1. Introduction

In this paper we consider systems of differential operators on $\mathbb{R}^n$ whose coefficients have certain asymptotic properties as $|x| \to \infty$. These elliptic operators define continuous maps between weighted function spaces on $\mathbb{R}^n$ which are locally modeled on Sobolev, Hölder or other types of spaces, but which contain a derivative dependent weight that controls behaviour as $|x| \to \infty$. In this setting we obtain results of the following types, which parallel those of the standard theory for elliptic operators on compact manifolds:

- A priori estimates for solutions.
- Regularity results relating solutions in different weighted spaces.
- The Fredholm property for operators acting between certain weighted spaces.
- Dependence of the Fredholm index on the weighted spaces.

In order to avoid a large number of lengthy definitions in the Introduction we presently restrict our attention to the formulation of our results for the case of scalar operators acting on weighted function spaces modeled on Sobolev spaces of integral order. At the end of the Introduction we indicate where the corresponding results for the general case can be found.

For the purpose of defining our class of elliptic operators we introduce the following symbol classes (see Section 1.1 for basic notation).

Definition 1.1. For any $\beta \in \mathbb{R}$ define Sc^{β} to be the set of those functions $p \in C^{\infty}_{\mathrm{loc}}$ for which there exist $a \in C^{\infty}(S^{n-1})$ and $q \in C^{\infty}_{\mathrm{loc}}$ satisfying the following conditions:

(i) Writing $x \in \mathbb{R}^n_*$ in polar coordinates as $x = (r, \omega)$ we have

$$p(x) = a(\omega) r^{-\beta} + q(x)$$

whenever $|x| = r \geq 1$.

(ii) For any multi-index α we have an estimate of the form

$$D^{\alpha}_x q(x) = o(|x|^{-\beta-|\alpha|}) \quad \text{as } |x| \to \infty.$$

The function $a(\omega) r^{-\beta}$ defined on $\mathbb{R}^n_*$ will be called the *principal part* of p.

Let $A(x, D_x)$ be a differential operator on $\mathbb{R}^n$ of order m. Thus we can write

$$A(x, D_x) = \sum_{|\alpha| \leq m} p^{\alpha}(x) D^{\alpha}_x \tag{1}$$

where $p^{\alpha} \in C^{\infty}_{\mathrm{loc}}$ for each multi-index α with $|\alpha| \leq m$. We say that A is an *admissible elliptic operator* provided $p^{\alpha} \in \mathrm{Sc}^{m-|\alpha|}$ for each $|\alpha| \leq m$ and A is uniformly elliptic on

$\mathbb{R}^n$ in the sense that

$$\Big|\sum_{|\alpha|=m} p^\alpha(x)\xi^\alpha\Big| \geq C|\xi|^m \tag{2}$$

for all $x, \xi \in \mathbb{R}^n$ (where C is some positive constant). Examples of admissible elliptic operators include the Laplacian $-\varDelta$ and the Schrödinger operator $-\varDelta + V$ whenever $V \in \mathrm{Sc}^2$; in particular, V must decay at least as quickly as $|x|^{-2}$.

In order to define the function spaces on which A shall act we need to introduce the weight function $\varLambda$ defined on $\mathbb{R}^n$ by $\varLambda(x) = (1+|x|^2)^{1/2}$.

Definition 1.2. For $p \in [1,\infty)$, $k \in \mathbb{Z}$ and $\beta \in \mathbb{R}$ define a norm $\|\cdot\|_{H^{p,k}_\beta}$ on C^∞_0 by

$$\|u\|^p_{H^{p,k}_\beta} = \sum_{|\alpha|\leq k} \int \varLambda^{p(\beta+|\alpha|)}(x)|D^\alpha_x u(x)|^p \, d^n x,$$

and let $H^{p,k}_\beta$ denote the Banach space obtained by taking the completion of C^∞_0 with respect to this norm.

For $p \in (1,\infty)$, $k \in \mathbb{Z}\setminus\mathbb{N}_0$ and $\beta \in \mathbb{R}$ let $q \in (1,\infty)$ be given by $1/p+1/q=1$ and define $H^{p,k}_\beta$ to be the Banach space obtained by taking the dual of $H^{q,-k}_{-\beta}$ with respect to the L^2 pairing on $\mathbb{R}^n$.

If A is an admissible elliptic operator of order m then A defines a continuous map $H^{p,k+m}_{\beta-m} \to H^{p,k}_\beta$ for any $p \in (1,\infty)$, $k \in \mathbb{Z}$ and $\beta \in \mathbb{R}$. We obtain the following regularity result relating solutions of the equation $Au = f$ for some different values of p, k and β.

Theorem 1.3. *Let $p, q \in (1,\infty)$, $k, l \in \mathbb{Z}$, $\beta, \gamma \in \mathbb{R}$ and suppose we have either $\beta + n/p < \gamma + n/q$ or $\beta + n/p \leq \gamma + n/q$ and $p \geq q$. If $Au \in H^{p,k}_\beta \cap H^{q,l}_\gamma$ for some $u \in H^{q,l+m}_{\gamma-m}$ then we also have $u \in H^{p,k+m}_{\beta-m}$. Furthermore,*

$$\|u\|_{H^{p,k+m}_{\beta-m}} \leq C(\|Au\|_{H^{p,k}_\beta} + \|u\|_{H^{q,l+m}_{\gamma-m}})$$

for all such u.

In order to proceed with further regularity results and Fredholm properties for the map $A : H^{p,k+m}_{\beta-m} \to H^{p,k}_\beta$ we must eliminate a countable set of values of β. These values are related to the eigenvalues of an associated spectral problem which we now introduce.

Suppose A is an admissible elliptic operator of order m given by (1). We define the *principal part* of A to be the operator A_0 on $\mathbb{R}^n_*$ given by

$$A_0(x, D_x) = \sum_{|\alpha|\leq m} a^\alpha(\omega) r^{|\alpha|-m} \, D^\alpha_x,$$

where, for each $|\alpha| \leq m$, $a^\alpha(\omega)r^{|\alpha|-m}$ is the principal part of p^α. It is easy to see that the ellipticity estimate (2) for A implies that A_0 is elliptic on $\mathbb{R}^n_*$.

The principal part of A can be rewritten in the form

$$A_0(x, D_x) = \sum_{j=0}^m A^{m-j}(\omega, D_\omega)(rD_r)^j(r^{-m}\,\cdot),$$

where, for $j = 0, \ldots, m$, $A^j(\omega, D_\omega)$ is a differential operator on S^{n-1} of order at most j.

We associate with A an operator pencil $\mathfrak{B}_A : \mathbb{C} \to \mathscr{L}(H^m(S^{n-1}), L^2(S^{n-1}))$, which is defined by

$$\mathfrak{B}_A(\lambda) = \sum_{j=0}^{m} A^{m-j}(\omega, D_\omega)\lambda^j$$

for each $\lambda \in \mathbb{C}$. The spectrum of this operator pencil is the set

$$\sigma(\mathfrak{B}_A) = \{\lambda \in \mathbb{C} \mid \mathfrak{B}_A(\lambda) : H^m(S^{n-1}) \to L^2(S^{n-1}) \text{ is not invertible}\}.$$

The geometric and algebraic multiplicities of any $\lambda_0 \in \sigma(\mathfrak{B}_A)$ can be respectively defined as $\dim \operatorname{Ker} \mathfrak{B}_A(\lambda_0)$ and the sum of the lengths of a set of maximal Jordan chains corresponding to λ_0 (see Section 3.3 or [GGK] for more details).

By using the ellipticity of A_0 it can be shown that $\sigma(\mathfrak{B}_A)$ consists of isolated points of finite algebraic multiplicity and that any strip of finite width parallel to the real axis contains at most finitely many points of $\sigma(\mathfrak{B}_A)$ (see Theorem 5.2.1 in [KMR] or Theorem 1.2.1 in [NP], for example).

The projection of $\sigma(\mathfrak{B}_A)$ onto the imaginary axis is of particular importance and will be denoted by $\mathit{\Gamma}(A)$; that is,

$$\mathit{\Gamma}(A) = \{\operatorname{Im} \lambda \mid \lambda \in \sigma(\mathfrak{B}_A)\} \subset \mathbb{R}.$$

In particular, the above discussion implies that $\mathit{\Gamma}(A)$ consists of isolated points and, given $\gamma \in \mathit{\Gamma}(A)$, the total algebraic multiplicity of all those $\lambda \in \sigma(\mathfrak{B}_A)$ with $\operatorname{Im} \lambda = \gamma$ is finite.

The Fredholm property for A is related to the spectrum of the associated operator pencil through the set $\mathit{\Gamma}(A)$ as follows.

Theorem 1.4. *Let $p \in (1, \infty)$, $k \in \mathbb{Z}$ and $\beta \in \mathbb{R}$. If $\beta + n/p \notin \mathit{\Gamma}(A)$ then the map $A : H^{p,k+m}_{\beta-m} \to H^{p,k}_{\beta}$ is Fredholm.*

We also obtain a $\mathit{\Gamma}(A)$ dependent regularity result complementing Theorem 1.3.

Theorem 1.5. *Let $p, q \in (1, \infty)$, $k, l \in \mathbb{Z}$, $\beta, \gamma \in \mathbb{R}$ and suppose $\beta + n/p$ and $\gamma + n/q$ belong to the same component of $\mathbb{R} \setminus \mathit{\Gamma}(A)$. If $Au \in H^{p,k}_{\beta} \cap H^{q,l}_{\gamma}$ for some $u \in H^{q,l+m}_{\gamma-m}$ then we also have $u \in H^{p,k+m}_{\beta-m}$. Furthermore,*

$$\|u\|_{H^{p,k+m}_{\beta-m}} \leq C(\|Au\|_{H^{p,k}_{\beta}} + \|u\|_{H^{q,l+m}_{\gamma-m}})$$

for all such u.

As a consequence of Theorems 1.4 and 1.5 we also obtain a stability result for the Fredholm index of A.

Theorem 1.6. *Let $p, q \in (1, \infty)$, $k, l \in \mathbb{Z}$, $\beta, \gamma \in \mathbb{R}$ and suppose $\beta + n/p$ and $\gamma + n/q$ belong to the same component of $\mathbb{R} \setminus \mathit{\Gamma}(A)$. Then the Fredholm maps $A : H^{p,k+m}_{\beta-m} \to H^{p,k}_{\beta}$ and $A : H^{q,l+m}_{\gamma-m} \to H^{q,l}_{\gamma}$ have the same index.*

If the parameter β is varied so that $\beta + n/p$ moves between components of $\mathbb{R} \setminus \mathit{\Gamma}(A)$ then the index of the corresponding map will change. This change is related to more detailed information about the spectrum of the operator pencil $\mathfrak{B}_A$.

Theorem 1.7. *Let $p \in (1, \infty)$, $k \in \mathbb{Z}$ and $\beta_1, \beta_2 \in \mathbb{R}$ with $\beta_1 \leq \beta_2$ and $\beta_i + n/p \notin \mathit{\Gamma}(A)$ for $i = 1, 2$. Set $\mathit{\Sigma} = \{\lambda \in \sigma(\mathfrak{B}) \mid \operatorname{Im} \lambda \in [\beta_1, \beta_2]\}$ and, for each $\lambda \in \mathit{\Sigma}$, let m_λ denote* *the algebraic multiplicity of* λ. *Then we have*

$$\operatorname{Index} A^{(\beta_1)} = \operatorname{Index} A^{(\beta_2)} + \sum_{\lambda \in \Sigma} m_\lambda,$$

where $A^{(\beta_i)}$ *denotes the map* $A : H^{p,k+m}_{\beta_i - m} \to H^{p,k}_{\beta_i}$ *for* $i = 1, 2$.

Let $A^*(x, D)$ denote the differential operator obtained by taking the formal adjoint of $A(x, D_x)$ (with respect to the standard Lebesgue measure on $\mathbb{R}^n$). By using the definition of the symbol classes Sc^β it is straightforward to check that A^* is also an admissible elliptic operator. Furthermore, we have

$$\Gamma(A^*) = (n + m) - \Gamma(A). \tag{3}$$

In the case when A is formally self-adjoint we can determine the Fredholm index entirely from knowledge of the spectrum of the operator pencil $\mathfrak{B}_A$.

Theorem 1.8. *Let* $p \in (1, \infty)$, $k \in \mathbb{Z}$ *and, for any* $\gamma \in \mathbb{R}$, *let* $A^{(\gamma)}$ *denote the map* $A : H^{p,k+m}_{\gamma - m} \to H^{p,k}_\gamma$. *Now suppose* A *is formally self-adjoint. Then* $\Gamma(A)$ *is symmetric about* $(n + m)/2$ *and, for any* $\beta \in \mathbb{R}$ *with* $\beta + n/p \notin \Gamma(A)$, *we have*

$$\operatorname{Index} A^{(n+m-\beta-2n/p)} = -\operatorname{Index} A^{(\beta)}.$$

In particular, we have either $(n+m)/2 \notin \Gamma(A)$, *in which case* $\operatorname{Index} A^{(m/2+n/2-n/p)} = 0$, *or* $(n + m)/2 \in \Gamma(A)$, *in which case the sum of the algebraic multiplicities of those* $\lambda \in \sigma(\mathfrak{B}_A)$ *with* $\operatorname{Im} \lambda = (n + m)/2$ *is even (say* $2d$ *for some* $d \in \mathbb{N}$*) and*

$$\operatorname{Index} A^{(m/2+n/2-n/p-\varepsilon)} = d = -\operatorname{Index} A^{(m/2+n/2-n/p+\varepsilon)}$$

for all sufficiently small $\varepsilon > 0$.

The paper is arranged as follows. In Section 2 we define the general class of weighted function spaces on which our elliptic operators act, as well as related weighted function spaces for the associated "model operators". The majority of that section is devoted to establishing the basic properties of these spaces that are necessary in order to work with them. In particular, Section 2.3.5 gives details of how some previously defined weighted function spaces (including the weighted Sobolev spaces of Definition 1.2) arise in this general setting.

The full class of elliptic operators on $\mathbb{R}^n$ to which our results apply is introduced at the beginning of Section 4 (see Definition 4.3). This class is basically a generalisation of the class of (scalar) admissible elliptic operators introduced above to cover the case of systems with Douglis–Nirenberg type ellipticity. The generalisations of Theorems 1.3 to 1.8 are given in Theorems 4.12, 4.22, 4.18, 4.19, 4.23 and 4.26 respectively (Remarks 2.22 and 2.36 provide the details needed to derive the results given above from their counterparts in Section 4). Additionally, it is shown that the finite dimensionality of the kernel implied by Theorem 1.4 (or Theorem 4.22) remains valid without restriction on the parameter β (see Theorem 4.17). Finally, at the end of Section 4, we give some index formulae for elliptic operators whose principal part is homogeneous with constant coefficients (see Theorems 4.29, 4.30 and 4.31).

The main results are obtained from results for "model operators" on Π^n and $\mathbb{R}^n_*$. These operators provide the necessary generalisation of the operators $\mathfrak{B}_A$ and A_0 introduced above and are dealt with in Section 3. The isomorphism results contained in Sections 3.4 and 3.5 (for Π^n and $\mathbb{R}^n_*$ respectively) are also of interest in their own right.

Broadly speaking, there appear to be two areas of research related to the results presented here. The older of these areas appears principally in the Russian literature and was centred around the problem of elliptic operators on bounded domains with conical singularities on the boundary. The motivation for a lot of this work came from applications to the mechanics and electrodynamics of continua and to numerical methods.

The difference between the problems of elliptic operators on $\mathbb{R}^n$ and on domains with conical singularities on the boundary is a largely superficial one—in some sense ∞ can be regarded as a one-point boundary of $\mathbb{R}^n$ which is of conical type. Both problems can be split into the study of standard elliptic problems on bounded domains and of elliptic problems in neighbourhoods of the singular points. In turn, the latter can be reduced to the study of model elliptic problems on conical or cylindrical domains. The model problems and the arguments used to go from results for these to results for the actual problems are similar in many respects. In particular, the study of the model problems suggests a natural choice of weighted function spaces, which, in turn, gives rise to a natural choice of weighted function spaces for the original problems.

Early work on the problem of elliptic operators on domains with conical singularities on the boundary includes [Es], [K1], [Lop], and the fundamental paper [K2], in which the general approach to these problems was refined and applied to scalar operators of arbitrary order in weighted Sobolev spaces based on H^k, $k \in \mathbb{N}_0$. Aspects of the theory were developed by authors including V. G. Maz'ya, B. A. Plamenevskiĭ, S. A. Nazarov, V. A. Kozlov, J. Rossmann and M. Dauge (see [KMR] for a comprehensive list of references). The paper [MP] and the monograph [Da] are of particular relevance to the work presented here; in [MP] generalisations were made to systems of operators on weighted L^p Sobolev spaces of integral order and weighted Hölder spaces, whilst in [Da] weighted L^2 Sobolev spaces of fractional order were considered. The monographs [Gr], [Da], [NP] and [KMR], as well as the overview paper [Pl], provide expositions of (aspects of) this theory, further results and references, and some notes on the historical development of this work.

Although the presumed existence of parallel results for elliptic operators on $\mathbb{R}^n$ has been remarked upon by several authors (see in particular Remark 4.1.5 in [NP]) the problem was considered explicitly in only a handful of papers from this area, the most significant of these being [BK]. Here Theorem 4.22 was established for scalar operators and the model spaces $E = H^k$, $k \in \mathbb{N}_0$, whilst Theorem 4.17 was given for scalar operators and the model spaces $E = L^p$, $p \in (1, \infty)$, when $\beta = n/p$. The possibility of generalisation to systems of operators was also observed in [BK]. However, since [BK], no detailed study of the problem on $\mathbb{R}^n$ appears to have been carried out by authors in this area.

The second area of research related to the results presented here is connected to the study of elliptic problems on non-compact manifolds, with motivation coming from applications to global analysis on non-compact manifolds, especially in questions related to general relativity. Initial work in this area centred on $\mathbb{R}^n$ (as the simplest of a class of non-compact manifolds) and elliptic operators of the form $A_0 + Q$ where A_0 is a homogeneous operator with constant coefficients and Q is a perturbation with variable coefficients which have suitable decay at infinity. The approach used in this work was fundamentally different from that taken here; it was based on establishing mapping properties for an explicit class of related convolution operators. This technique was essentially introduced in the key paper [NW] where it was shown that such operators have a finite-dimensional kernel when acting on $H^{p,k}$, $p \in (1, \infty)$, $k \in \mathbb{N}_0$.

The weighted function spaces suggested by the estimates in [NW] (i.e. the spaces $H^{p,k}_\beta$ appearing in Definition 1.2 above) were explicitly defined in [C1] and the theory was developed steadily in a series of papers thereafter; these included [C2], [C3], [Loc], [M1], [M2], [CC], [Mu], [LM1], [Ben], [De] and [BP]. In particular, Theorem 4.29 was given for the model spaces $E = H^{p,k}$, $p \in (1, \infty)$, $k \in \mathbb{N}_0$ in [LM1] (see also [LM2]), and for the model spaces $E = C^{l+\sigma}$, $l \in \mathbb{N}_0$, $\sigma \in (0, 1)$ in [BP]. It should be remarked that the papers [C2], [C3], [CC], [Loc] and [De] also considered operators of the above type on non-compact manifolds which are "asymptotically Euclidean". The essential modifications needed to consider such problems relate to the non-Euclidean part of the manifold—this is bounded and can be treated using the well developed theory for elliptic operators on compact manifolds.

The techniques of the above papers seem to be well suited to operators which are appropriate perturbations of homogeneous constant coefficient operators and even allow for the computation of the index and the characterisation of kernels and cokernels in certain cases. However, these techniques do not appear to generalise easily to cover operators with arbitrarily varying coefficients. In [LM3] a switch was made to techniques similar to those employed here, and Theorems 4.22, 4.23 and 4.26 were established for the model spaces $E = H^{p,k}$, $p \in (1, \infty)$, $k \in \mathbb{N}_0$. In fact, these theorems were established for an appropriately defined class of operators on non-compact manifolds which have finitely many cylindrical ends ($\mathbb{R}^n$ being the special case of a manifold with a single cylindrical end homeomorphic to $\mathbb{R}^+ \times S^{n-1}$). Along with [BK], the results of [LM3] appear to be the closest existing results to those presented here.

The work on non-compact manifolds cited above can be viewed as being part of a broader collection of work on elliptic operators on non-compact manifolds. Apart from the inherent interest of such problems, results parallel to the well known ones for elliptic operators on compact manifolds have useful applications in many areas of geometry and analysis. However, it is not possible to develop a theory as comprehensive as that for compact manifolds; to obtain useful results one is forced to restrict to classes of problems where there is some kind of relationship between the asymptotic behaviour of the operators and the asymptotic properties of the function spaces on which they are acting. Of the large and diverse collection of work in this area we mention only a brief (and necessarily subjective) selection. In [CH] and [M3] necessary and sufficient conditions for certain types of (pseudo-)differential operators to be Fredholm when acting between Sobolev spaces on a complete Riemannian manifold were formulated in terms of objects related to the operator's symbol. In [Ei] several approaches to the study of the spectral theory of certain self-adjoint differential operators on non-compact manifolds were presented, whilst in [An] manifolds with asymptotically negative curvature were considered.

Finally we mention the very general works [MM], [Ma] and [Sc] which gave results related to various special classes of (pseudo-)differential operators on compact manifolds with boundaries (including "totally characteristic operators" and "edge operators"). By using appropriate coordinate transformations these results can be reformulated for certain operators on non-compact manifolds; in particular, the general framework used in these works allows one to consider operators similar to those considered here, at least when they are classically elliptic and act between weighted spaces modeled on H^k, $k \in \mathbb{N}_0$.

The present paper extends existing work in its consideration of much more general types of function spaces. Apart from filling numerous gaps in the existing collection of results (in particular for weighted L^p Sobolev spaces of negative integral order and weighted Hölder spaces) numerous new types of spaces are considered; perhaps the most important of these are the weighted L^p Sobolev spaces of arbitrary real order. The key technique employed to achieve this lies in a characterisation of pseudo-differential operators given in [Bea] which allows the generalisation of results about the model operators (see Section 3).

The inclusion of spaces of "negative order" in our general setting means we can consider dual problems as well as operators arising from problems in variational form. In particular, the use of duality simplifies the argument needed to go from the semi-Fredholm property to the Fredholm property (i.e. from Theorem 4.16 to Theorem 4.22). In most existing work in this area some type of parametrix or regulariser is constructed for the corresponding step (n.b. duality is also used in [KMR]).

In this paper, as in the majority of the literature cited above, the necessary weighted function space results are proved locally. Due to the more general function space setting of the present work a more complete set of related weighted function space results has been obtained here. However, it should be pointed out that many of these results probably appear in the function space literature.

One application of the results presented here is to the study of zero modes (or zero energy bound states) of the Dirac–Weyl operator $\sigma.(D - A)$ on $\mathbb{R}^3$ (here σ is the vector of Pauli matrices and A is a real vector potential); this will appear in [El].

1.1. Notation. In this section we introduce some (not necessarily standard) notation and conventions that will be used throughout the paper.

We define $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ to be the set of non-negative integers; thus a multi-index α is simply an element of $\mathbb{N}_0^n$ with $|\alpha| := \alpha_1 + \ldots + \alpha_n$.

The sets $\mathbb{R}^n \setminus 0$ and $\mathbb{R} \times S^{n-1}$ are denoted by $\mathbb{R}^n_*$ and $\varPi^n$ respectively. The letter ω is used to denote a point on S^{n-1}, whilst (r, ω) and (t, ω) denote polar coordinates on $\mathbb{R}^n_*$ and cylindrical coordinates on $\varPi^n$ respectively. When necessary, S^{n-1}, $\mathbb{R}^n_*$ and $\varPi^n$ will be considered as Riemannian manifolds with the obvious choice of Riemannian metrics (i.e. that given by the standard embedding into $\mathbb{R}^n$ as the unit sphere for S^{n-1}, the restriction of the Euclidean metric for $\mathbb{R}^n_*$ and the product metric for $\varPi^n$).

For $i \in \{1, \ldots, n\}$ we use D_i to denote the differential operator $-i\partial/\partial x_i$ on $\mathbb{R}^n$. By D_x we mean the vector differential operator $(D_1, \ldots, D_n)$ whilst D_r and D_t are used to denote the differential operators $-i\partial/\partial r$ and $-i\partial/\partial t$ on $\mathbb{R}^+$ and $\mathbb{R}$ respectively. Finally, the notation $A(\omega, D_\omega)$ is used to mean that A is a differential operator on S^{n-1}.

For any manifold M with volume measure dM let $(\cdot,\cdot)_M$ denote the L^2 pairing on M; that is, $(u,v)_M = \int_M uv\,dM$ for all appropriate u and v. The associated sesquilinear pairing is then $\langle u,v\rangle_M := (u,\overline{v})_M$. We use these pairings with M being $\mathbb{R}^n$, S^{n-1}, $\mathbb{R}^n_*$ or Π^n. In all cases dM is the volume measure induced by our choice of the Riemannian metric; in particular, the volume measures on $\mathbb{R}^n$, $\mathbb{R}^n_*$ and Π^n are d^nx (the Lebesgue measure), $d^nx = r^{n-1}drdS^{n-1}$ and $dtdS^{n-1}$ respectively, where dS^{n-1} is the standard volume measure on S^{n-1} inherited from its embedding into $\mathbb{R}^n$ as the unit sphere.

Let $\mathscr{D}'$ denote the set of distributions on $\mathbb{R}^n$ and $\mathscr{D}'(M)$ the set of distributions on an arbitrary manifold M (see Section 6.3 of [H1] for further details). The pairing $(u,v)_M$ can be defined for all $u \in C_0^\infty(M)$ and $v \in \mathscr{D}'(M)$; in particular, this pairing can be viewed as a way of identifying elements of $\mathscr{D}'(M)$ with distributional densities on M.

If $E \subset \mathscr{D}'(M)$ for some manifold M we use E_{loc} to denote the set of $u \in \mathscr{D}'(M)$ with $\phi u \in E$ for all $\phi \in C_0^\infty(M)$. On the other hand, for open $U \subseteq M$, $E(U)$ denotes the set of (equivalence classes of) restrictions of elements $u \in E$ to U. If $\phi \in C_0^\infty(U)$ and $u \in E(U)$ we can extend ϕu by 0 outside U to enable us to consider it as an element of E.

We use Λ to denote the weight function defined on $\mathbb{R}^n$ by $\Lambda(x) = (1+|x|^2)^{1/2}$. Let $\mathscr{S}$ and $\mathscr{S}'$ denote respectively the locally convex spaces of Schwartz class functions on $\mathbb{R}^n$ and its dual, the set of tempered distributions on $\mathbb{R}^n$. The topology of the former is provided by the semi-norms

$$p_l(u) := \sum_{|\alpha|\le l} \sup_{x\in\mathbb{R}^n} \Lambda^l(x)|D_x^\alpha u(x)|$$

for any $l \in \mathbb{N}_0$, whilst we choose the weak dual topology for the latter.

For any $l \in \mathbb{N}_0$ we use C^l to denote the set of bounded l times continuously differentiable functions on $\mathbb{R}^n$, provided with the norm

$$\|u\|_{C^l} = \sum_{|\alpha|\le l} \sup_{x\in\mathbb{R}^n} |D_x^\alpha u(x)|.$$

We put $C^\infty = \bigcap_{l\in\mathbb{N}_0} C^l$ and provide this set with the locally convex topology induced by the collection of semi-norms $\{\|\cdot\|_{C^l} \mid l \in \mathbb{N}_0\}$. The set of smooth functions on $\mathbb{R}^n$ (without restrictions on growth at infinity) is then denoted by C^∞_{loc}. We also use C_0^∞ for the set of smooth functions on $\mathbb{R}^n$ with compact support.

For $p \in [1,\infty]$ and $s \in \mathbb{R}$ we use $H^{p,s}$ to denote the Sobolev space on $\mathbb{R}^n$ of "functions with s p-integrable derivatives". This notation will be simplified to H^s in the case $p = 2$ and L^p in the case $s = 0$. Other spaces appearing as examples include the Hölder spaces $C^{l+\sigma}$ for $l \in \mathbb{N}_0$ and $\sigma \in [0,1)$ (n.b. we set $C^{l+0} = C^l$) and the Zygmund spaces $\mathcal{C}^s$ for $s \in \mathbb{R}^+$. A detailed account of all these spaces can be found in [T1].

Let BS denote the set of $\mathbb{R}$-valued functions $\zeta \in C^\infty$ which are constant in a neighbourhood of 0 and ∞. We also use BS_{01} to denote the subset of BS containing those ζ with $\zeta = 0$ in a neighbourhood of 0 and $\zeta = 1$ in a neighbourhood of ∞.

If χ_1 and χ_2 are $\mathbb{R}$-valued functions we write $\chi_1 \prec \chi_2$ (or, alternatively, $\chi_2 \succ \chi_1$) provided $\chi_2 = 1$ on $\mathrm{supp}(\chi_1)$. If $\chi_1 \prec \chi_2$ then it clearly follows that $\chi_1\chi_2 = \chi_1$.

We use C to denote any positive constant whose exact value is not important but which may depend only on the things it is allowed to in a given problem (i.e. parameters defining function spaces but not the actual element of the space under consideration etc.). Constants depending on something extra are indicated with appropriate function type notation whilst subscripts are added if we need to keep track of the value of a particular constant (e.g. $C_1(u)$ etc.).

We use the notation $\kappa(u) \asymp \varrho(u)$ to indicate that the quantities $\kappa(u)$ and $\varrho(u)$ satisfy the inequalities $C\kappa(u) \leq \varrho(u) \leq C\kappa(u)$ for all relevant u (possibly including parameter values). Generally κ and ϱ will be norms of some description.

2. Function spaces

In this section we introduce classes of weighted function spaces for our elliptic operators and associated "model" operators. In order to define these spaces and establish the basic results necessary to work with them, we use general constructions and arguments applied to specific "model spaces".

DEFINITION 2.1. A *model space* E is a Banach space of functions with the following properties:

(A1) We have continuous inclusions $\mathscr{S} \hookrightarrow E \hookrightarrow \mathscr{S}'$.

(A2) Multiplication defines a continuous bilinear map $C^\infty \times E \to E$.

(A3) The norm $\|\cdot\|_E$ is translationally invariant.

(A4) If ψ is a diffeomorphism on $\mathbb{R}^n$ which is linear outside some compact set then the pull-back $\psi^* : \mathscr{S}' \to \mathscr{S}'$ restricts to give an isomorphism on E.

Throughout this paper the letters E, F and G are used to denote model spaces.

REMARK 2.2. Examples of model spaces include the Sobolev spaces $H^{p,s}$ for $p \in [1,\infty]$ and $s \in \mathbb{R}$, the Hölder spaces $C^{k+\sigma}$ for $k \in \mathbb{N}_0$ and $\sigma \in [0,1)$ and the Zygmund spaces $\mathcal{C}^s$ for $s \in \mathbb{R}^+$.

LEMMA 2.3. *Suppose $\psi : U \to V$ is a diffeomorphism between open subsets of $\mathbb{R}^n$ and $\chi \in C_0^\infty(V)$. Then $\|\psi^*(\chi u)\|_E \asymp \|\chi u\|_E$ for all $u \in E(V)$.*

Proof. Any point $x \in U$ has a neighbourhood in U outside which ψ can be extended linearly to give a diffeomorphism of $\mathbb{R}^n$. Choose a finite collection $\{U_i\}_{i\in I}$ of such neighbourhoods which cover $\psi^{-1}(\operatorname{supp}(\chi))$ and, for each $i \in I$, let $\psi_i : \mathbb{R}^n \to \mathbb{R}^n$ denote a diffeomorphism which is linear outside a compact region and satisfies $\psi_i(x) = \psi(x)$ for all $x \in U_i$. Thus $\{\psi_i(U_i)\}_{i\in I} \cup \{\mathbb{R}^n \setminus \operatorname{supp}(\chi)\}$ is an open cover of $\mathbb{R}^n$. Choosing any partition of unity $\{\phi_i\}_{i\in I} \cup \{\phi_\infty\}$ subordinate to this cover we clearly have $\sum_{i\in I} \phi_i = 1$ on $\operatorname{supp}(\chi)$. Conditions (A2) and (A4) now give

$$\begin{aligned}\|\psi^*(\chi u)\|_E &\leq \sum_{i\in I} \|\psi^*(\phi_i \chi u)\|_E = \sum_{i\in I} \|\psi_i^*(\phi_i \chi u)\|_E \\ &\leq C \sum_{i\in I} \|\phi_i \chi u\|_E \leq C \sup_{i\in I} \|\phi_i \chi u\|_E \leq C\|\chi u\|_E\end{aligned}$$

for all $u \in E(V)$. Symmetry completes the result. ∎

Definition 2.4. Suppose E is a model space and $l \in \mathbb{N}_0$. We define the space E^l to be the set of all $u \in \mathscr{S}'$ satisfying $D_x^\alpha u \in E$ for each multi-index α with $|\alpha| \le l$, equipped with the norm

$$\|u\|_{E^l} = \sum_{|\alpha| \le l} \|D_x^\alpha u\|_E.$$

Remark 2.5. It is straightforward to check that E^l is again a model space. Furthermore, the differential operator D_x^α clearly defines a continuous map $E^l \to E$ whenever $|\alpha| \le l$.

Definition 2.6. For any model space E we define E_0 to be the separable subspace of E obtained by taking the closure of $\mathscr{S}$ in E.

Remark 2.7. It is straightforward to check that E_0 is again a model space. Furthermore, C_0^∞ is a dense subset of $\mathscr{S}$ (see Proposition VI.1.3 in [Yo], for example) so $E_0 = \mathrm{Cl}(\mathscr{S}) = \mathrm{Cl}(C_0^\infty)$; in particular, $\mathscr{S}$ is dense in E (i.e. $E = E_0$) iff C_0^∞ is dense in E.

2.1. Weighted function spaces on $\varPi^n$.

For any chart (ψ, U) on S^{n-1} we can define a corresponding chart $(\varPsi, \mathbb{R} \times U)$ on $\varPi^n$ by setting $\varPsi(t, \omega) = (t, \psi(\omega)) \in \mathbb{R} \times \psi(U) \subseteq \mathbb{R}^n$ for all $(t, \omega) \in \mathbb{R} \times U$. Now suppose $\{(\psi_i, U_i)\}_{i \in I}$ is a finite atlas for S^{n-1} and let $\{(\varPsi_i, \mathbb{R} \times U_i)\}_{i \in I}$ be the corresponding atlas for $\varPi^n$. Choose a partition of unity $\{\chi_i\}_{i \in I}$ which is subordinate to the cover $\{U_i\}_{i \in I}$ of S^{n-1}. We also consider $\{\chi_i\}_{i \in I}$ to be a partition of unity subordinate to the cover $\{\mathbb{R} \times U_i\}_{i \in I}$ of $\varPi^n$ by regarding each χ_i as a function on $\varPi^n$ which is independent of t.

Let E and F be model spaces on $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$ respectively. We define $E(\varPi^n)$ to be the set of $u \in \mathscr{D}'(\varPi^n)$ with $(\varPsi_i^{-1})^*(\chi_i u) \in E$ for each $i \in I$. On this set we define a norm

$$\|u\|_{E(\varPi^n)} = \sum_{i \in I} \|(\varPsi_i^{-1})^*(\chi_i u)\|_E. \tag{4}$$

We define the normed space $F(S^{n-1})$ in a similar fashion with $\varPsi_i$ replaced by ψ_i. Standard calculations using conditions (A2) and (A4) show that $E(\varPi^n)$ and $F(S^{n-1})$ are Banach spaces which are independent of the choice of atlas $\{(\psi_i, U_i)\}_{i \in I}$ and partition of unity $\{\chi_i\}_{i \in I}$ (up to equivalent norms); the next result is a somewhat more general statement of this fact for the space $E(\varPi^n)$.

Lemma 2.8. *Suppose $\{(\phi_j, V_j)\}_{j \in J}$ is a finite atlas for S^{n-1} and $\{\zeta_j\}_{j \in J}$ is a collection of functions in $C^\infty(S^{n-1})$ with $\mathrm{supp}(\zeta_j) \subset V_j$ for each $j \in J$ and $|\sum_{j \in J} \zeta_j| \ge C > 0$ on S^{n-1}. Let $\{(\varPhi_j, \mathbb{R} \times V_j)\}_{j \in J}$ be the corresponding atlas for $\varPi^n$ and consider ζ_j as a function on $\varPi^n$ which is independent of t. Then $u \in E(\varPi^n)$ iff $u \in \mathscr{D}'(\varPi^n)$ and $(\varPhi_j^{-1})^*(\zeta_j u) \in E$ for each $j \in J$. Furthermore,*

$$\|u\|_{E(\varPi^n)} \asymp \sum_{j \in J} \|(\varPhi_j^{-1})^*(\zeta_j u)\|_E.$$

Proof. For each $i \in I$ choose $\chi_i' \in C_0^\infty(U_i)$ with $\chi_i' \succ \chi_i$. Then, for each $j \in J$, set $\widetilde{\zeta}_{ij} = (\zeta_j \chi_i') \circ \varPsi_i^{-1} \in C^\infty$ (here we are considering $\zeta_j \chi_i'$ to be a function on $\varPi^n$ which is independent of t and extending $\widetilde{\zeta}_{ij}$ by 0 outside $\mathbb{R} \times \psi_i(U_i)$). For $u \in E(\varPi^n)$ and $j \in J$,

$$(\varPhi_j^{-1})^*(\zeta_j u) = \sum_{i \in I} (\varPhi_j^{-1})^*(\zeta_j \chi_i' \chi_i u) = \sum_{i \in I} (\varPsi_i \circ \varPhi_j^{-1})^*(\widetilde{\zeta}_{ij} (\varPsi_i^{-1})^*(\chi_i u)).$$

Using Lemma 2.3 (with the obvious minor modification) and condition (A2) we now get $(\Phi_j^{-1})^*(\zeta_j u) \in E$ and

$$\|(\Phi_j^{-1})^*(\zeta_j u)\|_E \leq C \sum_{i\in I} \|(\Psi_i^{-1})^*(\chi_i u)\|_E \leq C\|u\|_{E(\Pi^n)}.$$

Our assumption on the ζ_j's implies $\zeta := (\sum_{j\in J} \zeta_j)^{-1} \in C^\infty(S^{n-1})$. By using the partition of unity on S^{n-1} given by $\{\zeta\zeta_j\}_{j\in J}$, the remainder of the result can be completed with an argument similar to that above. ∎

Remark 2.9. We note the following technically useful consequence of Lemma 2.8. For each $i \in I$ choose $\chi_i' \in C_0^\infty(U_i)$ with $\chi_i' \succ \chi_i$. Then

$$\|u\|_{E(\Pi^n)} = \sum_{i\in I} \|(\Psi_i^{-1})^*(\chi_i u)\|_E \asymp \sum_{i\in I} \|(\Psi_i^{-1})^*(\chi_i' u)\|_E$$

for all $u \in E(\Pi^n)$.

Lemma 2.10. *Suppose (ψ, U) is a chart for S^{n-1} and let $(\Psi, \mathbb{R}\times U)$ be the corresponding chart for Π^n. Also suppose $\chi \in C_0^\infty(U)$ (considered as a function on Π^n which is independent of t) and define $\widetilde{\chi} = \chi\circ\Psi^{-1} \in C^\infty$. Then $\chi u \in E(\Pi^n)$ iff $(\Psi^{-1})^*(\chi u) \in E$ whilst $\|\chi u\|_{E(\Pi^n)} \asymp \|(\Psi^{-1})^*(\chi u)\|_E$. On the other hand, $\widetilde{\chi} v \in E$ iff $\Psi^*(\widetilde{\chi} v) \in E(\Pi^n)$ whilst $\|\widetilde{\chi} u\|_E \asymp \|\Psi^*(\widetilde{\chi} u)\|_{E(\Pi^n)}$.*

Proof. The first part of this result follows easily from Lemma 2.8 applied to the atlas $\{(\psi, U)\}\cup\{(\psi_i, U_i)\}$ for S^{n-1} and any partition of unity $\{\zeta\}\cup\{\zeta_i\}_{i\in I}$ which is subordinate to the covering $\{U\} \cup \{U_i \setminus \operatorname{supp}(\chi)\}_{i\in I}$ of S^{n-1} (n.b. $\zeta\chi = \chi$ and $\zeta_i\chi = 0$ for each $i \in I$ in this case).

Choose $\chi' \in C_0^\infty(U)$ with $\chi' \succ \chi$ and set $\widetilde{\chi}' = \chi' \circ \Psi^{-1} \in C^\infty$. The second part of the result follows from the first part by taking $u = \Psi^*(\widetilde{\chi}' v)$. ∎

Remark 2.11. Suppose K is a locally convex space which satisfies conditions (A1), (A2) and (A4). By applying (4) to individual semi-norms we can obviously define a new locally convex space $K(\Pi^n) \subset \mathscr{D}'(\Pi^n)$. In particular, we shall need the locally convex spaces $\mathscr{S}(\Pi^n)$, $\mathscr{S}'(\Pi^n)$ and $C^\infty(\Pi^n)$. It is straightforward to check that condition (A1) gives us continuous inclusions $\mathscr{S}(\Pi^n) \hookrightarrow E(\Pi^n) \hookrightarrow \mathscr{S}'(\Pi^n)$ for any model space E. It is also clear that $C^\infty(\Pi^n) = \bigcap_{l\in\mathbb{N}_0} C^l(\Pi^n)$ whilst the topology on $C^\infty(\Pi^n)$ is that induced by the collection of semi-norms $\{\|\cdot\|_{C^l(\Pi^n)} \mid l \in \mathbb{N}_0\}$.

Lemma 2.12. *For any model space E multiplication defines a continuous bilinear map $C^\infty(\Pi^n) \times E(\Pi^n) \to E(\Pi^n)$.*

Proof. Condition (A2) for E means we can find $l \in \mathbb{N}_0$ such that

$$\|\phi u\|_E \leq C\|\phi\|_{C^l}\|u\|_E$$

for all $\phi \in C^\infty$ and $u \in E$. With the notation of Remark 2.9 it follows that

$$\|(\Psi_i^{-1})^*(\chi_i\phi u)\|_E = \|(\Psi_i^{-1})^*(\chi_i\phi)(\Psi_i^{-1})^*(\chi_i' u)\|_E \leq C\|(\Psi_i^{-1})^*(\chi_i\phi)\|_{C^l}\|(\Psi_i^{-1})^*(\chi_i' u)\|_E$$

for any $\phi \in C^\infty(\Pi^n)$ and $u \in E(\Pi^n)$. Combining this with Remark 2.9 and the fact that

I is finite, we then get

$$\begin{aligned}\|\phi u\|_{E(\Pi^n)} &\leq C\sum_{i\in I}\|(\Psi_i^{-1})^*(\chi_i\phi u)\|_E \\ &\leq C\sum_{i\in I}\|(\Psi_i^{-1})^*(\chi_i\phi)\|_{C^l}\|(\Psi_i^{-1})^*(\chi_i' u)\|_E \leq C\|\phi\|_{C^l(\Pi^n)}\|u\|_{E(\Pi^n)}\end{aligned}$$

for all $\phi\in C^\infty(\Pi^n)$ and $u\in E(\Pi^n)$. ∎

Given a model space E and $\beta\in\mathbb{R}$ we define $\mathsf{Z}_\beta E$ to be the set of $u\in\mathscr{D}'(\Pi^n)$ with $e^{\beta t}u\in E(\Pi^n)$. On this set we define a norm

$$\|u\|_{\mathsf{Z}_\beta E} = \|e^{\beta t}u\|_{E(\Pi^n)}. \tag{5}$$

Clearly, $\mathsf{Z}_\beta E$ is a Banach space and $\mathsf{Z}_\beta E\subset E(\Pi^n)_{\mathrm{loc}}$.

Remark 2.13. An easy consequence of Lemma 2.12 is that multiplication defines a continuous bilinear map $C^\infty(\Pi^n)\times\mathsf{Z}_\beta E\to\mathsf{Z}_\beta E$ for any model space E and $\beta\in\mathbb{R}$.

Remark 2.14. With K as in Remark 2.11 it is clear that we can define a locally convex space $\mathsf{Z}_\beta K\subset\mathscr{D}'(\Pi^n)$ for any $\beta\in\mathbb{R}$. It is straightforward to check that condition (A1) gives us continuous inclusions $\mathsf{Z}_\beta\mathscr{S}\hookrightarrow\mathsf{Z}_\beta E\hookrightarrow\mathsf{Z}_\beta\mathscr{S}'$ for any model space E and $\beta\in\mathbb{R}$.

In order to work with the spaces $\mathsf{Z}_\beta E$ (and other spaces to be defined below) we need to introduce a set of auxiliary functions. Choose $\phi_0\in C^\infty(\mathbb{R})$ with $\phi_0=0$ on $(-\infty,-2/3]$, $\phi_0=1$ on $[-1/3,\infty)$ and $\operatorname{Ran}\phi_0=[0,1]$. For any $i,j\in\mathbb{Z}\cup\{\pm\infty\}$ with $j\geq i$ define $\phi_{ij}\in C^\infty(\Pi^n)$ by

$$\phi_{ij}(t,\omega)=\phi_0(t-i)-\phi_0(t-j-1).$$

Therefore ϕ_{ij} is non-negative, $\operatorname{supp}(\phi_{ij})\subseteq(i-1,j+1)\times S^{n-1}$ and $\phi_{ij}=1$ on $[i,j]\times S^{n-1}$. We also set $\phi_i=\phi_{i\infty}$ so $\phi_{ij}=\phi_i-\phi_{j+1}$.

Remark 2.15. Suppose $\phi\in C^\infty(\Pi^n)$ satisfies $\operatorname{supp}(\phi)\subseteq[i,j]\times S^{n-1}$ for some $i,j\in\mathbb{Z}\cup\{\pm\infty\}$ with $i\leq j$. Then we have

$$\|\phi u\|_{\mathsf{Z}_\beta E}=\Big\|\sum_{k=i}^{j}\phi_{kk}\phi u\Big\|_{\mathsf{Z}_\beta E}\leq\sum_{k=i}^{j}\|\phi_{kk}\phi u\|_{\mathsf{Z}_\beta E}$$

for all $u\in E(\Pi^n)_{\mathrm{loc}}$. On the other hand, $\{\phi_{kk}\}_{k\in\mathbb{Z}}$ is a bounded subset of $C^\infty(\Pi^n)$ so Remark 2.13 implies

$$\sup_{i\leq k\leq j}\|\phi_{kk}\phi u\|_{\mathsf{Z}_\beta E}\leq C\|\phi u\|_{\mathsf{Z}_\beta E}$$

for all $u\in E(\Pi^n)_{\mathrm{loc}}$.

For any $k\in\mathbb{Z}$ we can write $e^{\beta t}=e^{\beta k}e^{\beta(t-k)}$ where $e^{\beta(t-k)}$ and its derivatives can be bounded independently of $k\in\mathbb{Z}$ on $\operatorname{supp}(\phi_{kk})\subseteq(k-1,k+1)$. Lemma 2.12 then gives

$$\|\phi_{kk}u\|_{\mathsf{Z}_\beta E}\asymp e^{\beta k}\|\phi_{kk}u\|_{E(\Pi^n)}, \tag{6}$$

where the equivalence constants are independent of k.

2.2. Weighted function spaces on $\mathbb{R}^n_*$ and $\mathbb{R}^n$. Let $\Theta:\Pi^n\to\mathbb{R}^n_*$ denote the diffeomorphism defined by $\Theta(t,\omega)=(r,\omega)$ where $r=e^t$. Under the pull-back Θ^* we clearly have

$$r \to e^t, \quad rD_r \to D_t \quad \text{and} \quad r^{-1}\,dr \to dt. \tag{7}$$

Now Θ^* defines an isomorphism $\mathscr{D}'(\mathbb{R}^n_*) \to \mathscr{D}'(\Pi^n)$. For any model space E and $\beta \in \mathbb{R}$ we define $\mathsf{Y}_\beta E \subset \mathscr{D}'(\mathbb{R}^n_*)$ to be the preimage of $\mathsf{Z}_\beta E$ under Θ^* with the induced norm. Therefore the restricted map

$$\Theta^* : \mathsf{Y}_\beta E \to \mathsf{Z}_\beta E \tag{8}$$

is an isomorphism, and

$$\|u\|_{\mathsf{Y}_\beta E} = \|\Theta^* u\|_{\mathsf{Z}_\beta E} \quad \text{for any } u \in \mathsf{Y}_\beta E. \tag{9}$$

Choose $\theta \in \mathrm{BS}_{01}$ with $\theta = 1$ on $\{|x| \geq 2\}$, $\theta = 0$ on $\{|x| \leq 1\}$ and $\operatorname{Ran}\theta = [0,1]$. For any model space E and $\beta \in \mathbb{R}$ we define $\mathsf{X}_\beta E$ to be the set of $u \in \mathscr{D}'$ with $(1-\theta)u \in E$ and $\theta u \in \mathsf{Y}_\beta E$. On this set we define a norm

$$\|u\|_{\mathsf{X}_\beta E} = \|(1-\theta)u\|_E + \|\theta u\|_{\mathsf{Y}_\beta E}.$$

Straightforward calculations show that $\mathsf{X}_\beta E \subset E_{\mathrm{loc}}$, $\mathsf{X}_\beta E$ is a Banach space and the definition is independent of the choice of θ (up to equivalence of norms). These and other observations are summarised in the following result.

Lemma 2.16. *Suppose $\beta \in \mathbb{R}$, $\zeta \in \mathrm{BS}_{01}$ and $\eta \in C_0^\infty$ is a non-negative function for which $\eta + \zeta$ is bounded away from 0. Then $u \in \mathsf{X}_\beta E$ iff $\eta u \in E$ and $\zeta u \in \mathsf{Y}_\beta E$. Furthermore, all such u satisfy an estimate of the from*

$$\|u\|_{\mathsf{X}_\beta E} \asymp \|\eta u\|_E + \|\zeta u\|_{\mathsf{Y}_\beta E}.$$

Also, for $u \in E(\mathbb{R}^n_*)_{\mathrm{loc}}$ we have $\zeta u \in \mathsf{X}_\beta E$ iff $\zeta u \in \mathsf{Y}_\beta E$ whilst all such u satisfy an estimate of the form $\|\zeta u\|_{\mathsf{X}_\beta E} \asymp \|\zeta u\|_{\mathsf{Y}_\beta E}$.

Remark 2.17. Suppose $f \in C^\infty$ is constant on a neighbourhood of 0 and satisfies $f \circ \Theta \in C^\infty(\Pi^n)$ (we could take $f \in \mathrm{BS}$, for example). For any $\beta \in \mathbb{R}$, Remark 2.13 and (8) then imply that multiplication by f defines a continuous map $\mathsf{Y}_\beta E \to \mathsf{Y}_\beta E$. By coupling this observation with Lemma 2.16 and the fact that f is constant on a neighbourhood of 0, it follows easily that multiplication by f also defines a continuous map $\mathsf{X}_\beta E \to \mathsf{X}_\beta E$.

Remark 2.18. Suppose $\beta, \gamma \in \mathbb{R}$ and $\zeta \in \mathrm{BS}_{01}$. Making straightforward applications of Lemma 2.16 (with $\eta = 1 - \zeta$) we can obtain the following:

(i) If $\zeta u \in \mathsf{X}_\gamma E$ for some $u \in \mathsf{X}_\beta E$ then we also have $u \in \mathsf{X}_\gamma E$.

(ii) If $\zeta u \in \mathsf{Y}_\gamma E$ and $u = 0$ in a neighbourhood of 0 for some $u \in \mathsf{Y}_\beta E$ then we also have $u \in \mathsf{Y}_\gamma E$.

For each $i, j \in \mathbb{Z} \cup \{\pm\infty\}$ with $j \geq i$ define $\zeta_i, \zeta_{ij} \in C^\infty$ by $\zeta_i = \phi_i \circ \Theta^{-1}$ and $\zeta_{ij} = \phi_{ij} \circ \Theta^{-1}$ (n.b. there are no problems with smoothness at 0 since ϕ_i and ϕ_{ij} are constant in a neighbourhood of $-\infty \times S^{n-1}$). Also set $\eta_i = \zeta_{-\infty i-1}$. Therefore $\zeta_i \in \mathrm{BS}_{01}$, $\eta_i \in C_0^\infty$ and $\eta_i + \zeta_i = 1$. The next result follows from Lemma 2.16 with $\zeta = \zeta_0$, $\eta = \eta_0$ and the observation that $\zeta_0\zeta_i = \zeta_i$, $\eta_0\zeta_i = 0$ for any $i \in \mathbb{N}$.

Lemma 2.19. *Given $i \in \mathbb{N}$ we have $\|\zeta_i u\|_{\mathsf{X}_\beta E} \asymp \|\zeta_i u\|_{\mathsf{Y}_\beta E}$ for any $u \in \mathsf{X}_\beta E$, where the equivalence constants are independent of i.*

2.3. Basic properties

2.3.1. *Isomorphisms*

Proposition 2.20. *Suppose E is a model space and $\beta, \gamma \in \mathbb{R}$. Then multiplication by $e^{\gamma t}$, r^{γ} and Λ^{γ} defines isomorphisms $\mathsf{Z}_{\beta+\gamma}E \to \mathsf{Z}_{\beta}E$, $\mathsf{Y}_{\beta+\gamma}E \to \mathsf{Y}_{\beta}E$ and $\mathsf{X}_{\beta+\gamma}E \to \mathsf{X}_{\beta}E$ respectively.*

Proof. The first isomorphism is an immediate consequence of (5) whilst the second then follows from (7) and (8). Now define a smooth function f on $\mathbb{R}^n$ by $f = \zeta_0 \Lambda^{\gamma} r^{-\gamma}$. It follows that $f \circ \Theta$ is a smooth function on Π^n with $f(\Theta(t,\omega)) = 0$ for $t \le -1$ and $f(\Theta(t,\omega)) = (1+e^{-2t})^{\gamma/2}$ for $t \ge 1$. Hence $f \circ \Theta \in C^{\infty}(\Pi^n)$, so multiplication by f defines a continuous map $\mathsf{Y}_{\beta}E \to \mathsf{Y}_{\beta}E$ by Remark 2.17. Combining this observation with Lemma 2.16, the identity $\zeta_1 \Lambda^{\gamma} = f \zeta_1 r^{\gamma}$ and the second part of the present result, we now have

$$\|\Lambda^{\gamma}u\|_{\mathsf{X}_{\beta}E} \le C(\|\eta_1\Lambda^{\gamma}u\|_E + \|f\zeta_1 r^{\gamma}u\|_{\mathsf{Y}_{\beta}E}) \le C(\|\eta_1\Lambda^{\gamma}u\|_E + \|\zeta_1 u\|_{\mathsf{Y}_{\beta+\gamma}E}) \le C\|u\|_{\mathsf{X}_{\beta+\gamma}E}$$

for all $u \in \mathsf{X}_{\beta+\gamma}E$ (n.b. $\eta_1\Lambda^{\gamma} \in C_0^{\infty}$ is non-negative whilst $\eta_1\Lambda^{\gamma} + \zeta_1$ is bounded away from 0). A similar argument for $\Lambda^{-\gamma}$ now completes the result. ∎

2.3.2. *Inclusions*

Remark 2.21. It is easy to see that any continuous inclusion $E \hookrightarrow F$ between model spaces E and F induces continuous inclusions $\mathsf{Z}_{\beta}E \hookrightarrow \mathsf{Z}_{\beta}F$, $\mathsf{Y}_{\beta}E \hookrightarrow \mathsf{Y}_{\beta}F$ and $\mathsf{X}_{\beta}E \hookrightarrow \mathsf{X}_{\beta}F$ for any $\beta \in \mathbb{R}$.

By a *local inclusion* $E_{\mathrm{loc}} \hookrightarrow F_{\mathrm{loc}}$ we mean that $E_{\mathrm{loc}} \subseteq F_{\mathrm{loc}}$ and $\|\phi u\|_F \le C\|\phi u\|_E$ for $\phi \in C_0^{\infty}$ and all $u \in E_{\mathrm{loc}}$ (where C may depend on ϕ).

Remark 2.22. We have a local inclusion $H^{p_1,s_1}_{\mathrm{loc}} \hookrightarrow H^{p_2,s_2}_{\mathrm{loc}}$ whenever $p_1, p_2 \in [1,\infty)$ and $s_1, s_2 \in \mathbb{R}$ satisfy $s_1 \ge s_2$ and $s_1 - n/p_1 \ge s_2 - n/p_2$. The additional condition $p_1 \le p_2$ is needed in order to get a continuous inclusion $H^{p_1,s_1} \hookrightarrow H^{p_2,s_2}$. We also have continuous inclusions $H^{p_1,s_1} \hookrightarrow \mathcal{C}^{s_2}$ whenever $s_1 - n/p_1 \ge s_2 > 0$ and $H^{p_1,s_1} \hookrightarrow C^k$ whenever $k \in \mathbb{N}_0$ satisfies $s_1 - n/p_1 > k$. Further details can be found in Sections 2.3.2, 2.7.1 and 3.3.1 of [T1].

Obviously, a continuous inclusion $E \hookrightarrow F$ leads to a local inclusion $E_{\mathrm{loc}} \hookrightarrow F_{\mathrm{loc}}$. Although the converse does not hold in general we can obtain the following related results, the first of which is an easy consequence of the compactness of S^{n-1}.

Lemma 2.23. *A local inclusion $E_{\mathrm{loc}} \hookrightarrow F_{\mathrm{loc}}$ for model spaces E and F on $\mathbb{R}^{n-1}$ induces a continuous inclusion $E(S^{n-1}) \hookrightarrow F(S^{n-1})$.*

Lemma 2.24. *Suppose we have a local inclusion $E_{\mathrm{loc}} \hookrightarrow F_{\mathrm{loc}}$ for some model spaces E and F. If $\beta \in \mathbb{R}$, $\varepsilon > 0$, $l \in \mathbb{Z}$ and $\phi_{\pm} \in C^{\infty}(\Pi^n)$ with* $\operatorname{supp}(\phi_{\pm}) \subseteq \pm[l,+\infty) \times S^{n-1}$, *then*

$$\|\phi_{\pm}u\|_{\mathsf{Z}_{\beta}F} \le C(\varepsilon,l)\|\phi_{\pm}u\|_{\mathsf{Z}_{\beta\pm\varepsilon}E}$$

for any $u \in E(\Pi^n)_{\mathrm{loc}}$. Furthermore, $C(\varepsilon,l)$ can be chosen independently of $\phi_{\pm}$.

Proof. Clearly, condition (A3) implies that the norm $\|\cdot\|_{F(\Pi^n)}$ is invariant under translations with respect to the first variable of Π^n. With the help of (6) it follows that the local inclusion $E_{\mathrm{loc}} \hookrightarrow F_{\mathrm{loc}}$ leads to an estimate

$$\|\phi_{kk}v\|_{\mathsf{Z}_\beta F} \le Ce^{-\varepsilon k}\|\phi_{kk}v\|_{\mathsf{Z}_{\beta+\varepsilon}E} \tag{10}$$

for any $k \in \mathbb{Z}$ and $v \in E(\varPi^n)_{\mathrm{loc}}$, where C is independent of k. On the other hand, $\sum_{k\ge l}\phi_{kk} = 1$ on $\mathrm{supp}(\phi_+)$. Combined with Remark 2.15 and (10) this implies

$$\begin{aligned}\|\phi_+u\|_{\mathsf{Z}_\beta F} &\le \sum_{k\ge l}\|\phi_{kk}\phi_+u\|_{\mathsf{Z}_\beta F} \le C\sum_{k\ge l}e^{-\varepsilon k}\|\phi_{kk}\phi_+u\|_{\mathsf{Z}_{\beta+\varepsilon}E}\\ &\le C(\sup_{k\ge l}\|\phi_{kk}\phi_+u\|_{\mathsf{Z}_{\beta+\varepsilon}E})\sum_{k\ge l}e^{-\varepsilon k} \le C\|\phi_+u\|_{\mathsf{Z}_{\beta+\varepsilon}E}\end{aligned}$$

for any $u \in E(\varPi^n)_{\mathrm{loc}}$. Clearly, a similar argument can be used for ϕ_-. ∎

Remark 2.25. If $\phi_\pm$ are as in the previous lemma it is easy to check that multiplication by $\phi_\pm$ defines continuous maps $\mathsf{Z}_\beta\mathscr{S} \to \mathsf{Z}_{\beta\mp\varepsilon}\mathscr{S}$ and $\mathsf{Z}_\beta\mathscr{S}' \to \mathsf{Z}_{\beta\mp\varepsilon}\mathscr{S}'$ for any $\beta \in \mathbb{R}$ and $\varepsilon \ge 0$.

Proposition 2.26. *Suppose we have a local inclusion $E_{\mathrm{loc}} \hookrightarrow F_{\mathrm{loc}}$ for some model spaces E and F. If $\beta, \gamma \in \mathbb{R}$ with $\beta < \gamma$ then we have a continuous inclusion $\mathsf{X}_\gamma E \hookrightarrow \mathsf{X}_\beta F$.*

This result obviously implies that we have a continuous inclusion

$$\mathsf{X}_\gamma E \hookrightarrow \mathsf{X}_\beta E \quad \text{for } \beta \le \gamma. \tag{11}$$

Proof. Let $u \in \mathsf{X}_\gamma E$. Therefore $u \in E_{\mathrm{loc}} \subseteq F_{\mathrm{loc}}$. Now Lemma 2.16 gives

$$\|u\|_{\mathsf{X}_\beta F} \le C(\|\eta_1 u\|_F + \|\zeta_1 u\|_{\mathsf{Y}_\beta F}). \tag{12}$$

However, (9) and Lemma 2.24 (with $\phi_+ = \phi_1$) combine to give

$$\|\zeta_1 u\|_{\mathsf{Y}_\beta F} = \|\phi_1\varTheta^*u\|_{\mathsf{Z}_\beta F} \le C\|\phi_1\varTheta^*u\|_{\mathsf{Z}_\gamma E} = \|\zeta_1 u\|_{\mathsf{Y}_\gamma E}.$$

Together with the definition of local inclusion and (12) this shows that

$$\|u\|_{\mathsf{X}_\beta F} \le C(\|\eta_1 u\|_E + \|\zeta_1 u\|_{\mathsf{Y}_\gamma E}) \le C\|u\|_{\mathsf{X}_\gamma E},$$

the last inequality following from a further application of Lemma 2.16. ∎

We finish this section with some results which are direct consequences of conditions (A1) to (A3) for model spaces. We will make use of the collection of semi-norms $\{p_l \mid l \in \mathbb{N}_0\}$ for $\mathscr{S}$ introduced in Section 1.1; in particular, we observe that $p_{l'}(u) \le p_l(u)$ whenever $0 \le l' \le l$.

Lemma 2.27. *We have $C^l_{\mathrm{loc}} \hookrightarrow E_{\mathrm{loc}}$ for all sufficiently large $l \in \mathbb{N}_0$.*

Proof. The continuous inclusion $\mathscr{S} \hookrightarrow E$ given by condition (A1) simply means that we can find $l' \in \mathbb{N}_0$ such that $\|u\|_E \le Cp_{l'}(u)$ for all $u \in \mathscr{S}$. Now let $l \in \mathbb{N}_0$ with $l \ge l'$. Also let $\phi \in C^\infty_0$ and choose $\phi_1 \in C^\infty_0$ with $\phi_1 \succ \phi$. Therefore $D^\alpha_x(\phi u) = \phi_1 D^\alpha_x(\phi u)$ for any multi-index α whilst $\varLambda^l\phi_1 \in C^\infty_0$. Hence

$$\|\phi u\|_E \le Cp_l(\phi u) = C\sum_{|\alpha|\le l}\|\varLambda^l\phi_1 D^\alpha_x(\phi u)\|_{L^\infty} \le C\sum_{|\alpha|\le l}\|D^\alpha_x(\phi u)\|_{L^\infty} = C\|\phi u\|_{C^l}$$

for all $u \in \mathscr{S}$. The result now follows from the fact that if $u \in C^l$ then we can approximate ϕu arbitrarily closely (in the C^l norm) by $\phi u'$ for some $u' \in \mathscr{S}$. ∎

LEMMA 2.28. *Suppose L is a locally convex space whose topology is given by a countable collection of semi-norms $\{q_l \mid l \in \mathbb{N}\}$. Also suppose X and Y are Banach spaces and $T : L \times X \to Y$ is a bilinear mapping which is continuous in each variable. Then we can find $l \in \mathbb{N}$ and a constant C such that*

$$\|T(a,x)\|_Y \leq C\|x\|_X \sum_{i=1}^{l} q_i(a) \tag{13}$$

for all $(a,x) \in L \times X$.

Proof. Without loss of generality we may assume that the semi-norms are defined so that $q_i \leq q_j$ for all $i, j \in \mathbb{N}$ with $i \leq j$. Therefore (13) can be rewritten as

$$\|T(a,x)\|_Y \leq C\|x\|_X \, q_l(a)$$

for all $(a,x) \in L \times X$. Suppose an estimate of this form is not valid. It follows that we can find a sequence $\{a_i\}_{i\in\mathbb{N}}$ in L such that $q_i(a_i) \leq 1$ and $\|T(a_i,\cdot)\|_{\mathscr{L}(X,Y)} \to \infty$ as $i \to \infty$. Now the continuity of the map $T(\cdot,x) : L \to Y$ gives us $j \in \mathbb{N}$ and a constant C such that $\|T(a_i,x)\|_Y \leq Cq_j(a_i)$ for all $i \in \mathbb{N}$. It follows that the set $\{T(a_i,x) \mid i \in \mathbb{N}\}$ is bounded in Y (by the maximum of $Cq_j(a_1), \ldots, Cq_j(a_{j-1})$ and C). The Uniform Boundedness Theorem (see Section II.1 in [Yo], for example) then implies $\{\|T(a_i,\cdot)\|_{\mathscr{L}(X,Y)} \mid i \in \mathbb{N}\}$ must also be bounded. The result now follows by contradiction. ■

LEMMA 2.29. *We have $E^l \hookrightarrow C^0$ for all sufficiently large $l \in \mathbb{N}_0$.*

Proof. Using the inclusion $E \hookrightarrow \mathscr{S}'$ and the dual pairing $\mathscr{S} \times \mathscr{S}' \to \mathbb{C}$ we can define a continuous bilinear map $\mathscr{S} \times E \to \mathbb{C}$ by $(\phi, u) \mapsto (\phi, u)_{\mathbb{R}^n}$. By Lemma 2.28 we can thus find $j \in \mathbb{N}$ and a constant C such that

$$|(\phi,u)_{\mathbb{R}^n}| \leq C\|u\|_E \, p_j(\phi)$$

for all $\phi \in \mathscr{S}$ and $u \in E$. Now suppose $\chi \in C_0^\infty$ and choose $\chi_1, \chi_2 \in C_0^\infty$ with $\chi_2 \succ \chi_1 \succ \chi$. Thus $D_x^\alpha(\chi_1\phi) = \chi_2 D_x^\alpha(\chi_1\phi)$ and $\Lambda^j\chi_2 \in C_0^\infty$ so

$$p_j(\chi_1\phi) = \sum_{|\alpha|\leq j} \|\Lambda^j \chi_2 D_x^\alpha(\chi_1\phi)\|_{L^\infty} \leq C\|\chi_1\phi\|_{C^j} \leq C\|\phi\|_{H^{j+n}}$$

for all $\phi \in \mathscr{S}$, where we have used the continuous inclusion $H^{j+n} \hookrightarrow C^j$ in the last inequality. Therefore

$$|(\phi,\chi u)_{\mathbb{R}^n}| = |(\chi_1\phi,\chi u)_{\mathbb{R}^n}| \leq C\|\chi u\|_E \, \|\phi\|_{H^{j+n}}$$

for all $u \in E$ and $\phi \in \mathscr{S}$. Now let $l \in \mathbb{N}_0$ with $l > 2n + j$. Choose $l' \in 2\mathbb{N}_0$ so that $2n + j \leq l' \leq l$. Now the Fourier multiplier $\Lambda^{-l'}(D)$ defines an isomorphism on $\mathscr{S}$ whose inverse is simply the constant coefficient differential operator $\Lambda^{l'}(D)$ of order l' (recall that l' is even). Therefore

$$\begin{aligned} |(\phi,\chi u)_{\mathbb{R}^n}| &= |(\Lambda^{l'}(D)\Lambda^{-l'}(D)\phi,\chi u)_{\mathbb{R}^n}| = |(\Lambda^{-l'}(D)\phi,\Lambda^{l'}(D)(\chi u))_{\mathbb{R}^n}| \\ &\leq C\|\Lambda^{l'}(D)(\chi u)\|_E \|\Lambda^{-l'}(D)\phi\|_{H^{j+n}} \leq C\|\chi u\|_{E^l}\|\phi\|_{H^{-n}} \end{aligned} \tag{14}$$

for all $u \in E^l$ and $\phi \in \mathscr{S}$, where the last inequality follows from Remark 2.5, the fact that $\Lambda^{-l'}(D)$ defines an isomorphism $H^{j+n-l'} \to H^{j+n}$, and the inequality $j + n - l' \leq -n$. Since $\mathscr{S}$ is dense in H^{-n}, (14) implies that for any $u \in E^l$ we have $\chi u \in H^n$ (the dual space of H^{-n}) with a corresponding norm estimate. Since we have a continuous inclusion $H^n \hookrightarrow C^0$ whilst $\chi \in C_0^\infty$ was arbitrary, we finally arrive at a local inclusion $E^l_{\mathrm{loc}} \hookrightarrow C^0_{\mathrm{loc}}$.

Let $\{\chi_I\}_{I\in\mathbb{Z}^n}$ be a partition of unity of $\mathbb{R}^n$ where $\chi_0 \in C_0^\infty$ and, for each $I \in \mathbb{Z}^n$, $\chi_I(x) = \chi_0(x - I)$. Using conditions (A2) and (A3) we thus have $\|\chi_I u\|_{E^l} \le C\|u\|_{E^l}$ for all $u \in E^l$, where C is independent of $I \in \mathbb{Z}^n$. On the other hand, it is clear that $\|u\|_{C^0} \le C \sup_{I\in\mathbb{Z}^n} \|\chi_I u\|_{C^0}$ (where for C we can take $\#\{I \in \mathbb{Z}^n \mid \mathrm{supp}(\chi_0) \cap \mathrm{supp}(\chi_I) \neq \emptyset\}$). The fact that we have a continuous inclusion $E^l \hookrightarrow C^0$ now follows from the existence of a local inclusion. ■

Clearly, a continuous inclusion $E \hookrightarrow F$ for model spaces E and F leads to a continuous inclusion $E^l \hookrightarrow F^l$ for any $l \in \mathbb{N}_0$. Coupling this observation with Lemmas 2.27 and 2.29 we immediately get the following.

COROLLARY 2.30. *For any model spaces E and F we have a local inclusion $E^l_{\mathrm{loc}} \hookrightarrow F_{\mathrm{loc}}$ for all sufficiently large $l \in \mathbb{N}_0$.*

2.3.3. *Derivatives.* For each $i \in \{1, \dots, n\}$ we can write

$$D_i = r^{-1}(b_i(\omega)(rD_r) + P_i(\omega, D_\omega)) \tag{15}$$

where $b_i(\omega) = x_i/r \in C^\infty(S^{n-1})$ and $P_i(\omega, D_\omega)$ is a first order differential operator on S^{n-1}. Now any first order differential operator A on S^{n-1} can be written as a (not necessarily unique) linear combination

$$A = a_0 + \sum_{i=1}^{n} a_i P_i$$

where $a_0, \dots, a_n \in C^\infty(S^{n-1})$. On the other hand, $b_1^2 + \dots + b_n^2 = 1$ while $b_1 P_1 + \dots + b_n P_n = 0$. Defining differential operators B_i on $\varPi^n$ by $B_i = b_i(\omega)D_t + P_i(\omega, D_\omega)$ for $i = 1, \dots, n$ we thus arrive at the following result.

LEMMA 2.31. *If $A(\omega, D_\omega, D_t)$ is a first order differential operator on $\varPi^n$ whose coefficients do not depend upon t then we can write*

$$A = a_0 + \sum_{i=1}^{n} a_i B_i$$

for some $a_0, \dots, a_n \in C^\infty(S^{n-1})$.

LEMMA 2.32. *Suppose E is a model space and $\beta \in \mathbb{R}$. Then*

$$\|u\|_{\mathsf{Z}_\beta E^1} \asymp \|u\|_{\mathsf{Z}_\beta E} + \sum_{i=1}^{n} \|B_i u\|_{\mathsf{Z}_\beta E}$$

for all $u \in \mathscr{D}'(\varPi^n)$ (where we define the norm of a function not belonging to the relevant space to be $+\infty$). In particular, B_i defines a continuous map $\mathsf{Z}_\beta E^1 \to \mathsf{Z}_\beta E$ for $i = 1, \dots, n$.

Proof. Consider the notation of Remark 2.9 and let $i \in I$ and $j \in \{1, \dots, n\}$. Now $(\varPsi_i^{-1})^* \chi_i B_j \chi_i' \varPsi_i^*$ is a first order differential operator on $\mathbb{R}^n$ whose coefficients are contained in C^∞ (in fact, the coefficients do not depend upon the first variable and are compactly supported with respect to the rest). Thus we have

$$\|(\varPsi_i^{-1})^*\chi_i(B_j+i\beta b_j)\chi_i'\varPsi_i^* v\|_E \leq C\|v\|_{E^1}$$

for any $v \in \mathscr{D}'$. On the other hand, $e^{\beta t}B_j u = (B_j+i\beta b_j)e^{\beta t}u$ so

$$(\varPsi_i^{-1})^*\chi_i e^{\beta t}B_j u = (\varPsi_i^{-1})^*\chi_i(B_j+i\beta b_j)\chi_i'\varPsi_i^*(\varPsi_i^{-1})^*\chi_i' e^{\beta t}u$$

for any $i \in I$, $j \in \{1,\dots,n\}$ and $u \in \mathscr{D}'(\varPi^n)$. Since we clearly have $\|u\|_{\mathsf{X}_\beta E} \leq C\|u\|_{\mathsf{X}_\beta E^1}$, the above results combine to give

$$\|u\|_{\mathsf{Z}_\beta E} + \sum_{i=1}^n \|B_i u\|_{\mathsf{Z}_\beta E} \leq C\|u\|_{\mathsf{Z}_\beta E^1}$$

for all $u \in \mathscr{D}'(\varPi^n)$.

Let $i \in I$. Using Definition 2.4 we get

$$\|(\varPsi_i^{-1})^*\chi_i e^{\beta t}u\|_{E^1} \leq \|(\varPsi_i^{-1})^*\chi_i e^{\beta t}u\|_E + \sum_{j=1}^n \|D_j(\varPsi_i^{-1})^*\chi_i e^{\beta t}u\|_E$$

for all $u \in \mathscr{D}'(\varPi^n)$. On the other hand, we can write

$$D_j(\varPsi_i^{-1})^*\chi_i e^{\beta t}u = (\varPsi_i^{-1})^*\chi_i' e^{\beta t}A_{ij}u$$

where $A_{ij} = \chi_i'\varPsi^*(D_j - i\beta\delta_{j1})(\varPsi^{-1})^*\chi_i$ is a first order differential operator on $\varPi^n$ whose coefficients are independent of t. By Lemma 2.10, Remark 2.9 and (5) we thus have

$$\|u\|_{\mathsf{Z}_\beta E^1} \leq C\Big(\|u\|_{\mathsf{Z}_\beta E} + \sum_{i\in I}\sum_{j=1}^n \|A_{ij}u\|_{\mathsf{Z}_\beta E}\Big)$$

for all $u \in \mathscr{D}'(\varPi^n)$. On the other hand, using Lemma 2.31 and the fact that multiplication by an element of $C^\infty(S^{n-1})$ defines a continuous map on $\mathsf{Z}_\beta E$ (see Remark 2.13), we get

$$\|A_{ij}u\|_{\mathsf{Z}_\beta E} \leq C\Big(\|u\|_{\mathsf{Z}_\beta E} + \sum_{j'=1}^n \|B_{j'}u\|_{\mathsf{Z}_\beta E}\Big)$$

for all $u \in \mathscr{D}'(\varPi^n)$. Clearly, the last two estimates complete the result. ∎

For any $i \in \{1,\dots,n\}$, (7) and (15) give us $D_i = r^{-1}(\varTheta^{-1})^*B_i\varTheta^*$. With the help of (8) and Proposition 2.20, Lemma 2.32 now implies that

$$\|u\|_{\mathsf{Y}_\beta E^1} \asymp \|u\|_{\mathsf{Y}_\beta E} + \sum_{i=1}^n \|D_i u\|_{\mathsf{Y}_{\beta+1}E} \tag{16}$$

for all $u \in \mathscr{D}'(\mathbb{R}^n_*)$.

Proposition 2.33. *Let E be a model space, $l \in \mathbb{N}_0$ and $\beta \in \mathbb{R}$. Given $u \in \mathscr{D}'$ we have $u \in \mathsf{X}_\beta E^l$ iff $D_x^\alpha u \in \mathsf{X}_{\beta+|\alpha|}E$ for all multi-indices α with $|\alpha| \leq l$. Furthermore,*

$$\|u\|_{\mathsf{X}_\beta E^l} \asymp \sum_{|\alpha|\leq l} \|D_x^\alpha u\|_{\mathsf{X}_{\beta+|\alpha|}E}$$

for all $u \in \mathsf{X}_\beta E^l$. In particular, the differential operator D_x^α defines a continuous map $\mathsf{X}_\beta E^l \to \mathsf{X}_{\beta+|\alpha|}E$ whenever $|\alpha| \leq l$.

Proof. Induction clearly reduces the proof to the case $l = 1$. Now let $i \in \{1, \dots, n\}$ and $u \in \mathscr{D}'$. Since $\zeta_0, \eta_2 \succ D_i\zeta_1 = -D_i\eta_1$ several applications of Lemma 2.16 and condition (A2) give us

$$\begin{aligned}\|(D_i\zeta_1)u\|_{\mathsf{Y}_{\beta+1}E} &= \|\zeta_0(D_i\zeta_1)u\|_{\mathsf{Y}_{\beta+1}E} \le C\|(D_i\zeta_1)u\|_{\mathsf{X}_{\beta+1}E} \\ &\le C(\|\eta_2(D_i\zeta_1)u\|_E + \|\zeta_2(D_i\zeta_1)u\|_{\mathsf{Y}_{\beta+1}E}) \\ &= C\|(D_i\eta_1)u\|_E \le C\|\eta_2 u\|_E \le C\|u\|_{\mathsf{X}_\beta E}.\end{aligned}$$

Combining this with Definition 2.4, Lemma 2.16 and (16) we thus have

$$\begin{aligned}\|D_iu\|_{\mathsf{X}_{\beta+1}E} &\le C(\|\eta_1 D_iu\|_E + \|\zeta_1 D_iu\|_{\mathsf{Y}_{\beta+1}E}) \\ &\le C(\|D_i(\eta_1u)\|_E + \|D_i(\zeta_1u)\|_{\mathsf{Y}_{\beta+1}E} + \|u\|_{\mathsf{X}_\beta E}) \\ &\le C(\|\eta_1u\|_{E^1} + \|\zeta_1u\|_{\mathsf{Y}_\beta E^1} + \|u\|_{\mathsf{X}_\beta E}) \le C\|u\|_{\mathsf{X}_\beta E^1}\end{aligned}$$

and

$$\begin{aligned}\|u\|_{\mathsf{X}_\beta E^1} &\le C(\|\eta_1u\|_{E^1} + \|\zeta_1u\|_{\mathsf{Y}_\beta E^1}) \\ &\le C\Big(\|\eta_1u\|_E + \|\zeta_1u\|_{\mathsf{Y}_\beta E} + \sum_{i=1}^n(\|D_i(\eta_1u)\|_E + \|D_i(\zeta_1u)\|_{\mathsf{Y}_{\beta+1}E})\Big) \\ &\le C\Big(\|u\|_{\mathsf{X}_\beta E} + \sum_{i=1}^n(\|\eta_1D_iu\|_E + \|\zeta_1D_iu\|_{\mathsf{Y}_{\beta+1}E})\Big) \\ &\le C\Big(\|u\|_{\mathsf{X}_\beta E} + \sum_{i=1}^n\|D_iu\|_{\mathsf{X}_{\beta+1}E}\Big)\end{aligned}$$

for all $u \in \mathscr{D}'$. The result follows. ■

2.3.4. *Separable subspaces.* Suppose E is a model space and let E_0 denote the separable subspace obtained by taking the closure of C_0^∞ (see Definition 2.6 and Remark 2.7).

LEMMA 2.34. *For any $\beta \in \mathbb{R}$, $\mathsf{Z}_\beta E_0$ is the closure of $C_0^\infty(\Pi^n)$ in $\mathsf{Z}_\beta E$, $\mathsf{Y}_\beta E_0$ is the closure of $C_0^\infty(\mathbb{R}^n_*)$ in $\mathsf{Y}_\beta E$ and $\mathsf{X}_\beta E_0$ is the closure of C_0^∞ in $\mathsf{X}_\beta E$.*

Proof. For any $\beta \in \mathbb{R}$ it is easy to check that $C_0^\infty(\Pi^n) \subset \mathsf{Z}_\beta E$ whilst the isometric inclusion $E_0 \hookrightarrow E$ leads to an isometric inclusion $\mathsf{Z}_\beta E_0 \hookrightarrow \mathsf{Z}_\beta E$. Since $\mathsf{Z}_\beta E_0$ is complete it thus remains to show that $C_0^\infty(\Pi^n)$ is dense in $\mathsf{Z}_\beta E_0$. In turn, by the isomorphism $e^{\beta t} : \mathsf{Z}_\beta E_0 \to E_0(\Pi^n)$ (see (5)) and the definition of the norm on $E_0(\Pi^n)$, it is clear that the first part of the result is completed by the following.

Claim: *Suppose $(\Psi, \mathbb{R} \times U)$ and χ are as in Lemma 2.10. If $u \in E_0(\Pi^n)$ and $\varepsilon > 0$ then there exists $u_\varepsilon \in C_0^\infty(\Pi^n)$ with $\|\chi u - u_\varepsilon\|_{E_0(\Pi^n)} < \varepsilon$.* Indeed, choose $\chi_1 \in C_0^\infty(U)$ with $\chi_1 \succ \chi$ and set $\widetilde{\chi}_1 = \chi_1 \circ \Psi^{-1} \in C^\infty$. Now Lemma 2.10 shows that $(\Psi^{-1})^*(\chi u) \in E_0$ so, given any $\delta > 0$, we can find $v_\delta \in C_0^\infty$ with $\|(\Psi^{-1})^*(\chi u) - v_\delta\|_E < \delta$. On the other hand, $\widetilde{\chi}_1 \in C^\infty$ so condition (A2) implies

$$\|(\Psi^{-1})^*(\chi u) - \widetilde{\chi}_1 v_\delta\|_E = \|\widetilde{\chi}_1((\Psi^{-1})^*(\chi u) - v_\delta)\|_E < C\delta$$

where C is independent of δ. Setting $u_\delta = \Psi^*(\widetilde{\chi}_1 v_\delta) \in C_0^\infty(\Pi^n)$ we deduce from Lemma 2.10 that $\|\chi u - u_\delta\|_{E(\Pi^n)} < C\delta$ where C is again independent of δ. This completes the claim.

The second part of the result follows from (8) and the fact that $\Theta^*(C_0^\infty(\mathbb{R}^n_*)) = C_0^\infty(\Pi^n)$.

By arguments as above it is clear that $\mathsf{X}_\beta E_0$ is a closed subspace of $\mathsf{X}_\beta E$ which contains C_0^∞. Now let $u \in \mathsf{X}_\beta E_0$. By Lemma 2.16, $\eta_1 u \in E_0$ and $\zeta_1 u \in \mathsf{Y}_\beta E_0$ so, given $\delta > 0$, we can find $v \in C_0^\infty$ and $w \in C_0^\infty(\mathbb{R}^n_*)$ with $\|\eta_1 u - v\|_E, \|\zeta_1 u - w\|_{\mathsf{Y}_\beta E} < \delta$. Thus $\eta_2 v + \zeta_0 w \in C_0^\infty$. On the other hand, condition (A2) gives

$$\|\eta_1 u - \eta_2 v\|_E = \|\eta_2(\eta_1 u - v)\|_E < C\delta, \tag{17}$$

whilst from Remark 2.17 we have

$$\|\zeta_1 u - \zeta_0 w\|_{\mathsf{Y}_\beta E} = \|\zeta_0(\zeta_1 u - w)\|_{\mathsf{Y}_\beta E} < C\delta. \tag{18}$$

Combining (17), (18) and Lemma 2.16 we get

$$\begin{aligned}\|u - (\eta_2 v + \zeta_0 w)\|_{\mathsf{X}_\beta E} &\le \|\eta_1 u - \eta_2 v\|_{\mathsf{X}_\beta E} + \|\zeta_1 u - \zeta_0 w\|_{\mathsf{X}_\beta E}\\ &\le C(\|\eta_1 u - \eta_2 v\|_E + \|\zeta_1 u - \zeta_0 w\|_{\mathsf{Y}_\beta E}) < C\delta.\end{aligned}$$

The fact that C is independent of δ completes the result. ■

2.3.5. *Equivalent norms for some model spaces.* In this section we consider some equivalent norms on the spaces $\mathsf{X}_\beta E$ when E is either a Sobolev space of positive integral order or a Hölder space. In particular, this will allow us to identify the spaces $H^{p,k}_\beta$ given in Definition 1.2 in the Introduction with $\mathsf{X}_\beta E$ for appropriate β and E.

PROPOSITION 2.35. *Suppose $\beta \in \mathbb{R}$, $p \in [1,\infty)$ and $k \in \mathbb{N}_0$. Given $u \in \mathscr{D}'$ we have $u \in \mathsf{X}_\beta H^{p,k}$ iff $D^\alpha_x u$ is a measurable function for all multi-indices α with $|\alpha| \le k$ and*

$$\sum_{|\alpha|\le k} \int \Lambda^{p(\beta+|\alpha|)-n}(x)|D^\alpha_x u(x)|^p\, d^n x < +\infty. \tag{19}$$

Furthermore, the quantity on the left hand side of (19) is equivalent to $\|u\|^p_{\mathsf{X}_\beta H^{p,k}}$.

REMARK 2.36. Let $\beta \in \mathbb{R}$, $p \in [1,\infty)$ and $k \in \mathbb{N}_0$. Since the Sobolev space $H^{p,k}$ contains C_0^∞ as a dense subset, Lemma 2.34 and Proposition 2.35 immediately imply that the space $H^{p,k}_\beta$ given in Definition 1.2 in the Introduction is simply $\mathsf{X}_{\beta+n/p} H^{p,k}$ (up to equivalent norms).

Proof of Proposition 2.35. Propositions 2.20 and 2.33 reduce our task to proving the result under the assumptions that $\beta = 0$ and $k = 0$.

Let I, ψ_i, Ψ_i, U_i and χ_i be as given in the introduction to Section 2.1. Now the pull-back of the density $d^n x$ on $\mathbb{R} \times \psi_i(U_i) \subseteq \mathbb{R}^n$ is a density on $\mathbb{R} \times U_i \subset \Pi^n$ so we can write $(\Psi_i)^*(d^n x) = J_i\, dt\, dS^{n-1}$ for some positive function J_i defined on $\mathbb{R} \times U_i$. In particular,

$$\int (\Psi_i^{-1})^* v\, d^n x = \int v J_i\, dt\, dS^{n-1}$$

for any measurable function v which is supported on $\mathbb{R} \times U_i$. Now J_i is independent of t whilst $\operatorname{supp}(\chi_i) \subset U_i$ is compact. Therefore J_i is bounded and bounded away from 0 on $\mathbb{R} \times \operatorname{supp}(\chi_i)$. Hence we have

$$\|(\Psi_i^{-1})^*(\chi_i u)\|_{L^p}^p = \int |(\Psi_i^{-1})^*(\chi_i u)|^p \, d^n x = \int |\chi_i u|^p J_i \, dt \, dS^{n-1} \asymp \int |\chi_i u|^p \, dt \, dS^{n-1}$$

for any measurable function u on Π^n. From the definition of $\mathsf{Z}_0 L^p$ (see (4) and (5)) and the finiteness of I it follows that

$$\|u\|_{\mathsf{Z}_0 L^p}^p \asymp \int \Big(\sum_{i \in I} |\chi_i u| \Big)^p \, dt \, dS^{n-1} = \int |u|^p \, dt \, dS^{n-1},$$

where we have used the identity

$$\sum_{i \in I} |\chi_i u| = |u| \tag{20}$$

(recall that $\{\chi_i\}_{i \in I}$ is a partition of unity). From (7) we have

$$(\Theta^{-1})^* dt dS^{n-1} = r^{-1} dr dS^{n-1} = r^{-n} r^{n-1} dr dS^{n-1} = |x|^{-n} d^n x$$

so (9) now gives

$$\|u\|_{\mathsf{Y}_0 L^p}^p \asymp \int |\Theta^* u|^p dt \, dS^{n-1} = \int |u|^p \, (\Theta^{-1})^* (dt \, dS^{n-1}) = \int |x|^{-n} |u|^p d^n x$$

for all measurable functions u on $\mathbb{R}^n_*$. Finally, Lemma 2.16 gives

$$\begin{aligned} \|u\|_{\mathsf{X}_0 L^p}^p &\asymp \|\eta_0 u\|_{L^p}^p + \|\zeta_0 u\|_{\mathsf{Y}_0 L^p}^p \\ &\asymp \int (|\eta_0(x)|^p + |x|^{-n} |\zeta_0(x)|^p) |u(x)|^p \, d^n x \asymp \int \Lambda^{-n}(x) |u(x)|^p \, d^n x \end{aligned}$$

for all measurable functions u on $\mathbb{R}^n$, where the last line follows from the existence of constants $C_1, C_2 > 0$ such that

$$C_1 \Lambda^{-n}(x) \le |\eta_0(x)|^p + |x|^{-n} |\zeta_0(x)|^p \le C_2 \Lambda^{-n}(x)$$

for all $x \in \mathbb{R}^n$. ■

A simpler version of the previous argument can be used to find an equivalent norm for the spaces $\mathsf{X}_\beta C^l$ when $\beta \in \mathbb{R}$ and $l \in \mathbb{N}_0$.

PROPOSITION 2.37. *Suppose $\beta \in \mathbb{R}$ and $l \in \mathbb{N}_0$. Given $u \in \mathscr{D}'$ we have $u \in \mathsf{X}_\beta C^l$ iff u is l times continuously differentiable and*

$$\sum_{|\alpha| \le l} \sup_{x \in \mathbb{R}^n} \Lambda^{\beta + |\alpha|}(x) |D_x^\alpha u(x)| < +\infty. \tag{21}$$

Furthermore, the quantity on the left hand side of (21) *is equivalent to* $\|u\|_{\mathsf{X}_\beta C^l}$.

Proof. Once again Propositions 2.20 and 2.33 reduce our task to proving the result in the case $\beta = 0$ and $l = 0$.

If u is a continuous function on Π^n then (4) and (5) give

$$\|u\|_{\mathsf{Z}_0 C^0} = \sum_{i \in I} \sup_{x \in \mathbb{R} \times \psi_i(U_i)} |(\chi_i u)(\Psi_i^{-1}(x))| \asymp \sup_{(t,\omega) \in \Pi^n} \sum_{i \in I} |(\chi_i u)(t, \omega)| = \sup_{(t,\omega) \in \Pi^n} |u(t, \omega)|$$

where we have used the finiteness of I and (20) in the second last and last steps respectively. From (9) we now get

$$\|u\|_{\mathsf{Y}_0 C^0} \asymp \sup_{(t,\omega) \in \Pi^n} |u(\Theta(t, \omega))| = \sup_{x \in \mathbb{R}^n_*} |u(x)|,$$

for all continuous functions u on $\mathbb{R}^n_*$, whilst Lemma 2.16 finally gives

$$\begin{aligned}\|u\|_{\mathsf{X}_0C^0} &\asymp \sup_{x\in\mathbb{R}^n}|\eta_0(x)u(x)| + \sup_{x\in\mathbb{R}^n}|\zeta_0(x)u(x)| \\ &\asymp \sup_{x\in\mathbb{R}^n}(|\eta_0(x)u(x)|+|\zeta_0(x)u(x)|) = \sup_{x\in\mathbb{R}^n}|u(x)|\end{aligned}$$

for all continuous functions u on $\mathbb{R}^n$. ∎

It is clear from the proof of Proposition 2.37 that we have

$$\|u\|_{\mathsf{Y}_\beta C^l} \asymp \sum_{|\alpha|\le l}\sup_{x\in\mathbb{R}^n_*}|x|^{\beta+|\alpha|}|D^\alpha_x u(x)| \tag{22}$$

for any $\beta\in\mathbb{R}$ and $l\in\mathbb{N}_0$. This observation will be useful below.

For any $l\in\mathbb{N}_0$ and $\sigma\in(0,1)$ the Hölder space $C^{l+\sigma}$ can be defined as the collection of all those functions $u\in C^l$ for which

$$\sum_{|\alpha|\le l}\sup_{x\in\mathbb{R}^n}|D^\alpha_x u(x)| + \sum_{|\alpha|=l}\sup_{\substack{x,y\in\mathbb{R}^n\\ x\ne y}}|D^\alpha_x u(x)-D^\alpha_x u(y)|\cdot|x-y|^{-\sigma} < +\infty.$$

We can use this sum to define the norm $\|\cdot\|_{C^{l+\sigma}}$ on $C^{l+\sigma}$.

Remark 2.38. Suppose $V\subseteq\mathbb{R}^n$ is open and $\{B_x\}_{x\in V}$ is a collection of open subsets of V for which there exists a constant $\kappa>0$ such that $V\cap\{y \mid |x-y|<\kappa\}\subseteq B_x$ for all $x\in V$. It is straightforward to check that

$$\|u\|_{C^{0+\sigma}} \asymp \sup_{x\in V}|u(x)| + \sup_{\substack{x\in V\\ y\in B_x\setminus\{x\}}}|u(x)-u(y)|\cdot|x-y|^{-\sigma}$$

for all continuous functions u with $\operatorname{supp}(u)\subseteq V$.

As was the case for the model spaces $H^{p,k}$ and C^l we can obtain an explicit description of the space $\mathsf{X}_\beta C^{l+\sigma}$.

Proposition 2.39. *Suppose* $\beta\in\mathbb{R}$, $l\in\mathbb{N}_0$ *and* $\sigma\in(0,1)$. *Given* $u\in\mathscr{D}'$ *we have* $u\in\mathsf{X}_\beta C^{l+\sigma}$ *iff* u *is* l *times continuously differentiable and*

$$\begin{aligned}&\sum_{|\alpha|\le l}\sup_{x\in\mathbb{R}^n}|(\Lambda^{\beta+|\alpha|}D^\alpha_x u)(x)| \\ &\quad+\sum_{|\alpha|=l}\sup_{\substack{x,y\in\mathbb{R}^n\\ x\ne y}}|(\Lambda^{\beta+l+\sigma}D^\alpha_x u)(x)-(\Lambda^{\beta+l+\sigma}D^\alpha_x u)(y)|\cdot|x-y|^{-\sigma}<+\infty.\end{aligned}\tag{23}$$

Furthermore, the quantity on the left hand side of (23) *is equivalent to* $\|\cdot\|_{\mathsf{X}_\beta C^{l+\sigma}}$.

The proof of Proposition 2.39 will be preceded by some technical results for which we introduce the following notation: if u is a continuous function on $\mathbb{R}^n$ let $\|u\|_\sigma$ denote the left hand side of (23) when $\beta=0$ and $l=0$.

Lemma 2.40. *For continuous functions* u *on* $\mathbb{R}^n$,

$$\|u\|_\sigma \asymp \sup_{x\in\mathbb{R}^n}|u(x)| + \sup_{\substack{x,y\in\mathbb{R}^n\\ 0<|x-y|<\Lambda(x)}}\Lambda^\sigma(x)|u(x)-u(y)|\cdot|x-y|^{-\sigma}.$$

Proof. Since $\sigma \in (0,1)$ we have $1 - t^\sigma \le 1 - t \le (1-t)^\sigma$ for all $t \in [0,1]$. It follows that

$$|\Lambda^\sigma(x) - \Lambda^\sigma(y)| \le |\Lambda(x) - \Lambda(y)|^\sigma \le |x-y|^\sigma \tag{24}$$

for all $x, y \in \mathbb{R}^n$. If u is any continuous function on $\mathbb{R}^n$ we thus have

$$\big||(\Lambda^\sigma u)(x) - (\Lambda^\sigma u)(y)| - \Lambda^\sigma(x)|u(x) - u(y)|\big| \le |\Lambda^\sigma(x) - \Lambda^\sigma(y)| \cdot |u(y)| \le |x-y|^\sigma |u(y)|$$

for all $x, y \in \mathbb{R}^n$. Hence

$$\|u\|_\sigma \asymp \sup_{x \in \mathbb{R}^n} |u(x)| + \sup_{\substack{x,y \in \mathbb{R}^n \\ x \ne y}} \Lambda^\sigma(x)|u(x) - u(y)| \cdot |x-y|^{-\sigma}.$$

On the other hand, $|x-y|^{-\sigma} \le \Lambda^{-\sigma}(x)$ whenever $|x-y| \ge \Lambda(x)$. Therefore

$$\sup_{\substack{x,y \in \mathbb{R}^n \\ |x-y| \ge \Lambda(x)}} \Lambda^\sigma(x)|u(x) - u(y)| \cdot |x-y|^{-\sigma} \le 2 \sup_{x \in \mathbb{R}^n} |u(x)|.$$

The result now follows. ■

Lemma 2.41. *We have $\|uv\|_\sigma \le C\|u\|_{\mathsf{X}_0C^1}\|v\|_\sigma$ for all $u \in \mathsf{X}_0C^1$ and continuous functions v on $\mathbb{R}^n$.*

Proof. For any $u \in \mathsf{X}_0C^1$ Proposition 2.37 gives us

$$|u(x)| \le C\|u\|_{\mathsf{X}_0C^1} \tag{25}$$

and

$$1|D_i u(x)| \le C\|u\|_{\mathsf{X}_0C^1}\Lambda^{-1}(x), \qquad i = 1, \ldots, n, \tag{26}$$

for all $x \in \mathbb{R}^n$. Now suppose $x, y \in \mathbb{R}^n$ with $0 < |x-y| \le \Lambda(y)$. Thus x and y both belong to the ball of radius $2\Lambda(y)$ centred at the origin. Combining this observation with (26) and the fact that $\Lambda^{\sigma-1}(y)|x-y|^{1-\sigma} \le 1$ we then get

$$\Lambda^\sigma(y)|u(x) - u(y)| \cdot |x-y|^{-\sigma} \le C\Lambda^\sigma(y)|x-y|^{1-\sigma}\|u\|_{\mathsf{X}_0C^1}\Lambda^{-1}(y) \le C\|u\|_{\mathsf{X}_0C^1}.$$

On the other hand, if $x, y \in \mathbb{R}^n$ with $|x-y| \ge \Lambda(y)$ then $\Lambda^\sigma(y)|x-y|^{-\sigma} \le 1$ so

$$\Lambda^\sigma(y)|u(x) - u(y)| \cdot |x-y|^{-\sigma} \le C\|u\|_{\mathsf{X}_0C^1}$$

by (25). Combining the above estimates we thus get

$$\Lambda^\sigma(y)|u(x) - u(y)| \cdot |x-y|^{-\sigma} \le C\|u\|_{\mathsf{X}_0C^1} \tag{27}$$

for all $x, y \in \mathbb{R}^n$ with $x \ne y$.

Suppose v is a continuous function and $x, y \in \mathbb{R}^n$ with $x \ne y$. Using (25) and (27) we thus have

$$\begin{aligned}
&|(\Lambda^\sigma uv)(x) - (\Lambda^\sigma uv)(y)| \cdot |x-y|^{-\sigma} \\
&\quad \le |u(x)| \cdot |(\Lambda^\sigma v)(x) - (\Lambda^\sigma v)(y)| \cdot |x-y|^{-\sigma} + |v(y)|\Lambda^\sigma(y)|u(x) - u(y)| \cdot |x-y|^{-\sigma} \\
&\quad \le C\|u\|_{\mathsf{X}_0C^1}(|(\Lambda^\sigma v)(x) - (\Lambda^\sigma v)(y)| \cdot |x-y|^{-\sigma} + |v(y)|).
\end{aligned}$$

The result now follows from the definition of $\|\cdot\|_\sigma$. ■

Proof of Proposition 2.39. Propositions 2.20 and 2.33 reduce our task to proving the result in the case $\beta = 0$ and $l = 0$.

Consider the notation introduced in the first paragraph of Section 2.1 and let Θ be as given in Section 2.2. Now, for each $i \in I$, define open sets $V_i, W_i \subset \mathbb{R}^n$ by $V_i =$

$(0,\infty)\times\psi_i(U_i)$ and $W_i=\Theta(\Psi_i^{-1}(V_i))$. Thus the map $\Phi_i:=\Theta\circ\Psi_i^{-1}:V_i\to W_i$ is a diffeomorphism. Also define a function ϱ_i by $\varrho_i(x)=\zeta_1(x)\chi_i(\Phi_i^{-1}(x))$ for $x\in W_i$ and $\varrho_i(x)=0$ for $x\notin W_i$. It is easy to see that ϱ_i is smooth and independent of $|x|$ for sufficiently large $|x|$; it follows from Proposition 2.37 that $\varrho_i\in\mathsf{X}_0C^1$. Finally, set $V=\{|x|<3\}\subset\mathbb{R}^n$, so $\eta_1\in C_0^\infty(V)\subset\mathsf{X}_0C^1$.

Since $\{\chi_i\}_{i\in I}$ is a partition of unity on $\varPi^n$ we have $\eta_1+\sum_{i\in I}\varrho_i=\eta_1+\zeta_1=1$. Together with Lemma 2.41 and the fact that I is finite this gives

$$\|u\|_\sigma\asymp\|\eta_1u\|_\sigma+\sum_{i\in I}\|\varrho_iu\|_\sigma$$

for all continuous functions u on $\mathbb{R}^n$. On the other hand, the definition of $\mathsf{X}_0C^{0+\sigma}$ means we have

$$\|u\|_{\mathsf{X}_0C^{0+\sigma}}\asymp\|\eta_1u\|_{C^{0+\sigma}}+\sum_{i\in I}\|\Phi_i^*(\varrho_iu)\|_{C^{0+\sigma}}$$

for all $u\in\mathsf{X}_0C^{0+\sigma}$. The following claims thus complete the result.

Claim (i): *We have* $\|v\|_{C^{0+\sigma}}\asymp\|v\|_\sigma$ *for all continuous functions v with* $\operatorname{supp}(v)\subseteq V$. This is a straightforward consequence of Remark 2.38, Lemma 2.40 and the fact that $\varLambda(x)\asymp1$ for $x\in V$ (n.b. V is bounded).

Claim (ii): *If $i\in I$ then* $\|\Phi_i^*v\|_{C^{0+\sigma}}\asymp\|v\|_\sigma$ *for all continuous functions v with* $\operatorname{supp}(v)\subseteq V_i$. If we write $y\in\mathbb{R}^n$ in the form $y=(t,w)$ with $t\in\mathbb{R}$ and $w\in\mathbb{R}^{n-1}$ then $\Phi_i(y)=e^t\psi_i^{-1}(w)$ (where we are considering S^{n-1} to be the unit sphere in $\mathbb{R}^n$). Thus, for all $y_1,y_2\in V_i$ with $|y_1-y_2|<1$,

$$\begin{aligned}|\Phi_i(y_1)-\Phi_i(y_2)|&\asymp e^{t_1}(|1-e^{t_2-t_1}|\cdot|\psi_i^{-1}(w_2)|+|\psi_i^{-1}(w_1)-\psi_i^{-1}(w_2)|)\\&\asymp e^{t_1}(|t_1-t_2|+|w_1-w_2|)\\&\asymp\varLambda(\Phi_i(y_1))|y_1-y_2|,\end{aligned}\tag{28}$$

where the inequalities $|t_1-t_2|\le|y_1-y_2|<1$ and $|\Phi_i(y_1)|=e^{t_1}\ge1$ have been used in the second last and last lines respectively. For each $y_1\in V_i$ set

$$B_{y_1}=\{y_2\in V_i\mid|\Phi_i(y_1)-\Phi_i(y_2)|<\varLambda(\Phi_i(y_1))\}$$

Estimate (28) implies there exists $\kappa>0$ such that $V_i\cap\{y_2\mid|y_1-y_2|<\kappa\}\subseteq B_{y_1}$ for all $y_1\in V_i$. Further use of (28) together with Remark 2.38 and Lemma 2.40 then gives

$$\begin{aligned}\|\Phi_i^*v\|_{C^{0+\sigma}}&\asymp\sup_{y\in V_i}|v(\Phi_i(y))|+\sup_{\substack{y_1\in V_i\\y_2\in B_{y_1}\setminus\{y_1\}}}|v(\Phi_i(y_1))-v(\Phi_i(y_2))|\cdot|y_1-y_2|^{-\sigma}\\&\asymp\sup_{x\in W_i}|v(x)|+\sup_{\substack{x_1,x_2\in W_i\\0<|x_1-x_2|<\varLambda(x_1)}}\varLambda^\sigma(x_1)|v(x_1)-v(x_2)|\cdot|x_1-x_2|^{-\sigma}\asymp\|v\|_\sigma\end{aligned}$$

for all continuous functions v with $\operatorname{supp}(v)\subset V_i$. ■

REMARK 2.42. Let $\beta\in\mathbb{R}$, $l\in\mathbb{N}_0$ and $\sigma\in(0,1)$. By using Proposition 2.39 it can be seen that $\mathsf{X}_\beta C^{l+\sigma}$ coincides with the space $C_\beta^{\sigma+l}(\mathbb{R}^n)$ defined in [Ben]. Also, by Proposition 2.20, Lemma 2.34 and the obvious modification of Proposition 2.39 for the spaces $\mathsf{Y}_\beta C^{l+\sigma}$, it can be seen that $\mathsf{Y}_\beta C_0^{l+\sigma}$ coincides with the space $\varLambda_{\beta+l+\sigma}^{l,\sigma}(\mathbb{R}_*^n)$ defined in Section 3.6.4 of [NP] (here $C_0^{l+\sigma}$ denotes the separable subspace of $C^{l+\sigma}$ obtained by taking the completion of C_0^∞). The non-separable space $\mathsf{Y}_\beta C^{l+\sigma}$ contains elements which behave as $O(|x|^{-\beta})$ for $|x| \to 0, \infty$ and is strictly larger than $\mathsf{Y}_\beta C_0^{l+\sigma}$ (see Section 2.3.9 for a related discussion).

2.3.6. *Dual spaces.* Suppose E is a model space. By definition $\mathscr{S}$ is dense in E_0 so any element in the dual of E_0 is uniquely determined by its action on $\mathscr{S}$. We can thus uniquely identify elements of the dual of E_0 with tempered distributions on $\mathbb{R}^n$, i.e. elements of $\mathscr{S}'$. Furthermore, a norm can be defined on E_0^* by the expression

$$\|v\|_{E_0^*} = \sup_{0 \neq u \in \mathscr{S}} \frac{|(u,v)_{\mathbb{R}^n}|}{\|u\|_E} = \sup_{0 \neq u \in C_0^\infty} \frac{|(u,v)_{\mathbb{R}^n}|}{\|u\|_E}, \tag{29}$$

where the second equality follows from the density of C_0^∞ in E_0 (see Remark 2.7). Applying standard duality arguments to conditions (A1) to (A4) we obtain the following.

Lemma 2.43. *If E is a model space then so is E_0^*.*

If E and F are model spaces we shall write $E_0^* = F$ provided these spaces agree as subsets of $\mathscr{S}'$ and $\|\cdot\|_F$ is equivalent to the norm given by (29). In other words, $E_0^* = F$ iff there are constants $C_1, C_2 > 0$ such that the following hold:

(D1) For each $u \in C_0^\infty$ and $v \in F$ we have $|(u,v)_{\mathbb{R}^n}| \le C_1 \|u\|_E \|v\|_F$.
(D2) For each $v \in F$ there exists $0 \neq u \in C_0^\infty$ with $|(u,v)_{\mathbb{R}^n}| \ge C_2 \|u\|_E \|v\|_F$.

Suppose E is a model space and $\beta \in \mathbb{R}$. Using Lemma 2.34 and an argument similar to that above, we can identify the dual spaces $(\mathsf{Z}_\beta E_0)^*$, $(\mathsf{Y}_\beta E_0)^*$ and $(\mathsf{X}_\beta E_0)^*$ with subspaces of $\mathscr{D}'(\Pi^n)$, $\mathscr{D}'(\mathbb{R}^n_*)$ and $\mathscr{D}'$ respectively. The pairings $(\cdot,\cdot)_{\Pi^n}$, $(\cdot,\cdot)_{\mathbb{R}^n_*}$ and $(\cdot,\cdot)_{\mathbb{R}^n}$ then allow us to define norms on these dual spaces (as in the second part of (29)) and compare them with existing spaces.

Remark 2.44. For any $\beta \in \mathbb{R}$ the pairing $(\cdot,\cdot)_{\Pi^n}$ on Π^n extends to a dual pairing $\mathsf{Z}_\beta \mathscr{S}' \times \mathsf{Z}_{-\beta} \mathscr{S} \to \mathbb{C}$.

Lemma 2.45. *If E is a model space and $\beta \in \mathbb{R}$ then $(\mathsf{Z}_\beta E_0)^* = \mathsf{Z}_{-\beta}(E_0^*)$.*

Here "equality" is understood in the sense of equivalent norms; that is, we have expressions similar to (D1) and (D2) above.

Proof. Let $F = E_0^*$ and consider the notation of Remark 2.9. Now, for $i \in I$, the pullback under Ψ_i^{-1} of the density $\chi_i' dt dS^{n-1}$ is a smooth density on $\mathbb{R}^n$. Thus we can write $(\Psi_i^{-1})^*(\chi_i'\, dt dS^{n-1}) = J_i\, d^n x$ where $J_i \in C^\infty$ (in fact, J_i is independent of t and compactly supported in the remaining variables). It follows that

$$(u, \chi_i v)_{\Pi^n} = (\chi_i' \chi_i' e^{\beta t} u, \chi_i e^{-\beta t} v)_{\mathbb{R}^n} = (J_i (\Psi_i^{-1})^*(\chi_i' e^{\beta t} u), (\Psi_i^{-1})^*(\chi_i e^{-\beta t} v))_{\mathbb{R}^n}$$

for all $u \in C_0^\infty(\Pi^n)$ and $v \in \mathscr{D}'(\Pi^n)$. Using (D1), (A2) (to deal with J_i), Remark 2.9 and (5), we therefore have

$$\begin{aligned} |(u,v)_{\Pi^n}| &\le \sum_{i \in I} |(u, \chi_i v)_{\Pi^n}| \le C \sum_{i \in I} \|J_i (\Psi_i^{-1})^*(\chi_i' e^{\beta t} u)\|_E \|(\Psi_i^{-1})^*(\chi_i e^{-\beta t} v)\|_F \\ &\le C \sum_{i \in I} \|(\Psi_i^{-1})^*(\chi_i' e^{\beta t} u)\|_E \sum_{i \in I} \|(\Psi_i^{-1})^*(\chi_i e^{-\beta t} v)\|_F \le C \|u\|_{\mathsf{Z}_\beta E} \|v\|_{\mathsf{Z}_{-\beta} F} \end{aligned}$$

for all $u \in C_0^\infty(\Pi^n)$ and $v \in \mathsf{Z}_{-\beta} F$.

Now let $v \in \mathsf{Z}_{-\beta}F$. By definition

$$\|v\|_{\mathsf{Z}_{-\beta}F} = \sum_{i\in I} \|(\varPsi_i^{-1})^*(\chi_i e^{-\beta t} v)\|_F.$$

Hence we can find $i \in I$ such that

$$\|v\|_{\mathsf{Z}_{-\beta}F} \leq C_1 \|(\varPsi_i^{-1})^*(\chi_i e^{-\beta t} v)\|_F;$$

n.b. i may depend on v but C_1 does not (we can define C_1 to be the number of elements in I). Using (D2) we can now choose $0 \neq \phi \in C_0^\infty$ such that

$$\begin{aligned} |(\phi, (\varPsi_i^{-1})^*(\chi_i e^{-\beta t} v))_{\mathbb{R}^n}| &\geq C_2 \|\phi\|_E \|(\varPsi_i^{-1})^*(\chi_i e^{-\beta t} v)\|_F \\ &\geq C_1^{-1} C_2 \|\phi\|_E \|v\|_{\mathsf{Z}_{-\beta}F}. \end{aligned} \tag{30}$$

Define $J_i' \in C^\infty(\varPi^n)$ by $\chi_i' \varPsi_i^*(d^n x) = J_i' dt dS^{n-1}$ and set $u = \chi_i J_i' e^{-\beta t} \varPsi_i^* \phi \in C_0^\infty(\varPi^n)$. By (5), Lemma 2.10 and condition (A2) (n.b. $(\varPsi_i^{-1})^*(\chi_i J_i') \in C^\infty$), it follows that

$$\|u\|_{\mathsf{Z}_\beta E} = \|\chi_i J_i' \varPsi_i^* \phi\|_{E(\varPi^n)} \leq C \|(\varPsi_i^{-1})^*(\chi_i J_i')\, \phi\|_E \leq C_3 \|\phi\|_E. \tag{31}$$

On the other hand, the definition of J_i' gives us

$$\begin{aligned} (u, v)_{\varPi^n} &= (J_i' \chi_i e^{-\beta t} \varPsi_i^* \phi,\, v)_{\varPi^n} \\ &= (J_i' \varPsi_i^* \phi,\, \chi_i e^{-\beta t} v)_{\varPi^n} = (\phi,\, (\varPsi_i^{-1})^*(\chi_i e^{-\beta t} v))_{\mathbb{R}^n}. \end{aligned} \tag{32}$$

Combining (30), (31) and (32), we then get

$$|(u,v)_{\varPi^n}| \geq C_1^{-1} C_2 \|\phi\|_E \|v\|_{\mathsf{Z}_{-\beta}F} \geq C_1^{-1} C_2 C_3^{-1} \|u\|_{\mathsf{Z}_\beta E} \|v\|_{\mathsf{Z}_{-\beta}F}. \;\blacksquare$$

Under the pull-back induced by the diffeomorphism $\varTheta : \varPi^n \to \mathbb{R}^n_*$ (from Section 2.2) we have $\varTheta^* d^n x = e^{nt} dt dS^{n-1}$. It follows that $(u, v)_{\mathbb{R}^n_*} = (\varTheta^* u, e^{nt} \varTheta^* v)_{\varPi^n}$ for any $u \in C_0^\infty(\mathbb{R}^n_*)$ and $v \in \mathscr{D}'(\mathbb{R}^n_*)$. Now suppose E is a model space and $\beta \in \mathbb{R}$. With the help of (5) and (8), Lemma 2.45 then gives

$$\begin{aligned} \|v\|_{\mathsf{Y}_{n-\beta}(E_0^*)} &= \|e^{nt} \varTheta^* v\|_{\mathsf{Z}_{-\beta}(E_0^*)} \asymp \sup_{0 \neq w \in C_0^\infty(\varPi^n)} \frac{|(w, e^{nt} \varTheta^* v)_{\varPi^n}|}{\|w\|_{\mathsf{Z}_\beta E}} \\ &= \sup_{0 \neq u \in C_0^\infty(\mathbb{R}^n_*)} \frac{|(\varTheta^* u, e^{nt} \varTheta^* v)_{\varPi^n}|}{\|\varTheta^* u\|_{\mathsf{Z}_\beta E}} = \sup_{0 \neq u \in C_0^\infty(\mathbb{R}^n_*)} \frac{|(u, v)_{\mathbb{R}^n_*}|}{\|u\|_{\mathsf{Y}_\beta E}} = \|v\|_{(\mathsf{Y}_\beta E_0)^*} \end{aligned}$$

for all $v \in \mathsf{Y}_{n-\beta}(E_0^*)$; that is, we have

$$(\mathsf{Y}_\beta E_0)^* = \mathsf{Y}_{n-\beta}(E_0^*). \tag{33}$$

Proposition 2.46. *If E is a model space and $\beta \in \mathbb{R}$ then $(\mathsf{X}_\beta E_0)^* = \mathsf{X}_{n-\beta}(E_0^*)$.*

Proof. Let $F = E_0^*$ and choose $0 \neq v \in \mathsf{X}_{n-\beta}F$. Now $\eta_2 \eta_1 = \eta_1$, $\zeta_0 \zeta_1 = \zeta_1$ and $\eta_2 + \zeta_0 \geq \eta_1 + \zeta_1 = 1$ so, for any $u \in C_0^\infty$,

$$(u, v)_{\mathbb{R}^n} = (\eta_2 u, \eta_1 v)_{\mathbb{R}^n} + (\zeta_0 u, \zeta_1 v)_{\mathbb{R}^n_*}$$

while

$$\|u\|_{\mathsf{X}_\beta E} \asymp \|\eta_2 u\|_E + \|\zeta_0 u\|_{\mathsf{Y}_\beta E} \quad \text{and} \quad \|v\|_{\mathsf{X}_{n-\beta}F} \asymp \|\eta_1 v\|_F + \|\zeta_1 v\|_{\mathsf{Y}_{n-\beta}F}, \tag{34}$$

by Lemma 2.16. Together with (33) this implies

$$\begin{aligned} |(u,v)_{\mathbb{R}^n}| &\leq |(\eta_2 u, \eta_1 v)_{\mathbb{R}^n}| + |(\zeta_0 u, \zeta_1 v)_{\mathbb{R}^n_*}| \leq C(\|\eta_2 u\|_E \|\eta_1 u\|_F + \|\zeta_0 u\|_{\mathsf{Y}_\beta E} \|\zeta_1 v\|_{\mathsf{Y}_{n-\beta}F}) \\ &\leq C(\|\eta_2 u\|_E + \|\zeta_0 u\|_{\mathsf{Y}_\beta E})(\|\eta_1 v\|_F + \|\zeta_1 v\|_{\mathsf{Y}_{n-\beta}F}) \leq C \|u\|_{\mathsf{X}_\beta E} \|v\|_{\mathsf{X}_{n-\beta}F} \end{aligned}$$

for any $u \in C_0^\infty$.

By the second part of (34) we have

$$\|v\|_{\mathsf{X}_{n-\beta}F} \le C_1(\|\eta_1 v\|_F + \|\zeta_1 v\|_{\mathsf{Y}_{n-\beta}F}).$$

Consider the following cases.

Case (i): $\|v\|_{\mathsf{X}_{n-\beta}F} \le 2C_1\|\eta_1 v\|_F$. Using (D2) choose $w \in C_0^\infty$ with $\|w\|_E = 1$ and $\|v\|_{\mathsf{X}_{n-\beta}F} \le 4C_1|(w,\eta_1 v)_{\mathbb{R}^n}|$. Set $u = \eta_1 w$, so $u \in C_0^\infty$, $\|v\|_{\mathsf{X}_{n-\beta}F} \le 4C_1|(u,v)_{\mathbb{R}^n}|$ and

$$\|u\|_{\mathsf{X}_\beta E} \le C(\|\eta_2 u\|_E + \|\zeta_2 u\|_{\mathsf{Y}_\beta E}) = C\|\eta_1 w\|_E \le C_2$$

by Lemma 2.16, where C_2 is independent of u and v.

Case (ii): $\|v\|_{\mathsf{X}_{n-\beta}F} \le 2C_1\|\zeta_1 v\|_{\mathsf{Y}_{n-\beta}F}$. Using (33) choose $w \in C_0^\infty(\mathbb{R}^n_*)$ with $\|w\|_{\mathsf{Y}_\beta E} = 1$ and $\|v\|_{\mathsf{X}_{n-\beta}F} \le 4C_1|(w,\zeta_1 v)_{\mathbb{R}^n}|$. Set $u = \zeta_1 w$, so $u \in C_0^\infty$, $\|v\|_{\mathsf{X}_{n-\beta}F} \le 4C_1|(u,v)_{\mathbb{R}^n}|$ and

$$\|u\|_{\mathsf{X}_\beta E} \le C(\|\eta_0 u\|_E + \|\zeta_0 u\|_{\mathsf{Y}_\beta E}) = C\|\zeta_1 w\|_{\mathsf{Y}_\beta E} \le C_2$$

by Lemma 2.16, where C_2 is independent of u and v.

By combining the two cases it follows that we can find $0 \ne u \in C_0^\infty$ with

$$\|u\|_{\mathsf{X}_\beta E}\,\|v\|_{\mathsf{X}_{n-\beta}F} \le 4C_1C_2|(u,v)_{\mathbb{R}^n}|. \ \blacksquare$$

2.3.7. *Multiplication*

PROPOSITION 2.47. *Suppose multiplication defines a continuous bilinear map $E\times F \to G$ for some model spaces E, F and G. Then multiplication also defines continuous bilinear maps $\mathsf{Z}_\beta E \times \mathsf{Z}_\gamma F \to \mathsf{Z}_{\beta+\gamma}G$, $\mathsf{Y}_\beta E \times \mathsf{Y}_\gamma F \to \mathsf{Y}_{\beta+\gamma}G$ and $\mathsf{X}_\beta E \times \mathsf{X}_\gamma F \to \mathsf{X}_{\beta+\gamma}G$ for any $\beta,\gamma \in \mathbb{R}$.*

Proof. We can prove that multiplication defines a continuous bilinear map $E(\Pi^n) \times F(\Pi^n) \to G(\Pi^n)$ by an argument identical to that given for Lemma 2.12. The first two parts of the result now follow from (5) and (8) respectively.

Now $\eta_2\eta_1 = \eta_1$ and $\zeta_1\zeta_0 = \zeta_1$ so $\zeta_1 u\zeta_0 v = \zeta_1 uv$ and $\eta_2 u\eta_1 v = \zeta_1 uv$. However, $\eta_2 + \zeta_1, \eta_1 + \zeta_0 \ge 1$, so Lemma 2.16 and the continuity of multiplication as a map $E \times F \to G$ and as a map $\mathsf{Y}_\beta E \times \mathsf{Y}_\gamma F \to \mathsf{Y}_{\beta+\gamma}G$ gives

$$\begin{aligned}\|uv\|_{\mathsf{X}_{\beta+\gamma}G} &\le C(\|\eta_1 uv\|_G + \|\zeta_1 uv\|_{\mathsf{Y}_{\beta+\gamma}G}) \le C(\|\eta_2 u\|_E\|\eta_1 v\|_F + \|\zeta_1 u\|_{\mathsf{Y}_\beta E}\|\zeta_0 v\|_{\mathsf{Y}_\gamma F})\\ &\le C(\|\eta_2 u\|_E + \|\zeta_1 u\|_{\mathsf{Y}_\beta E})(\|\eta_1 v\|_F + \|\zeta_0 v\|_{\mathsf{Y}_\gamma F}) \le C\|u\|_{\mathsf{X}_\beta E}\|v\|_{\mathsf{X}_\gamma F}\end{aligned}$$

for any $u \in \mathsf{X}_\beta E$ and $v \in \mathsf{X}_\gamma F$. $\blacksquare$

REMARK 2.48. Suppose multiplication defines a continuous bilinear map $E \times F \to G$ for some model spaces E, F and G. A straightforward consequence of Definition 2.4 and the Leibniz rule is that multiplication also defines a continuous bilinear map $E^l \times F^l \to G^l$ for any $l \in \mathbb{N}_0$.

Let E' denote the closure of $E \cap C^\infty_{\mathrm{loc}}$ in E with the induced norm (n.b. we have $E_0 \subseteq E' \subseteq E$ although both inclusions could be strict in general). If $u \in E \cap C^\infty_{\mathrm{loc}}$ and $v \in G_0^*$ then $(uv,f)_{\mathbb{R}^n} = (v,fu)_{\mathbb{R}^n}$ and $\|uf\|_G \le C\|u\|_E\|f\|_F$ for all $f \in C_0^\infty$. Using (29) we thus get

$$\|uv\|_{F_0^*} = \sup_{0\ne f\in C_0^\infty}\frac{|(uv,f)_{\mathbb{R}^n}|}{\|f\|_F} \le C\|u\|_E \sup_{\substack{f\in C_0^\infty\\ uf\ne 0}}\frac{|(v,uf)_{\mathbb{R}^n}|}{\|uf\|_G} \le C\|u\|_E\|v\|_{G_0^*}.$$

By taking completions we finally see that multiplication defines a continuous bilinear map $E' \times G_0^* \to F_0^*$.

For any model space E, condition (A2) ensures that multiplication defines a continuous bilinear map $C^\infty \times E \to E$. Since the topology on C^∞ is defined by the collection of semi-norms $\{\|\cdot\|_{C^l} \mid l \in \mathbb{N}_0\}$, we can find $l \in \mathbb{N}_0$ and a constant C such that $\|\phi u\|_E \leq C\|\phi\|_{C^l}\|u\|_E \leq C\|\phi\|_{C^{l+1}}\|u\|_E$ for all $u \in E$ and $\phi \in C^\infty$. The fact that the closure of C^∞ in C^l includes C^{l+1} now completes the following result.

LEMMA 2.49. *For all sufficiently large $l \in \mathbb{N}_0$, multiplication defines a continuous bilinear map $C^l \times E \to E$.*

We conclude this section with an immediate consequence of Proposition 2.47 and Lemma 2.49 which will be used later in dealing with symbols.

PROPOSITION 2.50. *Suppose E is a model space and $\beta, \gamma \in \mathbb{R}$. Then, for all sufficiently large $l \in \mathbb{N}_0$, multiplication defines continuous bilinear maps $\mathsf{Z}_\gamma C^l \times \mathsf{Z}_\beta E \to \mathsf{Z}_{\beta+\gamma} E$, $\mathsf{Y}_\gamma C^l \times \mathsf{Y}_\beta E \to \mathsf{Y}_{\beta+\gamma} E$ and $\mathsf{X}_\gamma C^l \times \mathsf{X}_\beta E \to \mathsf{X}_{\beta+\gamma} E$.*

2.3.8. *Compactness.* Let E and F be model spaces. We say *E is locally compact in F* if multiplication by any $\phi \in C_0^\infty$ defines a compact map $E \to F$; in particular, it follows that we have a local inclusion $E_{\mathrm{loc}} \hookrightarrow F_{\mathrm{loc}}$.

PROPOSITION 2.51. *Suppose E, F and G are model spaces with E locally compact in G and for which multiplication defines a continuous bilinear map $E \times F \to G$. Let $\beta, \gamma \in \mathbb{R}$ and $v \in \mathsf{X}_\gamma F_0$. Then multiplication by v defines a compact map $\mathsf{X}_\beta E \to \mathsf{X}_{\beta+\gamma} G$.*

Proof. Initially suppose $v \in C_0^\infty$. Let $\{u_i\}_{i\in\mathbb{N}} \subset \mathsf{X}_\beta E$ be a bounded sequence. By Lemma 2.16 it follows that $\{vu_i\}_{i\in\mathbb{N}}$ is a bounded sequence in E. By local compactness we can thus find a subsequence $\{vu_{i(j)}\}_{j\in\mathbb{N}}$ which is convergent in G. Lemma 2.16 then implies this subsequence must also be convergent in $\mathsf{X}_{\beta+\gamma} G$. It follows that multiplication by v defines a compact map $\mathsf{X}_\beta E \to \mathsf{X}_{\beta+\gamma} G$.

Now let v be an arbitrary element of $\mathsf{X}_\gamma F_0$. Let $\varepsilon > 0$ and, using Lemma 2.34, choose $v_\varepsilon \in C_0^\infty$ with $\|v - v_\varepsilon\|_{\mathsf{X}_\gamma F} < \varepsilon$. Proposition 2.47 then implies that multiplication by $v - v_\varepsilon$ defines a map in $\mathscr{L}(\mathsf{X}_\beta E, \mathsf{X}_{\beta+\gamma} G)$ with norm at most $C\varepsilon$, where C is independent of ε. The result now follows from the fact that the set of compact maps in $\mathscr{L}(\mathsf{X}_\beta E, \mathsf{X}_{\beta+\gamma} G)$ is closed (see Theorem III.4.7 in [Ka] for example). ■

2.3.9. *Some results relating to symbols.* Let $l \in \mathbb{N}_0$. Clearly, $\phi_0 \in C^\infty(\Pi^n) \subset C^l(\Pi^n)$, whilst ϕ_i is simply a translation of ϕ_0 for any $i \in \mathbb{Z}$. Condition (A3) for $C^l(\Pi^n) = \mathsf{Z}_0 C^l$ then implies $\|\phi_i\|_{\mathsf{Z}_0 C^l}$ is independent of $i \in \mathbb{Z}$. However, $\zeta_i = \phi_i \circ \Theta^{-1}$ (by definition) so (9) now implies $\|\zeta_i\|_{\mathsf{Y}_0 C^l}$ is also independent of $i \in \mathbb{Z}$. Since $1 \in \mathsf{X}_0 C^l$ (by Proposition 2.37), Lemma 2.19 now completes the following result.

LEMMA 2.52. *Let $l \in \mathbb{N}_0$. Then $\|\zeta_i\|_{\mathsf{Y}_0 C^l}$ and $\|\zeta_i\|_{\mathsf{X}_0 C^l}$ are bounded uniformly for $i \in \mathbb{N}$.*

Proposition 2.47 and the fact that multiplication defines a continuous bilinear map $C^l \times C^l \to C^l$ show that multiplication also defines a continuous bilinear map $\mathsf{X}_0 C^l \times \mathsf{X}_\beta C^l \to \mathsf{X}_\beta C^l$ for any $\beta \in \mathbb{R}$. For a given $u \in \mathsf{X}_\beta C^l$, Lemmas 2.19 and 2.52 now imply $\|\zeta_i u\|_{\mathsf{X}_\beta C^l}$ and $\|\zeta_i u\|_{\mathsf{Y}_\beta C^l}$ are bounded uniformly for $i \in \mathbb{N}$.

Lemma 2.53. *Let $l \in \mathbb{N}_0$, $\beta \in \mathbb{R}$ and $u \in \mathsf{X}_\beta C^l$. Then the following are equivalent:*

(i) $\|\zeta_i u\|_{\mathsf{X}_\beta C^l} \to 0$ *as* $i \to \infty$.

(ii) $\|\zeta_i u\|_{\mathsf{Y}_\beta C^l} \to 0$ *as* $i \to \infty$.

(iii) $D_x^\alpha u(x) = o(|x|^{-\beta-|\alpha|})$ *as* $x \to \infty$ *for each* $|\alpha| \le l$.

Proof. The equivalence of (i) and (ii) follows from Lemma 2.19.

(iii)⇒(ii). Let $i \in \mathbb{N}_0$. Using (22) and the Leibniz rule we have

$$\begin{aligned}\|\zeta_i u\|_{\mathsf{Y}_\beta C^l} &\le C \sum_{|\alpha|\le l} \sup_{x\in\mathbb{R}^n_*} |x|^{\beta+|\alpha|}|D_x^\alpha(\zeta_i u)(x)| \\ &\le C\Big(\sum_{|\alpha|\le l} \sup_{x\in\mathbb{R}^n_*} |x|^{|\alpha|}|D_x^\alpha \zeta_i(x)|\Big)\Big(\sum_{|\alpha|\le l} \sup_{x\in \operatorname{supp}(\zeta_i)} |x|^{\beta+|\alpha|}|D_x^\alpha u(x)|\Big) \\ &\le C\|\zeta_i\|_{\mathsf{Y}_0 C^l} \sum_{|\alpha|\le l} \sup_{x\in \operatorname{supp}(\zeta_i)} |x|^{\beta+|\alpha|}|D_x^\alpha u(x)|,\end{aligned}$$

where the constants are independent of i. With the help of Lemma 2.52 and the fact that $\operatorname{supp}(\zeta_i) \subset \{|x| > e^{i-1}\}$ we now get (iii)⇒(ii).

(ii)⇒(iii). Suppose (iii) is not satisfied. Thus we can find some multi-index α with $|\alpha| \le l$ and a sequence of points $\{x_j\}_{j\in\mathbb{N}}$ with $x_j \to \infty$ such that

$$|x_j|^{\beta+|\alpha|}|D_x^\alpha u(x_j)| \ge C > 0 \quad \text{for all } j \in \mathbb{N}. \tag{35}$$

Choose a sequence $\{j(i)\}_{i\in\mathbb{N}}$ with $j(i) \to \infty$ as $i \to \infty$ and $|x_{j(i)}| > e^i$. Therefore $\zeta_i = 1$ on a neighbourhood of $x_{j(i)}$ so (22) gives

$$\|\zeta_i u\|_{\mathsf{Y}_\beta C^l} \ge C \sum_{|\alpha|\le l} \sup_{x\in\mathbb{R}^n_*} |x|^{\beta+|\alpha|}|D_x^\alpha(\zeta_i u)(x)| \ge C|x_{j(i)}|^{\beta+|\alpha|}|D_x^\alpha u(x_{j(i)})|.$$

The fact that (ii) is not satisfied now follows from (35). ■

By Lemma 2.34 we know that, for any $\beta \in \mathbb{R}$, $\mathsf{X}_\beta C^l_0$ is the separable subspace of $\mathsf{X}_\beta C^l$ obtained by taking the closure of C_0^∞ with respect to the norm $\|\cdot\|_{\mathsf{X}_\beta C^l}$. Elements of $\mathsf{X}_\beta C^l_0$ can be given an alternative characterisation as follows.

Proposition 2.54. *Suppose $l \in \mathbb{N}_0$, $\beta \in \mathbb{R}$ and $u \in \mathsf{X}_\beta C^l$. Then $u \in \mathsf{X}_\beta C^l_0$ iff $D_x^\alpha u(x) = o(|x|^{-\beta-|\alpha|})$ as $x \to \infty$ for each multi-index α with $|\alpha| \le l$.*

Remark 2.55. It follows that $\Lambda^{-s} \in \mathsf{X}_\beta C^l_0$ for any $l \in \mathbb{N}_0$ and $\beta, s \in \mathbb{R}$ with $\beta < s$.

Proof. By Lemma 2.53 it suffices to show $u \in \mathsf{X}_\beta C^l_0$ iff $\|\zeta_i u\|_{\mathsf{X}_\beta C^l} \to 0$ as $i \to \infty$.

Let $u \in \mathsf{X}_\beta C^l_0$ and $\varepsilon > 0$. Thus we can find $u_\varepsilon \in C_0^\infty$ with $\|u - u_\varepsilon\|_{\mathsf{X}_\beta C^l} \le \varepsilon$. Now suppose $i \in \mathbb{N}$ is sufficiently large so that $\operatorname{supp}(u_\varepsilon) \subseteq \{|x| < e^{i-1}\}$. Since $\zeta_i = 0$ on the latter set we have $\zeta_i u = \zeta_i(u - u_\varepsilon)$. Lemma 2.52 and the continuity of multiplication as a bilinear map $\mathsf{X}_0 C^l \times \mathsf{X}_\beta C^l \to \mathsf{X}_\beta C^l$ now imply $\|\zeta_i u\|_{\mathsf{X}_\beta C^l} \le C\varepsilon$ for some C which is independent of ε. It follows that $\|\zeta_i u\|_{\mathsf{X}_\beta C^l} \to 0$ as $i \to \infty$.

On the other hand, suppose $\|\zeta_i u\|_{\mathsf{X}_\beta C^l} \to 0$ as $i \to \infty$. Let $\varepsilon > 0$ and choose $I \in \mathbb{N}$ so that $\|\zeta_I u\|_{\mathsf{X}_\beta C^l} \le \varepsilon$. Now Lemma 2.16 gives us a constant C_1 such that

$$\|\eta_{I+1} v\|_{\mathsf{X}_\beta C^l} \le C_1\|\eta_{I+1} v\|_{C^l} \tag{36}$$

for all $v \in \mathsf{X}_\beta C^l$. Since $\eta_I u \in C^l$ with $\mathrm{supp}(\eta_I u) \subset \{|x| < e^{I+1}\}$ we can find $u_\varepsilon \in C_0^\infty$ with $\mathrm{supp}(u_\varepsilon) \subset \{|x| < e^{I+1}\}$ and $\|\eta_I u - u_\varepsilon\|_{C^l} \leq \varepsilon / C_1$. It follows that $\eta_{I+1}(\eta_I u - u_\varepsilon) = \eta_I u - u_\varepsilon$ so (36) gives $\|\eta_I u - u_\varepsilon\|_{\mathsf{X}_\beta C^l} \leq \varepsilon$. However, $u = \eta_I u + \zeta_I u$ so $\|u - u_\varepsilon\|_{\mathsf{X}_\beta C^l} \leq 2\varepsilon$. ■

From Definition 1.1 and Proposition 2.37 it is clear that we have

$$\mathrm{Sc}^\gamma \subset \mathsf{X}_\gamma C^l \tag{37}$$

for any $\gamma \in \mathbb{R}$ and $l \in \mathbb{N}_0$. Together with Proposition 2.50 this implies the following.

Proposition 2.56. *If E is a model space and $\beta, \gamma \in \mathbb{R}$ then multiplication by any $p \in \mathrm{Sc}^\gamma$ defines a continuous map $\mathsf{X}_\beta E \to \mathsf{X}_{\beta+\gamma} E$.*

Remark 2.57. Suppose $\gamma \in \mathbb{R}$ and $p \in \mathrm{Sc}^\gamma$ with principal part $r^{-\gamma} a(\omega)$. For any $f \in \mathrm{BS}$ define a function p_f by

$$p_f(x) = f(x)p(x) + (1 - f(x))r^{-\gamma} a(\omega).$$

If $f = 1$ on a neighbourhood of 0 it is clear that $p_f \in \mathrm{Sc}^\gamma$ with the same principal part as p. On the other hand, if $f = 0$ on a neighbourhood of 0 then Lemma 2.16 and (22) imply p_f is contained in $\mathsf{Y}_\gamma C^l$ for any $l \in \mathbb{N}_0$.

Finally, condition (ii) of Definition 1.1 and Lemma 2.53 give the following result.

Lemma 2.58. *Suppose $\gamma \in \mathbb{R}$ and $p \in \mathrm{Sc}^\gamma$ with principal part $r^{-\gamma} a(\omega)$. Then, for any $l \in \mathbb{N}_0$,*

$$\lim_{i \to \infty} \|\zeta_i (p - a(\omega) r^{-\gamma})\|_{\mathsf{Y}_\gamma C^l} = 0 = \lim_{i \to \infty} \|\zeta_i (p - a(\omega) r^{-\gamma})\|_{\mathsf{X}_\gamma C^l}.$$

2.4. Admissible spaces

Definition 2.59. For any $m \in \mathbb{R}$ let Sym^m denote the set of functions on $a \in C^\infty_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R}^n)$ which satisfy estimates of the form

$$|D_x^\alpha D_\xi^{\alpha'} a(x, \xi)| \leq C_{\alpha,\alpha'} \Lambda^{m - |\alpha'|}(\xi) \tag{38}$$

for all multi-indices α and α'. The best constants in (38) provide Sym^m with a collection of semi-norms making it into a locally convex space.

For any $a \in \mathrm{Sym}^m$ we shall use $a(x, D_x)$ to denote the pseudo-differential operator defined by the symbol $a(x, \xi)$. The set of all pseudo-differential operators of order m (i.e. the set of all operators defined by symbols in Sym^m) will be denoted by $\Psi\mathrm{Op}^m$.

Remark 2.60. For any $m \in \mathbb{R}$ the pairing $(a, u) \mapsto a(x, D_x)u$ defines a continuous bilinear map $\mathrm{Sym}^m \times \mathscr{S} \to \mathscr{S}$ and a bilinear map $\mathrm{Sym}^m \times \mathscr{S}' \to \mathscr{S}'$ which is separately continuous in each variable. If $m, l \in \mathbb{R}$ it can also be shown that the composition of operators in $\Psi\mathrm{Op}^m$ and $\Psi\mathrm{Op}^l$ gives an operator in $\Psi\mathrm{Op}^{m+l}$, whilst the adjoint of an operator in $\Psi\mathrm{Op}^m$ is again in $\Psi\mathrm{Op}^m$. Furthermore, the corresponding symbol maps $\mathrm{Sym}^m \times \mathrm{Sym}^l \to \mathrm{Sym}^{l+m}$ and $\mathrm{Sym}^m \to \mathrm{Sym}^m$ are continuous bilinear and continuous anti-linear respectively. Further details can be found in Section 18.1 of [H2], for example.

If $a \in \mathrm{Sym}^m$ for some $m \in \mathbb{R}$ then we can write

$$a(x, D_x)u = \int K_a(x, y) u(y) \, d^n y \tag{39}$$

for all $u \in \mathscr{S}'$, where $K_a \in \mathscr{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ is the Schwartz kernel of $a(x, D_x)$; that is, $K_a(x,y) = (2\pi)^{-n}\widehat{a}(x, y-x)$ where $\widehat{a}$ is the Fourier transform of $a(x,\xi)$ with respect to the second variable (see Section 18.1 of [H2], for example).

Lemma 2.61. *We have* $K_a \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$ *iff* $a \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$. *Furthermore, in this case* $a(x, D_x)$ *defines a continuous map* $\mathscr{S}' \to \mathscr{S}$.

Proof. The first part of the result follows from the fact that the Fourier transform defines an isomorphism $\mathscr{S} \to \mathscr{S}$. Now suppose $K_a \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$ and choose any multi-index α and $s \in \mathbb{R}$. Then (39) gives

$$|\Lambda^s(x) D_x^\alpha a(x, D_x) u| \le |(\Lambda^s(x) D_x^\alpha K_a(x,\cdot),\, u)_{\mathbb{R}^n}| \tag{40}$$

for all $u \in \mathscr{S}'$. The assumption that $K_a \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$ implies that $\Lambda^s(x) D_x^\alpha K_a(x,\cdot)$ forms a bounded subset of $\mathscr{S}$ as x varies over $\mathbb{R}^n$. The second part of the result now follows from (40) and the continuity of the dual pairing of $\mathscr{S}$ and $\mathscr{S}'$. ■

As a corollary of this result we have the following.

Corollary 2.62. *Let* $s \in \mathbb{R}$ *and* $\phi_1, \phi_2 \in C^\infty$ *with* $\operatorname{supp}(\phi_1) \cap \operatorname{supp}(\phi_2) = \emptyset$ *and either* $\phi_1 \in C_0^\infty$ *or* $\phi_2 \in C_0^\infty$. *Then the pseudo-differential operator* $\phi_1(x)\Lambda^s(D_x)\phi_2(x)$ *defines a continuous map* $\mathscr{S}' \to \mathscr{S}$.

Proof. The Schwartz kernel of the operator $\phi_1(x)\Lambda^s(D_x)\phi_2(x)$ is

$$K(x,y) = (2\pi)^{-n}\phi_1(x)\psi(y-x)\phi_2(y)$$

where $\psi \in \mathscr{S}'$ is the Fourier transform of Λ^s. By standard properties of the Fourier transform of symbols (see Proposition VI.4.1 in [St], for example) ψ is smooth and rapidly decaying away from 0, along with all its derivatives. On the other hand, ϕ_1 and ϕ_2 have disjoint supports, at least one of which is compact. It follows that $K \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$. Lemma 2.61 now completes the result. ■

Using the mapping properties of pseudo-differential operators we can now single out a special class of model spaces which will provide the natural function space setting for our main results.

Definition 2.63. An *admissible space* is a model space E satisfying the following additional condition:

(B) We have a continuous bilinear map $\mathrm{Sym}^0 \times E \to E$ which sends (a,u) to $a(x, D_x)u$.

Remark 2.64. Condition (A2) for a model space is a special case of condition (B).

Example 2.65. A rich class of admissible spaces is provided by the Besov spaces B^s_{pq} for $s \in \mathbb{R}$ and $p, q \in [1,\infty]$, and the Triebel–Lizorkin spaces F^s_{pq} for $s \in \mathbb{R}$ and $p, q \in [1,\infty]$ with $q \ne 1$ if $p = \infty$; see [T1] and [T2] for the definitions of these spaces and the justification of conditions (A1) to (A4) and (B) (1).

---

(1) In the cited literature the continuity of the bilinear map in condition (B) is established explicitly only for the second variable. However, the full continuity of this map can be obtained easily from the proof of the relevant result (Theorem 6.2.2 in [T2]) by using symbol norms of a to make simple estimates of the constants appearing therein.

The classes of Besov spaces and Triebel–Lizorkin spaces include a large number of the “standard” function spaces as follows (see [T1]):

$F^s_{p2} = H^{p,s}$ (the Sobolev or Bessel-potential spaces) for $s \in \mathbb{R}$ and $p \in (1,\infty)$,
$F^0_{p2} = h_p$ (the local Hardy spaces) for $p \in [1,\infty)$,
$F^0_{\infty 2} = \mathrm{bmo}$ (the inhomogeneous version of BMO),
$B^s_{\infty\infty} = \mathcal{C}^s$ (the Zygmund spaces) for $s > 0$,
$B^s_{pp} = F^s_{pp} = W^s_p$ (the Slobodetskiĭ spaces) for $s \in \mathbb{R}^+ \setminus \mathbb{N}$ and $p \in [1,\infty)$,
$B^s_{pq} = \Lambda^s_{p,q}$ (the Lipschitz spaces) for $s > 0$, $p \in [1,\infty)$ and $q \in [1,\infty]$.

The most notable omissions from this list are the spaces C^l for $l \in \mathbb{N}_0$; these spaces do not satisfy condition (B) (n.b. although the Zygmund space $\mathcal{C}^s$ coincides with the Hölder space $C^{l+\sigma}$ whenever $l \in \mathbb{N}_0$, $\sigma \in (0,1)$ and $s = l+\sigma$, we only have a *strict* inclusion $\mathcal{C}^l \subset C^l$ when $l \in \mathbb{N}$).

Definition 2.66. Suppose E is an admissible space and let $s \in \mathbb{R}$. We define E^s to be the set of $u \in \mathscr{S}'$ for which $\Lambda^s(D_x)u \in E$. Furthermore, we give this set a norm $\|\cdot\|_{E^s}$ defined by $\|u\|_{E^s} = \|\Lambda^s(D_x)u\|_E$.

Remark 2.67. If $E = H^{p,s}$ for some $s \in \mathbb{R}$ and $p \in (1,\infty)$ then $E^{s'} = H^{p,s+s'}$ for any $s' \in \mathbb{R}$. Likewise, if $E = \mathcal{C}^s$ for some $s > 0$ then $E^{s'} = \mathcal{C}^{s+s'}$ for any $s' \in \mathbb{R}$, provided $s + s' > 0$.

Proposition 2.68. *Suppose $s, m \in \mathbb{R}$. Then the assignment $(a,u) \mapsto a(x,D_x)u$ defines a continuous bilinear map $\mathrm{Sym}^m \times E^s \to E^{s-m}$. In particular, for any $s \in \mathbb{R}$ and multi-index α, the differential operator D^α_x defines a continuous map $E^{s+|\alpha|} \to E^s$.*

Proof. Given $a \in \mathrm{Sym}^m$ define a new symbol $b \in \mathrm{Sym}^0$ as the symbol of the pseudo-differential operator $b(x,D_x) = \Lambda^{s-m}(D_x)a(x,D_x)\Lambda^{-s}(D_x)$. Standard results on the calculus of pseudo-differential operators (see Remark 2.60) imply that the assignment $a \mapsto b$ defines a continuous map $\mathrm{Sym}^m \to \mathrm{Sym}^0$. On the other hand, $\Lambda^{s-m}(D_x)a(x,D_x)u = b(x,D_x)\Lambda^s(D_x)u$ for any $u \in \mathscr{S}'$. The result now follows from Definition 2.66 and condition (B) on the admissible space E. ■

Proposition 2.69. *Any admissible spaces E and F have the following properties:*

(i) *If $s \in \mathbb{R}$ then E^s is again an admissible space.*

(ii) *If $s, s' \in \mathbb{R}$ then $(E^s)^{s'} = E^{s+s'}$.*

(iii) *If $s = l \in \mathbb{N}_0$ then $E^s = E^l$ up to equivalent norms (where E^l is as given by Definition 2.4).*

(iv) *We have a continuous inclusion $E \hookrightarrow F$ iff we have a continuous inclusion $E^s \hookrightarrow F^s$ for any $s \in \mathbb{R}$.*

(v) *We have a local inclusion $E_{\mathrm{loc}} \hookrightarrow F_{\mathrm{loc}}$ iff we have a local inclusion $E^s_{\mathrm{loc}} \hookrightarrow F^s_{\mathrm{loc}}$ for any $s \in \mathbb{R}$.*

(vi) *The space E_0 and its dual E^*_0 are admissible spaces.*

(vii) *If $s \in \mathbb{R}$ then $(E^s)_0 = (E_0)^s$ and $(E^s_0)^* = (E^*_0)^{-s}$.*

Proof. (i) The only non-trivial conditions are (B) and (A4) (n.b. condition (A2) is covered by condition (B)). The former is established in Proposition 2.68 whilst the latter uses the fact that conjugation of a pseudo-differential operator by a diffeomorphism which is linear outside a compact region gives another pseudo-differential operator of the same order (see Theorem 18.1.17 in [H2], for example (2)).

(ii) This is an easy consequence of the identity $\Lambda^s(D_x)\Lambda^{s'}(D_x) = \Lambda^{s+s'}(D_x)$.

(iii) Suppose $u \in E^s$ and let α be a multi-index with $|\alpha| \le l = s$. Thus $\Lambda^s(D_x)u \in E$ whilst $\xi^\alpha\Lambda^{-s}(\xi)$ defines a symbol in Sym^0 so $D_x^\alpha\Lambda^{-s}(D_x)$ defines a continuous map on E (by condition (B) for the admissible space E). Hence $D_x^\alpha u = D_x^\alpha\Lambda^{-s}(D_x)\Lambda^s(D_x)u \in E$ and we have a norm estimate of the form

$$\|D_x^\alpha u\|_E \le C\|\Lambda^s(D_x)u\|_E = C\|u\|_{E^s}.$$

The existence of a continuous inclusion $E^s \hookrightarrow E^l$ now follows from the definition of E^l.

On the other hand, we can write

$$\Lambda^{2s}(D_x) = \Lambda^{2l}(D_x) = \Big(1 + \sum_{i=1}^{n} D_i^2\Big)^l = \sum_{|\alpha|\le l} a^\alpha D_x^\alpha D_x^\alpha \tag{41}$$

for some constants $a^\alpha \in \mathbb{C}$. Now suppose $u \in E^l$. Therefore $D_x^\alpha u \in E$ with $\|D_x^\alpha u\|_E \le C_{1,\alpha}\|u\|_{E^l}$ (see Remark 2.5) whilst $\Lambda^{-s}(\xi)\xi^\alpha \in \mathrm{Sym}^0$ so $\Lambda^{-s}(D_x)D_x^\alpha$ defines a continuous map on E. Hence $\Lambda^{-s}(D_x)D_x^\alpha D_x^\alpha u \in E$ and

$$\|\Lambda^{-s}(D_x)D_x^\alpha D_x^\alpha u\|_E \le C_{2,\alpha}\|u\|_{E^l}.$$

With the help of (41) it follows that

$$\Lambda^s(D_x)u = \sum_{|\alpha|\le l} a^\alpha\Lambda^{-s}(D_x)D_x^\alpha D_x^\alpha u \in E$$

and $\|u\|_{E^s} = \|\Lambda^s(D_x)u\|_E \le C\|u\|_{E^l}$. This completes the proof of part (iii).

(iv) The fact that a continuous inclusion $E \hookrightarrow F$ induces a continuous inclusion $E^s \hookrightarrow F^s$ is trivial. Part (ii) now gives the converse.

(v) Let $\phi \in C_0^\infty$ and choose $\phi_1, \phi_2 \in C_0^\infty$ with $\phi_1 \succ \phi$ and $\phi_2 = 1$ on a neighbourhood of $\mathrm{supp}(\phi_1)$. Setting $\phi_3 = 1 - \phi_2$ we deduce that $\phi_3 \in C^\infty$ and $\mathrm{supp}(\phi_1) \cap \mathrm{supp}(\phi_3) = \emptyset$. Now let $u \in E^s_{\mathrm{loc}}$. Therefore $\phi u \in E^s$ or, equivalently, $\Lambda^s(D_x)\phi u \in E$. By the local inclusion $E_{\mathrm{loc}} \hookrightarrow F_{\mathrm{loc}}$ it follows that $\phi_2\Lambda^s(D_x)\phi u \in F$ and

$$\|\phi_2\Lambda^s(D_x)\phi u\|_F \le C_1\|\phi u\|_{E^s}, \tag{42}$$

where C_1 may depend on ϕ (through ϕ_2) but not on u. On the other hand, Corollary 2.62 implies that $\phi_3\Lambda^s(D_x)\phi_1$ defines a continuous map $\mathscr{S}' \to \mathscr{S}$. Condition (A1) for the model spaces E^s and F then shows that $\phi_3\Lambda^s(D_x)\phi_1$ defines a continuous map $E^s \to F$. Therefore $\phi_3\Lambda^s(D_x)\phi u = \phi_3\Lambda^s(D_x)\phi_1\phi u \in F$ and

$$\|\phi_3\Lambda^s(D_x)\phi u\|_F \le C_2\|\phi u\|_{E^s}, \tag{43}$$

where C_1 may depend on ϕ (through ϕ_1 and ϕ_3) but not on u. Since $\phi_2 + \phi_3 = 1$, (42) and (43) combine to establish the existence of a local inclusion $E^s_{\mathrm{loc}} \hookrightarrow F^s_{\mathrm{loc}}$. Part (ii) now gives the converse.

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(2) Technically the result in [H2] only gives a local version of what we need; however, it is easy to see how the proof can be modified to give the conjugation result as stated above.

(vi) Condition (B) for E_0 follows from the same condition for E and the fact that pseudo-differential operators preserve the set $\mathscr{S}$ (which is dense in E_0). For the dual space we can use Lemma 2.43 and the fact that the map which sends a pseudo-differential operator $a(x, D_x) \in \Psi\mathrm{Op}^0$ to its adjoint induces an (anti-linear) isomorphism on Sym^0 (see Remark 2.60).

(vii) Clearly, $(E_0)^s \subseteq E^s$ whilst $\Lambda^s(D_x) : (E_0)^s \to E_0$ is an isomorphism which preserves the set $\mathscr{S}$. The fact that $\mathscr{S}$ is dense in E_0 now implies the same is true for $(E_0)^s$, with the first identity following immediately. The second identity can be obtained from (29) and the expression $(u, v)_{\mathbb{R}^n} = (\Lambda^s(D_x)u, \Lambda^{-s}(D_x)v)_{\mathbb{R}^n}$, which is valid for all $u \in \mathscr{S}$ and $v \in \mathscr{S}'$. ■

For the proofs of the next three results let ψ denote a choice of function in C_0^∞ with $\psi = 1$ in a neighbourhood of 0 and $\operatorname{Ran}\psi = [0, 1]$. Also, for each $j \in \mathbb{N}$, define $\psi_j \in C_0^\infty$ by $\psi_j(x) = \psi(x/j)$.

Lemma 2.70. *Suppose $f \in E$ for some admissible space E. Then we can find a sequence $\{f_j\}_{j\in\mathbb{N}} \subset \bigcap_{s\in\mathbb{R}} E^s$ with $f_j \to f$ in $E^{-\delta}$ for any $\delta > 0$.*

Proof. For each $j \in \mathbb{N}$ set $f_j = \psi_j(D_x)f$. Since $\psi_j(\xi) \in \mathrm{Sym}^m$ for any $m \in \mathbb{R}$, Proposition 2.68 gives $f_j \in E^s$ for any $s \in \mathbb{R}$. On the other hand, a straightforward check shows $\Lambda^{-\delta}(\xi)\psi_j(\xi) \to \Lambda^{-\delta}(\xi)$ in Sym^0 for any $\delta > 0$. Condition (B) for the admissible space E then implies $\Lambda^{-\delta}(D_x)f_j \to \Lambda^{-\delta}(D_x)f$ in E for any $\delta > 0$; by definition this means $f_j \to f$ in $E^{-\delta}$. ■

Although the space $\mathsf{X}_\beta E$ need not contain C_0^∞ as a dense subset for a general admissible space E, Lemma 2.70 leads to the following slightly weaker result.

Lemma 2.71. *Suppose $f \in \mathsf{X}_\beta E$ for some admissible space E and $\beta \in \mathbb{R}$. Then we can find a sequence $\{f_i\}_{i\in\mathbb{N}} \subset C_0^\infty$ such that $f_i \to f$ in $\mathsf{X}_{\beta-\varepsilon}E^{-\delta}$ for any $\varepsilon, \delta > 0$.*

Proof.

Claim: *Given $f \in \mathsf{Z}_\beta E$ we can find a sequence $\{f_j\}_{j\in\mathbb{N}} \subset \bigcap_{s\in\mathbb{R}} \mathsf{Z}_\beta E^s$ with $f_j \to f$ in $\mathsf{Z}_\beta E^{-\delta}$ for any $\delta > 0$.* Since multiplication by $e^{\beta t}$ defines an isomorphism $\mathsf{Z}_\beta E^s \to \mathsf{Z}_0 E^s = E^s(\Pi^n)$ for any $s \in \mathbb{R}$, we may prove the Claim assuming $\beta = 0$. Consider the notation of Remark 2.9 and define $\widetilde{\chi}'_i = \chi_i \circ \Psi_i^{-1} \in C^\infty$ for each $i \in I$. Now Lemma 2.10 implies $g_i := (\Psi_i^{-1})^*(\chi_i f) \in E$ so Lemma 2.70 gives us a sequence $\{g_{ij}\}_{j\in\mathbb{N}} \subset \bigcap_{s\in\mathbb{R}} E^s$ with $g_{ij} \to g_i$ in $E^{-\delta}$ for any $\delta > 0$. Condition (A2) and the fact that $\chi'_i \succ \chi_i$ then give $\widetilde{\chi}'_i g_{ij} \to \widetilde{\chi}'_i g_i = g_i$ in $E^{-\delta}$ for any $\delta > 0$. Setting

$$f_j = \sum_{i\in I} \Psi_i^*(\widetilde{\chi}'_i g_{ij})$$

for any $j \in \mathbb{N}$, we deduce from Lemma 2.10 that $\{f_j\}_{j\in\mathbb{N}} \subset \bigcap_{s\in\mathbb{R}} E^s(\Pi^n)$ whilst

$$f_j \to \sum_{i\in I} \Psi_i^* g_i = \sum_{i\in I} \chi_i f = f$$

in $E^{-\delta}(\Pi^n)$ for any $\delta > 0$. This completes the Claim.

Let $f \in \mathsf{X}_\beta E$. By Lemma 2.16 we have $\eta_0 f \in E$ and $\zeta_0 f \in \mathsf{Y}_\beta E$ so Lemma 2.70 and the above Claim (coupled with (8)) give us sequences $\{g_j\}_{j\in\mathbb{N}} \subset \bigcap_{s\in\mathbb{R}} E^s$ and $\{h_j\}_{j\in\mathbb{N}} \subset$

$\bigcap_{s\in\mathbb{R}} \mathsf{Y}_\beta E^s$ with $g_j \to \eta_0 f$ in $E^{-\delta}$ and $h_j \to \zeta_0 f$ in $\mathsf{Y}_\beta E^{-\delta}$ for any $\delta > 0$. For any $j \in \mathbb{N}$ define further functions by $f'_j = \eta_1 g_j + \zeta_{-1} h_j$ and $f_j = \psi_j f'_j$. Another application of Lemma 2.16 now shows $\{f'_j\}_{j\in\mathbb{N}} \subset \bigcap_{s\in\mathbb{R}} \mathsf{X}_\beta E^s$ and $f'_j \to (\eta_1\eta_0 + \zeta_{-1}\zeta_0)f = f$ in $\mathsf{X}_\beta E^{-\delta}$ for any $\delta > 0$.

By Remark 2.21 and Lemma 2.29 we can find $k \in \mathbb{N}_0$ so that we have a continuous inclusion $\mathsf{X}_\beta E^{k+l} \hookrightarrow \mathsf{X}_\beta C^l$ for all $l \in \mathbb{N}_0$. It follows that $\{f'_j\}_{j\in\mathbb{N}} \subset \bigcap_{l\in\mathbb{N}} \mathsf{X}_\beta C^l \subset C^\infty_{\mathrm{loc}}$. Therefore $\{f_j\}_{j\in\mathbb{N}} \subset C^\infty_0$. Now let $\varepsilon, \delta > 0$. A straightforward application of Proposition 2.37 shows $\psi_j \to 1$ in $\mathsf{X}_{-\varepsilon} C^l$ for any $l \in \mathbb{N}_0$. Proposition 2.50 and the convergence $f'_j \to f$ in $\mathsf{X}_\beta E^{-\delta}$ then imply $f_j \to f$ in $\mathsf{X}_{\beta-\varepsilon} E^{-\delta}$. ■

Lemma 2.72. *The set C^∞_0 is dense in both $\mathscr{S}$ and $\mathscr{S}'$.*

Proof. For the density of C^∞_0 in $\mathscr{S}$ see Proposition VI.1.3 in [Yo]. Now let $f \in \mathscr{S}'$ and, for each $j \in \mathbb{N}$, set $f_j = \psi_j(x)\psi_j(D_x)f$. Thus f_j has compact support (contained in $\mathrm{supp}(\psi_j)$) whilst the symbol $\psi_j(x)\psi_j(\xi)$ is contained in $C^\infty_0(\mathbb{R}^n \times \mathbb{R}^n)$ so Lemma 2.61 gives $f_j \in \mathscr{S}$. Therefore $\{f_j\}_{j\in\mathbb{N}} \subset C^\infty_0$.

A straightforward check shows $\Lambda^{-1}(x)\psi_j(x)\psi_j(D_x) \to \Lambda^{-1}(x)$ in Sym^1 so Remark 2.60 gives $\Lambda^{-1}(x)f_j \to \Lambda^{-1}(x)f$ in $\mathscr{S}'$. Since the operator (of multiplication by) $\Lambda(x)$ defines a continuous map on $\mathscr{S}'$ we now get $f_j \to f$ in $\mathscr{S}'$. ■

The next result follows from Lemma 2.72 and an argument similar to that used to prove the Claim in the proof of Lemma 2.71.

Lemma 2.73. *For any $\beta \in \mathbb{R}$ the set $\mathsf{Z}_\beta \mathscr{S}$ is dense in $\mathsf{Z}_\beta \mathscr{S}'$.*

The compactness results given in Section 2.3 can be refined for admissible spaces. We begin with two technical lemmas, the first of which is essentially the Ascoli–Arzelà Theorem (see Section III.3 of [Yo]).

Lemma 2.74. *Any bounded subset of $\mathscr{S}$ is pre-compact.*

Lemma 2.75. *If E is an admissible space and $s > 0$ then E^s is locally compact in E.*

Proof. Let $\phi \in C^\infty_0$ and choose a sequence $\{\psi_i\}_{i\in\mathbb{N}} \subset C^\infty_0$ such that $\psi_i \to \Lambda^{-s}$ in $\mathsf{X}_0 C^\infty$ (which is possible by Remark 2.55 since $s > 0$). It follows that the sequence of symbols $\phi(x)\psi_i(\xi)$ converges to $\phi(x)\Lambda^{-s}(\xi)$ in Sym^0 and so the pseudo-differential operator $\phi(x)\psi_i(D_x)$ converges to $\phi(x)\Lambda^{-s}(D_x)$ in $\mathscr{L}(E,E)$ (by condition (B) for the admissible space E). However, the map $E^s \to E$ given by multiplication by ϕ can be written as $\phi(x)\Lambda^{-s}(D_x)\Lambda^s(D_x)$ where $\Lambda^s(D_x)$ acts as an isomorphism $E^s \to E$. By the fact that the set of compact operators is closed in $\mathscr{L}(E,E)$ it therefore suffices to show that the operator $\phi(x)\psi_i(\xi)$ defines a compact map $E \to E$ for any $i \in \mathbb{N}$.

Now $\phi(x)\psi_i(\xi) \in C^\infty_0(\mathbb{R}^n \times \mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$ so Lemma 2.61 implies that $\phi(x)\psi_i(D_x)$ defines a continuous map $\mathscr{S}' \to \mathscr{S}$. On the other hand, condition (A1) for the admissible space E gives us continuous inclusions $E \hookrightarrow \mathscr{S}'$ and $\mathscr{S} \hookrightarrow E$. By composing these maps and using Lemma 2.74 it follows that $\phi(x)\psi_i(D_x)$ defines a compact map $E \to E$. ■

Lemma 2.75 and Proposition 2.51 (with E, F and G replaced by E^s, F and E respectively) immediately lead to the following useful result.

Proposition 2.76. *Let $\beta, \gamma, s \in \mathbb{R}$ with $s > 0$ and suppose that, for an admissible space E and a model space F, multiplication defines a continuous bilinear map $E \times F \to E$ (or, more generally, $E^s \times F \to E$). Then multiplication by any $\phi \in \mathsf{X}_\gamma F_0$ defines a compact map $\mathsf{X}_\beta E^s \to \mathsf{X}_{\beta+\gamma} E$.*

Corollary 2.77. *Suppose E is an admissible space and $\beta, \gamma, s \in \mathbb{R}$ with $\gamma > \beta$ and $s > 0$. Then the inclusion $\mathsf{X}_\gamma E^s \hookrightarrow \mathsf{X}_\beta E$ is compact.*

Proof. By Lemma 2.49 we know that multiplication defines a continuous map $E \times C^l \to E$ for all sufficiently large $l \in \mathbb{N}_0$. Furthermore, $\beta - \gamma < 0$ so $1 \in \mathsf{X}_{\beta-\gamma} C^l_0$ by Remark 2.55. The result now follows directly from an application of Proposition 2.76. ■

For any admissible space E, $\beta \in \mathbb{R}$, $k \in \mathbb{N}$ and vector $\kappa = (\kappa_1, \ldots, \kappa_k)$ of non-negative integers, we define the following product spaces:

$$E^\kappa = \prod_{i=1}^k E^{\kappa_i}, \quad \mathsf{Z}^\kappa_\beta E = \prod_{i=1}^k \mathsf{Z}_\beta E^{\kappa_i}, \quad \mathsf{Y}^\kappa_\beta E = \prod_{i=1}^k \mathsf{Y}_{\beta-\kappa_i} E^{\kappa_i}, \quad \mathsf{X}^\kappa_\beta E = \prod_{i=1}^k \mathsf{X}_{\beta-\kappa_i} E^{\kappa_i}.$$

Let $r^{\pm\kappa}$ denote the matrix operator given as multiplication by $\operatorname{diag}(r^{\pm\kappa_1}, \ldots, r^{\pm\kappa_k})$. Thus (8) and Proposition 2.20 imply that we have an isomorphism

$$\Theta^* r^{-\kappa} : \mathsf{Y}^\kappa_\beta E \xrightarrow{r^{-\kappa}} \prod_{i=1}^k \mathsf{Y}_\beta E^{\kappa_i} \xrightarrow{\Theta^*} \mathsf{Z}^\kappa_\beta E \tag{44}$$

with inverse

$$r^\kappa (\Theta^{-1})^* : \mathsf{Z}^\kappa_\beta E \xrightarrow{(\Theta^{-1})^*} \prod_{i=1}^k \mathsf{Y}_\beta E^{\kappa_i} \xrightarrow{r^\kappa} \mathsf{Y}^\kappa_\beta E. \tag{45}$$

Finally, let π_κ denote the projection defined on any of the above product spaces by

$$(\pi_\kappa u)_i = \begin{cases} u_i & \text{if } \kappa_i > 0, \\ 0 & \text{if } \kappa_i = 0. \end{cases}$$

Clearly, π_κ commutes with multiplication by a scalar or a diagonal matrix.

2.5. Elliptic operators.

Unless otherwise stated, we shall assume that the coefficients of any differential operator on $\mathbb{R}^n$ are contained in C^∞. In particular, condition (A2) and Remark 2.5 imply that any differential operator $A(x, D_x)$ on $\mathbb{R}^n$ of order m defines a continuous map $E^m \to E$ for any model space E.

Definition 2.78. Let $\mathcal{A}(x, D_x)$ be a $k \times k$ system of differential operators on $\mathbb{R}^n$ with entries $A_{ij}(x, D_x)$ for $i, j = 1, \ldots, k$. We say that $\mathcal{A}$ is *elliptic* on some open set $U \subseteq \mathbb{R}^n$ if the following conditions are satisfied:

(i) There exist vectors $\mu = (\mu_1, \ldots, \mu_k)$ and $\nu = (\nu_1, \ldots, \nu_k)$ of non-negative integers such that $\operatorname{ord} A_{ij} = \mu_j - \nu_i$ (with $A_{ij} = 0$ whenever $\mu_j - \nu_i < 0$). Furthermore, $\min_i \nu_i = 0$.

(ii) Let $a_{ij}(x, \xi)$ denote the principal symbol of $A_{ij}(x, \xi)$ (so $a_{ij}(x, \xi)$ is a homogeneous polynomial of degree $\mu_j - \nu_i$ in ξ) and define $\det_{\mathcal{A}}(x, \xi)$ to be the determinant of the $k \times k$ matrix with entries $a_{ij}(x, \xi)$. Then $\det_{\mathcal{A}}(x, \xi) \neq 0$ for any $(x, \xi) \in U \times \mathbb{R}^n_*$.

We say that $\mathcal{A}$ is *uniformly elliptic* on U if condition (ii) can be replaced by the following stronger condition:

(ii)′ Set $[\mu-\nu]=\sum_{i=1}^{k}(\mu_i-\nu_i)$ and let $\det_{\mathcal{A}}(x,\xi)$ be defined as in condition (ii) above. Then $|\det_{\mathcal{A}}(x,\xi)|\geq C|\xi|^{[\mu-\nu]}$ for all $(x,\xi)\in U\times\mathbb{R}^n$.

We shall refer to the pair (μ,ν) as the *order* of $\mathcal{A}$.

In the case of scalar operators (i.e. when $k=1$) this definition of ellipticity clearly reduces to the usual one. Furthermore, $\nu_1=0$ and $\mu_1=m$, the usual order of the operator. On the other hand, if $\mathcal{A}$ is elliptic on some open set U then $\mathcal{A}$ is uniformly elliptic on any bounded open set V with $\mathrm{Cl}(V)\subseteq U$.

Remark 2.79. If $n\geq 3$ then ellipticity implies that $[\mu-\nu]$ is even (say $2l$) and the polynomial equation $\det_{\mathcal{A}}(x,\xi+t\eta)$, $\xi,\eta\in\mathbb{R}^n_*$, has exactly l roots with $\operatorname{Im}t>0$ (and hence exactly l roots with $\operatorname{Im}t<0$). These conditions need not be satisfied when $n=2$ and must be imposed as extra assumptions when working with boundary value problems. However, we do *not* need to impose such assumptions here.

Remark 2.80. Suppose $\mathcal{A}$ and $\mathcal{B}$ are $k\times k$ systems of differential operators of order (μ,ν) on $U\subseteq\mathbb{R}^n$. Furthermore, suppose the coefficients of $\mathcal{B}$ are bounded (pointwise on U) by some function $b(x)$. Then we have

$$|\det_{\mathcal{A}+\mathcal{B}}(x,\xi)-\det_{\mathcal{A}}(x,\xi)|\leq C(b(x)+b^k(x))|\xi|^{[\mu-\nu]}$$

for all $(x,\xi)\in U\times\mathbb{R}^n$, where C may depend on $\mathcal{A}$ but not on $\mathcal{B}$, b or (x,ξ). It follows that if $\mathcal{A}$ is uniformly elliptic on U and b is bounded by a sufficiently small constant (depending on $\mathcal{A}$) then $\mathcal{A}+\mathcal{B}$ is also uniformly elliptic on U.

Suppose $\mathcal{A}$ is an operator of order (μ,ν) which is uniformly elliptic on an open set $U\subseteq\mathbb{R}^n$. Clearly, $\mathcal{A}$ defines a continuous map $E^\mu\to E^\nu$ for any admissible space E. Condition (B) for admissible spaces and the calculus of pseudo-differential operators also allow us to derive the following regularity result and elliptic estimates.

Theorem 2.81. *Suppose $\mathcal{A}$ is a $k\times k$ system of differential operators on $\mathbb{R}^n$ of order (μ,ν) which is uniformly elliptic on some open set $U\subseteq\mathbb{R}^n$. Suppose further that $\chi_1,\chi_2\in C^\infty$ with $\chi_1\prec\chi_2$ and $\mathrm{supp}(\chi_1)\subseteq U$. If $u\in\mathscr{S}'$ satisfies $\chi_2\mathcal{A}u\in E^\nu$ and $\pi_\mu\chi_2u\in(E^{-1})^\mu$ for some admissible space E, then we also have $\chi_1u\in E^\mu$. Furthermore,*

$$\|\chi_1u\|_{E^\mu}\leq C(\|\chi_2\mathcal{A}u\|_{E^\nu}+\|\pi_\mu\chi_2u\|_{(E^{-1})^\mu})$$

for any such u.

Proof. Without loss of generality we may assume $\chi_2=1$ on a neighbourhood of $\mathrm{supp}(\chi_1)$ (if this were not the case we could simply replace χ_2 with $\chi_3\chi_2$ where $\chi_3\in C^\infty$ is chosen so that $\chi_3\chi_2=1$ on $\chi_2^{-1}(1/2,+\infty)$). It follows that we can find $\chi\in C^\infty$ with $\chi_1\prec\chi\prec\chi_2$ and $\mathrm{supp}(\chi)\subseteq U$.

For $i,j\in\{1,\dots,k\}$ let $A_{ij}(x,D_x)$ denote the ijth entry of $\mathcal{A}(x,D_x)$ and $A_{ij}(x,\xi)\in\mathrm{Sym}^{\mu_j-\nu_i}$ its (full) symbol. Also, let $\mathcal{A}(x,\xi)$ denote the $k\times k$ matrix with entries $A_{ij}(x,\xi)$ and $A(x,\xi)$ the determinant of this matrix; in particular, $A(x,\xi)\in\mathrm{Sym}^m$ where $m=[\mu-\nu]$. Now, by the definition of uniform ellipticity, we can find $B(x,\xi)\in\mathrm{Sym}^{-m}$ such that $\chi_1(x)A(x,\xi)B(x,\xi)\equiv\chi_1(x)\pmod{\mathrm{Sym}^{-1}}$. Let $A^\dagger_{ij}(x,\xi)$ denote the ijth cofactor of the matrix $\mathcal{A}(x,\xi)$. Hence $A^\dagger_{ij}(x,\xi) \in \mathrm{Sym}^{m-\mu_j+\nu_i}$. Also, set $B_{ij}(x,\xi) = (-1)^{i+j}B(x,\xi)A^\dagger_{ji}(x,\xi) \in \mathrm{Sym}^{\nu_j-\mu_i}$, define $B_{ij}(x,D_x) \in \Psi\mathrm{Op}^{\nu_j-\mu_i}$ to be the pseudo-differential operator with symbol $B_{ij}(x,\xi)$ and let $\mathcal{B}$ denote the $k\times k$ system of pseudo-differential operators with entries B_{ij}.

If $i,j,l \in \{1,\dots,k\}$ the differential operator $B_{il}\chi_1A_{lj}$ is contained in $\Psi\mathrm{Op}^{\mu_j-\mu_i}$ and has symbol $\chi_1(x)B_{il}(x,\xi)A_{lj}(x,\xi)$ $(\mathrm{mod}\,\mathrm{Sym}^{\mu_j-\mu_i-1})$. Therefore the ijth entry of $\mathcal{B}\chi_1\mathcal{A}$ is contained in $\Psi\mathrm{Op}^{\mu_j-\mu_i}$ and, modulo an element of $\mathrm{Sym}^{\mu_j-\mu_i-1}$, has symbol

$$\sum_{l=1}^k \chi_1(x)B_{il}(x,\xi)A_{lj}(x,\xi) = \chi_1(x)\delta_{ij}A(x,\xi)B(x,\xi) \equiv \chi_1(x)\delta_{ij}.$$

Hence

$$\mathcal{B}\chi_1\mathcal{A} = \chi_1 + \mathcal{C} \tag{46}$$

where $\mathcal{C}$ is a $k\times k$ matrix of pseudo-differential operators whose ijth entry is contained in $\Psi\mathrm{Op}^{\mu_j-\mu_i-1}$. In particular, $\mathcal{B}$ and $\mathcal{C}$ define continuous maps

$$\mathcal{B} : E^\nu \to E^\mu \quad \text{and} \quad \mathcal{C} : (E^{-1})^\mu \to E^\mu. \tag{47}$$

Using the assumption of uniform ellipticity on $\mathcal{A}$ we can choose $\xi_0 \in \mathbb{R}^n$ so that $A(x,\xi_0) = \det\mathcal{A}(x,\xi_0)$ is bounded away from 0 uniformly for $x \in U$. For each $i,j \in \{1,\dots,k\}$ set $D_{ij}(x) = (-1)^{i+j}\chi(x)A^\dagger_{ji}(x,\xi_0)/A(x,\xi_0) \in C^\infty$ and let $\mathcal{D}$ denote the $k\times k$ matrix of multiplication operators whose ijth entry is given by

$$\begin{cases} D_{ij}(x) & \text{if } \mu_i = 0,\\ 0 & \text{if } \mu_i > 0. \end{cases} \tag{48}$$

Now, as matrices, $\mathcal{D}(x)\chi(x)\mathcal{A}(x,\xi_0) = \chi(x)(I-\pi_\mu)$ (where π_μ is simply the diagonal matrix with a 1 in the ith position if $\mu_i > 0$ and 0 otherwise). Since $(I-\pi_\mu)^2 = I-\pi_\mu$ while $\mathcal{A}(x,\xi)(I-\pi_\mu)$ is independent of ξ (n.b. if $\mu_j = 0$ then A_{ij} is a zeroth order differential operator) we have $\mathcal{D}\chi\mathcal{A}(I-\pi_\mu) = \chi(I-\pi_\mu)$ as differential operators. Coupled with the relation $\chi \prec \chi_2$ and the fact that $\mathcal{A}$ is a differential operator this gives

$$\chi = \chi\pi_\mu + \mathcal{D}\chi\mathcal{A}(I-\pi_\mu) = \chi\pi_\mu\chi_2 + \mathcal{D}\chi\chi_2\mathcal{A} - \mathcal{D}\chi\mathcal{A}\pi_\mu\chi_2. \tag{49}$$

On the other hand, condition (A2), (48) and the fact that $\nu_j \geq 0$ clearly imply that $\mathcal{D}$ defines a continuous map

$$\mathcal{D} : (E^{-1})^\nu \to (E^{-1})^\mu. \tag{50}$$

Using (46) and (49), the relations $\chi_1 \prec \chi \prec \chi_2$ and the fact that $\mathcal{A}$ is a differential operator, we get

$$\chi_1 = \mathcal{B}\chi_1\mathcal{A}\chi - \mathcal{C}\chi = \mathcal{B}\chi_1\chi_2\mathcal{A} - \mathcal{C}\chi\pi_\mu\chi_2 - \mathcal{C}\mathcal{D}\chi\chi_2\mathcal{A} + \mathcal{C}\mathcal{D}\chi\mathcal{A}\pi_\mu\chi_2. \tag{51}$$

We also observe that $\mathcal{A}$ defines a continuous map

$$\mathcal{A} : (E^{-1})^\mu \to (E^{-1})^\nu. \tag{52}$$

The first part of the result now follows if we apply (51) to u and use condition (A2), the continuous inclusion $E \hookrightarrow E^{-1}$ and the mapping properties given by (47), (50) and (52). On the other hand, we can combine norm estimates for the various continuous maps to get

$$\begin{aligned}\|\chi_1 u\|_{E^\mu} &\le C(\|\chi_2\mathcal{A}u\|_{E^\nu} + \|\pi_\mu\chi_2 u\|_{(E^{-1})^\mu} + \|\mathcal{D}\chi\chi_2\mathcal{A}u\|_{(E^{-1})^\mu} + \|\mathcal{D}\chi\mathcal{A}\pi_\mu\chi_2 u\|_{(E^{-1})^\mu})\\ &\le C(\|\chi_2\mathcal{A}u\|_{E^\nu} + \|\pi_\mu\chi_2 u\|_{(E^{-1})^\mu} + \|\chi_2\mathcal{A}u\|_{(E^{-1})^\nu} + \|\mathcal{A}\pi_\mu\chi_2 u\|_{(E^{-1})^\nu})\\ &\le C(\|\chi_2\mathcal{A}u\|_{E^\nu} + \|\pi_\mu\chi_2 u\|_{(E^{-1})^\mu})\end{aligned}$$

for all such u. ■

Suppose ψ is a diffeomorphism on $\mathbb{R}^n$ which is linear outside some compact set. By considering the standard rules for transforming principal symbols of differential operators it is clear that a system of differential operators $\mathcal{A}$ is elliptic on an open set U iff the operator $(\psi^{-1})^*\mathcal{A}\psi^*$ is elliptic on the open set $\psi(U)$. It follows that the definition of ellipticity can be applied to systems of differential operators on a smooth manifold. The next result is used to justify a later remark (n.b. S^{n-1} can be replaced by any compact manifold without boundary).

THEOREM 2.82. *Suppose $\mathcal{A}$ is a $k\times k$ system of differential operators on S^{n-1} of order (μ,ν). If $\mathcal{A}$ is elliptic and $u\in\mathscr{D}'(S^{n-1})$ satisfies $\mathcal{A}u\in E^\nu(S^{n-1})\cap F^\nu(S^{n-1})$ and $\pi_\mu u\in F^\mu(S^{n-1})$ for some admissible spaces E and F (on $\mathbb{R}^{n-1}$) then we also have $u\in E^\mu(S^{n-1})$. Furthermore,*

$$\|u\|_{E^\mu(S^{n-1})} \le C(\|\mathcal{A}u\|_{E^\nu(S^{n-1})} + \|\pi_\mu u\|_{F^\mu(S^{n-1})})$$

for all such u.

Proof. We can obtain the result with $F = E^{-1}$ by applying Theorem 2.81 on coordinate charts and patching the conclusion together with the help of observations similar to Lemma 2.10 and Remark 2.9. Induction, Lemma 2.23, Corollary 2.30 and Proposition 2.69 then complete the result for general F. ■

3. Model problems

3.1. Elliptic operators on Π^n and a priori estimates

DEFINITION 3.1. A differential operator $\mathcal{B} = \mathcal{B}(t,\omega,D_t,D_\omega)$ on Π^n is said to be *uniform* if its coefficients are contained in $C^\infty(\Pi^n)$.

Suppose B is a uniform scalar operator on Π^n of order m and E is an admissible space. By Lemma 2.32 and Proposition 2.50 (n.b. $C^\infty(\Pi^n) = \bigcap_{l\in\mathbb{N}_0}\mathsf{Z}_0C^l$) it is clear that B defines a continuous map $\mathsf{Z}_\beta E^m\to\mathsf{Z}_\beta E$ for any $\beta\in\mathbb{R}$. We also have

$$e^{\beta t}B(t,\omega,D_t,D_\omega) = B(t,\omega,D_t+i\beta,D_\omega)(e^{\beta t}\,\cdot\,) \tag{53}$$

for any $\beta\in\mathbb{R}$, where $B(t,\omega,D_t+i\beta,D_\omega)$ is once again a uniform operator on Π^n. This observation essentially allows us to take $\beta=0$ when studying the properties of the map $B:\mathsf{Z}_\beta E^m\to\mathsf{Z}_\beta E$ for a uniform operator B.

Let $\mathcal{B}$ be a uniform operator on Π^n of order (μ,ν) with entries B_{ij} for $i,j\in\{1,\dots,k\}$. As above, let $\det_{\mathcal{B}}(x,\xi)$ denote the determinant of the $k\times k$ matrix formed from the principal symbols of the operators B_{ij}. We shall say that $\mathcal{B}$ is a *uniform elliptic operator on Π^n* if we can find $C>0$ such that $|\det_{\mathcal{B}}(x,\xi)|\ge C|\xi|_{\Pi^n}^{[\mu-\nu]}$ for all $(x,\xi)\in T^*\Pi^n$, where $|\cdot|_{\Pi^n}$ denotes the norm on the fibres of the cotangent bundle $T^*\Pi^n$ given by the Riemannian metric on Π^n.

For $i, j \in \{1, \dots, k\}$, B_{ij} is a uniform scalar operator on Π^n of order $\mu_j - \nu_i$, so the above discussion implies that B_{ij} defines a continuous map $\mathsf{Z}_\beta E^{\mu_j} \to \mathsf{Z}_\beta E^{\nu_i}$ for any $\beta \in \mathbb{R}$. It follows that $\mathcal{B}$ defines a continuous map $\mathsf{Z}_\beta^\mu E \to \mathsf{Z}_\beta^\nu E$. We shall denote this map by $\mathcal{B}^{(E,\beta)}$ or $\mathcal{B}^{(\beta)}$ when we need to make clear the spaces $\mathcal{B}$ is acting between.

Lemma 3.2. *Suppose E is an admissible space and $\beta \in \mathbb{R}$. If $\mathcal{B}u \in \mathsf{Z}_\beta^\nu E$ for some $u \in \mathsf{Z}_\beta^\mu E^{-1}$ then we also have $u \in \mathsf{Z}_\beta^\mu E$. Furthermore,*

$$\|u\|_{\mathsf{Z}_\beta^\mu E} \le C(\|\mathcal{B}u\|_{\mathsf{Z}_\beta^\nu E} + \|\pi_\mu u\|_{\mathsf{Z}_\beta^\mu E^{-1}})$$

for all $u \in \mathsf{Z}_\beta^\mu E$.

Proof. Suppose $\chi_1 \in C^\infty(S^{n-1})$ with $\operatorname{supp}(\chi_1) \neq S^{n-1}$. Thus we can choose a chart (ψ, U) of S^{n-1} with $\operatorname{supp}(\chi_1) \subset U$. Let $(\Psi, \mathbb{R} \times U)$ be the corresponding chart of Π^n and choose further functions $\chi_2, \chi_3 \in C_0^\infty(U)$ with $\chi_3 \succ \chi_2 \succ \chi_1$. Also, define $\widetilde{\chi}_i \in C^\infty$ by $\widetilde{\chi}_i = \chi_i \circ \Psi^{-1}$ (extended by 0) for $i = 1, 2, 3$.

Using the transformation properties of the principal symbol of a differential operator under diffeomorphisms (see Section 18.1 in [H2], for example) it is straightforward to check that the operator $(\Psi^{-1})^*\chi_3\mathcal{B}\chi_3\Psi^*$ is uniformly elliptic on the open set $\widetilde{\chi}_3^{-1}(1/2, \infty)$. Theorem 2.81 then gives us the following: if $\widetilde{\chi}_2(\Psi^{-1})^*\chi_3\mathcal{B}\chi_3\Psi^*v \in E^\nu$ for some $v \in (E^{-1})^\mu$ then $\widetilde{\chi}_1 v \in E^\mu$ and we have an estimate of the form

$$\|\widetilde{\chi}_1 v\|_{E^\mu} \le C(\|\widetilde{\chi}_2(\Psi^{-1})^*\chi_3\mathcal{B}\chi_3\Psi^*v\|_{E^\nu} + \|\pi_\mu\widetilde{\chi}_2 v\|_{(E^{-1})^\mu}).$$

Putting $v = (\Psi^{-1})^*(\chi_3 u)$ and using the fact that $\mathcal{B}$ is a differential operator we get

$$\widetilde{\chi}_2(\Psi^{-1})^*\chi_3\mathcal{B}\chi_3\Psi^*v = (\Psi^{-1})^*\chi_2\chi_3\mathcal{B}\chi_3^2 u = (\Psi^{-1})^*\chi_2\mathcal{B}u.$$

Combining these observations with Lemma 2.10 we now get the following: if $\chi_2\mathcal{B}u \in \mathsf{Z}_0^\nu E$ for some $u \in \mathsf{Z}_0^\mu E^{-1}$ then $\chi_1 u \in \mathsf{Z}_0^\mu E$ and we have an estimate of the form

$$\|\chi_1 u\|_{\mathsf{Z}_0^\mu E} \le C(\|\chi_2\mathcal{B}u\|_{\mathsf{Z}_0^\nu E} + \|\pi_\mu\chi_2 u\|_{\mathsf{Z}_0^\mu E^{-1}}) \le C(\|\mathcal{B}u\|_{\mathsf{Z}_0^\nu E} + \|\pi_\mu u\|_{\mathsf{Z}_0^\mu E^{-1}}),$$

the second inequality coming from Remark 2.13 and the fact that $\chi_2 \in C^\infty(\Pi^n)$. The result now follows by allowing χ_1 to vary through the elements of a suitable partition of unity on S^{n-1}. ∎

We need to make a couple of general observations before we proceed to the next regularity result.

Remark 3.3. By Proposition 2.50 multiplication defines a continuous bilinear map $\mathsf{Z}_0C^l \times \mathsf{Z}_\beta E \to \mathsf{Z}_\beta E$ for all sufficiently large $l \in \mathbb{N}_0$. Using the translational invariance of the norm on $\mathsf{Z}_0C^l = C^l(\Pi^n)$ and the fact that $\phi_{ij} = \phi_i - \phi_{j+1}$ we can bound $\|\phi_{ij}\|_{\mathsf{Z}_0C^l}$ independently of $i, j \in \mathbb{Z} \cup \{\pm\infty\}$ with $i \le j$. It follows that multiplication by ϕ_{ij} defines a continuous map $\mathsf{Z}_\beta E \to \mathsf{Z}_\beta E$ whose operator norm can be bounded independently of i, j.

Remark 3.4. Suppose $\mathcal{B}$ is a uniform operator on Π^n of order (μ, ν) and $\phi \in C^\infty(\Pi^n)$. Then the ijth entry of the commutator $[\mathcal{B}, \phi]$ is the uniform scalar operator $[B_{ij}, \phi]$ of order at most $\mu_j - \nu_i - 1$; in particular, $[B_{ij}, \phi] = 0$ whenever $\nu_i \ge \mu_j$ whilst $[\mathcal{B}, \phi]$ defines a continuous map $\mathsf{Z}^\mu_\beta E^{-1} \to \mathsf{Z}^\nu_\beta E$. Furthermore, the operator norm of this map can be estimated by the $\mathsf{Z}_0 C^l$ norm of ϕ and the coefficients of $\mathcal{B}$ for some sufficiently large $l \in \mathbb{N}_0$.

Lemma 3.5. *Suppose E is an admissible space, $\beta \in \mathbb{R}$, $i, j \in \mathbb{Z} \cup \{\pm\infty\}$ with $i \le j$ and $l \in \mathbb{N}_0$. If $\phi_{i-l,j+l}\mathcal{B}u \in \mathsf{Z}^\nu_\beta E$ and $\phi_{i-l,j+l}u \in \mathsf{Z}^\mu_\beta E^{-l}$ for some $u \in \mathscr{D}'(\Pi^n)$ then we also have $\phi_{ij}u \in \mathsf{Z}^\mu_\beta E$. Furthermore,*

$$\|\phi_{ij}u\|_{\mathsf{Z}^\mu_\beta E} \le C(\|\phi_{i-l,j+l}\mathcal{B}u\|_{\mathsf{Z}^\nu_\beta E} + \|\pi_\mu \phi_{i-l,j+l}u\|_{\mathsf{Z}^\mu_\beta E^{-l}})$$

for all such u, where C is independent of i, j.

Proof. Induction clearly reduces the result to the case $l = 1$. Now suppose we have $\phi_{i-1,j+1}\mathcal{B}u \in \mathsf{Z}^\nu_\beta E$ and $\phi_{i-1,j+1}u \in \mathsf{Z}^\mu_\beta E^{-1}$ for some $u \in \mathscr{D}'(\Pi^n)$. Therefore $\phi_{ij}u \in \mathsf{Z}^\mu_\beta E^{-1}$ whilst $\phi_{ij} \prec \phi_{i-1,j+1}$ and $\mathcal{B}$ is a differential operator so

$$\mathcal{B}\phi_{ij}u = \phi_{ij}\phi_{i-1,j+1}\mathcal{B}u + [\mathcal{B}, \phi_{ij}]\phi_{i-1,j+1}u. \tag{54}$$

By Remark 3.3 we know that multiplication by ϕ_{ij} defines a continuous map $\mathsf{Z}^\nu_\beta E \to \mathsf{Z}^\nu_\beta E$ with norm bounded independently of i, j. On the other hand, Remark 3.4 implies $[\mathcal{B}, \phi_{ij}]$ maps $\mathsf{Z}^\mu_\beta E^{-1} \to \mathsf{Z}^\nu_\beta E$ continuously with operator norm bounded independently of i, j. Combining these observations with (54) and our original hypothesis we thus get $\mathcal{B}\phi_{ij}u \in \mathsf{Z}^\nu_\beta E$ with a norm estimate of the form

$$\|\mathcal{B}\phi_{ij}u\|_{\mathsf{Z}^\nu_\beta E} \le C(\|\phi_{i-1,j+1}\mathcal{B}u\|_{\mathsf{Z}^\nu_\beta E} + \|\pi_\mu \phi_{i-1,j+1}u\|_{\mathsf{Z}^\mu_\beta E^{-1}}),$$

where C is independent of i, j. Lemma 3.2 (applied to $\phi_{ij}u$) now implies $\phi_{ij}u \in \mathsf{Z}^\mu_\beta E$ and

$$\|\phi_{ij}u\|_{\mathsf{Z}^\mu_\beta E} \le C(\|\phi_{i-1,j+1}\mathcal{B}u\|_{\mathsf{Z}^\nu_\beta E} + \|\pi_\mu \phi_{i-1,j+1}u\|_{\mathsf{Z}^\mu_\beta E^{-1}} + \|\pi_\mu \phi_{ij}u\|_{\mathsf{Z}^\mu_\beta E^{-1}}),$$

where C is independent of i, j. A further application of Remark 3.3 clearly completes the result. ■

3.2. Adjoint operators. Suppose $\mathcal{B}$ is a uniform elliptic operator on Π^n of order (μ, ν) with entries B_{ij} for $i, j \in \{1, \dots, k\}$. Let $\mathcal{B}^*$ denote the formal adjoint of $\mathcal{B}$ (with respect to the volume measure $dt dS^{n-1}$ on Π^n); that is, $\mathcal{B}^*$ is the $k \times k$ system of differential operators with entries

$$(B^*)_{ij}(t, \omega, D_t, D_\omega) = (B_{ji}(t, \omega, D_t, D_\omega))^* \tag{55}$$

for $i, j = 1, \dots, k$. Define $m \in \mathbb{N}_0$ and vectors of integers $\overline{\mu}$ and $\overline{\nu}$ by

$$m = \max_i \mu_i \quad \text{and} \quad \overline{\mu}_i = m - \nu_i, \quad \overline{\nu}_i = m - \mu_i \quad \text{for } i = 1, \dots, k. \tag{56}$$

Therefore $\min_i \overline{\mu}_i \ge \min_i \overline{\nu}_i = 0$ (n.b. if the first inequality were not valid then we would have an i such that $\mu_j - \nu_i < 0$ for all $j \in \{1, \dots, k\}$; the ith row of $\mathcal{B}$ would then be 0, contradicting the assumption that $\mathcal{B}$ is elliptic). Furthermore,

$$\operatorname{ord}(B^*)_{ij} = \operatorname{ord} B_{ji} = \mu_i - \nu_j = \overline{\mu}_j - \overline{\nu}_i$$

whilst it is easily seen that $\det_{\mathcal{B}^*}(x, \xi) = \overline{\det_{\mathcal{B}}(x, \xi)}$ for all $(x, \xi) \in T^*\Pi^n$. It follows that $\mathcal{B}^*$ is a uniform elliptic operator on Π^n of order $(\overline{\mu}, \overline{\nu})$.

Let E be an admissible space and set $F = E_0^*$. By Lemma 2.45 and Proposition 2.69 we know that F is also an admissible space whilst

$$(\mathsf{Z}^{\mu}_{\beta}E_0)^* = \prod_{i=1}^{k} \mathsf{Z}_{-\beta}F^{-\mu_i} = \mathsf{Z}^{\overline{\nu}}_{-\beta}F^{-m} \quad \text{and} \quad (\mathsf{Z}^{\nu}_{\beta}E_0)^* = \prod_{i=1}^{k} \mathsf{Z}_{-\beta}F^{-\nu_i} = \mathsf{Z}^{\overline{\mu}}_{-\beta}F^{-m}$$

for any $\beta \in \mathbb{R}$. Therefore the adjoint of the map $\mathcal{B}^{(E_0,\beta)} : \mathsf{Z}^{\mu}_{\beta}E_0 \to \mathsf{Z}^{\nu}_{\beta}E_0$ is the map

$$(\mathcal{B}^{(E_0,\beta)})^* = (\mathcal{B}^*)^{(F^{-m},-\beta)} : \mathsf{Z}^{\overline{\mu}}_{-\beta}F^{-m} \to \mathsf{Z}^{\overline{\nu}}_{-\beta}F^{-m} \quad \text{where } F = E_0^*.$$

3.3. Operator pencils

Definition 3.6. A *model operator on Π^n* is a differential operator $\mathcal{B} = \mathcal{B}(\omega, D_t, D_\omega)$ on Π^n whose coefficients are independent of t. If $\mathcal{B}$ is also elliptic we shall refer to it as a *model elliptic operator on Π^n*.

Clearly, any model operator on Π^n is a uniform differential operator whilst any model elliptic operator is also a uniform elliptic operator. Now let $\mathcal{B}$ be a model elliptic operator on Π^n of order (μ, ν). For $i, j \in \{1, \ldots, k\}$ we can write the ijth entry of $\mathcal{B}$ in the form

$$B_{ij}(\omega, D_t, D_\omega) = \sum_{l=0}^{\mu_j-\nu_i} B_{ij}^{\mu_j-\nu_i-l}(\omega, D_\omega) D_t^l \tag{57}$$

where $B_{ij}^l(\omega, D_\omega)$ is a differential operator on S^{n-1} of order at most l.

For each $\lambda \in \mathbb{C}$ let $\mathfrak{B}(\lambda)$ be the differential operator on S^{n-1} defined by $\mathfrak{B}(\lambda)(\omega, D_\omega) = \mathcal{B}(\omega, \lambda, D_\omega)$; that is, $\mathfrak{B}(\lambda)$ is the $k \times k$ system of differential operators with entries given by

$$B_{ij}(\omega, \lambda, D_\omega) = \sum_{l=0}^{\mu_j-\nu_i} B_{ij}^{\mu_j-\nu_i-l}(\omega, D_\omega) \lambda^l. \tag{58}$$

Given any admissible space E on $\mathbb{R}^{n-1}$, the operator B_{ij}^l defines a continuous map $E^{\mu_j}(S^{n-1}) \to E^{\nu_i}(S^{n-1})$ for $l = 0, \ldots, \mu_j - \nu_i$. Thus $\mathfrak{B}$ defines an operator-valued function (or *operator pencil*)

$$\mathfrak{B} : \mathbb{C} \to \mathscr{L}(E^{\mu}(S^{n-1}), E^{\nu}(S^{n-1})). \tag{59}$$

The *spectrum* $\sigma(\mathfrak{B})$ of the operator pencil $\mathfrak{B}$ is defined to be the set of all $\lambda \in \mathbb{C}$ for which $\mathfrak{B}(\lambda) : E^{\mu}(S^{n-1}) \to E^{\nu}(S^{n-1})$ is not an isomorphism. Suppose $\lambda_0 \in \sigma(\mathfrak{B})$. Any non-zero element in the kernel of $\mathfrak{B}(\lambda_0)$ is called an *eigenfunction* of $\mathfrak{B}$ and the dimension of the kernel is the *geometric multiplicity* of λ_0. A collection of functions $\phi_0, \ldots, \phi_{M-1} \in E^{\mu}(S^{n-1})$ is called a *Jordan chain* corresponding to $\lambda_0 \in \sigma(\mathfrak{B})$ iff ϕ_0 is an eigenfunction corresponding to λ_0 and the meromorphic function $\Phi(\lambda)$ defined by

$$\Phi(\lambda) = \sum_{j=0}^{M-1} \frac{\phi_j}{(\lambda - \lambda_0)^{M-j}} \tag{60}$$

satisfies

$$\mathfrak{B}(\lambda)\Phi(\lambda) = O(1) \quad \text{as } \lambda \to \lambda_0. \tag{61}$$

It can be seen that this is equivalent to the condition

$$\sum_{j=0}^{M'} \frac{1}{j!}\,\partial_\lambda^j \mathfrak{B}(\lambda_0)\,\phi_{M'-j} = 0 \tag{62}$$

for $M' = 0, \dots, M-1$. The functions $\phi_1, \dots, \phi_{M-1}$ are called *generalised eigenfunctions*. The *algebraic multiplicity* of λ_0 is defined to be the dimension of the space of all functions of the form (60) which satisfy (61).

REMARK 3.7. It is straightforward to check that $\mathfrak{B}(\lambda)$ is an elliptic operator of order (μ, ν) on S^{n-1} for any $\lambda \in \mathbb{C}$ (n.b. the principal symbol of each entry of $\mathfrak{B}(\lambda)$ is independent of λ). Standard elliptic regularity results (see Theorem 2.82) then imply that the spectrum, eigenfunctions and generalised eigenfunctions of the operator pencil (59) are independent of the admissible space E. Furthermore, the eigenfunctions and generalised eigenfunctions are all contained in $C^\infty(S^{n-1})$.

Using the a priori estimates for the elliptic operator $\mathcal{B}$ on Π^n and general results for Fredholm operator pencils we can obtain general information about the structure of the spectrum of $\mathfrak{B}$. The following result was obtained in [AN] (see Chapter V) and [AV] for scalar operators, and appears in Section 1.2.1 of [NP] for systems of operators (see also [GGK] or Appendix A to [KM] for details about the algebraic multiplicities of the isolated points in $\sigma(\mathfrak{B})$).

THEOREM 3.8. *The spectrum of* $\mathfrak{B}$ *consists of isolated points of finite algebraic multiplicity. Furthermore, there exist constants* $C_1, C_2 > 0$ *such that*

$$\sigma(\mathfrak{B}) \subset \{\lambda \in \mathbb{C} \mid |\mathrm{Re}\,\lambda| \le C_1 |\mathrm{Im}\,\lambda| + C_2\}.$$

REMARK 3.9. Theorem 3.8 implies that $\mathrm{Im}\,\sigma(\mathfrak{B})$ (the projection of $\sigma(\mathfrak{B})$ onto the imaginary axis) consists of isolated points. Furthermore, for $\gamma \in \mathrm{Im}\,\sigma(\mathfrak{B})$, the total algebraic multiplicity of all those $\lambda \in \sigma(\mathfrak{B})$ with $\mathrm{Im}\,\lambda = \gamma$ is finite.

A system of Jordan chains

$$\{\phi_{j,0}, \dots, \phi_{j,M_j-1} \mid j = 1, \dots, J\} \tag{63}$$

is called *canonical* if $\{\phi_{1,0}, \dots, \phi_{J,0}\}$ forms a basis for $\mathrm{Ker}\,\mathfrak{B}(\lambda_0)$ (i.e. the geometric eigenspace of λ_0), $M_1 \ge \dots \ge M_J$ and $M_1 + \dots + M_J = M$, the algebraic multiplicity of λ_0. It follows that J is simply the geometric multiplicity of λ_0 whilst $M_1, \dots, M_J$ are known as the *partial algebraic multiplicities* of λ_0. It is well known (see [GGK] or Appendix A to [KM], for example) that a canonical system of Jordan chains exists for any $\lambda_0 \in \sigma(\mathfrak{B})$ of finite algebraic multiplicity.

Define functions $u_{j,m} \in C^\infty(\Pi^n)_{\mathrm{loc}}$ by

$$u_{j,m}(t,\omega) = e^{i\lambda_0 t} \sum_{l=0}^{m} \frac{1}{l!}\,(it)^l\,\phi_{j,m-l}(\omega) \tag{64}$$

for $j = 1, \dots, J$, $m = 0, \dots, M_j - 1$. This collection of functions forms a basis for the set of "power-exponential" solutions of the equation $\mathfrak{B}(\lambda_0)u = 0$.

Let $\mathfrak{B}^*$ denote the operator pencil associated with the model elliptic operator $\mathcal{B}^*$ (the formal adjoint of $\mathcal{B}$). For any $\lambda \in \mathbb{C}$ and $i, j \in \{1, \dots, k\}$ the ijth entry of $\mathfrak{B}^*(\lambda)$ is thus given by $(B^*)_{ij}(\omega,\lambda,D_\omega)$, where $(B^*)_{ij}(\omega,D_t,D_\omega)$ is the operator given by (55). On the other hand, by taking formal adjoints of (57) and (58) (on Π^n and S^{n-1} respectively) we have

$$(B^*)_{ij}(\omega,\lambda,D_\omega) = (B_{ji}(\omega,\overline{\lambda},D_\omega))^*.$$

Therefore $\mathfrak{B}^*(\lambda) = (\mathfrak{B}(\overline{\lambda}))^*$, the formal adjoint of the operator $\mathfrak{B}(\overline{\lambda})$ on S^{n-1}. It follows that $\lambda_0 \in \sigma(\mathfrak{B})$ iff $\overline{\lambda}_0 \in \sigma(\mathfrak{B}^*)$ with full agreement of geometric, algebraic and partial algebraic multiplicities. In fact, given a canonical system of Jordan chains (63) corresponding to $\lambda_0 \in \sigma(\mathfrak{B})$ we can find a unique canonical system of Jordan chains

$$\{\psi_{j,0},\dots,\psi_{j,M_j-1} \mid j = 1,\dots,J\} \tag{65}$$

corresponding to $\overline{\lambda}_0 \in \sigma(\mathfrak{B}^*)$ which satisfy the biorthogonality conditions

$$\sum_{l=0}^{m}\sum_{l'=0}^{m'} \frac{1}{(l+l'+1)!}\langle \partial_\lambda^{l+l'+1}\mathfrak{B}(\lambda_0)\phi_{j,m-l},\psi_{j',m'-l'}\rangle_{S^{n-1}} = \delta_{j,j'}\delta_{M_j-1-m,m'} \tag{66}$$

for $j,j' = 1,\dots,J$, $m = 0,\dots,M_j-1$ and $m' = 0,\dots,M_{j'}-1$ (see [GGK] or Appendix A to [KM], for example; n.b. by (62) and a similar expression for the $\psi_{j,m}$'s it is possible to rewrite the biorthogonality conditions in a number of different ways—in particular, it is enough to consider (66) for $m = M_j-1$, $j,j' = 1,\dots,J$ and $m' = 0,\dots,M_{j'}-1$).

As above, the power-exponential solutions of $\mathfrak{B}^*(\overline{\lambda}_0)v = 0$ can be expanded in terms of a basis given by the functions

$$v_{j,m}(t,\omega) = e^{i\overline{\lambda}_0 t}\sum_{l=0}^{m}\frac{1}{l!}\,(it)^l\,\psi_{j,m-l}(\omega) \tag{67}$$

for $j = 1,\dots,J$, $m = 0,\dots,M_j-1$.

3.4. Isomorphisms. The study of Fredholm properties of a certain class of elliptic operators on $\mathbb{R}^n$ (to be introduced in Section 4.1) can be largely reduced to the study of model elliptic operators on Π^n. Theorems 3.10 and 3.11 provide the key results concerning the latter. These (or closely related) results have been established by a number of authors, at least for the admissible spaces $E = H^{p,k}$ with $p \in (1,\infty)$ and $k \in \mathbb{Z}$, and $E = \mathcal{C}_0^s$ with $s \in \mathbb{R}^+ \setminus \mathbb{N}$ (in particular, see [AN], [AV], [K2] and [MP]; see also [KM] where such results are obtained as part of a general theory of differential equations with operator coefficients). Below we give a general argument to derive Theorems 3.10 and 3.11 for an arbitrary admissible space E from the case $E = L^2$.

THEOREM 3.10. *Let E be an admissible space, let $\mathcal{B}$ be a model elliptic operator on Π^n of order (μ,ν) and define $\mathfrak{B}$ to be the associated operator pencil. If $\beta \in \mathbb{R}\setminus\operatorname{Im}\sigma(\mathfrak{B})$ then $\mathcal{B} : \mathsf{Z}_\beta^\mu E \to \mathsf{Z}_\beta^\nu E$ is an isomorphism.*

THEOREM 3.11. *Let E, $\mathcal{B}$ and $\mathfrak{B}$ be as in Theorem 3.10. Suppose $\beta_1,\beta_2 \in \mathbb{R}$ satisfy $\beta_1,\beta_2 \notin \operatorname{Im}\sigma(\mathfrak{B})$ and $\beta_1 < \beta_2$. Set $\Sigma = \{\lambda \in \sigma(\mathfrak{B}) \mid \operatorname{Im}\lambda \in [\beta_1,\beta_2]\}$ and, for each $\lambda \in \Sigma$, let $\{\phi^\lambda_{j,0},\dots,\phi^\lambda_{j,M_{\lambda,j}-1} \mid j = 1,\dots,J_\lambda\}$ and $\{\psi^\lambda_{j,0},\dots,\psi^\lambda_{j,M_{\lambda,j}-1} \mid j = 1,\dots,J_\lambda\}$ be canonical systems of Jordan chains corresponding to $\lambda \in \sigma(\mathfrak{B})$ and $\overline{\lambda} \in \sigma(\mathfrak{B}^*)$ respectively, which satisfy the biorthogonality condition (66). Define $u^\lambda_{j,m}$ and $v^\lambda_{j,m}$ to be the corresponding power-exponential functions given by (64) and (67) respectively. Now* *let* $f \in \mathsf{Z}^\nu_{\beta_1}E \cap \mathsf{Z}^\nu_{\beta_2}E$ *and let* $u^{(i)} \in \mathsf{Z}^\mu_{\beta_i}E$ *be the corresponding solutions of* $\mathcal{B}u = f$ *for* $i = 1, 2$. *Then we have*

$$u^{(1)}(t,\omega) - u^{(2)}(t,\omega) = \sum_{\lambda\in\Sigma}\sum_{j=1}^{J_\lambda}\sum_{m=0}^{M_{\lambda,j}-1} c^\lambda_{j,m}(f)\, u^\lambda_{j,M_{\lambda,j}-1-m}(t,\omega) \tag{68}$$

where the coefficient functions are given by

$$c^\lambda_{j,m}(f) = \langle f, iv^\lambda_{j,m}\rangle_{\Pi^n} \tag{69}$$

for $\lambda \in \Sigma$, $j = 1, \ldots, J_\lambda$ *and* $m = 0, \ldots, M_{\lambda,j} - 1$.

We begin by using a straightforward regularity argument to obtain Theorem 3.10 for the spaces H^s, $s \in \mathbb{R}$, from the case $s = 0$.

Proposition 3.12. *Theorem* 3.10 *holds with* $E = H^s$ *for any* $s \in \mathbb{R}$.

Proof. Theorem 3.10 is known to hold for $E = L^2 = H^0$; see Theorem 3.1.1 in [NP], for example. Duality and induction now reduce our task to proving that if Theorem 3.10 holds for $E = H^s$, $s \in \mathbb{R}$, then it also holds for $E = H^{s+\delta}$ where $\delta \in [0, 1]$.

Denote the maps $\mathcal{B} : \mathsf{Z}^\mu_\beta H^s \to \mathsf{Z}^\nu_\beta H^s$ and $\mathcal{B} : \mathsf{Z}^\mu_\beta H^{s+\delta} \to \mathsf{Z}^\nu_\beta H^{s+\delta}$ by $\mathcal{B}_0$ and $\mathcal{B}_\delta$ respectively, and assume $\mathcal{B}_0$ is an isomorphism. Since $\mathsf{Z}^\mu_\beta H^{s+\delta} \subseteq \mathsf{Z}^\mu_\beta H^s$, $\mathcal{B}_\delta$ must also be injective. Now let $f \in \mathsf{Z}^\mu_\beta H^{s+\delta} \subseteq \mathsf{Z}^\mu_\beta H^s$ and set $u = (\mathcal{B}_0)^{-1}f$. Therefore $u \in \mathsf{Z}^\mu_\beta H^s \subseteq \mathsf{Z}^\mu_\beta H^{s+\delta-1}$ whilst $\mathcal{B}u = f \in \mathsf{Z}^\nu_\beta H^{s+\delta}$. Lemma 3.2 then implies $u \in \mathsf{Z}^\mu_\beta H^{s+\delta}$, thereby establishing the surjectivity of $\mathcal{B}_\delta$. The Open Mapping Theorem now shows that $\mathcal{B}_\delta$ is an isomorphism. ∎

Let $\mathscr{O}^{0,1}$ denote the set of those scalar differential operators on Π^n of order 0 or 1 which can be written as the linear combination of a model operator and (multiplication by) t. In what follows we use the notation $[P, Q] = PQ - QP$ for the commutator of operators P and Q.

Remark 3.13. (i) If $P \in \mathscr{O}^{0,1}$ and $\chi \in C^\infty(S^{n-1})$ then $[P, \chi] \in C^\infty(S^{n-1})$; that is, the operator $[P, \chi]$ is given as multiplication by some function in $C^\infty(\Pi^n)$ which is independent of t. Furthermore, $[P, \chi] = 0$ if $\operatorname{ord} P = 0$.

(ii) If $m \in \mathbb{N}$ and $P_1, \ldots, P_m \in \mathscr{O}^{0,1}$ then $\mathcal{B}' := [P_1, \ldots, [P_m, \mathcal{B}] \ldots]$ is a $k \times k$ matrix of differential operators on Π^n whose coefficients are independent of t and whose ijth entry has order at most $\mu_j - \nu_i - m'$ where

$$m' = \sum_{k=1}^m (1 - \operatorname{ord} P_k).$$

It follows that $\mathcal{B}'$ defines a continuous map $\mathsf{Z}^\mu_\beta E \to \mathsf{Z}^\nu_\beta E^{m'}$ for any admissible space E and $\beta \in \mathbb{R}$.

Let $\beta \in \mathbb{R} \setminus \operatorname{Im}\sigma(\mathfrak{B})$ and define $\mathcal{C}$ to be the continuous map $\mathsf{Z}_\beta\mathscr{S} \to \mathsf{Z}_\beta\mathscr{S}'$ obtained by taking the restriction of the inverse of the isomorphism $\mathcal{B} : \mathsf{Z}^\mu_\beta L^2 \to \mathsf{Z}^\nu_\beta L^2$. For any $m \in \mathbb{N}_0$ let $\mathscr{O}^m_{\mathcal{C}}$ denote the set of operators obtained by taking finite sums of operators of the form

$$\chi_1\mathcal{C}\mathcal{Q}_1\mathcal{C}\mathcal{Q}_2 \ldots \mathcal{C}\mathcal{Q}_l\mathcal{C}\chi_2 \tag{70}$$

where $\chi_1, \chi_2 \in C^\infty(S^{n-1})$, $l \in \mathbb{N}_0$ and, for $i = 1, \ldots, l$, we can write

$$\mathcal{Q}_i = [P_{i1}, \ldots, [P_{im_i}, \mathcal{B}] \ldots]$$

for some $m_i \in \mathbb{N}_0$ and $P_{i1}, \dots, P_{im_i} \in \mathscr{O}^{0,1}$ with

$$\sum_{i=1}^{l} \sum_{k=1}^{m_i} (1 - \operatorname{ord} P_{ik}) = m.$$

Remark 3.14. For any $s \in \mathbb{R}$, Lemma 2.34 and Proposition 3.12 imply that $\mathcal{C}$ has a unique extension to a continuous map $\mathsf{Z}^{\nu}_{\beta} H^s \to \mathsf{Z}^{\mu}_{\beta} H^s$. This observation can be coupled with Remark 3.13(ii) to show that the operator given by the formal expression (70) defines a continuous map $\mathsf{Z}^{\nu}_{\beta} H^s \to \mathsf{Z}^{\mu}_{\beta} H^{s+m}$ for any $s \in \mathbb{R}$. Taking linear combinations we deduce that any element of $\mathscr{O}^m_{\mathcal{C}}$ has similar mapping properties.

Lemma 3.15. *Let* $\chi_1, \chi_2 \in C^\infty(S^{n-1})$, $m \in \mathbb{N}$ *and* $P_1, \dots, P_m \in \mathscr{O}^{0,1}$. *Set*

$$m' = \sum_{k=1}^{m} (1 - \operatorname{ord} P_k) \in \mathbb{N}_0.$$

Then the operator

$$[P_1, \dots, [P_m, \chi_1 \mathcal{C} \chi_2] \dots] \tag{71}$$

has a unique extension to a continuous map $\mathsf{Z}^{\nu}_{\beta} H^s \to \mathsf{Z}^{\mu}_{\beta} H^{s+m'}$ *for any* $s \in \mathbb{R}$.

The operator $\mathcal{C}$ defines a continuous map $\mathsf{Z}_\beta \mathscr{S} \to \mathsf{Z}_\beta \mathscr{S}'$ whilst any P_i defines continuous maps $\mathsf{Z}_\beta \mathscr{S} \to \mathsf{Z}_\beta \mathscr{S}$ and $\mathsf{Z}_\beta \mathscr{S}' \to \mathsf{Z}_\beta \mathscr{S}'$. It follows that the operator (71) can be initially defined as a continuous map $\mathsf{Z}_\beta \mathscr{S} \to \mathsf{Z}_\beta \mathscr{S}'$.

Proof of Lemma 3.15. By Lemma 2.34 and Remark 3.14 it suffices to prove that the operator (71) is contained in $\mathscr{O}^{m'}_{\mathcal{C}}$. Since we obviously have $\chi_1 \mathcal{C} \chi_2 \in \mathscr{O}^0_{\mathcal{C}}$ the following Claim completes the result.

Claim: *If* $\mathcal{R} \in \mathscr{O}^m_{\mathcal{C}}$ *and* $P \in \mathscr{O}^{0,1}$ *then* $[P, \mathcal{R}] \in \mathscr{O}^{m+1-\operatorname{ord} P}_{\mathcal{C}}$. Clearly, it suffices to prove the Claim assuming $\mathcal{R}$ is given by (70). Then

$$\begin{aligned} [P, \mathcal{R}] &= [P, \chi_1] \mathcal{C} \mathcal{Q}_1 \dots \mathcal{Q}_l \mathcal{C} \chi_2 + \sum_{i=1}^{l+1} \chi_1 \mathcal{C} \mathcal{Q}_1 \dots \mathcal{Q}_{i-1} [P, \mathcal{C}] \mathcal{Q}_i \dots \mathcal{Q}_l \mathcal{C} \chi_2 \\ &\quad + \sum_{i=1}^{l} \chi_1 \mathcal{C} \mathcal{Q}_1 \dots \mathcal{C} [P, \mathcal{Q}_i] \mathcal{C} \dots \mathcal{Q}_l \mathcal{C} \chi_2 + \chi_1 \mathcal{C} \mathcal{Q}_1 \dots \mathcal{Q}_l \mathcal{C} [P, \chi_2]. \end{aligned} \tag{72}$$

Now, by Remark 3.13(i), we have $[P, \chi_1] = \chi_1'$ and $[P, \chi_2] = \chi_2'$ for some $\chi_1', \chi_2' \in C^\infty(S^{n-1})$, with $\chi_1' = \chi_2' = 0$ if $\operatorname{ord} P = 0$. On the other hand, $[P, \mathcal{C}] = -\mathcal{C}[P, \mathcal{B}]\mathcal{C}$. Therefore each term in (72) is contained in $\mathscr{O}^{m+1-\operatorname{ord} P}_{\mathcal{C}}$. ■

Suppose (ψ, U) is a chart for S^{n-1} and let $(\Psi, \mathbb{R} \times U)$ be the corresponding chart for Π^n. Also, choose $\chi_1, \chi_2 \in C_0^\infty(U)$; we shall use the same letters to denote the extensions of these functions to Π^n which are independent of t. Define a $k \times k$ matrix of operators by $\mathcal{D} := (\Psi^{-1})^* e^{\beta t} \chi_1 \mathcal{C} \chi_2 e^{-\beta t} \Psi^*$.

Remark 3.16. Suppose R is a scalar operator on Π^n which is supported on $\mathbb{R} \times U$ (in the sense that $R = \chi R \chi$ for some $\chi \in C_0^\infty(U)$). If E and F are admissible spaces then Lemma 2.10 and (5) show that R defines a continuous map $R : \mathsf{Z}_\beta E \to \mathsf{Z}_\beta F$ iff $(\Psi^{-1})^* e^{\beta t} R e^{-\beta t} \Psi^*$ defines a continuous map $E \to F$. Clearly, similar statements hold when E or F is replaced by $\mathscr{S}$ or $\mathscr{S}'$, or when R is replaced by a matrix of operators.

LEMMA 3.17. *For each $i, j \in \{1, \dots, n\}$, the ijth entry of the operator $\mathcal{D}$ is contained in $\Psi\mathrm{Op}^{\nu_j-\mu_i}$.*

Proof. Since we know that $\mathcal{D}$ defines a continuous map $\mathscr{S} \to \mathscr{S}'$ (see Remark 3.16) we can apply a characterisation of pseudo-differential operators given in [Bea]. Let $m \in \mathbb{N}$ and suppose, for $k = 1, \dots, m$, that Q_k is either the operator D_i or (multiplication by) x_i for some $i \in \{1, \dots, n\}$. Let $m' \in \mathbb{N}_0$ be the number of Q_k which are of zero order. Choose $\chi_0 \in C^\infty(U)$ with $\chi_0 \succ \chi_1, \chi_2$ and define $\widetilde{\chi}_i \in C^\infty$ by $\widetilde{\chi}_i = \chi_i \circ \Psi^{-1}$ (extended by 0) for $i = 0, 1, 2$. Now the fact that $Q_1, \dots, Q_m$ are differential operators implies

$$[Q_1, \dots, [Q_m, \mathcal{D}] \dots] = [\widetilde{\chi}_0 Q_1 \widetilde{\chi}_0, \dots [\widetilde{\chi}_0 Q_m \widetilde{\chi}_0, \mathcal{D}] \dots].$$

Setting $P_k = e^{-\beta t}\Psi^* \widetilde{\chi}_0 Q_k \widetilde{\chi}_0 (\Psi^{-1})^* e^{\beta t}$ for $k = 1, \dots, m$ allows us to rewrite this as

$$[Q_1, \dots, [Q_m, \mathcal{D}] \dots] = (\Psi^{-1})^* e^{\beta t}[P_1, \dots, [P_m, \chi_1 \mathcal{C} \chi_2] \dots] e^{-\beta t}\Psi^*. \tag{73}$$

If Q_k is not given as multiplication by x_1 then P_k is a model operator on Π^n of order $\operatorname{ord} Q_k$, and thus $P_k \in \mathscr{O}^{0,1}$. On the other hand, if $Q_k = x_1$ then $P_k = \chi_0^2 t$; although this operator is not contained in $\mathscr{O}^{0,1}$, the fact that $P_1, \dots, P_m$ are differential operators whilst $\chi_0 \succ \chi_1, \chi_2$ means that (73) will be unchanged if we replace P_k by the operator t. It follows that we may assume (73) holds with $P_1, \dots, P_m \in \mathscr{O}^{0,1}$. By combining (73) with Lemma 3.15 and Remark 3.16 we then get that the ijth entry of $[Q_1, \dots, [Q_m, \mathcal{D}] \dots]$ defines a continuous map $H^{s+\nu_j} \to H^{s+\mu_i+m'}$ for any $s \in \mathbb{R}$. Theorem 2.9 in [Bea] now completes the result. ∎

Proof of Theorem 3.10. Let $\{\chi_i\}_{i\in I}$ be a finite partition of unity for S^{n-1} such that $\operatorname{supp}(\chi_i) \cup \operatorname{supp}(\chi_j) \neq S^{n-1}$ for any $i, j \in I$. Thus we can write

$$\mathcal{C} = \sum_{i,j\in I} \chi_i \mathcal{C} \chi_j. \tag{74}$$

Now let $i, j \in I$ and choose a chart (ψ, U) of S^{n-1} with $\operatorname{supp}(\chi_i) \cup \operatorname{supp}(\chi_j) \subset U$. Let $(\Psi, \mathbb{R} \times U)$ be the corresponding chart of Π^n. Set $\mathcal{D} = (\Psi^{-1})^* e^{\beta t} \chi_i \mathcal{C} \chi_j e^{-\beta t} \Psi^*$, initially defined as a continuous map $\mathscr{S} \to \mathscr{S}'$. By Lemma 3.17, Remark 2.60 and Proposition 2.68, $\mathcal{D}$ defines a continuous map $\mathscr{S} \to \mathscr{S}$ which extends to give continuous maps $\mathscr{S}' \to \mathscr{S}'$ and $E^\nu \to E^\mu$ for any admissible space E. Furthermore, any two of these extensions agree on functions common to their domains. By (74) and Remark 3.16 it follows that $\mathcal{C}$ defines a continuous map $\mathcal{C}^{(\mathscr{S})} : \mathsf{Z}_\beta \mathscr{S} \to \mathsf{Z}_\beta \mathscr{S}$ which has continuous extensions $\mathcal{C}^{(\mathscr{S}')} : \mathsf{Z}_\beta \mathscr{S}' \to \mathsf{Z}_\beta \mathscr{S}'$ and $\mathcal{C}^{(E)} : \mathsf{Z}^\nu_\beta E \to \mathsf{Z}^\mu_\beta E$ for any admissible space E. Furthermore,

$$\mathcal{C}^{(E)} f = \mathcal{C}^{(\mathscr{S}')} f \quad \text{for any } f \in \mathsf{Z}^\nu_\beta E. \tag{75}$$

From the definition of $\mathcal{C}$ it clearly follows that the compositions $\mathcal{C}^{(\mathscr{S})}\mathcal{B}^{(\mathscr{S})}$ and $\mathcal{B}^{(\mathscr{S})}\mathcal{C}^{(\mathscr{S})}$ are both equal to the identity on $\mathsf{Z}_\beta \mathscr{S}$; that is, $\mathcal{B}^{(\mathscr{S})}$ is an isomorphism with inverse $\mathcal{C}^{(\mathscr{S})}$. On the other hand, $\mathcal{B}^{(\mathscr{S}')}$ and $\mathcal{C}^{(\mathscr{S}')}$ are extensions of $\mathcal{B}^{(\mathscr{S})}$ and $\mathcal{C}^{(\mathscr{S})}$ so the compositions $\mathcal{C}^{(\mathscr{S}')}\mathcal{B}^{(\mathscr{S}')}$ and $\mathcal{B}^{(\mathscr{S}')}\mathcal{C}^{(\mathscr{S}')}$ must also give the identity map when restricted to $\mathsf{Z}_\beta \mathscr{S}$. Lemma 2.73 then implies that $\mathcal{B}^{(\mathscr{S}')}$ is an isomorphism with inverse $\mathcal{C}^{(\mathscr{S}')}$. Now let E be any admissible space. Using (75) and the obvious fact that $\mathcal{B}^{(E)} u = \mathcal{B}^{(\mathscr{S}')} u$ for any $u \in \mathsf{Z}^\mu_\beta E$, we finally see that $\mathcal{B}^{(E)}$ is an isomorphism with inverse $\mathcal{C}^{(E)}$. ∎

We now turn our attention to Theorem 3.11, firstly establishing that the coefficient functions $c^\lambda_{j,m}$ given by (69) are in fact well defined.

LEMMA 3.18. *Consider the notation of Theorems* 3.10 *and* 3.11 *and let* $\lambda \in \Sigma$, $j \in \{1, \dots, J_\lambda\}$ *and* $m \in \{0, \dots, M_{\lambda,j} - 1\}$. *Then the function* $c^\lambda_{j,m}$ *given by* (69) *defines a continuous map* $\mathsf{Z}_{\beta_1}\mathscr{S}' \cap \mathsf{Z}_{\beta_2}\mathscr{S}' \to \mathbb{C}$. *By the continuous inclusions* $\mathsf{Z}^\nu_{\beta_j}E \hookrightarrow \mathsf{Z}_{\beta_j}\mathscr{S}'$ *for* $j = 1, 2$ (*see Remark* 2.14), *it follows that* $c^\lambda_{j,m}$ *also defines a continuous map* $\mathsf{Z}^\nu_{\beta_1}E \cap \mathsf{Z}^\nu_{\beta_2}E \to \mathbb{C}$.

The topology on $\mathsf{Z}^\nu_{\beta_1}E \cap \mathsf{Z}^\nu_{\beta_2}E$ is smallest making the inclusion $\mathsf{Z}^\nu_{\beta_1}E \cap \mathsf{Z}^\nu_{\beta_2}E \hookrightarrow \mathsf{Z}^\nu_{\beta_j}E$ continuous for $j = 1, 2$. A similar remark applies to $\mathsf{Z}_{\beta_1}\mathscr{S}' \cap \mathsf{Z}_{\beta_2}\mathscr{S}'$.

Proof of Lemma 3.18. Using (67) we can write

$$v^\lambda_{j,m}(t, \omega) = e^{i\bar{\lambda}t} \sum_{l=0}^{m} \frac{1}{l!}\,(it)^l\,\psi^\lambda_{j,m-l}(\omega),$$

where $\psi^\lambda_{j,0}, \dots, \psi^\lambda_{j,m} \in C^\infty(S^{n-1})$ (the fact that these functions are $\mathbb{C}^k$-valued is unimportant in what follows and will not be mentioned explicitly). Now, by assumption, $\operatorname{Im}\lambda \in (\beta_1, \beta_2)$ so $\operatorname{Re}(i\bar{\lambda} - \beta_2) < 0$ and $\operatorname{Re}(i\bar{\lambda} - \beta_1) > 0$. Since $\operatorname{supp}(\phi_0) \subseteq (-1, +\infty)$ and $\operatorname{supp}(1 - \phi_0) \subseteq (-\infty, 0)$ it follows that

$$\phi_0 v^\lambda_{j,m} \in \mathsf{Z}_{-\beta_2}\mathscr{S} \quad \text{and} \quad (1 - \phi_0)v^\lambda_{j,m} \in \mathsf{Z}_{-\beta_1}\mathscr{S}.$$

On the other hand, for any $f \in \mathsf{Z}_{\beta_1}\mathscr{S}' \cap \mathsf{Z}_{\beta_2}\mathscr{S}'$, (69) can be rewritten as

$$c^\lambda_{j,m}(f) = \langle f, i\phi_0 v^\lambda_{j,m}\rangle_{\Pi^n} + \langle f, i(1 - \phi_0)v^\lambda_{j,m}\rangle_{\Pi^n}.$$

The result now follows from the continuity of the dual pairings $\mathsf{Z}_{\beta_j}\mathscr{S}' \times \mathsf{Z}_{-\beta_j}\mathscr{S} \to \mathbb{C}$ for $j = 1, 2$ (see Remark 2.44). ∎

Proof of Theorem 3.11. From the proof of Theorem 3.10 we know that, for $i = 1, 2$, the inverse of $\mathcal{B}^{(E,\beta_i)}$ is the restriction of the inverse of $\mathcal{B}^{(\mathscr{S}',\beta_i)}$ to $\mathsf{Z}^\nu_\beta E$. It therefore suffices to prove the result with E replaced by $\mathscr{S}'$. Now Theorem 3.11 holds with $E = L^2$; see Theorems 3.1.4 and 3.2.1 in [NP], for example. It follows that (68) is valid for all $f \in \mathsf{Z}_{\beta_1}\mathscr{S} \cap \mathsf{Z}_{\beta_2}\mathscr{S}$. Lemma 3.18 together with the following Claim thus completes the proof.

Claim: *The set* $\mathsf{Z}_{\beta_1}\mathscr{S} \cap \mathsf{Z}_{\beta_2}\mathscr{S}$ *is dense in* $\mathsf{Z}_{\beta_1}\mathscr{S}' \cap \mathsf{Z}_{\beta_2}\mathscr{S}'$. Let $f \in \mathsf{Z}_{\beta_1}\mathscr{S}' \cap \mathsf{Z}_{\beta_2}\mathscr{S}'$. Therefore $(1 - \phi_1)f = \phi_{-\infty 0}f \in \mathsf{Z}_{\beta_1}\mathscr{S}'$ and $\phi_1 f \in \mathsf{Z}_{\beta_2}\mathscr{S}'$. By Lemma 2.73 we can then find sequences $\{f_{ji}\}_{i\in\mathbb{N}} \subset \mathsf{Z}_{\beta_j}\mathscr{S}$ for $j = 1, 2$ such that $f_{1i} \to \phi_{-\infty 0}f$ in $\mathsf{Z}_{\beta_1}\mathscr{S}'$ and $f_{2i} \to \phi_1 f$ in $\mathsf{Z}_{\beta_2}\mathscr{S}'$. Remark 2.25 then gives $\phi_{-\infty 1}f_{1i} \in \mathsf{Z}_{\beta_1}\mathscr{S} \cap \mathsf{Z}_{\beta_2}\mathscr{S}$ and $\phi_{-\infty 1}f_{1i} \to \phi_{-\infty 1}\phi_{-\infty 0}f = \phi_{-\infty 0}f$ in $\mathsf{Z}_{\beta_1}\mathscr{S}' \cap \mathsf{Z}_{\beta_2}\mathscr{S}'$. Similarly, $\phi_0 f_{2i} \in \mathsf{Z}_{\beta_1}\mathscr{S} \cap \mathsf{Z}_{\beta_2}\mathscr{S}$ and $\phi_0 f_{2i} \to \phi_1 f$ in $\mathsf{Z}_{\beta_1}\mathscr{S}' \cap \mathsf{Z}_{\beta_2}\mathscr{S}'$. Setting $f_i = \phi_{-\infty 1}f_{1i} + \phi_0 f_{2i}$ for each $i \in \mathbb{N}$, we deduce that $\{f_i\}_{i\in\mathbb{N}}$ is a sequence in $\mathsf{Z}_{\beta_1}\mathscr{S} \cap \mathsf{Z}_{\beta_2}\mathscr{S}$ which converges to f in $\mathsf{Z}_{\beta_1}\mathscr{S}' \cap \mathsf{Z}_{\beta_2}\mathscr{S}'$. ∎

3.5. Model problems on $\mathbb{R}^n_*$

DEFINITION 3.19. A scalar differential operator $A(x, D_x)$ on $\mathbb{R}^n_*$ of order m will be called *uniform* if we can write

$$A(x, D_x) = r^{-m}(\Theta^{-1})^* B(t, \omega, D_t, D_\omega)\Theta^* = r^{-m} B(\ln r, \omega, rD_r, D_\omega) \tag{76}$$

for some uniform scalar operator $B(t, \omega, D_t, D_\omega)$ on Π^n.

Suppose $\mathcal{A}(x, D_x)$ is a $k \times k$ system of differential operators on $\mathbb{R}^n_*$ of order (μ, ν). We will call $\mathcal{A}$ a *uniform* (*elliptic*) *operator on* $\mathbb{R}^n_*$ if we can write

$$\mathcal{A}(x, D_x) = r^{\nu}(\varTheta^{-1})^* \mathcal{B}(t, \omega, D_t, D_\omega)\varTheta^* r^{-\mu} = r^{\nu}\mathcal{B}(\ln r, \omega, rD_r, D_\omega) r^{-\mu} \tag{77}$$

for some uniform (elliptic) operator $\mathcal{B}(t, \omega, D_t, D_\omega)$ on $\varPi^n$ (where r^ν and $r^{-\mu}$ are the isomorphisms defined at the end of Section 2.4).

REMARK 3.20. If $B(t, \omega, D_t, D_\omega)$ is a differential operator on $\varPi^n$ and $m \in \mathbb{R}$ then (7) and (53) give

$$\begin{aligned} r^{-m}(\varTheta^{-1})^* B(t, \omega, D_t, D_\omega)\varTheta^* &= (\varTheta^{-1})^* e^{-mt} B(t, \omega, D_t, D_\omega)\varTheta^* \\ &= (\varTheta^{-1})^* B(t, \omega, D_t - im, D_\omega)\varTheta^* r^{-m}. \end{aligned} \tag{78}$$

Now suppose $\mathcal{A}$ is a uniform operator on $\mathbb{R}^n_*$ of order (μ, ν) and let $\mathcal{B}$ be the associated uniform operator on $\varPi^n$ given by (77). Then, for each $i, j \in \{1, \dots, k\}$, the ijth entries of $\mathcal{A}$ and $\mathcal{B}$ satisfy the relationship

$$A_{ij}(x, D_x) = r^{\nu_i}(\varTheta^{-1})^* B_{ij}(t, \omega, D_t, D_\omega)\varTheta^* r^{-\mu_j} = r^{\nu_i - \mu_j}(\varTheta^{-1})^* B_{ij}(t, \omega, D_t + i\mu_j, D_\omega)\varTheta^*.$$

Therefore A_{ij} is a uniform scalar operator on $\mathbb{R}^n_*$ of order $\mu_j - \nu_i$.

Suppose A is a uniform scalar operator on $\mathbb{R}^n_*$ of order m. By (8), Proposition 2.20 and the mapping properties of uniform scalar operators on $\varPi^n$ (see Section 3.1), it is clear that A defines a continuous map $\mathsf{Y}_{\beta-m}E^m \to \mathsf{Y}_\beta E$ for any admissible space E and $\beta \in \mathbb{R}$. On the other hand, if $\mathcal{A}$ is a uniform operator on $\mathbb{R}^n_*$ of order (μ, ν), then (44), (45) and the mapping properties of uniform operators on $\varPi^n$ imply that $\mathcal{A}$ defines a continuous map $\mathcal{A} : \mathsf{Y}^\mu_\beta E \to \mathsf{Y}^\nu_\beta E$ for any admissible space E and $\beta \in \mathbb{R}$. We shall denote this map by $\mathcal{A}^{(E,\beta)}$ or $\mathcal{A}^{(\beta)}$ when we need to make clear the spaces $\mathcal{A}$ is acting between.

PROPOSITION 3.21. *A scalar differential operator A on $\mathbb{R}^n_*$ of order m is uniform iff the coefficient of D^α_x is contained in $\mathsf{Y}_{m-|\alpha|}C^l$ for all $l \in \mathbb{N}_0$ and multi-indices α with $|\alpha| \le m$.*

Proof. It is straightforward to check that the set of uniform scalar operators on $\varPi^n$ can be generated from $C^\infty(\varPi^n)$ and the set of first order model operators by taking linear combinations of products. With the help of Lemma 2.31 and the fact that scalar differential operators commute to leading order, we can thus write any uniform scalar operator B of order m on $\varPi^n$ in the form

$$\begin{aligned} B(t, \omega, D_t, D_\omega) &= \sum_{|\alpha| \le m} f_\alpha(t, \omega)(B_1(\omega, D_t, D_\omega))^{\alpha_1} \dots (B_n(\omega, D_t, D_\omega))^{\alpha_n} \\ &= \sum_{|\alpha| \le m} g_\alpha(t, \omega) e^{|\alpha|t}(e^{-t}B_1(\omega, D_t, D_\omega))^{\alpha_1} \dots (e^{-t}B_n(\omega, D_t, D_\omega))^{\alpha_n} \end{aligned}$$

where, for each multi-index α with $|\alpha| \le m$, $f_\alpha, g_\alpha \in C^\infty(\varPi^n)$. However, $g \in C^\infty(\varPi^n)$ iff $g \circ \varTheta^{-1} \in \mathsf{Y}_0 C^l$ for all $l \in \mathbb{N}_0$, whilst $D_i = (\varTheta^{-1})^* e^{-t} B_i \varTheta^*$ for $i = 1, \dots, n$. Proposition 2.20 and (7) thus complete the result. ∎

PROPOSITION 3.22. *Suppose $\mathcal{A}$ is a $k \times k$ system of differential operators on $\mathbb{R}^n_*$. Then $\mathcal{A}$ is a uniform elliptic operator iff $\mathcal{A}$ is a uniform operator which is uniformly elliptic on $\mathbb{R}^n_*$ in the sense of Definition 2.78.*

Proof. Suppose $\mathcal{A}$ is a uniform operator (on $\mathbb{R}^n_*$) and let $\mathcal{B}$ be the uniform operator on Π^n given by (77). We thus have to show that

$$|\det_{\mathcal{A}}(x,\xi)| \geq C|\xi|^{[\mu-\nu]} \quad \text{for all } (x,\xi) \in \mathbb{R}^n_* \times \mathbb{R}^n \tag{79}$$

is equivalent to

$$|\det_{\mathcal{B}}(y,\eta)| \geq C|\eta|^{[\mu-\nu]}_{\Pi^n} \quad \text{for all } (y,\eta) \in T^*\Pi^n \tag{80}$$

(where $|\cdot|_{\Pi^n}$ denotes the norm on the fibres of the cotangent bundle $T^*\Pi^n$ given by the Riemannian metric on Π^n).

For each $i,j \in \{1,\ldots,k\}$ let a_{ij} and b_{ij} denote the principal symbols of the ijth entries of $\mathcal{A}$ and $\mathcal{B}$ respectively. Using (77) and the transformation properties of principal symbols under diffeomorphisms (see Section 18.1 in [H2], for example) we have

$$a_{ij}(\Theta(y),\xi) = r^{\nu_i-\mu_j}\, b_{ij}(y,(D\Theta(y))^*\xi)$$

for all $y \in \Pi^n$ and $\xi \in \mathbb{R}^n$, where $r = |\Theta(y)|$ and $(D\Theta(y))^*$ denotes the adjoint of the linear map $D\Theta(y) : T_y\Pi^n \to \mathbb{R}^n$. By taking determinants of the $k \times k$ matrices with entries a_{ij} and b_{ij} we now get

$$\det_{\mathcal{A}}(\Theta(y),\xi) = r^{-[\mu-\nu]}\det_{\mathcal{B}}(y,(D\Theta(y))^*\xi) = \det_{\mathcal{B}}(y,r^{-1}(D\Theta(y))^*\xi) \tag{81}$$

for all $y \in \Pi^n$ and $\xi \in \mathbb{R}^n$, where we have made use of the fact that $\det_{\mathcal{B}}(y,\eta)$ is homogeneous of degree $[\mu-\nu]$ in η. On the other hand, the definition of Θ gives

$$r^{-1}|(D\Theta(y))^*\xi|_{\Pi^n} = |\xi|$$

for all $y \in \Pi^n$ and $\xi \in \mathbb{R}^n$. The equivalence of (79) and (80) follows from this identity and (81). ■

The relationship between $\mathcal{A}$ and $\mathcal{B}$ given by (77), in combination with (44) and (45), allows us to obtain results for uniform elliptic operators on $\mathbb{R}^n_*$ from results for uniform elliptic operators on Π^n. The next result comes directly from Lemma 3.5 (with $j = +\infty$ and $l = 1$).

LEMMA 3.23. *Suppose $\mathcal{A}$ is a uniform elliptic operator on $\mathbb{R}^n_*$ of order (μ,ν), E is an admissible space, $\beta \in \mathbb{R}$ and $i \in \mathbb{Z} \cup \{-\infty\}$. If $\zeta_{i-1}\mathcal{A}u \in \Upsilon^\nu_\beta E$ and $\zeta_{i-1}u \in \Upsilon^\mu_\beta E^{-1}$ for some $u \in \mathscr{D}'(\mathbb{R}^n_*)$ then we also have $\zeta_i u \in \Upsilon^\mu_\beta E$. Furthermore,*

$$\|\zeta_i u\|_{\Upsilon^\mu_\beta E} \leq C(\|\zeta_{i-1}\mathcal{A}u\|_{\Upsilon^\nu_\beta E} + \|\pi_\mu\zeta_{i-1}u\|_{\Upsilon^\mu_\beta E^{-1}})$$

for all such u, where C is independent of i.

We also need to transfer the results of Section 3.4 from Π^n to $\mathbb{R}^n_*$. With this in mind we firstly make the following definitions.

DEFINITION 3.24. We say that a uniform scalar operator A on $\mathbb{R}^n_*$ is a *model scalar operator on* $\mathbb{R}^n_*$ if the operator B given by (76) is a model scalar operator on Π^n (i.e. the coefficients of B do not depend on t). Likewise, we say that a uniform elliptic operator $\mathcal{A}$ on $\mathbb{R}^n_*$ is a *model elliptic operator on* $\mathbb{R}^n_*$ if the operator $\mathcal{B}$ given by (77) is a model elliptic operator on Π^n (i.e. the coefficients of $\mathcal{B}$ do not depend on t). We shall denote the associated operator pencil by $\mathfrak{B}_{\mathcal{A}}$ and put

$$\Gamma(\mathcal{A}) = \operatorname{Im}\sigma(\mathfrak{B}_{\mathcal{A}}). \tag{82}$$

Theorem 3.10 can be immediately rewritten as follows.

THEOREM 3.25. *Let E be an admissible space and let $\mathcal{A}$ be a model elliptic operator on $\mathbb{R}^n_*$ of order (μ,ν). If $\beta \in \mathbb{R}\setminus\Gamma(\mathcal{A})$ then $\mathcal{A} : \mathsf{Y}^\mu_\beta E \to \mathsf{Y}^\nu_\beta E$ is an isomorphism.*

Finally, Theorem 3.11 can be coupled with basic properties of the power-exponential solutions given by their explicit form (see (64)) and Remark 3.7 to give the following results.

THEOREM 3.26. *Let E and $\mathcal{A}$ be as in Theorem 3.25 and define $\mathfrak{B}_{\mathcal{A}}$ to be the associated spectral pencil. Suppose $\beta_1,\beta_2 \in \mathbb{R}\setminus\Gamma(\mathcal{A})$ satisfy $\beta_1<\beta_2$. Set $\Sigma = \{\lambda \in \sigma(\mathfrak{B}_{\mathcal{A}}) \mid \operatorname{Im}\lambda \in [\beta_1,\beta_2]\}$ and let M denote the sum of the algebraic multiplicities of all the $\lambda\in\Sigma$. Then there exists a vector space $X_\Sigma \subset C^\infty_{\mathrm{loc}}(\mathbb{R}^n_*)$ such that:*

- (i) *For each $f \in \mathsf{Y}^\nu_{\beta_1}E \cap \mathsf{Y}^\nu_{\beta_2}E$ we have $(\mathcal{A}^{(\beta_1)})^{-1}f - (\mathcal{A}^{(\beta_2)})^{-1}f \in X_\Sigma$.*
- (ii) $\dim X_\Sigma = M$ *and $\mathcal{A}u = 0$ for each $u \in X_\Sigma$.*
- (iii) *Given any $u\in X_\Sigma$ and $\zeta \in \mathrm{BS}_{01}$ we have $\zeta u \in \mathsf{Y}^\mu_{\beta_1}E$ and $(1-\zeta)u \in \mathsf{Y}^\mu_{\beta_2}E$.*

COROLLARY 3.27. *If $[\beta_1,\beta_2]\cap\Gamma(\mathcal{A}) = \emptyset$ then $(\mathcal{A}^{(\beta_1)})^{-1}f = (\mathcal{A}^{(\beta_2)})^{-1}f$ for all $f \in \mathsf{Y}^\nu_{\beta_1}E \cap \mathsf{Y}^\nu_{\beta_2}E$.*

4. Main results

4.1. The class of operators

DEFINITION 4.1. Let $A(x,D_x)$ be a scalar differential operator on $\mathbb{R}^n$ of order m. We call A an *admissible operator* if we can write

$$A(x,D_x) = \sum_{|\alpha|\le m} p^\alpha(x) D^\alpha_x, \tag{83}$$

where $p^\alpha \in \mathrm{Sc}^{m-|\alpha|}$ for each multi-index α with $|\alpha|\le m$.

Suppose A is an admissible operator of order m. By Propositions 2.33 and 2.56 it is clear that A defines a continuous map $\mathsf{X}_{\beta-m}E^m \to \mathsf{X}_\beta E$ for any admissible space E and $\beta\in\mathbb{R}$. As in the Introduction, we define the *principal part of* A to be the operator on $\mathbb{R}^n_*$ which is given by

$$A_0(x,D_x) = \sum_{|\alpha|\le m} r^{|\alpha|-m} a^\alpha(\omega) D^\alpha_x,$$

where $a^\alpha \in C^\infty(S^{n-1})$ is the principal part of p^α for each multi-index α. We can rewrite A_0 in the form

$$A_0(x,D_x) = r^{-m}\sum_{j=0}^m A_0^{m-j}(\omega,D_\omega)(rD_r)^j,$$

where, for $j=0,\dots,m$, $A_0^j(\omega,D_\omega)$ is a differential operator on S^{n-1} of order at most j. It follows that A_0 is a model operator on $\mathbb{R}^n_*$.

REMARK 4.2. Suppose A is a scalar admissible operator with principal part A_0. For any $f\in\mathrm{BS}$ set $A_f = fA + (1-f)A_0$. If $f=1$ on a neighbourhood of 0 then Remark 2.57 implies A_f is a scalar admissible operator with the same principal part as A. On the other hand, if $f = 0$ on a neighbourhood of 0 then Remark 2.57 and Proposition 3.21 imply A_f is a uniform scalar operator on $\mathbb{R}^n_*$.

We now define the class of elliptic differential operators on $\mathbb{R}^n$ to which our main results apply.

DEFINITION 4.3. Let $\mathcal{A}$ be a $k \times k$ system of differential operators on $\mathbb{R}^n$ of order (μ, ν). We call $\mathcal{A}$ an *admissible elliptic operator on* $\mathbb{R}^n$ if it satisfies the following conditions:

(i) For $i, j \in \{1, \dots, k\}$ the ijth entry of $\mathcal{A}$ is a scalar admissible operator on $\mathbb{R}^n$ (of order $\mu_j - \nu_i$).

(ii) $\mathcal{A}$ is a uniformly elliptic operator on $\mathbb{R}^n$ (in the sense of Definition 2.78).

For $i, j \in \{1, \dots, k\}$ let $(A_0)_{ij}$ denote the principal part of the ijth entry of $\mathcal{A}$. By the *principal part of* $\mathcal{A}$ we mean the $k \times k$ system of differential operators on $\mathbb{R}^n_*$ with entries $(A_0)_{ij}$; this operator will be denoted by $\mathcal{A}_0$. By the *operator pencil associated with* $\mathcal{A}$ we mean $\mathfrak{B}_{\mathcal{A}_0}$; this will also be denoted by $\mathfrak{B}_{\mathcal{A}}$. Finally, define $\Gamma(\mathcal{A}) \subset \mathbb{R}$ by

$$\Gamma(\mathcal{A}) = \Gamma(\mathcal{A}_0) = \operatorname{Im} \sigma(\mathfrak{B}_{\mathcal{A}}).$$

Suppose $\mathcal{A}$ satisfies condition (i) of Definition 4.3 and, for $i, j \in \{1, \dots, k\}$, let A_{ij} denote the ijth entry of $\mathcal{A}$. From the discussion after Definition 4.1 we know that A_{ij} defines a continuous map $\mathsf{X}_{\beta-\mu_j} E^{\mu_j} \to \mathsf{X}_{\beta-\nu_i} E^{\nu_i}$ for any admissible space E and $\beta \in \mathbb{R}$. It follows that $\mathcal{A}$ defines a continuous map $\mathsf{X}^{\mu}_{\beta} E \to \mathsf{X}^{\nu}_{\beta} E$. We shall denote this map by $\mathcal{A}^{(E,\beta)}$ or $\mathcal{A}^{(\beta)}$ when we need to make clear the spaces $\mathcal{A}$ is acting between.

REMARK 4.4. Suppose $\mathcal{A}$ satisfies condition (i) of Definition 4.3. For any $f \in \mathrm{BS}$ set $\mathcal{A}_f = f\mathcal{A} + (1-f)\mathcal{A}_0$. By Remark 4.2 it is straightforward to check that $\mathcal{A}_f$ is a uniform operator on $\mathbb{R}^n_*$ if $f = 0$ on neighbourhood of 0 whilst $\mathcal{A}_f$ satisfies condition (i) of Definition 4.3 provided $f = 1$ on a neighbourhood of 0; in the latter case, the principal part of $\mathcal{A}_f$ is simply $\mathcal{A}_0$.

REMARK 4.5. Suppose $\mathcal{A}$ satisfies condition (i) of Definition 4.3. Using condition (ii) of Definition 1.1 we can find a bounded non-negative function b on $\mathbb{R}^n$ with $b(x) \to 0$ as $|x| \to \infty$ such that the coefficients of all the entries of the operator $\mathcal{A} - \mathcal{A}_0$ are bounded by $b(x)$ when $|x| \geq 1$. If $f, g \in \mathrm{BS}$ then $\mathcal{A}_f - \mathcal{A}_g = (f-g)(\mathcal{A} - \mathcal{A}_0)$ so Remark 2.80 now leads to the estimate

$$|\det_{\mathcal{A}_f}(x, \xi) - \det_{\mathcal{A}_g}(x, \xi)| \leq C b(x) |(f-g)(x)| \cdot |\xi|^{[\mu-\nu]}$$

for all $|x| \geq 1$ and $\xi \in \mathbb{R}^n$, where C may depend on $\mathcal{A}$ and $\mathcal{A}_0$ but not on (x, ξ).

REMARK 4.6. If an operator $\mathcal{A}$ satisfies condition (i) of Definition 4.3 then condition (ii) is equivalent to the following:

(ii)′ $\mathcal{A}$ is an elliptic operator on $\mathbb{R}^n$ whilst $\mathcal{A}_0$ is a model elliptic operator on $\mathbb{R}^n_*$.

In order to see this first observe that condition (ii) means

$$|\det_{\mathcal{A}}(x, \xi)| \geq C |\xi|^{[\mu-\nu]} \quad \text{for all } (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n.$$

On the other hand, $\mathcal{A}_0$ is a model operator (by condition (i) and the discussion preceding Remark 4.2) so Proposition 3.22 implies $\mathcal{A}_0$ is a model elliptic operator on $\mathbb{R}^n_*$ iff $\mathcal{A}_0$

is uniformly elliptic on $\mathbb{R}^n_*$ in the sense of Definition 2.78. Thus condition (ii)′ means $\det_{\mathcal{A}}(x,\xi) \neq 0$ for any $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n_*$ and

$$|\det_{\mathcal{A}_0}(x,\xi)| \geq C|\xi|^{[\mu-\nu]} \quad \text{for all } (x,\xi) \in \mathbb{R}^n_* \times \mathbb{R}^n.$$

The equivalence of conditions (ii) and (ii)′ now follows from Remark 4.5 (with $f = 1$ and $g = 0$) and the fact that $\det_{\mathcal{A}_0}(x,\xi)$ is homogeneous of degree 0 in x.

Remark 4.7. If $\mathcal{A}$ is an admissible elliptic operator, then the associated operator pencil $\mathfrak{B}_{\mathcal{A}}$ can be equivalently defined using the expression

$$\mathfrak{B}_{\mathcal{A}}(\lambda)\phi(\omega) = r^{-i\lambda}r^{-\nu}\mathcal{A}_0(r^{i\lambda}r^{\mu}\phi(\omega)) \tag{84}$$

for any $\lambda \in \mathbb{C}$ and function (or distribution) ϕ on S^{n-1}. In order to see this let $\mathcal{B}(\omega, D_t, D_\omega)$ denote the model elliptic operator on Π^n associated with $\mathcal{A}_0$ by (77) (see also Remark 4.6). Thus

$$\mathcal{A}_0(x, D_x) = r^{\nu}\mathcal{B}(\omega, rD_r, D_\omega)r^{-\mu}$$

and so

$$r^{-i\lambda}r^{-\nu}\mathcal{A}_0(r^{i\lambda}r^{\mu}\phi(\omega)) = r^{-i\lambda}\mathcal{B}(\omega, rD_r, D_\omega)(r^{i\lambda}\phi(\omega)) = \mathcal{B}(\omega, \lambda, D_\omega)\phi(\omega).$$

However, $\mathfrak{B}_{\mathcal{A}}(\lambda)(\omega, D_\omega) = \mathcal{B}(\omega, \lambda, D_\omega)$ by definition.

For the remainder of this section we suppose that $\mathcal{A}$ is an admissible elliptic operator on $\mathbb{R}^n$ with principal part $\mathcal{A}_0$.

Remark 4.8. By Remark 4.4 we know that $\mathcal{A}_{\zeta_i}$ is a uniform operator on $\mathbb{R}^n_*$ for all $i \in \mathbb{N}$. On the other hand, condition (ii)′ for $\mathcal{A}_0$ and Remark 4.5 (with $f = 0$ and $g = \zeta_i$) imply that $\mathcal{A}_{\zeta_i}$ is uniformly elliptic on $\mathbb{R}^n_*$ for all sufficiently large $i \in \mathbb{N}$. Coupled with Proposition 3.22 this shows that $\mathcal{A}_{\zeta_i}$ is a uniform elliptic operator on $\mathbb{R}^n_*$ for all sufficiently large $i \in \mathbb{N}$.

Remark 4.9. By Remark 4.4 we know that $\mathcal{A}_{\eta_i}$ satisfies condition (i) of Definition 4.3 for all $i \in \mathbb{N}$. On the other hand, condition (ii) for $\mathcal{A}$ and Remark 4.5 (with $f = 1$ and $g = \eta_i = 1 - \zeta_i$) imply that $\mathcal{A}_{\eta_i}$ is uniformly elliptic on $\mathbb{R}^n$ for all sufficiently large $i \in \mathbb{N}$. Thus $\mathcal{A}_{\eta_i}$ is an admissible elliptic operator on $\mathbb{R}^n$ for all sufficiently large $i \in \mathbb{N}$. Furthermore, the coefficients of $\mathcal{A}_{\eta_i}$ and $\mathcal{A}_0$ agree on a neighbourhood of ∞. It follows that the principal part of $\mathcal{A}_{\eta_i}$ is simply $\mathcal{A}_0$, the principal part of $\mathcal{A}$.

Suppose E is an admissible space and $\beta \in \mathbb{R}$. Then, for any $i \in \mathbb{N}$, the operator $\mathcal{A}_{\zeta_i} - \mathcal{A}_0$ defines a continuous map $\mathsf{Y}^{\mu}_{\beta}E \to \mathsf{Y}^{\nu}_{\beta}E$ (see Remark 4.8) whilst $\mathcal{A}_{\eta_i} - \mathcal{A}$ defines a continuous map $\mathsf{X}^{\mu}_{\beta}E \to \mathsf{X}^{\nu}_{\beta}E$ (see Remark 4.9).

Lemma 4.10. *We have* $\|\mathcal{A}_{\zeta_i} - \mathcal{A}_0\|_{\mathscr{L}(\mathsf{Y}^{\mu}_{\beta}E, \mathsf{Y}^{\nu}_{\beta}E)}$, $\|\mathcal{A}_{\eta_i} - \mathcal{A}\|_{\mathscr{L}(\mathsf{X}^{\mu}_{\beta}E, \mathsf{X}^{\nu}_{\beta}E)} \to 0$ *as* $i \to \infty$.

Proof. If A is a scalar admissible operator of order m on $\mathbb{R}^n$ then (16), Propositions 2.33 and 2.50, and Lemma 2.58 give

$$\lim_{i\to\infty} \|\zeta_i(A - A_0)\|_{\mathscr{L}(\mathsf{Y}_{\beta-m_1}E^{m_1}, \mathsf{Y}_{\beta-m_2}E^{m_2})} = 0 = \lim_{i\to\infty} \|\zeta_i(A - A_0)\|_{\mathscr{L}(\mathsf{X}_{\beta-m_1}E^{m_1}, \mathsf{X}_{\beta-m_2}E^{m_2})} \tag{85}$$

for any $m_1, m_2 \in \mathbb{Z}$ with $m = m_1 - m_2$ (n.b. the operator $\zeta_i(A - A_0)$ can be interpreted as acting on either $\mathsf{Y}_{\beta-m_1}E^{m_1}$ or $\mathsf{X}_{\beta-m_1}E^{m_1}$, by using Lemma 2.16 and the expression

$\zeta_i(A - A_0) = \zeta_i(A - A_0)\zeta_{i-1})$. On the other hand, we have

$$\mathcal{A}_{\zeta_i} - \mathcal{A}_0 = \zeta_i(\mathcal{A} - \mathcal{A}_0) = \mathcal{A} - \mathcal{A}_{\eta_i}.$$

The result follows by applying (85) to the individual entries of $\zeta_i(\mathcal{A} - \mathcal{A}_0)$. ∎

4.2. General estimates and some regularity. In this section we give some general estimates and regularity results for the admissible elliptic operator $\mathcal{A}$; here "general" refers to the fact that these results hold without restriction on β (the index appearing in the weighted spaces $\mathsf{X}_\beta E$).

PROPOSITION 4.11. *Suppose E is an admissible space, $l \in \mathbb{N}$ and $\beta \in \mathbb{R}$. If we have $\mathcal{A}u \in \mathsf{X}^\nu_\beta E$ for some $u \in \mathsf{X}^\mu_\beta E^{-l}$ then we also have $u \in \mathsf{X}^\mu_\beta E$. Furthermore,*

$$\|u\|_{\mathsf{X}^\mu_\beta E} \leq C(\|\mathcal{A}u\|_{\mathsf{X}^\nu_\beta E} + \|\pi_\mu u\|_{\mathsf{X}^\mu_\beta E^{-l}}) \tag{86}$$

for all such u.

Proof. By induction it clearly suffices to prove the result in the case $l = 1$. Now, by Remark 4.8, we can choose $I \in \mathbb{N}$ sufficiently large so that the operator $\mathcal{A}_{\zeta_{I-2}}$ is a uniform elliptic operator on $\mathbb{R}^n_*$. However, $\zeta_{I-1}\mathcal{A}_{\zeta_{I-2}}u = \zeta_{I-1}\mathcal{A}u$ whilst Lemma 2.16 gives us $\zeta_{I-1}\mathcal{A}u \in \mathsf{Y}^\nu_\beta E$ and $\zeta_{I-1}u \in \mathsf{Y}^\mu_\beta E^{-1}$. Lemma 3.23 then implies $\zeta_I u \in \mathsf{Y}^\mu_\beta E$ and

$$\|\zeta_I u\|_{\mathsf{Y}^\mu_\beta E} \leq C(\|\zeta_{I-1}\mathcal{A}u\|_{\mathsf{Y}^\nu_\beta E} + \|\pi_\mu \zeta_{I-1}u\|_{\mathsf{Y}^\mu_\beta E^{-1}}).$$

On the other hand, $\mathcal{A}$ is an elliptic operator (on $\mathbb{R}^n$) whilst Lemma 2.16 gives us $\eta_{I+1}\mathcal{A}u \in E^\nu$ and $\eta_{I+1}u \in (E^{-1})^\mu$. Theorem 2.81 then implies $\eta_I u \in E^\mu$ and

$$\|\eta_I u\|_{E^\mu} \leq C(\|\eta_{I+1}\mathcal{A}u\|_{E^\nu} + \|\pi_\mu \eta_{I+1}u\|_{(E^{-1})^\mu})$$

(n.b. $\eta_I \prec \eta_{I+1}$). The result now follows from Lemma 2.16. ∎

The next result gives the strongest form of the elliptic regularity which can be achieved for an arbitrary $\beta \in \mathbb{R}$.

THEOREM 4.12. *Let E and F be admissible spaces and $\beta, \gamma \in \mathbb{R}$. Suppose that either $\beta < \gamma$ or $\beta \leq \gamma$ and there exists a continuous inclusion $F^l \hookrightarrow E$ for some $l \in \mathbb{N}_0$. If we have $\mathcal{A}u \in \mathsf{X}^\nu_\beta E \cap \mathsf{X}^\nu_\gamma F$ for some $u \in \mathsf{X}^\mu_\gamma F$ then we also have $u \in \mathsf{X}^\mu_\beta E$. Furthermore,*

$$\|u\|_{\mathsf{X}^\mu_\beta E} \leq C(\|\mathcal{A}u\|_{\mathsf{X}^\nu_\beta E} + \|\pi_\mu u\|_{\mathsf{X}^\mu_\gamma F}) \tag{87}$$

for all such u.

Proof. Using the hypothesis, Corollary 2.30 and Proposition 2.69, we can choose $l \in \mathbb{N}$ sufficiently large so that we either have $\beta < \gamma$ and a local inclusion $F_{\mathrm{loc}} \hookrightarrow E^{-l}_{\mathrm{loc}}$ or $\beta = \gamma$ and a continuous inclusion $F \hookrightarrow E^{-l}$. Proposition 2.26 or Remark 2.21 then gives us a continuous inclusion $\mathsf{X}^\mu_\gamma F \hookrightarrow \mathsf{X}^\mu_\beta E^{-l}$. The result now follows from Proposition 4.11. ∎

4.3. A semi-Fredholm property, further regularity and stability of the index. Throughout this section $\mathcal{A}$ is an admissible elliptic operator on $\mathbb{R}^n$. Let $\mathcal{A}_0$ and $\mathfrak{B}_{\mathcal{A}}$ denote the principal part of $\mathcal{A}$ and the associated operator pencil respectively. We start with a result which provides a key step for both Fredholm properties and further regularity results for $\mathcal{A}$.

PROPOSITION 4.13. *Suppose E is an admissible space and $\beta_1, \beta_2 \in \mathbb{R}$ satisfy $\beta_1 \le \beta_2$ and $[\beta_1, \beta_2] \cap \varGamma(\mathcal{A}) = \emptyset$. If $\mathcal{A}u \in \mathsf{X}^\nu_{\beta_1}E \cap \mathsf{X}^\nu_{\beta_2}E$ for some $u \in \mathsf{X}^\mu_{\beta_1}E$ then we also have $u \in \mathsf{X}^\mu_{\beta_2}E$. Furthermore,*

$$\|u\|_{\mathsf{X}^\mu_{\beta_2}E} \le C(\|\mathcal{A}u\|_{\mathsf{X}^\nu_{\beta_2}E} + \|u\|_{\mathsf{X}^\mu_{\beta_1}E})$$

for all such u.

For the proof of this result we make use of the operators $\mathcal{A}_{\zeta_i} = \zeta_i\mathcal{A} + (1-\zeta_i)\mathcal{A}_0$ for $i \in \mathbb{N}$. In particular, Lemma 4.10 implies $\mathcal{A}_{\zeta_i} \to \mathcal{A}_0$ in $\mathscr{L}(\mathsf{Y}^\mu_\beta E, \mathsf{Y}^\nu_\beta E)$ as $i \to \infty$. Since the set of invertible elements in $\mathscr{L}(\mathsf{Y}^\mu_\beta E, \mathsf{Y}^\nu_\beta E)$ is open, Theorem 3.25 immediately implies the following result.

LEMMA 4.14. *If $\beta \in \mathbb{R} \setminus \varGamma(\mathcal{A})$ then $\mathcal{A}_{\zeta_i} : \mathsf{Y}^\mu_\beta E \to \mathsf{Y}^\nu_\beta E$ is an isomorphism for all sufficiently large $i \in \mathbb{N}$.*

Using Neumann series we can also obtain the following extension to Corollary 3.27.

LEMMA 4.15. *If β_1, β_2 are as in Proposition 4.13 then $(\mathcal{A}^{(\beta_1)}_{\zeta_i})^{-1}f = (\mathcal{A}^{(\beta_2)}_{\zeta_i})^{-1}f$ for all $f \in \mathsf{Y}^\nu_{\beta_1}E \cap \mathsf{Y}^\nu_{\beta_2}E$ and sufficiently large $i \in \mathbb{N}$.*

Proof. From Lemma 4.10 we know that for all sufficiently large $i \in \mathbb{N}$ we have

$$\|\mathcal{A}_{\zeta_i} - \mathcal{A}_0\|_{\mathscr{L}(\mathsf{Y}^\mu_{\beta_j}E, \mathsf{Y}^\nu_{\beta_j}E)} \le \tfrac{1}{2}\|\mathcal{A}_0\|_{\mathscr{L}(\mathsf{Y}^\mu_{\beta_j}E, \mathsf{Y}^\nu_{\beta_j}E)}$$

for $j = 1, 2$. Choose $i \in \mathbb{N}$ for which this is true and set $\mathcal{P} = \mathcal{A}_0 - \mathcal{A}_{\zeta_i}$. It follows that we have a Neumann series expansion

$$(\mathcal{A}^{(\beta_j)}_{\zeta_i})^{-1} = \sum_{l=0}^{\infty}((\mathcal{A}^{(\beta_j)}_0)^{-1}\mathcal{P})^l(\mathcal{A}^{(\beta_j)}_0)^{-1}, \tag{88}$$

which is convergent in $\mathscr{L}(\mathsf{Y}^\nu_{\beta_j}E, \mathsf{Y}^\mu_{\beta_j}E)$ for $j = 1, 2$.

Let $f \in \mathsf{Y}^\nu_{\beta_1}E \cap \mathsf{Y}^\nu_{\beta_2}E$ and set $u = (\mathcal{A}^{(\beta_1)}_0)^{-1}f$. Then

$$u = (\mathcal{A}^{(\beta_1)}_0)^{-1}f = (\mathcal{A}^{(\beta_2)}_0)^{-1}f \in \mathsf{Y}^\mu_{\beta_1}E \cap \mathsf{Y}^\mu_{\beta_2}E$$

by Corollary 3.27. Now let $l \in \mathbb{N}_0$ and suppose

$$((\mathcal{A}^{(\beta_1)}_0)^{-1}\mathcal{P})^l u = ((\mathcal{A}^{(\beta_2)}_0)^{-1}\mathcal{P})^l u \in \mathsf{Y}^\mu_{\beta_1}E \cap \mathsf{Y}^\mu_{\beta_2}E. \tag{89}$$

Therefore

$$\mathcal{P}((\mathcal{A}^{(\beta_1)}_0)^{-1}\mathcal{P})^l u = \mathcal{P}((\mathcal{A}^{(\beta_2)}_0)^{-1}\mathcal{P})^l u \in \mathsf{Y}^\nu_{\beta_1}E \cap \mathsf{Y}^\nu_{\beta_2}E,$$

so Corollary 3.27 implies that (89) also holds with l replaced by $l+1$. The result now follows from induction and (88). ■

Proof of Proposition 4.13. Choose $I \in \mathbb{N}$ so that the conclusion of Lemma 4.15 holds with $i = I$. Now suppose $\mathcal{A}u \in \mathsf{X}^\nu_{\beta_1}E \cap \mathsf{X}^\nu_{\beta_2}E$ for some $u \in \mathsf{X}^\mu_{\beta_1}E$. Set $v = \zeta_I u$ and $g = \mathcal{A}_{\zeta_I}v$. By Lemma 2.16 we have $v \in \mathsf{Y}^\mu_{\beta_1}E$ and thus $g \in \mathsf{Y}^\nu_{\beta_1}E$ (by the mapping properties of uniform operators on $\mathbb{R}^n_*$; see also Remark 4.4). On the other hand, $g = 0$ on a neighbourhood of 0 whilst

$$\zeta_{I+1}g = \zeta_{I+1}(\zeta_I\mathcal{A} + (1-\zeta_I)\mathcal{A}_0)\zeta_I u = \zeta_{I+1}\mathcal{A}u \in \mathsf{Y}^\nu_{\beta_1}E \cap \mathsf{Y}^\nu_{\beta_2}E \tag{90}$$

by the hypothesis on $\mathcal{A}u$ and Lemma 2.16. Remark 2.18 thus gives $g \in \mathsf{Y}^{\nu}_{\beta_1}E \cap \mathsf{Y}^{\nu}_{\beta_2}E$ and so $\zeta_I u = v = (\mathcal{A}^{(\beta_1)}_{\zeta_I})^{-1}g = (\mathcal{A}^{(\beta_2)}_{\zeta_I})^{-1}g$ by Lemma 4.15. From (9), Lemma 2.24 and Remark 2.17 we obtain

$$\|g\|_{\mathsf{Y}^{\nu}_{\beta_2}E} \le C(\|\zeta_{I+1}g\|_{\mathsf{Y}^{\nu}_{\beta_2}E} + \|\eta_{I+1}g\|_{\mathsf{Y}^{\nu}_{\beta_1}E}) \le C(\|\zeta_{I+1}g\|_{\mathsf{Y}^{\nu}_{\beta_2}E} + \|g\|_{\mathsf{Y}^{\nu}_{\beta_1}E}).$$

Using (90) and the fact that $\mathcal{A}^{(\beta_j)}_{\zeta_I}$ is an isomorphism for $j = 1, 2$ we thus get a norm estimate of the form

$$\|\zeta_I u\|_{\mathsf{Y}^{\mu}_{\beta_2}E} \le C(\|\zeta_{I+1}\mathcal{A}u\|_{\mathsf{Y}^{\nu}_{\beta_2}E} + \|\zeta_I u\|_{\mathsf{Y}^{\mu}_{\beta_1}E}).$$

Lemma 2.16 now completes the result. ■

The next result establishes a slightly weaker form of the Fredholm property for $\mathcal{A}$ in which "Fredholm" is replaced with "semi-Fredholm"; by the latter we mean an operator which has a closed range and for which either the kernel or the cokernel is finite-dimensional. The key steps in establishing this semi-Fredholm property are provided by Corollary 2.77 and Propositions 4.11 and 4.13.

THEOREM 4.16. *Suppose E is an admissible space and let $\beta \in \mathbb{R} \setminus \Gamma(\mathcal{A})$. Then the map $\mathcal{A} : \mathsf{X}^{\mu}_{\beta}E \to \mathsf{X}^{\nu}_{\beta}E$ is semi-Fredholm with a finite-dimensional kernel.*

Proof. Choose $I \in \mathbb{N}$ so that the conclusion to Lemma 4.14 holds for $i = I$. Now suppose we have a sequence satisfying

$$\{u_i\}_{i\in\mathbb{N}} \subset \mathsf{X}^{\mu}_{\beta}E, \qquad \|u_i\|_{\mathsf{X}^{\mu}_{\beta}E} = 1, \qquad \mathcal{A}u_i \to 0 \quad \text{in } \mathsf{X}^{\nu}_{\beta}E. \tag{91}$$

Let $i \in \mathbb{N}$. Lemma 2.16 gives us $\zeta_I\mathcal{A}u_i \in \mathsf{Y}^{\nu}_{\beta}E$ so $\mathcal{A}^{-1}_{\zeta_I}(\zeta_I\mathcal{A}u_i) \in \mathsf{Y}^{\mu}_{\beta}E$. Setting $v_i = \zeta_I\mathcal{A}^{-1}_{\zeta_I}(\zeta_I\mathcal{A}u_i)$ we then have $v_i \in \mathsf{X}^{\mu}_{\beta}E$. Furthermore, by combining norm estimates given by Lemma 2.16 and the fact that $\mathcal{A}_{\zeta_I} : \mathsf{Y}^{\mu}_{\beta}E \to \mathsf{Y}^{\nu}_{\beta}E$ is an isomorphism, we get

$$\|v_i\|_{\mathsf{X}^{\mu}_{\beta}E} \le C\|\mathcal{A}u_i\|_{\mathsf{X}^{\nu}_{\beta}E} \le C\|u_i\|_{\mathsf{X}^{\mu}_{\beta}E}. \tag{92}$$

Also, $\zeta_{I+1} \prec \zeta_I$ so $\zeta_{I+1}\mathcal{A}\zeta_I = \zeta_{I+1}\mathcal{A}_{\zeta_I}$ and hence

$$\zeta_{I+1}\mathcal{A}v_i = \zeta_{I+1}\mathcal{A}(\zeta_I\,\mathcal{A}^{-1}_{\zeta_I}(\zeta_I\mathcal{A}u_i)) = \zeta_{I+1}\mathcal{A}_{\zeta_I}(\mathcal{A}^{-1}_{\zeta_I}(\zeta_I\mathcal{A}u_i)) = \zeta_{I+1}\mathcal{A}u_i. \tag{93}$$

Now define w_i by $w_i = u_i - v_i$ for all $i \in \mathbb{N}$. By the second inequality in (92), $\{w_i\}_{i\in\mathbb{N}}$ is a bounded sequence in $\mathsf{X}^{\mu}_{\beta}E$ whilst (93) gives us $\zeta_{I+1}\mathcal{A}w_i = 0$ for all $i \in \mathbb{N}$. Choose $\gamma > \beta$ so that $[\beta, \gamma] \cap \Gamma(\mathcal{A}) = \emptyset$. From Lemma 2.16 we have that $\{\mathcal{A}w_i\}_{i\in\mathbb{N}}$ is a bounded sequence in $\mathsf{X}^{\nu}_{\gamma}E$, so Proposition 4.13 now implies that $\{w_i\}_{i\in\mathbb{N}}$ is a bounded sequence in $\mathsf{X}^{\mu}_{\gamma}E$. However, $\mathsf{X}^{\mu}_{\gamma}E \hookrightarrow \mathsf{X}^{\mu}_{\beta}E^{-1}$ is a compact map by Corollary 2.77, so we can choose a subsequence $\{w_{i(j)}\}_{j\in\mathbb{N}}$ which is convergent in $\mathsf{X}^{\mu}_{\beta}E^{-1}$. On the other hand, the last part of (91) and the first inequality in (92) imply $u_i - w_i = v_i \to 0$ in $\mathsf{X}^{\mu}_{\beta}E$. Since $\mathsf{X}^{\mu}_{\beta}E \hookrightarrow \mathsf{X}^{\mu}_{\beta}E^{-1}$ continuously we then get the convergence of $\{u_{i(j)}\}_{j\in\mathbb{N}}$ in $\mathsf{X}^{\mu}_{\beta}E^{-1}$. By combining Proposition 4.11 and the last part of (91) it follows that $\{u_{i(j)}\}_{j\in\mathbb{N}}$ is convergent in $\mathsf{X}^{\mu}_{\beta}E$.

Summarising, we have shown that any sequence satisfying (91) has a subsequence which is convergent in $\mathsf{X}^{\mu}_{\beta}E$. A standard argument (see Proposition 19.1.3 in [H2] or Theorems IV.5.9, IV.5.10 and IV.5.11 in [Ka]) can now be used to show that $\mathcal{A} : \mathsf{X}^{\mu}_{\beta}E \to \mathsf{X}^{\nu}_{\beta}E$ has a finite-dimensional kernel and a closed range. ■

If we are only interested in the kernel of the map $\mathcal{A}^{(E,\beta)}$ then the restriction on β in Theorem 4.16 can be dropped.

THEOREM 4.17. *For any admissible space E and $\beta \in \mathbb{R}$ the map $\mathcal{A} : \mathsf{X}^{\mu}_{\beta}E \to \mathsf{X}^{\nu}_{\beta}E$ has a finite-dimensional kernel.*

Proof. Choose $\gamma \in \mathbb{R} \setminus \varGamma(\mathcal{A})$ with $\gamma \leq \beta$. By (11) we have a continuous inclusion $\mathsf{X}^{\mu}_{\beta}E \hookrightarrow \mathsf{X}^{\mu}_{\gamma}E$ so $\operatorname{Ker}\mathcal{A}^{(\beta)} \subseteq \operatorname{Ker}\mathcal{A}^{(\gamma)}$. On the other hand, $\gamma \in \mathbb{R} \setminus \varGamma(\mathcal{A})$ so $\operatorname{Ker}\mathcal{A}^{(\gamma)}$ must be finite-dimensional by Theorem 4.16. ∎

We complete this section with two further consequences of Proposition 4.13, the first of which gives a stronger form of elliptic regularity than Theorem 4.12 but under some restrictions on β.

THEOREM 4.18. *Suppose E and F are admissible spaces and let $\beta, \gamma \in \mathbb{R}$ belong to the same component of $\mathbb{R} \setminus \varGamma(\mathcal{A})$. If $\mathcal{A}u \in \mathsf{X}^{\nu}_{\beta}E \cap \mathsf{X}^{\nu}_{\gamma}F$ for some $u \in \mathsf{X}^{\mu}_{\gamma}F$ then we also have $u \in \mathsf{X}^{\mu}_{\beta}E$. Furthermore,*

$$\|u\|_{\mathsf{X}^{\mu}_{\beta}E} \leq C(\|\mathcal{A}u\|_{\mathsf{X}^{\nu}_{\beta}E} + \|\pi_{\mu}u\|_{\mathsf{X}^{\mu}_{\gamma}F})$$

for all such u.

Proof. Choose any $\lambda \in \mathbb{R}$ which lies in the same component of $\mathbb{R} \setminus \varGamma(\mathcal{A})$ as β and γ, and satisfies $\lambda < \beta, \gamma$. Now $u \in \mathsf{X}^{\mu}_{\gamma}F$ and $\mathcal{A}u \in \mathsf{X}^{\nu}_{\beta}E \cap \mathsf{X}^{\nu}_{\gamma}F$ so we also have $\mathcal{A}u \in \mathsf{X}^{\nu}_{\lambda}E$ and $\|\mathcal{A}u\|_{\mathsf{X}^{\nu}_{\lambda}E} \leq C\|\mathcal{A}u\|_{\mathsf{X}^{\nu}_{\beta}E}$ by (11). Theorem 4.12 then gives $u \in \mathsf{X}^{\mu}_{\lambda}E$ and

$$\|u\|_{\mathsf{X}^{\mu}_{\lambda}E} \leq C(\|\mathcal{A}u\|_{\mathsf{X}^{\nu}_{\lambda}E} + \|\pi_{\mu}u\|_{\mathsf{X}^{\mu}_{\gamma}F}) \leq C(\|\mathcal{A}u\|_{\mathsf{X}^{\nu}_{\beta}E} + \|\pi_{\mu}u\|_{\mathsf{X}^{\mu}_{\gamma}F}).$$

Applying Proposition 4.13 (with $\beta_1 = \lambda$ and $\beta_2 = \beta$) now gives $u \in \mathsf{X}^{\mu}_{\beta}E$ and $\|u\|_{\mathsf{X}^{\mu}_{\beta}E} \leq C(\|\mathcal{A}u\|_{\mathsf{X}^{\nu}_{\beta}E} + \|u\|_{\mathsf{X}^{\mu}_{\lambda}E})$, completing the result. ∎

For semi-Fredholm maps we can define a $\mathbb{Z} \cup \{\pm\infty\}$-valued index; in particular, a semi-Fredholm map is Fredholm iff its index is finite (this will be established for $\mathcal{A}$ in Theorem 4.22). The next result establishes the stability of this index over particular ranges of the weighted spaces on which $\mathcal{A}$ is acting; in fact, Theorem 4.23 (below) shows these ranges to be maximal.

THEOREM 4.19. *Suppose E and F are admissible spaces and let $\beta, \gamma \in \mathbb{R}$ belong to the same component of $\mathbb{R} \setminus \varGamma(\mathcal{A})$. Then the semi-Fredholm maps $\mathcal{A} : \mathsf{X}^{\mu}_{\beta}E \to \mathsf{X}^{\nu}_{\beta}E$ and $\mathcal{A} : \mathsf{X}^{\mu}_{\gamma}F \to \mathsf{X}^{\nu}_{\gamma}F$ have the same index.*

Proof. We get $\operatorname{Ker}\mathcal{A}^{(E,\beta)} = \operatorname{Ker}\mathcal{A}^{(F,\gamma)}$ as a direct consequence of Theorem 4.18. It therefore remains to show $\operatorname{codim}(\operatorname{Ran}\mathcal{A}^{(E,\beta)}, \mathsf{X}^{\nu}_{\beta}E) = \operatorname{codim}(\operatorname{Ran}\mathcal{A}^{(F,\gamma)}, \mathsf{X}^{\nu}_{\gamma}F)$. By symmetry, Corollary 2.30, Proposition 2.69 and the obvious inclusion $E^{l} \hookrightarrow E$ for any $l \geq 0$, it suffices to prove this equality under the assumption that either $\beta > \gamma$ and we have a local inclusion $E_{\mathrm{loc}} \hookrightarrow F_{\mathrm{loc}}$, or $\beta \geq \gamma$ and we have a continuous inclusion $E \hookrightarrow F$. Combining this assumption with Remark 2.21 and Proposition 2.26, we get a continuous inclusion $\mathsf{X}_{\beta}E \hookrightarrow \mathsf{X}_{\gamma}F$.

Suppose V is a subspace of $\mathsf{X}^{\nu}_{\beta}E$ with $V \cap \operatorname{Ran}\mathcal{A}^{(E,\beta)} = 0$.

Claim (i): *We have* $V \cap \operatorname{Ran} \mathcal{A}^{(F,\gamma)} = 0$. If $f \in V \cap \operatorname{Ran} \mathcal{A}^{(F,\gamma)} \subseteq \mathsf{X}^{\nu}_{\beta}E \cap \mathsf{X}^{\nu}_{\gamma}F$ then $f = \mathcal{A}u$ for some $u \in \mathsf{X}^{\mu}_{\gamma}F$. Theorem 4.18 then gives $u \in \mathsf{X}^{\mu}_{\beta}E$ and so $f = \mathcal{A}u \in V \cap \operatorname{Ran} \mathcal{A}^{(E,\beta)} = 0$, completing the claim.

If $\operatorname{codim}(\operatorname{Ran} \mathcal{A}^{(E,\beta)}, \mathsf{X}^{\nu}_{\beta}E) = \infty$ then $\operatorname{codim}(\operatorname{Ran} \mathcal{A}^{(F,\gamma)}, \mathsf{X}^{\nu}_{\gamma}F) = \infty$ by an easy application of Claim (i). Now suppose $\operatorname{codim}(\operatorname{Ran} \mathcal{A}^{(E,\beta)}, \mathsf{X}^{\nu}_{\beta}E) < \infty$ and choose a finite-dimensional space $V \subset \mathsf{X}^{\nu}_{\beta}E$ which satisfies

$$V + \operatorname{Ran} \mathcal{A}^{(E,\beta)} = \mathsf{X}^{\nu}_{\beta}E \quad \text{and} \quad V \cap \operatorname{Ran} \mathcal{A}^{(E,\beta)} = 0. \tag{94}$$

Claim (i), (94) and the inclusions $V \subset \mathsf{X}^{\nu}_{\beta}E \subseteq \mathsf{X}^{\nu}_{\gamma}F$ give

$$V + \operatorname{Ran} \mathcal{A}^{(F,\gamma)} \subseteq \mathsf{X}^{\nu}_{\gamma}F \quad \text{and} \quad V \cap \operatorname{Ran} \mathcal{A}^{(F,\gamma)} = 0. \tag{95}$$

The following claim thus completes the proof.

Claim (ii): *We have* $\mathsf{X}^{\nu}_{\gamma}F \subseteq V + \operatorname{Ran} \mathcal{A}^{(F,\gamma)}$. Let $f \in \mathsf{X}^{\nu}_{\gamma}F$ and choose $\lambda < \gamma$ so that λ and γ lie in the same component of $\mathbb{R} \setminus \Gamma(\mathcal{A})$. By Lemma 2.71 we can find a sequence $\{f_i\}_{i \in \mathbb{N}} \subset C^{\infty}_0$ such that $f_i \to f$ in $\mathsf{X}^{\nu}_{\lambda}F^{-1}$. Now (94) and the inclusions $\mathsf{X}^{\mu}_{\beta}E \subseteq \mathsf{X}^{\mu}_{\gamma}F \subseteq \mathsf{X}^{\mu}_{\lambda}F^{-1}$ give

$$f_i \in C^{\infty}_0 \subset \mathsf{X}^{\nu}_{\beta}E = V + \operatorname{Ran} \mathcal{A}^{(E,\beta)} \subseteq V + \operatorname{Ran} \mathcal{A}^{(F^{-1},\lambda)}. \tag{96}$$

However, Theorem 4.16 implies that $\operatorname{Ran} \mathcal{A}^{(F^{-1},\lambda)}$ is a closed subspace of $\mathsf{X}^{\nu}_{\lambda}F^{-1}$ whilst $V \subset \mathsf{X}^{\nu}_{\beta}E \subseteq \mathsf{X}^{\nu}_{\lambda}F^{-1}$. Hence $V + \operatorname{Ran} \mathcal{A}^{(F^{-1},\lambda)}$ is also a closed subspace of $\mathsf{X}^{\nu}_{\lambda}F^{-1}$. It follows from (96) that $f \in V + \operatorname{Ran} \mathcal{A}^{(F^{-1},\lambda)}$, which in turn means we can find $g \in V$ and $u \in \mathsf{X}^{\mu}_{\lambda}F^{-1}$ satisfying $f = \mathcal{A}u + g$. Since $f - g \in \mathsf{X}^{\nu}_{\gamma}F$, Theorem 4.18 then gives $u \in \mathsf{X}^{\mu}_{\gamma}F$. Therefore $f \in V + \operatorname{Ran} \mathcal{A}^{(F,\gamma)}$, completing the claim. ■

4.4. Adjoint operators and the Fredholm property. Suppose $\mathcal{A}$ is an admissible elliptic operator on $\mathbb{R}^n$ of order (μ, ν) and let $\mathcal{A}^*$ denote the formal adjoint of $\mathcal{A}$ (with respect to the Lebesgue measure on $\mathbb{R}^n$). Also, let m, $\overline{\mu}$ and $\overline{\nu}$ be defined as in (56). It follows easily from the definition of an admissible operator that $\mathcal{A}^*$ is a $k \times k$ system of differential operators on $\mathbb{R}^n$ of order $(\overline{\mu}, \overline{\nu})$ which satisfies condition (i) of Definition 4.3. Furthermore, $\det_{\mathcal{A}^*}(x,\xi) = \overline{\det_{\mathcal{A}}(x,\xi)}$ for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$, from which it is clear that $\mathcal{A}^*$ also satisfies condition (ii) of Definition 4.3. Hence $\mathcal{A}^*$ is an admissible elliptic operator on $\mathbb{R}^n$ of order $(\overline{\mu}, \overline{\nu})$.

Let $\mathcal{A}_0$ denote the principal part of $\mathcal{A}$ and let $\mathcal{B}(\omega, D_t, D_\omega)$ denote the corresponding model elliptic operator on Π^n given by (77); that is,

$$\mathcal{A}_0(x, D_x) = r^{\nu} \mathcal{B}(\omega, rD_r, D_\omega) r^{-\mu} \tag{97}$$

whilst $\mathfrak{B}_{\mathcal{A}}$, the operator pencil associated with $\mathcal{A}$, is defined by

$$\mathfrak{B}_{\mathcal{A}}(\lambda)(\omega, D_\omega) = \mathcal{B}(\omega, \lambda, D_\omega). \tag{98}$$

Now it is straightforward to check that the principal part of $\mathcal{A}^*$ is simply $\mathcal{A}^*_0$, the formal adjoint of $\mathcal{A}_0$ (with respect to the Lebesgue measure on $\mathbb{R}^n_*$). Let $\mathcal{B}^*$ denote the formal adjoint of $\mathcal{B}$ (with respect to the measure $dt dS^{n-1}$ on Π^n). Since the Lebesgue measure on $\mathbb{R}^n_*$ can be written as $d^n x = r^{n-1} dr dS^{n-1}$, we have

$$(rD_r)^* = (r^{n-1})^{-1} D_r (r r^{n-1} \cdot) = rD_r - in$$

with respect to this measure. Combined with (78) and (97) this gives

$$\begin{aligned}\mathcal{A}_0^*(x,D_x) &= r^{-\mu}\mathcal{B}^*(\omega, rD_r - in, D_\omega)r^{\nu} = r^m r^{-\mu}\mathcal{B}^*(\omega, rD_r - i(n+m), D_\omega)r^{-m}r^{\nu}\\ &= r^{\overline{\nu}}\mathcal{B}^*(\omega, rD_r - i(n+m), D_\omega)r^{-\overline{\mu}}.\end{aligned}$$

Thus the model operator on Π^n associated with $\mathcal{A}_0^*$ by (77) is $\mathcal{B}^*(\omega, D_t - i(n+m), D_\omega)$, and so the operator pencil associated with $\mathcal{A}^*$ is given by

$$\mathfrak{B}_{\mathcal{A}^*}(\lambda)(\omega, D_\omega) = \mathcal{B}^*(\omega, \lambda - i(n+m), D_\omega). \tag{99}$$

Taken together (98) and (99) complete the proof of the following result.

Proposition 4.20. *The operator pencils $\mathfrak{B}_{\mathcal{A}}$ and $\mathfrak{B}_{\mathcal{A}^*}$ associated with $\mathcal{A}$ and $\mathcal{A}^*$ respectively satisfy the relationship*

$$\mathfrak{B}_{\mathcal{A}^*}(\lambda) = (\mathfrak{B}_{\mathcal{A}}(\overline{\lambda} + i(n+m)))^*$$

for all $\lambda \in \mathbb{C}$, where the right hand side denotes the formal adjoint of the operator $\mathfrak{B}_{\mathcal{A}}(\overline{\lambda} + i(n+m))$ on S^{n-1}. It follows that $\lambda \in \sigma(\mathfrak{B}_{\mathcal{A}^*})$ iff $\overline{\lambda} + i(n+m) \in \sigma(\mathfrak{B}_{\mathcal{A}})$ with full agreement of geometric, algebraic and partial algebraic multiplicities. In particular,

$$\Gamma(\mathcal{A}^*) = (n+m) - \Gamma(\mathcal{A}).$$

Let E be an admissible space and set $F = E_0^*$. By Propositions 2.46 and 2.69 we know that F is also an admissible space whilst

$$(\mathsf{X}_\beta^\mu E_0)^* = \prod_{i=1}^k \mathsf{X}_{n-\beta+\mu_i} F^{-\mu_i} = \mathsf{X}_{n+m-\beta}^{\overline{\nu}} F^{-m},$$

$$(\mathsf{X}_\beta^\nu E_0)^* = \prod_{i=1}^k \mathsf{X}_{n-\beta+\nu_i} F^{-\nu_i} = \mathsf{X}_{n+m-\beta}^{\overline{\mu}} F^{-m}$$

for any $\beta \in \mathbb{R}$. Therefore the adjoint of the map $\mathcal{A}^{(E_0,\beta)} : \mathsf{X}_\beta^\mu E_0 \to \mathsf{X}_\beta^\nu E_0$ is the map

$$(\mathcal{A}^{(E_0,\beta)})^* = (\mathcal{A}^*)^{(F^{-m}, n+m-\beta)} : \mathsf{X}_{n+m-\beta}^{\overline{\mu}} F^{-m} \to \mathsf{X}_{n+m-\beta}^{\overline{\nu}} F^{-m}, \qquad F = E_0^*. \tag{100}$$

Remark 4.21. Let $\beta \in \mathbb{R}$. By Theorem 4.16 we see that $\mathcal{A}^{(E_0,\beta)}$ is semi-Fredholm provided $\beta \notin \Gamma(\mathcal{A})$ and $(\mathcal{A}^*)^{(F^{-m}, n+m-\beta)}$ is semi-Fredholm provided $n+m-\beta \notin \Gamma(\mathcal{A}^*)$. These conditions can be seen to be equivalent by Proposition 4.20, which is expected since the maps $\mathcal{A}^{(E_0,\beta)}$ and $(\mathcal{A}^*)^{(F^{-m}, n+m-\beta)}$ are each other's adjoints.

Theorem 4.22. *Suppose $\mathcal{A}$ is an admissible elliptic operator on $\mathbb{R}^n$ of order (μ,ν), let E be an admissible space and choose $\beta \in \mathbb{R} \setminus \Gamma(\mathcal{A})$. Then the map $\mathcal{A} : \mathsf{X}_\beta^\mu E \to \mathsf{X}_\beta^\nu E$ is Fredholm.*

Proof. Since Theorem 4.16 shows that the map $\mathcal{A}^{(E,\beta)}$ is semi-Fredholm we simply have to establish that $\operatorname{Index} \mathcal{A}^{(E,\beta)}$ is finite. Furthermore, Theorem 4.19 shows this index to be stable under changes in E. It thus suffices to prove the result for a particular admissible space E; we take this to be $E = L^2$ so $E = E_0 = E_0^*$.

By (100) we have $(\mathcal{A}^{(E,\beta)})^* = (\mathcal{A}^*)^{(E^{-m}, n+m-\beta)}$. Furthermore, Proposition 4.20 implies $n+m-\beta \notin \Gamma(\mathcal{A}^*)$. Theorem 4.16 then shows that the maps $\mathcal{A}^{(E,\beta)}$ and $(\mathcal{A}^{(E,\beta)})^*$ are both semi-Fredholm with finite-dimensional kernels. However, we have the general identity

$$\operatorname{codim}(\operatorname{Ran}\mathcal{A}^{(E,\beta)}, \mathsf{X}^{\nu}_{\beta}E) = \dim\operatorname{Ker}(\mathcal{A}^{(E,\beta)})^*$$

for semi-Fredholm operators (see Theorem IV.5.13 in [Ka], for example). Therefore

$$\begin{aligned}\operatorname{Index}\mathcal{A}^{(E,\beta)} &= \dim\operatorname{Ker}\mathcal{A}^{(E,\beta)} - \operatorname{codim}(\operatorname{Ran}\mathcal{A}^{(E,\beta)}, \mathsf{X}^{\nu}_{\beta}E),\\ &= \dim\operatorname{Ker}\mathcal{A}^{(E,\beta)} - \dim\operatorname{Ker}(\mathcal{A}^*)^{(E^{-m},n+m-\beta)},\end{aligned}$$

which is finite. ∎

4.5. The change in index formula. This section is devoted to establishing the following result, which shows how the index of the Fredholm map $\mathcal{A} : \mathsf{X}^{\mu}_{\beta}E \to \mathsf{X}^{\nu}_{\beta}E$ varies when we change β and E over a greater range than is permitted in Theorem 4.19.

THEOREM 4.23. *Suppose E is an admissible space and $\beta_1, \beta_2 \in \mathbb{R} \setminus \Gamma(\mathcal{A})$ with $\beta_1 \le \beta_2$. Set $\Sigma = \{\lambda \in \sigma(\mathfrak{B}_{\mathcal{A}}) \mid \operatorname{Im}\lambda \in [\beta_1, \beta_2]\}$ and, for each $\lambda \in \Sigma$, let m_λ denote the algebraic multiplicity of λ. Then we have*

$$\operatorname{Index}\mathcal{A}^{(\beta_1)} = \operatorname{Index}\mathcal{A}^{(\beta_2)} + \sum_{\lambda\in\Sigma} m_\lambda.$$

By Remark 4.9 and Lemma 4.10, $\mathcal{A}$ can be approximated arbitrarily closely (simultaneously in $\mathscr{L}(\mathsf{X}^{\mu}_{\beta_j}E, \mathsf{X}^{\nu}_{\beta_j}E)$ for $j = 1, 2$) by another admissible elliptic operator $\mathcal{A}'$ whose coefficients agree with those of $\mathcal{A}_0$ on a neighbourhood of ∞. Since the set of Fredholm operators of a given index is open (in operator norm) and $\mathfrak{B}_{\mathcal{A}_0}$ is the operator pencil associated with both $\mathcal{A}$ and $\mathcal{A}'$, it suffices to prove the result assuming the coefficients of $\mathcal{A}$ and $\mathcal{A}_0$ agree on a neighbourhood of ∞. Choose $I \in \mathbb{N}$ so that

$$\zeta_I(\mathcal{A} - \mathcal{A}_0) = 0. \tag{101}$$

Set $M = \sum_{\lambda\in\Sigma} m_\lambda$ and let $\{w_1, \ldots, w_M\}$ denote any basis of the vector space X_Σ given by Theorem 3.26 for the operator $\mathcal{A}_0$.

LEMMA 4.24. *Let $f \in \mathsf{X}^{\nu}_{\beta_2}E$ and suppose $\mathcal{A}u = f$ for some $u \in \mathsf{X}^{\mu}_{\beta_1}E$. Then*

$$u = v + \zeta_I \sum_{j=1}^{M} z_j w_j \tag{102}$$

for some $v \in \mathsf{X}^{\mu}_{\beta_2}E$ and $z_1, \ldots, z_M \in \mathbb{C}$.

Proof. By Lemma 2.16, $\zeta_{I-1}u \in \mathsf{Y}^{\mu}_{\beta_1}E$. By setting $g = \mathcal{A}_0(\zeta_{I-1}u)$ it follows that $g \in \mathsf{Y}^{\nu}_{\beta_1}E$ and $g = 0$ on a neighbourhood of 0. On the other hand, (101) gives

$$\zeta_I g = \zeta_I \mathcal{A}_0(\zeta_{I-1}u) = \zeta_I \mathcal{A}u = \zeta_I f.$$

Thus Lemma 2.16 and Remark 2.18 imply $g \in \mathsf{Y}^{\nu}_{\beta_2}E$.

For $i = 1, 2$ set $u_i = (\mathcal{A}_0^{(\beta_i)})^{-1}g \in \mathsf{Y}^{\mu}_{\beta_i}E$. By Theorem 3.26 it follows that

$$u_1 - u_2 = \sum_{j=1}^{M} z_j w_j \tag{103}$$

for some $z_1, \ldots, z_M \in \mathbb{C}$. Now $u_2 \in \mathsf{Y}^{\mu}_{\beta_2}E$ so, setting $v = \eta_I u + \zeta_I u_2$, we get $v \in \mathsf{X}^{\mu}_{\beta_2}E$ by Lemma 2.16. On the other hand, $u_1 = \zeta_{I-1}u$ (this follows from the definition of g) so $\zeta_I u_1 = \zeta_I\zeta_{I-1}u = \zeta_I u$. Combining this with the definition of v and (103) we get (102). ∎

We have $\beta_1 \le \beta_2$ so $\mathsf{X}^{\mu}_{\beta_2}E \subseteq \mathsf{X}^{\mu}_{\beta_1}E$ (see (11)) and hence $\operatorname{Ker}\mathcal{A}^{(\beta_2)} \subseteq \operatorname{Ker}\mathcal{A}^{(\beta_1)}$. Since both kernels are finite-dimensional we can therefore choose $d \in \mathbb{N}_0$ and $u_1, \dots, u_d \in \operatorname{Ker}\mathcal{A}^{(\beta_1)} \subset \mathsf{X}^{\mu}_{\beta_1}E$ such that $u_1, \dots, u_d$ are linearly independent over $\mathsf{X}^{\mu}_{\beta_2}E$ and

$$\operatorname{Ker}\mathcal{A}^{(\beta_1)} = \operatorname{Ker}\mathcal{A}^{(\beta_2)} + \operatorname{Sp}\{u_1, \dots, u_d\}. \tag{104}$$

In particular,

$$\dim\operatorname{Ker}\mathcal{A}^{(\beta_1)} = \dim\operatorname{Ker}\mathcal{A}^{(\beta_2)} + d. \tag{105}$$

By Lemma 4.24 we can find $z_{ij} \in \mathbb{C}$ for $i \in \{1, \dots, d\}$ and $j \in \{1, \dots, M\}$ such that

$$u_i \;\equiv\; \zeta_I \sum_{j=1}^{M} z_{ij} w_j \ (\operatorname{mod} \mathsf{X}^{\mu}_{\beta_2}E)$$

for $i = 1, \dots, d$. Furthermore, by the linear independence of $u_1, \dots, u_d$ over $\mathsf{X}^{\mu}_{\beta_2}E$, the $d \times M$ matrix with entries z_{ij} must have rank d. Therefore $d \le M$ and we can choose the basis $\{w_1, \dots, w_M\}$ of X_Σ so that

$$u_i \;\equiv\; \zeta_I w_i \ (\operatorname{mod} \mathsf{X}^{\mu}_{\beta_2}E) \tag{106}$$

for $i = 1, \dots, d$. We will stick to such a choice of $\{w_1, \dots, w_M\}$ for the remainder of this section.

Let $Y = \operatorname{Ran}\mathcal{A}^{(\beta_1)} \cap \mathsf{X}^{\nu}_{\beta_2}E$.

LEMMA 4.25. *The space Y is closed in $\mathsf{X}^{\nu}_{\beta_2}E$ and satisfies*

$$\operatorname{codim}(Y, \mathsf{X}^{\nu}_{\beta_2}E) = \operatorname{codim}(\operatorname{Ran}\mathcal{A}^{(\beta_1)}, \mathsf{X}^{\nu}_{\beta_1}E).$$

Proof. Since $\mathsf{X}^{\mu}_{\beta_2}E \subseteq \mathsf{X}^{\mu}_{\beta_1}E$ we get $\operatorname{Ran}\mathcal{A}^{(\beta_2)} \subseteq Y \subseteq \mathsf{X}^{\nu}_{\beta_2}E$. However, $\operatorname{Ran}\mathcal{A}^{(\beta_2)}$ is closed in $\mathsf{X}^{\nu}_{\beta_2}E$ with finite codimension (by Theorem 4.22) so the same must be true for Y. Now choose a finite-dimensional space $V \subset \mathsf{X}^{\nu}_{\beta_2}E$ so that $\mathsf{X}^{\nu}_{\beta_2}E = V + Y$ and $V \cap Y = 0$.

Claim: $\mathsf{X}^{\nu}_{\beta_1}E = V + \operatorname{Ran}\mathcal{A}^{(\beta_1)}$. The inclusion $V \subset \mathsf{X}^{\nu}_{\beta_2}E \subseteq \mathsf{X}^{\nu}_{\beta_1}E$ immediately gives the $\supseteq$ inclusion. Now let $f \in \mathsf{X}^{\nu}_{\beta_1}E$. Put $v = \zeta_{I-1}(\mathcal{A}_0^{(\beta_1)})^{-1}(\zeta_I f) \in \mathsf{X}^{\mu}_{\beta_1}E$ and $g = f - \mathcal{A}v \in \mathsf{X}^{\nu}_{\beta_1}E$ (n.b. Lemma 2.16 and Theorem 3.25 ensure that v is well defined). With the help of (101) we thus have

$$\zeta_{I+1}g = \zeta_{I+1}f - \zeta_{I+1}\mathcal{A}_0(\zeta_{I-1}(\mathcal{A}_0^{(\beta_1)})^{-1}(\zeta_I f)) = 0.$$

Hence $g \in \mathsf{X}^{\nu}_{\beta_2}E$ by Remark 2.18. On the other hand, by the definition of V, we have $g = \mathcal{A}w + h$ for some $w \in \mathsf{X}^{\mu}_{\beta_1}E$ and $h \in V$. Setting $u = v + w \in \mathsf{X}^{\mu}_{\beta_1}E$ we then get

$$f = g + \mathcal{A}v = h + \mathcal{A}v + \mathcal{A}w = h + \mathcal{A}u \;\in\; V + \operatorname{Ran}\mathcal{A}^{(\beta_1)}.$$

Therefore $\mathsf{X}^{\nu}_{\beta_1}E \subseteq V + \operatorname{Ran}\mathcal{A}^{(\beta_1)}$, completing the Claim.

Now $V \subset \mathsf{X}^{\nu}_{\beta_2}E$ so

$$V \cap \operatorname{Ran}\mathcal{A}^{(\beta_1)} = V \cap \mathsf{X}^{\nu}_{\beta_2}E \cap \operatorname{Ran}\mathcal{A}^{(\beta_1)} = V \cap Y = 0.$$

Combined with the above Claim this now completes the result. ■

From Lemma 4.25 and the observation that $\operatorname{Ran}\mathcal{A}^{(\beta_2)} \subseteq Y \subseteq \mathsf{X}^{\nu}_{\beta_2}E$ we get

$$\begin{aligned}\operatorname{codim}(\operatorname{Ran}\mathcal{A}^{(\beta_1)}, \mathsf{X}^{\nu}_{\beta_1}E) &= \operatorname{codim}(Y, \mathsf{X}^{\nu}_{\beta_2}E) \\ &= \operatorname{codim}(\operatorname{Ran}\mathcal{A}^{(\beta_2)}, \mathsf{X}^{\nu}_{\beta_2}E) - \operatorname{codim}(\operatorname{Ran}\mathcal{A}^{(\beta_2)}, Y).\end{aligned} \tag{107}$$

Proof of Theorem 4.23. For $i = 1, \ldots, M-d$ set $f_i = \mathcal{A}(\zeta_I w_{i+d}) \in \operatorname{Ran} \mathcal{A}^{(\beta_1)} \subseteq \mathsf{X}^\nu_{\beta_1} E$. As $\zeta_{I+1}\mathcal{A}(\zeta_I w_{i+d}) = \zeta_{I+1}\mathcal{A}_0 w_{i+d} = 0$ (by (101) and Theorem 3.26; recall that $w_{i+d} \in X_\Sigma$) we immediately get $f_i \in \mathsf{X}^\nu_{\beta_2} E$ using Remark 2.18. Now let $W = \operatorname{Sp}\{f_1, \ldots, f_{M-d}\} \subset Y$.

Claim (i): *$f_1, \ldots, f_{M-d}$ are linearly independent over* $\operatorname{Ran} \mathcal{A}^{(\beta_2)}$. Suppose

$$\sum_{i=1}^{M-d} z_{i+d} f_i = \mathcal{A}u$$

for some $u \in \mathsf{X}^\mu_{\beta_2} E$ and $z_{d+1}, \ldots, z_M \in \mathbb{C}$. Thus

$$\zeta_I \sum_{i=1}^{M-d} z_{i+d} w_{i+d} - u \in \operatorname{Ker} \mathcal{A}^{(\beta_1)}$$

and so, by (104), there exists $v \in \operatorname{Ker} \mathcal{A}^{(\beta_2)}$ and $z_1, \ldots, z_d$ such that

$$\sum_{i=1}^{d} z_i u_i + \zeta_I \sum_{i=1}^{M-d} z_{i+d} w_{i+d} = u + v.$$

Now $u + v \in \mathsf{X}^\mu_{\beta_2} E$ so (106) implies $\zeta_I w \in \mathsf{X}^\mu_{\beta_2} E$ where $w = \sum_{i=1}^M z_i w_i$. Lemma 2.16 then implies $\zeta_I w \in \mathsf{Y}^\mu_{\beta_2} E$. On the other hand, $w \in X_\Sigma$ so $(1-\zeta_I)w \in \mathsf{Y}^\mu_{\beta_2} E$ and $\mathcal{A}_0 w = 0$ by Theorem 3.26. However, $\mathcal{A}_0 : \mathsf{Y}^\mu_{\beta_2} E \to \mathsf{Y}^\nu_{\beta_2} E$ is an isomorphism by Theorem 3.25. Thus $w = \sum_{i=1}^M z_i w_i = 0$. The fact that $\{w_1, \ldots, w_M\}$ is a basis for X_Σ now completes Claim (i).

Claim (ii): $Y = W + \operatorname{Ran} \mathcal{A}^{(\beta_2)}$. The inclusion $W + \operatorname{Ran} \mathcal{A}^{(\beta_2)} \subseteq Y$ is trivial. Now let $f \in Y$ so $f \in \mathsf{X}^\nu_{\beta_2} E$ and $f = \mathcal{A}u$ for some $u \in \mathsf{X}^\mu_{\beta_1} E$. By Lemma 4.24 and (106) we can thus find $v \in \mathsf{X}^\mu_{\beta_2} E$ and $z_1, \ldots, z_M \in \mathbb{C}$ such that

$$u = v + \sum_{i=1}^{d} z_i u_i + \sum_{i=1}^{M-d} z_{i+d} \zeta_I w_{i+d}.$$

Using the fact that $u_1, \ldots, u_d \in \operatorname{Ker} \mathcal{A}^{(\beta_1)}$ we then get

$$f = \mathcal{A}u = \mathcal{A}v + \sum_{i=1}^{M-d} z_{i+d} \mathcal{A}(\zeta_I w_{i+d}) = \mathcal{A}v + \sum_{i=1}^{M-d} z_{i+d} f_i \in \operatorname{Ran} \mathcal{A}^{(\beta_2)} + W.$$

Hence $Y \subseteq W + \operatorname{Ran} \mathcal{A}^{(\beta_2)}$, completing Claim (ii).

By Claim (i) we have $\dim W = M - d$ and $W \cap \operatorname{Ran} \mathcal{A}^{(\beta_2)} = 0$. Combined with Claim (ii) this implies that

$$\operatorname{codim}(\operatorname{Ran} \mathcal{A}^{(\beta_2)}, Y) = M - d. \tag{108}$$

By combining (105), (107) and (108) we finally get

$$\begin{aligned} \operatorname{Index} \mathcal{A}^{(\beta_1)} &= \dim \operatorname{Ker} \mathcal{A}^{(\beta_1)} - \operatorname{codim}(\operatorname{Ran} \mathcal{A}^{(\beta_1)}, \mathsf{X}^\nu_{\beta_1} E) \\ &= \dim \operatorname{Ker} \mathcal{A}^{(\beta_2)} + d - \operatorname{codim}(\operatorname{Ran} \mathcal{A}^{(\beta_2)}, \mathsf{X}^\nu_{\beta_2} E) + M - d \\ &= \operatorname{Index} \mathcal{A}^{(\beta_2)} + M, \end{aligned}$$

completing the result. ∎

4.6. Self-adjoint operators. Let $\mathcal{A}$ be an admissible elliptic operator on $\mathbb{R}^n$ of order (μ,ν). If we can determine the set $\sigma(\mathfrak{B}_{\mathcal{A}})$ together with the algebraic multiplicities of each point $\lambda \in \sigma(\mathfrak{B}_{\mathcal{A}})$, then Theorems 4.19 and 4.23 allow us to compute the change in the index of the map $\mathcal{A} : \mathsf{X}^{\mu}_{\beta}E \to \mathsf{X}^{\nu}_{\beta}E$ whenever we change the admissible space E or the parameter β within the set $\mathbb{R} \setminus \Gamma(\mathcal{A})$. In turn, this allows us to compute the actual index for any such pair (E,β) provided we know the index for one pair. In general, this requires further computation; however, if $\mathcal{A}$ is formally self-adjoint we can use symmetry to compute the index of $\mathcal{A}^{(E,\beta)}$ from knowledge of the spectrum of $\mathfrak{B}_{\mathcal{A}}$ alone.

THEOREM 4.26. *Suppose $\mathcal{A}$ is a formally self-adjoint admissible elliptic operator of order (μ,ν) and let $m = \max_i \mu_i$. Then $\Gamma(\mathcal{A})$ is symmetric about $(n+m)/2$. Furthermore, if E is an admissible space and $\beta \in \mathbb{R} \setminus \Gamma(\mathcal{A})$, then*

$$\operatorname{Index} \mathcal{A}^{(n+m-\beta)} = -\operatorname{Index} \mathcal{A}^{(\beta)}. \tag{109}$$

In particular, we either have $(n+m)/2 \notin \Gamma(\mathcal{A})$, in which case $\operatorname{Index} \mathcal{A}^{(n/2+m/2)} = 0$, or $(n+m)/2 \in \Gamma(\mathcal{A})$, in which case the sum of the algebraic multiplicities of those $\lambda \in \sigma(\mathfrak{B}_{\mathcal{A}})$ with $\operatorname{Im}\lambda = (n+m)/2$ is even (say $2d$ for some $d \in \mathbb{N}$) and

$$\operatorname{Index} \mathcal{A}^{(n/2+m/2-\varepsilon)} = d = -\operatorname{Index} \mathcal{A}^{(n/2+m/2+\varepsilon)}$$

for all sufficiently small $\varepsilon > 0$.

Proof. For the sake of convenience we set $l = (n+m)/2$. The symmetry of $\Gamma(\mathcal{A})$ about l follows directly from Proposition 4.20. Now let E be an admissible space and $\beta \in \mathbb{R} \setminus \Gamma(\mathcal{A})$. Using (100) and the assumption $\mathcal{A}^* = \mathcal{A}$, we deduce that the adjoint of the map $\mathcal{A}^{(E_0,\beta)}$ is $\mathcal{A}^{(F^{-m},2l-\beta)}$ where $F = E_0^*$. Using Theorem 4.19 together with the fact that the index of a Fredholm map changes sign when we take its adjoint (see Corollary IV.5.14 in [Ka], for example) we now get

$$\operatorname{Index} \mathcal{A}^{(E,2l-\beta)} = \operatorname{Index} \mathcal{A}^{(F^{-m},2l-\beta)} = -\operatorname{Index} \mathcal{A}^{(E_0,\beta)} = -\operatorname{Index} \mathcal{A}^{(E,\beta)},$$

which establishes (109).

If $l \notin \Gamma(\mathcal{A})$ then we get $\operatorname{Index} \mathcal{A}^{(l)} = 0$ by setting $\beta = l$ in (109). Now suppose $l \in \Gamma(\mathcal{A})$. Since $\Gamma(\mathcal{A})$ consists of isolated points (see Remark 3.9) it follows that we can find some $\delta > 0$ such that $(l, l+\delta) \cap \Gamma(\mathcal{A}) = \emptyset$. Theorems 4.19 and 4.22 together with (109) then imply the existence of $d \in \mathbb{Z}$ such that

$$\operatorname{Index} \mathcal{A}^{(l-\varepsilon)} = d = -\operatorname{Index} \mathcal{A}^{(l+\varepsilon)}$$

for all $0 < \varepsilon < \delta$. By Theorem 4.23 we also know that

$$2d = \operatorname{Index} \mathcal{A}^{(l-\varepsilon)} - \operatorname{Index} \mathcal{A}^{(l+\varepsilon)} = \sum_{\lambda \in \Sigma} m_\lambda,$$

where $\Sigma = \{\lambda \in \sigma(\mathfrak{B}_{\mathcal{A}}) \mid \operatorname{Im}\lambda = l\}$ and m_λ is the algebraic multiplicity of a given $\lambda \in \Sigma$. ∎

4.7. Homogeneous operators with constant coefficients. Let $\mathcal{A}$ be a $k\times k$ system of differential operators on $\mathbb{R}^n$ of order (μ,ν). We say that $\mathcal{A}$ is a *homogeneous constant coefficient operator* if, for each $i,j\in\{1,\dots,k\}$, the ijth entry of $\mathcal{A}$ is a constant coefficient scalar differential operator which is homogeneous of order $\mu_j-\nu_i$ (or equal to 0 if $\mu_j-\nu_i<0$).

Remark 4.27. Suppose $\mathcal{A}$ is an elliptic homogeneous constant coefficient operator on $\mathbb{R}^n$ of order (μ,ν). It follows immediately that $\mathcal{A}$ is uniformly elliptic on $\mathbb{R}^n$. On the other hand, $A_{ij}(x,D_x)$, the ijth entry of $\mathcal{A}$, is a constant coefficient scalar differential operator which is homogeneous of order $\mu_j-\nu_i$. Thus $A_{ij}(x,D_x)$ is an admissible scalar operator with principal part $A_{ij}(x,D_x)$ (see Definition 4.1). It follows that $\mathcal{A}$ is an admissible elliptic operator on $\mathbb{R}^n$ with principal part $\mathcal{A}_0=\mathcal{A}$.

Let $\mathcal{A}$ be an admissible elliptic operator on $\mathbb{R}^n$ of order (μ,ν) whose principal part $\mathcal{A}_0$ is a homogeneous constant coefficient operator. Set $\mathcal{Q}=\mathcal{A}-\mathcal{A}_0$ and let Q_{ij} denote the entries of $\mathcal{Q}$ for $i,j\in\{1,\dots,k\}$. Then Q_{ij} is a scalar admissible operator on $\mathbb{R}^n$ of order $\mu_j-\nu_i$ whose principal part is equal to 0. We say that $\mathcal{A}$ *has constant leading order coefficients* if, for each $i,j\in\{1,\dots,k\}$, $\operatorname{ord}Q_{ij}<\mu_j-\nu_i$ (or $Q_{ij}=0$ if $\mu_j-\nu_i\le 0$). The next result indicates the importance of such operators.

Proposition 4.28. *Let $\mathcal{A}$ be an admissible elliptic operator on $\mathbb{R}^n$ of order (μ,ν) whose principal part $\mathcal{A}_0$ is a homogeneous constant coefficient operator. Furthermore, suppose $\mathcal{A}$ has constant leading order coefficients. Then, for any admissible space E and $\beta\in\mathbb{R}$, the operator $\mathcal{Q}:=\mathcal{A}-\mathcal{A}_0$ defines a compact map $\mathsf{X}^{\mu}_{\beta}E\to\mathsf{X}^{\nu}_{\beta}E$.*

Proof. Consider the notation introduced before the statement of the result and let $i,j\in\{1,\dots,k\}$. Therefore we can write

$$Q_{ij}(x,D_x)=\sum_{|\alpha|<\mu_j-\nu_i} q^{\alpha}(x)\,D_x^{\alpha} \tag{110}$$

where, for each multi-index α with $|\alpha|<\mu_j-\nu_i$, $q^{\alpha}\in\mathrm{Sc}^{\mu_j-\nu_i-|\alpha|}$ has principal part equal to 0. By (37), Proposition 2.54 and the definition of the principal part (see Definition 1.1) it follows that $q^{\alpha}\in\mathsf{X}_{\mu_j-\nu_i-|\alpha|}C^l_0$ for any $l\in\mathbb{N}_0$. Lemma 2.49, Proposition 2.76 and the fact that $\mu_j-|\alpha|>\nu_i$ can now be combined to show that multiplication by q^{α} defines a compact map $\mathsf{X}_{\beta-\mu_j+|\alpha|}E^{\mu_j-|\alpha|}\to\mathsf{X}_{\beta-\nu_i}E^{\nu_i}$. Together with Proposition 2.33 and (110) this finally implies that Q_{ij} defines a compact map $\mathsf{X}_{\beta-\mu_j}E^{\mu_j}\to\mathsf{X}_{\beta-\nu_i}E^{\nu_i}$. The result now follows. ■

A straightforward application of Theorems 4.19 and 4.23 allows us to generalise an index formula given in [LM1] to cover arbitrary admissible spaces. Before giving the result we need to introduce the following notation. For any $\beta\in\mathbb{R}$ let $P_n(\beta)$ denote the dimension of the set of polynomials in n variables whose degree does not exceed β (with $P_n(\beta)=0$ when $\beta<0$). Thus

$$P_n(\beta)=\begin{cases}P_n(l) & \text{if } \beta\in[l,l+1) \text{ for some } l\in\mathbb{N}_0,\\ 0 & \text{if } \beta<0.\end{cases} \tag{111}$$

For any $\mu, \nu \in \mathbb{N}_0^k$ and $\beta \in \mathbb{R}$ we set

$$P_n^{\mu,\nu}(\beta) = \sum_{i=1}^{k} (P_n(-\beta + \mu_i) - P_n(-\beta + \nu_i) - P_n(\beta - \nu_i - n) + P_n(\beta - \mu_i - n)).$$

It is clear from (111) that the function $P_n^{\mu,\nu}$ is constant on the components of $\mathbb{R} \setminus \mathbb{Z}$.

THEOREM 4.29. *Suppose $\mathcal{A}$ is an admissible elliptic operator on $\mathbb{R}^n$ of order (μ, ν) whose principal part is a homogeneous constant coefficient operator. Then $\Gamma(\mathcal{A}) = \{\beta \in \mathbb{R} \mid P_n^{\mu,\nu}(\beta^-) \neq P_n^{\mu,\nu}(\beta^+)\} \subseteq \mathbb{Z}$ and there exists $d \in \mathbb{Z}$ such that* $\operatorname{Index} \mathcal{A}^{(E,\beta)} = P_n^{\mu,\nu}(\beta) + d$ *for any admissible space E and $\beta \in \mathbb{R} \setminus \Gamma(\mathcal{A})$. In particular, $\Gamma(\mathcal{A}) \cap (m, n) = \emptyset$ and* $\operatorname{Index} \mathcal{A}^{(E,\beta)} = d$ *for any $\beta \in (m, n)$, where $m = \max_i \mu_i$.*

Proof. With the help of Remark 4.27 we see that $\mathcal{A}$ and its principal part $\mathcal{A}_0$ are two admissible elliptic operators with the same principal part. It follows that these operators have the same associated operator pencil. Theorem 4.19 then implies that $\operatorname{Index} \mathcal{A}^{(E,\beta)}$ and $\operatorname{Index} \mathcal{A}_0^{(E,\beta)}$ are both constant (with possibly different values) when β remains within a single component of $\mathbb{R} \setminus \Gamma(\mathcal{A}) = \mathbb{R} \setminus \Gamma(\mathcal{A}_0)$. On the other hand, Theorem 4.23 implies that $\operatorname{Index} \mathcal{A}^{(E,\beta)}$ and $\operatorname{Index} \mathcal{A}_0^{(E,\beta)}$ change by the same amount as β changes between the components of $\mathbb{R} \setminus \Gamma(\mathcal{A})$. It follows that there exists $d \in \mathbb{Z}$ such that $\operatorname{Index} \mathcal{A}^{(E,\beta)} = \operatorname{Index} \mathcal{A}_0^{(E,\beta)} + d$ for all $\beta \in \mathbb{R} \setminus \Gamma(\mathcal{A})$. Furthermore, Theorem 4.19 implies that d cannot depend on the admissible space E.

The preceding argument means it suffices to prove the result for $\mathcal{A}_0$; that is, we can assume the admissible elliptic operator $\mathcal{A}$ to be a homogeneous constant coefficient operator. In this case Theorem 3 of [LM1] shows that $\mathcal{A}^{(L^2,\beta)}$ is a Fredholm map with index $P_n^{\mu,\nu}(\beta)$ whenever $\beta \notin \mathbb{Z}$. On the other hand, $\Gamma(\mathcal{A})$ is a discrete subset of $\mathbb{R}$ (see Remark 3.9) whilst Theorems 4.19 and 4.23 imply that the function $\beta \mapsto \operatorname{Index} \mathcal{A}_0^{(E,\beta)}$ is independent of the admissible space E, is constant on the components of $\mathbb{R} \setminus \Gamma(\mathcal{A})$, and satisfies

$$\operatorname{Index} \mathcal{A}^{(E,\beta^-)} \neq \operatorname{Index} \mathcal{A}^{(E,\beta^+)}$$

for all $\beta \in \Gamma(\mathcal{A})$. It follows that $\Gamma(\mathcal{A})$ is precisely the set of points at which $P_n^{\mu,\nu}$ is discontinuous, whilst $\operatorname{Index} \mathcal{A}^{(E,\beta)} = P_n^{\mu,\nu}(\beta)$ for any admissible space E and $\beta \in \mathbb{R} \setminus \Gamma(\mathcal{A})$.

If $\nu_i > m$ for some i then the ith row of the matrix operator $\mathcal{A}$ would be 0. As this contradicts the assumption that $\mathcal{A}$ is elliptic we get $\max_i \nu_i \le m$. Coupled with (111) and the fact that μ and ν are vectors of non-negative integers, this clearly shows that all the terms in the sum defining $P_n^{\mu,\nu}(\beta)$ are equal to 0 when $\beta \in (m, n)$. ■

In general, the constant d appearing in Theorem 4.29 may be non-zero—see [LM2] for an example of an operator for which this is the case. On the other hand, the next result gives two useful special cases in which we can show $d = 0$.

THEOREM 4.30. *Let $\mathcal{A}$ and d be as in Theorem 4.29. Then we have $d = 0$ if $\mathcal{A}$ is either formally self-adjoint or has constant leading order coefficients.*

Proof. Let (μ, ν) be the order of $\mathcal{A}$ and define m, $\overline{\mu}$ and $\overline{\nu}$ as in (56). It is then straightforward to check that $P_n^{\mu,\nu}(m + n - \beta) = -P_n^{\overline{\mu},\overline{\nu}}(\beta)$ for all $\beta \in \mathbb{R} \setminus \mathbb{Z}$. If $\mathcal{A}$ is formally self-adjoint then $\overline{\mu} = \mu$ and $\overline{\nu} = \nu$ so Theorems 4.26 and 4.29 now give

$$P_n^{\mu,\nu}(\beta) + d = \operatorname{Index} \mathcal{A}^{(\beta)} = -\operatorname{Index} \mathcal{A}^{(n+m-\beta)} = -P_n^{\mu,\nu}(n+m-\beta) - d = P_n^{\mu,\nu}(\beta) - d$$

for all $\beta \in \mathbb{R} \setminus \mathbb{Z}$. From this it clearly follows that $d = 0$.

On the other hand, if $\mathcal{A}$ has constant leading order coefficients then Proposition 4.28 implies that $\mathcal{A}$ is a compact perturbation of its principal part $\mathcal{A}_0$. However, Theorem 4.29 holds with $d = 0$ for the homogeneous constant coefficient operator $\mathcal{A}_0$ (this is clear from the proof of that theorem). The result is now completed by the fact that a compact perturbation does not affect the index of a Fredholm operator (see Theorem IV.5.26 in [Ka], for example). ∎

Theorem 4.29 can be made more explicit for several special classes of operators; these classes include Dirac type operators and the Laplacian.

THEOREM 4.31. *Suppose $\mathcal{A}$ is an admissible elliptic operator on $\mathbb{R}^n$ of order (μ, ν) whose principal part is a homogeneous constant coefficient operator. Also suppose $n \geq 2$, $\nu_1 = \ldots = \nu_k = 0$, $\mu_1 = \ldots = \mu_k = m$ for some $m \in \{1, 2\}$, and that $\mathcal{A}$ is either formally self-adjoint or has constant leading order coefficients. Then $\Gamma(\mathcal{A}) = \mathbb{Z} \setminus \{m+1, \ldots, n-1\}$ (with $\Gamma(\mathcal{A}) = \mathbb{Z}$ if $n < m+2$). Furthermore, for any admissible space E and $\beta \in \mathbb{R} \setminus \Gamma(\mathcal{A})$,*

$$\operatorname{Index} \mathcal{A}^{(E,\beta)} = -\frac{k}{(n-1)!}\,(l-1) \prod_{j=2}^{n-1} |l-j| \tag{112}$$

if $m = 1$ and

$$\operatorname{Index} \mathcal{A}^{(E,\beta)} = -\frac{k}{(n-1)!}\,(2l-n-1) \prod_{j=2}^{n-1} |l-j| \tag{113}$$

if $m = 2$, where $l \in \mathbb{Z}$ is chosen so that $\beta \in [l, l+1)$ and the product terms are defined to be equal to 1 if $n = 2$. In particular, $\operatorname{Index} \mathcal{A}^{(E,\beta)} = 0$ *for any $\beta \in (m, n)$.*

REMARK 4.32. The special form of μ and ν means that the domain and codomain of the map $\mathcal{A}^{(E,\beta)}$ are given as

$$\mathsf{X}_\beta^\mu E = \prod_{i=1}^k \mathsf{X}_{\beta-m} E^m \quad \text{and} \quad \mathsf{X}_\beta^\nu E = \prod_{i=1}^k \mathsf{X}_\beta E$$

respectively.

Proof of Theorem 4.31. We have $P_1(l) = l+1$ for any $l \in \mathbb{N}_0$ and $P_n(0) = 1$ for any $n \in \mathbb{N}$. On the other hand, the set of polynomials in $n+1$ variables which are homogeneous of degree l can be seen to have dimension $P_n(l)$. Hence $P_{n+1}(l) = P_{n+1}(l-1) + P_n(l)$ for all $n, l \in \mathbb{N}$. The resulting recurrence relations have the unique solution

$$P_n(l) = \frac{(l+n)!}{n!\,l!} \tag{114}$$

for all $n \in \mathbb{N}$ and $l \in \mathbb{N}_0$. A straightforward calculation using (111) and (114) now shows that if $\beta \in (l, l+1)$ for some $l \in \mathbb{Z}$ then $P_n^{\mu,\nu}(\beta)$ is given explicitly by the right hand side of (112) or (113) when $m = 1$ or 2 respectively. The result then follows from Theorems 4.29 and 4.30, and the fact that the right hand sides of (112) and (113) are non-increasing functions of $l \in \mathbb{Z}$ which are strictly decreasing outside $\{m+1, \ldots, n-1\}$. ∎

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