1. Introduction

Plan of the Introduction

Section 1.1
The section starts with a statement of two theorems which exemplify the type of results proved in this work. The notions of a faithful class and of a determining category are then introduced. A class of topological spaces is said to be faithful if its members are reconstructible from their homeomorphism groups. Example 1.2 contains a short survey of older reconstructibily theorems, and Example 1.3 mentions several determining categories. We then describe the precise forms of the theorems which will be proved in this work.
Section 1.2
This section summarizes Chapter 2. The theorems described in 1.2 have the form: If for $i = 1, 2$, $G_i \leq H(X_i)$ and φ is an isomorphism between G_1 and G_2, then there is a homeomorphism τ between X_1 and X_2 such that τ induces φ.
Section 1.3
This section is a summary of Chapters 3 and 4. It starts with the definition of a modulus of continuity. A modulus of continuity Γ is a set of functions from $[0, \infty)$ to $[0, \infty)$ which serves as a measure for the continuity of a uniformly continuous function. With Γ one associates the group $H_\Gamma^{\mathrm{LC}}(X)$ of locally Γ-bicontinuous homeomorphisms of X. The reconstruction result for groups of type $H_\Gamma^{\mathrm{LC}}(X)$ says that any isomorphism between $H_\Gamma^{\mathrm{LC}}(X)$ and $H_\Gamma^{\mathrm{LC}}(Y)$ is induced by a locally Γ-bicontinuous homeomorphism between X and Y.
Section 1.4
Section 1.4 summarizes the reconstruction theorems for the group $\mathrm{UC}(X)$ of uniformly bicontinuous homeomorphisms of X. These theorems appear in Chapter 5.
Section 1.5
The previous sections dealt mainly with spaces which are an open subset of a normed vector space. This section describes the reconstruction theorems for spaces which are the closure of an open subset of a normed vector space. These theorems appear in Chapter 6. Section 1.5 also includes a survey of the results of Chapter 7.
Section 1.6
Let X be the closure of an open subset of a normed space. Chapters 8–12 deal with the group $H_\Gamma^{\mathrm{LC}}(X)$ when X is such a space. Section 1.6 describes the results obtained in these chapters.
Section 1.7
This section contains a discussion and open problems.
Section 1.8
This section contains a short historical survey.

1.1. General description. This work concerns groups of auto-homeomorphisms of open subsets of normed vector spaces and of manifolds over normed vector spaces. Mainly, we consider groups whose definition is based on the metric of the normed space, for example, the group of all bilipschitz auto-homeomorphisms of such a space.

Two types of results are proved. The following statement is an example of the first type.

1. Suppose that X_1, X_2 are open subsets of the Banach spaces spaces E_1 and E_2 respectively. For $i = 1, 2$ let G_i be a group of auto-homeomorphisms of X_i such that every bilipschitz homeomorphism of X_i belongs to G_i. Suppose that φ is a group isomorphism between G_1 and G_2. Then there is a homeomorphism τ between X_1 and X_2 such that for every $g \in G_1$, $\varphi(g) = \tau \circ g \circ \tau^{-1}$.

An example of the second type of results is as follows.

2. $\mathrm{BL}(E)$ denotes the group of all auto-homeomorphisms f of a Banach space E such that f and f^{-1} are Lipschitz on every bounded set, and $\mathrm{BUC}(E)$ denotes the group of all auto-homeomorphisms f of E such that f and f^{-1} are uniformly continuous on every bounded set. These groups determine the spaces they act upon in the following sense.

(a) Suppose that E_1 and E_2 are Banach spaces, and φ is a group isomorphism between $\mathrm{BL}(E_1)$ and $\mathrm{BL}(E_2)$. Then there is a unique homeomorphism τ between E_1 and E_2 such that for every $f \in \mathrm{BL}(E_1)$, $\varphi(f) = \tau \circ f \circ \tau^{-1}$. Also, τ and τ^{-1} are Lipschitz on every bounded set (τ is BL).

(b) The same holds for groups of the type $\mathrm{BUC}(E)$. That is, the statement obtained from (a) by replacing BL by BUC is true.

(c) For every E_1 and E_2, $\mathrm{BL}(E_1)$ and $\mathrm{BUC}(E_2)$ are not isomorphic.

Terminology. The notation $f : X \cong Y$ means that f is a homeomorphism between the topological spaces X and Y. That is, f is bijective, and f and f^{-1} are continuous. Let $H(X) = \{f \mid f : X \cong X\}$. If G, H are groups, then $\varphi : G \cong H$ means that φ is an isomorphism between G and H. The ordered pair with elements a and b is denoted by $\langle a, b \rangle$.

Definition 1.1. (a) A pair $\langle X, G \rangle$ consisting of a topological space X and a group G of auto-homeomorphisms of X is called a *space-group pair*. Let K be a class of space-group pairs. K is *faithful* if for every $\langle X_1, G_1 \rangle, \langle X_2, G_2 \rangle \in K$ and $\varphi : G_1 \cong G_2$ there exists $\tau : X_1 \cong X_2$ which induces φ. That is, for every $f \in G_1$, $\varphi(f) = \tau \circ f \circ \tau^{-1}$.

A class K of topological spaces is *faithful* if $\{\langle X, H(X) \rangle \mid X \in K\}$ is faithful.

(b) A *restricted topological category* is a category $\boldsymbol{K}$ whose objects are topological spaces, in which every morphism between two objects X and Y of $\boldsymbol{K}$ is a homeomorphism from X onto Y, and in which for every morphism g of $\boldsymbol{K}$, g^{-1} also belongs to $\boldsymbol{K}$. For every $X, Y \in \boldsymbol{K}$ let $\mathrm{Iso}_{\boldsymbol{K}}(X, Y)$ denote the set of morphisms between X and Y and $\mathrm{Aut}_{\boldsymbol{K}}(X) = \mathrm{Iso}_{\boldsymbol{K}}(X, X)$.

We say that $\boldsymbol{K}$ is a *determining category* if for every $X, Y \in \boldsymbol{K}$ and a group isomorphism $\varphi : \mathrm{Aut}_{\boldsymbol{K}}(X) \cong \mathrm{Aut}_{\boldsymbol{K}}(Y)$ there is $\tau \in \mathrm{Iso}_{\boldsymbol{K}}(X, Y)$ such that $\varphi(g) = \tau \circ g \circ \tau^{-1}$ for every $g \in \mathrm{Aut}_{\boldsymbol{K}}(X)$.

Let $\boldsymbol{K}, \boldsymbol{L}$ be restricted topological categories. $\boldsymbol{K}, \boldsymbol{L}$ are said to be *distinguishable* if for every $X \in \boldsymbol{K}$ and $Y \in \boldsymbol{L}$: if $\mathrm{Aut}_{\boldsymbol{K}}(X) \cong \mathrm{Aut}_{\boldsymbol{L}}(Y)$, then

$$X \in \boldsymbol{L} \text{ and } \mathrm{Aut}_{\boldsymbol{K}}(X) = \mathrm{Aut}_{\boldsymbol{L}}(X) \quad \text{or} \quad Y \in \boldsymbol{K} \text{ and } \mathrm{Aut}_{\boldsymbol{L}}(Y) = \mathrm{Aut}_{\boldsymbol{K}}(Y). \ \square$$

The above notions provide a convenient way for stating the second type of results in this work. However, we shall not use other notions or any techniques from category theory.

Some faithful classes of topological spaces and some determining categories are listed in the next two examples. The lists are not exhaustive.

EXAMPLES 1.2. The following classes are faithful.

(a) The class of Euclidean manifolds. This was proved by J. Whittaker [W] (published 1963).

(b) The class of manifolds over the Hilbert cube. This was proved by R. McCoy [McC] (published 1972).

(c) The class Euclidean manifolds with boundary. This was proved by M. Rubin [Ru1] (published 1989).

(d) The class of all spaces $\langle X, \tau \rangle$ such that:

(1) X is a polyhedron, and τ is either the metric or the coherent topology of X,
(2) the simplicial complex defining X does not have an infinite increasing (with respect to inclusion) sequence of simplexes,
(3) for every $x \in X$, $\{h(x) \mid h \in H(X)\}$ has no isolated points.

This was proved by M. Rubin [Ru1].

(e) The class of all manifolds over normed vector spaces. This was proved by M. Rubin [Ru1].

(f) The class of manifolds over the class of real topological vector spaces which are locally convex, normal and have a nonempty open set which intersects every straight line in a bounded set. This was proved by A. Leiderman and M. Rubin [LR] (published 1999). $\square$

EXAMPLES 1.3. The following are determining categories.

(a) For $n \leq \infty$ let $\boldsymbol{K}_n^C$ be the category of C^k-smooth manifolds. The morphisms of $\boldsymbol{K}_n^C$ are the homeomorphisms f such that f and f^{-1} are k times continuously differentiable. This was proved in [Fi] (R. Filipkiewicz 1982), but was earlier proved by W. Ling in [Lg1] and [Lg2] (unpublished preprint, 1980). See the topic "Reconstruction questions for related groups" in Subsection 1.7 of the Introduction.

(b) The categories arising from C^k-smooth Euclidean manifolds carrying various types of additional structure, the morphisms being the C^k-diffeomorphisms which preserve that structure. These are determining categories. This includes e.g. foliated manifolds (Ling [Lg1] and [Lg2]) and symplectic manifolds (Banyaga [Ba1] 1997). See the topic "Reconstruction questions for related groups" in Subsection 1.7 for more details.

(c) The category of open subsets of $\mathbb{R}^n$ with quasi-conformal homeomorphisms as morphisms. This was proved by V. Gol'dshtein and M. Rubin [GR] (1995). $\square$

Continuing the investigaton of faithful classes and determining categories, we consider topological spaces with extra structure. The spaces considered in this work are open subsets of a normed vector space, and more generally, manifolds over normed vector spaces. We also consider sets which are the closures of open subsets of a normed space.

If X is an open subset of a normed space E, the "extra structure" attached to X is usually the object $\langle X, \mathrm{bd}^E(X), d\rangle$, where $\mathrm{bd}^E(X)$ is the boundary of X in E, and d is the metric on $\mathrm{cl}^E(X)$ inherited from E ($\mathrm{cl}^E(X)$ denotes the closure of X in E). The methods of this work can be applied to more general "extra structures". See Remarks 6.25 and 6.28.

This extra structure is used to define various subgroups of $H(X)$. The groups $\mathrm{BL}(X)$ and $\mathrm{BUC}(X)$ defined at the beginning of Subsection 1.1 are examples of such subgroups. Another typical example is as follows. Let X, Y be open subsets of the normed spaces E and F respectively. A homeomorphism $h : X \cong Y$ is said to be *extendible* if there is a continuous function $\bar{h} : \mathrm{cl}(X) \to \mathrm{cl}(Y)$ such that $\bar{h}$ extends h. We consider the group $\mathrm{EXT}(X) := \{h \in H(X) \mid h \text{ and } h^{-1} \text{ are extendible}\}$.

A homeomorphism $h : X \cong Y$ is said to be *completely locally uniformly continuous* (*CMP.LUC*) if h is extendible, and for every $x \in \mathrm{cl}(X)$ there is a neighborhood U of x in $\mathrm{cl}(X)$ such that $h \restriction (U \cap X)$ is uniformly continuous. We also consider the group

$$\mathrm{CMP.LUC}(X) := \{h \in H(X) \mid h \text{ and } h^{-1} \text{ are CMP.LUC}\}.$$

The setting is thus as follows. We shall have a class $\mathcal{M}$ of topological spaces. Usually this class consists of spaces X such that either X is an open subset or the closure of an open subset of a normed vector space, or even more generally, X can be the closure of an open subset of a manifold over a normed vector space. $\mathcal{P}$ and $\mathcal{Q}$ are properties of maps between X and Y defined for objects of the form $\langle X, \mathrm{bd}(X), d\rangle$. The set $\mathcal{P}(X)$ of all homeomorphisms $f \in H(X)$ such that f and f^{-1} have property $\mathcal{P}$ is a subgroup of $H(X)$, and the same holds for $\mathcal{Q}(Y)$. The final results have the following form.

If $X, Y \in \mathcal{M}$ and $\varphi : \mathcal{P}(X) \cong \mathcal{Q}(Y)$, then

(1) φ is induced by a unique homeomorphism $\tau : X \cong Y$,
(2) $\mathcal{P}(X) = \mathcal{Q}(X)$ and τ and τ^{-1} have property $\mathcal{Q}$, or $\mathcal{P}(Y) = \mathcal{Q}(Y)$ and τ and τ^{-1} have property $\mathcal{P}$.

Let $\boldsymbol{K}_{\mathcal{M},\mathcal{P}}$ be the following category.

(a) The class of objects of $\boldsymbol{K}_{\mathcal{M},\mathcal{P}}$ is $\mathcal{M}$.
(b) The class of morphisms of $\boldsymbol{K}_{\mathcal{M},\mathcal{P}}$ is $\{g : X \cong Y \mid X, Y \in \mathcal{M} \text{ and } g \text{ and } g^{-1} \text{ have property } \mathcal{P}\}$.

Conclusion (1)–(2) is the same as saying that $\boldsymbol{K}_{\mathcal{M},\mathcal{P}}$ and $\boldsymbol{K}_{\mathcal{M},\mathcal{Q}}$ are determining categories and $\boldsymbol{K}_{\mathcal{M},\mathcal{P}}$ and $\boldsymbol{K}_{\mathcal{M},\mathcal{Q}}$ are distinguishable.

This work uses only elementary facts. It is self-contained except for Theorem 2.3 which is taken from [Ru5]; it is stated there as Corollary 1.4 on page 122, and it is proved there in Corollary 2.10 on page 131.

Theorem 2.3 says that given a pair (X, G), where G is a subgroup of $H(X)$ satisfying certain weak transitivity requirements, it is possible to recover from G the Boolean algebra $\mathrm{Ro}(X)$ of regular open subsets of X, together with the action of G on $\mathrm{Ro}(X)$. (A set U is *regular open* if U is equal to the interior of its closure.)

Consider the structures $(G, \mathrm{Ro}(X); \lambda_G^{\mathrm{Ro}(X)})$ and $(G, X; \lambda_G^X)$, where $\lambda_G^{\mathrm{Ro}(X)}$ and λ_G^X denote the action of G on $\mathrm{Ro}(X)$ and on X respectively. The essence of Chapter 2 is showing that for appropriate classes of (X, G)'s, $(G, X; \lambda_G^X)$ can be recovered from $(G, X; \lambda_G^{\mathrm{Ro}(X)})$. This kind of argument appears in Theorems 2.5, 2.8, 2.30 and 8.8.

1.2. Faithfulness of classes of space-group pairs. Chapter 2 deals with the faithfulness of classes of space-group pairs. We introduce some terminology.

DEFINITION 1.4. (a) A homeomorphism h between two metric spaces (X, d^X) and (Y, d^Y) is *Lipschitz* if there is $K > 0$ such that $d^Y(h(u), h(v)) \leq K d^X(u, v)$ for every $u, v \in X$. We say that h is *bilipschitz* if both h and h^{-1} are Lipschitz homeomorphisms. Define

$$\mathrm{LIP}(X) := \{h \in H(X) \mid h \text{ is bilipschitz}\}.$$

(b) Let X, Y be metric spaces. A homeomorphism h between X and Y is *locally Lipschitz* if for every $u \in X$ there is a neighborhood U of u such that $h\restriction U$ is Lipschitz. h is *locally bilipschitz* if both h and h^{-1} are locally Lipschitz. Define

$$\mathrm{LIP}^{\mathrm{LC}}(X) := \{h \in H(X) \mid h \text{ is locally bilipschitz}\}.$$

(c) If $S \subseteq X$ is open, then

$$\mathrm{LIP}(X, S) := \{h \in \mathrm{LIP}(X) \mid h\restriction(X - S) = \mathrm{Id}\}.$$

(d) Let E be a normed vector space, F be dense linear subspace of E, and X be an open subset of E. Set

$$\mathrm{LIP}(X; F) := \{h \in \mathrm{LIP}(X) \mid h(X \cap F) = X \cap F\}.$$

(e) For E, F, X, S as above we define

$$\mathrm{LIP}(X; S, F) := \mathrm{LIP}(X; F) \cap \mathrm{LIP}(X, S).$$

(f) $\mathrm{LIP}^{\mathrm{LC}}(X, S)$, $\mathrm{LIP}^{\mathrm{LC}}(X; F)$ and $\mathrm{LIP}^{\mathrm{LC}}(X; S, F)$ are defined analogously.

(g) Let $G \leq H$ mean that G is a subgroup of H.

(h) For a normed vector space E, $x \in E$ and $r > 0$ let

$$B^E(x, r) = \{y \in E \mid \|y - x\| < r\}.$$

Note that $\mathrm{LIP}(X, S)$ and $\mathrm{LIP}(X; F)$ are subgroups of $H(X)$. □

The main result of Chapter 2 is part (c) of the next theorem. It is restated as Theorem 2.8(b). Parts (a) and (b) of Theorem 1.5 are special cases of (c). They are more frequently used, and are more readable.

THEOREM 1.5. (a) *Let K be the class of all pairs $\langle X, G\rangle$ such that X is an open subset of some Banach space and* $\mathrm{LIP}(X) \leq G \leq H(X)$. *Then K is faithful.*

(b) *Let K be the class of all pairs $\langle X, G\rangle$ such that X is an open subset of some normed vector space and* $\mathrm{LIP}^{\mathrm{LC}}(X) \leq G \leq H(X)$. *Then K is faithful.*

(c) *The class K of all pairs $\langle X, G\rangle$ which satisfy* (1) *and* (2), *or* (3) *and* (4) *below is faithful.*

(1) *X is an open subset of some Banach space E and $G \leq H(X)$.*
(2) *For every $x \in X$ there are an open set $S \subseteq X$ containing x and a dense linear subspace $F \subseteq E$ such that* $\mathrm{LIP}(X; S, F) \leq G$.
(3) *X is an open subset of some normed vector space E and $G \leq H(X)$.*
(4) *For every $x \in X$ there are an open set $S \subseteq X$ containing x and a dense linear subspace $F \subseteq E$ such that* $\mathrm{LIP}^{\mathrm{LC}}(X; S, F) \leq G$.

Compare parts (a) and (b) of Theorem 1.5. Part (a) deals with Banach spaces, and assumes that $\mathrm{LIP}(X) \le G$. Part (b) deals with normed spaces, but assumes that $\mathrm{LIP}^{\mathrm{LC}}(X) \le G$. It is unknown whether in (b), assuming only that $\mathrm{LIP}(X) \le G$ suffices. The following theorem contains the strongest known fact regarding this question. It is restated as Corollary 2.26.

For a metric space Z, $x \in Z$ and $r > 0$ let $B^Z(x,r)$ denote the open ball in Z determined by x and r. Let X be an open subset of a normed space E. Let $\bar{E}$ denote the completion of E. Define $\overline{\mathrm{int}}(X) = \bigcup\{B^{\bar{E}}(x,r) \mid B^E(x,r) \subseteq X\}$ and

$$\mathrm{IXT}(X) = \{h{\restriction}X \mid h \in H(\overline{\mathrm{int}}(X)) \text{ and } h(X) = X\}.$$

Theorem 1.6. *Let K be the class of all space-group pairs $\langle X, G\rangle$ such that*

(1) *X is an open subset of a Banach space, or X is an open subset of a normed vector space which is a topological space of the first category,*
(2) $\mathrm{LIP}(X) \le G \le \mathrm{IXT}(X)$.

Then K is faithful.

Theorem 1.5 deals with open subsets of normed spaces. However, the method of proof transfers without substantial change to the more cumbersome setting of manifolds over normed vector spaces (*normed manifolds*). This is dealt with in Theorem 2.30. In fact, Theorem 2.30 deals even with normed manifolds with boundary and with spaces which are the closures of open subsets of normed spaces. For such spaces Theorem 2.30 says that the "extended normed interior" of the space can be reconstructed from the group. See Definition 2.29. An additional step is needed in order to recover the entire space. This step is carried out under various assumptions in Theorems 5.2, 6.22, 6.24, 6.27(a) and 6.30.

For reasons of exposition and accessibility we include in Chapter 2 a theorem from [Ru1]. It says that K_{LCM} is faithful, where K_{LCM} is the class of all space-group pairs $\langle X, G\rangle$ which satisfy:

(i) X is a locally compact Hausdorff space without isolated points.
(ii) G has the property that for every nonempty open subset U of X and $x \in U$ the closure of the set $\{g(x) \mid g \in G \text{ and } g{\restriction}(X - U) = \mathrm{Id}\}$ has a nonempty interior.

This result appears here as Theorem 2.5.

1.3. Moduli of continuity and groups of locally uniformly continuous homeomorphisms. Chapters 3, 4 and 5 deal with groups consisting of uniformly continuous homeomorphisms. The uniform continuity of a function f can be measured by a real function which determines the bound of $d(f(x), f(y))$ as a function of $d(x,y)$. Using semigroups of such real functions we obtain a hierarchy of subgroups of $H(X)$.

Definition 1.7. MC denotes the set of functions $\alpha \in H([0,\infty))$ such that for every $x, y \in [0,\infty)$ and $0 \le \lambda \le 1$,

$$\alpha(\lambda x + (1-\lambda)y) \ge \lambda\alpha(x) + (1-\lambda)\alpha(y).$$

That is, MC is the set of all concave homeomorphisms of $[0,\infty)$. □

It is trivial that if $\alpha \in \mathrm{MC}$, then $\alpha(cx) \geq c\alpha(x)$ and $\alpha(dx) \leq d\alpha(x)$, for every $0 \leq c \leq 1$ and $d \geq 1$.

Definition 1.8. Let f be a function from a metric space (X, d^X) to a metric space (Y, d^Y). Let $\alpha \in \mathrm{MC}$. We say that f is *α-continuous* if $d^Y(f(u), f(v)) \leq \alpha(d^X(u,v))$ for every $u, v \in X$.

If $f, g : A \to \mathbb{R} \cup \{\infty\}$, then $f \leq g$ means that $f(a) \leq g(a)$ for every $a \in A$.

Let $\alpha, \beta : [0,\infty) \to \mathbb{R} \cup \{\infty\}$. Then $\alpha \preceq \beta$ means that there is $a > 0$ such that $\alpha\restriction[0,a] \leq \beta\restriction[0,a]$.

For $\Gamma \subseteq \mathrm{MC}$ we define

$$\mathrm{cl}_{\preceq}(\Gamma) = \{\alpha \in \mathrm{MC} \mid \text{ for some } \gamma \in \Gamma,\ \ \alpha \preceq \gamma\}. \ \square$$

Note that if $K > 0$, then the function $y = Kx$ belongs to MC. Also, if $\alpha, \beta \in \mathrm{MC}$, then $\alpha + \beta,\ \alpha \circ \beta \in \mathrm{MC}$.

Definition 1.9. Let Γ denote a subset of MC containing $\mathrm{Id}_{[0,\infty)}$. We define the following properties of Γ.

- M1 For every $\alpha \in \Gamma$ and $\beta \in \mathrm{MC}$: if $\beta \preceq \alpha$, then $\beta \in \Gamma$.
- M2 For every $\alpha \in \Gamma$ and $K > 0$: $K\alpha, \alpha(Kx) \in \Gamma$.
- M3 For every $\alpha, \beta \in \Gamma$: $\alpha + \beta \in \Gamma$.
- M4 For every $\alpha, \beta \in \Gamma$: $\alpha \circ \beta \in \Gamma$.
- M5 Γ is *countably generated*. This means that there is a countable set $\Gamma_0 \subseteq \Gamma$ such that $\Gamma \subseteq \mathrm{cl}_{\preceq}(\Gamma_0)$.
- M6 Let $\alpha^{\circ n}$ denote $\alpha \circ \cdots \circ \alpha$, n times. We say that Γ is *principal* if there is $\alpha \in \Gamma$ such that $\Gamma \subseteq \mathrm{cl}_{\preceq}(\{\alpha^{\circ n} \mid n \in \mathbb{N}\})$. $\square$

Example 1.10. (a) The set $\Gamma^{\mathrm{LIP}} := \{\alpha \in \mathrm{MC} \mid \alpha \preceq Kx \text{ for some } K > 0\}$ satisfies M1–M6, and it is called the *Lipschitz modulus*.

(b) For $0 < r \leq 1$ the set $\Gamma_r^{\mathrm{HLD}} := \{\alpha \in \mathrm{MC} \mid \alpha \preceq Kx^r \text{ for some } K > 0\}$ is called the *r-Hölder set*, and it satisfies M1–M3 and M5.

(c) The set $\Gamma^{\mathrm{HLD}} := \bigcup\{\Gamma_r^{\mathrm{HLD}} \mid r \in (0,1]\}$ is called the *Hölder modulus*, and it satisifies M1–M6. $\square$

Proposition 1.11. (a) *If $\Gamma \supseteq \Gamma^{\mathrm{LIP}}$ and Γ satisfies M1 and M4, then it satisfies M3.*

(b) *If Γ satisfies M1 and M3, then it satisfies M2.*

Proof. Left to the reader. ∎

Definition 1.12. (a) Let $\Gamma \subseteq \mathrm{MC}$ and f be a function from a metric space X to a metric space Y. Then f is *locally Γ-continuous* if for every $x \in X$ there is a neighborhood U of x and $\alpha \in \Gamma$ such that $f\restriction U$ is α-continuous. f is *locally Γ-bicontinuous* if f is a homeomorphism between X and $\mathrm{Rng}(f)$, and both f and f^{-1} are locally Γ-continuous.

(b) Let $\Gamma \subseteq \mathrm{MC}$. Then Γ is called a *modulus of continuity* if $\mathrm{Id}_{[0,\infty)} \in \Gamma$ and Γ satisfies M1–M4. Hence $\Gamma^{\mathrm{LIP}} \subseteq \Gamma$.

(c) Let Γ be a modulus of continuity, and X be a metric space. $H_\Gamma^{\mathrm{LC}}(X)$ denotes the set of locally Γ-bicontinuous homeomorphisms from X onto X. $\square$

Obviously, $\langle H_\Gamma^{\mathrm{LC}}(X), \circ \rangle$ is a group.

Chapters 3 and 4 deal with groups of type $H_{\Gamma}^{\mathrm{LC}}(X)$. The main result on such groups is stated in Theorem 4.1(a), and is proved at the end of Chapter 4. The part of that theorem which deals with moduli of continuity different from MC appears in Corollary 3.42(a).

The following theorem captures much of the content of 4.1(a). The full statement of 4.1(a) requires more terminology.

THEOREM 1.13. *For $\ell = 1,2$ let Γ_ℓ be a modulus of continuity such that either Γ_ℓ is countably generated or $\Gamma_\ell = \mathrm{MC}$; let E_ℓ be a normed space and X_ℓ be a nonempty open subset of E_ℓ. Let $\varphi : H_{\Gamma_1}^{\mathrm{LC}}(X_1) \cong H_{\Gamma_2}^{\mathrm{LC}}(X_2)$. Then $\Gamma_1 = \Gamma_2$, and there is a locally Γ_1-bicontinuous homeomorphism τ such that τ induces φ. That is, $\varphi(f) = \tau \circ f \circ \tau^{-1}$ for every $f \in H_{\Gamma_1}^{\mathrm{LC}}(X)$.*

Let $\boldsymbol{K}_\Gamma$ denote the restricted topological category in which the objects are open subsets of normed vector spaces, and the morphisms are locally Γ-bicontinuous homeomorphisms between such sets. The above theorem says that for every Γ as above $\boldsymbol{K}_\Gamma$ is a determining category, that $\boldsymbol{K}_{\Gamma_1}$ and $\boldsymbol{K}_{\Gamma_2}$ are distinguishable, and that for every nonempty open subset of a normed vector space X and distinct Γ_1 and Γ_2, $H_{\Gamma_1}^{\mathrm{LC}}(X) \neq H_{\Gamma_2}^{\mathrm{LC}}(X)$.

The proof of 1.13 has two main steps. In the first step we apply Theorem 1.5 and deduce that there is $\tau : X \cong Y$ such that τ induces φ. This part of the argument is used repeatedly for the other groups which are dealt with in this work.

The following statement constitutes the second step in the proof of 1.13.

THEOREM 1.14. *Let X and Y be open subsets of the normed spaces E and F respectively and $\tau : X \cong Y$. Let Γ be a countably generated modulus of continuity. If $\mathrm{LIP}(X)^\tau \subseteq H_{\Gamma}^{\mathrm{LC}}(Y)$, then τ is locally Γ-bicontinuous.*

The above theorem is restated as Theorem 3.27.

REMARK 1.15. (a) Theorem 1.13 is stated only for open subsets of normed spaces. But it is also true for normed manifolds. See Definitions 2.29 and 3.46 and Corollary 3.48(a). In fact, if $\langle X, \Phi \rangle$ is a normed manifold with an atlas Φ such that for every $\varphi_1, \varphi_2 \in \Phi$, $\varphi_1 \circ \varphi_2^{-1}$ is locally Γ-continuous, then $H_{\Gamma}^{\mathrm{LC}}(X)$ can be defined, and Theorem 1.13 remains true. The proof remains essentially unchanged.

(b) Theorem 1.13 has the obvious shortcoming of assuming that Γ is countably generated. In fact, the assumption on Γ in Theorem 4.1(a) is weaker. For example, for open subsets $X, Y \subseteq \ell_\infty$ the conclusion of Theorem 1.13 is true for every modulus of continuity. Note though that the two natural moduli which motivated 1.13, the Lipschitz and the Hölder moduli, are countably generated, and hence are covered by 1.13. But the question of whether Theorem 1.13 is true for every modulus of continuity remains open. □

1.4. Other groups of uniformly continuous homeomorphisms. A priori it seems natural to deal with the group $\mathrm{UC}(X)$ of all uniformly bicontinuous homeomorphisms of X rather than with $H_{\mathrm{MC}}^{\mathrm{LC}}(X)$. (A homeomorphism h is *uniformly bicontinuous* if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $d(x,y) < \delta$, then $d(h(x), h(y)) < \varepsilon$, and if $d(h(x), h(y)) < \delta$, then $d(x,y) < \varepsilon$.)

Similarly, the group $H_\Gamma(X)$ of all Γ-bicontinuous homeomorphisms of X seems to be more natural than $H_\Gamma^{\mathrm{LC}}(X)$. (A homeomorphism h is *Γ-bicontinuous* if there is $\gamma \in \Gamma$ such that h and h^{-1} are γ-continuous.) It turns out that $\mathrm{UC}(X)$ and $H_\Gamma(X)$ pose more problems than their counterparts. Chapter 5 addresses these groups and some related groups.

Let $\mathcal{P}$ be a property of maps and X, Y be topological spaces. Define $\mathcal{P}(X,Y) = \{h \mid h : X \cong Y$ and h has property $\mathcal{P}\}$. If H is a set of 1-1 functions, then $H^{-1} := \{h^{-1} \mid h \in H\}$. Define $\mathcal{P}^{\pm}(X,Y) = \mathcal{P}(X,Y) \cap (\mathcal{P}(Y,X))^{-1}$ and $\mathcal{P}(X) = \mathcal{P}^{\pm}(X,X)$. We consider only $\mathcal{P}$'s such that $\mathcal{P}(X)$ is a group. The final results of Chapter 5 have the following form.

$(*)$ Suppose that $\varphi : \mathcal{P}(X) \cong \mathcal{P}(Y)$. Then there is $\tau \in \mathcal{P}^{\pm}(X,Y)$ such that τ induces φ.

A class $\mathcal{M}$ of topological spaces is called *$\mathcal{P}$-determined* if $(*)$ holds for every $X, Y \in K$, that is, if the category $\boldsymbol{K}_{\mathcal{M},\mathcal{P}}$ whose objects are the members of $\mathcal{M}$ and whose morphisms are the members of $\mathcal{P}^{\pm}(X,Y)$ for $X, Y \in \mathcal{M}$ is a determining category.

The first result in Chapter 5 is about groups of type $\mathrm{UC}(X)$. Denote the *diameter* of a subset A of a metric space by $\mathrm{diam}(A)$. A metric space $\langle X, d\rangle$ is *uniformly-in-diameter arcwise-connected* if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $x, y \in X$: if $d(x,y) < \delta$, then there is an arc $L \subseteq X$ connecting x and y such that $\mathrm{diam}(L) < \varepsilon$. The following statement is the main result on groups of type $\mathrm{UC}(X)$. It is restated as Corollary 5.6.

THEOREM 1.16. *Let X be an open subset of a Banach space or of a normed vector space of the first category. Suppose that the same holds for Y. Suppose further that X and Y are uniformly-in-diameter arcwise-connected. Let $\varphi : \mathrm{UC}(X) \cong \mathrm{UC}(Y)$. Then there is $\tau \in \mathrm{UC}^{\pm}(X,Y)$ such that τ induces φ.*

The following theorem restated later as 5.2 is a corollary of 1.16.

THEOREM 1.17. *Let F and K be the closures of uniformly-in-diameter arcwise-connected open bounded subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Let $\varphi : H(F) \cong H(K)$. Then φ is induced by a homeomorphism between F and K.*

Theorem 1.17 is considerably stronger than the analogous statement for Euclidean manifolds with boundary. This is so, since uniformly-in-diameter arcwise-connected open subsets of $\mathbb{R}^n$ may have a boundary which is more complicated than the boundary of a manifold with boundary.

$\mathrm{UC}(X)$ is a special case of the groups $H_\Gamma(X)$. But the analogue of Theorem 1.16 is not true for $H_\Gamma(X)$. In Example 5.11 it is shown that for every normed space E there is $\tau \in H(E)$ such that $(\mathrm{LIP}(E))^\tau = \mathrm{LIP}(E)$ but $\tau \notin \mathrm{LIP}(E)$.

Chapter 5 proves $\mathcal{P}$-determinedness for several other $\mathcal{P}$'s. Definition 5.4 lists eight types of groups for which $\mathcal{P}$-determinedness can be proved. But we have chosen to deal only with properties $\mathcal{P}$ which occur in other mathematical contexts.

DEFINITION 1.18. (a) Let $\mathrm{BUC}(X,Y)$ denote the set of homeomorphisms $g : X \cong Y$ such that g takes bounded sets to bounded sets and for every bounded $B \subseteq X$, $g{\restriction}B$ is uniformly continuous.

(b) Let X be a metric space. X is *boundedly uniformly-in-diameter arcwise-connected* if for every bounded set $B \subseteq X$ and $\varepsilon > 0$ there is $\delta > 0$ such that for every $x, y \in B$: if $d(x, y) < \delta$, then there is an arc $L \subseteq X$ connecting x and y such that $\mathrm{diam}(L) < \varepsilon$.

(c) If $h : [0,1] \times X \to X$ and $t_0 \in [0,1]$, then the function f from X to X defined by $f(x) = h(t_0, x)$ is denoted by h_{t_0}. X has *Property MV1* if for every bounded $B \subseteq X$ there are $r = r_B > 0$ and $\alpha = \alpha_B \in \mathrm{MC}$ such that for every $x \in B$ and $0 < s \le r$, there is an α-continuous function $h : [0,1] \times X \to X$ such that: for every $t \in [0,1]$, $h_t(x) \in H(X)$ and h_t^{-1} is α-continuous; $h_0 = \mathrm{Id}$ and $d(x, h_1(x)) = s$; and $h_t \restriction (X - B(x, 2s)) = \mathrm{Id}$ for every $t \in [0,1]$. $\square$

The following $\mathcal{P}$-determinedness theorem is restated as Theorem 5.20.

THEOREM 1.19. *Let K be the class of all X such that X is an open subset of a Banach space or X is an open subset of a normed space of the first category, X is boundedly uniformly-in-diameter arcwise-connected, and X has Property MV1. Then K is BUC-determined.*

There is of course the Γ variant of $\mathrm{BUC}(X)$. For a modulus of continuity Γ define

$$H_{\Gamma}^{\mathrm{BD}}(X) = \{h \in H(X) \mid \text{for every bounded } A \subseteq X \text{ there is } \gamma \in \Gamma \text{ such that } h \restriction A \text{ is } \gamma\text{-bicontinuous}\}.$$

When X is a subset of a finite-dimensional normed space and Γ is principal, then Theorem 8.4 provides a faithfulness result for this type of groups.

We do not know a more general theorem in this direction.

The last type of groups considered in Chapter 5 are groups of homeomorphisms g such that $g \restriction B$ is uniformly continuous for every $B \subseteq X$ such that B is bounded, and the distance of B from the boundary of X is positive. The $\mathcal{P}$-determinedness in this situation is proved in Theorems 5.32 and 5.36.

These theorems are not quoted here because their statement requires terminology that has not yet been introduced.

Throughout Chapter 5 one encounters two types of intermediate results.

(1) Let $\tau : X \cong Y$ be such that $(\mathcal{P}(X))^{\tau} = \mathcal{P}(Y)$. Then $\tau \in \mathcal{P}^{\pm}(X, Y)$.
(2) Let $\tau : X \cong Y$ be such that $(\mathcal{P}(X))^{\tau} \subseteq \mathcal{P}(Y)$. Then $\tau \in \mathcal{P}^{\pm}(X, Y)$.

Results of type (2) are stronger, but they are not true for all $\mathcal{P}$'s which we consider. Results of type (2) are needed in order to show that $\mathcal{P}(X)$ cannot be isomorphic to $\mathcal{Q}(Y)$ when $\mathcal{P}$ is different from $\mathcal{Q}$.

1.5. Groups of extendible homeomorphisms and the group of homeomorphisms of the closure of an open set. Chapter 6 is concerned with the faithfulness of groups of the form $H(\mathrm{cl}(X))$ and with groups of the form $\mathrm{EXT}(X)$, where X is an open subset of a normed vector space. The group $\mathrm{EXT}(X)$ is defined below.

Let X, Y be open subsets of the normed spaces E and F. A continuous function $g : X \to Y$ is called *extendible* if there is a continuous function $\hat{g} : \mathrm{cl}(X) \to \mathrm{cl}(Y)$ such that $\hat{g}$ extends g. The set of extendible homeomorphisms between X and Y is denoted by $\mathrm{EXT}(X, Y)$. Accordingly, $\mathrm{EXT}(X) = \{g \in H(X) \mid g \text{ and } g^{-1} \text{ are extendible}\}$. Note that if X is a regular open subset of $\mathbb{R}^n$, then $\mathrm{EXT}(X) = H(\mathrm{cl}(X))$. Recall that a set is called *regular open* if it is equal to the interior of its closure.

The goal is to find large classes K of open subsets of a normed space containing the commonly encountered open sets and containing also exotic open sets for which $\{\mathrm{cl}(X) \mid X \in K\}$ is faithful. It is not true, though, that for any open subsets of $X, Y \subseteq \mathbb{R}^n$, if $\varphi : H(\mathrm{cl}(X)) \cong H(\mathrm{cl}(Y))$, then there is $\tau : \mathrm{cl}(X) \cong \mathrm{cl}(Y)$ such that τ induces φ. Example 5.8 demonstrates this phenomenon in two different ways.

The following theorem gives the flavor of the type of results proved in Chapter 6.

THEOREM 1.20. *Let X, Y be open bounded subsets of the Banach spaces E and F. Assume that:*

(1) *There is d such that for every $u, v \in X$ there is a rectifiable arc $L \subseteq X$ connecting u and v such that* $\mathrm{length}(L) \leq d$.
(2) *For every point w in the boundary of X and for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $u, v \in X$: if $\|u - w\|, \|v - w\| < \delta$, then there is an arc $L \subseteq X$ connecting u and v such that* $\mathrm{diam}(L) < \varepsilon$.
(3) *Conditions* (1) *and* (2) *hold for Y.*

Then

(a) *If $\varphi : H(\mathrm{cl}(X)) \cong H(\mathrm{cl}(Y))$, then there is $\tau : \mathrm{cl}(X) \cong \mathrm{cl}(Y)$ such that τ induces φ.*
(b) *If $\varphi : \mathrm{EXT}(X) \cong \mathrm{EXT}(Y)$, then there is $\tau \in \mathrm{EXT}^{\pm}(X, Y)$ such that τ induces φ.*

Part (a) of the above theorem is an excerpt from Theorem 6.22, and (b) is an excerpt from Theorem 6.3(a).

The class of spaces defined in Theorem 1.20 contains some spaces whose boundary is quite complicated. Also, such spaces may have boundary points which are fixed under $H(\mathrm{cl}(X))$. Here is an example of a possibly not well-behaved set which is covered by Theorem 6.22.

EXAMPLE 1.21. Let B and S be the open unit ball and the unit sphere in a Banach space E, and $\{\bar{B}_i \mid i \in I\}$ be a family of pairwise disjoint closed balls such that $\bar{B}_i \subseteq B$ for every $i \in I$. Suppose that for every $x \in E$: if every neighborhood of x intersects infinitely many B_i's, then $x \in S$. Then the set $X := B - \bigcup_{i \in I} \bar{B}_i$, satisfies clauses (1) and (2) of Theorem 1.20. Note that even in the case of $E = \mathbb{R}^n$, the boundary of X can be complicated. □

Clause (2) in Theorem 1.20 implies that $\mathrm{cl}(X)$ is arcwise connected. Consider the open set X described in the following example. Its closure is not locally arcwise connected.

EXAMPLE 1.22. Let $X = \left\{(r, \theta) \mid \theta \in (\pi, \infty) \text{ and } 1 - \frac{1}{\theta - \pi/2} < r < 1 - \frac{1}{\theta + \pi/2}\right\}$ (in polar coordinates). Note that X is an open spiral strip converging to the circle $S(0, 1)$. □

Example 1.22 is not covered by Theorem 1.20 but it is included in the class considered in the following theorem.

THEOREM 1.23. *Let X, Y be open bounded subsets of the normed spaces E and F. Assume that:*

(1) *For every sequence* $\vec{x} = \{x_n \mid n \in \mathbb{N}\} \subseteq X$ *there are a subsequence* $\vec{y}$ *of* $\vec{x}$, *a sequence* $\vec{z}$ *such that* $\vec{z}$ *is convergent in* E *and a sequence of rectifiable arcs* $L_n \subseteq X$, $n \in \mathbb{N}$, *such that* $\sup_{n\in\mathbb{N}} \text{length}(L_n) < \infty$ *and* L_n *connects* y_n *and* z_n.

(2) *For every* $x \in \text{bd}(X)$ *and* $r > 0$ *there is a continuous function* $h_t(x) : [0,1] \times \text{cl}(X) \to \text{cl}(X)$ *such that* $h_0 = \text{Id}$, $h_1(x) \neq x$, *and for every* $t \in [0,1]$, $h_t{\restriction}X \in \text{EXT}(X)$ *and* $h_t{\restriction}(\text{cl}(X) - B(x,r)) = \text{Id}$.

(3) *Conditions* (1) *and* (2) *hold for* Y.

Then

(a) *If* $\varphi : H(\text{cl}(X)) \cong H(\text{cl}(Y))$, *then there is* $\tau : \text{cl}(X) \cong \text{cl}(Y)$ *such that* τ *induces* φ.

(b) *If* $\varphi : \text{EXT}(X) \cong \text{EXT}(Y)$, *then there is* $\tau \in \text{EXT}^{\pm}(X,Y)$ *such that* τ *induces* φ.

Theorem 1.23(a) is an excerpt from Theorem 6.24, and 1.23(b) is an excerpt from 6.18. Example 1.22 is restated as 6.15(a). Other examples which are covered by Theorems 6.24 and 6.18, but have a non-locally arcwise connected closure appear in 6.8 and 6.15(b). Another EXT-determined class is described in Theorem 6.12.

Chapter 6 also deals with groups of type $\text{CMP.LUC}(X)$ defined in Subsection 1.1. CMP.LUC-determinedness is proved in Theorem 6.20(a). It completes the picture given in Chapters 8–12. The following is a special case of 6.20(a).

THEOREM 1.24. *Let* X, Y *be open bounded subsets of the normed spaces* E *and* F. *Assume that*:

(1) *For every sequence* $\vec{x} = \{x_n \mid n \in \mathbb{N}\} \subseteq X$ *there are a subsequence* $\vec{y}$ *of* $\vec{x}$, *a sequence* $\vec{z}$ *such that* $\vec{z}$ *is convergent in* E *and a sequence of rectifiable arcs* $L_n \subseteq X$, $n \in \mathbb{N}$, *such that* $\sup_{n\in\mathbb{N}} \text{length}(L_n) < \infty$ *and* L_n *connects* y_n *and* z_n.

(2) *For every* $x \in \text{bd}(X)$ *there is* $r > 0$ *such that for every* $\varepsilon > 0$ *there is* $\delta > 0$ *such that for every* $u, v \in B^E(x,r) \cap X$: *if* $d(u,v) < \delta$, *then there is an arc* $L \subseteq X$ *connecting* u *and* v *such that* $\text{diam}(L) < \varepsilon$.

(3) *Conditions* (1) *and* (2) *hold for* Y.

Then if $\varphi : \text{CMP.LUC}(X) \cong \text{CMP.LUC}(Y)$, *then there is* $\tau \in \text{CMP.LUC}^{\pm}(X,Y)$ *such that* τ *induces* φ.

Two extensions of the results of Chapter 6 are presented at the end of that chapter. These extensions cover some natural spaces which are not covered by the original classes. Also, the faithful class dealt with in Extension 2 contains $2^{2^{\aleph_0}}$ subsets of $\mathbb{R}^3$.

(1) The original classes considered in Chapter 6 consist of open subsets of normed vector spaces, and the closures of such sets. However, all the results obtained for these classes translate to the class of open subsets of manifolds over normed vector spaces and the closures of such sets. See Example 6.28 and Theorem 6.30.

(2) The results obtained for the class of closures of open subsets of a normed vector space extend to the class of all subsets Z of a normed vector space which satisfy $Z \subseteq \text{cl}(\text{int}(Z))$. See Example 6.26 and Theorem 6.27.

Chapter 7 contains theorems of the following type. Suppose that $\varphi : \mathcal{P}(X) \cong \mathcal{Q}(Y)$. Then

(i) There is $\tau : X \cong Y$ such that τ induces φ.
(ii) $\mathcal{P}(X) = \mathcal{Q}(X)$ and $\tau \in \mathcal{Q}^{\pm}(X,Y)$, or $\mathcal{P}(Y) = \mathcal{Q}(Y)$ and $\tau \in \mathcal{P}^{\pm}(X,Y)$.

These results appear in Corollary 7.11. As an example of such results we quote 7.11(e).

THEOREM 1.25. *If X and Y are nonempty open subsets of an infinite-dimensional Banach space, then* $\mathrm{UC}(X) \not\cong \mathrm{EXT}(Y)$.

1.6. Local uniform continuity at the boundary of an open set. Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be open sets and suppose that $\varphi : \mathrm{LIP}(\mathrm{cl}(X)) \cong \mathrm{LIP}(\mathrm{cl}(Y))$. Can we conclude that there is $\tau : \mathrm{cl}(X) \cong \mathrm{cl}(Y)$ such that τ is bilipschitz and τ induces φ? This question motivates the work presented in Chapters 8–12. Indeed, if the boundaries of X and Y are well-behaved, then the answer to the above question is positive.

Let X, Y be open subsets of the normed spaces E and F, and Γ be a modulus of continuity. For $g \in \mathrm{EXT}(X,Y)$ let g^{cl} denote the continuous extension of g to $\mathrm{cl}(X)$. Define

$$H_{\Gamma}^{\mathrm{CMP.LC}}(X,Y) = \{g \in \mathrm{EXT}(X,Y) \mid g^{\mathrm{cl}} \text{ is locally } \Gamma\text{-continuous}\}$$

and $H_{\Gamma}^{\mathrm{CMP.LC}}(X) = (H_{\Gamma}^{\mathrm{CMP.LC}})^{\pm}(X,X)$.

Note that the group $\mathrm{CMP.LUC}(X)$ discussed in Subsection 1.5 is a special case of groups of the form $H_{\Gamma}^{\mathrm{CMP.LC}}(X)$. Indeed, $\mathrm{CMP.LUC}(X) = H_{\mathrm{MC}}^{\mathrm{CMP.LC}}(X)$. In the special case that $X \subseteq \mathbb{R}^n$ is a regular open bounded set we have $\mathrm{LIP}(\mathrm{cl}(X)) = H_{\Gamma^{\mathrm{LIP}}}^{\mathrm{CMP.LC}}(X)$. More generally, $H_{\Gamma}(\mathrm{cl}(X)) = H_{\Gamma}^{\mathrm{CMP.LC}}(X)$. So a determiningness result for the property $\mathcal{P} = \mathrm{CMP.LC}_{\Gamma^{\mathrm{LIP}}}$ implies such a result for the class $\boldsymbol{K}_{\mathcal{M},\mathcal{P}}$, where $\mathcal{P} = \mathrm{LIP}$ and $\mathcal{M}$ is the class of bounded regular open subsets of finite-dimensional spaces.

Chapters 8–12 are devoted to the proof of the following statement about $H_{\Gamma}^{\mathrm{CMP.LC}}(X)$.

(∗) If $\varphi : H_{\Gamma}^{\mathrm{CMP.LC}}(X) \cong H_{\Delta}^{\mathrm{CMP.LC}}(Y)$, then $\Gamma = \Delta$, and there is $\tau \in (H_{\Gamma}^{\mathrm{CMP.LC}})^{\pm}$ (X,Y) such that τ induces φ.

Statement (∗) is proved for X, Y, Γ and Δ which satisfy the following assumptions.

(1) Γ is principal (see M6 in Definition 1.9).
(2) X is locally Γ-LIN-bordered, and Y is locally Δ-LIN-bordered (see Definition 8.1(b)).

The exact definition of local LIN-borderedness is a bit long, but a main special case is the class of open sets whose closure is a manifold with boundary with a Γ-bicontinuous atlas.

Statement (∗) is restated in Theorem 8.4(a). The proof of 8.4(a) has four steps. The two major steps are Steps 3 and 4, which are stated as Theorems 8.8 and 12.19. The following theorem is the conclusion of the first three steps combined together. The prinicipality of Γ is not needed here. It is needed only at Step 4.

THEOREM 1.26. *Let Γ, Δ be countably generated moduli of continuity, E and F be normed spaces and $X \subseteq E$, $Y \subseteq F$ be open. Suppose that X is locally Γ-LIN-bordered, and Y is locally Δ-LIN-bordered. Let $\varphi : H_{\Gamma}^{\mathrm{CMP.LC}}(X) \cong H_{\Delta}^{\mathrm{CMP.LC}}(Y)$. Then there is $\tau \in \mathrm{EXT}^{\pm}(X,Y)$ such that τ induces φ.*

The proof of Theorem 1.26 requires much technical work. This work is carried out in Chapters 9 and 10. The proof of 1.26 appears at the end of Chapter 11.

Step 4 of the proof of Theorem 8.4(a) says that if in Theorem 1.26, Γ is principal, then the homeomorphism τ obtained in 1.26 belongs to $(H_{\Gamma}^{\mathrm{CMP.LC}})^{\pm}(X,Y)$.

It should be pointed out that the results mentioned above are true for open subsets of normed manifolds. The final result for manifolds is stated in Theorem 8.4(b).

As a byproduct of the proof of the main theorem of Chapters 8–12, we also obtain a determiningness result for the group defined below. Let X be an open subset of a normed space E. Define

$$H_{\Gamma}^{\mathrm{BDR.LC}}(X) = \{g \in \mathrm{EXT}(X) \mid \text{every } x \in \mathrm{cl}(X) - X \text{ has a neighborhood } U \text{ in } \mathrm{cl}(X) \text{ such that } g^{\mathrm{cl}} \restriction U \text{ is } \Gamma\text{-bicontinuous}\}.$$

Theorem 12.20(b) contains a determiningness result for the property $\mathcal{P} = \mathrm{BDR.LC}_{\Gamma}$.

1.7. Further questions and discussion. This work leaves many unsolved questions, which we mention at the point where they naturally arise. In what follows we highlight the questions we regard to be more central.

The countable generatedness of Γ

Question 1.27. Can Theorem 1.13 be proved for every pair of moduli of continuity, regardless of whether they are countably generated or not? That is, we ask if the following statement true:

For $\ell = 1,2$ let Γ_ℓ be a modulus of continuity. Let E_ℓ be a normed space and X_ℓ be an open subset of E_ℓ. Let $\varphi : H_{\Gamma_1}^{\mathrm{LC}}(X_1) \cong H_{\Gamma_2}^{\mathrm{LC}}(X_2)$. Then $\Gamma_1 = \Gamma_2$, and there is a locally Γ_1-bicontinuous homeomorphism τ such that τ induces φ. □

Note that the assumption in Theorem 4.1 is in fact somewhat weaker than countable generatedness. We ask Question 1.27 also for the other theorems in which Γ is required to be countably generated. See e.g. parts (a) and (b) of Theorem 5.24.

The principality of Γ in the theorem about $H_{\Gamma}^{\mathrm{CMP.LC}}(X)$

Question 1.28. Is Theorem 12.20(a) true without the assumption that Γ is principal? That is, we ask if the following statement is true:

Let X, Y be open subsets of a normed space, and Γ, Δ be moduli of continuity. Assume that X is locally Γ-LIN-bordered, and Y is locally Δ-LIN-bordered. If $\varphi : H_{\Gamma}^{\mathrm{CMP.LC}}(X) \cong H_{\Delta}^{\mathrm{CMP.LC}}(Y)$, then $\Gamma = \Delta$, and there is $\tau \in (H_{\Gamma}^{\mathrm{CMP.LC}})^{\pm}(X,Y)$ such that τ induces φ. □

Obviously, the case that Γ and Δ are countably generated is also unknown.

A possible stronger way of distinguishing between the $H_{\Gamma}^{\mathrm{LC}}(X)$'s. The fact that $H_{\Gamma}^{\mathrm{LC}}(X) \not\cong H_{\Delta}^{\mathrm{LC}}(Y)$ for $\Gamma \neq \Delta$ may have a stronger reason. That is, maybe there is a locally Δ-bicontinuous homeomorphism which is not conjugate to any locally Γ-bicontinuous homeomorphism. So a positive answer to the following question together with the faithfulness result of Theorem 1.5(a) will imply the distinguishability of the $\boldsymbol{K}_{\Gamma}$'s.

Question 1.29. Let $\varGamma, \varDelta$ be moduli of continuity such that $\varDelta \not\subseteq \varGamma$ and let X be a nonempty open subset of a normed space of dimension > 1. Is there a locally $\varDelta$-bicontinuous homeomorphism g of X such that g is not conjugate to any $\varGamma$-bicontinuous homeomorphism? $\square$

In the space $\mathbb{R}$, every homeomorphism is conjugate to a Lipschitz homeomorphism.

Relaxing the assumption on the boundary in the theorem about $H_{\varGamma}^{\mathrm{CMP.LC}}(X)$. Let $X_0 = \{(x, y) \in \mathbb{R}^2 \mid x > 0, -x^2 < y < x^2\}$. The set X_0 is not $\varGamma^{\mathrm{LIP}}$-LIN-bordered. Our general question is whether Theorem 12.20(a) can be strengthened to classes which include sets similar to X_0. We may ask the following concrete question.

Question 1.30. Let $\varphi \in \mathrm{Aut}(H_{\varGamma^{\mathrm{LIP}}}^{\mathrm{CMP.LC}}(X_0))$. Is φ an inner automorphism? $\square$

Question 8.11 introduces the notion of a locally $\varGamma$-almost-linearly-bordered set (locally $\varGamma$-ALIN-bordered set). It seems that Theorem 12.20(a) can be extended to the class of locally $\varGamma$-ALIN-bordered sets. This requires a more detailed technical analysis similar to the work carried out in Chapters 9–11.

However, we do not know how to handle the type of singularity at the boundary point $(0, 0)$ of X_0 above.

A variant of the group $H_{\varGamma}^{\mathrm{CMP.LC}}(X)$. Let X, Y be open subsets of the normed spaces E and F, $f : X \to Y$ and $\varGamma$ be a modulus of continuity. f is *completely weakly $\varGamma$-continuous* (*CMP.WK $\varGamma$-continuous*) if f is extendible, and there is $\gamma \in \varGamma$ such that for every $x \in \mathrm{cl}(X)$ there is a neighborhood U of x such that $f^{\mathrm{cl}} \restriction U$ is γ-continuous. As usual,

$$H_{\varGamma}^{\mathrm{CMP.WK}}(X, Y) := \{f \mid f \text{ is a homeomorphism between } X \text{ and } Y \text{ and } f \text{ is CMP.WK } \varGamma\text{-continuous}\}.$$

Question 1.31. Prove the analogue of Theorem 12.20(a) for the groups $H_{\varGamma}^{\mathrm{CMP.WK}}(X)$. $\square$

Naturally, the definition of local $\varGamma$-LIN-borderedness has to be replaced by the analogous notion of weak $\varGamma$-LIN-borderedness.

It seems that the main difficulty in proving $\mathrm{CMP.WK}_{\varGamma}$-determinedness is the counterpart of Theorem 1.26.

Groups which fit into the framework but have not been investigated

Definition 1.32. Let $\varGamma$ be a modulus of continuity and $f : X \to Y$.

(a) f is *regionally $\varGamma$-continuous* if for every nonempty open $U \subseteq X$ there is a nonempty $V \subseteq U$ and $\alpha \in \varGamma$ such that $f \restriction V$ is α-continuous.

(b) f is *pointwise $\varGamma$-continuous* if for every $x \in X$ there is a neighborhood V of x and $\alpha \in \varGamma$ such that $d(f(y), f(x)) \le \alpha(d(y, x))$ for every $y \in V$. Note that "pointwise MC-continuous" is just "continuous".

(c) f is *boundedly $\varGamma$-continuous* if for every bounded set $V \subseteq X$ there is $\alpha \in \varGamma$ such that $f \restriction V$ is α-continuous.

Let $H_{\varGamma}^{\mathrm{RG}}(X)$, $H_{\varGamma}^{\mathrm{PW}}(X)$ and $H_{\varGamma}^{\mathrm{BD}}(X)$ denote the groups of homeomorphisms corresponding to the notions introduced in (a)–(c). $\square$

PROPOSITION 1.33. (a) *Let X be a metric space and Γ be a modulus of continuity. Then*

(i) $H_{\Gamma}^{\mathrm{BD}}(X) \subseteq H_{\Gamma}^{\mathrm{LC}}(X) \subseteq H_{\Gamma}^{\mathrm{PW}}(X)$;
(ii) $H_{\Gamma}^{\mathrm{LC}}(X) \subseteq H_{\Gamma}^{\mathrm{RG}}(X)$.

(b) *Let X be an open subset of a Banach space and Γ be a countably generated modulus of continuity. Then $H_{\Gamma}^{\mathrm{PW}}(X) \subseteq H_{\Gamma}^{\mathrm{RG}}(X)$.*

Proof. (a) Part (a) follows from the definitions.

(b) Suppose that $f : X \to Y$ is not regionally Γ-continuous. Let $\{\alpha_i \mid i \in \mathbb{N}\}$ generate Γ. Let $U \subseteq X$ be an open ball which shows that f is not regionally Γ-continuous. We define by induction $x_i, y_i \in U$. Let x_0, y_0 be such that $d(f(x_0), f(y_0)) > 2\alpha_0(d(x_0, y_0))$. Suppose that x_i, y_i have been defined. Let $x_{i+1}, y_{i+1} \in B\left((x_i + y_i)/2,\ d(x_i, y_i)/2^i\right)$ be such that $d(f(x_{i+1}), f(y_{i+1})) > 2\alpha_{i+1}(d(x_{i+1}, y_{i+1}))$. Since $\{x_i \mid i \in \mathbb{N}\}$ is a Cauchy sequence, it converges, say to z. Hence $\lim_i y_i = z$. We may assume that $d(f(z), f(x_i)) \geq d(f(x_i), f(y_i))/2$ for every $i \in \mathbb{N}$. So for $i \in \mathbb{N}$,

$$d(f(z), f(x_i)) \geq \tfrac{1}{2} d(f(x_i), f(y_i)) > \tfrac{1}{2} \cdot 2\alpha_i(d(x_i, y_i)) > \alpha_i(d(z, x_i)).$$

Hence z shows that f is not pointwise Γ-continuous. ∎

Let

$$K = \{X \mid X \text{ is an open subset of a separable normed space of the second category}\}.$$

Using an argument similar to the one used in Theorem 3.41, one can prove the analogues of 1.13 and 1.14 for the class

$$\{H_{\Gamma}^{\mathrm{RG}}(X) \mid X \in K \text{ and } \Gamma \text{ is a countably generated modulus of continuity}\}.$$

It is not known whether other arguments used for $H_{\Gamma}^{\mathrm{LC}}(X)$ can be applied to $H_{\Gamma}^{\mathrm{RG}}(X)$.

QUESTION 1.34. Prove the analogues of 1.13 and 1.14 for the class $\{H_{\Gamma}^{\mathrm{RG}}(X) \mid X$ is an open subset of a normed space, and Γ is a countably generated modulus of continuity$\}$. □

It is easy to see that a reconstruction theorem for the class of $H_{\Gamma}^{\mathrm{RG}}(X)$'s implies reconstruction theorems for the classes of $H_{\Gamma}^{\mathrm{WK}}(X)$'s and $H_{\Gamma}^{\mathrm{BD}}(X)$'s.

1.8. Some more facts about reconstruction theorems

Reconstruction questions for related groups. Much work has been done on the analogous problems for diffeomorphism groups. It seems that the first work in this direction was carried out by F. Takens [Ta].

Soon afterwards there was an unpublished extensive work by W. Ling [Lg1] and [Lg2]. Ling proved that many types of structures on a Euclidean manifold give rise to a determining category (or to an appropriate variant of this notion). Some of these categories are:

(1) The category of k-smooth Euclidean manifolds with k-smooth diffeomorphisms.
(2) The category of k-smooth Euclidean manifolds with a k-smooth volume form with diffeomorphisms preserving the form.
(3) The category of k-smooth foliated Euclidean manifolds with the foliation preserving diffeomorphisms.

(4) Differentiable manifolds with a contact form.
(5) Manifolds with a piecewise linear structure, and homeomorphisms preserving this structure.

The authors in [RY] (unpublished) reproved result (1) from Ling's work, and proved some additional facts. For example, they showed that the category of Euclidean differentiable manifolds with diffeomorphisms that have a locally Γ-continuous kth derivative is a determining category, for every countably generated modulus of continuity Γ.

The next work was by R. Filipkiewicz [Fi]. He proved that the category of k-smooth manifolds with k-smooth diffeomorphisms is a determining category.

Further work on this subject has been done more recently by a number of authors.

A. Banyaga [Ba1], [Ba2] proved the determiningness for the categories arising from differentiable structures, unimodular structures, symplectic structures, and contact structures. Also, he established an analogous result for measure preserving homeomorphisms.

T. Rybicki [Ryb] presented an axiomatic approach to groups of C^∞ diffeomorphisms which determine a C^∞ manifold.

Recent progress on reconstruction problems was obtained by J. Borzellino and V. Brunsden [BB]. They proved faithfulness for the class of spaces which are locally compact orbifolds.

Results on differentiabilty obtained by the authors of this work which refine older results and which also deal with Fréchet differentiabilty in infinite-dimensional spaces, will appear in a subsequent work.

V. Gol'dshtein and M. Rubin obtained analogous results for quasi-conformal homeomorphism groups. Part of these results appeared in [GR]. The results for quasi-conformal homeomorphism groups apply to finite- and infinite-dimensional spaces. The full account of this subject will be presented in a separate article.

Another interesting theorem on a determining category appears in the works of M. G. Brin and of Brin and F. Guzmán on the Thompson group. Let $G \leq H([0,1])$ be the group of all homeomorphisms h such that: (1) h is piecewise linear; (2) every slope of h is an integral power of 2; (3) every breakpoint of h is a diadic number. It is clear that $G \in K_{\mathrm{LCM}}$ (see 2.4 and 2.5). Hence $\{\langle [0,1], G\rangle\}$ is faithful. Interestingly, G is a finitely presented group.

One of Brin's results from [Br1] is as follows.

- Every automorphism of G is induced by a homeomorphism $f \in H([0,1])$ such that for every $a < b$ in $[0,1]$, $f \restriction [a,b]$ satisfies (1)–(3) above.
- Every such homeomorphism induces an automorphism of G.

Denote by G^+ the group of all $f \in H([0,1])$ such that conjugation by f is an automorphism of G. Brin also proves that $\{\langle [0,1], G^+\rangle\}$ is a determining category. See also Brin [Br2] and Brin and F. Guzmán [BG].

Reconstruction theorems in other areas. The theme of reconstructing a structure from its automorphism group was investigated in several other areas.

The recovery of a vector space from its group of linear isomorphisms has a long history. Mackey [Mac] proved in 1942 that a normed vector space X can be reconstructed from its group $L(X)$ of isomorphisms (that is, bijective bounded linear transformations from the space to itself). More precisely, Mackey showed that if X is finite-dimensional and $L(X) \cong L(Y)$, then $\dim(X) = \dim(Y)$. In the case that X is infinite-dimensional an isomorphism between $L(X)$ and $L(Y)$ is induced by an isomorphism between X and Y. In the case that X is reflexive an isomorphism between $L(X)$ and $L(Y)$ can also be induced by an isomorphism between X^* and Y.

Let F_1, F_2 be division rings and $n_1, n_2 > 2$ be integers. If the linear groups $\mathrm{GL}(n_1, F_1)$ and $\mathrm{GL}(n_2, F_2)$ are isomorphic, then $n_1 = n_2$ and either $F_1 \cong F_2$ or $F_1 \cong F_2^{\mathrm{op}}$, where F^{op} is the division ring obtained from F by reversing the multiplication. That is, $a \cdot^{F^{\mathrm{op}}} b = b \cdot^{F} a$. This fact is due to J. Dieudonné [Di1] (1947) and [Di2] (1951).

For infinite-dimensional vector spaces, V_1 over F_1 and V_2 over F_2, every isomorphism between $\mathrm{Aut}(V_1)$ and $\mathrm{Aut}(V_2)$ is induced by isomorphisms between F_1 and F_2 and between V_1 and V_2. A strong theorem concerning this, but not exactly this fact, was proved by C. E. Rickart in [Ri1]–[Ri3] (1950–1951). The theorem of Dieudonné for finite dimensions is a special case of Rickart's Theorem. O. O'Meara [Om] (1977) proved the reconstruction theorem for infinite dimensions. Another proof was found by V. Tolstykh [To1] (2000).

Free groups are also reconstructible from their automorphism groups. That $\mathrm{Aut}(F_n) \not\cong \mathrm{Aut}(F_m)$ for $n \neq m$ can be deduced from the work of J. Dyer and G. P. Scott [DS] (1975). F_n denotes the free group with n generators (in the variety of all groups). E. Formanek in [Fo] (1990) proved that $\mathrm{Inn}(F_n)$ is the only normal free subgroup of rank n of $\mathrm{Aut}(F_n)$. This implies immediately the reconstruction result for finitely generated free groups. V. Tolstykh in [To2] (2000) proved that if λ is an infinite cardinal then $\mathrm{Inn}(F_\lambda)$ is definable in $\mathrm{Aut}(F_\lambda)$. This implies the reconstruction result for free groups with infinite rank.

Another body of reconstruction results for groups of linear transformations is due to M. Droste and M. Göbel [DG1] (1995) and [DG2] (1996). Given a ring R with unity and a poset P one can define the generalized McLain group $G(R, P)$ of R and P. Droste and Göbel reconstruct R and P from $G(R, P)$.

The symmetric group is another important instance. It is the automorphism group of a structure with no relations and no operations. $\mathrm{Sym}(6)$ is the only symmetric group which has outer automorphisms. A proof that A is recoverable from $\mathrm{Sym}(A)$ appears in McKenzie [McK] (1971). This had been known before. See Scott [Sc, p. 311].

Automorphism groups of various types of ordered structures were also extensively investigated. We mention some of the more recent references. Reconstruction theorems for trees appear in Rubin [Ru3] (1993). Linear orders and related structures are considered in Rubin [Ru5] (1996) and in [MR]. And Boolean algebras are reconstructed in Rubin [Ru2] (1989).

The reconstruction of measure algebras is dealt with in [Ru2]. The group of measure preserving transformations of $[0, 1]$ is considered by S. Eigen in [Ei] (1982).

Rubin [Ru4] (1994) deals with the reconstruction of $\aleph_0$-categorical structures.

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The fact that the principal modulus of continuity generated by α is α-star-closed was proved by Kubis. See Definition 12.11(d) and Proposition 12.12(a).

We also thank Vladimir Tolstykh for his help in surveying reconstruction theorems in algebra. His thorough survey was a great help. We also thank Yoav Benyamini and Edmund Ben Ami for helpful discussions.

2. Obtaining a homeomorphism from a group isomorphism

2.1. Capturing the action of the group on the regular open sets. Let $G \leq H(X)$. In order to prove that X is reconstructible from G, we shall first show that the action of G on the set of regular open subsets of X is reconstructible from G.

We next introduce some notations, recall some basic definitions, and present some notions specific to this work.

DEFINITION 2.1. Let X be a topological space $U \subseteq X$ and $G \leq H(X)$.

(a) Let $\mathrm{int}^X(U)$, $\mathrm{cl}^X(U)$, $\mathrm{bd}^X(U)$ and $\mathrm{acc}^X(U)$ denote respectively the interior, closure, boundary and the set of accumulation points of U in X. The boundary, $\mathrm{bd}^X(U)$, is defined by $\mathrm{bd}^X(U) := \mathrm{cl}^X(U) \cap \mathrm{cl}^X(X - U)$. The superscript X is omitted when X is understood from the context.

(b) U is *regular open* if $U = \mathrm{int}(\mathrm{cl}(U))$. $\mathrm{Ro}(X)$ denotes the set of regular open subsets of X. We equip $\mathrm{Ro}(X)$ with the operations: $U + V := \mathrm{int}(\mathrm{cl}(U \cup V))$, $U \cdot V := U \cap V$ and $-U := \mathrm{int}(X - U)$. Then $\langle \mathrm{Ro}(X), +, \cdot, - \rangle$ is a complete Boolean algebra. Obviously, $0^{\mathrm{Ro}(X)} = \emptyset$, $1^{\mathrm{Ro}(X)} = X$, and the induced partial ordering of $\mathrm{Ro}(X)$ is $\leq^{\mathrm{Ro}(X)} = \subseteq$. We regard $\mathrm{Ro}(X)$ both as a set and as a Boolean algebra.

(c) If $g : X \cong Y$ then g induces an isomorphism g^{Ro} between $\mathrm{Ro}(X)$ and $\mathrm{Ro}(Y)$: $g^{\mathrm{Ro}}(U) = \{g(x) \mid x \in U\}$. For $G \leq H(X)$ let $G^{\mathrm{Ro}} := \{g^{\mathrm{Ro}} \mid g \in G\}$. Then $G^{\mathrm{Ro}} \leq \mathrm{Aut}(\mathrm{Ro}(X))$ and if X is Hausdorff, then $g \mapsto g^{\mathrm{Ro}}$ is an embedding of G into $\mathrm{Aut}(\mathrm{Ro}(X))$. We assume that X is Hausdorff and identify G with G^{Ro}. So $H(X)$ is regarded as a subgroup of $\mathrm{Aut}(\mathrm{Ro}(X))$.

(d) G is a *locally moving* subgroup of $H(X)$ if for every nonempty open $V \subseteq X$ there is $g \in G - \{\mathrm{Id}\}$ such that $g{\restriction}(X - V) = \mathrm{Id}$. In that case $\langle X, G \rangle$ is called a *topological local movement system.*

(e) Let $\mathrm{Ap} : G \times \mathrm{Ro}(X) \to X$ be the *application function.* That is, $\mathrm{Ap}(g, V) = g(V)$. The structure $\mathrm{MR}(X, G)$ is defined as follows:

$$\mathrm{MR}(X, G) = \langle \mathrm{Ro}(X), G, +, \cdot, -, \mathrm{Ap} \rangle.$$

(f) $\eta : \mathrm{MR}(X, G) \cong \mathrm{MR}(Y, H)$ means that η is an isomorphism between $\mathrm{MR}(X, G)$ and $\mathrm{MR}(Y, H)$. That is, η is a bijection between $\mathrm{Ro}(X) \cup G$ and $\mathrm{Ro}(Y) \cup H$, $\eta(G) = H$, and η preserves $+, \cdot, -$ and Ap.

(g) If $\eta : A \to B$ is a bijection and $g : A \to A$, then the *conjugation* of g by η is defined as $g^{\eta} := \eta \circ g \circ \eta^{-1}$. $\square$

PROPOSITION 2.2. *Let X, Y be Hausdorff spaces, $G \leq H(X)$ and $H \leq H(Y)$. Suppose that $\varphi : G \cong H$ and $\eta : \mathrm{Ro}(X) \cong \mathrm{Ro}(Y)$. Then $\varphi \cup \eta : \mathrm{MR}(X, G) \cong \mathrm{MR}(Y, H)$ iff $\varphi(g) = g^\eta$ for every $g \in G$.*

The next theorem says that for topological local movement systems the action of G on $\mathrm{Ro}(X)$ can be reconstructed from G. This theorem is proved in [Ru5].

THEOREM 2.3 (The reconstruction theorem for topological local movement systems). *Let $\langle X, G \rangle$ and $\langle Y, H \rangle$ be topological local movement systems and $\varphi : G \cong H$. Then there is a unique $\eta : \mathrm{Ro}(X) \cong \mathrm{Ro}(Y)$ such that $\varphi \cup \eta : \mathrm{MR}(X, G) \cong \mathrm{MR}(Y, H)$. That is, there is a unique $\eta : \mathrm{Ro}(X) \cong \mathrm{Ro}(Y)$ such that $\varphi(g) = g^\eta$ for every $g \in G$.*

Proof. See [Ru5, Definition 1.2, Corollary 1.4 or Corollary 2.10 and Proposition 1.8]. ■

2.2. Faithfulness in locally compact spaces. The first faithfulness theorem to be presented is about locally compact spaces. It is taken from [Ru1] and brought here for the sake of completeness. It is the conjunction of parts (a), (b) and (c) of Theorem 3.5 there.

DEFINITION 2.4. (a) For $G \leq H(X)$, $g \in H(X)$ and $x \in X$, let $G(x) := \{g(x) \mid g \in G\}$. A set $A \subseteq X$ is *somewhere dense* if $\mathrm{int}(\mathrm{cl}(A)) \neq \emptyset$. X is a *perfect space* if there is no $x \in X$ such that $\{x\}$ is open. Suppose that G is a set of permutations of a set A and $B \subseteq A$. Define $G\lfloor \underline{B} \rfloor := \{g \in G \mid g \restriction (A - B) = \mathrm{Id}\}$.

(b) Let

$$K_{\mathrm{LCM}} := \{\langle X, G \rangle \mid X \text{ is a perfect locally compact Hausdorff space, and}$$
$$\text{for every open } V \subseteq X \text{ and } x \in V,\ G\lfloor \underline{V} \rfloor(x) \text{ is somewhere dense}\}. \ \square$$

THEOREM 2.5 (Rubin [Ru1] 1989). *K_{LCM} is faithful.*

Proof. It follows easily from the definitions that for every $\langle X, G \rangle \in K_{\mathrm{LCM}}$, $\langle X, G \rangle$ is a topological local movement system.

A subset p of a Boolean algebra B is called an *ultrafilter* if: (i) $0 \notin p$; (ii) if $a_1, \ldots, a_n \in p$, then $\prod_{i=1}^n a_i \in p$; (iii) if $a \in p$ and $b \geq a$, then $b \in p$; (iv) for every $a \in B$ either $a \in p$ or $-a \in p$.

By Zorn's lemma, every subset of B satisfying (i)–(ii) is contained in an ultrafilter. For an ultrafilter p in $\mathrm{Ro}(X)$, let $A_p := \bigcap\{\mathrm{cl}(V) \mid V \in p\}$. Let $\langle X, G \rangle \in K_{\mathrm{LCM}}$. We say that an ultrafilter p in $\mathrm{Ro}(X)$ is *good* if A_p is a singleton. If p is good and $A_p = \{x\}$, then we write $x = x_p$. The following facts can be easily checked.

(a) $A_p = \{x\}$ iff p contains all regular open neighborhoods of x.

(b) p is good iff there is $W \in \mathrm{Ro}(X) - \{\emptyset\}$ such that for every $V \in \mathrm{Ro}(X) - \{\emptyset\}$: if $V \subseteq W$, then there is $g \in G$ such that $g(V) \in p$.

(c) Let p and q be good ultrafilters. Then $x_p \neq x_q$ iff

$$(\exists U \in p)(\exists V \in q)((U \cap V = \emptyset) \wedge (\forall U_1 \subseteq U)(U_1 \neq \emptyset \rightarrow$$
$$(\exists f \in G)(V \in f(q) \wedge U_1 \in f(p)))).$$

(d) Let p be a good ultrafilter, and $U \in \mathrm{Ro}(X)$. Then $x_p \in U$ iff for every good ultrafilter q: if $x_q = x_p$, then $U \in q$.

(e) Let p, q be good ultrafilters, and $g \in G$. Then $g(x_p) = x_q$ iff $x_{g(p)} = x_q$.
(f) If p is a good ultrafilter and $g \in G$, then $g(p)$ is a good ultrafilter.
(g) If $x \in X$, then there is a good ultrafilter p such that $x_p = x$.

Clearly, the fact that p is an ultrafilter is expressible in terms of the operations of $\langle \mathrm{Ro}(X), +, \cdot, - \rangle$.

(1) By (b), the fact that p is a good ultrafilter is expressible in terms of the operations of $\mathrm{MR}(X, G)$.

(2) By (c), for good ultrafilters p and q, the fact that $x_p = x_q$ is expressible in terms of the operations of $\mathrm{MR}(X, G)$.

(3) By (d), for a good ultrafilter p and $U \in \mathrm{Ro}(X)$, the fact that $x_p \in U$ is expressible in terms of the operations of $\mathrm{MR}(X, G)$.

(4) By (e), for good ultrafilters p and q and $g \in G$, the fact that $g(x_p) = x_q$ is expressible in terms of the operations of $\mathrm{MR}(X, G)$.

Let $\langle X, G \rangle, \langle Y, H \rangle \in K_{\mathrm{LCM}}$, and let $\varphi : G \cong H$. By Theorem 2.3, there is $\eta : \mathrm{Ro}(X) \cong \mathrm{Ro}(Y)$ such that $(\varphi \cup \eta) : \mathrm{MR}(X, G) \cong \mathrm{MR}(Y, H)$. Let $\psi = \varphi \cup \eta$. We define $\tau : X \to Y$. Let $x \in X$. By (g), there is an ultrafilter p such that $x_p = x$. By (1), $\psi(p)$ is a good ultrafilter.

We define $\tau(x) = x_{\psi(p)}$. If q is a good ultrafilter such that also $x_q = x$, then by (2), $x_{\psi(q)} = x_{\psi(p)}$. So the definition of τ is valid.

We check that τ is a bijection between X and Y. Suppose that $x_p \neq x_q$. By (2), $\tau(x_p) = x_{\psi(p)} \neq x_{\psi(q)} = \tau(x_q)$. So τ is injective.

Let $y \in Y$. By (g), there is an ultrafilter q such that $x_q = y$. By (1), $p := \psi^{-1}(q)$ is a good ultrafilter. So $\tau(x_p) = x_{\psi(p)} = x_q = y$. So τ is surjective.

Let $\tau(A)$ denote $\{\tau(a) \mid a \in A\}$. In order to show that τ is a homeomorphism, it suffices to show that for some open base $\mathcal{B}$ of X, $\{\tau(U) \mid U \in \mathcal{B}\}$ is an open base for Y. Since X and Y are locally compact, they are regular spaces. So $\mathrm{Ro}(X)$ and $\mathrm{Ro}(Y)$ are open bases of X and Y repectively. So it suffices to show that $\{\tau(U) \mid U \in \mathrm{Ro}(X)\} = \mathrm{Ro}(Y)$. Let $x \in X$ and $U \in \mathrm{Ro}(X)$. Let p be an ultrafilter such that $x_p = x$. By (3), $x_p \in U$ iff $x_{\psi(p)} \in \psi(U)$. That is, $x \in U$ iff $\tau(x) \in \psi(U)$. So $\tau(U) = \psi(U)$ for every $U \in \mathrm{Ro}(X)$. Hence $\{\tau(U) \mid U \in \mathrm{Ro}(X)\} = \{\psi(U) \mid U \in \mathrm{Ro}(X)\} = \mathrm{Ro}(Y)$. So τ is a homeomorphism.

It remains to show that τ induces φ. Let $g \in G$ and $y \in Y$. Let q be an ultrafilter in $\mathrm{Ro}(Y)$ such that $x_q = y$. Then $g^{\tau}(y) = \tau \circ g \circ \tau^{-1}(x_q) = \tau \circ g(x_{\psi^{-1}(q)}) = \tau(x_{g(\psi^{-1}(q))}) = x_{\psi(g(\psi^{-1}(q)))} = x_{\eta(g(\eta^{-1}(q)))} = x_{g^{\eta}(q)}$. But by Proposition 2.2, $g^{\eta} = \varphi(g)$. So $x_{g^{\eta}(q)} = x_{\varphi(g)(q)}$. However, if $x_q = y$, then trivially $x_{h(q)} = h(y)$ for every $h \in H$. In particular, $x_{\varphi(g)(q)} = \varphi(g)(y)$.

We have shown that $g^{\tau}(y) = \varphi(g)(y)$ for every $y \in Y$. So $g^{\tau} = \varphi(g)$. ■

Remark. In the above proof the existence of the inducing homeomorphism τ was deduced from facts (b)–(e) which showed that point representation, equality, belonging and application were expressible in $\mathrm{MR}(X, G)$. The toil of deducing the existence of τ from (b)–(e) could have been spared by using a certain general machinery called the method of interpretation. The notion of interpretation is not introduced here, since it is used only twice. Interpretations are described e.g. in [Ru2, Section 2] or in [MR, Section 6].

Theorem 2.5 has many applications in the Euclidean case. For example, it applies to m times continuously differentiable Euclidean manifolds.

COROLLARY 2.6 ([Ru1]). *Let* $K_D = \{\langle X, G\rangle \mid$ *for some* $0 \leq m \leq \infty$, X *is a Euclidean* C^m*-manifold and* G *contains all homeomorphisms* f *such that both* f *and* f^{-1} *are* C^m *homeomorphisms*$\}$. *Then* K_D *is faithful.*

Proof. $K_D \subseteq K_{\mathrm{LCM}}$. ■

Theorem 2.5 also applies to Hilbert cube manfolds, and in fact to manifolds over $[0,1]^\lambda$ for any cardinal λ.

The class of Menger manifolds is also a subclass of K_{LCM}, and hence it is faithful. See Kawamura [K].

The finitely presented subgroups of $H(\mathbb{R})$ defined by R. Thompson (see [Br1], [Br2] and [BG]) also belong to K_{LCM}.

2.3. Faithfulness in normed and Banach spaces. We now turn to the context of normed vector spaces and Banach spaces.

To avoid notational complications, we shall mainly deal with open subsets of normed and Banach spaces and not with manifolds over such spaces. Nevertheless, all theorems and proofs transfer (with a correct translation) to manifolds. In this section, Definition 2.29 and Theorem 2.30 deal with the setting of manifolds (and indeed with a somewhat more general setting).

Manifolds are considered again at the end of Chapter 3 starting from Definition 3.46.

Recall that for a metric space X, $\mathrm{LIP}(X) = \{h \in H(X) \mid h \text{ is bilipschitz}\}$ and $\mathrm{LIP}^{\mathrm{LC}}(X) = \{h \in H(X) \mid h \text{ is locally bilipschitz}\}$.

For a normed space E, an open set $S \subseteq E$ and a dense linear subspace $F \subseteq E$, we shall use the notations $\mathrm{LIP}(X; S, F)$, $\mathrm{LIP}^{\mathrm{LC}}(X; S, F)$, $\mathrm{LIP}(X; F)$ and $\mathrm{LIP}^{\mathrm{LC}}(X, F)$ introduced in Definition 1.4.

We shall prove the faithfulness of the classes K_{B} and K_{N} defined below. However, these faithfulness results do not suffice for some of the continuations. To this end we define the bigger class K_{BNO} and prove its faithfulness.

DEFINITION 2.7. Let E be a normed space, $X \subseteq E$ be open, $\mathcal{S}$ be a set of open subsets of X and $\mathcal{F} = \{F_S \mid S \in \mathcal{S}\}$ be a family of dense linear subspaces of E indexed by $\mathcal{S}$. Then $\mathcal{F}$ is called a *subspace choice* for $\mathcal{S}$. If $\mathcal{S}$ is a cover of X, then $\langle E, X, \mathcal{S}, \mathcal{F}\rangle$ is called a *subspace choice system*.

(a) $\mathrm{LIP}(X; \mathcal{S}, \mathcal{F})$ is the subgroup of $H(X)$ generated by $\bigcup\{\mathrm{LIP}(X; S, F_S) \mid S \in \mathcal{S}\}$. $\mathrm{LIP}^{\mathrm{LC}}(X; \mathcal{S}, \mathcal{F})$ denotes the subgroup of $H(X)$ generated by $\bigcup\{\mathrm{LIP}^{\mathrm{LC}}(X; S, F_S) \mid S \in \mathcal{S}\}$. Also, $\mathrm{LIP}(X, \mathcal{S})$ denotes the subgroup of $H(X)$ generated by $\bigcup\{\mathrm{LIP}(X, S) \mid S \in \mathcal{S}\}$. The group $\mathrm{LIP}^{\mathrm{LC}}(X, \mathcal{S})$ is defined analogously.

(b) Let K_{B} be the class of all $\langle X, G\rangle$'s such that X is an open subset of some Banach space, and $\mathrm{LIP}(X) \leq G \leq H(X)$.

Let K_{N} be the class of all $\langle X, G\rangle$'s such that X is an open subset of some normed space, and $\mathrm{LIP}^{\mathrm{LC}}(X) \le G \le H(X)$.

Let K_{BO} be the class of all $\langle X, G\rangle$'s such that:

(1) X is an open subset of some Banach space E,
(2) there are an open cover $\mathcal{S}$ of X and a subspace choice $\mathcal{F}$ for $\mathcal{S}$ such that we have $\mathrm{LIP}(X; \mathcal{S}, \mathcal{F}) \le G \le H(X)$.

Let K_{NO} be the class of all $\langle X, G\rangle$'s such that:

(1) X is an open subset of some normed space E,
(2) there are an open cover $\mathcal{S}$ of X and a subspace choice $\mathcal{F}$ for $\mathcal{S}$ such that we have $\mathrm{LIP}^{\mathrm{LC}}(X; \mathcal{S}, \mathcal{F}) \le G \le H(X)$.

Let $K_{\mathrm{BNO}} = K_{\mathrm{BO}} \cup K_{\mathrm{NO}}$. If $\langle X, G\rangle \in K_{\mathrm{BNO}}$ and $E, \mathcal{S}, \mathcal{F}$ are as above, then the system $\langle E, X, \mathcal{S}, \mathcal{F}, G\rangle$ is called a *BNO-system.* $\square$

THEOREM 2.8. (a) $K_{\mathrm{B}} \cup K_{\mathrm{N}}$ *is faithful.*

(b) K_{BNO} *is faithful.*

Note that $K_{\mathrm{B}} \cup K_{\mathrm{N}} \subseteq K_{\mathrm{BNO}}$. So only (b) has to be proved.

REMARK 2.9. (a) Dealing with the larger but less natural classes of groups $\mathrm{LIP}(X; \mathcal{S}, \mathcal{F})$ and $\mathrm{LIP}^{\mathrm{LC}}(X; \mathcal{S}, \mathcal{F})$ needs justification. Certainly the groups $\mathrm{LIP}(X)$ and $\mathrm{LIP}^{\mathrm{LC}}(X)$ are those that come to mind first. There are two classes of groups which merit attention for which Theorem 2.8(a) does not suffice, but Theorem 2.8(b) does.

Let E be a normed vector space and $X \subseteq E$ be open. The group of extendible homeomorphisms of X is defined as follows:

$$\mathrm{EXT}^{E}(X) = \{h{\restriction}X \mid h \in H(\mathrm{cl}^{E}(X)) \text{ and } h{\restriction}X \in H(X)\}.$$

If E is a Banach space, then $\mathrm{LIP}(X) \subseteq \mathrm{EXT}^{E}(X)$. However, if E is not complete, then $\mathrm{LIP}(X) \not\subseteq \mathrm{EXT}^{E}(X)$.

For $h \in \mathrm{EXT}(X)$ let h^{cl} denote the extension of h to $\mathrm{cl}^{E}(X)$. Let Γ be a modulus of continuity. Define

$$H_{\Gamma}^{\mathrm{CMP.LC}}(X) = \{h \in \mathrm{EXT}(X) \mid \text{for some } \alpha \in \Gamma,\ h^{\mathrm{cl}} \text{ is locally } \alpha\text{-bicontinuous}\}.$$

Then Theorem 2.8(a) does not apply to $H_{\Gamma}^{\mathrm{CMP.LC}}(X)$, but Theorem 2.8(b) does.

Another such example is the following group. Let E be a finite-dimensional normed space, $X \subseteq E$ be open and

$$H = \{h \in H(X) \mid \mathrm{cl}(\{x \in X \mid h(x) \ne x\}) \text{ is compact}\}.$$

Then $G \not\supseteq \mathrm{LIP}(X)$, but nevertheless X is reconstructible from G.

The reason for introducing the group $\mathrm{LIP}(X; F)$ is as follows. For an incomplete normed space X, we give a proof that X is reconstructible from G's which contain $\mathrm{LIP}^{\mathrm{LC}}(X)$. But we do not know whether X is reconstructible from $\mathrm{LIP}(X)$. In fact, every member of $\mathrm{LIP}(X)$ can be uniquely extended to a homeomorphism of $\bar{X}$, the completion of X. So $\mathrm{LIP}(X)$ can be regarded as a subgroup of $H(\bar{X})$. By considering $\mathrm{LIP}(\bar{X}; X)$ we prove that $\bar{X}$ is reconstructible from $\mathrm{LIP}(X)$. It remains open (except for spaces of the first category) whether X is reconstructible from $\mathrm{LIP}(X)$.

(b) The groups $\mathrm{LIP}(X;\mathcal{S},\mathcal{F})$ and $\mathrm{LIP}(X)$ in Theorem 2.8 can be replaced by the following smaller groups. Suppose that a normed or a Banach space E has an equivalent norm which is C^m, $m \leq \infty$, that is, a norm which is m times continuously Fréchet differentiable at every $x \neq 0$. We define $\mathrm{Diff}^m(X)$ to be the group of all homeomorphisms g of X such that g, g^{-1} are C^m, and whose first derivative is bounded. The group $\mathrm{Diff}^m(X;\mathcal{F},\mathcal{S})$ is defined in analogy to $\mathrm{LIP}(X;\mathcal{S},\mathcal{F})$, and the classes K_{BD^m}, K_{ND^m} and K_{BNOD^m} are defined in analogy to K_{B}, K_{N} and K_{BNO}. Then Theorem 2.8 remains true. The proof remains the same. The only difference is that the homeomorphisms which are constructed in the proof of Theorem 2.8 have to be in this case C^m and not just bilipschitz.

This variant of Theorem 2.8 will be needed in a subsequent work where groups of Fréchet differentiable homeomorphisms will be considered. $\square$

An explanation of the method of proof of Theorem 2.8. We show that there is a property $P(x,y)$ of pairs $\langle x,y\rangle$ which is expressible in terms of the operations of $\mathrm{MR}(X,G)$ such that for every $\langle X,G\rangle \in K_{\mathrm{BNO}}$ and $U, V \in \mathrm{Ro}(X)$:

$$P(U,V) \text{ holds in } \mathrm{MR}(X,G) \quad \text{iff} \quad \mathrm{cl}(U)\cap\mathrm{cl}(V) \text{ is a singleton.}$$

A pair $\langle U,V\rangle$ satisfying P is called a *point representing pair*.

We shall then prove two similar facts.

(1) There is a property $Q(x_1,y_1,x_2,y_2)$ expressible in terms of the operations of $\mathrm{MR}(X,G)$ such that for every $\langle X,G\rangle \in K_{\mathrm{BNO}}$ and point representing pairs $\langle U_1,V_1\rangle, \langle U_2,V_2\rangle \in (\mathrm{Ro}(X))^2$:

$$Q(U_1,V_1,U_2,V_2) \text{ holds in } \mathrm{MR}(X,G) \quad \text{iff} \quad \mathrm{cl}(U_1)\cap\mathrm{cl}(V_1) = \mathrm{cl}(U_2)\cap\mathrm{cl}(V_2).$$

(2) There is a property $S(x,y,z)$ expressible in terms of the operations of $\mathrm{MR}(X,G)$ such that for every $\langle X,G\rangle \in K_{\mathrm{BNO}}$, a point representing pair $\langle U,V\rangle \in (\mathrm{Ro}(X))^2$ and $W \in \mathrm{Ro}(X)$:

$$S(U,V,W) \text{ holds in } \mathrm{MR}(X,G) \quad \text{iff} \quad \mathrm{cl}(U)\cap\mathrm{cl}(V) \subseteq W.$$

As in the proof of 2.5, the existence of properties P, Q and S implies that every isomorphism between $\mathrm{MR}(X,G)$ and $\mathrm{MR}(Y,H)$ is induced by a homeomorphism between X and Y.

The following conventions are kept through Lemma 2.23 and the proof of Theorem 2.8.

(a) In what follows, $\langle E,X,\mathcal{S},\mathcal{F},G\rangle$ denotes a BNO-system. That is, E denotes a normed vector space, X is an open subset of E, $\mathcal{S}$ is a cover of X, $\mathcal{F}$ is a subspace choice for $\mathcal{S}$ and $G \leq H(X)$. If E is a Banach space, then $\mathrm{LIP}(X;\mathcal{S},\mathcal{F}) \leq G$, and if E is incomplete, then $\mathrm{LIP}^{\mathrm{LC}}(X;\mathcal{S},\mathcal{F}) \leq G$.

If X is an open subset of E and $\langle X,G\rangle \in K_{\mathrm{B}} \cup K_{\mathrm{N}}$, then $\langle X,G\rangle$ is regarded as a BNO-system with $\mathcal{S} = \{X\}$ and $F_X = E$.

(b) Also, U,V,W denote members of $\mathrm{Ro}(X)$. If $A \subseteq X$, then $\mathrm{cl}^X(A)$ and $\mathrm{int}^X(A)$ are abbreviated by $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ respectively.

Definition 2.10. (a) For U, $V \in \mathrm{Ro}(X)$ let $U \cong V$ denote that $(\exists g \in G)(g(U) = V)$.

(b) U is a *small set* if there is $W \neq \emptyset$ such that for every $\emptyset \neq W' \subseteq W$ there is $U' \cong U$ such that $U' \subseteq W'$.

(c) U is *strongly small* in V ($U \prec V$) if there is $\emptyset \neq W \subseteq V$ such that for every $\emptyset \neq W_1 \subseteq W$ there is $g \in G\lfloor V \rfloor$ such that $g(U) \subseteq W_1$.

(d) U is *strongly separated* from W ($U \leftarrow\!\!\Vdash\!\!\rightarrow W$) if there is $V \in \mathrm{Ro}(X)$ such that $U \prec V$ and $V \cap W = \emptyset$. □

Remark 2.11. The properties "$U \cong V$", "U is a small set", "$U \prec V$" and "$U \leftarrow\!\!\Vdash\!\!\rightarrow W$" are expressible in terms of the operations of $\mathrm{MR}(X, G)$. Formally this means the following statements.

(1) Let $\chi_{\cong}(x, y) \equiv (\exists z \in G)(\mathrm{Ap}(z, x) = y)$. Then U, V satisfy $\chi_{\cong}$ in $\mathrm{MR}(X, G)$ iff $U \cong V$.
(2) Let $\chi_{\subseteq}(x, y) \equiv x \cdot y = x$. Then U, V satisfy $\chi_{\subseteq}$ in $\mathrm{MR}(X, G)$ iff $U \subseteq V$.
(3) Let $\chi_{\emptyset}(x) \equiv (\forall y \in \mathrm{Ro}(X))(x \cdot y = x)$. Then U satisfies $\chi_{\emptyset}$ in $\mathrm{MR}(X, G)$ iff $U = \emptyset$.
(4) Let
$$\chi_{Sml}(x) \equiv (\exists y \in \mathrm{Ro}(X))(\neg\chi_{\emptyset}(y) \wedge (\forall y' \in \mathrm{Ro}(X))((\chi_{\subseteq}(y', y) \wedge \neg\chi_{\emptyset}(y')) \rightarrow (\exists x' \in \mathrm{Ro}(X))(\chi_{\cong}(x', x) \wedge \chi_{\subseteq}(x', y')))).$$
Then U satisfies χ_{Sml} in $\mathrm{MR}(X, G)$ iff U is small.
(5) Let $\chi_{Spprtd}(x, y) \equiv (\forall z \in \mathrm{Ro}(X))(\chi_{\emptyset}(z \cdot y) \rightarrow (\mathrm{Ap}(x, z) = z))$. Then g, V satisfy χ_{Spprtd} in $\mathrm{MR}(X, G)$ iff $g \in G\lfloor V \rfloor$.

Similar formulas $\chi_{\prec}$ and $\chi_{\leftarrow\Vdash\rightarrow}$ can be written for $U \prec V$ and for $U \leftarrow\!\!\Vdash\!\!\rightarrow V$. The above formulas use only the operations $+$, $\cdot$, $-$ and Ap. So if χ is any of the above formulas, $\psi : \mathrm{MR}(X, G) \cong \mathrm{MR}(Y, H)$ and $U, V \in \mathrm{Ro}(X)$, then U, V satisfy χ in $\mathrm{MR}(X, G)$ iff $\psi(U), \psi(V)$ satisfy χ in $\mathrm{MR}(Y, H)$. So smallness, $\prec$, $\leftarrow\!\!\Vdash\!\!\rightarrow$ etc. are preserved under isomorphisms. □

Definition 2.12. (a) For a metric space (Z, d), $x \in Z$ and $r > 0$ we define $B^Z(x, r) := \{y \in Z \mid d(x, y) < r\}$, $S^Z(x, r) := \{y \in Z \mid d(x, y) = r\}$ and $\bar{B}^Z(x, r) := \{y \in Z \mid d(x, y) \leq r\}$. If $A \subseteq Z$, then $B^Z(A, r) := \bigcup_{x \in A} B^Z(x, r)$.

In the context of this section there are two metric spaces involved: a normed space E and an open subset $X \subseteq E$. We use $B(x, r)$, $S(x, r)$ and $\bar{B}(x, r)$ as abbreviations of $B^X(x, r)$, $S^X(x, r)$ and $\bar{B}^X(x, r)$.

For $x, y \in E$, $[x, y]$ denotes the line segment connecting x and y. For $v \in E$ let $\mathrm{tr}_v^E : E \to E$ be the translation by v, that is, $\mathrm{tr}_v^E(x) = v + x$. Whenever E can be understood from the context, tr_v^E is abbreviated by tr_v.

(b) Let $\mathcal{N} = \langle E, X, \mathcal{S}, \mathcal{F}, G \rangle$ be a BNO-system and $B = B^E(x, r)$ be a ball of E. Then B is a *manageable ball* of X (with respect to $\mathcal{N}$) if there are $S \in \mathcal{S}$ and $\varepsilon > 0$ such that $x \in S \cap F_S$ and $B^E(x, r + \varepsilon) \subseteq S$. In such a case we say that B *is based on* S. Note that if $B = B^E(x, r)$ is a manageable ball, then $B^E(x, r) = B^X(x, r)$ and $\mathrm{cl}^E(B) = \mathrm{cl}^X(B)$.

(c) For a topological space Y and $h \in H(Y)$, the support of h is defined as
$$\mathrm{supp}(h) = \{y \in Y \mid h(y) \neq y\}. \ \square$$

PROPOSITION 2.13. (a) *Suppose that Y is any topological space, and let $H \le H(Y)$. For $k \in H$ let $\psi_k : \mathrm{MR}(Y,H) \to \mathrm{MR}(Y,H)$ be defined as follows. For every $h \in H$, $\psi_k(h) = h^k$, and for every $U \in \mathrm{Ro}(Y)$, $\psi_k(U) = \{h(x) \mid x \in U\}$. Then $\psi_k \in \mathrm{Aut}(\mathrm{MR}(Y,H))$.*

(b) *Let Y be any topological space.*

(i) *If $F \subseteq Y$ is closed, then $\mathrm{int}(F) \in \mathrm{Ro}(Y)$.*

(ii) *$\mathrm{int}(\mathrm{cl}(A)) \in \mathrm{Ro}(Y)$ for every $A \subseteq Y$.*

(iii) *$\mathrm{int}(\mathrm{cl}(A))$ is the minimal regular open set containing A.*

(iv) *If $T, S \subseteq Y$ are disjoint open sets, then $\mathrm{int}(\mathrm{cl}(T)) \cap S = \emptyset$.*

Proof. Trivial. ■

We shall next construct certain homeomorphisms in $\mathrm{LIP}(X;\mathcal{S},\mathcal{F})$. Geometrically, their existence is quite obvious. However, the formal proof requires some computation.

All balls mentioned in the next lemma are manageable. For such balls we write $B^E(x,r) = B(x,r)$. Part (b)(ii) of the lemma will be used in Chapter 3. See Proposition 3.4.

LEMMA 2.14. (a) *Suppose that $B = B(x_0, r_0)$ is a manageable ball based on S, $x_0 \in F_S$ and $0 < s_0 < s_1 < r_0$. Then there is $h \in \mathrm{LIP}(X;\mathcal{S},\mathcal{F})\lfloor B \rfloor$ such that $h(B(x_0,s_1)) = B(x_0,s_0)$.*

(b) *Suppose that $B = B(x_0,r_0)$ is a manageable ball based on S, $x_0, v \in B \cap F_S$, $0 < r < r_0$ and $0 < s < r_0 - \|v - x_0\|$. Then*

(i) *There is $h \in \mathrm{LIP}(X;\mathcal{S},\mathcal{F})\lfloor B \rfloor$ such that $h(B(x_0,r)) = B(v,s)$.*

(ii) *If also $r = s$, then h is $(1 + \frac{\|v\|}{r_0 - r - \|v\|})$-bilipschitz and $h{\restriction}B(x_0,r) = \mathrm{tr}_v{\restriction}B(x_0,r)$.*

(c) *Let B be a manageable ball based on S, $x, y \in B \cap F_S$ and $r > 0$. Assume that $B([x,y],r) \subseteq B$. Then there is $h \in \mathrm{LIP}(X;\mathcal{S},\mathcal{F})\lfloor B([x,y],r) \rfloor$ such that $h{\restriction}B(x,2r/3) = \mathrm{tr}_{y-x}{\restriction}B(x,2r/3)$. Moreover, there is a function $K_{\mathrm{seg}}(\ell,t)$ increasing in ℓ and decreasing in t such that the above h is $K_{\mathrm{seg}}(\|x-y\|,r)$-bilpschitz.*

(d) *Let $U \subseteq X$ be open, $\gamma : [0,1] \to U$ be continuous and 1-1 and $s \in (0,1]$. Then there is $h \in \mathrm{LIP}(X)$ such that $h(\gamma(0)) = \gamma(0)$, $h(\gamma(s)) = \gamma(1)$ and $\mathrm{supp}(h) \subseteq U$.*

Proof. (a) Assume for simplicity that $x_0 = 0$. Let $g \in H([0,\infty))$ be the piecewise linear function with breakpoints at s_0 and r_0 such that $g(s_0) = s_1$ and $g(t) = t$ for every $t \ge r_0$. Then g is K-bilipschitz with $K = \max\left(\frac{s_1}{s_0}, \frac{r_0 - s_0}{r_0 - s_1}\right)$.

We define $h : E \to E$

$$h(x) = g(\|x\|)\frac{x}{\|x\|} \quad \text{if } x \ne 0 \quad \text{and} \quad h(0) = 0.$$

Let $x, y \in E$. We may assume that $0 \ne \|y\| \le \|x\|$. Let $z = \|y\|\frac{x}{\|x\|}$. Then $\|x - z\| = \|x\| - \|y\| \le \|x - y\|$ and $\|z - y\| \le \|z - x\| + \|x - y\| \le 2\|x - y\|$. So

$$\begin{aligned}\|h(x) - h(y)\| &\le \|h(x) - h(z)\| + \|h(z) - h(y)\| \le K\|x - z\| + \frac{g(\|y\|)}{\|y\|}\|z - y\| \\ &\le K\|x - y\| + K \cdot 2\|x - y\| = 3K\|x - y\|.\end{aligned}$$

An identical argument shows that h^{-1} is $3K$-Lipschitz.

It is obvious that $h(F) = F$ and that $h(B(0, s_0)) = B(0, s_1)$. So $h^{-1} \restriction X$ is as required.

(b) Assume for simplicity that $x_0 = 0$. By (a), we may assume that $r = s$. Define $g : [0, \infty) \to [0, 1]$ as follows:

$$g(t) = \begin{cases} 1, & 0 \le t \le r, \\ \frac{r_0 - t}{r_0 - r}, & r \le t \le r_0, \\ 0, & r_0 \le t. \end{cases}$$

Suppose that $a > r_0$ and $\bar{B}(0, a) \subseteq X$. We define $h : \bar{B}(0, a) \to E$ by $h(x) = x + g(\|x\|) \cdot v$. Obviously, $h(B(0, r)) = B(v, r)$.

We show that h is Lipschitz. At first we see that $h \restriction (\bar{B}(0, r_0) - B(0, r))$ is Lipschitz. Let $x, y \in \bar{B}(0, r_0) - B(0, r)$. Then $h(x) - h(y) = x - y + \frac{\|y\| - \|x\|}{r_0 - r} \cdot v$. It follows that

$$\|h(x) - h(y)\| \le \|x - y\| + \frac{|\,\|y\| - \|x\|\,|}{r_0 - r} \cdot \|v\| \le \|x - y\| + \frac{\|x - y\|}{r_0 - r} \cdot \|v\| = \left(1 + \frac{\|v\|}{r_0 - r}\right) \cdot \|x - y\|.$$

Let $x, y \in \bar{B}(0, a)$. If $x, y \in B(0, r)$ or $x, y \in \bar{B}(0, r_0) - B(0, r)$ or $x, y \in \bar{B}(0, a) - B(0, r_0)$, then $\|h(x) - h(y)\| \le (1 + \frac{\|v\|}{r_0 - r}) \cdot \|x - y\|$.

If $x \in B(0, r)$ and $y \in \bar{B}(0, r_0) - B(0, r)$, let $z \in [x, y] \cap S(0, r)$. Then

$$\begin{aligned} \|h(x) - h(y)\| &\le \|h(x) - h(z)\| + \|h(z) - h(y)\| \le \|x - z\| + \left(1 + \frac{\|v\|}{r_0 - r}\right) \cdot \|z - y\| \\ &\le \left(1 + \frac{\|v\|}{r_0 - r}\right) \cdot (\|x - z\| + \|z - y\|) = \left(1 + \frac{\|v\|}{r_0 - r}\right) \cdot \|x - y\|. \end{aligned}$$

The other cases are dealt with similarly. So h is $(1 + \frac{\|v\|}{r_0 - r})$-Lipschitz.

In order to show that h is 1-1 and that h^{-1} is Lipschitz, we first check that there is K such that $\|x - y\| \le K \cdot \|h(x) - h(y)\|$ for every $x, y \in \bar{B}(0, r_0) - B(0, r)$. Indeed,

$$\begin{aligned} \|h(x) - h(y)\| &\ge \|x - y\| - \frac{|\,\|y\| - \|x\|\,|}{r_0 - r} \cdot \|v\| \ge \|x - y\| - \frac{\|y - x\|}{r_0 - r} \cdot \|v\| \\ &= \left(1 - \frac{\|v\|}{r_0 - r}\right) \cdot \|x - y\| = \frac{r_0 - r - \|v\|}{r_0 - r} \cdot \|x - y\|. \end{aligned}$$

Clearly, $\frac{r_0 - r - \|v\|}{r_0 - r} > 0$. Let $K = \frac{r_0 - r}{r_0 - r - \|v\|}$. Then $\|x - y\| \le K \cdot \|h(x) - h(y)\|$. This implies that $h \restriction (\bar{B}(0, r_0) - B(0, r))$ is 1-1.

We next check that $h(\bar{B}(0, r_0) - B(0, r)) = \bar{B}(0, r_0) - B(v, r)$. Let $x \in \bar{B}(0, r_0) - B(0, r)$. There are $x_1, x_2 \in \operatorname{bd}(\bar{B}(0, r_0) - B(0, r))$ such that $x \in [x_1, x_2] \subseteq \bar{B}(0, r_0) - B(0, r)$, and $x_2 = x_1 + \lambda v$ for some $\lambda \ge 0$. Suppose first that $x_1, x_2 \in S(0, r_0)$. Clearly, $h([x_1, x_2])$ is a line segment. Since $h \restriction [x_1, x_2]$ is 1-1 and $h(x_i) = x_i,\ i = 1, 2$, we have $h([x_1, x_2]) = [x_1, x_2]$.

A similar argument shows that if $x_1 \in S(0, r_0)$ and $x_2 \in S(0, r)$, then $h([x_1, x_2]) = [x_1, x_2 + v] \subseteq \bar{B}(0, r_0) - B(0, r)$. Also if $x_1 \in S(0, r)$ and $x_2 \in S(0, r_0)$, then $h([x_1, x_2]) = [x_1 + v, x_2] \subseteq \bar{B}(0, r_0) - B(0, r)$.

It follows that $h(\bar{B}(0, r_0) - B(0, r)) \subseteq \bar{B}(0, r_0) - B(v, r)$. A similar consideration shows that $\bar{B}(0, r_0) - B(v, r) \subseteq h(\bar{B}(0, r_0) - B(0, r))$. Also, $h(B(0, r)) = B(0, v)$, $h(\bar{B}(0, a) - \bar{B}(0, r_0)) = \bar{B}(0, a) - \bar{B}(0, r_0)$ and $h \restriction ((\bar{B}(0, a) - \bar{B}(0, r_0) \cup B(0, r))$ is 1-1. So h is a bijection and $\mathrm{Rng}(h) = \bar{B}(0,a)$. We have proved that $h^{-1}\restriction(\bar{B}(0,r_0)-B(0,r))$ is $\frac{r_0-r}{r_0-r-\|v\|}$-Lipschitz. The argument that h^{-1} is $\frac{r_0-r}{r_0-r-\|v\|}$-Lipschitz is the same one used to show that h is Lipschitz.

Clearly, $\frac{r_0-r}{r_0-r-\|v\|} = 1 + \frac{\|v\|}{r_0-r-\|v\|}$ and $1 + \frac{\|v\|}{r_0-r} \le 1 + \frac{\|v\|}{r_0-r-\|v\|}$. So h is $1 + \frac{\|v\|}{r_0-r-\|v\|}$-bilipschitz. As in the preceding arguments, this implies that $h \cup \mathrm{Id}\restriction(X - \bar{B}(0,a))$ is $1 + \frac{\|v\|}{r_0-r-\|v\|}$-bilipschitz.

For every $x \in \bar{B}(0,a)$, $h(x) - x \in \mathrm{span}(\{v\}) \subseteq F$. So $x \in F$ iff $h(x) \in F$. Hence $h \cup \mathrm{Id}\restriction(X - \bar{B}(0,a)) \in \mathrm{LIP}(X;\mathcal{F},\mathcal{S})$. Note also that $h\restriction B(0,r) = \mathrm{tr}_v\restriction B(0,r)$. So $h \cup \mathrm{Id}\restriction(X - \bar{B}(0,a))$ fulfills the requirements of (i) and (ii).

(c) Let $x_0,\ldots,x_n \in [x,y]$ be such that $x_0 = x$, $x_1 = y$ and $\|x_i - x_{i+1}\| < r/4$ for every $i < n$. By (b), for every $i < n$ there is $h_i \in \mathrm{LIP}(X;\mathcal{S},\mathcal{F})\boxed{B(x_i,r)}$ such that $h_i\restriction B(x_i,2r/3)) = \mathrm{tr}_{x_{i+1}-x_i}\restriction B(x_i,2r/3)$. Let $h = h_0 \circ \cdots \circ h_{n-1}$. Then h is as required. Note that n can be chosen to be $[4\|x-y\|/r]+1$. By (b) each h_i is $(1 + \frac{r/4}{r-2r/3-r/4})$-bilipschitz. That is, h_i is 4-bilipschitz. Hence $K_{\mathrm{seg}}(\ell,t) = 4^{[4\ell/t]+1}$.

(d) Let $x = \gamma(s)$, $y = \gamma(1)$, $L = \gamma([s,1])$ and $r = d(L,(X-U)\cup\{s(0)\})$. There is a sequence of balls $B(x_1,r),\ldots,B(x_n,r)$ such that $x_1,\ldots,x_n \in L$ and $\bigcup_{i=1}^n B(x_i,r) \supseteq L$. We may assume that $x \in B(x_1,r)$, $y \in B(x_n,r)$, and $B(x_i,r)\cap B(x_{i+1},r) \neq \emptyset$ for every $i < n$. For every $i < n$ let $y_i \in B(x_i,r)\cap B(x_{i+1},r)$. Set $y_0 = x$ and $y_n = y$. By (b), for every $i = 1,\ldots,n$ there is $h_i \in \mathrm{LIP}(X)$ such that $h_i(y_{i-1}) = y_i$ and $\mathrm{supp}(h_i) \subseteq B(x_i,r)$. Clearly, $h_n \circ \cdots \circ h_1$ is as required. ∎

The following observation will be used in many arguments. Its proof is left to the reader.

Proposition 2.15. (a) *Let X be a metric space, and $\vec{x}$ be a sequence in X. Then either $\vec{x}$ has a Cauchy subsequence, or there are $r > 0$ and a subsequence $\{y_n \mid n \in \mathbb{N}\}$ of $\vec{x}$ such that for distinct $i,j \in \mathbb{N}$, $d(y_i,y_j) \ge r$.*

(b) *Let X be a metric space and $\{x_i \mid i \in \mathbb{N}\} \subseteq X$ be a bounded sequence. Then either $\{x_i \mid i \in \mathbb{N}\}$ has a Cauchy subsequence, or there is a subsequence $\{y_i \mid i \in \mathbb{N}\}$ of $\{x_i \mid i \in \mathbb{N}\}$ and $r > 0$ such that for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $|d(y_i,y_j) - r| < \varepsilon$ for distinct $n,m > N$.*

Proposition 2.16. (a) *If $U_1 \subseteq U \prec V \subseteq V_1$, then $U_1 \prec V_1$.*

(b) *If $U \prec V$ for some V, then U is small.*

(c) *Let $B(x,r)$ and $B(y,s)$ be manageable balls based on the same S. If $\mathrm{cl}(B(x,r)) \subseteq B(y,s)$, then $B(x,r) \prec B(y,s)$.*

(d) *If $U \in \mathrm{Ro}(X)$ is a subset of a manageable ball, then U is small.*

(e) *If $U \prec V$, then $\mathrm{cl}(U) \subseteq V$.*

(f) *If B is a manageable ball of X, then $B \in \mathrm{Ro}(X)$ and B is small.*

Proof. Parts (a) and (b) follow trivially from the definitions.

(c) Note that if $\mathrm{cl}(B(x,r)) \subseteq B(y,s)$, then $\|x-y\|+r < s$. So (c) follows from Lemma 2.14(b).

(d) Suppose that $U \subseteq B$, and B is a manageable ball. There is a manageable ball B' with the same center as B such that $\mathrm{cl}(B) \subseteq B'$. Obviously, B and B' are based on the same S. So by (c), $B \prec B'$. By (a), $U \prec B'$. By (b), U is small.

(e) Suppose that $x \in \mathrm{cl}(U) - V$. Let $\emptyset \neq W \subseteq V$. Then there is $\emptyset \neq W' \subseteq W$ such that $\mathrm{cl}(W') \subseteq W$. Let $g \in G\lfloor \underline{V} \rfloor$. Then $g(x) = x$. Suppose by contradiction that $g(U) \subseteq W'$. Then $g(x) \in g(\mathrm{cl}(U)) \subseteq \mathrm{cl}(W') \subseteq W \not\ni x$. A contradiction.

(f) $B \in \mathrm{Ro}(E)$ and $\mathrm{int}(\mathrm{cl}(B)) = \mathrm{int}^E(\mathrm{cl}^E(B))$. So $B \in \mathrm{Ro}(X)$. ∎

Let $\mathcal{U} \subseteq \mathrm{Ro}(X)$. We use $\sum \mathcal{U}$ to denote the supremum of $\mathcal{U}$ in the complete Boolean algebra $\mathrm{Ro}(X)$. It is easy to check that $\sum \mathcal{U} = \mathrm{int}(\mathrm{cl}(\bigcup \mathcal{U}))$.

Definition 2.17. (a) Let $U \subseteq V$ and $\mathcal{U} \subseteq \mathrm{Ro}(X)$. $\mathcal{U}$ is called a *V-small semicover* of U if $\sum \mathcal{U} = U$ and $U' \prec V$ for every $U' \in \mathcal{U}$.

(b) Let $\mathcal{U}$ be a V-small semicover of U, and let $\{U_i \mid i \in I\}$ be a 1-1 enumeration of $\mathcal{U}$. We say that $\mathcal{U}$ is a *V-good semicover* of U if the following holds. For every $J \subseteq I$ and $\{W_j \mid j \in J\} \subseteq \mathrm{Ro}(X)$: if J is infinite and $\emptyset \neq W_j \subseteq U_j$ for every $j \in J$, then there are pairwise disjoint infinite $J_1, J_2 \subseteq J$ and $\{W'_j \mid j \in J_1 \cup J_2\} \subseteq \mathrm{Ro}(X)$ such that $\emptyset \neq W'_j \subseteq W_j$ for every $j \in J_1 \cup J_2$ and $\sum_{j \in J_1} W'_j \dashv\vdash \sum_{j \in J_2} W'_j$.

(c) For a normed vector space E let $\bar{E}$ denote the completion of E. So $\bar{E}$ is a Banach space.

(d) Let Z be a topological space. Suppose that $F \subseteq H(Z)$ and $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \emptyset$ for distinct $f, g \in F$. We define

$$\circ F := \bigcup \{f \restriction \mathrm{supp}(f) \mid f \in F\} \cup \mathrm{Id} \restriction (Z - \bigcup \{\mathrm{supp}(f) \mid f \in F\}).$$

Let $F = \{f_n \mid n \in \mathbb{N}\} \subseteq H(Z)$ be such that for any distinct $m, n \in \mathbb{N}$, Then $\circ_{n \in \mathbb{N}} f_n := \circ F$. □

Lemma 2.18. *Let V be a small set. Then for every $U \in \mathrm{Ro}(X)$: $\mathrm{cl}(U) \subseteq V$ iff U has a V-good semicover.*

Proof. Suppose that $\mathrm{cl}(U) \not\subseteq V$. Let $\mathcal{U}$ be a V-small semicover of U; we show that $\mathcal{U}$ is not V-good. The fact that V is small is not used in the proof of this direction. Let $x \in \mathrm{cl}(U) - V$. If $U' \in \mathcal{U}$, then by 2.16(e), $\mathrm{cl}(U') \subseteq V$. By induction on $i \in \mathbb{N}$ we define $U_i \in \mathcal{U}$ and $W_i \subseteq U_i$. Let U_0 be any member of $\mathcal{U}$ and $W_0 = U_0$. Suppose $U_0, \ldots, U_{i-1}$ and $W_0, \ldots, W_{i-1}$ have been defined. Let B_i be a ball with center at x and radius $< 1/i$ such that $B_i \cap \bigcup_{j<i} U_j = \emptyset$. Let $U_i \in \mathcal{U}$ be such that $B_i \cap U_i \neq \emptyset$, and let $W_i = U_i \cap \mathrm{int}(\mathrm{cl}(B_i))$. So $W_i \in \mathrm{Ro}(X)$. For every infinite $J' \subseteq \mathbb{N}$ and $\{W'_j \mid j \in J'\} \subseteq \mathrm{Ro}(X)$: if $\emptyset \neq W'_j \subseteq W_j$ for every $j \in J'$, then $x \in \mathrm{cl}(\sum_{j \in J'} W'_j)$. Suppose by contradiction that $\mathcal{U}$ is V-good. The family $\{U_i \mid i \in \mathbb{N}\}$ is an infinite subset of $\mathcal{U}$, and $W_i \subseteq U_i$ for every $i \in \mathbb{N}$. So let J_1, J_2 and $\{W'_j \mid j \in J_1 \cup J_2\}$ be as required in the definition of V-goodness for $\{U_i \mid i \in \mathbb{N}\}$ and $\{W_i \mid i \in \mathbb{N}\}$, and let W strongly separate $\sum_{j \in J_1} W'_j$ from $\sum_{j \in J_2} W'_j$. Since $x \in \mathrm{cl}(\sum_{j \in J_2} W'_j)$ and $W \cap \sum_{j \in J_2} W'_j = \emptyset$, it follows that $x \notin W$. But $x \in \mathrm{cl}(\sum_{j \in J_1} W'_j)$. So by 2.16(e), $\sum_{j \in J_1} W'_j \not\prec W$. A contradiction.

Assume next that V is small and that $\mathrm{cl}(U) \subseteq V$; we will construct a V-good semicover $\mathcal{U}$ of U. Since V is small, there is $g \in G$ such that $g(V)$ is contained in a manageable ball. Obviously $\mathrm{cl}(g(U)) \subseteq g(V)$. Clearly, $g(U)$ has a $g(V)$-good semicover iff U has a

V-good semicover. In fact, this follows from Proposition 2.13(a). We may thus assume that V is contained in a manageable ball. This means that $\mathrm{cl}(U) = \mathrm{cl}^E(U)$.

We may further assume that there is a manageable ball $B^* = B^E(x^*, r^*)$ such that $V \subseteq B^E(x^*, r^*/16)$. Suppose that B^* is based on S^*, and denote F_{S^*} by F^*. We may assume that $x^* \in F^*$. For every $x \in \mathrm{cl}(U)$ let $W_x \in \mathrm{Ro}(X)$ be such that $x \in W_x \prec V$. The existence of W_x follows from Proposition 2.16(c), (a) and (f). Since $\mathrm{cl}(U)$ is paracompact, there is an open locally finite refinement $\mathcal{T}$ of $\{W_x \mid x \in \mathrm{cl}(U)\}$ such that $\mathrm{cl}(U) \subseteq \bigcup \mathcal{T}$. Let $\mathcal{U} = \{\mathrm{int}(\mathrm{cl}(T)) \cap U \mid T \in \mathcal{T}\}$. By Proposition 2.13(b)(ii), $\mathcal{U} \subseteq \mathrm{Ro}(X)$. Clearly, $\bigcup \mathcal{U} = U$. So $\sum \mathcal{U} = U$.

We show that for every $x \in \mathrm{cl}(U)$ there is a neighborhood S_x such that $\{U' \in \mathcal{U} \mid U' \cap S_x \neq \emptyset\}$ is finite. For $x \in \mathrm{cl}(U)$ let S_x be an open neighborhood of x such that $\{T \in \mathcal{T} \mid T \cap S_x \neq \emptyset\}$ is finite. By Proposition 2.13(b)(iv), $\{T \in \mathcal{T} \mid \mathrm{int}(\mathrm{cl}(T)) \cap S_x \neq \emptyset\}$ is finite. So $\{T \in \mathcal{T} \mid (\mathrm{int}(\mathrm{cl}(T)) \cap U) \cap S_x \neq \emptyset\}$ is finite. That is, $\{U' \in \mathcal{U} \mid U' \cap S_x \neq \emptyset\}$ is finite.

We show that $\mathcal{U}$ is V-good. Let $\{U_i \mid i \in \mathbb{N}\} \subseteq \mathcal{U}$ be such that $U_i \neq U_j$ for every $i \neq j$; and let $\emptyset \neq W_i \subseteq U_i$. We shall find J_1, J_2 and $\{W'_j \mid j \in J_1 \cup J_2\}$ as required in the definition of V-goodness. For every $i \in \mathbb{N}$ let $x_i \in W_i \cap F^*$.

Claim 1. $\{x_i \mid i \in \mathbb{N}\}$ *does not contain a convergent subsequence.*

Proof. If x is a limit of such a subsequence, then $x \in \mathrm{cl}(U)$, but then S_x intersects only finitely many U_i's. So $\{i \in \mathbb{N} \mid x_i \in S_x\}$ is finite. A contradiction, so the claim is proved.

By Claim 1 and Proposition 2.15(b), either (i) or (ii) below happen:

(i) E is incomplete, there is an infinite $J \subseteq \mathbb{N}$ such that $\{x_i \mid i \in J\}$ is a Cauchy sequence, and $\{x_i \mid i \in J\}$ is not convergent in $\mathrm{cl}^E(X)$.

(ii) There is infinite $J \subseteq \mathbb{N}$ and an $r > 0$ such that for any distinct $i, j \in J$, $r < \|x_i - x_j\| < 9r/8$.

Case (i). Let $\bar{x} = \lim^{\bar{E}}_{i \in J} x_i$. Hence $\bar{x} \in \mathrm{cl}^{\bar{E}}(V) - X$. Since $V \subseteq B^E(x^*, r^*/16)$, there is $r > 0$ such that $B^{\bar{E}}(\bar{x}, r) \cap E \subseteq B^E(x^*, r^*/8)$. So $\bar{x} \notin E$. We may assume that $x_i \in B^{\bar{E}}(\bar{x}, r/8)$ for every $i \in J$. Let $v \in F^*$ and $\|v\| = r/2$. Let $L_i = [x_i, x_i + v]$ and $L = [\bar{x}, \bar{x} + v]$. So $L_i \subseteq B^{F^*}(x^*, r^*/8)$ for every $i \in J$. Also, $L \subseteq \bar{E} - E$. One can choose an infinite subset $J_0 \subseteq J$ and a sequence $\{r_i \mid i \in J_0\} \subseteq (0, r/8)$ such that $B^E(x_i, r_i) \subseteq W_i$ for every $i \in J_0$, and $\mathrm{cl}^E(B(L_i, r_i)) \cap \mathrm{cl}^E(B(L_j, r_j)) = \emptyset$ for distinct $i, j \in J_0$.

For every $i \in J_0$ let $W'_i = B(x_i, r_i/3)$. Let $J_1 \subseteq J_0$ be such that J_1 and $J_0 - J_1$ are infinite, and let $J_2 = J_0 - J_1$. For $\ell = 1, 2$ let $W^\ell = \sum_{i \in J_\ell} W'_i$. We shall show that $W^1 \dashv\vdash W^2$.

For every $i \in J_1$, $\|x_i - \bar{x}\| < r/8$ and $r_i < r/8$, and for every $u \in L_i$, we have $\|u - x_i\| \leq \|(x_i + v) - x_i\| = r/2$. It follows that for every $u \in B(L_i, r_i)$, $\|u - \bar{x}\| < r/8 + r/2 + r/8 = 3r/4$. So $B(L_i, r_i) \subseteq B(\bar{x}, r) \subseteq B(x^*, r^*) \subseteq S^*$.

By Lemma 2.14(c), for every $i \in J_1$ there is $h_i \in \mathrm{LIP}(X; \mathcal{S}, \mathcal{F}) \lfloor B(L_i, r_i) \rfloor$ such that $h_i(B(x_i, r_i/3)) = B(x_i + v, r_i/3)$. Let $h = \circ_{i \in J_1} h_i$. We show that $h \in \mathrm{LIP}^{\mathrm{LC}}(X; \mathcal{S}, \mathcal{F})$. Clearly, $\mathrm{supp}(h) = \bigcup_{i \in J_1} \mathrm{supp}(h_i) \subseteq S^*$. We show that for every $u \in E$, there is a neighborhood V_u of u such that $|\{i \in J_1 \mid B(L_i, r_i) \cap V_u \neq \emptyset\}| \leq 1$. Suppose that u is a counter-example. Since $\{x_i \mid i \in \mathbb{N}\}$ is a Cauchy sequence and the $B(x_i, r_i)$'s are pairwise disjoint, $\lim_i r_i = 0$. Since for $i \neq j$, $\mathrm{cl}^E(B(L_i, r_i)) \cap \mathrm{cl}^E(B(L_j, r_j)) = \emptyset$, there is at most one i such that $u \in \mathrm{cl}^E(B(L_i, r_i))$. Hence there is an infinite set $J_3 \subseteq J_1$ and a sequence $\{u_i \mid i \in J_3\}$ such that $u_i \in B(L_i, r_i)$ for every $i \in J_3$, and $\lim_{i \in J_3} u_i = u$. There are $y_i \in L_i$ such that $\|y_i - u_i\| < r_i$. Hence $\lim_{i \in J_3} y_i = u$. Let $y_i = x_i + t_i v$. Since $\{x_i\}$ and $\{y_i\}$ converge in $\bar{E}$, $\lim_{i \in J_3} t_i$ exists. Also, $\lim_{i \in J_3} t_i \in [0, 1]$. So $u \in [\bar{x}, \bar{x} + v]$. Hence $u \notin E$, a contradiction.

Let $u \in X$. Then there is $i \in J_1$ such that $h \restriction V_u = h_i \restriction V_u$. So $h \restriction V_u$ is bilipschitz. This means that $h \in \mathrm{LIP}^{\mathrm{LC}}(X; \mathcal{S}, \mathcal{F})$. Since E is incomplete, $\mathrm{LIP}^{\mathrm{LC}}(X; \mathcal{S}, \mathcal{F}) \subseteq G$. So $h \in G$.

We shall prove that $h(W^1) \dashv\vdash W^2$. Let us first see that $h(W^1) \subseteq B^{\bar{E}}(\bar{x} + v, r/6)$. We have $h(W^1) = \bigcup_{i \in J_1} h_i(W_i') = \bigcup_{i \in J_1} h_i(B(x_i, r_i/3)) = \bigcup_{i \in J_1} B(x_i + v, r_i/3)$. Also, $\|(x_i + v) - (\bar{x} + v)\| = \|x_i - \bar{x}\| < r/8$. Since $\|x_i - \bar{x}\| < r/8$ and $\bar{x} \notin B^{\bar{E}}(x_i, r_i)$, it follows that $r_i < r/8$. So $B(x_i + v, r_i/3) \subseteq B^{\bar{E}}(\bar{x} + v, r/6)$. That is, $h(W_i') \subseteq B^{\bar{E}}(\bar{x} + v, r/6)$. Hence $h(W^1) \subseteq B^{\bar{E}}(\bar{x} + v, r/6)$.

Similarly, $W^2 \subseteq B^{\bar{E}}(\bar{x}, r/6)$. Since $\|(\bar{x} + v) - \bar{x}\| = r/2 > r/3$, there are $\hat{x} \in E$ and $0 < s_0 < s_1$ such that $B^{\bar{E}}(\bar{x} + v, r/6) \subseteq B^{\bar{E}}(\hat{x}, s_0)$ and $B^{\bar{E}}(\hat{x}, s_1) \cap B^{\bar{E}}(\bar{x}, r/6) = \emptyset$. So $h(W^1) \subseteq B^E(\hat{x}, s_0)$. By Propositions 2.16(c) and 2.16(a), $h(W^1) \prec B^E(\hat{x}, s_1)$. Since $B^E(\hat{x}, s_1) \cap W^2 = \emptyset$, it follows that $h(W^1) \dashv\vdash W^2$.

Note that $h(W^2) = W^2$. By Proposition 2.13(a), $h^{-1}(h(W^1)) \dashv\vdash h^{-1}(W^2)$. But $h^{-1}(h(W^1)) = W^1$ and $W^2 = h^{-1}(h(W^2))$. So $W^1 \dashv\vdash W^2$.

Case (ii). Since the x_i's belong to $B^E(x^*, r^*/16)$ and $r < \|x_i - x_j\|$, it follows that $r < r^*/8$. Let $i_0 \in J$ and J_1 and J_2 be disjoint infinite subsets of J not containing i_0. For every $i \in J_1 \cup J_2$ let $B_i = B^E(x_i, r/8)$ and $W_i' = B_i \cap W_i$. Clearly, $B_i \subseteq B^E(x^*, 3r^*/16)$. So $B_i \subseteq X$, and hence $W_i' \in \mathrm{Ro}(X)$. For $\ell = 1, 2$ let $W^\ell = \sum_{i \in J_\ell} W_i'$, and let $W = B(x_{i_0}, 2r)$.

We shall show that:

$(*)$ There is $h \in \mathrm{LIP}(E; B^E(x_{i_0}, 3r), F^*)$ such that $h \restriction W^1 = \mathrm{Id}$ and $h(W^2) \cap W = \emptyset$.

But first we prove that $(*)$ implies that $W^1 \dashv\vdash W^2$. If $x \in B^E(x_{i_0}, 3r)$, then $\|x - x^*\| \leq \|x - x^*\| + 3r < r^*/16 + 3r^*/8 = 7r^*/16$. So $B^E(x_{i_0}, 3r) \subseteq B^* \subseteq S^*$. Hence $h \restriction X \in \mathrm{LIP}(X; S^*, F^*) \subseteq \mathrm{LIP}(X; \mathcal{S}, \mathcal{F})$. Now, $W^1 \subseteq B(0, 5r/4)$, so by 2.16(c) and 2.16(a), $W^1 \prec W$. Also $h(W^2) \cap W = \emptyset$. Hence W strongly separates W^1 from $h(W^2)$. That is, $W^1 \dashv\vdash h(W^2)$. By Proposition 2.13(a), $h^{-1}(W^1) \dashv\vdash h^{-1}(h(W^2))$. But $h^{-1}(W^1) = W^1$ and $W^2 = h^{-1}(h(W^2))$. So $W^1 \dashv\vdash W^2$.

To complete the proof, it remains to show that $(*)$ holds. For simplicity let us assume that $x_{i_0} = 0$ and that $r = 1$. We define a function $g : [0, 3] \times [0, \infty) \to \mathbb{R}$ as follows. For every $s_0 \in [0, 3]$, $g(s_0, t)$ will be a piecewise linear homeomorphism of $[0, \infty]$. Let $a(s)$ be the linear function such that $a(3/8) = 3/4$ and $a(5/8) = 2$.

If $s_0 \leq 3/8$, then $g(s_0, t) = t$. If $3/8 \leq s_0 \leq 5/8$, then

$$g(s_0, t) = \begin{cases} t, & t \leq \frac{1}{2}, \\ \frac{a(s_0) - \frac{1}{2}}{\frac{3}{4} - \frac{1}{2}}(t - \frac{1}{2}) + \frac{1}{2}, & \frac{1}{2} \leq t \leq \frac{3}{4}, \end{cases}$$

$$g(s_0,t) = \begin{cases} \frac{3-a(s_0)}{3-\frac{3}{4}}(t-3)+3, & \frac{3}{4} \le t \le 3, \\ t, & 3 \le t. \end{cases}$$

If $5/8 \le s_0 \le 3$, then $g(s_0,t) = g(5/8,t)$.

Let $F = \{x_i \mid i \in J_1\}$ and

$$h(x) = \begin{cases} g(d(\frac{x}{\|x\|},F),\|x\|) \cdot \frac{x}{\|x\|}, & x \ne 0, \\ 0, & x = 0. \end{cases}$$

We leave it to the reader to check that $h \in \mathrm{LIP}(E; B^E(0,3), F_{S^*})$.

If $i \in J_1 \cup J_2$ and $x \in B_i$, then $\|\frac{x}{\|x\|} - x_i\| \le \|\frac{x}{\|x\|} - x\| + \|x - x_i\| < 1/4 + 1/8 = 3/8$. Let $x \in W^1$. There is $i \in J_1$ such that $x \in B_i$. Hence $d(\frac{x}{\|x\|},F) \le \|\frac{x}{\|x\|} - x_i\| < 3/8$. So $g(d(\frac{x}{\|x\|},F),\|x\|) = \|x\|$, and hence $h(x) = x$. Let $x \in W^2$. There is $i \in J_2$ such that $x \in B_i$. So $d(\frac{x}{\|x\|},F) \ge d(x_i,F) - \|\frac{x}{\|x\|} - x_i\| > 1 - 3/8 = 5/8$. Also, $\|x\| > 7/8$. Hence

$$\begin{aligned} \|h(x)\| &= \left\| g\left(d\left(\frac{x}{\|x\|},F\right),\|x\|\right) \cdot \frac{x}{\|x\|} \right\| \\ &= g\left(d\left(\frac{x}{\|x\|},F\right),\|x\|\right) = g(5/8,\|x\|) > g(5/8,3/4) = 2. \end{aligned}$$

We have proved $(*)$, so the proof of the lemma is complete. ∎

LEMMA 2.19. *Let V be a small set. Then for every U: $\mathrm{cl}(U) \cap \mathrm{cl}(V) \ne \emptyset$ iff for every small V_1: if $\mathrm{cl}(V) \subseteq V_1$, then $V_1 \cap U \ne \emptyset$.*

Proof. If $\mathrm{cl}(U) \cap \mathrm{cl}(V) \ne \emptyset$, then clearly $V_1 \cap U \ne \emptyset$ for every $V_1 \supseteq \mathrm{cl}(V)$. Conversely, suppose that V is small and $\mathrm{cl}(V) \cap \mathrm{cl}(U) = \emptyset$. Let V' be a small set such that $\mathrm{cl}(V) \subseteq V'$, and let $V_1 = V' \cap \mathrm{int}(X - U)$. Since $\mathrm{int}(X-U) \supseteq \mathrm{cl}(V)$, $V_1 \supseteq \mathrm{cl}(V)$, hence V_1 is as required. ∎

LEMMA 2.20. *Let U and V be small sets. Then $|\mathrm{cl}(U) \cap \mathrm{cl}(V)| = 1$ iff the following holds.*

(i) $\mathrm{cl}(U) \cap \mathrm{cl}(V) \ne \emptyset$,

(ii) *for every small W_1 and W_2: if $\mathrm{cl}(U \cap W_1) \cap \mathrm{cl}(V \cap W_1) \ne \emptyset$ and $\mathrm{cl}(U \cap W_2) \cap \mathrm{cl}(V \cap W_2) \ne \emptyset$, then $\mathrm{cl}(W_1) \cap \mathrm{cl}(W_2) \ne \emptyset$.*

Proof. Suppose that $x_1, x_2 \in \mathrm{cl}(U) \cap \mathrm{cl}(V)$ and $x_1 \ne x_2$. For $i = 1,2$ let $W_i \in \mathrm{Ro}(X)$ be a neighborhood of x_i such that W_i is small and $W_i \subseteq B^X(x_i, \frac{1}{3}\|x_2 - x_1\|)$. Then $\mathrm{cl}(U \cap W_i) \cap \mathrm{cl}(V \cap W_i) \ne \emptyset$ for $i = 1,2$, but $\mathrm{cl}(W_1) \cap \mathrm{cl}(W_2) = \emptyset$.

Suppose that $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \{x\}$ and let W_i, $i = 1,2$, be such that $\mathrm{cl}(U \cap W_i) \cap \mathrm{cl}(V \cap W_i) \ne \emptyset$. Hence $x \in \mathrm{cl}(W_1) \cap \mathrm{cl}(W_2)$. ∎

LEMMA 2.21. *For $i = 1,2$ let U_i, V_i be small sets such that $|\mathrm{cl}(U_i) \cap \mathrm{cl}(V_i)| = 1$. Then $\mathrm{cl}(U_1) \cap \mathrm{cl}(V_1) = \mathrm{cl}(U_2) \cap \mathrm{cl}(V_2)$ iff $(*)$ for any small W_1, W_2: if $\mathrm{cl}(U_i \cap W_i) \cap \mathrm{cl}(V_i \cap W_i) \ne \emptyset$, $i = 1,2$, then $\mathrm{cl}(W_1) \cap \mathrm{cl}(W_2) \ne \emptyset$.*

Proof. Similar to 2.20. ∎

LEMMA 2.22. *Let U, V be small sets such that $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \{x\}$ and $W \in \mathrm{Ro}(X)$. Then $x \in W$ iff $(*)$ for any small U', V': if $\mathrm{cl}(U') \cap \mathrm{cl}(V') = \mathrm{cl}(U) \cap \mathrm{cl}(V)$, then $U' \cap W \ne \emptyset$.*

Proof. It is trivial that if $x \in W$, then $(*)$ holds.

Suppose that $x \notin W$. Since W is regular open, $x \in \mathrm{cl}(X - \mathrm{cl}(W))$. Let B be a manageable ball containing x. So let $\{x_i \mid i \in \mathbb{N}\} \subseteq B$ be a 1-1 sequence converging to x and disjoint from $\mathrm{cl}(W)$. Let $r_i = \frac{1}{3}\min(1/i, d(x_i, \{x_j \mid j \neq i\} \cup W \cup (X - B)))$. Let $U' = \bigcup\{B^E(x_i, r_i) \mid i \text{ is odd}\}$ and $V' = \bigcup\{B^E(x_i, r_i) \mid i \text{ is even}\}$. Then U', $V' \subseteq B \subseteq X$. It is easy to see that U', $V' \in \mathrm{Ro}(X)$. Also, since U', $V' \subseteq B$, they are small. We have $\mathrm{cl}(U') \cap \mathrm{cl}(V') = \{x\} = \mathrm{cl}(U) \cap \mathrm{cl}(V)$, and $U' \cap W = \emptyset$. So $(*)$ does not hold. ■

LEMMA 2.23. *For every $x \in X$ there are small U, V such that* $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \{x\}$.

Proof. Use the construction of 2.22. ■

Proof of Theorem 2.8. Recall that 2.8(a) is a special case of 2.8(b). We prove (b). Let $\langle X_1, G_1\rangle, \langle X_2, G_2\rangle \in K_{\mathrm{BNO}}$ and $\varphi : G_1 \cong G_2$. It is trivial that $\langle \mathrm{Ro}(X_i), G_i\rangle$ are topological local movement systems. Indeed, this follows from Lemma 2.14(a). Hence by Theorem 2.3, there is $\eta : \mathrm{Ro}(X_1) \cong \mathrm{Ro}(X_2)$ such that $\varphi \cup \eta : \mathrm{MR}(X_1, G_1) \cong \mathrm{MR}(X_2, G_2)$. Let $\psi = \varphi \cup \eta$.

As in Remark 2.11 the property of $\mathcal{U}$ being a V-small semicover of U is expressed in terms of the operations of $\mathrm{MR}(X, G)$. That is, there is a formula $\varphi_{\text{sm-sc}}(\mathcal{X}, x, y)$ expressed in terms of the operations of $\mathrm{MR}(X, G)$ such that for every $\langle X, G\rangle \in K_{\mathrm{BNO}}$, $\mathcal{U} \subseteq \mathrm{Ro}(X)$ and $U, V \in \mathrm{Ro}(X)$, $\langle \mathcal{U}, U, V\rangle$ satisfies $\varphi_{\text{sm-sc}}(\mathcal{X}, x, y)$ in $\mathrm{MR}(X, G)$ iff $\mathcal{U}$ is a V-small semicover of U. Hence, if $\mathcal{U}$ is a V-small semicover of U in $\mathrm{MR}(X_1, G_1)$, then $\psi(\mathcal{U}) := \{\psi(U') \mid U' \in \mathcal{U}\}$ is a $\psi(V)$-small semicover of $\psi(U)$ in $\mathrm{MR}(X_2, G_2)$.

The same fact is true for the property of being a V-good semicover.

Lemmas 2.18–2.22, and the existence of the formulas χ_{Sml} etc. of Remark 2.11 imply that the following properties are expressible in terms of the operations of $\mathrm{MR}(X, G)$.

(1) U and V are small, and $\mathrm{cl}(U) \cap \mathrm{cl}(V)$ is a singleton.
(2) U_1, V_1, U_2, V_2 are small, $\mathrm{cl}(U_1) \cap \mathrm{cl}(V_1)$ is a singleton, and $\mathrm{cl}(U_1) \cap \mathrm{cl}(V_1) = \mathrm{cl}(U_2) \cap \mathrm{cl}(V_2)$.
(3) U and V are small, $\mathrm{cl}(U) \cap \mathrm{cl}(V)$ is a singleton, and $\mathrm{cl}(U) \cap \mathrm{cl}(V) \subseteq W$.

A word of caution. In (1)–(3) smallness cannot be omitted. This is so, since in Lemmas 2.18–2.22 the equivalence of (1)–(3) to the expressible properties mentioned there was proved only under the assumption that the sets in question are small.

We are ready to define $\tau : X_1 \to X_2$. Let $x \in X_1$. By Lemma 2.23, there are small U and V such that $\{x\} = \mathrm{cl}(U) \cap \mathrm{cl}(V)$. Since ψ is an isomorphism between $\mathrm{MR}(X_1, G_1)$ and $\mathrm{MR}(X_2, G_2)$, and by the expressibility of (1) above, $\mathrm{cl}(\psi(U)) \cap \mathrm{cl}(\psi(V))$ is a singleton. Denote it by $\{y\}$ and define $\tau(x) = y$.

By the expressibility of (2) above: if U', V' are small and $\{x\} = \mathrm{cl}(U') \cap \mathrm{cl}(V')$, then $\mathrm{cl}(\psi(U') \cap \mathrm{cl}(\psi(V')) = \{y\}$. So the definition of τ is valid. As in the proof of Theorem 2.5, Lemma 2.23 and the expressibility of (1) and (2) imply that τ is 1-1 and onto. As in the proof of Theorem 2.5, the expressibility of (3) implies that τ is a homeomorphism and that τ induces φ. This completes the proof of Theorem 2.8. ■

Consider the class

$$K_{\mathrm{NL}} = \{\langle X, G\rangle \mid X \text{ is an open subset of a normed space and } \mathrm{LIP}(X) \le G \le H(X)\}.$$

It is not known whether K_{NL} is faithful. But we can show the faithfulness of the subclass of K_{NL} consisting of those $\langle X, G\rangle$'s in which X is a first category topological space and G is internally extendible. (See below.) To this end we have strengthened the original statement of Theorem 2.8, and included G's which are required to contain $\mathrm{LIP}(X;F)$ rather than $\mathrm{LIP}(X)$. Since $\mathrm{LIP}(X;F) \subseteq \mathrm{LIP}(X)$, this is a stronger result.

DEFINITION 2.24. Suppose that E is a normed vector space, and that $X \subseteq E$ is open.

(a) The *complete interior* of X in E is defined by

$$\overline{\mathrm{int}}^E(X) = \bigcup\{B^{\bar{E}}(x,r) \mid x \in E \text{ and } B^E(x,r) \subseteq X\}.$$

Note that $\overline{\mathrm{int}}^E(X)$ is open in $\bar{E}$.

(b) Let $h \in H(X)$. We say that h is *internally extendible in* E if there is $\bar{h} \in H(\overline{\mathrm{int}}^E(X))$ such that $\bar{h}$ extends h. Let $\mathrm{IXT}^E(X)$ denote the group of internally extendible homeomorphisms of X.

(c) Let X be an open subset of a normed space E, and $\mathcal{U}$ be a set of open subsets of X. Then $\mathcal{U}$ is a *complete cover* of X if $\bigcup\{\overline{\mathrm{int}}(U) \mid U \in \mathcal{U}\} = \overline{\mathrm{int}}(X)$.

(d) For a subset A of a metric space denote the *diameter* of A by $\mathrm{diam}(A)$. That is, $\mathrm{diam}(A) = \sup_{x,y\in A} d(x,y)$. So $\mathrm{diam}(A) \in \mathbb{R} \cup \{\infty\}$. $\square$

The following proposition is known. See [BP], the chapter on incomplete norms. We present a proof here.

PROPOSITION 2.25. (a) *Let E be a normed space and $x, y \in B^{\bar{E}}(0,a) - E$. Then there is $h \in \mathrm{LIP}(\bar{E};E)\lfloor B^{\bar{E}}(0,a)\rfloor$ such that $h(x) = y$.*

(b) *Let E be a normed space, $x \in B^E(0,a)$ and $y \in B^{\bar{E}}(0,a) - E$. Then there is $h \in \mathrm{LIP}(\bar{E})\lfloor B^{\bar{E}}(0,a)\rfloor$ such that $h(E - \{x\}) = E$ and $h(x) = y$.*

Proof. (a) We leave the straightforward proof of the following claim to the reader.

CLAIM 1. *Let E be a normed space. Let $\{K_n \mid n \in \mathbb{N}\} \subseteq (1,\infty)$ be such that $\prod_{n\in\mathbb{N}} K_n < \infty$ and $\{g_n \mid n \in \mathbb{N}\} \subseteq \mathrm{LIP}(\bar{E};E)$ be such that*:

(1) *g_n is K_n-bilipschitz*;

(2) *$\sum_{n\in\mathbb{N}} \mathrm{diam}(\mathrm{supp}(g_n)) < \infty$*;

(3) *there is $x_0 \in \bar{E} - E$ and a sequence $\{r_n \mid n \in \mathbb{N}\} \subseteq (0,\infty)$ converging to 0 such that for every $n \in \mathbb{N}$, $\mathrm{supp}(g_n) \subseteq g_{n-1} \circ \cdots \circ g_0(B^{\bar{E}}(x_0, r_n))$.*

Let $h_n = g_{n-1} \circ \cdots \circ g_0$. Then for every $x \in \bar{E}$, $\lim_{n\to\infty} h_n(x)$ exists. Define $h(x) = \lim_{n\to\infty} h_n(x)$. Then $h \in \mathrm{LIP}(\bar{E};E)$.

We construct g_n's which satisfy the assumptions of Claim 1. Let $\{M_n \mid n \in \mathbb{N}\} \subseteq (3,\infty)$ be such that $\prod_{n\in\mathbb{N}}(1 + 1/(M_n - 3)) < \infty$. We may assume that $\|x - y\| \cdot M_0 < a$. Set $x = x_0$ and $\|x - y\| = d_0$. Define d_n by induction as follows: $d_{n+1} = d_n/M_{n+1}$.

We shall apply Proposition 2.14(b)(ii). The normed space E of 2.14 is taken to be $\bar{E}$, $\mathcal{S} = \{\bar{E}\}$, $F_{\bar{E}} = \bar{E}$ and a of 2.14(b) is a here. The homeomorphism h constructed in Proposition 2.14(b) depended on the vectors x_0 and v and on the radii r_0 and r. Denote that h by $h_{x_0,v,r_0,r}$.

We define g_n and x_{n+1} by induction. Suppose that x_n has been defined. Let

$$u_n = d_{n+1} \cdot \frac{y - x_n}{\|y - x_n\|} \quad \text{and} \quad f_n = h_{x_n, u_n, M_n d_n, 2d_n}.$$

So $\operatorname{supp}(f_n) \subseteq B(x_n, M_n d_n)$. Note that f_n is $(1 + \frac{d_n}{M_n d_n - 2d_n - d_{n+1}})$-bilipschitz. Since $d_{n+1} < d_n$, we have $\frac{d_n}{M_n d_n - 2d_n - d_{n+1}} > \frac{1}{M_n - 3}$. So

(1.1) $\|y - f_n(x_n)\| = d_{n+1} < 2d_{n+1}$,
(1.2) $f_n \restriction B(x_n, 2d_n) = \mathrm{tr}_{u_n} \restriction B(x_n, 2d_n)$,
(1.3) for some $\varepsilon > 0$, f_n is $(1 + \frac{1}{M_n - 3} + \varepsilon)$-bilipschitz,
(1.4) if $n > 0$, then for some $\varepsilon > 0$, $\operatorname{supp}(f_n) \subseteq B(x_n, d_{n-1} - \varepsilon)$.

Choose $y_n, v_n \in E$ close enough to x_n and u_n respectively so that for g_n defined by $g_n = h_{y_n, v_n, M_n d_n, 2d_n}$ the following holds:

(2.1) $\|y - g_n(x_n)\| < 2d_{n+1}$,
(2.2) $g_n \restriction B(x_n, d_n) = \mathrm{tr}_{v_n} \restriction B(x_n, d_n)$,
(2.3) g_n is $(1 + \frac{1}{M_n - 3})$-bilipschitz,
(2.4) if $n > 0$, then $\operatorname{supp}(g_n) \subseteq B(x_n, d_{n-1})$.

Let $x_{n+1} = g_n(x_n)$. So $x_{n+1} = x_n + v_n$. Also, $g_n \in \mathrm{LIP}(\bar{E}; E)$

We check that (1)–(3) of Claim 1 are fulfilled. Clearly, $K_n = 1 + \frac{1}{M_n - 3}$, $n \in \mathbb{N}$ fulfill clause (1). Since $d_{n+1} < d_n/3$, we have $\sum_{n \in \mathbb{N}} d_n < \infty$. So $\sum_{n \in \mathbb{N}} \operatorname{diam}(\operatorname{supp}(g_n)) < \sum_{n \in \mathbb{N}} 2d_n < \infty$, proving (2).

Let $h_n = g_n \circ \cdots \circ g_0$ and $w_n = \sum_{i \le n} v_i$. We show by induction that

(2.5) $h_n \restriction B(x_0, d_n) = \mathrm{tr}_{w_n} \restriction B(x_0, d_n)$ for every $n \in \mathbb{N}$.

By (2.2), this is true for $n = 0$. Assume it is true for n. Hence $x_{n+1} = h_n(x_0) = x_0 + w_n$. For $n + 1$ we have

$$\begin{aligned} h_{n+1} \restriction B(x_0, d_{n+1}) &= (g_{n+1} \circ h_n) \restriction B(x_0, d_{n+1}) = g_{n+1} \restriction h_n(B(x_0, d_{n+1})) \circ \mathrm{tr}_{w_n} \restriction B(x_0, d_{n+1}) \\ &= g_{n+1} \restriction B(x_0 + w_n, d_{n+1}) \circ \mathrm{tr}_{w_n} \restriction B(x_0, d_{n+1}) = g_{n+1} \restriction B(x_{n+1}, d_{n+1}) \circ \mathrm{tr}_{w_n} \restriction B(x_0, d_{n+1}) \\ &= \mathrm{tr}_{v_{n+1}} \restriction B(x_{n+1}, d_{n+1}) \circ \mathrm{tr}_{w_n} \restriction B(x_0, d_{n+1}) = \mathrm{tr}_{w_{n+1}} \restriction B(x_0, d_{n+1}). \end{aligned}$$

It follows from (2.4) and (2.5) that $\operatorname{supp}(g_{n+1}) \subseteq B(x_{n+1}, d_n) = h_n(B(x_0, d_n))$. Since $\lim_{n \to \infty} d_n = 0$, clause (3) of Claim 1 holds. Let h be as constructed in Claim 1. So $h \in \mathrm{LIP}(\bar{E}; E)$.

Since $\|y - x_n\| = d_n$ and $\lim_{n \to \infty} d_n = 0$, we have $h(x) = y$. We show that $\operatorname{supp}(g_n) \subseteq B(x, a)$ for every $n \in \mathbb{N}$. For $n = 0$, $\operatorname{supp}(g_0) \subseteq B(x, M_0 d_0) \subseteq B(x, a)$. Suppose that $n > 0$. Then $\operatorname{supp}(g_n) \subseteq B(x_n, M_n d_n) \subseteq B(x, M_n d_n + \|x_n - x\|)$. Since

$$M_n d_n + \|x_n - x\| \le M_n d_n + \|x_n - y\| + \|y - x\| < d_{n-1} + 2d_n + d_0 < 3d_0 < M_0 d_0 < a,$$

we have $\operatorname{supp}(g_n) \subseteq B(x, a)$. It follows that $\operatorname{supp}(h) \subseteq B(x, a)$. So h is as required.

(b) The proof is very similar to the proof of (a). ■

Corollary 2.26. *Let K_{NFCB} be the class of all space-group pairs $\langle X, G \rangle$ for which there is a normed space E such that X is an open subset of E and*

(1) *E is of the first category, or E is a Banach space*;

(2) *There is a complete cover* $\mathcal{U}$ *of* X *such that* $\mathrm{LIP}(X,\mathcal{U}) \le G \le \mathrm{IXT}(X)$.

Then K_{NFCB} *is faithful.*

Proof. Let $\langle X, G\rangle \in K_{\mathrm{NFCB}}$. For $g \in G$ let $\bar{g}$ be the extension of g to $\overline{\mathrm{int}}(X)$ and $\bar{G} = \{\bar{g} \mid g \in G\}$. Then $\langle \overline{\mathrm{int}}^E(X), \bar{G}\rangle \in K_{\mathrm{BO}}$.

Let $\bar{\mathcal{O}}(X,G)$ be the set of orbits of $\bar{G}$. That is, $\bar{\mathcal{O}}(X,G) = \{\bar{G}(x) \mid x \in \overline{\mathrm{int}}(X)\}$. It follows from Proposition 2.25(a) that if X is an open subset of an incomplete normed space, then for every $O \in \bar{\mathcal{O}}(X,G)$ there is a set $\mathcal{C}$ of connected components of $\overline{\mathrm{int}}(X)$ such that $O = E \cap \bigcup \mathcal{C}$ or $O = (\bar{E} - E) \cap \bigcup \mathcal{C}$. Clearly, if X is an open subset of a Banach space, then for every $O \in \bar{\mathcal{O}}(X,G)$ there is a set of connected components of X such that $O = \bigcup \mathcal{C}$. Let $\mathrm{FC}(X,G) = \bigcup\{O \in \bar{\mathcal{O}}(X,G) \mid O \text{ is a first category set}\}$. If X is of the first category, then $X = \mathrm{FC}(X,G)$.

For $i = 1,2$ let $\langle E_i, G_i\rangle \in K_{\mathrm{NFCB}}$, and let $\varphi : G_1 \cong G_2$. Let $\bar{\varphi} : \bar{G}_1 \to \bar{G}_2$ be defined by $\bar{\varphi}(\bar{g}) = \overline{\varphi(g)}$. Then $\bar{\varphi} : \bar{G}_1 \cong \bar{G}_2$. By Theorem 2.8(b), there is $\bar{\tau} : \bar{E}_1 \cong \bar{E}_2$ which induces $\bar{\varphi}$. Obviously, $\bar{\tau}$ takes orbits of $\bar{G}_1$ to orbits of $\bar{G}_2$. So $\bar{\mathcal{O}}(X, G_1)$ contains members of the first category iff $\bar{\mathcal{O}}(X, G_2)$ contains members of the first category.

It is obvious that $\bar{\tau}$ takes every first category orbit of $\bar{G}_1$ to a first category orbit of $\bar{G}_2$. So if X_1 is of the first category, then $\bar{\tau}(X_1) = \bar{\tau}(\mathrm{FC}(X_1, G_1)) = \mathrm{FC}(X_2, G_2) = X_2$, and hence $\tau : X_1 \cong X_2$. If X_1 is an open subset of a Banach space, then $\bar{\tau} = \tau$ and hence $\tau : X_1 \cong X_2$. ■

Remark 2.27. If E has a countable Hamel basis, then it is of the first category. The space ℓ_1 is a linear subspace of ℓ_2, and it is of the first category in ℓ_2.

This is a special case of the following fact. If $T : F \to E$ is a bounded linear operator from a Banach space F to a Banach space E, and $\mathrm{Rng}(T)$ is a proper dense subset of E, then $\mathrm{Rng}(T)$ is of the first category in E. This follows from the proof of the Open Mapping Theorem. If $\mathrm{Rng}(T)$ is of the second category, then for some ball $B = B^F(0,n)$, $T(B)$ is somewhere dense. Hence $T(B)$ is dense in some ball of the form $B^E(0,r)$. It can then be proved that $T(B) \supseteq B^E(0,r)$. This implies that $\mathrm{Rng}(T) = E$. □

In Corollary 2.26 the assumptions that E is of the first category, and that G is completely extendible are undesirable. We do not know whether they can be dispensed with.

The final reconstruction results of Chapter 5 are proved for open subsets of first category normed vector spaces and for open subsets of Banach spaces. The proofs of all intermediate theorems are valid for open subsets of any normed space. If Parts (c) or (d) of the following question have a negative answer, then the final results of Chapter 5 will be true for open subsets of any normed vector space.

On the other hand, examples answering (c) or (d) below in the affirmative imply that certain results in Chapter 5 are not true for arbitrary normed spaces.

Question 2.28. (a) Is K_{NL} faithful?

(b) Let K_{NLIX} be the subclass of K_{NL} consisting of all $\langle E, G\rangle$'s in which G is internally extendible. Is K_{NLIX} faithful?

(c) Are there normed spaces E and F and a homeomorphism $\tau : \bar{E} \cong \bar{F}$ such that $\tau(E) = \bar{F} - F$?

Note that the answer to (b) is positive iff the answer to (c) is negative.

(d) Are there normed spaces E and F and a uniformly bicontinuous homeomorphism $\tau : \bar{E} \cong \bar{F}$ such that $\tau(E) = \bar{F} - F$? $\square$

2.4. Faithfulness of normed manifolds. As has been mentioned, the proof of Theorem 2.8 extends without change to manifolds over normed vector spaces. This class contains some new instances. The unit sphere of a normed space is one, and spaces which are a finite product of manifolds are another.

We extend the results a bit further, in order to allow the inclusion of manifolds with boundary over a normed vector space. To this end we introduce the notion of a "regionally normed manifold". By combining Remark 2.31 with the various results on extendible homeomorphism groups appearing in Chapter 5, one obtains reconstruction theorems for manifolds with boundary.

It should be pointed out that no new arguments are needed for this new framework,

DEFINITION 2.29. (a) Let X be a topological space. A family of mappings Φ is called a *regional normed atlas* for X if the following holds.

(1) $\bigcup\{\text{Rng}(\varphi) \mid \varphi \in \Phi\}$ is a dense subset of X.

(2) For every $\varphi \in \Phi$ there is a normed space $E = E_\varphi$, $x = x_\varphi \in E$ and $r = r_\varphi > 0$ such that:

(i) $\varphi : \bar{B}^E(x, r) \to X$,

(ii) φ is a homeomorphism between $\text{Dom}(\varphi)$ and $\text{Rng}(\varphi)$,

(iii) $\text{Rng}(\varphi)$ is closed in X, and $\varphi(B^E(x, r))$ is open in X.

If Φ is a regional normed atlas for X, then $\langle X, \Phi \rangle$ is called a *regionally normed manifold* (*RNM*). If $X = \bigcup\{\varphi(B^{E_\varphi}(x_\varphi, r_\varphi)) \mid \varphi \in \Phi\}$, then $\langle X, \Phi \rangle$ is called a *normed manifold*. Let $\langle X, \Phi \rangle$ be an RNM. If for every $\varphi \in \Phi$, E_φ is a Banach space, then $\langle X, \Phi \rangle$ is said to be a *regional Banach manifold* (*RBM*). A normed manifold which is an RBM is called a *Banach manifold*.

(b) Recall that for a metric space (Y, d), $x \in Y$ and $r > 0$, $S^Y(x, r)$ denotes $\{y \in Y \mid d(x, y) = r\}$. For a normed space E, $x \in E$ and $r > 0$ let

$$L_1(E, x, r) := \{h \in H(\bar{B}^E(x, r)) \mid h \text{ is bilipschitz, and } h{\restriction}S(x, r) = \text{Id}\},$$
$$L_1^{\text{LC}}(E, x, r) := \{h \in H(\bar{B}^E(x, r)) \mid h \text{ is locally bilipschitz, and } h{\restriction}S(x, r) = \text{Id}\}.$$

Let F be a dense linear subspace of E. Define

$$L_1(E, x, r; F) := \{h \in L_1(E, x, r) \mid h(\bar{B}^E(x, r) \cap F) = \bar{B}^E(x, r) \cap F\},$$
$$L_1^{\text{LC}}(E, x, r; F) := \{h \in L_1^{\text{LC}}(E, x, r) \mid h(\bar{B}^E(x, r) \cap F) = \bar{B}^E(x, r) \cap F\}.$$

If $\langle X, \Phi \rangle$ is an RNM, $\varphi \in \Phi$ and $h \in L_1^{\text{LC}}(E_\varphi, x_\varphi, r_\varphi)$, then $h^{[\varphi]} := h^\varphi \cup \text{Id}{\restriction}(X - \text{Rng}(\varphi)) \in H(X)$. Suppose that $\mathcal{F} := \{F_\varphi \mid \varphi \in \Phi\}$ is a family of linear spaces such that for every $\varphi \in \Phi$, F_φ is a dense subspace of E_φ. Then $\mathcal{F}$ is called a *subspace choice for* $\langle X, \Phi \rangle$.

Let $\text{LIP}(X; \Phi, \mathcal{F})$ denote the subgroup of $H(X)$ generated by $\{h^{[\varphi]} \mid \varphi \in \Phi,\ h \in L_1(E_\varphi, x_\varphi, r_\varphi; F_\varphi)\}$. Let $\text{LIP}^{\text{LC}}(X; \Phi, \mathcal{F})$ denote the subgroup of $H(X)$ generated by $\{h^{[\varphi]} \mid \varphi \in \Phi,\ h \in L_1^{\text{LC}}(E_\varphi, x_\varphi, r_\varphi; F_\varphi)\}$. If $F_\varphi = E_\varphi$ for every $\varphi \in \Phi$, then $\text{LIP}(X; \Phi, \mathcal{F})$ and $\text{LIP}^{\text{LC}}(X; \Phi, \mathcal{F})$ are denoted by $\text{LIP}(X; \Phi)$ and $\text{LIP}^{\text{LC}}(X; \Phi)$ respectively.

Remark: Even though the groups considered below contain $\mathrm{LIP}(X; \varPhi, \mathcal{F})$, we do not have to require at this point that the transition maps in the atlas be Lipschitz. That is, we do not require that $\varphi^{-1} \circ \psi$ is bilipschitz for every $\varphi, \psi \in \varPhi$.

(c) Let K_{BM} be the class of all $\langle X, G \rangle$'s which satisfy the following: There are $\varPhi$ and $\mathcal{F}$ such that

(1) $\langle X, \varPhi \rangle$ is a Banach manifold and $\mathcal{F}$ is a subspace choice for $\varPhi$,
(2) $\mathrm{LIP}(X; \varPhi, \mathcal{F}) \leq G \leq H(X)$.

Let K_{NM} be the class of all $\langle X, G \rangle$'s which satisfy the following: There are $\varPhi$ and $\mathcal{F}$ such that

(1) $\langle X, \varPhi \rangle$ is a normed manifold and $\mathcal{F}$ is a subspace choice for $\varPhi$,
(2) $\mathrm{LIP}^{\mathrm{LC}}(X; \varPhi, \mathcal{F}) \leq G \leq H(X)$.

Let $K_{\mathrm{BNM}} = K_{\mathrm{BM}} \cup K_{\mathrm{NM}}$.

(d) Let $\langle X, \varPhi \rangle$ be an RNM. The set $\mathrm{NI}(X, \varPhi) := \bigcup\{\varphi(B^{E_\varphi}(x_\varphi, r_\varphi)) \mid \varphi \in \varPhi\}$ is called the *normed interior* of $\langle X, \varPhi \rangle$.

Let $G \leq H(X)$. The *extended normed interior* of $\langle X, \varPhi, G \rangle$ is defined as

$$\mathrm{ENI}(X, \varPhi, G) := \{g(x) \mid x \in \mathrm{NI}(X, \varPhi) \text{ and } g \in G\}.$$

Also, $\mathrm{ENI}(X, \varPhi, H(X))$ is denoted by $\mathrm{ENI}(X, \varPhi)$.

If X is a subset of a normed space E and $\mathrm{int}^E(X)$ is dense in X, then X is a regional normed manifold. As a regional normed atlas for X we take the set $\varPhi = \{\mathrm{Id} \restriction \bar{B}^E(x, r) \mid \bar{B}^E(x, r) \subseteq X\}$. We denote $\mathrm{ENI}(X, \varPhi)$ by $\mathrm{ENI}(X)$. Hence we have $\mathrm{ENI}(X) = \{h(x) \mid x \in \mathrm{int}^E(X),\ h \in H(X)\}$. $\square$

THEOREM 2.30. (a) *K_{BNM} is faithful.*

(b) *For $i = 0, 1$ let $\langle X_i, \varPhi_i \rangle$ be an RNM and $\mathcal{F}_i$ be a subspace choice for $\langle X_i, \varPhi_i \rangle$. Let $G_i \leq H(X_i)$. Suppose that for $i = 0, 1$:*

(1) *if $\langle X_i, \varPhi_i \rangle$ is an RBM, then $\mathrm{LIP}(X_i, \varPhi_i; \mathcal{F}_i) \leq G_i$,*
(2) *if $\langle X_i, \varPhi_i \rangle$ is not an RBM, then $\mathrm{LIP}^{\mathrm{LC}}(X_i, \varPhi_i; \mathcal{F}_i) \leq G_i$.*

Let $\varphi : G_1 \cong G_2$. Then there is $\tau : \mathrm{ENI}(X_1, \varPhi_1, G_1) \cong \mathrm{ENI}(X_2, \varPhi_2, G_2)$ such that τ induces φ. That is, $\varphi(g) \restriction \mathrm{ENI}(X_2, \varPhi_2, G_2) = (g \restriction \mathrm{ENI}(X_1, \varPhi_1, G_1))^\tau$ for every $g \in G_1$.

(c) *Let X be a subset of a normed space E and Y be a subset of a normed space F such that $\mathrm{int}^E(X)$ is dense in X and $\mathrm{int}^F(Y)$ is dense in Y. Suppose that $\varphi : H(X) \cong H(Y)$. Then there is $\tau : \mathrm{ENI}(X) \cong \mathrm{ENI}(Y)$ such that τ induces φ. That is, for every $g \in H(X)$, $\varphi(g) \restriction \mathrm{ENI}(Y) = (g \restriction \mathrm{ENI}(X))^\tau$.*

Proof. (a) If $\langle X, G \rangle \in K_{\mathrm{BNM}}$ and $\varPhi$ is a normed regional atlas for X which demonstrates that X is a normed manifold, then $\mathrm{NI}(X, \varPhi) = X$. So $\mathrm{ENI}(X, \varPhi, G) = X$. Hence (b) implies (a).

(b) The proof of Theorem 2.8 applies without change.

(c) This is a special case of (b). ∎

REMARK 2.31. The proof of the above theorem applies to RNM's too. The statement that is proved for RNM's is as follows. If $\varphi : G_1 \cong G_2$, then there is τ :

$\mathrm{ENI}(X_1, \Phi_1, G_1) \cong \mathrm{ENI}(X_2, \Phi_2, G_2)$ such that τ induces φ. That is, for every $g \in G_1$, $\varphi(g) \restriction \mathrm{ENI}(X_2, \Phi_2, G_2) = (g \restriction \mathrm{ENI}(X_1, \Phi_1, G_1))^{\tau}$. □

Manifolds with boundary, closures of open subsets of a normed space and closures of open subsets of a normed manifold are obviously RNM's. Note that in the above theorem, the groups G_i are not assumed to preserve the boundary of X_i. Indeed, when the X_i's are infinite-dimensional, it may happen that their boundary is not preserved.

2.5. The faithfulness of some smaller subgroups. The homeomorphisms constructed in Lemma 2.14(b) suggest some new types of subgroups of $H(X)$ which may be interesting in the context of reconstruction and in other contexts involving homeomorphisms of infinite-dimensional spaces.

DEFINITION 2.32. Let X be an open subset of a normed vector space E and $g \in H(X)$.

(a) We call g a *"finite-dimensional difference" homeomorphism* if there is a finite-dimensional subspace F of E such that $g(x) - x \in F$ for every $x \in X$.

Let $\mathrm{FD}(X)$ denote the set of "finite-dimensional difference" homeomorphisms of X and $\mathrm{FD.LIP}(X) := \mathrm{FD}(X) \cap \mathrm{LIP}(X)$.

(b) We call g a *weakly "finite-dimensional difference" homeomorphism*, if there is a finite-dimensional subspace F of E such that for every $x \in X$ there is $a \in \mathbb{R} - \{0\}$ such that $g(x) - ax \in F$.

Let $\mathrm{WFD}(X)$ denote the set of weakly "finite-dimensional difference" homeomorphisms of X and $\mathrm{WFD.LIP}(X) := \mathrm{WFD}(X) \cap \mathrm{LIP}(X)$. For a subspace choice system $\langle E, X, \mathcal{S}, \mathcal{F} \rangle$ define $\mathrm{WFD.LIP}(X; \mathcal{S}, \mathcal{F})$ and $\mathrm{WFD.LIP}^{\mathrm{LC}}(X; \mathcal{S}, \mathcal{F})$ in analogy to the definition of $\mathrm{LIP}(X; \mathcal{S}, \mathcal{F})$. See Definition 2.7(a). Also, define $K_{\mathrm{WFD.BNO}}$ in analogy to the definition of K_{BNO}. □ □

It is easy to check that $\mathrm{FD}(X)$ and $\mathrm{WFD}(X)$ are groups. The following is a corollary of the proof of Theorem 2.8.

COROLLARY 2.33. *$K_{\mathrm{WFD.BNO}}$ is faithful.*

Proof. The proof of Theorem 2.8 applies, since it uses only homeomorphisms belonging to $\mathrm{WFD}(X)$. ■

By Lemma 2.14(b), $\mathrm{FD.LIP}(X)$ is locally moving. In fact, the construction of 2.14(b) can be used to show that $\mathrm{FD.LIP}(X)$ is transitive in the following sense. There is an open base $\mathcal{B}$ of X such that for every $B \in \mathcal{B}$ and for every finite injective function ϱ whose domain and range are subsets of B there is $g \in G\lfloor \underline{B} \rfloor$ such that g extends ρ. In fact, $\mathcal{B}$ can be taken to be $\{B^E(x,r) \mid B^E(x,r) \subseteq X\}$.

QUESTION 2.34. Are any of the classes related to $\mathrm{FD}(X)$ faithful? For example, is the class $K_{\mathrm{BFD}} := \{\langle E, G \rangle \mid E$ is a Banach space, and $\mathrm{FD}(E) \leq G \leq H(E)\}$ faithful? □

3. The local Γ-continuity of a conjugating homeomorphism

3.1. General description. The Main Result of this section is the statement that if X_1, X_2 are open subsets of normed spaces E_1 and E_2 respectively, Γ_1 and Γ_2 are countably generated moduli of continuity, and $\tau : X_1 \cong X_2$ is such that $(H^{\mathrm{LC}}_{\Gamma_1}(X_1))^\tau = H^{\mathrm{LC}}_{\Gamma_2}(X_2)$, then $\Gamma_1 = \Gamma_2$ and τ is locally Γ_1-bicontinuous. This is proved in Theorem 3.19(a). Equally central are the four results stated in Corollary 3.43.

The conjunction of the final results of Chapters 2 and 3 is stated in Theorem 3.42. It says that the existence of an isomorphism φ between the groups $H^{\mathrm{LC}}_{\Gamma_1}(X_1)$ and $H^{\mathrm{LC}}_{\Gamma_2}(X_2)$ implies that $\Gamma_1 = \Gamma_2$, and that φ is induced by a locally Γ_1-bicontinuous homeomorphism τ between X_1 and X_2.

As in Chapter 2, the results quoted above are in fact special cases of a more general setting. The groups which are actually being considered are of the type $H^{\mathrm{LC}}_{\Gamma}(X; \mathcal{S}, \mathcal{F})$. See Definition 3.17.

There are two methods of proving the Main Result. The central intermediate lemma in Method I roughly says that if X_1, X_2 are normed vector spaces, $\tau : X_1 \cong X_2$, and for every translation tr_v of X_1, $(\mathrm{tr}_v)^\tau \in H^{\mathrm{LC}}_{\Gamma_2}(X_2)$, then τ^{-1} is locally Γ_2-continuous. This is in fact the hidden content of Theorem 3.15. A variant of this statement which works only for second category spaces, but yields a slightly stronger result is proved in Theorem 3.26.

The main lemma in Method II says roughly that if X_1, X_2 are normed vector spaces, $\tau : X_1 \cong X_2$, and for every bounded affine isomorphism T of X_1, $T^\tau \in H^{\mathrm{LC}}_{\Gamma_2}(X_2)$, then τ is locally Γ_2-bicontinuous.

Going back to the Main Result, we in fact prove a stronger statement. Suppose that $\langle E, X, \mathcal{S}, \mathcal{E} \rangle$ and $\langle F, Y, \mathcal{T}, \mathcal{F} \rangle$ are subspace choice systems, Γ, Δ are countably generated moduli of continuity, $\tau : X \cong Y$, and the following holds:

$$(H_\Gamma(X; \mathcal{S}, \mathcal{F}))^\tau \subseteq H^{\mathrm{LC}}_\Delta(Y) \quad \text{and} \quad (H_\Delta(Y; \mathcal{T}, \mathcal{F}))^{\tau^{-1}} \subseteq H^{\mathrm{LC}}_\Gamma(X).$$

Then $\Gamma = \Delta$ and τ is locally Γ-bicontinuous. This is proved in Theorem 3.19(b). See Definitions 2.7 and 3.17(a).

Part of this strengthening is needed in the proof that if $\tau : \mathrm{cl}(X) \cong \mathrm{cl}(Y)$ and $(H^{\mathrm{LC}}_\Gamma(\mathrm{cl}(X)))^\tau = H^{\mathrm{LC}}_\Delta(Y)$, then $\Gamma = \Delta$ and τ is locally Γ-bicontinuous.

There are two situations in which we use Method I and we cannot use Method II. The first one appears in Chapter 11, where the reconstruction of the closure of an open set is considered. Method I is used again in the proof that the derivative of a conjugating homeomorphism is Γ-continuous. Such results will appear in a subsequent work.

3.2. Partial actions and decayability. If X is a proper open subset of a normed space E, then X is not closed under the group of translations $\mathbb{T}(E)$ of E. So there is no natural action of $\mathbb{T}(E)$ on X. But for every $x \in X$ there are neighborhoods B_x and V_x of x in X and Id^E in $\mathbb{T}(E)$ respectively such that the action of every $\mathrm{tr}_v \in V_x$ on B_x is defined. Moreover, $H(X)$ contains a homeomorphism which coincides with tr_v on B_x and which is the identity outside some bigger neighborhood of x. Indeed, even $\mathrm{LIP}(X)$ contains such a homeomorphism. We shall use such homeomorphisms. To this end we introduce two notions: the notion of a partial action of a topological group on a topological space, and the notion of decayability of partial actions.

Definition 3.1. (a) Let X be a topological space and $x \in X$. Set $\mathrm{Nbr}^X(x) := \{U \mid x \in U \subseteq X \text{ and } U \text{ is open}\}$ and $\mathrm{MBC} = \{\alpha \in \mathrm{MC} \mid \mathrm{Id}_{[0,\infty)} \leq \alpha\}$. Let $\alpha \in \mathrm{MBC}$, X, Y be metric spaces and $\tau : X \cong Y$. We say that τ is *α-bicontinuous* if τ and τ^{-1} are α-continuous. Let $x \in X$. We say that τ is *α-continuous at x* if for some $U \in \mathrm{Nbr}(x)$, $\tau{\restriction}U$ is α-continuous. Also, τ is said to be *α-bicontinuous at x* if for some $U \in \mathrm{Nbr}(x)$, $\tau{\restriction}U$ is α-bicontinuous. Let $\Gamma \subseteq \mathrm{MC}$. We say that τ is *Γ-continuous (resp. Γ-bicontinuous) at x* if for some $\alpha \in \Gamma$, τ is α-continuous (resp. α-bicontinuous) at x.

If H is a group, then e_H denotes the unit of H.

(b) Let H be a topological group, X be a topological space and λ be a function such that $\mathrm{Dom}(\lambda) \subseteq H \times X$ and $\mathrm{Rng}(\lambda) \subseteq X$. We say that λ is a *partial action* of H on X if the following conditions hold.

(1) λ is continuous.
(2) $\mathrm{Dom}(\lambda)$ is open in $H \times X$.
(3) For $g \in H$ let g_λ be the function defined by $g_\lambda(x) = \lambda(g, x)$. Then g_λ is a homeomorphism between $\mathrm{Dom}(g_\lambda)$ and $\mathrm{Rng}(g_\lambda)$.
(4) $(e_H)_\lambda = \mathrm{Id}_{\mathrm{Dom}((e_H)_\lambda)}$.
(5) For every $g \in H$, $(g^{-1})_\lambda = (g_\lambda)^{-1}$.
(6) For every $g, h \in H$ and $x \in X$: if $g_\lambda(x)$ and $h_\lambda(g_\lambda(x))$ are defined, then $(hg)_\lambda(x)$ is defined and $(hg)_\lambda(x) = h_\lambda(g_\lambda(x))$.

Define $\mathrm{Fld}(\lambda) := \mathrm{Dom}((e_H)_\lambda)$. Note that by (5) and (6), $\mathrm{Dom}(g_\lambda) \subseteq \mathrm{Fld}(\lambda)$ for every $g \in H$.

(c) Let $\alpha \in \mathrm{MBC}$, $a \in (0,1)$, H be a topological group, λ be a partial action of H on a metric space X, $G \leq H(X)$ and $x \in \mathrm{Fld}(\lambda)$. Then λ is called an *(a, α, G)-decayable action at x* if there is $r_x > 0$ such that for every $r \in (0, r_x)$ there is $V = V_{x,r} \in \mathrm{Nbr}(e_H)$ such that:

(i) $V \times B(x, ar) \subseteq \mathrm{Dom}(\lambda)$;
(ii) for every $h \in V$ there is $g \in G$ such that: g is α-bicontinuous, $g{\restriction}B(x, ar) = h_\lambda{\restriction}B(x, ar)$ and $\mathrm{supp}(g) \subseteq B(x, r)$.

Let $A \subseteq \mathrm{Fld}(\lambda)$. We say that λ is an *(a, α, G)-decayable action in A* if it is (a, α, G)-decayable at every $x \in A$; λ is *(a, α, G)-decayable* if it is (a, α, G)-decayable in $\mathrm{Fld}(\lambda)$. Suppose that Γ is a modulus of continuity. Then λ is called *(a, Γ, G)-decayable* if λ is (a, α, G)-decayable for some $\alpha \in \Gamma$.

If in the above $a = 1/2$, then we omit its mention. So "λ is (α, G)-decayable at x" means "λ is $(1/2, \alpha, G)$-decayable at x" etc. If $a = 1/2$ and $G = H(X)$, then we omit the mention of a and G. So "λ is α-decayable at x" means "λ is $(1/2, \alpha, H(X))$-decayable at x", "λ is α-decayable in A" means "λ is $(1/2, \alpha, H(X))$-decayable in A" etc.

(d) Let λ be a partial action of a topological group H on a topological space X, $A \subseteq H$ and $x \in X$. We write $A_\lambda(x) = \{h_\lambda(x) \mid h \in A\}$. We say that x is a *λ-limit-point* if $x \in \operatorname{acc}(V_\lambda(x))$ for every $V \in \operatorname{Nbr}(e_H)$. $\square$

Note that if λ is (a, α, G) decayable partial action of H at x, then there are $V \in \operatorname{Nbr}(e^H)$ and $U \in \operatorname{Nbr}(x)$ such that $h_\lambda \restriction U$ is α-bicontinuous for every $h \in V$.

The partial actions appearing in this section are obtained by restricting a full group action on a space E to an open subset of E. This is described in (a) below.

Proposition 3.2. (a) *Suppose that λ is a partial action of a topological group H on a topological space E. Let $X \subseteq \operatorname{Fld}(\lambda)$ be open, and define $\lambda \upharpoonleft X$ by setting* $\operatorname{Dom}(\lambda \upharpoonleft X) = \{\langle h, x\rangle \mid h \in H$ *and* $x, h_\lambda(x) \in X\}$ *and* $(\lambda \upharpoonleft X)(h, x) = \lambda(h, x)$. *Then $\lambda \upharpoonleft X$ is a partial action of H on X.*

(b) *Let λ be a partial action of H on X, $G \leq H(X)$, $D \subseteq C \subseteq \operatorname{Fld}(\lambda)$, $a \in (0, 1)$, $\alpha \in \mathrm{MBC}$, $r_0 > 0$ and let $V_r \in \operatorname{Nbr}(e_H)$ for every $r \in (0, r_0)$. Assume that*: (i) *D is a dense subset of C,* (ii) *λ is (a, α, G)-decayable in D,* (iii) *$r_x \geq r_0$ for every $x \in D$,* (iv) *$V_{x,r} \supseteq V_r$ for every $x \in D$ and $r \in (0, r_0)$. Then λ is (a, α, G)-decayable in C, $r_x \geq r_0$ for every $x \in C$, and $V_{x,r} \supseteq V_r$ for every $x \in C$ and $r \in (0, r_0)$.*

Proof. The proof of both parts is trivial. ∎

Suppose that X is an open subset of a normed space E. We shall be interested in two partial actions on X: the partial action of the group $\mathbb{T}(E)$ of translations of E, and the partial action of the group $\mathbb{A}(E)$ of affine transformations of E. We need to know that these partial actions are decayable. In fact, we shall show that $\mathbb{A}(E)$ is (α, G)-decayable, where $\alpha(t) = 15t$, and G is any group containing $\operatorname{LIP}(X)$.

Obviously, the decayability of $\mathbb{A}(E)$ implies the decayability of both $\mathbb{T}(E)$ and the group of bounded linear automorphisms of E. Because we deal with groups containing $\operatorname{LIP}(X; F)$, we shall really need to show that $\{T \in \mathbb{A}(E) \mid T(F) = F\}$ is decayable with respect to any group G containing $\operatorname{LIP}(X; F)$.

Definition 3.3. (a) Let E be a normed space and $v \in E$. Define $\operatorname{tr}_v^E(x) := v + x$ and $\mathbb{T}(E) = \{\operatorname{tr}_v^E \mid v \in E\}$. Whenever E can be understood from the context, we abbreviate tr_v^E by tr_v. We define $d(\operatorname{tr}_u, \operatorname{tr}_v) = \|u - v\|$.

(b) Let E be a normed space and $x \in X$. Denote the group of bounded linear automorphisms of E by $\mathbb{L}(E)$ and set $\mathbb{L}(E, x) = (\mathbb{L}(E))^{\operatorname{tr}_x^E}$. For $S, T \in \mathbb{L}(E)$ define $d(S, T) = \|S - T\| + \|S^{-1} - T^{-1}\|$. Let $\mathbb{A}(E) := \{\operatorname{tr}_v^E \circ T \mid v \in E,\ T \in \mathbb{L}(E)\}$. That is, $\mathbb{A}(E)$ is the group of bounded affine transformations of E. Suppose that $A = \operatorname{tr}_v^E \circ T \in \mathbb{A}(E)$. Then v and T are uniquely determined by A. We set $v = v_A$ and $T = T_A$. We may thus define

$$d(A_1, A_2) = \|v_{A_1} - v_{A_2}\| + \|T_{A_1} - T_{A_2}\| + \|T_{A_1}^{-1} - T_{A_2}^{-1}\|.$$

Then d is a metric on $\mathbb{A}(E)$, $\langle \mathbb{A}(E), d\rangle$ is a topological group, and the action of $\mathbb{A}(E)$ on E is continuous. Note that $\mathbb{L}(E,x) \leq \mathbb{A}(E)$ and the function $T \mapsto T^{\mathrm{tr}_x}$, $T \in \mathbb{L}(E)$, is a topological isomorphism between $\mathbb{L}(E)$ and $\mathbb{L}(E,x)$.

Let $\lambda_{\mathbb{T}}^{E}$, $\lambda_{\mathbb{L}}^{E}$, $\lambda_{\mathbb{L}}^{E,x}$, $\lambda_{\mathbb{A}}^{E,x}$ denote respectively the natural actions of $\mathbb{T}(E)$, $\mathbb{L}(E)$, $\mathbb{L}(E,x)$ and $\mathbb{A}(E)$ on E.

(c) Suppose that E is a normed space, F is a linear subspace of E and $x \in F$. Define

$$\mathbb{T}(E;F) = \{\mathrm{tr}_v^E \mid v \in F\}, \quad \mathbb{L}(E;F) = \{T \in \mathbb{L}(E) \mid T(F) = F\},$$
$$\mathbb{A}(E;F) = \{A \in \mathbb{A}(E) \mid A(F) = F\}, \quad \mathbb{L}(E,x;F) = (\mathbb{L}(E;F))^{\mathrm{tr}_x^E}.$$

The groups $\mathbb{T}(E;F)$, $\mathbb{L}(E;F)$, $\mathbb{L}(E,x;F)$ and $\mathbb{A}(E;F)$ equipped with the metric they inherit from $\mathbb{T}(E)$, $\mathbb{L}(E)$, $\mathbb{L}(E,x)$ and $\mathbb{A}(E)$ respectively are metric topological groups.

If λ is a partial action of H on X and $H_1 \leq H$, let $\lambda \restriction H_1$ denote the restriction of λ to H_1. Let $\lambda_{\mathbb{T}}^{E;F} = \lambda_{\mathbb{T}}^{E} \restriction \mathbb{T}(E;F)$; $\lambda_{\mathbb{L}}^{E;F}$, $\lambda_{\mathbb{L}}^{E,x;F}$ and $\lambda_{\mathbb{A}}^{E;F}$ are defined in a similar way.

(d) Suppose that X is a topological space and F is a set. Define

$$H(X;F) := \{h \in H(X) \mid h(X \cap F) = X \cap F\}. \ \square$$

Proposition 3.4. *Let E be a normed space, $X \subseteq E$ be open, $\mathcal{S}$ be an open cover of X, $\mathcal{F}$ be a subspace choice for $\mathcal{S}$, $S \in \mathcal{S}$, $G = \mathrm{LIP}(X;S,F_S)$ and $\alpha(t) = 3t$. Then $\lambda_{\mathbb{T}}^{E;F_S} \restriction S$ is $(5/8,\alpha,G)$-decayable. In particular, $\lambda_{\mathbb{T}}^{E;F_S} \restriction S$ is (α,G)-decayable.*

Proof. We show that if $x \in S \cap F_S$, then $\lambda_{\mathbb{T}}^{E;F_S} \restriction S$ is $(5/8,\alpha,G)$-decayable at x, $r_x = d(x, E - S)$, and for every $r \in (0, r_x)$, $V_{x,r} = B^{\mathbb{T}(E;F_S)}(\mathrm{Id}_E, r/4)$.

Let $r < r_x$. Let $\mathrm{tr}_v^E \in V_{x,r}$. So $v \in F_S$ and $\|v\| < r/4$. We apply Lemma 2.14(b). Choose r_0 of 2.14(b) to be r, choose r and s of 2.14(b) to be $5r/8$ and v of 2.14(b) to be v. Let h be as ensured by 2.14(b). By 2.14(b)(ii), h is $(1 + \frac{\|v\|}{r-5r/8-\|v\|})$-bilipschitz. $(1 + \frac{\|v\|}{r-5r/8-\|v\|}) < 3$. Hence h is 3-bilipschitz. It follows from 2.14(b)(ii) that h is as required. By Proposition 3.2(b), $\lambda_{\mathbb{T}}^{E;F_S} \restriction S$ is (α,G)-decayable. ■

Proposition 3.5. *Let $\eta : [0,\infty) \to [0,1]$. Suppose that η is K-Lipschitz and that $\eta(t) = a$ for every $t \geq a$. Let E be a normed space. Define $g : E \to E$ by $g(x) = \eta(\|x\|) \cdot x$. Then g is $(1 + Ka)$-Lipschitz.*

Proof. Let $x, y \in E$. If $\|x\|, \|y\| \geq a$, then $g(x) = x$ and $g(y) = y$, and hence $\|g(x) - g(y)\| = \|x - y\|$. Assume that $\|x\| \leq a$ or $\|y\| \leq a$. Without loss of generality $\|y\| \leq a$. Hence

$$\begin{aligned}
\|g(x) - g(y)\| &= \|\eta(\|x\|) \cdot x - \eta(\|y\|) \cdot y\| \\
&\leq \|\eta(\|x\|) \cdot x - \eta(\|x\|) \cdot y\| + \|\eta(\|x\|) \cdot y - \eta(\|y\|) \cdot y\| \\
&= \eta(\|x\|) \cdot \|x - y\| + |\eta(\|x\|) - \eta(\|y\|)| \cdot \|y\| \\
&\leq \|x - y\| + K \cdot \|x - y\| \cdot \|y\| \leq (1 + Ka) \cdot \|x - y\|. \ ■
\end{aligned}$$

Proposition 3.6. *Let E be a normed space, $T \in \mathbb{L}(E)$, $\eta : [0,\infty) \to [0,1]$ and $a > 0$. Set $\mathrm{Id}_E = I$. Suppose that η is K-Lipschitz, $\eta(t) = t$ for every $t \geq a$ and $\|I - T\|(1 + Ka) < 1$. Define $h : E \to E$ by*

$$h(x) = (1 - \eta(\|x\|)) \cdot T(x) + \eta(\|x\|) \cdot x.$$

Then

(i) $h \in H(E)$, *h is* $(\|T\| + \|I - T\| \cdot (1 + Ka))$*-Lipschitz, and* h^{-1} *is* $\max(\frac{\|T^{-1}\|}{1-\|I-T\|\cdot(1+Ka)}, 1)$*-Lipschitz.*

(ii) *If F is a linear subspace of E, and* $T \in \mathbb{L}(E; F)$, *then* $h \in H(E; F)$.

Proof. (i) We prove that h is Lipschitz. Let $x, y \in E$. Then

$$\begin{aligned} h(x) - h(y) &= (1 - \eta(\|x\|)) \cdot T(x) + \eta(\|x\|) \cdot x - ((1 - \eta(\|y\|)) \cdot T(y) + \eta(\|y\|) \cdot y) \\ &= T(x - y) + (I - T)(\eta(\|x\|) \cdot x - \eta(\|y\|) \cdot y). \end{aligned}$$

By Proposition 3.5,

$$\|h(x)-h(y)\| \le \|T\|\cdot\|x-y\|+\|I-T\|\cdot(1+Ka)\cdot\|x-y\| \le (\|T\|+\|I-T\|\cdot(1+Ka))\cdot\|x-y\|.$$

Hence h is $(\|T\| + \|I - T\| \cdot (1 + Ka))$-Lipschitz.

We prove that h^{-1} is Lipschitz. Let $x, y \in E$. By the above,

$$\begin{aligned} T^{-1}(h(x) - h(y)) &= (x - y) + T^{-1}(I - T)(\eta(\|x\|) \cdot x - \eta(\|y\|) \cdot y) \\ &= (x - y) + (T - I)(\eta(\|x\|) \cdot x - \eta(\|y\|) \cdot y). \end{aligned}$$

So

$$\begin{aligned} \|T^{-1}\|\cdot\|h(x)-h(y)\| &\ge \|T^{-1}(h(x)-h(y))\| \ge \|x-y\|-\|(T-I)(\eta(\|x\|)\cdot x-\eta(\|y\|)\cdot y)\| \\ &\ge \|x - y\| - \|(T - I)\| \cdot (1 + Ka) \cdot \|x - y\| = (1 - \|T - I\| \cdot (1 + Ka)) \cdot \|x - y\|. \end{aligned}$$

That is, $\|x - y\| \le \frac{\|T^{-1}\|}{1-\|T-I\|\cdot(1+Ka)} \cdot \|h(x) - h(y)\|$.

(ii) Let $x \in F$. Set $T_x = (1-\eta(\|x\|)T+\eta(\|x\|)I$. Then $h(x) = T_x(x)$ and $T_x(F) = F$. ∎

Lemma 3.7. *Let E be a normed space,* $X \subseteq E$ *be open,* $\mathcal{S}$ *be an open cover of X,* $\mathcal{F}$ *be a subspace choice for* $\mathcal{S}$, $S \in \mathcal{S}$, $x \in S \cap F_S$, $G = \mathrm{LIP}(X; S, F_S)$ *and* $\alpha(t) = 5t$. *Then* $\lambda_{\mathbb{L}}^{E,x;F_S}\restriction S$ *is* (α, G)*-decayable at x,* $r_x = d(x, E - S)$, *and* $V_{x,r} = (B^{\mathbb{L}(E;F)}(\mathrm{Id}_E, 1/4)^{\mathrm{tr}_x^E}$ *for every* $r \in (0, r_x)$.

Proof. We may assume that $0^E \in S$ and $x = 0^E$. Set $I = \mathrm{Id}_E$. Let $r_0 = d(0^E, E - S)$ and $V = B^{\mathbb{L}(E;F_S)}(I, 1/4)$. Let $r < r_0$ and $T \in V$. We show that T is "decayable". Define $\eta(t) : [0, \infty) \to [0, 1]$ to be the following piecewise linear function. The breakpoints of η are $r/2$ and r; $\eta(t) = 0$ for every $t \in [0, r/2]$ and $\eta(t) = 1$ for every $t \ge r$. Clearly, η is $2/r$-Lipschitz.

Define $h : E \to E$ by $h(y) = (1 - \eta(\|y\|)) \cdot T(y) + \eta(\|y\|) \cdot y$. We check that Proposition 3.6 applies to h. Set $K = 2/r$. So η is K-Lipschitz. Since $\|I - T\| < 1/4$ and $Ka = \frac{2}{r} \cdot r = 2$, it follows that $\|I - T\| \cdot (1 + Ka) < \frac{1}{4} \cdot (1 + 2) = 3/4 < 1$. It thus follows from 3.6(i) that $h \in H(E)$ and h is $\|T\| + \|I - T\| \cdot (1 + Ka)$-Lipschitz. By the above, $\|T\|+\|I-T\|\cdot(1+Ka) < 5/4+3/4 = 2$. So h is 2-Lipschitz. Since $\|T^{-1}\| < 5/4$, it follows that $\frac{\|T^{-1}\|}{1-\|I-T\|\cdot(1+Ka)} < \frac{5/4}{1-3/4} = 5$. By 3.6(i), h^{-1} is 5-Lipschitz. So h is 5-bilipschitz.

Clearly, $\mathrm{supp}(h) \subseteq B(0^E, r) \subseteq X$. So $h\restriction X \in H(X)$. Also, $h\restriction B(0^E, r/2) = T\restriction B(0^E, r/2)$. By 3.6(ii), $h(E \cap F_S) = F_S$. Hence $h\restriction X$ is as required. ∎

Lemma 3.8. *Let E be a normed space, $X \subseteq E$ be open, $\mathcal{S}$ be an open cover of X, $\mathcal{F}$ be a subspace choice for $\mathcal{S}$, $\mathrm{LIP}(X;\mathcal{S},\mathcal{F}) \leq G \leq H(X)$ and $\alpha(t) = 15t$. Let $S \in \mathcal{S}$. Then $\lambda_{\mathbb{A}}^{E;F_S} \upharpoonright S$ is (α, G)-decayable.*

Proof. Set $I = \mathrm{Id}_E$. Let $x \in S \cap F_S$, $r_x = d(x, E - S)$ and $r \in (0, r_x)$. If $x \neq 0^E$ let $a_r = \min(1/4, r/8, \frac{r}{8\|x\|})$ and if $x = 0^E$ let $a_r = \min(1/4, r/8)$. Let $V_{x,r} = B^{\mathbb{A}(E;F)}(I, a_r)$. We show that

$$(*) \qquad V_{x,r} \subseteq B^{\mathbb{T}(E;F_S)}(I, r/4) \circ (B^{\mathbb{L}(E;F_S)}(I, 1/4))^{\mathrm{tr}_x^E}.$$

If $A \in \mathbb{A}(E;F)$, then A can be uniquely represented in the form $A = \mathrm{tr}_{u_{A,x}} \circ (T_{A,x})^{\mathrm{tr}_x}$, where $T_{A,x} \in \mathbb{L}(E;F)$. Let $A = \mathrm{tr}_{v_A} \circ T_A$, where $T_A \in \mathbb{L}(E;F)$. Then $T_{A,x} = T_A$ and $u_{A,x} = v_A + (T_A - I)(x)$. Set $T = T_A$, $v = v_A$ and $u = u_{A,x}$. Suppose that $A \in V_{x,r}$. Then $d(T, I) < a_r < 1/4$. So $T \in B^{\mathbb{L}(E;F_S)}(I, 1/4)$. Hence $T^{\mathrm{tr}_x} \in (B^{\mathbb{L}(E;F_S)}(I, 1/4))^{\mathrm{tr}_x}$. Suppose that $x \neq 0$. Then $\|u\| \leq \|v\| + \|T - I\| \cdot \|x\| \leq r/8 + \frac{r}{8\|x\|} \cdot \|x\| = r/4$. If $x = 0$, then $u = v$. So $\|u\| < r/4$. In both cases $u \in B^{\mathbb{T}(E;F_S)}(I, r/4)$. This proves $(*)$.

Let $A \in V_{x,r}$. Let T and u be as above. By Lemma 3.7, there is $h_1 \in H(X;F_S)\lfloor B(x,r)\rfloor$ such that $h_1 \upharpoonright B(x, r/2) = T^{\mathrm{tr}_x} \upharpoonright B(x, r/2)$ and h_2 is 5-bilipschitz. By Proposition 3.4, there is $h_2 \in H(X;F_S)\lfloor B(x,r)\rfloor$ such that $h_2 \upharpoonright B(x, 5r/8) = \mathrm{tr}_u \upharpoonright B(x, 5r/8)$ and h_1 is 3-bilipschitz. Let $h = h_2 \circ h_1$. So $h \in H(X;F_S)$, $\mathrm{supp}(h) \subseteq B(x,r)$ and h is 15-bilipschitz. It remains to show that $h \upharpoonright B(x, r/2) = A \upharpoonright B(x, r/2)$. Let $y \in B(x, r/2)$. Then $h_1(y) = T^{\mathrm{tr}_x}(y)$. Since $T \in B^{\mathbb{L}(E;F_S)}(I, 1/4)$, $\|T\| \leq 5/4$. So $\|T(y - x)\| \leq \frac{5}{4}\|y - x\|$. That is, $d(T(y-x), 0) \leq \frac{5}{4}\|y - x\|$. Since tr_x is an isometry, $d(T^{\mathrm{tr}_x}(\mathrm{tr}_x(y-x)), \mathrm{tr}_x(0)) \leq \frac{5}{4}\|y - x\|$. That is, $\|T^{\mathrm{tr}_x}(y) - x\| \leq \frac{5}{4}\|y - x\|$. Since $y \in B(x, r/2)$, $\|T^{\mathrm{tr}_x}(y) - x\| \leq 5r/8$. Hence $h_2(T^{\mathrm{tr}_x}(y)) = \mathrm{tr}_u(T^{\mathrm{tr}_x}(y))$. So $h(y) = h_2(h_1(y)) = A(y)$. We have shown that if $x \in S \cap F_S$, then $\lambda_{\mathbb{A}}^{E;F_S} \upharpoonright S$ is (α, G)-decayable at x.

Let $x \in S - F_S$. Then $x \in \mathrm{acc}(S \cap F_S)$. Define $r_x = \frac{1}{2} d(x, E - S)$. For $r \in (0, r_x)$ let $a_r = \frac{1}{2}\min(1/4, r/8, \frac{r}{8\|x\|})$ and $V_{x,r} = B^{\mathbb{A}(E;F)}(x, a_r)$. Let $D = B(x, r/3) \cap F_S$. By the above argument, for every $y \in D$: $\lambda_{\mathbb{A}}^{E;F_S} \upharpoonright S$ is (α, G)-decayable at y, $r_y \geq r_x$, and $V_{y,r} \supseteq V_{x,r}$ for every $r \in (0, r_x)$. By Proposition 3.2(b), $\lambda_{\mathbb{A}}^{E;F_S} \upharpoonright S$ is (α, G)-decayable at x. ∎

Recall that in this section we prove that if $(H_{\Gamma}^{\mathrm{LC}}(E))^{\tau} = H_{\Gamma}^{\mathrm{LC}}(F)$, then τ is locally Γ-bicontinuous. If Γ is countably generated or if $\Gamma = \mathrm{MC}$, then the above is true for any E and F. For Γ's which are not countably generated, we have only a partial answer. We know how to prove that τ is locally Γ-bicontinuous only for Γ's which are $\kappa(E)$-generated. See the definition below.

Definition 3.9. (a) Let X be a metric space and $r > 0$. A family $\mathcal{A}$ of subsets of X is *r-spaced* if $d(A, B) \geq r$ for any distinct $A, B \in \mathcal{A}$. A subset $C \subseteq X$ is *r-spaced* if $\{\{x\} \mid x \in C\}$ is r-spaced. A set C is *spaced* if C is r-spaced for some $r > 0$.

(b) Let X be a metric space $x \in X$ and $A \subseteq X$. We define the set of cardinals $K^X(x, A)$ as follows: $\kappa \in K^X(x, A)$ iff for every $U \in \mathrm{Nbr}(x)$ there is $B \subseteq A \cap U$ such that $|B| = \kappa$ and B is spaced. Let

$$\kappa^X(x, A) = \sup(K^X(x, A)), \qquad \kappa(X) = \min_{x \in X} \kappa^X(x, X).$$

(c) Let Γ be a modulus of continuity. We say that Γ_0 *generates* Γ if $\Gamma = \mathrm{cl}_{\preceq}(\Gamma_0)$. We say that Γ is $(\leq\kappa)$*-generated* if there is Γ_0 such that $|\Gamma_0| \leq \kappa$ and $\Gamma = \mathrm{cl}_{\preceq}(\Gamma_0)$.

(d) Let $\gamma \in \mathrm{MC}$ and $a, b \in [0, \infty)$. Then $a \approx^{\gamma} b$ means that $a \leq \gamma(b)$ and $b \leq \gamma(a)$.

(e) Let X be a metric space, $x \in X$, $G \leq H(X)$ and $\alpha \in \mathrm{MBC}$. We say that G is α*-infinitely-closed at* x if there is $U \in \mathrm{Nbr}(x)$ such that if $F \subseteq G$ and F satisfies:

(1) for every $f \in F$, f is α-bicontinuous,
(2) for every $f \in F$, $\mathrm{supp}(f) \subseteq U$ and $x \notin \mathrm{cl}(\mathrm{supp}(f))$,
(3) for any distinct $f, g \in G$, $\mathrm{cl}(\mathrm{supp}(f)) \cap \mathrm{cl}(\mathrm{supp}(g)) = \emptyset$,
(4) $\mathrm{cl}(\bigcup_{f\in F} \mathrm{supp}(f)) = \{x\} \cup \bigcup_{f\in F} \mathrm{cl}(\mathrm{supp}(f))$,

then $\circ F \in G$.

Note that if F is as above, then $\circ F \in H(X)$. So $H(X)$ is α-infinitely-closed at x for every $\alpha \in \mathrm{MBC}$.

(f) When dealing with partial actions, we often wish to perform a composition $g \circ f$, where $\mathrm{Rng}(f) \not\subseteq \mathrm{Dom}(g)$. Such a composition is considered to be legal. The domain of the resulting function is $f^{-1}(\mathrm{Rng}(f) \cap \mathrm{Dom}(g))$.

If f, g are functions and ϱ is a 1-1 function, then $f \sim^{\varrho} g$ means that

$$\mathrm{Dom}(f) \cup \mathrm{Rng}(f) \subseteq \mathrm{Dom}(\varrho), \qquad g = \varrho \circ f \circ \varrho^{-1}. \ \square$$

Proposition 3.10. (a) *If X is a metric space, $A \subseteq X$ and $x \in \mathrm{acc}(A)$, then $\kappa(x, A) \geq \aleph_0$.*

(b) *If E is a normed space, then $\kappa(x, E) = \min(\{|D| \mid D$ is a dense subset of $E\})$ for every $x \in E$.*

(c) *If $E = \ell_\infty$, then $\kappa(E) = 2^{\aleph_0}$.*

(d) *If E is a Hilbert space with an orthonormal base of cardinality ν, then $\kappa(E) = \nu$.*

Proof. The proof is trivial. ∎

The next lemma says roughly that if for every $h \in H$, $(h_\lambda)^\tau$ is Γ-bicontinuous at x, then there are $\gamma \in \Gamma$ and neighborhoods T, V of x and e_H respectively such that $(h_\lambda)^\tau \restriction T$ is γ-bicontinuous for every $h \in V$. This is proved under the assumption that H is G-decayable, where G is an infinitely-closed subgroup of $H(X)$.

For countably generated Γ's the conclusion of the lemma is true for every metric space X. If however, Γ is not countably generated, then we need to assume that Γ has a generating set of size $\leq \kappa(X)$. The lemma will be applied to $\mathbb{T}(E; F)$ and $\mathbb{A}(E; F)$.

Lemma 3.11. *Suppose that*:

(i) *X is a metric space, $G \leq H(X)$, H is a topological group, λ is a partial action of H on X, $x \in \mathrm{Fld}(\lambda)$, x is a λ-limit-point, $\alpha \in \mathrm{MBC}$, G is α-infinitely-closed at x, and for some $N \in \mathrm{Nbr}(x)$, λ is (α, G)-decayable at every point $y \in H_\lambda(x) \cap N$. Set*

$$\kappa = \min(\{\kappa(x, V_\lambda(x)) \mid V \in \mathrm{Nbr}(e_H)\}).$$

(ii) *Y is a metric space and $\tau : X \cong Y$.*
(iii) *Γ is a modulus of continuity, and Γ is $(\leq\kappa)$-generated.*
(iv) *There is $U \in \mathrm{Nbr}(x)$ such that for every $g \in G\lfloor U \rfloor$: if g is $\alpha \circ \alpha$-bicontinuous, then g^τ is Γ-bicontinuous at $\tau(x)$.*

Then $P(x)$ holds, where

$P(x)$: *There are $T \in \mathrm{Nbr}(x)$, $V \in \mathrm{Nbr}(e_H)$ and $\gamma \in \Gamma$ such that for every $h \in V$, $T \subseteq \mathrm{Dom}(h_\lambda)$ and $(h_\lambda)^\tau \restriction \tau(T)$ is γ-bicontinuous.*

Proof. Let $U_1 \in \mathrm{Nbr}(x)$ be as ensured by the α-infinite-closedness of G at x. Let r_x be as ensured by the decayability of H at x. Let $r \in (0, r_x)$ be such that $B(x,r) \subseteq U_1 \cap U \cap N$, and $W = V_{x,r}$ be as ensured by the decayability of H at x. So $W \in \mathrm{Nbr}(e_H)$, $W \times B(x,r) \subseteq \mathrm{Dom}(\lambda)$ and $W_\lambda(x) \subseteq B(x,r)$. First we prove the following claim.

Claim 1. *There is $y \in B(x, r/2) \cap W_\lambda(x)$ such that $P(y)$ holds.*

Proof. Suppose by contradiction that there is no such y. Let Γ_0 be as ensured by clause (iii). We distinguish two cases.

Case 1: $|\Gamma_0| = \aleph_0$. Let $\vec{x} = \{x_i \mid i \in \mathbb{N}\}$ be a 1-1 sequence tending to x and contained in $B(x,r/2) \cap W_\lambda(x) - \{x\}$. Let $\{\gamma_i \mid i \in \mathbb{N}\}$ be an enumeration of Γ_0 such that $\{j \mid \gamma_j = \gamma_i\}$ is infinite for every i. Let $r_{x_i} > 0$ be as ensured by the decayability of λ at x_i. Let $\{r_i \mid i \in \mathbb{N}\}$ be a sequence such that for any distinct $i, j \in \mathbb{N}$ we have $0 < r_i < r_{x_i}$, $B(x_i, r_i) \subseteq B(x,r)$, $d(x_i, x) > r_i$ and $\mathrm{cl}(B(x_i,r_i)) \cap \mathrm{cl}(B(x_j,r_j)) = \emptyset$.

Let $W_i = V_{x_i,r_i}$ be as ensured by the decayability of λ at x_i. That is, $W_i \in \mathrm{Nbr}(e_H)$ and $\mathrm{Dom}(h_\lambda) \supseteq B(x_i, r_i/2)$ for every $h \in W_i$, and there is $g \in G$ such that g is α-bicontinuous, $g \restriction B(x_i, r_i/2) = h_\lambda \restriction B(x_i, r_i/2)$ and $\mathrm{supp}(g) \subseteq B(x_i, r_i)$.

Let $V_i = B(x_i, r_i/2)$. Then $\mathrm{Dom}(h_\lambda) \supseteq V_i$ for every $h \in W_i$. Since $\neg P(x_i)$ holds, there is $h_i \in W_i$ such that $((h_i)_\lambda)^\tau \restriction \tau(V_i)$ is not γ_i-bicontinuous. Let $g_i \in G$ be such that g_i is α-bicontinuous, $g_i \restriction B(x_i, r_i/2) = (h_i)_\lambda \restriction B(x_i, r_i/2)$ and $\mathrm{supp}(g_i) \subseteq B(x_i, r_i)$. Clearly, $F := \{g_i \mid i \in \mathbb{N}\}$ satisfies clauses (1)–(4) in the definition of α-infinite-closedness, so $g := \circ_{i\in\mathbb{N}}\, g_i \in G$. For every $u, v \in X$ there are $i, j \in \mathbb{N}$ such that $g(u) = g_i \circ g_j(u)$ and $g(v) = g_i \circ g_j(v)$. So g is $\alpha \circ \alpha$-continuous. Similarly, g^{-1} is $\alpha \circ \alpha$-continuous. Since $\mathrm{supp}(g) \subseteq U$, by clause (iv), g^τ is Γ-bicontinuous at $\tau(x)$. That is, there are $\gamma \in \Gamma$ and $T \in \mathrm{Nbr}(\tau(x))$ such that

$$g^\tau \restriction T \text{ is } \gamma\text{-bicontinuous.} \tag{1.1}$$

Let i be such that $\gamma \preceq \gamma_i$, and let $t > 0$ be such that $\gamma \restriction [0,t] \le \gamma_i \restriction [0,t]$. There is j such that $\gamma_j = \gamma_i$, $\tau(B(x_j, r_j)) \subseteq T$ and

$$\mathrm{diam}(\tau(B(x_j, r_j))) < t. \tag{†}$$

Set $k = (h_j)_\lambda$. Now, $g \restriction V_j = g_j \restriction V_j = k \restriction V_j$. So

$$g^\tau \restriction \tau(V_j) = (g_j)^\tau \restriction \tau(V_j) = k^\tau \restriction \tau(V_j). \tag{1.2}$$

Recall that $k^\tau \restriction \tau(V_j)$ is not γ_j-bicontinuous. So there are $u, v \in \tau(V_j)$ such that $d^Y(k^\tau(u), k^\tau(v)) \not\precsim^{\gamma_j} d^Y(u,v)$. By (1.2),

$$d^Y(g^\tau(u), g^\tau(v)) \not\precsim^{\gamma_j} d^Y(u,v). \tag{1.3}$$

Let $u_1 = \tau^{-1}(u)$ and $v_1 = \tau^{-1}(v)$. So $u_1, v_1 \in B(x_j, r_j/2)$. Since $k \restriction B(x_j, r_j/2) = g_j \restriction B(x_j, r_j/2)$ and $\mathrm{supp}(g_j) \subseteq B(x_j, r_j)$, we have $k(u_1), k(v_1) \in B(x_j, r_j)$. By (†), $d^Y(\tau(k(u_1)), \tau(k(v_1))) < t$. Also, $\tau(k(u_1)) = k^\tau(u)$, and the same holds for v and v_1. So

$d^Y(u,v) < t$ and $d^Y(k^\tau(u), k^\tau(v)) < t$. By (1.2),

$$d^Y(u,v) < t, \quad d^Y(g^\tau(u), g^\tau(v)) < t. \tag{1.4}$$

Recall that $\gamma{\restriction}[0,t] \leq \gamma_j{\restriction}[0,t]$. Hence by (1.3) and (1.4),

$$d^Y(g^\tau(u), g^\tau(v)) \not\precsim^\gamma d^Y(u,v). \tag{1.5}$$

Recall that $u, v \in \tau(V_j) \subseteq T$. Hence (1.1) and (1.5) are contradictory. So there is $y \in B(x, r/2) \cap W_\lambda(x)$ such that $P(y)$ holds.

Case 2: $|\Gamma_0| > \aleph_0$. Let $L = W_\lambda(x)$ and $\kappa = \kappa^X(x, L)$. We prove that there are sequences $\{r_i \mid i \in \mathbb{N}\} \subseteq (0, \infty)$ and $\{L_i \mid i \in \mathbb{N}\}$ such that:

(i) $r_0 = r/2$ and $\{r_i \mid i \in \mathbb{N}\}$ is a strictly decreasing sequence converging to 0;
(ii) for every $i \in \mathbb{N}$, $L_i \subseteq L \cap (B(x, r_i) - B(x, r_{i+1}))$ and L_i is spaced;
(iii) $|\bigcup\{L_i \mid i \in \mathbb{N}\}| = \kappa$.

Suppose first that $\mathrm{cf}(\kappa) = \aleph_0$. (That is, there is a countable set of cardinals K such that for every $\kappa' \in K$, $\kappa' < \kappa$ and $\sum K = \kappa$.) Let $K = \{\kappa_i \mid i \in \mathbb{N}\}$ and $r_0 = r/2$. We may assume that each κ_i is infinite. We define L_i and r_{i+1} by induction on i. Suppose that r_i has been defined. Since $\kappa_i < \kappa^X(x, L)$ there is $L_i \subseteq L \cap B(x, r_i)$ such that L_i is spaced and $|L_i| = \kappa_i$. Suppose that L_i is s_i-spaced. There is at most one member $y \in L_i$ such that $d(x, y) < s_i/2$. So by removing this member we may assume that $d(L_i, x) \geq s_i/2$. Let $r_{i+1} = \min(\frac{s_i}{2}, \frac{1}{i+1})$. Evidently, $\{r_i \mid i \in \mathbb{N}\}$, $\{L_i \mid i \in \mathbb{N}\}$ fulfill (i)–(iii).

Suppose that $\mathrm{cf}(\kappa) > \aleph_0$. First we show that

$(*)$ For every $s > 0$ there is $M \subseteq L \cap B(x, s)$ such that $|M| = \kappa$ and M is spaced.

Suppose not, and let s be a counter-example. For every $n > 0$ let K_n be the set of all κ' such that there is $M \subseteq L \cap B(x, s)$ such that $|M| = \kappa'$ and M is $1/n$-spaced. Then there is n such that K_n is unbounded in κ. Let N be a maximal $\frac{1}{2n}$-spaced subset of $L \cap B(x, s)$. Then $|N| < \kappa$. So there is $\kappa' \in K_n$ such that $|N| < \kappa'$. Let M be a $1/n$-spaced subset of $L \cap B(x, s)$ of cardinality κ'. Then there are $y \in N$ and $z_1, z_2 \in M$ such that $z_1, z_2 \in B(y, \frac{1}{2n})$. A contradiction, so $(*)$ holds.

As in the case that $\mathrm{cf}(\kappa) = \aleph_0$ we define a sequence $\{\kappa_i \mid i \in \mathbb{N}\}$. Indeed, we set $\kappa_i = \kappa$ for every $i \in \mathbb{N}$. The L_i's and r_i's are now constructed as in the case $\mathrm{cf}(\kappa) = \aleph_0$, and they obviously fulfill clauses (i)–(iii).

We really need sequences $\{r_i \mid i \in \mathbb{N}\} \subseteq (0, \infty)$ and $\{L_i \mid i \in \mathbb{N}\}$ which fulfill the following conditions:

(i) $r_0 = r/2$ and $\{r_i \mid i \in \mathbb{N}\}$ is a strictly decreasing sequence converging to 0;
(ii) for every $i \in \mathbb{N}$, $L_i \subseteq L \cap (B(x, r_i/2) - B(x, 2r_{i+1}))$ and L_i is spaced, and $|L_i| \leq |L_j|$ for every $i < j$;
(iii) $|\bigcup\{L_i \mid i \in \mathbb{N}\}| = |\Gamma_0|$.

Such sequences can be obtained from the original $\{r_i \mid i \in \mathbb{N}\}$ and $\{L_i \mid i \in \mathbb{N}\}$ by taking an appropriate subsequence of $\{r_i \mid i \in \mathbb{N}\}$ and by replacing L_i by a subset of L_i if necessary.

Let $s_i > 0$ be such that L_i is s_i-spaced. Set $M = \bigcup\{L_i \mid i \in \mathbb{N}\}$, and let $\iota : M \to \Gamma_0$ be a function such that for every $\gamma \in \Gamma_0$ there is $n \in \mathbb{N}$ such that $\gamma \in \iota(L_m)$ for every

$m \geq n$. Define $\gamma_y = \iota(y)$. Let r_y be as ensured by the decayability of H at y. For every $y \in M$ we define $s_y > 0$. If $y \in L_i$, choose $s_y < \min(r_y, r_{i+1}, s_i/3)$. Note that for distinct $y, z \in L_i$, $B(y, s_y) \subseteq B(x, r_i) - B(x, r_{i+1})$ and $\mathrm{cl}(B(y, s_y)) \cap \mathrm{cl}(B(z, s_z)) = \emptyset$. So for distinct $y, z \in M$, $\mathrm{cl}(B(y, s_y)) \cap \mathrm{cl}(B(z, s_z)) = \emptyset$.

For every $y \in M$ let $W_y = V_{y,s_y}$ be as ensured by the decayability of λ at y. That is, $W_y \in \mathrm{Nbr}(e_H)$, $\mathrm{Dom}(h_\lambda) \supseteq B(y, s_y/2)$ for every $h \in W_y$, and there is $g \in G$ such that g is α-bicontinuous, $g \restriction B(y, s_y/2) = h_\lambda \restriction B(y, s_y/2)$ and $\mathrm{supp}(g) \subseteq B(y, s_y)$.

Let $V_y = B(y, s_y/2)$. So $\mathrm{Dom}(h_\lambda) \supseteq V_y$ for every $h \in W_y$. Since $\neg P(y)$ holds, there is $h_y \in W_y$ such that $((h_y)_\lambda)^\tau \restriction \tau(V_y)$ is not γ_y-bicontinuous. Let $g_y \in G$ be such that g_y is α-bicontinuous, $g_y \restriction B(y, s_y/2) = (h_y)_\lambda \restriction B(y, s_y/2)$ and $\mathrm{supp}(g_y) \subseteq B(y, s_y)$. For any distinct $y, z \in M$, $\mathrm{supp}(g_y) \cap \mathrm{supp}(g_z) = \emptyset$. Clearly, $F := \{g_y \mid y \in M\}$ satisfies clauses (1)–(4) in the definition of α-infinite-closedness, so $g = \circ_{y \in M}\, g_y \in G$. The rest of the argument is identical to the one given in Case 1. We have proved Claim 1.

Let y be as ensured by Claim 1. Since $y \in W_\lambda(x)$, there is $\hat{h} \in W$ such that $y = \hat{h}_\lambda(x)$. Since $W = V_{x,r}$, there is $g \in G$ such that g is α-bicontinuous, $g \restriction B(x, r/2) = \hat{h}_\lambda \restriction B(x, r/2)$ and $\mathrm{supp}(g) \subseteq B(x, r)$. So $g(x) = y$. Since $\alpha \in \mathrm{MBC}$, we have $\alpha \leq \alpha \circ \alpha$, and hence g is $\alpha \circ \alpha$-bicontinuous. The bicontinuity of g and the fact $\mathrm{supp}(g) \subseteq B(x, r) \subseteq U$ imply that g^τ is Γ-bicontinuous at $\tau(x)$. Let $R \in \mathrm{Nbr}(\tau(x))$ and $\beta \in \Gamma$ be such that $g^\tau \restriction R$ is β-bicontinuous. We may assume that

$$\tau^{-1}(R) \subseteq B(x, r/2). \tag{2.1}$$

Hence $g^\tau \restriction R = (\hat{h}_\lambda)^\tau \restriction R$. So

$$(\hat{h}_\lambda)^\tau \restriction R \text{ is } \beta\text{-bicontinuous.} \tag{2.2}$$

Note that if T', V', γ' fulfill the requirements of $P(y)$ and $T' \supseteq T'' \in \mathrm{Nbr}(y)$, then T'', V', γ' fulfill the requirements of $P(y)$. Since $P(y)$ holds, there are $S_1 \in \mathrm{Nbr}(y)$, $V_1 \in \mathrm{Nbr}(e_H)$ and $\gamma_1 \in \Gamma$ such that for every $h \in V_1$,

$$S_1 \subseteq \mathrm{Dom}(h_\lambda), \qquad (h_\lambda)^\tau \restriction \tau(S_1) \text{ is } \gamma_1\text{-bicontinuous.} \tag{2.3}$$

Since $\hat{h}_\lambda(x) = y$ and $\tau^{-1}(R) \in \mathrm{Nbr}(x)$, we may assume that

$$S_1 \subseteq \hat{h}_\lambda(\tau^{-1}(R)). \tag{2.4}$$

So $S_1 \subseteq \hat{h}_\lambda(B(x, r/2))$. Let $S_2 \in \mathrm{Nbr}(y)$ and $V_2 \in \mathrm{Nbr}(e_H)$ be such that

$$S_2 \subseteq S_1, \qquad V_2 \subseteq V_1, \qquad \lambda(V_2 \times S_2) \subseteq S_1. \tag{2.5}$$

Note that $S_2 \subseteq \mathrm{Rng}(\hat{h}_\lambda)$. We define $T = (\hat{h}_\lambda)^{-1}(S_2)$, $V = \hat{h}^{-1} \cdot V_2 \cdot \hat{h}$ and $\gamma = \beta \circ \gamma_1 \circ \beta$ and show that T, V, γ satisfy the requirements of $P(x)$. Since $\beta, \gamma_1 \in \Gamma$, we have

$$\gamma \in \Gamma. \tag{2.6}$$

We verify that if $h \in V$, then

$$T \subseteq \mathrm{Dom}(h_\lambda) \quad \text{and} \quad (h^{\hat{h}})_\lambda \restriction S_2 \sim^{\varrho^{-1}} h_\lambda \restriction T, \quad \text{where} \quad \varrho = \hat{h}_\lambda \restriction \tau^{-1}(R). \tag{2.7}$$

Let $\bar{h} = h^{\hat{h}}$. Then $\bar{h} \in V_2$ and $h = \hat{h}^{-1} \cdot \bar{h} \cdot \hat{h}$. We show that $\hat{h}_\lambda(z)$, $\bar{h}_\lambda(\hat{h}_\lambda(z))$ and $(\hat{h}^{-1})_\lambda(\bar{h}_\lambda(\hat{h}_\lambda(z)))$ are defined for every $z \in T$. Clearly, $T \subseteq \mathrm{Dom}(\hat{h}_\lambda)$ and $\hat{h}_\lambda(T) = S_2$. So by (2.5),

(i) for every $z \in T$, $\bar{h}_\lambda(\hat{h}_\lambda(z))$ is defined and $\bar{h}_\lambda(\hat{h}_\lambda(z)) \in S_1$.

By (2.4), $S_1 \subseteq \mathrm{Rng}(\hat{h}_\lambda)$. So $(\hat{h}_\lambda)^{-1}(\bar{h}_\lambda(\hat{h}_\lambda(z)))$ is defined. Since $h = \bar{h}^{\hat{h}}$ and by the definition of a partial action, it follows that

(ii) for every $z \in T$, $h_\lambda(z)$ is defined and $h_\lambda(z) = (\hat{h}_\lambda)^{-1} \circ \bar{h}_\lambda \circ \hat{h}_\lambda(z)$.

By (ii), $T \subseteq \mathrm{Dom}(h_\lambda)$, and by (2.1), $\tau^{-1}(R) \subseteq \mathrm{Dom}(\hat{h}_\lambda)$. So $\mathrm{Dom}(\varrho^{-1}) = \mathrm{Rng}(\varrho) = \hat{h}_\lambda(\tau^{-1}(R))$. Since $\bar{h} \in V_2$, we have $S_2 \subseteq \mathrm{Dom}(\bar{h}_\lambda)$, hence $\mathrm{Dom}(\bar{h}_\lambda \restriction S_2) = S_2$. By (2.4) and (2.5), $S_2 \subseteq \hat{h}_\lambda(\tau^{-1}(R))$. So $\mathrm{Dom}(\bar{h}_\lambda \restriction S_2) \subseteq \mathrm{Dom}(\varrho^{-1})$. We have $\mathrm{Rng}(\bar{h}_\lambda \restriction S_2) = \bar{h}_\lambda(S_2)$, and from (2.5) and the fact that $\bar{h} \in V_2$, it follows that $\bar{h}_\lambda(S_2) \subseteq S_1$. By (2.4), $S_1 \subseteq \hat{h}_\lambda(\tau^{-1}(R))$, so $\mathrm{Rng}(\bar{h}_\lambda \restriction S_2) \subseteq \mathrm{Dom}(\varrho^{-1})$. Note that $T \subseteq \tau^{-1}(R)$; indeed, this follows from the definition of T, (2.4) and (2.5). So

(iii) for every $z \in T$, $\hat{h}_\lambda(z) = (\hat{h}_\lambda \restriction \tau^{-1}(R))(z) = \varrho(z)$.

Also,

(iv) for every $z \in T$, $\bar{h}_\lambda(\hat{h}_\lambda(z)) = (\bar{h}_\lambda \restriction S_2)(\hat{h}_\lambda(z))$.

Let $z \in T$ and denote $u = \bar{h}_\lambda(\hat{h}_\lambda(z))$. By (i) and (2.4), $u \in S_1 \subseteq \hat{h}_\lambda(\tau^{-1}(R)) = \mathrm{Dom}(\varrho^{-1})$. Hence $(\hat{h}_\lambda)^{-1}(u) = \varrho^{-1}(u)$. We conclude that

(v) for every $z \in T$, $(\hat{h}_\lambda)^{-1}(\bar{h}_\lambda(\hat{h}_\lambda(z))) = \varrho^{-1}(\bar{h}_\lambda(\hat{h}_\lambda(z)))$.

It follows from (ii)–(v) that $h_\lambda \restriction T = \varrho^{-1} \circ (\bar{h}_\lambda \restriction S_2) \circ \varrho$. We have verified (2.7). Next conjugate (2.7) by τ. We obtain

$$((h^{\hat{h}})_\lambda \restriction S_2)^\tau \sim^{(\varrho^{-1})^\tau} (h_\lambda \restriction T)^\tau. \tag{2.8}$$

Clearly, $((h^{\hat{h}})_\lambda \restriction S_2)^\tau = ((h^{\hat{h}})_\lambda)^\tau \restriction \tau(S_2)$. Since $h \in V$, we have $h^{\hat{h}} \in V^{\hat{h}} = V_2$. So by (2.3),

$$((h^{\hat{h}})_\lambda \restriction S_2)^\tau \text{ is } \gamma_1\text{-bicontinuous.} \tag{2.9}$$

Fact (2.8) has the form $f \sim^{\sigma^{-1}} k$, where $f = ((h^{\hat{h}})_\lambda \restriction S_2)^\tau$, $k = (h_\lambda \restriction T)^\tau$ and $\sigma = \varrho^\tau = (\hat{h}_\lambda)^\tau \restriction R$. By (2.9), f is γ_1-bicontinuous, and by (2.2) σ is β-bicontinuous. Since $k = \sigma^{-1} \circ f \circ \sigma$, it follows that k is $\beta \circ \gamma_1 \circ \beta$-bicontinuous. Recall that $\gamma = \beta \circ \gamma_1 \circ \beta$ and $k = (h_\lambda \restriction T)^\tau = (h_\lambda)^\tau \restriction \tau(T)$. Hence $(h_\lambda)^\tau \restriction \tau(T)$ is γ-bicontinuous.

We have shown that for every $h \in V$, $\mathrm{Dom}(h_\lambda) \supseteq T$ and $(h_\lambda)^\tau \restriction \tau(T)$ is γ-bicontinuous. So T, V, γ satisfy the requirements of the lemma. ∎

3.3. Translation-like partial actions. We have isolated the properties of $\mathbb{T}(E)$ and $\mathbb{A}(E)$ which are used in the proof that τ is Γ-continuous. The following definition deals with the properties of $\mathbb{T}(E)$. Partial actions having these properties are called translation-like partial actions. In fact, the definition captures the properties of $\mathbb{T}(E; F)$, where F is any dense linear subspace of E. The properties of $\mathbb{A}(E)$ to be used appear in Definition 3.28(b).

DEFINITION 3.12. Suppose that X is a metric space, H is a topological group, and λ is a partial action of H on X. Let $x \in \mathrm{Fld}(\lambda)$. We say that λ is *a translation-like partial action at x* if for every $V \in \mathrm{Nbr}(e_H)$ there are:

(i) $U = U_{x,V} \in \mathrm{Nbr}(x)$, and a dense subset of U, $D = D_{x,V}$,
(ii) a radius $r = r_{x,V} > 0$ and a constant $K = K_{x,V} > 0$,

such that the following holds. For any distinct $\bar{x}_0, \bar{x}_1 \in D$ there are $n \le K \cdot \frac{r}{d(\bar{x}_0,\bar{x}_1)}$, a sequence $\bar{x}_0 = x_0, x_1, \ldots, x_n \in X$ and $h_1, \ldots, h_n \in V$ such that $x_n \notin B(x,r)$, and for every $i = 1, \ldots, n$, $\bar{x}_0, \bar{x}_1 \in \mathrm{Dom}((h_i)_\lambda)$, $(h_i)_\lambda(\bar{x}_0) = x_{i-1}$ and $(h_i)_\lambda(\bar{x}_1) = x_i$.

A partial action λ is *translation-like* if λ is translation-like at x for all $x \in \mathrm{Fld}(\lambda)$. $\square$

PROPOSITION 3.13. *Let E be a normed space, F be a dense linear subspace of E and $X \subseteq E$ be open. Then $\lambda_{\mathbb{T}}^{E;F} \restriction X$ is a translation-like partial action.*

Proof. For $x \in X$ and $V \in \mathrm{Nbr}^{\mathbb{T}(E;F)}(\mathrm{Id})$ we define $U = U_{x,V}$, $D = D_{x,V}$ etc. as follows. Let $r_0 > 0$ be such that $B^E(x, r_0) \subseteq X$ and $\{\mathrm{tr}_v \mid v \in B^F(0, r_0)\} \subseteq V$. Now define $U = B(x, r_0/4)$, $D = F \cap U$, $r = r_0/2$ and $K = 2$.

For distinct $\bar{x}_0, \bar{x}_1 \in D$ we define n, $x_0, \ldots, x_n$ and $h_1, \ldots, h_n$ as required in Definition 3.12. Let n be the least integer such that $n \cdot \|\bar{x}_1 - \bar{x}_0\| \ge r$. For $i = 0, \ldots, n$ let $x_i = \bar{x}_0 + i(\bar{x}_1 - \bar{x}_0)$ and for $i = 1, \ldots, n$ let $h_i = \mathrm{tr}_{(i-1)(\bar{x}_1 - \bar{x}_0)}$. It is easily checked that n, the x_i's and the h_i's are as required. ∎

We let X and Y denote metric spaces. Their metrics are denoted by d^X and d^Y. However, in most cases we write $d(x,y)$ as an abbreviation of both $d^X(x,y)$ and $d^Y(x,y)$.

LEMMA 3.14. *Let X be a metric space and λ be a partial action of H on X. Suppose that $x \in \mathrm{Fld}(\lambda)$ and λ is translation-like at x. Let Y be a metric space and $\tau : X \cong Y$. Let $\Gamma \subseteq \mathrm{MC}$, and suppose that for every $\gamma \in \Gamma$ and $K > 0$, $K \cdot \gamma \in \Gamma$. Suppose that $P(x)$ of Lemma 3.11 holds. That is, there are $T \in \mathrm{Nbr}(x)$, $V \in \mathrm{Nbr}(e_H)$ and $\gamma \in \Gamma$ such that for every $h \in V$, $T \subseteq \mathrm{Dom}(h_\lambda)$ and $(h_\lambda)^\tau \restriction \tau(T)$ is γ-bicontinuous. Then τ^{-1} is Γ-continuous at $\tau(x)$.*

Proof. Let $U = U_{x,V}$, $D = D_{x,V}$, $r = r_{x,V}$ and $K = K_{x,V}$ be as ensured by the translation-likeness of H at x. Set $y = \tau(x)$, $B = B(x,r)$ and $C = \tau(B)$. Since $C \in \mathrm{Nbr}(y)$, we have $e := d(y, Y - C) > 0$. Let $R = \tau(T \cap U) \cap B(y, e/2)$ and $M = 2Kr/e$. Since $\gamma \in \Gamma$, we have $M \cdot \gamma \in \Gamma$.

We show that $\tau^{-1} \restriction R$ is $M \cdot \gamma$-continuous. Suppose by way of contradiction that this is not true. Hence there are $\bar{y}_0, \bar{y}_1 \in R$ such that $d(\tau^{-1}(\bar{y}_0), \tau^{-1}(\bar{y}_1)) > M \cdot \gamma(d(\bar{y}_0, \bar{y}_1))$. Since D is dense in U and $\bar{y}_0, \bar{y}_1 \in \tau(U)$, we may assume that $\bar{y}_0, \bar{y}_1 \in \tau(D)$. For every $h \in H$ let $\hat{h}$ denote h_λ, and for $\ell = 0, 1$ let $\bar{x}_\ell = \tau^{-1}(\bar{y}_\ell)$. Hence $\bar{x}_0, \bar{x}_1 \in D$. So there are $n \le Kr/d(\bar{x}_0, \bar{x}_1)$, $\bar{x}_0 = x_0, x_1, \ldots, x_n$ and $h_1, \ldots, h_n \in V$ such that $x_n \notin B$, and for every $i = 1, \ldots, n$, $\bar{x}_0, \bar{x}_1 \in \mathrm{Dom}(h_i)$, $\hat{h}_i(\bar{x}_0) = x_{i-1}$ and $\hat{h}_i(\bar{x}_1) = x_i$. For $i = 1, \ldots, n$ let $y_i = \tau(x_i)$.

In the space Y we thus have the following situation:

(i) $d(y, y_0) < e/2$;
(ii) for every $i = 1, \ldots, n$, $\hat{h}_i^\tau(\bar{y}_0) = y_{i-1}$ and $\hat{h}_i^\tau(\bar{y}_1) = y_i$;
(iii) $y_n \notin C$.

Every h_i belongs to V, hence $\hat{h}_i^\tau \restriction \tau(T)$ is γ-bicontinuous. Also, $\bar{y}_0, \bar{y}_1 \in \tau(T)$, so

(iv) $d(y_{i-1}, y_i) \le \gamma(d(\bar{y}_0, \bar{y}_1))$.

Hence

$$\begin{aligned} e = d(y, Y - C) &\le d(y, y_n) \le d(y, y_0) + \sum_{i=1}^{n} d(y_{i-1}, y_i) < e/2 + n \cdot \gamma(d(\bar{y}_0, \bar{y}_1)) \\ &\le e/2 + \frac{Kr}{d(\bar{x}_0, \bar{x}_1)} \cdot \gamma(d(\bar{y}_0, \bar{y}_1)) < e/2 + \frac{Kr}{M \cdot \gamma(d(\bar{y}_0, \bar{y}_1))} \cdot \gamma(d(\bar{y}_0, \bar{y}_1)) \\ &= e/2 + \frac{Kr}{2Kr/e} = e. \end{aligned}$$

A contradiction, so the lemma is proved. ∎

The following theorem is the conjunction of Lemmas 3.11 and 3.14. It will be used in Theorem 3.16. The statement of Theorem 3.15 is rather technical. So it seems worthwhile to explain its main application. Let X be an open subset of a normed space E and $G \le H(X)$. Suppose that for every $x \in X$ and $r > 0$ there are $s \in (0, r)$ and $K > 0$ such that for every $v \in B_E(0, s)$ there is $g \in G$ such that $g \restriction B(x, s) = \mathrm{tr}_v \restriction B(x, s)$, g is K-bilipschitz and $\mathrm{supp}(g) \subseteq B(x, r)$. Assume further that G is α-infinitely-closed for every α of the form $y = Mt$. Then if τ is a homeomorphism between X and a metric space Y, Γ is a countably generated modulus of continuity and $G^\tau \subseteq \mathrm{LIP}_\Gamma^{\mathrm{LC}}(Y)$, then τ^{-1} is locally Γ-continuous.

THEOREM 3.15. *Suppose that*:

(i) X *is a metric space*, $G \le H(X)$, H *is a topological group*, λ *is a partial action of* H *on* X, $x \in \mathrm{Fld}(\lambda)$ *and* $\alpha \in \mathrm{MBC}$;
(ii) G *is* α*-infinitely-closed at* x;
(iii) x *is a* λ*-limit-point*;
(iv) *for some* $N \in \mathrm{Nbr}(x)$, λ *is* (α, G)*-decayable in* $H_\lambda(x) \cap N$;
(v) λ *is translation-like at* x;
(vi) Γ *is a modulus of continuity and* Γ *is* $(\le\kappa)$*-generated*, *where* $\kappa = \min(\{\kappa(x, V_\lambda(x)) \,|\, V \in \mathrm{Nbr}(e_H)\})$;
(vii) Y *is a metric space and* $\tau : X \cong Y$;
(viii) *there is* $U \in \mathrm{Nbr}(x)$ *such that for every* $g \in G\lfloor U \rfloor$: *if* g *is* $\alpha \circ \alpha$*-bicontinuous*, *then* g^τ *is* Γ*-bicontinuous at* $\tau(x)$.

Then τ^{-1} *is* Γ*-continuous at* $\tau(x)$.

Proof. Combine Lemmas 3.11 and 3.14. ∎

The above lemma will be used in the proof that the derivative of a diffeomorphism τ is locally Γ-continuous. For groups of type $H_\Gamma^{\mathrm{LC}}(X)$, Theorem 3.15 yields a result which is slightly weaker than the result obtained in Theorem 3.27, where the action is assumed to be “affine-like” rather than just “translation-like”.

THEOREM 3.16. *Let* $\langle E, X, \mathcal{S}, \mathcal{F} \rangle$ *be a subspace choice system*, Γ *be a* $(\le\kappa(E))$*-generated modulus of continuity*, Y *be a metric space and* $\tau : X \cong Y$. *Suppose that* $(\mathrm{LIP}(X; \mathcal{S}, \mathcal{F}))^\tau \subseteq H_\Gamma^{\mathrm{LC}}(Y)$. *Then* τ^{-1} *is locally* Γ*-continuous.*

Proof. Let $x \in X$ and $S \in \mathcal{S}$ be such that $x \in S$. Write $H = \mathbb{T}(E; F_S)$, $\lambda = \lambda_{\mathbb{T}}^{E;F_S} \restriction S$, $G = \mathrm{LIP}(X; S, F_S)$ and $\alpha(t) = 3t$. We shall apply Theorem 3.15.

By Lemma 3.4, λ is (α, G)-decayable. So 3.15(iv) holds. Let $V \in \mathrm{Nbr}(e_H)$. Then there is $r > 0$ such that $V_\lambda(x) \supseteq B^{F_S}(x, r)$. Since F_S is dense in E, $\kappa(F_S) = \kappa(E)$. So $\kappa(x, V_\lambda(x)) = \kappa(F_S) = \kappa(E)$. It follows that $\min(\{\kappa(x, V_\lambda(x)) \mid V \in \mathrm{Nbr}(e_H)\}) = \kappa(E)$. Since Γ is $(\leq\kappa(E))$-generated, 3.15(vi) holds.

Take U in the definition of α-infinite-closedness to be S. Let L be a subset of G which satisfies clauses (1)–(4) in the definition of α-infinite-closedness (see Definition 3.9(e)). Then $\circ L$ is $\alpha \circ \alpha$-bicontinuous, which implies that $\circ L \in G$. So G is α-infinitely-closed at x. That is, 3.15(ii) holds.

Since for every $V \in \mathrm{Nbr}(e_H)$ there is $r > 0$ such that $V_\lambda(x) \supseteq B^{F_S}(x, r)$, x is a λ-limit-point. That is, 3.15(iii) holds. By Proposition 3.13, λ is translation-like at x. That is, 3.15(v) holds. By the assumptions of this theorem, 3.15(vii) and (viii) hold.

We have seen that all the assumptions of Theorem 3.15 are fulfilled, so τ^{-1} is Γ-continuous at $\tau(x)$. ∎

Definition 3.17. (a) Let E be a normed space, $S \subseteq X \subseteq E$ be open subsets and F be a dense linear subspace of E. Let Γ be a modulus of continuity. We define

$$H_\Gamma(X) = \{h \in H(X) \mid \text{there is } \gamma \in \Gamma \text{ such that } h \text{ is } \gamma\text{-bicontinuous}\},$$
$$H_\Gamma(X, S) = H_\Gamma(X)\lfloor S \rfloor,$$
$$H_\Gamma(X; F) = \{h \in H_\Gamma(X) \mid h(X \cap F) = X \cap F\}$$

and

$$H_\Gamma(X; S, F) = H_\Gamma(X, S) \cap H_\Gamma(X; F).$$

Similarly, let $H_\Gamma^{\mathrm{LC}}(X, S) = H_\Gamma^{\mathrm{LC}}(X)\lfloor S \rfloor$, $H_\Gamma^{\mathrm{LC}}(X; F) = \{h \in H_\Gamma^{\mathrm{LC}}(X) \mid h(X \cap F) = X \cap F\}$ and $H_\Gamma^{\mathrm{LC}}(X; S, F) = H_\Gamma^{\mathrm{LC}}(X, S) \cap H_\Gamma^{\mathrm{LC}}(X; F)$.

Let $\langle E, X, \mathcal{S}, \mathcal{F} \rangle$ be a subspace choice system. We define $H_\Gamma(X; \mathcal{S}, \mathcal{F})$ to be the subgroup of $H(X)$ generated by $\bigcup\{H_\Gamma(X; S, F_S) \mid S \in \mathcal{S}\}$. Analogously, the group $H_\Gamma^{\mathrm{LC}}(X; \mathcal{S}, \mathcal{F})$ is defined to be the subgroup of $H(X)$ generated by $\bigcup\{H_\Gamma^{\mathrm{LC}}(X; S, F_S) \mid S \in \mathcal{S}\}$.

(b) Let E be a normed space, $z \in E$ and $\eta \in H([0, \infty))$. Define $h = \mathrm{Rad}_{\eta,z}^E$ as follows:

$$h(x) = z + \eta(\|x - z\|)\frac{x - z}{\|x - z\|}, \qquad x \neq z,$$

and $h(z) = z$. Clearly, $h \in H(E)$. We call h the *radial homeomorphism based on* η, z. Also, denote $\mathrm{Rad}_{\eta,0^E}^E$ by Rad_η^E, and call it the *radial homeomorphism based on* η. □

Remark. Note the following facts.

(1) $H_\Gamma(X)$ is a special case of $H_\Gamma(X; \mathcal{S}, \mathcal{F})$, where $\mathcal{S} = \{X\}$ and $F_X = E$. The same holds for $H_\Gamma^{\mathrm{LC}}(X)$.
(2) $H_\Gamma(X, S), H_\Gamma(X; F), H_\Gamma(X; S, F), H_\Gamma(X; \mathcal{S}, \mathcal{F}) \subseteq H_\Gamma(X)$.
(3) $H_\Gamma^{\mathrm{LC}}(X, S), H_\Gamma^{\mathrm{LC}}(X; F), H_\Gamma^{\mathrm{LC}}(X; S, F), H_\Gamma^{\mathrm{LC}}(X; \mathcal{S}, \mathcal{F}) \subseteq H_\Gamma^{\mathrm{LC}}(X)$. □

Proposition 3.18. *Let E be a normed space, $z \in E$ and $\eta \in H([0, \infty))$. Suppose that η is α-bicontinuous. Then $h_{\eta,z}$ is $3 \cdot \alpha$-bicontinuous.*

Proof. Set $h = \mathrm{Rad}_{\eta,z}$. We may assume that $z = 0$. Note that $\eta(t) \le \alpha(t)$ for every $t \ge 0$. Since α is concave, it follows that $\frac{\alpha(t)}{t} \cdot s \le \alpha(s)$ for every $0 < s \le t$.

Let $u, v \in E - \{0\}$. Assume that $\|u\| \le \|v\|$ and set $w = \frac{\|u\|}{\|v\|}v$. Then $\|w - u\| \le \|u\| + \|w\| = 2\|u\|$. So $\|\frac{w-u}{2}\| \le \|u\|$. Also, $\|v - w\| = \|v\| - \|u\| \le \|v - u\|$. So $\|w - u\| \le \|v - u\| + \|v - w\| \le 2\|v - u\|$. Hence

$$\begin{aligned}
\|h(v) - h(u)\| &\le \|h(v) - h(w)\| + \|h(w) - h(u)\| \\
&= (\eta(\|v\|) - \eta(\|w\|)) + \frac{\eta(\|u\|)}{\|u\|}\|w - u\| = (\eta(\|v\|) - \eta(\|u\|)) + 2 \cdot \frac{\eta(\|u\|)}{\|u\|}\left\|\frac{w - u}{2}\right\| \\
&\le \alpha(\|v\| - \|u\|) + 2\alpha\left(\frac{\|w - u\|}{2}\right) \le \alpha(\|v - u\|) + 2\alpha(\|v - u\|) = 3\alpha(\|v - u\|).
\end{aligned}$$

So h is 3α-continuous. Since $h^{-1} = \mathrm{Rad}_{\eta^{-1},z}$, it follows that h^{-1} is 3α-continuous. ■

The main result of the next theorem is part (a). It is a more readable special case of (b). Part (b) is a trivial corollary of (c). The proof of (c) is more than just collecting some of the previous lemmas together. It requires an additional argument.

THEOREM 3.19. (a) *Let X, Y be open subsets of the normed spaces E and F respectively. Write $\kappa = \kappa(E)$ and let Γ, Δ be $(\le\kappa)$-generated moduli of continuity. Let $\tau : X \cong Y$, and suppose that $(H_\Gamma^{\mathrm{LC}}(X))^\tau = H_\Delta^{\mathrm{LC}}(Y)$. Then $\Gamma = \Delta$ and τ is locally Γ-bicontinuous.*

(b) *Let $\langle E, X, \mathcal{S}, \mathcal{E}\rangle$ and $\langle F, Y, \mathcal{T}, \mathcal{F}\rangle$ be subspace choice systems. Write $\kappa = \kappa(E)$ and let Γ, Δ be $(\le \kappa)$-generated moduli of continuity. Let $\tau : X \cong Y$, and suppose that*:

$$\text{(i) } (H_\Gamma(X; \mathcal{S}, \mathcal{F}))^\tau \subseteq H_\Delta^{\mathrm{LC}}(Y), \qquad \text{(ii) } (H_\Delta(Y; \mathcal{T}, \mathcal{F}))^{\tau^{-1}} \subseteq H_\Gamma^{\mathrm{LC}}(X).$$

Then $\Gamma = \Delta$ and τ is locally Γ-bicontinuous.

(c) *Let $\langle E, X, \mathcal{S}, \mathcal{E}\rangle$ and $\langle F, Y, \mathcal{T}, \mathcal{F}\rangle$ be subspace choice systems. Write $\kappa = \kappa(E)$ and let Γ, Δ be $(\le\kappa)$-generated moduli of continuity. Let $\tau : X \cong Y$, and suppose that*:

(i) $(\mathrm{LIP}(X; \mathcal{S}, \mathcal{F}))^\tau \subseteq H_\Delta^{\mathrm{LC}}(Y)$,
(ii) $(H_\Delta(Y; \mathcal{T}, \mathcal{F}))^{\tau^{-1}} \subseteq H_\Gamma^{\mathrm{LC}}(X)$.

Then $\Delta \subseteq \Gamma$ and τ is locally Γ-bicontinuous.

Proof. Part (a) is a special case of (b), and (b) is concluded by applying (c) twice: once to X, Y and once to Y, X. So it suffices prove (c).

(c) Since X and Y are homeomorphic, $\kappa(F) = \kappa(E) = \kappa$. Suppose by way of contradiction that $\Delta \not\subseteq \Gamma$. Pick any $T \in \mathcal{T}$ and $y \in T \cap F_T$, and set $x = \tau^{-1}(y)$. (Recall that F_T denotes the dense subspace of F assigned to T by the subspace choice system). Let $x \in S \in \mathcal{S}$. By Theorem 3.16 and clause (c)(i), for some $\delta \in \Delta$, τ^{-1} is δ-continuous at $\tau(x)$. There is $\alpha \in (\Delta - \Gamma) \cap \mathrm{MBC}$ such that $\delta \preceq \alpha$. So τ^{-1} is α-continuous at $\tau(x)$. Choose $r > 0$ be such that $\tau^{-1}\restriction B^F(y, r)$ is α-continuous and $B^F(y, r) \subseteq \tau(S) \cap T$, and let e be such that $\alpha \circ \alpha(e) = r/2$. We define $\eta : [0, \infty) \to [0, \infty)$ as follows. For $t \in [0, e]$, $\eta(t) = \alpha \circ \alpha(t)$, for $t \in [r, \infty)$, $\eta(t) = t$, $\eta\restriction[e, r]$ is a linear function, and η is continuous. Clearly, $\eta \in H([0, \infty))$, and it is easily seen that η is $4 \cdot \alpha \circ \alpha$-continuous and that η^{-1} is 2-Lipschitz. So η is $4 \cdot \alpha \circ \alpha$-bicontinuous. Let $h = \mathrm{Rad}_{\eta,y}\restriction Y$. By Proposition 3.18, h is $12 \cdot \alpha \circ \alpha$-bicontinuous, hence $h \in H_\Delta(Y)$. Since $y \in F_T$, we have $h(Y \cap F_T) = Y \cap F_T$, and so $h \in H_\Delta(Y; \mathcal{T}, \mathcal{F})$. By clause (c)(ii), $g := h^{\tau^{-1}}$ is locally Γ-bicontinuous, and by Theorem 3.16 and clause (c)(ii), τ is locally Γ-continuous. This implies that $\tau \circ g$ is locally Γ-continuous. Since $h \circ \tau = \tau \circ g$, we conclude that $h \circ \tau$ is locally Γ-continuous. Let $\gamma \in \Gamma$ be such that $h \circ \tau$ is γ-continuous at x, and choose s such that $h \circ \tau \restriction B^E(x, s)$ is γ-continuous. We may assume that $\tau(B^E(x, s)) \subseteq B^F(y, r/2)$.

Since $\alpha \notin \Gamma$, there is $t < s$ such that $\alpha(t) > \gamma(t)$. Choose w such that $\|w - x\| = t$ and set $z = \tau(w)$. Then $z \in B^F(y, r/2)$ and hence $\|h(z) - h(y)\| = \alpha \circ \alpha(\|z - y\|)$. Now, $\|w - x\| = \|\tau^{-1}(z) - \tau^{-1}(y)\| \le \alpha(\|z - y\|)$. So $\alpha^{-1}(\|w - x\|) \le \|z - y\|$ and hence

$$\|h(z) - h(y)\| = \alpha \circ \alpha(\|z - y\|) \ge \alpha \circ \alpha(\alpha^{-1}(\|w - x\|)) = \alpha(\|w - x\|).$$

That is, $\|h \circ \tau(w) - h \circ \tau(x)\| \ge \alpha(\|w - x\|) > \gamma(\|w - x\|)$. This contradicts the fact that $h \circ \tau \restriction B^E(x, s)$ is γ-continuous. So $\Delta \subseteq \Gamma$.

Since τ^{-1} is locally Δ-continuous, τ^{-1} is locally Γ-continuous. Recall also that τ is locally Γ-continuous. So τ is locally Γ-bicontinuous. ∎

Remark 3.20. The assumptions of Theorem 3.19(c) probably imply that τ is locally Δ-bicontinuous. We do not know how to prove this fact. However, the final result is not affected. We also do not know how to prove Theorem 3.19(a) without the assumption that Γ, Δ are $(\le\kappa(E))$-generated. □

There is a variant of translation-likeness which we shall use in the context of diffeomorphisms. Suppose that $f, g \in \mathrm{Diff}([0, 1])$. If the derivative f' of f is α-continuous and g' is β-continuous, then (i) for some $K, L > 0$, $(f \circ g)'$ is $(K \cdot \alpha + L \cdot \beta)$-continuous. Also, (ii) for some $M > 0$, $(f^{-1})'$ is $M \cdot \alpha$-continuous. (iii) A similar fact holds for higher derivatives.

Let $\Gamma \subseteq \mathrm{MC}$, and assume that $K \cdot \alpha + L \cdot \beta \in \Gamma$ for every $\alpha, \beta \in \Gamma$ and $K, L > 0$. Consider $G_\Gamma = \{f \in \mathrm{Diff}([0, 1]) \mid \text{for some } \alpha \in \Gamma,\ f' \text{ is } \alpha\text{-continuous}\}$. By (i)–(ii), G_Γ is a group, and by (iii), the analogous fact for $\mathrm{Diff}^n([0, 1])$ is also true. So Γ need not be a modulus of continuity in order for G_Γ to be a group. Let us call such a Γ a *modulus of differentiability.*

We do not deal with differentiability in this work, but we shall show here that if Γ is a modulus of differentiability and $(\mathrm{LIP}(X))^\tau \subseteq H_\Gamma^{\mathrm{LC}}(Y)$, then τ^{-1} is locally Γ-continuous. This is the analogue of Theorem 3.16, and Theorem 3.15 has an analogue too. The proofs use the additional assumptions that X is of the second category, and that Γ is countably generated. On the other hand, the infinite-closedness of G is not needed, and the assumption of decayability is replaced by a much weaker property.

Definition 3.21. Let X be a topological space, λ be a partial action of a topological group H on X and $G \le H(X)$. Let $x \in X$. We say that λ is *compatible with G at x* if there is $W \in \mathrm{Nbr}(e_H)$ such that for every $h \in W$ there are $U \in \mathrm{Nbr}(x)$ and $g \in G$ such that $U \subseteq \mathrm{Dom}(h_\lambda)$ and $h_\lambda \restriction U = g \restriction U$.

We say that λ is *compatible with G* if λ is compatible with G at every $x \in \mathrm{Fld}(\lambda)$. □

The following lemma replaces Lemma 3.11.

Lemma 3.22. *Suppose that*:

(i) *X is a metric space, $G \leq H(X)$, H is a topological group and H is of the second category, λ is a partial action of H on X, $x \in \mathrm{Fld}(\lambda)$, and λ is compatible with G at x.*
(ii) *Y is a metric space and $\tau : X \cong Y$.*
(iii) *Γ is a countably generated subset of* MC, $\mathrm{cl}_{\preceq}(\{\gamma\}) \subseteq \Gamma$ *and $K \cdot \gamma \in \Gamma$ for every $\gamma \in \Gamma$ and $K > 0$.*
(iv) *For every $g \in G$, g^τ is Γ-bicontinuous at $\tau(x)$.*

Then $Q(x)$ holds, where

$Q(x)$: *For every $W \in \mathrm{Nbr}(e_H)$ there are $T \in \mathrm{Nbr}(x)$, a nonempty open subset $V \subseteq W$ and $\gamma \in \Gamma$ such that for every $h \in V$: $T \subseteq \mathrm{Dom}(h_\lambda)$ and $(h_\lambda)^\tau \restriction \tau(T)$ is γ-bicontinuous.*

Proof. For every $h \in H$ denote h_λ by $\hat{h}$. Let $W \in \mathrm{Nbr}(e_H)$. We may assume that for every $h \in W$ there are $U_h \in \mathrm{Nbr}(x)$ and $g_h \in G$ such that $U_h \subseteq \mathrm{Dom}(\hat{h})$ and $h_\lambda \restriction U = g_h \restriction U$.

We verify that $(*)$ for every $h \in W$ there are $r_h > 0$ and $\gamma_h \in \Gamma$ such that $B(x, r_h) \subseteq \mathrm{Dom}(\hat{h})$ and $\hat{h}^\tau \restriction \tau(B(x, r_h))$ is γ_h-bicontinuous. Let U_h, g_h be as above. Then $(g_h)^\tau$ is Γ-bicontinuous at $\tau(x)$. Let $\gamma_h \in \Gamma$ and $T \in \mathrm{Nbr}(\tau(x))$ be such that $(g_h)^\tau \restriction T$ is γ_h-bicontinuous, and let $r_h > 0$ be such that $B(x, r_h) \subseteq U_h$ and $\tau(B(x, r_h)) \subseteq T$. Obviously,

$$\hat{h}^\tau \restriction \tau(B(x, r_h)) = (\hat{h} \restriction B(x, r_h))^\tau = (g_h \restriction B(x, r_h))^\tau = (g_h)^\tau \restriction \tau(B(x, r_h)).$$

So $\hat{h}^\tau \restriction \tau(B(x, r_h)$ is γ-bicontinuous. That is, $(*)$ holds.

Let $\Gamma_0 = \{\gamma_i \mid i \in \mathbb{N}\}$ be such that $\Gamma = \mathrm{cl}_{\preceq}(\Gamma_0)$, and assume that $\{j \mid \gamma_j = \gamma_i\}$ is infinite for every $i \in \mathbb{N}$. Set

$$K_i = \left\{ h \in W \;\middle|\; B\left(x, \frac{1}{i+1}\right) \subseteq \mathrm{Dom}(\hat{h}) \text{ and } \hat{h}^\tau \restriction \tau\left(B\left(x, \frac{1}{i+1}\right)\right) \text{ is } \gamma_i\text{-bicontinuous} \right\}.$$

By $(*)$, $\bigcup_{i \in \mathbb{N}} K_i = W$. We show that for every i, K_i is closed in W. Set $B_i = B(x, \frac{1}{i+1}))$. Let $h \in W - K_i$. So there are $y_1, y_2 \in \tau(B_i)$ such that (i) $d(\hat{h}^\tau(y_1), \hat{h}^\tau(y_2)) > \gamma_i(d(y_1, y_2))$ or (ii) $d(\hat{h}^\tau(y_1), \hat{h}^\tau(y_2)) < \gamma_i^{-1}(d(y_1, y_2))$. We may assume that (i) happens. For $\ell = 1, 2$ let T_ℓ be an open neighborhood of $\hat{h}^\tau(y_\ell)$ such that $d(T_1, T_2) > \gamma_i(d(y_1, y_2))$. Set $S_\ell = \tau^{-1}(T_\ell)$ and $x_\ell = \tau^{-1}(y_\ell)$. Let $V_0 = \{k \in W \mid x_1, x_2 \in \mathrm{Dom}(\hat{k}),\ \hat{k}(x_1) \in S_1 \text{ and } \hat{k}(x_2) \in S_2\}$. So V_0 is open. We show that V_0 contains h and is disjoint from K_i. Clearly, $\hat{h}(x_\ell) = \tau^{-1}(\hat{h}^\tau(y_\ell)) \in \tau^{-1}(T_\ell) = S_\ell$, hence $h \in V_0$. If $k \in V_0$, then $\hat{k}(x_\ell) \in S_\ell$ and so $\hat{k}^\tau(y_l) \in \tau(S_\ell) = T_\ell$. Hence $\hat{k}^\tau \restriction \tau(B_i)$ is not γ_i-continuous, namely, $k \notin K_i$. Since W is of the second category and every K_n is closed, there is n such that $\mathrm{int}(K_n) \neq \emptyset$. Define $V = \mathrm{int}(K_n)$, $T = B(x, \frac{1}{n+1})$ and $\gamma = \gamma_n$. Then V, T and γ are as required in the lemma. ∎

Definition 3.23. Let X be a metric space, H be a topological group λ be a partial action of H on X and $x \in \mathrm{Fld}(\lambda)$. The action λ is said to be *regionally translation-like at x* if there is $W_x \in \mathrm{Nbr}(e_H)$ such that for every nonempty open $V \subseteq W_x$ such that $V \times \{x\} \subseteq \mathrm{Dom}(\lambda)$ there are:

(i) $U = U_{x,V} \in \mathrm{Nbr}(x)$ and a dense subset of U, $D = D_{x,V}$;
(ii) a point $z = z_{x,V}$, a radius $r = r_{x,V} > 0$, and a constant $K = K_{x,V} > 0$;

such that for any distinct $\bar{x}_0, \bar{x}_1 \in U \cap D$ there are $n \le K \cdot \frac{r}{d(\bar{x}_0,\bar{x}_1)}$, a sequence $z = z_0, z_1, \dots, z_n \in X$ and $h_1, \dots, h_n \in V$ such that $z_n \notin B(z,r)$, and for every $i = 1, \dots, n$, $\bar{x}_0, \bar{x}_1 \in \mathrm{Dom}((h_i)_\lambda)$, $(h_i)_\lambda(\bar{x}_0) = z_{i-1}$ and $(h_i)_\lambda(\bar{x}_1) = z_i$.

If λ is regionally translation-like at every $x \in \mathrm{Fld}(\lambda)$, then λ is said to be a *regionally translation-like action.* $\square$

The next proposition is a counterpart of Proposition 3.13.

PROPOSITION 3.24. *Let E be a normed vector space, F be a dense linear subspace of E and X be an open subset of E. Then $\lambda_{\mathbb{T}}^{E;F} \restriction X$ is regionally translation-like.*

Proof. Write $\lambda = \lambda_{\mathbb{T}}^{E;F} \restriction X$ and define $W_x = \mathbb{T}(E;F)$. Let $V \subseteq W_x$ be open and nonempty, and suppose that $V \times \{x\} \subseteq \mathrm{Dom}(\lambda)$. Choose $v \in F$ and $s > 0$ such that $V_1 := \{\mathrm{tr}_u^E \mid u \in B^F(v,s)\} \subseteq V$ and $V_1 \times B^E(x,s) \subseteq \mathrm{Dom}(\lambda)$. Define $z_{x,V} = v + x$, $r = r_{x,V} = s/2$, $U_{x,V} = B(x, s/4)$, $D_{x,V} = U_{x,V} \cap (x + F)$ and $K_{x,V} = 2$. It is left to the reader to verify that the above satisfy the requirements of regional translation-likeness of λ at x. ■

The following lemma is a counterpart of Lemma 3.14.

LEMMA 3.25. *Let X be a metric space, and λ be a partial action of H on X. Suppose that $x \in \mathrm{Fld}(\lambda)$, and λ is regionally translation-like at x. Let Y be a metric space and $\tau : X \cong Y$. Let $\Gamma \subseteq \mathrm{MC}$, and suppose that for every $\gamma \in \Gamma$ and $K > 0$, $K \cdot \gamma \in \Gamma$. Also assume that $Q(x)$ of Lemma 3.22 holds. That is, for every $W \in \mathrm{Nbr}(e_H)$ there are $U \in \mathrm{Nbr}(x)$, a nonempty open subset $V \subseteq W$ and $\gamma \in \Gamma$ such that $U \subseteq \mathrm{Dom}(h_\lambda)$ and $(h_\lambda)^\tau \restriction \tau(U)$ is γ-bicontinuous for every $h \in V$. Then τ^{-1} is Γ-continuous at $\tau(x)$.*

Proof. Let W_x be as ensured by the regional translation-likeness of λ at x. By $Q(x)$, there are $U \in \mathrm{Nbr}(x)$, a nonempty open $V \subseteq W_x$ and $\gamma \in \Gamma$ such that for every $h \in V$: $U \subseteq \mathrm{Dom}(h_\lambda)$ and $(h_\lambda)^\tau \restriction \tau(U)$ is γ-bicontinuous. So $V \subseteq W_x$ and $V \times \{x\} \subseteq \mathrm{Dom}(\lambda)$. We apply the definition of regional translation-likeness to V. Write $S = U_{x,V}$, $D = D_{x,V}$, $z = z_{x,V}$, $r = r_{x,V}$ and $K = K_{x,V}$.

Let $w = \tau(z)$, $B = B(z,r)$ and $C = \tau(B)$. Since $C \in \mathrm{Nbr}(w)$, we conclude that $e := d(w, Y - C) > 0$. Let $R = \tau(U \cap S)$ and $M = Kr/e$. Since $\gamma \in \Gamma$, we have $M \cdot \gamma \in \Gamma$.

We show that $\tau^{-1} \restriction R$ is $M \cdot \gamma$-continuous. Suppose by contradiction that this is not true. For $h \in H$ denote h_λ by $\hat{h}$. Hence there are $\bar{y}_0, \bar{y}_1 \in R$ such that $d(\tau^{-1}(\bar{y}_0), \tau^{-1}(\bar{y}_1)) > M \cdot \gamma(d(\bar{y}_0, \bar{y}_1))$. Since D is dense in S and $\bar{y}_0, \bar{y}_1 \in \tau(S)$, we may assume that $\bar{y}_0, \bar{y}_1 \in \tau(D)$. For $\ell = 0, 1$ let $\bar{x}_\ell = \tau^{-1}(\bar{y}_\ell)$. Hence $\bar{x}_0, \bar{x}_1 \in D$. So there are $n \le \frac{Kr}{d(\bar{x}_0,\bar{x}_1)}$, $z = z_0, z_1, \dots, z_n$ and $h_1, \dots, h_n \in V$ such that $z_n \notin B$, and for every $i = 1, \dots, n$, $\bar{x}_0, \bar{x}_1 \in \mathrm{Dom}(\hat{h}_i)$, $\hat{h}_i(\bar{x}_0) = z_{i-1}$ and $\hat{h}_i(\bar{x}_1) = z_i$. For $i = 1, \dots, n$ let $w_i = \tau(z_i)$.

In the space Y we have: $w_0 = w$; for every $i = 1, \dots, n$, $\hat{h}_i^\tau(\bar{y}_0) = w_{i-1}$ and $\hat{h}_i^\tau(\bar{y}_1) = w_i$; and $w_n \notin C$. Every h_i belongs to V, hence $\hat{h}_i^\tau \restriction \tau(U)$ is γ-bicontinuous. Also, $\bar{y}_0, \bar{y}_1 \in \tau(U)$, so $d(w_{i-1}, w_i) \le \gamma(d(\bar{y}_0, \bar{y}_1))$. Hence

$$\begin{aligned} e = d(w, Y - C) &\le d(w, w_n) = d(w_0, w_n) \le \sum_{i=1}^{n} d(w_{i-1}, w_i) \le n \cdot \gamma(d(\bar{y}_0, \bar{y}_1)) \\ &\le \frac{Kr}{d(\bar{x}_0, \bar{x}_1)} \cdot \gamma(d(\bar{y}_0, \bar{y}_1)) < \frac{Kr}{M \cdot \gamma(d(\bar{y}_0, \bar{y}_1))} \cdot \gamma(d(\bar{y}_0, \bar{y}_1)) = \frac{Kr}{Kr/e} = e. \end{aligned}$$

A contradiction, so the lemma is proved. ■

THEOREM 3.26. *Assume the following facts.*

(i) *X is a metric space, $G \leq H(X)$, H is a topological group and H is of the second category, λ is a partial action of H on X and $x \in \mathrm{Fld}(\lambda)$.*
(ii) *λ is compatible with G at x.*
(iii) *λ is regionally translation-like at x.*
(iv) *Γ is a countably generated subset of* MC, $\mathrm{cl}_{\preceq}(\{\gamma\}) \subseteq \Gamma$, *and $K \cdot \gamma \in \Gamma$ for every $\gamma \in \Gamma$ and $K > 0$.*
(v) *Y is a metric space and $\tau : X \cong Y$.*
(vi) *For every $g \in G$, g^τ is Γ-bicontinuous at $\tau(x)$.*

Then τ^{-1} is Γ-continuous at $\tau(x)$.

Proof. Combine Lemmas 3.22 and 3.25. ∎

3.4. Affine-like partial actions. The goal of this part of the chapter is the following final theorem.

THEOREM 3.27. *Let $\langle E, X, \mathcal{S}, \mathcal{E}\rangle$ be a subspace choice system with* $\dim(E) > 1$, *Y be an open subset of a normed space F, Γ be a $(\leq\kappa(E))$-generated modulus of continuity and $\tau : X \cong Y$. Suppose that* $(\mathrm{LIP}(X, \mathcal{S}, \mathcal{E}))^\tau \subseteq H^{\mathrm{LC}}_{\Gamma}(Y)$. *Then τ is locally Γ-bicontinuous.*

This parallels Theorem 3.16, but has a stronger conclusion. Whereas in 3.16 the conclusion is that τ^{-1} is locally Γ-continuous, 3.27 says that τ is locally Γ-bicontinuous.

DEFINITION 3.28. (a) A subset D of a metric space X is called a *metrically dense subset* of X if for any $x, y \in X$ and $\varepsilon > 0$ there are $x_1 \in B(x, \varepsilon) \cap D$ and $y_1 \in B(y, \varepsilon) \cap D$ such that $d(x_1, y_1) = d(x, y)$. Note that metric density implies density.

(b) Let X be a metric space, H be a topological group and λ be a partial action of H on X. For $h \in H$ denote h_λ by $\hat{h}$. Let $x \in X$. We say that λ is an *affine-like partial action at x* if the following holds. For every $V \in \mathrm{Nbr}(e_H)$ and $U \in \mathrm{Nbr}(x)$ there are $n = n(x, V, U) \in \mathbb{N}$, $U_0 = U_0(x, V, U) \in \mathrm{Nbr}(x)$ and $D = D(x, V, U) \subseteq U_0$ such that $U_0 \subseteq U$, D is metrically dense in U_0, and for every $x_1, y_1, x_2, y_2 \in D$: if $d(x_1, y_1) = d(x_2, y_2)$, then there are $h_1, \ldots, h_n \in V$ such that $\hat{h}_1 \circ \cdots \circ \hat{h}_n(x_1) = x_2$, $\hat{h}_1 \circ \cdots \circ \hat{h}_n(y_1) = y_2$ and $\hat{h}_i \circ \hat{h}_{i+1} \circ \cdots \circ \hat{h}_n(\{x_1, y_1\}) \subseteq U$ for every $1 \leq i \leq n$.

If λ is affine-like at every $x \in \mathrm{Fld}(\lambda)$, then λ is said to be an *affine-like partial action.*

(c) If H is a group, $A \subseteq H$ and $n \in \mathbb{N}$, then $A^n = \{a_1 \cdots a_n \mid a_1, \ldots, a_n \in A\}$. Let λ be a partial action of a topological group H on a topological space X. If $h \in H$ then h_λ is denoted by $\hat{h}$. For $U \subseteq H$ and $W_1, W_2 \subseteq X$ define

$$U^{[n;W_1,W_2]} = \{h_1 \cdots h_n \mid h_1, \ldots, h_n \in U,\ W_1 \subseteq \mathrm{Dom}(\hat{h}_i \circ \cdots \circ \hat{h}_n) \text{ and } \hat{h}_i \circ \cdots \circ \hat{h}_n(W_1) \subseteq W_2 \text{ for every } i = 1, \ldots, n\}. \ \square$$

We shall prove two intermediate main facts. They roughly say the following.

(a) If X is an open subset of a normed space E, and F is a dense linear subspace of E, then $\lambda^{E;F}_{\mathbb{A}} \restriction X$ is affine-like.
(b) Suppose that λ is a decayable affine-like partial action of H on X, $\tau : X \cong Y$, Γ is a countably generated modulus of continuity, and $(h_\lambda)^\tau$ is locally Γ-bicontinuous for every $h \in H$. Then τ is locally Γ-bicontinuous.

We start with the proof of (a). When proving the affine-likeness of $\lambda_{\mathbb{A}}^{E;F}\restriction X$ at x, it is easier to deal first with x's which belong to $F \cap X$. To conclude that $\lambda_{\mathbb{A}}^{E;F}\restriction X$ is affine-like at every $x \in X$, we use the observation that if λ is affine-like at every $x \in C$, and $U_0(x, V, U)$ and $n(x, V, U)$ depend on $x \in C$ and V in some uniform way, then λ is affine-like at every $x \in \mathrm{cl}(C)$.

PROPOSITION 3.29. *Assume the following facts.*

(i) *X is a metric space, λ is a partial action of H on X, $C \subseteq \mathrm{Fld}(\lambda)$, $r_0 > 0$, $\iota : \mathrm{Nbr}(e_H) \times C \to \mathrm{Nbr}(e_H)$, $\bar{n} : \mathrm{Nbr}(e_H) \times (0, r_0) \to \mathbb{N}$ and $\bar{s} : \mathrm{Nbr}(e_H) \times (0, r_0) \to (0, \infty)$. Denote $\iota(V, y)$ by V_y.*

(ii) *For every $y \in C$, λ is affine-like at y, and for every $V \in \mathrm{Nbr}(e_H)$ and $r \in (0, r_0)$, $n(y, V_y, B(y, r)) \le \bar{n}(V, r)$ and $U_0(y, V_y, B(y, r)) \supseteq B(y, \bar{s}(V, r))$.*

(iii) *For every $x \in \mathrm{cl}(C)$ and $W \in \mathrm{Nbr}(e_H)$ there are $U_1 \in \mathrm{Nbr}(x)$ and $V \in \mathrm{Nbr}(e_H)$ such that for every $y \in C \cap U_1$, $V_y \subseteq W$.*

Then for every $x \in \mathrm{cl}(C)$, λ is affine-like at x. Also, if $r < r_0$, then $n(x, V, B(x, r))$ and $U_0(x, V, B(x, r))$ can be taken to be $\bar{n}(V, r/2)$ and $B(x, \frac{1}{2}\bar{s}(V, r/2))$ respectively.

Proof. Let $x \in \mathrm{cl}(C)$, $W \in \mathrm{Nbr}(e_H)$, $r \in (0, r_0)$ and $U = B(x, r)$. There is $V \in \mathrm{Nbr}(e_H)$ and $U_1 \in \mathrm{Nbr}(x)$ such that for every $y \in U_1 \cap C$, $V_y \subseteq W$. Define $U_0 = U_0(x, W, U)$ to be $B(x, \frac{1}{2}\bar{s}(V, r/2))$. Let $y \in C \cap U_0 \cap U_1$. Then $U_0 \subseteq B(y, \bar{s}(V, r/2)) \subseteq U_0(y, V_y, B(y, r/2))$. Hence $D(y, V_y, B(y, r/2)) \cap U_0$ is metrically dense in U_0. Let $D = D(x, W, V) = D(y, V_y, B(y, r/2)) \cap U_0$ and $n = n(x, W, U) = \bar{n}(V, r/2)$. We show that U_0, D and n fullfill the requirements of affine-likeness.

Let $x_1, x_2, y_1, y_2 \in D$ be such that $d(x_1, y_1) = d(x_2, y_2)$. Let $h_1, \dots, h_n \in V_y$ be as ensured by the affine-likeness of λ at y. So for every $i = 1, \dots, n$, $\hat{h}_i \circ \dots \circ \hat{h}_n(\{x_1, y_1\}) \subseteq B(y, r/2)$. Clearly, $\bar{s}(V, r/2) \le r/2$ and $d(x, y) < \bar{s}(V, r/2)$. So $B(y, r/2) \subseteq B(x, r) = U$. Since $y \in U_1$, $V_y \subseteq W$. So $h_1, \dots, h_n$ fulfill the requirements needed to demonstrate that λ is affine-like at x. ∎

If X is an open subset of $\mathbb{R}$, then $\mathbb{A}(\mathbb{R})\restriction X$ is not affine-like. So in what follows we assume that $\dim(E) > 1$.

The group $\mathbb{L}(E)$ has a property similar to affine-likeness. But the "affine-likeness" of $\mathbb{L}(E)$ applies only to pairs of pairs x_1, y_1, x_2, y_2 in which $x_1 = x_2 = 0^E$.

LEMMA 3.30. *Let E be a normed space with dimension > 1, E_1 be a dense linear subspace of E and $V \in \mathrm{Nbr}^{\mathbb{L}(E;E_1)}(\mathrm{Id})$. Then there is $n = n(V) \in \mathbb{N}$ with the following property:* $(*)$ *For every $W_1 \in \mathrm{Nbr}^E(0)$ there is $W_2 \in \mathrm{Nbr}^E(0)$ such that $W_2 \subseteq W_1$ and for every $x_1, x_2 \in W_2 \cap E_1$: if $\|x_1\| = \|x_2\|$, then there is $S \in V^{[n; W_2, W_1]}$ such that $S(x_1) = x_2$.*

Moreover, if in the above $V = B^{\mathbb{L}(E;E_1)}(\mathrm{Id}, r)$ and $W_1 = B^E(0, s)$, then W_2 can be taken to be $B^E(0, s/(1 + r)^{n(V)})$.

Proof. The proof of the lemma relies on three easy claims.

CLAIM 1. *Let $\boldsymbol{H}^2$ be the 2-dimensional Hilbert space. For every $K \ge 1$ and $V \in \mathrm{Nbr}^{\mathbb{L}(\boldsymbol{H}^2)}(\mathrm{Id})$ there is $n = n(V, K) \in \mathbb{N}$ such that for every $x_1, x_2 \in \boldsymbol{H}^2$: if $1/K \le \|x_1\|/\|x_2\| \le K$, then there is $T \in V^n$ such that $T(x_1) = x_2$.*

Proof. We may assume that $V = V^{-1}$. For some angle $\gamma_0 > 0$, U contains all rotations Rot_γ, $\gamma \in [0, \gamma_0]$. For some $\varepsilon_0 > 0$, U contains all isomorphisms $T_\varepsilon(x) = (1+\varepsilon)x$ where $\varepsilon \in [0, \varepsilon_0]$. It is left to the reader to verify that $n(U, K) = [\pi/\gamma_0] + \log K/\log(\varepsilon_0 + 1) + 2$ is as required.

We do not prove Claim 2 which is well-known and easy. In fact, the best possible constant in Claim 2 is $\sqrt{2}$.

CLAIM 2. *For every 2-dimensional normed space E there is an isomorphism T between E and the 2-dimensional Hilbert space $\boldsymbol{H}^2$ such that $\|T\| \le 1$ and $\|T^{-1}\| \le 3\sqrt{2}$.*

CLAIM 3. *Let E be a normed space, E_1 be a dense linear subspace of E, F be a 2-dimensional linear subspace of E_1 and $T \in \mathbb{L}(F)$, then there is $T_1 \in \mathbb{L}(E; E_1)$ extending T such that $d(T_1, \mathrm{Id}_E) \le 3d(T, \mathrm{Id}_F)$.*

Proof. Let x_1, x_2 be a basis for F such that $\|x_1\| = d(x_1, \mathrm{span}(\{x_2\}))$. For $i = 1, 2$ let $\varphi_1, \varphi_2 \in F^*$ be such that $\varphi_i(x_j) = \delta_{i,j} \cdot \|x_j\|$, and let $\psi_i \in E^*$ be such that ψ_i extends φ_i and $\|\psi_i\| = \|\varphi_i\|$. Set $F_1 = \bigcap_{i=1}^2 \ker(\varphi_i)$, hence $F \oplus F_1 = E$. For $x \in E$ let $\hat{x} \in F$ and $\bar{x} \in F_1$ denote the components of x in F and F_1 respectively. If $\hat{x} = ax_1 + bx_2$, denote ax_1 and bx_2 by $\hat{x}^1$ and $\hat{x}^2$ respectively. Let $x \in F$. Then $|\varphi_1(\hat{x})| = \|\hat{x}^1\| = d(\hat{x}, \mathrm{span}(\{x_2\})) \le \|\hat{x}\|$. So $\|\varphi_1\| \le 1$. Hence $\|\psi_1\| \le 1$. It follows that $\|\hat{x}^1\| = |\psi_1(x)| \le \|x\|$. Also, $\|\hat{x}^2\| \le \|\hat{x}\| + \|\hat{x}^1\| \le 2\|\hat{x}\|$. Hence $|\varphi_2(\hat{x})| = \|\hat{x}^2\| \le 2\|\hat{x}\|$. So $\|\psi_2\| = \|\varphi_2\| \le 2$. Hence $\|\hat{x}^2\| = |\psi_2(x)| \le 2\|x\|$. So $\|\hat{x}\| \le \|\hat{x}^1\| + \|\hat{x}^2\| \le 3\|x\|$.

Let T_1 be defined by $T_1(x) = T(\hat{x}) + \bar{x}$. Hence $T_1^{-1}(x) = T^{-1}(\hat{x}) + \bar{x}$. Then for every $x \in E$, $\|(T_1 - \mathrm{Id}_E)(x)\| = \|(T - \mathrm{Id}_F)(\hat{x})\| \le \|T - \mathrm{Id}_F\| \cdot \|\hat{x}\| \le 3\|T - \mathrm{Id}_F\| \cdot \|x\|$. That is, $\|T_1 - \mathrm{Id}_E\| \le 3\|T - \mathrm{Id}_F\|$. A similar computation shows that $\|T_1^{-1} - \mathrm{Id}_E\| \le 3\|T^{-1} - \mathrm{Id}_F\|$. So $d(T_1, \mathrm{Id}_E) \le 3d(T, \mathrm{Id}_F)$.

Also for every $x \in E$, $T_1(x) - x \in F \subseteq E_1$. So $T_1(E_1) = E_1$, that is, $T_1 \in \mathbb{L}(E; E_1)$. This proves Claim 3.

We return to the proof of the lemma. Let $V \in \mathrm{Nbr}^{\mathbb{L}(E;E_1)}(\mathrm{Id})$. We may assume that $V = B^{\mathbb{L}(E;E_1)}(\mathrm{Id}_E, r)$. Let $n = n(B^{\mathbb{L}(\boldsymbol{H}^2)}(\mathrm{Id}_{\boldsymbol{H}^2}, r/9\sqrt{2}), 3\sqrt{2})$ be as ensured by Claim 1.

Let $x_1, x_2 \in E_1$ be such that $\|x_1\| = \|x_2\| \ne 0$. We show that there is $S \in V^n$ such that $S(x_1) = x_2$. Let F be a 2-dimensional subspace of E_1 containing x_1 and x_2, and $T : F \to \boldsymbol{H}^2$ be as ensured by Claim 2. Since $\|T\| \le 1$ and $\|T^{-1}\| \le 3\sqrt{2}$, it follows that $1/3\sqrt{2} \le \|T(x_1)\|/\|T(x_2)\| \le 3\sqrt{2}$. Hence there is $S_0 \in (B^{\mathbb{L}(\boldsymbol{H}^2)}(\mathrm{Id}_{\boldsymbol{H}^2}, r/9\sqrt{2}))^n$ such that $S_0(T(x_1)) = T(x_2)$. Let $S_0 = S_{0,1} \circ \cdots \circ S_{0,n}$, where $S_{0,i} \in B^{\mathbb{L}(\boldsymbol{H}^2)}(\mathrm{Id}_{\boldsymbol{H}^2}, r/9\sqrt{2})$, and define $S_1 = T^{-1}S_0T$ and $S_{1,i} = T^{-1}S_{0,i}T$. Then $S_1(x_1) = x_2$ and $S_1 = S_{1,1} \circ \cdots \circ S_{1,n}$. Clearly, $S_{1,i} - \mathrm{Id}_F = T^{-1}(S_{0,i} - \mathrm{Id}_{\boldsymbol{H}^2})T$, and hence

$$\|S_{1,i} - \mathrm{Id}_F\| \le \|T^{-1}\| \cdot \|(S_{0,i} - \mathrm{Id}_{\boldsymbol{H}^2})\| \cdot \|T\| \le 3\sqrt{2} \cdot \|(S_{0,i} - \mathrm{Id}_{\boldsymbol{H}^2})\|.$$

The same inequality holds for $(S_{1,i})^{-1}$. So

$$\begin{aligned} d(\mathrm{Id}_F, S_{1,i}) &= \|S_{1,i} - \mathrm{Id}_F\| + \|(S_{1,i})^{-1} - \mathrm{Id}_F\| \\ &\le 3\sqrt{2} \cdot \|(S_{0,i} - \mathrm{Id}_{\boldsymbol{H}^2})\| + 3\sqrt{2} \cdot \|((S_{0,i})^{-1} - \mathrm{Id}_{\boldsymbol{H}^2})\| \\ &= 3\sqrt{2} \cdot d(S_{0,i}, \mathrm{Id}_{\boldsymbol{H}^2}) < r/3. \end{aligned}$$

By Claim 3, there are $S_{2,i} \in \mathbb{L}(E; E_1)$ extending $S_{1,i}$ such that $d(\mathrm{Id}_E, S_{2,i}) \le 3 \cdot d(\mathrm{Id}_F, S_{1,i})$. Hence $S_{2,i} \in B^{\mathbb{L}(E;E_1)}(\mathrm{Id}_E, r)$, and so $S := S_{2,1} \circ \cdots \circ S_{2,n} \in (B^{\mathbb{L}(E;E_1)}(\mathrm{Id}_E, r))^n = V^n$.

Let $W_1 \in \mathrm{Nbr}^E(0)$, and suppose that $W_1 \supseteq B^E(0, s)$. Set $W_2 = B^E(0, s/(1+r)^n)$. For any $L \in V$, $\|L\| < 1 + r$, hence for every $i \le n$ and $L' \in V^i$, $\|L'\| < (1+r)^i$. So $L'(W_2) \subseteq W_1$ for every $i \le n$ and $L' \in V^i$. This proves that n fulfills the requirements of the lemma. ■

The following lemma is analogous to Proposition 3.13.

LEMMA 3.31. *Let E be a normed space with dimension > 1, F be a dense linear subspace of E and $X \subseteq E$ be open. Then $\lambda_{\mathbb{A}}^{E;F} \restriction X$ is an affine-like partial action.*

Proof. First we show that for every $x \in X \cap F$, $\lambda_{\mathbb{A}}^{E;F} \restriction X$ is affine-like at x. Let $Y = X - x$. The function χ from $\mathbb{A}(E; F) \cup X$ to $\mathbb{A}(E; F) \cup Y$ defined by: $\chi(u) = u - x$, $x \in X$; and $\chi(h) = h^{\mathrm{tr}_{-x}}$, $h \in \mathbb{A}(E; F)$, is an isomorphism between the partial actions $\lambda_{\mathbb{A}}^{E;F} \restriction X$ and $\lambda_{\mathbb{A}}^{E;F} \restriction Y$. Also, $\chi \restriction X$ is an isometry. So it suffices to prove that $\lambda_{\mathbb{A}}^{E;F} \restriction Y$ is affine-like at 0^E. We rename Y and call it X.

Denote $\mathbb{A}(E; F)$ by $\mathbb{A}$, $\mathbb{T}(E; F)$ by $\mathbb{T}$ and $\mathbb{L}(E; F)$ by $\mathbb{L}$. Let $r, s > 0$, $V = B^{\mathbb{A}}(\mathrm{Id}, r)$, $U = B^E(0, s)$, and assume that $U \subseteq X$. We shall find $n = n(0^E, V, U)$, $U_0 = U_0(0^E, V, U)$ and $D = D(0^E, V, U)$ which demonstrate that $\mathbb{A}$ is affine-like at 0^E. Let $m = n(B^{\mathbb{L}}(\mathrm{Id}, r))$ be as ensured by Lemma 3.30. Define $t = \min(r, s)/2$, $W_1 = B^E(0, t)$ and $W_2 = B^E(0, t/(1+r)^m)$, and set $n = m + 2$, $U_0 = \frac{1}{2} W_2$ and $D = U_0 \cap F$.

It is obvious that D is metrically dense in U_0. Let $x_1, y_1, x_2, y_2 \in D$ be such that $\|x_1 - y_1\| = \|x_2 - y_2\|$. For $\ell = 1, 2$ let $g_\ell = \mathrm{tr}^E_{-x_\ell}$. Since $\|x_1\|, \|x_2\| < \frac{r}{2}$, it follows that $g_1, g_2 \in B^{\mathbb{T}}(\mathrm{Id}, r)$. Clearly, $g_\ell(x_\ell) = 0$, and since $x_\ell, y_\ell \in U_0 = \frac{1}{2} W_2$, it follows that $g_\ell(y_\ell) \in W_2$. By Lemma 3.30, there are $h_1, \ldots, h_m \in B^{\mathbb{L}}(\mathrm{Id}, r)$ such that $h_1 \circ \cdots \circ h_m(g_1(y_1)) = g_2(y_2)$ and for every $i = 1, \ldots, m$, $h_i \circ \cdots \circ h_m(g_1(y_1)) \in W_1$. It follows that $g_2^{-1}, h_1, \ldots, h_m, g_1$ are as required in the definition of affine-likeness.

To show that $\mathbb{A}$ is affine-like at points that do not belong to F we shall apply Proposition 3.29. Let $x \in X$. Choose $r_0 > 0$ such that $B(x, 2r_0) \subseteq X$ and set $C = B(x, r_0) \cap F$. By the preceding argument, $\mathbb{A}$ is affine-like at every $y \in C$. For $y \in C$ and $V \in \mathrm{Nbr}^{\mathbb{A}}(\mathrm{Id})$ we define $V_y = V^{\mathrm{tr}_y}$.

We next define functions $\bar{n} : \mathrm{Nbr}^{\mathbb{A}}(\mathrm{Id}) \times (0, r_0) \to \mathbb{N}$ and $\bar{s} : \mathrm{Nbr}^{\mathbb{A}}(\mathrm{Id}) \times (0, r_0) \to (0, \infty)$ as needed in 3.29. Let $V = B^{\mathbb{A}}(\mathrm{Id}, r)$ and $s \in (0, r_0)$. Set $m = n(B^{\mathbb{L}}(\mathrm{Id}, r))$, where $n(B^{\mathbb{L}}(\mathrm{Id}, r))$ is as ensured by Lemma 3.30. Define $\bar{n}(V, s) = m + 2$, set $t = \min(r, s)/2$ and define $\bar{s}(V, s) = t/2(1+r)^m$. It was proved in the preceding argument that

$$n(0^E, V, B(0^E, s)) = \bar{n}(V, s) \quad \text{and} \quad U_0(0^E, V, B(0^E, s)) = B(0^E, \bar{s}(V, s)).$$

Since tr_y defines an isomorphism of partial actions, which is an isometry on X, and since tr_y takes 0^E to y, it can be concluded that

$$n(y, V^{\mathrm{tr}_y}, B^E(y, s)) = \bar{n}(V, s) \quad \text{and} \quad U_0(y, V^{\mathrm{tr}_y}, B^E(y, s)) = B(y, \bar{s}(V, s)).$$

We have shown that clauses (i) and (ii) of Proposition 3.29 hold.

Recall that $x \in X$, $B(x, 2r_0) \subseteq X$ and $C = B(x, r_0) \cap F$. Let $r > 0$ and $W = B^{\mathbb{A}}(\mathrm{Id}, r)$. We shall find U_1 and V as required in clause (iii) of 3.29. Let $\overline{\mathbb{A}} = \mathbb{T}(E) \cdot \mathbb{L}(E; F)$. Clearly, $\overline{\mathbb{A}} \le \mathbb{A}(E)$. Also, $\mathbb{A}$ is dense in $\overline{\mathbb{A}}$. Let $\overline{W} = B^{\overline{\mathbb{A}}}(\mathrm{Id}, r)$, $g = \mathrm{tr}_x$ and $\overline{V}_1 = \overline{W}^{g^{-1}}$. Note that

$W = \overline{W} \cap \mathbb{A}$. Let $t > 0$ be such that $(B^{\bar{\mathbb{A}}}(\mathrm{Id}, t))^3 \subseteq \overline{V}_1$ and set $\overline{V} = B^{\bar{\mathbb{A}}}(\mathrm{Id}, t)$. Define

$$V = B^{\mathbb{A}}(\mathrm{Id}, t) \quad \text{and} \quad U_1 = x + B^E(0, t).$$

Let $y \in U_1$. Then $\mathrm{tr}_y \in g \cdot \overline{V}$ and so

$$(\overline{V})^{\mathrm{tr}_y} \subseteq g \cdot \overline{V} \cdot \overline{V} \cdot (\overline{V})^{-1} \cdot g^{-1} = g \cdot (\overline{V})^3 \cdot g^{-1} \subseteq (\overline{V}_1)^g = \overline{W}.$$

That is, $(\overline{V})^{\mathrm{tr}_y} \subseteq \overline{W}$. If $y \in F$, then $V^{\mathrm{tr}_y} \subseteq \mathbb{A}$. In particular, if $y \in U_1 \cap F$, then $V^{\mathrm{tr}_y} \subseteq \overline{W} \cap \mathbb{A} = W$. This implies that clause (iii) of Proposition 3.29 holds. By Proposition 3.29, $\mathbb{A}$ is affine-like at x. ■

DEFINITION 3.32. (a) Let X be a metric space and $x \in X$. We say that X has the *discrete path property at x (X is DPT at x)* if the following holds. There is $U \in \mathrm{Nbr}(x)$ and $K \geq 1$ such that $(*)$ for every $y, z \in U$ and $d \in (0, d(y, z))$ there are $n \in \mathbb{N}$ and $u_0, \ldots, u_n \in X$ such that $n \leq K \cdot d(y, z)/d$, $d(y, u_0)$, $d(u_n, z) < d$ and $d(u_{i-1}, u_i) = d$ for every $i = 1, \ldots, n$.

If X is DPT at every $x \in X$, then X is called a *DPT space*.

(b) Let X be a metric space and $x \in X$. X has *connectivity property 1 at x (X is CP1 at x)* if for every $r > 0$ there is $r^* \in (0, r)$ such that for every $x' \in X$ and $r' > 0$: if $B(x', r') \subseteq B(x, r^*)$ and C is a connected component of $B(x, r) - B(x', r')$, then $C \cap (B(x, r) - B(x, r^*)) \neq \emptyset$.

If X is CP1 at every $x \in X$, then X is called a *CP1 space.* □

PROPOSITION 3.33. *Let X be an open subset of a normed space E. Then X is DPT and CP1.*

Proof. Let $x \in X$ and $s > 0$ be such that $B^E(x, s) \subseteq X$. First we show that X is DPT at x. Let $y, z \in B^E(x, s)$ and $d \in (0, \|z - y\|)$. The points $u_i = y + i \cdot d(z - y)/\|z - y\|$, $i = 0, \ldots, [\|z - y\|/d]$ demonstrate the DPT-ness at x. So $K = 1$.

Let $r > 0$. Take r^* to be any member of $(0, \min(r, s))$. Let x' and $r' < r^*$ be such that $B^E(x', r') \subseteq B^E(x, r^*)$. It is trivial that $B^E(x, s) - B^E(x', r')$ is connected. So there is only one component C of $B(x, r) - B^E(x', r')$ which intersects $B^E(x, s)$, and C contains $B^E(x, s) - B^E(x', r')$. So C intersects $B(x, r) - B^E(x^*, r)$. Trivially, any connected component of $B(x, r) - B^E(x', r')$ which is disjoint from $B^E(x, s)$ intersects $B(x, r) - B^E(x^*, r)$. ■

Suppose that X is an open subset of a normed space E, $G \leq H(X)$, $\tau : X \cong Y$ and $G^\tau \subseteq H_\Gamma^{\mathrm{LC}}(Y)$. Loosely speaking we shall prove that if (†) $\mathbb{A}(E){\restriction}X \subseteq G$, then τ is locally Γ-bicontinuous. Obviously, (†) is flawed because $\mathbb{A}(E){\restriction}X$ is not a set of homeomorphisms of X, and hence not a subset of G. The correct statement which replaces (†) has the assumption that $\lambda_{\mathbb{A}}^E$ is compatible with G. We do not know if this assumption suffices unless E is a normed space of the second category, or in particular, a Banach space. Instead we assume that $\lambda_{\mathbb{A}}^E$ is G-decayable, and that G is infinitely closed. These assumptions work for every normed space E.

The following remains open.

QUESTION 3.34. Let E, F be normed spaces of the first category, $\tau : E \cong F$ and Γ be a countably generated modulus of continuity. Suppose that $\mathbb{A}(E)^{\tau} \subseteq H^{\mathrm{LC}}_{\Gamma}(F)$. Are τ or τ^{-1} or both locally Γ-continuous? $\square$

The core fact that leads to the final result of 3.27 is stated in the following theorem.

THEOREM 3.35. *Assume the following facts.*

(i) *X and Y are metric spaces, $x \in X$ and $\tau : X \cong Y$. Also, X is DPT at x, and Y is DPT and CP1 at $\tau(x)$.*

(ii) *$G \leq H(X)$, λ is a partial action of a topological group H on X, $\alpha \in$ MBC, $x \in \mathrm{Fld}(\lambda)$, x is a λ-limit-point, G is α-infinitely-closed at x and for some $N \in \mathrm{Nbr}(x)$, λ is (α, G)-decayable in $H_{\lambda}(x) \cap N$.*

(iii) *Γ is a modulus of continuity, and Γ is $(\leq\kappa)$-generated, where $\kappa = \min(\{\kappa(x, V_{\lambda}(x)) \mid V \in \mathrm{Nbr}(e_H)\})$.*

(iv) *There is $U \in \mathrm{Nbr}(x)$ such that for every $g \in G\lfloor U \rfloor$: if g is $\alpha \circ \alpha$-bicontinuous, then g^{τ} is Γ-bicontinuous at $\tau(x)$.*

Then τ is Γ-bicontinuous at x.

We next introduce the notion of almost Γ-continuity. The proof of Theorem 3.35 is broken into two claims. The first one, Lemma 3.37(b), says that if G fulfills assumptions (i)–(iv) of 3.35 and $G^{\tau} \subseteq H^{\mathrm{LC}}_{\Gamma}(Y)$, then τ is locally almost Γ-continuous. This part of the proof does not use the DPT-ness or the CP1-ness of X or Y. The second claim is stated in Theorem 3.40. It says that if X and Y are DPT and PC1 metric spaces, and $\tau : X \cong Y$ is locally almost Γ-continuous, then τ is locally Γ-bicontinuous.

DEFINITION 3.36. (a) Let X, Y be metric spaces, $\alpha \in$ MC, Γ be a modulus of continuity and $f : X \to Y$. We say that f is *almost α-continuous* if f is continuous, and for every $x_1, y_1, x_2, y_2 \in X$: if $d(x_1, y_1) = d(x_2, y_2)$, then $d(f(x_2), f(y_2)) \leq \alpha(d(f(x_1), f(y_1)))$. The notion f is *almost α-continuous at x* means that there is $U \in \mathrm{Nbr}(x)$ such that $f \restriction U$ is almost α-continuous. We say that f is *almost Γ-continuous at x* if for some $\gamma \in \Gamma$, f is almost γ-continuous at x, and f is said to be *locally almost Γ-continuous* if for every $x \in X$, f is almost Γ-continuous at x.

(b) If $g : A \to A$, then $g^{\circ n}$ denotes $\overbrace{g \circ \cdots \circ g}^{n}$. $\square$

The following lemma has also a variant in which H is assumed to be of the second category, but decayability is replaced by compatibility, and infinite-closedness is dropped.

LEMMA 3.37.

(a) *Suppose that the following facts hold.*

(i) *X and Y are metric spaces, $x \in X$ and $\tau : X \cong Y$.*

(ii) *λ is a partial action of a topological group H on X, $x \in \mathrm{Fld}(\lambda)$ and λ is affine-like at x.*

(iii) *Γ is a modulus of continuity and $\gamma \in \Gamma$.*

(iv) *$T \in \mathrm{Nbr}(x)$, $V \in \mathrm{Nbr}(e_H)$, $V \times T \subseteq \mathrm{Dom}(\lambda)$ and for every $h \in V$, $(h_{\lambda})^{\tau} \restriction \tau(T)$ is γ-bicontinuous.*

Then τ is almost Γ-continuous at x.

(b) *Suppose that the following facts hold.*

(i) X *and* Y *are metric spaces*, $x \in X$ *and* $\tau : X \cong Y$.

(ii) $G \leq H(X)$, λ *is a partial action of a topological group* H *on* X *and* $\alpha \in \mathrm{MBC}$. *Also*, $x \in \mathrm{Fld}(\lambda)$, x *is a* λ*-limit-point*, G *is* α*-infinitely-closed at* x, *for some* $N \in \mathrm{Nbr}(x)$, λ *is* (α, G)*-decayable in* $H_\lambda(x) \cap N$, *and* λ *is affine-like at* x.

(iii) Γ *is a* $(\leq\kappa)$*-generated modulus of continuity*, *where* $\kappa = \min(\{\kappa(x, V_\lambda(x)) \mid V \in \mathrm{Nbr}(e_H)\})$.

(iv) *There is* $U \in \mathrm{Nbr}(x)$ *such that for every* $g \in G\lfloor U \rfloor$: *if* g *is* $\alpha \circ \alpha$*-bicontinuous*, *then* g^τ *is* Γ*-bicontinuous at* $\tau(x)$.

Then τ *is almost* Γ*-continuous at* x.

Proof. (a) Let $n = n(x, V, T)$, $U_0 = U_0(x, V, T)$ and $D = D(x, V, T)$ be as ensured by the definition of affine-likeness (Definition 3.28(a)). For $h \in H$ denote h_λ by $\hat{h}$. Set $\beta = \gamma^{\circ n}$, so $\beta \in \Gamma$. Suppose that $x_1, y_1, x_2, y_2 \in D$ and $d(x_1, y_1) = d(x_2, y_2)$. Choose $h_1, \dots, h_n \in V$ as ensured by the definition affine-likeness, and define $h = \circ_{i=1}^n h_i$. So $\hat{h}(x_1) = x_2$, $\hat{h}(y_1) = y_2$ and $\hat{h}_i \circ \dots \circ \hat{h}_n(\{x_1, x_2\}) \subseteq T$ for every $i = 1, \dots, n$. Also, for every $i = 1, \dots, n$, $(\hat{h}_i)^\tau \restriction \tau(T)$ is γ-continuous. Hence $d(\tau(x_2), \tau(y_2)) = d((\tau(x_1))^{\hat{h}}, (\tau(y_2))^{\hat{h}})$ $\leq \beta(d(\tau(x_2), \tau(y_2)))$. We have shown that $\tau \restriction D$ is almost β-continuous. Relying on the fact that D is metrically dense in U_0 we conclude that $\tau \restriction U_0$ is almost β-continuous. So τ is almost Γ-continuous at x.

(b) By Lemma 3.11, there are $T \in \mathrm{Nbr}(x)$, $V \in \mathrm{Nbr}(e_H)$ and $\gamma \in \Gamma$ such that for every $h \in V$: $T \subseteq \mathrm{Dom}(h_\lambda)$ and $(h_\lambda)^\tau \restriction \tau(T)$ is γ-bicontinuous. By part (a), τ is almost Γ-continuous at x. ■

The next two propositions are ingredients in the proof of Theorem 3.40.

PROPOSITION 3.38. *Let* x *belong to a metric space* X, *and suppose that* X *is DPT at* x, *that* K *and* U *satisfy condition* $(*)$ *of Definition* 3.32(a) *and that* $W \in \mathrm{Nbr}(x)$. *Then there is* $T \in \mathrm{Nbr}(x)$ *such that*: $(**)$ $T \subseteq W$, *and for every* $y, z \in T$ *and* $d \in (0, d(y, z))$ *there are* $n \in \mathbb{N}$ *and* $u_0, \dots, u_n \in W$ *such that* $n \leq K \cdot d(x, y)/d$, $d(x, u_0)$, $d(u_n, y) < d$, *and* $d(u_i, u_{i+1}) = d$ *for every* $i = 0, \dots, n-1$.

Proof. Let $s > 0$ be such that $B(x, (2K+3)s) \subseteq U \cap W$. We show that $T := B(x, s)$ is as required. Let $y, z \in T$ and $d \in (0, d(y, z))$. Let $n \in \mathbb{N}$ and $u_0, \dots, u_n$ be as ensured in $(*)$ of 3.32(a). Then for every $i = 1, \dots, n$,

$$\begin{aligned} d(u_i, x) &\leq d(u_i, u_0) + d(u_0, y) + d(y, x) < id + d + s \leq nd + d + s \\ &\leq Kd(x, y) + 2s + s < K \cdot 2s + 2s + s < (2K+3)s. \end{aligned}$$

So $u_i \in W$. ■

PROPOSITION 3.39. *Let* X, Y *be metric spaces and* $\tau : X \cong Y$. *Suppose that* $x \in X$, τ *is almost* α*-continuous at* x, *and* Y *is CP1 at* $\tau(x)$. *Then there is* $U \in \mathrm{Nbr}(x)$ *such that for all* $x_1, y_1, x_2, y_2 \in U$: *if* $d(x_2, y_2) \leq d(x_1, y_1)$, *then* $d(\tau(x_2), \tau(y_2)) \leq \alpha(d(\tau(x_1), \tau(y_1)))$.

Proof. Let $T \in \mathrm{Nbr}(x)$ be such that $\tau \restriction T$ is almost α-continuous, and $s > 0$ be such that $B(\tau(x), s) \subseteq \tau(T)$. Choose $s^* \in (0, s)$ such that for every $y \in Y$ and $t > 0$: if

$B(y,t) \subseteq B(\tau(x), s^*)$, then every connected component of $B(\tau(x), s) - B(y,t)$ intersects $B(\tau(x), s) - B(\tau(x), s^*)$. Let $r^* > 0$ be such that

(i) $\tau(B(x, r^*)) \subseteq B(\tau(x), s^*)$,

and let $r \in (0, r^*/3)$ be such that $U := B(x,r)$ satisfies the following condition:

(ii) $\mathrm{diam}(\tau(U)) + \alpha(\mathrm{diam}(\tau(U))) < s^*$.

We show that U is as required. Let $x_1, y_1, x_2, y_2 \in U$ and $d(x_2, y_2) \le d(x_1, y_1)$. If $d(x_2, y_2) = d(x_1, y_1)$, then by the choice of T, and since $U \subseteq T$, $d(\tau(x_2), \tau(y_2)) \le \alpha(d(\tau(x_1), \tau(y_1)))$. Suppose next that $d(x_2, y_2) < d(x_1, y_1)$. Let $r_1 = d(x_1, y_1)$, and set $s_1 = \alpha(d(\tau(x_1), \tau(y_1)))$. By the almost α-continuity of $\tau \restriction T$,

(iii) $\tau(S(x_2, r_1)) \subseteq B(\tau(x_2), s_1 + \varepsilon)$ for every $\varepsilon > 0$.

Since $r < r^*/3$, $d(x_2, x) < r$ and $r_1 = d(x_1, y_1) < 2r$, we have

(iv) $B(x_2, r_1) \subseteq B(x, r^*)$.

The following three facts: $d(\tau(x), \tau(x_2)) \le \mathrm{diam}(\tau(U))$, $s_1 \le \alpha(\mathrm{diam}(\tau(U)))$ and $\mathrm{diam}(\tau(U)) + \alpha(\mathrm{diam}(\tau(U))) < s^*$, imply that

(v) for all sufficiently small ε, $B(\tau(x_2), s_1 + \varepsilon) \subseteq B(\tau(x), s^*)$.

Let $z \in Y - B(\tau(x_2), s_1 + \varepsilon)$. We show that $\tau^{-1}(z) \notin B(x_2, r_1)$. If $z \notin B(\tau(x), s)$, then $\tau^{-1}(z) \notin B(x, r^*) \supseteq B(x_2, r_1)$. Suppose that $z \in B(\tau(x), s)$, and let C be the connected component of z in $B(\tau(x), s) - B(\tau(x_2), s_1 + \varepsilon)$. Hence

(vi) $C \cap (B(\tau(x), s) - B(\tau(x), s^*)) \neq \emptyset$.

Since $\tau(B(x_2, r_1)) \subseteq \tau(B(x, r^*)) \subseteq B(\tau(x), s^*)$, it follows that

(vii) $\tau^{-1}(C) \cap (X - B(x_2, r_1)) \neq \emptyset$.

From the facts: $\tau(S(x_2, r_1)) \subseteq B(\tau(x_2), s_1 + \varepsilon)$ and $C \cap B(\tau(x_2), s_1 + \varepsilon) = \emptyset$, we conclude that

(viii) $\tau^{-1}(C) \cap S(x_2, r_1) = \emptyset$.

The connectedness of C and hence of $\tau^{-1}(C)$ and facts (vii) and (viii) imply that

(ix) $\tau^{-1}(C) \cap B(x_2, r) = \emptyset$.

This implies that $\tau^{-1}(z) \notin B(x_2, r_1)$. Since the above argument holds for all sufficiently small ε, it follows that for every $z \in Y$: if $z \notin \bar{B}(\tau(x_2), s_1)$, then $\tau^{-1}(z) \notin B(x_2, r_1)$. But $y_2 \in B(x_2, r_1)$, so $\tau(y_2) \in \bar{B}(\tau(x_2), s_1)$. That is, $d(\tau(x_2), \tau(y_2)) \le s_1 = \alpha(d(x_1, y_1))$. ∎

THEOREM 3.40. *Let X and Y be metric spaces, $x_0 \in X$, $\tau : X \cong Y$ and $\alpha \in$ MBC. Suppose that X is DPT at x_0, Y is DPT and CP1 at $\tau(x_0)$, and τ is almost α-continuous at x_0. Then there is $M > 0$ such that τ is $M \cdot \alpha$-bicontinuous at x_0.*

Proof. We first show that there is some $M > 0$ such that τ^{-1} is $M \cdot \alpha$-continuous at $\tau(x_0)$. By Proposition 3.39, by the fact that Y is CP1, and since τ is almost α-continuous at x_0, there is $U \in \mathrm{Nbr}(x_0)$ such that for every $x_1, y_1, x_2, y_2 \in U$: if $d(x_2, y_2) \le d(x_1, y_1)$, then $d(\tau(x_2), \tau(y_2)) \le \alpha(d(\tau(x_1), \tau(y_1)))$. It is assumed that X is DPT at x_0, so there are $W \in \mathrm{Nbr}(x_0)$ and $K \ge 1$ such that $W \subseteq U$, and W, K satisfy condition $(*)$ of

Definition 3.32(a). Let $V \subseteq W$ be an open neighborhood of x_0 satisfying condition $(**)$ of Proposition 3.38. Fix any distinct $x_1, y_1 \in V$ and set $d_1 = d(x_1, y_1)$, $e_1 = d(\tau(x_1), \tau(y_1))$, $V_1 = B(x_0, d_1/2) \cap V$ and $V_2 = \tau(V_1)$.

We show that $\tau^{-1} \restriction V_2$ is $d_1/e_1 \cdot (K+2) \cdot \alpha$-continuous. Let $u, v \in V_2$ be distinct and set $d = d(\tau^{-1}(u), \tau^{-1}(v))$. Since $\tau^{-1}(u), \tau^{-1}(v) \in V_1$, $d < d_1 = d(x_1, y_1)$. So there are $n \le K \cdot \frac{d(x_1,y_1)}{d}$ and $z_0, \dots, z_n \in U$ such that $d(x_1, z_0),\ d(z_n, y_1) < d$ and $d(z_i, z_{i+1}) = d$ for all $i = 0, \dots, n-1$. By the choice of U,

$$d(\tau(x_1), \tau(z_0)),\ d(\tau(z_n), \tau(y_1)),\ d(\tau(z_i), \tau(z_{i+1})) \le \alpha(d(\tau\tau^{-1}(u), \tau\tau^{-1}(v))) = \alpha(d(u,v)).$$

Hence

$$\begin{aligned} d(\tau(x_1), \tau(y_1)) &\le d(\tau(x_1), \tau(z_0)) + \sum_{i=0}^{n-1} d(\tau(z_i), \tau(z_{i+1})) + d(\tau(z_n), \tau(y_1)) \\ &\le (n+2)\alpha(d(u,v)) \le \left(K \frac{d(x_1, y_1)}{d(\tau^{-1}(u), \tau^{-1}(v))} + 2 \right) \alpha(d(u,v)). \end{aligned}$$

It follows from the above inequality that

$$\begin{aligned} d(\tau^{-1}(u), \tau^{-1}(v)) &\le \frac{Kd(x_1, y_1) + 2d(\tau^{-1}(u), \tau^{-1}(v))}{d(\tau(x_1), \tau(y_1))} \alpha(d(u,v)) \\ &\le \frac{Kd_1 + 2d_1}{e_1} \cdot \alpha(d(u,v)) = \frac{d_1}{e_1} \cdot (K+2)\alpha(d(u,v)). \end{aligned}$$

So $\tau^{-1} \restriction V_2$ is $\frac{d_1}{e_1} \cdot (K+2) \cdot \alpha$-continuous, and hence τ^{-1} is locally Γ-continuous.

Note that in the above proof we only used the facts that X is DPT at x_0, and that Y is CP1 at $\tau(x_0)$.

We now turn to the proof that there is $M > 0$ such that τ is $M \cdot \alpha$-continuous at x_0. In this part we use the facts that Y is DPT and CP1 at $\tau(x_0)$. Let $U_1 \in \mathrm{Nbr}(x_0)$ and $K \ge 1$ be such that $\tau(U_1)$ and K satisfy condition $(*)$ of 3.32(a) applied to $\tau(x_0)$. By Proposition 3.39, there is $U_0 \in \mathrm{Nbr}(x_0)$ such that $U_0 \subseteq U_1$, and

(1) for every $x_1, y_1, x_2, y_2 \in U_0$: if $d(x_2, y_2) \le d(x_1, y_1)$, then $d(\tau(x_2), \tau(y_2)) \le \alpha(d(\tau(x_1), \tau(y_1)))$.

We apply Proposition 3.38 to $\tau(x_0)$ and $\tau(U_0)$, and obtain $T \subseteq Y$ satisfying condition $(**)$ of Proposition 3.38. Let $U = \tau^{-1}(T)$. We may assume that

(2) $K \ge 2$.

Let $x, y \in U$ be distinct. Set $N = 4Kd(\tau(x), \tau(y))/d(x,y)$ and $M = \max(1, N)$. We show that if $x', y' \in U$ and $d(x', y') < d(x,y)/4K$, then $d(\tau(x'), \tau(y')) \le M \cdot \alpha(d(x', y'))$. Obviously, this implies that $\tau \restriction (B(x_0, d(x,y)/8K) \cap U)$ is $M \cdot \alpha$-continuous.

Let $x', y' \in U$ be such that $d(x', y') < d(x,y)/4K$ and $n = [d(x,y)/Kd(x', y')] - 2$. Hence $n \ge 2$. Let $d = d(\tau(x), \tau(y))/n$. So there are $m \le Kn$ and $z_0, \dots, z_m \in \tau(U_0)$ such that $d(\tau(x), z_0),\ d(z_m, \tau(y)) < d$ and $d(z_{i-1}, z_i) = d$, $i = 1, \dots, m$. Let $x_i = \tau^{-1}(z_i)$. Denote x by x_{-1} and y by x_{m+1}. For $\ell \in \{-1, m+1\}$ let $z_\ell = \tau(x_\ell)$. The number of x_j's is $m+3$. So the number of distances between consecutive x_j's is $m+2$. Hence for some $i \in \{0, \dots, m+1\}$,

(3) $d(x_{i-1}, x_i) \geq \dfrac{d(x,y)}{m+2}$.

It follows from (3) and (2) that

$$d(x_{i-1}, x_i) \geq \frac{d(x,y)}{m+2} \geq \frac{d(x,y)}{K([\frac{d(x,y)}{Kd(x',y')}] - 2) + 2}$$
$$\geq \frac{d(x,y)}{K(\frac{d(x,y)}{Kd(x',y')} + 1 - 2) + K} \geq \frac{d(x,y)}{K \cdot \frac{d(x,y)}{Kd(x',y')}} = d(x',y').$$

That is,

(4) $d(x', y') \leq d(x_{i-1}, x_i)$.

Since the z_i's belong to $\tau(U_0)$, the x_i's belong to U_0. This is also true for $x_{-1} = x$ and $x_{m+1} = y$ because they belong to $U \subseteq U_0$. By (1) and (4),

(5) $d(\tau(x'), \tau(y')) \leq \alpha(d(z_{i-1}, z_i)) = \alpha(d)$.

Also,

$$d = \frac{1}{n} d(\tau(x), \tau(y)) = \frac{1}{[\frac{d(x,y)}{Kd(x',y')}] - 2} d(\tau(x), \tau(y)) \leq \frac{1}{\frac{d(x,y)}{Kd(x',y')} - 1 - 2} d(\tau(x), \tau(y))$$
$$= \frac{Kd(x',y')}{d(x,y) - 3Kd(x',y')} d(\tau(x), \tau(y)) \leq \frac{Kd(x',y')}{d(x,y) - 3K\frac{d(x,y)}{4K}} d(\tau(x), \tau(y))$$
$$= \frac{4Kd(\tau(x), \tau(y))}{d(x,y)} d(x',y') = Nd(x',y').$$

By (5), by the fact $M \geq 1, N$ and by the concavity of α,

$$d(\tau(x'), \tau(y')) \leq \alpha(d) \leq \alpha(Nd(x',y')) \leq \alpha(M \cdot d(x',y')) \leq M \cdot \alpha(d(x',y')).$$

We have thus shown that $\tau \restriction (B(x_0, \frac{d(x,y)}{8K}) \cap U)$ is $M \cdot \alpha$-continuous. ■

Proof of Theorem 3.35. Let X, x, Y, τ, Γ etc. fulfill the premises of 3.35. Then the assumptions of Lemma 3.37(b) are satisfied. So τ is almost Γ-continuous at x. By Theorem 3.40, τ is Γ-bicontinuous at x. ■

Proof of Theorem 3.27. Let $\langle E, X, \mathcal{S}, \mathcal{E} \rangle$ be a subspace choice system, Y be an open subset of a normed space F, Γ be a $(\leq\kappa(E))$-generated modulus of continuity and $\tau : X \cong Y$. Suppose that $(\mathrm{LIP}(X, \mathcal{S}, \mathcal{E}))^\tau \subseteq H_\Gamma^{\mathrm{LC}}(Y)$, and we prove that τ is locally Γ-bicontinuous.

For $x \in X$ choose $S \in \mathcal{S}$ such that $x \in S$ and denote E_S by D. We wish to apply Theorem 3.35 to $G = \mathrm{LIP}(X, \mathcal{S}, \mathcal{E})$, $H = \mathbb{A}(E; D)$, $\alpha(t) = 15t$ and $\lambda = \lambda_{\mathbb{A}}^{E;D} \restriction S$, so we check that clauses (i)–(iv) of Theorem 3.35 hold.

In clause (i) we have to check that X is DPT at x and that Y is DPT and CP1 at $\tau(x)$, and this was proved in Proposition 3.33. In clause (ii) we have to check: (1) x is a λ-limit-point; (2) G is α-infinitely-closed at x; (3) for some $N \in \mathrm{Nbr}(x)$, λ is (α, G)-decayable in $N \cap H_\lambda(x)$.

(1) Obviously, for every $V \in \mathrm{Nbr}^H(\mathrm{Id})$, $V_\lambda(x)$ contains a ball with center at x. So x is a λ-limit-point.

(2) Suppose that $\beta \in \mathrm{MC}$, $K \subseteq H_{\{\beta\}}(Z)$ and for any distinct $k_1, k_2 \in K$, $\mathrm{supp}(k_1) \cap \mathrm{supp}(k_2) = \emptyset$. Then $k := \circ K \in H(Z)$, and k is $\beta \circ \beta$-bicontinuous. Also, if $M \subseteq Z$,

and $k'(M) = M$ for every $k' \in K$, then $k(M) = M$. These observations imply that G is α-infinitely-closed.

(3) The (α, G)-decayability of λ at every point of S was proved in Lemma 3.8.

Clause (iii) is given, and clause (iv) holds, since it is assumed that $G^\tau \subseteq H_\Gamma^{\mathrm{LC}}(Y)$.

By Theorem 3.35, τ is Γ-bicontinuous at x. We have shown that τ is locally Γ-bicontinuous. ∎

In Theorem 3.26 we have presented an alternative argument for showing the local Γ-continuity of τ^{-1}. This method used the Baire Category Theorem, but did not require the assumptions of decayability of λ and the infinite-closedness of G. The same alternative argument can be employed in the context of affine-like partial actions. It is presented in the following theorem.

THEOREM 3.41. *Assume that the following facts hold.*

(i) *X is a metric space, $G \le H(X)$, H is a topological group and H is of the second category, λ is a partial action of H on X and $x \in \mathrm{Fld}(\lambda)$.*
(ii) *λ is compatible with G at x.*
(iii) *λ is affine-like at x.*
(iv) *Γ is a countably generated modulus of continuity.*
(v) *Y is a metric space and $\tau : X \cong Y$.*
(vi) *For every $g \in G$, g^τ is Γ-bicontinuous at $\tau(x)$.*
(vii) *X is DPT at x and Y is DPT and CP1 at $\tau(x)$.*

Then τ is Γ-bicontinuous at x.

Proof. For $h \in H$ write $h_\lambda = \hat{h}$. The assumptions of Lemma 3.22 hold, so there are $T \in \mathrm{Nbr}(x)$, a nonempty open subset $V \subseteq H$ and $\gamma \in \Gamma$ such that $V \times T \subseteq \mathrm{Dom}(\lambda)$ and $\hat{h}^\tau \restriction \tau(T)$ is γ-bicontinuous for every $h \in V$. Note that $(\hat{h}^{-1})^\tau \restriction \tau(\hat{h}(T))$ is γ-bicontinuous for every $h \in V$.

Let $h_0 \in V$. There are $S \in \mathrm{Nbr}(x)$ and $V_1 \in \mathrm{Nbr}(h_0)$ such that $V_1 \subseteq V$, $S \subseteq T$ and $\lambda(V_1 \times S) \subseteq \hat{h}_0(T)$. Set $W = h_0^{-1} \cdot V_1$. Clearly, $W \in \mathrm{Nbr}(e_H)$ and $W \times S \subseteq \mathrm{Dom}(\lambda)$. Let $h \in W$. So for some $h_1 \in V_1$ we have $h = h_0^{-1} \cdot h_1$. From the facts $h_1 \in V_1 \subseteq V$ and $S \subseteq T$, it follows that (1) $(\hat{h}_1)^\tau \restriction \tau(S)$ is γ-bicontinuous, and since $\hat{h}_1(S) \subseteq \hat{h}_0(T)$ and $h_1^{-1} \in V^{-1}$, we conclude that (2) $(\hat{h}_0^{-1})^\tau \restriction \tau(\hat{h}_1(S))$ is γ-bicontinuous. (1) and (2) imply that $\hat{h}^\tau \restriction \tau(S)$ is $\gamma \circ \gamma$-bicontinuous.

We have shown that there are $W \in \mathrm{Nbr}(e_H)$ and $S \in \mathrm{Nbr}(x)$ such that $W \times S \subseteq \mathrm{Dom}(\lambda)$, and for every $h \in W$, $\hat{h}^\tau \restriction \tau(W)$ is $\gamma \circ \gamma$-bicontinuous. By Lemma 3.37(a), τ is almost Γ-continuous at x, and by Theorem 3.40, τ is Γ-bicontinuous at x. ∎

3.5. Summary and questions. The following final theorem combines the results of Chapters 2 and 3. Note that part (a) of 3.42 is not a special case of (b).

THEOREM 3.42. (a) *Let X, Y be open subsets of the normed spaces E and F respectively, Γ, Δ be moduli of continuity and $\varphi : H_\Gamma^{\mathrm{LC}}(X) \cong H_\Delta^{\mathrm{LC}}(Y)$. Suppose that Γ is $(\le\kappa(E))$-generated. Then $\Gamma = \Delta$, there is $\tau : X \cong Y$ such that $\varphi(h) = h^\tau$ for every $h \in H_\Gamma^{\mathrm{LC}}(X)$, and τ is locally Γ-bicontinuous.*

(b) *Let $\langle E, X, \mathcal{S}, \mathcal{E}\rangle$ and $\langle F, Y, \mathcal{T}, \mathcal{F}\rangle$ be subspace choice systems, Γ, Δ be moduli of continuity and $\varphi : H^{\rm LC}_{\Gamma}(X;\mathcal{S},\mathcal{E}) \cong H^{\rm LC}_{\Delta}(Y;\mathcal{T},\mathcal{F})$. Suppose that Γ and Δ are $(\leq\kappa(E))$-generated. Then $\Gamma = \Delta$, there is $\tau : X \cong Y$ such that $\varphi(h) = h^{\tau}$ for every $h \in H^{\rm LC}_{\Gamma}(X;\mathcal{S},\mathcal{E})$, and τ is locally Γ-bicontinuous.*

Proof. (a) $\mathrm{LIP}^{\rm LC}(X) \subseteq H^{\rm LC}_{\Gamma}(X) \subseteq H(X)$ and the same holds for Y. So by Theorem 2.8(a) there is $\tau : X \cong Y$ such that τ induces φ. Hence $(H^{\rm LC}_{\Delta}(Y))^{\tau^{-1}} = H^{\rm LC}_{\Gamma}(X)$. In particular, $(\mathrm{LIP}(Y))^{\tau^{-1}} \subseteq H^{\rm LC}_{\Gamma}(X)$. Since $X \cong Y$, $\kappa(F) = \kappa(E)$. So Γ is $(\leq\kappa(F))$-generated. By Theorem 3.27, τ^{-1} is locally Γ-bicontinuous. That is, τ is locally Γ-bicontinuous. Hence $H^{\rm LC}_{\Delta}(Y) = (H^{\rm LC}_{\Gamma}(X))^{\tau} \subseteq H^{\rm LC}_{\Gamma}(Y)$. It is easy to see that if $\alpha \in \Delta - \Gamma$, then there is $h \in H(Y)$ such that h is α-bicontinuous and h is not locally Γ-continuous. This implies that $\Delta \subseteq \Gamma$.

Suppose by contradiction that $\Gamma - \Delta \neq \emptyset$. It is easy to see that there is $h \in H^{\rm LC}_{\Gamma}(Y) - H^{\rm LC}_{\Delta}(Y)$. So $g := h^{\tau^{-1}} \in H^{\rm LC}_{\Gamma}(X)$. However, $g^{\tau} = h \notin H^{\rm LC}_{\Delta}(Y)$. A contradiction. So $\Gamma = \Delta$.

(b) $\mathrm{LIP}^{\rm LC}(X;\mathcal{S},\mathcal{E}) \subseteq H^{\rm LC}_{\Gamma}(X;\mathcal{S},\mathcal{E}) \subseteq H(X)$ and the same holds for Y. So by Theorem 2.8(b) there is $\tau : X \cong Y$ such that τ induces φ. Hence $(H^{\rm LC}_{\Gamma}(X;\mathcal{S},\mathcal{E}))^{\tau} = H^{\rm LC}_{\Gamma}(Y;\mathcal{T},\mathcal{F})$. In particular, $(\mathrm{LIP}(X;\mathcal{S},\mathcal{E}))^{\tau} \subseteq H^{\rm LC}_{\Delta}(Y)$ and $(\mathrm{LIP}(Y;\mathcal{T},\mathcal{F}))^{\tau^{-1}} \subseteq H^{\rm LC}_{\Gamma}(X)$. By Theorem 3.19(b), $\Gamma = \Delta$ and τ is locally Γ-bicontinuous. ■

The technical and abstract formulation of Theorems 3.15, 3.26, 3.35 and 3.41 hinders the understanding of their essence. The above theorems are better understood through their application to normed spaces, as stated in the following corollary.

COROLLARY 3.43. *Suppose that*

(1) *$\langle E, X, \mathcal{S}, \mathcal{E}\rangle$ is a subspace choice system and $G \leq H(X)$,*
(2) *$\alpha \in \mathrm{MBC}$ and $\Gamma \subseteq \mathrm{MC}$,*
(3) *F is a normed space, $Y \subseteq F$ is open and $\tau : X \cong Y$,*
(4) *for every $g \in G$, g^{τ} is locally Γ-bicontinuous.*

(a) *Assume that in addition to (1)–(4) the following conditions are fulfilled.*

(a1) *For every $x \in X$, if $x \in S \in \mathcal{S}$, then $\lambda^{E;E_S}_{\mathbb{T}} \restriction S$ is (α, G)-decayable at x.*
(a2) *For every $x \in X$, G is α-infinitely-closed at x.*
(a3) *Γ is a modulus of continuity.*
(a4) *Γ is $(\leq\kappa(E))$-generated.*

Then τ^{-1} is locally Γ-continuous.

(b) *Assume that in addition to (1)–(4) the following conditions are fulfilled.*

(b1) *For every $x \in X$, if $x \in S \in \mathcal{S}$, then $\lambda^{E;E_S}_{\mathbb{T}} \restriction S$ is compatible with G at x.*
(b2) *For every $S \in \mathcal{S}$, E_S is of the second category.*
(b3) *For every $\gamma \in \Gamma$ and $K > 0$, $K \cdot \gamma \in \Gamma$.*
(b4) *Γ is countably generated.*

Then τ^{-1} is locally Γ-continuous.

(c) *Assume that in addition to (1)–(4) the following conditions are fulfilled.*

(c1) *For every $x \in X$, if $x \in S \in \mathcal{S}$, then $\lambda^{E;E_S}_{\mathbb{A}} \restriction S$ is (α, G)-decayable at x.*
(c2) *For every $x \in X$, G is α-infinitely closed at x.*

(c3) *Γ is a modulus of continuity.*
(c4) *Γ is $(\leq\kappa(E))$-generated.*

Then τ is locally Γ-bicontinuous.

(d) *Assume that in addition to* (1)–(4) *the following conditions are fulfilled.*

(d1) *For every $x \in X$, if $x \in S \in \mathcal{S}$, then $\lambda_{\mathbb{A}}^{E;E_S} \restriction S$ is compatible with G at x.*
(d2) *For every $S \in \mathcal{S}$, E_S is of the second category.*
(d3) *Γ is a modulus of continuity.*
(d4) *Γ is countably generated.*

Then τ is locally Γ-bicontinuous.

Proof. Parts (a), (b), (c) and (d) follow respectively from Theorems 3.15, 3.26, 3.35 and 3.41. ■

There are cases in which the action is translation-like but not affine-like. In such situations parts (a) or (b) are applicable but (c) and (d) are not, and hence we can only prove the Γ-continuity of τ^{-1}.

For spaces of the first category only (a) and (c) are applicable. Part (c) has a conclusion stronger than that of (a). However, the final theorem about groups of the form $H_{\Gamma}^{\mathrm{LC}}(X)$ (Theorem 3.19) can be inferred from either (a) or (c).

The conclusion of (c) is stronger than that of (d). But the assumptions of (c) are stronger in some respects than those of (d). Nevertheless, we do not know how to construct a group G to which the reconstruction methods of Chapter 2 apply, and for which (d) can be applied but (c) cannot.

There are two outstanding open questions. The first is whether the assumption that Γ is $(\leq\kappa(E))$-generated is needed. The second is whether translation-likeness of the action implies the Γ-continuity of τ.

QUESTION 3.44. Let X, Y be open subsets of the normed spaces E, F, and Γ be a modulus of continuity. Suppose that $\tau : X \cong Y$ and that $(H_{\Gamma}^{\mathrm{LC}}(X))^{\tau} = H_{\Gamma}^{\mathrm{LC}}(Y)$. Is τ locally Γ-bicontinuous? □

QUESTION 3.45. Let E and F be normed space, $\tau : X \cong Y$ and Γ be a countably generated modulus of continuity. Suppose that $(\mathbb{T}(E))^{\tau} \subseteq H_{\Gamma}^{\mathrm{LC}}(Y)$. Is τ locally Γ-continuous? Is the above true when E, F are Banach spaces? □

3.6. Normed manifolds. As in Chapter 2, the results of this section extend to normed manifolds. Also, the proofs presented to this point transfer without change to this new context. We now state some of these results explicitly.

DEFINITION 3.46. (a) Let $\langle X, \Phi \rangle$ be a normed manifold. We say that $\langle X, \Phi \rangle$ is a *locally Lipschitz normed manifold* if for every $\varphi, \psi \in \Phi$, $\varphi^{-1} \circ \psi$ is a bilipschitz function.

(b) Let $\langle X, \Phi \rangle$ and $\langle Y, \Psi \rangle$ be locally Lipschitz normed manifolds and $\tau : X \cong Y$. We say that τ is *Lipschitz with respect to Φ and Ψ* if there is K such that for every $x \in X$ there are $\varphi \in \Phi$ and $\psi \in \Psi$ such that $x \in \mathrm{int}(\mathrm{Rng}(\varphi))$, $\tau(x) \in \mathrm{int}(\mathrm{Rng}(\psi))$ and $\psi^{-1} \circ \tau \circ \varphi$ is K-Lipschitz. We say that τ is *bilipschitz with respect to Φ and Ψ* if both τ and τ^{-1} are Lipschitz. Define

$$\mathrm{LIP}(X, \Phi) = \{h \in H(X) \mid h \text{ is bilipschitz with respect to } \Phi\}.$$

(c) Let $\langle X, \Phi \rangle$ and $\langle Y, \Psi \rangle$ be locally Lipschitz normed manifolds and Γ be a modulus of continuity. A homeomorphism $\tau : X \cong Y$ is *locally Γ-continuous with respect to Φ and Ψ* if for every $x \in X$ there are $\varphi \in \Phi$, $\psi \in \Psi$, $U \in \mathrm{Nbr}(\varphi^{-1}(x))$ and $\gamma \in \Gamma$ such that $x \in \mathrm{int}(\mathrm{Rng}(\varphi))$, $\tau(x) \in \mathrm{int}(\mathrm{Rng}(\psi))$, $U \subseteq \mathrm{Dom}(\varphi)$ and $(\psi^{-1} \circ \tau \circ \varphi) \restriction U$ is γ-continuous. We say that τ is *locally Γ-bicontinuous* if τ and τ^{-1} are locally Γ-continuous. Define

$$H_\Gamma^{\mathrm{LC}}(X, \Phi) = \{h \in H(X) \mid h \text{ is locally } \Gamma\text{-bicontinuous with respect to } \Phi\}.$$

(d) Let $\langle X, \Phi \rangle$ be a locally Lipschitz normed manifold, $\mathcal{S}$ be an open cover of X and Γ be a modulus of continuity. Define $\mathrm{LIP}(X, \Phi, \mathcal{S})$ to be the group generated by $\bigcup\{\mathrm{LIP}(X, \Phi)\lfloor S \rfloor \mid S \in \mathcal{S}\}$ and $H_\Gamma^{\mathrm{LC}}(X, \Phi, \mathcal{S})$ to be the group generated by $\bigcup\{H_\Gamma^{\mathrm{LC}}(X, \Phi)$ $\lfloor S \rfloor \mid S \in \mathcal{S}\}$. □

THEOREM 3.47. *Let $\langle X, \Phi \rangle$ and $\langle Y, \Psi \rangle$ be normed manifolds with locally Lipschitz atlases and $\tau : X \cong Y$. Let Γ be a countably generated modulus of continuity.*

(a) *Suppose that* $(\mathrm{LIP}(X, \Phi))^\tau \subseteq H_\Gamma^{\mathrm{LC}}(Y, \Psi)$. *Then τ is locally Γ-bicontinuous with respect to Φ and Ψ.*

(b) *Let $\mathcal{S}$ be an open cover of X, and suppose that* $(\mathrm{LIP}(X, \Phi, \mathcal{S}))^\tau \subseteq H_\Gamma^{\mathrm{LC}}(Y, \Psi)$. *Then τ is locally Γ-bicontinuous with respect to Φ and Ψ.*

Note that (a) is a special case of (b).

We simplify the notation below by omitting the mention of Φ and Ψ.

COROLLARY 3.48. *Let $\langle X, \Phi \rangle$ and $\langle Y, \Psi \rangle$ be normed manifolds with locally Lipschitz atlases.*

(a) *Let Γ and Δ be countably generated moduli of continuity, and $\varphi : H_\Gamma^{\mathrm{LC}}(X) \cong H_\Delta^{\mathrm{LC}}(Y)$. Then $\Gamma = \Delta$ and there is $\tau : X \cong Y$ such that τ induces φ, and τ is locally Γ-bicontinuous.*

(b) *Let Γ be a countably generated modulus of continuity, $\mathcal{S}$ an open cover of X, and $G \le H(X)$. Assume that if $\langle X, \Phi \rangle$ is a Banach manifold, then* $\mathrm{LIP}(X, \mathcal{S}) \le G$, *and if $\langle X, \Phi \rangle$ is not a Banach manifold, then* $\mathrm{LIP}^{\mathrm{LC}}(X, \mathcal{S}) \le G$. *Suppose that $\varphi : G \cong H_\Gamma^{\mathrm{LC}}(Y)$.*

Then $G = H_\Gamma^{\mathrm{LC}}(X)$ and there is $\tau : X \cong Y$ such that τ induces φ, and τ is locally Γ-bicontinuous.

Proof. (a) Note that if $H_\Gamma^{\mathrm{LC}}(X) = H_\Delta^{\mathrm{LC}}(X)$, then $\Gamma = \Delta$. Hence (a) can be concluded from (b).

(b) We shall apply Theorem 2.30(a). Clearly, $\mathrm{LIP}^{\mathrm{LC}}(Y; \Psi) \le H_\Gamma^{\mathrm{LC}}(Y)$ (see Definition 2.29(b)). There is an atlas Φ' for X such that if $\langle X, \Phi \rangle$ is a Banach manifold, then $\mathrm{LIP}(X, \Phi') \le G$, and if $\langle X, \Phi \rangle$ is not a Banach manifold, then $\mathrm{LIP}^{\mathrm{LC}}(X, \Phi') \le G$. Indeed, $\Phi' = \{\psi \restriction \bar{B}(x, r) \mid \psi \in \Phi,\ \bar{B}(x, r) \subseteq \mathrm{Dom}(\psi) \text{ and there is } U \in \mathcal{S} \text{ with } \psi(\bar{B}(x, r)) \subseteq U\}$. By Theorem 2.30(a), there is $\tau : X \cong Y$ such that τ induces φ. So $G^\tau = H_\Gamma^{\mathrm{LC}}(Y)$. In particular, $(\mathrm{LIP}(X, \mathcal{S}))^\tau \subseteq H_\Gamma^{\mathrm{LC}}(Y)$. By Theorem 3.47(b), τ is locally Γ-bicontinuous. So $G = H_\Gamma^{\mathrm{LC}}(X)$. ■

QUESTION. In the above theorem does it suffice to assume that $\mathrm{LIP}(X, \mathcal{S}) \le G$, regardless of whether $\langle X, \Phi \rangle$ is a Banach manifold? □

4. The local uniform continuity of conjugating homeomorphisms

To complete the picture of the local Γ-bicontinuity of conjugating homeomorphisms, we now deal with the group $H^{\mathrm{LC}}_{\mathrm{MC}}(X)$ of locally bi-uniformly-continuous homeomorphisms. (Note that MC is a modulus of continuity, so the notation $H^{\mathrm{LC}}_{\mathrm{MC}}(X)$ is a special case of Definition 1.12(c).) The methods employed in dealing with $H^{\mathrm{LC}}_{\mathrm{MC}}(X)$ are quite different from those used in the previous section.

We shall prove the following extension of Theorem 3.42:

THEOREM 4.1. (a) *Let X, Y be open subsets of the normed spaces E and F respectively, Γ, Δ be moduli of continuity and $\varphi : H^{\mathrm{LC}}_{\Gamma}(X) \cong H^{\mathrm{LC}}_{\Delta}(Y)$. Suppose that Γ is $(\kappa(E))$-generated or $\Gamma = \mathrm{MC}$. Then $\Gamma = \Delta$, there is $\tau : X \cong Y$ such that $\varphi(h) = h^{\tau}$ for every $h \in H^{\mathrm{LC}}_{\Gamma}(X)$, and τ is locally Γ-bicontinuous.*

(b) *Let $\langle E, X, \mathcal{S}, \mathcal{E}\rangle$ and $\langle F, Y, \mathcal{T}, \mathcal{F}\rangle$ be subspace choice systems, Γ, Δ be moduli of continuity and $\varphi : H^{\mathrm{LC}}_{\Gamma}(X; \mathcal{S}, \mathcal{E}) \cong H^{\mathrm{LC}}_{\Delta}(Y; \mathcal{T}, \mathcal{F})$. Suppose that Γ is $(\leq\kappa(E))$-generated or $\Gamma = \mathrm{MC}$, and the same holds for Δ. Then $\Gamma = \Delta$, there is $\tau : X \cong Y$ such that $\varphi(h) = h^{\tau}$ for every $h \in H^{\mathrm{LC}}_{\Gamma}(X; \mathcal{S}, \mathcal{E})$, and τ is locally Γ-bicontinuous.*

Note that (a) is not a special case of (b), since in (b) Δ is assumed to be $(\leq\kappa(E))$-generated or equal to MC, and this is not assumed in (a). The key intermediate step in the proof of Theorem 4.1 is Theorem 4.8.

There are several ways of defining uniform continuity. We sort this matter out in the next definition and proposition.

DEFINITION 4.2. (a) Let $\langle X, d^X\rangle$ and $\langle Y, d^Y\rangle$ be metric spaces, and $f : X \to Y$.

We say that f is *uniformly continuous* (f is UC) if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $x, y \in X$: if $d^X(x, y) < \delta$, then $d^Y(f(x), f(y)) < \varepsilon$. If $f : X \cong f(X)$ and both f and f^{-1} are uniformly continuous, then f is said to be *bi-uniformly-continuous* (*bi-UC*).

(b) Let $\alpha \in \mathrm{MC}$ and $r > 0$. We say that $f : X \to Y$ is (r, α)*-continuous* if for every $x, y \in X$: if $d^X(x, y) < r$, then $d^Y(f(x), f(y)) \leq \alpha(d^X(x, y))$.

(c) We say that $f : X \to Y$ is *uniformly continuous for all distances* if there is $\alpha \in \mathrm{MC}$ such that f is α-continuous.

(d) Let $f : X \to Y$ and $x \in X$. Say that f *is uniformly continuous at x* (*f is UC at x*) if there is $U \in \mathrm{Nbr}(x)$ such that $f{\restriction}U$ is UC, and f *is bi-uniformly-continuous at x* (*bi-UC at x*) if there is $U \in \mathrm{Nbr}(x)$ such that $f{\restriction}U$ is bi-UC.

(e) Let $f : X \to Y$. Say that f is *locally uniformly continuous* (*locally UC*) if f is UC at every $x \in X$, and f is *locally bi-uniformly-continuous* (*locally bi-UC*) if f is bi-UC at every $x \in X$.

(f) Let $\langle X, d\rangle$ be a metric space. *The discrete path property for large distances* is the following property of X. There are $a, b > 0$ such that for every $x, y \in X$ and $r > 0$ there are $n \in \mathbb{N}$ and $x = x_0, x_1, \ldots, x_n = y$ in X such that for every $i < n$, $d(x_i, x_{i+1}) < r$ and $\sum_{i<n} d(x_i, x_{i+1}) \leq ad(x, y) + b$. $\square$

PROPOSITION 4.3. (a) *Let $f : X \to Y$. Then f is UC iff for some $\alpha \in \mathrm{MC}$ and $r > 0$, f is (r, α)-continuous.*

(b) *Suppose that X has the discrete path property for large distances. Let $f : X \to Y$. Then f is UC iff f is uniformly continuous for all distances.*

(c) *Suppose that $f : X \to Y$, f is UC and $\mathrm{Rng}(f)$ is bounded. Then f is uniformly continuous for all distances.*

(d) *Let $f : X \to Y$ and $x \in X$. Then f is UC at x iff for some $\alpha \in \mathrm{MC}$, f is α-continuous at x.*

Proof. All parts are trivial. However, the proof of the implication $\Rightarrow$ in (a) requires the following fact. If $\eta : (0, a] \to [0, \infty)$, and $\lim_{tt \to 0} \eta(t) = 0$, then there is $\alpha \in \mathrm{MC}$ such that $\eta \leq \alpha \restriction (0, a]$. The verification of this fact is left to the reader. $\blacksquare$

DEFINITION 4.4. (a) Suppose that X, Y are topological spaces $D \subseteq X$. Define $H(X, Y) = \{h \mid h : X \cong Y\}$ and $H(X; D) = \{h \in H(X) \mid h(D) = D\}$.

(b) For metric spaces X, Y define $\mathrm{UC}(X, Y) = \{h \in H(X, Y) \mid h \text{ is UC}\}$, $\mathrm{UC}^{\pm}(X, Y) = \{h \in H(X, Y) \mid h \text{ is bi-UC}\}$ and $\mathrm{UC}(X) = \mathrm{UC}^{\pm}(X, X)$. For $x \in X$ let $\mathrm{PNT.UC}(X, x) = \{h \in H(X) \mid h(x) = x \text{ and } h \text{ is bi-UC at } x\}$.

(c) Let X be an open subset of a normed space E, $S \subseteq X$ be open, and F be a dense linear subspace of E. Define $\mathrm{UC}(X; F) = \{h \in \mathrm{UC}(X) \mid h(X \cap F) = X \cap F\}$ and $\mathrm{UC}(X; S, F) = \mathrm{UC}(X)\lfloor\underline{S}\rfloor \cap \mathrm{UC}(X; F)$. For $x \in S$ let $\mathrm{UC}(X; S, F, x) = \{h \in \mathrm{UC}(X; S, F) \mid h(x) = x\}$.

(d) Let $\langle E, X, \mathcal{S}, \mathcal{F}\rangle$ be a subspace choice system. Then $\mathrm{UC}(X, \mathcal{S})$ denotes the subgroup of $H(X)$ generated by $\bigcup\{\mathrm{UC}(X)\lfloor\underline{S}\rfloor \mid S \in \mathcal{S}\}$, and $\mathrm{UC}(X; \mathcal{S}, \mathcal{F})$ denotes the subgroup of $H(X)$ generated by $\bigcup\{\mathrm{UC}(X; S, F_S) \mid S \in \mathcal{S}\}$.

(e) For metric spaces X, Y let $\mathrm{LUC}(X, Y) = \{h \in H(X, Y) \mid h \text{ is locally UC}\}$. As usual we define $\mathrm{LUC}^{\pm}(X, Y) = \{h \in H(X, Y) \mid h \text{ is locally bi-UC}\}$ and $\mathrm{LUC}(X) = \mathrm{LUC}^{\pm}(X, X)$. $\square$

REMARK. Note that $H_{\mathrm{MC}}(X) \leq \mathrm{UC}(X)$ but equality need not hold. See Proposition 4.3. It is the group $H_{\mathrm{MC}}(X)$ that fits into the framework better, but the group which has been traditionally considered is $\mathrm{UC}(X)$. We based the above definitions on $\mathrm{UC}(X)$ rather than on $H_{\mathrm{MC}}(X)$. As for local uniform continuity, the two ways of defining this notion are equivalent. Hence $\mathrm{LUC}(X) = H^{\mathrm{LC}}_{\mathrm{MC}}(X)$ for every metric space X. This fact is a triviality. $\square$

The following easy proposition will be used extensively.

PROPOSITION 4.5. *Let X be a metric space and $\{U_n \mid n \in \mathbb{N}\}$ be a sequence of open sets in X such that $\lim_{n\to\infty} \mathrm{diam}(U_n) = 0$, and for any distinct $m, n \in \mathbb{N}$, $d(U_m, U_n) > 0$. For every $n \in \mathbb{N}$ let $h_n \in \mathrm{UC}(X)$ be such that $\mathrm{supp}(h_n) \subseteq U_n$. Then $\circ_{n\in\mathbb{N}} h_n \in \mathrm{UC}(X)$.*

Proof. Let $h = \circ_{n\in\mathbb{N}} h_n$. Let $\varepsilon > 0$. Let $N \in \mathbb{N}$ be such that for every $m \geq N$, $\operatorname{diam}(U_m) < \varepsilon/3$. Let $\delta_1 > 0$ be such that for every $i < N$ and $x, y \in X$: if $d(x,y) < \delta_1$, then $d(h_i(x), h_i(y)) < \varepsilon/3$. Let $\delta_2 = \min(\{d(U_i, U_j) \mid i < j < N\})$, and let $\delta = \min(\delta_1, \delta_2, \varepsilon/3)$.

Suppose that $d(x,y) < \delta$, and we show that $d(h(x), h(y)) < \varepsilon$. Since for any distinct $i, j < N$, $d(x,y) < d(U_i, U_j)$, there are no distinct $i, j < N$ such that $x \in U_i$ and $y \in U_j$. So we may assume that one of the following occurs: (i) for some $i < N$, $x \in U_i$ and $y \notin \bigcup\{U_j \mid j \neq i\}$; (ii) for some $i < N$ and $j \geq N$, $x \in U_i$ and $y \in U_j$; (iii) for some $i \geq N$, $x \in U_i$ and $y \notin \bigcup\{U_j \mid j \neq i\}$; (iv) for some distinct $i, j \geq N$, $x \in U_i$ and $y \in U_j$; (v) $x, y \notin \bigcup\{U_i \mid i \in \mathbb{N}\}$.

In case (i), $h(x) = h_i(x)$ and $h(y) = h_i(y)$, so since $d(x,y) < \delta_1$, it follows that $d(h(x), h(y)) < \varepsilon$. In case (ii),

$$d(h(x), h(y)) \leq d(h(x), y) + d(y, h(y)) = d(h_i(x), h_i(y)) + d(y, h_j(y)) < \varepsilon/3 + \varepsilon/3 < \varepsilon.$$

In case (iii),

$$d(h(x), h(y)) = d(h_i(x), h_i(y)) \leq d(h_i(x), x) + d(x,y) + d(y, h_i(y)) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Case (iv) is similar to case (iii), and case (v) is trivial. ∎

Definition 4.6. Let M be a topological space and N be a Hausdorff space.

(a) Let $A \subseteq M$ and $g : A \to N$ be continuous. For every $x \in \operatorname{cl}^M(A)$ there is at most one $y \in N$ such that $g \cup \{\langle x, y\rangle\}$ is a continuous function. Let

$$g^{\mathrm{cl}}_{M,N} = \{\langle x, y\rangle \mid x \in \operatorname{cl}^M(A),\ y \in N \text{ and } g \cup \{\langle x, y\rangle\} \text{ is a continuous function}\}.$$

Obviously, $g^{\mathrm{cl}}_{M,N}$ extends g, and $\operatorname{Rng}(g^{\mathrm{cl}}_{M,N}) \subseteq \operatorname{cl}^N(\operatorname{Rng}(g))$. When possible, $g^{\mathrm{cl}}_{M,N}$ is abbreviated by g^{cl}, and if $M = N$, then $g^{\mathrm{cl}}_{M,N}$ is denoted by g^{cl}_M. If H is a set of continuous functions from A to B, then H^{cl} denotes $\{h^{\mathrm{cl}} \mid h \in H\}$.

(b) Let $X \subseteq M$ and $Y \subseteq N$. We define

$$\mathrm{EXT}^{M,N}(X,Y) = \{h \in H(X,Y) \mid \operatorname{Dom}(h^{\mathrm{cl}}_{M,N}) = \operatorname{cl}^M(X)\}.$$

When possible, we abbreviate $\mathrm{EXT}^{M,N}(X,Y)$ by $\mathrm{EXT}(X,Y)$. The notation $\mathrm{EXT}^M(X)$ stands for $(\mathrm{EXT}^{M,M})^{\pm}(X,X)$. □

Proposition 4.7. (a) (i) *Let X be a topological space, $D \subseteq X$ be dense, Y be a regular topological space and $h : D \to Y$ be continuous. Suppose that for every $x \in X$ there is a continuous function $h_x : D \cup \{x\} \to Y$ extending h. Then $\bigcup\{h_x \mid x \in X\}$ is continuous.*

(ii) *Let M be a topological space, N be a regular space $A \subseteq M$ and $g : A \to N$ be continuous. Then $g^{\mathrm{cl}}_{M,N}$ is continuous.*

(b) *Let X be a metric space, Y be a complete metric space, $A \subseteq X$, and $g : A \to Y$ be a uniformly continuous function. Then* $\operatorname{Dom}(g^{\mathrm{cl}}) = \operatorname{cl}(A)$.

(c) *Let E be a normed space, D be a dense linear subspace of E, $X \subseteq E$ be open, $u \in D$, $B^E(u,p) \subseteq X$, $x, y \in D \cap B^E(u,p)$, $z \in B^{\bar{E}}(u,p)$, $\varepsilon > 0$, $0 < s < \min(\|x - z\|, \|y - z\|)$ and $\max(\|x - z\|, \|y - z\|) < t < \|z - u\| + p$. Then there is $h \in \mathrm{UC}(X; D)$ such that*: (i) $\operatorname{supp}(h^{\mathrm{cl}}_{\bar{E}}) \subseteq B^{\bar{E}}(z,t) - B^{\bar{E}}(z,s)$, (ii) $h(x) = x$ *and* (iii) $h(y) \in B(x, \varepsilon)$.

Proof. The proofs of parts (a) and (b) are trivial.

(c) Write $r' = \|z-u\|+t$. For every $a \in (0,1)$ there is $h \in \mathrm{LIP}(X;D)\lfloor B(u,p) \rfloor$ such that $h \restriction B^E(u,r')$ is the multiplication by the scalar a/r', that is, $h(w) = \frac{a}{r'}w$ for every $w \in B^E(u,r')$. So we may assume that $B^{\bar{E}}(z,t) \subseteq B^{\bar{E}}(u,ap)$. Let $s < \bar{s} < \min(\|x-z\|,\|y-z\|)$, $t > \bar{t} > \max(\|x-z\|,\|y-z\|)$ and $\bar{z} \in D$ be such that $\|\bar{z}-z\| < \min(t-\bar{t},\bar{s}-s)$. Since $\mathrm{tr}^E_{\bar{z}}$ is an isometry belonging to $H(E;D)$, we may shift $\bar{z}$ to the origin. That is, we may assume that $\bar{z}=0$. We have $\|x\| \geq \|x-z\|-\|z\| > s-(s-\bar{s}) = \bar{s}$. The same computation applies to y. We conclude that $\|x\|,\|y\| > \bar{s}$. Another similar computation shows that $\|x\|,\|y\| < \bar{t}$. It is also obvious that $B^{\bar{E}}(z,s) \subseteq B^{\bar{E}}(0,\bar{s})$ and that $B^{\bar{E}}(z,t) \supseteq B^{\bar{E}}(0,\bar{t})$. It thus remains to show that there is $h \in \mathrm{UC}(X;D)$ such that $\mathrm{supp}(h) \subseteq B(0,\bar{t}) - B(0,\bar{s})$, and h fulfills clauses (ii) and (iii). The construction of such a homeomorphism is routine but long, so we skip some details.

In the inclusion $B^{\bar{E}}(z,t) \subseteq B^{\bar{E}}(u,ap)$, choose a so small that $B^E(0,6\max(\|x\|,\|y\|)) \subseteq X$. By an argument similar to the choice of a above, we may also assume that (1) $\bar{t} > 5\max(\|x\|,\|y\|)$ and $\bar{s} < \frac{1}{5}\min(\|x\|,\|y\|)$. Let $F = \mathrm{span}(\{x,y\})$. As in the proof of Claim 3 in Lemma 3.30, there is E_1 such that $F \oplus E_1 = E$, and $\|v_0\|+\|v_1\| \leq 3\|v_0+v_1\|$ for every $v_0 \in F$ and $v_1 \in E_1$. Let $\| \ \|^{\mathbf{H}}$ be a Hilbert norm on F such that (2) $\|v\| \leq \|v\|^{\mathbf{H}} \leq 3\sqrt{2}\|v\|$ for every $v \in F$.

For $v \in E$ let v_F and v_{E_1} be such that $v = v_F + v_{E_1}$ and define $|v| = \|v_F\|^{\mathbf{H}} + \|v_{E_1}\|$. We may assume that $\|y\|^{\mathbf{H}} \neq \|x\|^{\mathbf{H}}$. Let $S = \{v \in F \mid \|v\|^{\mathbf{H}} = \|y\|^{\mathbf{H}}\}$. By (1) and (2), $S \subseteq B^E(0,\bar{t}) - \bar{B}^E(0,\bar{s})$. So there is $b > 0$ such that $x \notin B^{\langle E, | \, | \rangle}(S,b) \subseteq B^E(0,\bar{t}) - \bar{B}^E(0,\bar{s})$.

Suppose that the angle between x and y in $\langle F, \| \ \|^{\mathbf{H}} \rangle$ is θ. Let $\eta : [0,\infty) \to [0,\infty)$ be the piecewise linear function with a unique breakpoint at b such that $\eta(0) = \theta$ and $\eta(b) = 0$. For $v \in X$ define $h_1(v) = \mathrm{Rot}_{\eta(|v|)}(v_F) + v_{E_1}$, where Rot_ϕ is rotation through angle ϕ in F. Obviously, $h_1 \in \mathrm{LIP}(E;D)$, $\mathrm{supp}(h_1) \subseteq B^E(0,\bar{t}) - \bar{B}^E(0,\bar{s})$, $h_1(x) = x$ and for some $c > 0$, $h_1(y) = cx$. It is easy to construct a radial homeomorphism $h_2 \in \mathrm{LIP}(E;D)$ such that $\mathrm{supp}(h_2) \subseteq B^E(0,\bar{t}) - \bar{B}^E(0,\bar{s})$, $h_2(x) = x$ and $h_2(cy) \in B(x,\varepsilon)$. So $h = h_2 \circ h_1$ is as required. ∎

Theorem 4.8 is phrased in a way that part (a) is easiest to read, (b) is the main statement of the theorem, and (c) is the "pointwise" version of (b). So (c)⇒(b)⇒(a), and we actually prove (c).

Note that Theorem 4.8(b) is analogous to Theorem 3.27, but the assumption here is that $(\mathrm{UC}(X))^\tau \subseteq \mathrm{LUC}(Y)$, whereas in 3.27 the weaker assumption that $(\mathrm{LIP}(X))^\tau \subseteq H^{\mathrm{LC}}_\Gamma(Y)$ did suffice.

THEOREM 4.8. (a) *Let X, Y be open subsets of the normed spaces E and F, and $\tau \in H(X,Y)$ be such that* $(\mathrm{UC}(X))^\tau \subseteq \mathrm{LUC}(Y)$. *Then* $\tau \in \mathrm{LUC}^{\pm}(X,Y)$.

(b) *Let $\langle E, X, \mathcal{S}, \mathcal{D} \rangle$ be a subspace choice system, Y an open subset of a normed space F and $\tau \in H(X,Y)$. Suppose that* $(\mathrm{UC}(X;\mathcal{S},\mathcal{D}))^\tau \subseteq \mathrm{LUC}(Y)$. *Then* $\tau \in \mathrm{LUC}^{\pm}(X,Y)$.

(c) *Let X, Y be open subsets of the normed spaces E and F, $S \subseteq X$ be open, D be a dense linear subspace of E, $x^* \in S$ and $\tau \in H(X,Y)$. Suppose that* $(\mathrm{UC}(X;S,D,x^*))^\tau \subseteq \mathrm{PNT.UC}(Y,\tau(x^*))$. *Then τ is bi-UC at x^*.*

Proof. (c) Let X, Y etc. be as in (c).

PART 1. *τ is UC at x^*.*

Suppose by contradiction that for every $U \in \mathrm{Nbr}^X(x^*)$, $\tau \restriction U$ is not UC. The trivial proof of the following claim is left to the reader.

CLAIM 1. *For every $r > 0$ there are sequences $\vec{x}, \vec{y}$ and $d, e > 0$ such that:*

(1) $\mathrm{Rng}(\vec{x}) \cup \mathrm{Rng}(\vec{y}) \subseteq B^X(x^*, r/2) \cap D$;
(2) $\lim_{n\to\infty} \|x_n - y_n\| = 0$;
(3) *either* (i) *for any distinct $m, n \in \mathbb{N}$, $d(\{x_m, y_m\}, \{x_n, y_n\}) \geq e$, or* (ii) *$\vec{x}$ is a Cauchy sequence*;
(4) $d(\mathrm{Rng}(\vec{x}) \cup \mathrm{Rng}(\vec{y}), x^*) > e$;
(5) *for every $n \in \mathbb{N}$, $\|\tau(x_n) - \tau(y_n)\| \geq d$.*

Let $e_{-1} > 0$ be such that $B^E(x^*, e_{-1}) \subseteq S$. It is easy to define by induction on $i \in \mathbb{N}$ a radius r_i, sequences $\vec{x}^i = \{x^i_n \mid n \in \mathbb{N}\}$, $\vec{y}^i = \{y^i_n \mid n \in \mathbb{N}\}$ and $d_i, e_i > 0$ such that $r_i = e_{i-1}/8$ and such that $\vec{x}^i, \vec{y}^i, d_i, e_i$ satisfy (1)–(5) of Claim 1 for $r = r_i$. By deleting, if necessary, initial segments from each of the $\vec{x}^{\,i}$'s and $\vec{y}^{\,i}$'s, we may further assume that for every $i, n \in \mathbb{N}$, $\|x^i_n - y^i_n\| < e_i/4$. We may further assume that either for every $i \in \mathbb{N}$, clause (3)(i) of Claim 1 holds, or for every $i \in \mathbb{N}$, clause (3)(ii) of Claim 1 holds.

CASE 1: Clause (3)(i) of Claim 1 holds. Let $\{\langle i(k), n(k)\rangle \mid k \in \mathbb{N}\} \subseteq \mathbb{N}^2$ be a 1-1 sequence of pairs such that $\lim_{k\to\infty} \|x^{i(k)}_{n(k)} - y^{i(k)}_{n(k)}\| = 0$, and for every $i \in \mathbb{N}$, $\{k \mid i(k) = i\}$ is infinite. For every $k \in \mathbb{N}$ set $u_k = x^{i(k)}_{n(k)}$, $v_k = y^{i(k)}_{n(k)}$, $s_k = 2\|u_k - v_k\|$ and $B_k = B(u_k, s_k)$. Then it can be easily checked that for any distinct $k, l \in \mathbb{N}$, $B_k \subseteq S$ and $d(B_k, B_l) > e_{i(k)}/4$. Also, $\lim_{k\to\infty} \mathrm{diam}(B_k) = 0$. Let $w_k \in [u_k, v_k] - \{u_k\}$ be such that $\|\tau(w_k) - \tau(u_k)\| < 1/(k+1)$. So $w_k \in B_k \cap D$. By Lemma 2.14(c), there is $h_k \in \mathrm{LIP}(X; S, D)$ such that $\mathrm{supp}(h_k) \subseteq B_k$, $h_k(u_k) = u_k$ and $h_k(w_k) = v_k$.

By Proposition 4.5, $h := \circ_{k\in\mathbb{N}} h_k \in \mathrm{UC}(X)$, and indeed $h \in \mathrm{UC}(X; S, D, x^*)$. However, we shall now see that for every $V \in \mathrm{Nbr}^Y(\tau(x^*))$, $h^\tau \restriction V$ is not uniformly continuous and hence $h^\tau \notin \mathrm{PNT.UC}(Y, \tau(x^*))$ which is a contradiction.

Write $h^\tau = \hat{h}$, $h(u_k) = \hat{u}_k$, $h(v_k) = \hat{v}_k$ and $h(w_k) = \hat{w}_k$. Then $\hat{h}(\hat{u}_k) = \hat{u}_k$ and $\hat{w}_k = \hat{v}_k$. There is i such that for every n, $\tau([x^i_n, y^i_n]) \subseteq V$. Define $\sigma = \{k \in \mathbb{N} \mid i(k) = i\}$. Then $\hat{u}_k, \hat{v}_k, \hat{w}_k \in V$ for every $k \in \sigma$. Now, $\lim_{k\in\sigma} \|\hat{u}_k - \hat{w}_k\| = 0$, but $\|\hat{h}(\hat{u}_k) - \hat{h}(\hat{w}_k)\| = \|\hat{u}_k - \hat{v}_k\| \geq d_i$ for every $k \in \sigma$. So $\hat{h} \restriction V$ is not uniformly continuous.

CASE 2: Clause (3)(ii) of Claim 1 holds. Let $\bar{z}_i = \lim \vec{x}^i$. Note that $\bar{z}_i \in \bar{E} - E$. Clearly, $\bar{z}_i \in B^{\bar{E}}(x^*, r_i) - B^{\bar{E}}(x^*, e_i)$. Fix $i \in \mathbb{N}$ and for $j \in \mathbb{N}$ set $t_{i,j} = \max(\|x^i_j - \bar{z}_i\|, \|y^i_j - \bar{z}_i\|)$ and $s_{i,j} = \min(\|x^i_j - \bar{z}_i\|, \|y^i_j - \bar{z}_i\|)$. By taking a subsequence of $\{\langle x^i_j, y^i_j\rangle \mid j \in \mathbb{N}\}$, we may assume that for every j, $t_{i,j+1} < s_{i,j}$. Let $\varepsilon_{i,j} > 0$ be such that for every $u \in B(x^i_j, \varepsilon_{i,j})$, $\|\tau(u) - \tau(x^i_j)\| < \frac{1}{j+1}$. Choose $\bar{s}_{i,j}, \bar{t}_{i,j}$ such that for every j, $s_{i,j} > \bar{s}_{i,j} > \bar{t}_{i,j+1} > t_{i,j+1}$. We may also assume that for any distinct i and i', $d(B^{\bar{E}}(\bar{z}_i, \bar{t}_{i,0}), B^{\bar{E}}(\bar{z}_{i'}, \bar{t}_{i',0})) > 0$ and that $B^{\bar{E}}(\bar{z}_0, \bar{t}_{0,0}) \subseteq \mathrm{cl}^{\bar{E}}(S)$.

By Proposition 4.7(c), for every i, j there is $h_{i,j} \in \mathrm{UC}(X; D)$ such that $\mathrm{supp}(h_{i,j}) \subseteq B^{\bar{E}}(\bar{z}_i, \bar{t}_i) - \bar{B}^{\bar{E}}(\bar{z}_i, \bar{s}_i)$, $h_{i,j}(x^i_j) = x^i_j$ and $h_{i,j}(y^i_j) \in B(x^i_j, \varepsilon_{i,j})$. Let $h_i = \circ_{j\in\mathbb{N}} h_{i,j}$. By Proposition 4.5, $h_i \in \mathrm{UC}(X)$. So $h_i \in \mathrm{UC}(X; D)$. Also, $\mathrm{supp}(h_i) \subseteq S$. Let $h = \circ_{i\in\mathbb{N}} h_i$. Applying again Proposition 4.5, we conclude that $h \in \mathrm{UC}(X; S, D, x^*)$.

We check that h^{τ} is not bi-UC at $\tau(x^*)$. Let $V \in \mathrm{Nbr}^Y(\tau(x^*))$. For some i, $\mathrm{supp}((h_i)^{\tau}) \subseteq V$. Define $u^i_j = \tau(x^i_j)$ and $v^i_j = \tau(y^i_j)$. So

(1) for every j, $\|u^i_j - v^i_j\| > d_i$.

Since $h_i(y^i_j) \in B(x^i_j, \varepsilon_{i,j})$, it follows that $\lim_{j\to\infty} \|\tau(x^i_j) - \tau(h_i(y^i_j)\| = 0$. That is, $\lim_{j\to\infty} \|(h_i)^{\tau}(u^i_j) - (h_i)^{\tau}(v^i_j)\| = 0$. Hence

(2) $\lim_{j\to\infty} \|h^{\tau}(u^i_j) - h^{\tau}(v^i_j)\| = 0$.

(1) and (2) imply that $h^{\tau}\restriction V$ is not bi-UC. That is, $h^{\tau} \notin \mathrm{PNT.UC}(Y, \tau(x^*))$. A contradiction. We have reached a contradiction in both Case 1 and Case 2. So τ is UC at x^*.

PART 2. *τ^{-1} is UC at $\tau(x^*)$.*

Suppose by contradiction that this is not true. So for every $V \in \mathrm{Nbr}^Y(\tau(x^*))$, $\tau^{-1}\restriction V$ is not UC.

CLAIM 2. *For every $k \in \mathbb{N}$ there are positive numbers $r^k_1, \dots, r^k_5$ and sequences $\vec{x}^k$ and $\vec{y}^k$ which fulfill the following requirements.*

(1) $r^k_1 > r^k_2 \geq r^k_3 > r^k_4 > r^k_5 = 2r^{k+1}_1$.
(2) $\lim_{i\to\infty} \|x^k_i - x^*\| = r^k_2$ *and* $\lim_{i\to\infty} \|y^k_i - x^*\| = r^k_3$.
(3) *There is $e_k > 0$ such that $\|x^k_i - y^k_i\| > e_k$ for every $i \in \mathbb{N}$.*
(4) $\mathrm{Rng}(\vec{x}^k) \cup \mathrm{Rng}(\vec{y}^k) \subseteq D$.
(5) *Define* $s_k = \sup(\{\|\tau(x) - \tau(x^*)\| \mid x \in B(x^*, r^k_4)\})$ *and* $t_k = \|\tau(x^*) - \tau(\vec{x}^k)\|$. *Then $s_k < t_k$.*
(6) $\lim_{i\to\infty} \|\tau(x^k_i) - \tau(y^k_i)\| = 0$.
(7) *Either $\vec{x}^k$ is a Cauchy sequence or $\vec{x}^k$ is spaced, and either $\vec{y}^k$ is a Cauchy sequence or $\vec{y}^k$ is spaced.*

Proof. Let $r^0_1 > 0$ be such that $\bar{B}(x^*, r^0_1) \subseteq S$. Suppose that r^k_1 has been defined, and we define $r^k_2, \dots, r^k_5$ and r^{k+1}_1. Let $r = r^k_1/2$. Since $\tau^{-1}\restriction\tau(B(x^*, r))$ is not uniformly continuous, there are $e_k > 0$ and sequences $\vec{x}, \vec{y} \subseteq B(x^*, r)$ such that for every $i \in \mathbb{N}$, $\|x_i - y_i\| > e_k$ and $\lim_{i\to\infty} \|\tau(x_i) - \tau(y_i)\| = 0$. Since $D \cap S$ is dense in S, we may assume that $\mathrm{Rng}(\vec{x}) \cup \mathrm{Rng}(\vec{y}) \subseteq D$. We may also assume that $x^* \notin \mathrm{Rng}(\vec{x}) \cup \mathrm{Rng}(\vec{y})$.

By interchanging some x_i's with their corresponding y_i's, we may assume that $\|x_i - x^*\| \geq \|y_i - x^*\|$. Taking subsequences we may assume that $r^k_2 := \lim_{i\to\infty} \|x_i - x^*\|$ and $r^k_3 := \lim_{i\to\infty} \|y_i - x^*\|$ exist. Hence $r^k_3 \leq r^k_2$. Taking subsequences again, we may assume that either $\vec{x}$ is a Cauchy sequence or $\vec{x}$ is spaced, and that either $\vec{y}$ is a Cauchy sequence or $\vec{y}$ is spaced.

Note that $\vec{x}$ does not contain a convergent subsequence, since if x' is a limit of a subsequence of $\vec{x}$, then τ^{-1} is not continuous at $\tau(x')$. Also recall that $x^* \notin \mathrm{Rng}(\vec{x})$. It thus follows that $t_k := \|\tau(x^*), \tau(\vec{x}^k)\| > 0$. Next define $\vec{x}^k = \vec{x}$ and $\vec{y}^k = \vec{y}$. Let $r^k_4 < r^k_3$ be such that $s_k := \sup(\{\|\tau(x) - \tau(x^*)\| \mid x \in B(x^*, r^k_4)\}) < t_k$. Finally, let $r^k_5 = r^k_4/2$ and $r^{k+1}_1 = r^k_5/2$. This concludes the construction which proves Claim 2.

Since $\lim_{i\to\infty} \|x^k_i\| = r^k_2$ and $\lim_{i\to\infty} \|y^k_i\| = r^k_3$, we may assume that

(8) for every $i \in \mathbb{N}$, $r^k_4 < \|x^k_i - x^*\| < (r^k_2 + r^k_1)/2$ and $r^k_4 < \|y^k_i - x^*\| < (r^k_2 + r^k_1)/2$.

We may also assume that either for every $k \in \mathbb{N}$, $\vec{y}^k$ is spaced, or for every $k \in \mathbb{N}$, $\vec{y}^k$ is a Cauchy sequence.

Case 1: For every $k \in \mathbb{N}$, $\vec{y}^k$ is spaced. Fix $k \in \mathbb{N}$ and denote r_i^k, $\vec{x}^k$, $\vec{y}^k$ and e_k by r_i, $\vec{x}$, $\vec{y}$ and e respectively.

Claim 3. *There are subsequences* $\{x_{i_n} \mid n \in \mathbb{N}\}$ $\{y_{i_n} \mid n \in \mathbb{N}\}$ *of* $\vec{x}$ *and* $\vec{y}$ *respectively, such that* $d(\{x_{i_n} \mid n \in \mathbb{N}\}, \{y_{i_n} \mid n \in \mathbb{N}\}) > 0$.

Proof. The claim is trivial if $\vec{x}$ is a Cauchy sequence. So suppose $\vec{x}$ is spaced. We show that there is a sequence $\{i_n \mid n \in \mathbb{N}\}$ such that (i) $\lim_{n>m\to\infty} \|x_{i_m} - y_{i_n}\|$ exists, and (ii) $\lim_{n>m\to\infty} \|y_{i_m} - x_{i_n}\|$ exists. By repeatedly applying the Ramsey Theorem, we obtain a decreasing sequence $A_0 \supseteq A_1 \supseteq \cdots$ of infinite subsets of $\mathbb{N}$ such that for every $\ell \in \mathbb{N}$ and $m, n, m', n' \in A_\ell$: if $m < n$ and $m' < n'$, then $|\|x_m - y_n\| - \|x_{m'} - y_{n'}\|| < 2^{-\ell}$. Let $\{i_n \mid n \in \mathbb{N}\}$ be a 1-1 sequence such that for every $n \in \mathbb{N}$, $i_n \in A_n$. Then (i) holds for $\{i_n \mid n \in \mathbb{N}\}$. The same argument is applied to show that (ii) holds.

Let $\bar{s}_1 = \lim_{n>m\to\infty} \|x_{i_m} - y_{i_n}\|$ and $\bar{s}_2 = \lim_{n>m\to\infty} \|y_{i_m} - x_{i_n}\|$. It is easy to see that if $\bar{s}_1 = 0$ or $\bar{s}_2 = 0$, then $\vec{x}$ is a Cauchy sequence. So $\bar{s}_1, \bar{s}_2 > 0$. By removing an initial segment from the sequences $\{x_{i_n}\}_{n\in\mathbb{N}}$ and $\{y_{i_n}\}_{n\in\mathbb{N}}$ we may assume that for every $n > m$, $\|x_{i_m} - y_{i_n}\| > \bar{s}_1/2$ and $\|x_{i_n} - y_{i_m}\| > \bar{s}_2/2$. Recall also that $\|x_i - y_i\| > e$ for every $i \in \mathbb{N}$. So $d(\{x_{i_n} \mid n \in \mathbb{N}\}, \{y_{i_n} \mid n \in \mathbb{N}\}) \geq \min(\bar{s}_1/2, \bar{s}_2/2, e)$. So Claim 3 is proved.

We may thus assume that $d_k := d(\mathrm{Rng}(\vec{x}^k), \mathrm{Rng}(\vec{y}^k)) > 0$.

Claim 4. *For every* $k \in \mathbb{N}$ *there is* $h_k \in \mathrm{LIP}(X; D)$ *with the following properties*: (i) $\mathrm{supp}(h_k) \subseteq B(x^*, r_1^k) - B(x^*, r_5^k)$; *and* (ii) *there is* $n_k \in \mathbb{N}$ *such that for every* $i \geq n_k$, $h_k(x_i^k) = x_i^k$ *and* $h_k(y_i^k) \in B(x^*, r_4^k)$.

Proof. Fix k, for $j = 1, \ldots, 5$ set $r_j^k = r_j$, write $\vec{x}^k = \vec{x}$, $\vec{y}^k = \vec{y}$, $x_i^k = x_i$, $y_i^k = y_i$ and define $w_i = x_i - x^*$, $z_i = y_i - x^*$, $u_i = z_i/\|z_i\|$. Note that $\lim_{i\in\mathbb{N}} \|(x^* + r_3 u_i) - y_i\| = 0$, and recall that $d(\mathrm{Rng}(\vec{x}), \mathrm{Rng}(\vec{y})) > 0$. From these facts it follows that by removing an initial segment of $\vec{x}$ and of $\vec{y}$, we may assume that there is $a > 0$ such that $\|x_i - (x^* + r_3 u_j)\| \geq a$ for every $i, j \in \mathbb{N}$. Similarly, since $\vec{y}$ is spaced, we may assume that $\{x^* + r_3 u_i\}_{i\in\mathbb{N}}$ is spaced too. Certainly we may choose a to be smaller than $r_3 - r_4$ and $r_1 - r_3$, and we may assume that for every i, $\|w_i\| \geq r_3 - a/8$ and $r_3 - a/4 < \|z_i\| < r_3 + a/4$. Let $L_i = [x^* + r_4 u_i, x^* + (r_3 + a/4) u_i]$. Note that $y_i \in L_i$. We show that for every i, j, $d(x_i, L_j) > a/4$. Let $y \in L_j$. If $y \in [x^* + (r_3 - a/2) u_j, x^* + (r_3 + a/4) u_j]$, then

$$\|x_i - y\| \geq \|x_i - (x^* + r_3 u_j)\| - \|(x^* + r_3 u_j) - y\| \geq a - a/2 = a/2,$$

and if $y \in [x^*, x^* + (r_3 - a/2) u_i]$, then

$$\|x_i - y\| \geq \|x_i - x^*\| - \|y - x^*\| \geq r_3 - a/8 - (r_3 - a/2) = 3a/8.$$

It follows that $d(x_i, L_j) > a/4$.

Let $v_i = x^* + r_4 u_i$, and let $b > 0$ be such that for every $i \neq j$, $\|v_i - v_j\| > b$. We show that if $i \neq j$, then $d(L_i, L_j) \geq b/2$. It is easy to see that $d(L_i, L_j) = d(v_i, L_j)$. Let $x^* + t u_j \in L_j$. If $t \in [r_4, r_4 + b/2]$, then

$$\|v_i - (x^* + t u_j)\| \geq \|v_i - v_j\| - \|x^* + t u_j - v_j\| > b - b/2 = b/2.$$

If $t > r_4 + b/2$, then

$$\|v_i - (x^* + tu_j)\| \geq \|tu_j\| - \|v_i - x^*\| > r_4 + b/2 - r_4 = b/2.$$

It follows that there is $d > 0$ such that:

(1) for every $i \neq j$, $2d < d(L_i, L_j)$;
(2) for every $i \neq j$, $d < d(x_i, L_j)$;
(3) $r_3 + a/4 + d < r_1$;
(4) $r_4 - d > r_5$.

Let $L_i^1 = [v_i, y_i]$. So $L_i^1 \subseteq L_i$. Hence

(1.1) for every $i \neq j$, $2d < d(L_i^1, L_j^1)$;
(1.2) for every $i \neq j$, $d < d(x_i, L_j^1)$;
(1.3) $\|y_i - v_i\| < r_3 - r_4 + a/4$.

By (3), $d(B(L_i^1, d), X - B(x^*, r_1)) > r_1 - (r_3 + a/4 + d) > 0$ and by (4), $d(B(L_i^1, d), B(x^*, r_5)) > r_4 - r_5 - d > 0$. So

(1.4) $d(B(L_i^1, d), X - (B(x^*, r_1) - B(x^*, r_5))) > 0$ for every $i \in \mathbb{N}$.

Recall that $y_i \in D$, but v_i need not be in D. For every i, choose $v_i' \in D$ sufficiently close to v_i and define $L_i' = [v_i', y_i]$. This can be done in such a way that L_i' satisfy (1.1)–(1.4). So indeed choose $v_i' \in D \cap B(x^*, r_4)$ in such a way that the L_i''s fulfill (1.1)–(1.4). Write $v_{k,i} = v_i'$.

Let $K = K_{\mathrm{seg}}(r_3 - r_4 + a/4, d)$ be as in 2.14(c) and $i \in \mathbb{N}$. By 2.14(c), there is $h_i' \in \mathrm{LIP}(X; D)$ such that: $\mathrm{supp}(h_i') \subseteq B(L_i', d)$, h_i' is K-bilipschitz, and $h_i'(y_i) = v_i'$. Since the L_i''s satisfy (1.1), it follows that for every $i \neq j$, $d(\mathrm{supp}(h_i'), \mathrm{supp}(h_j')) > 0$. So $h_k := \circ_{i \in \mathbb{N}} h_i'$ is well defined. Also, h_k is $2K$-bilipschitz.

For every i, $h_k(y_i) = h_i'(y_i) = v_i' \in B(x^*, r_4)$. By (1.2) applied to the L_j''s, $x_i \notin \mathrm{supp}(h_k)$. So $h_k(x_i) = x_i$. By (1.4) applied to L_i', for every i, $\mathrm{supp}(h_j') \subseteq B(x^*, r_1) - B(x^*, r_5)$. So $\mathrm{supp}(h_k) \subseteq B(x^*, r_1) - B(x^*, r_5)$. Recall that for every i, $h_i' \in H(X; D)$. So $h_k \in H(X; D)$. We have shown that h_k fulfills the requirements of Claim 4.

Let $h = \circ_{k \in \mathbb{N}} h_k$. By Proposition 4.5, $h \in \mathrm{UC}(X)$. Since $B(x^*, r_1^0) \subseteq S$, we obtain that $\mathrm{supp}(h) \subseteq S$, and since for every k, $h_k \in H(X; D)$, we conclude that $h \in H(X; D)$. Also for every k, $x^* \notin \mathrm{supp}(h_k)$. So $h(x^*) = x^*$, that is, $h \in \mathrm{UC}(X; S, D, x^*)$.

We shall reach a contradiction by showing that $h^\tau \notin \mathrm{PNT.UC}(Y \tau(x^*))$. Let $V \in \mathrm{Nbr}^Y(\tau(x^*))$. Let k be such that $\tau(B(x^*, r_1^k)) \subseteq V$. Hence

(i) for every $i \in \mathbb{N}$, $\tau(x_i^k), \tau(y_i^k) \in V$, and $\lim_{i \to \infty} \|\tau(x_i^k) - \tau(y_i^k)\| = 0$.

Now $h^\tau(\tau(x_i^k)) = \tau(x_i^k)$ and $h^\tau(\tau(y_i^k)) = \tau(h(y_i^k)) \in \tau(B(x^*, r_4^k))$. So for every $i \in \mathbb{N}$,

$$(\dagger) \quad \begin{aligned} &\|(h^\tau(\tau(x_i^k)) - \tau(x^*)) - (h^\tau(\tau(y_i^k)) - \tau(x^*))\| \\ &= \|(\tau(x_i^k) - \tau(x^*)) - (\tau(h(y_i^k)) - \tau(x^*))\| \geq \|\tau(x_i^k) - \tau(x^*)\| - \|\tau(h(y_i^k)) - \tau(x^*)\|. \end{aligned}$$

Recall that $h(y_i^k) = v_{k,i} \in B(x^*, r_4^k)$. Let s_k, t_k be as in clause (5) of Claim 2. Then $\|\tau(h(y_i^k)) - \tau(x^*)\| \leq s_k$ and $\|\tau(x_i^k) - \tau(x^*)\| \geq t_k$. Denote the right hand side of $(\dagger)$ by A. So $A \geq t_k - s_k$. By clause (5) in Claim 2, $t_k - s_k > 0$. We have proved that

(ii) for every $i \in \mathbb{N}$, $\|h^\tau(\tau(x_i^k)) - h^\tau(\tau(y_i^k))\| \geq t_k - s_k > 0$.

(i) and (ii) demonstrate that $h^\tau \restriction V$ is not bi-UC. We have shown that for every $V \in \mathrm{Nbr}(\tau(x^*))$, $h^\tau \restriction V$ is not UC. That is, $h^\tau \notin \mathrm{PNT.UC}(Y, \tau(x^*))$. A contradiction.

Case 2: For every $k \in \mathbb{N}$, $\vec{y}^k$ is a Cauchy sequence.

Claim 5. *For every $k \in \mathbb{N}$ there is $h_k \in \mathrm{LIP}(X; D)$ with the following properties*: (i) $\mathrm{supp}(h_k) \subseteq B(x^*, r_1^k) - B(x^*, r_5^k)$; *and* (ii) *there is $n_k \in \mathbb{N}$ such that for every $i \geq n_k$, $h_k(x_i^k) = x_i^k$ and $h_k(y_i^k) \in B(x^*, r_4^k)$.*

Proof. Fix k, and denote $\vec{x}^k$, $\vec{x}^k$, r_j^k etc. by $\vec{x}$, $\vec{y}$, r_j etc. Let $\bar{y} = \lim^{\bar{E}} \vec{y}$. Since τ^{-1} is continuous, $\bar{y} \in \mathrm{cl}^{\bar{E}}(S) - S$. Also, $\|\bar{y} - x^*\| = r_3$. Since $\bar{y} \notin E$ and $\mathrm{Rng}(\vec{x}) \subseteq E$, $\mathrm{Rng}(\vec{x}) \cap [x^*, \bar{y}]$ contains at most one element. By removing this element we may assume that $\hat{e} := d(\mathrm{Rng}(\vec{x}), [x^*, \bar{y}]) > 0$. Let $b = (r_4 + r_5)/2$, $a = (r_4 - r_5)/2$ and $c = \min(a, \hat{e}, r_1 - r_3)$. Let $w \in [x^*, \bar{y}]$ be such that $\|w - x^*\| = b$. Let $u, v \in D$ be such that $\|u - \bar{y}\|, \|v - w\| < c/12$. By Lemma 2.14(c), there is $h \in \mathrm{LIP}(X; D)$ such that $\mathrm{supp}(h) \subseteq B([u, v], c/4)$, $h(u) = v$ and $h(B(u, c/12)) = B(v, c/12)$. Since h is bilipschitz, $\mathrm{Dom}(h^{\mathrm{cl}}) = \mathrm{cl}^{\bar{E}}(X)$. Denote $\hat{h} = h^{\mathrm{cl}}$. We show that $\hat{h}(\bar{y}) \in B^{\bar{E}}(x^*, r_4)$. Since $\bar{y} \in B^{\bar{E}}(u, c/12)$, $\hat{h}(\bar{y}) \in B^{\bar{E}}(v, c/12)$. So

$$\|\hat{h}(\bar{y}) - x^*\| \leq \|\hat{h}(\bar{y}) - v\| + \|v - w\| + \|w - x^*\| < c/12 + c/12 + b \leq b + a/6 < b + a = r_4.$$

It follows that

(1) for all but finitely many i's, $h(y_i) \in B(x^*, r_4)$.

For every i, $d(x_i, [u, v]) \geq d(x_i, [\bar{y}, w]) - (c/12 + c/12) \geq \hat{e} - c/6 \geq c/4$. So $x_i \notin \mathrm{supp}(h)$ and hence

(2) $h(x_i) = x_i$ for all $i \in \mathbb{N}$.

$\|u - x^*\| \leq c/12 + r_3 < r_1 - c/4$. It easily follows that $B([u, v], c/4) \subseteq B(x^*, r_1)$. $\|v - x^*\| \geq b - c/12 > r_5 + a/4$. Next we have

$$d(B([u, v], c/4), x^*) \geq d(B([\bar{y}, w], c/4), x^*) - c/6 - c/4 = b - 5c/12 > r_5.$$

So $B([u, v], c/4) \cap B(x^*, r_5) = \emptyset$. Similarly, for every $y \in B([u, v], c/4)$,

$$\|y\| \leq \max(\|u\|, \|v\|) + c/4 \leq \max(\|\bar{y}\|, \|w\|) + c/12 + c/4 = r_3 + 5c/12 < r_1.$$

That is, $\mathrm{supp}(h) \subseteq B(x^*, r_1)$. So

(3) $\mathrm{supp}(h) \subseteq B(x^*, r_1) - B(x^*, r_5)$.

It follows that $h_k := h$ fulfills the requirements of Claim 5. So Claim 5 is proved.

The remaining steps in the proof are identical to those in Case 1. So both Case 1 and Case 2 lead to a contradiction. This means that τ^{-1} is UC at $\tau(x^*)$. ∎

Question 4.9. Let X, Y be open subsets of the normed spaces E and F and $\tau \in H(X, Y)$ be such that $(\mathrm{LIP}(X))^\tau \subseteq \mathrm{LUC}(Y)$. Is τ locally UC? Is τ^{-1} locally UC? □

Note that by Theorem 3.27, the answer to both parts of the question is positive for E's such that $\kappa(E) \geq 2^{\aleph_0}$. Hence the answer is positive for open subsets of ℓ_∞.

Proof of Theorem 4.1. (a) Let X, Y, Γ, Δ and φ be as in part (a). Suppose that Γ is $(\leq\kappa(E))$-generated. Then by Theorem 3.42, $\Gamma = \Delta$ and there is $\tau \in H(X, Y)$ as required.

Note that for every metric space X, $\mathrm{LUC}(X) = H^{\mathrm{LC}}_{\mathrm{MC}}(X)$.

Suppose that $\Gamma = \mathrm{MC}$. By Theorem 2.8(a), there is $\tau \in H(X,Y)$ such that τ induces φ. We have $(\mathrm{UC}(X))^{\tau} \subseteq \mathrm{LUC}(Y)$. So by Theorem 4.8(a), τ is locally bi-UC. So $(\mathrm{LUC}(X))^{\tau} = \mathrm{LUC}(Y)$. Hence $H^{\mathrm{LC}}_{\mathrm{MC}}(X) = H^{\mathrm{LC}}_{\Delta}(Y)$. We have seen that the above equality implies that $\mathrm{MC} = \Delta$. So (a) is proved.

(b) Let $\langle E, X, \mathcal{S}, \mathcal{E}\rangle$, $\langle F, Y, \mathcal{T}, \mathcal{F}\rangle$, Γ, Δ be and φ be as in (b). If both Γ and Δ are $(\leq\kappa(E))$-generated, then by Theorem 3.42, $\Gamma = \Delta$, and there is τ which induces φ.

Suppose that Δ or Γ are not $(\leq\kappa(E))$-generated. By Theorem 2.8(a), there is $\tau \in H(X,Y)$ such that τ induces φ.

Suppose by contradiction that $\Gamma = \mathrm{MC}$ and $\Delta \neq \mathrm{MC}$. Hence Δ is $(\leq\kappa(E))$-generated. Clearly, $(\mathrm{LIP}(X;\mathcal{S},\mathcal{E}))^{\tau} \subseteq H^{\mathrm{LC}}_{\Delta}(Y)$. By Theorem 3.27, τ is locally Δ-bicontinuous. Hence $(H^{\mathrm{LC}}_{\Delta}(Y;\mathcal{T},\mathcal{F}))^{\tau^{-1}} \subseteq H^{\mathrm{LC}}_{\Delta}(X)$. However, $(H^{\mathrm{LC}}_{\Delta}(Y;\mathcal{T},\mathcal{F}))^{\tau^{-1}} = H^{\mathrm{LC}}_{\mathrm{MC}}(X;\mathcal{S},\mathcal{E})$. Hence $H^{\mathrm{LC}}_{\mathrm{MC}}(X;\mathcal{S},\mathcal{E}) \subseteq H^{\mathrm{LC}}_{\Delta}(X)$. A contradiction. It follows that $\Gamma = \Delta = \mathrm{MC}$. ∎

As in Chapter 3, the analogous statement for manifolds is also true.

Corollary 4.10. *Let $\langle X, \Phi\rangle$ and $\langle Y, \Psi\rangle$ be normed manifolds with locally Lipschitz atlases. Let Γ and Δ be moduli of continuity, Suppose that Γ is countably generated or* $\Gamma = \mathrm{MC}$, *and the same holds for Δ.*

(a) *If $\varphi : H^{\mathrm{LC}}_{\Gamma}(X, \Phi) \cong H^{\mathrm{LC}}_{\Delta}(Y)$. Then $\Gamma = \Delta$ and there is $\tau : X \cong Y$ such that τ induces φ, and τ is locally Γ-bicontinuous.*

(b) *Let $\mathcal{S}$ be an open cover of X, $\mathcal{T}$ be an open cover of Y and $\varphi : H^{\mathrm{LC}}_{\Gamma}(X, \Phi, \mathcal{S}) \cong H^{\mathrm{LC}}_{\Delta}(Y, \Psi, \mathcal{T})$. Then $\Gamma = \Delta$, there is $\tau : X \cong Y$ such that $\varphi(h) = h^{\tau}$ for every $h \in H^{\mathrm{LC}}_{\Gamma}(X;\mathcal{S},\mathcal{E})$, and τ is locally Γ-bicontinuous.*

5. Other groups defined by properties related to uniform continuity

5.1. General description. The results we have obtained on groups of type $H_{\Gamma}^{\mathrm{LC}}(X)$ are more comprehensive than those obtained for other types of groups. We have presented the results on $H_{\Gamma}^{\mathrm{LC}}(X)$ in the quite general framework of "subspace choice systems". We now abandon this framework, and restrict the discussion to the class of open subsets of normed spaces.

Recall the following notations which were introduced in the introduction.

Definition 5.1. (a) For a set F of 1-1 functions let $F^{-1} = \{f^{-1} \mid f \in F\}$. Suppose that $\mathcal{P}$ is used as an abbreviation for some property of maps, and let X and Y be topological spaces. We shall use the notation $\mathcal{P}(X,Y)$ to denote the set of all homeomorphisms between X and Y which have property $\mathcal{P}$. We define

$$\mathcal{P}^{\pm}(X,Y) := \mathcal{P}(X,Y) \cap (\mathcal{P}(Y,X))^{-1} \quad \text{and} \quad \mathcal{P}(X) := \mathcal{P}^{\pm}(X,X).$$

Usually but not always this convention will be used for $\mathcal{P}$'s which are "closed under composition". ($\mathcal{P}$ is *closed under composition* if for every $f : X \to Y$ and $g : Y \to Z$: if f and g fulfill $\mathcal{P}$, then $g \circ f$ fulfills $\mathcal{P}$.) In such cases $\mathcal{P}(X)$ is a group.

(b) Let $\langle X, d\rangle$ be a metric space. X is *uniformly-in-diameter arcwise-connected* (*UD.AC*) if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $x, y \in X$: if $d(x,y) < \delta$, then there is an arc $L \subseteq X$ connecting x and y such that $\operatorname{diam}(L) < \varepsilon$.

(c) Let $K_{\mathrm{NRM}}^{\mathcal{O}}$ be the class of all spaces X such that X is an open subset of a normed space. Let $K_{\mathrm{BNC}}^{\mathcal{O}}$ be the class of all spaces X such that X is an open subset of a Banach space. Let $K_{\mathrm{NFCB}}^{\mathcal{O}}$ be the class of all spaces X such that X is an open subset of a normed space of the first category, or X is an open subset of a Banach space. $\square$

Note that a disconnected space may be UD.AC. The space $[0,1] \cup [2,3]$ is such an example.

The following statement is a typical example of some of the final results obtained in this chapter. It is restated in Corollary 5.6.

Theorem A. *Let $X, Y \in K_{\mathrm{NFCB}}^{\mathcal{O}}$. Suppose that X and Y are UD.AC spaces. Let $\varphi : \mathrm{UC}(X) \cong \mathrm{UC}(Y)$. Then there is $\tau \in \mathrm{UC}^{\pm}(X,Y)$ such that τ induces φ.*

The reason that Theorem A can be proved only for members of $K_{\mathrm{NFCB}}^{\mathcal{O}}$ and not for all members of $K_{\mathrm{NRM}}^{\mathcal{O}}$ is that Theorem 2.8 cannot be used. This is so, since in Theorem 2.8 we need to know that $\mathrm{LIP}^{\mathrm{LC}}(X) \le G$. However, $\mathrm{LIP}^{\mathrm{LC}}(X) \not\le \mathrm{UC}(X)$.

Theorem A assumes that the open sets X and Y are UD.AC. Different extra assumptions on the open sets in question are often used in proving other reconstruction results. We make sure, though, that these extra assumptions do not exclude the known well-behaved open subsets of a normed space. For example, convex bounded open sets are always included. Usually the classes for which reconstruction is proved do contain some complicated open sets.

Theorem A has the following corollary.

THEOREM 5.2. *Let F and K be the closures of UD.AC bounded open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Let $\varphi : H(F) \cong H(K)$. Then φ is induced by a homeomorphism between F and K.*

The proof of Theorem 5.2 appears after Example 5.7. The boundedness of F and K above is necessary: see Example 5.8. The analogue of Theorem 5.2 for open subsets of infinite-dimensional normed spaces is proved in 6.22. The boundedness of F and K is not required in the infinite-dimensional case.

Let us point out that the closure of a UD.AC open subset of $\mathbb{R}^n$ does not have to be a Euclidean manifold with boundary, neither does it have to be a polyhedron. The reconstruction theorems for polyhedra and for Euclidean manifolds with boundary were proved in [Ru1, 3.34 and 3.43]. Theorem 5.2 is not a special case of these theorems.

DEFINITION 5.3. (a) Throughout this section, if not otherwise stated, X and Y denote nonempty open subsets of normed spaces E and F respectively. The metrics d^E and d^F are both abbreviated by d. For $A \subseteq X$, $\mathrm{cl}(A)$, $\mathrm{bd}(A)$, $\mathrm{acc}(A)$, $B(A,r)$ etc. are abbreviations for $\mathrm{cl}^E(A)$, $\mathrm{bd}^E(A)$, $\mathrm{acc}^E(A)$, $B^E(A,r)$ etc. Let $\vec{x}$, $\vec{y}$, $\vec{x}^0$ etc. denote the infinite sequences $\{x_n \mid n \in \mathbb{N}\}$, $\{y_n \mid n \in \mathbb{N}\}$, $\{x^0_n \mid n \in \mathbb{N}\}$ etc. So $\vec{x} \subseteq X$ means that $\{x_n \mid n \in \mathbb{N}\} \subseteq X$.

(b) For $A \subseteq X$ define $\delta^X(A) := d(A, E-X)$. The notation $\delta^X(x)$ abbreviates $\delta^X(\{x\})$ and $\delta^X(A)$ and $\delta^X(x)$ are abbreviated by $\delta(A)$ and $\delta(x)$.

(c) If L is a rectifiable arc, then $\mathrm{lngth}(L)$ denotes the length of L.

(d) Let $A \subseteq X$. We say that A is a *positively distanced set* (*PD set*) if $\delta(A) > 0$. A bounded PD set is called a *BPD set.* A sequence $\vec{x}$ is a *BPD sequence* if $\mathrm{Rng}(\vec{x})$ is a BPD set.

(e) Let $\{A_i \mid i \in \mathbb{N}\}$ be a sequence of sets. We define $\lim_{i\to\infty} A_i = x$ if for every $U \in \mathrm{Nbr}(x)$ there is i_0 such that for every $i > i_0$, $A_i \subseteq U$.

(f) Let $f : X \to Y$. We say that f is *positive distance preserving* (*f is PD.P*) if for every PD set $A \subseteq X$, $f(A)$ is a PD subset of Y. The function f is *boundedness preserving* (*f is BDD.P*) if for every bounded $A \subseteq X$, $f(A)$ is a bounded set, and f is *boundedness positive distance preserving* (*f is BPD.P*) if for every bounded PD set $A \subseteq X$, $f(A)$ is a bounded PD subset of Y.

(h) Let $\mathrm{UC}_0(X) := \{f \in \mathrm{UC}(X) \mid \mathrm{Dom}(f^{\mathrm{cl}}) = \mathrm{cl}(X) \text{ and } f^{\mathrm{cl}}\restriction\mathrm{bd}(X) = \mathrm{Id}\}$. □

The following definition lists some subgroups of $H(X)$ for which reconstruction can be proved.

DEFINITION 5.4. Let $f : X \to Y$.

(a) f is *boundedly UC* (*f is BUC*) if f is boundedness preserving, and for every bounded set $B \subseteq X$, $f\restriction B$ is UC. According to Definition 5.1(a), $\mathrm{BUC}(X,Y) = \{f \in H(X,Y) \mid f \text{ is BUC}\}$.

(b) f is *extendible* if $\mathrm{Dom}(f^{\mathrm{cl}}) = \mathrm{cl}(X)$. According to Definition 4.6(b), $\mathrm{EXT}(X,Y) := \{f \in H(X,Y) \mid f \text{ is extendible}\}$.

(c) *f is bounded positive distance UC* (*f is BPD.UC*) if f is BPD.P, and for every BPD set $A \subseteq X$, $f{\restriction}A$ is UC.

(d) *f is positive distance UC* (*f is PD.UC*) if f is PD.P, and for every PD set $A \subseteq X$, $f{\restriction}A$ is UC.

(e) *f is LUC on* $\mathrm{bd}(X)$ (*f is BR.LUC*) if f is extendible, and for every $x \in \mathrm{bd}(X)$ there is $U \in \mathrm{Nbr}^{\mathrm{cl}(X)}(x)$ such that $f^{\mathrm{cl}}{\restriction}U$ is UC.

(f) *f is completely LUC* (*f is CMP.LUC*) if f is extendible, and f^{cl} is UC at every $x \in \mathrm{cl}(X)$. That is, for every $x \in \mathrm{cl}(X)$ there is $U \in \mathrm{Nbr}^{\mathrm{cl}(X)}(x)$ such that $f^{\mathrm{cl}}{\restriction}U$ is UC.

(g) *f is UC around* $\mathrm{bd}(X)$ (*f is BDR.UC*) if f is extendible, and for some $d > 0$, $f^{\mathrm{cl}}{\restriction}\{x \in \mathrm{cl}(X) \mid \delta(x) < d\}$ is UC.

(h) Let $A, B \subseteq X$. We say that f is *(A,B)-UC* if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $x \in A$ and $y \in B$: if $d(x,y) < \delta$, then $d(f(x), f(y)) < \varepsilon$. The function *$f$ is BI.UC* if f is extendible, and f^{cl} is $(\mathrm{bd}(X), X)$-UC. Note that f is BI.UC iff for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $x, y \in X$: if $\delta(x), d(x,y) < \delta$, then $d(f(x), f(y))) < \varepsilon$. $\square$

Note that if $\mathcal{P}$ is one of the properties defined in (a)–(h), that is, if

$$\mathcal{P} = \mathrm{BUC}, \mathrm{EXT}, \mathrm{BPD.UC}, \mathrm{PD.UC}, \mathrm{BR.LUC}, \mathrm{CMP.LUC}, \mathrm{BDR.UC}, \mathrm{BI.UC},$$

then $\mathcal{P}(X)$ is a group.

For each $\mathcal{P}$ appearing above we can prove the following statement. If $\varphi : \mathcal{P}(X) \cong \mathcal{P}(Y)$, then there is $\tau \in \mathcal{P}^{\pm}(X,Y)$ such that τ induces φ. More precisely, the above statement can be proved, provided that some additional restrictions are imposed on X and Y.

We shall prove the above statement only for $\mathrm{UC}(X)$ and the groups $\mathrm{BUC}(X)$, $\mathrm{EXT}(X)$, $\mathrm{BPD.UC}(X)$ and $\mathrm{CMP.LUC}(X)$ defined in 5.4(a), (b), (c) and (f). Recall that the group $\mathrm{LUC}(X)$ has already been dealt with in Chapter 4. We omit the proof for the remaining groups, since the arguments used are similar to those employed in the proofs that we do present fully. Also, the groups that we do deal with are defined by properties that seem to have played a role in other contexts in analysis and topology.

The group $\mathrm{UC}(X)$ and each of the groups in Definition 5.4 except for $\mathrm{EXT}(X)$ has a generalization in which "uniform continuity" is replaced by "Γ-continuity". This type of generalization is demonstrated by the following three examples.

Example 1. The generalization of $\mathrm{UC}(X)$ is defined as follows. Let Γ be a modulus of continuity. We say that $f : X \to Y$ is nearly Γ-continuous if there are $\alpha \in \Gamma$ and $r > 0$ such that f is (r, α)-continuous. Let $H_{\Gamma}^{\mathrm{NR}}(X,Y)$ be the set of $f \in H(X,Y)$ such that f is nearly Γ-continuous. In view of Proposition 4.3(a), $\mathrm{UC}(X) = H_{\mathrm{MC}}^{\mathrm{NR}}(X)$. $\square$

Example 2. The generalization of $\mathrm{CMP.LUC}(X)$ is defined as follows. For a modulus of continuity Γ let $H_{\Gamma}^{\mathrm{CMP.LC}}(X) = \{h \in \mathrm{EXT}(X) \mid \text{for every } x \in \mathrm{cl}(X),\ h^{\mathrm{cl}} \text{ is } \Gamma\text{-bicontinuous at } x\}$. $\square$

Example 3. The generalization of BPD.UC(X) is the following group. For a modulus of continuity Γ let

$$H_\Gamma^{\mathrm{NBPD}}(X) = \{h \in H(X) \mid h \text{ and } h^{-1} \text{ are BPD.P, and for every BPD set } A \subseteq X,$$
$$h \restriction A \text{ is nearly } \Gamma\text{-bicontinuous}\}. \ \square$$

The reconstruction problem for these generalizations has not been investigated thoroughly. However, an answer for the groups in Example 3 is given in Theorem 5.32. Example 2 is considerably more difficult to sort out. It is dealt with in Chapters 8–12. The generalization in Example 1 is not true. A counter-example is presented in Example 5.11.

So far, the reconstruction question arising from Example 2 has only a partial answer. It is proved only for principal moduli of continuity (see M6 in Definition 1.9), and only for X's with a "well-behaved" boundary. This is proved in Theorem 12.20.

5.2. The group of uniformly continuous homeomorphisms. The first group to be considered is UC(X). The final reconstruction theorem for such groups is stated in Corollary 5.6. The following is the main intermediate theorem.

Theorem 5.5. *Let $X, Y \in K_{\mathrm{NRM}}^{\mathcal{O}}$. Suppose that X is UD.AC. Let $\tau \in H(X,Y)$ be such that* $(\mathrm{UC}_0(X))^\tau \subseteq \mathrm{UC}(Y)$. *Then* $\tau \in \mathrm{UC}(X,Y)$.

Proof. Variants of the argument used in this proof will be applied in several other proofs.

Suppose by contradiction that $\tau \notin \mathrm{UC}(X,Y)$. Let $d > 0$ and $\vec{x}, \vec{y} \subseteq X$ be such that $\lim_{n\to\infty} d(x_n, y_n) = 0$, and for every $n \in \mathbb{N}$, $d(\tau(x_n), \tau(y_n)) \geq d$. Since τ is continuous, there is no $z \in X$ such that $\{n \mid x_n = z\}$ is infinite. So we may assume that $\vec{x}$ is 1-1. We may further assume that for any distinct $m, n \in \mathbb{N}$, $\{x_m, y_m\} \cap \{x_n, y_n\} = \emptyset$. By 2.15(a), we may assume that either (i) $\vec{x}$ is Cauchy sequence, or (ii) there is $e > 0$ such that $\vec{x}$ is e-spaced.

Case 1: (i) holds. Let $x^* = \lim^{\bar{E}} \vec{x}$. So $x^* \in \bar{E} - X$. Note that either $x^* \in \overline{\mathrm{int}}^E(X)$ or $x^* \in \mathrm{cl}^{\bar{E}}(\mathrm{bd}(X))$. By the UD.AC-ness of X and since $\lim_{n\to\infty} d(x_n, y_n) = 0$, we may assume that for every $n \in \mathbb{N}$ there is an arc $L_n \subseteq X$ connecting x_n and y_n such that $\lim_{n\to\infty} \mathrm{diam}(L_n) = 0$. By induction on k, we define $n_k \in \mathbb{N}$ and $r_k > 0$. Let $n_0 = 0$. Suppose that n_k has been defined. Let $r_k = \frac{1}{4} d^{\bar{E}}(L_{n_k}, \{x^*\} \cup (E - X))$ and n_{k+1} be such that $L_{n_{k+1}} \subseteq B^{\bar{E}}(x^*, r_k)$. We denote x_{n_k}, y_{n_k} and L_{n_k} by u_k, v_k and J_k respectively.

Let $U_k = B^X(J_k, r_k)$. Clearly, $\lim_{k\to\infty} \mathrm{diam}(U_k) = 0$, and for every $k \in \mathbb{N}$, $\delta(U_k) > r_k$ and $d(U_k, \bigcup\{U_m \mid m \neq k\}) > r_k$. Let $w_k \in J_k - \{u_k\}$ be such that $d(\tau(u_k), \tau(w_k)) < 1/(k+1)$. By Lemma 2.14(d), there is $h_k \in \mathrm{LIP}(X)$ such that $\mathrm{supp}(h_k) \subseteq U_k$, $h_k(u_k) = u_k$ and $h_k(w_k) = v_k$.

Let $h = \circ_{k\in\mathbb{N}} h_k$. By Proposition 4.5, $h \in \mathrm{UC}(X)$. Since $\delta(\mathrm{supp}(h_k)) > 0$, $h \in \mathrm{UC}_0(X)$. We check that $h^\tau \notin \mathrm{UC}(Y)$. Clearly, $h^\tau(\tau(u_k)) = \tau(u_k)$ and $h^\tau(\tau(w_k)) = \tau(v_k)$. However, $\lim_{k\to\infty} d(\tau(u_k), \tau(w_k)) = 0$, whereas for every $k \in \mathbb{N}$, $d(\tau(u_k), \tau(v_k)) \geq d$. So $h^\tau \notin \mathrm{UC}(Y)$.

Case 2: (ii) holds. By the UD.AC-ness of X, and since $\lim_{n\to\infty} d(x_n, y_n) = 0$, there is $N \in \mathbb{N}$ such that for every $n \geq N$ there is an arc $L_n \subseteq X$ connecting x_n and y_n such that $\mathrm{diam}(L_n) < e/6$ and $\lim_{n\to\infty} \mathrm{diam}(L_n) = 0$. We may assume that $N = 0$. Let $r_n = \min(\mathrm{diam}(L_n), \delta(L_n)/2)$ and $U_n = B(L_n, r_n)$. So $\delta(U_n) > 0$, $\lim_{n\to\infty} \mathrm{diam}(U_n) = 0$, and for any distinct $m, n \in \mathbb{N}$, $d(U_m, U_n) \geq e/3$. The proof now proceeds as in Case 1. ∎

The final result for groups of type $\mathrm{UC}(X)$ is at this stage as follows.

Corollary 5.6. *Let $X, Y \in K^{\mathcal{O}}_{\mathrm{NFCB}}$. Suppose that X and Y are UD.AC spaces. Let $\varphi : \mathrm{UC}(X) \cong \mathrm{UC}(Y)$. Then there is $\tau \in \mathrm{UC}^{\pm}(X, Y)$ such that τ induces φ.*

Proof. Combine Corollary 2.26 and Theorem 5.5. ∎

In the case of local uniform continuity, we deduced from the fact that $(\mathrm{UC}(X))^{\tau} \subseteq \mathrm{LUC}(Y)$ that both τ and τ^{-1} are LUC. The analogue of this fact for uniform continuity is not true.

Example 5.7. (a) *Let $X = Y = (1, \infty)$, and $\tau : X \to Y$ be defined by $\tau(x) = \sqrt{x}$. Then $(\mathrm{UC}(X))^{\tau} \subseteq \mathrm{UC}(Y)$, but τ^{-1} is not UC.*

(b) *There are bounded open subsets X and Y of the Hilbert space ℓ_2 and $\tau \in H(X, Y)$ such that $(\mathrm{UC}(X))^{\tau} \subseteq \mathrm{UC}(Y)$, but τ^{-1} is not uniformly continuous. The boundary of both X and Y is the union of a spaced family of spheres.*

Proof. (a) Clearly $\tau^{-1} \notin \mathrm{UC}(X)$. Let $f \in \mathrm{UC}(X)$. By Proposition 4.3(b), f is α-continuous for some $\alpha \in \mathrm{MC}$. By the uniform continuity of f^{-1}, there is C such that for every $y \in X$, $f^{-1}(y+1) - f^{-1}(y) \leq C$. Set $K = C + 1$. We check that $f(x) \geq x/K$ for every $x \in X$. Let $y \in X$. Then $f^{-1}(y) - 1 \leq f^{-1}([y]+1) - f^{-1}(1) \leq [y] \cdot C \leq y \cdot C$. Hence $f^{-1}(y) \leq Cy + 1 \leq (C+1)y$. That is, $y \leq f((C+1)y)$. Write $x = (C+1)y$. We conclude that if $x \geq C+1$, then $x/K \leq f(x)$. The above inequality holds automatically for $x \leq C+1$ since $f(x) \geq 1$.

We show that f^{τ} is $(1, 2\sqrt{K}\alpha)$-continuous. This trivially implies that f^{τ} is UC. Let $y > x \geq 1$ be such that $y - x \leq 1$. We have $\tau^{-1}(y) - \tau^{-1}(x) = y^2 - x^2 \leq 2y(y-x)$. So $f(\tau^{-1}(y)) - f(\tau^{-1}(x)) \leq \alpha(2y(y-x)) \leq 2y\alpha(y-x)$. The last inequality follows from the fact that $2y \geq 1$. Now, $\tau f \tau^{-1}(y) - \tau f \tau^{-1}(x) = \sqrt{f(y^2)} - \sqrt{f(x^2)}$. There is $c \in (f(x^2), f(y^2))$ such that $\sqrt{f(y^2)} - \sqrt{f(x^2)} = \frac{1}{2\sqrt{c}}(f(y^2) - f(x^2))$. Recall that $f(x^2) \geq x^2/K$. So

$$\begin{aligned} f^{\tau}(y) - f^{\tau}(x) &= \tau f \tau^{-1}(y) - \tau f \tau^{-1}(x) = \frac{1}{2\sqrt{c}}(f(y^2) - f(x^2)) \leq \frac{1}{2\sqrt{f(x^2)}} \cdot 2y\alpha(y-x) \\ &\leq \frac{1}{\sqrt{x^2/K}} \cdot y\alpha(y-x) \leq \frac{1}{\sqrt{x^2/K}} \cdot 2x\alpha(y-x) = 2\sqrt{K}\alpha(y-x). \end{aligned}$$

(b) In ℓ_2 let $e_i = (0, \ldots, 0, \overset{i}{1}, 0, \ldots)$ and $a_i = 3\sqrt{2}e_i$. Let $X = B(0,6) - \bigcup_{n>0} B(a_i, 1)$ and $Y = B(0,6) - \bigcup_{n>0} B(a_i, 1/n)$. For every $n > 0$ let $h_n : [0, \infty) \to [0, \infty)$ be the piecewise linear function with two breakpoints which takes 0 to 0, 1 to $1/n$, and such that $h_n(t) = t$ for every $t \geq 2$. Let $\tau_n : X \to Y$ be defined by

$$\tau_n(x) = a_n + h_n(\|x - a_n\|)\frac{x - a_n}{\|x - a_n\|},$$

and $\tau = \circ_{n>0}\, \tau_n$. It is left to the reader to check that τ is as required. ∎

We shall later see a finite-dimensional example in which $(\mathrm{UC}(X))^\tau \subseteq \mathrm{UC}(Y)$, but τ^{-1} is not uniformly continuous. In Example 6.7(a) we construct two bounded domains $X, Y \subseteq \mathbb{R}^2$ and $\tau \in H(X,Y)$ with these properties.

However, for some sets X, which are very well behaved, the fact that $(\mathrm{UC}(X))^\tau \subseteq \mathrm{UC}(Y)$ does imply that τ^{-1} is uniformly continuous. Theorems 7.1 and 7.7(a) and Remark 7.8(b) and (c) prove the above implication in some special cases involving subsets of a Banach space or a Banach manifold. For example, the above implication holds when X and Y are spheres of a Banach space.

Proof of Theorem 5.2. Let X' and Y' be UD.AC open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, $F = \mathrm{cl}(X')$, $K = \mathrm{cl}(Y')$ and $\varphi : H(F) \cong H(K)$. Let $X = \mathrm{int}(F)$ and $Y = \mathrm{int}(K)$. Clearly, X and Y are regular open sets, $F = \mathrm{cl}(X)$ and $K = \mathrm{cl}(Y)$. It is trivial to check that X and Y are UD.AC. It is also trivial to check that if Z is a bounded regular open subset of $\mathbb{R}^k$, then $H(\mathrm{cl}(Z)) = \{f^{\mathrm{cl}} \mid f \in \mathrm{UC}(Z)\}$. Let $\psi : H(X) \to H(Y)$ be defined by $\psi(f) = \varphi(f^{\mathrm{cl}}) \restriction Y$. So $\psi : \mathrm{UC}(X) \cong \mathrm{UC}(Y)$.

By Theorem 2.8, there is $\tau \in H(X,Y)$ such that for every $h \in \mathrm{UC}(X)$, $\psi(h) = h^\tau$. Obviously, $(\mathrm{UC}(X))^\tau = \mathrm{UC}(Y)$. Applying Theorem 5.5 to τ and τ^{-1} one concludes that τ and τ^{-1} are uniformly continuous. It follows that $\tau^{\mathrm{cl}} : F \cong K$. It is trivial that for every $h \in H(F)$, $\varphi(h) = h^{\tau^{\mathrm{cl}}}$. ■

Part (a) of the next example shows that in Theorem 5.2, the requirement that F and K are bounded cannot be dropped, and (b) shows that in Theorem 5.2, the requirement that F and K are closures of UD.AC open sets cannot be dropped.

EXAMPLE 5.8. (a) *There are regular open connected subsets $X, Y \subseteq \mathbb{R}^2$ such that X, Y are UD.AC, X is bounded,* $\mathrm{cl}(X) \ncong \mathrm{cl}(Y)$ *but* $H(\mathrm{cl}(X)) \cong H(\mathrm{cl}(Y))$.

(b) *There are regular open connected subsets $X, Y \subseteq \mathbb{R}^2$ such that X is UD.AC, X and Y are bounded,* $\mathrm{cl}(X) \ncong \mathrm{cl}(Y)$ *but* $H(\mathrm{cl}(X)) \cong H(\mathrm{cl}(Y))$.

Proof. (a) Let $x \in S(0,1)$ and $B_i = B(x/2^{2i+2}, 1/2^{2i+3})$. So $\bigcup_{i\in\mathbb{N}} B_i \subseteq B(0,1/2)$, for every $i \neq j$, $\mathrm{cl}(B_i) \cap \mathrm{cl}(B_j) = \emptyset$ and $\lim_{i\to\infty} B_i = 0$.

Let $F = \mathrm{cl}(B(0,1)) - \bigcup_{i\in\mathbb{N}} B_i$. Let $\tau(x) := x/\|x\|^2$ be the inversion map in $\mathbb{R}^2$ and $K = \tau(F - \{0\})$. Let $X = \mathrm{int}(F)$ and $Y = \mathrm{int}(K)$. Then $F = \mathrm{cl}(X)$ and $K = \mathrm{cl}(Y)$. Clearly, X, Y are UD.AC. It is easy to see that $H(K) = \{(h\restriction(F - \{0\}))^\tau \mid h \in H(F)\}$. So $H(F) \cong H(K)$. It is obvious $F \ncong K$.

(b) Let

$$X_0 = \{(\theta - \pi, t) \mid \theta \in (0, 2\pi),\ t \in (1 - \tfrac{1}{4}|\sin(\theta/2)|, 1 + \tfrac{1}{4}|\sin(\theta/2)|)\}$$

and

$$Y_0 = \{t \cdot (\cos\theta, \sin\theta) \mid \theta \in (0, 2\pi),\ t \in (1 - \tfrac{1}{4}|\sin(\theta/2)|, 1 + \tfrac{1}{4}|\sin(\theta/2)|)\}.$$

Note that X_0 is a strip surrounding the line segment $((-\pi,0),(\pi,0))$ with width tending to 0 as $(\theta,0)$ approaches $(-\pi,0)$ and $(\pi,0)$, and Y_0 is a strip surrounding the circular arc $\{(\cos\theta,\sin\theta) \mid \theta \in (0,2\pi)\}$ with width tending to 0 as θ approaches 0 and 2π. Let $\tau : X_0 \to Y_0$ be defined by $\tau((\theta - \pi, t)) = t \cdot (\cos\theta, \sin\theta)$. Then $\tau \in H(X_0, Y_0)$.

For every $n \in \mathbb{Z}$ let $x_n = (\frac{n}{|n|+1} \cdot \pi, 0)$, $r_n = \frac{1}{3}\min(\delta^{X_0}(x_n), d(x_n, \{x_i \mid i \in \mathbb{Z} - \{n\}\}))$ and $\bar{B}_n = \bar{B}(x_n, r_n)$. So $\bar{B}_n \subseteq X_0$, for $n \neq m$, $\bar{B}_n \cap \bar{B}_m = \emptyset$, $\lim_{n\to\infty} \bar{B}_n = (\pi, 0)$ and

$\lim_{n\to-\infty}\bar{B}_n = (-\pi, 0)$. Let $X = X_0 - \bigcup_{n\in\mathbb{Z}}\bar{B}_n$ and $Y = \tau(X)$. Clearly, X and Y are bounded, connected and regular open. Hence $H(\mathrm{cl}(X)) = (H(X))^{\mathrm{cl}}$, and the same holds for Y. It is also obvious that $\mathrm{cl}(X) \not\cong \mathrm{cl}(Y)$. Note that for every $h \in H(\mathrm{cl}(X))$, $h((\pi,0)) \in \{(\pi,0),(-\pi,0)\}$ and the same holds for $(-\pi,0)$. Also, for every $h \in H(\mathrm{cl}(Y))$, $h((1,0)) = (1,0)$. It follows that $h^{\mathrm{cl}} \mapsto (h^\tau)^{\mathrm{cl}},\ h \in H(X)$, is an isomorphism between $H(\mathrm{cl}(X))$ and $H(\mathrm{cl}(Y))$. ∎

Example 5.8(b) leads to the following questions.

QUESTION 5.9. A topological space Z has the *Perfect Orbit Property* if for every $z \in Z$, $z \in \mathrm{acc}(\{h(z) \mid h \in H(Z)\})$. Is it true that for every open $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$: if $\mathrm{cl}(X)$ and $\mathrm{cl}(Y)$ have the Perfect Orbit Property and $\varphi : H(\mathrm{cl}(X)) \cong H(\mathrm{cl}(Y))$, then there is $\tau \in H(\mathrm{cl}(X), \mathrm{cl}(Y))$ such that τ induces φ?

If the above is not true, is the conclusion in the above question true for open subsets of $\mathbb{R}^n$ that have the following stronger property: For every $x \in \mathrm{bd}(X)$ the orbit of x under $H(\mathrm{cl}(X))$ is locally arcwise connected?

Is the same true for open subsets of infinite-dimensional normed spaces? □

The generalization of Corollary 5.6 is not true for all moduli of continuity. As shown in the next example, Γ^{LIP} is a counter-example. The question whether Theorem 5.6 is true for any countably generated Γ is open.

QUESTION 5.10. Is there a countably generated modulus of continuity Γ such that for every normed space E and $\tau \in H(E)$: if $(H_\Gamma(E))^\tau = H_\Gamma(E)$, then $\tau \in H_\Gamma(E)$? □

EXAMPLE 5.11. *Let E be a normed space and $\tau \in H(E)$ be defined by: $\tau(x) = x$ if $\|x\| \le 1$ and $\tau(x) = \|x\| \cdot x$ if $\|x\| > 1$. Then $(\mathrm{LIP}(E))^\tau = \mathrm{LIP}(E)$ and $\tau \notin \mathrm{LIP}(E,E)$.*

Proof. Let $g \in \mathrm{LIP}(X,X)$. We show that g^τ is Lipschitz. Let r be such that $r \ge 1, \|g(0)\|$ and $g(B(0,r)) \supseteq B(0,1)$. We show that $g^\tau \restriction (E - B(0,r^2))$ is Lipschitz. Suppose that g is K-Lipschitz. Let $u \in E - B(0,r)$. Then

$$\|g(u)\| \le \|g(u) - g(0)\| + \|g(0)\| \le K\|u\| + \|g(0)\| \le K\|u\| + \|u\| = (K+1)\|u\|.$$

That is,

(i) $\|g(u)\| \le (K+1)\|u\|$.

For $u, v \in E - \{0\}$ write $w(u,v) = \frac{\|v\|}{\|u\|}u$, and for $u, v \ne g^{-1}(0)$ set $w_g(u,v) = w(g(u), g(v))$. Clearly,

(ii) $\|u - w(u,v)\| = |\|u\| - \|v\|| \le \|u - v\|$,
(iii) $\|w(u,v) - v\| \le \|w(u,v) - u\| + \|u - v\| \le 2\|u - v\|$,

and it follows that

(iv) $\|g(u) - w_g(u,v)\| \le K\|u - v\|$,
(v) $\|w_g(u,v) - g(v)\| \le 2K\|u - v\|$.

CLAIM 1. *There is M such that for every $x, y \in E - B(0,r^2)$: if $y = \lambda x$ for some $\lambda > 1$, then $\|g^\tau(y) - g^\tau(x)\| \le M\|y - x\|$.*

Proof. Let $x = az$ and $y = (a+e)z$, where $\|z\| = 1$ and $a > 0$. Clearly, $e > 0$ and hence $\|y - x\| = e$. Also, $a \ge r^2$. Then $\|\tau^{-1}((a+e)z) - \tau^{-1}(az)\| = \sqrt{a+e} - \sqrt{a} \le e/\sqrt{a+e}$.

Set $u = \tau^{-1}((a+e)z)$ and $v = \tau^{-1}(az)$. So $\|u - v\| \le e/\sqrt{a+e}$. The next inequality uses the definitions of τ and w_g, the K-Lipschitzness of g and (i):

$$\begin{aligned}\|\tau(g(u)) - \tau(w_g(u,v))\| &= |\|g(u)\|^2 - \|w_g(u,v)\|^2| = |\|g(u)\|^2 - \|g(v)\|^2| \\ &= (\|g(u)\| + \|g(v)\|) \cdot |\|g(u)\| - \|g(v)\|| \le (\|g(u)\| + \|g(v)\|) \cdot \|g(u) - g(v)\| \\ &\le (\|g(u)\| + \|g(v)\|) \cdot K\|u - v\| \le (\|g(u)\| + \|g(v)\|) \cdot \frac{Ke}{\sqrt{a+e}} \\ &\le (K+1)(\|u\| + \|v\|) \cdot \frac{Ke}{\sqrt{a+e}} = (K+1)(\sqrt{a+e} + \sqrt{a}) \cdot \frac{Ke}{\sqrt{a+e}} \\ &\le 2(K+1)^2\sqrt{a+e} \cdot \frac{e}{\sqrt{a+e}} = 2(K+1)^2 e = 2(K+1)^2\|y - x\|.\end{aligned}$$

We next find a bound for $\|\tau(w_g(u,v)) - \tau(g(v))\|$. Since g is K-Lipschitz and by (v),

$$\begin{aligned}\|\tau(w_g(u,v)) - \tau(g(v))\| &= \|g(v)\| \cdot \|w_g(u,v) - g(v)\| \le (K+1) \cdot \|v\| \cdot 2K \cdot \|u - v\| \\ &\le (K+1) \cdot \sqrt{a} \cdot 2K \cdot \frac{e}{\sqrt{a+e}} \le 2(K+1)^2 \cdot \|y - x\|.\end{aligned}$$

Note that $g^\tau(y) = \tau(g(u))$ and $g^\tau(x) = \tau(g(v))$. It follows that

$$\|g^\tau(y) - g^\tau(x)\| \le \|\tau(g(u)) - \tau(w_g(u,v))\| + \|\tau(w_g(u,v)) - \tau(g(v))\| \le 4(K+1)^2 \cdot \|y - x\|.$$

So Claim 1 is proved.

CLAIM 2. *There is M such that for every $x, y \in E - B(0, r^2)$: if $\|x\| = \|y\|$, then $\|g^\tau(x) - g^\tau(y)\| \le M\|x - y\|$.*

Proof. Let $\|x\| = \|y\| = a \ge r^2$. Set $u = \tau^{-1}(x)$ and $v = \tau^{-1}(y)$. Then by (iv), $\|g(u) - w_g(u,v)\| \le K\|u - v\|$. So

$$\begin{aligned}\|\tau(g(u)) - \tau(w_g(u,v))\| &= |\|g(u)\|^2 - \|w_g(u,v)\|^2| = |\|g(u)\|^2 - \|g(v)\|^2| \\ &= (\|g(u)\| + \|g(v)\|) \cdot |\|g(u)\| - \|g(v)\|| \le (K+1)(\|u\| + \|v\|) \cdot \|g(u) - g(v)\| \\ &\le 2(K+1)\sqrt{a} \cdot K\|u - v\| = 2(K+1)K\sqrt{a} \cdot \frac{\|x - y\|}{\sqrt{a}} \le 2(K+1)^2\|x - y\|.\end{aligned}$$

We next find a bound for $\|\tau(w_g(u,v)) - \tau(g(v))\|$. By (iv) we have $\|w_g(u,v) - g(v)\| \le 2K\|u - v\|$. So

$$\begin{aligned}\|\tau(w_g(u,v)) - \tau(g(v))\| &= \|g(v)\| \cdot \|w_g(u,v) - g(v)\| \le (K+1)\sqrt{a} \cdot \|w_g(u,v) - g(v)\| \\ &\le (K+1)\sqrt{a} \cdot 2K\|u - v\| = (K+1)\sqrt{a} \cdot 2K \cdot \frac{\|x - y\|}{\sqrt{a}} \le 2(K+1)^2\|x - y\|.\end{aligned}$$

It follows that $\|g^\tau(x) - g^\tau(y)\| \le 4(K+1)^2\|x - y\|$. We have proved Claim 2.

Let $x, y \in E - B(0, r^2)$. By Claims 1 and 2 and by (ii) and (iii),

$$\begin{aligned}\|g^\tau(x) - g^\tau(y)\| &\le \|g^\tau(x) - g^\tau(w(x,y))\| + \|g^\tau(w(x,y)) - g^\tau(y)\| \\ &\le 4(K+1)^2\|x - w(x,y)\| + 4(K+1)^2\|w(x,y) - y\| \le 12(K+1)^2\|x - y\|.\end{aligned}$$

We have shown that if g is Lipschitz, then $g^\tau \restriction (E - B(0, r^2))$ is Lipschitz. Since for every bounded set B, $\tau \restriction B$ is bilipschitz, it follows that $g^\tau \restriction \bar{B}(0, r^2)$ is Lipschitz. It is now esay to conclude that g^τ is Lipschitz.

The proof that $(\mathrm{LIP}(E))^{\tau^{-1}} \subseteq \mathrm{LIP}(E)$ is slightly different. Denote τ^{-1} by η. We prove that if g is bilipschitz, then g^η is Lipschitz. Let $g \in \mathrm{LIP}(X)$, suppose that g is K-bilipschitz and let r be such that $r \geq \max(1, 2K\|g(0)\|)$ and $g(B(0,r)) \supseteq B(0,1)$. We show that $g^\eta \restriction (E - B(0,\sqrt{r}))$ is Lipschitz.

We shall use facts (ii)–(v) from the preceding part of the proof. In addition, we need the following fact. Let $u \in E - B(0,r)$. Then

$$\|g(u)\| \geq \|g(u) - g(0)\| - \|g(0)\| \geq \|u\|/K - \|g(0)\| \geq \|u\|/K - \|u\|/2K = \|u\|/(2K).$$

That is,

(vi) $\|g(u)\| \geq \|u\|/(2K)$.

Claim 3. *There is M such that for every $x, y \in E - B(0,\sqrt{r})$: if $y = \lambda x$ for some $\lambda > 1$, then $\|g^\eta(y) - g^\eta(x)\| \leq M\|y - x\|$.*

Proof. Let $x = az$ and $y = (a+e)z$, where $\|z\| = 1$ and $a, e > 0$. Then $\|y - x\| = e$ and $a \geq \sqrt{r}$. Set $u = \eta^{-1}((a+e)z)$ and $v = \eta^{-1}(az)$. We skip the verification of the following facts:

$$\|g^\eta(x) - \eta(w_g(v,u))\| \leq \sqrt{2}K^{3/2}\|x - y\|, \tag{1}$$

$$\|\eta(w_g(v,u)) - g^\eta(y)\| \leq 4\sqrt{2}K^{3/2}\|x - y\|. \tag{2}$$

From (1) and (2) it follows that

$$\|g^\eta(x) - g^\eta(y)\| \leq \|\eta(g(v)) - \eta(w_g(v,u))\| + \|\eta(w_g(v,u)) - \eta(g(u))\| \leq 5\sqrt{2}K^{3/2}\|x - y\|.$$

This proves Claim 3.

Claim 4. *There is M such that for every $x, y \in E - B(0,\sqrt{r})$: if $\|x\| = \|y\|$, then $\|g^\eta(x) - g^\eta(y)\| \leq M\|x - y\|$.*

Proof. Let $\|x\| = \|y\| \geq \sqrt{r}$. Set $u = \eta^{-1}(x)$ and $v = \eta^{-1}(y)$. We skip the verification of the following facts:

$$\|\eta(g^\eta(y)) - \eta(w_g(v,u))\| \leq (\sqrt{2}/2)K^{3/2}\|y - x\|, \tag{3}$$

$$\|\eta(w_g(v,u)) - g^\eta(x)\| \leq 2\sqrt{2}K^{3/2}\|y - x\|. \tag{4}$$

We conclude that

$$\|g^\eta(y) - g^\eta(x)\| \leq (5\sqrt{2}/2)K^{3/2}\|y - x\|.$$

This proves Claim 4.

The rest of the argument is the same as in the preceding part of the proof. ∎

5.3. The group of homeomorphisms which are uniformly continuous on every bounded set. We now turn to the group $\mathrm{BUC}(X)$ of all homeomorphisms f of X such that f and f^{-1} are boundedness preserving, and f and f^{-1} are uniformly continuous on every bounded subset of X. The final reconstruction result for such groups is stated in Theorem 5.20. The conclusion of 5.20 is the statement: $(*)$ if $\varphi : \mathrm{BUC}(X) \cong \mathrm{BUC}(Y)$, then there is $\tau \in \mathrm{BUC}^{\pm}(X,Y)$ such that τ induces φ. However, $(*)$ is not true for general open subsets of a normed space, so we shall make some extra assumptions on X and Y. These assumptions are (roughly): (1) X and Y are uniformly-in-diameter arcwise-connected; (2) the orbit of every member of $\mathrm{bd}(X)$ under the action of $\mathrm{BUC}(X)$ contains an arc, and the same holds for Y.

Let $\mathrm{ABUC}(X,Y) = \{h \in H(X,Y) \mid \text{for every bounded set } A \subseteq X,\ h{\restriction}A \text{ is UC}\}$. Recall that $\mathrm{ABUC}(X) = \mathrm{ABUC}^{\pm}(X,X)$. While $\mathrm{BUC}(X)$ is a group, it is not always true that $\mathrm{ABUC}(X)$ is a group. It is easy to construct an open set X in a normed space and $f \in \mathrm{ABUC}(X)$ such that f takes a bounded set to an unbounded set. We can then choose another $g \in \mathrm{ABUC}(X)$ such that $g \circ f \notin \mathrm{ABUC}(X)$. However, if X has the discrete path property for large distances (see 4.2(f)), then every member of $\mathrm{ABUC}(X)$ is boundedness preserving, and hence $\mathrm{ABUC}(X) = \mathrm{BUC}(X)$. So $\mathrm{ABUC}(X)$ is a group.

Proposition 5.12. *Let X have the discrete path property for large distances.*

(a) *There are $a_1, b_1 > 0$ such that, for every $x, y \in X$ and $0 < t < d(x,y)$, there are $n \in \mathbb{N}$ and $x = x_0, x_1, \dots, x_n = y$ such that $n \leq (a_1 d(x,y) + b_1)/t$, and for every $i < n$, $d(x_i, x_{i+1}) \leq t$.*

(b) *If Y is a metric space, and $\tau \in \mathrm{ABUC}(X,Y)$, then τ is boundedness preserving. (Hence $\tau \in \mathrm{BUC}(X,Y)$.)*

(c) $\mathrm{BUC}(X) = \mathrm{ABUC}(X)$.

Proof. (a) Let $x = z_0, z_1, \dots, z_m = y$ be such that $d(z_i, z_{i+1}) < t/2$ for every $i < m$, and $\sum_{i<m} d(z_i, z_{i+1}) \leq ad(x,y) + b$. There are $n \in \mathbb{N}$ and $0 = i_0 < \cdots < i_n \leq m$ such that for every $j < n$, $t/2 \leq d(z_{i_j}, z_{i_{j+1}}) < t$ and $d(z_{i_n}, z_m) \leq t/2$. It follows that $n \cdot \frac{t}{2} \leq \sum_{j<i_n} d(z_j, z_{j+1}) \leq ad(x,y) + b$. Hence $n \leq (2ad(x,y) + 2b)/t$ and so $n + 1 \leq ((2a+1)d(x,y) + 2b)/t$. For $j \leq n$ define $x_j = z_{i_j}$ and define $x_{n+1} = z_m$. Then $n+1$ and $x_0, \dots, x_{n+1}$ are as required. That is, we may take a_1 and b_1 to be $2a+1$ and $2b$. So (a) is proved.

(b) Let a_1, b_1 be the numbers obtained by applying (a) to X. Let $C \subseteq X$ be bounded. Define $r = \mathrm{diam}(C)$ and $B = B(C, a_1 r + b_1)$. Since B is bounded, there is $\delta > 0$ such that for every $x, y \in B$: if $d(x,y) \leq \delta$, then $d(\tau(x), \tau(y)) \leq 1$. Let $x, y \in C$. If $d(x,y) \leq \delta$, then $d(\tau(x), \tau(y)) \leq 1$. Otherwise, let $n \in \mathbb{N}$ and $x = z_0, \dots, z_n = y$ be such that $n \leq (a_1 d(x,y) + b_1)/\delta$ and $d(z_i, z_{i+1})) \leq \delta$ for every $i < n$. So for every $i \leq n$, $d(x, z_i) \leq n\delta \leq \frac{a_1 d(x,y) + b_1}{\delta} \cdot \delta \leq a_1 r + b_1$. So $z_i \in B$ and hence $d(\tau(z_i), \tau(z_{i+1})) \leq 1$. Then $d(\tau(x), \tau(y)) \leq \sum_{i<n} d(\tau(z_i), \tau(z_{i+1})) \leq n \leq (a_1 d(x,y) + b_1)/\delta \leq (a_1 \cdot \mathrm{diam}(C) + b_1)/\delta$. So $\tau(C)$ is bounded.

(c) By (b), if $f \in \mathrm{ABUC}(X,X)$, then $f \in \mathrm{BUC}(X,X)$. So $\mathrm{ABUC}(X) = \mathrm{BUC}(X)$. ∎

Remark. Part (b) of the above proposition follows trivially from Proposition 4.3(b). However, the proof of 4.3 was left to the reader. □

Suppose that $\tau \in H(X,Y)$ and $(\mathrm{UC}(X))^{\tau} \subseteq \mathrm{ABUC}(Y)$. Assuming that τ is boundedness preserving, the proof that $\tau \in \mathrm{ABUC}(X,Y)$ is just as the proof of 5.5. This is the content of the next lemma. The main problem will be to deduce that τ is boundedness preserving.

Definition 5.13. Let X be a metric space. X is *boundedly uniformly-in-diameter arcwise-connected* (X *is BUD.AC*) if for every bounded set $B \subseteq X$ and $\varepsilon > 0$ there is $\delta > 0$ such that for every $x, y \in B$: if $d(x,y) < \delta$, then there is an arc $L \subseteq X$ connecting x and y such that $\mathrm{diam}(L) < \varepsilon$. □

LEMMA 5.14. *Let X be BUD.AC, and $\tau \in H(X,Y)$ be boundedness preserving. Suppose that* $(\mathrm{UC}(X))^{\tau} \subseteq \mathrm{BUC}(Y)$. *Then* $\tau \in \mathrm{BUC}(X,Y)$.

Proof. The proof is the same as that of 5.5. ■

The following example is a preparation for Theorem 5.18. It shows that the assumptions of that theorem are "correct".

EXAMPLE 5.15. (a) Let $X = B^E(0,1) - \{0\}$, $Y = E - \mathrm{cl}(B^E(0,1))$, and $\tau(x) := \frac{x}{\|x\|^2}$ be the inversion map from X to Y. Then $(\mathrm{BUC}(X))^{\tau} = \mathrm{BUC}(Y)$, but τ is not ABUC. Note that $0 \in \mathrm{bd}(X)$ and for every $h \in \mathrm{BUC}(X)$, $h^{\mathrm{cl}}(0) = 0$. In part (b) we get rid of this pathology.

(b) Let X, Y and τ be as in part (a). Let $X_1 = X \times \mathbb{R}$, $Y_1 = Y \times \mathbb{R}$ and $\tau_1(x,y) = (\tau(x), y)$. Then $(\mathrm{BUC}(X_1))^{\tau_1} \subseteq \mathrm{BUC}(Y_1)$, but τ_1 is not ABUC. In this example, X does not have boundary points fixed under $\mathrm{BUC}(X)$, but we have containment and not equality between $(\mathrm{BUC}(X_1))^{\tau_1}$ and $\mathrm{BUC}(Y_1)$. □

We next formulate the movability property of X, which will be used in the proof that τ is boundedness preserving. It is rather technical but it includes many open sets whose boundary is not so well-behaved.

DEFINITION 5.16. For $h : [0,1] \times X \to X$ and $t \in [0,1]$ we define $h_t(x) := h(t,x)$. We say that X has *Property MV1* if for every bounded $B \subseteq X$ there are $r = r_B > 0$ and $\alpha = \alpha_B \in \mathrm{MC}$ such that for every $x \in B$ and $0 < s \le r$, there is an α-continuous function $h : [0,1] \times X \to X$ such that: (1) for every $t \in [0,1]$, $h_t \in H(X)$ and h_t^{-1} is α-continuous; (2) $h_0 = \mathrm{Id}$ and $d(x, h_1(x)) = s$; and (3) for every $t \in [0,1]$, $\mathrm{supp}(h_t) \subseteq B(x, 2s)$. □

Note that if there is $x \in \mathrm{bd}(X)$ such that $f(x) = x$ for every $f \in \mathrm{BUC}(X)$, then X does not have Property MV1. On the other hand, Property MV1 holds for sets whose boundary is, in a certain sense, well-behaved. Open half spaces, open balls, and complements of closed subspaces fulfill MV1.

The following family of examples contains open sets X such that $\mathrm{cl}(X)$ is not a manifold with boundary. Let U be any nonempty open subset of a normed space E_0 and $X = U \times \mathbb{R}$. Then X has Property MV1. More generally, X has Property MV1 if the following happens. Let E_0 be a normed space, $E = E_0 \times \mathbb{R}$, $s > 0$ and $\alpha \in \mathrm{MBC}$. Suppose that X is an open subset of E with the following property. For every $x \in \mathrm{bd}(X)$ there are: an open subset $U \subseteq E_0$, $x_0 \in \mathrm{bd}(U)$ and a homeomorphism φ from $\bar{B}^{E_0}(x_0, s) \times [-1,1]$ into E, such that:

(1) $\varphi(x_0, 0) = x$,
(2) $\mathrm{Rng}(\varphi)$ is closed in E, and $\varphi(B^{E_0}(x_0,s) \times (-1,1))$ is open in E,
(3) $X \cap \mathrm{Rng}(\varphi) = \varphi((U \cap \bar{B}^{E_0}(x_0,s)) \times [-1,1])$,
(4) φ is α-bicontinuous.

PROPOSITION 5.17. (a) *Let X be a metric space,* $\alpha \in \mathrm{MC}$ *and* $\{h_n \mid n \in \mathbb{N}\} \subseteq H(X)$. *Suppose that for any distinct $m, n \in \mathbb{N}$, h_m is α-continuous and* $\mathrm{supp}(h_m) \cap \mathrm{supp}(h_n) = \emptyset$. *Then $\circ_{n \in \mathbb{N}} h_n$ is $\alpha \circ \alpha$-continuous.*

(b) *Let X be a subset of a normed space E, $\alpha \in \mathrm{MC}$ and $\{h_n \mid n \in \mathbb{N}\} \subseteq H(X)$. Suppose that for any distinct $m, n \in \mathbb{N}$, h_m is α-continuous, $\mathrm{cl}^E(\mathrm{supp}(h_n)) \subseteq X$ and $\mathrm{supp}(h_m) \cap \mathrm{supp}(h_n) = \emptyset$. Then $\circ_{n\in\mathbb{N}} h_n$ is 2α-continuous.*

Proof. (a) Define $h = \circ_{n\in\mathbb{N}} h_n$. Let $x, y \in X$. Then there are $m, n \in \mathbb{N}$ such that $x, y \in \mathrm{supp}(h_m) \cup \mathrm{supp}(h_n) \cup (X - \bigcup_{i\in\mathbb{N}} \mathrm{supp}(h_i))$. So $h(x) = h_m \circ h_n(x)$ and $h(y) = h_m \circ h_n(y)$. Since $h_m \circ h_n$ is $\alpha \circ \alpha$-continuous, $d(h(x), h(y)) \le \alpha \circ \alpha(d(x, y))$.

(b) Define $h = \circ_{n\in\mathbb{N}} h_n$. Let $x, y \in X$. Then there are $m, n \in \mathbb{N}$ such that $x, y \in \mathrm{supp}(h_m) \cup \mathrm{supp}(h_n) \cup (X - \bigcup_{i\in\mathbb{N}} \mathrm{supp}(h_i))$. If x or y belong to $X - \bigcup_{i\in\mathbb{N}} \mathrm{supp}(h_i)$, or $x, y \in \mathrm{supp}(h_m)$, or $x, y \in \mathrm{supp}(h_n)$, then either $d(h(x), h(y)) = d(h_m(x), h_m(y)) \le \alpha(d(x, y))$, or $d(h(x), h(y)) = d(h_n(x), h_n(y)) \le \alpha(d(x, y))$.

So we may assume that $x \in \mathrm{supp}(h_m)$ and $y \in \mathrm{supp}(h_n)$. Let $z \in [x, y] \cap \mathrm{bd}(\mathrm{supp}(h_m))$. Then $z \in X$ and $z \notin \mathrm{supp}(h_n)$. Hence $h_m(z) = h_n(z) = z$. So

$$\begin{aligned} d(h(x), h(y)) &\le d(h(x), h(z)) + d(h(z), h(y)) = d(h_m(x), h_m(z)) + d(h_n(z), h_n(y)) \\ &\le \alpha(d(x, z)) + \alpha(d(z, y)) \le 2\alpha(d(x, y)). \ \blacksquare \end{aligned}$$

Theorem 5.18. *Let $X, Y \in K^{\mathcal{O}}_{\mathrm{NRM}}$. Suppose that X has Property MV1, and let $\tau \in H(X, Y)$ be such that $(\mathrm{UC}(X))^\tau \subseteq \mathrm{BUC}(Y) \subseteq (\mathrm{BUC}(X))^\tau$. Then τ is boundedness preserving.*

Proof. Suppose otherwise. Let $\vec{x} \subseteq X$ be a bounded sequence such that $\tau(\vec{x})$ is unbounded. We may assume that either $\vec{x}$ is a Cauchy sequence or $\vec{x}$ is spaced.

Case 1: $\vec{x}$ is a Cauchy sequence. Applying MV1 to the bounded set $\mathrm{Rng}(\vec{x})$ we obtain $r = r_{\mathrm{Rng}(\vec{x})} > 0$ and $\alpha = \alpha_{\mathrm{Rng}(\vec{x})} \in \mathrm{MC}$. Set $x^* = \lim^{\bar{E}} \vec{x}$, and choose $\delta > 0$ such that $\delta, \alpha(\delta) < r/4$, and m such that $d(x_m, x^*) < \delta$. Let $h : [0, 1] \times X \to X$ be the isotopy provided by MV1 when x and s are taken to be x_m and r, and let $\bar{h} = h^{\mathrm{cl}}_{[0,1]\times\bar{E}}$. (See Definition 4.6.) From the fact that h is α-continuous it follows that $\bar{h} : \mathrm{cl}^{[0,1]\times\bar{E}}([0, 1] \times X) \to \mathrm{cl}^{\bar{E}}(X)$ and $\bar{h}$ is α-continuous. Since $\bar{h}_1$ is α-continuous, $d(\bar{h}_1(x^*), \bar{h}_1(x_m)) \le \alpha(d(x^*, x_m)) < \alpha(\delta) < r/4$. So $d(x^*, \bar{h}_1(x^*)) \ge d(x_m, \bar{h}_1(x_m)) - d(x_m, x^*) - d(\bar{h}_1(x_m), \bar{h}_1(x^*)) > r - r/4 - r/4 = r/2$. That is, $d(x^*, \bar{h}_1(x^*)) > r/2$. For $n \in \mathbb{N}$ define $L_n = h(x_n, [0, 1])$.

Claim 1. $\lim_{n\to\infty} d(\tau(L_n), 0) = \infty$.

Proof. Suppose otherwise. Then there are a 1-1 sequence $\{n_k \mid k \in \mathbb{N}\}$ and a sequence $\{t_k \mid k \in \mathbb{N}\} \subseteq [0, 1]$ such that $\{\tau(h(x_{n_k}, t_k)) \mid k \in \mathbb{N}\}$ is bounded. We may assume that $\{t_k \mid k \in \mathbb{N}\}$ converges to t^*. Since $h_{t^*} \in \mathrm{UC}(X)$, $(h_{t^*})^\tau \in \mathrm{BUC}(Y)$. In particular, $(h_{t^*})^\tau \in \mathrm{BDD.P}(Y)$. It follows that $\{\tau(h_{t^*}(x_{n_k})) \mid k \in \mathbb{N}\} = (h_{t^*})^\tau(\{\tau(x_{n_k}) \mid k \in \mathbb{N}\})$ is unbounded. Let I_k be the interval whose endpoints are t_k and t^* and $L'_k = h(I_k \times \{x_{n_k}\})$. By the α-continuity of h, $\lim_{k\to\infty} \mathrm{diam}(L'_k) = 0$. Proceeding as in the proof of Case 1 of Theorem 5.5, we construct a 1-1 sequence $\{k_i \mid i \in \mathbb{N}\}$ and $g \in \mathrm{UC}(X)$ such that $g(h(t_{k_i}, x_{n_{k_i}})) = h(t^*, x_{n_{k_i}})$. The fact that $g \in \mathrm{UC}(X)$ implies that $g^\tau \in \mathrm{BUC}(Y)$, so in particular, g^τ is boundedness preserving. However, g^τ takes the bounded sequence $\tau(h(t_{k_i}, x_{n_{k_i}}))$ to the unbounded sequence $\tau(h(t^*, x_{n_{k_i}}))$. A contradiction, so Claim 1 is proved.

Let $u_n = h(1, x_n)$ and $U_n = B^Y(\tau(L_n), 1)$. There is a subsequence $\{U_{n_k} \mid k \in \mathbb{N}\}$ of $\{U_n \mid n \in \mathbb{N}\}$ such that for every $k \in \mathbb{N}$, $U_{n_k} \subseteq B(0, d(0, U_{n_{k+1}}))/2$. For every $k \in \mathbb{N}$, let $g_k \in \mathrm{UC}(Y)$ be such that $\mathrm{supp}(g_k) \subseteq U_{n_k}$ and $g_k(\tau(x_{n_k})) = \tau(u_{n_k})$. Let $g = \circ_{k\in\mathbb{N}}\, g_{2k}$ and $f = g^{\tau^{-1}}$.

Clearly, $g \in \mathrm{BUC}(Y)$. So f must belong to $\mathrm{BUC}(X)$. Note that $\lim_{n\in\mathbb{N}} u_n = \bar{h}_1(x^*) \neq x^* = \lim_{n\in\mathbb{N}} x_n$. So since $f(x_{n_{2k}}) = u_{n_{2k}}$ and $f(x_{n_{2k+1}}) = x_{n_{2k+1}}$, $\{f(x_{n_k}) \mid k \in \mathbb{N}\}$ is not convergent in $\bar{E}$. However, $\{x_{n_k} \mid k \in \mathbb{N}\}$ is convergent in $\bar{E}$. Hence f takes a Cauchy sequence to a sequence which is not a Cauchy sequence. So $f \notin \mathrm{BUC}(X)$, a contradiction.

Case 2: $\vec{x}$ is spaced. Let $r_0 > 0$ be such that $\vec{x}$ is $5r_0$-spaced. Applying MV1 to the bounded set $\mathrm{Rng}(\vec{x})$ we obtain $r_1 = r_{\mathrm{Rng}(\vec{x})} > 0$ and $\alpha = \alpha_{\mathrm{Rng}(\vec{x})} \in \mathrm{MC}$. Let $s = \min(r_0, r_1)$. For every $n \in \mathbb{N}$ let $h_n : [0,1] \times X \to X$ be the function ensured by MV1 for x_n and s. Recall that for $t \in [0,1]$, $h_{n,t}(x)$ is the homeomorphism of X taking every $x \in X$ to $h_n(t, x)$. Set $L_n = h_n([0,1] \times \{x_n\})$.

Claim 2. $\lim_{n\to\infty} d(\tau(L_n), 0) = \infty$.

Proof. Suppose otherwise. Then there are a 1-1 sequence $\{n_k \mid k \in \mathbb{N}\}$ and a sequence $\{t_k \mid k \in \mathbb{N}\} \subseteq [0,1]$ such that $\{\tau(h_{n_k}(t_k, x_{n_k})) \mid k \in \mathbb{N}\}$ is bounded. Clearly, for any distinct $m, n \in \mathbb{N}$ and $q, t \in [0,1]$, $d(\mathrm{supp}(h_{m,q}), \mathrm{supp}(h_{n,t})) \geq r_0$. So by 5.17(a), $f := \circ_{k\in\mathbb{N}}\, h_{n_k,t_k} \in \mathrm{UC}(X)$. So $f^\tau \in \mathrm{BUC}(Y) \subseteq \mathrm{BDD.P}(Y)$. We shall reach a contradiction by showing that f^τ takes an unbounded sequence to a bounded sequence. $\{\tau(x_{n_k}) \mid k \in \mathbb{N}\}$ is unbounded, whereas $f^\tau(\{\tau(x_{n_k}) \mid k \in \mathbb{N}\}) = \{\tau(h_{n_k}(t_k, x_{n_k})) \mid k \in \mathbb{N}\}$ is bounded. Claim 2 is thus proved.

Let $u_n = h_n(1, x_n)$, $v_n = h_n(1/n, x_n)$ and $U_n = B^Y(\tau(L_n), 1)$. There is a subsequence $\{U_{n_k} \mid k \in \mathbb{N}\}$ of $\{U_n \mid n \in \mathbb{N}\}$ such that for every $k \in \mathbb{N}$, $U_{n_k} \subseteq B(0, d(0, U_{n_{k+1}}))/2$. For every $k \in \mathbb{N}$, let $g_k \in \mathrm{UC}(Y)$ be such that $\mathrm{supp}(g_k) \subseteq U_{n_k}$, $g_k(\tau(x_{n_k})) = \tau(x_{n_k})$ and $g_k(\tau(v_{n_k})) = \tau(u_{n_k})$. Let $g = \circ_{k\in\mathbb{N}}\, g_k$ and $f = g^{\tau^{-1}}$.

Clearly, $g \in \mathrm{BUC}(Y)$. So f must belong to $\mathrm{BUC}(X)$. By the α-continuity of all h_n's, $\lim_{k\to\infty} d(x_{n_k}, v_{n_k}) = 0$, whereas for every $k \in \mathbb{N}$, $d(f(x_{n_k}), f(v_{n_k})) = d(x_{n_k}, u_{n_k}) = s$. So $f \notin \mathrm{BUC}(X)$, a contradiction. ∎

Recall the convention that X and Y denote open subsets of the normed spaces E and F.

Corollary 5.19. *Let $X, Y \in K^{\mathcal{O}}_{\mathrm{NRM}}$. Suppose that X is BUD.AC, and X has Property MV1. Let $\tau \in H(X,Y)$ be such that $(\mathrm{UC}(X))^\tau \subseteq \mathrm{BUC}(Y)$ and $(\mathrm{BUC}(Y))^{\tau^{-1}} \subseteq \mathrm{BUC}(X)$. Then $\tau \in \mathrm{BUC}(X,Y)$.*

Proof. Combine Lemma 5.14 and Theorem 5.18. ∎

The following theorem is the final result for groups of type $\mathrm{BUC}(X)$.

Theorem 5.20. *Let $X, Y \in K^{\mathcal{O}}_{\mathrm{NFCB}}$. Suppose that X and Y are BUD.AC, and X and Y have Property MV1. Let $\varphi : \mathrm{BUC}(X) \cong \mathrm{BUC}(Y)$. Then there is $\tau \in \mathrm{BUC}^{\pm}(X,Y)$ such that τ induces φ.*

Proof. Combine Corollaries 2.26 and 5.19. ∎

5.4. Groups of homeomorphisms which are uniformly continuous on every bounded positively distanced set. We next deal with the group $\mathrm{BPD.UC}(X)$ and with some related groups. Recall that $\mathrm{BPD.UC}(X)$ is the group of all homeomorphisms f such that f and f^{-1} take every subset of X whose distance from the boundary of X is positive to a set whose distance from the boundary of X is positive, and such that f and f^{-1} are uniformly continuous on every such set. The generalization of $\mathrm{BPD.UC}(X)$ to arbitrary moduli of continuity is denoted by $H_{\Gamma}^{\mathrm{NBPD}}(X)$. That is, $\mathrm{BPD.UC}(X)$ is the group $H_{\Gamma}^{\mathrm{NBPD}}(X)$ when $\Gamma = \mathrm{MC}$. These groups are explained in the next definition. The final reconstruction result for such groups appears in Theorem 5.32, and this result is obtained for countably generated Γ's and for $\Gamma = \mathrm{MC}$. The main intermediate result for countably generated Γ's appears in Theorem 5.24(b), and it says that if $(\mathrm{LIP}_{00}(X))^{\tau} \subseteq H_{\Gamma}^{\mathrm{NBPD}}(X)$, then $\tau \in H_{\Gamma}^{\mathrm{NBPD}}(X,Y)$. The intermediate result fot $\Gamma = \mathrm{MC}$ appears in Theorem 5.31. The analogous statement here is: if $(\mathrm{UC}_{00}(X))^{\tau} \subseteq \mathrm{BPD.UC}(Y)$, then $\tau \in \mathrm{BPD.UC}(X,Y)$. The groups $\mathrm{LIP}_{00}(X)$ and $\mathrm{UC}_{00}(X)$ are defined in 5.23.

For open subsets of a Banach space we can also conclude that $\tau^{-1} \in \mathrm{BPD.UC}(X,Y)$. That is, if $(\mathrm{BUC}(X))^{\tau} \subseteq \mathrm{BPD.UC}(Y)$, then $\tau^{-1} \in \mathrm{BPD.UC}(Y,X)$. This is done in Theorem 5.41(a).

A weaker variant of uniform continuity pops up, and is also dealt with. Groups arising from this variant are defined in 5.21(c) and are denoted by $H_{\Gamma}^{\mathrm{WBPD}}(X)$. The final result for such groups is stated in Theorem 5.36. The main intermediate results for such groups appear in Theorem 5.24(a) and Proposition 5.35.

We next define the groups $H_{\Gamma}^{\mathrm{BPD}}(X)$, $H_{\Gamma}^{\mathrm{NBPD}}(X)$ and $H_{\Gamma}^{\mathrm{WBPD}}(X)$.

Definition 5.21. (a) Define

$$H_{\Gamma}^{\mathrm{BPD}}(X,Y) = \{f \in \mathrm{BPD.P}(X,Y) \mid \text{for every BPD set } A \subseteq X,\ f{\restriction}A \text{ is } \Gamma\text{-continuous}\}.$$

(b) Let Γ be a modulus of continuity and $f : X \to Y$. We say that f is *nearly Γ-continuous on BPD sets* if for every BPD set $A \subseteq X$ there are $\alpha \in \Gamma$ and $r > 0$ such that $f{\restriction}A$ is (r,α)-continuous. See Definition 4.2(b). We denote by $H_{\Gamma}^{\mathrm{NBPD}}(X,Y)$ the set of all $h \in \mathrm{BPD.P}(X,Y)$ such that h is nearly Γ-continuous on BPD sets.

(c) Let $\alpha \in \mathrm{MC}$, and $f : X \to Y$ be a function between metric spaces. Recall that according to Definition 1.12(a), f is locally $\{\alpha\}$-continuous if for every $x \in X$ there is $U \in \mathrm{Nbr}^X(x)$ such that $f{\restriction}U$ is α-continuous. Let $f : X \to Y$ be a function between metric spaces and Γ be a modulus of continuity. Call f *weakly Γ-continuous* if there is $\alpha \in \Gamma$ such that f is locally $\{\alpha\}$-continuous. If $f \in H(X,Y)$ and both f and f^{-1} are weakly Γ-continuous, then f is said to be *weakly Γ-bicontinuous.*

Let X and Y be open subsets of normed spaces E and F respectively, Γ be a modulus of continuity and $f : X \to Y$. Call f *weakly Γ-continuous on BPD sets* if for every BPD set $A \subseteq X$, $f{\restriction}A$ is weakly Γ-continuous. We denote by $H_{\Gamma}^{\mathrm{WBPD}}(X,Y)$ the set of all $h \in \mathrm{BPD.P}(X,Y)$ such that h is weakly Γ-continuous on BPD sets.

(d) Let X be a subset of a metric space E. X has *the discrete path property for BPD sets* if for every BPD subset $A \subseteq X$ there are $d > 0$ and $K \geq 1$ such that for every $x,y \in A$ and $r > 0$ there are $n \in \mathbb{N}$ and $x = x_0,\dots,x_n = y \in X$ such that $n \leq K \cdot d(x,y)/r$, and for every $i < n$, $\delta(x_i) > d$ and $d(x_i,x_{i+1}) \leq r$. $\square$

Note that $H_\Gamma^{\mathrm{BPD}}(X)$, $H_\Gamma^{\mathrm{NBPD}}(X)$ and $H_\Gamma^{\mathrm{WBPD}}(X)$ are groups. It is easy to check that for X's which are open subsets of a finite-dimensional normed space, X has the discrete path property for BPD sets iff X is connected. For infinite-dimensional normed spaces neither of the above implications is true. In any case, "well-behaved" open subsets of a normed space have the discrete path property for BPD sets. For example, an open ball has this property. We first observe the following easy facts. Part (a) follows from Proposition 4.3(a), and the proof of (b) is left to the reader.

PROPOSITION 5.22. (a) $\mathrm{BPD.UC}(X) = H_{\mathrm{MC}}^{\mathrm{NBPD}}(X)$.

(b) *Suppose that X has the discrete path property for BPD sets. Then* $H_\Gamma^{\mathrm{BPD}}(X) = H_\Gamma^{\mathrm{NBPD}}(X)$.

DEFINITION 5.23. (a) X is *BPD-arcwise-connected* (X *is BPD.AC*) if for every BPD set $A \subseteq X$ there are $C, D > 0$ such that for every $x, y \in A$ there is a rectifiable arc $L \subseteq X$ connecting x and y such that $\mathrm{lngth}(L) \le D$ and $\delta(L) \ge C$.

(b) In some of the subsequent lemmas it will be convenient to regard a sequence as a function whose domain is an infinite subset of $\mathbb{N}$. So if $\sigma \subseteq \mathbb{N}$ is infinite, then the object $\{x_i \mid i \in \sigma\}$ is considered to be a sequence. The notions of a subsequence, a convergent sequence etc. are easily modified to fit into this setting.

(c) Let $\mathrm{LIP}_{00}(X) = \{h \in \mathrm{LIP}(X) \mid \mathrm{supp}(h) \text{ is a BPD set}\}$ and $\mathrm{UC}_{00}(X) = \{h \in \mathrm{UC}(X) \mid \mathrm{supp}(h) \text{ is a BPD set}\}$.

(d) For $x \in X$ let $\delta_1^X(x) = \max(\|x\|, 1/\delta^X(x))$. We abbreviate $\delta_1^X(x)$ by $\delta_1(x)$.

(e) Let $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$. Define $A^{\ge n} = \{m \in A \mid m \ge n\}$. The notations $A^{>n}$, $A^{\le n}$, $A^{<n}$ etc. are defined analogously. □

Note that if X is BPD.AC, then X is connected. Note that a subset $A \subseteq X$ is BPD iff $\sup(\{\delta_1^X(x) \mid x \in A\}) < \infty$.

THEOREM 5.24. *Let Γ be a countably generated modulus of continuity. Suppose that X and Y are open subsets of normed spaces E and F respectively, X is BPD.AC and $\tau \in H(X, Y)$.*

(a) *If* $(\mathrm{LIP}_{00}(X))^\tau \subseteq H_\Gamma^{\mathrm{WBPD}}(Y)$, *then* $\tau \in H_\Gamma^{\mathrm{WBPD}}(X, Y)$.

(b) *If* $(\mathrm{LIP}_{00}(X))^\tau \subseteq H_\Gamma^{\mathrm{NBPD}}(Y)$, *then* $\tau \in H_\Gamma^{\mathrm{NBPD}}(X, Y)$.

The argument of Claim 3 in the proof below is repeated in some other proofs.

LEMMA 5.25. *Suppose that X is BPD.AC, $\tau \in H(X, Y)$ and* $(\mathrm{LIP}_{00}(X))^\tau \subseteq \mathrm{BPD.P}(Y)$. *Then* $\tau \in \mathrm{BPD.P}(X, Y)$.

Proof. Let X, Y and τ be as in the lemma.

CLAIM 1. *Suppose that $u \in X$, $0 < r < s$, $B(u, s) \subseteq X$ and $\vec{x} \subseteq B(u, r)$. Then $\tau(\vec{x})$ is BPD in Y.*

Proof. Suppose by contradiction that $\tau(\vec{x})$ is not BPD in Y. Let $a \in (0, 1)$ be such that $\tau(B(u, ar))$ is BPD in Y. Let $\eta : [0, \infty) \to [0, \infty)$ be the piecewise linear function with breakpoints at ar and $(r + s)/2$ such that $\eta(ar) = r$ and for every $t \ge (r + s)/2$, $\eta(t) = t$. Let $h = \mathrm{Rad}_{\eta,u}^E \restriction X$. (See Definition 3.17(b).) Then $h \in \mathrm{LIP}_{00}(X)$. Let $\vec{v} = h^{-1}(\vec{x})$. Clearly, $\vec{v} \subseteq B(u, ar)$. So $\tau(\vec{v})$ is BPD in Y. Obviously, $h^\tau(\tau(\vec{v})) = \tau(\vec{x})$. Hence h^τ takes a BPD set to a set which is not BPD. That is, $h^\tau \notin \mathrm{BPD.P}(Y)$, a contradiction.

Claim 2. *If $\vec{x}$ is a BPD sequence in X and $\vec{x}$ is a Cauchy sequence, $\tau(\vec{x})$ is a BPD sequence in Y.*

Proof. Suppose by contradiction that $\vec{x}$ is a counter-example. Let $x^* = \lim^{\bar{E}}(\vec{x})$. Clearly, $x^* \in \overline{\mathrm{int}}(X)$. Let $u \in X$ and $r > 0$ be such that $x^* \in B^{\bar{E}}(u,r)$ and $B^E(u,2r) \subseteq X$. Let $\vec{y}$ be a final segment of $\vec{x}$ such that $\vec{y} \subseteq B(u,r)$. Then $\vec{y}$ is a counter-example to Claim 1. This proves Claim 2.

Suppose by contradiction that $\tau \notin \mathrm{BPD.P}(X,Y)$. Let $\vec{x}$ be a BPD 1-1 sequence such that $\tau(\vec{x})$ is not BPD. We may assume that $\lim_{n\to\infty} \delta_1(\tau(x_n)) = \infty$. Hence for every subsequence $\vec{y}$ of $\vec{x}$, $\tau(\vec{y})$ is not BPD.

It follows from Claim 2 that $\vec{x}$ has no Cauchy subsequences. Let $x^* \in X - \mathrm{Rng}(\vec{x})$ and $A = \mathrm{Rng}(\vec{x}) \cup \{x^*\}$. Let C an D be as ensured by the property BPD.AC. For every $n \in \mathbb{N}$ let $L_n \subseteq X$ be a rectifiable arc connecting x^* and x_n such that $\delta(L_n) \geq C$ and $\mathrm{lngth}(L_n) \leq D$. Note that $\bigcup_{n\in\mathbb{N}} L_n$ is a BPD set. Let $\gamma_n : [0,1] \to L_n$ be a parametrization of L_n such that $\gamma_n(0) = x^*$, $\gamma_n(1) = x_n$, and for every $t \in [0,1]$, $\mathrm{lngth}(\gamma_n([0,t])) = t \cdot \mathrm{lngth}(L_n)$.

For every infinite $\sigma \subseteq \mathbb{N}$ and $t \in [0,1]$ let $A[\sigma,t] = \{\gamma_n(t) \mid n \in \sigma\}$. We regard $A[\sigma,t]$ as a sequence whose domain is σ. Clearly, for every $t \in [0,1]$, $A[\mathbb{N},t] \subseteq \mathrm{cl}(B(x^*,tD))$. So by the continuity of τ, there is $t_0 > 0$ such that for every $t \in [0,t_0]$, and $\sigma \subseteq \mathbb{N}$, $\tau(A[\sigma,t])$ is a BPD set. For every infinite $\sigma \subseteq \mathbb{N}$ let $s_\sigma = \inf(\{t \in [0,1] \mid \tau(A[\sigma,t])$ is not a BPD set$\})$. So $s_\sigma > 0$.

For $\sigma,\eta \subseteq \mathbb{N}$ let $\sigma \subsetapprox \eta$ mean that $\sigma - \eta$ is finite.

Claim 3. *There is an infinite $\sigma \subseteq \mathbb{N}$ such that for every infinite $\eta \subseteq \sigma$, $s_\eta = s_\sigma$.*

Proof. Suppose by contradiction that no such σ exists. Clearly if $\eta \subsetapprox \sigma$, then $s_\eta \geq s_\sigma$. We define by transfinite induction on $\nu < \omega_1$ an infinite subset $\sigma_\nu \subseteq \mathbb{N}$ such that for every $\nu < \mu$: $\sigma_\mu \subsetapprox \sigma_\nu$ and $s_{\sigma_\mu} > s_{\sigma_\nu}$. If σ_ν has been defined, let $\sigma_{\nu+1} \subseteq \sigma_\nu$ be such that $s_{\sigma_{\nu+1}} > s_{\sigma_\nu}$. If μ is a limit ordinal, and σ_ν has been defined for every $\nu < \mu$, let σ_μ be an infinite set such that for every $\nu < \mu$, $\sigma_\mu \subsetapprox \sigma_\nu$. By the induction hypothesis, if $\nu < \mu$, then $s_{\sigma_{\nu+1}} > s_{\sigma_\nu}$. Hence $s_{\sigma_\mu} \geq s_{\sigma_\nu} > s_{\sigma_\nu}$. So the induction assertion holds. The set $\{s_{\sigma_\nu} \mid \nu < \omega_1\}$ is a subset of $\mathbb{R}$ order isomorphic to ω_1, a contradiction. Claim 3 is proved.

Let σ be as ensured by Claim 3 and write $s = s_\sigma$.

Claim 4. *$A[\sigma,s]$ does not have Cauchy subsequences.*

Proof. Suppose by contradiction that $\eta \subseteq \sigma$ is infinite, and $A[\eta,s]$ is a Cauchy sequence. Since $A[\mathbb{N},1] = \vec{x}$ does not contain Cauchy subsequences, $s < 1$. Let $\hat{x} = \lim A[\eta,s]$. Since $A[\eta,s]$ is a BPD sequence $\hat{x} \in \overline{\mathrm{int}}(X)$. So there are $u \in X$ and $r > 0$ such that $\hat{x} \in B^{\bar{E}}(u,r)$ and $B^E(u,3r) \subseteq X$. We may assume that $A[\eta,s] \subseteq B(u,r)$. For every i and t, $\|\gamma_i(t) - \gamma_i(s)\| \leq (t-s) \cdot D$. So for every $t \in (s, s+r/D)$, $A[\eta,t] \subseteq B(u,2r)$. By the definition of σ, $s_\eta = s_\sigma = s$. So there is $t \in (s, s+r/D)$ such that $\tau(A[\eta,t])$ is not a BPD subset of Y. But $A[\eta,t] \subseteq B(u,2r)$ and $B(u,3r) \subseteq X$. This contradicts Claim 1. So Claim 4 is proved.

By Proposition 2.15(a) and Claim 4, we may assume that there is $d > 0$ such that $A[\sigma, s]$ is d-spaced. Let $r = \min(C, d)/4$. $\delta(A[\sigma, s]) \ge C$, and so $B^E(A[\sigma, s], r) \subseteq X$ and $\delta(B^E(A[\sigma, s], r)) > 0$. Also for any distinct $m, n \in \sigma$, $d(B(\gamma_m(s), r), B(\gamma_n(s), r)) \ge d/2$. Let $t_1 \in (s - \frac{r}{2D}, s)$. Since $t_1 < s$, it follows that $\tau(A[\sigma, t_1])$ is a BPD set. Let $t_2 \in [s, s + \frac{r}{2D})$ be such that $\tau(A[\sigma, t_2])$ is not a BPD set.

By Lemma 2.14(b), there is $K \ge 1$ such that for every normed space E, $u \in E$, $r > 0$ and $x, y \in B^E(u, r/2)$ there is $h \in H(E)$ such that h is K-bilipschitz, $\mathrm{supp}(h) \subseteq B^E(u, r)$ and $h(x) = y$.

Clearly, for every $n \in \sigma$, $\gamma_n(t_1), \gamma_n(t_2) \in B(\gamma_n(s), r/2)$. So by the above fact, there is $h_n \in H(X)$ such that h_n is K-bilipschitz, $\mathrm{supp}(h_n) \subseteq B(\gamma_n(s), r)$ and $h_n(\gamma_n(t_1)) = \gamma_n(t_2)$.

By Proposition 5.17(b), $h := \circ_{n \in \sigma} h_n \in \mathrm{LIP}(X)$. Since $\mathrm{supp}(h) \subseteq B^E(A[\sigma, s], r)$, and $\delta(B^E(A[\sigma, s], r)) > 0$, $h \in \mathrm{LIP}_{00}(X)$. Hence $h^\tau \in \mathrm{BPD.P}(Y)$. However, $\tau(A[\eta, t_1])$ is a BPD set, $\tau(A[\eta, t_2])$ is not a BPD set, and $h^\tau(\tau(A[\eta, t_1])) = \tau(A[\eta, t_2])$. A contradiction. ■

PROPOSITION 5.26. *For a compact metric space C and $t > 0$ let $\nu_C(t)$ denote the minimal cardinality of a cover of C consisting of subsets of C with diameter $\le t$. Let $\vec{C} = \{C_i \mid i \in \mathbb{N}\}$ be a sequence of compact subsets of a metric space X, and let $\nu : (0, \infty) \to \mathbb{N}$. Suppose that for every $i \in \mathbb{N}$, $\nu_{C_i} \le \nu$. Suppose further that there is no infinite set $\eta \subseteq \mathbb{N}$ and a sequence $\{c_i \mid i \in \eta\}$ such that for every $i \in \eta$, $c_i \in C_i$, and $\{c_i \mid i \in \eta\}$ is a Cauchy sequence. Then there is a subsequence $\vec{D}$ of $\vec{C}$ such that $\vec{D}$ is spaced.*

Proof. Suppose that $\vec{C}$ has no spaced subsequences, and we show that there are an infinite set $A \subseteq \mathbb{N}$ and a Cauchy sequence $\vec{c} = \{c_i \mid i \in A\}$ such that for every $i \in A$, $c_i \in C_i$. There are a subsequence $\vec{C}^1$ of $\vec{C}$ and $r \in \mathbb{R} \cup \{\infty\}$ such that $\lim_{i,j \to \infty} d(C^1_i, C^1_j) = r$. Since $\vec{C}$ has no spaced subsequences, $r = 0$. We may assume that $\vec{C} = \vec{C}^1$.

For $\vec{p} \subseteq \mathbb{N}$ let $T_{\vec{p}}$ be the tree of finite sequences $\vec{n}$ such that for every $i < \mathrm{lngth}(\vec{n})$, $n_i < p_i$. Let $S_{\vec{p}} = \prod_{i \in \mathbb{N}} \mathbb{N}^{<p_i}$.

Let $p_i = \prod_{j \le i} \nu(1/j)$, $T = T_{\vec{p}}$ and $S = S_{\vec{p}}$. Then for every $i \in \mathbb{N}$ there is $\{C_{i,\vec{n}} \mid \vec{n} \in T\}$ such that for every $\vec{n} \in T$, $C_{i,\vec{n}}$ is closed and $\mathrm{diam}(C_{i,\vec{n}}) \le 1/\mathrm{lngth}(\vec{n})$; for every $\ell \in \mathbb{N}$, $C_i = \bigcup\{C_{i,\vec{n}} \mid \vec{n} \in T \text{ and } \mathrm{lngth}(\vec{n}) = \ell\}$; and for every $\vec{m}, \vec{n} \in T$: if $\vec{m}$ is an initial segment of $\vec{n}$, then $C_{i,\vec{n}} \subseteq C_{i,\vec{m}}$.

By the Ramsey Theorem, there are a sequence of infinite subsets of $\mathbb{N}$, $A_0 \supseteq A_1 \supseteq \cdots$, and $\vec{q}, \vec{r} \in S$ such that for every ℓ and $i, j \in A_\ell$: if $i < j$, then $d(C_{i,\vec{q} \restriction \mathbb{N}^{\le \ell}}, C_{j,\vec{r} \restriction \mathbb{N}^{\le \ell}}) = d(C_i, C_j)$.

Let $A \subseteq \mathbb{N}$ be an infinite set such that for every i, $A - A_i$ is finite. For every $i \in A$ let $D_i = \bigcap_{j \in \mathbb{N}} C_{i,\vec{q} \restriction \mathbb{N}^{\le j}}$ and $E_i = \bigcap_{j \in \mathbb{N}} C_{i,\vec{r} \restriction \mathbb{N}^{\le j}}$. Clearly, D_i, E_i are singletons, denote them by x_i and y_i respectively. We check that $\lim_{i \to \infty, i < j} d(x_i, y_j) = 0$. Let $\varepsilon > 0$. Then there is N_1 such that for every $i, j > N_1$, $d(C_i, C_j) < \varepsilon/3$. Let N_2 be such that $1/N_2 < \varepsilon/3$, N_3 be such that $A^{\ge N_3} \subseteq A_{N_2}$ and $N = \max(N_1, N_3)$. Let $i < j$ and $i, j \in A^{\ge N}$. So $i, j \in A_{N_2}$. Hence $d(C_{i,\vec{q} \restriction \mathbb{N}^{\le N_2}}, C_{i,\vec{r} \restriction \mathbb{N}^{\le N_2}}) = d(C_i, C_j) < \varepsilon/3$. It follows that

$$d(x_i, y_j) \le \mathrm{diam}(C_{i,\vec{q} \restriction \mathbb{N}^{\le N_2}}) + d(C_i, C_j) + \mathrm{diam}(C_{j,\vec{r} \restriction \mathbb{N}^{\le N_2}}) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

We have proved that $\lim_{i\to\infty, i<j} d(x_i, y_j) = 0$. Let $\varepsilon > 0$. Choose N such that for every $i, j \in A^{\geq N}$: if $i < j$, then $d(x_i, y_j) < \varepsilon/2$. Suppose that $i_1, i_2 \in A^{\geq N}$ and let j be such that $i_1, i_2 < j \in A$. Then $d(x_{i_1}, x_{i_2}) \leq d(x_{i_1}, y_j) + d(y_j, x_{i_2}) < \varepsilon$. So $\{x_i \mid i \in A\}$ is a Cauchy sequence. ∎

LEMMA 5.27. *There is $K_{\rm arc}(\ell, t) > 0$ such that for every normed space E, $L, r > 0$, and a rectifiable arc $\gamma \subseteq E$ with endpoints x, y: if $\operatorname{lngth}(\gamma) \leq L$, then there is $h \in H(E)$ such that*:

(1) $\operatorname{supp}(h) \subseteq B(\gamma, r)$;
(2) $h{\restriction}B(x, r/2) = \operatorname{tr}_{y-x}{\restriction}B(x, r/2)$;
(3) *h is $K_{\rm arc}(L, r)$-bilipschitz.*

Proof. Let $n = [\frac{L}{r/2}] + 1$. Suppose that $\gamma : [0,1] \to X$. There are $0 = t_0, t_1, \ldots, t_n = 1$ such that for every $i < n$, $\operatorname{lngth}(\gamma{\restriction}[t_i, t_{i+1}]) < r/2$. Let $x_i = \gamma(t_i)$. Then for every $z \in [x_i, x_{i+1}]$, $d(z, \gamma{\restriction}[t_i, t_{i+1}]) < r/4$. So $\bigcup_{i<n} B([x_i, x_{i+1}], 3r/4) \subseteq B(\gamma, r)$. By Lemma 2.14(c), there are $h_1, \ldots, h_n \in H(E)$ such that for every $i = 1, \ldots, n$:

(1.1) $\operatorname{supp}(h_i) \subseteq B([x_{i-1}, x_i], 3r/4)$;
(1.2) $h_i{\restriction}B(x_{i-1}, \frac{2}{3} \cdot \frac{3r}{4}) = \operatorname{tr}_{x_i - x_{i-1}}{\restriction}B(x_{i-1}, \frac{2}{3} \cdot \frac{3r}{4})$.
(1.3) h_i is $K_{\rm seg}(r/2, 3r/4)$-bilipschitz.

Let $h = h_n \circ \cdots \circ h_1$. Then h satisfies requirements (1) and (2) in the lemma. Also, h is $K_{\rm seg}(r/2, 3r/4)^n$-bilipschitz. Since $n = [2L/r] + 1$, we may define $K_{\rm arc}(\ell, t) = K_{\rm seg}(t/2, 3t/4)^{[2\ell/t]+1}$. ∎

If L is a rectifiable arc let $\gamma_L : [0,1] \to L$ be a parametrization of L such that for every $t \in [0,1]$, $\operatorname{lngth}(\gamma_L{\restriction}[0,t]) = t \cdot \operatorname{lngth}(L)$.

LEMMA 5.28. *Let X be an open subset of a normed space E. For $n \in \mathbb{N}$ let $L_n \subseteq X$ be a rectifiable arc with $\operatorname{lngth}(L_n) \leq M$ and $\delta(L_n) \geq d > 0$. Let $\gamma_n = \gamma_{L_n}$ and $x_n = \gamma_n(0)$. Suppose that $\{x_n \mid n \in \mathbb{N}\}$ is spaced and 1-1 and that there is $x^* \in X$ such that for every $n \in \mathbb{N}$, $\gamma_n(1) = x^*$. Then there are $\hat{x} \in X$, $r > 0$, an infinite $\eta \subseteq \mathbb{N}$ and $t \in (0,1]$ such that*:

(1) *$B(\hat{x}, r) \subseteq X$, $B(\hat{x}, r)$ is a BPD set, and for every $n \in \eta$, $x_n \notin \operatorname{cl}^E(B(\hat{x}, r))$*;
(2) *for every $n \in \eta$, $\gamma_n(t) \in B(\hat{x}, r)$*;
(3) *$\{\gamma_n{\restriction}[0,t] \mid n \in \eta\}$ is spaced.*

Proof. For $\eta \subseteq \mathbb{N}$ and $t \in [0,1]$ let $A[\eta, t] = \{\gamma_n(t) \mid n \in \mathbb{N}\}$. We regard $A[\eta, t]$ both as a set and as a sequence. For every infinite $\eta \subseteq \mathbb{N}$ let

$$s_\eta = \inf(\{s \in [0,1] \mid A[\eta, s] \text{ contains a Cauchy sequence}\}).$$

Since for every $n \in \mathbb{N}$, $\gamma_n(1) = x^*$, s_η is well defined. Clearly, if $\eta \subseteq \sigma$, then $s_\eta \geq s_\sigma$.

As in 5.25, there is an infinite $\sigma \subseteq \mathbb{N}$ such that for every infinite $\eta \subseteq \sigma$, $s_\eta = s_\sigma$. Let $s = s_\sigma$. We show that if $t \in [0, s)$, then

($*$) there is no infinite set $\eta \subseteq \sigma$ and a sequence $\{t_i \mid i \in \eta\}$ such that for every $i \in \eta$, $t_i \in [0, t]$, and $\{\gamma_i(t_i) \mid i \in \eta\}$ is a Cauchy sequence.

Suppose otherwise. We may assume that $\{t_i \mid i \in \eta\}$ is a convergent sequence. Let t^* be the limit of this sequence. So $t^* < s$. Let I_i be the interval whose endpoints are t_i and t^*. Recall that $\text{lngth}(\gamma_i \restriction I_i) = |t^* - t_i| \cdot \text{lngth}(\gamma_i) \le |t^* - t_i| \cdot M$. So $\lim_{i \in \eta} d(\gamma_i(t_i), \gamma_i(t^*)) = 0$. Hence $\{\gamma_i(t^*) \mid i \in \eta\}$ is a Cauchy sequence. This contradicts the definition of s.

Suppose by contradiction that there is an infinite $\eta \subseteq \sigma$ such that $A[\eta, s]$ is spaced. Let $e > 0$ be such that $A[\eta, s]$ is e-spaced. Then for every $t \in [s, s + e/3M]$, $A[\eta, t]$ is spaced. So $s_\eta > s_\sigma$. This contradicts the definition of σ.

It follows that $A[\sigma, s]$ contains a Cauchy sequence. Hence we may assume that $A[\sigma, s]$ is a Cauchy sequence. Let $\bar{x} = \lim^{\bar{E}} A[\sigma, s]$. Since $\{x_n \mid n \in \sigma\}$ is 1–1, we may assume that for every n, $\bar{x} \ne x_n$. Since $\delta(L_n) \ge d > 0$, $d^{\bar{E}}(\bar{x}, E - X) \ge d > 0$. Since $\{x_n \mid n \in \mathbb{N}\}$ is spaced, there is $0 < r < d$ such that $\{x_n \mid n \in \sigma\} \cap B^{\bar{E}}(\bar{x}, r) = \emptyset$. Let $t = s - r/2M$. There is i_0 such that for every $i_0 \le i \in \sigma$, $d(\gamma_i(s), \bar{x}) < r/4$. We may assume that $i_0 = 0$. So for every $i \in \sigma$, $d(\gamma_i(t), \bar{x}) \le d(\gamma_i(t), \gamma_i(s)) + d(\gamma_i(s), \bar{x}) < \text{lngth}(\gamma_i \restriction [t, s]) + r/4 \le (s - t) \cdot M + r/4 \le 3r/4$. Let $\hat{x} \in E \cap B^{\bar{E}}(\bar{x}, r/8)$. So for every $i \in \sigma$, $d(\gamma_i(t), \hat{x}) < 7r/8$.

By $(*)$ and Proposition 5.26, there is an infinite $\eta \subseteq \sigma$ such that $\{\gamma_i \restriction [0, t] \mid i \in \eta\}$ is spaced. Also, since $\delta(\hat{x}) \ge d - r/8$, $\delta(B(\hat{x}, 7r/8)) \ge d - r > 0$. So $B(\hat{x}, 7r/8)$ is a BPD set. Hence $\hat{x}$, r, η and t are as required in the lemma. ∎

Proposition 5.29. *Let Γ be a countably generated modulus of continuity, and let $a > 0$. Then there is $\{\alpha_n \mid n \in \mathbb{N}\} \subseteq \Gamma$ such that*

(1) *for every $\alpha \in \Gamma$ there is $n \in \mathbb{N}$ such that $\alpha \preceq \alpha_n$, that is, $\{\alpha_n \mid n \in \mathbb{N}\}$ generates Γ;*
(2) *for every $m < n$, $\alpha_m \restriction [0, a] \le \alpha_n \restriction [0, a]$.*

Proof. Let $\{\beta_n \mid n \in \mathbb{N}\}$ be a generating set for Γ such that for every $m < n$, $\beta_m \preceq \beta_n$. We define by induction $K_n > 0$ and $\alpha_n \in \Gamma$. We assume by induction that $\alpha_n = K_n \beta_n$. Let $K_0 = 1$ and $\alpha_0 = \beta_0$. Suppose that K_n and α_n have been defined. Let $i \le n$. Since $\beta_i \preceq \beta_{n+1}$ and $\alpha_i = K_i \beta_i$, it follows that $M_i := \sup_{x \in [0,a]} \alpha_i(x)/\beta_{n+1}(x) < \infty$. Let $K_{n+1} = \max(M_0, \ldots, M_n) + 1$ and $\alpha_{n+1} = K_{n+1} \beta_{n+1}$. Obviously, $\{\alpha_n \mid n \in \mathbb{N}\} \subseteq \Gamma$ and $\{\alpha_n \mid n \in \mathbb{N}\}$ is as required. ∎

Proof of Theorem 5.24. (a) Let Γ, X, Y and τ be as in (a). We have $\text{LIP}_{00}(X) \subseteq H_\Gamma^{\text{WBPD}}(X)$ and $H_\Gamma^{\text{WBPD}}(Y) \subseteq \text{BPD.P}(Y)$, hence $(\text{LIP}_{00}(X))^\tau \subseteq \text{BPD.P}(Y)$. So by Lemma 5.25, $\tau \in \text{BPD.P}(X, Y)$.

Using the notation of Definition 2.7(a), $\text{LIP}_{00}(X) = \text{LIP}(X; \mathcal{U})$, where $\mathcal{U}$ is the set of all open BPD subsets of X. Clearly, $H_\Gamma^{\text{WBPD}}(Y) \subseteq H_\Gamma^{\text{LC}}(Y)$ so $(\text{LIP}(X; \mathcal{U}))^\tau \subseteq H_\Gamma^{\text{LC}}(Y)$. Hence by Theorem 3.27, τ is locally Γ-continuous.

Suppose by contradiction that there is an open BPD set $U \subseteq X$ such that for no $\alpha \in \Gamma$, $\tau \restriction U$ is locally $\{\alpha\}$-continuous. Let $\{\alpha_n \mid n \in \mathbb{N}\}$ generate Γ. We may assume that for every $m < n \in \mathbb{N}$, $\alpha_m \preceq \alpha_n$. For every $n \in \mathbb{N}$ let $\beta_n = \alpha_n \circ \alpha_n$ and $x_n \in U$ be such that for every $V \in \text{Nbr}^X(x_n)$, $\tau \restriction V$ is not β_n-continuous. Let $\vec{x} = \{x_n \mid n \in \mathbb{N}\}$.

Suppose by contradiction that $\vec{x}$ has a Cauchy subsequence $\vec{y}$. Let $\bar{y} = \lim^{\bar{E}} \vec{y}$. Since U is a BPD set and $\text{Rng}(\vec{y}) \subseteq U$, $\bar{y} \in \overline{\text{int}}(X)$. Let $u \in X$ and $r > 0$ be such that $B^E(u, 2r) \subseteq X$ and $\bar{y} \in B^{\bar{E}}(u, r)$. Since τ is locally Γ-continuous, there are $V \in \text{Nbr}^X(u)$

and $\beta \in \Gamma$ such that $\tau \restriction V$ is β-continuous. There is $h \in \mathrm{LIP}(X)\lfloor B(u,r) \rfloor$ such that $h^{\mathrm{cl}}_{\bar{E}}(\bar{y}) \in \overline{\mathrm{int}}(V)$. Since $h \in \mathrm{LIP}_{00}(X)$, $h^{\tau} \in H^{\mathrm{WBPD}}_{\Gamma}(Y)$.

Recall that $\tau \in \mathrm{BPD.P}(X,Y)$. Since $B(u,r)$ is a BPD set in X, $W := \tau(B(u,r))$ is a BPD set in Y. So there is $\alpha \in \Gamma$ such that $(h^{\tau})^{-1} \restriction W$ is locally $\{\alpha\}$-continuous. Since $\lim \vec{y} = \bar{y}$ and $h^{\mathrm{cl}}_{\bar{E}}(\bar{y}) \in \overline{\mathrm{int}}(V)$, we may assume that $h(\vec{y}) \subseteq V$. Let K be such that h is K-bilipschitz, and define $\gamma(t) = Kt$. So $\gamma \in \Gamma$. We show that for every $n \in \mathbb{N}$, τ is $\alpha \circ \beta \circ \gamma$-bicontinuous at y_n. Note that $\tau = (h^{\tau})^{-1} \circ \tau \circ h$. We have

(i) h is γ-bicontinuous at y_n.

Since $h(y_n) \in B(u,r)$, we have

(ii) τ is β-bicontinuous at $h(y_n)$.

Also, $\tau(h(y_n)) \in \tau(B(u,r)) = W$. So

(iii) $(h^{\tau})^{-1}$ is α-bicontinuous at $\tau(h(y_n))$.

It follows from (i)–(iii) that τ is $\alpha \circ \beta \circ \gamma$-bicontinuous at y_n. Clearly, $\alpha \circ \beta \circ \gamma \in \Gamma$, so there is n such that $\alpha \circ \beta \circ \gamma \preceq \beta_n$. Hence τ is β_n-bicontinuous at y_n. This contradicts the choice of y_n. So $\vec{x}$ does not have Cauchy subsequences.

We may thus assume that $\vec{x}$ is spaced. Let $x^* \in U$. Since X is BPD.AC, there are $M, d > 0$ and rectifiable arcs $\{L_n \mid n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}$, L_n connects x_n with x^*, $\delta(L_n) \geq d$ and $\mathrm{lngth}(L_n) \leq M$. Applying Lemma 5.28 to x^* and $\{L_n \mid n \in \mathbb{N}\}$ we obtain $\hat{x} \in X$, $r > 0$, an infinite $\eta \subseteq \mathbb{N}$ and $t \in (0,1]$ as ensured by that lemma. So for the parametrization γ_n of L_n defined in Lemma 5.28 the following holds:

(1.1) $B(\hat{x},r) \subseteq X$, $B(\hat{x},r)$ is a BPD set, and for every $n \in \eta$, $x_n \notin \mathrm{cl}^E(B(\hat{x},r))$;
(1.2) for every $n \in \eta$, $\gamma_n(t) \in B(\hat{x},r)$;
(1.3) $\{\gamma_n \restriction [0,t] \mid n \in \eta\}$ is spaced.

We may assume that $\eta = \mathbb{N}$. For every $n \in \mathbb{N}$ let t_n be the least t' such that $\gamma_n(t') \in \mathrm{cl}^E(B(\hat{x},r))$. Let $\gamma'_n = \gamma_n \restriction [0,t_n]$ and $y_n = \gamma_n(t_n)$. So $d(y_n,\hat{x}) = r$ and $\mathrm{Rng}(\gamma'_n) \cap B(\hat{x},r) = \emptyset$.

Since τ is locally Γ-continuous, there is $\alpha^* \in \Gamma$ and $r_1 < r$ such that $\tau \restriction B(\hat{x},r_1)$ is α^*-continuous. Let $z_n = \hat{x} + \frac{r_1}{2} \cdot (y_n - \hat{x})/\|y_n - \hat{x}\|$ and $L^*_n = \mathrm{Rng}(\gamma'_n) \cup [y_n,z_n]$. So L^*_n is a rectifiable arc. Clearly, there are $M^*, d^*, D^* > 0$ such that for any distinct $m,n \in \mathbb{N}$,

(2.1) $\mathrm{lngth}(L^*_m) \leq M^*$;
(2.2) $\delta(L^*_m) \geq d^*$;
(2.3) $d(L^*_m, L^*_n) \geq D^*$.

Let $r^* > 0$ be such that $r^* < d^*/2, D^*/3, r_1/2$. For every $n \in \mathbb{N}$ we apply Lemma 5.27 with $L = M^*$, $r = r^*$, $\gamma = L^*_n$, $x = x_n$ and $y = z_n$. We obtain $h_n \in H(X)$ such that:

(3.1) $\mathrm{supp}(h_n) \subseteq B(L^*_n, r^*)$;
(3.2) $h_n \restriction B(x_n, r^*/2) = \mathrm{tr}_{z_n - x_n} \restriction B(x_n, r^*/2)$;
(3.3) h_n is $K_{\mathrm{arc}}(M^*, r^*)$-bilipschitz.

Clearly, $\{h_n \mid n \in \mathbb{N}\}$ and $\{h_n^{-1} \mid n \in \mathbb{N}\}$ satisfy the conditions of Proposition 5.17(b) with $\alpha(x) = K_{\mathrm{arc}}(M^*, r^*) \cdot x$. Define $h = \bigcirc_{n \in \mathbb{N}} h_n$ and $g = h^{-1}$. So by Proposition 5.17(b), h and g are $2K(M^*, r^*)$-Lipschitz. Also $\delta(\mathrm{supp}(h)) \geq d^* - r^* > 0$. So $h, g \in \mathrm{LIP}_{00}(X)$.

Since $\tau \in \mathrm{BPD.P}(X,Y)$, $\tau(U)$ is a BPD subset of Y. We shall thus reach a contradiction by proving the following statement:

($*$) There is no $\alpha \in \Gamma$ such that $g^{\tau} \restriction \tau(U)$ is locally $\{\alpha\}$-continuous.

Let $\alpha \in \Gamma$. Choose n such that $\alpha, \alpha^* \preceq \alpha_n$ and set $u = \tau(z_n)$. For $s > 0$ define $U_s = B(u,s)$, $T_s = \tau^{-1}(U_s)$ and $S_s = h^{-1}(T_s)$. There is $s > 0$ such that:

(4.1) $\alpha \restriction [0,2s] \le \alpha_n \restriction [0,2s]$;
(4.2) $T_s \subseteq B(z_n, r^*/2)$;
(4.3) $\alpha^* \restriction [0, \mathrm{diam}(T_s)] \le \alpha_n \restriction [0, \mathrm{diam}(T_s)]$.

Let $s' < s$. We show that $h^{\tau} \restriction B(u,s')$ is not α-continuous. Since $S_{s'}$ is a neighborhood of x_n, there are $x^1, x^2 \in S_{s'}$ such that

(5.1) $d(\tau(x^1), \tau(x^2)) > \beta_n(d(x^1,x^2))$.

For $i = 1,2$ let $z^i = h(x^i)$ and $u^i = \tau(z^i)$. So $z^1, z^2 \in T_{s'}$ and so $u^1, u^2 \in U_{s'}$. By (4.2), the choice of z_n and the choice of r^*, $T_{s'} \subseteq B(z_n, r^*/2) \subseteq B(\hat{x}, r_1)$. So $\tau \restriction T_{s'}$ is α^*-continuous. Hence $\alpha^*(d(z^1,z^2)) \ge d(u^1,u^2)$. By (4.3), $\alpha_n(d(z^1,z^2)) \ge \alpha^*(d(z^1,z^2))$. So $\alpha_n(d(z^1,z^2)) \ge d(u^1,u^2)$. Hence

(5.2) $d(z^1,z^2) \ge (\alpha_n)^{-1}(d(u^1,u^2))$.

Since $T_{s'} \subseteq B(z_n, r^*/2)$ and by property (3.2) of h_n, $h^{-1} \restriction U_{s'}$ is an isometry. So

(5.3) $d(z^1,z^2) = d(x^1,x^2)$.

By (5.1) and (5.3),

(5.4) $d(\tau(x^1), \tau(x^2)) > \beta_n(d(z^1,z^2))$.

Combining (5.2) and (5.4) we obtain

(5.5) $d(\tau(x^1), \tau(x^2)) > \beta_n((\alpha_n)^{-1}(d(u^1,u^2)))$.

But $\beta_n = \alpha_n \circ \alpha_n$. So

(5.6) $d(\tau(x^1), \tau(x^2)) > \alpha_n(d(u^1,u^2))$.

By clause (4.1) in the definition of s, and since $u_1, u_2 \in B(u,s)$,

(5.7) $d(\tau(x^1), \tau(x^2)) > \alpha(d(u^1,u^2))$.

But $\tau(x^i) = (h^{-1})^{\tau}(u^i) = g^{\tau}(u^i)$. So

(5.8) $d(g^{\tau}(u^1), g^{\tau}(u^2)) > \alpha(d(u^1,u^2))$.

We have proved ($*$), and this contradicts the fact that $g^{\tau} \in H_{\Gamma}^{\mathrm{WBPD}}(Y)$. So (a) is proved.

(b) Let Γ, X, Y and τ be as in (b). As in the proof of (a), we conclude that $\tau \in \mathrm{BPD.P}(X,Y)$ and τ is locally Γ-continuous.

Suppose by contradiction that there is an open BPD set $U \subseteq X$ such that for no $\alpha \in \Gamma$ and $r > 0$, $\tau \restriction U$ is (r,α)-continuous. By Proposition 5.29, there is a set $\{\alpha_n \mid n \in \mathbb{N}\}$ which generates Γ and such that $\alpha_m \restriction [0,1] \le \alpha_n \restriction [0,1]$ for every $m < n$. For every $n \in \mathbb{N}$ let $\beta_n = \alpha_n \circ \alpha_n$, and $x_n, x'_n \in U$ be such that $d(x_n, x'_n) < 1/n$ and $d(\tau(x_n), \tau(x'_n)) > \beta_n(d(x_n x'_n))$. Let $\vec{x} = \{x_n \mid n \in \mathbb{N}\}$.

Suppose by contradiction that $\{x_{n_i} \mid i \in \mathbb{N}\}$ is a Cauchy subsequence $\vec{x}$. Set $y_i = x_{n_i}$ and $y'_i = x'_{n_i}$. Let $\bar{y} = \lim^{\bar{E}} \vec{y}$. Since U is a BPD set and $\mathrm{Rng}(\vec{y}) \subseteq U$, $\bar{y} \in \overline{\mathrm{int}}(X)$. Let $u \in X$ and $r > 0$ be such that $B^E(u, 2r) \subseteq X$ and $\bar{y} \in B^{\bar{E}}(u, r)$. Since τ is locally Γ-continuous, there are $V \in \mathrm{Nbr}^X(u)$ and $\beta \in \Gamma$ such that $\tau{\restriction}V$ is β-continuous. There is $h \in \mathrm{LIP}(X)\lfloor B(u,r) \rfloor$ such that $h^{\mathrm{cl}}_{\bar{E}}(\bar{y}) \in \overline{\mathrm{int}}(V)$. Since $h \in \mathrm{LIP}_{00}(X)$, $h^\tau \in H^{\mathrm{NBPD}}_{\Gamma}(Y)$.

Recall that $\tau \in \mathrm{BPD.P}(X, Y)$. Since $B(u, r)$ is a BPD set in X, it follows that $W := \tau(B(u, r))$ is a BPD set in Y. So there are $\alpha \in \Gamma$ and $s > 0$ such that $h^\tau{\restriction}W$ is (s, α)-continuous, and $(h^\tau)^{-1}{\restriction}W$ is (s, α)-continuous. Since $\lim \vec{y} = \lim \vec{y}' = \bar{y}$ and $h^{\mathrm{cl}}_{\bar{E}}(\bar{y}) \in \overline{\mathrm{int}}(V)$, we may assume that $h(\vec{y}), h(\vec{y}') \subseteq V$.

From the fact $h \in \mathrm{LIP}(X)$ it follows that $\lim_{i\to\infty} d(h(y_i), h(y'_i)) = 0$. Set $u_i = h(y_i)$ and $u'_i = h(y'_i)$. Since $h(\vec{y}), h(\vec{y}') \subseteq V$ and $\tau{\restriction}V$ is β-continuous, it follows that $\lim_{i\to\infty} d(\tau(u_i), \tau(u'_i)) = 0$. We may thus assume that for every $i \in \mathbb{N}$, $d(\tau(u_i), \tau(u'_i)) < s$.

Let K be such that h is K-bilipschitz, define $\gamma(t) = Kt$ and $\varrho = \alpha \circ \beta \circ \gamma$. So $\gamma \in \Gamma$ and hence $\varrho \in \Gamma$. We show that for every $i \in \mathbb{N}$

(†) $d(\tau(y_i), \tau(y'_i)) \le \varrho(d(y_i, y'_i))$.

Note that $\tau(y_i) = (h^\tau)^{-1} \circ \tau \circ h(y_i)$, and the same holds for y'_i. So

(1) $d(h(y_i), h(y'_i) \le \gamma(d(y_i, y'_i)$.

Now, $h(y_i), h(y'_i) \in V$ and $\tau{\restriction}V$ is β-continuous, so

(2) $d(\tau(h(y_i)), \tau(h(y'_i))) \le \beta(\gamma(d(y_i, y'_i)))$.

Since $d(\tau(u_i), \tau(u'_i)) < s$ and $\tau(u_i), \tau(u'_i) \in W$, it follows that

(3) $d((h^\tau)^{-1}(\tau(u_i)), (h^\tau)^{-1}(\tau(u'_i))) \le \alpha(d(\tau(u_i), \tau(u'_i))$.

Obviously, (1)–(3) imply (†).

Define $\hat{\beta}_i = \beta_{n_i}$. There is j such that $\varrho \preceq \hat{\beta}_j$. Let $\ell \in \mathbb{N}$ be such that $\varrho{\restriction}[0, 1/\ell] \le \hat{\beta}_j{\restriction}[0, 1/\ell]$. Let $i = \max(j, \ell)$. So $d(y_i, y'_i) \le 1/n_i \le 1/\ell$. From (†) and the fact $\hat{\beta}_j{\restriction}[0, 1] \le \hat{\beta}_i{\restriction}[0, 1]$ we conclude that $d(\tau(y_i), \tau(y'_i)) \le \varrho(d(y_i, y'_i)) \le \hat{\beta}_i(d(y_i, y'_i))$. That is,

$$d(\tau(x_{n_i}), \tau(x'_{n_i})) \le \beta_{n_i}(d(x_{n_i}, x'_{n_i})).$$

This contradicts the way that x_{n_i} and x'_{n_i} were chosen. So $\vec{x}$ has no Cauchy subsequences.

We may thus assume that $\vec{x}$ is spaced. Let $x^* \in U$. Since X is BPD.AC, there are $M, d > 0$ and rectifiable arcs $\{L_n \mid n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}$, L_n connects x_n with x^*, $\delta(L_n) \ge d$ and $\mathrm{lngth}(L_n) \le M$. From Lemma 5.28 we obtain $\hat{x} \in X$, $r > 0$, an infinite $\eta \subseteq \mathbb{N}$ and $t \in (0, 1]$ such that for the parametrization γ_n of L_n defined in Lemma 5.28 the following holds: $B(\hat{x}, r)$ is a BPD subset of X, for every $n \in \eta$, $x_n \notin \mathrm{cl}^E(B(\hat{x}, r))$ and $\gamma_n(t) \in B(\hat{x}, r)$ and the set of arcs $\{\gamma_n{\restriction}[0, t] \mid n \in \eta\}$ is spaced. We may assume that $\eta = \mathbb{N}$.

For every $n \in \mathbb{N}$ let t_n be the least t' such that $\gamma_n(t') \in \mathrm{cl}^E(B(\hat{x}, r))$. Let $\gamma'_n = \gamma_n{\restriction}[0, t_n]$ and $y_n = \gamma_n(t_n)$. So $d(y_n, \hat{x}) = r$ and $\mathrm{Rng}(\gamma'_n) \cap B(\hat{x}, r) = \emptyset$.

Since τ is locally Γ-continuous, there is $\alpha^* \in \Gamma$ and $r_1 < r$ such that $\tau{\restriction}B(\hat{x}, r_1)$ is α^*-continuous. Let $z_n = \hat{x} + \frac{r_1}{2} \cdot (y_n - \hat{x})/\|y_n - \hat{x}\|$ and $L^*_n = \mathrm{Rng}(\gamma'_n) \cup [y_n, z_n]$.

So L_n^* is a rectifiable arc. Clearly, there are $M^*, d^*, D^* > 0$ such that for any distinct $m, n \in \mathbb{N}$, $\mathrm{lngth}(L_m^*) \le M^*$, $\delta(L_m^*) \ge d^*$ and $d(L_m^*, L_n^*) \ge D^*$. Let $r^* > 0$ be such that $r^* < d^*/2$, $D^*/3$, $r_1/2$.

For every $n \in \mathbb{N}$ we apply Lemma 5.27 with $L = M^*$, $r = r^*$, $\gamma = L_n^*$, $x = x_n$ and $y = z_n$. We obtain $h_n \in H(X)$ such that $\mathrm{supp}(h_n) \subseteq B(L_n^*, r^*)$, $h_n \restriction B(x_n, r^*/2) = \mathrm{tr}_{z_n - x_n} \restriction B(x_n, r^*/2)$ and h_n is $K_{\mathrm{arc}}(M^*, r^*)$-bilipschitz.

The families $\{h_n \mid n \in \mathbb{N}\}$ and $\{h_n^{-1} \mid n \in \mathbb{N}\}$ satisfy the conditions of Proposition 5.17(b) with $\alpha(x) = K_{\mathrm{arc}}(M^*, r^*) \cdot x$. Let $h = \circ_{n \in \mathbb{N}} h_n$ and $g = h^{-1}$. So by Proposition 5.17(b), h is $2K_{\mathrm{arc}}(M^*, r^*)$-bilipschitz. Also, $\delta(\mathrm{supp}(h)) \ge d^* - r^* > 0$, and hence $h, g \in \mathrm{LIP}_{00}(X)$. Since $\tau \in \mathrm{BPD.P}(X, Y)$, $\tau(U)$ is a BPD subset of Y. From the fact $(\mathrm{LIP}_{00}(X))^\tau \subseteq H_\Gamma^{\mathrm{NBPD}}(Y)$ it follows that for some $\alpha \in \Gamma$ and $r > 0$, $g^\tau \restriction \tau(U)$ is (r, α)-bicontinuous. We shall thus reach a contradiction by proving the following statement:

(∗) There are no $r > 0$ and $\alpha \in \Gamma$ such that $g^\tau \restriction \tau(U)$ is (r, α)-continuous.

Let $r > 0$ and $\alpha \in \Gamma$. For $n \in \mathbb{N}$ set $z'_n = h(x'_n)$, $u_n = \tau(z_n)$ and $u'_n = \tau(z'_n)$. Choose $m \in \mathbb{N}$ and $b \in (0,1)$ such that $\alpha \restriction [0,b], \alpha^* \restriction [0,b] \le \alpha_m \restriction [0,b]$. So for every $n \ge m$,

(1) $\alpha \restriction [0,b] \le \alpha_n \restriction [0,b]$;
(2) $\alpha^* \restriction [0,b] \le \alpha_n \restriction [0,b]$.

There is $n \ge m$ such that:

(3) $1/n < b$;
(4) $\alpha^*(1/n) < r$;
(5) $\alpha^*(1/n) < b$;
(6) $1/n < r^*/2$.

By the choice of z_n and r^*, $B(z_n, r^*) \subseteq B(\hat{x}, r_1)$. So $\tau \restriction B(z_n, r^*)$ is α^*-continuous. Since $d(x_n, x'_n) \le 1/n < r^*/2$ and by the definition of h_n and h,

(7) $d(x_n, x'_n) = d(z_n, z'_n)$.

Hence $z'_n \in B(z_n, r^*)$, and so

(8) $d(u_n, u'_n) \le \alpha^*(d(z_n, z'_n))$.

By (3) and (7), $d(z_n, z'_n) \le 1/n < b$, so by (2) and (8), $d(u_n, u'_n) \le \alpha_n(d(z_n, z'_n))$. It follows that

(9) $d(z_n, z'_n) \ge \alpha_n^{-1}(d(u_n, u'_n))$.

By (7) and (9), $d(x_n, x'_n) \ge \alpha_n^{-1}(d(u_n, u'_n))$. By the definition of β_n, x_n and x'_n, $d(\tau(x_n), \tau(x'_n)) > \alpha_n \circ \alpha_n(d(x_n, x'_n))$. So

(10) $d(\tau(x_n), \tau(x'_n)) > \alpha_n(d(u_n, u'_n))$.

Note that $\tau(x_n) = g^\tau(u_n)$ and $\tau(x'_n) = g^\tau(u'_n)$. So

(11) $d(g^\tau(u_n), g^\tau(u'_n)) > \alpha_n(d(u_n, u'_n))$.

Since $d(z_n, z'_n) \le 1/n$, by (8) and (5), $d(u_n, u'_n) \le b$. So by (1), $\alpha_n(d(u_n, u'_n)) \ge \alpha(d(u_n, u'_n))$. It now follows from (11) that

(12) $d(g^\tau(u_n), g^\tau(u'_n)) > \alpha(d(u_n, u'_n))$.

By (8), $d(u_n, u'_n) \le \alpha^*(1/n)$. So by (4),

(13) $d(u_n, u'_n) < r$.

Facts (12), (13) mean that $g^\tau \restriction \tau(U)$ is not (r,α)-continuous. This was proved for arbitrary r and α, that is, we have proved $(*)$. We have a contradiction to the fact that $g^\tau \in H_\Gamma^{\mathrm{NBPD}}(Y)$. So (b) is proved. ■

QUESTION 5.30. Does Theorem 5.24 remain true when the assumption that Γ is countably generated is dropped or replaced by the assumption that Γ is generated by a set whose cardinality is $\le \kappa(X)$? □

Note that the use of the countable generatedness of Γ in the proof of 5.24 was essential.

THEOREM 5.31. *Let $X, Y \in K_{\mathrm{NRM}}^{\mathcal{O}}$. Suppose that X is BPD.AC. Let $\tau \in H(X,Y)$ be such that $(\mathrm{UC}_{00}(X))^\tau \subseteq \mathrm{BPD.UC}(Y)$. Then $\tau \in \mathrm{BPD.UC}(X,Y)$.*

Proof. By definition, $\mathrm{BPD.UC}(Y) \subseteq \mathrm{BPD.P}(Y)$, hence by Lemma 5.25, $\tau \in \mathrm{BPD.P}(X,Y)$.

Suppose by contradiction that $\tau \notin \mathrm{BPD.UC}(X,Y)$. Then there are $d > 0$ and $\vec{x}, \vec{y} \subseteq X$ such that $\mathrm{Rng}(\vec{x}) \cup \mathrm{Rng}(\vec{y})$ is a BPD set, $\lim_{n\to\infty} d(x_n, y_n) = 0$, and for every $n \in \mathbb{N}$, $d(\tau(x_n), \tau(y_n)) \ge d$.

Suppose by contradiction that $\vec{x}$ has a Cauchy subsequence. We may then assume that $\vec{x}$ is a Cauchy sequence. Let $\bar{x} = \lim^{\bar{E}} \vec{x}$. Since $\mathrm{Rng}(\vec{x})$ is a BPD set, $\bar{x} \in \overline{\mathrm{int}}(X)$. Let $u \in X$ and $r > 0$ be such that $B^E(u, 2r) \subseteq X$ and $\bar{x} \in B^{\bar{E}}(u, r)$.

We have $\mathrm{BPD.UC}(X) \subseteq \mathrm{LUC}(X)$ and $\mathrm{UC}_{00}(X) = \mathrm{UC}(X, \mathcal{U})$, where $\mathcal{U}$ is the set of all open BPD subsets of X. So by Theorem 4.8(b), $\tau \in \mathrm{LUC}(X,Y)$. So there is $V \in \mathrm{Nbr}^X(u)$ such that $\tau \restriction V$ is uniformly continuous. There is $h \in \mathrm{LIP}(X) \lfloor B(u,r) \rfloor$ such that $h^{\mathrm{cl}}_{\bar{E}}(\bar{y}) \in \overline{\mathrm{int}}(V)$. Since $h \in \mathrm{UC}_{00}(X)$, $h^\tau \in \mathrm{BPD.UC}(X)$.

Recall that $\tau \in \mathrm{BPD.P}(X,Y)$. Since $B(u,r)$ is a BPD set in X, $W := \tau(B(u,r))$ is a BPD set in Y. So $h^\tau \restriction W$ is bi-UC. Since $\lim \vec{x} = \lim \vec{y} = \bar{x}$ and $h^{\mathrm{cl}}_{\bar{E}}(\bar{x}) \in \overline{\mathrm{int}}(V)$, we may assume that $h(\vec{x}), h(\vec{y}) \subseteq V$. Since h is uniformly continuous and $\tau \restriction V$ is uniformly continuous,

(1) $\lim_{i\to\infty} d(\tau(h(x_i)), \tau(h(y_i))) = 0$.

Note that $(h^\tau)^{-1}(\tau(h(x_i))) = \tau(x_i)$, and the same holds for y_i. So for every i,

(2) $d((h^\tau)^{-1}(\tau(h(x_i))), (h^\tau)^{-1}(\tau(h(y_i)))) \ge d$.

(1) and (2) contradict the fact that $h^\tau \restriction W$ is bi-UC. So $\vec{x}$ has no Cauchy subsequences.

We may thus assume that there is $s > 0$ such that $\vec{x}$ is s-spaced. Let $r = \min(s, \delta(\vec{x}))/3$. We may assume that for every $n \in \mathbb{N}$, $d(y_n, x_n) < r/3$. Let $r_n = 2d(y_n, x_n)$. Hence $B^E(x_n, r_n) \subseteq X$, and $\lim_{n\to\infty} \mathrm{diam}(B^E(x_n, r_n)) = 0$. Also, for any distinct $m, n \in \mathbb{N}$, $d(B^E(x_m, r_m), B^E(x_n, r_n)) \ge s/3$.

For every $n \in \mathbb{N}$, let $z_n \in [x_n, y_n]$ be such that $d(\tau(z_n), \tau(x_n)) \le d/(n+2)$, and $h_n \in \mathrm{UC}(X)$ be such that $\mathrm{supp}(h_n) \subseteq B(x_n, r_n)$, $h_n(x_n) = x_n$ and $h_n(z_n) = y_n$. By Proposition 4.5, $h := \circ_{n\in\mathbb{N}} h_n \in \mathrm{UC}(X)$. Also $\delta(\mathrm{supp}(h)) \ge r/3$. So $h \in \mathrm{UC}_{00}(X)$. Hence $h^\tau \in \mathrm{BPD.UC}(Y)$. $\vec{x} \cup \vec{y} \cup \vec{z}$ is a BPD set. So since $\tau \in \mathrm{BPD.P}(X,Y)$, it follows that $\tau(\vec{x}) \cup \tau(\vec{y}) \cup \tau(\vec{z})$ is a BPD set. However, $h^\tau \restriction (\tau(\vec{x}) \cup \tau(\vec{y}) \cup \tau(\vec{z}))$ is not UC. This is so, because $\lim_{n\to\infty} d(\tau(x_n),\tau(z_n)) = 0$, whereas for every $n \in \mathbb{N}$, $d(h^\tau(\tau(x_n)), h^\tau(\tau(z_n))) = d(\tau(x_n),\tau(y_n)) \geq d$. A contradiction. ■

THEOREM 5.32. *Let Γ, Δ be moduli of continuity. Suppose that Γ is countably generated or $\Gamma = \mathrm{MC}$, and that the same holds for Δ. Let $X, Y \in K^{\mathcal{O}}_{\mathrm{NFCB}}$, and assume that X and Y are BPD.AC. Suppose that $\varphi : H^{\mathrm{NBPD}}_\Gamma(X) \cong H^{\mathrm{NBPD}}_\Delta(Y)$. Then $\Gamma = \Delta$ and there is $\tau \in (H^{\mathrm{NBPD}}_\Gamma)^{\pm}(X,Y)$ which induces φ.*

Proof. Let $\mathcal{U}$ denote the set of all open BPD subsets of X. Note that

(1) $\mathrm{LIP}_{00}(X) \leq H^{\mathrm{NBPD}}_\Gamma(X) \leq \mathrm{IXT}(X)$ and $\mathrm{LIP}_{00}(X) = \mathrm{LIP}(X,\mathcal{U})$.

Hence by Corollary 2.26, there is $\tau \in H(X,Y)$ such that τ induces φ. Suppose that Δ is countably generated. Clearly,

(2) $H^{\mathrm{NBPD}}_\Delta(Y) \subseteq H^{\mathrm{LC}}_\Delta(Y)$.

By (1) and (2), $(\mathrm{LIP}(X,\mathcal{U}))^\tau \subseteq H^{\mathrm{LC}}_\Delta(Y)$. By Theorem 3.27, τ is locally Δ-bicontinuous. Suppose by contadiction that $\alpha \in \Delta - \Gamma$. Let B be an open ball in E such that B is a BPD subset of X and such that for some $\beta \in \Delta$, $\tau \restriction B$ is β-bicontinuous. There is $g \in H(X)\lfloor B \rfloor$ such that g is α-bicontinuous, and for every $\gamma \in \Gamma$, g is not γ-bicontinuous. So $g \notin H^{\mathrm{NBPD}}_\Gamma(X)$, but $g^\tau \in H^{\mathrm{NBPD}}_\Delta(Y)$, a contradiction. So $\Delta \subseteq \Gamma$. An identical argument shows that $\Gamma \subseteq \Delta$. Hence $\Gamma = \Delta$. Applying Theorem 5.24 to τ and τ^{-1}, we conclude that $\tau \in (H^{\mathrm{NBPD}}_\Gamma)^{\pm}(X,Y)$.

Suppose next that $\Gamma = \Delta = \mathrm{MC}$. Since $\mathrm{UC}_{00}(X) \leq H^{\mathrm{NBPD}}_{\mathrm{MC}}(X)$, we have $(\mathrm{UC}_{00}(X))^\tau \subseteq H^{\mathrm{NBPD}}_{\mathrm{MC}}(X)$, and the same holds for Y. Hence Theorem 5.31 may be applied to τ and τ^{-1}. We conclude that $\tau \in \mathrm{BPD.UC}^{\pm}(X,Y)$. That is, $\tau \in (H^{\mathrm{NBPD}}_{\mathrm{MC}})^{\pm}(X,Y)$. ■

We now turn to the group $H^{\mathrm{WBPD}}_{\mathrm{MC}}(X)$. We shall reach the same final result as for the groups of type $H^{\mathrm{NBPD}}_{\mathrm{MC}}(X)$. But here we need the extra assumption that X is fillable. This notion is defined below.

DEFINITION 5.33. Let X be a topological space and $G \leq H(X)$. A sequence $\vec{x} \subseteq X$ is called a *G-filling* of X if the following holds. For every sequence $\{U_i \mid i \in \mathbb{N}\}$ such that for every i, $U_i \in \mathrm{Nbr}(x_i)$, there is a sequence $\{g_i \mid i \in \mathbb{N}\} \subseteq G$ such that $\bigcup_{i\in\mathbb{N}} g_i(U_i) = X$. We say that X is *G-fillable* if X has a G-filling. □

The trivial verification of the following observation is left to the reader.

PROPOSITION 5.34. *Let E be a normed space.*

(a) *If E is separable and $X \subseteq E$ is open, then X is $\mathrm{LIP}_{00}(X)$-fillable.*
(b) *If $r > 0$, then $B^E(0,r)$ is $\mathrm{LIP}_{00}(X)$-fillable.*

The following observation gives some answer for the groups of type $H^{\mathrm{WBPD}}_{\mathrm{MC}}(X)$.

PROPOSITION 5.35. *Suppose that X is BPD.AC, $\mathrm{UC}_{00}(X) \leq G \leq H^{\mathrm{WBPD}}_{\mathrm{MC}}(X)$ and X is G-fillable. Let $\tau \in H(X,Y)$ be such that $G^\tau \subseteq H^{\mathrm{WBPD}}_{\mathrm{MC}}(Y)$. Then $\tau \in H^{\mathrm{WBPD}}_{\mathrm{MC}}(X,Y)$.*

Proof. Let $\mathcal{U}$ be the set of all open BPD subsets of X. Then $\mathrm{UC}_{00}(X) = \mathrm{UC}(X,\mathcal{U})$. Note that $H^{\mathrm{WBPD}}_{\mathrm{MC}}(Y) \subseteq \mathrm{LUC}(Y)$. So $(\mathrm{UC}(X,\mathcal{U}))^\tau \subseteq \mathrm{LUC}(Y)$. By Theorem 4.8(b), $\tau \in \mathrm{LUC}^{\pm}(X,Y)$. Similarly, $(\mathrm{LIP}_{00}(X))^\tau \subseteq (\mathrm{UC}_{00}(X))^\tau \subseteq H^{\mathrm{WBPD}}_{\mathrm{MC}}(Y) \subseteq \mathrm{BPD.P}(Y)$. So by Lemma 5.25, $\tau \in \mathrm{BPD.P}(X,Y)$.

Let $\vec{x}$ be a G-filling for X. For every $i \in \mathbb{N}$ let $U_i \in \mathrm{Nbr}(x_i)$ and α_i be such that $\tau \restriction U_i$ is α_i-bicontinuous. Let $\{g_i \mid i \in \mathbb{N}\} \subseteq G$ be such that $\bigcup\{g_i(U_i) \mid i \in \mathbb{N}\} = X$.

Let $A \subseteq X$ be a BPD set. We show that $\tau \restriction A$ is weakly MC-bicontinuous. Since $\tau \in \mathrm{BPD.P}(X, Y)$, $\tau(A)$ is a BPD set. For every $i \in \mathbb{N}$ let β_i be such that $g_i \restriction A$ is locally $\{\beta_i\}$-bicontinuous and γ_i be such that $g_i^\tau \restriction \tau(A)$ is locally $\{\gamma_i\}$-bicontinuous. Next note that

$$\tau \restriction g_i(U_i) = (g_i^\tau \restriction \tau(U_i)) \circ (\tau \restriction U_i) \circ ((g_i)^{-1} \restriction g_i(U_i)).$$

Hence $\tau \restriction (g_i(U_i) \cap A)$ is locally $\{\gamma_i \circ \alpha_i \circ \beta_i\}$-bicontinuous.

There is $\varrho \in \mathrm{MC}$ such that for every $i \in \mathbb{N}$, $\gamma_i \circ \alpha_i \circ \beta_i \preceq \varrho$. Hence for every $i \in \mathbb{N}$, $\tau \restriction (g_i(U_i) \cap A)$ is locally $\{\varrho\}$-bicontinuous, and from the fact $\bigcup_{i \in \mathbb{N}} (g_i(U_i) \cap A) = A$ we conclude that $\tau \restriction A$ is $\{\varrho\}$-bicontinuous. So $\tau \in H_{\mathrm{MC}}^{\mathrm{WBPD}}(X, Y)$. ■

THEOREM 5.36. *Let Γ, Δ be moduli of continuity. Suppose that Γ is countably generated or $\Gamma = \mathrm{MC}$, and that the same holds for Δ. Let $X, Y \in K_{\mathrm{NFCB}}^{\mathcal{O}}$. Assume that*

(1) *X and Y are BPD.AC;*
(2) *If $\Gamma = \mathrm{MC}$, then X is $H_{\mathrm{MC}}^{\mathrm{WBPD}}(X)$-fillable, and the same holds for Δ and Y.*

Suppose that $\varphi : H_\Gamma^{\mathrm{WBPD}}(X) \cong H_\Delta^{\mathrm{WBPD}}(Y)$. Then $\Gamma = \Delta$ and there is $\tau \in H_\Gamma^{\mathrm{WBPD}}(X, Y)$ which induces φ.

Proof. The proof is very similar to the proof of Theorem 5.32. ■

In some cases we reach a final reconstruction result of the following strong form:

(1) If $\varphi : \mathcal{P}(X) \cong \mathcal{Q}(Y)$, then either $\mathcal{P}(X) = \mathcal{Q}(X)$ and there is $\tau \in \mathcal{Q}^{\pm}(X, Y)$ which induces φ, or $\mathcal{P}(Y) = \mathcal{Q}(Y)$ and there is $\tau \in \mathcal{P}^{\pm}(X, Y)$ which induces φ.

In other cases we are able to reach only the following weaker conclusion:

(2) If $\varphi : \mathcal{P}(X) \cong \mathcal{P}(Y)$, then there is $\tau \in \mathcal{P}^{\pm}(X, Y)$ which induces φ.

Roughly speaking, in order to prove results of the first form, we need to prove the following intermediate claim:

(3) If $\tau \in H(X, Y)$ and $(\mathcal{P}(X))^\tau \subseteq \mathcal{P}(Y)$, then $\tau \in \mathcal{P}^{\pm}(X, Y)$,

and in order to prove a result of the second form, the following intermediate claim suffices:

(4) If $\tau \in H(X, Y)$ and $(\mathcal{P}(X))^\tau \subseteq \mathcal{P}(Y)$, then $\tau \in \mathcal{P}(X, Y)$.

For example, Theorem 4.8 which deals with the group $\mathrm{LUC}(X)$ has the stronger form (3), and Theorem 5.5 which deals with the group $\mathrm{UC}(X)$ has the weaker form (4).

The strong intermediate claim is not always true. Example 5.7 shows that (3) is false for $\mathrm{UC}(X)$, and also false for $\mathrm{BPD.UC}(X)$, as is shown in Example 5.38(a). However, if X is an open subset of a Banach space, and X fulfills some additional requirements, then the implication

$$(\mathrm{BPD.UC}(X))^\tau \subseteq \mathrm{BPD.UC}(Y) \ \Rightarrow\ \tau \in \mathrm{BPD.UC}^{\pm}(X, Y)$$

is true. This will be proved in Theorem 5.41(a). Later, in Theorem 7.7 we shall prove an analogous statement for $\mathrm{UC}(X)$. Namely, if X satisfies certain additional requirements, then $(\mathrm{UC}(X))^\tau \subseteq \mathrm{UC}(Y) \Rightarrow \tau \in \mathrm{UC}^{\pm}(X, Y)$.

We need yet another notion of weak uniform arcwise connectedness. This will be the additional assumption in Theorem 5.41(a).

Definition 5.37. Let E be a metric space, $X \subseteq E$ and $x \in \mathrm{bd}^E(X)$. We say that X is *locally arcwise connected at* x if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $y, z \in X$: if $d(x, y), d(x, z) < \delta$, then there is an arc $L \subseteq X$ connecting y and z such that $\mathrm{diam}(L) < \varepsilon$. We then call x a *simple boundary point* of X. We say that X is *locally arcwise connected at its boundary with respect to* E (*BR.LC.AC with respect to* E) if every boundary point of X is simple. $\square$

An equivalent formulation of simplicity is as follows. For every $\varepsilon > 0$ there is $\delta > 0$ such that for every $y, z \in X \cap B(x, \delta)$ there is an arc L connecting y and z such that $L \subseteq X \cap B(x, \varepsilon)$. Note that being locally arcwise connected at $x \in \mathrm{bd}(X)$ implies but is not equivalent to the fact that $X \cup \{x\}$ is locally arcwise connected at x.

The following example shows that the completeness requirement in Lemma 5.39 cannot be dropped.

Example 5.38. *Let E be an incomplete normed space, $K \subseteq [0, 1/2)$ be a closed nowhere dense perfect set containing* 0, $X' = B^E(0, 2) - \bar{B}^E(0, 1)$, $u \in S^E(0, 1)$, $C = \{(1 + t) \cdot u \mid t \in K - \{0\}\}$, $X = X' - C$, $Y' = B^E(0, 1)$, $D = \{(1 - t) \cdot u \mid t \in K - \{0\}\}$, *and* $Y = Y' - D$.

(a) *X and Y are BPD.AC, BR.LC.AC and UD.AC.*

(b) *There is $\tau \in H(X, Y)$ such that* $(\mathrm{BPD.UC}(X))^\tau \subseteq \mathrm{BPD.UC}(Y)$ *and* $\tau^{-1} \notin \mathrm{BPD.UC}(Y, X)$.

(c) *There is $\tau \in H(X, Y)$ such that*

(1) $(\mathrm{BPD.UC}(X))^\tau \subseteq \mathrm{BPD.UC}(Y)$,

(2) $\tau^{-1} \notin \mathrm{BPD.P}(Y, X)$,

(3) *for every BPD set $A \subseteq X$, $\tau \restriction A$ is bilipschitz.*

Proof. (a) This part is trivial, so we leave its verification to the reader. In any case, (a) shows that the fact that the boundaries of X and Y are well-behaved does not by itself imply that $\tau^{-1} \in \mathrm{BPD.UC}(Y, X)$.

(b) This follows from (c). So it suffices to prove (c).

(c) Note the following facts: $C \subseteq B^E(0, 3/2) - \bar{B}^E(0, 1)$, $D \subseteq B^E(0, 1) - \bar{B}^E(0, 1/2)$, $u \in \mathrm{acc}(C)$ and $u \in \mathrm{acc}(D)$.

Let $y \in B^{\bar{E}}(0, 1/2) - B^E(0, 1/2)$. Proposition 2.25(b) yields $\varrho \in \mathrm{LIP}(\bar{E}) \lfloor B^{\bar{E}}(0, 1/2) \rfloor$ such that $\varrho(0) = y$ and $\varrho(E - \{0\}) = E$. So $\varrho(D) = D$ and hence $\varrho(Y - \{0\}) = Y$. Let $\eta : X \to Y - \{0\}$ be defined by $\eta(x) = (2 - \|x\|) \cdot \frac{x}{\|x\|}$ and $\tau = \varrho \circ \eta$. Clearly, $\tau \in H(X, Y)$, and it is easy to check that τ satisfies clause (3).

Let $r > 0$ be such that $B^{\bar{E}}(y, r) \subseteq B^{\bar{E}}(0, 1/2)$ and $M = B^{\bar{E}}(y, r) \cap E$. Then M is a BPD subset of Y. However, $\tau^{-1}(M)$ contains a set of the form $B^E(0, 2) - \bar{B}(0, 2 - \varepsilon)$, where $\varepsilon > 0$. So $\tau^{-1}(M)$ is not a BPD subset of X. Hence clause (2) is fulfilled.

We show that τ fulfills clause (1). It is easy to check that $(\mathrm{BPD.P}(X))^\tau \subseteq \mathrm{BPD.P}(Y)$. So it remains to show that if $h \in \mathrm{BPD.UC}(X)$ and $M \subseteq Y$ is a BPD set, then $h^\tau \restriction M$ is bi-UC.

Since ϱ is bilipschitz it suffices to show that for every $h \in \text{BPD.UC}(X)$ and $M \subseteq Y - \{0\}$: if $d(M, D \cup S(0,1)) > 0$, then $h^\eta \restriction M$ is bi-UC. (Indeed we show that $h^\eta \restriction M$ is bi-UC, even for M's which satisfy $M \subseteq Y - \{0\}$ and $d(M, D) > 0$.)

CLAIM 1. *Let Z, W be metric spaces, $z \in Z$, and $f : Z \to W$. Suppose that f is continuous at z, and for every $r > 0$, $f \restriction (Z - B(z,r))$ is UC. Then f is UC.*

Proof. Let $\varepsilon > 0$. There is $\delta_1 > 0$ such that $\text{diam}(f(B(z,\delta_1))) < \varepsilon$. Let $\delta_2 > 0$ be such that for every $x, y \in Z - B(z, \delta_1/2)$: if $d(x,y) < \delta_2$, then $d(f(x), f(y)) < \varepsilon$. Let $\delta = \min(\delta_1/2, \delta_2)$. Suppose that $d(x,y) < \delta$. Either $x, y \in Z - B(z, \delta_1/2)$ or $x, y \in B(z, \delta_1)$. In either case $d(f(x), f(y)) < \varepsilon$. Claim 1 is proved.

CLAIM 2. *Let $h \in \text{BPD.P}(X)$ and $\vec{x} \subseteq X$ be such that $\lim_{n\to\infty} \|x_n\| = 2$. Then $\lim_{n\to\infty} \|h(x_n)\| = 2$.*

Proof. Suppose by way of contradiction that this is not true, and let $\vec{x}$ be a counterexample. Since $h \in \text{BPD.P}(X)$, for every subsequence $\vec{x}\,'$ of $\vec{x}$, $h(\vec{x}\,')$ is not a BPD sequence. It follows easily that either $\vec{x}$ has a subsequence $\vec{x}\,'$ such that $\lim_{n\to\infty} \|h(x'_n)\| = 1$, or $\vec{x}$ has a subsequence $\vec{x}\,'$ which converges to a member of C. Taking $\vec{x}$ to be $\vec{x}\,'$ we may assume that either (i) $\lim_{n\to\infty} \|h(x_n)\| = 1$ or (ii) for some $\hat{u} \in C$, $\lim h(\vec{x}) = \hat{u}$.

Suppose that (i) happens. Then for every $n \in \mathbb{N}$ there are $u_n \in C$, $s_n > r_n > 0$ and an arc $L_n \subseteq X$ such that the following hold:

(1) $h(x_n) \in L_n$ and L_n intersects both $S(u_n, r_n)$ and $S(u_n, s_n)$.
(2) Set $S_n = S(u_n, r_n) \cup S(u_n, s_n)$. Then $\delta^X(S_n) > 0$. (Hence $S_n \subseteq X$.)
(3) Define $d_n = \sup(\{d(z, S(0,1)) \mid z \in L_n \cup S_n\})$. Then $\lim_{n\to\infty} d_n = 0$.
(4) $(B(u_n, s_n) - \bar{B}(u_n, r_n)) \cap C \neq \emptyset$.

Suppose that (ii) happens. Then for every $n \in \mathbb{N}$ there are $s_n > r_n > 0$ and an arc $L_n \subseteq X$ such that the following hold:

(5) $h(x_n) \in L_n$, and L_n intersects both $S(\hat{u}, r_n)$ and $S(\hat{u}, s_n)$.
(6) Set $S_n = S(\hat{u}, r_n) \cup S(\hat{u}, s_n)$. Then $\delta^X(S_n) > 0$. (Hence $S_n \subseteq X$.)
(7) Define $d_n = \sup(\{d(z, \hat{u}) \mid z \in L_n \cup S_n\})$. Then $\lim_{n\to\infty} d_n = 0$.
(8) $(B(\hat{u}, s_n) - \bar{B}(\hat{u}, r_n)) \cap C \neq \emptyset$.

In both case (i) and case (ii) set $A_n = L_n \cup S_n$ and $B_n = h^{-1}(A_n)$. Let $\vec{z}$ be a sequence such that $z_n \in B_n$ for every $n \in \mathbb{N}$. By (3) and (7), $\lim_{n\to\infty} \delta^X(h(z_n)) = 0$. From the fact that $h \in \text{BPD.P}(X)$ it follows that $\lim_{n\to\infty} \delta^X(z_n) = 0$. There is a subsequence $\{n_i \mid i \in \mathbb{N}\}$ such that either $\lim_{n\to\infty} d(z_{n_i}, S(0,2)) = 0$ or $\lim_{n\to\infty} d(z_{n_i}, S(0,1) \cup C) = 0$. We may assume that $n_i = i$ for every i. Suppose by contradiction that the latter happens. Now, $x_n, z_n \in B_n$, B_n is connected and $\lim_{n\to\infty} d(x_n, S(0,2)) = 0$. We also have $d(S(0,2), S(0,1) \cup C) > 0$. Choose $y_n \in B_n$ such that $\|y_n - x_n\| = \|z_n - x_n\|/2$. Then $d(\{y_n \mid n \in \mathbb{N}\}, \text{bd}(X)) > 0$, a contradiction. So $\lim_{n\to\infty} d(z_n, S(0,2)) = 0$.

Let $e_n = \sup(\{d(z, S(0,2)) \mid z \in B_n\})$. It follows that $\lim_{n\to\infty} e_n = 0$. Let n be such that $e_n \leq 1/4$. Define $S = S_n$ and $T = h^{-1}(S)$. Since S is a BPD set, T is a BPD set. Let $d = d(T, S(0,2))$. It is obvious that $X - S$ has three connected components, and neither of them is a BPD set. So the same holds for T. However, since $e_n \leq 1/4$, $T \subseteq B(0,2) - \bar{B}(0,3/2)$ and so $X \cap B(0,3/2)$ is contained in a component of $X - T$, and

$B(0,2) - \bar{B}(0, 2-d)$ is also contained in a component of $X - T$. It follows that one of the components of $X - T$ is contained in $W := \bar{B}(0, 2-d) - B(0, 3/2)$. But W is a BPD subset of X. A contradiction, so Claim 2 is proved.

Let $h \in \mathrm{BPD.UC}(X)$ and define $g = h^{\eta}$.

Claim 3. $0 \in \mathrm{Dom}(g^{\mathrm{cl}})$ *and* $g^{\mathrm{cl}}(0) = 0$.

Proof. Let $\vec{x} \subseteq B(0,1) - \{0\}$ be such that $\lim \vec{x} = 0$. Then $\lim_{n\to\infty} \|\eta^{-1}(x_n)\| = 2$. Note that $h \in \mathrm{BPD.P}(X)$. Applying Claim 2 to h, we conclude that $\lim_{n\to\infty} \|h(\eta^{-1}(x_n))\| = 2$. Hence $\lim_{n\to\infty} \|\eta(h(\eta^{-1}(x_n)))\| = 0$. That is, $\lim_{n\to\infty} \|g(x_n)\| = 0$. So Claim 3 is proved.

Let $M \subseteq Y - \{0\}$ be such that $d(M, D) > 0$. Let $r > 0$ and $N = \eta^{-1}(M - B(0,r))$. Then $d(N, S(0,2)) \geq r$. So $\eta\restriction N$ is bilipschitz, hence (i) $\eta^{-1}\restriction(M - B(0,r))$ is bilipschitz. N is a BPD subset of X. So (ii) $h\restriction N$ is bi-UC. Also, $h(N)$ is a BPD subset of X. In particular, $d(h(N), S(0,2)) > 0$. So (iii) $\eta\restriction h(N)$ is bilipschitz.

$$\begin{aligned} g\restriction(M - B(0,r)) &= \eta \circ h \circ \eta^{-1}\restriction(M - B(0,r)) \\ &= (\eta\restriction h(\eta^{-1}(M - B(0,r)))) \circ (h\restriction\eta^{-1}(M - B(0,r))) \circ (\eta^{-1}\restriction(M - B(0,r))) \\ &= \eta\restriction h(N) \circ (h\restriction N) \circ (\eta^{-1}\restriction(M - B(0,r))). \end{aligned}$$

By (i)–(iii), $g\restriction(M - B(0,r))$ is bi-UC. By Claim 3 and Claim 1, $g^{\mathrm{cl}}\restriction M$ is UC. Applying the same argument to h^{-1} we conclude that $(g^{\mathrm{cl}})^{-1}\restriction g(M)$ is UC. So $g\restriction M$ is bi-UC. That is, $h^{\eta}\restriction M$ is bi-UC. It has already been argued that this implies that $h^{\tau} \in \mathrm{BPD.UC}(Y)$. ■

Lemma 5.39. *Suppose that X is an open subset of a Banach space E.*

(a) $\mathrm{BUC}(X) \subseteq \mathrm{BPD.UC}(X)$.

(b) *Suppose that X is BR.LC.AC, $\tau \in H(X,Y)$ and* $(\mathrm{BUC}(X))^{\tau} \subseteq \mathrm{BPD.P}(Y)$. *Then* $\tau^{-1} \in \mathrm{BPD.P}(Y, X)$.

Proof. (a) Let $h \in \mathrm{BUC}(X)$. Suppose that $x \in \mathrm{bd}(X)$, $\vec{x} \subseteq X$ and $\lim \vec{x} = x$. Then $h(\vec{x})$ is a Cauchy sequence. Let $y = \lim h(\vec{x})$. Clearly, $y \in \mathrm{bd}^{\bar{E}}(X) \cup \overline{\mathrm{int}}(X)$ and $y \notin X$. Since E is complete, $\overline{\mathrm{int}}(X) = X$. Hence $y \in \mathrm{bd}(X)$. We have shown that $\mathrm{Dom}(h^{\mathrm{cl}}) = \mathrm{cl}(X)$ and that $h^{\mathrm{cl}}(\mathrm{bd}(X)) \subseteq \mathrm{bd}(X)$. Applying the same argument to h^{-1} one concludes that (†) $h^{\mathrm{cl}}(\mathrm{bd}(X)) = \mathrm{bd}(X)$. It is trivial that $h^{\mathrm{cl}} \in \mathrm{BUC}(\mathrm{cl}(X))$.

Suppose by contradiction that A is a BPD set and $h(A)$ is not a BPD set. By definition, h is boundedness preserving. So $h(A)$ is bounded and hence $\delta(h(A)) = 0$. Let $\vec{x} \subseteq h(A)$ and $\vec{y} \subseteq \mathrm{bd}(X)$ be such that $\lim_{n\to\infty} d(x_n, y_n) = 0$. By (†), $(h^{\mathrm{cl}})^{-1}(\vec{y}) \subseteq \mathrm{bd}(X)$. So for every n, $d(h^{-1}(x_n), h^{-1}(y_n)) \geq \delta(A) > 0$. Hence $(h^{\mathrm{cl}})^{-1}\restriction(\mathrm{Rng}(\vec{x}) \cup \mathrm{Rng}(\vec{y}))$ is not uniformly continuous. A contradiction.

(b) Let X, E, Y and τ be as in part (b), and suppose that Y is an open subset of the normed space F. Then F is a Banach space. To see this note that an open ball B of F is homeomorphic to an open subset of E. So B is completely metrizable. But $F \cong B$, so F is completely metrizable. So F is a dense G_{δ} subset of $\bar{F}$, and so is every coset of F in $\bar{F}$. Since $\bar{F}$ has no disjoint dense G_{δ} subsets, $F = \bar{F}$. Suppose by contradiction $\tau^{-1} \notin \mathrm{BPD.P}(Y, X)$. Then there is a 1-1 sequence $\vec{x} \subseteq X$ such that $\vec{x}$ is not a BPD sequence, but $\tau(\vec{x})$ is a BPD sequence. We may assume that $\lim_{n\to\infty} \delta_1^X(x_n) = \infty$.

Since $\tau(\vec{x})$ is a BPD set, it does not have convergent subsequences in F, hence $\tau(\vec{x})$ does not have a Cauchy subsequence. So we may assume that there is $d > 0$ such that $\tau(\vec{x})$ is d-spaced.

Claim 1. *$\vec{x}$ is not a Cauchy sequence.*

Proof. Suppose otherwise, and let $x^* = \lim \vec{x}$. Then $x^* \in \mathrm{bd}(X)$, for if $x^* \notin X$, then $\vec{x}$ is a BPD sequence.

By the simplicity of x^*, we can find a subsequence $\vec{y}$ of $\vec{x}$, arcs $\{L_n \mid n \in \mathbb{N}\}$ and open sets $\{U_n \mid n \in \mathbb{N}\}$ such that $y_{2n}, y_{2n+1} \in L_n \subseteq U_n \subseteq \mathrm{cl}(U_n) \subseteq X$, for any distinct $m, n \in \mathbb{N}$, $d(U_m, U_n) > 0$, and $\lim_{n\to\infty} \mathrm{diam}(U_n) = 0$.

Let $z_n \in L_n - \{y_{2n}\}$ be such that $\lim_{n\to\infty} d(\tau(y_{2n}), \tau(z_n)) = 0$. It follows that

(1) $\tau(\vec{y}) \cup \tau(\vec{z})$ is a BPD set.

Let $h_n \in \mathrm{UC}(X)$ be such that $h_n(y_{2n}) = y_{2n}$, $h_n(z_n) = y_{2n+1}$ and $\mathrm{supp}(h_n) \subseteq U_n$. By Proposition 4.5, $h := \circ_{n\in\mathbb{N}} h_n \in \mathrm{UC}(X)$. However,

(2) $h^\tau \restriction (\tau(\vec{y}) \cup \tau(\vec{z}))$ is not UC.

To see this recall that $\lim_{n\to\infty} d(\tau(y_{2n}), \tau(z_n)) = 0$. However, $d(h^\tau(\tau(y_{2n})), h^\tau(\tau(z_n))) = d(\tau(y_{2n}), \tau(y_{2n+1})) \geq d$. Facts (1) and (2) mean that $h^\tau \notin \mathrm{BPD.UC}(Y)$. A contradiction, so Claim 1 is proved.

Claim 2. *It is not true that* $\lim_{n\to\infty} \delta(x_n) = 0$.

Proof. Suppose otherwise. By Claim 1, we may assume that there is $e_1 > 0$ such that $\vec{x}$ is e_1-spaced. For every $n \in \mathbb{N}$ let $b_n \in \mathrm{bd}(X)$ be such that $d(x_n, b_n) \leq 2\delta(x_n)$, and $[x_n, b_n) \subseteq X$.

For every $n \in \mathbb{N}$ let $\vec{x}^n = \{x^n_i \mid i \in \mathbb{N}\} \subseteq [x_n, b_n)$ be a sequence converging to b_n. By Claim 1, $\tau(\vec{x}^n)$ is not a BPD set. It follows that there is a sequence $\{i_n \mid n \in \mathbb{N}\}$ such that $\{\tau(x^n_{i_n}) \mid n \in \mathbb{N}\}$ is not a BPD set. Let $y_n = x^n_{i_n}$. Since $\vec{x}$ is e_1-spaced and $\lim_{n\to\infty} d(x_n, y_n) = 0$, we may assume that there is $e > 0$ such that $\{[x_n, y_n] \mid n \in \mathbb{N}\}$ is e-spaced.

Let $\{U_n \mid n \in \mathbb{N}\}$ be a sequence of open subsets of X such that $[x_n, y_n] \subseteq U_n$, $\lim_{n\to\infty} \mathrm{diam}(U_n) = 0$ and for any distinct $m, n \in \mathbb{N}$, $d(U_m, U_n) > 0$. Let $h_n \in \mathrm{UC}(X)$ be such that $\mathrm{supp}(h_n) \subseteq U_n$ and $h_n(x_n) = y_n$. By Proposition 4.5, $h := \circ_{n\in\mathbb{N}} h_n \in \mathrm{UC}(X)$, but $h^\tau \notin \mathrm{BPD.P}(Y)$. This is so, because $h^\tau(\tau(\vec{x})) = \tau(\vec{y})$, and $\tau(\vec{x})$ is a BPD set, whereas $\tau(\vec{y})$ is not. A contradiction. This proves Claim 2.

From Claims 1 and 2 and the fact that $\vec{x}$ is not a BPD sequence, it follows that $\vec{x}$ is unbounded. So we may assume that $\{\|x_n\| \mid n \in \mathbb{N}\}$ is a strictly increasing sequence converging to ∞. Recall also that $\tau(\vec{x})$ is d-spaced. We now deal with two cases.

Case 1: $E - X$ is bounded. We may assume that $E - X \subseteq B(0, \|x_0\|/2)$. Set $x_{-1} = 0$. Choose $y_n \in (x_{2n}, x_{2n+1}]$ with $d(\tau(x_{2n}), \tau(y_n)) < 1/(n+1)$. Define $r_n = \min(\|x_{2n} - x_{2n-1}\|, \|x_{2n+2} - x_{2n+1}\|)/2$ and let $h_n \in \mathrm{UC}(X)$ be such that $h_n(x_{2n}) = x_{2n}$, $h_n(y_n) = x_{2n+1}$ and $\mathrm{supp}(h_n) \subseteq B([x_{2n}, x_{2n+1}], r_n)$. Clearly, $\mathrm{supp}(h_m) \cap \mathrm{supp}(h_n) = \emptyset$ for every $n \neq m$ and hence $h := \circ_{n\in\mathbb{N}} h_n \in \mathrm{BUC}(X)$. Since $\lim_{n\to\infty} d(\tau(x_{2n}), \tau(y_n)) = 0$, it follows that

$\tau(\vec{x}) \cup \tau(\vec{y})$ is a BPD set. But $h^\tau \restriction (\tau(\vec{x}) \cup \tau(\vec{y}))$ is not UC. So $h^\tau \notin \mathrm{BPD.UC}(Y)$. A contradiction, so Case 1 does not occur.

Case 2: $E - X$ is unbounded. We define by induction on $n \in \mathbb{N}$: $u_n \in \mathrm{Rng}(\vec{x})$, $v_n \in X$, $h_n \in \mathrm{UC}(X)$ and $r_n > 0$. Let $r_{-1} = 0$. Suppose that r_{n-1} has been defined. Let $u_n \in \mathrm{Rng}(\vec{x}) - \mathrm{cl}(B(0, r_{n-1}))$ and Let $b_n \in \mathrm{bd}(X) - \mathrm{cl}(B(0, r_{n-1}))$. We may assume that there is an arc $L_n \subseteq (X \cup \{b_n\}) - \mathrm{cl}(B(0, r_{n-1}))$ connecting u_n and b_n. Let $\vec{v}^n := \{v^{n,i} \mid i \in \mathbb{N}\} \subseteq L_n - \{b_n\}$ be a sequence converging to b_n. So $\vec{v}^n$ is a Cauchy sequence. So by Claim 1, $\tau(\vec{v}^n)$ is not a BPD set. Hence there is $v_n \in L_n - \{b_n\}$ such $\delta_1(\tau(v_n)) > n$.

Let r_n be such that $L_n \subseteq B(0, r_n)$ and $h_n \in \mathrm{UC}(X)$ be such that $h_n(u_n) = v_n$ and $\mathrm{supp}(h_n) \subseteq B(0, r_n) - \mathrm{cl}(B(0, r_{n-1}))$. Clearly, $\mathrm{supp}(h_m) \cap \mathrm{supp}(h_n) = \emptyset$ for every $m \neq n$, and hence $h := \circ_{n \in \mathbb{N}} h_n \in \mathrm{BUC}(X)$. However, since $\tau(\vec{u})$ is a BPD sequence, $\tau(\vec{v})$ is not a BPD sequence, and $h^\tau(\tau(\vec{u})) = \tau(\vec{v})$, $h^\tau \notin \mathrm{BPD.P}(Y)$. A contradiction, so Case 2 does not happen. It follows that $\tau^{-1} \in \mathrm{BPD.P}(Y, X)$. ∎

If X is BPD.AC, and we remove from X a spaced set, then the resulting open set is also BPD.AC. This is proved in the next proposition. Although this fact is quite trivial, a complete proof requires much writing.

Proposition 5.40. (a) *Let E be a normed space which is not 1-dimensional. Let $u, v, w \in E$ be such that $\|u - w\| = \|v - w\| = r > 0$. Then there is an arc $L \subseteq X$ connecting u and v such that $L \cap B(w, r) = \emptyset$, and* $\mathrm{lngth}(L) \leq 8r$.

(b) *Suppose that X is BPD.AC, and is not 1-dimensional. If $A \subseteq X$ is spaced, then $X - A$ is BPD.AC.*

Proof. (a) We may assume that E is 2-dimensional, $w = 0$ and $r = 1$. Let $z \in S(0, 1)$ be such that $\ell := \{u + tz \mid t \in \mathbb{R}\}$ is a supporting line for $B(0, 1)$. Represent v as $v = au + bz$, and choose z in such a way that $b > 0$. Let $L_1 = [u, u + 2z]$, $L_2 = [2z + u, 2z - u]$, $L_3 = [2z - u, -u]$ and $L_0 = L_1 \cup L_2 \cup L_3$. Since ℓ is a supporting line of $B(0, 1)$ it follows that L_1 and L_3 are disjoint from $B(0, 1)$. Suppose that $w \in L_2$. So $w = 2z + tu$, where $|t| \leq 1$. We may assume that $t \geq 0$. Then $\|w\| \geq 2\|z\| - t\|u\| \geq 1$. So $L_2 \cap B(0, 1) = \emptyset$. Recall that $v = au + bz \in S(0, 1)$. From the fact that ℓ supports $B(0, 1)$ it follows that $a \leq 1$. Then $1 = \|v\| \geq b - a \geq b - 1$. So $b \leq 2$. Let $\lambda = \min(1/|a|, 2/b)$ and $L_v = [v, \lambda v]$. Clearly, $L_v \cap B(0, 1) = \emptyset$. Either $\lambda v = u + b_1 z$, where $b_1 \in [0, 2]$, or $\lambda v = -u + b_1 z$, where $b_1 \in [0, 2]$, or $\lambda v = a_1 u + 2z$, where $a_1 \in [-1, 1]$. Hence $\lambda v \in L_1 \cup L_3 \cup L_2 = L_0$. The set $L_0 \cup L_v$ is disjoint from $B(0, 1)$ and contains an arc L connecting u and v. Obviously, for $i = 1, \ldots, 3$, $\mathrm{lngth}(L_i) = 2$ and $\mathrm{lngth}(L_v) = \|\lambda v\| - \|v\| \leq 2\|z\| + \|u\| - 1 = 2$. So $\mathrm{lngth}(L) \leq 8$.

(b) We prove Claim 1 stated below, and leave it to the reader to verify that (b) is implied by Claim 1.

Claim 1. *For every $r, C, D > 0$ there are $r_1, C_1, D_1 > 0$ such that for every normed space E, an open subset $X \subseteq E$ and an r-spaced subset $A \subseteq X$ the following holds. If $x, x^* \in X - A$ are such that $d(\{x, x^*\}, A) \geq r$, and $L \subseteq X$ is an arc connecting x and x^* such that $\delta^X(L) \geq C$ and* $\mathrm{lngth}(L) \leq D$, *then there is an arc $M \subseteq X - A$ connecting x and x^* such that $d(M, A) \geq r_1$, $\delta^X(M) \geq C_1$ and* $\mathrm{lngth}(M) \leq D_1$.

Proof. Let $D_1 = 8D$, $C_1 = C/2$ and $r_1 = \min(r, C)/64$. Let E, X, A, x, x^* and L be as in the claim and $\gamma : [0,1] \to L$ be a parametrization of L which satisfies $\mathrm{lngth}(\gamma\restriction[0,t]) = t \cdot \mathrm{lngth}(L)$ for every $t \in [0,1]$. For every $a \in A$ let $T_a = \{t \in [0,1] \mid \gamma(t) \in B(w, 2r_1)\}$. Clearly, T_a is an open subset of $(0,1)$, and $\mathrm{cl}(T_a) \cap \mathrm{cl}(T_b) = \emptyset$ for any distinct $a, b \in A$. Define $T = \bigcup\{T_a \mid a \in A\}$, and let $\mathcal{I}$ be a set of pairwise disjoint open intervals of $(0,1)$ such that $\bigcup \mathcal{I} = T$. For an open interval I in $(0,1)$ denote by s_I and t_I the left and right endpoints of I, and if $I \in \mathcal{I}$ denote by a_I that member of A such that $I \subseteq T_a$. Clearly, $s_I, t_I \in S(a_I, 2r_1)$. For every $I \in \mathcal{I}$ let $L_I = \gamma([s_I, t_I])$ and M_I be a rectifiable arc connecting a_I and b_I such that $M_I \cap B(a_I, 2r_1) = \emptyset$ and $\mathrm{lngth}(M_I) \le 16r_1$. The existence of M_I is ensured by part (a). Let $\mathcal{I}_0 = \{I \in \mathcal{I} \mid d(L_I, a_I) \le r_1\}$. Let

$$M = L - \bigcup_{I \in \mathcal{I}_0} L_I \cup \bigcup_{I \in \mathcal{I}_0} M_I.$$

Certainly, M is an arc whose endpoints are x and x^*. It is trivial that if $I \in \mathcal{I}_0$, then $\mathrm{lngth}(L_I) \ge 2r_1$, and so for every $I \in \mathcal{I}_0$, $\mathrm{lngth}(M_I)/\mathrm{lngth}(L_I) \le 8$. It follows that M is rectifiable and that $\mathrm{lngth}(M) \le 8 \cdot \mathrm{lngth}(L) \le 8D$.

Let $w \in M$. If $w \in L - \bigcup_{I \in \mathcal{I}_0} L_I$, then $d(w, A) \ge 2r_1$. If there is $I \in \mathcal{I}_0$ such that $w \in M_I$, then $d(w, a_I) \ge 2r_1$ and for every $b \in A - \{a_I\}$,

$$d(w, b) \ge d(b, a_I) - d(w, a_I) \ge r - 8r_1 - 2r_1 \ge 64r_1 - 10r_1 = 54r_1.$$

It follows that $d(M, A) \ge r_1$.

It remains to show that $\delta^X(M) \ge C/2$. Obviously, $\delta^X(L - \bigcup_{I \in \mathcal{I}_0} L_I) \ge \delta^X(L) \ge C$. Let $I \in \mathcal{I}_0$ and be such that $w \in M_I$. Then

$$d(w, E - X) \ge d(a_I, E - X) - d(w, a_I) \ge C - 8r_1 - 2r_1 = C - 10r_1 \ge C - 16r_1 \ge C/2.$$

It follows that $\delta^X(M) \ge C/2$. We have proved Claim 1. ■

We are ready to prove that for open subsets of Banach spaces, if $(\mathrm{BUC}(X))^\tau \subseteq \mathrm{BPD.UC}(Y)$, then $\tau^{-1} \in \mathrm{BPD.UC}(Y, X)$. This is the content of part (a) of the next theorem. The main argument lies though in part (b), and once it is proved, (a) follows easily. So we shall start with the proof of (b).

THEOREM 5.41. *Let E be a Banach space and X be an open subset of E.*

(a) *Suppose that X is BPD.AC and BR.LC.AC, and that $\tau \in H(X, Y)$ is such that $(\mathrm{BUC}(X))^\tau \subseteq \mathrm{BPD.UC}(Y)$. Then $\tau^{-1} \in \mathrm{BPD.UC}(Y, X)$.*

(b) *Suppose that X is BPD.AC, and that $\tau \in H(X, Y)$ is such that $(\mathrm{LIP}_{00}(X))^\tau \subseteq \mathrm{BPD.UC}(Y)$. Assume further that $\tau^{-1} \in \mathrm{BPD.P}(Y, X)$. Then $\tau^{-1} \in \mathrm{BPD.UC}(Y, X)$.*

Proof. (b) We shall see that the proof of (b) can be reduced to an instance of Lemma 5.25. Suppose by contradiction that $\tau^{-1} \notin \mathrm{BPD.UC}(Y, X)$. So there are sequences $\vec{x}', \vec{y}'$ in Y and $e > 0$ such that $\mathrm{Rng}(\vec{x}') \cup \mathrm{Rng}(\vec{y}')$ is a BPD subset of Y, $\lim_{n\to\infty} d(x'_n, y'_n) = 0$, and $d(\tau^{-1}(x'_n), \tau^{-1}(y'_n)) > e$ for every $n \in \mathbb{N}$. We may assume that $\vec{x}'$ is either a Cauchy sequence or $\vec{x}'$ is spaced. However, $\vec{x}'$ cannot be a Cauchy sequence because in that case its limit belongs to Y, and this violates the continuity of τ^{-1}. So we may assume that $\vec{x}'$ is spaced. Set $\vec{x} = \tau^{-1}(\vec{x}')$ and $\vec{y} = \tau^{-1}(\vec{y}')$. From the fact that $\tau^{-1} \in \mathrm{BPD.P}(Y, X)$ it follows that $\mathrm{Rng}(\vec{x})$ is a BPD set. We may assume that $\vec{x}$ is either spaced or is a Cauchy sequence. But if it is a Cauchy sequence then its limit belongs to X, and by the continuity of τ at x, $\vec{x}'$ is a Cauchy sequence, which we have already excluded. So we may assume that $\vec{x}$ is spaced. Let $d > 0$ be such that $\vec{x}$ is d-spaced. Then for every $n \in \mathbb{N}$ there is at most one m such that $\|y_n - x_m\| < d/2$. It follows that there is an infinite set $\eta \subseteq \mathbb{N}$ such that $\|y_n - x_m\| \geq \min(e, d/2)$ for every $m, n \in \eta$. We may thus assume that $d(\mathrm{Rng}(\vec{x}), \mathrm{Rng}(\vec{y})) > 0$.

We denote $\mathrm{Rng}(\vec{x}), \mathrm{Rng}(\vec{y}), \mathrm{Rng}(\vec{x}')$ and $\mathrm{Rng}(\vec{y}')$ by A, B, A' and B' respectively. Let $\widehat{X} = X - A$, $\widehat{Y} = Y - A'$ and $\hat{\tau} = \tau \restriction \widehat{X}$. So $\hat{\tau} \in H(\widehat{X}, \widehat{Y})$. We shall prove that

(i) $\widehat{X}$ is BPD.AC,
(ii) $(\mathrm{LIP}_{00}(\widehat{X}))^{\hat{\tau}} \subseteq \mathrm{BPD.P}(\widehat{Y})$,
(iii) B is a BPD subset of $\widehat{X}$, whereas $\hat{\tau}(B)$ is not a BPD subset of $\widehat{Y}$.

Facts (i)–(iii) contradict Lemma 5.25.

(i) By Proposition 5.40(b), $\widehat{X}$ is BPD.AC.

(ii) Let $h \in \mathrm{LIP}_{00}(\widehat{X})$. Then h is extendible, and $h^{\mathrm{cl}} \restriction \mathrm{bd}(\widehat{X}) = \mathrm{Id}$. So $h^{\mathrm{cl}}(A) = A$. Hence $h^* := h^{\mathrm{cl}} \restriction X \in H(X)$ and clearly, $h^* \in \mathrm{LIP}_{00}(X)$. So $(h^*)^\tau \in \mathrm{BPD.UC}(Y)$. We show that if C is a BPD subset of $\widehat{Y}$, then $h^{\hat{\tau}}(C)$ is a BPD subset of $\widehat{Y}$. Clearly, $h^{\hat{\tau}} = (h^*)^\tau \restriction \widehat{Y}$. Obviously, $C \cup A'$ is a BPD subset of Y, and hence $(h^*)^\tau \restriction (C \cup A')$ is bi-UC. So since $d(C, A') > 0$, $d((h^*)^\tau(C), (h^*)^\tau(A')) > 0$. Since $(h^*)^\tau(A') = A$, it follows that (†) $d((h^*)^\tau(C), A') > 0$. Since $(h^*)^\tau \in \mathrm{BPD.P}(Y)$, and C is a BPD subset of Y, we also have (††) $(h^*)^\tau(C)$ is a BPD subset of Y. From (†) and (††) it follows that $(h^*)^\tau(C)$ is a BPD subset of $\widehat{Y}$. That is, $h^{\hat{\tau}}(C)$ is a BPD subset of $\widehat{Y}$. We have shown that for every $h \in \mathrm{LIP}_{00}(\widehat{X})$, $h^{\hat{\tau}}$ is BPD.P. The same holds for h^{-1}, so $(\mathrm{LIP}_{00}(\widehat{X}))^{\hat{\tau}} \subseteq \mathrm{BPD.P}(\widehat{Y})$.

(iii) Since $\tau^{-1} \in \mathrm{BPD.P}(Y, X)$ and B' is a BPD subset of Y, we see that B is a BPD subset of X. From the fact that $d(A, B) > 0$ we conclude that B is a BPD subset of $X - A = \widehat{X}$. On the other hand, $d(A', B') = d(\mathrm{Rng}(\vec{x}'), \mathrm{Rng}(\vec{y}')) = 0$, so B' is not a BPD subset of $\widehat{Y}$.

Facts (i)–(iii) contradict Lemma 5.25, so $\tau^{-1} \in \mathrm{BPD.UC}(Y, X)$. Part (b) is thus proved.

(a) Let X, Y, τ be as in (a). Then $(\mathrm{BUC}(X))^\tau \subseteq \mathrm{BPD.P}(Y)$. So by Lemma 5.39(b), $\tau^{-1} \in \mathrm{BPD.P}(Y, X)$. We also have $(\mathrm{LIP}_{00}(X))^\tau \subseteq \mathrm{BPD.UC}(Y)$. So by part (b) of this theorem, $\tau^{-1} \in \mathrm{BPD.UC}(Y, X)$. ∎

6. Groups of extendible homeomorphisms and reconstruction of the closure of open sets

6.1. General description. This chapter deals with the homeomorphism groups of closed sets which are the closures of open subsets of a normed space and with groups of extendible homeomorphisms. Under appropriate assumptions on the open sets X and Y we prove that if $\varphi : H(\mathrm{cl}(X)) \cong H(\mathrm{cl}(Y))$, then there is $\tau \in H(\mathrm{cl}(X), \mathrm{cl}(Y))$ which induces φ. Under the same assumptions we also prove that if $\varphi : \mathrm{EXT}(X) \cong \mathrm{EXT}(Y)$, then there is $\tau \in \mathrm{EXT}^{\pm}(X, Y)$ which induces φ. The definitions of $\mathrm{EXT}(X, Y)$ and $\mathrm{EXT}(X)$ appear in 4.6(b) and 5.1(a).

The results about $H(\mathrm{cl}(X))$ appear in Theorems 6.22 and 6.24, and those about $\mathrm{EXT}(X)$ appear in Theorems 6.3, 6.12 and 6.18. These theorems cover open subsets of a normed space whose boundary may be quite complicated. So they go far beyond the class of open sets whose closure is a manifold with boundary. Nevertheless, the statements

$$\text{Every } \varphi : H(\mathrm{cl}(X)) \cong H(\mathrm{cl}(Y)) \text{ is induced by some } \tau \in H(\mathrm{cl}(X), \mathrm{cl}(Y))$$

and

$$\text{Every } \varphi : \mathrm{EXT}(X) \cong \mathrm{EXT}(Y) \text{ is induced by some } \tau \in \mathrm{EXT}^{\pm}(X, Y)$$

are not true for every pair of open subsets of a normed space, not even in the finite-dimensional case. Example 5.8 exhibits two different trivial reasons why the above statements are not true in their full generality.

The proofs of the theorems about $\mathrm{EXT}(X)$ and about $H(\mathrm{cl}(X))$ are essentially identical. Moreover, for finite-dimensional normed spaces the question about the faithfulness of $\{H(\mathrm{cl}(X)) \mid X \text{ is open}\}$ is a special case of the question about the EXT-determinedness of $\{X \mid X \text{ is open}\}$. To see this, notice the following facts.

(1) If U is a regular open subset of $\mathbb{R}^n$, then $\mathrm{EXT}(U) = H(\mathrm{cl}(U))$.

(2) If $X \subseteq \mathbb{R}^n$ is open and $\widehat{X} = \mathrm{int}(\mathrm{cl}(X))$, then $\widehat{X}$ is regular open and $\mathrm{cl}(X) = \mathrm{cl}(\widehat{X})$.

Suppose now that $\varphi : H(\mathrm{cl}(X)) \cong H(\mathrm{cl}(Y))$. By (2), $\varphi : H(\mathrm{cl}(\widehat{X})) \cong H(\mathrm{cl}(\widehat{Y}))$, and by (1), $\varphi : \mathrm{EXT}(\widehat{X}) \cong \mathrm{EXT}(\widehat{Y})$. So if it can be proved that there is $\tau \in \mathrm{EXT}^{\pm}(\widehat{X}, \widehat{Y})$ which induces φ, then this τ indeed belongs to $H(\mathrm{cl}(X), \mathrm{cl}(Y))$.

Theorems 6.3 and 6.18 prove the EXT-determinedness of certain classes. In 6.3 it is assumed that the members of the EXT-determined class are BR.LC.AC (see 5.37). This property is a weakening of uniform-in-diameter arcwise connectedness. It may happen though that every point in the boundary of such a set is fixed under $\mathrm{EXT}(X)$. In 6.18, on the other hand, the EXT-determinedness is derived from the property that the $\mathrm{EXT}(X)$-orbit of every member of $\mathrm{bd}(X)$ contains an arc, but X need not be BR.LC.AC.

In Corollary 6.6(a) we prove that if X and Y satisfy certain weak assumptions on arcwise connectedness, and $(\mathrm{EXT}(X))^{\tau} = \mathrm{EXT}(Y)$, then $\tau \in \mathrm{EXT}(X, Y)$. A statement of the form: "$(\mathrm{EXT}(X))^{\tau} \subseteq \mathrm{EXT}(X) \Rightarrow \tau \in \mathrm{EXT}(X, Y)$" is also proved, but only under rather restrictive assumptions on X and Y. See Corollary 6.6(b).

Suppose that X is an open subset of $\mathbb{R}^n$. Then $\mathrm{EXT}(X) = \mathrm{BUC}(X)$. If in addition, X is bounded, then $\mathrm{EXT}(X) = \mathrm{UC}(X)$. So for finite-dimensional bounded X's Corollary 5.6 which deals with $\mathrm{BUC}(X)$ is indeed about $\mathrm{EXT}(X)$. However, Theorems 6.12 and 6.18 are stronger than 5.6 even for finite-dimensional bounded X's.

Groups of completely locally uniformly continuous homeomorphisms are dealt with in Theorem 6.20. (See Definition 5.3(f).) The Γ-continuous version of these groups is the subject of Chapters 8–12.

At the end of this chapter in items 6.25–6.30, we discuss two generalizations of these results. The first generalization deals with subsets Z of a normed space such that $Z \subseteq \mathrm{cl}(\mathrm{int}(Z))$. The second generalization deals with sets which are the closures of open subsets in a normed manifold.

Recall that unless otherwise stated, X and Y denote respectively open subsets of the normed spaces E and F.

6.2. Groups of extendible homeomorphisms. The following definition contains some notions related to arcwise connectedness. These notions are used in the statement of Theorem 6.3 which deals with EXT-determinedness. In the next definition only, E denotes a general metric space.

DEFINITION 6.1. Let E be a metric space and $X \subseteq E$.

(a) A set of pairwise disjoint sets is called a *pairwise disjoint family*. Let $\mathcal{A}$ be a pairwise disjoint family of subsets of X. $\mathcal{A}$ is *completely discrete* with respect to E if for every $x \in E$ there is $U \in \mathrm{Nbr}(x)$ such that $\{A \in \mathcal{A} \mid A \cap U \neq \emptyset\}$ is finite. A set $A \subseteq X$ is *completely discrete* with respect to E if A does not have accumulation points in E. The mention of E in the above definition is often omitted, since E is usually understood from the context. A sequence $\vec{x} \subseteq X$ is a *completely discrete sequence* if it is 1-1, and its range is completely discrete.

(b) X is said to be *boundedly arcwise connected* (*BD.AC*) if for every bounded $A \subseteq X$ there is $d > 0$ such that for every $x, y \in A$ there is a rectifiable arc $L \subseteq X$ connecting x and y such that $\mathrm{lngth}(L) \leq d$.

(c) X is said to be a *wide set* if for every infinite completely discrete set $A \subseteq X$ there is an infinite $B \subseteq A$, a set $\{y_b \mid b \in B\}$ and a set of arcs $\{L_b \mid b \in B\}$ such that: $\{y_b \mid b \in B\}$ is bounded; for every $b \in B$, $y_b, b \in L_b \subseteq X$; and $\{L_b \mid b \in B\}$ is completely discrete.

(d) Let $\vec{x} \subseteq X$ be a completely discrete sequence. Let $x^* \in \mathrm{cl}(X)$, $\{L_n \mid n \in \mathbb{N}\}$ be a sequence of arcs and $\vec{y} \subseteq X$. Assume that

(1) $L_n \subseteq X$ for every $n \in \mathbb{N}$,
(2) L_n connects x_n with y_n for every $n \in \mathbb{N}$,
(3) $\lim \vec{y} = x^*$,

(4) $L_m \cap L_n = \emptyset$ for any distinct $m, n \in \mathbb{N}$,

(5) for every $r > 0$, $\{L_n - B^E(x^*, r) \mid n \in \mathbb{N}\}$ is completely discrete.

Then $\langle \vec{x}, x^*, \{L_n \mid n \in \mathbb{N}\}, \vec{y} \rangle$ is called a *joining system for $\vec{x}$ with respect to E.*

(e) X is *jointly arcwise connected* (*JN.AC*) *with respect to E* if for every completely discrete sequence $\vec{x} \subseteq X$ there is a subsequence $\vec{x}'$ of $\vec{x}$ such that $\vec{x}'$ has a joining system. □

In (a)–(d) of the next proposition we infer joint arcwise connectedness from various simpler properties of X. Part (e) is a trivial observation, so we do not prove it.

Proposition 6.2. (a) *Suppose that $\vec{x} \subseteq X$ is a Cauchy sequence and $\lim^{\bar{E}} \vec{x} \in \overline{\mathrm{int}}(X) - X$. Then $\vec{x}$ has a subsequence $\vec{x}'$ such that $\vec{x}'$ has a joining system.*

(b) *Suppose that X is an open subset of a finite-dimensional normed space. Then X is JN.AC iff X is bounded.*

(c) *Suppose that X is an open subset of a Banach space and X is BD.AC. Then every bounded completely discrete sequence $\vec{x} \subseteq X$ has a subsequence $\vec{x}'$ such that $\vec{x}'$ has a joining system. In particular, if in addition X is bounded, then X is JN.AC.*

(d) *If X is an open subset of a Banach space, X is wide and X is BD.AC, then X is JN.AC.*

(e) *Let X be a bounded subset of a finite-dimensional normed space. Then X is BR.LC.AC iff X is UD.AC.*

Proof. (a) Let $\bar{x} = \lim^{\bar{E}} \vec{x}$. Let $u \in E$ and $r > 0$ be such that $B(u, r) \subseteq E$ and $\bar{x} \in B^{\bar{E}}(u, r)$. Let $v \in B(u, r)$. There is a subsequence $\vec{y}$ of $\vec{x}$ such that $\vec{y} \subseteq B(u, r)$ and $\{[y_n, v) \mid n \in \mathbb{N}\}$ is a pairwise disjoint family. Let $v_n \in [y_n, v)$ be such that $\lim \vec{v} = v$. Then $\langle \vec{y}, v, \{[y_n, v_n] \mid n \in \mathbb{N}\}, \vec{v} \rangle$ is a joining system for $\vec{y}$.

(b) If X is a bounded open subset of a finite-dimensional space, then X does not contain an infinite completely discrete set. So X is JN.AC.

Suppose that X is an unbounded open subset of a finite-dimensional space, Let $\vec{x} \subseteq X$ be a 1-1 sequence such that $\lim_{n\to\infty} \|x_n\| = \infty$. Then $\vec{x}$ is completely discrete, and it is trivial that $\vec{x}$ has no joining system.

(c) Let X be as in (c). Let $\vec{x} \subseteq X$ be completely discrete. Since X is an open subset of a Banach space, we may assume that $\vec{x}$ is spaced. Let $u \in X$. For every $n \in \mathbb{N}$ let $L_n \subseteq X$ be a rectifiable arc connecting x_n with u such that $\mathrm{lngth}(L_n) \le d$. Let $\gamma_n(t)$ be the parametrization of L_n satisfying $\gamma_n(0) = u$, $\gamma_n(1) = x_n$ and $\mathrm{lngth}(\gamma_n([0, t])) = t \cdot \mathrm{lngth}(L_n)$.

For every $\sigma \subseteq \mathbb{N}$ and $t \in [0, 1]$ set $A[\sigma, t] = \{\gamma_n(t) \mid n \in \sigma\}$, and if σ is infinite define $t_\sigma = \inf(\{t \mid A[\sigma, t] \text{ is spaced}\})$. There is an infinite σ such that for every infinite $\eta \subseteq \sigma$, $t_\eta = t_\sigma$. It is easy to see that there is no infinite $\eta \subseteq \sigma$ such that $A[\eta, t_\sigma]$ is spaced. So there is $\eta \subseteq \sigma$ such that $A[\eta, t_\sigma]$ is a Cauchy sequence. Then $A[\eta, 1]$ is a subsequence of $\vec{x}$ and $\langle A[\eta, 1], \lim A[\eta, t_\eta], \{\gamma_n([t_\eta, 1]) \mid n \in \eta\}, A[\eta, t_\eta] \rangle$ is a joining system for $A[\eta, 1]$.

(d) This part follows easily from (c). ∎

In the next theorem, (a) is a special case of (b). It seems worthwhile to state (a) separately, because the class considered there is more understandable than the class dealt with in (b).

THEOREM 6.3. (a) *Let* $K^{\mathcal{O}}_{\mathrm{BCX}}$ *denote the class of all* $X \in K^{\mathcal{O}}_{\mathrm{BNC}}$ *such that* X *is wide, BR.LC.AC and BD.AC. Suppose that* $X, Y \in K^{\mathcal{O}}_{\mathrm{BCX}}$ *and* $\varphi : \mathrm{EXT}(X) \cong \mathrm{EXT}(Y)$. *Then there is* $\tau \in \mathrm{EXT}^{\pm}(X, Y)$ *which induces* φ. *Note that* $K^{\mathcal{O}}_{\mathrm{BCX}}$ *contains the class of all bounded members of* $K^{\mathcal{O}}_{\mathrm{BNC}}$ *which are BR.LC.AC and BD.AC.*

(b) *Let* $K^{\mathcal{O}}_{\mathrm{NMX}}$ *denote the class of all* $X \in K^{\mathcal{O}}_{\mathrm{NRM}}$ *such that* X *is BR.LC.AC and JN.AC. Let* $X, Y \in K^{\mathcal{O}}_{\mathrm{NMX}}$. *Suppose that* $\varphi : \mathrm{EXT}(X) \cong \mathrm{EXT}(Y)$. *Then there is* $\tau \in \mathrm{EXT}^{\pm}(X, Y)$ *which induces* φ.

The proof of Theorem 6.3 appears after Corollary 6.6.

REMARK. (a) By Proposition 6.2(c), $K^{\mathcal{O}}_{\mathrm{BCX}} \subseteq K^{\mathcal{O}}_{\mathrm{NMX}}$. So 6.3(b) is a special case of 6.3(a).

(b) Note that all members of $K^{\mathcal{O}}_{\mathrm{BCX}}$ which are subsets of a finite-dimensional normed space are bounded. This is so, since for finite-dimensional spaces, wideness implies boundedness. Yet $K^{\mathcal{O}}_{\mathrm{BCX}}$ contains unbounded subsets of infinite-dimensional Banach spaces.

(c) There is a regular open subset $X \subseteq \mathbb{R}^3$ such that $X \in K^{\mathcal{O}}_{\mathrm{BCX}}$ and $g^{\mathrm{cl}} \restriction \mathrm{bd}(X) = \mathrm{Id}$ for every $g \in \mathrm{EXT}(X)$. This is maybe somewhat unexpected, since it means that $\mathrm{bd}(X)$ is recoverable from $\mathrm{EXT}(X)$ even though every member of $\mathrm{EXT}(X)$ is the identity on $\mathrm{bd}(X)$. See Example 6.7(d). $\square$

Recall that $\mathrm{UC}_0(X) = \{f \in \mathrm{UC}(X) \mid \mathrm{Dom}(f^{\mathrm{cl}}) = \mathrm{cl}(X) \text{ and } f^{\mathrm{cl}} \restriction \mathrm{bd}(X) = \mathrm{Id}\}$.

PROPOSITION 6.4. *Suppose that* X *is BR.LC.AC, and let* $\tau \in H(X, Y)$ *be such that* $(\mathrm{UC}_0(X))^{\tau} \subseteq \mathrm{EXT}(Y)$. *Let* $x \in \mathrm{bd}(X)$, $y \in \mathrm{bd}(Y)$ *and* $\vec{x} \subseteq X$ *be such that* $\lim \vec{x} = x$ *and* $\lim \tau(\vec{x}) = y$. *Then* $\tau \cup \{\langle x, y\rangle\}$ *is continuous.*

Proof. Let $\vec{u} \subseteq X$ be such that $\lim \vec{u} = x$. Suppose by contradiction that $\tau(\vec{u})$ does not converge to y. We may assume that y is not a limit point of $\tau(\vec{u})$.

We now repeat the construction appearing in the proof of Case 1 in Theorem 5.5. Using the fact that X is BR.LC.AC, by induction on $i \in \mathbb{N}$ we construct $n_i \in \mathbb{N}$ and $L_i \subseteq X$ such that: (i) L_i is an arc connecting x_{n_i} and u_{n_i}; (ii) $\lim_{i\to\infty} \mathrm{diam}(L_i) = 0$; and (iii) for every $i \in \mathbb{N}$, $d(L_i, \bigcup_{j\neq i} L_j) > 0$. For every $i \in \mathbb{N}$ let $U_i \subseteq X$ be an open set such that $L_i \subseteq U_i$, $\lim_{i\to\infty} \mathrm{diam}(U_i) = 0$, and for every $i \neq j$, $d(U_i, U_j) > 0$.

Let $h_i \in \mathrm{UC}(X)$ be such that $\mathrm{supp}(h_i) \subseteq U_{2i}$ and $h_i(x_{n_{2i}}) = u_{n_{2i}}$. By Proposition 4.5, $h := \bigcirc_{i\in\mathbb{N}} h_i \in \mathrm{UC}(X)$. It is also obvious that $h \in \mathrm{UC}_0(X)$. However, h^{τ} is not exendible, since $\tau(\vec{x})$ is convergent, whereas $h^{\tau}(\tau(\vec{x}))$ is not convergent. A contradiction. ∎

Our next goal is to show that if $(\mathrm{EXT}(X))^{\tau} \subseteq \mathrm{EXT}(Y)$, then for every $y \in \mathrm{bd}(Y)$ there is a sequence $\vec{y}$ converging to y such that $\tau^{-1}(\vec{y})$ is a convergent sequence. This holds automatically when X is bounded and finite-dimensional, but in that case extendibility is equivalent to uniform continuity, and so Theorem 5.2 already answers our question. In the general case we have to make an additional arcwise connectedness assumption on X.

For a metric space E and $X \subseteq E$ define

$$\mathrm{LUC}_{01}(X) = \{h \in \mathrm{LUC}(X) \mid \text{ there is an } E\text{-open set } U \supseteq \mathrm{bd}(X) \text{ such that } h\restriction(U \cap X) = \mathrm{Id}\}.$$

LEMMA 6.5. *Assume that* X *is JN.AC,* $\tau \in H(X, Y)$ *and* $(\mathrm{LUC}_{01}(X))^{\tau} \subseteq \mathrm{EXT}(Y)$, *and let* $y \in \mathrm{bd}(Y)$.

(a) *Suppose that* $\vec{x} \subseteq X$ *is completely discrete,* $\langle \vec{x}, x^*, \{L_n \mid n \in \mathbb{N}\}, \vec{x}' \rangle$ *is a joining system for* $\vec{x}$ *and* $\lim \tau(\vec{x}) = y$. *Then there is a sequence* $\vec{u} \subseteq X$ *such that* $\lim \vec{u} = x^*$ *and* $\lim \tau(\vec{u}) = y$.

(b) *There is a sequence* $\vec{u} \subseteq X$ *such that* $\vec{u}$ *converges to a member of* $\mathrm{bd}(X)$ *and* $\lim \tau(\vec{u}) = y$.

Proof. (a) Suppose that $\vec{x}$ is completely discrete, $\langle \vec{x}, x^*, \{L_n \mid n \in \mathbb{N}\}, \vec{x}' \rangle$ is a joining system for $\vec{x}$, and $\tau(\vec{x})$ converges to y. We may assume that $x^* \notin \{x_n \mid n \in \mathbb{N}\}$. Hence since $\vec{x}$ is completely discrete, $d := d(\vec{x}, x^*) > 0$. Also assume that $L_n(0) = x_n$ and $L_n(1) = x'_n$.

CLAIM 1. *For every* $r > 0$ *there is a sequence* $\vec{u}^r \subseteq B(x^*, r) \cap X$ *such that* $\tau(\vec{u}^r)$ *converges to* y.

Proof. Let $r \in (0, d)$. For every $n \in \mathbb{N}$ we define v_n. If n is even and $d(x'_n, x^*) \le r/2$, let $t_n = \min\{t \in [0,1] \mid d(L_n(t), x^*) = r/2\}$ and $v_n = L_n(t_n)$. Otherwise, let $v_n = x_n$. Let $\vec{v} = \{v_n \mid n \in \mathbb{N}\}$. Let L'_n be the subarc of L_n connecting x_n with v_n. Clearly $L'_n \cap B(x^*, r/2) = \emptyset$, and hence by Definition 6.1(d)(5), $\{L'_n \mid n \in \mathbb{N}\}$ is completely discrete. It is easy to see that there is a completely discrete family of open sets $\{U_n \mid n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}$, $L'_n \subseteq U_n \subseteq \mathrm{cl}(U_n) \subseteq X$. Let $h_n \in \mathrm{UC}(X)$ be such that $\mathrm{supp}(h_n) \subseteq U_n$ and $h_n(x_n) = v_n$. It is easy to see that $h := \circ\{h_n \mid n \in \mathbb{N}\} \in \mathrm{LUC}_{01}(X)$. Hence $h^\tau \in \mathrm{EXT}(Y)$.

The facts that $\tau(\vec{x})$ is convergent in $\mathrm{cl}(Y)$ and that $h^\tau \in \mathrm{EXT}(Y)$ imply that $h^\tau(\tau(\vec{x}))$ is also convergent in $\mathrm{cl}(Y)$. Note that $h^\tau(\tau(\vec{x})) = \tau(\vec{v})$. So $\tau(\vec{v})$ is convergent in $\mathrm{cl}(Y)$. Recall that for every $n \in \mathbb{N}$, $v_{2n+1} = x_{2n+1}$. So $\lim \tau(\vec{v}) = \lim \tau(\vec{x}) = y$. Let $N_r \in \mathbb{N}$ be such that for every $n > N_r$, $d(x'_n, x^*) \le r/2$ and define $\vec{u}^r = \{v_{2n} \mid 2n > N_r\}$. Then $\vec{u}^r \subseteq B(x^*, r) \cap X$ and hence $\vec{u}^r$ is as required in Claim 1.

Let $r_n = 1/n$. For every $n \in \mathbb{N}$ let k_n be such that $d(y, \tau(u^{r_n}_{k_n})) < 1/n$. Then $\vec{u} := \{u^{r_n}_{k_n} \mid n \in \mathbb{N}\}$ converges to x^* and $\lim \tau(\vec{u}) = y$.

(b) Suppose by contradiction that y is a counter-example to the claim of (b). Let $\vec{y} \subseteq Y$ be a 1-1 sequence converging to y and $\vec{z} = \tau^{-1}(\vec{y})$. If $\vec{z}$ has a convergent subsequence, then this subsequence converges to a member of $\mathrm{bd}(X)$, so y is not a counter-example. Hence $\vec{z}$ is completely discrete.

Since X is JN.AC, there is a subsequence $\vec{x}$ of $\vec{z}$ such that $\vec{x}$ has a joining system $\langle \vec{x}, x^*, \{L_n \mid n \in \mathbb{N}\}, \vec{x}' \rangle$. By (a) there is a sequence $\vec{u} \subseteq X$ such that $\lim \vec{u} = x^*$ and $\lim \tau(\vec{u}) = y$. If $x^* \in X$, then $y = \lim \tau(\vec{u}) = \tau(x^*) \in Y$, a contradiction. So $x^* \in \mathrm{bd}(X)$. This means that y is not a counter-example to (b). A contradiction, so (b) is proved. ■

The fact $(\mathrm{EXT}(X))^\tau \subseteq \mathrm{EXT}(X)$ does not imply that $\tau \in \mathrm{EXT}(X, Y)$. To deduce that $\tau \in \mathrm{EXT}(X, Y)$, we need to assume that $(\mathrm{EXT}(X))^\tau = \mathrm{EXT}(X)$. This is shown in part (a) of the next corollary. In (b) we show that if $\mathrm{EXT}(X)$ acts transitively on $\mathrm{bd}(X)$, then the assumption $(\mathrm{EXT}(X))^\tau \subseteq \mathrm{EXT}(X)$ does suffice.

COROLLARY 6.6. (a) *Suppose that* X *is BR.LC.AC, and* Y *is JN.AC. Let* $\tau \in H(X, Y)$ *be such that* (†) $(\mathrm{UC}_0(X))^\tau \subseteq \mathrm{EXT}(Y)$ *and* (††) $(\mathrm{LUC}_{01}(Y))^{\tau^{-1}} \subseteq \mathrm{EXT}(X)$. *Then* $\tau \in \mathrm{EXT}(X, Y)$.

(b) *Suppose that X is BR.LC.AC, X is JN.AC, and that the boundary of X has the following transitivity property*: $(*)$ *for every* $x, y \in \mathrm{bd}(X)$ *there is* $h \in \mathrm{EXT}(X)$ *such that* $h^{\mathrm{cl}}(x) = y$. *Let* $\tau \in H(X,Y)$ *be such that* $(\mathrm{EXT}(X))^\tau \subseteq \mathrm{EXT}(Y)$. *Then* $\tau \in \mathrm{EXT}(X,Y)$.

Proof. The two parts of the corollary will be proved by combining Lemma 6.5(b) and Propositions 6.4 and 4.7(a).

(a) Let $x \in \mathrm{bd}(X)$. By Lemma 6.5(b) applied to τ^{-1}, there is $\vec{x} \subseteq X$ converging to x such that $\tau(\vec{x})$ converges to a point in $\mathrm{bd}(Y)$. Let $y = \lim \tau(\vec{x})$. By Proposition 6.4, $\tau \cup \{\langle x, y\rangle\}$ is continuous. So by Proposition 4.7(a), τ is extendible.

(b) By Lemma 6.5(b) applied to τ, there are $x_0 \in \mathrm{bd}(X)$ and $\vec{x} \subseteq X$ converging to x_0 such that $\tau(\vec{x})$ converges to a member of $\mathrm{bd}(Y)$. Let $x \in \mathrm{bd}(X)$. There is $h \in \mathrm{EXT}(X)$ such that $h(x_0) = x$. Since $h^\tau \in \mathrm{EXT}(Y)$, $h^\tau(\tau(\vec{x}))$ converges to a member of $\mathrm{bd}(Y)$. But $\tau(h(\vec{x})) = h^\tau(\tau(\vec{x}))$. It follows that for every $x \in \mathrm{bd}(X)$ there is a sequence $\vec{u}$ converging to x such that $\tau(\vec{u})$ is convergent. By Propositions 6.4 and 4.7(a), $\tau \in \mathrm{EXT}(X,Y)$. ∎

Proof of Theorem 6.3. (a) This is a special case of (b), because by Proposition 6.2(d), a BD.AC wide open subset of a Banach space is JN.AC.

(b) $\mathrm{LIP}_{00}(X) \subseteq \mathrm{EXT}(X)$ and $\mathrm{LIP}_{00}(X) = \mathrm{LIP}(X, \mathcal{S})$, where $\mathcal{S}$ is the set of all open BPD subsets of X. The same holds for Y. So by Theorem 2.8(b), there is $\tau \in H(X,Y)$ such that τ induces φ. From the fact that $\mathrm{UC}_0(X) \subseteq \mathrm{EXT}(X)$ we conclude that $(\mathrm{UC}_0(X))^\tau \subseteq \mathrm{EXT}(Y)$. So 6.6(a) can be applied to τ and τ^{-1}. We conclude that $\tau \in \mathrm{EXT}^{\pm}(X,Y)$. This proves (b). ∎

Part (a) of the next example is designed to show that the condition (†) of 6.6(a) is needed. Indeed, for Y, X and τ^{-1} of (a), (††) holds but the conclusion of 6.6(a) does not. Part (b) shows that assumption (††) in Corollary 6.6(a) cannot be omitted. The example is infinite-dimensional. Indeed, for finite-dimensional normed spaces (†) *does* suffice. This follows from Theorem 5.5 and Proposition 6.2(e). Part (c) shows that the transitivity assumption $(*)$ in Corollary 6.6(b) is indeed needed. Part (d) shows that there is $X \in K^{\mathcal{O}}_{\mathrm{BCX}}$ such that $\mathrm{EXT}(X)$ fixes $\mathrm{bd}(X)$ pointwise. The set X is a regular open subset of $\mathbb{R}^3$, therefore $\mathrm{EXT}(X) = H(\mathrm{cl}(X))$.

Let $\mathrm{Cmp}(X)$ denote the set of connected components of a topological space X.

Example 6.7. (a) *There are bounded regular open connected sets X and Y in $\mathbb{R}^2$ and $\tau \in H(X,Y)$ such that X and Y are BR.LC.AC*, $(\mathrm{EXT}(X))^\tau \subseteq \mathrm{EXT}(Y)$, *but* $\tau^{-1} \notin \mathrm{EXT}(Y,X)$. *Note that by Proposition* 6.2(b), *X and Y are JN.AC.*

(b) *There are regular open bounded domains X and Y in an infinite-dimensional Banach space and $\tau \in H(X,Y)$ such that X and Y are BR.LC.AC and JN.AC*, $(\mathrm{EXT}(X))^\tau \subseteq \mathrm{EXT}(Y)$, *but* $\tau \notin \mathrm{EXT}(X,Y)$.

(c) *There are bounded domains X and Y in an infinite-dimensional Banach space and $\tau \in H(X,Y)$ such that X and Y are BR.LC.AC and JN.AC*, $\mathrm{bd}(X)$ *has two connected components*, $\mathrm{bd}(Y)$ *is connected*, $\mathrm{EXT}(X)$ *and* $\mathrm{EXT}(Y)$ *act very transitively on* $\mathrm{bd}(X)$ *and* $\mathrm{bd}(Y)$ *respectively*, $(\mathrm{EXT}(X))^\tau \subseteq \mathrm{EXT}(Y)$, *but* $\tau \notin \mathrm{EXT}(X,Y)$.

(d) *There is* $X \in K^{\mathcal{O}}_{\mathrm{BCX}}$ *such that X is a regular open bounded subset of* $\mathbb{R}^3$, *and* $g^{\mathrm{cl}} \restriction \mathrm{bd}(X) = \mathrm{Id}$ *for every* $g \in \mathrm{EXT}(X)$.

Proof. (a) Let $X' \subseteq \mathbb{R}^2$ be the open square whose vertices are $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$, and $Y' \subseteq \mathbb{R}^2$ be the open triangle whose vertices are $(0,0)$, $(0,1)$ and $(1,1)$. Let $\tau' \in H(X',Y')$ be defined by $\tau'((x,y)) = (xy,y)$. Let $A = [(0,0),(1,0)]$.

Clearly, $\tau' \in \mathrm{EXT}(X',Y')$, $(\tau')^{\mathrm{cl}}\restriction(\mathrm{cl}(X) - A) \in H(\mathrm{cl}(X') - A,\ \mathrm{cl}(Y') - \{(0,0)\})$ and $(\tau')^{\mathrm{cl}}(A) = \{(0,0)\}$. Also, if $g \in \mathrm{EXT}(X',X')$ and $g^{\mathrm{cl}}(A) = A$, then $g^{\tau'} \in \mathrm{EXT}(Y')$.

For $n > 1$ and $1 \le k < n$ let $x_{n,k} = (k/2^n, 1/2^n)$, $B_{n,k} = \mathrm{cl}(B(x_{n,k}, 1/8^n))$ and $\mathcal{B} = \{B_{n,k} \mid n > 1,\ 1 \le k < n\}$. Note that $\mathcal{B}$ is a pairwise disjoint family of closed balls contained in X' and $\mathrm{cl}(\bigcup\mathcal{B}) - \bigcup\mathcal{B} = A$. Let $X = X' - \bigcup\mathcal{B}$, $Y = \tau'(X)$ and $\tau = \tau'\restriction X$. Clearly, for every $g \in \mathrm{EXT}(X)$, $g^{\mathrm{cl}}(A) = A$. It follows that X, Y and τ are as required. Note also that for every $x,y \in A - \{(0,0),(1,0)\}$ there is $g \in \mathrm{EXT}(X)$ such that $g(x) = y$.

(b) Let E be the Hilbert space ℓ_2, Y' be the open cylinder defined by

$$Y' = \Big\{(x_0,x_1,\ldots)\,\Big|\, |x_0| < 3 \text{ and } \sum_{i=1}^{\infty} x_i^2 < 9\Big\}$$

and $X' = Y' - \bar{B}^E(0,1)$. Let $\tau_1 : X' \cong Y' - \{0\}$ be such that $\tau_1\restriction(Y' - B^E(0,2)) = \mathrm{Id}$. Let $\tau_2 : Y' - \{0\} \cong Y'$ be such that $\tau_2\restriction(Y' - B^E(0,2)) = \mathrm{Id}$ and $\tau' = \tau_2 \circ \tau_1$. The existence of τ_2 follows from the facts that a point in $\mathbb{R}^{\mathbb{N}}$ is a strongly negligible set, and that $\ell_2 \cong \mathbb{R}^{\mathbb{N}}$. See [BP, Chapter IV, Definition 5.1 and Chapter V, Proposition 2.2(c)] and Theorem 6.4.

Note that τ' cannot be continued to a continuous function defined on $S(0,1)$. Hence $\tau' \notin \mathrm{EXT}(X',Y')$. It is trivial that $\mathrm{bd}(Y')$ is homeomorphic to a sphere, and that $\mathrm{bd}(X')$ has two components: $\mathrm{bd}(Y')$ and $S(0,1)$. It can be easily checked that for every $h \in \mathrm{EXT}(X')$: if $h^{\mathrm{cl}}(S(0,1)) = S(0,1)$, then $h^{\tau'} \in \mathrm{EXT}(Y')$. However, there is $h \in \mathrm{EXT}(X')$ such that $h^{\mathrm{cl}}(S(0,1)) = \mathrm{bd}(Y')$. This implies that $(\mathrm{EXT}(X'))^{\tau} \not\subseteq \mathrm{EXT}(Y')$, contrary to what is required in this example.

For a pairwise disjoint family $\mathcal{C}$ of subsets a topological space Z define

$$\mathrm{acc}^Z(\mathcal{C}) = \{z \in Z \mid \text{for every } U \in \mathrm{Nbr}^Z(z),\ \{C \in \mathcal{C} \mid U \cap C \neq \emptyset\} \text{ is infinite}\}.$$

To define X we construct a pairwise disjoint family $\mathcal{F}$ of closed sets such that (i) $\bigcup\mathcal{F} \subseteq Y' - \bar{B}(0,2)$ and (ii) $\mathrm{acc}(\mathcal{F}) \subseteq \mathrm{bd}(Y') \cup \bigcup\mathcal{F}$. We then define X, Y and τ to be respectively $X' - \bigcup\mathcal{F}$, $\tau'(X)$ and $\tau'\restriction X$. It follows from (ii) that X is open, and the construction of $\mathcal{F}$ will ensure that $S(0,1)$ is the unique connected component of $\mathrm{bd}(X)$ which is clopen in $\mathrm{bd}(X)$ and which is also strongly connected (a notion to be defined later). It will thus follow that for every $h \in \mathrm{EXT}(X)$, $h^{\mathrm{cl}}(S(0,1)) = S(0,1)$, and this in turn implies that $(\mathrm{EXT}(X))^{\tau} \subseteq \mathrm{EXT}(Y)$.

Let $\{e_i \mid i \in \mathbb{N}\}$ be the standard basis of ℓ_2, denote by T the set of finite sequences of natural numbers, let $f : T \to \mathbb{N} - \{0\}$ be a 1-1 function, and for $\eta \in T$ define $d_\eta = e_{f(\eta)}$. Let Λ denote the empty sequence and $T^* = T - \{\Lambda\}$. The relation "ν is a proper initial segment of η" is denoted by $\eta \precdot \nu$. Suppose that $\eta = \nu\hat{\ }\langle i\rangle$, $\zeta = \nu\hat{\ }\langle j\rangle$ and $i \neq j$. In that case we write $\nu = \mathrm{pred}(\eta)$, $\eta \in \mathrm{Suc}(\nu)$ and $\zeta \in \mathrm{Brthr}(\eta)$.

Let $<^T$ be the relation on T defined by $\nu <^T \eta$ if either $\eta \precdot \nu$ or there is $n \in \mathrm{Dom}(\nu) \cap \mathrm{Dom}(\eta)$ such that $\nu\restriction\mathbb{N}^{<n} = \eta\restriction\mathbb{N}^{<n}$ and $\nu(n) < \eta(n)$. It is easy to check that $<^T$ is a dense linear ordering with maximum Λ and with no minimum. Denote by T_n the set of all $\eta \in T$ such that $\mathrm{Dom}(\eta) = \mathbb{N}^{<m}$ for some $m \le n$. Then T_n is well-ordered by $<^T$.

We define a line segment L_η for every $\eta \in T^*$. If $\eta = \nu\hat{\ }\langle m\rangle$, then L_η has the form $[d_\nu + a_\eta \cdot e_0, d_\eta + a_\eta \cdot e_0]$, where $2 < a_\eta < 3$. So for L_η to be defined we need to define a_η. We define a_η by induction. Let $\{\eta_n \mid n \in \mathbb{N}\}$ be a 1-1 enumeration of T such that for every $n \in \mathbb{N}$ and $\nu \lessdot \eta_n$ there is $m < n$ such that $\eta_m = \nu$. Define $S_n = \{\eta_m\hat{\ }\langle i\rangle \mid m < n \text{ and } i \in \mathbb{N}\}$. We define by induction on n the set $\{a_\nu \mid \nu \in S_n\}$. So at stage n we need to define the set $\{a_{\eta_n\hat{\ }\langle i\rangle} \mid i \in \mathbb{N}\}$. Since $\{\nu \mid \nu \lessdot \eta_n\} \subseteq \{\eta_m \mid m < n\}$ for every n, it follows that $\eta_0 = \Lambda$. Let $\{a_{\langle i\rangle}\}_{i\in\mathbb{N}}$ be a strictly increasing sequence converging to 3 such that $a_{\langle 0\rangle} = 5/2$. So

$$L_{\langle i\rangle} = [d_\Lambda + a_{\langle i\rangle} \cdot e_0, d_{\langle i\rangle} + a_{\langle i\rangle} \cdot e_0].$$

Let $n > 0$ and suppose that a_ν has been defined for every $\nu \in S_n$. Let $\bar{0} = \langle 0, \dots\rangle$ denote the infinite sequence of 0's. It is convenient to define $a_{\bar{0}} = 2$. We assume by induction that

(1) $2 < a_\nu < 3$ for every $\nu \in S_n$,
(2) $\{a_{\eta_m\hat{\ }\langle i\rangle} \mid i \in \mathbb{N}\}$ is a strictly increasing sequence converging to a_{η_m} for every $0 < m < n$,
(3) if $\nu, \varrho \in S_n$ and $\nu <^T \varrho$, then $a_\nu < a_\varrho$.

Note that for $n = 1$ the induction hypotheses hold. Clearly, $S_n \subseteq T_{n+1}$, so $\{a_\nu \mid \nu \in S_n\}$ is well-ordered. Obviously, $\eta_n \in S_n$. If $\eta_n = \langle 0, \dots, 0\rangle$, then $\eta_n = \min(S_n)$. In this case set $\varrho_n = \bar{0}$. Otherwise, write η_n as $\nu\hat{\ }\langle k\rangle\hat{\ }\langle 0, \dots, 0\rangle$, where $k > 0$, and the sequence of 0's at the end of η_n may be the empty sequence. Define $\varrho_n = \nu\hat{\ }\langle k-1\rangle$. It is easy to check that in this case ϱ_n is the predecessor of η_n in S_n. Choose $\{a_{\eta_n\hat{\ }\langle i\rangle} \mid i \in \mathbb{N}\}$ to be a strictly increasing sequence converging to a_{η_n} such that $a_{\eta_n\hat{\ }\langle 0\rangle} = (a_{\varrho_n} + a_{\eta_n})/2$. It is left to the reader to verify that the induction hypotheses hold.

Let $\mathcal{L} = \{L_\eta \mid \eta \in T^*\}$, set $a_\Lambda = 3$, for $\eta \in T$ define $c_\eta = d_\eta + a_\eta e_0$ and let $C = \{c_\eta \mid \eta \in T\}$. Note that $c_\Lambda \in \mathrm{bd}(Y')$. For $\eta = \nu\hat{\ }\langle i\rangle \in T^*$ define $b_\eta = d_\nu + a_\eta e_0$. So $L_\eta = [b_\eta, c_\eta]$.

We first establish some facts about the distance between the members of $\mathcal{L}$.

Claim 1. *If $\nu \neq \mathrm{pred}(\eta)$, $\eta \neq \mathrm{pred}(\nu)$ and $\mathrm{pred}(\nu) \neq \mathrm{pred}(\eta)$, then $d(L_\nu, L_\eta) > 1$.*

Proof. L_ν and L_η can be written as $L_\nu = a_\nu e_0 + [b, c]$ and $L_\eta = a_\eta e_0 + [d, e]$, where $b, c, d, e \in \{e_i \mid i \in \mathbb{N}^{\geq 1}\}$ and $\{b, c\} \cap \{d, e\} = \emptyset$. So $(d(L_\nu, L_\eta))^2 = (a_\nu - a_\eta)^2 + 4 \cdot \frac{1}{4} > 1$.

Claim 2. *Suppose that $\nu = \mathrm{pred}(\eta)$ or $\eta = \mathrm{pred}(\nu)$ or $\nu \in \mathrm{Brthr}(\eta)$ and write $L_\nu = a_\nu e_0 + [b, c]$ and $L_\eta = a_\eta e_0 + [b, d]$, where $b, c, d \in \{e_i \mid i \in \mathbb{N}^{\geq 1}\}$. Let $x \in L_\nu$ and write $x = a_\nu e_0 + b + e$. Then $d(x, L_\eta) > \frac{\sqrt{3}}{2}\|e\|$.*

Proof. Clearly, e can be written as $e = t(c - b)$ and so

$$d(x, L_\eta)^2 = (a_\eta - a_\nu)^2 + d(b + e, [b, d])^2 > d(b + e, [b, d])^2 = d(t(c - b), [0, d - b])^2.$$

Also,

$$d(t(c - b), [0, d - b]) \geq d(t(c - b), \{s(d - b) \mid s \in \mathbb{R}\}) = \|t(c - b)\| \cdot \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}\|e\|.$$

So $d(x, L_\eta) > \frac{\sqrt{3}}{2}\|e\|$. This proves Claim 2.

If we define $X_0 = X' - \bigcup \mathcal{L}$, $Y_0 = \tau'(X_0)$ and $\tau_0 = \tau' \restriction X_0$, then all the requirements of part (b) are fulfilled except that X_0 is not regular open. To achieve that X be regular open, we replace every L_η by a set F_η such that $F_\eta = \mathrm{cl}(\mathrm{int}(F_\eta))$. This will ensure that X is regular open. The verification of the following trivial fact is left to the reader.

Claim 3. *C is $\sqrt{2}$-spaced.*

Let $\eta, \nu \in T^*$. For distinct $x, y \in \ell_2$ define $H_{x,y} = (\{t(y-x) \mid t \in \mathbb{R}\})^\perp$. Let θ be such that $\tan\theta = 1/8$ and define the "closed double cone" of x, y to be

$$\mathrm{dcone}(x,y) = \{z \in [x,y] + H_{x,y} \mid d(z,[x,y]) \le d(z,\{x,y\}) \cdot \sin\theta\}.$$

Note that $\mathrm{dcone}(x,y)$ is the union of two cones with vertices x, y. The common base of the two cones is $B((x+y)/2, r) \cap ((x+y)/2 + H_{x,y})$, where $r = \frac{1}{2}\|y-x\| \cdot \tan\theta$, and the opening angle of the cones is θ. The verification of the following fact is omitted.

Claim 4. *There is $K > 1$ such that for any distinct $x, y, u, v \in \ell_2$ and $\varepsilon > 0$: if $u, v \notin \mathrm{dcone}(x,y)$ and $d(u, \mathrm{dcone}(x,y)), d(v, \mathrm{dcone}(x,y)) \le \varepsilon$, then there is a rectifiable arc J connecting u, v such that $J \subseteq \{z \mid d(z, \mathrm{dcone}(x,y)) \le \varepsilon\} - \mathrm{dcone}(x,y)$, $d(J,\{x,y\}) = d(\{u,v\},\{x,y\})$ and $\mathrm{lngth}(J) \le K\|u-v\|$.*

Note that in order to prove Claim 4 it suffices to consider the affine subspace of ℓ_2 generated by x, y, u, v. So the proof can be carried out in a 3-dimensional Euclidean space.

Define $F_\eta = \mathrm{dcone}(b_\eta, c_\eta)$, $\mathcal{F} = \{F_\eta \mid \eta \in T^*\}$, $\widehat{F} = \bigcup \mathcal{F}$, $X = X' - \widehat{F}$, $Y = Y' - \widehat{F}$ and $\tau = \tau' \restriction X$. Clearly, $\tau \in H(X,Y)$. Since τ' cannot be continued to a continuous function defined on $S(0,1)$, neither can τ. Hence $\tau \notin \mathrm{EXT}(X,Y)$. The next claim contains the central fact about $\mathcal{F}$.

Claim 5. *Let $\eta \in T^*$ and $r > 0$. Then $d(F_\eta - B(c_\eta, r), \widehat{F} - F_\eta) > 0$.*

Proof. Let $\eta = \nu\hat{\ }\langle i\rangle$. If $i > 0$ define $\delta_\eta = \min(a_{\nu\hat{\ }\langle i+1\rangle} - a_{\nu\hat{\ }\langle i\rangle}, a_{\nu\hat{\ }\langle i\rangle} - a_{\nu\hat{\ }\langle i-1\rangle})$ and if $i = 0$ define $\delta_\eta = a_{\nu\hat{\ }\langle i+1\rangle} - a_{\nu\hat{\ }\langle i\rangle}$. Let $\varepsilon_{\eta,r} = \min(3/4, 3r/4, \delta_\eta/3)$. Let $\zeta \in T^* - \{\eta\}$. We show that $d(F_\eta - B(c_\eta, r), F_\zeta) \ge \varepsilon_{\eta,r}$. If $\zeta \notin \mathrm{Brthr}(\eta) \cup \mathrm{Suc}(\eta) \cup \{\mathrm{pred}(\eta)\}$, then by Claim 1, $d(L_\eta, L_\zeta) > 1$. So $d(F_\eta, F_\zeta) > 1 - 2 \cdot \frac{1}{8}\frac{\sqrt{2}}{2} > 3/4$.

Suppose that $\zeta \in \mathrm{Suc}(\eta)$. Recall that $c_\eta = a_\eta e_0 + d_\eta$. Let $x \in F_\eta - B(c_\eta, r)$ and let y be the nearest point to x in L_η. Then $y = a_\eta e_0 + d_\eta + e$, where e has the form $e = s(d_\nu - d_\eta)$. Since $\|x - y\| \le \|e\|/8$ and $\|x - (a_\eta e_0 + d_\eta)\| \ge r$, we have $\|e\| \ge 8r/9$. Take a point $z \in F_\zeta$, let w be the nearest point to z in L_ζ and suppose that $\|w - (a_\zeta e_0 + d_\eta)\| = t$. Then $\|y - w\| > \sqrt{\|e\|^2 + t^2}$ and hence $\|y - z\| > \sqrt{\|e\|^2 + t^2} - t/8$. The minimal value of the function $g(t) = \sqrt{\|e\|^2 + t^2} - t/8$ is $\ge \|e\| - \|e\|/56$. This implies that $d(y, F_\zeta) \ge \|e\| - \|e\|/56$. Since $\|x - y\| \le \|e\|/8$, it follows that $d(x, F_\zeta) \ge \|e\| - \|e\|/56 - \|e\|/8 = 6\|e\|/7$. Hence $d(x, F_\zeta) \ge \frac{6}{7} \cdot \frac{8}{9} r \ge 3r/4$.

Assume that $\zeta \in \mathrm{Brthr}(\eta) \cup \{\mathrm{pred}(\eta)\}$. Define $f = a_\eta e_0 + d_\nu$. Let $x \in F_\eta$ and suppose first that $\|x - f\| \le \delta_\eta/2$. If $w \in L_\zeta$ and $\|w - (a_\zeta e_0 + d_\nu)\| = t$, then $d(f, w) \ge \sqrt{\delta_\eta^2 + t^2}$, So the distance between f and a general point in F_ζ is $\ge \sqrt{\delta_\eta^2 + t^2} - t/8$. So $d(f, F_\zeta) \ge \delta_\eta - \delta_\eta/56$ and hence $d(x, F_\zeta) \ge \delta_\eta - \delta_\eta/56 - \delta_\eta/2 > \delta_\eta/3$.

Suppose that $x \in F_\eta$ and $\|x - f\| \geq \delta_\eta/2$. Let y be the nearest point to x in L_η and $\delta = \|y - f\|$. Then $d(y, F_\zeta) \geq \delta - \delta/56$ and hence $d(x, F_\zeta) \geq \delta - \delta/56 - \delta/8 = 6\delta/7$. Also, $\delta \geq \frac{8}{9} \cdot \frac{\delta_\eta}{2}$. So $d(x, F_\zeta) \geq \frac{6}{7} \cdot \frac{8}{9} \cdot \frac{\delta_\eta}{2} > \frac{\delta_\eta}{3}$. The proof of Claim 5 is complete.

Claim 6. (i) *$\mathcal{F}$ is a pairwise disjoint family and* $\mathrm{acc}^E(\mathcal{F}) = C$.

(ii) *Let $\eta \in T$, $\{F_n \mid n \in \mathbb{N}\} \subseteq \mathcal{F}$ be a* 1-1 *sequence, $x_n \in F_n$ and* $\lim_{n\to\infty} x_n = c_\eta$. *Then $\{F_n \mid n \in \mathbb{N}\} - \{F_{\eta\hat{}\langle i \rangle} \mid i \in \mathbb{N}\}$ is finite.*

Proof. By Claim 5, $(F_\eta - \{c_\eta\}) \cap F_\zeta = \emptyset$ for any distinct $\eta, \zeta \in T^*$. Since $c_\eta \neq c_\zeta$ for any $\eta \neq \zeta$, it follows that $\mathcal{F}$ is pairwise disjoint.

We show that $C \subseteq \mathrm{acc}(\mathcal{F})$. Recall that $C = \{c_\eta \mid \eta \in T\}$, where $c_\eta = d_\eta + a_\eta e_0$ and $a_\Lambda = 3$. We start with c_Λ. By the construction, $a_{\langle n \rangle} \cdot e_0 + d_\Lambda \in L_{\langle n \rangle} \subseteq F_{\langle n \rangle}$ and $c_\Lambda = 3e_0 + d_\Lambda = \lim_{n\to\infty} a_{\langle n \rangle} \cdot e_0 + d_\Lambda$. So $c_\Lambda \in \mathrm{acc}(\mathcal{F})$. Suppose now that $\eta \neq \Lambda$. Then $a_{\eta\hat{}\langle n \rangle} \cdot e_0 + d_\eta \in L_{\eta\hat{}\langle n \rangle} \subseteq F_{\eta\hat{}\langle n \rangle}$ and $c_\eta = a_\eta e_0 + d_\eta = \lim_{n\to\infty} a_{\eta\hat{}\langle n \rangle} \cdot e_0 + d_\eta$. So $c_\eta \in \mathrm{acc}(\mathcal{F})$. We have shown that $C \subseteq \mathrm{acc}(\mathcal{F})$.

Let $\{\nu_i \mid i \in \mathbb{N}\} \subseteq T^*$ be a 1-1 sequence, $x_i \in F_{\nu_i}$, and suppose that $\{x_i\}_{i\in\mathbb{N}}$ is convergent. Let $x = \lim_{i\to\infty} x_i$. We shall show that for some $\eta \in T$, $x = c_\eta$ and $\{F_{\nu_i} \mid i \in \mathbb{N}\} - \{F_{\eta\hat{}\langle i \rangle} \mid i \in \mathbb{N}\}$ is finite. This will imply both that $\mathrm{acc}(\mathcal{F}) \subseteq C$ and (ii). We color the unordered pairs of $\mathbb{N}$ in three colors. The pair $\{i, j\}$ has Color 1 if $\nu_i \in \mathrm{Brthr}(\nu_j)$, and $\{i, j\}$ has Color 2 if $\nu_i = \mathrm{pred}(\nu_j)$ or $\nu_j = \mathrm{pred}(\nu_i)$. The remaining unordered pairs have Color 3. By the Ramsey Theorem we may assume that $\mathbb{N}$ is monochromatic. Color 2 has no infinite monochromatic sets, and if $\mathbb{N}$ has Color 3, then by the first paragraph in the proof of Claim 5 the sequence $\{x_i\}_{i\in\mathbb{N}}$ is $\frac{3}{4}$-spaced. It follows that for some $\eta \in T$, $\{\nu_i \mid i \in \mathbb{N}\} \subseteq \mathrm{Suc}(\eta)$.

Let y_i be the nearest point to x_i in L_{ν_i}, and write $y_i = a_{\nu_i} \cdot e_0 + d_\eta + f_i$, where $f_i = t_i(d_{\eta\hat{}\langle n_i \rangle} - d_\eta)$ for some $t_i \in [0, 1]$. We may assume that $\{f_i\}_{i\in\mathbb{N}}$ is convergent and let $f = \lim_{i\to\infty} f_i$. Suppose by way of contradiction that $f \neq 0$. Let n be such that for every $i, j \geq n$, $\|x_i - x_j\| < \varepsilon$, where ε is to be chosen later, and $\frac{4}{5}\|f\| < \|f_i\| < 2\|f\|$. Let $i, j \geq n$ be distinct. Then $\|x_i - y_i\| \leq \|f_i\|/8 \leq \|f\|/4$ and $\|x_j - y_j\| \leq \|f\|/8 \leq \|f\|/4$. So

$$\|y_i - y_j\| \leq \|y_i - x_i\| + \|x_i - x_j\| + \|x_j - y_j\| \leq \|f_i\|/8 + \|f_j\|/8 + \varepsilon < \|f\|/2 + \varepsilon.$$

On the other hand, by Claim 2,

$$\|y_i - y_j\| \geq d(y_i, L_{\nu_j}) \geq \frac{\sqrt{3}}{2}\|f_i\| \geq \frac{2\sqrt{3}}{5}\|f\|.$$

If ε is sufficiently small, then the last two inequalities are contradictory. So $f = 0$. Now, $\|x_i - y_i\| \leq \|f_i\|/8$. So $\lim_{i\to\infty} \|x_i - y_i\| = 0$ and hence

$$\lim_{i\to\infty} x_i = \lim_{i\to\infty} y_i = \lim_{i\to\infty} a_{\nu_i} e_0 + d_\eta + f_i = \lim_{i\to\infty} a_{\nu_i} e_0 + d_\eta = a_\eta e_0 + d_\eta = c_\eta \in C.$$

We have proved that $\mathrm{acc}(\mathcal{F}) \subseteq C$. We have also shown that if $\{F_n \mid n \in \mathbb{N}\} \subseteq \mathcal{F}$ is a 1-1 sequence, $x_n \in F_n$ and $\lim_{n\to\infty} x_n = c_\eta$, then $\{F_n \mid n \in \mathbb{N}\} \cap \{F_{\eta\hat{}\langle i \rangle} \mid i \in \mathbb{N}\}$ is infinite. Obviously, this implies (ii). This completes the proof of Claim 6.

Denote $\widehat{F} \cup \{c_\Lambda\}$ by $\widetilde{F}$. Since every member of $\mathcal{F}$ is closed and $\mathrm{acc}(\mathcal{F}) = C \subseteq \widetilde{F}$, it follows that $\widetilde{F}$ is closed. Recall that $c_\Lambda \in \mathrm{bd}(Y')$ and hence $c_\Lambda \notin X'$. It follows that

$X = X' - \widetilde{F}$, so X is open. Clearly, $F = \mathrm{cl}(\mathrm{int}(F))$ for every $F \in \mathcal{F}$. So $\widetilde{F} = \mathrm{cl}(\mathrm{int}(\widetilde{F}))$. This implies that $E - \widetilde{F}$ is regular open, and hence $X = X' \cap (E - \widetilde{F})$ is regular open. An identical argument shows that Y is regular open in E.

Claim 7. *Let K be the constant mentioned in Claim 4. Then for every $x, y \in Y$ there is a rectifiable arc $J \subseteq Y$ connecting x and y such that* $\mathrm{lngth}(J) \le 2K\|x - y\|$. *Similarly, let $K_1 = \max(2K, \pi)$. Then for every $x, y \in X$ there is a rectifiable arc $J \subseteq X$ connecting x and y such that* $\mathrm{lngth}(J) \le K_1\|x - y\|$.

Proof. Let $x, y \in Y$. By Claim 3, C is spaced, so for every $\varepsilon > 0$ there is $z \in B(y, \varepsilon)$ such that $[x, z] \cap C = \emptyset$. Choose such a z for a small ε which will be determined later. Since Y is open, we may choose z such that $[z, y] \subseteq Y$, and since Y' is convex, $[x, z] \subseteq Y'$. Since $[x, z] \cap C = \emptyset$ and $\mathrm{acc}(\mathcal{F}) = C$, $\mathcal{F}_0 := \{F \in \mathcal{F} \mid F \cap [x, z] \ne \emptyset\}$ is finite. The fact that C is spaced implies that $r := d([x, z], C) > 0$. Let $\mathcal{F}_0 = \{F_0, \ldots, F_{n-1}\}$, $F_i = F_{\eta_i}$, $b_i = b_{\eta_i}$, $c_i = c_{\eta_i}$, $F_i \cap [x, z] = [x_{i,0}, x_{i,1}]$ and

$$\delta_i = \tfrac{1}{2} \min(d(F_i - B(c_i, r/2), \widehat{F} - F_i), r, \delta^{Y'}(\textstyle\bigcup \mathcal{F}_0)).$$

By Claim 5, $\delta_i > 0$. Let $\hat{x}_{i,j} \in [x, z]$ be such that $\|\hat{x}_{i,j} - x_{i,j}\| \le \delta_i$ and $[\hat{x}_{i,j}, x_{i,j}) \cap F_i = \emptyset$. By Claim 4, there is a rectifiable arc J_i connecting $\hat{x}_{i,0}$ and $\hat{x}_{i,1}$ such that $\mathrm{lngth}(J_i) \le K\|\hat{x}_{i,0} - \hat{x}_{i,1}\|$, $J_i \subseteq \{z \in \ell_2 \mid d(z, F_i) \le \delta_i\} - F_i$ and $d(J_i, \{b_i, c_i\}) = d(\{\hat{x}_{i,0}, \hat{x}_{i,1}\}, \{b_i, c_i\})$. Since $d(\{\hat{x}_{i,0}, \hat{x}_{i,1}\}, \{b_i, c_i\}) \ge r$, it follows that $d(J_i, c_i) \ge r$. Let $u \in J_i$ and v be the nearest point to u in F_i. Then $\|c_i - v\| \ge \|c_i - u\| - \|u - v\| \ge r/2$. So $v \in F_i - B(c_i, r/2)$, and hence $d(v, \widehat{F} - F_i) \ge 2\delta_i$. From the fact that $\|u - v\| \le \delta_i$ it follows that $u \notin \widehat{F} - F_i$, so $J_i \cap \widehat{F} = \emptyset$. Also, since for every $u \in J_i$, $d(u, \bigcup \mathcal{F}_0) < \delta^{Y'}(\bigcup \mathcal{F}_0)$, we have $J_i \subseteq Y'$. Let $J' = [x, z] \cup \bigcup_{i<n} J_i - \bigcup_{i<n} [\hat{x}_{i,0}, \hat{x}_{i,1}]$ and $J = J' \cup [z, y]$. It is easily seen that J' and J are rectifiable arcs, and it follows that $J \subseteq Y' - \widehat{F} = Y$. From the fact that $\mathrm{lngth}(J_i) \le K\|\hat{x}_{i,0} - \hat{x}_{i,1}\|$, it follows that $\mathrm{lngth}(J') \le K\|z - x\|$. Recall that $\|y - z\| < \varepsilon$. So if ε is sufficiently small, then $\mathrm{lngth}(J) < 2K\|y - x\|$.

The proof of the analogous fact for X is almost identical. We have proved Claim 7.

We now show that X and Y are BR.LC.AC and JN.AC. Claim 7 implies that Y is UD.AC and BD.AC. It follows directly from the definitions that if F is any metric space, $Z \subseteq F$ and Z is UD.AC, then Z is BR.LC.AC with respect to F. Hence Y is BR.LC.AC with respect to ℓ_2. The bounded arcwise connectedness of Y and Proposition 6.2(c) imply that Y is JN.AC. The same arguments apply to X, hence X too is BR.LC.AC and JN.AC.

Our next goal is to show $(*)$ $h(S(0,1)) = S(0,1)$ for every $h \in \mathrm{EXT}(X)$. It may very well be true that $(\dagger)$ $S(0,1)$ is the only clopen component of $\mathrm{bd}(X)$. This would imply $(*)$, but we do not know how to prove this. So instead we prove $(\dagger\dagger)$ $S(0,1)$ is the only clopen component of $\mathrm{bd}(X)$ which is strongly connected in $\mathrm{bd}(X)$. This also implies $(*)$.

Let Z be a connected space. We say that Z is *strongly connected* if for every $z \in Z$ and $U \in \mathrm{Nbr}(z)$, there is $V \in \mathrm{Nbr}(z)$ such that $V \subseteq U$ and $Z - V$ is connected. Clearly, $S(0,1)$ is strongly connected.

For $\eta \in T^*$ let $S_\eta = \mathrm{bd}^{\ell_2}(F_\eta)$. It is easy to see that $\mathrm{bd}(X) = S(0,1) \cup S(0,3) \cup \bigcup_{\eta \in T^*} S_\eta$. Obviously, $S(0,1)$ is a component of $\mathrm{bd}(X)$, and $S(0,1)$ is clopen in $\mathrm{bd}(X)$.

Let $\mathcal{K}$ denote the set of components of $\mathrm{bd}(X)$ which are clopen in $\mathrm{bd}(X)$ and which are different from $S(0,1)$. Let $\eta \in T$ and $T' \subseteq T$. We say that T' is *η-large* if $\eta \in T' \subseteq T^{\geq \eta}$, and for every $\nu \in T'$, $\{i \mid \nu\hat{\ }\langle i\rangle \notin T'\}$ is finite. Define $S_\Lambda = S(0,3)$ and for $T' \subseteq T$ set $S_{T'} = \bigcup_{\nu \in T'} S_\nu$.

CLAIM 8. *For every $K \in \mathcal{K}$ there are a finite set $\sigma \subseteq T$ and a family $\{T_\nu \mid \nu \in \sigma\}$ such that T_ν is ν-large for every $\nu \in \sigma$, and $K = \bigcup_{\nu\in\sigma} S_{T_\nu}$.*

Proof. Note that S_η is connected for every $\eta \in T$. Hence for every $K \in \mathcal{K}$ and $\eta \in T$, either $S_\eta \subseteq K$ or $S_\eta \cap K = \emptyset$. Also, for every $\eta \in T$ and an infinite $\sigma \subseteq \mathbb{N}$, $S_\eta \cap \mathrm{acc}(\{S_{\eta\hat{\ }\langle i\rangle} \mid i \in \sigma\}) \neq \emptyset$. This implies that (†) if $K \in \mathcal{K}$ and $S_\eta \cap K \neq \emptyset$, then $S_\eta \subseteq K$ and $\{i \mid S_{\eta\hat{\ }\langle i\rangle} \not\subseteq K\}$ is finite. The fact that the members of $\mathcal{K}$ are closed implies that (††) if $K \in \mathcal{K}$ and $\{i \mid S_{\eta\hat{\ }\langle i\rangle} \subseteq K\}$ is infinite, then $S_\eta \subseteq K$. Facts (†) and (††) imply that Claim 8 is true.

Let $K \in \mathcal{K}$ and suppose that $\sigma \subseteq T$ and $\{T_\nu \mid \nu \in \sigma\}$ are as ensured by Claim 8. So there are $\eta \in T^*$ and an infinite $T' \subseteq T$ such that $S_\eta \subseteq K = S_{T'}$. By Claim 5, $d(S_\eta - B(c_\eta, r), \widehat{F} - F_\eta) > 0$ for every $r > 0$. Since S_η and S_Λ are closed and disjoint, it follows that $d(S_\eta, S_\Lambda) > 0$, and from the facts that $K \subseteq \widehat{F} \cup S_\Lambda$ and $S_\eta \subseteq F_\eta$ we conclude that $d(S_\eta - B(c_\eta, r), K - S_\eta) > 0$. So $S_\eta - B(c_\eta, r)$ is clopen in K. This implies that K is not strongly connected. We have shown that $S(0,1)$ is the unique clopen strongly connected component of $\mathrm{bd}(X)$. Hence $h(S(0,1)) = S(0,1)$ for every $h \in \mathrm{EXT}(X)$. It follows that $(\mathrm{EXT}(X))^\tau \subseteq \mathrm{EXT}(Y)$. This completes the proof of (b).

(c) Let $S \subseteq \ell_2$ be a two-dimensional sphere with radius 1 and center at 0. Let $X = B(0,3) - S$ and $Y = B(0,3)$. Then there is $\tau \in H(X,Y)$ such that $\tau\restriction(B(0,3) - \bar{B}(0,2)) = \mathrm{Id}$. It is trivial that X and Y are BR.LC.AC and JN.AC, and it is easy to see that $(\mathrm{EXT}(X))^\tau \subseteq \mathrm{EXT}(Y)$ and $\tau \notin H(X,Y)$.

(d) We construct a set X with the following properties:

(1) X is a regular open bounded subset of $\mathbb{R}^3$,
(2) there is $K > 1$ such that for every $x, y \in X$ there is a rectifiable arc $L \subseteq X$ such that $\mathrm{lngth}(L) \leq K\|x - y\|$,
(3) for every $g \in \mathrm{EXT}(X)$, $g^{\mathrm{cl}}\restriction\mathrm{bd}(X) = \mathrm{Id}$.

It is easy to verify that if X satisfies (1)–(3), then it fulfills the requirements of the example.

We turn to the construction of X. Let $\widehat{R}_n$ be the n-fold solid torus and $\widehat{T}_n$ denote its boundary. A subset $A \subseteq \mathbb{R}^3$ is *K-bypassable* if for every $x, y \in \mathbb{R}^3 - A$ there is a rectifiable arc $L \subseteq \mathbb{R}^3 - A$ connecting x and y such that $\mathrm{lngth}(L) \leq K\|x - y\|$ and $d(z, A) \leq d(x, A), d(y, A)$ for every $z \in L$. Obviously, there is $K > 1$ such that for every n there is a K-bypassable $F \subseteq \mathbb{R}^3$ such that $F \cong \widehat{R}_n$. Let D be a countable dense subset of $B(0,1)$, E be a countable dense subset of $S(0,1)$ and $\{\{a_n, b_n\} \mid n \in \mathbb{N}\}$ be a list of all 2-element subsets of D and all singletons from $D \cup E$. Also assume that $a_0 = b_0 \in E$. We define by induction a finite family of open sets $\mathcal{U}_n$ and a finite family of closed sets $\mathcal{F}_n$ such that for any distinct $A \in \mathcal{U}_n \cup \mathcal{F}_n$ and $F \in \mathcal{F}_n$, $\mathrm{cl}(A) \subseteq B(0,1)$, $\mathrm{cl}(A) \cap F = \emptyset$ and F is K-bypassable. Let $\mathcal{U}_0 = \mathcal{F}_0 = \emptyset$. Suppose that $\mathcal{U}_n$ and $\mathcal{F}_n$ have been defined.

Case 1: $a_n \neq b_n$. If $\{a_n, b_n\} \cap \bigcup \mathcal{F}_n \neq \emptyset$ define $\mathcal{U}_{n+1} = \mathcal{U}_n$ and $\mathcal{F}_{n+1} = \mathcal{F}_n$. Suppose otherwise. Define $\mathcal{F}_{n+1} = \mathcal{F}_n$. Since $\mathcal{F}_n$ is a finite pairwise disjoint family of closed K-bypassable sets there is a rectifiable arc $L_n \subseteq B(0,1) - \bigcup \mathcal{F}_n$ connecting a_n and b_n such that $\mathrm{lngth}(L_n) \leq K\|a_n - b_n\|$. Let $r = d(L_n, S(0,1) \cup \bigcup \mathcal{F}_n)$ and $\mathcal{U}_{n+1} = \mathcal{U}_n \cup \{B(L_n, r/2)\}$.

Case 2: $a_n = b_n$. If $a_n \in D$ let $c_n \in \bigcup_{F \in \mathcal{F}_n} \mathrm{bd}(F))$ be such that $\|c_n - a_n\| = d(a_n, \bigcup_{F \in \mathcal{F}_n} \mathrm{bd}(F))$ and $H_n \in \mathcal{F}_n$ be such that $c_n \in \mathrm{bd}(H_n)$. If $a_n \in E$ let $c_n = a_n$ and $H_n = S(0,1)$. Let $F_n \subseteq B(c_n, \frac{1}{n+1}) \cap (B(0,1) - \bigcup \mathcal{F}_n - \bigcup_{U \in \mathcal{U}_n} \mathrm{cl}(U))$ be such that $F_n \cong \widehat{R}_n$ and F_n is K-bypassable. Define $\mathcal{F}_{n+1} = \mathcal{F}_n \cup \{F_n\}$. Let $r_n = d(H_n, S(0,1) \cup \bigcup \mathcal{F}_{n+1} - H_n)$ and $U_{n,0} = B(0,1) \cap (B(H_n, r_n/2) - \mathrm{cl}(B(H_n, r_n/4)))$. Let $x_n \in B(0,1) \cap (B(c_n, r_n/2) - H_n)$, $s_n \in (0, r_n/2)$ be such that $U_{n,1} := B(x_n, s_n)$ is disjoint from H_n and $\mathcal{U}_{n+1} = \mathcal{U}_n \cup \{U_{n,0}, U_{n,1}\}$. This concludes the inductive construction.

Let $X = B(0,1) - \mathrm{cl}(\bigcup_{n \in \mathbb{N}} \mathcal{F}_n)$. Since any two members of $D \cap X$ lie in the same member of $\mathrm{Cmp}(X)$ and $D \cap X$ is dense in X, it follows that X is connected.

Set $A = \{n \mid a_n = b_n\}$, for every $n \in A$ let $f_n : \widehat{R}_n \cong F_n$ and $T_n = f_n(\widehat{T}_n)$ and define $T = S(0,1) \cup \bigcup_{n \in A} T_n$. The verification of the following facts is left to the reader.

(1) $\mathrm{bd}(X) = \mathrm{cl}(T)$ and $T \subseteq \mathrm{cl}(\mathrm{int}(\mathbb{R}^3 - X))$.
(2) For every $n \in A$, $T_n \in \mathrm{Cmp}(\mathrm{bd}(X))$, and $S(0,1) \in \mathrm{Cmp}(\mathrm{bd}(X))$.
(3) For every $C \in \mathrm{Cmp}(\mathrm{bd}(X)) - \{T_n \mid n \in A\} - \{S(0,1)\}$, $\mathbb{R}^3 - C$ is connected.

Fact (1) implies that X is regular open. It follows from (3) and Alexander's Duality Theorem that for every $C \in \mathrm{Cmp}(\mathrm{bd}(X)) - \{T_n \mid n \in A\} - \{S(0,1)\}$ and $n \in \mathbb{N}$, $C \not\cong \widehat{T}_n$. Let $x \in T$. Then there is a sequence $\{k_n \mid n \in \mathbb{N}\} \subseteq A$ such that $\lim_{k_n \to \infty} T_n = x$. Hence x has the following property:

> There is a sequence $\{C_n \mid n \in \mathbb{N}\}$ of members of $\mathrm{Cmp}(\mathrm{bd}(X))$ such that $C_n \cong T_{k_n}$ and $\lim_{n \to \infty} C_n = x$.

However, if $y \in \mathrm{bd}(X) - \{x\}$, then y does not have this property. Since $\mathrm{bd}(X)$ is invariant under $\mathrm{EXT}(X)$, it follows that $g(x) = x$ for every $g \in \mathrm{EXT}(X)$. That is, $g \restriction T = \mathrm{Id}$ for every $g \in \mathrm{EXT}(X)$. Since T is dense in $\mathrm{bd}(X)$, it follows that $g \restriction \mathrm{bd}(X) = \mathrm{Id}$ for every $g \in \mathrm{EXT}(X)$. ∎

Remark. Recall that in Corollary 6.6(b) it was assumed that for every $x, y \in \mathrm{bd}(X)$ there is $h \in \mathrm{EXT}(X)$ such that $h^{\mathrm{cl}}(x) = y$. In part (c) of the above example $\mathrm{bd}(X)$ has two connected components K_0, K_1, neither is a singleton, and for every $i = 0, 1$ and $x, y \in K_i$ there is $h \in \mathrm{EXT}(X)$ such that $h^{\mathrm{cl}}(x) = y$. The space Y in the above example has the property that $\mathrm{bd}(Y)$ is connceted, $\mathrm{bd}(Y)$ is not a singleton, and for every $x, y \in \mathrm{bd}(Y)$, there is $h \in \mathrm{EXT}(X)$ such that $h^{\mathrm{cl}}(x) = y$. These transitivity properties of $\mathrm{bd}(X)$ and $\mathrm{bd}(Y)$, though quite strong, do not imply the conclusion of 6.6(b). □

In Theorem 6.3 it was shown that if $\varphi : \mathrm{EXT}(X) \cong \mathrm{EXT}(Y)$, then φ is induced by some $\tau \in \mathrm{EXT}^{\pm}(X, Y)$. But Theorem 6.3 applies only to sets X with finitely many connected components. To see this let X be BR.LC.AC and JN.AC as was assumed in 6.3, and suppose by contradiction that X has infinitely many connected components. Let $\vec{z}$ be a sequence of members of X which lie in distinct components of X. Let $\langle \vec{x}, x^*, \{L_n \mid$

$n \in \mathbb{N}\}, \vec{x}'\,\rangle$ be a joining system for some subsequence $\vec{x}$ of $\vec{z}$. Then $\vec{x}'$ is a convergent sequence, but each member of $\mathrm{Rng}(\vec{x}')$ lies in a different component of X. This contradicts the fact that X is BR.LC.AC. So X has only finitely many connected components.

Our next goal is to extend 6.3 to sets X that may have infinitely many connected components. We have four test cases X for which $\mathrm{EXT}(X)$ seems to be sufficiently well behaved to imply a reconstruction theorem for $\mathrm{EXT}(X)$, but which are not covered by Theorem 6.3. The first example which is defined below, has infinitely many components. The three others appear in Example 6.15, and they are connected.

Example 6.8. Let E be a Banach space. We define

$$R_1^E = \bigcup_{n\in\mathbb{N}} \left(B^E\left(0, 1 - \frac{1}{2n+3}\right) - \bar{B}^E\left(0, 1 - \frac{1}{2n+2}\right)\right)$$

The set R_1^E is the union of a sequence of pairwise disjoint open rings converging to $S^E(0,1)$. $\square$

We shall prove a reconstruction theorem for a class which contains R_1^E. The definition of this class is rather technical, but it contains quite complicated sets. This class will be denoted by $K_{\mathrm{BX}}^{\mathcal{O}}$. For simplicity, we consider only subsets of Banach spaces and not subsets of general normed spaces. Hence only 6.3(a) is extended. That is, $K_{\mathrm{BCX}}^{\mathcal{O}} \subseteq K_{\mathrm{BX}}^{\mathcal{O}}$.

Definition 6.9. (a) Recall that $\mathrm{Cmp}(X)$ denotes the set of connected components of a topological space X. For $x, y \in X$, $x \simeq^X y$ denotes that x and y lie in the same connected component of X. The notation $\vec{x} \simeq^X \vec{y}$ means that $x_n \simeq^X y_n$ for every $n \in \mathbb{N}$.

(b) Let X be a metric space. We say that X *is boundedly component-wise arcwise connected* (*BD.CW.AC*) if for every bounded set $A \subseteq X$ there is $d = d_A$ such that for every $x, y \in A$: if $x \simeq^X y$, then there is a rectifiable arc $L \subseteq X$ connecting x and y such that $\mathrm{lngth}(L) \le d$.

(c) Let $X \in K_{\mathrm{NRM}}^{\mathcal{O}}$ and $x \in \mathrm{bd}(X)$. We say that X is *component-wise locally arcwise connected* at x if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $y, z \in B(x,\delta) \cap X$: if $y \simeq^X z$, then there is an arc $L \subseteq B(x,\varepsilon) \cap X$ connecting y and z. We say that X is *component-wise locally arcwise connected at its boundary* (*BR.CW.LC.AC*) if X is component-wise locally arcwise connected at every $x \in \mathrm{bd}(X)$.

(d) Let $X \in K_{\mathrm{NRM}}^{\mathcal{O}}$. Call X a *component-wise wide space* if for every $r > 0$, $\bigcup\{C \in \mathrm{Cmp}(X) \mid C \cap B(0,r) \neq \emptyset\}$ is wide.

(e) Let $X \subseteq E$. A point $x \in \mathrm{bd}(X)$ is called a *multiple boundary point* of X if for every $C \in \mathrm{Cmp}(X)$, $x \in \mathrm{bd}(X - C)$, and x is a *double boundary point* of X if there are distinct $C_1, C_2 \in \mathrm{Cmp}(X)$ such that $x \in \mathrm{bd}(C_1) \cap \mathrm{bd}(C_2)$.

(f) A subspace $X \subseteq E$ is *locally movable at its multiple boundary* if for every $\vec{x} \subseteq X$ which converges in E to a multiple boundary point and $U \in \mathrm{Nbr}^{\mathrm{cl}(X)}(\lim \vec{x})$ there is a subsequence $\vec{x}'$ of $\vec{x}$ and $g \in \mathrm{EXT}(X)$ such that: $g(\vec{x}') \simeq^X \vec{x}'$, $g^{\mathrm{cl}}(\lim \vec{x}) \neq \lim \vec{x}$ and $\mathrm{supp}(g) \subseteq U$.

(g) Let $K_{\mathrm{BX}}^{\mathcal{O}}$ be the class of all $X \in K_{\mathrm{BNC}}^{\mathcal{O}}$ such that:

(1) X is component-wise wide, BR.CW.LC.AC and BD.CW.AC,
(2) X is locally movable at its multiple boundary. $\square$

PROPOSITION 6.10. (a) *Let* R_1^E *be as defined in Example* 6.8. *Then* $R_1^E \in K_{\mathrm{BX}}^{\mathcal{O}}$.

(b) $K_{\mathrm{BCX}}^{\mathcal{O}} \subseteq K_{\mathrm{BX}}^{\mathcal{O}}$.

Proof. The proofs of both parts are trivial. Anyway, we indicate the proof of (b). Suppose that $X \in K_{\mathrm{BCX}}^{\mathcal{O}}$. It is easily seen that the multiple boundary of X is empty, hence X is locally movable at its multiple boundary. The fact that X is wide implies that it is component-wise wide. Similarly, since X is BR.LC.AC and BD.AC, it is BR.CW.LC.AC and BD.CW.AC. So $X \in K_{\mathrm{BX}}^{\mathcal{O}}$. ∎

PROPOSITION 6.11. (a) *Let* $X \in K_{\mathrm{BX}}^{\mathcal{O}}$. *Then for every* $C \in \mathrm{Cmp}(X)$, C *is BR.LC.AC and JN.AC.*

(b) *Let* $X, Y \in K_{\mathrm{BX}}^{\mathcal{O}}$ *and* $\tau \in H(X,Y)$ *be such that* $(\mathrm{EXT}(X))^{\tau} = \mathrm{EXT}(Y)$. *Let* $C \in \mathrm{Cmp}(X)$, $D = \tau(C)$ *and* $\eta = \tau{\restriction}C$. *Then* $D \in \mathrm{Cmp}(Y)$ *and* $\eta \in \mathrm{EXT}^{\pm}(C,D)$.

Proof. (a) The fact that X is component-wise wide implies that C is wide. The fact that X is BD.CW.AC implies that C is BD.AC. So by Proposition 6.2(d), C is JN.AC.

Let $x \in \mathrm{bd}(C)$. The fact that X is component-wise locally arcwise connected at x implies that C is locally arcwise connected at x. So C is BR.LC.AC.

(b) It is trivial that C is an open subset of E and that $D \in \mathrm{Cmp}(Y)$. So by (a), C is JN.AC and BR.LC.AC, and the same holds for D. We wish to apply Corollary 6.6(a) to η, so we need to check that $(\mathrm{UC}_0(C))^{\eta} \subseteq \mathrm{EXT}(D)$ and that $(\mathrm{LUC}_{01}(D))^{\eta^{-1}} \subseteq \mathrm{EXT}(C)$. Let $g \in \mathrm{UC}_0(C)$. Set $h = g \cup \mathrm{Id}{\restriction}(X - C)$. Then $h \in \mathrm{UC}_0(X) \subseteq \mathrm{EXT}(X)$. So $h^{\tau} \in \mathrm{EXT}(Y)$. Hence $g^{\eta} = h^{\tau}{\restriction}D \in \mathrm{EXT}(D)$. A similar argument shows that $(\mathrm{LUC}_{01}(D))^{\eta^{-1}} \subseteq \mathrm{EXT}(C)$. By Corollary 6.6(a), $\eta \in \mathrm{EXT}(C,D)$. The same argument can be applied to η^{-1}. Hence $\eta \in \mathrm{EXT}^{\pm}(C,D)$. ∎

THEOREM 6.12. *Let* $X, Y \in K_{\mathrm{BX}}^{\mathcal{O}}$ *and* $\varphi : \mathrm{EXT}(X) \cong \mathrm{EXT}(Y)$. *Then there is* $\tau \in \mathrm{EXT}^{\pm}(X,Y)$ *such that* τ *induces* φ.

Proof. By Theorem 2.8(b), there is $\tau \in H(X,Y)$ such that τ induces φ.

CLAIM 1. *Let* $\vec{x}, \vec{u} \subseteq X$. *Suppose that* $\vec{x}, \vec{u}, \tau(\vec{x}), \tau(\vec{u})$ *are convergent sequences and* $\lim \vec{x} = \lim \vec{u} \in \mathrm{bd}(X)$. *Then* $\lim \tau(\vec{x}) = \lim \tau(\vec{u})$.

Proof. Let $x = \lim \vec{x}$, $y = \lim \tau(\vec{x})$ and $v = \lim \tau(\vec{u})$, and suppose by contradiction that $y \neq v$. Clearly, $y, v \in \mathrm{bd}(Y)$. Assume first that either y or v is a multiple boundary point of Y, and assume without loss of generality that y is such a point. Since Y is locally movable at its multiple boundary, there are $h \in \mathrm{EXT}(Y)$ and a subsequence $\vec{y}'$ of $\tau(\vec{x})$ such that $h^{\mathrm{cl}}(y) \neq y$, $h(\vec{y}') \simeq^{Y} \vec{y}'$ and for some $W \in \mathrm{Nbr}^{\mathrm{cl}(Y)}(v)$, $h{\restriction}(W \cap Y) = \mathrm{Id}$. By removing an initial segment of $\tau(\vec{u})$ we may assume that $\tau(\vec{u}) \subseteq W$. So h, $\vec{y}'$ and W satisfy

(∗) $h \in \mathrm{EXT}(Y)$, $\vec{y}'$ is a subsequence of $\tau(\vec{x})$, $W \in \mathrm{Nbr}^{\mathrm{cl}(Y)}(v)$, $h^{\mathrm{cl}}(y) \neq y$, $\tau(\vec{u}) \subseteq W$ and $h^{\mathrm{cl}}{\restriction}W = \mathrm{Id}$.

Now assume that y, v are not multiple boundary points of Y. Then there are $C_1, C_2 \in \mathrm{Cmp}(X)$ such that all but finitely members of $\vec{x}$ belong to C_1, and all but finitely members of $\vec{u}$ belong to C_2. From Proposition 6.11(b) and the fact that $\lim \tau(\vec{x}) \neq \lim \tau(\vec{u})$ it follows that $C_1 \neq C_2$. So x is a double boundary point of X. Let $D_1 = \tau(C_1)$

and set $\widehat{D} = Y - D_1$. Then by 6.11(b), $D_1 \in \mathrm{Cmp}(Y)$, and since y is not a multiple boundary point of Y, it follows that $y \in \mathrm{bd}(D_1) - \mathrm{cl}(\widehat{D})$. Let $V \in \mathrm{Nbr}^F(y)$ be such that $\mathrm{cl}(V) \cap \mathrm{cl}(\widehat{D}) = \emptyset$, and let $U \in \mathrm{Nbr}^E(x)$ be such that $\tau(U \cap C_1) \subseteq V$. Since X is locally movable at its multiple boundary, there is $k \in \mathrm{EXT}(X)$ and a subsequence $\vec{z}'$ of $\vec{x}$ such that $k^{\mathrm{cl}}(x) \neq x$, $\mathrm{supp}(k) \subseteq U$ and $k(\vec{z}') \simeq \vec{z}'$. Let $h = (k{\restriction}C_1)^\tau \cup \mathrm{Id}{\restriction}(Y - D_1)$. Then $h{\restriction}D_1 \in \mathrm{EXT}(D_1)$. Also,

$$\mathrm{supp}(h) = \mathrm{supp}(h{\restriction}D_1) = \tau(\mathrm{supp}(k{\restriction}C_1)) \subseteq \tau(U \cap C_1) \subseteq V.$$

So $\mathrm{supp}((h{\restriction}D_1)^{\mathrm{cl}}) \subseteq \mathrm{cl}(V)$. From the fact that $\mathrm{cl}(V) \cap \mathrm{cl}(\widehat{D}) = \emptyset$, it follows that $h \in \mathrm{EXT}(Y)$. Let $\vec{y}' = \tau(\vec{z}')$. Then $h^{\mathrm{cl}}(y) \neq y$ and $h(\vec{y}') \simeq^Y \vec{y}'$. Clearly, $v \in \mathrm{cl}(\tau(C_2))$ and $\tau(C_2) \subseteq \widehat{D}$. So $v \in \mathrm{cl}(\widehat{D})$, and hence for some $W \in \mathrm{Nbr}^{\mathrm{cl}(Y)}(v)$, $h{\restriction}(W \cap Y) = \mathrm{Id}$. By removing an initial segment of $\tau(\vec{u})$ we may assume that $\tau(\vec{u}) \subseteq W$. It follows that h, $\vec{y}'$ and W satisfy $(*)$. So whether or not $\{u, v\}$ contains a multiple boundary point, we have found $h, \vec{y}'$ and W satisfying $(*)$.

Let $g = h^{\tau^{-1}}$ and $\vec{x}' = \tau^{-1}(\vec{y}')$. So $g \in \mathrm{EXT}(X)$ and $g{\restriction}\vec{u} = \mathrm{Id}$. Since $\vec{u} \cup \vec{x}'$ converges to x and $g \in \mathrm{EXT}(X)$, $\lim g(\vec{x}') = x$. Since $h(\vec{y}') \simeq^Y \vec{y}'$, it follows that $g(\vec{x}') \simeq^X \vec{x}'$. Since X is BR.CW.AC, there is $\{f_k \mid k \in \mathbb{N}\} \subseteq \mathrm{UC}(X)$ and subsequences $\{n_k\}_{k\in\mathbb{N}}$ and $\{m_k\}_{k\in\mathbb{N}}$ such that: (i) for every k, $f_k(x'_{n_k}) = g(x'_{n_k})$, $\mathrm{cl}(\mathrm{supp}(f_k)) \subseteq X$ and $f_k{\restriction}\{x'_{m_k} \mid k \in \mathbb{N}\} = \mathrm{Id}$, (ii) $\lim_{k\to\infty} \mathrm{diam}(\mathrm{supp}(f_k)) = 0$, (iii) for any $\ell \neq k$, $\mathrm{supp}(f_\ell) \cap \mathrm{supp}(f_k) = \emptyset$. Let $f = \circ_{k\in\mathbb{N}} f_k$. So $f \in \mathrm{UC}_0(X) \subseteq \mathrm{EXT}(X)$, and hence f^τ must belong to $\mathrm{EXT}(Y)$. Let us see that this does not happen. Recall that $\lim \vec{y}' = y$. However, $\lim_k f^\tau(y'_{n_k}) = \lim_k h(y'_{n_k}) = h(y) \neq y$, and on the other hand, $\lim_k f^\tau(y'_{m_k}) = \lim_k y'_{m_k} = y$. So $\vec{y}'$ is convergent, but $f^\tau(\vec{y}')$ is not, and hence $f^\tau \notin \mathrm{EXT}(Y)$. A contradiction, so Claim 1 is proved.

Claim 2. *Let $\vec{x} \subseteq X$ be a convergent sequence in E. Then there is a subsequence $\vec{x}'$ of $\vec{x}$ such that $\tau(\vec{x}')$ is convergent in F.*

Proof. Let $x = \lim \vec{x}$. We may assume that $x \in \mathrm{bd}(X)$. If for some $C \in \mathrm{Cmp}(X)$, $\{n \mid x_n \in C\}$ is infinite, then by Proposition 6.11(b), there is a subsequence as required in the claim.

Hence we may assume that for every $m \neq n$, $x_m \not\simeq^X x_n$, and so x is a multiple boundary point. For every n let $y_n = \tau(x_n)$, and C_n and D_n be such that $x_n \in C_n \in \mathrm{Cmp}(X)$ and $y_n \in D_n \in \mathrm{Cmp}(Y)$.

Suppose by contradiction that $\{D_n \mid n \in \mathbb{N}\}$ is completely discrete. Let $\vec{u} \in \prod_{n\in\mathbb{N}} C_n$. Define $\vec{v} = \tau(\vec{u})$. There is $k \in \mathrm{EXT}(Y)$ such that for every n, $k(y_{2n}) = v_{2n}$ and $k(y_{2n+1}) = y_{2n+1}$. Let $g = k^{\tau^{-1}}$. Then $g \in \mathrm{EXT}(X)$. Since $\vec{x}$ is convergent, $g(\vec{x})$ is convergent. For every n, $g(x_{2n}) = u_{2n}$ and $g(x_{2n+1}) = x_{2n+1}$. So $\lim_{n\to\infty} u_{2n} = \lim_{n\to\infty} x_{2n+1} = x$. This implies that $\lim_{n\to\infty} C_{2n} = x$. Hence for every $f \in \mathrm{EXT}(X)$: if $\{n \in \mathbb{N} \mid f(x_{2n}) \simeq^X x_{2n}\}$ is infinite, then $f(x) = x$. Clearly, x is a multiple boundary point. So the above fact is in contradiction with the fact that X is locally movable at its multiple boundary. It follows that $\{D_n \mid n \in \mathbb{N}\}$ is not completely discrete. By choosing a subsequence of $\vec{x}$ we may assume that there is $\vec{v} \in \prod_{i\in\mathbb{N}} D_n$ such that $\vec{v}$ is convergent in F. Let $v = \lim \vec{v}$.

Suppose by way of contradiction that $\vec{y}$ does not contain a convergent subsequence. We show that if $\vec{y}$ is unbounded, then there is another counter-example to Claim 2 in which $\vec{y}$ is bounded. Let r be such that $v \in B^F(0,r)$. Then for every n, $D_n \cap B^F(0,r) \neq \emptyset$. Since Y is component-wise wide, there are a subsequence $\vec{y}'$ of $\vec{y}$, $s > 0$ and a completely discrete sequence of arcs $\{L_n \mid n \in \mathbb{N}\}$ such that for every n, $L_n \subseteq D_n$ and L_n connects y'_n with a member of $B^F(0,s)$. We may assume that $\vec{y}' = \vec{y}$.

Denote the endpoint of L_{2n} which is not y_{2n} by $\hat{w}_n$. Let $\hat{k} \in \mathrm{EXT}(Y)$ be such that for every n, $\hat{k}(y_{2n}) = \hat{w}_n$ and $\hat{k}(y_{2n+1}) = y_{2n+1}$ and set $\hat{g} = \hat{k}^{\tau^{-1}}$. Then $\hat{g} \in \mathrm{EXT}(X)$ and hence $\lim \hat{g}(\vec{x})$ exists. So $\lim_{n\to\infty} \hat{g}(x_{2n+1}) = \lim_{n\to\infty} \hat{g}(x_{2n}) = x$. Since $\hat{k}(y_{2n}) = \hat{w}_n$, it follows that $\hat{g}(x_{2n}) = \tau^{-1}(\hat{w}_n)$. That is, $\tau(\hat{g}(x_{2n})) = \hat{w}_n$. So $\{\tau(g(x_{2n})) \mid n \in \mathbb{N}\}$ is bounded and completely discrete. By replacing $\vec{x}$ by $\{\hat{g}(x_{2n}) \mid n \in \mathbb{N}\}$ we obtain a counter-example to Claim 2 in which $\vec{y}$ is bounded. Since E is a Banach space, we may also assume that $\vec{y}$ is spaced.

Since Y is BR.CW.AC, there are d and rectifiable arcs $L_n \subseteq D_n$ such that L_n connects y_n with v_n and $\mathrm{lngth}(L_n) \leq d$. Let $\gamma_n(t)$ be a parametrization of L_n such that $\gamma_n(1) = y_n$, $\gamma_n(0) = v_n$, and for every t, $\mathrm{lngth}(\gamma_n([0,t])) = t \cdot \mathrm{lngth}(L_n)$. For every infinite $\sigma \subseteq \mathbb{N}$ let $s_\sigma = \inf(\{t \in [0,1] \mid \{\gamma_n([t,1]) \mid n \in \sigma\} \text{ is spaced}\})$. Let σ be an infinite set such that for every infinite $\eta \subseteq \sigma$, $s_\eta = s_\sigma$. Then $\{\gamma_n(s_\sigma) \mid n \in \sigma\}$ contains a Cauchy sequence, and for every $t > s_\sigma$, $\{\gamma_n([t,1]) \mid n \in \sigma\}$ is spaced. Set $s = s_\sigma$. It can be assumed that $\{\gamma_n(s) \mid n \in \sigma\}$ is a Cauchy sequence, that $\sigma = \mathbb{N}$ and that $s = 0$. So $\gamma_n(1) = y_n$ for every $n \in \mathbb{N}$, $\{\gamma_n(0) \mid n \in \mathbb{N}\}$ is a Cauchy sequence, and $\{\gamma_n([t,1]) \mid n \in \mathbb{N}\}$ is spaced for every $t \in (0,1]$. Let $w_n = \gamma_n(0)$ and $w = \lim \vec{w}$.

For every $t > 0$ let $\vec{w}^t = \{\gamma_{2n}(t) \mid n \in \mathbb{N}\}$. Let $\vec{y}^0 = \{y_{2n} \mid n \in \mathbb{N}\}$ and $\vec{y}^1 = \{y_{2n+1} \mid n \in \mathbb{N}\}$. For every $t > 0$ there is $k_t \in \mathrm{EXT}(Y)$ such that $k_t(\vec{y}^0) = \vec{w}^t$ and $k_t(\vec{y}^1) = \vec{y}^1$. This follows from the fact that for $t > 0$, $\{\gamma_n([t,1]) \mid n \in \mathbb{N}\}$ is completely discrete. We check that for every $t \in (0,1]$, $\lim \tau^{-1}(\vec{w}^t) = x$. Let $h_t = k_t^{\tau^{-1}}$. Then $h_t(x_{2n+1}) = x_{2n+1}$ and $h_t(x_{2n}) = \tau^{-1}(w^t_n)$. Clearly, $h_t \in \mathrm{EXT}(X)$, so h_t takes $\vec{x}$ to a convegent sequence. But $h_t(x_{2n}) = x_{2n}$, hence $\lim h_t(\vec{x}) = \lim_n x_{2n} = x$. So $\lim_n \tau^{-1}(w^t_n) = x$.

Note that for every $\varepsilon > 0$ there are $t_\varepsilon > 0$ and m_ε such that for every $t \leq t_\varepsilon$ and $n \geq m_\varepsilon$, $\|w^t_n - w\| < \varepsilon$. Also, $x_{2n} \simeq^X \tau^{-1}(w^t_n)$ for every n and t. It follows that there are sequences $\vec{z}$ and $\{n_k\}_{k=1}^\infty$ such that $\lim \vec{z} = x$, $\lim \tau(\vec{z}) = w$, and for every k, $z_k \simeq^X x_{2n_k}$. To see this, take z_k to be $\tau^{-1}(w^{t_k}_{n_k})$, where $\{t_k\}_{k=1}^\infty$ is any sequence converging to 0 and n_k is such that $n_k \geq m_{1/k}$ and $\|\tau^{-1}(w^{t_k}_{n_k}) - x\| < 1/k$.

From the facts X is BR.CW.AC, $z_k \simeq^X x_{2n_k}$ and $\lim \vec{z} = \lim_k x_{2n_k}$, we conclude that there is $g \in \mathrm{EXT}(X)$ such that for infinitely many k's, $g(x_{2n_k}) = z_k$. We now check that $g^\tau \notin \mathrm{EXT}(Y)$, and this is of course a contradiction. Using the fact that $\tau(\vec{x}) = \vec{y}$, it is evident that g^τ takes an infinite subsequence of $\vec{y}$ to an infinite subsequence of $\tau(\vec{z})$. However, $\vec{y}$ is spaced, and $\tau(\vec{z})$ is converges to w, that is, g^τ takes a spaced sequence to a convergent sequence. Hence $g^\tau \notin \mathrm{EXT}(Y)$. A contradiction. This proves Claim 2.

We prove that $\tau \in \mathrm{EXT}(X,Y)$. Suppose by contradiction that $\vec{x} \subseteq X$ is a convergent sequence and $\tau(\vec{x})$ is not a convergent sequence. By Claim 2, there is a subsequence $\vec{x}^0$ of $\vec{x}$ such that $\tau(\vec{x}^0)$ is convergent. Since $\tau(\vec{x})$ is not convergent, there is a subsequence $\vec{x}^2$ of $\vec{x}$ such that $d(\tau(\vec{x}^2), \tau(\vec{x}^0)) > 0$. By Claim 2, there is a subsequence $\vec{x}^1$ of $\vec{x}^2$ such that $\tau(\vec{x}^1)$ is convergent. But $\lim \tau(\vec{x}^0) \neq \lim \tau(\vec{x}^1)$. This contradicts Claim 1. So

$\tau \in \mathrm{EXT}(X, Y)$. The assumptions on X, Y and τ were symmetric with respect to X and Y. So $\tau \in \mathrm{EXT}^{\pm}(X, Y)$. ■

REMARK 6.13. The requirement that X be locally movable at its multiple boundary, which appears in Theorem 6.12 is stronger than what is really needed in the proof of that theorem. However, the exact assumption needed in that proof is longer and more complicated, so we include it only as a remark. Thus in Theorem 6.12 the assumption that X is locally movable at its multiple boundary can be replaced by the following weaker requirement. The proof remains essentially unchanged.

Let $X \subseteq E$. Then

(1) For every $\vec{x} \subseteq X$ which is convergent to a multiple boundary point and $z \in \mathrm{bd}(X) - \{\lim \vec{x}\}$, there is a subsequence $\vec{x}'$ of $\vec{x}$ and $g \in \mathrm{EXT}(X)$ such that: $g(\vec{x}') \simeq^X \vec{x}'$, $g^{\mathrm{cl}}(\lim \vec{x}') \neq \lim \vec{x}'$ and for some $U \in \mathrm{Nbr}^E(z)$, $g \restriction (U \cap X) = \mathrm{Id}$.

(2) For every $\vec{x} \subseteq X$ which converges to a double boundary point and $U \in \mathrm{Nbr}^E(\lim \vec{x})$ there is a subsequence $\vec{x}'$ of $\vec{x}$ and $g \in \mathrm{EXT}(X)$ such that: $g(\vec{x}') \simeq^X \vec{x}'$, $g^{\mathrm{cl}}(\lim \vec{x})$ $\neq \lim \vec{x}$ and $\mathrm{supp}(g) \subseteq U$. □

The requirement that X be locally movable at its multiple boundary which appears in Definition 6.9(g) cannot be entirely omitted. This is demonstrated by the following trivial example.

EXAMPLE 6.14. *There are regular open subsets $X, Y \subseteq \mathbb{R}^2$ which satisfy clause* 1 *in the definition of $K^{\mathcal{O}}_{\mathrm{BX}}$ such that* $\mathrm{EXT}(X) \cong \mathrm{EXT}(Y)$ *and* $\mathrm{cl}(X) \not\cong \mathrm{cl}(Y)$.

Proof. Let $u \in \mathbb{R}^2$ and $F_0, \ldots, F_3 \subseteq \mathbb{R}^2$ be closed solid triangles such that for any $i \neq j$, $F_i \cap F_j = \{u\}$. For $i = 1, 2, 3$ let $\{D_{i,j} \mid j < i\}$ be a set of pairwise disjoint closed balls such that $D_{i,j} \subseteq \mathrm{int}(F_i)$ for every $j < i$. Let $X = \bigcup_{i<4} \mathrm{int}(F_i) - \bigcup\{D_{i,j} \mid i = 1, 2, 3,\ j < i\}$.

Let $v, w \in \mathbb{R}^2$ and $G_0, \ldots, G_3 \subseteq \mathbb{R}^2$ be closed solid triangles such that $G_0 \cap G_1 = \{v\}$, $G_2 \cap G_3 = \{w\}$ and $G_i \cap G_\ell = \emptyset$ for every $i \in \{0, 1\}$ and $\ell \in \{2, 3\}$. For $i = 1, 2, 3$ let $\{E_{i,j} \mid i = 1, 2, 3,\ j < i\}$ be a set of pairwise disjoint closed balls such that $E_{i,j} \subseteq \mathrm{int}(G_i)$ for every $j < i$. Let $Y = \bigcup_{i<4} \mathrm{int}(G_i) - \bigcup\{E_{i,j} \mid i = 1, 2, 3,\ j < i\}$. Then X and Y are as required in the example. ■

For open subsets of finite-dimensional spaces we have Theorem 5.2 which says that the class of bounded sets which are the closures of open UD.AC subsets of a Euclidean space is faithful. We shall next define another faithful class of spaces which are not required to be UD.AC. This class, denoted by $K^{\mathcal{O}}_{\mathrm{IMX}}$, is defined in 6.16(b). Loosely speaking, we replace the assumption that X is UD.AC by the assumption that the orbit of every $x \in \mathrm{bd}(X)$ under $\mathrm{EXT}(X)$ contains an arc. This gives rise to a rather large class. See Proposition 6.17.

The next example contains finite- and infinite-dimensional sets which belong to $K^{\mathcal{O}}_{\mathrm{IMX}}$ but do not belong to any of the previously defined EXT-determined classes. The three examples are connected. The first example is a subset of $\mathbb{R}^2$ which is not UD.AC. The second set is infinite-dimensional. It is quite similar to the set R^E_{I} defined in 6.8, yet it does not belong to $K^{\mathcal{O}}_{\mathrm{BX}}$. Note the second example is BD.AC, and the first two examples are regular open.

EXAMPLE 6.15. (a) Let $R_2 = \{(r,\theta) \mid \theta \in (\pi,\infty) \text{ and } 1-\frac{1}{\theta-\pi/2} < r < 1-\frac{1}{\theta+\pi/2}\}$ (R_2 is described in polar coordinates). So R_2 is an open spiral strip converging to $S(0,1)$. Note that R_2 is connected, R_2 is not UD.AC and $R_2 \not\in K^{\mathcal{O}}_{\mathrm{BX}}$.

(b) Let $E = \ell_2$ and R^E_1 be as in Example 6.8. So the set R^E_1 is the union of a sequence of pairwise disjoint open rings converging to $S^E(0,1)$. We connect any two consecutive rings by an open tube whose closure is disjoint from the closure of any other ring. The set of tubes is to be spaced. Let $\{e_n\}_{n\in\mathbb{N}}$ be the standard basis of ℓ_2 and $L_n = [(1-\frac{1}{2n+3})e_n, (1-\frac{1}{2n+4})e_n]$. So each L_n connects two consecutive rings in R^E_1. For some $d>0$, $\{L_n \mid n \in \mathbb{N}\}$ is d-spaced. Let $s_n = \frac{1}{2n+4} - \frac{1}{2n+5}$ and $r_n = \min(d/3, s_n)$ and $R_3 = R^E_1 \cup \bigcup_{n\in\mathbb{N}} B(L_n, r_n)$. It follows that R_3 is connected, R_3 is not UD.AC and $R_3 \not\in K^{\mathcal{O}}_{\mathrm{BX}}$. However, R_3 is JN.AC.

(c) Let E be a normed space with dimension >2 and F be a subspace of E with co-dimension 1. Let $R^E_4 = B^E(0,2) - \bar{B}^F(0,1)$. □

DEFINITION 6.16. (a) Let $h : [a,b]\times Z_1 \to Z_2$ and $t \in [a,b]$. We denote by h_t the function $g(z) = h(t,z)$. Let $X \in K^{\mathcal{O}}_{\mathrm{NRM}}$ and $x \in \mathrm{bd}(X)$. We say that x is *isotopically movable* with respect to X if for every $r>0$ there is a continuous function $h : [0,1]\times \mathrm{cl}(X) \to \mathrm{cl}(X)$ such that $h_0 = \mathrm{Id}$, $h_1(x) \neq x$, and for every $t \in [0,1]$, $h_t\restriction X \in \mathrm{EXT}(X)$ and $\mathrm{supp}(h_t) \subseteq B(x,r)$. We say that X is *isotopically movable at its boundary* (*BR.IS.MV*) if every $x \in \mathrm{bd}(X)$ is isotopically movable with respect to X.

(b) Let $K^{\mathcal{O}}_{\mathrm{IMX}}$ be the class of all open subsets X of a normed space such that X is JN.AC and BR.IS.MV. □

The next observation and Proposition 6.2 show that $K^{\mathcal{O}}_{\mathrm{IMX}}$ is a large class. Let E be a normed space and $X \subseteq E\times(0,\infty)$ be open and $Z = \{z \in E \mid \exists a\ ((z,a) \in X)\}$. The *body of revolution* of X is defined as follows:

$$\mathrm{revb}(X) = \{(z,u,v) \mid (z, \sqrt{u^2+v^2}) \in X\}.$$

So $\mathrm{revb}(X)$ is an open subset of $E \times \mathbb{R}^2$. If $\inf(\{a \mid (z,a)) \in X\}) > 0$ for every $z \in Z$, then $\mathrm{revb}(X)$ is called a *hollow body of revolution.* Clearly if $\mathrm{revb}(X)$ is hollow, then $\mathrm{revb}(X) \cong X \times S^1$.

PROPOSITION 6.17. *Let $X, Y \in K^{\mathcal{O}}_{\mathrm{NRM}}$.*

(1) *If Y is BR.IS.MV, then $X\times Y$ is BR.IS.MV.*
(2) *If X and Y are JN.AC, then $X\times Y$ is JN.AC.*
(3) *If $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$, X, Y are bounded, and Y is BR.IS.MV, then $X\times Y \in K^{\mathcal{O}}_{\mathrm{IMX}}$.*
(4) *If X and Y are JN.AC and Y is BR.IS.MV, then $X\times Y \in K^{\mathcal{O}}_{\mathrm{IMX}}$.*
(5) *If $X \subseteq \mathbb{R}^n$, X is bounded and* $\mathrm{revb}(X)$ *is hollow, then* $\mathrm{revb}(X) \in K^{\mathcal{O}}_{\mathrm{IMX}}$.

Proof. The proof is trivial. For (3) and (5) see 6.2(b). ■

REMARK. The class $K^{\mathcal{O}}_{\mathrm{IMX}}$ does not contain any of the classes $K^{\mathcal{O}}_{\mathrm{NMX}}$, $K^{\mathcal{O}}_{\mathrm{BCX}}$ and $K^{\mathcal{O}}_{\mathrm{BX}}$ defined in 6.3 and 6.9(g). Recall that $K^{\mathcal{O}}_{\mathrm{BCX}} \subseteq K^{\mathcal{O}}_{\mathrm{BX}}, K^{\mathcal{O}}_{\mathrm{NMX}}$. Example 6.8 belongs to $K^{\mathcal{O}}_{\mathrm{BCX}}$ but not to $K^{\mathcal{O}}_{\mathrm{IMX}}$. □

THEOREM 6.18. *Suppose that $X, Y \in K^{\mathcal{O}}_{\mathrm{IMX}}$ and $\varphi : \mathrm{EXT}(X) \cong \mathrm{EXT}(Y)$. Then there is $\tau \in \mathrm{EXT}^{\pm}(X,Y)$ which induces φ.*

Proof. By Theorem 2.8(b), there is $\tau \in H(X,Y)$ which induces φ.

Claim 1. *For every $x \in \mathrm{bd}(X)$ there is a sequence $\vec{x}$ converging to x such that $\tau(\vec{x})$ converges to a member of* $\mathrm{bd}(Y)$.

Proof. This claim follows from Lemma 6.5(b) applied to τ^{-1}.

Claim 2. *Let $x \in \mathrm{bd}(X)$ and $\vec{x}, \vec{u} \subseteq X$. Suppose that* $\lim \vec{x} = \lim \vec{u} = x$ *and that* $\tau(\vec{x})$ *and* $\tau(\vec{u})$ *are convergent. Then* $\lim \tau(\vec{x}) = \lim \tau(\vec{u})$.

Proof. Set $\vec{y} = \tau(\vec{x})$, $\vec{v} = \tau(\vec{u})$, $y = \lim \vec{y}$, $v = \lim \vec{v}$, and suppose by contradiction that $y \neq v$. Obviously, $y, v \in \mathrm{bd}(Y)$. Let $r = \|y - v\|/2$. We may assume that $\vec{v} \subseteq B(v,r)$ and that $\vec{y} \cap B(v,r) = \emptyset$. Let $h : [0,1] \times \mathrm{cl}(Y) \to \mathrm{cl}(Y)$ be an isotopy as ensured by the fact that v is isotopically movable with respect to Y, and such that for every $t \in [0,1]$, $\mathrm{supp}(h_t) \subseteq B(v,r)$.

For every $t \in [0,1]$ let $u_{n,t} = \tau^{-1}(h(t, v_n))$. We first prove the following fact. $(*)$ For every $t \in [0,1]$, $\lim_{n\to\infty} u_{n,t} = x$. Let $t \in [0,1]$. Let $\bar{h} = h_t \restriction Y$ and $\bar{g} = \bar{h}^{\tau^{-1}}$. Then $\bar{g} \in \mathrm{EXT}(X)$. Also $\bar{g} \restriction \vec{x} = \mathrm{Id}$. So $\bar{g}^{\mathrm{cl}}(x) = x$. Hence $\lim_{n\to\infty} u_{n,t} = \lim_{n\to\infty} \bar{g}(u_n) = \bar{g}(\lim_{n\to\infty} u_n) = \bar{g}(x) = x$. So $(*)$ is proved.

Let $L_n = h([0,1] \times \{v_n\})$ and $K_n = \tau^{-1}(L_n)$. We prove that $\lim_{n\to\infty} K_n = x$. Suppose by contradiction that this is not true. Then there are $d > 0$, $\vec{t} \subseteq [0,1]$ and a 1-1 sequence $\{n_i \mid i \in \mathbb{N}\}$ such that $d(x, u_{n_i,t_i}) \geq d$ for every $i \in \mathbb{N}$. We may assume that $\vec{t}$ is convergent. Set $t^* = \lim \vec{t}$. Let I_i be the closed interval whose endpoints are t_i and t^* and $J_i = h(I_i \times \{v_{n_i}\})$. Then $\lim_{i\to\infty} J_i = h(t^*, v)$. Since for every $t \in [0,1]$, $h_t \restriction Y \in \mathrm{EXT}(Y)$ and $v \in \mathrm{bd}(Y)$, it follows that $h(t^*, v) \in \mathrm{bd}(Y)$. The fact $\vec{v} \subseteq Y$ implies that $J_i \subseteq Y$, and hence $h(t^*, v) \notin J_i$ for every $i \in \mathbb{N}$. Since J_i is compact, $d(J_i, h(t^*, v)) > 0$. We may thus replace $\{n_i\}_{i\in\mathbb{N}}$ by a subsequence and deduce that $\max(\{d(z, h(t^*, v)) \mid z \in J_{i+1}\}) < d(J_i, h(t^*, v))$ for every $i \in \mathbb{N}$. There is a sequence $\{V_i\}_{i\in\mathbb{N}}$ of open sets such that for any distinct $i, j \in \mathbb{N}$, $J_i \subseteq V_i \subseteq \mathrm{cl}(V_i) \subseteq Y \cap B(v,r)$, $V_i \cap V_j = \emptyset$ and $\lim_{i\to\infty} V_i = \lim_{i\to\infty} J_i$. From the fact that J_i connects $h(v_{n_i}, t^*)$ and $h(v_{n_i}, t_i)$, it follows that there is $h_i \in \mathrm{UC}(Y) \lfloor V_i \rfloor$ such that $h_i(h(v_{n_i}, t^*)) = h(v_{n_i}, t_i)$. Let $\hat{h} = \circ_{i\in\mathbb{N}} h_i$. Then by Proposition 4.5, $\hat{h} \in \mathrm{UC}_0(Y) \subseteq \mathrm{EXT}(Y)$. Clearly, $\mathrm{supp}(\hat{h}) \subseteq B(v,r)$ and so $\hat{h} \restriction \vec{y} = \mathrm{Id}$. Let $\hat{g} = \hat{h}^{\tau^{-1}}$. So $\hat{g} \in \mathrm{EXT}(X)$. Since $\hat{h} \restriction \vec{y} = \mathrm{Id}$, it follows that $\hat{g} \restriction \vec{x} = \mathrm{Id}$ and hence $\hat{g}^{\mathrm{cl}}(x) = x$. Clearly, for every i, $\hat{g}(u_{n_i,t^*}) = u_{n_i,t_i}$, and from $(*)$ it follows that $\lim_{i\to\infty} u_{n_i,t^*} = x$. So

$$\lim_{i\to\infty} u_{n_i,t_i} = \lim_{i\to\infty} \hat{g}(u_{n_i,t^*}) = \hat{g}^{\mathrm{cl}}(\lim_{i\to\infty} u_{n_i,t^*}) = \hat{g}^{\mathrm{cl}}(x) = x.$$

This contradicts the fact that $d(x, u_{n_i,t_i}) \geq d$. So $\lim_{n\to\infty} K_n = x$.

There is an infinite set $\sigma \subseteq \mathbb{N}$ such that $K_i \cap K_j = \emptyset$ for any distinct $i, j \in \sigma$. Let $\{U_i \mid i \in \sigma\}$ be such that $K_i \subseteq U_i \subseteq X$, U_i is open, $U_i \cap U_j = \emptyset$ for any $i \neq j$ and $\lim_{i\in\sigma} U_i = x$. Let $\eta \subseteq \sigma$ be such that η and $\sigma - \eta$ are infinite. For every $i \in \eta$ let $g_i \in \mathrm{UC}(X) \lfloor U_i \rfloor$ be such that $g_i(u_i) = u_{i,1}$. Let $\bar{g} = \circ_{i\in\eta} g_i$ and $\bar{h} = \bar{g}^{\tau}$. By Proposition 4.5, $\bar{g} \in \mathrm{UC}_0(X) \subseteq \mathrm{EXT}(X)$, hence it follows that $\bar{h} \in \mathrm{EXT}(Y)$.

For every $i \in \eta$, $\bar{h}(v_i) = h(v_i, 1)$, so $\lim_{i\in\eta} \bar{h}(v_i) = h(v, 1)$. For every $i \in \sigma - \eta$, $\bar{h}(v_i) = v_i$, so $\lim_{i\in\sigma-\eta} \bar{h}(v_i) = v$. Recall that $h(v,1) \neq v$. Also, $\lim_{i\to\infty} v_i = v$. So $\vec{v}$ is convergent and $\bar{h}(\vec{v})$ is not convergent. Hence $\bar{h} \notin \mathrm{EXT}(Y)$. A contradiction, so Claim 2 is proved.

Suppose by contradiction that $x \in \mathrm{bd}(X)$ and $x \notin \mathrm{Dom}(\tau^{\mathrm{cl}})$. By Claim 1, there is a sequence $\vec{x} \subseteq X$ such that $\lim \vec{x} = x$ and $\tau(\vec{x})$ is convergent. Set $y = \lim \tau(\vec{x})$. There are a 1-1 sequence $\vec{u} \subseteq X$ and $d > 0$ such that $\lim \vec{u} = x$ and $d(\tau(\vec{u}), y) \geq d$. Define $\vec{v} = \tau(\vec{u})$. By Claim 2, $\vec{v}$ does not have a convergent subsequence. That is, $\vec{v}$ is completely discrete. Since Y is JN.AC, there is a subsequence $\vec{w}$ of $\vec{v}$ such that $\vec{w}$ has a joining system. Let $\langle \vec{w}, w^*, \{L_n \mid n \in \mathbb{N}\}, \vec{w}' \rangle$ be a joining system for $\vec{w}$. We may assume that $w^* \notin \mathrm{Rng}(\vec{w})$.

We show that it can be assumed that $w^* \neq y$. Suppose that $w^* = y$. Let $r = d(\vec{w}, y)$. Since Y is BR.IS.MV and $y \in \mathrm{bd}(Y)$, there is $h \in \mathrm{EXT}(Y)$ such that $\mathrm{supp}(h) \subseteq B(y, r)$ and $h^{\mathrm{cl}}(y) \neq y$. So $h \restriction \vec{w} = \mathrm{Id}$. It follows that $\langle \vec{w}, h^{\mathrm{cl}}(y), \{h(L_n) \mid n \in \mathbb{N}\}, h(\vec{w}') \rangle$ is a joining system for $\vec{w}$. So we may assume that $w^* \neq y$.

Recall that Y is JN.AC. So we may apply Lemma 6.5(b) to τ^{-1}. Recall also that $\lim \tau^{-1}(\vec{w}) = \lim \tau^{-1}(\vec{v}) = x$. Hence there is $\vec{z} \subseteq Y$ such that $\lim \vec{z} = w^*$ and $\lim \tau^{-1}(\vec{z}) = x$. We now have two sequences: $\vec{x}$ and $\tau^{-1}(\vec{z})$, both converge to x, and $\tau(\vec{x})$ and $\tau(\tau^{-1}(\vec{z}))$ are convergent, but not to the same point. This contradicts Claim 2, so $x \in \mathrm{Dom}(\tau^{\mathrm{cl}})$.

We have shown that $\tau \in \mathrm{EXT}(X, Y)$, and an identical argument shows that $\tau^{-1} \in \mathrm{EXT}(Y, X)$. That is, $\tau \in \mathrm{EXT}^{\pm}(X, Y)$. ■

6.3. Completely locally uniformly continuous homeomorphism groups. Having obtained the results about $\mathrm{EXT}(X)$ and $\mathrm{LUC}(X)$, only little extra work is needed to prove CMP.LUC-determinedness. See Definition 5.3(f). This faithfulness result will complete the picture of groups of type $H_{\Gamma}^{\mathrm{CMP.LC}}(X)$ discussed in Chapters 8–12.

The following is a strengthening of property BR.LC.AC.

DEFINITION 6.19. X is *locally uniformly-in-diameter arcwise connected* (*LC.UD.AC*) if for every $x \in \mathrm{bd}(X)$ there is $U \in \mathrm{Nbr}(x)$ such that for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $u, v \in U$: if $d(u, v) < \delta$, then there is an arc $L \subseteq X$ connecting u and v such that $\mathrm{diam}(L) < \varepsilon$. □

THEOREM 6.20. (a) *Let $X, Y \in K_{\mathrm{NRM}}^{\mathcal{O}}$. Suppose that X and Y are LC.UD.AC and JN.AC. Let $\varphi : \mathrm{CMP.LUC}(X) \cong \mathrm{CMP.LUC}(Y)$. Then there is $\tau \in \mathrm{CMP.LUC}^{\pm}(X, Y)$ which induces φ.*

(b) *Suppose that X is LC.UD.AC and Y is JN.AC, and let $\tau \in H(X, Y)$ be such that $(\mathrm{UC}_0(X))^{\tau} \subseteq \mathrm{CMP.LUC}(Y)$ and $(\mathrm{LUC}_{01}(Y))^{\tau^{-1}} \subseteq \mathrm{CMP.LUC}(X)$. Then $\tau \in \mathrm{CMP.LUC}(X, Y)$.*

Proof. We shall see that (b) implies (a). So we start by proving (b).

(b) It is trivial that X is BR.LC.AC. We first show that $\tau \in \mathrm{EXT}(X, Y)$. By definition, $\mathrm{CMP.LUC}(X) \subseteq \mathrm{EXT}(X)$. So $(\mathrm{UC}_0(X))^{\tau} \subseteq \mathrm{EXT}(Y)$ and $(\mathrm{LUC}_{01}(Y))^{\tau^{-1}} \subseteq \mathrm{EXT}(X)$. By Corollary 6.6(a), $\tau \in \mathrm{EXT}(X, Y)$.

We show that $\tau \in \mathrm{LUC}(X,Y)$. Let $\mathcal{S}$ be the set of BPD-subsets of X. Then $\mathrm{UC}(X,\mathcal{S}) \subseteq \mathrm{UC}_0(X)$ and $\mathrm{CMP.LUC}(Y) \subseteq \mathrm{LUC}(Y)$. So $(\mathrm{UC}(X,\mathcal{S}))^\tau \subseteq \mathrm{LUC}(Y)$. By Theorem 4.8(b), $\tau \in \mathrm{LUC}^{\pm}(X,Y)$.

Let $x^* \in \mathrm{bd}(X)$. We show that there is $U \in \mathrm{Nbr}(x^*)$ such that $\tau\restriction(U \cap X)$ is UC. The proof is very much a repetition of the proof of part 1 of Theorem 4.8(c).

Suppose by contradiction that for every $U \in \mathrm{Nbr}^X(x^*)$, $\tau\restriction U$ is not UC. The following claim is an easy consequence of the fact that $\tau\restriction B(x^*,r) \cap X$ is not UC. Its proof is left to the reader.

Claim 1. *For every $r > 0$ there are sequences $\vec{x}, \vec{y}$ and $d, e > 0$ such that:*

(1) $\mathrm{Rng}(\vec{x}) \cup \mathrm{Rng}(\vec{y}) \subseteq B^X(x^*, r/2)$;
(2) $\lim_{n\to\infty} \|x_n - y_n\| = 0$;
(3) *either* (i) *for any distinct $m,n \in \mathbb{N}$, $d(\{x_m,y_m\},\{x_n,y_n\}) \geq e$, or* (ii) *$\vec{x}$ is a Cauchy sequence*;
(4) $d(\mathrm{Rng}(\vec{x}) \cup \mathrm{Rng}(\vec{y}), x^*) > e$;
(5) *for every $n \in \mathbb{N}$, $\|\tau(x_n) - \tau(y_n)\| \geq d$.*

Let $U \in \mathrm{Nbr}(x^*)$ be as ensured by the LC.UD.AC-ness of X. There is $a > 0$ and a function $\eta : (0,a] \to \mathbb{R}$ such that $\lim_{t\to 0} \eta(t) = 0$ and for every $u, v \in U \cap X$, if $\|u-v\| \leq t$, then there is an arc $L \subseteq X$ connecting u and v sucu that $\mathrm{diam}(L) \leq \eta(t)$.

Let $e_{-1} > 0$ be such that $B^E(x^*, e_{-1}) \subseteq U$. It is easy to define by induction on $i \in \mathbb{N}$, $r_i > 0$, sequences $\vec{x}^{\,i}, \vec{y}^{\,i}$ and $d_i, e_i > 0$ such that: (i) $\vec{x}^{\,i}, \vec{y}^{\,i}, d_i, e_i$ satisfy the conclusion of Claim 1 for r_i; and (ii) for every $i \in \mathbb{N}$, $r_i = e_{i-1}/8$. Clearly $e_{i+1} \leq e_i/4$. By deleting initial segments from the $\vec{x}^{\,i}$'s and $\vec{y}^{\,i}$'s, we may further assume that for every $i,n \in \mathbb{N}$, $\eta(\|x^i_n - y^i_n\|) < e_i/8$. We may further assume that either for every $i \in \mathbb{N}$ clause (3)(i) of Claim 1 holds, or for every $i \in \mathbb{N}$ clause (3)(ii) of Claim 1 holds.

Case 1: Clause (3)(i) of Claim 1 holds. Let $\{\langle i(k), n(k)\rangle \mid k \in \mathbb{N}\}$ be a 1-1 enumeration of $\mathbb{N}^2$. Then $\lim_{k\to\infty} \|x^{i(k)}_{n(k)} - y^{i(k)}_{n(k)}\| = 0$. Set $u_k = x^{i(k)}_{n(k)}$, $v_k = y^{i(k)}_{n(k)}$ and let $L_k \subseteq X$ be an arc connecting u_k and v_k such that $\mathrm{diam}(L_k) \leq \eta(\|u_k - v_k\|)$. Let $B_k = B(L_k, e_{i(k)+1}/4)$. Then

$$\begin{aligned}\mathrm{diam}(B_k) &\leq \mathrm{diam}(L_k) + e_{i(k)+1}/2 \leq \eta(\|u_k - v_k\|) + e_{i(k)+1}/2\\ &\leq e_{i(k)}/8 + e_{i(k)+1}/2 \leq e_{i(k)}/4.\end{aligned}$$

It follows that if $i(k) = i(\ell)$, then $d(B_k, B_\ell) \geq e_{i(k)}/2$. Suppose that $i(k) < i(\ell)$. Then $\|u_\ell - u_k\| \geq 7e_{i_\ell}/8$, $\mathrm{diam}(B_k) \leq e_{i(k)}/4 \leq e_{i(\ell)}/4$ and $\mathrm{diam}(B_\ell) \leq e_{i(\ell)}/4$. So $d(B_k, B_\ell) \geq 3e_{i_\ell}/8$. Obviously, $\lim_{k\to\infty} \mathrm{diam}(B_k) = 0$. Let $w_k \in L_k - \{u_k\}$ be such that $\|\tau(w_k) - \tau(u_k)\| < \frac{1}{k+1}$. By Lemma 2.14(d), there is $h_k \in \mathrm{LIP}(X)$ such that $\mathrm{supp}(h_k) \subseteq B_k$, $h_k(u_k) = u_k$ and $h_k(w_k) = v_k$. By Proposition 4.5, $h := \circ_{k\in\mathbb{N}} h_k \in \mathrm{UC}(X)$ and indeed $h \in \mathrm{UC}_0(X)$.

Let us see that for every $V \in \mathrm{Nbr}(\tau^{\mathrm{cl}}(x^*))$, $h^\tau\restriction(V \cap Y)$ is not UC. For $i \in \mathbb{N}$ define $\sigma_i = \{k \mid i(k) = i\}$. So if $k \in \sigma_i$, then $L_k \subseteq B(x^*, \eta(2r_k))$. Since $\lim_{i\to\infty} \eta(2r_i) = 0$, and since τ^{cl} is continuous at x^*, there is i such that for every $k \in \sigma_i$, $\tau(L_k) \subseteq V$.

For every $k \in \sigma_i$, $\tau(u_i), \tau(w_i) \in V$. Clearly, $\lim_{k\in\sigma_i} \|\tau(u_k) - \tau(w_k)\| = 0$. However, for every $k \in \sigma_i$, $\|h^\tau(\tau(u_k)) - h^\tau(\tau(w_k))\| = \|\tau(u_i)) - \tau(v_i))\| \geq d_i$. So $h^\tau\restriction(V \cap Y)$ is not UC. Hence $h^\tau \notin \mathrm{CMP.LUC}(Y)$ even though $h \in \mathrm{UC}_0(X)$, a contradiction.

Case 2: Clause (3)(ii) of Claim 1 holds. Let $\bar{z}_i = \lim \vec{x}^{\,i}$. Clearly, $\bar{z}_i \in B^{\bar{E}}(x^*, r_i) - B^{\bar{E}}(x^*, e_i)$. So $\{\bar{z}_i \mid i \in \mathbb{N}\}$ is 1-1 and $\lim_{i\to\infty} \bar{z}_i = x^*$. Also, $\bar{z}_i \in \bar{E} - E$. This is so, because if $\bar{z}_i \in E$, then either $\bar{z}_i \in X$ and τ is not continuous at $\bar{z}_i$, or $\bar{z}_i \in \mathrm{bd}^E(X)$ and $\bar{z}_i \notin \mathrm{Dom}(\tau^{\mathrm{cl}})$. Both situations are impossible. For every i and n let $L_{i,n} \subseteq X$ be an arc connecting x^i_n and y^i_n such that $\mathrm{diam}(L_{i,n}) \le \eta(\|x^i_n - y^i_n\|)$. Note that for every i, $\lim_{n\to\infty} L_{i,n} = \bar{z}_i$. From the facts $\bar{z}_i \notin E$ and $L_{i,n} \subseteq E$ we conclude that $d(\bar{z}_i, L_{i,n}) > 0$.

It follows easily that there is a sequence $\{\langle i(k), n(k)\rangle \mid k \in \mathbb{N}\}$ such that

(1) for every $i \in \mathbb{N}$, $\{k \mid i(k) = i\}$ is infinite,

(2) for every $k \in \mathbb{N}$, $c_k := d(L_{i(k),n(k)}, \bigcup_{m\ne k} L_{i(m),n(m)}) > 0$.

It is also clear from the construction that

(3) $\lim_{k\to\infty} \mathrm{diam}(L_{i(k),n(k)}) = 0$.

Set $L_k = L_{i(k),n(k)}$, $u_k = x^{i(k)}_{n(k)}$, $v_k = y^{i(k)}_{n(k)}$ and $B_k = B(L_k, c_k/3)$. Clearly, for every $\ell \ne k$, $d(B_\ell, B_k) \ge c_k$ and $\lim_{k\to\infty} \mathrm{diam}(B_k) = 0$. From this point on the proof proceeds exactly as in Case 1. So in Case 2 too, a contradiction is reached.

It follows that there is $U \in \mathrm{Nbr}(x^*)$ such that $\tau\restriction(U \cap X)$ is UC, and this implies that τ^{cl} is UC at x^*. Recall that we have already shown before that $\tau \in \mathrm{EXT}(X,Y)$ and that $\tau \in \mathrm{LUC}(X,Y)$. So $\tau \in \mathrm{CMP.LUC}(X,Y)$.

(a) Let $\varphi : \mathrm{CMP.LUC}(X) \cong \mathrm{CMP.LUC}(Y)$. Clearly, $\mathrm{LIP}^{\mathrm{LC}}(X) \le \mathrm{CMP.LUC}(X) \le H(X)$, and the same holds for Y. So by Theorem 2.8(a), there is $\tau \in H(X,Y)$ such that τ induces φ. Hence $(\mathrm{CMP.LUC}(X))^\tau = \mathrm{CMP.LUC}(Y)$. Obviously, $\mathrm{UC}_0(X) \subseteq \mathrm{CMP.LUC}(X)$ and $\mathrm{LUC}_{01}(Y) \subseteq \mathrm{CMP.LUC}(Y)$. So part (b) of this lemma can be applied. Hence $\tau \in \mathrm{CMP.LUC}(X,Y)$. Similarly, $\tau^{-1} \in \mathrm{CMP.LUC}(Y,X)$. That is, $\tau \in \mathrm{CMP.LUC}^{\pm}(X,Y)$. ∎

6.4. The reconstruction of $\mathrm{cl}(X)$ **from** $H(\mathrm{cl}(X))$. The next two theorems 6.22 and 6.24 deal with the reconstruction of F from $H(F)$, when F is the closure of an open subset of a normed space. The sets to which these theorems apply may have rather complicated boundaries. It is not true though that for any F, K which are the closures of open subsets of a normed space, $H(F) \cong H(K)$ implies that $F \cong K$. See Example 5.8.

Recall that if $A \subseteq E$ has a nonempty interior, then $\mathrm{ENI}(A) := \{h(x) \mid x \in \mathrm{int}^E(A)$ and $h \in H(A)\}$. For $f \in \mathrm{UC}_0(X)$, define $f^{\mathrm{eni}} = f^{\mathrm{cl}}\restriction\mathrm{ENI}(\mathrm{cl}(X))$. Hence $f^{\mathrm{eni}} \in H(\mathrm{ENI}(\mathrm{cl}(X)))$. Also define $\mathrm{UC}^{\mathrm{eni}}_0(X) = \{f^{\mathrm{eni}} \mid f \in \mathrm{UC}_0(X)\}$.

Parts (a) and (b) of the next proposition are analogous to Proposition 6.4 and Lemma 6.5(a). The proofs of (a) and (b) are essentially identical to the proofs of their counterparts, so they are omitted. Part (c) is analogous to Lemma 6.5(b), but (c) is stated for η^{-1} rather than for η.

Proposition 6.21. (a) *Let X be BR.LC.AC and $\tau \in H(\mathrm{ENI}(\mathrm{cl}(X)), \mathrm{ENI}(\mathrm{cl}(Y)))$. Assume that $(\mathrm{UC}^{\mathrm{eni}}_0(X))^\tau \subseteq \mathrm{EXT}(\mathrm{ENI}(\mathrm{cl}(Y)))$. Let $x \in \mathrm{bd}(X) - \mathrm{ENI}(\mathrm{cl}(X))$, $y \in \mathrm{bd}(Y)$ and $\vec{x} \subseteq X$ be such that $\lim \vec{x} = x$ and $\lim \tau(\vec{x}) = y$. Then $(\tau\restriction X) \cup \{\langle x, y\rangle\}$ is continuous.*

(b) *Let X be JN.AC and $\tau \in H(\mathrm{ENI}(\mathrm{cl}(X)), \mathrm{ENI}(\mathrm{cl}(Y)))$ be such that $(\mathrm{LUC}_{01}(X))^\tau \subseteq H(\mathrm{ENI}(\mathrm{cl}(Y)))$. Let $y \in \mathrm{bd}(Y) - \mathrm{ENI}(\mathrm{cl}(Y))$. Suppose that $\vec{x} \subseteq X$ is completely discrete,*

$\langle \vec{x}, x^*, \{L_n \mid n \in \mathbb{N}\}, \vec{x}' \rangle$ is a joining system for $\vec{x}$ and $\lim \tau(\vec{x}) = y$. Then there is a sequence $\vec{u} \subseteq X$ such that $\lim \vec{u} = x^*$ and $\lim \tau(\vec{u}) = y$.

(c) *Let $X, Y \in K^{\mathcal{O}}_{\mathrm{NRM}}$. Assume that Y is JN.AC. Set $K = \mathrm{cl}(X)$ and $M = \mathrm{cl}(Y)$, and let $\eta \in H(\mathrm{ENI}(K), \mathrm{ENI}(M))$ be such that for every $h \in H(M)$, $((h{\restriction}\mathrm{ENI}(M))^{\eta^{-1}})^{\mathrm{cl}} \in H(K)$. Then for every $x \in K - \mathrm{ENI}(K)$ there is a sequence $\vec{x} \subseteq X$ converging to x such that $\eta(\vec{x}) \subseteq Y$, and $\eta(\vec{x})$ is convergent in M.*

Proof. (c) Let $x \in K - \mathrm{ENI}(K)$. Let $\vec{x}' \subseteq X$ be a sequence converging to x. For every $n \in \mathbb{N}$ let $r_n = \min(\delta(x'_n), d(x'_n, x))$. So $B^E(x'_n, r_n)$ is a nonempty open subset of $\mathrm{ENI}(K)$. Clearly, $\mathrm{bd}(Y) \cap \mathrm{ENI}(M)$ is nowhere dense in $\mathrm{ENI}(M)$. So there is $x_n \in B^E(x'_n, r_n)$ such that $\eta(x_n) \notin \mathrm{bd}(Y) \cap \mathrm{ENI}(M)$. That is, $\eta(x_n) \in Y$. So $\vec{x} \subseteq X$, $\lim \vec{x} = x$ and $\eta(\vec{x}) \subseteq Y$.

Define $\vec{y} = \eta(\vec{x})$. Suppose that $\vec{y}$ has a subsequence $\vec{y}'$ such that $\vec{y}'$ is convergent in $\mathrm{cl}(Y)$. Then $\eta^{-1}(\vec{y}')$ is as required in the proposition. Suppose that such a $\vec{y}'$ does not exist. Hence $\vec{y}$ is completely discrete.

Let $\langle \vec{y}, y^*, \{L_n \mid n \in \mathbb{N}\}, \vec{y}' \rangle$ be a joining system for $\vec{y}$. By 6.21(b) applied to $\vec{y}$ and η^{-1}, there is $\vec{v} \subseteq Y$ such that $\lim \vec{v} = y^*$ and $\lim \eta^{-1}(\vec{v}) = x$. It is obvious that $y^* \in \mathrm{bd}(Y) - \mathrm{ENI}(\mathrm{cl}(Y))$.

As at the beginning of the proof, there is a sequence $\vec{v}' \subseteq Y$ such that $\lim \vec{v}' = y^*$, $\eta^{-1}(\vec{v}') \subseteq X$ and $\lim \eta^{-1}(\vec{v}') = \lim \eta^{-1}(\vec{v}) = x$. So $\eta^{-1}(\vec{v}')$ is as required. ■

The following theorem is analogous to Theorem 6.3(b). The proofs are essentially the same.

THEOREM 6.22. *Let $X, Y \in K^{\mathcal{O}}_{\mathrm{NMX}}$* (see 6.3(b)). *If $\varphi : H(\mathrm{cl}(X)) \cong H(\mathrm{cl}(Y))$, then there is $\tau : \mathrm{cl}(X) \cong \mathrm{cl}(Y)$ such that τ induces φ.*

Proof. Let $K = \mathrm{cl}(X)$ and $M = \mathrm{cl}(Y)$. From Theorem 2.30(c) it follows that there is $\eta \in H(\mathrm{ENI}(K), \mathrm{ENI}(M))$ which induces φ.

For every $x \in \mathrm{bd}(X) - \mathrm{ENI}(\mathrm{cl}(X))$ let $\vec{x} \subseteq X$ be such that $\lim \vec{x} = x$ and $\eta(\vec{x})$ is convergent in M. The existence of $\vec{x}$ is ensured by Proposition 6.21(c). Let $y_x = \lim \eta(\vec{x})$. Since $\mathrm{Rng}(\eta) \supseteq Y$, $y_x \in \mathrm{bd}(Y)$. Since η induces φ, for every $g \in H(K)$, $((g{\restriction}\mathrm{ENI}(K))^{\eta})^{\mathrm{cl}} \in \mathrm{EXT}(\mathrm{ENI}(M))$. In particular, $(\mathrm{UC}^{\mathrm{eni}}_0(X))^{\eta} \subseteq \mathrm{EXT}(\mathrm{ENI}(M))$. Hence by Proposition 6.21(a), $\eta{\restriction}X \cup \{\langle x, y_x \rangle\}$ is continuous. Also, for every $x \in \mathrm{bd}(X) \cap \mathrm{ENI}(\mathrm{cl}(X))$, $\eta{\restriction}X \cup \{\langle x, \eta(x) \rangle\}$ is continuous. We thus have

(1) for every $x \in \mathrm{bd}(X) - \mathrm{ENI}(\mathrm{cl}(X))$, $\eta{\restriction}X \cup \{\langle x, y_x \rangle\}$ is continuous,
(2) for every $x \in \mathrm{bd}(X) \cap \mathrm{ENI}(\mathrm{cl}(X))$, $\eta{\restriction}X \cup \{\langle x, \eta(x) \rangle\}$ is continuous.

So by Proposition 4.7(a), $\eta \cup \{\langle x, y_x \rangle \mid x \in \mathrm{bd}(X) - \mathrm{ENI}(\mathrm{cl}(X))\}$ is continuous. So η can be extended to a continuous function τ from $\mathrm{cl}(X)$ to $\mathrm{cl}(Y)$.

Similarly, η^{-1} can be extended to a continuous function ϱ from $\mathrm{cl}(Y)$ to $\mathrm{cl}(X)$. It follows easily that τ is 1-1 and that $\tau^{-1} = \varrho$. So $\tau \in H(\mathrm{cl}(X), \mathrm{cl}(Y))$. Since η induces φ and $\mathrm{Dom}(\eta)$ is dense in $\mathrm{Dom}(\tau)$, it follows that τ induces φ. ■

PROPOSITION 6.23. (a) *Let $X \in K^{\mathcal{O}}_{\mathrm{NRM}}$, $K = \mathrm{cl}(X)$, $U \subseteq \mathrm{ENI}(K)$ be open in K, $L \subseteq U$ be an arc and x, y be the endpoints of L. Then there is $h \in H(K)\lfloor U \rfloor$ such that $h(x) = y$.*

(b) *Let Z be a topological space $z \in Z$ and $\{h_i \mid i \in \mathbb{N}\} \subseteq H(Z)$ be such that for any $i \neq j$, $\mathrm{supp}(h_i) \cap \mathrm{supp}(h_j) = \emptyset$ and $\lim_{i \to \infty} \mathrm{supp}(h_i) = z$. Then $\circ_{i \in \mathbb{N}} h_i \in H(Z)$.*

Proof. (a) Let $\gamma : [0,1] \to L$ be a parametrization of L such that $\gamma(0) = x$ and $\gamma(1) = y$. There are $n \in \mathbb{N}$, $\{U_i \mid i < n\}$ and $0 = t_0 < \cdots < t_n = 1$ such that for every $i < n$: U_i is open in K, U_i is homeomorphic to an open ball of a normed space, $U_i \subseteq U$ and $\gamma([t_i, t_{i+1}]) \subseteq U_i$. So for every $i < n$ there is $h_i \in H(K)\lfloor U_i \rfloor$ such that $h_i(z_i) = z_{i+1}$. Clearly, $h_{n-1} \circ \cdots \circ h_0$ is as required.

(b) The proof is trivial. ∎

The following theorem is analogous to Theorem 6.18. The proofs are essentially the same.

THEOREM 6.24. *Let $X, Y \in K^{\mathcal{O}}_{\mathrm{IMX}}$ and $\varphi : H(\mathrm{cl}(X)) \cong H(\mathrm{cl}(Y))$. Then there is $\tau \in H(\mathrm{cl}(X), \mathrm{cl}(Y)))$ which induces φ.*

Proof. Set $K = \mathrm{cl}(X)$ and $M = \mathrm{cl}(Y)$. Then by Theorem 2.30(c), there is $\eta \in H(\mathrm{ENI}(K), \mathrm{ENI}(M))$ which induces φ. So for every $g \in H(K)$, $((g{\restriction}\mathrm{ENI}(K))^{\eta})^{\mathrm{cl}} = \varphi(g) \in H(M)$. We shall prove that $\eta^{\mathrm{cl}} \in H(K, M)$.

CLAIM 1. *Let $x \in K - \mathrm{ENI}(K)$ and $\vec{x}, \vec{u} \subseteq X$. Suppose that $\lim \vec{x} = \lim \vec{u} = x$ and that $\eta(\vec{x})$ and $\eta(\vec{u})$ are convergent in M. Then $\lim \eta(\vec{x}) = \lim \eta(\vec{u})$.*

Proof. Let $\vec{y} = \eta(\vec{x})$, $\vec{v} = \eta(\vec{u})$, $y = \lim \vec{y}$, $v = \lim \vec{v}$, and suppose by contradiction that $y \neq v$. Obviously, $y, v \in \mathrm{bd}(Y)$. Let $r = \|y - v\|/2$. We may assume that $\vec{v} \subseteq B(v, r)$ and that $\vec{y} \cap B(v, r) = \emptyset$. Let $h : [0,1] \times \mathrm{cl}(Y) \to \mathrm{cl}(Y)$ be an isotopy as ensured by the fact that v is isotopically movable with respect to Y, and such that for every $t \in [0,1]$, $\mathrm{supp}(h_t) \subseteq B(v, r)$.

For every $t \in [0,1]$ let $u_{n,t} = \eta^{-1}(h(t, v_n))$. We prove the following fact. $(*)$ For every $t \in [0,1]$, $\lim_{n\to\infty} u_{n,t} = x$. Let $t \in [0,1]$. Let $\bar{h} = h_t{\restriction}\mathrm{ENI}(M)$ and $\bar{g} = \bar{h}^{\eta^{-1}}$. Then $\bar{g} \in \mathrm{EXT}(\mathrm{ENI}(K))$. Clearly, $\bar{g}{\restriction}\vec{x} = \mathrm{Id}$ and so $\bar{g}^{\mathrm{cl}}(x) = x$. Hence $\lim_{n\to\infty} u_{n,t} = \lim_{n\to\infty} \bar{g}(u_n) = \bar{g}(\lim_{n\to\infty} u_n) = \bar{g}(x) = x$. So $(*)$ is proved.

Let $L_n = h([0,1] \times \{v_n\})$ and $K_n = \eta^{-1}(L_n)$. We prove that $\lim_{n\to\infty} K_n = x$. Suppose by contradiction that this is not true. Then there are $d > 0$, $\vec{t} \subseteq [0,1]$ and a 1-1 sequence $\{n_i \mid i \in \mathbb{N}\}$ such that for every $i \in \mathbb{N}$, $d(x, u_{n_i,t_i}) \geq d$. We may assume that $\vec{t}$ is convergent. Let $t^* = \lim \vec{t}$. Let I_i be the closed interval whose endpoints are t_i and t^* and $J_i = h(I_i \times \{v_{n_i}\})$. Then $\lim_{i\to\infty} J_i = h(t^*, v)$. Since for every $t \in [0,1]$, $h_t{\restriction}Y \in \mathrm{EXT}(Y)$ and $v \in \mathrm{bd}(Y)$, it follows that $h(t^*, v) \in \mathrm{bd}(Y)$. The fact that $v_{n_i} \in Y$ implies that $J_i \subseteq Y$. Hence for every $i \in \mathbb{N}$, $h(t^*, v) \notin J_i$. We may thus assume that for any $i \neq j$, $J_i \cap J_j = \emptyset$.

There is a sequence $\{V_i\}_{i\in\mathbb{N}}$ of pairwise disjoint open sets such that for every $i \in \mathbb{N}$, $J_i \subseteq V_i \subseteq \mathrm{cl}(V_i) \subseteq Y \cap B(v, r)$ and $\lim_{i\to\infty} V_i = h(t^*, v)$. Let $h_i \in \mathrm{UC}(Y)\lfloor V_i \rfloor$ be such that $h_i(h(v_{n_i}, t^*)) = h(v_{n_i}, t_i)$ and $\tilde{h} = \circ_{i\in\mathbb{N}}\, h_i$. Then $\tilde{h} \in \mathrm{UC}_0(Y)$. Hence $\hat{h} := \tilde{h}^{\mathrm{eni}} \in \mathrm{EXT}(\mathrm{ENI}(M))$. Let $\hat{g} = \hat{h}^{\eta^{-1}}$. So $\hat{g} \in \mathrm{EXT}(\mathrm{ENI}(K))$. Clearly, $\hat{g}{\restriction}\vec{x} = \mathrm{Id}$ and hence $\hat{g}^{\mathrm{cl}}(x) = x$. Also, for every $i \in \mathbb{N}$, $\hat{g}(u_{n_i,t^*}) = u_{n_i,t_i}$. It follows from $(*)$ that $\lim_{i\to\infty} u_{n_i,t^*} = x$ and so

$$\lim_{i\to\infty} u_{n_i,t_i} = \lim_{i\to\infty} \hat{g}(u_{n_i,t^*}) = \hat{g}^{\mathrm{cl}}(\lim_{i\to\infty} u_{n_i,t^*}) = \hat{g}^{\mathrm{cl}}(x) = x.$$

This contradicts the fact that $d(x, u_{n_i,t_i}) \geq d$, so $\lim_{n\to\infty} K_n = x$.

Recall that $x \in K - \mathrm{ENI}(K)$, and note that $K_i = \eta^{-1}(L_i) \subseteq \eta^{-1}(Y) \subseteq \mathrm{ENI}(K)$. So $x \notin K_i$. Hence there is an infinite set $\sigma \subseteq \mathbb{N}$ such that for any distinct $i, j \in \sigma$, $K_i \cap K_j = \emptyset$. There is a sequence $\{U_i \mid i \in \sigma\}$ of pairwise disjoint sets such that $K_i \subseteq U_i \subseteq \mathrm{ENI}(K)$, U_i is open in $\mathrm{ENI}(K)$ and $\lim_{i \in \sigma} U_i = x$. Let $\varrho \subseteq \sigma$ be such that ϱ and $\sigma - \varrho$ are infinite.

By Proposition 6.23(a), for every $i \in \varrho$ there is $g_i \in H(K)\lfloor U_i \rfloor$ such that $g_i(u_i) = u_{i,1}$. By Proposition 6.23(b), $\hat{g} := \circ_{i \in \varrho}\, g_i \in H(K)$. Let $\bar{g} = \hat{g}{\restriction}\mathrm{ENI}(K)$ and $\bar{h} = \bar{g}^{\eta}$. Then $\bar{g}^{\mathrm{cl}} = \hat{g} \in H(K)$. From the fact that η induces φ it follows that $\bar{h}^{\mathrm{cl}} \in H(M)$.

For every $i \in \varrho$, $\bar{h}(v_i) = h(v_i, 1)$. So $\lim_{i \in \varrho} \bar{h}(v_i) = h(v, 1)$. For every $i \in \sigma - \varrho$, $\bar{h}(v_i) = v_i$. So $\lim_{i \in \sigma - \varrho} \bar{h}(v_i) = v$. Recall that $h(v, 1) \neq v$ and that $\lim_{i \to \infty} v_i = v$. So $\vec{v}$ is convergent and $\bar{h}(\vec{v})$ is not convergent. Hence $\bar{h}^{\mathrm{cl}} \notin H(M)$. A contradiction, so Claim 1 is proved.

Suppose by contradiction that $x \in K - \mathrm{ENI}(K)$ and $x \notin \mathrm{Dom}((\eta{\restriction}X)^{\mathrm{cl}})$. Recall that $Y \in K^{\mathcal{O}}_{\mathrm{IMX}}$ and hence Y is JN.AC. So by Proposition 6.21(c), for every $x \in K - \mathrm{ENI}(K)$ there is a sequence $\vec{x} \subseteq X$ converging to x such that $\eta(\vec{x}) \subseteq Y$, and $\eta(\vec{x})$ is convergent in M. Set $y = \lim \eta(\vec{x})$. Obviously, $y \in \mathrm{bd}(Y)$. Since $x \notin \mathrm{Dom}((\eta{\restriction}X)^{\mathrm{cl}})$, there are a 1-1 sequence $\vec{u} \subseteq X$ and $d > 0$ such that $\lim \vec{u} = x$ and $d(\eta(\vec{u}), y) \geq d$. Define $\vec{v} = \eta(\vec{u})$. Then by Claim 1, $\vec{v}$ does not have a convergent subsequence. That is, $\vec{v}$ is completely discrete. Since Y is JN.AC, there is a subsequence $\vec{w}$ of $\vec{v}$ such that $\vec{w}$ has a joining system. Let $\langle \vec{w}, w^*, \{L_n \mid n \in \mathbb{N}\}, \vec{w}' \rangle$ be a joining system for $\vec{w}$. We may assume that $w^* \notin \mathrm{Rng}(\vec{w})$.

It can be assumed that $w^* \neq y$. For suppose that $w^* = y$. Let $r = d(\vec{w}, y)$. Since Y is BR.IS.MV and $y \in \mathrm{bd}(Y)$, there is $h \in \mathrm{EXT}(Y)$ such that $\mathrm{supp}(h) \subseteq B(y, r)$ and $h^{\mathrm{cl}}(y) \neq y$. So $h{\restriction}\vec{w} = \mathrm{Id}$. It follows that $\langle \vec{w}, h^{\mathrm{cl}}(y), \{h(L_n) \mid n \in \mathbb{N}\}, h(\vec{w}') \rangle$ is a joining system for $\vec{w}$, and if we redefine w^* to be $h^{\mathrm{cl}}(y)$, then $w^* \neq y$.

Recall that Y is JN.AC. So we may apply Lemma 6.21(b) to η^{-1}. Recall also that $\lim \eta^{-1}(\vec{w}) = \lim \eta^{-1}(\vec{v}) = x$. Hence there is $\vec{z} \subseteq Y$ such that $\lim \vec{z} = w^*$ and $\lim \eta^{-1}(\vec{z}) = x$. The two sequences $\vec{x}$ and $\eta^{-1}(\vec{z})$ converge to x, however, $\eta(\vec{x})$ and $\eta(\eta^{-1}(\vec{z}))$ are convergent, but they do not converge to the same point. This contradicts Claim 1, so $\mathrm{Dom}((\eta \restriction X)^{\mathrm{cl}}) \supseteq K - \mathrm{ENI}(K)$. Since $\mathrm{Dom}(\eta) = \mathrm{ENI}(K)$, we have $\mathrm{Dom}(\eta^{\mathrm{cl}}) = K$.

We have shown that $\eta \in \mathrm{EXT}(\mathrm{ENI}(X), \mathrm{ENI}(Y))$. An identical argument shows that $\eta^{-1} \in \mathrm{EXT}(\mathrm{ENI}(Y), \mathrm{ENI}(X))$. Hence $\eta^{\mathrm{cl}} \in H(K, M)$. Since η induces φ, η^{cl} induces φ. ∎

6.5. Generalizations to manifolds and to nearly open sets. The results of this chapter are true in two other settings, which are more general than the present setting. The proofs remain exactly the same.

REMARK 6.25. (a) Let Z be a subset of the normed space E. Z is a *nearly open set* if $Z \subseteq \mathrm{cl}^E(\mathrm{int}^E(Z))$. The results of this chapter can be extended to the class of nearly open subsets of a normed space. Let

$$K^{\mathcal{NO}}_{\mathrm{NRM}} = \{\langle X, Z \rangle \mid X \in K^{\mathcal{O}}_{\mathrm{NRM}} \text{ and } X \subseteq Z \subseteq \mathrm{cl}(X)\}.$$

Note that $\{\langle X, \mathrm{cl}(X) \rangle \mid X \in K^{\mathcal{O}}_{\mathrm{NRM}}\} \subseteq K^{\mathcal{NO}}_{\mathrm{NRM}}$.

(b) The analogy with $K^{\mathcal{O}}_{\mathrm{NRM}}$ is as follows. Let $\langle X, Z\rangle \in K^{\mathcal{NO}}_{\mathrm{NRM}}$. The group

$$\mathrm{EXT}^{Z}(X) = \{h{\restriction}X \mid h \in H(Z) \text{ and } h(X) = X\}$$

is the analogue of $\mathrm{EXT}^{E}(X)$, and the group $H(Z)$ is the analogue of $H(\mathrm{cl}(X))$.

(c) Suitable reformulations of Theorem 6.3, Corollary 6.6 and Theorems 6.18, 6.20, 6.22 and 6.24 are true for $K^{\mathcal{NO}}_{\mathrm{NRM}}$. $\square$

We demonstrate the generalization discussed in Remark 6.25 by describing the analogues of Theorem 6.3(b) and 6.22. The faithful class captured by this generalization contains $2^{2^{\aleph_0}}$ subsets of $\mathbb{R}^3$.

Let $K^{\mathcal{NO}}_{\mathrm{NMX}}$ be the class of all $\langle X, Z\rangle \in K^{\mathcal{NO}}_{\mathrm{NRM}}$ such that X is BR.LC.AC with respect to Z, and X is JN.AC with respect to Z. Evidently, this is the analogue of $K^{\mathcal{O}}_{\mathrm{NMX}}$ defined in 6.3(b). Let us first see that $K^{\mathcal{NO}}_{\mathrm{NMX}}$ is a large class. Write $X = (0,1)^3$, that is, X is an open cube in $\mathbb{R}^3$. We construct sets Z such that $\langle X, Z\rangle \in K^{\mathcal{NO}}_{\mathrm{NMX}}$, and in fact, we show that $|\{Z \mid \langle X, Z\rangle \in K^{\mathcal{NO}}_{\mathrm{NMX}}\}| = 2^{2^{\aleph_0}}$. We skip the easy proof of part (b) of the next example.

EXAMPLE 6.26. *Let* $X = (0,1)^3$.

(a) *For* $x, y \in \mathbb{R}$ *let* $L_{x,y} = [(x,0,0),(x,y,0)]$. *Let* $\emptyset \neq A \subseteq [0,1]$ *and* $\varrho : A \to [0,1)$. (*We do not assume that* ϱ *is continuous.*) *Let* $Z_\varrho = X \cup \bigcup_{x\in A} L_{x,\varrho(x)}$. *Then* $\langle X, Z_\varrho\rangle \in K^{\mathcal{NO}}_{\mathrm{NMX}}$.

(b) *Let* F *be a closed nonempty subset of* $\mathrm{bd}^{\mathbb{R}^3}(X)$. *Then* $\langle X, X\cup F\rangle \in K^{\mathcal{NO}}_{\mathrm{NMX}}$.

Proof. (a) Let X, A, ϱ and Z be as above. It is trivial that X is BR.LC.AC with respect to Z. We show that X is JN.AC with respect to Z. Let $\vec{u} = \{u_n\}_{n\in\mathbb{N}} \subseteq X$ be a completely discrete sequence with respect to Z. It may be assumed that $\vec{u}$ is convergent in $\mathbb{R}^3$, and we denote its limit by $\hat{u}$. So $\hat{u} \in \mathrm{cl}^{\mathbb{R}^3}(X) - Z$. Write $u_n = (x_n, y_n, z_n)$ and $\hat{u} = (\hat{x}, \hat{y}, \hat{z})$.

CASE 1: Assume that $\hat{z} = 0$. Suppose first that there is $a \in A$ such that $\{n \mid x_n = a\}$ is infinite. So we may assume that $x_n = a$ for every $n \in \mathbb{N}$. It follows that for some $b > \varrho(a)$, $\lim \vec{u} = (a, b, 0)$. Hence $\vec{u}$ has a subsequence $\vec{v}$ such that $[v_m, (a, \varrho(a), 0)] \cap [v_m, (a, \varrho(a), 0)] = \{(a, \varrho(a), 0)\}$ for any $m \neq n$. Choose $w_n \in [v_n, (a, b, 0))$ such that $\lim_{n\to\infty} w_n = (a, b, 0)$ and define $L_n = [v_n, w_n]$. It is easy to see that $\langle \vec{v}, (a, \varrho(a), 0), \vec{L}, \{w_n\}_{n\in\mathbb{N}}\rangle$ is a joining system for $\vec{v}$

Suppose next that for every $a \in A$, $\{n \mid x_n = a\}$ is finite. Choose any $a \in A$ and remove from $\vec{u}$ all u_n's such that $x_n = a$. Then $a \neq x_n$ for every $n \in \mathbb{N}$. We may also assume that $z_0 < 1/2$ and that $\{z_n\}_{n\in\mathbb{N}}$ is strictly decreasing. Let $y'_n = \max(1 - z_n, y_n)$, $u'_n = (x_n, y'_n, z_n)$ and $L^0_n = [u_n, u'_n]$. We show that $\vec{L}^0 := \{L^0_n\}_{n\in\mathbb{N}}$ is completely discrete with respect to Z. Since $\{z_n\}_{n\in\mathbb{N}}$ is 1-1, $\vec{L}^0$ is a pairwise disjoint sequence, that is, $L^0_m \cap L^0_n = \emptyset$ for any $m \neq n$. If $(x, y, z) \in \mathrm{acc}^{\mathbb{R}^3}(\vec{L}^0)$, then $x = \hat{x}$, $z = 0$ and $y \geq \hat{y}$, and since $(\hat{x}, \hat{y}, 0) \notin Z$, it follows that $(\hat{x}, y, 0) \notin Z$. The sequence $\{y'_n\}_{n\in\mathbb{N}}$ converges to 1, so we may assume that it is strictly increasing. Let $v_n = (x_n, y'_n, 1/2)$ and $L^1_n = [u'_n, v_n]$. It is trivial that $\vec{L}^1 := \{L^1_n\}_{n\in\mathbb{N}}$ is a pairwise disjoint sequence. If $(x, y, z) \in \mathrm{acc}^{\mathbb{R}^3}(\vec{L}^1)$, then $y = 1$ and so $(x, y, z) \notin Z$. So $\vec{L}^1$ is completely discrete with respect to Z. Suppose that $m < n$. Then $L^0_m \cap L^1_n = \emptyset$, since the y-coordinate of any member of L^0_m is $\leq y'_m$, and the y-coordinate of any member of L^1_n is equal to y'_n which is $> y'_m$. Similarly,

$L^1_m \cap L^0_n = \emptyset$, since members of L^1_m and L^0_n differ in their z-coordinate. We conclude that $(L^0_m \cup L^1_m) \cap (L^0_n \cap L^1_n) = \emptyset$ for any $m \neq n$.

Let $w_n = (a, y'_n, 1/2)$ and $L^2_n = [v_n, w_n]$. The sequence $\vec{L}^2 := \{L^2_n\}_{n\in\mathbb{N}}$ is a pairwise disjoint sequence, since members of L^2_m and L^2_n differ in their y-coordinate. Also, $L^2_n \cap (L^0_m \cup L^1_m) = \emptyset$ for any $m \neq n$. This follows from the fact that the only point in $L^0_m \cup L^1_m$ whose z-coordinate is $1/2$ is v_m and $v_m \notin L^2_n$. The y-coordinate of any member of $\mathrm{acc}^{\mathbb{R}^3}(\vec{L}^2)$ is 1, so $\mathrm{acc}^{\mathbb{R}^3}(\vec{L}^2) \cap Z = \emptyset$ and hence $\vec{L}^2$ is completely discrete with respect to Z. Let $w^* = (a, \varrho(a), 0)$, choose $w'_n \in [w_n, w^*)$ such that $\lim_{n\to\infty} w'_n = w^*$ and define $L^3_n = [w_n, w'_n]$. Clearly, $\vec{L}^3 := \{L^3_n\}_{n\in\mathbb{N}}$ is a pairwise disjoint sequence. Since $\lim_{n\to\infty} w_n = (a, 1, 0)$, it follows that $\mathrm{acc}^{\mathbb{R}^3}(\vec{L}^3) = [w^*, (a, 1, 0)]$. So $\mathrm{acc}^Z(\vec{L}^3) = \{w^*\}$. It follows that for every $r > 0$, $\{L^3_n - B(w^*, r) \mid n \in \mathbb{N}\}$ is completely discrete with respect to Z. Note that w_m is the only point in $\bigcup_{i\leq 2} L^i_m$ whose x-coordinate is a. So since for $n \neq m$, $w_m \notin L^3_n$, $L^3_n \cap (\bigcup_{i\leq 2} L^i_m) = \emptyset$. Define $L_n = \bigcup_{i\leq 3} L^i_m$, $\vec{w}' = \{w'_n\}_{n\in\mathbb{N}}$ and $\vec{L} = \{L_n\}_{n\in\mathbb{N}}$. It follows that $\vec{L}$ is a pairwise disjoint sequence and that for every $r > 0$, $\{L_n - B(w^*, r) \mid n \in \mathbb{N}\}$ is completely discrete with respect to Z. So $\langle \vec{u}, w^*, \vec{L}, \vec{w}' \rangle$ is a joining system for $\vec{u}$.

The case that $\hat{z} \neq 0$ is divided into several subcases. Their proofs are similar to the proof of Case 1, but simpler. ∎

THEOREM 6.27. *For $\ell = 1, 2$ let $\langle X_\ell, Z_\ell \rangle \in K^{\mathcal{NO}}_{\mathrm{NMX}}$.*

(a) *If $\varphi : Z_1 \cong Z_2$, then there is $\tau \in H(Z_1, Z_2)$ which induces φ.*

(b) *If $\varphi : \mathrm{EXT}^{Z_1}(X_1) \cong \mathrm{EXT}^{Z_2}(X_2)$, then there is $\tau \in \mathrm{EXT}^{Z_1,Z_2}(X_1, X_2)$ which induces φ.*

Proof. The proof of (a) is identical to the proof of Theorem 6.22. The proof of (b) is identical to the proof of Theorem 6.3. ∎

REMARK 6.28. The second generalization is motivated by the following example. Let $E = \mathbb{R} \times S^{\mathbb{R}^2}(0, 1)$, $Y = [0, 1] \times S^{\mathbb{R}^2}(0, 1)$ and $X = (0, 1) \times S^{\mathbb{R}^2}(0, 1)$. X is a normed manifold. So its reconstruction from subgroups of $H(X)$ is included in Theorem 2.30(a). The local Γ-continuity of conjugating homeomorphisms of X is proved in 3.47(a), 3.48(a) and 4.10. The space Y, however, is not covered by any of the above theorems because it is not a normed manifold. Also, Y is not the closure of an open subset of a normed space. So the theorems proved so far in Chapter 6 do not apply to Y. However, Y is a well-behaved space and is very similar to the spaces which have already been dealt with. □

The above remark calls for the setting in which E is a normed manifold, X is an open subset of E and $Y = \mathrm{cl}^E(X)$. This setting will yield reconstruction results for Y.

DEFINITION 6.29. (a) Let $\langle X, \Phi, d \rangle$ be such that $\langle X, \Phi \rangle$ is a normed manifold, $\langle X, d \rangle$ is a metric space, and there is K such that for every $\varphi \in \Phi$, φ is K-bilipschitz. Then $\langle X, \Phi, d \rangle$ is called a *normed Lipschitz manifold.*

(b) Let $K^{\mathcal{O}}_{\mathrm{NLPM}} = \{Y \mid Y$ is an open subset of a normed Lipschitz manifold$\}$. □

Chapter 6 in its entirety can be proved for $K^{\mathcal{O}}_{\mathrm{NLPM}}$.

THEOREM 6.30. *In Definitions* 6.1, 6.9, 6.16, 6.19 *and in Remark* 6.25 *change every mention of* $K^{\mathcal{O}}_{\mathrm{NRM}}$ *to a mention of* $K^{\mathcal{O}}_{\mathrm{NLPM}}$. *Then the variants obtained in this way from Theorem* 6.3 *and Theorems* 6.12, 6.18, 6.20, 6.22, 6.24 *and* 6.27 *are true.*

Proof. The proofs of all the above theorems are identical to the proofs of their counterparts. ∎

7. Groups which are not of the same type are not isomorphic

In the previous chapters we considered several properties of homeomorphisms, for instance, UC homeomorphisms, LUC homeomorphisms, extendible homeomorphisms and homeomorphisms which are uniformly continuous on every bounded positively distanced set. In this chapter we prove that for properties $\mathcal{P}$ and $\mathcal{Q}$ as above, if $\mathcal{P}(X) \cong \mathcal{Q}(Y)$, then either $\mathcal{P}(X) = \mathcal{Q}(X)$ or $\mathcal{P}(Y) = \mathcal{Q}(Y)$. But before we deal with these questions, we prove some additional facts about the group $\mathrm{UC}(X)$.

7.1. The group $\mathrm{UC}(X)$ **revisited.** We have seen in Theorem 5.5 that if $X, Y \in K_{\mathrm{NRM}}^{\mathcal{O}}$, X is UD.AC and $(\mathrm{UC}(X))^{\tau} \subseteq \mathrm{UC}(Y)$, then τ is uniformly continuous. We next reconsider the problem of deducing that τ^{-1} is uniformly continuous from the fact that $(\mathrm{UC}(X))^{\tau} \subseteq \mathrm{UC}(Y)$. Recall that the implication

$$(\dagger) \qquad (\mathrm{UC}(X))^{\tau} \subseteq \mathrm{UC}(Y) \;\Rightarrow\; \tau^{-1} \text{ is uniformly continuous}$$

is not true for every $X, Y \in K_{\mathrm{NRM}}^{\mathcal{O}}$. Counter-examples appear in 5.7 and 6.7(a). Yet, $(\dagger)$ holds when X and Y are well-behaved. Theorem 7.1 below deals with finite-dimensional spaces for which $(\dagger)$ is true. The infinite-dimensional case is considered in 7.7. The result of 7.7 is needed in the proof of Corollary 7.11(d) and (e).

THEOREM 7.1. *Let $X, Y \in K_{\mathrm{NRM}}^{\mathcal{O}}$. Suppose that X is finite-dimensional and bounded, X is UD.AC, $|\mathrm{Cmp}(\mathrm{bd}(X))| \le \aleph_0$ and $(*)$ for every $C \in \mathrm{Cmp}(\mathrm{bd}(X))$, distinct $x, y \in C$, and $z \in \mathrm{bd}(X) - \{x, y\}$, there is $f \in \mathrm{UC}(X)$ such that either $f^{\mathrm{cl}}(x) = y$ and $f^{\mathrm{cl}}(z) = z$, or $f^{\mathrm{cl}}(z) = y$ and $f^{\mathrm{cl}}(x) = x$. Suppose that for every $C \in \mathrm{Cmp}(\mathrm{bd}(Y))$, $|C| > 1$. Let $\tau \in H(X, Y)$ be such that $(\mathrm{UC}(X))^{\tau} \subseteq \mathrm{UC}(Y)$. Then τ^{-1} is uniformly continuous.*

Proof. By Theorem 5.5, τ is uniformly continuous, and hence τ^{cl} maps $\mathrm{cl}(X)$ onto $\mathrm{cl}(Y)$. It thus suffices to show that τ^{cl} is injective. Suppose otherwise. For $x \in \mathrm{bd}(X)$ let C_x denote the connected component of $\mathrm{bd}(X)$ containing x. It follows from $(*)$ that if for some $z \ne x$, $\tau^{\mathrm{cl}}(x) = \tau^{\mathrm{cl}}(z)$, then for every $y \in C_x$, $\tau^{\mathrm{cl}}(y) = \tau^{\mathrm{cl}}(x)$. The argument is as follows. Suppose indeed that $z \ne x$, $\tau^{\mathrm{cl}}(x) = \tau^{\mathrm{cl}}(z)$ and $y \in C_x - \{x, z\}$. Let $f \in \mathrm{UC}(X)$ be as ensured by $(*)$. We assume first that $f^{\mathrm{cl}}(x) = x$ and $f^{\mathrm{cl}}(z) = y$, Let $\vec{x}, \vec{y} \subseteq X$ converge respectively to x and y, and let $\vec{x}' = f^{-1}(\vec{x})$ and $\vec{z} = f^{-1}(\vec{y})$. Then

$$\tau^{\mathrm{cl}}(y) = \lim \tau(\vec{y}) = \lim f^{\tau} \circ \tau \circ f^{-1}(\vec{y}) = \lim f^{\tau}(\tau(\vec{z})) = (f^{\tau})^{\mathrm{cl}}(\lim \tau(\vec{z})) = (f^{\tau})^{\mathrm{cl}}(\tau^{\mathrm{cl}}(z)).$$

Similarly,

$$\begin{aligned} \tau^{\mathrm{cl}}(x) &= \lim \tau(\vec{x}) = \lim f^{\tau} \circ \tau \circ f^{-1}(\vec{x}) = \lim f^{\tau}(\tau(\vec{x}')) \\ &= (f^{\tau})^{\mathrm{cl}}(\lim \tau(\vec{x}')) = (f^{\tau})^{\mathrm{cl}}(\tau^{\mathrm{cl}}(\lim \vec{x}')). \end{aligned}$$

Since $f^{\mathrm{cl}}(x) = x$ and $\lim \vec{x} = x$, we have $\lim \vec{x}' = x$. So

$$\tau^{\mathrm{cl}}(x) = (f^{\tau})^{\mathrm{cl}}(\tau^{\mathrm{cl}}(\lim \vec{x}')) = (f^{\tau})^{\mathrm{cl}}(\tau^{\mathrm{cl}}(x)) = (f^{\tau})^{\mathrm{cl}}(\tau^{\mathrm{cl}}(z)) = \tau^{\mathrm{cl}}(y).$$

The same argument applies to the case that $f^{\mathrm{cl}}(z) = z$ and $f^{\mathrm{cl}}(x) = y$. It follows that for any distinct $C, D \in \mathrm{Cmp}(\mathrm{bd}(X))$, either $\tau^{\mathrm{cl}}(C) = \tau^{\mathrm{cl}}(D)$ and $\tau^{\mathrm{cl}}(C)$ is a singleton, or $\tau^{\mathrm{cl}}(C) \cap \tau^{\mathrm{cl}}(D) = \emptyset$.

Let x and y be distinct members of $\mathrm{bd}(X)$ such that $\tau^{\mathrm{cl}}(x) = \tau^{\mathrm{cl}}(y)$, and C be the component of $\tau^{\mathrm{cl}}(x)$ in bd(Y). The family $\{\tau^{\mathrm{cl}}(C_u) \cap C \mid u \in \mathrm{bd}(X)\}$ is a partition of C into more than one and at most countably many closed sets. This contradicts the theorem of Sierpiński that a continuum cannot be partitioned into countably many nonempty closed sets. See [En, Theorem 6.1.27]. ∎

We do not know whether in the above theorem, the requirement that $\mathrm{bd}(X)$ has at most countably many components can be dropped. Here is an easy example of a bounded regular open subset $X \subseteq \mathbb{R}^3$ such that X is UD.AC, X satisfies $(*)$ of Theorem 7.1, every connected component of $\mathrm{bd}(X)$ has cardinality > 1, and $\mathrm{bd}(X)$ has $2^{\aleph_0}$ connected components.

Example 7.2. Let $C \subseteq [0,1]$ be the Cantor set. Let $K = C \times \{1\}$. So $K \subseteq B^{\mathbb{R}^2}(0,2)$ and $B^{\mathbb{R}^2}(0,2) - K$ is connected. Let $A = \{a_n \mid n \in \mathbb{N}\} \subseteq B^{\mathbb{R}^2}(0,2)$ be such that $\mathrm{cl}(A) - A = K$, and every member of A is an isolated point in A. Let $r_n > 0$ and $D_n = \bar{B}(a_n, r_n)$. Assume that $D_n \subseteq B(0,2) \cap \{(x,y) \mid x > 0\}$ and $B(a_m, 2r_m) \cap B(a_n, 2r_n) = \emptyset$ for any $m \neq n$, and that $\mathrm{cl}(\bigcup_{n\in\mathbb{N}} D_n) - \bigcup_{n\in\mathbb{N}} D_n = K$. Let $U = B(0,3) - \mathrm{cl}(\bigcup_{n\in\mathbb{N}} D_n)$. Let $X \subseteq \mathbb{R}^3$ be the set obtained by rotating U about the x-axis. Note that if $x, y \in U$, then there is an arc $L \subseteq U$ connecting x and y such that $\mathrm{lngth}(L) \leq 2\pi \cdot \|x - y\|$. It follows easily that X is as required. □

We next deal with infinite-dimensional open sets for which the fact that $(\mathrm{UC}(X))^{\tau} \subseteq \mathrm{UC}(Y)$ implies that τ^{-1} is uniformly continuous.

Definition 7.3. (a) For $A \subseteq X$ define $\varDelta^{X,E}(A) = \sup_{a\in A} d(a, E - X)$. As usual, we abbreviate $\varDelta^{X,E}(A)$ by $\varDelta(A)$.

(b) Let $h \in H(X)$. We say that h is *strongly extendible* if for every $\varepsilon > 0$ there is $\tilde{h} \in H(E)$ such that $\tilde{h}$ extends h and $\mathrm{supp}(\tilde{h}) \subseteq B(\mathrm{supp}(h), \varepsilon)$. Define $\mathrm{UC_e}(X) := \{h \in \mathrm{UC}(X) \mid h \text{ is strongly extendible}\}$.

(c) A *simple arc* is a space homeomorphic to $[0,1]$. For a simple arc L and $x, y \in L$ let $[x,y]^L$ denote the subarc of L whose endpoints are x and y. Let $\alpha \in \mathrm{MBC}$ and $\eta : (0,\infty) \to (0,\infty)$ be such that η is monotonic and $\lim_{tt\to 0} \eta(t) = 0$. Let X be a metric space and $L \subseteq X$ be a simple arc. We say that L is an *$\langle \alpha, \eta \rangle$-track* if for every $x, y \in L$ there is $h \in \mathrm{UC}(X)$ such that h is α-bicontinuous, $h(x) = y$ and $\mathrm{supp}(h) \subseteq B([x,y]^L, r)$, where $r = \eta(\mathrm{diam}([x,y]^L)$. If in the above definition we require that $h \in \mathrm{UC_e}(X)$, then L is called an *$\langle \alpha, \eta \rangle$-e-track.*

(d) We define the notion of a track system for $\vec{x}$. Let $\vec{x} \subseteq X$ be a completely discrete sequence, $y^* \in \mathrm{bd}(X)$, $\vec{y} \subseteq X$ and $\vec{L} = \{L_n \mid n \in \mathbb{N}\}$ be a sequence of simple arcs such that $\lim \vec{y} = y^*$, $L_n \subseteq X$, L_n connects x_n with y_n and $\bigcup_{n\in\mathbb{N}} L_n$ is bounded. Assume that

(1) there are α and η such that L_n is an $\langle \alpha, \eta \rangle$-track for every $n \in \mathbb{N}$,

(2) there are $\beta \in \mathrm{MC}$ and for every n a parametrization γ_n of L_n such that $\mathrm{Dom}(\gamma_n) = [0,1]$, γ_n is β-UC for every $n \in \mathbb{N}$ and $\gamma_n(0) = y_n$ and $\gamma_n(1) = x_n$.

Then $T = \langle \vec{x}, y^*, \vec{L}, \vec{y} \rangle$ is called a *track system for* $\vec{x}$, and γ_n is called a *legal parametrization* of L_n in T. Note that (2) just means that $\{\gamma_n \mid n \in \mathbb{N}\}$ is equicontinuous. If in (1) we require that L_n be an e-track, then T is called an *e-track system*.

Let $T = \langle \vec{x}, y^*, \vec{L}, \vec{y} \rangle$ be a track system. If for every $r > 0$, $\{L_n - B(y^*, r) \mid n \in \mathbb{N}\}$ is completely discrete, then T is called a *completely discrete track system*. If for every $r > 0$, $\{L_n - B(y^*, r) \mid n \in \mathbb{N}\}$ is spaced, then T is called a *spaced track system*.

(f) X is *jointly track connected* (*JN.TC*) if for every completely discrete bounded sequence $\vec{x} \subseteq X$: if $\lim_{n\to\infty} \delta(x_n) = 0$, then $\vec{x}$ has a subsequence $\vec{y}$ such that $\vec{y}$ has a track system. X is *jointly e-track connected* (*JN.ETC*) if the above subsequence $\vec{y}$ is required to have an e-track system. $\square$

Remark 7.4. We explain the notion of a track system by an example. Let X be the unit ball of the Hilbert space ℓ_2 and S be the unit sphere. Let $\vec{x}$ be a completely discrete sequence in X such that $\delta(\vec{x}) = 0$. We construct a track system for a subsequence of $\vec{x}$. Let $e_0 = (1, 0, 0, \ldots)$. Take a subsequence $\vec{y}$ of $\vec{x}$ such that $\{e_0\} \cup \mathrm{Rng}(\vec{y})$ is an independent set. For $n \in \mathbb{N}$ let $z_n = \|y_n\| e_0$, $S_n = S(0, \|y_n\|) \cap \mathrm{span}(\{y_n, e_0\})$ and L_n be any of the two subarcs of S_n connecting y_n with z_n. Then $T = \langle \vec{y}, e_0, \{L_n\}_{n\in\mathbb{N}}, \{z_n\}_{n\in\mathbb{N}} \rangle$ is a track system for $\vec{y}$. Indeed, T is an e-track system.

The property JN.ETC is needed in the proof that $\mathrm{UC}(X) \not\cong \mathrm{EXT}(X)$. $\square$

Proposition 7.5. (a) *Let* $\{h_n \mid n \in \mathbb{N}\} \subseteq \mathrm{UC_e}(X)$, *and suppose that* $\{\mathrm{supp}(h_n) \mid n \in \mathbb{N}\}$ *is spaced. Then* $\circ_{n\in\mathbb{N}} h_n \in \mathrm{UC_e}(X)$.

(b) *Let* $x, y \in E$ *be such that* $\|x\| = \|y\|$ *and* $\|x - y\| = d > 0$. *Let* $L = \{tx \mid t \ge 0\}$. *Then* $d(y, L) \ge d/2$.

(c) *If* $T = \langle \vec{x}, y^*, \vec{L}, \vec{y} \rangle$ *a track system, then the following hold.*

(i) *For every* $t \in (0,1)$, $T_t := \langle \{\gamma_n(t)\}_{n\in\mathbb{N}}, y^*, \{\gamma_n([0,t])\}_{n\in\mathbb{N}}, \vec{y} \rangle$ *is a track system, and if* T *is completely discrete, so is* T_t.

(ii) $\lim_{n\to\infty} \Delta(L_n) = 0$.

(d) *Let* $\langle \vec{x}, y^*, \vec{L}, \vec{y} \rangle$ *be a completely discrete track system. Then there is an infinite* $\sigma \subseteq \mathbb{N}$ *such that* $\langle \vec{x} \restriction \sigma, y^*, \vec{L} \restriction \sigma, \vec{y} \restriction \sigma \rangle$ *is a spaced track system.*

(e) *Let* $T = \langle \vec{x}, y^*, \vec{L}, \vec{y} \rangle$ *be a track system. Let* γ_n *be legal parametrization of* L_n *in* T. *Then there are* $t \in [0,1)$, $z^* \in \mathrm{bd}(X)$ *and an infinite* $\sigma \subseteq \mathbb{N}$ *such that* $\langle \vec{x} \restriction \sigma, z^*, \{\gamma_n([t,1]) \mid n \in \sigma\}, \{\gamma_n(t) \mid n \in \sigma\} \rangle$ *is a spaced track system.*

(f) *Let* $T = \langle \vec{x}, y^*, \vec{L}, \vec{y} \rangle$ *be a completely discrete track system and* $C \in \mathrm{Cmp}(\mathrm{bd}(X))$ *be such that* $d(\vec{x}, C) = 0$. *Then* $y^* \in C$.

(g) *Let* $T = \langle \vec{x}, y^*, \vec{L}, \vec{y} \rangle$ *be a track system,* $h \in \mathrm{UC}(X)$ *and* $T' := \langle h(\vec{x}), h^{\mathrm{cl}}(y^*), h(\vec{L}), h(\vec{y}) \rangle$. *Then* T' *is a track system.*

Proof. (a) The proof is trivial and is left to the reader.

(b) We may assume that $\|x\| = 1$. Let $tx \in L$. If $|1 - t| \le d/2$, then use the triangle with vertices x, tx and y to conclude that $\|y - tx\| \ge \|y - x\| - \|x - tx\| \ge d/2$; and if $|1 - t| \ge d/2$, then use the triangle with vertices 0, tx and y to conclude that $\|y - tx\| \ge |\, \|y - 0\| - \|tx - 0\| \,| = |1 - t| \ge d/2$.

(c) The first part of (c) follows from the definition of a track system. To prove the second part, suppose by way of contradiction that for some $d > 0$, $\{n \mid \Delta(L_n) > d\}$ is infinite. Let α and η be as ensured by the fact that T is a track system. Since $\lim \vec{y} = y^* \in \mathrm{bd}(X)$, there is n such that $\alpha(\delta(y_n)) < d$ and $\Delta(L_n) > d$. Choose $z \in L_n$ such that $\delta(z) > d$ and $w \in \mathrm{bd}(X)$ such that $\alpha(\|y_n - w\| < d$. Since L_n is an $\langle \alpha, \eta \rangle$-track, there is $h \in H(X)$ such that h is α-bicontinuous and $h(y_n) = z$. Then

$$\|h(y_n) - h(w)\| = \|z - h(w)\| \geq d(z, \mathrm{bd}(X)) > d > \alpha(\|y_n - w\|),$$

and this contradicts the α-continuity of h.

(d) For every $r > 0$, $\{L_i - B(y^*, r) \mid i \in \mathbb{N}\}$ is completely discrete. So by Proposition 5.26, for every $r > 0$ and an infinite $\eta \subseteq \mathbb{N}$ there is an infinite $\nu \subseteq \eta$ such that $\{L_i - B(y^*, r) \mid i \in \nu\}$ is spaced. We define by induction $\varrho_n \subseteq \mathbb{N}$. Let $\varrho_0 = \mathbb{N}$. For every $n \in \mathbb{N}$ let ϱ_{n+1} be an infinite subset of ϱ_n such that $\{L_i - B(z^*, \frac{1}{n+1}) \mid i \in \varrho_{n+1}\}$ is spaced. Let $\sigma = \{\min(\varrho_n \cap \mathbb{N}^{\geq n}) \mid n \in \mathbb{N}\}$. It is easy to see that for every $r > 0$, $\{L_i - B(z^*, r) \mid i \in \sigma\}$ is spaced. So $\langle \vec{x} \restriction \sigma, y^*, \vec{L} \restriction \sigma, \vec{y} \restriction \sigma \rangle$ is a spaced track system.

(e) For every infinite $\eta \subseteq \mathbb{N}$ and $t \in [0,1]$ let $A[\eta, t] = \{\gamma_n(t) \mid n \in \eta\}$. Let $s_\eta = \sup(\{t \mid A[\eta, t]$ is not completely discrete$\})$. Let $\varrho \subseteq \mathbb{N}$ be an infinite set such that for every infinite $\eta \subseteq \varrho$, $s_\eta = s_\varrho$. Set $s = s_\varrho$. Suppose by contradiction that $A[\varrho, s]$ does not contain a Cauchy sequence. Then for some infinite $\eta \subseteq \varrho$ and $d > 0$, $A[\eta, s]$ is d-spaced. There is $\varepsilon > 0$ such that for every $t > s - \varepsilon$, $A[\eta, t]$ is spaced. The existence of ε follows from the equicontinuity of $\{\gamma_n \mid n \in \mathbb{N}\}$, that is, from the existence of β appearing in clause (2) of the definition of a track system. So $s_\eta < s$. A contradiction. So $A[\varrho, s]$ contains a Cauchy sequence. We may thus assume that $A[\varrho, s]$ is a Cauchy sequence. Let $z^* = \lim A[\varrho, s]$.

Let $J_i = \gamma_i([s, 1])$. We show that there are no $r > 0$, an infinite $\eta \subseteq \varrho$ and $\vec{u} \in \prod_{i \in \eta}(J_i - B(z^*, r))$ such that $\vec{u}$ is a Cauchy sequence. Suppose otherwise. Let $t_i \in [s, 1]$ be such that $u_i = \gamma_i(t_i)$. We may assume that $\vec{t} = \{t_i \mid i \in \eta\}$ is a Cauchy sequence. Let $t^* = \lim \vec{t}$. Since $\mathrm{Rng}(\vec{u}) \cap B(z^*, r) = \emptyset$, it follows that $t^* \neq s$, and since $\lim_{i \in \eta} d(\gamma_i(t_i), \gamma_i(t^*)) = 0$, we find that $\{\gamma_i(t^*) \mid i \in \eta\}$ is a Cauchy sequence. That is, $s_\eta > s$, a contradiction. We have shown that $\langle \{x_n \mid n \in \varrho\}, z^*, \{\gamma_n([s, 1]) \mid n \in \varrho\}, A[\varrho, s] \rangle$ is a completely discrete track system. By (d), there is an infinite $\sigma \subseteq \varrho$ such that $\langle \{x_n \mid n \in \sigma\}, z^*, \{\gamma_n([s, 1]) \mid n \in \sigma\}, A[\sigma, s] \rangle$ is a spaced track system.

(f) Suppose by contradiction that $y^* \notin C$. By (d), we may assume that T is a spaced track system. Let α, η be as ensured by the fact that T is a track system. Clearly, $a := d(y^*, C) > 0$. Choose $u \in C$, and for every $n \in \mathbb{N}$ choose $z_n \in (B(y^*, a/2) - B(y^*, a/4)) \cap L_n$ and set $J_n = [x_n, z_n]^{L_n}$. Then $b_1 := d(u, \bigcup_{n \in \mathbb{N}} J_n) > 0$, and there is b_2 such that $\{J_n \mid n \in \mathbb{N}\}$ is b_2-spaced. Set $b = \min(b_1, b_2)/3$, and let $c > 0$ be such that $c + \eta(c) < b$. From the equicontinuity of $\{\gamma_n\}_{n \in \mathbb{N}}$ it follows that there is $k \in \mathbb{N}$ and $\{z_{n,i} \mid n \in \mathbb{N},\ i \leq k\}$ such that for every $n \in \mathbb{N}$, $z_{n,0} = x_n$, $z_{n,k} = z_n$ and $z_{n,i} \in L_n$, and $\mathrm{diam}([z_{n,i}, z_{n,i+1}]^{L_n}) < c$ for every $i < k$. So for every $n \in \mathbb{N}$ and $i < k$ there is $h_{n,i} \in \mathrm{UC}(X)$ such that $h_{n,i}$ is α-bicontinuous, $h_{n,i}(z_{n,i}) = z_{n,i+1}$ and $\mathrm{supp}(h_{n,i}) \subseteq B([z_{n,i}, z_{n,i+1}]^{L_n}, c)$. Let $h_n = \circ_{i<k} h_{n,i}$. Clearly, $h_n \in \mathrm{UC}(X)$, and it is easily seen that $\{\mathrm{supp}(h_n) \mid n \in \mathbb{N}\}$ is $b_2/3$-spaced and $d(u, \mathrm{supp}(h_n)) > b_1/2 > 0$. It follows that $h := \circ_{i<k} h_{n,i} \in \mathrm{UC}(X)$, $h^{\mathrm{cl}}(u) = u$ and $h(x_n) = z_n$ for every $n \in \mathbb{N}$. Since $h(u) = u$,

it follows that $h(C) = C$. However, $d(\vec{x}, C) = 0$ and $d(h(\vec{x}), h(C)) = d(\vec{z}, C) > a/2 > 0$. This contradicts the fact that h is uniformly continuous.

(g) By Proposition 4.3(c), there is $\gamma \in \mathrm{MBC}$ such that h is γ-bicontinuous. Let α, η and β be as in the definition of a track system. Define $\alpha' = \gamma \circ \alpha \circ \gamma$, $\eta' = \gamma \circ \eta \circ \gamma$ and $\beta' = \gamma \circ \beta$. Then α', η' and β' demonstrate that T' is a track system. ■

Proposition 7.6. *Let Z be a metric space, and $\{F_n \mid n \in \mathbb{N}\}$ and $\{K_n \mid n \in \mathbb{N}\}$ be sequences of compact subsets of Z such that*: (i) *$\{F_n \mid n \in \mathbb{N}\}$ is spaced*; (ii) *for every $\varepsilon > 0$ there is $\ell_\varepsilon \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ and a subset $A \subseteq K_n$, if $|A| \geq \ell_\varepsilon$, then there are distinct $x, y \in A$ such that $d(x, y) < \varepsilon$; and* (iii) $\inf(\{d(F_n, K_n) \mid n \in \mathbb{N}\}) > 0$. *Then there is an infinite $\sigma \subseteq \mathbb{N}$ such that $d(\bigcup\{F_n \mid n \in \sigma\}, \bigcup\{K_n \mid n \in \sigma\}) > 0$.*

Proof. Write $\mathbb{N}^+ = \{n \in \mathbb{N} \mid n > 0\}$. We define by induction on $i \in \mathbb{N}^+$ a sequence of infinite subsets of $\mathbb{N}$, $\sigma_0 \supseteq \sigma_1 \supseteq \cdots$. Let $\sigma_0 = \mathbb{N}$. Suppose that σ_i has been defined. We color the increasing pairs $\langle m, n\rangle$ of members of σ_i in four colors, according to whether $d(F_m, K_n) < 1/i$ or not, and according to whether $d(K_m, F_n) < 1/i$ or not. By the Ramsey Theorem, there is a monochromatic infinite $\sigma_{i+1} \subseteq \sigma_i$. If there is $i \in \mathbb{N}^+$ such that for any distinct $m, n \in \sigma_i$, $d(F_m, K_n) \geq 1/i$ and $d(K_m, F_n) \geq 1/i$, then $\sigma := \sigma_i$ is as required. Otherwise, for every $i \in \mathbb{N}$ either (1) for every $m < n$ in σ_i, $d(F_m, K_n) < 1/i$, or (2) for every $m < n$ in σ_i, $d(K_m, F_n) < 1/i$.

Let $i \in \mathbb{N}$ and $\ell = \ell_{1/i}$ be as ensured by clause (ii). Let $k_0 < \cdots < k_\ell$ be members of σ_i. Suppose that case (1) occurs. For every $j < \ell$ let $x_j \in F_j$ and $y_j \in K_\ell$ be such that $d(x_j, y_j) < 1/i$. Hence for some $j_1 < j_2 < \ell$, $d(y_{j_1}, y_{j_2}) < 1/i$. So $d(F_{j_1}, F_{j_2}) < 3/i$. The same argument is repeated in case (2). Hence for every $i \in \mathbb{N}^+$ there are distinct j_1 and j_2 such that $d(F_{j_1}, F_{j_2}) < 3/i$, contradicting the fact that $\{F_n \mid n \in \mathbb{N}\}$ is spaced. ■

The properties that X is required to fulfill in the next theorem are quite restrictive. However, they are shared by "well-behaved" open sets. For example, if $X = B - \bigcup_{i<k} \bar{B}_i$, where B is an open ball and $\{\bar{B}_0, \ldots, \bar{B}_{k-1}\}$ is a pairwise disjoint family of closed balls contained in B, then X fulfills the requirements of the theorem. Part (b) of the theorem is a slight modification of its first part. This modification is needed in the proof that $\mathrm{UC}(X)$ and $\mathrm{EXT}(X)$ are not isomorphic unless they coincide.

Theorem 7.7. (a) *Let $X \in K^{\mathcal{O}}_{\mathrm{BNC}}$. Suppose that the following hold.*

(1) *X is bounded and X is UD.AC*,
(2) $\mathrm{bd}(X)$ *has finitely many connected components*,
(3) *if $C \in \mathrm{Cmp}(\mathrm{bd}(X))$, $x, y \in C$ are distinct and $z \in \mathrm{bd}(X) - \{x, y\}$, then there is $f \in \mathrm{UC}(X)$ such that either $f^{\mathrm{cl}}(x) = y$ and $f^{\mathrm{cl}}(z) = z$, or $f^{\mathrm{cl}}(z) = y$ and $f^{\mathrm{cl}}(x) = x$*,
(4) *X is JN.TC*,

Let $Y \in K^{\mathcal{O}}_{\mathrm{BNC}}$ and assume that

(5) *if C is a component of* $\mathrm{bd}(Y)$, *then $|C| > 1$.*

Let $\tau \in H(X, Y)$ be such that $(\mathrm{UC}(X))^\tau \subseteq \mathrm{UC}(Y)$. Then τ^{-1} is uniformly continuous.

(b) *Modify clause* (3) *of* (a) *by requiring that* $f \in \mathrm{UC_e}(X)$, *and modify* (4) *by requiring that* X *is JN.ETC. Let* $\tau \in H(X,Y)$ *be such that* $(\mathrm{UC_e}(X))^\tau \subseteq \mathrm{UC}(Y)$. *Then* τ^{-1} *is uniformly continuous.*

Proof. The proofs of (a) and (b) are identical. We prove (a). Recall that X and Y are subsets of the Banach spaces E and F respectively. By Theorem 5.5, τ is uniformly continuous.

Claim 1. *Let* $\vec{x} \subseteq X$ *be a completely discrete sequence such that* $\tau(\vec{x})$ *is a Cauchy sequence. Then there is a sequence* $\vec{x}\,' \subseteq X$ *such that* $\lim_{n\to\infty} \delta(x'_n) = 0$, $\vec{x}\,'$ *is completely discrete*, *and* $\lim_{n\to\infty} \tau(\vec{x}\,') = \lim_{n\to\infty} \tau(\vec{x})$.

Proof. If $\delta(\vec{x}) = 0$, then we take $\vec{x}\,'$ to be a subsequence of $\vec{x}$ such that $\lim_{n\to\infty} \delta(x'_n) = 0$. Suppose otherwise. Since $X \in K^{\mathcal{O}}_{\mathrm{BNC}}$, we may assume that for some $d > 0$, $\vec{x}$ is d-spaced, and since X is bounded, we may also assume that for every $n \in \mathbb{N}^+$, $d(x_n, x_0) \le d + d/8$. Without loss of generality, $x_0 = 0$. For every $n \in \mathbb{N}^+$ let $t_n = \min(\{t > 1 \mid t x_n \in \mathrm{bd}(X)\})$, $y_n = t_n x_n$, $L_n = [x_n, y_n]$ and $\gamma_n(t) = x_n + t(y_n - x_n)$, $t \in [0,1]$. If $m \ne n$, then

$$\left\| d \cdot \frac{x_m}{\|x_m\|} - d \cdot \frac{x_n}{\|x_n\|} \right\| \ge \|x_m - x_n\| - \left\| x_m - d \cdot \frac{x_m}{\|x_m\|} \right\| - \left\| x_n - d \cdot \frac{x_n}{\|x_n\|} \right\| \ge \frac{3d}{4}.$$

Hence by Proposition 7.5(b), $d(L_m, L_n) \ge 3d/8$.

Define $\eta(t) = \delta(\{\gamma_n(t) \mid n \in \mathbb{N}^+\})$. Since $\{\|x_n - y_n\| \mid n \in \mathbb{N}\}$ is bounded, η is continuous. Also, $\eta(1) = 0$. Let $s = \min(\eta^{-1}(0))$. We may assume that for every $n \in \mathbb{N}^+$, $\delta(\gamma_n(s)) < 1/n$. It follows that for every $t \in (0,s)$, the family $\{\gamma_n([0,t]) \mid n \in \mathbb{N}^+\}$ is spaced, and $\delta(\bigcup\{\gamma_n([0,t]) \mid n \in \mathbb{N}^+\}) > 0$. Also, since X is bounded, $\{d(x_n, \gamma_n(t)) \mid n \in \mathbb{N}^+\}$ is bounded. So for every $t < s$ there is $h_t \in \mathrm{UC}(X)$ such that for every $n \in \mathbb{N}^+$, $h_t(x_{2n}) = \gamma_{2n}(t)$ and $h_t(x_{2n-1}) = x_{2n-1}$. Let $z^* = \lim \tau(\vec{x})$. Let $t \in (0,s)$. Clearly, $\tau(\{\gamma_{2n}(t) \mid n \in \mathbb{N}^+\} \cup \{x_{2n-1} \mid n \in \mathbb{N}^+\}) = (h_t)^\tau(\vec{x})$, and since $(h_t)^\tau \in \mathrm{UC}(Y)$ and $\tau(\vec{x})$ is a Cauchy sequence, $\tau(\{\gamma_{2n}(t) \mid n \in \mathbb{N}^+\} \cup \{x_{2n-1} \mid n \in \mathbb{N}^+\})$ is a Cauchy sequence. Denote this sequence by $\vec{u}$. Then $\tau(\{x_{2n-1} \mid n \in \mathbb{N}^+\})$ is a subsequence of $\vec{u}$ converging to z^*. So $\vec{u}$ converges to z^*, and hence $\tau(\{\gamma_{2n}(t) \mid n \in \mathbb{N}^+\})$ converges to z^*. Let $\vec{s} \subseteq (0,s)$ be a sequence converging to s. For every $n \in \mathbb{N}^+$ let $k_n \ge n$ be such that $d(\tau(\gamma_{2k_n}(s_n)), z^*) < 1/n$. Let $x'_n = \gamma_{2k_n}(s_n)$. So $\lim \tau(\vec{x}\,') = z^*$, $\lim_{n\to\infty} \delta(x'_n) = 0$ and $\vec{x}\,'$ is spaced. Claim 1 is thus proved.

Claim 2. *Let* $T = \langle \vec{y}, y^*, \{L_n \mid n \in \mathbb{N}\}, \vec{z}\rangle$ *be a completely discrete track system in* X, *and suppose that* $\lim \tau(\vec{y}) = w^*$. *Then* $\tau^{\mathrm{cl}}(y^*) = w^*$.

Proof. Suppose by contradiction that $\tau^{\mathrm{cl}}(y^*) \ne w^*$. Let γ_n be a legal parametrization of L_n, and $\beta \in \mathrm{MC}$ be such that for every $t_1, t_2 \in [0,1]$ and $n \in \mathbb{N}$, $\gamma_n(t_1) - \gamma_n(t_2) \le \beta(|t_1 - t_2|)$.

We now follow the proof of Lemma 5.25. For every infinite $\sigma \subseteq \mathbb{N}$ and $t \in [0,1]$ let $A[\sigma, t] = \{\gamma_n(t) \mid n \in \sigma\}$ and $s_\sigma = \inf(\{t \in [0,1] \mid \tau(A[\sigma,t])$ converges to $w^*\})$. Since $\tau^{\mathrm{cl}}(y^*) \ne w^*$, there is $U \in \mathrm{Nbr}(y^*)$ such that $d(w^*, \tau(U \cap X)) > 0$. Thus there is $t_0 > 0$ such that for every $t < t_0$, $d(w^*, \tau(A[\mathbb{N}, t])) > 0$. So for every infinite $\sigma \subseteq \mathbb{N}$, $s_\sigma > 0$. As in Lemma 5.25, there is an infinite $\sigma \subseteq \mathbb{N}$ such that for every infinite $\eta \subseteq \sigma$, $s_\eta = s_\sigma$. Write $s = s_\sigma$.

Suppose by contradiction that $d(A[\sigma,s],y^*)=0$. We may assume that $\lim A[\sigma,s]=y^*$. Let $r>0$. Then there is m such that $A[\sigma^{\geq m},s]\subseteq B(y^*,r/2)$. By the definition of s, there is $t\geq s$ such that $\beta(t-s)<r/2$ and $\lim \tau(A[\sigma,t])=w^*$. Then $A[\sigma^{\geq m},t]\subseteq B(y^*,r)$. Hence for every $r,\varepsilon>0$ there are $m\in\mathbb{N}$ and $t\in[s,s+\varepsilon)$ such that $A[\sigma^{\geq m},t]\subseteq B(y^*,r)$ and $\lim\tau(A[\sigma,t])=w^*$. It follows that there is a sequence $\vec{u}\subseteq X$ such that $\lim\vec{u}=y^*$ and $\lim\tau(\vec{u})=w^*$, and hence $\tau^{\mathrm{cl}}(y^*)=w^*$. A contradiction, so $d(A[\sigma,s],y^*)>0$.

From the fact that $\{L_n-B(y^*,r)\mid n\in\mathbb{N}\}$ is completely discrete for every $r>0$, it follows that $A[\sigma,s]$ is completely discrete. So we may assume that for some $d>0$, $A[\sigma,s]$ is d-spaced. Let α and η be as ensured by the fact that T is a track system. It follows from the equicontinuity of $\{\gamma_n\}_{n\in\mathbb{N}}$ that there is $\delta>0$ such that for every $n\in\mathbb{N}$ and $t_1,t_2\in[0,1]$: if $0<t_2-t_1<\delta$, then

$$\mathrm{diam}(\gamma_n([t_1,t_2]))+\eta(\mathrm{diam}(\gamma_n([t_1,t_2])))<d/3.$$

Choose $t_1\in[s,s+\delta/2)\cap[0,1]$ such that $\lim\tau(A[\sigma,t_1])=w^*$ and $t_2\in(s-\delta/2,s)\cap[0,1]$. For every $n\in\sigma$ let $x_n=\gamma_n(t_1)$, $u_n=\gamma_n(t_2)$ and $J_n=[x_n,u_n]^{L_n}$, that is, $J_n=\gamma_n([t_2,t_1])$. Since $|t_1-t_2|<\delta$, it follows that

$$\mathrm{diam}(B(J_n,\eta(\mathrm{diam}(J_n))))\leq\mathrm{diam}(J_n)+\eta(\mathrm{diam}(J_n))\leq d/3.$$

We may assume that $\sigma=\mathbb{N}$. Since T is a track system, there is $h_n\in H(X)$ such that $h_n(x_n)=u_n$, $\mathrm{supp}(h_n)\subseteq B(J_n,\eta(\mathrm{diam}(J_n)))$ and h_n is α-bicontinuous. We check that $\{\mathrm{supp}(h_n)\mid n\in\mathbb{N}\}$ is $d/3$-spaced. Let $m\neq n$. Then $\gamma_m(s),\gamma_n(s)\in A[\sigma,s]$ and so $\|\gamma_m(s)-\gamma_n(s)\|\geq d$. Since $\gamma_m(s)\in J_m$ and the same holds for n, it follows that

$$d(B(J_n,\eta(\mathrm{diam}(J_n))),B(J_n,\eta(\mathrm{diam}(J_n))))\geq d-2d/3=d/3.$$

So $\{\mathrm{supp}(h_n)\mid n\in\mathbb{N}\}$ is $d/3$-spaced.

By Proposition 5.17(a), $h:=\circ_{n\in\mathbb{N}}h_{2n}\in\mathrm{UC}(X)$. It follows that $h^\tau\in\mathrm{UC}(Y)$. Let $w_n=x_n$ if n is odd, and $w_n=u_n$ if n is even. Hence $h^\tau(\tau(\vec{x}))=\tau(\vec{w})$. By the choice of t_1, $\tau(\vec{x})$ converges to w^*. By the choice of σ and t_2, $\tau(\{u_{2n}\mid n\in\mathbb{N}\})$ does not converge to w^*. So $\tau(\vec{w})$ is not a Cauchy sequence. This contradicts the fact that $h^\tau\in\mathrm{UC}(Y)$. We have thus proved Claim 2.

CLAIM 3. $\mathrm{bd}(Y)\subseteq\mathrm{Rng}(\tau^{\mathrm{cl}})$.

Proof. Suppose by contradiction that $z^*\in\mathrm{bd}(Y)-\mathrm{Rng}(\tau^{\mathrm{cl}})$. Let $\vec{z}\subseteq Y$ converge to z^*. So $\vec{x}:=\tau^{-1}(\vec{z})$ is completely discrete. By Claim 1, we may assume that $\lim_{n\to\infty}\delta(x_n)=0$. Let $\vec{y}$ be a subsequence of $\vec{x}$ which has a track system. By Proposition 7.5(e), $\vec{y}$ has a completely discrete track system $\langle\vec{y},y^*,\{L_n\mid n\in\mathbb{N}\},\vec{y}'\rangle$. By Claim 2, $\tau^{\mathrm{cl}}(y^*)=z^*$. A contradiction, so Claim 3 is proved.

CLAIM 4. *If $C\in\mathrm{Cmp}(\mathrm{bd}(X))$, then $\tau^{\mathrm{cl}}(C)$ is closed in F.*

Proof. Let $C\in\mathrm{Cmp}(\mathrm{bd}(X))$ and $v\in\mathrm{cl}(\tau^{\mathrm{cl}}(C))$. Let $\vec{x}'\subseteq C$ be such that $\lim_{n\to\infty}\tau^{\mathrm{cl}}(x'_n)=v$. If $\vec{x}'$ has a Cauchy subsequence $\vec{y}$, then $\lim\vec{y}\in C$ and $\tau^{\mathrm{cl}}(\lim\vec{y})=v$. Suppose that $\vec{x}'$ does not have Cauchy subsequences, that is, $\vec{x}'$ is completely discrete. There is $\vec{x}\subseteq X$ such that $\lim_{n\to\infty}d(x_n,x'_n)=0$ and $\lim_{n\to\infty}\tau(x_n)=v$. So $\vec{x}$ is completely discrete. Since X is JN.TC, there are a subsequence $\vec{y}$ of $\vec{x}$ and a track system $T=\langle\vec{y},z^*,\vec{L},\vec{z}\rangle$.

By Proposition 7.5(e), we may assume that T is a spaced track system, and by 7.5(f), $z^* \in C$. By Claim 2, $\tau^{\rm cl}(z^*) = v$, so $\tau^{\rm cl}(C)$ is closed.

CLAIM 5. *$\tau^{\rm cl}$ is* 1-1.

Proof. By (3), for every component $C \in \mathrm{Cmp}(\mathrm{bd}(X))$, either $\tau^{\rm cl} \restriction C$ is 1-1 or $\tau^{\rm cl}(C)$ is a singleton; and for any distinct $C, D \in \mathrm{Cmp}(\mathrm{bd}(X))$, either $\tau^{\rm cl}(C) = \tau^{\rm cl}(D)$ and $\tau^{\rm cl}(C)$ is a singleton, or $\tau^{\rm cl}(C) \cap \tau^{\rm cl}(D) = \emptyset$. The argument is as in the proof of Theorem 7.1.

Suppose by contradiction that $\tau^{\rm cl}$ is not 1-1. Then there is $C \in \mathrm{Cmp}(\mathrm{bd}(X))$ and $y \in \mathrm{bd}(Y)$ such that $\tau^{\rm cl}(C_0) = \{y\}$. Let D be the component of y in $\mathrm{bd}(Y)$. Then $|D| > 1$. By Claims 3 and 4, $\{\tau^{\rm cl}(C) \mid C \in \mathrm{Cmp}(\mathrm{bd}(X))$ and $\tau^{\rm cl}(C) \subseteq D\}$ is a partition of D into finitely many and more than 1 closed sets. This contradicts the connectivity of D.

CLAIM 6. *Let $T = \langle \vec{x}, y^*, \vec{L}, \vec{y} \rangle$ be a track system in X. Then for every $d > 0$ there is $h \in \mathrm{UC}(X)$ such that $h^{\rm cl}(y^*) \neq y^*$ and* $\mathrm{supp}(h) \subseteq B(y^*, d)$.

Proof. Let α and η be as ensured by the fact that T is a track system. We may assume that $y^* \notin \mathrm{Rng}(\vec{x})$, and hence we may also assume that $d < d(\vec{x}, y^*)$. Let $a > 0$ be such that $2a + \eta(a) < d$ and b be such that $\alpha(b) < a - b$. Clearly, $b < a$. Let n be such that $\|y_n - y^*\| < b$. Then $\|x_n - y_n\| \geq d - b > a$, and hence there is $z \in L_n$ such that $\|z - y_n\| = \mathrm{diam}([z, y_n]^{L_n}) = a$. Since L_n is an $\langle \alpha, \eta \rangle$-track, there is $h \in H(X)$ such that h is α-bicontinuous, $h(y_n) = z$ and $\mathrm{supp}(h) \subseteq B([z, y_n]^{L_n}, \eta(a))$. Clearly, $B([z, y_n]^{L_n}, \eta(a)) \subseteq B(y^*, b + a + \eta(a)) \subseteq B(y^*, d)$. So $\mathrm{supp}(h) \subseteq B(y^*, d)$. Suppose by way of contradiction that $h(y^*) = y^*$. Then $\|z - y^*\| = \|h(y_n) - h(y^*)\| \leq \alpha(\|y_n - y^*\| < \alpha(b)$. However, $\|z - y^*\| \geq \|z - y_n\| - \|y_n - y^*\| \geq a - b$. That is, $\alpha(b) > a - b$, a contradiction. So $h(y^*) \neq y^*$. So Claim 6 is proved.

CLAIM 7. *There is no sequence $\vec{x} \subseteq X$ such that $\vec{x}$ is completely discrete, and $\tau(\vec{x})$ is a Cauchy sequence.*

Proof. Suppose otherwise, and let $\vec{x}$ be a counter-example to the claim. By Claim 1, we may assume that $\lim_{n\to\infty} \delta(x_n) = 0$. Since X is JN.TC, there are a subsequence $\vec{y}$ of $\vec{x}$, y^*, $\vec{L}$ and $\vec{z}$ such that $T = \langle \vec{y}, y^*, \vec{L}, \vec{z} \rangle$ is a track system. By Proposition 7.5(e), we may assume that T is a spaced track system. Let $w = \lim \tau(\vec{x})$. So $w = \lim \tau(\vec{y})$. By Claim 2, (i) $\tau^{\rm cl}(y^*) = w$. Since $y^* \in \mathrm{bd}(X)$ and $\vec{y} \subseteq X$, it follows that $y^* \notin \mathrm{Rng}(\vec{y})$, and since $\vec{y}$ is completely discrete, $d(\vec{y}, y^*) > 0$. By Claim 6, there is $h \in \mathrm{UC}(X)$ such that (ii) $h^{\rm cl}(y^*) \neq y^*$ and $\mathrm{supp}(h) \subseteq B(y^*, d(\vec{y}, y^*))$. So $h \restriction \vec{y} = \mathrm{Id}$. By Proposition 7.5(g), $T' := \langle h(\vec{y}), h^{\rm cl}(y^*), h(\vec{L}), h(\vec{z}) \rangle$ is a track system. Since T is spaced and $h \in \mathrm{UC}(X)$ it follows that T' is also spaced. Recall that $h(\vec{y}) = \vec{y}$ and so $\lim h(\vec{y}) = w$. So by Claim 2 applied to T', (iii) $\tau^{\rm cl}(h^{\rm cl}(y^*)) = w$. Facts (i)–(iii) contradict the fact that $\tau^{\rm cl}$ is 1-1. This proves Claim 7.

Suppose by contradiction that τ^{-1} is not uniformly continuous. Then there are sequences $\vec{x}, \vec{y} \subseteq X$ and $d > 0$ such that for every $n \in \mathbb{N}$, $d(x_n, y_n) \geq d$ and $\lim_{n\to\infty} d(\tau(x_n), \tau(y_n)) = 0$. We may assume that each of the sequences $\vec{x}$, $\vec{y}$, $\tau(\vec{x})$ and $\tau(\vec{y})$ is either spaced or is a Cauchy sequence.

CLAIM 8. *The sequences $\vec{x}$, $\vec{y}$, $\tau(\vec{x})$ and $\tau(\vec{y})$ are spaced.*

Proof. Suppose by contradiction that $\vec{x}$ is a Cauchy sequence. Since $\tau^{\rm cl}$ is uniformly continuous and $\mathrm{Dom}(\tau^{\rm cl}) = \mathrm{cl}(X)$, it follows that $\tau(\vec{x})$ is a Cauchy sequence. Hence $\tau(\vec{y})$ is also a Cauchy sequence. If $\vec{y}$ is a Cauchy sequence, then $\tau^{\rm cl}$ is not 1-1, contradicting Claim 5; and if $\vec{y}$ is completely discrete, then Claim 7 is contradicted. So $\vec{x}$ is not a Cauchy sequence. The same is true for $\vec{y}$. By Claim 7, $\tau(\vec{x})$ and $\tau(\vec{y})$ are completely discrete. Claim 8 is proved.

We call a pair of sequences $\langle \vec{u}, \vec{v} \rangle$ in X a *counter-example* if $\vec{u}$ and $\vec{v}$ are spaced, $\inf(\{d(u_n, v_n) \mid n \in \mathbb{N}\}) > 0$ and $\lim_{n\to\infty} d(\tau(u_n), \tau(v_n)) = 0$.

Claim 9. *There is a counter-example $\langle \vec{u}, \vec{v} \rangle$ such that $\delta(\vec{u}) = 0$.*

Proof. By Claim 8, there is a counter-example $\langle \vec{x}, \vec{y} \rangle$. If $\delta(\vec{x}) = 0$ or $\delta(\vec{y}) = 0$, then there is nothing to prove. Suppose otherwise. By Proposition 7.6, we may assume that $d(\vec{x}, \vec{y}) > 0$. Let $d > 0$ be such that $\vec{x}$ is d-spaced and $d(\vec{x}, \vec{y}) \geq d$. By possibly interchanging $\vec{x}$ and $\vec{y}$, we may also assume that there are $e_1 \geq e_2 > 0$ such that $\lim_{n\to\infty} \|x_n\| = e_1$ and $\lim_{n\to\infty} \|y_n\| = e_2$. Let $x'_n = (e_1/\|x_n\|)x_n$. Since $\delta(\vec{x}) > 0$, there is $a > 0$ such that for every $n \in \mathbb{N}$, $B(x'_n, a) \subseteq X$. We may further assume that $a < d/8$, and that for every $n \in \mathbb{N}$, $d(x_n, x'_n) < a/2$. So $(\bigcup\{B(x'_n, a) \mid n \in \mathbb{N}\}) \cap \{y_n \mid n \in \mathbb{N}\} = \emptyset$, and for any distinct $m, n \in \mathbb{N}$, $d(B(x'_m, a), B(x'_n, a)) > d/2$. Let $x''_n = (1 + a/2)x'_n$. It follows that there is $h \in \mathrm{LIP}(X)$ such that for every $n \in \mathbb{N}$, $h(x_n) = x''_n$ and $\mathrm{supp}(h) \subseteq \bigcup\{B(x'_n, a) \mid n \in \mathbb{N}\}$. Since $h(\vec{x}) = \vec{x}''$ and $h(\vec{y}) = \vec{y}$, it follows that $\langle \vec{x}'', \vec{y} \rangle$ is a counter-example. So we may assume that $e_1 > e_2$, and that $\|x_n\| = e_1$ for every $n \in \mathbb{N}$. We still assume that $\vec{x}$ is d-spaced and that $d(\vec{x}, \vec{y}) \geq d$.

We now proceed as in the proof of Claim 1. For $n \in \mathbb{N}^+$ let $t_n = \min(\{t > 1 \mid tx_n \in \mathrm{bd}(X)\})$, $z_n = t_n x_n$, $L_n = [x_n, z_n]$ and $\gamma_n(t) = x_n + t(z_n - x_n)$, $t \in [0,1]$. By Proposition 7.5(b), for any distinct $m, n \in \mathbb{N}$, $d(L_m, L_n) \geq d/2$, and clearly, $d(L_m, \vec{y}) \geq e_1 - e_2$.

Let $s = \min(\{t \mid \delta(\{\gamma_n(t) \mid n \in \mathbb{N}^+\}) = 0\})$. We may assume that for every $n \in \mathbb{N}^+$, $\delta(\gamma_n(s)) < 1/n$. It follows that for every $t \in (0, s)$, the family $\{\gamma_n([0,t]) \mid n \in \mathbb{N}^+\}$ is spaced, $d(\bigcup\{\gamma_n([0,t]) \mid n \in \mathbb{N}^+\}, \vec{y}) > 0$ and $\delta(\bigcup\{\gamma_n([0,t]) \mid n \in \mathbb{N}^+\}) > 0$. Also, since X is bounded, $\{d(x_n, \gamma_n(t)) \mid n \in \mathbb{N}^+\}$ is bounded. So for every $t < s$ there is $h_t \in \mathrm{UC}(X)$ such that for every $n \in \mathbb{N}^+$, $h_t(x_n) = \gamma_n(t)$ and $h_t(y_n) = y_n$. Since $h_t^\tau \in \mathrm{UC}(Y)$, $\lim_{n\to\infty} d(\tau(x_n), \tau(y_n)) = 0$ and $h_t^\tau(\tau(x_n)) = \tau(\gamma_n(t))$, it follows that $\lim_{n\to\infty} d(\tau(\gamma_n(t)), \tau(y_n)) = 0$.

Let $\vec{s} \subseteq (0, s)$ be a sequence converging to s. For every $n \in \mathbb{N}^+$ let $k_n \geq n$ be such that $d(\tau(\gamma_{k_n}(s_n)), \tau(y_n)) < 1/n$. Define $x'_n = \tau(\gamma_{k_n}(s_n))$. It follows that $d(\vec{x}', \vec{y}) > 0$, $\lim_{n\to\infty} d(\tau(x'_n), \tau(y_n)) = 0$, $\lim_{n\to\infty} \delta(x'_n) = 0$ and $\vec{x}'$ is spaced. Claim 9 is thus proved.

Let $T = \langle \vec{y}, y^*, \vec{L}, \vec{z} \rangle$ be a track system, and γ_n be a legal parametrization of L_n. We say that T is *good* if for every $t \in [0,1)$, $\inf(\{d(y_n, \gamma_n([0,t])) \mid n \in \mathbb{N}\}) > 0$.

Claim 10. *If $\langle \vec{y}, y^*, \vec{L}, \vec{z} \rangle$ is a track system, and γ_n is a legal parametrization of L_n, then there is $s \in (0,1]$ and an infinite $\sigma \subseteq \mathbb{N}$ such that $\langle \{\gamma_n(s) \mid n \in \sigma\}, y^*, \{\gamma_n([0,s]) \mid n \in \sigma\}, \vec{z} \rangle$ is a good track system, and $\lim_{n\in\sigma} d(\tau(y_n), \tau(\gamma_n(s))) = 0$.*

Proof. For every infinite $\eta \subseteq \mathbb{N}$ let $s_\eta = \inf(\{t \in [0,1] \mid \lim_{n\in\eta} d(y_n, \gamma_n(t)) = 0\})$. As in previous analogous arguments, there is an infinite $\eta \subseteq \mathbb{N}$ such that for every infinite $\zeta \subseteq \eta$, $s_\zeta = s_\eta$. Let $s = s_\eta$ and $\vec{t}$ be a sequence converging to s such that for every $i \in \mathbb{N}$, $\lim_{n\in\eta} d(y_n, \gamma_n(t_i)) = 0$. Let $\sigma = \{n_i \mid i \in \mathbb{N}\} \subseteq \eta$ be an increasing sequence such that $\lim_{i\to\infty} d(y_{n_i}, \gamma_{n_i}(t_i)) = 0$. By the equicontinuity of $\{\gamma_n\}_{n\in\mathbb{N}}$, $\lim_{i\to\infty} d(\gamma_{n_i}(t_i), \gamma_{n_i}(s)) = 0$. So $\lim_{n\in\sigma} d(y_n, \gamma_n(s)) = 0$. Hence since τ is uniformly continuous, $\lim_{n\in\sigma} d(\tau(y_n), \tau(\gamma_n(s))) = 0$. Now suppose by contradiction that there is $t < s$ such that $\liminf_{n\in\sigma} d(y_n, \gamma_n([0,t])) = 0$. So there is an increasing sequence $\zeta = \{k_i \mid i \in \mathbb{N}\} \subseteq \sigma$ and $\vec{t} \subseteq [0,t]$ such that $\lim_{i\to\infty} d(y_{k_i}, \gamma_{k_i}(t_i)) = 0$. We may assume that $\vec{t}$ converges, say to s^*. Hence $s^* \le t < s$, and $\lim_{n\in\zeta} d(y_n, \gamma_n(s^*)) = 0$. So $s_\zeta \le s^* < s$, a contradiction. So for every $t \in [0,s)$, $\liminf_{n\in\sigma} d(y_n, \gamma_n([0,t])) > 0$. Since $\lim_{n\in\sigma} d(y_n, \gamma_n(s)) = 0$, it follows that for every $t \in [0,s)$, $\liminf_{n\in\sigma} d(\gamma_n(s), \gamma_n([0,t])) > 0$; and the fact that L_n is a simple arc implies that $\gamma_n(s) \notin \gamma_n([0,s))$. So $\inf(\{d(\gamma_n(s), \gamma_n([0,t])) \mid n \in \sigma\}) > 0$. Claim 10 is proved.

CLAIM 11. *There are a counter-example $\langle \vec{u}, \vec{v} \rangle$ and a completely discrete track system $\langle \vec{u}, u^*, \vec{J}, \vec{u}' \rangle$ such that* $\inf_{n\in\mathbb{N}} d(J_n, v_n) > 0$.

Proof. By Claim 9, there is a counter-example $\langle \vec{x}, \vec{y} \rangle$ such that $\delta(\vec{x}) = 0$. Let $T = \langle \vec{x}, x^*, \vec{L}, \vec{x}' \rangle$ be a completely discrete track system for $\vec{x}$. By Claim 10, we may assume that T is a good track system.

Suppose first that $d := \liminf_{n\to\infty} d(L_n, y_n) > 0$. Let $\{\ell_i \mid i \in \mathbb{N}\}$ be a subsequence of $\mathbb{N}$ such that $d(L_{\ell_i}, y_{\ell_i}) \ge d/2$. Hence $\vec{u} = \{x_{\ell_i} \mid i \in \mathbb{N}\}$, $u^* = x^*$, $\vec{v} = \{y_{\ell_i} \mid i \in \mathbb{N}\}$ and $\vec{J} = \{L_{\ell_i} \mid i \in \mathbb{N}\}$ are as required in the claim.

Assume next that $\liminf_{n\to\infty} d(L_n, y_n) = 0$. So we may assume that $\lim_{n\to\infty} d(L_n, y_n) = 0$. Let γ_n be a legal parametrization of L_n. Hence there is $\vec{t} \subseteq [0,1]$ such that $\lim_{n\to\infty} d(\gamma_n(t_n), y_n) = 0$. We may assume that $\vec{t}$ is convergent. Let $t = \lim \vec{t}$. It easily follows that $\lim_{n\to\infty} d(\gamma_n(t), y_n) = 0$. Clearly $t < 1$, for otherwise $\lim_{n\to\infty} d(x_n, y_n) = 0$. For every $n \in \mathbb{N}$ let $u_n = \gamma_n(t)$, $v_n = x_n$ and $J_n = \gamma_n([0,t])$.

Since τ is uniformly continuous, we know that $\lim_{n\to\infty} d(\tau(u_n), \tau(y_n)) = 0$. Also, $\lim_{n\to\infty} d(\tau(v_n), \tau(y_n)) = 0$. Hence $\lim_{n\to\infty} d(\tau(u_n), \tau(v_n)) = 0$. Since $\langle \vec{x}, x^*, \vec{L}, \vec{x}' \rangle$ is a good track system, $\inf_{n\in\mathbb{N}} d(x_n, \gamma_n([0,t])) > 0$. That is, $\inf_{n\in\mathbb{N}} d(v_n, J_n) > 0$. By Proposition 7.5(c)(i) applied to T and t, $\langle \vec{u}, x^*, \vec{J}, \vec{x}' \rangle$ is a track system. So $\vec{u}$, $\vec{v}$, x^* and $\vec{J}$ are as required. Claim 11 is proved.

CLAIM 12. *There are a counter-example $\langle \vec{u}, \vec{v} \rangle$ and a completely discrete track system $\langle \vec{u}, u^*, \vec{J}, \vec{u}' \rangle$ such that $d(\bigcup\{J_n \mid n \in \mathbb{N}\}, \vec{v}) > 0$.*

Proof. Let $\langle \vec{u}, \vec{v} \rangle$ and $\langle \vec{u}, u^*, \vec{J}, \vec{u}' \rangle$ be as ensured by the previous claim. We show that there is an infinite $\sigma \subseteq \mathbb{N}$ such that $\langle \vec{u}{\restriction}\sigma, \vec{v}{\restriction}\sigma \rangle$ and $\langle \vec{u}{\restriction}\sigma, u^*, \vec{J}{\restriction}\sigma, \vec{u}'{\restriction}\sigma \rangle$ are as required in the claim. We shall apply Proposition 7.6 with F_n taken to be $\{v_n\}$ and K_n taken to be J_n. By our assumptions, clauses (i) and (iii) of 7.6 hold. We show that (ii) holds. Let γ_n be a legal parametrization of J_n. Suppose that $\varepsilon > 0$. Then by the equicontinuity of $\{\gamma_n\}_{n\in\mathbb{N}}$, there is $\delta > 0$ such that for every $n \in \mathbb{N}$ and $t_1, t_2 \in [0,1]$: if $|t_1 - t_2| < \delta$, then $\|\gamma_n(t_1) - \gamma_n(t_2)\| < \varepsilon$. Define $\ell_\varepsilon = [1/\delta] + 1$. Then ℓ_ε fulfills the requirement of clause (ii) of 7.6. The set σ obtained from 7.6 is as required. This proves Claim 12.

Conclusion of the proof of the theorem. Let $\langle \vec{x}, \vec{y} \rangle$ and $T = \langle \vec{x}, x^*, \vec{L}, \vec{x}' \rangle$ be as ensured by Claim 12. By Claim 7, $\tau(\vec{y})$ is completely discrete. So we may assume that $\tau(\vec{y})$ is spaced.

Write $d_1 = d(\bigcup\{L_n \mid n \in \mathbb{N}\}, \vec{y})$. Let γ_n be a legal parametrization of L_n. For every infinite $\sigma \subseteq \mathbb{N}$ let $s_\sigma = \inf(\{t \in [0,1] \mid \lim_{n\in\sigma} d(\tau(\gamma_n(t)), \tau(y_n)) = 0\})$. Let σ be such that for every infinite $\eta \subseteq \sigma$, $s_\eta = s_\sigma$. Since $\tau(\vec{x}')$ is convergent and $\tau(\vec{y})$ is spaced, $s := s_\sigma > 0$. As in previous analogous arguments, $\{\gamma_n(s) \mid n \in \sigma\}$ is completely discrete. So we may assume that for some $d_2 > 0$, $\{\gamma_n(s) \mid n \in \sigma\}$ is d_2-spaced. Set $d = \min(d_1, d_2)$. Let α, η be as ensured by the fact that T is a track system. Let $a > 0$ be such that $a + \eta(a) < d/3$. By the equicontinuity of $\{\gamma_n\}_{n\in\mathbb{N}}$, there is $\delta > 0$ such that for every $n \in \mathbb{N}$ and $t_1, t_2 \in [0,1]$: if $|t_1 - t_2| < \delta$, then $\|\gamma_n(t_1) - \gamma_n(t_2)\| < a$. By the choice of s, there is $t_1 \in [s, s+\delta/2)$ such that $\lim_{n\in\sigma} d(\tau(\gamma_n(t_1)), \tau(y_n)) = 0$. Also, choose $t_2 \in (s-\delta/2, s)$. Then by the choice of σ and s, $\inf_{n\in\sigma} d(\tau(\gamma_n(t_2)), \tau(y_n)) > 0$. For $n \in \sigma$ write $u_n = \gamma_n(t_1)$, $v_n = \gamma_n(t_2)$ and $J_n = \gamma_n([t_2, t_1])$. Let $n \in \sigma$. Then since L_n is an $\langle \alpha, \eta \rangle$-track, there is $h_n \in H(X)$ such that h_n is α-bicontinuous, $h_n(u_n) = v_n$ and $\operatorname{supp}(h_n) \subseteq B(J_n, \eta(\operatorname{diam}(J_n)))$. Since $|t_1 - t_2| < \delta$, it follows that $\operatorname{diam}(J_n) < a$. So for every $x \in \operatorname{supp}(h_n)$, $\|x - \gamma_n(s)\| < a + \eta(a) < d_2/3$. This implies that $d(\operatorname{supp}(h_m), \operatorname{supp}(h_n)) > d_2/3$ for any $m \neq n$. We conclude that $h := \circ_{n\in\sigma_1} h_n$ is well defined and belongs to $\mathrm{UC}(X)$. Clearly, $\operatorname{supp}(h) \subseteq B(\bigcup_{n\in\mathbb{N}} L_n, \eta(a))$. Since $d(\bigcup_{n\in\mathbb{N}} L_n, \vec{y}) = d_1$ and $\eta(a) < d_1$, we infer that $\operatorname{supp}(h) \cap \operatorname{Rng}(\vec{y}) = \emptyset$ and hence $h\restriction\vec{y} = \mathrm{Id}$. It follows that $\inf_{n\in\sigma} d(h^\tau(\tau(y_n)), h^\tau(\tau(u_n))) = \inf_{n\in\sigma} d(\tau(y_n), \tau(v_n))) > 0$. But $\lim_{n\in\sigma} d(\tau(y_n), \tau(u_n)) = 0$. So $h^\tau \notin \mathrm{UC}(Y)$. A contradiction. ∎

Remark 7.8. (a) Clause (2) in Theorem 7.7 can be relaxed. In that case (5) has to be strengthened. Replace (2) and (5) by (2.1) and (5.1) stated below.

(2.1) $\operatorname{bd}(X)$ has countably many components.

(5.1) If C is a component of $\operatorname{bd}(Y)$, then C is not a singleton, and either C is arcwise connected or C is locally connected.

The proof of 7.7 is changed only in one place. In the proof of Claim 5, the component D of $\operatorname{bd}(Y)$ is partitioned into countably many closed sets. By (5.1), this is impossible. So a contradiction is reached.

There are spaces X which satisfy (1), (2.1), (3) and (4), but do not satisfy (2). However, such examples are rare.

(b) Let $K^{\mathcal{O}}_{\mathrm{BLPM}} = \{Y \mid Y$ is an open subset of a Banach Lipschitz manifold$\}$ (see Definition 6.29). In Theorem 7.7 replace the assumption that $X \in K^{\mathcal{O}}_{\mathrm{BNC}}$ by the assumption that $X \in K^{\mathcal{O}}_{\mathrm{BLPM}}$. Then parts (a) and (b) of 7.7 remain true, and the proof remains as is.

(c) The sphere of a Banach space satisfies the assumptions of (b). See Remark 7.4. □

Question 7.9. (a) Prove Theorem 7.7 for incomplete normed spaces.

(b) Let E be a Banach space. Let $\{\bar{B}_n \mid n \in \mathbb{N}\}$ be a spaced set of closed balls such that for every n, $\bar{B}_n \subseteq B^E(0,1)$. Let $X = B^E(0,2) - \bigcup_{n\in\mathbb{N}} \bar{B}_n$. Let $Y \in K^{\mathcal{O}}_{\mathrm{BNC}}$ and $\tau \in H(X,Y)$. Suppose that $(\mathrm{UC}(X))^\tau \subseteq \mathrm{UC}(Y)$. Is τ^{-1} uniformly continuous?

Note that X is not JN.TC, but it satisfies all the other assumptions of Theorem 7.7. □

Proposition 7.10. *Suppose that X is an open ball of a Banach space. Then X satisfies clauses* (1)–(4) *of Theorem* 7.7(b).

Proof. The proof is easy and is left to the reader. ∎

7.2. The nonexistence of isomorphisms between groups of different types. In the previous chapters we considered groups of various types. We now show that groups of different types cannot be isomorphic unless they coincide. We shall deal with the groups $\mathrm{UC}(X)$, $\mathrm{LUC}(X)$, $\mathrm{BUC}(X)$, $\mathrm{BPD.UC}(X)$ and $\mathrm{EXT}(X)$, and we add to this list the group $H(X)$. Let $\mathcal{P}, \mathcal{Q}$ denote one of the above properties and $\mathcal{P}(X), \mathcal{Q}(X)$ be the groups they define. We describe the situation precisely. It may happen that for distinct properties $\mathcal{P}$ and $\mathcal{Q}$, there is φ such that $\varphi : \mathcal{P}(X) \cong \mathcal{Q}(Y)$. But in that case either $\mathcal{P}(X) = \mathcal{Q}(X)$ and φ is induced by a homeomorphism belonging to $\mathcal{Q}^{\pm}(X, Y)$, or $\mathcal{P}(Y) = \mathcal{Q}(Y)$, and φ is induced by a homeomorphism belonging to $\mathcal{P}^{\pm}(X, Y)$. The situation with regard to such questions is not sorted out completely, and we only state results which follow directly from the theorems that have been proved so far. Only some of the possible consequences are stated and proved.

Let $X \in K^{\mathcal{O}}_{\mathrm{NRM}}$ and $h \in H(X)$. Recall that h is said to be internally extendible if there is $\bar{h} \in H(\overline{\mathrm{int}}(X))$ such that $\bar{h} \supseteq h$. Denote $\bar{h}$ by $h^{\overline{\mathrm{int}}}$. If $\mathcal{P} = \mathrm{UC}, \mathrm{BUC}, \mathrm{BPD.UC}$, then $\mathcal{P}(X) \subseteq \mathrm{IXT}(X)$. See Definition 2.24(b). For these $\mathcal{P}$'s define $X^{\mathcal{P}} = \overline{\mathrm{int}}(X)$ and $\mathcal{P}^{\mathrm{BNO}}(X) = \{h^{\overline{\mathrm{int}}} \mid h \in \mathcal{P}(X)\}$. So $\langle X^{\mathcal{P}}, \mathcal{P}^{\mathrm{BNO}}(X)\rangle \in K_{\mathrm{BO}}$. See Definition 2.7(b). For $\mathcal{P} = \mathrm{LUC}, \mathrm{EXT}$, write $X^{\mathcal{P}} = X$ and $\mathcal{P}^{\mathrm{BNO}}(X) = \mathcal{P}(X)$. So $\langle X^{\mathcal{P}}, \mathcal{P}^{\mathrm{BNO}}(X)\rangle \in K_{\mathrm{NO}}$.

Corollary 7.11. *Let $X, Y \in K^{\mathcal{O}}_{\mathrm{NRM}}$.*

(a) *If $\varphi : \mathrm{LUC}(X) \cong \mathcal{P}(Y)$, then $\mathcal{P}(Y) = \mathrm{LUC}(Y)$, and there is $\tau \in \mathrm{LUC}^{\pm}(X, Y)$ which induces φ.*

(b) *Let $X, Y \in K^{\mathcal{O}}_{\mathrm{NFCB}}$. Assume that X is BUD.AC and MV1, Y is UD.AC and that $\varphi : \mathrm{UC}(X) \cong \mathrm{BUC}(Y)$. Then $\mathrm{BUC}(X) = \mathrm{UC}(X)$, and there is $\tau \in \mathrm{BUC}^{\pm}(X, Y)$ which induces φ. (X may be unbounded, and X need not be UC-equivalent to Y.)*

(c) *Let $X, Y \in K^{\mathcal{O}}_{\mathrm{NFCB}}$. Suppose that X is BPD.AC, Y is UD.AC, and Y has the discrete path property for large distances. Let $\varphi : \mathrm{UC}(X) \cong \mathrm{BPD.UC}(Y)$. Then $\mathrm{BPD.UC}(X) = \mathrm{UC}(X)$, and there is $\tau \in \mathrm{BPD.UC}^{\pm}(X, Y)$ which induces φ.*

(d) *Let $X, Y \in K^{\mathcal{O}}_{\mathrm{BNC}}$. Suppose that X is BPD.AC and BR.LC.AC. Let $\varphi : \mathrm{BUC}(X) \cong \mathrm{BPD.UC}(Y)$. Then $\mathrm{BUC}(X) = \mathrm{BPD.UC}(X)$, and there is $\tau \in \mathrm{BPD.UC}^{\pm}(X, Y)$ which induces φ.*

(e) *Suppose that $X, Y \in K^{\mathcal{O}}_{\mathrm{BNC}}$, and X or Y is infinite-dimensional. Then there is no $\varphi : \mathrm{UC}(X) \cong \mathrm{EXT}(Y)$. (Since $\mathrm{EXT}(X) = \mathrm{BUC}(X)$ whenever X is finite-dimensional, such cases are included in* (c).*)*

(f) *Suppose that $X, Y \in K^{\mathcal{O}}_{\mathrm{BNC}}$, and X or Y is infinite-dimensional. Then there is no $\varphi : \mathrm{UC}(X) \cong H(Y)$.*

Proof. (a) Since $\mathcal{P}^{\mathrm{BNO}}(Y) \cong \mathcal{P}(Y)$, there is $\bar{\varphi} : \mathrm{LUC}(X) \cong \mathcal{P}^{\mathrm{BNO}}(Y)$. We have $\langle Y^{\mathcal{P}}, \mathcal{P}^{\mathrm{BNO}}(Y)\rangle \in K_{\mathrm{BNO}}$. Also $\langle X, \mathrm{LUC}(X)\rangle \in K_{\mathrm{BNO}}$. So by Theorem 2.8(b), there is $\tau \in H(X, Y^{\mathcal{P}})$ which induces $\bar{\varphi}$. Since $\langle X, \mathrm{LUC}(X)\rangle$ is transitive, $\langle Y^{\mathcal{P}}, \mathcal{P}^{\mathrm{BNO}}(Y)\rangle$ is transitive. Since Y is an orbit of $\langle Y^{\mathcal{P}}, \mathcal{P}^{\mathrm{BNO}}(Y)\rangle$, $Y^{\mathcal{P}} = Y$. Hence $\bar{\varphi} = \varphi$, and hence τ induces φ.

Note that if $\mathcal{P} = \mathrm{UC}, \mathrm{LUC}, \mathrm{BUC}, \mathrm{BPD.UC}$, then $\mathrm{UC}_{00}(Y) \subseteq \mathcal{P}(Y)$. So $(\mathrm{UC}_{00}(Y))^{\tau^{-1}} \subseteq (\mathcal{P}(Y))^{\tau^{-1}} \subseteq \mathrm{LUC}(X)$. Also, $\mathrm{UC}_{00}(Y) = \mathrm{UC}(Y, \mathcal{U})$, where $\mathcal{U}$ is the set of all open

BPD subsets of Y. So by Theorem 4.8(b), $\tau^{-1} \in \mathrm{LUC}^{\pm}(Y,X)$, that is, $\tau \in \mathrm{LUC}^{\pm}(X,Y)$. So $\mathcal{P}(Y) = (\mathrm{LUC}(X))^{\tau} = \mathrm{LUC}(Y)$.

(b) By Corollary 2.26 there is $\tau \in H(X,Y)$ which induces φ. So (†) $(\mathrm{UC}(X))^{\tau} = \mathrm{BUC}(Y)$. We show that $\tau \in \mathrm{BUC}(X,Y)$. By (†), $(\mathrm{UC}(X))^{\tau} \subseteq \mathrm{BUC}(Y)$ and $(\mathrm{BUC}(Y))^{\tau^{-1}} \subseteq \mathrm{BUC}(X)$. Recall that X is BUD.AC and MV1. So by Corollary 5.19, $\tau \in \mathrm{BUC}(X,Y)$.

We show that $\tau^{-1} \in \mathrm{UC}(Y,X)$. By (†), $\mathrm{UC}_0(Y))^{\tau^{-1}} \subseteq \mathrm{UC}(X)$. Recall that Y is UD.AC. So by Theorem 5.5, $\tau^{-1} \in \mathrm{UC}(Y,X)$, and hence $\tau \in \mathrm{BUC}^{\pm}(X,Y)$. Then $\mathrm{UC}(X) = (\mathrm{BUC}(Y))^{\tau^{-1}} = \mathrm{BUC}(X)$.

(c) Let $\varphi : \mathrm{UC}(X) \cong \mathrm{BPD.UC}(Y)$. By Corollary 2.26, there is $\tau \in H(X,Y)$ which induces φ. So $(*)$ $(\mathrm{UC}(X))^{\tau} = \mathrm{BPD.UC}(Y)$. By $(*)$, $(\mathrm{UC}_{00}(X))^{\tau} = \mathrm{BPD.UC}(Y)$. Recall that X is BPD.AC. Hence by Theorem 5.31, $\tau \in \mathrm{BPD.UC}(X,Y)$.

Obviously, $\mathrm{UC}_0(Y) \subseteq \mathrm{BPD.UC}(Y)$. So by $(*)$, $(\mathrm{UC}_0(Y))^{\tau^{-1}} \subseteq \mathrm{UC}(X)$. Recall that Y is UD.AC. Hence by Theorem 5.5, $\tau^{-1} \in \mathrm{UC}(Y,X)$. Since Y has the discrete path property for large distances, by Proposition 4.3(b), τ^{-1} is uniformly continuous for all distances. That is, for some $\alpha \in \mathrm{MC}$, τ^{-1} is α-continuous. In particular, τ^{-1} is boundedness preserving. So $\tau^{-1} \in \mathrm{BPD.UC}(Y,X)$. In summary, $\tau^{-1} \in \mathrm{BPD.UC}^{\pm}(Y,X)$. It follows that $\mathrm{UC}(X) = (\mathrm{BPD.UC}(Y))^{\tau^{-1}} = \mathrm{BPD.UC}(X)$.

(d) By Theorem 2.8(a), there is $\tau \in H(X,Y)$ which induces φ. This means that $(\mathrm{BUC}(X))^{\tau} = \mathrm{BPD.UC}(Y)$. By Theorem 5.31, $\tau \in \mathrm{BPD.UC}(X,Y)$, and by Theorem 5.41(a), $\tau^{-1} \in \mathrm{BPD.UC}(Y,X)$. Hence $\tau^{-1} \in \mathrm{BPD.UC}^{\pm}(Y,X)$. It follows that $\mathrm{BUC}(X) = (\mathrm{BPD.UC}(Y))^{\tau^{-1}} = \mathrm{BPD.UC}(X)$.

(e) Suppose by contradiction that $\varphi : \mathrm{UC}(X) \cong \mathrm{EXT}(Y)$. By Theorem 2.8(a), there is $\tau \in H(X,Y)$ which induces φ. So $(\mathrm{UC}(X))^{\tau} = \mathrm{EXT}(Y)$.

Suppose that Y is an open subset of the Banach space F. Let B be a ball in F such that $\mathrm{cl}^F(B) \subseteq Y$. Clearly, for every $h \in \mathrm{UC}_{\mathrm{e}}(B)$ there is $\tilde{h} \in \mathrm{EXT}(Y)$ which extends h. Let $\eta = \tau^{-1} \restriction B$ and $C = \eta(B)$. Since $(\mathrm{EXT}(Y))^{\tau^{-1}} \subseteq \mathrm{UC}(X)$, $(\mathrm{UC}_{\mathrm{e}}(B))^{\eta} \subseteq \mathrm{UC}(C)$. So also $(\mathrm{UC}_0(B))^{\eta} \subseteq \mathrm{UC}(C)$. So by Theorem 5.5, η is UC. It follows that C is bounded, and hence $\mathrm{bd}(C)$ is not a singleton. Clearly, $\mathrm{bd}(C) = \eta^{\mathrm{cl}}(\mathrm{bd}(B))$, and so $\mathrm{bd}(C)$ is connected. So no component of $\mathrm{bd}(C)$ is a singleton. By Proposition 7.10, B satisfies clauses (1)–(4) of Theorem 7.7(b). By Theorem 7.7(b) applied to B, C and η, η^{-1} is UC. In summary, $\eta \in \mathrm{UC}^{\pm}(B,C)$.

Choose $h \in H(B) - \mathrm{UC}(B)$ which is strongly extendible. So there is $\tilde{h} \in \mathrm{EXT}(Y)$ extending h. So $\tilde{h}^{\tau^{-1}} \in \mathrm{UC}(X)$. Hence $h^{\eta} = \tilde{h}^{\tau^{-1}} \restriction C \in \mathrm{UC}(C)$. Since $\eta^{-1} \in \mathrm{UC}^{\pm}(C,B)$, $h = (h^{\eta})^{\eta^{-1}} \in \mathrm{UC}(B)$. A contradiction.

(f) The proof is identical to that of (e). ■

The following trivial examples show that the conclusions of Corollary 7.11(b), (c) and (f) cannot be strengthened.

Example 7.12. (a) *There are regular open sets $X, Y \subseteq \mathbb{R}^2$ such that*

(1) $\mathrm{UC}(X) = \mathrm{BUC}(X) \cong \mathrm{BUC}(Y) \not\cong \mathrm{UC}(Y)$.
(2) *X is BUD.AC and MV1, and Y is UD.AC.*

(b) *Let $X = (0,1)$. Then* $\mathrm{UC}(X) = \mathrm{BPD.UC}(X)$.

(c) *Let E be a Banach space. Let $Y = B^E(0,1)$. Let $\tau : E \to Y$ be defined by $\tau(x) = x/(1 + \|x\|)$. Then $\tau \in \mathrm{BPD.UC}^{\pm}(E, Y)$, $\mathrm{BUC}(E) = \mathrm{BPD.UC}(E)$ and $\mathrm{BPD.UC}(Y) \supsetneqq \mathrm{BUC}(Y)$.*

Proof. (a) For $n \in \mathbb{N}$ we define an open set B_n by

$$B_n = B(0,1) - \bigcup_{i<n} \bar{B}((i/n, 0), 1/3n).$$

So B_n is obtained by removing from $B(0,1)$ n pairwise disjoint closed balls each of which contained in $B(0,1)$. For every $n \in \mathbb{N}$ let $X_n = (n,0) + \frac{1}{n+4} \cdot B_n$ and $Y_n = (n,0) + \frac{1}{4} \cdot B_n$. Let $X = \bigcup_{n\in\mathbb{N}} X_n$ and $Y = \bigcup_{n\in\mathbb{N}} Y_n$. Note that for every $n \neq m$, $d(X_n, X_m), d(Y_n, Y_m) \geq 1/2$ and $X_n \cong Y_n \ncong Y_m$. Note that $\lim_{n\to\infty} \mathrm{diam}(X_n) = 0$ and for every n, $\mathrm{diam}(Y_n) = 1/2$. It is easy to check that X and Y have the required properties.

The proofs of (b) and (c) are trivial. ∎

QUESTION 7.13. For $n > 1$, construct an open subset $X \subseteq \mathbb{R}^n$ such that $\mathrm{UC}(X) = \mathrm{BPD.UC}(X)$. Note that if X is such an example, then every connected component of X is an example. Note that every example which is a connected set is bounded. □

8. The group of locally Γ-continuous homeomorphisms of the closure of an open set

8.1. General description. Lipschitz equivalence between open subsets of $\mathbb{R}^n$ is relevant in the theory of function spaces. Suppose that U, V are open subsets of $\mathbb{R}^n$. The fact that U, V are homeomorphic by a bilipschitz homeomorphism or by a quasiconformal homeomorphism is equivalent to the fact that certain Sobolev spaces of functions from U to $\mathbb{R}$ and from V to $\mathbb{R}$ are isomorphic as lattice ordered vector spaces. These results appear in [GV1], [GV2] and [GRo]. We consider the analogous question for the setting in which the Sobolev function spaces are replaced by homeomorphism groups.

The simplest question of this kind is as follows. Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be open sets. Suppose that $\varphi : \mathrm{LIP}(\mathrm{cl}(X)) \cong \mathrm{LIP}(\mathrm{cl}(Y))$. Prove that there is $\tau \in \mathrm{LIP}^{\pm}(X, Y)$ which induces φ.

We shall prove the above statement for bounded open subsets of $\mathbb{R}^n$ which have a well-behaved boundary. In fact, we shall deal with a different group of homeomorphisms, namely, the group $\mathrm{LIP}^{\mathrm{LC}}(\mathrm{cl}(X))$ of locally bilipschitz homeomorphisms of $\mathrm{cl}(X)$. But for bounded subsets of $\mathbb{R}^n$ this group coincides with $\mathrm{LIP}(\mathrm{cl}(X))$.

The group of bilipschitz homeomorphisms is only a special case. It is generalized to the setting of Γ-bicontinuous homeomorphisms, where Γ is any principal modulus of continuity. (See Property M6 in Definition 1.9.)

The open sets for which we can prove such results at this point, have a very well-behaved boundary. They are called locally Γ-LIN-bordered sets. See Definition 8.1(c). Essentially these are the open subsets of a normed space whose closure is a manifold with boundary. For such sets we define the group of completely locally Γ-bicontinuous homeomorphisms. This group is denoted by $H_{\Gamma}^{\mathrm{CMP.LC}}(X)$, and is defined in Definition 8.2. We give here an equivalent definition. Let X be an open subset of a metric space E and Γ be a modulus of continuity. Define

$$H_{\Gamma}^{\mathrm{CMP.LC}}(X) = \{g \in H(\mathrm{cl}(X)) \mid g \text{ is locally } \Gamma\text{-bicontinuous and } g(X) = X\}.$$

Suppose that Γ, Δ are moduli of continuity and Γ is principal, E, F are normed spaces, $X \subseteq E$, $Y \subseteq F$ and X, Y are locally Γ-LIN-bordered sets. We shall prove that if $\varphi : H_{\Gamma}^{\mathrm{CMP.LC}}(X) \cong H_{\Delta}^{\mathrm{CMP.LC}}(Y)$, then $\Gamma = \Delta$ and there is $\tau : \mathrm{cl}(X) \cong \mathrm{cl}(Y)$ such that $\tau(X) = Y$, τ is locally Γ-bicontinuous and $\varphi(g) = \tau \circ g \circ \tau^{-1}$ for every $g \in H_{\Gamma}^{\mathrm{CMP.LC}}(X)$.

The above statement is also true when X and Y are open subsets of a normed Lipschitz manifold; see Theorem 8.4(b). The argument for manifolds is essentially identical, so proofs will be given only for the class of open subsets of normed spaces.

8.2. Statement of the main theorems and the plan of the proof. We shall now define the class of open sets with a well-behaved boundary.

Definition 8.1. (a) Let E be a normed space, $A \subseteq E$ and $r > 0$. The set $\mathrm{BCD}^E(A,r) := B^E(0,r) - A$ is called the *boundary chart domain based on E and A with radius r*. We say that $A \subseteq E$ is a *closed half space* of E if there is $\varphi \in E^*$ such that $A = \{x \in E \mid \varphi(x) \geq 0\}$. Suppose that $\dim(E) > 1$, and A is either a closed subspace of E different from $\{0\}$ or a closed half space of E. Then $\mathrm{BCD}^E(A,r)$ is called a *linear boundary chart domain.*

(b) Let $\langle Y, \Phi, d\rangle$ be a normed manifold, $X \subseteq Y$ be open, $x \in \mathrm{bd}(X)$ and $\alpha \in \mathrm{MBC}$. We say that X is *α-linearly-bordered at x* (*α-LIN-bordered*) if there are a linear boundary chart domain $\mathrm{BCD}^E(A,r)$ and a function $\psi : B^E(0,r) \to Y$ such that:

(i) $\psi : B^E(0,r) \cong \mathrm{Rng}(\psi)$,
(ii) ψ takes open subsets of E to open subsets of Y and closed subsets of E to closed subsets of Y,
(iii) $\psi(\mathrm{BCD}^E(A,r)) = \mathrm{Rng}(\psi) \cap X$,
(iv) $\psi{\restriction}\mathrm{BCD}^E(A,r)$ is α-bicontinuous,
(v) $\psi(0) = x$.

$\langle \psi, A, r\rangle$ is called a *boundary chart element* for x.

(c) Let $\Gamma \subseteq \mathrm{MC}$. We say that X is *locally Γ-LIN-bordered* if for every $x \in \mathrm{bd}(X)$ there is $\alpha \in \Gamma$ such that X is α-LIN-bordered at x. □

The open sets that we had in mind when defining LIN-borderedness are described below. Take an open subset U of $\mathbb{R}^n$ whose boundary is a smooth submanifold. Let $K_1, \ldots, K_n$ be pairwise disjoint subsets of U, and assume that for every i, K_i is a compact smooth submanifold of $\mathbb{R}^n$ which is not a singleton. Then $U - \bigcup_{i=1}^n K_i$ is Γ^{LIP}-LIN-bordered.

We recall the definition of the group $H_\Gamma^{\mathrm{CMP.LC}}(X)$.

Definition 8.2. Suppose that E, F are metric spaces, $X \subseteq E$ and $Y \subseteq F$, $\Gamma \subseteq \mathrm{MC}$.

Let $f : X \to Y$. Then f is *completely locally Γ-continuous* if $f \in \mathrm{EXT}^{E,F}(X,Y)$, and for every $x \in \mathrm{cl}^E(X)$ there are $\alpha \in \Gamma$ and $T \in \mathrm{Nbr}^E(x)$ such that $f{\restriction}(T \cap X)$ is α-continuous. *Complete local Γ-bicontinuity* is defined analogously.

$H_\Gamma^{\mathrm{CMP.LC}}(X,Y;E,F)$ denotes the set of completely locally Γ-continuous homeomorphisms between X and Y. We use the notation $H_\Gamma^{\mathrm{CMP.LC}}(X,Y)$ as an abbreviation of $H_\Gamma^{\mathrm{CMP.LC}}(X,Y;E,F)$. The notations $(H_\Gamma^{\mathrm{CMP.LC}})^{\pm}(X,Y)$ and $H_\Gamma^{\mathrm{CMP.LC}}(X)$ are derived in the usual way. □

Remark 8.3. (a) Note that in the above definition, if E and F are complete metric spaces, then the requirement that $f \in \mathrm{EXT}(X,Y)$ is not needed.

(b) In the above definition assume that E, F are finite-dimensional normed spaces, and X, Y are bounded. Let $g \in H(X,Y)$. Then $g \in (H_\Gamma^{\mathrm{CMP.LC}})^{\pm}(X,Y)$ iff there is $\alpha \in \Gamma$ such that g^{cl} is α-bicontinuous.

(c) The motivation for dealing with groups of the type $H_\Gamma^{\mathrm{CMP.LC}}(X)$ is the finite-dimensional special case described in (b). However, the proof of Theorem 8.4 below covers other types of groups. The following is an example. Let E be a normed space, and $\bar{E}$ be its completion. Let $X \subseteq E$ be open. Write

$$\overline{\mathrm{bd}}(X) = \mathrm{cl}^{\bar{E}}(X) - \overline{\mathrm{int}}(X).$$

See Definition 2.24(a). Let $\overline{\mathrm{cl}}(X) = X \cup \overline{\mathrm{bd}}(X)$. Let $\bar{H}_{\Gamma}^{\mathrm{CMP.LC}}(X) = H_{\Gamma}^{\mathrm{CMP.LC}}(X; \overline{\mathrm{cl}}(X))$.

The proof of Theorem 8.4 carries over to the group $\bar{H}_{\Gamma}^{\mathrm{CMP.LC}}(X)$ except for a slight change in the construction of homeomorphisms in Chapter 11. □

The next theorem is our main final goal. It is proved in 12.20(a).

THEOREM 8.4. (a) *Let Γ be a principal modulus of continuity and Δ be a modulus of continuity. Let E, F be normed spaces, $X \subseteq E$ be a locally Γ-LIN-bordered open set, and $Y \subseteq F$ be a locally Δ-LIN-bordered open set. Suppose that $\varphi : H_{\Gamma}^{\mathrm{CMP.LC}}(X) \cong H_{\Delta}^{\mathrm{CMP.LC}}(Y)$. Then $\Gamma = \Delta$, and there is $\tau \in (H_{\Gamma}^{\mathrm{CMP.LC}})^{\pm}(X, Y)$ such that $\varphi(g) = g^{\tau}$ for every $g \in H_{\Gamma}^{\mathrm{CMP.LC}}(X)$.*

(b) *In* (a) *assume that E and F are normed Lipschitz manifolds. Then the claim of* (a) *is true.*

Part (a) is a special case of (b). But we shall prove only (a), since the setting of (b) is more complicated and the proofs are essentially identical.

In the special case of bounded finite-dimensional spaces, Theorem 8.4 has a more natural formulation, which we state in the next corollary.

COROLLARY 8.5. *Let Γ be a principal modulus of continuity, Δ be a modulus of continuity and $\langle X, d \rangle$ and $\langle Y, e \rangle$ be compact metric Euclidean manifolds with boundary. Assume that $\langle X, d \rangle$ has an atlas consisting of Γ-bicontinuous charts, $\langle Y, e \rangle$ has an atlas consisting of Δ-bicontinuous charts and $\varphi : H_{\Gamma}(X) \cong H_{\Delta}(Y)$. Then $\Gamma = \Delta$ and there is $\tau : X \cong Y$ such that τ is Γ-bicontinuous and $\varphi(g) = g^{\tau}$ for every $g \in H_{\Gamma}(X)$.*

Proof. The corollary follows from Theorem 8.4(b) and Remark 8.3(b). ■

Plan of the proof of Theorem 8.4(a). The proof of Theorem 8.4(a) has four main steps:

Step 1: There is $\tau \in H(X, Y)$ such that $\varphi(g) = g^{\tau}$ for every $g \in H_{\Gamma}^{\mathrm{CMP.LC}}(X)$.
Step 2: $\Gamma = \Delta$, and τ is locally Γ-bicontinuous.
Step 3: $\tau \in \mathrm{EXT}^{\pm}(X, Y)$.
Step 4: τ is completely locally Γ-bicontinuous.

The first two steps have already been accomplished. Step 1 follows from Theorem 2.8 and Step 2 from Theorem 3.27. The exact statement of Step 3 is given in Theorem 8.8. The proof of this theorem takes all of Chapters 8–11, and the conclusion of the proof appears at the end of Chapter 11. Chapter 12 is devoted to the proof of Step 4.

Theorem 8.8 has two variants. Part (a) is indeed the main goal. However, the strength of the argument is partially lost when dealing only with groups of the type $H_{\Gamma}^{\mathrm{CMP.LC}}(X)$. Part (b) is stated in order to later reveal the full strength of the argument. See further explanation after the statement of Theorem 8.8.

DEFINITION 8.6. (a) Suppose that $X \subseteq E$ is open. A subset $H \subseteq \mathrm{EXT}^{E}(X)$ is *E-discrete* if $\{\mathrm{supp}(h) \mid h \in H\}$ is completely discrete with respect to E. (See Definition 6.1(a).) Note that if H is E-discrete, then $\circ\{h \mid h \in H\} \in \mathrm{EXT}^{E}(X)$.

(b) A subgroup $G \leq \mathrm{EXT}(X)$ is *closed under E-discrete composition* if $\circ\{h \mid h \in H\} \in G$ for every E-discrete set $H \subseteq G$.

(c) Let E be a metric space, $X \subseteq E$ be open, and $G \leq \mathrm{EXT}(X)$. We say that G is of *boundary type Γ* if for every $x \in \mathrm{bd}(X)$:

(i) there is $U \in \mathrm{Nbr}^E(x)$ such that $G\lfloor \underline{U \cap X} \rfloor \supseteq H_\Gamma^{\mathrm{CMP.LC}}(X)\lfloor \underline{U \cap X} \rfloor$,

(ii) for every $g \in G$, there is $V \in \mathrm{Nbr}^E(x)$ such that $g\restriction(V \cap X)$ is Γ-bicontinuous.

A subgroup $G \leq \mathrm{EXT}(X)$ is *Γ-appropriate* if G is closed under E-discrete composition, and G is of boundary type Γ.

(d) Let $H_\Gamma^{\mathrm{BDR.LC}}(X) = \{g \in \mathrm{EXT}(X) \mid$ for every $x \in \mathrm{bd}(X)$, g is Γ-bicontinuous at $x\}$. Let Δ be a modulus of continuity. Define $H_{\Delta,\Gamma}^{\mathrm{CMP.LC}}(X) = H_\Delta^{\mathrm{LC}}(X) \cap H_\Gamma^{\mathrm{BDR.LC}}(X)$. □

EXAMPLE 8.7. $H_\Gamma^{\mathrm{CMP.LC}}(X)$ and $H_\Gamma^{\mathrm{BDR.LC}}(X)$ are Γ-appropriate, and if $\Gamma \subseteq \Delta$, then $H_{\Delta,\Gamma}^{\mathrm{CMP.LC}}(X)$ is Γ-appropriate. □

THEOREM 8.8. *Let Γ, Δ be countably generated moduli of continuity, E and F be normed spaces and $X \subseteq E$, $Y \subseteq F$ be open. Suppose that X is locally Γ-LIN-bordered, and Y is locally Δ-LIN-bordered and let $\tau \in H(X,Y)$.*

(a) *If $(H_\Gamma^{\mathrm{CMP.LC}}(X))^\tau = H_\Delta^{\mathrm{CMP.LC}}(Y)$, then $\tau \in \mathrm{EXT}^{\pm}(X,Y)$.*

(b) *Suppose that $G \leq \mathrm{EXT}(X)$, $H \leq \mathrm{EXT}(Y)$ are respectively Γ- and Δ-appropriate and $G^\tau = H$. Then $\tau \in \mathrm{EXT}^{\pm}(X,Y)$.*

The proof of Theorem 8.8 appears at the end of Chapter 11.

Explanation. Suppose that $(H_\Gamma^{\mathrm{CMP.LC}}(X))^\tau = H_\Delta^{\mathrm{CMP.LC}}(Y)$. Then $\Gamma = \Delta$. This is easily concluded in the following way. Let $U \subseteq X$ be an open set such $\mathrm{cl}(U) \subseteq X$ and $\mathrm{cl}(\tau(U)) \subseteq Y$. Since $\mathrm{cl}(U) \subseteq X$, $H_\Gamma^{\mathrm{CMP.LC}}(X)\lfloor \underline{U} \rfloor = H_\Gamma^{\mathrm{LC}}(X)\lfloor \underline{U} \rfloor$. Since $\mathrm{cl}(\tau(U)) \subseteq Y$, $H_\Delta^{\mathrm{CMP.LC}}(Y)\lfloor \underline{\tau(U)} \rfloor = H_\Delta^{\mathrm{LC}}(Y)\lfloor \underline{\tau(U)} \rfloor$. So $(H_\Gamma^{\mathrm{LC}}(X)\lfloor \underline{U} \rfloor)^\tau = H_\Delta^{\mathrm{LC}}(Y)\lfloor \underline{\tau(U)} \rfloor$. It now follows easily from Theorem 3.27 or from Theorem 3.42(b) that $\Gamma = \Delta$.

When dealing with $H_\Gamma^{\mathrm{BDR.LC}}(X)$, the above argument is no longer valid. Instead one has to infer that $\Gamma = \Delta$ from the behavior of τ at $\mathrm{bd}(X)$. This is more difficult, and we have a proof only in special cases. Part (b) of 8.8 prepares the ground for this argument.

As a consequence of Step 2, at the time that we reach Step 4, we already know that $\Gamma = \Delta$. So the statement of Step 4 is as follows.

THEOREM 8.9. *Let Γ be a principal modulus of continuity, $X \subseteq E$ and $Y \subseteq F$ be open subsets of the normed spaces E and F and $\tau \in \mathrm{EXT}^{\pm}(X,Y)$. Suppose that X and Y are Γ-LIN-bordered and $(H_\Gamma^{\mathrm{CMP.LC}}(X))^\tau = H_\Gamma^{\mathrm{CMP.LC}}(Y)$. Then $\tau \in (H_\Gamma^{\mathrm{CMP.LC}})^{\pm}(X,Y)$.*

Chapter 12 is devoted to the proof of Theorem 8.9. Actually, the main result of Chapter 12 is Theorem 12.19, and 8.9 is just a corollary of that theorem. At the end of Chapter 12 we prove Theorem 8.4(a). At that point it is only a matter of combining the intermediate results from Chapters 11 and 12. This is done in Theorem 12.20, and 8.4(a) is the first part of that theorem.

Certain types of boundary points have to be treated differently than others. These types are defined below.

Definition 8.10. If in part (b) of Definition 8.1, A is a closed subspace of E and $\dim(A) = 1$, or $\dim(E) = 2$ and A is a half space of E, then we say that $\mathrm{bd}(X)$ *is 1-dimensional at* x.

If in part (b) of Definition 8.1, A is a closed subspace of E and $\text{co-dim}(A) = 1$, or A is a half space of E, then we say that $\mathrm{bd}(X)$ *has co-dimension* 1 *at* x.

If in part (b) of 8.1, A is a closed subspace of E with co-dimension 1, then we say that X *is two-sided at* x. Hence $\mathrm{Rng}(\psi) \cap X$ has two connected components. Let $u, v \in \mathrm{Rng}(\psi) \cap X$. We say that $u, v \in X$ are *on the same side of* $\mathrm{bd}(X)$ *with respect to* $\langle \psi, A, r \rangle$ if u, v are in the same connected component of $\mathrm{Rng}(\psi) \cap X$. We say that $u, v \in X$ are *on different sides of* $\mathrm{bd}(X)$ *with respect to* $\langle \psi, A, r \rangle$ if u, v are in different connected components of $\mathrm{Rng}(\psi) \cap X$.

If in part (b) of 8.1, (i) $\dim(E) > 2$, and (ii) A is a closed subspace of E of dimension > 1 or A is a closed half space of E, then we say that X is *α-simply-linearly-bordered* (*α-SLIN-bordered*) *at* x. □

Let $x \in \mathrm{bd}(X)$. Note that if $\mathrm{bd}(X)$ is 1-dimensional at x, and $\langle \psi, A, r \rangle$ is any boundary chart element for x, then either (i) A is a 1-dimensional subspace, or (ii) $\dim(E) = 2$ and A is a closed half space. Similarly, if X is two-sided at x, and $\langle \psi, A, r \rangle$ is any boundary chart element for x, then A is a closed subspace with co-dimension 1.

Question 8.11. A subset $A \subseteq E$ is called a *closed half subspace* of E if there is a closed subspace F of E such that $F \neq \{0\}$ and A is a half space of F. Let $\mathrm{BCD}^E(A, r)$ be a boundary chart domain. We call $\mathrm{BCD}^E(A, r)$ an *almost linear boundary chart domain* if either it is a linear boundary chart domain, or A is a closed half subspace of E. Let $\Gamma \subseteq \mathrm{MC}$. Define the notion *"$X$ is locally Γ-almost-linearly-bordered"* (*locally Γ-ALIN-bordered*) in analogy with Definition 8.1(c).

Are Theorems 8.8 and 8.9 true for locally ALIN-bordered sets?

In order to prove the analogues of 8.8 and 8.9 for locally ALIN-bordered sets, only Lemma 9.13 needs to be generalized. All other ingredients in the proof remain essentially the same. □

Some ALIN-bordered sets are described below. Take an open subset U of $\mathbb{R}^n$ whose boundary is a smooth submanifold. Let $K_1, \ldots, K_n$ be pairwise disjoint subsets of U, and assume that for every i, K_i is a compact manifold with a boundary which is not a singleton, and K_i is smoothly embedded in $\mathbb{R}^n$. Then $U - \bigcup_{i=1}^n K_i$ is Γ^{LIP}-ALIN-bordered.

9. The Uniform Continuity Constant

9.1. Preliminary lemmas about the existence of certain constants. In preparing the ground for the proof of Theorem 8.8, we need to characterize the pairs of convergent sequences $\vec{x}, \vec{y}$ in X for which there is an α-bicontinuous homeomorphism $g \in H(X)$ and subsequences $\vec{x}', \vec{y}'$ of $\vec{x}$ and $\vec{y}$ such that $g(\vec{x}') = \vec{y}'$. Stated more precisely, let $z \in \mathrm{bd}(X)$ and $\lim \vec{x} = \lim \vec{y} = z$, and assume that for every $n \in \mathbb{N}$,

(1) $\|x_n - z\| \le \alpha(\|y_n - z\|)$ and $\|y_n - z\| \le \alpha(\|x_n - z\|)$,
(2) $d(x_n, \mathrm{bd}(X)) \le \alpha(d(y_n, \mathrm{bd}(X)))$ and $d(y_n, \mathrm{bd}(X)) \le \alpha(d(x_n, \mathrm{bd}(X)))$.

We shall prove that there are $g \in H(X)$ and subsequences $\vec{x}'$ and $\vec{y}'$ of $\vec{x}$ and $\vec{y}$ respectively such that $g(\vec{x}') = \vec{y}'$ and g is $N \cdot \alpha \circ \alpha \circ \alpha \circ \alpha$-bicontinuous. In fact, this is only an approximation of what we really prove. The exact statement to be proved is the equivalence between the conjunction of (1) and (2) above and the fact that $\vec{x} \precapprox^{N\alpha^4} \vec{y}$. The relation $\precapprox^{\alpha}$ is defined in 11.1(c), and in Proposition 11.3(a) we prove this equivalence.

The Uniform Continuity Constant Lemma 9.13 is the main fact needed in the proof of the above. It says that there is $K > 0$ for which $\mathbf{A} \Rightarrow \mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ are the following statements.

($\mathbf{A}$) E is a normed vector space, F is a closed subspace of E with dimension > 1, $\alpha \in \mathrm{MBC}$, $x, y \in E - F$, $\|x\| \le \|y\| \le \alpha(\|x\|)$ and $\alpha^{-1}(d(x, F)) \le d(y, F) \le \alpha(d(x, F))$.

($\mathbf{B}$) There is an $K \cdot \alpha \circ \alpha$-bicontinuous homeomorphism g such that: $g(x) = y$, $g(F) = F$ and $\mathrm{supp}(g) \subseteq B(0, 2\|y\|) - \bar{B}(0, \|x\|/2)$.

This chapter is devoted to the proof of this lemma. The geometric content of the lemma is simple, but a detailed proof seems to require much work. When the claim of the lemma is restricted to pre-Hilbert spaces and not to general normed spaces, the proof is easier.

We shall also need a statement analogous to $\mathbf{A} \Rightarrow \mathbf{B}$ for subspaces F of E with $\dim(F) = 1$. In this case statements $\mathbf{A}$ and $\mathbf{B}$ need to be slightly modified. Chapter 10 deals with this situation.

Before turning to the proof of the Uniform Continuity Constant Lemma we quote some well-known basic facts from functional analysis, and we also establish the existence of various types of homeomorphisms which will be used in the proof of 9.13. These preparations are carried out in 9.1–9.10. We start with some notation.

Notations 9.1. (a) For $K \ge 1$ and $a, b \ge 0$ let $a \approx^K b$ mean that $a/K \le b \le Ka$. If $\| \ \|^1, \| \ \|^2$ are norms on a vector space E, then $\| \ \|^1 \approx^K \| \ \|^2$ means that $\|u\|^1 \approx^K \|u\|^2$ for every $u \in E$.

(b) The notation $E = L \oplus^{\rm alg} S$ means that $L + S = E$ and $L \cap S = \{0\}$. If $E = L \oplus^{\rm alg} S$, then $(x)_{L,S}, (x)_{S,L}$ denote the components of x in L and S respectively. In what follows we sometimes abbreviate $(x)_{L,S}$ by $(x)_L$ and $(x)_{S,L}$ by $(x)_S$. Suppose that $E = L \oplus^{\rm alg} S$. We define $\|u\|^{L,S} = \|(u)_S\| + \|(u)_L\|$. The notation $E = L \oplus S$ means that $E = L \oplus^{\rm alg} S$, and that for some $K \geq 1$, $\| \ \|^{L,S} \approx^K \| \ \|$. In such a case S is called a *complement* of L in E.

(c) Let L be a linear subspace of E. Then $\text{co-dim}^E(L)$ denotes the co-dimension of L in E. This is abbreviated by $\text{co-dim}(L)$.

(d) Let F and H be linear subspaces of a normed space E and $M \geq 1$. We define $H \perp^M F$ if $d(h, F) \geq \|h\|/M$ for every $h \in H$.

(e) Let $E = F \oplus^{\rm alg} H$. Then $\text{Proj}_{F,H}$ denotes the function $u \mapsto (u)_{F,H}$, $u \in E$.

(f) Let X be a metric space, $x \in X$ and $0 < r < s$. The ring with center at x and with radii r, s is defined as

$$B(x; r, s) = \{y \in X \mid r < d(x, y) < s\}.$$

We quote without proof some basic and well-known facts from functional analysis.

PROPOSITION 9.2. (a) *For every $n > 0$ there is $M = M^{\rm aoc}(n) \geq 1$ such that for every normed space E and an n-dimensional subspace L of E there is a complement S of L in E such that $M\|x\| \geq \|(x)_{L,S}\| + \|(x)_{S,L}\|$ for every $x \in E$. A subspace S satisfying the above is called an* almost orthogonal complement *of L.*

(b) *For every $n > 0$ there is $M = M^{\rm thn}(n) \geq 1$ such that for every normed n-dimensional space E there is a Hilbert norm $\| \ \|^{\mathbf{H}}$ on E such that $\|x\| \leq \|x\|^{\mathbf{H}} \leq M\|x\|$ for every $x \in E$. The norm $\| \ \|^{\mathbf{H}}$ is called a* tight Hilbert norm *on E. We denote $M^{\rm thn}(2)$ by $M^{\rm thn}$.*

(c) *For every $n > 0$ there is $M = M^{\rm hlb}(n) \geq 1$ such that for every normed space E and an n-dimensional linear subspace L of E there are a Euclidean norm $\| \ \|^{\mathbf{H}}$ on L and a complement S of L such that for every $x \in E$,*

$$\|(x)_{L,S}\|^{\mathbf{H}} + \|(x)_{S,L}\| \approx^M \|x\|.$$

Also, if $m < n$, then $M^{\rm hlb}(m) \leq M^{\rm hlb}(n)$. A pair $\langle \| \ \|^{\mathbf{H}}, S \rangle$ satisfying the above is called a tight Hilbert complementation *for L. We denote $M^{\rm hlb}(2)$ by $M^{\rm hlb}$.*

(d) *Let $E = F \oplus H$ and $M \geq 1$. Then $H \perp^M F$ iff $\|\text{Proj}_{H,F}\| \leq M$.*

(e) *Let $E = F \oplus H$ and suppose that $H \perp^M F$. Then $F \perp^{M+1} H$.*

(f) *Let $E = F \oplus H$ and suppose that $H \perp^M F$. Then $\| \ \|^{F,H} \approx^{2M+1} \| \ \|$.*

(g) *Let $E = F \oplus H$ and suppose that $\| \ \|^{F,H} \approx^M \| \ \|$. Then $H \perp^M F$.*

(h) *Let $T : E \to E$ be a bounded linear projection with a closed range. Then* $\ker(T) \perp^{\|T\|+1} \text{Rng}(T)$.

(i) *Let $x, y \in E - \{0\}$ be such that $\|x\| \leq \|y\|$. Let $z = \frac{\|x\|}{\|y\|} y$. Then $\|y - z\| \leq \|y - x\|$ and $\|x - z\| \leq 2\|y - x\|$.*

PROPOSITION 9.3. *Let F be a closed subspace of a normed vector space E, $x, y \in E - F$ and $\varepsilon > 0$. Then there is a closed subspace H of E such that $F \subseteq H$, $\text{span}(H \cup \{x, y\}) = E$, $d(x, H) \geq \frac{1}{1+\varepsilon} d(x, F)$ and $d(y, H) \geq \frac{1}{1+\varepsilon} d(y, F)$.*

Proof. Let $\Delta = 1+\varepsilon$ and $\hat{x} \in F$ be such that $\|x-\hat{x}\| \le \Delta d(x,F)$. Write $x^{\perp} = x-\hat{x}$. Let ψ be the linear functional on $\operatorname{span}(F \cup \{x\})$ defined by $\psi(x^{\perp}) = \|x^{\perp}\|$ and $\psi(F) = \{0\}$. We check that $\|\psi\| \le \Delta$. Let $z \in \operatorname{span}(F \cup \{x\})$. If $z \in F$, then $|\psi(z)| = 0 \le \Delta\|z\|$. Suppose that $z = u + \lambda x^{\perp}$, where $u \in F$ and $\lambda \neq 0$. We may assume that $\lambda = 1$. Then

$$|\psi(z)| = \|x^{\perp}\| \le \Delta d(x,F) \le \Delta\|(u-\hat{x})+x\| = \Delta\|u+x^{\perp}\| = \Delta\|z\|.$$

Let $\varphi \in E^*$ extend ψ and $\|\varphi\| = \|\psi\|$. Let $H_1 = \ker(\varphi)$. So $F \subseteq H_1$. Since $x = \hat{x} + x^{\perp}$ and $\hat{x} \in H_1$, $d(x,H_1) = d(x^{\perp},H_1)$. Let $u \in H_1$. Then

$$\|x^{\perp} - u\| \ge \frac{|\varphi(x^{\perp}-u)|}{\Delta} = \frac{\|x^{\perp}\|}{\Delta} \ge \frac{d(x,F)}{\Delta}.$$

Hence $d(x,H_1) = d(x^{\perp},H_1) \ge \frac{d(x,F)}{1+\varepsilon}$.

Similarly, there is a closed linear subspace H_2 with co-dimension 1 such that $d(y,H_2) \ge \frac{d(y,F)}{1+\varepsilon}$. Let $H = H_1 \cap H_2$. Then H is as required. ■

The next proposition contains some additional basic and well-known facts from functional analysis. The proofs are again omitted.

PROPOSITION 9.4. (a) *For every $n \in \mathbb{N}$ there is $M^{\mathrm{prj}}(n)$ such that for every normed space E and a closed linear subspace $F \subseteq E$: if* co-$\dim^E(F) = n$, *then there is a projection $T : E \to F$ such that $\|T\| \le M^{\mathrm{prj}}(n)$.*

(b) *For every $n \in \mathbb{N}$ there is $M = M^{\mathrm{ort}}(n)$ such that for every normed space E and a closed linear subspace $F \subseteq E$: if* co-$\dim^E(F) \le n$, *then there is a closed linear subspace $H \subseteq E$ such that $F \oplus H = E$ and $H \perp^M F$. One can take $M^{\mathrm{ort}}(n)$ to be $2^n - 1 + \varepsilon$ for any $\varepsilon > 0$. We denote $M^{\mathrm{ort}}(2)$ by M^{ort}.*

(c) *Let $M^{\mathrm{fdn}}(n) = (1 + M^{\mathrm{thn}}(n)) \cdot M^{\mathrm{ort}}(n) + 1$. Let E be a normed space, $F \subseteq E$ be a closed subspace with co-dimension $\le n$ and H be such that $F \oplus H = E$ and $H \perp^{M^{\mathrm{ort}}(n)} F$. Let $\|\ \|^{\mathbf{H}}$ be a Hilbert norm on H such that $\|\ \|^{\mathbf{H}} \approx^{M^{\mathrm{thn}}(n)} \|\ \| \restriction H$. Define a new norm on E by $\|u\|^{\mathbf{N}} = \|(u)_F\| + \|(u)_H\|^{\mathbf{H}}$. Then $\|\ \|^{\mathbf{N}} \approx^{M^{\mathrm{fdn}}(n)} \|\ \|$. We denote $M^{\mathrm{fdn}}(2)$ by M^{fdn}.*

DEFINITION 9.5. (a) Let H be a 2-dimensional Hilbert space and $\theta \in \mathbb{R}$. Then Rot_θ^H denotes the rotation by an angle of θ in H. Let $E = F \oplus H$ be normed spaces. Suppose that H is a 2-dimensional Hilbert space. For $\theta \in \mathbb{R}$ let $\mathrm{Rot}_\theta^{F,H} \in H(E)$ be defined by

$$\mathrm{Rot}_\theta^{F,H}(u) = (u)_F + \mathrm{Rot}_\theta^H((u)_H), \quad u \in E.$$

(b) Let $h = \mathrm{Rad}_{\eta,z}^E$ be a radial homeomorphism. (See Definition 3.17(b).) We say that h is *piecewise linearly radial* if η is piecewise linear. □

Part (a) of the following proposition is a variant of Lemma 2.14(c).

PROPOSITION 9.6. (a) *There is $M^{\mathrm{seg}} > 1$ such that for every normed space E, $x, y \in E$ and $r > 0$, there is $h \in H(E)$ such that*

(1) $\operatorname{supp}(h) \subseteq B([x,y],r)$,
(2) $h(x) = y$,
(3) *h is $M^{\mathrm{seg}} \cdot (\|x-y\|/r + 1)$-bilipschitz.*

(b) *For every $t > 0$ there is $M^{\mathrm{arc}}(t) > 1$ such that for every normed space E, a rectifiable arc $L \subseteq E$ with endpoints x, y and $r > 0$ there is $h \in H(E)$ such that*

(1) $\text{supp}(h) \subseteq B(L, r)$,
(2) $h(x) = y$,
(3) *h is $M^{\text{arc}}(\text{lngth}(L)/r)$-bilipschitz.*

(c) *There is $M^{\text{rot}} \geq 1$ such that the following holds. Let $E = F \oplus H$ be normed spaces. Suppose that H is a 2-dimensional Hilbert space, and that for every $u \in E$, $\|u\| = \|(u)_F\| + \|(u)_H\|$. Let S be a closed subset of E, $\eta : [0, \infty) \to \mathbb{R}$, and $K, r > 0$ be such that*: (i) $S \subseteq \bar{B}(0, r)$; (ii) *for every $u \in S$ and $\theta \in \mathbb{R}$, $\text{Rot}_\theta^{F,H}(u) \in S$*; (iii) *$\eta$ is K-Lipschitz*; (iv) *$\eta(s) = 0$ for every $s \geq r$. Let $g : E \to E$ be defined by $g(u) = \text{Rot}_{\eta(d(u,S))}^{F,H}(u)$. Then $g \in H(E)$ and g is $(M^{\text{rot}}Kr + 1)$-bilipschitz.*

(d) *Suppose that F, H are normed spaces, $E = F \oplus H$, and $\|u + v\| = \|u\| + \|v\|$ for every $u \in F$ and $v \in H$. Let $\hat{x} \in F$, $x \in H$, $a > 1$, $x' = \hat{x} + x$ and $x'' = \hat{x} + ax$. Then there is $g \in H(E)$ such that*

(1) $g(x') = x''$,
(2) $g \restriction F = \text{Id}$,
(3) *for every $u \in F$ we have $\text{supp}(g) \subseteq B(u; s, t)$, where $s = \|x' - u\|/2$ and $t = 3\|x'' - u\|/2$.*
(4) *g is $2M^{\text{seg}}a$-bilipschitz.*

Proof. (a) Set $\bar{x} = x/\|x\|$ and $a = \|x - y\|$. We may place the origin in such a way that $x = (r/2) \cdot \bar{x}$ and $y = (r/2 + a) \cdot \bar{x}$. We may assume that $r < a$. Write $M = M^{\text{aoc}}(1)$. Let $L = \text{span}(\{x\})$ and S be a complement of L such that $M\|u\| \geq \|(u)_{L,S}\| + \|(u)_{S,L}\|$ for every $u \in E$. So for every $u \in S$, $\|u\| \leq M \cdot d(u, L)$. Write $(u)_{L,S} = \hat{u}$ and $(u)_{S,L} = u^\perp$. For every $u \in E$ let λ_u be such that $\hat{u} = \lambda_u \bar{x}$. So $u = \lambda_u \bar{x} + u^\perp$.

Let $g(s, t) = g_s(t)$, $s \geq 0$, $t \in \mathbb{R}$, be defined as follows. For every $s \geq 0$, $g_s(t)$ is a piecewise linear function satisfying the following.

(1) The breakpoints of $g_s(t)$ are 0, $r/2$ and $a + r$.
(2) If $s \in [0, r/2M]$, then $g_s(r/2) = \frac{r/2M - s}{r/2M} \cdot (a + r/2)$, and if $s \geq r/2M$, then $g_s(r/2) = r/2$.
(3) If $t \leq 0$ or $t \geq a + r$, then $g_s(t) = t$.

So $g_0(r/2) = a + r/2$ and $g_s = \text{Id}$ for every $s \geq r/2M$. Define

$$h(u) = u^\perp + g_{d(u,L)}(\lambda_u) \cdot \bar{x}.$$

Clearly, $h(x) = y$. Let $u \in E - B([x, y], r)$, and we prove that $h(u) = u$. If $d(u, L) \geq r/2M$, then $g_{d(u,L)} = \text{Id}$. So $h(u) = u^\perp + \lambda_u \bar{x} = u$. Assume that $d(u, L) < r/2M$. If $\lambda_u \leq 0$, then for every s, $g_s(\lambda_u) = \lambda_u$ and hence $h(u) = u$. Assume that $\lambda_u > 0$. Since $d(u, L) < r/2M$, it follows that $\|u^\perp\| < r/2$. Hence

$$|\lambda_u - (a + r/2)| = \|\hat{u} - y\| \geq \|u - y\| - \|u^\perp\| > r - r/2 = r/2.$$

That is, either (i) $\lambda_u - (a + r/2) > r/2$ or (ii) $\lambda_u - (a + r/2) < -r/2$. Suppose by contradiction that (ii) happens. Then $0 < \lambda_u < a$. If $\lambda_u \geq r/2$, then $\hat{u} = \lambda_u \bar{x} \in [x, y]$, and hence $d(u, [x, y]) \leq \|u - \hat{u}\| = \|u^\perp\| < r/2$. So $u \in B([x, y], r)$, a contradiction. If $\lambda_u < r/2$, then $d(u, [x, y]) \leq \|x - u\| \leq \|x - \hat{u}\| + \|u^\perp\| < r/2 + r/2 = r$. So $u \in B([x, y], r)$.

A contradiction. Hence $\lambda_u - (a + r/2) > r/2$. So $\lambda_u > a + r$, and hence for every s, $g_s(\lambda_u) = \lambda_u$. So $h(u) = u$. We have shown that $h\restriction(E - B([x, y], r)) = \mathrm{Id}$.

For every $s \geq 0$ let $f_s = g_s^{-1}$, and let $f(s,t) = f_s(t)$. Note that for every $u \in E$, $u^{\perp} = (h(u))^{\perp}$, and hence $d(h(u), L) = d(u, L)$. So if $w = h(u)$, then $u = w^{\perp} + f_{d(w,L)}(\lambda_w) \cdot \bar{x}$. Hence h^{-1} exists and is continuous, and so $h \in H(E)$.

We show that h and h^{-1} are Lipschitz. Note that for every s, the three slopes of g_s are $\leq \frac{a+r/2}{r/2}$. Also, for every $s_1, s_2 \geq 0$ and $t \in \mathbb{R}$,

$$|g_{s_1}(t) - g_{s_2}(t)| \leq \frac{a}{r/2M} \cdot |s_1 - s_2|.$$

For f_s, the maximal slope is again $\frac{a+r/2}{r/2}$ and

$$|f_{s_1}(t) - f_{s_2}(t)| \leq \frac{a}{r/2M} \cdot |s_1 - s_2|.$$

Now

$$h(u) - h(v) = u^{\perp} - v^{\perp} + (g_{d(u,L)}(\lambda_u) - g_{d(u,L)}(\lambda_v))\bar{x} + (g_{d(u,L)}(\lambda_v) - g_{d(v,L)}(\lambda_v))\bar{x}.$$

Write $w = u - v$. So

$$\begin{aligned}
\|h(u) - h(v)\| &\leq \|u^{\perp} - v^{\perp}\| + |g_{d(u,L)}(\lambda_u) - g_{d(u,L)}(\lambda_v)| + |g_{d(u,L)}(\lambda_v) - g_{d(v,L)}(\lambda_v)| \\
&\leq \|w^{\perp}\| + \frac{a + r/2}{r/2}\|\hat{u} - \hat{v}\| + \frac{a}{r/2M} \cdot (d(u, L) - d(v, L)) \\
&\leq \|w^{\perp}\| + (2a/r + 1) \cdot \|\hat{w}\| + \|u - v\| \\
&\leq M\|u - v\| + (2a/r + 1)M\|u - v\| + \|u - v\| \leq (3M + 1)(a/r + 1)\|u - v\|.
\end{aligned}$$

An identical computation shows that h^{-1} is $(3M + 1)(a/r + 1)$-Lipschitz. So $M^{\mathrm{seg}} = 3M + 1$.

(b) Let E be a normed space, $L \subseteq E$ be a rectifiable arc with endpoints x, y and $r > 0$. Set $\ell = \mathrm{lngth}(L)$ and $n = [\ell/r] + 1$. There are $x_i \in L$, $i = 0, \ldots, n$, such that $x_0 = x$, $x_n = y$ and for every $i < n$, $\|x_i - x_{i+1}\| \leq r$. For $i < n$ let $L_i = [x_i, x_{i+1}]$. Then $B(L_i, r/2) \subseteq B(L, r)$. By (a), there is $g_i \in H(E)$ such that

(1) $\mathrm{supp}(g_i) \subseteq B(L_i, r/2)$,
(2) $g_i(x_i) = x_{i+1}$,
(3) g_i is $M^{\mathrm{seg}} \cdot (\frac{\|x_i - x_{i+1}\|}{r/2} + 1)$-bilipschitz.

Since $\|x_i - x_{i+1}\| \leq r$ and by (3), g_i is $3M^{\mathrm{seg}}$-bilipschitz. Let $M^{\mathrm{arc}}(t) = (3M^{\mathrm{seg}})^{[t]+1}$. Define $g = g_0 \circ \cdots \circ g_{n-1}$. It is easily seen that $g(x) = y$, $\mathrm{supp}(g) \subseteq B(L, r)$ and g is $M^{\mathrm{arc}}(\ell/r)$-bilipschitz.

(c) Suppose that some function $f : E \to E$ has the property that for some $a > 0$, $\|f(u) - f(v)\| \leq M\|u - v\|$ for every $u, v \in E$ such that $\|u - v\| \leq a$. Then f is M-Lipschitz. For the function g we take a to be r. Let $u, v \in E$ be such that $\|u - v\| \leq r$. If $v \notin B(0, 3r)$, then $u \notin B(0, 2r)$. So $g(u) = u$ and $g(v) = v$. We may thus assume that $\|v\| < 3r$. Denote $(u)_H, (u)_F, (v)_H, (v)_F$ by u_1, u_2, v_1, v_2 respectively and $\theta(w) := \eta(d(w, S))$. Then

$$g(v) - g(u) = (\mathrm{Rot}^H_{\theta(v)}(v_1) - \mathrm{Rot}^H_{\theta(u)}(v_1)) + (\mathrm{Rot}^H_{\theta(u)}(v_1) - \mathrm{Rot}^H_{\theta(u)}(u_1)) + (v_2 - u_2).$$

So

$$\begin{aligned}\|g(v)-g(u)\| &\le \|(\mathrm{Rot}^H_{\theta(v)}(v_1)-\mathrm{Rot}^H_{\theta(u)}(v_1))\| \\ &\quad + \|(\mathrm{Rot}^H_{\theta(u)}(v_1)-\mathrm{Rot}^H_{\theta(u)}(u_1))+(v_2-u_2)\| \\ &= \|(\mathrm{Rot}^H_{\theta(v)}(v_1)-\mathrm{Rot}^H_{\theta(u)}(v_1))\|+\|v-u\|.\end{aligned}$$

We estimate the first summand in the last expression:

$$\begin{aligned}\|(\mathrm{Rot}^H_{\theta(v)}(v_1)-\mathrm{Rot}^H_{\theta(u)}(v_1))\| &\le |\theta(v)-\theta(u)|\cdot\|v_1\| \le |\theta(v)-\theta(u)|\cdot\|v\| \\ &= |\eta(d(v,S))-\eta(d(u,S))|\cdot\|v\| \le K\cdot|d(v,S)-d(u,S)|\cdot\|v\| \\ &\le K\cdot\|v-u\|\cdot\|v\| \le 3Kr\cdot\|v-u\|.\end{aligned}$$

It follows that $\|g(v)-g(u)\| \le (3Kr+1)\cdot\|v-u\|$.

Note that $g^{-1}(u)=\mathrm{Rot}^{F,H}_{-\eta(d(u,S))}(u)$. Since (iii) and (iv) of (c) hold for $-\eta$, we also find that g^{-1} is $(3Kr+1)$-Lipschitz. So $M^{\mathrm{rot}}=3$.

(d) Let $E,F,H,\hat{x},x,a$ be as in (d). It suffices to prove (d) for $\hat{x}=0$, since if g satisfies the requirements of (d) for $E,F,0,x,a$, then $g^{\mathrm{tr}_{\hat{x}}}$ satisfies those requirements for $E,F,\hat{x},x,a$. So $x'=x$ and $x''=ax$. Let $L=[x,ax]$ and $r=\|x\|/2$. So

$$\frac{\mathrm{lngth}(L)}{r}+1 \le \frac{(a-1)\|x\|}{\|x\|/2}+1=2(a-1)+1\le 2a.$$

It follows from (a) that there is $g\in H(E)$ such that $\mathrm{supp}(g)\subseteq B(L,r)$, $g(x)=ax$ and g is $2aM^{\mathrm{seg}}$-bilipschitz. A trivial computation shows that g fulfills (d)(2) and (d)(3). ∎

Proposition 9.7. *For every $K\ge 1$ there is $M^{\mathrm{bnd}}(K)\ge 1$ such that the following holds. Suppose that E is a normed space and F is a closed linear subspace of E. Let $x\in E-F$ be such that $d(x,F)>\|x\|/K$ and $y\in F-\{0\}$. Then there is $g\in H(E)$ and $a,b>0$ such that*

(1) $g(x)=ax+by$,
(2) $\|g(x)\|=\|x\|$,
(3) $d(g(x),F)=\|g(x)\|/K$,
(4) $g\restriction F=\mathrm{Id}$,
(5) $\mathrm{supp}(g)\subseteq B(0;\|x\|/2,3\|x\|/2)$,
(6) *g is $M^{\mathrm{bnd}}(K)$-bilipschitz.*

Proof. Let x,y be as in the proposition. We may assume that $\|y\|=\|x\|$. Let $L_1=[x,y]$. We find $D(K)$ such that $d(L_1,0)\ge D(K)\|x\|$. Let $E_1=\mathrm{span}(\{x,y\})$ and $F_1=\mathrm{span}(\{y\})$. So $\|x\|\le Kd(x,F_1)$. Set $M=M^{\mathrm{thn}}(2)$, and let $\|\ \|^{\mathbf{H}}$ be a Hilbert norm on E_1 such that $\|u\|\le\|u\|^{\mathbf{H}}\le M\|u\|$ for every $u\in E_1$, Hence $\|x\|^{\mathbf{H}}/M\le Kd^{\mathbf{H}}(x,E_1)$. Also, $\|x\|^{\mathbf{H}},\|y\|^{\mathbf{H}}\ge\|x\|/M$. Let α be the angle between x and F_1. Hence $\sin(\alpha)=d^{\mathbf{H}}(x,E_1)/\|x\|^{\mathbf{H}}\ge 1/MK$. It follows that

$$d(0,L_1)\ge\frac{d^{\mathbf{H}}(0,L_1)}{M}=\frac{\sin(\alpha/2)\|x\|^{\mathbf{H}}}{M}\ge\frac{\sin(\alpha)}{2M}\|x\|\ge\frac{\|x\|}{2M^2K}.$$

So $D(K)=1/2M^2K$.

Since $d(x,F)/\|x\|>1/K$ and $d(y,F)/\|y\|=0<1/K$, there is $z_0\in[x,y]$ such that $d(z_0,F)/\|z_0\|=1/K$. Obviously, $\|z_0\|\le\|x\|$. Let $z=(\|x\|/\|z_0\|)\cdot z_0$. So $\|z\|=\|x\|$ and $d(z,F)/\|z\|=1/K$. Obviously, for some $a,b>0$, $z=ax+by$. Let $L=[x,z]$. For some $\lambda \geq 1$, $z = \lambda z_0$. This implies that for every $u \in L$ there are $v \in [x, z_0]$ and $\mu \geq 1$ such that $u = \mu v$. It follows that $d(L, 0) \geq d([x, z_0], 0)$, and since $[x, z_0] \subseteq L_1$, we have $d(L, 0) \geq d(L_1, 0) \geq \|x\|/2M^2K$.

Obviously, $\|x - z\| \leq 2\|x\|$. Let $r = \|x\|/4M^2K$. By Proposition 9.6(a), there is $h \in H(E)$ such that $h(x) = z$, $\mathrm{supp}(h) \subseteq B(L, r)$ and h is $M^{\mathrm{seg}} \cdot (\|x - z\|/r + 1)$-bilipschitz. By the above,

$$\frac{\|x - z\|}{r} + 1 \leq \frac{2\|x\|}{\|x\|/4M^2K} + 1 = 8M^2K + 1 \leq 9M^2K.$$

So h is $9M^{\mathrm{seg}}M^2K$-bilipschitz.

Recall that $d(z, F) = \|z\|/K$, $d(x, F) > \|x\|/K$ and for some $u \in F$ and $c > 0$, $x = u + cz$. This implies that $d(L, F) = \|z\|/K$. Hence

$$d(B(L, r), F) = \frac{\|x\|}{K} - r = \frac{\|x\|}{K} - \frac{\|x\|}{4M^2K} > 0.$$

So $h{\restriction}F = \mathrm{Id}$.

From the fact that $\|z\| = \|x\|$, it follows that $L \subseteq \bar{B}(0, \|x\|)$. Therefore $B(L, r) \subseteq B(0, \|x\| + r)$. Hence $\mathrm{supp}(h) \subseteq B(0, (1 + 1/4M^2K)\|x\|)$. But $1 + 1/4M^2K < 3/2$, so $\mathrm{supp}(h) \subseteq B(0, \frac{3}{2}\|x\|)$. Clearly, $h{\restriction}B(0, d(L, 0) - r) = \mathrm{Id}$. Hence $h{\restriction}B(0, \|x\|/4M^2K) = \mathrm{Id}$.

Let $\eta \in H([0, \infty))$ be the piecewise linear function such that: (i) the breakpoints of η are $\|x\|/4M^2K$ and $\|x\|$; (ii) $\eta(\|x\|/4M^2K) = \|x\|/2$, and $\eta(t) = t$ for every $t \geq \|x\|$. The slopes of the pieces of η are $2M^2K$, $4M^2K/2(4M^2K - 1)$ and 1. So η is $2M^2K$-bilipschitz.

Let k be the radial homeomorphism based on η. Then by Proposition 3.18, k is $6M^2K$-bilipschitz. Also, $k(B(0, \|x\|/4M^2K)) = B(0, \|x\|/2)$, $k(B(0, 3\|x\|/2)) = B(0, 3\|x\|/2)$, $k(F) = F$, $k(x) = x$ and $k(z) = z$.

Let $g = h^k$. Then $g(x) = z$, $\mathrm{supp}(g) \subseteq B(0; \|x\|/2, 3\|x\|/2)$, $g{\restriction}F = \mathrm{Id}$, and g has bilipschitz constant $(6M^2K)^2 \cdot 9M^{\mathrm{seg}}M^2K$. So $M^{\mathrm{bnd}}(K) = 324M^{\mathrm{seg}}M^6K^3$. ■

PROPOSITION 9.8. *There is $M^{\mathrm{cmp}} \geq 1$ such that the following holds. Suppose that $E = F \oplus H$, $\dim(H) \leq 2$ and $H \perp^{M^{\mathrm{ort}}} F$. Let $x \in E - F$, $x = \hat{x} + x^{\perp}$, $\hat{x} \in F$, $\|x^{\perp}\| \leq \frac{4}{3}d(x, F)$ and $d(x, F) \leq \frac{1}{16}\|x\|$. Then there is $g \in H(E)$ such that*

(1) *g is M^{cmp}-bilipschitz,*
(2) *$g(x) = \hat{x} + (x)_H$,*
(3) *$g{\restriction}F = \mathrm{Id}$,*
(4) *$\mathrm{supp}(g) \subseteq B(0; \|x\|/2, 3\|x\|/2)$.*

Proof. Note that $\hat{x} + x^{\perp} = x = (x)_F + (x)_H$. So $(x)_H - x^{\perp} = \hat{x} - (x)_F \in F$. Hence $d(x + \lambda((x)_H - x^{\perp}), F) = d(x, F)$ for every $\lambda \in \mathbb{R}$. Consider the interval $L = [x, \hat{x} + (x)_H]$. Then $L = \{x + \lambda((x)_H - x^{\perp}) \mid \lambda \in [0, 1]\}$ and so $d(L, F) = d(x, F)$. It follows that

$$\mathrm{lngth}(L) = \|(x)_H - x^{\perp}\| \leq \|(x)_H\| + \|x^{\perp}\| \leq M^{\mathrm{ort}} \cdot d(x, F) + \tfrac{4}{3}d(x, F) = (M^{\mathrm{ort}} + \tfrac{4}{3})d(x, F)$$

and hence $\mathrm{lngth}(L)/d(x, F) + 1 \leq M^{\mathrm{ort}} + 4/3 + 1 \leq M^{\mathrm{ort}} + 3$. We shall now find $\min_{u \in L} \|u\|$ and $\max_{u \in L} \|u\|$. Let $u \in L$. Then for some $\lambda \in [0, 1]$, $u = x + \lambda((x)_H - x^{\perp})$. Recall that

$d((x)_H, F) \geq \|(x)_H\|/M^{\mathrm{ort}}$. So

$$\begin{aligned}\|u\| &\geq \|x\| - \|x^{\perp}\| - \|(x)_H\| \geq \|x\| - \frac{4}{3}d(x,F) - M^{\mathrm{ort}} \cdot d((x)_H, F) \\ &= \|x\| - \frac{4}{3}d(x,F) - M^{\mathrm{ort}} \cdot d(x,F) = \|x\| - \left(M^{\mathrm{ort}} + \frac{4}{3}\right)d(x,F) \\ &\geq \|x\| - \left(M^{\mathrm{ort}} + \frac{4}{3}\right)\frac{\|x\|}{16} \geq \frac{9}{16}\|x\|.\end{aligned}$$

For the maximum of $\|u\|$ we have

$$\|u\| \leq \|x\| + \|x^{\perp}\| + \|(x)_H\| \leq \|x\| + \frac{4/3}{16}\|x\| + \frac{M^{\mathrm{ort}}}{16}\|x\| < 1\frac{7}{16}\|x\|.$$

It follows that $B(L, d(x,F)) \subseteq B(L, \|x\|/16) \subseteq B(0; \|x\|/2, 3\|x\|/2)$. So by Proposition 9.6(a), there is $g \in H(E)$ such that $\mathrm{supp}(g) \subseteq B(L, d(x,F))$, $g(x) = \hat{x} + (x)_H$ and g is $M^{\mathrm{seg}} \cdot (M^{\mathrm{ort}} + 3)$-bilipschitz. It follows that g satisfies requirements (3)–(4) of the proposition. So we may define $M^{\mathrm{cmp}} = M^{\mathrm{seg}}(M^{\mathrm{ort}} + 3)$. ■

DEFINITION 9.9. (a) Let $\alpha \in \mathrm{MBC}$ and $s, t \in [0, \infty)$. Then $s \approx^{\alpha} t$ means that $t \leq \alpha(s)$ and $s \leq \alpha(t)$.

(b) Let $\alpha \in \mathrm{MBC}$, $n \in \mathbb{N}$ and $\varrho : [0,\infty) \to [0,\infty)$ be continuous. We say that ϱ is (n, α)-*continuous* if there are $0 = a_0 < \cdots < a_{n-1} < a_n = \infty$ such that

$$\varrho_i(t) := \varrho(t + a_{i-1}), \qquad t \in [0, a_i - a_{i-1}),$$

is α-continuous for every $0 < i \leq n$. □

The four parts of the next proposition are trivial. Their proofs are omitted.

PROPOSITION 9.10. (a) *Let* $\alpha \in \mathrm{MBC}$, $n \in \mathbb{N}$ *and* $\varrho : [0,\infty) \to [0,\infty)$. *If* ϱ *is* (n,α)-*continuous, then* ϱ *is* $n \cdot \alpha$-*continuous.*

(b) *Let* $\varrho : [0,\infty) \to [0,\infty)$ *and* $a > 0$. *Define* $\eta(s,t)$ *as follows. If* $s \geq a$, *then* $\eta(s,t) = t$; *and if* $s \in [0,a]$, *then* $\eta(s,t) = (1 - s/a)\varrho(t) + (s/a)t$. *Suppose that* $\beta \in \mathrm{MC}$ *and* ϱ *is* β-*continuous. Then* $\eta_s(t) := \eta(s,t)$ *is* β-*continuous for every* $s \in [0,\infty)$. *We denote* $\eta(s,t)$ *by* $\eta_{(\varrho,a)}(s,t)$.

(c) *Let* $\beta \in \mathrm{MC}$, $a > 0$ *and* $0 < m \leq \beta(a)/a$. *Then the function* $f(t) = mt$, $t \in [0,a]$, *is* β-*continuous.*

(d) *If* $\beta \in \mathrm{MC}$, $M \geq 1$, *and* γ *is the function defined by* $\gamma(t) = \beta(Mt)$, *then* $\gamma \leq M\beta$.

9.2. The main construction

DEFINITION 9.11. (a) Let $0 < a < 1$ and $b, M > 1$. We say that M *is a Uniform Continuity constant for* $\langle a, b\rangle$ (M *is UC-constant* for $\langle a, b\rangle$) if the following holds.

Suppose that E, F, α, x, y satisfy the following assumptions.

- A1 E is a normed space, F is a closed linear proper subspace of E, $\dim(F) > 1$, $\alpha \in \mathrm{MBC}$ and $x, y \in E - F$,
- A2 $\|x\| \leq \|y\|$ and $\|x\| \approx^{\alpha} \|y\|$,
- A3 $d(x,F) \approx^{\alpha} d(y,F)$,
- A4 if $\text{co-dim}^E(F) = 1$, then x, y are on the same side of F.

Then there are $g_1, g_2 \in H(E)$ such that

B1 g_1, g_2 are $M\alpha$-bicontinuous,
B2 $g_2 \circ g_1(x) = y$,
B3 $g_1(F) = F$ and $g_2(F) = F$,
B4 for every $i = 1, 2$, $\mathrm{supp}(g_i) \subseteq B(0; a\|x\|, b\|y\|)$.

(b) We define a relation $R(u, v, g; \alpha, a, b, F)$. Let F be a closed linear subspace of a normed space E, $u, v \in E - F$, $g \in H(E)$, $0 < a < 1$, $b > 1$ and $\alpha \in \mathrm{MBC}$. The notation $R(u, v, g; \alpha, a, b, F)$ means that

R1 $g(u) = v$,
R2 g is α-bicontinuous,
R3 $g(F) = F$,
R4 $g{\restriction}B(0; a\|u\|, b\|v\|) = \mathrm{Id}$.

Let $M \geq 1$. Then $R(u, v, g; M, a, b, F)$ means that $R(u, v, g; M \cdot \mathrm{Id}_{[0,\infty)}, a, b, F)$ holds. □

The trivial proof of part (b) in the next proposition is omitted.

Proposition 9.12.

(a) $(R(u, v, g; \alpha, a, b, F) \wedge R(v, w, h; \beta, c, d, F)) \Rightarrow R(u, w, h \circ g; \beta \circ \alpha, ac, bd, F)$.
(b) $R(u, v, g; M, a, b, F) \Rightarrow R(v, u, g^{-1}; M, a/M, Mb, F)$.

Proof. (a) It is obvious that $h \circ g$ is $\beta \circ \alpha$-bicontinuous, $h \circ g(u) = w$ and $h \circ g(F) = F$.

If $v = u$, then $ca\|u\| < c\|u\| = c\|v\|$. So $h{\restriction}B(0, ca\|u\|) = \mathrm{Id}$. If $v \neq u$, then $v \in \mathrm{supp}(g)$. This implies that $\|v\| > a\|u\|$ and hence $c\|v\| > ca\|u\|$. So $h{\restriction}B(0, ca\|u\|) = \mathrm{Id}$. Clearly, $ca\|u\| < a\|u\|$. So $g{\restriction}B(0, ca\|u\|) = \mathrm{Id}$. It follows that $h \circ g{\restriction}B(0, ac\|u\|) = \mathrm{Id}$.

If $v = w$, then $bd\|w\| = bd\|v\| > b\|v\|$. So $\mathrm{supp}(g) \subseteq B(0, bd\|w\|)$. If $v \neq w$, then $v \in \mathrm{supp}(g) \subseteq B(0, d\|w\|)$. This implies that $\|v\| < d\|w\|$ and hence $b\|v\| < bd\|w\|$. So $\mathrm{supp}(g) \subseteq B(0, b\|v\|) \subseteq B(0, bd\|w\|)$. It follows that $\mathrm{supp}(g) \subseteq B(0, bd\|w\|)$. From the fact that $bd > d$ it follows that $\mathrm{supp}(h) \subseteq B(0, bd\|w\|)$. So $\mathrm{supp}(h \circ g) \subseteq B(0, bd\|w\|)$. We have shown that $\mathrm{supp}(h \circ g \subseteq B(0; ac\|u\|, bd\|w\|)$. So $R(u, w, h \circ g; \beta \circ \alpha, ac, bd, F)$ holds. ■

Lemma 9.13 (The Uniform Continuity Constant Lemma).

(a) *There are* $0 < a < 1$, $b > 1$ *and* $M > 1$ *such that* M *is a UC-constant for* $\langle a, b \rangle$.
(b) *For every* $0 < a < 1$, $b > 1$ *there is* $M > 1$ *such that* M *is a UC-constant for* $\langle a, b \rangle$.

Proof. (a) The proof is long and has many steps. The survey below may help guide the reader through the proof.

Plan of the proof. Let E, F, α, x_0, y_0 satisfy conditions A1–A4 in the definition of a UC-constant. We construct two bilipschitz homeomorphisms e and h. Set $e(x_0) = x$ and $y = h^{-1}(y_0)$. Next we construct $N \cdot \alpha$-bicontinuous homeomorphisms f_1, f_2 and $v \in E$ such that $f_1(x) = v$ and $f_2(v) = y$. Here N is a fixed number independent of E, F, α, x_0 and , y_0. So we have

$$e(x_0) = x, \;\; f_1(x) = v, \;\; f_2(v) = y, \;\; h(y) = y_0.$$

The homeomorphisms $g_1 := f_1 \circ e$ and $g_2 := h \circ f_2$ are the ones required in the definition of a UC-constant. To explain what each homeomorphism does, we take the simpler situation in which E is a pre-Hilbert space. Let F be a closed linear subspace of E. For any $z \in E$, denote $(z)_{F,F^\perp}$ by $\hat{z}$ and $(z)_{F^\perp,F}$ by $z^\perp$. The homeomorphism e is a composition of four actions. So $e = e_4 \circ \cdots \circ e_1$. Similarly, h is a composition of two actions. We shall define homeomorphisms h_1 and h_2, and h will be $h_1^{-1} \circ h_2^{-1}$.

The first action e_1 is needed only if $d(x_0, F) > \|x_0\|/3$. Otherwise, $e_1 = \mathrm{Id}$. If the former happens, then $e_1(x_0) = x_1$, where $d(x_1, F) = \|x_1\|/3$ and $\|x_1\| = \|x_0\|$. A similar action is performed by the homeomorphism h_1 on y_0, and we denote $h_1(y_0)$ by y_1. We now have the points x_1 and y_1 with the properties $\|x_1\| = \|x_0\|$, $d(x_1, F) \leq \|x_1\|/3$, $\|y_1\| = \|y_0\|$ and $d(y_1, F) \leq \|y_1\|/3$.

Now, e_2 takes x_1 to $\lambda \hat{y}_1 + x_1^\perp$, where $\lambda > 0$ and $\|\lambda \hat{y}_1\| = \|\hat{x}_1\|$. The action of e_2 can be roughly described as a rotation in the plane F_1 generated by $\hat{x}_1$ and $\hat{y}_1$ and the identity on $F_1^\perp$. It is at this stage that we need F to be of dimension ≥ 2. Denote $e_2(x_1)$ by x_2.

The homeomorphism e_3 takes x_2 to a vector x_3 of the form $a\hat{x}_2 + bx_2^\perp$, where $a, b > 0$, $\|x_3\| = \|x_2\|$, $d(x_3, F) \leq \|x_3\|/\Delta$, and Δ is a fixed number > 1 independent of E, F, α, x_0 and y_0. Similarly, h_2 takes y_1 to a vector y of the form $c\hat{y}_1 + dy_1^\perp$, where $c, d > 0$, $\|y\| = \|y_1\|$ and $d(y, F) \leq \|y\|/\Delta$. Denote y by y_2.

Note that the subspace $K := \mathrm{span}(x_3^\perp, y^\perp)$ is orthogonal to F. (This is not true when E is a general normed space.) Set $x^\vee = (\|x_3^\perp\|/\|y^\perp\|)y^\perp$ and define $x = \hat{x}_3 + x^\vee$. Clearly, $x^\perp = x^\vee$. The homeomorphism e_4 takes x_3 to x. The action of e_4 can be roughly described as a rotation in the plane $\hat{x}_3 + K$ and the identity on $K^\perp$. Define $x_4 = x$.

We have the following situation: $x = \hat{x} + x^\perp$, $y = \hat{y} + y^\perp$, $\hat{x}, \hat{y} \in F$ and for some $\lambda, \mu > 0$, $\hat{y} = \lambda \hat{x}$ and $y^\perp = \mu x^\perp$. If $\|\hat{y}\| \geq \|\hat{x}\|$ define $v = \hat{y} + x^\perp$, and if $\|\hat{y}\| < \|\hat{x}\|$ define $v = \lambda x$. We shall define f_1 such that $f_1(x) = v$ and f_1 is $N \cdot \alpha$-bicontinuous for some fixed N. If $v = \hat{y} + x^\perp$, then f_1 has the form $f_1(z) = z + a(z) \cdot \hat{x}$, and $a(z)$ tends to zero as $d(z, [x, v])$ tends to λ. In the case that $v = \lambda x$, f_1 is a piecewise linearly radial homeomorphism and f_1 is N-bilipschitz. This of course implies that f_1 is $N \cdot \alpha$-bicontinuous.

Now we have $v = \hat{y} + v^\perp$ and $y = \hat{y} + y^\perp$, where for some $\nu > 0$, $y^\perp = \nu v^\perp$. We shall define f_2 which takes v to y. The homeomorphism f_2 will have the form $f_2(z) = z + a(z) \cdot v^\perp$, and it will be $N \cdot \alpha$-bicontinuous.

Along the construction described above, but independently of the particular choice of E, F, α, x_0, y_0, we shall define numbers

$M_{1,i}, a_{1,i}, b_{1,i}$ for $i = 1, \ldots, 4$;
$M_{2,i}, a_{2,i}, b_{2,i}$ for $i = 1, 2$;
$M_{3,i}, a_{3,i}, b_{3,i}$ for $i = 1, 2$.

These numbers satisfy the following conditions.

C1 for every $i = 1, \ldots, 4$, $R(x_{i-1}, x_i, e_i; M_{1,i}, a_{1,i}, b_{1,i}, F)$;
C2 for every $i = 1, 2$, $R(y_{i-1}, y_i, h_i; M_{2,i}, a_{2,i}, b_{2,i}, F)$;
C3 $R(x_4, v, f_1; M_{3,1} \cdot \alpha, a_{3,1}, b_{3,1}, F)$;
C4 $R(v, y_2, f_2; M_{3,2} \cdot \alpha, a_{3,2}, b_{3,2}, F)$.

We thus have the following conclusion. There are $M_{i,j}, a_{i,j},\ b_{i,j}$ such that for every E, F, α, x_0, y_0 satisfying conditions A1–A4 in the definition of a UC-constant, there are $e_i \in H(E)$, x_i, $i = 1, \dots, 4$; $h_i \in H(E)$, y_i, $i = 1, 2$; $f_1, f_2 \in H(E)$ and v such that C1–C4 hold.

We now find a, b, M such that M is a UC-constant for $\langle a, b \rangle$. Let E, F, α, x_0, y_0 fulfill conditions A1–A4 in the definition of a UC-constant. Then there are e_i's, f_i's, h_i, etc. which satisfy C1–C4. Define $e = e_4 \circ \cdots \circ e_1$, $h = h_1^{-1} \circ h_2^{-1}$, $g_1 = f_1 \circ e$ and $g_2 = h \circ f_2$.

Let $M_1 = \prod_{i=1}^4 M_{1,i}$, $A_1 = \prod_{i=1}^4 a_{1,i}$ and $B_1 = \prod_{i=1}^4 b_{1,i}$. Then by Proposition 9.12(a), $R(x_0, x_4, e; M_1, A_1, B_1, F)$ holds. By 9.12(b), $R(y_1, y_0, h_1^{-1}; M_{2,1}, a_{2,1}/M_{2,1}, M_{2,1}b_{2,1}, F)$ and $R(y_2, y_1, h_2^{-1}; M_{2,2}, a_{2,2}/M_{2,2}, M_{2,2}b_{2,2}, F)$ hold. Let $A_2 = (a_{2,2}/M_{2,2})(a_{2,1}/M_{2,1})$, $B_2 = M_{2,2}b_{2,2}M_{2,1}b_{2,1}$ and $M_2 = M_{2,2}M_{2,1}$. Then by Proposition 9.12(a), $R(y_2, y_0, h; M_2, A_2, B_2, F)$ holds. Let $M' = M_1 M_{3,1}$, $A' = A_1 a_{3,1}$ and $B' = B_1 b_{3,1}$. Note that if $\alpha \in \mathrm{MC}$ and $M \geq 1$, then $\alpha(Mt) \leq M\alpha$. So by Proposition 9.12(a),

$$R(x_0, v, g_1; M' \cdot \alpha, A', B', F) \text{ holds.} \tag{1}$$

Let $M'' = M_{3,2}M_2$, $A'' = a_{3,2}A_2$ and $B''' = b_{3,2}B_2$. Then by Proposition 9.12(a),

$$R(v, y_0, g_2; M'' \cdot \alpha, A'', B'', F) \text{ holds.} \tag{2}$$

Let $M = \max(M', M'')$, $a = A'A''$ and $b = B'B''$. Then (1) and (2) imply that B1–B4 of Definition 9.11(a) hold. So M is a UC-constant for $\langle a, b \rangle$.

C1 is the conjunction of four requirements. Denote them by C1.1, ..., C1.4. Similarly, denote the two conjuncts of C2 by C2.1 and C2.2.

The construction

PART 1: The construction of e_1 and h_1. Let E, F, α, x_0, y_0 satisfy conditions A1–A4 in the definition of a UC-constant. Write $x = x_0$ and $y = y_0$. If $d(x, F) \leq \|x\|/3$, let $e_1 = \mathrm{Id}$. Otherwise let $u \in F - \{0\}$ and $e_1 \in H(E)$ be such that

(1) $e_1(x) \in \mathrm{span}(\{x, u\})$,
(2) $\|e_1(x)\| = \|x\|$,
(3) $d(e_1(x), F) = \|e_1(x)\|/3$,
(4) $e_1 \restriction F = \mathrm{Id}$;
(5) $\mathrm{supp}(e_1) \subseteq B(0; \|x\|/2, 3\|x\|/2)$,
(6) e_1 is $M^{\mathrm{bnd}}(3)$ bilipschitz.

The existence of e_1 is ensured by Proposition 9.7. Let $x_1 = f(x)$, $M_{1,1} = M^{\mathrm{bnd}}(3)$, $a_{1,1} = 1/2$ and $b_{1,1} = 3/2$. Recall that $x_0 = x$. By (1)–(6), $R(x_0, x_1, e_1; M_{1,1}, a_{1,1}, b_{1,1}, F)$ holds. So C1.1 is fulfilled.

Let $h_1 \in H(E)$ have the same role for y as e_1 had for x. Let $y_1 = h_1(y)$, $M_{2,1} = M^{\mathrm{bnd}}(3)$, $a_{2,1} = 1/2$ and $b_{2,1} = 3/2$. Recall that $y_0 = y$. Then $R(y_0, y_1, h_1; M_{2,1}, a_{2,1}, b_{2,1}, F)$ holds. So C2.1 is fulfilled.

PART 2: The construction of e_2. Since $\|e_1(x)\| = \|x\|$ and $\|h_1(y)\| = \|y\|$, $\|e_1(x)\| \approx^\alpha \|h_1(y)\|$. We check that

$$d(e_1(x), F) \approx^\alpha d(h_1(y), F).$$

If $e_1 = h_1 = \mathrm{Id}$, then there is nothing to check. Suppose that $e_1 \neq \mathrm{Id} \neq h_1$. Then $d(e_1(x), F) = \|x\|/3$ and $d(h_1(y), F) = \|y\|/3$. So

$$\frac{d(h_1(y), F)}{d(e_1(x), F)} = \frac{\|y\|}{\|x\|} \leq \frac{\alpha(\|x\|)}{\|x\|} \leq \frac{\alpha(\|x\|/3)}{\|x\|/3} = \frac{\alpha(d(e_1(x), F))}{d(e_1(x), F)}.$$

Hence $d(h_1(y), F) \leq \alpha(d(e_1(x), F))$. Since $\|x\| \leq \|y\|$, $d(h_1(x), F) \leq d(e_1(y), F) \leq \alpha(d(e_1(y), F))$.

Suppose that $e_1 \neq \mathrm{Id} = h_1$. Then $d(h_1(y), F) \leq \|y\|/3$ and $d(e_1(x), F) = \|x\|/3$. So $d(h_1(y), F) \leq \alpha(\|x\|)/3 \leq \alpha(\|x\|/3) = \alpha(d(e_1(x), F))$. Also, $d(e_1(x), F) \leq d(x, F) \leq \alpha(d(y, F)) = \alpha(d(h_1(y), F))$. The argument in the case $e_1 = \mathrm{Id} \neq h_1$ is identical.

Let $e_1(x)$ take the role of x and $h_1(y)$ take the role of y. That is, $e_1(x), h_1(y)$ are renamed and are now denoted by x and y. Hence $d(x, F) \leq \|x\|/3$ and $d(y, F) \leq \|y\|/3$. Let $\hat{x}, \hat{y} \in F$ be such that $\|x - \hat{x}\| \leq (1+\varepsilon)d(x, F)$ and $\|y - \hat{y}\| \leq (1+\varepsilon)d(y, F)$. ε will be determined later. Let $x^{\perp} = x - \hat{x}$ and $y^{\perp} = y - \hat{y}$. Then e_2 will take x to a vector of the form $\lambda\hat{y} + x^{\perp}$, where $\lambda > 0$. It is in this part that F needs to be of dimension > 1. We may assume that:

2.1 $x = \hat{x} + x^{\perp}$ and $y = \hat{y} + y^{\perp}$,
2.2 $\hat{x}, \hat{y} \in F$,
2.3 $\|x^{\perp}\| \leq (1+\varepsilon)d(x, F)$ and and $\|y^{\perp}\| \leq (1+\varepsilon)d(y, F)$,
2.4 $d(x, F) \leq \|x\|/3$ and $d(y, F) \leq \|y\|/3$,
2.5 $\|x\| \approx^{\alpha} \|y\|$ and $d(x, F) \approx^{\alpha} d(y, F)$,
2.6 if $\text{co-dim}^{E}(F) = 1$, then x and y are on the same side of F.

We define a functional ψ on $\mathrm{span}(F \cup \{x^{\perp}\})$: $\psi(x^{\perp}) = \|x^{\perp}\|$, and $\psi(u) = 0$ for every $u \in F$. Let $\varphi \in E^*$ extend ψ and $\|\varphi\| = \|\psi\|$. Let $L = \mathrm{span}(\{x^{\perp}\})$ and $H = \ker(\varphi)$. So $F \subseteq H$. For every $u \in F$,

$$|\psi(u + x^{\perp})| = \|x^{\perp}\| \leq (1+\varepsilon)d(x, F) = (1+\varepsilon)d(x^{\perp}, F) \leq (1+\varepsilon)\|u + x^{\perp}\|.$$

So $\|\varphi\| = \|\psi\| \leq 1 + \varepsilon$.

Let $u \in E$. Define $v = u - \varphi(u)x^{\perp}/\|x^{\perp}\|$. Then $(u)_H = v$ and $(u)_L = \varphi(u)x^{\perp}/\|x^{\perp}\|$. So

$$\begin{aligned}\|(u)_H\| = \|v\| = \left\| u - \varphi(u)\frac{x^{\perp}}{\|x^{\perp}\|} \right\| &\leq \|u\| + |\varphi(u)| \left\| \frac{x^{\perp}}{\|x^{\perp}\|} \right\| = \|u\| + |\varphi(u)| \\ &\leq \|u\| + \|\varphi\|\|u\| \leq (2+\varepsilon)\|u\|\end{aligned}$$

and $\|(u)_L\| = \|\varphi(u)\frac{x^{\perp}}{\|x^{\perp}\|}\| = |\varphi(u)| \leq \|\varphi\|\|u\| \leq (1+\varepsilon)\|u\|$. So

$$\|(u)_H\| + \|(u)_L\| \leq (3 + 2\varepsilon)\|u\|.$$

Let F_1 be a 2-dimensional subspace of F such that $\hat{x}, \hat{y} \in F_1$. Such a subspace exists since F is not 1-dimensional. Let H_1 be an almost orthogonal complement of F_1 in H. That is, $H_1 \oplus F_1 = H$, and for every $u \in H$, $\|(u)_{F_1}\| + \|(u)_{H_1}\| \leq M^{\mathrm{aoc}}(2) \cdot \|u\|$. Let $\| \, \|^{\mathbf{H}}$ be a tight Hilbert norm on F_1. So $\| \, \|^{\mathbf{H}} \approx^{M^{\mathrm{thn}}} \| \, \|^{F_1}$.

We define an equivalent norm $\| \, \|^{\mathbf{N}}$ on E. Let $u \in E$ and suppose that $u = u_1 + u_2 + u_3$, where $u_1 \in F_1$, $u_2 \in H_1$ and $u_3 \in L$. Define $\|u\|^{\mathbf{N}} := \|u_1\|^{\mathbf{H}} + \|u_2\| + \|u_3\|$. Then $\|u\| \approx^{3+2\varepsilon} \|u_1 + u_2\| + \|u_3\|$ and $\|u_1 + u_2\| \approx^{M^{\mathrm{hlb}}} \|u_1\|^{\mathbf{H}} + \|u_2\|$. Note that if $E =$

$E_1 \oplus E_2$, for $\ell, i = 1,2$, $\| \; \|^{\ell,i}$ is a norm on E_ℓ and $\| \; \|^{\ell,1} \approx^{M_\ell} \| \; \|^{\ell,2}$, then for every $u \in E$, $\|(u)_{E_1}\|^{1,1} + \|(u)_{E_2}\|^{2,1} \approx^{\max(M_1,M_2)} \|(u)_{E_1}\|^{1,2} + \|(u)_{E_2}\|^{2,2}$. So $\|u_1 + u_2\| + \|u_3\| \approx^{\max(M^{\rm hlb},1)} \|u_1\|^{\mathbf{H}} + \|u_2\| + \|u_3\|$. That is, $\|u_1 + u_2\| + \|u_3\| \approx^{M^{\rm hlb}} \|u_1\|^{\mathbf{H}} + \|u_2\| + \|u_3\|$. Let $M^{\rm sp} = (3 + 2\varepsilon)M^{\rm hlb}$. Then $\|u\| \approx^{M^{\rm sp}} \|u_1\|^{\mathbf{H}} + \|u_2\| + \|u_3\| = \|u\|^{\mathbf{N}}$. Let $d^{\mathbf{N}}$ denote the metric on E obtained from $\| \; \|^{\mathbf{N}}$.

Let $\hat{z} = (\|\hat{x}\|^{\mathbf{H}}/\|\hat{y}\|^{\mathbf{H}})\hat{y}$. Then $\|\hat{z}\|^{\mathbf{H}} = \|\hat{x}\|^{\mathbf{H}}$. The homeomorphism e_2 will take x to $\hat{z} + x^{\perp}$. Let $r = \|\hat{x}\|^{\mathbf{H}}$, $S_1 = S^{\mathbf{H}}(0,r)$ and $S = \{u + \mu \cdot (x)^{\perp} \mid u \in S_1 \text{ and } 0 \le \mu \le 1\}$. Let θ_0 be the angle from $\hat{x}$ to $\hat{y}$. That is, $\mathrm{Rot}^{F_1}_{\theta_0}(\hat{x}) = \hat{z}$.

Let $E_1 = H_1 + L$. Then $F_1 \oplus E_1 = E$. We first define a function $\eta : [0,\infty) \to [0,\theta_0]$, and the homeomorphism e_2 will be defined by means of η as follows:

$$e_2(u) = \mathrm{Rot}^{F_1}_{\eta(d^{\mathrm{N}}(u,S))}((u)_{F_1}) + (u)_{E_1}.$$

Define η to be the piecewise linear function with one breakpoint at $r/2$ such that $\eta(0) = \theta_0$ and $\eta(s) = 0$ for every $s \ge r/2$.

Note that $\hat{x} \in F_1$, $x^{\perp} \in L$ and $x = \hat{x} + x^{\perp}$. So $(x)_{F_1} = \hat{x}$ and $(x)_{E_1} = x^{\perp}$. Also, $x \in S$. It follows that $e_2(x) = \hat{z} + x^{\perp}$. Hence for some $\lambda > 0$, $e_2(x) = \lambda\hat{y} + x^{\perp}$. Obviously, $e_2(F_1) = F_1$. We verify that

$$e_2(F) = F. \tag{2.1}$$

Suppose that $u \in F$. So $u = (u)_{F_1} + (u)_{E_1}$. Hence $(u)_{E_1} \in F$. For some angle β, $e_2(u) = \mathrm{Rot}^{F_1}_{\beta}((u)_{F_1}) + (u)_{E_1}$. Since $F_1 \subseteq F$, $\mathrm{Rot}^{F_1}_{\beta}((u)_{F_1}) \in F$. So $e_2(u) \in F$.

Note that $d^{\mathbf{N}}(B^{\mathbf{N}}(u,s), S) = r/2$. Hence $e_2 \restriction B^{\mathbf{N}}(0,r/2) = \mathrm{Id}$. By *2.3* and *2.4*,

$$\begin{aligned} r = \|\hat{x}\|^{\mathbf{N}} &\ge \frac{\|\hat{x}\|}{M^{\rm sp}} \ge \frac{1}{M^{\rm sp}}\left(\|x\| - \|x^{\perp}\|\right) \ge \frac{1}{M^{\rm sp}}(\|x\| - (1+\varepsilon)d(x,F)) \\ &\ge \frac{1}{M^{\rm sp}}\left(\|x\| - (1+\varepsilon)\frac{\|x\|}{3}\right) = \frac{1}{M^{\rm sp}}\left(\frac{2}{3} - \varepsilon\right)\|x\| > \frac{1}{2M^{\rm sp}}\|x\| \end{aligned}$$

The last inequality holds when ε is sufficiently small. So $e_2 \restriction B^{\mathbf{N}}(0, \frac{1}{4M^{\rm sp}}\|x\|) = \mathrm{Id}$.

Recall that $\| \; \|^{E} \approx^{M^{\rm sp}} \| \; \|^{\mathbf{N}}$. So $B(0, s/M^{\rm sp}) \subseteq B^{\mathbf{N}}(0,s)$ for every s. It follows that $e_2 \restriction B(0, \frac{1}{4(M^{\rm sp})^2}\|x\|) = \mathrm{Id}$. Let $a_1 = 1/4(M^{\rm sp})^2$. We have shown that

$$e_2 \restriction B(0, a_1\|x\|) = \mathrm{Id}. \tag{2.2}$$

Now, $\mathrm{supp}(e_2) \subseteq B^{\mathbf{N}}(0, \|x\|^{\mathbf{N}} + r/2) \subseteq B(0, M^{\rm sp}(\|x\|^{\mathbf{N}} + r/2))$ and $r/2 = \|\hat{x}\|^{\mathbf{N}}/2 \le M^{\rm sp}\|\hat{x}\|/2 \le M^{\rm sp} \cdot \frac{4}{3}\|x\|/2 = \frac{2}{3}M^{\rm sp}\|x\|$. So $\mathrm{supp}(e_2) \subseteq B(0, 2(M^{\rm sp})^2\|x\|)$. Define $b_1 = 2(M^{\rm sp})^2$. Then

$$e_2 \restriction (E - B(0, b_1\|x\|)) = \mathrm{Id}. \tag{2.3}$$

We next show that there is $M_1 > 0$ which is independent of x, F and θ_0 such that e_2 is M_1-bilipschitz. Indeed, we shall find M_1' such that for every $u, v \in E$: if $\|u - v\|^{\mathbf{N}} \le r/2$, then $\|e_2(u) - e_2(v)\|^{\mathbf{N}} \le M_1' \cdot \|u - v\|^{\mathbf{N}}$. This fact implies that e_2 is M_1'-Lipschitz in the metric $d^{\mathbf{N}}$.

Obviously, $|\eta(t) - \eta(s)| \le \frac{\theta_0}{r/2}|t - s| \le \frac{2\pi}{r}|t - s|$ for every $s, t \in [0,\infty)$. Define $\theta(u) = \eta(d^{\mathbf{N}}(u,S))$. So $|\theta(u) - \theta(v)| = |\eta(d^{\mathbf{N}}(u,S)) - \eta(d^{\mathbf{N}}(v,S))| \le \frac{2\pi}{r}\|u - v\|^{\mathbf{N}}$.

Clearly, $\|x^{\perp}\| < \|x\|/2$. So $\|x\| < 2\|\hat{x}\|$. Hence $\|x^{\perp}\| < \|\hat{x}\|$. It follows that $\|x^{\perp}\|^{\mathbf{N}} < (M^{\mathrm{sp}})^2\|\hat{x}\|^{\mathbf{N}}$. Hence $\max(\{\|u\|^{\mathbf{N}} \mid u \in S\}) \le (1+(M^{\mathrm{sp}})^2)\cdot\|\hat{x}\|^{\mathbf{N}} = 2(1+(M^{\mathrm{sp}})^2)\cdot r$.

Let $u, v \in E$ be such that $\|u-v\|^{\mathbf{N}} \le r/2$. If $\|u\|^{\mathbf{N}} > 2(1+(M^{\mathrm{sp}})^2)\cdot r + r$, then $\|v\|^{\mathbf{N}} > 2(1+(M^{\mathrm{sp}})^2)\cdot r + r/2$. So $e_2(u) = u$ and $e_2(v) = v$. Suppose that $\|u\|^{\mathbf{N}} \le 2(1+(M^{\mathrm{sp}})^2)\cdot r + r$. Define $M^{\mathrm{sp1}} = 4 + 2(M^{\mathrm{sp}})^2$. Then $\|u\|^{\mathbf{N}}, \|v\|^{\mathbf{N}} < M^{\mathrm{sp1}}\cdot r$. We have

$$\begin{aligned} e_2(v) - e_2(u) &= (\mathrm{Rot}^{F_1}_{\theta(v)}((v)_{F_1}) - \mathrm{Rot}^{F_1}_{\theta(u)}((v)_{F_1})) \\ &\qquad + (\mathrm{Rot}^{F_1}_{\theta(u)}((v)_{F_1}) - \mathrm{Rot}^{F_1}_{\theta(u)}((u)_{F_1})) + ((v)_{E_1} - (u)_{E_1}). \end{aligned}$$

So

$$\begin{aligned} \|e_2(v) - e_2(u)\|^{\mathbf{N}} &\le \|\mathrm{Rot}^{F_1}_{\theta(v)}((v)_{F_1}) - \mathrm{Rot}^{F_1}_{\theta(u)}((v)_{F_1})\|^{\mathbf{N}} \\ &\qquad + \|(\mathrm{Rot}^{F_1}_{\theta(u)}((v)_{F_1}) - \mathrm{Rot}^{F_1}_{\theta(u)}((u)_{F_1})) + ((v)_{E_1} - (u)_{E_1})\|^{\mathbf{N}} \\ &= \|\mathrm{Rot}^{F_1}_{\theta(v)}((v)_{F_1}) - \mathrm{Rot}^{F_1}_{\theta(u)}((v)_{F_1})\|^{\mathbf{N}} + \|v-u\|^{\mathbf{N}}. \end{aligned}$$

We deal with the first summand in the last expression:

$$\begin{aligned} \|\mathrm{Rot}^{F_1}_{\theta(v)}((v)_{F_1}) - \mathrm{Rot}^{F_1}_{\theta(u)}((v)_{F_1})\|^{\mathbf{N}} &\le |\theta(v)-\theta(u)|\cdot\|v\|^{\mathbf{N}} \le \frac{2\pi}{r}\|v-u\|^{\mathbf{N}}\cdot\|v\|^{\mathbf{N}} \\ &\le \frac{2\pi}{r}\|v-u\|^{\mathbf{N}}\cdot M^{\mathrm{sp1}}\cdot r = 2\pi\|v-u\|^{\mathbf{N}}\cdot M^{\mathrm{sp1}}. \end{aligned}$$

It follows that for every $u, v \in E$, $\|e_2(v) - e_2(u)\|^{\mathbf{N}} \le (2\pi M^{\mathrm{sp1}} + 1)\cdot\|v-u\|^{\mathbf{N}}$.

Obviously, for every $u \in E$, $e_2^{-1}(u) = \mathrm{Rot}^{F_1}_{-\eta(d^{\mathrm{N}}(u,S))}((u)_{F_1}) + (u)_{E_1}$. So

$$\|e_2^{-1}(v) - e_2^{-1}(u)\|^{\mathbf{N}} \le (2\pi M^{\mathrm{sp1}} + 1)\cdot\|v-u\|^{\mathbf{N}}.$$

Let $M_1 = (2\pi M^{\mathrm{sp1}} + 1)\cdot(M^{\mathrm{sp}})^2$. Then

$$\text{(2.4)} \qquad e_2 \text{ is } M_1\text{-bilipschitz in the norm } \|\ \|^E.$$

Set $x_2 = e_2(x)$ and recall that $x_1 = x$. Hence by (2.1)–(2.4), $R(x_1, x_2, e_2; M_1, a_1, b_1, F)$ holds. That is, C1.2 is fulfilled with $M_{1,2} = M_1$, $a_{1,2} = a_1$ and $b_{1,2} = b_1$.

Since e_2 is M_1-bilipschitz and $e_2(0) = 0$, it follows that $\|e_2(x)\| \approx^{M_1} \|x\|$. From the fact that $e_2(F) = F$, it follows that $d(e_2(x), F) \approx^{M_1} d(x, F)$. So

$$\text{(2.5)} \qquad \|e_2(x)\| \approx^{M_1\cdot\alpha} \|y\| \quad\text{and}\quad d(e_2(x), F) \approx^{M_1\cdot\alpha} d(y, F).$$

PART 3: The construction of e_3, h_2 and e_4. Recall that x_2 has the form $\lambda\hat{y} + x^{\perp}$. Rename x_2 and call it x, and denote $\lambda\hat{y}$ by $\hat{x}$. We now have:

- *3.1** $x = \hat{x} + x^{\perp}$ and $y = \hat{y} + y^{\perp}$,
- *3.2** $\hat{x}, \hat{y} \in F$ and for some $\lambda > 0$, $\hat{x} = \lambda\hat{y}$,
- *3.3** $\|x^{\perp}\| \le (1+\varepsilon)d(x,F)$ and $\|y^{\perp}\| \le (1+\varepsilon)d(y,F)$,
- *3.5** $\|x\| \approx^{M_1\cdot\alpha} \|y\|$ and $d(x,F) \approx^{M_1\cdot\alpha} d(y,F)$,
- *3.6** if $\text{co-dim}^E(F) = 1$, then x and y are on the same side of F.

Property *3.4* which is analogous to *2.4* is missing. Only after applying e_3 to x and h_2 to y, we shall retain this property.

For the next step in the construction we choose some $\Delta > 1$. The value of Δ will be determined later, and it will be independent of E, F, α, x_0 and y_0. The definition of e_3 and h_2 depends on Δ.

We first define e_3. If $d(x,F) \leq \|x\|/\varDelta$, then define $e_3 = \mathrm{Id}$. Suppose that $d(x,F) > \|x\|/\varDelta$. Then there are $e_3 \in H(E)$ and $a,b > 0$ such that

(1) $e_3(x) = a\hat{x} + bx$,
(2) $\|e_3(x)\| = \|x\|$,
(3) $d(e_3(x),F) = \|e_3(x)\|/\varDelta$,
(4) $e_3 \restriction F = \mathrm{Id}$,
(5) $\mathrm{supp}(e_3) \subseteq B(0;\|x\|/2, 3\|x\|/2)$,
(6) e_3 is $M^{\mathrm{bnd}}(\varDelta)$-bilipschitz.

The existence of e_3 follows from Proposition 9.7.

Recall that $x_2 = x$ and define $x_3 = e_3(x)$. Then $R(x_2, x_3, e_3; M^{\mathrm{bnd}}(\varDelta), 1/2, 3/2, F)$ holds. That is, C1.3 is fulfilled with $M_{1,3} = M^{\mathrm{bnd}}(\varDelta)$, $a_{1,3} = 1/2$ and $b_{1,3} = 3/2$.

There is $h_2 \in H(E)$ which acts on y in the way that e_3 acts on x. That is, if $d(y,F) \leq \|y\|/\varDelta$, then $h_2 = \mathrm{Id}$, and if $d(y,F) > \|y\|/\varDelta$, then there are $c,d > 0$ such that (1)–(6) above hold when y, h_2, c, d replace x, e_3, a, b. Recall that $y_1 = y$ and define $y_2 = h_2(y)$. Then $R(y_1, y_2, h_2; M^{\mathrm{bnd}}(\varDelta), 1/2, 3/2, F)$ holds. That is, C2.2 is fulfilled with $M_{2,2} = M^{\mathrm{bnd}}(\varDelta)$, $a_{2,2} = 1/2$ and $b_{2,2} = 3/2$.

Suppose that $e_3 \neq \mathrm{Id}$. Then $(\star)$ $e_3(x) = a\lambda\hat{y} + b(\lambda\hat{y} + x^{\perp}) = (a+b)\lambda\hat{y} + bx^{\perp}$. By *3.1*–3.3*, $\|x^{\perp}\| \leq (1+\varepsilon)d(x^{\perp},F)$. So from $(\star)$ it follows that $\|bx^{\perp}\| \leq (1+\varepsilon)d(e_3(x),F)$. Denote $(a+b)\lambda\hat{y}$ by $\hat{x}_3$ and $bx^{\perp}$ by $x_3^{\perp}$. In *3.1*–3.3** and in *3.6** replace x, $\hat{x}$ and $x^{\perp}$ by x_3, $\hat{x}_3$ and $x_3^{\perp}$, and denote the resulting statements by *3.1**(x_3,y) etc. Then *3.1**(x_3,y)–*3.3**(x_3,y) and *3.6**(x_3,y) hold. Also,

$$(\dagger) \qquad d(x_3,F) \leq \|x_3\|/\varDelta.$$

If $e_3 = \mathrm{Id}$ and we define $\hat{x}_3$ to be $\hat{x}$ and $x_3^{\perp}$ to be $x^{\perp}$, then again $(\dagger)$ holds.

Applying the same argument to y_2 and defining $\hat{y}_2$ and $y_2^{\perp}$ in analogy with $\hat{x}_3$ and $x_3^{\perp}$ we conclude that *3.1**(x,y_2)–*3.3**(x,y_2) and *3.6**(x,y_2) hold. Also, $(\dagger)$ holds for y_2.

From *3.5** and from (6) applied to e_3 and h_2 it follows that

$$\|x_3\| \approx^{M^{\mathrm{bnd}}(\varDelta)} \|x\| \approx^{M_1 \cdot \alpha} \|y\| \approx^{M^{\mathrm{bnd}}(\varDelta)} \|y_2\|$$

and so $(\dagger\dagger)$ $\|x_3\| \approx^{M_1(M^{\mathrm{bnd}}(\varDelta))^2 \cdot \alpha} \|y\|$. Similarly, $(\dagger\dagger\dagger)$ $d(x_3,F) \approx^{M_1(M^{\mathrm{bnd}}(\varDelta))^2 \cdot \alpha} d(y_2,F)$. We now rename $x_3, \hat{x}_3, x_3^{\perp}, y_2, \hat{y}_2, y_2^{\perp}$ as $x, \hat{x}, x^{\perp}, y, \hat{y}, y^{\perp}$. We also denote $M_1(M^{\mathrm{bnd}}(\varDelta))^2 \cdot \alpha$ by α_1. From the above we conclude that

3.1 $x = \hat{x} + x^{\perp}$ and $y = \hat{y} + y^{\perp}$,
3.2 $\hat{x}, \hat{y} \in F$ and for some $\lambda > 0$, $\hat{x} = \lambda\hat{y}$,
3.3 $\|x^{\perp}\| \leq (1+\varepsilon)d(x,F)$ and $\|y^{\perp}\| \leq (1+\varepsilon)d(y,F)$,
3.4 $d(x,F) \leq \|x\|/\varDelta$ and $d(y,F) \leq \|y\|/\varDelta$,
3.5 $\|x\| \approx^{\alpha_1} \|y\|$ and $d(x,F) \approx^{\alpha_1} d(y,F)$,
3.6 if $\text{co-dim}^E(F) = 1$, then x and y are on the same side of F.

Property *3.1* follows from *3.1**(x_3,y) and *3.1**(x,y_2), and the same is true for Properties *3.2*, *3.3* and *3.6*. Property *3.4* is the conjunction of $(\dagger)$ applied to x_3 and to y_2 and *3.6* is the conjunction of $(\dagger\dagger)$ and $(\dagger\dagger\dagger)$.

Set $z^{\perp} = \|x^{\perp}\| \cdot \frac{y^{\perp}}{\|y^{\perp}\|}$ and $z = \hat{x} + z^{\perp}$. We next define e_4. It will take x to z. So after applying e_4 we shall reach the following situation: $x_4 = \hat{x}_4 + x_4^{\perp}$, $y_2 = \hat{y}_2 + y_2^{\perp}$, $\hat{x}_4 = \lambda \hat{y}_2$ for some $\lambda > 0$ and $x_4^{\perp} = \mu y_2^{\perp}$ for some $\mu > 0$.

There are two cases: $\text{co-dim}^E(F) = 1$ and $\text{co-dim}^E(F) > 1$.

Case 1: $\text{co-dim}^E(F) = 1$. Since x and y are on the same side of F, there are $\nu > 0$ and $u \in F$ such that $z^{\perp} = u + \nu x^{\perp}$. Let $L = [x, \hat{x} + z^{\perp}]$. We may assume that in *3.3*, $\varepsilon \le 1/2$. We show that $\text{lngth}(L)/d(L,F) + 1 \le 19$. Clearly, $\text{lngth}(L) = \|\hat{x} + z^{\perp} - x\| = \|z^{\perp} - x^{\perp}\| \le 2\|x^{\perp}\|$. So

$$\text{lngth}(L) \le 2\|x^{\perp}\|. \tag{3.1}$$

Since for some t, $z^{\perp} = ty^{\perp}$, we have $\|z^{\perp}\| \le (1+\varepsilon) d(z^{\perp}, F)$. So

$$\|x^{\perp}\| = \|z^{\perp}\| \le (1+\varepsilon) d(u + \nu x^{\perp}, F) = (1+\varepsilon)\nu d(x^{\perp}, F) \le (1+\varepsilon)\nu \|x^{\perp}\|.$$

Hence $1 \le (1+\varepsilon)\nu$. In the above argument we interchange the roles of $x^{\perp}$ and $z^{\perp}$. That is, for some $u' \in F$, $x^{\perp} = u' + \frac{1}{\nu} z^{\perp}$, and hence $1 \le (1+\varepsilon)\frac{1}{\nu}$. We conclude that $\frac{1}{1+\varepsilon} \le \nu \le 1 + \varepsilon$. Let $v \in L$. Then for some $t \in [0,1]$, $v = \hat{x} + x^{\perp} + t(z^{\perp} - x^{\perp}) = \hat{x} + x^{\perp} + t((u + \nu x^{\perp} - x^{\perp})$. So

$$\begin{aligned} d(v,F) &= d((1 + t(\nu - 1))x^{\perp}, F) = |1 + t(\nu-1)| \cdot d(x^{\perp}, F) \ge (1 - t|\nu - 1|) \cdot d(x^{\perp}, F) \\ &\ge (1 - |\nu - 1|) \cdot d(x^{\perp}, F) \ge \left(1 - \left(1 + \varepsilon - \frac{1}{1+\varepsilon}\right)\right) \cdot d(x^{\perp}, F) \\ &= \left(\frac{1}{1+\varepsilon} - \varepsilon\right) d(x^{\perp}, F) \ge \frac{1}{6} d(x^{\perp}, F) \ge \frac{1}{6(1+\varepsilon)} \|x^{\perp}\| \ge \frac{1}{9}\|x^{\perp}\|. \end{aligned}$$

Hence

$$d(L,F) \ge \|x^{\perp}\|/9. \tag{3.2}$$

It follows from (3.1) and (3.2) that $\text{lngth}(L)/d(L,F) + 1 \le 19$.

Set $\Delta = 8$. Then $d(L,F) \le d(x,F) \le \|x\|/8$. Hence $\|x^{\perp}\| \le \frac{3}{2} d(x,F) \le \frac{3}{16}\|x\|$. So $\text{lngth}(L) \le \frac{3}{8}\|x\|$. Let $B = B(L, d(L,F))$. Then

$$\min_{v \in B} \|v\| \ge \|x\| - \text{lngth}(L) - d(L,F) \ge \|x\|/2. \tag{3.3}$$

Similarly,

$$\max_{v \in B} \|v\| \le \|x\| + \text{lngth}(L) + d(L,F) \le 3\|x\|/2. \tag{3.4}$$

The endpoints of L are x and $\hat{x} + z^{\perp}$, so by Proposition 9.6(a), there is $e_4 \in H(E)$ such that

(3.5) $\text{supp}(e_4) \subseteq B(L, d(L,F))$,
(3.6) $e_4(x) = \hat{x} + z^{\perp}$,
(3.7) e_4 is $19M^{\text{seg}}$-bilipschitz.

By (3.5), $e_4 \restriction F = \text{Id}$. By (3.3), (3.4) and (3.5), $\text{supp}(e_4) \subseteq B(0; \|x\|/2, 3\|x\|/2)$. Recall that $x_3 = x$ and define $x_4 = e_4(x)$. It follows that $R(x_3, x_4, e_4; 19M^{\text{seg}}, 1/2, 3/2, F)$ holds.

CASE 2: $\text{co-dim}^E(F) > 1$. Let $\Upsilon > 1$. By Proposition 9.3, there is a closed subspace F_1 of E such that $F \subseteq F_1$, $\text{span}(F_1 \cup \{x, y\}) = E$, $d(x, F_1) \geq \frac{1}{\Upsilon} d(x, F)$ and $d(y, F_1) \geq \frac{1}{\Upsilon} d(y, F)$. Obviously, either $\text{co-dim}^E(F_1) = 1$ or $\text{co-dim}^E(F_1) = 2$. If $\text{co-dim}^E(F_1) = 1$, let $F \subseteq F_2 \subseteq F_1$ be a closed subspace such that $\text{co-dim}^E(F_2) = 2$. Otherwise let $F_2 = F_1$. It follows that $\text{co-dim}^E(F_2) = 2$, $d(x, F_2) \geq \frac{1}{\Upsilon} d(x, F)$ and $d(y, F_2) \geq \frac{1}{\Upsilon} d(y, F)$.

In *3.4*, choose $\Delta = 24$. Hence $d(x, F_2) \leq d(x, F) \leq \|x\|/24$. In *3.3*, choose $\varepsilon = 1/9$, and choose $\Upsilon = 1\frac{1}{9}$. So $\|x^{\perp}\| \leq (1+\varepsilon) d(x, F) \leq \Upsilon(1+\varepsilon) d(x, F_2) \leq (1\frac{1}{9})^2 d(x, F_2) \leq \frac{4}{3} d(x, F_2)$. In summary,

$$\|x^{\perp}\| \leq 4d(x, F_2)/3 \quad \text{and} \quad d(x, F_2) \leq \|x\|/24. \tag{3.8}$$

Recall that $z^{\perp} = (\|x^{\perp}\|/\|y^{\perp}\|) y^{\perp}$ and $z = \hat{x} + z^{\perp}$. We have $\|y^{\perp}\| \leq \frac{4}{3} d(y, F_2)$. This is shown in the same way as the analogous fact for x. Obviously, $d(y, F_2) = d(y^{\perp}, F_2)$. So $\|y^{\perp}\| \leq \frac{4}{3} d(y^{\perp}, F_2)$. Since $z^{\perp}$ is a multiple of $y^{\perp}$, $\|z^{\perp}\| \leq \frac{4}{3} d(z^{\perp}, F_2)$. Also, $d(z, F_2) = d(z^{\perp}, F_2)$. So $\|z^{\perp}\| \leq \frac{4}{3} d(z, F_2)$.

Note that $z = x - x^{\perp} + z^{\perp}$. So $\|z\| \geq \|x\| - \|x^{\perp}\| - \|z^{\perp}\| = \|x\| - 2\|x^{\perp}\|$. Also, $\|x^{\perp}\| \leq \frac{4}{3} d(x, F_2) \leq \frac{4}{3} \cdot \frac{1}{24} \|x\| = \frac{1}{18} \|x\|$. Hence

$$\frac{d(z, F_2)}{\|z\|} \leq \frac{\|z^{\perp}\|}{\|z\|} \leq \frac{\|x^{\perp}\|}{\|x\| - 2\|x^{\perp}\|} \leq \frac{\|x\|/18}{\|x\| - \|x\|/9} = \frac{1}{16}.$$

In summary,

$$\|z^{\perp}\| \leq 4d(z, F_2)/3 \text{ and } d(z, F_2) \leq \|z\|/16. \tag{3.9}$$

Let H be such that $E = F_2 \oplus H$ and $H \perp^{M^{\text{ort}}} F_2$. We apply Proposition 9.8 to x and to z. Note that by (3.8) and (3.9), x and z satisfy the assumptions of 9.8. So there is $f_1 \in H(E)$ such that: f_1 is M^{cmp}-bilipschitz, $f_1(x) = \hat{x} + (x)_H$, $f_1 \restriction F_2 = \text{Id}$ and $\text{supp}(f_1) \subseteq B(0; \|x\|/2, 3\|x\|/2)$. Similarly, there is $h_1 \in H(E)$ such that: h_1 is M^{cmp}-bilipschitz, $h_1(z) = \hat{x} + (z)_H$, $h_1 \restriction F_2 = \text{Id}$ and $\text{supp}(h_1) \subseteq B(0; \|z\|/2, 3\|z\|/2)$.

We now translate what we have obtained for f_1 and h_1 to statements of the form $R(.,., f_1; \ldots)$ and $R(.,., h_1; \ldots)$. Since f_1 is M^{cmp}-bilipschitz $f_1(x) = \hat{x} + (x)_H$ and $f_1(0) = 0$, it follows that $\|x\| \leq M^{\text{cmp}} \|\hat{x} + (x)_H\|$. So $\text{supp}(f_1) \subseteq B(0; \frac{1}{2}\|x\|, \frac{3M^{\text{cmp}}}{2} \|\hat{x} + (x)_H\|)$. This implies that

$$R(x, \hat{x} + (x)_H, f_1; M^{\text{cmp}}, 1/2, 3M^{\text{cmp}}/2, F) \text{ holds.} \tag{3.10}$$

Similarly,

$$R(z, \hat{x} + (z)_H, h_1; M^{\text{cmp}}, 1/2, 3M^{\text{cmp}}/2, F) \text{ holds.} \tag{3.11}$$

Let $\| \ \|^{\mathbf{H}}$ be a tight equivalent Hilbert norm on H, and define a new norm on E by $\|u\|^{\mathbf{N}} = \|(u)_{F_2}\| + \|(u)_H\|^{\mathbf{H}}$. So $\| \ \| \approx^{M^{\text{fdn}}} \| \ \|^{\mathbf{N}}$. This follows from Proposition 9.4(c). Let $d^{\mathbf{N}}$ denote the metric induced by $\| \ \|^{\mathbf{N}}$ on E.

Set $x^* = (x)_H$, $z^* = (z)_H$ and $z^{\#} = (\|x^*\|^{\mathbf{N}}/\|z^*\|^{\mathbf{N}}) z^*$. We define a homeomorphism $g_{2,1}$ which takes $\hat{x} + x^*$ to $\hat{x} + z^{\#}$. A second homeomorphism $g_{2,2}$ will take $\hat{x} + z^{\#}$ to $\hat{x} + z^*$. So

$$x = \hat{x} + x^{\perp} \xrightarrow{f_1} \hat{x} + (x)_H \xrightarrow{g_{2,1}} \hat{x} + z^{\#} \xrightarrow{g_{2,2}} \hat{x} + (z)_H \xrightarrow{h_1^{-1}} \hat{x} + z^{\perp} = z.$$

Finally, we shall define $\mathsf{e}_4 := h_1^{-1} \circ g_{2,2} \circ g_{2,1} \circ f_1$.

Let θ be the angle from x^* to $z^{\#}$. That is, $\theta \in [0, \pi]$ and $\mathrm{Rot}^H_\theta(x^*) = z^{\#}$. Let $\eta : [0, \infty) \to [0, \theta]$ be the piecewise linear function with one breakpoint at $s_0 = \|x^*\|^{\mathbf{N}}/2M^{\mathrm{thn}}$ such that $\eta(0) = \theta$ and $\eta(s) = 0$ for every $s \geq s_0$. Let S_0 be the circle in $\langle H, \| \, \|^{\mathbf{H}} \rangle$ with center at 0 and radius $\|x^*\|^{\mathbf{H}}$, and let $S = \hat{x} + S_0$. Let $g_{2,1}$ be defined as follows. For $u \in E$ set $u_1 = (u)_H$ and $u_2 = (u)_{F_2}$. Define

$$g_{2,1}(u) = u_2 + \mathrm{Rot}^H_{\eta(d^{\mathbf{N}}(u,S))}(u_1).$$

Since for every $u \in E$, $d^{\mathbf{N}}(u, S) = d^{\mathbf{N}}(g_{2,1}(u), S)$, it follows that $g_{2,1} \in H(E)$. Clearly,

$$g_{2,1}(\hat{x} + x^*) = \hat{x} + z^{\#}.$$

Also, $\mathrm{supp}(g_{2,1}) \subseteq B^{\mathbf{N}}(S, s_0)$. If $u \in F_2$ then $d^{\mathbf{N}}(u, S) = \|u - \hat{x}\| + \|x^*\|^{\mathbf{H}} > s_0$ and so $g_{2,1}(u) = u$. That is, $g_{2,1} \restriction F_2 = \mathrm{Id}$. Since $F \subseteq F_2$,

$$g_{2,1} \restriction F = \mathrm{Id}.$$

Note that $s_0 = \|x^*\|^{\mathbf{N}}/2M^{\mathrm{thn}} \leq \|x^*\|/2$. So $\mathrm{supp}(g_{2,1}) \subseteq B^{\mathbf{N}}(S, \|x^*\|/2)$.

Let $u \in B^{\mathbf{N}}(0, \|\hat{x}\| - \|x^*\|/2)$. So $\|u_2\| \leq \|\hat{x}\| - \|x^*\|/2$. Then

$$\begin{aligned} d^{\mathbf{N}}(u, S) &= \|u_2 - \hat{x}\| + d^{\mathbf{N}}(u_1, S_0) \geq \|u_2 - \hat{x}\| \geq \|\hat{x}\| - \|u_2\| \\ &\geq \|\hat{x}\| - (\|\hat{x}\| - \|x^*\|/2) = \|x^*\|/2. \end{aligned}$$

It follows that $g_{2,1} \restriction B^{\mathbf{N}}(0, \|\hat{x}\| - \|x^*\|/2) = \mathrm{Id}$.

Let $r = \|\hat{x}\| + 2\|x^*\|^{\mathbf{N}}$. Suppose that $u \in E - B^{\mathbf{N}}(0, r)$. Either $\|u_1\| \geq 3\|x^*\|^{\mathbf{N}}/2$ or $\|u_2\| \geq \|\hat{x}\| + \|x^*\|^{\mathbf{N}}/2$. If $v \in S$ then $v = \hat{x} + w$, where $w \in H$ and $\|w\|^{\mathbf{N}} = \|x^*\|^{\mathbf{N}}$. Hence $\|u - v\|^{\mathbf{N}} = \|u_1 - w\|^{\mathbf{N}} + \|u_2 - \hat{x}\|$. If $\|u_1\| \geq 3\|x^*\|^{\mathbf{N}}/2$, then $\|u - v\|^{\mathbf{N}} \geq \|u_1 - w\|^{\mathbf{N}} \geq 3\|x^*\|^{\mathbf{N}}/2 - \|x^*\|^{\mathbf{N}} = \|x^*\|^{\mathbf{N}}/2$. So $u \notin \mathrm{supp}(g_{2,1})$. If $\|u_2\| \geq \|\hat{x}\| + \|x^*\|^{\mathbf{N}}/2$, then $\|u - v\|^{\mathbf{N}} \geq \|u_2 - \hat{x}\| \geq \|\hat{x}\| + \|x^*\|^{\mathbf{N}}/2 - \|\hat{x}\| = \|x^*\|^{\mathbf{N}}/2$. So $u \notin \mathrm{supp}(g_{2,1})$. It follows that $\mathrm{supp}(g_{2,1}) \subseteq B^{\mathbf{N}}(0, r)$.

By (3.8), $\|x^{\perp}\| \leq \frac{1}{18}\|x\|$, and since $x = \hat{x} + x^{\perp}$, we have $\frac{17}{18}\|x\| \leq \|\hat{x}\| \leq \frac{19}{18}\|x\|$. Since $H \perp^{M^{\mathrm{ort}}} F_2$, $\|x^*\| \leq M^{\mathrm{ort}} d(x^*, F_2)$. Also, $M^{\mathrm{ort}} < 4$. By the above and (3.8), $\|x^*\| \leq M^{\mathrm{ort}} d(x^*, F_2) = M^{\mathrm{ort}} d(x, F_2) \leq \frac{4}{24}\|x\|$. Hence $\|\hat{x}\| - \|x^*\|/2 \geq \frac{17-2}{24}\|x\|$ and $r = \|\hat{x}\| + 2\|x^*\|^{\mathbf{N}} \leq (1 + M^{\mathrm{thn}}/3)\|x\|$. It follows that

$$\mathrm{supp}(g_{2,1}) \subseteq B(0; \|x\|/2, 2M^{\mathrm{thn}}\|x\|).$$

Next we find a Lipschitz constant for $g_{2,1}$. By its definition, η is $\frac{\theta}{\|x^*\|^{\mathbf{N}}/2M^{\mathrm{thn}}}$-Lipschitz. So η is $2\pi M^{\mathrm{thn}}/\|x^*\|^{\mathbf{N}}$-Lipschitz. Obviously, $S \subseteq \hat{x} + \bar{B}^{\mathbf{N}}(0, \|x^*\|^{\mathbf{N}})$. By 9.6(c), $g_{2,1}$ is $(M^{\mathrm{rot}} \cdot \frac{2\pi M^{\mathrm{thn}}}{\|x^*\|^{\mathbf{N}}} \cdot \|x^*\|^{\mathbf{N}} + 1)$-Lipschitz in the norm $\| \, \|^{\mathbf{N}}$. That is, $g_{2,1}$ is $(2\pi M^{\mathrm{rot}} \cdot M^{\mathrm{thn}} + 1)$-Lipschitz in the norm $\| \, \|^{\mathbf{N}}$. The same is true for $g_{2,1}^{-1}$. So $g_{2,1}$ is $(2\pi M^{\mathrm{rot}} \cdot M^{\mathrm{thn}} + 1)$-bilipschitz in the norm $\| \, \|^{\mathbf{N}}$. Recall that $\| \, \| \approx^{M^{\mathrm{fdn}}} \| \, \|^{\mathbf{N}}$. Write $\widehat{M}_{2,1} = (M^{\mathrm{fdn}})^2(2\pi M^{\mathrm{rot}} \cdot M^{\mathrm{thn}} + 1)$. Then $g_{2,1}$ is $\widehat{M}_{2,1}$-bilipschitz.

We may now write an $R(\ldots)$ statement for $g_{2,1}$. Since f_1 is M^{cmp}-bilipschitz, $f_1(x) = \hat{x} + (x)_H$ and $f_1(0) = 0$, it follows that $\|x\| \geq \|\hat{x} + (x)_H\|/M^{\mathrm{cmp}}$. Similarly, $g_{1,2} \circ f_1(x) = \hat{x} + z^{\#}$, $g_{1,2} \circ f_1(0) = 0$ and $g_{2,1} \circ f_1$ is $\widehat{M}_{2,1} M^{\mathrm{cmp}}$-bilipschitz. Consequently, $\|x\| \leq$

$\widehat{M}_{2,1}M^{\rm cmp}\|\hat{x}+z^{\#}\|$. It follows that

$$\operatorname{supp}(g_{2,1}) \subseteq B\left(0; \frac{1}{2M^{\rm cmp}}\|\hat{x}+(x)_H\|, 2M^{\rm thn}\widehat{M}_{2,1}M^{\rm cmp}\|\hat{x}+z^{\#}\|\right).$$

Hence

$$(3.12) \qquad R(\hat{x}+(z)_H, \hat{x}+z^{\#}, g_{2,1}; \widehat{M}_{2,1}, 1/2M^{\rm cmp}, 2M^{\rm thn}\widehat{M}_{2,1}M^{\rm cmp}, F) \text{ holds.}$$

Our next goal is to define $g_{2,2}$. Recall that $f_1(\hat{x}) = \hat{x}$ and $f_1(\hat{x}+x^{\perp}) = \hat{x}+x^*$. Also, f_1 is $M^{\rm cmp}$-bilipschitz. So $\|x^*\| \approx^{M^{\rm cmp}} \|x^{\perp}\|$. Similarly, $\|z^*\| \approx^{M^{\rm cmp}} \|z^{\perp}\|$. Also, $\|x^{\perp}\| = \|z^{\perp}\|$. Let $M_{2,1} = (M^{\rm cmp})^2$ and $M_{2,2} = M_{2,1}\cdot(M^{\rm fdn})^2$. It follows that $\|x^*\| \approx^{M_{2,1}} \|z^*\|$. By Proposition 9.4(c), $\| \; \| \approx^{M^{\rm fdn}} \| \; \|^{\mathbf{N}}$, and hence $\|x^*\|^{\mathbf{N}} \approx^{M_{2,2}} \|z^*\|^{\mathbf{N}}$. Since $\|z^{\#}\|^{\mathbf{N}} = \|x^*\|^{\mathbf{N}}$, $\|z^{\#}\|^{\mathbf{N}} \approx^{M_{2,2}} \|z^*\|^{\mathbf{N}}$. Let $a = \|z^*\|^{\mathbf{N}}/\|x^*\|^{\mathbf{N}}$. So

(i) $z^* = az^{\#}$,
(ii) $E = F_2 \oplus H$ and $\|u+v\|^{\mathbf{N}} = \|u\| + \|v\|^{\mathbf{H}}$ for every $u \in F_2$ and $v \in H$,
(iii) $\hat{x} \in F_2$ and $z^{\#} \in H$,
(iv) $1/M_{2,2} \le a \le M_{2,2}$.

Assume first that $a \ge 1$. Let $\hat{x}, z^{\#}, a, 0$ take the roles of $\hat{x}, x, a$ and u in Proposition 9.6(d). By (i)–(iii), the assumptions of 9.6(d) are fulfilled. So relying also on (iv), we conclude that there is $g_{2,2} \in H(E)$ such that (1) $g_{2,2}(\hat{x}+z^{\#}) = \hat{x}+z^*$; (2) $g_{2,2}\restriction F_2 = \mathrm{Id}$; (3) $\operatorname{supp}(g_{2,2}) \subseteq B^{\mathbf{N}}(0; \|\hat{x}+z^{\#}\|^{\mathbf{N}}/2, 3\|\hat{x}+z^*\|^{\mathbf{N}}/2)$; (4) $g_{2,2}$ is $2M^{\rm seg}\cdot M_{2,2}$-bilipschitz in the norm $\| \; \|^{\mathbf{N}}$.

If $a < 1$ then we apply 9.6(d) to $\hat{x}, z^*, 1/a$ and 0, thus obtaining a homeomorphism $g'_{2,2} \in H(E)$ such that $g'_{2,2}(\hat{x}+z^*) = \hat{x}+z^{\#}$. Define $g_{2,2} = (g'_{2,2})^{-1}$. Then (1), (2) and (4) remain true. Instead of (3) we now have $\operatorname{supp}(g_{2,2}) \subseteq B^{\mathbf{N}}(0; \|\hat{x}+z^*\|^{\mathbf{N}}/2, 3\|\hat{x}+z^{\#}\|^{\mathbf{N}}/2)$. Note that by (i)–(iv), $\|\hat{x}+z^{\#}\|^{\mathbf{N}} \le M_{2,2}\|\hat{x}+z^*\|^{\mathbf{N}}$. So $\operatorname{supp}(g_{2,2}) \subseteq B^{\mathbf{N}}(0; \|\hat{x}+z^{\#}\|^{\mathbf{N}}/2M_{2,2}, 3M_{2,2}\|\hat{x}+z^*\|^{\mathbf{N}}/2)$. Recall that $z^* = (z)_H$. What we have shown implies that

$$(3.13) \quad R\left(\hat{x}+z^{\#}, \hat{x}+(z)_H, g_{2,2}; 2(M^{\rm fdn})^2M^{\rm seg}M_{2,2}, \frac{1}{2M^{\rm fdn}M_{2,2}}, 2M^{\rm fdn}M_{2,2}, F\right) \text{ holds.}$$

Note that in deducing (3.13) we used the fact that $\| \; \|^{\mathbf{N}} \approx^{M^{\rm fdn}} \| \; \|$. This concludes the construction of $g_{2,2}$.

Define $e_4 = h_1^{-1}\circ g_{2,2}\circ g_{2,1}\circ f_1$ and $x_4 = z$. Recall that $x_3 = x$. So $x_4 = z = e_4(x_3)$. We now apply Proposition 9.12(a) and (b). It follows from (3.10)–(3.13) and from 9.12 that there are M'_2, A'_2, B'_2 which do not depend on E, F, α, x_0, y_0 such that $R(x_3, x_4, e_4; M'_2, A'_2, B'_2, F)$ holds.

In Case 1 too, we found M'_1, A'_1, B'_1 such that $R(x_3, x_4, e_4; M'_1, A'_1, B'_1, F)$ holds. Define $M_{1,4} = \max(M'_1, M'_2)$, $a_{1,4} = \min(A'_1, A'_2)$ and $b_{1,4} = \max(B'_1, B'_2)$. Then $M_{1,4}, a_{1,4}, b_{1,4}$ fulfill C1.4 in both Case 1 and Case 2.

PART 4: The construction of f_1. We have shown that for $i = 1, \dots, 4$ there is $M_{1,i}$ which does not depend on E, F, α, x_0, y_0 such that e_i is $M_{1,i}$-bilipschitz. We define $e = e_4\circ\cdots\circ e_1$. Then $e(x_0) = x_4 = z$ and $e(0) = 0$. Let $M_{3,1} = \prod_{i=1}^4 M_{1,i}$. So e is $M_{3,1}$-bilipschitz. It follows that $\|z\| \approx^{M_{3,1}} \|x_0\|$. Similarly, for $i = 1, 2$ there is $M_{2,i}$ such that h_i is $M_{2,i}$-bilipschitz. We define $h = h_2 \circ h_1$. Then $h(y_0) = y_2 = y$ and $h(0) = 0$. Let $M_{3,2} = M_{2,1} M_{2,2}$. So h is $M_{3,2}$-bilipschitz. Let $M_{3,0} = M_{3,1} M_{3,2}$. Then

4.1 $\|z\| \approx^{M_{3,0}} \|x_0\|$.

Since $e(F) = F$, we have $d(z,F) \approx^{M_{3,1}} d(x_0,F)$. Similarly, $d(y,F) \approx^{M_{3,2}} d(y_0,F)$. Hence

4.2 $\|z\| \approx^{M_{3,0}\cdot\alpha} \|y\|$ and $d(z,F) \approx^{M_{3,0}\cdot\alpha} d(y,F)$.

The construction also implies that

4.3 $z = \hat{z} + z^{\perp}$, $y = \hat{y} + y^{\perp}$, where $\hat{z}, \hat{y} \in F$, and for some $\lambda, \mu > 0$, $\hat{y} = \lambda\hat{z}$ and $y^{\perp} = \mu z^{\perp}$.

If Case 1 of Part 3 happens, let $\widehat{F} = F$. Suppose that Case 2 of Part 3 happens. Let F_2 be as defined in Case 2 of Part 3. So by (3.9), $\|z^{\perp}\| \leq \frac{4}{3} d(z, F_2)$. By Proposition 9.3 applied to F_2 and taking x and y to be $z^{\perp}$, there is a closed subspace $\widehat{F}$ such that $\|z^{\perp}\| \leq \frac{3}{2} d(z^{\perp}, \widehat{F})$, $F_2 \subseteq \widehat{F}$ and $\operatorname{span}(E \cup \{z^{\perp}\}) = E$. In both cases we have

4.4 $F \subseteq \widehat{F}$, $\widehat{F} \oplus \operatorname{span}(\{z^{\perp}\}) = E$ and $\|z^{\perp}\| \leq 1\frac{1}{2} d(z^{\perp}, \widehat{F})$.

Case 1: $\|\hat{y}\| \geq \|\hat{z}\|$. In this case $\lambda \geq 1$. Let $v = \hat{y} + z^{\perp}$. We shall construct a homeomorphism f_1 such that $f_1(z) = v$. (Recall that $z = x_4$.) Denote v by v. So $v = \lambda\hat{z} + z^{\perp}$. If $\lambda = 1$ let $f_1 = \mathrm{Id}$. Assume that $\lambda > 1$.

Let $H = \operatorname{span}(\{\hat{y}, y^{\perp}\})$, $H_1 = \operatorname{span}(\{\hat{y}\})$ and $H_2 = \operatorname{span}(\{y^{\perp}\})$. Let F_3 be a subspace of $\widehat{F}$ such that for some $\varphi \in \widehat{F}^*$, $\|\varphi\| = 1$, $\varphi(\hat{z}) = \|\hat{z}\|$ and $F_3 = \ker(\varphi)$. It follows that $H_1 \oplus F_3 = \widehat{F}$, $\widehat{F} \oplus H_2 = E$ and $\widehat{F} = H_1 \oplus H_2 \oplus F_3$. Clearly, $\|\mathrm{Proj}_{H_1,F_3}\| = \|\varphi\| = 1$. So by Proposition 9.2(d), $H_1 \perp^1 F_3$.

Let $S = \{a\hat{z} + bz^{\perp} \mid a \in \mathbb{R},\ b \in [0,1]\}$. We define $\eta : [0,\infty) \times [0,\infty) \to [0,\infty)$. For every s, $\eta_s(t) := \eta(s,t)$ is a piecewise linear function of t. For $s \geq (\lambda - 1)\|\hat{z}\|$, $\eta_s = \mathrm{Id}$. If $s < (\lambda-1)\|\hat{z}\|$, then $\eta_s(t)$ has breakpoints at $\|\hat{z}\|/2$, $\|\hat{z}\|$ and $2\lambda\|\hat{z}\|$; $\eta_s(t) = t$ for every $t \in [0, \|\hat{z}\|/2) \cup [2\lambda\|\hat{z}\|, \infty)$; and

$$\eta_s(\|\hat{z}\|) = \left(1 - \frac{s}{(\lambda-1)\|\hat{z}\|}\right) \cdot \lambda\|\hat{z}\| + \frac{s}{(\lambda-1)\|\hat{z}\|} \cdot \|\hat{z}\|.$$

Denote $(\lambda-1)\|\hat{z}\|$ by a. Then in particular, $\eta_0(\|\hat{z}\|) = \lambda\|\hat{z}\|$ and $\eta_a(\|\hat{z}\|) = \|\hat{z}\|$.

For $u \in E$ we denote $(u)_{H_1}, (u)_{H_2}, (u)_{F_3}$ by $(u)_1, (u)_2$ and $(u)_3$ respectively, and we abbreviate $(u)_i$ by u_i when the notation $(u)_i$ is too cumbersome. Set $E^+ = \{t\hat{z} + w \mid t \geq 0,\ w \in H_2 \oplus F_3\}$. Let f_1 be defined by

$$f_1(u) = \begin{cases} \eta(d(u,S), \|u_1\|)\dfrac{\hat{z}}{\|\hat{z}\|} + u_2 + u_3, & u \in E^+, \\ u, & u \in E - E^+. \end{cases}$$

Note that $f_1 \restriction H_2 \oplus F_3 = \mathrm{Id}$, so $f_1 \in H(E)$. We shall define the constants mentioned in C3 and show that C3 holds. Recall that $\mathrm{C3} \equiv R(x_4, v, f_1; M_{3,1}\cdot\alpha, a_{3,1}, b_{3,1}, F)$. We verify R1–R4 in the definition of $R(\ldots)$.

R1: Clearly, $f_1(x_4) = f_1(z) = v = v$.

R3: We verify that $f_1(F) = F$. For every $u \in E$ and in particular for every $u \in F$, $f_1(u) - u \in H_1 = \mathrm{span}(\{\hat{y}\}) \subseteq F$. So $f_1(u) = u + (f_1(u) - u) \in F$. An identical argument shows that $f_1^{-1}(F) \subseteq F$. Hence R3 holds.

R2: We find $M_{3,1}$ and prove that f_1 is $M_{3,1} \cdot \alpha$-bicontinuous. Note that if $g \in H(E)$, $K \subseteq E$ is closed, $\mathrm{supp}(g) \subseteq K$ and $g \restriction K$ is β-continuous, then g is 2β-continuous. Since $\mathrm{supp}(f_1) \subseteq E^+$, we may consider only points which belong to E^+. Let $u, w \in E^+$. Then

$$\begin{aligned}\|f_1(w) - f_1(u)\| &\leq |\eta(d(w,S), \|w_1\|) - \eta(d(u,S), \|u_1\|)| + \|(w-u)_2\| + \|(w-u)_3\| \\ &\leq |\eta(d(w,S), \|w_1\|) - \eta(d(u,S), \|w_1\|)| + |\eta(d(u,S), \|w_1\|) - \eta(d(u,S), \|u_1\|)| \\ &\quad + \|(w-u)_2\| + \|(w-u)_3\|.\end{aligned}$$

That is,

$$\begin{aligned}\text{(4.1)} \qquad \|f_1(w) - f_1(u)\| \leq\ & |\eta(d(w,S), \|w_1\|) - \eta(d(u,S), \|w_1\|)| \\ &+ |\eta(d(u,S), \|w_1\|) - \eta(d(u,S), \|u_1\|)| + \|(w-u)_2\| + \|(w-u)_3\|.\end{aligned}$$

The first summand on the right hand side of (4.1) has the form $|\eta(s_1,t) - \eta(s_2,t)|$. If $s_1, s_2 \in [0, (\lambda - 1)\|\hat{z}\|]$, then

$$\begin{aligned}|\eta(s_1,t) - \eta(s_2,t)| &= \frac{|s_1 - s_2|}{(\lambda-1)\|\hat{z}\|} \cdot (\eta(0,t) - \eta((\lambda-1)\|\hat{z}\|, t)) \\ &\leq \frac{\lambda\|\hat{z}\| - \|\hat{z}\|}{(\lambda-1)\|\hat{z}\|} \cdot |s_1 - s_2| = |s_1 - s_2|.\end{aligned}$$

The inequality between the first and last expression above is true for every $s_1, s_2 \in [0, \infty)$. So $|\eta(d(w,S), \|w_1\|) - \eta(d(u,S), \|w_1\|)| \leq |d(w,S) - d(u,S)| \leq \|w - u\|$. That is,

$$\text{(4.2)} \qquad |\eta(d(w,S), \|w_1\|) - \eta(d(u,S), \|w_1\|)| \leq |d(w,S) - d(u,S)| \leq \|w - u\|.$$

The next computations are needed in order to estimate the second summand on the right hand side of (4.1). We find A, B, C such that $A\|z\| \leq \|\hat{z}\| \leq B\|z\|$ and $\|z^{\perp}\| \leq C\|z\|$. There are different computations corresponding to Cases 1 and 2 of Part 3.

In Case 1 of Part 3, $\varDelta = 8$ and $\varepsilon = 1/2$. So $d(x,F) \leq \|x\|/8$ and $\|x^{\perp}\| \leq 1\frac{1}{2}d(x,F)$. Hence $\|z^{\perp}\| = \|x^{\perp}\| \leq \frac{3}{2} \cdot \frac{1}{8}\|x\| = \frac{3}{16}\|x\|$. We have $z = x - x^{\perp} + z^{\perp}$. Hence $\|z\| \geq \|x\| - \|x^{\perp}\| - \|z^{\perp}\| = \|x\| - 2\|x^{\perp}\|$. Hence $\|z\| \geq \|x\| - \frac{3}{8}\|x\| = \frac{5}{8}\|x\|$. It follows that $\|z^{\perp}\| \leq \frac{3}{16} \cdot \frac{8}{5}\|z\|$. That is,

$$\text{(4.4.1)} \qquad \|z^{\perp}\| \leq \tfrac{3}{10}\|z\|.$$

From the fact that $\hat{z} = z - z^{\perp}$, we conclude

$$\text{(4.5.1)} \qquad \tfrac{7}{10}\|z\| \leq \|\hat{z}\| \leq \tfrac{13}{10}\|z\|.$$

Recall that in Case 2 of Part 3, $\varDelta = 24$ and $\varepsilon = 1/9$. We carry out a computation similar to the one in Case 1. So $d(x,F) \leq \|x\|/24$ and $\|x^{\perp}\| \leq 1\frac{1}{9}d(x,F)$. So $\|z^{\perp}\| = \|x^{\perp}\| \leq \frac{10}{9 \cdot 24}\|x\| = \frac{5}{108}\|x\|$. We have $\|z\| \geq \|x\| - 2\|x^{\perp}\| \geq \|x\| - \frac{5}{54}\|x\| = \frac{49}{54}\|x\|$ and hence $\|z^{\perp}\| \leq \frac{5}{108} \cdot \frac{54}{49}\|z\|$. That is,

$$\text{(4.4.2)} \qquad \|z^{\perp}\| \leq \tfrac{5}{98}\|z\|$$

and hence

$$\text{(4.5.2)} \qquad \tfrac{93}{98}\|z\| \leq \|\hat{z}\| \leq \tfrac{103}{98}\|z\|.$$

By (4.4.1) and (4.4.2),

$$\|z^{\perp}\| \le \tfrac{3}{10}\|z\|, \tag{4.4}$$

and by (4.5.1) and (4.5.2),

$$\tfrac{7}{10}\|z\| \le \|\hat{z}\| \le \tfrac{13}{10}\|z\|. \tag{4.5}$$

Since y also obeys *3.3, 3.4*, in Case 1 of Part 3 we obtain $\frac{13}{16}\|y\| \le \|\hat{y}\| \le \frac{19}{16}\|y\|$ and in Case 2, $\frac{103}{108}\|y\| \le \|\hat{y}\| \le \frac{113}{108}\|y\|$. The following is thus true in both cases:

$$\tfrac{13}{16}\|y\| \le \|\hat{y}\| \le \tfrac{19}{16}\|y\|. \tag{4.6}$$

By *4.3*, (4.6), *4.2*, (4.5), the monotonicity of α and the fact that $\alpha(At) \le A\alpha(t)$ for $A \ge 1$,

$$\begin{aligned}\lambda\|\hat{z}\| = \|\hat{y}\| &\le \tfrac{19}{16}\|y\| \le \tfrac{19}{16}M_{3,0}\cdot\alpha(\|z\|) \le \tfrac{19}{16}M_{3,0}\cdot\alpha(\tfrac{10}{7}\|\hat{z}\|)\\ &\le \tfrac{10}{7}\cdot\tfrac{19}{16}M_{3,0}\cdot\alpha(\|\hat{z}\|) \le 2M_{3,0}\cdot\alpha(\|\hat{z}\|).\end{aligned}$$

So

$$\lambda\|\hat{z}\| \le 2M_{3,0}\cdot\alpha(\|\hat{z}\|). \tag{4.7}$$

Let $\varrho = \eta_0$. So ϱ is the piecewise linear function with breakpoints at $\|\hat{z}\|/2$, $\|\hat{z}\|$ and $2\lambda\|\hat{z}\|$; $\varrho(t) = t$ for every $t \in [0,\|\hat{z}\|/2) \cup [2\lambda\|\hat{z}\|,\infty)$; and $\varrho(\|\hat{z}\|) = \lambda\|\hat{z}\|$. Clearly, $\varrho \in H([0,\infty))$. Using the notations of Proposition 9.10(b), $\eta = \eta_{(\varrho,(\lambda-1)\cdot\|\hat{z}\|)}$.

We show that ϱ is $16M_{3,0}\cdot\alpha$-continuous. The linear pieces of ϱ have the slopes: 1, $\frac{\lambda\|\hat{z}\|-\|\hat{z}\|/2}{\|\hat{z}\|/2}$, $\frac{2\lambda\|\hat{z}\|-\lambda\|\hat{z}\|}{2\lambda\|\hat{z}\|-\|\hat{z}\|}$ and 1. That is, the slopes of the linear pieces of ϱ are 1, $2\lambda-1$ and $\frac{\lambda}{2\lambda-1}$. We use the notations of Definition 9.9(b). Let $a_0,\dots,a_4$ denote 0, $\|\hat{z}\|/2$, $\|\hat{z}\|$, $2\lambda\|\hat{z}\|$ and ∞. Then $\varrho_1,\dots,\varrho_4$ are the functions

$$\begin{aligned}&\mathrm{Id}\restriction[0,\|\hat{z}\|/2],\\ &y = (2\lambda-1)t + \|\hat{z}\|/2, && t\in[0,\|\hat{z}\|/2],\\ &y = \frac{\lambda}{2\lambda-1}t + \lambda\|\hat{z}\|, && t\in[0,(2\lambda-1)\|\hat{z}\|],\\ &y = t + 2\lambda\|\hat{z}\|, && t\in[0,\infty).\end{aligned}$$

For $i = 1,3,4$, for every t_1,t_2, $|\varrho_i(t_1)-\varrho_i(t_2)| \le |t_1-t_2| \le 4M_{3,0}\cdot\alpha(|t_1-t_2|)$. Hence ϱ_i is $4M_{3,0}\cdot\alpha$-continuous. We deal with ϱ_2. By (4.7), $2\lambda-1 \le 2\lambda \le 4M_{3,0}\cdot\alpha(\|\hat{z}\|)/\|\hat{z}\|$. So $(2\lambda-1)/4M_{3,0} \le \alpha(\|\hat{z}\|)/\|\hat{z}\|$. Let $\varrho_2^*(t)$ be the function

$$y = \frac{2\lambda-1}{4M_{3,0}}t, \qquad t\in[0,\|\hat{z}\|].$$

Then by Proposition 9.10(c), $\varrho_2^*(t)$ is α-continuous. Clearly, $\varrho_2(t) = 4M_{3,0}\cdot\varrho_2^*(t) + \|\hat{z}\|/2$. So ϱ_2 is $4M_{3,0}\cdot\alpha$-continuous. We have shown that ϱ is $(4,4M_{3,0}\cdot\alpha)$-continuous. By Proposition 9.10(a), ϱ is $16M_{3,0}\cdot\alpha$-continuous. Define $\gamma = 16M_{3,0}\cdot\alpha$.

We next deal with the second summand on the right hand side of inequality (4.1). It has the form $|\eta(s,t_1)-\eta(s,t_2)|$. Recall that $\eta = \eta_{(\varrho,(\lambda-1)\cdot\|\hat{z}\|)}$. Then by Proposition 9.10(b), for every $s\in[0,\infty)$, η_s is γ-continuous. So

$$|\eta(d(u,S),\|w_1\|) - \eta(d(u,S),\|u_1\|)| \le \gamma(|\|w_1\|-\|u_1\||) = \gamma(\|(w-u)_1\|).$$

That is,

$$|\eta(d(u,S),\|w_1\|) - \eta(d(u,S),\|u_1\|)| \le \gamma(\|(w-u)_1\|). \tag{4.8}$$

We shall now bound the expressions $\|(w-u)_i\|$ appearing in (4.1) and (4.8) by a multiple of $\|w-u)\|$. For $\bar{u} \in E$ let $\bar{u}_{1,3} = \bar{u}_1 + \bar{u}_3$. Recall that $H_2 = \mathrm{span}(\{y^{\perp}\})$. By *4.4*, $\|y^{\perp}\| \le \frac{3}{2} d(y^{\perp}, \widehat{F})$. Hence $\|\bar{u}_2\| \le \frac{3}{2} d(\bar{u}_2, \widehat{F}) \le \frac{3}{2}\|\bar{u}\|$. From the fact that $\bar{u}_{1,3} = \bar{u} - \bar{u}_2$, it follows that $\|\bar{u}_{1,3}\| \le \|\bar{u}\| + \|\bar{u}_2\| \le \frac{5}{2}\|\bar{u}\|$. So we have

$$\|\bar{u}_{1,3}\| \le \tfrac{5}{2}\|\bar{u}\|. \tag{$\star$}$$

From the fact that $H_1 \perp^1 F_3$, it follows that $\|\bar{u}_1\| \le \|\bar{u}_{1,3}\|$, and this implies that $\|\bar{u}_3\| \le 2\|\bar{u}_{1,3}\|$. It follows that

$$\|\bar{u}_1\| \le \tfrac{5}{2}\|\bar{u}\|, \qquad \|\bar{u}_2\| \le \tfrac{3}{2}\|\bar{u}\|, \qquad \|\bar{u}_3\| \le 5\|\bar{u}\|. \tag{4.9}$$

Substituting (4.2) and (4.8) into (4.1), we obtain

$$\|f_1(w) - f_1(u)\| \le \|w-u\| + \gamma(\|(w-u)_1\|) + \|(w-u)_2\| + \|(w-u)_3\|. \tag{4.10}$$

We substitute (4.9) into (4.10) and use Proposition 9.10(d). So

$$\|f_1(w) - f_1(u)\| \le 7\tfrac{1}{2}\|w-u\| + 2\tfrac{1}{2}\gamma(\|w-u\|).$$

This means that $f_1 \restriction E^+$ is $(40M_{3,0} \cdot \alpha + 7\frac{1}{2}\mathrm{Id})$-continuous. Hence $f_1 \restriction E^+$ is $50M_{3,0} \cdot \alpha$-continuous. It follows that f_1 is $100M_{3,0} \cdot \alpha$-continuous.

The computation which shows that for some M, f_1^{-1} is $M \cdot \alpha$-continuous is analogous. However, for f_1^{-1} there is M which does not depend on E, F, α, x_0, y_0 such that f_1^{-1} is even M-Lipschitz. For this M it is also true that f_1^{-1} is $M \cdot \alpha$-continuous. We now carry out the computation for f_1^{-1}. For $s \in [0,\infty)$ let $\theta_s = \eta_s^{-1}$. Write $\theta(s,t) = \theta_s(t)$. Note that for every $u \in E$, $d(f_1(u), S) = d(u,S)$. This implies that

$$f_1^{-1}(u) = \theta_{d(u,S)}(\|u_1\|)\,\frac{u_1}{\|u_1\|} + u_2 + u_3.$$

The analogues (4.1*) of (4.1) and (4.2*) of (4.2) obtained by replacing η by θ are still true. Let $\mu = \theta_0$. So $\mu = \varrho^{-1}$ and $\theta = \eta_{(\mu,(\lambda-1)\cdot\|\hat{z}\|)}$. The slopes of the linear pieces of μ are the inverses of the slopes of the linear pieces of ϱ. Hence the slopes of the linear pieces of μ are: 1, $\frac{1}{2\lambda-1}$ and $\frac{2\lambda-1}{\lambda}$. Clearly, $1, \frac{1}{2\lambda-1}, \frac{2\lambda-1}{\lambda} \le 2$. So μ is 2-Lipschitz. By Proposition 9.10(b), for every $s \in [0,\infty)$, θ_s is 2-Lipschitz. Hence

$$|\theta(d(u,S),\|w_1\|) - \theta(d(u,S),\|u_1\|)| \le 2 \cdot |\|w_1\| - \|u_1\|| = 2 \cdot \|(w-u)_1\|.$$

So (4.8) is replaced by

$$|\theta(d(u,S),\|w_1\|) - \theta(d(u,S),\|u_1\|)| \le 2 \cdot \|(w-u)_1\|. \tag{4.11}$$

Substituting (4.9) into (4.11) we get

$$|\theta(d(u,S),\|w_1\|) - \theta(d(u,S),\|u_1\|)| \le 5\|w-u\|. \tag{4.12}$$

Replace the first summand of the right hand side of (4.1*) by (4.2*) and the second summand by (4.12). Use (4.9) to estimate the last two summands of (4.1*). So

$$\|f_1^{-1}(w) - f_1^{-1}(u)\| \le \|w-u\| + 5\|w-u\| + \tfrac{3}{2}\|w-u\| + 5\|w-u\|.$$

That is, $\|f_1^{-1}(w) - f_1^{-1}(u)\| \leq 12\frac{1}{2}\|w - u\|$. From the fact that $\alpha \geq \mathrm{Id}$, it follows that $f_1^{-1}\restriction E^+$ is $12\frac{1}{2} \cdot \alpha$-continuous. It follows that f_1^{-1} is $25 \cdot \alpha$-continuous. Hence f_1 is $100M_{3,0} \cdot \alpha$-bicontinuous. So $M_{3,1} = 100M_{3,0}$.

R4: We next find $a_{3,1}$ such that $f_1\restriction B(0, a_{3,1}\|z\|) = \mathrm{Id}$. For every $t \leq \|\hat{z}\|/2$ and for every s, $\eta(s,t) = t$. So for every $u \in E$, if $\|u_1\| \leq \|\hat{z}\|/2$, then $f_1(u) = u$. By (4.9) and the above, if $\|u\| \leq \frac{1}{5}\|\hat{z}\|$, then $f_1(u) = u$. By (4.5), $\frac{7}{10}\|z\| \leq \|\hat{z}\|$. So if $\|u\| \leq \frac{7}{50}\|z\|$, then $f_1(u) = u$. Let $a_{3,1} = \frac{7}{50}$, then $f_1\restriction B(0, a_{3,1}\|z\|) = \mathrm{Id}$.

We now find $b_{3,1}$ such that $\mathrm{supp}(f_1) \subseteq B(0, b_{3,1}\|v\|)$. We shall find A_i, $i = 1, 2, 3$, such that for every $u \in E$: if $\|u_i\| \geq A_i$, then $f_1(u) = u$. For every $t \geq 2\lambda\hat{z}$ and every s, $\eta(s,t) = t$. So

$$\text{If } \|u_1\| \geq 2\lambda\|\hat{z}\|, \text{ then } f_1(u) = u. \tag{4.13}$$

For every $s \geq (\lambda - 1)\|\hat{z}\|$, $\eta_s = \mathrm{Id}$. So for every $u \in E$, if $d(u, S) \geq (\lambda - 1)\|\hat{z}\|$, then $f_1(u) = u$. Let $u \in E$. By the second part of (4.9), $\|u\| \geq \frac{2}{3}\|u_2\|$. Let $a > 0$. If $\|u_2\| \geq a + \|z^{\perp}\|$, then for every $w \in S$, $\|(u - w)_2\| \geq a$. Hence $\|u - w\| \geq \frac{2}{3}\|(u - w)_2\| \geq \frac{2}{3}a$. Take $a = \frac{3}{2}(\lambda - 1)\|\hat{z}\|$. So if $\|u_2\| \geq \frac{3}{2}(\lambda - 1)\|\hat{z}\| + \|z^{\perp}\|$, then for every $w \in S$, $\|u - w\| \geq (\lambda - 1)\|\hat{z}\|$. That is, if $\|u_2\| \geq \frac{3}{2}(\lambda - 1)\|\hat{z}\| + \|z^{\perp}\|$, then $d(u, S) \geq (\lambda - 1)\|\hat{z}\|$. Hence

$$\text{If } \|u_2\| \geq \tfrac{3}{2}(\lambda - 1)\|\hat{z}\| + \|z^{\perp}\|, \text{ then } f_1(u) = u. \tag{4.14}$$

The third part of (4.9) says that $\|\bar{u}\| \geq \frac{1}{5}\|\bar{u}_3\|$ for every $\bar{u} \in E$. Let $u \in E$ be such that $\|u_3\| \geq 5(\lambda - 1)\|\hat{z}\|$. For every $w \in S$, $(u - w)_3 = u_3$. So $\|(u - w)\| \geq \frac{1}{5}\|(u - w)_3\| = \frac{1}{5}\|u_3\| \geq (\lambda - 1)\|\hat{z}\|$. That is, $d(u, S) \geq (\lambda - 1)\|\hat{z}\|$. Hence

$$\text{If } \|u_3\| \geq 5(\lambda - 1)\|\hat{z}\|, \text{ then } f_1(u) = u. \tag{4.15}$$

Combining (4.13)–(4.15) we conclude that

$$\text{If } \|u_1\| + \|u_2\| + \|u_3\| \geq (8\tfrac{1}{2}\lambda - 6\tfrac{1}{2})\|\hat{z}\| + \|z^{\perp}\|, \text{ then } f_1(u) = u. \tag{4.16}$$

By *4.4*, $\|z^{\perp}\| \leq 1\frac{1}{2}d(z^{\perp}, \widehat{F}) = 1\frac{1}{2}d(z, \widehat{F}) \leq 1\frac{1}{2}\|z\|$, and by (4.5), $\|\hat{z}\| \leq \frac{13}{10}\|z\|$. So

$$(8\tfrac{1}{2}\lambda - 6\tfrac{1}{2})\|\hat{z}\| + \|z^{\perp}\| \leq (\tfrac{13}{10} \cdot (8\tfrac{1}{2}\lambda - 6\tfrac{1}{2}) + 1\tfrac{1}{2})\|z\| \leq 10\lambda\|z\|. \tag{4.17}$$

Note that $z = \hat{z} + z^{\perp} = \frac{1}{\lambda}v - \frac{1}{\lambda}z^{\perp} + z^{\perp} = \frac{1}{\lambda}v + (1 - \frac{1}{\lambda})z^{\perp}$. Hence $\|z\| \leq \frac{1}{\lambda}\|v\| + \|z^{\perp}\|$. By (4.4), $\|z\| \leq \frac{1}{\lambda}\|v\| + \frac{3}{10}\|z\|$. So

$$\|z\| \leq \tfrac{10}{7\lambda}\|v\|. \tag{4.18}$$

From (4.16), (4.17) and (4.18) we conclude that

$$\text{If } \|u\| \geq \tfrac{100}{7} \cdot \|v\|, \text{ then } f_1(u) = u. \tag{4.19}$$

That is, $\mathrm{supp}(f_1) \subseteq B(0, \frac{100}{7} \cdot \|v\|)$. So $b_{3,1} := \frac{100}{7}$ is as required in R4.

Case 2: $\|\hat{y}\| < \|\hat{z}\|$. So $\lambda < 1$. Let $v = v = \lambda z$, and we construct f_1 such that $f_1(z) = v$. By (4.6), $\|\hat{y}\| \geq \frac{13}{16}\|y\|$, and by (4.5), $\|\hat{z}\| \leq \frac{13}{10}\|z\|$. So (i) $\lambda = \|\hat{y}\|/\|\hat{z}\| \geq \frac{5}{8}\|y\|/\|z\|$. By the construction of h_1 and h_2, (ii) $\|y\| = \|y_0\|$. By *4.1*, (iii) $\|z\| \approx^{M_{3,0}} \|x_0\|$. Since x_0, y_0 satisfy conditions A1–A4 appearing in the definition of a UC-constant, (iv) $\|y_0\| \geq \|x_0\|$.

So by (i)–(iv),

$$\lambda \geq \frac{5}{8}\frac{\|y_0\|}{M_{3,0}\|x_0\|} \geq \frac{1}{2M_{3,0}}. \tag{4.20}$$

Let $\eta : [0,\infty) \to [0,\infty)$ be a piecewise linear function with breakpoints at $\lambda\|z\|/2$, $\|z\|$ and $2\|z\|$ such that $\eta\restriction([0,\lambda\|z\|/2] \cup [2\|z\|,\infty)) = \mathrm{Id}$ and $\eta(\|z\|) = \lambda\|z\|$. Define f_1 to be the piecewise linearly radial homeomorphism based on η. (See Definition 9.5(b).) Recall that $z = x_4$, $v = v$. We shall define $M'_{1,3}$, $a'_{1,3}$ and $b'_{1,3}$ such that $R(x_4, v, f_1, M'_{1,3} \cdot \alpha, a'_{1,3}, b'_{1,3}, F)$ holds.

R1 and **R3:** Obviously, $f_1(x_4) = v$ and $f_1(F) = F$.

R2: The slopes of the linear pieces of η are 1, $\frac{\frac{1}{2}\lambda\|z\|}{\|z\|-\frac{1}{2}\lambda\|z\|}$, and $\frac{2\|z\|-\lambda\|z\|}{\|z\|}$. That is, they are 1, $\frac{\lambda}{2-\lambda}$ and $2-\lambda$. Now, $\frac{\lambda}{2-\lambda} \leq 1$ and by (4.20), $\frac{1}{4M_{3,0}} \leq \frac{\lambda}{2-\lambda}$. That is, $\frac{1}{4M_{3,0}} \leq \frac{\lambda}{2-\lambda} \leq 1$. Also, $1 \leq 2-\lambda \leq 2$. Hence the slopes of all linear pieces of η and η^{-1} are $\leq 4M_{3,0}$. So η is $4M_{3,0}$-bilipschitz. By Proposition 3.18, f_1 is $12M_{3,0}$-bilipschitz. Since $\alpha \geq \mathrm{Id}$, f_1 is $12M_{3,0}\cdot\alpha$-bicontinuous. We may thus define $M'_{3,1} = 12M_{3,0}$.

R4: Obviously, $\mathrm{supp}(f_1) \subseteq B(0; \frac{\lambda\|z\|}{2}, 2\|z\|)$. By (4.20), $B(0, \frac{1}{4M_{3,0}}\|z\|) \subseteq B(0, \frac{\lambda\|z\|}{2})$. So we may define $a'_{3,1} = \frac{1}{4M_{3,0}}$. Recall that $v = \lambda z$. So by (4.20), $\|v\| = \lambda\|z\| \geq \frac{1}{2M_{3,0}}\|z\|$. Hence $2\|z\| \leq 4M_{3,0}\|v\|$. It follows that $B(0, 2\|z\|) \subseteq B(0, 4M_{3,0}\|v\|)$. So we may take $b'_{3,1} = 4M_{3,0}$.

We have shown that $R(x_4, v, f_1, M'_{1,3}\cdot\alpha, a'_{1,3}, b'_{1,3}, F)$ holds. Taking in account Case 1 and Case 2, we define $M''_{3,1} = \max(M_{3,1}, M'_{3,1})$, $a''_{3,1} = \min(a_{3,1}, a'_{3,1})$ and $b''_{3,1} = \max(b_{3,1}, b'_{3,1})$. Then $M''_{3,1}, a''_{3,1}, b''_{3,1}$ are as required in C3.

Part 5: The construction of f_2. Let v be as in Part 4. Remember that v was defined in two different ways. In the case that $\|\hat{z}\| \leq \|\hat{y}\|$, $v = \hat{y} + z^{\perp}$, and in the case that $\|\hat{z}\| > \|\hat{y}\|$, $v = \lambda z$. Define $v^{\perp} = v - \hat{y}$. The following holds:

5.1 $y = \hat{y} + y^{\perp}$, $v = \hat{y} + v^{\perp}$, $y^{\perp} = \nu v^{\perp}$, $\hat{y} \in F$ and $\nu > 0$.

If $\nu = 1$ let $f_2 = \mathrm{Id}$. Assume that $\nu \neq 1$. The vector $y^{\perp}$ is as in Part 4, and in both Cases 1 and 2 of Part 4, $v^{\perp}$ is a multiple of $y^{\perp}$. So the analogue of clause *4.4* in Part 4 holds for $y^{\perp}$ and $v^{\perp}$. That is,

5.2 $F \subseteq \widehat{F}$, $\widehat{F} \oplus \mathrm{span}(\{y^{\perp}\}) = E$ and $\|y^{\perp}\| \leq 1\frac{1}{2}d(y^{\perp}, \widehat{F})$ and equivalently $\|v^{\perp}\| \leq 1\frac{1}{2}d(v^{\perp}, \widehat{F})$.

Recall that $g_1 = f_1 \circ e$. We shall next show that there is N_1 which does not depend on E, F, α, x_0, y_0 such that

($\clubsuit$) for every $u \in E$, $d(g_1(u), F) \approx^{N_1} d(u, F)$. In particular, $d(v, F) \approx^{N_1} d(x_0, F)$,

Recall that $M_{3,1} = \prod_{i=1}^{4} M_{1,i}$. Then by C1, $d(e(u), F) \approx^{M_{3,1}} d(u, F)$ for every $u \in E$. In Case 1 of Part 4, $f_1(u) - u \in F$ for every $u \in E$, so $d(f_1(u), F) = d(u, F)$. So in Case 1 of Part 4, $d(g_1(u), F) \approx^{N_1} d(u, F)$ for every $u \in E$.

In Case 2 of Part 4, f_1 is the piecewise linearly radial homeomorphism based on η, and for any slope a of a piece of η, $1/4M_{3,0} \leq a \leq 2 \leq 4M_{3,0}$. So for every $u \in E$, $d(u, F) \approx^{4M_{3,0}} d(f_1(u), F)$. Now, define $N_1 = 4M_{3,1}M_{3,0}$. Then in both Case 1 and

Case 2 of Part 4, $d(\mathsf{g}_1(u), F) \approx^{\mathsf{N}_1} d(u, F)$ for every $u \in E$. The fact $d(v, F) \approx^{\mathsf{N}_1} d(\mathsf{x}_0, F)$ is a special case of the above, since $v = \mathsf{g}_1(\mathsf{x}_0)$.

It is given that $d(\mathsf{x}_0, F) \approx^{\alpha} d(\mathsf{y}_0, F)$. Let $\mathsf{N}_2 = \mathsf{M}_{2,1}\mathsf{M}_{2,2}$. Then from C2 it follows that $d(\mathsf{y}_0, F) \approx^{\mathsf{N}_2} d(\mathsf{y}, F)$. So

(♣♣) $d(\mathsf{x}_0, F) \approx^{\mathsf{N}_2 \cdot \alpha} d(y, F)$.

Let $\mathsf{N} = \mathsf{N}_1\mathsf{N}_2$ and $\beta = \mathsf{N} \cdot \alpha$. It follows from (♣) and (♣♣) that $d(v, F) \approx^{\beta} d(y, F)$. By *5.1*, $d(v, F) = d(v^{\perp}, F)$ and $d(y, F) = d(y^{\perp}, F)$. Hence

$$d(y^{\perp}, F) \approx^{\beta} d(v^{\perp}, F).$$

Clause *3.3* in Part 3 says that $\|y^{\perp}\| \le (1 + \varepsilon)d(y, F)$. In Cases 1 and 2 of Part 3, ε was taken to be 1/2 and 1/9 respectively. So $\|y^{\perp}\| \le \frac{3}{2}d(y^{\perp}, F)$. Hence

$$\|y^{\perp}\| \le \tfrac{3}{2} \cdot \beta(d(v^{\perp}, F)) \le \tfrac{3}{2} \cdot \beta(\|v^{\perp}\|).$$

Since $v^{\perp}$ is a multiple of $y^{\perp}$, it follows that $\|v^{\perp}\| \le \frac{3}{2}d(v^{\perp}, F)$. So

$$\|v^{\perp}\| \le \tfrac{3}{2} \cdot \beta(d(v^{\perp}, F)) \le \tfrac{3}{2} \cdot \beta(d(y^{\perp}, F)) \le \tfrac{3}{2} \cdot \beta(\|y^{\perp}\|).$$

Let $\gamma = 3\beta/2$. Hence

(5.1) $$\|y^{\perp}\| \approx^{\gamma} \|v^{\perp}\|.$$

From the fact that $y^{\perp} = \nu v^{\perp}$ and (5.1), it follows that

(5.2) If $\nu > 1$, then $\nu \cdot \|v^{\perp}\| \le \gamma(\|v^{\perp}\|)$; and if $\nu < 1$, then $\frac{1}{\nu} \cdot \|y^{\perp}\| \le \gamma(\|y^{\perp}\|)$.

Let $L = \{\hat{y} + ty^{\perp} \mid t \in \mathbb{R}\}$. So L is the straight line connecting y and v. Recall that $H_2 = \mathrm{span}(\{y^{\perp}\})$. By *5.2*, $H_2 \perp^{1\frac{1}{2}} \widehat{F}$. So by Proposition 9.2(f), $\|\ \|^{\widehat{F}, H_2} \approx^{2\frac{1}{2}} \|\ \|$. By *5.1* and *5.2*, $\hat{y} \in \widehat{F}$. So for every $t \in \mathbb{R}$, $\|\hat{y} + ty^{\perp}\| \ge \frac{2}{5} \cdot (\|\hat{y}\| + |t|\|y^{\perp}\|) \ge \frac{2}{5} \cdot \|\hat{y}\|$. That is,

(5.3) $$d(L, 0) \ge \tfrac{2}{5} \cdot \|\hat{y}\|.$$

We show that

(5.4) $$\|v^{\perp}\| \le \tfrac{3}{7}\|\hat{y}\|.$$

Let $\hat{y}, \hat{z}$ be as in Part 4. Suppose first that $\|\hat{y}\| \ge \|\hat{z}\|$. In this case $v^{\perp} = z^{\perp}$. By (4.4), $\|z^{\perp}\| \le \frac{3}{10}\|z\|$. Since $z = \hat{z} + z^{\perp}$, $\|z^{\perp}\| \le \frac{3}{7}\|\hat{z}\|$, and since $\|\hat{y}\| \ge \|\hat{z}\|$, $\|z^{\perp}\| \le \frac{3}{7}\|\hat{y}\|$. That is, if $\|\hat{y}\| \ge \|\hat{z}\|$, then $\|v^{\perp}\| \le \frac{3}{7}\|\hat{y}\|$. Next suppose that $\|\hat{y}\| < \|\hat{z}\|$. In this case $\hat{y} + v^{\perp} = v = \lambda z = \lambda(\hat{z} + z^{\perp}) = \hat{y} + \lambda z^{\perp}$. That is, $v^{\perp} = \lambda z^{\perp}$ and $\hat{y} = \lambda\hat{z}$. Hence $\|v^{\perp}\|/\|\hat{y}\| = \|z^{\perp}\|/\|\hat{z}\|$. By (4.4), $\|v^{\perp}\|/\|\hat{y}\| = \|z^{\perp}\|/\|\hat{z}\| \le \frac{3}{7}$. So, if $\|\hat{y}\| < \|\hat{z}\|$, then $\|v^{\perp}\| \le \frac{3}{7}\|\hat{y}\|$. We conclude that (5.4) holds in both cases.

Since $v = \hat{y} + v^{\perp}$, it follows that $\|v\| \le \|\hat{y}\| + \|v^{\perp}\|$. So by (5.4), $\|v\| \le \frac{10}{7}\|\hat{y}\|$. Similarly, $\|\hat{y}\| \le \|v\| + \|v^{\perp}\| \le \|v\| + \frac{3}{7}\|\hat{y}\|$. So $\frac{4}{7}\|\hat{y}\| \le \|v\|$. Hence

(5.5) $$\tfrac{7}{10}\|v\| \le \|\hat{y}\| \le \tfrac{7}{4}\|v\|.$$

Fact (5.3) and the first inequality in (5.5) imply that

(5.6) $$d(L, 0) \ge \tfrac{14}{50}\|v\|.$$

In Case 1 of Part 3 we chose $\varepsilon = \frac{1}{2}$ and $\varDelta = 8$. So by *3.3* and *3.4*, $\|y^{\perp}\| \le \frac{3}{2}d(y, F) \le \frac{3}{2} \cdot \frac{1}{8}\|y\|$. That is, $\|y^{\perp}\| \le \frac{3}{16}\|y\|$. Since $y = \hat{y} + y^{\perp}$, $\|\hat{y}\| \ge \frac{13}{16}\|y\|$. Hence in Case 1,

$\|y^{\perp}\| \leq \frac{3}{13}\|\hat{y}\|$. In Case 2 of Part 3 we follow the same computation with $\varepsilon = \frac{1}{9}$ and $\Delta = \frac{1}{24}$. We obtain $\|y^{\perp}\| \leq \frac{10}{108}\|y\|$ and hence $\|y^{\perp}\| \leq \frac{10}{98}\|\hat{y}\|$. So in both cases

$$\|y^{\perp}\| \leq \tfrac{3}{13}\|\hat{y}\|. \tag{5.7}$$

We shall next define g_4. The required f_2 will be either g_4 or g_4^{-1}. Recall that $\widehat{F}$ and H_2 were defined in Part 4, and that ν was defined in *5.1*. For $u \in E$ set $u_1 := (u)_{\widehat{F}}$ and $u_2 := (u)_{H_2}$.

If $\nu > 1$ let

$$\bar{\nu} = \nu, \quad \bar{v}^{\perp} = v^{\perp}, \quad \bar{y}^{\perp} = y^{\perp}, \quad \bar{v} = v, \quad \bar{y} = y,$$

and if $\nu < 1$ let

$$\bar{\nu} = \frac{1}{\nu}, \quad \bar{v}^{\perp} = y^{\perp}, \quad \bar{y}^{\perp} = v^{\perp}, \quad \bar{v} = y, \quad \bar{y} = v.$$

So $\bar{\nu} > 0$, $\bar{y}^{\perp} = \bar{\nu} \cdot \bar{v}^{\perp}$ and by (5.2),

$$\bar{\nu} \leq \frac{\gamma(\|\bar{v}^{\perp}\|)}{\|\bar{v}^{\perp}\|}. \tag{5.8}$$

Let $\varrho \in H([0,\infty))$ be the piecewise linear function with breakpoints at $\|\bar{v}^{\perp}\|/2$, $\|\bar{v}^{\perp}\|$ and $2\bar{\nu}\|\bar{v}^{\perp}\|$ such that $\varrho{\restriction}([0,\|\bar{v}^{\perp}\|/2] \cup [2\bar{\nu}\|\bar{v}^{\perp}\|,\infty)) = \mathrm{Id}$ and $\varrho(\|\bar{v}^{\perp}\|) = \bar{\nu}\|\bar{v}^{\perp}\|$. Define $\eta(s,t)$ to be the function

$$\eta(s,t) = \begin{cases} \left(1 - \dfrac{s}{\|\hat{y}\|/5}\right)\varrho(t) + \dfrac{s}{\|\hat{y}\|/5}\, t, & s \in [0, \|\hat{y}\|/5], \\ t, & s \geq \|\hat{y}\|/5. \end{cases}$$

So $\eta = \eta_{(\varrho,\|\hat{y}\|/5)}$ as defined in Proposition 9.10(b). Let $E^{\hat{}} = \{u \in E \mid u_2 \geq 0\}$. Define

$$g_4(u) = \begin{cases} u_1 + \eta(d(u,L), \|u_2\|) \cdot \dfrac{\bar{v}^{\perp}}{\|\bar{v}^{\perp}\|}, & u \in E^{\hat{}}, \\ u, & u \in E - E^{\hat{}}. \end{cases}$$

If $u_2 = 0$ then $g_4(u) = u$, so $g{\restriction}\widehat{F} = \mathrm{Id}$. and hence $g_4 \in H(E)$. Note that if $\nu > 1$, then $g_4(v) = y$, and if $\nu < 1$, then $g_4^{-1}(v) = y$. Next we find $M_{3,2}, a_{3,2}, b_{3,2}$ independent of E, F, α, x_0, y_0 such that $R(v, y, g_4; M_{3,2} \cdot \alpha, a_{3,2}, b_{3,2}, F)$ holds or $R(y, v, g_4; M_{3,2} \cdot \alpha, a_{3,2}, b_{3,2}, F)$ holds.

R3: Clearly, $g_4{\restriction}F = \mathrm{Id}$ and hence $g_4(F) = g_4^{-1}(F) = F$.

R2: We shall next find $M_{3,2}$ such that g_4 is $M_{3,2} \cdot \alpha$-bicontinuous. The slopes of the linear pieces of ϱ are: 1, $\frac{\bar{\nu}\|\bar{v}^{\perp}\| - \|\bar{v}^{\perp}\|/2}{\|\bar{v}^{\perp}\| - \|\bar{v}^{\perp}\|/2}$, $\frac{2\bar{\nu}\|\bar{v}^{\perp}\| - \bar{\nu}\|\bar{v}^{\perp}\|}{2\bar{\nu}\|\bar{v}^{\perp}\| - \|\bar{v}^{\perp}\|}$ and 1. That is, the four slopes of ϱ are 1, $2\bar{\nu} - 1$, $\frac{\bar{\nu}}{2\bar{\nu}-1}$ and 1. We apply Proposition 9.10(a) to ϱ taking a_0 to be 0, a_1, a_2, a_3 to be the breakpoints of ϱ, and a_4 to be ∞. Using the notation of Definition 9.9(b), the functions ϱ_1, ϱ_3 and ϱ_4 are linear function with slopes 1, $\frac{\bar{\nu}}{2\bar{\nu}-1}$ and 1 respectively. So they are 1-Lipschitz. Clearly, $\varrho_2(t) = (2\bar{\nu} - 1)t + c$, $t \in [0, \|\bar{v}^{\perp}\|/2)$. By (5.8) and Proposition 9.10(c), ϱ_2 is $2 \cdot \gamma$-continuous, and so ϱ is $(4, 2 \cdot \gamma)$-continuous. By Proposition 9.10(a),

$$\varrho \text{ is } 8 \cdot \gamma\text{-continuous}. \tag{5.9}$$

Let $u, w \in E^{\wedge}$. Then

$$\|g_4(w) - g_4(u)\| \le \|(w-u)_1\| + |\eta(d(w,L), \|w_2\|) - \eta(d(u,L), \|w_2\|)| + |\eta(d(u,L), \|w_2\|) - \eta(d(u,L), \|u_2\|)|. \tag{5.10}$$

Denote the three summands on the right hand of inequality (5.10) by D_1, D_2 and D_3. If $d(w,L), d(u,L) \in [0, \|\hat{y}\|/5)$, then

$$\begin{aligned} D_2 &\le \frac{|d(w,L) - d(u,L)|}{\|\hat{y}\|/5} \cdot (\varrho(\|w_2\|) - \|w_2\|) \le \frac{\|w-u\|}{\|\hat{y}\|/5} \cdot (\varrho(\|w_2\|) - \|w_2\|) \\ &\le \frac{\|w-u\|}{\|\hat{y}\|/5} \cdot (\bar{\nu} - 1)\|\bar{v}^{\perp}\| \le \frac{\|w-u\|}{\|\hat{y}\|/5} \cdot \bar{\nu} \cdot \|\bar{v}^{\perp}\| := D_2'. \end{aligned}$$

The above is true for every $u, w \in E^{\wedge}$. Since $\bar{\nu} \cdot \bar{v}^{\perp} = v^{\perp}$ or $\bar{\nu} \cdot \bar{v}^{\perp} = y^{\perp}$, by (5.4) and (5.7), $\bar{\nu} \cdot \|\bar{v}^{\perp}\|/\|\hat{y}\| \le \frac{3}{7}$. Hence, $D_2' \le \frac{15}{7} \cdot \|w-u\|$. That is,

$$|\eta(d(w,L), \|w_2\|) - \eta(d(u,L), \|w_2\|)| \le \tfrac{15}{7} \cdot \|w-u\|. \tag{5.11}$$

By (5.9) and Proposition 9.10(b), $D_3 \le 8 \cdot \gamma(|\|w_2\| - \|u_2\||) \le 8 \cdot \gamma(\|(w-u)_2\|) := D_3'$, and by the second inequality in (4.9) and Proposition 9.10(d), $D_3' \le \frac{3}{2} \cdot 8 \cdot \gamma(\|w-u\|$. Hence

$$|\eta(d(u,L), \|w_2\|) - \eta(d(u,L), \|u_2\|)| \le 12 \cdot \gamma(\|w-u\|). \tag{5.12}$$

Note that for every $\bar{u} \in E$, $\bar{u}_1$ of Part 5 is $\bar{u}_1 + \bar{u}_3$ of Part 4. So by the first and third inequalities in (4.9),

$$\|(w-u)_1\| \le 7\tfrac{1}{2}\|w-u\|. \tag{5.13}$$

Substitute into (5.10) inequalities (5.13), (5.11) and (5.12). We obtain the inequality $\|g_4(w) - g_4(u)\| \le 9\frac{9}{14}\|w-u\| + 12 \cdot \gamma(\|w-u\|)$. Recall that $\gamma = \frac{3}{2}\beta$ and that $\beta = N\alpha$. Hence, since $\alpha \ge \mathrm{Id}$,

$$g_4 \restriction E^{\wedge} \text{ is } (18N + 10) \cdot \alpha\text{-continuous.}$$

The computation which shows that for some M independent of E, F, α, x_0, y_0, $g_4^{-1} \restriction E^{\wedge}$ is $M \cdot \alpha$-continuous is analogous. But for g_4^{-1} there is M which does not depend on E, F, α, x_0, y_0 such that $g_4^{-1} \restriction E^{\wedge}$ is M-Lipschitz. So we conclude that $g_4^{-1} \restriction E^{\wedge}$ is $M \cdot \alpha$-continuous. This computation is analogous to the proof that f_1^{-1} is Lipschitz.

For $s \in [0, \infty)$ let $\theta_s = \eta_s^{-1}$. Write $\theta(s,t) = \theta_s(t)$. As in Part 4, for every $u \in E$,

$$g_4^{-1}(u) = u_1 + \theta_{d(u,S)}(\|u_2\|)\,\frac{\bar{v}^{\perp}}{\|\bar{v}^{\perp}\|}.$$

The analogues (5.10*) of (5.10) and (5.11*) of (5.11) obtained by replacing η by θ are true. Let $\mu = \theta_0$. So $\mu = \varrho^{-1}$ and $\theta = \eta_{(\mu, \|\hat{y}\|/5)}$. The slopes of the linear pieces of μ are the inverses of the slopes of the linear pieces of ϱ. Hence the slopes are 1, $\frac{1}{2\bar{\nu}-1}$ and $\frac{2\bar{\nu}-1}{\bar{\nu}}$. The first two slopes are ≤ 1 and the third is ≤ 2. So μ is 2-Lipschitz. By Proposition 9.10(b), for every $s \in [0, \infty)$, θ_s is 2-Lipschitz. Hence

$$|\theta(d(u,L), \|w_2\|) - \theta(d(u,L), \|u_2\|)| \le 2 \cdot |\|w_2\| - \|u_2\|| = 2 \cdot \|(w-u)_2\|.$$

Applying the second inequality in (4.9) we conclude that

$$|\theta(d(u,L), \|w_2\|) - \theta(d(u,L), \|u_2\|)| \le 3\|w-u\|. \tag{5.14}$$

Substituting (5.13), (5.11*) and (5.14) into (5.10*) we conclude that

$$\|g_4^{-1}(w) - g_4^{-1}(u)\| \le (7\tfrac{1}{2} + \tfrac{15}{7} + 3)\|w-u\| \le 13\|w-u\|.$$

Since $13 \cdot \mathrm{Id} \le (18N+10)\cdot\alpha$, $g_4^{-1}\restriction E^\wedge$ is $(18N+10)\cdot\alpha$-continuous. Hence $g_4\restriction E^\wedge$ is $(18N+10)\cdot\alpha$-bicontinuous and so g_4 is $2(18N+10)\cdot\alpha$-bicontinuous. So $M_{3,2} := 60N$ is as required. That is, g and g^{-1} are $M_{3,2}\cdot\alpha$-bicontinuous.

R4: We shall find a' and b' independent of E, F, α, x_0 and y_0 such that $\mathrm{supp}(g_4) \subseteq B(0; a'\|\hat{y}\|, b'\|\hat{y}\|)$. Let $u \in B(0, \|\hat{y}\|/5)$. By (5.3), $d(u,L) > \|\hat{y}\|/5$. So for every $t \in [0,\infty)$, $\eta(d(u,L),t) = t$. In particular, $\eta(d(u,L),\|u_2\|) = \|u_2\|$. Hence

$$g_4(u) = u_1 + \eta(d(u,L),\|u_2\|)\cdot\frac{\bar{v}^\perp}{\|\bar{v}^\perp\|} = u_1 + u_2 = u.$$

That is, $g_4\restriction B(0,\|\hat{y}\|/5) = \mathrm{Id}$ and hence $a' = 1/5$.

Let $u \in E$. If $d(u,L) \ge \|\hat{y}\|/5$ or $\|u_2\| \ge 2\bar{\nu}\|\bar{v}^\perp\|$, then $\eta(d(u,L),\|u_2\|) = \|u_2\|$ and hence $g_4(u) = u$. Recall that $\bar{\nu}\cdot\bar{v}^\perp = v^\perp$ or $\bar{\nu}\cdot\bar{v}^\perp = y^\perp$. So if $\|u_2\| \ge 2\|\bar{v}^\perp\|$ and $\|u_2\| \ge 2\|\bar{y}^\perp\|$, then $g_4(u) = u$. So by (5.4) and (5.7),

(5.15) $\qquad$ If $\|u_2\| \ge \frac{6}{7}\|\hat{y}\|$, then $g_4(u) = u$.

Fact ($\star$) in Part 4 (which precedes (4.9)) says that $\bar{u}_{1,3} \le 2\frac{1}{2}\bar{u}$ for every $\bar{u} \in E$. But $\bar{u}_{1,3}$ of Part 4 is $\bar{u}_1$ of Part 5. So $\|\bar{u}_1\| \le \frac{5}{2}\|\bar{u}\|$ for every $\bar{u} \in E$. We show that

(5.16) $\qquad$ If $\|u_1\| \ge 1\frac{1}{2}\|\hat{y}\|$, then $g_4(u) = u$.

Suppose that $\|u_1\| \ge 1\frac{1}{2}\|\hat{y}\|$ and let $w \in L$. Then $(u-w)_1 = u_1 - \hat{y}$ and hence $\|(u-w)_1\| \ge \|u_1\| - \|\hat{y}\| \ge \frac{1}{2}\|\hat{y}\|$. So $\|u-w\| \ge \frac{2}{5}\|(u-w)_1\| \ge \frac{2}{5}\cdot\frac{1}{2}\|\hat{y}\| = \|\hat{y}\|/5$. Hence $d(u,L) \ge \|\hat{y}\|/5$. This implies that $g_4(u) = u$. Suppose that $\|u\| \ge 3\|\hat{y}\|$ and we show that $g_4(u) = u$. Clearly, $\|u_1\| + \|u_2\| \ge \|u\| \ge 3\|\hat{y}\|$. So either $\|u_1\| \ge 1\frac{1}{2}\|\hat{y}\|$ or $\|u_2\| \ge \frac{6}{7}\|\hat{y}\|$. By (5.16) and (5.15), $g_4(u) = u$. It follows that $g_4\restriction(E - B(0,3\|\hat{y}\|)) = \mathrm{Id}$. So $b' := 3$ is as desired.

Recall that (4.6) said that $\|\hat{y}\| \le \frac{19}{16}\|y\|$, and that (5.5) said that $\frac{7}{10}\|v\| \le \|\hat{y}\|$. It follows that $\mathrm{supp}(g_4) \subseteq B(0; \frac{1}{5}\cdot\frac{7}{10}\|v\|, 3\cdot\frac{19}{16}\|y\|)$. That is, $\mathrm{supp}(g_4) \subseteq B(0; \frac{7}{50}\|v\|, \frac{57}{16}\|y\|)$, and the same is true for g_4^{-1}. Let $a_{3,2} = 7/50$ and $b_{3,2} = 57/16$. Then $\mathrm{supp}(g_2), \mathrm{supp}(g_2^{-1}) \subseteq B(0; a_{3,2}\|v\|, b_{3,2}\|y\|)$. So R3 is proved.

The definition of f_2: If $\nu > 1$ define $f_2 = g_4$, and if $\nu < 1$ define $f_2 = g_4^{-1}$.

R1: Clearly, $f_2(v) = y$, and since $V = v$ and $Y_2 = y$, we have $f_2(V) = Y_2$.

We have found $M_{i,j}$'s, $a_{i,j}$'s and $b_{i,j}$'s which fulfill C1–C4. It follows from the first part of the proof of the lemma that there exist M, a, b such that M is a UC-constant for $\langle a, b\rangle$.

(b) Let M, a, b be as ensured by (a), and let $a' < 1$ and $b' > 1$. We may assume that $a' > a$ and that $b' < b$. Let $x, y \in E - F$ be as in the definition of a UC-constant. Let g_1, g_2 be as ensured in (a) for the numbers a and b. (See Definition 9.11(a).) Let $\eta \in H([0,\infty))$ be a piecewise linear function with breakpoints at: $a\cdot\|x\|, \|x\|, \|y\|, b\cdot\|y\|, 2b\cdot\|y\|$ and such that: $\eta(0) = 0$; $\eta(a\cdot\|x\|) = a'\cdot\|x\|$; $\eta(\|x\|) = \|x\|$; $\eta(\|y\|) = \|y\|$; $\eta(b\cdot\|y\|) = b'\cdot\|y\|$; $\eta\restriction[2b\cdot\|y\|,\infty) = \mathrm{Id}$. The slopes of the linear pieces of η are: $\frac{a'}{a}$, $\frac{1-a'}{1-a}$, 1, $\frac{b'-1}{b-1}$, $\frac{2b-b'}{b}$ and 1. These slopes depend only on a, a', b, b' and not on x and y. Let L be the maximum of all the above slopes and their inverses. So η is L-bilipschitz.

Let k be the piecewise linearly radial homeomorphism based on η. That is, for every $u \in E - \{0\}$, $k(u) = \eta(\|u\|)\frac{u}{\|u\|}$ and $k(0) = 0$. By Proposition 3.18, k is $3\mathsf{L}$-bilipschitz. For $i = 1, 2$, let $\mathsf{g}'_i = k \circ \mathsf{g}_i \circ k^{-1}$. Then g'_i is $(3\mathsf{L} \cdot \mathrm{Id}) \circ \alpha \circ (3\mathsf{L} \cdot \mathrm{Id})$-bicontinuous. So by Proposition 9.10(d), g'_i is $9\mathsf{L}^2\mathsf{M} \cdot \alpha$-bicontinuous. Define $\mathsf{M}' = 9\mathsf{L}^2\mathsf{M}$. It is easy to verify that clauses B1–B4 in the definition of a UC-constant (Definition 9.11(a)) are fulfilled by a', b', g'_1, g'_2 and M'. Hence M' is a UC-constant for $\langle a', b' \rangle$. ■

10. 1-dimensional boundaries

Chapter 9 dealt with the following situation. E is a normed space, F is a closed subspace of E with dimension ≥ 2, $x, y \in E - F$, and $\|x\| \approx^{\alpha} \|y\|$ and $d(x, F) \approx^{\alpha} d(y, F)$. It was shown that there is an $M \cdot \alpha \circ \alpha$-bicontinuous $g \in H(E)$ such that $g(x) = y$, $g(F) = F$ and $\mathrm{supp}(g)$ is contained in the ring $B(0; a\|x\|, b\|y\|)$. When F is 1-dimensional, such a g does not always exist. The reason for this is that in order to move x to y we need to rotate x about an axis perpendicular to F. See the construction of g_1 in Part 2 of the proof of Lemma 9.13(a). When F is 1-dimensional, such a rotation does not exist.

Whereas Part 2 of the proof of Lemma 9.13(a) fails for a 1-dimensional subspace, Parts 1 and 3–5 remain without change. In these parts, the fact that $\dim(F) \geq 2$ is not used. By skipping Part 2 in the proof of Lemma 9.13(a) one obtains the following lemma.

Let F, K be linear subspaces of a normed space E and $u \in E$. Then $u \perp F$ denotes the fact that $\|u\| = d(u, F)$, and $K \perp F$ means that $u \perp F$ for every $u \in K$.

LEMMA 10.1. *Let M be a UC-constant for $\langle a, b \rangle$. Let E be a normed space and F be a 1-dimensional linear subspace of E. Let $\alpha \in$ MBC and $x, y \in E - F$ be such that*:

(i) $\|x\| \leq \|y\|$ *and* $\|x\| \approx^{\alpha} \|y\|$,
(ii) $d(x, F) \approx^{\alpha} d(y, F)$,
(iii) $x = \hat{x} + x^{\perp}$, $y = \hat{y} + y^{\perp}$, $\hat{x}, \hat{y} \in F$, $x^{\perp}, y^{\perp} \perp F$, *and for some* $\lambda > 0$, $\hat{x} = \lambda \hat{y}$,
(iv) *if* $\dim(E) = 2$, *then* x, y *are on the same side of* F.

Then there are $g_1, g_2 \in H(E)$ such that:

(1) g_1, g_2 *are $M\alpha$-bicontinuous*,
(2) $g_1 \circ g_2(x) = y$,
(3) $g_1(F) = F$ *and* $g_2(F) = F$,
(4) *for every* $i = 1, 2$, $\mathrm{supp}(g_i) \subseteq B(0; a\|x\|, b\|y\|)$.

Proof. Parts 1, 3–5 of the proof of Lemma 9.13(a) constitute the proof of this lemma. ∎

DEFINITION 10.2. Let $0 < a < 1$, $b > 1$ and $M \geq 1$. We say that M is a *1-dimensional Uniform Continuity constant for a and b* (abbreviated by "M is a *1UC-constant* for $\langle a, b \rangle$") if the following holds. Suppose that E, F, α, x, y satisfy the following assumptions.

A1 E is a normed space and F is a proper linear subspace of E such that $\dim(F) = 1$, $\alpha \in$ MBC and $x, y \in E - F$,
A2 $\|x\| \leq \|y\| \leq \alpha(\|x\|)$,
A3 $d(x, F) \approx^{\alpha} d(y, F)$,
A4 $\|x\| \leq \alpha(d(x, F))$ and $\|y\| \leq \alpha(d(y, F))$,
A5 if $\dim(E) = 2$, then x, y are on the same side of F.

Then there are $g_1, g_2, g_3 \in H(E)$ such that:

B1 for every $i = 1, 2, 3$, g_i is $M \cdot \alpha$-bicontinuous,
B2 $g_3 \circ g_2 \circ g_1(x) = y$,
B3 for every $i = 1, 2, 3$, $g_i(F) = F$,
B4 for every $i = 1, 2, 3$, $\mathrm{supp}(g_i) \subseteq B(0; a\|x\|, b\|y\|)$. □

REMARK. Note that in the definition of a 1UC-constant there is an extra assumption on x and y which did not appear in the definition of a UC-constant, namely, Assumption A4 which says that $\|x\| \le \alpha(d(x, F))$ and $\|y\| \le \alpha(d(y, F))$. □

The rest of the chapter is devoted to the proof of the following lemma.

LEMMA 10.3. (a) *There are a, b, M such that M is a 1UC-constant for a and b.*

(b) *For every $0 < a < 1$ and $b > 1$ there is M such that M is a 1UC-constant for a and b.*

Items 10.4–10.9 are needed in the proof of the above lemma.

PROPOSITION 10.4. *Let F be a finite-dimensional linear subspace of a normed space E and $u \notin F$. Then there is a 1-dimensional subspace $L \subseteq \mathrm{span}(F \cup \{u\})$ such that $L \perp F$.*

PROPOSITION 10.5. *Let X be a metric space, $\alpha \in \mathrm{MBC}$, $c > 0$, $D, K \ge 1$, $g \in H(X)$, $\mathrm{diam}(\mathrm{supp}(g)) \le D\alpha(c)$ and g is $K \cdot \alpha(c)/c$-Lipschitz. Then g is $(D + K + 1) \cdot \alpha$-continuous.*

Proof. Note that if $\alpha \in \mathrm{MC}$, then the function $\alpha(t)/t$ is decreasing. Let $x, y \in X$. Suppose first that $d(x, y) \le c$. Then

$$\begin{aligned} d(g(x), g(y)) &\le K\frac{\alpha(c)}{c} d(x, y) \le K\frac{\alpha(d(x, y))}{d(x, y)} d(x, y) = K\alpha(d(x, y)) \\ &\le (D + K + 1) \cdot \alpha(d(x, y)). \end{aligned}$$

Next assume that $d(x, y) > c$. If $x, y \in \mathrm{supp}(g)$, then

$$d(g(x), g(y)) \le D\alpha(c) < D\alpha(d(x, y)) \le (D + K + 1) \cdot \alpha(d(x, y)).$$

If $x \notin \mathrm{supp}(g)$ and $y \in \mathrm{supp}(g)$, then

$$\begin{aligned} d(g(x), g(y)) &\le d(x, y) + d(y, g(y)) \le \alpha(d(x, y)) + D\alpha(c) \\ &< \alpha(d(x, y)) + D\alpha(d(x, y)) = (D + K + 1)\alpha(d(x, y)). \end{aligned}$$

The case that $x \in \mathrm{supp}(g)$ and $y \notin \mathrm{supp}(g)$ is identical, and the case that $x, y \notin \mathrm{supp}(g)$ is trivial. ∎

PROPOSITION 10.6. *There are $b > 1$, $0 < a < 1$ and $M > 1$ such that the following holds. Suppose that:*

(1) $\alpha \in \mathrm{MBC}$,
(2) *E is a normed space, and L is a 1-dimensional linear subspace of E,*
(3) *$u \in E - L$ and $\|u\| \le \alpha(d(u, L))$,*
(4) *$u = \hat{u} + u^{\perp}$, where $\hat{u} \in L$ and $u^{\perp} \perp L$, and $v = (\|u\|/\|u^{\perp}\|)u^{\perp}$.*

Then there is $g \in H(E)$ such that:

(1) $g(u) = v$,
(2) g *is* $M \cdot \alpha$*-bicontinuous*,
(3) $\mathrm{supp}(g) \subseteq B(0; a\|u\|, b\|u\|)$,
(4) $g(L) = L$.

Define $M^{\mathrm{lift}} = M$, $a^{\mathrm{lift}} = a$ *and* $b^{\mathrm{lift}} = b$. *Note that the conjunction of clauses* (1)–(4) *is the relation* $R(u, v, g; M \cdot \alpha, a, b, L)$ *defined in Definition* 9.11(b).

Proof. Let $A = [u, v]$. Clearly, $d(u, L) = \|u^{\perp}\|$. So $\|u^{\perp}\| \le \|u\|$. We find an upper bound for $\|u - v\|$:

$$\begin{aligned}\|u - v\| &\le \|u - u^{\perp}\| + \|u^{\perp} - v\| = \|\hat{u}\| + (\|u\| - d(u, L))\\ &\le (\|u\| + \|u^{\perp}\|) + (\|u\| - d(u, L))\\ &= (\|u\| + d(u, L)) + (\|u\| - d(u, L)) = 2\|u\| \le 2\alpha(d(u, L)).\end{aligned}$$

We show that $d(A, L) = d(u, L)$. For every $z \in A$ there are $\lambda \in [0, 1]$ and $\mu \ge 1$ such that $z = \lambda\hat{u} + \mu u^{\perp}$. So $d(z, L) = \mu\|u^{\perp}\| \ge \|u^{\perp}\| = d(u, L)$. Since $u \in A$, we have $d(A, L) = d(u, L)$. We show that $d(A, 0) \ge \|u\|/4$. Let $w = \hat{u} + v$ and $C = [u, w] \cup [w, v]$. We first show that $d(C, 0) \ge \|u\|/2$. If $z \in [v, w]$, then for some $t \in \mathbb{R}$, $z = v + t\hat{u}$. So $\|z\| \ge d(z, L) = d(v, L) = \|v\| = \|u\|$. Hence $d([v, w], 0) = \|u\|$.

Note that $[u, w] = \{u + tv \mid 0 \le t \le 1 - \|u^{\perp}\|/\|v\|\}$. Let $z = u + tv \in [u, w]$. If $t \le 1/2$, then $\|u + tv\| \ge \|u\| - t\|v\| \ge \|u\| - \|u\|/2 = \|u\|/2$. If $t \ge 1/2$, then

$$\begin{aligned}\|u + tv\| &\ge d(u + tv, L) = d(\hat{u} + u^{\perp} + tv, L) = d(u^{\perp} + tv, L)\\ &= d((t + \|u^{\perp}\|/\|v\|)v, L) \ge d(tv, L) \ge d(v, L)/2 = \|u\|/2.\end{aligned}$$

Hence $d([u, w], 0) \ge \|u\|/2$. It follows that $d(C, 0) \ge \|u\|/2$.

We next prove that $(*)$ for every $x \in A$ there are $z \in C$ and $\mu \in [1/2, 1]$ such that $x = \mu z$. Recall that $w = \hat{u} + v$. The equation $\mu w = \lambda u + (1 - \lambda)v$ has the solution $\mu = \lambda = \frac{\|u\|}{2\|u\| - \|u^{\perp}\|}$. So $\mu \in (0, 1)$. That is, there are $x \in A$, $z \in C$ and $\mu \in (0, 1)$ such that $x = \mu z$, and hence for every $x \in A$ there are $z \in C$ and $\mu \in (0, 1)$ such that $x = \mu z$.

Let $z \in [u, w]$. Then $z = u + t\frac{\|u\|}{\|u^{\perp}\|}u^{\perp}$, where $0 \le t \le 1 - \frac{\|u^{\perp}\|}{\|u\|}$. The equation $\mu z = \lambda u + (1 - \lambda)v$ has the solution $\lambda = \mu = \frac{1}{1+t}$. Since $t \in (0, 1)$, $\mu \in [1/2, 1]$. Let $z \in [v, w]$. Then $z = \frac{\|u\|}{\|u^{\perp}\|}u^{\perp} + t(u - u^{\perp})$, where $0 \le t \le 1$. The equation $\mu z = \lambda u + (1 - \lambda)v$ has the solution

$$\mu = \frac{\|u\|}{\|u\| + t(\|u\| - \|u^{\perp}\|)}, \qquad \lambda = t\mu.$$

It follows that $\mu \in (1/2, 1]$. So $(*)$ is proved. Hence $d(A, 0) \ge d(C, 0)/2 \ge \|u\|/4$.

Let $r = d(u, L)/8$. By Proposition 9.6(a), there is $g \in H(E)$ such that $g(u) = v$, $\mathrm{supp}(g) \subseteq B(A, r)$ and g is $M^{\mathrm{seg}} \cdot (\mathrm{lngth}(A)/r + 1)$-bilipschitz. Hence requirement (1) holds. Moreover

$$M^{\mathrm{seg}} \cdot \left(\frac{\mathrm{lngth}(A)}{r} + 1\right) \le M^{\mathrm{seg}} \cdot \left(\frac{16\,\alpha(d(u, L))}{d(u, L)} + 1\right) \le M^{\mathrm{seg}} \cdot \frac{17\,\alpha(d(u, L))}{d(u, L)}.$$

So g is $17M^{\mathrm{seg}} \cdot \frac{\alpha(d(u,L))}{d(u,L)}$-bilipschitz. Also,

$$\mathrm{diam}(B(A, r)) \le \mathrm{lngth}(A) + 2r \le 2\alpha(d(u, L)) + d(u, L)/4 \le 3\alpha(d(u, L)).$$

We apply Proposition 10.5 to g and to g^{-1} with $c = d(u,L)$, $D = 3$ and $K = 17M^{\rm seg}$. It follows that g is $(4 + 17M^{\rm seg}) \cdot \alpha$-bicontinuous. So requirement (2) holds with $M = 4 + 17M^{\rm seg}$. Since $d(A,L) = d(u,L)$ and $r < d(u,L)$, it follows that $d(B(A,r),L) > 0$. So $g{\restriction}L = \mathrm{Id}$. Requirement (4) thus holds.

We find the a and b of requirement (3). Let $r_0 = d(B(A,r),0)$. So $g{\restriction}B(0,r_0) = \mathrm{Id}$. But $r_0 = d(A,0) - r \geq \|u\|/4 - d(u,L)/8 \geq \|u\|/8$. So $a = 1/8$. Let $r_1 = \sup_{x \in B(A,r)} \|x\|$. Then $\mathrm{supp}(g) \subseteq B(0,r_1)$. For every $x \in A$, $\|x\| \leq \max(\|u\|,\|v\|) = \|u\|$. So $r_1 \leq \|u\| + r < 2\|u\|$. Define $b = 2$. Then $\mathrm{supp}(g) \subseteq B(0; a\|u\|, b\|u\|)$. So requirement (3) is fulfilled with $a = 1/8$ and $b = 2$. ∎

PROPOSITION 10.7. *Let E be a 3-dimensional Hilbert space, L be a 1-dimensional sybspace of E, $u,v \in E - L$ and $M \geq 1$. Suppose that $\|u\|,\|v\| \leq Md(u,L)$ and $\|u\|,\|v\| \leq Md(v,L)$. Then there is a rectifiable arc A connecting u and v such that*:

(1) $\mathrm{lngth}(A) \leq (4+\pi)M\|u\|$,
(2) $d(A,L) \geq \|u\|/M$,
(3) $\max(\{\|x\| \mid x \in A\}) \leq M\|u\|$.

Proof. Let $w_1 = u^{\perp}$, $w_2 = v^{\perp}$ and $w_3 = (\|u^{\perp}\|/\|v^{\perp}\|)v^{\perp}$. Let S be a subarc of $S(0,\|w_1\|) \cap L^{\perp}$ whose endpoints are w_1 and w_3 and such that $\mathrm{lngth}(S) \leq \pi\|w_1\|$. Define $A = [u,w_1] \cup S \cup [w_3,w_2] \cup [w_2,v]$. Then $d(A,L) = \min(d(u,L),d(v,L)) \geq \|u\|/M$. It is obvious that $\max(\{\|x\| \mid x \in A\}) = \max(\|u\|,\|v\|) \leq M\|u\|$. Now,

$$\begin{aligned}\mathrm{lngth}(A) &\leq \|(u)_L\| + \pi\|u^{\perp}\| + |\|u^{\perp}\| - \|v^{\perp}\|| + \|(v)_L\|\\ &\leq \|u\| + \pi\|u\| + \|u\| + \|v\| \leq (4+\pi)M\|u\|.\end{aligned}$$

So A is as required. ∎

PROPOSITION 10.8. *There are $M > 1$, $0 < a < 1$ and $b > 1$ such that the following holds. Suppose that*:

(1) *E is a normed space, and L is a 1-dimensional linear subspace of E*,
(2) *$u,v \in E - L$, $\|u\| = \|v\|$, $u \perp L$ and $v \perp L$*,
(3) *if E is 2-dimensional, then u,v are on the same side of L.*

Then there is $g \in H(E)$ such that $R(u,v,g;M,a,b,L)$ holds. (See Definition 9.11(b).) *We write $M^{\rm perp} = M$, $a^{\rm perp} = a$ and $b^{\rm perp} = b$.*

Proof. If E is 2-dimensional, then $[u,v] \subseteq S(0,\|u\|)$. So $d([u,v],L) = \|u\|$ and $\mathrm{lngth}([u,v]) \leq \|u\| + \|v\| = 2\|u\|$. By Proposition 9.6(a), there is $g \in H(E)$ such that: $g(u) = v$, $\mathrm{supp}(g) \subseteq B([u,v],\|u\|/2)$, and g is $M^{\rm seg} \cdot \frac{2\|u\|}{\|u\|/2}$-bilipschitz. So for 2-dimensional E's, M,a,b can be taken to be $4M^{\rm seg}$, $1/2$ and $3/2$.

Suppose that $\dim(E) > 2$. Let F be a 3-dimensional linear subspace of E containing L, u and v, and let $\|\ \|^{\mathbf{H}}$ be a tight Hilbert norm on F. Define $N = M^{\rm thn}(3)$. (See Proposition 9.2(b).) So for every $x \in F$, $\|x\| \leq \|x\|^{\mathbf{H}} \leq N\|x\|$. Obviously, $\|u\|^{\mathbf{H}},\|v\|^{\mathbf{H}} \leq Nd^{\mathbf{H}}(u,L)$, and $\|u\|^{\mathbf{H}},\|v\|^{\mathbf{H}} \leq Nd^{\mathbf{H}}(v,L)$. By Proposition 10.7, there is a rectifiable arc A in F connecting u and v such that: $\mathrm{lngth}^{\mathbf{H}}(A) \leq (4+\pi)N\|u\|^{\mathbf{H}}$, $d^{\mathbf{H}}(A,L) \geq \frac{1}{N}\|u\|^{\mathbf{H}}$ and $\max(\{\|x\|^{\mathbf{H}} \mid x \in A\}) \leq N\|u\|^{\mathbf{H}}$. So

(1) $\mathrm{lngth}(A) \leq (4+\pi)N^2\|u\|$,
(2) $d(A,L) \geq \frac{1}{N^2}\|u\|$,
(3) $\max(\{\|x\| \mid x \in A\}) \leq N^2\|u\|$.

Let $r = \frac{1}{2N^2}\|u\|$, By Proposition 9.6(b), there is $g \in H(E)$ such that:

(4) $\mathrm{supp}(g) \subseteq B(A,r)$,
(5) $g(u) = v$,
(6) g is $M^{\mathrm{arc}}(\frac{(4+\pi)N^2\|u\|}{\|u\|/(2N^2)})$-bilipschitz.

So g is $M^{\mathrm{arc}}(16N^4)$-bilipschitz.

Since $d(B(A,r),L) \geq (1/2N^2)\|u\|$, $g \restriction L = \mathrm{Id}$. Define $M = M^{\mathrm{arc}}(16N^4)$, $a = 1/2N^2$ and $b = N^2+1$. Then M,a,b are as required in the proposition. ∎

Proposition 10.9. *There are $M > 1$, $0 < a < 1$ and $b > 1$ such that the following holds. Suppose that E is a normed space, $u \in E - \{0\}$, $\alpha \in \mathrm{MBC}$, $1 \leq \lambda \leq \alpha(\|u\|)/\|u\|$ and $v = \lambda u$. Then there is a radial homeomorphism $g \in H(E)$ such that: $g(u) = v$, g is $M \cdot \alpha$-bicontinuous and* $\mathrm{supp}(g) \subseteq B(0;a\|u\|,b\|v\|)$. *Note that this implies that $R(u,v,g;M\cdot\alpha,a,b,L)$ holds. Denote M,a,b by $M^{\mathrm{dlt}},a^{\mathrm{dlt}}$ and b^{dlt}.*

Proof. Let $\eta \in H([0,\infty))$ be the piecewise linear function which is determined by the following equalities: $\eta(0) = 0$, $\eta(\|u\|/2) = \|u\|/2$, $\eta(\|u\|) = \lambda\|u\|$, and for every $t \geq \lambda\|u\|+\|u\|$, $\eta(t) = t$. The slopes of the linear parts of η are 1, 2λ and $1/\lambda$. Since $1 \leq \lambda \leq \alpha(\|u\|)/\|u\|$, η is $2\cdot\alpha(\|u\|)/\|u\|$-bilipschitz. Let g be the radial homeomorphism of E based on η. By Proposition 3.18, g is $6\cdot\alpha(\|u\|)/\|u\|$-bilipschitz. Also, $\lambda\|u\|+\|u\| \leq 2\|v\|$, hence $\mathrm{supp}(g) \subseteq B(0,2\|v\|)$. By Proposition 10.5, g is $(6+2+1)\cdot\alpha$-bicontinuous. So we may define $M = 9$, $a = 1/2$ and $b = 2$. ∎

Proof of Lemma 10.3. (a) Let E,F,x,y be as in the definition of a 1UC-constant (Definition 10.2). There are $\hat{x}$ and $x^\perp$ such that $x = \hat{x}+x^\perp$, $\hat{x} \in F$ and $x^\perp \perp F$. Similarly, there are $\hat{y}$ and $y^\perp$ such that $y = \hat{y}+y^\perp$, $\hat{y} \in F$ and $y^\perp \perp F$. Let $x_1 = (\|x\|/\|x^\perp\|)x^\perp$ and $y_1 = (\|y\|/\|y^\perp\|)y^\perp$. By Proposition 10.6, there are $f_1,h_1 \in H(E)$ such that

$$R(x,x_1,f_1;M^{\mathrm{lift}}\cdot\alpha,a^{\mathrm{lift}},b^{\mathrm{lift}},F) \quad \text{and} \quad R(y,y_1,h_1;M^{\mathrm{lift}}\cdot\alpha,a^{\mathrm{lift}},b^{\mathrm{lift}},F).$$

Let $y_2 = (\|x_1\|/\|y_1\|)y_1$. Note that $\|x_1\| = \|y_2\|$, $x_1 \perp F$ and $y_2 \perp F$, and if E is 2-dimensional then x_1, y_2 are on the same side of F. By Proposition 10.8, there is $f_2 \in H(E)$ such that

$$R(x_1,y_2,f_2;M^{\mathrm{perp}},a^{\mathrm{perp}},b^{\mathrm{perp}},F).$$

Since $\|y_2\| = \|x\|$ and $\|y_1\| = \|y\|$, it follows that $\|y_2\| \leq \|y_1\| \leq \alpha(\|y_2\|)$. So by Proposition 10.9, there is $g_2 \in H(E)$ such that

$$R(y_2,y_1,g_2;M^{\mathrm{dlt}}\cdot\alpha,a^{\mathrm{dlt}},b^{\mathrm{dlt}},F).$$

Let $g_1 = f_2\circ f_1$ and $g_3 = h_1^{-1}$. Clearly, $g_3\circ g_2\circ g_1(x) = y$. Let $M = \max(M^{\mathrm{lift}}M^{\mathrm{perp}},M^{\mathrm{dlt}})$. Note that $\|x_1\| = \|x\|$, so $f_2 \restriction B(0,a^{\mathrm{perp}}\|x\|) = \mathrm{Id}$. Set $a = \min(a^{\mathrm{lift}},a^{\mathrm{perp}},a^{\mathrm{dlt}})$ and $b = \max(b^{\mathrm{lift}},b^{\mathrm{perp}},b^{\mathrm{dlt}})$. It is obvious that clauses B1–B4 in the definition of a 1UC-constant hold for M,a,b,x,y,g_1,g_2,g_3 and F.

(b) Part (b) is deduced from (a) in the same way that part (b) of Lemma 9.13 is deduced from (a) of that lemma. ∎

11. Extending the inducing homeomorphism to the boundary

A *sequence* means a function whose domain is an infinite subset of $\mathbb{N}$. If $\sigma \subseteq \mathbb{N}$ is infinite, then $\{x_i \mid i \in \sigma\}$ is abbreviated by $\vec{x}^{(\sigma)}$. Suppose that $\vec{x}^{(\sigma)}$ is a sequence in X and $g \in H(X, Y)$. Then $g(\vec{x}^{(\sigma)})$ denotes the sequence $\{g(x_i) \mid i \in \sigma\}$. For $n \in \mathbb{N}$ and an infinite $\sigma \subseteq \mathbb{N}$ let $\sigma^{\geq n} := \{k \in \sigma \mid k \geq n\}$. For a sequence $\vec{x}$ let $\vec{x}^{\geq n} := \vec{x}{\restriction}\mathrm{Dom}(\vec{x})^{\geq n}$.

Recall that if $\alpha : A \to A$, then $\alpha^{\circ n}$ denotes $\alpha \circ \cdots \circ \alpha$, n times. Let X, Y be open sets in metric spaces E and F respectively and $g : X \to Y$. If $x \in \mathrm{Dom}(g^{\mathrm{cl}})$, then we sometimes abbreviate $g^{\mathrm{cl}}(x)$ by $g(x)$.

DEFINITION 11.1. (a) Let X, Y be open sets in metric spaces E and F respectively. Suppose that $x \in \mathrm{cl}(X)$ and $g \in H(X, Y)$. We say that g is *α-continuous at x* if there is $T \in \mathrm{Nbr}(x)$ such that $g{\restriction}(T \cap X)$ is α-continuous.

Obviously, if F is a complete metric space, and g is α-continuous at x, then $x \in \mathrm{Dom}(g^{\mathrm{cl}})$. We say that *$g$ is α-bicontinuous at x* if g is α-continuous at x, $x \in \mathrm{Dom}(g^{\mathrm{cl}})$ and g^{-1} is α-continuous at $g^{\mathrm{cl}}(x)$. We say that *g is Γ-bicontinuous at x* if for some $\alpha \in \Gamma$, g is α-bicontinuous at x.

(b) Suppose that E is a metric space $X \subseteq E$ is open, $b \in \mathrm{bd}(X)$, $\alpha \in \mathrm{MBC}$ and $x, y \in X$. Recall that we write $\delta^{X,E}(x) = d(x, E - X)$. Superscripts ^{E} and ^{X} are omitted when they are understood from the context. The notation $x \approx^{(\alpha,b)}_{(X,E)} y$ means that

$$d(x, b) \approx^{\alpha} d(y, b) \quad \text{and} \quad \delta^X(x) \approx^{\alpha} \delta^X(y).$$

Suppose that $\vec{x}^{(\sigma)}$ and $\vec{y}^{(\sigma)}$ are sequences in X. Then $\vec{x}^{(\sigma)} \approx^{(\alpha,b)}_{(X,E)} \vec{y}^{(\sigma)}$ means that for every $n \in \sigma$, $x_n \approx^{(\alpha,b)}_{(X,E)} y_n$. We abbreviate $\approx^{(\alpha,b)}_{(X,E)}$ by $\approx^{(\alpha,b)}$. Note that the notation $\vec{x} \approx^{(\alpha,b)} \vec{y}$ entails that $\mathrm{Dom}(\vec{x}) = \mathrm{Dom}(\vec{y})$.

(c) Let X be a topological space, $A \subseteq H(X)$, $\varrho \subseteq \mathbb{N}$ be infinite and $\vec{x}^{(\varrho)}, \vec{y}^{(\varrho)}$ be sequences in X. We define the relation $\vec{x}^{(\varrho)} \approxeq^A \vec{y}^{(\varrho)}$. The relation $\vec{x}^{(\varrho)} \approxeq^A \vec{y}^{(\varrho)}$ means that for any infinite $\sigma, \eta \subseteq \varrho$ there is $g \in A$ such that $\{i \in \sigma \mid g(x_i) = y_i\}$ and $\{i \in \eta \mid g(x_i) = x_i\}$ are infinite. If $\alpha \in \mathrm{MBC}$, then $\vec{x}^{(\varrho)} \approxeq^{\alpha} \vec{y}^{(\varrho)}$ means that $\vec{x}^{(\varrho)} \approxeq^A \vec{y}^{(\varrho)}$, where $A = \{g \in H(X) \mid g \text{ is } \alpha\text{-bicontinuous}\}$.

(d) Let E be a metric space, $X \subseteq E$ be open, $\alpha \in \mathrm{MBC}$ and Γ be a modulus of continuity. A sequence $\vec{x}$ in X is called an *α-abiding* sequence if

(i) $\vec{x}$ is convergent and $b := \lim \vec{x} \in \mathrm{bd}(X)$;
(ii) there is $n = n(\vec{x}, \alpha) \in \mathbb{N}$ such that for every $k \in \mathrm{Dom}(\vec{x})^{\geq n}$, $d(x_n, b) \leq \alpha(\delta(x_n))$.

A sequence $\vec{x}$ in X is called a *Γ-evasive sequence* if

(i) $\vec{x}$ is convergent and $b := \lim \vec{x} \in \mathrm{bd}(X)$;

(ii) for every subsequence $\vec{y}$ of $\vec{x}$ and $\alpha \in \Gamma$, $\vec{y}$ is not α-abiding.

Equivalently, $\vec{x}$ is Γ-evasive iff (i) holds and for every $\alpha \in \Gamma$ there is $n \in \mathbb{N}$ such that for every $m \in \mathrm{Dom}(\vec{x})^{\geq n}$, $d(x_m, b) > \alpha(\delta(x_m))$.

(e) Let X be an open subset of a normed space E, and $x \in \mathrm{bd}(X)$. Suppose that X is two-sided at x, and let $\langle \psi, A, r \rangle$ be a boundary chart element for x. Let $U, V \in \mathrm{Nbr}(x)$ and $h \in \mathrm{EXT}^{\pm}(U \cap X, V \cap X)$ be such that $h^{\mathrm{cl}}(x) = x$. We say that h is *side preserving* at x if there is $U' \in \mathrm{Nbr}(x)$ such that for every $u \in U' \cap X$, u and $h(u)$ are on the same side of $\mathrm{bd}(X)$ with respect to $\langle \psi, A, r \rangle$. See Definition 8.10. We say that h is *side reversing* at x if there is $U' \in \mathrm{Nbr}(x)$ such that for every $u \in U' \cap X$, u and $h(u)$ are on different sides of $\mathrm{bd}(X)$ with respect to $\langle \psi, A, r \rangle$. Note that the properties of being side preserving or side reversing do not depend on the choice of $\langle \psi, A, r \rangle$.

(f) Let X be an open subset of a normed space E, and $x \in \mathrm{bd}(X)$. Suppose that $\mathrm{bd}(X)$ is 1-dimensional at x, and let $\langle \psi, A, r \rangle$ be a boundary chart element for x. Let $L = \mathrm{bd}(X) \cap \mathrm{Rng}(\psi)$. So L is an open arc. Let $U, V \in \mathrm{Nbr}(x)$ and $h \in \mathrm{EXT}^{\pm}(U \cap X, V \cap X)$ be such that $h^{\mathrm{cl}}(x) = x$. We say that h is *order preserving* at x if there is $U' \in \mathrm{Nbr}(x)$ such that for every $u \in U' \cap L$, u and $h^{\mathrm{cl}}(u)$ are in the same connected component of $L - \{x\}$. We say that h is *order reversing* at x if there is $U' \in \mathrm{Nbr}(x)$ such that for every $u \in U' \cap X$, u and $h(u)$ are in different connected components of $L - \{x\}$. Note that the properties of being order preserving or order reversing are independent of the choice of $\langle \psi, A, r \rangle$.

Let $G \leq \mathrm{EXT}(X)$. We say that $\mathrm{bd}(X)$ *is G-order-reversible at x* if there is $g \in G$ such that g is order reversing at x, and if X is two-sided at x, then g is side preserving. If such a g does not exist, then we say that $\mathrm{bd}(X)$ *is G-order-irreversible at x*. $\square$

PROPOSITION 11.2. *Let E, F normed spaces. Suppose that $X \subseteq E$, $Y \subseteq F$ are open, $\alpha \in \mathrm{MBC}$ and $g \in \mathrm{EXT}^{\pm}(X, Y)$. Let $b \in \mathrm{bd}(X)$, and suppose that g is α-bicontinuous at x.*

(a) *There is $r_0 > 0$ such that for every $x \in B(b, r_0) \cap X$, $\delta(x) \approx^{\alpha} \delta(g(x))$.*

(b) *Assume that $E = F$, $Y = X$ and $g(b) = b$. Suppose that $\vec{x}$ is a sequence in X converging to b. Then for some $n \in \mathbb{N}$, $\vec{x}^{\geq n} \approx^{(\alpha, b)} g(\vec{x})^{\geq n}$.*

(c) *Assume that $E = F$, $Y = X$ and $g(b) = b$. Suppose that X is two-sided at b. Let $\langle \psi, A, r \rangle$ be a boundary chart element for b. Then there is $U \in \mathrm{Nbr}(b)$ such that $U, g(U) \subseteq \mathrm{Rng}(\psi)$, and for every $u, v \in U \cap X$: u, v are on the same side of $\mathrm{bd}(X)$ with respect to $\langle \psi, A, r \rangle$ iff $g(u), g(v)$ are on the same side $\mathrm{bd}(X)$ with respect to $\langle \psi, A, r \rangle$.*

Proof. (a) Let $r > 0$ be such that $g \restriction (B(b, r) \cap X)$ is α-continuous. Choose $s > 0$ such that $g^{-1} \restriction (B(g(b), s) \cap Y)$ is α-continuous, and let r_0 be such that $r_0 < r/2$ and $g(B(b, r_0) \cap X) \subseteq B(g(b), s/2) \cap Y$. Let $x \in B(b, r_0) \cap X$. Suppose that $\varepsilon \in (0, r/2 - \|x - b\|)$. Let $u \in \mathrm{bd}(X)$ be such that $\|u - x\| < \delta(x) + \varepsilon$. Since $\delta(x) < \|x - b\| < r_0$, it follows that

$$\|u - b\| \leq \|u - x\| + \|x - b\| < \delta(x) + r/2 - \|x - b\| + \|x - b\| \leq r_0 + r/2 < r.$$

Hence $g^{\mathrm{cl}} \restriction \{x, u\}$ is α-continuous. So

$$\delta(g(x)) \leq \|g^{\mathrm{cl}}(x) - g^{\mathrm{cl}}(u)\| \leq \alpha(\|x - u\|) < \alpha(\delta(x) + \varepsilon).$$

Since this argument is valid for any $\varepsilon \in (0, r/2 - \|x-b\|)$, it follows that $\delta(g(x)) \le \alpha(\delta(x))$. We apply the analogous argument to $g(x)$. This can be done, since $g(x) \in B(g(b), s/2) \cap Y$. So $\delta(g^{-1}(g(x))) \le \alpha(\delta(g(x)))$. That is, $\delta(x) \le \alpha(\delta(g(x)))$. We conclude that

$$\delta(x) \approx^{\alpha} \delta(g(x)).$$

(b) This follows trivially from (a).

(c) There is $s \in (0, r)$ such that $g(\psi(B(0,s))) \subseteq \mathrm{Rng}(\psi)$. Let $U = \psi(B(0,s))$. Let $u, v \in U \cap X$ be on the same side of $\mathrm{bd}(X)$. Let $L = [\psi^{-1}(u), \psi^{-1}(v)]$. Then $L \subseteq B(0,r) - A$ and thus $\psi(L) \subseteq X$. So $g(\psi(L)) \subseteq X$. Hence $\psi^{-1}(g(\psi(L))) \subseteq B(0,r) - A$. That is, there is an arc in $B(0,r) - A$ connecting $\psi^{-1}(g(u))$ and $\psi^{-1}(g(v))$. So $\psi^{-1}(g(u))$ and $\psi^{-1}(g(v))$ are on the same side of A. This means that $g(u)$ and $g(v)$ are on the same side of $\mathrm{bd}(X)$. ■

PROPOSITION 11.3. (a) *There is $\mathsf{N} > 1$ such that* (a1) *and* (a2) *below hold. Let $\alpha, \beta \in$* MBC, *X be an open subset of a normed space. Suppose that $b \in \mathrm{bd}(X)$, X is β-LIN-bordered at b, and $\mathrm{bd}(X)$ is not 1-dimensional at b. Define $\bar{\alpha} = \beta \circ \alpha \circ \beta$.*

(a1) *Let $\vec{x}, \vec{y}$ be sequences in X converging to b. Suppose that $\vec{x} \approx^{(\alpha,b)} \vec{y}$. Also assume that if X is two-sided at b, then for every $n \in \mathrm{Dom}(\vec{x})$, x_n and y_n are on the same side of* $\mathrm{bd}(X)$. *Then $\vec{x} \approxeq^{\mathsf{N}\cdot\beta \circ \bar{\alpha}^{\circ 4} \circ \beta} \vec{y}$.*

(a2) *Let $g \in \mathrm{EXT}(X)$ be α-bicontinuous at b. Suppose that $g(b) = b$. Suppose further that if X is two-sided at b, then g is side preserving at b. Let $\vec{x}$ be a sequence in X converging to b. Then $\vec{x} \approxeq^{\mathsf{N}\cdot\beta \circ \bar{\alpha}^{\circ 4} \circ \beta} g(\vec{x})$.*

(b) *Let X be an open subset of a normed space and $\beta \in$* MBC. *Suppose that $b \in \mathrm{bd}(X)$, X is β-LIN-bordered at b, and X is two-sided at b. Let $g \in \mathrm{EXT}(X)$ be such that $g(b) = b$, and g is side reversing at b. Let $\vec{x}$ be a sequence in X converging to b. Then $\vec{x} \not\approxeq^{\mathrm{EXT}(X)} g(\vec{x})$.*

Proof. (a) Let M be a UC-constant for $\langle 1/2, 2\rangle$, $\mathsf{M} = M^2$ and $\mathsf{N} = \mathsf{M}^2$. (See Definition 9.11(a).) We shall prove that N is as required in (a).

(a1) Let $X, b, \vec{x}, \vec{y}$ and α be as in (a1). Let $\langle \psi, A, r\rangle$ be a boundary chart element for b, and assume that ψ is β-bicontinuous. We show that $\vec{x} \approxeq^{\mathsf{N}\cdot\beta \circ \bar{\alpha}^{\circ 4} \circ \beta} \vec{y}$. We may assume that $\vec{x}, \vec{y} \subseteq \mathrm{Rng}(\psi)$. Set $\vec{w} = \psi^{-1}(\vec{x})$ and $\vec{z} = \psi^{-1}(\vec{y})$. Clearly, $\vec{w} \approx^{(\bar{\alpha},0)} \vec{z}$. Let $\sigma, \eta \subseteq \mathbb{N}$ be infinite. We may assume that either for every $i \in \sigma$, $\|w_i\| \le \|z_i\|$, or for every $i \in \sigma$, $\|z_i\| < \|w_i\|$. Let us assume that the former happens. The case that $\|z_i\| < \|w_i\|$ is dealt with in a similar way. Let $\{m_i \mid i \in \mathbb{N}\}$ and $\{m_i^1 \mid i \in \mathbb{N}\}$ be respectively 1-1 enumerations of σ and η and set $u_i = w_{m_i}$, $v_i = z_{m_i}$ and $u_i^1 = w_{m_i^1}$. So $\vec{u} \approx^{(\bar{\alpha},0)} \vec{v}$.

We define by induction $i_n, j_n \in \mathbb{N}$ and $h_n \in H(B^E(0,r))$ such that:

(1) $\|v_{i_0}\| < r/2$,
(2) $h_n(u_{i_n}) = v_{i_n}$,
(3) h_n is $\mathsf{M} \cdot \bar{\alpha} \circ \bar{\alpha}$-bicontinuous,
(4) $\mathrm{supp}(h_n) \subseteq B(0; \frac{1}{2}\|u_{i_n}\|, 2\|v_{i_n}\|)$,
(5) $\|u_{j_n}^1\| < \|u_{i_n}\|/2$ and $\|v_{i_{n+1}}\| < \|u_{j_n}^1\|/2$,
(6) $h_n(A) = A$.

That the construction is possible follows from Lemma 9.13(b). Facts (4) and (5) imply that $\mathrm{supp}(h_m) \cap \mathrm{supp}(h_n) = \emptyset$ for any $m \neq n$. So $h := \circ_n h_n$ is well defined.

Let $\gamma = M \cdot \bar{\alpha} \circ \bar{\alpha}$. We verify that h is $\gamma \circ \gamma$-bicontinuous. Let $u, v \in B^E(0,r)$. Then there are $m, n \in \mathbb{N}$ such that $u, h_m(u) \in B^E(0,r) - \bigcup_{k \neq m} \mathrm{supp}(h_k)$ and $v, h_n(u) \in B^E(0,r) - \bigcup_{k \neq n} \mathrm{supp}(h_k)$. If $m \neq n$, then $h(u) = h_m \circ h_n(u)$ and $h(v) = h_m \circ h_n(v)$, and if $m = n$, then $h(u) = h_m(u)$ and $h(v) = h_m(v)$. Since $h_m \circ h_n$ and h_m are $\gamma \circ \gamma$-continuous, $\|h(u) - h(v)\| \leq \gamma \circ \gamma(\|u - v\|)$. So h is $\gamma \circ \gamma$-continuous. The same argument holds for h^{-1}. It follows that h is $\gamma \circ \gamma$-bicontinuous. Since $\gamma \circ \gamma \leq M^2 \cdot \bar{\alpha}^{\circ 4}$, we infer that h is $M^2 \cdot \bar{\alpha}^{\circ 4}$-bicontinuous. By (4) and (5), $h(u^1_{j_n}) = u^1_{j_n}$ for every $n \in \mathbb{N}$. Let $g' = \psi \circ h \circ \psi^{-1} \restriction \mathrm{BCD}^E(A,r)$. Then $\mathrm{Dom}(g') = \mathrm{Rng}(\psi) \cap X$. Clearly, g' is $\beta \circ (M^2 \cdot \bar{\alpha}^{\circ 4}) \circ \beta$-bicontinuous, and hence g' is $M^2 \cdot \beta \circ \bar{\alpha}^{\circ 4} \circ \beta$-bicontinuous. Define $g = g' \cup \mathrm{Id} \restriction (X - \mathrm{Rng}(\psi))$. From (1) and (4) it follows that $g \in H(X)$. The fact that $M^2 \cdot \beta \circ \bar{\alpha}^{\circ 4} \circ \beta \in \mathrm{MBC}$ implies that g too is $M^2 \cdot \beta \circ \bar{\alpha}^{\circ 4} \circ \beta$-bicontinuous. Clearly, $x'_n := \psi(u^1_{j_n}) \in \{x_i \mid i \in \eta\}$ and $h(x'_n) = x'_n$. For every $n \in \mathbb{N}$ there is $k(n) \in \sigma$ such that $\psi(u_{i_n}) = x_{k(n)}$ and $\psi(v_{i_n}) = y_{k(n)}$. From the fact that $h(u_{i_n}) = v_{i_n}$ it follows that $g(x_{k(n)}) = y_{k(n)}$. So g fulfills the requirements which are needed in order to show that $\vec{x} \approxeq^{N \cdot \beta \circ \bar{\alpha}^{\circ 4} \circ \beta} \vec{y}$.

(a2) It follows trivially from Proposition 11.2(b) and (a1) that N is as required.

(b) Suppose by contradiction that $\vec{x} \approxeq^{\mathrm{EXT}(X)} g(\vec{x})$. Then $(*)$ there is $h \in \mathrm{EXT}(X)$ such that $\{i \in \mathbb{N} \mid h(x_i) = g(x_i)\}$ and $\{i \in \mathbb{N} \mid h(x_i) = x_i\}$ are infinite. Since $\lim \vec{x} = b$ and g is side reversing, $(*)$ contradicts Proposition 11.2(c). ∎

PROPOSITION 11.4. *There is $N > 1$ such that the following holds. Let $\alpha, \beta \in \mathrm{MBC}$, and X be an open subset of a normed space E. Suppose that $b \in \mathrm{bd}(X)$, X is β-LIN-bordered at b, and $\mathrm{bd}(X)$ is 1-dimensional at b. Define $\bar{\alpha} = \beta \circ \alpha \circ \beta$.*

(a) *Let $\vec{x}, \vec{y}$ be α-abiding sequences in X converging to b and $\vec{x} \approx^{(\alpha,b)} \vec{y}$. Also assume that if X is two-sided at b, then for every $n \in \mathrm{Dom}(\vec{x})$, x_n and y_n are on the same side of $\mathrm{bd}(X)$. Then $\vec{x} \approxeq^{N \cdot \beta \circ \bar{\alpha}^{\circ 6} \circ \beta} \vec{y}$.*

(b) *Let $g \in \mathrm{EXT}(X)$ be α-bicontinuous at b. Suppose that $g(b) = b$. Suppose further that if X is two-sided at b, then g is side preserving at b. Let $\vec{x}$ be an α-abiding sequence in X converging to b. Then $\vec{x} \approxeq^{N \cdot \beta \circ \bar{\alpha}^{\circ 6} \circ \beta} g(\vec{x})$.*

Proof. (a) The proof follows the same steps as the proof of Proposition 11.3(a1). But here Lemma 10.3 replaces the use of Lemma 9.13 in the proof of 11.3(a1).

(b) The proof follows the same steps as the proof of Proposition 11.3(a2). ∎

PROPOSITION 11.5. (a) *There is $N > 1$ such that* (a1) *and* (a2) *below hold. Let $\alpha, \beta \in$ MBC, and X be an open subset of a normed space E. Suppose that $b \in \mathrm{bd}(X)$, X is β-LIN-bordered at b, and $\mathrm{bd}(X)$ is 1-dimensional at b. Let $\langle \psi, A, r \rangle$ be boundary chart element for b with ψ being β-bicontinuous. If A is a subspace of E, let $F = A$. If $\dim(E) = 2$ and A is a half space of E, let $F = \mathrm{bd}(A)$. (So F is a 1-dimensional subspace of E.) Define $\bar{\alpha} = \beta \circ \alpha \circ \beta$.*

(a1) *Let $\vec{x}, \vec{y} \subseteq \mathrm{Rng}(\psi)$ be sequences which converge to b, and set $\vec{u} = \psi^{-1}(\vec{x})$ and $\vec{v} = \psi^{-1}(\vec{y})$. Assume that*

(i) $\vec{x} \approx^{(\alpha,b)} \vec{y}$,

(ii) *for every* $n \in \mathrm{Dom}(\vec{x})$ *there are* $\hat{u}_n, u_n^{\perp}, \hat{v}_n, v_n^{\perp}$ *and* λ_n *such that* $u_n = \hat{u}_n + u_n^{\perp}$, $v_n = \hat{v}_n + v_n^{\perp}$, $\hat{u}_n, \hat{v}_n \in F$, $u_n^{\perp}, v_n^{\perp} \perp F$, $\lambda_n > 0$ *and* $\hat{v}_n = \lambda_n \hat{u}_n$,

(iii) *if* X *is two-sided at* b, *then for every* $n \in \mathrm{Dom}(\vec{x})$, x_n *and* y_n *are on the same side of* $\mathrm{bd}(X)$.

Then $\vec{x} \precapprox^{N \cdot \beta \circ \bar{\alpha}^{\circ 4} \circ \beta} \vec{y}$.

(a2) *Let* Γ *be a modulus of continuity,* $\alpha, \beta \in \Gamma$, *and* $\vec{x}$ *be a* Γ*-evasive sequence in* X *converging to* b. *Let* $g \in \mathrm{EXT}(X)$ *be* α*-bicontinuous at* b, *and assume that*: $g(b) = b$, g *is order preserving at* b, *and if* X *is two-sided at* b *then* g *is side preserving at* b. *Then* $\vec{x} \precapprox^{N \cdot \beta \circ \bar{\alpha}^{\circ 4} \circ \beta} g(\vec{x})$.

In parts (b)–(d) *below we assume that* Γ *is a modulus of continuity,* $\beta \in \Gamma \cap \mathrm{MBC}$, X *is an open subset of a normed space* E, $b \in \mathrm{bd}(X)$, X *is* β*-LIN-bordered at* b, *and* $\mathrm{bd}(X)$ *is* 1*-dimensional at* b. *We also assume that* $G \leq \mathrm{EXT}^{\pm}(X)$, *and* G *is of boundary type* Γ.

(b) *Let* $\langle \psi, A, r \rangle$ *be boundary chart element for* b *with* ψ *being* β*-bicontinuous. If* A *is a subspace of* E *set* $F = A$, *and if* $\dim(E) = 2$ *and* A *is a half space of* E, *set* $F = \mathrm{bd}(A)$. (*So* F *is a* 1*-dimensional subspace of* E.) *Let* $\vec{x}, \vec{y} \subseteq \mathrm{Rng}(\psi)$ *be sequences which converge to* b, *and set* $\vec{u} = \psi^{-1}(\vec{x})$ *and* $\vec{v} = \psi^{-1}(\vec{y})$. *Assume that*

(i) $\vec{x}, \vec{y}$ *are* Γ*-evasive,*

(ii) *for every* $n \in \mathrm{Dom}(\vec{x})$ *there are* $\hat{u}_n, u_n^{\perp}, \hat{v}_n, v_n^{\perp}$ *and* λ_n *such that* $u_n = \hat{u}_n + u_n^{\perp}$, $v_n = \hat{v}_n + v_n^{\perp}$, $\hat{u}_n, \hat{v}_n \in F$, $u_n^{\perp}, v_n^{\perp} \perp F$, $\lambda_n < 0$ *and* $\hat{v}_n = \lambda_n \hat{u}_n$.

Then $\vec{x} \not\precapprox^{G} \vec{y}$.

(c) *Let* $\vec{x}$ *be a* Γ*-evasive sequence in* X *converging to* b. *Let* $g \in G$. *Suppose that* $g(b) = b$, *and* g *is order reversing at* b. *Then* $\vec{x} \not\precapprox^{G} g(\vec{x})$.

(d) *Let* $\vec{x}$ *be a sequence in* X *converging to* b. *Let* $g \in G$ *be such that* $g(b) = b$ *and* g *is order preserving at* b. *Assume further that if* X *is two-sided at* b, *then* g *is side preserving. Then* $\vec{x} \precapprox^{\Gamma} g(\vec{x})$.

Proof. (a1) The proof follows the same steps as the proof of Proposition 11.3(a1). But here Lemma 10.1 replaces the use of Lemma 9.13 in the proof of 11.3(a1).

(a2) Let $\langle \psi, A, r \rangle$ be boundary chart element for b such that ψ is β-bicontinuous. If A is a half space set $F = \mathrm{bd}(A)$. Otherwise, set $F = A$. Let B be an open ball with center at b such that $g^{\mathrm{cl}} \restriction (\mathrm{cl}(B) \cap \mathrm{cl}(X))$ is α-bicontinuous, and $\mathrm{cl}(B)$, $g^{\mathrm{cl}}(\mathrm{cl}(B) \cap \mathrm{cl}(X)) \subseteq \mathrm{Rng}(\psi)$. Let $U = \psi^{-1}(B \cap X)$ and $h = (g \restriction (B \cap X))^{\psi^{-1}}$.

We may assume that $\vec{x} \subseteq B$ and that $\mathrm{Dom}(\vec{x}) = \mathbb{N}$. Set $\vec{u} = \psi^{-1}(\vec{x})$, and for every $n \in \mathbb{N}$ let $u_n = \hat{u}_n + u_n^{\perp}$, where $\hat{u}_n \in F$ and $u_n^{\perp} \perp F$. Denote $h(\vec{u})$ by $\vec{v}$, and for every $n \in \mathbb{N}$ let $v_n = \hat{v}_n + v_n^{\perp}$, where $\hat{v}_n \in F$ and $v_n^{\perp} \perp F$. Let $s > 0$ be such that $B(0,s) \cap (E - A) \subseteq U, h(U)$. We may assume that $u_n, \hat{u}_n, u_n^{\perp}, v_n, \hat{v}_n, v_n^{\perp} \in B(0,s)$ for every $n \in \mathbb{N}$. In order to apply (a1), we need to show that $\hat{v}_n = \lambda_n \hat{u}_n$, where $\lambda_n > 0$. From Proposition 11.2(a) and the facts that $\vec{x}$ is Γ-evasive, $\beta \in \Gamma$ and ψ is β-bicontinuous, it follows that $\vec{u}$ is Γ-evasive.

Define $\bar{\alpha} = \beta \circ \alpha \circ \beta$. Then h is $\bar{\alpha}$-bicontinuous. This implies that $\vec{v}$ too is Γ-evasive. So $\lim d(u_n, F)/\|u_n\| = 0$ and $\lim d(v_n, F)/\|v_n\| = 0$. We may thus assume that $d(u_n, F) <$

$\|u_n\|/2$ and $d(v_n, F) < \|v_n\|/2$ for every $n \in \mathbb{N}$. It follows that for every n, $\hat{u}_n \neq 0$. Let λ_n be such that $\hat{v}_n = \lambda_n \hat{u}_n$. It is trivial that h^{cl} is $\bar{\alpha}$-bicontinuous, and that $h^{\mathrm{cl}} \restriction (F \cap B(0,s))$ is order preserving, that is, for every $u \in F \cap B(0,s)$, u and $h^{\mathrm{cl}}(u)$ are on the same side of 0. It follows that for every n there is $\mu_n > 0$ such that $h^{\mathrm{cl}}(\hat{u}_n) = \mu_n \hat{u}_n$. Suppose by contradiction that for infinitely many n's, $\lambda_n \leq 0$. Take such an n. Then

$$\|h^{\mathrm{cl}}(u_n) - h^{\mathrm{cl}}(\hat{u}_n)\| \leq \bar{\alpha}(\|u_n - \hat{u}_n\|) = \bar{\alpha}(\|u_n^{\perp}\|).$$

But

$$h^{\mathrm{cl}}(u_n) - h^{\mathrm{cl}}(\hat{u}_n) = v_n - \mu_n \hat{u}_n = v_n^{\perp} + \lambda_n \hat{u}_n - \mu_n \hat{u}_n = v_n^{\perp} - (\mu_n - \lambda_n)\hat{u}_n.$$

So

$$\begin{aligned}\|h^{\mathrm{cl}}(u_n) - h^{\mathrm{cl}}(\hat{u}_n)\| &= \|v_n^{\perp} - (\mu_n - \lambda_n)\hat{u}_n\| \geq (\mu_n - \lambda_n)\|\hat{u}_n\| - \|v_n^{\perp}\| \\ &\geq \mu_n\|\hat{u}_n\| - \|v_n^{\perp}\| = \|h(\hat{u}_n)\| - \|v_n^{\perp}\| \geq \bar{\alpha}^{-1}(\|\hat{u}_n\|) - \|v_n^{\perp}\| \geq \bar{\alpha}^{-1}(\|u_n\|/2) - \bar{\alpha}(\|u_n^{\perp}\|).\end{aligned}$$

Note that $\bar{\alpha}(\|u_n^{\perp}\|) = \bar{\alpha}(\|u_n - \hat{u}_n\|) \geq \|h^{\mathrm{cl}}(u_n) - h^{\mathrm{cl}}(\hat{u}_n)\|$. It follows that

$$\bar{\alpha}^{-1}(\|u_n\|/2) - \bar{\alpha}(\|u_n^{\perp}\|) \leq \bar{\alpha}(\|u_n^{\perp}\|).$$

So $\|u_n\| \leq 2\bar{\alpha} \circ \bar{\alpha}(\|u_n^{\perp}\|) + 2\bar{\alpha}(\|u_n^{\perp}\|) \leq 4\bar{\alpha} \circ \bar{\alpha}(\|u_n^{\perp}\|)$. That is, $\|u_n\| \leq 4\bar{\alpha} \circ \bar{\alpha}(d(u_n, F))$. Since $4\bar{\alpha} \circ \bar{\alpha} \in \Gamma$, and since the above holds for infinitely many n's, $\vec{u}$ is not Γ-evasive. A contradiction. Hence for all but finitely many n's, $\lambda_n > 0$. Recall that $\vec{v} = h(\vec{u})$. So $\vec{v} = \psi^{-1}(g(\vec{x}))$. Obviously, $\vec{x} \approx^{(\alpha,b)} g(\vec{x})$. Hence by (a), $\vec{x} \approxeq^{N \cdot \beta \circ \bar{\alpha}^{\circ 4} \circ \beta} g(\vec{x})$.

(b) Suppose by contradiction that there are infinite $\sigma, \eta \subseteq \mathrm{Dom}(\vec{x})$ and $g \in G$ such that for every $i \in \sigma$, $g(x_i) = y_i$, and for every $i \in \eta$, $g(x_i) = x_i$. Let $h = g^{\psi^{-1}}$. So for some $\gamma \in \Gamma$, h is γ-bicontinuous at 0. Let $Y = E - A$. Then $\vec{u}$ is Γ-evasive with respect to Y and E. Note that for every $i \in \sigma$, $h(u_i) = v_i$, and for every $i \in \eta$, $h(u_i) = u_i$. We abbreviate h^{cl} by h. Define $h(\hat{u}_i) = \mu_i \hat{u}_i$. Assume by contradiction that for infinitely many i's in η, $\mu_i \leq 0$. Since $\vec{u}$ is Γ-evasive, there is n such that for every $i \in \eta^{\geq n}$, $\|u_i^{\perp}\| \leq \frac{1}{4}\|u_i\|$. Let $i \in \eta^{\geq n}$, and assume that $\mu_i \leq 0$. Then

$$\begin{aligned}\gamma(\delta(u_i)) = \gamma(\|u_i^{\perp}\|) = \gamma(\|u_i^{\perp}\|) &= \gamma(\|u_i - \hat{u}_i\|) \geq \|h(u_i) - h(\hat{u}_i)\| \\ &= \|u_i - \mu_i \hat{u}_i\| = \|u_i^{\perp} + \hat{u}_i - \mu_i \hat{u}_i\| \geq (1 - \mu_i)\|\hat{u}_i\| - \|u_i^{\perp}\| \\ &\geq \|\hat{u}_i\| - \|u_i^{\perp}\| \geq \tfrac{3}{4}\|u_i\| - \tfrac{1}{4}\|u_i\| = \tfrac{1}{2}\|u_i\|.\end{aligned}$$

So $\vec{u}$ is not Γ-evasive, a contradiction. It follows that there is i such that $\mu_i > 0$. This implies that h is order preserving at 0. In particular, for every $i \in \sigma$, $\mu_i > 0$. We claim that $\vec{v}$ is Γ-evasive.

This is so, since (i) $\vec{v} = h(\vec{u})$, (ii) $\gamma \in \Gamma$, (iii) h is γ-continuous and (iv) $\vec{u}$ is Γ-evasive. Let n be such that for every $i \in \mathrm{Dom}(\vec{v})^{\geq n}$, $\delta(v_i) \leq \frac{1}{4}\|v_i\|$. Let $i \in \sigma^{\geq n}$. Then

$$\begin{aligned}\gamma^{\circ 2}(\delta(v_i)) \geq \gamma(\delta(u_i)) = \gamma(\|u_i - \hat{u}_i\|) &\geq \|h(u_i) - h(\hat{u}_i)\| = \|v_i - \mu_i \hat{u}_i\| \\ &= \|v_i^{\perp} + (\lambda_i - \mu_i)\hat{u}_i\| \geq |\lambda_i - \mu_i|\,\|\hat{u}_i\| - \|v_i^{\perp}\| \geq |\lambda_i|\,\|\hat{u}_i\| - \|v_i^{\perp}\| \\ &= \|\hat{v}_i\| - \|v_i^{\perp}\| \geq \tfrac{3}{4}\|v_i\| - \tfrac{1}{4}\|v_i\| = \tfrac{1}{2}\|v_i\|.\end{aligned}$$

So $\vec{v} \restriction \sigma^{\geq n}$ is $2 \cdot \gamma \circ \gamma$-abiding. This contradicts the fact that $\vec{v}$ is Γ-evasive.

(c) Let $\langle \psi, A, r \rangle$ be a boundary chart element for b such that ψ is β-bicontinuous. Since $g \in G$ there is $\alpha \in \Gamma$ and $U \in \mathrm{Nbr}^E(b)$ such that $g \restriction (U \cap X)$ is α-bicontinuous. We may assume that $\vec{x} \subseteq \mathrm{Rng}(\psi) \cap U$. Let $h = g^{\psi^{-1}}$ and $\gamma = \beta \circ \alpha \circ \beta$. Then h is γ-bicontinuous. Let $\vec{u} = \psi^{-1}(\vec{x})$ and $\vec{v} = h(\vec{u})$. So $\vec{v} = \psi^{-1}(g(\vec{x}))$. Also, let $u_i = \hat{u}_i + u_i^{\perp}$ and $v_i = \hat{v}_i + v_i^{\perp}$, where $\hat{u}_i, \hat{v}_i \in F$ and $u_i^{\perp}, v_i^{\perp} \perp F$. Since $\vec{u}$ is Γ-evasive, and $\vec{v} = h(\vec{u})$, $\vec{v}$ is Γ-evasive. We may thus assume that for every $i \in \mathrm{Dom}(\vec{u})$, $\|u_i^{\perp}\| \le \|u_i\|/4$ and $\|v_i^{\perp}\| \le \|v_i\|/4$. Let λ_i be such that $\hat{v}_i = \lambda \hat{u}_i$. Suppose by contradiction that for infinitely many i's, $\lambda_i \ge 0$. We abbreviate h^{cl} by h. Let μ_i be such that $h(\hat{u}_i) = \mu_i \hat{u}_i$. Since g is order reversing at b, h is order reversing at 0. So $\mu_i < 0$. Let i be such that $\lambda_i \ge 0$. Then

$$\begin{aligned}\gamma(\|u_i^{\perp}\|) &\ge \|h(u_i) - h(\hat{u}_i)\| = \|v_i^{\perp} + \lambda_i \hat{u}_i - \mu_i \hat{u}_i\| \ge (\lambda_i - \mu_i)\|\hat{u}_i\| - \|v_i^{\perp}\| \\ &\ge |\mu_i|\,\|\hat{u}_i\| - \|v_i^{\perp}\| = \|\hat{v}_i\| - \|v_i^{\perp}\| \ge \|v_i\|/2.\end{aligned}$$

But $\|u_i^{\perp}\| = \delta(u_i) \le \gamma(\delta(v_i))$. So $2 \cdot \gamma \circ \gamma(\delta(v_i)) \ge \|v_i\|$. That is, $\vec{v}$ is not Γ-evasive, a contradiction. It follows that for all but finitely many i's, $\lambda_i < 0$. By (b), $\vec{x} \not\approx^{G} g(\vec{x})$.

(d) Let σ, η be infinite subsets of $\mathrm{Dom}(\vec{x})$. Either (i) there is an infinite $\varrho \subseteq \sigma$ and $\gamma \in \Gamma$ such that $\vec{x} \restriction \varrho$ is γ-abiding; or (ii) there is an infinite $\varrho \subseteq \sigma$ such that $\vec{x} \restriction \varrho$ is Γ-evasive.

Suppose that case (i) happens. To get an $f \in G$ such that $\{i \in \varrho \mid f(x_i) = g(x_i)\}$ and $\{i \in \eta \mid f(x_i) = x_i\}$ are infinite, follow the construction in Proposition 11.3(a). However, Lemma 9.13 which was used in 11.3(a) is replaced here by Lemma 10.3. In case (ii), follow the proof of (a2) in this proposition. ∎

Recall that we deal with the setting where we have a normed space E and an open subset $X \subseteq E$. In this setting, when we write $\mathrm{cl}(A)$ we mean $\mathrm{cl}^{E}(A)$. If we wish to denote the closure of A with respect to other sets, e.g. the closure of A with respect to X, then we write $\mathrm{cl}^{X}(A)$.

PROPOSITION 11.6. *For a topological space X and a subgroup $G \le H(X)$, we define the property $P_{\mathrm{cmpct}}(\vec{x})$ of sequences $\vec{x}$ in X as follows.*

$P_{\mathrm{cmpct}}(\vec{x}) \equiv$ *For every infinite $\sigma \subseteq \mathrm{Dom}(\vec{x})$ and a sequence $\{U_i \mid i \in \sigma\} \in \prod\{\mathrm{Nbr}(x_i) \mid i \in \sigma\}$ consisting of pairwise disjoint sets, there is a sequence $\{g_i \mid i \in \sigma\} \in \prod\{G\lfloor U_i \rfloor \mid i \in \sigma\}$ such that $\circ\{g_i \mid i \in \sigma\} \notin G$.*

Let E be a normed space and $X \subseteq E$ be open. Let Γ be a countably generated modulus of continuity and $G \le \mathrm{EXT}(X)$ be Γ-appropriate. (*See Definition* 8.6(c).) *Let $\vec{x}$ be a* 1-1 *sequence in X. Then* $\mathrm{cl}(\mathrm{Rng}(\vec{x}))$ *is compact iff $P_{\mathrm{cmpct}}(\vec{x})$ holds.*

Proof. Suppose first that $\mathrm{cl}(\mathrm{Rng}(\vec{x}))$ is not compact. Then there is an infinite $\sigma \subseteq \mathrm{Dom}(\vec{x})$ such that either $\{x_i \mid i \in \sigma\}$ is spaced, or $\{x_i \mid i \in \sigma\}$ is a nonconvergent Cauchy sequence. For every $i \in \sigma$ let $r_i = \frac{1}{3}\inf\{\|x_j - x_i\| \mid j \in \sigma - \{i\}\}$ and $U_i = B^{X}(x_i, r_i)$. Hence $d(U_i, U_j) \ge r_i/3$ for any $i \ne j$ in σ. It is easily seen that $\{U_i \mid i \in \mathbb{N}\}$ is $\mathrm{cl}(X)$-discrete. Let $\{g_i \mid i \in \sigma\} \in \prod\{G\lfloor U_i \rfloor \mid i \in \sigma\}$. So $\{g_i \mid i \in \sigma\}$ is $\mathrm{cl}(X)$-discrete. Since G is Γ-appropriate, $\circ\{g_i \mid i \in \sigma\} \in G$. So $\neg P_{\mathrm{cmpct}}(\vec{x})$ holds.

Suppose that $\vec{x}$ is 1-1 and that $\mathrm{cl}(\mathrm{Rng}(\vec{x}))$ is compact. Let $\{\alpha_i \mid i \in \mathbb{N}\} \subseteq \Gamma$ be a generating sequence for Γ. That is, for every $\alpha \in \Gamma$ there is $i \in \mathbb{N}$ such that $\alpha \preceq \alpha_i$. We also assume that for every $i \in \mathbb{N}$, $\{j \mid \alpha_j = \alpha_i\}$ is infinite. Let $\sigma \subseteq \mathrm{Dom}(\vec{x})$ be infinite, and for every $i \in \sigma$ let $U_i \in \mathrm{Nbr}^{X}(x_i)$. Assume that for any $i \ne j$, $U_i \cap U_j = \emptyset$. Since $\mathrm{cl}(\mathrm{Rng}(\vec{x}))$ is compact, $\{x_i \mid i \in \sigma\}$ contains a 1-1 convergent subsequence $\{x_{i_n} \mid$

$n \in \mathbb{N}\}$. Define $y_n = x_{i_n}$ and $V_n = U_{i_n} \cap B(y_n, \frac{1}{n+1})$. For every n let $g_{i_n} \in G\lfloor V_n \rfloor$ be such that $g_{i_n} \restriction V_n$ is not α_n-continuous. It is easy to see that such a g_{i_n} exists. For $i \in \sigma - \{i_n \mid n \in \mathbb{N}\}$ let $g_i = \mathrm{Id}$. Let $y = \lim_n y_n$ and $g = \circ\{g_i \mid i \in \sigma\}$. Then there is no $\alpha \in \Gamma$ and $U \in \mathrm{Nbr}(y)$ such that $g\restriction(U \cap X)$ is α-continuous. We justify this claim. Let $\alpha \in \Gamma$. Then for some $i \in \mathbb{N}$, $\alpha \preceq \alpha_i$. Let $r > 0$ be such that $\alpha\restriction[0,r) \leq \alpha_i\restriction[0,r)$. There is n such that $\alpha_{i_n} = \alpha_i$, $\mathrm{diam}(V_n) < r$ and $V_n \subseteq U$. There are $u, v \in V_n$ such that $\|g_{i_n}(u) - g_{i_n}(v)\| > \alpha_i(\|u - v\|)$. Since $\|u - v\| < r$, we have $\alpha_i(\|u - v\|) \geq \alpha(\|u - v\|)$. So $\|g_{i_n}(u) - g_{i_n}(v)\| > \alpha(\|u - v\|)$. That is, $\|g(u) - g(v)\| > \alpha(\|u - v\|)$. Hence $g\restriction(U \cap X)$ is not α-continuous. It follows that $g \notin G$. So $P_{\mathrm{cmpct}}(\vec{x})$ holds. ∎

Explanation. For a topological space $\langle X, \tau^X \rangle$ and $G \leq H(X)$ let $\mathrm{Ap} : G \times X \to X$ be the application function, that is, $\mathrm{Ap}(g, x) = g(x)$ and let $M(X, G)$ be the structure $\langle X, \tau^X, G; \in, \circ, \mathrm{Ap}\rangle$. Note that $P_{\mathrm{cmpct}}(\vec{x})$ is a property of $\vec{x}$ which can be expressed in $M(X, G)$. Hence if $\vec{x} \subseteq X$, $P_{\mathrm{cmpct}}(\vec{x})$ holds and $\psi : M(X, G) \cong M(Y, H)$, then $P_{\mathrm{cmpct}}(\psi(\vec{x}))$ holds. So in the case that X is an open subset of a normed space E and G is Γ-appropriate, and a similar fact holds for Y, then the property "$\mathrm{cl}(\mathrm{Rng}(\vec{x}))$ is compact" is preserved under ψ. In what follows we shall define additional properties of $\vec{x}$ which are expressible in $M(X, G)$. So they too are preserved under isomorphisms between $M(X, G)$ and $M(Y, H)$.

Definition 11.7. Let X be a topological space, $G \leq H(X)$ and $\vec{x}$ be a sequence in X.

(a) Let $P_{\mathrm{prerep}}(\vec{x})$ be the following property of $\vec{x}$:

- (i) $\mathrm{Dom}(\vec{x}) = \mathbb{N}$ and $\vec{x}$ is 1-1,
- (ii) no subsequence of $\vec{x}$ is convergent in X,
- (iii) $P_{\mathrm{cmpct}}(\vec{x})$ holds.

A sequence $\vec{x}$ which fulfills P_{prerep} is called a *point pre-representative.*

(b) Let $P_{\mathrm{cnvrg}}(\vec{x})$ and $P_{\mathrm{pnt}}(\vec{x})$ be the following properties:

$$P_{\mathrm{cnvrg}}(\vec{x}) \equiv \text{ For every infinite } \sigma \subseteq \mathrm{Dom}(\vec{x}) \text{ and } g \in G,$$
$$\text{if } \vec{x}\restriction\sigma \approxeq^G g(\vec{x})\restriction\sigma, \text{ then } \vec{x} \approxeq^G g(\vec{x}).$$

$$P_{\mathrm{pnt}}(\vec{x}) \equiv P_{\mathrm{prerep}}(\vec{x}) \wedge P_{\mathrm{cnvrg}}(\vec{x}). \ \square$$

Lemma 11.8. *Let Γ be a countably generated modulus of continuity. Suppose that E is a normed space, $X \subseteq E$ is open, X is locally Γ-LIN-bordered, and $G \leq \mathrm{EXT}(X)$ is Γ-appropriate. Let $\vec{x}$ be a point pre-representative in X. Then $P_{\mathrm{cnvrg}}(\vec{x})$ holds iff $\vec{x}$ is convergent, and* (i), (ii), (iii), (iv) *or* (v) *below happen. Set $b = \lim \vec{x}$.*

- (i) *For some $\beta \in \Gamma$, X is β-SLIN-bordered at b.*
- (ii) *For some $\beta \in \Gamma$, X is β-LIN-bordered at b, X is two-sided at b, and* $\mathrm{bd}(X)$ *is not* 1*-dimensional at b.*
- (iii) $\mathrm{bd}(X)$ *is* 1*-dimensional and G-order-reversible at b, and for some $\alpha \in \Gamma$, $\vec{x}$ is α-abiding.*
- (iv) $\mathrm{bd}(X)$ *is* 1*-dimensional and G-order-reversible at b, and $\vec{x}$ is Γ-evasive.*
- (v) $\mathrm{bd}(X)$ *is* 1*-dimensional and G-order-irreversible at b.*

Proof. We shall use the following trivial facts.

Claim 1. *If* $\vec{y} \precapprox^{A} \vec{z}$, *then for every infinite* $\sigma \subseteq \mathrm{Dom}(\vec{y})$, $\vec{y}{\restriction}\sigma \precapprox^{A} \vec{z}{\restriction}\sigma$.

Claim 2. *Suppose that* $\vec{y}$ *is a sequence in* X *converging to a point in* $\mathrm{bd}(X)$. *Assume further that* $\mathrm{bd}(X)$ *is* 1*-dimensional at* $\lim \vec{y}$. *Then either* $\vec{y}$ *is* Γ*-evasive, or for some* $\alpha \in \Gamma$, $\vec{y}$ *has an* α*-abiding subsequence.*

Claim 3. *Suppose that* $\vec{y}$ *is a sequence in* X *converging to a point in* $\mathrm{bd}(X)$. *Assume further that* $\mathrm{bd}(X)$ *is two-sided at* $\lim \vec{y}$. *Let* $g \in \mathrm{EXT}(X)$ *be such that* $g^{\mathrm{cl}}(\lim \vec{y}) = \lim \vec{y}$, *and suppose that* g *is side reversing. Then* $g(\vec{y}) \not\precapprox^{\mathrm{EXT}(X)} \vec{y}$.

Proof. The claim follows trivially from Proposition 11.2(c).

Claim 4. *Let* $\vec{y}$ *be a sequence in* X *such that* $\vec{y}$ *is convergent in* $\mathrm{cl}(X)$. *Suppose that* $g \in \mathrm{EXT}(X)$ *and* $g^{\mathrm{cl}}(\lim \vec{y}) \neq \lim \vec{y}$. *Then* $g(\vec{y}) \not\precapprox^{\mathrm{EXT}(X)} \vec{y}$.

The following fact does require a proof.

Claim 5. *Let* $\vec{x}$ *be a point pre-representative. If* $P_{\mathrm{cnvrg}}(\vec{x})$ *holds, then* $\vec{x}$ *is convergent.*

Proof. Suppose that $\vec{x}$ is not convergent. Let $\vec{y}, \vec{z}$ be convergent subsequences of $\vec{x}$ such that $\lim \vec{y} \neq \lim \vec{z}$. Assume further that $(*)$ if $\mathrm{bd}(X)$ is 1-dimensional at $\lim \vec{y}$, then either $\vec{y}$ is Γ-evasive, or for some $\alpha \in \Gamma$, $\vec{y}$ is α-abiding. Since X is locally Γ-LIN-bordered, there is $g \in G$ such that

(1) $g^{\mathrm{cl}}(\lim \vec{y}) = \lim \vec{y}$ and $g^{\mathrm{cl}}(\lim \vec{z}) \neq \lim \vec{z}$,
(2) if X is two-sided at $\lim \vec{y}$, then g is side preserving,
(3) if $\mathrm{bd}(X)$ is 1-dimensional at $\lim \vec{y}$, then g is order preserving.

By Propositions 11.3(a2), 11.4(b) and 11.5(a2) and by $(*)$, $g(\vec{y}) \precapprox^{G} \vec{y}$. By Claim 4, $g(\vec{z}) \not\precapprox^{G} \vec{z}$, and by Claim 1, $g(\vec{x}) \not\precapprox^{G} \vec{x}$. Hence $\neg P_{\mathrm{cnvrg}}(\vec{x})$ holds. This proves Claim 5.

Suppose that $\vec{x}$ satisfies clause (i) in the statement of the lemma. We show that $P_{\mathrm{cnvrg}}(\vec{x})$ holds. Let $g \in G$. If $g^{\mathrm{cl}}(b) \neq b$, then by Claim 4, $g(\vec{x}\,') \not\precapprox^{G} \vec{x}\,'$, for every subsequence of $\vec{x}\,'$ of $\vec{x}$. If $g^{\mathrm{cl}}(b) = b$, then by Proposition 11.3(a2), $g(\vec{x}) \precapprox^{G} \vec{x}$. So $P_{\mathrm{cnvrg}}(\vec{x})$ holds.

Suppose that $\vec{x}$ satisfies clause (ii) in the statement of the lemma. Let $g \in G$. If $g^{\mathrm{cl}}(b) \neq b$, then by Claim 4, $g(\vec{x}\,') \not\precapprox^{G} \vec{x}\,'$ for every subsequence of $\vec{x}\,'$ of $\vec{x}$. Suppose that $g^{\mathrm{cl}}(b) = b$. If g is side reversing, then by Claim 3, $g(\vec{x}\,') \not\precapprox^{G} \vec{x}\,'$ for every subsequence of $\vec{x}\,'$ of $\vec{x}$. If g is side preserving, then by Proposition 11.3(a2), $g(\vec{x}) \precapprox^{G} \vec{x}$. So $P_{\mathrm{cnvrg}}(\vec{x})$ holds.

Suppose that $\vec{x}$ satisfies clause (iii) above. Let $g \in G$. The case $g^{\mathrm{cl}}(b) \neq b$, is treated as in (i) and (ii). Suppose that $g^{\mathrm{cl}}(b) = b$. If X is two-sided at x and g is side reversing, then by Claim 3, $g(\vec{x}\,') \not\precapprox^{G} \vec{x}\,'$, for every subsequence of $\vec{x}\,'$ of $\vec{x}$. Suppose that either X is not two-sided at b, or X is two-sided at b and g is side preserving. Then by Proposition 11.4(b), $g(\vec{x}) \precapprox^{G} \vec{x}$. So $P_{\mathrm{cnvrg}}(\vec{x})$ holds.

Suppose that $\vec{x}$ satisfies clause (iv). As above, we may assume that $g^{\mathrm{cl}}(b) = b$, and that if X is two-sided at b, then g is side preserving. If g is order reversing at b, then by Proposition 11.5(c), $g(\vec{x}\,') \not\precapprox^{G} \vec{x}\,'$, for every subsequence of $\vec{x}\,'$ of $\vec{x}$. If g is order preserving at b, then by Proposition 11.5(a2), $g(\vec{x}) \precapprox^{G} \vec{x}$. So $P_{\mathrm{cnvrg}}(\vec{x})$ holds.

Suppose that $\vec{x}$ satisfies clause (v). We may assume that $g^{\mathrm{cl}}(b) = b$, and that if X is two-sided at b, then g is side preserving. Since $\mathrm{bd}(X)$ is G-order-irreversible at b, g must be order preserving at b. Then by Proposition 11.5(d), $g(\vec{x}) \precapprox^G \vec{x}$. So $P_{\mathrm{cnvrg}}(\vec{x})$ holds.

We have shown that if $\vec{x}$ is point pre-representative, $\vec{x}$ is convergent, and $\vec{x}$ satisfies one of the clauses (i)–(v), then $P_{\mathrm{cnvrg}}(\vec{x})$ holds.

Let $\vec{x}$ be a point pre-representative, and suppose that $P_{\mathrm{cnvrg}}(\vec{x})$ holds. By Claim 5, $\vec{x}$ is convergent. Suppose by contradiction that $\vec{x}$ does not satisfy any of the clauses (i)–(v). Let $b = \lim \vec{x}$. Then $\mathrm{bd}(X)$ is 1-dimensional and G-order-reversible at b, and (1) $\vec{x}$ is not Γ-evasive; (2) there is no $\alpha \in \Gamma$ such that $\vec{x}$ is α-abiding. There is $\gamma \in \Gamma$ and a subsequence $\vec{y}$ of $\vec{x}$ such that $\vec{y}$ is γ-abiding. Since Γ is countably generated, there is a subsequence $\vec{z}$ of $\vec{x}$ such that $\vec{z}$ is Γ-evasive. Let $g \in G$ be such that g is order reversing at b, and if X is two-sided at x, then g is side preserving. By Proposition 11.5(c), $g(\vec{z}) \not\precapprox^G \vec{z}$. So $g(\vec{x}) \not\precapprox^G \vec{x}$. By Proposition 11.4(b), $g(\vec{y}) \precapprox^G \vec{y}$. So $\neg P_{\mathrm{cnvrg}}(\vec{x})$ holds. A contradiction. ■

We represent points in $\mathrm{bd}(X)$ by sequences $\vec{x}$ in X which satisfy $P_{\mathrm{pnt}}(\vec{x})$. Such sequences are called *point representatives*. By the above proposition, for every $x \in \mathrm{bd}(X)$, there is $\vec{x}$ such that $\lim \vec{x} = x$ and $P_{\mathrm{pnt}}(\vec{x})$ holds. So every point of $\mathrm{bd}(X)$ is represented.

We shall find a property $\varphi_{\mathrm{pnteq}}(\vec{x}, \vec{y})$ which for point representatives $\vec{x}, \vec{y}$ expresses the fact that $\lim \vec{x} = \lim \vec{y}$. Let $\vec{x}$ be a point representative. The *weak stabilizer* of $\vec{x}$ is defined as follows:

$$\mathrm{wstab}(\vec{x}) = \{g \in G \mid g(\vec{x}) \precapprox^G \vec{x}\}.$$

Define

$$P_{\mathrm{pnteq}}(\vec{x}, \vec{y}) \equiv (\mathrm{wstab}(\vec{x}) \subseteq \mathrm{wstab}(\vec{y})) \vee (\mathrm{wstab}(\vec{y}) \subseteq \mathrm{wstab}(\vec{x})).$$

For an open subset U of X define $\mathrm{opcl}(U) = U \cup (\mathrm{bd}(X) - \mathrm{acc}^{\mathrm{cl}(X)}(X - U))$. Then $\mathrm{opcl}(U)$ is open in $\mathrm{cl}(X)$. Also, if $V \in \mathrm{Ro}(\mathrm{cl}(X))$, then $V = \mathrm{opcl}(V \cap X)$. Let $\mathcal{B} = \{\mathrm{opcl}(U) \mid U \text{ is open in } X\}$. Hence $\mathrm{Ro}(\mathrm{cl}(X)) \subseteq \mathcal{B}$, and so $\mathcal{B}$ is an open base for $\mathrm{cl}(X)$. Every open subset U of X will represent $\mathrm{opcl}(U)$. So the set of open subsets of $\mathrm{cl}(X)$ which are represented forms an open base for $\mathrm{cl}(X)$.

We next define property $P_{\mathrm{blng}}(\vec{x}, U)$. For a point representative $\vec{x}$ and an open subset U of X, $P_{\mathrm{blng}}(\vec{x}, U)$ will express the fact that $\lim \vec{x} \in \mathrm{opcl}(U)$. Let

$P_{\mathrm{blng}}(\vec{x}, U) \equiv$ For every sequence $\vec{y}$: if $P_{\mathrm{pnt}}(\vec{y})$ and $P_{\mathrm{pnteq}}(\vec{x}, \vec{y})$, then $\mathrm{Rng}(\vec{y})U$ is finite.

PROPOSITION 11.9. *Let Γ be a countably generated modulus of continuity. Suppose that E is a normed space, $X \subseteq E$ is open, and X is locally Γ-LIN-bordered. Let G be a Γ-appropriate subgroup of* $\mathrm{EXT}(X)$.

(a) *Suppose that $\vec{x}, \vec{y}$ are point representatives. Then* $\lim \vec{x} = \lim \vec{y}$ *iff* $P_{\mathrm{pnteq}}(\vec{x}, \vec{y})$ *holds.*

(b) *Let $\vec{x}$ be a point representative, and $U \subseteq X$ be open. Then* $\lim \vec{x} \in \mathrm{opcl}(U)$ *iff* $P_{\mathrm{blng}}(\vec{x}, U)$ *holds.*

Proof. (a) Let $\vec{x}, \vec{y}$ be point representatives. If $\lim \vec{x} \neq \lim \vec{y}$, then there is $g \in G$ such that g is the identity on some neighborhood of $\lim \vec{x}$ and $g(\lim \vec{y}) \neq \lim \vec{y}$. So $g \in$

$\mathrm{wstab}(\vec{x}) - \mathrm{wstab}(\vec{y})$. Similarly, $\mathrm{wstab}(\vec{y}) \not\subseteq \mathrm{wstab}(\vec{x})$. So $\mathrm{wstab}(\vec{x})$ and $\mathrm{wstab}(\vec{y})$ are incomparable.

Suppose that $\lim \vec{x} = \lim \vec{y}$. Define $b = \lim \vec{x}$. If for some $\alpha \in \Gamma$, $\mathrm{bd}(X)$ is α-SLIN-bordered at b, then by Proposition 11.3(a2), $\mathrm{wstab}(\vec{x}) = \mathrm{wstab}(\vec{y}) = \{g \in G \mid g(b) = b\}$.

Suppose that X is two-sided at b and $\mathrm{bd}(X)$ is not 1-dimensional at b. Then $\mathrm{wstab}(\vec{x}) = \mathrm{wstab}(\vec{y}) = \{g \in G \mid g(b) = b$ and g is side preserving at $b\}$. This follows from Proposition 11.3(a2) and (b).

Suppose that $\mathrm{bd}(X)$ is 1-dimensional at b. If $\mathrm{bd}(X)$ is G-order-irreversible at b, and X is not two-sided at b, then $\mathrm{wstab}(\vec{x}) = \mathrm{wstab}(\vec{y}) = \{g \in G \mid g(b) = b\}$. This follows from Proposition 11.5(d). Next assume that $\mathrm{bd}(X)$ is G-order-irreversible at b, and X is two-sided at b. Then $\mathrm{wstab}(\vec{x}) = \mathrm{wstab}(\vec{y}) = \{g \in G \mid g(b) = b$ and g is side preserving at $b\}$. This follows from Propositions 11.5(d) and 11.3(b).

Suppose that G-order-reversible at b. Then by Lemma 11.8, $\vec{x}$ is Γ-evasive, or there is $\alpha \in \Gamma$ such that $\vec{x}$ is α-abiding. The same holds for $\vec{y}$. If both $\vec{x}$ and $\vec{y}$ are evasive or both are abiding, then $\mathrm{wstab}(\vec{x}) = \mathrm{wstab}(\vec{y})$. This follows from Propositions 11.4(b), 11.5(a2), 11.5(c) and 11.3(b). Suppose that $\vec{x}$ is evasive and $\vec{y}$ is abiding. Then $\mathrm{wstab}(\vec{x})$ consists of all $g \in G$ such $g(b) = b$, g is order preserving at b, and if X is two-sided at b, then g is side preserving at b. $\mathrm{wstab}(\vec{y})$ consists of all $g \in G$ such that $g(b) = b$, and if X is two-sided at b, then g is side preserving at b. So $\mathrm{wstab}(\vec{x}) \subseteq \mathrm{wstab}(\vec{y})$. We have shown that if $\lim \vec{x} = \lim \vec{y}$, then $\mathrm{wstab}(\vec{x})$ and $\mathrm{wstab}(\vec{y})$ are comparable.

(b) Let $\vec{x}$ be a point representative, $U \subseteq X$ be open in X and $b = \lim \vec{x}$. If $b \in \mathrm{opcl}(U)$, then for every sequence $\vec{y}$ in X such that $\lim \vec{y} = b$ there is n such that $\mathrm{Rng}(\vec{y}^{\geq n}) \subseteq U$. So $P_{\mathrm{blng}}(\vec{x}, U)$ holds. If $b \notin \mathrm{opcl}(U)$, then there is a sequence $\vec{y}$ in X which converges to b and such that $\mathrm{Rng}(\vec{y})$ is disjoint from U. There is a subsequence $\vec{z}$ of $\vec{y}$ such that $P_{\mathrm{pnt}}(\vec{z})$ holds. So $P_{\mathrm{pnteq}}(\vec{x}, \vec{z})$ holds. Hence $\neg P_{\mathrm{blng}}(\vec{x}, U)$ holds. ∎

Proof of Theorem 8.8. Part (a) of 8.8 is a special case of (b), so we prove (b). Let X, Y, G, H and τ be as in (b). Then τ induces an isomorphism $\tilde{\tau}$ between $M(X, G)$ and $M(Y, H)$. Clearly, properties $P_{\mathrm{pnt}}(\vec{x})$, $P_{\mathrm{pnteq}}(\vec{x})$ and $P_{\mathrm{blng}}(\vec{x})$ are preserved by $\tilde{\tau}$. This implies the bi-extendability of τ. ∎

12. The complete Γ-bicontinuity of the inducing homeomorphism

In the previous chapter we have shown that if $(H_{\Gamma}^{\mathrm{CMP.LC}}(X))^{\tau} = H_{\Delta}^{\mathrm{CMP.LC}}(Y)$, then $\tau \in \mathrm{EXT}^{\pm}(X, Y)$. Further, by Theorem 3.27, τ is locally Γ-bicontinuous. In this chapter we finally conclude that τ is completely locally Γ-bicontinuous. However, at this point we can only show this for principal Γ's.

12.1. Γ-continuity in directions parallel to the boundary of X

Definition 12.1. (a) Let S be a set and $\mathcal{P}$ be a partition of S, that is, $\mathcal{P}$ is a pairwise disjoint family whose union is S. Denote S by $S_{\mathcal{P}}$. For $T \subseteq S$ let $\mathcal{P} \restriction T := \{P \cap T \mid P \in \mathcal{P}\}$. Let $a \sim^{\mathcal{P}} b$ mean that there is $P \in \mathcal{P}$ such that $a, b \in P$. If X is a topological space, and $S \subseteq X$ is an open set, then $\mathcal{P}$ is called an *open sum partition with respect to X*.

In (b)–(d) assume that $\langle X, d\rangle, \langle Y, e\rangle$ are metric spaces, $\tau : X \cong Y$, $\alpha \in \mathrm{MC}$ and $\Gamma \subseteq \mathrm{MC}$. Let $\mathcal{P}$ be an open sum partition with respect to X and $S = S_{\mathcal{P}}$.

(b) Call τ an *$\langle \alpha, \mathcal{P}\rangle$-continuous* function if for every $P \in \mathcal{P}$ and $x_1, x_2 \in P$, $e(\tau(x_1), \tau(x_2)) \le \alpha(d(x_1, x_2))$, and call τ an *$\langle \alpha, \mathcal{P}\rangle$-inversely-continuous* if for every $P \in \mathcal{P}$ and $x_1, x_2 \in P$, $d(x_1, x_2) \le \alpha(e(\tau(x_1), \tau(x_2)))$. We say that τ is *$\langle \alpha, \mathcal{P}\rangle$-bicontinuous* if for every $P \in \mathcal{P}$ and $x_1, x_2 \in P$, $e(\tau(x_1), \tau(x_2)) \le \alpha(d(x_1, x_2))$ and $d(x_1, x_2) \le \alpha(e(\tau(x_1), \tau(x_2)))$.

(c) We say that τ is *$\langle \alpha, \mathcal{P}\rangle$-continuous at x* if there is $T \in \mathrm{Nbr}(x)$ such that $T \subseteq S$ and τ is $\langle \alpha, \mathcal{P} \restriction T\rangle$-continuous, and τ is said to be *$\langle \Gamma, \mathcal{P}\rangle$-continuous at x* if there is $\alpha \in \Gamma$ such that τ is $\langle \alpha, \mathcal{P}\rangle$-continuous at x. The notions of $\langle \alpha, \mathcal{P}\rangle$-inverse-continuity at x, $\langle \alpha, \mathcal{P}\rangle$-bicontinuity at x, $\langle \Gamma, \mathcal{P}\rangle$-inverse-continuity at x and $\langle \Gamma, \mathcal{P}\rangle$-bicontinuity at x are defined analogously.

(d) Call τ a *locally $\langle \Gamma, \mathcal{P}\rangle$-continuous* function if for every $x \in S$, τ is $\langle \Gamma, \mathcal{P}\rangle$-continuous at x. The notions of local $\langle \Gamma, \mathcal{P}\rangle$-inverse-continuity and local $\langle \Gamma, \mathcal{P}\rangle$-bicontinuity are defined analogously. $\square$

The partitions $\mathcal{P}$ that will be used here are of the following form. Let F be a closed linear subspace of E. Then $\mathcal{P}$ is the partition of E into the cosets of F.

The next goal is to show that if $(H_{\Gamma}^{\mathrm{CMP.LC}}(X))^{\tau} \subseteq H_{\Gamma}^{\mathrm{CMP.LC}}(Y)$, then for every $x \in \mathrm{bd}(X)$ there is $\alpha \in \Gamma$ and a neighborhood of the identity in the group of translations parallel to the boundary of X such that for every h in this neighborhood, h^{τ} is α-bicontinuous at $\tau^{\mathrm{cl}}(x)$.

Recall that the notion of decayability was defined in Definition 3.1(c). We shall use it now again for the following situation. Let $\mathrm{BCD}^{E}(A, r)$ be a linear boundary chart domain, $X = \mathrm{cl}^{B(0,r)}(\mathrm{BCD}^{E}(A, r))$, $H = \{\mathrm{tr}_v \mid v \in \mathrm{bd}^{E}(A)\}$ and λ be the natural action of H on X. Then λ is decayable.

When dealing with partial actions, it is often the case that we wish to perform a composition $g \circ f$, where $\mathrm{Rng}(f) \not\subseteq \mathrm{Dom}(g)$. Such a composition is considered to be legal. The domain of the resulting function is $f^{-1}(\mathrm{Rng}(f) \cap \mathrm{Dom}(g))$.

PROPOSITION 12.2. (a) *Suppose that* $\mathrm{BCD}^E(A, r')$ *is a linear boundary chart domain and* $L = \mathrm{bd}^E(A)$. *So* L *is a closed subspace of* E. *Let* $L' = L \cap B(0, r')$. *So* $L' = \mathrm{bd}^{B(0,r')}(\mathrm{BCD}^E(A, r'))$. *Let* $X = \mathrm{BCD}^E(A, r') \cup L'$ *and* $H = \{\mathrm{tr}_v^E \mid v \in L\}$. *We equip* H *with the norm topology of* L. *Let* λ *be defined as follows*:

$$\mathrm{Dom}(\lambda) = \{\langle h, z\rangle \mid h \in H \textit{ and } z, h(z) \in X\} \quad \textit{and} \quad \lambda(h, z) = h(z).$$

Then λ *is a partial action of* H *on* X.

(b) *Let* $\mathrm{BCD}^E(A, r')$ *etc. be as in* (a) *and* $\alpha(t) = 2t$. *Then* λ *is* α*-decayable in* X.

(c) *Let* $\mathrm{BCD}^E(A, r')$ *etc. be as in* (a). *Then for every* $x \in L'$, x *is a* λ*-limit-point.*

Proof. (a) This is trivial.

(b) It suffices to check that λ is α-decayable at 0. We take r_0 to be r'. For $r \in (0, r')$ we take $V = V_{0,r}$ to be $\{\mathrm{tr}_v^E \mid v \in B^L(0, r/4\}$. So indeed $V \times B(0, ar) \subseteq \mathrm{Dom}(\lambda)$. (Recall that $a = 1/2$.) It thus suffices to show that for every normed space E, $r > 0$ and $v \in B(0, r/4)$ there is $g \in H(E)$ such that

(i) $\mathrm{supp}(g) \subseteq B(0, r)$,

(ii) for every $x \in E$, $g(x) - x \in \mathrm{span}(\{v\})$,

(iii) $g{\restriction}B(0, r/2) = \mathrm{tr}_v{\restriction}B(0, r/2)$,

(iv) g is 2-bilipschitz.

Let $k : [0, \infty) \to [0, \infty)$ be the piecewise linear function such that $k(t) = 1$ for $t \in [0, r/2]$, $k(t) = 0$ for $t \geq r$, and k is linear in $[r/2, r]$. So $(k{\restriction}[r/2, r])(t) = 2 - 2t/r$. Let

$$g(x) = x + k(\|x\|) \cdot v.$$

It is trivial that (i)–(iii) hold. We check that (iv) holds. Let $x, y \in E$. If $\|x\|, \|y\| \geq r$ or $\|x\|, \|y\| \leq r/2$, then $\|g(x) - g(y)\| = \|x - y\|$. Let $u = g(x)$ and $w = g(y)$. Assume first that $\|x\|, \|y\| \in [r/2, r]$. Then $u - w = (x - y) - \frac{2}{r}(\|x\| - \|y\|) \cdot v$. So

$$\|u - w\| \leq \|x - y\| + \frac{2}{r}\|x - y\| \cdot \|v\| < \left(1 + \frac{2}{r} \cdot \frac{r}{4}\right)\|x - y\| = \frac{3}{2}\|x - y\|$$

and

$$\|u - w\| \geq \|x - y\| - \frac{2}{r}\|x - y\| \cdot \|v\| > \left(1 - \frac{2}{r} \cdot \frac{r}{4}\right)\|x - y\| = \frac{1}{2}\|x - y\|.$$

That is, $\|x - y\| < 2\|u - w\|$.

Suppose that $r/2 < \|x\| \leq r$ and $\|y\| < r/2$. Let $z \in [x, y]$ be such that $\|z\| = r/2$. Let $f \in \{g, g^{-1}\}$. Then

$$\begin{aligned}\|f(x) - f(y)\| &\leq \|f(x) - f(z)\| + \|f(z) - f(y)\| \leq 2\|x - z\| + \|z - y\| \\ &< 2(\|x - z\| + \|z - y\|) = 2\|x - y\|.\end{aligned}$$

The case that $r/2 < \|x\| \leq r$ and $\|y\| > r$ is dealt with in a similar way. The case that $\|x\| < r/2$ and $\|y\| > r$ too is dealt with in a similar way.

(c) It is trivial that every $x \in X$, and in particular every $x \in L'$, is a λ-limit-point. ∎

Definition 12.3. Let $\langle X, d\rangle$ be a metric space, $\mathcal{P}$ be an open sum partition with $S = S_{\mathcal{P}}$, H be a topological group and λ be a partial action of H on X. Denote the unit of H by e_H, and for $g \in H$ set $\hat{g} = g_\lambda$.

(a) Let $x \in S$. We say that $\langle H, \lambda\rangle$ is *$\mathcal{P}$-translation-like at x* if for every $M \in \mathrm{Nbr}(e_H)$ and $U \in \mathrm{Nbr}(x)$ there are:

(i) $N \in \mathrm{Nbr}(e_H)$ such that $N \subseteq M$,
(ii) $T, B \in \mathrm{Nbr}(x)$ such that $T \subseteq B \subseteq S \cap U$ and $N \times B \subseteq \mathrm{Dom}(\lambda)$,
(iii) $K > 0$;

such that for every $P \in \mathcal{P}$ and distinct $x_0, x_1 \in P \cap T$ there are $n \leq K/d(x_0, x_1)$ and $g_1, \ldots, g_n \in N$ which satisfy:

(1) $g_1 = e_H$,
(2) for every $i = 1, \ldots, n-1$, $\hat{g}_i(x_1) = \hat{g}_{i+1}(x_0)$,
(3) $\hat{g}_n(x_1) \notin B$.

(b) Let $L \subseteq S$. We say that $\langle H, \lambda\rangle$ is *$\mathcal{P}$-translation-like in L* if for every $x \in L$, $\langle H, \lambda\rangle$ is $\mathcal{P}$-translation-like at x. □

The notion of a $\mathcal{P}$-translation-like action will be used in the following setting. Let $\mathrm{BCD}^E(A, r)$ be a linear boundary chart domain, $X = \mathrm{cl}^{B(0,r)}(\mathrm{BCD}^E(A, r))$ and $H = \{\mathrm{tr}_v \mid v \in \mathrm{bd}^E(A)\}$. The natural partial action of H on X is translation-like.

Proposition 12.4. *Let $\mathrm{BCD}^E(A, r)$ be a linear boundary chart domain, $L = \mathrm{bd}^E(A)$. So L is a closed subspace of E. Let $L' = L \cap B(0, r)$. So $L' = \mathrm{bd}^{B(0,r)}(\mathrm{BCD}^E(A, r))$. Let $X = \mathrm{BCD}^E(A, r) \cup L'$, $\mathcal{P} = \{X \cap (v + L) \mid v \in X\}$ and $H = \{\mathrm{tr}^E_v \mid v \in L\}$. We equip H with the norm topology of L. Let λ be the following partial action of H on X:*

$$\mathrm{Dom}(\lambda) = \{\langle h, z\rangle \mid h \in H \text{ and } z, h(z) \in X\} \quad \text{and} \quad \lambda(h, z) = h(z).$$

Then λ is $\mathcal{P}$-translation-like in X.

Proof. The proof is trivial. ■

The following lemma will be applied to the group of translations in a direction parallel to the boundary of a linear boundary chart domain. This lemma captures the main argument in the proof of Lemma 12.6.

Lemma 12.5. *Let $\langle X, d^X\rangle$ and $\langle Y, d^Y\rangle$ be metric spaces, and $\tau : X \cong Y$. Let Γ be a countably generated modulus of continuity, and let $\alpha \in \mathrm{MBC}$. Let $S \subseteq X$ be open, and $\mathcal{P}$ be a partition of S. Let H be a topological group and λ be a partial action of H on X. Let $x \in S$. Assume that:*

(i) $S \subseteq \mathrm{Fld}(\lambda)$,
(ii) *λ is $\mathcal{P}$-translation-like at x,*
(iii) *λ is α-decayable in S,*
(iv) *x is a λ-limit-point,*
(v) *there is $U \in \mathrm{Nbr}(x)$ such that for every $g \in H(X)$, if $\mathrm{supp}(g) \subseteq U$ and g is $\alpha \circ \alpha$-bicontinuous, then g^τ is Γ-bicontinuous at $\tau(x)$.*

Then τ is inversely $\langle \Gamma, \mathcal{P}\rangle$-continuous at x.

Proof. Suppose by contradiction that τ is not inversely $\langle \Gamma, \mathcal{P} \rangle$-continuous at x. The conditions of Lemma 3.11 hold for x, according to the following correspondence. The group G of 3.11 is $H(X)$ here, and N of 3.11 is S here. Also, since x is a λ-limit-point, $\kappa := \min(\{\kappa(x, V_\lambda(x)) \mid V \in \mathrm{Nbr}(e_H)\}) \geq \aleph_0$. Hence Γ is $(\leq\kappa)$-generated. It follows from 3.11 that there are $V \in \mathrm{Nbr}(x)$, $M \in \mathrm{Nbr}(e_H)$ and $\gamma \in \Gamma$ such that $M \times V \subseteq \mathrm{Dom}(\lambda)$, and

(i) for every $h \in M$, $(h_\lambda)^\tau \restriction \tau(V)$ is γ-bicontinuous.

For $g \in H$ denote $\hat{g} = g_\lambda$. Since λ is $\mathcal{P}$-translation-like at x, there are:

(ii) $N \in \mathrm{Nbr}(e_H)$ such that $N \subseteq M$,
(iii) $T, B \in \mathrm{Nbr}(x)$ such that $T \subseteq B \subseteq S \cap V$,
(iv) $K > 0$,

such that for every $P \in \mathcal{P}$ and distinct $x_0, x_1 \in P \cap T$ there are $n \leq K/d(x_0, x_1)$ and $e_H = g_1, \ldots, g_n \in N$ which satisfy: $\hat{g}_i(x_1) = \hat{g}_{i+1}(x_0)$ for every $i = 1, \ldots, n-1$, and $\hat{g}_n(x_1) \notin B$.

Let $C = \tau(B)$ and $y = \tau(x)$. Since C is a neighborhood of y, $d := d(y, Y - C) > 0$. Let $t > 0$ be such that $\tau(B(x,t)) \subseteq B(y, d/2)$ and $B(x,t) \subseteq T$. Set $K^* = 2K/d$. By clause M2 in Definition 1.9, $K^* \cdot \gamma \in \Gamma$. We have assumed that τ^{-1} is not $\langle \Gamma, \tau(\mathcal{P}) \rangle$-continuous at y. Hence there are $P \in \mathcal{P}$ and $y_0, y_1 \in \tau(B(x,t) \cap P)$ such that

$$d(\tau^{-1}(y_0), \tau^{-1}(y_1)) > K^* \gamma(d(y_0, y_1)).$$

For $\ell = 0, 1$ let $x_\ell = \tau^{-1}(y_\ell)$, hence $x_0, x_1 \in B(x,t) \subseteq T$. So there are $n \leq K/d(x_0, x_1)$ and $e_H = g_1, \ldots, g_n \in N$ such that for every $i = 1, \ldots, n-1$, $g_i(x_1) = g_{i+1}(x_0)$ and $g_n(x_1) \notin B$. For $i = 2, \ldots, n$ let $x_i = g_i(x_1)$ and $y_i = \tau(x_i)$. Since $y_0 \in \tau(B(x,t)) \subseteq B(y, d/2)$, we have $d(y_0, y) < d/2$. Note that

(1) For every $i = 1, \ldots, n$, $g_i^\tau(y_0) = y_{i-1}$ and $g_i^\tau(y_1) = y_i$,

and recall that

(2) $y_0, y_1 \in \tau(B(x,t)) \subseteq \tau(V)$,
(3) $g_1, \ldots, g_n \in N \subseteq M$.

So by (i) and (1)–(3), $d(y_{i-1}, y_i) \leq \gamma(d(y_0, y_1))$ for every $i = 1, \ldots, n$. Recall that $d(x_0, x_1) > K^* \gamma(d(y_0, y_1))$. Also, $x_n \notin B$ and hence $y_n \notin C$. So

$$\begin{aligned} d(y, Y - C) \leq d(y, y_n) &\leq d(y, y_0) + \sum_{1=1}^{n} d(y_{i-1}, y_i) < d/2 + n\gamma(d(y_0, y_1)) \\ &\leq d/2 + \frac{K}{d(x_0, x_1)} \cdot \gamma(d(y_0, y_1)) \\ &< d/2 + \frac{K}{K^* \gamma(d(y_0, y_1))} \cdot \gamma(d(y_0, y_1)) \\ &= d/2 + \frac{K}{\frac{2K}{d} \gamma(d(y_0, y_1))} \cdot \gamma(d(y_0, y_1)) = d. \end{aligned}$$

But $d(y, Y - C) = d$, a contradiction. ∎

Lemma 12.6. *Assume the following situation.*

(1) *Γ, Δ are countably generated moduli of continuity.*
(2) *$X \subseteq E$ and $Y \subseteq F$ are open subsets of the normed spaces E and F, X is Γ-LIN-bordered and Y is Δ-LIN-bordered.*
(3) *$\tau \in \mathrm{EXT}^{\pm}(X, Y)$, G is a Γ-appropriate subgroup of $\mathrm{EXT}(X)$, H is a Δ-appropriate subgroup of $\mathrm{EXT}(Y)$ and $G^{\tau} = H$.*
(4) *$x \in \mathrm{bd}(X)$, $\langle \varphi, A, r\rangle$ is a boundary chart element for x, $\gamma \in \Gamma$ and φ is γ-bicontinuous.*
(5) *$y \in \mathrm{bd}(Y)$, $\langle \psi, B, s\rangle$ is a boundary chart element for y, $\delta \in \Delta$ and ψ is δ-bicontinuous.*
(6) *$\tau^{\mathrm{cl}}(x) = y$ and $\tau(\varphi(\mathrm{BCD}^E(A, r))) \subseteq \psi(\mathrm{BCD}^F(B, s))$.*
(7) *Set $L = \mathrm{bd}(A)$, $\widehat{X} = \mathrm{BCD}^E(A, r) \cup (L \cap B(0, r))$, $\hat{\tau} = \psi^{-1} \circ \tau^{\mathrm{cl}} \circ \varphi$, $\widehat{Y} = \hat{\tau}(\widehat{X})$ and $\mathcal{P} = \{(v + L) \cap \widehat{X} \mid v \in \widehat{X}\}$.*

Then $\hat{\tau}$ is inversely $\langle \Delta, \mathcal{P}\rangle$-continuous at 0.

Proof. We may assume that $X - \mathrm{Rng}(\varphi) \neq \emptyset$. From the fact that G has boundary type Γ it follows that there is $Z \in \mathrm{Nbr}^E(x)$ such that $G\lfloor \underline{Z \cap X}\rfloor \supseteq H_{\Gamma}^{\mathrm{CMP.LC}}(X)\lfloor \underline{Z \cap X}\rfloor$. We may also assume that $\varphi(\mathrm{BCD}^E(A, r)) \subseteq Z$.

We wish to apply Lemma 12.5 to $\widehat{X}, \widehat{Y}$ and $\hat{\tau}$. More specifically, the roles of the objects mentioned in 12.5 are taken by the following objects here. The role of Γ in 12.5 is taken by Δ here, the spaces X, Y in 12.5 are $\widehat{X}, \widehat{Y}$ here, τ of 12.5 is $\hat{\tau}$, α of 12.5 is the function $y = 2x$, S is $\widehat{X}$ and $\mathcal{P}$ of 12.5 is $\mathcal{P}$ here. The topological group H appearing in 12.5 is $\{\mathrm{tr}_v^E \mid v \in L\}$ equipped with the norm topology of L, and λ is the natural partial action of $\{\mathrm{tr}_v^E \mid v \in L\}$ on $\widehat{X}$.

Our next goal is to define the open set U appearing in clause (v) of 12.5. We first check that $\varphi(\widehat{X}) = \mathrm{cl}(X) \cap \mathrm{Rng}(\varphi)$ and that $\varphi(\widehat{X})$ is open in $\mathrm{cl}(X)$. Clearly, $\widehat{X} \subseteq \mathrm{cl}^E(\mathrm{BCD}^E(A, r))$. So if $u \in \widehat{X}$, then by the continuity of φ, $\varphi(u) \in \mathrm{cl}^E(\varphi(\mathrm{BCD}^E(A, r))) \subseteq \mathrm{cl}(X)$. That is, $\varphi(\widehat{X}) \subseteq \mathrm{cl}(X)$. Now, $\widehat{X}$ is closed in $B(0, r)$ and so $B(0, r) - \widehat{X}$ is open in $B(0, r)$. So $B(0, r) - \widehat{X}$ is open in E. Since φ takes open subsets of E to open subsets of E, $\varphi(B(0, r) - \widehat{X})$ is open in E. Also, $\varphi(B(0, r) - \widehat{X}) \cap X = \emptyset$. So $\varphi(B(0, r) - \widehat{X}) \cap \mathrm{cl}(X) = \emptyset$. It follows that $\mathrm{Rng}(\varphi) \cap \mathrm{cl}(X) = \varphi(\widehat{X})$. From the fact that $\mathrm{Rng}(\varphi)$ is open in E it follows that $\varphi(\widehat{X})$ is open in $\mathrm{cl}(X)$.

Since $x = \varphi(0)$ and $0 \in \widehat{X}$, it follows that $\varphi(\widehat{X}) \in \mathrm{Nbr}^{\mathrm{cl}(X)}(x)$. So $d(x, \mathrm{cl}(X) - \varphi(\widehat{X})) > 0$. Let $r' \in (0, r)$ be such that $\mathrm{diam}(\varphi(\widehat{X} \cap B(0, r'))) < d(x, \mathrm{cl}(X) - \varphi(\widehat{X}))/2$. The open set U appearing in clause (v) of 12.5 is $\widehat{X} \cap B^E(0, r')$.

We have to show that clauses (i)–(v) of 12.5 hold. It follows from Proposition 12.2(b) that λ is α-decayable in $\widehat{X}$, and from Proposition 12.2(c) that 0 is a λ-limit-point. It follows from Proposition 12.4 that λ is $\mathcal{P}$-translation-like at 0.

We check that U satisfies clause (v) of 12.5. Note that $\widehat{X} = \mathrm{cl}^{B(0,r)}(\mathrm{BCD}^E(A, r))$. We shall also use the fact that if $\mathrm{cl}^E(A) \subseteq \mathrm{Dom}(\varphi)$, then $\mathrm{cl}^E(\varphi(A)) = \varphi(\mathrm{cl}^E(A))$. This follows from the fact that φ takes closed subsets of E to closed subsets of E.

Let $\beta = \alpha \circ \alpha$. So $\beta(t) = 4t$. Let $g \in H(\widehat{X})$ be β-bicontinuous and $\mathrm{supp}(g) \subseteq U$. In order to prove that clause (v) is fulfilled, it has to be shown that $g^{\hat{\tau}}$ is Δ-bicontinuous at

$\hat{\tau}(0^E)$. Recall that $\hat{\tau} = \psi^{-1} \circ \tau^{\mathrm{cl}} \circ \varphi$. So $g^{\hat{\tau}} = ((g^{\varphi})^{\tau^{\mathrm{cl}}})^{\psi^{-1}}$. Set $\hat{h} = g^{\varphi}$ and $\varrho = \gamma \circ \beta \circ \gamma$. Since g is β-bicontinuous and φ is γ-bicontinuous, it follows that $\hat{h}$ is ϱ-bicontinuous. Also, $\beta \in \Gamma^{\mathrm{LIP}} \subseteq \Gamma$ and $\gamma \in \Gamma$, so $\varrho \in \Gamma$.

Note that $\mathrm{Dom}(g) = \widehat{X} \subseteq \mathrm{Dom}(\varphi)$. So $\mathrm{Dom}(\hat{h}) = \varphi(\widehat{X})$ and hence $\mathrm{Dom}(\hat{h})$ is open in $\mathrm{cl}(X)$. It follows trivially from the definitions of $\widehat{X}$ and U that $\mathrm{cl}^E(U) \subseteq \widehat{X}$. Hence $\mathrm{cl}^E(\mathrm{supp}(g)) \subseteq \widehat{X} \subseteq \mathrm{Dom}(\varphi)$ and so $\mathrm{cl}^E(\varphi(\mathrm{supp}(g))) = \varphi(\mathrm{cl}^E(\mathrm{supp}(g)))$. So

$$\mathrm{cl}^E(\mathrm{supp}(\hat{h})) = \mathrm{cl}^E(\varphi(\mathrm{supp}(g))) = \varphi(\mathrm{cl}^E(\mathrm{supp}(g))) \subseteq \varphi(\mathrm{cl}^E(U) \subseteq \varphi(\widehat{X}) = \mathrm{Dom}(\hat{h}).$$

Let $\bar{h} = \hat{h} \cup \mathrm{Id}\restriction(\mathrm{cl}(X) - \mathrm{Dom}(\hat{h}))$. We show that $\bar{h} \in H_{\Gamma}(\mathrm{cl}(X))$. That is, $\bar{h} \in H(\mathrm{cl}(X))$ and $\bar{h}$ is Γ-bicontinuous. Let $u \in \mathrm{cl}(X)$. If $u \in \mathrm{Dom}(\hat{h})$, then since $\mathrm{Dom}(\hat{h})$ is open in $\mathrm{cl}(X)$ and $\hat{h}$ is continuous, we infer that $\bar{h}$ is continuous at u. If $u \notin \mathrm{Dom}(\hat{h})$, then since $\mathrm{cl}^E(\mathrm{supp}(\hat{h})) \subseteq \mathrm{Dom}(\hat{h})$, it follows that $u \in \mathrm{cl}(X) - \mathrm{cl}^E(\mathrm{supp}(\hat{h}))$. So there is $V \in \mathrm{Nbr}^{\mathrm{cl}(X)}(u)$ such that $\bar{h}\restriction V = \mathrm{Id}$. Hence $\bar{h}$ is continuous at u. The same argument applies to $\bar{h}^{-1}$. So $\bar{h} \in H(\mathrm{cl}(X))$.

We now show that $\bar{h}$ is Γ-bicontinuous. Recall that $X - \mathrm{Rng}(\varphi) \neq \emptyset$ and hence $X - \mathrm{Dom}(\hat{h}) \neq \emptyset$. Since φ is γ-continuous, it follows that $\mathrm{Dom}(\hat{h})$ and hence $\mathrm{supp}(\hat{h})$ are bounded. Set $c = d(\mathrm{supp}(\hat{h}), \mathrm{cl}(X) - \mathrm{Dom}(\hat{h}))$ and $e = \mathrm{diam}(\mathrm{supp}(\hat{h}))$. Clearly, $e < \infty$. We show that $c > 0$. Recall that $\mathrm{supp}(g) \subseteq U$, and hence $\mathrm{supp}(\hat{h}) = \varphi(\mathrm{supp}(g)) \subseteq \varphi(U)$. Also, $x = \varphi(0) \in \varphi(U)$. So

$$\begin{aligned} c = d(\mathrm{supp}(\hat{h}), \mathrm{cl}(X) - \mathrm{Dom}(\hat{h})) &\geq d(\varphi(U), \mathrm{cl}(X) - \varphi(\widehat{X})) \\ &\geq d(x, \mathrm{cl}(X) - \varphi(\widehat{X})) - \mathrm{diam}(\varphi(U)) \\ &\geq d(x, \mathrm{cl}(X) - \varphi(\widehat{X})) - d(x, \mathrm{cl}(X) - \widehat{X})/2 = d(x, \mathrm{cl}(X) - \widehat{X})/2 > 0. \end{aligned}$$

Let $u, v \in \mathrm{cl}(X)$. If $u, v \in \mathrm{supp}(\hat{h})$, then $\|\bar{h}(u) - \bar{h}(v)\| \leq \varrho(\|u - v\|)$. If $u, v \in \mathrm{cl}(X) - \mathrm{supp}(\hat{h})$, then $\|\bar{h}(u) - \bar{h}(v)\| = \|u - v\|$. Suppose that $u \in \mathrm{supp}(\hat{h})$ and $v \notin \mathrm{supp}(\hat{h})$. If $v \in \mathrm{Dom}(\hat{h})$, then $\|\bar{h}(u) - \bar{h}(v)\| \leq \varrho(\|u - v\|)$. Otherwise,

$$\begin{aligned} \|\bar{h}(u) - \bar{h}(v)\| &\leq \|\bar{h}(u) - u\| + \|u - v\| \leq e + \|u - v\| \\ &= \frac{e}{c} \cdot c + \|u - v\| \leq \frac{e}{c} \cdot \|u - v\| + \|u - v\| = \frac{e + c}{c} \cdot \|u - v\|. \end{aligned}$$

It follows that $\bar{h}$ is $(1 + e/c) \cdot \varrho$-continuous. The same argument applies to $\bar{h}^{-1}$. Since $(1 + e/c) \cdot \varrho \in \Gamma$, it follows that $\bar{h}$ is Γ-bicontinuous.

Let $h = \bar{h}\restriction X$. Then $\mathrm{supp}(h) \subseteq Z \cap X$. Hence $h \in H_{\Gamma}^{\mathrm{CMP.LC}}(X)\lfloor \underline{Z \cap X} \rfloor$. It follows that $h \in G$. By assumption (3) in the statement of the lemma, $h^{\tau} \in H$. So $h^{\tau} \in \mathrm{EXT}(Y)$ and h^{τ} is Δ-bicontinuous at y. That is, for some $\nu \in \Delta$, h^{τ} is ν-bicontinuous at y. So $(h^{\tau})^{\mathrm{cl}}$ is ν-bicontinuous at y. Now, $\bar{h} = h^{\mathrm{cl}}$, hence $\bar{h}^{\tau^{\mathrm{cl}}} = (h^{\tau})^{\mathrm{cl}}$ and so $\bar{h}^{\tau^{\mathrm{cl}}}$ is ν-bicontinuous at y. Recall that ψ is δ-bicontinuous, where $\delta \in \Delta$. Also, $\psi^{-1}(y) = 0^F$. It follows that $(\bar{h}^{\tau^{\mathrm{cl}}})^{\psi^{-1}}$ is $\delta \circ \nu \circ \delta$-bicontinuous at 0^F. That is, $(\bar{h}^{\tau^{\mathrm{cl}}})^{\psi^{-1}}$ is Δ-bicontinuous at 0^F. Finally, $g^{\varphi} = \hat{h} \subseteq \bar{h}$ and $y \in \mathrm{Dom}((g^{\varphi})^{\tau^{\mathrm{cl}}})$. So $((g^{\varphi})^{\tau^{\mathrm{cl}}})^{\psi^{-1}}$ is Δ-bicontinuous at 0^F. That is, $g^{\hat{\tau}}$ is Δ-bicontinuous at $\hat{\tau}(0^E)$.

We have checked that the conditions of Lemma 12.5 hold. So $\hat{\tau}$ is inversely $\langle \Delta, \mathcal{P} \rangle$-continuous at 0. ∎

12.2. ***Γ*-continuity for submerged pairs and the star operation.** The next intermediate goal is to show that in the above setting, $\hat{\tau}$ is inversely $\varDelta$-continuous at 0^E (Lemma 12.17(b)). Unfortunately, we are able to prove this only under additional asumptions on $\varGamma$ and $\varDelta$. The assumptions $\varGamma = \varDelta$ and $\varGamma$ principal suffice. (See clause M6 in Definition 1.9.) The exact extra assumptions use the notion of star-closedness which is defined in Definition 12.11(d). They are: $\varGamma \subseteq \varDelta$ and $\varDelta$ is $\varGamma$-star-closed.

PROPOSITION 12.7. *Recall that for $\varrho \in H([0,\infty))$ and a normed space E, the homeomorphism $\mathrm{Rad}_\varrho^E \in H(E)$ was defined as follows: for $u \neq 0$, $\mathrm{Rad}_\varrho^E(u) = \varrho(\|u\|) \cdot u/\|u\|$ and $\mathrm{Rad}_\varrho^E(0) = 0$. If $\alpha \in \mathrm{MC}$ and ϱ is α-continuous, then Rad_ϱ^E is $5 \cdot \alpha$-continuous.*

Proof. Let $x, y \in E$ and $y \neq 0$. Define $z = \|x\| \cdot y/\|y\|$. Then $\|y - z\| = |\|y\| - \|x\|| \leq \|y - x\|$. So $\|x - z\| \leq \|x - y\| + \|y - z\| \leq 2\|y - x\|$. Let $h = \mathrm{Rad}_\varrho^E$. Suppose that $x \neq z$. Then

$$\|h(y) - h(x)\| \leq \|h(y) - h(z)\| + \|h(z) - h(x)\| \leq \alpha(\|y - z\|) + \frac{\varrho(\|x\|)}{\|x\|} \cdot \|x - z\|$$

$$\leq \alpha(\|y - x\|) + \frac{\alpha(\|x\|)}{\|x\|} \cdot \|x - z\| \leq \alpha(\|y - x\|) + \frac{\alpha(\|\frac{x-z}{2}\|)}{\|\frac{x-z}{2}\|} \cdot \|x - z\|$$

$$= \alpha(\|y - x\|) + 2\alpha\left(\left\|\frac{x - z}{2}\right\|\right) \leq \alpha(\|y - x\|) + 2\alpha(\|x - z\|) \leq \alpha(\|y - x\|) + 2\alpha(2\|y - x\|)$$

$$\leq \alpha(\|y - x\|) + 4\alpha(\|y - x\|) = 5\alpha(\|y - x\|).$$

If $x = z$, then $\|h(y) - h(x)\| \leq \alpha(\|y - x\|)$. So Rad_ϱ^E is $5 \cdot \alpha$-continuous. ■

PROPOSITION 12.8. *There is M^{rtn} such that the following holds. Let $\alpha \in \mathrm{MBC}$ and $a > 0$. Let E be a normed space, $x, y \in E$ and $\|x\| = \|y\| = \alpha(a)$. Then there is $g \in H(E)$ such that $g(0) = 0$, $g(x) = y$, $\mathrm{supp}(g) \subseteq B(0, \alpha(a) + a/2)$, and g is $M^{\mathrm{rtn}} \cdot \alpha \circ \alpha$-bicontinuous.*

Proof. Let $b = \alpha(a)$, $c = \alpha(a) + a/2$ and $N = M^{\mathrm{hlb}}$. (See Proposition 9.2(c).) Let $\varrho \in H([0,\infty))$ be the piecewise linear function with breakpoints at b and c such that $\varrho(b) = b/2N$ and $\varrho(t) = t$ for every $t \geq c$. The slope of ϱ on $[0, \alpha(a)]$ is $1/2N < 1$. The slope of ϱ on $[\alpha(a), \alpha(a) + a/2]$ is

$$\frac{c - b/2N}{\alpha(a)/2} = \frac{2\alpha(a) + a - \alpha(a)/N}{a} \leq \frac{3\alpha(a)}{a}.$$

The slope of ϱ on $[\alpha(a) + \alpha/2, \infty)$ is 1. So ϱ is $(3, 3\alpha)$-continuous. (See Definition 9.9(b).) By Proposition 9.10(a), ϱ is 9α-continuous. By Proposition 12.7, Rad_ϱ^E is $45 \cdot \alpha$-continuous. Clearly, $(\mathrm{Rad}_\varrho^E)^{-1} = \mathrm{Rad}_{\varrho^{-1}}^E$. The slope of ϱ^{-1} on $[0, \alpha(a)/2N]$ is $2N$. The slope of ϱ^{-1} on $[\alpha(a)/2N, \alpha(a) + \alpha/2]$ is $\leq a/\alpha(a) \leq 1$. So ϱ^{-1} is $(3, 2N\alpha)$-continuous. It follows that $(\mathrm{Rad}_\varrho^E)^{-1}$ is $30N \cdot \alpha$-continuous. Let $M_1 = \max(30N, 45)$. Then Rad_ϱ^E is $M_1 \cdot \alpha$-bicontinuous. Let $h = \mathrm{Rad}_\varrho^E$. Then

(1) $\mathrm{supp}(h) \subseteq B(0, \alpha(a) + a/2)$,
(2) $h(x) = x/2N$,
(3) h is $M_1 \cdot \alpha$-bicontinuous.

Let $L = \mathrm{span}(\{x, y\})$. By Proposition 9.2(c), there are a Euclidean norm $\| \; \|^{\mathbf{H}}$ on L and a complement S of L such that for every $u \in E$, $\|(u)_L\|^{\mathbf{H}} + \|(u)_S\| \approx^{M^{\mathrm{hlb}}} \|u\|$. Define $|u| = \|(u)_L\|^{\mathbf{H}} + \|(u)_S\|$. We shall apply Proposition 9.6(c). Let $\hat{x} = x/2N$, $\hat{y} = |\hat{x}|/|y|y$, and θ be the angle from $\hat{x}$ to $\hat{y}$ in $\langle L, \| \; \|^{\mathbf{H}} \rangle$. So $|\hat{y}| = |\hat{x}|$. Let $S = \bar{B}^L(0, |\hat{x}|)$. Let η be the piecewise linear function with breakpoint at $|\hat{x}|$ such that $\eta(0) = \theta$ and $\eta(|\hat{x}|) = 0$. So η is $\theta/|\hat{x}|$-Lipschitz. Hence the conditions of Proposition 9.2(c) hold with $r = |\hat{x}|$ and $K = \theta/|\hat{x}|$. Let $\bar{d}$ denote the distance function obtained from $| \; |$. Let g_1 be defined by $g_1(u) = \mathrm{Rot}^{F,H}_{\eta(\bar{d}(u,S))}(u)$. Then $g_1 \in H(E)$ and g_1 is $(M^{\mathrm{rot}} \cdot Kr + 1)$-bilipschitz with respect to $\bar{d}$. Note that $Kr = \theta \le \pi$. So g_1 is $M^{\mathrm{rot}}(\pi + 1)$-bilipschitz with respect to $\bar{d}$. Write $M_2 = (M^{\mathrm{hlb}})^2 M^{\mathrm{rot}}(\pi + 1)$. Hence

(4) g_1 is M_2-bilipschitz in $\langle E, \| \; \| \rangle$.

Let $u \in E - B(0, \|x\|)$. Then $|u| \ge \|u\|/M^{\mathrm{hlb}} \ge \|x\|/N$. So $\bar{d}(u, S) \ge |\hat{x}|$. Hence $g_1(u) = u$. That is,

(5) $\mathrm{supp}(g_1) \subseteq B(0, \|x\|)$.

It is also obvious that

(6) $g_1(\hat{x}) = \hat{y}$.

Let $\bar{y} = y/2N$. Then $\|\bar{y}\| = \|\hat{x}\|$. Recall that $|\hat{y}| = |\hat{x}|$. Since $|\hat{x}| \approx^{M^{\mathrm{hlb}}} \|\hat{x}\|$, $\|\hat{y}\| \approx^{M^{\mathrm{hlb}}} \|\bar{y}\|$. That is, $(1/M^{\mathrm{hlb}}) \cdot \|\hat{y}\| \le \|\bar{y}\| \le M^{\mathrm{hlb}} \cdot \|\hat{y}\|$. We construct g_2 which takes $\hat{y}$ to $\bar{y}$. Let $\varrho : [0, \infty) \to [0, \infty)$ be the piecewise linear function with breakpoints $\|\hat{y}\|$ and $\|x\|$ such that $\varrho(0) = 0$, $\varrho(\|\hat{y}\|) = \|\bar{y}\|$ and $\varrho(t) = t$ for every $t \ge \|x\|$. Since $\|\hat{y}\|, \|\bar{y}\| < \|x\|$, $\varrho \in H([0, \infty))$. The slopes of ϱ are $\frac{\|\bar{y}\|}{\|\hat{y}\|}$, $\frac{\|x\| - \|\bar{y}\|}{\|x\| - \|\hat{y}\|}$ and 1, and the slopes of ϱ^{-1} are $\frac{\|\hat{y}\|}{\|\bar{y}\|}$, $\frac{\|x\| - \|\hat{y}\|}{\|x\| - \|\bar{y}\|}$ and 1. Clearly, $\|\bar{y}\|/\|\hat{y}\| \le M^{\mathrm{hlb}} = N$. Note that $\|\hat{y}\| \le \|\hat{y}\|^{\mathbf{H}} = |\hat{y}| = |\hat{x}| \le M^{\mathrm{hlb}} \cdot \|\hat{x}\| = N \cdot \frac{\|x\|}{2N} = \|x\|/2$. So

$$\frac{\|x\| - \|\bar{y}\|}{\|x\| - \|\hat{y}\|} = \frac{(1 - 1/2N)\|x\|}{\|x\| - \|\hat{y}\|} \le \frac{\|x\|}{\|x\| - \|x\|/2} = 2.$$

Hence ϱ is $\max(N, 2)$-Lipschitz.

As to the slopes of ϱ^{-1}, clearly, $\|\hat{y}\|/\|\bar{y}\| \le N$ and

$$\frac{\|x\| - \|\hat{y}\|}{\|x\| - \|\bar{y}\|} \le \frac{\|x\|}{(1 - 1/2N)\|x\|} \le 2.$$

So ϱ^{-1} is $\max(N, 2)$-Lipschitz. Let $M_3 = 3\max(N, 2)$ and $g_2 = \mathrm{Rad}^E_{\varrho}$. By Proposition 3.18,

(7) g_2 is M_3-bilipschitz.

It follows trivially from the definitions of ϱ and g_2 that

(8) $g_2(\hat{y}) = \bar{y}$,

(9) $\mathrm{supp}(g_2) \subseteq B(0, \|x\|)$.

Let $g = h^{-1} \circ g_2 \circ g_1 \circ h$. Note that

(10) $h^{-1}(\bar{y}) = h^{-1}(y/2N) = y$.

It follows from (1)–(10) that g is $M_1^2 M_2 M_3 \cdot \alpha \circ \alpha$-bicontinuous, $g(x) = y$ and $\mathrm{supp}(g) \subseteq B(0, \alpha(a) + a/2)$. Define $M^{\mathrm{rtn}} = M_1^2 M_2 M_3$. Then M^{rtn} is as required. ■

DEFINITION 12.9. (a) Let E be a metric space, $x, y \in X \subseteq E$ and $\alpha \in \mathrm{MC}$. We say that $\langle x, y \rangle$ is *α-submerged* in X with respect to E if $\delta^X(x) \ge \|x - y\| + \alpha^{-1}(\|x - y\|)$.

(b) Let $X \subseteq E$, $Y \subseteq F$ be open subsets of the metric spaces E, F, $V \subseteq X$, $x \in \mathrm{bd}(X)$, $\alpha, \beta \in \mathrm{MC}$, $\Gamma, \Delta \subseteq \mathrm{MC}$ and $\tau \in \mathrm{EXT}^{\pm}(X, Y)$. We say that τ is *β-continuous for α-submerged pairs* in V if for every α-submerged pair $\langle y, z \rangle$ in V, $d^Y(\tau(y), \tau(z)) \le \beta(d^X(y, z))$.

We say that τ is *β-continuous for α-submerged pairs* at x (τ is *$(\beta; \alpha)$-continuous* at x) if there is $U \in \mathrm{Nbr}^E(x)$ such that τ is β-continuous for α-submerged pairs in $U \cap X$. We say that τ is *Δ-continuous for Γ-submerged pairs at x* (τ is *$(\Delta; \Gamma)$-continuous* at x) if for any $\alpha \in \Gamma$ there is $\beta \in \Delta$ such that τ is $(\beta; \alpha)$-continuous at x.

(c) Let $X \subseteq E$, $Y \subseteq F$ be open subsets of the metric spaces E, F, $V \subseteq X$, $\alpha, \beta \in \mathrm{MC}$ and $\tau \in \mathrm{EXT}^{\pm}(X, Y)$. We say that τ *is almost β-continuous for α-submerged pairs* in V (τ is *$(\beta; \alpha)$-almost-continuous* in V) if for any α-submerged pairs $\langle y, z_1 \rangle, \langle y, z_2 \rangle$ in V: if $d(y, z_1) = d(y, z_2)$, then $d^Y(\tau(y), \tau(z_2)) \le \beta(d^Y(\tau(y), \tau(z_1)))$. □

Under assumptions similar to Lemma 12.6, we prove the submerged continuity of τ^{-1}.

LEMMA 12.10. *Assume the following facts.*

(1) *Γ, Σ are countably generated moduli of continuity, and Ω is the modulus of continuity generated by $\Gamma \cup \Sigma$.*
(2) *$X \subseteq E$ and $Y \subseteq F$ are open subsets of the normed spaces E and F, X is Γ-LIN-bordered and Y is Σ-LIN-bordered.*
(3) *$\tau \in \mathrm{EXT}^{\pm}(X, Y)$, G is a Γ-appropriate subgroup of $\mathrm{EXT}(X)$, H is a Δ-appropriate subgroup of $\mathrm{EXT}(Y)$ and $G^{\tau} = H$.*
(4) *$x \in \mathrm{bd}(X)$, $\langle \varphi, A, r \rangle$ is a boundary chart element for x, $\gamma \in \Gamma$ and φ is γ-bicontinuous.*
(5) *$y \in \mathrm{bd}(Y)$, $\langle \psi, B, s \rangle$ is a boundary chart element for y, $\sigma \in \Sigma$ and ψ is σ-bicontinuous.*
(6) *$\tau^{\mathrm{cl}}(x) = y$ and $\tau(\varphi(\mathrm{BCD}^E(A, r))) \subseteq \psi(\mathrm{BCD}^F(B, s))$.*
(7) *Set $\widetilde{X} = \mathrm{BCD}^E(A, r)$, $\tilde{\tau} = \psi^{-1} \circ \tau \circ \varphi$ and $\widetilde{Y} = \tilde{\tau}(\widetilde{X})$.*

Then $\tilde{\tau}^{-1}$ is $(\Omega; \Sigma)$-continuous at $\tilde{\tau}(0)$.

Proof. There is $Z \in \mathrm{Nbr}^F(y)$ such that $H\lfloor \underline{Z \cap Y}\rfloor \supseteq H_{\Sigma}^{\mathrm{CMP.LC}}(Y)\lfloor \underline{Z \cap Y}\rfloor$, and we may assume that $\psi(\mathrm{BCD}^F(B, s)) \subseteq Z$. Set $L = \mathrm{bd}(A)$, $\widehat{X} = \mathrm{BCD}^E(A, r) \cup (L \cap B^E(0, r))$, $\hat{\tau} = \psi^{-1} \circ \tau^{\mathrm{cl}} \circ \varphi$, $\widehat{Y} = \hat{\tau}(\widehat{X})$ and $\mathcal{P} = \{(v + L) \cap \widehat{X} \mid v \in \widehat{X}\}$. Note that $\hat{\tau} = \tilde{\tau}^{\mathrm{cl}}_{B^E(0,r), B^F(0,s)}$. By Lemma 12.6, $\hat{\tau}$ is inversely $\langle \Sigma, \mathcal{P} \rangle$-continuous at 0. Let $r_0 \in (0, r)$ and $\sigma \in \Sigma$ be such that $\hat{\tau} \restriction (B^E(0, r_0) \cap \widehat{X})$ is inversely $\langle \sigma, \mathcal{P} \rangle$-continuous. Let $L_0 \subseteq L$ be any ray whose endpoint is 0. For every $u \in B(0, r_0) \cap \widehat{X}$ let x_u be the intersection point of the ray $u + L_0$ with the sphere $S(0, r_0)$. Clearly, $\lim_{u \to 0} x_u = x_{0^E}$. So $\lim_{u \to 0} d^F(\hat{\tau}(u), \hat{\tau}(x_u)) = d^F(\hat{\tau}(0^E), \hat{\tau}(x_{0^E})) > 0$. Also, $\lim^{\widetilde{Y}}_{u \to 0} \delta^{\widetilde{Y}}(\tilde{\tau}(u)) = 0$. Hence there is $r_1 \in (0, r_0)$ such that for every $u \in B(0, r_1) \cap \widetilde{X}$, $d^F(\tilde{\tau}(u), \tilde{\tau}(x_u)) > \delta^{\widetilde{Y}}(\hat{\tau}(u))$. Let $V = \tilde{\tau}(B(0, r_1) \cap \widetilde{X})$. So for every $v \in V$ and $t \in [0, \delta^{\widetilde{Y}}(v)]$ there is $y(v, t) \in \tilde{\tau}([\tilde{\tau}^{-1}(v), x_{\tilde{\tau}^{-1}(v)}])$ such that

$d^F(y(v,t),v) = t$. Denote $\tilde{\tau}^{-1}$ by $\tilde{\eta}$. By the inverse $\langle \sigma, \mathcal{P} \rangle$-bicontinuity of $\hat{\tau}$, for every v and t as above $d^E(\tilde{\eta}(y(v,t)), \tilde{\eta}(v)) \le \sigma(d^F(y(v,t),v))$.

Claim 1. *Let $\alpha \in \Sigma \cap \mathrm{MBC}$. Then there are $W \in \mathrm{Nbr}^F(0)$ and $\gamma \in \Gamma$ such that $\tilde{\eta}$ is $(\gamma;\alpha)$-almost-continuous in $W \cap \widetilde{Y}$.*

Proof. Suppose by contradiction this is not so. Let $\{\gamma_i \mid i \in \mathbb{N}\}$ be a generating set for Γ, and assume that for every i, $\{j \mid \gamma_j = \gamma_i\}$ is infinite. There is a sequence $\{\langle y_i, u_i, v_i \rangle \mid i \in \mathbb{N}\}$ such that: (i) for every i, $\langle y_i, u_i \rangle$ is α-submerged in $\widetilde{Y}$ and $\|u_i - y_i\| = \|v_i - y_i\|$; (ii) $\lim_i y_i = 0^F$; (iii) $\delta^{\widetilde{Y}}(y_{i+1}) < \alpha^{-1}(\|y_i - u_i\|)/4$; (iv) $\|\tilde{\eta}(v_i) - \tilde{\eta}(y_i)\| > \gamma_i(\|\hat{\eta}(u_i) - \hat{\eta}(y_i)\|)$. Let $r_i = \|u_i - y_i\| + \alpha^{-1}(\|u_i - y_i\|)/2$. Note that from (iii) and the fact that $\langle y_i, u_i \rangle$ is α-submerged it follows that $B(y_i, r_i) \cap B(y_i, r_j) = \emptyset$ for any $i \ne j$. By Proposition 12.8, there is $g_i \in H(\widetilde{Y})$ such that $g_i(y_i) = y_i$, $g(u_i) = v_i$, $\mathrm{supp}(g_i) \subseteq B(y_i, r_i)$, and g_i is $M^{\mathrm{rtn}} \cdot \alpha \circ \alpha$-bicontinuous. Since $\mathrm{supp}(g_i) \cap \mathrm{supp}(g_j) = \emptyset$ for any $i \ne j$, we infer that $\tilde{g} = \circ_i\, g_i$ is well-defined, and $\tilde{g}$ is $(M^{\mathrm{rtn}})^2 \cdot \alpha^{\circ 4}$-bicontinuous. We shall reach a contradiction by showing that $\tilde{g}$ is Σ-bicontinuous at 0^F, whereas $\tilde{g}^{\tilde{\tau}^{-1}}$ is not Γ-bicontinuous at 0^E.

Define $\tilde{h} = \tilde{g}^{\psi}$ and $h = \tilde{h} \cup \mathrm{Id}\restriction(Y - \psi(\widetilde{Y}))$. We shall show that $h \in H$. Recall that $y = \psi(0^F)$ and set $h_i = g_i^{\psi}$. Then $\tilde{h} = \circ_{i \in \mathbb{N}}\, h_i$. Recall that $\mathrm{supp}(g_i) \subseteq B^F(y_i, r_i)$ and note that $\lim_{i \in \mathbb{N}} \bar{B}^F(y_i, r_i) = 0^F$. Since $\{0^F\} \cup \bigcup_{i \in \mathbb{N}} \bar{B}^F(y_i, r_i) \subseteq \mathrm{Dom}(\psi)$, it follows that $\lim_{i \in \mathbb{N}} \psi(\bar{B}^F(y_i, r_i)) = y$. Also, $\mathrm{supp}(h_i) = \psi(\mathrm{supp}(g_i))$. Hence $\mathrm{cl}(\mathrm{supp}(h_i)) = \psi(\mathrm{cl}(\mathrm{supp}(g_i))) \subseteq \psi(\bar{B}^F(y_i, r_i))$ and so $\lim_{i \in \mathbb{N}} \mathrm{cl}(\mathrm{supp}(h_i)) = y$. We thus conclude that: (1) $\mathrm{cl}(\mathrm{supp}(\tilde{h})) = \{y\} \cup \bigcup_{i \in \mathbb{N}} \mathrm{cl}(\mathrm{supp}(h_i))$. It also follows that: (2) if $\vec{z} \subseteq Y$ and $\lim \vec{z} = y$, then $\lim h(\vec{z}) = y$. Note that: (3) for every $i \in \mathbb{N}$, $\mathrm{cl}(\mathrm{supp}(h_i)) \subseteq \psi(\bar{B}^F(y_i, r_i)) \subseteq \psi(\widetilde{Y})$. Let $z \in \mathrm{cl}(Y)$. If $z \notin \mathrm{cl}(\mathrm{supp}(h))$, then $h \cup \{\langle z, z \rangle\}$ is continuous. If $z \in \mathrm{cl}(\mathrm{supp}(h))$, then $z \in \mathrm{cl}(\mathrm{supp}(\tilde{h}))$. So by (1) and (3), either $z = y$ or $z \in \psi(\widetilde{Y})$. If $z = y$, then by (2), $h \cup \{\langle z, z \rangle\}$ is continuous. If $z \in \psi(\widetilde{Y})$, then $h(z) = \tilde{h}(z)$. From the facts: $\tilde{h}$ is continuous, $h\restriction\widetilde{Y} = \tilde{h}$ and $\psi(\widetilde{Y})$ is open in F, it follows that h is continuous at z. We have shown that h is extendible in F. The same argument applies to h^{-1}, so $h \in \mathrm{EXT}(Y)$. Clearly, $\mathrm{supp}(h) = \mathrm{supp}(\tilde{h}) \subseteq \psi(\widetilde{Y}) \subseteq \psi(\mathrm{BCD}^F(B, s)) \subseteq Z$. That is, (4) $\mathrm{supp}(h) \subseteq Z$.

We now show that $h \in H_{\Sigma}^{\mathrm{CMP.LC}}(Y)$. Write $\bar{\alpha} = (M^{\mathrm{rtn}})^2 \cdot \alpha^{\circ 4}$ and $\beta = \sigma \circ \bar{\alpha} \circ \sigma$. Then $\beta \in \Sigma$. We have seen that $\tilde{g}$ is $\bar{\alpha}$-bicontinuous. So since ψ is σ-bicontinuous, it follows that $\tilde{h}$ is β-bicontinuous. This implies that $\tilde{h}^{\mathrm{cl}}$ is β-bicontinuous. We show that for every $z \in \mathrm{cl}(Y)$, h is β-bicontinuous at z. This is certainly true if $z \notin \mathrm{cl}(\mathrm{supp}(h))$. So suppose that $z \in \mathrm{cl}(\mathrm{supp}(h))$. Then $z \in \mathrm{cl}(\mathrm{supp}(\tilde{h}))$. By (1) and (3), either $z \in \psi(\widetilde{Y})$ or $z = y$. If $z \in \psi(\widetilde{Y})$, then $\psi(\widetilde{Y}) \in \mathrm{Nbr}^F(z)$ and $h\restriction\psi(\widetilde{Y}) = \tilde{h}\restriction\psi(\widetilde{Y})$. So h is β-bicontinuous at z.

Assume that $z = y$. Recall that $x = \varphi(0^E)$ and $y = \psi(0^F)$ and define $X_0 = X \cup \{x\}$ and $Y_0 = Y \cup \{y\}$. Note that $\psi(\widetilde{Y}) = \tau(\varphi(\mathrm{BCD}^E(A, r)))$. Since $\varphi(\mathrm{BCD}^E(A, r)) = \varphi(B^E(0, r)) \cap X$ and $\varphi(B^E(0, r))$ is open in E, it follows that $\varphi(\mathrm{BCD}^E(A, r)) \cup \{x\} \in \mathrm{Nbr}^{X_0}(x)$. From the fact that $\tau \in \mathrm{EXT}^{\pm}(X, Y)$ it follows that $\tau(\varphi(\mathrm{BCD}^E(A, r))) \cup \{y\} \in \mathrm{Nbr}^{Y_0}(y)$. That is, $\psi(\widetilde{Y}) \cup \{y\} \in \mathrm{Nbr}^{Y_0}(y)$. So there is $W \in \mathrm{Nbr}^F(y)$ such that $W \cap Y = \psi(\widetilde{Y})$. Thus $h\restriction W = \tilde{h}\restriction W$. It follows that h is β-bicontinuous at y. So $h \in H_{\Sigma}^{\mathrm{CMP.LC}}(Y)$. By (4), $h \in H_{\Sigma}^{\mathrm{CMP.LC}}(Y)\lfloor \underline{Z \cap Y} \rfloor$. Also recall that $H\lfloor \underline{Z \cap Y} \rfloor \supseteq H_{\Sigma}^{\mathrm{CMP.LC}}(Y)\lfloor \underline{Z \cap Y} \rfloor$. So $h \in H$.

We conclude that $h^{\tau^{-1}} \in G$. Now, G is of boundary type Γ, so $h^{\tau^{-1}}$ is Γ-bicontinuous at x. Since φ is Γ-bicontinuous and $\varphi(0^E) = x$, we see that $(h^{\tau^{-1}})^{\varphi^{-1}}$ is Γ-bicontinuous at 0^E. The following steps show that $(h^{\tau^{-1}})^{\varphi^{-1}} = \tilde{g}^{\tilde{\eta}}$:

$$h^{\tau^{-1}} = (\tilde{h} \cup \mathrm{Id}\restriction(Y - \mathrm{Dom}(\tilde{h})))^{\tau^{-1}} = (\tilde{g}^{\psi} \cup \mathrm{Id}\restriction(Y - \psi(\widetilde{Y})))^{\tau^{-1}} = (\tilde{g}^{\psi})^{\tau^{-1}} \cup \mathrm{Id}\restriction(X - \varphi(\widetilde{X})).$$

Since $\mathrm{Rng}(\varphi)$ is disjoint from $X - \varphi(\widetilde{X})$,

$$((\tilde{g}^{\psi})^{\tau^{-1}} \cup \mathrm{Id}\restriction(X - \varphi(\widetilde{X})))^{\varphi^{-1}} = ((\tilde{g}^{\psi})^{\tau^{-1}})^{\varphi^{-1}}.$$

That is,

$$(h^{\tau^{-1}})^{\varphi^{-1}} = ((\tilde{g}^{\psi})^{\tau^{-1}})^{\varphi^{-1}} = \tilde{g}^{\tilde{\eta}}.$$

We conclude that $\tilde{g}^{\tilde{\eta}}$ is Γ-bicontinuous at 0^E.

We shall now show that $\tilde{g}^{\tilde{\eta}}$ is not Γ-continuous at 0^E, thus reaching a contradiction. Let $T \in \mathrm{Nbr}^E(0)$ and $\gamma' \in \Gamma$. Then there are $i \in \mathbb{N}$ and $a > 0$ such that $\gamma'\restriction[0,a] \le \gamma_i\restriction[0,a]$, $\tilde{\eta}(u_i), \tilde{\eta}(y_i) \in T$ and $\|\tilde{\eta}(u_i) - \tilde{\eta}(y_i)\| \le a$. So

$$\|g^{\tilde{\eta}}(\tilde{\eta}(u_i)) - g^{\tilde{\eta}}(\tilde{\eta}(y_i))\| = \|\tilde{\eta}(v_i) - \tilde{\eta}(y_i)\| > \gamma_i(\|\tilde{\eta}(u_i) - \tilde{\eta}(y_i)\|) \ge \gamma'(\|\tilde{\eta}(u_i) - \tilde{\eta}(y_i)\|).$$

This shows that $g^{\tilde{\eta}}$ is not Γ-continuous at 0^E. A contradiction, so Claim 1 is proved.

Let W and γ be as in Claim 1. We may assume that $W \subseteq V$. There is $U \in \mathrm{Nbr}^F(0)$ such that for every $u, v \in U \cap \widetilde{Y}$: if $\langle u, v \rangle$ is α-submerged in $\widetilde{Y}$, then $B(u, \|v - u\|) \subseteq W$. Let $u, v \in U \cap \widetilde{Y}$ be such that $\langle u, v \rangle$ is α-submerged in $\widetilde{Y}$. Let $w = y(u, \|v - u\|)$. Then $w \in U$. Hence

$$\|\tilde{\eta}(v) - \tilde{\eta}(u)\| \le \gamma(\|\tilde{\eta}(w) - \tilde{\eta}(u)\|) \le \gamma \circ \sigma(\|w - u\|) = \gamma \circ \sigma(\|v - u\|).$$

Clearly, $\gamma \circ \sigma \in \Omega$, and we have just shown that $\tilde{\eta}$ is $(\gamma \circ \sigma; \alpha)$-continuous at 0^F. ■

Definition 12.11. (a) Let $\alpha \in H([0,\infty))$. For every $t \in [0,\infty)$ we define a sequence $\vec{t} = \{t_n \mid n \in \mathbb{N}\}$. Define $t_0 = t$ and for every $n \in \mathbb{N}$, let t_{n+1} satisfy the equation

$$t_{n+1} + \alpha(t_{n+1}) = t_n$$

and define

$$p_{\alpha,n}(t) = t_n \quad \text{and} \quad q_{\alpha,n}(t) = t_n - t_{n+1}.$$

Note that $p_{\alpha,0} = \mathrm{Id}$.

(b) Let $\alpha, \beta \in H([0,\infty))$. We define the function $\beta \star \alpha : [0,\infty) \to [0,\infty) \cup \{\infty\}$ by

$$\beta \star \alpha(t) = \sum_{n=0}^{\infty} \beta(q_{\alpha,n}(t)).$$

(c) For $\alpha \in \mathrm{MC}$ let $\Gamma_\alpha = \mathrm{cl}_{\preceq}(\{\alpha^{\circ n} \mid n \in \mathbb{N}\})$.

(d) Let $\Gamma \subseteq \mathrm{MC}$ and $\alpha \in \mathrm{MC}$. We say that Γ is *α-star-closed* if for every $\beta \in \Gamma$ there is $\gamma \in \Gamma$ such that $\beta \star \alpha \preceq \gamma$. Let $\Delta \subseteq \mathrm{MC}$. We say that Γ is *Δ-star-closed* if there is $\delta \in \Delta$ such that Γ is δ-star-closed. □

The next proposition contains some trivial observations about the operation "$\star$". For the continuation of the proof of the main theorems we need only parts (a)–(c) of the proposition. The other parts are mentioned in order to familiarize the reader with this operation. Part (a) was proved by Wiesław Kubis.

PROPOSITION 12.12. *Let $\alpha, \beta, \gamma \in H([0,\infty))$.*

(a) *For every $n \in \mathbb{N}$, $\alpha^{\circ n} \star \alpha \leq n\alpha^{\circ n} + \mathrm{Id}$.*
(b) *If $\gamma \preceq \beta$, then $\gamma \star \alpha \preceq \beta \star \alpha$.*
(c) *For every $n \in \mathbb{N}$, $q_{\alpha,n}$ and $p_{\alpha,n+1}$ are strictly increasing functions.*
(d) *If $s < t$, then $\beta \star \alpha(s) \leq \beta \star \alpha(t)$.*
(e) *Either $\beta \star \alpha \restriction (0,\infty)$ is the constant function $f(t) = \infty$, or $\beta \star \alpha \in H([0,\infty))$.*

Proof. (a) Let $t \in [0,\infty)$. Define $p_{\alpha,n}(t) = p_n$ and $q_{\alpha,n}(t) = q_n$. Hence $q_n = \alpha(p_n)$ and $p_n + q_n = p_{n-1}$. Let $k \geq n \geq 1$. Then

$$\alpha^{\circ n}(q_k) \leq \alpha^{\circ n}(p_{k-1}) = \alpha^{\circ(n-1)}(q_{k-1}) \leq \cdots \leq \alpha^{\circ(n-(n-1))}(p_{k-n})) = \alpha(p_{k-n}) = q_{k-n}.$$

Note that $\sum_{i=0}^{\infty} q_i = t$. Let $n \geq 1$. Then

$$\begin{aligned} \alpha^{\circ n} \star \alpha(t) &= \sum_{k=0}^{\infty} \alpha^{\circ n}(q_k) = \sum_{k<n} \alpha^{\circ n}(q_k) + \sum_{k\geq n} \alpha^{\circ n}(q_k) \\ &\leq \sum_{k<n} \alpha^{\circ n}(t) + \sum_{k\geq n} q_{k-n} = n\alpha^{\circ n}(t) + t. \end{aligned}$$

(b) This is immediate.

(c) Note that $p_{\alpha,n+1} + q_{\alpha,n} = p_{\alpha,n}$. This equality together with the facts that α is strictly increasing and $p_{\alpha,0} = \mathrm{Id}$ implies by induction that $q_{\alpha,n}$ and $p_{\alpha,n+1}$ are strictly increasing for every $n \in \mathbb{N}$.

(d) This follows from the facts that $q_{\alpha,n}$ and β are increasing functions.

(e) Note that $q_{\alpha,k}(p_{\alpha,n}(t)) = q_{\alpha,k+n}(t)$. Hence $\beta \star \alpha(p_{\alpha,n}(t))$ is a tail of $\beta \star \alpha(t)$. So for every n, $\beta \star \alpha(p_{\alpha,n}(t)) < \infty$ iff $\beta \star \alpha(t) < \infty$. Note also that $\lim_n p_{\alpha,n}(t) = 0$. Suppose that for some t, $\beta \star \alpha(t) = \infty$ and let $s > 0$. Then there is n such that $p_{\alpha,n}(t) < s$. So $\infty = \beta \star \alpha(p_{\alpha,n}(t)) \leq \beta \star \alpha(s)$. Hence $\beta \star \alpha \restriction (0,\infty)$ is the constant function with value ∞.

Suppose that $\beta \star \alpha \restriction (0,\infty)$ is not the constant ∞. So $\mathrm{Rng}(\beta \star \alpha) \subseteq [0,\infty)$. Note that $q_{\alpha,0} = \alpha \circ p_{\alpha,0} = \alpha \circ (\mathrm{Id} + \alpha)^{-1}$. So $\lim_{t\to\infty} q_{\alpha,0}(t) = \infty$. For β we have $\lim_{t\to\infty} \beta(t) = \infty$. It follows that $\lim_{t\to\infty} \beta \star \alpha(t) \geq \lim_{t\to\infty} \beta(q_{\alpha,0}(t)) = \infty$.

The strict increasingness of β and all the $q_{\alpha,n}$'s together with the fact that $\beta \star \alpha(t) < \infty$ for every t, implies that $\beta \star \alpha$ is strictly increasing.

It remains to show that $\beta \star \alpha$ is continuous. Let $a \in (0,\infty)$, and we show that $\sum_n \beta(q_{\alpha,n}(t))$ is uniformly convergent in $[0,a]$. Let $\varepsilon > 0$. There is n such that $\sum_{k\geq n} \beta(q_{\alpha,k}(a)) < \varepsilon$. From the increasingness of β and all the $q_{\alpha,n}$'s it follows that $\sum_{k\geq n} \beta(q_{\alpha,k}(t)) < \varepsilon$ for all $t \in [0,a]$. So $\sum_n \beta(q_{\alpha,n}(t))$ is uniformly convergent in $[0,a]$. Hence $\beta \star \alpha$ is continuous. ■

QUESTION 12.13. (a) Let $\alpha, \beta \in \mathrm{MC}$. Is it true that either $\beta \star \alpha \restriction (0,\infty)$ is the constant function ∞, or $\beta \star \alpha$ belongs to MC?

(b) Let $\alpha_1, \alpha_2, \beta \in \mathrm{MC}$. Is the following statement true: if $\alpha_1 \preceq \alpha_2$, then $\beta \star \alpha_2 \preceq \beta \star \alpha_1$?

(c) Let $\alpha \in \mathrm{MC}$. Is there $\beta \in \mathrm{MC} - \Gamma_\alpha$ such that Γ_β is α-star-closed? □

PROPOSITION 12.14. *Let $K > 0$, $r \in (0,1)$, $\alpha(t) = Kt$ and $\beta(t) = t^r$. Then there is C such that $\beta \star \alpha = C \cdot \beta$.*

Proof. Abbreviate $q_{\alpha,n}(t)$ by q_n. Let $t \geq 0$. Then

$$q_n = \left(\frac{1}{(1+K)^n} - \frac{1}{(1+K)^{n+1}}\right) \cdot t = \frac{1}{(1+K)^n} \cdot \frac{Kt}{1+K}$$

and hence

$$\beta\star\alpha = \sum_{n=0}^{\infty} \frac{1}{(1+K^r)^n} \cdot \left(\frac{Kt}{1+K}\right)^r = \frac{(1+K)^r}{(1+K)^r - 1} \cdot \frac{K^r}{(1+K)^r} \cdot t^r = \frac{K^r}{(1+K)^r - 1} \cdot \beta(t).$$

So $C = K^r/((1+K)^r - 1)$. ∎

Lemma 12.17(b) is our next main step. It is preceded by two propositions. Part (a) of 12.17 is also a step in the proof of 12.17(b). For $\alpha \in \mathrm{MC}$, a normed space E and $x, y \in E$ let $\mathrm{prt}_\alpha(x,y)$ be the point z in the line segment $[x,y]$ such that $\alpha(\|z-y\|) = \|x-z\|$.

PROPOSITION 12.15. *Let $\alpha \in \mathrm{MC}$ and $a > 0$. Then there is $\varepsilon = \varepsilon_{\alpha,a}$ such that the following holds. If F is a normed space, M is a closed subspace of F or a closed half space of F, $x \in F - M$ and $d(x,M) = a$, then for every $y \in \mathrm{bd}(M)$: if $d(x,y) < a + \varepsilon$, then $\langle x, \mathrm{prt}_\alpha(x,y)\rangle$ is 2α-submerged in $F - M$.*

Proof. Let $q(t) = q_{\alpha,0}(t)$ and $f(t) = q(t) + (2\alpha)^{-1}(q(t))$. Then $f(t) = q(t) + \alpha^{-1}(\frac{1}{2}q(t)) < q(t) + \alpha^{-1}(q(t))$. In particular, $f(a) < q(a) + \alpha^{-1}(q(a)) = a$. So there is $\varepsilon > 0$ such that for every t: if $|t-a| < \varepsilon$, then $f(t) < (f(a)+a)/2$. Let $y \in \mathrm{bd}(M)$ be such that $d(x,y) < a + \varepsilon$. Then

$$\begin{aligned}\|x - \mathrm{prt}_\alpha(x,y)\| + (2\alpha)^{-1}(\|x - \mathrm{prt}_\alpha(x,y)\|) &= q(\|x-y\|) + (2\alpha)^{-1}(q(\|x-y\|)) \\ &= f(\|x-y\|) < \frac{f(a)+a}{2} < a = \delta^{F-M}(x).\end{aligned}$$

So $\langle x, \mathrm{prt}_\alpha(x,y)\rangle$ is 2α-submerged in $F - M$. ∎

PROPOSITION 12.16. *Let $\alpha \in \mathrm{MC}$, F be a normed space, M be a closed subspace of F or a closed half space of F, $x \in F - M$ and $y \in M$. Then there is a sequence $\{x_i \mid i \in \mathbb{N}\}$ such that:*

(i) $x_0 = x$,
(ii) *for every* $i \in \mathbb{N}$, $\langle x_i, x_{i+1}\rangle$ *is* 2α*-submerged in* $F - M$,
(iii) *for every* $i \in \mathbb{N}$, $\|x_i - x_{i+1}\| \leq q_{\alpha,i}(\|x-y\|)$,
(iv) $\lim_i x_i$ *exists and* $\lim_i x_i \in \mathrm{bd}(M)$,
(v) $\|\lim_i x_i - y\| \leq 2\|x-y\|$.

Note that the convergence of $\{x_i \mid i \in \mathbb{N}\}$ follows from (iii), *and need not be required.*

Proof. Write $p_{\alpha,i} = p_i$ and $q_{\alpha,i} = q_i$. Note that $p_1 \circ p_i = p_{i+1}$ and that $q_0 \circ p_i = q_i$. Let $x_0 = x$ and $y_0 = y$. We define by induction $x_i \in F - M$ and $y_i \in \mathrm{bd}(M)$. Suppose that x_i, y_i have been defined. Let $y_{i+1} \in \mathrm{bd}(M)$ be such that $\|x_i - y_{i+1}\| \leq \|x_i - y_i\|$ and $\langle x_i, \mathrm{prt}_\alpha(x_i, y_{i+1})\rangle$ is 2α-submerged in $F - M$. The existence of such y_{i+1} is ensured by Proposition 12.15. Let $x_{i+1} = \mathrm{prt}_\alpha(x_i, y_{i+1})$. (Note that if for some $\bar{y} \in M$, $d(x,M) = \|x - \bar{y}\|$, then y_i can be chosen to be $\bar{y}$ for every $i \geq 1$.)

By the definitions, clauses (i) and (ii) hold. We prove (iii). We prove by induction on i that $\|x_i - x_{i+1}\| \leq q_i(\|x-y\|)$ and $\|x_{i+1} - y_{i+1}\| \leq p_{i+1}(\|x-y\|)$. It is trivial that the induction hypotheses hold for $i = 0$. Suppose that the induction hypotheses hold for $i - 1$. Then

$$\|x_i - x_{i+1}\| = q_0(\|x_i - y_{i+1}\|) \le q_0(\|x_i - y_i\|) \le q_0(p_i(\|x - y\|)) = q_i(\|x - y\|),$$
$$\|x_{i+1} - y_{i+1}\| = p_1(\|x_i - y_{i+1}\|) \le p_1(\|x_i - y_i\|) \le p_1(p_i(\|x - y\|)) = p_{i+1}(\|x - y\|).$$

So (iii) holds.

We prove (iv). Obviously, $\sum_{i=0}^{\infty} q_i(\|x-y\|) = \|x-y\|$. Since $\|x_i - x_{i+1}\| \le q_i(\|x-y\|)$, it follows that $\sum_{i=0}^{\infty} \|x_i - x_{i+1}\|$ is convergent. So $\{x_i \mid i \in \mathbb{N}\}$ is convergent. Let $\bar{x} = \lim_i x_i$. The facts $\lim_i p_i(\|x - y\|) = 0$ and $\|x_i - y_i\| \le p_i(\|x - y\|)$ imply that $\lim_i \|x_i - y_i\| = 0$. Since $y_i \in \mathrm{bd}(M)$, it follows that $\bar{x} \in \mathrm{bd}(M)$.

We prove (v):

$$\|\bar{x} - x\| \le \sum_{i=0}^{\infty} \|x_i - x_{i+1}\| \le \sum_{i=0}^{\infty} q_i(\|x - y\|) = \|x - y\|.$$

So $\|\bar{x} - y\| \le \|\bar{x} - x\| + \|x - y)\| \le 2\|x - y\|$. ■

Lemma 12.17. *Assume that clauses* (1)–(7) *of Lemma* 12.10 *hold. That is*,

(1) *Γ, Σ are countably generated moduli of continuity, and Ω is the modulus of continuity generated by $\Gamma \cup \Sigma$.*

(2) *$X \subseteq E$ and $Y \subseteq F$ are open subsets of the normed spaces E and F, X is Γ-LIN-bordered and Y is Σ-LIN-bordered.*

(3) *$\tau \in \mathrm{EXT}^{\pm}(X, Y)$, G is a Γ-appropriate subgroup of $\mathrm{EXT}(X)$, H is a Δ-appropriate subgroup of $\mathrm{EXT}(Y)$ and $G^{\tau} = H$.*

(4) *$x \in \mathrm{bd}(X)$, $\langle \varphi, A, r\rangle$ is a boundary chart element for x, $\gamma \in \Gamma$ and φ is γ-bicontinuous.*

(5) *$y \in \mathrm{bd}(Y)$, $\langle \psi, B, s\rangle$ is a boundary chart element for y, $\sigma \in \Sigma$ and ψ is σ-bicontinuous.*

(6) *$\tau^{\mathrm{cl}}(x) = y$ and $\tau(\varphi(\mathrm{BCD}^E(A, r))) \subseteq \psi(\mathrm{BCD}^F(B, s))$.*

(7) *Set $L = \mathrm{bd}(A)$, $\widehat{X} = \mathrm{BCD}^E(A, r) \cup (L \cap B(0, r))$, $\hat{\tau} = \psi^{-1} \circ \tau^{\mathrm{cl}} \circ \varphi$, $\widehat{Y} = \hat{\tau}(\widehat{X})$, $\widetilde{Y} = \hat{\tau}(\mathrm{BCD}^E(A, r))$ and $\mathcal{P} = \{(v + L) \cap \widehat{X} \mid v \in \widehat{X}\}$.*

Assume further that

(8) *Ω is Σ-star-closed.*

The the following hold:

(a) *Let $M = \mathrm{bd}(B)$. Then there is $W \in \mathrm{Nbr}^F(0)$ and $\omega \in \Omega$ such that for every $x \in (\widehat{Y} - M) \cap W$ and $y \in \widehat{Y} \cap M \cap W$, $\|\hat{\tau}^{-1}(x) - \hat{\tau}^{-1}(y)\| \le \omega(\|x - y\|)$.*

(b) *$\hat{\tau}^{-1}$ is Ω-continuous at $\hat{\tau}(0)$.*

Proof. (a) Let $\alpha \in \Sigma$ be such that Ω is α-star-closed. It is easy to see that α may be chosen to be in MBC. Note that $\widetilde{Y} = \widehat{Y} - M$. Let $\hat{\eta} = \hat{\tau}^{-1}$. By Lemma 12.10, there are $\varrho \in \Omega$ and $W_1 \in \mathrm{Nbr}^F(0^F)$ such that for every $u, v \in W_1 \cap \widetilde{Y}$: if $\langle u, v\rangle$ is 2α-submerged in $\widetilde{Y}$, then $\|\hat{\eta}(u) - \hat{\eta}(v)\| \le \varrho(\|u - v\|)$. Let $\nu \in \Omega$ and $a > 0$ be such that $\varrho \star \alpha \restriction [0, a] \le \nu \restriction [0, a]$.

Let $\mathcal{P} = \{(v + L) \cap \widehat{X} \mid v \in \widehat{X}\}$. By Lemma 12.6, $\hat{\tau}$ is inversely $\langle \Delta, \mathcal{P} \rangle$-continuous at 0^E. Note that $\hat{\tau}(L \cap \widehat{X}) = M \cap \widehat{Y}$, that is, $M \cap \widehat{Y} \in \hat{\tau}(\mathcal{P})$. So there are $\sigma \in \Sigma$ and $W_2 \in \mathrm{Nbr}^F(0^F)$ such that for every $u, v \in W_2 \cap M \cap \widehat{Y}$, $\|\hat{\eta}(u) - \hat{\eta}(v)\| \le \sigma(\|u - v\|)$. Choose $s_0 \in (0, a/2)$ such that $\bar{B}(0^F, 6s_0) \cap \mathrm{BCD}^F(B, s) \subseteq \widehat{Y} \cap W_1 \cap W_2$ and let $W = B(0^F, s_0)$.

Let $x \in (\widehat{Y} - M) \cap W$ and $y \in \widehat{Y} \cap M \cap W$. Let $\{x_i \mid i \in \mathbb{N}\}$ be the sequence ensured by Proposition 12.16 and $\bar{x} = \lim_i x_i$. Note that by (iii) of 12.16, $\sum_{i \in \mathbb{N}} \|x_i - x_{i+1}\| \le \|x - y\| < 2s_0$. So $\|x_n\| \le \|x\| + \sum_{i=0}^{n-1} \|x_i - x_{i+1}\| < 3s_0$ for every $n \in \mathbb{N}$. Similarly, $\|\bar{x}\| < 3s_0$. Hence $\{x_i \mid i \in \mathbb{N}\} \subseteq W_1 \subseteq \mathrm{Dom}(\hat{\eta})$ and $\bar{x} \in W_2 \subseteq \mathrm{Dom}(\hat{\eta})$. We conclude that

$$\|\hat{\eta}(x) - \hat{\eta}(y)\| \le \sum_{i=0}^{\infty} \|\hat{\eta}(x_i) - \hat{\eta}(x_{i+1})\| + \|\hat{\eta}(\bar{x}) - \hat{\eta}(y)\| := \mathsf{A}.$$

Since $\bar{x}, y \in W_2 \cap M \cap \widehat{Y}$, we have $\|\hat{\eta}(\bar{x}) - \hat{\eta}(y)\| \le \sigma(\|\bar{x} - y\|)$.

By (ii) of 12.16, $\langle x_i, x_{i+1} \rangle$ is 2α-submerged in $F - M$. Using the facts that $x_i \in B(0, 3s_0)$ and that $B(0^F, 6s_0) \cap \mathrm{BCD}^F(B, s) \subseteq \widehat{Y}$, it is easily seen that $\delta^{\widetilde{Y}}(x_i) = \delta^{F-M}(x_i)$ for every $i \in \mathbb{N}$. So $\langle x_i, x_{i+1} \rangle$ is 2α-submerged in $\widetilde{Y}$. This, together with the fact that $x_i, x_{i+1} \in W_1$, implies that $\|\hat{\eta}(x_i) - \hat{\eta}(x_{i+1})\| \le \varrho(\|x_i - x_{i+1}\|)$. Hence

$$\mathsf{A} \le \sum_{i=0}^{\infty} \varrho(\|x_i - x_{i+1}\|) + \sigma(\|\bar{x} - y\|) := \mathsf{B}.$$

By the increasingness of $q_{\alpha,i}$ and clause (iii) in Proposition 12.16,

$$\sum_{i=0}^{\infty} \varrho(\|x_i - x_{i+1}\|) \le \sum_{i=0}^{\infty} \varrho(q_{\alpha,i}(\|x - y\|)) = \varrho{\star}\alpha(\|x - y\|).$$

Clause (v) in 12.16 implies that $\sigma(\|\bar{x} - y\|) \le \sigma(2\|x - y\|)$. Hence

$$\mathsf{B} \le \varrho{\star}\alpha(\|x - y\|) + \sigma \circ (2 \cdot \mathrm{Id})(\|x - y\|).$$

Recall that $\nu \in \Omega$, $\varrho{\star}\alpha{\restriction}[0, a] \le \nu{\restriction}[0, a]$ and $s_0 < a/2$. Let $\omega = \nu + \sigma \circ (2 \cdot \mathrm{Id})$. It follows from the above that $\omega \in \Omega$ and $\|\hat{\eta}(x) - \hat{\eta}(y)\| \le \omega(\|x - y\|)$. This proves (a).

(b) We use the notations of (a). Let $x, y \in W \cap \widehat{Y}$. If $x, y \in M$, then $\|\hat{\eta}(x) - \hat{\eta}(y)\| \le \sigma(\|x - y\|)$. If $x \notin M$ and $y \in M$ or vice versa, then $\|\hat{\eta}(x) - \hat{\eta}(y)\| \le \omega(\|x - y\|)$. Suppose that $x, y \notin M$ and write $\beta = 2\alpha$. If $\langle x, y \rangle$ is β-submerged in $\widetilde{Y}$ or $\langle y, x \rangle$ is β-submerged in $\widetilde{Y}$, then $\|\hat{\eta}(x) - \hat{\eta}(y)\| \le \varrho(\|x - y\|)$.

Suppose that neither $\langle x, y \rangle$ nor $\langle y, x \rangle$ are β-submerged in $\widetilde{Y}$. Since $x, y \in B(0, s_0)$ and $B(0, 6s_0) \cap \mathrm{BCD}^F(B, s) \subseteq \widehat{Y}$, $\delta^{F-M}(x) = \delta^{\widetilde{Y}}(x)$ and $\delta^{F-M}(y) = \delta^{\widetilde{Y}}(y)$. So by the non-submergedness of $\langle x, y \rangle$ and $\langle y, x \rangle$, $\delta^{F-M}(x), \delta^{F-M}(y) < \|x - y\| + \beta^{-1}(\|x - y\|)$. Since $\beta \in \mathrm{MBC}$, $\beta^{-1}(t) \le t$ for every t. So $\delta^{F-M}(x), \delta^{F-M}(y) < 2\|x - y\|$.

Let $\bar{x}, \bar{y} \in M$ be such that $\|x - \bar{x}\| < 2\delta^{F-M}(x)$ and $\|y - \bar{y}\| < 2\delta^{F-M}(y)$. Clearly, $\|\bar{x}\| < 3\|x\| < 3s_0$. Hence $\bar{x} \in W_2 \cap M \cap \widehat{Y}$. Similarly, $\bar{y} \in W_2 \cap M \cap \widehat{Y}$. We also have

$$\|\bar{x} - \bar{y}\| \le \|\bar{x} - x\| + \|x - y\| + \|y - \bar{y}\| \le 2\delta^{F-M}(x) + \|x - y\| + 2\delta^{F-M}(y) \le 9\|x - y\|$$

and $\|x - \bar{x}\|, \|y - \bar{y}\| < 4\|x - y\|$. The final estimate is

$$\begin{aligned}\|\hat{\eta}(x) - \hat{\eta}(y)\| &\le \|\hat{\eta}(x) - \hat{\eta}(\bar{x})\| + \|\hat{\eta}(\bar{x}) - \hat{\eta}(\bar{y})\| + \|\hat{\eta}(\bar{y}) - \hat{\eta}(y)\| \\ &\le \omega(\|x - \bar{x}\|) + \sigma(\|\bar{x} - \bar{y}\|) + \omega(\|\bar{y} - y\|) \\ &\le \omega(4\|x - y\|) + \sigma(9\|x - y\|) + \omega(4\|x - y\|) \le 8\omega(\|x - y\|) + 9\sigma(\|x - y\|).\end{aligned}$$

Clearly, $\gamma := 8\omega + 9\sigma \in \Omega$. Obviously, $\sigma, \omega, \varrho \le \gamma$. We have thus shown that for every $x, y \in W \cap \widehat{Y}$, $\|\hat{\eta}(x) - \hat{\eta}(y)\| \le \gamma(\|x - y\|)$. So $\hat{\eta}$ is Ω-continuous at 0^F. ∎

We make a last trivial observation before proving the main theorem.

PROPOSITION 12.18. (a) *Let Γ be a modulus of continuity and $\alpha \in \mathrm{MBC} - \Gamma$. Let X be an open subset of a normed space E and $x \in \mathrm{bd}(X)$. Then there is $g \in H(E)\lfloor X \rfloor$ such that g is $9 \cdot \alpha \circ \alpha$-bicontinuous and g is not Γ-bicontinuous at x.*

(b) *Let Γ, Δ be moduli of continuity, E, F be normed spaces, $X \subsetneqq E$ be an open Γ-LIN-bordered set, $Y \subseteq F$ be an open Δ-LIN-bordered set, $G \le \mathrm{EXT}(X)$ and $H \le \mathrm{EXT}(Y)$ be respectively Γ-appropriate and Δ-appropriate, $\tau \in (H_{\Delta}^{\mathrm{BDR.LC}})^{\pm}(X, Y)$ and $G^{\tau} = H$. Then $\Gamma = \Delta$.*

Proof. (a) For $r > 0$ define $g_r : E \cong E$ as follows: $g_r(0) = 0$,

$$g_r(z) = \frac{r}{\alpha(r)} \cdot \alpha(\|z\|) \cdot \frac{z}{\|z\|} \quad \text{if } \|z\| \in (0, r),$$

and $g_r(z) = z$ if $\|z\| \ge r$. Obviously, $\mathrm{supp}(g_r) = B(0, r)$, and it is left to the reader to check that g_r is $\frac{3r}{\alpha(r)} \cdot \alpha$-bicontinuous, and that if $\gamma \in \mathrm{MC}$ is such that g_r is γ-bicontinuous, then $\gamma \restriction [0, r] \ge \frac{r}{\alpha(r)} \cdot \alpha \restriction [0, r]$. For $y \in E$ define $g_{y,r} = g_r^{\mathrm{tr}_y}$. Let $\{B(x_i, r_i) \mid i \in \mathbb{N}\}$ be a sequence of pairwise disjoint balls such that for every i, $B(x_i, r_i) \subseteq X$ and $\lim_i x_i = x$, and let $g = \circ_i\, g_{x_i, r_i} \restriction X$. Then g is as required.

(b) First we show that $\Delta \subseteq \Gamma$. Suppose otherwise. Let $x \in \mathrm{bd}(X)$ and $y = \tau^{\mathrm{cl}}(x)$. So $y \in \mathrm{bd}(Y)$. There are $W \in \mathrm{Nbr}(y)$ and $\beta \in \Delta$ such that $\tau^{-1} \restriction (W \cap Y)$ is β-bicontinuous. Let $V \in \mathrm{Nbr}(y)$ such that $V \subseteq W$ and $H_{\Delta}^{\mathrm{CMP.LC}}(Y)\lfloor V \cap Y \rfloor \subseteq H$. Choose $\alpha \in \Delta \cap \mathrm{MBC} - \Gamma$ and define $\bar{\alpha} = 9 \cdot \alpha \circ \alpha$ and $\delta = \beta \circ \bar{\alpha} \circ \beta$. Let $U = \tau^{-1}(V \cap Y)$. Hence $x \in \mathrm{bd}(U)$.

Let X' be an open subset of $U \cap X$ such that $\mathrm{cl}(X') \cap \mathrm{bd}(X) = \{x\}$. By (a), there is $g' \in H(E)\lfloor X' \rfloor$ such that g' is $\bar{\alpha}$-bicontinuous, and g' is not Γ-bicontinuous at x. Let $g = g' \restriction X$ and $h = g^{\tau}$. Since g is E-biextendible and τ is (E, F)-biextendible, h is F-biextendible. From the fact that $\tau \restriction (U \cap X)$ is β-bicontinuous, it follows that $h \restriction (V \cap Y)$ is δ-bicontinuous. We wish to conclude that h is δ-bicontinuous. Indeed, this follows from the facts: $\mathrm{cl}^F(\mathrm{supp}(h)) \subseteq (V \cap Y) \cup \{y\}$ and $y \in \mathrm{cl}(V \cap Y)$. (The same argument appears in the proof 12.10, where it is proved that $h \in H_{\Sigma}^{\mathrm{CMP.LC}}(Y)$.) Obviously, $\delta \in \Delta$, so $h \in H_{\Delta}^{\mathrm{CMP.LC}}(Y)\lfloor V \cap Y \rfloor \subseteq H$. Recall that $G^{\tau} = H$, hence $g = h^{\tau^{-1}} \in G$. But g is not Γ-bicontinuous at x. This contradicts the fact that G is Γ-appropriate. Hence $\Delta \subseteq \Gamma$.

It follows that $\tau \in (H_{\Gamma}^{\mathrm{BDR.LC}})^{\pm}(X, Y)$ and hence $\tau^{-1} \in (H_{\Gamma}^{\mathrm{BDR.LC}})^{\pm}(Y, X)$. We now repeat the above argument for τ^{-1}. So the roles of Γ and Δ are interchanged, and we conclude that $\Gamma \subseteq \Delta$. ∎

12.3. Final results

THEOREM 12.19 (Main Theorem of Chapter 12). *Assume that*

(1) Γ, Δ *are countably generated moduli of continuity*, $\Gamma \subseteq \Delta$ *and* Δ *is* Γ*-star-closed.* (*Or assume the special cases*: (i) Γ *is principal and* $\Delta = \Gamma$, *or* (ii) $\Gamma = \Gamma^{\mathrm{LIP}}$ *and* $\Delta = \Gamma^{\mathrm{HLD}}$.)
(2) $X \subsetneqq E$ *and* $Y \subseteq F$ *are open subsets of the normed spaces* E *and* F, X *is* Γ*-LIN-bordered*, *and* Y *is* Δ*-LIN-bordered.*
(3) $G \leq \mathrm{EXT}(X)$ *is* Γ*-appropriate*, *and* $H \leq \mathrm{EXT}(Y)$ *is* Δ*-appropriate.*
(4) $\tau \in \mathrm{EXT}^{\pm}(X, Y)$ *and* $G^{\tau} = H$.

Then $\Gamma = \Delta$ *and* $\tau \in (H_{\Gamma}^{\mathrm{BDR.LC}})^{\pm}(X, Y)$.

Proof. That (i) is a special case of (1) follows from Proposition 12.12(a) and (b), and that (ii) is a special case follows from Proposition 12.14.

Since $\Gamma \subseteq \Delta$, the modulus of continuity Ω which is generated by $\Gamma \cup \Delta$ is Δ, and since Δ is Γ-star-closed and $\Gamma \subseteq \Delta$, we see that Δ is Δ-star-closed. So Ω is Δ-star-closed. Let $x \in \mathrm{bd}(X)$. There are a boundary chart element for x, $\langle \varphi, A, r \rangle$, and $\gamma \in \Gamma$ such that φ is γ-bicontinuous. Let $y = \tau^{\mathrm{cl}}(x)$. Choose a boundary chart element for y, $\langle \psi, B, s \rangle$, and $\sigma \in \Sigma$ such that ψ is σ-bicontinuous. Also assume $\tau(\varphi(\mathrm{BCD}^{E}(A, r))) \subseteq \psi(\mathrm{BCD}^{F}(B, s))$. Set $L = \mathrm{bd}(A)$, $\widehat{X} = \mathrm{BCD}^{E}(A, r) \cup (L \cap B(0, r))$, $\hat{\tau} = \psi^{-1} \circ \tau^{\mathrm{cl}} \circ \varphi$ and $\widehat{Y} = \hat{\tau}(\widehat{X})$.

By Theorem 12.17(b), $\hat{\tau}^{-1}$ is Ω-continuous at 0^{F}. That is, $\hat{\tau}^{-1}$ is Δ-continuous at 0^{F}. Since φ, ψ are Δ-bicontinuous at 0^{E} and 0^{F} respectively, $\varphi \circ \hat{\tau}^{-1} \circ \psi^{-1}$ is Δ-continuous at y. Note that there is $V \in \mathrm{Nbr}^{F}(y)$ such that $\mathrm{Dom}(\varphi \circ \hat{\tau}^{-1} \circ \psi^{-1}) \supseteq V \cap Y$. Also, $\varphi \circ \hat{\tau}^{-1} \circ \psi^{-1} \restriction (V \cap Y) = \tau^{-1} \restriction (V \cap Y)$. Hence τ^{-1} is Δ-continuous at y. Since it is also given that $\tau \in \mathrm{EXT}^{\pm}(X, Y)$, it follows that $\tau^{-1} \in H_{\Delta}^{\mathrm{BDR.LC}}(Y, X)$.

We now reverse the roles of X and Y. Let $\eta = \tau^{-1}$. So $\eta : Y \cong X$, $H^{\eta} = G$ and the modulus of continuity Ω generated by $\Delta \cup \Gamma$ is again Δ. So Ω is Γ-star-closed.

Let $y \in \mathrm{bd}(Y)$ and $x = \eta(y)$. We choose ψ and φ and define $\hat{\eta}$ in the same way that φ, ψ and $\hat{\tau}$ were defined in the preceding argument. We thus conclude that $\hat{\eta}^{-1}$ is Ω-continuous at x. That is, $\hat{\eta}^{-1}$ is Δ-continuous at x. There is $U \in \mathrm{Nbr}^{E}(x)$ such that $\psi \circ \hat{\eta} \circ \varphi^{-1} \restriction (U \cap X) = \tau \restriction (U \cap X)$. Hence τ is Δ-continuous at x. We also need to know that $\tau \in \mathrm{EXT}^{\pm}(X, Y)$, and this is indeed given. Hence $\tau \in H_{\Delta}^{\mathrm{BDR.LC}}(X, Y)$. We proved that $\tau \in (H_{\Delta}^{\mathrm{BDR.LC}})^{\pm}(X, Y)$. By Proposition 12.18(b), $\Gamma = \Delta$. ∎

Proof of Theorem 8.9. If $X = E$ then $Y = F$ and hence $H_{\Gamma}^{\mathrm{CMP.LC}}(X) = H_{\Gamma}^{\mathrm{LC}}(X)$, and the same holds for Y. So in this case the claim of 8.9 is implied by Theorem 3.27.

Assume that $X \neq E$. We apply Theorem 12.19 to the special case that $\Gamma = \Delta$ and Γ is principal, and take G, H to be $H_{\Gamma}^{\mathrm{CMP.LC}}(X)$ and $H_{\Delta}^{\mathrm{CMP.LC}}(Y)$ respectively. So $\tau \in (H_{\Gamma}^{\mathrm{BDR.LC}})^{\pm}(X, Y)$. By Theorem 3.27, τ is locally Γ-bicontinuous. Hence $\tau \in (H_{\Gamma}^{\mathrm{CMP.LC}})^{\pm}(X, Y)$. ∎

The final reconstruction theorems of Chapters 8–12. Combining the results of the previous sections in different ways, one obtains various reconstruction theorems. Parts (a) and (b) of the following theorem are such corollaries. Part (a) is a restatement of Theorem 8.4(a). Indeed, the special case of (a) in which $\Gamma = \Gamma_{\mathrm{LIP}}$ motivated the whole work presented in Chapters 8–12.

The reconstruction theorem for the group $H_{\Gamma}^{\mathrm{BDR.LC}}(X)$ which appears in (b) is a byproduct of the proof of the main result. We thought it was worth mentioning.

In (c) we tried to capture the essence of the argument. Part (c) can be further strengthened. But it seems to be a natural stopping point.

THEOREM 12.20. *Let Γ, Δ be moduli of continuity, E and F be normed spaces and $X \subseteq E$, $Y \subseteq F$ be open. Suppose that X is locally Γ-LIN-bordered, and Y is locally Δ-LIN-bordered.*

(a) *Suppose that Γ is principal. If $\varphi : H_{\Gamma}^{\mathrm{CMP.LC}}(X) \cong H_{\Delta}^{\mathrm{CMP.LC}}(Y)$. Then $\Gamma = \Delta$ and there is $\tau \in (H_{\Gamma}^{\mathrm{CMP.LC}})^{\pm}(X,Y)$ such that $\varphi(g) = g^{\tau}$ for every $g \in H_{\Gamma}^{\mathrm{CMP.LC}}(X)$.*

(b) *Suppose that Γ is principal. If $\varphi : H_{\Gamma}^{\mathrm{BDR.LC}}(X) \cong H_{\Gamma}^{\mathrm{BDR.LC}}(Y)$. Then there is $\tau \in (H_{\Gamma}^{\mathrm{BDR.LC}})^{\pm}(X,Y)$ such that $\varphi(g) = g^{\tau}$ for every $g \in H_{\Gamma}^{\mathrm{BDR.LC}}(X)$.*

(c) *Suppose that Γ and Δ are countably generated, $\Gamma \subseteq \Delta$ and Δ is Γ-star-closed. Let $G \leq \mathrm{EXT}(X)$ be Γ-appropriate and $H \leq \mathrm{EXT}(Y)$ be Δ-appropriate. Assume further that $\mathrm{LIP}^{\mathrm{LC}}(X) \leq G$ and $\mathrm{LIP}^{\mathrm{LC}}(Y) \leq H$, and suppose that $\varphi : G \cong H$. Then $\Gamma = \Delta$, and there is $\tau \in (H_{\Gamma}^{\mathrm{BDR.LC}})^{\pm}(X,Y)$ such that $\varphi(g) = g^{\tau}$ for every $g \in G$.*

Proof. (a) By Theorem 2.8(b), there is $\tau \in H(X,Y)$ such that τ induces φ. By Theorem 3.27, $\Gamma = \Delta$ and $\tau \in (H_{\Gamma}^{\mathrm{LC}})^{\pm}(X,Y)$. By Theorem 8.8(a), $\tau \in \mathrm{EXT}^{\pm}(X,Y)$. By Theorem 8.9, $\tau \in (H_{\Gamma}^{\mathrm{CMP.LC}})^{\pm}(X,Y)$.

(b) The proof is similar to the proof of (a). However, we use Theorem 8.8(b) and not 8.8(a).

(c) By Theorem 2.8(b), there is $\tau \in H(X,Y)$ which induces φ. By Theorem 8.8(b), $\tau \in \mathrm{EXT}^{\pm}(X,Y)$. By Theorem 12.19, $\Gamma = \Delta$ and $\tau \in (H_{\Gamma}^{\mathrm{BDR.LC}})^{\pm}(X,Y)$. ∎

Proof of Theorem 8.4(a). Theorem 8.4(a) is restated as part (a) of 12.20 above. ∎

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Index of symbols

(in order of appearance)

Index of notations

(in alphabetical order)

Index of definitions

(in alphabetical order)