# Part 1

# Leśniewski's Protothetic

CHAPTER 1

# Definitions, theses and hypotheses

## 1. Definitions in antiquity

*Definitions*, in one form or another, have been with us since time immemorial. On the other hand it took the Socratic psyche to study them, classify them and make good use of them.

Socrates and Plato first confronted the problem. For example in *Laws X 895d* we find[1]:

> ATHENIAN: *In heaven's name, then, hold. You will grant, I presume, that there are three points to be noted about anything?*
>
> CLINIAS: *You mean?*
>
> ATHENIAN: *I mean, for one, the reality of the thing, what it* is, *for another the* definition *of this reality, for another its* name. *And thus you see that there are two questions we can ask about everything which is.*
>
> CLINIAS: *And what are the two?*
>
> ATHENIAN: *Sometimes a man propounds the bare name and demands the definition; sometimes, again, he propounds the definition by itself and asks for the corresponding name. In other words, we mean something to this effect, do we not?*
>
> CLINIAS: *To what effect?*
>
> ATHENIAN: *There is, as you know, bisection in numbers, as in other things. Well, in the case of a number, the* name *of the thing is 'even', and the definition, 'number divisible into two equal parts'.*
>
> CLINIAS: *Certainly.*
>
> ATHENIAN: *That is the sort of case I have in mind. We are denoting the same thing, are we not, in either case, whether we are asked about the definition and reply with the name, or about the name, and reply with the definition? It is the same thing we describe indifferently by the name 'even,' and the definition 'number divided into equal parts'?*
>
> CLINIAS: *Identically the same.*

And what Socrates and Plato started, Aristotle made into a philosophical doctrine. For example consider the following passage from *Anal. post.* I. 2, 72a 14–24:

> *Among immediate syllogistic principles, I call that a* thesis *which it is neither possible to prove nor essential for any one to hold who is to learn anything; but that which it is necessary for any one to hold who is to learn anything whatever is an* axiom*: for there are some principles of this kind, and that is the most usual name*

[1] From the translations given in [Hamilton and Cairns, 1996].

> *by which we speak of them. But, of* theses*, one kind is that which assumes one or other side of a predication, as, for instance, that something exists or does not exist, and this is a* hypothesis*; the other, which makes no such assumption, is a* definition*. For a definition is a thesis; thus the arithmetician posits* ($\tau\iota\theta\epsilon\tau\alpha\iota$) *that a unit is that which is indivisible in respect of quantity; but this is not a hypothesis, since what is meant by a unit and the fact that a unit exists are different things*[2].

And then Euclid based the 13 books of the "Elements" on the 23 Definitions with which he started.[3]

## 2. Mathematical Logic

Aristotle's variety of Logic and Euclid's version of geometry, so innovative at first, dominated, and eventually suffocated, their fields for close to 2000 years. This domination continued until a new variety of logic, nowadays called *Mathematical Logic*, was introduced.

G. W. Leibniz (1646–1716) is recognized as the originator of Mathematical Logic, although he cannot be considered as the *founder* since most of his logical works were not published until long after his death. The logicians who founded significant schools, and thus could be considered as the founders, were George Boole (*The Mathematical Analysis of Logic, 1847*) and G. Peano (*Arithmetices principia, novo methodo exposita. 1889*)[4].

However, the philosopher who did the most to introduce this new variety of logic to both philosophers and mathematicians, was Bertrand Russell. His landmark work, which showed that mathematical logic had come of age and had freed itself from the Aristotelian domination, was *Principia Mathematica* (co-authored with A. N. Whitehead[5]). Now, although there was unanimous agreement on the importance, scope and magnitude of Whitehead and Russell's *Principia*, it eventually came to be realized that *Principia* was not as tightly constructed as it first appeared to be.

One of the criticisms (and by no means the only one) was Russell's treatment of *definitions*. In the *Principia* definitions are always considered to be formulas *outside of* the theory. Furthermore, there were never any instructions for *substitutions* although substitutions were freely done in the extra-logical definitions and the results were considered as part of theory.

One person who noticed that such cavalier attitude towards *definitions* could lead to contradiction was Stanisław Leśniewski (1886–1939). He believed that *definitions* should be stated, whenever possible, in the form of *sentences of the theory* and that they should have the same status as *axioms*; in other words, that definitions were to be nothing less

[2]From the Introduction to "Euclid's Elements", by Sir Thomas L. Heath, Dover Publ. Co.

[3]A few more definitions were added at the beginning of the later books.

[4]These two paragraphs cannot in any way convey the story of the development of Mathematical Logic from its trial beginnings in the Middle Ages; for a comprehensive study the reader is referred to N. I. Styazhkin's *History of Mathematical Logic from Leibniz to Peano*, [Styazhkin, 1969].

[5][Whitehead and Russell, 1925].

(nor more) than theses of the theory[6]. This view was shared by Hiz, Łukasiewicz, Meredith, Sobociński, Tarski and most of the logicians in Warsaw. Leśniewski, being one of those who felt that *Principia Mathematica* could be improved (and not just because of the problem with *definitions*), proposed in his *New Foundations of Mathematics*[7] [translated by Srzednicki–Stachniak [8]]:

> *My system of the foundations of mathematics... consists of three deductive theories, whose union forms one of the possible bases of the whole structure of mathematics. The theories in question are the following: (1) What I call* Protothetic, *which is the result of a certain peculiar enlargement of the well known theory which goes by the name of the 'propositional calculus', or 'theory of deduction'. (2) What I call* Ontology, *which forms a type of modernized 'traditional logic' and which most closely resembles in its content and power Schröder's 'logic of classes', regarded as including the theory of 'individuals'. (3) What I call* Mereology, *whose first outline was published by me in a work of 1916 entitled* Foundations of a general set theory.

The Holocaust interrupted Leśniewski's project, the Warsaw fire of 1944 destroyed most of Leśniewski's manuscripts and the surviving Polish logicians shifted their attention to other fundamental areas of mathematical logic; thus Leśniewski's thesis that definitions should be theorems was all but forgotten. Even the few references to Leśniewski that can be found are often antagonistic, for example in A. N. Prior's *Formal Logic*[9], one finds on page 97:

> *I have outlined this account of definitions as assertions of equivalence partly because the logicians who have espoused it—including Leśniewski, Tarski, Sobociński, and with modifications Łukasiewicz and Meredith—are about as distinguished a group as any theory could muster, and partly because it has had such fruitful by-products (the definition of 'K' in terms of 'E' and '*Π*', and the metalogical theorem reproduced in the last paragraph, are solid achievements). But I shall not conceal my own belief that it is wrong-headed. A grave objection to it is that it makes it difficult to distinguish between the definitions of the system and the additional axioms; and the use of '=' can be defended against the charge of being a surreptitious introduction of a new primitive symbol. The authors of PM, for example, argue that definitions are not genuine parts of the deductive system to which they are attached, but simply indicate alternate ways of symbolizing the same thing within the system. On this view—and this consequence of it is made very explicit by Whitehead and Russell—all defined symbols are in principle superfluous; the entire system could be set forth without them, only it would then be insufferably cumbrous.*

### 3. On the primitive term of logistic

The result mentioned in the above quote was Tarski's reduction of all the classical logical atoms to the *equivalence propositional connective* (and the *universal quantifier*). Leśniewski had incorporated Tarski's result into his Protothetic and consequently was

[6]Although the words *thesis* and *definition* have undergone some variation in their meanings, Aristotle was also of the opinion that *definitions are theses*; see Aristotle's quote in the previous section.

[7][Leśniewski, 1916].

[8][Srzednicki and Stachniak, 1988].

[9][Prior, 1962].

able to give a very satisfying axiomatization for the Protothetic; but in spite of such positive augurs the project did not survive the war.

This phenomenon of a theory being intensively studied for a period of time and then relegated to oblivion is not an uncommon event. One of the earliest examples is Parmenides' *One* which flourished around the beginning of Socrates' time has now made a comeback in modern Cosmology, see for example T. Ferris: *The Whole Sheebang. The State of the Universe(s)*[10]. Now, long before the Big Bang Theory had become popular, B. Russell, discussing Parmenides' contributions, remarks in *History of Western Philosophy*[11]:

> *I have put the argument here to remind the reader that philosophical theories, if they are important, can generally be revived in a new form after being refuted as originally stated. Refutations are seldom final; in most cases, they are only a prelude to further refinements.*

## 4. Reviving and refining the Protothetic

Now although the Leśniewski/Tarski thesis has not been refuted (or ever will be), it had suffered an even worst fate: it had been forgotten! Since we believe that the Leśniewski/Tarski classification is indeed an important one and, motivated by Russell's observation, we decided to develop a refinement of the Protothetic suitable for modern times.

In the 75 years since the publication of Tarski's result there has been an increasing trend for Mathematics to be more constructive. Thus we decided that an appropriate way to acknowledge the work of Leśniewski and Tarski was to develop, *ab ovo*, a Constructive Protothetic, which we call the **New Protothetic**; furthermore not only the formalization should reflect constructive intuition, but the metatheory should also be constructively acceptable.

In Part Two of the monograph we set up the New Protothetic, simultaneously explaining why we chose that particular formalization. Then we prove some general results about it, results which further legitimize the system; for example, the completeness with respect to Beth models *and* proven in an intuitionistic metatheory (which is the version of constructive Mathematics that we are adopting) and the *normalization property*[12]. To complete Part Two we show that Tarski's reduction, that *conjunction* is definable in terms of equivalence and the universal quantifier, is also applicable in the New Protothetic. Thus Part Two may be considered as an extension of the Leśniewski/Tarski project to constructive logics.

As Russell's observation predicted, not only have we revived the ideas of Leśniewski and Tarski, but also refined in the sense that it can be used to solve *new* problems. In the New Protothetic we are able to offer an answer concerning the intuitionistic connectives.

A question raised by G. Kreisel[13], in the early 60's, about Intuitionism was:

[10] [Ferris, 1996].

[11] [Russell, 1945], page 52.

[12] Actually the proof of normalization is in Appendix A.

[13] [Kreisel, 1962a].

> *More generally,* what is an intuitionistic propositional connective? *As is well known, for the classical case the corresponding question has been satisfactorily settled by the identification of propositional connectives with truth functions.*

Many answers have been put forward. Now although all the proposed answers have been mathematically interesting, they nevertheless still have an aura of arbitrariness about them. In Part Three we use the New Protothetic to give yet another answer to Kreisel's question. The New Protothetic is not just another *ad hoc* formal system; the *raison d'être* of the New Protothetic is that it is constructed with the absolute minimum required to have any kind of theory, namely *definitions*. Thus the New Protothetic *bootstrapped* itself from the concept of a *definition* and it has the capability to introduce, at will, definitions of propositional functions, as well as to quantify over them. The propositional functions which are equivalence-invariant merit, in our view, the name of *propositional connectives*. Since the New Protothetic is *Intuitionistic compliant*, both with respect to the choice of formalization as well as with respect to the metatheory, we believe that the place to look for *intuitionistic propositional connectives* is indeed in the New Protothetic[14]. Thus we have strong reasons to be optimistic that our answer avoids some of the arbitrariness of previous answers.

As an example of an intuitionistic monadic connective, which is *not* definable in the Extended Intuitionistic Propositional Calculus (a.k.a. Second Order), but which is definable in the New Protothetic we take Kaminski's monadic connective. This is developed in Part Three.

## 5. Traditional systems in the New Protothetic

In Part Four of the monograph we consider two traditional systems (that is, without *Leśniewskian definitions* or *quantifiers* ) contained within the New Protothetic. These are: MEC, the **Minimal Equivalence Calculus** whose only *primitive term* is the connective for equivalence ($\equiv$) and BCC, the **Bi-conditional Calculus** which in addition has a connective for conjunction[15].

Another advantage of having chosen a Gentzenian Natural Deduction system for the New Protothetic, in which each logical atom has its own pair of **I-E** rules of inference, is that the general results for the New Protothetic, such as completeness, normalization, can be trivially adapted to MEC (and BCC). Furthermore since in MEC and BCC there are no quantifiers, the normalization property can be sharpened. As a consequence we can easily show that in MEC there are infinitely many inequivalent formulas in two propositional parameters.

At first sight it would appear that the *Lindenbaum algebras* for MEC should be algebras with a single binary operation which is not associative, but is commutative and every

[14] At least those intuitionistic propositional connectives based purely on logical grounds. Connectives which depend on the natural number sequence, such as Goad's, generated from Nishimura's lattice [Nishimura, 1960], require an extension of the New Protothetic which encompasses the natural numbers.

[15] The name "Bi-conditional" was chosen because having equivalence and conjunction it can be stated that equivalence is equivalent to a conjunction of conditionals.

element is invertible. In the classical situation, since then the binary operation is associative, we obtain groups and thus an algebraic completeness is easily obtained. On second thoughts on MEC it appears that we should instead consider *Lindenbaum structures* in which in addition to the binary operation corresponding to the equivalence connective, there is also a partial ordering corresponding to inference (which unfortunately is not first order definable from the binary operation). Even so, since the partial ordering is a *binary* relation and the inference relation is between *finitely many* and a singleton and in addition in MEC there is no conjunction, the Lindenbaum structures for MEC appear not to be finitely axiomatizable using just the properties of MEC. On the other hand the Lindenbaum structures for BCC have a more tame partial ordering and thus can be used as a check for the MEC structures.

In Part Five we reconsider the Lindenbaum structures from a purely mathematical point of view. One of the questions of interest is to determine the exact relationship between the category of Equivalence structures and the category of complete Heyting algebras.

CHAPTER 2

# Tarski's contributions to the Protothetic

## 1. Alfred Tarski's 1923 reduction

In his Doctoral Thesis of 1923, presented at the University of Warsaw, Alfred Tarski[1] shows that he can offer a solution to the following problem:

> *Is it possible to construct a system of logistic[2] in which the sign of equivalence is the only primitive sign (in addition of course to the quantifiers)?*

This reduction was much more than just a simple reduction on the number of connectives needed to formalize the (extended) classical propositional logic; as Tarski himself wrote:

> *We know that it is possible to construct the system of logistic by means of a single primitive term, employing for this purpose either the sign of implication, if we wish to follow the example of Russell, or by making use of the idea of Sheffer, who adopts as the primitive term the sign of incompatibility, especially introduced for that purpose. Now, in order really to attain our goal, it is necessary to guard against the entry of any constant special term into the wording of the definitions involved, if this special term is at the same time distinct from the primitive term, from terms previously defined, and from the term to be defined[3]. The sign of equivalence, if we employ it as our primitive term, presents from this standpoint the advantage that it permits us to observe the above rule quite strictly and at the same time to give our definitions a form as natural as it is convenient, that is to say the form of equivalences.*

Not surprisingly, the concept of *definition* has undergone some (but not too many) changes in the 2000+ years since the Stagirite. What Aristotle would have called the *name* and the *definition* have now become the *definiendum* and the *definiens*, respectively, of

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[1]See also [Tarski, 1956], [Tarski, 1923a] and [Tarski, 1923b].

[2]We shall sometimes use the older *logistic*, instead of the more modern *logic*. According to *The Universal Dictionary of the English Language*, published by Standard American Corporation, Chicago 1938:

**logistic**, adj & n. Gk. *logistikós*. skilled in reasoning.

**logistics** n. pl., fr Fr. *logistique*, fr. *loger* to lodge. (mil.) Science and practice of moving, lodging and supplying troops.

[3]Tarski continues in a footnote: *In this article we regard definitions as sentences belonging to the system of logistic. If therefore we were to use some special symbol in formulating definitions we could hardly claim that only one symbol is accepted in our system as a primitive term. It may be mentioned that, in the work of Whitehead and Russell cited above all the definitions have the form 'a = b Df.' and thus actually contain a special symbol which occurs neither in the axioms nor in theorems; it seems, however, that these authors do not treat definitions as sentences belonging to the system.*

the definition; for example: the name *Even number* and the definition *divisible in two equal parts* of the Socratics, is now rendered as the single *definition*:

$$\forall n[\ n \text{ is an Even number} \quad \text{iff} \quad \exists m(n = m + m)\ ].$$

An advantage of having the definitions as sentences belonging to the system is that they may be introduced as required, for example, to clarify concepts. Since in a definition the *definiendum* must involve a new symbol[4] the interpretations of the symbols of the Protothetic are not fixed (except for the primitive ones, that is to say the equivalence sign "$\equiv$" and the universal quantifier "$\bigwedge$"). Thus the same symbol could be attached to different definitions in different developments of the Protothetic. Now although this is quite common in informal mathematics and in computer languages, at the beginning of the Century this was looked with mistrust; so much so that people doubted that a completeness proof for the Protothetic was possible[5].

Tarski's method to show that the classical propositional connectives could be defined in terms of $\equiv$ and $\bigwedge$, was to first show that *conjunction* could be so defined and then, since *falsum*, $\perp$, may be defined by the equivalence

$$\perp \equiv \bigwedge xx,$$

the usual truth-table analysis shows that all the other classical propositional connectives can be given definitions in the Protothetic.

## 2. The number of primitives

The title of Tarski's article gives the impression that there is only one *primitive term* and even when stating the problem the reference to the quantifiers is parenthetical. However in the body of the article it is clear that the universal quantifier plays a fundamental role; thus, strictly speaking the title should have been: *On the primitive terms of logistic*. On the other hand, if Tarski had considered *concepts* instead of *terms*, then he would have realized that there is indeed a *primitive concept for the classical logics*, namely that of a *definition*.

The reason is that the universal quantifier and equivalence are the minimal terms required in order to have a definition (for a propositional function). The propositional connective of equivalence is needed to relate the *definiendum* to the *definiens*, the universal quantifier is needed in order for the relation between the definiendum and the definiens to hold universally and also so that the definition be a *sentence*[6]. Thus in our view, Tarski's result, which shows that all the classical connectives and quantifiers may be specified by definitions involving only the universal quantifier and the connective of equivalence, clearly demonstrates *the primordial role of definitions in classical logic*.

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[4]Otherwise it is not a definition but rather a sentence expressing some property of a previously introduced symbol.

[5]See [Henkin, 1963].

[6]We should also mention the fundamental role played by the variables; which could, perhaps, be avoided by use of H. B. Curry's *Combinators*.

### 3. Leśniewski's Protothetic

Leśniewski had started the development of the Protothetic before Gödel showed that *truth* and *derivation* need not be co-extensive. Consequently the *axioms* for the propositional connectives were determined purely on their truth-value interpretation; furthermore only two truth values were considered. On the whole *derivations* were considered as secondary objects; they were just a means to discover the *truths.* Almost universally, the only rule of inference was *modus ponens.* The particular choice of axioms was motivated mainly by aesthetical considerations (the least, shortest etc.).

After Gödel's incompleteness theorems it became evident that *derivations* could, and should, be studied in their own right. Gentzen's systems of Natural Deduction showed that *pure logic* could be adequately analyzed through the use of various *rules of inference* rather than relying on cleverly chosen *axioms.* In fact Gentzen went much further and showed that the Natural Deduction Systems could be set up so that each *logical atom* (i.e. each connective, quantifier etc.) had its own set of rules (in which no other logical atom was explicitly mentioned); in addition he observed that the rules of inference for the logical atoms could be separated into two types. One type, now traditionally called an Introduction rule, acted as a *definition*[7] of the logical atom; the other type, called an Elimination rule, gives sufficient conditions in order to infer from a formula with the logical atom[8].

### 4. Intuitionism

Another event that took place at the turn of the last century that reduced the over-emphasis on *truth values* was the introduction of L. E. J. Brouwer's *Intuitionistic Mathematics.* Now although Brouwer insisted that his interest lay in the study of Mathematics and not in the study of this or that particular logistic, Arend Heyting observed that in intuitionistic mathematics the linguistic expressions *and, or, for all* etc. were being used with certain regularities[9]. Thus in spite of Brouwer's dislike of formal logic, there arose a well defined *Intuitionistic Logic* and in particular an *Intuitionistic Propositional Calculus.* Gentzen's investigations on logical inference include intuitionistic logic—the *NJ* and *LJ* systems—and in fact he found that as far as the Natural Deduction Systems were concerned the intuitionistic inference was far more amenable to his analysis in that he was able to derive his famous *Hauptsatz* for *NJ* [10]. With 20/20 hindsight this is not surprising since both intuitionism and Gentzen's systems place much more importance

[7]That is, gave sufficient conditions to derive a formula with the logical atom as principal connective (quantifier).

[8]We shall follow this principle in dealing with the Leśniewskian definitions. We turn the definitional sentence into a pair of inference rules for the *definiens.* The Introduction rule gives us the way to go from the *definiendum* to the *definiens* and the Elimination rule does the converse.

[9]For the history of the formalization the reader is recommended to read Troelstra's [Troelstra, 1989] and Ruitenburg's [Ruitenburg, 1991].

[10]And 30 years passed before D. Prawitz, in [Prawitz, 1965], obtained it for the classical system *NK.*

on *proofs* than on *truth values*; although in intuitionism the proofs are in the (ideal) mathematician's mind, while in Gentzen's systems the proofs are represented by finite trees of formulas.

Eventually it came to be recognized[11] that the abstract notion of proof could be made the subject of mathematical analysis—just as it had occurred with the abstract concept of set. Consequently in the middle of the twentieth century different theories of *constructions* were put forward[12]; the paradigm usually being to show that a sentence is an intuitionistic theorem iff there is a(n abstract) construction justifying it.

Now although the idea of an abstract proof arose in the intuitionistic mathematics, it need not be so restricted; it can be used whenever one is more interested in a *dynamic* rather than *static* viewpoint[13]. And even in classical mathematics, the problem of the *identity of proofs* is wide open; in particular there are conflicting views of the relation between *proofs* and their linguistic representations as *derivations*.

## 5. The role of definitions in the Protothetic

As already mentioned, one of Leśniewski's concern was the role of definitions in a logistic; he wanted them to have the same status as theorems. This created a conflict with the prevalent view of formal logical systems because of Leśniewski's requirement of having what might be called *open systems*, that is, systems in which it was not specified which axiomatic definitions would be introduced. In particular, the syntactical characterizations of the symbols (except for the primitive ones) would not be specified when setting up the system. Instead he gave careful instructions for defining new symbols and the way to use the definitions. Unfortunately Leśniewski was ahead of his time and his suggestions were not followed[14]. We mentioned that Leśniewski was ahead of his times because nowadays (a modification of) his views on definitions have been rediscovered in computer science! Typically in a computer language there are few restrictions on what symbols may be used and it is up to the individual programmer to specify their use.

In addition, probably also because of the availability of the computer, there has been a shift from a static two-valued view of mathematics to one in which the inference method is often as important as the result itself.

## 6. Modeling inference

If one views mathematics just as a set of (true) statements, then one might be able to avoid analyzing inferences. On the other hand, already in the early 50's it was shown by Hiz[15] that it was possible to give a complete axiomatization, with finitely many axiom schemas

[11] Principally because of the many articles by G. Kreisel supporting that view.

[12] See, for example, [Goodman, 1970], [Läuschi, 1970] and [Scott, 1970].

[13] A viewpoint shared by the Categorists, see [Lawvere, 1975].

[14] The only mathematician to apply Leśniewski's philosophy was A. P. Morse in the book *A Theory of Sets*, [Morse, 1986], and in his lectures at the University of California, Berkeley, in the 50's.

[15] [Hiz, 1960].

and rules of inference, of the classical propositional calculus[16] in which the *deduction theorem* does not hold. But the deduction theorem is one of the most common inference methods in mathematics; thus Hiz' example shows that a system may be complete w.r.t. to *truth* but need not be so with respect to *inferences.*

Logic, from its earliest times, has been concerned both with *truth* as well as with *inferences.* However, the axiomatic method at first was biased towards *truth.* Probably one of the first references in which the axiomatic method is viewed as an *inference engine* can be found in Hilbert and Ackermann[17]:

> *The purpose of the symbolic language in mathematical logic is to achieve in logic what it has achieved in mathematics, namely, an exact scientific treatment of its subject-matter. The logical relations which hold with regard to judgements, concepts, etc., are represented by formulas whose interpretation is free from the ambiguities of common language. The transition from statements to their logical consequences, as occurs in the drawing of conclusions, is analyzed into its primitive elements, and appears as a formal transformation of the initial formulas in accordance with certain rules, similar to the rules of algebra; logical thinking is reflected in a logical calculus. This calculus makes possible a successful attack on problems whose nature precludes their solution by purely intuitive logical thinking. Among these for instance, is the problem of characterizing those statements which can be deduced from given premises.*

## 7. Natural Deduction Systems

Now although Hilbert encouraged the inferential method, it was soon recognized that there was a large disparity between formal derivations, in the style of Hilbert, and informal proofs. Łukasiewicz was well aware of the difference; in S. Jaśkowski[18]:

> *In 1926 Professor J. Łukasiewicz called attention to the fact that mathematicians in their proofs do not appeal to the theses of the theory of deduction, but make use of other methods of reasoning. The chief means employed in their method is that of an arbitrary supposition. The problem raised by Mr. Łukasiewicz was to put those methods under the form of structural rules and analyze their relation to the theory of deduction.*

Jaśkowski and (later, although independently) Gentzen proposed solutions to Łukasiewicz' problem. Although both solutions are similar, Gentzen used his systems to obtain results about first-order number theory and the derivations are in the form of finite rooted trees; consequently the Natural Deduction Systems of Gentzen became the system of choice amongst mathematical logicians. Dag Prawitz, whose book: *Natural Deduction. A Proof Theoretical Study* sparked a renewed interest in the Natural Deduction Systems of Gentzen compares, in [Prawitz, 1971], the work of Gentzen to that of Turing. In both cases a characterization of a mathematical concept was obtained by considering how a human

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[16]Complete in the sense that exactly the tautologies are theorems.
[17][Hilbert and Ackermann, 1950], page 1.
[18][Jaśkowski, 1934].

would carry out certain tasks; Turing analyzed the smallest steps that a person would do in carrying out a computation with paper and pencil (and eraser), while Gentzen considered the atomic steps, both from the logical as well as from the mathematical viewpoint, involved in the proof of Euclid's theorem on the number of primes.

Given that modern logic is as much concerned with *inferences* as with *truth*, any foundational formal system which completely neglects traditional inferences, such as the deduction theorem (when valid in the discipline in question) will likely be destined to oblivion. We shall adopt the Gentzen style of formalization since we find it to be just about the simplest one to represent inferences. We are also in complete agreement with Gentzen and Turing that in the early stages of a discipline it is best to concentrate on one "object" at a time; in the case of inferences the "objects" are formulas. Thus the Gentzen's systems that use trees of formulas[19] seem the most suitable for *setting up* a logistic.

*After* the system has been set up, it is usually advantageous to look at the consequence relation without being encumbered by *all* the formulas in the tree. This can be achieved through the associated relation "$\vdash$" where

$$\Gamma \vdash \mathcal{A}$$

is to hold just in case there is a derivation-tree of formulas in which the end-formula is $\mathcal{A}$ and all the undischarged assumption formulas are in $\Gamma$.

## 8. Development of this monograph

One of the authors had, in the late 50's, taken a course in Measure Theory given by A. P. Morse. Morse followed Leśniewski's principle that definitions were to be theorems of the system and introduced as required and thus, although the presentation of the course was strictly formal, it followed the traditional informal way of doing mathematics.

About 40 years later, the same author was glancing at Tarski's *On the primitive term of the logistic* when the question arose whether Tarski's result would be valid in a constructive setting, say in the intuitionistic calculus. After verifying that that indeed was the case, memories of Morse's course resurfaced. Then came the realization that some of Leśniewski's suggestions were now being carried out both in informal mathematics and in the most formal of mathematics, namely mathematical computing. And of course no mention is ever made of Leśniewski!

The other author observed that the Lindenbaum algebras associated with the New Protothetic were related to *Special Groups* and thus undertook the task of defining and developing the theory of Equivalence algebras.

## 9. www.math.umd.edu/research/books

We are adopting Proclus' interpretation of "*lemma*" as a proposition which is required in the course of a demonstration, but whose proof, if included, would break the thread

[19]In contradistinction to the trees of sequents which involve finitely many formulas at a time.

of the demonstration. Consequently the proofs of the lemmas will not be included in the monograph. However, we have decided to make use of the ubiquitous computer and establish a Web Page to include most of the proofs of the lemmas.

The lemmas and their proofs can be found at:

http://www.math.umd.edu/research/books/Lopez-Escobar/Definitions.htm

# Part 2

# The New Protothetic

CHAPTER 3

# The language of the New Protothetic

Remark. For the reader already familiar with the subtleties of Natural Deduction Systems we have included Section 5 in Chapter 4: "**The New Protothetic in a nutshell**".

## 1. Syntax of the New Protothetic

Towards the beginning of the twentieth century Leśniewski introduced his *New Foundations for Mathematics*. It consisted of three parts: the Protothetic, the Ontology and the Mereology. The Protothetic was the part that dealt with *Pure Logic* while the Ontology and Mereology concerned themselves much more with *elements* and the *subset* relation[1].

On the surface, the Protothetic was similar to Russell's (Extended) Calculus of Propositions. However there was a fundamental difference. For Russell *definitions* were simply shorthand devices—and thus not part of the formal theory—in which substitutions could be freely carried out, although no specific rules were ever laid down on what were permissible substitutions. On the other hand, for Leśniewski definitions were part and parcel of the formal theory and had the same status as the *theorems*. Since definitions had the same status as theorems (or more accurately: as axioms), Leśniewski gave precise instructions on what formal expressions could be classified as *definitions*.

The traditional requirements on definitions of logical concepts are that:

- *The definition is to consist of four parts:*
  - *the binding universal quantifiers,*
  - *the* definiendum*,*
  - *the symbol for* equivalence*, typically:* $\equiv$*,*
  - *and the* definiens*.*
- *The "symbol" being defined should occur exactly once and only in the definiendum.*

We shall refine Leśniewski's Protothetic into the **New Protothetic** under the following general principles:

- *the New Protothetic has to be reformulated in an updated style, nevertheless still retaining the ability to add definitions as it develops,*
- *the theory AND the underlying metatheory has to be intuitionistically acceptable*[2]*.*

[1]This is an oversimplification, and in fact, misleading interpretation of Leśniewski's aims for the Ontology and Mereology; for an accurate interpretation the reader is referred to E. C. Luschei's [Luschei, 1962].

[2]This does not exclude that it may *also* be interest to consider it from a classical viewpoint.

We consider the Leśniewski/Tarski thesis to be that *definitions* are the basic building blocks of Logic, thus in the New Protothetic the set of **primitive symbols** will be the minimal set required to be able to state a definition, that is:

- *The symbol for equivalence: "* $\equiv$*",*
- *The symbol for the universal quantifier: "*$\bigwedge$*".*

Most of the other symbols of the New Protothetic have a supporting role. For example, since we are not planning to use Polish notation, we need some parsing symbols. Thus we include:

- *The parsing symbols*: "(", ")", ",".

Unlike Leśniewski, we are not trying to develop a foundation for *all* of Mathematics, our aim is much more modest; namely to obtain a better understanding of, amongst other things, the intuitionistic connectives. Now the essence of a, say dyadic, propositional connective is that out of two propositions it produces a third proposition. In other words, a propositional connective is an instance of a propositional function. Thus in the New Protothetic there have to be symbols for *propositions* and *propositional functions*. Hence we include in the Syntax for the New Protothetic:

- *The symbols for propositional parameters*: $p_0, p_1, \ldots$
- *The symbols for propositional variables*: $x_0, x_1, \ldots$
- For each natural number $n$, *the symbols for parameters for n-ary propositional functions*: $F_0^n, F_1^n, \ldots$
- For each natural number $n$, *the symbols for variables for n-ary propositional functions*: $f_0^n, f_1^n, \ldots$

The symbols just described will be called the **reserved symbols** of the New Protothetic.

## 2. The identifiers of the New Protothetic

The other symbols of the New Protothetic are to be the symbols which may be freely used in the Leśniewskian definitions. Leśniewski, and A. P. Morse[3], chose to start their systems by discussing the physical representation of mathematical symbols, consequently they had detailed and fairly complicated instructions on what merited the name of a *symbol.* We need not get involved with such a hornet's nest since thanks to the almost universal presence of the computer we may use the more or less accepted convention that *mathematical symbols are (or can be determined by) finite sequences of consecutive keyboard entries.* In addition we shall make use of D. Knuth's TEX to extend the QWERTY keyboard (so that we may have such symbols as "$\bigwedge$").

Thus we choose for the remaining symbols of the New Protothetic, which will be called *identifiers,* certain finite sequences of TEX symbols[4]:

[3][Morse, 1986].

[4]Or, if you prefer, the corresponding keyboard sequences.

- *The* **identifiers of the New Protothetic** *are (denoted by) arbitrary finite, non-empty sequences of upper case roman letters in the "math bold font" of TEX.*

Since in this (restricted) version of the New Protothetic we will only use definitions for propositional functions and quantifiers, we will partition the identifiers into two classes:

- *The* **identifiers for quantifiers** *are those which start with "* **Q** *".*
- *The* **identifiers for propositional functions** *(also called* **functional identifiers***,* **defined operators** *or simply* **operators***) are those identifiers which are not identifiers for quantifiers.*

**Notational conventions.** First of all, from now on we shall usually not bother to distinguish between *use* and *mention* of linguistic expressions. Secondly, we shall use *meta-variables* for the parameters and variables of the New Protothetic. By and large we follow the convention that:

- $p, q, r, \ldots$ *stand for propositional parameters.*
- $x, y, z, \ldots$ *stand for propositional variables.*
- $F, G, H, \ldots$ *stand for propositional function parameters (a.k.a. functional parameters).*
- $f, g, \ldots$ *stand for functional variables.*
- $\mathbb{F}, \mathbb{G}, \mathbb{H}, \ldots$ *stand for functional identifiers (defined operators).*
- *We may use other parsers in addition to the parentheses (such as brackets); and on occasion we may leave out some of them.*

## 3. The Minimal Protothetic

The subsystem obtained by using only the primitive and reserved symbols of the New Protothetic, that is: $\bigwedge, \equiv$, (, ), parameters and variables, will be called the **Minimal Protothetic**.

The **proto-formulas** of the Minimal Protothetic are exhaustively obtained as follows:

- *Every propositional parameter and variable is a* proto-formula*.*
- *If $\phi$ is an $n$-ary functional parameter or variable and $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are proto-formulas, then the expression $\phi(\mathcal{A}_1, \ldots, \mathcal{A}_n)$, is a* proto-formula*; also called a* prime-proto-formula*.*
- *If $\mathcal{A}$ and $\mathcal{B}$ are proto-formulas, then so is $(\mathcal{A} \equiv \mathcal{B})$; it is called an* equivalence*.*
- *If $\mathcal{A}$ is a proto-formula and $x$ a propositional variable then $\bigwedge x \mathcal{A}$ is a proto-formula in which all occurrences of the propositional variable $x$ are* **bound occurrences***. The displayed occurrence of $x$ is called an* **indicial occurrence***. $\bigwedge x \mathcal{A}$ is a* propositional quantification*.*
- *If $\mathcal{A}$ is a proto-formula and $f$ a functional variable, then $\bigwedge f \mathcal{A}$ is a proto-formula in which all occurrences of the functional variable $f$ are* **bound occurrences***. The displayed occurrence of $f$ is called an* **indicial occurrence***. $\bigwedge f \mathcal{A}$ is a* functional quantification*.*

An occurrence of a variable in a proto-formula which is not a bound occurrence is called a **free occurrence**. A **formula** is a proto-formula in which there are no free occurrences of variables. A **sentence** is a formula in which there are no occurrences of parameters.

## 4. Replacement and substitution

If $\mathcal{A}, \mathcal{B}$ are proto-formulas and if $t$ is either a propositional parameter or variable, then

$$[t = \mathcal{B}]\mathcal{A}$$

is the result of replacing all (free) occurrences of $t$ in $\mathcal{A}$ by $\mathcal{B}$. If in addition no free occurrence of a variable in $\mathcal{B}$ becomes bound in $[t{=}\mathcal{B}]\mathcal{A}$, then the replacement is a **(propositional) substitution**.

Similarly if $\mathcal{A}$ is a proto-formula, $\tau, \sigma$ are propositional function parameters or variables of the same arity, then

$$[\tau{=}\sigma]\mathcal{A}$$

is the result of replacing all (free) occurrences of $\tau$ in $\mathcal{A}$ by $\sigma$. If in addition no occurrence of $\sigma$ in $[\tau = \sigma]\mathcal{A}$ is a bound occurrence, then it is a **(functional) substitution**.

From now on the use of the expression $[{*}{=}{*}]\mathcal{A}$ is to represent a *substitution.* It can be extended, and we shall assume that it has been done, to **simultaneous substitution**.

**More notational conventions.** In order to be closer to more traditional presentations of Logic, we may write $\mathcal{F}\ulcorner q/\mathcal{B}\urcorner$, even $\mathcal{F}\ulcorner\mathcal{B}\urcorner$, to represent the substitution[5] $[q = \mathcal{B}]\mathcal{F}$. In addition, if we wish to call attention that the variables that have a free occurrence in the proto-formula $\mathcal{F}$ are included in $\{x_0, \ldots, x_{n-1}, f_0, \ldots, f_{m-1}\}$, we may write the cumbersome $\mathcal{F}\ulcorner x_0, \ldots, x_{n-1}, f_0, \ldots, f_{m-1}\urcorner$ instead of $\mathcal{F}$.

## 5. Leśniewskian definitions of propositional functions

By a *Leśniewskian proto-definition* (for a propositional function[6]) we understand an expression of the form

$$\bigwedge x_0 \bigwedge x_1 \ldots \bigwedge x_{n-1}(\ \mathbb{F}(x_0, \ldots, x_{n-1})\ \equiv\ \mathcal{D}\ ),$$

where:

- $\mathbb{F}$ *is functional identifier,*
- $\mathcal{D}$ *is like a proto-formula of the Minimal Protothetic, with the exception that functional identifiers may occur in the place of functional parameters,*
- *none of the variables* $x_0, \ldots, x_{n-1}$ *may have a bound occurrence in* $\mathcal{D}$*,*
- *only the variables* $x_0, \ldots, x_{n-1}$ *may have a free occurrence in* $\mathcal{D}$*,*
- *there are no parameters (neither functional nor propositional) occurring in* $\mathcal{D}$*,*
- *the identifier* $\mathbb{F}$ *does not occur in* $\mathcal{D}$*.*

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[5] And correspondingly for simultaneous substitutions.

[6] We shall not introduce the concept of a Leśniewskian definition for a quantifier until Chapter 8, thus we will omit the qualifier *for a propositional function* in this and the next few chapters.

$\mathcal{D}$ is the **definiens** and $\mathbb{F}(x_0, \ldots, x_{n-1})$ the **definiendum** of the proto-definition. $\mathbb{F}$ is the **(defined) functional identifier**, or **operator** and it is **assigned** (by the proto-definition) the arity $n$. Thus the syntactical properties of *arity* and being a *operator* are determined by the proto-definition.

Next we consider finite sequences of Leśniewskian proto-definitions:

*By an* LPD-scheme *(Leśniewskian proto-definitional-scheme) we understand a finite sequence* $\mathfrak{S} = (\mathfrak{D}_1, \ldots, \mathfrak{D}_k)$ *of Leśniewskian proto-definitions such that for each* $1 \leq j \leq k$*, the defined operator of* $\mathfrak{D}_j$ *does not occur in any of the Leśniewskian proto-definitions* $\mathfrak{D}_i$ *where* $i < j$*.*

Given an LPD-scheme $\mathfrak{S}$, the (proto-)formulas of the $\mathfrak{S}$-Syntax are obtained by adding the following clause:

*if* $\mathbb{F}$ *is one of the defined operators of the LPD-scheme* $\mathfrak{S}$ *and its assigned arity is* $n$*, then for any* $\mathfrak{S}$*-proto-formulas* $\mathcal{A}_1, \ldots, \mathcal{A}_n$*:*

$$\mathbb{F}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$$

*is a* $\mathfrak{S}$*-proto-formula.*

Finally:

*A* **Leśniewskian Definitional Scheme** *is an LPD-scheme* $\mathfrak{S} = (\mathfrak{D}_1, \ldots, \mathfrak{D}_k)$ *such that for all* $i$*,* $1 \leq i \leq k$*;* $\mathfrak{D}_i$ *is a sentence in the* $\mathfrak{S}$*-syntax.*

## 6. Syntactical complexity of formulas

Very often we have to carry out an induction on the *uninterpreted* formulas. A useful measure of that kind of complexity we call **syntactical complexity**:

*The* syntactical complexity *of a proto-formula* $\mathcal{A}$ *of the* $\mathfrak{S}$*-syntax,* $\mathrm{Sc}(\mathcal{A})$*, is characterized as follows:*

- *If* $\mathcal{A}$ *is either a propositional variable or parameter, or a* $0$*-ary defined operator of* $\mathfrak{S}$*, then* $\mathrm{Sc}(\mathcal{A}) = 1$*.*
- *If* $\mathcal{A} = (\mathcal{B} \equiv \mathcal{C})$*, then* $\mathrm{Sc}(\mathcal{A}) = \mathrm{Sc}(\mathcal{B}) + \mathrm{Sc}(\mathcal{C}) + 1$*.*
- *If* $\mathcal{A} = \bigwedge x\mathcal{B}$ *or if* $\mathcal{A} = \bigwedge f\mathcal{B}$ *then* $\mathrm{Sc}(\mathcal{A}) = \mathrm{Sc}(\mathcal{B}) + 1$*.*
- *If* $\mathcal{A} = X(\mathcal{B}_1, \ldots \mathcal{B}_n)$ *then*

$$\mathrm{Sc}(\mathcal{A}) = \Big(\sum_{i=1}^{i=n} \mathrm{Sc}(\mathcal{B}_i)\Big) + 1.$$

CHAPTER 4

# The inferences of the New Protothetic

## 1. The inferences of the Minimal Protothetic

The crucial step, which by and large determines the inferential nature of the Protothetic, is to determine under what conditions one may *assert* a formula of the form

$$(\mathcal{A} \equiv \mathcal{B}).$$

Following the Brouwer–Heyting–Kolmogorov interpretation[1] we assert such a formula only when there is an abstract proof (a.k.a. construction) that justifies it. Keeping in mind the view of equivalence as a bi-conditional, one argues that a *construction $c$ proves* or *justifies* the formula $(\mathcal{A} \equiv \mathcal{B})$ when $c$ consists of a pair of constructions $(c_1, c_2)$ where $c_1$ takes any proof of $\mathcal{A}$ into a proof of $\mathcal{B}$ and $c_2$ takes a proof of $\mathcal{B}$ into a proof of $\mathcal{A}$ [2].

The simplest formalization[3] to represent such an interpretation is one in the style of Gentzen's *NJ*. That is, the formal derivations (which may be considered as representations of the *proofs* or *inferences*) are in the form of trees of formulas and the structure of the tree is completely determined by the *rules of inference*. The formulas at the top of the tree are *assumptions* and (some) of the rules of inference may *close* or *discharge* assumption formulas[4]. As Gentzen himself observed, the beauty of his Natural Deduction Systems was that the rules of inference dealt explicitly with one logical atom at a time. Furthermore the rules on inference for any given logical atom can be partitioned into two classes, one that acts as a *definition* (called the Introduction or **I**-rules) and the other which gives sufficient conditions for drawing inferences from formulas containing the logical atom (called the Elimination, Educing, or just simply **E**-rules).

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[1]For a history of the BHK interpretation the reader is referred to [Troelstra, 1989] and [Ruitenburg, 1991]. In the authors' view it would be more accurate to call it the *BHK*[2] *interpretation* because of the many contributions of Kreisel, for example, [Kreisel, 1962a].

[2]Kreisel would probably require that there be also a proof that $c_1$ and $c_2$ do indeed have those properties; however at this stage we are only using the abstract notion of proof to *suggest* a formalization for the calculus. Note also that one could put additional conditions, *e.g.* that $c_1$ and $c_2$ be in some sense *equivalent* (equal complexity, similar structure, built up from the same components and so on).

[3]But by no means the only one. Nevertheless we find it to be the simplest one that does not explicitly involve terms from a construction calculus.

[4]The following is a small partial list of more detailed developments of Natural Deduction Systems: [Gentzen, 1936], [Curry, 1963], [Prawitz, 1965], [Prawitz, 1971] and [Troelstra, 1973].

**Introduction rule for $\equiv$.** There is only one **I**-rule of inference for $\equiv$. It *defines*[5] the propositional connective $\equiv$ and it represents a possible reading of the above interpretation of the equivalence connective using a pair of constructions. Following Gentzen, Prawitz *et al.* we represent the rule of inference by the diagram

$$\frac{\begin{matrix}[\mathcal{A}]\\ \mathcal{B}\end{matrix}\qquad\qquad \begin{matrix}[\mathcal{B}]\\ \mathcal{A}\end{matrix}}{(\mathcal{A}\equiv\mathcal{B})}$$

We are following the convention of enclosing within [ ] the formula occurrences discharged by the rule.

An application of the rule would then be of the form

$$\frac{\begin{matrix}[\mathcal{A}]\\ \Pi_0\\ \mathcal{B}\end{matrix}\qquad\qquad \begin{matrix}[\mathcal{B}]\\ \Pi_1\\ \mathcal{A}\end{matrix}}{(\mathcal{A}\equiv\mathcal{B})}$$

where $\Pi_0$ and $\Pi_1$ are derivations with *end-formulas* $\mathcal{B}$ and $\mathcal{A}$ respectively. Note that $\mathcal{A}$ may have an *undischarged assumption occurrence* in the derivation $\Pi_0$ and correspondingly for $\Pi_1$.

**Elimination rules for $\equiv$.** Keeping in mind that equivalence can be interpreted as a bi-conditional, the following two **E**-rules suggest themselves:

$$\frac{(\mathcal{A}\equiv\mathcal{B})\qquad \mathcal{A}}{\mathcal{B}}$$

and

$$\frac{(\mathcal{A}\equiv\mathcal{B})\qquad \mathcal{B}}{\mathcal{A}}$$

The formula explicitly mentioning the logical atom $\equiv$ is known as the *major premise*. An important characteristic of the elimination rules for $\equiv$ is that the conclusion (of an application) of the rule is a proper subformula of the major premise.

**Introduction rule for propositional quantifier.** An application of the propositional $\bigwedge$-**I**-rule would be of the following form:

$$\frac{\begin{matrix}\Pi\\ \mathcal{A}\end{matrix}}{\bigwedge x[p{=}x]\mathcal{A}}$$

In the above diagram $\Pi$ stands for a derivation whose end-formula is $\mathcal{A}$. The formula $\bigwedge x[p{=}x]\mathcal{A}$ is the formula obtained by an application of the rule of inference. In the $\bigwedge$-**I**-rule it is further **required** that the propositional parameter $p$ does not occur in any open (undischarged) assumption formula of the subderivation $\Pi$.

The propositional parameter $p$ is known as the **(propositional) eigen-parameter** of the application.

[5] In the sense of Gentzen's [Gentzen, 1936].

The essence of the $\bigwedge$-**I**-rule is that in order to derive the quantificational formula $\bigwedge x[p{=}x]\mathcal{A}$ it requires that there be **one** method that generates for each formula $\mathcal{B}$, by substitution of parameters, a derivation of: $[p{=}\mathcal{B}]\mathcal{A}$. Clearly this is a very strong restriction on the derivation of formulas of the form $\bigwedge x[p{=}x]\mathcal{A}$. This is even stricter than the intuitionistic requirement that in order to assert $\bigwedge x[p{=}x]\mathcal{A}$ there has to be a constructive method $M$ such that for each formula $\mathcal{B}$: $M(\mathcal{B})$ is a proof of $[p{=}\mathcal{B}]\mathcal{A}$. That makes the completeness theorem of any calculus with a universal quantifier even more interesting.

A more suggestive (but ambiguous) representation of the $\bigwedge$-**I**-rule is the following:

$$\frac{\mathcal{A}\ulcorner p\urcorner}{\bigwedge x\mathcal{A}\ulcorner x\urcorner}$$

Note that the eigen-parameter $p$ no longer occurs in the conclusion.

**Elimination rule for propositional quantifier.** The Elimination rule for the propositional $\bigwedge$ tells us what can be deduced from $\bigwedge x\mathcal{A}\ulcorner x\urcorner$. Thus the following representation:

$$\frac{\bigwedge x[p{=}x]\mathcal{A}}{[p{=}\mathcal{B}]\mathcal{A}}$$

where $\mathcal{B}$ is any formula, called the *instantiated formula* of the application.

Using the more suggestive notation we have

$$\frac{\bigwedge x\mathcal{A}\ulcorner x\urcorner}{\mathcal{A}\ulcorner\mathcal{B}\urcorner}$$

Note that, unlike the case of the $\equiv$-**E**-rule(s), the conclusion of an application of the propositional $\bigwedge$-**E**-rule may be more complex than the (major) premise.

**Introduction rule for functional quantifier.** The functional $\bigwedge$-**I**-rule, in the *Minimal Protothetic*, is *mutatis mutandis* as for the propositional case. In the long form we have

$$\frac{\begin{array}{c}\Pi\\ \mathcal{A}\end{array}}{\bigwedge f[F{=}f]\mathcal{A}}$$

where $F$ is a functional parameter and $f$ is a functional variable of the same arity. Furthermore the functional parameter $F$ must not occur in any undischarged assumption formula of the derivation $\Pi$. $F$ is the **(functional) eigen-parameter** of the application.

In the short form we write

$$\frac{\mathcal{A}\ulcorner F\urcorner}{\bigwedge f\mathcal{A}\ulcorner f\urcorner}$$

**Elimination rule for functional quantifier.** The functional $\bigwedge$-**E**-rule, *in the Minimal Protothetic*, may be represented thus:

$$\frac{\bigwedge f[F{=}f]\mathcal{A}}{[F{=}G]\mathcal{A}}$$

where $F$ and $G$ are functional parameters of the same arity as the variable $f$.

The short representation of the rule is

$$\frac{\bigwedge f \mathcal{A}\ulcorner f \urcorner}{\mathcal{A}\ulcorner G \urcorner}$$

where $G$ is a functional parameter. $G$ is the *instantiated functional.*

## 2. Standardized eigen-parameters

Given a derivation $\Pi$ we shall say that the eigen-parameters of $\Pi$ are **standardized** or alternatively that $\Pi$ is a **standardized derivation** iff the following conditions are met:

- *Each different application of a* $\bigwedge$-**I** *rule in* $\Pi$ *has a different eigen-parameter.*
- *The eigen-parameters of* $\Pi$ *do not occur in* $\Pi$ *in any of the formulas appearing below the application of its* $\bigwedge$-**I**-*rule.*

Since the number of parameters is not limited, by judicious replacement of parameters one can show:

PROPOSITION 4.1. *To every derivation* $\Pi$ *one can effectively associate a standardized derivation* $\Pi^*$ *with the same conclusion and undischarged assumption formulas. Furthermore* $\Pi^*$ *is obtained from* $\Pi$ *by replacement of eigen-parameters in some of the formulas of* $\Pi$*.*

*Proof.* By induction on the length of the derivation. ∎

In view of the above proposition we shall always assume that the derivations are in standardized form; if they are not, say as a result of some operation on a derivation, then it will be implicitly assumed that the first order of business is to standardize it!

## 3. The $\mathfrak{S}$-Protothetic

If $\mathfrak{S} = (\mathfrak{D}_1, \ldots, \mathfrak{D}_k)$ is a Leśniewskian definitional scheme then by the $\mathfrak{S}$-Protothetic we understand the extension of the Minimal Protothetic obtained by (a) allowing arbitrary formulas of the $\mathfrak{S}$-Syntax in the rules of inference of the Minimal Protothetic, (b) in the functional $\bigwedge$-**E**-rule, allowing the instantiated functional to be one of the defined operators of $\mathfrak{S}$ and (c) introducing a pair of **I-E** rules of inference for each of the defined operators.

**Rules for defined operators.** Suppose that the defined operator $\mathbb{F}$ has the following Leśniewskian definition in $\mathfrak{S}$:

$$\bigwedge x_0 \ldots \bigwedge x_{n-1}(\ \mathbb{F}(x_0, \ldots, x_{n-1})\ \equiv\ \mathcal{D}\ulcorner x_0, \ldots, x_{n-1} \urcorner\ )$$

Then the Leśniewskian definition[6] generates the following two inference rules for the operator $\mathbb{F}$:

[6]The **I**-rule being thus Gentzen's version of the *definition* of the "connective" $\mathbb{F}$. Furthermore the **E**-rule is making use of how $\mathbb{F}$ was introduced. It would thus appear that "Leśniewskian definition" and "Gentzenian definition" are consistent with each other.

**I-rule for $\mathbb{F}$:**

$$\frac{\mathcal{D}\ulcorner \mathcal{A}_0, \ldots, \mathcal{A}_{n-1} \urcorner}{\mathbb{F}(\mathcal{A}_0, \ldots, \mathcal{A}_{n-1})}$$

where $\mathcal{A}_0, \ldots, \mathcal{A}_{n-1}$ are arbitrary $\mathfrak{S}$ formulas.

**E-rule for $\mathbb{F}$:**

$$\frac{\mathbb{F}(\mathcal{A}_0, \ldots, \mathcal{A}_{n-1})}{\mathcal{D}\ulcorner \mathcal{A}_0, \ldots, \mathcal{A}_{n-1} \urcorner}$$

It should be observed that if one then adds a new Leśniewskian definition, thus extending $\mathfrak{S}$ to a new definitional scheme $\mathfrak{S}^+$, then all the rules of inference should be interpreted so as to apply to the $\mathfrak{S}^+$ formulas.

## 4. Logical complexity in the $\mathfrak{S}$-Protothetic

Suppose that the defined dyadic operator $\mathbb{F}$ is given by the definition:

$$\bigwedge x \bigwedge y[\ \mathbb{F}(x, y) \equiv \bigwedge f[x \equiv (f(x) \equiv f(y))]\ ].$$

Then it is clear[7] that the logical complexity, with respect to $\mathfrak{S}$, of the formula $\mathbb{F}(\mathcal{A}, \mathcal{B})$, $\mathrm{lc}_{\mathfrak{S}}(\mathbb{F}(\mathcal{A}, \mathcal{B}))$, should be strictly greater than that of

$$\bigwedge f[\mathcal{A} \equiv (f(\mathcal{A}) \equiv f(\mathcal{B}))]$$

(and correspondingly for any other defined operators of $\mathfrak{S}$). One way to achieve this, in the example under consideration, is to require that

$$\mathrm{lc}_{\mathfrak{S}}(\mathbb{F}(\mathcal{A}, \mathcal{B})) = (2\times \mathrm{lc}_{\mathfrak{S}}(\mathcal{A}) + \mathrm{lc}_{\mathfrak{S}}(\mathcal{B}) + 5) + 1.$$

For the other proto-formulas one may proceed as usual:

- *If $a$ is either a propositional variable or parameter:* $\mathrm{lc}_{\mathfrak{S}}(a) = 1$.
- *If $\mathcal{A}$ is a (proto)-formula of the form $X(\mathcal{A}_1, \ldots, \mathcal{A}_n)$, where $X$ is either a functional parameter or variable, then*

$$\mathrm{lc}_{\mathfrak{S}}(\mathcal{A}) = 1 + \sum_{i=1}^{i=n} \mathrm{lc}_{\mathfrak{S}}(\mathcal{A}_i).$$

- $\mathrm{lc}_{\mathfrak{S}}((\mathcal{A}{\equiv}\mathcal{B})) = \mathrm{lc}_{\mathfrak{S}}(\mathcal{A}) + \mathrm{lc}_{\mathfrak{S}}(\mathcal{B}) + 1$.
- $\mathrm{lc}_{\mathfrak{S}}(\bigwedge x \mathcal{A}) = \mathrm{lc}_{\mathfrak{S}}(\mathcal{A}) + 1$.
- $\mathrm{lc}_{\mathfrak{S}}(\bigwedge f \mathcal{A}) = \mathrm{lc}_{\mathfrak{S}}(\mathcal{A}) + 1$.

Such a measure of logical complexity has the property that all the **I**-rules of inference, including the rules corresponding to the Leśniewskian definitions, increase the complexity of the formulas and for all the **E**-rules, with the exception of the $\bigwedge$-**E**-rules, the conclusion of the inference is of smaller complexity than that of the major premise.

[7]If we are to have any success with inductive proofs on the logical complexity of the end-formulas of derivations.

**Derivability in the $\mathfrak{S}$-Protothetic.** If $\Gamma \cup \{\mathcal{B}\}$ is a set of $\mathfrak{S}$ formulas then by

$$\Gamma \vdash \mathcal{B}$$

is to be understood that there is a (standardized) derivation $\Pi$ in the $\mathfrak{S}$-Protothetic of the formula $\mathcal{B}$ in which all the formulas which have an open assumption occurrence in $\Pi$ belong to $\Gamma$.

The following observations express the principal properties of the derivability relation. In order to shorten the statements, we sometimes use notation reminiscent of Gentzen's Calculus of Sequents. Now although the *results* are essential to the rest of the monograph, the actual *derivations* of the results are of lesser importance (other that it be done in a constructive manner) and thus we often did not include the proofs of the lemmas within the monograph. However most of the proofs can be found on the Web Page:

http://www.math.umd.edu/research/books/Lopez-Escobar/Definitions.htm.

LEMMA 4.1 (**Finiteness**). *If* $\Gamma \vdash \mathcal{B}$ *then* $\exists\Delta \subseteq_{Finite} \Gamma[\,\Delta \vdash \mathcal{B}\,]$.

LEMMA 4.2 (**Monotonicity**).

$$\frac{\Gamma \vdash \mathcal{B}}{\Gamma, \Delta \vdash \mathcal{B}}$$

LEMMA 4.3 (**Transitivity of Deduction**).

$$\frac{\Gamma \vdash \mathcal{B} \qquad\qquad \Delta, \mathcal{B} \vdash \mathcal{F}}{\Gamma, \Delta \vdash \mathcal{F}}$$

LEMMA 4.4 (**Deduction Theorem**).

$$\frac{\Gamma, \mathcal{F} \vdash \mathcal{C} \qquad\qquad \Delta, \mathcal{C} \vdash \mathcal{F}}{\Gamma, \Delta \vdash (\mathcal{F} \equiv \mathcal{C})}$$

LEMMA 4.5 (**Modus Ponens**).

$$\frac{\Gamma \vdash (\mathcal{F}{\equiv}\mathcal{C}) \qquad\qquad \Delta \vdash \mathcal{F}}{\Gamma, \Delta \vdash \mathcal{C}}$$

LEMMA 4.6.

(1) $\vdash \bigwedge x(x{\equiv}x)$.
(2) $\vdash \bigwedge x \bigwedge y((x{\equiv}y) \equiv (y{\equiv}x))$.
(3) $\vdash \bigwedge x \bigwedge y((x{\equiv}(y{\equiv}y)) \equiv x)$.

LEMMA 4.7. *If neither of the parameters $p$ nor $F$ occur in $\Gamma$, then:*

(1) *If* $\Gamma \vdash \mathcal{A}\ulcorner p \urcorner$ *then* $\Gamma \vdash \bigwedge x \mathcal{A}\ulcorner p \urcorner$.
(2) *If* $\Gamma \vdash \mathcal{A}\ulcorner F \urcorner$ *then* $\Gamma \vdash \bigwedge f \mathcal{A}\ulcorner f \urcorner$.

LEMMA 4.8 (**Transitivity of** $\equiv$). *For any formulas $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$:*

$$(\mathcal{A} \equiv \mathcal{B}), (\mathcal{B} \equiv \mathcal{C}) \vdash (\mathcal{A} \equiv \mathcal{C})$$

LEMMA 4.9 (**Invariance**). *If $\Pi\ulcorner q \urcorner$ is a standardized derivation of $\mathcal{F}\ulcorner q \urcorner$ from $\Gamma\ulcorner q \urcorner$, then for all formulas $\mathcal{B}$ which do not contain any of the eigen-parameters of $\Pi$: $\Pi\ulcorner \mathcal{B} \urcorner$ is a derivation of $\mathcal{F}\ulcorner \mathcal{B} \urcorner$ from $\Gamma\ulcorner \mathcal{B} \urcorner$.*

Corollary. *If* $\Gamma\ulcorner q\urcorner \vdash \mathcal{F}\ulcorner q\urcorner$ *then for all formulas* $\mathcal{B}$*:* $\Gamma\ulcorner\mathcal{B}\urcorner \vdash \mathcal{F}\ulcorner\mathcal{B}\urcorner$*.*

*Proof.* Given the formula $\mathcal{B}$, rename the eigen-parameters of the derivation $\Pi$ justifying $\Gamma\ulcorner q\urcorner \vdash \mathcal{F}\ulcorner q\urcorner$ so that they are distinct from all the parameters occurring in $\mathcal{B}$. Then apply Lemma 4.9. □

Lemma 4.10. *Each of the Leśniewskian definitions in* $\mathfrak{S}$ *is a thesis of the* $\mathfrak{S}$*-Protothetic.*

## 5. The New Protothetic in a nutshell

It is our contention that the New Protothetic is both an extremely simple and yet very powerful natural deduction system. But then the reader might ask why has it taken so many pages to describe it?

The reason is that in addition to describing it we have tried to both give some historical perspective as well as explain our reasons for choosing this particular formalization.

For those readers which are already familiar with Natural Deduction Formalizations we now present the core of the New Protothetic (as well as throwing caution and precision overboard).

**Language of the New Protothetic.** A higher order propositional calculus with quantification over propositions and propositional functions (of any finite arity).

The primitive terms are $\equiv$ (equivalence) and $\bigwedge$ (universal quantifier).

There is an unlimited supply of identifiers which can be used in definitions, which may be introduced at will.

**Rules of inference for the primitive terms.**

**I-Rules:**

$$\frac{\begin{matrix}[\mathcal{A}] \\ \mathcal{B}\end{matrix} \qquad \begin{matrix}[\mathcal{B}] \\ \mathcal{A}\end{matrix}}{\mathcal{A}\equiv\mathcal{B}}$$

$$\frac{\mathcal{A}\ulcorner p\urcorner}{\bigwedge x\mathcal{A}\ulcorner x\urcorner} \qquad \frac{\mathcal{B}\ulcorner F\urcorner}{\bigwedge f\mathcal{B}\ulcorner f\urcorner}$$

**E-rules:**

$$\frac{\mathcal{A}\equiv\mathcal{B} \qquad \mathcal{A}}{\mathcal{B}} \qquad \frac{\mathcal{A}\equiv\mathcal{B} \qquad \mathcal{B}}{\mathcal{A}}$$

$$\frac{\bigwedge x\mathcal{A}\ulcorner x\urcorner}{\mathcal{A}\ulcorner\mathcal{B}\urcorner} \qquad \frac{\bigwedge f\mathcal{A}\ulcorner f\urcorner}{\mathcal{A}\ulcorner X\urcorner}$$

**Rules of inference for Leśniewskian definitions.** Given the Leśniewskian definition

$$\bigwedge x_0\bigwedge x_1\ldots[\,\mathbb{F}(x_0,x_1,\ldots)\equiv\mathcal{D}\ulcorner x_0,x_1\ldots\urcorner\,],$$

we have the following **I**- and **E**-rules:

$$\frac{\mathcal{D}\ulcorner\mathcal{A}_0,\mathcal{A}_1,\ldots\urcorner}{\mathbb{F}(\mathcal{A}_0,\mathcal{A}_1\ldots.)} \qquad \frac{\mathbb{F}(\mathcal{A}_0,\mathcal{A}_1,\ldots)}{\mathcal{D}\ulcorner\mathcal{A}_0,\mathcal{A}_1,\ldots\urcorner}$$

respectively.

### 6. A condensed form for derivations

So far we have only the extreme representations for derivations, that is, we can either produce the full Gentzenian tree of formulas, or else use $\Gamma \vdash \mathcal{F}$, in which we only display the undischarged assumption formulas and the End-formula. We now introduce an intermediary method.

When we write an array of the form

$$\begin{array}{rcl} \mathcal{A}_1, \ldots, \mathcal{A}_{n-1} & \Rightarrow & \mathcal{F}_1 \\ & \Rightarrow & \mathcal{F}_2 \\ & \Rightarrow & \mathcal{F}_3 \\ & \ldots & \ldots \\ & \ldots & \ldots \\ & \Rightarrow & \mathcal{F}_m \end{array}$$

we understand that for each $i$ such that $1 \leq i \leq m$:

$$\mathcal{A}_1, \ldots, \mathcal{A}_{n-1}, \mathcal{F}_1, \ldots, \mathcal{F}_{i-1} \vdash \mathcal{F}_i.$$

We call such an array a **condensed derivation of** $\mathcal{F}_m$ from $\mathcal{A}_1, \ldots, \mathcal{A}_{n-1}$.

As an example[8] of the method we give a condensed derivation of the following proposition:

PROPOSITION 4.2.

(1) $\bigwedge x[\mathcal{A}\ulcorner x \urcorner \equiv \mathcal{B}\ulcorner x \urcorner] \vdash \bigwedge x \mathcal{A}\ulcorner x \urcorner \equiv \bigwedge x \mathcal{B}\ulcorner x \urcorner$.
(2) $\bigwedge f[\mathcal{A}\ulcorner f \urcorner \equiv \mathcal{B}\ulcorner f \urcorner] \vdash \bigwedge f \mathcal{A}\ulcorner f \urcorner \equiv \bigwedge f \mathcal{B}\ulcorner f \urcorner$.

*Proof.* (1)

$$\begin{array}{rcl} \bigwedge x[\mathcal{A}\ulcorner x \urcorner \equiv \mathcal{B}\ulcorner x \urcorner], \bigwedge x \mathcal{A}\ulcorner x \urcorner & \Rightarrow & \mathcal{A}\ulcorner q \urcorner \\ & \Rightarrow & \mathcal{A}\ulcorner q \urcorner \equiv \mathcal{B}\ulcorner q \urcorner \\ & \Rightarrow & \mathcal{B}\ulcorner q \urcorner \end{array}$$

Thus $\bigwedge x[\mathcal{A}\ulcorner x \urcorner \equiv \mathcal{B}\ulcorner x \urcorner], \bigwedge x \mathcal{A}\ulcorner x \urcorner \vdash \mathcal{B}\ulcorner q \urcorner$. It follows that

$$\bigwedge x[\mathcal{A}\ulcorner x \urcorner \equiv \mathcal{B}\ulcorner x \urcorner], \bigwedge x \mathcal{A}\ulcorner x \urcorner \vdash \bigwedge x \mathcal{B}\ulcorner x \urcorner.$$

Similarly one obtains

$$\bigwedge x[\mathcal{A}\ulcorner x \urcorner \equiv \mathcal{B}\ulcorner x \urcorner], \bigwedge x \mathcal{B}\ulcorner x \urcorner \vdash \bigwedge x \mathcal{A}\ulcorner x \urcorner.$$

From the last two derivations one then obtains the desired result

$$\bigwedge x[\mathcal{A}\ulcorner x \urcorner \equiv \mathcal{B}\ulcorner x \urcorner] \vdash \bigwedge x \mathcal{A}\ulcorner x \urcorner \equiv \bigwedge x \mathcal{B}\ulcorner x \urcorner.$$

(2) Analogous to (1). ■

### 7. Equivalence-invariance of functions

In *Appendix C: "Truth-functions and others"* of *Principia*, Whitehead and Russell[9] propose the name of "*truth-function*" for those propositional functions $F$ which are

[8] Which shows that an advantage of the condensed derivation is that each of the $\Rightarrow$ steps is fairly evident.

[9] [Whitehead and Russell, 1925].

equivalence-invariant, that is, such that

$$p \equiv q \vdash F(p) \equiv F(q).$$

Furthermore, they provide an intuitive argument why one should not expect all propositional functions to be equivalence-invariant. Tarski shows in [Tarski, 1956], using Łukasiewicz' three-valued interpretation, that the statement that all propositional functions are truth-functions is independent of the axioms of Leśniewski's Protothetic[10].

The condition that all the (monadic) propositional functions be equivalence-invariant (*i.e.* truth-functions) is rendered by Tarski by the following sentence $\mathcal{LS}$ (which he calls the **Law of Substitution**):

$$\mathcal{LS} = \bigwedge f \bigwedge x \bigwedge y [\ x \equiv y \ \supset \ f(x) \equiv f(y)\ ].$$

Tarski went then to give many equivalents to the Law of Substitution; for example the equivalence of $\mathcal{LS}$ to

$$\bigwedge f \bigwedge x [\ f(x) \equiv [(f(\top) \wedge x) \vee (f(\bot) \wedge \neg x)\ ].$$

The latter goes by the name of **The Law of Development**[11].

Now although Tarski[12] made strong use of both **The Law of the Excluded Middle** and the associativity of equivalence, nevertheless we can, *mutatis mutandis*, reproduce (most, but obviously not all) his results in the New Protothetic.

We start with the following characterization:

*The* **truth-functions** *of the New Protothetic are those defined operators which are equivalence-invariant (in the New Protothetic).*

In order to characterize those defined operators that are equivalence-invariant we must consider the definiens of their corresponding Leśniewskian definitions; and since they will be proto-formulas we propose:

- *By a* **propositional assertion on the propositional variables** $\vec{x}$ *we understand a proto-formula whose free propositional variables all belong to* $\vec{x}$*.*
- *A propositional assertion* $\mathcal{F}\ulcorner\vec{x}\,\urcorner$*, where* $\vec{x} = (x_0, \ldots, x_{n-1})$*, is* **equivalence-invariant** *iff for all sequences of formulas* $\vec{\mathcal{A}} = (\mathcal{A}_0, \ldots, \mathcal{A}_{n-1})$*,* $\vec{\mathcal{B}} = (\mathcal{B}_0, \ldots, \mathcal{B}_{n-1})$*:*

$$\mathcal{A}_0 \equiv \mathcal{B}_0, \ldots, \mathcal{A}_{n-1} \equiv \mathcal{B}_{n-1} \ \vdash \ \mathcal{F}\ulcorner\vec{\mathcal{A}}\,\urcorner \equiv \mathcal{F}\ulcorner\vec{\mathcal{B}}\,\urcorner\ .$$

- *An n-ary functional parameter*[13] *or defined operator* $X$ *is* **truth-functional** *iff the propositional assertion* $X(x_0, \ldots, x_{n-1})$ *is equivalence-invariant.*

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[10]Actually Tarski never explicitly mentions the Protothetic in the cited article, preferring to always talk about the "Logistic"; the only indirect reference to the Protothetic is in a footnote, in the first page, in which he states: "*Such a theory was developed in 1920 by S. Leśniewski in his course on the principles of arithmetic at the University of Warsaw; an exposition of the foundations of a system of logistic based upon his theory of types can be found in Leśniewski* [Leśniewski, 1929], [Leśniewski, 1930] *and* [Leśniewski, 1938]."

[11]See also [Schröder, 1890], or [Boole, 1854].

[12][Tarski, 1956].

[13]Clearly functional parameters will not be truth-functional unless under certain assumptions.

**Towards characterizing equivalence invariance.** We now solve the problem of finding *sufficient syntactical conditions* for a propositional assertion to be equivalence-invariant; we postpone the problem of finding necessary conditions to Part 3.

We start with the following characterizations:

- *A propositional variable $x$ is* **linked to the functional $X$ in the proto-formula** *$\mathcal{F}$ iff $x$ has a free occurrence in a proto-subformula of $\mathcal{F}$ of the form*
$$X(\dots, \mathcal{A}\ulcorner x \urcorner, \dots).$$
- *A propositional assertion $\mathcal{F}\ulcorner \vec{x} \urcorner$ is* **Tarskian** *iff for each propositional variable $x$ of $\vec{x}$:*
  1. *$x$ is not linked to any of the functional parameters or functional variables of $\mathcal{F}\ulcorner \vec{x} \urcorner$.*
  2. *Should $x$ be linked to a defined operator, then the defined operator must be truth-functional.*
- *A Leśniewskian definition in which the definiens is a Tarskian assertion will be called a* **Tarskian definition***.*

Our first result along these lines is:

Proposition 4.3. *Every Tarskian assertion is equivalence-invariant.*

*Proof.* The proof given below is by induction on the syntactical complexity, $\mathrm{Sc}(\mathcal{F}\ulcorner \vec{x} \urcorner)$, of the Tarskian assertion $\mathcal{F}\ulcorner \vec{x} \urcorner$:

*Basis step:* $\mathcal{F}\ulcorner \vec{x} \urcorner$ is the proto-formula $x$. This case is trivial.

*Inductive steps.* First observe that $\mathcal{F}$, since it is a Tarskian assertion on $\vec{x}$, cannot be of the form $X(\mathcal{A}_1, \dots, \mathcal{A}_n)$, where $X$ is a functional parameter. If $X$ is a defined operator, then it must be a truth-function and thus the result is immediate.

Let us thus consider the case where
$$\mathcal{F}\ulcorner \vec{x} \urcorner = \bigwedge f \mathcal{A}\ulcorner \vec{x}, f \urcorner.$$
Then $\mathcal{A}\ulcorner \vec{x}, F \urcorner$ is a Tarskian assertion of lower syntactical complexity. Using the induction hypothesis one finds that
$$\vec{p} \equiv \vec{q} \;\vdash\; \mathcal{A}\ulcorner \vec{p}, F \urcorner \equiv \mathcal{A}\ulcorner \vec{q}, F \urcorner.$$
Then by judicious choice of $F$ we get
$$\vec{p} \equiv \vec{q} \;\vdash\; \bigwedge f [\mathcal{A}\ulcorner \vec{p}, f \urcorner \equiv \mathcal{A}\ulcorner \vec{q}, f \urcorner].$$
The result, for this case, then follows by applying Proposition 4.2. The remaining cases are even simpler. ∎

Corollary. *If the operator $\mathbb{O}$ has a Tarskian definition, then the operator $\mathbb{O}$ is a truth-function.*

**Unfoldings of the Protothetic, the New Protothetic.** When we do not wish to call attention to the definitional scheme $\mathfrak{S}$, then we may call the $\mathfrak{S}$-Protothetic **an unfolding of the** (Minimal) **Protothetic**. By the **New Protothetic** we understand, depending on the context:

- *Either the collection of* all *possible unfoldings of the Protothetic, or*
- *a given unfolding of the Protothetic.*

## 8. The Leśniewskian Protothetic

One can obtain the Protothetic of Leśniewski/Tarski (or the "Old Protothetic") by making sure that the sentence expressing the **associativity of equivalence**:

$$\bigwedge x \bigwedge y \bigwedge z [\,[x \equiv (y \equiv z)] \equiv [(x \equiv y) \equiv z]\,],$$

is a thesis of the system.

The simplest way is to add it as an axiom. The following proposition gives us another possible axiom:

Proposition 4.4. *The following sentences are equivalent in the Minimal Protothetic:*

(1) $\bigwedge x \bigwedge y \bigwedge z [\,[x \equiv (y \equiv z)] \equiv [(x \equiv y) \equiv z]\,]$.
(2) $\bigwedge x \bigwedge y [\,x \equiv ((x \equiv y) \equiv y)\,]$.

*Proof.* That (1) yields (2), in the New Protothetic, is fairly obvious. Thus assume (2). Then we will show that

$$p \equiv (q \equiv r) \vdash (p \equiv q) \equiv r.$$

First observe that under the assumption of (2),

$$\begin{array}{rcl} p \equiv (q \equiv r), p \equiv q & \Rightarrow & q \equiv (q \equiv r) \\ & \Rightarrow & (r \equiv q) \equiv q \\ & \Rightarrow & r \end{array}$$

Thus (2) yields

$$(*) \qquad p \equiv (q \equiv r), p \equiv q \vdash r.$$

On the other hand:

$$\begin{array}{rcl} p \equiv (q \equiv r), r, p & \Rightarrow & q \equiv r \\ & \Rightarrow & q \end{array}$$

$$\begin{array}{rcl} p \equiv (q \equiv r), r, q & \Rightarrow & q \equiv r \\ & \Rightarrow & p \end{array}$$

$$p \equiv (q \equiv r), r \quad \vdash \quad p \equiv q$$

Hence, combining it with $(*)$:

$$p \equiv (q \equiv r) \quad \vdash \quad (p \equiv q) \equiv r$$

It is then a small step to complete the proof that (1) and (2) are equivalent. ∎

The disadvantage of adding those sentences as axioms is that they require the quantifier and thus would not be formulas of subsystems in which there are no quantifiers. Now instead of axioms we could add *rules of inference.* The rule of inference corresponding to the sentence of the associativity of equivalence is:

$$\begin{array}{ll} \text{From:} & ((\mathcal{A} \equiv \mathcal{B}) \equiv \mathcal{C}) \\ \text{To obtain:} & (\mathcal{A} \equiv (\mathcal{B} \equiv \mathcal{C})) \end{array}$$

Note that the rule is neither an **I**- nor an **E**-rule.

The rule of inference corresponding to the other possible axiom can be represented by the following:

$$\frac{\begin{array}{c}[(\mathcal{A} \equiv \mathcal{B})]\\ \Pi_0\\ \mathcal{B}\end{array} \qquad \begin{array}{c}[\mathcal{B}]\\ \Pi_1\\ (\mathcal{A} \equiv \mathcal{B})\end{array}}{\mathcal{A}}$$

And although it is neither an **I**- nor **E**-rule of inference, in the case that $\mathcal{B}$ is the Intuitionistic Absurdity $\perp$ (equivalently: $\bigwedge xx$) it simplifies to

$$\frac{\begin{array}{c}[\mathcal{A} \equiv \perp]\\ \Pi_0\\ \perp\end{array}}{\mathcal{A}}$$

which is in essence the way that D. Prawitz handles the classical law of *Double Negation*. Consequently, we propose that the sentence

$$\bigwedge x \bigwedge y[\, x \equiv ((x \equiv y) \equiv y)\,]$$

be called the **law of universal double negation**. For the corresponding rule we shall use the name: **rule of universal double negation**.

CHAPTER 5

# Semantics for the New Protothetic

## 1. Another of Tarski's contributions

In an address to the Third Polish Mathematical Congress in Warsaw on 30 September 1937, A. Tarski stated[1]:

> *...I shall point out certain formal connexions between the sentential calculus and topology (as well as some other mathematical theories). I am concerned in the first place with a topological interpretation of two systems of the sentential calculus, namely the ordinary (two-valued) and the intuitionistic (Brouwer–Heyting) system. With every sentence $\mathfrak{A}$ of the sentential calculus we correlate, in a one-one fashion, a sentence $\mathfrak{A}_1$ of topology in such a way that $\mathfrak{A}$ is provable in the two-valued calculus if and only if $\mathfrak{A}_1$ holds in every topological space. An analogous correlation is set up for the intuitionistic calculus.*

Part of the above result of Tarski is that the set of open subsets of a topological space can be made into an algebra having the same properties as the intuitionistic propositional calculus (very much in the same manner that a field of subsets of a set has the *same* properties as the classical two-valued propositional calculus).

The Tarskian interpretation of the intuitionistic propositional calculus was extended to the intuitionistic functional calculus[2], and for a fairly comprehensive development of Tarski's interpretation—up to the early 1960's—there is the monograph of H. Rasiowa and R. Sikorski[3]. After the advent of sheaves and toposes, there has been an exponential explosion on the subject. One by-product is that the lattice of open subsets of a topological space is now recognized as an example of a *complete Heyting algebra*, **cHa**; in fact nowadays it is not uncommon to express results about intuitionistic logic in terms of **cHa** (or in terms of the categorical dual: **locales**).

## 2. Beth Semantics

Since the rules of inference for the New Protothetic are motivated by a constructive interpretation of the logical atoms, it is natural that we should prefer to choose a *semantics* which can easily be expressed in an intuitionistic metatheory. The semantics that appears

[1]See [Tarski, 1938].
[2]See, for example, [Mostowski, 1948].
[3][Rasiowa and Sikorski, 1963].

most suitable for an intuitionistic development is *Beth Semantics*, specially as modified by W. Veldman, and H. de Swart[4].

### 3. Some intuitionistic concepts

*An* **infinitely proceeding sequence** *(abbreviated:* ips*) is a sequence (usually of natural numbers) that can be continued* ad infinitum.

*A* **binary infinitely proceeding sequence** *(abbreviated:* bips*) is an ips of* $0$*'s and* $1$*'s.*

*A* **spread** $\mathfrak{M}$ *is determined by two laws; the first, which is called the* spread-law, $\Lambda_{\mathfrak{M}}$*, regulates the choices of natural numbers, while the second or* complementary law, $\Delta_{\mathfrak{M}}$*, assigns a sequence of mathematical entities to any ips of natural numbers which is generated according to the first law.*

*The spread-law* $\Lambda_{\mathfrak{M}}$ *satisfies the following requirements:*

- *The spread-law is a rule which divides the finite sequences of natural numbers into* **admissible** *and* **inadmissible** *sequences*[5]*.*
- *It can be decided by* $\Lambda$ *for every natural number* $k$ *whether it is a one-member admissible sequence or not.*
- *Every admissible sequence* $(a_0, \dots, a_n, a_{n+1})$ *is an* immediate descendant *of the admissible sequence* $(a_0, \dots, a_n)$*.*
- *If an admissible sequence* $(a_0, \dots, a_{n-1})$ *is given,* $\Lambda$ *allows one to decide for every natural number* $k$ *whether* $(a_0, \dots, a_{n-1}, k)$ *is an admissible sequence or not.*
- *For any admissible sequence* $(a_0, \dots, a_{n-1})$ *at least one natural number* $k$ *can be found such that* $(a_0, \dots, a_{n-1}, k)$ *is an admissible sequence.*

*The complementary law* $\Delta_{\mathfrak{M}}$ *of a spread* $\mathfrak{M}$ *assigns a definite mathematical entity to any finite sequence which is admissible according to the spread-law* $\Lambda_{\mathfrak{M}}$*.*

*An ips* $\alpha = (a_0, a_1, \dots)$ *such that for each* $n$*,* $(a_0, \dots, a_n)$ *is admissible sequence according to the spread-law* $\Lambda_{\mathfrak{M}}$ *is called an* **admissible ips** *of* $\mathfrak{M}$*, or simply* an element of $\mathfrak{M}$ *and denoted:* $\alpha \in \mathfrak{M}$*.*

*A spread* $\mathfrak{M}$ *is* **finitary** *(***fan***) if the spread-law* $\Lambda_{\mathfrak{M}}$ *is such that only a finite number of one-member sequences are admissible and such that every admissible sequence has only a finite number of immediate descendants.*

*A* **Cantor fan** *is a fan such that all finite sequences of* $0$*'s and* $1$*'s are admissible and only such sequences are admissible.*

*The* **functional formulas** *are those formulas of the form*

$$F(\mathcal{A}_0, \dots, \mathcal{A}_{n-1}),$$

*where* $F$ *is a functional parameter.*

[4]See [Swart, 1977] and [Veldman, 1976].

[5]We shall assume that the empty sequence is always admissible.

*The* **operator formulas** *are those formulas of the form*

$$\mathbb{F}(\mathcal{A}_0, \ldots, \mathcal{A}_{n-1}),$$

*where* $\mathbb{F}$ *is a defined operator*[6].

*The* **prime formulas** *of the New Protothetic*[7] *consist of the propositional parameters, the functional formulas and the operator formulas.*

Our basic *semantical construct* is specified as follows:

*A* **Beth structure** *is a spread* $\mathfrak{B}$ *such that the complementary law* $\Delta_{\mathfrak{B}}$ *assigns to each* admissible sequence $\vec{n}$, *also called a* **node** *of* $\mathfrak{B}$, *a finite set of prime formulas. Furthermore if* $\vec{n}$ *is an initial segment of* $\vec{m}$, *in symbols:* $\vec{n} \preceq \vec{m}$, *then* $\Delta_{\mathfrak{B}}(\vec{n}) \subseteq \Delta_{\mathfrak{B}}(\vec{m})$ *(we say then that the complementary law is* monotonic*).*

If $\alpha$ is an admissible ips of some predetermined spread, then we let

$$\tilde{\alpha}(n) = (\alpha(0), \ldots, \alpha(n-1)).$$

Furthermore, when there is no risk of confusion we will write $\tilde{\alpha}n$ instead of $\tilde{\alpha}(n)$; analogously for $\alpha n$.

If $\vec{n}$ is a node such that for some $t$: $\vec{n} = \tilde{\alpha}t$, then we say that the ips $\alpha$ **passes through the node** $\vec{n}$ and write

$$\vec{n} \in \alpha.$$

Using the syntactical complexity of the formulas one can prove the following proposition:

Proposition 5.1. *There is a relation* $\Vdash$, *which holds between Beth structures* $\mathfrak{B}$, *nodes* $\vec{n}$ *of the structure and formulas* $\mathcal{A}$ *of the New Protothetic such that* $\Vdash_{\mathfrak{B},\vec{n}}\mathcal{A}$[8] *iff:*

- $\mathcal{A}$ *is a prime formula and*

$$\forall \alpha_{\vec{n} \in \alpha} \exists t[\ \mathcal{A} \in \Delta_{\mathfrak{B}}(\tilde{\alpha}t)\ ].$$

- $\mathcal{A} = (\mathcal{B} \equiv \mathcal{C})$ *and*

$$\forall \vec{m}_{\vec{n} \preceq \vec{m}}[\ \Vdash_{\mathfrak{B},\vec{m}}\mathcal{B} \ \ \textit{iff} \ \ \Vdash_{\mathfrak{B},\vec{m}}\mathcal{C}\ ].$$

- $\mathcal{A} = \bigwedge x \mathcal{B}\ulcorner x \urcorner$ *and for all propositional parameters* $p$*:*

$$\Vdash_{\mathfrak{B},\vec{n}}\mathcal{B}\ulcorner p \urcorner.$$

- $\mathcal{A} = \bigwedge f \mathcal{B}\ulcorner f \urcorner$ *and for all functionals*[9] $X$*:*

$$\Vdash_{\mathfrak{B},\vec{n}}\mathcal{B}\ulcorner X \urcorner.$$

We read $\Vdash_{\mathfrak{B},\vec{n}}\mathcal{A}$ (and often abbreviate it by $\Vdash_{\vec{n}}\mathcal{A}$) as *the structure* $\mathfrak{B}$ *forces (or, satisfies or justifies) the formula* $\mathcal{A}$ *at the node* $\vec{n}$.

Since the complementary law of a Beth structure is monotonic, a simple induction on the syntactical complexity of the formula gives:

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[6]We also include the 0-ary operators.
[7]Strictly speaking, of an *unfolding* determined by a definitional scheme.
[8]Abbreviation for $(\mathfrak{B}, \vec{n}, \mathcal{A}) \in \Vdash$.
[9]That is, either functional parameters or defined operators of the appropriate arity.

LEMMA 5.1. *In any given Beth structure and formula $\mathcal{A}$:*

$$\text{If } \Vdash_{\vec{n}}\mathcal{A} \text{ and } \vec{n} \preceq \vec{m} \text{ then } \Vdash_{\vec{m}}\mathcal{A}.$$

The following explains, we believe, the underlying motivation of the Beth structures:

PROPOSITION 5.2. *Given a Beth structure $\mathfrak{B}$, for all formulas $\mathcal{A}$ and nodes $\vec{n}$, the following conditions are equivalent:* (a) $\Vdash_{\mathfrak{B},\vec{n}}\mathcal{A}$. (b) $\forall\alpha_{\vec{n}\in\alpha}\exists t[\ \Vdash_{\mathfrak{B},\tilde{\alpha}t}\mathcal{A}\ ]$.

*Proof.* First of all observe that the implication (a)$\Rightarrow$(b) is immediate. We show the converse by an induction on the syntactical complexity of the formula $\mathcal{A}$.

For prime formulas the result is fairly straightforward. Let us next consider the case of an equivalence formula. Hence assume that

$$(*) \qquad \forall\alpha_{\vec{n}\in\alpha}\exists t[\ \Vdash_{\tilde{\alpha}t}(\mathcal{B}\equiv\mathcal{C})\ ].$$

Now suppose that: $\vec{n} \preceq \vec{m}$ and $\Vdash_{\vec{m}}\mathcal{B}$. Then every admissible $\beta$ such that $\vec{m} \in \beta$ also satisfies $\vec{n} \in \beta$. Now using $(*)$ and Lemma 5.1 we obtain that

$$\forall\beta_{\vec{m}\in\beta}\exists u[\ \Vdash_{\tilde{\beta}u}\mathcal{C}\ ].$$

And then the induction hypothesis gives us $\Vdash_{\vec{m}}\mathcal{C}$. Interchanging $\mathcal{B}$ with $\mathcal{C}$ leads us to $\Vdash_{\vec{m}}(\mathcal{B}\equiv\mathcal{C})$.

The remaining cases are similar (in fact even simpler). ∎

Forcing with respect to *ips* is specified as follows:

*Given a Beth structure $\mathfrak{B}$, an admissible ips $\alpha$ and formula $\mathcal{A}$, we say that $\mathcal{A}$* **is forced by the ips** $\alpha$ **in** $\mathfrak{B}$, $\Vdash_{\mathfrak{B},\alpha}\mathcal{A}$ *(or simply: $\Vdash_{\alpha}\mathcal{A}$) iff*

$$\exists k\forall\beta_{\tilde{\alpha}k\ \in\ \beta}\exists t[\ \Vdash_{\mathfrak{B},\tilde{\beta}t}\mathcal{A}\ ].$$

Because of Proposition 5.2 we see that the ips $\alpha$ forces the formula $\mathcal{A}$ in the structure $\mathfrak{B}$ if and only if

$$\exists k[\ \Vdash_{\mathfrak{B},\tilde{\alpha}k}\mathcal{A}\ ].$$

In order to make a Beth structure a *Beth model* for the unfolding of the New Protothetic we must impose further conditions in order to deal with the impredicative nature of the New Protothetic as well as with the Leśniewskian definitions:

*A Beth structure $\mathfrak{B}$ is a* **Beth model** *of (an unfolding of) the New Protothetic iff*

(1) *For all its Leśniewskian definitions $\mathcal{D}$ and all nodes $\vec{n}$ of the Beth structure:*

$$\Vdash_{\vec{n}}\mathcal{D}.$$

(2) *For all nodes $\vec{n}$ and for all formulas $\bigwedge x\mathcal{A}\ulcorner x\urcorner$ the following are equivalent:*

- $\Vdash_{\mathfrak{B},\vec{n}}\bigwedge x\mathcal{A}\ulcorner x\urcorner$
- *for* **all** *formulas $\mathcal{B}$:* $\Vdash_{\mathfrak{B},\vec{n}}\mathcal{A}\ulcorner\mathcal{B}\urcorner$.

In order to eliminate the explicit reference to the nodes of a Beth model we introduce:

*A Beth model $\mathfrak{B}$* **validates** *or* **justifies** *or* **forces** *a formula $\mathcal{A}$ iff the formula is forced at all the nodes of the structure. We express it in symbols:*

$$\Vdash_{\mathfrak{B}}\mathcal{A}.$$

The **soundness** of the New Protothetic is that every provable formula (derivable from the empty set of assumptions) is validated in every Beth model. As expected, in a Natural Deduction System one must also handle undischarged assumptions; hence the following:

*A Beth model* $\mathfrak{B}$ **justifies that** $\mathcal{A}$ **is a** $\mathfrak{B}$ **semantical consequence** *of the finite set* $\Gamma$ *of formulas at the node* $\vec{n}$*, in symbols:*

$$\Gamma \Vdash_{\mathfrak{B},\vec{n}} \mathcal{A},$$

*iff for all nodes* $\vec{m}$ *after* $\vec{n}$*, that is,* $\vec{n} \preceq \vec{m}$*:*

$$\textit{if} \quad \forall \mathcal{B}_{\mathcal{B} \in \Gamma} [\, \Vdash_{\mathfrak{B},\vec{m}} \mathcal{B}\, ] \quad \textit{then} \quad \Vdash_{\mathfrak{B},\vec{m}} \mathcal{A}.$$

We then say that $\mathcal{A}$ **is a Beth consequence of** $\Gamma$ *iff for all Beth models* $\mathfrak{B}$ *and all nodes* $\vec{n}$*:* $\quad \Gamma \Vdash_{\mathfrak{B},\vec{n}} \mathcal{A}$.

Using induction on the length of the derivation one obtains:

Lemma 5.2 (Soundness w.r.t. Beth models). *If* $\mathcal{A}$ *is derivable from* $\Gamma$ *in the New Protothetic, then* $\mathcal{A}$ *is a Beth consequence of* $\Gamma$*.*

CHAPTER 6

# Completeness of the New Protothetic

## 1. A universal Beth model

Let $\Gamma$ be a finite set of formulas of the New Protothetic[1]. We will show how to construct a Beth $\mathfrak{B}_\Gamma$ such that for all formulas $\mathcal{A}$:

$$\Gamma \vdash \mathcal{A} \quad \text{iff} \quad \Vdash_{\mathfrak{B}_\Gamma} \mathcal{A}.$$

We call such a Beth model a **universal Beth model** (for the set $\Gamma$).

In addition to being dependent on the (finite) set $\Gamma$, the construction also depends on:

- *An effective enumeration* $\mathcal{F}_0, \mathcal{F}_1, \ldots$ *of all the formulas of the unfolding.*
- *An effective enumeration* $\Pi_0, \Pi_1, \ldots$ *of all the derivations in the unfolding.*
- *A polynomial function* $\varpi$ *function mapping* $\mathbb{N}\times\mathbb{N}$ *one-one and onto* $\mathbb{N}$ *such that* $\varpi(m, n) \geq \max(m, n)$.

Since the universal Beth model is to be a Cantor fan, all finite sequences of 0's and 1's are admissible by the spread-law. Thus the crucial step is the construction of the complementary law $\Delta$[2]. That is, for each finite sequence $\vec{b}$, $\Delta(\vec{b})$ is to be a finite set of *prime formulas*. We shall first define a function $T$ that gives, for each $\vec{b}$, a finite set of formulas $T(\vec{b})$.

**Inductive definition of $T_\Gamma$.**

**Basis step.** $T(()) = \Gamma$.

**Inductive step.** Let $\vec{b}$ be a binary sequence and assume that $T(\vec{b})$ has been defined. Then determine natural numbers $s, l$ such that

$$\operatorname{length}(\vec{b}) = i = \varpi(s, l).$$

We proceed by cases.

**Case 1:** For some $j \leq i$, $\Pi_j$ is a derivation of $\mathcal{F}_s$ from $T(\vec{b})$. Then we set[3]

$$T(\vec{b}\,0) = T(\vec{b}) \cup \{\mathcal{F}_s\}, \quad T(\vec{b}\,1) = T(\vec{b}) \cup \{\mathcal{F}_s\}.$$

**Case 2:** For all $j \leq i$, $\Pi_j$ is not a derivation of $\mathcal{F}_s$ from $T(\vec{b})$. Then we set

$$T(\vec{b}\,0) = T(\vec{b}), \quad T(\vec{b}\,1) = T(\vec{b}) \cup \{\mathcal{F}_s\}.$$

**End of definition of $T_\Gamma$.**

[1]Here is another instance where by New Protothetic we understand any given unfolding.
[2]We often omit the subscript $\Gamma$.
[3]We express concatenation by juxtaposition.

**The complementary law $\Delta_\Gamma$.** For each finite binary sequence $\vec{b}$ we set

$$\Delta_\Gamma(\vec{b}) = \{\mathcal{P} \mid \mathcal{P} \in T_\Gamma(\vec{b}) \text{ and } \mathcal{P} \text{ is a prime formula}\}.$$

**End of definition of $\Delta_\Gamma$.**

## 2. The Universal Beth Structure $\mathfrak{B}_\Gamma$

The Universal Beth Structure $\mathfrak{B}_\Gamma$ has as its *fan law* the law generating all the *bips* and $\Delta_\Gamma$ as its complementary law.

## 3. Properties of the Universal Beth Structure $\mathfrak{B}_\Gamma$

It is useful to mentally collect all the formulas produced by $T$ along a bips. Thus we set for an arbitrary bips $\alpha$:

$$T_\alpha = \bigcup_k T(\tilde{\alpha}k).$$

First of all note that the construction of $T$ gives

$$\text{if} \quad T(\vec{b}) \vdash \mathcal{A} \quad \text{then} \quad \forall\alpha_{\vec{b}\in\alpha}\exists t[\ \mathcal{A} \ \in \ T(\tilde{\alpha}t)\ ].$$

In order to prove the converse we need the following:

PROPOSITION 6.1. *To every node $\vec{b}$ there corresponds a subfan $\mathfrak{S}$ such that*

(1) $\forall\beta_{\beta\in\mathfrak{S}}[\ \vec{b} \ \in \ \beta\ ]$.
(2) *For all formulas $\mathcal{A}$:*

$$\forall\beta_{\beta\in\mathfrak{S}}\exists t[\mathcal{A} \in T(\tilde{\beta}t)] \quad \textit{iff} \quad T(\vec{b}) \ \vdash \ \mathcal{A}.$$

*Proof.* Let $k$ be the length of $\vec{b}$. Then the subfan $\mathfrak{S}$ consist of all the bips $\beta$ such that:

- $\tilde{\beta}k = \vec{b}$.
- For all $i \ \geq \ k$: $\beta(i) = 0$.

In order to show that the subfan $\mathfrak{S}$ has the required properties we proceed as follows.

*Part 1.* Assume that $\mathcal{A}$ is a formula such that

$$T(\vec{b}) \ \vdash \ \mathcal{A}.$$

Then determine a natural number $s$ such that $\mathcal{F}_s = \mathcal{A}$ and then a natural number $l$ so that $\Pi_l$ justifies the statement $T(\vec{b}) \ \vdash \ \mathcal{F}_s$. Then let

$$t = \varpi(s, l).$$

Then from the definition of $\mathfrak{S}$ we conclude that $\forall\beta_{\beta\in\mathfrak{S}}[\ \mathcal{A} \ \in \ T(\tilde{\beta}(t+1))\ ]$.

*Part 2.* Assume this time that

$$\forall\beta_{\beta\in\mathfrak{S}}\exists t[\ \mathcal{A} \in T(\tilde{\beta}t)\ ].$$

Then by applying the fan theorem (see [Swart, 1977]) we obtain a $t_0$ such that

$$\forall\beta_{\beta\in\mathfrak{S}}[\ \mathcal{A} \ \in \ T(\tilde{\beta}t_0)\ ].$$

If $t_0 \leq k$, then the result is immediate. Thus let us assume that $k < t_0$. We then use induction on $r \leq (t_0 - k)$ to show that

$$T(\tilde{\beta}(t_0 - r)) \ \vdash \ \mathcal{A}.$$

For $r = 0$ this is immediate. For the inductive steps we use the definitions of $T$ and $\mathfrak{S}$[4]. ■

Now we are ready to prove that:

PROPOSITION 6.2. *For any formula $\mathcal{A}$ and node $\vec{b}$ we have the equivalence of:*

(a) $T(\vec{b}) \ \vdash \ \mathcal{A}$.
(b) $\forall\alpha_{\vec{b}\in\alpha}\exists t[\ \mathcal{A} \ \in \ T(\tilde{\alpha}t)\ ]$.

*Proof.* (b)⇒(a) is an immediate consequence of the previous proposition; the converse is trivial. ■

Making use of the complementary law $\Delta_\Gamma$ we obtain

COROLLARY. *For* **prime formulas** *$\mathcal{P}$ and nodes $\vec{b}$ the following are equivalent:*

(1) $T(\vec{b}) \ \vdash \ \mathcal{P}$.
(2) $\Vdash_{\mathfrak{B}_\Gamma,\vec{b}}\mathcal{P}$.

Eventually we shall prove that the above corollary applies to *all formulas* and not just to the prime formulas.

PROPOSITION 6.3. *To every formula $\mathcal{A}$ and node $\vec{b}$ there corresponds a subfan $\mathfrak{F}$ and a node $\vec{c}$ such that:*

(1) $\vec{b} \ \preceq \ \vec{c}$.
(2) $\forall\beta_{\beta\in\mathfrak{F}}[\ \vec{c}\in\beta \quad \textit{and} \quad \mathcal{A}\in T(\vec{c})\ ]$.
(3) *For all formulas $\mathcal{B}$ and bips $\beta\in\mathfrak{F}$:*

$$\mathcal{B} \ \in \ T_\beta \quad \textit{iff} \quad T(\vec{b}),\mathcal{A} \ \vdash \ \mathcal{B}.$$

*Proof* Let $k$ be the length of $\vec{b}$. Then determine a natural number $s_0$ such that $\mathcal{A} = \mathcal{F}_{s_0}$, a natural number $l$ such that $k \ \leq \ \varpi(s_0, l)$. Then let $t = \varpi(s_0, l)$ and $\vec{c}$ the extension of $\vec{b}$ by 0's so as to be of length $t$. The subfan $\mathfrak{F}$ consists of all those bips $\beta$ such that:

. $\tilde{\beta}k = \vec{b}$.
. For all $i$ such that $k \ \leq \ i \ < t$: $\beta(i) = 0$.
. $\beta(t) = 1$.
. For all $i > t$: $\beta(i) = 0$.

Then proceeding as in Proposition 6.1 one shows that the subfan has the required properties. ■

PROPOSITION 6.4. *For every pair of formulas $\mathcal{A}$, $\mathcal{B}$ and node $\vec{b}$ the following conditions are equivalent:*

(1) $T(\vec{b}),\mathcal{A} \ \vdash \mathcal{B}$.
(2) $\forall\vec{c}_{\vec{b}\preceq\vec{c}}[\ T(\vec{c})\vdash\mathcal{A} \ \rightarrow \ T(\vec{c})\vdash\mathcal{B}\ ]$.

[4]Observe that a formula is added to a $\mathfrak{S}$ path only when there is a derivation at hand from earlier entries in the path.

*Proof.* (1)⇒(2) is immediate. Thus assume (2). Let the subfan $\mathfrak{F}$ and the nodes $\vec{b} \preceq \vec{c}$ be as in the previous proposition. Then since $\mathcal{A} \in T(\vec{c})$ we immediately see that $T(\vec{c}) \vdash \mathcal{A}$. From (2) we then obtain $T(\vec{c}) \vdash \mathcal{B}$, and then from the properties of $\mathfrak{F}$ we obtain (1). ∎

COROLLARY. *For any pair of formulas $\mathcal{A}, \mathcal{B}$ and node $\vec{b}$, the following conditions are equivalent:*

(1) $T(\vec{b}) \vdash (\mathcal{A} \equiv \mathcal{B})$.

(2) $\forall \vec{c}_{\vec{b} \preceq \vec{c}}[\, T(\vec{c}) \vdash \mathcal{A}$ *iff* $T(\vec{c}) \vdash \mathcal{B}\,]$.

Next we consider the quantifiers; here the essential point is that since each of the $T(\vec{b})$ is a finite set of formulas there are always parameters which do not occur in any of the formulas in $T(\vec{b})$.

LEMMA 6.1. *For every formula $\bigwedge x \mathcal{B}\ulcorner x \urcorner$ and node $\vec{b}$ the following are equivalent:*

(1) $T(\vec{b}) \vdash \bigwedge x \mathcal{B}\ulcorner x \urcorner$.

(2) *For all formulas $\mathcal{C}$: $T(\vec{b}) \vdash \mathcal{B}\ulcorner \mathcal{C} \urcorner$.*

LEMMA 6.2. *For every formula $\bigwedge f \mathcal{B}\ulcorner f \urcorner$ and node $\vec{b}$ the following are equivalent:*

(1) $T(\vec{b}) \vdash \bigwedge f \mathcal{B}\ulcorner f \urcorner$.

(2) *For all functionals $F$: $T(\vec{b}) \vdash \mathcal{B}\ulcorner F \urcorner$.*

## 4. Completeness of Beth Semantics

Combining the above results and using an induction on the syntactical complexity of a formula $\mathcal{A}$ we obtain:

PROPOSITION 6.5. *For any formula $\mathcal{A}$ and node $\vec{b}$, the following conditions are equivalent:*

(1) $T(\vec{b}) \vdash \mathcal{A}$.

(2) $\Vdash_{\mathfrak{B}_\Gamma, \vec{b}} \mathcal{A}$.

*Proof.* *Case 1:* $\mathcal{A}$ is a prime formula. Then the result follows from the Corollary to Proposition 6.2.

*Case 2:* $\mathcal{A} = (\mathcal{B} \equiv \mathcal{C})$.
(2a) $T(\vec{b}) \vdash \mathcal{B} \equiv \mathcal{C}$. Then by the Corollary to Proposition 6.4 we obtain

$$\forall \vec{c}_{\vec{b} \preceq \vec{c}}[T(\vec{c}) \vdash \mathcal{B} \quad \text{iff} \quad T(\vec{c}) \vdash \mathcal{C}\,].$$

Using the induction hypothesis:

$$\forall \vec{c}_{\vec{b} \preceq \vec{c}}[\Vdash_{\mathfrak{B}_\Gamma, \vec{c}} \mathcal{B} \quad \text{iff} \quad \Vdash_{\mathfrak{B}_\Gamma, \vec{c}} \mathcal{C}\,].$$

Thus $\Vdash_{\mathfrak{B}_\Gamma, \vec{b}} (\mathcal{B} \equiv \mathcal{C})$.
(2b) $\Vdash_{\mathfrak{B}_\Gamma, \vec{b}} (\mathcal{B} \equiv \mathcal{C})$. Then reversing the previous argument we obtain $T(\vec{b}) \vdash (\mathcal{B} \equiv \mathcal{C})$.

*Case 3:* $\mathcal{A} = \bigwedge x \mathcal{B}\ulcorner x \urcorner$.
(3a) $T(\vec{b}) \vdash \bigwedge x \mathcal{B}\ulcorner x \urcorner$. Then for all propositional parameters $p$: $T(\vec{b}) \vdash \mathcal{B}\ulcorner p \urcorner$. Then the induction hypothesis gives us that for all propositional parameters $p$:

$$\Vdash_{\mathfrak{B}_\Gamma, \vec{b}} \mathcal{B}\ulcorner p \urcorner,$$

and thus $\Vdash_{\mathfrak{B}_\Gamma, \vec{b}} \bigwedge x \mathcal{B}\ulcorner x \urcorner$.

(3b) $\Vdash_{\mathfrak{B}_\Gamma,\vec{b}} \bigwedge x\mathcal{B}\ulcorner x\urcorner$. Then for some propositional parameter $q$, which does not occur in $T(\vec{b})$,

$$\Vdash_{\mathfrak{B}_\Gamma,\vec{b}} \mathcal{B}\ulcorner q\urcorner.$$

But then the induction hypothesis gives $T(\vec{b}) \vdash \mathcal{B}\ulcorner q\urcorner$. Then using the assumption that $q$ does not occur in $T(\vec{b})$: $T(\vec{b}) \vdash \bigwedge x\mathcal{B}\ulcorner x\urcorner$.

*Case 4:* $\mathcal{A} = \bigwedge f\mathcal{B}\ulcorner f\urcorner$. Analogous to Case 3. ∎

Then using the fact that $T(()) = \Gamma$ one obtains

$$\Gamma \vdash \mathcal{A} \quad \text{iff} \quad \Vdash_{\mathfrak{B}_\Gamma} \mathcal{A}.$$

It still remains to be shown that the Universal Beth Structure is a Beth model.

PROPOSITION 6.6. *The Beth structure $\mathfrak{B}_\Gamma$ is a Beth model.*

*Proof.* That $\mathfrak{B}_\Gamma$ validates the Leśniewskian definitions is an immediate consequence of Proposition 6.5 and Lemma 4.10, which states that the Leśniewskian definitions are theses of the unfolding. The other requirement for a Beth structure to be a Beth model is given by Lemma 6.1. ∎

Now that we know that the Universal Beth Structure is also a Beth model we may conclude:

COROLLARY. *If the formula $\mathcal{A}$ is forced in all Beth models, then $\mathcal{A}$ is a theorem of the New Protothetic.*

## 5. A brief history of completeness

E. W. Beth was the first person who attempted to give an intuitionistic proof of the completeness of the Intuitionistic Predicate Calculus. Dyson and Kreisel[5] found an error in Beth's argument and then Kreisel[6] showed that completeness along the lines envisioned by Beth[7] yields Markov's principle for primitive recursive predicates. Heyting remarked that he believed that there could not be a proof of completeness for intuitionism. And even though Heyting meant complete with respect to all methods of proof, nevertheless his remark, combined with Kreisel's result, discouraged any further attempts at an intuitionistic completeness.

On the other hand the intuitionists at the Catholic University of Nijmegen, under the leadership of Professor Johann de Iongh, refused to accept that. In the 70's, Wim Veldman finally succeeded in giving an intuitionistic proof of the completeness of the IPC using a very simple generalization of the models of Beth. The generalization consisted in allowing the possibility that $\perp$ (The Absurd, The False, The Unsatisfiable) be satisfied at a node of a Beth model[8]. Classically, one could then cut out such nodes and obtain

[5][Dyson and Kreisel, 1961].

[6][Kreisel, 1962b].

[7]On closer analysis it was discovered that Kreisel's proof was about a species interpretation and not about Beth models.

[8]And then all the formulas would also be satisfied there.

one of the original models of Beth. However, from the intuitionistic viewpoint, it is the possible existence of such nodes that prevents the derivation of Markov's principle from the completeness property!

H. C. M. de Swart, also from Nijmegen, gave other intuitionistic proofs of the completeness. He also introduced the idea of defining the satisfaction with respect to the paths (bips) instead of the nodes. At about the same time, Lopez-Escobar and Veldman[9] obtained an intuitionistic completeness for a restricted second order logic.

The intuitionistic acceptable completeness for the New Protothetic given here is a combination of the above three proofs. What appears surprising about the completeness given for the New Protothetic is its stark simplicity[10] even though the New Protothetic can be shown to include the full impredicative second order intuitionistic propositional calculus!

The reason for such simplicity is not so much the sparsity of connectives as the role of absurdity (and thus also negation) in the New Protothetic.

In both Classical and Intuitionistic logics, *absurdity* is given the inferential property that every other formula (in the system) is obtainable from it; there is also an implied understanding that *absurdity* is never semantically satisfied. It would thus appear that there is a leap of faith between the inferential and semantical properties of absurdity.

The advantage of having universal quantification over propositions is that the sentence $\bigwedge xx$ has all the inferential properties ascribed to *absurdity* and thus it is natural to use the equivalence

$$(\perp \equiv \bigwedge xx).$$

Now in the sentence $\bigwedge xx$ there is nothing to associate it with the idea that something does **not** happen. It thus may be much more palatable to have "$\bigwedge xx$" satisfiable at a node than having "The Unsatisfiable" satisfiable.

[9] [López-Escobar and Veldman, 1974].

[10] After all the definition of the universal Beth model is barely half a page long!

CHAPTER 7

# Conjunction in the New Protothetic

## 1. 1923 revisited

In this chapter we re-establish, in the New Protothetic, Tarski's 1923 result for the Leśniewskian Protothetic. It may be of interest to observe that our proof uses neither the associativity of equivalence nor the concept of *Absurd* ($\perp$) or *Negation*.

Before we start on Tarski's result, we obtain some results that hold in the Minimal Protothetic (that is, without any defined operators):

PROPOSITION 7.1. $\vdash [\bigwedge z(p \equiv z) \equiv \bigwedge z(q \equiv z)]$.

*Proof.* We provide condensed derivations:

$$\begin{array}{rcl} \bigwedge z(p \equiv z), q & \Rightarrow & p \equiv q \\ & \Rightarrow & p \\ & \Rightarrow & p \equiv r \\ & \Rightarrow & r \end{array}$$

Thus

$$\bigwedge z(p \equiv z), q \vdash r.$$

Analogously

$$\bigwedge z(p \equiv z), r \vdash q.$$

Hence

$$\bigwedge z(p \equiv z) \vdash q \equiv r.$$

From which one obtains

$$\bigwedge z(p \equiv z) \vdash \bigwedge z(q \equiv z).$$

Interchanging $p$ and $q$, we obtain the missing steps to conclude

$$\vdash [\bigwedge z(p \equiv z) \equiv \bigwedge z(q \equiv z)]. \ \blacksquare$$

In a similar way we obtain, again in the Minimal Protothetic:

LEMMA 7.1.

(1) $p, q \vdash [\bigwedge z(p \equiv F(z)) \equiv \bigwedge z(q \equiv F(z))]$.
(2) $p, q \vdash p \equiv [\bigwedge z(p \equiv F(z)) \equiv \bigwedge z(q \equiv F(z))]$.
(3) $p, q \vdash \bigwedge f[\, p \equiv [\bigwedge z(p \equiv f(z)) \equiv \bigwedge z(q \equiv f(z))]\,]$.

Our proof, of the definability of conjunction in the New Protothetic, will be carried out in the unfolding of the Minimal Protothetic containing the following Leśniewskian definitions (where for aesthetical reasons we are not writing down the outermost propositional universal quantifiers):

$$
\begin{array}{rcl}
\top & \equiv & \bigwedge x(x \equiv x), \\
\mathbf{T}(x_0) & \equiv & \top, \\
\mathbf{I}(x_0) & \equiv & x_0, \\
\mathbf{K}(x_0, x_1) & \equiv & \bigwedge f[\, x_0 \equiv [\bigwedge z(x_0{\equiv}f(z)) \equiv \bigwedge z(x_1{\equiv}f(z))]\, ].
\end{array}
$$

Observe that each of the above definitions is in fact Tarskian. Hence applying the Corollary to Proposition 4.3, we may conclude that **K** is equivalence-invariant. We show that it is the connective of conjunction in the sense that it satisfies Gentzen's **I**- and **E**-rules.

THEOREM 7.2.

(1) $\mathcal{A}, \mathcal{B} \vdash \mathbf{K}(\mathcal{A}, \mathcal{B})$.
(2) $\mathbf{K}(\mathcal{A}, \mathcal{B}) \vdash \mathcal{A}$.
(3) $\mathbf{K}(\mathcal{A}, \mathcal{B}) \vdash \mathcal{B}$.

*Proof.* First of all observe that it suffices to prove the above when the formulas $\mathcal{A}$ and $\mathcal{B}$ are propositional parameters $p$ and $q$ respectively.

*Proof of* (1). This is essentially Lemma 7.1.

*Proof of* (2). $\mathbf{K}(p, q) \vdash p$. From Proposition 7.1 we obtain

$$\vdash \bigwedge z(p{\equiv}z) \equiv \bigwedge z(q{\equiv}z).$$

Then since

$$\vdash \mathbf{K}(p, q) \equiv \bigwedge f[\, p \equiv [\bigwedge z(p{\equiv}f(z)) \equiv \bigwedge z(q{\equiv}f(z))]\, ],$$

the proof is completed by instantiating the functional variable $f$ by the defined identity operator **I**.

*Proof of* (3). $\mathbf{K}(p, q) \vdash q$. This time one makes use of (2) and instantiates the functional variable $f$ by the defined constant operator **T**.

## 2. Creative definitions?

One of the criticisms leveled against Leśniewski's insistence that the definitions be part of the system is (the claim) that it is not necessary; after all (so they say) the definiendum could always be replaced by the definiens. Now although that may be the case in some systems, it is not the case in the (New) Protothetic! For example, it is not possible to show that

$$\mathbf{K}(p, q) \vdash q$$

in an unfolding of the Minimal Protothetic in which the only defined operator is **K**. Of course the detractors of Leśniewski might then argue that the operators **I** and **T** should also be included as primitives; but then where does one stop? How many intuitionistic connectives should be considered as primitive?

Thus it is fair to say that in the New Protothetic the Leśniewskian definitions do not just have a simplification role (even though they are excellent in that role), but that they also may have a creative role.

### 3. Traditional intuitionistic connectives

Let us now add some additional Tarskian definitions (where once again we omit writing the outermost universal quantifiers).

$$\begin{array}{rcl} \bot & \equiv & \bigwedge x x, \\ \mathbf{F}(x_0) & \equiv & \bot, \\ \mathbf{N}(x_0) & \equiv & (x_0 \equiv \bot), \\ \mathbf{C}(x_0, x_1) & \equiv & (x_0 \equiv \mathbf{K}(x_0, x_1)), \\ \mathbf{A}(x_0, x_1) & \equiv & \bigwedge z \mathbf{C}(\mathbf{K}(\mathbf{C}(x_0, z), \mathbf{C}(x_1, z)), z), \\ \mathbf{E}(x_0, x_1) & \equiv & (x_0 \equiv x_1), \\ \mathbf{DN}(x_0) & \equiv & \mathbf{N}(\mathbf{N}(x_0)). \end{array}$$

The reason for adding $\mathbf{E}$ is that equivalence becomes an operator and thus may be used in applications of the functional $\bigwedge$-$\mathbf{E}$-rule.

LEMMA 7.2. *The operators $\bot, \mathbf{N}, \mathbf{C}$ and $\mathbf{A}$ are truth-functional and satisfy, as derived rules, the traditional intuitionistic rules of inference*[1].

Then, proceeding as in the intuitionistic propositional calculus, one obtains the following correspondences between the connectives and the inference relation:

LEMMA 7.3.

(1) $\Gamma, \mathbf{K}(\mathcal{A}, \mathcal{B}) \vdash \mathcal{C}$ *iff* $\Gamma, \mathcal{A}, \mathcal{B} \vdash \mathcal{C}$.
(2) $\Gamma, \mathbf{K}(\mathcal{A}, \mathcal{B}) \vdash \mathcal{C}$ *iff* $\Gamma, \mathcal{A} \vdash \mathbf{C}(\mathcal{B}, \mathcal{C})$.
(3) $\Gamma, \mathbf{K}(\mathcal{A}, \mathcal{B}) \vdash \bot$ *iff* $\Gamma, \mathcal{A} \vdash \mathbf{N}(\mathcal{B})$.
(4) $\vdash \mathbf{A}(\mathcal{A}, \mathcal{B}) \equiv \mathbf{A}(\mathcal{B}, \mathcal{A})$.
(5) $\vdash \mathbf{C}(\mathcal{A}, \mathbf{A}(\mathcal{A}, \mathcal{B}))$.

### 4. The explicit disjunction property

Combining the above properties of the intuitionistic connectives with the Normalization Property of the New Protothetic (given in Appendix A) we shall prove the *explicit disjunction property*, that is, whenever a disjunction is a theorem of the New Protothetic, then at least one of the disjuncts is also a theorem. But first:

PROPOSITION 7.3. *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are formulas in which the propositional parameter $p$ does not occur and*

$$\mathcal{A}_1 \equiv \mathbf{K}(\mathcal{A}_1, p), \mathcal{A}_2 \equiv \mathbf{K}(\mathcal{A}_2, p) \vdash p,$$

*then for $i = 1$ or $i = 2$,*

$$\mathcal{A}_1 \equiv \mathbf{K}(\mathcal{A}_1, p), \mathcal{A}_2 \equiv \mathbf{K}(\mathcal{A}_2, p) \vdash \mathcal{A}_i.$$

*Proof.* Assume the hypothesis of the proposition; then according to the Normalization Property there is a normal derivation $\Pi$ justifying

$$\mathcal{A}_1 \equiv \mathbf{K}(\mathcal{A}_1, p), \mathcal{A}_2 \equiv \mathbf{K}(\mathcal{A}_2, p) \vdash p.$$

[1]For the traditional intuitionistic rules of inference see, for example, [Prawitz, 1965].

Clearly the end-rule must be an **E**-rule of inference. But since $\Pi$ is a normal derivation, the *main path* (see Appendix A) of $\Pi$ must consist entirely of **E**-rules. Since in the New Protothetic the only rule that discharges assumption formulas is the $\equiv$-Introduction rule, the formula at the top (beginning) of the main path of $\Pi$ has to be either $\mathcal{A}_1 \equiv \mathbf{K}(\mathcal{A}_1, p)$ or $\mathcal{A}_2 \equiv \mathbf{K}(\mathcal{A}_2, p)$. Let $i$ be such that the formula $\mathcal{A}_i \equiv \mathbf{K}(\mathcal{A}_i, p)$ occurs at the top of the main path.

Now the only **E**-rule that may be applied with $\mathcal{A}_i \equiv \mathbf{K}(\mathcal{A}_i, p)$ as the major premise is an $\equiv$-**E**-rule of inference and thus the minor premise has to be either $\mathcal{A}_i$ or $\mathbf{K}(\mathcal{A}_i, p)$.

In either case we obtain

$$\mathcal{A}_1 \equiv \mathbf{K}(\mathcal{A}_1, p), \mathcal{A}_2 \equiv \mathbf{K}(\mathcal{A}_2, p) \ \vdash \mathcal{A}_i,$$

as required. ∎

Theorem 7.4. *If the formula $\mathbf{A}(\mathcal{A}_1, \mathcal{A}_2)$ is a theorem of the New Protothetic, then either $\mathcal{A}_1$ or $\mathcal{A}_2$ is a theorem.*

*Proof.* Let $\Pi$ be a derivation of the formula $\mathbf{A}(\mathcal{A}_1, \mathcal{A}_2)$. Then using the Normalization Property once more, we may assume that $\Pi$ is a normal derivation. Since there are no undischarged assumption formulas, the end-rule of $\Pi$ must be the **A**-Introduction rule. Thus the formula

$$\bigwedge x \mathbf{C}(\mathbf{K}(\mathbf{C}(\mathcal{A}_1, x), \mathbf{C}(\mathcal{A}_2, x)), x),$$

is also a theorem (with a normal derivation). Repeating the argument we obtain a propositional parameter $p$, not occurring in $\mathcal{A}_1$ nor $\mathcal{A}_2$ such that

$$\mathbf{C}(\mathbf{K}(\mathbf{C}(\mathcal{A}_1, p), \mathbf{C}(\mathcal{A}_2, p)), p),$$

is a theorem. Continuing in this manner, and making use of the properties of the defined connectives one obtains

$$\mathbf{C}(\mathcal{A}_1, p), \mathbf{C}(\mathcal{A}_2, p) \ \vdash \ p.$$

Applying the previous lemma we conclude that for $i$ equal to either 1 or 2:

$$\mathbf{C}(\mathcal{A}_1, p), \mathbf{C}(\mathcal{A}_2, p) \ \vdash \ \mathcal{A}_i.$$

Substituting $\mathbf{A}(\mathcal{A}_1, \mathcal{A}_2)$ for the propositional parameter $p$ one obtains $\vdash \ \mathcal{A}_i$. ∎

## 5. Characterization of equivalence-invariance

Since we now have available the conditional connective we are able to characterize (up to equivalence) the propositional assertions which are equivalence-invariant. However we should first note:

Lemma 7.4. *If $\mathcal{B}\ulcorner\vec{x}\urcorner$ is an equivalence-invariant assertion and $\mathcal{A}\ulcorner\vec{x}\urcorner$ is such that*

$$\vdash \bigwedge \vec{x}(\ \mathcal{A}\ulcorner\vec{x}\urcorner \ \equiv \ \mathcal{B}\ulcorner\vec{x}\urcorner\ ),$$

*then $\mathcal{A}\ulcorner\vec{x}\urcorner$ is also equivalence-invariant.*

Theorem 7.5. *If $\mathcal{A}\ulcorner\vec{x}\ \urcorner$ is an equivalence-invariant propositional assertion[2] in which all the defined operators are equivalence-invariant, then there is a Tarskian assertion $\mathcal{B}\ulcorner\vec{x}\ \urcorner$*

[2]In which there are no occurrences of parameters.

*such that*

$$\bigwedge\vec{x}\ (\mathcal{A}\ulcorner\vec{x}\,\urcorner \equiv \mathcal{B}\ulcorner\vec{x}\,\urcorner)$$

*is a thesis of the New Protothetic.*

*Proof.* In order to emphasize the method of the proof, let us assume that $\vec{x}$ consists of just one propositional variable $x$. Also let us write $\mathcal{F} \supset \mathcal{G}$ instead of $\mathbf{C}(\mathcal{F}, \mathcal{G})$.

The hypothesis that $\mathcal{A}\ulcorner x\urcorner$ is equivalence-invariant tells us that

$$p{\equiv}q\ \vdash\ \mathcal{A}\ulcorner p\urcorner \equiv \mathcal{A}\ulcorner q\urcorner.$$

For $\mathcal{B}\ulcorner x\urcorner$ we may take the proto-formula $\bigwedge z[\ z{\equiv}x \supset \mathcal{A}\ulcorner z\urcorner\ ]$, where $z$ is the first propositional variable that does not occur in $\mathcal{A}\ulcorner x\urcorner$.

Now even if $x$ had been linked to a functional variable in $\mathcal{A}\ulcorner x\urcorner$ it is no longer linked to any functional variable in $\mathcal{B}\ulcorner x\urcorner$; thus it is a Tarskian assertion.

The inference $\mathcal{B}\ulcorner p\urcorner\ \vdash\ \mathcal{A}\ulcorner p\urcorner$ is a consequence of the $\bigwedge$-**E**-rule. The inference $\mathcal{A}\ulcorner p\urcorner\ \vdash$ $\mathcal{B}\ulcorner p\urcorner$ is a consequence of the assumption that $\mathcal{A}\ulcorner x\urcorner$ is equivalence-invariant. ∎

COROLLARY. *The propositional assertions that are equivalence-invariant are precisely those that are equivalent to Tarskian assertions.*

CHAPTER 8

# Definitions of quantifiers

## 1. Quantifiers in the New Protothetic

In the previous chapter it was shown that the New Protothetic is an extension of intuitionistic propositional calculus since all the traditional intuitionistic propositional connectives have Tarskian definitions. Since it is well known that the existential intuitionistic quantifier $\bigvee$ may expressed[1] using the universal quantifier $\bigwedge$ and the conditional $\supset$, it follows that the *Extended Intuitionistic Propositional Calculus* (also known as the *Second Order Intuitionistic Propositional Calculus*) may be *translated* into the New Protothetic.

In A. P. Morse's system (which had its roots in Leśniewski's Protothetic) it is possible to define all kinds of variable binding operators. We do not plan to achieve the generality of Morse's system, but if we are to claim that the New Protothetic *extends* the Second Order Intuitionistic Calculus and not just that the latter can be *translated* into it, then we must expand the Leśniewskian definitions of the New Protothetic so that the intuitionistic existential quantifier is definable (within the system).

For the time being we shall only be concerned with singularly quantifiers, that is, quantifiers which bind a single variable in a single formula.

*A proto-definition of a (singularly propositional) quantifier is an expression of the form*

$$\bigwedge x_1(\mathbb{Q}x_0x_1 \equiv \mathcal{Q}\ulcorner x_1\urcorner),$$

*where:*

- $\mathbb{Q}$ *is an identifier for quantifiers.*
- $\mathcal{Q}\ulcorner x_1\urcorner$ *is like a proto-formula except perhaps for identifiers for quantifiers.*
- $\mathbb{Q}$ *does not occur in* $\mathcal{Q}\ulcorner x_1\urcorner$*.*
- *There are no bound occurrences of the propositional variable* $x_1$ *in* $\mathcal{Q}\ulcorner x_1\urcorner$*.*
- *There are no occurrences of parameters in* $\mathcal{Q}\ulcorner x_1\urcorner$*.*
- *Only the variable* $x_1$ *may have a free occurrence in* $\mathcal{Q}\ulcorner x_1\urcorner$*.*
- *Every occurrence of* $x_1$ *in* $\mathcal{Q}\ulcorner x_1\urcorner$ <u>*must*</u> *be within the scope of a quantifier with indicial variable* $x_0$*.*

## 2. Syntax associated with defined quantifiers

Suppose that we are given a proto-definition for a quantifier identifier $\mathbb{Q}$. Then the class of proto-formulas is enlarged by adding the following:

[1]More specifically the formula $\bigvee x\mathcal{F}\ulcorner x\urcorner$ is intuitionistically equivalent to the formula $\bigwedge y[\,\bigwedge x(\mathcal{F}\ulcorner x\urcorner \supset y) \supset y\,]$.

- *If $\mathcal{F}$ is a proto-formula, then for each natural number $n$ the expression $\mathbb{Q}x_n\mathcal{F}$ is a proto-formula in which every occurrence of the variable $x_n$ is a* **bound occurrence***. Furthermore the displayed occurrence of $x_n$ is an* **indicial occurrence***.*
- *The expression $\mathbb{Q}x_n$ is known as a* **quantifier phrase** *and the proto-formula $\mathbb{Q}x_n\mathcal{F}$ as a $\mathbb{Q}x_n$-***quantification***.*

The Leśniewskian definitional schemes are henceforth expanded to allow definitions for quantifiers.

## 3. Rules of inference for quantifier phrases

Suppose that the identifier $\mathbb{Q}$ has the following definition:

$$\bigwedge x_1(\mathbb{Q}x_0x_1 \equiv \mathcal{Q}\ulcorner x_1\urcorner).$$

Then the natural pair of rules of inference for the quantifier phrase $\mathbb{Q}x_0$ are as follows:

**I-rule for $\mathbb{Q}x_0$:**

$$\frac{\mathcal{Q}\ulcorner x_1/\mathcal{P}\urcorner}{\mathbb{Q}x_0\mathcal{P}}$$

where $\mathcal{P}$ is a proto-formula whose only free variable may be $x_0$.

**E-rule for $\mathbb{Q}x_0$:**

$$\frac{\mathbb{Q}x_0\mathcal{P}}{\mathcal{Q}\ulcorner x_1/\mathcal{P}\urcorner}$$

where $\mathcal{P}$ is a proto-formula whose only free variable may be $x_0$.

In order to give the inference rules corresponding to the quantifier phrase $\mathbb{Q}x_n$, with $n > 0$, we need to apply a *shift* to the propositional variables occurring in a proto-formula.

*If $\mathcal{F}$ is a proto-formula and $n$ a natural number, then the $n$-***shift of** *$\mathcal{F}$, in symbols: $\mathsf{S}_n(\mathcal{F})$, is the proto-formula obtained by replacing* every occurrence *of a* propositional variable *by its $n$-th translate, that is, $x_i$ is replaced by $x_{i+n}$.*

Now consider a quantifier definition, say

$$\mathfrak{D} = \bigwedge x_1[\mathbb{Q}x_0x_1 \equiv \mathcal{Q}].$$

Then $\mathsf{S}_n(\mathfrak{D})$ is the sentence

$$\bigwedge x_{1+n}[\mathbb{Q}x_nx_{1+n} \equiv \mathsf{S}_n(\mathcal{Q})]$$

where in $\mathsf{S}_n(\mathcal{Q})$ the only variable that may have a free occurrence is $x_{1+n}$. This suggests that for the rules of inference for the quantifier phrase $\mathbb{Q}x_n$ we should use:

**I-rule for $\mathbb{Q}x_n$:**

$$\frac{[x_{1+n}{=}\mathcal{P}]\mathsf{S}_n(\mathcal{Q})}{\mathbb{Q}x_n\mathcal{P}}$$

where the only variable that may occur free in the proto-formula $\mathcal{P}$ is $x_n$.

**E-rule for $\mathbb{Q}x_n$:**

$$\frac{\mathbb{Q}x_n\mathcal{P}}{[x_{1+n}{=}\mathcal{P}]\mathsf{S}_n(\mathcal{Q})}$$

with the analogous proviso on the proto-formula $\mathcal{P}$.

By an unfolding of the Minimal Protothetic we shall now understand the system obtained from a *finite definitional scheme* in which there may be definitions for singularly quantifiers.

## 4. The existential quantifier

Let the quantifier identifier **QE** be set by the following definition:

$$\bigwedge x_1[\ \mathbf{QE}x_0x_1 \equiv \bigwedge x_2\mathbf{C}(\bigwedge x_0\mathbf{C}(x_1,x_2),x_2)\ ].$$

In order to follow tradition, we shall write $\bigvee$ instead of the more cumbersome **QE**.

LEMMA 8.1. $\bigvee$ *has the inferential properties of the* intuitionistic existential quantifier*, that is:*

(1) $\mathcal{A}\ulcorner\mathcal{B}\urcorner \vdash \bigvee x\mathcal{A}\ulcorner x\urcorner$.
(2) *If* $\Gamma_1 \vdash \bigvee x\mathcal{A}\ulcorner x\urcorner$ *and* $\Gamma_2, \mathcal{A}\ulcorner q\urcorner \vdash \mathcal{B}$*, where the eigen-parameter* $q$ *does not occur in* $\Gamma_2 \cup \{\mathcal{B}\}$ *then* $\Gamma_1, \Gamma_2 \vdash \mathcal{B}$.

## 5. The Explicit Definability Property

The Normalization Property of the New Protothetic allows us to prove the *Explicit Definability Property* for the existential quantifier $\bigvee$, that is, if an existential sentence $\bigvee x\mathcal{F}\ulcorner x\urcorner$ is a theorem of the New Protothetic, then there is a formula $\mathcal{A}$ such that $\mathcal{F}\ulcorner\mathcal{A}\urcorner$ is also a theorem.

And just as in the case of the Explicit Disjunction Property we start with

PROPOSITION 8.1. *If the propositional parameter* $q$ *does not occur in the formula* $\mathcal{F}\ulcorner p\urcorner$ *and*

$$(+)\qquad \bigwedge x(\mathcal{F}\ulcorner x\urcorner \equiv \mathbf{K}(\mathcal{F}\ulcorner x\urcorner, q)) \vdash q,$$

*then there is a formula*[2] $\mathcal{B}$ *such that*

$$\bigwedge x(\mathcal{F}\ulcorner x\urcorner \equiv \mathbf{K}(\mathcal{F}\ulcorner x\urcorner, q)) \vdash \mathcal{F}\ulcorner\mathcal{B}\urcorner.$$

*Proof.* Let $\Pi$ be a normal derivation justifying (+). Then the main path of $\Pi$ must consist of **E**-rule applications. Furthermore the top formula of the main path must be $\bigwedge x(\mathcal{F}\ulcorner x\urcorner \equiv \mathbf{K}(\mathcal{F}\ulcorner x\urcorner, q))$. The only applicable **E**-rule is then $\bigwedge$-**E**. Thus there is a formula $\mathcal{B}\ulcorner q\urcorner$, in which $q$ may possibly occur, such that the second formula in the main path is

$$(\mathcal{F}\ulcorner\mathcal{B}\urcorner \equiv \mathbf{K}(\mathcal{F}\ulcorner\mathcal{B}\urcorner, q)).$$

Since the end-formula of the main path is $q$ and the only way to break down the former formula is by an $\equiv$-**E** application, it follows that

$$\bigwedge x(\mathcal{F}\ulcorner x\urcorner \equiv \mathbf{K}(\mathcal{F}\ulcorner x\urcorner, q)) \vdash \mathcal{F}\ulcorner\mathcal{B}\urcorner,$$

as required. ■

Now we have all the necessary prerequisites to show that

[2] In which the propositional parameter $q$ may occur.

THEOREM 8.2 (Explicit Definability). *If the sentence $\bigvee x\mathcal{F}\ulcorner x\urcorner$ is a theorem of the New Protothetic, then for some formula $\mathcal{A}$:*

$$\vdash\ \mathcal{F}\ulcorner\mathcal{A}\urcorner.$$

*Proof.* From the hypothesis that $\bigvee x\mathcal{F}\ulcorner x\urcorner$ is a theorem one finds first that there is a normal derivation of it; then analyzing the normal derivation one obtains a *new* propositional parameter $q$ such that

$$\bigwedge x(\mathcal{F}\ulcorner x\urcorner \equiv \mathbf{K}(\mathcal{F}\ulcorner x\urcorner, q))\ \vdash\ q.$$

Applying the previous proposition we obtain a formula $\mathcal{B}\ulcorner q\urcorner$ such that

$$\bigwedge x\mathbf{C}(\mathcal{F}\ulcorner x\urcorner, q)\ \vdash\ \mathcal{F}\ulcorner\,\mathcal{B}\ulcorner q\urcorner\,\urcorner.$$

Then substituting the formula $\bigvee x\mathcal{F}\ulcorner x\urcorner$ for the propositional parameter $q$, one obtains

$$\vdash\ \mathcal{F}\ulcorner\,\mathcal{B}\ulcorner\bigvee x\mathcal{F}\ulcorner x\urcorner\,\urcorner\,\urcorner.$$

The proof of the theorem is completed by setting $\mathcal{A} = \mathcal{B}\ulcorner\bigvee x\mathcal{F}\ulcorner x\urcorner\,\urcorner$. ∎

**Remark on the Explicit Definability Property.** We proved the explicit disjunction and definability properties because it is traditionally a necessary (but not sufficient) requirement for a (formal) system to bear the intuitionistic imprimatur. However, for systems with propositional quantifiers the explicit definability property is not particularly useful as can be seen by considering the sentence that instantiates (according to the proof of the above theorem) the existential sentence $\bigvee xx$ using the following normal derivation (where we are using the infix $\supset$ instead of the prefix $\mathbf{C}$):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\bigwedge x(x\supset q)]}{q\supset q} \qquad \begin{array}{c}[\bigwedge x(x\supset q)]\\ \Psi\\ q\end{array}}{q}}{\bigwedge x(x\supset q)\supset q}}{\bigwedge y[\bigwedge x(x\supset y)\supset y]}}{\bigvee xx}$$

Following the argument one obtains that the formula that instantiates $\bigvee xx$ is $\bigvee xx$ itself!

## 6. The Extended Intuitionistic Propositional Calculus

Now that we have the propositional quantifier $\bigvee$ we may truly claim that the New Protothetic is indeed an extension of the Extended Intuitionistic Propositional Calculus, **EIPC**, (a.k.a. the Second Order Intuitionistic Propositional Calculus).

The **EIPC** is known to be a powerful and complex system[3] and thus so is the New Protothetic. Furthermore we will show that there are intuitionistic connectives which are definable in the New Protothetic but are not definable in the **EIPC**.

[3]See, for example, [Gabbay, 1981].

## 7. There is more to definitions than meets the eye

As the Socratic philosophers aptly argued, *definitions* are essential for any coherent discussion. What we have shown in the last few chapters, using the ideas of Beth, Gentzen, Leśniewski and Tarski, is that in a certain sense **definitions are all you need** to generate very powerful and complex logical theories. Thus A. N. Prior's criticisms, see Section 2 of Chapter 1, are not universally valid!

# Part 3

# Intuitionistic propositional connectives

CHAPTER 9

# The Intuitionistic Protothetic

**A home for the intuitionistic propositional connective.** In previous chapters we showed how using $\equiv$, $\bigwedge$ and Leśniewskian definitions we can obtain an unfolding of the Minimal Protothetic which contains the traditional intuitionistic propositional connectives and quantifiers of the Extended Intuitionistic Propositional Calculus, **EIPC**. The purpose of doing that was to show, in a very explicit way, that the Leśniewski/Tarski thesis that *definitions are a primordial concept for logics* is wellfounded not only in classical logic but also in constructive disciplines, such as Intuitionism.

Another of the purposes of introducing the New Protothetic was to provide an answer to Kreisel's question: *What is an intuitionistic propositional connective?*

Previous attempts to answer the question often came with a set of conditions which, in our opinion, were *ad hoc*, see for example Gabbay[1]. Our offer is extremely simple:

**Intuitionistic propositional connectives.** *The intuitionistic propositional connectives*[2] *are given by the truth-functional operators of the New Protothetic.*

We strongly believe that the previous chapters of the monograph are ample proof that the equivalence-invariant operators of the New Protothetic merit the label of *intuitionistic propositional connective.*

The converse is naturally more problematic. We shall restrict ourselves to showing that there are intuitionistic propositional connectives which are not definable in the Second Order Intuitionistic Propositional Calculus, but which are definable in the New Protothetic.

## 1. The Intuitionistic Protothetic

We could of course continue using the New Protothetic which has $\equiv$ as the sole *primitive* propositional connective. However, just as no one would insist that every discussion on computable functions has *always* to be carried in the language of Turing machines, we shall elevate the (definable) propositional connectives of *conjunction* and the *conditional* to the level of *primitive terms.*

We shall use infix notation for both of them; for conjunction we shall use $\wedge$, and for the conditional we shall use $\supset$.

What we understand by having *conjunction* and the *conditional* as *primitive terms* is that their rules of inference will not be obtained from the Tarskian definitions, but

[1] [Gabbay, 1981].

[2] Which do not depend on arithmetic for their formulation.

rather that they will be stipulated at the beginning, that is, in the corresponding Minimal Protothetic. Following Gentzen we add an **I-E** pair of rules of inference for each of them[3]. Note that both of the **E**-rules for $\supset$ and $\wedge$ have the property that they do indeed *eliminate* the main propositional connective and consequently the conclusion of an application of the rule is always of smaller logical complexity than that of the major premise (which would no longer be the case if we introduced the disjunction propositional connective $\vee$ as a primitive term).

*By the (an)* **Intuitionistic Protothetic, IP,** *we understand any unfolding of the Minimal Protothetic (with* $\equiv$*,* $\wedge$*,* $\supset$ *and* $\bigwedge$ *as primitives) which includes the following Tarskian definitions (where once again we omit to write down the outermost universal propositional quantifiers):*

$$
\begin{array}{rcl}
\top & \equiv & \bigwedge x(x{\equiv}x), \\
\bot & \equiv & \bigwedge x x, \\
\mathbf{I}(x) & \equiv & x, \\
\mathbf{T}(x) & \equiv & \top, \\
\mathbf{F}(x) & \equiv & \bot, \\
\mathbf{N}(x) & \equiv & (x{\equiv}\bot), \\
\mathbf{DN}(x) & \equiv & \mathbf{N}(\mathbf{N}(x)), \\
\mathbf{E}(x,y) & \equiv & (x{\equiv}y), \\
\mathbf{C}(x,y) & \equiv & (x{\supset}y), \\
\mathbf{K}(x,y) & \equiv & (x \wedge y), \\
\mathbf{A}(x,y) & \equiv & \bigwedge z[\ (x{\supset}z){\wedge}(y{\supset}z) \supset z\ ], \\
\bigvee xy & \equiv & \bigwedge z[\ \bigwedge x(y{\supset}z){\supset}z\ ].
\end{array}
$$

## 2. Intuitionistic truth-values

We have already introduced the two (0-ary) operators $\top$ and $\bot$ using as definiens the sentences $\bigwedge x(x{\equiv}x)$ and $\bigwedge xx$ respectively. In fact any sentence $\mathcal{S}$ may be used to define an operator:

$$(\mathbb{V}_{\mathcal{S}} \equiv \mathcal{S}).$$

Thus the operator $\mathbb{V}_{\mathcal{S}}$ can be considered as an **intuitionistic truth-value**. Now clearly there is no need to introduce the operator in order to have the truth-value; in order words, we may identify sentences[4] of the Intuitionistic Protothetic with truth-values. It is simple to verify that the set of truth-values forms a Heyting algebra (a.k.a. *pseudo Boolean Algebra*).

Although there are many ways to show that the Heyting Algebra of truth-values is denumerably infinite, one particular interesting way is to show that to any finite Heyting Algebra $\mathfrak{H}$ there corresponds a sentence $\mathcal{S}_{\mathfrak{H}}$ in the **IP** which *describes* the algebra $\mathfrak{H}$ and such that non-isomorphic (finite) algebras produce non-equivalent sentences (and thus different truth-values).

[3] See, for example, [Prawitz, 1965] or [Troelstra, 1973].

[4] Or more accurately, equivalence classes of sentences.

In the Intuitionistic Protothetic there are also sentences which are not equivalent to any of the sentences defining finite Heyting algebras; for example:

$$\mathcal{LS} = \bigwedge f \bigwedge x \bigwedge y(\ x{\equiv}y \supset f(x){\equiv}f(y)\ ).$$

The sentence (truth-value) $\mathcal{LS}$ is called by Tarski[5] the **Law of Substitution**.

OPEN PROBLEM 9.1. *Characterize the Heyting Algebra of the Intuitionistic Protothetic.*

We are not aware whether the Boolean algebra of Leśniewski's Protothetic has been characterized.

### 3. The Law of Development and other laws

Tarski shows that, in Leśniewski's Protothetic, the Law of Substitution (*i.e.* that all the monadic propositional functionals are equivalence-invariant) is equivalent to each of the following:

(1) $\bigwedge f \bigwedge x[\,f(x) \equiv (f(\top) \wedge x) \vee (f(\bot) \wedge \neg x)\,]$ *(The Law of Development)*[6].
(2) $\bigwedge f \bigwedge x[\,(f(\top) \wedge f(\bot)) \supset f(x) \supset (f(\top) \vee f(\bot))\,]$ *(The first theorem on the bounds of a function).*

Our principal result, along those lines, is that in **IP** the Law of Substitution is equivalent to either of (and hence both of)

(1) $\bigwedge f \bigwedge x[\,f(x) \equiv \bigwedge y(x \equiv y \supset f(y))\,]$.
(2) $\bigwedge f \bigwedge x[\,f(x) \equiv \bigvee y(x \equiv y \wedge f(y))\,]$.

We can prove a slightly stronger result, namely that the equivalence applies to individual propositional functions. For that purpose let $\mathcal{LS}_F$ be an abbreviation for

$$\bigwedge x \bigwedge y(\,x \equiv y \supset F(x) \equiv F(y)\,).$$

PROPOSITION 9.2. *$\mathcal{LS}_F$ is equivalent, in* **IP**, *to* $\bigwedge x[\,F(x) \equiv \bigwedge y(\,x \equiv y \supset F(y)\,)\,]$.

*Proof. Part 1.* Assume $\mathcal{LS}_F$. Then we obtain:

$$\begin{array}{rcl} F(p), p \equiv q & \vdash & F(q) \\ F(p) & \vdash & p \equiv q \supset F(q) \\ F(p) & \vdash & \bigwedge y(p \equiv y \supset F(y)) \end{array}$$

On the other hand

$$\begin{array}{rcl} \bigwedge y(p \equiv y \supset F(y)) & \Rightarrow & p \equiv p \supset F(p) \\ & \Rightarrow & F(p) \end{array}$$

Thus, under the stated assumption:

$$\begin{array}{rl} \vdash & F(p) \equiv \bigwedge y(p \equiv y \supset F(y)) \\ \vdash & \bigwedge x[\,F(x) \equiv \bigwedge y(x \equiv y \supset F(y))\,] \end{array}$$

[5] [Tarski, 1956].
[6] [Boole, 1854], [Schröder, 1890].

*Part 2.* Assume that $\bigwedge x[\,F(x) \equiv \bigwedge y(x \equiv y \supset F(y))\,]$. Now

$$\begin{aligned}\bigwedge x[\,F(x) \equiv \bigwedge y(x \equiv y \supset F(y))\,], F(p), p \equiv q \;&\Rightarrow\; F(p) \equiv \bigwedge y(p \equiv y \supset F(y))\\ &\Rightarrow\; \bigwedge y(p \equiv y \supset F(y))\\ &\Rightarrow\; p \equiv q \supset F(q)\\ &\Rightarrow\; F(q)\end{aligned}$$

Thus, under the given assumption,

$$F(p), p \equiv q \;\vdash\; F(q).$$

And similarly

$$F(q), p \equiv q \;\vdash\; F(p).$$

Hence

$$p \equiv q \;\vdash\; F(p) \equiv F(q),$$

which is but a small step from the desired conclusion. ■

In an analogous fashion one obtains:

PROPOSITION 9.3. *$\mathcal{LS}_F$ is equivalent, in* **IP**, *to* $\bigwedge x[\,F(x) \equiv \bigvee y(\,x \equiv y \wedge F(y)\,)\,]$.

In order to shorten some of the written statements we introduce the following abbreviations:

| | | |
|---|---|---|
| $F_\top$ | abbreviates | $F(\top)$. |
| $F_\bot$ | ... | $F(\bot)$. |
| $(\mathcal{A} \vee \mathcal{B})$ | ... | $\mathbf{A}(\mathcal{A}, \mathcal{B})$. |
| $\neg\mathcal{A}$ | ... | $\mathbf{N}(\mathcal{A})$ |
| $\neg\neg\mathcal{A}$ | ... | $\mathbf{DN}(\mathcal{A})$. |
| $\mathcal{A}\not\equiv\mathcal{B}$ | ... | $\mathbf{N}(\mathcal{A}\equiv\mathcal{B})$. |
| $\bigvee f \mathcal{A}\ulcorner f \urcorner$ | ... | $\bigwedge z(\bigwedge f(\mathcal{A}\ulcorner f\urcorner \supset z) \supset z)$. |

An application of the above two propositions is the following:

THEOREM 9.4 (Bounds on a truth-function). *The following is a consequence, in* **IP**, *of* $\mathcal{LS}_F$*:*

$$\bigwedge x[\,(x \wedge F_\top) \vee (\neg x \wedge F_\bot) \quad\supset\quad F(x) \quad\supset\quad (x \supset F_\top) \wedge (\neg x \supset F_\bot)\,].$$

*Proof.* First note that:

$$\begin{aligned}\mathcal{LS}_F, F(\top), p \;&\Rightarrow\; (p \equiv \top) \wedge F(\top)\\ &\Rightarrow\; \bigvee y(p \equiv y \wedge F(y))\\ &\Rightarrow\; F(p)\end{aligned}$$

Thus

$$p \wedge F_\top \;\vdash\; F(p).$$

Similarly

$$\neg p \wedge F_\bot \;\vdash\; F(p).$$

Hence

$$\vdash\; (p \wedge F_\top) \vee (\neg p \wedge F_\bot) \supset F(p).$$

In a similar way one proves:

$$
\begin{array}{rcl}
F(p) & \vdash & p \supset F(\top) \\
F(p) & \vdash & \neg p \supset F(\bot) \\
 & \vdash & F(p) \supset (p \supset F_\top) \wedge (\neg p \supset F_\bot) \; \blacksquare
\end{array}
$$

Next we recall the following result from the Intuitionistic Propositional Calculus, and hence also of **IP**:

LEMMA 9.1. $[(p \supset q) \wedge (\neg p \supset r)] \supset \neg\neg[(p \wedge q) \vee (\neg p \wedge r)]$.

Then combining Theorem 9.4 with Lemma 9.1 we obtain the **IP**-version of the Law of Development:

THEOREM 9.5 (IP-Development).[7]

$$
\mathcal{LS}_F \;\vdash\; \bigwedge x[\,(x \wedge F_\top) \vee (\neg x \wedge F_\bot) \;\supset\; F(x) \;\supset\; \neg\neg(\,(x \wedge F_\top) \vee (\neg x \wedge F_\bot)\,)\,].
$$

Observe that if the associativity of equivalence is derivable (as it is in Leśniewski's Protothetic) the double negation is equivalent to identity. Hence, in the presence of associativity, we would recapture Boole's[8] Law of Development.

Another reading of the **IP**-Law of Development is that the truth-functions of **IP** are bounded between a truth-table (called the **bounding truth-table**) and its double negation.

On the other hand, since it is not required of a truth-function $F$ that

$$
(*) \qquad (F(\top) \equiv \top \vee F(\top) \equiv \bot) \wedge (F(\bot) \equiv \top \vee F(\bot) \equiv \bot),
$$

the bounding truth-table would not necessarily be the familiar two-valued one.

However, just about all the propositional connectives considered in the literature have the property $(*)$ (or the corresponding $n$-ary version), thus we propose:

*A monadic*[9] *operator* $\mathbb{M}$ *is a* **pseudo-boolean monadic connective** *iff it satisfies (in* **IP***) the following:*

(1) $\vdash \bigwedge x \bigwedge y[\,x \equiv y \;\supset\; \mathbb{M}(x) \equiv \mathbb{M}(y)\,]$.
(2) $\vdash (\mathbb{M}_\top \equiv \top \vee \mathbb{M}_\top \equiv \bot)$.
(3) $\vdash (\mathbb{M}_\bot \equiv \top \vee \mathbb{M}_\bot \equiv \bot)$.

Note that because of the Explicit Disjunction Property, (1), (2) and (3) can be replaced by:

(1) $\mathcal{LS}_\mathbb{M}$.
(2) $\vdash \mathbb{M}_\top \equiv \top$ *or* $\vdash \mathbb{M}_\top \equiv \bot$.
(3) $\vdash \mathbb{M}_\bot \equiv \top$ *or* $\vdash \mathbb{M}_\bot \equiv \bot$.

---

[7] Corresponding results hold for $n$-ary propositional functions; the objection of writing them out is that there are $2^{2^n}$ disjuncts (on each side of the conditional).

[8] [Boole, 1854].

[9] Correspondingly for $n$-ary ones.

An immediate consequence of the above is that there are only 4 bounding truth-tables for the monadic pseudo-boolean connectives of **IP**.[10] In slightly more detail we have:

PROPOSITION 9.6. *If $\mathbb{M}$ is a monadic pseudo-boolean connective, then one of the following cases hold:*

**Case 1.** $\vdash \mathbb{M}_\top \equiv \top \ \wedge\ \vdash \mathbb{M}_\bot \equiv \top$.

$$\vdash \bigwedge x[\, x \vee \neg x \supset \mathbb{M}(x) \supset \top\,].$$

**Case 2.** $\vdash \mathbb{M}_\top \equiv \top \ \wedge\ \vdash \mathbb{M}_\bot \equiv \bot$.

$$\vdash \bigwedge x[\, x \supset \mathbb{M}(x) \supset \neg\neg x\,].$$

**Case 3.** $\vdash \mathbb{M}_\top \equiv \bot \ \wedge\ \vdash \mathbb{M}_\bot \equiv \top$.

$$\vdash \bigwedge x[\, \mathbb{M}(x) \equiv \neg x\,].$$

**Case 4.** $\vdash \mathbb{M}_\top \equiv \bot \ \wedge\ \vdash \mathbb{M}_\bot \equiv \bot$.

$$\vdash \bigwedge x[\, \mathbb{M}(x) \equiv \bot\,].$$

*Proof.* From the assumption that $\mathbb{M}$ is a monadic truth-function we obtain

$$\vdash (p \wedge \mathbb{M}_\top) \vee (\neg p \wedge \mathbb{M}_\bot) \supset \mathbb{M}(p) \supset \neg\neg[\,(p \wedge \mathbb{M}_\top) \vee (\neg p \wedge \mathbb{M}_\bot)\,]$$

The proof is completed using the assumption that $\mathbb{M}$ is pseudo-boolean. ∎

REMARK. Interesting consequences of Cases 3 and 4 are that the only monadic pseudo-boolean connectives bounded by the negation truth-table [the false truth-table] are the negation connective [the false connective] respectively.

On the other hand, Cases 1 and 2 tell us that we have some room to maneuver in the case of pseudo-boolean connectives bounded by the true truth-table and the identity truth-table.

## 4. Weak identity propositional connectives

*Let us call a monadic pseudo-boolean propositional connective $\mathbb{I}$ bounded by the identity truth-table a* **weak identity propositional connective** *or simply a* **weak identity** *iff it satisfies the following additional conditions:*

(1) $\vdash \bigwedge x[\, \mathbb{I}(\mathbb{I}(x)) \equiv \mathbb{I}(x)\,]$.
(2) $\vdash \bigwedge x \bigwedge y[\, \mathbb{I}(x \supset y) \supset (\mathbb{I}(x) \supset \mathbb{I}(y))\,]$.

LEMMA 9.2. *If $\mathbb{I}$ is a weak identity then it satisfies the following:*

(1) $\mathbb{I}(\top) \equiv \top$.
(2) $\mathbb{I}(\bot) \equiv \bot$.
(3) $p \supset \mathbb{I}(p) \supset \neg\neg p$.
(4) $\mathbb{I}(\mathbb{I}(p)) \equiv \mathbb{I}(p)$.
(5) $\mathbb{I}(\neg p) \equiv \neg p \equiv \neg\mathbb{I}(p)$.
(6) $\mathbb{I}(p \supset q) \supset (\mathbb{I}(p) \supset \mathbb{I}(q))$.
(7) $\mathbb{I}(p \wedge q) \supset (\mathbb{I}(p) \wedge \mathbb{I}(q))$.

[10] And 2 for the 0-ary, 16 for the dyadic etc.

(8) $\mathbb{I}(p{\equiv}q) \;\supset\; (\mathbb{I}(p) \;\equiv\; \mathbb{I}(q))$.
(9) $\mathbb{I}(\bigwedge x\mathcal{A}\ulcorner x\urcorner) \;\supset\; \bigwedge x\mathbb{I}(\mathcal{A}\ulcorner x\urcorner)$.
(10) $\mathbb{I}(\bigwedge f\mathcal{A}\ulcorner f\urcorner) \;\supset\; \bigwedge f\mathbb{I}(\mathcal{A}\ulcorner f\urcorner)$.

## 5. The Kaminski propositional connective

Kaminski[11] introduced a monadic propositional connective which is *not* definable in the Second Order Intuitionistic Propositional Calculus (our **EIPC**). We will show that it is definable in the Intuitionistic Protothetic and although it is not **provable** in the **IP** that it is distinct from the Identity and Double Negation propositional connectives, it is *consistent* to assume that it is distinct from them.

The Kaminski propositional connective # can be axiomatized by adding the following axioms to the Intuitionistic Propositional Calculus:

**K 1** $\#(p{\supset}q) \supset (\#p{\supset}\#q)$.
**K 2** $\#\#p \supset \#p$.
**K 3** $p \supset \#p$.
**K 4** $\#p \supset \neg\neg p$.

An immediate consequence of the Kaminski axioms is that the following sentences are provable:

- $\#\top \equiv \top$.
- $\#\bot \equiv \bot$.

In other words, # is a *weak identity propositional connective.*

We shall give a Leśniewskian definition for a monadic propositional function **KAM** and show that it has the properties of the Kaminski propositional connective.

In order to simplify the exposition we introduce the following abbreviations:

$K0_f$ $[\bigwedge y(y \equiv \top \supset f(y) \equiv \top) \;\wedge\; \bigwedge y(y \equiv \bot \supset f(y) \equiv \bot)\,]$.
$K1_f$ $\bigwedge z\bigwedge y[\,f(z{\supset}y) \;\supset\; (f(z){\supset}f(y))\,]$.
$K2_f$ $\bigwedge z[\,f(f(z)) \;\supset\; f(z)\,]$.
$K3_f$ $\bigwedge z[\,z \;\supset\; f(z)\,]$.
$K4_f$ $\bigwedge z[\,f(z) \;\supset\; \neg\neg z\,]$.
$TK_f$ $\neg\neg[\bigwedge z(f(z) \;\equiv\; z) \vee \bigwedge z(f(z) \;\equiv\; \neg\neg z)]$.
$KA_f$ $[\,K0_f \;\wedge\; K1_f \;\wedge\; K2_f \;\wedge\; K3_f \;\wedge\; K4_f\,]$.

*The Kaminski monadic operator* **KAM** *of* **IP** *is given by the following Leśniewskian definition:*

$$\bigwedge x[\,\mathbf{KAM}(x) \equiv \bigwedge f(\mathrm{KA}_f \;\wedge\; \neg\mathrm{TK}_f \;\supset\; \bigwedge y[(x \;\equiv\; y) \supset f(y)])\,].$$

Because the *definiens* of **KAM** is a Tarskian assertion we obtain:

LEMMA 9.3. **KAM** *is a truth-function of* **IP***; that is:* $\vdash \mathcal{LS}_{\mathbf{KAM}}$.

We now turn to show that **KAM** satisfies the Kaminski axioms.

[11][Kaminski, 1988].

PROPOSITION 9.7. $\vdash \bigwedge x \bigwedge y[\mathbf{KAM}(x \supset y) \supset (\mathbf{KAM}(x) \supset \mathbf{KAM}(y))]$.

*Proof.* The crucial step is to show that

$$\mathbf{KAM}(p \supset q), \mathbf{KAM}(p), KA_F \vdash F(q).$$

The latter is a consequence of $KA_F, F(p \supset q), F(p) \vdash F(q)$. ■

PROPOSITION 9.8. $\vdash \mathbf{KAM}(\mathbf{KAM}(p)) \supset \mathbf{KAM}(p)$.

*Proof.* What needs to be shown is that: $\mathbf{KAM}(\mathbf{KAM}(p)), KA_F \vdash F(p)$. Clearly it suffices to show that $KA_F, F(\mathbf{KAM}(p)) \vdash F(p)$.

We achieve the latter in a sequence of steps:

$$\begin{array}{rcl}
KA_F & \vdash & \mathbf{KAM}(p) \supset F(p), \\
KA_F & \vdash & (\mathbf{KAM}(p) \supset F(p)) \equiv \top, \\
KA_F & \vdash & F(\top), \\
KA_F & \vdash & F(\mathbf{KAM}(p) \supset F(p)), \\
KA_F & \vdash & F(\mathbf{KAM}(p)) \supset F(F(p)), \\
KA_F, F(\mathbf{KAM}(p)) & \vdash & F(F(p)), \\
KA_F, F(\mathbf{KAM}(p)) & \vdash & F(F(p)) \supset F(p), \\
KA_F, F(\mathbf{KAM}(p)) & \vdash & F(p).\ ■
\end{array}$$

Using similar methods one can show that:

LEMMA 9.4.

(1) $\vdash \top \equiv \mathbf{KAM}(\top)$.
(2) $\vdash \bot \supset \mathbf{KAM}(\bot)$.

If we could prove that $(\mathbf{KAM}(\bot) \supset \bot)$, then we would have shown that **KAM** is a weak identity and hence the Kaminski propositional connective. Unfortunately the proof of $(\mathbf{KAM}(\bot) \supset \bot)$ requires the assumption that there be at least one $F$ such that $KA_F \wedge \neg TK_F$; in other words $\bigvee f[KA_f \wedge \neg TK_f]$.

OPEN PROBLEM 9.9. *Is the sentence $\bigvee f KA_f$ derivable in* **IP**?

Although we do not know if $\bigvee f[KA_f \wedge \neg TK_f]$ is provable, we can show that it is consistent and will do that in the next section. We close this section by collecting the results about the operator **KAM**:

THEOREM 9.10. *Under the (consistent) assumption that $\bigvee f[KA_f \wedge \neg TK_f]$ the following are derivable properties of the monadic propositional connective* **KAM**:

(1) $\mathbf{KAM}(\top) \equiv \top \wedge \mathbf{KAM}(\bot) \equiv \bot$.
(2) $\mathbf{KAM}(p \supset q) \supset [\mathbf{KAM}(p) \supset \mathbf{KAM}(q)]$.
(3) $\mathbf{KAM}(\mathbf{KAM}(p)) \equiv \mathbf{KAM}(p)$.
(4) $p \supset \mathbf{KAM}(p)$.
(5) $\mathbf{KAM}(p) \supset \neg\neg p$.

## 6. Consistency

The consistency of $\bigvee f[\,KA_f \wedge \neg TK_f\,]$ will be shown through the use of Kripke models. This is not surprising since the Kaminski propositional connective is often introduced by the following forcing condition on Kripke models for intuitionistic propositional systems:

$$(+) \qquad \Vdash_{\vec{n}} \#\mathcal{F} \quad \text{iff} \quad \forall \alpha_{\vec{n}\in\alpha} \exists t[\,\Vdash_{\tilde{\alpha}t} \mathcal{A}\,].$$

As the reader will notice, the forcing condition in Kripke models for the Kaminski propositional connective is analogous to the forcing condition on Beth models. It is a routine matter to verify that such a forcing condition on a Kripke model results in the validation of the 4 Kaminski axioms.

The method that we use to show the consistency of $\bigvee f[\,KA_f \wedge \neg TK_f\,]$ is to show that the sentence

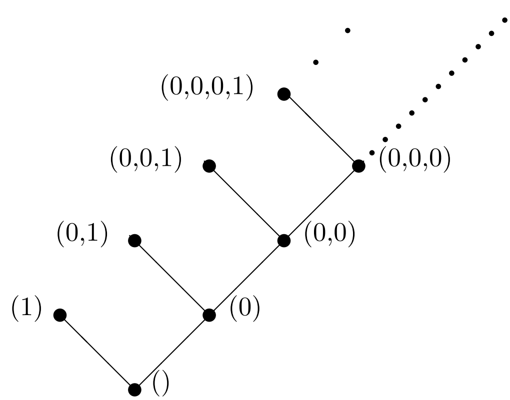
$$\bigwedge f(KA_f \supset TK_f)$$

is not provable in IP. A moment's thought shows us that it suffices to find a Kripke model $\mathfrak{K}$ which forces $KA_F$ and yet fails to force $TK_F$.

If the Kripke structure $\mathfrak{K}$ is constructed so that a prime formula $F(\mathcal{A})$ belongs to the valuation domain at a node $\vec{n}$ just in case that every path through $\vec{n}$ forces $\mathcal{A}$; then $\mathfrak{K}$ will force the formula $KA_F$ at $\vec{n}$. Consequently $F$ will behave as a Kaminski connective.

The Kripke model $\mathfrak{K}$ will be constructed in an infinite sequence of steps.

For the first step we take the following infinite bi-branching tree:




In the above tree we name the nodes by appropriate binary sequences, with the empty sequence corresponding to the root. The rightmost infinite branch corresponds to the constant function zero, that is the nodes on that path are finite sequences of 0's. The terminal nodes are those binary sequences that terminate with a 1 and all the other entries are 0.

To each of the nodes of the above tree we associate a functional domain consisting of $F$. Then we assign propositional parameters to the nodes of the above tree according to the following recipe:

At the root we have the empty set as the valuation domain. At the node $(0)$ we place $p_1$. At the node $(0,0)$ we use $\{p_1, p_3\}$; at $(0,0,0)$ $\{p_1, p_3, p_5\}$ *etc.* At the terminal node $(1)$ we place $\{p_0, p_1\}$; at the terminal node $(0,1)$ we insert $\{p_0, p_1, p_3\}$, and so on.

Then based on the above and the Kaminski evaluation in Kripke models we add all the required $F(p)$ in the valuation domain (for our purposes it suffices to consider only the nodes in the rightmost infinite path); we thus obtain

| | |
|---|---|
| () | $F(p_1)$. |
| (0) | $F(p_1), F(p_3)$. |
| (0,0) | $F(p_1), F(p_3), F(p_5)$. |

And so on.

Combining the latter with what we already know about the valuation domains we deduce that the following hold on the nodes of the rightmost path:

| | |
|---|---|
| () | $\Vdash_{()} F(p_1)$ and $\nVdash_{()} p_1$. |
| (0) | $\Vdash_{(0)} F(p_3)$ and $\nVdash_{(0)} p_3$. |
| (0,0) | $\Vdash_{(0,0)} F(p_5)$ and $\nVdash_{(0,0)} p_5$. |

And so on.

In other words, for all nodes $\vec{n}$ <u>in the rightmost path</u>:

(∗) $$\exists p[\nVdash_{\vec{n}} [F(p) \supset p]].$$

Analogously and again only for the nodes $\vec{n}$ in <u>the rightmost path</u>:

$$\nVdash_{\vec{n}} [\neg\neg p_0 \supset F(p_0)],$$

and hence

(∗∗) $$\exists p[\nVdash_{\vec{n}} [\neg\neg p \supset F(p)]].$$

If we could extend (∗) and (∗∗) to apply to all the nodes $\vec{n}$ of the Kripke model $\mathfrak{K}$ then we would have shown that $\mathfrak{K}$ failed to force $TK_F$. Unfortunately (∗) and (∗∗) do not hold at the terminal nodes of the above mentioned Kripke model.

The solution is to graft homologous trees at the terminal nodes; of course we then have a new set of terminal nodes, so the procedure must be repeated infinitely often. For example at the node $(1)$ which contains the propositional parameters $p_0$ and $p_1$, in the "grafted" tree we would use only the parameters $p_0, p_1, p_2, p_3, p_5, p_7, \ldots$ to populate the additional nodes.

CHAPTER 10

# Propositional operators as connectives

## 1. Propositional connectives of IP

In a previous chapter we required that in order for a defined operator to be called a propositional connective it had to be *equivalence-invariant*; that is, in case $\mathbb{F}$ were a monadic operator, the following sentence had to be a thesis of IP:

$$\bigwedge x \bigwedge y[x \equiv y \supset \mathbb{F}(x) \equiv \mathbb{F}(y)].$$

Observe that the quickest way to guarantee that all defined operators (and in fact all the formulas) are equivalence-invariant is to restrict the propositional functions to just the equivalence-invariant ones, in other words we could add the Law of Substitution $\mathcal{LS}$ (and all its $n$-ary analogues) as additional axioms of the Protothetic. Now from the intuitionistic viewpoint that would be a Draconian simplification, even though from a mathematical viewpoint it might still be of interest to consider IP with such restrictions.

## 2. A necessity modal operator?

It has been recognized by some that in spite of the syntactical and semantical similarities between the modal logics and propositional logics, they are distinct disciplines; for example, according to Ian Hacking[1]:

> *"Modal logics provide yet another direction in which to extend logic."*

Consider the monadic operator $\mathbb{O}$ given by the following Leśniewskian definition:

$$\begin{aligned}\bigwedge x[\,\mathbb{O}(x) \quad \equiv \quad & \bigwedge f[\,(f(\top) \equiv \top) \wedge (f(\bot) \equiv \bot) \wedge \\ & \bigwedge x \bigwedge y(f(x \wedge y) \equiv (f(x) \wedge f(y))) \wedge \\ & \bigwedge y(f(f(y)) \equiv f(y)) \\ & \supset \\ & f(x)\,]\,].\end{aligned}$$

Then the following formulas are theorems of the New Protothetic:

Lemma 10.1. *For any formulas $\mathcal{A}$ and $\mathcal{B}$:*

(1) $\mathbb{O}(\top) \;\equiv\; \top.$
(2) $\mathcal{O}(\bot) \;\equiv\; \bot.$
(3) $\mathbb{O}(\mathcal{A}) \supset \mathcal{A}.$
(4) $\mathbb{O}(\mathcal{A} \wedge \mathcal{B}) \;\equiv\; (\mathbb{O}(\mathcal{A}) \wedge \mathbb{O}(\mathcal{B})).$
(5) $\mathbb{O}(\mathbb{O}(\mathcal{A})) \;\equiv\; \mathbb{O}(\mathcal{A}).$

[1] [Hacking, 1994], page 24.

Note that the above are the typical properties expected of most necessity modal operators. And although we have the following derived rule of inference:

$$\begin{array}{lcl} \text{From} & : & \vdash \mathbb{O}(\mathcal{A}) \\ \text{To infer} & : & \vdash \mathcal{A}, \end{array}$$

we do not have the following as a derived rule of inference:

$$\begin{array}{lcl} \text{From} & : & \vdash \mathcal{A} \\ \text{To infer} & : & \vdash \mathbb{O}(\mathcal{A}). \end{array}$$

Since such a rule would be consistent, we could add it to the New Protothetic; however if we did so then, in our opinion, it would no longer be the New Protothetic but rather an **Applied Protothetic**.

## 3. The role of propositional functions in the modal operator $\mathbb{O}$

Suppose that in the Leśniewskian definition of $\mathbb{O}$ we restrict the functional variables to those that are equivalence-invariant on $\top$ (in other words we replace in the definiens: $\bigwedge f[\,f(\top)\equiv\top\ldots)$ by $\bigwedge f[\bigwedge y(y\equiv\top \supset f(y)\equiv\top)\ldots))$. Then $\mathbb{O}$ becomes indistinguishable from the identity operator[2] **I**:

PROPOSITION 10.1. *If* $\mathbb{O}$ *is modified as above then*

$$\vdash \bigwedge x(\mathbb{O}(x) \equiv x).$$

*Proof.* In view of Lemma 10.1 it suffices to show that

$$p \vdash \mathbb{O}(p).$$

In other words,

$$p, \bigwedge y(y\equiv\top \supset F(y)\equiv\top), \ldots \vdash F(p);$$

but the latter is an immediate consequence of

$$p \vdash p \equiv \top. \blacksquare$$

[2]And thus it is equivalence-invariant.

# Part 4

# The equivalence calculus

CHAPTER 11

# Equivalence in the New Protothetic

## 1. The Minimal Equivalence Calculus, MEC

Although *both* primitive terms $\equiv$ and $\bigwedge$ are essential for the development of the New Protothetic, it is clear that $\equiv$ has the major role. It is thus of some interest to try to isolate the contributions of $\equiv$ to the New Protothetic. Because our formalization of the New Protothetic was in a Gentzen style in which each logical particle has its very own **I**- and **E**-rules of inference, a simple way to obtain a better understanding of the role of *equivalence* in the New Protothetic is to isolate it from the other logical particles (and their corresponding rules of inference).

Thus we shall next consider the **Minimal Equivalence Calculus**, MEC, obtained by eliminating everything from the New Protothetic except the propositional parameters and the propositional connective $\equiv$. Therefore the only rules of inferences of MEC are $\equiv$**-I** and $\equiv$**-E**.

As it turns out there is a certain incompleteness in the system MEC. The inferential strength of

$$\mathcal{A}_1, \ldots, \mathcal{A}_n \ \vdash \ \mathcal{B},$$

is basically equivalent to that of

$$\mathcal{A}_1 \wedge \ldots \wedge \mathcal{A}_n \ \vdash \ \mathcal{B};$$

but in MEC, unlike in the New Protothetic, conjunction $\wedge$ is not definable. Thus later on we shall also include the propositional connective $\wedge$[1]. The rules of inference for BCC (Bi-conditional Constructive Calculus) are the **I** and **E** rules for $\equiv$ and $\wedge$.

**Completeness of the Beth Semantics for** MEC**.** In the Beth structures (viewed as Cantor fans) for MEC, the complementary law, $\Delta$, assigns to each finite sequence of 0's and 1's a finite set of *propositional parameters*.

The completeness proof of the New Protothetic can trivially be restricted to give the completeness (and soundness) for MEC in an intuitionistic metatheory.

**The Classical Minimal Equivalence Calculus, CEC.** If we add to MEC the following rule of inference[2]:

$$\begin{array}{rcl} \text{From} & : & \mathcal{A} \equiv (\mathcal{B} \equiv \mathcal{C}) \\ \text{To obtain} & : & (\mathcal{A} \equiv \mathcal{B}) \equiv \mathcal{C} \end{array}$$

[1]Having conjunction, we can express, by a formula, that *equivalence* is in fact a *bi-conditional.*

[2]Which is neither an **I** nor an **E** rule and thus destroys the important and useful symmetry of the system.

then we obtain the Classical Minimal Equivalence Calculus, CEC. This Calculus has been extensively studied by the Warsaw School. One of the most memorable results is the algorithm[3] for the provable formulas in terms of the parity of the number of occurrences of the propositional parameters.

*A formula $\mathcal{A}$ in the language of* MEC *is*

- *of* even parity *iff* every *propositional parameter occurring in $\mathcal{A}$ occurs an even number of times; and*
- *of* uneven parity *iff* at least one *propositional parameter occurring in $\mathcal{A}$ occurs an odd number of times.*

**Formalizations of** CEC**.** The traditional formalizations of the Classical Equivalence Calculus are of the Hilbertian type. For example, let **HEC** be the following Hilbertian formalization for CEC:

**Axiom schemas:**

$$\begin{array}{rcl} \mathcal{A} & \equiv & \mathcal{A} \\ (\mathcal{A} \equiv \mathcal{B}) & \equiv & (\mathcal{B} \equiv \mathcal{A}) \\ \mathcal{A} \equiv (\mathcal{B} \equiv \mathcal{C}) & \equiv & (\mathcal{A} \equiv \mathcal{B}) \equiv \mathcal{C} \\ (\mathcal{A} \equiv \mathcal{C}) \equiv (\mathcal{B} \equiv \mathcal{C}) & \equiv & (\mathcal{A} \equiv \mathcal{B}) \end{array}$$

**Rule of inference** (Modus Ponens):

$$\frac{\mathcal{A} \equiv \mathcal{B} \qquad \mathcal{A}}{\mathcal{B}}$$

Let $\mathrm{CON}(\Gamma)$ be the set of all those formulas which can be derived in **HEC** using modus ponens, the axioms and the formulas in $\Gamma$. Using the Leśniewski algorithm concerning the parity of formulas it can easily be shown that

$$\emptyset \vdash_{\mathrm{CEC}} \mathcal{A} \quad \text{iff} \quad \mathcal{A} \in \mathrm{CON}(\emptyset).$$

We will show that the above result also holds when the empty set $\emptyset$ is replaced by an arbitrary set $\Gamma$ of formulas; however since in CEC there is a form of the *deduction theorem*:

$$\frac{\Gamma, \mathcal{A} \vdash_{\mathrm{CEC}} \mathcal{B} \qquad\qquad \Delta, \mathcal{B} \vdash_{\mathrm{CEC}} \mathcal{A}}{\Gamma, \Delta \vdash_{\mathrm{CEC}} (\mathcal{A} \equiv \mathcal{B})}$$

and it is not obvious that similar operations can be carried out[4] with CON(), the proof requires some additional steps. First a simple induction, using the associativity of equivalence, gives us:

PROPOSITION 11.1. *If $\mathcal{A}_1, \ldots, \mathcal{A}_n, \mathcal{B}$ are equivalence formulas, then the following conditions are equivalent*[5]*:*

(a) $\mathcal{A}_1, \ldots, \mathcal{A}_n \vdash_{\mathrm{CEC}} \mathcal{B}$.

[3] Attributed to Leśniewski by A. Church in [Church, 1956].

[4] For example, when $\Gamma = \emptyset$, is $(\mathcal{A} \equiv \mathcal{B}) \in \mathrm{CON}(\emptyset)$ naturally obtainable from $\mathcal{A} \in \mathrm{CON}(\mathcal{B})$ and $\mathcal{B} \in \mathrm{CON}(\mathcal{A})$?

[5] Because of the Associativity axiom we often leave out parentheses when considering formulas in a classical situation.

(b) *For some subset* $\{\mathcal{A}_{i_0}, \ldots, \mathcal{A}_{i_k}\} \subseteq \{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$*:*

$$\vdash_{\mathrm{CEC}} \mathcal{A}_{i_0} \equiv \cdots \equiv \mathcal{A}_{i_k} \equiv \mathcal{B}.$$

Corollary. *There is a decision method for* $\Gamma \vdash_{\mathrm{CEC}} \mathcal{A}$*, where* $\Gamma$ *is a finite set, in terms of the parity of formulas.*

Proposition 11.2. *In* CEC *the following conditions are equivalent:*

(a) $\Gamma \vdash_{\mathrm{CEC}} \mathcal{A}$.
(b) $\mathcal{A} \in \mathrm{CON}(\Gamma)$.

*Proof.* Once again, that (b)⇒(a) is immediate. For the converse, assume that $\Gamma \vdash_{\mathrm{CEC}} \mathcal{A}$. Then there are $\mathcal{G}_1, \ldots, \mathcal{G}_k \in \Gamma$ such that

$$\vdash_{\mathrm{CEC}} \mathcal{G}_1 \equiv \ldots \equiv \mathcal{G}_k \equiv \mathcal{A}.$$

But then $\mathcal{G}_1 \equiv \ldots \equiv \mathcal{G}_k \equiv \mathcal{A} \in \mathrm{CON}(\emptyset)$, and thus $\mathcal{A} \in \mathrm{CON}(\Gamma)$.

The case for ≡-Elimination is even more straightforward. ∎

**Normalization Property for** MEC**.** A by-product of the Normalization Property for the Intuitionistic Protothetic is the Normalization Property for MEC. Making use of the results, and terminology, in Appendix A, but restricting them to MEC, we obtain:

Lemma 11.1. *In a normal derivation in* MEC *whose End-Rule is* ≡-**E***, the top formula of the major path is an undischarged formula occurrence. Consequently, such a derivation is an open derivation.*

Corollary. *Every closed, normal derivation in* MEC *ends with an application of the* ≡-**I***-rule and the End-Formula cannot be a propositional parameter.*

Corollary. *If* $\mathcal{A}$ *is a thesis of* MEC *then there must be at least one occurrence of* ≡ *in* $\mathcal{A}$*.*

## 2. Non-equivalent formulas in MEC

L. Rieger[6] (and, independently, Nishimura[7]) showed that in the full intuitionistic propositional calculus there are infinitely many non-equivalent formulas in one propositional parameter. On the other hand A. Diego[8] showed that in the subsystem of intuitionistic logic using only the conditional connective, the number of non-equivalent[9] formulas using finitely many propositional parameters is finite.

The situation for MEC is as follows:

(a) *There are two non-equivalent formulas in one propositional parameter* $p$*, namely* $p$ *and* $(p \equiv p)$*.*

(b) *There are infinitely many non-equivalent formulas in two propositional parameters.*

---

[6] [Rieger, 1949].
[7] [Nishimura, 1960].
[8] [Diego, 1954].
[9] That is, not inter-derivable.

In order to show (b) we shall define the following infinite sequence of formulas of MEC:

$$\begin{array}{lcl}\mathcal{N}_0 & = & (((p{\equiv}q){\equiv}p){\equiv}q).\\ \mathcal{N}_{n+1} & = & ((\mathcal{N}_n{\equiv}p){\equiv}q).\end{array}$$

Note that $\mathcal{N}_n$ has exactly the same number (of occurrences) of $p$'s as of $q$'s, namely $n{+}2$. In the classical Equivalence Calculus it can be quickly shown, using the associativity of classical equivalence, that the odd indexed $\mathcal{N}$'s are equivalent to $(p \equiv q)$ and the even ones to $(p \equiv p)$.

In MEC they are all non-equivalent. First of all observe that any formula of the form $(\mathcal{N}_{2i}{\equiv}\mathcal{N}_{2j+1})$ is of uneven parity and thus unprovable in MEC.

Since we will be going back and forth between MEC and CEC, we will use $\vdash_C$ for derivability in CEC and $\vdash_M$ for derivability in MEC.

PROPOSITION 11.3. *For all n:* $\mathcal{N}_n \nvdash_M p$ *and* $\mathcal{N}_n \nvdash_M q$.

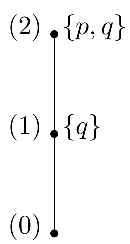
*Proof.* Suppose that $\mathcal{N}_n \vdash_M p$. Then $\mathcal{N}_n \vdash_C p$. But in the classical system $\mathcal{N}_n$ is either equivalent to $(p \equiv q)$ or to $(p \equiv p)$; neither of which has $p$ as a consequence. Similarly for $\mathcal{N}_n \nvdash_M q$. ■

PROPOSITION 11.4. *For all n:* $\mathcal{N}_{2n} \nvdash_M \mathcal{N}_{2n-1} \equiv p$.

*Proof.* Again if it were derivable in MEC, then it would be classically derivable. But then $\vdash_C (p{\equiv}q){\equiv}p$; which is not the case. ■

PROPOSITION 11.5. *For each n:* $\nvdash_M \mathcal{N}_{2n}$.

*Proof.* Let $\mathfrak{K}$ be the following propositional Kripke model (see Appendix C):




One then proves by induction on $n$ that the root of $\mathfrak{K}$ fails to force $\mathcal{N}_{2n}$. As is usual with inductive proofs one has to show some additional steps. What we will show is that:

(a) $(((\mathcal{N}_{2n-2} \equiv p) \equiv q) \equiv p)$ is forced at the root (0) (and hence also at (1) and (2)).
(b) $\mathcal{N}_{2n}$ is not forced at (0), while it is forced at (1) and (2).

*Basis step* $n = 0$. Then $(((\mathcal{N}_{2n-2} \equiv p) \equiv q) \equiv p)$ is $((p \equiv q) \equiv p)$ and $\mathcal{N}_{2n} = (((p \equiv q) \equiv p) \equiv q)$. In this case a moment's thought gives us (a) and (b).

*Inductive step* $n \geq 0$. Consider now $(((\mathcal{N}_{2n-2} \equiv p) \equiv q) \equiv p)$. $p$ is only forced at (2) and clearly $((\mathcal{N}_{2n-2} \equiv p) \equiv q)$ is also forced there.

Now $(\mathcal{N}_{2n-2} \equiv p)$ is forced only at (2). Consequently $((\mathcal{N}_{2n-2} \equiv p) \equiv q)$ is also only forced at (2). Thus $(((\mathcal{N}_{2n-2} \equiv p) \equiv q) \equiv p)$ is forced at the root (0). Thus $\mathcal{N}_{2n}$ is not forced at (0) while it is forced at (1) and (2). ■

Using the Normalization Property of MEC we obtain:

LEMMA 11.2. *If* $n > m$ *then:* $\mathcal{N}_{2n} \nvdash_M \mathcal{N}_{2m}$.

COROLLARY. *If* $n \neq m$, *then:* $\nvdash_M \mathcal{N}_{2n} \equiv \mathcal{N}_{2m}$.

LEMMA 11.3. *For any formulas* $\mathcal{A}$ *and* $\mathcal{B}$, *if* $\vdash_M \mathcal{A}\equiv\mathcal{B}$, *then*

$$\vdash_M ((\mathcal{A}\equiv p)\equiv q) \equiv ((\mathcal{B}\equiv p)\equiv q).$$

LEMMA 11.4. *For any* $n \neq m$: $\nvdash_M \mathcal{N}_{2n+1} \equiv \mathcal{N}_{2m+1}$.

**Non-definability of conjunction.** Conjunction would be definable in MEC if there existed a formula $\mathcal{K}\ulcorner p, q\urcorner$ of MEC such that:

$$p, q \;\vdash\; \mathcal{K}\ulcorner p, q\urcorner \qquad \mathcal{K}\ulcorner p, q\urcorner \;\vdash\; p \qquad \mathcal{K}\ulcorner p, q\urcorner \;\vdash\; q.$$

If the above were provable in MEC then they would also be provable in the classical system. But in the classical system, $\mathcal{K}\ulcorner p, q\urcorner$ would be equivalent to either: $p$, $q$, $(p\equiv q)$ or $(p\equiv p)$. Consequently conjunction is not definable in MEC.

**Decidability of** MEC**.** Since the Normalization property applies to the Intuitionistic Propositional Calculus and MEC may also be considered as a subsystem of the Natural Deduction formalization of the Intuitionistic Propositional Calculus and the latter is decidable, it follows that MEC is a decidable system.

**Pseudo-negation in** MEC**.** In the system MEC there is no propositional operator for *falsum* nor any connective for *negation.* Nevertheless if we abbreviate $(\mathcal{A} \equiv r)$ by $\neg_r\mathcal{A}$, and call it a *pseudo-negation* of $\mathcal{A}$, then we find that $\neg_r$ has many of the properties of intuitionistic negation.

LEMMA 11.5. *The following are derivable in* MEC*.*

$$\begin{array}{rcl}
\neg_r p, p & \vdash & r, \\
p & \vdash & \neg_r\neg_r p, \\
 & \vdash & (\neg_r\neg_r\neg_r p \equiv \neg_r p), \\
(p \equiv q) & \vdash & (\neg_r p \equiv \neg_r q), \\
\neg_r p, \neg_r q & \vdash & (p \equiv q), \\
 & \vdash & (\neg_r p \equiv \neg_r q) \equiv (\neg_r\neg_r p \equiv \neg_r\neg_r q), \\
\neg_r(p \equiv q) & \vdash & (\neg_r\neg_r p \equiv \neg_r q), \\
(p \equiv q) \equiv r & \vdash & \neg_r\neg_r p \equiv (q \equiv r).
\end{array}$$

## 3. A Hilbert formalization for MEC?

The early formalizations of the Classical Equivalence Calculus consisted of finitely many axiom schemas, typically: reflexivity, symmetry, transitivity, associativity and the single rule of *modus ponens.* In our formalization of MEC we can prove reflexivity, symmetry and transitivity and *modus ponens* is the **E**-≡-rule of inference. And although the associativity of equivalence is not provable in MEC, the **I**-≡-rule of inference, the bi-conditional deduction theorem, more than makes up for the deficiency.

From our point of view the formalization chosen for (constructive) equivalence is indeed the correct one since it was obtained by analyzing under what conditions one could assert formulas of the form $(\mathcal{A} \equiv \mathcal{B})$; and thus whether there exist other types of formalizations is of no foundational importance. Nevertheless, since the original Classical Equivalence Calculus had a (finite) Hilbert type formalization, that is, one in which none of the rules of inference discharge assumptions, the question naturally arose on whether there is a Hilbert axiomatization for the *inferences* of MEC?

Open Problem. *Is there an axiomatization for the inferences of* MEC *which has finitely many axiom schemas and finitely many rules of inference, none of which discharge assumptions*[10]*?*

## 4. The BCC calculus

It has been our contention throughout this monograph that there is foundational significance that the **only** primitive terms of the New Protothetic are equivalence, $\equiv$, and the universal quantifier $\bigwedge$. On the other hand, *foundational significance*[11] is not automatically inherited by the subsystems of the New Protothetic. In the previous chapter we considered the subsystem consisting of the propositional system MEC involving only the connective $\equiv$ and although as a formal system it is an interesting theory[12], no particular foundational significance was claimed for it. In fact, as already mentioned, there is a certain mismatch between the inferential strength of MEC and the expressive power of its formulas. It is interesting to note that a *completeness theorem* (as for example that of MEC) does not guarantee the complete equivalence between the chosen semantics and derivation.

One way to try to equalize the expressive power of the formulas with the inferences is to add the connective $\wedge$ for conjunction to MEC. In other words, BCC (Bi-conditional Constructive Calculus) is the Natural Deduction System for the intuitionistic propositional calculus using only the connectives: $\equiv$ and $\wedge$ (and their corresponding **I**- and **E**-rules of inference).

BCC is a subsystem of the Intuitionistic Protothetic. Consequently the Normalization property for the Intuitionistic Protothetic (given in Appendix A) can be trivially simplified for BCC. As a consequence we obtain:

Theorem 11.6. BCC *is a conservative extension of* MEC*.*

[10] For if you allow one rule of inference to discharge assumptions, then the Gentzen style axiomatization that we have chosen for MEC is the simplest possible one!

[11] In contradistinction to mathematical interest.

[12] Specially in comparison to its classical counterpart.

# Part 5

# The algebras of equivalence

CHAPTER 12

# Lindenbaum algebras

Another tradition started by Lindenbaum, Łukasiewicz, Tarski *et al.* is the association of algebraic structures to logical theories[1]. This technique has been extensively developed so that it is now possible to construct complete algebraic semantics for many important logics, see for example H. Rasiowa [Rasiowa, 1974] or D. Scott [Scott, 1974]. In this chapter we construct the Lindenbaum algebras of the Minimal Equivalence Calculus (MEC) and of an extension of it that contains the rule for intuitionistic negation, called the *Minimal Equivalence Calculus with negation*, MECn. In Chapter 13 we shall discuss the mathematical structures that correspond to the Lindenbaum algebras of MEC and MECn and prove algebraic completeness results for these calculi.

## 1. The Minimal Equivalence Calculus with Negation

We **add** to the propositional language of MEC a propositional constant for *intuitionistic absurdity*, "$\perp$". The rules for constructing formulas are exactly as for MEC, with $\perp$ adjoined to the alphabet. The essence of intuitionistic absurdity is that all formulas are derivable from $\perp$. Hence, we set down

**Rule for Intuitionistic Absurdity:** $\dfrac{\perp}{\mathcal{A}}$

where $\mathcal{A}$ is any formula.

Definition 12.1. *The extension of* MEC *obtained by adding $\perp$ to the language (as a propositional constant) and the rule of intuitionistic absurdity to the rules of inference, is called the* Minimal Equivalence Calculus with Negation*, in symbols:* MECn.

**Abbreviation.** We will abbreviate "$(\mathcal{F}\equiv\perp)$" by "$\neg\mathcal{F}$".

The following results contain some of the basic inferences in MECn, where "$\vdash$" represents provability in MECn.

Lemma 12.1. (a) $\vdash \neg\perp$ *and* $\vdash (\neg\perp\equiv\top)$.
(b) $\vdash (\neg\mathcal{A}\equiv\neg\mathcal{B})\equiv(\neg\neg\mathcal{A}\equiv\neg\neg\mathcal{B})$.
(c) $\vdash (\neg\mathcal{A}\equiv\neg\neg\mathcal{B})\equiv(\neg\mathcal{B}\equiv\neg\neg\mathcal{A})$.

Since any formula may be derived from $\perp$, we obtain the following:

Theorem 12.1. *If* $\Gamma, \mathcal{F} \vdash \perp$*, then* $\Gamma \vdash \neg\mathcal{F}$.

[1] At first this was quite distinct from the *algebrization of Logic*, a method started by G. Boole and continued by Tarski, Halmos *et al.* However, with the advent of *categories*, the demarcation line has just about disappeared.

Corollary. (a) $\mathcal{A}, \neg\mathcal{A} \vdash \mathcal{B}$.

(b) *If* $\Gamma, \mathcal{F} \vdash \mathcal{G}$ *then* $\Gamma, \neg\mathcal{G} \vdash \neg\mathcal{F}$.

(c) $(\mathcal{A}{\equiv}\neg\mathcal{A}) \vdash \neg\mathcal{A}$.

(d) $\vdash \neg(\mathcal{A}{\equiv}\neg\mathcal{A})$.

(e) $\vdash \neg(\mathcal{A}{\equiv}\mathcal{B}){\equiv}\neg(\neg\mathcal{A}{\equiv}\neg\mathcal{B})$.

The following require a little more work:

Lemma 12.2. (a) $\vdash \neg(\mathcal{A}{\equiv}\mathcal{B}){\equiv}(\neg\mathcal{A}{\equiv}\neg\neg\mathcal{B})$.

(b) $\vdash \neg(\mathcal{A}{\equiv}\mathcal{B}){\equiv}\neg(\neg\mathcal{A}{\equiv}\neg\mathcal{B})$.

(c) $\vdash \neg\neg\mathcal{A}{\equiv}((\neg\neg\mathcal{A}{\equiv}\mathcal{B}){\equiv}\mathcal{B})$.

(d) $\vdash \neg\mathcal{A}{\equiv}\neg(\mathcal{B}{\equiv}(\mathcal{A}{\equiv}\mathcal{B}))$.

(e) $\vdash \neg\neg\mathcal{A}{\equiv}(\neg\mathcal{B}{\equiv}\neg(\mathcal{A}{\equiv}\mathcal{B}))$.

(f) $\vdash \neg\neg(\mathcal{A}{\equiv}\mathcal{B}){\equiv}(\neg\neg\mathcal{A}{\equiv}\neg\neg\mathcal{B})$.

The next theorem could be called **associativity under negation**:

Theorem 12.2. $\vdash \neg(\mathcal{A}{\equiv}(\mathcal{B}{\equiv}\mathcal{C})){\equiv}\neg((\mathcal{A}{\equiv}\mathcal{B}){\equiv}\mathcal{C})$.

We shall present algebraic proofs of Lemma 12.2 and Theorem 12.2 when we discuss **equivalence algebras with negation** in Chapter 13 (see Proposition 13.66).

## 2. Lindenbaum algebras of MEC and MECn

Let $\Gamma$ be a set of MEC (respectively, MECn) formulas.

Definition 12.1. *If* $\mathcal{A}$ *is a formula of* MEC *or* MECn, *set:*

* $\mathcal{A}_\Gamma = \{\mathcal{B} : \Gamma \vdash (\mathcal{A} \equiv \mathcal{B})\}$;
* $\mathcal{A}_\Gamma \leq \mathcal{B}_\Gamma$ *iff* $\Gamma, \mathcal{A} \vdash \mathcal{B}$;
* $\mathcal{L}_\Gamma = \{\mathcal{B}_\Gamma : \mathcal{B}$ *is a formula in* MEC$\}$;
* $\mathcal{L}_{\perp\Gamma} = \{\mathcal{B}_\Gamma : \mathcal{B}$ *is a formula in* MECn$\}$;
* $\top = (p \equiv p)_\Gamma$;
* *In the case of* MECn, *write* $\perp$ *for* $\perp_\Gamma$.

When $\Gamma = \emptyset$, write $\overline{\mathcal{A}}$ for $\mathcal{A}_\Gamma$, $\mathcal{L}$ for $\mathcal{L}_\Gamma$ and $\mathcal{L}_\perp$ for $\mathcal{L}_{\perp\Gamma}$.

Using the derivability properties of MEC and MECn, the following are straightforward:

Lemma 12.2. *With notation as above:*

(a) $\mathcal{A}_\Gamma = \mathcal{B}_\Gamma$ *iff* $\Gamma \vdash (\mathcal{A} \equiv \mathcal{B})$.

(b) $\leq$ *is a partial order on* $\mathcal{L}_\Gamma$ *and* $\top$ *is its largest element.*

(c) *For* MECn, $\perp$ *is the smallest element in the partial order* $\leq$.

(d) *If* $\mathcal{A}_{1\Gamma} = \mathcal{A}_{2\Gamma}$ *and* $\mathcal{B}_{1\Gamma} = \mathcal{B}_{2\Gamma}$, *then* $(\mathcal{A}_1 \equiv \mathcal{B}_1)_\Gamma = (\mathcal{A}_2 \equiv \mathcal{B}_2)_\Gamma$.

Because of 12.2(d), we may define a **non-associative binary operation $*$ on $\mathcal{L}_\Gamma$** by

$$\mathcal{A}_\Gamma * \mathcal{B}_\Gamma = (\mathcal{A} \equiv \mathcal{B})_\Gamma.$$

Note that because neither in MEC nor in MECn do we have a propositional connective for the *conditional*, there is a fundamental distinction in $\mathcal{L}_\Gamma$ between $\leq$ and $*$. Thus, we consider it appropriate that Lindenbaum algebras include both of them.

DEFINITION 12.3. *With notation as above:*

(a) *The* **Lindenbaum algebra of** MEC (MECn) **relative to a set of formulas** $\Gamma$ *is the structure* $\mathcal{L}_\Gamma = \langle \mathcal{L}_\Gamma, \leq, \top, * \rangle$, *(resp.,* $\mathcal{L}_{\perp\Gamma} = \langle \mathcal{L}_{\perp\Gamma}, \leq, \top, \perp, * \rangle$*).*

(b) *The* **Lindenbaum algebra of** MEC (MECn)*,* $\mathcal{L}$ *(resp.,* $\mathcal{L}_\perp$*), is the Lindenbaum algebra relative to the empty set of formulas.*

From the derivability properties of MEC and MECn, including the results on negation (see Section 1), one can show

LEMMA 12.4. *The Lindenbaum algebras* $\mathcal{L}_\Gamma$ *and* $\mathcal{L}_{\perp\Gamma}$ *of* MEC *and* MECn *relative to a set of formulas* $\Gamma$ *satisfy the following rules:*

(a) $\leq$ *is a partial order, with* $\top$ *as its largest element and, in the case of* MECn*, with* $\perp$ *as its least element.*

(b) $*$ *is a binary operation, such that the following hold universally:*

[$*$ 1] $x * y = y * x$.
[$*$ 2] $x * \top = x$.
[$*$ 3] $x * y = \top$ *iff* $x = y$.
[$*$ 4] *If* $a \leq x * y$ *and* $a \leq b * c$, *then,* $a \leq (x * b) * (y * c)$.

(c) *In the case of* $\mathcal{L}_{\perp\Gamma}$ *we also have*

[*neg*] $x * (x * \perp) = \perp$.

*Proof.* We comment only on [$*$ 4] in (b). Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{F}$, $\mathcal{G}$ are formulas in MEC or MECn such that

$$\Gamma, \mathcal{A} \vdash (\mathcal{B} \equiv \mathcal{C}) \quad \text{and} \quad \Gamma, \mathcal{A} \vdash (\mathcal{F} \equiv \mathcal{G}).$$

It is readily verified that

$$\Gamma, \ \mathcal{A}, \ (\mathcal{B} \equiv \mathcal{C}), \ (\mathcal{F} \equiv \mathcal{G}), \ (\mathcal{B} \equiv \mathcal{F}), \ \mathcal{C} \ \vdash \ \mathcal{G}.$$

From this it follows that

$$\Gamma, \ \mathcal{A}, \ (\mathcal{B} \equiv \mathcal{C}), \ (\mathcal{F} \equiv \mathcal{G}) \ \vdash \ (\mathcal{B} \equiv \mathcal{F}) \equiv (\mathcal{C} \equiv \mathcal{G}),$$

which, in turn, using the symmetry of the argument, yields [$*$ 4]. ∎

## 3. Theories in $\mathcal{L}$ and in $\mathcal{L}_\perp$

If $\Lambda$ is a set of formulas in MEC or MECn, write $\overline{\Lambda} = \{\overline{\mathcal{A}} \in \mathcal{L} : \mathcal{A} \in \Lambda\}$.[2] Recall that $\mathcal{L}$ and $\mathcal{L}_\perp$ are the Lindenbaum algebras of MEC and MECn, respectively. The relation of provability induces a relation from $2^{\mathcal{L}}$ to $\mathcal{L}$, indicated by the same symbol, and defined as follows:

[2]In the case of MECn, $\overline{\Lambda} = \{\overline{\mathcal{A}} \in \mathcal{L}_\perp : \mathcal{A} \in \Lambda\}$.

For $S \cup x \subseteq \mathcal{L}$ (or $\mathcal{L}_\perp$),

$$S \vdash x \quad \text{iff} \quad \begin{cases} \text{There is } \Lambda \cup \{\mathcal{A}\} \subseteq \text{MEC (resp., MECn)} \\ \quad \text{such that } \overline{\Lambda} \subseteq S,\ \overline{\mathcal{A}} = x \text{ and } \Lambda \vdash \mathcal{A}. \end{cases}$$

For $x \in \mathcal{L}$, set

$$x^{\rightarrow} = \{y \in \mathcal{L} : x \vdash y\}.$$

An analogous notation applies also to $\mathcal{L}_\perp$.

Definition 12.5. *A subset $T$ of $\mathcal{L}$ (resp., $\mathcal{L}_\perp$) is a* **theory** *iff it is closed under $\vdash$, that is, if $T \vdash x$, then $x \in T$. A theory is* **proper** *if $T \neq \mathcal{L}$ (resp., $\mathcal{L}_\perp$).*

Note that a theory in $\mathcal{L}_\perp$ is proper iff $\perp \notin T$.

Proposition 12.6. *With notation as above, let $T$ be a theory in $\mathcal{L}$ (resp., $\mathcal{L}_\perp$) and let $x, y, z \in \mathcal{L}$ (resp., $\mathcal{L}_\perp$).*

(a) (i) $\top \in T$;
(ii) $x * y,\ y * z \in T$ *implies* $x * z \in T$.
(iii) $x,\ x * y \in T$ *implies* $y \in T$.

(b) *For all $x \in \mathcal{L}$ (resp., $\mathcal{L}_\perp$), $x^{\rightarrow}$ is a theory.*

(c) *The property of being a theory is preserved under arbitrary intersections and directed unions.*

(d) *For $U \subseteq \mathcal{L}$ (resp., $\mathcal{L}_\perp$), let $U^t$ be the intersection of all theories containing $U$ (the theory generated by $U$). Then*

$$U^t = \{x \in \mathcal{L} : U \vdash x\},$$

*and analogously for $\mathcal{L}_\perp$. In particular, $\{x\}^t = x^{\rightarrow}$, for all $x \in \mathcal{L}$ (resp., $\mathcal{L}_\perp$).*

(e) *If $U \subseteq \mathcal{L}$ (resp., $\mathcal{L}_\perp$), then*

$$(U \cup \{x\})^t = (U \cup \{y\})^t \quad \textit{iff} \quad x * y \in U^t.$$

(f) *The operation of taking theories satisfies the following properties, where $U, V \subseteq \mathcal{L}$ (resp., $\mathcal{L}_\perp$):*

$*$ $U \subseteq U^t$ *(inflationary);*
$*$ $U \subseteq V$ *implies* $U^t \subseteq V^t$ *(increasing);*
$*$ $(U^t)^t = U^t$ *(idempotent).*

*Proof.* We work in MEC, but all arguments apply to MECn.

Items (a) and (b) are straightforward. For (c), let $T_i$, $i \in I$, be theories and write $T = \bigcap_{i \in I} T_i$. If $T \vdash x$, there is $\Lambda \cup \{\mathcal{A}\}$ such that $\overline{\Lambda} \subseteq T$, $\overline{\mathcal{A}} = x$ and $\Lambda \vdash \mathcal{A}$. It follows immediately that $x \in T_i$, for all $i \in I$, and so, $x \in T$.

Let $\{T_i : i \in I\}$ be up-directed, with $T \vdash x$ and $T = \bigcup_{i \in I} T_i$. Thus, there is $\Lambda \cup \{\mathcal{A}\}$ such that $\overline{\Lambda} \subseteq T$, $\overline{\mathcal{A}} = x$ and $\Lambda \vdash \mathcal{A}$. By the compactness of MEC, there is a finite $\Gamma \subseteq \Lambda$ such that $\Gamma \vdash \mathcal{A}$ in MEC. Since $\Gamma$ is finite, $\overline{\Gamma} \subseteq T_i$, for some $i \in I$. But then $x \in T_i \subseteq T$, as desired.

(d) It is sufficient to show that the right-hand side of the equality is a theory. Write $T = \{x \in \mathcal{L} : U \vdash x\}$ and assume that $T \vdash y$. As above, there is $\Lambda \cup \{\mathcal{A}\}$ such that

$\overline{\Lambda} \subseteq T$, $\overline{\mathcal{A}} = x$ and $\Lambda \vdash \mathcal{A}$ in MEC. Let $\Sigma$ be a set of formulas such that $\overline{\Sigma} \subseteq U$ and $\Sigma \vdash \Lambda$. But then, in MEC,

$$\Sigma \vdash \Lambda \quad \text{and} \quad \Lambda \vdash \mathcal{A},$$

and so transitivity of proof yields $\Sigma \vdash \mathcal{A}$, i.e., $x \in T$, as desired.

(e) By (b) above, the left-hand side of the equivalence implies its right-hand side. For the converse, the hypothesis means that $U, x \vdash y$ and $U, x \vdash y$. It follows easily from the definition of $\vdash$ in $\mathcal{L}$ and the $\equiv$ introduction rule that $U \vdash x * y$, as needed. Item (f) is left to the reader. ∎

DEFINITION 12.7. *Let $S$ be a set, $P$ a subset of $S$ and $x$, $y$ be distinct elements of $S$. We say that $P$* **separates** $x$ **and** $y$ *if both $P$ and its complement have non-empty intersection with $\{x, y\}$, that is,*

$$\textit{Either } (x \in P \textit{ and } y \notin P) \quad \textit{or} \quad (y \in P \textit{ and } x \notin P).$$

*A collection $\mathcal{U}$ of subsets of $S$* **separates points** *in $S$ iff all distinct points in $S$ can be separated by elements of $\mathcal{U}$.*

One of the most important properties of theories is described by

THEOREM 12.8. *Let $T$ be a theory in $\mathcal{L}$ or $\mathcal{L}_\perp$. If $x, y \in \mathcal{L}$ (or $\mathcal{L}_\perp$) are such that $x * y \notin T$, then there is a proper theory that extends $T$ and separates $x$ and $y$.*

*Proof.* By 12.6(e) we have $(T \cup \{x\})^t \neq (T \cup \{y\})^t$, that is, either

$$x \notin (T \cup \{y\})^t \quad \text{or} \quad y \notin (T \cup \{x\})^t.$$

If the first alternative holds, $(T \cup \{y\})^t$ is a proper extension of $T$, separating $x$ and $y$; if the second alternative holds, then $(T \cup \{x\})^t$ is the extension of $T$ separating $x$ and $y$. ∎

## 4. The Lindenbaum algebra of BCC

Recall from Section 11.4 that BCC (the Bi-conditional Constructive Calculus) is obtained from MECn (or MEC) by adding the (infix) binary propositional connective $\wedge$, together with the appropriate introduction and elimination rules.

In analogy with Section 12.2, if $\Sigma \cup \{\mathcal{A}\}$ is a set of formulas in BCC let

$$\mathcal{A}^\Sigma = \{\mathcal{B} \in \text{BCC} : \Sigma \vdash (\mathcal{A} \equiv \mathcal{B})\},$$

where $\vdash$ indicates provability in BCC. Define

$$\mathcal{L}^\wedge_\Sigma = \{\mathcal{A}^\Sigma : \mathcal{A} \in \text{BCC}\}.$$

Define operations $*$, $\wedge$ and a relation $\leq$ in $\mathcal{L}^\wedge_\Sigma$ by

$$(*) \qquad \begin{cases} \mathcal{A}^\Sigma * \mathcal{B}^\Sigma & = & (\mathcal{A} \equiv \mathcal{B})^\Sigma; \\ \mathcal{A}^\Sigma \wedge \mathcal{B}^\Sigma & = & (\mathcal{A} \wedge \mathcal{B})^\Sigma; \\ \mathcal{A}^\Sigma \leq \mathcal{B}^\Sigma & \text{iff} & \Sigma, \mathcal{A} \vdash \mathcal{B}. \end{cases}$$

It is straightforward to check that this is independent of representatives. Write

$$\top =_{def} (p \equiv p)^\Sigma \quad \text{and} \quad \bot =_{def} \bot^\Sigma.$$

When $\Sigma = \emptyset$, we adopt the following conventions:

– $\mathcal{A}^{\Sigma}$ is written as $\overline{\mathcal{A}}$; [3] – $\mathcal{L}^{\wedge}$ stands for $\mathcal{L}^{\wedge}_{\emptyset}$.

Definition 12.9. *With notation as above:*

(a) *The structure* $\langle \mathcal{L}^{\wedge}_{\Sigma}, *, \wedge, \top, \bot \rangle$ *is the* **Lindenbaum algebra** *of* BCC *relative to the set of formulas* $\Sigma$.

(b) *The structure* $\langle \mathcal{L}^{\wedge}, \wedge, *, \top, \bot \rangle$ *is the* **Lindenbaum algebra** *of* BCC.

In distinction with the case of MEC and MECn, **the order has been removed from the specification of the Lindenbaum algebras originated by** BCC. We have

Proposition 12.10. *With notation as above,*

(a) *The relation* $\leq$, *defined in* $(*)$ *above, is a partial order in* $\mathcal{L}^{\wedge}_{\Sigma}$, *with* $\top$ *and* $\bot$ *as its top and bottom elements, respectively. Moreover, for all* $x, y \in \mathcal{L}^{\wedge}_{\Sigma}$,

$$x \leq y \quad \textit{iff} \quad x \wedge y = x.$$

(b) *The following hold universally in* $\mathcal{L}^{\wedge}_{\Sigma}$:

| | |
|---|---|
| [$\wedge$ 1] | $x \wedge y = x \wedge y$; |
| [$\wedge$ 2] | $x \wedge (y \wedge z) = (x \wedge y) \wedge z$; |
| [$\wedge$ 3] | $x \wedge x = x = x \wedge \top$; $\;x \wedge \bot = \bot$; |
| [$*$ 1] | $x * y = y * x$; |
| [$*$ 2] | $x * \top = x$; |
| [*exp* 2] | $x * x = \top$; |
| [*bca* 0] | $x \wedge (x * y) = y \wedge (x * y)$; |
| [*bca*] | $x \wedge (y * z) = x \wedge [(x \wedge y) * (x \wedge z)]$. |

*Proof.* Item (a) is a direct consequence of the introduction and elimination rules for the connective $\wedge$. The clauses in (b) involving $\wedge$ are algebraic expressions of standard properties of conjunction. The clauses concerning $*$ follow from the fact that BCC is a conservative extension of MECn (Theorem 11.6). To prove [*bca* 0], let $\mathcal{A}$, $\mathcal{B}$ be formulas in BCC; it must be shown that

$$\begin{cases} \Sigma,\ \ \mathcal{A} \wedge (\mathcal{A} \equiv \mathcal{B}) \ \vdash\ \mathcal{B} \wedge (\mathcal{A} \equiv \mathcal{B}) \\ \Sigma,\ \ \mathcal{B} \wedge (\mathcal{A} \equiv \mathcal{B}) \ \vdash\ \mathcal{A} \wedge (\mathcal{A} \equiv \mathcal{B}), \end{cases}$$

being, by symmetry, enough to verify one of these sequents. We have

$$\begin{aligned} \Sigma,\ \ \mathcal{A} \wedge (\mathcal{A} \equiv \mathcal{B}) &\vdash \mathcal{A}, (\mathcal{A} \equiv \mathcal{B}) \\ &\vdash \mathcal{A}, \mathcal{B} \\ &\vdash \mathcal{B}, (\mathcal{A} \equiv \mathcal{B}) \\ &\vdash \mathcal{B} \wedge (\mathcal{A} \equiv \mathcal{B}), \end{aligned}$$

as desired. The proof of [*bca*] is analogous. ■

There is a simple relationship between the Lindenbaum algebras of MECn and BCC, relative to a set of formulas $\Sigma$ in MECn. With notation as in Section 12.2, define a map

$$\iota_{\Sigma} : \mathcal{L}_{\bot\Sigma} \to \mathcal{L}^{\wedge}_{\Sigma}, \qquad \mathcal{A}_{\Sigma} \longmapsto \mathcal{A}^{\Sigma}.$$

[3]Just as in the case of $\mathcal{L}$ or $\mathcal{L}_{\bot}$.

Proposition 12.11. *If* $\Sigma \subseteq$ MECn, *the map* $\iota_\Sigma$ *is an embedding of* $\mathcal{L}_{\perp\Sigma}$ *into* $\mathcal{L}^{\wedge}_{\Sigma}$, *that is,* $\iota_\Sigma$ *is an injection such that for all* $x, y \in \mathcal{L}_{\perp\Sigma}$,

$$\begin{cases} x \leq y & \textit{iff} & \iota_\Sigma(x) \leq \iota_\Sigma(y); \\ \iota_\Sigma(x * y) & = & \iota_\Sigma(x) * \iota_\Sigma(y); \\ \iota_\Sigma(\top) & = & \top; \\ \iota_\Sigma(\perp) & = & \perp. \end{cases}$$

*Proof.* Let $\mathcal{A}$, $\mathcal{B}$ be formulas in MECn such that $\iota_\Sigma(\mathcal{A}_\Sigma) = \iota_\Sigma(\mathcal{B}_\Sigma)$. Then $\vdash_{BCC} (\mathcal{A} \equiv \mathcal{B})$; since BCC is a conservative extension of MECn (Theorem 11.6), we conclude that $\vdash_{MECn} (\mathcal{A} \equiv \mathcal{B})$. Hence $\mathcal{A}_\Sigma = \mathcal{B}_\Sigma$, and $\iota_\Sigma$ is injective. Using the same principles, it can be verified that $\iota_\Sigma$ preserves $\leq$, $*$, $\top$ and $\perp$, as asserted. ∎

CHAPTER 13

# Equivalence algebras

In this chapter we start discussing the algebraic structures suggested by the logical constructions expounded above. Since Heyting and complete Heyting algebras (or frames) will be important in what follows, we have included a section which reviews their basic properties. We then introduce the notions of *weak equivalence algebra* and *equivalence algebras*, that are the mathematical structures associated to MEC and MECn. As will become clear, there are important distinctions between the systems developed here and the usual ones associated to logic:

$*$ Although called *algebras*, equivalence algebras are, in fact, relational structures, because the partial order that represents provability **is not** definable in terms of its basic operations, as is the case with lattices, Boolean or Heyting algebras. Hence, we shall have work harder to lay hands on the analogues of filters.

$*$ The binary operation representing equivalence, in general, **is not associative**, a characteristic of intuitionistic systems (see Proposition 13.7).

## 1. Heyting and complete Heyting algebras

The aim of this section is to set down the basic terminology for the topics in the title for the convenience of the reader and later reference. Proofs of our statements can be found in [Balbes and Dwinger, 1974] and [Fourman and Scott, 1979].

If $L$ is a partially ordered set and $x, y \in L$, write

$*$ $x^{\rightarrow} = \{y \in L : x \leq y\}$;
$*$ $x^{\leftarrow} = \{y \in L : y \leq x\}$;
$*$ $x \wedge y$ (the meet of $x$ and $y$) for inf $\{x, y\}$, whenever it exists;
$*$ $x \vee y$ (the join of $x$ and $y$) for sup $\{x, y\}$, whenever it exists;
$*$ $\top$ and $\perp$ for the largest and smallest element of $L$, whenever these exist.

A partially ordered set $L$ is a **lattice** if every pair of elements in $L$ has a meet and a join. A lattice is **distributive** iff

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

It is easily verified that this last condition is equivalent to its dual,

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Let $L$ be a distributive lattice with $\perp$ and $\top$. An element $x \in L$ is **complemented** if there is $y \in L$ such that

[C] $$x \wedge y = \perp \quad \text{and} \quad x \vee y = \top;$$

such a $y$ is *unique* in $L$, and called the **complement** of $x$ in $L$. A **Boolean algebra** is a distributive lattice with $\bot$, $\top$ in which all elements are complemented.

Let $L$ be lattice with $\top$. A subset $F$ of $L$ is a **filter** if $\top \in F$, it is closed under finite meets and $x \in F$ implies $x^{\rightarrow} \subseteq F$. A filter is **proper** iff it is distinct from $L$. Hence, if $L$ has $\bot$, $F$ is proper iff $\bot \notin F$. The property of being a filter is preserved by arbitrary intersections and directed unions. If $S \subseteq L$, the **filter generated** by $S$ is the intersection of all filters in $L$ that contain $S$.

LEMMA 13.1. *Let $L$ be a lattice with $\top$. Let $S$ be a subset of $L$. Then the filter generated by $S$ in $L$ is given by*

$$[S] = \{x \in L : \exists\, a_1, \ldots, a_n \subseteq S \text{ such that } x \geq a_1 \wedge \ldots \wedge a_n\}.$$

*If $L$ has $\bot$, $[S]$ is proper iff $S$ has the finite intersection property (fip), that is, the meet of any finite subset of $S$ is distinct from $\bot$.*

13.2. A **Heyting algebra (Ha)** is a distributive lattice with $\top$ and $\bot$, $H$ such that for all $x, y \in H$,

[Ha] The set $\{z \in H : x \wedge z \leq y\}$ has a *maximum* in $H$.

We write

$$x \rightarrow y =_{def} \max \{z \in H : x \wedge z \leq y\},$$

called the **implication operation**[1] in $H$. Hence, for all $x, y, z \in H$,

[$\rightarrow$] $$x \wedge y \leq z \quad \text{iff} \quad x \leq (y \rightarrow z).$$

For all $x \in H$,

$$\neg x = (x \rightarrow \bot)$$

is the **pseudo-complement** of $x$ in $H$. In view of [$\rightarrow$], $\neg x$ is the largest element of $H$ whose meet with $x$ is $\bot$.

LEMMA 13.3. *Let $H$ be a Ha and let $x, y, z \in H$. Then:*

(a) $x \leq y$ *iff* $x \rightarrow y = \top$.
(b) $x \wedge (x \rightarrow y) = x \wedge y$.
(c) $x \wedge (y \rightarrow z) = x \wedge ((x \wedge y) \rightarrow (x \wedge z))$.
(d) *If $F$ is a filter in $H$, then*

  (i) $x \in F$ *and* $(x \rightarrow y) \in F$ *imply* $y \in F$.
  (ii) $x \in F$ *implies* $y \rightarrow x \in F$.

(e) *If $F$ is a proper filter in $H$ and $(x \rightarrow y) \notin F$, then there is a proper filter $G$ in $H$ such that $x \in G$ and $y \notin G$.*

*Proof.* We prove only (e). If $y \in F \cup \{x\}$, Lemma 13.1 yields $t \in F$ such that $x \wedge t \leq y$, and the adjointness relation [$\rightarrow$] implies $t \leq (x \rightarrow y)$. Since $t \in F$, we get $(x \rightarrow y) \in F$, a contradiction. Hence, the filter generated by $F$ and $x$ is a proper extension of $F$ separating $x$ and $y$. ■

[1]Sometimes *relative pseudo-complement* of $x$ in $y$.

We consider Boolean algebras as particular Heyting algebras. If $B$ is a Boolean algebra, the classical definition of implication,

$$x \to y \quad = \quad \neg\, x \vee y,$$

defines an implication operation in $B$ that satisfies the fundamental adjunction $[\to]$ (13.2). Here is a collection of well known equivalent conditions for a Heyting algebra to be a Boolean algebra:

LEMMA 13.4. *A Heyting algebra $H$ is a Boolean algebra iff any of the following conditions is satisfied:*

(1) *For all $x \in H$, $\;x \vee \neg\, x = \top$;*
(2) *For all $x \in H$, $\;\neg\,\neg\, x = x$;*
(3) *For all $x \in H$, $\;\neg\,\neg\, x = \top \;$ iff $\; x = \top$;*
(4) *For all $x, y \in H$, $\;x \to y = \neg\, x \vee y$;*
(5) *For all $x, y \in H$, $\;\neg\,(x \wedge y) = \neg\, x \vee \neg\, y$.*

REMARK 13.5. Let $B$ be a Boolean algebra. For $x, y \in B$, define

$$x \vartriangle y = (x \wedge \neg\, y) \vee (y \wedge \neg\, x),$$

called the **symmetric difference** of $x$ and $y$. This operation has the following properties:

[$\vartriangle$ 1] $\quad x \vartriangle y = y \vartriangle x$;
[$\vartriangle$ 2] $\quad x \vartriangle (y \vartriangle z) = (x \vartriangle y) \vartriangle z$;
[$\vartriangle$ 3] $\quad x \vartriangle x = \bot$; $\neg\, x = \top \vartriangle x$;
[$\vartriangle$ 4] $\quad x \wedge (y \vartriangle z) = (x \wedge y) \vartriangle (x \wedge z)$.

The structure $\mathcal{B} = \langle B, \wedge, \vartriangle, \bot, \top\rangle$ is a commutative ring with identity, where $\wedge$ plays the role of multiplication and $\vartriangle$ that of addition. Moreover, all elements in $\mathcal{B}$ are idempotent and have additive exponent two (i.e., $x \wedge x = x$ and $x \vartriangle x = \bot$). Commutative rings that satisfy these properties are called *Boolean*. It is well known that there is a natural equivalence between Boolean rings and Boolean algebras. ■

If $x, y$ are elements of a Ha $H$, define

$[\leftrightarrow]$ $$x \leftrightarrow y = (x \to y) \wedge (y \to x),$$

called the **equivalence operation** in $H$; its basic properties are stated in

LEMMA 13.6. *Let $H$ be a Ha and let $x$, $y$, $z$ be elements of $H$. Then:*

(a) *$x = y \;$ iff $\; x \leftrightarrow y = \top$.*
(b) *$x \le (y \leftrightarrow z) \;$ iff $\; x \wedge y = x \wedge z$; $\; x \wedge y \le (x \leftrightarrow y)$.*
(c) *$x \wedge (y \leftrightarrow z) = x \wedge ((x \leftrightarrow y) \leftrightarrow (y \leftrightarrow z))$.*
(d) *If $F$ is a filter in $H$, then*

 (i) *$x \in F$ and $(x \leftrightarrow y) \in F \;$ imply $\; y \in F$.*
 (ii) *$x \in F$ and $y \in F \;$ imply $\; (x \leftrightarrow y) \in F$.*

(e) *If $F$ is a filter in $H$, then*

$$(x \leftrightarrow y) \in F \quad \textit{iff} \quad [F \cup \{x\}] = [F \cup \{y\}].$$

(f) *If $F$ is a proper filter in $H$ and $(x \leftrightarrow y) \notin F$, then there is a proper filter $G$ in $H$, such that $F \subseteq G$ and $x \notin G$ or $y \notin G$.*

*Proof.* We prove only (e). Suppose that $[F \cup \{x\}] = [F \cup \{y\}]$; by 13.1, there are $a, b \in F$ such that

$$a \wedge x \leq y \quad \text{and} \quad b \wedge y \leq x.$$

But then $a \wedge b \wedge x = a \wedge b \wedge y$ and (b) yields $(a \wedge b) \leq x \leftrightarrow y$, as desired. The converse is immediate from item (i) in (d). ■

The lack of associativity of the operation $\leftrightarrow$ is a characteristic of intuitionistic systems. The next proposition shows that even the mildest of assumptions concerning the associative rule for $\leftrightarrow$ is equivalent to classical logic.

Proposition 13.7. *The following are equivalent for a Heyting algebra $H$:*

(1) *$H$ is a Boolean algebra (Ba).*
(2) $\forall\, p, q, r \in H, \quad [p \leftrightarrow (q \leftrightarrow r)] \leq [(p \leftrightarrow q) \leftrightarrow r]$.
(3) $\forall\, p, q, r \in H, \quad [(p \leftrightarrow q) \leftrightarrow r] \leq [p \leftrightarrow (q \leftrightarrow r)]$.
(4) $\forall\, p, q, r \in H, \quad [p \leftrightarrow (q \leftrightarrow r)] = [(p \leftrightarrow q) \leftrightarrow r]$.
(5) $\forall\, p, q \in H, \quad p = [(p \leftrightarrow q) \leftrightarrow q]$.

*Proof.* By 13.4, $H$ is a Ba iff $x = \neg\neg x$, for all $x \in H$. Thus, taking $p = q = \bot$ shows that that (2)$\Rightarrow$(1), while $q = r = \bot$ yields (3)$\Rightarrow$(1). It is clear that (4)$\Rightarrow$(2), (3).

(5)$\Rightarrow$(3). We must verify the following two conditions:

$$(*) \qquad p \wedge [(p \leftrightarrow q) \leftrightarrow r] \leq q \leftrightarrow r,$$

$$(**) \qquad (q \leftrightarrow r) \wedge [(p \leftrightarrow q) \leftrightarrow r] \leq p.$$

*Proof of* $(*)$. It is enough to check that

$$\text{(a) } q \wedge p \wedge [(p \leftrightarrow q) \leftrightarrow r] \leq r \qquad \text{and} \qquad \text{(b) } r \wedge p \wedge [(p \leftrightarrow q) \leftrightarrow r] \leq q.$$

For (a), we have, recalling that $p \wedge q \leq (p \leftrightarrow q)$ (13.6(b)),

$$q \wedge p \wedge [(p \leftrightarrow q) \leftrightarrow r] \leq (p \leftrightarrow q) \wedge [(p \leftrightarrow q) \leftrightarrow r] = r \wedge [(p \leftrightarrow q) \leftrightarrow r] \leq r.$$

For (b), we have

$$\begin{aligned} p \wedge r \wedge [(p \leftrightarrow q) \leftrightarrow r] &= p \wedge (p \leftrightarrow q) \wedge [(p \leftrightarrow q) \leftrightarrow r] \\ &= q \wedge (p \leftrightarrow q) \wedge [(p \leftrightarrow q) \leftrightarrow r] \leq q, \end{aligned}$$

ending the proof of $(*)$. For $(**)$, we first remark

Fact 1. *For all $p, q, r \in H$, $\quad (q \leftrightarrow r) \wedge [(p \leftrightarrow q) \leftrightarrow r] \leq (p \leftrightarrow q) \leftrightarrow q$.*

*Proof.* We must show that

(a) $(p \leftrightarrow q) \wedge (q \leftrightarrow r) \wedge [(p \leftrightarrow q) \leftrightarrow r] \ \leq\ q$,
(b) $q \wedge (q \leftrightarrow r) \wedge [(p \leftrightarrow q) \leftrightarrow r] \ \leq\ p \leftrightarrow q$.

For (a), we have

$$\begin{aligned}(p \leftrightarrow q) \wedge (q \leftrightarrow r) \wedge [(p \leftrightarrow q) \leftrightarrow r] &= r \wedge (q \leftrightarrow r) \wedge [(p \leftrightarrow q) \leftrightarrow r] \\ &= q \wedge (q \leftrightarrow r) \wedge [(p \leftrightarrow q) \leftrightarrow r] \\ &\leq q.\end{aligned}$$

For (b), we get

$$q \wedge (q \leftrightarrow r) \wedge [(p \leftrightarrow q) \leftrightarrow r] \leq r \wedge [(p \leftrightarrow q) \leftrightarrow r] \leq p \leftrightarrow q,$$

completing the proof of Fact 1.

Now, $(**)$ follows from Fact 1 and assumption (5). Similarly, one gets (5)$\Rightarrow$(2). To complete the proof, it remains to check that (1)$\Rightarrow$(5). To this end, we first observe

Fact 2. *If $H$ is a Ba, then $\forall\, p, q \in H$, $p \leftrightarrow q = \top \triangle (p \triangle q)$, where $\triangle$ stands for symmetric difference* (13.5).

*Proof.* We shall use the properties of $\triangle$ in 13.5, without explicit reference. The distributive laws in $H$ yield

$$p \leftrightarrow q = (\neg\, p \vee q) \wedge (\neg\, q \vee p) = \neg\,(p \triangle q) = \top \triangle (p \triangle q),$$

as claimed.

With Fact 2 we then obtain

$$(p \leftrightarrow q) \leftrightarrow q = \top \triangle (p \leftrightarrow q) \triangle q = \top \triangle \top \triangle p \triangle q \triangle q = p,$$

ending the proof of Proposition 3.7. ■

To end this section, we recall the notions of complete lattice and of complete Heyting algebra.

13.8. A lattice $L$ is **complete** if all subsets of $L$ have meets and joins. In particular, a complete lattice has $\perp$ and $\top$. For $S \subseteq L$, write $\bigvee S$ and $\bigwedge S$ for the join and meet, respectively, of $S$ in $L$.

A **complete Heyting algebra (cHa)** is a complete lattice that satisfies the following distributive law:

$[\wedge, \bigvee]$ For all $a \in L$ and $S \subseteq L$, $a \wedge \bigvee S = \bigvee_{s \in S} a \wedge s$.

Some authors use *frame* or *locale* for what here is called a cHa. Note that all cHa's are distributive lattices. Important examples of cHa's are furnished by topologies on any set.

If $L$ is a cHa, the implication operation in $L$ is defined by

$$a \rightarrow b = \bigvee \{x \in L : x \wedge a \leq b\}.$$

Because $L$ satisfies $[\wedge, \bigvee]$, it is easily established that the defining property of implication, the adjunction $[\rightarrow]$ in 13.2, is satisfied in $L$.

If $L$, $R$ are cHa's, a map $f : L \rightarrow R$ is a **cHa-morphism** or a $[\wedge, \bigvee]$-morphism if it preserves finite meets and arbitrary joins. We write **cHa** for the category of cHa's and $[\wedge, \bigvee]$-morphisms.

## 2. Weak equivalence algebras

Weak equivalence algebras constitute the setting in which our constructions, leading to the definition of **equivalence algebra**, will take place. The following definition should be compared with 12.4:

DEFINITION 13.9. *A* **weak equivalence algebra (wEa)***,*

$$\langle L, \leq, \top, * \rangle,$$

*is a set $L$ together with a partial order $\leq$ and a binary operation $*$ on $L$ such that, for all $a, x, y, t, z \in L$:*

[$*$ 0] $\top$ *is the largest element of $L$ in the partial order $\leq$;*
[$*$ 1] $x * y = y * x$;
[$*$ 2] $x * \top = x$;
[$*$ 3] $x * y = \top$ *iff* $x = y$;
[$*$ 4] $a \leq x * y$ *and* $a \leq t * z$ *implies* $a \leq (x * t) * (y * z)$.

*A* **weak equivalence algebra with negation (wEan)** *is a wEa, $L$, that has a least element, $\bot$, satisfying*

[*neg* 1] *For all* $x \in L$, $x * (x * \bot) = \bot$.

*If $L$ and $R$ are wEa's a map $f : L \to R$ is a* **wEa-morphism** *if it is increasing and preserves $\top$ and $*$. We say that $f$ is a* **wEa-embedding** *if it is a wEa-morphism such that for all $x, y \in L$,*

$$x \leq y \quad \textit{iff} \quad f(x) \leq f(y).$$

*The definitions of morphism and embedding of wEan's are analogous, adding the requirement that $f$ take $\bot$ to $\bot$. Write $Hom(L, R)$ for the set of wEa-morphisms (or wEan-morphisms) from $L$ to $R$.*

*Whenever convenient, we write $xy$ for $x * y$ and $\neg x$ for $x * \bot$.*

REMARK 13.10. Note that, in general, the operation $*$ **is not** associative. One of the main distinctions between the equivalence algebras and the usual algebraic structures associated to Logic is that the partial order $\leq$ **is not** definable by a connective. This introduces the need to be careful in defining concepts such as embedding. In the category of meet-semilattices (or join-semilattices), an injective increasing map is an isomorphism onto its image; for, in this case we have

$$x \leq y \quad \text{iff} \quad x \wedge y = x \text{ (resp., } x \vee y = y).$$

In the category of posets and increasing maps, it is an entirely different matter. Just consider $P = \{\{1\}, \{2\}, \{1, 2\}\}$ with its natural partial order (containment) and $Q = \{1, 2, 3\}$ with the order induced by the natural numbers. Then, $f(\{1\}) = 1$, $f(\{2\}) = 2$ and $f(\{1, 2\}) = 3$ is a bijective increasing map, but it is clear that $P$ and $Q$ are not isomorphic. This fact is at the root of the definition of *embedding of wEa*: it must be required that order be strictly preserved, for an injective increasing map to be an isomorphism onto its image. ■

From Lemmas 12.4 and 13.6 we get

COROLLARY 13.11. (a) *The Lindenbaum algebra* $\mathcal{L}$ *of* MEC *is a wEa, while* $\mathcal{L}_\perp$ *is a wEan.*

(b) *Let* $H$ *be a Heyting algebra and let* $L$ *be a subset of* $H$ *which contains* $\top$ *and is closed under the equivalence operation* $\leftrightarrow$ *of* $H$*. Then, with the partial order and operations induced by* $H$*,* $L$ *is a wEa. Moreover, if* $\perp \in L$*, then* $L$ *is a wEan. In particular,* $\langle H, \leq, \top, \leftrightarrow\rangle$ *is a wEan.*

LEMMA 13.12. *For* $x, y, z$ *in a wEa* $L$*:*

(a) $x \leq y$ *and* $x \leq z$ *implies* $x \leq y * z$.
(b) $x \leq (x * y) * y$.
(c) $x * y \leq (x * z) * (y * z)$.
(d) $x * y \leq y$ *iff* $x * y \leq x$.
(e) $x * z \leq x * y$ *iff* $x * z \leq y * z$.

*Proof.* (a) Using [$*$ 1] and [$*$ 2], we may write our hypothesis as

$$x \leq y * \top \quad \text{and} \quad x \leq \top * z.$$

An application of [$*$ 4] yields the desired conclusion.

(b) From [$*$ 4], $x \leq x * \top$ and $x \leq y * y$, we conclude

$$x \leq (x * y) * (\top * y) = (x * y) * y.$$

(c) Since $x * y \leq x * y$ and $x * y \leq z * z$, [$*$ 4] yields the desired result.

(d) From [$*$ 4], $xy \leq \top * y$ and $xy \leq x * y$, we get $xy \leq x$. The converse is clear.

(e) [$*$ 4], $x * z \leq x * y$ and $x * z \leq x * z$ yield

$$x * z \leq (x * x) * (y * z) = (y * z).$$

The converse is similar, ending the proof. ∎

DEFINITION 13.13. *A subset* $F$ *of a wEa* $L$ *is a* **filter** *if for all* $x, y, z \in L$*:*

[$fil$ 1] $\top \in F$;
[$fil$ 2] $x \in F$ *implies* $x^{\rightarrow} \subseteq F$;
[$fil$ 3] $x * y \in F$ *and* $y * z \in F$ *implies* $x * z \in F$.

*A filter is* **proper** *iff* $F \neq L$.

Note that if $L$ has a least element $\perp$, then $F$ is proper iff $\perp \notin F$.

LEMMA 13.14. *If* $F$ *is a filter in a wEa* $L$ *and* $x, y, t, z \in L$*, then*

(a) $x \in F$ *and* $x * y \in F$ *implies* $y \in F$.
(b) $x, y \in F$ *implies* $x * y \in F$.
(c) $x * y \in F$ *and* $t * z \in F$ *implies* $(xt) * (yz) \in F$.

*Proof.* (a) and (b) are a consequence of [$fil$ 3] and [$*$ 2]. For (c), note that [$fil$ 2] and 13.12(b) yield

$$\begin{cases} xy \in F & \text{implies} \quad (xt) * (yt) \in F; \\ tz \in F & \text{implies} \quad (yt) * (yz) \in F, \end{cases}$$

and so [$fil$ 3] yields $(xt) * (yz) \in F$, as desired. ∎

Example 13.15. 1. The set $\{\top\}$ is a filter in any wEa. The only condition that needs verification is [$fil$ 3]; recalling that $x * y = \top$ iff $x = y$ ([$*$ 3]), it is easily seen that it satisfies [$fil$ 3]. $\{\top\}$ is the smallest filter (with respect to containment) in any wEa.

2. More generally, let $x$ be an element of a wEa $L$. It is clear that $x^{\rightarrow}$ satisfies [$fil$ 1] and [$fil$ 2]. That it is, in fact, a filter, is a consequence of [$*$ 4]. The filter $x^{\rightarrow}$ is the **principal filter** generated by $x$.

3. It is easily established that in the Lindenbaum algebras $\mathcal{L}$ and $\mathcal{L}_{\perp}$,

*filters are exactly the theories.*

4. If $F_i$, $i \in I$, is a family of filters in $L$, then $\bigcap_{i \in I} F_i$ is a filter in $L$.

5. If $F_i$, $i \in I$, is a right-directed family of filters, that is,

For all $i$, $j \in I$, there is $k \in I$ such that $F_i, F_j \subseteq F_k$,

then $\bigcup_{i \in I} F_i$ is a filter in $L$. ■

Because the property of being a filter is preserved by intersections, we may define **the filter generated** by a subset $S$ of a wEa $L$ as

$$[S] = \bigcap \{F\colon F \text{ is a filter in } L \text{ and } S \subseteq F\}.$$

Clearly, $S \subseteq T$ implies $[S] \subseteq [T]$. The following is straightforward:

Lemma 13.16. *Let $S_i$, $i \in I$, be a right-directed collection of subsets of $L$. If $S = \bigcup_{i \in I} S_i$, then $[S] = \bigcup_{i \in I} [S_i]$. In particular, if $A \subseteq L$ and $2^A_\omega$ is the set of finite subsets of $A$, then*

$$[A] = \textstyle\bigcup_{\alpha \in 2^A_\omega} [\alpha].$$

The situations in which inverse image by a wEa-morphism preserves *proper filters* are described by

Lemma 13.17. *Let $f\colon L \to R$ be a wEa-morphism and $F$ a proper filter in $R$. Then:*

(a) *If $f$ is surjective, then $f^{-1}(F)$ is a proper filter in $L$.*
(b) *If $L$, $R$ are wEan's and $f$ is a wEa-morphism, then $f^{-1}(F)$ is a proper filter in $L$.*

If $L$ is a wEan, let $\mathcal{F}il(L)$ be the set of filters in $L$. The proof of the next result is left to the reader.

Lemma 13.18. *Let $L$ be a wEan.*

(a) *Partially ordered by inclusion, $\mathcal{F}il(L)$ is a complete lattice, where meets are given by (set-theoretical) intersection and joins are given by*

$$\textstyle\bigvee_{i \in I} F_i \quad = \quad [\bigcup_{i \in I} F_i],$$

*where $\{F_i\colon i \in I\} \subseteq \mathcal{F}il(L)$. Moreover, $\{\top\}$ is the bottom of $\mathcal{F}il(L)$, while $L$ is its top element.*

(b) *Let $f\colon L \to K$ be a wEan-morphism. Then $f$ induces a map $f^* : \mathcal{F}il(K) \to \mathcal{F}il(L)$, given by $f^*(G) = f^{-1}(G)$, which preserves all meets and arbitrary directed joins.*

## 3. $T$-operators and $*$-filters

The property that will distinguish **equivalence algebras** among their weak counterparts is the possibility of separating elements by filters. The basic model is Theorem 12.8, which shows that *theories* in the Lindenbaum algebras of MEC and MECn have that property. In this section we develop methods to identify the class of filters in a wEa that have a similar property, i.e., that generalize the idea of a *theory.* The method consists in defining this class as the set of fixed points of certain operators on a wEa. Some of the results presented here first appeared in [López-Escobar and Miraglia, 1999].

Let $D$ be a set. Recall that a map $\mu$: $2^D \to 2^D$ is

– **Inflationary** if $A \subseteq \mu(A)$, for all $A \subseteq D$;
– **Increasing** if for all $A, B \in 2^D$, $A \subseteq B$ implies $\mu(A) \subseteq \mu(B)$;
– **Idempotent** if $\mu \circ \mu = \mu$.

Let $L$ be a wEa. Define $\mu_0$: $2^L \to 2^L$ by

$$\mu_0(A) = \bigcup \{(x * y)^{\to}: \exists\, t \in L \text{ such that } (x * t),\, (t * y) \in A \cup \{\top\}\}.$$

LEMMA 13.19. *With notation as above, $\mu_0$ is increasing and inflationary. Moreover, for all $x \in L$ and $A \subseteq L$,*

$$x \in \mu_0(A) \quad \textit{implies} \quad x^{\to} \subseteq \mu_0(A).$$

*Proof.* Clearly $\mu_0$ is increasing and $x \in \mu_0(A)$ implies $x^{\to} \subseteq \mu_0(A)$. For $a \in A$, note that $a = a * \top$, with $(a * \top)$ and $(\top * \top)$ both in $A \cup \{\top\}$. Hence, $A \subseteq \mu_0(A)$. ■

For $A \subseteq L$, define a sequence of subsets of $L$, $\sigma_n^0(A)$, $n \geq 0$, by induction on $n$, as follows:

$$\sigma_0^0(A) = A \quad \text{and} \quad \sigma_{n+1}^0(A) = \mu_0(\sigma_n^0(A)).$$

Now, set $\tau_0(A) = \bigcup_{n\geq 0} \sigma_n^0(A)$. Then

PROPOSITION 13.20. *For all $A \subseteq L$, $\tau_0(A)$ is the filter generated by $A$ in $L$.*

*Proof.* It is clear that any filter containing $A$ must contain $\tau_0(A)$. By 13.19, $A \cup \{\top\} \subseteq \mu_0(A) \subseteq \tau_0(A)$ and $x \in \mu_0(A)$ implies $x^{\to} \subseteq \tau_0(A)$. To verify [$fil$ 2], let $(x * t)$ and $(t * y)$ be in $\tau_0(A)$. Since the sequence $\sigma_n^0(A)$ is increasing, there is $n \geq 0$ such that $(x * t), (t * y) \in \sigma_n^0(A)$. But then $x * y \in \sigma_{n+1}^0(A) \subseteq \tau_0(A)$. ■

From here on we shall use, interchangeably, the notation $[A]$ and $\tau_0(A)$ for the filter generated by $A$.

DEFINITION 13.21. *Let $L$ be a wEa. A $T$-**operator** on $L$ is a map $\beta$: $2^L \to 2^L$ satisfying:*

[$T$ 1] *$\beta$ is inflationary, increasing and idempotent;*
[$T$ 2] *For all $A \subseteq L$, $\beta(A)$ is a filter in $L$;*
[$T$ 3] *For all $x, y \in L$ and $A \subseteq L$,*

$$\beta(A \cup \{x\}) = \beta(A \cup \{y\}) \quad \textit{iff} \quad (x * y) \in \beta(A).$$

*Write $T_{op}(L)$ for the set of $T$-operators on $L$. Define a partial order in $T_{op}(L)$ by*

$$\alpha \leq \beta \quad \textit{iff} \quad \textit{For all } A \subseteq L, \quad \alpha(A) \subseteq \beta(A).$$

*For $\beta_i \in T_{op}(L)$, $i \in I$, and $A \subseteq L$, set*

$[T\wedge]$ $$[\bigwedge_{i\in I}\beta_i](A) = \bigcap_{i\in I}\beta_i(A).$$

REMARK 13.22. Let $L$ be a wEa. With notation as above,

(a) The map $A \in 2^L \mapsto L \in 2^L$ is the largest $T$-operator on $L$.
(b) $T_{op}(L)$ is a complete lattice, with meets as in $[T\wedge]$ of 13.21. ∎

It will be important in what follows to obtain an explicit description of the bottom of the complete lattice $T_{op}(L)$. To this end, we construct an increasing sequence of inflationary and increasing maps,

$$\tau_0 \leq \tau_1 \leq \ldots \leq \tau_n \leq \tau_{n+1} \leq \ldots$$

such that for all $A \subseteq L$, all $n \geq 0$ and all $x, y \in L$:

[c 1] $\tau_n(A)$ is a filter in $L$;
[c 2] $\tau_n(A \cup \{x\}) = \tau_n(A)$ iff $x \in \tau_n(A)$;
[c 3] $\tau_n(A \cup \{x\}) = \tau_n(A \cup \{y\})$ implies $(x * y) \in \tau_{n+1}(A)$.

For $n = 0$, $\tau_0(A)$ is the filter generated by $A$ in $L$. Assume that $\tau_n$ has been constructed, and define $\mu_{n+1}\colon 2^L \to 2^L$ as follows:

$$\mu_{n+1}(A) = \bigcup \{(x * y)^{\rightarrow}\colon \tau_n(A \cup \{x\}) = \tau_n(A \cup \{y\})\}.$$

LEMMA 13.23. *With notation as above, $\mu_{n+1}$ is increasing and for all $x \in L$ and $A \subseteq L$, $x \in \mu_{n+1}(A)$ implies $x^{\rightarrow} \subseteq \mu_{n+1}(A)$. Moreover, for all $A \subseteq L$, $\tau_n(A) \subseteq \mu_{n+1}(A)$.*

*Proof.* It is clear that $x \in \mu_{n+1}(A)$ implies $x^{\rightarrow} \subseteq \mu_{n+1}(A)$. To show that $\mu_{n+1}$ is increasing, let $A \subseteq B \subseteq L$. It is enough to verify that for $x, y \in L$,

(I) $$\tau_n(A \cup \{x\}) = \tau_n(A \cup \{y\}) \Rightarrow \tau_n(B \cup \{x\}) = \tau_n(B \cup \{y\}).$$

Since $\tau_n$ is inflationary and increasing, we have

$$x \in \tau_n(A \cup \{y\}) \subseteq \tau_n(B \cup \{y\}).$$

It follows from [c 2] that

$$\tau_n(B \cup \{y\}) = \tau_n(B \cup \{y\} \cup \{x\}).$$

Similarly, one shows that $\tau_n(B \cup \{x\}) = \tau_n(B \cup \{x\} \cup \{y\})$, proving (I). It remains to check that $\tau_n(A) \subseteq \mu_{n+1}(A)$. Since $\tau_n(A)$ is a filter, $\top \in \tau_n(A)$, and so [c 2] yields $\tau_n(A \cup \{\top\}) = \tau_n(A)$. Similarly, [c 2] yields

$$x \in \tau_n(A) \quad \text{implies} \quad \tau_n(A \cup \{x\}) = \tau_n(A) = \tau_n(A \cup \{\top\}).$$

Thus, $x = x * \top \in \mu_{n+1}(A)$, ending the proof. ∎

For $A \subseteq L$, define, by induction on $k \geq 0$, a sequence of subsets of $L$, $\sigma_k^{n+1}(A)$, as follows:

$$\sigma_0^{n+1}(A) = A \quad \text{and} \quad \sigma_{k+1}^{n+1}(A) = \mu_{n+1}(\sigma_k^{n+1}(A)).$$

It is clear that $\sigma_k^{n+1}(A)$ is increasing. Now set

$$\tau_{n+1}(A) = \bigcup_{k\geq 0} \sigma_k^{n+1}(A).$$

Clearly, $\tau_n(A) \subseteq \tau_{n+1}(A)$ (i.e., $\tau_n \leq \tau_{n+1}$) and $\tau_{n+1}$ is increasing.

PROPOSITION 13.24. *The map $\tau_{n+1}$ satisfies* [c 1], [c 2] *and* [c 3].

*Proof.* [c 1] Fix $A \subseteq L$. Clearly, $\tau_n(A)$ satisfies [$fil$ 1] and $\top \in \tau_{n+1}(A)$. Now assume that $(x * t)$, $(t * y) \in \tau_{n+1}(A)$. Then there is $k \geq 0$ such that $(x * t)$, $(t * y) \in \sigma_k^{n+1}(A)$. It is enough to show that

$$\text{(I)} \qquad \tau_n(\sigma_k^{n+1}(A) \cup \{x\}) = \tau_n(\sigma_k^{n+1}(A) \cup \{y\}),$$

because then $(x * y) \in \mu_{n+1}(\sigma_k^{n+1}(A)) = \sigma_{k+1}^{n+1}(A) \subseteq \tau_{n+1}(A)$.

Since $\tau_n$ satisfies [c 2], to get (I) it is sufficient to check that

$$\begin{cases} \text{(i) } y \in \tau_n(\sigma_k^{n+1}(A) \cup \{x\}), \\ \text{(ii) } x \in \tau_n(\sigma_k^{n+1}(A) \cup \{y\}). \end{cases}$$

Since $(x * t)$, $(t * y) \in \sigma_k^{n+1}(A)$ and $\tau_n(\sigma_k^{n+1}(A) \cup \{x\})$ is a filter containing $x$, 13.14(a) implies that (i) is verified. A similar argument proves (ii), completing the proof of [c 1].

[c 2] First notice that for all $k \geq 0$,

$$\text{(II)} \qquad \tau_{n+1}(\sigma_k^{n+1}(A)) = \tau_{n+1}(A),$$

because for all $l \geq 0$,

$$\sigma_l^{n+1}(\sigma_k^{n+1}(A)) = \mu_{n+1}^l(\mu_{n+1}^k(A)) = \mu_{n+1}^{k+l}(A) = \sigma_{k+l}^{n+1}(A) \subseteq \tau_{n+1}(A).$$

Hence, if $x \in \tau_{n+1}(A)$, then there is $k \geq 0$ such that $x \in \sigma_k^{n+1}(A)$ and so (II) yields

$$\tau_{n+1}(A \cup \{x\}) \subseteq \tau_{n+1}(\sigma_k^{n+1}(A)) \subseteq \tau_{n+1}(A),$$

and equality follows from the fact that $\tau_{n+1}$ is increasing.

That $\tau_{n+1}$ satisfies [c 3] follows from the fact that

$$\tau_n(A \cup \{x\}) = \tau_n(A \cup \{y\}) \ \Rightarrow\ (x * y) \in \mu_{n+1}(A) \subseteq \tau_{n+1}(A),$$

ending the proof. ■

Proposition 13.25. *With notation as above:*

(a) *For all integers $l > n \geq 0$, $\tau_l \circ \tau_n = \tau_l$.*
(b) *For all $n \geq 0$ and right-directed families of subsets of $L$, $B_i$, $i \in I$,*

$$\tau_n(\bigcup B_i) = \bigcup_{i \in I} \tau_n(B_i).$$

*In particular, for all $A \subseteq L$ and $n \geq 0$, $\tau_n(A) = \bigcup_{\alpha \in 2^A_\omega} \tau_n(\alpha)$.*

*Proof.* (a) We first verify that for all $n \geq 0$ and $A \subseteq L$,

$$(*) \qquad \tau_{n+1}(\tau_n(A)) = \tau_{n+1}(A).$$

Since $A \subseteq \tau_n(A)$, it follows that $\tau_{n+1}(A) \subseteq \tau_{n+1}(\tau_n(A))$. On the other hand, by Lemma 13.23, $\tau_n(A) \subseteq \mu_{n+1}(A) = \sigma_1^{n+1}(A)$. But in the proof of 13.24 (see (II)), we have shown that $\tau_{n+1}(\sigma_1^{n+1}(A)) = \tau_{n+1}(A)$, and the equality in $(*)$ follows. Then induction on $k \geq 1$ yields

$$\tau_{n+k+1} = \tau_{n+k+1} \circ \tau_{n+k} = \tau_{n+k+1} \circ (\tau_{n+k} \circ \tau_n) = (\tau_{n+k+1} \circ \tau_{n+k}) \circ \tau_n = \tau_{n+k+1} \circ \tau_n,$$

ending the proof of (a).

(b) Write $A = \bigcup_{i \in I} B_i$. It is enough to verify that if $x \in \tau_n(A)$, there is $i \in I$ such that $x \in \tau_n(B_i)$. Proceed by induction on $n \geq 0$. For $n = 0$, the result follows from 13.16. Assume the result is true for $n$. We first prove

FACT. *For all $k \geq 0$,* $\sigma_k^{n+1}(A) = \bigcup_{i \in I} \sigma_k^{n+1}(B_i)$.

*Proof.* For $k = 0$ there is nothing to prove. Assume the result is true for $k$ and that $y \in \sigma_{k+1}^{n+1}(A)$. Then there are $u, v \in L$ such that

$$\tau_n(\sigma_k^{n+1}(A) \cup \{u\}) = \tau_n(\sigma_k^{n+1}(A) \cup \{v\}),$$

and $y \geq (u * v)$. Note that the family $(\sigma_k^{n+1}(B_i) \cup \{u\})$, $i \in I$, is directed. Moreover, by induction,

$$\sigma_k^{n+1}(A) \cup \{u\} = \bigcup_{i \in I} (\sigma_k^{n+1}(B_i) \cup \{u\})$$

Since $v \in \tau_n(\sigma_k^{n+1}(A) \cup \{u\})$, the (first) induction hypothesis yields $k \in I$ such that $v \in \tau_n(\sigma_k^{n+1}(B_k) \cup \{u\})$. By a similar argument, there is $j \in I$ such that $u \in \tau_n(\sigma_j^{n+1}(B_i) \cup \{u\})$. If we select $i \in I$ such that $B_j,\ B_k \subseteq B_i$, we conclude that

$$u \in \tau_n(\sigma_k^{n+1}(B_i) \cup \{v\}) \quad \text{and} \quad v \in \tau_n(\sigma_k^{n+1}(B_i) \cup \{u\}).$$

It follows that

$$\tau_n(\sigma_k^{n+1}(B_i) \cup \{u\}) = \tau_n(\sigma_k^{n+1}(B_i) \cup \{v\}),$$

and so $y \in (u * v)^{\rightarrow} \subseteq \sigma_{k+1}^{n+1}(B_i)$, ending the proof of the Fact.

If $x \in \tau_{n+1}(A)$, then $x \in \sigma_k^{n+1}(A)$, for some $k \geq 0$. By the Fact, there is $i \in I$ such that $x \in \sigma_k^{n+1}(B_i) \subseteq \tau_{n+1}(B_i)$, as desired. ∎

For $A \subseteq L$, set $\tau_L(A) = \bigcup_{n \geq 0} \tau_n(A)$.

Whenever clear from context, we omit the name of the wEa $L$ from the notation. Since this is a directed union of filters containing $A$, $\tau(A)$ is a filter containing $A$. Moreover, $A \mapsto \tau(A)$ is increasing and inflationary, because the same is true of each $\tau_n$.

PROPOSITION 13.26. *For $A \subseteq L$ and $x, y \in L$:*

(a) $x \in \tau(A)$ *iff* $\tau(A) = \tau(A \cup \{x\})$.
(b) $\tau(A \cup \{x\}) = \tau(A \cup \{y\})$ *iff* $x \in \tau(A \cup \{y\})$ *and* $y \in \tau(A \cup \{x\})$.
(c) *The operation $A \mapsto \tau(A)$ satisfies* $[T\ 2]$ *and* $[T\ 3]$.
(d) *For all $n \geq 0$ and all $B \subseteq L$,*

$$B \subseteq \tau(A) \quad \textit{implies} \quad \tau(B) \subseteq \tau(A).$$

(e) *For all $A \subseteq L$, $\tau(\tau(A)) = \tau(A)$ and $\tau$ is a $T$-operator on $L$.*

*Proof.* (a) If $x \in \tau(A)$, there is $n \geq 0$ such that $x \in \tau_n(A)$. By [c 2], $\tau_n(A \cup \{x\}) = \tau_n(A)$. It follows from 13.25 that for all $l > n$,

$$\tau_l(A \cup \{x\}) = \tau_l(\tau_n(A \cup \{x\})) = \tau_l(\tau_n(A)) = \tau_l(A),$$

and so $\tau(A \cup \{x\}) = \tau(A)$. The converse is clear.

(b) If the right-hand side of the equivalence holds, then (a) yields

$$\tau(A \cup \{x\}) = \tau(A \cup \{x\} \cup \{y\}) = \tau(A \cup \{y\}),$$

while the converse is obvious.

(c) We have already observed that $\tau(A)$ is a filter. If $(x * y) \in \tau(A)$, since $\tau(A) \subseteq \tau(A \cup \{x\})$ and this last set is a filter, we get $y \in \tau(A \cup \{x\})$. Similarly, $x \in \tau(A \cup \{y\})$ and equality follows from (b). Conversely, if $\tau(A \cup \{x\}) = \tau(A \cup \{y\})$, since the sequence

$\tau_n(A)$ is increasing, (b) yields $n \geq 0$ such that $x \in \tau_n(A \cup \{y\})$ and $y \in \tau_n(A \cup \{x\})$. Consequently, $\tau_n(A \cup \{x\}) = \tau_n(A \cup \{y\})$ and $(x * y) \in \tau_{n+1}(A) \subseteq \tau(A)$.

(d) By induction on $n \geq 0$, we prove

Fact 1. *For all $n \geq 0$ and all $B \subseteq L$:*

$$B \subseteq \tau(A) \quad \textit{implies} \quad \tau_n(B) \subseteq \tau(A).$$

*Proof.* For $n = 0$, since $\tau(A)$ is a filter, it must contain $\tau_0(B)$ (13.20). Assume the result true for $n \geq 0$. We then have

Fact 2. *For all $D \subseteq L$, $D \subseteq \tau(A)$ implies $\mu_{n+1}(D) \subseteq \tau(A)$.*

*Proof.* If $\tau_n(D \cup \{x\}) = \tau_n(D \cup \{y\})$, then $D \subseteq \tau(A) \subseteq \tau(A \cup \{x\})$ implies $D \cup \{y\} \subseteq \tau(A \cup \{y\})$. By induction, we get

$$x \in \tau_n(D \cup \{y\}) \subseteq \tau(A \cup \{y\}).$$

Similarly, one shows that $y \in \tau_n(D \cup \{x\}) \subseteq \tau(A \cup \{x\})$. By (b), $\tau(A \cup \{x\}) = \tau(A \cup \{y\})$, and so (c) yields $(x * y) \in \tau(A)$. Since $\tau(A)$ is a filter, we get $\mu_{n+1}(D) \subseteq \tau(A)$, ending the proof of Fact 2.

By Fact 2, if $B \subseteq \tau(A)$, then $\sigma_k^{n+1}(B) \subseteq \tau(A)$, for all $k \geq 0$. Hence, $\tau_{n+1}(B) \subseteq \tau(A)$, completing the induction step and the proof of Fact 1. Item (d) is now clear, while (e) is a consequence of (d) and 13.21. ∎

Proposition 13.25(b) yields

Corollary 13.27. *If $\{B_i \colon i \in I\}$ is a right-directed family of subsets of a wEa $L$ and $B = \bigcup_{i \in I} B_i$, then*

$$\tau(B) = \textstyle\bigcup_{i \in I} \tau(B_i).$$

*In particular, for all $A \subseteq L$,*

[*compactness*] $$\tau(A) = \textstyle\bigcup_{\alpha \in 2^A_\omega} \tau(\alpha),$$

*where $2^A_\omega$ is the set of finite subsets of $A$.*

Proposition 13.28. *If $L$ is a wEa and $\beta \in T_{op}(L)$, then $\tau \leq \beta$.*

*Proof.* By induction on $n \geq 0$, we check that for all $A, B \subseteq L$,

(I) $$B \subseteq \beta(A) \quad \text{implies} \quad \tau_n(B) \subseteq \beta(A).$$

For $n = 0$ there is nothing to prove because $\beta(A)$ is a filter. Assume the result true for $n \geq 0$; we then have

Fact. *$B \subseteq \beta(A)$ implies $\mu_{n+1}(B) \subseteq \beta(A)$.*

*Proof.* Since $\beta(A)$ is a filter, it is enough to verify that if $x, y \in L$ are such that $\tau_n(B \cup \{x\}) = \tau_n(B \cup \{y\})$, then $(x * y) \in \beta(A)$. The induction hypothesis guarantees that

$$\begin{cases} x \in \tau_n(B \cup \{y\}) \subseteq \beta(A \cup \{y\}), \\ y \in \tau_n(B \cup \{x\}) \subseteq \beta(A \cup \{x\}), \end{cases}$$

and so $\beta(A \cup \{x\}) = \beta(A \cup \{y\})$; but then $(x * y) \in \beta(A)$, as desired.

The Fact implies that if $B \subseteq \beta(A)$, then $\sigma_k^{n+1}(B) \subseteq \beta(A)$, for all $k \geq 0$. Hence, $\tau_{n+1}(B) \subseteq \beta(A)$, completing the induction step. ■

Our next result describes some of the basic properties of $T$-operators.

PROPOSITION 13.29. *Let $L$ be a wEa and let $\beta, \gamma \in T_{op}(L)$. Let $B_i$, $i \in I$, be a family in $2^L$.*

(a) $\gamma \leq \beta$ *iff* $\gamma \circ \beta = \beta$ *iff* $\beta \circ \gamma = \beta$.

(b) *Write $Fix(\beta) = \{A \in 2^L\colon \beta(A) = A\}$ for the set of fixed points of $\beta$ in $L$. Then $\gamma \leq \beta$ implies $Fix(\beta) \subseteq Fix(\gamma)$.*

(c) *If $D = \bigcap_{i \in I} \beta(B_i)$, then $D = \beta(D)$.*

*Proof.* (a) If $\gamma \leq \beta$ then for $A \subseteq L$, we have

$$\begin{cases} \beta(A) \subseteq \gamma(\beta(A)) \subseteq \beta(\beta(A)) = \beta(A), \\ \beta(A) \subseteq \beta(\gamma(A)) \subseteq \beta(\beta(A)) = \beta(A), \end{cases}$$

proving the stated identities. The converses are left to the reader.

(b) For $A \in Fix(\beta)$, (a) yields $\gamma(A) = \gamma(\beta(A)) = \beta(A) = A$, as needed.

(c) Since $D \subseteq \beta(B_i)$, we have $\beta(D) \subseteq \beta(\beta(B_i) = \beta(B_i)$, $i \in I$. Hence, $\beta(D) \subseteq D$ and equality follows. ■

DEFINITION 13.30. *Let $L$ be a wEa. A subset $A$ of $L$ is a* **$*$-filter** *if $A \in Fix(\tau)$. Write $S(L)$ for the set of* **proper** *$*$-filters on $L$.*

Proposition 13.29(b) yields

COROLLARY 13.31. *If $\beta \in T_{op}(L)$, then all fixed points of $\beta$ are $*$-filters on $L$.*

As an application we give a sufficient condition for all extensions of a filter to be $*$-filters (including itself). Recall that $[U]$ is the filter generated by $U \subseteq L$.

PROPOSITION 13.32. *Let $L$ be a wEa and let $F$ be a filter in $L$. Assume that $F$ satisfies the following condition:*

$$[**] \qquad \begin{cases} \text{For all finite subsets } S \cup \{x, y\} \subseteq L, \\ [F \cup S \cup \{x\}] = [F \cup S \cup \{y\}] \Rightarrow x * y \in [F \cup S]. \end{cases}$$

*Then all filters in $L$ containing $F$ are $*$-filters.*

*Proof.* For $A \subseteq L$ define $\beta\colon 2^L \to 2^L$ by $\beta(A) = [F \cup A]$. It is clear that $\beta$ satisfies conditions $[T\ 1]$ and $[T\ 2]$ in Definition 13.21. To check that it also has $[T\ 3]$, assume that for $x, y \in L$, we have

$$\beta(A \cup \{x\}) = \beta(A \cup \{y\}).$$

By Lemma 13.16, there are finite $S_1, S_2 \subseteq A$ such that

$$x \in [F \cup S_1 \cup \{y\}] \quad \text{and} \quad y \in [F \cup S_2 \cup \{x\}].$$

If $S = S_1 \cup S_2$, then $S$ is finite and we have

$$x \in [F \cup S \cup \{y\}] \quad \text{and} \quad y \in [F \cup S \cup \{x\}],$$

that is, $[F \cup S \cup \{x\}] = [F \cup S \cup \{y\}]$. By $[**]$, we conclude that

$$x * y \in [F \cup S] \subseteq [F \cup A] = \beta(A),$$

establishing that $\beta$ is a $T$-operator in $L$. By 13.31, all fixed points of $\beta$ are $*$-filters. It is clear that $F \in Fix(\beta)$. Now notice that if $S$ is a *fixed* finite set in $L$ and $G = [F \cup S]$, then $G$ also satisfies $[**]$, because for all finite sets $S' \subseteq L$,

$$[G \cup S'] \;=\; [F \cup S \cup S'].$$

The conclusion now follows from 13.16 and 13.27, ending the proof. ∎

PROPOSITION 13.33. *Let $f\colon L \to R$ be a wEa-morphism.*

(a) *For $\beta \in T_{op}(R)$ and $A \subseteq L$, define*

$$f^*\beta(A) \;=_{def}\; f^{-1}(\beta(f(A))).$$

*Then $f^*\beta \in T_{op}(L)$.*

(b) *The inverse image of a $*$-filter in $R$ is a $*$-filter in $L$. If $f$ is a wEan-morphism, then inverse image by $f$ takes $S(R)$ into $S(L)$.*

*Proof.* (a) Write $\alpha = f^*\beta$; it is clear that $\alpha$ is inflationary and increasing. Moreover, by 13.17, $\alpha(A)$ is a filter in $L$. To verify that $\alpha$ is idempotent, we have for $A \subseteq L$ and recalling that $f(f^{-1}(B)) \subseteq B$ $(B \subseteq R)$,

$$\begin{aligned}\alpha(\alpha(A)) &= f^{-1}\;\beta\;f\;\alpha(A) = f^{-1}\;\beta\;f\;f^{-1}\;\beta\;f(A)\\ &\subseteq f^{-1}\;\beta\;\beta\;f(A) = f^{-1}\;\beta\;f(A) = \alpha(A),\end{aligned}$$

where composition is written by superposition for ease of reading. Hence, $\alpha(A) = \alpha(\alpha(A))$, as desired. It remains to verify that $\alpha$ satisfies $[T\ 3]$ in 13.21. For $A \cup \{x\} \cup \{y\} \subseteq L$, assume that $\alpha(A \cup \{x\}) = \alpha(A \cup \{y\})$. Now observe that

$$f(x) \in \beta(f(A) \cup \{f(y)\}) \quad \text{and} \quad f(y) \in \beta(f(A) \cup \{f(x)\}),$$

which follow from $f(A \cup \{y\}) = f(A) \cup \{f(y)\}$ and the equality $\alpha(A \cup \{x\}) = \alpha(A \cup \{y\})$. Thus, $f(x) * f(y) = f(x * y) \in \beta(f(A))$ and $(x * y) \in \alpha(A)$, ending the proof of (a).

(b) Let $B$ be a $*$-filter in $R$. We check that $f^{-1}(B)$ is a fixed point of $\alpha$ and then 13.31 will guarantee that $f^{-1}(B)$ is a $*$-filter. We have

$$\alpha(f^{-1}(B)) = f^{-1}\;\tau_R\;f\;f^{-1}(B) \subseteq f^{-1}\tau_R(B) = f^{-1}(B),$$

and so $f^{-1}(B) \in Fix(f^*\tau_R)$, as claimed. ∎

We end this section with the concept of **cokernel** of wEa-morphisms.

DEFINITION 13.34. *Let $L \xrightarrow{f} R$ be a wEa-morphism. Define*

$$\text{coker } f \;=\; f^{-1}(\tau_R(\{\top\})) \;=\; f^*\tau_R(\{\top\}).$$

When $\{\top\}$ is a $*$-filter in $R$ (as will be the case if it is an equivalence algebra) coker $f$ takes the familiar form $\{x \in L\colon f(x) = \top\}$. The following is a straightforward consequence of the previous results:

COROLLARY 13.35. *If $L \xrightarrow{f} R$ is a wEa-morphism, then*

(a) coker $f$ *is a $*$-filter in $L$.*

(b) *If $\{\top\}$ is a $*$-filter in $R$, then $f$ is injective iff* coker $f$ $= \{\top\}$.

## 4. Equivalence algebras

We now introduce the structures that generalize the Lindenbaum algebras of MEC and MECn:

DEFINITION 13.36. *A wEa (wEan) $L$ is an* **equivalence algebra (Ea)** *if for all $x \in L$, $x^{\to}$ is a $*$-filter in $L$. An* **equivalence algebra with negation (Ean)** *is a wEan which is an Ea.*

*If $L$ and $R$ are Ea's, an* **Ea-morphism**, *$L \xrightarrow{f} R$, is a wEa-morphism. In case $L$ and $R$ are Ean's, $f$ is required to take $\perp$ to $\perp$. Write* **EA** *and* **EAn** *for the categories of equivalence algebras and equivalence algebras with negation, respectively.*

The following results will yield our first examples of equivalence algebras.

PROPOSITION 13.37. *In the Lindenbaum algebra $\mathcal{L}$ of* MEC, *the $*$-filters correspond to its theories. In particular, for all $x \in \mathcal{L}$, $x^{\to}$ is a $*$-filter.*

*Proof.* It follows from 12.4 that $\mathcal{L}$ is a wEa and from 12.6 that the operation

$$U \subseteq \mathcal{L} \mapsto U^t \text{ (the theory generated by } U)$$

is a $T$-operator on $\mathcal{L}$ (it is easily established that any theory in $\mathcal{L}$ is a filter). By 13.28, for all $U \subseteq \mathcal{L}$, we have $\tau(U) \subseteq U^t$. To prove the reverse containment, we proceed by induction on the length of proof trees to verify that for all $U, V \subseteq L$,

$$\text{(I)} \qquad U \subseteq \tau(V) \quad \text{and} \quad U \vdash x \quad \text{implies} \quad x \in \tau(V).$$

We may as well suppose that $x \notin U$. The following possibilities arise:

(i) $x$ comes from an application of the elimination rule. Then there are proofs of strictly smaller length that $U \vdash (x * y)$ and $U \vdash y$. By induction, $(x * y)$, $y \in \tau(V)$; since $\tau(V)$ is a filter, we get $x \in \tau(V)$, as needed.

(ii) $x$ comes from an application of the introduction rule. Then we have $x = (a * b)$ and there are proofs of strictly smaller length of $U,\ a \vdash b$ and $U,\ b \vdash a$. By induction, $b \in \tau(V \cup \{a\})$ and $a \in \tau(V \cup \{b\})$, that is, $\tau(V \cup \{a\}) = \tau(V \cup \{b\})$. But then $x = (a * b) \in \tau(V)$, completing the proof. ∎

If $H$ is a Heyting algebra (Section 13.1), consider the structure $H_{eq} = \langle H, \leq, \top, \leftrightarrow \rangle$. Then we have:

PROPOSITION 13.38. *Let $H$ be a Heyting algebra.*

- (a) *The following are equivalent for $F \subseteq H$:*
  - (1) *$F$ is a filter in $H_{eq}$;*
  - (2) *$F$ is a (lattice-theoretic) filter in $H$.*
- (b) *A subset of $H_{eq}$ is a $*$-filter iff it is a filter.*

*Proof.* (a) Clearly, (2) implies (1). For the converse, it is enough to verify that a filter $F$ in the wEa $H_{eq}$ is closed under meets. For $x, y \in F$, note that

$$y \leq (x \to y) = (x \leftrightarrow (x \wedge y)).$$

Thus, $(x \leftrightarrow (x \wedge y)) \in F$. But then, since $x \in F$ and $F$ is a wEa-filter, we get $(x \wedge y) \in F$, as needed.

(b) It is easily established that in any wEa $L$ the operation $A \mapsto [A]$ is inflationary, increasing and idempotent and obviously satisfies $[T\ 2]$. Since $[A] \subseteq \tau(A)$, for all $A \subseteq L$, it follows from 13.28 that to prove equality it suffices to show that the operation of "filter generated by" satisfies $[T\ 3]$. If $F$ is a filter in $H_{eq}$ (which by (a) is a filter in $H$) and $x, y \in H$ are such that $[F \cup \{x\}] = [F \cup \{y\}]$, since $F$ is closed under meets, by 13.1 there are $a, b \in F$ such that

$$y \geq a \wedge x \quad \text{and} \quad x \geq b \wedge y.$$

Hence, $x \wedge (a \wedge b) = y \wedge (a \wedge b)$; by 13.6(b), $(a \wedge b) \leq (x * y)$. Thus, $(x * y) \in F$, showing $[T\ 3]$ and ending the proof. ∎

From 13.37 and 13.38 we get

Corollary 13.39. (a) *The Lindenbaum algebra* $\mathcal{L}$ *of* MEC *is an Ea.*

(b) *The Lindenbaum algebra* $\mathcal{L}_\perp$ *of* MECn *is an Ean.*

(c) *If* $H$ *is a Ha, then* $H_{eq} = \langle H, \leq, \top, \leftrightarrow \rangle$ *is an Ean.*

Further examples come from the preservation of $*$-filters by inverse image of wEa-morphisms (Proposition 13.33), namely

Corollary 13.40. *Let* $L \xrightarrow{f} R$ *be a wEa-morphism.*

(a) *If* $R$ *is an Ea* (*Ean*) *and* $f$ *is an embedding, then* $L$ *is an Ea* (*resp., Ean*).

(b) *If* $H$ *is a Ha and* $L \subseteq H$ *is such that* $\top \in L$ ($\perp \in L$) *and* $L$ *is closed under* $\leftrightarrow$, *then, with the structure induced by* $H_{eq}$, $L$ *is an Ea* (*resp., Ean*).

*Proof.* For (a), note that for all $x \in L$, $f^{-1}(f(x)^{\rightarrow}) = x^{\rightarrow}$, and the conclusion follows from 13.33(b); (b) is immediate from (a). ∎

Example 13.41. If $L$ is a linear order, then $L$ is a Ha, where

$$x \rightarrow y \quad = \quad \begin{cases} \top & \text{if } x \leq y; \\ y & \text{if } y < x, \end{cases}$$

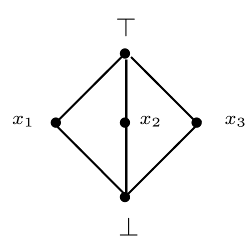
that is, this operation satisfies the fundamental adjunction $[\rightarrow]$ in 13.2. Consequently, equivalence in $L$ is given by

$$x \leftrightarrow y \quad = \quad (x \rightarrow y) \wedge (y \rightarrow x) \quad = \quad \begin{cases} \top & \text{if } x = y; \\ x \wedge y & \text{if } x \neq y. \end{cases} \qquad ∎$$

We now construct further examples of equivalence algebras, which will be useful in the future.

Example 13.42. Let $C_3 = \{\perp,\ x_1,\ x_2,\ x_3,\ \top\}$ be a set with five elements, partially ordered as follows:

$\top$ is its top element, $\perp$ is its bottom element and the $x_i$'s are unrelated.



Note that $C_3$ is a complete lattice, that is, all subsets of $C_3$ have sup and inf. However, $C_3$ is not distributive, because

$$x_1 \wedge (x_2 \vee x_3) = x_1 \wedge \top = x_1,$$

while

$$(x_1 \wedge x_2) \vee (x_1 \wedge x_3) = \bot \vee \bot = \bot.$$

Thus, $C_3$ cannot be embedded, as a lattice, into a Heyting algebra. We shall see shortly that $C_3$ has a natural structure of Ean, with which it can be embedded, as an Ean, into a cHa.

Define a binary operation $*$ on $C_3$ by the following rules:

(1) For all $z \in C_3$, $\; z * \top = \top * z = z \;$ and $\; z * z = \top$.

(2) If $i \neq j$, then $x_i * x_j = x_j * x_i =_{def} x_k$, where $k$ is the unique integer in $\{1, 2, 3\}$ distinct from $i$ and $j$.

(3) For all $z \neq \bot$ in $C_3$, $\; z * \bot = \bot * z = \bot$.

We shall prove that $\langle C_3, \leq, *, \top\rangle$ is an Ean. It is clear that $C_3$ satisfies axioms $[*\ i]$, $i = 1, 2, 3$ (13.9). We outline the argument to show that $C_3$ satisfies $[*\ 4]$; assume that

$$a \leq b * c \quad \text{and} \quad a \leq u * v.$$

We may as well suppose that $a \neq \bot, \top$; if $a = x_1$, since the roles of $b * c$ and $u * v$ are symmetrical, we have the following possibilities:

(i) $b * c = x_1 = u * v$; (ii) $b * c = \top \;$ and $\; x_1 = u * v$.

In case (i), a typical situation is $b = x_2$, $c = x_3$, $u = x_1$ and $v = \top$ or $v = x_1$. In these cases,

$$(b * u) * (c * v) = \begin{cases} x_1 * x_1 \ = \top & \text{if } v = x_1, \\ x_1 * \top \ = x_1 & \text{if } v = \top, \end{cases}$$

proving $[*\ 4]$. In case (ii), we must have $b = c$ and the conclusion follows from 13.12(c). The other cases can be treated similarly. Hence, $\langle C_3, *, \leq, \top, \bot\rangle$ is a wEan. It is easily established that the operation $*$ is not associative. Notice that the proper filters in $C_3$ are

$$x_j{}^{\rightarrow},\ j = 1,\ 2,\ 3, \text{ and } C_3 \setminus \{\bot\}.$$

It is straightforward (and a bit tedious) to check, using 13.32, that all filters in $C_3$ are $*$-filters. Alternatively, let $X = \{0, 1, 2, 3\}$ and consider the following collection of subsets of $X$:

$$O_0 = \{0\} \quad \text{and} \quad O_j = \{0, j\},\ j = 1,\ 2,\ 3.$$

Note that for $i \neq j$, $O_j \cap O_i = O_0 = O_1 \cap O_2 \cap O_3$. We take the empty set and the $O_j$'s as a basis for a topology $\Omega$ on $X$. For $j, i = 1, 2, 3$, we have

$$O_j \leftrightarrow O_i = \bigcup \{O_k : O_k \cap O_j = O_k \cap O_i\} \; = \begin{cases} X & \text{if } j = i, \\ O_k & \text{if } i \neq j \text{ and } k \neq i, j. \end{cases}$$

On the other hand,

$$O_j \leftrightarrow \emptyset = \bigcup \{O_k\colon O_j \cap O_k = \emptyset\} = \bigcup \emptyset = \emptyset.$$

By these observations the map $\sigma$: $C_3 \to \Omega$, defined by

$$\sigma(\perp) = \emptyset, \quad \sigma(\top) = X \quad \text{and} \quad \sigma(x_j) = O_j,\ j = 1, 2, 3,$$

is a wEan-embedding of $C_3$ in $\Omega$. We have

$$\begin{cases} x_j^{\rightarrow} & = \sigma^{-1}(O_j^{\rightarrow}),\ j = 1, 2, 3, \text{ and} \\ C_3 \setminus \{\perp\} & = \sigma^{-1}(O_0^{\rightarrow}), \end{cases}$$

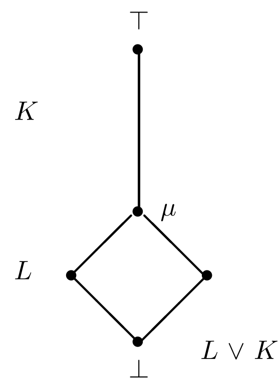
and so, by 13.38 and 13.33(b), *all filters* in $C_3$ are $*$-filters. Thus, $C_3$ is an Ean and the filter $C_3^t =_{def} C_3 \setminus \{\perp\}$ is an equivalence algebra. ■

We now introduce a method for constructing equivalence algebras that will provide further examples of these structures.

13.43. **Linear sums.** If $L$ is a wEa, set

$$L^t = L \setminus \{\top\} \quad \text{and} \quad L^b = L \setminus \{\perp\}.$$

We shall deal with wEan's, but the method applies just as well to weak equivalence algebras. For wEan's $L$ and $K$ define a partially ordered set, $\langle L \vee K, \leq \rangle$, constructed as follows:




Domain of $L \vee K$: The disjoint union of $L^t$, $K^b$ and a new element $\mu$. Equivalently, the disjoint union of $L$ and $K$, identifying $\top_L$ with $\perp_K$;

Partial order in $L \vee K$: Determined by the following rules:

1. In the copies of $L^t$ and $K^b$ inside $L \vee K$, $\leq$ coincides with the original orders in $L$ and $K$, respectively;

2. For all $l \in L$, $k \in K$, $\quad l \leq \mu \leq k$.

We shall write $\perp$, $\top$ for the least element of $L \vee K$. Note that $\perp$ is $\perp_L$, while $\top$ is $\top_K$. Define a binary operation $*$ on $L \vee K$, as follows:

$*$ For all $x \in L \vee K$, $\quad x * x = \top$;

$*$ For $a, b \in K^b$,

$$a * b = b * a = \begin{cases} a *_K b & \text{if } (a *_K b) \neq \perp_K; \\ \mu & \text{otherwise.} \end{cases}$$

$*$ For $a \neq b$ in $L^t$, $*$ coincides with the original operation in $L$;

$*$ For all $a \in L^t$ and $b \in K^b$,

$$\begin{cases} a * \mu = \mu * a = a \\ b * \mu = \mu * b = b *_K \perp_K; \\ a * b = b * a = a. \ ■ \end{cases}$$

Lemma 13.44. *Let $L$, $K$ be wEan's. With notation as above:*

(a) *The maps $L \xrightarrow{g} (L \vee K)$ and $K \xrightarrow{h} (L \vee K)$ defined by*

$$g(x) = \begin{cases} x & \textit{if } x \neq \top_L, \\ \top & \textit{if } x = \top_L, \end{cases} \qquad h(x) = \begin{cases} x & \textit{if } x \neq \bot_K, \\ \mu & \textit{if } x = \bot_L, \end{cases}$$

*are* isomorphisms *of* $L$ *onto the substructure* $L^t \cup \{\top\}$ *and of* $K$ *onto the substructure* $K^b \cup \{\mu\}$ *of* $(L \vee K)$*, respectively.*

(b) $\forall\ a, b \in (L \vee K), \quad a * b \in K^b \cup \{\mu\} \quad \Leftrightarrow \quad a, b \in K^b \cup \{\mu\}.$

*Proof.* (a) Item (a) is clear; regarding (b), note that the product of an element of $L^t$ by an element of $K^b \cup \{\mu\}$ belongs to $L^t$, while the product of two elements of $L^t$ is in $L^t \cup \{\top\}$ and $\top \neq \mu$. ∎

PROPOSITION 13.45. *With notation as above,* $L \vee K$ *is a wEan. Moreover, if* $L$ *and* $K$ *are Ean's, the same is true of* $L \vee K$.

*Proof.* The proof is rather long, because of the number of conditions to be verified. Let $K^\sharp = K^b \cup \{\mu\}$.

Axioms $[* \ i]$, $i = 1, 2, 3$ (13.9), are straightforward. To prove $[* \ 4]$, assume that

$$a \leq b * c \quad \text{and} \quad a \leq u * v.$$

If $\mu \leq a$, it follows from 13.44(b) that $b, c, u, v \in K^\sharp$. Hence, the isomorphism $h$ in 13.44(a) guarantees that $[* \ 4]$ holds. It remains to check the case in which $a \in L^t$, i.e., $a < \mu$. We have

(I) $b * c, u * v \in K^\sharp$: Then $b, c, u, v \in K^\sharp$, wherefrom we get $(b * u) * (u * v) \in K^\sharp$, proving $[* \ 4]$;

(II) $b * c \in K^\sharp$ and $u * v \in L^t$: Then $b, c \in K^\sharp$. Taking into account symmetry and (I), we are left with the following possibilities:

(i) $u, v \in L^t$: In this case, $u * b = u$, $v * c = v$ and there is nothing to prove.
(ii) $u \in L^t$ and $v \in K^\sharp$: In this case we have

$$u * b = u < v * c \in K^\sharp,$$

and so $a \leq u * v = u = (u * b) * (v * c)$, as needed.

(III) $b * c, u * v \in L^t$: Then (13.44(b)) at least one among $\{b, c\}$ and $\{u, v\}$ must be in $L^t$. Assume, to fix ideas, that $b, u \in L^t$. Further, we may also assume that one among $\{c, v\}$ is outside $L^t$, otherwise the conclusion follows from the fact that $L$ is a wEa. We are then left with

(iii) $v, c \in K^\sharp$: In this case

$$a \leq b * c = b \quad \text{and} \quad a \leq u * v = u,$$

and so $a \leq bu$ (apply 13.12(a) in $L$). On the other hand

$$(b * u) * (c * v) = b * u,$$

since $b * u \in L^t$ and $(c * v) \in K^\sharp$.

(iv) $c \in L^t$ and $v \in K^b$: In this case we have $a \leq u * v = u$, as well as $c = c * v$. But then, $[* \ 4]$ applied in $L$ to

$$a \leq b * c \quad \text{and} \quad a \leq u$$

leads to $a \leq (b * u) * c = (b * u) * (c * v)$, as needed.

This completes the verification that $L \vee K$ satisfies the axioms of wEa. It is clear that for all $x \in L \vee K$,

$$x * (x * \perp) = \perp,$$

and so $L \vee K$ is, in fact, a wEan.

Now assume that both $L$ and $K$ are equivalence algebras. We must verify that all principal filters in $L \vee K$ are $*$-filters. Let $\tau_L$ and $\tau_K$ be the bottom of the lattice of $T$-operators on $L$ and $K$, respectively (13.26(e)). By the isomorphism $h$ in 13.44(a), we can consider that $\tau_K$ is defined for all subsets of $K^\sharp$. Define $\gamma$: $2^{L^t} \to 2^{L^t}$ by

$$\gamma(A) = \tau_L(A) \setminus \{\top_L\}.$$

Since $\top_L \in \tau_L(A)$ for all $A$ (it is a filter in $L$), the fact that $\tau_L$ is increasing and inflationary implies that the same is true of $\gamma$. To check that $\gamma$ is idempotent, note that for all $A \subseteq L^t$,

$$\begin{aligned}\gamma(\gamma(A)) = \gamma(\tau_L(A) \setminus \{\top_L\}) &= \tau_L(\tau_L(A) \setminus \{\top_L\}) \ \setminus \ \{\top_L\} \\ &\subseteq \tau_L(\tau_L(A)) \ \setminus \ \{\top_L\} = \tau_L(A) \ \setminus \ \{\top_L\} = \gamma(A),\end{aligned}$$

as needed.

To simplify notation, set $\Sigma = L \vee K$. Define a map $\beta$: $2^\Sigma \to 2^\Sigma$ as follows: for $A \subseteq \Sigma$

$$\beta(A) \quad = \quad \begin{cases} \gamma(A \cap L^t) \cup K^\sharp & \text{if } A \cap L^t \neq \emptyset, \\ \tau_K(A) & \text{if } A \cap L^t = \emptyset. \end{cases}$$

It is clear that $\beta$ is increasing, inflationary and that $\beta(\beta(A)) = \beta(A)$ whenever $A \cap L^t = \emptyset$. If $A \cap L^t = B \neq \emptyset$, then, since $\beta(A) \cap L^t = \gamma(B)$, we get using the idempotency of $\gamma$,

$$\beta(\beta(A)) = \beta(\gamma(B) \cup K^\sharp) = \gamma(\gamma(B)) \cup K^\sharp = \gamma(B) \cup K^\sharp = \beta(A).$$

The same technique used to verify that $\Sigma$ is a wEan can be used to show that $\beta(A)$ is a filter in $\Sigma$. Thus, to prove that $\beta$ is a $T$-operator in $\Sigma$ it remains to prove condition [$T$ 3] in 13.21. We begin with

FACT 1. *If $A \cap L^t = \emptyset$ and $y \in K^\sharp$, then $\beta(A \cup \{y\}) \subseteq K^\sharp$.*

*Proof.* Just note that $(A \cup \{y\}) \cap L^t = \emptyset$, and so the value of $\beta$ at $A \cup \{y\}$ is equal to the value of $\tau_R$ at this set.

For $A \subseteq \Sigma$ and $x, y \in \Sigma$, assume that $\beta(A \cup \{x\}) = \beta(A \cup \{y\})$.

<u>Case 1. $A \cap L^t = \emptyset$:</u> If $x * y \in K^\sharp$, then $x, y \in K^\sharp$, and so the fact that $\tau_K$ is a $T$-operator in $K$ and the isomorphism $h$ in 13.44(a) imply that $x * y \in \beta(A)$.

If $x * y \in L^t$, at least one of them must be in $L^t$. Suppose, without loss of generality, that $x \in L^t$. But then, since $x \in \beta(A \cup \{y\})$, Fact 1 guarantees that $y$ is also in $L^t$. Now the definition of the partial order in $\Sigma$ yields $A \subseteq x^\rightarrow$ and $A \subseteq y^\rightarrow$. Since $x^\rightarrow \subseteq \beta(A \cup \{x\})$, we get

$$\beta(A \cup \{x\}) = \beta(x^\rightarrow) = \gamma(x^\rightarrow \cap L^t) \cup K^\sharp.$$

Next, note that $x^\rightarrow \cap L^t$ is simply the principal filter generated by $x$ in $L$, minus $\top_L$. Since $L$ is an equivalence algebra, we conclude that

$$\gamma(x^\rightarrow \cap L^t) = x^\rightarrow \cap L^t,$$

and hence $\beta(A \cup \{x\}) = x^{\rightarrow}$. Similarly, $\beta(A \cup \{y\}) = y^{\rightarrow}$ and so from $x^{\rightarrow} = y^{\rightarrow}$ we get $x = y$ and $x * y = \top \in \beta(A)$.

Case 2. $A \cap L^t \neq \emptyset$: Write $B = A \cap L^t$; 13.44(b) allows us to assume that $x * y \in L^t$, because $\beta(A)$ contains $K^\sharp$. As before, we may further suppose that $x \in L^t$. We have two subcases:

(i) $y \notin L^t$: Then $\beta(A \cup \{y\}) = \beta(A) = \beta(A \cup \{x\})$. Hence, $x, y \in \beta(A)$, and so $x * y \in \beta(A)$ because it is a filter in $\Sigma$.

(ii) $y \in L^t$: We may assume that $x \neq y$. In this case we have

$$(A \cup \{x\}) \cap L^t = B \cup \{x\} \quad \text{and} \quad (A \cup \{y\}) \cap L^t = B \cup \{y\}.$$

Hence, $\tau_L(B \cup \{x\}) = \tau_L(B \cup \{y\})$, from which we conclude (because $\tau_L$ satisfies [$T$ 3] in 13.21) that

$$x * y \in \tau_L(B) \setminus \{\top_L\} \quad \subseteq \quad \beta(A),$$

concluding the proof that $\beta$ satisfies [$T$ 3] and is a $T$-operator in $\Sigma$. Finally, for $x \notin L^t$, we have

$$\beta(x^{\rightarrow}) = \tau_K(x^{\rightarrow}) = x^{\rightarrow},$$

while if $x \in L^t$, recalling that $x^{\rightarrow} \cap L^t$ is the principal filter generated in $L$ by $x$, minus $\top_L$, we get

$$\beta(x^{\rightarrow}) = \gamma(x^{\rightarrow} \cap L^t) \cup K^\sharp = (x^{\rightarrow} \cap L^t) \cup K^\sharp = x^{\rightarrow},$$

completing the proof that $L \vee K$ is an Ean. ■

We now turn to a similar construction, which involves a special condition on $K$, contained in the following

DEFINITION 13.46. *A wEa $L$ is a* **dense wEa** *if $L^b$ is a wEa.*[2]

Any Ea without $\perp$ is a dense wEa. If $L$ is a wEan, then

$$\text{(D)} \qquad \left\{ \begin{array}{cc} L \text{ is a dense wEan} & \text{iff} \\ \perp \neq \top \text{ and } L^b \text{ is closed under } * & \text{iff} \\ \forall\, x, y \in L, \quad x * y = \perp \;\Leftrightarrow\; \perp \in \{x, y\} \text{ and } x \neq y. & \end{array} \right.$$

All linear orders with first and last element are dense Ea's. The Ean $C_3$ of 13.42 is also a dense Ean.

13.47. **p-Linear sums.** Notation is as in 13.43. Let $L$ be a wEan and $K$ be a dense wEa. We define a partially ordered set $\langle L \mathring{\vee} K, \leq \rangle$ by the following conditions:

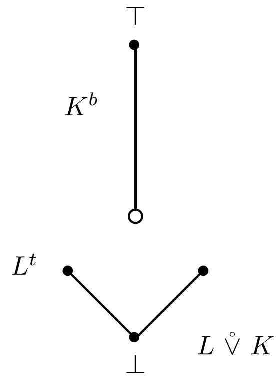
[2] The reason for the denomination appears in 13.69.

<u>Domain of $L \mathring{\vee} K$:</u> The disjoint union of $L^t$ and $K^b$;

<u>Partial order in $L \mathring{\vee} K$:</u> Determined by the following rules:

1. In the copies of $L^t$ and $K^b$ inside $L \mathring{\vee} K$, $\leq$ coincides with the original orders in $L$ and $K$, respectively;

2. For all $l \in L$, $k \in K$, $\; l \leq k$.




We shall write $\perp$, $\top$ for the least element of $L \mathring{\vee} K$. Note that $\perp$ is $\perp_L$, while $\top$ is $\top_K$. Define a binary operation $*$ on $L \mathring{\vee} K$, as follows:

$*$ For all $x \in L \mathring{\vee} K$, $\; x * x = \top$;

$*$ For all $x \neq y$ in $L^t$ or $K^b$, $*$ coincides with the original operations in $L$ and $K$, respectively. Note that this is well defined because $K^b$ is closed under $*_K$;

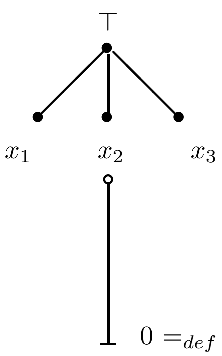
$*$ For all $a \in L^t$ and $b \in K^b$, $\; a * b = b * a = a$. ■

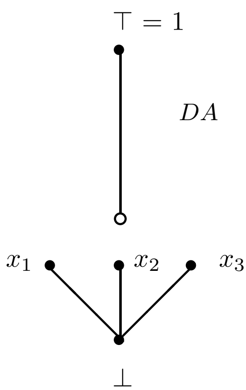
With the same techniques used to prove Proposition 13.45 one has

Proposition 13.48. *If $L$ is a wEan and $K$ is a dense wEa, then $L \mathring{\vee} K$ is a wEan. Moreover, if $L$ and $K$ are Ean's, the same is true of $L \mathring{\vee} K$.*

Example 13.49. Let $I = [0, 1]$ be the real unit interval, $C_3$ be the Ean of 13.42, $2 = \{\perp, \top\}$ be the two-element Boolean algebra and $B_4$ be the four-element Boolean algebra. Note that $I$, $C_3$ and 2 are dense Ean's. On the other hand, $B_4$ <u>is not</u> a dense Ean because if $a \neq \perp$ in $B_4$, then $\neg a$ (the complement of $a$) $\neq \perp$ and $a * \neg a = \perp$.

(I) Let $U = I \mathring{\vee} C_3$; a schematic diagram of $U$ appears on the left below. By 13.48, $U$ is an Ean. But note that $U$ is not closed under meets: for instance, the elements $x_1$, $x_2$ have no meet in $U$.




(II) Let $DA = C_3 \mathring{\vee} I$, a schematic diagram of which above right. Again, $DA$ is an Ean. Although closed under meets, $DA$ <u>is not</u> closed under joins, for sup $\{x_2, x_3\}$ does not exist in $DA$. As an example of application of 13.32 we prove

FACT 13.49.A. *All filters in $DA$ and $U$ are $*$-filters.*

*Proof.* We treat the case of $U$; the other is analogous. We show that for all finite sets $S \cup \{x, y\} \subseteq U$,

$$\text{(I)} \qquad [S \cup \{x\}] \ = \ [S \cup \{y\}] \ \Rightarrow \ x * y \in [S],$$

proving that the filter $\{\top\}$ has property $[**]$ of 13.32. We discuss two cases:

Case 1: $S \cap I = \emptyset$. If $\{x, y\} \subseteq C_3^t$, the conclusion follows because, as verified in 13.42, all filters in $C_3$ are $*$-filters. Assume then that $x \in I$. Then, $x$ is below all elements of $S$ and so

$$[S \cup \{x\}] = x^{\rightarrow}.$$

Moreover, since $x \in [S \cup \{y\}]$, we cannot have $y \in C_3^t$. Hence, $y \in I$ and so

$$[S \cup \{y\}] = y^{\rightarrow} = x^{\rightarrow},$$

from which we conclude that $x = y$ and $x * y = \top \in [S]$.

Case 2: $S \cap I \neq \emptyset$. Let $x_S = \textit{infimum}\ S \in I$; by 13.38(c), $[S] = x_S^{\rightarrow}$. If $\{x, y\} \subseteq C_3^t$, then $x * y \in C_3^t$ and there is nothing to prove. If $\{x, y\} \subseteq I$, the desired conclusion follows from the fact that the cHa $[0, 1]$ is an Ean. It remains to treat the case $x \in I$ and $y \in C_3^t$. But then, since $x_S < y$, we have

$$[S \cup \{y\}] = [S] = x_S^{\rightarrow} = [S \cup \{x\}],$$

and so $x_S \leq x$. Hence, from $x_S \leq x, y$, 13.12(a) yields $x_S \leq x * y$, concluding the proof of the Fact. Note that the argument above yields another proof that $U$ and $DA$ are Ean's.

(III) From the Ean's $I$ and $B_4$ we obtain three new Ean's,

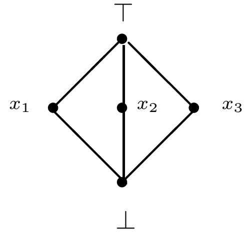
(i) $I \vee B_4$ and $B_4 \vee I$; (ii) $B_4 \mathring{\vee} I$.

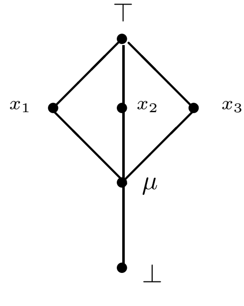
The schemes for $B_4 \vee I$ and $B_4 \mathring{\vee} I$ appear in the illustrative diagrams in 13.43 and 13.47, respectively. The reader can check that there is no binary operation on the partially ordered set $\langle I \mathring{\vee} B_4, \leq \rangle$ that satisfies the rules of a wEa.

(IV) If $K$ is an Ea without $\perp$, we may embed $K$ in an Ean by considering the Ean

$$ab(K) =_{def} 2 \mathring{\vee} K.$$

Observe that if $K = C_3^b$, then $ab(K)$ is *isomorphic* to $C_3$. Thus, this process destroys certain algebraic properties of the original algebra: $C_3^b$ is *associative*, but $C_3$ is not.



The natural embedding of $C_3$ in $2 \vee C_3$ (see 13.44(a)) is an example of a non-surjective wEa-morphism such that the inverse image of a proper filter is not proper, showing this hypothesis in 13.17(a) to be necessary. ∎

## 5. The embedding theorem. Applications

The $*$-filters on a wEa $L$ were constructed so as to have the following separation property (compare with 12.8):

THEOREM 13.50. *Let $L$ be a wEa and let $a, b \in L$. Let $A$ be a proper $*$-filter on $L$. If $(a * b) \notin A$, then there is a proper $*$-filter $B$, containing $A$, that separates $a$ and $b$.*

*Proof.* Since $A$ is a $*$-filter, $\tau(A \cup \{a\}) = \tau(A \cup \{b\})$ is impossible, otherwise $(a * b) \in \tau(A) = A$. Hence, either $\tau(A \cup \{a\})$ or $\tau(A \cup \{b\})$ separates $a$ and $b$. ∎

COROLLARY 13.51. *Let $L$ be an Ea with $\perp$. Suppose that $F$ is a $*$-filter in $L$ and $a$ an element of $L$ such that $a * \perp \notin F$. Then there is a proper $*$-filter $G$ containing $F$ such that $a \in G$.*

We may consider 13.14(a) as an algebraic counterpart of the $\equiv$-Elimination rule of MEC; here is an analog of the $\equiv$-Introduction rule:

COROLLARY 13.52. *For $a$, $b$, $c$ in an Ea $L$, the following are equivalent:*

(1) *$a \leq (b * c)$;*
(2) *No $*$-filter containing $a$ separates $b$ and $c$.*

*Proof.* (1) implies (2) comes from 13.14(a). For the converse, note that if $(b * c) \notin a^{\rightarrow}$, Theorem 13.50 yields a $*$-filter containing $a^{\rightarrow}$ that separates $b$ and $c$, contradicting (2). ∎

Let $L$ be a wEa. Recall (13.30) that $S(L)$ is the set of all proper $*$-filters on $L$. For $x \in L$, set

$$S_x = \{F \in S(L)\colon x \in F\}.$$

We take $\mathcal{B} = \{S_x\colon x \in L\}$ as a subbasis for a topology $\Omega(L)$ on $S(L)$, that is, for all $U \subseteq S(L)$,

$$U \in \Omega(L) \quad \text{iff} \quad \text{There is } K \subseteq 2^A_\omega \text{ such that } U = \textstyle\bigcup_{k\in K} \bigcap_{x\in k} S_x.$$

Thus, the empty set together with the finite intersections of elements in $\mathcal{B}$ constitute a basis for $\Omega(L)$. Note that $S_\top = S(L)$; if $L$ has a least element $\perp$, then $S_\perp = \emptyset$. It is well known that $\Omega(L)$ is a *complete Heyting algebra* (*cHa*) (see Section 13.1).

THEOREM 13.53. *If $L$ is an equivalence algebra, the map*

$$\sigma\colon L \to \Omega(L), \text{ given by } a \mapsto S_a,$$

*is an embedding of $L$ in $\Omega(L)$, satisfying:*

(a) *If $L$ has a least element $\perp$, then $\sigma$ takes $\perp$ to $\perp$ in $\Omega(L)$.*
(b) *If $F \in S(L)$, then the filter $G$ generated by $F$ in $\Omega(L)$ is a proper filter and*

$$\text{For all } a \in L, \quad a \in F \quad \text{iff} \quad S_a \in G,$$

*that is, $F = \sigma^{-1}(G)$.*

*Proof.* Write $\Omega$ for $\Omega(L)$. The definition of filter (13.13) yields $x \leq y$ implies $S_x \subseteq S_y$. For the converse, note that since $x^{\rightarrow}$ is a $*$-filter and $x^{\rightarrow} \in S_x$, it follows that $S_x \subseteq S_y$ implies $y \in x^{\rightarrow}$.

For $x, y \in L$, Lemma 13.14(a) yields

$$S_x \cap S_{x*y} = S_y \cap S_{x*y}.$$

Thus, $S_{x*y} \subseteq (S_x \equiv S_y)$ in the cHa $\Omega$. Since the intersection of finite subsets of $\mathcal{B}$ is a basis for $\Omega$ and intersection distributes over arbitrary joins in $\Omega$ (it is a cHa), to show that

$$S_{x*y} = (S_x \equiv S_y) \quad (\text{in } \Omega) \tag{1}$$

it is enough to verify that if $\{t_1, \ldots, t_n\} \subseteq L$ and $V = \bigcap_{i=1}^{n} S_{t_i}$, then

$$V \cap S_x = V \cap S_y \quad \text{implies} \quad V \subseteq S_{x*y}.$$

Assume that there is $F \in V$ such that $(x * y) \notin F$. By 13.50, there is $G \in S(L)$ satisfying $F \subseteq G$ and separating $a$ and $b$, that is,

$$\text{either} \quad (x \in G \text{ and } y \notin G) \quad \text{or} \quad (y \in G \text{ and } x \notin G). \tag{2}$$

Since $F \subseteq G$, we have $t_i \in G$, $1 \leq i \leq n$, that is, $G \in V$; but then, the alternatives in (2) imply that $S_x \cap V \neq S_y \cap V$, a contradiction. Therefore, (1) is true and $\sigma$ is an embedding. Item (a) is clear.

For (b), if $F$ be a proper $*$-filter in $L$, then

$$G = \{U \in \Omega\colon \text{There is a finite } K \subseteq F \text{ such that } \textstyle\bigcap_{t \in K} S_t \subseteq U\},$$

is the filter generated by $\sigma(F) = \{S_t\colon t \in F\}$ in $\Omega$. Since $F \in S_t$, i.e. $t \in F$, $\sigma(F)$ has the finite intersection property. Thus, $G$ is a proper filter in $\Omega$. Clearly, $\sigma(F) \subseteq G$. Conversely, if $S_a \in G$, then there is a finite $A \subseteq F$ such that $\bigcap_{t \in A} S_t \subseteq S_a$; hence, $F \in S_a$, ending the proof. ∎

Theorem 13.53 has a number of important consequences. Here is a sample.

COROLLARY 13.54. *If $L$ is an Ea, the embedding $\sigma\colon L \to \Omega(L)$ satisfies $\sigma^*\tau_{\Omega(L)} = \tau_L$. In particular, the $*$-filters in $L$ are precisely the inverse images of the filters in $\Omega(L)$ by $\sigma$.*

*Proof.* By 13.28, it is enough to show that $\sigma^*\tau_{\Omega(L)} \leq \tau_L$. Recall (13.38) that for $B \subseteq \Omega(L)$, $\tau_{\Omega(L)}(B) = [B]$, the filter generated by $B$. For $A \subseteq L$, since $\tau_L(A)$ is a $*$-filter, 13.53(b) yields

$$\tau_L(A) = \sigma^{-1}([\sigma(\tau_L(A))]) = \sigma^*\tau_{\Omega(L)},$$

as claimed. ∎

COROLLARY 13.55. (a) *An equivalence algebra with $\perp$ is an equivalence algebra with negation.*

(b) *Every equivalence algebra can be embedded in an equivalence algebra with negation.*

*Proof.* (b) is immediate from 13.53. For (a), we give two proofs. Let $L$ be an Ea with $\perp$.

<u>First Proof</u>: By 13.53, $L$ is isomorphic to a subalgebra $K$ of $\Omega$ with $\perp$ $(= \emptyset) \in K$. Since $K$ is an Ean (13.40(c)), so is $L$.

Second Proof: Suppose that for some $a \in L$, $b = a * (a * \perp) \neq \perp$. Then $b^{\rightarrow}$ is a proper $*$-filter in $L$; moreover, $(a * \perp)$ cannot be in $b^{\rightarrow}$, for otherwise it would not be proper. By 13.50, there is a proper $*$-filter $F$ in $L$, containing $b^{\rightarrow}$, such that $a \in F$. Then, from $b \in F$ and $a \in F$, we get $\perp \in F$, a contradiction. Thus, for all $a \in L$, $\perp = (a * \perp) * a$, and $L$ is an Ean. ∎

COROLLARY 13.56. *Let $u$ be an element of an Ea $L$. Then $u^{\rightarrow}$ is an Ean. In particular, for all $a \geq u$, $u = a * (a * u)$.*

*Proof.* Immediate from Corollary 13.55(a), once it is remarked that $u^{\rightarrow}$ is a subalgebra of $L$, with $\perp = u$. ∎

COROLLARY 13.57. *Every Ea satisfies the following axioms:*

[∗ 5] $x * y = ((x * y) * y) * y,$
[∗ 6] $x \leq y \leq z$ *implies* $x * z \leq x * y,$
[∗ 7] $(x * y) * z \leq [x * z] * [(y * z) * z],$
[∗ 8] $[(x * z) * (x * z)] * z \leq (x * y) * z,$
[∗ 9] $x \leq y$ *iff* $y * (x * y) \leq x * y$ *iff* $x = y * (x * y).$

*Proof.* The reader can check that the aforementioned rules hold in $H_{eq}$, for any Heyting algebra $H$. The result then follows from 13.53(a). ∎

DEFINITION 13.58. *A wEa $L$ is said to be* **special** *if it satisfies axioms* $[* \, i]$, $i = 5, 6, 7, 8$, *in Corollary* 13.57.

REMARK 13.59. It follows immediately from [$E$ 6] that a special wEan satisfies the *contrapositive positive law*

**[neg 2]** For all $x, y \in L$, $x \leq y$ implies $(y * \perp) \leq (x * \perp)$. ∎

If $L$ is a special wEa, define

$$A = \{x \in L\colon \text{For all } u \in L,\ x = (xu)u\},$$

called the set of **associative** elements of $L$. It is clear that $\top \in A$. If $A$ has negation (13.36), then $\perp \in A$. The name for $A$ is justified by

LEMMA 13.60. *Let $L$ be a special wEa. Then*

$$\textit{For all } x, z \in A \textit{ and all } y \in L, \quad (xy)z = x(yz).$$

*Proof.* (a) From [∗ 7] (13.57) and Lemma 13.12(d) comes

$$(xy)z \leq ((xz)z)\ (yz) = x(yz).$$

Similarly, one proves that $(yz)x \leq (xy)z$ and equality follows. ∎

OPEN PROBLEM 13.61. *Is $A$ a subalgebra of $L$? Or equivalently, is $A$ closed under $*$?*

Another important property of associative elements is that [∗ 7] and [∗ 8] become equalities.

PROPOSITION 13.62. *Let $L$ be a special wEa and let $t$ be an associative element of $L$. Then, for all $x, y \in L$,*

$$\begin{aligned}[(x * t) * (y * t)] * t \;=\; (x * y) * t \;&=\; (x * t) * [(y * t) * t] \\ &=\; (y * t) * [(x * t) * t].\end{aligned}$$

*Proof.* Applying [$*$ 7] and [$*$ 8], we get

$$(xt)\ [(yt)t] \leq [(xt)\ (yt)][(t\ (xt))\ (xt)] = [(xt)\ (yt)]\ t \leq (xy)\ t,$$

a relation that, together with the inequalities in [$*$ 7], [$*$ 8] and Lemma 13.12(d), yields the desired conclusion. ∎

The construction at the base of Theorem 13.53 yields two functors

$$S\colon \mathbf{Ean} \to \mathbf{Top} \quad \text{and} \quad \Omega\colon \mathbf{Ean} \to \mathbf{cHa},$$

which we now describe[3].

13.63. **The functor $S$: Ean → Top.** Let **Top** be the category of topological spaces and continuous maps. If $L \xrightarrow{f} K$ is an Ean-morphism, define, recalling Proposition 13.33(b),

$$f_*\colon S(K) \to S(L) \quad \text{by} \quad f_*(F) = f^{-1}(F).$$

FACT A. *For all $a \in L$, $f_*^{-1}(S_a) = S_{f(a)}$.*

*Proof.* If $G$ is a $*$-filter in $S(K)$, then

$$\begin{aligned} G \in S_{f(a)} \quad &\text{iff} \quad f(a) \in G \quad \text{iff} \quad a \in f^{-1}(G) \quad \text{iff} \quad a \in f_*(G) \\ &\text{iff} \quad f_*(G) \in S_a, \end{aligned}$$

as asserted.

Since $\{S_a\colon a \in L\}$ is a subbasis for the topology in the space of proper $*$-filters, Fact 13.63.A implies that $f_*$ is a continuous map from $S(K)$ to $S(L)$. It is easily established that

(A) $$(f \circ g)_* = g_* \circ f_* \quad \text{and} \quad (Id_L)_* = Id_{S(L)},$$

and so $S$ is a contravariant functor from **EAn** to **Top**. ∎

13.64. **The functor $\Omega$: Ean → cHa.** Let **cHa** be the category of complete Heyting algebras. Recall that morphisms in this category are the $[\wedge, \bigvee]$-morphisms (13.8), i.e., maps that preserve finite meets and arbitrary joins. For each Ean $L$, let $\Omega(L)$ be the cHa of opens in the space $S(L)$. If $L \xrightarrow{f} K$ is an Ean-morphism, define

$$\Omega(f)\colon \Omega(L) \to \Omega(K) \quad \text{by} \quad \Omega(f)(U) = f_*^{-1}(U),$$

where $f_*$ is the map described in 13.63. Because $f_*$ is continuous, $\Omega(f)$ is a $[\wedge, \bigvee]$-morphism from $\Omega(L)$ to $\Omega(K)$. It follows from (A) in 13.63, that $\Omega$ is a covariant functor from **EAn** to **cHa**. ∎

The fundamental property of the functor $\Omega$ is described in following result, whose proof is left to the reader:

LEMMA 13.65. *Let $\Omega$ be the functor of* 13.64. *Then, for all Ean-morphisms $L \xrightarrow{f} K$, the following diagram is commutative, where $\sigma$ is the embedding of* 13.53*:*

[3] **Ean** is the category of Ean's and Ean-morphisms.

$$
\begin{tikzcd}
L \arrow[r, "\sigma_L"] \arrow[d, "f"'] & \Omega(L) \arrow[d, "\Omega(f)"] \\
K \arrow[r, "\sigma_K"'] & \Omega(K)
\end{tikzcd}
$$
■

It follows from 13.65 that the family $\{\sigma_L\colon L \text{ is an Ean}\}$ of Ean-morphisms is a *natural transformation* from the identity functor in **Ean** to the functor $\Omega$.

## 6. Basic properties of negation

We now describe some of the properties of negation in an Ean. In fact, our results hold for *special wEan's*, as defined in 13.58. The most fundamental of these properties is item (c) in Proposition 13.66.

PROPOSITION 13.66. *For $x$, $y$, $z$ in a special wEan $L$, we have:*

(a) $x \le \neg\neg x$ *and* $\neg x = \neg\neg\neg x$; $x \le y$ *implies* $\neg\neg x \le \neg\neg y$.
(b) $x * y \le \neg x * \neg y = \neg\neg x * \neg\neg y$.
(c) $\neg(x * y) = \neg x * \neg\neg y = \neg y * \neg\neg x = \neg(\neg x * \neg y)$.
(d) $\neg\neg(x * y) = \neg\neg x * \neg\neg y$.
(e) $\neg x = \neg[y * (x * y)]$; $[y * (x * y)] \le \neg\neg x$.
(f) $\neg\neg x = \neg y * \neg(x * y)$.
(g) $\neg(x * (y * z)) = \neg((x * y) * z)$.
(h) $\neg\neg x * (\neg\neg y * \neg\neg z) = (\neg\neg x * \neg\neg y) * \neg\neg z$.
(i) $x * y = \perp$ *iff* $x * \neg\neg y = \perp$ *iff* $\neg x = \neg\neg y$.

*Proof.* (a) comes from 13.12(b), [$*$ 4] and **[neg 2]** (13.59). Item (b) follows from 13.12(c) and (a). Item (c) is just a restatement of 13.62, with $\perp = t$. For (d), we get, using (c),

$$\neg\neg(x * y) = \neg(\neg x * \neg\neg y) = \neg\neg x * \neg\neg(\neg\neg y) = \neg\neg x * \neg\neg y,$$

as desired. For (e), **[neg 2]** (13.59) and 13.12(b) yield

$$\neg[y * (x * y)] \le \neg x.$$

For the reverse inequality, from 13.12(b) and (c) we get

$$\neg x \le \neg\neg y * (\neg x * \neg\neg y) = \neg\neg y * \neg(x * y) = \neg[y * (x * y)],$$

establishing the first part of (e); the second follows from (a). For (f), the preceding results yield

$$\begin{aligned}\neg\neg x &= \neg\neg[y * (x * y] = \neg[\neg\neg y * \neg(x * y)]\\ &= \neg(\neg\neg y) * \neg\neg[\neg(x * y)] = \neg y * \neg(x * y),\end{aligned}$$

as needed. For (g), first note that it is enough to verify that

$$\text{(I)} \qquad \neg[x * (y * z)] \le \neg[(x * y) * z].$$

In fact, we have, applying (I) in succession

$$\neg[(x * y) * z] = \neg[z * (y * x)] \leq \neg[(z * y) * x] = \neg[x * (y * z)].$$

To prove (I), compute as follows, recalling [$*$ 7], (c), (d) and (f):

$$\begin{aligned}\neg[x * (y * z)] &= \neg x * \neg\neg(y * z) = \neg x * (\neg\neg y * \neg\neg z) \\ &\leq (\neg x * \neg\neg y) * [(\neg\neg z * \neg x) * \neg x] \\ &= (\neg x * \neg\neg y) * \neg\neg z = (\neg x * \neg\neg y) * \neg\neg z \\ &= \neg(x * y) * \neg\neg z = \neg[(x * y) * z],\end{aligned}$$

completing the proof of (g). For (h), we have, using (d) and (g):

$$\begin{aligned}\neg\neg x * (\neg\neg y * \neg\neg z) = \neg\neg x * \neg\neg(y * z) &= \neg\neg[x * (y * z)] \\ &= \neg\neg[(x * y) * z] = \neg\neg(x * y) * \neg\neg z \\ &= (\neg\neg x * \neg\neg y) * \neg\neg z.\end{aligned}$$

To verify (i), note that if $x * y = \bot$, then $(x * y) * \bot = \top$. Thus, by [$*$ 7], $\neg x * \neg\neg y = \top$, that is, $\neg x = \neg\neg y$. On the other hand, if this equation is true, then

$$\bot = \neg x * \neg y = \neg\neg x * \neg\neg y = \neg\neg(x * y),$$

and so (a) implies $x * y = \bot$, ending the proof. ■

## 7. Regular and dense elements

The purpose of this section is to discuss the generalizations to Ean's of the usual notions of *regular* and *dense* in Heyting algebras.

DEFINITION 13.67. *Let $L$ be an Ean and $x$ be an element of $L$.*

(a) *$x$ is* **regular** *iff $\neg\neg x = x$. Set $Reg(L) = \{x \in L\colon \neg\neg x = x\}$.*

(b) *$x$ is* **dense** *if $\neg\neg x = \top$. Let $D(L) = \{x \in L\colon \neg\neg x = \top\}$. Whenever $L$ is clear from context, write $D$ for $D(L)$.*

PROPOSITION 13.68. *Let $L$ be an Ean. Then:*

(a) *For all $x, y \in L$,* $\begin{cases} (i)\ x * y \in D \quad \textit{iff} \quad \neg\neg x = \neg\neg y. \\ (ii)\ x * \neg\neg x \in D. \end{cases}$

(b) *$D$ is a $*$-filter in $L$.*

*Proof.* (a) By Proposition 13.66(d), we have

$$\top = \neg\neg(x * y) = \neg\neg x * \neg\neg y,$$

and so $\neg\neg x = \neg\neg y$, verifying (i). Item (ii) comes directly from the distributivity of double negation over $*$.

(b) To keep the exposition self-contained, we first give a proof that $D$ is a filter in $L$. Recall that every Ean is a special wEa (13.57, 13.58).

Clearly, $\top \in D$. By 13.59, $x \leq y$ implies $\neg\neg x \leq \neg\neg y$; thus, $D$ satisfies [$fil$ 2]. Now suppose that $x * y \in D$ and $t * z \in D$. By Proposition 13.66(h), we have:

$$\neg\neg[(x * t) * (y * z)] = \neg\neg[(x * y) * (t * z)] = \neg\neg(x * y) * \neg\neg(t * z) = \top,$$

completing the verification of [*fil* 3]. In particular, if $L$ is Heyting algebra, then $D$ is a $*$-filter in $L$ (13.39). To finish the proof note that if $\sigma\colon L \to \Omega$ is the Ea-embedding of 13.53, then $D_L = \sigma^{-1}(D_\Omega)$, and the conclusion follows from 13.54. ■

Proposition 13.69. *Let $L$ be an Ean. With notation as in* 13.46 *and* 13.43*, we have:*

(a) *The following conditions are equivalent:*

(1) *$L$ is a dense Ean;*
(2) *For all $x \in L$, $\;x \neq \bot \;\Rightarrow \neg\neg x = \top$.*

(b) *If $L$ is a dense Ean, then there is a Heyting algebra $H$ such that $L^b$ can be wEa-embedded in $D(H)$.*

*Proof.* (a) (1)⇒(2): If $x \neq \bot$ in $L$, then $x * \bot = \bot$, otherwise $L^b$ would not be closed under $*$. Hence, $\neg\neg x = \neg\bot = \top$.

(2)⇒(1): By 13.66(i),

$$x * y = \bot \quad \text{implies} \quad \neg x = \neg\neg y.$$

Thus, if $x \neq \bot$, then $y \leq \neg\neg y = \bot$; similarly, $y \neq \bot$ implies $x = \bot$, and $L^b$ is closed under $*$.

(b) Let $\sigma\colon L \to \Omega$ be the embedding of 13.53. If $L$ is a dense Ean, it follows from (a) that $\sigma(L^b) \subseteq D(\Omega)$, as desired. ■

Definition 13.70. *A wEa $L$ is* **associative** *if the operation $*$ in $L$ is associative.*

Proposition 13.71. (a) *If $L$ is an Ean then $Reg(L)$ is an associative Ean.*

(b) *Every associative Ean can be embedded in a Boolean algebra.*

*Proof.* (a) It follows from [*neg* 1] that $\bot$ is regular, while items (a), (d) and (h) in Proposition 13.66, guarantee that $\top$ is regular, that $Reg(L)$ is closed under $*$ and that $*$ is associative in $Reg(L)$, respectively. Thus, $Reg(L)$ is an associative subalgebra of $L$.

(b) If $H$ is a Heyting algebra and $D$ is the filter of dense elements in $H$, it is well known that $B = H/D$ is a Boolean algebra (see Theorem VIII.4.3 (p. 157), [Balbes and Dwinger, 1974])[4].

If $L$ is an associative Ean, then for all $x \in L$, we have $x = \neg\neg x$. Let $\sigma\colon L \to \Omega(L)$ be the embedding of $L$ in a cHa given by Theorem 13.53. If $\pi_D\colon \Omega \to B = \Omega/D$ is the canonical quotient map, it is straightforward that $\pi_D \circ \sigma$ is an embedding of $L$ into $B$. ■

Item (b) in 13.71 yields a partial converse to 13.7 for Ean's. For associative Ea's in general we pose

Open Problem 13.72. *Does every associative equivalence algebra embed in a Boolean algebra?*

## 8. Algebraic completeness of MEC and MECn

From now on we shall follow the algebraic tradition of using the same symbol (and font) for a structure and the set of elements of the structure. Recall that $\leftrightarrow$ denotes the equivalence

[4] In fact, $H/D$ is the largest Boolean algebra quotient of $H$.

operation in a Heyting algebra (or a cHa), while $\mathcal{L}$ and $\mathcal{L}_\perp$ are the Lindenbaum algebras of MEC and MECn, respectively (Section 12.2).

DEFINITION 13.73. *Let $L$ be an weak equivalence algebra.*

$*$ *An* **assignment in** *$L$ is a function from the set of propositional parameters into $L$. In the case of* MECn*, the propositional parameter $\perp$ is assigned to $\perp \in L$.*

$*$ *A* MEC**-valuation in** *$L$ is a function $v$ from the set of all formulas of* MEC *into $L$, such that for all formulas $\mathcal{A}$, $\mathcal{B}$ of* MEC*:*

$$v(\mathcal{A} \equiv \mathcal{B}) = v(\mathcal{A}) * v(\mathcal{B}).$$

$*$ *A* MECn**-valuation** *in $L$ is a* MEC*-valuation $v$ such that $v(\perp) = \perp$.*

$*$ VAL$(L)$ *is the set of valuations in $L$.*

The usual induction on complexity yields

LEMMA 13.74. *If $L$ is a weak equivalence algebra, then* VAL$(L) \neq \emptyset$. *In fact, any assignment in $L$ can be extended to a unique valuation. This leads to a natural bijective correspondence between $Hom(\mathcal{L}, L)$ and* VAL$(L)$. *A similar result holds for $\mathcal{L}_\perp$ and* MECn.

As a consequence of 13.74 we get

COROLLARY 13.75. *$\mathcal{L}$ and $\mathcal{L}_\perp$ are, respectively, the free Ea and the free Ean on the set of propositional parameters that constitute their basic alphabet.*

We extend the concept of a valuation to sets of formulas; if $\Gamma$ is a set of formulas in MEC or MECn and $v \in$ VAL$(L)$, then we set

$$v(\Gamma) = \{v(\mathcal{F}) \colon \mathcal{F} \in \Gamma\}.$$

DEFINITION 13.76. *Let $L$ be an Ea and let $v$ be an $L$-valuation of* MEC *or* MECn. *Let $\Gamma \cup \{\mathcal{A}\}$ be a set of formulas of* MEC *or* MECn.

(1) *$\mathcal{A}$ is an* $L, v$**-algebraic consequence of** $\Gamma$, *in symbols*

$$\Gamma \models_L \mathcal{A}[\![v]\!],$$

*iff $v(\mathcal{A})$ belongs to the $*$-filter generated by $v(\Gamma)$ in $L$.*

(2) *$\mathcal{A}$ is an* $L$**-algebraic consequence of** $\Gamma$, *in symbols, $\Gamma \models_L \mathcal{A}$, if for all $L$-valuations $v$, $\Gamma \models_L \mathcal{A}[\![v]\!]$.*

(3) *$\mathcal{A}$ is an* **algebraic consequence of** $\Gamma$, *in symbols, $\Gamma \models \mathcal{A}$, iff for all equivalence algebras $L$, $\Gamma \models_L \mathcal{A}$.*

PROPOSITION 13.77 (Soundness). *If $L$ is an Ea and $\Gamma \cup \{\varphi\} \subseteq$* MEC *is such that $\Gamma \vdash \varphi$, then for all valuations $v$ of* MEC *in $L$ and all $*$-filters $F$ in $L$,*

$$v(\Gamma) \subseteq F \quad \textit{implies} \quad v(\varphi) \in F,$$

*where $v(\Gamma) = \{v(\psi) \colon \psi \in \Gamma\}$. A similar result holds for* MECn.

*Proof.* We shall treat the case of MEC, leaving the straightforward modifications needed for MECn to the reader. The proof is by induction on the *length of the derivation*, $\Pi$, of $\mathcal{A}$ from $\Gamma$. Recall that the length of $\Pi$ is the number of nodes in the tree corresponding to $\Pi$. If $\Pi$ has length 1, then $\mathcal{A}$ must be an assumption, that is, $\mathcal{A} \in \Gamma$ and there is nothing to prove.

Now suppose that $\Pi$ has length $n+1$; if $\mathcal{A} \in \Gamma$, there is nothing to prove. If not, $\mathcal{A}$ arises either from an application of the Introduction Rule, or from an application of the Elimination Rules. Thus, we have two cases to discuss:

(i) $\mathcal{A}$ arises from an Elimination Rule: In this case, $\mathcal{A}$ comes from an application of the E-Rules

$$\frac{(\mathcal{A} \equiv \mathcal{B}) \qquad \mathcal{B}}{\mathcal{A}} \qquad \text{or} \qquad \frac{(\mathcal{B} \equiv \mathcal{A}) \qquad \mathcal{B}}{\mathcal{A}}$$

Suppose that the rule was used in the first form. Let $F$ be a filter in $L$ such that $v(\Gamma) \subseteq F$. Since we have derivations of length $\leq n$ of $(\mathcal{A} \equiv \mathcal{B})$ and $\mathcal{B}$ from $\Gamma$, we know that $v(\mathcal{A} \equiv \mathcal{B}) = (v(\mathcal{A}) * v(\mathcal{B}))$ and $v(\mathcal{B})$ are both in $F$. It follows from 13.14(a) that $v(\mathcal{A}) \in F$. An application of the second form of the E-rule can be handled similarly.

(ii) $\mathcal{A}$ arises from the Introduction Rule: In this case, $\mathcal{A} = (\mathcal{B} \equiv \mathcal{C})$ and the following rule was applied:

$$\frac{\begin{matrix}[\mathcal{C}] & \qquad & [\mathcal{B}] \\ \mathcal{B} & \qquad & \mathcal{C}\end{matrix}}{\mathcal{B} \equiv \mathcal{C}}$$

In particular, there are derivations of length $\leq n$ of $\Gamma, \mathcal{C} \vdash \mathcal{B}$ and of $\Gamma, \mathcal{B} \vdash \mathcal{C}$. Let $F$ be a $*$-filter in $L$ such that $v(\Gamma) \subseteq F$. If $v(\mathcal{B} \equiv \mathcal{C}) = v(\mathcal{B}) * v(\mathcal{C})$ is not in $F$, Theorem 13.50 may be applied to yield a $*$-filter $G$, containing $F$, and separating $v(\mathcal{B})$ and $v(\mathcal{C})$, that is,

(1) $\qquad$ either $(v(\mathcal{B}) \in G$ and $v(\mathcal{C}) \notin G)$ or $(v(\mathcal{B}) \notin G$ and $v(\mathcal{C}) \in G)$.

Note that $v(\Gamma) \subseteq G$. Suppose the first alternative in (1) occurs; then,

$$v(\Gamma) \cup \{v(\mathcal{B})\} \subseteq G \quad \text{and} \quad v(\mathcal{C}) \notin G,$$

which contradicts the induction hypothesis, because $\Gamma, \mathcal{B} \vdash \mathcal{C}$ has a derivation of length $\leq n$. The same reasoning shows that the second alternative in (1) is not feasible. Hence, $v(\mathcal{A}) = v(\mathcal{B} \equiv \mathcal{C}) \in F$, completing the proof. ∎

From soundness we get

COROLLARY 13.78. *Let $L$ be an Ea algebra and let $\Gamma \cup \{\mathcal{A}, \mathcal{B}\}$ be a set of formulas in* MEC *or* MECn. *If $v$ is an $L$-valuation, then*

(a) $\mathcal{A} \vdash \mathcal{B}$ *implies* $v(\mathcal{A}) \leq v(\mathcal{B})$.
(b) *If $\Gamma \vdash \mathcal{A}$ and $v(\mathcal{F}) = \top$ for all $\mathcal{F} \in \Gamma$, then $v(\mathcal{A}) = \top$.*
(c) *If $\mathcal{B}$ is a* MEC *or* MECn *thesis, then $v(\mathcal{B}) = \top$.*

*Proof.* Apply Proposition 13.77 to the filter $v(\mathcal{A})^{\rightarrow}$ to get (a) and to the filter $\{\top\}$ to get (b) and (c). ∎

THEOREM 13.79 (Completeness). *Let $\Gamma \cup \{\mathcal{A}\}$ be a set of formulas in* MEC *or* MECn. *The following are equivalent:*

(1) $\Gamma \vdash \mathcal{A}$;
(2) $\Gamma \models \mathcal{A}$;
(3) *For all Heyting algebras $H$ and all valuations in $H$,* $\Gamma \models_H [\![\mathcal{A}]\!]$.

*Proof.* We shall treat the case of MEC, leaving the straightforward modifications needed for MECn to the reader.

The implication (1)⇒(2) comes from 13.77, while (2)⇒(3) is clear. It remains to check that (3)⇒(1). Let $\mathcal{L}$ be the Lindenbaum algebra of MEC; since $\mathcal{L}$ is an Ea (13.39), Theorem 13.53 furnishes a wEa-embedding $\varepsilon\colon \mathcal{L} \to \Omega$, where $\Omega$ is the cHa of opens in the space $S(L)$ of proper $*$-filters in $\mathcal{L}$, which we know to be its proper theories (13.37). Assume that $\Gamma \vdash \mathcal{A}$ is false. Then[5]

$$F_\Gamma = \{\overline{\mathcal{B}} \in \mathcal{L}\colon \Gamma \vdash \mathcal{B}\},$$

is a proper filter in $\mathcal{L}$, because $\overline{\mathcal{A}} \notin F_\Gamma$. By Theorem 13.53, there is a filter $G$ in $\Omega$ such that for all $\mathcal{C} \in$ MEC,

$$\varepsilon(\overline{\mathcal{C}}) \in G \quad \text{iff} \quad \overline{\mathcal{C}} \in F_\Gamma.$$

By Lemma 13.74, the map $v(\mathcal{C}) = \varepsilon(\overline{\mathcal{C}})$, $\mathcal{C}$ a formula in MEC, is an $\Omega$-interpretation of MEC. Hence, $v(\Gamma) \subseteq G$, while $v(\mathcal{A}) \notin G$, contradicting (3) and ending the proof. ■

## 9. Ea-quotients

Let $L$ be a wEa and $F$ be a proper filter on $L$. For $x, y \in L$, define

$$x \;\theta_F\; y \quad \text{iff} \quad (x * y) \in F.$$

LEMMA 13.80. *$\theta_F$ is a congruence on $L$.*

*Proof.* It must be verified that $\theta_F$ is an equivalence relation such that for all $x, y, t, z \in L$,

(I) $$x \;\theta_F\; t \ \text{ and } \ y \;\theta_F\; z \quad \text{implies} \quad (x * y) \;\theta_F\; (t * z).$$

Clearly, $\theta_F$ is reflexive and symmetric, while its transitivity follows from the fact that $F$ is a filter. Hence, $\theta_F$ is an equivalence relation on $L$. To show that it is a congruence with respect to $*$, we may apply 13.14(c) to get (I), ending the proof. ■

If $F$ is a proper filter in a wEa $L$ and $x, y \in L$,

* Write $x/F$ for the equivalence class of $x$ with respect to $\theta_F$;
* Write $L/F = \{x/F\colon x \in L\}$ for the set of equivalence classes of elements of $L$ by $\theta_F$;
* Write $\pi_L\colon L \to L/F$ for the canonical quotient map, $x \mapsto x/L$;
* Define an operation $*$ on $L/F$ by
$$x/F * y/F = (x * y)/F,$$

which is independent of representatives by 13.80. Clearly, the structure $\langle L/F, *, \top/F\rangle$ satisfies axioms [$*$ 1], [$*$ 2] and [$*$ 3] in 13.9.

If $F$ is a $*$-filter on the wEa $L$, define, for $x, y \in L$, [6]

[po] $$x/F \leq y/F \quad \text{iff} \quad y \in \tau(F \cup \{x\}).$$

[5] $\overline{\mathcal{A}}$ is the class of the formula $\mathcal{A}$ in $\mathcal{L}$, as in Section 12.2.

[6] $\tau$ is the least $*$-operator on $L$; see 13.26 and 13.28.

Proposition 13.81. *If $F$ is a $*$-filter on a wEa (wEan) $L$, then:*

(a) *The relation defined in* [po] *is independent of representatives and constitutes a partial order in $L/F$ whose largest element is $\top/F$ and, whenever $L$ has $\bot$, has $\bot/F$ as its least element.*

(b) *$L/F = \langle L/F, \leq, \top, *\rangle$ is a wEa (resp., wEan) and the quotient map $\pi_F$ is a wEa-morphism (resp., wEan-morphism).*

*Proof.* (a) Suppose $(xt), (yz) \in F$ and $y \in \tau(F \cup \{x\})$. Then

$$(yz),\ y \in \tau(F \cup \{x\}) = \tau(F \cup \{t\}),$$

and so $z \in \tau(F \cup \{t\})$ (13.14(a)). Clearly, $\leq$ is reflexive on $L/F$. Since $F$ is a $*$-filter,

$$x \in \tau(F \cup \{y\}) \text{ and } y \in \tau(F \cup \{x\})$$

implies $xy \in F$, showing that $\leq$ is antisymmetric in $L/F$. For transitivity, note that (13.26(a))

$$y \in \tau(F \cup \{z\}) \quad \text{implies} \quad \tau(F \cup \{y\}) \subseteq \tau(F \cup \{z\}),$$

and so $x \in \tau(F \cup \{y\})$ and $y \in \tau(F \cup \{z\})$ yields $x \in \tau(F \cup \{z\})$, verifying transitivity. Since

$$\tau(F \cup \{\top\}) = \tau(F) = F \quad \text{and} \quad \tau(F \cup \{\bot\}) = \tau(\{\bot\}) = L,$$

it follows that $\bot/F$ (when $L$ has $\bot$) and $\top/F$ are, respectively, the least and largest elements of $L/F$ in the partial order $\leq$.

(b) It remains to verify [$*$ 4] for $L/F$. Assume that $a/F \leq (x * y)/F$ and $a/F \leq (t * z)/F$. Set $G = \tau(F \cup \{a\})$. Hence, $xy, tz \in G$, which implies, since $G$ is a filter, that $(xt) * (yz) \in G$, as needed. Clearly, $\pi_F$ preserves $*$ and $\bot$ (if $L$ is a wEan); hence it will be a wEa-morphism if $x \leq y$ implies $\pi_F(x) \leq \pi_F(y)$. But if $x \leq y$, then $y$ must belong to any filter containing $x$ and so $y \in \tau(F \cup \{x\})$, which is equivalent to $x/F \leq y/F$, ending the proof. ■

Before proving that if $L$ is an Ea or an Ean, the same is true for the quotient of $L$ by a $*$-filter, we recall some basic facts about quotients of Heyting algebras. This will also lead to a version of the fundamental theorem for morphisms of Ea's and Ean's.

If $H$ is a Heyting algebra and $G$ is a filter on $H$, recall that the congruence $\theta_G$ defined by $G$ on $H$ may be described by

$$a\ \theta_G\ b \quad \text{iff} \quad \text{there is } t \in G \text{ such that } a \wedge t = b \wedge t.$$

The quotient $H/G$ is a Heyting algebra and the canonical quotient map,

$$\pi_G \colon H \to H/G, \quad a \mapsto a/G,$$

is a morphism of Heyting algebras, that is, it preserves $\bot$, $\top$ and the operations of meet ($\wedge$), join ($\vee$), negation ($\neg$), implication ($\to$) and equivalence ($\equiv$). In particular, $\pi_G$ is a morphism of Ean's. The next result is stated for equivalence algebras, but **is valid, *verbatim*, for Ean's**.

Theorem 13.82. *Let $L$ be an Ea and let $F$ be a proper $*$-filter in $L$. With notation as in* 13.53*, let $G$ be the (proper) filter generated by $\sigma(F)$ in $\Omega(L)$. Then:*

(a) *The map* $\alpha\colon L/F \to \Omega(L)/G$ *defined by* $\alpha(a/F) = S_a/G$ *is a wEa-embedding.*

(b) $L/F = \langle L/F, \leq, *, \top, * \rangle$ *is an Ea.*

(c) *If* $L \xrightarrow{f} R$ *is an Ea-morphism such that* $F \subseteq \operatorname{coker} f$*, then there is a unique Ea-morphism* $L/F \xrightarrow{g} R$ *such that the following diagram commutes:*

$$\begin{tikzcd} L \arrow[rr, "\pi_F"] \arrow[dr, "f"'] & & L/F \arrow[dl, "g"] \\ & R & \end{tikzcd}$$

*Moreover,* $g$ *is an embedding iff* $F = \operatorname{coker} f$.

*Proof.* We identify $L$ with its image via $\sigma\colon L \to \Omega(L)$. Under this identification, if $A \subseteq L$, write $[A]_H$ for the filter generated by $A$ in $\Omega(L)$. Note that for all $B \subseteq \Omega(L)$, $B \cap L$ stands for $\sigma^{-1}(B)$. Write $\tau$ for the $T$-operator $\tau_L$ on $L$ (13.26).

(a) Since $F \subseteq G$, $\alpha$ is well defined and $\alpha(\top/F) = \top/G$. To check preservation of $*$, let $x, y \in L$. Then

$$\begin{aligned} \alpha(x/F * y/F) &= \alpha((x * y)/F) = S_{x*y}/G = (S_x \equiv S_y)/G \\ &= (S_x/G \equiv S_y/G) = [\alpha(x/F) \equiv \alpha(y/F)], \end{aligned}$$

as needed. Now we observe

Fact. *For all* $x \in L$, $[\tau(F \cup \{x\})]_H = [F \cup \{x\}]_H$.

*Proof.* Clearly, $[F \cup \{x\}]_H \subseteq [\tau(F \cup \{x\})]_H$. For the reverse inclusion, recall that

- $*$ By 13.38(a), every filter in $\Omega(L)$ is an $*$-filter;
- $*$ 13.40(a) guarantees that $I = L \cap [F \cup \{x\}]_H$ is a $*$-filter on $L$.

Note that $F \cup \{x\} \subseteq I$. Hence, $\tau(F \cup \{x\}) \subseteq \tau(I) = I$. It follows that $[\tau(F \cup \{x\})]_H \subseteq [F \cup \{x\}]_H$, ending the proof of the Fact.

Now let $x, y \in L$ be such that $x/F \leq y/F$. Then the Fact yields

$$y \in \tau(F \cup \{x\}) \subseteq [\tau(F \cup \{x\})]_H = [F \cup \{x\}]_H,$$

and so there is $u \in F$ such that $S_x \cap S_u \subseteq S_y$. Hence,

$$S_x \cap S_y \cap S_u = S_x \cap S_u,$$

that is, $S_x/G \leq S_y/G$, showing that $\alpha$ is a wEa-morphism. Conversely, suppose that $S_x/G \leq S_y/G$ in $\Omega(L)$. Hence, there is $V \in G$ such that $S_x \cap V \subseteq S_y$. Since $G$ is the filter generated by $F$, there are $a_1, \ldots, a_n$ in $F$ satisfying $\bigcap_{i=1}^n S_{a_i} \subseteq V$. Therefore

$$S_x \cap \textstyle\bigcap_{i=1}^n S_{a_i} \subseteq S_y.$$

This last inclusion implies that $y \in [F \cup \{x\}]_H$, and so, by the Fact and 13.53(b), $y \in \tau(F \cup \{x\})$, and $x/F \leq y/F$. This completes the proof that $\alpha$ is a wEa-embedding. Item (b) follows immediately from 13.40(b).

(c) Uniqueness is clear. For $x \in L$, set $g(x/F) = f(x)$; since $\{\top\}$ is a $*$-filter in $R$ (it is an Ea) and $F \subseteq \operatorname{coker} f$, for all $x, y \in L$,

$$(1) \qquad (x * y) \in F \;\Rightarrow\; f(x * y) = \top \quad \text{iff} \quad f(x) = f(y),$$

and $g$ is well defined. If $F = \operatorname{coker} f$, then the first implication in (1) is an equivalence, and so $g$ will be injective iff $\operatorname{coker} f = F$. Clearly, $g$ preserves $*$. If $x/F \leq y/F$, then $y \in \tau(F \cup \{x\})$. It must be verified that $f(x) \leq f(y)$. Consider the $*$-filter $I = f(x)^{\rightarrow} \subseteq R$; by 13.40(a), $G = f^{-1}(I)$ is a $*$-filter on $L$. Since $\{f(x), \top\} \subseteq I$, we conclude that $F \cup \{x\} \subseteq \operatorname{coker} f \cup \{x\} \subseteq G$. Thus,

$$\tau(F \cup \{x\}) \subseteq \tau(G) = G,$$

whence $y \in G$, that is, $f(x) \leq f(y)$, ending the proof. ∎

When $L$ is an Ea or an Ean and $F$ is a $*$-filter on $L$, the structure $L/F$ constructed above is called the **quotient** of $L$ by $F$. It comes with a canonical quotient morphism, $\pi_F\colon L \to L/F$.

Since the Lindenbaum algebras of MEC and MECn, $\mathcal{L}_\perp$ and $\mathcal{L}$, are the free Ea and Ean, respectively, generated by their propositional parameters (13.75), we get

Corollary 13.83. *Any Ean (Ea) is a quotient of* $\mathcal{L}_\perp$ *(resp.,* $\mathcal{L}$*).*

*Proof.* If $L$ is an Ean, let $A$ be a set and $h\colon A \to L$ be a bijection. Let $\mathcal{L}_\perp(A)$ be the Lindenbaum algebra of the MECn calculus whose set of propositional parameters is $A$. By Lemma 13.74, the map $a \in A \;\mapsto\; h(a) \in L$ extends to an Ean-morphism $f\colon \mathcal{L}_\perp(A) \to L$. Clearly, $f$ is surjective. It follows from 13.82(c) that $f$ factors through an isomorphism $g\colon \mathcal{L}_\perp/U \to L$, where $U = \operatorname{coker} f$. ∎

The next result shows that the Lindenbaum algebra of MECn (or MEC) relative to a set of formulas is a quotient of its full Lindenbaum algebra. With notation as in Section 12.2, we have

Proposition 13.84. *If* $\Sigma$ *is a set of formulas in* MECn *and* $U = \overline{\Sigma}^{\,t}$ *is the theory generated by* $\Sigma$ *in* $\mathcal{L}_\perp$*, then* $\mathcal{L}_{\perp\Sigma}$ *is isomorphic to* $\mathcal{L}_\perp/U$*. In particular,* $\mathcal{L}_{\perp\Sigma}$ *is an Ean. A similar result holds for* $\mathcal{L}$ *and a set of formulas in* MEC.

*Proof.* Define $f\colon \mathcal{L}_\perp/U \to \mathcal{L}_{\perp\Sigma}$ by $f(\overline{\mathcal{A}}/U) = \mathcal{A}_\Sigma$. The properties of deducibility in MECn will show that $f$ is an isomorphism. ∎

As another application of Ea-quotients, it will be shown that every Ean has a *largest associative quotient*, generalizing the construction of the largest Boolean algebra quotient for Heyting algebras (Glivenko's Theorem, see Theorem VIII.4.3 (p. 157) in [Balbes and Dwinger, 1974]).

Recall from Section 13.7 that if $L$ is an Ean, $D$ is the $*$-filter of dense elements in $L$ (13.68).

Theorem 13.85. *Let* $L$ *be an Ean and* $D$ *be the filter of dense elements in* $L$*. Then:*

(a) *For* $x, y \in L$, $\;x/D \leq y/D\;$ *iff* $\;\neg\neg x \leq \neg\neg y$.

(b) $L/D$ *is an associative Ean.*

(c) *Let* $L \xrightarrow{f} K$ *be an Ean-morphism, with* $K$ *an associative Ean. Then there is a* <u>unique</u> *Ean-morphism* $\widehat{f}\colon L/D \to K$ *such that the following diagram is commutative:*

$$\begin{tikzcd} L \arrow[rr, "\pi_D"] \arrow[dr, "f"'] & & L/D \arrow[dl, "\widehat{f}"] \\ & K & \end{tikzcd}$$

(d) *The map* $\neg\neg x \mapsto x/D$ *is an isomorphism from* $Reg(L)$ *onto* $L/D$.

*Proof.* We prove (a), (b) and (c), leaving (d) to the reader. Recall that the definition of $x/D \le y/D$ is condition [po], just before the statement of 13.81.

(a) First assume that $\neg\neg x \le \neg\neg y$. Then, recalling that for all $z \in L$,

$$z \le \neg\neg z \quad \text{and} \quad (z * \neg\neg z) \in D$$

(see 13.66(a) and 13.68(a)(ii)), it follows that $y \in \tau(D \cup \{\neg\neg x\}) = \tau(D \cup \{x\})$. Hence, $x/D \le y/D$.

Conversely, assume that $x/D \le y/D$, i.e., $y \in \tau(D \cup \{x\})$. Let $\sigma\colon L \to \Omega$ be the embedding of Theorem 13.53. It was observed in the proof of 13.68(b) that $D(L) = \sigma^{-1}(D(\Omega))$. Let $G$ be the filter generated by $\sigma(D(L)) \cup \{\sigma(x)\}$ in $\Omega$. Since $\sigma^{-1}(G)$ is a $*$-filter in $L$ (13.54) containing $D \cup \{x\}$, we must have $\tau(D \cup \{x\}) \subseteq \sigma^{-1}(G)$. Thus, $\sigma(y) \in G$. By 13.1, there is $\{d_1, \ldots, d_n\} \subseteq D(L)$ such that

$$\sigma(d_1) \wedge \ldots \wedge \sigma(d_n) \wedge \sigma(x) \le \sigma(y). \tag{1}$$

From (1), and 13.6(b) we get, recalling that $\sigma$ preserves negation,

$$\begin{aligned}\sigma(\neg y) &\le \neg[\sigma(d_1) \wedge \ldots \wedge \sigma(d_n) \wedge \sigma(x)] \le \neg[\sigma(d_1) * \ldots * \sigma(d_n) * \sigma(x)] \\ &= \sigma(\neg(d_1 * \ldots * d_n * x)), {}^{7}\end{aligned}$$

and so we have $\neg y \le \neg(d_1 * \ldots * d_n * x)$, which in turn implies that

$$\neg\neg(d_1 * \ldots * d_n * x) \le \neg\neg y.$$

Because $\neg\neg$ distributes over $*$ (13.66(d)) and $\neg\neg d_k = \top$, the preceding inequality yields $\neg\neg x \le \neg\neg y$, completing the proof of (a).

(b) By Theorem 13.82, $L/D$ is an Ean. To show that $*$ is associative in $L/D$, let $a, b, c \in L$; by Proposition 13.66(h),

$$\neg\neg a * (\neg\neg b * \neg\neg c) = (\neg\neg a * \neg\neg b) * \neg\neg c,$$

and we conclude by 13.68(a)(i) that

$$a/D * (b/D * c/D) = (a/D * b/D) * c/D,$$

as needed.

(c) By 13.82(c), it is sufficient to check that $D \subseteq \text{coker } f$. But this immediate from the associativity of $K$, completing the proof. ∎

The reader will certainly recognize in Theorem 13.85 an adjoint functor situation. To make matters precise, write **AEan** for the category of associative Ean's and Ean-morphisms. If $L \xrightarrow{f} K$ is an Ean-morphism, by composing $f$ with the quotient map

[7] In any Ha, $\neg\neg$ distributes over finite meets. Since we have not shown this to be true, we chose a path that uses the results proven here.

from $K$ to $K/D(K)$, we get an Ean-morphism from $L$ to the associative Ean $K/D(K)$. Theorem 13.85 yields a unique Ean-morphism, $\mathcal{D}(f)\colon L/D(L) \to K/D(K)$, such that the following diagram is commutative:

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "\pi_{D(L)}"'] & K \arrow[d, "\pi_{D(K)}"] \\ L/D(L) \arrow[r, "\mathcal{D}(f)"'] & K/D(K) \end{tikzcd}$$

We can now state

Corollary 13.86. *The functor*

$$\mathcal{D}\colon \mathbf{Ean} \to \mathbf{AEan}, \quad \begin{cases} L \ \mapsto \ L/D, \\ f \in Hom(L, K) \ \mapsto \ \mathcal{D}(f), \end{cases}$$

*is left adjoint to the forgetful functor from* **AEan** *to* **Ean**.

## 10. A non-constructive embedding theorem

In this section we describe a non-constructive way of embedding an equivalence algebra in a complete Heyting algebra, making use of the concept of irreducible filter, as in Section 1 of Chapter II in [Rasiowa, 1974].

Definition 13.87. *A proper filter $F$ in a wEa $L$ is* **irreducible** *if for all filters $G_1$, $G_2$ in $L$,*

$$F = G_1 \cap G_2 \quad \textit{implies} \quad F = G_1 \textit{ or } F = G_2.$$

*Write $\mathcal{I}(L)$ for the set of irreducible $*$-filters in $L$. For $x \in L$, set*

$$I_x = \{F \in \mathcal{I}(L)\colon x \in F\}.$$

Note that $I_\top = \mathcal{I}(L)$, while if $L$ is a wEan, $I_\perp = \emptyset$.

Proposition 13.88. *Let $L$ be a wEa, $F$ a proper $*$-filter in $L$ and let $a, b$ be elements of $L$.*

(a) *If $a \notin F$, then there is an irreducible $*$-filter $G$ such that $F \subseteq G$ and $a \notin G$.*

(b) *If $(a * b) \notin F$, then there is an irreducible $*$-filter $G$ containing $F$ and separating $a$ and $b$.*

*Proof.* (a) Let $V = \{G \in S(L)\colon F \subseteq G$ and $a \notin G\}$, partially ordered by inclusion. Clearly, $V$ is non-empty and all chains in $V$ have an upper bound. By Zorn's Lemma, $V$ has a maximal element $G$, with $F \subseteq G$ and $a \notin G$. To show that $G$ is irreducible, assume that $G = H_1 \cap H_2$, for filters $H_1$, $H_2$ in $L$. Since $a \notin G$, $a$ must be outside one of the $H_i$'s, say $a \notin H_1$. But then $H_1 \in V$ and so the maximality of $G$ and $G \subseteq H_1$ imply that $G = H_1$.

(b) By 13.50, there is a proper $*$-filter $K$, containing $F$ and satisfying the alternative in the statement. If $a \in K$ and $b \notin K$, (a) yields an irreducible $*$-filter $G$, containing $K$, satisfying the same condition. The other possibility is handled similarly and the proof is complete. ■

Irreducibility generalizes primeness in distributive lattices:

LEMMA 13.89. *Let $L$ be a distributive lattice with $\top$ and let $F$ be a filter in $L$. The following are equivalent:*

(1) *$F$ is irreducible;*
(2) *$F$ is* **prime***, that is,*

$$\text{For all } a, b \in L, \quad (a \vee b) \in L \quad \text{implies} \quad a \in F \text{ or } b \in F.$$

*Proof.* (1)⇒(2). Suppose neither $a$ nor $b$ are in $F$. Define

$$G_1 = \{x \in L\colon x \geq a \wedge z, \text{ for some } z \in F\}.$$

It is straightforward to check that $G_1$ is a filter in $L$, that is, it satisfies

* $\top \in G_1$;
* $x \in G_1$ and $y \geq x$ implies $y \in G_1$;
* $x, y \in G_1$ implies $x \wedge y \in G_1$.

It is clear that $a \in G_1$ and that $F \subseteq G_1$. Thus, $F \neq G_1$. Similarly, we may define

$$G_2 = \{x \in L\colon x \geq b \wedge z, \text{ for some } z \in F\},$$

to get $b \in G_2 \setminus F$, with $F \subseteq G_2$. We now show that $G_1 \cap G_2 = F$, a contradiction that will end the proof of (1)⇒(2). For $x \in G_1 \cap G_2$, there are $t, z \in F$ such that

$$x \geq a \wedge t \quad \text{and} \quad x \geq b \wedge z.$$

Let $c = t \wedge z$ $(\in F)$; the inequalities above imply that $x$ is larger than $a \wedge c$ and $b \wedge c$. Thus,

$$x \geq (a \wedge c) \vee (b \wedge c) = c \wedge (a \vee b) \in F,$$

and so $x \in F$, as claimed.

(2)⇒(1). Suppose that $F = G_1 \cap G_2$, with $F \neq G_i$, $i = 1, 2$. Select $a \in G_1$ and $b \in G_2$, both outside $F$. Since $a, b \leq a \vee b$, we conclude that $a \vee b \in G_1 \cap G_2 = F$, a contradiction since neither $a$ nor $b$ are in $F$. ∎

REMARK 13.90. In spite of 13.89, there is an important difference between prime filters in distributive lattices and irreducible filters in Ea's: prime filters in distributive lattices are functorial, while irreducible filters in Ea's are not (see 14.47).

We take $\{\emptyset\} \cup \{I_x\colon x \in L\}$ as a subbasis for a topology on $\mathcal{I}(L)$; let $\Omega_{ir}(L)$ **be the cHa of opens of this topology**. The proof of Theorem 13.53, with Proposition 13.88 in place of Theorem 13.50, can be adapted to yield

THEOREM 13.91. *Let $L$ be an equivalence algebra. Then the map*

$$h\colon L \to \Omega_{ir}(L), \qquad x \mapsto I_x,$$

*is an Ea-embedding of $L$ into $\Omega_{ir}(L)$. Moreover, if $L$ has a least element $\perp$, then $h$ takes $\perp$ to $\perp$ in $\Omega_{ir}(L)$.*

CHAPTER 14

# Bi-conditional algebras

In this chapter we present another algebraic generalization of MEC and MECn, called *bi-conditional algebras*. These structures, a special kind of equivalence algebra, arise by observing that there is an implicit use of conjunction in the notion of proof. Thus, if $\Gamma = \{\mathcal{A}_0, \ldots, \mathcal{A}_n\}$ is a finite set of formulas in MEC (or MECn), $\Gamma \vdash \mathcal{B}$ functions as

$$\text{“ } \mathcal{A}_1 \wedge \ldots \wedge \mathcal{A}_n \vdash \mathcal{B} \text{ ”.}$$

## 1. Introduction. Basic properties

In this section we discuss the basic properties of equivalence algebras that have finite meets.

DEFINITION 14.1. *A partially ordered set $\langle P, \leq \rangle$ is a* $\wedge$**-semilattice** (*meet-semilattice*) *if for all $x, y \in P$*

$$x \wedge y =_{def} \textit{inf}\,\{x,\ y\} \textit{ exists in } P.$$

*We shall assume that all our $\wedge$-semilattices have a top element, $\top$. A* **morphism** *of $\wedge$-semilattices is a map preserving $\top$ and the operation $\wedge$.*[1]

*A $\wedge$-semilattice wEa ($\wedge$-wEa) is a weak equivalence algebra which is also a $\wedge$-semilattice.*

*For $a, b$ in a $\wedge$-wEa, $L$, define*

$$a \to b =_{def} a * (a \wedge b),$$

*called the* **implication** *operation in $L$.*

Note that for all $a, b$ in a $\wedge$-wEa $L$, we have

$$a \to b = a \to (a \wedge b).$$

PROPOSITION 14.2. *If $L$ is a $\wedge$-wEa and $a, b, c, d \in L$, then:*

(a) $(a * b) \wedge (c * d) \leq (a * c) * (b * d)$.
(b) $(a * b) \wedge (b * d) \leq (a * c)$ *and* $a \wedge c \leq a * c$.
(c) $a \wedge (a * b) = b \wedge (a * b) = a \wedge b$.
(d) $a \wedge (a \to b) = a \wedge b$.
(e) $a \leq (b * c)$ *implies* $a \wedge b = a \wedge c$.
(f) $a \leq b$ *iff* $(a \to b) = \top$.
(g) *If $L$ is a wEan, then* $(a \to \neg a) = \neg a$ *and* $(\neg a \to a) = \neg\neg a$.

[1]Similarly, one defines *join-semilattices* ($\vee$-semilattices).

*Proof.* (a) Since $(a * b) \wedge (c * d) \leq a * b, c * d$, axiom $[* \, 4]$ yields the desired conclusion.

(b) Taking $b = c$ in (a), we may write

$$(a * b) \wedge (b * d) = (b * a) \wedge (b * d) \leq (b * b) * (a * d) = (a * d),$$

verifying the first inequality in (b); the second follows from the first with $b = \top$.

(c) Since $a \wedge (a * b) \leq a, (a * b)$, axiom $[* \, 4]$ yields

$$a \wedge (a * b) \leq b. \tag{1}$$

The symmetry of the above argument gives

$$a \wedge (a * b) = b \wedge (a * b).$$

Inequality (1) guarantees that $a \wedge (a * b) \leq a \wedge b$, and so equality follows from the first inequality in (b). Item (d) is a direct consequence of (b) and the definition of implication.

(e) Since $a \leq b * c$, item (b) yields

$$b \wedge a \leq b \wedge (b * c) = b \wedge c,$$

and so $b \wedge a = a \wedge b \wedge c$. Similarly, $a \wedge c = a \wedge b \wedge c$, and the desired conclusion follows. Items (f) and (g) are clear. ■

DEFINITION 14.3. *A* **bi-conditional algebra (Bca)** *is a $\wedge$-wEa, $L$, satisfying, for all $x, y, z \in L$,*

$$x \wedge (y * z) \;=\; x \wedge [(x \wedge y) * (x \wedge z)]. \tag{$bca$}$$

*If $L$ has $\perp$, it is a* **bi-conditional algebra with negation (Bcan)** *if $L$ is a wEan.*

*If $L, K$ are Bca's, a map $L \xrightarrow{f} K$ is a morphism of Bca's if it is an Ea-morphism that preserves meets, that is, for $a, b \in L$,*

$$f(a \wedge b) \;=\; f(a) \wedge f(b).$$

*If $L, K$ are Bcan's, then $f$ is also required to preserve $\perp$. Write* **Bcan** *for the category of equivalence algebras with meets and negation.*

We shall be mostly interested in bi-conditional algebras with negation, although most of our proofs work for Bca's. Note that for $x$ in a $\wedge$-wEan $L$ we have

$$\neg\, x = x * \perp = x * (x \wedge \perp) = x \rightarrow \perp.$$

Therefore, any morphism of Bcan's preserves $\rightarrow$ and $\neg$.

PROPOSITION 14.4. *Let $L$ be a Bcan and $a$, $b$, $c$, $d$ be elements in $L$. Then:*

(a) $a \wedge b \leq c$ *iff* $a \leq (b \rightarrow c)$.
(b) $(a * b) = (a \rightarrow b) \wedge (b \rightarrow a)$. [2]
(c) $a \leq (b * c)$ *iff* $a \wedge b = a \wedge c$.
(d) $a * b \leq [(a \wedge c) * (b \wedge c)]$.
(e) $a \leq b$ *implies* $\begin{cases} b \rightarrow c \leq a \rightarrow c \\ c \rightarrow a \leq c \rightarrow b. \end{cases}$
(f) $a \rightarrow (b \rightarrow c) = (a \wedge b) \rightarrow c$; $(a \rightarrow b) \wedge (a \rightarrow c) = a \rightarrow (b \wedge c)$.

[2] This equation is the motivation for the name bi-conditional algebras.

(g) $(a * b) \wedge (c * d) \leq (a * c) * (b * d)$.

(h) $a, b \;\; \leq \;\; [(a \to b) \to b]$.

*Proof.* (a) Suppose that $a \leq (b \to c) = [b * (b \wedge c)]$; then 14.2(c) yields

$$a \wedge b = (a \wedge [b * (b \wedge c)]) \wedge b = a \wedge b \wedge [b * (b \wedge c)] = a \wedge b \wedge c \leq c,$$

verifying the "only if" part of the statement. For the converse, assume that $a \wedge b \leq c$, that is, $a \wedge b = a \wedge b \wedge c$. Then, by [bca],

$$a \wedge [b * (b \wedge c)] = a \wedge [(a \wedge b) * (a \wedge b \wedge c)] = a,$$

proving the first statement in (c); the second is an immediate consequence of the first.

(b) By 14.2(b), we have

$$(a \to b) \wedge (b \to a) = [a * (a \wedge b)] \wedge [b * (a \wedge b)] \leq a * b.$$

On the other hand, 14.2(c) and (b) say that $a * b$ is below $a \to b$ and $b \to a$. Item (c) comes directly from (a) and (b).

(d) We have

$$(a \wedge c) \wedge (a * b) = c \wedge (a \wedge (a * b)) = c \wedge (b \wedge (a * b)) = (c \wedge b) \wedge (a * b),$$

and we conclude by (c).

(e) We discuss the first inequality, leaving the second to the reader. By (a), it is enough to verify that $a \wedge (b \to c) \leq c$. But we have

$$a \wedge (b \to c) \leq b \wedge (b \to c) \leq c,$$

as needed. Item (f) can be proven similarly.

By (c), it is enough to verify that

$$(a * c) \wedge (a * b) \wedge (c * d) = (b * d) \wedge (a * b) \wedge (c * d),$$

or equivalently, that

$$(*) \qquad \begin{cases} (a * c) \wedge (a * b) \wedge (c * d) \leq b * d, \\ (b * d) \wedge (a * b) \wedge (c * d) \leq a * c. \end{cases}$$

Recalling 14.2(b), we have

$$(a * c) \wedge (a * b) \wedge (c * d) \leq (c * b) \wedge (c * d) \leq b * d,$$

proving the first inequality in $(*)$. The second is similar. Item (h) follows directly from (a) and 14.2(d). ∎

Example 14.5. Let $H$ be a Heyting algebra and let $L$ be a subset of $H$ which contains $\top$, and is closed under $\leftrightarrow$ and $\wedge$. By 13.6(c), $L$ is a Bca. If $\bot \in L$, then it is a Bcan. For instance, the interval $(0, 1]$ $(= [0, 1] - \{0\})$ is a Bca. ∎

Example 14.6. An Ean might be closed under meets without being a bi-conditional algebra. As an example, consider the Ean $C_3$ of 13.42. Although a complete lattice, $C_3$ does not satisfy [bca]. To see this, note that $x_1 \wedge (x_2 * \bot) = x_1 \wedge \bot = \bot$, while

$$x_1 \wedge [(x_1 \wedge x_2) * (x_1 \wedge \bot)] = x_1 \wedge [\bot * \bot] = x_1 * \top = x_1.$$

A similar computation will show that the Ean $DA$ in 13.49(I), which is closed under meets, is not a Bcan. ∎

However, we have

PROPOSITION 14.7. *Let $L$ and $K$ be Bcan's such that $K$ is a dense algebra. Then $L \mathring{\vee} K$ is a Bcan.*

*Proof.* The pertinent definitions and notation are in 13.47; by 13.48, $A =_{def} (L \mathring{\vee} K)$ is an Ean. Recall that the domain of $A$ is the disjoint union of $L^t$ and $K^b$. We begin with

FACT 14.7.A. *If $S$ is a Bcan and a dense algebra, then for all $x, y \in S$,*

$$x \wedge y = \bot \quad \textit{iff} \quad x = \bot \text{ or } y = \bot.$$

*Proof.* Suppose that $x \neq \bot$. From $x \wedge y = \bot$, we conclude (14.4(a)) $y \leq (x \to \bot) = \neg x$. Since $S$ is a dense algebra, $\neg x = \bot$ and so $y = \bot$. The converse is clear.

It follows from 14.7.A that $K^b$ is closed under meets. Recall that for $u \in K^b$ and $v \in L^t$, we have $u \wedge v = v$. Thus, $A$ is closed under meets.

The definition of $\mathring{\vee}$ and the fact that $L$ is a dense algebra yield

(A) $u * v \in K^b$ iff $u, v \in K^b$;

$$\text{(B) } u * v \in L^t \quad \text{iff} \quad \begin{cases} (B.1) \ \ u, v \in L^t \\ \qquad\quad \textit{or} \\ (B.2) \ \ u \in L^t \textit{ and } v \in K^b, \textit{with } u * v = u; \\ \qquad\quad \textit{or} \\ (B.3) \ \ u \in K^b \textit{ and } v \in L^t, \textit{with } u * v = v. \end{cases}$$

We now verify [bca]. Notation is as in Definition 14.3.

**Proof of [bca]**. Note that if $x = \bot$, $x = \top$ or $y = z$, there is nothing to prove. We now discuss the following cases:

<u>Case 1. $x \in K^b$.</u> If $y * z \in K^b$ (A), there is nothing to prove, because $K$ is a Bcan. We must now take care of the cases corresponding to (B): $y * z \in L^t$. Note that $x \wedge (y * z) = y * z$.

<u>1.B.1: $y, z \in L^t$.</u> Since $y, z < x$, we get

$$x \wedge [(x \wedge y) * (x \wedge z)] = x \wedge (y * z) = y * z,$$

as needed.

<u>1.B.2: $y \in L^t$ and $z \in K^b$.</u> Then, $y * z = y$; since $x \wedge z \in K^b$,

$$x \wedge [(x \wedge y) * (x \wedge z)] = x \wedge [y * (x \wedge z)] = x \wedge y = y \ ,$$

as needed. (B.3) can be treated exactly as (B.2), ending the discussion of case 1.

<u>Case 2. $x \in L^t$.</u> We have the following subcases:

<u>2.A: $y * z \in K^b$.</u> Then $x \wedge (y * z) = x$ and

$$x \wedge [(x \wedge y) * (x \wedge z)] = x \wedge (x * x) = x * \top = x,$$

as needed.

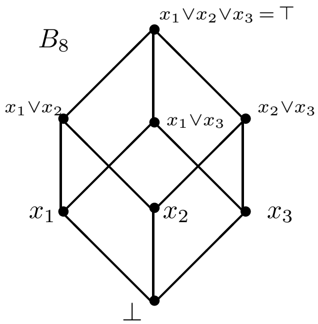
<u>2.B: $y \in z \in L^t$.</u> Notice that if $y, z \in L^t$ (B.1), there is nothing to prove, because $L$ is a Bcan. We are left with

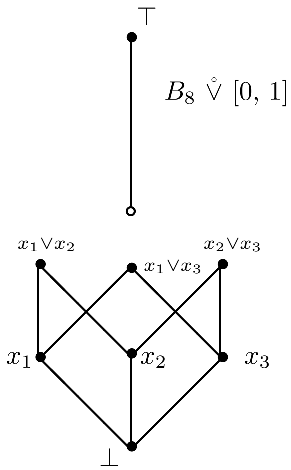
<u>2.B.2: $y \in L^t$ and $z \in K^b$.</u> Then $x \wedge (y * z) = x \wedge y$; since $x \in L^t$, we also have

$$x \wedge [(x \wedge y) * (x \wedge z)] = x \wedge [(x \wedge y) * x) = x \wedge y,$$

because of (c) in 14.2 and the fact that $L$ is Bcan. The case corresponding to (B.3) can be treated similarly. ■

Example 14.8. Let $B_{2^n}$ be the Boolean algebra generated by $n$ atoms. Since $B_{2^n}$ is a Boolean algebra and $I = [0, 1]$ is a cHa, it follows from 14.7 that $B_{2^n} \mathring{\vee} I$ is a Bcan. A schematic diagram of $B_4 \mathring{\vee} I$ appears in 13.47, while one for $B_8 \mathring{\vee} I$ is included below.




■

Example 14.6 exhibits an Ean which is a (complete) lattice and not a Bcan. With Bcan's is a different story:

Proposition 14.9. *Let $L$ be a Bcan. Then:*

(a) *$L$ is a Heyting algebra iff it is a lattice.*
(b) *$L$ is a complete Heyting algebra iff it is a complete lattice.*

*Proof.* We only prove (b); a similar reasoning will yield (a). By Example 14.5, every Heyting algebra is a Bcan in a natural way. To prove the converse, it must be shown, by 13.8, that if $L$ has arbitrary sup's and inf's, then, for all $T \subseteq L$ and all $x \in L$,

$[\wedge, \bigvee]$ $$x \wedge \bigvee T = \bigvee_{t \in T} (x \wedge t).$$

To prove $[\wedge, \bigvee]$, it suffices to show that

(1) $$x \wedge \bigvee T \;\leq\; \bigvee_{t \in T} (x \wedge t),$$

or equivalently, $\bigvee T \;\leq\; x \to [\bigvee_{t \in T} (x \wedge t)]$. Thus, (1) is equivalent to the following statement:

(2) $$\text{For all } t \in T,\quad t \;\leq\; x \to [\bigvee_{t \in T} (x \wedge t)].$$

Obviously $x \wedge t \leq \bigvee_{t \in T} (x \wedge t)$, and so 14.4(a) yields (2). ■

It will be shown in Theorem 14.18 that all *associative Bcan's* are Boolean algebras.

We end this section with the laws for the negation of meets and implication, complementing Proposition 13.66. One should recall

– Proposition 13.66(a) and the contra-positive property for negation, [**neg 2**], in Remark 13.59, that may be used without explicit reference.

– The concepts of dense and regular in Definition 13.67.

PROPOSITION 14.10. *For all $x, y$ in a Bcan $L$:*

(a) $x \le \neg y$ *iff* $x \wedge y = \bot$ *iff* $\neg\neg x \wedge y = \bot$.
(b) $\neg\neg(x \wedge y) = \neg\neg x \wedge \neg\neg y$.
(c) $\neg(x \to y) = \neg\neg x \wedge \neg y$.
(d) *If $y$ is regular in $L$, then $x \to y$ is regular in $L$.*
(e) $\neg\neg(x \to y) = x \to \neg\neg y = \neg\neg x \to \neg\neg y$.
(f) *If $x$, $y$ are dense in $L$, then $x \wedge y$ is dense in $L$.*

*Proof.* (a) Since $\neg y = y \to \bot$, the first equivalence follows from 14.4(a). For the second, since $x \le \neg\neg x$, the "if" part is clear. The converse can be obtained from the first equivalence, as follows:

$$x \wedge y = \bot \;\Rightarrow\; x \le \neg y \;\Rightarrow\; \neg\neg x \le \neg\neg\neg y = \neg y \;\Rightarrow\; \neg\neg x \wedge y = \bot.$$

(b) Since $x \wedge y \le x, y$, we certainly have $\neg\neg(x \wedge y) \le \neg\neg x \wedge \neg\neg y$. For the reverse inequality, since

$$(*) \qquad x \wedge y \wedge \neg(x \wedge y) = \bot,$$

applying (a) twice, we arrive at $\neg\neg x \wedge \neg\neg y \wedge \neg(x \wedge y) = \bot$, wherefrom we infer $\neg\neg x \wedge \neg\neg y \le \neg\neg(x \wedge y)$, as needed.

(c) We have $x \wedge \neg y \wedge (x \to y) = \neg y \wedge x \wedge y = \bot$, and so, by (a), we conclude $\neg\neg x \wedge \neg y \wedge (x \to y) = \bot$, that is ((a), once more), $\neg\neg x \wedge \neg y \le \neg(x \to y)$. For the reverse inequality, notice that

$$(1) \qquad y \le x \to y \quad \text{and} \quad \neg x \le x \to y,$$

because $x \wedge y \le y$ and $\neg x \wedge x \le y$. From (1) and the contra-positive law, we deduce $\neg(x \to y) \le \neg y \wedge \neg\neg x$, as desired.

(d) It must be shown that if $\neg\neg y = y$, then $\neg\neg(x \to y) = x \to y$. It is enough to verify that $\neg\neg(x \to y) \le x \to y$, or equivalently, that $x \wedge \neg\neg(x \to y) \le y$. But, from $(*)$ above, item (a) yields

$$x \wedge \neg\neg(x \wedge y) \wedge \neg y = \bot,$$

that is, $x \wedge \neg\neg(x \to y) \le \neg\neg y = y$, as needed.
(e) By 14.4(e), we have $(\neg\neg x \to \neg\neg y) \le (x \to \neg\neg y)$. It follows from $(*)$ and item (a) that

$$\neg\neg x \wedge (x \to y) \le \neg\neg y,$$

and so $(x \to y) \le (\neg\neg x \to \neg\neg y)$. Since both $(x \to \neg\neg y)$ and $(\neg\neg x \to \neg\neg y)$ are regular, to finish the proof of (e) it is enough to verify that $(x \to \neg\neg y) \le \neg\neg(x \to y)$, or equivalently, by (a) and (c),

$$(**) \qquad (x \to \neg\neg y) \wedge \neg(x \to y) = (x \to \neg\neg y) \wedge \neg\neg x \wedge \neg y = \bot.$$

But notice that we have

$$x \wedge \neg y \wedge (x \to \neg\neg y) = \neg y \wedge x \wedge \neg\neg y = \perp,$$

and so (**) follows from this last inequality and item (a). Item (f) is an immediate consequence of (b). ■

## 2. Characterizations of bi-conditional algebras

This section is dedicated to the proof of Theorem 14.11 and some applications.

THEOREM 14.11. *For a $\wedge$-wEa $L$, the following conditions are equivalent:*

(1) *$L$ is a Bca.*
(2) *For all $a, b, c \in L$, $a \leq (b * c)$ iff $a \wedge b = a \wedge c$.*
(3) *$L$ is an Ea and all Ea-filters in $L$* (13.13) *are closed under meets.*
(4) *All wEa-filters in $L$ are meet closed $*$-filters.*
(5) *For all $a, b, c \in L$, $[a \to (b \to c)] = [(a \wedge b) \to c]$.*
(6) *The structure $\langle L, *, \wedge, \top\rangle$ satisfies, for all $x$, $y$, $z$,*[3]

$$\bullet \begin{cases} [*\ 1]: & x * y = y * x; \\ [*\ 2]: & x * \top = x; \\ [exp\ 2]: & x * x = \top. \end{cases}$$

$$\bullet \begin{cases} [\wedge\ 1]: & x \wedge y = y \wedge x; \\ [\wedge\ 2]: & x \wedge (y \wedge z) = (x \wedge y) \wedge z; \\ [\wedge\ 3]: & x \wedge x = x \wedge \top = x. \end{cases}$$

$$\bullet \begin{cases} [bca]: & x \wedge (y * z) = x \wedge [(x \wedge y) * (x \wedge z)]; \\ [bca\ 0]: & x \wedge (x * y) = y \wedge (x * y). \end{cases}$$

*Proof.* The strategy is to prove that (1) through (5) are equivalent as follows:

$$(1) \Leftrightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (2),$$

and then that (6)$\Leftrightarrow$(2).

(1)$\Rightarrow$(2). By 14.2(e) it is enough to check that $a \wedge c = b \wedge c$ implies $a \leq b * c$. But $[bca]$ in 14.3 yields

$$a \wedge (b * c) = a \wedge [(a \wedge b) * (a \wedge c)] = a \wedge \top = a,$$

and so $a \leq b * c$, as needed.

(2)$\Rightarrow$(1). For $a, b, c \in L$, it must be shown that

$$\begin{cases} \text{(i) } a \wedge (b * c) \wedge (a \wedge b) = a \wedge (b * c) \wedge (a \wedge c), \\ \text{(ii) } a \wedge [(a \wedge b) * (a \wedge c)] \wedge b = a \wedge [(a \wedge b) * (a \wedge c)] \wedge c, \end{cases}$$

equalities which together with (2) imply $[bca]$. For (i), 14.2(c) yields

[3]Here it is not assumed that $L$ is a $\wedge$-wEa.

$$a \wedge (b * c) \wedge (a \wedge b) = a \wedge b \wedge (b * c) = a \wedge b \wedge c \ = (a \wedge c) \wedge [(a \wedge b) * (a \wedge c)],$$

as desired. The equality in (ii) may be proven analogously.

(2)⇒(3). We first show that wEa-filters in $L$ are meet closed. Let $F$ be a filter in $L$ and assume that $a, b \in F$. From $a \wedge b = a \wedge (a \wedge b)$ and (2) we get

$$a \leq b * (a \wedge b).$$

Hence, $b * (a \wedge b) \in F$ and 13.14(a) yields $a \wedge b \in F$, as desired. For $S \subseteq L$, define

$$\beta(S) = \{z \in L\colon \exists\ \widehat{A} \subseteq_f S \text{ such that } \textstyle\bigwedge A \leq z\}.^{4}$$

We shall verify that $\beta$ is a $T$-operator in $L$. It is clear that $\beta$ is inflationary and increasing. To verify idempotency, let $t \in \beta(\beta(S))$; then there is $A \subseteq_f \beta(S)$ such that $\bigwedge A \leq t$. For each $a \in A$, select a finite subset $C_a \subseteq S$ such that $\bigwedge C_a \leq a$. Then $C = \bigcup_{a \in A} C_a$ is a finite subset of $S$ and

$$\bigwedge C \leq \bigwedge A \leq t,$$

showing that $t \in \beta(S)$. Next, it must be shown that $\beta(S)$ is a filter in $L$. Clearly, $\top \in \beta(S)$ and $x \in \beta(S)$ implies $x^{\rightarrow} \subseteq \beta(S)$. It also clear that $\beta(S)$ is meet closed. Assume that $a * b$, $u * v \in \beta(S)$; by 14.2(a),

$$(a * b) \wedge (u * v) \leq (a * u) * (b * v),$$

and the meet closure of $\beta(S)$ guarantees that $(a * u) * (b * v) \in \beta(S)$, completing the verification that it is a filter in $L$.

To check that $\beta$ satisfies $[T\ 3]$ in 13.21, let $S \cup \{x, y\} \subseteq L$ be such that

$$\beta(S \cup \{x\}) = \beta(S \cup \{y\}).$$

Since $y \in \beta(S \cup \{x\})$ and $x \in \beta(S \cup \{y\})$, there are a finite subsets $A$, $B$ of $S$ such that

$$x \wedge \bigwedge A \leq y \quad \text{and} \quad y \wedge \bigwedge B \leq x.$$

Let $t = \bigwedge (A \cup B)$; it is clear that $x \wedge t = y \wedge t$ and that $t \in \beta(S)$. Since (2) guarantees that $t \leq x * y$, we conclude that $x * y \in \beta(S)$, as needed.

To finish the proof of (2)⇒(3), observe that if $x \in L$, then $x^{\rightarrow}$ is a fixed point of $\beta$. Thus, all principal filters in $L$ are $*$-filters and $L$ is an equivalence algebra.

(3)⇒(4). Let $F$ be a filter in $L$; since $F$ is closed under meets, $F$ is the *directed union* of principal filters. The conclusion follows from 13.27 and (3).

(4)⇒(5). We shall use Corollary 13.52, the algebraic analog of the $\equiv$-introduction rule of MEC and MECn. We start with

FACT 1. *For all $a, b, c \in L$, $\ a \wedge [(a \wedge b) \rightarrow c] \leq (b \rightarrow c)$.*

*Proof.* Let $F$ be a $*$-filter containing $a \wedge [(a \wedge b) \rightarrow c]$ and $b$. Then $a$ and $(a \wedge b) \rightarrow c$ must be in $F$. Since $F$ is closed under meets, we get $a \wedge b \in F$, and so $c \in F$ and $b \wedge c \in F$. Hence,

$$b * (b \wedge c) = (b \rightarrow c) \in F.$$

[4]The symbol $\subseteq_f$ indicates that $A$ is a <u>finite</u> subset of $S$ (possibly empty); $\bigwedge A$ is the meet of the elements in $A$.

Clearly, any $*$-filter containing $b \wedge c$ must contain $b$. We have shown that no $*$-filter to which $a \wedge [(a \wedge b) \to c]$ belongs can separate $b$ from $b \wedge c$; the desired conclusion follows from 13.52.

To show that $(a \wedge b) \to c \leq [a \to (b \to c)] = a * [a \wedge (b \to c)]$, let $F$ be a $*$-filter containing $(a \wedge b) \to c$ and $a$. Since $F$ is closed under meets, we get $a \wedge [(a \wedge b) \to c] \in F$, and Fact 1 yields $(b \to c) \in F$. Now, closure under meets yields $a \wedge (b \to c) \in F$. Clearly, any $*$-filter containing $a \wedge (b \to c)$ must contain $a$ and we conclude by 13.52. The proof of the inequality $[a \to (b \to c)] \leq (a \wedge b) \to c$ is analogous.

(5)⇒(2). By 14.2(e), it is enough to show the "if" part of the equivalence. From $a \wedge b = a \wedge c$, we get $a \wedge b = a \wedge b \wedge c$, and so

$$\top = (a \wedge b) * (a \wedge b \wedge c) = [(a \wedge b) \to c] = a \to (b \to c),$$

and 14.2(f) yields $a \leq (b \to c)$. Similarly, $a \leq (c \to b)$ and 14.2(a) yields

$$a \leq (b \to c) \wedge (c \to b) = [b * (b \wedge c)] \wedge [c * (b \wedge c)] \leq b * c,$$

as needed.

(2)⇒(6). It is easily established that conditions $[\wedge\, i]$, $i = 1, 2, 3$, are satisfied in any $\wedge$-semilattice. The remaining assertions are clear.

(6)⇒(2). Assume that $\langle L, *, \leq, \top\rangle$ is a structure satisfying the conditions in (6). For $x, y \in L$, define

$$x \leq y \quad \text{iff} \quad x \wedge y = x.$$

This gives a partial order in $L$ such that $x \wedge y = \inf\{x, y\}$.

Fact 2. *For all* $a, b, c \in L$, $\quad a \leq (b * c) \quad \Leftrightarrow \quad a \wedge b = a \wedge c.$

*Proof.* (⇐) From $[bca]$, $[exp\ 2]$ and $[*\ 2]$ comes

$$a \wedge (b * c) = a \wedge [(a \wedge b) * (a \wedge c)] = a * \top = a,$$

and $a \leq b * c$, as needed.

(⇒) From $[bca]$ we get

$$a = a \wedge (b * c) = a \wedge [(a \wedge b) * (a \wedge c)],$$

and so $a \leq (a \wedge b) * (a \wedge c)$. Now, an application of $[bca\ 0]$ yields

$$a \wedge b \leq (a \wedge b) \wedge [(a \wedge b) * (a \wedge c)] = (a \wedge c) \wedge [(a \wedge b) * (a \wedge c)],$$

wherefrom we conclude that $a \wedge b \leq a \wedge c$. By symmetry, it follows that $a \wedge b = a \wedge c$.

Since $[bca]$ is included in the axioms, to finish the proof it must be verified that $L$ is a wEa, that is, that it satisfies $[*\ 4]$ and (the other half of) $[*\ 3]$ in 13.9. For $x, y \in L$, suppose that $x * y = \top$. From Fact 2 (or $[bca\ 0]$) it follows that $x = x * \top = y * \top = y$. For $[*\ 4]$, suppose that $a \leq (x * y) \wedge (u * v)$. By Fact 2, $a \wedge x = a \wedge y$ and $a \wedge u = a \wedge v$. Hence

$$a \wedge (x * u) = a \wedge [(a \wedge x) * (a \wedge u)] = a \wedge [(a \wedge y) * (a \wedge v)] = a \wedge (y * v),$$

and another application of Fact 2 yields $a \leq (x * u) * (y * v)$, as needed. ■

We explicitly register some of the consequences of the statement and proof of Theorem 14.11 in

COROLLARY 14.12. (a) *Every Bca is an equivalence algebra.*

(b) *The statements in* 14.11(6) *yield a finite equational axiomatization of bi-conditional algebras in a language containing two binary operations* $(*, \wedge)$ *and a constant* $\top$. *An axiomatization of bi-conditional algebras with negation is obtained if a new constant* $\perp$ *is added to the language and axiom* [*neg* 1] *in* 13.9 *is added to the list in* 14.11(6).

(c) *If* $L$ *is a Bca then for all* $a, b, u, v \in L$*:*

(i) $(a * b) \wedge (u * v) \leq (a \wedge u) * (b \wedge v)$.
(ii) $b \leq a \rightarrow b$ *(i.e.,* $b \rightarrow (a \rightarrow b) = \top$*).*

COROLLARY 14.13. *Let* **Bca** *and* **Bcan** *be the categories of Bca's and Bcan's, respectively.*

(a) **Bca** *and* **Bcan** *are complete and co-complete categories.*
(b) **Bca** *and* **Bcan** *have free objects in any number of generators.*

*Proof.* Since both categories are equational, this is standard universal algebra. For (a) see [MacLane, 1971], Sections III.3 and III.4 (pp. 62ff), as well as 14.21(b); for (b) see, [Grätzer, 1979], Corollary IV.25.2 (p. 167) or [Balbes and Dwinger, 1974], Theorem I.12.4 (p. 19). ■

The presence of meets makes the theory of filters in a Bcan similar to the familiar theory in distributive lattices. Recall that $[S]$ is the (wEa-) filter generated by $S$.

COROLLARY 14.14. *Let* $L$ *be a Bcan.*

(a) *A subset of* $L$ *is a filter iff it satisfies* [$fil$ 1], [$fil$ 2] *in* 13.13 *and is closed under meets.*

(b) *If* $S$ *is a subset of* $L$*, then*

$$[S] = \{x \in L\colon \text{There is } A \subseteq_f S \text{ such that } \bigwedge A \leq x\},$$

*where* $\subseteq_f$ *denotes "finite subset of" and* $\bigwedge A$ *is the meet of the elements in* $A$. *In particular,* $[S]$ *is a proper filter iff* $S$ *has the fip (finite intersection property), that is, for all finite* $A \subseteq S$, $\bigwedge A \neq \perp$.

*Proof.* (a) By 14.11(3), all wEa-filters in $L$ are meet closed. Conversely, if $F \subseteq L$ is meet closed and satisfies

$$\top \in F \ ([fil\ 1]) \quad \text{and} \quad x \in F \text{ implies } x^{\rightarrow} \subseteq F \ ([fil\ 2]),$$

the same method used to show that $\beta(S)$ was a filter in the proof of 14.11 will show that $F$ satisfies [$fil$ 3] in 13.13. Item (b) is an immediate consequence of (a). ■

Our next theme is a constructive way to carve out Bcan's from a given Bcan.

DEFINITION 14.15. *Let* $L$ *be a Bcan. An* **interior operator** (*I-operator*) *on* $L$ *is a map* $\eta\colon L \rightarrow L$ *such that for* $a, b \in L$,

[$I$ 1] $\eta(a) \leq a$;
[$I$ 2] $\eta(\eta(a)) = \eta(a)$;
[$I$ 3] $\eta(a \wedge b) = \eta(a) \wedge \eta(b)$.

*If* $\eta$ *is an I-operator on* $L$*, write* $\eta L$ *for the image of* $\eta$ *in* $L$.

Proposition 14.16. *If $\eta$ is an $I$-operator on a Bcan $L$, then, with the partial order induced by $L$, $\eta L$ is a Bcan. Moreover, if $\widehat{\wedge}$, $\widehat{*}$ and $\widehat{\to}$ are the meet, equivalence and implication in $\eta L$, then, for all $x, y \in \eta L$,*

$$\begin{cases} x \mathbin{\widehat{\wedge}} y = x \wedge y \ (in\ L); \\ x \mathbin{\widehat{*}} y = \eta(x * y); \\ x \mathbin{\widehat{\to}} y = \eta(x \to y). \end{cases}$$

*Proof.* Note that $\eta L = \{x \in L\colon \eta(x) = x\}$ is precisely the set of fixed points of $\eta$. Also, any $I$-operator is increasing, that is,

$$x \leq y \quad \text{implies} \quad \eta(x) \leq \eta(y),$$

because $[I\ 3]$ yields $\eta(x) = \eta(x \wedge y) = \eta(x) \wedge \eta(y)$.

With the partial order induced by $L$, suppose that $z \in \eta L$ satisfies $z \leq x, y$. Then $z \leq x \wedge y$ and so

$$z = \eta(z) \leq \eta(x \wedge y) = \eta(x) \wedge \eta(y) = x \wedge y,$$

proving that $x \wedge y$ is the meet of $x$, $y$ in $\eta L$. Clearly, $\eta(\bot) = \bot$, while the top in $\eta L$ is $\eta(\top)$. It follows that $\langle \eta L, \leq \rangle$ is a $\wedge$-semilattice.

For $x, y \in \eta L$, set

$$(*) \qquad x \mathbin{\widehat{*}} y =_{def} \eta(x * y).$$

It is clear that this operation satisfies axioms $[*\ 1]$, $[*\ 2]$ and $[exp\ 2]$ in 14.11(6). To conclude the proof it is sufficient to check that $\widehat{*}$ and $\wedge$ satisfy axioms $[bca\ 0]$ and $[bca]$ in 14.11(6). For $a, b \in \eta L$,

$$\begin{aligned} a \wedge (a \mathbin{\widehat{*}} b) &= a \wedge \eta(a * b) = \eta(a) \wedge \eta(a * b) = \eta(a \wedge (a * b)) \\ &= \eta(b \wedge (a * b)) = \eta(b) \wedge \eta(a * b) = b \wedge (a \mathbin{\widehat{*}} b), \end{aligned}$$

as desired. The computations for $[bca]$ are analogous and are omitted. Finally, for implication in $\eta L$ we have

$$a \mathbin{\widehat{\to}} b = a \mathbin{\widehat{*}} (a \wedge b) = \eta(a * (a \wedge b) = \eta(a \to b),$$

ending the proof. ∎

Example 14.17. Let $L$ be a Bcan. If $\eta$ is an $I$-operator in $L$, in general, $\eta L$ is not a subalgebra of $L$. Not only the top of $\eta L$ might not be the top of $L$, but implication will not be preserved. Consider, for instance, the $I$-operator $m_x(y) = y \wedge x$, $x \in L$. Clearly, the set of fixed points of $m_x$ is $x^{\leftarrow}$, whose top element is $x$. Moreover, it follows from 14.16 that for all $a, b \leq x$,

$$a \mathbin{\widehat{\to}} b = x \wedge (a \to b),$$

where $\to$ is implication in $L$, while $\widehat{\to}$ denotes implication in $x^{\leftarrow}$.

As yet another example, consider the Bcan $L \times L$, with coordinatewise operations; $L \times L$ is a Bcan because of 14.13(a). Define an $I$-operator, $\eta$, on $L \times L$ by

$$\eta(a, b) = (a \wedge b, b).$$

Write $L^{[2]}$ for the set of fixed points of $\eta$. Clearly,

$$L^{[2]} = \{(a,\, b) \in L \times L\colon a \le b\}.$$

Here we have $\eta(\top,\, \top) = (\top,\, \top)$ and $\eta(\bot,\, \bot) = \bot$, but implication is not the implication in $L \times L$. For instance, for $(a,\, b) \in L^{[2]}$, 14.16 yields

$$\widehat{\neg}\,(a,\, b) = [(a,\, b) \mathrel{\widehat{\to}} (\bot,\, \bot)] = \eta([(a,\, b) \to (\bot,\, \bot)] = \eta(\neg\, a,\, \neg\, b) \ = (\neg\, b,\, \neg\, b),$$

because $a \le b$ implies $\neg\, b \le \neg\, a$ (14.4(e)). When $L$ is a complete Boolean algebra, $L^{[2]}$ is an important ingredient in the classification of injective Stone algebras (Theorem VIII.8.8 in [Balbes and Dwinger, 1974]). ■

We end this section with

THEOREM 14.18. *An associative Bcan is a Boolean algebra.*

*Proof.* The definition of associative wEa appears in 13.70. Let $L$ be an associative Bcan. We first show that binary join ($\vee$) is definable in $L$ and that $\wedge$ distributes over $\vee$. For $x, y \in L$, set

$$x \vee y = \neg\,(\neg\, x \wedge \neg\, y).$$

FACT. *For all $x, y, z \in L$:*

(a) $x \vee y = sup\ \{x,\, y\}$ *in the partial order of $L$.*
(b) $x \vee y = \neg\, x \to y$.
(c) $z \wedge (x \vee y) = (z \wedge x) \vee (z \wedge y)$.
(d) $x \vee \neg\, x = \top$.

*Proof.* (a) Since $\neg\, x \wedge \neg\, y \le \neg\, x$, we get

$$x = \neg\,\neg\, x \le \neg\,(\neg\, x \wedge \neg\, y) = x \vee y;$$

similarly, one has $y \le x \vee y$. Now assume that $x, y \le z$. Then the contra-positive law yields $\neg\, z \le \neg\, x \wedge \neg\, y$, and so, taking the contra-positive once more, we get $x \vee y \le \neg\,\neg\, z = z$, proving (a).

(b) By Proposition 14.10(c), $\neg\, x \wedge \neg\, y = \neg\,(\neg\, x \to y)$. Hence, taking negation on both sides, we get the desired conclusion.

(c) It is enough to verify that $z \wedge (x \vee y) \le (z \wedge x) \vee (z \wedge y)$. With (b), this amounts to

$$z \wedge (\neg\, x \to y) \le [\neg\,(z \wedge x) \to (z \wedge y)],$$

or equivalently,

$$z \wedge (\neg\, x \to y) \wedge \neg\,(z \wedge x) \le y. \tag{1}$$

By 14.2(d), to prove (1), it is enough to check that

$$z \wedge \neg\,(z \wedge x) \le \neg\, x.$$

But this is clear, since $x \wedge z \wedge \neg\,(z \wedge x) = \bot$, completing the verification of (c). Item (d) is clear, ending the proof of the Fact.

It follows from the Fact that $\langle L, \wedge, \vee, \neg\,, \bot, \top\rangle$ is a Boolean algebra, that is, a distributive lattice with $\bot$ and $\top$ in which all elements have a complement. Now notice that for all $x, y \in L$, 14.4(c) tells us that $*$ is the usual equivalence in a Boolean algebra. ■

## 3. Algebraic completeness of BCC

Recall from Section 11.4 that BCC is the calculus obtained from **IEC** (or MEC) by adding the (infix) binary propositional connective $\wedge$, together with the appropriate introduction and elimination rules. The Lindenbaum algebra $\mathcal{L}^{\wedge}$ of BCC was presented in Section 12.4. From 12.10 and 14.11(6) we get

LEMMA 14.19. *$\mathcal{L}^{\wedge}$ is a Bcan.*

With 13.8 as a model, we can also state a completeness result for **BCC**. We start with

DEFINITION 14.20. *Let $L$ be a Bcan. An $L$-**valuation** of* BCC *is a map $v$:* BCC $\to L$ *such that for all formulas $\varphi, \psi \in$* BCC*,*

$$v(\varphi \equiv \psi) = v(\varphi) * v(\psi), \qquad v(\varphi \wedge \psi) = v(\varphi) \wedge v(\psi) \qquad \textit{and} \qquad v(\perp) = \perp.$$

All the results in Section 13.8 hold for BCC and Bcan's, with essentially the same proofs. The new ingredient is to keep in mind that filters in a Bcan are closed under meets (14.11(4)). Consequently, we give full statements, omitting proofs.

THEOREM 14.21. (a) (*Completeness of* BCC) *Let $\Gamma \cup \{\sigma\}$ be a finite set of formulas in* BCC*. The following are equivalent:*

(1) $\Gamma \vdash_{BCC} \sigma$.

(2) *For all interpretations of* BCC *in a Bcan $L$,*

$$\textstyle\bigwedge_{\phi \in \Gamma} v(\varphi) \leq v(\sigma).$$

(3) *For all interpretations of* BCC *in a Heyting algebra $H$,*

$$\textstyle\bigwedge_{\phi \in \Gamma} v(\varphi) \leq v(\sigma).$$

(b) *Let $A$ be a set and $\mathcal{L}^{\wedge}(A)$ be the Lindenbaum algebra of* BCC *whose set of propositional parameters distinct from $\perp$ is $A$. Then $\mathcal{L}^{\wedge}(A)$ is the free Bcan in the generators $A$.*

(c) *Let $\mathcal{H}$ be the full intuitionistic propositional calculus, constructed from the same set of propositional parameters as* BCC*. Let $\Gamma \cup \{\varphi\}$ be a finite set formulas in* BCC*. We consider $\Gamma \cup \{\varphi\}$ as a set of formulas in $\mathcal{H}$, identifying $\equiv$ and $\wedge$ in* BCC *with the connectives $(\star \to \star') \wedge (\star' \to \star)$ and $\wedge$ in $\mathcal{H}$, respectively. Then*

$$\Gamma \ \vdash_{BCC} \ \varphi \quad \textit{iff} \quad \Gamma \ \vdash_{\mathcal{H}} \ \varphi,$$

*where $\vdash_{BCC}$ indicates proof in* BCC*, while $\vdash_{\mathcal{H}}$ denotes proof in $\mathcal{H}$.* ■

## 4. Quotients. The lattice of congruences in a Bcan

Just as in Section 13.9, if $F$ is a filter on a Bcan $L$, define a relation $\theta_F$ on $L^2$ by:

$$x \ \theta_F \ y \quad \text{iff} \quad x * y \in F.$$

We shall see that this relation has smoother properties than the corresponding relation in Ea's.

LEMMA 14.22. *Let $F$ be a filter in a Bcan $L$. Then $\theta_F$ is an equivalence relation in $L$ such that for all $x, y, u, v \in L$,*

(a) $x \; \theta_F \; y$ *iff there is* $t \in F$ *such that* $x \wedge t = y \wedge t$.

(b) $x \; \theta_F \; u$ *and* $y \; \theta_F \; v$ *implies* $\begin{cases} (x * y) \; \theta_F \; (u * v); \\ (x \wedge y) \; \theta_F \; (u \wedge v); \\ (x \to y) \; \theta_F \; (u \to v); \\ \neg x \; \theta_F \; \neg y. \end{cases}$

(c) $(x \wedge y) \; \theta_F \; x$ *iff* $(x \to y) \in F$.

*Proof.* By Lemma 13.14, $\theta_F$ is an equivalence relation. Item (a) is a direct consequence of 14.4(c). For (b), note that by 13.80, $\theta_F$ preserves $*$; next, since $\to$ and $\neg$ are definable from $\wedge$ and $*$, it is enough to verify that $\theta_F$ preserves $\wedge$. By (a), $x \; \theta_F \; u$ implies that there is $t \in F$ such that $x \wedge t = u \wedge t$. Similarly, there is $z \in F$ such that $y \wedge z = v \wedge z$. But then

$$(x \wedge y) \wedge (t \wedge z) = (x \wedge t) \wedge (y \wedge z) = (u \wedge t) \wedge (v \wedge z) = (u \wedge v) \wedge (t \wedge z),$$

with $t \wedge z \in F$, because $F$ is meet closed by 14.11(3).

(c) If $(x \wedge y) \; \theta_F \; x$, then $(x \to y) = x * (x \wedge y) \in F$. Conversely, if $(x \to y) \in F$, then by 14.4(a) we get

$$x \wedge y \wedge (x \to y) = x \wedge y = x \wedge (x \to y),$$

and so (a) yields $x \; \theta_F \; (x \wedge y)$. ∎

If $F$ is a filter in a Bcan $L$ and $x \in L$, write $x/F$ for the equivalence class of $x$ with respect to $\theta_F$. Let

$$L/F = \{x/F \colon x \in L\}$$

be the set of equivalence classes of elements of $L$ for $\theta_F$. Define binary operations $*$, $\wedge$ on $L/F$ by the rules

$$x/F \wedge y/F = (x \wedge y)/F, \qquad x/F * y/F = (x * y)/F.$$

THEOREM 14.23. (a) *Let* $F$ *be a filter in a Bcan* $L$. *With the operations* $*$, $\wedge$ *defined above* $L/F$ *is a Bcan, whose top element is* $\top/F$ *and whose bottom element is* $\bot/F$. *Moreover, the canonical quotient map* $\pi_F \colon L \to L/F$, $\pi_F(x) = x/F$, *is a Bcan morphism.*

(b) *Let* $f \colon L \to K$ *be a Bcan morphism. Let*

$$\text{coker } f = \{x \in L \colon f(x) = \top\}.$$

*Then* coker $f$ *is a filter in* $L$ *and there is a unique Bcan embedding,* $\widehat{f} \colon L/\text{coker } f \to K$, *such that the following diagram is commutative:*

$$\begin{tikzcd} L \arrow[rr, "\pi_F"] \arrow[dr, "f"'] & & L/\text{coker } f \arrow[dl, "\widehat{f}"] \\ & K & \end{tikzcd}$$

(c) *The map* $F \mapsto L/F$ *is a natural bijective correspondence between the set of filters in* $L$ *and the set of isomorphism types of Bcan's over* $L$ *which are images of* $L$.

*Proof.* (a) It follows from Lemma 14.22(b) that $\theta_F$ is a congruence with respect to $*$, $\wedge$, $\rightarrow$ and $\neg$. Consequently, $L/F$ satisfies all the axioms in 14.11(6), being therefore a Bcan. It is clear that the canonical quotient map $\pi_F$ is a Bcan morphism. Item (b) is a consequence of Theorem 13.82(c), recalling that all filters in a Bcan are $*$-filters (14.11(4)).

(c) By (b), if $f\colon L \rightarrow K$ is a surjective Bcan morphism, then $\widehat{f}\colon L/\mathrm{coker} f \rightarrow K$ is an isomorphism. Thus, to prove (c), it is enough to verify that if $F$, $G$ are distinct filters in $L$, the quotient algebras cannot be isomorphic over $L$. It is left to the reader to check that, if there is a Bcan isomorphism, $\gamma\colon L/F \rightarrow L/G$, such that the diagram

$$\begin{tikzcd} \mathrm{L} \arrow[r, "\pi_F"] \arrow[dr, "\pi_G"'] & L/F \arrow[d, "\gamma"] \\ & L/G \end{tikzcd}$$

is commutative, then $F = G$. ■

As a consequence of the above and preceding results we register

Corollary 14.24. *If $L$ is a Bcan and $D$ is the filter of dense elements in $L$, then $L/D$ is a Boolean algebra.*

*Proof.* By Theorems 14.23 and 13.85(b), $L/D$ is an associative Bcan and the result follows from 14.18. ■

Recall from Section 12.4 that $\mathcal{L}^{\wedge}_{\Sigma}$ is the Lindenbaum algebra of BCC relative to a set of formulas $\Sigma$. By 14.21(b), $\mathcal{L}^{\wedge}(A)$ is the free Bcan on the generators $A$. As was the case for Ean's, discussed in 13.83 and 13.84, and with essentially the same proofs, we have

Proposition 14.25.

(a) *Every Bcan is a quotient of $\mathcal{L}^{\wedge}$.*

(b) *If $\Sigma$ is a set of formulas in* BCC *and $U$ is the theory generated by $\Sigma$ in $\mathcal{L}^{\wedge}$, then $\mathcal{L}^{\wedge}_{\Sigma}$ is isomorphic to $\mathcal{L}^{\wedge}/U$. In particular, $\mathcal{L}^{\wedge}_{\Sigma}$ is a Bcan.*

Theorem 14.23 and Proposition 13.88 yield the following sharpening of 13.50, where the notion of **irreducible filter** is in 13.87:

Proposition 14.26. *Let $L$ be a Bcan and let $F$ be a proper filter in $L$. For $a, b \in L$ suppose that $a \rightarrow b \notin F$. Then:*

(a) *There is a proper filter $G$ in $L$ such that $F \subseteq G$, $a \in G$ and $b \notin G$.*
(b) *There is an irreducible filter $K$ in $L$ such that $F \subseteq K$, $a \in K$ and $b \notin K$.*

*Proof.* (a) Since $L/F$ is a Bcan, it follows from Lemma 14.22(c) that it is not true that $a/F \leq b/F$, that is, $b/F \notin (a/F)^{\rightarrow}$. Since $(a/F)^{\rightarrow}$ is a proper filter in $L/F$, $\pi_F^{-1}((a/F)^{\rightarrow}) = G$ is a proper filter in $L$ (14.23(d)). It is straightforward to verify that $G$ has the required properties.

(b) Just apply Proposition 13.88 to the filter $G$ constructed in (a). ■

Remark 14.27. By Lemma 13.89, the celebrated separation result of G. Birkhoff and M. Stone (see Theorem III.4.1 in [Balbes and Dwinger, 1974]) could be phrased for Bcan's as

(SB) *If $I$ is an ideal and $F$ is a filter such that $I \cap F = \emptyset$, then there is an irreducible filter $G$ containing $F$ and disjoint from $I$.*

However, (SB) is false for Bcan's in general. Let $L = B_4 \mathring{\vee} [0,1]$, discussed in 13.49(III) and 14.8. By 14.7, $L$ is a Bcan. With notation as in 13.49, $I = \{\bot, a, b\}$ is an ideal in $L$, disjoint from the filter $\widehat{F} = (0,1]$. However, there is no filter properly containing $F$, disjoint from $I$; moreover $F$ is not irreducible, because $F = a^{\rightarrow} \cap b^{\rightarrow}$, with both $a^{\rightarrow}$ and $b^{\rightarrow}$ distinct from $F$. ■

The next result describes the lattice of filters in a Bcan. Before its statement we set down some standard terminology from lattice theory. Recall that $A \subseteq_f B$ means that $A$ is a finite subset of $B$.

DEFINITION 14.28. *Let $L$ be a complete lattice and $a \in L$.*

(a) *$a$ is* **compact** *if for all $S \subseteq L$,*

[*compact*] $$a \leq \bigvee S \;\Rightarrow\; \exists\, A \subseteq_f S \text{ such that } a \leq \bigvee A.$$

(b) *A subset $\mathcal{B} \subseteq L$ is a* **basis** *for $L$ if for all $a \in L$, there is $S \subseteq \mathcal{B}$ such that $a = \bigvee S$.*

(c) *$L$ is* **algebraic** *if the set of compact elements in $L$ constitutes a basis for $L$.*

THEOREM 14.29. *If $L$ is a Bcan, then $\mathcal{F}il(L)$ is an algebraic complete Heyting algebra.*

*Proof.* We begin with

FACT 1. *A distributive algebraic lattice is a cHa.*

*Proof.* Suppose $L$ is as above and let $S \cup \{x\} \subseteq L$. To prove the $[\wedge, \bigvee]$-law (13.8), it is enough to verify that

(1) $$x \wedge \bigvee S \leq \bigvee_{s \in S} x \wedge s.$$

Let $c$ be a compact element of $L$ such that $c \leq x \wedge \bigvee S$. Then there is $A \subseteq_f S$ such that $c \leq \bigvee A$. From $c \leq x$, distributivity and the finiteness of $A$, we conclude that

(2) $$c \leq x \wedge \bigvee A = \bigvee_{a \in A} (x \wedge a) \leq \bigvee_{s \in S} x \wedge s.$$

Since the compact elements form a basis for $L$, relation (2) guarantees that (1) holds in $L$, ending the proof of Fact 1.

FACT 2. *All finitely generated filters in $\mathcal{F}il(L)$ are compact and $\mathcal{F}il(L)$ is an algebraic lattice.*

*Proof.* Let $A \subseteq_f L$ and $F = [A]$ be the filter generated by $A$. If $S = \{G_i \colon i \in I\}$ is a family of filters in $L$, write $B = \bigcup_{i \in I} F_i$. By 13.18(a), $\bigvee S = [B]$. If $S$ is such that

$$A \subseteq F \subseteq \bigvee S = [B],$$

then 13.16 implies that for each $a \in A$ there is $\alpha_a \subseteq_f B$ with $a \in [\alpha_a]$. Since $A$ and the $\alpha_a$ are finite, $a \in A$, there must be $J \subseteq_f I$ such that $\bigcup_{a \in A} \alpha_a \subseteq \bigcup_{i \in J} F_i$. Hence, $F = [A] \subseteq \bigvee\{F_i \colon i \in J\}$, proving $F$ to be compact in $\mathcal{F}il(L)$. The fact that the compact elements form a basis for $\mathcal{F}il(L)$ is now an immediate consequence of 13.16.

Taking into account Facts 1 and 2, the proof will be finished as soon as it is verified that $\mathcal{F}il(L)$ is a distributive lattice. To this end, we show

FACT 3. *Suppose that $a$, $b$, $c$ are elements of $L$ such that $a \wedge b \leq c$. Then* $[(a \to c) \to c] \wedge [(b \to c) \to c] \ \leq \ c$.

*Proof.* From $a \wedge b \leq c$ and 14.4(a) we get $a \leq (b \to c)$; this last relation and 14.4(e) yield

$$[(b \to c) \to c] \leq (a \to c). \tag{3}$$

It is clear that $c \leq (a \to c)$. From (3) and Lemma 14.2(d) comes

$$[(a \to c) \to c] \wedge [(b \to c) \to c] \leq [(a \to c) \to c] \wedge (a \to c) \ = (a \to c) \wedge c = c,$$

ending the proof of Fact 3.

To check that $\mathcal{F}il(L)$ is distributive, let $F$, $G$, $H$ be filters in $L$. It is enough to show that

$$F \cap (G \vee H) \subseteq (F \vee G) \cap (F \vee H),$$

where $\vee$ denotes the filter generated by the union of the sets involved. If $t \in F \cap (G \vee H)$, then, by 14.14, there are $x \in G$ and $y \in H$ such that $x \wedge y \leq t$. By Fact 3, we have

$$[(x \to t) \to t] \wedge [(y \to t) \to t] \ \leq \ t. \tag{4}$$

Since 14.4(h) gives

$$x,\, t \leq [(x \to t) \to t] \quad \text{and} \quad y,\, t \leq [(y \to t) \to t],$$

we get $[(x \to t) \to t] \in F \cap G$ and $[(y \to t) \to t] \in F \cap H$. Thus, (4) implies that $t \in [(F \cap G) \vee (F \cap H)]$, as desired. ■

Our next result shows that $\mathcal{F}il(L)$ yields all *congruences* in Bca's. Recall that a **congruence** on a Bcan $L$ is an equivalence relation $\theta$ on $L$ such that for all $x, y, u, v \in L$,

$$x \ \theta \ u \quad \text{and} \quad y \ \theta \ v \ \ \Rightarrow \ \ (x * y) \ \theta \ (u * v) \quad \text{and} \quad (x \wedge y) \ \theta \ (u \wedge v).$$

Write $Cong(L)$ for the set of congruences in $L$. If $\{\theta_i \colon i \in I\}$ is a family of congruences in $L$, then $\bigcap_{i \in I} \theta_i$ is a congruence in $L$. Define the **congruence generated by** $T \subseteq L^2$ as

$$\theta_T = \bigcap \{\theta \in Cong(L) : T \subseteq \theta\}.$$

It is clear that, partially ordered by inclusion, $Cong(L)$ is a complete lattice. If $\theta$ is a congruence in $L$, set

$$F_\theta = \{x \in L \colon x \ \theta \ \top\}.$$

PROPOSITION 14.30. *Let $L$ be a Bcan. The maps*

$$F \mapsto \theta_F \quad \textit{and} \quad \theta \mapsto F_\theta$$

*are inverse isomorphisms between the complete lattices $Cong(L)$ and $\mathcal{F}il(L)$. In particular, $Cong(L)$ is an algebraic cHa.*

*Proof.* We begin with

FACT. *If $\theta \in Cong(L)$, then $F_\theta$ is a filter on $L$ such that $\theta_{F_\theta} = \theta$.*

*Proof.* It is clear that $\top \in F_\theta$. If $x \ \theta \ \top$ and $y \geq x$, then $y \wedge x = x$. Since $\theta$ is a congruence, we have

$$x\ \theta\ \top \ \Rightarrow\ (y \wedge x)\ \theta\ (y \wedge \top)\ \Rightarrow\ x\ \theta\ y\ \Rightarrow y\ \theta\ \top,$$

where the last step comes from the transitivity of $\theta$; thus, $F_\theta$ satisfies $[fil\ 2]$ in 13.13. By 14.14(a), it is enough to check that $F_\theta$ is meet closed. If $x, y \in F_\theta$, then

$$x\ \theta\ \top\ \Rightarrow\ (x \wedge y)\ \theta\ y\ \Rightarrow\ (x \wedge y)\ \theta\ \top,$$

and $x \wedge y \in F_\theta$, as needed. It remains to check that $\theta = \theta_{F_\theta}$. To this end, we prove that for all $x, y \in L$,

(1) $$x\ \theta\ y \quad \text{iff} \quad (x * y) \in F_\theta,$$

and the conclusion follows from the definition of $\theta_{F_\theta}$. Since $\theta$ is a congruence with respect to $*$, the "only if" part of (1) is clear. For the converse, we have, recalling from 14.2(c) that $x \wedge (x * y) = x \wedge y$,

$$(x * y)\ \theta\ \top\ \Rightarrow\ x \wedge (x * y)\ \theta\ (x \wedge \top)\ \Rightarrow\ (x \wedge y)\ \theta\ x.$$

Similarly, $(x \wedge y)\ \theta\ y$, and so, by transitivity, $x\ \theta\ y$, as needed.

Write
$$\begin{cases} \alpha\colon \mathcal{F}il(L) \to Cong(L) \text{ for } F \mapsto \theta_F, \\ \beta\colon Cong(L) \to \mathcal{F}il(L) \text{ for } \theta \mapsto F_\theta; \end{cases}$$

to show that $\alpha$ and $\beta$ are inverse isomorphisms of complete lattices, it is enough to check that
$$\begin{cases} \text{(i) both are increasing;} \\ \text{(ii) } \alpha \circ \beta = Id_{Cong(L)} \quad \text{and} \quad \beta \circ \alpha = Id_{\mathcal{F}il(L)}. \end{cases}$$

It is straightforward that $\alpha$ and $\beta$ are increasing. For $F \in \mathcal{F}il(L)$, 13.14(c) gives $F_{\theta_F} = F$, and so $\beta(\alpha(F)) = F$. On the other hand, the Fact above assures $\alpha(\beta(\theta)) = \theta_{F_\theta} = \theta$, completing the proof. ■

## 5. Regular completions

One of the main results of this section is that a Bcan can be regularly embedded in a complete Heyting algebra. If $L$ is a Bcan, Theorem 14.37 shows that $L$ can be regularly embedded in the cHa of complete ideals in $L$, $\gamma L$. This result is related to Theorem XII.1.14 in [Balbes and Dwinger, 1974], due to P. Crawley, for lattices satisfying the distributive law in Lemma 14.33(a) (called $D_1$ in [Balbes and Dwinger, 1974]). For *complete Browerian lattices* (the duals of cHa's), the corresponding construction appears as Theorem 3.18 in [Rasiowa, 1951]. We also show that the *normal completion* of $L$ is isomorphic to $\gamma L$ (14.44), and so the regular Ea-embedding $L \xrightarrow{\gamma} \gamma L$ of Theorem 14.37 is called the *completion* of the Bcan $L$. The normal completion of a partially ordered set is described in Section 2, Chapter XII, of [Balbes and Dwinger, 1974]. To keep the exposition self-contained, we recall the main points needed for our constructions. Most of our results also apply to Bca's, but they are stated and proven only for Bcan's.

If $a$, $b$ are elements of a partially ordered set $\langle P, \leq \rangle$, $a \vee b$ and $a \wedge b$ are the sup and inf of $\{a, b\}$, whenever they exist in $P$; for $S \subseteq P$, $\bigvee S$ and $\bigwedge S$ denote the sup and inf of $S$ in $P$, respectively, whenever they exist.

Definition 14.31. *Let* $P \xrightarrow{f} Q$ *be a map, where* $P$ *and* $Q$ *are partially ordered sets. We say that* $f$ *is:*

(i) *a* **$\wedge$-morphism** *whenever it preserves the finite meets that exist in* $P$, *that is, if* $S \subseteq P$ *is finite and* $\bigwedge S$ *exists in* $P$, *then* $\bigwedge f(S)$ *exists in* $Q$ *and* $f(\bigwedge S) = \bigwedge f(S)$;

(ii) *a* **$[\wedge, \bigvee]$-morphism** *if it preserves finite meets and arbitrary joins in* $P$, *i.e., it is a* $\wedge$*-morphism such that for all* $S \subseteq P$, *if* $\bigvee S$ *exists in* $P$, *then* $\bigvee f(S)$ *exists in* $Q$ *and* $f(\bigvee S) = \bigvee f(S)$; [5]

(iii) *a* **regular morphism** *if it preserves all meets and joins that exist in* $P$.

Definition 14.32. *Let* $\langle P, \leq \rangle$ *be a partially ordered set. A non-empty subset* $I$ *of* $P$ *is an* **ideal** *if it satisfies, for all* $a, b \in L$,

[*id* 1] $\quad a \in I$ *implies* $a^{\leftarrow} \subseteq I$;
[*id* 2] $\quad$ *If* $a, b \in I$ *and* $a \vee b$ *exists in* $P$, *then* $a \vee b \in I$.

*An ideal is* **proper** *if it is distinct from* $P$. *Thus, an ideal* $I$ *is proper iff* $\top \notin I$. *Write* $\mathcal{I}d(P)$ *for the set of ideals in* $P$.

*An ideal* $I$ *is said to be* **complete** *if for all* $S \subseteq I$, *if* $\bigvee S$ *exists in* $P$, *then* $\bigvee S \in I$. *Write* $\gamma P$ *for the set of complete ideals in* $P$.

Note that if $P$ is a partially ordered set and $x \in P$, then $x^{\leftarrow}$ is a complete ideal in $P$.

Lemma 14.33. *Let* $L$ *be a Bcan.*

(a) *Let* $A \subseteq L$ *be such that* $\bigvee A$ *exists in* $L$. *Then, for all* $b \in L$, $\bigvee_{a \in A} b \wedge a$ *exists in* $L$ *and*

$$b \wedge \bigvee A = \bigvee\nolimits_{a \in A} b \wedge a.$$

(b) *Let* $A$, $B$ *be subsets of* $L$ *such that* $\bigvee A$ *and* $\bigvee B$ *exist in* $L$. *Then*

$$(\bigvee A) \wedge (\bigvee B) = \bigvee\nolimits_{(a,b) \in A \times B} a \wedge b.$$

*Proof.* (a) Let $\alpha = \bigvee_{a \in A} b \wedge a$; it suffices to check that

$$b \wedge \bigvee A \ \leq \ \alpha.$$

Since $b \wedge a \leq \alpha$ for all $a \in A$, 14.4(a) yields $a \leq (b \to \alpha)$ for all $a \in A$. Thus, $\bigvee A \leq (b \to \alpha)$ and so $b \wedge \bigvee A \leq \alpha$, as desired. Item (b) follows directly from (a). ∎

Lemma 14.34. *Let* $L$ *be a Bcan.*

(a) *The intersection of a family of complete ideals is a complete ideal. If* $S$ *is a subset of* $L$, *let*

$$c(S) = \bigcap \{K\colon K \text{ is a complete ideal in } L \text{ and } S \subseteq K\}$$

*be the complete ideal generated by* $S$ *in* $L$. *Then,*

$$c(S) = \{x \in L\colon \exists\ A \subseteq \bigcup\nolimits_{s \in S} s^{\leftarrow} \text{ such that } \bigvee A \text{ exists in } L \text{ and } x \leq \bigvee A\}.$$

(b) *If* $I$, $J$ *are ideals in* $L$, *then* $c(I \cap J) = c(I) \cap c(J)$.

(c) *Let* $I_u$, $u \in U$, *be a family of complete ideals in* $L$. *Then the supremum of the* $I_u$*'s in the order of containment in* $\gamma L$ *is given by*

$$\bigvee\nolimits_{u \in U} I_u = \{x \in L\colon \exists\ A \subseteq \bigcup\nolimits_{u \in U} I_u \text{ such that } \bigvee A \text{ exists in } L \text{ and } x \leq \bigvee A\}.$$

[5]This generalizes cHa-morphisms (13.8) to complete posets.

*Proof.* (a) It is clear that the intersection of complete ideals is a complete ideal. Write $J$ for the right side of the second equality in (a). Note that $S \subseteq J$ and that any complete ideal containing $S$ must contain $J$. To finish the proof we must show that $J$ is a complete ideal in $L$. It is clear that $J$ satisfies $[id\ 1]$. For $[id\ 2]$, let $B \subseteq J$ be such that $\bigvee B$ exists in $L$; for each $b \in B$, select $A_b \subseteq \bigcup_{s \in S} s^{\leftarrow}$ such that $b \leq \bigvee A_b$. Note that

$$C_b \ = \ \{b \wedge a \colon a \in A_b\} \ \subseteq \ \textstyle\bigcup_{s \in S} s^{\leftarrow}.$$

By 14.33(a), we have $b = \bigvee C_b$ for all $b \in B$. It is now straightforward to verify that

$$\textstyle\bigvee B = \bigvee(\bigcup_{b \in B} C_b),$$

with $\bigcup_{b \in B} C_b \subseteq \bigcup_{s \in S} s^{\leftarrow}$, as required to prove that $\bigvee B \in J$.

(b) Since it is clear that $I \subseteq c(I)$, to prove that completion preserves intersection, it is enough to verify that

$$c(I) \cap c(J) \subseteq c(I \cap J).$$

If $t \in c(I) \cap c(J)$, by (a) there are $A \subseteq I$ and $B \subseteq J$ such that $t \leq \bigvee A$ and $t \leq \bigvee B$. Since $I$ and $J$ are ideals, we have

$$C =_{def} \{a \wedge b \colon a \in A \text{ and } b \in B\} \subseteq I \cap J.$$

By 14.33(b), $t \leq (\bigvee A) \cap (\bigvee B) = \bigvee C$, and so $t \in c(I \cap J)$, as desired. Item (c) follows directly from (a), since the complete ideal generated by $\bigcup_{u \in U} I_u$ is the sup of the $I_u$'s in $\gamma L$. ■

By 14.34, $\gamma L$ is a complete lattice, where meets are set-theoretical intersection and joins are given by 14.34(c).

DEFINITION 14.35. *An equivalence algebra is a* **complete equivalence algebra (cEa)** *if it is complete in the associated partial order.*

*Let $L$ be an Ean and let $P$ be a cEa. A map $L \xrightarrow{f} P$ is said to be a* **regular Ean-morphism** *if it is a morphism of Ean's that preserves all meets and joins that exist in $L$.*

*A* **regular Ean-embedding** *is a regular Ean-morphism that is also an embedding of Ean's.*

Note that a cEa is an Ean (13.55). Also observe that cHa's are complete equivalence algebras; 13.42 gives an example of a cEa which is not even a Bcan (14.6). The next result is a close relative of 14.9:

PROPOSITION 14.36. *If $L$ is a cEa, the following are equivalent:*

(1) *$L$ is a cHa.*
(2) *For all families $\{(a_i, b_i) : i \ \in \ I\}$ in $L \times L$,*

$$\textstyle\bigwedge_{i \in I} \, (a_i \, * \, b_i) \leq (\bigwedge_{i \in I} \, a_i) \, * \, (\bigwedge_{i \in I} \, b_i).$$

THEOREM 14.37. *If $L$ is a Bcan, then $\gamma L$ is a complete Heyting algebra and the map*

$$\gamma \colon L \to \gamma L, \qquad \gamma(x) = x^{\leftarrow},$$

*is a regular Ea-embedding, whose image is a basis for $\gamma L$. Moreover, if $H$ is a cHa and $L \xrightarrow{f} H$ is a $[\wedge, \bigvee]$-morphism, then there is a unique $[\wedge, \bigvee]$-morphism, $\gamma f\colon \gamma L \to H$, making the following diagram commutative:*

$$\begin{tikzcd} L \arrow[rr, "\gamma"] \arrow[dr, "f"'] & & \gamma L \arrow[dl, "\gamma f"] \\ & H & \end{tikzcd}$$

*Proof.* To show that $\gamma L$ is a cHa, it is enough to verify that

$$J \cap \left(\bigvee_{u\in U} I_u\right) \;\subseteq\; \bigvee_{u\in U} J \cap I_u \tag{1}$$

where $J$ and $I_u$, $u \in U$, are complete ideals in $L$. If $t \in J \cap \left(\bigvee_{u\in U} I_u\right)$, by 14.34(c), there is $A \subseteq \bigcup_{u\in U} I_u$ such that $\alpha = \bigvee A$ exists in $L$ and $t \le \alpha$. Consider the set

$$B = \{t \wedge a\colon a \in A\}.$$

By 14.33(a), $\bigvee B$ exists in $L$ and

$$\bigvee B = t \wedge \bigvee A = t \wedge \alpha = t.$$

By 14.34(c), it is enough to show that $B \subseteq \bigcup_{u\in U} J \cap I_u$, for then $t \in \bigvee_{u\in U} J \cap I_u$, as needed to prove (1). For $a \in A$, select $u \in U$ such that $a \in I_u$; because $t \in J$ and $a \wedge t \le a, t$, condition $[id\ 1]$ implies that $a \wedge t \in J \cap I_u$. Thus, $B \subseteq \bigvee_{u\in U} J \cap I_u$, completing the proof that $\gamma L$ is a cHa.

It is clear that $\gamma$ is injective and that for all $x, y \in L$,

$$x \le y \quad \text{iff} \quad \gamma(x) \subseteq \gamma(y).$$

It is also clear that for all $I \in \gamma L$, $I = \bigvee_{x\in I} x^{\leftarrow}$, and so Im $\gamma$ is a basis for $\gamma L$. Next, we show that $\gamma$ preserves $*$, as well as all meets and joins that exist in $L$.

I. $\gamma$ preserves $*$: We must verify that for all $x, y \in L$,

$$(x * y)^{\leftarrow} = \bigvee\{J \in \gamma L\colon J \cap x^{\leftarrow} = J \cap y^{\leftarrow}\}, \tag{2}$$

corresponding to the equation $\gamma(x * y) = [\gamma(x) \leftrightarrow \gamma(y)]$, where $\leftrightarrow$ denotes equivalence in $\gamma L$. First note that if $t \in (x * y)^{\leftarrow} \cap x^{\leftarrow}$, then axiom $[*\ 4]$ (13.9) implies that $t \in y^{\leftarrow}$. Since the argument is symmetrical in $x$ and $y$, we conclude that

$$(x * y)^{\leftarrow} \cap x^{\leftarrow} = (x * y)^{\leftarrow} \cap y^{\leftarrow}.$$

Suppose that $J$ is a complete ideal satisfying $J \cap x^{\leftarrow} = J \cap y^{\leftarrow}$. For $t \in J$, note that $t \wedge x \in J \cap x^{\leftarrow}$. Thus, $(t \wedge x) \le y$. Analogously, one verifies that $t \wedge y \le x$, that is, $t \wedge x = t \wedge y$. Now, 14.4(c) yields $t \le x * y$, completing the proof of (2).

II. $\gamma$ preserves the meets in $L$: Suppose that for $A \subseteq L$, $\alpha = \bigwedge A$ exists in $L$. It is straightforward to check that

$$\alpha^{\leftarrow} = \bigcap_{a\in A} a^{\leftarrow},$$

that is, $\gamma(\alpha) = \bigwedge_{a\in A} \gamma(a)$.

III. $\gamma$ preserves the joins in $L$: Let $A \subseteq L$ be such that $\beta = \bigvee A$ exists in $L$. Clearly, $a^{\leftarrow} \subseteq \beta^{\leftarrow}$ for all $a \in A$, and so $\bigvee_{a \in A} a^{\leftarrow} \subseteq \beta^{\leftarrow}$. Conversely, if $J$ is a complete ideal containing $A$, it follows from [*id* 2] (14.32) that $\beta \in J$. Thus,

$$\gamma(\beta) = \bigvee_{a \in A} \gamma(a),$$

as desired. To end the proof, let $H$ be a cHa and $f\colon L \to H$ be a $[\wedge, \bigvee]$-morphism. For $I \in \gamma L$, define

$$\gamma f(I) = \bigvee_{x \in I} f(x).$$

Since $(\bigvee y^{\leftarrow}) = y$ exists in $L$ and $f$ preserves the joins that exist in $L$, we conclude that

$$\gamma f(y^{\leftarrow}) = \bigvee_{x \leq y} f(x) = f(y),$$

and the diagram in the statement is indeed commutative. It is clear that $\gamma f$ is increasing, that is, $I \subseteq J$ implies $\gamma f(I) \leq \gamma f(J)$. Thus, since $\gamma L$ is a lattice, to show that $\gamma f$ preserves finite meets in $\gamma L$, it is enough to verify that for $I, J \in \gamma L$,

$$\gamma f(I) \wedge \gamma f(J) \leq \gamma f(I \cap J). \tag{3}$$

14.33(b) yields, recalling that $\langle x, y\rangle \in I \times J \Rightarrow x \wedge y \in I \cap J$,

$$\begin{aligned}
\gamma f(I) \wedge \gamma f(J) &= \big(\bigvee_{x \in I} f(x)\big) \wedge (\bigvee_{y \in J} f(y)) \\
&= \bigvee_{(x,y) \in I \times J} f(x) \wedge f(y) = \bigvee_{(x,y) \in I \times J} f(x \wedge y) \\
&\subseteq \bigvee_{z \in I \cap J} f(z) = \gamma f(I \cap J),
\end{aligned}$$

ending the verification of (3). To check that $\gamma f$ preserves arbitrary joins, let $I_u$, $u \in U$, be a family of elements in $\gamma L$. Because $\gamma f$ is increasing, it is sufficient to verify that

$$\gamma f\left(\bigvee_{u \in U} I_u\right) \subseteq \bigvee_{u \in U} \gamma f(I_u).$$

Set $J = \bigvee_{u \in U} I_u$ and let $t \in J$. By 14.34(c), there is $A \subseteq \bigcup_{u \in U} I_u$ such that $\bigvee A$ exists in $L$ and $t \leq \bigvee A$. Since $f$ preserves the joins existing in $L$, we get $f(\bigvee A) = \bigvee f(A)$. But for all $a \in A$,

$$f(a) \leq \bigvee_{u \in U} \gamma f(I_u).$$

To see this, select, for each $a \in A$, $u \in U$ such that $a \in I_u$; then

$$f(a) = \gamma f(a^{\leftarrow}) \leq \gamma f(I_u) \leq \bigvee_{u \in U} \gamma f(I_U),$$

as claimed. It now follows that

$$f(t) \leq f(\bigvee A) = \bigvee f(A) \leq \bigvee_{u \in U} f\gamma(I_u). \tag{4}$$

Since $t$ is arbitrary in $J = \bigvee_{u \in I} I_u$, (4) yields

$$\gamma f(J) = \bigvee_{t \in J} f(t) \leq \bigvee_{u \in U} \gamma f(I_u),$$

ending the proof. ∎

Remark 14.38. Part of the proof of Theorem 14.37 is similar to that of Theorem XII.1.14 in [Balbes and Dwinger, 1974]. The main point in our proof is to show that it goes through without the lattice structure, but making essential use of the implication present in a Bcan. ∎

EXAMPLE 14.39. Let $L = [0,1] - \{1/2\}$, where $[0,1]$ is the real unit interval. $L$ is a Bcan, with the structure induced by the cHa $[0,1]$. It is straightforward to see that $\gamma L$ is (naturally isomorphic to) $[0,1]$, because the only complete ideal which is non-principal in $L$ is $[0,1/2)$. Let $H$ be the cHa $[0,1/4] \cup [3/4,1]$, the disjoint union of two closed intervals in the real line, with the structure induced by [0, 1]. Let $f : L \to H$ be the regular embedding given by

$$f(x) = \begin{cases} x/2 & \text{if } x < 1/2 \\ 1/2 + x/2 & \text{if } x > 1/2. \end{cases}$$

Since $\gamma f$ preserves joins, we must have $\gamma f(1/2) = 1/4$. But then $\gamma f$ cannot preserve meets: $1/2 = \bigwedge_{x>1/2} x$, while $\bigwedge_{x>1/2} f(x) = 3/4$, distinct from $f(1/2) = 1/4$. ∎

We now turn to a description of what is called the **normal completion** of a partially ordered set. This construction, a generalization of the "completion by cuts" that produces the reals from the rationals, is due to H. M. MacNeille. The basic terminology and results can be found in Section 2, Chapter XII, of [Balbes and Dwinger, 1974], particularly pages 235ff.

DEFINITION 14.40. *Let $\langle P, \leq \rangle$ be a partially ordered set. An ideal $I$ is said to be* **closed** *if $I = \bigcap_{t \in T} t^{\leftarrow}$ for some $T \subseteq P$.*

*For $x \in P$ and $A \subseteq P$, write*

$$A^{\rightarrow} = \{x \in P\text{: For all } a \in A,\ a \leq x\}$$

*for the set of upper bounds of $A$ in $P$. We may write $x \geq A$ to stand for $x \in A^{\rightarrow}$. The meaning of expressions such as $A^{\leftarrow}$ or $x \leq A$ should be clear.*

*Let $\overline{P}$ be the set of closed ideals in $P$, with the empty set adjoined if $P$ does not possess a bottom element.*

It is clear that any closed ideal is complete; in particular, any principal ideal $x^{\leftarrow}$ is closed. Partially ordered by set-theoretical inclusion, $\overline{P}$ is a complete lattice, where meets are given by set-theoretical intersections. For each $A \subseteq P$, set

$$(*) \qquad \overline{A} = \bigcap_{x \geq A} x^{\leftarrow},$$

called the **closure of** $A$ in $P$. Clearly, the operation of closure is increasing, i.e., $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$.

LEMMA 14.41. *Let $\langle P, \leq \rangle$ be a partially ordered set. Let $A$ be a subset of $P$ and $I$, $J$ be ideals in $P$.*

(a) *$I$ is a closed ideal iff $I = \overline{I}$.*
(b) *$\overline{A}$ is the least closed ideal containing $A$ ($\overline{A}$ is the closed ideal generated by $A$ in $P$).*
(c) *Let $I_u$, $u \in U$, be a collection of closed ideals in $P$. Then*

$$\bigvee_{u \in U} I_u = \overline{\bigcup_{u \in U} I_u}$$

*is the supremum of the $I_u$'s in the complete lattice $\overline{P}$.*
(d) *If $I$ is a closed ideal in $P$, then $I = \bigvee_{x \in I} x^{\leftarrow}$.*
(e) *If $P$ is a Bcan, then $\overline{I \cap J} = \overline{I} \cap \overline{J}$.*

*Proof.* (a) By the definition of closed ideal, it is enough to show that if $I$ is closed, then $\overline{I} \subseteq I$. Let $T \subseteq P$ be such that $I = \bigcap_{t \in T} t^{\leftarrow}$. Now observe that $T \subseteq I^{\rightarrow}$ and so

$$\overline{I} = \bigcap \{x^{\leftarrow} \colon x \in I^{\rightarrow}\} \ \subseteq \ \bigcap_{t \in T} t^{\leftarrow} = I,$$

as desired.

(b) It is clear that $\overline{A}$ is a closed ideal containing $A$. Now suppose that $J$ is a closed ideal such that $A \subseteq J$. Then $J^{\rightarrow} \subseteq A^{\rightarrow}$ and so, by (a),

$$\overline{A} = \bigcap \{x^{\leftarrow} \colon x \in A^{\rightarrow}\} \ \subseteq \ \bigcap \{y^{\leftarrow} \colon y \in J^{\rightarrow}\} = J,$$

proving that $\overline{A} \subseteq J$. Item (c) follows directly from (b).

(d) Since $I$ is an ideal, we have $I = \bigcup_{x \in I} x^{\leftarrow}$; thus, $t$ is an upper bound of $I$ iff it is an upper bound of $\bigcup_{x \in I} x^{\leftarrow}$. Since $I$ is assumed closed, item (a) yields

$$I = \bigcap \{t^{\leftarrow} \colon t \geq I\} = \bigcap \{t^{\leftarrow} \colon t \geq \big(\bigcup_{x \in I} x^{\leftarrow}\big)\} = \bigvee_{x \in I} x^{\leftarrow},$$

as desired.

(e) We shall employ 14.4(a) repeatedly, without explicit mention. Since the operation of closure is increasing, it is enough to show that

$$\overline{I} \cap \overline{J} \subseteq \overline{I \cap J}. \tag{1}$$

Let $t \in \overline{I} \cap \overline{J}$ and suppose that $\alpha \geq I \cap J$. If $a \in I$ and $b \in J$, then $a \wedge b \in I \cap J$, and so $a \wedge b \leq \alpha$. It follows that

$$\text{For all } a \in I \text{ and all } b \in I, \quad a \leq (b \rightarrow \alpha),$$

that is, $I \leq (b \rightarrow \alpha)$ for all $b \in J$. Consequently, $\overline{I} \leq (b \rightarrow \alpha)$ for all $b \in J$. Since $t \in \overline{I}$, we conclude that for all $b \in J$, $t \leq (b \rightarrow \alpha)$. Thus, for $b \in J$, $t \wedge b \leq \alpha$. Hence,

$$\text{For all } b \in J, \quad b \leq (t \rightarrow \alpha). \tag{2}$$

From (2) we see that $\overline{J} \leq (t \rightarrow \alpha)$ and, in particular, $t \leq (t \rightarrow \alpha)$, that is, $t \leq \alpha$, showing (1) and ending the proof. ■

REMARK 14.42. In Lemma XII.2.3 (p. 235) in [Balbes and Dwinger, 1974] it is proven that $A \mapsto \overline{A}$ is a *closure operator* on the power set of $P$. If $L$ is a Bcan, 14.41(e) and 14.34(b) indicate that we have a bit more: the maps

$$I \mapsto \overline{I} \quad \text{and} \quad I \mapsto c(I)$$

are $J$-operators on $\mathcal{I}d(L)$, the lattice of ideals of $L$, according to Definition 2.11 in [Fourman and Scott, 1979] (p. 324). Although it is defined there only for cHa's, the concept of $J$-operator may be defined for any $\wedge$-semilattice, $L$, as follows: A map $L \xrightarrow{J} L$ is a **$J$-operator** if it satisfies, for all $x, y \in L$,

[$J1$] $\quad x \leq J(x)$
[$J2$] $\quad J(J(x)) = J(x)$;
[$J3$] $\quad J(x \wedge y) = J(x) \wedge J(y)$.

Note that

$*$ If $J(I) = \overline{I}$, then [$J1$] comes from the definition of closure, while [$J2$] and [$J3$] follow from (b) and (e) in 14.41, respectively.

$*$ If $J(I) = c(I)$, then $[J1]$ and $[J2]$ follow from the definition of completion, while $[J3]$ follows from 14.34(b).

$J$-operators may be used to give a constructive version of quotients of complete Heyting algebras. Thus, if $\Omega$ is a cHa and $J$ is a $J$-operator on $\Omega$, then the set fixed points of $J$

$$Fix(J) = \{x \in \Omega\colon J(x) = x\}$$

is a cHa and the map $x \mapsto J(x)$ is a $[\wedge, \bigvee]$-morphism from $\Omega$ onto $Fix(J)$. Moreover, there is a natural bijective correspondence between onto $[\wedge, \bigvee]$-morphisms from $\Omega$ to a cHa and $J$-operators on $\Omega$. For more details, the interested reader may consult Section 2 in [Fourman and Scott, 1979].

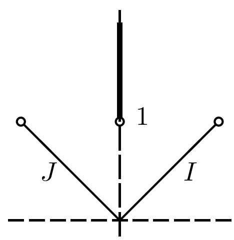
If $L$ is a Bcan, the set of fixed points of $I \mapsto \overline{I}$ is $\overline{L}$, while the set of fixed points of $I \mapsto c(I)$ is $\gamma L$. In 14.49 we show that if $L$ is a Bcan, then $\mathcal{I}d(L)$ is a cHa and so the results in [Fourman and Scott, 1979], mentioned above, give a constructive proof that $\gamma L$ and $\overline{L}$ are cHa quotients of $\mathcal{I}d(L)$. ■

Example 14.43. For Ean's, 14.41(e) is false. To see this, let $\mathbb{R}$ be the real line and let $L \subseteq \mathbb{R}^2$ be given by

$$\begin{aligned} L = {} & \{(x, y) \in \mathbb{R}^2\colon y = x,\, 0 \leq x < 1\} \cup \{(x, y) \in \mathbb{R}^2\colon y = -x,\, -1 < x \leq 0\} \\ & \cup \{(0, y) \in \mathbb{R}^2\colon 1 < y \leq 2\}, \end{aligned}$$

where the partial order is given by the second coordinate, that is,

$$(x, y) \leq (x', y') \quad \text{iff} \quad y \leq y'.$$




Clearly, $\top = (0, 2)$, while $\perp = (0, 0)$. It is left to the reader to check that with the operation defined by

$$(a, b) * (c, d) = \begin{cases} \top & \text{if } (a, b) = (c, d), \\ (a,b) & \text{if } b < 1 < d, \\ \perp & \text{if } ac < 0, \\ (a,b) & ac > 0 \text{ and } b \leq d, \\ (c,d) & ac > 0 \text{ and } d \leq b. \end{cases}$$

$\langle L, *, \leq, \top, \perp \rangle$ is an Ean. Observe that

- $I = \{(x, y) \in \mathbb{R}^2\colon y = x,\, 0 \leq x < 1\}$ and
- $J = \{(x, y) \in \mathbb{R}^2\colon y = -x,\, -1 < x \leq 0\}$,

are complete ideals such that $I \cap J = \{\perp\}$ and $\overline{I} = \overline{J} = I \cup J$, showing that $L$ does not satisfy 14.41(e). This example also shows that, in Ean's, there are complete ideals which are not closed. In contrast, it is proven in Theorem 14.44 that all complete ideals in a Bcan are closed. ∎

It is shown in Section 2, Chapter XII, of [Balbes and Dwinger, 1974] (p. 236) that the map

$$\nu\colon P \to \overline{P}, \quad \nu(x) = x^{\leftarrow},$$

is a regular embedding (14.31) of $P$ into the complete lattice $\overline{P}$. The pair $(\overline{P}, \nu)$ is called the **normal** (or **MacNeille**) **completion** of $P$.

THEOREM 14.44. *The normal completion of a Bcan $L$ is a complete Heyting algebra. Moreover:*

(a) *The map $\lambda\colon \gamma L \to \overline{L}$, $\lambda(I) = \overline{I}$, is an isomorphism, making commutative the following diagram:*

$$\begin{tikzcd} L \arrow[rr, "\gamma"] \arrow[dr, "\nu"'] & & \gamma L \arrow[dl, "\lambda"] \\ & \overline{L} & \end{tikzcd}$$

(b) *The image of $\gamma$ is both a basis for $\gamma L$ and meet dense in $\gamma L$.*

*Proof.* Let $L$ be a Bcan. We prove that $\overline{L}$ is a cHa by showing that $\lambda$ (in item (a)) is an isomorphism. To this end, we show

FACT. *Every complete ideal in $L$ is closed.*

*Proof.* Let $I \in \gamma L$ and $x \in \overline{I}$. Let $A = \{x \wedge t\colon t \in I\}$. Since $I$ is an ideal, we have $A \subseteq I$. We claim that $x = \bigvee A$. To see this, suppose $y \in L$ satisfies $y \geq A$, i.e., $x \wedge t \leq y$ for all $t \in A$. But then, $t \leq (x \to y)$, $t \in I$, that is, $I \leq (x \to y)$. Since $x \in \overline{I}$, we conclude that $x \leq (x \to y)$, or equivalently, $x \leq y$, proving our claim. Now, the fact that $I$ is a complete ideal guarantees that $x \in I$, as desired.

It follows immediately from the Fact that $\lambda$ is an isomorphism, making the displayed diagram commute. From the very definition of closed ideal, we see that the image of $\gamma$ is meet dense in $\gamma L$, that is, every complete ideal is the intersection of principal ideals. ∎

COROLLARY 14.45. *If $H$ is a Heyting algebra, then $\gamma H$ and $\overline{H}$ are isomorphic complete Heyting algebras.* ∎

DEFINITION 14.46. *Let $L$ be a Bcan. The diagram $L \xrightarrow{\gamma} \gamma L$ is called the* **cHa completion** *of $L$.*

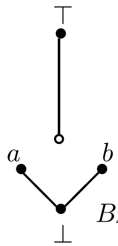
EXAMPLE 14.47. Let $L = B_4 \mathring{\vee} [0,1]$ be the Bcan mentioned in 14.8 (also 13.49(III); see figure below). Then

$$\gamma(L) = \overline{L} = B_4 \vee [0,1].$$

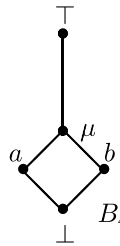
To see this, write $B_4 = \{\bot, a, b, \top\}$; then the only non-principal closed ideal in $L$ is

$$I = \{\bot, a, b\} = a^{\leftarrow} \vee b^{\leftarrow} \text{ (in the cHa } \overline{L}),$$

corresponding to $a \vee b$. The injection $\gamma\colon L \to \gamma(L)$ is just the canonical map from $L$ to $B_4 \vee [0, 1]$.




$B_4 \mathring{\vee} [0,\ 1]$




$B_4 \vee [0,\ 1]$

Now observe that:

$*$ The filter $F = L^b$ in $L$ is equal to $a^{\rightarrow} \cap b^{\rightarrow}$, but neither $a$ nor $b$ are in it. Thus, $F$ is not irreducible in $L$.

$*$ The filter $G = L^b$ <u>is irreducible</u> in $\gamma(L)$. To see this note that the only filters in $\gamma(L)$ properly containing it are the whole algebra, $a^{\rightarrow}$, $b^{\rightarrow}$ and $(a \vee b)^{\rightarrow}$; and $G$ cannot be written as an intersection of any two of these.

$*$ $\gamma^{-1}(G) = F$.

Hence, irreducibility is not preserved by inverse image, even of regular embeddings (see Remark 13.90). Finally, the reader will notice that this example is typical: if $\Omega$ is a cHa and $K$ is a complete linear order, then $\Omega \vee K$ is the cHa completion of $\Omega \mathring{\vee} K$. ■

For equivalence algebras in general we pose

Open Problem 14.48. *Can every Ea be regularly embedded in a cEa? Is the normal completion of an equivalence algebra an Ean?*

The next result shows that $\mathcal{I}d(L)$ is a cHa whenever $L$ is a Bcan.

Proposition 14.49. *Let $L$ be a Bcan.*

(a) *The intersection of any family of ideals is an ideal. For $S \subseteq L$, let*

$$I(S) = \bigcap \{I\colon I \text{ is an ideal in } L \text{ and } A \subseteq I\}$$

*be the ideal <u>generated by $S$</u> in $L$. Then*

$$I(S) = \{x \in L\colon \text{There exists a finite } A \subseteq S \text{ such that } \textstyle\bigvee A \text{ exists in } L \text{ and } x \leq \bigvee A\}.$$

(b) *If $I_u$, $u \in U$, is a family of ideals in $\mathcal{I}d(L)$, then the join of the $I_u$ in $\mathcal{I}d(L)$ is given by*

$$\textstyle\bigvee_{u \in U} I_u = \{x \in L\colon \text{There is a finite } A \subseteq \bigcup_{u \in U} I_u \text{ such that } \bigvee A \text{ exists in } L \text{ and } x \leq \bigvee A\}.$$

(c) *An ideal is compact in $\mathcal{I}d(L)$ iff it is finitely generated, that is, it is of the form $I(A)$ for some finite subset $A$ of $L$.*

(d) *$\mathcal{I}d(L)$ is an algebraic cHa.*

*Proof.* The proofs of (a) and (b) are similar to the corresponding statements in 14.34. Item (c) is left to the reader.

(d) Clearly, $\mathcal{I}d(L)$ is a complete lattice. Note that for all $I \in \mathcal{I}d(L)$, (a) yields $I = \bigvee_{x \in I} x^{\leftarrow}$, with each $x^{\leftarrow}$ compact. Thus, $\mathcal{I}d(L)$ is an algebraic lattice. Moreover, by Fact 1 in the proof of 14.29, to prove that $\mathcal{I}d(L)$ is a cHa, it is sufficient to verify that it is a distributive lattice, that is, for all $I, J, K \in \mathcal{I}d(L)$,

$$I \cap (J \vee K) \subseteq (I \cap J) \vee (I \cap K).$$

Let $t \in I \cap (J \vee K)$. By (a), there is a finite $A \subseteq J \cup K$ such that $t \leq \bigvee A$. Consider the set

$$B = \{t \wedge a \colon a \in A\}.$$

It is clear that $B$ is finite; by 14.33(a),

$$t = t \wedge (\bigvee A) = \bigvee_{a \in A} t \wedge a = \bigvee B.$$

By item (b), the proof will be finished if we show that

$$B \subseteq (I \cap J) \cup (I \cap K). \tag{1}$$

But if $a \in A \cap J$, then $t \wedge a \in I \cap J$; and if $a \in A \cap K$, then $t \wedge a \in I \cap K$. Since $A$ is contained in $J \cup K$, (1) is verified, concluding the proof. ∎

We now establish a useful description of the implication operation in $\mathcal{I}d(L)$, whenever the consequent is a closed ideal.

Proposition 14.50. *Let $L$ be a Bcan.*

(a) *For all $I \in \mathcal{I}d(L)$ and all $J \in \overline{L}$, $(I \to J) \in \overline{L}$ ($\to$ is implication in $\mathcal{I}d(L)$); further, we have*

$$(I \to J) = \bigcap \{(x \to y)^{\leftarrow} \colon x \in I \text{ and } y \geq J\}.$$

(b) *For all $I \in \mathcal{I}d(L)$, $\neg I = \bigcap \{(\neg x)^{\leftarrow} \colon x \in I\}$.*

(c) *For all $I$, $J$ in $\gamma L$ (or $\overline{L}$),*

$$(I \to J) = \bigcap \{(x \to y)^{\leftarrow} \colon x \in I \text{ and } y \geq J\}.$$

*In particular, $\neg I = \bigcap \{(\neg x)^{\leftarrow} \colon x \in I\}$.*

*Proof.* (a) Write $K$ for the right-hand side of the equality in the statement. We shall prove that

(i) $K \cap I \subseteq J$; (ii) If $P \in \mathcal{I}d(L)$ satisfies $P \cap I \subseteq J$, then $P \subseteq K$,

showing that $K$ is $(I \to J)$ in the cHa $\mathcal{I}d(L)$.

Proof of (i): Suppose $z \in K \cap I$ and let $y$ be an upper bound of $J$. Then $z \leq (z \to y)$, that is, $z \leq y$. Since $y$ is an arbitrary upper bound of the closed ideal $J$, we conclude that $z \in J$.

Proof of (ii): Let $P \in \mathcal{I}(L)$. Fix $x \in I$ and $y \geq J$; for $p \in P$, note that $p \wedge x \in P \cap I \subseteq J$. Therefore, $p \wedge x \leq y$, and so, $p \leq (x \to y)$. Hence, $p \in K$, as desired.

(b) Note that if $y \geq \bot$, then $x \to \bot \leq x \to y$ (14.4(e)). Consequently,

$$\neg I = (I \to \{\bot\}) = \bigcap \{(x \to y)^{\leftarrow} \colon x \in I \text{ and } y \geq \bot\} = \bigcap \{(\neg x)^{\leftarrow} \colon x \in I\},$$

as needed. Item (c) is an immediate consequence of (a) and (b). ∎

As an application of 14.50, 14.37 and 14.44, we give new proofs of some well known results about the normal completion of Boolean algebras (see, for instance, Sections 2 and 3, Chapter XII, in [Balbes and Dwinger, 1974]).

COROLLARY 14.51. *If $L$ is a Boolean algebra then the normal completion of $L$ is a complete Boolean algebra. Moreover, if $C$ is a complete Boolean algebra and $f\colon L \to C$ is a regular Boolean algebra morphism, then there is a* <u>*unique*</u> *regular Boolean algebra morphism, $\overline{f} : \overline{L} \to C$, making the following diagram commutative:*

$$\begin{tikzcd} L \arrow[rr, "\nu"] \arrow[dr, "f"'] & & \overline{L} \arrow[dl, "\overline{f}"] \\ & C & \end{tikzcd}$$

*Proof.* Since $\gamma L$ may be naturally identified with $\overline{L}$ (14.44), we reason with $\gamma L$. Recall that a Heyting algebra $H$ is a Boolean algebra iff $\neg\neg x \le x$ for all $x \in H$ (13.4).

Suppose $t \in \neg\neg I$ and $\alpha \ge I$. For $x \in I$, we have $x \wedge t \le \alpha$. Thus, $x \le (t \to \alpha)$ for all $x \in I$. Hence, $\neg(t \to \alpha) \le \neg x$ for all $x \in I$. From 14.50 we conclude that $\neg(t \to \alpha) \in \neg I$. Another application of 14.50 yields that

$$\neg\neg(t \to \alpha) = (t \to \alpha) \ge \neg\neg I,$$

and so, $t \le (t \to \alpha)$, that is, $t \le \alpha$. This shows that $t \in \overline{I} = I$ and $\gamma L$ is a complete Boolean algebra (cBa). To verify the extension property, let $\gamma f$ be the $[\wedge, \bigvee]$-morphism whose existence and uniqueness is guaranteed by Theorem 14.37. Clearly, $\gamma f$ makes the displayed diagram commute. Since $f(\top) = \top$ and $f(\bot) = \bot$, we have

$$\gamma f(\top) = \top \quad \text{and} \quad \gamma f(\bot) = \bot.$$

To finish the proof, we establish

FACT. *Suppose $D$ and $E$ are cBa's and $D \xrightarrow{h} E$ is a $[\wedge, \bigvee]$-morphism such that $h(\bot) = \bot$ and $h(\top) = \top$. Then $h$ is a regular Boolean algebra morphism, that is, it preserves all joins and meets, as well as implication and negation.*

*Proof.* Since implication can be defined in terms of joins and negation, and we have the de Morgan laws involving negation, joins and meets in a cBa, namely,

$$\neg(\textstyle\bigvee_{i\in I} x_i) = \bigwedge_{i\in I} \neg x_i \quad \text{and} \quad \neg(\bigwedge_{i\in I} x_i) = \bigvee_{i\in I} \neg x_i,$$

$h$ will certainly preserve implication and meets, once it is verified that it preserves negation. For $a \in D$, we have $(a \vee \neg a) = \top$ and so, since $h$ preserves joins and takes $\top$ to $\top$, we get

$$h(a) \vee h(\neg a) = \top. \tag{1}$$

Similarly, since $h$ preserves $\bot$ and finite meets, from $(a \wedge \neg a) = \bot$ comes

$$h(a) \wedge h(\neg a) = \bot. \tag{2}$$

Since $E$ is a Boolean algebra, it follows easily from (1) and (2) that $h(\neg a) = \neg h(a)$, completing the proof. ∎

# Part 6

# Appendices

APPENDIX A

# Normal derivations

## 1. Normalization Property of the New Protothetic

At the turn of the last century the interest lay much more on the *provable* than on the *proofs*, so it is not surprising that neither Leśniewski nor Tarski paid much attention to the problem as to whether every derivation (say in their Protothetic) could be transformed into another one of some particular form; or as one would say nowadays: *Normal Form Property for Derivations*.

In fact, before Gentzen[1] the style of formalization did not render itself to the concept of a normal derivation. One could say that the *Mathematical Theory of Proofs* (or as is more commonly called: *Proof Theory*) owes much of its existence to the works of Gebhard Gentzen and his systems of Natural Deduction; for details on the subject of *Normal Form Property* and *Normalization Property* of derivations the reader is referred to the various articles of Kreisel, Prawitz, Troelstra *et al.*[2]

In this appendix we wish to prove the *Normalization Property for the New Protothetic*. Actually we shall prove it for the Intuitionistic Protothetic (which has a couple of additional primitives and corresponding pairs of **I-E**-rules of inference) and thus obtain as a corollary the Normalization Property for the New Protothetic and its various subsystems, such as MEC and BCC.

## 2. Rules of inference

For the reader's convenience we state, *in a very condensed form*, the rules of inference of an unfolding of the Intuitionistic Protothetic.

**I-Rules:**

$$\frac{\begin{matrix}[\mathcal{A}] \\ \mathcal{B}\end{matrix} \qquad \begin{matrix}[\mathcal{B}] \\ \mathcal{A}\end{matrix}}{\mathcal{A}\equiv\mathcal{B}} \qquad \frac{\begin{matrix}[\mathcal{A}] \\ \mathcal{B}\end{matrix}}{\mathcal{A}\supset\mathcal{B}} \qquad \frac{\mathcal{A} \qquad \mathcal{B}}{\mathcal{A}\wedge\mathcal{B}}$$

$$\frac{\mathcal{A}\ulcorner p\urcorner}{\bigwedge x\mathcal{A}\ulcorner x\urcorner} \qquad \frac{\mathcal{B}\ulcorner F\urcorner}{\bigwedge f\mathcal{B}\ulcorner f\urcorner}$$

$$\frac{\mathcal{D}_i\ulcorner\mathcal{A}_0,\mathcal{A}_1,\ldots\urcorner}{\mathbb{F}_i(\mathcal{A}_0,\mathcal{A}_1\ldots)}$$

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[1][Gentzen, 1936].

[2]For example: [Kreisel, 1960], [Prawitz, 1971] and [Troelstra, 1973].

**E-rules:**

$$\frac{\mathcal{A}{\equiv}\mathcal{B} \qquad \mathcal{A}}{\mathcal{B}} \quad \frac{\mathcal{A}{\equiv}\mathcal{B} \qquad \mathcal{B}}{\mathcal{A}} \quad \frac{\mathcal{A}{\supset}\mathcal{B} \qquad \mathcal{A}}{\mathcal{B}}$$

$$\frac{\mathcal{A}\wedge\mathcal{B}}{\mathcal{A}} \qquad \frac{\mathcal{A}\wedge\mathcal{B}}{\mathcal{B}} \qquad \frac{\bigwedge x \mathcal{A}\ulcorner x \urcorner}{\mathcal{A}\ulcorner \mathcal{B} \urcorner} \qquad \frac{\bigwedge f \mathcal{A}\ulcorner f \urcorner}{\mathcal{A}\ulcorner X \urcorner}$$

$$\frac{\mathbb{F}_i(\mathcal{A}_0, \mathcal{A}_1, \ldots)}{\mathcal{D}_i\ulcorner \mathcal{A}_0, \mathcal{A}_1, \ldots \urcorner}$$

**Roundabout derivations.** Following Gentzen, let us call a derivation a *roundabout derivation*[3] if there is an application of an **E**-rule whose major premise is the conclusion of the corresponding **I**-rule. Then those derivations which are not roundabout are the *normal* ones.

If it were not for the quantifiers, that is, in a system whose only primitives were $\equiv$, $\supset$ and $\wedge$, then a simple induction would show that to every derivation there corresponds a normal derivation with the same conclusion and no additional assumptions.[4] However in IP not only do we have quantifiers, but the system is of an impredicative nature so we must use other methods introduced by D. Prawitz and J.-Y. Girard.

**Reminder.** *All the derivations are supposed to be standardized with respect to the eigenparameters.*

**Contractions and reductions.** We now follow Gentzen's method of changing a roundabout derivation into one which hopefully is less roundabout. It centers around removing a guilty **I-E** pair; the process is called a **contraction**:

$$\dfrac{\dfrac{\begin{matrix}[\mathcal{A}]\\ \Pi_0\\ \mathcal{B}\end{matrix} \qquad \begin{matrix}[\mathcal{B}]\\ \Pi_1\\ \mathcal{A}\end{matrix}}{\mathcal{A} \equiv \mathcal{B}} \qquad \begin{matrix}\Pi_2\\ \mathcal{A}\end{matrix}}{\mathcal{B}} \qquad \text{contracts to} \qquad \begin{matrix}\Pi_2\\ (\mathcal{A})\\ \Pi_0\\ \mathcal{B}\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}[\mathcal{A}]\\ \Pi_0\\ \mathcal{B}\end{matrix}}{\mathcal{A} \supset \mathcal{B}} \qquad \begin{matrix}\Pi_1\\ \mathcal{A}\end{matrix}}{\mathcal{B}} \qquad \text{contracts to} \qquad \begin{matrix}\Pi_1\\ (\mathcal{A})\\ \Pi_0\\ \mathcal{B}\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\Pi_0\\ \mathcal{A}\end{matrix} \qquad \begin{matrix}\Pi_1\\ \mathcal{B}\end{matrix}}{\mathcal{A} \wedge \mathcal{B}}}{\mathcal{B}} \qquad \text{contracts to} \qquad \begin{matrix}\Pi_1\\ \mathcal{B}\end{matrix}$$

---

[3] *Umwege* in [Gentzen, 1936].

[4] See [Prawitz, 1965].

$$\dfrac{\dfrac{\begin{array}{c}\Pi\ulcorner p\urcorner\\ \mathcal{A}\ulcorner p\urcorner\end{array}}{\bigwedge x\mathcal{A}\ulcorner x\urcorner}}{\mathcal{A}\ulcorner\mathcal{B}\urcorner} \quad \text{contracts to} \quad \begin{array}{c}\Pi\ulcorner\mathcal{B}\urcorner\\ \mathcal{A}\ulcorner\mathcal{B}\urcorner\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\Pi\ulcorner F\urcorner\\ \mathcal{A}\ulcorner F\urcorner\end{array}}{\bigwedge f\mathcal{A}\ulcorner f\urcorner}}{\mathcal{A}\ulcorner X\urcorner} \quad \text{contracts to} \quad \begin{array}{c}\Pi\ulcorner X\urcorner\\ \mathcal{A}\ulcorner X\urcorner\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\Pi\\ \mathcal{D}\ulcorner\vec{\mathcal{A}}\urcorner\end{array}}{\mathbb{F}\ulcorner\vec{\mathcal{A}}\urcorner}}{\mathcal{D}\ulcorner\vec{\mathcal{A}}\urcorner} \quad \text{contracts to} \quad \begin{array}{c}\Pi\\ \mathcal{D}\ulcorner\vec{\mathcal{A}}\urcorner\end{array}$$

A derivation $\Pi_1$ is an **immediate reduction** of $\Pi_2$, in symbols: $\Pi_1 \prec \Pi_2$ or $\Pi_2 \succ \Pi_1$, iff exactly one subderivation of $\Pi_2$ is replaced by its contraction. $\Pi_1$ is a **reduction** of $\Pi_2$, in symbols: $\Pi_2 \rhd \Pi_1$ or $\Pi_1 \lhd \Pi_2$ iff there is a finite (non-empty) sequence of immediate reductions between them.

Red($\Pi$) is the **reduction tree** of $\Pi$ whose nodes consist of all the reductions of $\Pi$ and the tree ordering is that of *immediate reduction.* A derivation $\Pi$ is **irreducible** or **normal** iff $\mathrm{Red}(\Pi) = \emptyset$.

Clearly Red($\Pi$) is a finitely branching tree. If it has a finite branch, then $\Pi$ is said to have a **normal form**. On the other hand if Red($\Pi$) is finite, and thus all of its branches are finite, then $\Pi$ is said to be **normalizable**.

**Normalization Property.** *An unfolding of the Intuitionistic Protothetic has the Normalization Property iff every derivation is normalizable.*

### 3. Girard assignments

In order to handle the impredicativity of the universal quantifier we will use the method developed by J.-Y. Girard.[5]

First of all let us agree that a non-empty set $S$ of derivations is a **regular set of derivations** iff (a) all the derivations in $S$ are normalizable and (b) $S$ is closed under reductions. Then a **Girard assignment** is a set of pairs of the form $(\mathcal{F}, S)$, where $\mathcal{F}$ is a formula and $S$ a regular set of derivations. We think of a Girard assignment[6], $\mathfrak{G}$, as an old fashioned *many-valued function* so that we may write $\mathfrak{G}(\mathcal{F}) = S$ instead of the more precise $(\mathcal{F}, S) \in \mathfrak{G}$.

*If $\mathfrak{G}$ is a Girard assignment and $\mathcal{A}$ a formula, then the* **degree of $\mathcal{A}$ relative to $\mathfrak{G}$**, *in symbols:* $\deg_{\mathfrak{G}}(\mathcal{A})$, *is defined to be the smallest of:*

1. *the syntactical complexity of $\mathcal{A}$ and*

[5][Girard, 1971].

[6]We shall reserve the fraktur letter $\mathfrak{G}$ for Girard assignments.

2. *minimum of the syntactical complexity of the formulas* $\mathcal{F}\ulcorner\vec{p}\urcorner$ *such that for formulas* $\vec{\mathcal{B}}$*, in the domain of* $\mathfrak{G}$*:* $\mathcal{A} = \mathcal{F}\ulcorner\vec{\mathcal{B}}\urcorner$.

Next we need to be able to extend the domain of a Girard assignment. Thus suppose that $\mathfrak{G}$ is a Girard assignment, $\mathcal{B}$ a formula and $S$ a regular set of derivations. Then we set

$$\mathfrak{G}\cdot(\mathcal{B}, S) = \mathfrak{G} \cup \{(\mathcal{B}, S)\}.$$

Finally let us agree to call a pair of the form $(\Pi, \mathfrak{G})$, where $\Pi$ is a derivation and $\mathfrak{G}$ a Girard assignment, a **relativized derivation**. By the **complexity** of the relativized derivation $(\Pi, \mathfrak{G})$, in symbols: $\mathrm{Comp}(\Pi, \mathfrak{G})$, we understand the natural number

$$\mathrm{Comp}(\Pi, \mathfrak{G}) = \deg_{\mathfrak{G}}(\mathrm{EndFor}(\Pi)),$$

where $\mathrm{EndFor}(\Pi)$ is the *end-formula* or *conclusion* of the derivation $\Pi$.

LEMMA A.1. *If* $\deg_{\mathfrak{G}}(\mathcal{A}) > 1$ *then:*

1. *If* $\mathcal{A} = (\mathcal{B} \equiv \mathcal{C})$ *then* $\deg_{\mathfrak{G}}(\mathcal{A}) > \deg_{\mathfrak{G}}(\mathcal{B}), \deg_{\mathfrak{G}}(\mathcal{C})$.
2. *If* $\mathcal{A} = \bigwedge x \mathcal{B}\ulcorner x\urcorner$ *then* $\deg_{\mathfrak{G}}(\mathcal{A}) > \deg_{\mathfrak{G}}(\mathcal{B}\ulcorner p\urcorner)$.
3. *Correspondingly for the other connectives and quantifier.*

LEMMA A.2. *For any formula* $\mathcal{B}$ *and regular set* $S$ *of derivations:*

$$\textit{if } \deg_{\mathfrak{G}}(\bigwedge x \mathcal{A}\ulcorner x\urcorner) > 1 \textit{ then } \deg_{\mathfrak{G}}(\bigwedge x \mathcal{A}\ulcorner x\urcorner) > \deg_{\mathfrak{G}\cdot(\mathcal{B},S)}(\mathcal{A}\ulcorner\mathcal{B}\urcorner).$$

**Inductively normalizable derivations.** We now proceed to define, for each natural number $n \geq 1$, the sets $\mathrm{GIR}^n$ of relativized derivations. As a preliminary step we define the property (or predicate) $\mathrm{PGR}_n()$ of sets of relativized derivations.

**Basis step.** *A set* $\mathsf{R}$ *of relativized derivations of complexity* $\leq 1$ *has the 1-Girard Property, in symbols:* $\mathrm{PGR}_1(\mathsf{R})$*, iff:*

1. *If* $\Pi$ *is a normalizable derivation* *then* $(\Pi, \mathfrak{G}) \in \mathsf{R}$.
2. *If* $\Pi \in \mathfrak{G}(\mathrm{EndFor}(\Pi))$*, then* $(\Pi, \mathfrak{G}) \in \mathsf{R}$.

Clearly the set of *all* relativized derivations of complexity less than or equal to 1 has the 1-Girard Property. Our aim is to choose the smallest one; but first we need to observe that:

PROPOSITION A.1. *The intersection of a family of sets having the 1-Girard Property also has the property.*

Then we set

$$\mathrm{GIR}^1 = \bigcap\{\mathsf{R} : \mathrm{PGR}_1(\mathsf{R})\},$$

COROLLARY. $\mathrm{PGR}_1(\mathrm{GIR}^1)$.

**Inductive step.** Assume that $\mathrm{GIR}^n$ has been defined and is the smallest set such that $\mathrm{PRG}_n(\mathrm{GIR}^n)$.

Recall that whenever we write $\Phi(\mathcal{F})\Pi$, we understand that $\Phi$ is a derivation whose end-formula is $\mathcal{F}$, that $\Pi$ is a derivation in which some (or all, or none) undischarged assumption occurrences of the formula $\mathcal{F}$ have been specified and that $\Phi(\mathcal{F})\Pi$ is the derivation obtained by replacing the specified occurrences of the formula $\mathcal{F}$ by the derivation $\Phi$ (and that the eigen-parameters have been standardized).

*A set* $\mathsf{R}$ *of relativized derivations of complexity* $\leq n+1$ *has the* $n+1$*-Girard Property, in symbols:* $\mathrm{PGR}_{n+1}(\mathsf{R})$ *iff:*

1. $\mathrm{GIR}^n \subseteq \mathsf{R}$.
*Now let us consider only those relativized derivations* $(\Pi, \mathfrak{G})$ *such that*

$$1 < \mathrm{Comp}(\Pi, \mathfrak{G}) \leq n + 1.$$

2. *If* $\Pi$ *is a normalizable derivation* *then* $(\Pi, \mathfrak{G}) \in \mathsf{R}$.
3. *If* $\mathrm{EndRule}(\Pi) \neq \mathbf{I}$*ntroduction and for all* $\Phi \prec \Pi$*:* $(\Phi, \mathfrak{G}) \in \mathsf{R}$*, then* $(\Pi, \mathfrak{G}) \in \mathsf{R}$.
4. *If* $\mathrm{EndRule}(\Pi) = \equiv$*-*$\mathbf{I}$*ntroduction so that*

$$\Pi = \cfrac{\begin{matrix}[\mathcal{A}_1] \\ \Pi_0 \\ \mathcal{A}_0\end{matrix} \qquad\qquad \begin{matrix}[\mathcal{A}_0] \\ \Pi_1 \\ \mathcal{A}_1\end{matrix}}{\mathcal{A}_0 \equiv \mathcal{A}_1}$$

*and for all appropriate* $(\Phi_i, \mathfrak{G}) \in \mathrm{GIR}^n$, $i = 0, 1$*:* $(\Phi_i(\mathcal{A}_{1-i})\Pi_i, \mathfrak{G}) \in \mathrm{GIR}^n$*, then* $(\Pi, \mathfrak{G}) \in \mathsf{R}$.
5. *Analogously for the Introduction rules for* $\supset$ *and* $\wedge$.
6. *If* $\mathrm{EndRule}(\Pi) = \mathbb{F}$*-*$\mathbf{I}$*ntroduction so that*

$$\Pi = \cfrac{\begin{matrix}\Pi_0 \\ \mathcal{D}\ulcorner\vec{\mathcal{A}}\urcorner\end{matrix}}{\mathbb{F}(\vec{\mathcal{A}})}$$

*and for all regular sets of derivations* $S$*:* $(\Pi_0, \mathfrak{G}\cdot(\mathcal{D}\ulcorner\vec{\mathcal{A}}\urcorner, S)) \in \mathrm{GIR}^n$*, then* $(\Pi, \mathfrak{G}) \in \mathsf{R}$.
7. *If* $\mathrm{EndRule}(\Pi) =$ *Propositional* $\bigwedge$*-*$\mathbf{I}$*ntroduction, so that*

$$\Pi = \cfrac{\begin{matrix}\Pi_1\ulcorner p\urcorner \\ \mathcal{A}\ulcorner p\urcorner\end{matrix}}{\bigwedge x\mathcal{A}\ulcorner x\urcorner}$$

*and for all formulas* $\mathcal{B}$ *and regular sets of derivations* $S$*:* $(\Pi_1\ulcorner\mathcal{B}\urcorner, \mathfrak{G} \cdot (\mathcal{B}, S)) \in, \mathrm{GIR}^n$*, then* $(\Pi, \mathfrak{G}) \in \mathsf{R}$.
8. *If* $\mathrm{EndRule}(\Pi) =$ *Functional* $\bigwedge$*-*$\mathbf{I}$*ntroduction, so that*

$$\Pi = \cfrac{\begin{matrix}\Pi_1\ulcorner F\urcorner \\ \mathcal{A}\ulcorner F\urcorner\end{matrix}}{\bigwedge f\mathcal{A}\ulcorner f\urcorner}$$

*and for all functionals*[7] $X$ *and all regular sets* $S$ $(\Pi_1\ulcorner X\urcorner, \mathfrak{S} \cdot (\mathcal{A}\ulcorner X\urcorner, S)) \in \mathrm{GIR}^n$*, then* $(\Pi, \mathfrak{G}) \in \mathsf{R}$.

Then we set (after verifying that the $(n + 1)$-Girard Property is closed under intersections)

$$\mathrm{GIR}^{n+1} = \bigcap\{\mathsf{R} : \mathrm{PGR}_{n+1}(\mathsf{R})\}.$$

[7] Parameters or defined operators (of appropriate arity).

$\mathrm{IND}_{\mathfrak{G}}$, the set of **inductively normalizable derivations** w.r.t. $\mathfrak{G}$, is defined to be the set

$$\mathrm{IND}_{\mathfrak{G}} = \{\Pi : (\Pi, \mathfrak{G}) \in \textstyle\bigcup_n \mathrm{GIR}^n\}.$$

Eventually we show (using the ideas of Prawitz and Girard) that (i) all the inductively normalizable derivations are normalizable and that (ii) all derivations are inductively normalizable.

A simple observation gives us:

Proposition A.2.

1. *For all natural numbers $n$:* $\mathrm{PGR}_n(\mathrm{GIR}^n)$.
2. *If* $(\Pi, \mathfrak{G}) \in \mathrm{GIR}^n$ *then* $\mathrm{Comp}(\Pi, \mathfrak{G}) \leq n$.

The sets of relativized derivations $\mathrm{GIR}^n$ are introduced so as to define the sets of derivations $\mathrm{IND}_{\mathfrak{G}}$ and then to be able to prove that the sets $\mathrm{IND}_{\mathfrak{G}}$ can be characterized as the smallest set having the following properties:

Proposition A.3 (Inductive Definition). *The derivation* $\Pi$ *belongs to* $\mathrm{IND}_{\mathfrak{G}}$ *iff:*

1. $\Pi$ *is a normalizable derivation, or*
2. $\mathrm{EndRule}(\Pi)$ *is not an Introduction Rule and all derivations* $\Phi$ *which are immediate reductions of* $\Pi$ *belong to* $\mathrm{IND}_{\mathfrak{G}}$*, or*
3. $\mathrm{EndRule}(\Pi)$ *is not an Elimination Rule and*
   (a) $\Pi \in \mathfrak{G}(\mathrm{EndFor}(\Pi))$*, or*
   (b) $\mathrm{EndRule}(\Pi) = \mathbf{I}\text{-}\supset$*, so that*

$$\Pi = \cfrac{\begin{matrix}[\mathcal{A}]\\ \Pi_0\\ \mathcal{B}\end{matrix}}{\mathcal{A} \supset \mathcal{B}}$$

*and for each* $\Phi \in \mathrm{IND}_{\mathfrak{G}}$ *such that* $\mathrm{EndFor}(\Phi) = \mathcal{A}$: $\Phi(\mathcal{A})\Pi_0 \in \mathrm{IND}_{\mathfrak{G}}$*, or*

   (c) *correspondingly for the* $\equiv$ *and* $\wedge$ *Introductions, or*
   (d) $\mathrm{EndRule}(\Pi) = \mathbb{F}$*-Introduction, so that*

$$\Pi = \cfrac{\begin{matrix}\Pi_0\\ \mathcal{D}\ulcorner\vec{\mathcal{A}}\urcorner\end{matrix}}{\mathbb{F}(\vec{\mathcal{A}})}$$

*and for all regular sets* $S$: $\Pi_0 \in \mathrm{IND}_{\mathfrak{G}\cdot(\mathcal{D}\ulcorner\vec{\mathcal{A}}\urcorner, S)}$*, or*

   (e) $\mathrm{EndRule}(\Pi) =$ *Propositional* $\mathbf{I}\text{-}\bigwedge$*, so that*

$$\Pi = \cfrac{\begin{matrix}\Pi_0\ulcorner p\urcorner\\ \mathcal{A}\ulcorner p\urcorner\end{matrix}}{\bigwedge x\mathcal{A}\ulcorner x\urcorner}$$

*and for all formulas* $\mathcal{B}$ *and all regular sets* $S$: $\Pi_0\ulcorner\mathcal{B}\urcorner \in \mathrm{IND}_{\mathfrak{G}\cdot(\mathcal{B}, S)}$*, or*

   (f) *correspondingly for the functional* $\bigwedge$ *Introduction Rule.*

The above proposition can then be used to obtain the following:

LEMMA A.3.

1. *If* $\Pi \in \mathrm{IND}_{\mathfrak{G}}$ *then* $\Pi$ *is normalizable.*
2. *If* $\Pi \in \mathrm{IND}_{\mathfrak{G}}$ *and* $\Phi \prec \Pi$ *then* $\Phi \in \mathrm{IND}_{\mathfrak{G}}$*.*
3. $\mathrm{IND}_{\mathfrak{G}}$ *is a regular set.*

Another important property of the sets $\mathrm{IND}_{\mathfrak{G}}$ derivable using their "inductive definition" is the following:

PROPOSITION A.4. *For any formula* $\mathcal{B}$*,*

$$\mathrm{IND}_{\mathfrak{G}} = \mathrm{IND}_{\mathfrak{G}\cdot(\mathcal{B},\mathrm{IND}_{\mathfrak{G}})}.$$

THEOREM A.5 (Pivotal Theorem for Normalization). *If* $\Pi$ *is a derivation whose EndRule is not an introduction rule then for all Girard assignments* $\mathfrak{G}$*, if the subderivations of the premises of* $\Pi$ *are in* $\mathrm{IND}_{\mathfrak{G}}$*, then so is* $\Pi$*.*

*Outline of proof.* Let us first define the **reduction rank** of a normalizable derivation to be the (finite) ordinal of its reduction tree.

Then let us assume the hypothesis of the theorem for a derivation $\Pi$. Using Lemma A.3 we obtain that the derivations of the premises of the conclusion of $\Pi$ are normalizable and hence of finite reduction rank. We define the **induction rank** of a derivation $\Pi$ whose EndRule is not an Introduction to be the sum of the reduction ranks of the premises.

It is clear that the essential step in the proof of the theorem is to show that if $\Phi$ is an immediate reduction of $\Pi$ then $\Phi \in \mathrm{IND}_{\mathfrak{G}}$.

The latter breaks into two cases. The first, in which the immediate reduction takes place within one of the premises of $\mathrm{EndFor}(\Pi)$, has the effect of reducing the induction rank and producing a derivation whose EndRule is not an Introduction and consequently the induction hypothesis may be used.

The second case is when $\Phi$ is the result of a contraction of $\Pi$. This case is further broken down into the subcases corresponding to each of the primitive terms and Leśniewskian definitions. Let us take just one of those subcases, more specifically the subcase corresponding to the Propositional $\bigwedge$-contraction. Thus assume that

$$\Pi = \cfrac{\cfrac{\begin{matrix}\Pi_0\ulcorner p\urcorner\\ \mathcal{A}\ulcorner p\urcorner\end{matrix}}{\bigwedge x\mathcal{A}\ulcorner x\urcorner}}{\mathcal{A}\ulcorner\mathcal{B}\urcorner},$$

and thus

$$\Phi = \begin{matrix}\Pi_0\ulcorner\mathcal{B}\urcorner\\ \mathcal{A}\ulcorner\mathcal{B}\urcorner\end{matrix}.$$

Define

$$\Pi_1 = \cfrac{\begin{matrix}\Pi_0\ulcorner p\urcorner\\ \mathcal{A}\ulcorner p\urcorner\end{matrix}}{\bigwedge x\mathcal{A}\ulcorner x\urcorner},$$

so that $\Pi_1$ is the premise of the end-formula of $\Pi$. Consequently $\Pi_1 \in \mathrm{IND}_{\mathfrak{G}}$. From the latter we obtain $\Phi \in \mathrm{IND}_{\mathfrak{G}\cdot(\mathcal{B},\mathrm{IND}_{\mathfrak{G}})}$. Then using Proposition A.4, we conclude that $\Phi \in \mathrm{IND}_{\mathfrak{G}}$. ■

The remaining steps in the proof that the Intuitionistic Protothetic has the Normalization Property are as follows.

- *A parameter occurring in an undischarged assumption formula of a derivation*[8] *and which is not an eigen-parameter is called a* **free parameter** *of the derivation.*
- *A derivation* $\Pi \in \mathrm{IND}_{\mathfrak{G}}$ *is* **hereditarily in** $\mathrm{IND}_{\mathfrak{G}}$ *iff*
  - *Substituting the free propositional parameters by arbitrary formulas results in a derivation in* $\mathrm{IND}_{\mathfrak{G}}$*.*
  - *Substituting the free functional parameters by either other (appropriate) functional parameters or defined operators results in a derivation in* $\mathrm{IND}_{\mathfrak{G}}$*.*
  - *Replacing undischarged assumption formula occurrences in* $\Pi$ *by derivations in* $\mathrm{IND}_{\mathfrak{G}}$ *results in a derivation in* $\mathrm{IND}_{\mathfrak{G}}$*.*

Then by an induction on the length of the derivation, and making use of the Pivotal Theorem, one can prove:

THEOREM A.6. *For any Girard assignment* $\mathfrak{G}$ *and for all derivations* $\Pi$ *of the unfolding of the Intuitionistic Protothetic* $\Pi$ *is hereditarily in* $\mathrm{IND}_{\mathfrak{G}}$*.*

COROLLARY. *The Intuitionistic Protothetic enjoys the Normalization Property.*

## 4. On the form of normal derivations

*A* **path** *in a derivation* $\Pi$ *is a sequence of formula occurrences* $\mathcal{A}_1, \ldots, \mathcal{A}_n$ *such that:*

- $\mathcal{A}_1$ *is an assumption occurrence.*
- *For* $i < n$*,* $\mathcal{A}_{i+1}$ *is the formula occurrence immediately below* $\mathcal{A}_i$*.*
- *For* $i < n$*,* $\mathcal{A}_i$ *is either a premise of an* **I***-rule or the major premise of an* **E***-rule.*
- $\mathcal{A}_n$ *is either the end-formula of* $\Pi$ *or a minor premise of an* **E***-rule.*

Note that every formula occurrence in a derivation $\Pi$ belongs to at least one path. A path $\mathcal{A}_1, \ldots, \mathcal{A}_n$ in which $\mathcal{A}_n$ is the end-formula of the derivation is called a **major path**.

LEMMA A.4. *In a normal derivation, a path* $\mathcal{A}_1, \ldots, \mathcal{A}_n$ *can be divided into two parts:*

1. *an* **E***-part* $\mathcal{A}_1, \ldots, \mathcal{A}_{j-1}$ *where for each* $i < j-1$*,* $\mathcal{A}_i$ *is the major premise of an application of an* **E***-rule with conclusion* $\mathcal{A}_{i+1}$*. If there are no applications of the* $\bigwedge$-**E***-rules, then each* $\mathcal{A}_{i+i}$ *is of smaller logical complexity than that of* $\mathcal{A}_i$*.*
2. *an* **I***-part* $\mathcal{A}_j, \ldots, \mathcal{A}_n$ *where for* $j \leq i < n$*,* $\mathcal{A}_i$ *is a premise of an application of an* **I***-rule with* $\mathcal{A}_{i+i}$ *as conclusion. Each* $\mathcal{A}_i$ *is of smaller logical complexity than that of* $\mathcal{A}_{i+1}$*.*

Let us partition the derivations into those that are *open* and those that are *closed*:

- *A derivation is an* **open derivation** *iff there is at least one undischarged assumption formula occurrence.*
- *A derivation which is not an open derivation is a* **closed derivation**.

[8] As usual assumed to be standardized w.r.t. eigen-parameters.

Since the only rules of inference that discharge assumptions are the Introduction rules for $\equiv$ and $\supset$ one immediately obtains from Lemma A.4:

COROLLARY. *Any normal derivation which ends with an application of an* **E**-*rule of inference for* $\equiv$ *or* $\supset$ *is an open derivation, in fact the top formula of the major path is an open assumption.*

APPENDIX B

# Classical completeness theorem for IP

## 1. Traditional Beth models

In the Beth models for **IP** it is not excluded that the sentence $\bigwedge xx$ may be forced in some of its nodes. Since in **IP**, $\bot \equiv \bigwedge xx$, it follows that $\bot$ may be satisfied in some Beth models for **IP**. Now, for a variety of reasons, some people are uncomfortable with such a state of affairs[1]. At the expense of destroying the constructiveness of the completeness theorem, it is possible to restrict oneself to Beth models which do not have such discomforting properties and that is the content of this appendix.

*A* **traditional Beth structure for IP** *is a Beth structure* $\mathfrak{B}$ *for* **IP** *such that each node* $n$ *of* $\mathfrak{B}$ *there corresponds a propositional parameter* $p$ *such that*

$$\not\Vdash_{\mathfrak{B},n} p.$$

The corresponding syntactical notion is that of *consistency.*

- *A finite set of formulas* $\Gamma$ *of* **IP** *is* **inconsistent** *iff every propositional parameter is derivable from* $\Gamma$*.*
- $\Gamma$ *is* **consistent** *iff it is not inconsistent.*

Lemma B.1. *If* $\Gamma$ *is a finite set of* **IP** *formulas, then:*

1. $\Gamma$ *is inconsistent iff every formula is derivable from* $\Gamma$*.*
2. $\Gamma$ *is inconsistent iff* $\bigwedge xx$ *is derivable from* $\Gamma$*.*

**Soundness of IP w.r.t. traditional Beth models.** An induction on the length of the derivation shows that if $\Gamma$ is a finite, consistent set of formulas and

$$\Gamma \vdash \mathcal{A},$$

then every traditional Beth model that satisfies all the formulas in $\Gamma$, satisfies $\mathcal{A}$.

**Universal traditional Beth model.** Assume that $\Gamma$ is a finite and *consistent* set of formulas of **IP**. Then we will construct a traditional Beth model $\mathfrak{B}_\Gamma$ such that for all formulas $\mathcal{A}$:

$$\Gamma \vdash \mathcal{A} \quad \text{iff} \quad \Vdash_{\mathfrak{B}_\Gamma} \mathcal{A}.$$

**Construction of the fan.** Although the mathematical constructions that we carry out in this section do not satisfy the intuitionistic requirements (because we have to make some undecidable choices), we shall continue to use the intuitionistic terminology such as *fans, spreads, ips* etc. *Caveat lector!*

[1] Probably because they read "$\bot$" as *the unsatisfiable.*

Let $\mathcal{F}_1, \ldots, \mathcal{F}_n, \ldots$ be an enumeration of all the formulas of **IP** in which every formula is repeated infinitely often.

**Construction of the function $T$.** $T$ will be defined on all finite sequences of 0's and 1's using an induction on the length of the finite sequence. Each $T(\vec{a})$ is a finite, consistent set of formulas.

**Basis step.** $T(()) = \Gamma$.

**Induction step.** Suppose that $T((a_0, \ldots, a_{n-1}))$ has been defined and let us abbreviate it by $\Delta$. We proceed by cases:

**Case 1: $\Delta \nvdash \mathcal{F}_n$ and $\Delta \nvdash (\mathcal{F}_n \equiv \bot)$.** Then

$$T((a_0, \ldots, a_{n-1}, 0)) = \Delta \quad \text{and} \quad T((a_0, \ldots, a_{n-1}, 1)) = \Delta \cup \{\mathcal{F}_n\}.$$

**Case 2: $\Delta \vdash \mathcal{F}_n$.** Then

$$T((a_0, \ldots, a_{n-1}, 0)) = \Delta \cup \{\mathcal{F}_n\} \quad \text{and} \quad T((a_0, \ldots, a_{n-1}, 1)) = \Delta \cup \{\mathcal{F}_n\}.$$

**Case 3: $\Delta \vdash (\mathcal{F}_n \equiv \bot)$.** Then

$$T((a_0, \ldots, a_{n-1}, 0)) = \Delta \quad \text{and} \quad T((a_0, \ldots, a_{n-1}, 1)) = \Delta.$$

The proof of the completeness then proceeds as for the intuitionistic Beth models.

APPENDIX C

# Kripke semantics

When using an intuitionistic metatheory we found that the Beth models for the New Protothetic are the most natural. On the other hand once we allow non-constructive reasoning (as we did in the completeness using traditional Beth models), we find that the **Kripke models** are simpler to use, although not necessarily simpler to describe since now we have to consider domains for the propositional and functional parameters. In addition we shall only consider Kripke structures which are the analogue of the *traditional* Beth structures, that is, $\perp$ is never satisfied.

*A* **Kripke structure** *for the New Protothetic is a structure:*

$$\mathfrak{K} = \langle \mathsf{K}, 0, \preceq, \mathsf{V}, \mathsf{D}, \mathsf{E} \rangle$$

*such that:*

1. $\langle \mathsf{K}, 0, \preceq \rangle$ *is a partial ordering with* $0$ *as the least element. The elements of* $\mathsf{K}$ *are called the* **nodes** *of* $\mathfrak{K}$.
2. $\mathsf{V}, \mathsf{D}$ *and* $\mathsf{E}$ *are* $\preceq$*-increasing functions.*
3. *For each node* $n \in \mathsf{K}$, $\mathsf{V}(n)$ *is a set of prime formulas.* $\mathsf{V}(n)$ *is the* **valuation domain** *at* $n$.
4. *For each node* $n \in \mathsf{K}$, $\mathsf{D}(n)$ *is a set of propositional parameters which includes all the propositional parameters occurring in the formulas of* $\mathsf{V}(n)$. $\mathsf{D}(n)$ *is the* **propositional domain** *at* $n$.
5. *For each node* $n \in \mathsf{K}$, $\mathsf{E}(n)$ *is a set of functional parameters which includes all the functional parameters occurring in the formulas of* $\mathsf{V}(n)$. $\mathsf{E}(n)$ *is the* **functional domain** *at* $n$.
6. *For each path* $\alpha$ *through* $\mathsf{K}$ *we define*

$$\mathsf{V}_\alpha = \bigcup\nolimits_{n \in \alpha} \mathsf{V}(n).$$

7. *For each path* $\alpha$ *there is at least one propositional parameter* $p$ *such that* $p \notin \mathsf{V}_\alpha$.

We characterize the various parameters occurring in a formula $\mathcal{A}$ as follows:

1. $\mathrm{Par}_0(\mathcal{A})$ *is the set of propositional parameters occurring in* $\mathcal{A}$.
2. $\mathrm{Par}_1(\mathcal{A})$ *is the set of functional parameters occurring in* $\mathcal{A}$.
3. $\mathrm{Par}(\mathcal{A}) = \mathrm{Par}_0(\mathcal{A}) \cup \mathrm{Par}_1(\mathcal{A})$.

Next we define the relation of **forcing** of a formula $\mathcal{A}$ at a node $n$ of a Kripke structure $\mathfrak{K}$, in symbols: $\Vdash_{\mathfrak{K},n}\mathcal{A}$ (or simply: $\Vdash_n\mathcal{A}$) by the following induction on the syntactical complexity of the formula $\mathcal{A}$ of the New Protothetic. Because the Kripke structure have varying domains we shall consider the notion of satisfaction only in the case that

$$\mathrm{Par}_0(\mathcal{A}) \subseteq \mathsf{D}(n) \quad \text{and} \quad \mathrm{Par}_1(\mathcal{A}) \subseteq \mathsf{E}(n).$$

With that proviso we proceed as follows:

- *$\mathcal{A}$ is a prime formula and*

$$\mathcal{A} \in \mathsf{V}(n).$$

- *$\mathcal{A} = (\mathcal{B} \supset \mathcal{C})$ and*

$$\forall m_{n \preceq m}[\ \textit{if}\ \ \Vdash_m \mathcal{B}\ \ \textit{then}\ \ \Vdash_m \mathcal{C}\ ].$$

- *$\mathcal{A} = (\mathcal{B} \wedge \mathcal{C})$ and*

$$\Vdash_n \mathcal{B}\ \ \textit{and}\ \ \Vdash_n \mathcal{C}.$$

- *$\mathcal{A} = (\mathcal{B} \equiv \mathcal{C})$ and*

$$\forall m_{n \preceq m}[\ \Vdash_m \mathcal{B}\ \ \textit{iff}\ \ \Vdash_m \mathcal{C}\ ].$$

- *$\mathcal{A} = \bigwedge x \mathcal{B}\ulcorner x \urcorner$ and for all $n \preceq m$ and all the propositional parameters $q$ in $\mathsf{D}(m)$:*

$$\Vdash_m \mathcal{B}\ulcorner q \urcorner.$$

- *$\mathcal{A} = \bigwedge f \mathcal{B}\ulcorner f \urcorner$ and for all $n \preceq m$ and for all functionals $X$ which are either defined operators or functional parameters in $\mathsf{E}(m)$:*

$$\Vdash_m \mathcal{B}\ulcorner X \urcorner.$$

REMARK. Whenever we write $\Vdash_{\mathfrak{K},n} \mathcal{A}$, it will be assumed that

$$\mathrm{Par}(\mathcal{A}) \subseteq \mathsf{D}(n) \cup \mathsf{E}(n).$$

Since $\mathsf{V}, \mathsf{D}$ and $\mathsf{E}$ are $\preceq$-monotonic we obtain:

LEMMA C.1. *If $\mathfrak{K}$ is a Kripke structure and $\mathcal{A}$ is a formula then*

$$\textit{if}\ \ \Vdash_{\mathfrak{K},n} \mathcal{A}\ \ \textit{and}\ \ n \preceq m\ \ \textit{then}\ \ \Vdash_{\mathfrak{K},m} \mathcal{A}.$$

We take care of the impredicative nature of the New Protothetic by the following:

*A* **Kripke model** *is a Kripke structure such that:*

1. *All the Leśniewskian definitions (of the unfolding) are forced at every node.*
2. *For every formula of the form $\bigwedge x \mathcal{B}\ulcorner x \urcorner$ and nodes $n$:*
   - *if $\Vdash_n \bigwedge x \mathcal{B}\ulcorner x \urcorner$*
   - *then for all $n \preceq m$ and for all formulas $\mathcal{F}$ such that $\mathrm{Par}(\mathcal{F}) \subseteq \mathsf{D}(m) \cup \mathsf{E}(n)$ : $\Vdash_m \mathcal{B}\ulcorner \mathcal{F} \urcorner$.*

**Soundness w.r.t. Kripke models.** Suppose that $\mathrm{Par}(\Gamma \cup \{\mathcal{A}\}) \subseteq \mathsf{D}(n) \cup \mathsf{E}(n)$. Then we set

$$\Gamma \Vdash_{\mathfrak{K},n} \mathcal{A}$$

iff

$$\forall m_{n \preceq m}[\ \forall \mathcal{G}_{\mathcal{G} \in \Gamma}(\Vdash_m \mathcal{G}) \to \Vdash_m \mathcal{A}\ ].$$

An induction on the length of the derivation gives:

LEMMA C.2. *For any Kripke model* $\mathfrak{K}$ *and node* $n$*: if* $\mathrm{Par}(\Gamma \cup \{\mathcal{A}\}) \subseteq \mathsf{D}(n) \cup \mathsf{E}(n)$*, then:*

$$\textit{if}\ \ \Gamma \vdash \mathcal{A} \quad \textit{then} \quad \Gamma \Vdash_{\mathfrak{K},n} \mathcal{A}.$$

COROLLARY. *For sets of sentences* $\Sigma \cup \{\mathcal{S}\}$*:*

$$\textit{if}\ \ \Sigma \vdash \mathcal{S} \quad \textit{then} \quad \Sigma \Vdash_{\mathfrak{K}} \mathcal{S}.$$

**Completeness w.r.t. Kripke models.** For the completeness of Kripke semantics we continue the method started with the completeness of Beth semantics. That is, we construct a spread[1] whose paths give us deductively closed set of formulas. One fundamental change is that the paths themselves now become the nodes of the Kripke model. There are also other modifications due to the varying domains of the Kripke models.

**Preliminary steps.**

- $\Gamma$ *is a consistent, finite set*[2] *of formulas of* **IP**.
- $\mathsf{P}_\emptyset$ *and* $\mathsf{F}_\emptyset$ *are the propositional parameters and functionals respectively, occurring in the formulas* $\Gamma$.
- $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots$ *is an enumeration of all the formulas of the Intuitionistic Protothetic in which each formula is repeated infinitely often.*
- $\mathsf{P}_0, \mathsf{P}_1, \mathsf{P}_2, \ldots$ *is a partition of all the propositional parameters in which each* $\mathsf{P}_i$ *is infinite. In addition it is assumed that* $\mathsf{P}_\emptyset \subset \mathsf{P}_0$.
- $\mathsf{F}_0, \mathsf{F}_1, \mathsf{F}_2, \ldots$ *is a partition of all the functional parameters in which each* $\mathsf{F}_i$ *contains infinitely many functional parameters of each arity. In addition it is assumed that* $\mathsf{F}_\emptyset \subset \mathsf{F}_0$.
- $p_{i,0}, p_{i,1}, p_{i,2}, \ldots$ *is an enumeration of* $\mathsf{P}_i$.
- $F_{i,0}, F_{i,1}, F_{i,2}, \ldots$ *is an enumeration of* $\mathsf{F}_i$.

**Construction of the spread.** For simplicity we shall allow *all* finite sequences of natural numbers. In addition to assigning to each finite sequence $\vec{n}$ a finite, *consistent* set $T(\vec{n})$ of formulas, we shall assign finite sets $D(\vec{n})$ and $E(\vec{n})$ of propositional parameters and functionals respectively.

**Basis step.**

$$T(()) = \Gamma, \qquad D(()) = \mathsf{P}_\emptyset, \qquad E(()) = \mathsf{F}_\emptyset.$$

Note that $\mathrm{Par}(T()) \subseteq D() \cup E()$.

**Inductive steps.** Suppose that $\vec{n} = (n_0, \ldots, n_{i-1})$, $T(\vec{n})$, $D(\vec{n})$ $E(\vec{n})$ are defined, $T(\vec{n})$ is a consistent set of formulas and that $\mathrm{Par}(T(\vec{n})) \subseteq D(\vec{n}) \cup E(\vec{n})$. Then for each natural number $k$ we define[3]:

$$\begin{aligned} D(\vec{n}k) &= D(\vec{n}) \cup \{p_{0,0}, \ldots, p_{0,k-1}\} \cup \ldots \cup \{p_{i,0}, \ldots, p_{i,k-1}\}, \\ E(\vec{n}k) &= E(\vec{n}) \cup \{F_{0,0}, \ldots, F_{0,k-1}\} \cup \ldots \cup \{F_{i,0}, \ldots, F_{i,k-1}\}. \end{aligned}$$

To extend $T$ we proceed by cases.

[1]Once again, we should really call it a tree since, at a couple of points, we will not be following intuitionistic principles.

[2]One can avoid the restriction to finite sets by introducing a new infinite set of parameters.

[3]$\vec{n}k$ stands for the concatenation of $\vec{n}$ and $(k)$.

**Case 1:** $\mathrm{Par}(\mathcal{F}_i) \not\subseteq D(\vec{n}) \cup E(\vec{n})$. Then for each natural number $k$ we set

$$T(\vec{n}k) = T(\vec{n}).$$

**Case 2:** $\mathrm{Par}(\mathcal{F}_i) \subseteq D(\vec{n}) \cup E(\vec{n})$.

*Subcase 2.1:* $T(\vec{n}) \cup \{\mathcal{F}_i\}$ is inconsistent. Then for each natural number $k$ we set

$$T(\vec{n}k) = T(\vec{n}).$$

*Subcase 2.2:* $T(\vec{n}) \vdash \mathcal{F}_i$. Then for each natural number $k$ we set

$$T(\vec{n}k) = T(\vec{n}) \cup \{\mathcal{F}_i\}.$$

*Subcase 2.3:* $T(\vec{n}) \nvdash \mathcal{F}_i$ and $T(\vec{n}) \nvdash \neg\mathcal{F}_i$. Then for each natural number $k$ we set

$$T(\vec{n}(2k)) = T(\vec{n}), \quad T(\vec{n}(2k+1)) = T(\vec{n}) \cup \{\mathcal{F}_i\}.$$

As before we use the concepts just defined to generate some new ones:

*For each ips $\alpha$ let*

$$T_\alpha = \bigcup_i T(\tilde{\alpha}i), \quad D(\alpha) = \bigcup_i D(\tilde{\alpha}i), \quad E(\alpha) = \bigcup_i E(\tilde{\alpha}i), \quad \underline{0} = (0, 0, \dots).$$

The construction then gives us that:

LEMMA C.3. *For each ips $\alpha$:*

1. *$T_\alpha$ is a consistent set of formulas.*
2. $\mathrm{Par}(T_\alpha) \subseteq D_\alpha \cup E_\alpha$.
3. *For each formula $\mathcal{A}$ such that* $\mathrm{Par}(\mathcal{A}) \subseteq D_\alpha \cup E_\alpha$*:*

$$T_\alpha \vdash \mathcal{A} \quad \textit{iff} \quad \mathcal{A} \in T_\alpha.$$

4. $T_{\underline{0}} \subseteq T_\alpha$.

We are now ready to define the Kripke structure:

1. $\alpha \preceq \beta$ *iff* $T_\alpha \subseteq T_\beta$.
2. $V(\alpha) = \{\mathcal{P} \mid \mathcal{P} \in T_\alpha$ *and* $\mathcal{P}$ *is a prime formula* $\}$.
3. *$K$ is the set of all ips.*
4. $\mathfrak{K}_\Gamma = \langle K, \preceq, \underline{0}, V, D, E\rangle$.

**Properties of the Universal Kripke model.** Adapting the techniques used for the completeness of Beth semantics it can be shown that the following conditions are equivalent for all ips $\alpha$, if $\mathrm{Par}(\mathcal{A}) \subseteq D(\alpha) \cup E(\alpha)$:

1. $\Vdash_{\mathfrak{K}_\Gamma, \alpha} \mathcal{A}$,
2. $\mathcal{A} \in T_\alpha$,
3. $T_\alpha \vdash \mathcal{A}$.

COROLLARY. *For any formula $\mathcal{A}$ such that* $\mathrm{Par}(\mathcal{A}) \subseteq \mathrm{Par}(\Gamma)$*:*

$$\Gamma \vdash \mathcal{A} \quad \textit{iff} \quad \Vdash_{\mathfrak{K}_\Gamma} \mathcal{A}.$$

Then since the Leśniewskian definitions are theorems of the unfolding it is a small step to obtain:

PROPOSITION C.1. *$\mathfrak{K}_\Gamma$ is a Kripke model.*

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# Index of terms and symbols

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