Summary

Our work is divided into five chapters. In Chapter I we introduce necessary notions and we present the most important facts that we shall use. We also present our main results. Chapter I covers the following topics:

- holomorphically contractible families of functions and pseudometrics, their basic properties, product property, Lempert Theorem, notion of geodesic, problem of finding effective formulas for invariant functions and pseudometrics and geodesics, completeness with respect to holomorphically contractible distances, its application in the study of the relation between norm balls and Carathéodory balls;
- pluricomplex Green function with a logarithmic pole as an example of a holomorphically contractible family of functions, problem of its symmetry, pluricomplex Green function with many poles as a natural generalization of the Green function with one pole;
- Bergman distance, Bergman completeness.

Chapter II is devoted to the problem of completeness with respect to Carathéodory, Kobayashi and Bergman distances in a class of pseudoconvex Reinhardt domains. First we recall well known geometric properties of pseudoconvex Reinhardt domains (Section 2.1). In Section 2.2 we deal with properties of real convex cones, objects closely related to pseudoconvex Reinhardt domains. Section 2.3 is devoted to the study of algebraic mappings, especially those inducing proper and biholomorphic mappings of $\mathbb{C}_*^n$ (Theorem 2.3.1). A special role in our study will be played by quasi-elementary Reinhardt domains (Section 2.4). Before we study completeness we give a precise description of hyperbolic (in different sense) pseudoconvex Reinhardt domains (Theorem 2.5.1). The solution of the problem which hyperbolic pseudoconvex Reinhardt domains are Kobayashi (respectively, Carathéodory) complete, is given in Theorem 2.6.5 (respectively, Theorem 2.6.6). Additionally, the problem when the Carathéodory distance tends to infinity as one variable is fixed and the other tends to a boundary point not lying on axis in bounded pseudoconvex Reinhardt domains is discussed (Theorem 2.6.1, Corollary 2.6.2, and Example 2.6.4). In contrast to the Carathéodory and Kobayashi distances no characterization of Bergman completeness is known. Nevertheless, it is known in dimension 2 (Corollary 2.7.4). Some partial results are given in Proposition 2.7.2 (a sufficient condition for not being Bergman complete) and Theorem 2.7.3 (a sufficient condition for Bergman completeness). A relation between good boundary behavior of the Green function and Bergman completeness in the class of bounded pseudoconvex Reinhardt domains (Lemma 2.8.2 and Proposition 2.8.5) and in planar domains (Corollary 2.8.8) is considered.

In Chapter III we find formulas for holomorphically contractible functions and pseudometrics in the class of elementary Reinhardt domains (Sections 3.1–3.5) and for the pluricomplex Green function of the unit ball with two poles (with equal weights) (Section 3.6). First we recall known formulas (Theorem 3.1). Then we present formulas for elementary Reinhardt domains not contained in $\mathbb{C}_*^n$ (Theorem 3.1.1). The proof of the theorem is contained in Sections 3.2–3.4. For elementary Reinhardt domains lying in $\mathbb{C}_*^n$ the proof of the formulas (Theorem 3.5.1) is much simpler. Theorem 3.6.1 gives a formula for the pluricomplex Green function of the unit ball with two poles of equal weights. The key role in the proof of the formula is played by Theorem 3.6.2 showing how the pluricomplex Green function with many poles behaves under proper holomorphic mappings.

In Chapter IV we deal with symmetry of the Green function. First we entirely solve the problem in the class of complex ellipsoids (Theorem 4.1.1). In Section 4.2 some kind of "infinitesimal" symmetry in the class of bounded hyperconvex domains is described (Corollary 4.2.4). This property is a consequence of regularity properties of the Azukawa pseudometric (Theorems 4.2.1 and 4.2.2, and Corollary 4.2.3). The results on regularity properties of the Azukawa pseudometric cannot be extended to the class of bounded pseudoconvex domains (Example 4.2.10). In Section 4.3 we discuss the problem of nonsymmetry of the Green function in pseudoconvex complete Reinhardt domains whose boundary contains some "exponential line". It turns out that in such domains the Green function is extremely nonsymmetric (Propositions 4.3.1 and 4.3.2, and Remark 4.3.3).

In Chapter V we consider the problem which Carathéodory balls are simultaneously norm balls in the class of convex ellipsoids. The ideas used in this chapter have been used lately in the study of the same problem for a wider class of domains.

Most of the properties that we use may be found in the following books: [Kob 70], [Kli 91], [Jar-Pfl 93], and [Kob 98]. If some result that we use is not quoted explicitly it may be found in one of these books.

Some of the results contained in the work may be found in the following papers: [Zwo 96], [Edi-Zwo 98], [Pfl-Zwo 98], [Zwo 97], [Zwo 98a], [Zwo 98b], [Zwo 98c], and [Zwo 99].

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I. Introduction

1.1. (Holomorphically) contractible families of functions and pseudometrics. Let us denote by E the unit disc in $\mathbb{C}$ (¹). Put

$$m(\lambda_1,\lambda_2) := \left|\frac{\lambda_1-\lambda_2}{1-\overline{\lambda}_1\lambda_2}\right|, \quad \lambda_1,\lambda_2\in E,$$

$$\gamma(\lambda;X) := \frac{|X|}{1-|\lambda|^2}, \qquad \lambda\in E,\ X\in\mathbb{C}.$$

We call m the *Möbius distance*. We define the following *Poincaré distance:*

$$p := \tanh^{-1}(m).$$

In what follows both functions (m and p) will be used. In general, the objects defined with the help of m will be more handy in calculations, whereas the ones defined with the help of p will be more regular.

Let us recall:

THEOREM 1.1.1 (Schwarz–Pick Lemma). *Let $f\in\mathcal{O}(E,E)$. Then*

(a) $$p(f(\lambda_1),f(\lambda_2)) \le p(\lambda_1,\lambda_2), \quad \lambda_1,\lambda_2\in E;$$

(b) $$\gamma(f(\lambda);f'(\lambda)) \le \gamma(\lambda;1), \qquad \lambda\in E.$$

Moreover, if in (a) *equality holds for some $\lambda_1\neq\lambda_2$ or in* (b) *equality holds for some $\lambda\in E$ then the inequalities in* (a) *and* (b) *become equalities.*

It would be nice if we could find an analogue of the function m (and γ) for which some version of the Schwarz–Pick Lemma would also be satisfied in other domains.

In the twenties Carathéodory defined for an arbitrary domain (²) D in $\mathbb{C}^n$ the following function (see [Car 27]):

$$c_D(w,z) := \sup\{p(f(w),f(z)) : f\in\mathcal{O}(D,E)\}, \quad w,z\in D.$$

Note that $c_E = p$ and $c_G(F(w),F(z)) \le c_D(w,z)$ for any $F\in\mathcal{O}(D,G)$, $w,z\in D$. We call c_D *the Carathéodory pseudodistance* (³) *of* D.

(¹) For convenience we list some standard notation in the section "List of symbols".

(²) Unless otherwise stated by D (and G) we shall always mean a domain in $\mathbb{C}^n$.

(³) We say that a function $d: X\times X\to[0,\infty)$ is a *pseudodistance* (X is a nonempty set) if (i) $d(x,x)=0$, $x\in X$; (ii) d is symmetric (i.e. $d(x,y)=d(y,x)$, $x,y\in X$); (iii) d satisfies the triangle inequality (i.e. $d(x,y)\le d(x,z)+d(z,y)$, $x,y,z\in X$).

In the sixties, S. Kobayashi defined the following pseudodistance (see [Kob 67], [Kob 70]):

$$k_D := \text{the largest pseudodistance not exceeding } \widetilde{k}_D,$$

where

$$\widetilde{k}_D(w,z) := \inf\{p(\lambda_1,\lambda_2) : \text{there is } f \in \mathcal{O}(E,D) \text{ with } f(\lambda_1)=w,\ f(\lambda_2)=z\},$$

$w,z \in D$ (4). It is immediate that $k_E = p$ and $k_G(F(w),F(z)) \le k_D(w,z)$ for any $F \in \mathcal{O}(D,G)$, $w,z \in D$. We call k_D the *Kobayashi pseudodistance of* D.

The above considerations lead us to definition of (holomorphically) contractible family of functions.

We say that $d := (d_D)_{D \text{ domain in } \mathbb{C}^n}$, where $d_D : D \times D \to [0,\infty)$, is *a (holomorphically) contractible family of functions* if

$$d_E = p; \tag{1.1.1}$$

$$d_G(F(w),F(z)) \le d_D(w,z) \quad \text{for any } F \in \mathcal{O}(D,G),\ w,z \in D. \tag{1.1.2}$$

The property (1.1.2) says that holomorphic mappings are *contractions* with respect to the functions d_D and d_G. One may interpret the inequality (1.1.2) as a generalized Schwarz–Pick Lemma. The property (1.1.1) plays a uniformization role.

It is easy to see that if $F : D \to G$ is biholomorphic then $d_G(F(w),F(z)) = d_D(w,z)$, $w,z \in D$.

Obviously, both the Carathéodory and the Kobayashi pseudodistances form holomorphically invariant families of functions. The functions $\widetilde{k}_D$ also form a holomorphically contractible family of functions. We call the function $\widetilde{k}_D$ the *Lempert function of* D.

In view of the Schwarz–Pick Lemma, the Carathéodory pseudodistance (respectively, Lempert function) is the "smallest" (respectively, the "largest") among all holomorphically contractible families of functions. Therefore, we have

$$c_D \le k_D \le \widetilde{k}_D.$$

One may define many other holomorphically contractible families of functions. Below we define only one of them, which will be of special importance for us.

For $w,z \in D$ we define the *pluricomplex Green function* (*with a logarithmic pole at* w) (see [Kli 85]):

$$g_D(w,z) := \sup\{u(z)\}$$

where the supremum is taken over all $u \in \mathrm{PSH}(D)$ (5), $u < 0$, such that $u(\,\cdot\,) - \log\|\cdot - w\|$ is bounded from above.

Put $\widetilde{g}_D := \exp g_D$. Then one may check that the family

$$(\tanh^{-1}(\widetilde{g}_D))_{D \text{ domain in } \mathbb{C}^n}$$

forms a holomorphically contractible family of functions.

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(4) One may verify that the infimum in the definition of $\widetilde{k}_D$ is taken over a nonempty set.

(5) By $\mathrm{PSH}(D)$ we denote the set of plurisubharmonic functions on D; we allow the plurisubharmonic function to be equal identically to $-\infty$. By $\mathrm{SH}(D)$ we denote the set of subharmonic functions.

In what follows, while considering the holomorphically contractible families of functions d, it will be often more convenient to use the functions $d_D^* := \tanh d_D$, which, in the case of the unit disc, correspond to the function m (instead of p). When we want to underline that both d_D^* and d_D may be used we write $d_D^{(*)}$ ([6]).

In the Schwarz–Pick Lemma two kinds of inequalities have been given. The first one involving the Poincaré distance and the other involving the function γ. Generalizing the property (b) of the Schwarz–Pick Lemma, in a similar way as we did it while generalizing (a), we arrive at the definition of a (holomorphically) contractible family of pseudometrics.

A family $\delta := (\delta_D)_{D \text{ domain in } \mathbb{C}^n}$, where $\delta_D : D \times \mathbb{C}^n \to [0, \infty)$ is a pseudometric ([7]), is *a holomorphically contractible family of pseudometrics* if

(1.1.3) $$\delta_E = \gamma;$$

(1.1.4) $$\delta_G(F(w); F'(w)X) \le \delta_D(w; X) \quad \text{for any } F \in \mathcal{O}(D, G),\ w \in D,\ X \in \mathbb{C}^n.$$

As previously the property (1.1.4) says that holomorphic mappings are *contractions* with respect to pseudometrics δ_D and δ_G.

Below we give some examples of holomorphically invariant families of pseudometrics:

- *the Carathéodory–Reiffen pseudometric* (see [Rei 65]):
$$\gamma_D(w; X) := \sup\{\gamma(\varphi(w); \varphi'(w)X) : \varphi \in \mathcal{O}(D, E)\};$$
- *the Kobayashi–Royden pseudometric* (see [Roy 71]):
$$\kappa_D(w; X) := \inf\{\gamma(\lambda; \alpha) : \exists \varphi \in \mathcal{O}(E, D), \exists \alpha \in \mathbb{C},\ \varphi(\lambda) = z, \alpha\varphi'(\lambda) = X\};$$
- *the Azukawa pseudometric* (see [Azu 86]):
$$A_D(w; X) := \limsup_{0 \ne \lambda \to 0} \frac{\widetilde{g}_D(w, w + \lambda X)}{|\lambda|}, \quad w \in D,\ X \in \mathbb{C}^n.$$

In view of the Schwarz–Pick Lemma γ (respectively, κ) is the "smallest" (respectively, the "largest") among all holomorphically contractible families of pseudometrics. Therefore, we have
$$\gamma_D \le A_D \le \kappa_D.$$
Other examples of holomorphically contractible functions are Möbius functions of order k ($k \ge 2$). Their infinitesimal versions form holomorphically contractible families of pseudometrics (for definitions see [Jar-Pfl 91c]).

Among many elementary properties let us recall the ones concerning continuity:

- c_D, k_D, γ_D are continuous (see e.g. [Jar-Pfl 93]), whereas
- $\widetilde{k}_D, g_D, \kappa_D, A_D$ are upper semicontinuous (see e.g. [Jar-Pfl 93], [Jar-Pfl 95b]).

The following simple result combined with the existence of balanced pseudoconvex domains with discontinuous Minkowski functions shows that one cannot hope to have better continuity properties of the four latter functions:

([6]) This does not apply to the Green function.

([7]) A function $\delta_D : D \times \mathbb{C}^n \to [0, \infty)$ is called a *pseudometric* if $\delta_D(z; \lambda X) = |\lambda|\delta_D(z; X)$ for any $(z; X) \in D \times \mathbb{C}^n$, $\lambda \in \mathbb{C}$.

PROPOSITION 1.1.2. *Let $D = D_h := \{h(z) < 1\}$ be a balanced pseudoconvex domain and let h be its Minkowski function. Then*

$$\kappa_D(0;X) = A_D(0;X) = h(X), \quad X \in \mathbb{C}^n,$$
$$\widetilde{k}_D^*(0,z) = \widetilde{g}_D(0,z) = h(z), \qquad z \in D.$$

It is easy to prove that $\log c_D^*(w,\cdot)$, $w \in D$, is a plurisubharmonic function.

It turns out that all the discussed contractible families of functions and pseudometrics are continuous with respect to increasing sequences of domains. More precisely, for any sequence of domains $\{D_j\}_{j=1}^\infty \subset \mathbb{C}^n$, $D_j \subset D_{j+1}$, $D = \bigcup_{j=1}^\infty D_j$ we have (see e.g. [Jar-Pfl 93], [Azu 86], [Azu 87]):

$$d_{D_j} \to d_D, \quad \delta_{D_j} \to \delta_D \quad \text{as } j \to \infty, \tag{1.1.5}$$

where $d = c, g, k$ or $\widetilde{k}$ and $\delta = \gamma, A$ or κ.

It is well known (see [Kob 70] and [Jar-Pfl 93]) that if $\pi : D \to G$ is a holomorphic covering (D and G are domains in $\mathbb{C}^n$), $w, z \in G$, $\pi(\widetilde{w}) = w$, $\pi'(\widetilde{w})X = Y$ then

$$\widetilde{k}_G(w,z) = \inf\{\widetilde{k}_D(\widetilde{w},\widetilde{z})\}, \quad k_G(w,z) = \inf\{k_D(\widetilde{w},\widetilde{z})\}, \tag{1.1.6}$$

$$\kappa_G(w;Y) = \kappa_D(\widetilde{w};X), \tag{1.1.7}$$

where the infimum in both cases of (1.1.6) is taken over all $\widetilde{z} \in D$ such that $\pi(\widetilde{z}) = z$.

The last result together with the Uniformization Theorem gives $\widetilde{k}_D = k_D$ for any domain D in $\mathbb{C}$. Therefore, the simplest possible example of the inequality $k_D \neq \widetilde{k}_D$ may be found in dimension 2. And this is really the case: for $\varepsilon > 0$ small enough the Lempert function of the domain $\{z \in \mathbb{C}^2 : |z_1 z_2| < \varepsilon\} \cap E^2$ does not satisfy the triangle inequality (see [Lem 81]).

The problem whether the infimum in (1.1.6) is always attained was posed in [Kob 70] (in the case of the Kobayashi pseudodistance) and in [Jar-Pfl 93] (in the case of the Lempert function). Note that in dimension one the infimum may always be replaced with minimum (use the Uniformization Theorem).

In Chapter III we provide an example giving a negative answer to this question based on elementary Reinhardt domains of irrational type not containing the origin (*and*, *therefore*, *we solve the problem posed above*).

More precisely, any elementary Reinhardt domain of irrational type with negative exponents gives us that kind of example.

There is a close relation between the Carathéodory–Reiffen pseudometric and the Carathéodory pseudodistance given by the formula

$$\gamma_D(w;X) = \lim_{w_1 \neq w_2,\, w_1,w_2 \to w,\, \frac{w_1 - w_2}{\|w_1 - w_2\|} \to X} \frac{c_D^{(*)}(w_1,w_2)}{\|w_1 - w_2\|}, \quad w \in D,\ \|X\| = 1,\ X \in \mathbb{C}^n.$$

An analogous result, but only in the class of bounded taut ([8]) domains, holds for the Lempert function and the Kobayashi–Royden pseudometric

([8]) A domain $D \subset \mathbb{C}^n$ is *taut* if for any sequence $\{\varphi_\nu\}_{\nu=1}^\infty \subset \mathcal{O}(E,D)$ either φ_ν diverges locally uniformly (i.e. for any compact sets $K \subset E$, $L \subset D$, $\varphi_\nu(K) \cap L = \emptyset$ for ν large enough) or it has a subsequence converging to a mapping $\varphi_0 \in \mathcal{O}(E,D)$ (see [Wu 67]).

PROPOSITION 1.1.3 (see [Pang 94]). *Let D be a bounded taut domain. Then*

$$\kappa_D(w;X) = \lim_{w_1\neq w_2,\, w_1,w_2\to w,\, \frac{w_1-w_2}{\|w_1-w_2\|}\to X} \frac{\widetilde{k}_D^{(*)}(w_1,w_2)}{\|w_1-w_2\|}, \quad w\in D,\ \|X\|=1,\ X\in\mathbb{C}^n.$$

We prove an analogous result for the Azukawa pseudometric and the Green function in the class of domains containing, among others, bounded hyperconvex domains.

THEOREM 1.1.4 (cf. Corollary 4.2.3). *Let D be a bounded hyperconvex domain. Then*

$$A_D(w;X) = \lim_{w_1\neq w_2,\, w_1,w_2\to w,\, \frac{w_1-w_2}{\|w_1-w_2\|}\to X} \frac{\widetilde{g}_D(w_1,w_2)}{\|w_1-w_2\|}, \quad w\in D,\ \|X\|=1,\ X\in\mathbb{C}^n.$$

We also give an example of a bounded pseudoconvex domain in $\mathbb{C}^2$ for which the formula above does not hold and, additionally, we cannot replace "lim sup" *in the definition of the Azukawa pseudometric with* "lim" (*see Example* 4.2.10) (9).

In view of the Removable Singularity Theorems the following properties hold:

$$(1.1.8)\qquad (g_D)_{|(D\setminus P)\times(D\setminus P)} = g_{D\setminus P}, \quad (A_D)_{|(D\setminus P)\times\mathbb{C}^n} = A_{D\setminus P},$$

$$(1.1.9)\qquad (c_D)_{|(D\setminus B)\times(D\setminus B)} = c_{D\setminus B}, \quad (\gamma_D)_{|(D\setminus B)\times\mathbb{C}^n} = \gamma_{D\setminus B},$$

where P is a relatively closed pluripolar subset of D and B is a proper analytic subset of D.

Combining (1.1.6)–(1.1.9), the Removable Singularity Theorems, and the Uniformization Theorem we get

$$\begin{gathered} c_D = \widetilde{k}_D \equiv 0, \quad \gamma_D = \kappa_D \equiv 0,\ D=\mathbb{C},\mathbb{C}_*,\\ c^*_{\mathbb{C}\setminus\{0,1\}} = \widetilde{g}_{\mathbb{C}\setminus\{0,1\}} \equiv 0, \quad k_{\mathbb{C}\setminus\{0,1\}}(\lambda_1,\lambda_2)>0,\ \lambda_1\neq\lambda_2\in\mathbb{C}\setminus\{0,1\},\\ \gamma_{\mathbb{C}\setminus\{0,1\}} = A_{\mathbb{C}\setminus\{0,1\}} \equiv 0, \quad \kappa_{\mathbb{C}\setminus\{0,1\}}(\lambda;1)>0,\ \lambda\in\mathbb{C}\setminus\{0,1\}. \end{gathered}$$

1.2. Product property. A family of holomorphically contractible functions d has the *product property* if for any domains D_1, D_2 and for any points $(w_1,w_2),(z_1,z_2)\in D_1\times D_2$ we have

$$(1.2.1)\qquad d_{D_1\times D_2}((w_1,w_2),(z_1,z_2)) = \max\{d_{D_1}(w_1,z_1), d_{D_2}(w_2,z_2)\}.$$

Similarly, a family of holomorphically contractible pseudometrics δ has the *product property* if for any domains D_1, D_2 and for any points $(w_1,w_2)\in D_1\times D_2$, $(X_1,X_2)\in\mathbb{C}^{n_1}\times\mathbb{C}^{n_2}$ we have

$$(1.2.2)\qquad \delta_{D_1\times D_2}((w_1,w_2);(X_1,X_2)) = \max\{\delta_{D_1}(w_1;X_1), \delta_{D_2}(w_2;X_2)\}.$$

Because of contractivity of projections, the inequalities "$\geq$" in (1.2.1) and (1.2.2) are always fulfilled.

It is easy to verify that the Kobayashi pseudodistance and the Kobayashi–Royden pseudometric have product property (see e.g. [Jar-Pfl 93]).

(9) In dimension 1 the formula from Theorem 1.1.4 holds for any domain (see Corollary 4.2.11).

The problem whether the Carathéodory pseudodistance (and the Carathéodory–Reiffen pseudometric) has the product property turned out to be more difficult. The complete (positive) solution of the problem can be found in [Jar-Pfl 89c].

The problem whether the Green function (and the Azukawa pseudometric) has the product property has remained unsolved until recently. The final (positive) solution of the problem for all domains was given by A. Edigarian in [Edi 97b]. The proof of A. Edigarian, in contrast to earlier partial solutions of the problem (see [Jar-Pfl 91c], [Jar-Pfl 95b]) does not make use of the definition of the Green function given by us (the supremum of some subclass of plurisubharmonic functions) but makes use of an alternate (but equivalent) definition of the pluricomplex Green function, employing analytic disks (see [Pol-Sch 89], [Pol 93] and [Edi 97a]). In what follows, we shall quote and use this alternate definition.

It is worth noting that there are holomorphically contractible families of functions and pseudometrics for which the product property fails to hold (see [Jar-Pfl 93], [Jar-Pfl 91c]).

1.3. Various notions of geodesics. Lempert Theorem. A mapping $\varphi \in \mathcal{O}(E, D)$ is called a *κ_D-geodesic* for $(z; X)$, $X \neq 0$, if $\varphi(\lambda) = z$, $\alpha\varphi'(\lambda) = X$ and $\gamma(\lambda; \alpha) = \kappa_D(z; X)$ for some $\lambda \in E$, $\alpha \in \mathbb{C}$.

A mapping $\varphi \in \mathcal{O}(E, D)$ is called a *$\widetilde{k}_D$-geodesic* for (w, z), $w \neq z$, if $\varphi(\lambda_1) = w$, $\varphi(\lambda_2) = z$ and $p(\lambda_1, \lambda_2) = \widetilde{k}_D(w, z)$ for some $\lambda_1, \lambda_2 \in E$ ([10]).

If it does not lead to misunderstanding we shall briefly write κ_D- or $\widetilde{k}_D$-geodesics.

If D is a taut domain then for any $w \neq z$, $w, z \in D$ (respectively, for any $(w; X) \in D \times \mathbb{C}^n$, $X \neq 0$) there is a $\widetilde{k}_D$-geodesic for (w, z) (respectively, $\widetilde{k}_D$-geodesic for $(w; X)$).

Similarly, we could define γ_D- and c_D-geodesics; but because of the following property (following easily from the Schwarz–Pick Lemma) we shall introduce a notion of a (complex) geodesic (see [Ves 81]):

PROPOSITION 1.3.1 (see [Ves 81]). *Let $\varphi \in \mathcal{O}(E, D)$. Let $w, z \in D$, $w \neq z$, $X \in \mathbb{C}^n$, $X \neq 0$. Assume that one of the following conditions holds*:

(i) $\varphi(\lambda^0) = w$, $\alpha\varphi'(\lambda^0) = X$ *and* $\gamma(\lambda^0; \alpha) = \gamma_D(w; X)$ *for some* $\lambda^0 \in E$, $\alpha \in \mathbb{C}$;
(ii) $\varphi(\lambda_1^0) = w$, $\varphi(\lambda_2^0) = z$ *and* $p(\lambda_1^0, \lambda_2^0) = c_D(w, z)$ *for some* $\lambda_1^0, \lambda_2^0 \in E$.

Then

$$\gamma_D(\varphi(\lambda); \varphi'(\lambda)) = \kappa_D(\varphi(\lambda); \varphi'(\lambda)) = \gamma(\lambda; 1);$$
$$c_D(\varphi(\lambda_1), \varphi(\lambda_2)) = \widetilde{k}_D(\varphi(\lambda_1), \varphi(\lambda_2)) = p(\lambda_1, \lambda_2) \quad \textit{for any } \lambda, \lambda_1, \lambda_2 \in E.$$

A mapping $\varphi \in \mathcal{O}(E, D)$ is called a (*complex*) *geodesic* (in D) if

$$c_D(\varphi(\lambda_1), \varphi(\lambda_2)) = p(\lambda_1, \lambda_2)$$

for any $\lambda_1, \lambda_2 \in E$.

In view of Proposition 1.3.1 any complex geodesic is a $\widetilde{k}_D$- (respectively, κ_D-) geodesic for $(\varphi(\lambda_1), \varphi(\lambda_2))$ (respectively, $(\varphi(\lambda); \varphi'(\lambda))$). The converse implication does not hold in general ([11]). Nevertheless, in the class of convex domains it is always the case. It follows

([10]) In other words, $\widetilde{k}_D$- and κ_D-geodesics are mappings for which the infimum in the definition of $\widetilde{k}_D$ and κ_D is attained.

([11]) The simplest example of that kind may be found for $D := E_*$.

from the most spectacular result in the theory of holomorphically invariant functions and pseudometrics, namely, the Lempert Theorem.

THEOREM 1.3.2 (Lempert Theorem—see [Lem 81], [Lem 84]). *If D is a convex domain, then*

$$\widetilde{k}_D = c_D \quad \textit{and} \quad \kappa_D = \gamma_D.$$

Moreover, if D is additionally bounded, then for any pair $(w,z) \in D \times D$, $w \neq z$ (respectively, $(w;X) \in D \times \mathbb{C}^n$, $X \neq 0$), there exists a complex geodesic φ such that $w, z \in \varphi(E)$ (respectively, $\varphi(\lambda) = w$ and $\alpha\varphi'(\lambda) = X$ for some $\lambda \in E$, $\alpha \in \mathbb{C}$). If, additionally, D is strongly convex then the geodesics are unique up to an automorphism of E ([12]).

The results of Lempert contain also additional pieces of information on possibility of the extension of complex geodesics onto $\bar{E}$ as well as regularity properties of the invariant functions and pseudometrics (regularity of the functions depends on the regularity of the domain).

It is easy to see that for any complex geodesic ($\widetilde{k}_D$-, κ_D-geodesic) its image cannot be a relatively compact subset of the domain (it must touch the boundary). Even more, for a wide class of bounded domains the radial limit must lie almost everywhere in the boundary of the domain (for details see e.g. [Lem 81], [Lem 84], [Edi 95] and [Pang 93]).

The problem of finding explicit formulas for complex geodesics (or κ_D-, $\widetilde{k}_D$-geodesics) is, in general, very difficult. Among the very few examples (except for several trivial ones ([13])) for which the formulas for complex geodesics are known completely are convex complex ellipsoids (see [Jar-Pfl-Zei 93] and [Jar-Pfl 95a]); without the assumption of convexity only necessary forms of κ_D- and $\widetilde{k}_D$-geodesics are known (see [Edi 95]).

A domain

$$\mathcal{E}(p_1,\ldots,p_n) := \{z \in \mathbb{C}^n : |z_1|^{2p_2} + \ldots + |z_n|^{2p_n} < 1\}, \quad p_1,\ldots,p_n > 0,\ n > 1,$$

is called a *complex ellipsoid*.

Observe that $\mathcal{E}(p)$ is convex iff $p_1,\ldots,p_n \geq 1/2$.

THEOREM 1.3.3. *Let $\mathcal{E}(p)$ be a complex ellipsoid.*

1 (see [Jar-Pfl-Zei 93] and [Jar-Pfl 95a]). *If $\mathcal{E}(p)$ is convex then a nonconstant mapping $\varphi : E \to \mathbb{C}^n$ is a complex geodesic in $\mathcal{E}(p)$ if and only if φ may be given in the following form*:

$$\varphi_j(\lambda) = a_j \left(\frac{\lambda - \alpha_j}{1 - \overline{\alpha}_j\lambda}\right)^{r_j} \left(\frac{1 - \overline{\alpha}_j\lambda}{1 - \overline{\alpha}_0\lambda}\right)^{1/p_j}, \tag{1.3.1}$$

([12]) Instead of strong convex domains the same theorem also holds for so called strongly linearly convex domains (see [Lem 84]). Unique up to an automorphism of E means that if φ and ψ are two complex geodesics for some pair (w,z), $w \neq z$ (or for some pair $(w;X)$, $X \neq 0$), then there is an automorphism a of the unit disk such that $\varphi = \psi \circ a$. Uniqueness of complex geodesics may be proven for a class of strictly convex bounded domains (see [Din 89]).

([13]) These trivial examples include the unit ball and the polydisk.

where $r_j \in \{0,1\}$, $a_j \in \mathbb{C}$, *for* $j=1,\dots,n$, $\alpha_0 \in E$, $\alpha_j \in E$ *for* j *such that* $r_j = 1$, $\alpha_j \in \bar{E}$ *for* j *such that* $r_j = 0$, *and the following relations hold*:

$$\alpha_0 = \sum_{j=1}^{n} |a_j|^{2p_j}\alpha_j, \quad 1+|\alpha_0|^2 = \sum_{j=1}^{n} |a_j|^{2p_j}(1+|\alpha_j|^2).$$

The branches of roots are taken so that $1^{1/p_j} = 1$. *Moreover*, *geodesics for a given pair are unique up to an automorphism of the unit disc* (14).

2 (see [Edi 95]). *In the general case* (*i.e. without the assumption of convexity*) *any* $\kappa_{\mathcal{E}(p)}$-*geodesic* φ *for some* $(z;X)$, $X \neq 0$ (*and any* $\widetilde{k}_{\mathcal{E}(p)}$-*geodesic for some* (w,z), $w \neq z$) *must be of the same form as in the first case.*

In the nonconvex case neither the uniqueness nor the sufficiency as in the convex case of the theorem holds (see [Pfl-Zwo 96]).

The formulas from Theorem 1.3.3 have found many applications. They have been used to describe the automorphism group of convex complex ellipsoids (see [Zwo 95a]). They have also been used to find the formulas for the Kobayashi–Royden metric for ellipsoids $\mathcal{E}(1,m)$ (see [BFKKMP 92] if $m \geq 1/2$ and [Pfl-Zwo 96] if $0 < m < 1/2$).

In what follows we shall use the formulas from Theorem 1.3.3 while calculating the Green function of the unit ball with two poles (see Section 3.6) and while considering the problem of symmetry of Green function for complex ellipsoids (see Section 4.1). We shall also use Theorem 1.3.3 in the study of the relation between Carathéodory balls and norm balls in convex ellipsoids.

The technique of $\widetilde{k}_D$-geodesics will be helpful while calculating the Lempert function for elementary Reinhardt domains (see Sections 3.2 and 3.3).

1.4. Effective formulas for invariant functions. Invariant functions (and pseudometrics) are also objects for which effective formulas are very difficult to find. First, note that because invariant functions and pseudometrics are preserved under biholomorphic mappings we may easily find (because of Proposition 1.1.2) formulas for invariant functions for the polydisk E^n and the unit ball $\mathbb{B}_n$ (remember that the automorphism groups of E^n and $\mathbb{B}_n$ are transitive). Among other domains for which all the formulas are known are E_* (15), the annulus (here all functions that we defined are different; see e.g. [Jar-Pfl 93]).

As already mentioned, with the help of complex geodesics (or κ_D-geodesics) one can find effective formulas for the Kobayashi–Royden metric for the ellipsoid $\mathcal{E}(1,m)$ (see [BFKKMP 92] for $m \geq 1/2$ and [Pfl-Zwo 96] for $0 < m < 1/2$).

In Chapter III we find invariant functions and pseudometrics for all elementary Reinhardt domains (see [Pfl-Zwo 98], [Zwo 98a] and [Zwo 99]).

(14) If $\mathcal{E}(p)$ is strictly convex (i.e. $p_j = 1/2$ for at most one j) then the uniqueness of geodesics follows from the general theory (see [Din 89]).

(15) Because of (1.1.8) and (1.1.9) the only problem in this case is with κ_{E_*} and $\widetilde{k}_{E_*} = k_{E_*}$, but in order to find the formulas it is sufficient to use (1.1.6) and (1.1.7).

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}_*^n$, $n > 1$, define the following *elementary Reinhardt domains:*

$$D_\alpha := \{z \in \mathbb{C}^n : |z_1|^{\alpha_1} \dots |z_n|^{\alpha_n} < 1, \text{ if } \alpha_j < 0 \text{ then } z_j \neq 0\}.$$

We say that α is of *rational type* if there are $t > 0$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_*^n$ such that $\alpha = t\beta$; otherwise, we say that α is of *irrational type*. Note that if α is of rational type then we may assume that all α_j's are relatively prime integers. We also define

$$\widetilde{D}_\alpha := \{z \in D_\alpha : z_1 \dots z_n \neq 0\}.$$

For $\alpha \in \mathbb{Z}_*^n$, $r \in \mathbb{N}$ we set

$$F^\alpha(z) := z^\alpha := z_1^{\alpha_1} \dots z_n^{\alpha_n},$$
$$F^\alpha_{(r)}(z)X := \sum_{\beta_1+\dots+\beta_n=r} \frac{1}{\beta_1! \dots \beta_n!} \frac{\partial^{\beta_1+\dots+\beta_n} F^\alpha(z)}{\partial z_1^{\beta_1} \dots \partial z_n^{\beta_n}} X^\beta,$$

where $X \in \mathbb{C}^n$, $z \in \mathbb{C}^n$, and if $\alpha_j < 0$ then $z_j \neq 0$.

Note that the domain D_α is always unbounded, Reinhardt, and pseudoconvex but not convex.

The formulas for the Carathéodory pseudodistance and the Carathéodory–Reiffen pseudometric as well as for the Green function for elementary Reinhardt domains of the rational type have been known for a long time (see [Jar-Pfl 93]).

THEOREM 1.4.1 (Theorem 3.1; see [Jar-Pfl 93]). *If $\alpha \in \mathbb{Z}_*^n$, where α_j's are relatively prime, then*

$$\begin{aligned} c_{D_\alpha}(w,z) &= p(w^\alpha, z^\alpha),\\ \widetilde{g}_{D_\alpha}(w,z) &= m(w^\alpha, z^\alpha)^{1/r},\\ \gamma_{D_\alpha}(w;X) &= \gamma(w^\alpha; (F^\alpha)'(w)X),\\ A_{D_\alpha}(w;X) &= (\gamma(w^\alpha; F^\alpha_{(r)}(w)X))^{1/r}, \quad (w,z) \in D_\alpha \times D_\alpha, (w;X) \in D_\alpha \times \mathbb{C}^n, \end{aligned}$$

where $r = r(w)$ is the order of vanishing of the function $F^\alpha(\,\cdot\,) - F^\alpha(w)$ at w. If α is of irrational type, then

$$\begin{aligned} c_{D_\alpha}(w,z) &= 0,\\ \gamma_{D_\alpha}(w;X) &= 0, \quad (w,z) \in D_\alpha \times D_\alpha, (w;X) \in D_\alpha \times \mathbb{C}^n. \end{aligned}$$

Below we present formulas in the remaining cases. We present the formulas for the Green function (and for the Azukawa pseudometric) in the irrational case and for the Lempert function, Kobayashi pseudodistance and Kobayashi–Royden pseudometric for all elementary Reinhardt domains.

We can assume that $\alpha_1, \dots, \alpha_l < 0$, $\alpha_{l+1}, \dots, \alpha_n > 0$, $l \in \{0, \dots, n\}$.

THEOREM 1.4.2 (Theorem 3.1.1). *Assume that $0 \le l < n$. Let $(w,z) \in D_\alpha \times D_\alpha$, $(w;X) \in D_\alpha \times \mathbb{C}^n$. Set $\mathcal{J} := \{j \in \{1,\dots,n\} : w_j = 0\} = \{j_1, \dots, j_k\}$ (16). Define $\widetilde{\alpha}_{l+1} := \min\{\alpha_{l+1}, \dots, \alpha_n\}$.*

(16) Obviously, $\mathcal{J} \subset \{l+1, \dots, n\}$.

1. *Assume that $\alpha\in\mathbb{Z}^n_*$ with α_j's relatively prime. Then*

$$\widetilde{k}_{D_\alpha}(w,z)=\begin{cases}\min\{p((w^\alpha)^{1/\widetilde{\alpha}_{l+1}},(z^\alpha)^{1/\widetilde{\alpha}_{l+1}})\} & \text{if } w,z\in\widetilde{D}_\alpha,\\ p(0,|z^\alpha|^{1/(\alpha_{j_1}+\ldots+\alpha_{j_k})}) & \text{if } \mathcal{J}\neq\emptyset,\end{cases}$$
$$k_{D_\alpha}(w,z)=\min\{p((w^\alpha)^{1/\widetilde{\alpha}_{l+1}},(z^\alpha)^{1/\widetilde{\alpha}_{l+1}})\},$$

where the minima are taken over all possible roots. In the infinitesimal case we have

$$\kappa_{D_\alpha}(w;X)=\begin{cases}\gamma\Big((w^\alpha)^{1/\widetilde{\alpha}_{l+1}},(w^\alpha)^{1/\widetilde{\alpha}_{l+1}}\dfrac{1}{\widetilde{\alpha}_{l+1}}\displaystyle\sum_{j=1}^n\frac{\alpha_jX_j}{w_j}\Big) & \text{if } \mathcal{J}=\emptyset,\\ (|w_1|^{\alpha_1}\ldots|X_{j_1}|^{\alpha_{j_1}}\ldots|X_{j_k}|^{\alpha_{j_k}}\ldots|w_n|^{\alpha_n})^{1/(\alpha_{j_1}+\ldots+\alpha_{j_k})} & \text{if } \mathcal{J}\neq\emptyset.\end{cases}$$

2. *Assume that α is of irrational type. Then*

$$\widetilde{k}_{D_\alpha}(w,z)=\begin{cases}p((|w_1|^{\alpha_1}\ldots|w_n|^{\alpha_n})^{1/\widetilde{\alpha}_{l+1}},(|z_1|^{\alpha_1}\ldots|z_n|^{\alpha_n})^{1/\widetilde{\alpha}_{l+1}}) & \text{if } w,z\in\widetilde{D}_\alpha,\\ p(0,(|z_1|^{\alpha_1}\ldots|z_n|^{\alpha_n})^{1/(\alpha_{j_1}+\ldots+\alpha_{j_k})}) & \text{if } \mathcal{J}\neq\emptyset;\end{cases}$$
$$k_{D_\alpha}(w,z)=p\Big(\Big(\prod_{j=1}^n|w_j|^{\alpha_j}\Big)^{1/\widetilde{\alpha}_{l+1}},\Big(\prod_{j=1}^n|z_j|^{\alpha_j}\Big)^{1/\widetilde{\alpha}_{l+1}}\Big),$$
$$\widetilde{g}_{D_\alpha}(w,z)=\begin{cases}0 & \text{if } \mathcal{J}=\emptyset,\\ (|z_1|^{\alpha_1}\ldots|z_n|^{\alpha_n})^{1/(\alpha_{j_1}+\ldots+\alpha_{j_k})} & \text{if } \mathcal{J}\neq\emptyset.\end{cases}$$

In the infinitesimal case we have

$$\kappa_{D_\alpha}(w;X)=\begin{cases}\gamma\Big(\Big(\displaystyle\prod_{j=1}^n|w_j|^{\alpha_j}\Big)^{1/\widetilde{\alpha}_{l+1}},\Big(\prod_{j=1}^n|w_j|^{\alpha_j}\Big)^{1/\widetilde{\alpha}_{l+1}}\frac{1}{\widetilde{\alpha}_{l+1}}\sum_{j=1}^n\frac{\alpha_jX_j}{w_j}\Big) & \text{if } \mathcal{J}=\emptyset,\\ (|w_1|^{\alpha_1}\ldots|X_{j_1}|^{\alpha_{j_1}}\ldots|X_{j_k}|^{\alpha_{j_k}}\ldots|w_n|^{\alpha_n})^{1/(\alpha_{j_1}+\ldots+\alpha_{j_k})} & \text{if } \mathcal{J}\neq\emptyset;\end{cases}$$
$$A_{D_\alpha}(w;X)=\begin{cases}0 & \text{if } \mathcal{J}=\emptyset,\\ (|w_1|^{\alpha_1}\ldots|X_{j_1}|^{\alpha_{j_1}}\ldots|X_{j_k}|^{\alpha_{j_k}}\ldots|w_n|^{\alpha_n})^{1/(\alpha_{j_1}+\ldots+\alpha_{j_k})} & \text{if } \mathcal{J}\neq\emptyset.\end{cases}$$

THEOREM 1.4.3 (Theorem 3.5.1). *Assume that $l=n$. Then*

1. *If α is of rational type then*

$$\widetilde{k}_{D_\alpha}(w,z)=k_{D_\alpha}(w,z)=k_{E_*}(w^\alpha,z^\alpha),\quad \kappa_{D_\alpha}(w;X)=\kappa_{E_*}\Big(w^\alpha;w^\alpha\sum_{j=1}^n\frac{\alpha_jX_j}{w_j}\Big).$$

2. *If α is of irrational type then*

$$\widetilde{k}_{D_\alpha}(w,z)=k_{D_\alpha}(w,z)=k_{E_*}(|w_1|^{\alpha_1}\ldots|w_n|^{\alpha_n},|z_1|^{\alpha_1}\ldots|z_n|^{\alpha_n}),$$
$$\kappa_{D_\alpha}(w;X)=\kappa_{E_*}\Big(|w_1|^{\alpha_1}\ldots|w_n|^{\alpha_n};|w_1|^{\alpha_1}\ldots|w_n|^{\alpha_n}\sum_{j=1}^n\frac{\alpha_jX_j}{w_j}\Big).$$

In the formulas above writing $a_1\ldots b_{j_1}\ldots b_{j_k}\ldots a_n$ we always mean the expression composed of n factors, $n-k$ (out of n) numbers a_j (with $a_{j_1},\ldots,a_{j_k}$ deleted) and k numbers b_j $(b_{j_1},\ldots,b_{j_k})$.

The formulas from Theorem 1.4.2 may seem incomplete (they do not cover the case $w\in\widetilde{D}_\alpha$, $z\notin\widetilde{D}_\alpha$); nevertheless, because of the symmetry of the relevant functions (not the Green function) they do cover the other cases.

The proof of Theorem 1.4.3 is quite simple and short, whereas that of Theorem 1.4.2 is long and tedious and is based on formulas (1.1.6) and (1.1.7) and the Kronecker Theorem (in the irrational case).

As already mentioned, elementary Reinhardt domains of irrational type with $l = n$ *give us a negative answer to the question posed by S. Kobayashi about the possibility of replacing the infimum with minimum in* (1.1.6).

Some other new formulas for other classes of Reinhardt domains (but only for the Green function, the Azukawa pseudometric, Carathéodory pseudodistance, and the Carathéodory–Reiffen pseudometric) have been found recently (see [Jar-Pfl 99]).

1.5. Finite compactness and completeness with respect to invariant distances. We say that a domain D is *d-hyperbolic* ($d = c, k$ or $\widetilde{k}$) if $d_D(w, z) > 0$ whenever $w \neq z$. It is trivial that any bounded domain is d-hyperbolic. We say that D is *Brody hyperbolic* if every holomorphic mapping $f : \mathbb{C} \to D$ is constant. It is trivial that

$$c\text{-hyperbolic} \Rightarrow k\text{-hyperbolic} \Rightarrow \widetilde{k}\text{-hyperbolic} \Rightarrow \text{Brody hyperbolic.}$$

In the case when the above mentioned functions are distances it is natural to introduce the notion of completeness. More precisely, assume that D is d-hyperbolic ($d = c$ or k); then we say that a domain D is *d-complete* if any d_D-Cauchy sequence $\{z^\nu\}_{\nu=1}^\infty \subset D$ is convergent to some $z^0 \in D$ with respect to the standard topology in D.

Another, closely related notion may also be introduced. Namely, we say that a d-hyperbolic domain D is *d-finitely compact* if for any $w \in D$, $r > 0$ we have $B_{d_D}(w, r) \subset\subset D$ (d equals c or k). It is easy to see that for a d-hyperbolic domain D the following implications hold:

$$d\text{-finite compact} \Rightarrow d\text{-complete},$$

$$c\text{-complete} \Rightarrow k\text{-complete}, \quad c\text{-finite compact} \Rightarrow k\text{-finite compact}.$$

Moreover, k-completeness implies k-finite compactness (compare [Rin 61]). The problem whether an analogous implication holds for the Carathéodory distance is not solved.

All strongly pseudoconvex domains are c-finitely compact (use the existence of peak functions—see e.g. [Kra 92]). There is an example of a bounded balanced pseudoconvex domain with the continuous Minkowski function, which is not k-complete (see [Jar-Pfl 91b]). On the other hand any k-complete domain must be taut. In the complex plane any taut domain is k-complete.

In Chapter II we deal with finite compactness and completeness of pseudoconvex Reinhardt domains. First we have to characterize the notion of hyperbolicity in this class of domains. It turns out that for such domains all the notions of hyperbolicity considered coincide and are trivial in the following sense: the domains are biholomorphic to bounded domains (see [Zwo 99]).

Recall that a domain $D \subset \mathbb{C}^n$ is called *Reinhardt* if $(\lambda_1 z_1, \ldots, \lambda_n z_n) \in D$ for all $z = (z_1, \ldots, z_n) \in D$ and $\lambda_1, \ldots, \lambda_n \in \partial E$. If, additionally, $(\lambda_1 z_1, \ldots, \lambda_n z_n) \in D$ for any $\lambda_1, \ldots, \lambda_n \in E$ then we say that D is *complete*.

Let us define

$$\log D := \{x \in \mathbb{R}^n : (e^{x_1}, \ldots, e^{x_n}) \in D\}.$$

THEOREM 1.5.1 (cf. Theorem 2.5.1). *Assume that D is a Reinhardt pseudoconvex domain in $\mathbb{C}^n$. Then the following conditions are equivalent*:

(i) *D is c-hyperbolic*;
(ii) *D is $\widetilde{k}$-hyperbolic*;
(iii) *D is Brody hyperbolic*;
(iv) *D is biholomorphic to a bounded Reinhardt domain.*

We give a full description of Kobayashi completeness and Carathéodory completeness in hyperbolic pseudoconvex Reinhardt domains.

THEOREM 1.5.2 (Theorem 2.6.5). *Let D be a hyperbolic* (*in the sense of any condition from Theorem* 1.5.1) *pseudoconvex Reinhardt domain. Then D is k-finitely compact* (*in particular*, *D is Kobayashi complete*).

THEOREM 1.5.3 (Theorem 2.6.6). *Let D be a hyperbolic pseudoconvex Reinhardt domain. Then the following conditions are equivalent*:

(i) *D is c-finitely compact*;
(ii) *D is c-complete*;
(iii) *D is bounded and for any $j \in \{1, \ldots, n\}$*,

$$\text{if } \bar{D} \cap V_j \neq \emptyset \text{ then } D \cap V_j \neq \emptyset. \tag{1.5.1}$$

It was P. Pflug who started the investigation of completeness of Reinhardt domains. It was proved in [Pfl 84] that all bounded pseudoconvex complete Reinhardt domains are c-finitely compact. Next, in [Fu 94], S. Fu proved Theorem 1.5.2 for bounded domains by using the methods from [Pfl 84] and applying the localization principle for the Kobayashi distance. In view of Theorem 1.5.1 this result extends immediately to hyperbolic domains.

As far as Theorem 1.5.3 is concerned, the implication (iii)⇒(i) comes from [Fu 94]. We prove the remaining implication (ii)⇒(iii).

It turns out that in the class of hyperbolic pseudoconvex Reinhardt domains k-completeness is equivalent to tautness, whereas c-completeness is equivalent to hyperconvexity ([17]) (see Corollaries 2.6.10 and 2.6.11).

Although bounded pseudoconvex Reinhardt domains not satisfying the condition (1.5.1) are not c-finitely compact it is often the case that the Carathéodory distance tends to infinity when one point is fixed and the other one tends to a boundary point not lying on an axis (see Proposition 2.6.1). In particular, it is always the case in $\mathbb{C}^2$:

PROPOSITION 1.5.4 (see Corollary 2.6.2). *If D is a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^2$, then for any $z^0 \in \partial D \cap \mathbb{C}^2_*$ and for any $w \in D$ we have $c_D(w, z) \to \infty$ as z tends to z^0.*

([17]) A domain $D \subset \mathbb{C}^n$ is *hyperconvex* if there is a plurisubharmonic continuous negative function u defined on D such that $\{u < c\} \subset\subset D$ for any $c < 0$. This definition differs from the standard one, where additionally the boundedness of the domain is required (see [Ste 75]). In view of our definition the biholomorphic image of a hyperconvex domain is hyperconvex. Any bounded hyperconvex domain is taut. Any taut domain is pseudoconvex. The converse implications do not hold.

In higher dimensions there are examples of domains for which this does not hold (see Example 2.6.4).

1.6. Pluricomplex Green function with a logarithmic pole. First, let us recall some well known properties of the pluricomplex Green function (see [Dem 87], [Kli 85] and [Kli 91]):

THEOREM 1.6.1. (i) *For any* $w \in D$, $g_D(w,\cdot) \in \mathrm{PSH}(D,[-\infty,0))$. *Moreover*, $g_D(w,z) - \log\|w-z\|$ *is bounded from above*;

(ii) $g_D(w,\cdot)$ *is the largest plurisubharmonic function not exceeding* $\log \widetilde{k}^*_D(w,\cdot)$;

(iii) *if* D *is a bounded hyperconvex domain*, *then* g_D *is continuous and* $g_D(w,z)\to 0$ *as* $z\to\partial D$, $w\in D$;

(iv) *if* D *is a bounded domain then* $g_D(w,\cdot)$ *is a maximal function on* $D\setminus\{w\}$ ([18]).

As to the property (iii) let us mention that despite much effort we have not been able to prove the point convergence of $g_D(z,w)$ to 0 as z tends to ∂D (when D is a bounded hyperconvex domain). Note that this holds for c-finitely compact domains (e.g. pseudoconvex Reinhardt domains fulfilling (1.5.1)). The problem was dealt with in [Com 98] and [Carl-Ceg-Wik 98], where some kinds of convergence were proven. In any case for any bounded hyperconvex domain we have (see [Bło-Pfl 98], [Her 99])

$$\mathrm{Vol}(\{g_D(w,\cdot)<-1\})\to 0 \quad \text{as } w \text{ tends to } \partial D.$$

The above convergence plays an important role in the study of Bergman completeness (for a more detailed discussion see Section 1.8).

In contrast to other contractible functions the Green function is not, in general, symmetric. The first example of a very regular domain (strongly pseudoconvex with real analytic boundary) without symmetric Green function comes from [Bed-Dem 88]. Note that for a domain $D\subset\mathbb{C}$ the Green function is symmetric (see e.g. [Ran 95]). We see from the Lempert Theorem that for D convex the Green function is symmetric, too.

In Chapter IV we prove that in a reasonable class of domains (containing bounded hyperconvex domains) some kind of "infinitesimal" symmetry holds for the Green function.

THEOREM 1.6.2 (cf. Corollary 4.2.4). *Let* D *be a bounded hyperconvex domain in* $\mathbb{C}^n$. *Then*

$$\lim_{w',\,w''\to w,\,w'\neq w''}(g_D(w',w'')-g_D(w'',w'))=0.$$

On the other hand we can find very regular domains (smooth, bounded, complete Reinhardt and pseudoconvex; see Remark 4.3.3) and sequences $z^\nu\to\partial D$ such that

$$g_D(w,z_\nu),\quad g_D(z_\nu,w)\to 0 \quad\text{and}\quad \lim_{\nu\to\infty}\frac{g_D(z_\nu,w)}{g_D(w,z_\nu)}=\infty.$$

In other words, the Green function is in that case (globally) extremely unsymmetric.

([18]) A plurisubharmonic function $u: D\to\mathbb{R}$ is called *maximal* if for any open relatively compact subset G of D and for any function v plurisubharmonic on G and upper semicontinuous on $\bar G$ the inequality $v\le u$ on ∂G implies $v\le u$ on G.

The "global" symmetry of the Green function is completely characterized for complex ellipsoids.

THEOREM 1.6.3 (cf. Theorem 4.1.1). *Let $\mathcal{E}(p)$ be a complex ellipsoid. Then the Green function $g_{\mathcal{E}(p)}$ is symmetric iff $\mathcal{E}(p)$ is convex.*

Some other partial results also suggest that more generally (e.g. in the class of complete bounded pseudoconvex Reinhardt domains) the symmetry of the Green function is equivalent to the convexity of the domain (for a discussion of this subject see Section 4.3).

1.7. The Green function with many poles. Analytic disks and the Green function. Let D be a domain in $\mathbb{C}^n$. Let $\emptyset \neq P \subset D$ be a finite set and let $\nu : P \to (0,\infty)$. We define the *pluricomplex Green function with poles in P with weights ν* as follows (see [Lel 89]):

$$g_D(P;\nu;z) := \sup\{u(z)\},$$

where the supremum is taken over all $u \in \mathrm{PSH}(D)$, $u < 0$, such that $u(\cdot) - \nu(p)\log\|\cdot - p\|$ is bounded from above near p for all $p \in P$.

Note that when $\#P = 1$ and $\nu \equiv 1$ then g_D is the pluricomplex Green function with a logarithmic pole.

It is well known that $g_D(P;\nu;\cdot)$ is a negative plurisubharmonic function. Recall that if D is bounded, then $g_D(P;\nu;\cdot)$ is maximal on $D \setminus P$; if D is a bounded hyperconvex domain, then $g_D(P,\nu,\cdot)$ is a continuous function, which extends continuously to 0 on the boundary; compare Theorem 1.6.1(iii) (see [Dem 87] and [Lel 89]).

It turns out that an equivalent definition using analytic disks is possible. Namely, the following equality has been obtained in [Lar-Sig 98] (for the Green function with one pole this equality may be found in [Edi 97a] and [Pol 93]):

THEOREM 1.7.1. *The following equality holds*:

$$\begin{aligned}(1.7.1)\qquad & g_D(P;\nu;z)\\ &= \inf\{g_E(\varphi^{-1}(P)\cap E,\widetilde{\nu},0),\ \varphi\in\mathcal{O}(\bar E,D),\ \varphi(0)=z, 0<\#(E\cap\varphi^{-1}(P))<\infty\}\\ &= \inf\{g_E(\varphi^{-1}(P)\cap E,\widetilde{\nu},\lambda),\ \varphi\in\mathcal{O}(\bar E,D),\varphi(\lambda)=z,\ 0<\#(E\cap\varphi^{-1}(P))<\infty\},\end{aligned}$$

where $\widetilde{\nu}(\lambda) := \mathrm{ord}_\lambda(\varphi-\varphi(\lambda))\cdot\nu(\varphi(\lambda))$, $\lambda\in\varphi^{-1}(P)$ ([19]).

The above formula has turned out to be useful for proving the product property for the Green function (with one pole).

It turns out that the Green function with many poles exhibits some kind of invariance with respect to proper holomorphic mappings. Namely, let $\pi : \widetilde{D} \to D$ be a proper holomorphic mapping and let P be a set of poles in D such that $\pi^{-1}(P)\cap\{\det\pi' = 0\} = \emptyset$. Define $\widetilde{\nu}(q) := \nu(\pi(q))$, $q\in\pi^{-1}(P)$.

The theorem below can be found in [Lar-Sig 98]; we give an alternate proof (cf. [Edi-Zwo 98]).

([19]) We know that $g_E(P,\nu,\lambda) = \sum_{p\in P}\nu(p)g_E(p,\lambda)$. In the case of one pole, (1.7.1) can be read as follows: $g_D(p,z) = \inf\{\sum_{\varphi(\lambda)=p}\mathrm{ord}_\lambda(\varphi-\varphi(\lambda))\log|\lambda| : \varphi\in\mathcal{O}(\bar E,D),\ \varphi(0)=z,$ $0<\#\varphi^{-1}(p)<\infty\}$.

THEOREM 1.7.2 (see Theorem 3.6.2). *Under the above assumptions, for any* $\widetilde{w} \in \widetilde{D}$,

$$g_{\widetilde{D}}(\pi^{-1}(P); \widetilde{\nu}; \widetilde{w}) = g_D(P; \nu; \pi(\widetilde{w})).$$

Except for dimension one ([20]) practically no formulas for the Green function with many poles have been known so far. We give the formula for the unit ball with two poles with equal weights. The key role in establishing it will be played by Theorem 1.7.2, which enables us to reduce the problem to calculating the Green function of the convex complex ellipsoid $\mathcal{E}(1, 1/2)$ with one pole, and then using Theorem 1.3.3.

In case $\nu \equiv 1$ we write $g_D(P; \cdot) := g_D(P; \nu; \cdot)$.

THEOREM 1.7.3 (Theorem 3.6.1). *Let* $0 < p < 1$ *and* $(z_1, z_2) \in \mathbb{B}_2$. *Then*

$$g_{\mathbb{B}_2}((0,p),(0,-p);(z_1,z_2))$$
$$= \begin{cases} \dfrac{1}{2}\log\left(1 - \dfrac{(1-p^2)(1-|z_1|^2-|z_2|^2)}{|1-pz_2|^2}\right) & \text{if } p|z_1| \geq |z_2 - p|, \\ \dfrac{1}{2}\log\left(1 - \dfrac{(1-p^2)(1-|z_1|^2-|z_2|^2)}{|1+pz_2|^2}\right) & \text{if } p|z_1| \geq |z_2 + p|, \\ \dfrac{1}{2}\log\dfrac{2(1-p^2\operatorname{Re} z_2^2)|z_1|^2 + |p^2 - p^2|z_1|^2 - z_2^2|^2 + \sqrt{\triangle}}{2|1-p^2z_2^2|^2} & \\ \qquad\qquad \text{if } p|z_1| < \min\{|z_2 - p|, |z_2 + p|\}, \end{cases}$$

where $\triangle := -4|z_1|^4(p^2\operatorname{Im} z_2^2)^2 + 4|z_1|^2(1-p^2\operatorname{Re} z_2^2)|p^2-p^2|z_1|^2-z_2^2|^2 + |p^2-p^2|z_1|^2-z_2^2|^4$.

Another proof of Theorem 1.7.3 comes from [Com 97], where an entirely different approach to the problem was applied.

Recall that even in the case of the bidisk E^2 the complete formula for the Green function with two poles with equal weights is not known.

It is easy to see that the following upper and lower bounds hold (see [Lel 89]):

$$\min\{\nu(p)g_D(p,z) : p \in P\} \geq g_D(P;\nu;z) \geq \sum_{p\in P} \nu(p) g_D(p,z), \quad z \in D. \tag{1.7.2}$$

Set (see [Lel 89])

$$\mathcal{E}(D,P,\nu) := \Big\{z \in D : g_D(P;\nu;z) = \sum_{p\in P}\nu(p)g_D(p,z)\Big\}.$$

Clearly, $P \subset \mathcal{E}(D,P,\nu)$. Lelong asked whether the set $\mathcal{E}(D,P,\nu)$ had nonempty interior at least for D two-dimensional. The answer is negative even in the case of the bidisk (see [Carl 95]).

We give a precise description of this set for the unit ball.

COROLLARY 1.7.4 (Corollary 3.6.8). *Let* $P \subset \mathbb{B}_n$, $\#P \geq 2$, $n \geq 2$. *Then* $\mathcal{E}(\mathbb{B}_n, P, \nu) = P \cup (L \cap \mathbb{B}_n)$, *where* L *is the complex straight line containing* P ($L = \emptyset$ *if such a line does not exist*).

([20]) In this case it is easy to see that $g_D(P;\nu;z) = \sum_{p\in P}\nu(p)g_D(p,z)$.

1.8. The Bergman distance. Bergman completeness. For $0 < p < \infty$ put

$$L_h^p(D) := \mathcal{O}(D) \cap L^p(D).$$

For any domain D we may find an orthonormal basis $\{\varphi_j\}_{j\in J}$ of $L_h^2(D)$ ($\#J \le \aleph_0$). Then we define

$$K_D(z) := \sum_{j\in J} |\varphi_j(z)|^2, \quad z \in D.$$

We call K_D the *Bergman kernel* of D. For domains such that for any $z \in D$ there is $f \in L_h^2(D)$ with $f(z) \neq 0$ (for example for D bounded) we have

$$K_D(z) = \sup\{|f(z)|^2/\|f\|^2_{L^2(D)} : f \in L_h^2(D),\ f \not\equiv 0\}.$$

One may check that if D is such that $K_D(z) > 0$, $z \in D$, then $\log K_D$ is a smooth plurisubharmonic function. In this case we define

$$\beta_D(z;X) := \bigg(\sum_{j,k=1}^n \frac{\partial^2 \log K_D(z)}{\partial z_j \partial \overline{z}_k} X_j \overline{X}_k\bigg)^{1/2}, \quad z \in D,\ X \in \mathbb{C}^n,$$

and we see that β_D is a pseudometric called the *Bergman pseudometric*.

For $w, z \in D$ we put

$$b_D(w,z) := \inf\{L_{\beta_D}(\alpha)\}$$

where the infimum is taken over piecewise C^1 curves $\alpha : [0,1] \to D$ joining w and z and $L_{\beta_D}(\alpha) := \int_0^1 \beta_D(\alpha(t);\alpha'(t))\,dt$. We call b_D the *Bergman pseudodistance of D*.

Obviously, the Bergman pseudodistance is not defined for all domains. In the class of bounded domains (where it is always defined) it does not have the contractivity property (see [Ber 36]). Nevertheless, the Bergman distance (as well as the Bergman metric) is invariant with respect to biholomorphic mappings. More precisely, for any biholomorphic mapping $F : D \to G$ ($D, G \subset\subset \mathbb{C}^n$) we have

$$b_G(F(w),F(z)) = b_D(w,z), \quad \beta_G(F(w);F'(w)X) = \beta_D(w;X), \quad w,z \in D,\ X \in \mathbb{C}^n.$$

We will consider only bounded domains.

As in the case of invariant pseudodistances we may define Bergman completeness. A bounded domain D is called *Bergman complete* (or *b-complete*) if any b_D-Cauchy sequence is convergent to some point in D with respect to the standard topology of D.

Any bounded b-complete domain is pseudoconvex (see [Bre 55]). The converse implication fails to hold. The problem of b-completeness has a long history. Let us list only some classes of domains which are b-complete:

- bounded C^1-pseudoconvex domains (see e.g. [Ohs 81]);
- bounded pseudoconvex balanced domains with the continuous Minkowski function (see [Jar-Pfl 89b]);
- bounded hyperconvex domains (see [Bło-Pfl 98], [Her 99]).

The last class of domains contains the two preceding ones (see [Ker-Ros 91]). It turns out that there are nonhyperconvex domains which are b-complete (see [Chen 98] and [Her 99]). The example of the latter paper helped us find a class of bounded pseudoconvex Reinhardt domains which are b-complete although they are not hyperconvex.

Before we formulate the results we have to introduce some notations.

For a pseudoconvex Reinhardt domain $D \subset \mathbb{C}^n$, $a \in \log D$ we define

$$\mathfrak{C}(D) := \{v \in \mathbb{R}^n : a + \mathbb{R}_+ v \subset \log D\},$$
$$\widetilde{\mathfrak{C}}(D) := \{v \in \mathfrak{C}(D) : \overline{(\exp(a + \mathbb{R}_+ v))} \subset D\},$$
$$\mathfrak{C}'(D) := \mathfrak{C}(D) \setminus \widetilde{\mathfrak{C}}(D), \quad H := H(D) := \operatorname{Span}(\mathfrak{C}(D)) \subset \mathbb{R}^n.$$

It is easy to see that the set $\mathfrak{C}(D)$ (as well as $\widetilde{\mathfrak{C}}(D)$ and $\mathfrak{C}'(D)$) does not depend on the choice of a.

Let $v^1, \dots, v^r \in H$ be a maximal linearly independent subset of vectors from $\mathbb{Z}^n$. Put $H_1 := H_1(D) := \operatorname{Span}\{v^1, \dots, v^r\}$.

Proposition 1.8.1 (Proposition 2.7.2). *Let D be a bounded pseudoconvex Reinhardt domain. If $\mathfrak{C}'(D) \cap \mathbb{Q}^n \neq \emptyset$ then D is not Bergman complete.*

Theorem 1.8.2 (Theorem 2.7.3). *Let D be a bounded pseudoconvex Reinhardt domain. If $H_1 \cap \mathfrak{C}(D) = \{0\}$ then D is Bergman complete.*

Although in general we do not have a precise description of b-complete pseudoconvex Reinhardt domains (it may happen that $\mathfrak{C}'(D) \cap \mathbb{Q}^n = \emptyset$ and $H_1 \cap \mathfrak{C}(D) \neq \{0\}$) in dimension 2 the problem is entirely solved.

Theorem 1.8.3 (Corollary 2.7.4). *For a bounded pseudoconvex Reinhardt domain D in $\mathbb{C}^2$ the following two conditions are equivalent*:

(i) *D is Bergman complete*,
(ii) *$\mathfrak{C}'(D) \cap \mathbb{Q}^n = \emptyset$.*

In any case a number of Bergman complete and not hyperconvex pseudoconvex bounded Reinhardt domains is given by the above results ([21]). It would be interesting to know whether Theorem 1.8.3 generalizes to higher dimensions. If this generalization fails to hold, the question what the right description of Bergman complete bounded Reinhardt domains is, seems to be interesting.

A relation between good boundary behavior of the Green function (understood as the convergence to 0 of volumes of sublevel sets of the Green function as the pole tends to the boundary) and Bergman completeness has been discovered by S. Chen and G. Herbort (see Theorem 2.8.1). It turns out that these two properties are equivalent on bounded pseudoconvex Reinhardt domains in $\mathbb{C}^2$:

Proposition 1.8.4 (Proposition 2.8.5). *Let D be a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^2$. Then the following conditions are equivalent*:

(i) *D is Bergman complete*,
(ii) *for any $\delta > 0$, $\operatorname{Vol}(\{g_D(p, \cdot) < -\delta\}) \to 0$ as $p \to \partial D$*,
(iii) *for any $z \in D \cap \mathbb{C}^2_*$ we have $g_D(p, z) \to 0$ as $p \to \partial D$.*

([21]) The simplest possible example is $D := \left\{z \in E^2 : \frac{1}{2}|z_1|^\alpha < |z_2| < 2|z_1|^\alpha\right\}$, where α is a positive irrational number.

Also in higher dimensions a similar relation seems probable (compare Lemma 2.8.2). Nevertheless, it is not the case for all domains. We find an example of a bounded Bergman complete domain in $\mathbb{C}$ such that the condition (ii) of Proposition 1.8.4 is not satisfied (for any $\delta > 0$)—see Corollary 2.8.8.

1.9. Carathéodory balls and norm balls. For $0 < r < 1$ recall the definition of *Carathéodory balls* with center at $w \in D$ and radius r:

$$B_{c_D^*}(w,r) := \{z \in D : c_D^*(w,z) < r\}.$$

Under the additional assumption that D is a bounded balanced domain with the Minkowski function h we define for $s > 0$, $w \in D$ the following ball, which in the case when D is bounded may be called a *norm ball*:

$$B_D(w,s) := \{z \in \mathbb{C}^n : h(w-z) < s\}.$$

These balls are closely related to the natural geometry of the domain. For bounded balanced convex domains let us consider the following problem: Which Carathéodory balls are also norm balls (with respect to the norm of the domain considered)?

Note that any Carathéodory ball with center at 0 is a norm ball (use Proposition 1.1.2). Are there any other Carathéodory balls which are norm balls? An example of the unit disk E shows that it may happen that all Carathéodory balls are norm balls. On the other hand in higher dimensions the only Carathéodory balls in the unit ball which are simultaneously norm balls (in this case norm balls are precisely the Euclidean balls) are the ones centered at the origin (see [Rud 80]).

As we shall see the latter phenomenon is more common.

Making use of the form of complex geodesics in convex ellipsoids (Theorem 1.3.3) we shall give a sketch of the following result (see [Zwo 96]):

Theorem 1.9.1 (Theorem 5.1). *Let $\mathcal{E}(p)$ be a convex ellipsoid. Then if $p_1,\dots,p_n \neq 1$ or $p_1 = \ldots = p_n = 1$ then a Carathéodory ball with center at w is a norm ball iff $w = 0$. If $n = 2$, $p_1 = 1/2$, $p_2 = 1$, then any ball $B_{c^*_{\mathcal{E}(p)}}((0,w_2),r)$ is a norm ball.*

The partial results of Theorem 1.9.1 may be found in [Sch 93], [Sre 95], [Zwo 95b], and [Sch-Sre 96]. As already mentioned, a description of complex geodesics plays a key role in the proof of the theorem. In particular, they enable us to reduce the problem to dimension two. By a good choice of geodesics we get much information about the structure of Carathéodory balls.

A generalization of Theorem 1.9.1 has been found recently (see [Vis 99]); namely, making use of the description of complex geodesics, a similar result is proven for a wider class of domains. Moreover, it is proven that in some class of domains (containing convex ellipsoids), the only Carathéodory balls with center different from the origin which are norm balls, are the ones with center at w (and the domain is necessarily an ellipsoid), where w is such that there is exactly one j with $p_j = 1$, $w_j \neq 0$ and $p_k = 1/2$, $w_k = 0$ for $k \neq j$. This result may be seen as a complement of the results obtained in Theorem 1.9.1.

II. Pseudoconvex Reinhardt domains—completeness

In this chapter we consider pseudoconvex Reinhardt domains. First, we recall basic notions and results in Section 2.1 and prove some results on convex cones and their relations with pseudoconvex Reinhardt domains in Section 2.2. Next, we study algebraic mappings (as proper mappings introduced in a natural way) and closely related quasi-elementary Reinhardt domains (Sections 2.3 and 2.4). In Section 2.5 we give a precise description of hyperbolic pseudoconvex Reinhardt domains (Theorem 2.5.1), which is necessary for the study of the Carathéodory and Kobayashi completeness of pseudoconvex Reinhardt domains (see Theorems 2.6.5 and 2.6.6). In Section 2.7 the problem which bounded pseudoconvex Reinhardt domains are Bergman complete is considered; in dimension two we give a precise description of such domains, in higher dimensions we get partial solutions. In Section 2.8 we study the relation between the convergence to zero of the volume of sublevel sets of the Green function as the pole tends to the boundary, and the Bergman completeness of the domain. Since in the class of bounded hyperconvex domains both conditions mentioned above hold, we are mainly interested in nonhyperconvex bounded domains. In the class of bounded pseudoconvex Reinhardt domains these two phenomena seem to be closely related (see e.g. Proposition 2.8.5 and Lemma 2.8.2) whereas for planar domains they are different (see Corollary 2.8.8).

2.1. Geometry of pseudoconvex Reinhardt domains. For a point $z \in \mathbb{C}_*^n$ we put $\log|z| := (\log|z_1|, \dots, \log|z_n|)$.

There is a one-to-one correspondence between Reinhardt domains in $\mathbb{C}_*^n$ and domains in $\mathbb{R}^n$ given by

$$\{\text{Reinhardt domains in } \mathbb{C}_*^n\} \ni D \mapsto \log D \in \{\text{domains in } \mathbb{R}^n\}.$$

There is a similarity between Reinhardt domains and tube domains. For a domain $\omega \subset \mathbb{R}^n$ we define a *tube domain* T_ω (*over* ω):

$$T_\omega := \{x + iy : x \in \omega, y \in \mathbb{R}^n\} = \omega + i\mathbb{R}^n.$$

Then the mapping $\omega \mapsto T_\omega$ gives a one-to-one correspondence between domains in $\mathbb{R}^n$ and tube domains in $\mathbb{C}^n$.

Set

$$V_j := \{z \in \mathbb{C}^n : z_j = 0\}, \quad j = 1, \dots, n;$$
$$V_I := V_{j_1} \cap \dots \cap V_{j_k}, \quad \text{where } I = \{j_1, \dots, j_k\},\ 1 \le j_1 < \dots < j_k \le n.$$

The following two results are well known.

Proposition 2.1.1 (see [Vla 66], [Jak-Jar 98]). *Let D be a Reinhardt domain. Then D is pseudoconvex if and only if $\log D$ is convex and for any $j \in \{1, \dots, n\}$,*

(2.1.1) *if $D \cap V_j \neq \emptyset$ and $(z', z_j, z'') \in D$ then $(z', \lambda z_j, z'') \in D$ for any $\lambda \in \bar{E}$.*

Proposition 2.1.2 (see [Kra 92], [Vla 66]). *For a domain $\omega \subset \mathbb{R}^n$ the following three conditions are equivalent*:

- *ω is convex*;
- *T_ω is convex*;
- *T_ω is pseudoconvex.*

Note that for a Reinhardt domain $D \subset \mathbb{C}^n_*$ we may define

$$\pi : T_{\log D} \ni z \mapsto (\exp(z_1), \ldots, \exp(z_n)) \in D.$$

This is a holomorphic covering of D. Therefore, in view of Propositions 2.1.1 and 2.1.2, formulas 1.1.6 and the Lempert Theorem we have

LEMMA 2.1.3. *Let $D \subset \mathbb{C}^n_*$ be a pseudoconvex Reinhardt domain. Then $\widetilde{k}_D = k_D$. In particular, $\widetilde{k}_D$ is continuous.*

From (2.1.1) we get the following result. Assume that D is a pseudoconvex Reinhardt domain and $D \cap V_j \neq \emptyset$ for some $j \in \{1, \ldots, n\}$. Then for the mapping

$$\pi_j : D \ni z \mapsto (z_1, \ldots, z_{j-1}, 0, z_{j+1}, \ldots, z_n) \in V_j$$

we have $\pi_j(D) = D \cap V_j$. In particular, $\pi_j(D)$ is a pseudoconvex Reinhardt domain in $\mathbb{C}^{n-1}$ (after trivial identification). We may go further and formulate the following result.

Assume that $D \cap V_I \neq \emptyset$, $I = \{j_1, \ldots, j_k\}$, $1 \le j_1 < \ldots < j_k \le n$, $k < n$. Define $(\pi_I(z))_j := 0$ if $j \in I$ and z_j otherwise. Then $\pi_I(D) = D \cap V_I$ and $\pi_I(D)$ is a pseudoconvex Reinhardt domain in $\mathbb{C}^{n-k}$.

2.2. Convex cones and pseudoconvex Reinhardt domains. We have already seen that in the study of pseudoconvex Reinhardt domains in $\mathbb{C}^n$ convex domains in $\mathbb{R}^n$ may play an important role. It turns out that while considering different classes of holomorphic functions a special role is played by cones associated with the logarithmic image of the domain.

We say that $C \subset \mathbb{R}^n$ is a *cone with vertex at a* if for any $v \in C$ we have $a + t(v - a) \in C$ whenever $t > 0$. If we do not specify the vertex of a cone, then we shall mean a cone with vertex at 0.

For a convex domain $\Omega \subset \mathbb{R}^n$ and a point $a \in \overline{\Omega}$ set

$$\mathfrak{C}(\Omega, a) := \{v \in \mathbb{R}^n : a + \mathbb{R}_+ v \subset \overline{\Omega}\}.$$

It is easy to see that $\mathfrak{C}(\Omega, a)$ is a closed convex cone (with vertex at 0). Notice that

$$\mathfrak{C}(\Omega, a) = \bigcup_{C + a \subset \overline{\Omega},\, C \text{ a cone}} C = \text{the largest cone contained in } \overline{\Omega} - a.$$

Moreover, $\mathfrak{C}(\Omega, a) = \mathfrak{C}(\Omega, b)$ for any $a, b \in \overline{\Omega}$. Therefore, we may define $\mathfrak{C}(\Omega) := \mathfrak{C}(\Omega, a)$ for some (any) $a \in \overline{\Omega}$.

Note that if $a \in \Omega$ then $a + \mathfrak{C}(\Omega) \subset \Omega$. If $0 \in \Omega$ then $\mathfrak{C}(\Omega) = h^{-1}(0)$, where h is the Minkowski function of Ω. It is also easy to see that

$$\mathfrak{C}(\Omega) = \{0\} \quad \text{if and only if} \quad \Omega \subset\subset \mathbb{R}^n.$$

Domains Ω not containing affine lines will play a key role. The following three conditions are equivalent:

- Ω contains no affine line;
- $\mathfrak{C}(\Omega)$ contains no affine line;
- $v, -v \in \mathfrak{C}(\Omega) \Rightarrow v = 0$.

For a pseudoconvex Reinhardt domain $D \subset \mathbb{C}^n$ we define $\mathfrak{C}(D) := \mathfrak{C}(\log D)$ ([22]) ([23]).

LEMMA 2.2.1. *Let D be a pseudoconvex Reinhardt domain. Let $\alpha \in \mathbb{Z}^n$, $p \in (0, \infty)$. Then*

- *$z^\alpha \in L_h^p(D)$ if and only if $\langle (p/2)\alpha + \mathbf{1}, v\rangle < 0$ for any $v \in \mathfrak{C}(D)$, $v \neq 0$,*
- *if $\langle \alpha, v\rangle < 0$ for any $v \in \mathfrak{C}(D)$, $v \neq 0$, then $z^\alpha \in H^\infty(D)$; on the other hand,*
- *if $z^\alpha \in H^\infty(D)$ then $\langle \alpha, v\rangle \leq 0$ for any $v \in \mathfrak{C}(D)$.*

PROOF. Assume that $a = (1, \dots, 1) \in D$. First, we prove the following

CLAIM. *Assume that $\mathfrak{C}(D) \neq \{0\}$. Then for any $\varepsilon > 0$ there is a cone T such that $(\log D) \setminus T$ is bounded and if $v \in T$, $\|v\| = 1$ then there exists $w \in \mathfrak{C}(D)$ such that $\|w\| = 1$ and $\|v - w\| < \varepsilon$.*

PROOF. Let h be the Minkowski function of $\log D$. $\log D$ is convex, so h is continuous. Recall that $h^{-1}(0) = \mathfrak{C}(D)$. From the continuity of h we see that for any $\varepsilon > 0$ there is $\delta > 0$ such that $\{w \in \mathbb{R}^n : h(w) \leq \delta, \|w\| = 1\} \subset \{w \in \mathbb{R}^n : \|w\| = 1$ and there is $v \in \mathfrak{C}(D)$, $\|v\| = 1$, $\|w - v\| < \varepsilon\}$.

Now take T to be the smallest cone containing $\{w \in \mathbb{R}^n : h(w) \leq \delta,\ \|w\| = 1\}$. Note that $(\log D) \setminus T$ is bounded. If this were not the case, then there would be $x_\nu \to \infty$ such that $x_\nu \in (\log D) \setminus T$, so $h(x_\nu) < 1$, consequently $h(x_\nu/\|x_\nu\|) < 1/\|x_\nu\|$, so $x_\nu \in T$ for ν large enough—a contradiction. ■

If $\mathfrak{C}(D) = \{0\}$ then the result of Lemma 2.2.1 is trivial. Assume that $\mathfrak{C}(D) \neq \{0\}$. Fix an $\alpha \in \mathbb{Z}^n$ such that $z^\alpha \in L_h^p(D)$. Let $v \in \mathfrak{C}(D)$, $v \neq 0$. We may assume that $|v_n| = 1$. There is an open bounded set $U \subset \mathbb{R}^{n-1}$ such that $0 \in U \times \{0\}$ and $U \times \{0\} + \mathbb{R}_+ v \subset \log D$. We have

$$\infty > \int_D |z^\alpha|^p = \int_{D \cap \mathbb{C}_*^n} |z^\alpha|^p = (2\pi)^n \int_{\log D} e^{2\langle (p/2)\alpha + \mathbf{1}, x\rangle}\, dx_1 \dots dx_n$$
$$\geq (2\pi)^n \int_0^\infty \Big(\int_{U \times \{0\} + x_n v} e^{2\langle (p/2)\alpha + \mathbf{1}, x\rangle}\, dx_1 \dots dx_{n-1}\Big)\, dx_n = M \int_0^\infty e^{2x_n \langle (p/2)\alpha + \mathbf{1}, v\rangle}\, dx_n,$$

from which we get the desired inequality $\langle (p/2)\alpha + \mathbf{1}, v\rangle < 0$.

Assume now that $\langle (p/2)\alpha + \mathbf{1}, v\rangle < 0$ for any $v \in \mathfrak{C}(D)$, $v \neq 0$. Then there is some $\delta > 0$ such that $\langle (p/2)\alpha + \mathbf{1}, v\rangle \leq -\delta$ for any $v \in \mathfrak{C}(D)$, $\|v\| = 1$. Now using the above Claim we get the existence of a cone T satisfying

$$\langle (p/2)\alpha + \mathbf{1}, v\rangle \leq -\delta/2, \quad v \in T,\ \|v\| = 1$$

([22]) Note that in contrast to Section 1.8 we change the definition of the set $\mathfrak{C}(D)$ a little. We do this because in this form it will be easier to formulate and prove some auxiliary results; however, the results from Section 1.8 also remain true for this new definition as we shall see in Section 2.7.

([23]) By $H^\infty(D)$ we denote the set of all bounded holomorphic functions on D.

or

$$\langle (p/2)\alpha + \mathbf{1}, v\rangle \le (-\delta/2)\|v\|, \quad v \in T.$$

It follows from the description of T ($(\log D)\setminus T$ is bounded) that

$$\int\limits_{\log D} e^{2\langle (p/2)\alpha+\mathbf{1},x\rangle}\,dx < \infty \quad \text{if and only if} \quad \int\limits_{T} e^{2\langle (p/2)\alpha+\mathbf{1},x\rangle}\,dx < \infty.$$

We estimate the last expression:

$$\int\limits_{T} e^{2\langle (p/2)\alpha+\mathbf{1},x\rangle}\,dx \le \int\limits_{T} e^{-\delta\|x\|}\,dx \le \int\limits_{\mathbb{R}^n} e^{-\delta\|x\|}\,dx < \infty,$$

which finishes the proof of the first part of the lemma.

Similar estimates lead us in the second case to the inequality

$$\langle \alpha, v\rangle \le (-\delta/2)\|v\|, \quad v \in T,$$

or $\langle \alpha, v\rangle \le 0$ on T and consequently $\langle \alpha, v\rangle \le M < \infty$ on $\log D$, from which we get boundedness of z^α on D.

The last claim of the lemma is a straightforward consequence of the definition of $\mathfrak{C}(D)$. ∎

LEMMA 2.2.2. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain containing no affine lines. Then there are linearly independent $A^1,\dots,A^n \in \mathbb{Z}^n$ and $C \in \mathbb{R}^n$ such that*

$$\Omega \subset \{x \in \mathbb{R}^n : \langle x, A^j\rangle < C_j,\ j = 1,\dots,n\} =: \bigcap_{j=1}^{n} H(A^j, C_j) =: H(A, C).$$

PROOF. Taking any supporting hyperplane H of Ω and any point v lying on the other side of H than Ω (we may assume that $v = 0$) we may define the following domain:

$$\widetilde{\Omega} := \{tw : w \in \Omega,\ t > 0\}.$$

Then $\widetilde{\Omega}$ is the smallest open convex cone (with vertex at 0) containing Ω. It is easy to verify that $\widetilde{\Omega}$ contains no straight line. Therefore, to finish the proof of the lemma it is sufficient to prove it for cones. This, however, follows from Lemma 6 in [Jar-Pfl 85] ([24]). ∎

LEMMA 2.2.3. *Let C be a closed convex cone containing no affine lines. Then there are vectors $v_1,\dots,v_n \in \mathbb{R}^n$ such that*

$$C \setminus \{0\} \subset \Big\{ \sum_{j=1}^{n} t_j v_j : t_j > 0 \Big\}.$$

PROOF. It follows from Lemma 6 in [Jar-Pfl 85] (see the proof of Lemma 2.2.2) that there are linearly independent vectors $w_1,\dots,w_n \in \mathbb{R}^n$ such that

$$C \subset \bigcap_{j=1}^{n} \{x \in \mathbb{R}^n : \langle x, w_j\rangle \le 0\}.$$

Put $w := w_1 + \dots + w_n$. Then $C \subset \{x \in \mathbb{R}^n : \langle x, w\rangle \le 0\}$. Moreover, if $x \in C$ and $\langle x, w\rangle = 0$ then $\langle x, w_j\rangle = 0$, $j = 1,\dots,n$, so $x = 0$. Therefore, $C\setminus\{0\} \subset \{x \in \mathbb{R}^n : \langle x, w\rangle < 0\}$.

([24]) *Lemma* (see [Jar-Pfl 85]). Let C be an open cone in $\mathbb{R}^n$ containing no affine line. Then there is a nonempty open set $U \subset \mathbb{R}^n$ such that for any $v \in U$, C is contained in $\{x \in \mathbb{R}^n : \langle x, v\rangle < 0\}$.

Note that $C \cap \{x \in \mathbb{R}^n : \langle x, w\rangle = -1\}$ is bounded. Indeed, suppose this does not hold. Then there is a sequence $\{x^\nu\} \subset C$ such that $\|x^\nu\| \to \infty$ and $\langle x^\nu, w\rangle = -1$. Choosing, if necessary, a subsequence we have $x^\nu/\|x^\nu\| \to x^0$. Clearly, $x^0 \in C$ and $\langle x^0, w\rangle = 0$. But $\|x^0\| = 1$, a contradiction.

There are linearly independent vectors $v_1, \dots, v_n \in \{x \in \mathbb{R}^n : \langle x, w\rangle = -1\}$ such that

$$C \cap \{x \in \mathbb{R}^n : \langle x, w\rangle = -1\} \subset \{t_1 v_1 + \ldots + t_n v_n : t_1, \dots, t_n > 0,\ t_1 + \ldots + t_n = 1\},$$

which finishes the proof. ■

LEMMA 2.2.4. *Let* $\{x^\nu\}_{\nu=1}^\infty \subset \bar{\Omega}$ *(*Ω *is a convex domain in* $\mathbb{R}^n$*),* $\|x^\nu\| \to \infty$, $x^\nu/\|x^\nu\| \to x^0$ *as* $\nu \to \infty$, *where* $\|\cdot\|$ *is some norm on* $\mathbb{R}^n$. *Then* $x^0 \in \mathfrak{C}(\Omega)$.

PROOF. This easily follows from the properties of $\mathfrak{C}(\Omega)$. ■

LEMMA 2.2.5. *Let* Ω *be a convex domain and let* $\alpha \in \mathbb{R}^n$ *be such that*

$$\langle \alpha, v\rangle < 0, \quad v \in \mathfrak{C}(\Omega),\ v \neq 0.$$

Then for any $M \in \mathbb{R}$ *the (convex) set* $\{t \in \Omega : \langle \alpha, t\rangle \geq M\}$ *is bounded.*

PROOF. Suppose that there is a sequence $\{t^\nu\}_{\nu=1}^\infty \subset \Omega$ such that $\|t^\nu\| \to \infty$ as ν tends to infinity and $\langle \alpha, t^\nu\rangle \geq M$, $\nu = 1, 2, \dots$ Therefore,

$$\langle \alpha, t^\nu/\|t^\nu\|\rangle \geq M/\|t^\nu\|. \tag{2.2.1}$$

Choosing, if necessary, a subsequence we may assume that $t^\nu/\|t^\nu\| \to t^0$. We have $t^0 \in \mathfrak{C}(\Omega)$ (use Lemma 2.2.4), and clearly, $\|t^0\| = 1$. Letting $\nu \to \infty$ in (2.2.1) we get $\langle \alpha, t^0\rangle \geq 0$, a contradiction. ■

LEMMA 2.2.6. *Let* Ω *be a convex domain in* $\mathbb{R}^n$, $y \in \partial\Omega$ *and let* $L, L_\nu : \mathbb{R}^n \to \mathbb{R}$, $\nu = 1, 2, \dots,$ *be linear functionals such that*

(2.2.6.) $\quad L(y) = b,\ \Omega \subset \{L < b\},\ \bar{\Omega} \cap \{L = b\}$ *is bounded and*

$$\nu L_\nu(x) - \nu L(x) \to 0, \quad x \in \mathbb{R}^n.$$

Then

$$\sup_{x \in \bar{\Omega}} (\nu L_\nu(x) - \nu b) \to 0 \quad \text{as } \nu \to \infty.$$

PROOF. Substituting $x := y$ we see that the lower limit of the sequence considered is at least 0. Note that $\nu L_\nu - \nu L$ tends locally uniformly to 0. Therefore, L_ν tends locally uniformly to L. Suppose that there are a sequence $\{x^\nu\}_{\nu=1}^\infty \subset \Omega$ and $\varepsilon > 0$ such that $\nu L_\nu(x^\nu) \geq \nu b + \varepsilon$ for ν large enough (we take a subsequence of $\{L_\nu\}_{\nu=1}^\infty$, if necessary). From the local uniform convergence we get $\|x^\nu\| \to \infty$. Then

$$L_\nu\left(\frac{x^\nu}{\|x^\nu\|}\right) \geq \frac{\nu b + \varepsilon}{\nu\|x^\nu\|} \tag{2.2.3}$$

for ν large enough. Consequently, taking a subsequence we get $x^\nu/\|x^\nu\| \to x^0 \in \mathfrak{C}(\Omega)$ (use additionally Lemma 2.2.4). From the uniform convergence of L^ν to L on the unit sphere and (2.2.3) we get

$$0 \leq \lim_{\nu\to\infty} L_\nu\left(\frac{x^\nu}{\|x^\nu\|}\right) = \lim_{\nu\to\infty} L\left(\frac{x^\nu}{\|x^\nu\|}\right) = L(x^0).$$

Consequently, $L(y+tx^0) = b+tL(x^0) \geq b$, $t \in \mathbb{R}_+$. But $y+\mathbb{R}_+x^0 \subset \overline{\Omega}$, so $L(y+tx^0) = b$ for any $t \in \mathbb{R}_+$. Since the set $y+\mathbb{R}_+x^0 \subset \overline{\Omega}$ is unbounded, this contradicts the assumption (2.2.2). ∎

Define $p_j(x) := x_j$, $x \in \mathbb{R}^n$, $j = 1, \ldots, n$.

LEMMA 2.2.7. *Let $\Omega \subset \mathbb{R}^n$ be an unbounded convex domain. Assume that*

$$\sup p_j(\Omega) < \infty, \quad j = 1, \ldots, n. \tag{2.2.4}$$

Then for any $a \in \Omega$ there are an open set U with $a \in U$ and $v \in \mathbb{R}^n_- \setminus \{0\}$ such that $U + \mathbb{R}_+ v \subset \Omega$.

PROOF. Condition (2.2.4) gives us $\mathfrak{C}(\Omega) \subset \mathbb{R}^n_-$. Unboundedness of Ω implies that $\mathfrak{C}(\Omega) \neq \{0\}$. Simple properties of convexity give us the existence of an open set U as desired. ∎

2.3. Algebraic mappings. For $\alpha \in \mathbb{Z}^n$, $z \in \mathbb{C}^n$ such that $z_j \neq 0$ if $\alpha_j < 0$ we define $z^\alpha := z_1^{\alpha_1} \ldots z_n^{\alpha_n}$. Consider matrices $A := (A_k^j)_{j=1,\ldots,m,k=1,\ldots,n} \in \mathbb{Z}^{m\times n}$ with rank $A = m$ ([25]) such that every row consists of elements which are relatively prime.

For $A = (A_k^j)_{j=1,\ldots,m,k=1,\ldots,n} \in \mathbb{Z}^{m\times n}$, we define

$$\Phi_A(z) := z^A := (z^{A^1}, \ldots, z^{A^m}),$$

where $z \in \mathbb{C}^n$ is such that z^{A^j} is well defined for any $j = 1, \ldots, m$ (A^j denotes the jth row of the matrix A). Then $\Phi_A \in \mathcal{O}(\mathbb{C}^n_*, \mathbb{C}^m_*)$ for all $A \in \mathbb{Z}^{m\times n}$. For $A \in \mathbb{Z}^{m\times n}$, $B \in \mathbb{Z}^{k\times m}$ the following property holds:

$$(z^A)^B = z^{BA}.$$

Consequently, $\Phi_B \circ \Phi_A = \Phi_{BA}$.

Let $A \in \mathbb{Z}^{n\times n}$ be invertible. Then obviously $A^{-1} \in \mathbb{Q}^{n\times n}$ (here A^{-1} denotes the inverse). Define

$$A^{\mathrm{inv}} := (|\det A|)A^{-1}.$$

From the definition we have $A^{\mathrm{inv}} \in \mathbb{Z}^{n\times n}$.

For a proper holomorphic mapping $F : D_1 \to D_2$ (D_1 and D_2 are domains in $\mathbb{C}^n$) denote by $\mu(F)$ the multiplicity of the mapping F (for definition see e.g. [Rud 80]).

In the theorem below we shall see how the algebraic properties of the matrix A correspond to the properness property of the mapping Φ_A.

THEOREM 2.3.1. *Let $A \in \mathbb{Z}^{n\times n}$. Then*

(i) *the mapping $\Phi_A : \mathbb{C}^n_* \to \mathbb{C}^n_*$ is proper iff* $\det A \neq 0$,

(ii) *if* $\det A \neq 0$ *then* $\mu(\Phi_A) = |\det A|$ (*in particular*, $\Phi_A : \mathbb{C}^n_* \to \mathbb{C}^n_*$ *is biholomorphic iff* $|\det A| = 1$).

PROOF. (i) The mapping

$$\log \Phi_A(x) := (\log |(\Phi_A(z))_1|, \ldots, \log |(\Phi_A(z))_n|),$$

([25]) In particular, $m \leq n$.

where z is any point such that $\log|z_j| = x_j$, $x = (x_1, \dots, x_n)$, is well defined and $\log \Phi_A = A$. The properness of Φ_A implies that A is surjective. This gives $\det A \neq 0$. Conversely, assume that $\det A \neq 0$. Note that

$$\Phi_{A^{\mathrm{inv}}} \circ \Phi_A = \Phi_{A^{\mathrm{inv}}A} = \Phi_{(|\det A|)I_n}.$$

The last mapping is proper, which implies that so are $\Phi_{A^{\mathrm{inv}}}$ and Φ_A.

(ii) By the Gauss elimination process there are $B_1, \dots, B_n, C \in \mathbb{Z}^{n\times n}$ such that

$$B_n \dots B_1 A = C,$$

where all B_j's are of the form $\widetilde{B}D$, where D is a matrix with at most n nonzero elements and $\widetilde{B}$ (as well as C) is a triangular matrix. We easily get $\mu(\Phi_{B_j}) = |\det B_j|$ and $\mu(\Phi_C) = |\det C|$. This together with elementary properties of the determinant and the multiplicity of proper mappings under compositions implies that $\mu(\Phi_A) = |\det A|$. ∎

Remark 2.3.2. From the proof of Theorem 2.3.1 we get a little more. Namely, if $\det A \neq 0$ then $\Phi_A : \mathbb{C}_*^n \to \mathbb{C}_*^n$ is a holomorphic covering. In particular, $\det \Phi_A'(z) \neq 0$ for all $z \in \mathbb{C}_*^n$.

We are interested mainly in matrices (and the corresponding mappings) $A \in \mathbb{Z}^{n\times n}$ such that one of the following conditions is satisfied:

(2.3.1) $$A^{-1} \in \mathbb{Z}^{n\times n} \text{ (which implies that } |\det A| = 1,\ A^{\mathrm{inv}} = \pm A^{-1}),$$

(2.3.2) $$\det A \neq 0.$$

Under the assumption (2.3.1) the mapping Φ_A is biholomorphic on $\mathbb{C}_*^n$ and the inverse mapping is given by the formula $\Phi_A^{-1}|_{\mathbb{C}_*^n} = \Phi_{A^{-1}}|_{\mathbb{C}_*^n}$.

2.4. Quasi-elementary Reinhardt domains. Following [Jar-Pfl 88] and [Jar-Pfl 93] we define for a Reinhardt domain D in $\mathbb{C}^n$ the following sets:

$$S := S(D) := \{\alpha \in \mathbb{Z}^n : z^\alpha \in H^\infty(D)\}, \quad B := B(D) := S \setminus (S+S),$$

Let again $A := (A_k^j)_{j=1,\dots,m,k=1,\dots,n} \in \mathbb{Z}^{m\times n}$ be such that $\operatorname{rank} A = m$ and every row consists of elements which are relatively prime. For a positive integer r consider the following condition:

(2.4.1) $$\text{for any } x \in \mathbb{Q}^m \quad (xA \in \mathbb{Z}^n \Rightarrow rx \in \mathbb{Z}^m).$$

We define

$$r(A) := \min\{r \in \mathbb{Z},\ r > 0 : (2.4.1) \text{ holds}\}.$$

With $A \in \mathbb{Z}^{m\times n}$ (or even more generally $A \in \mathbb{R}^{m\times n}$) with $\operatorname{rank} A = m$ and $C \in \mathbb{R}^m$ we associate the following *quasi-elementary Reinhardt domain*:

$$\begin{aligned} G := G(A,C) &:= G(A^1, C_1) \cap \dots \cap G(A^m, C_m) \\ &:= \bigcap_{j=1}^m \{z \in \mathbb{C}^n : (z_k \neq 0 \text{ if } A_k^j < 0) \text{ and } |z_1|^{A_1^j} \dots |z_n|^{A_n^j} < e^{C_j}\}. \end{aligned}$$

Note that $\log G$ is a convex cone with vertex C; moreover, if $n = m$ and $\det A \neq 0$ then $\log G = H(A,C)$ (compare Lemma 2.2.2).

Below we list two lemmas concerning the above defined sets, whose proofs go along the same lines as those for complete Reinhardt domains (see [Jar-Pfl 93], Lemma 2.7.1 and Lemma 2.7.6) with the only difference that instead of Taylor series we consider Laurent series.

LEMMA 2.4.1. *Let D be a Reinhardt domain. Let $f \in H^\infty(D)$. Then*

$$f(z) = \sum_{\alpha \in S(D)} a_\alpha z^\alpha, \quad z \in D.$$

LEMMA 2.4.2. *For $A \in \mathbb{R}^{m\times n}$ with* $\operatorname{rank} A = m$ *and $C \in \mathbb{R}^m$ we have*

$$S(G(A,C)) = \mathbb{Z}^n \cap (\mathbb{R}_+ A^1 + \ldots + \mathbb{R}_+ A^m).$$

Moreover, if $A \in \mathbb{Z}^{m\times n}$ then ([26])

$$S(G(A,C)) = \mathbb{Z}^n \cap (\mathbb{Q}_+ A^1 + \ldots + \mathbb{Q}_+ A^m),$$
$$B(G(A,C)) \subset \mathbb{Z}^n \cap ((\mathbb{Q}\cap[0,1))A^1 + \ldots + (\mathbb{Q}\cap[0,1))A^m) \cup \{A^1,\ldots,A^m\}.$$

In the remaining part of Section 2.4 we restrict ourselves to $A \in \mathbb{Z}^{m\times n}$.

In view of Lemma 2.4.2 we know that for $A \in \mathbb{Z}^{m\times n}$ (with $\operatorname{rank} A = m$) we have

$$B(G(A,C)) = \left\{\frac{p_1^j}{q_1^j}A^1 + \ldots + \frac{p_m^j}{q_m^j}A^m : j = 1,\ldots,N\right\}, \tag{2.4.2}$$

where $p_k^j, q_k^j \in \mathbb{Z}_+$ ($j = 1,\ldots,N$, $k = 1,\ldots,m$) and the pairs p_k^j, q_k^j are relatively prime (for fixed k and j).

Denote by $s(A)$ the least common multiple of the denominators $\{q_k^j\}_{1\le j\le N, 1\le k\le m}$. Note that

$$s(A) = 1 \quad \text{iff} \quad B(G(A,C)) = \{A^1,\ldots,A^m\}. \tag{2.4.3}$$

LEMMA 2.4.3. *For $A \in \mathbb{Z}^{m\times n}$,*

$$r(A) = s(A).$$

PROOF. Any element from $B(G(A,C))$ is of the form (see (2.4.2))

$$\left(\frac{p_1^j}{q_1^j},\ldots,\frac{p_m^j}{q_m^j}\right)A \in \mathbb{Z}^n.$$

From the definition of $r(A)$ all q_j^k's must divide $r(A)$. Hence $s(A)$ divides $r(A)$.

In view of Lemma 2.4.2 any $\beta \in S(G(A,C))$ equals tA for some $t \in \mathbb{Q}_+^m$. From the definitions of $s(A)$ and $B(G(A,C))$, and Lemma 2.4.2 we know that

$$s(A)t \in \mathbb{Z}^m \quad \text{for any } \beta = tA \in \mathbb{Z}^n,\ t \in \mathbb{Q}_+^m. \tag{2.4.4}$$

Take now any $x \in \mathbb{Q}^m$ with $xA \in \mathbb{Z}^n$. We have

$$xA = uA + vA,$$

([26]) Actually, from the considerations in [Jar-Pfl 93] we have the equality $S(G(A,C)) = \mathbb{Z}^n \cap (\mathbb{R}_+A^1 + \ldots + \mathbb{R}_+A^m)$, so any element from $S(G(A,C))$ is of the form tA, where $t \in (\mathbb{R}_+)^m$. We may assume that the matrix $\widetilde{A} := (A_k^j)_{j,k=1,\ldots,m}$ is invertible ($\operatorname{rank} A = m$). Since $t\widetilde{A} \in \mathbb{Z}^m$, we get $t \in (\widetilde{A})^{-1}(\mathbb{Z}^m) \subset \mathbb{Q}^m$, which gives the desired formula.

where $u_j = x_j - [x_j] \geq 0$, $v_j = [x_j] \in \mathbb{Z}$, $j = 1, \dots, m$ ($[x]$ denotes the largest integer smaller than or equal to x). Obviously, $uA \in \mathbb{Z}^n$, so in view of (2.4.4), $s(A)u \in \mathbb{Z}^m$, and then $s(A)x \in \mathbb{Z}^m$. This gives $s(A) \geq r(A)$, which completes the proof. ■

From now on we only consider the case $m = n$.

LEMMA 2.4.4. *Let $A \in \mathbb{Z}^{n\times n}$ with $\det A \neq 0$. Then $r(A)$ divides $\det A$ and $\det A$ divides $r(A)^n$.*

PROOF. Note that $(A^{-1})^j A \in \mathbb{Z}^n$, $j = 1, \dots, n$, so $r(A)(A^{-1})^j \in \mathbb{Z}^n$, $j = 1, \dots, n$. This implies that $r(A)A^{-1} \in \mathbb{Z}^{n\times n}$. Consequently,

$$r(A)^n = \det(r(A)A^{-1}) \det A.$$

Both factors on the right hand side are integers, which finishes the proof of the second property.

To prove the first property suppose that $r(A)$ does not divide $\det A$. In other words there is a prime number p occurring k times in the prime factorization of $r(A)$ and occurring l times in the prime factorization of $\det A$, $l < k$. Put

$$\mathbb{N} \ni r := p^l r(A)/p^k < r(A).$$

There is $y \in \mathbb{Z}^n$ such that $y = xA$, $rx \notin \mathbb{Z}^n$ and $r(A)x = p^{k-l}rx \in \mathbb{Z}^n$. So there is j such that $x_j = a/b$, a, b are relatively prime integers, p^{l+1} divides b. Note that in

$$\widetilde{r}x := r\frac{\det A}{p^l}x = \frac{r}{p^l}\det A(yA^{-1})$$

the last vector is from $\mathbb{Z}^n$ but p^{l+1} does not divide $\widetilde{r}$, so $\widetilde{r}x_j \notin \mathbb{Z}$, a contradiction. ■

As a conclusion from Lemma 2.4.4 we get

REMARK 2.4.5. (a) $|\det A| = 1$ iff $r(A) = 1$.

(b) If $\det A = p_1 \dots p_k$, where all p_j's are pairwise different primes then $r(A) = |\det A|$.

EXAMPLE 2.4.6. In general, we do not have the equality $r(A) = |\det A|$: if

$$A := \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix},$$

then $|\det A| = 4$ whereas $r(A) = 2$.

COROLLARY 2.4.7. *For $A \in \mathbb{Z}^{n\times n}$ (with $\operatorname{rank} A = n$) we have $B(G(A, C)) = \{A^1, \dots, A^n\}$ iff $|\det A| = 1$.*

PROOF. Use (2.4.3), Lemma 2.4.3 and Remark 2.4.5. ■

Let us finish this section with another estimate of the number of elements of $B(G(A, C))$.

COROLLARY 2.4.8. *Let $A \in \mathbb{Z}^{n\times n}$ with $\det A \neq 0$. Then*

$$n \leq \#B(G(A, C)) \leq |\det A| - 1 + n.$$

PROOF. It is easy to see that $\mu(\Phi_A) = \mu(\Phi_{A^T}) = k$ if and only if there are k different points $\lambda^1, \dots, \lambda^k \in (\partial E)^n$ such that $\Phi_{A^T}(\lambda^j) = (1, \dots, 1)$, $j = 1, \dots, k$ (see Theorem 2.3.1). Therefore, one may easily verify that there are exactly $k-1$ vectors $t^1, \dots, t^{k-1} \in$

$[0,1)^n \setminus \{(0,\dots,0)\}$ such that $t^j A \in \mathbb{Z}^n$. This together with Lemma 2.4.2 finishes the proof. ∎

Following the ideas of the proof of Corollary 2.4.8 we can obtain the essential part of Corollary 2.4.7 without the use of results of Section 2.4. Since the only result from the present section that we need in the next section is Corollary 2.4.7, this means that the proofs of forthcoming results (especially, Theorem 2.5.1) may be a little simplified. Below we formulate and prove that result.

COROLLARY 2.4.7′. *Let $A \in \mathbb{Z}^{n\times n}$, $\det A \neq 0$. Then the existence of $t \in [0,1)^n \setminus \{(0,\dots,0)\}$ such that $tA \in \mathbb{Z}^n$ is equivalent to $|\det A| > 1$.*

PROOF. The condition $|\det A| = |\det A^T| > 1$ is equivalent to the existence of $\lambda \in (\partial E)^n \setminus \{(1,\dots,1)\}$ such that $\Phi_{A^T}(\lambda) = (1,\dots,1)$ (use Theorem 2.3.1). But the last is equivalent to the existence of $t \in [0,1)^n \setminus \{(0,\dots,0)\}$ such that $tA \in \mathbb{Z}^n$. ∎

2.5. Hyperbolicity of pseudoconvex Reinhardt domains. Our aim in this section is the following characterization of pseudoconvex Reinhardt domains:

THEOREM 2.5.1. *Assume that D is a pseudoconvex Reinhardt domain in $\mathbb{C}^n$. Then the following conditions are equivalent*:

(i) *D is c-hyperbolic*;

(ii) *D is $\widetilde{k}$-hyperbolic*;

(iii) *D is Brody hyperbolic*;

(iv) (a) *$\log D$ contains no affine lines and*

(b) *$D \cap V_j$ is either empty or c-hyperbolic* (*viewed as a domain in* $\mathbb{C}^{n-1}$);

(v) *there are matrices $A \in \mathbb{Z}^{n\times n}$ with $|\det A| = 1$ and $C \in \mathbb{R}^n$ such that*:

(a) *$D \subset G(A,C)$*;

(b) *$D \cap V_j$ is either empty or c-hyperbolic* (*thought as a domain in* $\mathbb{C}^{n-1}$);

(vi) *D is algebraically biholomorphic to a bounded Reinhardt domain* ([27]).

For $n = 1$ the condition (iv)(b) (and (v)(b)) is understood to be always fulfilled.

In view of Theorem 2.5.1 all notions of hyperbolicity considered coincide in the class of pseudoconvex Reinhardt domains, so the notion *hyperbolic* without any prefix is well defined in this class of domains.

LEMMA 2.5.2. *Let $A \in \mathbb{Z}^{n\times n}$ with $\det A \neq 0$ and $C \in \mathbb{R}^n$. Then there are $B \in \mathbb{Z}^{n\times n}$, $|\det B| = 1$, and $\widetilde{C} \in \mathbb{R}^n$ such that*

$$G(A,C) \subset G(B,\widetilde{C}).$$

PROOF. Using induction it is sufficient to prove that if $|\det A| > 1$ then there are $\widetilde{A} \in \mathbb{Z}^{n\times n}$ and $\widetilde{C} \in \mathbb{R}^n$ such that $0 < |\det \widetilde{A}| < |\det A|$ and $G(A,C) \subset G(\widetilde{A},\widetilde{C})$.

([27]) In other words there is $A \in \mathbb{Z}^{n\times n}$, $|\det A| = 1$, such that $\Phi_A(D)$ is bounded and $(\Phi_A)_{|D}$ is a biholomorphism onto the image.

In view of Corollary 2.4.7 there is $\beta \in B(G(A,C))$, $\beta \neq A^l$, $l = 1,\dots,n$. In view of Lemma 2.4.2, $\beta = t_1A^1 + \dots + t_nA^n$, $t_j \in [0,1)$ and for some l (say $l = 1$), $t_1 \neq 0$. Put

$$\widetilde{A} := \begin{bmatrix} \beta \\ A^2 \\ \cdot \\ A^n \end{bmatrix}, \quad \widetilde{C}_1 := t_1C_1 + \dots + t_nC_n, \quad \widetilde{C}_j := C_j, \quad j = 2,\dots,n.$$

It follows from the definition that $G(A,C) \subset G(\widetilde{A},\widetilde{C})$. Note that

$$0 < |\det \widetilde{A}| = t_1|\det A| < |\det A|,$$

which finishes the proof. ∎

Remark 2.5.3. From the proof of Lemma 2.5.2 we get the following result. Let $A^1 \in (\mathbb{Z}^n)_*$ consist of relatively prime numbers. Then there are $A^2,\dots,A^n \in \mathbb{Z}^n$ such that the matrix A formed by the rows A^j satisfies $|\det A| = 1$. In fact, assuming that A is one of possible complements of A^1 with the smallest positive absolute value of the determinant we put $C = (0,\dots,0)$ and (under the assumption that $|\det A| > 1$) we may apply the reasoning from the proof of Lemma 2.5.2. We only have to show that we can define a new matrix $\widetilde{A}$ (from the reasoning above) so that the row A^1 does not change; the condition that A^1 consists of relatively prime integers implies that there are $t_1,\dots,t_n \geq 0$ and $t_j > 0$ for some $j > 1$ such that $t_1A^1 + \dots + t_nA^n \in \mathbb{Z}^n$, which gives the desired result.

Proof of Theorem 2.5.1. The proof is by induction. The case $n = 1$ is trivial. Let $n \geq 2$. The implications (i)⇒(ii) and (ii)⇒(iii) are trivial.

(iii)⇒(iv). Note that (iv)(b) follows from applying the theorem in dimension $n-1$. From the Brody hyperbolicity of D we conclude that $\log D$ contains no affine line (otherwise, there is a mapping $\mathbb{C} \ni \lambda \mapsto (\exp(c_1 + \alpha_1\lambda),\dots,\exp(c_n + \alpha_n\lambda)) \in D$, where $\alpha_j, c_j \in \mathbb{R}$, $j = 1,\dots,n$, $(\alpha_1,\dots,\alpha_n) \neq (0,\dots,0)$).

(iv)⇒(v). Use Lemmas 2.2.2 and 2.5.2.

(v)⇒(vi). It is sufficient to prove the following: there is $A \in \mathbb{Z}^{n\times n}$ such that

(2.5.1) Φ_A is well defined on D, $\Phi_A(D)$ is bounded, $(\Phi_A)_{|D}$ is biholomorphic onto the image.

If $D \subset \mathbb{C}^n_*$ then Φ_A (where A is as in (v)(a)) maps biholomorphically D onto a bounded Reinhardt domain.

Now consider the other case. We may assume that $D \cap V_n \neq \emptyset$. We claim that it is sufficient to verify the assertion for D satisfying

$$V_n \cap D \neq \emptyset, \quad \{z_j \in \mathbb{C} : z \in D\} \text{ is bounded}, \quad j = 1,\dots,n-1. \tag{2.5.2}$$

To make the desired reduction put $\widetilde{D} := D \cap V_n$. (v)(b) implies that $\widetilde{D}$ is c-hyperbolic (in $\mathbb{C}^{n-1}$), so applying the inductive assumption we find $\widetilde{A} \in \mathbb{Z}^{(n-1)\times(n-1)}$ such that (2.5.1) is satisfied with A, D replaced with $\widetilde{A}$, $\widetilde{D}$. Now define

$$B := \begin{bmatrix} \widetilde{A} & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{Z}^{n\times n}.$$

The mapping Φ_B maps biholomorphically D onto a domain satisfying (2.5.2).

So assume that D satisfies (2.5.2). We may assume that

$$V_j \cap D \neq \emptyset, \quad j = 1, \dots, k, \qquad V_j \cap D = \emptyset, \quad j = k+1, \dots, n-1, \tag{2.5.3}$$

where $0 \le k \le n-1$ (remember that $V_n \cap D \neq \emptyset$). Put $\widetilde{D} := V_1 \cap \dots \cap V_k \cap D$. $\widetilde{D}$ is not empty. There is $\alpha \in S(\widetilde{D})$ such that $\alpha = (0, \dots, 0, \alpha_{k+1}, \dots, \alpha_n)$, $\alpha_n \neq 0$ (in the case $k = 0$ this follows from the assumption (v)(a), if $k > 0$ then use the inductive assumption to see that $\widetilde{D}$ is c-hyperbolic and identify an element from $\mathbb{Z}^{n-k}$ with one from $\{0\}^k \times \mathbb{Z}^{n-k}$). But $\widetilde{D} \cap V_n \neq \emptyset$, so $\alpha_n > 0$. Note that, in view of (2.5.2), $e_j \in S(\widetilde{D})$, $j = k+1, \dots, n-1$, so

$$\widetilde{\alpha} := \frac{1}{\alpha_n}\alpha + \sum_{j=k+1}^{n-1} \left(\left[\frac{\alpha_j}{\alpha_n} \right] + 1 - \frac{\alpha_j}{\alpha_n} \right) e_j \in S(\widetilde{D}) \subset S(D).$$

Now put

$$A := \begin{bmatrix} & & & \mathbb{I}_{n-1} & & & 0 \\ 0 & \dots & 0 & \widetilde{\alpha}_{k+1} & \dots & \widetilde{\alpha}_{n-1} & 1 \end{bmatrix}.$$

The matrix A has all the required properties (remember (2.5.3)). ∎

2.6. Carathéodory and Kobayashi completeness of pseudoconvex Reinhardt domains. The first result below (Proposition 2.6.1) is an attempt to generalize the corresponding result for bounded pseudoconvex complete Reinhardt domains from [Pfl 84]. Although in many cases the Carathéodory distance for bounded pseudoconvex Reinhardt domains blows up to infinity as one of the points goes to the boundary (not lying on the axis), this is not always the case; a counterexample is given in Example 2.6.4.

Before we formulate the result let us make some preparations. For a bounded pseudoconvex Reinhardt domain D and a point $z^0 \in \partial D \cap \mathbb{C}_*^n$ we may find a supporting hyperplane for the convex domain $\log D$ at $|\log z^0|$. In other words there are $\xi \in \mathbb{R}^n \setminus \{0\}$ and $b > 0$ such that

$$\log D \subset \{x \in \mathbb{R}^n : \langle x, \xi \rangle < \log b\} \quad \text{and} \quad \langle \log |z^0|, \xi \rangle = \log b. \tag{2.6.1}$$

Now let $s = s(\xi, D)$ be the largest number of $\mathbb{Q}$-linearly independent elements in $\{\xi_1, \dots, \xi_n\}$. Clearly, $1 \le s \le n$.

Proposition 2.6.1. *Let D be a bounded pseudoconvex Reinhardt domain, $z^0 \in \partial D \cap \mathbb{C}_*^n$. Assume that one of the following conditions is satisfied (the notations are as above):*

(i) $s = 1$ *or* $s = n$,
(ii) $\overline{\log D} \cap \{x \in \mathbb{R}^n : \langle \xi, x \rangle = \log b\}$ *is bounded.*

Then for any $w \in D$ we have $c_D(z, w) \to \infty$ as $z \to z^0$.

Proof. Let us consider the second case. In view of the Dirichlet pigeon-hole theorem (see e.g. [Har-Wri 78]) for any $\varepsilon > 0$ there are $q_1, \dots, q_n, p \in \mathbb{Z}$, $p > 0$ such that

$$q_j - p\xi_j = \varepsilon_j \in (-\varepsilon, \varepsilon), \quad j = 1, \dots, n. \tag{2.6.2}$$

Moreover, as ε tends to 0 then p may be chosen to tend to infinity. Define $L(x) := \langle \xi, x \rangle$, $L_p(x) := (1/p)\langle (q_1, \dots, q_n), x \rangle$, $x \in \mathbb{R}^n$. In view of Lemma 2.2.6 we see that for any $\delta > 0$

we have

$$\sup_{x\in\overline{\log D}} pL_p(x) \le p\log b + \delta \tag{2.6.3}$$

for those (large enough) p whose existence is guaranteed by (2.6.2) for sufficiently small ε.

Define $f(z) := z_1^{q_1}\dots z_n^{q_n}/(b^p e^\delta)$, $z\in\mathbb{C}_*^n$ (q_j and p are as in (2.6.2) with some small ε such that (2.6.3) is satisfied). Then $|f(z)|<1$, $z\in D\cap\mathbb{C}_*^n$. The same inequality extends onto D. For any $z\in\mathbb{C}_*^n$ we have

$$|f(z)| = \left(\frac{|z_1|^{\xi_1}\dots|z_n|^{\xi_n}}{b}\right)^p \frac{|z_1|^{\varepsilon_1}\dots|z_n|^{\varepsilon_n}}{e^\delta},\quad z\in D\cap\mathbb{C}_*^n.$$

Notice that if $w\in D\cap\mathbb{C}_*^n$ is fixed then as δ tends to 0 so does $|f(w)|$ (because of (2.6.1) and the convergence of p to infinity and ε to 0). On the other hand for z close to z^0 we may make $|f(z)|$ arbitrarily close to 1 (we choose δ even closer to 0 and then we choose z close to z^0). This gives the desired convergence for $w\in D\cap\mathbb{C}_*^n$. The triangle inequality finishes the proof for all $w\in D$.

Now we consider the first case. If $s=1$ then we may assume that $(\xi_1,\dots,\xi_n)\in(\mathbb{Z}^n)_*$. Taking $f(z):=z_1^{\xi_1}\dots z_n^{\xi_n}/b$, $z\in\mathbb{C}_*^n$, we have $f(D\cap\mathbb{C}_*^n)\subset E$ and $|f(z)|\to 1$ as $z\to z^0$. Extending f to D and using the contractivity of c we finish the proof in this case.

In case $s=n$ we may assume that the set $\{\xi_1,\dots,\xi_n\}$ is $\mathbb{Z}$-linearly independent (multiply all ξ_j with some positive t) and, additionally, $D\subset E^n$.

In view of the multidimensional Kronecker Theorem (see e.g. [Har-Wri 78]) for any $\varepsilon>0$ there are $p,q_j\in\mathbb{Z}$, $j=1,\dots,n$, $p>0$, such that

$$0 < -p\xi_j + q_j < \varepsilon,\quad j=1,\dots,n. \tag{2.6.4}$$

Moreover, p chosen above tends to infinity as ε tends to 0.

For $\varepsilon>0$ we define

$$f(z) := \frac{z_1^{q_1}\dots z_n^{q_n}}{b^p},\quad z\in D\cap\mathbb{C}_*^n,$$

where p,q_j, $j=1,\dots,n$, are chosen as in (2.6.4). Note that in view of (2.6.1), (2.6.4) and the fact that $z_j\in E$ we get

$$|f(z)| \le \frac{(\prod_{j=1}^n |z_j|^{\xi_j})^p}{b^p} < 1,\quad z\in D\cap\mathbb{C}_*^n. \tag{2.6.5}$$

Obviously, f extends holomorphically onto D and the estimate in (2.6.5) remains true on D. As earlier, taking ε small enough (and consequently p large enough) we may make $|f(w)|$ arbitrarily small for fixed $w\in D\cap\mathbb{C}_*^n$ (see (2.6.1) and (2.6.5)). Therefore, to finish the proof it is sufficient to show that $|f(z)|$ may be chosen to be arbitrarily close to 1 when $z\in D$ is close to z^0.

In view of (2.6.4) we have

$$|f(z)| = \frac{\prod_{j=1}^n |z_j|^{q_j}}{b^p} \ge \frac{(\prod_{j=1}^n |z_j|)^\varepsilon(\prod_{j=1}^n |z_j|^{\xi_j})^p}{b^p},\quad z\in D\cap\mathbb{C}_*^n.$$

Consider now z close to z^0. Taking $\varepsilon>0$ sufficiently small we may make $(\prod_{j=1}^s |z_j|)^\varepsilon$ arbitrarily close to 1. Moreover, taking z even closer to z^0 we may make $(\prod_{j=1}^n |z_j|^{\xi_j}/b)^p$ arbitrarily close to 1 (see (2.6.1)). This finishes the proof. ∎

Corollary 2.6.2. *If D is a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^2$ then for any $z^0 \in \partial D \cap \mathbb{C}_*^2$ and for any $w \in D$ we have $c_D(w,z) \to \infty$ as $z \to z^0$.*

Proof. Note that $s = 1$ or $s = 2 = n$ and then use Proposition 2.6.1. ∎

One may ask whether the convergence as given by Proposition 2.6.1 holds for any $z^0 \in \partial D \cap \mathbb{C}_*^n$ (as it holds for $n = 2$). Below we shall see that there are examples of bounded pseudoconvex Reinhardt domains for which this convergence fails to hold.

Lemma 2.6.3. *Let $D := G(A, \mathbf{0})$, where $A \in \mathbb{R}_+^{n\times n}$ with $\operatorname{rank} A = n$ is such that $(\mathbb{R}A^1) \cap \mathbb{Z}^n = \{0\}$ and there are $\lambda_k \in \mathbb{Q}$, $k = 1, \ldots, n$, such that*

$$\sum_{k=1}^n \lambda_k A_k^1 = 0, \quad \sum_{k=1}^n \lambda_k A_k^j > 0, \quad j = 2, \ldots, n,$$

Then for any $w \in \mathbb{C}_*^n$ such that $|w^{A^1}| = 1$, $|w^{A^j}| < 1$, $j = 2, \ldots, n$, we have

$$\limsup_{z\to w} c_D(a,z) < \infty \quad \textit{for any } a \in D.$$

Proof. We may assume that $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}$. By the triangle inequality it is sufficient to prove the lemma for $a = 0$. We know that (see [Jar-Pfl 93])

$$c_D^*(0,z) = \sup\{|z^\alpha|/\|z^\alpha\|_D : z^\alpha \in H^\infty(D),\ \alpha \neq 0\}, \quad z \in D.$$

Since $\|z^\alpha\|_D \geq 1$, we have

$$\log c_D^*(0,z) \leq \sup\{\langle \log|z|, \alpha\rangle : z^\alpha \in H^\infty(D),\ \alpha \neq 0\}. \tag{2.6.6}$$

We know that (see Lemma 2.4.2)

$$\{\alpha \in \mathbb{N}^n : z^\alpha \in H^\infty(D)\} = \Big\{\sum_{j=1}^n t_j A^j : t_j \geq 0\Big\} \cap \mathbb{N}^n.$$

Suppose that the lemma does not hold. Then in view of (2.6.6) there is $x \in \mathbb{R}^n$ such that $\langle A^1, x\rangle = 0$, $\langle A^j, x\rangle < 0$, $j = 2, \ldots, n$, and there are $t_j^\nu \geq 0$, $t_1^\nu + \ldots + t_n^\nu > 0$, $x^\nu \in \mathbb{R}^n$ such that $\langle A^j, x^\nu\rangle < 0$, $x^\nu \to x$, $j = 1, \ldots, n$, $\nu = 1, 2, \ldots$, $\sum_{j=1}^n t_j^\nu A^j \in \mathbb{Z}^n$ and

$$\sum_{j=1}^n t_j^\nu \langle A^j, x^\nu\rangle = \Big\langle \sum_{j=1}^n t_j^\nu A^j, x^\nu \Big\rangle \to 0 \quad \text{as } \nu \to \infty.$$

As $\langle A^j, x\rangle < 0$, $j = 2, \ldots, n$, $\langle A^1, x^\nu\rangle < 0$ we have $t_j^\nu \to 0$, $j = 2, \ldots, n$. Since $\sum_{j=1}^n t_j^\nu A^j \in \mathbb{Z}^n$, we know that

$$\sum_{j=1}^n t_j^\nu \Big(\sum_{k=1}^n \lambda_k A_k^j\Big) = \sum_{k=1}^n \lambda_k \Big(\sum_{j=1}^n t_j^\nu A_k^j\Big) \in \mathbb{Z}.$$

Therefore, the convergence $t_j^\nu \to 0$, $j = 2, \ldots, n$, and the assumptions of the lemma show that for ν large enough $t_j^\nu = 0$, $j = 2, \ldots, n$, which implies that $t_1^\nu > 0$. Consequently, for ν large enough, $t_1^\nu A^1 \in \mathbb{Z}^n$ with $t_1^\nu > 0$, a contradiction. ∎

EXAMPLE 2.6.4. Matrices A satisfying the assumption of Lemma 2.6.3 do exist. For instance

$$A := \begin{bmatrix} 1 & \alpha & \alpha+1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix},$$

where α is a positive irrational number.

The intersection of a domain as in Lemma 2.6.3 with $\mathbb{C}_*^n$ may be mapped with the help of an algebraic biholomorphism onto a bounded domain (see Theorem 2.5.1). This gives us an example of a bounded pseudoconvex Reinhardt domain (already in $\mathbb{C}_*^3$) such that $\limsup_{z\to w} c_D(a,z) < \infty$ for any $a \in D$ and for some $w \in \partial D \cap \mathbb{C}_*^n$.

To visualize an example consider A as above. Then the mapping

$$z \mapsto (z_1 z_2^{[\alpha]+1} z_3^{[\alpha]+3}, z_2 z_3^2, z_3)$$

maps biholomorphically the domain $G(A,\mathbf{0}) \cap \mathbb{C}_*^3$ into a bounded domain

$$\{z \in \mathbb{C}_*^3 : |z_2|, |z_3| < 1, |z_1||z_2|^{\alpha-[\alpha]-1}|z_3|^{[\alpha]-\alpha} < 1\}.$$

Let us start the study of completeness with respect to different distances. As already mentioned it was P. Pflug who proved that all bounded pseudoconvex complete Reinhardt domains are c-finitely compact (and consequently, both c- and k-complete). Next, S. Fu extended the result on k-completeness to the class of bounded pseudoconvex Reinhardt domains (see [Fu 94]). By Theorem 2.5.1 we may replace boundedness with hyperbolicity. Therefore, we have:

THEOREM 2.6.5. *Let D be a hyperbolic pseudoconvex Reinhardt domain. Then D is k-finitely compact* (*in particular*, *D is Kobayashi complete*).

On the other hand, for the Carathéodory completeness we prove:

THEOREM 2.6.6. *Let D be a hyperbolic pseudoconvex Reinhardt domain. Then the following conditions are equivalent*:

(i) *D is c-finitely compact*;
(ii) *D is c-complete*;
(iii) *D is bounded and for any $j \in \{1,\dots,n\}$*,

$$\text{if } \bar{D} \cap V_j \neq \emptyset \text{ then } D \cap V_j \neq \emptyset. \tag{2.6.7}$$

The geometric condition (2.6.7) ([28]) comes from [Fu 94], where (iii)$\Rightarrow$(i) is proved with methods from [Pfl 84]. The proof of (ii)$\Rightarrow$(iii) comes from [Zwo 98b].

Note that the notions of c-completeness and c-finite compactness coincide on domains in $\mathbb{C}$; it is not known whether the same remains true in higher dimensions (see e.g. [Jar-Pfl 93]).

In the proof of Theorem 2.6.6 we need the following characterization:

PROPOSITION 2.6.7. *Let D be a hyperbolic pseudoconvex Reinhardt domain. Then the following conditions are equivalent*:

([28]) This condition may be described as follows: if the closure of the domain intersects some axis then so does the domain itself.

(i) *D is algebraically equivalent to an unbounded domain*;

(ii) *D is algebraically equivalent to a bounded domain $\widetilde{D}$ such that*

$$\text{there is } j \in \{1,\dots,n\} \text{ with } \overline{\widetilde{D}} \cap V_j \neq \emptyset \text{ and } \widetilde{D} \cap V_j = \emptyset. \tag{2.6.8}$$

Proof. (ii)⇒(i). The mapping

$$\widetilde{D} \ni z \mapsto (z_1,\dots,z_{j-1},1/z_j,z_{j+1},\dots,z_n) \in \mathbb{C}^n$$

is well defined on $\widetilde{D}$ and maps biholomorphically $\widetilde{D}$ onto an unbounded domain.

(i)⇒(ii). We may assume that D is unbounded. Let Φ_A be a biholomorphism of a bounded domain $\widetilde{D}$ onto D (see Theorem 2.5.1(vi)). We show that $\widetilde{D}$ has the desired property. Since $\widetilde{D}$ is bounded it is sufficient to show that there is $j \in \{1,\dots,n\}$ such that

$$\overline{\widetilde{D}} \cap V_j \neq \emptyset \quad \text{and} \quad \widetilde{D} \cap V_j = \emptyset.$$

Suppose this does not hold. Then we may assume that for some k, $0 \le k \le n$:

$$\widetilde{D} \cap V_j \neq \emptyset, \quad j=1,\dots,k \quad \text{and} \quad \overline{\widetilde{D}} \cap V_j = \emptyset, \quad j=k+1,\dots,n.$$

The above conditions imply that

$$A_j^r \ge 0, \quad j=1,\dots,k,\ r=1,\dots,n,$$

and there is $M > 0$ such that for any $z \in \widetilde{D}$, $|z_j| \ge M$, $j = k+1,\dots,n$ (here we also need boundedness of $\widetilde{D}$).

This yields that (remember that $\widetilde{D}$ is bounded) $\|z^{A^r}\|_{\widetilde{D}} < \infty$, $r = 1,\dots,n$, which implies that D is bounded, a contradiction. ∎

Proof of Theorem 2.6.6. The implication (i)⇒(ii) is trivial. The implication (iii)⇒(i) comes from [Fu 94]. Therefore, we only need to prove (ii)⇒(iii).

Suppose that (iii) does not hold. Then, in view of Proposition 2.6.7, we may assume, using an algebraic biholomorphism if necessary, that D is a bounded domain such that for some $j \in \{1,\dots,n\}$, $\bar{D} \cap V_j \neq \emptyset$ and $D \cap V_j = \emptyset$.

We may assume that there are $1 \le k \le l \le n$ such that

$$\bar{D} \cap V_j \neq \emptyset \quad \text{and} \quad D \cap V_j = \emptyset, \quad j=1,\dots,k,$$
$$\bar{D} \cap V_j = \emptyset, \quad j=k+1,\dots,l, \quad D \cap V_j \neq \emptyset, \quad j=l+1,\dots,n.$$

We may reduce our considerations to the case $l = n$. In fact, put $\widetilde{D} := D \cap V_{l+1} \cap \dots \cap V_n$. Clearly, $\widetilde{D}$ is also c-complete. Then, after identification, $\widetilde{D} \subset \mathbb{C}^l$, $\widetilde{D} \cap V_j = \emptyset$, $j = 1,\dots,l$, and $\overline{\widetilde{D}} \cap V_j = \emptyset$, $j = k+1,\dots,l$. Moreover, using the description of pseudoconvex Reinhardt domains (see Proposition 2.1.1), one may easily verify that $\overline{\widetilde{D}} \cap V_j \neq \emptyset$, $j = 1,\dots,k$.

We assume that D is bounded and

$$D \subset \mathbb{C}_*^n, \quad \bar{D} \cap V_j \neq \emptyset, \quad j=1,\dots,k, \quad \bar{D} \cap V_j = \emptyset, \quad j=k+1,\dots,n, \tag{2.6.9}$$

where $1 \le k \le n$.

We may assume that $(1,\dots,1) \in D$. Applying Lemma 2.2.7 to $\Omega := \log D$ and $a := (0,\dots,0)$ we see that there is $v \in \mathbb{R}_-^n \setminus \{0\}$ and a neighborhood U of a such that

$$x + tv \in \log D \quad \text{for any } x \in U,\ t > 0.$$

In view of (2.6.9) we lose no generality assuming that $v = (v_1, \dots, v_l, 0, \dots, 0)$, where $v_j < 0$ and $1 \le l \le k$ (l fixed). Put $\alpha_j := -v_j$, $j = 1, \dots, l$. We may also assume that $\alpha_1 = 1$. Then

$$(e^{x_1}\exp(t), e^{x_2}\exp(t\alpha_2), \dots, e^{x_l}\exp(t\alpha_l), e^{x_{l+1}}, \dots, e^{x_n}) \in D \quad \text{for } t < 0,\ x \in U.$$

In particular,

$$(\exp(\lambda), \mu_2\exp(\lambda\alpha_2), \mu_3\exp(\lambda\alpha_3), \dots, \mu_l\exp(\lambda\alpha_l), 1, \dots, 1) \in D$$

for $\lambda \in H_0$ (where $H_R := \{\operatorname{Re}\lambda < R\} \subset \mathbb{C}$, $0 \le R \le \infty$), $\mu_j \in P := \{e^{-\varepsilon} < |\mu| < e^{\varepsilon}\} \subset \mathbb{C}$, $j = 2, \dots, l$, and $\varepsilon > 0$ is suitably small.

For $(\lambda, \mu_2, \dots, \mu_l) \in H_R \times P^{l-1}$ we define

$$\Phi_R(\lambda, \mu_2, \dots, \mu_l) := (\exp(\lambda), \mu_2\exp(\lambda\alpha_2), \mu_3\exp(\lambda\alpha_3), \dots, \mu_l\exp(\lambda\alpha_l)) \in \mathbb{C}^l.$$

Put $G_R := \Phi_R(H_R \times P^{l-1})$. We have $G_R \subset G_{R'}$ if $R < R'$ and $\bigcup_{R<\infty} G_R = G_\infty$.

Since G_R is a pseudoconvex Reinhardt domain lying in $\mathbb{C}^l_*$, we know from Lemma 2.1.3 that $\widetilde{k}_{G_R}$ is continuous (for $0 \le R \le \infty$).

Note that $G_0 \times \{1\}^{n-l} \subset D$ and $(0, \dots, 0, 1, \dots, 1) \notin D$; therefore, to complete the proof it is sufficient to find for a given sequence $\{a_\nu\}_{\nu=1}^\infty$ with $a_\nu > 0$ and $\sum_{\nu=1}^\infty a_\nu < \infty$ a sequence $\{z^\nu\}_{\nu=1}^\infty \subset G_0$ with $z^\nu \to 0$ and

$$\log c^*_{G_0}(z^\nu, z^{\nu+1}) \le g_{G_0}(z^\nu, z^{\nu+1}) \le \log a_\nu.$$

For $0 \le R < \infty$ put

$$\Psi_R : G_0 \ni z \mapsto (\exp(R)z_1, \exp(\alpha_2 R)z_2, \dots, \exp(\alpha_l R)z_l) \in G_R.$$

Note that Ψ_R is a biholomorphism.

Define $\varphi_R(\lambda) := \Phi_R(\lambda, 1, \dots, 1)$, $\lambda \in H_R$. Notice that

$$\widetilde{k}_{G_\infty}(\varphi_\infty(-1), \varphi_\infty(\lambda)) = 0, \quad \lambda \in \mathbb{C}.$$

The continuity of $\widetilde{k}_{G_\infty}$ implies

$$\widetilde{k}_{G_\infty}(\varphi_\infty(-1), z) = 0 \quad \text{for any } z \in \overline{\varphi_\infty(\mathbb{C})}$$

(the closure above is taken in G_∞). Now Dini's Lemma implies that for any ν there is $R_\nu > 0$ ($\{R_\nu\}_{\nu=1}^\infty$ may be assumed to be strictly increasing and tend to infinity) such that

$$\widetilde{k}^*_{G_{R_\nu}}(\varphi_0(-1), z) < a_\nu \quad \text{for any } \exp(-2) \le |z_1| \le \exp(-1),\ z \in \overline{\varphi_\infty(\mathbb{C})},$$

which implies

$$\widetilde{k}^*_{G_{R_\nu}}(\varphi_0(-1), \varphi_0(\lambda)) < a_\nu \quad \text{for any } \lambda \in \mathbb{C} \text{ with } -2 \le \operatorname{Re}\lambda \le -1.$$

Applying the biholomorphism $\Psi_{R_\nu}^{-1}$ we get

$$\widetilde{k}^*_{G_0}(\varphi_0(-1-R_\nu), \varphi_0(\lambda)) < a_\nu \quad \text{for any } \lambda \in \mathbb{C} \text{ with } -2-R_\nu \le \operatorname{Re}\lambda \le -1-R_\nu. \tag{2.6.10}$$

Now define $u(\lambda) := g_{G_0}(\varphi_0(-1-R_\nu), \varphi_0(\lambda))$, $\lambda \in H_0$. Clearly, $u \in \mathrm{SH}(H_0)$, $u < 0$. In view of (2.6.10) we have $u(\lambda) < \log a_\nu$ for $-2 - R_\nu \le \operatorname{Re}\lambda \le -1 - R_\nu$, from which, in view of the extended maximum principle (see e.g. [Ran 95]), we conclude that

$$g_{G_0}(\varphi_0(-1-R_\nu), \varphi_0(\lambda)) = u(\lambda) \le \log a_\nu \quad \text{for any } \lambda \in \mathbb{C} \text{ with } \operatorname{Re}\lambda < -1 - R_\nu.$$

To finish the proof it is sufficient to define $z^\nu := \varphi_0(-1 - R_\nu)$. ∎

Remark 2.6.8. Note that if there is $\alpha \in \mathbb{R} \cdot \mathbb{Q}_+^l$ (α is from the proof of Theorem 2.6.6), e.g. when $l = 1$, then we may assume that $\alpha \in \mathbb{Z}_+^l$ and the proof of Theorem 2.6.6 is much simpler. Actually, define $\varphi : E_* \to D$ as follows:

$$\varphi(\lambda) := (\lambda^{\alpha_1}, \dots, \lambda^{\alpha_l}, 1 \dots, 1).$$

Take any c_{E_*}-Cauchy sequence $\{\lambda_\nu\}_{\nu=1}^\infty$ such that $\lambda_\nu \to 0$. Then

$$g_{E_*}(\lambda_\nu, \lambda_\mu) \geq g_D(\varphi(\lambda_\nu), \varphi(\lambda_\mu)) \geq \log c_D^*(\varphi(\lambda_\nu), \varphi(\lambda_\mu)),$$

from which we conclude the desired result.

The example $\{z \in E^2 : \frac{1}{2}|z_1|^\alpha < |z_2| < |z_1|^\alpha\}$, where $\alpha > 0$ is irrational, shows that we have to consider the case when the α cannot be assumed to be chosen from $\mathbb{Z}^l$, so it seems that the proof cannot be essentially simplified.

Remark 2.6.9. From the proof of Theorem 2.6.6 we can conclude that conditions (i)–(iii) are equivalent to

$$\text{for any } z_0 \in D,\ g_D(z_0, z) \to 0, \text{ as } z \to \partial D \cup \{\infty\}$$

(writing $z \to \infty$ we mean $\|z\| \to \infty$).

As simple conclusions from Theorems 2.6.5 and 2.6.6 we get a characterization of hyperconvex and taut Reinhardt domains. We say that a domain $D \subset \mathbb{C}^n$ is *hyperconvex* if there exists a continuous, negative, plurisubharmonic function u on D such that $\{u < -\varepsilon\} \subset\subset D$ for any $\varepsilon > 0$ (note that in contrast to other authors we allow hyperconvex domains to be unbounded—see [Ste 75]).

First, recall that any taut domain is $\widetilde{k}$-hyperbolic and any k-complete domain is taut; therefore, in view of Theorem 2.6.5 we get:

Corollary 2.6.10. *For a pseudoconvex Reinhardt domain the following conditions are equivalent*:

(i) D *is algebraically equivalent to a bounded domain*;
(ii) D *is taut.*

Corollary 2.6.11. *Let D be a pseudoconvex Reinhardt domain. Then the following conditions are equivalent*:

(i) D *is bounded and for any* $j \in \{1, \dots, n\}$,

$$\textit{if } \bar{D} \cap V_j \neq \emptyset \textit{ then } D \cap V_j \neq \emptyset;$$

(ii) D *is hyperconvex.*

Proof. (ii)$\Rightarrow$(i). Let $D = \{u < 0\}$, where u is from the definition of a hyperconvex domain. We first prove that D is hyperbolic. If it were not, then in view of Theorem 2.5.1(iii) there would exist a nonconstant holomorphic mapping $\varphi : \mathbb{C} \to D$. Then $u_{|\varphi(\mathbb{C})} \equiv C$ for some $C < 0$ but $\varphi(\mathbb{C})$ is not bounded, which contradicts the fact that $\{u < C/2\}$ is relatively compact.

Since for any bounded hyperconvex domain G and $z_0 \in G$, $g_G(z_0, z) \to 0$ as $z \to \partial G$, (see Theorem 1.6.1(iii)), we complete the proof by making use of Remark 2.6.9.

(i)⇒(ii). Fix $z_0 \in D$. Put $u(z) := \log c_D^*(z_0, z)$, $z \in D$. We know that $u \in \mathrm{PSH}(D) \cap \mathcal{C}(D)$. On the other hand c-finite compactness of D (see Theorem 2.6.6) implies that $\{u < -\varepsilon\} \subset\subset D$ for any $\varepsilon > 0$. ■

The problem of characterizing bounded hyperconvex Reinhardt domains was also dealt with in [Carl-Ceg-Wik 98].

REMARK 2.6.12. A simpler, direct proof of Corollary 2.6.11 was presented to the author by Professor P. Pflug. Namely, in view of the proof of Theorem 2.6.6, it is sufficient to disprove the hyperconvexity of G_0 (as in the proof of Theorem 2.6.6). We can proceed as follows. Let u be an exhausting function from the definition of hyperconvex domain. Define $v(z) := \sup\{u(z_1 e^{i\theta_1}, \dots, z_l e^{i\theta_l}),\quad \theta_j \in \mathbb{R}\}$. It is easy to see that v is an exhausting function from the definition of the hyperconvexity and, additionally, we have $v(z) = v(|z_1|, \dots, |z_l|)$. Therefore, the function $E_* \ni \lambda \mapsto v(|\lambda|, |\lambda|^{\alpha_2}, \dots, |\lambda|^{\alpha_l})$ is subharmonic and bounded from above by 0; hence, it can be continued subharmonically onto E but because of hyperconvexity the value at 0 would have to be 0, which is only possible for a constant function, a contradiction.

2.7. Bergman completeness of bounded Reinhardt domains. In the proofs of Bergman completeness, an important role is played by the Kobayashi Criterion.

THEOREM 2.7.1 (see [Kob 62]). *Let D be a bounded domain. If there is a subspace $\mathcal{E} \subset L_h^2(D)$ with $\bar{\mathcal{E}} = L_h^2(D)$ such that for any $f \in \mathcal{E}$ and for any $z^0 \in \partial D$ we have*

$$\text{(KC)} \qquad |f(z)|/\sqrt{K_D(z)} \to 0 \quad \text{as } z \to z^0$$

then D is Bergman complete.

Let D be a pseudoconvex Reinhardt domain, $a \in \overline{\log D}$. Let us recall the definition $\mathfrak{C}(D) := \{v \in \mathbb{R}^n : a + \mathbb{R}_+ v \subset \overline{\log D}\}$ (recall that $\mathfrak{C}(D)$ is well defined, i.e. it does not depend on the choice of $a \in \overline{\log D}$).

If D is bounded then $\mathfrak{C}(D) \subset \mathbb{R}_-^n$. For $a \in \log D$ (note that a is from $\log D$ and not from $\overline{\log D}$ as it was in the case of the definition of $\mathfrak{C}(D)$) put

$$\widetilde{\mathfrak{C}}(D) := \{v \in \mathfrak{C}(D) : \overline{(\exp(a + \mathbb{R}_+ v))} \subset D\}, \quad \mathfrak{C}'(D) := \mathfrak{C}(D) \setminus \widetilde{\mathfrak{C}}(D).$$

Note that the definition of $\widetilde{\mathfrak{C}}(D)$ (and, consequently, that of $\mathfrak{C}'(D)$) does not depend on the choice of $a \in \log D$ (exactly as in the case of $\mathfrak{C}(D)$). This follows easily from the properties of pseudoconvex Reinhardt domains (see Proposition 2.1.1 and remarks at the end of Section 2.1).

Let us introduce some additional notations. Given a pseudoconvex Reinhardt domain D put $H := \operatorname{Span} \mathfrak{C}(D)$. Let $\{v^1, \dots, v^r\} \subset H$ be a maximal set of linearly independent vectors such that $v^1, \dots, v^r \in \mathbb{Z}^n$. Let $H_1 := \operatorname{Span}\{v^1, \dots, v^r\}$ ($H_1 := \{0\}$ if $r = 0$). This definition of H_1 does not depend on the choice of $v^1, \dots, v^r$.

If D is a pseudoconvex Reinhardt domain then the system

$$\{z^\alpha / \|z^\alpha\|_{L^2(D)} : \alpha \in \mathbb{Z}^n,\ z^\alpha \in L_h^2(D)\}$$

is an orthonormal basis of $L_h^2(D)$.

Our two main results concerning b-completeness are:

PROPOSITION 2.7.2. *Let D be a bounded pseudoconvex Reinhardt domain. If $\mathfrak{C}'(D)\cap\mathbb{Q}^n\neq\emptyset$ then D is not Bergman complete.*

THEOREM 2.7.3. *Let D be a bounded pseudoconvex Reinhardt domain such that $H_1\cap\mathfrak{C}(D)=\{0\}$. Then D is Bergman complete.*

In particular, we get the following description of Bergman complete bounded Reinhardt domains in $\mathbb{C}^2$.

COROLLARY 2.7.4. *For a bounded pseudoconvex Reinhardt domain D in $\mathbb{C}^2$ the following two conditions are equivalent*:

(i) *D is Bergman complete*,
(ii) *$\mathfrak{C}'(D)\cap\mathbb{Q}^2=\emptyset$.*

Proof of Proposition 2.7.2. We assume that $a\in\log D$ from the definition of $\mathfrak{C}(D)$ is $(0,\dots,0)$. Take $v\in\mathfrak{C}'(D)\cap\mathbb{Q}^n$. We may assume that $v\in\mathbb{Z}^n_-$ and $v_1,\dots,v_n$ are relatively prime.

It is sufficient to show that the Bergman length L_{β_D} of the curve $(t^{-v_1},\dots,t^{-v_n})$, $0<t<1$, is finite.

Let $\varphi(\lambda):=(\lambda^{-v_1},\dots,\lambda^{-v_n})$, $\lambda\in E_*$. Clearly, $\varphi\in\mathcal{O}(E_*,D)$. Put $u(\lambda):=K_D(\varphi(\lambda))$. Then we have (use Lemma 2.2.1)

$$u(\lambda)=\sum_{\alpha\in\mathbb{Z}^n:\langle\alpha+\mathbf{1},v\rangle<0}a_\alpha|\lambda|^{-2\langle\alpha,v\rangle}=\sum_{j=j_0}^{\infty}b_j|\lambda|^{2j},$$

where $b_{j_0}\neq 0$ (note that $j_0>\langle\mathbf{1},v\rangle$ and it is possible that many of a_α's in the formula above vanish).

Note that

$$\beta_D^2(\varphi(\lambda);\varphi'(\lambda))=\frac{\partial^2\log u(\lambda)}{\partial\lambda\partial\overline{\lambda}}=\frac{\partial^2}{\partial\lambda\partial\overline{\lambda}}\Big(\log\sum_{j=j_0}^{\infty}b_j|\lambda|^{2j-2j_0}\Big).$$

The last expression tends to some constant $C\in\mathbb{R}$, which finishes the proof. ∎

Below we are only interested in *bounded pseudoconvex Reinhardt domains*.

Consider the following subspace:

$$\mathcal{E}_0:=\operatorname{Span}\{z^\alpha:z^\alpha\in L_h^2(D)\}.$$

We know that $\overline{\mathcal{E}}_0=L_h^2(D)$. In order to verify the property (KC) at some z^0 for $\mathcal{E}_0$ it is sufficient to show that it holds for all $z^\alpha\in L_h^2(D)$.

LEMMA 2.7.5. *Let D be a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^n$. Fix $z^0\in\partial D$ satisfying*

$$\textit{for any } j\in\{1,\dots,n\},\ \textit{if } z_j^0=0 \textit{ then } D\cap V_j\neq\emptyset$$

(*this condition is satisfied if*, *for instance*, $z^0\in\mathbb{C}^n_*$). *Then the condition* (KC) *is satisfied at z^0* (*for the subspace $\mathcal{E}_0$*).

PROOF. For any $\alpha\in\mathbb{Z}^n$ such that $z^\alpha\in L_h^2(D)$ we have $\alpha_j\geq 0$ if $z_j^0=0$. Therefore, it is sufficient to show that $K_D(z)\to\infty$ as $z\to z^0$. Let $I:=\{j:z_j^0=0\}$. We may assume that $I=\{1,\dots,s\}$. We easily see that $s<n$. Then $D\subset\mathbb{C}^s\times\pi_I(D)$ (we identify $\pi_I(D)$ with a subset of $\mathbb{C}^{n-s}$, if $s=0$ then $\pi_I:=\mathrm{id}$). Note that the assumptions of the criterion from [Pfl 75] ([29]) are satisfied for the domain $\widetilde{D}$ (and consequently also for D), where $\widetilde{D}$ is a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^{n-s}$, $\pi_I(D)\subset\widetilde{D}$, $\pi_I(z^0)\in\partial\widetilde{D}$ and $\partial\widetilde{D}$ is C^2 near $\pi_I(z^0)$, which finishes the proof. The existence of such $\widetilde{D}$ follows from the convexity of $\log\pi_I(D)$ and the fact that $\pi_I(z^0)\in\partial\pi_I(D)\cap\mathbb{C}_*^{n-s}$. ∎

In the proof of Theorem 2.7.2 we need the following lemma:

Lemma 2.7.6. *Let H be a k-dimensional vector subspace of $\mathbb{R}^n$ $(1\le k<n)$ such that $H\cap\mathbb{Q}^n=\{0\}$. Let $\{v^1,\dots,v^k\}$ be a vector basis of H. Then the set*

$$\{(\langle\alpha,v^1\rangle,\dots,\langle\alpha,v^k\rangle):\alpha\in\mathbb{Z}^n\}$$

is dense in $\mathbb{R}^k$.

Proof. It is easy to see that there is a vector subspace $\widetilde{H}\supset H$ of dimension $n-1$ such that $\widetilde{H}\cap\mathbb{Q}^n=\{0\}$. Therefore, we can assume that $k=n-1$.

Moreover, we lose no generality assuming that for the matrix

$$\widetilde{V}:=\begin{bmatrix} v_1^1 & \dots & v_1^{n-1}\\ \cdot & \dots & \cdot\\ v_{n-1}^1 & \dots & v_{n-1}^{n-1}\end{bmatrix}$$

we have $\det\widetilde{V}\neq 0$.

For $j=1,\dots,n-1$ we find $t^j\in\mathbb{R}^{n-1}$ such that $\widetilde{V}t^j=e^j\in\mathbb{R}^{n-1}$. Put $w^j:=\sum_{k=1}^{n-1}t_k^j v^k$, $j=1,\dots,n-1$. We have $w_l^j=\delta_{jl}$, $j,l=1,\dots,n-1$. Clearly, $w^j\in H$, $j=1,\dots,n-1$. By assumption the set $\{w_n^1,\dots,w_n^{n-1}\}$ is $\mathbb{Z}$-linearly independent. Then in view of the multidimensional Kronecker Theorem the set

$$\{(\alpha_n w_n^1-[\alpha_n w_n^1],\dots,\alpha_n w_n^{n-1}-[\alpha_n w_n^{n-1}]):\alpha_n\in\mathbb{Z}\}$$

is dense in $[0,1)^{n-1}$. But $\langle\alpha,w^j\rangle=\alpha_j+\alpha_n w_n^j$; therefore,

$$\{(\langle\alpha,w^1\rangle,\dots,\langle\alpha,w^{n-1}\rangle):\alpha\in\mathbb{Z}^n\}\quad\text{is dense in }\mathbb{R}^{n-1}. \tag{2.7.1}$$

Put $T:=[t^1,\dots,t^{n-1}]\in\mathbb{R}^{(n-1)\times(n-1)}$. We have $\det T\neq 0$. We also have

$$[w^1,\dots,w^{n-1}]=[v^1,\dots,v^{n-1}]T.$$

Consequently,

$$(\langle\alpha,v^1\rangle,\dots,\langle\alpha,v^{n-1}\rangle)=(\langle\alpha,w^1\rangle,\dots,\langle\alpha,w^{n-1}\rangle)T^{-1},$$

which, in view of (2.7.1), finishes the proof. ∎

For points from the boundary not lying on the axis the Kobayashi condition (for $\mathcal{E}_0$) is always satisfied (see Lemma 2.7.5), so the whole difficulty in the proof of the Bergman completeness (with the help of the Kobayashi Criterion) of a domain reduces to the proof of the Kobayashi condition for those points from the boundary which lie in $\mathbb{C}^n\setminus\mathbb{C}_*^n$.

The next result gives sufficient conditions for this property.

([29]) *Theorem* (see [Pfl 75]). Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let $z^0\in\partial D$. Suppose that there exist $r\in(0,1]$, $a\ge 1$ and a sequence $\{z^\nu\}_{\nu=1}^\infty\subset\mathbb{C}^n\setminus\bar{D}$ such that $\lim_{\nu\to\infty}z^\nu=z^0$ and $D\cap B(z^\nu,r\|z^\nu-z^0\|^a)=\emptyset$. Then $\lim_{z\to z^0}K_D(z)=\infty$.

PROPOSITION 2.7.7. *Let D be a bounded pseudoconvex Reinhardt domain, let $\mathcal{E}_0$ be as above and let $z^0 \in \partial D \cap (\mathbb{C}^n \setminus \mathbb{C}^n_*)$. Assume that one of the conditions is satisfied*:

(i) *there is $j \in \{1, \ldots, n\}$ such that $z^0_j = 0$ and $v_j = 0$ for any $v \in \mathfrak{C}(D)$*;

(ii) *$H_1 \cap \mathfrak{C}(D) = \{0\}$.*

Then D satisfies (KC) *at z^0 for $\mathcal{E}_0$.*

PROOF. Because $\bar{D} \not\subset \mathbb{C}^n_*$ we know that $\mathfrak{C}(D) \neq \{0\}$. First, we consider case (i). Assume that $j = 1$. Fix $\alpha \in \mathbb{Z}^n$ such that $\langle \alpha + \mathbf{1}, v \rangle < 0$ for any $v \in \mathfrak{C}(D)$, $v \neq 0$ (i.e. $z^\alpha \in L^2_h(D)$, use Lemma 2.2.1). Note that $\langle \alpha - e_1 + \mathbf{1}, v \rangle = \langle \alpha + \mathbf{1}, v \rangle < 0$, $v \in \mathfrak{C}(D)$, $v \neq 0$. Consequently, $z^{\alpha - e_1} \in L^2_h(D)$ (see Lemma 2.2.1). Therefore, we have

$$0 \leq \frac{|z^\alpha|}{\sqrt{K_D(z)}} \leq \frac{|z^\alpha| \cdot \|z^{\alpha - e_1}\|_{L^2(D)}}{|z^{\alpha - e_1}|} = \|z^{\alpha - e_1}\|_{L^2(D)} |z_1|.$$

And the last number tends to 0 as z tends to z^0, which finishes this case.

In case (ii) our aim will be to find for a given $\alpha \in \mathbb{Z}^n$ with $z^\alpha \in L^2_h(D)$ (in other words $\langle \alpha + \mathbf{1}, v \rangle < 0$ for any $v \in \mathfrak{C}(D)$, $v \neq 0$) an $\widetilde{\alpha} \in \mathbb{Z}^n$ such that

$$\langle \alpha + \mathbf{1}, v \rangle < \langle \widetilde{\alpha} + \mathbf{1}, v \rangle < 0 \tag{2.7.2}$$

for any $v \in \mathfrak{C}(D)$, $v \neq 0$.

Assume that this can be done. Then we claim that (KC) is satisfied at z^0. In fact, then $\langle \alpha - \widetilde{\alpha}, v \rangle < 0$ for any $v \in \mathfrak{C}(D)$, $v \neq 0$. Therefore,

$$\langle \alpha - \widetilde{\alpha}, v \rangle \leq -\delta < 0$$

for any $v \in \mathfrak{C}(D)$, $\|v\| = 1$. Assume that $z^0_1 = 0$. There is $N \in \mathbb{N}$ such that $\langle \alpha - \widetilde{\alpha} - 1/N e_1, v \rangle < 0$ for any $v \in \mathfrak{C}(D)$, $\|v\| = 1$. Consequently, the same holds for any $v \in \mathfrak{C}(D)$, $v \neq 0$. Therefore, $z^{N(\alpha - \widetilde{\alpha}) - e_1} \in H^\infty(D)$ (use Lemma 2.2.1). And, finally (remember that $z^{\widetilde{\alpha}} \in L^2_h(D)$, see (2.7.2)), we have

$$0 \leq \frac{|z^\alpha|}{\sqrt{K_D(z)}} \leq \|z^{\widetilde{\alpha}}\|_{L^2(D)} |z^{\alpha - \widetilde{\alpha}}| = \|z^{\widetilde{\alpha}}\|_{L^2(D)} (|z^{N(\alpha - \widetilde{\alpha}) - e_1}\| z_1|)^{1/N}.$$

The last number tends to 0 as z tends to z^0.

So we need to prove (2.7.2). First, we show the existence of a subspace H_2 of H and some basis $\{v_{r+1}, \ldots, v_s\}$ of H_2 such that $H_1 + H_2 = H$, $H_1 \cap H_2 = \{0\}$ and

$$\mathfrak{C}(D) \setminus \{0\} \subset \Big\{ \sum_{j=1}^s t_j v_j : t_{r+1}, \ldots, t_s > 0 \Big\}, \quad \langle \alpha + \mathbf{1}, v_j \rangle < 0, \quad j = r+1, \ldots, s. \tag{2.7.3}$$

LEMMA 2.7.8. *Let V and W be two subspaces of U (U is a subspace of $\mathbb{R}^n$) such that $V + W = U$ and $V \cap W = \{0\}$. Let C be a closed, convex cone (with vertex at 0) such that $C \cap V = \{0\}$, $\operatorname{Span}(C) = U$ and C contains no straight line. Assume that $\{v_1, \ldots, v_k\}$ is a basis of V. Then there is a basis $\{v_{k+1}, \ldots, v_l\}$ of W such that*

$$C \setminus \{0\} \subset \Big\{ \sum_{j=1}^l t_j v_j : t_{k+1}, \ldots, t_l > 0 \Big\}.$$

PROOF. Denote by $\pi := \pi_{V,W}$ the projection of U onto W in direction V.

First, we prove that $\pi(C)$ is a closed, convex cone containing no line. The only difficulty is with the proof of the nonexistence of straight lines in $\pi(C)$. Assuming the contrary we easily get the existence of $w \neq 0$ such that $w, -w \in \pi(C)$. Consequently, there are $u_1 = v_1 + w, u_2 = v_2 - w \in C$, $v_1, v_2 \in V$. But then $v_1 + v_2 \in C \cap V = \{0\}$, from which we get $v_1 = -v_2$, so $u_1 = -u_2 \neq 0$, a contradiction.

Now let I be an isometry of W onto $\mathbb{R}^{l-k}$. Then $I(\pi(C))$ is a closed, convex cone not containing straight lines. From Lemma 2.2.3 we get the existence of linearly independent vectors $w_{k+1}, \dots, w_l$ such that

$$I(\pi(C)) \setminus \{0\} \subset \Big\{ \sum_{j=k+1}^{l} t_j w_j : t_j > 0 \Big\},$$

which easily finishes the proof with $v_j := I^{-1}(w_j)$, $j = k+1, \dots, l$. ■

Put $\widetilde{H}_2 := \{v \in H : \langle \alpha + \mathbf{1}, v\rangle = 0\}$. It is easy to verify that $\dim \widetilde{H}_2 = s - 1$. Note also that $\mathfrak{C}(D) \cap \{v \in H : \langle \alpha + \mathbf{1}, v\rangle = -1\}$ is bounded (use Lemma 2.2.4).

If $H_1 \not\subset \widetilde{H}_2$ then we define $\widehat{H}_2$ to be a complement of $H_1 \cap \widetilde{H}_2$ in $\widetilde{H}_2$ (i.e. $\widehat{H}_2 + H_1 \cap \widetilde{H}_2 = \widetilde{H}_2$, $\widehat{H}_2 \cap H_1 \cap \widetilde{H}_2 = \{0\}$). In view of Lemma 2.7.8 there are vectors $\{\widetilde{v}_{r+1}, \dots, \widetilde{v}_s\}$ spanning $\widehat{H}_2$ such that

$$\mathfrak{C}(D) \setminus \{0\} \subset \Big\{ \sum_{j=1}^{r} t_j v_j + \sum_{j=r+1}^{s} t_j \widetilde{v}_j : t_{r+1}, \dots, t_s > 0 \Big\}.$$

Obviously, $\langle \alpha + \mathbf{1}, \widetilde{v}_j\rangle = 0$, $j = r+1, \dots, s$. Adding to $\widetilde{v}_j$ some small enough vector $w \in H$ with $\langle \alpha + \mathbf{1}, w\rangle < 0$ we get linearly independent vectors $\{v_{r+1}, \dots, v_s\}$ spanning H_2 satisfying (2.7.3).

If $H_1 \subset \widetilde{H}_2$, $H_1 \neq \widetilde{H}_2$ then we define $\widehat{H}_2$ to be a complement of H_1 in $\widetilde{H}_2$. There is $v \in H$ such that $\langle \alpha + \mathbf{1}, v\rangle = -1$ and $(\mathbb{R}v + H_1) \cap \mathfrak{C}(D) = \{0\}$. We clearly have $H_1 + \widehat{H}_2 + \mathbb{R}v = H$. In view of Lemma 2.7.8 (applied to $H_1 + \mathbb{R}v$ and $\widehat{H}_2$) we have the existence of vectors $\{\widetilde{v}_{r+2}, \dots, \widetilde{v}_s\} \subset \widehat{H}_2$ such that

$$\mathfrak{C}(D) \setminus \{0\} \subset \Big\{ \sum_{j=1}^{r} t_j v_j + tv + \sum_{j=r+2}^{s} t_j \widetilde{v}_j : t_{r+2}, \dots, t_s > 0 \Big\}.$$

Since $\mathfrak{C}(D) \setminus \{0\} \subset \widetilde{H}_2 + (0,\infty)v = H_1 + \widehat{H}_2 + (0,\infty)v$, adding to vectors $\widetilde{v}_j$, $j = r+2, \dots, s$, some small tv $(t > 0)$ and putting $v_{r+1} := v$ we finish the proof as in the preceding case.

If $H_1 = \widetilde{H}_2$ then we put $H_2 := \mathbb{R}v_s$, where v_s is a vector such that $\langle \alpha + \mathbf{1}, v_s\rangle = -1$.

Let us prove that for $\delta_{r+1}, \dots, \delta_s > 0$ (to be chosen later) we can find $\beta \in \mathbb{Z}^n$ such that

$$\begin{aligned} \langle \beta, v_j\rangle &= 0, \quad j = 1, \dots, r, \\ -\delta_j < \langle \beta, v_j\rangle &< 0, \quad j = r+1, \dots, s. \end{aligned} \tag{2.7.4}$$

Let $A \in \mathbb{Z}^{n\times n}$ be a linear isomorphism of $\mathbb{R}^n$ such that $A(e_j) = v_j$, $j = 1, \dots, r$. We want to have $\langle \beta, Ae_j\rangle = \langle A^*\beta, e_j\rangle = 0$, so $(A^*\beta)_j = 0$, $j = 1, \dots, r$. Put $\widetilde{\beta} := A^*\beta$. Note that $A^{-1}v_j$ are linearly independent in $A^{-1}H_2$, $j = r+1, \dots, s$, and $A^{-1}H_2 \cap (\mathbb{R}^r \times \mathbb{Q}^{n-r}) = \{0\}$. It is easy to see that the vectors $((A^{-1}v_j)_{r+1}, \dots, (A^{-1}v_j)_n)$, $j = r+1, \dots, s$, are linearly independent. Therefore, we get $\widetilde{\beta} \in \mathbb{Z}^n$, $\widetilde{\beta}_j = 0$, $j = 1, \dots, r$, such that

$-|\det A|\delta_j < \langle\widetilde{\beta}, |\det A|A^{-1}v_j\rangle = |\det A|\langle\beta, v_j\rangle < 0$, $j = r+1,\dots,s$ (see Lemma 2.7.6). This finishes the proof of (2.7.4c).

There is a constant $M < \infty$ such that for any $\|v\| = 1$, $v \in \mathfrak{C}(D)$, $v = \sum_{j=1}^s t_j v_j$, we have $|t_j| \le M$, $j = 1,\dots,s$. Put $\widetilde{\alpha} := \alpha - \beta$ (β is as above, in particular we have (2.7.4)). Then

$$\langle\widetilde{\alpha} + \mathbf{1}, v\rangle = \langle\alpha + \mathbf{1}, v\rangle - \langle\beta, v\rangle = \langle\alpha + \mathbf{1}, v\rangle - \sum_{j=r+1}^{s} t_j\langle\beta, v_j\rangle,$$

$v = \sum_{j=1}^s t_j v_j \in \mathfrak{C}(D)$, $\|v\| = 1$.

In view of (2.7.3) and (2.7.4), $t_j > 0$ and $\langle\beta, v_j\rangle < 0$, $j = r+1,\dots,s$, so the last expression is larger than $\langle\alpha + \mathbf{1}, v\rangle$.

There is $\delta > 0$ such that $\langle\alpha + \mathbf{1}, v\rangle < -\delta$ for any $v \in \mathfrak{C}(D)$, $\|v\| = 1$. Therefore, choosing small enough δ_j we easily get $\langle\widetilde{\alpha} + \mathbf{1}, v\rangle < 0$ for any $v \in \mathfrak{C}(D)$, $\|v\| = 1$, which finishes the proof of (2.7.2). ■

Proof of Theorem 2.7.3. The condition (KC) is satisfied for $z^0 \in \mathbb{C}_*^n \cap \partial D$ (use Lemma 2.7.5). For $z^0 \in \partial D \cap (\mathbb{C}^n \setminus \mathbb{C}_*^n)$ it is satisfied by Proposition 2.7.7(ii). Then the assumptions of Theorem 2.7.1 are satisfied for $\mathcal{E}_0$ at every $z^0 \in \partial D$. ■

Proof of Corollary 2.7.4. The implication (i)⇒(ii) follows immediately from Proposition 2.7.2. For the proof of (ii)⇒(i) consider three cases.

Case (I): $\dim H = 2$. The condition $\mathfrak{C}'(D) \cap \mathbb{Q}^2 = \emptyset$ easily implies that $\mathfrak{C}(D) = \mathbb{R}_-^2$; moreover, D is complete. Then it is sufficient to use Lemma 2.7.5 and Theorem 2.7.1.

Case (II): $\dim H = 0$, so $\mathfrak{C}(D) = \{0\}$ or $D \subset\subset \mathbb{C}_*^2$. Use Lemma 2.7.5 and Theorem 2.7.1.

Case (III): $\dim H = 1$. The condition $\mathfrak{C}'(D) \cap \mathbb{Q}^2 = \emptyset$ implies that $\mathfrak{C}'(D) = \mathfrak{C}(D) \setminus \{0\} = \mathbb{R}_-(1,t) \setminus \{0\}$, where $t > 0$ is irrational or $\widetilde{\mathfrak{C}}(D) = \mathfrak{C}(D) = \mathbb{R}_- e_j$ ($j = 1$ or $j = 2$). In the first case use Theorem 2.7.3. In the second case if $0 \in \partial D$ then use Lemma 2.7.5, Proposition 2.7.7(i), and Theorem 2.7.1, if $0 \notin \partial D$ then use Lemma 2.7.5 and Theorem 2.7.1. ■

Remark 2.7.9. In case $n = 2$ the proof of Proposition 2.7.7 (and consequently Corollary 2.7.4) is much simpler. In fact, if we exclude case (i) from Proposition 2.7.7 then $\mathfrak{C}(D) = \mathbb{R}_+(-1,-t)$, where $t > 0$ is irrational. Then the proof of Proposition 2.7.7 boils down to the proof of (2.7.2), so to the existence of $\widetilde{\alpha} \in \mathbb{Z}^2$ such that $\langle\alpha + \mathbf{1}, (-1,-t)\rangle < \langle\widetilde{\alpha} + \mathbf{1}, (-1,-t)\rangle < 0$, which follows directly from the one-dimensional Kronecker Theorem.

Remark 2.7.10. The example from [Her 99] ($D := \{z \in E_* \times E : |z_2|^2 \exp(1/|z_1|^2) < 1\}$) is a special case of a bounded pseudoconvex Reinhardt domain such that $\mathfrak{C}(D) = \{0\} \times (-\infty, 0]$. Our proof that this domain is Bergman complete is much simpler than that in [Her 99]. This is so because in this case Proposition 2.7.7 (for $z^0 = 0$) boils down to the proof of the very simple case (i).

Remark 2.7.11. The results of Chapter II (concerning completeness) may be summarized as follows (for bounded domains).

All bounded pseudoconvex Reinhardt domains are Kobayashi complete.

The Carathéodory complete bounded pseudoconvex Reinhardt domains are exactly those fulfilling (2.6.7).

The class of Bergman complete bounded pseudoconvex Reinhardt domains is different from both classes above ([30]). The fact that this class does not include all bounded pseudoconvex Reinhardt domains was well known (see for example the Hartogs triangle). The fact that except for domains satisfying (2.6.7) we have other Bergman complete bounded Reinhardt domains is, to some extent, surprising.

Example 2.7.12. As the simplest examples of Bergman complete Reinhardt domains which do not satisfy (2.6.7), we give the following class of domains. They will be contained in $\mathbb{C}_*^n$; therefore, their construction boils down to the construction of a convex domain in $\mathbb{R}^n$. Let $v_1, \ldots, v_k \in \mathbb{R}_-^n$ be linearly independent vectors spanning a subspace H satisfying $H \cap \mathbb{Q}^n = \{0\}$ $(k \geq 1)$. Let $\{v_1, \ldots, v_n\}$ be a basis of $\mathbb{R}^n$. Define

$$\log D := \sum_{j=1}^{k} (0, \infty) v_j + \sum_{j=k+1}^{n} (a_j, b_j) v_j,$$

where $-\infty < a_j < b_j < \infty$. Then $\mathfrak{C}(D) = \sum_{j=1}^{k} [0, \infty) v_j$, $H_1 = \{0\}$.

2.8. Boundary behavior of the Green function and Bergman completeness. In the proof of Bergman completeness of hyperconvex domains (see [Bło-Pfl 98] and [Her 99]) the key role is played by the boundary behavior of the Green function. More precisely, good boundary behavior of the Green function implies Bergman completeness.

Theorem 2.8.1 (see [Her 99]). *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$. Then there exists a constant $C > 0$ (depending only on the diameter of D) such that for any $f \in L_h^2(D)$, $f \not\equiv 0$, and for any $w \in D$ we have*

$$\frac{|f(w)|^2}{K_D(w)} \leq C \int_{\{g_D(w,\cdot) < -1\}} |f|^2 \, dV.$$

Therefore, to prove Bergman completeness of some domain, it seems reasonable to examine the behavior of sublevel sets $\{g_D(w, \cdot) < -\delta\}$, where $\delta > 0$ and w tends to the boundary. We do this below for bounded pseudoconvex Reinhardt domains and we show that in this class convergence of the volume of sublevel sets to 0 is very closely related to Bergman completeness. In particular, our results show that in dimension two these two properties are equivalent. On the other hand we find (in dimension 1) examples of domains which are Bergman complete but for which the above volume does not converge to 0, which shows that in some sense the theory of L_h^2 functions (represented by Bergman completeness) is different from the pluripotential theory (represented by the convergence of the relevant volumes to 0).

The idea of the proofs of the forthcoming results comes from Z. Błocki (personal communication). His idea was applied to the example of G. Herbort (see [Her 99]). With the help of results of the preceding section we may put the result in some more general

([30]) Any bounded c-complete domain is b-complete (see e.g. [Jar-Pfl 93]).

setting of many bounded pseudoconvex Reinhardt domains. And although a complete answer is not known, we can give it in the two-dimensional case.

LEMMA 2.8.2. *Let $D \subset E^n \cap \mathbb{C}_*^n$ be a bounded pseudoconvex Reinhardt domain such that $\{0\} = \partial D \cap (\mathbb{C}^n \setminus \mathbb{C}_*^n)$ (in particular, $\mathfrak{C}(D) \setminus \{0\} \subset (-\infty, 0)^n$). Let $\|\cdot\|$ be some norm on $\mathbb{R}^n$. Assume that for any $p \in D$, $g_D(p,z) \to 0$ as z tends to a boundary point different from 0 and for any $0 < \delta_1 < \delta_2$ there is $\alpha \in \mathbb{Z}^n$ such that*

$$-\delta_2 < \langle \alpha, v \rangle < -\delta_1, \quad v \in \mathfrak{C}(D), \|v\| = 1. \tag{2.8.1}$$

Then for any $z \in D$ we have $g_D(p,z) \to 0$ as $p \to 0$.

PROOF. Take $\alpha \in \mathbb{Z}^n$ such that $\langle \alpha, v \rangle < 0$ for $v \in \mathfrak{C}(D)$, $v \neq 0$. Let $0 < \delta_1 < \delta_2$ be such that

$$\langle \alpha, v \rangle \in (-\delta_2, -\delta_1), \quad v \in \mathfrak{C}(D), \quad \|v\| = 1.$$

Then $z^\alpha \in H^\infty(D)$ (use Lemma 2.2.1). We claim that

$$\xi^\alpha \to 0 \quad \text{as } D \ni \xi \to 0. \tag{2.8.2}$$

In fact, there is $N \in \mathbb{N}$ such that $\langle \alpha - e_1/N, v \rangle < 0$, $v \in \mathfrak{C}(D)$, $\|v\| = 1$, so

$$0 < |\xi^\alpha| = |\xi^{N\alpha - e_1}|^{1/N} |\xi^{e_1}|^{1/N}$$

and because $z^{N\alpha - e_1} \in H^\infty(D)$ (use once more Lemma 2.2.1) the last expression tends to 0 as ξ tends to 0.

Put $M_\alpha := \|z^\alpha\|_\infty < \infty$. Take $p \in D$ (close to 0). For $\xi \in D$, $|\xi^\alpha| = 2|p^\alpha|$, we have

$$g_D(p,\xi) \geq \log \frac{|p^\alpha|}{2M_\alpha}.$$

Put $C(\alpha,p) := \sup\{\sum_{j=1}^n \log|\xi_j| : \xi \in D, |\xi^\alpha| = 2|p^\alpha|\} \in (-\infty, 0)$ (for p close to 0, remember (2.8.2)). We claim that for $\xi \in D$ such that $|\xi^\alpha| \geq 2|p^\alpha|$,

$$g_D(p,\xi) \geq \log \frac{|p^\alpha|}{2M_\alpha} \cdot \frac{\sum_{j=1}^n \log|\xi_j|}{C(\alpha,p)}. \tag{2.8.3}$$

In fact, it is sufficient to see that the second factor on the right hand side is ≥ 1 for $|\xi^\alpha| = 2|p^\alpha|$, $\xi \in D$ (which is immediate) and then to make use of maximality of $g_D(p,\cdot)$ on the domain $\{\xi \in D : |\xi^\alpha| > 2|p^\alpha|\}$ (we know that $\lim_{\xi \to \widetilde{\xi}} g_D(p,\xi) = 0$ for $\widetilde{\xi} \in \partial D$, $|\widetilde{\xi}^\alpha| \geq 2|p^\alpha|$, because $\widetilde{\xi} \in \mathbb{C}_*^n$, by Lemma 2.2.5).

Now we let $p \to 0$. Our aim is to estimate from above the expression $\big(\log \frac{|p^\alpha|}{2M_\alpha}\big)/C(\alpha,p)$ as $p \to 0$. In other words we want to find the upper limit of the expression

$$\frac{-\langle \alpha, t \rangle + \log(2M_\alpha)}{\inf\{\sum_{j=1}^n |s_j| : s \in \log D,\ \langle \alpha, s \rangle = \log 2 + \langle \alpha, t \rangle\}} \tag{2.8.4}$$

as $t \in \log D$ and $\|t\| \to \infty$. Note that in view of Lemma 2.2.4 we have

$$\limsup_{\|t\| \to \infty,\, t \in \log D} (-\langle \alpha, t/\|t\| \rangle + \log(2M_\alpha)/\|t\|) < \delta_2.$$

On the other hand the lower limit of the denominator (divided by $\|t\|$) is not smaller than

$$\liminf_{\|t\| \to \infty,\, t \in \log D} \inf\{\widetilde{C}\|s\|/\|t\| : s \in \log D,\ \langle \alpha, s \rangle = \log 2 + \langle \alpha, t \rangle\},$$

where $\widetilde{C} > 0$ depends only on the norm $\|\cdot\|$. Take sequences $\{t^\nu\} \subset \log D$ with $\|t^\nu\| \to \infty$ and $\{s^\nu\} \subset \log D$ such that $\langle \alpha, s^\nu \rangle = \log 2 + \langle \alpha, t^\nu \rangle$. We may assume that $t^\nu / \|t^\nu\| \to t^0 \in \mathfrak{C}(D)$ (use Lemma 2.2.4). Therefore,

$$\lim_{\nu\to\infty} \langle \alpha, s^\nu/\|t^\nu\| \rangle = \lim_{\nu\to\infty} \langle \alpha, t^\nu/\|t^\nu\| \rangle = \langle \alpha, t^0 \rangle \in (-\delta_2, -\delta_1), \tag{2.8.5}$$

from which we see that $\|s^\nu\| \to \infty$. Consequently, $s^\nu/\|s^\nu\| \to s^0 \in \mathfrak{C}(D)$ (choosing, if necessary, a subsequence), so $\langle \alpha, s^\nu/\|s^\nu\| \rangle \to \langle \alpha, s^0 \rangle \in (-\delta_2, -\delta_1)$, which combined with (2.8.5) gives $\lim_{\nu\to\infty} \|s^\nu\|/\|t^\nu\| > \delta_1/\delta_2$ and the upper limit of (2.8.4) is not larger than $\delta_2^2/(\widetilde{C}\delta_1)$.

Fix $\xi \in D$. Taking $\delta_1 = 1/(\nu+1)$, $\delta_2 = 1/\nu$, choosing α for these δ_1 and δ_2 and letting $p \to 0$ we get

$$\liminf_{p\to 0} g_D(p,\xi) \geq \frac{\nu+1}{\widetilde{C}\nu^2} \sum_{j=1}^{n} \log|\xi_j|, \quad \xi \in D.$$

Letting $\nu \to \infty$ we finish the proof. ∎

Remark 2.8.3. It follows from the proof of Proposition 2.7.7(ii) that the assumption on the existence of α in Lemma 2.8.2 is satisfied if $H_1 = \{0\}$ (notation as in Section 2.7). In fact, in view of Lemma 2.2.3,

$$\mathfrak{C}(D) \setminus \{0\} \subset \Big\{ \sum_{j=1}^{s} t_j v_j : t_j > 0 \Big\},$$

where $\{v_1, \ldots, v_n\}$ is a basis of $\mathbb{R}^n$ complementing some basis of $H_2 = H = \operatorname{Span} \mathfrak{C}(D)$. Proceeding as in the proof of (2.7.4) we get existence of $\alpha \in \mathbb{Z}^n$ such that $\langle \alpha, v_j \rangle \in (-\delta_2, -\delta_1)$, $j = 1, \ldots, s$. Taking $\|\sum_{j=1}^{n} t_j v_j\| := \sum_{j=1}^{n} |t_j|$ on $\mathbb{R}^n$ we get $\langle \alpha, v \rangle \in (-\delta_2, -\delta_1)$ for any $v \in \mathfrak{C}(D)$, $\|v\| = 1$.

As earlier, in the two-dimensional case the proof of Lemma 2.8.2 is much easier.

Lemma 2.8.4. *Let D be a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^2$ such that $D \cap (\mathbb{C} \times \{0\}) = E_* \times \{0\}$ and $\mathfrak{C}(D) = \mathbb{R}_+(0,-1)$. Then for any $z \in D \cap \mathbb{C}_*^2$ we have*

$$g_D(p,z) \to 0 \quad \textit{as } p \textit{ tends to } 0.$$

Proof. We may assume that $D = \{z \in E \times E : |z_2| < \varrho(|z_1|)\}$, where $\varrho : [0,1) \mapsto [0,1]$, $\varrho(r) = 0$ iff $r = 0$, and

(2.8.6) for any $A > 0$ there is $B \in \mathbb{R}$ such that $\log \varrho(e^t) < At + B$ for any $t \in (-\infty, 0)$.

Take $p \in D$ close to 0. Then for $|\xi_1| = 2|p_1|$ we have $g_D(p,\xi) \geq \log(|p_1|/2)$. We claim that

$$g_D(p,\xi) \geq \log\frac{|p_1|}{2} \cdot \frac{\log|\xi_2|}{\log \varrho(2|p_1|)}$$

for any $\xi \in D$ such that $|\xi_1| \geq 2|p_1|$. The second factor above is at least 1 for $\xi \in D$ with $|\xi_1| = 2|p_1|$, additionally, $g_D(p,\xi) \to 0$ if $\xi \to \widetilde{\xi} \in \partial D \setminus \{0\} \subset \mathbb{C}_*^2$ (use Corollary 2.6.2). Now applying the maximality of the function $g_D(p,\cdot)$ on $\{\xi \in D : |\xi_1| > 2|p_1|\}$ we get the desired inequality ([31]).

([31]) To apply maximality we have to proceed a little delicately here. First, we have to shrink a little the domain in which we consider the inequality (in particular, we delete $\{\xi_2 = 0\} \cap D$) and then after some standard approximation procedure we get the desired property.

Put $v(t) := \log \varrho(e^t)$, $-\infty < t < 0$. To finish the proof it is sufficient to show that $\frac{t-\log 2}{v(t+\log 2)} \to 0$ as t tends to $-\infty$. In view of (2.8.6) we see that for any $A > 0$, $\liminf_{t\to-\infty} v(t)/t \geq A$, so $\lim_{t\to-\infty} v(t)/t = \infty$, consequently,

$$\lim_{t\to-\infty} \frac{t-\log 2}{t+\log 2} \cdot \frac{t+\log 2}{v(t+\log 2)} = 0. \ \blacksquare$$

PROPOSITION 2.8.5. *Let D be a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^2$. Then the following conditions are equivalent*:

(i) *D is Bergman complete*,
(ii) *for any $\delta > 0$, $\mathrm{Vol}(\{g_D(p,\cdot) < -\delta\}) \to 0$ as $p \to \partial D$*,
(iii) *for any $z \in D \cap \mathbb{C}^2_*$ we have $g_D(p,z) \to 0$ as $p \to \partial D$.*

PROOF. (iii)⇒(ii) is trivial. (ii)⇒(i) follows from Theorem 2.8.1. So we are left with (i)⇒(iii). In view of Corollary 2.7.4 either D satisfies (2.6.7) and then the result follows from Theorem 2.6.6 or $\mathfrak{C}(D) = \mathbb{R}_+(-1,-t)$, $t > 0$, t is irrational and then $D \subset \mathbb{C}^2_*$, $\partial D \cap (\mathbb{C}^2 \setminus \mathbb{C}^2_*) = \{0\}$ and the result follows from Corollary 2.6.2, Lemma 2.8.2 and Remark 2.8.3 or D is as in Lemma 2.8.4 (up to dilatations and permutation of coordinates) and then we use Lemma 2.8.4 and Corollary 2.6.2 and the contractivity of the Green function for the points $(e^{i\theta}, 0)$. $\blacksquare$

REMARK 2.8.6. By the considerations above, if $D \subset \mathbb{C}^2$ is such that $\mathfrak{C}(D) = \mathbb{R}_+(-1,-t)$, where $t > 0$ is irrational, then for any $z \in \partial D$, $g_D(p,z) \to 0$ as $p \to \partial D$. Note that D is not hyperconvex in this case (for $z^0 \in \partial D$ we have $g_D(p,z) \to 0$ as $z \to z^0$ iff $z^0 \neq 0$).

Now we deal with the relation between hyperconvexity of the domain and convergence of sublevel sets when the pole tends to the boundary in dimension one. The situation here is completely different from the case of pseudoconvex Reinhardt domains. In particular, there are domains which are Bergman complete but such that the relevant volume does not converge to 0.

Let D be a domain in $\mathbb{C}$ such that ∂D is not polar (if D is unbounded then $\infty \in \partial D$ in the usual sense).

Recall that there is a polar set $F \subset \partial D$ such that for some (any) $\lambda \in D$ we have for any $\xi_0 \in \partial D$,

$$\lim_{\xi\to\xi_0} g_D(\lambda,\xi) = 0 \quad \text{iff} \quad \xi_0 \notin F. \tag{2.8.7}$$

Moreover, for any $\lambda \in D$ we have

$$\liminf_{\xi\to\partial D} g_D(\lambda,\xi) > -\infty. \tag{2.8.8}$$

Belonging to the set F is a local property. We can assume that $\infty \notin F$.

PROPOSITION 2.8.7. *For any $\xi_0 \in F$ there is a sequence $\{\xi_k\}_{k=1}^{\infty} \subset D$, $\xi_k \to \zeta_0$, such that*

$$\lim_{k\to\infty} g_D(\xi_k,\lambda) = \limsup_{k\to\infty} g_D(\xi_k,\lambda) < 0 \quad \textit{for any } \lambda \in D; \tag{2.8.9}$$

and for any $M > 0$ there are k_0 and an open set $\emptyset \neq U$ such that

$$U \subset \{\lambda \in D : g_D(\xi_k,\lambda) < -M\} \quad \textit{for any } k \geq k_0. \tag{2.8.10}$$

PROOF. Fix $\lambda_0 \in D$. We deduce from (2.8.7) that there is a sequence $\{\xi_k\}_{k=1}^{\infty} \subset D$ such that $\xi_k \to \xi_0$ and

$$g_D(\xi_k, \lambda_0) = g_D(\lambda_0, \xi_k) \to \alpha < 0. \tag{2.8.11}$$

Clearly, $\alpha > -\infty$ (use (2.8.8)).

Recall that $g_D(\xi_k, \cdot)$ is a harmonic function on $D \setminus \{\xi_k\}$, $g_D(\xi_k, \cdot) < 0$ on D, $k = 1, 2, \dots$ Therefore, applying a Montel type theorem for harmonic functions (see e.g. [Ran 95]) we get $g_D(\xi_k, \cdot) \to -\infty$ (which is impossible by (2.8.11)) or (choosing a subsequence if necessary)

(2.8.12) $g_D(\xi_k, \cdot)$ tends locally uniformly on D to a harmonic function h.

Clearly, $h(\lambda_0) = \alpha$, $h \le 0$, so $h < 0$. This gives (2.8.9).

Consider any (small enough) open connected neighborhoods $V_1 \subset\subset V_2$ of ξ_0. Then $D \cup V_1$ is a domain with nonpolar boundary. Note that $g_D(\xi_k, \cdot) \ge g_{D\cup V_1}(\xi_k, \cdot)$ on D, so $h(\cdot) \ge g_{D\cup V_1}(\xi_0, \cdot)$. In connection with (2.8.8) applied to $D \cup V_1$ ($\lambda := \xi_0$) this gives

(2.8.13) for any small neighborhood V of ξ_0 there is $C < \infty$ such that $h \ge -C$ on $D \setminus V$

and

$$\lim_{\xi \to \widetilde{\xi}} h(\xi) = 0 \quad \text{for any } \widetilde{\xi} \in \partial D \setminus F. \tag{2.8.14}$$

We claim that

$$\liminf_{\xi \to \xi_0} h(\xi) = -\infty. \tag{2.8.15}$$

Suppose not. Then for some neighborhood V of ξ_0, $h_{|V\cap D} > -2M$, where V is chosen so small that (2.8.13) is satisfied. Then h is bounded. The extended maximum principle implies that $h \equiv 0$ (remember (2.8.14)), a contradiction.

From (2.8.15) we get for any $M > 0$ an open relatively compact subset $U \ne \emptyset$ of D such that $U \subset \{\lambda \in D : h(\lambda) < -2M\}$, which finishes the proof (use (2.8.12)). ∎

Proposition 2.8.7 shows that in dimension one Bergman completeness and convergence to 0 of the volume of sublevel sets of the Green function as the pole tends to the boundary are two different phenomena.

COROLLARY 2.8.8. *There are a bounded domain $D \subset \mathbb{C}$ and $\xi_0 \in \partial D$ such that D is Bergman complete and for any $M > 0$, $\operatorname{Vol}(\{g_D(\xi, \cdot) < -M\})$ does not tend to 0 as $\xi \to \xi_0$.*

PROOF. Use [Chen 98] and Proposition 2.8.7. ∎

The domain claimed to exist in Corollary 2.8.8 is any bounded domain in $\mathbb{C}$ which is Bergman complete but not hyperconvex. For an example recall that the following domain has this property (see [Ohs 93], [Chen 98], the set $F = \{0\}$):

$$D := E \setminus \overline{\bigcup_{k=1}^{\infty} \mathbb{B}(2^{-k}, 2^{-k(k^2+1)})}.$$

III. Effective formulas in Reinhardt domains

In general it is very difficult to find effective formulas for invariant functions. Here we present formulas for a special subclass of quasi-elementary Reinhardt domains, so-called elementary Reinhardt domains. Additionally, in Section 3.6, we also find formulas for the Green function with two poles of equal weights of the unit ball.

For $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{R}^n_*$, $n>1$, we set

$$D_\alpha := G(\alpha,\mathbf{0}) = \{z\in\mathbb{C}^n : |z_1|^{\alpha_1}\dots|z_n|^{\alpha_n}<1,\ \text{if } \alpha_j<0 \text{ then } z_j\neq 0\}.$$

We say that α is of *rational type* if there are $t>0$, $\beta=(\beta_1,\dots,\beta_n)\in\mathbb{Z}^n$ such that $\alpha=t\beta$. We say that α is of *irrational type* if α is not of rational type. Note that if α is of rational type we my assume that all α_j's are relatively prime integers. We also define

$$\widetilde{D}_\alpha := \{z\in D_\alpha : z_1\dots z_n\neq 0\}.$$

For $\alpha\in\mathbb{Z}^n$, $r\in\mathbb{N}$ we set

$$F^\alpha(z) := z^\alpha = z_1^{\alpha_1}\dots z_n^{\alpha_n},$$
$$F^\alpha_{(r)}(z)X := \sum_{\beta_1+\ldots+\beta_n=r}\frac{1}{\beta_1!\cdot\ldots\cdot\beta_n!}\frac{\partial^{\beta_1+\ldots+\beta_n}F^\alpha(z)}{\partial z_1^{\beta_1}\dots\partial z_n^{\beta_n}}X^\beta,$$

where $z,X\in\mathbb{C}^n$ and if $\alpha_j<0$ then $z_j\neq 0$.

Note that the domain D_α is always unbounded, Reinhardt, and pseudoconvex but not convex. Because of the product properties of the functions considered we can assume that $\alpha\in\mathbb{R}^n_*$.

The formulas for the Carathéodory pseudodistance and Carathéodory–Reiffen pseudometric as well as for the Green function for elementary Reinhardt domains of the rational type have been known for a long time (see [Jar-Pfl 93]) ([32]).

THEOREM 3.1. *If* $\alpha\in\mathbb{Z}^n_*$, *where* α_j*'s are relatively prime*, *then*:

$$\begin{aligned}
&c_{D_\alpha}(w,z)=p(w^\alpha,z^\alpha),\\
&\widetilde{g}_{D_\alpha}(w,z)=m(w^\alpha,z^\alpha)^{1/r},\\
&\gamma_{D_\alpha}(w;X)=\gamma(w^\alpha;(F^\alpha)'(w)X),\\
&A_{D_\alpha}(w;X)=(\gamma(w^\alpha;F^\alpha_{(r)}(w)X))^{1/r},\quad (w,z)\in D_\alpha\times D_\alpha,(w,X)\in D_\alpha\times\mathbb{C}^n,
\end{aligned}$$

where r is the order of vanishing of the function $F^\alpha(\cdot)-F^\alpha(w)$ at w. If α is of irrational type, *then*

$$c_{D_\alpha}(w,z)=0,\quad \gamma_{D_\alpha}(w;X)=0,\ (w,z)\in D\times D,\ (w,X)\in D\times\mathbb{C}^n.$$

We extend Theorem 3.1 to other invariant functions and pseudometrics and we find the remaining formulas for the Green function (and the Azukawa pseudometric) in the irrational case.

([32]) The results from [Jar-Pfl 93] were stated only for $\alpha\in\mathbb{N}^n_*$; nevertheless, the general case follows immediately. Assume that $\alpha_1,\dots,\alpha_l<0$, $\alpha_{l+1},\dots,\alpha_n>0$. The mapping $z\mapsto(1/z_1,\dots,1/z_l,z_{l+1},\dots,z_n)$ maps biholomorphically D_α onto a domain $D_{(-\alpha_1,\dots,-\alpha_l,\alpha_{l+1},\dots,\alpha_n)}\cap(\mathbb{C}^l_*\times\mathbb{C}^{n-l})$. Making use of (1.1.8) and (1.1.9) we reduce the problem to the case $\alpha\in\mathbb{N}^n_*$.

We assume in this chapter that

$$\alpha_1, \dots, \alpha_l < 0, \quad \alpha_{l+1}, \dots, \alpha_n > 0, \qquad \text{where } l \in \{0, \dots, n\}.$$

3.1. Elementary Reinhardt domains with $l < n$. First we deal with elementary Reinhardt domains not contained in $\mathbb{C}^n_*$, in other words such that $l < n$. For clarity we formulate the results separately for the rational and the irrational case. The case $l = 0$ was done in [Pfl-Zwo 98]. The general case goes exactly along the same lines.

THEOREM 3.1.1. *Assume that $l < n$. Let $(w, z) \in D_\alpha \times D_\alpha$, $(w; X) \in D_\alpha \times \mathbb{C}^n$. Let $\mathcal{J} := \{j \in \{1, \dots, n\} : w_j = 0\} = \{j_1, \dots, j_k\}$ (33). Let also $\widetilde{\alpha}_{l+1} := \min\{\alpha_{l+1}, \dots, \alpha_n\}$.*

1. *Assume that $\alpha \in \mathbb{Z}^n_*$ with α_j's relatively prime. Then*

$$\widetilde{k}_{D_\alpha}(w, z) = \begin{cases} \min\{p((w^\alpha)^{1/\widetilde{\alpha}_{l+1}}, (z^\alpha)^{1/\widetilde{\alpha}_{l+1}})\} & \text{if } w, z \in \widetilde{D}_\alpha, \\ p(0, |z^\alpha|^{1/(\alpha_{j_1} + \dots + \alpha_{j_k})}) & \text{if } \mathcal{J} \neq \emptyset, \end{cases}$$

$$k_{D_\alpha}(w, z) = \min\{p((w^\alpha)^{1/\widetilde{\alpha}_{l+1}}, (z^\alpha)^{1/\widetilde{\alpha}_{l+1}})\},$$

where the minima are taken over all possible roots. In the infinitesimal case we have

$$\kappa_{D_\alpha}(w; X) = \begin{cases} \gamma\Big((w^\alpha)^{1/\widetilde{\alpha}_{l+1}}, (w^\alpha)^{1/\widetilde{\alpha}_{l+1}} \dfrac{1}{\widetilde{\alpha}_{l+1}} \displaystyle\sum_{j=1}^{n} \dfrac{\alpha_j X_j}{w_j}\Big) & \text{if } \mathcal{J} = \emptyset, \\ (|w_1|^{\alpha_1} \dots |X_{j_1}|^{\alpha_{j_1}} \dots |X_{j_k}|^{\alpha_{j_k}} \dots |w_n|^{\alpha_n})^{1/(\alpha_{j_1} + \dots + \alpha_{j_k})} & \text{if } \mathcal{J} \neq \emptyset. \end{cases}$$

2. *Assume that α is of irrational type. Then*

$$\widetilde{k}_{D_\alpha}(w, z) = \begin{cases} p((|w_1|^{\alpha_1} \dots |w_n|^{\alpha_n})^{1/\widetilde{\alpha}_{l+1}}, (|z_1|^{\alpha_1} \dots |z_n|^{\alpha_n})^{1/\widetilde{\alpha}_{l+1}}) & \text{if } w, z \in \widetilde{D}_\alpha, \\ p(0, (|z_1|^{\alpha_1} \dots |z_n|^{\alpha_n})^{1/(\alpha_{j_1} + \dots + \alpha_{j_k})}) & \text{if } \mathcal{J} \neq \emptyset. \end{cases}$$

$$k_{D_\alpha}(w, z) = p\Big(\Big(\prod_{j=1}^{n} |w_j|^{\alpha_j}\Big)^{1/\widetilde{\alpha}_{l+1}}, \Big(\prod_{j=1}^{n} |z_j|^{\alpha_j}\Big)^{1/\widetilde{\alpha}_{l+1}}\Big),$$

$$\widetilde{g}_{D_\alpha}(w, z) = \begin{cases} 0 & \text{if } \mathcal{J} = \emptyset, \\ (|z_1|^{\alpha_1} \dots |z_n|^{\alpha_n})^{1/(\alpha_{j_1} + \dots + \alpha_{j_k})} & \text{if } \mathcal{J} \neq \emptyset. \end{cases}$$

In the infinitesimal case we have

$$\kappa_{D_\alpha}(w; X) = \begin{cases} \gamma\Big(\Big(\displaystyle\prod_{j=1}^{n} |w_j|^{\alpha_j}\Big)^{1/\widetilde{\alpha}_{l+1}}, \Big(\displaystyle\prod_{j=1}^{n} |w_j|^{\alpha_j}\Big)^{1/\widetilde{\alpha}_{l+1}} \dfrac{1}{\widetilde{\alpha}_{l+1}} \displaystyle\sum_{j=1}^{n} \dfrac{\alpha_j X_j}{w_j}\Big) & \text{if } \mathcal{J} = \emptyset, \\ (|w_1|^{\alpha_1} \dots |X_{j_1}|^{\alpha_{j_1}} \dots |X_{j_k}|^{\alpha_{j_k}} \dots |w_n|^{\alpha_n})^{1/(\alpha_{j_1} + \dots + \alpha_{j_k})} & \text{if } \mathcal{J} \neq \emptyset, \end{cases}$$

$$A_{D_\alpha}(w; X) = \begin{cases} 0 & \text{if } \mathcal{J} = \emptyset, \\ (|w_1|^{\alpha_1} \dots |X_{j_1}|^{\alpha_{j_1}} \dots |X_{j_k}|^{\alpha_{j_k}} \dots |w_n|^{\alpha_n})^{1/(\alpha_{j_1} + \dots + \alpha_{j_k})} & \text{if } \mathcal{J} \neq \emptyset. \end{cases}$$

The proof is tedious and long. The details are in Sections 3.2–3.4. The remaining case $l = n$, which is much simpler, will be considered in Section 3.5.

(33) Clearly, $\mathcal{J} \subset \{l + 1, \dots, n\}$.

3.2. Auxiliary results. For $z\in\mathbb{C}^n$ put

$$T_z:=\{(e^{i\theta_1}z_1,\dots,e^{i\theta_n}z_n):\theta_j\in\mathbb{R}\}.$$

Note that T_z is a group with multiplication defined as follows:

$$(e^{i\theta_1}z_1,\dots,e^{i\theta_n}z_n)\circ(e^{i\widetilde\theta_1}z_1,\dots,e^{i\widetilde\theta_n}z_n):=(e^{i(\theta_1+\widetilde\theta_1)}z_1,\dots,e^{i(\theta_n+\widetilde\theta_n)}z_n).$$

Define $T_{z,\alpha}$ to be the subgroup of T_z generated by the set

$$\{(e^{i(\alpha_{j_1}/\alpha_1)2k_1\pi}z_1,\dots,e^{i(\alpha_{j_n}/\alpha_n)2k_n\pi}z_n):j_1,\dots,j_n\in\{1,\dots,n\},\ k_1,\dots,k_n\in\mathbb{Z}\}.$$

If α is of rational type, then $T_{z,\alpha}$ is finite; more precisely, if we assume that $\alpha\in\mathbb{Z}_*^n$ and α_j's are relatively prime, then

$$T_{z,\alpha}=\{(\varepsilon_1z_1,\dots,\varepsilon_nz_n):\varepsilon_j^{\alpha_j}=1\}.$$

However, if α is of irrational type, then the one-dimensional version of Kronecker Theorem gives

$$\overline{T}_{z,\alpha}=T_z. \tag{3.2.1}$$

For $\mu\in E_*$ we define

$$\Phi_\mu:\mathbb{C}^{n-1}\ni(\lambda_1,\dots,\lambda_{n-1})\mapsto(e^{\alpha_n\lambda_1},\dots,e^{\alpha_n\lambda_{n-1}},\mu e^{-\alpha_1\lambda_1}\dots e^{-\alpha_{n-1}\lambda_{n-1}})\in D_\alpha.$$

Put

$$V_\mu:=\Phi_\mu(\mathbb{C}^{n-1}),\quad \mu\in E_*,\quad V_0:=\{z\in D_\alpha:z_1\dots z_n=0\}.$$

Note that $\bigcup_{\mu\in E}V_\mu=D_\alpha$.

REMARK 3.2.1. Let $\mu\in E_*$. Assume $w,z\in V_\mu$ and $X\in\mathbb{C}^n$ satisfy $\sum_{j=1}^n(\alpha_jX_j/w_j)=0$. Then

$$\widetilde{k}^*_{D_\alpha}(w,z)=0,\quad \kappa_{D_\alpha}(w;X)=0.$$

In fact, $w=\Phi_\mu(\lambda)$, $z=\Phi_\mu(\gamma)$ for some $\lambda,\gamma\in\mathbb{C}^{n-1}$, so

$$\widetilde{k}^*_{D_\alpha}(w,z)=\widetilde{k}^*_{D_\alpha}(\Phi_\mu(\lambda),\Phi_\mu(\gamma))\le\widetilde{k}^*_{\mathbb{C}^{n-1}}(\lambda,\gamma)=0.$$

For the second equality note that assuming $\Phi_\mu(\lambda)=w$ we have

$$\Phi'_\mu(\lambda)(Y)=\Big[\alpha_nw_1Y_1,\dots,\alpha_nw_{n-1}Y_{n-1},-\sum_{j=1}^{n-1}\alpha_jw_nY_j\Big],\quad Y\in\mathbb{C}^{n-1}.$$

One may easily verify that

$$\Phi'_\mu(\lambda)(\mathbb{C}^{n-1})=\Big\{X\in\mathbb{C}^n:\sum_{j=1}^n\frac{\alpha_jX_j}{w_j}=0\Big\}.$$

Note that

$$0=\kappa_{\mathbb{C}^{n-1}}(\lambda;Y)\ge\kappa_{D_\alpha}(\Phi_\mu(\lambda),\Phi'_\mu(\lambda)Y),\quad Y\in\mathbb{C}^{n-1},$$

which finishes the proof.

In the proof of Lemma 3.2.2 below we replace E in the definition of the Lempert function with $H:=\{x+iy:1>x>-1\}$([34]).

([34]) And then we replace p with $\widetilde{k}_H=k_H$.

LEMMA 3.2.2. *Fix* $w, z \in D_\alpha$. *Take any* $\widetilde{z} \in T_{z,\alpha}$. *Then for any* $\varphi \in \mathcal{O}(E, D_\alpha)$ *such that* $\varphi(\lambda_1) = w$, $\varphi(\lambda_2) = z$, $\lambda_1 \neq \lambda_2$ *there is* $\widetilde{\varphi} \in \mathcal{O}(E, D_\alpha)$ *such that* $\widetilde{\varphi}(\lambda_1) = w$ *and* $\widetilde{\varphi}(\lambda_2) = \widetilde{z}$. *Consequently*,

$$\widetilde{k}^*_{D_\alpha}(w, z) = \widetilde{k}^*_{D_\alpha}(w, \widetilde{z}) \quad \textit{for any } \widetilde{z} \in T_{z,\alpha}.$$

PROOF. Take any mapping $\varphi \in \mathcal{O}(H, D_\alpha)$ with $\varphi(0) = w$, $\varphi(it) = z$, $t > 0$.

Define (for $k_n \in \mathbb{Z}$ fixed)

$$\widetilde{\varphi} : H \ni \lambda \mapsto (\varphi_1(\lambda), \dots, \varphi_{n-2}(\lambda), e^{-2k_n\pi\lambda/t}\varphi_{n-1}(\lambda), e^{\alpha_{n-1}2k_n\pi\lambda/(\alpha_n t)}\varphi_n(\lambda)) \in D_\alpha.$$

We have

$$\widetilde{\varphi}(0) = w, \quad \widetilde{\varphi}(it) = (z_1, \dots, z_{n-1}, e^{i(\alpha_{n-1}/\alpha_n)2k_n\pi} z_n).$$

We may replace α_{n-1} above with any other α_j and z_n with $e^{i(\alpha_j/\alpha_n)2k_n\pi} z_n$, and we may continue the procedure with the next components z_j varying, which finishes the proof. ■

REMARK 3.2.3. From the proof of Lemma 3.2.2 we also have the following property:

Fix $\alpha \in \mathbb{Z}^n_*$ with α_j's relatively prime and $0 < \delta_1 \leq m(\lambda_1, \lambda_2) \leq \delta_2 < 1$. Take any $\psi \in \mathcal{O}(E, \mathbb{C}^n)$ with $\psi(E) \subset\subset \mathbb{C}^n_*$ and choose $z \in \mathbb{C}^n_*$ such that $z_j^{\alpha_j} = \psi_j^{\alpha_j}(\lambda_2)$, for $j = 1, \dots, n$. Then there is $\widetilde{\psi} \in \mathcal{O}(E, \mathbb{C}^n)$ such that $\widetilde{\psi}(E) \subset\subset \mathbb{C}^n_*$, $\psi(\lambda_1) = \widetilde{\psi}(\lambda_1)$, $\widetilde{\psi}(\lambda_2) = z$ and

$$\psi_1^{\alpha_1}(\lambda) \dots \psi_n^{\alpha_n}(\lambda) = \widetilde{\psi}_1^{\alpha_1}(\lambda) \dots \widetilde{\psi}_n^{\alpha_n}(\lambda), \quad \lambda \in E,$$
$$m\|\psi_j\|_E \leq \|\widetilde{\psi}_j\|_E \leq M\|\psi_j\|_E, \quad j = 1, \dots, n,$$

where $m, M > 0$ depend only on δ_1 and α.

LEMMA 3.2.4. *Fix* $L_1^1, L_1^2 \subset\subset E$, $L_2 \subset\subset \mathbb{C}_*$ *and* $\alpha \in \mathbb{R}^n_*$. *Assume that there is* $\delta > 0$ *such that for any* $\lambda_1 \in L_1^1$, $\lambda_2 \in L_1^2$ *we have* $m(\lambda_1, \lambda_2) \geq \delta$. *Then there is* $L_2 \subset K \subset\subset \mathbb{C}_*$ *such that for any* $z_1, z_2 \in L_2$ *and any* $\lambda_1 \in L_1^1$, $\lambda_2 \in L_1^2$ *there is* $\psi \in \mathcal{O}(E, \mathbb{C}_*)$ *with* $\psi(\lambda_j) = z_j$, $j = 1, 2$, *and* $\psi(E) \subset K$. *Moreover, there is* $\widetilde{K} \subset\subset \mathbb{C}_*$ *such that for any* $z_1, \dots, z_n \in L_2$, $w_1, \dots, w_k \in L_2$, $k < n$ *with*

$$|z_1|^{\alpha_1} \dots |z_n|^{\alpha_n} = 1$$

there are functions $\psi_j \in \mathcal{O}(E, \mathbb{C}_*)$ with $\psi_j(E) \subset \widetilde{K}$, $j = 1, \dots, n$,

$$\psi_1^{\alpha_1}(\lambda) \dots \psi_n^{\alpha_n}(\lambda) = e^{i\theta}, \quad \lambda \in E,$$
$$\psi_j(\lambda_1) = z_j, \quad j = 1, \dots, n, \quad \psi_j(\lambda_2) = w_j, \quad j = 1, \dots, k.$$

PROOF. For the first part it is sufficient to consider $L_1^1 = \{\lambda_1\}$, $L_1^2 = \{\lambda_2\}$ with $m(\lambda_1, \lambda_2) = \delta$. The general case is then obtained by composing the functions with automorphisms of E and the dilatation $R\lambda$, where $0 \leq R < 1$, since the images of new functions are contained in those of the original ones.

Define

$$L := \exp^{-1}(L_2) \cap (\mathbb{R} \times [0, 2\pi)) \subset\subset \mathbb{C}.$$

Now put

$$K := \{\exp(h(\lambda)) : \lambda \in E,\ h(\lambda) = a\lambda + b,\ a, b \in \mathbb{C},\ h(\lambda_1) = \widetilde{z}_1,\ h(\lambda_2) = \widetilde{z}_2,\ \widetilde{z}_1, \widetilde{z}_2 \in L\}.$$

Observe that $K \subset\subset \mathbb{C}_*$. The mappings we are looking for are $\exp \circ h$, where h appears in the definition of K.

For the second part of the lemma we set w_j for $j = k+1,\dots,n-1$ to be any number from L_2 and we take mappings $\psi_1,\dots,\psi_{n-1}$ as in the first part. Define

$$\psi_n(\lambda) := \frac{e^{i\widetilde{\theta}}}{(\psi_1^{\alpha_1}(\lambda)\dots\psi_{n-1}^{\alpha_{n-1}}(\lambda))^{1/\alpha_n}},$$

where the branches of powers are chosen arbitrarily and $\widetilde{\theta} \in \mathbb{R}$ is such that $\psi_n(\lambda_1) = z_n$. ■

LEMMA 3.2.5. *Let $L_1^1, L_1^2, L_2, \delta$ be as in Lemma 3.2.4. Fix $\alpha \in \mathbb{Z}_*^n$, where α_j's are relatively prime. Then there is $K \subset\subset \mathbb{C}_*$ such that for any $\psi_j \in \mathcal{O}(E,\mathbb{C}_*)$, $j = 1,\dots,n$, with*

$$\psi_1^{\alpha_1}\dots\psi_n^{\alpha_n} = 1, \quad \lambda \in E,$$

and $\psi_j(\lambda_1), \psi_j(\lambda_2) \in L_2$, where $\lambda_1 \in L_1^1$, $\lambda_2 \in L_1^2$ there are $\widetilde{\psi}_j \in \mathcal{O}(E,\mathbb{C}_*)$ such that

$$\widetilde{\psi}_1^{\alpha_1}\dots\widetilde{\psi}_n^{\alpha_n} = 1, \quad \lambda \in E,$$
$$\widetilde{\psi}_j(\lambda_1) = \psi_j(\lambda_1), \quad \widetilde{\psi}_j(\lambda_2) = \psi_j(\lambda_2), \quad \widetilde{\psi}_j(E) \subset K, \quad j = 1,\dots,n.$$

PROOF. Put $z_j := \psi_j(\lambda_1)$, $w_j := \psi_j(\lambda_2)$, $j = 1,\dots,n$. From Lemma 3.2.4 there are $\widetilde{\psi}_j$, $j = 1,\dots,n-1$, as desired. Put

$$\widetilde{\psi}_n(\lambda) := \frac{1}{(\widetilde{\psi}_1^{\alpha_1}(\lambda)\cdot\ldots\cdot\widetilde{\psi}_{n-1}^{\alpha_{n-1}}(\lambda))^{1/\alpha_n}}$$

We choose the branch of the power $1/\alpha_n$ so that $\widetilde{\psi}_n(\lambda_1) = z_n$, note also that $\widetilde{\psi}_n^{\alpha_n}(\lambda_2) = w_n^{\alpha_n}$. From Remark 3.2.3 we may change $\widetilde{\psi} := (\widetilde{\psi}_1,\dots,\widetilde{\psi}_n)$ so that all the desired properties are preserved and, additionally, $\widetilde{\psi}_n(\lambda_2) = w_n$. ■

Now we present a lemma which is a weaker infinitesimal version of Lemma 3.2.4.

LEMMA 3.2.6. *Let $w \in \mathbb{C}_*$, $X \in \mathbb{C}$ and $\lambda_1 \in E$. Then there is $\psi \in \mathcal{O}(E,\mathbb{C}_*)$ such that*

$$\psi(\lambda_1) = w, \quad \psi'(\lambda_1) = X.$$

Moreover, for $w_1,\dots,w_n \in \mathbb{C}_*$, $X_1,\dots,X_k \in \mathbb{C}$ $(k < n)$ and $\alpha \in \mathbb{R}_*^n$, where $|w_1|^{\alpha_1}\cdot\ldots\cdot|w_n|^{\alpha_n} = 1$, there are $\psi_j \in \mathcal{O}(E,\mathbb{C}_*)$, $j = 1,\dots,n$, such that

$$\psi_j(\lambda_1) = w_j, \quad j = 1,\dots,n, \quad \psi_j'(\lambda_1) = X_j, \quad j = 1,\dots,k,$$
$$\psi_1^{\alpha_1}(\lambda)\dots\psi_n^{\alpha_n}(\lambda) = e^{i\theta}, \quad \lambda \in E.$$

PROOF. The first part is similar to the proof of Lemma 3.2.4 (note that we do not demand so much about the mapping ψ as in Lemma 3.2.4). The mapping we are looking for is of the form $\exp(a\lambda + b)$.

For the second part let X_j $(j = k+1,\dots,n-1)$ be any complex number. Take ψ_j as in the first part (for $j = 1,\dots,n-1$) with w replaced with w_j and X replaced with X_j. Put

$$\psi_n(\lambda) := \frac{e^{i\widetilde{\theta}}}{(\psi_1^{\alpha_1}(\lambda)\cdot\ldots\cdot\psi_{n-1}^{\alpha_{n-1}}(\lambda))^{1/\alpha_n}},$$

where the branches of powers are chosen arbitrarily and $\widetilde{\theta} \in \mathbb{R}$ is such that $\psi_n(\lambda_1) = w_n$. ■

Now we are able to give formulas for the Lempert function and the Kobayashi–Royden metric for special points.

LEMMA 3.2.7. *Fix $w \in V_0$. Let $z \in D_\alpha$ and $X \in \mathbb{C}^n$, $\alpha_1, \dots, \alpha_l < 0$, $\alpha_{l+1}, \dots, \alpha_n > 0$. Then*

$$\widetilde{k}^*_{D_\alpha}(w,z) = (|z_1|^{\alpha_1} \dots |z_n|^{\alpha_n})^{1/(\alpha_{j_1}+\dots+\alpha_{j_k})},$$
$$\kappa_{D_\alpha}(w;X) = (|w_1|^{\alpha_1} \dots |X_{j_1}|^{\alpha_{j_1}} \dots |X_{j_k}|^{\alpha_{j_k}} \cdot \dots \cdot |w_n|^{\alpha_n})^{1/(\alpha_{j_1}+\dots+\alpha_{j_k})},$$

where $\mathcal{J} := \{j \in \{1,\dots,n\} : w_j = 0\} = \{j_1, \dots, j_k\}$.

PROOF. We may assume that

$$w_{n-k+1} = \dots = w_n = 0, \quad w_1, \dots, w_{n-k} \neq 0, \quad n-k \geq l.$$

We prove both equalities simultaneously.

First we consider the case $z \in \widetilde{D}_\alpha$ (respectively, $X_j \neq 0$ for all $j = n-k+1, \dots, n$). Take any $\varphi \in \mathcal{O}(\bar{E}, D_\alpha)$ such that $\varphi(0) = w$, $\varphi(t) = z$ (respectively, $\varphi(0) = w$, $t\varphi'(0) = X$) for some $t > 0$. We have

$$\varphi(\lambda) = (\psi_1(\lambda), \dots, \psi_{n-k}(\lambda), \lambda\psi_{n-k+1}(\lambda), \dots, \lambda\psi_n(\lambda)), \quad \psi_j \in \mathcal{O}(\bar{E}, \mathbb{C}), \quad j = 1, \dots, n.$$

Put

$$u(\lambda) := \prod_{j=1}^n |\psi_j(\lambda)|^{\alpha_j}.$$

We know that $\log u \in \mathrm{SH}(\bar{E})$ and $u \leq 1$ on ∂E, so the maximum principle for subharmonic functions implies that $u \leq 1$ on E. In particular, $u(t) \leq 1$ (respectively, $u(0) \leq 1$), so

$$\frac{\prod_{j=1}^n |z_j|^{\alpha_j}}{t^{\alpha_{n-k+1}+\dots+\alpha_n}} \leq 1 \quad \left(\text{respectively, } \frac{\prod_{j=n-k+1}^n |X_j|^{\alpha_j} \prod_{j=1}^{n-k} |w_j|^{\alpha_j}}{t^{\alpha_{n-k+1}+\dots+\alpha_n}} \leq 1\right),$$

which gives

$$t \geq \Big(\prod_{j=1}^n |z_j|^{\alpha_j}\Big)^{\frac{1}{\alpha_{n-k+1}+\dots+\alpha_n}}$$

$$\left(\text{respectively, } t \geq \Big(\prod_{j=n-k+1}^n |X_j|^{\alpha_j} \prod_{j=1}^{n-k} |w_j|^{\alpha_j}\Big)^{\frac{1}{\alpha_{n-k+1}+\dots+\alpha_n}}\right).$$

Therefore,

$$\widetilde{k}^*_{D_\alpha}(w,z) \geq \Big(\prod_{j=1}^n |z_j|^{\alpha_j}\Big)^{\frac{1}{\alpha_{n-k+1}+\dots+\alpha_n}}$$

$$\left(\text{respectively, } \kappa_{D_\alpha}(w;X) \geq \Big(\prod_{j=n-k+1}^n |X_j|^{\alpha_j} \prod_{j=1}^{n-k} |w_j|^{\alpha_j}\Big)^{\frac{1}{\alpha_{n-k+1}+\dots+\alpha_n}}\right).$$

To get equality put

$$t := \Big(\prod_{j=1}^n |z_j|^{\alpha_j}\Big)^{\frac{1}{\alpha_{n-k+1}+\dots+\alpha_n}}$$

$$\left(\text{respectively, } t := \Big(\prod_{j=n-k+1}^n |X_j|^{\alpha_j} \prod_{j=1}^{n-k} |w_j|^{\alpha_j}\Big)^{\frac{1}{\alpha_{n-k+1}+\dots+\alpha_n}}\right)$$

and consider the mapping

$$\varphi(\lambda) := (\psi_1(\lambda), \dots, \psi_{n-k}(\lambda), \lambda\psi_{n-k+1}(\lambda), \dots, \lambda\psi_n(\lambda)), \quad \lambda \in E,$$

where $\psi_j \in \mathcal{O}(E, \mathbb{C}_*)$, $j = 1, \dots, n$, $\prod_{j=1}^n \psi_j(\lambda)^{\alpha_j} = e^{i\theta}$ on E and

$$\begin{aligned} &\psi_j(t) = z_j/t, \quad j = n-k+1, \dots, n, \quad \psi_j(t) = z_j, \quad j = 1, \dots, n-k;\\ &\psi_j(0) = w_j, \qquad j = 1, \dots, n-k \quad \text{(see Lemma 3.2.4)}, \end{aligned}$$

(respectively,

$$\begin{aligned} &\psi_j(0) = X_j/t, \quad j = n-k+1, \dots, n, \quad \psi_j(0) = w_j, \quad j = 1, \dots, n-k,\\ &\psi_j'(0) = X_j/t, \quad j = 1, \dots, n-k, \end{aligned}$$

see Lemma 3.2.6).

Then $\varphi \in \mathcal{O}(E, D_\alpha)$, $\varphi(0) = w$, $\varphi(t) = z$ (respectively, $t\varphi'(0) = X$), which finishes this case.

We are left with the case $z \in V_0$ (respectively, $X_j = 0$ for some $n-k+1 \le j \le n$). If there is j such that $w_j = z_j = 0$ (respectively, $w_j = X_j = 0$), then the mapping (note that $j \ge l+1$)

$$\mathbb{C}_*^l \times \mathbb{C}^{n-l-1} \ni (z_1, \dots, \check{z}_j, \dots, z_n) \mapsto (z_1, \dots, 0, \dots, z_n) \in D_\alpha$$

gives

$$0 = \widetilde{k}^*_{\mathbb{C}_*^l \times \mathbb{C}^{n-l-1}}((w_1, \dots, \check{w}_j, \dots, w_n), (z_1, \dots, \check{z}_j, \dots, z_n)) \ge \widetilde{k}^*_{D_\alpha}(w, z),$$

(respectively,

$$0 = \kappa_{\mathbb{C}_*^{l-1} \times \mathbb{C}^{n-l-1}}((w_1, \dots, \check{w}_j, \dots, w_n); (X_1, \dots, \check{X}_j, \dots, X_n)) \ge \kappa_{D_\alpha}(w; X)).$$

Therefore, only the Lempert function remains and then we may assume that for all j we have $|w_j| + |z_j| > 0$.

For fixed $1 > \beta > 0$ define the mapping $\varphi := (\varphi_1, \dots, \varphi_n)$ as follows (the first and the second case below may occur only if $j \ge l+1$):

$$\varphi_j(\lambda) := \begin{cases} \dfrac{\lambda - \beta}{1 - \beta\lambda}\psi_j(\lambda) & \text{if } w_j = 0,\\ \dfrac{\lambda + \beta}{1 + \beta\lambda}\psi_j(\lambda) & \text{if } z_j = 0,\\ \psi_j(\lambda) & \text{if } w_j z_j \ne 0, \end{cases}$$

where $\psi_j \in \mathcal{O}(E, \mathbb{C}_*)$, $\prod_{j=1}^n \psi_j(\lambda)^{\alpha_j} = e^{i\theta}$ on E and $\varphi(\beta) = w$, $\varphi(-\beta) = z$ (the values of $\psi_j(\beta)$ and $\psi_j(-\beta)$ are prescribed only if $w_j z_j \ne 0$; for those j for which $w_j z_j = 0$ only one of the values $\psi_j(\beta)$ and $\psi_j(-\beta)$ is prescribed; more precisely, take j_1 such that $z_{j_1} = 0$, and define $\psi_{j_1}(-\beta)$ so that $|\psi_1(-\beta)|^{\alpha_1} \dots |\psi_n(-\beta)|^{\alpha_n} = 1$; there is j_2 such that $w_{j_2} = 0$, so $\psi_{j_2}(\beta)$ has no fixed value—that is why we can use Lemma 3.2.4). Note also that $\varphi \in \mathcal{O}(E, D_\alpha)$. As $\beta > 0$ may be chosen arbitrarily small this completes the proof. ∎

Next, we prove a formula for the Lempert function for the domain $D_{(1,\dots,1)}$. Following the ideas from [Jar-Pfl-Zei 93] and [Pfl-Zwo 96] we extend the formulas to the general case using what could be called transport of geodesics. Roughly speaking, the idea is to transport the formulas from simpler domains to more complex ones with the help of "good" mappings. In [Jar-Pfl-Zei 93] and [Pfl-Zwo 96] the Euclidean ball was a model domain. In our case it is the domain $D_{(1,\dots,1)}$.

LEMMA 3.2.8. *If* $w,z\in V_0$, *then* $\widetilde{k}^*_{D_{(1,\dots,1)}}(w,z)=0$. *Assume that* $w\in\widetilde{D}_{(1,\dots,1)}$. *Then*

$$\widetilde{k}^*_{D_{(1,\dots,1)}}(w,z)=m(w_1\dots w_n,z_1\dots z_n)^{1/k},$$

where $k:=\max\{\#\{j:z_j=0\},1\}$.

PROOF. The first part and the case $z\in V_0$ are consequences of Lemma 3.2.7.

Consider the case $w,z\in\widetilde{D}_{(1,\dots,1)}$. We assume that $w_1\dots w_n\neq z_1\dots z_n$ (otherwise use Remark 3.2.1).

Consider the mapping (see Lemma 3.2.4)

$$\varphi(\lambda):=(\psi_1(\lambda),\dots,\psi_{n-1}(\lambda),e^{-i\theta}\lambda\psi_n(\lambda)),$$

where

$$\lambda_1:=w_1\dots w_n,\quad \lambda_2:=z_1\dots z_n,$$
$$\psi_j\in\mathcal{O}(E,\mathbb{C}_*),\quad j=1,\dots,n,\quad \psi_1(\lambda)\dots\psi_n(\lambda)=e^{i\theta},\quad \lambda\in E,$$
$$\psi_j(\lambda_1)=w_j,\quad \psi_j(\lambda_2)=z_j,\quad j=1,\dots,n-1$$

(using Lemma 3.2.4 we may even assume that $\psi_j(E)\subset K\subset\subset\mathbb{C}_*$, $j=1,\dots,n$; compare Remark 3.2.9).

Note that

$$\varphi\in\mathcal{O}(E,D_{(1,\dots,1)}),\quad \varphi(\lambda_1)=w,\quad \varphi(\lambda_2)=z.$$

Combining this with the contractivity property of the Lempert function we have

$$m(w_1\dots w_n,z_1\dots z_n)\geq\widetilde{k}^*_{D(1,\dots,1)}(w,z)\geq m(w_1\dots w_n,z_1\dots z_n).$$

This completes the proof. ∎

REMARK 3.2.9. From the proof of Lemma 3.2.8 we see that for any $w,z\in\widetilde{D}_{(1,\dots,1)}$ with $w_1\dots w_n\neq z_1\dots z_n$ there is a $\widetilde{k}_{D_{(1,\dots,1)}}$-geodesic for (w,z) of the form

$$\left(\psi_1(\lambda),\dots,\psi_{n-1}(\lambda),e^{i\theta}\frac{\lambda-\beta}{1-\overline{\beta}\lambda}\psi_n(\lambda)\right)$$

with $\psi_1(\lambda)\dots\psi_n(\lambda)=1$ and $\psi_j(E)\subset\subset\mathbb{C}_*$.

The domains D_α, although very regular, do not have a property which is crucial in the theory of holomorphically invariant functions: they are not taut. Therefore, we have no guarantee that they admit $\widetilde{k}_{D_\alpha}$-geodesics. However, as Lemma 3.2.10 will show, they do admit them at least in the rational case and for points which are "separated" by the Lempert function. The existence of geodesics will play a great role in the proof of the formula for the Lempert function in the rational case.

LEMMA 3.2.10. *Assume that* $\alpha\in\mathbb{Z}^n_*$ *and* α_j*'s are relatively prime*, $\alpha_1,\dots,\alpha_l<0$, $\alpha_{l+1},\dots,\alpha_n>0$, $l<n$. *Let* $w,z\in\widetilde{D}_\alpha$ *with* $w^\alpha\neq z^\alpha$. *Then there is a* $\widetilde{k}_{D_\alpha}$*-geodesic* φ *for* (w,z) *such that* $\varphi_j=B_j\psi_j$, $j=1,\dots,n$, *where* B_j *is a Blaschke product* (*up to a constant*) $j=1,\dots,n$, B_j *is constant for* $j=1,\dots,l$ *and* $\psi_1^{\alpha_1}\dots\psi_n^{\alpha_n}=1$ *on* E. *Moreover*, $\psi_j(E)\subset\subset\mathbb{C}_*$, $j=1,\dots,n$.

Proof. We know that $t := \widetilde{k}^*_{D_\alpha}(w,z) \geq m(w^\alpha, z^\alpha) > 0$; consequently, there are mappings $\varphi^{(k)} = (\varphi_1^{(k)}, \dots, \varphi_n^{(k)})$, $k = 1, 2, \dots$, such that

$$\varphi^{(k)} \in \mathcal{O}(\bar{E}, D_\alpha), \quad \varphi^{(k)}(0) = w, \quad \varphi^{(k)}(t_k) = z, \quad \text{where } t_k \geq t_{k+1} \to t > 0.$$

We have

$$\varphi_j^{(k)} = B_j^{(k)} \psi_j^{(k)}, \quad j = 1, \dots, n,$$

where $B_j^{(k)}$ is a Blaschke product and $\psi_j^{(k)} \in \mathcal{O}(E, \mathbb{C}_*)$ ($B_j^{(k)}$ is constant for $j = 1, \dots, l$).

Put $\psi^{(k)} := (\psi_j^{(k)})_{j=1}^n$. There are two possibilities (due to the maximum principle for subharmonic functions—remember the pseudoconvexity of D_α):

$$\psi^{(k)}(E) \subset D_\alpha, \tag{3.2.2}$$

$$\psi^{(k)}(E) \subset \partial D_\alpha. \tag{3.2.3}$$

Below we prove that we may restrict our attention to a case which is a generalization of (3.2.3).

Take any k such that (3.2.2) is satisfied. First, notice that the mapping $\widetilde{\psi}^{(k)} := ((\psi_1^{(k)})^{\alpha_1/|\alpha_1 \dots \alpha_n|}, \dots, (\psi_n^{(k)})^{\alpha_n/|\alpha_1 \dots \alpha_n|})$ is in $\mathcal{O}(E, D_{(1,\dots,1)})$. From Remark 3.2.9 there is a $\widetilde{k}_{D_{(1,\dots,1)}}$-geodesic for $(\widetilde{\psi}^{(k)}(0), \widetilde{\psi}^{(k)}(t_k))$ of the form

$$\mu^{(k)} := \left(\widehat{\psi}_1^{(k)}, \dots, \widehat{\psi}_{n-1}^{(k)}, e^{i\theta_k} \frac{\lambda - \beta_k}{1 - \overline{\beta}_k \lambda} \widehat{\psi}_n^{(k)}\right),$$

where $\widehat{\psi}_1^{(k)} \dots \widehat{\psi}_n^{(k)} = 1$ on E, such that $\mu^{(k)}(0) = \widetilde{\psi}^{(k)}(0)$ and $\mu^{(k)}(R_k t_k) = \widetilde{\psi}^{(k)}(t_k)$, $\beta_k \in E$, $R_k \leq 1$.

Coming back to the domain D_α we see that instead of considering $\varphi^{(k)}$ with the property (3.2.2) we may consider the mapping (note that $|\alpha_1 \dots \alpha_n|/\alpha_j \in \mathbb{Z}$ and $|\alpha_1 \dots \alpha_n|/\alpha_n \in \mathbb{N}$)

$$\widetilde{\varphi}^{(k)}(\lambda) := (B_j^{(k)}(\lambda)(\mu_j^{(k)})^{|\alpha_1 \dots \alpha_n|/\alpha_j}(R_k \lambda))_{j=1}^n$$

because $\widetilde{\varphi}^{(k)} \in \mathcal{O}(E, D_\alpha)$, $\widetilde{\varphi}^{(k)}(0) = w$ and $\widetilde{\varphi}^{(k)}(t_k) = z$.

Therefore we may assume that (irrespective of which case we consider, (3.2.2) or (3.2.3))

$$\varphi_j^{(k)} = B_j^{(k)} \psi_j^{(k)}, \quad j = 1, \dots, n,$$

where $(\psi_1^{(k)})^{\alpha_1} \dots (\psi_n^{(k)})^{\alpha_n} = 1$, $|B_j^{(k)}| \leq 1$, $j = 1, \dots, n$ (although $B_j^{(k)}$'s need no longer be Blaschke products) and $|B_j^{(k)}| \equiv 1$, $j = 1, \dots, l$.

Choosing a subsequence if necessary, we may assume that for all $j = 1, \dots, n$ the sequence $\{B_j^{(k)}\}_{k=1}^\infty$ converges locally uniformly on E. Keeping in mind that $\varphi^{(k)}(0) = w$ and $\varphi^{(k)}(t_k) = z$ we see that there is $K \subset\subset \mathbb{C}_*$ such that $\psi_j^{(k)}(E) \subset K$ for any j, k (we may apply Lemma 3.2.5 because $L_2 := \{\psi_j^{(k)}(t_k), \psi_j^{(k)}(0)\}_{j,k} \subset\subset \mathbb{C}_*$, which follows from convergence and boundedness of $\{B_j^{(k)}\}_{k=1}^\infty$, the fact that $w_j z_j \neq 0$, $j = 1, \dots, n$, and $(\psi_1^{(k)})^{\alpha_1} \dots (\psi_n^{(k)})^{\alpha_n} = 1$), and then choosing a subsequence if necessary we deduce that $\varphi^{(k)}$ converges to a mapping $\varphi \in \mathcal{O}(E, \bar{D}_\alpha)$ with $\varphi(E) \subset\subset \mathbb{C}^n$ and $\varphi_j(E) \subset\subset \mathbb{C}_*$ $j = 1, \dots, l$, such that $\varphi(0) = w$ and $\varphi(t) = z$. The maximum principle for subharmonic functions implies that $\varphi(E) \subset D_\alpha$. The mapping φ is a $\widetilde{k}_{D_\alpha}$-geodesic for (w, z).

Take the representation of φ:

$$\varphi_j = B_j\psi_j, \quad j = 1, \dots, n,$$

where B_j is a Blaschke product (up to a constant), B_j is constant for $j = 1, \dots, l$.

Consider two cases. In case $\psi_1^{\alpha_1} \dots \psi_n^{\alpha_n} = e^{i\theta}$ we may assume that $\psi_j(E) \subset K \subset\subset \mathbb{C}_*$ for some K by Lemma 3.2.5 (and then we may assume that $e^{i\theta} = 1$).

If $\psi_1^{\alpha_1} \dots \psi_n^{\alpha_n}$ is not constant on E, then $\widetilde{\psi} \in \mathcal{O}(E, D_{(1,\dots,1)})$ ($\widetilde{\psi}_j := \psi_j^{\alpha_j/|\alpha_1\dots\alpha_n|}$) and it is a $\widetilde{k}_{D_{(1,\dots,1)}}$-geodesic for $(\widetilde{\psi}(\lambda_1), \widetilde{\psi}(\lambda_2))$: otherwise, there would be $\widehat{\psi} \in \mathcal{O}(E, D_{(1,\dots,1)})$ such that $\widehat{\psi}(\lambda_1) = \widetilde{\psi}(\lambda_1)$, $\widehat{\psi}(\lambda_2) = \widetilde{\psi}(\lambda_2)$ and $\widehat{\psi}(E) \subset\subset D_{(1,\dots,1)} \cap (\mathbb{C}_*^l \times \mathbb{C}^{n-l})$ (see Remark 3.2.9) and taking $\widehat{\varphi}(\lambda) := (B_j(\lambda)\widehat{\psi}_j^{|\alpha_1\dots\alpha_n|/\alpha_j}(\lambda))_{j=1}^n$ we get a mapping such that $\widehat{\varphi}(\lambda_1) = \varphi(\lambda_1)$, $\widehat{\varphi}(\lambda_2) = \varphi(\lambda_2)$ and $\widehat{\varphi}(E) \subset\subset D_\alpha$, a contradiction. By Remark 3.2.9 there is a $\widetilde{k}_{D_{(1,\dots,1)}}$-geodesic μ for $(\widetilde{\psi}(\lambda_1), \widetilde{\psi}(\lambda_2)) = (\mu(\lambda_1), \mu(\lambda_2))$, where $\widehat{\psi}_1 \dots \widehat{\psi}_n = 1$ and $\widehat{\psi}_j(E)$'s are relatively compact in $\mathbb{C}_*$ ($\widehat{\psi}_j$ are nonvanishing parts of the factorization of μ_j, μ_j does not vanish for $j = 1, \dots, n-1$). Taking now $(B_j(\lambda)(\mu_j(\lambda))^{|\alpha_1\dots\alpha_n|/\alpha_j})_{j=1}^n$ instead of φ we get the desired property. ■

3.3. Proof of Theorem 3.1.1 in the rational case. We start with the Lempert function, which is basic in the calculation of other functions.

We begin with a formula for the Möbius function, which seems to be known; nevertheless, we were not able to find any references in the literature.

LEMMA 3.3.1. *Fix $\delta \geq 1$. Then for any $\lambda_1 \in (0,1)$, $\lambda_2 \in E$ we have*

$$m(\lambda_1^\delta, \lambda_2^\delta) \leq m(\lambda_1, \lambda_2),$$

where $\lambda_1^\delta \in (0,1)$ and the power λ_2^δ is appropriately chosen.

PROOF. Let $\lambda_2 = |\lambda_2|\exp(i\theta)$, $\theta \in [-\pi, \pi)$. We claim that there are $k, l \in \mathbb{Z}$ such that

$$(*) \qquad \delta(\theta + 2k\pi) + 2l\pi \in [-\pi, \pi), \quad |\theta + 2k\pi + 2l\pi/\delta| \leq |\theta|.$$

In fact, if $\delta \notin \mathbb{Q}$, then the result follows from the one-dimensional Kronecker Theorem (density of $\{\delta k \bmod \mathbb{Z} : k \in \mathbb{Z}\}$ in $[0,1)$). If $\delta = p/q$, where p and q are relatively prime, $p \geq q \geq 1$ we easily get the desired property choosing $k, l \in \mathbb{Z}$ such that $\theta + 2k\pi + 2lq\pi/p \in [-\pi/p, \pi/p) \subset [-\pi, \pi)$.

Since $m(\lambda_1, \lambda_2) \geq m(\lambda_1, |\lambda_2|\exp(i(\theta + 2k\pi + 2l\pi/\delta))$ (use $(*)$ and simple geometric properties of the Möbius distance), in order to finish the proof it is sufficient to show for $t \in (0,1]$ the following inequality (put $r = \lambda_1^\delta$, $s = |\lambda_2|^\delta$, $t = 1/\delta$ and $\delta(\theta + 2k\pi) + 2l\pi$ in place of θ):

$$m(r, s\exp(i\theta)) \leq m(r^t, s^t\exp(it\theta)), \quad r, s \in (0,1),\ \theta \in [-\pi, \pi).$$

Therefore, we shall finish the proof if we show that for any fixed $\theta \in [0, \pi)$ the function

$$f(t, \theta) := \frac{r^{2t} + s^{2t} - 2r^t s^t \cos(t\theta)}{1 + r^{2t}s^{2t} - 2r^t s^t \cos(t\theta)}, \quad t \in (0,1],$$

is decreasing with respect to t.

First, we check this for $\theta = 0$. This follows from a straightforward (but a little tedious) calculation of the derivative.

Let
$$f(t,\theta)=\frac{\varphi_1(t)+\psi(t,\theta)}{\varphi_2(t)+\psi(t,\theta)},$$
where
$$\varphi_1(t):=r^{2t}+s^{2t}-2r^ts^t,\quad \varphi_2(t):=1+r^{2t}s^{2t}-2r^ts^t,\quad \psi(t,\theta):=2r^ts^t(1-\cos(t\theta)).$$

By the monotonicity of $f(t,0)$ we get

$$(**)\qquad \varphi_1'(t)\varphi_2(t)-\varphi_1(t)\varphi_2'(t)\le 0.$$

Our aim is to show that
$$(\varphi_1'(t)+\psi_t'(t,\theta))(\varphi_2(t)+\psi(t,\theta))-(\varphi_1(t)+\psi(t,\theta))(\varphi_2'(t)+\psi_t'(t,\theta))\le 0,$$
which will follow if we prove that (use $(**)$)
$$\psi(t,\theta)(\varphi_1'(t)-\varphi_2'(t))+\psi_t'(t,\theta)(\varphi_2(t)-\varphi_1(t))\le 0.$$
The last inequality is equivalent to
$$\begin{aligned}2r^ts^t(1-\cos(t\theta))[2r^{2t}\log r+2s^{2t}\log s-2r^{2t}s^{2t}\log(rs)]\\ +2r^ts^t[\log(rs)(1-\cos(t\theta))+\theta\sin(t\theta)](1+r^{2t}s^{2t}-r^{2t}-s^{2t})\le 0\end{aligned}$$
or
$$\begin{aligned}(1-\cos(t\theta))[(1+r^{2t})(1-s^{2t})\log r+(1+s^{2t})(1-r^{2t})\log s]\\ +\theta\sin(t\theta)(1-r^{2t})(1-s^{2t})\le 0\end{aligned}$$
and then

$$(***)\qquad (1-\cos(t\theta))\left(\frac{1+r^{2t}}{1-r^{2t}}\log r+\frac{1+s^{2t}}{1-s^{2t}}\log s\right)+\theta\sin(t\theta)\le 0.$$

It is easy to check that the function
$$(0,1)\ni u\mapsto\frac{1+u^{2t}}{1-u^{2t}}\log u$$
is increasing. The left hand side of $(***)$ is not larger than
$$2(1-\cos(t\theta))\lim_{u\to 1^-}\left(\frac{1+u^{2t}}{1-u^{2t}}\log u\right)+\theta\sin(t\theta).$$
Using the l'Hospital rule we see that the last limit equals $-1/t$. Therefore, it is sufficient to show that
$$\frac{-2}{t}(1-\cos(t\theta))+\theta\sin(t\theta)\le 0$$
for any $t\in(0,1]$, $\theta\in[0,\pi]$. Fix t and denote the left hand side by $g(\theta)$. It is easy to see that
$$g'(\theta)=-\sin(t\theta)+t\theta\cos(t\theta)\le 0,\quad \theta\in[0,\pi],$$
which finishes the proof because $g(0)=0$. ∎

Proof of the formula for $\widetilde{k}^*_{D_\alpha}$ *in the rational case.* The case $w_1\dots w_n=0$ is a consequence of Lemma 3.2.7. The case $w,z\in\widetilde{D}_\alpha$, $w^\alpha=z^\alpha$ follows from Remark 3.2.1. We are left with the case $w,z\in\widetilde D_\alpha$, $w^\alpha\neq z^\alpha$. By Lemma 3.2.10 there is a $\widetilde k_{D_\alpha}$-geodesic $\varphi\in\mathcal O(E,D_\alpha)$ for $(w,z)=(\varphi(\lambda_1),\varphi(\lambda_2))$ such that

$$\varphi_j=B_j\psi_j,\quad j=1,\dots,n,$$

where B_j is a Blaschke product (up to a constant $|c_j|=1$), $\psi_j(E)\subset K\subset\subset\mathbb C_*$, $\psi_1^{\alpha_1}\dots\psi_n^{\alpha_n}=1$ and B_j is constant for $j=1,\dots,l$.

Therefore, $\varphi(E)\subset\subset\mathbb C_*^l\times\mathbb C^{n-l}$. Consequently, $\varphi(E)$ is contained in some smooth bounded pseudoconvex Reinhardt domain $G\subset D_\alpha$ which arises from the domain D_α by "cutting the ends" and "smoothing the corners". Therefore, φ is a $\widetilde k_G$-geodesic for (w,z). Using the results of [Edi 95] (35) we find that there are $h_j\in H^\infty(E,\mathbb C)$, $j=1,\dots,n$, and $\varrho:\partial E\to(0,\infty)$ such that ($f^*(\lambda)$ denotes the nontangential limit of f at λ, $\lambda\in\partial E$)

$$\frac1\lambda h_j^*(\lambda)\varphi_j^*(\lambda)=\varrho(\lambda)\alpha_j|(\varphi^*(\lambda))^\alpha|,\quad j=1,\dots,n,\ \text{for almost all }\lambda\in\partial E$$

(we easily exclude the case $(\varphi^*(\lambda))^\alpha=0$ for λ from some subset of ∂E with nonzero Lebesgue measure: use the Identity Principle for functions from Hardy spaces, see e.g. [Dur 70], [Gar 81]). Using the result of Gentili (see [Gen 87] (36)) we deduce that for some $b_j\in\mathbb R_*$, $j=1,\dots,n$, $\beta\in E$,

$$\varphi_j(\lambda)h_j(\lambda)=b_j(1-\overline\beta\lambda)(\lambda-\beta),\quad j=1,\dots,n,\ \lambda\in E,$$

where $b_j/\alpha_j=b_k/\alpha_k$, $j,k=1,\dots,n$. Consequently, we may take

$$B_j(\lambda)=c_j\Big(\frac{\lambda-\beta}{1-\overline\beta\lambda}\Big)^{r_j},\quad |c_j|=1,$$

where $r_j\in\{0,1\}$ and not all r_j's are 0. We may assume that $\beta=0$ (we then change only λ_1 and λ_2).

Now we come back to the domain D_α. We may assume (permuting the coordinates $l+1,\dots,n$ if necessary) that $r_1=\dots=r_{l+k}=0$ and $r_{l+k+1}=\dots=r_n=1$ $(0\le k\le n-l-1)$. We want to have for some $\lambda_1,\lambda_2\in E$ (we may assume that $c_j=1$—if necessary we change w and z with the help of rotations of suitable components, so the Lempert function does not change)

$$\begin{aligned}\lambda_1\psi_j(\lambda_1)&=w_j,\quad j=l+k+1,\dots,n,\quad \psi_j(\lambda_1)=w_j,\quad j=1,\dots,l+k,\\ \lambda_2\psi_j(\lambda_2)&=z_j,\quad j=l+k+1,\dots,n,\quad \psi_j(\lambda_2)=z_j,\quad j=1,\dots,l+k.\end{aligned}$$

Taking the α_jth power and multiplying the equalities we get

$$\lambda_1^{\alpha_{l+k+1}+\dots+\alpha_n}=w^\alpha,\quad \lambda_2^{\alpha_{l+k+1}+\dots+\alpha_n}=z^\alpha.$$

The formulas above describe all possibilities which may yield candidates for the realization of the Lempert function. Now for all possible λ_1,λ_2 as above we find mappings which map λ_1 and λ_2 to w and z. Note that there are mappings $\psi_j\in\mathcal O(E,\mathbb C_*)$, $j=1,\dots,n-1$,

(35) *Theorem* (see [Edi 95]). Let $D=\{u<0\}$, $u\in\mathrm{PSH}(G)\cap C^1(G)$, where $D\subset\subset G$, $\partial D=\{u=0\}$. Let φ be a $\widetilde k_D$-geodesic. Then there are $\varrho\in L^\infty(\partial E)$, $\varrho>0$ and $h_j\in H^\infty(E)$, $j=1,\dots,n$ such that $(1/\lambda)h_j^*(\lambda)=\varrho(\lambda)(\partial u/\partial z_j)(\varphi^*(\lambda))$, for almost all $\lambda\in\partial E$, $j=1,\dots,n$.

(36) *Theorem* (see [Gen 87]). Let $f\in H^\infty(E)$ be such that $(1/\lambda)f^*(\lambda)>0$ for almost all $\lambda\in\partial E$. Then there are $b>0$, $\beta\in\overline E$ such that $\varphi(\lambda)=b(1-\overline\beta\lambda)(\lambda-\beta)$, $\lambda\in E$.

such that (see Lemma 3.2.4)

$$\psi_j(\lambda_1) = \frac{w_j}{(w^\alpha)^{1/(\alpha_{l+k+1}+\ldots+\alpha_n)}} = \frac{w_j}{\lambda_1}, \quad j = l+k+1, \ldots, n-1,$$

$$\psi_j(\lambda_2) = \frac{z_j}{(z^\alpha)^{1/(\alpha_{l+k+1}+\ldots+\alpha_n)}} = \frac{z_j}{\lambda_2}, \quad j = l+k+1, \ldots, n-1,$$

$$\psi_j(\lambda_1) = w_j, \quad j = 1, \ldots, l+k,$$

$$\psi_j(\lambda_2) = z_j, \quad j = 1, \ldots, l+k.$$

Define also

$$\psi_n(\lambda) := \frac{1}{(\psi_1^{\alpha_2}(\lambda)\ldots\psi_{n-1}^{\alpha_{n-1}}(\lambda))^{1/\alpha_n}}, \quad \lambda \in E.$$

Put

$$\varphi(\lambda) := (\psi_1(\lambda), \ldots, \psi_{l+k}(\lambda), \lambda\psi_{l+k+1}(\lambda), \ldots, \lambda\psi_n(\lambda)).$$

The $(1/\alpha_n)$th power in the definition of ψ_n is chosen so that $\varphi_n(\lambda_1) = w_1$, and we know that $\varphi_n^{\alpha_n}(\lambda_2) = z_n^{\alpha_n}$. One may also easily verify that $\varphi(\lambda_1) = w$ and $\varphi_j(\lambda_2) = z_j$, $j = 1, \ldots, n-1$, which, however, in view of Lemma 3.2.2 shows that there is also a mapping $\widetilde{\varphi} \in \mathcal{O}(E, D_\alpha)$ such that $\widetilde{\varphi}(\lambda_1) = w$, $\widetilde{\varphi}(\lambda_2) = z$. Thus we have proved that

$$\widetilde{k}^*_{D_\alpha}(w,z) = \min\{m(\lambda_1,\lambda_2) : \lambda_1,\lambda_2 \in E,\ \lambda_1^{\alpha_{j_1}+\ldots+\alpha_{j_s}} = w^\alpha,\ \lambda_2^{\alpha_{j_1}+\ldots+\alpha_{j_s}} = z^\alpha\},$$

where the minimum is taken over all nonempty subsets $\{j_1, \ldots, j_s\} \subset \{l+1, \ldots, n\}$. Now Lemma 3.3.1 finishes the proof (we may assume that $w_j > 0$, $j = 1, \ldots, n$). ■

Proof of the formula for k_{D_α} in the rational case. In view of the formula for the Lempert function, the definition of the Kobayashi pseudodistance and its continuity finish the proof. ■

It remains to compute the Kobayashi–Royden pseudometric κ_{D_α}. We do that by defining an operator which connects κ_{D_α} to the Kobayashi pseudodistance.

Following M. Jarnicki and P. Pflug (see [Jar-Pfl 93]), for a domain $D \subset \mathbb{C}^n$ we define

$$\mathfrak{D}k_D(w;X) := \limsup_{\lambda\neq 0,\, \lambda\to 0} \frac{k^*_{D_\alpha}(w, w+\lambda X)}{|\lambda|}, \quad w \in D,\ X \in \mathbb{C}^n.$$

This function differs from that in [Jar-Pfl 93], but is no larger, so the inequality below, which is crucial for our considerations, remains true:

$$\mathfrak{D}k_D(w;X) \le \kappa_D(w;X), \quad w \in D,\ X \in \mathbb{C}^n. \tag{3.3.1}$$

Lemma 3.3.2. *Let $\alpha \in \mathbb{Z}^n_*$, where α_j's are relatively prime, and define $\widetilde{\alpha}_{l+1} := \min\{\alpha_{l+1}, \ldots, \alpha_n\}$. Then*

$$\mathfrak{D}k_{D_\alpha}(w;X) = \gamma\biggl(\Bigl(\prod_{j=1}^n |w_j|^{\alpha_j}\Bigr)^{1/\widetilde{\alpha}_{l+1}}, \Bigl(\prod_{j=1}^n |w_j|^{\alpha_j}\Bigr)^{1/\widetilde{\alpha}_{l+1}} \frac{1}{\widetilde{\alpha}_{l+1}} \sum_{j=1}^n \frac{\alpha_j X_j}{w_j}\biggr)$$

for $w \in \widetilde{D}_\alpha$ and $X \in \mathbb{C}^n$.

Proof. We may assume that $w_j > 0$, $j = 1, \ldots, n$, and $\alpha_n = \widetilde{\alpha}_{l+1}$. Using the formula for $k^*_{D_\alpha}$ we get

$$(3.3.2)\qquad \mathfrak{D}k_{D_\alpha}(w;X)=\limsup_{\lambda\to 0,\,\lambda\neq 0}\frac{|\prod_{j=1}^n(w_j+\lambda X_j)^{\alpha_j/\alpha_n}-\prod_{j=1}^n w_j^{\alpha_j/\alpha_n}|}{|1-\prod_{j=1}^n(w_j+\lambda X_j)^{\alpha_j/\alpha_n}\overline{\prod w_j^{\alpha_j/\alpha_n}}|\cdot|\lambda|}.$$

Applying the Taylor formula we get for λ close to 0 (if $w_j=0$ then $\alpha_j/\alpha_n\geq 1$)

$$(w_j+\lambda X_j)^{\alpha_j/\alpha_n}=w_j^{\alpha_j/\alpha_n}+\frac{\alpha_j}{\alpha_n}w_j^{\alpha_j/\alpha_n}\frac{\lambda X_j}{w_j}+\varepsilon_j(\lambda),\quad j=1,\dots,n,$$

where $\varepsilon_j(\lambda)/\lambda\to 0$ as $\lambda\to 0$. Substituting this in (3.3.2) we get

$$\mathfrak{D}k_{D_\alpha}(w;X)=\limsup_{\lambda\neq 0,\,\lambda\to 0}\frac{(\prod_{j=1}^n|w_j^{\alpha_j}|^{1/\alpha_n})|\lambda|\cdot|\sum_{j=1}^n\alpha_jX_j/\alpha_nw_j|}{(1-\prod_{j=1}^n|w_j|^{2\alpha_j/\alpha_n})|\lambda|},$$

which equals the desired value. ∎

Proof of the formula for κ_{D_α} in the rational case. If $\mathcal{J}\neq\emptyset$, then in view of Lemma 3.2.7 we are done. The case $\sum_{j=1}^n\alpha_jX_j/w_j=0$ follows from Remark 3.2.1.

Take $w\in\widetilde{D}_\alpha$. We may assume that $w_j>0$, $j=1,\dots,n$ and $\alpha_n=\widetilde{\alpha}_{l+1}$. Below, for $X\in\mathbb{C}^n$ with $\sum_{j=1}^n\alpha_jX_j/w_j\neq 0$ we construct $\varphi\in\mathcal{O}(E,D_\alpha)$ such that

$$\varphi(\lambda_1)=w,\quad t\varphi'(\lambda_1)=X,$$

where $\lambda_1:=(w_1^{\alpha_1}\dots w_n^{\alpha_n})^{1/\alpha_n}>0$, $t:=(w_1^{\alpha_1}\dots w_n^{\alpha_n})^{1/\alpha_n}\sum_{j=1}^n\alpha_jX_j/(\alpha_nw_j)$. This finishes the proof by Lemma 3.3.2 and (3.3.1).

Define

$$\varphi(\lambda):=\left(\psi_1(\lambda),\dots,\psi_{n-1}(\lambda),\frac{\lambda}{(\psi_1^{\alpha_1}(\lambda)\dots\psi_{n-1}^{\alpha_{n-1}}(\lambda))^{1/\alpha_n}}\right),$$

where (see Lemma 3.2.6)

$$\psi_j(\lambda_1)=w_j,\ t\psi_j'(\lambda_1)=X_j,\quad j=1,\dots,n-1.$$

We choose the $(1/\alpha_n)$th power so that $\varphi_n(\lambda_1)=w_n$; after elementary computation we get $t\varphi_n'(\lambda_1)=X_n$, which finishes the proof. ∎

3.4. Proof of Theorem 3.1.1 in the irrational case. As in the rational case we start with the proof of the formula for the Lempert function. First, we make use of special properties of domains of irrational type.

LEMMA 3.4.1. *Let α be of irrational type. Then for any $w,z\in D_\alpha$,*

$$\widetilde{k}^*_{D_\alpha}(w,z)=\widetilde{k}^*_{D_\alpha}(\widetilde{w},\widetilde{z}),\quad \widetilde{w}\in T_w,\ \widetilde{z}\in T_z.$$

PROOF. It is enough to prove that

$$\widetilde{k}^*_{D_\alpha}(w,z)=\widetilde{k}^*_{D_\alpha}(w,\widetilde{z})\quad\text{whenever }\widetilde{z}\in T_z.$$

Assume that

$$(3.4.1)\qquad \widetilde{k}^*_{D_\alpha}(w,\widetilde{z}_1)<\widetilde{k}^*_{D_\alpha}(w,\widetilde{z}_2)=:\varepsilon$$

for some $\widetilde{z}_1,\widetilde{z}_2\in T_z$. Then in view of Lemma 3.2.2,

$$(3.4.2)\qquad \widetilde{k}^*_{D_\alpha}(w,\widetilde{z})=\varepsilon$$

for all $\widetilde{z} \in T_{\widetilde{z}_2,\alpha}$. Because of (3.2.1) we have $\widetilde{z}_1 \in T_z = T_{\widetilde{z}_2} = \overline{T}_{\widetilde{z}_2,\alpha}$. Together with (3.4.1) and (3.4.2), the last statement contradicts the upper semicontinuity of the Lempert function. ∎

Here is an immediate corollary:

COROLLARY 3.4.2. *Let α be of irrational type. Then for any $z \in D_\alpha$,*

$$\widetilde{k}^*_{D_\alpha}(z,\widetilde{z}) = 0 \quad \textit{for any } \widetilde{z} \in T_z.$$

Proof of the formula for $\widetilde{k}_{D_\alpha}$ in the irrational case. The case $\mathcal{J} \neq \emptyset$ is covered by Lemma 3.2.7. Consider now the remaining case. In view of Lemma 3.4.1 we have

$$\widetilde{k}^*_{D_\alpha}(w,z) = \widetilde{k}^*_{D_\alpha}((|w_1|,\ldots,|w_n|),(|z_1|,\ldots,|z_n|)).$$

Choose a sequence $\{\alpha^{(k)}\}_{k=1}^\infty \subset (\mathbb{Q}_*)^n$ such that

$$\alpha_1^{(k)},\ldots,\alpha_l^{(k)} < 0, \quad \alpha_{l+1}^{(k)},\ldots,\alpha_n^{(k)} > 0 \quad \text{and} \quad \alpha^{(k)} \to \alpha.$$

By Theorem 3.1.1 in the rational case, if $x,y \in \mathbb{R}^n_+ \cap D_{\alpha^{(k)}}$ then

$$\begin{aligned}(3.4.3)\quad & \widetilde{k}^*_{D_{\alpha^{(k)}}}(x,y) \\ &= m((x_1^{\alpha_1^{(k)}}\ldots x_n^{\alpha_n^{(k)}})^{1/\min\{\alpha_{l+1}^{(k)},\ldots,\alpha_n^{(k)}\}},(y_1^{\alpha_1^{(k)}}\ldots y_n^{\alpha_n^{(k)}})^{1/\min\{\alpha_{l+1}^{(k)},\ldots,\alpha_n^{(k)}\}}).\end{aligned}$$

We may assume that $\min\{\alpha_{l+1},\ldots,\alpha_n\} = \alpha_n$ and $\min\{\alpha_{l+1}^{(k)},\ldots,\alpha_n^{(k)}\} = \alpha_n^{(k)}$. First we prove that

$$\widetilde{k}^*_{D_\alpha}(w,z) \geq m((|w_1|^{\alpha_1}\ldots|w_n|^{\alpha_n})^{1/\alpha_n},(|z_1|^{\alpha_1}\ldots|z_n|^{\alpha_n})^{1/\alpha_n}).$$

Indeed, otherwise there is $\varphi \in \mathcal{O}(\bar{E},D_\alpha)$ such that $\varphi(\lambda_1) = (|w_1|,\ldots,|w_n|)$, $\varphi(\lambda_2) = (|z_1|,\ldots,|z_n|)$ and

$$m(\lambda_1,\lambda_2) < m((|w_1|^{\alpha_1}\ldots|w_n|^{\alpha_n})^{1/\alpha_n},(|z_1|^{\alpha_1}\ldots|z_n|^{\alpha_n})^{1/\alpha_n}).$$

We may choose k so large that $\varphi(E) \subset D_{\alpha^{(k)}}$ and

$$m(\lambda_1,\lambda_2) < m((|w_1|^{\alpha_1^{(k)}}\ldots|w_n|^{\alpha_n^{(k)}})^{1/\alpha_n^{(k)}},(|z_1|^{\alpha_1^{(k)}}\ldots|z_n|^{\alpha_n^{(k)}})^{1/\alpha_n^{(k)}}),$$

which contradicts (3.4.3).

To get equality consider the mapping $\varphi(\lambda) := (\psi_1(\lambda),\ldots,\psi_{n-1}(\lambda),\lambda\psi_n(\lambda))$, where (see Lemma 3.2.4)

$$\psi_j \in \mathcal{O}(E,\mathbb{C}_*), \quad j=1,\ldots,n-1,$$
$$\lambda_1 := (|w_1|^{\alpha_1}\ldots|w_n|^{\alpha_n})^{1/\alpha_n} > 0, \quad \lambda_2 := (|z_1|^{\alpha_1}\ldots|z_n|^{\alpha_n})^{1/\alpha_n} > 0;$$
$$\psi_j(\lambda_1) = |w_j|, \quad \psi_j(\lambda_2) = |z_j|, \quad j=1,\ldots,n-1.$$

Define also

$$\psi_n(\lambda) := \frac{1}{(\psi_1^{\alpha_1}(\lambda)\ldots\psi_{n-1}^{\alpha_{n-1}}(\lambda))^{1/\alpha_n}}, \quad \lambda \in E.$$

The α_nth root is chosen so that $\varphi_n(\lambda_1) = |w_n|$. One may also easily check from the form of ψ_j's in the proof of Lemma 3.2.4 that then $\varphi_n(\lambda_2) > 0$, so $\varphi_n(\lambda_2) = |z_n|$. This completes the proof. ∎

Just as in the rational case we have:

Proof of the formula for k_{D_α} in the irrational case. The continuity of the Kobayashi pseudodistance as well as the definition of the Kobayashi pseudodistance and the formula for the Lempert function finish the proof. ■

Having the formula for the Lempert function we get

Proof of the formula for g_{D_α} in the irrational case. Because of (1.1.8) we can assume that $l = 0$.

CASE I: $\mathcal{J} = \emptyset$. Corollary 3.4.2 implies that

$$g_{D_\alpha}(w, z) = -\infty \quad \text{for any } z \in T_w.$$

The maximum principle for plurisubharmonic functions (applied to $g_{D_\alpha}(w, \cdot)$) implies that

$$g_{D_\alpha}(w, z) = -\infty \quad \text{for any } z \text{ with } |z_j| \le |w_j|,$$

which means that $g_{D_\alpha}(w, \cdot)$ equals $-\infty$ on a set with nonempty interior ($w_1 \dots w_n \neq 0$) but $g_{D_\alpha}(w, \cdot)$ is plurisubharmonic, so it must vanish on D_α.

CASE II: $\mathcal{J} \neq \emptyset$. This case is a consequence of Lemma 3.2.7, the inequality $\widetilde{g} \le \widetilde{k}^*$, the definition of the Green function and the fact that $(|z_1|^{\alpha_1} \dots |z_n|^{\alpha_n})^{1/(\alpha_{j_1} + \dots + \alpha_{j_k})}$ is logarithmically plurisubharmonic on D_α. ■

Proof of the formula for A_{D_α} in the irrational case. The result follows from the formula for the Green function and the definition of the Azukawa pseudometric. ■

Now, we complete the proof by showing the formula for κ_{D_α}.

LEMMA 3.4.3. *Let α be of irrational type, $\widetilde{\alpha}_{l+1} := \min\{\alpha_{l+1}, \dots, \alpha_n\}$. Then*

$$\mathfrak{D}k_{D_\alpha}(w; X) = \gamma\bigg(\Big(\prod_{j=1}^n |w_j|^{\alpha_j}\Big)^{1/\widetilde{\alpha}_{l+1}}, \Big(\prod_{j=1}^n |w_j|^{\alpha_j}\Big)^{1/\widetilde{\alpha}_{l+1}} \frac{1}{\widetilde{\alpha}_{l+1}} \sum_{j=1}^n \frac{\alpha_j X_j}{w_j}\bigg)$$

for $w \in \widetilde{D}_\alpha$ and $X \in \mathbb{C}^n$.

PROOF. We may assume that $\alpha_n = \widetilde{\alpha}_{l+1}$. The formula for the Kobayashi pseudodistance gives

$$\mathfrak{D}k_{D_\alpha}(w; X) = \limsup_{\lambda \neq 0,\, \lambda \to 0} \frac{|\prod_{j=1}^n |w_j + \lambda X_j|^{\alpha_j/\alpha_n} - \prod_{j=1}^n |w_j|^{\alpha_j/\alpha_n}|}{|1 - \prod_{j=1}^n |w_j + \lambda X_j|^{\alpha_j/\alpha_n} \prod |w_j|^{\alpha_j/\alpha_n}| \cdot |\lambda|}. \tag{3.4.4}$$

Therefore, applying the Taylor formula we get, for λ close to 0 (note that if $w_j = 0$ then $\alpha_j/\alpha_n \ge 1$),

$$|w_j + \lambda X_j|^{\alpha_j/\alpha_n} = |w_j|^{\alpha_j/\alpha_n} + \frac{\alpha_j}{\alpha_n} |w_j|^{\alpha_j/\alpha_n} \bigg(\operatorname{Re}\Big(\frac{\lambda X_j}{w_j}\Big)\bigg) + \varepsilon_j(\lambda), \quad j = 1, \dots, n,$$

where $\varepsilon_j/\lambda \to 0$ as $\lambda \to 0$. Substituting this in (3.4.4) we get

$$\mathfrak{D}k_{D_\alpha}(w; X) = \limsup_{\lambda \neq 0,\, \lambda \to 0} \frac{\prod_{j=1}^n (|w_j|^{\alpha_j})^{1/\alpha_n} \operatorname{Re}\big(\lambda\big(\sum_{j=1}^n \frac{\alpha_j X_j}{\alpha_n w_j}\big)\big)}{(1 - \prod_{j=1}^n |w_j|^{2\alpha_j/\alpha_n})|\lambda|},$$

which equals the desired value. ■

Proof of the formula for κ_{D_α} in the irrational case. If $\mathcal{J}\neq\emptyset$, then, by Lemma 3.2.7, we are done. Also the case $\sum_{j=1}^n \alpha_j X_j/w_j=0$ follows from Remark 3.2.1. Below we deal with the remaining cases.

Take $w\in\widetilde{D}_\alpha$. We may assume that $w_j>0$, $j=1,\dots,n$ and $\alpha_n=\min\{\alpha_{l+1},\dots,\alpha_n\}$. Below, for $X\in\mathbb{C}^n$ with $\sum_{j=1}^n\alpha_jX_j/w_j\neq 0$ we construct $\varphi\in\mathcal{O}(E,D_\alpha)$ such that

$$\varphi(\lambda_1)=w,\quad t\varphi'(\lambda_1)=X,$$

where $\lambda_1:=(w_1^{\alpha_1}\dots w_n^{\alpha_n})^{1/\alpha_n}>0$, $t:=(w_1^{\alpha_1}\dots w_n^{\alpha_n})^{1/\alpha_n}\sum_{j=1}^n\alpha_jX_j/(\alpha_nw_j)$. This finishes the proof by Lemma 3.4.3 and (3.3.1).

Define

$$\varphi(\lambda):=\left(\psi_1(\lambda),\dots,\psi_{n-1}(\lambda),\frac{\lambda}{(\psi_1^{\alpha_1}(\lambda)\dots\psi_{n-1}^{\alpha_{n-1}}(\lambda))^{1/\alpha_n}}\right),$$

where (see Lemma 3.2.6)

$$\psi_j(\lambda_1)=w_j,\quad t\psi_j'(\lambda_1)=X_j,\quad j=1,\dots,n-1.$$

We choose the $(1/\alpha_n)$th power so that $\varphi_n(\lambda_1)=w_n$. After an elementary computation we get $t\varphi_n'(\lambda_1)=X_n$, which finishes the proof. ■

3.5. Elementary Reinhardt domains with $l=n$. In this section we deal with the case $l=n$ (equivalently, $D_\alpha\subset\mathbb{C}_*^n$). Because of (1.1.8) and (1.1.9) we restrict attention to the Lempert function, the Kobayashi pseudodistance and the Kobayashi–Royden pseudometric.

THEOREM 3.5.1. *Assume that $l=n$, $w,z\in D_\alpha$, $X\in\mathbb{C}^n$.*

1. *If α is of rational type then*

$$\widetilde{k}_{D_\alpha}(w,z)=k_{D_\alpha}(w,z)=k_{E_*}(w^\alpha,z^\alpha),\quad \kappa_{D_\alpha}(w;X)=\kappa_{E_*}\left(w^\alpha;w^\alpha\sum_{j=1}^n\frac{\alpha_jX_j}{w_j}\right).$$

2. *If α is of irrational type then*

$$\widetilde{k}_{D_\alpha}(w,z)=k_{D_\alpha}(w,z)=k_{E_*}(|w_1|^{\alpha_1}\dots|w_n|^{\alpha_n},|z_1|^{\alpha_1}\dots|z_n|^{\alpha_n}),$$
$$\kappa_{D_\alpha}(w;X)=\kappa_{E_*}\left(|w_1|^{\alpha_1}\dots|w_n|^{\alpha_n};|w_1|^{\alpha_1}\dots|w_n|^{\alpha_n}\sum_{j=1}^n\frac{\alpha_jX_j}{w_j}\right).$$

PROOF. By Proposition 2.1.3 we know that $\widetilde{k}_{D_\alpha}=k_{D_\alpha}$. Define

$$\begin{aligned}\Psi:\mathbb{C}^{n-1}\times E_*\ni(\lambda_1,\dots,\lambda_n)&\\ \mapsto\left(\exp(\alpha_n\lambda_1),\dots,\exp(\alpha_n\lambda_{n-1}),\frac{1}{\lambda_n}\exp(-(\alpha_1\lambda_1+\dots+\alpha_{n-1}\lambda_{n-1}))\right)&\in D_\alpha.\end{aligned}$$

The mapping Ψ is a holomorphic covering. Note that $\Psi(\lambda)=w$ iff

$$\lambda_j=\frac{1}{\alpha_n}\log|w_j|+\frac{i}{\alpha_n}(\operatorname{Arg}w_j+2l_j\pi),\quad j=1,\dots,n-1,$$
$$\frac{1}{\lambda_n}=w_n(|w_1|^{\alpha_1}\dots|w_{n-1}|^{\alpha_{n-1}})^{1/\alpha_n}\exp\left(\frac{i}{\alpha_n}\Big(\sum_{j=1}^{n-1}(\operatorname{Arg}w_j+2l_j\pi)\alpha_j\Big)\right),$$

where $l_1,\dots,l_{n-1}\in\mathbb{Z}$. Applying (1.1.6) and the product property of k we get

$$k_{D_\alpha}(w,z)$$
$$= \inf \Big\{ k_{E_*}\Big(w_n^{-1}(|w_1|^{\alpha_1}\dots|w_{n-1}|^{\alpha_{n-1}})^{-1/\alpha_n}\exp\Big(-\frac{i}{\alpha_n}\sum_{j=1}^{n-1}\operatorname{Arg} w_j\alpha_j\Big),$$
$$z_n^{-1}(|z_1|^{\alpha_1}\dots|z_{n-1}|^{\alpha_{n-1}})^{-1/\alpha_n}\exp\Big(-\frac{i}{\alpha_n}\sum_{j=1}^{n-1}(\operatorname{Arg} z_j+2l_j\pi)\alpha_j\Big)\Big)\Big\},$$

where the infimum is taken over all $l_1,\dots,l_{n-1}\in\mathbb{Z}$.

In the rational case the last expression equals $k_{E_*}(w^\alpha,z^\alpha)$ (37).

In the irrational case the last infimum equals, by the Kronecker Theorem, $k_{E_*}((|w_1|^{\alpha_1}\dots|w_n|^{\alpha_n})^{-1/\alpha_n},(|z_1|^{\alpha_1}\dots|z_n|^{\alpha_n})^{-1/\alpha_n})$. The last expression equals the desired value (38).

For the Kobayashi–Royden pseudometric we have $\Psi'(\lambda)Y=X$ iff

$$\alpha_n w_jY_j=X_j,\quad j=1,\dots,n-1,\quad -\bigg(\bigg(\sum_{j=1}^{n-1}\frac{\alpha_jY_j}{\lambda_n}\bigg)+\frac{Y_n}{\lambda_n^2}\bigg)\exp\Big(-\sum_{j=1}^{n-1}\alpha_j\lambda_j\Big)=X_n,$$

from which we get (use (1.1.7))

$$\kappa_{D_\alpha}(w;X)=\kappa_{E_*}\bigg((|w_1|^{\alpha_1}\dots|w_n|^{\alpha_n})^{-1/\alpha_n},\frac{(|w_1|^{\alpha_1}\dots|w_n|^{\alpha_n})^{-1/\alpha_n}}{\alpha_n}\sum_{j=1}^{n}\frac{\alpha_jX_j}{w_j}\bigg).$$

The last number equals the desired value (39). ∎

Remark 3.5.2. In case $0<l<n$ we may get the formulas for the Lempert function, the Kobayashi pseudodistance and the Kobayashi–Royden pseudometric similarly to the proof of Theorem 3.5.1 reducing the problem to lower dimensional elementary Reinhardt domains with $l=0$. Namely, we may define the following holomorphic covering:

$$\Phi:\mathbb{C}^l\times D_{(\alpha_{l+1},\dots,\alpha_n)}\ni(\lambda_1,\dots,\lambda_n)$$
$$\mapsto(\exp(\alpha_n\lambda_1),\dots,\exp(\alpha_n\lambda_l),\lambda_{l+1},\dots,\lambda_{n-1},\lambda_n\exp(-(\alpha_1\lambda_1+\dots+\alpha_l\lambda_l)))\in D_\alpha.$$

Remark 3.5.3. From the proof of Theorem 3.5.1 we see that in the case $l=n$ and α of irrational type the infimum in the formula (1.1.6) need not be attained (the covering mapping is Ψ). Similarly using the mapping Φ (from Remark 3.5.2) in a more general case $0<l<n$ we may find examples of that kind. These examples answer (negatively) the question posed by S. Kobayashi (see [Kob 70]) about the existence of minimum in the formula (1.1.6).

To visualize these examples take α of irrational type with $l=n$. Assume that $\alpha_n=-1$. Consider the holomorphic covering Ψ as in the proof of Theorem 3.5.1. Take $w\in D_\alpha\cap\mathbb{R}_+^n$ and $z\in D_\alpha$ such that $z_j=w_j$, $j=1,\dots,n-1$, $|z_n|=w_n$. We know that $k_{D_\alpha}(w,z)=0$. The infimum would be attained if there were $l_1,\dots,l_{n-1}\in\mathbb{Z}$ such that $(\operatorname{Arg} z_n)/(2\pi)+$

(37) From the formula for k_{E_*} (see [Jar-Pfl 93]) we have $k_{E_*}(w^t,z^t)=k_{E_*}(w,z)$, $w,z\in E_*$, $t\in\mathbb{N}_*$. Put $t:=-\alpha_n$.

(38) If $x,y\in(0,1)$, $t>0$ then from the formula for k_{E_*} we have $k_{E_*}(x^t,y^t)=k_{E_*}(x,y)$; we apply this for $t:=-\alpha_n$.

(39) By the formula for κ_{E_*} (see [Jar-Pfl 93]), $\kappa_{E_*}(x;1)=\kappa_{E_*}(x^t;tx^{t-1})$ for any $t>0$, $x\in(0,1)$. Put $t:=-\alpha_n$.

$\sum_{j=1}^{n-1} l_j \alpha_j \in \mathbb{Z}$. This need not hold: it is sufficient to take z_n such that $(\operatorname{Arg} z_n)/(2\pi)$ does not belong to the $\mathbb{Q}$-linear subspace of $\mathbb{R}$ spanned by $\{1, \alpha_1, \ldots, \alpha_{n-1}\}$.

3.6. The pluricomplex Green function of the unit ball with two poles. The last example of new effective formulas is the formula for the pluricomplex Green function with many poles of the unit ball. It may sound incredible but except for the one dimensional case no effective formulas for the Green function with (at least) two poles have been known. We deal with the most natural case i.e. the unit ball in $\mathbb{C}^2$ with two poles of equal weights.

THEOREM 3.6.1. *Let $0 < p < 1$ and $(z_1, z_2) \in \mathbb{B}_2$. Then*

$$g_{\mathbb{B}_2}((0,p),(0,-p);(z_1,z_2))$$
$$= \begin{cases} \dfrac{1}{2}\log\left(1 - \dfrac{(1-p^2)(1-|z_1|^2-|z_2|^2)}{|1-pz_2|^2}\right) & \text{if } p|z_1| \geq |z_2 - p|, \\ \dfrac{1}{2}\log\left(1 - \dfrac{(1-p^2)(1-|z_1|^2-|z_2|^2)}{|1+pz_2|^2}\right) & \text{if } p|z_1| \geq |z_2 + p|, \\ \dfrac{1}{2}\log \dfrac{2(1-p^2 \operatorname{Re} z_2^2)|z_1|^2 + |p^2 - p^2|z_1|^2 - z_2^2|^2 + \sqrt{\triangle}}{2|1-p^2z_2^2|^2} & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } p|z_1| < \min\{|z_2 - p|, |z_2 + p|\}, \end{cases}$$

where $\triangle := -4|z_1|^4(p^2 \operatorname{Im} z_2^2)^2 + 4|z_1|^2(1-p^2 \operatorname{Re} z_2^2)|p^2 - p^2|z_1|^2 - z_2^2|^2 + |p^2 - p^2|z_1|^2 - z_2^2|^4$.

The formulas above entirely solve the problem for the Green function with two poles with equal weights because of the transitivity of the automorphism group of $\mathbb{B}_2$. Moreover, because three points lie in one plane the formulas above actually give effective formulas for the Green function with two poles with equal weights in $\mathbb{B}_n$ for any $n \geq 2$.

A decisive role in the proof of Theorem 3.6.1 is played by a theorem which shows how the Green function behaves under proper holomorphic mappings. Before we formulate that result let us recall some notations.

Let $\pi : \widetilde{D} \to D$ be a proper holomorphic mapping (with multiplicity m) and let P be a set of poles in D but such that $\pi^{-1}(P) \cap \{\det \pi' = 0\} = \emptyset$. Define $\widetilde{\nu}(q) := \nu(\pi(q))$ for any $q \in \pi^{-1}(P)$.

Recall that $g_D(P;\cdot)$ denotes the pluricomplex Green function with poles at P with all weights equal to 1.

We formulate a theorem which may be found in [Lar-Sig 98]; we give another proof below.

THEOREM 3.6.2. *Under the above assumptions and notations, for any $\widetilde{w} \in \widetilde{D}$ the following formula holds*:

$$g_{\widetilde{D}}(\pi^{-1}(P); \widetilde{\nu}; \widetilde{w}) = g_D(P; \nu; \pi(\widetilde{w})).$$

The most natural proper holomorphic mappings from the unit ball in dimension at least two are

$$(z_1, \ldots, z_n) \mapsto (z_1^{p_1}, \ldots, z_n^{p_n}), \quad p_j \in \mathbb{N}.$$

These mappings lead us to the problem of calculating the Green function of the (convex) ellipsoid $\mathcal{E}(1,1/2)$ because in view of Theorem 3.6.2,

$$g_{\mathbb{B}_2}((0,p),(0,-p);(z_1,z_2)) = g_{\mathcal{E}(1,1/2)}((0,p^2),(z_1,z_2^2)),$$

so the proof of Theorem 3.6.1 reduces to finding the formula for the Green function with one pole (with weight 1) of the complex ellipsoid $\mathcal{E}(1,1/2)$. The most tedious part of our paper is devoted to the proof of that formula.

THEOREM 3.6.3. *Let* $(0,t),(z_1,z_2)\in\mathcal{E}(1,1/2)$, $t\geq 0$. *Then*

$$g_{\mathcal{E}(1,1/2)}((0,t),(z_1,z_2))$$
$$= \begin{cases} \dfrac{1}{2}\log\left(1-\dfrac{(1-t)(1-|z_1|^2-|z_2|)}{|1-t^{1/2}z_2^{1/2}|^2}\right) & \textit{if } t^{1/2}|z_1|\geq |z_2^{1/2}-t^{1/2}|, \\ \dfrac{1}{2}\log\dfrac{2(1-t\operatorname{Re}z_2)|z_1|^2+|t-t|z_1|^2-z_2|^2+\sqrt{\triangle}}{2|1-tz_2|^2} & \textit{if } t^{1/2}|z_1|< |z_2^{1/2}-t^{1/2}|, \end{cases}$$

where $\triangle := -4|z_1|^4(t\operatorname{Im}z_2)^2+4|z_1|^2(1-t\operatorname{Re}z_2)|t-t|z_1|^2-z_2|^2+|t-t|z_1|^2-z_2|^4$, $(0,t),(z_1,z_2)\in\mathcal{E}(1,1/2)$ (*writing* $\lambda^{1/2}$ *for* $\lambda\in\mathbb{C}\setminus\{0\}$ *we mean* μ *such that* $\mu^2=\lambda$ *and* $\operatorname{Arg}\mu\in[-\pi/2,\pi/2)$).

The formula from Theorem 3.6.1 has been obtained independently, with other methods, in [Com 97].

Proof of Theorem 3.6.2. Take any $u\in\operatorname{PSH}(D,[-\infty,0))$ such that

$$u(z)\leq \nu(p_j)\log\|z-p_j\|+M \tag{3.6.1}$$

for z near p_j and some $M\in\mathbb{R}$. Put $\widetilde{u}:=u\circ\pi$, $\pi^{-1}(p_j)=\{p_j^1,\dots,p_j^m\}$. We have $\widetilde{u}\in\operatorname{PSH}(\widetilde{D},[-\infty,0))$ and because π is locally biholomorphic near p_j^k for all possible j,k there is $\widetilde{M}\in\mathbb{R}$ such that

$$\widetilde{u}(\widetilde{z})\leq \nu(p_j)\log\|\widetilde{z}-p_j^k\|+\widetilde{M}$$

for $\widetilde{z}$ near p_j^k. This proves the inequality "$\geq$".

To prove the opposite inequality take any $\widetilde{u}\in\operatorname{PSH}(\widetilde{D},[-\infty,0))$ as in the definition of the Green function $g_{\widetilde{D}}(\pi^{-1}(P);\widetilde{\nu};\cdot)$. Then define

$$u(z):=\max\{\widetilde{u}(\widetilde{z}):\pi(\widetilde{z})=z\},\quad z\in D.$$

By Proposition 2.9.26 in [Kli 91], u is plurisubharmonic and <0. We may easily verify that u fulfills the condition as in (3.6.1). This completes the proof. ∎

By Theorem 1.3.3 applied to the ellipsoid $\mathcal{E}(1,1/2)$, any complex geodesic φ passing through $(0,t)$ and (z_1,z_2) (with $t>0$) is such that either $\varphi_2^{-1}(0)=\emptyset$ or $\#\varphi_2^{-1}(0)=1$. Our first aim is to decide for which pairs of points the complex geodesic joining these points is of the first type and for which of the second type. Although we shall need this only for $\mathcal{E}(1,1/2)$ it is no more difficult for arbitrary ellipsoids $\mathcal{E}(1,m)$ (with $m\geq 1/2$); therefore, we show it in this general case.

Let $z_2=|z_2|e^{i\theta}$, where $\theta\in[-\pi,\pi)$. Let $\varphi=(\varphi_1,\varphi_2):E\to\mathcal{E}(1,m)$ be the (unique up to an automorphism of E) geodesic joining $(0,t)$ to (z_1,z_2).

Lemma 3.6.4. *The following conditions are equivalent*:

(i) $|m\theta| < \pi/2$ *and* $|z_1|t^m \geq ||z_2|^m e^{im\theta} - t^m|$;

(ii) φ_2 *has no roots.*

Proof. (ii)⇒(i). The formulas for geodesics from Theorem 1.3.3 imply that there is a geodesic $\widetilde{\varphi} : E \to \mathbb{B}_2$ joining $(0, t^m)$ to (z_1, z_2^m) such that $\widetilde{\varphi}_2$ has no roots and z_2^m is chosen so that t and z_2 are in the image of $\widetilde{\varphi}_2^{1/m}$. The graphs of geodesics in $\mathbb{B}_2$ are the parts of complex lines lying in $\mathbb{B}_2$, therefore, the lack of roots of $\widetilde{\varphi}_2$ implies that

$$|t^m z_1| \geq |z_2^m - t^m|$$

or equivalently

$$1 > |z_1| \geq |(z_2/t)^m - 1|. \tag{3.6.2}$$

Since $z_2^m = |z_2|^m e^{i(m\theta + m2k\pi)}$ for some $k \in \mathbb{Z}$, we deduce from (3.6.2) that there is an $l \in \mathbb{Z}$ such that

$$|m(\theta + 2k\pi) - 2l\pi| < \pi/2. \tag{3.6.3}$$

We know that $\widetilde{\varphi}_2$ passes through t^m and z_2^m and its $(1/m)$th power $(\widetilde{\varphi}_2^{1/m} = \varphi_2)$ passes through t and z_2. Therefore,

$$(t^m)^{1/m} = (t^m)^{1/m} e^{i(2s\pi/m)} \quad \text{for some } s \in \mathbb{Z} \text{ and } s/m \in \mathbb{Z}. \tag{3.6.4}$$

The interval $[t^m, z_2^m]$ lies in the image of $\widetilde{\varphi}_2$, so continuity of the argument implies that

$$(z_2^m)^{1/m} = |z_2| e^{(i/m)(m\theta + m2k\pi - 2l\pi + 2s\pi)} = |z_2| e^{i(\theta + \frac{s-l}{m} 2\pi)},$$

which is to equal $|z_2|e^{i\theta}$. But that implies $(s - l)/m \in \mathbb{Z}$, so in view of (3.6.4) we have

$$l/m \in \mathbb{Z}. \tag{3.6.5}$$

On the other hand, property (3.6.3) implies

$$|\theta + 2k\pi - 2l\pi/m| < \pi/(2m) \leq \pi,$$

which in view of (3.6.5) gives $k = l/m$, so $|m\theta| < \pi/2$ and $z_2^m = |z_2|^m e^{im\theta}$.

(i)⇒(ii). Put $z_2^m := |z_2|^m e^{im\theta}$. (i) implies that for $(0, t^m), (z_1, z_2^m) \in \mathbb{B}_2$ there is a complex geodesic $\widetilde{\varphi}$ such that $\widetilde{\varphi}_2$ has no roots and $\widetilde{\varphi}_1 = \varphi_1$. Take $\widetilde{\varphi}^{1/m} := (\varphi_1, \widetilde{\varphi}_2^{1/m})$ such that $(t^m)^{1/m} = t$. We get $(z_2^m)^{1/m} = |z_2| e^{im\theta/m} = z_2$. So $\widetilde{\varphi}^{1/m} = \varphi$ and this completes the proof. ∎

Now we make some comments relating to Lemma 3.6.4 in the case $m = 1/2$.

Remark 3.6.5. If φ is a complex geodesic in $\mathcal{E}(1, 1/2)$ joining $(0, t)$ to (z_1, z_2) such that φ_2 has no zeros, then $(\varphi_1, \varphi_2^{1/2})$ is a complex geodesic in $\mathbb{B}_2$ joining $(0, t^{1/2})$ to $(z_1, z_2^{1/2})$ (see Theorem 1.3.3), where the root $z_2^{1/2}$ is chosen so that (as follows from the reasoning in Lemma 3.6.4)

$$\operatorname{Arg} z_2^{1/2} \in [-\pi/2, \pi/2). \tag{3.6.6}$$

Therefore, it is convenient to assume that the square root of a complex number is always chosen so that (3.6.6) is satisfied.

Keeping this in mind we may reformulate Lemma 3.6.4 in the case $m = 1/2$ as follows:

LEMMA 3.6.6. *The following conditions are equivalent*:

(i) $t^{1/2}|z_1| \geq |t^{1/2} - z_2^{1/2}|$;

(ii) φ_2 *has no roots.*

Proof of Theorem 3.6.3. It is easy to verify that if $t = 0$ or $(0,t) = (z_1, z_2)$, then the formulas hold. Assume that $t > 0$ and $(0,t) \neq (z_1, z_2)$. In the first case, i.e. the geodesic joining $(0,t)$ to (z_1, z_2) is such that φ_2 has no zeros, the formula follows from the fact that $(\varphi_1, \varphi_2^{1/2})$ is a geodesic in $\mathbb{B}_2$ and from Remark 3.6.5. Applying the formula for the Green function in the unit ball, we get the desired result.

Consider now the remaining case. We may assume that $z_1 \geq 0$. Let φ be a complex geodesic such that

$$\varphi(0) = (0,t), \quad \varphi(\tau) = (z_1, z_2), \quad \tau \in E \setminus \{0\}. \tag{3.6.7}$$

In this case φ_2 has a zero. From the formulas in Theorem 1.3.3 and (3.6.7), we get

$$\alpha_1 = 0, \quad -a_2\alpha_2 = t, \quad \frac{a_1\tau}{1-\overline{\alpha}_0\tau} = z_1, \quad \frac{a_2(\tau-\alpha_2)(1-\overline{\alpha}_2\tau)}{(1-\overline{\alpha}_0\tau)^2} = z_2, \tag{3.6.8}$$

$$\alpha_0 = |a_2|\alpha_2, \quad 1 + |\alpha_0|^2 = |a_1|^2 + |a_2|(1+|\alpha_2|^2). \tag{3.6.9}$$

We may additionally assume that $\alpha_2 < 0$. Then $a_2 > 0$ and $\alpha_0 = -t$. Consequently, using (3.6.8) and (3.6.9) we get

$$1 + t^2 = \frac{|z_1|^2|1+t\tau|^2}{|\tau|^2} + a_2(1 + t^2/a_2^2), \quad \frac{a_2(\tau + t/a_2)(1+(t\tau)/a_2)}{(1+t\tau)^2} = z_2. \tag{3.6.10}$$

The second equality is equivalent to

$$a_2(1+t^2/a_2^2) = (z_2(1+t\tau)^2)/\tau - t(1/\tau + \tau).$$

Substituting this in (3.6.10), we get

$$(1+t\tau)(t+\tau) - \frac{|z_1|^2|1+t\tau|^2}{\overline{\tau}} = z_2(1+t\tau)^2,$$

or

$$\overline{\tau}(t - t|z_1|^2 - z_2) = |\tau|^2(tz_2 - 1) + |z_1|^2.$$

Taking modules and squaring we get

$$|\tau|^4|1-tz_2|^2 - |\tau|^2(2|z_1|^2(1 - t\operatorname{Re} z_2) + |t - t|z_1|^2 - z_2|^2) + |z_1|^4 = 0.$$

We are interested in a solution $|\tau| \geq 0$. There are at most two such $|\tau|$'s. Their number depends on the sign of

$$\begin{aligned}\triangle(z_1,z_2) := {}& 4|z_1|^4(1-t\operatorname{Re} z_2)^2 + 4(1-t\operatorname{Re} z_2)|z_1|^2|t-t|z_1|^2-z_2|^2 \\ &+ |t-t|z_1|^2-z_2|^4 - 4|1-tz_2|^2|z_1|^4 \\ ={}& -4|z_1|^4(t\operatorname{Im} z_2)^2 + 4|z_1|^2(1-t\operatorname{Re} z_2)|t-t|z_1|^2-z_2|^2 + |t-t|z_1|^2-z_2|^4.\end{aligned}$$

Now fix $t > 0$ and consider all $(z_1, z_2) \neq (0,t)$ for which φ_2 has a root. We see that is a domain and, moreover, after tedious calculations, $\triangle(z_1,z_2) = (|t-z_2|^2 - t^2|z_1|^4)^2 + 4|z_1|^2(1-t^2)|t-z_2-t|z_1|^2|^2$, which is > 0 for all (z_1,z_2) under consideration. Consequently, because of the continuity of the Green function with one pole (fixed) for the domain $\mathcal{E}(1,1/2)$, the Green function is given by the desired formula if only it is given by that formula for at least one possible point (z_1,z_2) (here we need the continuity of the Green function with one pole in $\mathcal{E}(1,1/2)$). One may easily check that this is the case for $(z_1,z_2)=(0,0)$. ∎

Proof of Theorem 3.6.1. The theorem is a simple consequence of Remark 3.6.5, Lemma 3.6.6, Theorems 3.6.2 and 3.6.3. ∎

Let us close this section with some remarks on a set which to some extent tells us how close the relation is between the Green function with many poles and the Green function with one pole.

Below we deal with the upper and lower bound from the following formula (see [Lel 89]):

$$\min\{\nu(p)g_D(p,z):p\in P\}\ge g_D(P;\nu;z)\ge\sum_{p\in P}\nu(p)g_D(p,z),\quad z\in D. \tag{3.6.11}$$

We consider the case of the lower bound. Consider the sets (see [Lel 89])

$$\mathcal{E}(D,P,\nu):=\Big\{z\in D:g_D(P;\nu;z)=\sum_{p\in P}\nu(p)g_D(p,z)\Big\}.$$

Clearly, $P\subset\mathcal{E}(D,P,\nu)$.

LEMMA 3.6.7. *Let D and P be as above. Then for any $\mu,\nu:P\to(0,\infty)$,*

$$\mathcal{E}(D,P,\nu)=\mathcal{E}(D,P,\mu).$$

PROOF. Take $z\in\mathcal{E}(D,P,\nu)$, $z\notin P$. In view of (3.6.11) we may assume that $g_D(p,z)>-\infty$. Fix $\varepsilon>0$ so small that

$$\begin{aligned}\min\Big\{\sum_{p\in Q}\nu(p)g_D(p,z):\emptyset\neq Q\subset P,Q\neq P\Big\}&\\ >\sum_{p\in P}\nu(p)g_D(p,z)+\min\{\nu(p)/\mu(p):p\in P\}\frac{\varepsilon}{\#P}&.\end{aligned}\tag{3.6.12}$$

Because of (1.7.1) there is $\varphi\in\mathcal{O}(\overline{E},D)$ such that $\varphi(0)=z$, $\varphi^{-1}(P)\cap E\neq\emptyset$ and

$$g_E(\varphi^{-1}(P)\cap E;\widetilde{\nu};0)<\sum_{p\in P}\nu(p)g_D(p,z)+\min\{\nu(p)/\mu(p):p\in P\}\varepsilon/\#P. \tag{3.6.13}$$

First, in view of (3.6.11), (3.6.12) and Theorem 1.7.1 we get $\varphi^{-1}(p)\cap E\neq\emptyset$ for any $p\in P$. The left hand side in (3.6.13) equals

$$\begin{aligned}\sum_{\lambda\in E,\varphi(\lambda)\in P}\widetilde{\nu}(\lambda)\log|\lambda|&=\sum_{p\in P}\sum_{\lambda\in E,\varphi(\lambda)=p}\widetilde{\nu}(\lambda)\log|\lambda|\\&=\sum_{p\in P}g_E(\varphi^{-1}(p)\cap E;\widetilde{\nu}_{|\varphi^{-1}(p)\cap E};0).\end{aligned}$$

Each summand in the last expression is at least $\nu(p)g_D(p,z)$, which gives, in view of (3.6.13),

$$\begin{aligned}\frac{\nu(p)}{\mu(p)}g_E(\varphi^{-1}(p)\cap E,\widetilde{\mu}_{|\varphi^{-1}(p)\cap E},0)&=g_E(\varphi^{-1}(p)\cap E;\widetilde{\nu}_{|\varphi^{-1}(p)\cap E};0)\\&<\nu(p)g_D(p,z)+\min\Big\{\frac{\nu(p)}{\mu(p)}:p\in P\Big\}\varepsilon/\#P,\ p\in P,\end{aligned}$$

so

$$g_E(\varphi^{-1}(p)\cap E;\widetilde{\mu}_{|\varphi^{-1}(p)\cap E};0) < \mu(p)g_D(p,z)+\varepsilon/\#P.$$

Summing over p with $\varphi^{-1}(p)\cap E\neq\emptyset$ we get (see (3.6.11))

$$\sum_{p\in P}\mu(p)g_D(p,z)\le g_D(P,\mu,z)<\sum_{p\in P}\mu(p)g_D(p,z)+\varepsilon,$$

and, consequently, $z\in\mathcal{E}(D,P,\mu)$. ∎

Let us recall (see [Edi-Zwo 98] and [Com 97]):

PROPOSITION 3.6.8. *Let* $\#P\ge 2$ *and* $\nu\equiv\text{const}$. *Then*

$$\mathcal{E}(\mathbb{B}_n;P;\nu)=(L\cap\mathbb{B}_n)\cup P,$$

where L *is the complex line containing* P *and* $L=\emptyset$ *if such a line does not exist.*

As an immediate corollary from Proposition 3.6.8 and Lemma 3.6.7 we get

COROLLARY 3.6.9. *Let* $P\subset\mathbb{B}_n$, $\#P\ge 2$, $n\ge 2$. *Then* $\mathcal{E}(\mathbb{B}_n,P,\nu)=P\cup(L\cap\mathbb{B}_n)$, *where* L *is the complex straight line containing* P ($L=\emptyset$ *if such a line does not exist*).

IV. Symmetry of the pluricomplex Green function

The Green function of a plane domain is symmetric (see e.g. [Ran 95]). In view of the Lempert Theorem it is also the case for convex domains. Nevertheless, there are examples of very regular domains for which the symmetry of the Green function fails to hold (see e.g. [Bed-Dem 88]). There are, however, no general results describing when the Green function is symmetric. In this chapter we deal with this problem. For pseudoconvex complete Reinhardt domains we give some partial results. For complex ellipsoids we show that the symmetry of the Green function is equivalent to the convexity of the ellipsoid (see Theorem 4.1.1). For the proof of this result the formulas from Theorem 1.3.3 are helpful. To confirm the conjecture that in the whole class of bounded pseudoconvex complete Reinhardt domains the same equivalence holds (between the symmetry of the Green function and the convexity of the domain) we disprove the symmetry of the Green function in pseudoconvex complete Reinhardt domains having some analytic disk in the boundary (see Proposition 4.3.1). In the same section we show that the Green function for these domains may be “extremely” nonsymmetric. In contrast to this result in Section 4.2 we give some kind of “infinitesimal” symmetry of the Green function in some class of domains (including bounded hyperconvex domains) for points lying close to each other (see Corollary 4.2.4). In connection with this result we prove that in the same class of domains “lim sup” in the definition of the Azukawa pseudometric may be replaced with “lim” (Theorem 4.2.2). Additionally, we prove a result on continuity of the Azukawa pseudometric (Theorem 4.2.1).

4.1. Symmetry of the Green function of complex ellipsoids. As mentioned earlier, we completely solve the problem of symmetry of the Green function in the class of complex ellipsoids.

THEOREM 4.1.1. *For a complex ellipsoid $\mathcal{E}(p)$ the following conditions are equivalent*:

(i) *for any $b \in \partial\mathcal{E}(p)$, $\lambda_1, \lambda_2 \in E$ we have* $\widetilde{k}^*_{\mathcal{E}(p)}(\lambda_1 b, \lambda_2 b) = m(\lambda_1, \lambda_2)$,
(ii) *for any $b \in \partial\mathcal{E}(p)$, $\lambda \in E$ we have* $g_{\mathcal{E}(p)}(\lambda b, 0) = g_{\mathcal{E}(p)}(0, \lambda b)$,
(iii) $g_{\mathcal{E}(p)}$ *is symmetric*,
(iv) $\mathcal{E}(p)$ *convex.*

REMARK 4.1.2. Theorem 4.1.1 shows that the symmetry of the Green function is a rare phenomenon. Nonconvex ellipsoids turn out to be examples of very "regular" domains failing to have the symmetry property for the pluricomplex Green function (for other examples see e.g. [Bed-Dem 88], [Pol 93], and [Jar-Pfl 93]). Moreover, our result and the methods used in the proof suggest that in the class of bounded pseudoconvex complete Reinhardt domains, or even, in the class of bounded pseudoconvex balanced domains, the symmetry of the Green function is equivalent to the convexity of the domain.

First, we prove the following lemma, which is part of Exercise 8.1, page 290 in [Jar-Pfl 93];

LEMMA 4.1.3. *Let D be a domain in $\mathbb{C}^n$. Let $\varphi \in \mathcal{O}(E, D)$. Assume that for some $\lambda_0, \lambda_1 \in E$, $\lambda_0 \neq \lambda_1$,*

$$\widetilde{g}_D(\varphi(\lambda_0), \varphi(\lambda_1)) = m(\lambda_0, \lambda_1). \tag{4.1.1}$$

Then for any $\lambda \in E$,

$$\widetilde{g}_D(\varphi(\lambda_0), \varphi(\lambda)) = m(\lambda_0, \lambda).$$

PROOF. Define

$$a(\lambda) := \frac{\lambda_0 - \lambda}{1 - \overline{\lambda}_0 \lambda}, \quad \lambda \in E.$$

We obviously have $a \circ a = \mathrm{id}_E$. Let $u : E \ni \lambda \mapsto \widetilde{g}_D(\varphi(\lambda_0), \varphi(a(\lambda))) \in [0, 1)$. Clearly, $u(0) = 0$, $\log u \in \mathrm{SH}(E)$. Moreover,

$$u(\lambda) \le \widetilde{k}^*_D(\varphi(\lambda_0), \varphi(a(\lambda))) \le m(\lambda_0, a(\lambda)) = m(0, \lambda) = |\lambda|.$$

So

$$v(\lambda) := \log u(\lambda) - \log|\lambda| \in \mathrm{SH}(E) \quad \text{and} \quad v \le 0.$$

But, in view of (4.1.1) and the definition of u, $v(a(\lambda_1)) = 0$, so the maximum principle implies that $v \equiv 0$, and $u(\lambda) = |\lambda|$ for $\lambda \in E$. Finally,

$$\widetilde{g}_D(\varphi(\lambda_0), \varphi(\lambda)) = \widetilde{g}_D(\varphi(\lambda_0), \varphi(a(a(\lambda)))) = u(a(\lambda)) = |a(\lambda)| = m(\lambda_0, \lambda). \ \blacksquare$$

COROLLARY 4.1.4. *Let D be a bounded pseudoconvex balanced domain in $\mathbb{C}^n$, $b \in \partial D$, $\lambda_0 \in E$, $\lambda_0 \neq 0$. Then the following conditions are equivalent*:

(i) $g_D(\lambda_0 b, 0) = g_D(0, \lambda_0 b)$,
(ii) $\widetilde{g}_D(\lambda_0 b, \lambda b) = \widetilde{k}^*_D(\lambda_0 b, \lambda b) = m(\lambda_0, \lambda)$ *for any $\lambda \in E$.*

PROOF. This follows from $\widetilde{g}_D(0, \lambda_0 b) = \widetilde{k}^*_D(0, \lambda_0 b) = |\lambda_0|$, the inequality $\widetilde{g}_D \le \widetilde{k}^*_D$ and Lemma 4.1.3. $\blacksquare$

Before the proof of the main theorem let us collect some auxiliary results, which are similar to that in [Pfl-Zwo 96] (Lemmas 8 and 11) but are adapted to our situation.

LEMMA 4.1.5. *Let $\varphi : E \to \mathcal{E}(p)$ be a $\widetilde{k}_{\mathcal{E}(p)}$-geodesic for $(\varphi(\lambda_1), \varphi(\lambda_2))$. Assume that*

$$\varphi_j(\lambda) = B_j(\lambda)\psi_j(\lambda), \quad \varphi_j \not\equiv 0, \quad j = 1, \dots, n,$$

where ψ_j never vanishes on E and B_j is a Blaschke product (if φ_j never vanishes, then $B_j :\equiv 1$). Let $1 \le k < n$ and $t_{k+1}, \dots, t_n$ be positive natural numbers. Put $q_j := p_j$, $j = 1, \dots, k$ and $q_j := t_j p_j$, $j = k+1, \dots, n$. Define

$$\begin{aligned}\eta(\lambda) :=&(\varphi_1(\lambda), \dots, \varphi_k(\lambda), \psi_{k+1}(\lambda), \dots, \psi_n(\lambda)),\\ \mu(\lambda) :=&(\varphi_1(\lambda), \dots, \varphi_k(\lambda), (\psi_{k+1}(\lambda))^{1/t_{k+1}}, \dots, (\psi_n(\lambda))^{1/t_n}), \quad \lambda \in E.\end{aligned}$$

Then

- *if η is not constant, then η is a $\widetilde{k}_{\mathcal{E}(p)}$-geodesic for $(\eta(\lambda_1), \eta(\lambda_2))$,*
- *if μ is not constant, then μ is a $\widetilde{k}_{\mathcal{E}(q)}$-geodesic for $(\mu(\lambda_1), \mu(\lambda_2))$.*

PROOF. By Theorem 1.3.3 each B_j has at most one zero and φ extends continuously to $\bar{E}$. We have $\widetilde{h} \circ \eta(\lambda) \le 1$ for $\lambda \in \partial E$ ($\widetilde{h}(z) := \sum_{j=1}^n |z_j|^{2p_j}$, $z \in \mathbb{C}^n$). The maximum principle for subharmonic functions implies that $\eta(E) \subset \bar{\mathcal{E}}(p)$ or $\eta(E) \subset \mathcal{E}(p)$.

If η were not a $\widetilde{k}_{\mathcal{E}(p)}$-geodesic, then there would exist $\widetilde{\eta} \in \mathcal{O}(E, \mathcal{E}(p))$ such that $\widetilde{\eta}(E) \subset\subset \mathcal{E}(p)$ and $\widetilde{\eta}(\lambda_1) = \eta(\lambda_1)$, $\widetilde{\eta}(\lambda_2) = \eta(\lambda_2)$. But setting

$$\widehat{\eta} := (\widetilde{\eta}_1, \dots, \widetilde{\eta}_k, B_{k+1}\widetilde{\eta}_{k+1}, \dots, B_n\widetilde{\eta}_n)$$

we find that $\widehat{\eta}(E) \subset\subset \mathcal{E}(p)$ and $\widehat{\eta}(\lambda_1) = \varphi(\lambda_1)$ and $\widehat{\eta}(\lambda_2) = \varphi(\lambda_2)$, contrary to the fact that φ is a $\widetilde{k}_{\mathcal{E}(p)}$-geodesic.

With the second part of the lemma we proceed similarly. Clearly $\mu(E) \subset \mathcal{E}(q)$. If μ were not a $\widetilde{k}_{\mathcal{E}(q)}$-geodesic, then there would exist $\widetilde{\mu} \in \mathcal{O}(E, \mathcal{E}(q))$ such that $\widetilde{\mu}(E) \subset\subset \mathcal{E}(q)$ and $\widetilde{\mu}(\lambda_1) = \mu(\lambda_1)$, $\widetilde{\mu}(\lambda_2) = \mu(\lambda_2)$. But setting

$$\widehat{\mu} := (\widetilde{\mu}_1, \dots, \widetilde{\mu}_k, \dots, (\widetilde{\mu}_{k+1})^{t_{k+1}}, \dots, (\widetilde{\mu}_n)^{t_n}),$$

we see that $\widehat{\mu}(E) \subset\subset \mathcal{E}(p)$ and $\widehat{\mu}(\lambda_1) = \eta(\lambda_1)$ and $\widehat{\mu}(\lambda_2) = \eta(\lambda_2)$, contradicting the fact that η is a $\widetilde{k}_{\mathcal{E}(p)}$-geodesic. ∎

Lemma 4.1.5 may be proved without the use of the results of [Edi 95] (i.e. the formulas from Theorem 1.3.3). But in that case we have to proceed a little more delicately. For the details consult the proof of Lemma 8 in [Pfl-Zwo 96].

Below we present a special two-dimensional version of a result, which, to some extent, is analogous to Lemma 11 in [Pfl-Zwo 96].

LEMMA 4.1.6. *Let $(z, 0)$, (z, w) be distinct elements of $\mathcal{E}(p) \subset \mathbb{C}^2$. Then*

$$\widetilde{k}^*_{\mathcal{E}(p)}((z, 0), (z, w)) = \frac{|w|}{(1 - |z|^{2p_1})^{1/(2p_2)}}$$

and the mapping

$$E \ni \lambda \mapsto (z, (1 - |z|^{2p_1})^{1/(2p_2)}\lambda) \in \mathcal{E}(p)$$

is a $\widetilde{k}_{\mathcal{E}(p)}$-geodesic for $((z, 0), (z, w))$.

PROOF. Take any holomorphic mapping $\psi : E \to \mathcal{E}(p)$ such that $\psi(0) = (z, 0)$ and $\psi(t) = (z, w)$, $t > 0$. We may assume that ψ is continuous on $\bar{E}$. Write $\psi(\lambda) = (\psi_1(\lambda), \lambda^k\widetilde{\psi}_2(\lambda))$, where $\widetilde{\psi}_2(0) \ne 0$, $k \ge 1$. Put $\widetilde{\psi} := (\psi_1, \widetilde{\psi}_2)$. Clearly $|\psi_1(\lambda)|^{2p_1} + |\psi_2(\lambda)|^{2p_2} \le 1$ for $\lambda \in \partial E$, so $|\psi_1(\lambda)|^{2p_1} + |\widetilde{\psi}_2(\lambda)|^{2p_2} \leq 1$ for $\lambda \in \partial E$. The maximum principle for subharmonic functions implies

$$|\psi_1(\lambda)|^{2p_1} + |\widetilde{\psi}_2(\lambda)|^{2p_2} \leq 1, \quad \lambda \in E.$$

In particular, putting $\lambda := t$, we have

$$|z|^{2p_1} + \frac{|w|^{2p_2}}{t^{2p_2 k}} \leq 1.$$

So we obtain

$$t \geq t^k \geq \frac{|w|}{(1-|z|^{2p_1})^{1/(2p_2)}}.$$

This completes the proof. ∎

In connection with the last lemma observe that for any $(z,u),(z,v) \in \mathcal{E}(p) \subset \mathbb{C}^2$,

$$\widetilde{k}^*_{\mathcal{E}(p)}((z,u),(z,v)) \leq m\bigg(\frac{u}{(1-|z|^{2p_1})^{1/(2p_2)}}, \frac{v}{(1-|z|^{2p_1})^{1/(2p_2)}}\bigg).$$

It turns out that the sharp inequality above has far reaching consequences.

Lemma 4.1.7. *Let (z,u), $(z,v) \in \mathcal{E}(p) \subset \mathbb{C}^2$. Assume that*

$$\widetilde{k}^*_{\mathcal{E}(p)}((z,u),(z,v)) < m\bigg(\frac{u}{(1-|z|^{2p_1})^{1/(2p_2)}}, \frac{v}{(1-|z|^{2p_1})^{1/(2p_2)}}\bigg).$$

Then there are $b \in \partial\mathcal{E}(p)$ and $\lambda_1, \lambda_2 \in E$ such that

$$\widetilde{k}^*_{\mathcal{E}(p)}(\lambda_1 b, \lambda_2 b) < m(\lambda_1, \lambda_2). \tag{4.1.2}$$

Proof. Define

$$b := (b_1, b_2) := (z, (1-|z|^{2p_1})^{1/(2p_2)}) \in \partial\mathcal{E}(p).$$

If we had equality in (4.1.2) for any $\lambda_1, \lambda_2 \in E$, then the mapping $E \ni \lambda \mapsto \lambda b \in \mathcal{E}(p)$ would be a $\widetilde{k}_{\mathcal{E}(p)}$-geodesic for any pair of points from the image. But due to Lemma 4.1.5, so is the mapping $(b_1, b_2\lambda) = (z, b_2\lambda)$. This, however, contradicts the assumption of the lemma. ∎

Proof of Theorem 4.1.1. It is enough to prove the theorem in dimension two because by the contractivity of $\widetilde{k}_D$ we have $\widetilde{k}^*_{\mathcal{E}(p_1,p_2)} = \widetilde{k}^*_{\mathcal{E}(p)}|_{(\mathcal{E}(p_1,p_2)\times\{0\}^{n-2})^2}$.

By Proposition 1.1.2, Lempert Theorem, Corollary 4.1.4, and Lemma 4.1.7 it is sufficient to find, for any nonconvex ellipsoid $\mathcal{E}(p)$, points $(z,u),(z,v) \in \mathcal{E}(p)$ such that

$$\widetilde{k}^*_{\mathcal{E}(p)}((z,u),(z,v)) < m\bigg(\frac{u}{(1-|z|^{2p_1})^{1/(2p_2)}}, \frac{v}{(1-|z|^{2p_1})^{1/(2p_2)}}\bigg), \tag{4.1.3}$$

We consider two cases:

Case (I): $p_1, p_2 < 1/2$. For $t_1, t_2 \in (0,1)$ define, on E,

$$\varphi(\lambda) := \bigg(\Big(\frac{t_2}{(t_2+t_1)(1+t_1t_2)}\Big)^{1/(2p_1)}(1-t_1\lambda)^{1/p_1}, \Big(\frac{t_1}{(t_2+t_1)(1+t_1t_2)}\Big)^{1/(2p_2)}(1+t_2\lambda)^{1/p_2}\bigg).$$

Notice that φ is exactly of one of the forms from Theorem 1.3.3 (with

$$a_j = \Big(\frac{t_{3-j}}{(t_2+t_1)(1+t_1t_2)}\Big)^{1/(2p_j)},$$

$j=1,2$, $\alpha_1 = t_1$, $\alpha_2 = -t_2$, $\alpha_0 = 0$). One may easily verify that $\varphi(E) \subset \mathcal{E}(p)$.

The numbers t_1 and t_2 and consequently φ will be fixed later. Our aim is to find φ (or equivalently t_1, t_2), $\lambda_1 = x + iy \in E$, $\lambda_2 = \overline{\lambda}_1$ (with $x, y > 0$) such that

$$\varphi_1(\lambda_1) = \varphi_1(\lambda_2) =: z, \tag{4.1.4}$$

$$u := \varphi_2(\lambda_1) = \overline{\varphi(\lambda_2)} =: \overline{v}, \quad \operatorname{Arg}(\varphi_2(\lambda_1)) = \operatorname{Arg}(\lambda_1) \in (0, \pi/2), \tag{4.1.5}$$

$$\frac{|u|}{(1 - |z|^{2p_1})^{1/(2p_2)}} > |\lambda_1|. \tag{4.1.6}$$

In fact, assuming that the conditions (4.1.4)–(4.1.6) are satisfied, by elementary properties of m and the definition of $\widetilde{k}^*$, we have (remember $\lambda_1 = \overline{\lambda}_2$)

$$\begin{aligned} m\Big(\frac{u}{(1 - |z|^{2p_1})^{1/(2p_2)}}, \frac{v}{(1 - |z|^{2p_1})^{1/(2p_2)}}\Big) &> m(\lambda_1, \lambda_2) \\ &\geq \widetilde{k}^*_{\mathcal{E}(p)}(\varphi(\lambda_1), \varphi(\lambda_2)) = \widetilde{k}^*_{\mathcal{E}(p)}((z, u), (z, v)), \end{aligned}$$

which gives (4.1.3) and finishes the proof (in Case (I)).

To get properties (4.1.4) and (4.1.5) it is enough to have

$$\frac{1}{p_1} \operatorname{arctg} \frac{t_1 y}{1 - t_1 x} = \pi, \tag{4.1.7}$$

$$\operatorname{arctg} \frac{y}{x} = \frac{1}{p_2} \operatorname{arctg} \frac{t_2 y}{1 + t_2 x} \qquad (=: \alpha \in (0, \pi/2)), \tag{4.1.8}$$

which gives

$$y = x \operatorname{tg} \alpha =: a_3 x, \tag{4.1.9}$$

$$t_2 = \frac{\operatorname{tg}(p_2\alpha)}{y - x \operatorname{tg}(p_2\alpha)} = \frac{\operatorname{tg}(p_2\alpha)}{x(\operatorname{tg}\alpha - \operatorname{tg}(p_2\alpha))} =: \frac{a_2}{x}, \tag{4.1.10}$$

$$t_1 = \frac{\operatorname{tg}(p_1\pi)}{x(\operatorname{tg}\alpha + \operatorname{tg}(p_1\pi))} =: \frac{a_1}{x}. \tag{4.1.11}$$

Let us recall the restrictions imposed on the numbers involved:

$$x + iy \in E, \quad x, y > 0, \quad t_1, t_2 \in (0, 1), \quad \alpha \in (0, \pi/2).$$

Therefore, in particular, $x < 1/\sqrt{1 + \operatorname{tg}^2 \alpha}$.

We impose on t_2 the condition $t_2 < 1$. Substituting $x = 1/\sqrt{1 + \operatorname{tg}^2 \alpha}$ in (4.1.10) we have

$$\frac{\operatorname{tg}(p_2\alpha)\sqrt{1 + \operatorname{tg}^2 \alpha}}{\operatorname{tg}\alpha - \operatorname{tg}(p_2\alpha)} < \frac{\operatorname{tg}\frac{\alpha}{2}\sqrt{1 + \operatorname{tg}^2 \alpha}}{\operatorname{tg}\alpha - \operatorname{tg}\frac{\alpha}{2}} = 1,$$

since $p_2 < 1/2$. This implies that for $x < 1/\sqrt{1 + \operatorname{tg}^2 \alpha}$ close enough, t_2 given by (4.1.10) is smaller than 1.

But we also want $t_1 < 1$. Utilizing formula (4.1.11), after substituting as previously $x = 1/\sqrt{1 + \operatorname{tg}^2 \alpha}$ we have

$$\operatorname{tg}^2(p_1\pi) \operatorname{tg}\alpha < \operatorname{tg}\alpha + 2 \operatorname{tg}(p_1\pi).$$

The last inequality is satisfied for $\alpha > 0$ small enough, so for $\alpha > 0$ small enough $t_1 < 1$ for $x < 1/\sqrt{1 + \operatorname{tg}^2 \alpha}$ close enough.

We have proved so far the existence of x, y, t_1, t_2 such that (4.1.7) and (4.1.8) are satisfied (with $\alpha > 0$ small enough). In other words, to complete that case it is sufficient to prove that (4.1.6) holds for $\alpha > 0$ small enough, and $x < 1/\sqrt{1+\operatorname{tg}^2\alpha}$ close enough. More precisely, we want to show that (see (4.1.4)–(4.1.6))

$$\frac{\frac{t_1}{(t_1+t_2)(1+t_1t_2)}((1+t_2x)^2+t_2^2y^2)}{\big(1-\frac{t_2}{(t_1+t_2)(1+t_1t_2)}((1-t_1x)^2+t_1^2y^2)\big)} > (x^2+y^2)^{p_2},$$

which is equivalent to (use (4.1.9)–(4.1.11))

$$a_1((1+a_2)^2+a_2^2a_3^2) > x^{2p_2}(1+a_3^2)^{p_2}\bigg((a_1+a_2)\bigg(1+\frac{a_1a_2}{x^2}\bigg)-a_2((1-a_1)^2+a_1^2a_3^2)\bigg).$$

Equivalently,

$$\begin{aligned}0 > x^{2p_2}&(1+a_3^2)^{p_2}(a_1+2a_1a_2-a_1^2a_2-a_1^2a_2a_3^2)\\ &+x^{2p_2-2}(1+a_3^2)^{p_2}a_1a_2(a_1+a_2)-a_1((1+a_2)^2+a_2^2a_3^2) =: \psi(x).\end{aligned}$$

Our aim is to prove that if α is sufficiently small then for $x < 1/\sqrt{1+a_3^2}$ close enough, the above inequality holds.

One may easily verify that $\psi(1/\sqrt{1+a_3^2}) = 0$. It is sufficient to show that

$$\psi'\bigg(\frac{1}{\sqrt{1+a_3^2}}\bigg) > 0$$

if α is small enough. But this is equivalent to

$$p_2(a_1+2a_1a_2-a_1^2a_2-a_1^2a_2a_3^2)+(p_2-1)a_1a_2(a_1+a_2)(1+a_3^2) > 0,$$

or

$$p_2((1+a_2)^2+a_2^2a_3^2) > a_2(a_1+a_2)(1+a_3^2).$$

Substituting the formulas (4.1.9)–(4.1.11) we get

$$\begin{aligned}p_2&\bigg(\frac{\operatorname{tg}^2\alpha}{(\operatorname{tg}\alpha-\operatorname{tg}(p_2\alpha))^2}+\frac{\operatorname{tg}^2\alpha\operatorname{tg}^2(p_2\alpha)}{(\operatorname{tg}\alpha-\operatorname{tg}(p_2\alpha))^2}\bigg)\\ &> \frac{\operatorname{tg}(p_2\alpha)}{\operatorname{tg}\alpha-\operatorname{tg}(p_2\alpha)}\,\frac{\operatorname{tg}\alpha(\operatorname{tg}(p_2\alpha)+\operatorname{tg}(p_1\pi))}{(\operatorname{tg}\alpha+\operatorname{tg}(p_1\pi))(\operatorname{tg}\alpha-\operatorname{tg}(p_2\alpha))}(1+\operatorname{tg}^2\alpha)\end{aligned}$$

or equivalently

$$p_2\frac{\operatorname{tg}\alpha}{1+\operatorname{tg}^2\alpha}\,\frac{1+\operatorname{tg}^2(p_2\alpha)}{\operatorname{tg}(p_2\alpha)} > \frac{\operatorname{tg}(p_2\alpha)+\operatorname{tg}(p_1\pi)}{\operatorname{tg}\alpha+\operatorname{tg}(p_1\pi)}$$

and, finally,

$$\beta(\alpha) := p_2\sin(2\alpha)(\operatorname{tg}\alpha+\operatorname{tg}(p_1\pi))-\sin(2p_2\alpha)(\operatorname{tg}(p_2\alpha)+\operatorname{tg}(p_1\pi)) > 0.$$

Note that ($0 < p_2 < 1/2$)

$$\beta(0)=\beta'(0)=0,\quad \beta''(0)=4p_2(1-p_2)>0,$$

which implies that $\beta(\alpha) > 0$ for $\alpha > 0$ small enough. This completes the proof.

Case (II): $p_1 < 1/2 \le p_2$. There is an $n \in \mathbb{N}$ ($n \ge 2$) such that $q_2 := (1/n)p_2 < 1/2$ ($q_1 := p_1$). Then by the proof of Case (I) there are $(z,u),(z,v) \in \mathcal{E}(q)$ such that (see (4.1.3))

$$\widetilde{k}^*_{\mathcal{E}(q)}((z,u),(z,v)) < m\bigg(\frac{u}{(1-|z|^{2q_1})^{1/(2q_2)}},\frac{v}{(1-|z|^{2q_1})^{1/(2q_2)}}\bigg). \tag{4.1.12}$$

Let φ be a $\widetilde{k}_{\mathcal{E}(q)}$-geodesic for $((z,u),(z,v))$ with $\varphi(\lambda_1)=(z,u)$ and $\varphi(\lambda_2)=(z,v)$ and let B_2 be the Blaschke product associated with φ_2. We have $\varphi_1\not\equiv z$ (this is a consequence of the Schwarz–Pick Lemma). By Lemma 4.1.5,

$$\mu(\lambda):=\bigg(\varphi_1(\lambda),\Big(\frac{\varphi_2(\lambda)}{B_2(\lambda)}\Big)^{1/n}\bigg),\quad \lambda\in E,$$

is a $\widetilde{k}_{\mathcal{E}(p)}$-geodesic for $(\mu(\lambda_1),\mu(\lambda_2)):=((z,\widetilde{u}),(z,\widetilde{v}))$. It is enough to show that

$$\widetilde{k}^*_{\mathcal{E}(p)}((z,\widetilde{u}),(z,\widetilde{v}))<m\bigg(\frac{\widetilde{u}}{(1-|z|^{2p_1})^{1/(2p_2)}},\frac{\widetilde{v}}{(1-|z|^{2p_1})^{1/(2p_2)}}\bigg).\tag{4.1.13}$$

Consider the mapping

$$\psi:E\ni\lambda\mapsto(z,\lambda(1-|z|^{2p_1})^{1/(2p_2)})\in\mathcal{E}(p).$$

If (4.1.13) did not hold, then we would have equality there. Then ψ is a $\widetilde{k}_{\mathcal{E}(p)}$-geodesic for $((z,\widetilde{u}),(z,\widetilde{v}))=:(\psi(\lambda_3),\psi(\lambda_4))$ with some $\lambda_3,\lambda_4\in E$. Consequently, the mapping $\widetilde{\psi}(\lambda):=(z,(\psi_2(\lambda))^nB_2(\lambda))$ is a $\widetilde{k}_{\mathcal{E}(q)}$-geodesic for $((z,u),(z,v))$ (because $\widetilde{\psi}(\lambda_3)=\varphi(\lambda_1)=(z,u)$, $\widetilde{\psi}(\lambda_4)=\varphi(\lambda_2)=(z,v)$, $m(\lambda_1,\lambda_2)=m(\lambda_3,\lambda_4)$ and φ is a $\widetilde{k}_{\mathcal{E}(q)}$-geodesic for $((z,u),(z,v))$). This, however, contradicts the fact that no such geodesic has constant first component (remember (4.1.13) and apply the Schwarz–Pick Lemma)—one may alternatively exclude that case using the description of geodesics from Theorem 1.3.3; namely, no geodesic has components with more than one zero (counted with multiplicities), which happens here. This finishes Case (II) and the proof of Theorem 4.1.1. ■

4.2. Infinitesimal symmetry of the Green function. We restrict ourselves to the problem of symmetry of the Green function for points close to each other. This turns out to be closely related to the problem of continuity of the Green function (that is the reason why all results in this chapter may be applied in bounded hyperconvex domains). We also examine in this connection the problem when "lim sup" in the definition of the Azukawa pseudometric may be replaced with "lim". It turns out that this is always the case when D is a bounded hyperconvex domain. On the other hand one cannot extend this result to the class of bounded pseudoconvex domains (see Example 4.2.10). Some results on continuity of the Azukawa pseudometric are also given. The results in this section come from [Zwo 98c].

Below we list the main results of this section.

For fixed $w\in D$ we often consider the following number:

$$\varepsilon(w):=\liminf_{z\to\partial D}g_D(w,z).$$

It is easy to see that for any bounded D we have $\varepsilon(w)>-\infty$ for any $w\in D$. As we shall see later if $\varepsilon(w)>-\infty$ then $g_D(w,z)>-\infty$ for any $z\in D$, $z\neq w$.

Our aim is to prove the continuity of the Azukawa pseudometric.

THEOREM 4.2.1. *Let D be a domain such that $\varepsilon(w)>-\infty$ for any $w\in D$ and g_D is a continuous function. Then A_D is a continuous function* (*on $D\times\mathbb{C}^n$*).

Note that bounded hyperconvex domains fulfill the assumptions of Theorem 4.2.1 (as well as the assumptions of all theorems from this section)—see Theorem 1.6.1.

In many cases we can replace "lim sup" in the definition of the Azukawa pseudometric with "lim" as the next result shows.

THEOREM 4.2.2. *Let $w \in D$, where D is a domain in $\mathbb{C}^n$ such that $g_D(w,\cdot)$ is continuous and $\varepsilon(w) > -\infty$. Then*

$$A_D(w;X) = \lim_{0\neq\lambda\to 0} \frac{\widetilde{g}_D(w, w+\lambda X)}{|\lambda|}, \quad X \in \mathbb{C}^n.$$

Let us underline once more that we cannot generalize Theorem 4.2.2 to all domains—a counterexample, given in Example 4.2.10, is a bounded pseudoconvex domain in $\mathbb{C}^2$. However, for many domains a sharper version of Theorem 4.2.2 remains true.

COROLLARY 4.2.3. *Let D be a domain such that g_D is continuous and $\varepsilon(w) > -\infty$ for any $w \in D$. Then for any $w \in D$, $X \in \mathbb{C}^n$ with $\|X\| = 1$,*

$$A_D(w;X) = \lim_{w',\, w''\to w,\, w'\neq w'',\, \frac{w'-w''}{\|w'-w''\|}\to X} \frac{\widetilde{g}_D(w',w'')}{\|w'-w''\|}.$$

As a conclusion of the above results it turns out that the Green function is "almost" symmetric when both variables are close to each other. More precisely, we have:

COROLLARY 4.2.4. *Let D be as in Corollary 4.2.3 and let $w \in D$ be fixed. Then*

$$\lim_{w',\, w''\to w,\, w'\neq w''} (g_D(w',w'') - g_D(w'',w')) = 0.$$

For $\infty > \varepsilon \geq 0$, $p \in D$ consider the *sublevel sets*

$$D_\varepsilon := D_\varepsilon(p) := \{z \in D : g_D(p,z) < -\varepsilon\}.$$

Note that $D_\varepsilon(p)$ is open (the Green function is plurisubharmonic, and therefore upper semicontinuous).

LEMMA 4.2.5. *Let D be a domain. Then $D_\varepsilon(p)$ is connected.*

PROOF. Let $u(z) := g_D(p;z)$, $z \in D$. Suppose that $D_\varepsilon(p)$ is not connected. Let U be a connected component of D_ε, $p \notin U$. The upper semicontinuity of the Green function implies that U is open. Since $u < -\varepsilon$ on U and $u(z) \geq -\varepsilon$ for $z \in \partial U \cap D$, the function

$$v(z) := \begin{cases} -\varepsilon & \text{if } z \in U, \\ u(z) & \text{if } z \in D \setminus U, \end{cases}$$

is plurisubharmonic (see [Kli 91]). Moreover, from the definition of the Green function and the fact that $U \cap P = \emptyset$ we have $u \geq v$, which implies that $u(z) \geq -\varepsilon$ for $z \in U$, a contradiction. ∎

REMARK 4.2.6. Note that if D is pseudoconvex then the sets D_ε are pseudoconvex. In the general case (D not pseudoconvex) this need not always be the case. Nevertheless, if D is a bounded domain then for large ε the sublevel sets D_ε are pseudoconvex. This follows from the fact that for large ε the set $D_\varepsilon \subset \mathbb{B}(p,r) \subset D$ for some $r > 0$. Even more generally, for any domain D, if $D_\varepsilon \subset U \subset D$, where U is pseudoconvex, then D_ε is pseudoconvex.

Below for unbounded domains D we say that $\infty \in \partial D$, and writing $z \to \infty$ we mean $\|z\| \to \infty$.

The lemma below will play a fundamental role.

LEMMA 4.2.7. *Let D be a domain in $\mathbb{C}^n$, $p \in D$ ($D_\varepsilon = D_\varepsilon(p)$). Then*

$$g_{D_\varepsilon}(p,z) = g_D(p,z) + \varepsilon, \tag{4.2.1}$$

$$\log A_{D_\varepsilon}(p;X) = \log A_D(p;X) + \varepsilon, \tag{4.2.2}$$

Moreover, for any $\varepsilon_1, \varepsilon_2 \geq 0$, we have $D_{\varepsilon_1+\varepsilon_2} = (D_{\varepsilon_1})_{\varepsilon_2}$.

PROOF. Note that $g_D(p,z) + \varepsilon < 0$ for $z \in D_\varepsilon$. Consequently, we have "$\geq$" in (4.2.1). Additionally, because for $z \in \partial D_\varepsilon \cap D$ we have

$$g_D(p,z) \geq -\varepsilon \geq \limsup_{w\to z,\, w\in D_\varepsilon} (g_{D_\varepsilon}(p,w) - \varepsilon),$$

the function

$$\omega(z) := \begin{cases} g_{D_\varepsilon}(p,z) - \varepsilon & \text{if } z \in D_\varepsilon, \\ g_D(p,z) & \text{if } z \in D \setminus D_\varepsilon, \end{cases}$$

is plurisubharmonic (see e.g. [Kli 91]). Therefore, $\omega(z) \leq g_D(p,z)$, $z \in D$, which completes the proof of (4.2.1).

Property (4.2.2) as well as the last part of the lemma follow from (4.2.1) and the definition of the Azukawa pseudometric. ∎

For $w \in D$ recall the definition

$$\varepsilon(w) := \liminf_{D\ni z\to\partial D} g_D(w,z).$$

We are interested in the case $\varepsilon(w) > -\infty$. Note that then $g_D(w,z) > -\infty$ for any $z \in D$, $z \neq w$. In fact, take any $\varepsilon > -\varepsilon(w)$. Then the set $D_\varepsilon(w)$ is bounded, otherwise, in view of Lemma 4.2.7 there would be a sequence $z^\nu \to \infty$, $z^\nu \in D_\varepsilon(w) \subset D$, such that

$$\limsup_{\nu\to\infty} g_D(w,z^\nu) = \limsup_{\nu\to\infty} g_{D_\varepsilon(w)}(w,z^\nu) - \varepsilon \leq -\varepsilon < \varepsilon(w),$$

a contradiction with the definition of $\varepsilon(w)$. Take any $z \in D$, $w \neq z$, such that $g_D(w,z) = -\infty$. Take any $\varepsilon > -\varepsilon(w)$. Then $z \in D_\varepsilon(w)$, boundedness of $D_\varepsilon(w)$ implies that $g_{D_\varepsilon(w)}(w,z) > -\infty$, which in view of Lemma 4.2.7 implies that $g_D(w,z) > -\infty$, a contradiction. The same reasoning (the fact that D_ε is bounded) implies that $A_D(w;X) > 0$ for any $w \in D$, $X \in \mathbb{C}^n \setminus \{0\}$.

LEMMA 4.2.8. *Fix $w \in D$. Assume that $g_D(w,\cdot)$ is continuous on D and $\varepsilon(w) > -\infty$. Then for any $\varepsilon \geq -\varepsilon(w)$ the domain $D_\varepsilon(w)$ is hyperconvex. Moreover, if $\varepsilon' > \varepsilon \geq -\varepsilon(w)$ then $D_{\varepsilon'}(w) \subset\subset D_\varepsilon(w)$. Consequently, $D_{\varepsilon'}(w)$ is a bounded hyperconvex domain.*

PROOF. If $z_0 \in \partial D_\varepsilon(w) \cap \partial D$ (z_0 may be ∞) then by Lemma 4.2.7,

$$\begin{aligned} 0 &\geq \limsup_{z\in D_\varepsilon(w),\, z\to z_0} g_{D_\varepsilon(w)}(w,z) \geq \liminf_{z\in D_\varepsilon(w),\, z\to z_0} g_{D_\varepsilon(w)}(w,z) \\ &= \liminf_{z\in D_\varepsilon(w),\, z\to z_0} g_D(w,z) + \varepsilon \geq \varepsilon(w) + \varepsilon \geq 0. \end{aligned}$$

If $z_0 \in \partial D_\varepsilon(w) \cap D$ then by continuity of $g_D(w,\cdot)$ and Lemma 4.2.7,

$$\lim_{z\in D_\varepsilon(w),\, z\to z_0} g_{D_\varepsilon(w)}(w,z) = 0.$$

Note that the definition of $\varepsilon(w)$ implies that $D_\varepsilon(w)$ is bounded for any $\varepsilon > -\varepsilon(w)$, which in view of Lemma 4.2.7 gives the second statement. ∎

Proof of Theorem 4.2.1. Fix $(w;X) \in D \times \mathbb{C}^n$. It is sufficient to prove that for any sequence $(w_\nu;X_\nu) \to (w;X)$,

$$\lim_{\nu\to\infty} A_D(w_\nu;X_\nu) = A_D(w;X). \tag{4.2.3}$$

Since A_D is upper semicontinuous we may assume that $X, X_\nu \neq 0$.

Fix $\varepsilon' > \varepsilon > -\varepsilon(w)$. From the assumptions of the theorem, Lemma 4.2.8 and Theorem 1.6.1 we can choose a sequence of affine isomorphisms $\Phi_\nu : \mathbb{C}^n \to \mathbb{C}^n$ such that for large ν:

$$\Phi_\nu(w_\nu) = w, \quad \Phi_\nu'(w_\nu)(X_\nu) = X, \tag{4.2.4}$$

$$\Phi_\nu(D_{\varepsilon'}(w_\nu)) \subset D_\varepsilon(w), \tag{4.2.5}$$

$$D_{\varepsilon'}(w) \subset\subset \Phi_\nu(D_\varepsilon(w_\nu)). \tag{4.2.6}$$

From Lemma 4.2.7, (4.2.4) and (4.2.5) we get

$$\begin{aligned}\log A_D(w_\nu;X_\nu) + \varepsilon' &= \log A_{D_{\varepsilon'}(w_\nu)}(w_\nu;X_\nu)\\ &= \log A_{\Phi_\nu(D_{\varepsilon'}(w_\nu))}(\Phi_\nu(w_\nu);\Phi_\nu'(w_\nu)X_\nu) = \log A_{\Phi_\nu(D_{\varepsilon'}(w_\nu))}(w;X)\\ &\geq \log A_{D_\varepsilon(w)}(w;X) = \log A_D(w;X) + \varepsilon.\end{aligned}$$

Consequently,

$$\log A_D(w_\nu;X_\nu) - \log A_D(w;X) \geq \varepsilon - \varepsilon'.$$

Analogously, Lemma 4.2.7, (4.2.4) and (4.2.6) give

$$\log A_D(w_\nu;X_\nu) - \log A_D(w;X) \leq \varepsilon' - \varepsilon.$$

Letting $\varepsilon' \to \varepsilon$ in both inequalities above we get (4.2.3). ∎

Proof of Theorem 4.2.2. In view of Lemmas 4.2.7 and 4.2.8 we can assume that D is a bounded hyperconvex domain.

Suppose that the theorem does not hold, so there are a sequence $\{\lambda_k\}_{k=1}^\infty \subset E_*$, $\varepsilon > 0$ and $X \in \mathbb{C}^n \setminus \{0\}$ such that $\lambda_k \to 0$ and

$$g_D(w, w+\lambda_k X) - \log|\lambda_k| < \log A_D(w;X) - 2\varepsilon, \quad k = 1,2,\ldots. \tag{4.2.7}$$

For convenience we may assume that $w = 0$. We have $D_\varepsilon \subset\subset D$. There is $\theta_0 \in (0,\pi)$ such that

$$e^{i\theta} D_\varepsilon \subset\subset D \quad \text{for any } |\theta| < \theta_0. \tag{4.2.8}$$

Taking a subsequence if necessary we may assume that $\lambda_k X \in D_\varepsilon$, $k = 1,2,\ldots$ We know that (by (4.2.7), (4.2.8), contractivity of the Green function, and Lemma 4.2.7)

$$\begin{aligned}&g_D(0, e^{i\theta}\lambda_k X) - \log|\lambda_k|\\ &\quad\leq g_{e^{i\theta}D_\varepsilon}(0, e^{i\theta}\lambda_k X) - \log|\lambda_k| = g_{D_\varepsilon}(0,\lambda_k X) - \log|\lambda_k|\\ &\quad= g_D(0,\lambda_k X) + \varepsilon - \log|\lambda_k| < \log A_D(0;X) - \varepsilon, \quad |\theta| < \theta_0, \quad k = 1,2,\ldots\end{aligned} \tag{4.2.9}$$

For $\lambda \in U$, where U is a sufficiently small neighborhood of 0 in $\mathbb{C}$, define a subharmonic function u as follows:

$$u(\lambda) := g_D(0,\lambda X) - \log|\lambda| \quad \text{for } \lambda \neq 0, \quad u(0) := \log A_D(0;X).$$

We may assume that $\lambda_k \in U$ for any k. As $\limsup_{\lambda\to 0} u(\lambda) = u(0)$ and u is upper semicontinuous, for k large enough we have

$$u(e^{i\theta}\lambda_k) < u(0) + \frac{\varepsilon 2\theta_0}{2\pi - 2\theta_0} =: u(0) + \widetilde{\varepsilon}, \quad \theta \in [-\pi, \pi]. \tag{4.2.10}$$

On the other hand we know from (4.2.9) that

$$u(e^{i\theta}\lambda_k) < u(0) - \varepsilon \quad \text{for any } k \text{ and } |\theta| < \theta_0. \tag{4.2.11}$$

Subharmonicity of u combined with (4.2.10) and (4.2.11) gives, for large k,

$$\begin{aligned} 2\pi u(0) &\le \int_{-\pi}^{\pi} u(e^{i\theta}\lambda_k)\,d\theta < \int_{|\theta|<\theta_0} (u(0)-\varepsilon)\,d\theta + \int_{\pi \ge |\theta| > \theta_0} u(e^{i\theta}\lambda_k)\,d\theta \\ &< (u(0)-\varepsilon)2\theta_0 + (2\pi - 2\theta_0)(u(0)+\widetilde{\varepsilon}) = 2\pi u(0), \end{aligned}$$

a contradiction. ■

LEMMA 4.2.9. *Assume that D is a domain such that $\varepsilon(w) > -\infty$ for any $w \in D$ and g_D is continuous. Fix $w \in D$. Assume that sequences $\{w_j^\nu\}_{\nu=1}^\infty$, $\{z_j^\nu\}_{\nu=1}^\infty \subset D$, $j = 1, 2$, are such that*

$$w_j^\nu \ne z_j^\nu \to w, \quad j = 1,2, \quad \frac{w_1^\nu - w_2^\nu}{\|w_1^\nu - w_2^\nu\|} - \frac{z_1^\nu - z_2^\nu}{\|z_1^\nu - z_2^\nu\|} \to 0, \quad \frac{\|w_1^\nu - w_2^\nu\|}{\|z_1^\nu - z_2^\nu\|} \to 1.$$

Then $g_D(w_1^\nu, w_2^\nu) - g_D(z_1^\nu, z_2^\nu) \to 0$.

PROOF. Fix $\varepsilon' > \varepsilon > -\varepsilon(w)$. From the assumptions of the lemma, Theorem 1.6.1 and Lemma 4.2.8 we know that for ν large enough there is an affine isomorphism $\Phi_\nu : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$\Phi_\nu(w_j^\nu) = z_j^\nu, \quad j = 1,2, \quad \Phi_\nu(D_{\varepsilon'}(w_1^\nu)) \subset\subset D_\varepsilon(z_1^\nu).$$

Now we have, in view of Lemma 4.2.7, for ν large enough,

$$\begin{aligned} g_D(z_1^\nu, z_2^\nu) + \varepsilon &= g_{D_\varepsilon(z_1^\nu)}(z_1^\nu, z_2^\nu) \le g_{\Phi_\nu(D_{\varepsilon'}(w_1^\nu))}(\Phi_\nu(w_1^\nu), \Phi_\nu(w_2^\nu)) \\ &= g_{D_{\varepsilon'}(w_1^\nu)}(w_1^\nu, w_2^\nu) = g_D(w_1^\nu, w_2^\nu) + \varepsilon'. \end{aligned}$$

Consequently, for ν large enough,

$$g_D(z_1^\nu, z_2^\nu) \le g_D(w_1^\nu, w_2^\nu) + \varepsilon' - \varepsilon.$$

Similarly, for ν large enough,

$$g_D(w_1^\nu, w_2^\nu) \le g_D(z_1^\nu, z_2^\nu) + \varepsilon' - \varepsilon.$$

Letting $\varepsilon' \to \varepsilon$ we complete the proof. ■

Proof of Corollary 4.2.3. Take any sequences $\{w_j^\nu\}_{\nu=1}^\infty$ of different points from D such that $w_j^\nu \to w$ $(j = 1,2)$ and $(w_1^\nu - w_2^\nu)/\|w_1^\nu - w_2^\nu\| \to X$. Define

$$z_1^\nu := w, \quad z_2^\nu := w - \|w_1^\nu - w_2^\nu\| X.$$

Note that $\|z_1^\nu - z_2^\nu\| = \|w_1^\nu - w_2^\nu\|$ and $(z_1^\nu - z_2^\nu)/\|z_1^\nu - z_2^\nu\| \to X$. Therefore, in view of Lemma 4.2.9,

$$\begin{aligned} \lim_{\nu\to\infty} (g_D(w_1^\nu, w_2^\nu) - \log\|w_1^\nu - w_2^\nu\|) &= \lim_{\nu\to\infty} (g_D(z_1^\nu, z_2^\nu) - \log\|z_1^\nu - z_2^\nu\|) \\ &= \lim_{\nu\to\infty} (g_D(w, w - \|w_1^\nu - w_2^\nu\|X) - \log\|w_1^\nu - w_2^\nu\|); \end{aligned}$$

the last expression is, in view of Theorem 4.2.2, equal to $\log A_D(w;X)$. ■

Proof of Corollary 4.2.4. It is sufficient to consider two sequences $\{w_j^\nu\}_{\nu=1}^\infty$ $(j=1,2)$ tending to w such that $(w_1^\nu - w_2^\nu)/\|w_1^\nu - w_2^\nu\| \to X$ for some $X \in \mathbb{C}^n$, $\|X\|=1$. In view of Corollary 4.2.3,

$$\lim_{\nu\to\infty}(g_D(w_1^\nu, w_2^\nu) - \log\|w_1^\nu - w_2^\nu\|) = \log A_D(w;X)$$
$$= \log A_D(w;-X) = \lim_{\nu\to\infty}(g_D(w_2^\nu, w_1^\nu) - \log\|w_2^\nu - w_1^\nu\|),$$

from which we get $(\log A_D(w;X) > -\infty)$

$$\lim_{\nu\to\infty}(g_D(w_1^\nu, w_2^\nu) - g_D(w_2^\nu, w_1^\nu)) = 0. \ \blacksquare$$

EXAMPLE 4.2.10. There is a bounded pseudoconvex domain for which we cannot replace "lim sup" with "lim" in the definition of the Azukawa pseudometric. Let $D_h = \{z \in \mathbb{C}^2 : h(z) < 1\}$ be a bounded pseudoconvex balanced domain, where h is the Minkowski function of D_h, such that $h(1,1)=1$ and there are sequences $\{a_k\}_{k=1}^\infty$ and $\{b_k\}_{k=1}^\infty$ of points from E different from 0 such that $a_k \to 0$, $b_k \to 0$ and

$$\lim_{k\to\infty} h(1,\exp(a_k)) \to \delta < 1, \quad \lim_{k\to\infty} h(1,\exp(b_k)) = 1$$

(note that such a function and sequences exist). Define

$$\varPhi : \mathbb{C}^2 \ni (z_1,z_2) \mapsto (z_1, z_2\exp(z_1)) \in \mathbb{C}^2.$$

Note that $\varPhi$ is a biholomorphism. Put $D := \varPhi^{-1}(D_h)$. Observe that D is a bounded pseudoconvex domain. For k large we have (see Proposition 1.1.2)

$$g_D(0,(a_k,a_k)) - \log|a_k| = g_{D_h}(0,(a_k,\exp(a_k)a_k) - \log|a_k|$$
$$= \log h(a_k,\exp(a_k)a_k) - \log|a_k| = \log h(1,\exp(a_k))$$

and the last expression tends to $\log\delta < 0$ as k tends to infinity. Similarly, we get

$$g_D(0,(b_k,b_k)) \to 0 \text{ as } k \text{ tends to infinity},$$

which shows that there is no limit in the definition of the Azukawa metric of $A_D(0,(1,1))$.

COROLLARY 4.2.11. *If D is a domain in $\mathbb{C}$, then A_D is continuous and for any $w \in D$,*

$$A_D(w;1) = \lim_{w',\,w''\to w,\,w'\neq w''} \frac{\widetilde{g}_D(w',w'')}{|w'-w''|}.$$

PROOF. The result is trivial if ∂D is polar ([40]). Therefore, we may assume that ∂D is not polar. We know that g_D is continuous (see [Ran 95], [Hay-Ken 76]), therefore, in view of Theorems 4.2.1 and 4.2.2, and Corollary 4.2.3, it is sufficient to notice that for any $w \in D$, $\varepsilon(w) > -\infty$. $\blacksquare$

4.3. Nonsymmetry of the pluricomplex Green function of complete Reinhardt domains. Below we present some partial results, confirming our conjecture about the equivalence of convexity of the bounded pseudoconvex complete Reinhardt domain with symmetry of the Green function. We restrict ourselves to dimension two.

([40]) It is well known (see [Hay-Ken 76], [Ran 95]) that if $D \subset \mathbb{C}$, then either $g_D \equiv -\infty$ (if ∂D is polar) or $g_D(w,z) > -\infty$ for any $w \neq z$ (if ∂D is not polar).

Proposition 4.3.1. *Let D be a bounded pseudoconvex complete Reinhardt domain in $\mathbb{C}^2$ such that the boundary of $\log D$ contains a nontrivial interval not parallel to any axis. Then g_D is not symmetric.*

Proposition 4.3.1 follows directly from the following result (use contractivity of the Green function, Theorems 3.1 and 3.1.1 and Proposition 1.1.2):

Proposition 4.3.2. *Put*

$$D := D_{R_1,R_2} := \{(z_1,z_2) \in \mathbb{C}^2 : |z_1| < R_1,\ |z_2| < R_2,\ |z_1|\cdot|z_2|^\alpha < R_3\},$$

where $R_1R_2^\alpha > R_3$, $\alpha > 0$. Fix $b = (b_1,b_2) \in \partial D$ such that $|b_1| < R_1$, $|b_2| < R_2$.

- *If $\alpha = p/q$ (p and q are relatively prime positive integers) then $g_D(\lambda b, 0) = (p+q)\log|\lambda|$ for $\lambda \in E$ close enough to ∂E.*
- *If $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ then $\lim_{\lambda\to\partial E} g_D(\lambda b,0)/\log|\lambda| = \infty$.*

Remark 4.3.3. (a) The result of Proposition 4.3.2 gives much more than stated in Proposition 4.3.1. Namely, if a bounded pseudoconvex complete Reinhardt domain contains in the boundary an analytic disc of the "irrational" type (i.e. α above is irrational) then $\lim_{\lambda\to\partial E} g_D(\lambda b,0)/g_D(0,\lambda b) = \infty$ (41). In other words, the Green function is "extremely" nonsymmetric when one of the arguments goes to the boundary (and the other one is fixed) for a very regular domain (e.g. bounded pseudoconvex complete Reinhardt with smooth boundary) although for points close to each other it is "almost" symmetric (compare Corollary 4.2.4).

(b) In [Carl 97] the quotient

$$\frac{g_D(z_1,z_2)}{G_D(z_1,z_2)}, \quad z_1 \neq z_2 \in D,$$

where G_D denotes the classical Green function, was considered. Under the additional assumption that D is bounded balanced with C^2 boundary we know that $-G_D(\lambda b,0) \le C(1-|\lambda|)$ for $\lambda \in E$ close to ∂E, $b \in \partial D$ (see [Carl 97]). Therefore, from Proposition 4.3.2 (in the case of irrational α) we easily find a domain D (bounded, smooth and pseudoconvex complete Reinhardt) such that $g_D(\lambda b,0)/G_D(\lambda b,0)$ tends to infinity as λ tends to ∂E, which gives another example of such a domain (see [Carl 97]) with better regularity properties.

Define

$$g_D^N(w;z) := \inf\Big\{\sum_{j=1}^N \log|\lambda_j|\Big\},$$

where the infimum is taken over all possible $\varphi \in \mathcal{O}(E,D)$ with $\varphi(0) = z$ and $\varphi(\lambda_j) = w$; some of λ_j's may be the same (but they can be repeated no more than $\operatorname{ord}_{\lambda_j}(\varphi - w)$ times).

Recall that g_D^N tends decreasingly to g_D as $N \to \infty$ (see Theorem 1.7.1). Actually, we shall need only the fact that the limit is not smaller than the Green function, which is easy to obtain.

(41) We have $g_D(0,\lambda b), g_D(\lambda b,0) \to 0$ as λ tends to ∂E because of c-finite compactness of D (by Theorem 2.6.6).

Before we prove Proposition 4.3.2 we need a lemma.

LEMMA 4.3.4. *Let $N \in \mathbb{N}$, let $\mu_0, \dots, \mu_{N-1}$ be pairwise different points from E and let $\lambda_0, \dots, \lambda_{N-1}$ be pairwise different points from ∂E. Then for $0 < s < 1$ sufficiently close to 1 there is $\psi \in \mathcal{O}(E, E)$ such that*

$$\psi(s\lambda_k) = \mu_k, \quad k = 0, \dots, N-1.$$

PROOF. Take $0 < t < 1$ such that $\mu_0, \dots, \mu_{N-1} \in tE$. Put

$$\Omega := tE \setminus \{\mu_0, \dots, \mu_{N-1}\}.$$

Let $\pi : E \to \Omega$ be a holomorphic covering. Then there are points $\nu_0, \dots, \nu_{N-1} \in \partial E$ such that $\pi^*(\nu_k) = \mu_k$, $k = 0, \dots, N-1$ (see [Nos 60]).

There is a finite Blaschke product B satisfying $B(\lambda_k) = \nu_k$, $k = 0, \dots, N-1$ (see [You 80], [Abr-Fis 80]).

Define $\widetilde{\pi} := \pi \circ B$. Then $\widetilde{\pi}^*(\lambda_k) = \mu_k$, $k = 0, \dots, N-1$, and $\|\widetilde{\pi}\|_E = t$. Therefore, for $s < 1$ large enough

$$|\widetilde{\pi}(s\lambda_k) - \mu_k| < \frac{(1-t)^2}{N}, \quad \prod_{j \neq k} \left| \frac{s\lambda_k - s\lambda_j}{1 - s^2\overline{\lambda}_j\lambda_k} \right| > 1 - t, \quad k = 0, \dots, N-1.$$

Define

$$\psi(\lambda) := \widetilde{\pi}(\lambda) + \sum_{k=0}^{N-1} (\mu_k - \widetilde{\pi}(s\lambda_k)) \frac{\prod_{j \neq k} \frac{\lambda - s\lambda_j}{1 - s^2\overline{\lambda}_j\lambda}}{\prod_{j \neq k} \frac{s\lambda_k - s\lambda_j}{1 - s^2\overline{\lambda}_j\lambda_k}}.$$

The function ψ has the desired properties. ∎

Proof of Proposition 4.3.2. We want to show that if $\alpha \notin \mathbb{Q}$ then

(4.3.1) $$\lim_{|\lambda| \to 1} g_D(\lambda b; 0) / \log|\lambda| = \infty$$

and if $\alpha = p/q$ then

(4.3.2) $$g_D(\lambda b, 0) = (p+q)\log|\lambda|$$

for λ close to ∂E.

To prove (4.3.2) it is sufficient to show that $g_D^N(\lambda b, 0) \leq (p+q)\log|\lambda|$ for $N = p+q$ and λ close to ∂E (use Theorem 3.1 and the contractivity of the Green function).

We may assume $R_3 = 1$. Applying the mapping $(z_1, z_2) \mapsto (z_1/b_1, z_2/b_2)$ we may assume $b = (1,1)$.

Let $\pi : H \ni \lambda \mapsto \exp(\lambda) \in P$, where $H := \{x + iy : -\log R_2 < y < (1/\alpha)\log R_1\}$ and $P := \{\lambda \in \mathbb{C} : 1/R_2 < |\lambda| < R_1^{1/\alpha}\}$. The map π is a holomorphic covering. Fix N (if α is rational we fix $N = p+q$). Define

$$\lambda_k := \exp\left(\frac{ik2\pi}{1+\alpha}\right), \quad k = 0, \dots, N-1.$$

The points λ_k, $k = 0, \dots, N-1$, are pairwise different. Put $\mu_k := i2k\pi/(1+\alpha) \in H$, $k = 0, \dots, N-1$. We have $\pi(\mu_k) = \lambda_k$, $k = 0, \dots, N-1$. Applying Lemma 4.3.4 to

λ_k, μ_k we get for $s < 1$ sufficiently close to 1 a function $\psi \in \mathcal{O}(E, H)$ (remember the conformality of E and H) such that $\psi(s\lambda_k) = \mu_k$, $k = 0, \ldots, N-1$. Define

$$\varphi(\lambda) := \lambda\bigg(\exp(\alpha\psi(\lambda)), \frac{1}{\exp(\psi(\lambda))}\bigg), \quad \lambda \in E.$$

Then

$$\varphi \in \mathcal{O}(E, D),\ \varphi(0) = 0,\ \varphi(s\lambda_k) = sb, \quad k = 0, \ldots, N-1.$$

The last equalities easily imply

$$N \log s \geq g_D^N(sb, 0) \geq g_D(sb, 0) \quad \text{for } s \text{ sufficiently close to 1.}$$

In the rational case this finishes the proof, in the irrational case we let $N \to \infty$ to get (4.3.1).

V. Norm balls and Carathéodory balls in convex ellipsoids

Our aim is to sketch the proof of the following result:

THEOREM 5.1. *Let $\mathcal{E}(p)$ be a convex ellipsoid. If $p_1, \ldots, p_n \neq 1$ or $p_1 = \ldots = p_n = 1$ then a Carathéodory ball with center at w is a norm ball (of the form $B_{\mathcal{E}(p)}(\widetilde{w}, s)$) if and only if $w = 0$. On the other hand, if $n = 2$, $p_1 = 1/2$, $p_2 = 1$, then any ball $B_{c^*_{\mathcal{E}(p)}}((0, w_2), r)$ is a norm ball.*

The method of proof, involving the study of complex geodesics from Theorem 1.3.3, also turned out to be successful in the study of some more general domains (see [Vis 99]).

Directly from the form of complex geodesics in convex ellipsoids (Theorem 1.3.3) one may obtain the following result:

PROPOSITION 5.2. *Let z and w, $w_j = z_j$, $j = k+1, \ldots, n$, be distinct points in a convex ellipsoid $\mathcal{E}(p)$. Put $\gamma := 1 - \sum_{j=k+1}^n |z_j|^{2p_j}$. Let φ be a complex geodesic in $\mathcal{E}(p_1, \ldots, p_k)$ joining $(w_1/\gamma^{1/2p_1}, \ldots, w_k/\gamma^{1/2p_k})$ to $(z_1/\gamma^{1/2p_1}, \ldots, z_k/\gamma^{1/2p_k})$. Then*

$$\widetilde{\varphi}(\lambda) := (\gamma^{1/2p_1}\varphi_1, \ldots, \gamma^{1/2p_k}\varphi_k, z_{k+1}, \ldots, z_n)$$

is a complex geodesic joining w to z. Consequently,

$$c_{\mathcal{E}(p)}(w, z) = c_{\mathcal{E}(p_1, \ldots, p_k)}\bigg(\bigg(\frac{w_1}{\gamma^{1/2p_1}}, \ldots, \frac{w_k}{\gamma^{1/2p_k}}\bigg), \bigg(\frac{z_1}{\gamma^{1/2p_1}}, \ldots, \frac{z_k}{\gamma^{1/2p_k}}\bigg)\bigg).$$

One may read Proposition 5.2 as follows. If there are two points in a convex ellipsoid with some coordinates equal, then taking a complex geodesic in some lower dimensional ellipsoid (which is defined as the trace of the considered ellipsoid with constant coordinates) we get a complex geodesic in the higher dimensional ellipsoid.

The following technical lemma is needed in the proof:

LEMMA 5.3. *Let $\lambda_1, \lambda_2 > 0$, $r_1, r_2 > 0$, $\theta_1, \theta_2 \in \mathbb{R}$. Let $p = (p_1, p_2)$, $p_1, p_2 > 0$ and suppose p_1 and p_2 are not simultaneously 1. Assume that for $t \in \mathbb{R}$,*

$$\lambda_1|r_1e^{i(t+\theta_1)} - a_1|^{2p_1} + \lambda_2|r_2e^{i(t+\theta_2)} - a_2|^{2p_2} \equiv \text{const.}$$

Then $a_1 = a_2 = 0$.

The proof of Lemma 5.3 boils down to differentiating the expression considered.

Below we prove a generalization of a result from [Zwo 96] (see also [Vis 99]).

LEMMA 5.4. *Let D be a bounded convex domain. Let $w \in D$, $\widetilde{w} \in \mathbb{C}^n$, $\widetilde{r} > 0$, $1 > r > 0$. Assume that*

$$\partial B_{c_D^*}(w,r) \subset \partial B_G(\widetilde{w},\widetilde{r}),$$

where G is a bounded pseudoconvex balanced domain with the continuous Minkowski function h. Then

$$B_G(\widetilde{w},\widetilde{r}) = B_{c_D^*}(w,r).$$

PROOF. First we prove that

$$w \in B_G(\widetilde{w},\widetilde{r}), \quad \widetilde{w} \in B_{c_D^*}(w,r).$$

Suppose that $w \neq \widetilde{w}$. The continuity of the Carathéodory distance and the fact that the Carathéodory distance (of D) tends to infinity near the boundary gives the existence of $s < 0$ such that $\widetilde{w} + (s-1)(\widetilde{w} - w) = w + s(\widetilde{w} - w) \in \partial B_{c_D^*}(w,r) \subset \partial B_G(\widetilde{w},\widetilde{r})$.

Suppose that $w \notin B_G(\widetilde{w},\widetilde{r})$. Then $h(w - \widetilde{w}) \geq \widetilde{r}$ and consequently there is $t \in (0,1]$ such that $\widetilde{w} - t(\widetilde{w} - w) = tw + (1-t)\widetilde{w} \in \partial B_G(\widetilde{w},\widetilde{r})$. So there are two distinct points lying on $\widetilde{w} + \mathbb{R}_+(w - \widetilde{w})$, which belong to $\partial B_G(\widetilde{w},\widetilde{r})$, a contradiction.

Suppose that $\widetilde{w} \notin B_{c_D^*}(w,r)$. Then the continuity of the Carathéodory distance and the convergence of the Carathéodory distance to infinity near the boundary yield the existence of $t \in [0,1)$ such that $\widetilde{w} - t(\widetilde{w} - w) = tw + (1-t)\widetilde{w} \in \partial B_{c_D^*}(w,r) \subset \partial B_G(\widetilde{w},\widetilde{r})$. As before we get the existence of two points from $\partial B_G(\widetilde{w},\widetilde{r})$ lying on $\widetilde{w} + \mathbb{R}_+(w - \widetilde{w})$, a contradiction.

We now prove the inclusion

$$B_{c_D^*}(w,r) \subset B_G(\widetilde{w},\widetilde{r}).$$

Take $z \in B_{c_D^*}(w,r)$. Then there are $0 \leq s < r$ and a complex geodesic $\varphi : E \to D$ such that $\varphi(0) = w$, $\varphi(s) = z$. Define

$$\widetilde{h} : E \ni \lambda \mapsto h(\varphi(\lambda) - \widetilde{w}) \in \mathbb{R}.$$

Then $\widetilde{h}$ is a subharmonic function. Since $\varphi(0) = w \in B_G(\widetilde{w},\widetilde{r})$, we get $\widetilde{h}(0) < \widetilde{r}$. For λ with $|\lambda| = r$ we have $c_D^*(\varphi(\lambda), w) = r$, so $\varphi(\lambda) \in \partial B_{c_D^*}(w,r) \subset \partial B_G(\widetilde{w},\widetilde{r})$. Consequently, $\widetilde{h}(\lambda) = \widetilde{r}$ for $|\lambda| = r$ (because h is continuous). But the maximum principle for subharmonic functions implies that $\widetilde{h}(\lambda) < \widetilde{r}$ for $|\lambda| < r$. This completes the proof of the first inclusion.

To get the reverse inclusion, suppose that there is $z \in B_G(\widetilde{w},\widetilde{r}) \setminus B_{c_D^*}(w,r)$. Since $\widetilde{w} \in B_{c_D^*}(w,r)$ and the function c_D^* is continuous we get the existence of $t \in [0,1)$ such that

$$z' := t\widetilde{w} + (1-t)z \in \partial B_{c_D^*}(w,r) \subset \partial B_G(\widetilde{w},\widetilde{r}),$$

so $h(z' - \widetilde{w}) = (1-t)h(z - \widetilde{w}) < \widetilde{r}$, a contradiction. ∎

Sketch of proof of Theorem 5.1. We may restrict our attention to the case when $p_j \neq 1$, $j = 1, \ldots, n$.

First we reduce the n-dimensional case to the two-dimensional one.

In order to get this reduction we have to extend a little the range of domains for which the equality of the Carathéodory ball to a domain of that type implies the center of the ball is 0. In other words we prove a little more than stated in Theorem 5.1.

For $\widetilde{r},\widetilde{\widetilde{r}}>0$, $w\in\mathbb{C}^n$ we define the following *N_p-ellipsoids:*

$$\mathcal{E}_p(w,\widetilde{r},\widetilde{\widetilde{r}}):=\{z\in\mathbb{C}^n: N_p(z-w,\widetilde{r})<\widetilde{\widetilde{r}}\},$$

where $N_p(z,\widetilde{r}):=|z_1/\widetilde{r}|^{2p_1}+\ldots+|z_n/\widetilde{r}|^{2p_n}$.

The condition $N_p(z,\widetilde{r})=1$ means that the value of the Minkowski function of $\mathcal{E}(p)$ at z is $\widetilde{r}$ or $B_{\mathcal{E}(p)}(w,\widetilde{r})=\mathcal{E}_p(w,\widetilde{r},1)$. Clearly, $\mathcal{E}_p(w,\widetilde{r},\widetilde{\widetilde{r}})$ is a bounded pseudoconvex Reinhardt domain.

Assume that $w_1\neq 0$. Below we show how to reduce the general problem to the two-dimensional one. Assume that $n\geq 3$. The mapping

$$\begin{aligned}&\mathcal{E}(p_1,p_2)\ni(z_1,z_2)\\ &\qquad\mapsto\Big(z_1\Big(1-\sum_{k=3}^n|w_k|^{2p_k}\Big)^{1/(2p_1)},z_2\Big(1-\sum_{k=3}^n|w_k|^{2p_k}\Big)^{1/(2p_2)},w_3,\ldots,w_n\Big)\in\mathcal{E}(p)\end{aligned}$$

is a Carathéodory distance preserving function (use Proposition 5.2). In particular, the preimage of the intersection of the Carathéodory ball with center at w with $\mathbb{C}^2\times\{(w_3,\ldots,w_n)\}$ is a Carathéodory ball in $\mathcal{E}(p)$ with center at $(w_1/(1-\sum_{k=3}^n|w_k|^{2p_k})^{1/(2p_1)})$, $w_2/(1-\sum_{k=3}^n|w_k|^{2p_k})^{1/(2p_2)}))$. The preimage of the intersection of an N_p-ellipsoid with the same set is an $N_{(p_1,p_2)}$-ellipsoid ([42]).

How do we prove the desired result in dimension two?

Assume that $w\neq 0$ and there are $\widetilde{w}$, $r,\widetilde{r},\widetilde{\widetilde{r}}$ such that

$$\partial B_{c^*_{\mathcal{E}(p)}}(w,r)=\partial\mathcal{E}_p(\widetilde{w},\widetilde{r},\widetilde{\widetilde{r}}). \tag{5.1}$$

We consider one-dimensional subsets of $\partial B_{c^*_{\mathcal{E}(p)}}(w,r)$ which correspond to the following subsets of $\mathbb{C}$:

$$\begin{aligned}A_1&:=\{\zeta\in\mathbb{C}:(\zeta,w_2)\in\partial B_{c^*_{\mathcal{E}(p)}}(w,r)\},\\ A_2&:=\{\zeta\in\mathbb{C}:(w_1,\zeta)\in\partial B_{c^*_{\mathcal{E}(p)}}(w,r)\},\\ B&:=\{\zeta\in\mathbb{C}:\zeta w\in\partial B_{c^*_{\mathcal{E}(p)}}(w,r)\}.\end{aligned}$$

For $\zeta\in A_1$ we have, in view of Proposition 5.2,

$$\tanh^{-1}r=c_{\mathcal{E}(p)}((\zeta,w_2),w)=p\bigg(\frac{\zeta}{(1-|w_2|^{2p_2})^{1/2p_1}},\frac{w_1}{(1-|w_2|^{2p_2})^{1/(2p_1)}}\bigg),$$

which implies that $\zeta/(1-|w_2|^{2p_2})^{1/(2p_1)}$ lies on the hyperbolic circle with center at $w_1/(1-|w_2|^{2p_2})^{1/(2p_1)}$ and radius $\tanh^{-1}r$. But this means, in view of the description of the Carathéodory disks in E, that (we can proceed with A_2 analogously)

$$\begin{aligned}A_j=\bigg\{\zeta:\zeta=&\frac{(1-r^2)(1-|w_{3-j}|^{2p_{3-j}})^{1/p_j}}{(1-|w_{3-j}|^{2p_j})^{1/p_j}-r^2|w_j|^2}w_j\\ &+r(1-|w_{3-j}|^{2p_{3-j}})^{1/(2p_j)}\frac{(1-|w_{3-j}|^{2p_{3-j}})^{1/p_j}-|w_j|^2}{(1-|w_{3-j}|^{2p_{3-j}})^{1/p_j}-r^2|w_j|^2}e^{i\theta},0\leq\theta\leq 2\pi\bigg\}\end{aligned}$$

([42]) Here it is essential that we consider a more general class of domains (not only norm balls), because the preimages of norm balls need not be norm balls.

for $j=1,2$. But from equality (5.1) we know that for $\zeta\in A_j$, $j=1,2$,

$$\widetilde{\widetilde{r}}=\left|\frac{\zeta-\widetilde{w}_j}{\widetilde{r}}\right|^{2p_j}+\left|\frac{w_{3-j}-\widetilde{w}_{3-j}}{\widetilde{r}}\right|^{2p_{3-j}}.$$

From the form of A_j we get

$$\widetilde{w}_j=\frac{(1-r^2)(1-|w_{3-j}|^{2p_{3-j}})^{1/p_j}}{(1-|w_{3-j}|^{2p_{3-j}})^{1/p_j}-r^2|w_j|^2}w_j,\quad j=1,2. \tag{5.2}$$

In particular,

$$w_j=0\quad\text{if and only if}\quad\widetilde{w}_j=0.$$

Below we consider two cases.

Case (I): $w_1,w_2\neq 0$. In this case we get a contradiction. For $\zeta\in B$ we have

$$\tanh^{-1}r=c_{\mathcal{E}(p)}(\zeta w,w)=p(h(w)\zeta,h(w)),$$

where h is the Minkowski function of $\mathcal{E}(p)$. Consequently, the points $h(w)\zeta$, where $\zeta\in B$, lie on a hyperbolic circle in E, hyperbolically centered at $h(w)$. Therefore, this is a Euclidean circle. Moreover, B is the circle given by

$$B=\{\zeta=\zeta_0+Re^{i\theta}:0\le\theta\le 2\pi\},$$

where

$$\zeta_0=\frac{1-r^2}{1-r^2h(w)^2},\qquad R=\frac{r}{h(w)}\frac{1-h(w)^2}{1-r^2h(w)^2}.$$

Thus, for all $\zeta\in B$,

$$\zeta w_j=\zeta_0w_j+R_je^{i(t+\theta_j)},\quad t\in\mathbb{R},$$

where

$$R_j=R|w_j|,\quad \theta_j=\arg w_j,\quad j=1,2.$$

In view of the definition of B and (5.1) we have, for $\zeta\in B$,

$$\begin{aligned}\widetilde{\widetilde{r}}&=N_p(\zeta w-\widetilde{w},\widetilde{r})\\&=\left|\frac{\zeta_0w_1+R_1e^{i(t+\theta_1)}-\widetilde{w}_1}{\widetilde{r}}\right|^{2p_1}+\left|\frac{\zeta_0w_2+R_2e^{i(t+\theta_2)}-\widetilde{w}_2}{\widetilde{r}}\right|^{2p_2},\quad t\in\mathbb{R}.\end{aligned}$$

By Lemma 5.3 we get $\widetilde{w}=\zeta_0w$. Therefore, in particular

$$\frac{1-r^2}{(1-|w_2|^{2p_2})^{1/p_1}-r^2|w_1|^2}(1-|w_2|^{2p_2})^{1/p_1}w_1=\frac{1-r^2}{1-r^2h(w)^2}w_1.$$

From the last equality we get (remember that $0<r<1$, $w_1\neq 0$)

$$(1-|w_2|^{2p_2})^{1/(2p_1)}h(w)=|w_1|.$$

Equivalently

$$h\bigg(\frac{(1-|w_2|^{2p_2})^{1/(2p_1)}}{|w_1|}w\bigg)=1,$$

so that (remember that h is the Minkowski function of $\mathcal{E}(p)$)

$$1-|w_2|^{2p_2}+\frac{(1-|w_2|^{2p_2})^{2p_2/(2p_1)}}{|w_1|^{2p_2}}|w_2|^{2p_2}=1$$

and finally (remember $w_2 \neq 0$!)

$$1 = |w_1|^{2p_1} + |w_2|^{2p_2}$$

so $w \in \partial\mathcal{E}(p)$, a contradiction.

Case (II): $w_1 = 0$, $w_2 \neq 0$. We know that (see (5.2))

$$B_{c^*_{\mathcal{E}(p)}}((0,w_2),r) = \mathcal{E}_p((0,\widetilde{w}_2),\widetilde{r},\widetilde{\widetilde{r}}), \quad \text{where} \quad \widetilde{w}_2 = \frac{w_2(1-r^2)}{1-r^2|w_2|^2}.$$

Consider the geodesics (see Theorem 1.3.3)

$$\varphi_{\alpha_2}(\lambda) := \left(\frac{((1-|w_2|^{2p_2})(1-\alpha_2^2|w_2|^{2p_2}))^{1/(2p_1)}}{(1-|w_2|^{2p_2}\alpha_2\lambda)^{1/(p_1)}}\lambda, w_2\left(\frac{1-\alpha_2\lambda}{1-|w_2|^{2p_2}\alpha_2\lambda}\right)^{1/(p_2)}\right)$$

for $\alpha_2 \in [-1,1]$. We see that $\varphi_{\alpha_2}(0) = (0,w_2)$. Therefore the points $\varphi_{\alpha_2}(r)$ for all $\alpha_2 \in [-1,1]$ are in $\partial B_{c^*_{\mathcal{E}(p)}}((0,w_2),r) = \partial\mathcal{E}_p((0,\widetilde{w}_2),\widetilde{r},\widetilde{\widetilde{r}})$. Then for $\alpha_2 \in [-1,1]$ we get

$$\begin{aligned} \widetilde{\widetilde{r}} &= N_p(\varphi_{\alpha_2}(r) - (0,\widetilde{w}_2),\widetilde{r}) \\ &= \frac{(1-|w_2|^{2p_2})(1-\alpha_2^2|w_2|^{2p_2})r^{2p_1}}{(1-|w_2|^{2p_2}\alpha_2 r)^2\widetilde{r}^{2p_1}} \\ &\quad + \frac{|w_2|^{2p_2}}{\widetilde{r}^{2p_2}}\left|\left(\frac{1-\alpha_2 r}{1-|w_2|^{2p_2}\alpha_2 r}\right)^{1/p_2} - \frac{1-r^2}{1-r^2|w_2|^2}\right|^{2p_2}. \end{aligned} \tag{5.3}$$

The first summand on the right hand side of (5.3) increases in α_2 for $\alpha_2 < r$ and decreases for $\alpha_2 > r$. Since the expression in the second summand with exponent $1/p_2$ decreases in α_2 and the sum in (5.3) is constant we see that the second summand must be zero for $\alpha_2 = r$.

Let us differentiate (5.3) with respect to α_2 where it is possible (the only exceptions are the points $\alpha_2 = r$ if $p_2 = 1/2$). Then we get

$$\begin{aligned} &\frac{(1-|w_2|^{2p_2})r^{2p_1}}{\widetilde{r}^{2p_1}}\,\frac{2|w_2|^{2p_2}(r-\alpha_2)}{(1-|w_2|^{2p_2}\alpha_2 r)^3} \\ &\qquad \pm \frac{|w_2|^{2p_2}}{\widetilde{r}^{2p_2}}2p_2\left|\left(\frac{1-\alpha_2 r}{1-|w_2|^{2p_2}\alpha_2 r}\right)^{1/p_2} - \frac{1-r^2}{1-r^2|w_2|^2}\right|^{2p_2-1} \\ &\qquad \times \frac{1}{p_2}\left(\frac{1-\alpha_2 r}{1-|w_2|^{2p_2}\alpha_2 r}\right)^{1/p_2-1}\frac{r(|w_2|^{2p_2}-1)}{(1-|w_2|^{2p_2}\alpha_2 r)^2} \equiv 0 \end{aligned} \tag{5.4}$$

for all possible α_2, the sign being "+" for $\alpha_2 < r$ and "−" for $\alpha_2 > r$.

Since all the functions appearing in (5.4) are real analytic for $1/r > \alpha_2 > r$, the limit of (5.4) as $\alpha_2 \to 1/r$ must be zero, which however may hold only for $p_2 = 1$. This gives the first part of the theorem for $n = 2$.

To get the second part we proceed as follows. Keeping in mind that $n = 2$, $p_2 = 1$, $p_1 \geq 1/2$ we take a point $w = (0,w_2)$ and find conditions equivalent to $(z_1,z_2) \in \mathcal{E}(p)$ lying in $\partial B_{c^*_{\mathcal{E}(p)}}(w,r)$. We have (for the description of automorphisms of ellipsoids in $\mathbb{C}^2$ with $p_2 = 1$, which we use below, see e.g. [Jar-Pfl 93])

$$r = c^*_{B_{\mathcal{E}(p)}}((0,w_2),(z_1,z_2)) = c^*_{B_{\mathcal{E}(p)}}\left((0,0),\left(\left(\frac{1-|w_2|^2}{(1-\overline{w}_2 z_2)^2}\right)^{1/(2p_1)} z_1, \frac{z_2-w_2}{1-\overline{w}_2 z_2}\right)\right),$$

which is equivalent after some calculations to the fact that a point $(z_1, z_2) \in \partial B_{c^*_{\mathcal{E}(p)}}(w, r)$ is in $\partial\mathcal{E}_p(\widetilde{w}, \widetilde{r}, \widetilde{\widetilde{r}})$ ($\widetilde{w}$ is given before) for some $\widetilde{r} > 0$ if and only if

$$\widetilde{\widetilde{r}}\,\widetilde{r}^{2p_1} = \frac{r^{2p_1}(1 - |w_2|^2)}{1 - r^2|w_2|^2}, \quad \widetilde{\widetilde{r}}\,\widetilde{r}^2 = \frac{r^2(1 - |w_2|^2)^2}{(1 - r^2|w_2|^2)^2},$$

which gives the second part of the theorem (use Lemma 5.4). ∎

Remark 5.5. As already mentioned the following result was proven in [Vis 99]: In convex ellipsoids a Carathéodory ball with center $w \neq 0$ is a norm ball exactly if there is exactly one j such that $p_j = 1$ and $p_k = 1/2$, $w_k = 0$, $k \neq j$. Note that following the reasoning used in the proof of Theorem 5.1 we may obtain the same result.

List of symbols

General symbols

$\mathbb{C}$:= the field of complex numbers;
$\mathbb{R}$:= the field of real numbers;
$\mathbb{Q}$:= the field of rational numbers;
$\mathbb{Z}$:= the ring of integers;
$\mathbb{N}$:= the set of natural numbers ($0 \in \mathbb{N}$);
$\operatorname{Re}\lambda$:= the real part of $\lambda \in \mathbb{C}$;
$\operatorname{Im}\lambda$:= the imaginary part of $\lambda \in \mathbb{C}$;
$|\lambda|$:= the absolute value of $\lambda \in \mathbb{C}$;
$A_* := A \setminus \{0\}$;
$A_+ := A \cap [0, \infty)$;
$A_- := A \cap (-\infty, 0]$;
$A_*^n := (A_*)^n$;
$z^\alpha := z_1^{\alpha_1} \dots z_n^{\alpha_n}$, $z \in \mathbb{C}^n$, $\alpha \in \mathbb{C}^n$, $z_j \neq 0$ if $\alpha_j < 0$;
$\mathcal{O}(D, G)$:= the set of all holomorphic mappings $F : D \to G$;
$\mathcal{O}(D) := \mathcal{O}(D, \mathbb{C})$;
$\langle w, z\rangle := w_1\overline{z}_1 + \dots + w_n\overline{z}_n$ = the scalar product on $\mathbb{C}^n$, $w, z \in \mathbb{C}^n$;
$B_d(w, r) := \{z : d(w, z) < r\}$ = the ball with center w and radius $r > 0$, d is a distance;
$\mathbb{B}(w, r) := B_d(w, r)$, where d is the Euclidean distance;
$\mathbb{B}_n := \mathbb{B}(0, 1)$ the unit Euclidean ball in $\mathbb{C}^n$;
$E := \mathbb{B}_1$ the unit disc in $\mathbb{C}$;
$\operatorname{Span} A$:= the smallest vector subspace containing $A \subset \mathbb{R}^n$;
$\|f\|_A := \sup\{|f(w)| : w \in A\}$, $f : A \to \mathbb{C}$;
B_D — the norm ball.

Chapter I

$m(\lambda_1, \lambda_2) := |\lambda_1 - \lambda_2|/|1 - \overline{\lambda}_1\lambda_2|$ — the Möbius distance;
$\gamma(\lambda; X) := |X|/1 - |\lambda|^2$;
p — the Poincaré distance;
c_D — the Carathéodory pseudodistance of D;
k_D — the Kobayashi pseudodistance of D;
$\widetilde{k}_D$ — the Lempert function of D;
$d := (d_D)_{D \text{ domain in } \mathbb{C}^n}$ — holomorphically contractible family of functions, $d = c, k, \widetilde{k}$;
$d_D^* := \tanh d_D$;

g_D — the pluricomplex Green function of D;
PSH(D) — plurisubharmonic functions on D;
SH(D) — subharmonic functions on D;
$\widetilde{g}_D := \exp g_D$;
$\delta := (\delta_D)_{D \text{ domain in } \mathbb{C}^n}$ — holomorphically contractible family of pseudometrics;
γ_D — the Carathéodory–Reiffen pseudometric of D;
κ_D — the Kobayashi–Royden pseudometric of D;
A_D — the Azukawa pseudometric of D;
$\mathcal{E}(p_1, \ldots, p_n)$ — complex ellipsoid, $p_j > 0$, $j = 1, \ldots, n$, $n > 1$;
D_α — elementary Reinhardt domain, $\alpha \in \mathbb{Z}^n$;
$g_D(P; \nu; \cdot)$ — the pluricomplex Green function with poles at $P \subset D$ with weights ν;
$L^p_h(D) := \mathcal{O}(D) \cap L^p(D)$ the space of p integrable holomorphic functions on D;
$\mathcal{E}(D, P, \nu)$ the set where the pluricomplex Green function with many poles is the sum of the Green functions with one pole;
K_D — the Bergman kernel of D;
β_D — the Bergman pseudometric of D;
b_D — the Bergman pseudodistance of D;
L_{β_D} — the Bergman length (of a piecewise C^1 curve); B_D — the norm ball.

Chapter II

$\log|z| := (\log|z_1|, \ldots, \log|z_n|)$, $z \in \mathbb{C}^n_*$;
$\log D$ — logarithmic image of a Reinhardt domain D;
T_ω — tube domain over a domain $\omega \subset \mathbb{R}^n$;
$V_j := \{z_j = 0\} \subset \mathbb{C}^n$;
$V_I := V_{j_1} \cap \ldots \cap V_{j_k}$, $I = \{j_1, \ldots, j_k\}$;
π_j — the projection on V_j;
π_I — the projection on V_I;
$\mathfrak{C}(\Omega, a)$, maximal subcone of $\overline{\Omega}$, $\Omega \subset \mathbb{R}^n$ convex domain, $a \in \overline{\Omega}$;
$\mathfrak{C}(\Omega) := \mathfrak{C}(\Omega, a)$;
$\mathfrak{C}(D) := \mathfrak{C}(\log D)$, D is a Reinhardt pseudoconvex domain;
$\widetilde{\mathfrak{C}}(D)$ — the set of exponential halflines with closures in D, $\widetilde{\mathfrak{C}}(D) \subset \mathfrak{C}(D)$;
$\mathfrak{C}'(D) := \mathfrak{C}(D) \setminus \widetilde{\mathfrak{C}}(D)$;
$H^\infty(D) :=$ the space of bounded holomorphic functions on D;
$H(A, C) := \bigcap_{j=1}^n H(A^j, C_j)$;
$\Phi_A(z) := z^A := (z^{A^1}, \ldots, z^{A^m})$, $A \in \mathbb{Z}^{m\times n}$;
$A^{\text{inv}} := |\det A| A^{-1}$;
$S := S(D)$ — the set of admissible exponents, D is a Reinhardt domain;
$B := B(D) = S \setminus (S + S)$;
$r(A)$, $A \in \mathbb{Z}^{m\times n}$;
$G(A, C) := \bigcap_{j=1}^m G(A^j, C_j)$ — a cone domain, $A \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^m$;
$s(A)$, $A \in \mathbb{Z}^{m\times n}$;
$\|\cdot\|$ — some norm on $\mathbb{R}^n$.

Chapter III

$T_z := \{(e^{i\theta_1}z_1, \dots, e^{i\theta_n}z_n) : \theta_j \in \mathbb{R}\}$, $z \in \mathbb{C}^n$;
$T_{z,\alpha}$ — a subgroup of T_z, $z \in \mathbb{C}^n$, $\alpha \in \mathbb{Z}^n$;
V_μ — the level subset of D_α;
$f^*(\lambda)$ — the nontangential limit of $f \in H^\infty(E)$, $\lambda \in \partial E$;
$\mathfrak{D}k_D$ — the derivative of the Kobayashi pseudodistance.

Chapter IV

$D_\varepsilon := D_\varepsilon(p)$ — sublevel set of the Green function, $\varepsilon > 0$, $p \in D$.

Chapter V

N_p — a function related to the Minkowski function of the ellipsoid.

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