' in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ with $T \wedge T' \leq T''$, namely $e_X(\{a, b\})$ and $e_X(\{a, d\})$.

CASE 2. Suppose that $a \notin \{c, d\}$. Again by Lemma 4.16 we have

$$T \wedge T' = e_X\big(\{a, b\}\big) \wedge e_X\big(\{c, d\}\big) = e_X\big(\{a, b\}, \{c, d\}\big)$$

and thus as above there are only 2 anti-atoms T'' in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ with $T \wedge T' \leq T''$, namely $e_X(\{a, b\})$ and $e_X(\{c, d\})$.

Therefore (2) holds in either case.

(2) *implies* (1). Suppose to the contrary that $b \in \{c, d\}$, say $b = c$. Note that using Lemmas 2.13, 4.14 and 4.17 it follows that $a \neq d$, as $T \neq T'$, and thus there are exactly three anti-atoms T'' in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ with $T \wedge T' \leq T''$, namely $e_X(\{a, b\}), e_X(\{a, d\})$ and $e_X(\{b, d\})$. This is a contradiction. ∎

The following lemma gives an internal (to X) characterization of spaces X with $\operatorname{card}(\lambda_{\mathcal{P}}X \setminus X) \geq 2$. This assumption has been used before in several statements.

LEMMA 4.35. *Let X be a Tychonoff space and let $\mathcal{P}$ be a clopen hereditary finitely additive perfect topological property. The following are equivalent:*

(1) $\operatorname{card}(\lambda_{\mathcal{P}}X \setminus X) \geq 2$.
(2) *There exist a pair of disjoint non-compact zero-sets of X each contained in a cozero-set of X whose closure (in X) has $\mathcal{P}$.*

Proof. (1) *implies* (2). Let $z_i \in \lambda_{\mathcal{P}}X \setminus X$ where $i = 1, 2$ be distinct and let U_i be an open neighborhood of z_i in $\lambda_{\mathcal{P}}X$ (and therefore in βX, as $\lambda_{\mathcal{P}}X$ is open in βX) such that $U_1 \cap U_2 = \emptyset$. Let $f_i : \beta X \to \mathbb{I}$ be continuous with $f(z_i) = 0$ and $f_i[\beta X \setminus U_i] \subseteq \{1\}$ and let

$$Z_i = f_i^{-1}\big[[0, 1/3]\big] \cap X \in \mathscr{Z}(X) \quad \text{and} \quad C_i = f_i^{-1}\big[[0, 1/2)\big] \cap X \in \operatorname{Coz}(X).$$

Note that $Z_i \subseteq f_i^{-1}[[0, 1/3]] \subseteq U_i$ and thus $Z_1 \cap Z_2 = \emptyset$. Also, since

$$z_i \in f_i^{-1}\big[[0, 1/3)\big] \subseteq \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X}\big(f_i^{-1}\big[[0, 1/3]\big] \cap X\big) \subseteq \operatorname{cl}_{\beta X}\big(f_i^{-1}\big[[0, 1/3]\big] \cap X\big) = \operatorname{cl}_{\beta X} Z_i$$

the set $\operatorname{cl}_{\beta X} Z_i \setminus X$ is non-empty and therefore Z_i is non-compact. Then

$$\operatorname{cl}_{\beta X} C_i = \operatorname{cl}_{\beta X}\big(f_i^{-1}\big[[0, 1/2)\big] \cap X\big) = \operatorname{cl}_{\beta X} f_i^{-1}\big[[0, 1/2)\big] \subseteq f_i^{-1}\big[[0, 1/2]\big] \subseteq U_i \subseteq \lambda_{\mathcal{P}}X$$

and thus by Lemma 2.4 the closure $\operatorname{cl}_X C_i$ has $\mathcal{P}$.

(2) *implies* (1). Let $Z_i \in \mathscr{Z}(X)$ be non-compact with $Z_i \subseteq C_i$ where $C_i \in \operatorname{Coz}(X)$ and $\operatorname{cl}_X C_i$ has $\mathcal{P}$ and $Z_1 \cap Z_2 = \emptyset$. By Lemma 2.14 we have $\operatorname{cl}_{\beta X} Z_i \subseteq \lambda_{\mathcal{P}}X$. Since Z_i is non-compact, $\operatorname{cl}_{\beta X} Z_i \setminus X$ is non-empty. Also

$$\operatorname{cl}_{\beta X} Z_1 \cap \operatorname{cl}_{\beta X} Z_2 = \operatorname{cl}_{\beta X}(Z_1 \cap Z_2) = \emptyset.$$

Therefore $\operatorname{card}(\lambda_{\mathcal{P}}X \setminus X) \geq 2$. ∎

THEOREM 4.36. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X and Y be Tychonoff locally-$\mathcal{P}$ non-$\mathcal{P}$ spaces with $\mathcal{Q}$ such that each space contains a pair of disjoint non-compact zero-sets each contained in a cozero-set whose closure has $\mathcal{P}$. Consider the following:*

(1) *$(\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X), \leq)$ and $(\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(Y), \leq)$ are order-isomorphic.*
(2) *$\beta X \setminus \lambda_{\mathcal{P}}X$ and $\beta Y \setminus \lambda_{\mathcal{P}}Y$ are homeomorphic.*

Then (1) *implies* (2)*, while* (2) *does not necessarily imply* (1).

Proof. By Lemma 4.35 we have $\operatorname{card}(\lambda_{\mathcal{P}}X \setminus X) \geq 2$ and $\operatorname{card}(\lambda_{\mathcal{P}}Y \setminus Y) \geq 2$. To show that (1) implies (2) let

$$\Theta : \big(\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X), \leq\big) \to \big(\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(Y), \leq\big)$$

denote an order-isomorphism. By Lemma 4.33 we have $\Theta(\mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(X)) \subseteq \mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(Y)$. Now, since

$$\Theta^{-1} : \big(\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(Y), \leq\big) \to \big(\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X), \leq\big)$$

is also an order-isomorphism, again, using Lemma 4.33 we have $\Theta^{-1}(\mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(Y)) \subseteq \mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(X)$, or equivalently, $\mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(Y) \subseteq \Theta(\mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(X))$. Therefore $\Theta(\mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(X)) = \mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(Y)$. Thus

$$\Theta|\mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(X) : \big(\mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(X), \leq\big) \to \big(\mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(Y), \leq\big)$$

is an order-isomorphism. By Theorem 4.13 this now implies (2).

Next, by means of an example, we show that (2) does not necessarily imply (1). Let

$$X = \mathbb{N} \oplus \bigoplus_{i<\Omega} R_i \quad \text{and} \quad Y = \bigoplus_{i<\Omega} R_i$$

where R_i for any $i < \Omega$ is the subspace $[0, \infty)$ of $\mathbb{R}$. Let $\mathcal{P}$ be the Lindelöf property and let $\mathcal{Q}$ be regularity. Then $\mathcal{P}$ and $\mathcal{Q}$ is a pair of compactness-like topological properties (see Example 2.16) and X and Y are locally compact non-Lindelöf spaces each containing a pair of disjoint non-compact clopen Lindelöf subsets. (Just consider R_i and R_j for some distinct $i, j < \Omega$ as the desired pair.) Note that

$$\lambda_{\mathcal{P}} X = \operatorname{cl}_{\beta X} \mathbb{N} \cup \bigcup \Big\{ \operatorname{cl}_{\beta X} \Big(\bigcup_{i \in J} R_i \Big) : J \subseteq [0, \Omega) \text{ is countable} \Big\} \tag{4.9}$$

and

$$\lambda_{\mathcal{P}} Y = \bigcup \Big\{ \operatorname{cl}_{\beta Y} \Big(\bigcup_{i \in J} R_i \Big) : J \subseteq [0, \Omega) \text{ is countable} \Big\}, \tag{4.10}$$

since, for example, in the first case, for any Lindelöf $Z \in \mathscr{Z}(X)$ we have

$$Z \subseteq \mathbb{N} \cup \bigcup_{i \in J} R_i$$

for some countable $J \subseteq [0, \Omega)$, and conversely, if $J \subseteq [0, \Omega)$ is countable then

$$S = \mathbb{N} \cup \bigcup_{i \in J} R_i$$

is a clopen Lindelöf subset of X, and thus $\operatorname{cl}_{\beta X} S = \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} S \subseteq \lambda_{\mathcal{P}} X$. (Note that a clopen subset of a Tychonoff space has a clopen closure in its Stone–Čech compactification; see Corollary 3.6.5 of [5].) We now verify that $\beta X \setminus \lambda_{\mathcal{P}} X$ and $\beta Y \setminus \lambda_{\mathcal{P}} Y$ are homeomorphic. Since X contains Y as a closed subspace (and it is normal) the spaces $\operatorname{cl}_{\beta X} Y$ and βY are equivalent compactifications of Y (see Corollary 3.6.8 of [5]). Therefore for any countable $J \subseteq [0, \Omega)$ we have

$$\operatorname{cl}_{\beta Y} \Big(\bigcup_{i \in J} R_i \Big) = \operatorname{cl}_{(\operatorname{cl}_{\beta X} Y)} \Big(\bigcup_{i \in J} R_i \Big) = \operatorname{cl}_{\beta X} \Big(\bigcup_{i \in J} R_i \Big) \cap \operatorname{cl}_{\beta X} Y = \operatorname{cl}_{\beta X} \Big(\bigcup_{i \in J} R_i \Big).$$

Thus by (4.10) we have

$$\lambda_{\mathcal{P}} Y = \bigcup \Big\{ \operatorname{cl}_{\beta X} \Big(\bigcup_{i \in J} R_i \Big) : J \subseteq [0, \Omega) \text{ is countable} \Big\}. \tag{4.11}$$

Then by (4.9) we have

$$\lambda_{\mathcal{P}} X = \operatorname{cl}_{\beta X} \mathbb{N} \cup \lambda_{\mathcal{P}} Y$$

and also, since $X = \mathbb{N} \cup Y$,

$$\beta X = \operatorname{cl}_{\beta X}(\mathbb{N} \cup Y) = \operatorname{cl}_{\beta X} \mathbb{N} \cup \operatorname{cl}_{\beta X} Y = \operatorname{cl}_{\beta X} \mathbb{N} \cup \beta Y.$$

Note that for any countable $J \subseteq [0, \Omega)$ we have

$$\operatorname{cl}_{\beta X} \mathbb{N} \cap \operatorname{cl}_{\beta X} \Big(\bigcup_{i \in J} R_i \Big) = \emptyset$$

as $\mathbb{N}$ and $\bigcup_{i \in J} R_i$ are disjoint zero-sets (in fact clopen subsets) of X. Therefore by (4.11),

$$\operatorname{cl}_{\beta X} \mathbb{N} \cap \lambda_{\mathcal{P}} Y = \emptyset.$$

Also

$$\mathrm{cl}_{\beta X}\,\mathbb{N}\cap\beta Y=\mathrm{cl}_{\beta X}\,\mathbb{N}\cap\mathrm{cl}_{\beta X}\,Y=\emptyset.$$

It now follows that

$$\beta X\setminus\lambda_{\mathcal{P}}X=(\mathrm{cl}_{\beta X}\,\mathbb{N}\cup\beta Y)\setminus(\mathrm{cl}_{\beta X}\,\mathbb{N}\cup\lambda_{\mathcal{P}}Y)=\beta Y\setminus\lambda_{\mathcal{P}}Y.$$

This shows (2). Next, we show that the partially ordered sets $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ and $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(Y)$ are not order-isomorphic. But first, we need to prove the following.

Claim. *Let D be a non-empty clopen subset of $\beta Y\setminus Y$. Then $\mathrm{cl}_{\beta Y}\,R_i\setminus Y\subseteq D$ for some $i<\Omega$.*

Proof of the claim. Let $g:\beta Y\setminus Y\to\mathbb{I}$ be continuous with

$$g[D]\subseteq\{0\}\quad\text{and}\quad g\big[(\beta Y\setminus Y)\setminus D\big]\subseteq\{1\}.$$

Since Y is locally compact, $\beta Y\setminus Y$ is closed in (the normal space) βY and thus by the Tietze–Urysohn Theorem $g=G|(\beta Y\setminus Y)$ for some continuous $G:\beta Y\to\mathbb{I}$. Let

$$V=G^{-1}\big[[0,1/2)\big]\cap Y.$$

Then V is an open subset of Y. Since

$$\begin{aligned}G^{-1}\big[[0,1/2)\big]\setminus Y&\subseteq\mathrm{cl}_{\beta Y}\,G^{-1}\big[[0,1/2)\big]\setminus Y=\mathrm{cl}_{\beta Y}\big(G^{-1}\big[[0,1/2)\big]\cap Y\big)\setminus Y\\&\subseteq G^{-1}\big[[0,1/2]\big]\setminus Y\end{aligned}$$

and

$$G^{-1}\big[[0,1/2)\big]\setminus Y=g^{-1}\big[[0,1/2)\big]=D=g^{-1}\big[[0,1/2]\big]=G^{-1}\big[[0,1/2]\big]\setminus Y$$

it follows that

$$D=\mathrm{cl}_{\beta Y}\big(G^{-1}\big[[0,1/2)\big]\cap Y\big)\setminus Y=\mathrm{cl}_{\beta Y}\,V\setminus Y.$$

Also, $\mathrm{bd}_Y\,V$ is compact, as

$$\begin{aligned}\mathrm{bd}_Y\,V=\mathrm{cl}_Y\,V\setminus V&\subseteq\big(G^{-1}\big[[0,1/2]\big]\cap Y\big)\setminus\big(G^{-1}\big[[0,1/2)\big]\cap Y\big)\\&=\big(G^{-1}\big[[0,1/2]\big]\setminus G^{-1}\big[[0,1/2)\big]\big)\cap Y\\&=G^{-1}(1/2)\cap Y\subseteq G^{-1}(1/2),\end{aligned}$$

which implies that

$$\mathrm{cl}_{\beta Y}\,\mathrm{bd}_Y\,V\setminus Y\subseteq G^{-1}(1/2)\setminus Y=g^{-1}(1/2)=\emptyset.$$

Therefore $\mathrm{cl}_{\beta Y}\,\mathrm{bd}_Y\,V\subseteq Y$ and thus $\mathrm{bd}_Y\,V=\mathrm{cl}_{\beta Y}\,\mathrm{bd}_Y\,V\cap Y=\mathrm{cl}_{\beta Y}\,\mathrm{bd}_Y\,V$ is compact, as it is closed in βY. Let

$$H=\{i<\Omega:\mathrm{bd}_Y\,V\cap R_i\neq\emptyset\}.$$

Note that H is finite, as $\mathrm{bd}_Y\,V$ is compact. To prove the claim suppose to the contrary that $\mathrm{cl}_{\beta Y}\,R_i\setminus Y\not\subseteq D$ for any $i<\Omega$. Then

$$\mathrm{cl}_{\beta Y}\,R_i\setminus Y=\mathrm{cl}_{\beta Y}\,R_i\setminus R_i=\beta R_i\setminus R_i$$

as $\operatorname{cl}_{\beta Y} R_i$ and βR_i are equivalent compactifications of R_i, because R_i is closed in Y (and Y is normal), and therefore since $\beta R_i \setminus R_i$ is connected (see Problem 6L of [8]), we have

$$\operatorname{cl}_{\beta Y} R_i \cap D = (\operatorname{cl}_{\beta Y} R_i \setminus Y) \cap D = \emptyset \tag{4.12}$$

for any $i < \Omega$. Now let $i < \Omega$ be such that $V \cap R_i$ is non-empty. If $\operatorname{bd}_{R_i}(V \cap R_i) = \emptyset$ then $V \cap R_i$ is clopen in R_i, and since R_i is connected, $V \cap R_i = R_i$, that is, $R_i \subseteq V$. But then

$$\emptyset \neq \beta R_i \setminus R_i = \operatorname{cl}_{\beta Y} R_i \setminus Y \subseteq \operatorname{cl}_{\beta Y} V \setminus Y = D,$$

which by (4.12) cannot be true. Thus

$$\operatorname{bd}_Y V \cap R_i = \operatorname{bd}_{R_i}(V \cap R_i) \neq \emptyset,$$

that is, $i \in H$. Therefore $V \subseteq \bigcup_{i \in H} R_i$. Now

$$D = \operatorname{cl}_{\beta Y} V \setminus Y \subseteq \operatorname{cl}_{\beta Y} V \subseteq \operatorname{cl}_{\beta Y}\Big(\bigcup_{i \in H} R_i\Big) = \bigcup_{i \in H} \operatorname{cl}_{\beta Y} R_i,$$

which again contradicts (4.12), as D is non-empty. This proves the claim.

Now we prove that $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ and $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(Y)$ are not order-isomorphic. Suppose the contrary and let

$$\Theta : \big(\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X), \leq\big) \to \big(\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(Y), \leq\big)$$

denote an order-isomorphism. Let

$$C = (\beta X \setminus X) \setminus \operatorname{cl}_{\beta X} \mathbb{N}.$$

Then C is a clopen non-empty subset of $\beta X \setminus X$. (Note that $\beta X \setminus X$ is closed in βX, as X is locally compact, and $\operatorname{cl}_{\beta X} \mathbb{N}$ is clopen in βX, as $\mathbb{N}$ is clopen in X.) Let $T = e_C X$. Then T is a one-point Tychonoff extension of X, which by Lemmas 4.22 and 4.30 is locally compact. By Lemmas 2.8 and 4.22 and Theorem 2.11 we have $T \in \mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$. Let $S = \Theta(T)$. Then by Lemma 4.33 the element S is a one-point locally compact extension of Y. Denote by $\psi : \beta Y \to \beta S$ the continuous extension of id_Y. Then $D = \psi^{-1}(S \setminus Y)$ is clopen in $\beta Y \setminus Y$ by Lemma 4.30, and obviously $D \neq \beta Y \setminus Y$, as S is not the smallest element in $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(Y)$, because T is not the smallest element in $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$, since $C \neq \beta X \setminus X$. By the above claim, $\operatorname{cl}_{\beta Y} R_i \setminus Y \subseteq (\beta Y \setminus Y) \setminus D$ for some $i < \Omega$. Choose some distinct $b', c' \in \operatorname{cl}_{\beta Y} R_i \setminus Y$ (which exist, as by the above $\operatorname{cl}_{\beta Y} R_i \setminus Y = \beta R_i \setminus R_i$) and choose $a' \in \beta Y \setminus \lambda_{\mathcal{P}} Y$ (which exists, as Y is non-Lindelöf; see Lemma 4.10). Let

$$S' = e_Y\big(\{a', b'\}\big) \quad \text{and} \quad S'' = e_Y\big(\{a', c'\}\big)$$

and let

$$T' = \Theta^{-1}(S') \quad \text{and} \quad T'' = \Theta^{-1}(S'').$$

By Lemma 4.33 both T' and T'' are anti-atoms in $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(X)$ of type (I), as S' and S'' are anti-atoms in $\mathscr{M}^{\mathcal{Q}}_{\mathcal{P}}(Y)$ of type (I). Let

$$T' = e_X\big(\{a, b\}\big) \quad \text{and} \quad T'' = e_X\big(\{c, d\}\big)$$

where $b, d \in \lambda_{\mathcal{P}} X$. Note that $b' \notin \{a', c'\}$, which by Lemma 4.34 yields $b \notin \{c, d\}$. We have neither $T \leq T'$ nor $T \leq T''$, as neither

$$\Theta(T) = S \leq S' = \Theta(T') \ \text{ nor } \ \Theta(T) = S \leq S'' = \Theta(T'')$$

because $b', c' \notin D$; see Lemmas 2.13 and 4.14. Thus again by Lemmas 2.13, 4.14 and 4.22 neither $\{a, b\} \subseteq C$ nor $\{c, d\} \subseteq C$, which implies that $b, d \notin C$, or $b, d \in \mathrm{cl}_{\beta X} \mathbb{N}$, as $a, c \in \beta X \setminus \lambda_{\mathcal{P}} X \subseteq C$. But, since $\mathrm{cl}_{\beta X} \mathbb{N} \setminus \mathbb{N}$ is zero-dimensional, there exists a clopen subset E of $\beta X \setminus X$ containing b, but not d, such that it contains $\beta X \setminus \lambda_{\mathcal{P}} X$. This by Lemmas 4.22 and 4.33 corresponds to a one-point locally compact element of $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$, namely, $e_E X$. By Lemma 4.33 the element $\Theta(e_E X)$ is a one-point locally compact element in $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(Y)$. Now if $\mathscr{F}_Y(\Theta(e_E X)) = \{G\}$, then by Lemma 4.30 the set G is a clopen subset of $\beta Y \setminus Y$, and neither

$$\mathrm{cl}_{\beta Y} R_i \setminus Y \subseteq G \quad \text{nor} \quad (\mathrm{cl}_{\beta Y} R_i \setminus Y) \cap G = \emptyset$$

as $b' \in G$ and $c' \notin G$, because $\{a', b'\} \subseteq G$ and $\{a', c'\} \nsubseteq G$, since $\Theta(e_E X) \leq S'$ and $\Theta(e_E X) \nleq S''$, as

$$e_E X \leq \Theta^{-1}(S') = T' \quad \text{and} \quad e_E X \nleq \Theta^{-1}(S'') = T''$$

because $\{a, b\} \subseteq E$ and $\{c, d\} \nsubseteq E$, again by Lemmas 2.13, 4.14 and 4.22. This contradicts the fact that $\mathrm{cl}_{\beta Y} R_i \setminus Y$ is connected. ∎

In the final result of this chapter we introduces the largest (with respect to the partial order $\leq$) compactification-like $\mathcal{P}$-extension of a Tychonoff space X. This largest element (also introduced in the proof of Lemma 4.17) turns out to be a familiar subspace of the Stone–Čech compactification of X. We formally define this element and prove some of its properties which characterize it among all $\mathcal{P}$-extensions of X with compact remainder.

THEOREM 4.37. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff locally-$\mathcal{P}$ non-$\mathcal{P}$ space with $\mathcal{Q}$. Consider the subspace*

$$\zeta_{\mathcal{P}} X = X \cup (\beta X \setminus \lambda_{\mathcal{P}} X)$$

of βX. Then

(1) *$\zeta_{\mathcal{P}} X$ is the largest element (with respect to $\leq$) of either $\mathscr{E}_{\mathcal{P}}^{\mathcal{Q}}(X)$, $\mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$ or $\mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(X)$.*
(2) *For any $Y \in \mathscr{E}_{\mathcal{P}}^{\mathcal{Q}}(X)$ consider the following properties:*
   (a) *For any $S, Z \in \mathscr{Z}(X)$ where $S \cap Z \subseteq C$ for some $C \in \mathrm{Coz}(X)$ such that $\mathrm{cl}_X C$ has $\mathcal{P}$, we have $\mathrm{cl}_Y S \cap \mathrm{cl}_Y Z \subseteq X$.*
   (b) *Y satisfies the following:*
      (i) *For any $S, Z \in \mathscr{Z}(X)$ we have*

$$\mathrm{cl}_Y(S \cap Z) \setminus X = (\mathrm{cl}_Y S \cap \mathrm{cl}_Y Z) \setminus X.$$

      (ii) *For any $Z \in \mathscr{Z}(X)$ where $Z \subseteq C$ for some $C \in \mathrm{Coz}(X)$ such that $\mathrm{cl}_X C$ has $\mathcal{P}$, we have $\mathrm{cl}_Y Z \subseteq X$.*

Then $\zeta_{\mathcal{P}} X$ is characterized in $\mathscr{E}_{\mathcal{P}}^{\mathcal{Q}}(X)$ by either of the above properties.

Proof. (1). By Lemma 2.10 we have $X \subseteq \lambda_{\mathcal{P}} X$. Thus $\zeta_{\mathcal{P}} X$ is a Tychonoff extension of X with the compact remainder $\zeta_{\mathcal{P}} X \setminus X = \beta X \setminus \lambda_{\mathcal{P}} X$. Since $X \subseteq \zeta_{\mathcal{P}} X \subseteq \beta X$ we have $\beta \zeta_{\mathcal{P}} X = \beta X$ (see Corollary 3.6.9 of [5]). Therefore by Lemma 2.8 (with $f = \mathrm{id}_X$ and $\phi = \mathrm{id}_{\beta X}$) we have $\zeta_{\mathcal{P}} X \in \mathscr{E}_{\mathcal{P}}^{\mathcal{Q}}(X)$ and by Theorem 2.15 it follows that $\zeta_{\mathcal{P}} X \in \mathscr{O}_{\mathcal{P}}^{\mathcal{Q}}(X)$ and thus $\zeta_{\mathcal{P}} X \in \mathscr{M}_{\mathcal{P}}^{\mathcal{Q}}(X)$. Now let $Y \in \mathscr{E}_{\mathcal{P}}^{\mathcal{Q}}(X)$ and let $\phi : \beta X \to \beta Y$ be the continuous

extension of id_X. By Lemma 2.8 we have $\beta X \setminus \lambda_{\mathcal{P}} X \subseteq \phi^{-1}[Y \setminus X]$. Therefore

$$\begin{aligned}\phi[\zeta_{\mathcal{P}} X] &= \phi\big[X \cup (\beta X \setminus \lambda_{\mathcal{P}} X)\big] = \phi[X] \cup \phi[\beta X \setminus \lambda_{\mathcal{P}} X] \\ &= X \cup \phi[\beta X \setminus \lambda_{\mathcal{P}} X] \subseteq X \cup \phi\big[\phi^{-1}[Y \setminus X]\big] \subseteq X \cup (Y \setminus X) = Y\end{aligned}$$

and thus $\phi|\zeta_{\mathcal{P}} X : \zeta_{\mathcal{P}} X \to Y$. Since the latter fixes X pointwise this shows that $Y \leq \zeta_{\mathcal{P}} X$. Therefore Y is the largest element of $\mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$.

(2). Let $Y = \zeta_{\mathcal{P}} X$. We show that Y satisfies (2.a) and (2.b). To show (2.a) suppose that $S, Z \in \mathscr{Z}(X)$ are such that $S \cap Z \subseteq C$ for some $C \in \mathrm{Coz}(X)$ such that $\mathrm{cl}_X C$ has $\mathcal{P}$. Then by Lemma 2.14 we have $\mathrm{cl}_{\beta X}(S \cap Z) \subseteq \lambda_{\mathcal{P}} X$ and thus

$$\mathrm{cl}_Y S \cap \mathrm{cl}_Y Z = \mathrm{cl}_{\beta X} S \cap \mathrm{cl}_{\beta X} Z \cap Y = \mathrm{cl}_{\beta X}(S \cap Z) \cap Y \subseteq \lambda_{\mathcal{P}} X \cap Y \subseteq X.$$

To show (2.b) note that for any $S, Z \in \mathscr{Z}(X)$ we have

$$\mathrm{cl}_Y(S \cap Z) = \mathrm{cl}_{\beta X}(S \cap Z) \cap Y = \mathrm{cl}_{\beta X} S \cap \mathrm{cl}_{\beta X} Z \cap Y = \mathrm{cl}_Y S \cap \mathrm{cl}_Y Z.$$

Therefore (2.b.i) holds. Note that since (2.a) holds, (2.b.ii) holds as well.

Now suppose that some $Y \in \mathscr{E}^{\mathcal{Q}}_{\mathcal{P}}(X)$ satisfies (2.a). Let $\phi : \beta X \to \beta Y$ be the continuous extension of id_X. Recall the construction of βY and the representation of ϕ given in Lemma 2.9. Note that (2.a) in particular implies that $\mathrm{cl}_Y Z \cap (Y \setminus X) = \emptyset$ for any $Z \in \mathscr{Z}(X)$ such that $Z \subseteq C$ for some $C \in \mathrm{Coz}(X)$ such that $\mathrm{cl}_X C$ has $\mathcal{P}$. Thus by Theorem 2.15 we have $Y \in \mathscr{O}_{\mathcal{P}}(X)$ and therefore $\phi^{-1}[Y \setminus X] = \beta X \setminus \lambda_{\mathcal{P}} X$. We show that for any $p \in Y \setminus X$ the set $\phi^{-1}(p)$ consists of a single point, which implies that $Y = \zeta_{\mathcal{P}} X$. Suppose to the contrary that for some $p \in Y \setminus X$ there exist distinct $a, b \in \phi^{-1}(p)$. Let $f : \beta X \to \mathbb{I}$ be continuous with $f(a) = 0$ and $f(b) = 1$. Let

$$S = f^{-1}\big[[0, 1/3]\big] \cap X \quad \text{and} \quad Z = f^{-1}\big[[2/3, 1]\big] \cap X.$$

Then $S, Z \in \mathscr{Z}(X)$ and $S \cap Z = \emptyset$. Thus (and since X is a non-empty Tychonoff locally-$\mathcal{P}$ space, and $\mathcal{P}$ is hereditary with respect to closed subsets of Hausdorff spaces and thus containing some $C \in \mathrm{Coz}(X)$ such that $\mathrm{cl}_X C$ has $\mathcal{P}$) by our assumption $\mathrm{cl}_Y S \cap \mathrm{cl}_Y Z \subseteq X$. We show that $p \in \mathrm{cl}_Y S \cap \mathrm{cl}_Y Z$; this contradiction proves that $Y = \zeta_{\mathcal{P}} X$. Let V be an open neighborhood of p in Y and let the open subset V' of βY be such that $V = V' \cap Y$. Then $\phi^{-1}[V'] \cap f^{-1}[[0, 1/3)]$ and $\phi^{-1}[V'] \cap f^{-1}[(2/3, 1]]$ are open neighborhoods of a and b in βX, respectively, and therefore have non-empty intersection with X. Note that

$$\phi^{-1}[V'] \cap f^{-1}\big[[0, 1/3)\big] \cap X \subseteq S \cap V \quad \text{and} \quad \phi^{-1}[V'] \cap f^{-1}\big[(2/3, 1]\big] \cap X \subseteq Z \cap V.$$

Thus $p \in \mathrm{cl}_Y S \cap \mathrm{cl}_Y Z$. Finally, we show that (2.b) implies (2.a). This together with the above proves the theorem. Suppose that (2.b) holds. Let $S, Z \in \mathscr{Z}(X)$ be such that $S \cap Z \subseteq C$ for some $C \in \mathrm{Coz}(X)$ such that $\mathrm{cl}_X C$ has $\mathcal{P}$. Then by (2.b.ii) we have $\mathrm{cl}_Y(S \cap Z) \subseteq X$. Therefore using (2.b.i) we have

$$(\mathrm{cl}_Y S \cap \mathrm{cl}_Y Z) \setminus X = \mathrm{cl}_Y(S \cap Z) \setminus X = \emptyset$$

and thus $\mathrm{cl}_Y S \cap \mathrm{cl}_Y Z \subseteq X$. ∎

5. Applications

5.1. Tight $\mathcal{P}$-extensions. In [20] K. D. Magill Jr. proved the following theorem relating the order-structure of the set of all compactifications of a locally compact space X to the topology of $\beta X \setminus X$. Recall that order-isomorphic lattices are called *lattice-isomorphic*.

THEOREM 5.1 (Magill [20]). *Let X and Y be locally compact non-compact spaces. The following are equivalent:*

(1) $(\mathscr{K}(X), \leq)$ *and* $(\mathscr{K}(Y), \leq)$ *are lattice-isomorphic.*
(2) $\beta X \setminus X$ *and* $\beta Y \setminus Y$ *are homeomorphic.*

The idea of generalizing the above result led J. Mack, M. Rayburn and R. G. Woods [17] to introduce and study a new class of extensions. We state some definitions and results from [17] below. The reader may find it useful to compare these results with those we have already obtained in the previous chapter.

Let X be a Tychonoff space and let $\mathcal{P}$ be a topological property. A Tychonoff $\mathcal{P}$-extension of X is called *tight* if it does not contain properly any other $\mathcal{P}$-extension of X. Now let $\mathcal{P}$ be a topological property which is closed hereditary, productive and is such that if a Tychonoff space is the union of a compact space and a space with $\mathcal{P}$ then it has $\mathcal{P}$. Let X be a Tychonoff space. Define the *$\mathcal{P}$-reflection* $\gamma_{\mathcal{P}}X$ of X by

$$\gamma_{\mathcal{P}}X = \bigcap\{T : T \text{ has } \mathcal{P} \text{ and } X \subseteq T \subseteq \beta X\}.$$

If $\mathcal{P}$ is compactness then $\gamma_{\mathcal{P}}X = \beta X$ and if $\mathcal{P}$ is realcompactness then $\gamma_{\mathcal{P}}X = \upsilon X$ (the Hewitt realcompactification of X). Also, $\gamma_{\mathcal{P}}X$ has $\mathcal{P}$ by Corollary 2.4 of [17]. Denote by $\mathscr{P}(X)$ the set of all tight $\mathcal{P}$-extensions of X. As remarked in [17], for a Tychonoff locally-$\mathcal{P}$ non-$\mathcal{P}$ space X there is the largest one-point extension X^* in $\mathscr{P}(X)$. Let

$$\mathscr{P}^*(X) = \big\{T \in \mathscr{P}(X) : X^* \leq T\big\}$$

and for any $T \in \mathscr{P}^*(X)$, if $f_T : \beta X \to \beta T$ denotes the continuous extension of id_X, let

$$\mathscr{D}^*(X) = \big\{T \in \mathscr{P}^*(X) : f_T[\gamma_{\mathcal{P}}X] = T\big\}.$$

THEOREM 5.2 (Mack, Rayburn and Woods [17]). *Let X and Y be Tychonoff locally-$\mathcal{P}$ non-$\mathcal{P}$ spaces. If $(\mathscr{P}^*(X), \leq)$ and $(\mathscr{P}^*(Y), \leq)$ are lattice-isomorphic then $\gamma_{\mathcal{P}}X \setminus X$ and $\gamma_{\mathcal{P}}Y \setminus Y$ are homeomorphic.*

The following main result of [17] generalizes Magill's theorem [20, Theorem 5.1].

THEOREM 5.3 (Mack, Rayburn and Woods [17]). *Let X and Y be Tychonoff locally-$\mathcal{P}$ non-$\mathcal{P}$ spaces and suppose that $\mathscr{D}^*(X) = \mathscr{P}^*(X)$ and $\mathscr{D}^*(Y) = \mathscr{P}^*(Y)$. Suppose moreover that $\gamma_{\mathcal{P}}X \setminus X$ and $\gamma_{\mathcal{P}}Y \setminus Y$ are C^*-embedded in $\gamma_{\mathcal{P}}X$ and $\gamma_{\mathcal{P}}Y$, respectively. The following are equivalent:*

(1) $(\mathscr{P}^*(X), \leq)$ *and* $(\mathscr{P}^*(Y), \leq)$ *are lattice-isomorphic.*
(2) $\gamma_{\mathcal{P}}X \setminus X$ *and* $\gamma_{\mathcal{P}}Y \setminus Y$ *are homeomorphic.*

The topological properties considered in [17] are all assumed to be productive while the topological properties we have considered are hardly productive. (As shown in Example 2.16 specific examples of compactness-like topological properties are mostly cov-

ering properties which are normally not expected to be productive.) However, there are topological properties which satisfy the two sets of requirements. We need to know the relation between the classes of compactification-like $\mathcal{P}$-extensions of a Tychonoff space X and the class of its tight $\mathcal{P}$-extensions with compact remainder; in particular, we need to know if these two coincide. In this section we apply some of our previous results to obtain analogous results in the context of tight $\mathcal{P}$-extensions with compact countable remainder. Also, we give examples to show that the concepts of "tight $\mathcal{P}$-extension with compact remainder", "minimal $\mathcal{P}$-extension" and "optimal $\mathcal{P}$-extension" in general do not coincide.

We start with the following result which together with Lemma 3.7 and Theorem 3.13 characterizes spaces having a tight $\mathcal{P}$-extension with compact countable remainder. Note that by definitions, the notions of "n-point minimal $\mathcal{P}$-extension" and "n-point tight $\mathcal{P}$-extension" coincide for any $n \in \mathbb{N}$. Thus Lemma 3.7 and Theorem 3.13 also characterize spaces with an n-point tight $\mathcal{P}$-extension.

THEOREM 5.4. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff space with $\mathcal{Q}$. The following are equivalent:*

(1) *X has a countable-point minimal $\mathcal{P}$-extension with $\mathcal{Q}$.*
(2) *X has a countable-point optimal $\mathcal{P}$-extension with $\mathcal{Q}$.*
(3) *X has a countable-point tight $\mathcal{P}$-extension with compact remainder with $\mathcal{Q}$.*

Proof. The equivalence of (1) and (2) follows from Theorem 3.13.

(1) *implies* (3). By Lemma 3.7(2.c) the space X is locally-$\mathcal{P}$ and $\beta X \setminus \lambda_{\mathcal{P}} X$ contains an infinite bijectively indexed sequence $H_1, H_2, \ldots$ of pairwise disjoint non-empty clopen subsets. Let p_i's, T, q and Y be as in Lemma 3.7 ((2.c)⇒(2.b)). Then Y is a countable-point Tychonoff $\mathcal{P}$-extension of X with compact remainder with $\mathcal{Q}$. We show that Y is also a tight $\mathcal{P}$-extension. Suppose to the contrary that there exists a $\mathcal{P}$-extension Y' of X properly contained in Y. Choose $p_n \in Y \setminus Y'$ for $n \in \mathbb{N}$. The sets H_n and $(\beta X \setminus \lambda_{\mathcal{P}} X) \setminus H_n$ are closed in βX, as they are closed in $\beta X \setminus \lambda_{\mathcal{P}} X$. Let $f_n : \beta X \to \mathbb{I}$ be continuous with

$$f_n[H_n] \subseteq \{0\} \quad \text{and} \quad f_n\big[(\beta X \setminus \lambda_{\mathcal{P}} X) \setminus H_n\big] \subseteq \{1\}.$$

When $n = 1$ note that p_1 is obtained by contracting a set containing H_1. The set

$$Z_n = f_n^{-1}\big[[0, 1/2]\big] \cap X = q\big[f_n^{-1}\big[[0, 1/2]\big]\big] \cap Y' \in \mathscr{Z}(X)$$

has $\mathcal{P}$, as it is closed in Y'. Therefore

$$H_n \subseteq f_n^{-1}\big[[0, 1/2)\big] \subseteq \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X}\big(f_n^{-1}\big[[0, 1/2]\big] \cap X\big) = \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} Z_n \subseteq \lambda_{\mathcal{P}} X,$$

which is a contradiction. This shows that Y is also a tight $\mathcal{P}$-extension. That (3) implies (1) is trivial and follows from definitions. ∎

The following is a counterpart for Corollary 3.14. Thus it too (besides Corollary 3.14) may be considered as a generalization of Magill's theorem [19, Theorem 3.4].

THEOREM 5.5. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff space with $\mathcal{Q}$. The following are equivalent:*

(1) *X has an n-point tight $\mathcal{P}$-extension with $\mathcal{Q}$ for any $n \in \mathbb{N}$.*
(2) *X has a countable-point tight $\mathcal{P}$-extension with compact remainder with $\mathcal{Q}$.*

Proof. As noted before, the notions of "n-point minimal $\mathcal{P}$-extension" and "n-point tight $\mathcal{P}$-extension" coincide for any $n \in \mathbb{N}$. The result now follows from Theorems 3.13 and 5.4. ∎

THEOREM 5.6. *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of compactness-like topological properties. Let X be a Tychonoff space.*

(1) *Let $n \in \mathbb{N}$. If X has a perfect image with $\mathcal{Q}$ which has an n-point tight $\mathcal{P}$-extension with $\mathcal{Q}$, then so does X.*
(2) *If X has a perfect image with $\mathcal{Q}$ which has a countable-point tight $\mathcal{P}$-extension with compact remainder with $\mathcal{Q}$, then so does X.*

Proof. This follows from Theorems 3.15 and 5.4. ∎

In the following we give examples of a topological property $\mathcal{P}$ and Tychonoff spaces X for which the notion of "tight $\mathcal{P}$-extension with compact remainder" differs from both "minimal $\mathcal{P}$-extension" and "optimal $\mathcal{P}$-extension". For convenience, for a space X and a topological property $\mathcal{P}$, denote by $\mathscr{T}_{\mathcal{P}}(X)$ the set of all tight $\mathcal{P}$-extensions of X with compact remainder. Observe that by definition $\mathscr{T}_{\mathcal{P}}(X) \subseteq \mathscr{M}_{\mathcal{P}}(X)$.

EXAMPLE 5.7. Let $\mathcal{P}$ be $\aleph_0$-boundedness (see Example 2.16 for the definition) and let $X = D(\aleph_1)$ (the discrete space of cardinality $\aleph_1$). Note that $\mathcal{P}$ is closed hereditary, productive, finitely additive, perfect and satisfies Mrówka's condition (W) (thus by Corollary 2.6 of [17], if a Tychonoff space is the union of a compact space and a space with $\mathcal{P}$, then it has $\mathcal{P}$). Then $\lambda_{\mathcal{P}} X = X$, as any $\aleph_0$-bounded $Z \in \mathscr{Z}(X)$ is finite. Therefore

$$\mathscr{O}_{\mathcal{P}}(X) = \mathscr{M}_{\mathcal{P}}(X) = \mathscr{K}(X)$$

where as before $\mathscr{K}(X)$ is the set of all compactifications of X. In [41] it is shown that

$$\gamma_{\mathcal{P}} X = \bigcup \{ \mathrm{cl}_{\beta X} A : A \subseteq X \text{ is countable} \}.$$

Now $\gamma_{\mathcal{P}} X$ is a $\mathcal{P}$-extension of X (as $\gamma_{\mathcal{P}} X$ always has $\mathcal{P}$; see Corollary 2.4 of [17]) and obviously it is contained properly in βX. Therefore $\beta X \in \mathscr{O}_{\mathcal{P}}(X)$, while $\beta X \notin \mathscr{T}_{\mathcal{P}}(X)$.

EXAMPLE 5.8. Let $\mathcal{P}$ be $\aleph_0$-boundedness and let $X = [0, \Omega) \setminus \{\omega\}$. Note that X is locally compact; denote by X^* the one-point compactification of X. Then $X^* \in \mathscr{T}_{\mathcal{P}}(X)$, as the only extension of X contained properly in X^* is X itself, which does not have $\mathcal{P}$, because $\omega \notin X$. Now let $\phi : \beta X \to X^*$ be the continuous extension of id_X. Then

$$\phi^{-1}[X^* \setminus X] = \beta X \setminus X \neq \beta X \setminus \lambda_{\mathcal{P}} X \tag{5.1}$$

as $\lambda_{\mathcal{P}} X \setminus X$ is non-empty. To show the latter simply let $Z = (\omega, \Omega)$ and observe that Z is clopen in X (thus a zero-set in X) and it has $\mathcal{P}$. Since Z is non-compact,

$$\emptyset \neq \mathrm{cl}_{\beta X} Z \setminus X = \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} Z \setminus X \subseteq \lambda_{\mathcal{P}} X \setminus X.$$

Now by Theorem 2.15 from (5.1) it follows that $X^* \notin \mathscr{O}_{\mathcal{P}}(X)$.

5.2. On a question of S. Mrówka and J. H. Tsai. Let X and E be Hausdorff spaces. The space X is said to be *E-completely regular* if X is homeomorphic to a subspace of a product E^{α} for some cardinal α (see [6] and [26]). In [28] (see also [36]) the authors proved that for a topological property $\mathcal{P}$ which is regular-closed hereditary, finitely additive with

respect to closed subsets (that is, if a Hausdorff space is the finite union of its closed subsets with $\mathcal{P}$, then it has $\mathcal{P}$) and satisfy Mrówka's condition (W), every E-completely regular (where E is regular and subject to some restrictions) locally-$\mathcal{P}$ space has a one-point E-completely regular $\mathcal{P}$-extension (see [21] for related results). The authors then posed the following more general question: *For what pairs of topological properties $\mathcal{P}$ and $\mathcal{Q}$ is it true that every locally-$\mathcal{P}$ space with $\mathcal{Q}$ has a one-point extension with both $\mathcal{P}$ and $\mathcal{Q}$?* Indeed, the systematic study of this sort of question dates back to the times when P. Alexandroff proved that every locally compact (Hausdorff) non-compact space has a one-point compact (Hausdorff) extension (thus answering the above question in the case when $\mathcal{P}$ is compactness and $\mathcal{Q}$ is the property of being Hausdorff). Since then the question has been considered by various authors for specific choices of $\mathcal{P}$ and $\mathcal{Q}$. The following corollary of Lemma 2.8 provides an answer to the above question of S. Mrówka and J. H. Tsai (see also Theorem 4.1 of [16] for a related result).

THEOREM 5.9. *Let $\mathcal{P}$ be a compactness-like topological property. Let $\mathcal{Q}$ be a topological property which is either*

- *clopen hereditary, inverse invariant under perfect mappings and satisfying Mrówka's condition* (W)*, or*
- *strong zero-dimensionality.*

Let X be a Tychonoff non-$\mathcal{P}$ space with $\mathcal{Q}$. The following are equivalent:

(1) *X is locally-$\mathcal{P}$.*
(2) *X has a one-point Tychonoff extension with both $\mathcal{P}$ and $\mathcal{Q}$.*

Proof. That (2) implies (1) is obvious.

(1) *implies* (2). By Lemma 2.10 we have $X \subseteq \lambda_{\mathcal{P}} X$. Let T be the space obtained from βX by contracting the compact subset $\beta X \setminus \lambda_{\mathcal{P}} X$ to a point p (note that $\beta X \setminus \lambda_{\mathcal{P}} X$ is non-empty by Lemma 4.10). Then T is Tychonoff. Consider the subspace $Y = X \cup \{p\}$ of T.

CASE 1. Suppose that $\mathcal{Q}$ is hereditary with respect to clopen subsets, inverse invariant under perfect mappings and satisfies (W). Lemma 2.8 then implies that Y is a one-point Tychonoff extension of X with both $\mathcal{P}$ and $\mathcal{Q}$.

CASE 2. Suppose that $\mathcal{Q}$ is strong zero-dimensionality. By Lemma 2.8 (with $\mathcal{Q}$ being regularity in its statement) the space Y is a one-point Tychonoff $\mathcal{P}$-extension of X. We verify that Y is strongly zero-dimensional. Note that T is a compactification of Y. Let $\phi : \beta X \to \beta Y$ and $\gamma : \beta Y \to T$ be the continuous extensions of id_X and id_Y, respectively. Since $\gamma\phi : \beta X \to T$ agrees with q on X we have $\gamma\phi = q$. Since T is a compactification of Y (and $\gamma | Y = \mathrm{id}_Y$), by Theorem 3.5.7 of [5] we have $\gamma[\beta Y \setminus Y] = T \setminus Y$. Thus $\gamma^{-1}(p) = \{p\}$ and

$$\phi^{-1}(p) = \phi^{-1}\big[\gamma^{-1}(p)\big] = (\gamma\phi)^{-1}(p) = q^{-1}(p) = \beta X \setminus \lambda_{\mathcal{P}} X.$$

By Lemma 2.9 we have $\beta Y = T$ and $\phi = q$. Using zero-dimensionality of βX it is easy to verify that βY is zero-dimensional, that is, Y is strongly zero-dimensional.

Therefore (2) holds in either case. ∎

6. Question

We conclude with a question which naturally arose in connection with our study.

QUESTION 6.1. Let X be a space, let $\mathcal{P}$ be a topological property and let Y be a Tychonoff $\mathcal{P}$-extension of X with compact remainder. The extension Y of X is called *maximal* if the topology of Y is maximal (with respect to inclusion) among all topologies which turn Y into a Tychonoff $\mathcal{P}$-extension of X with compact remainder. Thus Y is an optimal $\mathcal{P}$-extension of X if it is both a minimal and a maximal $\mathcal{P}$-extension of X. Now to what extent and how the results of this article can be rephrased in order to remain valid in the new context?

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