Contents

Abstract

We give a classification of all the countable homogeneous coloured partial orders. This generalizes the similar result in the monochromatic case given by Schmerl.

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1. Introduction

Classifying all the structures in a given class is usually an infeasible task, particularly where these are infinite, and when they have unstable theory. To cut down the class of structures under consideration, one often restricts attention to those which are 'sufficiently symmetrical'. This is partly to enable any sort of 'classification' at all to be carried out, and also because the symmetrical structures are likely to be the most interesting ones anyway. They will have rich automorphism groups, which can repay study in their own right, for instance from the point of view of reconstruction results (the problem of recognizing a structure, given what its automorphism group is).

In common with some of the existing classifications [10, 9, 8, 4], the restrictions which we adopt are those of countability and 'homogeneity' (also called 'ultrahomogeneity'), which says that any isomorphism between finite substructures extends to an automorphism.

In [10], Schmerl gave a classification of all the countable homogeneous partial orders, which we now state without proof, and introduce some notation. He showed that a countable partial order is homogeneous if and only if it is (isomorphic to) one of the following:

- a finite or countably infinite antichain,
- a finite or countably infinite union of copies of the ordered rationals $\mathbb{Q}$, elements in distinct copies being incomparable, and we refer to this as an *antichain of chains*,
- a union indexed by $\mathbb{Q}$ of antichains A_q all of the same (finite or countably infinite) size, and ordered by $x < y$ if and only if for some $q < r$, $x \in A_q$ and $y \in A_r$, and we refer to this as a *chain of antichains*,
- the generic partial order.

We carry out here a similar classification, but this time for the class of all countable homogeneous coloured partial orders, where a coloured partial order $(P, <, F)$ is a partial order $(P, <)$ together with a function F from P to a set C of 'colours' (and it is understood that automorphisms must preserve colours, as well as the ordering relation). Unless stated otherwise we always write F for the colouring function, C for the colour set, and assume that all members of C actually occur (so that F is onto).

As in the directed graph case [4], there will be $2^{\aleph_0}$ such structures of size $\aleph_0$. Hence a classification may consist for instance of a suitable description of such structures in terms of a real number parameter. The classification this time is however rather more explicit in the following sense. There is a notion of (interdense) 'component', and any of our structures may be written as a union of (convex) components. Furthermore, there is only a limited number of possibilities for components, and a general countable coloured partial order is homogeneous if and only if all finite unions of its components are. Thus, structures in our classification are determined by those in an easily described family.

In Section 2, we start our analysis of countable homogeneous coloured partial orders $(P, <, F)$ by describing a natural way to obtain the components. These are convex subsets any two of which are coloured by disjoint subsets of C, and for any such component, coloured by $C' \subseteq C$ say, for each element c of C' and $x < y$ in the component, there is a point in between x and y coloured by c. The components will be identified as the equivalence classes under a suitable quasi-order on P. Thus the task of classifying all the countable homogeneous coloured partial orders is reduced to classifying all possible components, and how they can fit together.

Now there may be infinitely many components (though only if C is itself infinite). As a trivial example we could take infinitely many copies of the rationals with each copy coloured by a distinct colour, and with the copies ordered (for instance linearly) in any fashion. This clearly gives rise to $2^{\aleph_0}$ cases of the classification. However, there is the 'compactness principle' mentioned above, which says that a partial order built up in this way is homogeneous if and only if every finite union of components is homogeneous, so the possibility of infinitely many components is not a real issue. (See Lemma 2.2 for the precise statement.) What is much more interesting is that the whole structure is controlled by the substructures which are unions of at most three components. This means that the main body of our work is an analysis of the possible configurations of one, two, or three components. We also have to establish that the general case reduces to these ones.

In Section 3 we give the description of all the possibilities for up to three components. For one component the classification is based on Schmerl's list, and indeed we can derive our result fairly directly from his. There are essentially three types, chains of antichains, antichains of (trivial or non-trivial) chains, and generic as before, except that the presence of colours makes their description rather more involved. For two components P_1 and P_2, there are always 'trivial cases', namely empty and complete, meaning that either all points of P_1 are incomparable with all points of P_2 (empty), or all points of P_1 are below all points of P_2 (complete), or the other way round. Apart from these, there are a number of 'generic' cases, which are obtained by a suitable Fraïssé amalgamation. There are a few others, which we may describe as 'strongly non-orthogonal' relations. A typical example is a perfect matching (bijection) between two distinctly coloured antichains. This is really just a disguised version of a single antichain, and while in the full classification this case has of course to be included, it can still be encoded in a relatively simple way. The complement of this relation is also possible, and is the 'complement of a perfect matching'. Another in this category can be referred to as 'shuffling', which applies specifically to comparable chains of antichains. For three components, the essential cases are '3-chain lemmas' describing the possible configurations of three components which form a chain, and 'V-shape lemmas' where one of the three components is below the other two, and the top two are incomparable. (The duals of V-shape lemmas are 'Λ-shape lemmas'; and in the other cases, one of the components is incomparable with both the others, so it reduces to the case of two components.) For each of these we have in principle to consider what the possibilities are for the two-component substructures, and then see in what ways all three can fit together. There are quite a number of cases, but they can be described reasonably systematically.

'Officially' the classification we give is done by means of structures that we call 'skeletons'. The idea is that a skeleton will be a simplified structure, which encodes in a clear manner which of the structures in the list we are referring to. We shall first introduce what is meant by the *skeleton of* a countable homogeneous coloured partial order. This will be taken to be the set of components labelled by their isomorphism types, and with comparable pairs labelled by the isomorphism type of the corresponding 2-component restriction. This makes more sense once we know what the possibilities are for the 1- and 2-component structures. One can then consider a notion of 'abstract skeleton', which will be a (countable) partial order labelled by possible labels as just mentioned on points and comparable pairs. Not all such labelled partially ordered sets will actually be realized as the skeletons of countable homogeneous coloured partial orders, so we must impose some additional conditions. It turns out that to be so realized it is necessary and sufficient that all 3-element substructures should be realized, and we can give explicit conditions for when this occurs, so as to provide a complete and correct definition of abstract skeleton.

In practice we may represent skeletons by using suggestive symbols instead for the isomorphism types of 1- and 2-component structures in a natural way. Thus the components may be labelled first of all as CA (for chain of antichains) with further information termed its 'colour structure partition' describing how it is built up from coloured antichains, A (for antichain) with its cardinality and colour, AC (for antichain of chains) with number of antichains and the colour set, or Ge (for generic) with colour set. Given this it suffices to label the 2-component structures where the two are comparable, by C, SH, PM, CPM, G (for complete, shuffled chains of antichains, perfect matching or its complement, or generic, respectively).

In terms of skeletons, we may now state the main result (see Theorem 10.2):

Theorem. *Associated with any countable homogeneous coloured partial order $\mathcal{P}$ is a skeleton $\mathcal{Q}$. Conversely, given any abstract skeleton $\mathcal{Q}$, there is a countable homogeneous coloured partial order $\mathcal{P}$ having $\mathcal{Q}$ as its skeleton, and $\mathcal{P}$ is uniquely determined up to isomorphism.*

Thus the family of skeletons provides a classification of all the structures in our class.

Now describing the countable homogeneous structures of a particular relational similarity type is equivalent to determining all the amalgamation classes of finite structures in that class, as was shown by Fraïssé. Specifically, if $\mathcal{A}$ is a countable homogeneous structure in a countable relational language, then the *age* of $\mathcal{A}$, which is defined to be the class of all structures isomorphic to finite substructures of $\mathcal{A}$, has the following properties: it is closed under isomorphisms and substructures, it has at most countably many members up to isomorphism, and it has the joint embedding and amalgamation properties. Conversely, for any class of finite structures fulfilling these properties (an *amalgamation class*), there is a countable homogeneous structure unique up to isomorphism, whose age is the class that we started with. See [7] for instance. Often the homogeneous structure arising from a given amalgamation class in this way is referred to as the corresponding

Fraïssé generic or *Fraïssé limit*. So classifying a class of countable homogeneous structures is equivalent to classifying the corresponding amalgamation classes, and we shall pass freely between the two as the occasion arises.

A Fraïssé generic structure which will feature frequently in the paper is the 'C-coloured version of the rationals' $\mathbb{Q}_C$ where C is a finite non-empty or countable set. This has domain (isomorphic to) $\mathbb{Q}$, and is 'interdense', meaning that between any two points, there are points of all possible colours. It exists and is unique up to isomorphism, and it arises as the Fraïssé limit of the class of all finite C-coloured linear orders, so is homogeneous.

Let us further illustrate these ideas by demonstrating the existence of a 'generic coloured partial order' P_C for any non-empty finite or countable colour set C. This will be extensively needed throughout, and directly generalizes one of Schmerl's partial orders (his is the case where C has size 1, which gives the 'generic' or 'countable universal-homogeneous' partial order).

By Fraïssé's Theorem, provided we can show that the class of all finite C-coloured partial orders is an amalgamation class, then there is a corresponding homogeneous structure. The only clause causing any difficulty is amalgamation, which says that for any structures $\mathcal{A}_0$, $\mathcal{A}_1$, and $\mathcal{A}_2$ in the class, and embeddings of $\mathcal{A}_0$ into each of $\mathcal{A}_1$ and $\mathcal{A}_2$, there is a structure $\mathcal{B}$ in the class, and embeddings of $\mathcal{A}_1$ and $\mathcal{A}_2$ into $\mathcal{B}$ such that the 'diagram commutes', that is, the compositions of the maps from $\mathcal{A}_0$ to $\mathcal{B}$ via $\mathcal{A}_1$ and $\mathcal{A}_2$ are equal. In practice by taking suitable copies we may take the embeddings from $\mathcal{A}_0$ into $\mathcal{A}_1$ and $\mathcal{A}_2$ to be inclusions, and if necessary also assume that A_0 actually equals the intersection of A_1 and A_2.

In the present case, suppose therefore that $(A_i, <_i, F_i)$ are finite C-coloured partial orders such that $A_0 = A_1 \cap A_2$. Let $B = A_1 \cup A_2$ be coloured by $F = F_1 \cup F_2$, and be partially ordered by the transitive closure $<$ of $<_1 \cup <_2$. This is transitive by definition. To show that $<$ is irreflexive, suppose otherwise, so that $x < x$ for some x. This must mean that there is a finite sequence $x = x_0 R_0 x_1 R_1 x_2 \dots R_{n-1} x_n = x$ where each R_i is either $<_1$ or $<_2$, and $n \geq 1$. Suppose such a sequence is chosen with n minimal. Then by transitivity, and minimality of n, no two consecutive R_is are equal, so they must alternate between $<_1$ and $<_2$. But if $x_{i-1} <_1 x_i <_2 x_{i+1}$ it follows that $x_i \in A_1 \cap A_2 = A_0$. Therefore every x_i lies in A_0, and so actually $x = x_0 <_0 x_1 <_0 \cdots <_0 x_n = x$, which leads to $x <_0 x$, contradiction.

Finally we have to see that $(A_1, <_1, F_1)$ and $(A_2, <_2, F_2)$ embed in $(B, <, F)$. Supposing for instance that $x < y$ where $x, y \in A_1$, then again there is a sequence of minimal length establishing this. As before $<_1$ and $<_2$ must alternate, and all intermediate entries must lie in A_0, so actually the whole sequence lies in A_1, and $x <_1 y$ follows by transitivity of $<_1$.

The paper is organized as follows.

In Section 2 we define 'colour class components', 'components' for short, and define 'skeleton of a structure'.

In Section 3 we list all the structures in our classification having at most three components.

In Section 4 we define (abstract) skeleton, in two versions as mentioned above, i.e. via isomorphism types or conventional labels.

Section 5 gives a complete and direct description of all structures in the classification whose components are all chains of antichains and whose comparabilities are all 'SH' (shuffled version of $<$).

In Section 6 we define 'reduced skeleton' and show how the main result can be deduced from the special case in which the skeleton is reduced.

In Section 7 we show that the natural class of finite structures corresponding to any reduced skeleton is an amalgamation class. This gives existence in all cases by Fraïssé's Theorem.

In Sections 8 and 9 we give detailed proofs, in Section 8 justifying the restrictions earlier stated on which configurations can arise for up the three components, and in Section 9 demonstrating uniqueness.

Finally in Section 10 we complete the proof of the main theorem, and make a few other remarks.

We conclude the introduction by mentioning further definitions and notation we shall use throughout. As above, we usually take a partial order as being given by a 'strict' relation, that is, $<$ rather than $\leq$, though occasionally we may use $\leq$. We write $\|$ for the relation of incomparability on members of a partial order. We may use set notation such as $A <_c B$ (for A is 'completely below' B) or $A \parallel B$ to mean that every element of A is less than (incomparable with respectively) every member of B. A *chain* is a linearly ordered set (or a subset of a partially ordered set which is linearly ordered by the induced relation). An *antichain* is a partially ordered set in which every two distinct elements are incomparable. A 3-element partial order $\{x, y, z\}$ with $x > y, z$ and $y \parallel z$ is referred to as a Λ-*shape* (with colouring unspecified) and a 3-element partial order $\{x, y, z\}$ with $x < y, z$ and $y \parallel z$ is referred to as a V-*shape*. As usual we write $[x, y] = \{z : x \leq z \leq y\}$ and $(x, y) = \{z : x < z < y\}$ and also $[x, y)$ and $(x, y]$. Note that these are not necessarily linearly ordered sets, but since we are only dealing with partial orders, there is no danger of confusion. We often distinguish a structure, and its domain, by using a script letter such as $\mathcal{P}$ or $\mathcal{Q}$ for the former, and the corresponding plain letter, P, Q, for the latter.

2. Colour classes, components, and skeletons

Our initial goal here is to describe a natural way of dividing a countable homogeneous coloured partial order $(P, <, F)$ into pieces, called 'components'. The classification will be given in terms of what these components can be, and how they 'fit together'. We think of the components P' as being 'interdense' with respect to the set C' of colours of points of P', where a set is said to be *interdense* if between any two comparable members $x < y$ of P', points of all possible colours in C' occur. The route taken to describe these subsets is as follows.

Let $(P, <, F)$ be a countable homogeneous coloured partial order with colour set C. We define the relation $\preceq$ on C by $c_1 \preceq c_2$ if there are $x_1 \leq x_2$ in P coloured by c_1 and c_2

respectively. This is a quasi-order (reflexive and transitive). Reflexivity is trivial. To see that it is transitive, let $c_1 \preceq c_2 \preceq c_3$, and choose $x_1 \leq x_2$, $y_2 \leq y_3$ such that $F(x_1) = c_1$, $F(x_2) = F(y_2) = c_2$ and $F(y_3) = c_3$. By homogeneity there is an automorphism θ taking y_2 to x_2, and then $x_1 \leq \theta(y_3)$ and as $F(x_1) = c_1$ and $F(\theta(y_3)) = c_3$ we find that $c_1 \preceq c_3$. (Note that 1-transitivity is enough for this part of the argument.)

The equivalence classes of C determined by this quasi-order, under the relation given by $c_1 \approx c_2$ if $c_1 \preceq c_2 \preceq c_1$, are called *colour classes*. For any colour class C' we call the subset $\{x \in P : F(x) \in C'\}$ of P the corresponding *colour class component*, or 'component' for short. It is clear that if P has just one colour class component then it is interdense. Conversely, if $\mathcal{P}$ is interdense and is not an antichain, then it has just one component. Note that we shall occasionally refer to 'connected components' of a partial order. For these we shall always use the whole expression 'connected components', and they should not be confused with colour class components.

Associated with $(P, <, F)$ are thus two partial orders. One is the family of colour classes under the partial ordering induced from $\preceq$, and the other is the corresponding family of colour class components. We shall see that where one component P_1 is below another P_2 under this partial ordering, this may arise in more than one way; for instance it is at least possible for *every* element of P_1 to be below every element of P_2, and for this not to be the case (and the second case may further subdivide). If we write $P_1 < P_2$ to mean that some element of P_1 is below some element of P_2, then we need a separate notation to distinguish the different ways that this can happen. We shall write $P_1 <_c P_2$ to mean that *every* element of P_1 is below every element of P_2 (signifying that the relation is 'complete'). If $P_1 < P_2$ but not $P_1 <_c P_2$, then we say that P_1 is *partially below* P_2.

The classification will be given in terms of the partial ordering of components under the relation $<$ just mentioned. This will not however on its own be sufficient. For instance, it will at the very least be necessary to say which component occurs at each point of the partial order. So for this, there will be a 'label' on each vertex, giving this information. In addition, as just hinted, we also need to label comparable pairs to tell us what sort of relation applies between them, since there may be more than one, possibly several, which can arise.

More precisely, the *skeleton of* a countable homogeneous coloured partial order $\mathcal{P} = (P, <, F)$ is the partial order consisting of its family of colour class components, with each point labelled by the isomorphism type of the component it represents, and each comparable pair $P_1 < P_2$ labelled by the isomorphism type of the 2-component structure $(P_1 \cup P_2, <, F)$. The key point then is that a structure in the classification is uniquely determined by its skeleton. Furthermore, we can characterize abstractly which possible skeletons can arise. So there will be a classification of the possible interdense countable homogeneous coloured partial orders (the possible components), and the possible relations between pairs. This gives rise to a notion of 'abstract skeleton', which is a partial ordering labelled by the labels arising from some countable homogeneous coloured partial order. A consequence of the main result is then that a labelled partial order is the skeleton of a countable homogeneous coloured partial order if and only if all its $\leq$3-element substructures are (and we can spell out explicitly in the definition of 'abstract skeleton' precisely which ones arise in this way).

The following lemma will be quite useful in various places. For instance, it will help to give an efficient derivation of the classification of the countable homogeneous interdensely coloured partial orders from the monochromatic ones.

Lemma 2.1. *Let $\mathcal{P}$ be a homogeneous coloured partial order with colour set C, and let C' be a non-empty subset of C. Then the restriction $\mathcal{P}'$ of $\mathcal{P}$ to C' is also homogeneous. Furthermore, the skeleton of $\mathcal{P}'$ is the corresponding restriction of the skeleton of $\mathcal{P}$.*

Proof. Any finite partial automorphism of $\mathcal{P}'$ is also a partial automorphism of $\mathcal{P}$, so extends to an automorphism of $\mathcal{P}$ whose restriction to C' is the desired automorphism of $\mathcal{P}'$. ■

The following remark is the 'compactness principle' referred to in the introduction.

Lemma 2.2. *Let $\mathcal{P}$ be a countable coloured partial order which is expressible as a union of a family $\mathcal{F}$ of convex subsets which are coloured by pairwise disjoint colour sets and are each interdensely coloured. Then $\mathcal{P}$ is homogeneous if and only if the union of every finite subfamily of $\mathcal{F}$ is homogeneous.*

We would like to say that $\mathcal{P}$ is homogeneous if and only if every finite union of *components* is homogeneous, but we cannot because the justification that components exist required homogeneity, so we have to refer to the components indirectly.

Proof. By Lemma 2.1, if $\mathcal{P}$ is homogeneous, the union of any finite subfamily of $\mathcal{F}$ is homogeneous.

Conversely, we use a back-and-forth argument. Let p be a finite partial automorphism of $\mathcal{P}$ and $x \in P$. We shall show that p can be extended to a finite partial automorphism q having x in its domain. Since p is finite, there is a union Q of a finite subfamily of $\mathcal{F}$ such that p is a partial automorphism of $\mathcal{Q}$. Since $\mathcal{Q}$ is homogeneous, we can extend p to an automorphism f of $\mathcal{Q}$, and then $q = p \cup \{(x, f(x))\}$ is the desired finite partial automorphism of $\mathcal{P}$. This shows that any finite partial automorphism can be extended to include any specified element of $\mathcal{P}$ in its domain. A similar argument applies to the range. Since $\mathcal{P}$ is assumed countable, it follows by back-and-forth that $\mathcal{P}$ is homogeneous. ■

3. One, two and three components

In this section we describe what the possibilities are for countable homogeneous coloured partial orders having up to three colour class components. The proofs that configurations of up to three components not in our list are impossible is given in Section 8, and the proof that the classification is as we assert is completed in Section 9.

One component. Saying that $(P, <, F)$ has just one component is almost the same as saying that it is interdense. The only difference is in the case of antichains, where according to the definition of interdensity, any antichain, however coloured, is interdense, whereas an antichain component is necessarily monochromatic. This is a trivial difference, but the classification works more smoothly if we insist that the antichain components are monochromatic, so the definition from components is to be preferred. The classification for single components is a suitable modification of the monochromatic case treated in [10]. To help us describe the structures, we need to recall the situation for coloured linear orders. The following is given in [2], and is also implicit in [4].

LEMMA 3.1. *A countable coloured linear order is homogeneous if and only if it may be written as the disjoint union of convex subsets, each either a singleton, or isomorphic to some* $\mathbb{Q}_C$ *where the colour sets for different convex pieces are pairwise disjoint.*

In terms of P_C and $\mathbb{Q}_C$ we can now list the following partial orders, which are all easily seen to be countable homogeneous interdensely coloured partial orders, and will form the members of our classification in the interdense case.

Any antichain of cardinality n, where $1 \leq n \leq \aleph_0$.

Each $\mathbb{Q}_C$ is interdense and homogeneous, and any countable homogeneous interdense chain having more than one point is of this form.

Building on these two cases, we have an *antichain of chains*, which is a union of a finite or countable set of copies of some $\mathbb{Q}_C$, with elements in distinct copies incomparable, and we also have a *chain of antichains*, which is obtained from some $\mathbb{Q}_C$ by replacing all points coloured by the same colour by a finite or countable coloured antichain, where points of the same colour must be replaced by isomorphic antichains, and the colour sets of antichains replacing differently coloured points of $\mathbb{Q}_C$ must be disjoint. The ordering is given by $x < y$ if for some $q < r$ in $\mathbb{Q}_C$, x and y lie in the antichains replacing q and r respectively.

Finally we have the generics P_C.

It is easy to see that each of these structures is homogeneous. For P_C this follows from its description as a Fraïssé limit. For the others, homogeneity can be checked directly, though they too can be construed as Fraïssé limits of suitable amalgamation classes. Better, for antichains of chains and chains of antichains, is to view them as 'compositions' of simpler cases, known to be homogeneous, and this eases verification of homogeneity; see below. Finally, interdensity for all of these cases is clear. We shall give proofs of this classification in Sections 8 and 9.

Note that for a full description of a chain of antichains, it is not enough to give the colour sets D_c of the antichains replacing c-coloured points; we also have to specify for each c, how many elements of the antichain are coloured by each element of D_c. The fact that this is the same for each point corresponding to c is ensured by requiring that they are isomorphic. Once this choice has been made, then this description does uniquely determine $\mathcal{P}$ up to isomorphism. We call a partition $\{D_c : c \in C'\}$ of the colour set C of a chain of antichains $\mathcal{P}$, together with multiplicities between 1 and $\aleph_0$ of all members of C, its *colour-structure partition*.

Following on from this remark, we note that we can regard a chain of antichains (and also an antichain of non-trivial chains) as having been formed by composition. If H is a C'-coloured partial order, and for each $c \in C'$, K_c is a D_c-coloured partial order, where the D_c are pairwise disjoint colour sets, then we may form a composite C-coloured partial order, written $H[\{K_c : c \in C'\}]$, where $C = \bigcup_{c \in C'} D_c$ by replacing each point of

H coloured c by a copy of K_c. The colouring on this by colours in C is automatically given, and it is partially ordered by letting $K_1 <_c K_2$, $K_2 <_c K_1$, or $K_1 \parallel K_2$ where K_1 and K_2 are the structures replacing distinct $x, y \in H$ if $x < y$, $y < x$, $x \parallel y$ respectively.

Another remark is that in degenerate cases there is some ambiguity. For instance, a single dense chain ($\mathbb{Q}$) can be viewed as either a chain of (trivial) antichains, or a (trivial) antichain of chains. To make the final list tidier, we shall adopt the following conventions. The cases of a dense chain of antichains at least one of which has size at least 2, an antichain of at least 2 dense chains, and a generic, are unambiguous. A singleton is viewed as an antichain (rather than a chain, or a chain of antichains), and a non-trivial chain is viewed as a chain of (trivial) antichains (rather than an antichain of chains). In other words, chains of antichains are assumed to contain infinitely many antichains, and antichains of non-trivial chains are assumed to have at least 2 maximal chains, though antichains may have any number of points, possibly just one.

For the sake of clarity, we summarize what the possible components can be, including the restrictions we shall make:

- Antichain of cardinality n, where $1 \leq n \leq \aleph_0$, all points coloured c.
- Antichain of n chains, all isomorphic to $\mathbb{Q}_C$, where $2 \leq n \leq \aleph_0$. This may also be regarded as the composite of an n-element antichain with $\mathbb{Q}_C$.
- Chain of antichains given by a colour structure partition $\{D_c : c \in C'\}$, where D_c are disjoint colour sets, including multiplicities, whose union is C. This may also be regarded as the composite of $\mathbb{Q}_{C'}$ with the family of antichains given by the colour-structure partition.
- The Fraïssé-generic C-coloured partial order.

We remark in the style of [4] that of these, antichains and dense chains are 'deficient' (lacking a 2-type, $x < y$ or $x \parallel y$ respectively), non-trivial antichains of dense chains and dense chains of antichains, not all singletons, are imprimitive (admitting a non-trivial equivalence relation preserved by the automorphism group), and generic is primitive and realizing both 2-types.

Since each component is homogeneous, it is isomorphic to one of the structures described above. To describe the general case completely, we need to see how the different components can be 'fitted together'. In the later sections we carry out a systematic analysis of the different possible cases.

In the remainder of this section we consider structures in our list having two or three components. We shall see that these cases are sufficient to describe the whole picture.

Two components. Let $\mathcal{P}$ be a countable homogeneous coloured partial order having two components P_1 and P_2. If $P_1 \parallel P_2$ then there is no interaction between P_1 and P_2, and each of them can be any of the one-component coloured partial orders just listed, independently. So we may assume that some point of P_1 or P_2 is below some point of the other, in the notation introduced above, $P_1 < P_2$ (or the other way round). The definition of $\preceq$ on $\mathcal{P}$ ensures that if $P_1 < P_2$ then no point of P_1 can be greater than any member of P_2, though there may be incomparable $x \in P_1$ and $y \in P_2$ as we shall see. There is therefore the possibility that more than one relationship between P_1 and P_2 can arise.

How many depends on what type of component they are. We try to introduce a notation which reflects the nature of these relations. Several of them are of a 'generic' type, meaning that the 2-coloured structure $P_1 \cup P_2$ may be formed by Fraïssé amalgamation from the class of all partial orderings on $X \cup Y$, where X and Y are finite substructures of P_1 and P_2 respectively with the induced orderings, and no point of X is greater than any point of Y. We therefore write any such relation as $<_g$.

Other specific types of relation which occur are 'perfect matchings' $<_{pm}$ or their complements $<_{cpm}$, which may be viewed as 'linkings' between components. The easiest of these is a perfect matching (1-1 correspondence) between two antichain components, and a similar kind of relation can also arise between antichains of chains, where each maximal chain of each component is related in the same way to all elements in the other component, so that the structure amounts to a 'stretched' version of one between two antichains. In fact, in the classification of countable homogeneous bipartite graphs, which is essentially the same as that of the 2-component countable homogeneous coloured partial orders with two antichain components, there are just 5 cases, empty, complete, perfect matching and its complement, and generic (see [6] for instance). The last can only occur if both antichains are infinite. The other kind of relation that we mention here is 'shuffling' which arises specifically in the case of two chains of antichains components, and it is bidefinable with a chain of antichains on the union of the two colour sets. We write this one as $<_{sh}$.

In analyzing the possible configurations of two components, we subdivide into essentially three cases, chain of antichains, antichain of chains, and generic components, where, as they behave similarly, 'antichains' and 'antichains of chains' are often grouped together as just 'antichain of (possibly trivial) chains'. Following the conventions on degenerate cases mentioned above however, we officially consider four types of component: dense chains of antichains, antichains, antichains of (at least 2) dense chains, and generics.

Viewing antichains and antichains of non-trivial chains under the same heading, there are up to duality (that is, reversal of the ordering between the two components considered) just six cases we need to consider, and we first summarize what the possibilities are for each of these.

- P_1 and P_2 are both chains of antichains, when the relation can be $<_c$ or another one, written $<_{sh}$ ('shuffle').
- P_1 is a chain of antichains, and P_2 is an antichain of chains, in which case $<_c$ is the only possibility.
- P_1 is a chain of antichains, and P_2 is generic, when again we must have $<_c$.
- P_1 and P_2 are both antichains of chains, and the relation between P_1 and P_2 is $<_c$, $<_{pm}$, $<_{cpm}$, or $<_g$ (meaning respectively that the relation is complete, a perfect matching or its complement, or is generic).
- P_1 is an antichain of chains, and P_2 is generic, in which case the only possibilities are $<_c$ and another (generic) possibility, written $<_g$, and any two members of the same chain of P_1 are related in the same way to each element of P_2.
- P_1 and P_2 are both generic, in which case we can have $<_c$ or a generic relation, written $<_g$.

Note that the cases involving one or more antichains of non-trivial chains may be viewed as a composition of the corresponding structure having antichain(s) in place of antichains of chains, with some $\mathbb{Q}_{C'}$.

There is one case in which there is ambiguity of the labels $<_{pm}$, $<_{cpm}$, so to obtain uniqueness we make the following convention. If P_1 and P_2 are perfectly matched antichains of 2 chains, we write $P_1 <_{pm} P_2$ rather than $P_1 <_{cpm} P_2$ (noting that in this case the complement of a perfect matching is the same as a perfect matching). There is no need to distinguish $<_c$ and $<_{pm}$ between antichains of a single chain, since by the earlier convention they are viewed as chains (of trivial antichains) rather than antichains of chains.

Three components; V-shape lemmas. We now move on to the case where there are three components P_1, P_2 and P_3. If one of these is incomparable with the other two then the possibilities are completely controlled by how the other two components interact, so we read that off from the previous list. So we may assume that each of the three components is related to at least one of the others, and that gives us three possible configurations, which are that they form a chain, we may suppose $P_1 < P_2 < P_3$, that they form a V-shape, one is below the other two, which are incomparable, we may suppose $P_1 < P_2, P_3$ and $P_2 \parallel P_3$, or the dual of this, called a 'Λ-shape', and which we may disregard at this stage, since all arguments for Λ-shapes are exactly as for V-shapes under the reversed ordering. We list the possibilities for V-shapes first, as there are fewer of these than 3-chains, in terms of their restrictions to two components.

In the first case, we have $P_1 <_c P_2$ or $P_1 <_c P_3$, in which case the other pair (P_1, P_3 or P_1, P_2 respectively) can be any of the possible 2-chains. From now on we suppose that this is not the case.

If one of P_1, P_2, P_3 is a chain of antichains, then the other two must necessarily also be chains of antichains and $P_1 <_{sh} P_2$ and $P_1 <_{sh} P_3$.

Otherwise, each of P_1, P_2, P_3 is an antichain of chains or generic, and $P_1 <_g P_2$ and $P_1 <_g P_3$.

Three components; 3-chain lemmas. We give a list of all the possibilities for a countable homogeneous coloured partial order in which there are three components $P_1 < P_2 < P_3$. We remark that it follows by homogeneity that the relation between P_1 and P_3 is the transitive closure of those between P_1 and P_2, and P_2 and P_3. See the beginning of the proof of Theorem 8.4 for the argument.

Each of the pairs P_1, P_2, P_1, P_3, and P_2, P_3 must be one of the 2-chains given above, but not all possible combinations can arise. The complete list is as follows:

One complete relation between adjacent components

$P_1 <_c P_2$ and $P_1 <_c P_3$ and the relation between P_2 and P_3 is any allowed for 2-chains.

$P_1 <_c P_3$ and $P_2 <_c P_3$ and the relation between P_1 and P_2 is any allowed for 2-chains.

All components of the same type

P_1, P_2, and P_3 are all chains of antichains, and $P_1 <_{sh} P_2 <_{sh} P_3$ and $P_1 <_{sh} P_3$.

P_1, P_2 and P_3 are all antichains of chains and

(i) $P_1 <_{pm} P_2$ and the relation between P_2 and P_3 is one of $<_{pm}$, $<_{cpm}$, $<_g$, and the relation between P_1 and P_3 is the same as that between P_2 and P_3, or
(ii) $P_2 <_{pm} P_3$ and the relation between P_1 and P_2 is one of $<_{pm}$, $<_{cpm}$, $<_g$, in which case the relation between P_1 and P_3 is the same as the one between P_1 and P_2, or
(iii) $P_1 <_g P_2$ and $P_2 <_g P_3$, in which case the relation between P_1 and P_3 is $<_g$, $<_{cpm}$ or $<_c$.

P_1, P_2 and P_3 are all generics and $P_1 <_g P_2$, $P_2 <_g P_3$, and either $P_1 <_g P_3$ or $P_1 <_c P_3$.

Just two components are antichains of chains

(i) P_1 and P_3 are antichains of chains, P_2 is a generic, $P_1 <_g P_2 <_g P_3$, and the relation between P_1 and P_3 is $<_g$, $<_{cpm}$, or $<_c$, or
(ii) P_1 and P_2 are antichains of chains, P_3 is a generic, and $P_1 <_{pm} P_2$ and $P_1, P_2 <_g P_3$; or $P_1 <_g P_2 <_g P_3$ and $P_1 <_g P_3$ or $P_1 <_c P_3$, or
(iii) P_2 and P_3 are antichains of chains, P_1 is a generic, and $P_2 <_{pm} P_3$ and $P_1 <_g P_2$, P_3; or $P_1 <_g P_2 <_g P_3$ and $P_1 <_g P_3$ or $P_1 <_c P_3$.

Just two components are generics. Two of P_1, P_2, P_3 are generics and the third is an antichain of chains, $P_1 <_g P_2 <_g P_3$ and $P_1 <_g P_3$ or $P_1 <_c P_3$,

The fact that any 1-, 2- or 3-component structure in our list must be of one of these types will be proved in Section 8, and existence and uniqueness in Section 9.

4. Definition of abstract skeleton

Armed with the information given in the previous section, we can now give an abstract definition of 'skeleton', which captures the constraints implicit in the list given. These constraints only refer to up to three components, and one of the main things that we have to establish later is that this is sufficient information to give a complete description, for arbitrarily many components (even infinitely many).

An *abstract skeleton* is a finite or countable partially ordered set $\mathcal{Q} = (Q, \prec)$ with labels on points and comparable pairs fulfilling the following conditions:

(i) each vertex is labelled by (the isomorphism type of) a countable interdensely coloured partial order, which, if it is an antichain, is monochromatic,
(ii) if $q \prec r$ then (q, r) is labelled by (the isomorphism type of) one of the two-component countable homogeneous coloured partial orders given above, where the two structures are the ones labelling q and r,
(iii) if $q \prec r, s$ and $r \parallel s$ then the isomorphism types labelling the pairs (q, r) and (q, s) are of one of the following forms: $q <_c r$ or $q <_c s$; $q <_{sh} r$ and $q <_{sh} s$; $q <_g r$ and $q <_g s$,
(iv) if $q, r \prec s$ and $q \parallel r$ then the dual of one of the conditions under (iii) holds,
(v) if $q \prec r \prec s$, then the isomorphism types labelling the pairs (q, r), (q, s), and (r, s) are of one of the following forms: $q <_c r$ and $q <_c s$; $q <_c s$ and $r <_c s$; $q <_{sh} r <_{sh} s$

and $q <_{sh} s$; $q <_{pm} r\, R\, s$ and $q\, R\, s$ or $q\, R\, r <_{pm} s$ and $q\, R\, s$ where R is $<_{pm}$, $<_{cpm}$, or $<_g$; $q <_g r <_g s$ and $q <_g s$ or $q <_{cpm} s$ or $q <_c s$.

Here clause (ii) corresponds to 2-chain lemmas, clause (iii) to V-shape lemmas, clause (iv) to their dual ('Λ-shape lemmas'), and clause (v) to 3-chain lemmas.

If we wish to make the classification more explicit, then we may rephrase the definition in the following terms, where we number the clauses correspondingly.

An *abstract skeleton* is a finite or countable partially ordered set $\mathcal{Q} = (Q, \prec)$ with labels on points and comparable pairs fulfilling the following conditions:

(ia) each vertex q is labelled by CA, A, AC, or Ge, and also by a non-empty finite or countable set C_q (whose elements we think of as colours), so that $q \neq r \Rightarrow C_q \cap C_r = \emptyset$,
(ib) if q is labelled CA, then it has an extra label which is a colour structure partition $\{D_c : c \in C'\}$ where $\bigcup C' = C_q$,
(ic) if q is labelled A or AC then it is also labelled by $N(q)$, where $1 \leq N(q) \leq \aleph_0$, and if A, then $|C_q| = 1$ (antichain components are monochromatic), and if AC then $N(q) > 1$ (we count a single dense chain as a chain of antichains rather than an antichain of chains),

(iia) if $q \prec r$ then (q, r) is labelled by one of C, SH, PM, CPM, G (and if (q, r) is labelled R then we also write $q\, R\, r$),
(iib) if $q\, SH\, r$ then q and r are labelled CA,
(iic) if $q\, PM\, r$ or $q\, CPM\, r$ then q and r are labelled A or AC and $N(q) = N(r)$, and if $q\, PM\, r$ then $N(q) = N(r) > 1$, and if $q\, CPM\, r$ then $N(q) = N(r) > 2$,
(iid) if $q\, G\, r$ then one of the following holds:

- q and r are labelled A or AC and $N(q) = N(r) = \aleph_0$,
- q is labelled A or AC and r is labelled Ge and $N(q) = \aleph_0$,
- q is labelled Ge and r is labelled A or AC and $N(r) = \aleph_0$,
- q and r are both labelled Ge,

(iiia) if $q \prec r, s$ and $r \parallel s$, then one of the following must hold: $q\, C\, r$; $q\, C\, s$; $q\, SH\, r$ and $q\, SH\, s$; $q\, G\, r$ and $q\, G\, s$,

(iva) if $q, r \prec s$ and $q \parallel r$ then the dual of one of the conditions under (iiia) must hold,

(va) if $q \prec r \prec s$, then one of the following must hold: $q\, C\, r$ and $q\, C\, s$; $q\, C\, s$ and $r\, C\, s$; $q\, SH\, r\, SH\, s$ and $q\, SH\, s$; $q\, PM\, r\, R\, s$ and $q\, R\, s$ or $q\, R\, r\, PM\, s$ and $q\, R\, s$ where R is PM, CPM, or G; $q\, G\, r\, G\, s$ and $q\, G\, s$ or $q\, CPM\, s$ or $q\, C\, s$.

We observe that by the results stated in Section 3, it is clear that any skeleton of a countable homogeneous coloured partial order satisfies all these conditions, at any rate in the first version, and also when one understands the 'coding', in the more explicit one. The point is to establish the converse. Often we abbreviate 'abstract skeleton' (in either sense) as just 'skeleton'.

5. Chains of antichains

The particular case of countable coloured partial orders all of whose components are chains of antichains can be handled directly, without using Fraïssé amalgamation explicitly. We present this material in this section, which will enable us to simplify the general case, assuming the results for a single such component, which are given in Sections 8 and 9. All structures may be found by redefining the ordering relation on a single component. First we deal with the most straightforward case, and then generalize. To begin with then we have two components P_1 and P_2, both chains of antichains (on disjoint colour sets), with P_1 partially below P_2. This structure is unique up to isomorphism, given P_1 and P_2, as we now show, and we write $P_1 <_{sh} P_2$.

LEMMA 5.1. *Let C_1 and C_2 be non-empty finite or countable disjoint colour sets, and for $i = 1, 2$ let $\{D_c : c \in C'_i\}$ be a colour-structure partition of C_i. Then there is a countable homogeneous coloured partial order $\mathcal{P}$ of the form $P_1 \cup P_2$ where P_1, P_2 are (interdensely coloured) chains of antichains having colour-structure partitions $\{D_c : c \in C'_1\}$ and $\{D_c : c \in C'_2\}$ respectively, and such that for each $x \in P_1$, P_2 may be written as the disjoint union of non-empty subsets S and L (for 'smaller' and 'larger') such that $S \parallel x$, $x < L$; and there is no $y \in P_2$ such that $y < x$. Furthermore, if $\mathcal{P}'$ is a countable homogeneous coloured partial order having components P'_1 and P'_2, both chains of antichains with colour-structure partitions $\{D_c : c \in C'_i\}$ for $i = 1, 2$, and such that there are $x, x' \in P'_1$, $y, y' \in P'_2$ with $x < y$ and $x' \parallel y'$, then $\mathcal{P}'$ is isomorphic to $\mathcal{P}$.*

Proof. We start with the composite structure $\mathbb{Q}_{C'_1 \cup C'_2}[\{D_c : c \in C'_1 \cup C'_2\}]$ obtained from $\mathbb{Q}_{C'_1 \cup C'_2}$ by replacing each point coloured $c \in C'_i$ by an antichain of points coloured by the members of D_c so that each member of D_c has the correct number of elements of that colour as given by its multiplicity. Let $<$ be the resulting partial order. The desired structure $\mathcal{P}$ then has the same domain $P = \mathbb{Q}_{C'_1 \cup C'_2}[\{D_c : c \in C'_1 \cup C'_2\}]$, but it is ordered by a different partial ordering $<'$, defined from $<$ as follows. Let P_1 and P_2 be the subsets coloured by C_1 and C_2 respectively, and let $x <' y$ if $x < y$ and x and y lie in P_i and P_j respectively, where $i \le j$. This is clearly irreflexive. To see that it is transitive, suppose that $x <' y <' z$. Then automatically $x < y < z$, so $x < z$. If we let $x \in P_i$, $y \in P_j$, $z \in P_k$, then $i \le j \le k$, so $i \le k$ and hence $x <' z$.

Next we observe that each P_i is convex in $(P, <')$, for if $x <' y <' z$ where $x, z \in P_i$, $y \in P_j$, then $i \le j \le i$ so $i = j$ and $y \in P_i$. Also, the orderings $<$ and $<'$ agree on P_i (the only change is between different P_is), and P_1 is partially below P_2. Since $\mathbb{Q}_{C'_1 \cup C'_2}$ is interdensely $C'_1 \cup C'_2$-coloured, it is immediate that its restrictions to the colour sets C'_1 and C'_2 are isomorphic to $\mathbb{Q}_{C'_1}$ and $\mathbb{Q}_{C'_2}$ respectively. Consequently, P_1 and P_2 are interdensely coloured chains of antichains with the correct colour-structure partitions, and are the components of $(P, <')$. Next, given $x \in P_1$, let $S = \{y \in P_2 : y < x\}$ and $L = \{y \in P_2 : x < y\}$. These are non-empty, and as required in the definition.

To verify the other properties, it is easiest to begin by noting that the ordering $<$ on $\mathbb{Q}_{C'_1 \cup C'_2}[\{D_c : c \in C'_1 \cup C'_2\}]$ can be recovered from $<'$ since $x < y \Leftrightarrow [(x \in P_1 \vee y \in P_2) \wedge x <' y] \vee [x \in P_2 \wedge y \in P_1 \wedge x \parallel y \text{ under } <']$. The fact that the two orderings are interdefinable (relative to the named subsets P_1 and P_2) means that the automorphism groups are equal, and from this, homogeneity of $(P, <')$ follows at once from homogeneity of $(\mathbb{Q}_{C_1' \cup C_2'}[\{D_c : c \in C_1' \cup C_2'\}], <)$ (see the proof of Theorem 5.2 for the details; and the homogeneity of $(\mathbb{Q}_{C_1' \cup C_2'}[\{D_c : c \in C_1' \cup C_2'\}], <)$ is shown in Theorem 9.1).

We may also deduce uniqueness using the same idea. Suppose that $\mathcal{P}'$ is as stated, and write $<'$ for its ordering. Then we can define $<$ on $P_1' \cup P_2'$ to turn it into a chain of antichains having the colour-structure partition $\{D_c : c \in C_1' \cup C_2'\}$, by using the above definition, $x < y \Leftrightarrow [(x \in P_1' \vee y \in P_2') \wedge x <' y] \vee [x \in P_2' \wedge y \in P_1' \wedge x \parallel y$ under $<']$. We verify that $(P_1' \cup P_2', <, F) \cong \mathbb{Q}_{C_1' \cup C_2'}[\{D_c : c \in C_1' \cup C_2'\}]$, and by reversing the definition, we deduce that $(P', <', F) \cong (P, <', F)$.

First, for transitivity, suppose that $x < y < z$. If the first clause applies to both $x < y$ and $y < z$, then $x <' y <' z$ and $(x \in P_1' \vee y \in P_2') \wedge (y \in P_1' \vee z \in P_2')$, from which we deduce $x \in P_1' \vee z \in P_2'$ and so $x < z$. The second clause cannot apply to both $x < y$ and $y < z$. Suppose without loss of generality that it applies just to $x < y$. Then $x \in P_2' \wedge y \in P_1' \wedge x \parallel y <' z$. If $z \in P_1'$ then $z <' x$ would give $y <' x$, contrary to supposition. Hence $x \parallel z$ under $<'$ and so $x < z$. If $z \in P_2'$ and $x <' z$ is false then $z \le' x$ or x, z lie in the same antichain of P_2', from which $y <' x$ follows, contradiction. Hence $x <' z$ holds, which also gives $x < z$.

Next we see that any $x \in P_1'$ and $y \in P_2'$ are comparable under $<$. For if not, from the first clause and $x \not< y$ it follows that $x \parallel y$ under $<'$, and from the second that $y < x$.

To see that $(P_1' \cup P_2', <, F)$ is interdensely $C_1 \cup C_2$-coloured, let $x < y$ and $c \in C_1 \cup C_2$ be given. Suppose $c \in C_1$. If $x, y \in P_1'$, then the existence of z in between coloured c follows by interdensity there. Otherwise it suffices therefore to find two comparable points of P_1' between x and y. By assumption, and appealing to homogeneity, there are $x_1 \in P_1'$, $y_1, y_2 \in P_2'$ such that $x_1 \parallel y_1$ under $<'$ and $x_1 <' y_2$. Thus $y_1 < x_1 < y_2$. Also we may suppose that y_1 and y_2 have the same colour. Applying homogeneity twice more we get $y_1 < x_1 < y_2 < x_2 < y_3 < x_3 < y_4$ with $x_i \in P_1'$, $y_i \in P_2'$. Hence (using homogeneity again), between any two members of $P_1' \cup P_2'$ not both in P_1' there are at least two members of P_1' (if both in P_2' use y_1 and y_3 and then x_1 and x_2 are in between, if $x \in P_1'$ and $y \in P_2'$ use x_1 and y_4 with x_2, x_3 in between, and if $x \in P_2'$ and $y \in P_1'$ use y_1 and x_3 with x_1, x_2 in between). ■

We remark that another method to prove this result is to show that $\mathcal{P}$ is the Fraïssé limit of the class of finite coloured partial orders $X \cup Y$ such that X, Y are finite coloured chains of antichains such that, for each antichain of X, Y, its points are coloured compatibly with some D_c for $c \in C_1'$, ($c \in C_2'$ respectively) and $(\forall x \in X)(\forall y \in Y)(x \not> y)$, but the direct construction we have described is quicker in this case.

Now we can generalize this to an arbitrary skeleton. The lemma just given corresponded to a skeleton which is a 2-element chain with both points labelled by CA and the corresponding colour structure partitions, and the relation SH between the two points.

Theorem 5.2. *Let $\mathcal{Q}$ be any skeleton in which all points are labelled by CA. Then there is a countable homogeneous coloured partial order $\mathcal{P}$ having $\mathcal{Q}$ as skeleton, and any two countable coloured homogeneous partial orders with skeleton $\mathcal{Q}$ are isomorphic.*

Proof. Let C be the union of all the colour sets at vertices of $\mathcal{Q}$. Then C is countable. We turn $(\mathbb{Q}_{C'}[\{D_c : c \in C'\}], <)$, where $\{D_c : c \in C'\}$ is the overall colour structure partition (the union of all the individual ones), into a countable coloured homogeneous partial ordering $\mathcal{P}$ having skeleton $\mathcal{Q}$ by redefining the ordering on it. Let $x <' y$ if $x \in P_i = \mathbb{Q}_{C'_i}[\{D_c : c \in C'_i\}]$ and $y \in P_j = \mathbb{Q}_{C'_j}[\{D_c : c \in C'_j\}]$ where C_i and C_j and the corresponding colour structure partitions are the labels on vertices q_i and q_j of $\mathcal{Q}$ such that $q_i \preceq q_j$, and either $x < y$ or (q_i, q_j) is labelled C (or both).

We first show that this is a partial order. Irreflexivity is clear. For transitivity, suppose that $x <' y <' z$ and let $x \in P_i$, $y \in P_j$, $z \in P_k$, where $\mathbb{Q}_{C'_i}$, $\mathbb{Q}_{C'_j}$, $\mathbb{Q}_{C'_k}$ and the corresponding colour structure partitions are labels for $q_i, q_j, q_k \in Q$ respectively. If (q_i, q_k) is labelled C then $x <' z$ is immediate. Otherwise, by definition of 'skeleton', neither (q_i, q_j) nor (q_j, q_k) is labelled C. Hence $x < y < z$, and so $x < z$ by transitivity of the ordering on $\mathbb{Q}_{C'}$. Also, $q_i \preceq q_j \preceq q_k$, so $q_i \preceq q_k$, by transitivity of the ordering on $\mathcal{Q}$, which gives $x <' z$ as required.

Next we see that P_i is a convex subset of $\mathcal{P}$. For if $x <' y <' z$ with $x, z \in P_i$ and $y \in P_j$, then $q_i \preceq q_j \preceq q_i$ so $i = j$, and y also lies in P_i, establishing convexity. Since the orderings $<$ and $<'$ agree on P_i, and it is interdense, the components of $\mathcal{P}$ are precisely the P_i for $q_i \in Q$.

If $q_i \, SH \, q_j$ in $\mathcal{Q}$, then the definition of $<'$ restricted to $P_i \cup P_j$ is just the same as in Lemma 5.1, and so the edge label is correctly interpreted. If $q_i \, C \, q_j$ then $P_i <_c P_j$. If $q_i \parallel q_j$ in $\mathcal{Q}$, then the definition gives us that P_i and P_j are incomparable. Since SH and C are the only possible edge labels allowed between comparable points labelled CA, we deduce that $\mathcal{Q}$ *is* $\mathcal{P}$'s skeleton.

To demonstrate the homogeneity of $\mathcal{P}$, it is easiest to work with suitable substructures of $\mathcal{Q}$. Let SH^* be the symmetric and transitive closure of SH on $\mathcal{Q}$, and we shall observe that the process of forming $(\mathbb{Q}_{C'}[\{D_c : c \in C'\}], <')$ from $(\mathbb{Q}_{C'}[\{D_c : c \in C'\}], <)$ is reversible on each SH^*-class of $\mathcal{Q}$. This is because the two structures are interdefinable. For we may define $<$ from $<'$ as in the proof of Lemma 5.1 by noting that $<$ is the transitive closure of the relation given by $x < y \Leftrightarrow [(x <' y \wedge (q_i = q_j \vee q_i \, SH \, q_j)) \vee (x \parallel y \wedge (q_j \, SH \, q_i))]$ where $x \in P_i$ and $y \in P_j$. For if $x < y$, where $x \in P_i$ and $y \in P_j$, and for example $q_i \prec q_{i_1} \succ q_{i_2} \prec \cdots \prec q_{i_k} \succ q_j$ where all the relations are SH, then by density we can find $x < x_1 < x_2 < \cdots < x_k < y$ where $x_l \in P_{i_l}$, and then we have $x <' x_1 \parallel x_2 <' \cdots <' y$, which gives $x < y$ determined by the transitive closure as asserted.

Now we can see that $\mathcal{P}$ is homogeneous thus. Clearly it is homogeneous if and only if each SH^*-class is homogeneous (any automorphism must fix each component, since it respects the colouring, and automorphisms on different SH^*-classes can be patched to give an automorphism on their union, as the relation between components in distinct SH^*-classes is $\parallel$ or C), so we shall assume there is just one SH^*-class. Let p be a finite partial automorphism. Then as $(\mathbb{Q}_{C'}[\{D_c : c \in C'\}], <)$ is definable from $(\mathbb{Q}_{C'}[\{D_c : c \in C'\}], <')$, p is also a finite partial automorphism of $(\mathbb{Q}_{C'}[\{D_c : c \in C'\}], <)$, so by homogeneity here, extends to an automorphism θ of $(\mathbb{Q}_{C'}[\{D_c : c \in C'\}], <)$. Since $(\mathbb{Q}_{C'}[\{D_c : c \in C'\}], <')$ is definable from $(\mathbb{Q}_{C'}[\{D_c : c \in C'\}], <)$, θ is also an automorphism of $(\mathbb{Q}_{C'}, <') = \mathcal{P}$, extending p.

Finally for uniqueness, let $(P', <')$ be any other countable homogeneous coloured partial order with skeleton $\mathcal{Q}$. Again we only need establish uniqueness of each SH^*-class, so we shall assume that $\mathcal{Q}$ has just one SH^*-class. By uniqueness of $\mathbb{Q}_{C'_i}[\{D_c : c \in C'_i\}]$ on its colour set C_i, $\mathcal{P}'$ has the same components as $\mathcal{P}$. Now $(\mathbb{Q}_{C'}[\{D_c : c \in C'\}], <)$ is definable from $(\mathbb{Q}_{C'}[\{D_c : c \in C'\}], <')$. To show that the coloured linear order defined from it by the above method is isomorphic to $(\mathbb{Q}_{C'}[\{D_c : c \in C'\}], <)$, we have to verify the same properties as in the previous theorem. This time $<$ is transitive by definition (though on any two comparable components it is already transitive as in the previous proof). Next the proof of Theorem 5.1 shows that if q_i and q_j are comparable, then P'_i and P'_j are each dense in the other under the ordering $<$, and it follows, repeatedly using the definition as the transitive closure that the same is true in general, which establishes linearity, and also that $(P', <) \cong (\mathbb{Q}_{C'}[\{D_c : c \in C'\}], <)$. Now passing back to $<'$ again, we deduce that $\mathcal{P}'$ must be isomorphic to $\mathcal{P}$.

There is one subtle point which is overlooked in the above sketch, and which we need to check (see the end of Section 10 for an example of why this really is necessary), and that is that we must check that in forming $\mathcal{P}'$ under $<$, no two points x and y are identified. If they were, this would mean that they would be forced to occupy the same Dedekind cuts defined by the rest. Supposing for a contradiction that this happens, there must be a sequence, for instance of the form $q_i \prec q_{i_1} \succ q_{i_2} \prec \cdots \prec q_{i_k} \succ q_j$ where all the relations are SH, and such that the cuts in $P'_{i_1}, \ldots, P'_{i_k}$ determined by x as the identifications in the sequence are made, give rise to the same one determined by y. Since we know that this does not happen if q_i and q_j are directly SH-related, the relation between P'_i and P'_j is either complete or incomparable. Choose $y' > y$ in P'_j of the same colour as y. Then the partial automorphism fixing x, and taking y to y', has no extension to an automorphism (since it cannot preserve the Dedekind cut), and homogeneity is contradicted. ■

6. Reduced skeletons

The natural way in which to show that any skeleton $\mathcal{Q}$ is the skeleton of some countable homogeneous coloured partial order $\mathcal{P}_{\mathcal{Q}}$ is to form the partial order as the Fraïssé limit of a suitable amalgamation class $\mathcal{K}_{\mathcal{Q}}$ say (which will be the family of finite substructures of $\mathcal{P}_{\mathcal{Q}}$). This was the method followed in [11], and is possible (without circularity). However, there are some technical difficulties involved, and at the referee's suggestion, we modify the use of the amalgamation class corresponding to a skeleton to try to circumvent these. The idea is that there will be a 'reduced skeleton', in which occurrences of the edge relations PM, CPM, and SH are eliminated (these being the 'non-orthogonal' relations). The ones which remain for comparable pairs are then C and G, for which the principal difficulties arising are absent.

The trouble with PM for instance is illustrated in a simple case. The amalgamation class corresponding to a perfect matching between two (distinctly coloured) antichains $P_1 <_{pm} P_2$ cannot be taken to be the family of finite partial orders of the form $X <_{pm} Y$ because this does not have the hereditary property. Since $P_1 \cup P_2$ has substructures in which the two parts do not have the same size, to make amalgamation work directly, one instead has to capture the family of *substructures* of a perfect matching, which is possible, but tedious.

Instead, we shall take $\mathcal{K}$ to be a 'reduced' version of the natural one, but such that verification of the amalgamation property becomes much easier. The main point is to eliminate PM and CPM, but since the parts of the skeleton where the SH relation occurs can also be easily separated from the rest (in view of the definition of 'skeleton') and chains of antichains are best dealt with by direct homogeneity arguments rather than via amalgamation as in Section 5, we remove them too.

The definition of (abstract) *reduced skeleton* is therefore just that it is a skeleton in which no edges are labelled by PM, CPM, or SH. As we see in Section 7, amalgamation over reduced skeletons is relatively easy. Of course there is a price to pay, and we have to show how to relate an arbitrary skeleton to an associated reduced one.

To make this more formal, we first take the equivalence relation $\equiv$ on any abstract skeleton $\mathcal{Q}$ to be the symmetric and transitive closure of the union of the relations PM, CPM, and SH. We observe that in view of the V-shape and Λ-shape restrictions in the definition of abstract skeleton, each non-singleton $\equiv$-class then consists either of a point or points all labelled CA, which we call a *SH-class* (called 'SH^*-class' in Section 5), or of points all labelled A or AC, which we call a *PM-class*. Any SH-class is necessarily convex, and so this means that we can essentially treat it in isolation, since comparabilities between it and other points can only be labelled C. Any PM-class T is in fact a chain, and such that there are no points $x < y < z$ for which (x, y) and (y, z) are both labelled CPM. A slight complication here is that T need not be convex, and it is possible, between points related by CPM, to have one or more point with all generic relations. Furthermore, since we are not ruling out infinite skeletons (even though in view of Lemma 2.2 we could do so), there is not necessarily a pair of adjacent elements of T connected by a CPM edge. This means that we have to handle this case with some care.

A reduced skeleton *associated with* a skeleton $\mathcal{Q}$ is then taken to consist of a family of representatives of its $\equiv$-classes, under the induced labels and relations. Thus points not in a SH- or PM-class are left untouched, and points in either of these two kinds of class are replaced by representatives. Let us write $\iota(q)$ for the element chosen from the $\equiv$-class of Q containing q.

THEOREM 6.1. *Let $\mathcal{Q}$ be an abstract skeleton, and $\mathcal{Q}'$ an associated reduced skeleton. If $\mathcal{A}'$ is a countable homogeneous coloured partial order having $\mathcal{Q}'$ as skeleton, then there is a countable homogeneous coloured partial order $\mathcal{A}$ containing $\mathcal{A}'$ which has $\mathcal{Q}$ as skeleton. Furthermore, any isomorphism from $\mathcal{A}$ to another coloured partial order $\mathcal{B}$ restricts to an isomorphism from $\mathcal{A}'$ to the corresponding restriction $\mathcal{B}'$ of $\mathcal{B}$, and conversely, any isomorphism from $\mathcal{A}'$ to some $\mathcal{B}'$ extends to an isomorphism from $\mathcal{A}$ to any extension $\mathcal{B}$ of $\mathcal{B}'$ having skeleton $\mathcal{Q}$.*

Proof. For each $q \in Q'$ let A_q be the component of $\mathcal{A}'$ corresponding to q. We have to choose suitable A_q for $q \in Q - Q'$ so that $\mathcal{A}$ having domain $\bigcup_{q \in Q} A_q$ is a countable homogeneous coloured partial order with $\mathcal{Q}$ as skeleton. We first make these choices, and partially order them within each $\equiv$-class, and then show how the union of the choices over all $\equiv$-classes is partially ordered.

Suppose $q \in Q'$ is labelled CA. Then A_q is a chain of antichains with colour structure partition matching the label at q. Let T be the SH-class containing q. Then by Theorem 5.2 there is a (uniquely determined) countable homogeneous coloured partial order having T as skeleton. By uniqueness, the component in this structure labelled by q is isomorphic to A_q, so we may assume it *equals* A_q, and we now also know what $A_{q'}$ is for every other $q' \in T$. In addition, the partial ordering on $\bigcup\{A_{q'} : q' \in T\}$ is known.

Next suppose that $q \in Q'$ lies in a PM-class T. As noted above, T is a chain, so for each $q' \in T - \{q\}$, $q \prec q'$ or $q' \prec q$. We let $A_{q'}$ be an antichain or antichain of chains of the correct size and colour, and we choose an arbitrary bijection $\theta_{q'}$ from the set of maximal chains of A_q to the set of maximal chains of $A_{q'}$ (which are either singletons, or coloured versions of the rationals). This exists by clause (iic) in the definition of skeleton. For ease we also write θ_q for the identity map from A_q to itself, for any q, even if it is labelled Ge. Let q' and q'' be two members of T such that $q' \prec q''$. We give the relation between $A_{q'}$ and $A_{q''}$. If $q' \, PM \, q''$ then we let $U < \theta_{q''}\theta_{q'}^{-1}(U)$ for each maximal chain U of $A_{q'}$ and otherwise $x \not< y$ for $x \in A_{q'}$, $y \in A_{q''}$. If $q' \, CPM \, q''$ then we let $x \not< y$ for each x lying in a maximal chain U of $A_{q'}$ and $y \in \theta_{q''}\theta_{q'}^{-1}(U)$, and otherwise $x < y$ for $x \in A_{q'}$, $y \in A_{q''}$. This defines the ordering on $\bigcup_{q' \in T} A_{q'}$, and it is clear that it is in accordance with the labels so far.

It remains to define the relation between different $\equiv$-classes. Let $q, r \in Q$ with $q \not\equiv r$, and suppose that q, r do not both lie in Q' (as otherwise the relation between A_q and A_r is already known). If $q \parallel r$ then we let $A_q \parallel A_r$. Otherwise, we suppose that $q \prec r$, and let $q \, R \, r$. Then as $q \not\equiv r$, $R \neq SH, PM, CPM$, so $R = C$ or G. If $R = C$ then we let $A_q <_c A_r$. If $R = G$ then neither q nor r is labelled CA, so if $q \neq \iota(q)$, q must lie in a PM-class, and similarly for r, and at least one of these must happen. We show how to deduce from the definition of 'skeleton' that all of q, $\iota(q)$, r, and $\iota(r)$ are comparable. If $\iota(q) \preceq q$ then it is comparable with r. If $q \prec \iota(q)$ then as $q \, G \, r$ and $q \, PM \, \iota(q)$ or $q \, CPM \, \iota(q)$, by the V-shape conditions, $\iota(q)$ is comparable with r. By the 3-chain conditions, $(\iota(q), r)$ or its reverse is labelled G. Repeating the argument shows that $\iota(r)$ is comparable with both q and $\iota(q)$, and the relation between each of $q, \iota(q)$ and $r, \iota(r)$ is G if they are unequal. This gives the following possibilities:

1. $q, \iota(q) \prec r, \iota(r)$. There are nine cases here since we can have $q \, PM \, \iota(q)$, $q = \iota(q)$, or $\iota(q) \, PM \, q$ and similarly for r, but we do not need to distinguish them.
2. $q \prec r, \iota(r) \prec \iota(q)$. There are three cases here, but in each, $q \, CPM \, \iota(q)$ by the 3-chain conditions.
3. $\iota(r) \prec q, \iota(q) \prec r$. This is similar to Clause 2.
4. $q \prec r \prec \iota(q) \prec \iota(r)$, and three similar 'alternating' cases. These are not possible as we must have $q \, CPM \, \iota(q) \, G \, \iota(r)$, contrary to the 3-chain conditions.

To ease presentation of the many cases, we make the following convention. If A_q is an antichain of chains or an antichain, let U be a maximal chain of A_q (which in the latter case will just be a singleton). If A_q is generic, then let U just be a singleton from A_q. Similarly we choose V from A_r. We shall define the circumstances under which $U < V$

or $U \not< V$, in terms of the relation between $\theta_q^{-1}(U)$ and $\theta_r^{-1}(V)$ in $A_{\iota(q)}$ and $A_{\iota(r)}$. This will be defined in the same way for the whole chain, even in the antichain of chains case. In some of the verifications, we shall refer to 'choosing a maximal chain' of a component; this will only be strictly correct in the antichain of chains and antichain cases, but in the generic case we shall assume that at this point we just choose a singleton, and this enables us to push through the same arguments.

For Clause 1 we let $U < V \Leftrightarrow \theta_q^{-1}(U) < \theta_r^{-1}(V)$.

For Clauses 2 and 3 there will be a reversal, in view of the presence of a complement of a perfect matching (and the fact the ι is order-reversing on $\{q, r\}$), and we let $U < V \Leftrightarrow \theta_r^{-1}(V) \not< \theta_q^{-1}(U)$.

This defines the partial ordering on $\mathcal{A}$. Note that the edge labels of $\mathcal{Q}$ have been correctly interpreted, so that $\mathcal{Q}$ really is the skeleton of $\mathcal{A}$. Also $\mathcal{A}$ is countable. The main point here is to verify that $\mathcal{A}$ is indeed a homogeneous coloured partial order.

For 'partial order', we just need to verify transitivity (since irreflexivity is clear). Let $x < y < z$ where $x \in A_q$, $y \in A_r$, $z \in A_s$. Thus $q \preceq r \preceq s$. We have to show that $x < z$. Let U, V, W be the maximal chains of A_q, A_r, A_s containing x, y, and z respectively (with the above provisos observed).

Case 1: $q \equiv r \equiv s$. If $q = r = s$, then $x < z$ is immediate as $<$ is certainly transitive on each individual component. If they lie in a SH-class, the result follows from the construction. So if q, r, s are not all equal, we may suppose that they lie in a PM-class. Then $U \le V \le W$. If $q = r$ then $U = V$ and as q, r, s are not all equal, $r \prec s$, so $V < W$. Hence $U < W$, so $x < z$. A similar argument applies if $r = s$. If q, r, and s are all distinct, we have $q \prec r \prec s$, and by the 3-chain conditions (va) in the definition of 'skeleton', one of the following applies: $q\,PM\,r\,PM\,s$, $q\,CPM\,r\,PM\,s$, $q\,PM\,r\,CPM\,s$, and the relation between q and s is PM, CPM, CPM respectively.

If $q\,PM\,r\,PM\,s$, then by definition, $U < V \Leftrightarrow \theta_q^{-1}(U) = \theta_r^{-1}(V)$, and so it follows that $\theta_q^{-1}(U) = \theta_r^{-1}(V)$. Similarly from $V < W$ we find that $\theta_r^{-1}(V) = \theta_s^{-1}(W)$. Therefore $\theta_q^{-1}(U) = \theta_s^{-1}(W)$, which in turn implies that $U < W$ and hence that $x < z$.

If $q\,CPM\,r\,PM\,s$, then $U < V \Leftrightarrow \theta_q^{-1}(U) \neq \theta_r^{-1}(V)$, and so $\theta_q^{-1}(U) \neq \theta_r^{-1}(V)$. But still $\theta_r^{-1}(V) = \theta_s^{-1}(W)$ and therefore $\theta_q^{-1}(U) \neq \theta_s^{-1}(W)$, so as the label on (q, s) is CPM, we get $U < W$, so again $x < z$.

If $q\,PM\,r\,CPM\,s$, then a similar argument applies to that in the previous paragraph.

Case 2: $q \equiv r \equiv s$ is false. Then $q \neq s$ since $q = s$ implies that $q = r$, and hence certainly that $q \equiv r \equiv s$. Also we have neither $q\,SH\,s$ nor $q\,PM\,s$, as from each of these we deduce by the 3-chain restrictions that $q \equiv r \equiv s$. Hence $q\,C\,s$ or $q\,CPM\,s$ or $q\,G\,s$.

Case 2A: $q\,C\,s$. Then $A_q <_c A_s$ and so $x < z$ is immediate.

Case 2B: $q\,CPM\,s$. From the 3-chain restrictions, we must have one of $q = r$, $r = s$, $q\,PM\,r\,CPM\,s$, $q\,CPM\,r\,PM\,s$, $q\,G\,r\,G\,s$. The first four would give $q \equiv r \equiv s$, and so $q\,G\,r\,G\,s$ is the only possibility. Thus q and s are labelled A or AC, and r is labelled A, AC, or Ge. Suppose $\iota(q) = \iota(s) \prec r$. If $\iota(r) \prec q$ then $\iota(r)\,CPM\,r\,G\,s$, contrary to the 3-chain restrictions. Hence $q \prec \iota(r)$ and $x < y$ is defined by Clause 1, and $y < z$ by Clause 3, so we have $\theta_q^{-1}(U) < \theta_r^{-1}(V)$ and $\theta_s^{-1}(W) \not< \theta_r^{-1}(V)$. By transitivity in A', $\theta_s^{-1}(W) \not\le \theta_q^{-1}(U)$.

But this holds in $A_{\iota(q)} = A_{\iota(s)}$, so $\theta_s^{-1}(W) \neq \theta_q^{-1}(U)$ and as $q\, CPM\, s$, this implies that $x < z$. If $r \prec \iota(q)$ then $x < y$ is defined by Clause 2 and $y < z$ by Clause 1 and a similar argument applies.

Case 2C: $q\, G\, s$ and $q = r$. Since transitivity holds in $\mathcal{A}'$, we may suppose that at least one of q, s does not lie in A', and hence is labelled A or AC. Since $q = r$, it follows that $y \in U$ so $U = V$. By definition, if Clause 1 applies, $x < z \Leftrightarrow \theta_q^{-1}(U) < \theta_s^{-1}(W)$ and also $y < z \Leftrightarrow \theta_q^{-1}(U) < \theta_s^{-1}(W)$, from which it follows that $x < z \Leftrightarrow y < z$. Since we know that $x < y$ and $y < z$ it follows that $x < z$. We argue similarly if Clause 2 or 3 applies.

Case 2D: $q\, G\, s$ and $r = s$. This is similar to Case 2C, and is omitted.

Case 2E: $q\, G\, s$ and $q\, PM\, r\, G\, s$. As $q\, PM\, r$ and $x < y$, $\theta_q^{-1}(U) = \theta_r^{-1}(V)$. Also, Clause 1 is used to define $x < z$ if and only if it is used to define $y < z$. It follows that $x < z \Leftrightarrow y < z$, and hence that $x < z$.

Case 2F: $q\, G\, s$ and $q\, G\, r\, PM\, s$. This is similar to Case 2E, and is omitted.

Case 2G: $q\, G\, s$ and $q\, G\, r\, G\, s$. (From the 3-chain restrictions, it follows that this is the last case.) The possibilities for $q, \iota(q), r$ and $\iota(r)$ were detailed in Clauses 1, 2, and 3, and there are similar possibilities for r and s. We shall see that the following are the only possibilities: $\iota(q) \prec \iota(r) \prec \iota(s)$, $\iota(r) \prec \iota(s) \prec \iota(q)$, and $\iota(s) \prec \iota(q) \prec \iota(r)$. In the first of these, since $x < y$ and $y < z$ we deduce that $\theta_q^{-1}(U) < \theta_r^{-1}(V)$ and $\theta_r^{-1}(V) < \theta_s^{-1}(W)$, and so by transitivity in A', $\theta_q^{-1}(U) < \theta_s^{-1}(W)$, which yields $x < z$. In the second we have $\theta_r^{-1}(V) \not< \theta_q^{-1}(U)$ and $\theta_r^{-1}(V) < \theta_s^{-1}(W)$, from which we deduce by transitivity in A' that $\theta_s^{-1}(W) \not< \theta_q^{-1}(U)$, which again implies $x < z$. The third case is similar.

It remains to show that the possibilities listed are the only ones. First we remark that the same proof as before shows that all of $q, \iota(q), r, \iota(r), s, \iota(s)$ are comparable. Combining the three possibilities, $q, \iota(q) \prec r, \iota(r)$; $q \prec r, \iota(r) \prec \iota(q)$; and $\iota(r) \prec q, \iota(q) \prec r$ with the corresponding possibilities for r and s gives the following:

(i) $q, \iota(q) \prec r, \iota(r) \prec s, \iota(s)$,
(ii) $q, \iota(q) \prec r \prec s, \iota(s) \prec \iota(r)$,
(iii) $q, \iota(q), \iota(s) \prec r, \iota(r) \prec s$,
(iv) $q \prec r, \iota(r) \prec \iota(q), s, \iota(s)$,
(v) $q \prec r \prec s, \iota(s) \prec \iota(r) \prec \iota(q)$,
(vi) $q, \iota(s) \prec r, \iota(r) \prec s, \iota(q)$,
(vii) $\iota(r) \prec q, \iota(q) \prec r \prec s, \iota(s)$,
(viii) $\iota(r) \prec q, \iota(q) \prec r \prec s, \iota(s) \prec \iota(r)$,
(ix) $\iota(s) \prec \iota(r) \prec q, \iota(q) \prec r \prec s$.

Of these (viii) violates irreflexivity of $\prec$ so does not occur, and (i) has already been covered. The following are impossible since they give rise to G followed by CPM or vice versa, contrary to the definition of 'skeleton': (ii), (v), (vii), (ix), and this also applies to (vi) whether we have $s \prec \iota(q)$ or $\iota(q) \prec s$. Also, the same argument shows that (iii) and (iv) must actually give $\iota(s) \prec \iota(q) \prec \iota(r)$ and $\iota(r) \prec \iota(s) \prec \iota(q)$ respectively, and these have been covered in the above discussion.

Now we verify homogeneity. Let p be a finite partial automorphism of $\mathcal{A}$. Let $\mathcal{B}$ be the union of all the non-SH-classes. Then all members of any SH-class are related to all other points by $\parallel$ or C. Suppose that we can extend the restrictions of p to $\mathcal{B}$, and to each SH-class, to automorphisms. Then we take the union of all of these, and since there is 'no interaction' between each SH-class and the rest, this union is also automatically an automorphism. Now each SH-class is homogeneous in its own right (by Theorem 5.2) and so we can find such an extension. It suffices therefore to show how to extend p restricted to $\mathcal{B}$ to an automorphism of $\mathcal{B}$. For ease then, suppose that p is itself a partial automorphism of $\mathcal{B}$.

Next we extend p to a finite partial automorphism p' whose action on $\mathcal{B}'$ is sufficient to control what happens on the whole of $\mathcal{B}$. This is done on each $\equiv$-class separately, and we then take the union. The only non-trivial $\equiv$-classes remaining are PM-classes. For any such class T, whose canonically chosen element is q, define p_1 on (some of) the maximal chains of A_q by letting $p_1(U) = V$ if for some $q' \in T$, p maps some element of $\theta_{q'}(U)$ into $\theta_{q'}(V)$. Then p_1 is well-defined. For suppose that $p_1(U) = V$ and $p_1(U) = V'$. Then for some $q', q'' \in T$, p maps an element of $\theta_{q'}(U)$ into $\theta_{q'}(V)$ and an element of $\theta_{q''}(U)$ into $\theta_{q''}(V')$. If $q' = q''$ then as $\theta_{q'}(U)$ is a chain, the elements of $\theta_{q'}(V)$ and $\theta_{q''}(V')$ arising are comparable, and hence $\theta_{q'}(V) = \theta_{q''}(V') = \theta_{q'}(V')$ so $V = V'$. Otherwise assume that $q' \prec q''$. If $q' \, PM \, q''$ then $\theta_{q'}(U) < \theta_{q''}(U)$, so as p is a partial automorphism, some element of $\theta_{q'}(V)$ is below some element of $\theta_{q''}(V')$, so $\theta_{q'}(V) < \theta_{q''}(V')$, from which again $V = V'$ follows. If $q' \, CPM \, q''$ then $\theta_{q'}(U) \parallel \theta_{q''}(U)$, so as p is a partial automorphism, some element of $\theta_{q'}(V)$ is incomparable with an element of $\theta_{q''}(V')$, from which $\theta_{q'}(V) \parallel \theta_{q''}(V')$ follows, again giving $V = V'$.

A similar argument shows that p_1 is 1-1.

Now, by definition of p_1, the restriction of p to A_q is compatible with it, meaning that if $p(x) = y$ for $x, y \in A_q$, and U, V are maximal chains of A_q containing x and y respectively, then $p_1(U) = V$. We can choose p' by extending the action of p on each such A_q so that if $p_1(U) = V$ where U and V are maximal chains of A_q then $p'(x) = y$ for some $x \in U$ and $y \in V$. To do this, note that a suitable x already exists in $\operatorname{dom} p$ if and only if a suitable y exists in range p, and if neither does, then we may make arbitrary choices of $x \in U$ and $y \in V$ and let $p'(x) = y$. This preserves the ordering between different components $A_{q'}$ for $q' \in T$ because of the compatibility with p_1, and with other components by definition of the ordering on $\mathcal{A}$.

Since $\mathcal{A}'$ is known to be homogeneous, the restriction $p'|A'$ of p' to A' extends to an automorphism φ' of $\mathcal{A}'$. Finally we extend φ' to an automorphism φ of $\mathcal{A}$. For this, we first determine the action of φ on the maximal chains of each $A_{q'} \not\subseteq A'$. This is uniquely determined from the canonical representatives by $\theta_{q'}$. This is still compatible with $p_1|A_{q'}$, which was why we extended from p to p' in the first place. Then by homogeneity of each $A_{q'}$, the action of p there can be extended to φ.

It is immediate that any isomorphism from $\mathcal{A}$ to another coloured partial order restricts to an isomorphism from $\mathcal{A}'$ to the corresponding restriction, and any isomorphism with domain $\mathcal{A}'$ extends to $\mathcal{A}$, because the extension to the rest of the PM-classes is achieved as above, and the SH-classes may be extended by the argument in the second paragraph of this proof. ∎

7. Amalgamation over reduced skeletons

In Section 6 we explained the reasons for considering a reduced skeleton associated with any given (abstract) skeleton $\mathcal{Q}$. In this section we show that we can establish amalgamation for the natural class of finite structures corresponding a reduced skeleton. For any reduced skeleton $\mathcal{Q}$, the *corresponding class* $\mathcal{K} = \mathcal{K}_{\mathcal{Q}}$ is defined to be the family of finite coloured partial orders which can be written as a disjoint union $\bigcup_{q \in Q} X_q$ (with all but finitely many X_q empty) fulfilling the following conditions:

- if q is labelled CA and by a colour structure partition $\{D_{c'} : c' \in C'\}$, then X_q is a finite chain of antichains where each antichain has colours in some $D_{c'}$ (counting multiplicities),
- if q is labelled A, then X_q is a finite antichain of cardinality $\leq N(q)$ coloured by C_q,
- if q is labelled AC then X_q is a finite antichain of at most $N(q)$ finite chains coloured by C_q,
- if q is labelled Ge, then X_q is a finite partial order coloured by C_q,
- if $q \prec r$ and $x \in X_q$, $y \in X_r$, then $x \not\geq y$,
- if $q \parallel r$, then $X_q \parallel X_r$ and if $q\, C\, r$, then $X_q <_c X_r$,
- if q is labelled AC and $x, y \in X_q$ lie in the same maximal chain of X_q, and $z \in X_r$, $r \neq q$, then $x < z \leftrightarrow y < z$ and $x > z \leftrightarrow y > z$.

THEOREM 7.1. *The class $\mathcal{K}$ corresponding to any reduced skeleton $\mathcal{Q}$ is an amalgamation class.*

Proof. To show that $\mathcal{K}$ has the amalgamation property, take $X, Y, Z \in \mathcal{K}$ such that $X = Y \cap Z$. Write X as a disjoint union $\bigcup_{q \in Q} X_q$ where each X_q is as prescribed in the definition of $\mathcal{K}$; similarly for Y_q and Z_q. The amalgamated structure $T = \bigcup_{q \in Q} T_q$ will have domain $Y \cup Z$, and so we only have to decide how members of $Y - X$ and $Z - X$ are related. It suffices to cover the case in which each of these sets has just one member, as we may repeat, so let $Y - X = \{y\}$ and $Z - X = \{z\}$. Let $y \in Y_q$ and $z \in Z_r$. As each of Y and Z is to be a substructure, all relations between pairs of points in Y, and all relations between pairs of points in Z, are already determined, and we just have to decide how y and z are related. This is given by the following clauses.

(i) If $y < x$ in Y and $x < z$ in Z for some $x \in X$, we let $y < z$, and if $y > x'$ in Y and $x' > z$ in Z for some $x' \in X$, we let $y > z$. Note that these cannot both hold, as this would give $x < x'$ and $x' < x$.

From now on, we assume that this does not hold.

(ii) If $q = r$ is labelled CA, then since the corresponding dense chain of antichains is homogeneous, its age, which contains X_q, Y_q, and Z_q, has the amalgamation property, so we choose any amalgam of Y_q and Z_q over X_q, and this determines the relation between y and z. The resulting structure lies in $\mathcal{K}_{\mathcal{Q}}$ since by definition of reduced skeleton, the label on (q, s) or (s, q) for any $s \in Q$ comparable with q is C (so there is no clash if for instance y and z are identified in the amalgam).

(iii) If $q = r$ is labelled A where $N(q)$ is finite, let y and z be identified in the amalgam. This is possible, because y and z have the same colour, and the label on any (q, s) or (s, q) for s comparable with q is C, so that the amalgam lies in $\mathcal{K}_Q$.

(iv) If $q = r$ is labelled A where $N(q) = \aleph_0$, let $y \parallel z$.

(v) If $q = r$ is labelled AC and for some $x \in X_q$, $y < x$ in Y and $z < x$ in Z, let $y < z$ (this choice is arbitrary; it could equally well be $z < y$, but not $y = z$ since we do not know whether y and z have the same colour). Similarly if $x < y$ in Y and $x < z$ in Z. In this case, we know that y and z must be in the same maximal chain of T_q, so we must make them comparable. Since there is no point in X in between (as clause (i) is assumed false) they can be either way round. The truth of the final condition on a member of $\mathcal{K}_Q$ for points labelled AC follows from that for Y and Z 'via' x. For if $t \in X_s$ where $s \neq q$, then $y < t \Leftrightarrow x < t$ holds in Y, and $z < t \Leftrightarrow x < t$ holds in Z, and therefore $y < t \Leftrightarrow z < t$, and similarly $y > t \Leftrightarrow z > t$.

(vi) If $q = r$ is labelled AC and (v) does not apply, and y is comparable in Y with a member of X_q, or z is comparable in Z with a member of X, or both, let $y \parallel z$. The point here is that since (i) and (v) are assumed false, the data ensure that in any amalgam, y and z must lie in different maximal chains of T_q, so we have to stipulate that they are incomparable.

(vii) If $q = r$ is labelled AC and y, z are not comparable in Y, Z respectively with any member of X_q, let $y < z$ if $N(q)$ is finite, and $y \parallel z$ if $N(q) = \aleph_0$. We have to make y and z comparable in the finite case to avoid increasing the number of maximal chains, but since the label on any (q, s) or (s, q) for s comparable with q is C, this causes no problems.

(viii) If $q = r$ is labelled Ge, let $y \parallel z$.

(ix) If $q\,C\,r$, let $y < z$ and if $r\,C\,q$, let $z < y$.

(x) If $q \parallel r$, let $y \parallel z$.

(xi) If $q\,G\,r$ or $r\,G\,q$, let $y \parallel z$.

To summarize, we let y and z be incomparable if possible, as this eases the verification of transitivity, but in some cases, we are forced to decide comparability one way or the other, or even identification of y and z.

This defines the amalgam T in all cases. The fact that Y and Z are substructures is immediate, since no new relations are given between members of Y, or between members of Z.

Next we verify that T is partially ordered. Irreflexivity is clear, since though in some cases we identified y and z, in those we did *not* say that $y < z$ or $z < y$. For transitivity, suppose that $a < b < c$ in T, and we have to verify that $a < c$. If all of a, b, and c lie in the same one of Y or Z, then we appeal to transitivity in that structure. If not all of a, b, c lie in the same one of Y, Z then both y and z are members of $\{a, b, c\}$. By interchanging y and z if necessary (since their roles are symmetrical) we suppose that y occurs first. This leaves the cases $a = y, b = z$; $a = y, c = z$; and $b = y, c = z$. We just do the first two, since the argument for the third is the same as for the first.

In the first case, $y < z < c$. We consider the clauses in turn. Let $c \in X_s$.

(i) $y < x$ in Y and $x < z$ in Z. Hence $x < z < c$ holds in Z, so by transitivity there, $x < c$ in Z, and hence in X, and hence in Y. Thus $y < x < c$ holds in Y and hence so does $y < c$.

(ii) $y < z$ in T_q where q is labelled CA. If $c \in T_q$ too, then $y < c$ holds since T_q is an amalgam of Y_q and Z_q. Otherwise, by definition of 'skeleton', $q\,C\,s$, and from this, $y < c$ follows.

(iii), (iv), (vi), (viii), (x), (xi) do not apply since they do not give $y < z$.

(v) If $s \neq q$ then as $Y, Z \in \mathcal{K}$, $y < c \Leftrightarrow x < c$ and $z < c \Leftrightarrow x < c$. Since $z < c$ we deduce that $y < c$ as required. If $s = q$ then as z and x are in the same maximal chain of Z_q, and so are z and c, it follows that x and c are in the same maximal chain of X_q, so y, x, c are in the same maximal chain of Y_q, so $y < c$ or $c < y$. But $c < y$ gives $z < c < y$ so we are back in clause (i). We deduce that $y < c$.

(vii) Since y, z are not comparable with any member of X_q, $s \neq q$ and the label on (q, s) is C which gives $y < c$.

(ix) If $q\,C\,r$ then by definition of 'skeleton' also $q\,C\,s$, so $y < c$.

In the second case, $y < b < z$. Then since $b \neq y, z$, $b \in X$, and so clause (i) is invoked to give $y < z$ in T.

We remarked as we went along that in each case, $T \in \mathcal{K}_{\mathcal{Q}}$, and so this concludes the verification of the amalgamation property. ∎

8. Proofs of the restrictions

In this section we prove that the configuration of any of our structures having one, two or three components is included in the catalogue given in Section 3.

Theorem 8.1 (The interdense case). *Any countable homogeneous interdensely coloured partial order $\mathcal{P}$ is isomorphic to one of the following:*

- *an antichain,*
- *an antichain of chains each isomorphic to $\mathbb{Q}_C$,*
- *a chain of antichains obtained from $Q_{C'}$ by replacing each point by a coloured antichain, so that points coloured the same are replaced by isomorphic antichains, and the colour sets of antichains replacing differently coloured points are disjoint,*
- *the C-coloured generic.*

We have formulated the result for the interdense case. As remarked before, we get the result for the single-component case by insisting that an antichain must be monochromatic. It is possible to obtain this list by following Schmerl's method, which involved considering which of five particular partial orders embed (for appropriate colours on their points), and this approach was adopted in [11]. These are a 2-element chain, a 2-element antichain, the union of a 2-element chain and an incomparable point, and a Λ-shape and V-shape. Some of the initial lemmas and examples for this were also given in [1]. It is more efficient however to use Lemma 2.1 to derive the coloured case directly from [10].

Proof of Theorem 8.1. Let $\mathcal{P} = (P, <, F)$ be countable interdensely coloured by the colour set C, and suppose it is not an antichain. Pick $c \in C$. Then by Lemma 2.1 the set P_c of points coloured by c is itself homogeneous and so is of one of the forms in Schmerl's list. Since P is not an antichain there are $x < y$ in P, and by interdensity there is z coloured c with $x < z < y$. Repeating we get comparable elements coloured c, so P_c is also not an antichain.

Case 1: P_c is a chain of antichains. Suppose that $\mathcal{P}$ contains a 2-element chain and an incomparable point, and from this we shall deduce that the partial order $Q = \{x_1, x_2, x_3, x_4\}$ where $x_1 < x_2$, $x_3 < x_4$, $x_1 \parallel x_3, x_4$ and $x_2 \parallel x_3, x_4$ also embeds in $\mathcal{P}$ for some colouring of its points. Suppose not, for a contradiction.

By assumption, there are x, y, z with $x < y$ and $x, y \parallel z$, and clearly $s \parallel z$ for all $s \in [x, y]$.

We first remark that any $t > z$ is strictly above every element of $[x, y)$ (and dually, any $t < z$ is strictly below every element of $(x, y]$). Since Q does not embed, for every $s \in [x, y)$, $\{s, y, z, t\}$ is not an instance of Q. Since $z \parallel \{x, y\}$, t is comparable with s or y. As $z < t$ and $z \not\leq s, y$, also $t \not\leq s, y$, so $t > s$ or $t > y$, but as $y > s$, this must mean that $t > s$.

Now, by interdensity, there is $z' \in (x, y)$ such that $F(z') = F(z)$. By homogeneity, there are x' and y' such that $x' < z < y'$ with $F(x') = F(x)$ and $F(y') = F(y)$. By what we have just shown, $y' > [x, y)$ and $x' < (x, y]$. Also, $z \parallel z'$. By homogeneity, there is an automorphism f of $\mathcal{P}$ that interchanges z and z'. By interdensity, there is $y'' \in (z', y)$ such that $F(y'') = F(y') = F(y)$. Since $z' < y''$, it follows that $z < fy''$ and therefore, by our previous remark, $y'' < fy''$. But also $z = f^{-1}z' < f^{-1}y''$ and therefore $y'' < f^{-1}y''$, a contradiction.

Since Q embeds, it now follows by interdensity that even in P_c there is a 2-element chain and an incomparable point, and this is contrary to P_c being a chain of antichains.

The conclusion is that $\mathcal{P}$ contains no 2-element chain and an incomparable point, and therefore 'incomparability or equality' is an equivalence relation $\sim$ on $\mathcal{P}$. This means that $\mathcal{P}$ can be partitioned into maximal antichains. By homogeneity, any two maximal antichains sharing a colour are isomorphic. Let Y be the set of these antichains and for each $y \in Y$ let D_y be the set of colours in C occurring in that antichain. Let $C' = \{D_y : y \in Y\}$ and we view C' as a set of colours, and colour the points of Y accordingly. We order Y by letting $y_1 < y_2$ if some member of y_1 is below some member of y_2. It follows by the definition of $\sim$ that this is equivalent to saying that *every* member of y_1 is below every member of y_2, and $<$ is clearly a partial ordering of Y. It is linear, since if $y_1, y_2 \in Y$ were incomparable, then they would have to determine the same antichain, hence be equal. We show that $Y \cong \mathbb{Q}_{C'}$ and that $\mathcal{P}$ is obtained from Y in the way described.

To see that Y is interdensely coloured, let $y_1 < y_2$ in Y, and let $c \in C'$. Then there is $y \in Y$ such that $D_y = c$. Pick $x_1 \in y_1, x_2 \in y_2$ and $x \in y$. Since $\mathcal{P}$ is interdensely coloured there is $x' \in (x_1, x_2)$ coloured $F(x)$.

By Lemma 3.1, $Y \cong \mathbb{Q}_{C'}$. (Note that Y has at least two points, since $\mathcal{P}$ was assumed not to be an antichain.)

Case 2: P_c is an antichain of at least two chains. Then by interdensity, between each pair of distinct points of a maximal chain of P_c, there are points of P of all possible colours. By homogeneity it follows that for every $x \in P - P_c$ there are $y, z \in P_c$ such that $y < x < z$. Furthermore, any point z' of P_c that is comparable with x lies in the same maximal chain X of P_c as y and z do. For if $x < z'$ for instance, then $y < z, z'$ in P_c, and as P_c is an antichain of chains, z and z' are comparable.

We show that any two members of P each comparable with an element of X are themselves comparable. For suppose for a contradiction that $y' < x, x' < z'$ where $y', z' \in X$, and $x \parallel x'$. Let y'' lie in a different maximal chain of P_c and let f be an automorphism taking y' to y''. Then the map p defined by $px = x, px' = fx'$ is a finite partial automorphism of P which does not extend to an automorphism, giving a contradiction.

Hence the set of points of P comparable with members of X itself forms a chain, and it follows that P is the union of all these chains, so is itself an antichain of chains.

Case 3: P_c is generic. Thus every finite c-coloured partial order embeds in P_c. We have to show that every finite C-coloured partial order $(X, <)$ embeds in P. We form another finite coloured partial order $Y \supseteq X$ adding points coloured c so that for each point x of X there are points x_1 and x_2 coloured c such that $x_1 < x < x_2$ and there are no points between x_1 and x_2 in Y other than x, and also if $x_1 < x < x_2$ and $y_1 < y < y_2$ are two such sets of points and $x \parallel y$ then $x_1 \not\leq y_2$ and $y_1 \not\leq x_2$. Now let Z be the c-coloured subset of Y. By assumption, Z embeds in P_c, and hence in P. We may extend the embedding to the whole of Y using interdensity, and the configuration of the added points ensures that the ordering relations between the elements of X are respected. ■

Theorem 8.2 (2-chains). *Let $\mathcal{P}$ be a countable homogeneous coloured partial order having just two components. Then the components may be labelled P_1 and P_2 so that one of the following holds:*

(i) *every member of P_1 is incomparable with every member of P_2,*
(ii) *every member of P_1 is below every member of P_2,*
(iii) *P_1 and P_2 are both chains of antichains, and P_1 is partially below P_2, written $<_{sh}$,*
(iv) *P_1 and P_2 are both antichains, or antichains of chains, and there is a* 1-1 *correspondence between their sets of maximal chains such that $x < y$ if and only if x and y lie in chains which correspond; we write $P_1 <_{pm} P_2$ (for 'perfect matching'),*
(v) *P_1 and P_2 are both antichains, or antichains of chains, and there is a* 1-1 *correspondence between their sets of maximal chains such that $x < y$ if and only if x and y lie in chains which do not correspond; we write $P_1 <_{cpm} P_2$ (for 'complement of perfect matching'),*
(vi) *P_1 and P_2 are both antichains, or antichains of chains, and for any x_1, x_2 in the same maximal chain of P_1 and y_1, y_2 in the same maximal chain of P_2, $x_1 < y_1 \Leftrightarrow x_2 < y_2$, and for any finite disjoint unions U and V of maximal chains of P_1 there is $y \in P_2$ such that y is above all members of U and not above any member of V, and for any finite disjoint unions U and V of maximal chains of P_2 there is $x \in P_1$ such that x is below all members of U and not below any member of V; we write $P_1 <_g P_2$ (for 'generic'),*

(vii) *P_1 is an infinite antichain or antichain of infinitely many dense chains and P_2 is generic, and P_1 is partially below P_2 (written $<_g$) and for any x and y lying in the same maximal chain of P_1 and $z \in P_2$, $x < z \Leftrightarrow y < z$, or the same thing with P_1 and P_2 interchanged (also written $<_g$),*

(viii) *P_1 and P_2 are both generic, and P_1 is partially below P_2 (also written $<_g$).*

Proof. If some member of one of the components, P_1 say, is less than some member of the other, P_2, then from the definition and initial properties of components, we know that $P_1 < P_2$. This means that some member of P_1 is below some member of P_2, and no member of P_2 is below any member of P_1. Let us suppose that neither (i) not (ii) applies, so that P_1 is partially (but not completely) below P_2.

First suppose that P_1 is a chain of antichains and P_2 is an antichain, or an antichain of chains. Let $y_1, y_2 \in P_2$ have the same colour and lie in distinct maximal chains. By homogeneity there is an automorphism f interchanging them. Let $x \in P_1$, $x < y_1$. Then as $fy_1 = f^{-1}y_1 = y_2$, $fx, f^{-1}x < y_2$. If fx is comparable with x then either $x \le fx$ or $x \le f^{-1}x$, so in each case $x < y_2$. If $x \parallel fx$, then x and fx lie in the same antichain of P_1. In either case, y_2 lies above some member of the antichain containing x. We deduce that for each $x \in P_1$, if $x < y_1$ then y_2 lies above some member of the antichain containing x. It follows that this is true even if y_1 and y_2 (of the same colour) lie in the same maximal chain of P_2.

Now we know that $x < y$ for some $x \in P_1$, $y \in P_2$. Let $z \in P_1$ be arbitrary. Then there is some $t > z$ in P_1 having the same colour as x, and by homogeneity, some automorphism takes x to t. Hence some member of P_2 (namely the image of y under this automorphism) lies above t. By the previous paragraph, all elements of P_2 having the same colour as y lie above an element of the antichain containing t, and hence above z. But all points of P_2 are greater than or equal to some point of P_2 coloured the same as y, and so it follows that $P_1 <_c P_2$, contrary to supposition.

Next suppose that P_1 is a chain of antichains, and P_2 is the generic partial order for its colour set. Since P_1 is partially but not completely below P_2, there are $x, x' \in P_1$ and $y, y' \in P_2$ such that $x < y$ and $x' \parallel y'$. By replacing y' by a smaller element of P_2 if necessary, we may suppose that y and y' have the same colour. Since for any w such that $w < x$, we have $w < y$, and since P_1 is interdensely coloured, we may also assume that x and x' have the same colour, and by homogeneity, that $x = x'$. By genericity of P_2, there is $z \in P_2$ incomparable with both y and y' and of the same colour as y. If $x \parallel z$, let $t_1 = y$ and $t_2 = z$. If $x < z$ then let $t_1 = z$ and $t_2 = y'$. In each case, $x < t_1$, $x \parallel t_2$, and t_1 and t_2 are incomparable and have the same colour.

Choose $x_1 < x$ having the same colour as x. By homogeneity of $\mathcal{P}$ there is an automorphism fixing t_1 and taking x_1 to x, and the image x_2 of x satisfies $x < x_2 < t_1$, and since $x \parallel t_2$, also $x_2 \parallel t_2$. By homogeneity again, there is an automorphism f of $\mathcal{P}$ that interchanges t_1 and t_2. Thus $fx_2, f^{-1}x_2 < t_2$ and $fx_2, f^{-1}x_2 \parallel t_1$. If $fx_2 \le x_2$ then $fx_2 < t_1$, contradiction, and if $fx_2 \ge x_2$, then $f^{-1}x_2 \le x_2$ giving $f^{-1}x_2 < t_1$, again a contradiction. Hence fx_2 lies in the same antichain of P_1 as x_2, and hence lies above x, which contradicts $x \parallel t_2$.

We deduce from the above that if P_1 is a chain of antichains, then P_2 must also be a chain of antichains, and similarly, if P_2 is a chain of antichains, then P_1 is a chain of antichains. This covers clause (iii).

Next suppose that P_1 and P_2 are both antichains of chains (with more than one chain). First we show that for any maximal chains X of P_1 and Y of P_2, X is either incomparable with Y, or completely below it. If not, X is partially below Y and by homogeneity, the same will hold for any maximal chains X' of P_1 and Y' of P_2 (on taking $x' \in X'$ and $y' \in Y'$ to points of X, Y similarly related, adjusting up and down to ensure that the colours match).

Now suppose that for every $x \in X$, $x \parallel Y$ or $x <_c Y$. Then the same holds for any X', Y' (by the argument just given). Since X is partially below Y there are $x_1, x_2 \in X$ such that $x_1 \parallel Y$ and $x_2 <_c Y$. By decreasing x_2 if necessary, we may assume that it has the same colour as x_1. Clearly $x_2 < x_1$. By homogeneity there is an automorphism taking x_2 to x_1, and the image Y' of Y satisfies $x_1 <_c Y'$. Pick $y \in Y$ and $y' \in Y'$ of the same colour. Then $x_2 < y, y'$, so there is an automorphism f fixing x_2 and interchanging y and y'. Since $x_1 < y'$, $fx_1 < fy' = y$. As $x_1 \parallel Y$, $fx_1 < x_1$. Hence $x_1 = f^{-1}fx_1 < f^{-1}x_1 < f^{-1}y' = y$, giving a contradiction.

It follows that some $x \in X$ is partially below Y. By homogeneity this holds for every member of X. Similarly, every member of Y is partially above X. (In particular, it follows that P_1 and P_2 are antichains of non-trivial chains; that is, they are not actually antichains.) Now pick $x \in X$, $y_1, y_2 \in Y$ such that $x \parallel y_1$, $x < y_2$. Then $y_1 < y_2$. Let X' be another maximal chain of P_1 and pick $z_1, z_2 \in X'$ such that $z_1 < y_1$ and $z_2 \parallel y_2$. By decreasing z_1 if necessary, suppose that z_1 and z_2 have the same colour, and pick $y' \in Y$ greater than y_1 and z_2. Then by homogeneity, there is an automorphism g fixing x and y' and taking z_1 to z_2. As $z_1 < y_1$, $z_2 = gz_1 < gy_1$ and as $z_2 \parallel y_2$, $y_2 < gy_1$. Hence $gx = x < y_2 < gy_1$ giving $x < y_1$, a contradiction.

The conclusion is that for any maximal chains X of P_1 and Y of P_2, either $X <_c Y$, or $X \parallel Y$. In other words, $<$ induces a natural (well-defined) partial order on the set $\mathcal{Q}$ of maximal chains of the two components. Now this is still homogeneous. For suppose that a finite partial automorphism of $\mathcal{Q}$ is given. If we choose an element from each of the chains in its domain or range, so that the points chosen from P_1 all have the same colour, and so do all the points chosen from P_2, then this gives a finite partial automorphism of $\mathcal{P}$, which extends to an automorphism; and clearly this automorphism induces an automorphism of $\mathcal{Q}$ extending the one originally given.

Now we appeal to the classification of the countable homogeneous bipartite graphs, which is given for instance in [6]. There are five possible types, which are empty, complete, perfect matching (that is, 1-1 correspondence between the two parts) or its complement, and generic (characterized by the property in clause (vi)). Our 2-component partial order is (or may be viewed as) a bipartite graph, and so the same list applies here, which gives the cases listed in clauses (i), (ii), (iv), (v), (vi).

Next suppose that P_1 is an antichain of (at least two) chains and P_2 is the generic partial order for its colour set. We show that for each maximal chain X of P_1 and $y \in P_2$, $X \parallel y$ or $X <_c y$. If not, there are $x_1, x_2 \in X$ such that $x_1 < y$ and $x_2 \parallel y$. By decreasing

x_1 if necessary, we may assume that $F(x_1) = F(x_2)$. Let y' be any other member of P_2 having the same colour as y. If $x_1 < y'$ we can take (x_1, y) to (x_1, y') by an automorphism, and if $x_1 \parallel y'$ we can take (x_2, y) to (x_1, y'). So in each case there is an automorphism fixing $X \cup P_2$ setwise and taking y to y'. Hence every member of P_2 having the same colour as y is partially above X, and by interdensity the same holds for all members of P_2.

Now we show that $X \cup P_2$ is homogeneous. For let p be a finite partial automorphism of $X \cup P_2$. In the first case, the domain of p intersects X. Then as p is also a finite partial automorphism of $\mathcal{P}$, it extends to an automorphism, and this must fix X setwise, so restricts to an automorphism of $X \cup P_2$ extending p. In the second case, $\operatorname{dom} p \subseteq P_2$. We can apply the same argument provided we can extend p to a finite partial automorphism q of $X \cup P_2$ whose domain intersects X. By the above, each member of P_2 is above some member of X, and so for each member of $\operatorname{dom} p \cup \operatorname{range} p$ we can choose a point of X below it, and the least x of these will be below all members of $\operatorname{dom} p \cup \operatorname{range} p$. Then $q = p \cup \{(x, x)\}$ is a partial automorphism, and the same argument as before applies.

Since $X \cup P_2$ is homogeneous, it follows from the chain of antichains and generic case above that either $X \parallel P_2$ or $X <_c P_2$, giving a contradiction. This establishes that for each $y \in P_2$, $X \parallel y$ or $X <_c y$, as required.

Next suppose that P_1 is an antichain of finitely many chains and that $P_1 \parallel P_2$ does not hold. Then for some $x \in P_1$ and $y \in P_2$, $x < y$. By homogeneity, any member of P_1 having the same colour as x also lies below a member of P_2. Hence for every maximal chain X of P_1, $X <_c y$ for some $y \in P_2$. As there are only finite many such X, there is $y \in P_2$ above all members of P_1. By homogeneity and interdensity, $P_1 <_c P_2$.

The final possibility is that P_1 and P_2 are both generic, which is clause (viii). ∎

Theorem 8.3 (V-shapes). *Let $\mathcal{P}$ be a countable homogeneous coloured partial order having three components P_1, P_2, P_3, such that $P_2 \parallel P_3$ and $x_1 < y$, $x_2 < z$ for some $x_1, x_2 \in P_1, y \in P_2$, and $z \in P_3$. Then one of the following holds:*

(i) *$P_1 <_c P_2$ and the relation between P_1 and P_3 is one of $<_c$, $<_{sh}$, $<_{pm}$, $<_{cpm}$ or $<_g$,*
(ii) *$P_1 <_c P_3$ and the relation between P_1 and P_2 is one of $<_c$, $<_{sh}$, $<_{pm}$, $<_{cpm}$ or $<_g$,*
(iii) *P_1, P_2, and P_3 are all chains of antichains, and $P_1 <_{sh} P_2$ and $P_1 <_{sh} P_3$,*
(iv) *P_1, P_2, and P_3 are all antichains of chains or generic, and $P_1 <_g P_2$ and $P_1 <_g P_3$.*

Proof. By Lemma 2.1, $P_1 \cup P_2$ and $P_1 \cup P_3$ are homogeneous, so by Theorem 8.2, P_1 and P_2, and also P_1 and P_3, are related by $<_c$, $<_{sh}$, $<_{pm}$, $<_{cpm}$, or $<_g$. In view of the first two clauses, we suppose that neither $P_1 <_c P_2$ nor $P_1 <_c P_3$.

First suppose that at least one of P_1, P_2, P_3 is a chain of antichains. By Theorem 8.2, all three must in fact be chains of antichains, and the relation between P_1 and P_2, and between P_1 and P_3, must be $<_{sh}$, which is clause (iii).

Next suppose that P_1, P_2, and P_3 are all antichains of chains. By Theorem 8.2, the relations between P_1 and P_2, and P_1 and P_3, are $<_{pm}$, $<_{cpm}$, or $<_g$.

Suppose first that $P_1 <_{pm} P_2$ and $P_1 <_{pm} P_3$. Then P_1, P_2, and P_3 must each have the same number of maximal chains. By our convention this number is greater than 1 (otherwise each would be an antichain of just one chain, which is counted as a chain). We show that $\mathcal{P}$ is not homogeneous. Take distinct maximal chains X_1 and X_2 of P_1, and let

Y_1 and Y_2 be the maximal chains of P_2 matched with them, and Z_1 and Z_2 the maximal chains of P_3 matched with them. Let $x_1 \in X_1$ and $x_2 \in X_2$ have the same colour, and similarly $y_1 \in Y_1$, $y_2 \in Y_2$, and $z_1 \in Z_1$, $z_2 \in Z_2$. Consider the partial automorphism p given by $py_1 = y_2$ and $pz_1 = z_1$. If p extends to an automorphism f, then, since $X_1 < Y_1$, we have $fX_1 < Y_2$, giving $fX_1 = X_2$, and applying the same reasoning to P_3, $fX_1 < Z_1$ so $fX_1 = X_1$, contradiction. Hence p cannot extend to an automorphism, and therefore $\mathcal{P}$ is not homogeneous.

The case where $P_1 <_{pm} P_2$ and $P_1 <_{cpm} P_3$ is very similar. By our convention, the sets all have size ≥ 3 (otherwise the relation between P_1 and P_3 would be counted as $<_{pm}$), and we again show that $\mathcal{P}$ is not homogeneous. Take X_i, Y_i, Z_i, x_i, y_i, z_i, p as before. If p extends to an automorphism f, then $fX_1 < Y_2$ and hence $fX_1 = X_2$ but also, since $X_1 \parallel Z_1$, we have $X_2 = fX_1 \parallel Z_1$, a contradiction, so again $\mathcal{P}$ is not homogeneous.

Similar proofs show that if $P_1 <_{cpm} P_2$ and $P_1 <_{cpm} P_3$, or if $P_1 <_{cpm} P_2$ and $P_1 <_{pm} P_3$, then $\mathcal{P}$ is not homogeneous.

Now suppose that $P_1 <_g P_2$ and $P_1 <_{pm} P_3$. Then P_1, P_2, and P_3 all have infinitely many maximal chains. Let X_1 and X_2 be distinct maximal chains of P_1, matched with maximal chains Z_1 and Z_2 of P_3. Let Y be a maximal chain of P_2 above X_1 but not X_2, which exists by genericity, and pick $y \in Y$, and $z_1 \in Z_1$ and $z_2 \in Z_2$ of the same colour. Consider the partial automorphism p given by $py = y$ and $pz_1 = z_2$. If p extends to an automorphism f, since $X_1 < Y, Z_1$, we have $fX_1 < Z_2$, so that $fX_1 = X_2$, but also $X_2 = fX_1 < Y$, a contradiction. Similar arguments apply if instead $P_1 <_{cpm} P_2$, or if P_2 and P_3 are interchanged.

This shows that if P_1, P_2, and P_3 are all antichains of chains, then $P_1 <_g P_2$ and $P_1 <_g P_3$.

Finally suppose that at least one of P_1, P_2, P_3 is generic. Then by Theorem 8.2, all must be either generic or an antichain of chains, and the relation between any of them which is generic and another which is an antichain of chains is $<_g$ (if they are comparable). The only cases remaining to eliminate are those in which $P_1 <_{pm} P_2$ or $P_1 <_{cpm} P_2$ (and $P_1 <_g P_3$), or the same with P_2 and P_3 interchanged. We just do the first as the others are similar. Since $P_1 <_g P_3$, P_1 has infinitely many maximal chains. Choose two of these, X_1, X_2 say, matched with Y_1, Y_2 respectively in P_2. By genericity we may find $z \in P_3$ above X_1 but not X_2. Let p be a finite partial automorphism fixing z, and taking a point of Y_1 to point of Y_2. Then any extension to an automorphism must take X_1 to X_2 and fix z, contrary to $X_1 < z$ and $X_2 \parallel z$. ■

THEOREM 8.4 (3-chains). *Let $\mathcal{P}$ be a countable homogeneous coloured partial order having just three components $P_1 < P_2 < P_3$. Then one of the following holds:*

(i) *$P_1 <_c P_2$ and $P_1 <_c P_3$,*
(ii) *$P_1 <_c P_3$ and $P_2 <_c P_3$,*
(iii) *P_1, P_2, and P_3 are all chains of antichains and $P_1 <_{sh} P_2 <_{sh} P_3$ and $P_1 <_{sh} P_3$,*
(iv) *P_1, P_2 and P_3 are all antichains of chains and*

- *$P_1 <_{pm} P_2$ and the relation between P_2 and P_3 is one of $<_{pm}$, $<_{cpm}$, $<_g$, and the relation between P_1 and P_3 is the same as that between P_2 and P_3, or*

- *$P_2 <_{pm} P_3$ and the relation between P_1 and P_2 is one of $<_{pm}$, $<_{cpm}$, $<_g$, in which case the relation between P_1 and P_3 is the same as the one between P_1 and P_2, or*
- *$P_1 <_g P_2$ and $P_2 <_g P_3$, in which case the relation between P_1 and P_3 is $<_g$, $<_{cpm}$ or $<_c$,*

(v) *P_1, P_2 and P_3 are all generics and $P_1 <_g P_2$, $P_2 <_g P_3$, and either $P_1 <_g P_3$ or $P_1 <_c P_3$,*

(vi) *P_1 and P_3 are antichains of chains, P_2 is a generic, $P_1 <_g P_2 <_g P_3$, and the relation between P_1 and P_3 is $<_g$, $<_{cpm}$, or $<_c$,*

(vii) *P_1 and P_2 are antichains of chains, P_3 is a generic, and $P_1 <_{pm} P_2$ or $P_1 <_g P_2$, and $P_1, P_2 <_g P_3$,*

(viii) *P_2 and P_3 are antichains of chains, P_1 is a generic, and $P_2 <_{pm} P_3$ or $P_2 <_g P_3$, and $P_1 <_g P_2$, P_3,*

(ix) *P_1 and P_2 are antichains of chains, P_3 is a generic, and $P_1 <_g P_2 <_g P_3$ and $P_1 <_c P_3$,*

(x) *P_2 and P_3 are antichains of chains, P_1 is a generic, and $P_1 <_g P_2 <_g P_3$ and $P_1 <_c P_3$,*

(xi) *two of P_1, P_2, P_3 are generics and the third is an antichain of chains, $P_1 <_g P_2 <_g P_3$ and $P_1 <_g P_3$ or $P_1 <_c P_3$.*

Proof. First, we see that the relation between P_1 and P_3 is the transitive closure of the relations between P_1, P_2 and P_2, P_3. Certainly, as $<$ is transitive, if $x \in P_1$, $y \in P_2$, and $z \in P_3$, and $x < y$ and $y < z$, then $x < z$. Conversely, suppose that $x \in P_1$ and $z \in P_3$ are such that $x < z$. We show that there is $y \in P_2$ such that $x < y < z$. We have, by assumption, $x_1 \in P_1$, $y_2, y_3 \in P_2$ and $z_1 \in P_3$ such that $x_1 < y_2$, and $y_3 < z_1$. If P_2 is an antichain, then it is monochromatic, so by homogeneity we may suppose that $y_2 = y_3 = y_1$ say. Otherwise there is a point $y_4 > y_2$ in P_2 having the same colour as y_2, and by interdensity, a point y_1 between y_2 and y_4 having the same colour as y_3. By homogeneity we may suppose that $y_1 = y_3$. Thus $x_1 < y_1 < z_1$. As $x < z$ and $x_1 < z_1$, by homogeneity there is an automorphism taking x_1 to x, and z_1 to z, and the image of y_1 provides the desired y.

If $P_1 <_c P_2$, then as $P_2 < P_3$, it follows by transitivity that $P_1 <_c P_3$, and similarly, if $P_2 <_c P_3$ then it follows that $P_1 <_c P_3$. These are clauses (i) and (ii), and from now on we suppose that P_1 is not completely below P_2, and P_2 is not completely below P_3.

Next we look at cases in which all three components are of the same type, beginning with chains of antichains. Since P_1 is not completely below P_2, by Lemma 5.1, $P_1 <_{sh} P_2$, and similarly $P_2 <_{sh} P_3$. We just have to show that $P_1 <_c P_3$ is false. Let $x \in P_1$, $y \in P_2$, $z \in P_3$ be such that $x \parallel y \parallel z$. Suppose for a contradiction that $x < z$. Since the relation between P_1 and P_3 is the transitive closure of the relations between P_1 and P_2, and P_2 and P_3, there is $y_1 \in P_2$ such that $x < y_1 < z$. As $x \parallel y$, $y < y_1$, and as $y \parallel z$, $y_1 < y$, contradiction.

Suppose next that P_1, P_2, P_3 are all antichains of chains. Then by Theorem 8.2, the possible relations between P_1 and P_2, and P_2 and P_3, are $<_{pm}$, $<_{cpm}$, and $<_g$. If $P_1 <_{pm} P_2$ then it follows easily from the fact that the relation between P_1 and P_3

is the transitive closure of those between P_1 and P_2, and P_2 and P_3, that the relation between P_2 and P_3 is the same as that between P_1 and P_3. Similarly if we are given $P_2 <_{pm} P_3$.

If $P_1 <_g P_2$ and $P_2 <_g P_3$, then the relation between P_1 and P_3 cannot be $<_{pm}$, for given a maximal chain X_1 of P_1 there is a maximal chain $Y_1 > X_1$ in P_2, and there are at least two distinct maximal chains of P_3 above Y_1. Hence the relation between P_1 and P_3 must be $<_g$, $<_{cpm}$, or $<_c$.

The cases which remain to eliminate under clause (iv) are those in which the relation between at least one of P_1 and P_2, and P_2 and P_3, is $<_{cpm}$, and the other is either $<_{cpm}$ or $<_g$.

First suppose that $P_1 <_{cpm} P_2 <_{cpm} P_3$. Recall that by definition of $<_{cpm}$, all sets of maximal chains have size at least 3. We show that $P_1 \cup P_2 \cup P_3$ is not homogeneous. Pick distinct maximal chains X_1, X_2, X_3 of P_1, and let Y_i be the 'matched' maximal chains of P_2, that is, the unique maximal chains such that $X_i \not< Y_i$, and similarly let Z_i be the maximal chain of P_3 matched with Y_i. Then in particular $X_1 < Y_2 < Z_1$ and $X_2 < Y_1 < Z_3$. Pick $x_1 \in X_1$, $x_2 \in X_2$ of the same colour and $z_1 \in Z_1$, $z_3 \in Z_3$ of the same colour. Then $x_1 < z_1$ and $x_2 < z_3$, so the map p given by $px_1 = x_2$ and $pz_1 = z_3$ is a partial automorphism. If p extends to an automorphism f then it must take the unique maximal chain Y_1 of P_2 not above X_1 to the unique maximal chain Y_2 of P_2 not above X_2, and similarly it must take Z_1 to Z_2, contradiction. Hence p does not extend, so $P_1 \cup P_2 \cup P_3$ is not homogeneous.

Next suppose that $P_1 <_{cpm} P_2$ and $P_2 <_g P_3$. Pick distinct maximal chains X_1, X_2, X_3 of P_1, and matching Y_i in P_2 (so that $X_i \not< Y_i$). As $P_2 <_g P_3$ there is a maximal chain Z of P_3 such that $Y_1 \not< Z$ and $Y_2, Y_3 < Z$. Then $X_1, X_2 < Z$, so p given by $px_1 = x_2$, $pz = z$ where $x_1 \in X_1$ and $x_2 \in X_2$ have the same colour, and $z \in Z$, is a partial automorphism. Any extension f to an automorphism has to take Y_1 to Y_2, but this is contrary to $Y_1 \not< Z, Y_2 < Z$. So this cannot be homogeneous either, and similarly $P_1 <_g P_2$ and $P_2 <_{cpm} P_3$ is impossible. This completes clause (iv).

If all of P_1, P_2, P_3 are generic, then by Theorem 8.2, the relation between any two is $<_c$ or $<_g$, so all possibilities are covered by clauses (i), (ii), and (v).

Next we look at the three cases where just two components are of the same type. If two of them are chains of antichains, then it follows at once from Theorem 8.2 that $P_1 <_c P_2$ or $P_2 <_c P_3$, so we are back in clause (i) or (ii).

If just two of the components are antichains of chains, then the third cannot be a chain of antichains (by 8.2) and so it is generic. This gives the following possibilities (invoking 8.2 again): P_2 generic (vi); P_3 generic (vii) and (ix); and P_1 generic (viii) and (x). All the constraints are given by 8.2 except that we have to rule out $P_1 <_{cpm} P_2 <_g P_3$ and $P_1 <_g P_2 <_{cpm} P_3$. The former is typical and follows the same steps as for the similar proof where P_3 is an antichain of chains where we take a single element z in P_3 instead of a maximal chain.

If just two of the components are generic, then by Theorem 8.2 the third must be an antichain of chains, and invoking the same result, the only possibilities are the ones listed in clause (xi).

The final case is where all three components are of different types. But this means that one of them is a chain of antichains, and by 8.2 again, we are back in clause (i) or (ii). ■

9. Existence and uniqueness proofs

We now give the existence and uniqueness proofs for the countable homogeneous coloured partial orders having at most three components, which were already described in Section 3. In view of the results of Section 6, it suffices to consider reduced skeletons of sizes 1, 2, or 3. For size 1 this makes no difference, but for sizes 2 and 3 it considerably cuts down the number of cases to be considered. The case of one component is given in Theorem 9.1, of 2-chains in Theorem 9.2, of three components in Theorem 9.3 for V-shapes and Theorem 9.4 for 3-chains. As usual it suffices to consider just these cases. In Section 10 we treat the general case, but essentially all the techniques used are given in this section.

THEOREM 9.1. *Each of the following structures $\mathcal{P}$ is a countable homogeneous interdensely coloured partial order and is uniquely determined up to isomorphism (by its label in the skeleton):*

- *a finite or countable antichain,*
- *an antichain of at least two and at most $\aleph_0$ chains each isomorphic to $\mathbb{Q}_C$,*
- *a chain of antichains obtained from $\mathbb{Q}_{C'}$ by replacing each point by a finite or countable coloured antichain, so that points coloured the same are replaced by isomorphic antichains, and the colour sets of antichains replacing differently coloured points are disjoint,*
- *the C-coloured generic.*

Proof. The constructions of $\mathbb{Q}_C$ and $\mathcal{P}_C$ as Fraïssé limits were given in Section 1. All the other structures in the list are given explicitly in terms of these (for instance using composition), so existence is clear in all cases. Homogeneity is clear for $\mathbb{Q}_C$ and $\mathcal{P}_C$ for the same reason, and for antichains is trivial. We may establish homogeneity in the imprimitive cases thus. For an antichain of chains $A[\mathbb{Q}_C]$ say, any finite partial automorphism p induces a finite partial automorphism q of A. This extends to an automorphism φ of A, which in turn is induced by an automorphism θ of $A[\mathbb{Q}_C]$ extending p. The proof for chains of antichains is similar.

Uniqueness of these structures was shown in Theorem 8.1. ■

THEOREM 9.2. *For each reduced (abstract) skeleton $\mathcal{Q}$ having just two points there is a countable homogeneous coloured partial order having $\mathcal{Q}$ as skeleton, which is uniquely determined up to isomorphism.*

Proof. For existence we appeal to Theorem 7.1. There it was shown that the class $\mathcal{K}$ corresponding to any reduced skeleton is an amalgamation class, and it follows by Fraïssé's Theorem that there is a countable homogeneous coloured partial ordering whose age is equal to $\mathcal{K}$. This has skeleton $\mathcal{Q}$ (see the beginning of the proof of Theorem 10.1 for the general argument).

To prove uniqueness, Let $\mathcal{P}$ be a countable homogeneous coloured partial order whose skeleton is $\mathcal{Q}$. By Fraïssé's Theorem it suffices to show that $\mathcal{K}$ is equal to the age of $\mathcal{P}$. It follows from the facts that $\mathcal{Q}$ is the skeleton of $\mathcal{P}$, and $\mathcal{K}$ is the class corresponding to $\mathcal{Q}$, that all members of the age of $\mathcal{P}$ lie in $\mathcal{K}$. It is the converse that we have to prove. We consider the various possibilities in turn. Let the points of Q be q and r, and let the components of $\mathcal{P}$ be P_1 and P_2, corresponding to q and r respectively.

In the first case $q \parallel r$, and then uniqueness follows from Theorem 8.1. For if $P_1' \cup P_2'$ is another countable homogeneous coloured partial order having $\mathcal{Q}$ as skeleton, with components P_1' and P_2', then by Theorem 9.1, $P_1 \cong P_1'$ and $P_2 \cong P_2'$, and as $P_1 \parallel P_2$ and $P_1' \parallel P_2'$, we can patch the isomorphisms to get an isomorphism from $P_1 \cup P_2$ to $P_1' \cup P_2'$.

Otherwise we suppose that $q \prec r$. In the next case, (q, r) is labelled C, and exactly the same proof as given for $q \parallel r$ demonstrates uniqueness, as there is again 'no interaction' between the components. From now on we suppose that (q, r) is not labelled C.

Since the skeleton is reduced, we must have $q\,G\,r$, so that q and r are labelled A, AC, or Ge.

First suppose that both q and r are labelled A or AC. By Theorem 8.2, for any maximal chains U of P_1 and U' of P_2, either $U \parallel U'$ or $U <_c U'$, and the resulting relation between the sets of maximal chains of P_1 and P_2 is $\parallel$, $<_c$, $<_{pm}$, $<_{cpm}$, or $<_g$, so in this instance, the last must apply. Hence $N(q) = N(r) = \aleph_0$, and the 'genericity' property holds: for any finite disjoint unions U and V of maximal chains of P_1 there is a maximal chain of P_2 above all members of U and above no members of V, and the corresponding dual property. This property is sufficient to demonstrate uniqueness, by a standard back-and-forth argument. This is just the 'random' (or alternatively called 'generic') bipartite graph, which is one of the homogeneous bipartite graphs classified in [6].

Next we suppose that just one of q and r is labelled Ge, in which case the other is labelled A or AC. Suppose that it is r which is labelled Ge (the argument for the case where q is labelled Ge being similar). We have to show that any member of $\mathcal{K}$ can be embedded in $\mathcal{P}$.

Let $X \cup Y \in \mathcal{K}$ where X and Y are lower and upper respectively, and we show that $X \cup Y$ can be embedded in $\mathcal{P}$. We first show how to reduce to the special case in which X has just one maximal chain. For this we add (new) points $\{y_x : x \in X\}$ to Y related by a perfect matching to the maximal chains of X, so that $y_x = y_{x'} \Leftrightarrow x, x'$ are comparable, $x < y_x$ for each x, and $x' \not< y_x$ if $x' \parallel x$. These new points are arbitrarily coloured, and are pairwise incomparable, and are also incomparable with all points of Y. (These extra stipulations are unimportant, but are given for definiteness.) Let $Y' = Y \cup \{y_x : x \in X\}$. The outcome is that all maximal chains of X are differently related to the points of Y' (meaning that if $x, x' \in X$ are incomparable there is $y \in Y'$ such that $x < y$ and $x' \parallel y$). Now suppose that we can embed all structures of the form $X' \cup Y'$ into $\mathcal{P}$ where X' is a maximal chain of X. By homogeneity of $\mathcal{P}$, we may suppose that these embeddings all agree on Y'. But now, because of the way that the maximal chains of X were differently joined to the points of Y', no two of them are made comparable in the embeddings, and we can take the union of all the embeddings to embed the whole of $X \cup Y'$ into $\mathcal{P}$.

From now on we may therefore assume that X has just one maximal chain. Since P_2 is generic for its colour set, we may also suppose that $Y \subseteq P_2$.

The hypotheses ensure that there are $x_1, x_2 \in P_1$ and $y_1, y_2 \in P_2$ such that $x_1 < y_1$ and $x_2 \parallel y_2$. By decreasing x_1 if necessary we may suppose that $F(x_1) = F(x_2)$ and by homogeneity we may suppose that $x_1 = x_2$. Let $y_3 \in P_2$ be incomparable with both y_1 and y_2 (which exists by genericity). If $y_3 > x_1$, let $y_1' = y_3$ and $y_2' = y_2$. If $y_3 \parallel x_1$, let $y_1' = y_1$ and $y_2' = y_3$. Then in both cases, $y_1' \parallel y_2'$, and $x_1 < y_1'$, $x_1 \parallel y_2'$.

Let $A = \{y \in Y : X < y\}$ and $B = \{y \in Y : X \parallel y\}$. Then no point of A is below any point of B, so by genericity, there are incomparable $a < A$ and $b > B$ in P_2, having the same colours as y_1' and y_2' respectively. By homogeneity, there is an automorphism taking y_1' to a and y_2' to b, and the image of x_1 is as required to complete the embedding of $\{x_1\} \cup Y$ into $\mathcal{P}$. By Theorem 8.2(vii), if X' is the maximal chain of P_1 containing x_1, then for each element y of P_2, $X' <_c y$ or $X' \parallel y$, and from this it follows that any embedding of X into X' correctly embeds $X \cup Y$ into $\mathcal{P}$.

Next we consider the case where q and r are both labelled Ge, and we have to show that the property of there being $x, x' \in P_1$ and $y, y' \in P_2$ such that $x < y$ and $x' \parallel y'$ uniquely determines $\mathcal{P}$ up to isomorphism. For this, let $X \cup Y \in \mathcal{K}$ where X and Y are lower and upper respectively, and we have to show that $X \cup Y$ can be embedded in $\mathcal{P}$. Let C_i be the colour set for P_i.

We build up the embedding of $X \cup Y$ into $\mathcal{P}$ by means of a series of special cases.

First suppose that $X = \{x_1, x_2\}$ and $Y = \{y_1\}$ and $x_1 < y_1$, $x_2 \parallel x_1, y_1$. Since P_2 is generic for the colour set C_2, there are $y_2 \geq y$ and $y_3 \leq y'$ in P_2 such that $F(y_2) = F(y_3) = F(y_1)$, and we also have $x < y_2$ and $x' \parallel y_3$. In other words we may assume by replacing y by y_2 and y' by y_3 if necessary that $F(y) = F(y') = F(y_1)$. Similarly we may assume that $F(x) = F(x_1)$ and $F(x') = F(x_2)$. We may further suppose by homogeneity that $y = y'$. By genericity of P_1 there is $x'' \in P_1$ incomparable with both x and x' and such that $F(x'') = F(x')$. If $x'' \parallel y$ then $\{x, x'', y\}$ is the desired copy of $\{x_1, x_2, y_1\}$. If $x'' < y$ then again appealing to genericity of P_1, there is $x''' \leq x''$ in P_1 incomparable with x' such that $F(x''') = F(x)$, and this time $\{x''', x', y\}$ is the required copy of $\{x_1, x_2, y_1\}$.

Next suppose that $X = \{x_1, \ldots, x_k, x_1', \ldots, x_l'\}$ is an antichain, $Y = \{y_1\}$ and $x_i < y_1$ for $1 \leq i \leq k$, $x_j' \parallel y_1$ for $1 \leq j \leq l$. By the previous paragraph, there are $x, x' \in P_1$ and $y \in P_2$ such that $F(y) = F(y_1)$, $x < y$ and $x' \parallel x, y$. Since P_1 is generic, there is an antichain of elements $z_1, \ldots, z_k, z_1', \ldots, z_l'$ such that $z_i < x$ and $x' < z_j'$ for each i and j having the correct colours. Then for each i, j, $z_i < y$ and $z_j' \parallel y$, so this gives an embedding of $X \cup Y$ into $\mathcal{P}$.

Further generalizing this case, suppose that X and Y are both antichains. Add an antichain Z of new points to X, perfectly matched with the points of Y and each incomparable with all points of X. Now by what we have just shown, each $X \cup Z \cup \{y\}$ for $y \in Y$ embeds in $\mathcal{P}$, and by homogeneity, we may suppose that all of these embeddings agree on $X \cup Z$. It remains to observe that Y maps to an antichain. First, all its elements are distinct since they are differently joined to the points of $X \cup Z$. Furthermore, if two are comparable, then one would be above at least two points of Z, contrary to the relation there being a perfect matching.

Next we suppose that X may be written as the disjoint union of two antichains L_1 and L_2 such that no element of L_1 is above any element of L_2, and that Y may be expressed in a similar way as the disjoint union of L_3 and L_4 (there are at most 4 'levels'). If actually $X = L_1$ and $Y = L_3$, then both of X and Y are antichains, and this case has already been covered. For the next possibility, suppose that $|X| = |L_1| = 1$, $|L_4| = 1$, and $L_1 < L_4$ (so that $L_2 = \emptyset$). Now increase L_3 if necessary so that there is $y_1 \in L_3$ with $L_1 < y_1 < L_4$. By the case already covered, $L_1 \cup L_3$ embeds, and by genericity of P_2, so does $L_3 \cup L_4$. By homogeneity, we may suppose that the images of L_3 under these embeddings are equal. In view of the presence of y_1, under the union of these two embeddings we also have $L_1 < L_4$, so all relations are respected, showing that $X \cup Y$ embeds.

If on the other hand $X = L_1 = \{x_1\}$, and $L_4 = \{z\}$, and $L_1 \not< L_4$, let us write $L_3 = \{y_1, \ldots, y_l, y'_1, \ldots, y'_m, y''_1, \ldots, y''_n\}$ where $x < y_i \not< z$, $x \not< y'_j \not< z$, $x \not< y''_k < z$. Then $\{y_1, \ldots, y_k, y'_1, \ldots, y'_l, z\}$ is an antichain, so by the case already proved, $\{x_1, y_1, \ldots, y_k, y'_1, \ldots, y'_l, z\}$ embeds in $\mathcal{P}$. By genericity of P_2 there is a correctly coloured antichain of size n below z incomparable with $\{y_1, \ldots, y_k, y'_1, \ldots, y'_l\}$, providing the desired embedding of all of $X \cup Y$ into $\mathcal{P}$. This covers all cases in which $|X| = 1$ and $|L_4| = 1$. More generally, by using the trick of adding points to L_3 perfectly matched to those of L_1 or L_4, as in the two antichains proof above, we may also deduce that provided $X = L_1$ is an antichain, and $Y = L_3 \cup L_4$, then $X \cup Y$ embeds in $\mathcal{P}$.

A similar argument applies if we know that $X = L_1 \cup L_2$ and Y is an antichain. So from now on we assume that $X = L_1 \cup L_2$, $Y = L_3 \cup L_4$, and all of L_1, L_2, L_3, L_4 are disjoint and non-empty. By the methods already used (suitable addition of perfectly matched elements), we may assume that at most L_3 out of the four 'levels' is of size greater than 1. If $L_1 \not< L_2$, then $L_1 \cup L_2$ is an antichain, so we may replace L_2 by $\emptyset$ and L_1 by $L_1 \cup L_2$ to reduce to an earlier case. So we may suppose that $L_1 < L_2$.

If in addition, $L_1 < L_4$, we first enlarge L_3 by adding z such that $L_1 < z < L_4$. Since $L_1 \cup L_2 \cup L_3 \cup \{z\}$ and $L_2 \cup L_3 \cup L_4 \cup \{z\}$ each have just three levels, they can be embedded into $\mathcal{P}$ as we have just seen. By homogeneity, we may assume that these embeddings agree on $L_2 \cup L_3 \cup \{z\}$, and the presence of z ensures that the correct relation $L_1 < L_4$ holds under the union of these two embeddings.

Otherwise, $L_1 \not< L_4$, so by transitivity, $L_2 \not< L_4$. First we embed $L_1 \cup L_2 \cup L_4 \cup \{x \in L_3 : x \not< L_4\}$ into $\mathcal{P}$, as this is a 3-level structure, and then we add an appropriately coloured antichain below L_4 to embed the remainder of L_3.

Finally we show how to derive the general case from that in which there are at most four levels. We use induction on $|X \cup Y|$. Suppose first that there are $z_1 < z_2 < z_3$ in $Z = X \cup Y$. By induction hypothesis, we may embed each of $Z - \{z_1\}$ and $Z - \{z_3\}$ into $\mathcal{P}$, and by homogeneity we may suppose that the embeddings agree on $Z - \{z_1, z_3\}$. We just have to check that for all a and b in Z, the relation between a and b is the same in Z and in $\mathcal{P}$. The only case we have to look at is $a = z_1$, $b = z_3$. But now $a < b$ in Z and as $a < z_2$ and $z_2 < b$ hold in $Z - \{z_3\}$ and $Z - \{z_1\}$ respectively, and hence in $\mathcal{P}$, so does $a < b$ by transitivity in $\mathcal{P}$. If Z has no chains of length 3, then each of X and Y is expressible as a union of at most two levels, reducing to the case already considered. ∎

Theorem 9.3. *For each reduced (abstract) skeleton $\mathcal{Q}$ having just three points which form a V-shape, there is a countable homogeneous coloured partial order having $\mathcal{Q}$ as skeleton, which is uniquely determined up to isomorphism.*

Proof. Existence follows from Theorem 7.1, so we just concentrate on uniqueness. For this part we do not even need to assume that the skeleton is reduced, and the argument amounts to considering the two 2-chains that $\mathcal{Q}$ is built from. We remark that this result is a special case of the general and main theorem that we give in the final section, but we give it here as a warm-up for that, and also because the structures having up to three components are the main ones to consider (which control the whole), so merit individual consideration.

Let the components of $\mathcal{P}$ be P_1, P_2, P_3 where $P_1 < P_2, P_3$ and $P_2 \parallel P_3$, and let $\mathcal{P}'$ be another countable homogeneous coloured partial order with the same skeleton and corresponding components P_i'. By Lemma 2.1, $P_1 \cup P_2$ and $P_1' \cup P_2'$ are homogeneous with skeleton which is the restriction of $\mathcal{Q}$ to those two vertices, so by Theorem 9.2, $P_1 \cup P_2 \cong P_1' \cup P_2'$. Similarly, $P_1 \cup P_3 \cong P_1' \cup P_3'$.

To see that $\mathcal{P} \cong \mathcal{P}'$ it suffices to show that they have the same age (as they are known to be homogeneous). Let $X_1 \cup X_2 \cup X_3$ be a finite substructure of $\mathcal{P}$ with $X_i \subseteq P_i$. As $P_1 \cup P_2 \cong P_1' \cup P_2'$, $X_1 \cup X_2$ embeds in $P_1' \cup P_2'$ and similarly $X_1 \cup X_3$ embeds in $P_1' \cup P_3'$. By homogeneity of $\mathcal{P}'$ we may assume that the two embeddings agree on X_1. Since $P_2 \parallel P_3$ and $P_2' \parallel P_3'$, the union of the two embeddings is an embedding of $X_1 \cup X_2 \cup X_3$ into $\mathcal{P}'$. Thus the age of $\mathcal{P}$ is contained in the age of $\mathcal{P}'$. The same argument applies in reverse, so they have the same ages, as required. ∎

Theorem 9.4. *For each reduced skeleton $\mathcal{Q}$ having just three points $q_1 < q_2 < q_3$, there is a countable homogeneous coloured partial order having $\mathcal{Q}$ as skeleton, which is uniquely determined up to isomorphism.*

Proof. This result is likewise a special case of what follows in Section 10.

We begin by remarking on cases in which either (q_1, q_2) or (q_2, q_3) is labelled C. Then by the conditions on 'skeleton', (q_1, q_3) is also labelled C. We just deal with the case that (q_1, q_2) is labelled C, the other being similar. By Theorems 9.1 and 9.2, there are countable homogeneous coloured partial orders $\mathcal{P}_1$ and $\mathcal{P}_2$ having skeletons the restrictions of $\mathcal{Q}$ to $\{q_1\}$ and $\{q_2, q_3\}$ respectively, and they are unique up to isomorphism. Then $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ with $P_1 <_c P_2$ is the unique countable homogeneous coloured partial order having skeleton $\mathcal{Q}$.

From now on we therefore suppose that neither $q_1 \, C \, q_2$ nor $q_2 \, C \, q_3$. Since the skeleton is reduced and not $q_1 \, C \, q_2$ or $q_2 \, C \, q_3$, none of q_1, q_2 or q_3 is labelled CA. Hence they are all labelled A, AC, or Ge, and $q_1 \, G \, q_2 \, G \, q_3$ and either $q_1 \, G \, q_3$ or $q_1 \, C \, q_3$. Since existence is known from Theorem 7.1 we just have to verify uniqueness. Let $\mathcal{P}$ have skeleton $\mathcal{Q}$, with components P_i corresponding to q_i. As usual, the task is to show that any member $X_1 \cup X_2 \cup X_3$ of the class corresponding to $\mathcal{Q}$ embeds in $\mathcal{P}$. We refer to such partial orders as '3-level', with levels X_1, X_2, X_3. We remark that the relation between P_1 and P_3 is the transitive closure of the relations between P_1, P_2 and P_2, P_3. This is proved just as at the beginning of the proof of Theorem 8.4.

We argue by induction on $|X_1 \cup X_2 \cup X_3|$. The basis case is where some X_i is empty, in which case we can embed by using Theorem 9.2, so from now on we suppose that each X_i is non-empty. Next we remark that we may suppose that $X_1 \cup X_2 \cup X_3$ contains no 3-element chain. For if $a < b < c$ in $X_1 \cup X_2 \cup X_3$, we may embed each of $X_1 \cup X_2 \cup X_3 - \{a\}$ and $X_1 \cup X_2 \cup X_3 - \{c\}$ into $\mathcal{P}$ inductively. By homogeneity of $\mathcal{P}$ we may suppose that the embeddings agree on $X_1 \cup X_2 \cup X_3 - \{a, c\}$. Then the union of the two embeddings is an embedding, since the relationship between a and c is correct, using the presence of b, and transitivity of $\mathcal{P}$. In a similar way we may assume that there are no $a, b, c \in X_1 \cup X_2 \cup X_3$ such that either $a < b$ and $a, b \parallel c$ where $b \in X_i$, $c \in X_j$ with $i < j$, or $b < c$ and $a \parallel b, c$ where $a \in X_i$, $b \in X_j$ with $i < j$. For in the former case, for instance, we embed $X_1 \cup X_2 \cup X_3 - \{b\}$ and $X_1 \cup X_2 \cup X_3 - \{c\}$ inductively so that the embeddings agree on the overlap. Then the relation between b and c (incomparability) is correct in the union of the two embeddings. For as $b \in P_i$ and $c \in P_j$ we cannot have $c < b$. If $b \le c$, then as $a < b$, by transitivity in $\mathcal{P}$, $a < c$, contrary to $a \parallel c$ in $X_1 \cup X_2 \cup X_3 - \{b\}$.

Next, by the method of adding antichains perfectly matched with occurring antichains, used in the proof of Theorem 9.2, we may suppose that each of X_2 and X_3 is in fact a chain, which has size at most 2 (size 1 for antichain components). If $X_2 \parallel X_3$ then we add an extra point x to X_1 and stipulate that $x < X_2$, $x \parallel X_3$. Since $X_2 \parallel X_3$, this is a partial order. By the 2-chain case, we can embed each of $X_1 \cup \{x\} \cup X_2$ and $X_1 \cup \{x\} \cup X_3$ into $\mathcal{P}$, and by homogeneity we may suppose that the two embeddings agree on $X_1 \cup \{x\}$. Since $x < X_2$ and $x \parallel X_3$, transitivity of $\mathcal{P}$ implies that $X_2 \parallel X_3$ holds in the embedding.

From now on we therefore suppose that some member of X_2 lies below some member of X_3.

Case 1: $|X_2| = |X_3| = 1$. Let $X_2 = \{y\}$, $X_3 = \{z\}$. By the above remark, we have $y < z$. Since there are no chains of length 3, $X_1 \parallel y$. Add an extra point z' to X_3 such that $y < z'$ and $X_1 \parallel z'$. Embed each of $X_1 \cup \{z, z'\}$ and $X_2 \cup \{z, z'\}$ in $\mathcal{P}$ by the 2-chain cases so that the embeddings agree on $\{z, z'\}$. As $X_1 \parallel z'$ and $y < z'$ this ensures that $X_1 \parallel y$ in the embedding as desired.

Case 2: $|X_2| = 1$, $|X_3| = 2$. Let $X_2 = \{y\}$, $X_3 = \{z_1, z_2\}$ where $z_1 < z_2$. Since there are no chains of length 3, $y \parallel z_1$, and since we are assuming that $X_2 \parallel X_3$ is false, we must have $y < z_2$. As there are no chains of length 3, $X_1 \parallel y$. Add z to X_3 so that $X_1 \parallel z$ and $y < z$. Embedding each of $X_1 \cup X_3 \cup \{z\}$ and $\{y\} \cup X_3 \cup \{z\}$ so that the embeddings agree on $X_3 \cup \{z\}$, we deduce as before that $X_1 \parallel y$ in the embedding.

Case 3: $|X_2| = 2$, $|X_3| = 1$. Let $X_2 = \{y_1, y_2\}$, $X_3 = \{z\}$ where $y_1 < y_2$. As in Case 2 we must have $y_1 < z$, $y_2 \parallel z$, and as there are no chains of length 3, $X_1 \parallel y_1$.

Since there are no points a, b, c such that $a \parallel b, c$ and $b < c$ with $a \in X_1$, $b \in X_2$ we deduce that $X_1 < y_2$ on taking $a \in X_1$, $b = y_1$ and $c = y_2$, and also that $X_1 < z$ on taking $a \in X_1$, $b = y_1$, and $c = z$. Now we add a new point y to X_2 incomparable with y_1 and y_2 such that $X_1 < y < X_3$, and we can embed $X_1 \cup X_2 \cup \{y\} \cup X_3$ by embedding each of $X_1 \cup X_2 \cup \{y\}$ and $X_2 \cup \{y\} \cup X_3$ to agree on $X_2 \cup \{y\}$.

Case 4: $|X_2| = |X_3| = 2$. Let $X_2 = \{y_1, y_2\}$, $X_3 = \{z_1, z_2\}$ where $y_1 < y_2$ and $z_1 < z_2$. Since there are no chains of length 3, $y_1 \parallel z_1$ and $y_2 \parallel z_1, z_2$. Since some member of X_2 lies below some member of X_3, the only possibility is that $y_1 < z_2$. But now if we take $a = y_1$, $b = y_2$, and $c = z_1$, we have $a < b$, $a, b \parallel c$, and $b \in X_2$, $c \in X_3$, and we supposed that this did not occur. This case can therefore not arise. ∎

10. The main theorem and further remarks

The main technical result which remains for us to show is the uniqueness of a countable homogeneous coloured partial order having a specified reduced skeleton.

Theorem 10.1. *Let $\mathcal{Q}$ be a reduced skeleton and $\mathcal{K}$ its corresponding class. Then the Fraïssé limit of $\mathcal{K}$ is the unique countable homogeneous coloured partial order whose skeleton is $\mathcal{Q}$.*

Proof. Let $\mathcal{F}$ be the Fraïssé limit of $\mathcal{K}$. We first show that $\mathcal{Q}$ *is* (isomorphic to) the skeleton $\mathcal{Q}'$ of $\mathcal{F}$. For this we first note that $\mathcal{F}$ is a union of members of $\mathcal{K}$, so has the form $\bigcup_{q \in Q} X_q$.

Suppose that q is labelled by A. If $N(q)$ is finite, then $\mathcal{K}$ contains the structure consisting just of an antichain of size $N(q)$ coloured $F(q)$, and if $N(q)$ is infinite, then $\mathcal{K}$ contains, for each finite n, the structure consisting just of an antichain of size n coloured $F(q)$. Since $\mathcal{F}$ embeds all members of $\mathcal{K}$, it follows that X_q is an antichain of size $N(q)$ coloured $F(q)$.

Next suppose that q is labelled AC. Once again, any finite disjoint union of at most $N(q)$ correctly coloured finite chains lies in $\mathcal{K}$ so embeds in $\mathcal{F}$, and it follows that X_q is an antichain of $N(q)$ correctly coloured versions of the rationals.

Similar arguments show that if q is labelled CA or Ge, then X_q is a dense chain of antichains having the correct colour structure partition, or a generic for the correct colour set, respectively. This is just because $\mathcal{K}$ was chosen to be the family of all finite approximations to such.

It follows that $\mathcal{Q}$ and $\mathcal{Q}'$ have vertices which are in natural 1-1 correspondence. By the choice of labels on vertices and edges, which were sufficient to characterize components, and comparable pairs of components, and by the definition of the corresponding class, it follows that all the labels in $\mathcal{Q}$, and the corresponding ones in $\mathcal{Q}'$, agree, so that $\mathcal{Q}$ and $\mathcal{Q}'$ are isomorphic.

To show the uniqueness of $\mathcal{F}$ with respect to $\mathcal{Q}$, let $\mathcal{P}$ be a countable homogeneous coloured partial order whose skeleton is $\mathcal{Q}$. We show that $\mathcal{P}$ and $\mathcal{F}$ are isomorphic, for which it suffices to show that they have the same age.

Let S be a finite substructure of $\mathcal{P}$. By definition of $\mathcal{K}$, S lies in $\mathcal{K}$ and hence is a finite substructure of $\mathcal{F}$.

Conversely, let S be a finite substructure of $\mathcal{F}$ (so $S \in \mathcal{K}$). Then S has the form $\bigcup_{q \in Q} Y_q$ where $Y_q \subseteq X_q$. Let Q' be the set of all $q \in Q$ such that $Y_q \neq \emptyset$, so that Q' is finite. We show that S embeds in $\mathcal{P}$ by induction on the size of Q'.

By the definition of $\mathcal{K}$, by definition of skeleton of $\mathcal{P}$, and by the results on two components, it is easy to see that if Q' has at most two elements, then S embeds in $\mathcal{P}$.

We first reduce to the case in which Q' is linearly ordered. Suppose not, and let q_1 and q_2 be incomparable elements of Q'. Then $Q_1 = Q' - \{q_1\}$ and $Q_2 = Q' - \{q_2\}$ are proper subsets of Q', and $S_1 = \bigcup_{q \in Q_1} Y_q$ and $S_2 = \bigcup_{q \in Q_2} Y_q$ are proper subsets of S. Furthermore, if $x \in S_1 - S_2$ and $y \in S_2 - S_1$ then $x \in Y_{q_2}$ and $y \in Y_{q_1}$, so $x \parallel y$. By induction hypothesis, S_1 and S_2 both embed in $\mathcal{P}$, and by homogeneity we may assume that the embeddings θ_1 and θ_2 agree on $S_1 \cap S_2$, and so we obtain a well-defined map θ from $S = S_1 \cup S_2$ into $\mathcal{P}$ by taking the union of θ_1 and θ_2. To see that θ is also an embedding, let $x, y \in S$. Then if x, y both lie in S_1, $x < y \Leftrightarrow \theta_1(x) < \theta_1(y) \Leftrightarrow \theta(x) < \theta(y)$, and similarly if they both lie in S_2. Otherwise suppose without loss of generality that $x \in S_1 - S_2$ and $y \in S_2 - S_1$. Then $x \parallel y$, and since $q_1 \parallel q_2$, also $\theta(x) \parallel \theta(y)$. We deduce that θ is also an embedding.

Since Q' is now a chain, let us write it as $q_1 \prec \cdots \prec q_n$, and by Theorem 9.4 we may assume that $n > 3$.

First if $q_i \, C \, q_{i+1}$ for some i, then we may embed each of $\bigcup_{1 \le j \le i} Y_{q_j}$ and $\bigcup_{i+1 \le j \le n} Y_{q_j}$ by induction hypothesis, and take the union of the two embeddings, so from now on we assume that this is not the case.

If any of the labels on the vertices are CA, then by definition of reduced skeleton, there must be i such that $q_i \, C \, q_{i+1}$, contrary to assumption. Hence all vertex labels are A, AC, or Ge, and all consecutive pairs are labelled G. We can now follow the steps in Theorem 9.4. As there we can reduce to the case in which Y_{q_n} and $Y_{q_{n-1}}$ are chains of size at most 2. We can follow the four cases at the end of the proof of that theorem, where X_1 is replaced in the argument by $\bigcup_{1 \le i \le n-2} Y_{q_i}$ throughout. (In other words, the fact that X_1 is a component is not actually used.)

Therefore $\mathcal{F}$ is the unique countable coloured partial order whose skeleton is $\mathcal{Q}$. ∎

The following theorem sums up the sense in which we have achieved a 'classification' of the countable homogeneous coloured partial orders.

THEOREM 10.2. *For any skeleton $\mathcal{Q}$ there is a unique countable homogeneous coloured partial order whose skeleton is $\mathcal{Q}$. Thus the correspondence between countable homogeneous coloured partial orders and their skeletons is* 1-1.

Proof. Let $\mathcal{Q}'$ be a reduced skeleton associated with $\mathcal{Q}$. By Theorem 7.1, the class $\mathcal{K}$ corresponding to $\mathcal{Q}'$ is an amalgamation class, so by Fraïssé's Theorem there is a countable homogeneous coloured partial order $\mathcal{P}'$ whose age is equal to $\mathcal{K}$, and this has skeleton $\mathcal{Q}'$. Moreover by Theorem 10.1 this is unique up to isomorphism. By Theorem 6.1 there is a countable homogeneous coloured partial order $\mathcal{P}$ containing $\mathcal{P}'$ having skeleton $\mathcal{Q}$, and this is uniquely determined from $\mathcal{P}'$, so is also unique up to isomorphism. ∎

Next we substantiate the earlier remark that to tell whether a labelled partial order is a skeleton, it suffices to look at its substructures of size at most three. This is immediate from the definition, since the only constraints on the labelling were on which forms were taken by the 1-, 2-, and 3-element substructures. An alternative way to formulate the result is as follows. One can describe the class of (finite or countable) partial orders which have labels on the vertices of the kinds given in the definition of 'skeleton' but without specifying any retrictions on the 2- or 3-element substructures. The main result can then be restated as saying that such a labelled partial order is the skeleton of one of the structures in our class if and only if every substructure of size at most 3 is.

We conclude by reading off from what we have done a simpler, but on its own, not completely straightforward case, namely the classification of all the finite homogeneous coloured partial orders.

THEOREM 10.3. *Any finite homogeneous coloured partial order has a finite skeleton $\mathcal{Q}$ in which all points are labelled A, with $N(q)$ finite, and all relations between comparable points are C, PM, or CPM. Conversely, any skeleton fulfilling these restrictions is the skeleton of a unique finite homogeneous coloured partial order.*

We remark that a corresponding result holds for countable homogeneous coloured partial orders in which all components are finite.

Now we have remarked that a labelled partial order is a skeleton if and only if every ≤ 3-element substructure is. We would like this to be true also for the encoded structures, but at present can only establish this in a special case.

THEOREM 10.4. *Suppose that $\mathcal{P}$ is a countable coloured partial order which can be written as the disjoint union of a family $\mathcal{F}$ of convex subsets coloured by pairwise disjoint colour sets such that each member of $\mathcal{F}$ is an antichain or antichain of chains, and such that the union of any ≤ 3 members of $\mathcal{F}$ is homogeneous, and no relation between members of $\mathcal{F}$ is generic. Then $\mathcal{P}$ is homogeneous.*

Proof. We may replace any antichain members of $\mathcal{F}$ by the union of distinctly coloured monochromatic subsets without changing the hypotheses, which ensures that $\mathcal{F}$ with the partial order and the relations between comparable points is a skeleton. It follows that there is a unique countable homogeneous coloured partial order $\mathcal{P}'$ having this as skeleton, so the result follows if we can show that $\mathcal{P} \cong \mathcal{P}'$.

Now the only relations between comparable components can be C, PM, and CPM. We cut into $\equiv$-classes as in Section 6, and since the relation between elements of distinct $\equiv$-classes is $\parallel$ or $<_c$, it suffices to prove the result for each $\equiv$-class on its own.

Let T be a $\equiv$-class. Then as all members of $\mathcal{F}$ are antichains or antichains of chains, it is a PM-class. Let $\mathcal{P}_1$ and $\mathcal{P}_1'$ be the unions of the components of $\mathcal{P}$, $\mathcal{P}'$ respectively labelled by members of T. Let X_q, X_q' be their components labelled by q. Choose fixed $q \in T$. Then for each $q' \neq q$ in T, (q, q') or (q', q) is labelled PM or CPM, so as all 2-component substructures of $\mathcal{P}$, $\mathcal{P}'$ are correctly labelled, there are corresponding bijections $\theta_{q'}$ from X_q to $X_{q'}$ and $\theta'_{q'}$ from X_q' to $X_{q'}'$ (and for ease we let θ_q, θ_q' be the identity). In other words, if $q\,PM\,q'$, then for each maximal chain A of X_q, $A < \theta_{q'}A$ and $A \not< B$ if $B \neq \theta_{q'}A$, and if $q\,CPM\,q'$, then $A \not< \theta_{q'}A$ and $A < B$ if $B \neq \theta_{q'}A$ (and similarly if $q' \prec q$). As $X_q \cong X_q'$ we may choose an isomorphism φ from X_q to X_q', and extend it to $\mathcal{P}_1$ by letting $\varphi(\theta_{q'}x) = \theta'_{q'}(\varphi x)$ for each $x \in X_q$ and $q' \in T$. The fact that φ is an isomorphism follows from the fact that $\mathcal{P}_1$ and $\mathcal{P}_1'$ agree on the 3-component subsets of Q. ∎

We conclude by remarking that this result cannot be proved if we allow $\mathcal{F}$ to include chains of antichains, as the following example shows. Let $\mathcal{F} = \{\mathbb{Q}_{\{c_0\}}, \mathbb{Q}_{\{c_1\}}, \mathbb{Q}_{\{c_2\}}, \mathbb{Q}_{\{c_3\}}\}$ and $\mathbb{Q}_{\{c_0\}} <_{sh} \mathbb{Q}_{\{c_1\}}$, $\mathbb{Q}_{\{c_2\}} <_{sh} \mathbb{Q}_{\{c_1\}}$, $\mathbb{Q}_{\{c_2\}} <_{sh} \mathbb{Q}_{\{c_3\}}$, with no other relations. Then this corresponds to a skeleton on four points and there is a unique countable homogeneous structure satisfying these requirements. We can however give an alternative definition of the ordering on $\mathcal{P}$ as follows, using a modification of the method in Section 5, still fulfilling all these requirements, so that the union of any ≤ 3 components is homogeneous, but $\mathcal{P}$ itself is *not*.

Start with the 3-coloured version of the rationals $\mathbb{Q}_{\{c_0,c_1,c_2\}}$ and let the four components of $\mathcal{P}$ be $\mathbb{Q}_{\{c_0\}} \times \{0\}$, $\mathbb{Q}_{\{c_1\}}$, $\mathbb{Q}_{\{c_2\}}$ and $\mathbb{Q}_{\{c_0\}} \times \{1\}$, coloured by c_0, c_1, c_2, c_3 respectively, each ordered in the natural way induced from $\mathbb{Q}_{\{c_0,c_1,c_2\}}$. Thus we take two copies of $\mathbb{Q}_{\{c_0\}}$, coloured by c_0 and c_3 (so that we change the colour on $\mathbb{Q}_{\{c_0\}} \times \{1\}$ to c_3). Now partially order $\mathcal{P}$ by $<'$ where

$$x <' y \text{ if } \begin{cases} x < y \text{ and } x \text{ and } y \text{ lie in the same component, or} \\ x \in \mathbb{Q}_{\{c_2\}} \text{ and } y \in \mathbb{Q}_{\{c_1\}}, \text{ or} \\ x = (x', 0) \text{ where } x' \in \mathbb{Q}_{\{c_0\}} \text{ and } y \in \mathbb{Q}_{\{c_1\}} \text{ and } x' < y, \text{ or} \\ y = (y', 1) \text{ where } y' \in \mathbb{Q}_{\{c_0\}} \text{ and } x \in \mathbb{Q}_{\{c_2\}} \text{ and } x < y'. \end{cases}$$

Then it is clear that the union of any three members of $\mathcal{F}$ is homogeneous, since either $\mathbb{Q}_{\{c_0\}} \times \{0\}$ or $\mathbb{Q}_{\{c_0\}} \times \{1\}$ is missing, and the construction of the other three is as originally given, or if $\mathbb{Q}_{\{c_1\}}$ or $\mathbb{Q}_{\{c_2\}}$ is missing, it is not even connected. The whole of $\mathcal{P}$ is however not homogeneous, since if we choose $x_1 < x_2$ in $\mathbb{Q}_{\{c_0\}}$, then the finite partial automorphism fixing $(x_1, 0)$ and taking $(x_1, 1)$ to $(x_2, 1)$ does not extend to an automorphism. The reason is that although $\mathbb{Q}_{\{c_0\}} \times \{0\}$ and $\mathbb{Q}_{\{c_0\}} \times \{1\}$ are incomparable, the obvious isomorphism between them is 'remembered' by the Dedekind cuts in the intermediate components $\mathbb{Q}_{\{c_1\}}$ and $\mathbb{Q}_{\{c_2\}}$. (This is analogous to the proof that there is no V-shape in which both comparabilities are perfect matchings.)

References

[1] G. Campero-Arena, *Transitivity properties of countable coloured orderings*, PhD thesis, Univ. of Leeds, 2002.

[2] G. Campero-Arena and J. K. Truss, *Countable, 1-transitive, coloured linear orders I*, J. Combin. Theory Ser. A 105 (2004), 1–13.

[3] —, —, *Countable, 1-transitivee, coloured linear orders II*, Fund. Math. 183 (2004), 185–213.

[4] G. Cherlin, *The classification of countable homogeneous directed graphs and countable n-tournaments*, Mem. Amer. Math. Soc. 131 (1998), no. 621.

[5] M. Droste, *Structure of partially ordered sets with transitive automorphism groups*, Mem. Amer. Math. Soc. 57 (1985), no. 334.

[6] M. Goldstern, R. Grossberg and M. Kojman, *Infinite homogeneous bipartite graphs with unequal sides*, Discrete Math. 149 (1996), 69–82.

[7] W. Hodges, *A Shorter Model Theory*, Cambridge Univ. Press, 1997.

[8] A. H. Lachlan, *Countable homogeneous tournaments*, Trans. Amer. Math. Soc. 284 (1984), 431–461.

[9] A. H. Lachlan and R. Woodrow, *Countable ultrahomogeneous undirected graphs*, ibid. 262 (1980), 51–94.

[10] J. H. Schmerl, *Countable homogeneous partially ordered sets*, Algebra Universalis 9 (1979), 317–321.

[11] S. Torrezão de Sousa, *Countable homogeneous coloured partial orders*, PhD thesis, Univ. of Leeds, 2005.