1. INTRODUCTION

In 1973, Victor Klee posed the following question [43]: *How many guards are always sufficient and sometimes necessary to see every point in an art gallery with* n *walls?* The guard is assumed to be a stationary point which can *see* any other point that can be connected to it by a line segment within the art gallery, i.e., the region bounded by a simple polygon. In 1975, Vasek Chvátal established that $\lfloor n/3 \rfloor$ guards are occasionally necessary and always sufficient to cover a polygon with n vertices [19]; Fig. 1.1 illustrates this with a 15-sided gallery (in the form of a comb) which requires five guards. Chvátal's proof is entirely combinatorial, starting with an arbitrary triangulation of the polygon and cutting off a small piece for the induction step. Three years later, in 1978, Steve Fisk offered a simpler proof of Chvátal's result based upon a 3-coloring of a triangulation graph [28]. This proof is briefly described below.

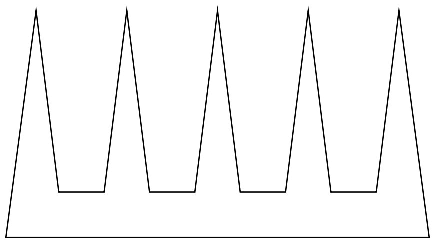

Fig. 1.1. A *comb*-polygon that requires $\lfloor n/3 \rfloor$ guards; here $n = 15$, and the polygon requires 5 guards.

Let us consider the polygon P with $n = 13$ vertices shown in Fig. 1.2(a-b), its triangulation T, and its triangulation graph G_T. The graph G_T can be 3-colored using the three colors $\{1, 2, 3\}$. We have four vertices of color "1", 5 of color "2", and four of color "3". Now, let us place guards at every "1"-vertex. Since each triangle of the triangulation has all three colors at its vertices, it has a guard in one of its corners. Since the triangles form a partition of P, and triangles are convex, every point of P is covered by a "1"-guard. Thus the "1"-guards cover the polygon, and there are at most $\lfloor 13/3 \rfloor$ of them. Notice that after removing all internal diagonals incident to any "1"-vertex we get a partition of P into four star-shaped polygons (see Fig. 1.2(c)). As the same method of placing guards works for an arbitrary polygon, $\lfloor n/3 \rfloor$ guards are always sufficient to cover the interior of an n-vertex polygon, and guards can be placed at the vertices of the polygon only.

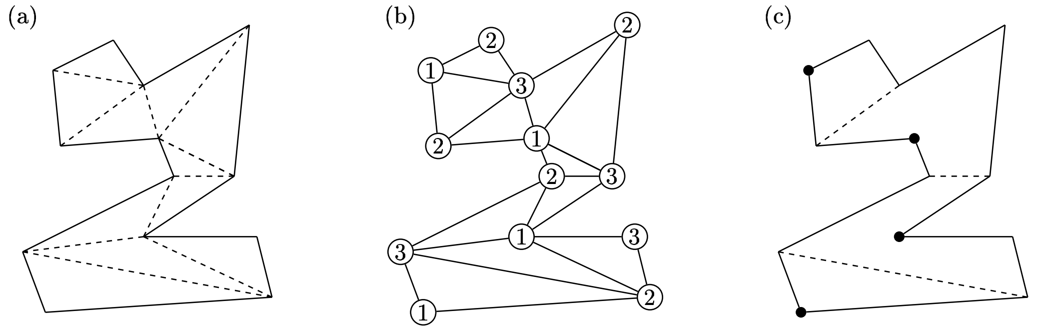

Fig. 1.2. (a) A polygon P and its triangulation T. (b) 3-coloring of the triangulation graph G_T. (c) A guard set for P (guards are placed at the black vertices); after removing all internal diagonals incident to any "1"-vertex, we get a partition of P into four star-shaped polygons.

Since Chvátal's result, many different variations of the art gallery problem have been studied, including mobile guards, guards with limited visibility or mobility, guarding rectilinear polygons, and others; many of these interesting results can be found in O'Rourke's monograph [73] and in survey articles by Shermer [85] and Urrutia [88].

1.1. Cooperative guards problem

In this dissertation, we investigate one variation of the art gallery problem posed by Liaw, Huang and Lee [58]: the *cooperative guards problem* (CG problem for short), i.e., one wants to determine the minimum number of guards sufficient to see every point of the interior of an n-vertex simple polygon, and such that the visibility graph of the set of guards is connected; the *visibility graph* of a set S of points in a polygon is the graph whose vertex set is S and two vertices are adjacent if the points see each other. The idea behind this concept is that if something goes wrong with one guard, all the others can be informed. It is worth pointing out that a variation of the CG problem had already been raised before: in 1992, Ahlfeld and Hecker [3] studied the problem of determining the minimum link number for polygons. These two problems are closely related, which will be discussed in Section 1.3 in detail.

Chapter 2 investigates the cooperative guards problem for various classes of polygons. Liaw *et al.* [58] established that the *minimum cooperative guards problem* for simple polygons is NP-hard, but for spiral and 2-spiral polygons this problem can be solved in linear time. Combinatorial bounds for arbitrary polygons were independently given by Ahlfeld and Hecker [3], and Hernández-Peñalver [39], who proved that $\lfloor n/2 \rfloor - 1$ cooperative guards are always sufficient and occasionally necessary to guard a polygon with n vertices. The necessity is established by the snake-polygon with $n = 2m$ vertices in Fig. 1.3. It is clear that it requires $m - 1$ guards. In addition, Chapter 2 deals with the CG problem in polygons with holes, that is, polygons enclosing several other polygons.

Some sufficiency bounds for both vertex and point guards are provided. For one-hole polygons, these bounds are tight.

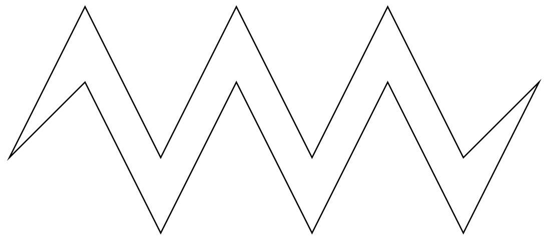

Fig. 1.3. A *snake*-polygon that requires $\lfloor n/2 \rfloor - 1$ cooperative guards; here $n = 14$, and the polygon requires 6 guards.

Chapter 3 is devoted to the study of the *weakly cooperative guards problem* (WCG problem for short) in which we require the visibility graph of a set of guards to have no isolated vertices [59]. In [39] Hernández-Peñalver claimed that $\lfloor 2n/5 \rfloor$ weakly cooperative guards always suffice to guard any polygon with n vertices. Unfortunately, this is not true. Fig. 1.4 shows a polygon with 12 vertices that requires 5 weakly cooperative guards. Each prong requires a guard, and these guards form a hidden set: they do not see each other. As any additional guard will see at most two of these three guards, a fifth guard is needed.

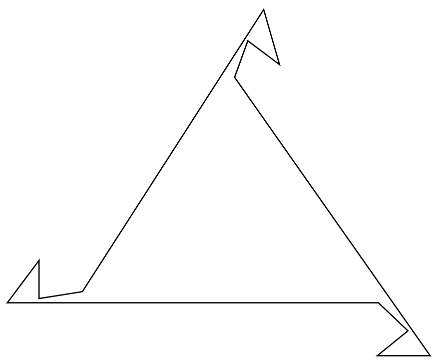

Fig. 1.4. A polygon with 12 vertices that requires 5 weakly cooperative guards.

The weakly cooperative guards problem for general simple polygons was completely settled by Michael and Pinciu [65, 66], and independently by Żyliński [92], who proved that $\lfloor (3n-1)/7 \rfloor$ is a tight bound. The WCG problem for orthogonal polygons was solved by Hernández-Peñalver [40], and by Michael and Pinciu [67], who proved the $\lfloor n/3 \rfloor$ bound to be tight. Combinatorial bounds for star-shaped, spiral and monotone polygons were given by Żyliński [97].

The *k-guarded guards problem* (k-GG problem for short) is a generalization of the weakly cooperative guards problem, and it was raised by Michael and Pinciu [67]. A set of guards is called *k-guarded* if each guard is seen by at least k of its colleagues (the minimum degree of the visibility graph of the set of guards is at least k). Of course, the 1-guarded guards problem is equivalent to the weakly cooperative guards problem. Chapter 4 investigates the k-GG problem for $k \geq 2$. In 2001, Michael and Pinciu established that $(k\lfloor n/6 \rfloor + \lfloor (n+2)/6 \rfloor)$ k-guarded guards always suffice and are sometimes necessary to guard any orthogonal polygon with n vertices. The k-guarded guards problem for general simple polygons was completely settled by Żyliński [98], who showed the $(k\lfloor n/5 \rfloor + \lfloor (n+2)/5 \rfloor)$-bound to be tight. The author [95] also established that the *minimum k-guarded guards problem* for simple polygons is NP-hard.

Possible extensions of the cooperative guards problem to fortresses and grids are discussed in Chapters 5 and 6, respectively. The *fortress problem*, independently posed by Joseph Malkelvitch and Derick Wood, asks about the number of guards sufficient to cover the exterior of a polygon P. In [73] O'Rourke and Wood solved the fortress problem for vertex guards—they showed that $\lceil n/2 \rceil$ vertex guards are sometimes necessary and always sufficient. A tight bound of $\lceil n/3 \rceil$ point guards was given by O'Rourke and Aggarwal [73]. In Chapter 5, we provide tight bounds for both vertex and point cooperative guards in the fortress problem.

In Chapter 6, we study the cooperative guards problem for grids, a special restricted class of polygons introduced by Ntafos [71]. A *grid* P is a connected union of vertical and horizontal line segments. Ntafos established that a minimum cover for a 2D-grid of n segments has $n-m$ guards, where m is the size of the maximum matching in the intersection graph of the grid, and it may be found in $O(n^{2.5})$ time [71]. However, in the case of 3D-grids, the problem of finding the minimum guard set is NP-hard [71]. For cooperative guards, we show that the minimum cooperative guards problem (MinCG problem for short) can be solved in polynomial time for both 2-dimensional and 3-dimensional grids. In the first case, the MinCG problem corresponds to the problem of finding a minimum spanning tree in the intersection graph of a grid, thus an $O(n+k)$ time algorithm is obtained, where n is the number of segments and k is the number of intersections in the grid. In the latter case, an algorithm uses $O(kn^{2.5})$ time; the solution is obtained from a spanning set of a 2-polymatroid constructed from the intersection graph of the grid. When considering weakly cooperative guards, we show that a minimum coverage for a grid of n segments has exactly $n - p_3$ weakly cooperative guards, where p_3 is the size of the maximum P_3-matching in the intersection graph of the grid. Consequently, it makes the problem of determining the minimum number of weakly cooperative guards NP-hard, as we prove that the maximum P_3-matching problem in intersection graphs is NP-hard. In Chapter 6, we also study cooperative mobile guards, where we allow a guard to move along a selected grid segment. By reduction to domination problems, we show that both the minimum cooperative mobile guards problem and the minimum weakly cooperative mobile guards problem are NP-hard. However, we will show that there exist certain classes of grids for which the latter problem can be solved in polynomial time.

In Chapter 7, we conclude with some remarks and indicate directions of the future research. In the remainder of the present chapter, we give precise definitions of most objects and some preliminary results that will be used throughout the dissertation.

1.2. Basic definitions

An *art gallery* is a simple polygon P, i.e., the region bounded by a simple polyline $\bar{P}$ (together with $\bar{P}$). P is said to be *in general position* if no three vertices are collinear. For the sake of simplicity, we shall assume that the vertices of P are in general position. All results hold even if the vertices are in special position; however, proving this is arduous because the number of degenerate cases becomes quite large. Next, since we only deal with simple polygons in the following, we will use the term *polygon* for a simple polygon. A *polygonal chain* is a single chain of consecutive vertices of a polyline $\bar{P}$.

A *guard* g is any point of P. A point $x \in P$ is said to be *seen* by a guard g if the line segment with endpoints x and g is a subset of P: $xg \subseteq P$. A collection of guards $S = \{g_1, \ldots, g_k\}$ is said to *cover* P if every point $x \in P$ can be seen by some guard $g \in S$. A *vertex* guard is one that is placed at a vertex of the polygon.

Let S be a guard set of a polygon. We define the *visibility graph* $\mathrm{VG}(S)$ as follows: the vertex set is S and two vertices v_1 and v_2 are adjacent if they see each other. A guard set is said to be *cooperative* if its visibility graph is connected; recall that a graph G is *connected* if for any pair of vertices v, w, there is a path connecting v and w in G. For a polygon P, define $\mathrm{CG}(P)$ to be the minimum cardinality of a cooperative guard set for P. Next, define $\mathrm{cg}(n)$ to be the maximum value of $\mathrm{CG}(P)$ over all polygons with n vertices. The function $\mathrm{cg}(n)$ represents the maximum number of cooperative guards that are ever needed for an n-gon—$\mathrm{cg}(n)$ cooperative guards always suffice, and $\mathrm{cg}(n)$ cooperative guards are necessary for at least one polygon with n vertices.

DEFINITION 1.1. The *cooperative guards problem* is to determine $\mathrm{cg}(n)$.

Similarly, a guard set is said to be *k-guarded* if the minimum degree of the visibility graph is at least k; Liaw *et al.* [59] refer to a 1-guarded guard set as a *weakly cooperative* guard set. For a polygon P, define $\mathrm{GG}(P, k)$ to be the minimum cardinality of a k-guarded guard set for P. Next, define $\mathrm{gg}(n, k)$ to be the maximum value of $\mathrm{GG}(P, k)$ over all polygons with n vertices.

DEFINITION 1.2. The *k-guarded guards problem* is to determine $\mathrm{gg}(n, k)$.

1.2.1. Guards in triangulation graphs. A *triangulation* T of a polygon P is a partitioning of P into a set of triangles with pairwise disjoint interiors in such a way that the edges of those triangles are either edges or diagonals of P joining pairs of vertices.

THEOREM 1.1 (Triangulation Theorem). *A polygon with n vertices can be partitioned into $n - 2$ triangles by adding $n - 3$ internal diagonals.*

Proof. The proof is by induction on n. The theorem is trivial for $n = 3$. Let P be a polygon with $n \geq 4$ vertices. Let v_2 be a convex vertex of P, and consider the three consecutive vertices v_1, v_2, v_3. (We take it as obvious that there must exist at least one convex vertex.) We seek an internal diagonal d.

If the segment v_1v_3 is completely interior to P (i.e., does not intersect the polyline $\bar{P}$), then let $d = \{v_1, v_3\}$. Otherwise, the closed triangle (v_1, v_2, v_3) must contain at least one vertex of P. Let x be the vertex of P closest to v_2, where the distance is measured perpendicularly to v_1v_3 (see Fig. 1.5), and let $d = \{v_2, x\}$.

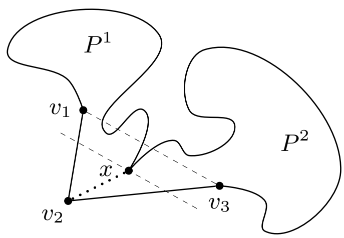

Fig. 1.5. The line segment v_2x is an internal diagonal.

In either case, d divides P into two smaller polygons P^1 and P^2. If P^i has n_i vertices, $i = 1, 2$, then $n_1 + n_2 = n + 2$, since both endpoints of d are shared between P^1 and P^2. Clearly, $n_i \geq 3$, $i = 1, 2$, which implies that $n_i < n$, $i = 1, 2$. Applying the induction hypothesis to each polygon results in a triangulation for P of $(n_1 - 2) + (n_2 - 2) = n - 2$ triangles, and $(n_1 - 3) + (n_2 - 3) + 1 = n - 3$ diagonals, including d. ∎

Next, let us make an important observation about the way the triangles in a triangulation fit together. Recall that a *tree* is a connected graph without cycles, and the *degree* $\deg(v)$ of a vertex $v \in V$ in a graph $G = (V, E)$ is the number of edges incident to v.

LEMMA 1.2. *The weak dual graph of a triangulation of a polygon, that is, the graph with a vertex for each triangle of the triangulation and an edge connecting two vertices whose triangles share a diagonal, is a tree with each vertex of degree at most* 3.

Proof. That each vertex has degree at most 3 follows immediately from the fact that a triangle has three sides. Next, suppose that the weak dual graph is not a tree. Then it must have a cycle. As this cycle encloses some vertices of the polygon, it encloses points in the exterior of the polygon, but this contradicts the definition of a (simple) polygon. ∎

Note that the technical term used in the above lemma is the "weak dual", as no vertex is assigned to the exterior face—that is, the exterior of the polygon. However, as we shall always encounter weak dual graphs, we will henceforth simply call them dual graphs.

A *triangulation graph* G_T of an n-vertex polygon P is the graph whose vertices correspond to the n vertices of P and whose edges correspond to the edges of P and the diagonals of a triangulation T. We say that three consecutive vertices v_1, v_2, v_3 form an *ear* in G_T at v_2 if $\{v_1, v_3\}$ is an internal diagonal of the triangulation of P. Lemma 1.2 yields an easy proof of the "two ears" theorem of Meister [64].

THEOREM 1.3 ([64]). *Every triangulation graph of a polygon with* $n \geq 4$ *vertices has at least two ears.*

Proof. Leaves in the dual graph of a triangulation correspond to ears, and every tree of two or more vertices must have at least two leaves (vertices of degree one). ■

A *vertex guard* in a triangulation graph G_T is a single vertex of G_T. A set S of guards is said to *dominate* G_T if every triangular face of G_T has at least one of its vertices selected as a guard. A collection of guards S is said to be *cooperative* if the subgraph of G_T induced by S is connected, and k-*guarded* if the minimum degree in the induced subgraph is at least k; recall that for a graph $G = (V, E)$, the *subgraph induced by* $V' \subseteq V$ is the graph with vertex set V' and with all edges of G whose endpoints are in V'. Guards in a graph are called *combinatorial* to distinguish them from *geometric* guards introduced earlier.

The reason for introducing triangulation graphs is that a proof of the sufficiency of a certain number of combinatorial guards establishes the sufficiency of the same number of geometric guards in a polygon, regardless of the cooperation model we consider.

LEMMA 1.4 ([39]). *Let P be an n-vertex polygon, and let G_T be a triangulation graph of P. If G_T can be dominated by $f(n)$ cooperative guards, then P can be covered by $f(n)$ geometric cooperative vertex guards.*

Proof. Since G_T is dominated, each triangle has at least one combinatorial guard at one of its vertices. Placing geometric guards at the corresponding vertices of P ensures that each triangular region is covered, and so is P. To establish that these guards are cooperative observe that the connectedness of the subgraph induced by S implies the connectedness of the visibility graph $\mathrm{VG}(S)$ in P as well. ■

LEMMA 1.5 ([98]). *Let P be an n-vertex polygon, and let G_T be a triangulation graph of P. If G_T can be dominated by $f(n,k)$ k-guarded guards, then P can be covered by $f(n,k)$ geometric k-guarded vertex guards.*

Proof. Following the proof of Lemma 1.4, all we need is to show that a guard set S is k-guarded, but this follows from the fact that the minimum degree in the subgraph induced by S is no greater than the minimum degree in the visibility graph $\mathrm{VG}(S)$ in P. ■

Thus in general, the idea of most of the proofs is to solve guarding problems on triangulation graphs, and then to extend these results to polygons.

1.3. Note on the link number of a polygon

Consider the plane embedding $\mathrm{SG}(S)$ of the visibility graph $\mathrm{VG}(S)$ of a cooperative guard set S: the vertex set $V(\mathrm{SG})$ consists of the guards and the edge set $E(\mathrm{SG})$ consists of segments (edges) connecting two guards that see each other; the graph SG is called the *segment graph* of S. The *link number* $\mathrm{lk}(\mathrm{SG})$ of the segment graph SG with exactly one segment is 1. Let SG be a segment graph with link number k. Let $V = \{v_1, \ldots, v_n\}$ be the set of vertices of SG and let $E = \{s_1, \ldots, s_m\}$ be the set of segments (edges) of SG. Let SG$'$ be the segment graph obtained from SG by inserting a new edge s_{m+1} (such that s_{m+1} meets endpoints of the edges of SG only). If one endpoint v_{m+1} of s_{m+1} is not a vertex of SG, then let v_j be the endpoint of s_{m+1} which is in V. If there is an edge

$s_l = \{v_i, v_j\} \in E$ such that v_i, v_j and v_{m+1} are collinear, then $\text{lk}(\text{SG}')$ is k, otherwise we define $\text{lk}(\text{SG}') = k+1$. If both endpoints of s_{m+1} are in V, then let v_i and v_j be these vertices. If there exist two edges $s_l = \{v_f, v_i\}$ and $s_t = \{v_g, v_j\}$ such that v_f, v_g, v_i and v_j are collinear, then we define $\text{lk}(\text{SG}') = k-1$. If there exists exactly one edge (s_l or s_t) such that v_f, v_i and v_j (v_g, v_i and v_j, respectively) are collinear, then let $\text{lk}(\text{SG}')$ be equal k. If there is no edge $\{v', v''\}$ in E with $v' \in \{v_i, v_j\}$ such that v'', v_i and v_j are collinear, then we define $\text{lk}(\text{SG}') = k+1$. A cooperative guard set S for a polygon P is said to be *minimal* if its link number is minimal among all cooperative guard sets for P; for a polygon P, the cardinality of a minimal cooperative guard set is denoted by $\text{lk}(P)$.

Fig. 1.6(a-b) explains the difference between a minimum cooperative guard set and a minimal cooperative guard set. It is easy to see that S_1 forms a minimum cooperative guard set with $\text{lk}(S_1) = 4$, but there exists a cooperative guard set S_2 with a larger number of guards, but with the smaller link number $\text{lk}(S_2) = 3$. Throughout this dissertation we are interested in the minimum cooperative guard set problem rather than the minimal cooperative guard set problem.

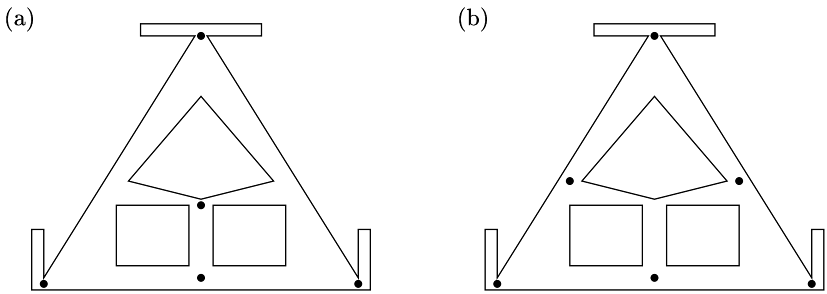

Fig. 1.6. (a) A minimum cooperative guard set S_1 has 5 guards, and $\text{lk}(S_1) = 4$. (b) A minimal cooperative guard set S_2 with $\text{lk}(S_2) = 3$ has 6 guards.

The idea of the link number for polygons was introduced by Ahlfeld and Hecker [3]. The key observation is that for any polygon P, $\text{lk}(P) \leq \text{cg}(P) - 1$. According to this observation, by using the concept of cooperative guards in triangulation graphs, Ahlfeld and Hecker first showed that $\lfloor n/2 \rfloor - 1$ is a tight bound for cooperative guards in triangulation graphs, and then they constructed a class of n-vertex polygons with the minimum link number $\text{lk}(\cdot) = \lfloor n/2 \rfloor - 2$, thus getting the following theorem.

THEOREM 1.6 ([3]). *For any n-vertex polygon P without holes, we have* $\text{lk}(P) \leq \lfloor n/2 \rfloor - 2$, *and there exists a class of n-vertex polygons with* $\text{lk}(\cdot) = \lfloor n/2 \rfloor - 2$.

Finally, by reduction to 3-SAT [33], Ahlfeld and Hecker showed that the problem of determining $\text{lk}(P)$ for a given polygon P with holes is NP-hard (for the complexity terminology see Section 1.5). This reduction is similar to an analogous reduction by Fowler and Tanimoto [29], and Suppowit [86].

1.4. Note on the pursuit and evasion problem

In general, in this dissertation we consider only guards which are not allowed to move, thus here we only mention the *pursuit and evasion* problem. In contrast to stationary guards, in the pursuit and evasion problem both a guard (pursuer) and a robber (evader) are allowed to move freely within a polygon. The two primary questions which arise in relation to capturing evaders are:

1. For a given polygon, what is the minimum number of pursuers necessary to capture the evader?
2. Can we efficiently compute a successful strategy to guarantee that the pursuers will capture the evader?

Capture is typically defined as occurring when a pursuer occupies the same position as the evader, or comes to within a predefined distance of him. Interest in problems involving pursuit and evasion is very recent, and variants of this game have been considered in the literature—[13, 36, 37, 44, 68, 78, 80], to cite a few—and involve multiple pursuers, evaders, and various constraints on both the visibility and the space in which pursuit takes place.

Herein we shall briefly discuss only one variation in which a cooperation restriction on a set of pursuers is assumed. In [25] Efrat *et al.* consider the cooperative version of the pursuit and evasion problem, i.e., they assume that guards must always form a simple polygonal chain through a polygon; the guards at the ends of the chain are always on the two edges of the polygon, while the rest are at internal vertices of the chain. All links in the chain are segments inside the polygon. Thus the guards are mutually visible in pairs and are linked together, that is, they form a cooperative guard set. The main results of [25] are the following algorithms.

- An algorithm to compute the minimum number r^* of guards needed to sweep an n-vertex polygon that runs in $O(n^3)$ time and uses $O(n^3)$ working space.
- A faster algorithm, using $O(n \log n)$ time and $O(n)$ space, to compute an integer r such that $\max(r - 16, 2) \leq r^* \leq r$ and the polygon can be swept with a chain of r guards.

The first algorithm is based upon the *link diagram* of a polygon which encodes the link distance between all pairs of points on the boundary of the polygon. In the latter case, the algorithm uses the *link width* of a polygon (see [25] for more details).

Recently, Tan [87] improved the first algorithm by a linear factor, namely, he presented an $O(n^2)$ time algorithm to compute the minimum number r^* of guards required to detect the target, no matter how fast the target moves; a sweep schedule may be reported in $O(n^2 r^*)$ time.

1.5. Algorithmic complexity

The definitions and terminology presented in this section are all standard in complexity theory and have been included for the sake of completeness.

DEFINITION 1.3. Let $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$ be any functions. Then $f(n)$ is $O(g(n))$ if there are constants c and N such that

$$|f(n)| \leq c|g(n)| \quad \text{for all } n > N.$$

For example, the function $f(n) = \sum_{i=1}^{n} i$ is $O(n^2)$, because for all $n > 0$,

$$f(n) = \frac{n^2 + n}{2} < n^2 + n \leq 2n^2.$$

DEFINITION 1.4. Let $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$. Then $f(n)$ is $\Omega(g(n))$ if $g(n)$ is $O(f(n))$.

DEFINITION 1.5. Let $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$. Then $f(n)$ is $\Theta(g(n))$ if $f(n)$ is both $\Omega(g(n))$ and $O(g(n))$, that is, there are constants c_1, c_2 and N such that

$$c_1|g(n)| \leq |f(n)| \leq c_2|g(n)| \quad \text{for all } n > N.$$

In the above example, the function $f(n) = \sum_{i=1}^{n} i$ is $\Theta(n^2)$, as for all $n > 0$ we have $n^2/2 \leq f(n) \leq 2n^2$.

An algorithm performs a sequence of operations on its *input*, the initial symbols and real numbers that specify the instance of the problem to be solved. If I denotes the input, we can let the function $T_A(I)$ denote the number of operations the algorithm A uses until it halts, with I as input. Thus we can associate a *time complexity function* $T_A : \mathbb{N} \to \mathbb{N}$ to every algorithm A by

$$T_A(n) = \max_{|I|=n}\{T_A(I)\},$$

that is, the worst case number of computation steps over all input instances that have size n. The *size* of the input is the number of symbols and real numbers used to specify the problem instance.

DEFINITION 1.6. A *polynomial time* algorithm is an algorithm whose time complexity function (or just running time) is $O(p(n))$ for some polynomial $p(n)$.

Thus a *linear time* algorithm is an algorithm with $O(n)$ running time, and a *quadratic time* algorithm is an algorithm with $O(n^2)$ running time.

DEFINITION 1.7. A *decision problem* is a problem that requires only a 'yes' or 'no' answer regarding whether some element of its domain has a particular property.

DEFINITION 1.8. A decision problem class belongs to the *class P* if there is a polynomial time algorithm to solve the problem.

For example, "Is a given graph connected?" is a decision problem and it is in the class P.

DEFINITION 1.9. A decision problem belongs to the *class NP* if, for every 'yes'-instance of the problem, one can verify in polynomial time that the instance is indeed a 'yes'-instance.

For example, "Does a given graph has a path which traverses all its vertices?" is in the class NP. Clearly, P $\subseteq$ NP, and it is believed that NP is much larger than P, but there is not a single problem in NP for which it has been proved that the problem is not in P. No polynomial time algorithms are known for many problems in NP, but no lower bounds larger than polynomial ones have been proved for these problems.

DEFINITION 1.10. A decision problem R is *polynomially reducible* to a decision problem Q if there is a polynomial time transformation of each instance I_R of problem R to an instance I_Q of problem Q such that the instances I_R and I_Q have the same answer ('yes' or 'no').

DEFINITION 1.11. A decision problem is *NP-hard* if every problem in the class NP is polynomially reducible to it.

DEFINITION 1.12. An NP-hard problem R is *NP-complete* if R is in the class NP.

The first major theorem establishing that the class of NP-complete problems is non-empty, is due to Stephen Cook [20]. Cook proved that SATISFIABILITY is such a problem. SATISFIABILITY takes as input a logical formula expressed as a list of clauses; each clause is a collection of literals (variables or their negations). A clause is considered to be true when at least one of the literals is true. A formula is *satisfiable* if there is an assignment of truth values to the variables that makes every clause true. The question is whether such an assignment exists.

THEOREM 1.7 ([20]). SATISFIABILITY *is NP-complete.*

SATISFIABILITY remains NP-complete even when restricted by requiring that every clause have three literals. The restricted problem is called 3SAT.

THEOREM 1.8 ([49]). 3SAT *is NP-complete.*

2. COOPERATIVE GUARDS

In this chapter, we investigate the cooperative guards problem. First, we give combinatorial bounds for arbitrary polygons. Both Ahlfeld and Hecker's proof [3] and Hernández-Peñalver's proof [39] of the sufficiency of $\lfloor n/2 \rfloor - 1$ cooperative guards are based upon combinatorial cooperative guards in triangulation graphs. In the first case, the proof follows from the deep analysis of all degree 2 vertex chains of maximal length in dual graphs of triangulations, whereas Hernández-Peñalver uses a simple induction, starting with an arbitrary triangulation of a polygon and cutting off a small piece for the induction step (see [3, 39] for more details). Herein we shall present two other proofs: the first one, due to the author [93], based upon a vertex cover of a diagonal graph, and another one, due to Pinciu [76], based upon 3-coloring of a triangulation graph. Additionally, we consider the cooperative guards problem in orthogonal, monotone, spiral and star-shaped polygons. Next, we discuss the complexity of the problem of finding the minimum number of cooperative guards in a given (arbitrary) polygon. Finally, we investigate the cooperative guards problem in polygons with holes.

2.1. Arbitrary polygons

The *diagonal graph* G_D of a triangulation of an n-vertex polygon P is the graph obtained only from $n - 3$ internal diagonals of the triangulation: the edges correspond to the diagonals and the vertices correspond to all endpoints of the diagonals (see Fig. 2.1 for an example).

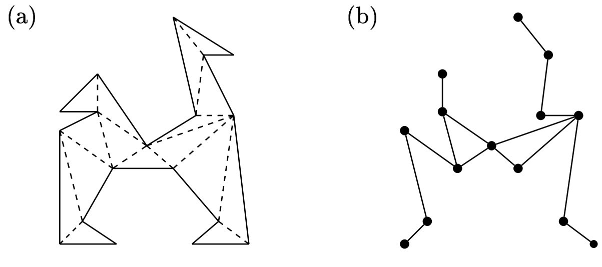

Fig. 2.1. (a) A triangulation of a polygon. (b) Its diagonal graph.

Lemma 2.1 ([93]). *Let T be a triangulation of a polygon, and let G_D be the diagonal graph of T. Then G_D is connected.*

Proof. The proof is by induction on n. As for any quadrilateral its triangulation requires only one internal diagonal, the validity of the assertion for $n = 4$ is established. So assume that $n \geq 4$, and that the assertion holds for all $4 \leq \hat{n} < n$. Let $d = \{x, y\}$ be an edge of G_D. We have two distinct cases: either d cuts off only one vertex from P or it partitions P into two polygons, each with at least four vertices.

CASE 1. Since the polygon P' that results from cutting off one vertex has $n-1$ vertices, the diagonal graph G'_D of any of its triangulations is connected by the induction hypothesis. As in any triangulation of P' there must exist a diagonal with one of its endpoints either at x or at y, the graph G_D with the diagonal $d = \{x, y\}$ is connected (see Fig. 2.2(a)).

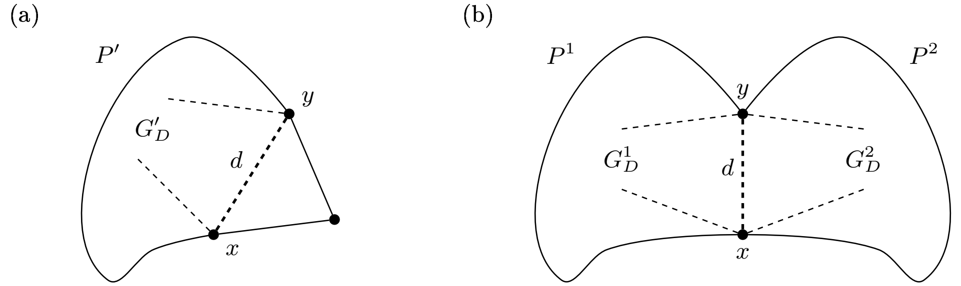

Fig. 2.2. The diagonal graph of a triangulation is connected.

CASE 2. Let P^1 and P^2 be the polygons that result from cutting P along the diagonal d. By the induction hypothesis, the relevant diagonal graphs G^1_D and G^D_2 are connected. As above, in any triangulation T^1 of P^1 there must exist a diagonal with one of its endpoints either at x or at y; analogously, in any triangulation T^2 of P^2 there must exist a diagonal with one of its endpoints either at x or at y. As the diagonal d connects G^1_D and G^2_D into G_D, the graph G_D with the diagonal $d = \{x, y\}$ is connected (see Fig. 2.2(b)). ∎

Recall that a subset $C \subset V$ of vertices is a *vertex cover* of a graph $G = (V, E)$ if each edge in E is incident to at least one vertex in C.

LEMMA 2.2 ([93]). *Let m be the number of edges of a graph G. Then there exists a vertex cover of cardinality at most $\lfloor (m+1)/2 \rfloor$.*

Proof. Let G^* be a tree on n^* vertices that results from splitting some vertices of $G = (V, E)$ without destroying the connectedness of the graph (if G is a tree, then $G^* = G$). For any vertex v of G, let $S(v)$ denote the set of vertices of G^* that results from splitting the vertex v. As any tree is a bipartite graph, there exists a vertex cover C^* of G^* with $|C^*| \leq \lfloor n^*/2 \rfloor$; recall that a graph is *bipartite* if its vertex set can be partitioned into two sets such that no two vertices in the same set are adjacent. Let

$$C = \{v \in V : (v \in C^*) \text{ or } (\exists_{x \in S(v)}\ x \in C^*)\}.$$

Of course, C is a vertex cover of G, and $|C| \leq |C^*|$. As $n^* = m+1$, $|C| \leq \lfloor (m+1)/2 \rfloor$. ∎

Thus by Lemma 2.2, we get

Corollary 2.3 ([93]). *Let G_D be the diagonal graph of a triangulation of an n-vertex polygon. Then there exists a vertex cover of cardinality at most $\lfloor n/2 \rfloor - 1$.*

The crucial fact is the relation between a vertex cover of a diagonal graph and a cooperative guard set of a triangulation graph.

Theorem 2.4 ([93]). *Let T, G_T, G_D be a triangulation of a polygon with $n \geq 4$ vertices, the triangulation graph of T, and the diagonal graph of T, respectively. Then any vertex cover of G_D is a cooperative guard set in G_T.*

Proof. Let $C = \{g_1, \ldots, g_k\}$ be a vertex cover of G_D. To show that C is a guard set in G_T, all we need is to observe that for each (bounded) triangular face in G_T at least one of its edges is an edge of G_D. As C is a vertex cover, any diagonal of G_D has at least one of its endpoints in C, and thus each triangular face of G_T has at least one of its vertices in C. Consequently, C is a guard set.

The next step is to show that the subgraph induced by C is connected, that is, for any two guards $g_i, g_j \in C$, there exists a path $p = (g_i = p_{i_1}, p_{i_2}, \ldots, p_{i_l - 1}, p_{i_l} = g_j)$ in G_T such that all $p_{i_t} \in C$, $t = 1, \ldots, l$. Without loss of generality, let the selected guards be g_1 and g_2. As G_D is connected, there exists a path $p^D = (g_1, v_1^D, \ldots, v_l^D, g_2)$ in G_D. Our purpose is to construct the required path p from p^D.

As C covers G_D, either $v_1^D \in C$ or $v_1^D \notin C$ and $v_2^D \in C$.

Case 1. If $v_1^D \in C$, we do not need any modification of p^D between g_1 and v_1^D.

Case 2: $v_1^D \notin C$ and $v_2^D \in C$. Since G_T is a triangulation graph of a polygon, either g_1 and v_2^D are connected in G_T, or there are some edges in G_T incident to v_1^D.

Subcase 2a. If g_1 and v_2^D are connected in G_T, we modify p^D by removing the vertex v_1^D.

Subcase 2b. Exactly two edges incident to v_1^D are edges of the polygon, and they are located on the same side of p^D (see Fig. 2.3). All the other edges belong to the set of edges of G_D. Let $d_1 = \{v_1^D, x_1\}, \ldots, d_k = \{v_1^D, x_k\}$ be the diagonals of the triangulation T lying on the opposite side to the edges of the polygon. As C covers G_D and $v_1^D \notin C$, all vertices $x_1, \ldots, x_k$ are selected as guards. As G_T is a triangulation graph, there is a path $p^T = (g_1, x_1, \ldots, x_k, v_2^D)$ from g_1 to v_2^D in G_T. Now, we replace the subpath (g_1, v_1^D, v_2^D) in p^D with p^T.

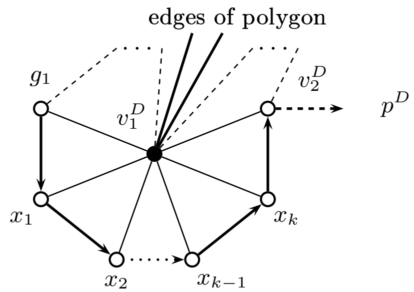

Fig. 2.3. A modification of the path p^D.

Applying the same construction to all successive vertices, we construct the required path $p = p^D$ from g_1 to g_2 that consists only of guards in C. Thus C is a cooperative guard set in G_T. ∎

We note in passing that any cooperative guard set in a triangulation graph is a vertex cover of the diagonal graph.

LEMMA 2.5 ([93]). *Let T, G_T, G_D be any triangulation of a polygon with $n \geq 4$ vertices, the triangulation graph of T, and the diagonal graph of T, respectively. Then any cooperative guard set C in G_T is a vertex cover of G_D.*

Proof. Let C be a cooperative guard set in G_T, and let $d = \{v_1, v_2\}$ be an edge of G_D. Suppose, contrary to our claim, that neither $v_1 \in C$ nor $v_2 \in C$. The edge d partitions G_T into two graphs G_T^1 and G_T^2. As both G_T^1 and G_T^2 are guarded by some guards in C, there must be a path p from a guard in G_T^1 to a guard in G_T^2 that consists only of guards. Clearly, every path from G_T^1 to G_T^2 must consist of either v_1 or v_2, and thus, if there are no guards at v_1 nor v_2, C is not a cooperative guard set in G_T—a contradiction. ∎

We are thus led to the following theorem.

THEOREM 2.6 ([3, 39, 76, 93]). *A triangulation graph of a polygon with $n \geq 4$ vertices can always be dominated by $\lfloor n/2 \rfloor - 1$ cooperative guards.*

Proof. Let P be a polygon with n vertices, and let G_D be the diagonal graph of a triangulation of P. By Corollary 2.3, there exists a vertex cover C in G_D of cardinality at most $\lfloor n/2 \rfloor - 1$, and by Theorem 2.4, C is a cooperative guard set dominating the triangulation graph. ∎

Thus by the theorem above and Lemma 1.4, we have

THEOREM 2.7 ([3, 39, 76, 93]). *For $n \geq 4$, $\mathrm{cg}(n) = \lfloor n/2 \rfloor - 1$, and the guards can be restricted to vertices of a polygon only.*

2.1.1. 3-coloring argument. In this section, we present another proof of Theorem 2.7 proposed by Pinciu [76]. The proof is based upon 3-coloring and it is constructive, that is, it can be easily converted into an algorithm. The algorithm is as follows:

1. Find a triangulation graph G_T of an n-vertex polygon P.
2. Find a 3-coloring for G_T, that is, a map from the vertex set to the color set $\{1, 2, 3\}$ such that adjacent vertices receive different colors (for the existence of such a coloring, see for example [73]).
3. Delete every diagonal in G_T that connects a color 2 vertex with a color 3 vertex. Let G'_T be the resulting graph. G'_T gives a partition of P into triangles and quadrilaterals. Let T and Q be the sets of triangles and quadrilaterals in this partition, respectively.
4. Find the dual graph of G'_T.
5. Since the dual graph of G'_T is a tree (Lemma 1.2), we can assign a + or − sign to each vertex so that any two adjacent vertices have opposite sings.

6. Assign + or − signs to each bounded face of G'_T, where each face is given the sign of the corresponding vertex in the dual tree of G'_T. We obtain the partitions $T = T^+ \cup T^-$, and $Q = Q^+ \cup Q^-$.
7. Define a function $f : T \to \{1, 2, 3\}$ such that for all $t \in T$, $f(t) = 1$ if t is adjacent to another triangle of T, and $f(t)$ is the color of a 2- or 3-colored vertex that t shares with an adjacent quadrilateral and is not colored 1, otherwise.
8. Find the partition $f^{-1}(\{2, 3\}) = T_1 \cup T_2$, where

$$T_1 = (f^{-1}(\{2\}) \cap T^+) \cup (f^{-1}(\{3\}) \cap T^-),$$
$$T_2 = (f^{-1}(\{2\}) \cap T^-) \cup (f^{-1}(\{3\}) \cap T^+).$$

   Note that we can assume that $|T_1| \leq |T_2|$, otherwise we switch the + and − signs for all faces of G'_T.
9. For every triangle $t \in f^{-1}(\{1\}) \cup T_1$, we define $g(t)$ to be the unique vertex of t that has color 1. For every quadrilateral $q \in Q^+$, we define $g(q)$ to be the unique vertex of Q that has color 2. And, for every quadrilateral $q \in Q^-$, we define $g(q)$ to be the unique vertex of Q that has color 3.
10. Let $S = g(f^{-1}(\{1\}) \cup T_1 \cup Q)$.

THEOREM 2.8 ([76]). *S is a cooperative guard set in the polygon P, and $|S| \leq \lfloor (n-2)/2 \rfloor$.*

Proof. First, note that by construction the triangles and quadrilaterals in the partition of P satisfy the following properties:

(i) Every triangle $t \in T$ has three vertices, colored 1, 2, and 3, respectively, and any point inside t is visible from any vertex of t.

(ii) Every quadrilateral $q \in Q$ has two vertices of color 1, one vertex of color 2, and one vertex of color 3. Every point inside q is visible from a vertex of color 2 or 3. In particular, every point inside q is visible from $g(q)$.

(iii) As there are no diagonals connecting vertices of color 2 and 3 in G'_T, it follows that if two triangles are adjacent, then they share a vertex of color 1.

(iv) By the same argument, if a triangle and a quadrilateral are adjacent, they must share a vertex of color 1 and a vertex of color 2 or 3.

(v) Finally, if two quadrilaterals are adjacent, they must share a vertex of color 1 and a vertex of color 2 or 3.

Let x be a point in P. Then x is either in a triangle $t \in T$ or in a quadrilateral $q \in Q$. If $x \in Q$, then x is visible from $g(q) \in S$. If $x \in t$, where $t \in f^{-1}(\{1\}) \cup T_1$, then x is visible from $g(t) \in S$. If $x \in t$, where $t \in T_2$, then t must be adjacent to a quadrilateral q that has opposite sign to that of t. From the way T_2 and $g(q)$ were defined, it is easy to see that $g(q)$ is a vertex of t, so x is visible from $g(t) \in S$. Therefore S is a guard set.

Next, we will show that any two adjacent faces of G'_T share a vertex g such that $g \in S$. This together with properties (i) and (ii), and the fact that the dual graph of G'_T is connected, will imply that S is a cooperative guard set. Indeed, if both faces are triangles, then they must be in $f^{-1}(\{1\})$, thus by (iii), they share a vertex of color 1, and this vertex is in S. If both faces are quadrilaterals, say q_1 and q_2, they share a vertex of color 2 or 3 by property (v). Since q_1 and q_2 are adjacent, they must have opposite signs, so the two quadrilaterals must share either $g(q_1)$ or $g(q_2)$, which are in S. Finally, assume that one is a triangle t, and the other is a quadrilateral q. If $t \in T_1$, then the vertex of color 1 shared by t and q is $g(t) \in S$. Otherwise, if $t \in T_2$, then t and q share $g(q)$. Therefore S is a cooperative guard set.

We finish the proof by showing that $S \le \lfloor (n-2)/2 \rfloor$. Indeed,

$$\begin{aligned}|S| &= |g(f^{-1}(\{1\}) \cup T_1 \cup Q)| \le |g(f^{-1}(\{1\}))| + |g(T_1)| + |g(Q)| \\ &\le \frac{1}{2}|f^{-1}(\{1\})| + |g(T_1)| + |g(Q)| \le \frac{1}{2}|f^{-1}(\{1\})| + |T_1| + |Q| \\ &\le \frac{1}{2}|f^{-1}(\{1\})| + \frac{1}{2}|f^{-1}(\{1,2\})| + |Q| = \frac{1}{2}|T| + |Q| = \frac{1}{2}(|T| + 2|Q|) = \frac{1}{2}(n-2). \ \blacksquare\end{aligned}$$

2.2. Miscellaneous shapes

Five generic shapes of polygons have been distinguished in the literature: convex, orthogonal, monotone, spiral, and star-shaped. If any straight line segment joining two interior points of a polygon lies entirely within the polygon, then the polygon is called *convex*. Convex polygons obviously do not lead to interesting theorems: two cooperative guards always suffice ([1]).

2.2.1. Orthogonal polygons. If each edge of a polyline $\bar{P}$ is parallel either to the x-axis or to the y-axis, then the polygon P is called *orthogonal* (see Fig. 2.4(a)).

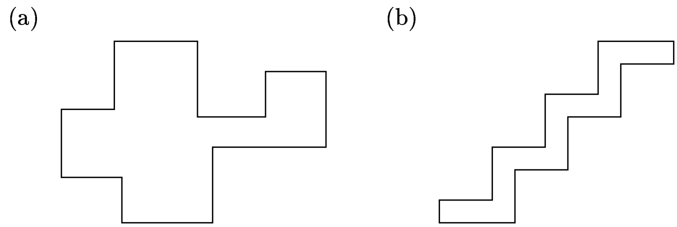

Fig. 2.4. (a) An orthogonal polygon. (b) An orthogonal polygon with n vertices that requires $\lfloor n/2 \rfloor - 2$ cooperative guards; here $n = 16$, and the polygon needs 6 guards.

THEOREM 2.9 ([45]). *Every orthogonal polygon (with or without holes) ([2]) is convexly quadrilateralizable, that is, it may be partitioned by internal diagonals between vertices into interior disjoint convex quadrilaterals.*

LEMMA 2.10 ([73]). *For any quadrilateralization of an orthogonal polygon with n vertices into q quadrilaterals, $n = 2q + 2$.*

Proof. The sum of interior angles of an orthogonal polygon with n vertices is $180(n-2)$ degrees. But since there are q quadrilaterals, each of 360 degrees, $360q = 180(n-2)$, and thus $n = 2q + 2$. ■

([1]) We assume that a polygon requires at least two cooperative guards.

([2]) A formal definition of a *polygon with holes* is introduced in Section 2.4.1.

The CG problem in orthogonal polygons was solved by Hernández-Peñalver [39].

THEOREM 2.11 ([39]). *For $n \geq 6$, $\mathrm{cg}_{\perp}(n) = \lfloor n/2 \rfloor - 2$, that is, $\lfloor n/2 \rfloor - 2$ cooperative guards always suffice and are sometimes necessary to cover an n-vertex orthogonal polygon, and the guards can be restricted to vertices only.*

Proof. The necessity is established by the polygon from Fig. 2.4(b), an orthogonal version of the snake-polygon from Fig. 1.3. For sufficiency, consider a convex quadrilateralization Q of a polygon P, and let q be the number of quadrilaterals (the existence of Q is guaranteed by Theorem 2.9). Let S be the set of guards obtained by placing a guard at every diagonal of P that is shared by two quadrilaterals. Clearly, S covers all of P, as quadrilaterals form a convex partition of the polygon. And it is easy to see that the visibility graph $\mathrm{VG}(S)$ is connected. By Lemma 2.10, the cardinality of S is $q - 1 = \lfloor n/2 \rfloor - 2$. ■

2.2.2. Monotone polygons. A polygonal chain $v_1, \ldots, v_n$ is called *monotone with respect to a line* L if the projections of $v_1, \ldots, v_n$ onto L are in the same order as in the chain, that is, there is no "doubling back" in the projection as the chain is traversed. Two adjacent vertices p_i and p_{i+1} may project to the same point on L without destroying monotonicity. A chain is called *monotone* if it is monotone with respect to at least one line. We will use the convention that the line of monotonicity is the x-axis. A polygon P is *monotone* if its polyline $\bar{P}$ can be partitioned into two chains monotone with respect to the same line (see Fig. 2.5); we will call them the *bottom* and *top* chains.

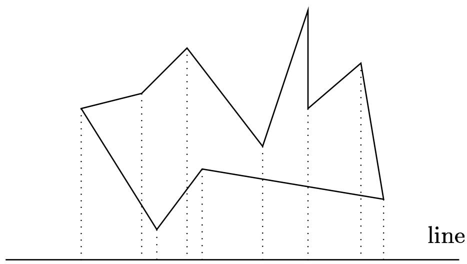

Fig. 2.5. A monotone polygon.

As Hernández-Peñalver's polygon requires as many as $\lfloor n/2 \rfloor - 1$ cooperative guards, we have the following corollary.

COROLLARY 2.12 ([95]). *For $n \geq 4$, $\mathrm{cg}_{\mathrm{monotone}}(n) = \lfloor n/2 \rfloor - 1$, that is, $\lfloor n/2 \rfloor - 1$ cooperative guards always suffice and are sometimes necessary to cover an n-vertex monotone polygon, and the guards can be restricted to vertices only.*

2.2.3. Spiral polygons. Recall that a vertex is called *reflex* if its interior angle is greater than $\pi/2$; otherwise, it is called *convex*. A chain $(x, r_1, \ldots, r_k, y)$, where $(r_1, \ldots, r_k)$ is a maximal chain of reflex vertices, is called a *reflex chain*. Note that x and y are convex vertices. Similarly, a chain of vertices $(x, c_1, \ldots, c_l, y)$ is called a *convex chain* if $(c_1, \ldots, c_l)$ is a maximal chain of convex vertices (x and y are reflex vertices). A k-*spiral* polygon is a polygon whose boundary chain has exactly k reflex chains and k convex chains (see Fig. 2.6(b)).

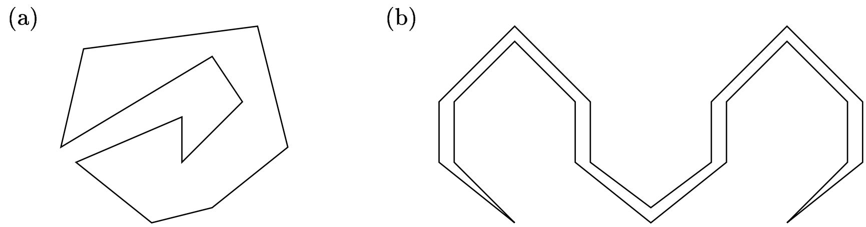

Fig. 2.6. (a) A 1-spiral polygon. (b) A k-spiral polygon with n vertices that requires $\lfloor n/2 \rfloor - 1$ cooperative guards; here $n = 24$, $k = 3$, and the polygon needs 11 guards.

Fig. 2.6(b) shows a k-spiral polygon that requires $\lfloor n/2 \rfloor - 1$ cooperative guards. Thus by Theorem 2.7, we have the following corollary.

COROLLARY 2.13 ([95]). *For $n \geq 4$, $\text{cg}_{\text{spiral}}(n) = \lfloor n/2 \rfloor - 1$, that is, $\lfloor n/2 \rfloor - 1$ cooperative guards always suffice and are sometimes necessary to cover a k-spiral polygon with n vertices, and the guards can be restricted to vertices only.*

2.2.4. Star-shaped polygons. A *star-shaped* polygon is a polygon whose kernel is non-empty. The *kernel* of a polygon is the set of points in the interior of the polygon from which the entire polygon is visible (see Fig. 2.7(a)). It is easy to see that the kernel is the intersection of all interior half-planes determined by the edges of a polygon (*interior half-planes* are towards the left in a counterclockwise traversal of the boundary). Thus the kernel is a convex polygon.

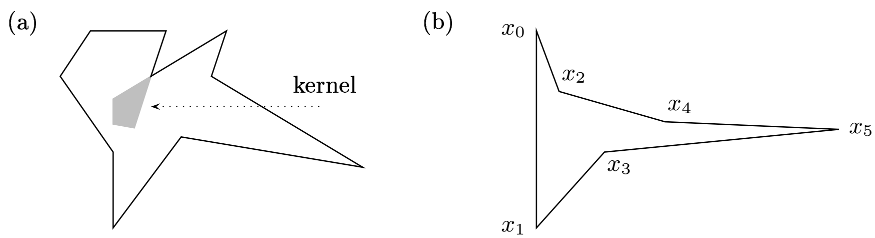

Fig. 2.7. (a) A star-shaped polygon. (b) A 6-vertex star-shaped polygon that requires 2 cooperative vertex guards.

Although every star-shaped polygon may be covered by two cooperative guards, a more interesting question arises if guards are restricted to be vertex guards. Of course, the proof of Theorem 2.7 gives us that $\lfloor n/2 \rfloor - 1$ cooperative vertex guards always suffice to cover any star-shaped polygon. Fig. 2.7(b) shows a star-shaped polygon P with six vertices with $\text{cg}_{\text{star}}(P) = 2$ if guards are restricted to vertices. A simple extension of this polygon—the vertex x_{2i} does not see any x_j, for $j \geq 2i + 3$, and x_{2i+1} does not see any vertex x_j, for $j \geq 2i + 4$—leads to a class of star-shaped polygons that require as many as $\lfloor n/2 \rfloor - 1$ cooperative vertex guards.

THEOREM 2.14 ([95]). *For every $n \geq 6$, there exists a star-shaped polygon that requires $\lfloor n/2 \rfloor - 1$ cooperative vertex guards.*

Proof. First, note that we only have to treat the case $n \equiv 0 \pmod 2$, as this is the critical value of n for which $\lfloor n/2 \rfloor - 1 > \lfloor (n-1)/2 \rfloor - 1$; we may always add one vertex to our polygon to deal with $n \equiv 1 \pmod 2$.

The case of $n = 6$ is established by the polygon shown in Fig. 2.7(b). Before constructing a star-shaped polygon P_{star} with $n = 2m$ vertices ($m \geq 3$) that satisfies the $m - 1$ bound, we need some definitions.

Let P be a star polygon, x be a point of P, and $d = \{v_1, v_2\}$ be an edge of P that is entirely visible from x (see Fig. 2.8(a)). Denote by $\alpha(x, d)$ the set of all points from the exterior of P that are visible from x after deleting d; $\alpha(x, d)$ is the open trapezoid delimited by d and two lines with endpoints v_1 and v_2.

Let r_1, v_2, r_3 be three consecutive vertices of P, in counterclockwise order, of which only v_2 is convex. Let $d = \{v_2, r_3\}$. Denote by $\beta(d)$ the set of all vertices of P that are visible from some point of d, excluding the endpoint r_3 (see Fig. 2.8(b)).

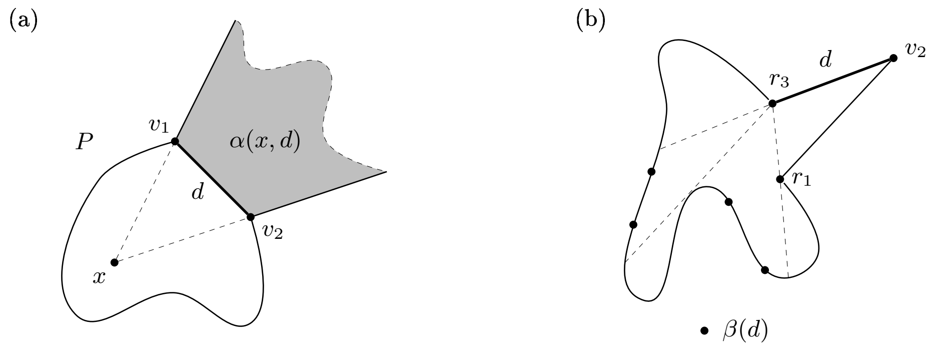

Fig. 2.8. An illustration of the definitions of (a) open trapezoid $\alpha(x, d)$, and (b) $\beta(d)$.

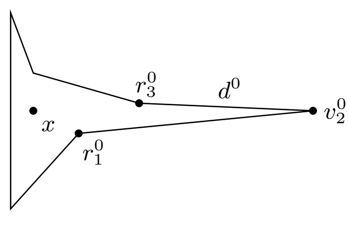

Fig. 2.9. A polygon P^0.

Again consider the polygon P^0 from Fig. 2.9. The idea of the construction of P_{star} starts with some observations:

- $\beta(d^0) = \{r_1^0, v_2^0, r_3^0\}$.
- P^0 requires two cooperative vertex guards, and one of them must be located either at r_1^0 or at r_3^0.

- None of these two cooperative vertex guards covering P^0 can be located at the endpoint v_2^0 of d^0, otherwise, a third guard is needed.
- Even with an additional point guard at an internal point of d^0, the polygon P^0 still requires two cooperative vertex guards.

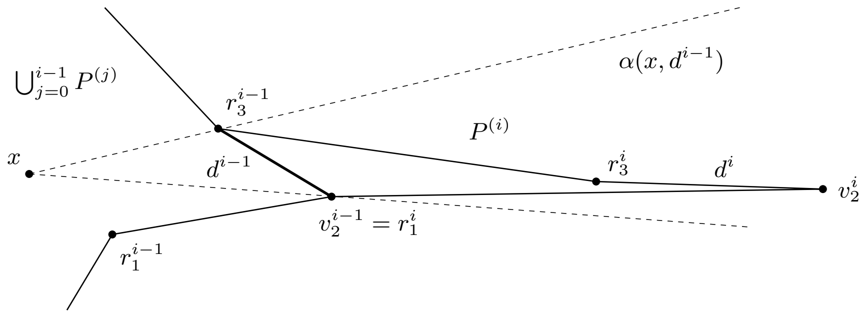

Fig. 2.10. An illustration of the construction of the polygon P_{star}.

Next, let x be a point from the kernel of P^0. For each $i = 1, \ldots, m-3$ in turn, we adjoin a 4-vertex polygon $P^{(i)} \subset \alpha(x, d^{i-1})$ at the diagonal d^{i-1} of the polygon $P^{(i-1)}$ (see Fig. 2.10). Each $P^{(i)}$ can be guarded from x, thus the polygon $P_{\text{star}} = \bigcup_{i=0}^{m-3} P^{(i)}$ is star-shaped, and it has $2m$ vertices.

The necessity of $m-1$ cooperative vertex guards is established by induction. Let S be a minimum cooperative vertex guard set for $P_{\text{star}} = \bigcup_{i=0}^{m-4} P^{(i)} \cup P^{(m-3)}$, $m \geq 4$. The following claim is crucial:

$$\left| S \cap \bigcup_{i=0}^{m-4} P^{(i)} \right| \geq (m-1) - 1.$$

Reason: suppose that there is a $g \in S$ located at v_2^{m-3}. Consider the guard set resulting from moving g along the line $l \supseteq xv_2^{m-3}$ towards x to the new location $p = l \cap d^{m-4}$. Clearly, such a move increases the visibility area of g in $\bigcup_{i=0}^{m-4} P^{(i)}$. However, by the induction hypothesis, with the new guard g at the point $p \neq r_3^{m-4}$ of d^{m-4}, the polygon $\bigcup_{i=0}^{m-4} P^{(i)}$ still requires $(m-1)-1$ cooperative vertex guards located in $\bigcup_{i=0}^{m-4} P^{(i)}$. A similar argument can be applied for a guard at r_3^{m-3}.

Consequently, by the induction hypothesis and by the above claim:

- $\beta(d^{m-4}) = \{r_1^{m-4}, v_2^{m-4}, r_3^{m-4}\}$, thus no point of the edge d^{m-3} is seen from a vertex of the polygon $\bigcup_{i=0}^{m-4} P^{(i)}$, except for $v_2^{m-4} = r_1^{m-3}$, hence $\beta(d^{m-3}) = \{r_1^{m-3}, v_2^{m-3}, r_3^{m-3}\}$.
- We do not need any additional guard only if there is a guard at v_2^{m-4}, but this case requires $[(m-1)-1]+1$ cooperative vertex guards for $\bigcup_{i=0}^{m-4} P^{(i)}$, thus we get $m-1$ cooperative vertex guards for P_{star}.
- We need one additional guard for $P^{(m-3)}$ either at $r_1^{m-3} = v_2^{m-4}$ or r_3^{m-3}. Together with $m-2$ cooperative vertex guards for $\bigcup_{i=0}^{m-4} P^{(i)}$, we have $m-1$ cooperative vertex guards for P_{star}.

- If we require a guard at v_2^{m-3}, then two additional guards are needed, as the guard at v_2^{m-3} requires a guard either at $r_1^{m-3} = v_2^{m-4}$ or r_3^{m-3}. Together with $m-2$ cooperative vertex guards for $\bigcup_{i=0}^{m-4} P^{(i)}$, we have m cooperative vertex guards for P_{star}.
- By similar arguments to the proof of the above claim, an additional point guard at an internal point of the edge d^{m-3} does not change the necessary number of cooperative vertex guards for $\bigcup_{i=0}^{m-3} P^{(i)}$.

Hence P_{star} with $2m$ vertices requires $m-1 = \lfloor n/2 \rfloor - 1$ cooperative vertex guards. ■

Corollary 2.15 ([95]). *For $n \geq 6$, $\text{cg}_{\text{star-shaped}}^{\text{vertex}}(n) = \lfloor n/2 \rfloor - 1$, that is, $\lfloor n/2 \rfloor - 1$ cooperative vertex guards always suffice and are sometimes necessary to cover an n-vertex star-shaped polygon.*

2.3. Minimum cooperative guards problem

The proof of Theorem 2.7 can be converted into an algorithm that covers an n-vertex polygon with $\lfloor n/2 \rfloor - 1$ cooperative vertex guards:

1. Triangulate the polygon ($O(n)$, [16]);
2. Find a minimum vertex cover of the diagonal graph ($O(n)$, [8]).

Although $\lfloor n/2 \rfloor - 1$ cooperative guards are necessary in some cases, this is often much more than needed to cover a particular polygon.

Example. Let G_D be a diagonal graph of an n-vertex polygon P. Let $S(P)$ and $S_{\text{OPT}}(P)$ denote the number of cooperative guards obtained by the above algorithm, and the minimum number of cooperative guards that cover P, respectively. It is natural to ask about

$$\lim_{n\to\infty} \max \frac{S(P)}{S_{\text{OPT}}(P)},$$

that is, how the result obtained can differ from the optimal solution.

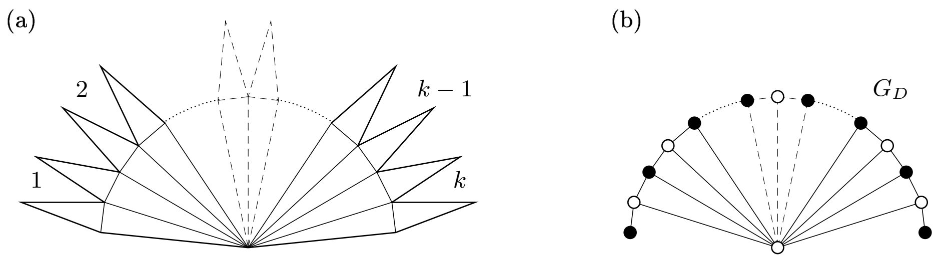

Fig. 2.11. A $(4k+2)$-vertex star-shaped polygon requires 2 cooperative vertex guards; (a) its triangulation T, (b) a minimum vertex cover of G_D is of cardinality $\Theta(k)$.

Consider the polygon P with $4k+2$ vertices, its triangulation T, and the corresponding diagonal graph G_D shown in Fig. 2.11. It is clear that any minimal vertex cover of G_D is of cardinality $\Theta(k)$, and as P is a star-shaped polygon with one of its vertices in the kernel, it can be guarded by two cooperative vertex guards. Thus

$$\lim_{n\to\infty} \max \frac{S(P)}{S_{\mathrm{OPT}}(P)} = \infty,$$

that is, this algorithm performs arbitrarily badly.

It is then natural to seek a placement of the minimum number of cooperative guards that cover a given polygon.

DEFINITION 2.1. Let P be a polygon. The *minimum cooperative guards* (MinCG for short) *problem* is to find a cooperative guard set for P of the minimum cardinality.

The MinCG problem is fundamentally intractable: it is NP-hard. This fact immediately follows from the proof in [56] establishing the NP-hardness of the original art gallery problem. In [56] an instance of the Boolean three satisfiability problem (3SAT) is reduced to the minimum guard problem for simple polygons. There is a set S of minimum number of guards stationed in the transformed polygon P if and only if the relevant instance of 3SAT is satisfiable. It can be seen that the visibility graph $\mathrm{VG}(S)$ is connected due to the transformation itself.

THEOREM 2.16 ([58]). *The minimum cooperative guards problem is NP-hard.*

As in Chapter 4 we shall use Lee and Lin's technique to prove NP-hardness of the minimum k-guarded guards problem, we will not present the original proof here (for the proof we refer the reader to [2] or [56]).

2.3.1. MinCG problem in spiral polygons. As the MinCG problem is NP-hard for general polygons, it is natural to try to solve it on a restricted class of polygons. Liaw *et al.* show that the minimum cooperative guards problem for 1-spiral and 2-spiral polygons can be solved in linear time [58].

Let P be a 1-spiral polygon, and let RC and CC be its reflex and convex chains, respectively. Traversing the boundary of P counterclockwise, the *starting* (*ending*) vertex of RC is v_s (v_e). Starting from v_s (v_e), let us draw a line along the first (last) edge of RC, until it hits the boundary of P at l_1 (r_1). This line segment $v_s l_1$ ($v_e r_1$) and the first (last) part of CC starting from v_s (v_e) form a region called the *starting* (*ending*) *region*. Fig. 2.12 shows an example. Note that there must be a guard stationed in both starting and ending regions.

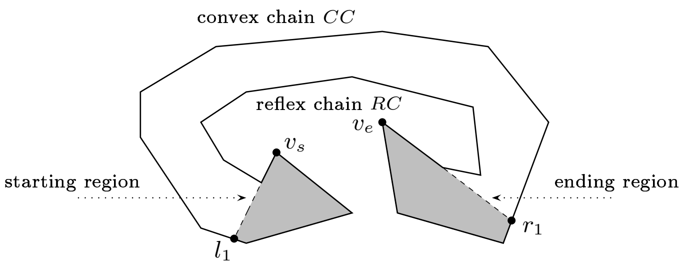

Fig. 2.12. A 1-spiral polygon.

Let x be a point in P. Given a reflex chain RC, we can draw two tangents with respect to RC from x. If the exterior of RC lies entirely to the right (left) of the tangent drawn from x, we call it the *left* (*right*) *tangent* of x with respect to RC. Now, let us draw the left (right) tangent of x with respect to RC until it hits the boundary of P at y. We call xy the *left* (*right*) *supporting line segment* with respect to x, and we call y the *ending point.*

The greedy algorithm MinCGA proceeds as follows. It starts by placing a guard at l_1. Then we find the point l_2 on the convex chain such that $l_1 l_2$ is the left supporting line segment with respect to l_1—the point l_2 is selected as a guard. If l_2 is in the ending region, then we are done. Otherwise, we repeat the process until the ending point of our newly created left supporting line segment is in the ending region (see Fig. 2.13).

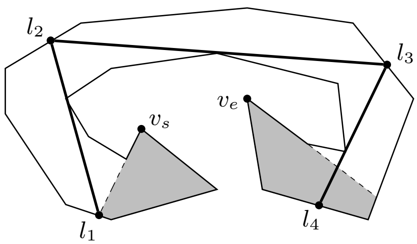

Fig. 2.13. A minimum set of cooperative guards $\{l_1, l_2, l_3, l_4\}$ in a 1-spiral polygon.

Theorem 2.17 ([58]). *The algorithm* MinCGA *is optimal for the* MinCG *problem on any* 1*-spiral polygon* P.

Before commencing the proof, let us recall the following lemma.

Lemma 2.18 ([70]). *There exists an optimal guard placement for a* 1*-spiral polygon in which all guards are stationed on the convex chain.*

Proof. For an edge $e = \{x, y\}$ from the reflex chain, we define *essential line segments* as the line segments that are extensions of e (see Fig. 2.14). The extension defines two essential line segments, the *forward* and *backward* essential line segments, xx_l and yy_r, respectively.

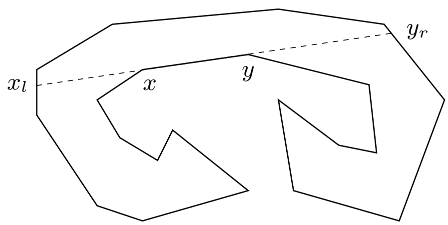

Fig. 2.14. The forward and backward essential line segments.

Now, let g be a guard taken from a cooperative guard cover. Then g is either to the left of some forward essential line segment and to the right of some backward line segment or to the right of some forward essential line segment and to the left of some backward line segment (these essential line segments are provided by two different edges). The line passing through g and the intersection point of these two essential line segments intersects the convex chain at some point p. Let us move g to p. It is easy to see that g sees all points that were seen from the previous location of g. Therefore the whole polygon remains covered, and the visibility graph of the transformed guard placement is connected. ∎

Proof of Theorem 2.17. Let P be a 1-spiral polygon, and let RC and CC be its reflex and convex chains, respectively. Let $S = \{g_1, \dots, g_m\}$ be a minimum cooperative guard set of P with all guards located at CC, as Lemma 2.18 guarantees, and with $g_1 < \cdots < g_m$. In any minimum cooperative guard set, there is a guard g_1 in the starting region of P. It is easy to see that the guard set $S^1 = \{g_1', g_2, \dots, g_m\}$, where the guard g_1' is located at l_1, is a cooperative guard set for P, as the region covered by g_1' is at least as large as that covered by g_1. Consequently, by a simple induction proof, it is easy to show that for all $i = 1, \dots, m$, the guard set $S^{(i)} = \{g_1', \dots, g_i', g_{i+1}, \dots, g_m\}$, where $g_1', \dots, g_i'$ are located at $l_1, \dots, l_i$, is a cooperative guard set for P, as the region covered by $g_1', \dots, g_i'$ is at least as large as that covered by $g_1, \dots, g_i$. ∎

COROLLARY 2.19 ([58]). *The algorithm* MinCGA *runs in linear time.*

Note that by the symmetry of the starting and ending regions, we can start from r_1 and successively find the right supporting line segments until the ending point of the last one is in the starting region.

Liaw *et al.* [58] also proposed an optimal linear algorithm for 2-spiral polygons; the algorithm is based upon the observation that we can divide a 2-spiral polygon into three specific subpolygons, two of which are 1-spiral polygons, and positions of guards can be determined by matching the partial solutions for these 1-spiral polygons. The complete analysis of all cases can be found in [23].

THEOREM 2.20 ([23, 58]). *The minimum cooperative guards problem for 2-spiral polygons can be solved in linear time.*

Finally, let us mention that Liaw *et al.* [58] also considered the *constrained* MinCG problem for 1-spiral polygons. The constrained version of the problem is the same as the original one except that a specified point must be included in the solution. By a slight modification of the algorithm for the non-constrained case, they obtained the following theorem.

THEOREM 2.21 ([58]). *There is a linear time algorithm to solve the constrained* MinCG *problem for 1-spiral polygons.*

2.4. Polygons with holes

A *polygon with holes* is a polygon P enclosing several other polygons $H_1, \dots, H_h$, known as *holes.* $H_1, \dots, H_h$, $\bar{P}$ are mutually disjoint. Similarly to polygons without holes, define $\mathrm{CG}(P)$ to be the minimum cardinality of a cooperative guard set for P. Next, define cg(n, h) to be the maximum value of CG(P) over all polygons with h holes and n vertices in total, i.e., counting vertices on the holes as well as on the outer boundary of P. The function cg(n, h) represents the maximum number of cooperative guards that are ever needed for an n-gon with h holes.

2.4.1. Polygons with one hole. In this section, we give a tight bound for the case of $h = 1$. We show that cg$(n, 1) = \lfloor (n-1)/2 \rfloor$, even for vertex guards. The idea of the proof of the sufficiency of $\lfloor (n-1)/2 \rfloor$ cooperative vertex guards for polygons with one hole follows the main outline of Shermer's proof for vertex (arbitrary) guards [82, 83].

In [82, 83] Shermer establishes that $\lfloor (n+1)/3 \rfloor$ is the tight bound for arbitrary vertex guards in any one-hole polygon. To show the sufficiency of this bound, Shermer uses an arbitrary triangulation (3) of the polygon. This triangulation must contain a cycle of triangles, i.e., the cycle of triangles corresponding to the cycle in the dual graph surrounding the hole. Shermer first shows that to prove the sufficiency of $\lfloor (n+1)/3 \rfloor$ vertex guards for any triangulation, it is enough to provide a proof for a reduced triangulation. A *reduced triangulation* is a triangulation such that every subgraph of its dual graph G that can be disconnected from G by the removal of a single edge, has exactly one vertex. In some of these triangulations, it is not possible to pick $\lfloor (n+1)/3 \rfloor$ vertex guards so that every triangle has a guard at one of its vertices. Shermer calls these configurations *tough triangulations* and makes a case study to show that in each situation $\lfloor (n+1)/3 \rfloor$ vertex guards are still sufficient. Our approach is similar. Before commencing the proof, let us establish the following lemma.

LEMMA 2.22 ([96]). *Let G_T be a triangulation graph of a hole-free polygon P with $n \geq 3$ vertices, and let $e = \{v_1, v_2\}$ be an edge of P. Then:*

(a) *if n is odd, then $\lfloor (n-1)/2 \rfloor$ cooperative guards with one guard placed at any endpoint of e suffice to dominate G_T (4);*
(b) *otherwise, $\lfloor (n-1)/2 \rfloor$ cooperative guards with one guard placed either at v_1 or at v_2 suffice to dominate G_T.*

Proof. The validity of the assertion for odd n follows immediately from Theorem 2.6 establishing that $\lfloor n/2 \rfloor - 1$ cooperative vertex guards suffice to dominate any triangulation graph of an n-vertex polygon. If n is odd, then $\lfloor n/2 \rfloor - 1 = \lfloor (n-1)/2 \rfloor - 1$ cooperative guards dominate G_T, and with one additional guard at any endpoint of e, we get a domination by $\lfloor (n-1)/2 \rfloor$ cooperative guards.

Now, assume n to be even. Let $G_T^\star$ be the graph that results from adjoining a triangle t at the edge e in G_T. It is clear that $G_T^\star$ is a triangulation graph of an $(n+1)$-vertex polygon, and by Theorem 2.6, it can be dominated by $\lfloor (n-1)/2 \rfloor$ cooperative vertex guards. Any triangular face of $G_T^\star$ has at least one of its vertices selected as a guard, thus there is a guard either at v_1 or at v_2. The same guard placement in G_T will dominate G_T. ∎

(3) The existence of a triangulation of a polygon with holes is established by Lemma 2.32 in Section 2.4.2, so in this section we take it for granted.

(4) Here we assume that a triangle may be covered by one cooperative guard.

We say that a triangulation of a one-hole polygon is *basic* if the dual graph of the triangulation is a cycle (surrounding the hole) (see Fig. 2.15(a)). Let P be a polygon with one hole, and let T be one of its triangulations. Suppose T to be basic. A cycle triangle of T is *based* on the inner boundary if it has exactly one vertex, its apex, on the outer boundary, and based on the outer boundary if its apex vertex is on the inner boundary. Let us label a cycle triangle "t". Then T is represented by a string of characters over the alphabet $\{t, /\}$, formed by concatenating all the labels of the cycle triangles, and inserting "/" between labels t_1 and t_2 if the triangle t_1 is based on the inner boundary and the triangle t_2 is based on the outer boundary, or vice versa. Thus each "/" records a switch in basing. This string of characters will be called the *string associated with T*.

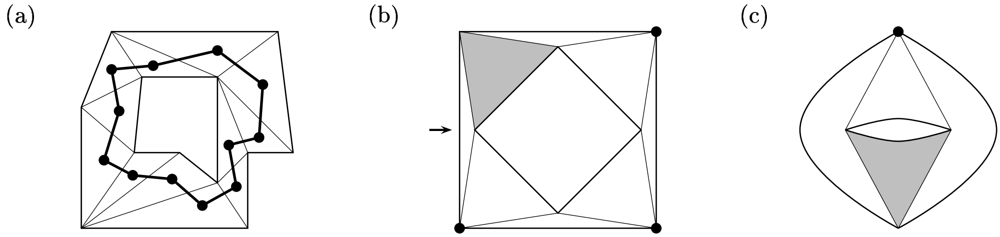

Fig. 2.15. (a) A basic triangulation—its dual graph is a cycle. (b) A triangulation graph on 8 vertices with string $(t/)^8$ that requires four combinatorial cooperative guards: the 3 shown (dots) do not cover the shaded triangle. (c) A triangulation graph on 4 vertices with string $(t/)^4$ that requires 2 combinatorial cooperative guards: the 1 shown (dot) does not cover the shaded triangle.

Fig. 2.15(b) shows an example. Starting at the indicated leftmost triangle and proceeding counterclockwise, we obtain the string $t/t/t/t/t/t/t/t/$. We employ the standard regular expression notation to condense the strings: s^k for k repetitions of a string s. Thus the above string is equivalent to $(t/)^8$. We consider two strings s_1 and s_2 to be equivalent if s_1 is a cyclic shift of s_2, or a cyclic shift of the reverse of s_2. Finally, note that the strings make no distinction between the inner and outer boundaries.

The main difficulty in the sufficiency proof is the existence of triangulation graphs that require as many as $\lfloor n/2 \rfloor$ combinatorial cooperative guards for a complete domination.

LEMMA 2.23 ([96]). *The triangulation graph of a basic triangulation T of an n-vertex polygon with one hole requires $\lfloor n/2 \rfloor$ combinatorial cooperative guards for a complete domination if and only if the string for T has the form $(t/)^{2k+4}$, $k \geq 0$.*

We will call a string that is an instance of $(t/)^{2k+4}$ *tough*. Fig. 2.15(b) satisfies the conditions of the lemma: $n = 8$ and it requires $\lfloor 8/2 \rfloor = 4$ combinatorial cooperative guards; an attempted cover with three guards is shown in the figure. Note that even triangulations whose strings are tough but do not correspond to any non-degenerate polygon require $\lfloor n/2 \rfloor$ combinatorial cooperative guards. Fig. 2.15(c) shows the smallest possible instance, $(t/)^4$, which requires $\lfloor 4/2 \rfloor = 2$ combinatorial cooperative guards.

Proof of Lemma 2.23. We will first prove that a triangulation graph G_T with a tough string requires $\lfloor n/2 \rfloor$ combinatorial cooperative guards.

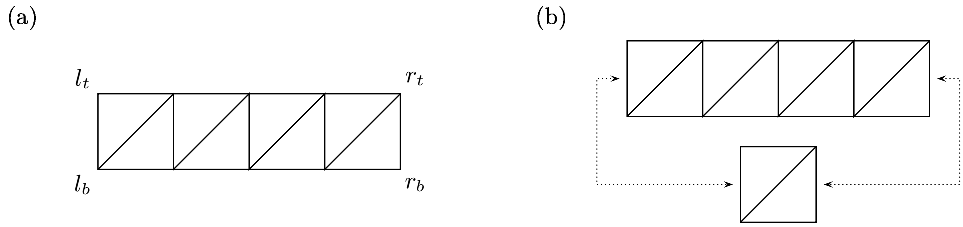

Fig. 2.16. A tough triangulation requires $\lfloor n/2 \rfloor$ combinatorial cooperative guards.

Let us consider the sequence s of triangles that is shown in Fig. 2.16(a). It is easy to see that it has the following properties:

- it requires $\lfloor m/2 \rfloor$ combinatorial cooperative guards, where m is the number of triangles;
- if a guard is required either at the vertex l_t or r_b, then the sequence requires $\lfloor (m+2)/2 \rfloor$ combinatorial cooperative guards, where m is the number of triangles;
- if guards are required at both l_b and r_t, then the sequence requires $\lfloor (m+2)/2 \rfloor$ combinatorial cooperative guards, where m is the number of triangles.

Next, if we close the sequence s with two triangles (see Fig. 2.16(b)), we will get a tough string. The above properties ensure that at least $\lfloor (m+2)/2 \rfloor$ combinatorial cooperative guards are required for a complete domination of the new string. As $m+2=n$, we are done.

Now we will prove the assertion in the other direction, in the contrapositive form: if a triangulation T is not an instance of a tough string, then fewer than $\lfloor n/2 \rfloor$ combinatorial cooperative guards suffice for a domination.

Each t in a tough triangulation must be followed by /. Thus any non-tough triangulation must contain a fragment of the form tt with the apex at some vertex v. Without loss of generality, we can assume that the sequence $t/$ is followed by tt. Otherwise, we can remove triangles ttt from the graph G_T and split the vertex v into two. The resulting hole-free triangulation graph can be dominated by $\lfloor (n-1)/2 \rfloor - 1$ combinatorial cooperative guards. The same guard placement in G_T with one additional guard at v yields a domination of G_T by $\lfloor (n-1)/2 \rfloor$ combinatorial cooperative guards.

Fig. 2.17. An existence of a tt-fragment leads to domination by $\lfloor (n-1)/2 \rfloor$ combinatorial cooperative guards.

Therefore let the triangles from the fragment tt/t be labeled t_1, t_2 and t_3, respectively, and let $\{x, v\}$ be the diagonal shared by the triangles t_1 and t_2. Removing t_2 and t_3

from G_T results in the triangulation graph G'_T of a hole-free polygon with n vertices (see Fig. 2.17). By Lemma 2.22, G'_T can be dominated by $\lfloor (n-1)/2 \rfloor$ combinatorial cooperative guards with one guard located either at x or at v. As t_1 and t_2 form the *tt*-fragment in G_T, without loss of generality, we can assume v to be selected as a guard. Then the same guard placement in G_T yields a domination of G_T by $\lfloor (n-1)/2 \rfloor$ combinatorial cooperative guards, as the triangles t_2 and t_3 are dominated by the guard at v. ■

Now, let P be a one-hole polygon with n vertices, and let T be one of its triangulations. The next lemma shows that the only triangulations "hard" to dominate by $\lfloor (n-1)/2 \rfloor$ combinatorial cooperative guards are the tough ones.

LEMMA 2.24 ([96]). *Let P be a polygon with n vertices and one hole, and suppose that there exists a non-tough triangulation T of P, that is, P has either a triangulation whose dual graph is a cycle with at most one tree attached to the cycle, or P has a non-tough basic triangulation. Then the triangulation graph of T can be dominated by $\lfloor (n-1)/2 \rfloor$ cooperative guards.*

Proof. The proof is by induction on the number of trees attached to the cycle of the dual graph of T. The initial step is established by Lemma 2.23: $\lfloor (n-1)/2 \rfloor$ combinatorial cooperative guards are sufficient to dominate a non-tough basic triangulation graph, which by definition has no attached trees in the dual graph. For the inductive step, assume that $\lfloor (n-1)/2 \rfloor$ combinatorial cooperative guards suffice for any non-tough triangulation with the dual graph of $s' < s$ attached trees.

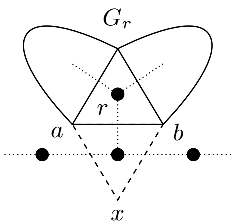

Fig. 2.18. The graph G_r attached at the diagonal $\{a, b\}$ to a cycle triangle.

Let G_T be the triangulation graph of T, and let G_r be a triangulation graph whose dual graph corresponds to a tree detachable from G_T by the removal of one arc r. This situation is illustrated in Fig. 2.18. Let a and b be the endpoints of the diagonal whose dual is r. Let m be the number of vertices in G_r, *not including* a and b. The proof proceeds in three cases, depending on the values $n \pmod 2$ and $m \pmod 2$. The easiest cases are considered first.

CASE 1: $n = 2i + 1$. The sufficiency of the $\lfloor (n-1)/2 \rfloor$ bound follows immediately from Corollary 2.35 (see Section 2.4.2): if n is odd, then $\lfloor n/2 \rfloor = \lfloor (n-1)/2 \rfloor$.

CASE 2: $n = 2i$ and $m = 2l$. Augment G_r to G_r^x by adding the triangle on the other side of $\{a, b\}$, whose apex is x. Next, by cutting G_T along the diagonals $\{x, a\}$ and $\{x, b\}$, we get two triangulation graphs G'_T and G_r^x, with $n - m + 1$ and $m + 3$ vertices, respectively. By Theorem 2.6, G'_T can be dominated by $\lfloor(n-m-1)/2\rfloor$ combinatorial cooperative guards. With the same arguments, G_r^x can be dominated by $\lfloor(m+3)/2\rfloor - 1$ combinatorial cooperative guards, with a guard at one of the vertices a, b or x. As m is even, $\lfloor(m+3)/2\rfloor - 1 = \lfloor m/2\rfloor$. The same guard placement in G_T yields a domination by $\lfloor(n-m-1)/2\rfloor + \lfloor m/2\rfloor \leq \lfloor(n-1)/2\rfloor$ combinatorial cooperative guards.

Case 3: $n = 2i$ and $m = 2l+1$. Again augment G_r to G_r^x by adding a triangle on the other side of $\{a, b\}$, whose apex is x. By Lemma 2.22, G_r^x can be dominated by $\lfloor(m+2)/2\rfloor$ combinatorial cooperative guards with one guard located either at a or at x. If x is selected as a guard, it may be moved to a. Thus we can assume a to be selected as a guard in G_r^x. Let G'_T be the result of removing all triangles of G_r^x and all triangles incident to a. G'_T has $n-m-1$ vertices, since it misses m vertices of G_r^x and the vertex a. Note that G'_T is not necessarily a triangulation graph of a polygon, as pieces may be attached at vertices only. But now connect each vertex of G'_T that was adjacent to a in G_T to b. In Fig. 2.19 the vertices $v_1, \ldots, v_4$ are connected. These connections are not always geometrically possible, but for this case we are only concerned with the combinatorial structure of the graph. The reconnections do not increase the number of vertices, but G'_T becames a triangulation graph of a polygon with one hole, with a smaller number of trees attached to the cycle of the dual graph of G'_T.

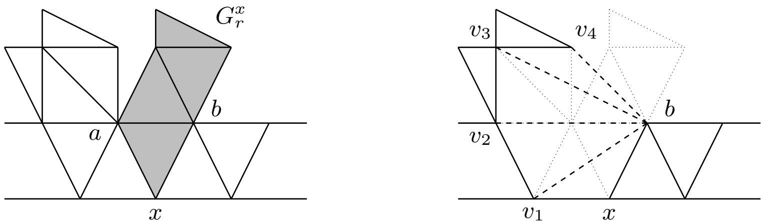

Fig. 2.19. Case 3: $n = 2i$ and $m = 2l+1$.

If G'_T is not tough, then G'_T can be dominated by

$$\left\lfloor \frac{n-m-2}{2} \right\rfloor = \left\lfloor \frac{n-m-3}{2} \right\rfloor$$

combinatorial cooperative guards by the induction hypothesis, as $n-m$ is odd. And it is easy to see that placing guards at vertices of G_T selected as guards either in G'_T or in G_r^x yields a combinatorial cooperative domination of G_T by at most $\lfloor(n-1)/2\rfloor$ guards. Otherwise, the toughness of G'_T implies the following properties of G_T:

- the dual graph of G_T is a single cycle with one tree attached (corresponding to the graph G_r^x);
- since connecting b to the vertices to which a was adjacent results in a tough string $(t/)^{2k+4}$, by simple enumeration of cases, the cycle triangles of G_T (without triangles of G_r) must have been of the form $tt/t/(t/)^{2k+2}$, and either a or b has belonged only to triangles from the tt-fragment. Therefore without loss of generality, either the configuration shown in Fig. 2.20(a) or Fig. 2.20(b) must hold.

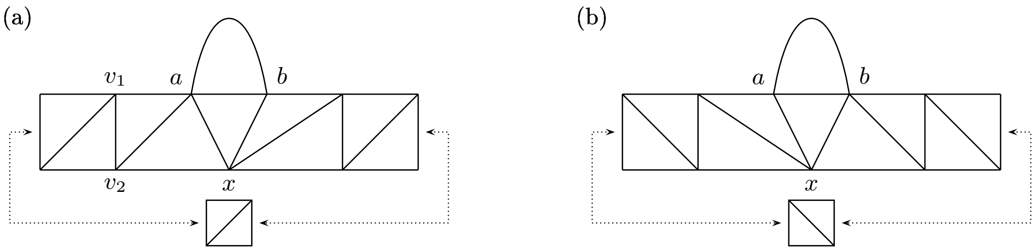

Fig. 2.20. Subcases 3.a and 3.b.

SUBCASE 3.a: G_T is of the form shown in Fig. 2.20(a). Then instead of connecting b to v_2, let us connect v_1 with x, that is, we have to flip a diagonal in the quadrilateral (b, v_1, v_2, x) in G'_T. Again note that this is only a combinatorial "procedure". This flipping results in a non-tough triangulation, and thus we can proceed as in the case of G'_T non-tough.

SUBCASE 3.b: G_T is of the form shown in Fig. 2.20(b). Then instead of considering the vertex a in the first step of the proof, we have to consider b. It is easy to see that this will lead to Subcase 3.a. ∎

The next step is to invoke the geometry of the triangulation and use geometric guards in the case of a tough triangulation. In particular, if a tough triangulation contains either a "c-pair" or a "c-triplet", then $\lfloor (n-1)/2 \rfloor$ vertex cooperative guards suffice. The final step is to show that every tough triangulation contains one of these two structures.

DEFINITION 2.2. A *c-pair* is a pair of adjacent cycle triangles that together form a convex quadrilateral.

LEMMA 2.25 ([96]). *A polygon with a tough triangulation containing a c-pair can be covered by $\lfloor (n-1)/2 \rfloor$ vertex cooperative guards.*

Proof. Flipping the diagonal of the c-pair will change the structure of the triangulation to non-tough. By Lemma 2.24, the resulting triangulation graph can be dominated by $\lfloor (n-1)/2 \rfloor$ combinatorial cooperative guards, and hence all of P can be covered by $\lfloor (n-1)/2 \rfloor$ vertex cooperative guards. ∎

DEFINITION 2.3. A *c-triplet* is a triple (A, B, C) of consecutive triangles such that the union of three triangles may be partitioned into two convex pieces.

LEMMA 2.26 ([96]). *A polygon with a tough triangulation containing a c-triplet can be covered by $\lfloor (n-1)/2 \rfloor$ vertex cooperative guards.*

Proof. Let a be a vertex common to the c-triplet triangles A, B and C, as shown in Fig. 2.21(a). Deleting B and splitting the vertex a into two results in the hole-free polygon with $n + 1$ vertices, which may therefore be covered by $\lfloor (n-1)/2 \rfloor$ vertex guards by Theorem 2.7. In particular, both A and C must have a guard at one of its corners. Now put back B. Because the three triangles form a c-triplet, B is also covered by the guards covering A and C. Note that if the triangles did not form a c-triplet, as in Fig. 2.21(b), B would not necessarily be covered. ∎

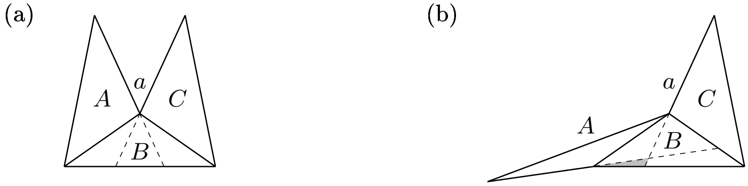

Fig. 2.21. A c-triplet is covered if A and C are covered (a), but B would not be necessarily covered if the triangles do not form a c-triplet (b).

We finally come to the last step of the proof. For a triangle t_i, let $\alpha(i)$ denote the open cone delimited by the two edges of t_i passing through the apex, and denote by $\beta(i)$ the similar region of the right-hand base vertex (see Fig. 2.22).

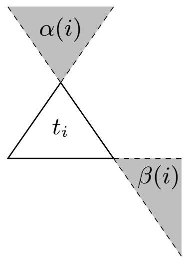

Fig. 2.22. The apex cone α and the base cone β for a triangle.

Lemma 2.27 ([96]). *Any tough triangulation of a polygon contains either a c-pair or a c-triplet.*

Proof. The proof is by contradiction and it is taken from [83]. Assume that a tough triangulation T contains neither a c-pair nor a c-triplet. Then we will show that it cannot close into a cycle, and so it is not the triangulation of a one-hole polygon.

Let us identify two adjacent cycle triangles of the form t/t; such a fragment must exist, because the general form is $(t/)^{2k+4}$. We will identify triangles by subscripts on their labels. The selected fragment is labeled t_1/t_2. Let a string s end on the right with t_i, and let v_i be the vertex at the tip of t_i. An embedding of s is said to be *nesting* if v_i is in the base cone $\beta(i-1)$ of the triangle t_{i-1} adjacent to t_i.

The general form of the fragment t_1/t_2 is as shown in Fig. 2.23(a). In order to avoid a c-pair, either the configuration shown in Fig. 2.23(b) or Fig. 2.23(c) must hold. In Fig. 2.23(c) we have $v_2 \in \beta(1)$, and so the nesting condition is satisfied. As Fig. 2.23(c) is just Fig. 2.23(b) reflected and rotated, we assume without loss of generality that Fig. 2.23(c) holds.

The string t_1/t_2 may be extended only with $/t$ while remaining compatible with the tough form; the general form is shown in Fig. 2.24(a). In order to avoid a c-pair in t_2/t_3,

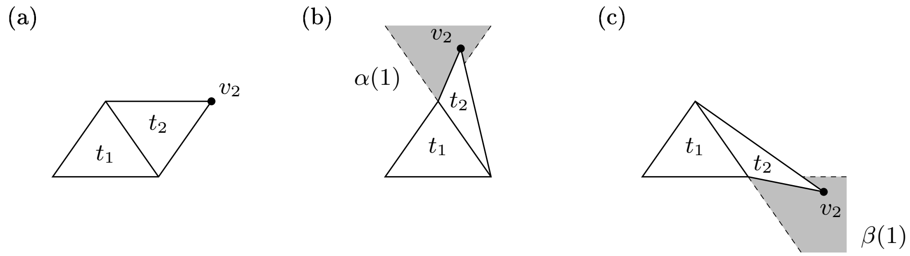

Fig. 2.23. t_1/t_2 is nesting.

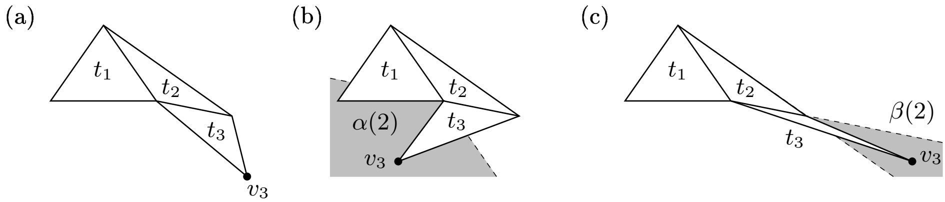

Fig. 2.24. $t_1/t_2/t_3$ is nesting.

either $v_3 \in \alpha(2)$ or $v_3 \in \beta(2)$. The former choice (Fig. 2.24(b)) leads to a c-triplet, and the latter (Fig. 2.23(c)) is a nesting configuration.

Now the contradiction is immediate. By applying the above observation to all triangles in turn, we conclude that every embedding of the string compatible with a tough string having no c-pairs and no c-triplets is nesting. The repeated nesting forces $v_i \in \beta(i-1)$, and since these base cones are clearly inside one another (cf. Fig. 2.23(c) and Fig. 2.24(c)), the embedding cannot wrap back around to permit $v_{n+1} = v_1$. ■

THEOREM 2.28 ([96]). $\lfloor (n-1)/2 \rfloor$ *vertex cooperative guards suffice to cover any* n*-vertex polygon with one hole.*

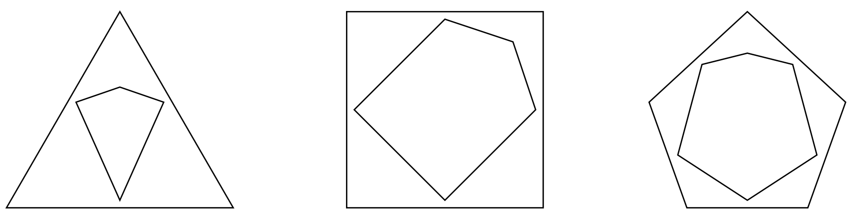

Fig. 2.25. A one-hole polygon may require as many as $\lfloor (n-1)/2 \rfloor$ cooperative guards.

Proof. Lemma 2.24 establishes that if there exists a non-tough triangulation, then $\lfloor (n-1)/2 \rfloor$ vertex cooperative guards suffice. So we only need to consider polygons with tough triangulations. Lemmas 2.25 and 2.26 show that if a tough triangulation contains either a c-pair or a c-triplet, then $\lfloor (n-1)/2 \rfloor$ vertex cooperative guards suffice. And Lemma 2.27 shows that every tough triangulation contains one of these structures. ∎

THEOREM 2.29 ([96]). *For all* $n \geq 6$, $\mathrm{cg}(n, 1) = \lfloor (n-1)/2 \rfloor$, *even for vertex guards.*

Proof. The necessity is established by the class of polygons shown in Fig. 2.25. It is easy to see that they require as many as $\lfloor (n-1)/2 \rfloor$ cooperative guards. Therefore $\mathrm{cg}(n, 1) \geq \lfloor (n-1)/2 \rfloor$. For sufficiency, we apply Theorem 2.28. ∎

2.4.2. Arbitrary number of holes. The cooperative guards problem for polygons with an arbitrary number of holes remains unsolved. And it does not seem easy to extend the proof of Theorem 2.28 to more than one hole. However, in this section we provide upper bounds for $\mathrm{cg}(n, h)$ for both point and vertex guards.

In [12] Bjorling-Sachs and Souvaine show that $\lfloor (n+h)/3 \rfloor$ arbitrary guards are sufficient in any polygon with n vertices and h holes. Their approach is to connect each hole to the exterior by cutting away a quadrilateral *channel* c_i, $i = 1, \dots, h$, such that one vertex is introduced for each channel, and there is a triangle T_i in the remaining polygon such that any point in it sees all of the channel c_i, $i = 1, \dots, h$. This triangle is then forced to be in a triangulation of the hole-free version of the polygon. A guard assignment based on 3-coloring will cover the hole-free polygon and all the channels as well. Since the new polygon has $n + h$ vertices, the number of guards is $\lfloor (n+h)/3 \rfloor$. These guards are vertex guards in the hole-free polygon, but point guards in the original polygon, since new vertices were added during the channel constructions.

The main result of Bjorling-Sachs and Souvaine's paper is the following theorem, which we will use to provide an upper bound for cooperative guards. (As the proof involves a long cascade of cases, we refer the reader to [12].)

THEOREM 2.30 ([12]). *In any polygon P with n vertices and h holes, all channels c_i, $i = 1, \dots, h$, can be removed in such a way that the remaining polygon has*:

- $n + h$ *vertices;*
- *no holes;*
- *a triangulation T with (disjoint) triangles t_i, $i = 1, \dots, h$, as leaves in the dual graph of T from whose vertices the areas of the removed channels are visible in P.*

Let P be a polygon with n vertices and h holes, and let T be a triangulation whose existence is guaranteed by the above theorem. By Theorem 2.6, G_T can be dominated by $\lfloor (n+h)/2 \rfloor - 1$ combinatorial cooperative guards, and thus all of the hole-free polygon can be covered by the same number of cooperative guards. Since each of the triangles t_i of T, $i = 1, \dots, h$, has a guard at one of its vertices, these guards see all of the channels by Theorem 2.30. Thus all of P is covered by $\lfloor (n+h)/2 \rfloor - 1$ cooperative guards. Note that the guards are point guards, since the hole-free polygon has vertices not present in P. This proves the following theorem.

THEOREM 2.31 ([96]). *For all* $h \geq 0$ *and* $n \geq 3 + 3h$, $\mathrm{cg}(n, h) \leq \lfloor (n+h)/2 \rfloor - 1$, *that is,* $\lfloor (n+h)/2 \rfloor - 1$ *cooperative guards are sufficient to cover the interior of a polygon with n vertices and h holes.*

The best sufficiency result for vertex guards follows the main outlines of O'Rourke's proof for arbitrary vertex guards in polygons with holes (cf. the proof of Theorem 5.1 of [73]), and it was established by Ahlfeld and Hecker [3].

LEMMA 2.32 ([73]). *An n-vertex polygon P with h holes can be partitioned into $t = n + 2h - 2$ triangles by adding $n + 3h - 3$ internal diagonals.*

Proof. The proof is by a double induction: first on h, and then on n. Theorem 1.1 establishes the case of $h = 0$. For the inductive step, let d be an internal diagonal, whose existence can be guaranteed by the same argument as used in Theorem 1.1. If d has one endpoint on a hole, then cutting the polygon along this diagonal increases n by 2, but decreases h by 1. If d has both endpoints on the outer boundary of P, then it partitions P into two polygons P^i with $n_i < n$ vertices and $h_i \leq h$ holes, $i = 1, 2$. In either case, the induction hypothesis applies and establishes the existence of a triangulation.

The number of triangles is obtained from Euler's theorem. There are $V = n$ vertices, $F = t+h+1$ faces, one for each triangle and hole, plus the exterior face, and $E = (3t+n)/2$. Then $V - E + F = 2$ yields $t = n + 2h - 2$. ■

Thus by the above lemma, the triangulation graph of a triangulation is well defined. Applying the same arguments as in the proof of Lemma 1.4, we have the following lemma.

LEMMA 2.33 ([95]). *Let P be an n-vertex polygon with h holes, and let G_T be one of its triangulation graphs. If G_T can be dominated by $f(n, h)$ cooperative guards, then P can be covered by $f(n, h)$ geometric cooperative vertex guards.*

THEOREM 2.34 ([3]). *A triangulation graph of an n-vertex polygon with h holes can always be dominated by $\lfloor n/2 \rfloor + h - 1$ cooperative guards.*

Proof. Let G_T be a triangulation graph of a polygon P with n vertices and h holes. The idea is to cut the polygon along internal diagonals so as to remove each hole by connecting it to the exterior of P. Cutting along any such diagonal either merges the hole with another or connects it to the outside. In either case, each cut reduces the number of holes by one. All we need is to choose the cuts so that the result is a single polygon.

Let D_T be the (weak) dual graph of the triangulation T. Then D_T is a planar graph of the maximum degree three, which, in its planar embedding, has h bounded faces $F_1, \ldots, F_h$, one per hole of P. Let F_0 be the exterior unbounded face. Choose any face F_i that shares at least one edge e with F_0. Note that there must be such a face because there must be a diagonal of T from the outer boundary to some hole, and the dual of this diagonal in D_T is e. Removal of e from D_T merges F_i with F_0 without disconnecting the new graph D_T. See Fig. 2.26 for an example. Of course, removal of an edge in D_T is equivalent to cutting P along the corresponding diagonal of T. By continuing to remove edges of D_T shared with the exterior face in this manner, a tree is obtained, and thus a polygon without holes.

Let P' be the polygon that results after all holes are cut in the above manner. Then P' has $n + 2h$ vertices, since two new vertices are introduced per each cut, but because the cuts do not create new triangles, the resulting triangulation graph G'_T has $n + 2h - 2$

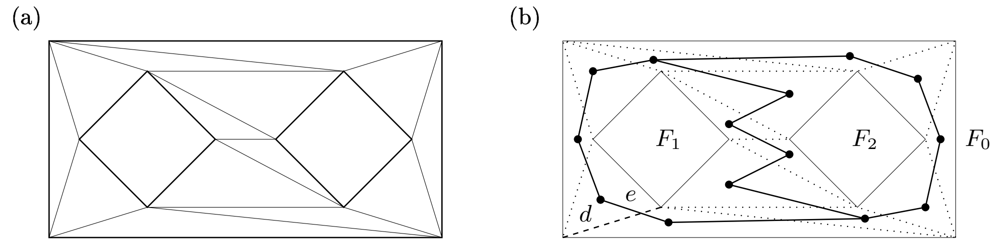

Fig. 2.26. (a) A triangulation graph of a polygon with holes, and (b) its dual. Each hole in (a) is surrounded by a cycle in (b).

triangular faces and $n + 2h$ vertices. Applying Theorem 2.6 yields a domination of G'_T by $\lfloor n/2 \rfloor + h + 1$ cooperative guards. The same guard placement in G_T will dominate all triangular faces of G_T. ∎

COROLLARY 2.35 ([3]). *For vertex guards,* $\mathrm{cg}(n, h) \leq \lfloor n/2 \rfloor + h - 1$.

But this easily obtained corollary is weak as no one so far has found examples of polygons that require so many guards. Fig. 2.27 shows an example from the class of n-vertex polygons with h holes that requires $\lfloor n/2 \rfloor$ vertex cooperative guards. We conjecture that this bound is tight.

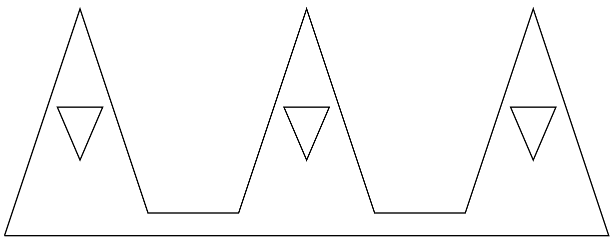

Fig. 2.27. A polygon with holes may require $\lfloor n/2 \rfloor$ cooperative vertex guards.

CONJECTURE 2.36. $\lfloor n/2 \rfloor$ *cooperative vertex guards are sometimes necessary and always sufficient to guard an n-vertex polygon with h holes.*

When considering point guards, surprisingly, it seems that increasing the number of holes decreases the number of required cooperative guards. The best necessity bound follows from the case of orthogonal polygons, and it will be discussed in the next section.

2.4.3. Orthogonal polygons with holes. We define an *orthogonal polygon with holes* to be an orthogonal polygon with orthogonal holes, with all edges aligned with the same pair of orthogonal axes. The case of one-hole orthogonal polygons was solved by the author [95], who proved the following theorem.

THEOREM 2.37 ([95]). $\mathrm{cg}_\perp(n, 1) = \lfloor (n - (n \bmod 4))/2 \rfloor - 1$.

Proof. First, consider the case $n \equiv 0 \pmod 4$. The necessity is established by the polygon shown in Fig. 2.28(a). For sufficiency, consider a quadrilateralization Q of a one-hole orthogonal polygon P. Now, by cutting P along a diagonal d in order to connect the hole to the outside, we get a new hole-free polygon P' with a new set D of diagonals. Of course, $|D| = \lfloor n/2 \rfloor - 1$. Now, by placing a guard on each diagonal from D, we get a guard set for P' of cardinality $\lfloor n/2 \rfloor - 1$. It is easy to see that S is cooperative, and it covers the original polygon P as well (cf. the proof of Theorem 2.11).

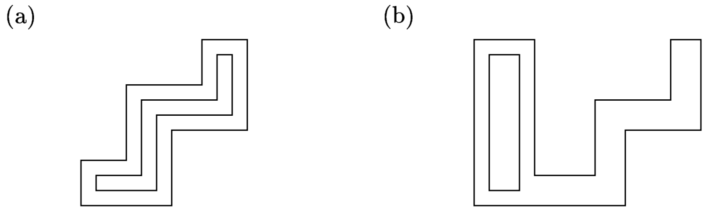

Fig. 2.28. (a) If $n \equiv 0 \pmod 4$ then $\mathrm{cg}_{\perp}(n, 1) \geq \lfloor n/2 \rfloor - 1$; here $n = 20$, and the polygon needs 9 guards. (b) Otherwise, $\mathrm{cg}_{\perp}(n, 1) \geq \lfloor n/2 \rfloor - 2$; here $n = 16$, and the polygon needs 6 guards.

If $n \equiv 2 \pmod 4$, the necessity is established by the polygon shown in Fig. 2.28(b) (due to Pinciu [77]). For sufficiency, all we need is to notice that if $n \equiv 2 \pmod 4$, then in any quadrilateralization Q of P, there exist two diagonals d_1 and d_2 that share a vertex. So if we cut P along a diagonal d, where $d \neq d_1$ and $d \neq d_2$, we can put a guard at the common vertex of d_1 and d_2. Hence the construction used in the case $n \equiv 0 \pmod 4$ will result in a cooperative guard set of cardinality $\lfloor n/2 \rfloor - 2$. ■

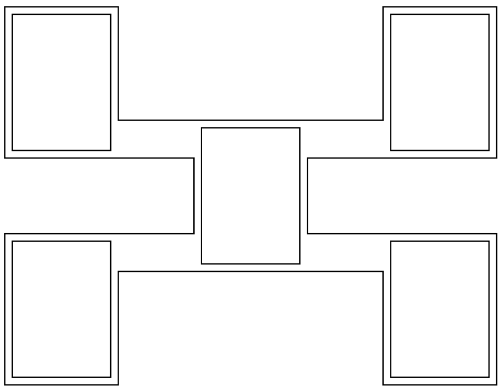

Fig. 2.29. A 40-vertex orthogonal polygon with 5 holes can require as many as $16 > \lfloor 40/2 \rfloor - 5 = 15$ cooperative guards.

The case of orthogonal polygons with at least two holes remains open. In 2003, Pinciu [77] sketched a proof that $\lfloor n/2 \rfloor - h$ cooperative guards always suffice to cover an n-vertex orthogonal polygon with h holes. However, Fig. 2.29 shows an 40-vertex orthogonal polygon with five holes that requires as many as $16 > \lfloor 40/2 \rfloor - 5 = 15$ cooperative guards, thus Pinciu's theorem is false. A simple extension of this polygon leads to a class of orthogonal polygons that require as many as $\lfloor n/2 \rfloor - h + \lfloor (h-1)/4 \rfloor$ cooperative guards (see Fig. 2.30).

Corollary 2.38 ([95]). $\mathrm{cg}_{\perp}(n, h) \geq \lfloor n/2 \rfloor - h + \lfloor (h-1)/4 \rfloor$, $h \geq 2$.

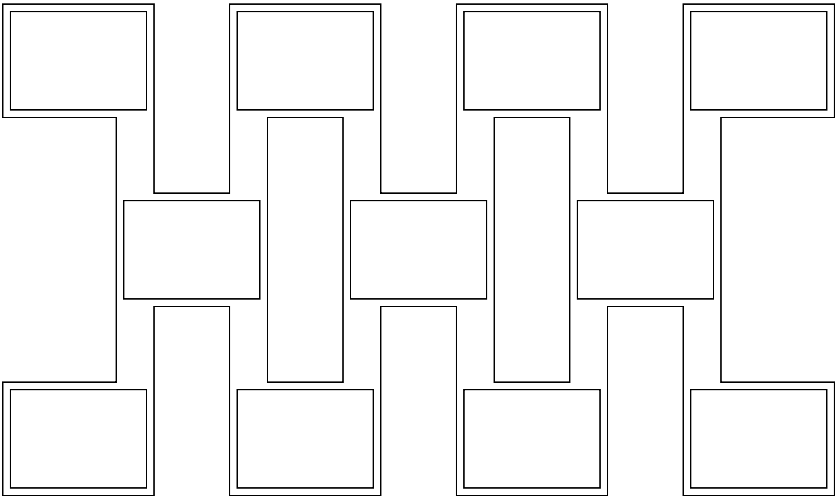

Fig. 2.30. An n-vertex orthogonal polygon with h holes can require as many as $\lfloor n/2 \rfloor - h + \lfloor (h-1)/4 \rfloor$ cooperative guards; here $n = 88$, $h = 13$, and the polygon requires 34 cooperative guards.

The best sufficiency result for point guards follows from the result established by Hoffman [42], who proved the following theorem.

THEOREM 2.39 ([42]). *Any orthogonal polygon, possibly with holes, can be partitioned into at most* $\lfloor n/4 \rfloor$ *rectilinear stars, each of size at most* 12.

Recall that a *rectilinear star* is a union of rectangles with non-empty intersection. So let P be an n-vertex orthogonal polygon with h holes, and consider a partitioning $\{S_1, \ldots, S_k\}$ of P into rectilinear stars with $k \leq \lfloor n/4 \rfloor$, as guaranteed by the theorem above. Let s_i, $i = 1, \ldots, k$, denote a point from the kernel of S_i. Of course, $S = \{s_1, \ldots, s_k\}$ is a guard set for P. Let $G = (S, E)$ be a graph whose vertex set is S and whose two vertices s_i and s_j are adjacent if their stars S_i and S_j have a point in common; for an edge $e \in E$, let $p(e)$ denote any such point. Let $T = (S, E')$ be a spanning tree of graph G. Then, by the same arguments as in the proof of Theorem 2.11, $S \cup p(E')$ is a cooperative guard set for P. Hence we have

THEOREM 2.40 ([95]). $\mathrm{cg}_{\perp}(n, h) \leq \lfloor n/2 \rfloor - 1$.

The best sufficiency result for vertex guards follows from the proof of Theorem 2.34. Let Q be a quadrilateralization of an n-vertex orthogonal polygon P with h holes. By cutting P along internal diagonals in order to connect the holes to the outside, we get a new hole-free polygon P' with a convex quadrilateralization of $\lfloor (n + 2h)/2 \rfloor - 2$ diagonals (Lemma 2.10). By applying the same reasoning as in the proof of Theorem 2.11, we get

COROLLARY 2.41 ([95]). *For vertex guards,* $\mathrm{cg}_{\perp}(n, h) \leq \lfloor n/2 \rfloor - 2 + h$.

3. WEAKLY COOPERATIVE GUARDS

In this chapter, we investigate the weakly cooperative guards problem [59] in which we require the visibility graph of a set of guards to have no isolated vertices; the WCG problem is also called the 1-*guarded guards problem* or *watched guards problem*. In [39] Hernández-Peñalver claimed that $\lfloor 2n/5 \rfloor$ weakly cooperative guards always sufficed to guard any polygon with n vertices. However, Michael and Pinciu [65, 66], and independently Żyliński [97], presented a class of polygons that required more than $\lfloor 2n/5 \rfloor$ weakly cooperative guards and they established a new tight bound for weakly cooperative guards: $\lfloor (3n-1)/7 \rfloor$. In [40] Hernández-Peñalver proved that $\lfloor n/3 \rfloor$ is a tight bound for orthogonal polygons, and tight bounds for polygons of miscellaneous shapes were provided by Żyliński [97]: $\lfloor 2n/5 \rfloor$ watched guards for monotone and spiral polygons, and $\lfloor (3n-1)/7 \rfloor$ vertex watched guards for star polygons.

3.1. Arbitrary polygons

First, for every $n \geq 5$, we will construct a polygon P_n that requires as many as $\lfloor (3n-1)/7 \rfloor$ weakly cooperative guards; the construction is taken from [66]. Next, we will show this bound to be tight. From now on, we refer to a weakly cooperative guard as a *watched* guard.

3.1.1. Necessity of $\lfloor (3n-1)/7 \rfloor$ watched guards. For convenience, write $f(n) = \lfloor (3n-1)/7 \rfloor$, and note that we only have to treat the cases $n \equiv 1, 3, 5 \pmod 7$, as these are the critical values of n for which $f(n) > f(n-1)$; we may always add one or two vertices to our polygons to deal with $n \equiv 0, 2, 4, 6 \pmod 7$.

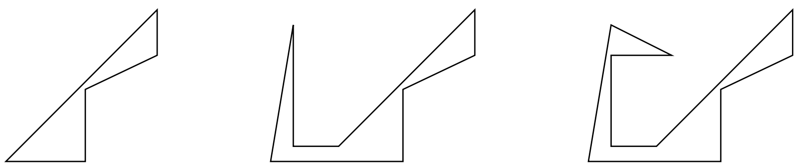

Fig. 3.1. Polygons with $n = 5, 8, 10$ vertices, respectively, that require $\lfloor (3n-1)/7 \rfloor$ watched guards.

Fig. 3.1 shows polygons with $n = 5, 8, 10$ vertices, respectively, that require $f(n)$ watched guards. A polygon P_n with n vertices is constructed from a polygon P_{n-7} by adjoining a special decagon P'_{10} with vertices $x'_0, \ldots, x'_7, x_1, x_2$ on the side x_1x_2 with a suitable orientation (see Fig. 3.2). At each step P_n has the following properties:

(i) The segments $x'_1x'_2$ and x_1x_2 are congruent and parallel.
(ii) The angles $\angle x'_1x'_2x'_3$ and $\angle x_1x_2x'_0$ are supplementary.
(iii) The line through x'_3 and x'_4 intersects the interior of the segment $x'_0x'_1$.
(iv) No point in P_n is simultaneously visible from any two of x_2, x'_2 and x'_6.

Properties (i) and (ii) guarantee that the construction is feasible at each step, while (iii) and (iv) are crucial in the induction proof of the necessity of $f(n)$ watched guards.

LEMMA 3.1 ([66]). *Let P_n be the n-vertex polygon defined inductively in the manner described above. Then* $\mathrm{gg}(P_n, 1) \geq \lfloor (3n-1)/7 \rfloor$ *for* $n \geq 5$.

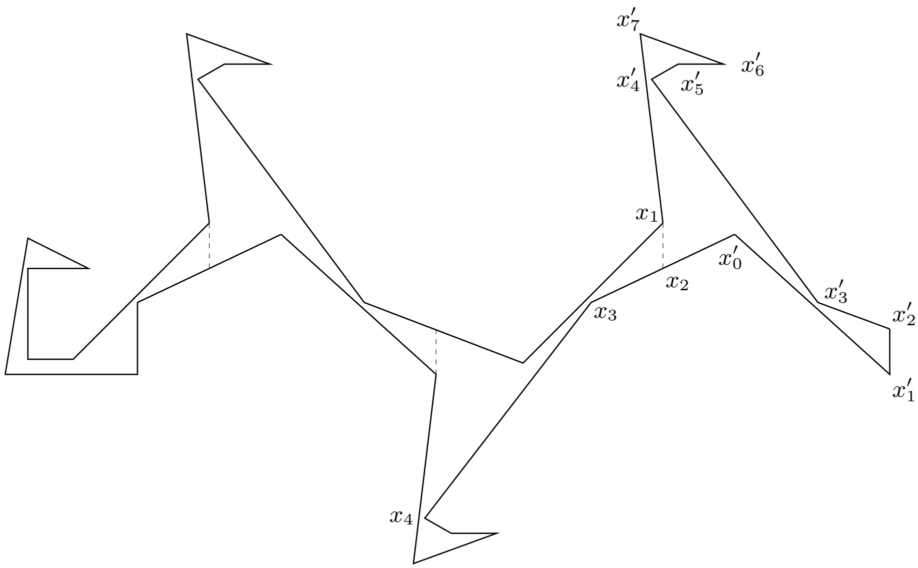

Fig. 3.2. The construction of a polygon P_n that requires $\lfloor (3n-1)/7 \rfloor$ watched guards.

Proof. The proof is by induction on n. The polygons from Fig. 3.1 establish the cases $n = 5, 8, 10$. So assume that $n \geq 12$, and that the assertion holds for polygons with at most $n - 7 \geq 5$ vertices. Let $S_n = S_{n-7} \cup S'$ be a watched guard set for P_n, where $S_{n-7} = S_n \cap P_{n-7}$ consists of all guards of S_n that are in P_{n-7}, and S' consists of all guards in the decagon P'_{10}, excluding the segment x_1x_2. Then $|S_n| = |S_{n-7}| + |S'|$, of course. We have the following claims:

CLAIM 1. $|S'| \geq 3$. Reason: There exist distinct guards g'_2 and g'_6 in S' from which the points x'_2 and x'_6, respectively, are visible. Moreover, there exist (not necessarily distinct) points g' and g'' in S_n from which g'_2 and g'_6, respectively, are visible. Note that $g' \in S'$.

CLAIM 2. $|S_{n-7}| \geq f(n-7) - 1$. Reason: First, observe that if a point x in P_{n-7} is visible from a point of P'_{10}, then x is also visible from both x_3 and x_1. Next, as points on x_3x_4

that are near x_3 are visible from no point in P'_{10}, they must be visible to a guard g in S_{n-7}. Of course, x_3 is visible to g. Therefore $S_{n-7} \cup \{z\}$ is a watched guard set for P_{n-7}, where $z = x_3$. By the induction hypothesis, $|S_{n-7} \cup \{z\}| \geq f(n-7)$, which establishes the claim.

If $|S'| \geq 4$, then by Claim 2, $|S_n| \geq f(n)$. Otherwise, if $|S'| = 3$, then we must have $g' = g''$ in the proof of Claim 1, and thus $S' = \{g'_2, g'_6, g'\}$. As points on x_2x_3 near x_3 are not visible from any point in S', there is a guard $g \in S_{n-7}$ that covers such points. Now, $S_{n-7} \setminus \{g\} \cup \{z\}$ is a watched guard set for P_{n-7}, where z is defined as in the proof of Claim 2. Therefore $|S_{n-7}| \geq f(n-7)$, and this yields $|S_n| \geq f(n)$. ∎

3.1.2. Sufficiency proof. The proof is by induction and it follows the outline of O'Rourke's proof for mobile guards [72]. Before commencing the proof, we recall certain facts that will be used in various cases of the proof.

Let P be a polygon, let G_T be a triangulation graph for P, let e be an edge of P, and let u and v be the two vertices of G_T corresponding to the endpoints of e. The *contraction* of e is a transformation that alters G_T by removing the vertices u and v and replacing them with a new vertex adjacent to every vertex to which u or v was adjacent. Note that an edge contraction is a graph transformation, not a polygon transformation: the geometric equivalent could result in self-crossing polygons. Edge contractions are nevertheless useful because of the following lemma.

Lemma 3.2 ([72]). *Let G_T be a triangulation graph of a polygon P with $n \geq 4$ vertices, and let G'_T be the graph resulting from an edge contraction of G_T. Then G'_T is a triangulation graph of an $(n-1)$-vertex polygon P'.*

Proof. The idea of the proof is to construct a figure with curved edges corresponding to G'_T, and then straighten it to obtain P'.

Let P_T be a planar figure corresponding to the triangulation T, let e be the contracted edge, and u and v its two endpoints in P_T. Let the vertices to which u and v are connected by diagonals and edges be labeled $u_0, \ldots, u_i$ and $v_0, \ldots, v_j$, respectively, with $u_0 = v$ and $v_0 = u$, and the remaining ones labeled according to their sorted angular order (see Fig. 3.3(a)). Note that $u_1 = v_1$ is the apex of the triangle supported by e.

Next, introduce a new vertex x in the interior of e and connect the u-vertices and v-vertices to x by the following procedure. Connect u_1 to x; this can be done without crossing any diagonals because u_1 is the apex of the triangle on whose base x lies. Remove the diagonal $\{u, u_1\}$. Connect u_2 to x within the region bounded by (x, u_1, u_2, u); the line may need to be curved but again no crossings are necessary. Remove the diagonal $\{u, u_2\}$. Continue in this manner (see Fig. 3.3(b)) until all u-vertices have been connected to x. Then apply a similar procedure to all v-vertices. The result is a planar figure whose connections are the same as those of T' (see Fig. 3.3(c)).

Finally, we apply Fáry's theorem [35]: for any planar graph drawn in the plane, perhaps with curved lines, there is a homeomorphism in the plane onto a straight-line graph such that vertices are mapped to vertices and edges to edges. Applying such a homeomorphism to the figure constructed above yields P', a polygon that has T' as one of its triangulations (see Fig. 3.3(d)). ∎

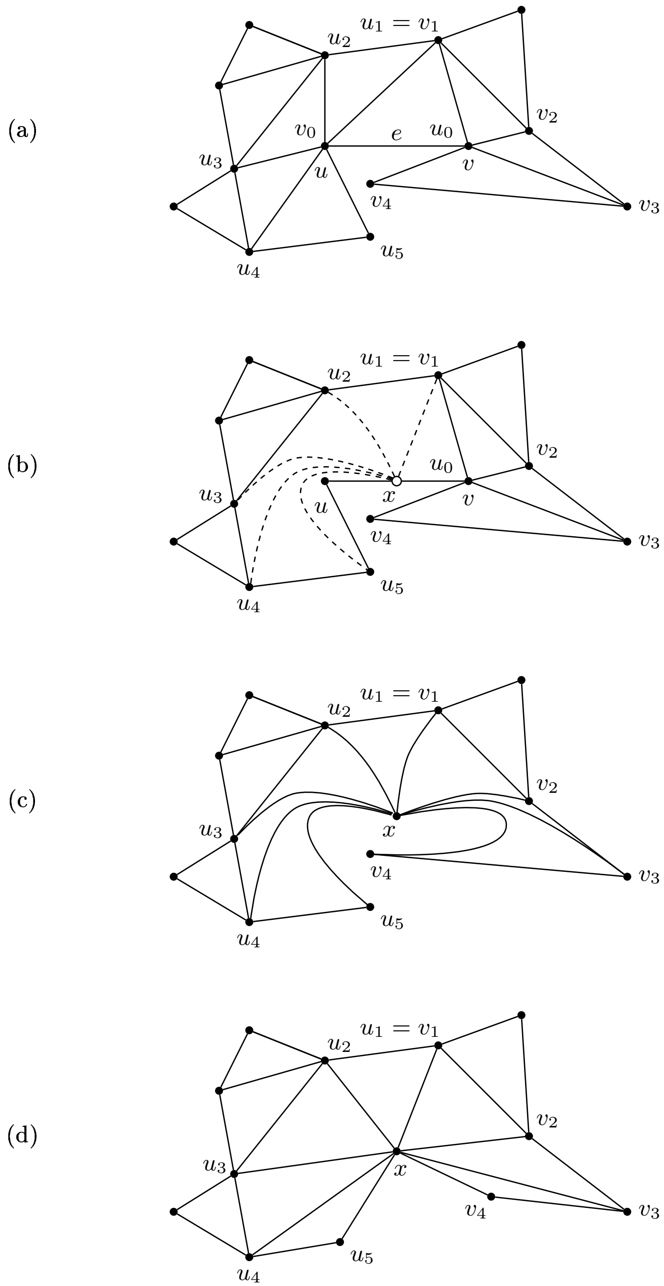

Fig. 3.3. If all the edges in a triangulation graph (a) incident to u and v are made incident to x ((b) and (c)), the resulting graph may be transformed into a straight-line graph (d).

Next, we recall a lemma that establishes the existence of a special diagonal in a triangulation graph that will allow us to make the induction step in most of the proofs.

LEMMA 3.3 ([84]). *Given a polygon triangulation graph G_T of n vertices and some positive integer $t \leq n-2$, there exists an edge d of G which separates G_T into two pieces G_1 and G_2 (with d in both pieces) such that G_1 has between t and $2t-1$ triangles, inclusive. The degenerate case $G_2 = d$ is allowed.*

Proof. Let e be an arbitrary fixed edge of G_T that corresponds to an edge of a polygon. Let t' be the minimum number, greater than or equal to t, of triangles in any piece cut off by an edge, and let d be an edge which cuts off a piece with t' triangles (we use the phrase 'piece cut off by the edge d' to indicate whichever piece that does not contain e). Such a d exists as e cuts off $n-2$ triangles, and $t \leq n-2$. Of the triangles cut off, let U be the one containing d. Note that t' is the total number of triangles cut off by the other edges of U, plus one (for U). Each of the other edges may cut off at most $t-1$ triangles (otherwise, t' is not minimum), thus $t' \leq 2(t-1)+1 = 2t-1$. ■

Now, following O'Rourke's proof, we must establish the sufficiency of the $\lfloor (3n-1)/7 \rfloor$ bound for small triangulation graphs.

LEMMA 3.4 ([84]). *Every triangulation graph of a pentagon can be dominated by two watched guards with one guard placed at any selected vertex.*

Proof. The proof is adapted from [72]. Let G_T be a triangulation graph of a pentagon, and let the selected vertex be labeled 1. As there are only five distinct triangulations, by a simple enumeration of cases, it is easy to see that two watched guards are always sufficient to dominate G_T with one guard at vertex 1 (see Fig. 3.4). ■

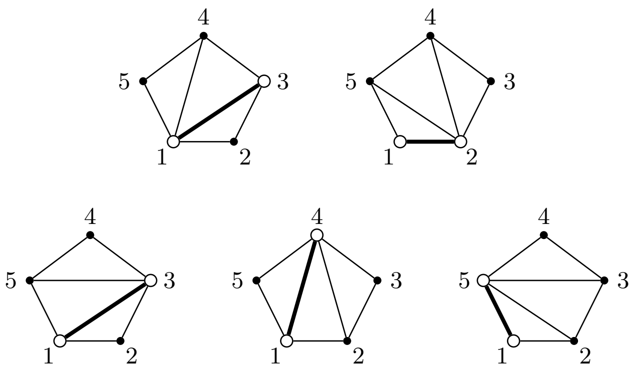

Fig. 3.4. A triangulation graph of a pentagon can be dominated by two watched guards with one guard placed at any vertex.

LEMMA 3.5 ([66, 92]). *Let G_T be a triangulation graph of a hexagon, and let x be a vertex of degree at least 3. Then G_T can be dominated by two watched guards with one guard placed at x.*

Proof. As x is of degree 3, there is a diagonal d with one of its endpoints at x. This diagonal partitions the six boundary edges of G_T according to either $2+4=6$ or $3+3=6$.

Case 1: $2 + 4 = 6$. Let $d = \{1, 3\}$. Then $(1, 3, 4, 5, 6)$ is a triangulation graph of a pentagon (see Fig. 3.5), and by Lemma 3.4, this graph can be dominated by two watched guards with one placed at 1. The guard at 1 dominates the triangle $(1, 2, 3)$.

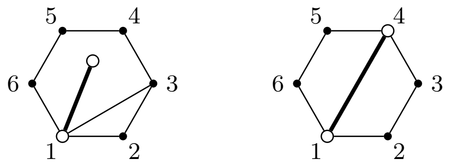

Fig. 3.5. A triangulation graph of a hexagon can be dominated by two watched guards with one guard placed at any vertex of degree at least 3.

Case 2: $3 + 3 = 6$. Let $d = \{1, 4\}$. Then $(1, 2, 3, 4)$ and $(1, 4, 5, 6)$ are triangulation graphs of quadrilaterals (see Fig. 3.5). Placing guards at vertices 1 and 4 will dominate all triangles, regardless of how the quadrilaterals are triangulated. ∎

Lemma 3.6 ([66, 92]). *Every triangulation graph of a septagon can be dominated by two watched guards.*

Proof. By Theorem 1.3, in any triangulation graph G_T of a septagon, there is at least one vertex of degree 2. Let the vertices of the septagon be labeled $1, \ldots, 7$, in a counterclockwise manner, and assume vertex 2 to be of degree 2. By cutting off this vertex, more precisely, the triangle $\triangle = (1, 2, 3)$, we get the triangulation graph G_T^* of a hexagon (see Fig. 3.6). By Lemma 3.5, the graph G_T^* can be dominated by two watched guards

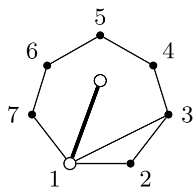

Fig. 3.6. A triangulation graph of a septagon can be dominated by two watched guards.

with one guard placed either at vertex 1 or 3. This yields a domination of G_T by two combinatorial watched guards, as the triangle $\triangle$ is dominated. ∎

Lemma 3.7 ([66, 92]). *Let G_T be a triangulation graph of an octagon P, and let x be a vertex of degree at least* 3. *Then G_T can be dominated by three watched guards, with one guard placed at x.*

Proof. As x is of degree 3, there is a diagonal d with one of its endpoints at x. This diagonal partitions the eight boundary edges of G_T according to either $2+6 = 8$, $3+5 = 8$ or $4 + 4 = 8$.

Case 1: $2 + 6 = 8$. Let $d = \{1, 3\}$. Then $(1, 3, 4, 5, 6, 7, 8)$ is a triangulation graph of a septagon (see Fig. 3.7), and by Lemma 3.6, the septagon can be dominated by two watched guards. Since one of them dominates vertex 1, by placing the additional guard at 1, we will dominate the triangle $(1, 2, 3)$, and thus G_T will be dominated by three watched guards, with one guard placed at 1.

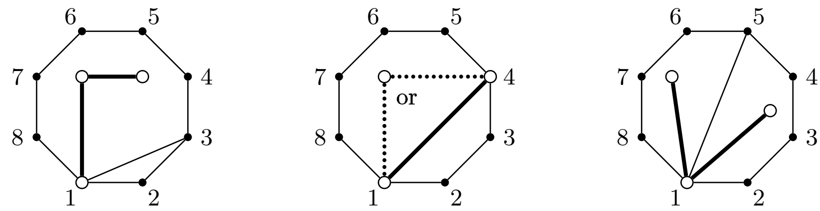

Fig. 3.7. A triangulation graph of an octagon can be dominated by three watched guards with one guard placed at any vertex of degree at least 3.

CASE 2: $3+5=8$. Let $d=\{1,4\}$. Then $Q=(1,2,3,4)$ and $H=(1,4,5,6,7,8)$ are triangulation graphs of a quadrilateral and a hexagon, respectively (see Fig. 3.7). Place guards at 1 and 4. Since in H either vertex 1 or 4 is of degree at least 3, one additional watched guard will dominate H by Lemma 3.5. Regardless of how the quadrilateral Q is triangulated, the guards at 1 and 4 will dominate it. Thus we get a domination of G_T by three watched guards, with one guard at 1.

CASE 3: $4+4=8$. Let $d=\{1,5\}$. Then $P_1=(1,2,3,4,5)$ and $P_2=(1,5,6,7,8)$ are triangulation graphs of pentagons (see Fig. 3.7). Dominate P_1 by two watched guards, with one guard at 1, and dominate P_2 by two watched guards, with one guard at 1, thus getting a domination of G_T by three watched guards, with one guard at 1. ■

LEMMA 3.8 ([66, 92]). *Every triangulation graph of an enneagon can be dominated by three watched guards.*

Proof. The proof follows the idea of the proof of Lemma 3.6. By Theorem 1.3, in any triangulation graph G_T of an enneagon there is at least one vertex of degree 2. Let the vertices of the enneagon be labeled $1, \ldots, 9$, in counterclockwise manner, and assume vertex 2 to be of degree 2. By cutting off this vertex, more precisely, the triangle $\triangle = (1, 2, 3)$, we get the triangulation graph G_T^* of an octagon. By Lemma 3.7, G_T^* can be dominated by three watched guards with one guard either at 1 or 3. This yields a domination of G_T by three watched guards, as the triangle $\triangle$ is also dominated. ■

LEMMA 3.9 ([66, 92]). *Every triangulation graph G_T of a decagon can be dominated by four watched guards with one guard placed at any vertex of degree at least* 3.

Proof. As x is of degree 3, there is a diagonal d with one of its endpoints at x. This diagonal partitions the ten boundary edges of G_T according to either $2+8=10$, $3+7=10$, $4+6=10$ or $5+5=10$. Assume that d cuts off the minimal number of vertices.

CASE 1: $2+8=10$. Let $d=\{1,3\}$. Then $E_9=(1,3,4,5,6,7,8,9,10)$ is a triangulation graph of an enneagon (see Fig. 3.8). By Lemma 3.8, E_9 can be dominated by three watched guards. One of these guards dominates vertex 1. Now, by placing one additional guard at 1, we will dominate the triangle $(1, 2, 3)$, and the resulting guard set is watched.

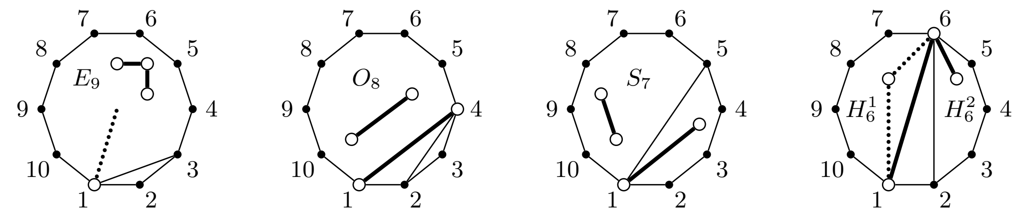

Fig. 3.8. A triangulation graph of a decagon can be dominated by four watched guards with one guard placed at any vertex of degree at least 3.

CASE 2: $3 + 7 = 10$. Let $d = \{1, 4\}$. Then $O_8 = (1, 4, 5, 6, 7, 8, 9, 10)$ is a triangulation graph of an octagon, and the minimality of d ensures that the quadrilateral $(1, 2, 3, 4)$ has diagonal $\{2, 4\}$ (see Fig. 3.8). By Lemma 3.7, O_8 can be dominated by three watched guards, with one guard either at 1 or at 4.

(1) Placing one additional guard at 4 will dominate all triangles in G_T, and the resulting guard set is watched.

(4) All triangles of G_T are dominated. Now, we place one additional guard at vertex 0; this guard is watched, as there is a guard at 4.

CASE 3: $4+6 = 10$. Let $d = \{1, 5\}$. Then $S_7 = (1, 5, 6, 7, 8, 9, 10)$ is a triangulation graph of a septagon (see Fig. 3.8). By Lemmas 3.4 and 3.6, a triangulation graph $(1, 2, 3, 4, 5)$ of a pentagon can be dominated by two watched guards, with one guard at 1, and S_7 can be dominated by two watched guards.

CASE 4: $5+5 = 10$. Let $d = \{1, 6\}$. Then $H_6^1 = (1, 2, 3, 4, 5, 6)$, and $H_6^2 = (1, 6, 7, 8, 9, 10)$ are triangulation graphs of hexagons, and the minimality of d ensures that H_6^2 has diagonal $\{2, 6\}$ (see Fig. 3.8). By Lemma 3.5, H_6^2 can be dominated by two watched guards, with one guard at vertex 6. Place one guard at 1. Now, again by Lemma 3.5, we need at most one additional guard for hexagon H_6^1, as either vertex 1 or 6 is of degree 3 in H_6^1. ∎

LEMMA 3.10 ([66, 92]). *Every triangulation graph of an 11-vertex polygon can be dominated by four watched guards.*

Proof. By Theorem 1.3, in any triangulation graph G_T of an 11-vertex polygon P there is at least one vertex of degree 2. Let the vertices of P be labeled $1, \ldots, 11$, in counterclockwise manner, and assume vertex 2 to be of degree 2. By cutting off the triangle $\triangle = (1, 2, 3)$, we get a triangulation graph G_T^* of a decagon. By Lemma 3.9, G_T^* can be dominated by four watched guards with one guard placed either at vertex 1 or 3. This yields a domination of G_T by four watched guards, as the triangle $\triangle$ is dominated. ∎

Finally, with all preceding lemmas available, the induction proof is a straightforward enumeration of cases.

THEOREM 3.11 ([66, 92]). *Every triangulation graph G_T of a polygon with $n \geq 5$ vertices can be dominated by $\lfloor (3n - 1)/7 \rfloor$ watched guards.*

Proof. Lemmas 3.4–3.10 establish the validity of the assertion for $n = 5, \ldots, 11$, so assume that $n \geq 12$, and that the assertion holds for all $5 \leq \hat{n} < n$. We have the following lemma.

LEMMA 3.12 ([39]). *Suppose that $f(m)$ watched guards are always sufficient to dominate any m-vertex triangulation graph, with $m < n$. Then if G'_T is a triangulation graph of a polygon with n' vertices, with $n' < n$, then a guard g placed at a vertex of G'_T with $f(n'-1)$ additional watched guards are sufficient to dominate G'_T (but, perhaps, g is not watched).*

Proof of Lemma 3.12. Suppose that $f(m)$ watched guards are always sufficient to dominate any m-vertex triangulation graph, with $m < n$, and let G'_T be a triangulation graph of a polygon P' with n' vertices, with $n' < n$. Let u be the vertex at which a guard g is placed, and let v be a vertex adjacent in G'_T to u across the edge e corresponding to an edge of P'. Edge-contraction of G'_T across e produces a graph G^*_T of $n' - 1$ vertices. By Lemma 3.2, G^*_T is a triangulation graph, and so it can be dominated by $f(n'-1)$ watched guards, as $n' - 1 < n$. Let x be the vertex that replaced u and v. Suppose that no guard is placed at x in a domination of G^*_T. Then the same guard placement, with one guard at u, will dominate all of G'_T, since the guard at u dominates the triangle supported by e, and the remaining triangles of G'_T have dominated counterparts in G^*_T. Otherwise, if a guard is used at x in the domination of G^*_T, then he can be assigned to v in G_T, with the remaining guards maintaining their positions. Again, with one additional guard at u, every triangle of G'_T is dominated. Note that all guards that were watched in G^*_T, are watched in G'_T as well. Thus the only guard that may be unwatched is the one at u. ■

Now, we return to the proof of the theorem. Lemma 3.3 guarantees the existence of a diagonal d that partitions G_T into two graphs G^1_T and G^2_T, where G^1_T contains k boundary edges of G with $5 \leq k \leq 8$. Assume k to be minimal. We must consider each value of k separately.

CASE 1: $k = 5$. Let $d = \{0, 5\}$. Then $G^1_T = (0, 1, 2, 3, 4, 5)$ is a triangulation graph of a hexagon. In G^1_T, either vertex 0 or 5, say 0, is of degree at least 3. By Lemma 3.5, G^1_T can be dominated by two watched guards, with one placed at 0. Next, by Lemma 3.12, the guard at 0 permits the remainder of G^2_T to be dominated by $f(n-4-1) = f(n-5)$ watched guards, where $f(m)$ specifies the number of watched guards that are always sufficient to dominate a triangulation graph on $m < n$ vertices. By the induction hypothesis,

$$f(n-5) = \left\lfloor \frac{3(n-5)-1}{7} \right\rfloor \leq \left\lfloor \frac{3n-1}{7} - 2 \right\rfloor = \left\lfloor \frac{3n-1}{7} \right\rfloor - 2$$

watched guards suffice to dominate the remainder of G^2_T. Together with the two watched guards allocated to G^1_T, all of G_T is dominated by at most $\lfloor (3n-1)/7 \rfloor$ watched guards.

CASE 2: $k = 6$. Let $d = \{0, 6\}$. Then $G^1_T = (0, 1, \ldots, 5, 6)$ is a triangulation graph of a septagon. By Lemma 3.6, G^1_T can be dominated by two watched guards. Since G^2_T has $n - 5$ vertices, it can be dominated by $\lfloor (3(n-5)-1)/7 \rfloor \leq \lfloor (3n-1)/7 \rfloor - 2$ watched guards by the induction hypothesis. This yields a domination of G_T by $\lfloor (3n-1)/7 \rfloor$ watched guards.

Case 3: $k = 7$. Let $d = \{0, 7\}$. Then $G_T^1 = (0, 1, \ldots, 6, 7)$ is a triangulation graph of an octagon. In G_T^1, either vertex 0 or 7, say 0, is of degree at least 3. By Lemma 3.7, G_T^1 can be dominated by three watched guards with one placed at 0. Next, we proceed as in Case 1 above. By Lemma 3.12, one guard at 0 permits the remainder of G_T^2 to be dominated by $f(n-6-1) = f(n-7)$ watched guards. By the induction hypothesis,

$$f(n-7) = \left\lfloor \frac{3(n-7)-1}{7} \right\rfloor = \left\lfloor \frac{3n-1}{7} \right\rfloor - 3$$

watched guards suffice to dominate the remainder of G_T^2. Together with the three watched guards allocated to G_T^1, all of G_T is dominated by $\lfloor (3n-1)/7 \rfloor$ watched guards.

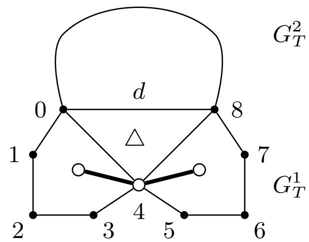

Fig. 3.9. Case $k = 8$: three watched guards are sufficient to dominate the triangulation graph G_T^1.

Case 4: $k = 8$ (see Fig. 3.9). Let $d = \{0, 8\}$. The presence of any of the diagonals $\{0, 7\}$, $\{1, 8\}$, $\{0, 6\}$, $\{2, 8\}$, $\{0, 5\}$ or $\{3, 8\}$ would violate the minimality of k. Consequently, the triangle $\triangle$ in G_T^1 bounded by d is $(0, 4, 8)$. Dominate a 5-vertex triangulation graph $(0, 1, 2, 3, 4)$ by two watched guards with one at 4, and dominate a 5-vertex triangulation graph $(4, 5, 6, 7, 8)$ by two watched guards with one at 4, thus getting a domination of G_T^1 by three watched guards (the triangle $\triangle$ is dominated by the guard at 4). Since G_T^2 has $n-7$ vertices, it can be dominated by $\lfloor (3(n-7)-1)/7 \rfloor = \lfloor (3n-1)/7 \rfloor - 3$ watched guards by the induction hypothesis. This yields a domination of G_T by $\lfloor (3n-1)/7 \rfloor$ watched guards. ■

Thus by Lemma 1.5, we have

Theorem 3.13 ([39]). *For all $n \geq 5$, $\mathrm{gg}(n, 1) = \lfloor (3n-1)/7 \rfloor$, and guards can be located at the vertices of a polygon.*

3.1.3. The minimum watched guards problem

Definition 3.1. Let P be a polygon. The *minimum watched guards* (MinWCG for short) *problem* is to find a watched guard set for P of the minimum cardinality.

As the connectedness of the visibility graph VG implies that VG has no isolated vertices, and of course, the cardinality of a minimum watched guard set is at least the cardinality of a minimum guard set, the transformation procedure in the NP-hardness proof of the minimum guards problem can be directly applied in the NP-hardness proof of the MinWCG problem [56, 58], which results in the following theorem.

Theorem 3.14 ([59]). *The minimum watched guards problem is NP-hard.*

In 1994, following the main idea of the paper [58], Liaw and Lee [59] considered the MinWCG problem for 1-spiral polygons, and they proposed a linear optimal algorithm for this problem. In general, their algorithm was based upon the following observations: in a 1-spiral polygon, there exists an optimal guard placement such that:

- all watched guards are located on the convex chain (cf. Lemma 2.18);
- no connected component of the visibility graph contains more than three watched guards (cf. the proof of Theorem 2.17).

THEOREM 3.15 ([59]). *The* MinWCG *problem for* 1*-spiral polygons can be solved in linear time.*

3.2. Orthogonal polygons

In 1995, Hernández-Peñalver [40] gave the tight bound for the function $\mathrm{gg}_{\perp}(n,1)$. He proved that $\lfloor n/3 \rfloor$ watched guards are always sufficient and occasionally necessary to guard any orthogonal polygon with n vertices. This result was established by induction, using a similar idea to the proof of Theorem 3.11. A few years later, Michael and Pinciu [66] proposed an entirely new proof based upon 3-coloring of a special triangulation graph. This is the proof we shall present here.

Let $G_Q = (V, E)$ be the quadrilateralization graph of a quadrilateralization Q of an orthogonal polygon. We say that $S \subset V$ is a *watched guard set* for G_Q if each quadrilateral face of G_Q has a vertex in S and each vertex v in S is in a quadrilateral face with another element of S. By arguments as in the proof of Lemmas 1.4 and 1.5, one can prove the following lemma.

LEMMA 3.16 ([66]). *Let P be an n-vertex orthogonal polygon, and G_Q be any of its quadrilateralization graphs. If G_Q can be dominated by $f(n)$ watched guards, then P can be covered by $f(n)$ geometric watched vertex guards.*

THEOREM 3.17 ([66]). *For every orthogonal polygon P with $n \geq 6$ vertices, any of its quadrilateralization graphs can be dominated by $\lfloor n/3 \rfloor$ watched guards.*

Proof. As any quadrilateralization graph is bipartite, and its (weak) dual is a tree with each vertex of degree at most 4 (cf. Lemma 1.2), there is a partition $V = V^+ \cup V^-$ of the vertex set of G_Q and a partition $F = F^+ \cup F^-$ of the quadrilateral (face) set F of G_Q. Then in G_Q, each edge joins a vertex in V^+ and a vertex in V^-, and each face (quadrilateral) f contains two vertices in V^+ and two vertices in V^-. Next, we construct a triangulation graph G_T as follows. If $f \in F^+$, then we connect the two vertices of f in V^+ by an edge, otherwise, if $f \in F^-$, we connect the two vertices of f in V^- by an edge. The resulting graph is a triangulation graph G_T of P. Fig. 3.10(a-b) shows an example. We have two crucial properties of G_T; both follow directly from the construction.

(1) Let E_{diag} denote the set of diagonals added to G_Q to obtain G_T, and suppose that two diagonals $\{v, w\}$ and $\{v, w'\}$ in E_{diag} are incident at the vertex v in G_T. Then the two faces of G_Q that contain the diagonals do not share an edge.

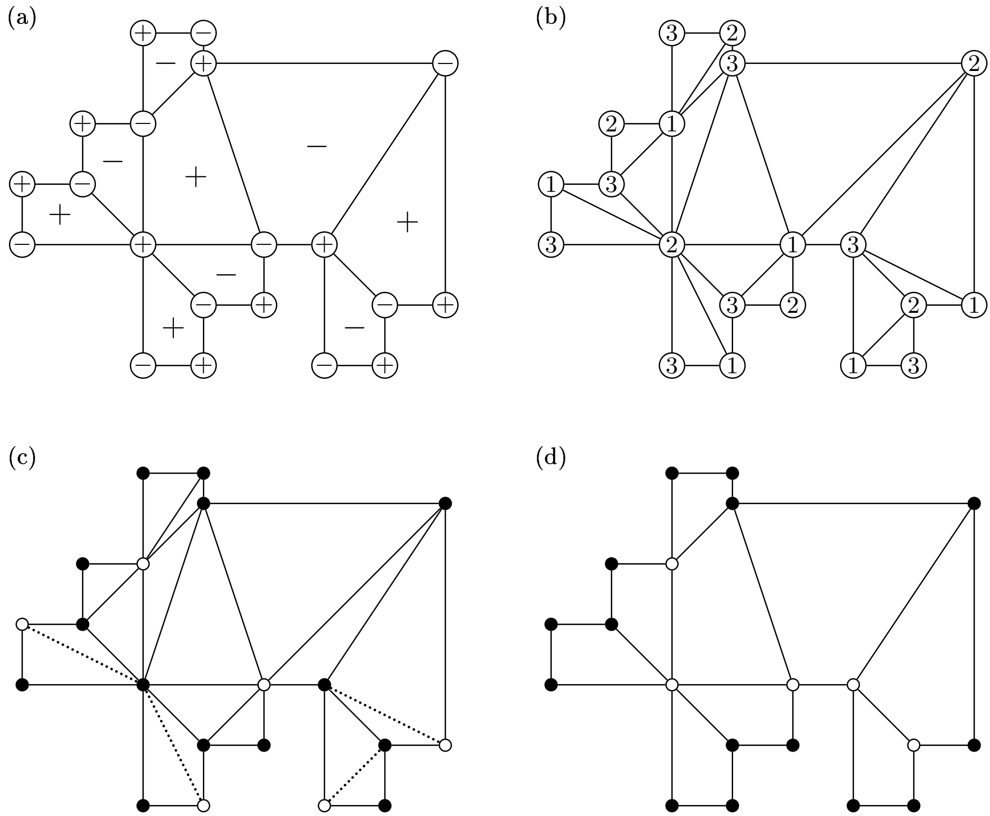

Fig. 3.10. (a) A quadrilateralization graph G_Q with vertex and face bipartitions indicated by $+$ and $-$. (b) The triangulation graph resulting from G_Q and its 3-coloring. (c) The guard set S; guards at vertices of degree 3 have to be shifted along diagonals of quadrilaterals. (d) The final watched guard set.

(2) Any vertex of degree 3 in G_T belongs to exactly one quadrilateral in G_Q.

Following Fisk's proof, the next step is to 3-color the vertices of G_T. Let S be the set of vertices assigned the least frequently used color. Of course, $|S| \leq \lfloor n/3 \rfloor$. Next, as S dominates G_T, S dominates G_Q as well. And if S is watched in G_Q, we are done. Otherwise, we have to *shift* some vertices of S along edges in E_{diag} to obtain a watched guard set; again (see Fig. 3.10(c-d)) for an example.

Let Y denote the set of vertices in S that are of degree 3 in G_T, and let X be the complement of Y in S. By property (2), for each $y \in Y$ there is exactly one edge $\{y, y^*\} \in E_{\text{diag}}$ incident to y, and the vertex y^* is unique. Let $Y^* = \{y^* : y \in Y\}$, and let $S^* = X \cup Y^*$. Of course, $|S^*| \leq \lfloor n/3 \rfloor$. Next, as we have shifted $y \in Y$ to another vertex y^* along an internal diagonal of the unique quadrilateral (property (2)), S^* is a guard set for the quadrilateralization graph G_Q. It remains to show that any vertex v in S^* is in a quadrilateral face of G_Q with another element of S^*. We have to consider two cases.

Case 1: $v \in X$. Then by (1), v is contained in a quadrilateral f of G_Q with a diagonal edge in E_{diag} that is not incident to v. Let $\overline{v}$ be a vertex opposite v in f. Then $\overline{v}$ is assigned the same color as v in our 3-coloring, and thus $\overline{v} \in S^*$. As $\overline{v}$ cannot be of degree 3 in G_T, $\overline{v} \in X \subseteq S^*$.

Case 2: $v \in Y^*$. Then v is the conjugate y^* for some vertex y of degree 3 in G_T. By (2), the diagonal $\{y, v\}$ is in exactly one quadrilateral face of G_Q, say $f_1 = (y, x, v, z)$. As y is of degree 3, both $\{y, x\}$ and $\{y, z\}$ are edges of the polygon, and either $\{v, x\}$ or $\{v, z\}$, say $\{v, x\}$, is a diagonal of the quadrilateralization, as $n \geq 6$. Then $\{v, x\}$ is incident to some other quadrilateral face of G_Q, say $f_2 = (v, x, u, w)$ (see Fig. 3.11). By (1), $\{x, w\} \in E_{\text{diag}}$, and then the vertex w is assigned the same color as y in our 3-coloring of G_T. Hence $w \in S = X \cup Y$. If $w \in X$ then $w \in S^*$, and both $v, w \in f_2$. Otherwise, if $w \in Y$ then $w^* \in S^*$, and both $v, w^* \in f_2$. ■

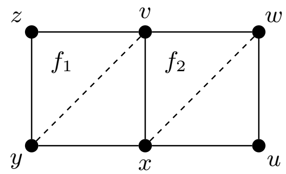

Fig. 3.11. Case 2: $v \in Y^*$.

Theorem 3.18 ([66]). *For all $n \geq 6$, $\mathrm{gg}_\perp(n, 1) = \lfloor n/3 \rfloor$, and guards can be located at the vertices of a polygon.*

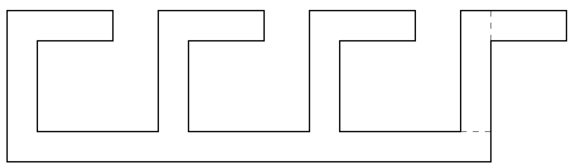

Fig. 3.12. An orthogonal polygon P with n vertices with $\mathrm{gg}_\perp(P, 1) = \lfloor n/3 \rfloor$; here $n = 20$, and $\mathrm{gg}_\perp(P, 1) = 8$.

Proof. The necessity is established by the orthogonal gallery P shown in Fig. 3.12. Each wave requires two watched guards, and it is clear that for $n \equiv 0 \pmod 6$, $\mathrm{gg}_\perp(P, 1) = \lfloor n/3 \rfloor$. The case of $n \equiv 2, 4 \pmod 6$ is indicated with dashed lines. For sufficiency, we apply Lemma 3.16 and Theorem 3.17. ■

3.3. Monotone polygons

In this section, we prove that $\lfloor 2n/5 \rfloor$ watched guards always suffice to cover any monotone polygon with n vertices. Hernández-Peñalver's polygon [39] with $5k$ vertices requiring $2k$

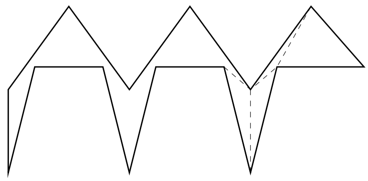

Fig. 3.13. Hernández-Peñalver's polygons of $5k$ vertices requiring $2k$ watched guards; here $k = 3$, and the polygon requires 6 watched guards.

watched guards is a monotone polygon (see Fig. 3.13), thus this bound is tight. The case of $n \equiv 1, 2, 3, 4 \pmod 5$ is indicated with dashed lines.

3.3.1. Monotone mountains. A *monotone mountain* is a monotone polygon one of whose chains is a single edge, called the *base edge.* Although this is a severely restricted class of polygons, it deserves our attention, for we use them to solve the problem for general monotone polygons. Guards for monotone mountains are point guards.

LEMMA 3.19 ([97]). *Suppose that $f(n)$ watched guards placed at the base edge are always sufficient to cover any n-vertex monotone mountain P. Then one vertex guard placed at any endpoint of the base edge with an additional $f(n-1)$ watched guards are sufficient to cover P.*

Proof. We only give the proof for the left-hand vertex of the base edge, the other case can be solved analogously. Let $e_b = \{x_L, x_R\}$ be the base edge, and let $(x_L, v_2, \ldots, v_{n-1}, x_R)$ be the top chain of P.

CASE 1: v_2 is convex (see Fig. 3.14(a)). Then there is an ear at the vertex v_2, otherwise P is not a monotone mountain. Cutting off the triangle (x_L, v_3, v_2) results in an $(n-1)$-vertex monotone mountain $\widehat{P}$ with the same base edge e_b requiring at most $f(n-1)$ watched guards, all placed at e_b. With one additional guard at x_L, we get a coverage of P by $f(n-1)+1$ watched guards, and all guards are placed at the base edge of P.

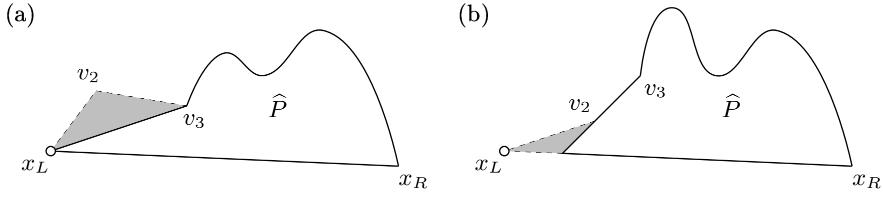

Fig. 3.14. (a) Case 1: there is an ear at v_2. (b) Case 2: v_2 is reflex.

CASE 2: v_2 is reflex (see Fig. 3.14(b)). By moving v_2 to the base edge e_b along the line enclosing the edge $\{v_2, v_3\}$, we get an $(n-1)$-vertex monotone mountain $\widehat{P}$ with base edge $\hat{e}_b$ requiring at most $f(n-1)$ watched guards, all placed at $\hat{e}_b$. With one additional guard at x_L, we get a coverage of P by $f(n-1)+1$ watched guards, and all guards are placed at e_b, as $\hat{e}_b \subset e_b$. ∎

THEOREM 3.20 ([97]). *$\lfloor n/3 \rfloor$ watched guards always suffice to cover a monotone mountain with $n \geq 6$ vertices, and all guards can be placed at the base edge.*

Proof. Let P be a monotone mountain with n vertices. We leave it to the reader to verify the validity of the assertion for $n = 6, 7, 8$. Assume that $n \geq 9$, and the theorem holds for all $6 \leq \hat{n} < n$. Without loss of generality, assume $e_b = \{x_L, x_R\}$ to be the base edge (the bottom chain), and $U = (x_L, v_2, v_3, \ldots, v_{n-1}, x_R)$ to be the top chain of P. Let x be the projection of v_3 onto the base edge e_b along the line perpendicular to the x-axis.

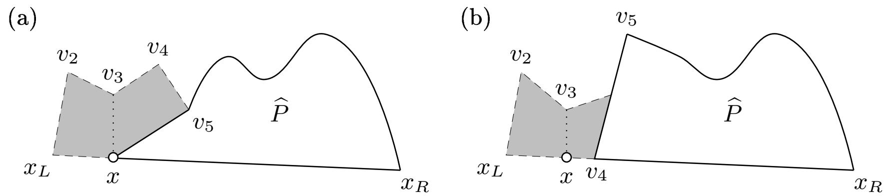

Fig. 3.15. (a) Case 1: x sees v_5. (b) Case 2: x does not see v_5.

CASE 1: x sees v_5 (see Fig. 3.15(a)). Replacing the polyline $(x_L, v_2, v_3, v_4, v_5)$ with the edge $\{x, v_5\}$ results in an $(n-3)$-vertex monotone mountain $\widehat{P}$ with base edge $\hat{e}_b = \{x, x_R\}$. By the induction hypothesis, $\widehat{P}$ can be covered by $\lfloor (n-3)/3 \rfloor$ watched guards, all placed at $\hat{e}_b$. As one additional guard at x covers the hexagon $(x_L, x, v_5, v_4, v_3, v_2)$, we get a coverage of P by $\lfloor (n-3)/3 \rfloor + 1 = \lfloor n/3 \rfloor$ watched guards, and all guards are placed at e_b, as $\hat{e}_b \subset e_b$.

CASE 2: x does not see v_5 (see Fig. 3.15(b)). By moving v_4 to e_b along the line enclosing the edge $\{v_4, v_5\}$, we get an $(n-3)$-vertex monotone mountain $\widehat{P}$ with base edge $\hat{e}_b = \{v_4, x_R\}$. $\widehat{P}$ requires $\lfloor (n-3)/3 \rfloor$ watched guards, all placed at $\hat{e}_b$ by the induction hypothesis. With one additional guard at x we get a coverage of P by $\lfloor (n-3)/3 \rfloor + 1 = \lfloor n/3 \rfloor$ watched guards, and all guards are placed at the edge e_b, as $\hat{e}_b \subset e_b$. ∎

Note that if a monotone mountain has five vertices, then it can be guarded by one guard located at its base edge; this is a degenerate case, because the guard is unwatched. Chvátal's comb-polygons are examples of monotone mountains of $3k$ vertices requiring $k = \lfloor n/3 \rfloor$ watched guards, thus this bound is tight. Moreover, the proof of Theorem 3.20 shows that there is always a cooperative guard set of cardinality $\lfloor n/3 \rfloor$ for any monotone mountain with all guards placed at the base edge.

COROLLARY 3.21 ([97]). *For all $n \geq 6$, $\mathrm{cg}_{\mathrm{mountain}}(n) = \lfloor n/3 \rfloor$, that is, $\lfloor n/3 \rfloor$ cooperative guards are sometimes necessary and always sufficient to cover a monotone mountain, and all of them can be located at its base edge.*

3.3.2. Monotone polygons. Now, we return to monotone polygons. Let P be a monotone polygon. Call a vertex of P the *left-hand* or *right-hand* vertex if there are no other vertices of P to the left or to the right of it, respectively. If there are two left-hand (right-hand) vertices, we choose anyone of them.

THEOREM 3.22 ([97]). *For all* $n \geq 5$, $\text{gg}_{\text{monotone}}(n,1) = \lfloor 2n/5 \rfloor$, *that is,* $\lfloor 2n/5 \rfloor$ *watched guards always suffice to cover a monotone polygon with* n *vertices.*

Proof. Let P be a monotone polygon with n vertices. By Theorem 3.13, the assertion holds for $n = 5, \ldots, 11$, so assume that $n \geq 12$, and that the assertion holds for all $5 \leq \hat{n} < n$. We have the following lemma (we omit the proof as it involves a long cascade of cases; for details see [97]).

LEMMA 3.23 ([97]). *Let* $\widehat{P}$ *be a monotone polygon with* $\hat{n}$ *vertices,* $5 \leq \hat{n} < n$, *let* x *be any vertex of* $\widehat{P}$, *and let* $\hat{x}$ *be either the left-hand or the right-hand vertex of* $\widehat{P}$. *Then*:

- (a) *If we place one guard* g *at* x, *then* $\lfloor 2(\hat{n}-1)/5 \rfloor$ *additional watched guards will be sufficient to cover all of* $\widehat{P}$ *(but, perhaps* g *is unwatched).*
- (b) *If we place two guards* g_1 *and* g_2 *at the endpoints of any edge* $d = \{\hat{x}, y\}$, *then* $\lfloor 2(\hat{n}-2)/5 \rfloor$ *additional watched guards will be sufficient to cover all of* $\widehat{P}$.

By Lemma 3.3, there is a diagonal d that partitions P into two polygons P_1 and P_2, where P_1 contains k edges of P, with $5 \leq k \leq 8$. Observe that P_1 and P_2 are monotone polygons as well. Assume k to be minimal. We have to consider each value of k separately.

CASE 1: $k = 5$. Let $d = \{v_1, v_6\}$. $P_1 = (v_1, v_2, v_3.v_4, v_5, v_6)$ is a hexagon, and by Lemma 3.5, it can be covered by two watched guards, with one placed either at v_1 or v_6. Next, by Lemma 3.23, the guard either at v_1 or v_6 permits the remainder of P_2 to be covered by $\lfloor 2[(n-4)-1]/5 \rfloor = \lfloor 2n/5 \rfloor - 2$ watched guards by the induction hypothesis. Together with the two guards allocated to P_1, all of P is covered by $\lfloor 2n/5 \rfloor$ watched guards.

CASE 2: $k = 6$. P_1 is a septagon. By Lemma 3.6, it can be covered by two watched guards. Since P_2 has $n-5$ vertices, it can be covered by at most $\lfloor 2n/5 \rfloor - 2$ watched guards by the induction hypothesis. This yields a coverage of P by $\lfloor 2n/5 \rfloor$ watched guards.

CASE 3: $k = 7$. Let $d = \{v_1, v_8\}$. Then $P_1 = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8)$ is an octagon. Let x_L and x_R be the left-hand and right-hand vertices of P, respectively. If both x_L and x_R are in P_2, then P_1 is a monotone mountain. By the induction hypothesis, P_2 can be covered by $\lfloor 2(n-6)/5 \rfloor$ watched guards, and by Theorem 3.20, P_1 can be covered by two watched guards. Thus all of P can be covered by at most $\lfloor 2n/5 \rfloor$ watched guards.

If both x_L and x_R are in P_1, then P_2 is a monotone mountain with base edge d. By Lemma 3.7, P_1 can be covered by three watched guards with one guard placed either at v_1 or v_8, and by Lemma 3.19 and Theorem 3.20, the guard either at v_1 or v_8 permits the remainder of P_2 to be covered by $\lfloor ((n-6)-1)/3 \rfloor$ watched guards. Since for all $n \geq 12$, $\lfloor (n-7)/3 \rfloor + 3 \leq \lfloor 2n/5 \rfloor$, $\lfloor 2n/5 \rfloor$ watched guards will cover all of P.

Finally, we can assume that x_R is in P_1 and x_L is in P_2. The minimality of k ensures that both v_1 and v_8 see a vertex v of P_1, more precisely, the diagonals $d_1 = \{v_1, v_4\}$ and $d_2 = \{v_4, v_8\}$ cut off a quadrilateral and a pentagon from P_1, respectively (see Fig. 3.16).

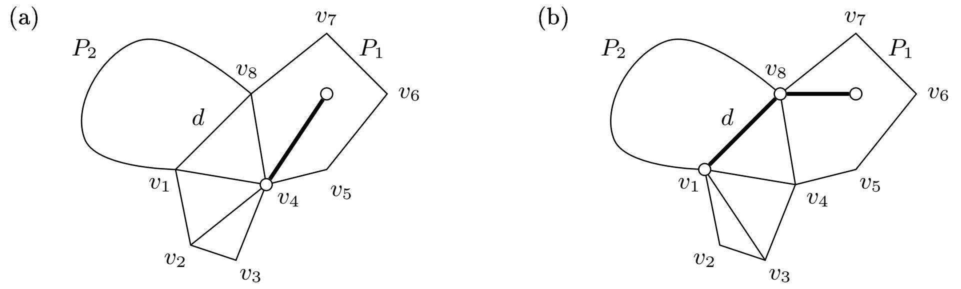

Fig. 3.16. (a) Case 3.a, (b) Case 3.b.

SUBCASE 3.a: v_4 sees v_2. Then by Lemma 3.4, two watched guards will cover the pentagon $(v_4, v_5, v_6, v_7, v_8)$, and one of them can be placed at v_4. As the guard at v_4 will cover both the quadrilateral (v_1, v_2, v_3, v_4) and the triangle (v_1, v_4, v_8), we get a complete coverage of P_1. Together with $\lfloor 2(n-6)/5 \rfloor$ watched guards for P_2 by the induction hypothesis, we get a coverage of P by at most $\lfloor 2n/5 \rfloor$ watched guards.

SUBCASE 3.b: v_4 does not see v_2, but v_1 sees v_3. Then by Lemma 3.4, two watched guards will cover the pentagon $(v_4, v_5, v_6, v_7, v_8)$, and one of them can be placed at v_8. One additional guard at v_1 will cover the quadrilateral (v_1, v_2, v_3, v_4) and the triangle (v_1, v_4, v_8), so we get a complete coverage of P_1, with two guards placed at the edge $\{v_1, v_8\}$. As P_2 is now a monotone polygon with the right-hand vertex either at v_1 or at v_8, two guards at the endpoints of the edge $\{v_1, v_8\}$ permit the remainder of P_2 to be covered by $\lfloor 2(n-6-2)/5 \rfloor \leq \lfloor 2n/5 \rfloor - 3$ watched guards by Lemma 3.23 and by the induction hypothesis. Together with the three watched guards allocated to P_1, all of P is covered by $\lfloor 2n/5 \rfloor$ watched guards.

CASE 4: $k = 8$ (see Fig. 3.17). Let $d = \{v_1, v_9\}$. Then $P_1 = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9)$ is an enneagon. The minimality of k ensures that v_1 and v_9 sees a vertex v of P_1, more precisely, the diagonals $d_1 = \{v_1, v_5\}$ and $d_1 = \{v_5, v_9\}$ cut off respectively two pentagons from P_1: $P_5^1 = (v_1, v_2, v_3, v_4, v_5)$ and $P_5^2 = (v_5, v_6, v_7, v_8, v_9)$. Let x_L and x_R be the left-hand and right-hand vertices of P, respectively.

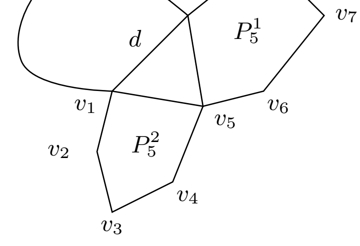

Fig. 3.17. Case 4.

If both x_L and x_R are in P_2, then P_1 is a monotone mountain with base edge d. By the induction hypothesis, P_2 can be covered by $\lfloor 2(n-7)/5 \rfloor$ watched guards. As P_5^1 and P_5^2 are monotone mountains with base edges $\{v_1, v_5\}$ and $\{v_5, v_9\}$, respectively, they can be guarded by two guards g_1 and g_2 placed at some point of $\{v_1, v_5\}$ and $\{v_5, v_9\}$, respectively. As g_1 and g_2 see each other, and cover the triangle (v_1, v_5, v_9), the whole of P_1 is covered by two watched guards. Thus all of P is covered by $\lfloor 2(n-7)/5 \rfloor + 2 \leq \lfloor 2n/5 \rfloor$ watched guards (note that these guards may be point guards).

If both x_L and x_R are in P_1, then P_2 is a monotone mountain with base edge d. Cover the pentagon P_5^1 by two watched guards with one guard at v_5, and cover P_5^2 by two watched guards with one guard at v_5, thus getting a coverage of P_1 by three watched guards (the triangle (v_1, v_5, v_9) is covered by the guard placed at v_5). Since the monotone mountain P_2 has $n-7$ vertices, it can be covered by $\lfloor (n-7)/3 \rfloor$ watched guards by Theorem 3.20. This yields a coverage of P by at most $\lfloor 2n/5 \rfloor$ watched guards, as $n \geq 12$.

Finally, we can assume that x_R is in P_1 and x_L is in P_2. It is obvious that either P_5^1 or P_5^2 is a monotone mountain, so it can be guarded by one guard g placed at its base edge e_b. Cutting off this pentagon results in a monotone polygon $\widehat{P}$ with $n-3$ vertices that can be covered by $\lfloor 2(n-3)/5 \rfloor$ watched guards by the induction hypothesis. As e_b is one of the edges of $\widehat{P}$, each of its points is covered. Consequently, there is a guard in $\widehat{P}$ that sees g. Hence P can be covered by $\lfloor 2(n-3)/5 \rfloor + 1 \leq \lfloor 2n/5 \rfloor$ watched guards. ■

Guards for monotone polygons are point guards, so it is natural to ask if we can restrict guards to be located at vertices only. A slight change of the construction in the case of star-shaped polygons presented in Section 3.4 shows that $\lfloor (3n-1)/7 \rfloor$ vertex guards are sometimes needed for monotone polygons: the polygon $P^{(i)}$ we add has to be monotone with respect to the line L, too. Note that the star-shaped polygon with five vertices shown in Fig. 3.20 is a monotone polygon with respect to the y-axis, and it requires two vertex watched guards.

COROLLARY 3.24 ([97]). *If guards are restricted to be located at the vertices of a polygon only, then for each $n \geq 5$, $\mathrm{gg}_{\mathrm{monotone}}^{\mathrm{vertex}}(n, 1) = \lfloor (3n-1)/7 \rfloor$.*

3.3.3. Spiral polygons. For every $k \geq 1$, there is a spiral polygon with $5k$ vertices requiring $2k$ watched guards (see Fig. 3.18). The case of $n \equiv 1, 2, 3, 4 \pmod 5$ is indicated with dashed lines. Thus $\lfloor 2n/5 \rfloor$ is a lower bound for watched guards in spiral polygons. We will show this bound to be also an upper bound. First, we recall an important property of any spiral polygon.

LEMMA 3.25 ([2]). *There exists a triangulation of a spiral polygon whose dual graph is a path.*

Proof. Let P be an n-vertex spiral polygon, and let $(x, r_1, \ldots, r_k, y)$ and $(y, c_1, \ldots, c_t, x)$ be the reflex chain and the convex chain of P, respectively, traversing the boundary of P in counterclockwise manner (see Chapter 2, Section 2.2.3). Then the segment $r_1 c_t$ lies entirely within P. Thus by joining r_1 and c_t and deleting x (and its associated edges) from P, we get a spiral polygon P' with $n-1$ vertices. This polygon either has $(c_1, r_1, \ldots, r_k, y)$

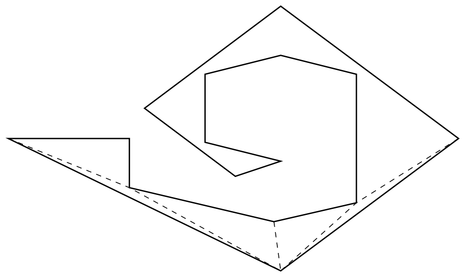

Fig. 3.18. Spiral polygons of $5k$ vertices requiring $2k$ watched guards; here $k = 3$, and the polygon requires 6 watched guards.

as its reflex chain and $(c_1, \ldots, c_t, y)$ as its convex chain or has $(r_1, \ldots, r_k, y)$ as its reflex chain and $(r_1, c_1, \ldots, c_t, y)$ as its convex chain (see Fig. 3.19). In both cases, this triangulation process can be continued until the resulting polygon is a triangle. And it is easy to see that all diagonals will form a triangulation whose dual graph is a path. ∎

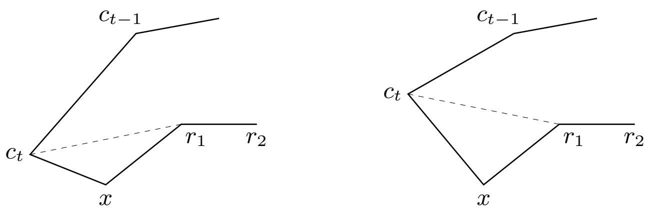

Fig. 3.19. Every spiral polygon has a triangulation whose dual graph is a path.

THEOREM 3.26 ([97]). *For every $n \geq 5$, $\mathrm{gg}_{\mathrm{spiral}}(n, 1) = \lfloor 2n/5 \rfloor$, that is, $\lfloor 2n/5 \rfloor$ watched guards always suffice to cover a spiral polygon with $n \geq 5$ vertices, and the guards can be vertex guards.*

Proof. The proof uses a similar method to that for diagonal guards in spiral polygons [2]. By Theorem 3.13, the assertion holds for $n = 5, 6, 7, 8, 9$, so assume that $n \geq 10$, and that the assertion holds for polygons with fewer than n vertices. Now, by choosing a triangulation whose dual graph is a path on $n - 2$ vertices as guaranteed by Lemma 3.25, there exists a diagonal d cutting off a septagon from a spiral polygon P. By Lemma 3.6, this septagon can always be covered by two watched guards. By the induction hypothesis, the remainder of P with $n - 5$ vertices can be covered by $\lfloor 2(n-5)/5 \rfloor$ watched guards, therefore $\lfloor 2n/5 \rfloor$ watched guards suffice to cover all of P. ∎

3.4. Star-shaped polygons

Similarly to the case of the cooperative guards problem in star-shaped polygons, we ask about the number of watched vertex guards that are sometimes necessary but always sufficient to cover an n-vertex star-shaped polygon. Again, set $f(n) = \lfloor (3n-1)/7 \rfloor$, and recall that we only have to treat the cases $n \equiv 1, 3, 5 \pmod 7$, as these are the critical values of n for which $f(n) > f(n-1)$; we can always add one or two vertices to our polygons to deal with $n \equiv 0, 2, 4, 6 \pmod 7$.

Figs. 3.20–3.21 show star-shaped polygons with $n = 5, 8, 10$ vertices, respectively, that require $\lfloor (3n-1)/7 \rfloor$ watched vertex guards. Similarly to the case of cooperative guards, a polygon P_n with n vertices is constructed from a polygon P_{n-7} by adjoining a special enneagon P at the distinguished edge d^{k-1} with a suitable orientation (see Fig. 3.23). The correctness proof for the construction uses the idea of the proof of Theorem 2.14.

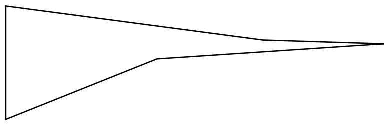

Fig. 3.20. A 5-vertex star-shaped polygon may require 2 vertex watched guards.

(a) (b)

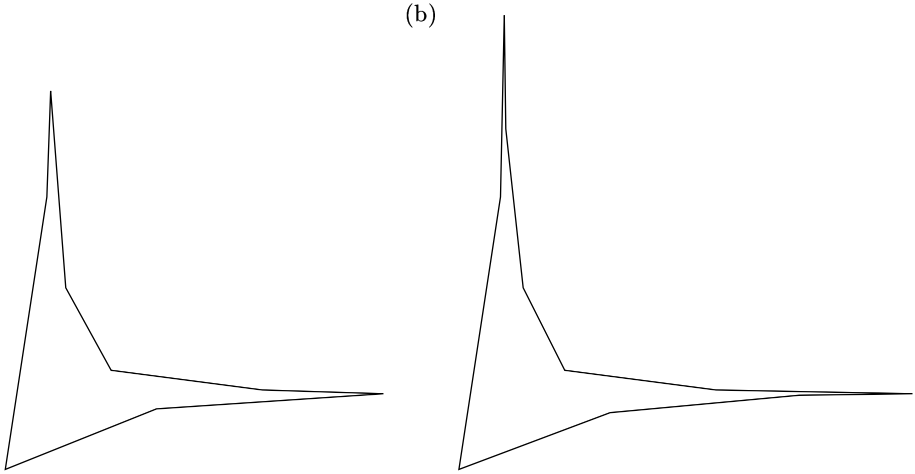

Fig. 3.21. (a) An 8-vertex star-shaped polygon that requires 3 watched vertex guards. (b) A 10-vertex star-shaped polygon that requires 4 vertex watched guards.

THEOREM 3.27 ([97]). *For every $n \geq 5$, there exists a star-shaped polygon that requires $\lfloor (3n-1)/7 \rfloor$ watched vertex guards.*

Proof. Here we only deal with the case $n = 7k + 5$, the other two cases can be solved in a similar way. Consider the 5-vertex polygon P_0 shown in Fig. 3.22. The construction of a star-shaped polygon P_{star} with $n = 7k+5$ vertices starts with the following observations:

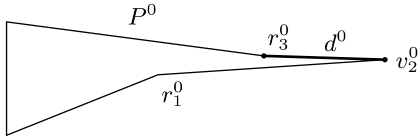

Fig. 3.22. None of the two watched vertex guards covering P^0 can be simultaneously located at the endpoints v_2^0, r_3^0 of the edge d^0, otherwise a third guard is needed.

- $\beta(d^0) = \{r_1^0, v_2^0, r_3^0\}$.
- P^0 requires two watched vertex guards.
- Two watched vertex guards covering P^0 cannot be simultaneously located at the endpoints v_2^0, r_3^0 of the edge d^0, otherwise a third guard is needed.
- Even with an additional point guard at an internal point of the edge d^0, the polygon P^0 still requires two watched vertex guards.

Next, let x be a point from the kernel of P^0. For each $i = 1, \ldots, k$, we sequentially adjoin a 9-vertex polygon $P^{(i)} \subset \alpha(x, d^{i-1})$ at the diagonal d^{i-1} of the polygon $P^{(i-1)}$ (see Fig. 3.23). Each of the polygons $P^{(i)}$ can be guarded from x, hence the polygon $P_{\text{star}} = \bigcup_{i=0}^{k} P^{(i)}$ is star-shaped.

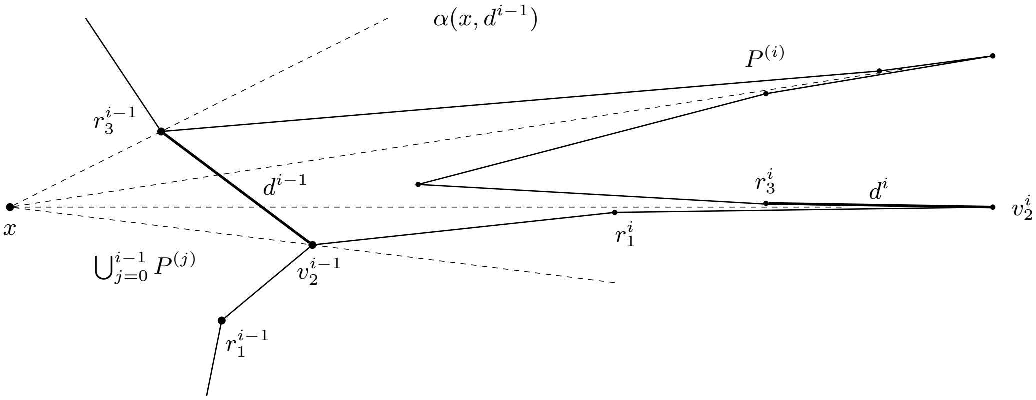

Fig. 3.23. Illustration of the construction of the polygon P_n.

The necessity of $\lfloor (3n-1)/7 \rfloor$ vertex watched guards is established by induction. Let S be a minimum watched vertex guard set for $P_{\text{star}} = \bigcup_{i=0}^{k-1} P^{(i)} \cup P^{(k)}$. Similarly to the case of cooperative vertex guards, the following claim is crucial:

$$\left| S \cap \bigcup_{i=0}^{k-1} P^{(i)} \right| \geq 3(k-1) + 2.$$

Reason: we may suppose that there is a guard $g \in S$ at r_1^k. Consider the guard set resulting by moving g along the line $l \supseteq xr_1^k$ towards x to the new location $p = l \cap d^{k-1}$. Clearly, such a move increases the visibility area of g in $\bigcup_{i=0}^{k-1} P^{(i)}$. However, by the induction hypothesis, with the new guard g at the point $p \neq r_3^{k-1}$ of d^{k-1}, the polygon $\bigcup_{i=0}^{k-1} P^{(i)}$ still requires $3(k-1)+2$ watched vertex guards located in $\bigcup_{i=0}^{k-1} P^{(i)}$.

Consequently, by the induction hypothesis and by the above claim:

- $\beta(d^{k-1}) = \{r_1^{k-1}, v_2^{k-1}, r_3^{k-1}\}$, thus no point of d^k is seen from any vertex of $\bigcup_{i=0}^{k-1} P^{(i)}$. Hence $\beta(d^k) = \{r_1^k, v_2^k, r_3^k\}$ and $P^{(k)}$ needs additional guards.
- We need two new guards only if there are guards at both endpoints of d^{k-1}, but this requires $[3(k-1)+2]+1$ watched guards for $\bigcup_{i=0}^{k-1} P^{(i)}$, thus we get $3k+2$ watched guards for P_{star}, and no pair of them can be located at both endpoints of d^k.
- Otherwise, we need three guards for $P^{(k)}$, and no two of them can be located at the endpoints of d^k. Together with $3(k-1)+2$ watched guards for $\bigcup_{i=0}^{k-1} P^{(i)}$, we have $3k+2$ watched guards for P_{star}.
- If we require two guards at both endpoints of d^k, $3k+3$ watched vertex guards for P_{star} are needed.
- By similar arguments to the proof of the above claim, an additional point guard at any internal point of edge d^k does not change the necessary number of watched vertex guards for $\bigcup_{i=0}^{k} P^{(i)}$.

Therefore P_{star} requires $3k+2 = \lfloor (3n-1)/7 \rfloor$ vertex watched guards. ■

Corollary 3.28 ([97]). *For every $n \geq 5$, $\mathrm{gg}_{\text{star-shaped}}^{\text{vertex}}(n,1) = \lfloor (3n-1)/7 \rfloor$, that is, $\lfloor (3n-1)/7 \rfloor$ vertex watched guards always suffice and are sometimes necessary to cover an n-vertex star-shaped polygon.*

4. ART GALLERIES WITH k-GUARDED GUARDS

The k-guarded guards problem is a generalization of the weakly cooperative guards problem, and it was raised by Michael and Pinciu [67]. A set of guards is called *k-guarded* if each guard is himself seen by at least k of its colleagues (the minimum degree of the visibility graph of the set of guards is at least k). Clearly, the 1-guarded guards problem is equivalent to the weakly cooperative guards problem, and it was discussed in Chapter 3, thus herein we shall only deal with the case of $k \geq 2$, and we will show the $(k\lfloor n/5 \rfloor + \lfloor (n+2)/5 \rfloor)$-bound to be tight.

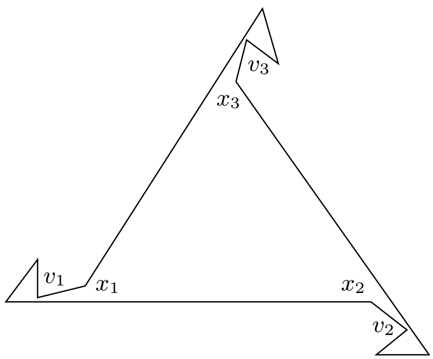

Fig. 4.1. For $k \geq 2$, a polygon with 12 vertices can be guarded by $(2k+2)$ k-guards.

Recall the 12-vertex polygon P shown in Fig. 4.1. It requires five watched guards, but if we consider the k-guarded guards problem with $k \geq 2$, then $(2k+2)$ k-guards are enough: we have to place $k-1$ guards at x_1, $k-1$ guards at x_2, and one guard per each of the vertices v_1, v_2, v_3 and x_3. This explains the discrepancy between the $\lfloor (3n-1)/7 \rfloor$- and $(k\lfloor n/5 \rfloor + \lfloor (n+2)/5 \rfloor)$-bound.

Note that as we shall allow choosing a vertex many times for the location of a guard, we have to modify the definition of the subgraph induced by a (multi)set S.

Definition 4.1. Let $G = (V, E)$ be a graph. The *subgraph induced by a multiset S* (with elements in V) is the graph with vertex set S where two vertices v_1 and v_2 are adjacent if and only if either they correspond to the same vertex or they are adjacent in G.

Thus placing t guards at a vertex of a triangulation graph yields a clique of size t spanned on all copies of this vertex in the induced subgraph. From now on, a k-guarded guard is simply referred to as a *k-guard*.

4.1. Arbitrary polygons

The idea of the proof follows the proof of Theorem 3.13: first, we have to establish the sufficiency bound for small triangulation graphs.

Lemma 4.1 ([98]).

(a) *Every triangulation graph of a pentagon can be dominated by* $(k+1)$ k*-guards with* k *guards placed at any selected vertex.*
(b) *Let* G_T *be a triangulation graph of a hexagon, and let* x *be a vertex of degree at least* 3*. Then* G_T *can be dominated by* $(k+1)$ k*-guards, with* k *guards at* x*.*
(c) *Every triangulation graph of a septagon can be dominated by* $(k+1)$ k*-guards.*
(d) *Let* G_T *be a triangulation graph of an octagon, and let* x *be a vertex of degree at least* 3*. Then* G_T *can be dominated by* $(k+2)$ k*-guards, with one guard at* x*.*
(e) *Every triangulation graph of an enneagon can be dominated by* $(k+2)$ k*-guards.*

Proof. We omit the proof, as it follows closely the lines of the proofs of Lemmas 3.4–3.10, respectively. ■

Lemma 4.2 ([98]). *Let* G_T *be a triangulation graph of an octagon and let* x *be any degree* 2 *vertex. Then one guard* g *at* x *with an additional* $k+1$ *combinatorial* k*-guards are sufficient to dominate* G_T *(but, perhaps,* g *is not adjacent to any other guard).*

Proof. Let the vertices of the octagon be labeled $1, \ldots, 8$, in counterclockwise manner, and assume 1 to be of degree 2. Placing a guard at 1 and cutting off the triangle $(1,2,8)$ from G_T results in the triangulation graph G_T^* of a septagon. By Corollary 4.1, G_T^* can be dominated by $k+1$ combinatorial k-guards, which completes the proof. ■

Lemma 4.3 ([98]). *Every triangulation graph* G_T *of a decagon can be dominated by* $(2k+2)$ k*-guards with a guard placed at any selected vertex.*

Proof. Let the vertices of the decagon be labeled counterclockwise, assuming that 1 is the selected vertex. First, suppose that vertex 1 is of degree at least 3. Then there is a diagonal d with one of its endpoints at 1. This diagonal partitions the ten boundary edges of G_T according to either $2+8=10$, $3+7=10$, $4+6=10$ or $5+5=10$. Assume that d cuts off the minimal number of vertices.

Case 1: $2+8=10$. Let $d=\{1,3\}$. Then $E_9=(1,3,4,5,6,7,8,9,10)$ is a triangulation graph of an enneagon (see Fig. 4.2(a)). By Lemma 4.1(d), E_9 can be dominated by $k+2$ combinatorial k-guards. One of these guards dominates vertex 1. By placing k additional guards at 1, we get a domination of the triangle $(1,2,3)$, and the resulting guard set is k-guarded.

Case 2: $3+7=10$. Let $d=\{1,4\}$. Then $O_8=(1,4,5,6,7,8,9,10)$ is a triangulation graph of an octagon, and the minimality of d ensures that the quadrilateral $(1,2,3,4)$ has the diagonal $\{2,4\}$ (see Fig. 4.2(b)). By Lemma 4.1(d), O_8 can be dominated by $k+2$ combinatorial k-guards, with one guard either at 1 or at 4.

(1) By placing $k-1$ additional guards at 1 and one guard at 4, we get a domination of all triangles in G_T, and the resulting guard set is k-guarded.

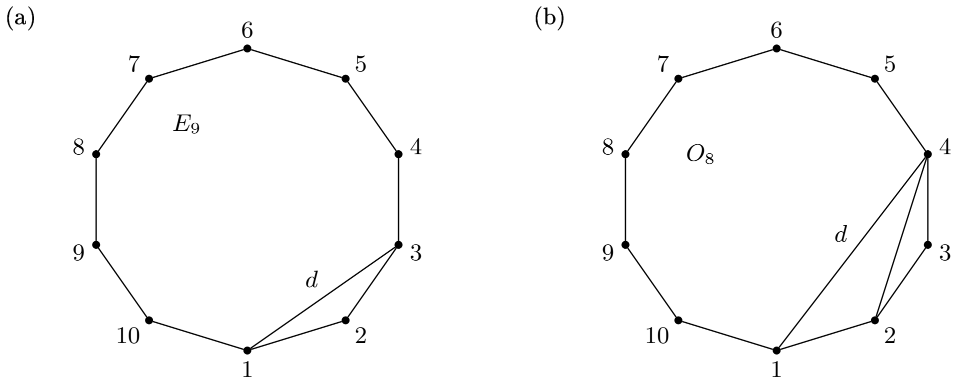

Fig. 4.2. A domination of a 10-vertex triangulation graph—Cases 1 and 2.

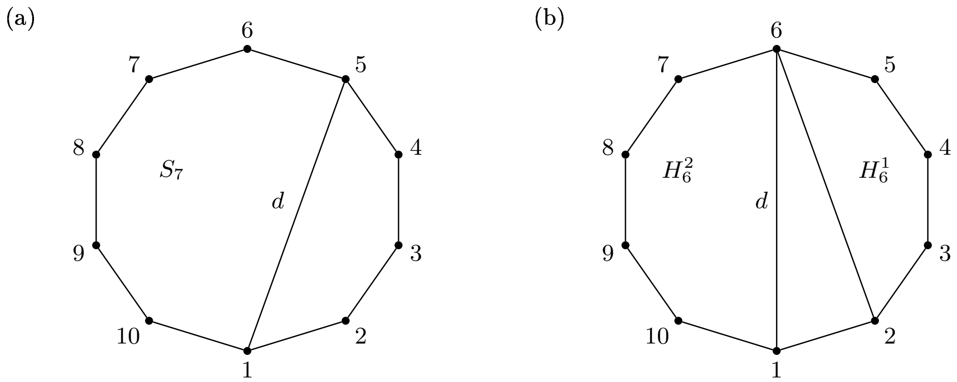

Fig. 4.3. A domination of a 10-vertex triangulation graph—Cases 3 and 4.

(4) All triangles of G_T are dominated. Now, we place k additional guards at 1; they are k-guarded, as there is a guard at 4.

Case 3: $4 + 6 = 10$. Let $d = \{1, 5\}$. Then $S_7 = (1, 5, 6, 7, 8, 9, 10)$ is a triangulation graph of a septagon (see Fig. 4.3(a)). By Lemma 4.1(c), S_7 can be dominated by $k + 1$ combinatorial k-guards, and a 5-vertex triangulation graph $(1, 2, 3, 4, 5)$ can be dominated by $k + 1$ combinatorial k-guards, with k guards at 1.

Case 4: $5 + 5 = 10$. Let $d = \{1, 6\}$. Then $H_6^1 = (1, 2, 3, 4, 5, 6)$ and $H_6^2 = (1, 6, 7, 8, 9, 10)$ are triangulation graphs of hexagons, and the minimality of d ensures that H_6^1 has diagonal $\{2, 6\}$ (see Fig. 4.3(b)). By Lemma 4.1(b), H_6^1 can be dominated by $k+1$ combinatorial k-guards, with k guards at 6. Place k guards at 1. As either vertex 1 or 6 is of degree 3 in H_6^2, we need at most one additional guard for H_6^2 to be k-guarded by Lemma 4.1.

Thus the lemma holds for all vertices of degree at least 3. Now, assume vertex 1 to be of degree 2. $E_9 = (2, 3, 4, 5, 6, 7, 8, 9, 10)$ is a triangulation graph of an enneagon. We proceed in four cases, depending on the triangle $\triangle$ in E_9 bounded by the diagonal $\{2, 10\}$.

CASE 5: $\triangle = (2,3,10)$ (see Fig. 4.4(a)). $O_8 = (3,4,5,6,7,8,9,10)$ is a triangulation graph of an octagon. Place k guards at vertex 1 and one guard at 10. By Lemma 4.2, the guard at 10 permits the remainder of O_8 to be dominated by at most $(k+1)$ k-guards.

CASE 6: $\triangle = (2,4,10)$ (see Fig. 4.4(b)). Then $S_7 = (4,5,6,7,8,9,10)$ is a triangulation graph of a septagon, and by Lemma 4.1(c), S_7 can be dominated by $k+1$ combinatorial k-guards. Place k guards at 1 and one guard at 2; all of G_T is k-guarded.

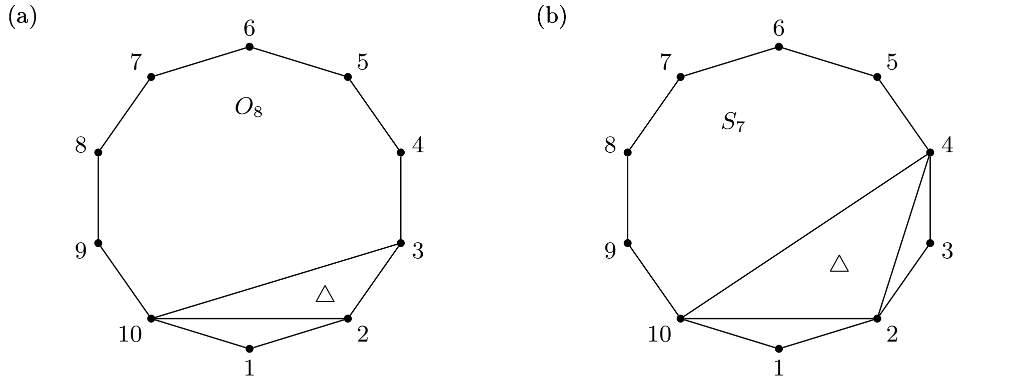

Fig. 4.4. A domination of a 10-vertex triangulation graph—Cases 5 and 6.

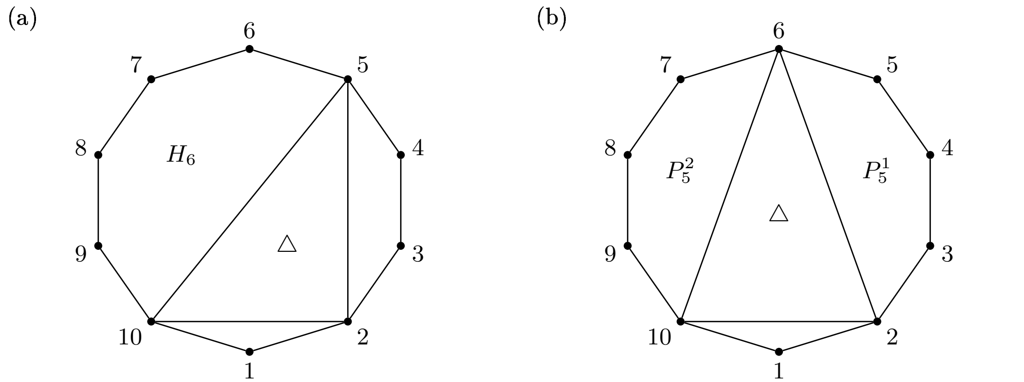

Fig. 4.5. A domination of a 10-vertex triangulation graph—Cases 7 and 8.

CASE 7: $\triangle = (2,5,10)$ (see Fig. 4.5(a)). Then $H_6 = (5,6,7,8,9,10)$ is a triangulation graph of a hexagon, and by Lemma 4.1(b), H_6 can be dominated by $k+1$ combinatorial k-guards, with k guards either at 5 or at 10.

(5) By placing k additional guards at 1 and one guard at 2, we get a domination of all triangles in G_T, regardless of how the quadrilateral $(2,3,4,5)$ is triangulated; the resulting guard set is k-guarded.

(10) By placing k additional guards at 1 and one guard at either 2 or 5, depending on how the quadrilateral $(2, 3, 4, 5)$ is triangulated, we get a domination of all triangles in G_T, and the resulting guard set is k-guarded.

CASE 8: $\triangle = (2, 6, 10)$ (see Fig. 4.5(b)). Then $P_5^1 = (2, 3, 4, 5, 6)$ and $P_5^2 = (6, 7, 8, 9, 10)$ are triangulation graphs of pentagons. By placing one guard at 1, $k-1$ guards at 2, $k-1$ guards at 10, and one guard at 6, with one additional guard for P_5^1, and one additional guard for P_5^2, we get a k-guarded domination of G_T. ∎

LEMMA 4.4 ([98]). *Every triangulation graph of an 11-vertex polygon can be dominated by $(2k+2)$ k-guards.*

Proof. In any triangulation graph G_T of a polygon, there is at least one vertex of degree 2. Let the vertices of an 11-vertex polygon be labeled $1, \ldots, 11$, in counterclockwise manner, and assume vertex 1 to be of degree 2. Cutting off the triangle $\triangle = (1, 2, 11)$ from G_T results in the triangulation G_T^* of a 10-vertex polygon. By Lemma 4.3, G_T^* can be dominated by $(2k+2)$ k-guards, with one guard placed at 2. This yields a domination of G_T by $2k+2$ combinatorial k-guards, as the triangle $\triangle$ is dominated. ∎

LEMMA 4.5 ([98]). *Let G_T be a triangulation graph of a 12-vertex polygon. Then G_T can be dominated by $(2k+2)$ k-guards.*

Proof. Lemma 3.3 guarantees the existence of a diagonal d that splits G_T into two graphs G_T^1 and G_T^2, where G_T^1 contains l boundary edges of G_T with $5 \leq l \leq 8$. Assume that l is minimal. We consider each value of l separately.

CASE 1: $l = 5$. Let $d = \{1, 6\}$. Then G_T^1 and G_T^2 are triangulation graphs of a hexagon $(1, 2, 3, 4, 5, 6)$ and an octagon $(1, 6, 7, 8, 9, 10, 11, 12)$, respectively. By Lemma 4.1, G_T^1 can be dominated by $(k+1)$ k-guards, with k guards either at 1 or at 6. By Lemma 4.2, those k guards permit the remainder of O_8 to be dominated by at most $(k+1)$ k-guards.

CASE 2: $l = 6$. Let $d = \{1, 7\}$. Then G_T^1 and G_T^2 are triangulation graphs of septagons, and by Lemma 4.1, they can together be dominated by $2k+2$ combinatorial k-guards.

CASE 3: $l = 7$. This case is equivalent to Case 1.

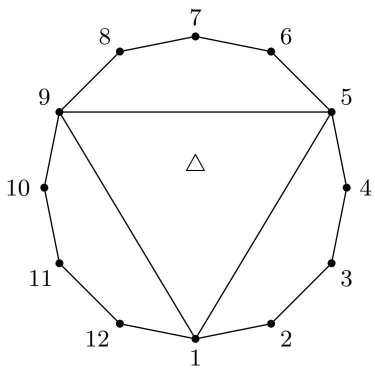

Fig. 4.6. A domination of a 12-vertex triangulation graph—Case 4.

CASE 4: $l = 8$. Let $d = \{1, 9\}$. The minimality of l ensures that the triangle $\triangle$ in G_T^1 bounded by d is $(1, 5, 9)$ (see Fig. 4.6). By placing one guard at 1, $k-1$ guards at 5, $k-1$ guards at 9, and one additional guard for $(1, 2, 3, 4, 5)$, one additional guard for $(5, 6, 7, 8, 9)$, and one additional guard for $(9, 10, 11, 12, 1)$, we get a k-guarded domination of G_T. ■

Thus, with all preceding lemmas available, we have the following corollary.

COROLLARY 4.6 ([98]). *Let G_T be a triangulation graph of an n-vertex polygon, $5 \leq n \leq 12$. For all $k \geq 2$, G_T can be dominated by $k\lfloor n/5 \rfloor + \lfloor (n+2)/5 \rfloor$ combinatorial k-guards, with at most k guards at a vertex.*

THEOREM 4.7 ([98]). *For all $n \geq 5$ and $k \geq 2$, every triangulation graph G_T of an n-vertex polygon can be dominated by $k\lfloor n/5 \rfloor + \lfloor (n+2)/5 \rfloor$ combinatorial k-guards with at most k guards at a vertex.*

Proof. Corollary 4.6 establishes the validity of the assertion for $n = 5, \ldots, 12$, so assume that $n \geq 13$, and that the assertion holds for all $5 \leq \hat{n} < n$. The following lemma is obtained by the same method as Lemma 3.12.

LEMMA 4.8 ([98]). *Suppose that for all $m < n$, $f(m, k)$ combinatorial k-guards are always sufficient to dominate any m-vertex triangulation graph, with at most k guards at a vertex. Then if G'_T is any triangulation graph of a polygon with n' vertices, $n' < n$, then:*

- (a) *k guards $g_1, \ldots, g_k$ placed at any vertex of G'_T with $f(n'-1, k)$ additional combinatorial k-guards are sufficient to dominate G'_T (but, perhaps, $g_1, \ldots, g_k$ are only $(k-1)$-guarded).*
- (b) *There are at most k guards at a vertex of G'_T.*

Proof of Lemma 4.8. Suppose that for all $m < n$, $f(m, k)$ watched guards are always sufficient to dominate any m-vertex triangulation graph, with at most k guards at a vertex, and let G'_T be a triangulation graph of a polygon P' with n' vertices, where $n' < n$. Let u be the vertex at which k guards are placed, and let v be a vertex adjacent in G'_T to u across an edge e corresponding to the edge of P'. Edge-contraction of G'_T across e produces the graph G^*_T on $n'-1$ vertices. By Lemma 3.2, G^*_T is a triangulation graph, and it can be dominated by $f(n'-1, k)$ combinatorial k-guards by the induction hypothesis, as $n'-1 < n$. Let x be the vertex that replaced u and v. Suppose that no guard is placed at x in a domination of G^*_T. Then the same guard placement with k guards at u will dominate all of G'_T, since the guards at u dominate the triangle supported by e, and the remaining triangles of G'_T have dominated counterparts in G^*_T. Otherwise, if a guard is used at x in the domination of G^*_T, more precisely, at most k guards are used by the induction hypothesis, then these guards can be assigned to v in G_T, with the remaining guards maintaining their positions. Again with k guards at u, every triangle of G'_T is dominated. Note that all guards that were k-guarded in G^*_T are k-guarded in G'_T as well. The only guards that could be non-k-guarded are the ones at u. And it is clear that there are at most k guards at any vertex of G'_T. ■

Now, we return to the proof of the theorem. Lemma 3.3 guarantees the existence of a diagonal d that partitions G_T into two graphs G_T^1 and G_T^2, where G_T^1 contains l

boundary edges of G_T with $5 \leq l \leq 8$. Assume that l is minimal. We consider each value of l separately.

CASE 1: $l = 5$. Let $d = \{0, 5\}$. Then G_T^1 is a triangulation graph of the hexagon $(0, 1, 2, 3, 4, 5)$. In G_T^1 either vertex 0 or 5, say 0, is of degree at least 3. By Lemma 4.6, G_T^1 can be dominated by $k+1$ combinatorial k-guards, with k guards at 0. Next, by Lemma 4.8, the k guards at 0 permit the remainder of G_T^2 to be dominated by $f(n-4-1, k) = f(n-5, k)$ k-guards, where $f(\hat{n}, k)$ specifies the number of k-guards that are always sufficient to dominate a triangulation graph on $\hat{n}$ vertices. By the induction hypothesis,

$$f(n-5, k) = k\left\lfloor \frac{n-5}{5} \right\rfloor + \left\lfloor \frac{(n-5)+2}{5} \right\rfloor = k\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n+2}{5} \right\rfloor - k - 1$$

k-guards suffice to dominate the remainder of G_T^2. Together with the $(k+1)$ k-guards allocated to G_T^1, all of G_T is dominated by at most $(k\lfloor n/5 \rfloor + \lfloor (n+2)/5 \rfloor)$ k-guards.

CASE 2: $l = 6$. Let $d = \{0, 6\}$. Then G_T^1 is a triangulation graph of the septagon $(0, 1, \ldots, 5, 6)$. By Corollary 4.6, G_T^1 can be dominated by $(k+1)$ k-guards. Since G_T^2 has $n-5$ vertices, it can be dominated by $(k\lfloor (n-5)/5 \rfloor + \lfloor ((n-5)+2)/5 \rfloor)$ k-guards by the induction hypothesis. This yields a k-guarded domination of G_T by $k\lfloor n/5 \rfloor + \lfloor (n+2)/5 \rfloor$ guards.

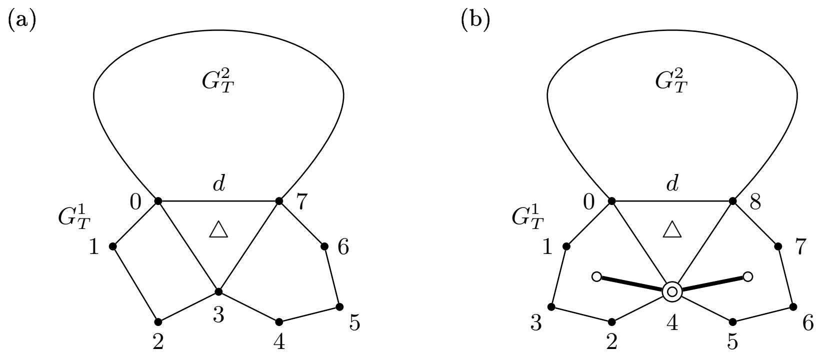

Fig. 4.7. (a) Case $k = 7$. (b) Case $k = 8$.

CASE 3: $l = 7$. Let $d = \{0, 7\}$ (see Fig. 4.7(a)). The presence of any of the diagonals $\{0, 6\}$, $\{1, 6\}$, $\{0, 5\}$, $\{2, 5\}$ would violate the minimality of l. Consequently, the triangle $\triangle$ in G_T^1 bounded by d is either $(0, 3, 7)$ or $(0, 4, 7)$. Without loss of generality, let $\triangle = (0, 3, 7)$. Form a graph G_T^0 by adjoining $\triangle$ to G_T^2. As G_T^0 has $n - 6 + 1$ vertices, it can be dominated by $(k\lfloor (n-5)/5 \rfloor + \lfloor ((n-5)+2)/5 \rfloor)$ k-guards by the induction hypothesis. In such a domination, at least one vertex of $\triangle$ must be assigned a guard. There are three possibilities:

(0) If there is a guard at 0, then by Corollary 4.6, $k + 1$ additional k-guards with k guards at 3 suffice to dominate a triangulation graph of the pentagon $(3, 4, 5, 6, 7)$. Regardless of how the quadrilateral $(0, 1, 2, 3)$ is triangulated, the guards at 0 and 3 will dominate it.

(3) If there is a guard at 3, then we can move it to 0 without destroying the k-guardness, thus getting case (0).

(7) If there is a guard at 7, then place one guard at 0, and $k-1$ guards at 3. Regardless of how $(0,1,2,3)$ is triangulated, the guards at 0 and 3 will dominate it. Next, it is easy to check that there is a vertex v in a 5-vertex triangulation graph $(3,4,5,6,7)$ such that v is adjacent to both 3 and 7, and by placing one guard at v, together with the $k-1$ guards at 3 and one guard at 7, we get a domination of G_T^1.

Thus all but the quadrilateral $(0,1,2,3)$ and pentagon $(3,4,5,6,7)$ can be dominated by $(k\lfloor n/5\rfloor+\lfloor(n+2)/5\rfloor-k-1)$ k-guards, and the pentagon and the quadrilateral merely require together $k+1$ guards. As these $k+1$ guards with one guard either at 0 or 7 in G_T^0 are k-guarded, all of G_T is dominated by $(k\lfloor n/5\rfloor+\lfloor(n+2)/5\rfloor)$ k-guards.

CASE 4: $k=8$. Let $d=\{0,8\}$ (see Fig. 4.7(b)). The presence of any of the diagonals $\{0,7\}$, $\{1,8\}$, $\{0,6\}$, $\{2,8\}$, $\{0,5\}$ or $\{3,8\}$ would violate the minimality of l. Consequently, the triangle $\triangle$ in G_T^1 bounded by d is $(0,4,8)$. Dominate a 5-vertex triangulation graph $(0,1,2,3,4)$ by $(k+1)$ k-guards with k guards at 4, and dominate a 5-vertex triangulation graph $(4,5,6,7,8)$ by $(k+1)$ k-guards with k guards at 4, thus getting a domination of G_T^1 by $(k+2)$ k-guards (the triangle $\triangle$ is dominated by the guards at 4). Next, the proof proceeds in five cases, depending on the value of $n \pmod 5$.

(3) $n=5t+3,\ t\geq 2$. The graph G_T^2 has $5(t-2)+3$ vertices, and it can be dominated by

$$k\left\lfloor\frac{5(t-1)+1}{5}\right\rfloor+\left\lfloor\frac{[5(t-1)+1]+2}{5}\right\rfloor=k(t-1)+t-1=tk+t-k-1$$

k-guards by the induction hypothesis. Together with $(k+2)$ k-guards allocated to G_T^1, we get a domination of G_T by $tk+t+1=\big(k\lfloor n/5\rfloor+\lfloor(n+2)/5\rfloor\big)$ k-guards.

(4) $n=5t+4,\ t\geq 2$.

$$k\left\lfloor\frac{5(t-1)+2}{5}\right\rfloor+\left\lfloor\frac{[5(t-1)+2]+2}{5}\right\rfloor+k+2=tk+t+1=k\left\lfloor\frac{n}{5}\right\rfloor+\left\lfloor\frac{n+2}{5}\right\rfloor.$$

(0) $n=5t,\ t\geq 3$.

$$k\left\lfloor\frac{5(t-2)+3}{5}\right\rfloor+\left\lfloor\frac{[5(t-2)+3]+2}{5}\right\rfloor+k+2$$
$$=tk+t-k+1\leq tk+t=k\left\lfloor\frac{n}{5}\right\rfloor+\left\lfloor\frac{n+2}{5}\right\rfloor,\qquad\text{as } k\geq 2.$$

(1) $n=5t+1,\ t\geq 3$.

$$k\left\lfloor\frac{5(t-2)+4}{5}\right\rfloor+\left\lfloor\frac{[5(t-2)+4]+2}{5}\right\rfloor+k+2$$
$$=tk+t-k+1\leq tk+t=k\left\lfloor\frac{n}{5}\right\rfloor+\left\lfloor\frac{n+2}{5}\right\rfloor,\qquad\text{as } k\geq 2.$$

(2) $n=5t+2,\ t\geq 3$. First, consider the case $t=3$, that is, $n=17$. Then G_T^2 is a triangulation graph of a 10-vertex polygon, but the minimality of l ensures that

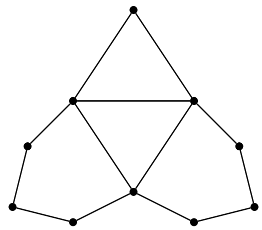

Fig. 4.8. A 10-vertex triangulation graph without diagonals cutting off 4, 5 or 6 vertices.

G_T^2 has the form shown in Fig. 4.8. If we place $k-1$ guards at x, and one guard at each of x_1 and x_2, then it is easy to see that together with at most two additional guards, we get a k-guarded guard set of G_T^2 of cardinality at most $k+3$. This yields a domination of G_T by at most $2k+5 \leq (3k+3)$ k-guards, as $k \geq 2$.
Now, suppose that $t \geq 4$. The minimality of k and Shermer's proof of Lemma 3.3 give more, namely there is a diagonal d' in G_T^2 such that d' partitions G_T^2 into two pieces G_T^{21} and G_T^{22}, one of which contains eight edges corresponding to the external edges of G_T^2, and vertices 0 and 8 are left in the remainder of G_T^2. Again by the minimality of l, the piece G_T^{21} can be dominated by $(k+2)$ k-guards. Note that G_T^{22}, the remainder of G_T^2, is now on $n-14$ vertices. Thus we get

$$k\left\lfloor\frac{5(t-3)+3}{5}\right\rfloor + \left\lfloor\frac{[5(t-3)+3]+2}{5}\right\rfloor + 2k+4$$
$$= tk+t-k+2 \leq tk+t = k\left\lfloor\frac{n}{5}\right\rfloor + \left\lfloor\frac{n+2}{5}\right\rfloor, \quad \text{as } k \geq 2.$$

That the number of guards at a vertex does not exceed k is established by noticing that in the above construction either we have to place at most k guards at an "empty" vertex or we just have to increase the number of guards at a vertex to k. ∎

THEOREM 4.9 ([98]). *For all $n \geq 5$ and $k \geq 2$, $\mathrm{gg}(n,k) = k\lfloor n/5\rfloor + \lfloor (n+2)/5\rfloor$, even for vertex guards.*

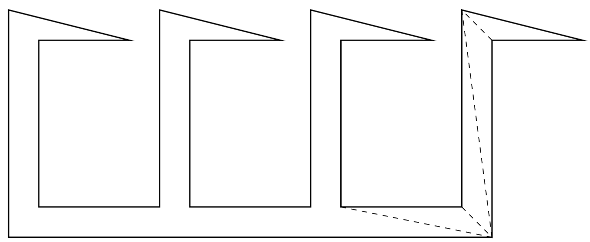

Fig. 4.9. A polygon P with n vertices with $\mathrm{gg}(P,k) = k\lfloor n/5\rfloor + \lfloor (n+2)/5\rfloor$; here $n=20$, and $\mathrm{gg}(P,k) = 4k+4$.

Proof. The necessity is established by the polygon P shown in Fig. 4.9. Each wave requires $(k+1)$ k-guards, and it is clear that for $n \equiv 0 \pmod 5$, $\mathrm{gg}(P,k) = k\lfloor n/5\rfloor + \lfloor (n+2)/5\rfloor$.

The case of $n \equiv 1, 2, 3, 4 \pmod 5$ is indicated with dashed lines. For sufficiency, we apply Lemma 1.5 and Theorem 4.7. Note that if $n = 3$ or $n = 4$, then the $(k\lfloor n/5 \rfloor + \lfloor (n+2)/5 \rfloor)$-bound fails. Clearly, $\text{gg}(3, k) = \text{gg}(4, k) = k + 1$. ■

4.1.1. Guards at disjoint points. All k-guarded sets constructed in the previous section are multisets, and they are satisfactory if we consider the graph theory only. But geometrically speaking, they are not, as guards should not be placed at the same point. Nevertheless, we will now show that guards at the same vertex can always be separated without destroying the k-guardness.

Let S be a k-guarded guard set for a polygon P obtained by the method of proof of Theorem 4.7, and let $V(S)$ denote the set of vertices of P at which our k-guards are placed. We start with simple observations.

(i) We have actually proved that for any vertex $v \in V(S)$, there are only three possibilities: there is one guard at v, or there are either $k-1$ or k guards at v.

(ii) Let $n(v)$ denote the number of guards at v; we have to split off only those guards that are located at those vertices v for which $n(v) \geq k - 1 \geq 2$.

(iii) Let C_v be a set of guards located at the same vertex v. Choose a vertex from C_v, say $l(C_v)$, and call it the *leader* of C_v. Now, note that for any $v, w \in V(S)$ with $n(v) \geq k - 1 \geq 2$, $v \neq w$, a guard $g \in C_v$ has to see at most $l(C_w)$ to be k-guarded. Thus by moving guards from C_w, except for $l(C_w)$, we do not destroy the k-guardness of g.

(iv) For any polygon, there exists a non-degenerate triangulation (there are no triangles with three vertices on a line).

Now, with all preceding observations, it remains to prove the following lemma.

LEMMA 4.10 ([98]). *Let G_T, S, and $V(S)$ be the triangulation graph of a non-degenerate triangulation of a polygon P, a k-guarded guard set for P, and the set of vertices of P at which our k-guards are placed, respectively. Let $v \neq w$ be two vertices from $V(S)$ such that $n(v) \geq 2$, and v is adjacent to w in G_T (v sees w). Then there exists a region R of points in P, close to v, such that $l(C_w)$ is visible from any point of R, that is, all guards from C_v, except for the leader $l(C_v)$, can be moved to R, and $l(C_w)$ is still visible to any moved guard.*

Proof. Let v be a vertex from $V(S)$ such that $n(v) \geq 2$, and let $N(v)$ be the set of vertices of G_T adjacent to v in G_T. Let $\gamma(v)$ be the sector interior to P delimited by the lines enclosing the edges of P with endpoints at v (see Fig. 4.10); the radius r of $\gamma(v)$ is such that $\gamma(v) \cap P = \gamma(v)$. Let $\{l_1, \ldots, l_m\} \subset N(v)$ be a set of leaders that are adjacent to v in G_T, ordered counterclockwise. For each l_i, $i = 1, \ldots, m$, in turn:

- rotate P in such a way that the line s_i enclosing the line segment vl_i is parallel to the y-axis, and v is below l_i;
- consider the vertex $y_i \in N(v)$, closest to the right of s_i, such that l_i is visible in P from y_i, and consider the vertex $x_i \in N(v)$, closest to the left of s_i, such that l_i is visible in P from x_i, respectively;

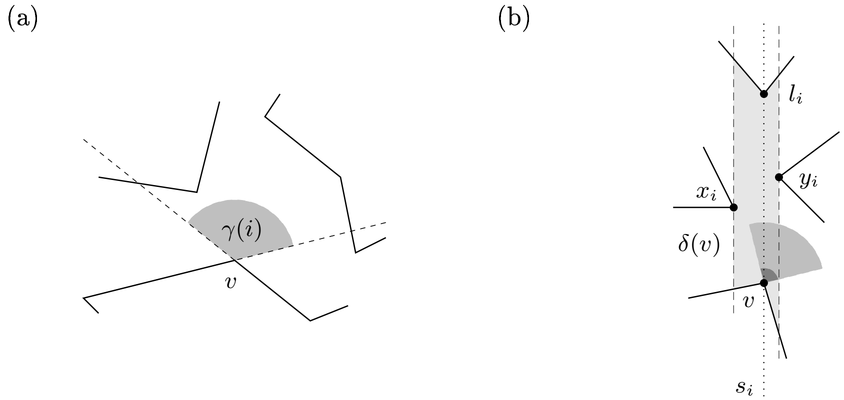

Fig. 4.10. (a) The definition of the sector $\gamma(v)$. (b) The idea of the construction of the region R.

- let $\delta(l_i)$ be the strip, interior to P, delimited by the lines enclosing x_i and y_i, respectively, and parallel to s_i;
- if $\gamma(v) \subset \delta(l_i)$, then l_i is visible from any point of $\gamma(v)$ in P, otherwise, decrease the radius $r > 0$ of $\gamma(v)$, thus getting $\gamma(v) \subset \delta(l_i)$.

Finally, all leaders are visible from any point of the sector $\gamma(v)$ in P. ■

4.2. The minimum k-guarded guards problem

Let k be a constant integer. We are concerned with the following decision problem.

BOUNDED k-GUARDED GUARDS PROBLEM (Bk-GG problem for short)

Instance: A polygon P and a positive integer I.

Question: Does there exist a subset $S \subset P$ such that S is a k-guarded guard set for P and $|S| \leq I$?

We will show that 3SAT is polynomially reducible to the Bk-GG problem. The goal is to accept an instance of 3SAT as an input and construct, in polynomial time, a polygon P such that there is a k-guarded guard set of cardinality at most I in P if and only if the instance of 3SAT is satisfiable. The proof is based upon the constructions proposed by Aggarwal [2], and Lee and Lin [56] in the case of arbitrary guards.

We first introduce some basic constructions on which the polygon is built and identify a number of distinguished points in this polygon such that no two different distinguished points are visible from the same point. In the following construction, the dotted lines in the figures indicate where these basic gadgets are adjoined to the main polygon. Let the bound I used in the Bk-GG problem be $(m+1)(k+1)+3m+n$.

Gutter Pattern: The gutter pattern is shown in Fig. 4.11(a). The black dot is a distinguished point associated with the pattern. Let $[p_1, \ldots, p_h]$ indicate that the points

$p_1, \ldots, p_h$ are collinear. Thus in the pattern of Fig. 4.11(a), we have $[z_3, z_4, z_5']$, $[z_2, z_5', z_6']$, and the lines l^T, l^M and l^B include the line segments $z_6'z_1$, z_6z_7, and z_2z_7, respectively.

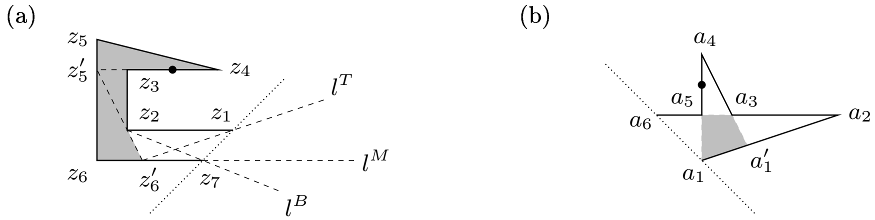

Fig. 4.11. (a) A gutter pattern. (b) A literal pattern.

The important characteristic of the pattern is that the shaded region requires $k+1$ guards, as a guard covering the distinguished point has to be guarded by k of its colleagues. Moreover, at most k guards can see a point from the cone delimited by l^T and l^B. One pattern per clause will exist in the final construction.

Literal Pattern: The literal pattern is shown in Fig. 4.11(b). The black dot is a distinguished point associated with the pattern. Note that $[a_1, a_4, a_5]$ and $[a_2, a_3, a_5, a_6]$. Consequently, the kernel of the pattern is the shaded quadrilateral (a_1, a_1', a_3, a_5), and any guard located outside this pattern (to the left of the segment a_1a_6) cannot cover this pattern entirely. These three patterns per clause will exist in the final construction, each of which will correspond to one literal.

Clause Pattern: Let $U = \{u_1, \ldots, u_n\}$ be a set of Boolean variables, and let $\phi = \{C_1, \ldots, C_m\}$ be a set of clauses over U such that $C_i \in \phi$ is a disjunction of precisely three literals, $i = 1, \ldots, m$. Without loss of generality, let us consider the clause $C_h \in F$, where $C_h = A \vee B \vee C$, $A \in \{u_i, \overline{u}_i\}$, $B \in \{u_j, \overline{u}_j\}$, $C \in \{u_l, \overline{u}_l\}$ are literals, and u_i, u_j and u_l are variables in U. The basic pattern for the clause pattern C_h is shown in Fig. 4.12. For simplification, let $\triangle abc$ denote the triangle determined by points a, b and c.

In the clause pattern in Fig. 4.12, we have $[x_{h1}, x_{h1}', c_{h1}, c_{h6}, b_{h1}, b_{h6}, a_{h1}, a_{h6}, x_{h2}', x_{h2}]$, $[x_{h2}', x_{h4}, x_{h5}]$, $[x_{h3}, x_{h4}, x_{h7}, x_{h1}]$, $[x_{h6}, x_{h7}, x_{h1}']$, $[z_{h2}, z_{h7}, x_{h1}']$, $[z_{h6}, z_{h7}, c_{h1}]$, $[z_{h6}', z_{h1}, x_{h2}']$, and a guard located at z_6' can see all points from the quadrilaterals $(a_{h1}, a_{h1}', a_{h3}, a_{h5})$, $(b_{h1}, b_{h1}', b_{h3}, b_{h5})$ and $(c_{h1}, c_{h1}', c_{h3}, c_{h5})$, respectively. Finally, $|x_{h1}x_{h1}'| = |x_{h1}'c_{h1}|$ and $|a_{h6}x_{h2}'| = |x_{h2}'x_{h2}|$, where $|uv|$ denotes the length of the line segment uv.

Since no point from the gutter pattern or from the interior of $\triangle x_{h1}x_{h2}x_{h3}$ can cover $\triangle a_{h1}a_{h1}'a_{h4}$, $\triangle b_{h1}b_{h1}'b_{h4}$ and $\triangle c_{h1}c_{h1}'c_{h4}$, they cannot be used to cover these triangles. Hence we need at least three guards for distinguished points in literal patterns, as no guard can see simultaneously two distinguished points. Furthermore, the gutter pattern requires at least $k+1$ guards. Therefore we have the following corollary.

COROLLARY 4.11 ([95]). *At least $(k+4)$ k-guarded guards are required to cover the region defined by the pattern C_h shown in Fig.* 4.12.

Next, it is easy to see that a point p from $\triangle x_{h1}x_{h1}'x_{h7}$ close to x_{h1} can only be seen from a point in $\triangle x_{h1}x_{h2}x_{h3}$. Moreover, no point outside the clause pattern can see p.

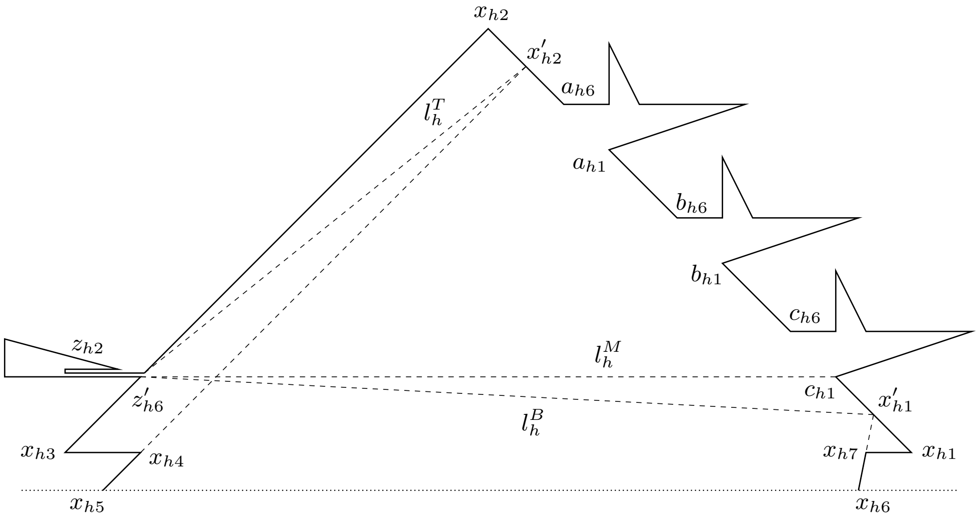

Fig. 4.12. A clause pattern C_h.

Next, as none of the $k+1$ guards covering the gutter pattern can see a point below the line l_h^B, we have the following corollary.

COROLLARY 4.12 ([95]). *In every minimum k-guarded guard set, among $k+4$ guards for C_h, there is a guard at one of the vertices a_{h1}, b_{h1} or c_{h1}, respectively.*

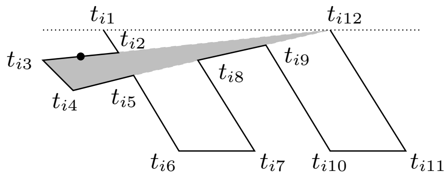

Fig. 4.13. Variable pattern.

Variable patterns: The pattern of a literal consists of two quadrilaterals called *bins* and a quadrilateral with the distinguished point as shown in Fig. 4.13. Note that $[t_{i2}, t_{i3}, t_{i12}]$ and $[t_{i4}, t_{i5}, t_{i8}, t_{i9}, t_{i12}]$ so that all points of the quadrilateral $(t_{i2}, t_{i3}, t_{i4}, t_{i5})$ are visible only from the shaded region. Consequently, one guard has to be located at a point of the shaded region to cover the associated quadrilateral. In the final polygon, there are n variable patterns, one for each variable. These are denoted by $t_1, \ldots, t_n$. Now, we will consider how to put variable patterns and clause patterns together.

Complete Construction: Two steps are needed.

STEP 1. We put variable patterns and clause patterns together as shown in Fig. 4.14. In Fig. 4.14:

(i) a guard at the vertex w'_6 of the gutter pattern w can cover all bins of the variable patterns $t_1, \ldots, t_n$, that is, we have $[w'_6 t_{i1}, t_{i2}, t_{i5}, t_{i6}]$, $[w'_6, t_{i7}, t_{i8}]$, $[w'_6, t_{i9}, t_{i10}]$, $[w'_6, t_{i11}, t_{i12}]$, for all $i = 1, \ldots, n$, and $[w'_6, w_1, t_{n11}, t_{n12}]$;

(ii) $[w_1, x_{15}, x_{16}, \ldots, x_{m5}, x_{m6}]$, $[t_{11}, x_{h5}, x_{h4}, x'_{h2}]$, $[t_{n12}, x_{h6}, x_{h7}, x'_{h1}]$, for $h = 1, \ldots, m$.

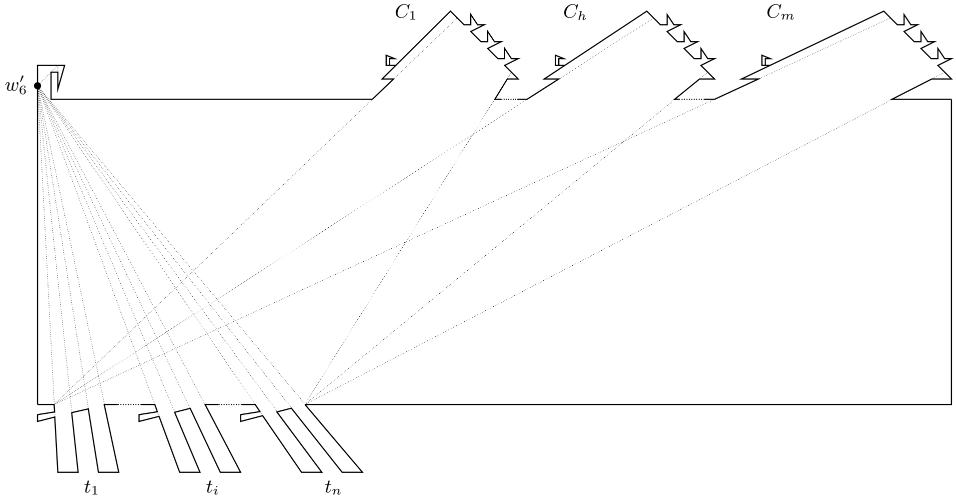

Fig. 4.14. Putting variable patterns and clause patterns together.

STEP 2: augmenting variable patterns with "spikes". Suppose that a literal $A \in \{u_i, \overline{u}_i\}$ appears in a clause C_h. If u_i is itself in C_h, then we add two spikes (s_1, s_2, s_3, s_4) and (q_1, q_2, q_3, q_4) to t_i as shown in Fig. 4.15(a), where $[s_1, s_2, a_{h1}]$ and $[s_3, s_4, t_{i8}, a_{h1}]$, and $[q_1, q_2, a_{h5}]$ and $[q_3, q_4, t_{i12}, a_{h5}]$. If u_i is negated (and occurs in C_h), then we add two spikes (s_1, s_2, s_3, p_4) and (q_1, q_2, q_3, q_4) to t_i as shown in Fig. 4.15(b), where $[s_1, s_2, a_{h5}]$ and $[s_3, s_4, t_{i8}, a_{h5}]$, and $[q_1, q_2, a_{h1}]$ and $[q_3, q_4, t_{i12}, a_{h1}]$. We call these spikes the *consistency-check patterns.*

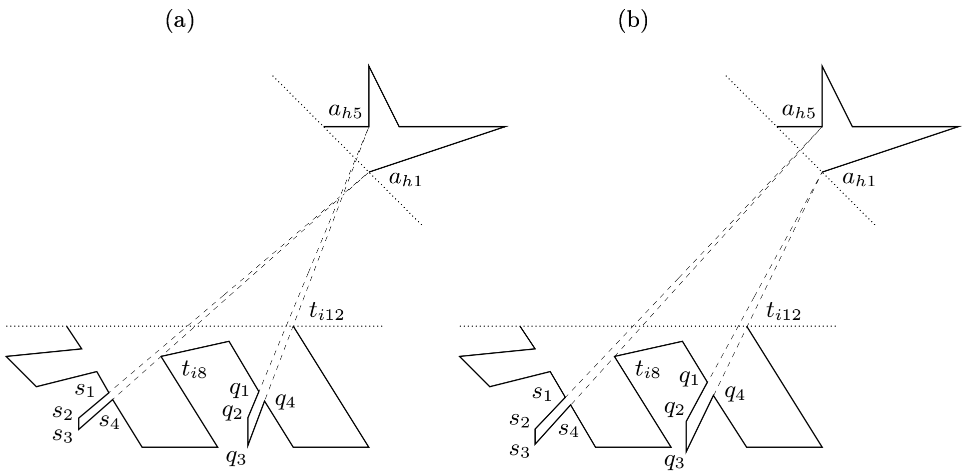

Fig. 4.15. (a) Spikes when $A = u_i$. (b) Spikes when $A = \overline{u}_i$.

It follows that a guard placed close to t_{i8} covers all the spikes of the left-hand bin and the quadrilateral $(t_{i2}, t_{i3}, t_{i4}, t_{i5})$, and a guard placed close to t_{i12} covers all the spikes of the right-hand bin and the quadrilateral $(t_{i2}, t_{i3}, t_{i4}, t_{i5})$. Note that these cases are disjoint. This is equivalent to labeling t_{i8} as F and t_{i12} as T. That is, the vertices t_{i8} and t_{i12} will represent the values *false* and *true* for the variable u_i, respectively.

LEMMA 4.13 ([95]). *At least $I = ((m+1)(k+1) + 3m + n)$ k-guarded guards are needed to cover the constructed polygon.*

Proof. At least $3m + (k+1)m$ guards are needed to cover m clause patterns with their gutter patterns. Next, we need at least n guards for the distinguished points in the n variable patterns. Finally, at least $k+1$ guards are needed for the gutter pattern w. ■

LEMMA 4.14 ([95]). *The constructed polygon can be covered with $I = ((m+1)(k+1) + 3m + n)$ k-guarded guards if and only if ϕ is satisfiable, that is, there exists a truth assignment to the n variables in U such that the conjunctive normal form $C_1 \wedge \cdots \wedge C_m$ is true.*

Proof. ($\Leftarrow$) If ϕ is satisfiable, then there exists a truth assignment to the variables such that each of the clauses in ϕ is true. Next, suppose a literal $A \in \{u_i, \overline{u}_i\}$ is in a clause C_h. If u_i is true, then we place a guard at the vertex t_{i12} of the variable pattern t_i. Also, we put a guard either at a_{h1} or a_{h5} of the literal pattern A depending on whether $A = u_i$ or $A = \overline{u}_i$, respectively. Otherwise, if u_i is false, then we place a guard at the vertex t_{i8} of the variable pattern t_i, and also either at a_{h1} or a_{h5} of the literal pattern A depending on whether $A = \overline{u}_i$ or $A = u_i$, respectively. Next, we put $(k+1)$ k-guarded guards per each gutter pattern, with k guards at z'_{h6}, $h = 1, \ldots, m$. From the construction it follows that the regions defined by the consistency-check patterns and literal patterns are covered by $(m(k+1)+3m+n)$ k-guarded guards. Finally, the remaining bins defined by the variable patterns can be covered by $(k+1)$ k-guarded guards in the last gutter pattern w with k guards at w'_6. Note that the k guards at w'_6 guarantee that the guards in the variable patterns are k-guarded.

($\Rightarrow$) Suppose there is a cover S with $I = ((m+1)(k+1)+3m+n)$ k-guarded guards. The gutter pattern w requires $k+1$ guards and all gutter patterns in the clause patterns require $m(k+1)$ guards. Thus $I - (k+1) - m(k+1) = 3m + n$ locations of guards are left to be considered.

In the polygon, there exist $3m$ literal patterns and n quadrilaterals $(t_{i2}, t_{i3}, t_{i4}, t_{i5})$, $i = 1, \ldots, n$, each of which contains a distinguished point. We know that a guard that covers a distinguished point cannot cover any other distinguished point. Therefore at least $3m + n$ guards are needed to cover the above $3m + n$ subregions. However, we cannot arbitrarily include in S any $3m$ points in these $3m$ literal patterns for they make the n variable patterns *inconsistent*. The definition of consistency is as follows.

We say that any variable pattern L is *consistent* if all consistency-check patterns connected to one of its two bins are covered by the $3m$ points in literal patterns and those connected to the other bin are not covered at all by the same $3m$ vertices; it is *inconsistent* otherwise.

In any variable pattern L, the number of consistency-check patterns connected to the first bin and the second bin is the same, say q. The total number of consistency-check patterns in the variable pattern L is $2q$. From the construction, the $3m$ points from literal patterns included in S can cover only q consistency-check patterns. If L is not consistent, then some of the consistency-check patterns connected to the first bin and some connected to the second bin will not be covered. Then at least two extra points of L are needed to cover L. On the other hand, if L is consistent, one extra guard is enough. Therefore, to get an I-cover, the consistency of all variable patterns is required.

Next, the consistency of variable patterns implies that a guard that covers simultaneously a spike and a literal pattern A must be located either at a_{h1} or at a_{h3}. Let us label the vertices a_{h1} and a_{h3} as follows. If $A = u_i$, then a_{h1} and a_{h3} are labeled as T and F (true and false), respectively; if $A = \overline{u}_i$, then a_{h1} and a_{h3} are labeled as F and T, respectively. In other words, a_{h1} represents *true* for A, and a_{h3} represents *false*. The vertices b_{h1} and b_{h3} of B, and c_{h1} and c_{h3} of C are labeled in the same manner.

We say that the truth assignment to variables u_i, u_j and u_l in a clause C_h is a *true assignment* if the resulting value of C_h is true; otherwise it is a *false assignment.* The key idea in the labeling of vertices in a clause pattern C_h is that the vertices that are selected according to the truth assignment of $\{u_i, u_j, u_l\}$ can cover the region $\triangle x_{h1}x_{h2}x_{h3}$ "for free", that is, without increasing the number of guards needed to cover the region defined by the pattern C_h, if and only if the assignment is a true assignment.

LEMMA 4.15 ([95]). *Suppose that all variable patterns are consistent. Then three vertices selected from the literal patterns A, B and C of a clause pattern $C_h = A \vee B \vee C$ cover the region $\triangle x_{h1}x_{h2}x_{h3}$ if and only if the truth values represented by the labels of the vertices give a true assignment for C_h.*

Proof of Lemma 4.15. C_h is false if and only if the truth values of the literals A, B and C are false. According to the vertex labels, this implies that the vertices selected must be a_{h3}, b_{h3} and c_{h3}. And this implies that $\triangle x_{h1}x_{h2}x_{h3}$ is not covered. ∎

Now, since our $3m$ guards in the literal patterns must satisfy the consistency requirement, the remaining n points chosen from the variable patterns can be determined, according to the sequence of spike alignments. We know that these guards, and guards from all gutter patterns, cannot cover any point from $\triangle x_{h1}x_{h2}x_{h3}$ close to x_{h1}, $h = 1, \ldots, m$. Therefore $\triangle x_{h1}x_{h2}x_{h3}$, $h = 1, \ldots, m$, must be covered by these guards. By Lemma 4.15, if they are covered for free, then each c_i is satisfiable. This implies ϕ is satisfiable. Once it has been known that the instance of 3SAT is satisfiable, the truth value assignment to the variables can easily be determined from the consistency property possessed by the minimum cover. ∎

Fig. 4.16 is an example for converting the Boolean formula

$$F = (u_1 \vee u_2 \vee u_3) \wedge (\overline{u}_1 \vee u_2 \vee u_3) \wedge (u_1 \vee \overline{u}_2 \vee \overline{u}_3),$$

and the dots are guards in the minimum cover. From the minimum cover we can conclude that the truth values of u_1, u_2, and u_3 are true, true, and false, respectively.

THEOREM 4.16 ([95]). *The bounded k-guarded guards problem is NP-complete.*

Proof. A polynomial non-deterministic algorithm that solves the bounded k-guarded guards problem is given below.

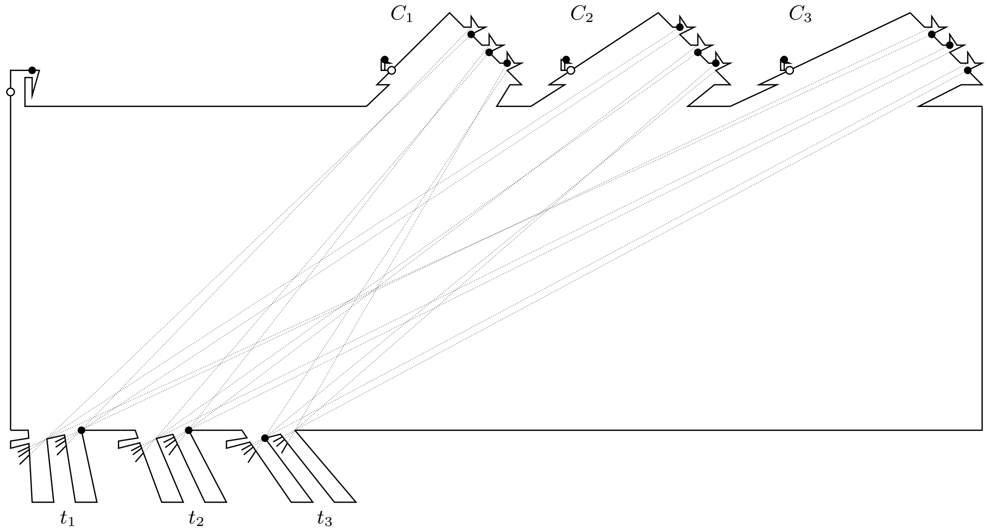

Fig. 4.16. The complete polygon for $F = (u_1 \vee u_2 \vee u_3) \wedge (\overline{u}_1 \vee u_2 \vee u_3) \wedge (u_1 \vee \overline{u}_2 \vee \overline{u}_3)$ and the minimum k-guarded cover.

ALGORITHM Bk-GG(P, I)

1. If $I \geq k(n-2)$, then put k guards at every triangle of a triangulation of the input polygon P. Clearly, these guards will form a k-guarded guard set S for P with $|S| \leq I$ (the number of triangles is $n-2$, Theorem 1.1).
2. For $1 \leq i \leq I$, choose a point p_i of P for the location of a guard g_i. Let S be the set of points chosen for guard locations.
3. For each p in S, compute its visibility polygon, that is, compute the region $R(p)$ of the given polygon that is visible from that point.
4. If $\bigcup_{p \in S} R(p) = P$, go to Step 4. Otherwise, S is not a guard set for P.
5. Compute the visibility graph VG(S). If the minimum degree of the resulting graph is at least k, then S is a k-guarded guard set, and $|S| \leq I$. Otherwise, S is not k-guarded.

Steps 1–3 take $O(nI)$ non-deterministic time, as the visibility polygon from a point inside an n-vertex polygon can be computed in linear time [73]. Step 4 takes $O(nI \log^2(I + n))$ time [18]. As the visibility graph can be computed in $O(nI^2)$ time using a greedy algorithm (comparing all pairs of vertices), Step 5 takes $O(nI^2)$ time. Finally, Step 1 guarantees that $I = O(n)$ (k is constant), thus the entire algorithm takes at most $O(n^3)$ time steps.

The NP-completeness follows from Lemma 4.14 and the fact that the polygon described above can be constructed in a time that is proportional to the multiplication of the number of vertices and clauses in a given instance of 3SAT. ■

COROLLARY 4.17 ([95]). *The minimum k-guarded guards problem is NP-hard.*

4.3. Miscellaneous shapes

4.3.1. Orthogonal polygons. Combinatorial bounds for k-guards in orthogonal polygons were given by Michael and Pinciu [67]. They proved the following theorem.

THEOREM 4.18 ([67]). *For all $k \geq 6$, $\mathrm{gg}_{\perp}(n,k) = k\lfloor n/6 \rfloor + \lfloor (n+2)/6 \rfloor$.*

Proof. Let S, $\mathrm{VG}(S)$, and F be a 1-guarded guard set for a given n-vertex polygon, the visibility graph of S, and a spanning forest of $\mathrm{VG}(S)$, respectively, as Theorem 3.18 guarantees. Let S^+ be a vertex cover of F. Of course, $|S^+| \leq \lfloor |S|/2 \rfloor$, as F is bipartite. Now, we insert $k-1$ additional guards at each vertex of S^+ to obtain a set S^*. The cardinality of S^* satisfies

$$|S^*| = |S| + |S^+| \leq \left\lfloor \frac{n}{3} \right\rfloor + (k-1)\left\lfloor \frac{\lfloor n/3 \rfloor}{2} \right\rfloor = k\left\lfloor \frac{n}{6} \right\rfloor + \left\lfloor \frac{n+2}{6} \right\rfloor,$$

and S^* is k-guarded. The necessity is established by the orthogonal polygon shown in Fig. 3.12. ∎

The authors [67] also proved that guards can be located at disjoint points. Before commencing the proof, let us present a lemma detailing the relationship between a diagonal of quadrilateralization and the local structure of an orthogonal polygon. Recall that the *orientation* of an edge is horizontal or vertical, and we say that edges a and b are *to the same side* of a diagonal d if they are in the same piece of P partitioned off by d; note that a and b may be in opposite half-planes defined by d, but still to the same side.

LEMMA 4.19 ([2]). *Let a and b be edges of an orthogonal polygon incident to a diagonal d and to the same side of d. Then a and b have the same orientation.*

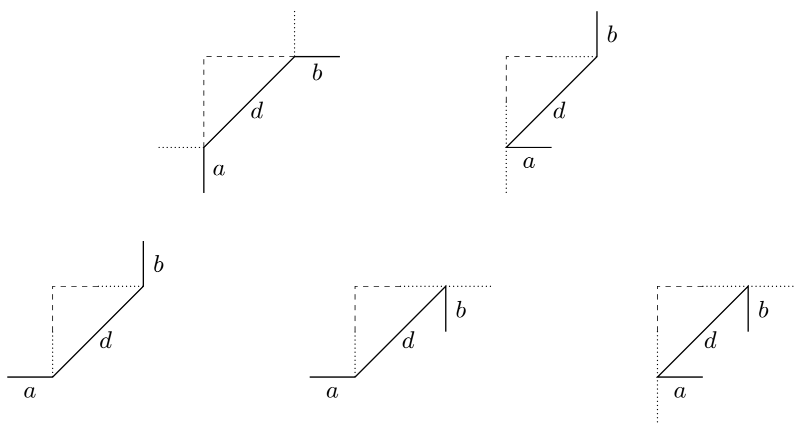

Fig. 4.17. The five possible arrangements when a and b have opposite orientations. The dotted lines represent possible orientations of the edges to the other side of d; the dashed lines indicate an added right angle that forms a subpolygon.

Proof. Without loss of generality, let us orient d with the positive slope with the polygon P_1 containing a and b below, and assume, contrary to our claim, that a and b have different orientations. Then there are five distinct possible combinations of a and b: hanging up, down, left, or right from the endpoints of d, as shown in Fig. 4.17. The other three possible combinations force a and b to be located to the different sides of d. Now, let us assign types $+$ and $-$ to each vertex in such a way that they alternate in a traversal of the boundary. In all five cases of Fig. 4.17, d connects two vertices of the same type, thus P_1 has an odd number of vertices. But this contradicts the assumption that P_1 is quadrilateralizable, since any polygon partitioned into quadrilaterals must have an even number of vertices. ■

The above lemma is useful in proving the following theorem.

THEOREM 4.20 ([66]). *The k-guarded guard multiset S^* constructed in Theorem* 4.18 *can be relocated to disjoint points.*

Proof. Let v be a vertex with k guards. Similarly to the proof of Lemma 4.10, we only have to show that all $k-1$ guards can be relocated close to the leader $l(v)$ without loosing the visibility with all other leaders, located only at vertices of quadrilaterals having v as a corner.

If there are no degenerate quadrilaterals (with three points on a line) at v, then we can proceed as for arbitrary polygons. Otherwise, we proceed in the following way.

If there is a 90° angle at v, one can easily check that there are no degenerate quadrilaterals at v. So assume that there is a 270° angle at v. Next, without loss of generality, let us assume that v is at the origin in the Cartesian plane, and let us order the quadrilaterals $q_1, \ldots, q_m$ that contain v counterclockwise, as shown in Fig. 4.18. Let v, x, y, z be

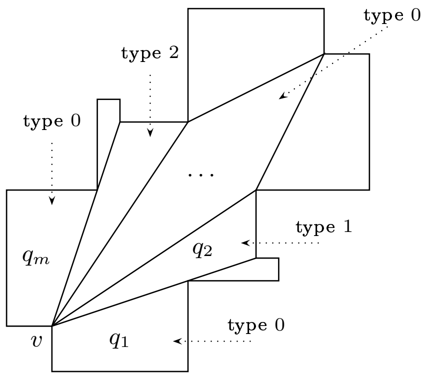

Fig. 4.18. Types of quadrilaterals.

the vertices (ordered counterclockwise) of a quadrilateral q having v as a corner. There are three types of quadrilaterals: *type* 0: neither x nor z lies on the segment vy; *type* 1: x lies on vy; *type* 2: z lies on vy. All we need is to show that if both quadrilaterals of type 1 and 2 are incident to v, then they are unique and the quadrilateral of type 1 occurs before the quadrilateral of type 2.

By Lemma 4.19, we have the following properties:

(i) The slope of the line segment vy in a quadrilateral of type 1 is in $[0°, 90°)$.
(ii) The slope of vy in a quadrilateral of type 2 is in $(0°, 90°]$.

Now, assume that a quadrilateral of type 2 with vertices v, x_2, y_2, z_2 precedes a quadrilateral of type 1 with vertices v, x_1, y_1, z_1 in the list $q_1, \ldots, q_m$. Then the above properties imply that the points y_1 and y_2 are both in the interior of Quadrant I and the segment vy_1 is above vy_2, as depicted in Fig. 4.19(a). Moreover, Lemma 4.19 implies that the

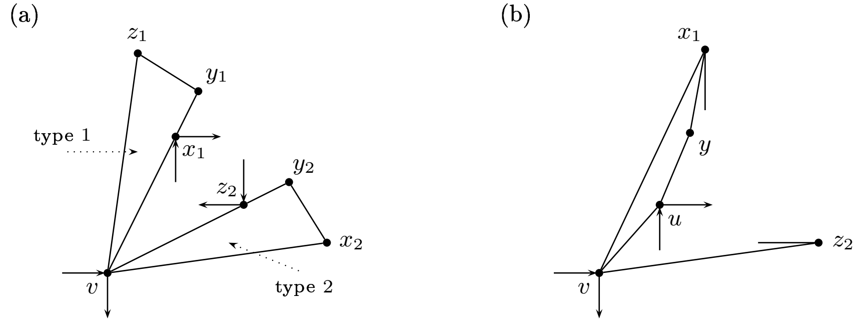

Fig. 4.19. (a) A quadrilateral of type 2 cannot precede a quadrilateral of type 1. (b) Illustration for the proof of Lemma 4.21.

diagonals vx_1 and vz_2 partition our polygon P into three polygons, each of which has a convex quadrilateralization. Let P' denote the polygon that has x_1, v and z_2 as consecutive vertices. Then the angles at x_1, v and z_2 in P' must be acute. Thus P' has a convex quadrilateralization and each interior angle is either 90° or 270°, except for three consecutive acute angles at x_1, v and z_2. However, by showing that such a polygon does not exist, we contradict that a quadrilateral of type 2 precedes a quadrilateral of type 1. Note that the same arguments can be applied to prove that the quadrilaterals of type 1 and 2 are unique.

LEMMA 4.21 ([66]). *Let P' be a polygon with each interior angle equal to 90° or 270°, except for three consecutive acute angles. Then P' does not have a convex quadrilateralization.*

Proof of Lemma 4.21. Assume that P' does have a convex quadrilateralization. We shall obtain a contradiction by induction.

Suppose that $m = 4$. Then the one non-acute angle of P' is equal to 270°, and hence P' does not have a convex quadrilateralization. So assume that $m \geq 6$. Notice that the sum a of these three acute angles must be 90°. For suppose that P' has r angles equal to 270°. Then $180(m-2) = 270r + 90(m-3-r) + a$, and hence $a = 90(m-2r-1)$. The parity of m yields $a = 90°$.

Next, let the convex quadrilateral q containing the side vz_1 of P' have vertices v, z_1, y, u (see Fig. 4.19(b)), and let us partition the vertices of P' into two alternating sets V^+ and V^-, with $v \in V^+$, as described in the proof of Lemma 4.19. Note that $u \in V^-$ and $u \notin \{z_1, x_2\}$. Now, let us orient the edges of P' counterclockwise so that the interior of P' lies to the left of each edge. It is easy to see that each vertex in V^- is exited horizontally (except for z_1) and is entered vertically (except for x_2). By Lemma 4.19, y is entered vertically and exited horizontally in a counterclockwise traversal of the boundary of P', and hence the only possibility is that y is entered from below and exited to the right. Now, the diagonal vy partitions P' into smaller convex-quadrilateralizable polygons. One of these smaller polygons contains three consecutive acute angles at u, v, z_2, with all other angles equal to 90° or 270°, and this contradicts the inductive hypothesis. ∎

Hence the construction from Lemma 4.10 can also be applied to the orthogonal gallery, thus yielding a disjoint k-guarded guard set. ∎

4.3.2. Star-shaped polygons. Theorem 4.9 shows that $k\lfloor n/5 \rfloor + \lfloor (n+2)/5 \rfloor$ vertex k-guards always suffice to cover any star-shaped polygon. Fig. 4.20(a) shows a star-shaped polygon P on 15 vertices with $\mathrm{gg}_{\mathrm{star}}(P,k) = 3k+3$ if guards are restricted to vertices. A simple extension of this polygon leads to a class of star-shaped polygons with $\mathrm{gg}_{\mathrm{star}}(\cdot,k) = k\lfloor n/5 \rfloor + \lfloor (n+2)/5 \rfloor$ for vertex guards.

COROLLARY 4.22 ([95]). *For all* $n \geq 5$, $\mathrm{gg}^{\mathrm{vertex}}_{\mathrm{star\text{-}shaped}}(n,k) = k\lfloor n/5 \rfloor + \lfloor (n+2)/5 \rfloor$.

4.3.3. Monotone and spiral polygons—open problems. The k-guarded guards problem for monotone polygons remains open. Of course, if guards are restricted to vertices (but we allow guards to be placed at the same point), then as many as $\big(k\lfloor n/5 \rfloor + \lfloor (n+2)/5 \rfloor\big)$ k-guards may be required to cover a monotone polygon: Fig. 4.20(b) shows a monotone version of the wave-polygon (cf. Fig. 4.9) with $\mathrm{gg}(\cdot,n) = k\lfloor n/5 \rfloor + \lfloor (n+2)/5 \rfloor$.

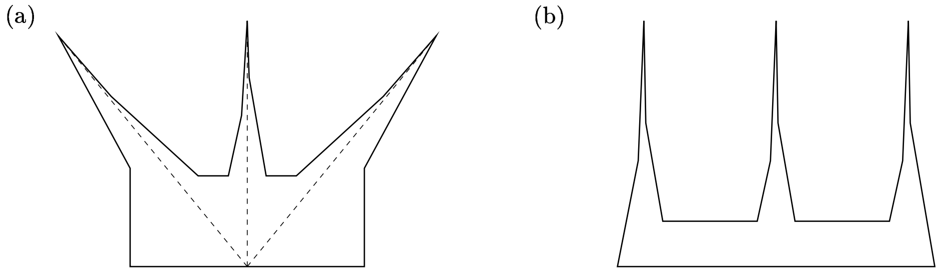

Fig. 4.20. (a) A star-shaped polygon P with $\mathrm{gg}(P,k) = k\lfloor n/5 \rfloor + \lfloor (n+2)/5 \rfloor$ for vertex guards; here $n = 15$, and the polygon requires $3k+3$ vertex k-guards. (b) A monotone polygon P with $\mathrm{gg}(P,k) = k\lfloor n/5 \rfloor + \lfloor (n+2)/5 \rfloor$ for vertex guards; here $n = 15$, and the polygon requires $3k+3$ vertex k-guards.

COROLLARY 4.23 ([95]). *For all* $n \geq 5$, $\mathrm{gg}^{\mathrm{vertex}}_{\mathrm{monotone}}(n,k) = k\lfloor n/5 \rfloor + \lfloor (n+2)/5 \rfloor$.

However, if guards are point guards, then no tight bound is known. Also, the k-guarded guards problem for spiral polygons remains unsolved.

5. THE FORTRESS PROBLEM

A *fortress* is a (simple) polygon P. Let $F(P)$ denote the set of all points of the plane exterior to P or on the boundary of P. A *guard* is any point of $F(P)$. A point $x \in F(P)$ is said to be *seen* by a guard g if the line segment $xg \subseteq F(P)$. A collection S of guards is said to *cover* the fortress P if every point $x \in F(P)$ can be seen by some guard $g \in S$.

The *fortress problem*, independently posed by Joseph Malkelvitch and Derick Wood, asks about the number of guards sufficient to cover a fortress. In 1983, O'Rourke and Wood [73] solved the fortress problem for vertex guards—they showed that $\lceil n/2 \rceil$ vertex guards are sometimes necessary and always sufficient. A tight bound of $\lceil n/3 \rceil$ point guards was given by O'Rourke and Aggarwal [73].

As we remember, Hernández-Peñalver [39] proved that $\lfloor n/2 \rfloor - 1$ cooperative guards are sometimes necessary and always sufficient to cover the interior of an n-vertex polygon (Chapter 2, Theorem 2.7). One may ask whether this is still true for the fortress problem; however, a convex n-gon requires $n-1$ cooperative vertex guards.

In [91] Yiu considers the number of k-consecutive vertex guards that are required to solve the fortress problem. A *k-consecutive vertex guard* is a set of vertex guards located at k consecutive vertices of the polygon. Yiu shows that $\lceil n/(k+1) \rceil$ k-consecutive vertex guards always suffice to cover the exterior of any n-vertex polygon. Thus we have

Corollary 5.1 ([94]). *$n-1$ cooperative vertex guards always suffice to cover an n-vertex fortress.*

However, convex polygons constitute a severely restricted class of polygons, so it is natural to investigate the fortress problem for cooperative guards as a function of a variable other than n, which is the number of vertices of a polygon. First, we shall consider the case of vertex guards; another case will be explored in Section 5.3.

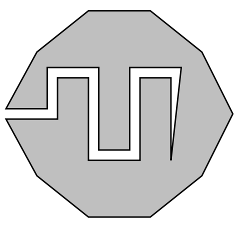

Fig. 5.1. A fortress that requires $c - 1 + (\lfloor (n_p - 1)/2 \rfloor - 1)$ cooperative vertex guards; here we have $c = 11$, $n_p = 17$, and the polygon requires 17 cooperative vertex guards.

Define a *pocket* p of a polygon as an exterior polygon interior to the convex hull of the polygon and bounded by a hull edge; recall that the *convex hull* of a polygon is the smallest-area convex polygon which encloses the original polygon. Consider the polygon P with one pocket p shown in Fig. 5.1; let n_p denote the number of vertices of p. One can easily check that P requires $c-1+(\lfloor (n_p-1)/2 \rfloor - 1)$ cooperative vertex guards, where c is the number of vertices of the convex hull of P. A simple extension of this polygon leads to a class of polygons of k pockets $p_1, \ldots, p_k$ that require

$$c-1+\sum_{i=1}^{k}\left(\left\lfloor \frac{n_i-1}{2} \right\rfloor - 1\right)$$

cooperative vertex guards, where n_i is the number of vertices of p_i, $i=1,\ldots,k$ (see Fig. 5.2).

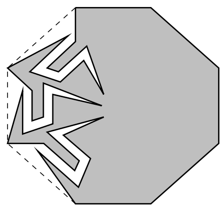

Fig. 5.2. A fortress that requires $c-1+\sum_{i=1}^{k}(\lfloor (n_i-1)/2 \rfloor - 1)$ cooperative vertex guards; here the polygon has three pockets, each of 11 vertices, $c=8$, and it requires 19 cooperative vertex guards.

LEMMA 5.2 ([94]). *Let $c \geq 3$ and $0 \leq k \leq c$ be integers. Then there exists a fortress with k pockets $p_1, \ldots, p_k$ and c vertices of the convex hull that requires $c-1+\sum_{i=1}^{k}(\lfloor (n_p-1)/2 \rfloor$ $-1)$ cooperative vertex guards, where n_i is the number of vertices of p_i, $i=1,\ldots,k$.*

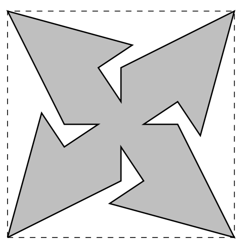

Fig. 5.3. A fortress with no edges on the boundary of the convex hull that requires $\sum_{i=1}^{k}\lfloor (n_i-2)/2 \rfloor$ cooperative vertex guards; here the polygon has four pockets, each of 6 vertices, and it requires 8 cooperative vertex guards.

But if $c=k$ and all n_i, $i=1,\ldots,k$, are even, then more than $c-1+\sum_{i=1}^{k}(\lfloor (n_i-1)/2 \rfloor - 1)$ cooperative vertex guards may be required. Consider the fortress in Fig. 5.3: here we have

$c = k = 4$, all $n_i = 6$, $i = 1, \dots, 4$, and the fortress requires

$$8 = 2 + 2 + 2 + 2 > c - 1 + \sum_{i=1}^{k} \left(\left\lfloor \frac{n_p - 1}{2} \right\rfloor - 1 \right)$$

cooperative vertex guards. Thus we have

LEMMA 5.3 ([94]). *Let k be an integer, $k \geq 3$. Then there exists a fortress with k pockets $p_1, \dots, p_k$ and with no edges on the boundary of the convex hull that requires $\sum_{i=1}^{k} \lfloor (n_i - 2)/2 \rfloor$ cooperative vertex guards, where n_i is the number of vertices of p_i, $i = 1, \dots, k$.*

5.1. Sufficiency proof

We will show that the bounds in Lemmas 5.2 and 5.3 are tight. By Lemma 2.22, we have

COROLLARY 5.4 ([94]). *Let p be a pocket of n_p vertices, and let $d = \{x_1, x_2\}$ be the pocket lid of p. Then:*

(a) *if n_p is odd, then $\lfloor (n_p - 1)/2 \rfloor$ cooperative vertex guards with one guard placed at any endpoint of d suffice to cover p;*
(b) *otherwise, $\lfloor (n_p - 1)/2 \rfloor$ cooperative vertex guards with one guard placed either at x_1 or at x_2 suffice to cover p.*

Note that if n_p is 3 or 4, we assume that the set of cooperative guards may consist of one guard only.

THEOREM 5.5 ([94]). *Let P be a fortress of k pockets $p_1, \dots, p_k$, and let c be the number of vertices of the convex hull of P. Then:*

(a) *if $c = k$ and all k pockets have an even number of vertices, then $\sum_{i=1}^{k} \lfloor (n_i - 2)/2 \rfloor$ cooperative vertex guards always suffice to cover $F(P)$;*
(b) *otherwise, $c - 1 + \sum_{i=1}^{k} (\lfloor (n_i - 1)/2 \rfloor - 1)$ cooperative vertex guards always suffice to cover $F(P)$.*

Proof. The proof is by induction on k, the number of pockets. Corollary 5.1 establishes the validity of the theorem for $k = 0$, so assume that $k \geq 1$, and that the assertion holds for all $0 \leq \hat{k} < k$. We need to consider three cases.

CASE 1: $c = k$ and all k pockets have an even number of vertices. By Corollary 5.4, k guards at all vertices of the convex hull permits the remainders of all k pockets to be covered by

$$\sum_{i=1}^{k} \left(\left\lfloor \frac{n_i - 1}{2} \right\rfloor - 1 \right) = \sum_{i=1}^{k} \left(\left\lfloor \frac{n_i - 2}{2} \right\rfloor - 1 \right)$$

cooperative vertex guards, as all n_i are even, $i = 1, \dots, k$. Therefore $F(P)$ can be covered by

$$\sum_{i=1}^{k} \left\lfloor \frac{n_i - 2}{2} \right\rfloor$$

cooperative vertex guards in total.

Case 2: $c \neq k$ and all k pockets have an even number of vertices.

Subcase 2.a: there are two consecutive edges of the polygon on the boundary of the convex hull. Let these edges be labeled $e_1 = \{x_1, x_2\}$ and $e_2 = \{x_2, x_3\}$. Then by placing $c-1$ guards at all vertices of the convex hull, except for x_2, and applying an argument similar to that in Case 1, we get a coverage of $F(P)$ by

$$c - 1 + \sum_{i=1}^{k} \left(\left\lfloor \frac{n_i - 1}{2} \right\rfloor - 1 \right)$$

cooperative vertex guards.

Subcase 2.b. Let the vertices of the convex hull be labeled $x_1, \ldots, x_c$, in a counterclockwise manner. Without loss of generality, we can assume $e_1 = \{x_c, x_1\}$ to be an edge of the polygon on the boundary of the convex hull and $\{x_1, x_2\}$ to be the pocket lid of the pocket p_1.

By Corollary 5.4, p_1 can be covered by $\lfloor (n_1 - 1)/2 \rfloor$ cooperative vertex guards, with one guard either at x_1 or at x_2. If there is a guard at x_2, then by placing $c-2$ additional guards at the vertices $x_3, \ldots, x_c$ of the convex hull, and applying an argument similar to that in Case 1, we get a coverage of $F(P)$ by

$$c - 1 + \sum_{i=1}^{k} \left(\left\lfloor \frac{n_i - 1}{2} \right\rfloor - 1 \right)$$

cooperative vertex guards. Otherwise, if there are no guards at x_2 and there is a guard at x_1, then let us consider the vertex x_2:

(1) x_2 is one of the endpoints of the pocket lid $\{x_2, x_3\}$ of the next pocket p_2, in a counterclockwise order. Again by Corollary 5.4, p_2 can be covered by $\lfloor (n_2 - 1)/2 \rfloor$ cooperative vertex guards, with one guard either at x_2 or at x_3. If there is a guard at x_3, then by placing $c-3$ additional guards at the vertices $x_4, \ldots, x_c$ of the convex hull, together with $\lfloor (n_1 - 1)/2 \rfloor$ guards allocated to the pocket p_1, $\lfloor (n_2 - 1)/2 \rfloor$ guards allocated to p_2, and by Lemma 5.4, $\sum_{i=3}^{k} (\lfloor (n_i - 1)/2 \rfloor - 1)$ guards allocated to the remainders of p_i, $i = 3, \ldots, k$, we get a coverage of $F(P)$ by

$$c - 1 + \sum_{i=1}^{k} (\lfloor (n_i - 1)/2 \rfloor - 1)$$

cooperative vertex guards. Otherwise, we apply (1) at the pocket lid $\{x_3, x_4\}$ or (2) at the edge $\{x_3, x_4\}$.

(2) $\{x_2, x_3\}$ is an edge of P on the boundary of the convex hull. Then we place the next guard at x_3, and apply the reasoning used in (1) in the case of a guard at x_3.

It is clear that the above construction will stop either at (1) or when we have considered the last pocket p_k with the pocket lid $\{x_{c-1}, x_c\}$, and a guard at x_{c-1} in the pocket p_k is needed. But in this case, we have $c-1$ guards at the vertices $x_1, x_2, \ldots, x_{c-1}$ of the convex hull and $\sum_{i=1}^{k} (\lfloor (n_i - 1)/2 \rfloor - 1)$ guards in the remainders of pockets. Again,

$F(P)$ is covered by

$$c - 1 + \sum_{i=1}^{k} \left(\left\lfloor \frac{n_i - 1}{2} \right\rfloor - 1 \right)$$

cooperative vertex guards.

CASE 3: there is a pocket with an odd number of vertices. Let the pocket be p_k, and let $d = \{x_1, x_2\}$ be its pocket lid. Then replacing p_k with a new edge d results in the fortress $\widehat{P}$ of $\hat{k} = k - 1$ pockets and the same number c of vertices of the convex hull. By the induction hypothesis, $F(\widehat{P})$ can be covered by $c-1+\sum_{i=1}^{k-1}(\lfloor (n_i - 1)/2 \rfloor - 1)$ cooperative vertex guards. As there is a guard either at x_1 or at x_2, Corollary 5.4 shows that

$$c - 1 + \sum_{i=1}^{k-1} \left(\left\lfloor \frac{n_i - 1}{2} \right\rfloor - 1 \right) + \left\lfloor \frac{n_k - 1}{2} \right\rfloor - 1 = c - 1 + \sum_{i=1}^{k} \left\lfloor \frac{n_i - 1}{2} \right\rfloor$$

cooperative vertex guards suffice to cover $F(P)$. ■

5.2. Orthogonal fortresses

Before considering the cooperative guards problem in orthogonal fortresses, let us focus on covering the interior of 1-orthogonal polygons.

5.2.1. 1-orthogonal polygons. A 1-*orthogonal polygon* is a hole-free polygon with a distinguished edge e called the *slanted edge* such that the polygon satisfies four conditions:

(1) The number of edges is even.
(2) Except for possibly e, the edges are alternately horizontal and vertical in a traversal of the boundary.
(3) All interior angles are less than or equal to 270°.
(4) The nose of the slanted edge contains no vertices.

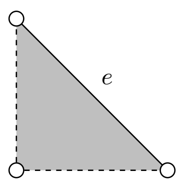

Fig. 5.4. The nose of a slanted edge.

The *nose* of a slanted edge is the triangle towards the inside of the polygon whose hypothenuse is e; the nose includes the interior of e but excludes the remainder of the boundary (see Fig. 5.4). The concept of 1-orthogonal polygons was introduced by Lubiv [61].

THEOREM 5.6 ([61]). *Any* 1-*orthogonal polygon is convexly quadrilateralizable.*

The existence of a convex quadrilateralization for any 1-orthogonal polygon leads to the following theorem.

THEOREM 5.7 ([94]). *$\lfloor n/2 \rfloor - 2$ cooperative vertex guards always suffice to cover the interior of any 1-orthogonal polygon with n vertices.*

Proof. We omit the proof as it follows the lines of the proof of Theorem 2.11. ∎

5.2.2. T-pockets, F-pockets and S-pockets. The convex hull of an orthogonal polygon is bounded by four *extremal* edges (northernmost, westernmost, southernmost, easternmost). As the pocket lid of a pocket with an even number of vertices is one of the extremal edges, there are at most four pockets with an even number of vertices; all other pockets have an odd number of vertices. Let p be a pocket with an odd number of vertices. If p has three vertices, then p is of type T; if p has five vertices, it is of type F; it is otherwise, of type S.

REMARK 5.8 ([94]). *Any F-pocket can be covered by two cooperative guards located at the endpoints of its pocket lid.*

LEMMA 5.9 ([94]). *Any S-pocket with n vertices can be covered by $\lfloor (n-1)/2 \rfloor - 1$ cooperative guards with one guard placed at one of the endpoints of its pocket lid.*

Proof. Let p be a pocket with $n \geq 7$ vertices. We leave it to the reader to verify that the assertion holds for $n = 7$. Assume that it holds for all pockets with $\hat{n}$ vertices, $7 \leq \hat{n} < n$. Let $d = \{x, y\}$ be the pocket lid of p, and let $\{x, x_1\}$ and $\{y, y_1\}$ be the edges of the pocket incident to d. We have to consider two cases:

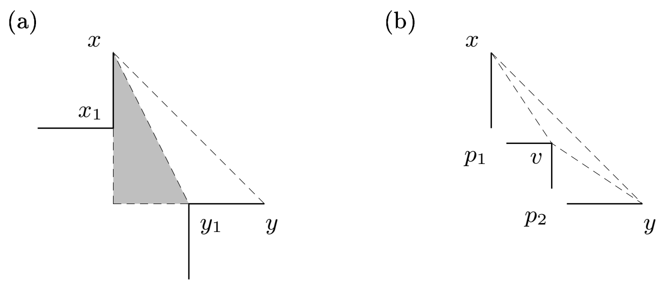

Fig. 5.5. (a) Case 1: the nose of a slanted edge $\{x, y_1\}$ is empty. (b) Case 2: there is a vertex in the nose of $\{x, y\}$.

CASE 1: x sees y_1 and the nose of $\{y_1, x\}$ is empty (see Fig. 5.5(a)). By replacing the polyline (y_1, y, x) with the new edge $\{y_1, x\}$, we get a 1-orthogonal polygon P' with $n - 1$ vertices, with the slanted edge $\{y_1, x\}$. By Theorem 5.7, P' can be covered by $\lfloor (n-1)/2 \rfloor - 2$ cooperative vertex guards. The same guard placement in the pocket p with one additional guard at x will cover p (the triangle (y_1, y, x) is covered by the guard at x), and the resulting guard set is cooperative.

Case 2: there is a vertex in the nose of $\{x, y\}$ (see Fig. 5.5(b)). Let v be the closest vertex to $\{x, y\}$. As v sees both x and y, the diagonals $\{x, v\}$ and $\{y, v\}$ partition the pocket p into the triangle (x, v, y) and two pockets p_1 and p_2 with n_1 and n_2 vertices, respectively.

Subcase 2.a: p_1 and p_2 are both F-pockets; $n_1 = 5$ and $n_2 = 5$. Due to Remark 5.8, by placing three guards at x, v and y, we get a coverage of p by $\lfloor (9-1)/2 \rfloor - 1$ cooperative vertex guards, as $n = n_1 + n_2 - 1 = 9$.

Subcase 2.b: p_1 is an S-pocket; $n_1 \geq 7$ and $n_2 \geq 3$. By the induction hypothesis, p_1 can be covered by $\lfloor (n_1 - 1)/2 \rfloor - 1$ cooperative vertex guards with one guard placed either at x or at v. By Corollary 5.4, p_2 can be covered by $\lfloor (n_2 - 1)/2 \rfloor$ cooperative vertex guards with one guard placed at y. With the same guard placement in p, we get a coverage of p by

$$\left\lfloor \frac{n_1 - 1}{2} \right\rfloor - 1 + \left\lfloor \frac{n_2 - 1}{2} \right\rfloor \leq \left\lfloor \frac{n - 1}{2} \right\rfloor - 1$$

cooperative vertex guards, as $n_1 + n_2 = n + 1$, with one guard placed at y. ∎

5.2.3. Tight bounds. First, assume that there are no pockets with an even number of vertices. Fig. 5.6(a) shows a class of orthogonal fortresses that require $3 + t + f + \sum_{i=1}^{s} (\lfloor (n_i - 1)/2 \rfloor - 1)$ cooperative vertex guards, where t is the number of T-pockets, f is the number of F-pockets, s is the number of S-pockets, and the S-pockets p_i have n_i vertices, $i = 1, \ldots, s$. Nevertheless, if $f \geq 4$ and $t + s \leq f - 4$, then more guards may be required: Fig. 5.6(b) shows a class of orthogonal fortresses that

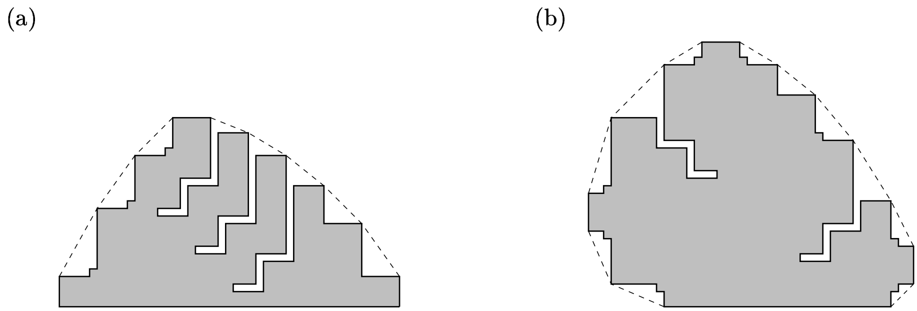

Fig. 5.6. (a) An orthogonal fortress that requires $3+t+f+\sum_{i=1}^{s}(\lfloor (n_i - 1)/2 \rfloor - 1)$ cooperative vertex guards; here $t = 2$, $f = 3$, and $s = 3$, each S-pocket has 11 vertices, and the polygon requires 20 cooperative vertex guards. (b) An orthogonal fortress that requires $4 + t + f + \sum_{i=1}^{s} (\lfloor (n_i - 1)/2 \rfloor - 1)$ cooperative vertex guards, with $f \geq 4$ and $t + s \leq f - 4$; here $t = 1$, $f = 8$, and $s = 2$, each S-pocket has 11 vertices, and the polygon requires 21 cooperative vertex guards.

require $4 + t + f + \sum_{i=1}^{s} (\lfloor (n_i - 1)/2 \rfloor - 1)$ cooperative vertex guards (note that any T-pocket or S-pocket is between two F-pockets). We will show these bounds to be tight.

THEOREM 5.10 ([94]). *Let P be an orthogonal fortress. Let t, f and s be the number of T-pockets, F-pockets and S-pockets in P, respectively, and let the S-pockets have n_i vertices, $i = 1, \ldots, s$. Then:*

(a) *if either $f \leq 3$ or $s+t > f-4$, then $3+t+f+\sum_{i=1}^{s}(\lfloor(n_i-1)/2\rfloor-1)$ cooperative vertex guards always suffice to cover $F(P)$,*
(b) *otherwise, $4+t+f+\sum_{i=1}^{s}(\lfloor(n_i-1)/2\rfloor-1)$ cooperative vertex guards always suffice to cover $F(P)$.*

Proof. Let e_n, e_e, e_s and e_w be the four extremal edges of P.

(a) As either $f \leq 3$ or $s+t > f-4$, there are two "consecutive" extremal edges, say e_w and e_n, such that between them there are t_1, f_1 and s_1 pockets of type T, F and S, respectively, and either $f_1 = 0$ or $s_1 + t_1 > f_1 - 4$. Now, by applying similar arguments to those in Case 2 of the proof of Theorem 5.5, we can show that there is a vertex v of the convex hull between the edges e_w and e_n at which we do not need to place a guard when we want to guard the whole of $F(P)$ between e_w and e_n. Therefore, with $c-1$ guards placed at all vertices of the convex hull, except at the vertex v, together with $\sum_{i=1}^{s}\lfloor(n_i-1)/2\rfloor - 2$ cooperative guards for the remainders of all S-pockets (by Lemma 5.9), we get a coverage of $F(P)$ by

$$3+t+f+\sum_{i=1}^{s}\left(\left\lfloor\frac{n_i-1}{2}\right\rfloor-1\right)$$

cooperative guards, as $c-1 = 3+t+f+s$.

(b) If $f \geq 4$ and $s+t \leq f-4$, then with c guards at all vertices of the convex hull, together with $\sum_{i=1}^{s}\lfloor(n_i-1)/2\rfloor - 2$ cooperative guards for the remainders of all S-pockets (by Lemma 5.9), we get a coverage of $F(P)$ by $4+t+f+\sum_{i=1}^{s}(\lfloor(n_i-1)/2\rfloor-1)$ cooperative guards, as $c = 4+t+f+s$. ∎

If there are (at most four) m pockets with an even number of vertices, then by similar arguments to those in the case of no "even" pockets, by Corollary 5.4, and by the induction on m, we get the following theorem.

THEOREM 5.11 ([94]). *Let P be an orthogonal fortress. Let t, f, s and m be the number of T-pockets, F-pockets, S-pockets, and pockets with an even number of vertices, respectively, and let the S-pockets have n_i vertices, $i = 1, \ldots, s$, and let the even pockets have $\hat{n}_i$ vertices, $i = 1, \ldots, m$. Then:*

(a) *if either $f \leq 3$ or $s+t > f-4$, then*
$$3+t+f+\sum_{i=1}^{s}\left(\left\lfloor\frac{n_i-1}{2}\right\rfloor-1\right)+\sum_{i=1}^{m}\left(\left\lfloor\frac{\hat{n}_i}{2}\right\rfloor-2\right)$$
cooperative vertex guards are sometimes necessary but always sufficient to cover $F(P)$,

(b) *otherwise,*
$$4+t+f+\sum_{i=1}^{s}\left(\left\lfloor\frac{n_i-1}{2}\right\rfloor-1\right)+\sum_{i=1}^{m}\left(\left\lfloor\frac{\hat{n}_i}{2}\right\rfloor-2\right)$$
cooperative vertex guards are sometimes necessary but always sufficient to cover $F(P)$.

5.3. Point guards

We have restricted guards to be placed at vertices of a fortress. However, we can allow guards to be placed at any point of $F(P)$. First, let us prove a reduced form of the theorem.

LEMMA 5.12 ([94]). *An n-vertex fortress with at most one triangle-pocket can be covered by two cooperative guards.*

Proof. Let P be a convex polygon. Rotate P so that a vertex a is uniquely highest and a vertex b uniquely lowest. By adding two guards below the lowest vertex of P, both of them far enough to see a, we will cover $F(P)$ by two cooperative guards.

Now, suppose P to be non-convex and that P has only one pocket (x, y, z) of three vertices, with $\{x, z\}$ as the pocket lid. Rotate P so that the edge $\{z, y\}$ is horizontal, and let d be the first edge, in clockwise order, that is not seen from any point of the line l collinear to $\{z, y\}$ (see Fig. 5.7). We have to consider two cases.

CASE 1: d is not parallel to l. Then by adding two guards, one at l, and the other one at the line collinear to d, both of them sufficiently far away to see each other and all the edges of P, we will get a cover of $F(P)$ by two cooperative guards (see Fig. 5.7(a)).

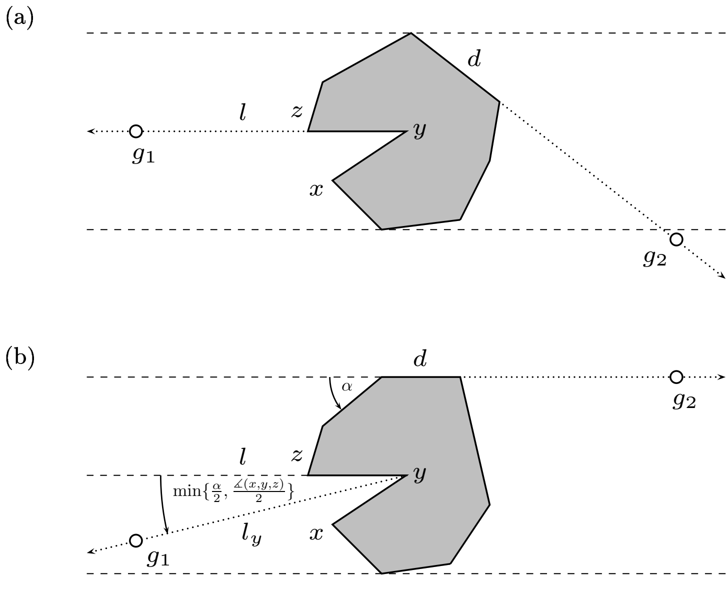

Fig. 5.7. Two cooperative guards always suffice in a fortress with one triangular pocket.

CASE 2: d is parallel to l. Let α be the angle between the last edge visible from a point at l and the line collinear to the edge d. Let l_y be the line with angle at y equal to $\min\{\alpha/2, \measuredangle(x, y, z)/2\}$ (see Fig. 5.7(b)). Again, by adding two guards, one at l_y, and the other one at the line collinear to d, both of them far enough to see each other and to see all the edges of P, we will get a cover of $F(P)$ by two cooperative guards. ■

THEOREM 5.13 ([94]). *Let P be a non-convex fortress of k pockets $p_1, \ldots, p_k$, of respectively n_i vertices, $i = 1, \ldots, k$. Then $1 + \sum_{i=1}^{k} \lfloor (n_i - 1)/2 \rfloor$ cooperative point guards always suffice to cover $F(P)$.*

Proof. Let us consider the pocket p_1. If it has three vertices, then by Lemma 5.12, p_1 and the convex hull of P can be covered by two cooperative guards. Together with $\sum_{i=2}^{k} \lfloor (n_i - 1)/2 \rfloor$ cooperative guards for the other pockets, with one guard per each pocket lid (by Corollary 5.4), we get a cooperative coverage of $F(P)$ of cardinality

$$2 + \sum_{i=2}^{s} \left\lfloor \frac{n_i - 1}{2} \right\rfloor = 1 + \sum_{i=1}^{k} \left\lfloor \frac{n_i - 1}{2} \right\rfloor.$$

Next, suppose that $n_1 > 3$ and n_1 is odd. Let T be a triangulation of p_1. Consider the triangulation graph G_T of T, and the triangle t of T with the pocket lid of p_1 as one of its edges. By Theorem 2.7, $\lfloor (n_1 - 2)/2 \rfloor$ cooperative guards suffice to dominate G_T, and there is a guard at a vertex of t. Again by Lemma 5.12, the pocket p_1 and the convex hull of P can be covered by $2 + \lfloor (n_{p_1} - 2)/2 \rfloor = 1 + \lfloor (n_1 - 1)/2 \rfloor$ cooperative guards, as n_1 is odd (these guards are cooperative, as there is a guard in t). Together with $\sum_{i=2}^{k} \lfloor (n_i - 1)/2 \rfloor$ cooperative guards for the other pockets, with one guard per each pocket lid (by Corollary 5.4), we get a cooperative coverage of $F(P)$ of cardinality

$$1 + \sum_{i=1}^{k} \left\lfloor \frac{n_i - 1}{2} \right\rfloor.$$

Finally, suppose that n_1 is even, and let $x_1, \ldots, x_s$ be the consecutive vertices of the pocket p_1, in a clockwise manner, with $\{x_s, x_1\}$ as the pocket lid ($s = n_1$). Let us consider the quadrilateral $Q = (x_1, x_2, x_{s-1}, x_s)$. We have to consider three cases.

CASE 1: Q is empty and convex (see Fig. 5.8(a)). Then the subpocket $(x_2, \ldots, x_{s-1})$ has $n_1 - 2$ vertices, and by Corollary 5.4, it can be covered by $\lfloor (n_1 - 2 - 1)/2 \rfloor$ cooperative guards, with one guard at x_2 or x_{s-1}, say at x_{s-1}. By Lemma 5.12, the triangle (x_1, x_{s-1}, x_s) and the convex hull of P can be covered by two cooperative guards. As the guard at x_{s-1} covers the triangle (x_1, x_2, x_{s-1}), the whole pocket p_1 is covered. As was the case before, this leads to a cooperative coverage of $F(P)$ of cardinality

$$2 + \left\lfloor \frac{n_1 - 2 - 1}{2} \right\rfloor + \sum_{i=2}^{k} \left\lfloor \frac{n_i - 1}{2} \right\rfloor = 1 + \sum_{i=1}^{k} \left\lfloor \frac{n_i - 1}{2} \right\rfloor.$$

CASE 2: Q is empty and non-convex. Assume the vertex x_{s-1} to be reflex. As in Case 1, we cover the subpocket $(x_2, \ldots, x_{s-1})$ by applying Corollary 5.4. If there is a guard at x_{s-1} in a coverage of $(x_2, \ldots, x_{s-1})$, we proceed in the same way as in the above case: the guard at x_{s-1} will cover all of Q. Otherwise, all we need is to notice that a guard at the line collinear to the edge $\{x_1, x_2\}$ (or close enough to it, the proof of Lemma 5.12, Case 2) will always cover the triangle (x_1, x_{s-1}, x_s), thus all of Q will be covered (see Fig. 5.8(b)).

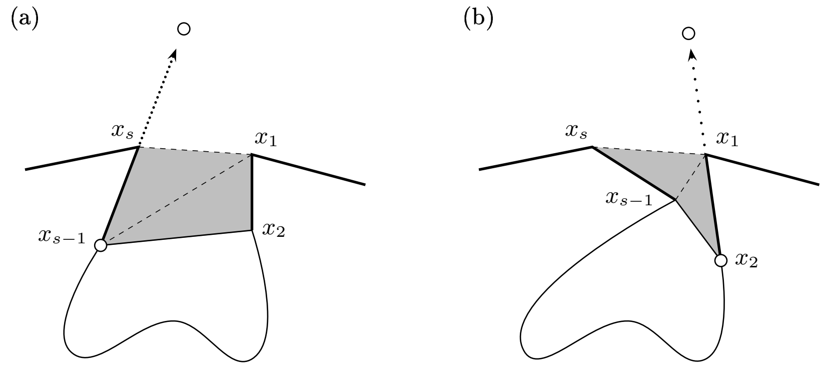

Fig. 5.8. (a) The quadrilateral $Q = (x_1, x_2, x_{s-1}, x_s)$ is empty and convex. (b) The quadrilateral Q is empty and non-convex.

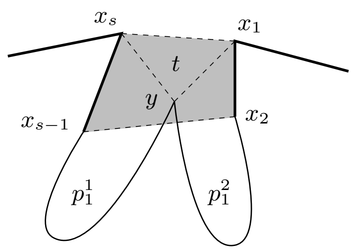

Fig. 5.9. The quadrilateral $Q = (x_1, x_2, x_{s-1}, x_s)$ is not empty.

Case 3: there is a vertex in Q. Let y be the closest vertex in Q to the pocket lid $\{x_s, x_1\}$. The pocket p_1 can be partitioned into two subpockets p_1^1 and p_1^2, of n_1^1 and n_1^2 vertices, respectively, and the triangle $t = (x_1, y, x_s)$ (see Fig. 5.9). As n_1 is even, either n_1^1 or n_1^2, say n_1^1, is odd. By Corollary 5.4, p_1^2 can be covered by $\lfloor (n_1^2 - 1)/2 \rfloor$ cooperative guards, with one guard either at x_1 or at y. If there is a guard at y, then by Corollary 5.4, the remainder of p_1^1 can be covered by $\lfloor (n_1^1 - 1)/2 \rfloor - 1$ cooperative guards, as n_1^1 is odd. The same construction as in the proof of Lemma 5.12 (we consider the triangle t as a pocket) leads to a coverage of t (thus the vertex y as well) and the convex hull of P by two cooperative guards. As before, this leads to a coverage of $F(P)$ by

$$2 + \left\lfloor \frac{n_1^1 - 1}{2} \right\rfloor - 1 + \left\lfloor \frac{n_1^2 - 1}{2} \right\rfloor + \sum_{i=2}^{k} \left\lfloor \frac{n_i - 1}{2} \right\rfloor \leq 1 + \sum_{i=1}^{k} \left\lfloor \frac{n_i - 1}{2} \right\rfloor$$

cooperative guards, as $n_1^1 + n_1^2 = n_1 + 1$. If there is a guard at x_1, then we have to consider two subcases.

Subcase 3.a: the lines l_1 and l_s, pointing outward from the convex hull and collinear respectively to the edges $\{x_1, x_2\}$ and $\{x_{s-1}, x_s\}$, intersect at a point $x^* \in F(P)$ (see

Fig. 5.10(a)). Note that x^* must see y. Consider the polygon p_1^* that results from replacing the polyline (x_{s-1}, x_s, x_1, x_2) in p_1 by (x_{s-1}, x^*, x_2), p_1^* has $n_1 - 1$ vertices. Next, consider a triangulation of p_1^* with $\{y, x^*\}$ as one of its internal diagonals. Then by Theorem 2.7, p_1^* can be covered by $\lfloor (n_1 - 1 - 2)/2 \rfloor$ cooperative guards, with one guard either at y or at x^*. If there is a guard at y, again we can proceed in a way similar to that in the proof of Lemma 5.12 (we consider the triangle (x_1, y, x_s) as a pocket). Otherwise, if there is a guard at x^*, then it is clear that two additional cooperative guards will cover the whole of the convex hull of P (and x^*). By Corollary 5.4, we will get a cooperative coverage of $F(P)$ of cardinality $1 + \sum_{i=1}^{k} \lfloor (n_i - 1)/2 \rfloor$.

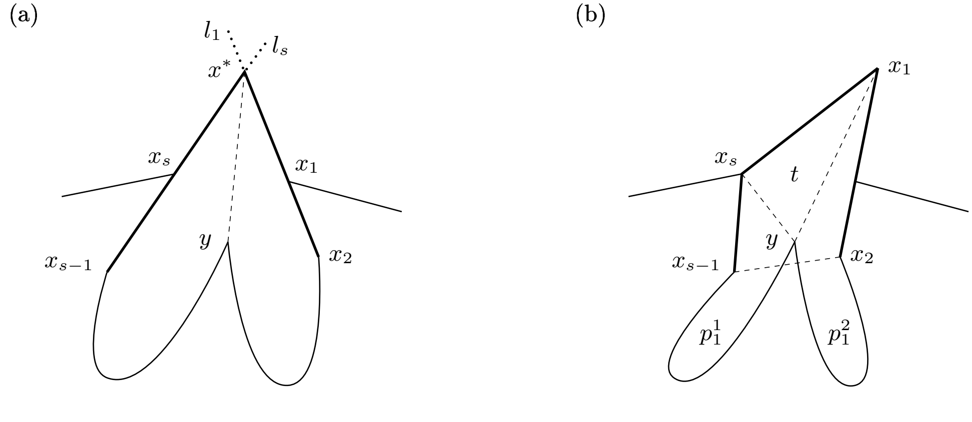

Fig. 5.10. (a) Subcase 3.a. (b) Subcase 3.b.

SUBCASE 3.b: the relevant lines l_1 and l_s, pointing outward from the convex hull, do not cross. Move x_1 along the line collinear to $\{x_1, x_2\}$ far enough to see all possible edges, thus transforming the subpocket p_1^2, still with n_1^2 vertices (see Fig. 5.10(b)). By Corollary 5.4, the new p_1^2 can be covered by $\lfloor (n_1^2 - 1)/2 \rfloor$ cooperative guards, with one guard either at y or at the new x_1. If there is a guard at y, then we can proceed as above. So assume there is a guard at x_1, and consider the polygon $p_1^1 \cup (x_1, y, x_s)$ of $n_1^1 + 1$ vertices. By Theorem 2.7, it can be covered by $\lfloor (n_1^1 + 1 - 2)/2 \rfloor$ cooperative guards, with one guard either at x_s or at y, and this guard is seen by the guard at the new x_1, of course. Thus the whole $p_1^1 \cup (x_1, y, x_s) \cup p_1^2$ can be covered by at most $\lfloor (n_1 - 1)/2 \rfloor$ cooperative guards. As the guard at x_1 is located far enough, with one additional guard we will cover the pocket p_1 and all of the convex hull of P by $1 + \lfloor (n_1 - 1)/2 \rfloor$ cooperative guards, provided that the first edge which is not visible from the new x_1 is not parallel to $\{x_1, x_2\}$ (cf. the proof of Lemma 5.12, Case 1). And again, $F(P)$ can be covered by $1 + \sum_{i=1}^{k} \lfloor (n_i - 1)/2 \rfloor$ cooperative guards.

Otherwise, if we consider the proof of Lemma 5.12, Case 2, then all we need is the possibility of moving the guard at x_1 a small distance $\epsilon > 0$ from x_1 along the edge $\{y, x_1\}$ without destroying the cooperation of the guards in the new p_1^2 (and hence without destroying the cooperation of the guards in $p_1^1 \cup (x_1, y, x_s) \cup p_1^2$). This can be done by the following argument.

Let G_T be the triangulation graph of a non-degenerate triangulation of an n-vertex polygon (there are no triangles with three points on a line), and let S be a guard coverage of this polygon, with $|S| \leq \lfloor (n-2)/2 \rfloor$, constructed from a cooperative domination of G_T (Theorem 2.7). Let x be a convex vertex with a guard at it. The vertex x with all triangles t_i incident to it forms a *fan*. Let $\{x_l, x\}$ and $\{x, x_r\}$ be the edges of P incident to x, and let $g_1, \ldots, g_k$ be the guards incident to x in the fan f. As S was obtained from a cooperative domination of G_T, it is clear that we have to show that the guard at x can be moved without destroying connectivity with these guards only.

For each g_i, $i = 1, \ldots, k$, in a sequence:

- rotate P in such a way that the line s_i collinear with the line segment xg_i is parallel to the y-axis, and x is below g_i;
- consider the vertex $r \in f$, closest to the right of the line s_i, and the vertex $l \in f$, closest to the left of s_i (if $g_i = x_l$ or $g_i = x_r$, then assume $r = x_l$, and $l = x_r$);
- let $\alpha(g_i)$ be the strip, interior to P, delimited by the lines enclosing the vertices l and r, respectively, and parallel to the line s_i (see Fig. 5.11).

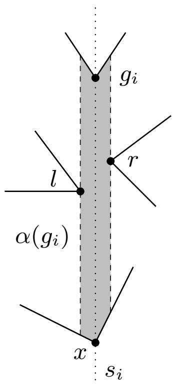

Fig. 5.11. The idea of the construction of the strip $\alpha(g_i)$.

It is obvious that $\bigcap_{i=1}^{k} \alpha(g_i) \neq \{x\}$, and all g_i are visible from any point in $\bigcap_{i=1}^{k} \alpha(g_i)$. Furthermore, $\bigcap_{i=1}^{k} \alpha(g_i) \cap \{x_l, x\} \neq \{x\}$, and $\bigcap_{i=1}^{k} \alpha(g_i) \cap \{x, x_r\} \neq \{x\}$. Thus the guard g at x can be moved a small distance $\epsilon > 0$ either along the edge $\{x_l, x\}$ or $\{x, x_r\}$, and the guard set S will still remain cooperative.

Thus $F(P)$ can be covered by $1 + \sum_{i=1}^{k} \lfloor (n_i - 1)/2 \rfloor$ cooperative guards. ■

5.4. Open problems

We have considered the situation when a guard g_1 sees another guard g_2 if they can be connected with a line segment outside the polygon. Nevertheless, we can restrict guards (and only guards) to see each other only when they can be connected with a line segment within the polygon (guards are located at the vertices, of course). This problem seems to be rather different from the one we have considered, and more realistic.

CONJECTURE 5.14. *If guards can see each other only within a polygon P, then $\lceil n/2 \rceil$ cooperative guards always suffice to cover $F(P)$.*

Weakly cooperative guards. Recall that a set S of guards is called *weakly cooperative* if the visibility graph $\mathrm{VG}(S)$ has no isolated vertices. A convex n-gon requires $\lceil 2n/3 \rceil$ watched vertex guards. From [91] we have

COROLLARY 5.15 ([94]). $\lceil 2n/3 \rceil$ *weakly cooperative vertex guards always suffice to cover an n-vertex fortress.*

But, as in the case of cooperative guards, it would be desirable to find a more accurate measure of the number of watched guards other than a function of n, the number of vertices.

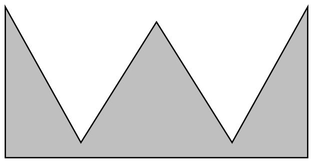

Fig. 5.12. If guards can see each other only within the polygon, then a non-convex fortress may require as many as $\lceil n/2 \rceil$ cooperative guards; here $n = 7$ and the fortress requires 4 cooperative guards.

The prison yard problem. Finally, it would be interesting to investigate the concept of cooperative guards for the *prison yard problem*, i.e., one wants to determine the number of cooperative guards always sufficient to cover both the interior and the exterior of a polygon. The original problem was solved in 1992 by Füredi and Kleitman [30], who proved that $\lceil n/2 \rceil$ vertex guards (respectively $\lfloor n/2 \rfloor$) are always sufficient and occasionally necessary to simultaneously guard the interior and the exterior of a convex (respectively non-convex) polygon with n vertices.

6. COOPERATIVE GUARDS IN GRIDS

In this chapter, we study the cooperative guards problem in grids, a special restricted class of polygons introduced by Ntafos [71]. A *grid* P is a connected union of vertical and horizontal line segments, an example of which is shown in Fig. 6.1. A grid can be thought

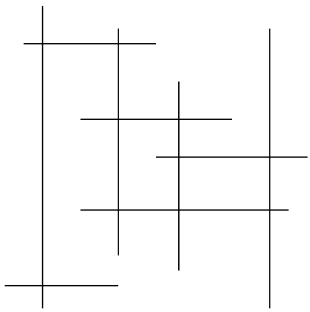

Fig. 6.1. A grid of 9 segments.

of as an orthogonal polygon with holes, consisting of very thin corridors. A point $x \in P$ can see a point $y \in P$ if the line segment $xy \subseteq P$. Ntafos established that a minimum cover for a (two-dimensional) grid of n segments has $n - m$ guards, where m is the size of the maximum matching in the intersection graph of the grid, and it can be found in $O(n^{2.5})$ time [71]. However, in the case of three-dimensional grids, the problem of finding the minimum guard set is NP-hard [71].

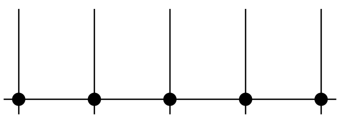

Fig. 6.2. A comb-grid of n segments that requires $n - 1$ cooperative guards (dots).

If we require a guard set to be cooperative, then more than $n - m$ guards may be required to cover a 2D-grid. Fig. 6.2 shows an example of an n-segment grid that requires $n-1$ cooperative guards, and we will prove this bound to be tight. But first, let us discuss an elementary property of a minimum cooperative guard set in a grid.

Lemma 6.1 ([69]). *In any minimum cooperative guard set of an n-segment grid with $n \geq 2$, guards are located only at the intersections of the line segments.*

Proof. Let S be a minimum cooperative guard set for a grid P, and let $\mathrm{VG}(S)$ be the visibility graph of S. Suppose, contrary to our claim, that there is a guard g located at a segment l, but not at an intersection of any segments. As the visibility graph $\mathrm{VG}(S)$ is connected, there is at least one guard at some intersection on l. Moreover, all guards visible to g are visible to each other as well. Consequently, the guard g is redundant, and this contradicts the minimality of S. ∎

Now, let $G = (P, E)$ be the intersection graph of a grid: each vertex of G corresponds to a line segment and two vertices are connected by an edge if their corresponding segments intersect. Fig. 6.3(a) shows the intersection graph for the grid in Fig. 6.1. By Lemma 6.1, any minimum cooperative guard set S is equivalent to a subset of E, denoted by E_S, and let $G[E_S]$ denote the induced subgraph of G formed by the edges from E_S (and their endpoints). By the definition of the minimum cooperative guard set S, it is easy to check that any such subset E_S has the property that (i) any vertex (line segment) of the intersection graph G must be an endpoint of some edge $e \in E_S$, and (ii) the graph $G[E_S]$ must be connected and acyclic. This implies that $G[E_S]$ is a spanning tree of the intersection graph G. Thus we have

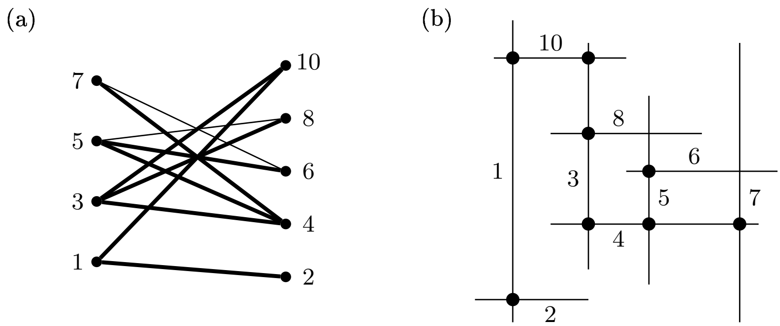

Fig. 6.3. (a) The intersection graph of the grid in Fig. 6.1, its spanning tree. (b) The cooperative guard set corresponding to the edges of the spanning tree.

Theorem 6.2 ([69]). *For any* 2D-*grid of* $n \geq 2$ *segments,* $n-1$ *cooperative guards are necessary and always sufficient, and there is a one-to-one correspondence between a minimum cooperative guard set and a spanning tree of the intersection graph of the grid.*

The above theorem leads to a polynomial algorithm for finding a minimum cooperative cover of an n-segment grid P:

1. Construct the intersection graph G of P;
2. Find a spanning tree T of G;
3. Place guards at crossings of P that correspond to edges of T.

By the definition, P can be partitioned into two sets of segments, say red segments (horizontal segments) and blue segments (vertical segments), such that no two segments from the same set intersect each other. This so-called red-blue line intersection problem can be solved in $O(n+k)$ time [27], as the grid is connected, where k is the number of intersections. Thus the intersection graph G can be constructed in $O(n+k)$ time. Next, a spanning tree T of G can be found in $O(n+k)$ time by the breadth-first search (BFS) algorithm. Finally, placing guards at all intersections that correspond to edges of T will result in $O(n+k)$ overall time complexity.

THEOREM 6.3 ([69]). *The problem of determining the minimum number of cooperative guards sufficient to guard an n-segment grid, $n \geq 2$, can be solved in $O(n+k)$ time and space, where k is the number of crossings in the grid.*

Finally, it follows from Theorem 6.2 that we can always construct a minimum cooperative guard cover of a grid with a guard placed at any selected crossing of the grid.

6.1. Cooperative guards in 3D-grids

Similarly to the two-dimensional case, a *three-dimensional* grid is a connected union of vertical and horizontal line segments in the three-dimensional space, but we additionally require that any two segments are either parallel or orthogonal; an example of a 3D-grid is shown in Fig. 6.4(a). When considering cooperative guards in 3D-grids, by similar arguments to the proof of Lemma 6.1, it is easy to see that guards can be restricted to be located at the vertices of a 3D-grid, which again motivates introducing intersection graphs.

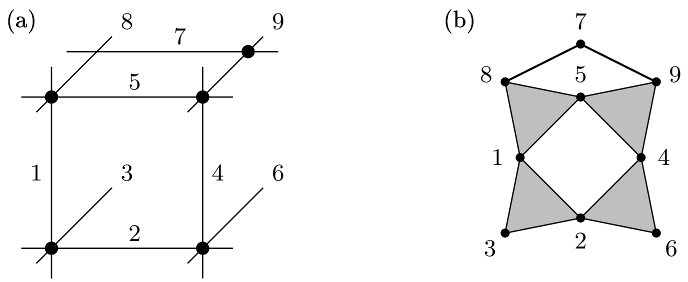

Fig. 6.4. (a) A three-dimensional grid P, and one of its cooperative guard sets (dots). (b) The intersection hypergraph $G(P, H)$.

Consider an exemplary three-dimensional grid P and the corresponding intersection graph $G = (P, E)$, shown in Fig. 6.4. As there are four intersections of three line segments, the intersection graph G has four hyperedges: $\{1,2,3\}$, $\{1,5,8\}$, $\{2,4,6\}$, $\{4,5,9\}$, respectively. Next, it is easy to check that there are four different minimum cooperative guard sets, each of five guards; one of them,

$$S = \{\{1, 2, 3\}, \{1, 5, 8\}, \{2, 4, 6\}, \{4, 5, 9\}, \{7, 9\}\},$$

is shown in our exemplary grid ($\{i, j\}$ and $\{l, k, t\}$ correspond to the intersections of segments i, j and segments l, k, t, respectively). Now, if we consider a subset $E_S \subset E$ of edges that correspond to all crossings in S, we get a minimum spanning subhypergraph of the intersection graph G.

By definition, the intersection graph G of a 3D-grid is a *3-restricted hypergraph*: the vertices correspond to segments and the edges correspond to intersections, but it may now happen that three line segments have a point in common, thus such an intersection corresponds to a *hyperedge* of rank 3 (in other words, the size of an edge, which is a subset of vertices, is at most 3). The analysis similar to that in the case of two-dimensional grids shows that any minimum cooperative guard set in P corresponds to a minimum spanning subhypergraph of the intersection graph G. Note that in a hypergraph, a minimum spanning subgraph need not be acyclic any longer.

LEMMA 6.4 ([69]). *There is a one-to-one correspondence between the minimum cooperative guards problem in a three-dimensional grid and the minimum spanning subhypergraph problem in the intersection hypergraph of the grid.*

Thus, to find a minimum cooperative guards set, all we need is to find a minimum spanning subhypergraph of the intersection hypergraph.

LEMMA 6.5 ([69]). *Let G be the intersection hypergraph of an n-segment grid. Then the minimum spanning subhypergraph problem for the hypergraph G can be solved in $O(kn^{2.5})$ time and $O(kn)$ space, where k is the number of edges in G.*

The proof of the above lemma is based upon the reduction to the minimum spanning set problem in 2-polymatroids [7], and it will be presented in Subsection 6.1.1, where polymatroids and the minimum spanning set problem are introduced.

As the intersection hypergraph can be constructed in $O(n^2)$ time by using the greedy technique (by considering all pairs of segments), we get the following theorem.

THEOREM 6.6 ([69]). *The minimum cooperative guards problem in a three-dimensional grid can be solved in $O(kn^{2.5})$ time and $O(kn)$ space, where n is the number of segments and k is the number of crossings.*

6.1.1. Polymatroids and hypergraphs—proof of Lemma 6.5. First, we show that the minimum spanning subhypergraph problem in 3-restricted hypergraphs can be easily reduced to the minimum spanning subset problem in 2-polymatroids.

6.1.1.1. *Reduction.* A *polymatroid* (H, d) consists of a finite set H and a *dimension function* d. The dimension function maps each subset of H to an integer. It is non-negative, increasing, and submodular, i.e.,

$$\forall X, Y \subset H : d(X \cap Y) + d(X \cup Y) \leq d(X) + d(Y).$$

If $d(\{h\}) \leq k$ for all $h \in H$, then H is called a *k-polymatroid*. A set $X \subseteq H$ is called a *spanning set* if $d(X) = d(H)$; the minimum cardinality of a spanning set of the polymatroid (H, d) is denoted by $\rho(H, d)$. A set $X \subseteq H$ is called a *matching* if $d(X) = k|X|$; the size of the maximum matching of the polymatroid (H, d) is denoted by $\nu(H, d)$.

Consider now a 3-restricted hypergraph $HG = ([n], H)$ with vertex set $[n] = \{1, \ldots, n\}$ and with edge set H. Let $G = ([n], E)$ be the complete graph on the same vertex set, with $E = \binom{[n]}{2}$; recall that for a set S, $\binom{S}{2}$ denotes the set of all 2-element subsets of S. Define now the 1-polymatroid (E, r), where for each $E' \subset E$, $r(E')$ is the number of edges in a spanning tree for E' ((E, r) is called the *graphic matroid* of G). Then, for a hyperedge $h \in H$, regarded as the edge subset $\binom{h}{2}$ of E, we have $r(\binom{h}{2}) = |h| - 1$, the number of edges in a spanning tree for h. Therefore we obtain a 2-polymatroid (H, d) if we put

$$d(X) := r\left(\bigcup_{h \in X} \binom{h}{2}\right) \quad \text{for } X \subseteq H.$$

By the definition of d, if $([n], H)$ is connected, then $d(H) = n - 1$, and it is clear that there is a one-to-one correspondence between minimum spanning sets of the 2-polymatroid (H, d) and those of $HG([n], H)$.

Algorithm. Let us recall the following fact, which generalizes Gallai's identity [31]: *For any 2-polymatroid (H, d), we have $\nu(H, d) + \rho(H, d) = d(H)$* [60]. Consequently, we get the following lemma.

LEMMA 6.7 ([7]). *A minimum spanning subhypergraph of a 3-restricted connected hypergraph $HG = ([n], H)$ has $n - \nu(H, d) - 1$ edges.*

Of course, we would like to know not only the number of edges, but also to determine the minimum spanning subhypergraph. This can be done by the following algorithm:

1. $S := M$, where M is a maximum matching in the input hypergraph $HG = ([n], H)$;
2. While $d(S) < n - 1$ do
   consider all edges $\notin M$ in turn:
   if the number of connected components in $HG[S \cup \{e\}]$
   is greater than in $HG[S]$, then $S := S \cup \{e\}$;

3 Return S.

All we need is to show that the output of this algorithm is a set of edges of a minimum spanning subhypergraph of HG. It is clear that the spanning property follows from the "while" condition $d(S) < n - 1$ (recall that HG is connected, and thus $d(H) = n - 1$). The minimality is established with the following observations. First, note that the initial induced subhypergraph $HG[S]$ (S is a maximum matching) has exactly $n - 2\nu(H, d)$ connected components by the definition of the matching in a polymatroid. Secondly, any edge added in the "while" step decreases the number of components exactly by one, thus exactly $n - 2\nu(H, d) - 1$ edges are added to the initial set S. Hence the output is a spanning set and it has $\nu(H, d) + n - 2\nu(H, d) - 1 = n - \nu(H, d) - 1$ elements.

COROLLARY 6.8 ([69]). *When the above algorithm halts, the output is the set of edges of a minimum spanning subhypergraph of the input 3-restricted (connected) hypergraph.*

Again consider an exemplary three-dimensional n-segment grid P and the corresponding intersection graph $HG = (P, H)$, shown in Fig. 6.4. It is easy to see that HG has four different maximal matchings; suppose that in the first step of the algorithm we have $M = \{\{1,2,3\},\{1,5,8\},\{2,4,6\}\}$. Then in the "while" step, either the edges $\{7,8\},\{8,9\}$, or $\{7,8\},\{4,5,9\}$, or $\{8,9\},\{4,5,9\}$ are added to the initial set S, thus yielding three different spanning subgraphs, and consequently, three different cooperative guard sets; the third one, corresponding to the spanning set

$$S = \{\{1,2,3\},\{1,5,8\},\{2,4,6\},\{4,5,9\},\{8,9\}\}$$

is shown in Fig. 6.4(a). Note that if we got the set $\{\{1,2,3\},\{1,5,8\},\{2,4,6\},\{7,8\},\{8,9\}\}$, then the resulting spanning subhypergraph would be acyclic. Thus in general, there is no guarantee that the algorithm returns an acyclic spanning subhypergraph of the intersection graph, even if such a subgraph exists.

The algorithm spends most of the time on finding the maximum matching M in a 2-polymatroid (H,d). As far as we know, the best known algorithm solves this problem in $O(|H|d(H)^{2.5})$ time and $O(|H|d(H))$ space [32], since by the following lemma, our 2-polymatroid (H,d) is linear.

LEMMA 6.9 ([69]). *Every 2-polymatroid (H,d) is linear.*

Proof. Recall that a polymatroid (H,d) is *linear* if there is a vector space V (over a field F) and a mapping ϕ of H into V that preserves the rank. We will construct a mapping ϕ that maps (H,d) into $V = \mathbb{R}^n$.

Let $e_i = (0,\dots,0,1,0,\dots,0)$ denote the ith unit vector from $V = \mathbb{R}^n$. The mapping ϕ is defined as follows. For each $h = \{i,j\} \in H$, $i<j$, let $\phi(h) := \{e_j - e_i\}$; and, for each $h = \{i,j,k\} \in H$, $i<j<k$, let $\phi(h) := \{e_j - e_i, e_k - e_i\}$. Next, for each $X \subseteq H$ we put $\phi(X) := \bigcup_{h\in X}\phi(h)$.

CLAIM. *Let $X \subseteq H$. Then the subgraph T of $G = ([n],E)$ induced by the set $\bigcup_{h\in X}\binom{h}{2}$ is a forest if and only if $\phi(X)$ is linearly independent.*

($\Rightarrow$) The proof is by induction on m, the number of edges in the forest T. The case $m=1$ is trivial. So assume that the claim holds for all forests with $1 \le m' \le m$ edges. As T is a forest with at least two edges, it has a leaf v. We assume that $v=1$ and $h=\{1,2\}$ is an edge of T. Then $v_1 = \phi(h) = e_2 - e_1$ is an element of $\phi(X)$. Let $v_2 = e_{j_2} - e_{i_2}, \dots, v_m = e_{j_m} - e_{i_m}$ be all vectors of $\phi(X)\setminus\{v_1\}$ (notice that $|\phi(X)| = m$). Consider now the equation

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0,$$

which, as $v_1 = e_2 - e_1, v_2 = e_{j_2} - e_{i_2}, \dots, v_m = e_{j_m} - e_{i_m}$, becomes

$$\alpha_1(e_2 - e_1) + \alpha_2(e_{j_2} - e_{i_2}) + \cdots + \alpha_m(e_{j_m} - e_{i_m}) = 0. \tag{6.1}$$

As v is a leaf, we have $e_{i_t} \ne e_1$ (and $e_{j_t} \ne e_1$, as $j_t > i_t$) for all $t = 2,\dots,m$. Consequently, this forces $\alpha_1 = 0$ in (6.1). As $X\setminus\{h\}$ is a forest with $m-1$ edges, and

$$\phi(X\setminus\{h\}) = \{v_2,\dots,v_m\}$$

by the definition, the vectors $v_2,\dots,v_m$ are linearly independent by the induction hypothesis. Therefore $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0$ if and only if $\alpha_i = 0$, $i = 1,\dots,m$.

($\Leftarrow$) On the contrary, suppose that $\bigcup_{h\in X}\binom{h}{2}$ has a cycle C. We can assume that $C=(1,2,3,\ldots,j)$, $3\le j\le n$. Consider the vector subset

$$\{v_1=e_2-e_1, v_2=e_3-e_2,\ldots,v_{m-1}=e_j-e_{j-1}, v_m=e_j-e_1\}\subseteq\phi(X).$$

As $(e_2-e_1)+(e_3-e_2)+\cdots+(e_j-e_{j-1})=e_j-e_1$, the vectors $v_1,\ldots,v_m$ are not linearly independent, which completes the proof of the claim.

Consider now $X\subset H$, and assume that $d(X)=c$, the number of edges in a spanning forest of the subgraph induced by $\bigcup_{h\in X}\binom{h}{2}$ in the complete graph $G=([n],E)$. We have to show that the rank of $\phi(X)$ is equal to c; recall that for a subset S of a vector space, the *rank* of S, denoted by $\mathrm{rank}(S)$, is the maximum number of linearly independent vectors in S.

If the subgraph $([n],E_X)$ induced by $\bigcup_{h\in X}\binom{h}{2}$ is a forest then $\mathrm{rank}(\phi(X))=c$ by the above claim. Otherwise, remove the vector $v=e_j-e_i$ from $\phi(X)$, where $\{i,j\}$ is on a cycle in $([n],E_X)$. By similar arguments to the proof of the above claim, v is a linear combination of all other cycle edges, and thus its removal does not change the rank, i.e., $\mathrm{rank}(\phi(X))=\mathrm{rank}(\phi(X)\setminus\{v\})$. We iterate this procedure until there are no more cycles. Since $d(X)=c$, the graph $([n],E_X)$ has $n-c$ connected components, and thus exactly $n-(n-c)=c$ vectors are left in $\phi(X)$. As these vectors correspond to the edges of a forest, they are linearly independent by the above claim, and thus $\mathrm{rank}(\phi(X))=c$. ∎

COROLLARY 6.10 ([69]). *Let $HG=(P,H)$ be the intersection hypergraph of an n-segment grid P. A minimum spanning subhypergraph of HG can be found in $O(kn^{2.5})$ time and $O(kn)$ space, where k is the number of crossings in the grid.*

Consequently, we get the following theorem.

THEOREM 6.11 ([69]). *A minimum cover of an n-segment 3D-grid has $n-\nu-1$ cooperative guards, where ν is the size of the maximum matching in the 2-polymatroid constructed from the intersection hypergraph of the grid.*

We emphasize that Theorem 6.11 generalizes the result of Theorem 6.2: if there are no hyperedges, then $\nu=0$.

6.2. Weakly cooperative guards

Recall that a set S of guards is called *weakly cooperative* if the visibility graph $\mathrm{VG}(S)$ has no isolated vertices. An n-segment comb-grid requires as many as $n-1$ weakly cooperative guards (see Fig. 6.2), but in general, this is far too many. In this section, we show that a minimum coverage for a grid of n segments has $n-p_3$ weakly cooperative guards, where p_3 is the size of the maximum P_3-matching in the intersection graph of the grid. Consequently, it makes the minimum weakly cooperative guards problem in grids NP-hard, as we prove that the maximum P_3-matching problem in subcubic bipartite planar graphs is NP-hard.

6.2.1. P_3-matching in subcubic bipartite planar graphs. As the minimum weakly cooperative guards problem in grids is solved using the matching techniques, in this section we will look more closely at the P_3-matching problem in bipartite planar graphs.

Generalized matching problems have been studied in a wide variety of contexts [5, 11, 38, 46, 52, 89]. One of the generalizations is to find the maximum number of vertex-disjoint copies of some fixed graph H in a given graph G (*maximum H-matching*). In [38] Kirkpatrick and Hell showed that any *perfect* matching problem is NP-complete for any connected H with at least three vertices (the same holds for the maximum H-matching problem). In [11] the authors showed that the maximum H-matching problem remains NP-complete even if we restrict to planar graphs.

The perfect H-matching problem is also known as the *$\{H\}$-factor problem.* A natural generalization is the *$\mathcal{H}$-factor problem*, where $\mathcal{H} = \{H_1, \ldots, H_h\}$ is a family of connected graphs. In the $\mathcal{H}$-factor problem, for a given graph G we ask whether there exists a spanning subgraph F of G such that every component of F is isomorphic to some H_i, $i = 1, \ldots, h$. A special case is if $\mathcal{H}$ consists only of all paths of order at least 3 (so called $\{P_{n\geq 3}\}$-factor). Observe that a graph has a $\{P_{n\geq 3}\}$-factor if and only if it has a $\{P_3, P_4, P_5\}$-factor, and the criterion for a graph to have such a factor was obtained by Kaneko [46] (for bipartite graphs, such a criterion was given by Wang [89]). Moreover, Kaneko proved that every r-regular graph with $r \geq 3$ has a $\{P_{n\geq 3}\}$-factor. For 2-connected cubic graphs, Kawarabayashi *et al.* [52] showed that any such graph of at least five vertices has a $\{P_3, P_4\}$-factor. However, the following conjecture, posed by Akiyama and Kano [5], is still open: *Every 3-connected cubic graph of order divisible by three has a $\{P_3\}$-factor.* Finally, recall that Kaneko *et al.* [48] established that every claw-free graph on n vertices has the maximum P_3-matching of cardinality $\lfloor n/3 \rfloor$. For more details concerning factor problems we refer the reader to [5, 47].

In this section, we show that the maximum P_3-matching problem in subcubic bipartite planar graphs is NP-hard; recall that a graph G is *subcubic* if its maximum degree $\Delta(G) \leq 3$. The proof proceeds by reduction from the three-dimensional matching problem (3DM for short) [33], which can be formulated in the way described below. Dyer and Frieze [24] proved that the 3DM problem remains NP-complete even for bipartite planar graphs. Hence from [24, 33], the following restricted 3DM problem is NP-complete.

Instance: A subcubic bipartite planar graph $G = (V \cup M, E)$, where $V = X \cup Y \cup Z$, $|X| = |Y| = |Z| = q$. For every vertex $m \in M$ we have $\deg(m) = 3$ and m is adjacent to exactly one vertex from each of the sets X, Y and Z.

Question: Is there a subset $M' \subseteq M$ of cardinality q covering all vertices in V?

THEOREM 6.12 ([69]). *The* 3DM *problem in subcubic bipartite planar graphs is* NP-*complete.*

Now, using the above result, we will show that the perfect P_3-matching problem in subcubic bipartite planar graphs is NP-complete. Let $G = (V \cup M, E)$ be a subcubic bipartite planar graph, where $V = X \cup Y \cup Z$, $|X| = |Y| = |Z| = q$, every vertex $m \in M$ has degree 3, and is adjacent to exactly one vertex from each of the sets X, Y and Z.

Let $G^* = (V^*, E^*)$ be the graph obtained from G by replacing each vertex $v^i \in M$, $i = 1, \ldots, |M|$, (and all edges incident to it) with the graph $G_i = (V_i, E_i)$ of Fig. 6.5. Formally:

- $V_i = \{p^i_j\}_{j=1,\ldots,9} \cup \{x^i_j, y^i_j, z^i_j\}_{j=1,2,3}$;
- $V^* = V \cup \bigcup_{i=1,\ldots,|M|} V_i$;
- $E^* = E \setminus E^- \cup E^+$, where

$$E^- = \bigcup_{i=1}^{|M|} \{\{x^i, v^i\}, \{y^i, v^i\}, \{z^i, v^i\}\},$$

$$E^+ = \bigcup_{i=1}^{|M|} (E_i \cup \{\{x^i, x^i_2\}, \{y^i, y^i_2\}, \{z^i, z^i_2\}\}),$$

and x^i, y^i and z^i are neighbours of the vertex v^i in G.

Clearly, G^* has $|V| + 18|M|$ vertices and $|E| + 17|M|$ edges, and $\Delta(G^*) = 3$.

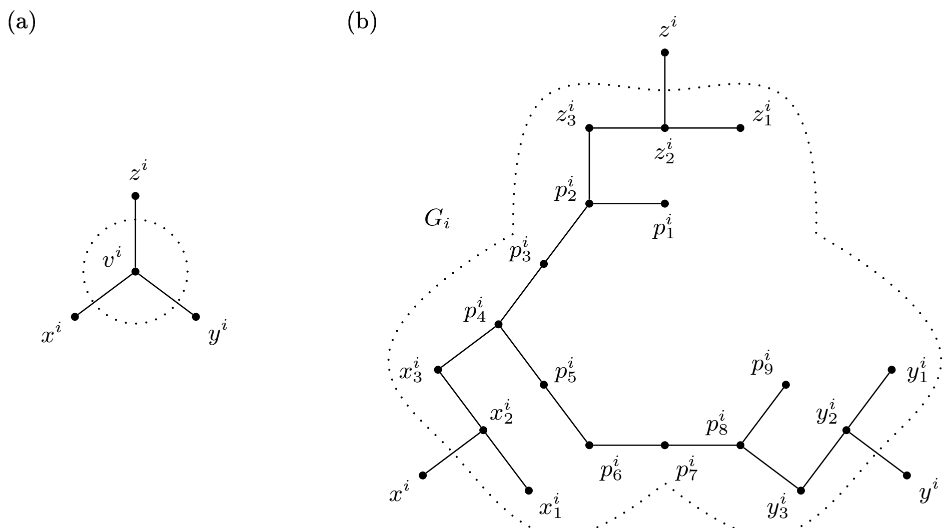

Fig. 6.5. The vertex v^i of (a) is replaced the graph G_i of (b).

Lemma 6.13 ([62]). *There exists a solution of the* 3DM *problem in the graph G if and only if there exists a perfect P_3-matching (of cardinality $q + 6|M|$) in the graph G^*.*

Proof. ($\Rightarrow$) Let M' be a solution of the 3DM problem in G with $|M'| = q$. A perfect P_3-matching in G^* consists of the following 3-vertex paths:

- if the vertex v^i corresponding to the graph G_i is in M', then in G_i with attached vertices x^i, y^i and z^i (see Fig. 6.5) we choose the following 3-vertex paths: $x^i_1 x^i_2 x^i$, $y^i_1 y^i_2 y^i$, $z^i_1 z^i_2 z^i$, $x^i_3 p^i_4 p^i_3$, $y^i_3 p^i_8 p^i_9$, $z^i_3 p^i_2 p^i_1$, $p^i_5 p^i_6 p^i_7$;

- otherwise, if the vertex v^j corresponding to the graph G_j is not in M', then in G_j with attached vertices x^j, y^j and z^j we choose the following 3-vertex paths: $x_1^j x_2^j x_3^j$, $y_1^j y_2^j y_3^j$, $z_1^j z_2^j z_3^j$, $p_1^j p_2^j p_3^j$, $p_4^j p_5^j p_6^j$, $p_7^j p_8^j p_9^j$.

($\Leftarrow$) Let P be a maximum P_3-matching of cardinality $q + 6|M|$ in G^* (P is perfect, of course). Consider any subgraph G_i, together with vertices x^i, y^i and z^i. Then either there exist paths $x_1^i x_2^i x^i$, $y_1^i y_2^i y^i$ and $z_1^i z_2^i z^i$ in P or none of them, which is forced by the following claims.

Claim 1. In any perfect P_3-matching, x^i cannot be the center of a 3-vertex path covering x_i^2, otherwise it is not possible to cover the vertex x_1^i. The same observation can be drawn for the vertices y^i and z^i.

Claim 2. If x_2^i is in a 3-vertex path in a perfect P_3-matching covering G^*, then x_1^i is in the same path, otherwise it is not possible to cover x_1^i. The same observation can be drawn for y_2^i and z_2^i.

Claim 3. It is not possible that x^i is covered by a 3-vertex path p and $x_2^i \notin p$, while either y^i or z^i (by Claims 1 and 2) is covered by $y_1^i y_2^i y^i$ or $z_1^i z_2^i z^i$, respectively. Otherwise, without loss of generality, assume that only y^i is covered by $y_1^i y_2^i y^i$ (by Claim 1, z^i is covered from "outside" of G_i). Then there does not exist a perfect P_3-matching that covers $V_i \setminus \{y_1^i, y_2^i\}$, as $|V_i \setminus \{y_1^i, y_2^i\}| = 16$. The same conclusion can be drawn when only z^i is covered by $z_1^i z_2^i z^i$ or for the case when both y^i and z^i are covered by $y_1^i y_2^i y^i$ and $z_1^i z_2^i z^i$, respectively. In general, similar observations apply to y^i and z^i.

Therefore our perfect P_3-matching P of G^* either (1) consists of the paths $x_1^i x_2^i x^i$, $y_1^i y_2^i y^i$ and $z_1^i z_2^i z^i$ or (2) none of them is in P. Hence the set

$$M' = \{v^i \in M : G_i \text{ is covered in the manner } (1)\}$$

is a solution to the 3DM problem in G. ■

A non-deterministic polynomial algorithm for the perfect P_3-matching problem just guesses triples of vertices and checks whether the subgraphs induced by these triples have no isolated vertices. Consequently, by the above lemma and Theorem 6.12, we get the following theorem.

Theorem 6.14 ([62]). *The perfect P_3-matching problem in subcubic bipartite planar graphs is NP-complete.*

As a consequence, we get the following theorem.

Theorem 6.15 ([62]). *The maximum P_3-matching problem in subcubic bipartite planar graphs is NP-hard.*

6.2.2. Weakly cooperative guards and P_3-matching. Following the proof of Lemma 6.1, it is easy to see that weakly cooperative guards can be restricted to be located at the crossings of a grid, and therefore the placement of a guard at the crossing of two segments s_1 and s_2 in a grid corresponds to the edge $\{s_1, s_2\}$ in the intersection graph G. Hence there is a one-to-one correspondence between a minimum weakly cooperative guard set in the grid and a subset E_S of edges in G. The weak cooperation of a guard set implies that the corresponding subset E_S of edges satisfies the following conditions:

(1) E_S covers all vertices in V;
(2) in the induced subgraph $G[E_S]$ there are no connected components isomorphic to a single edge.

Consequently, finding a minimum weakly cooperative guard set for a grid is equivalent to finding a minimum edge subset E_S of G satisfying (1) and (2). Note that such a subset always exists, as conditions (1) and (2) hold for any spanning tree of G.

As we are looking for a minimum subset E_S of edges satisfying (1) and (2), it is natural to ask about the structure of a connected component in the graph $G[E_S]$. A complete characterization is given by the following lemma. Recall that $\text{diam}(G)$ is the maximum of the shortest path distance over all vertex pairs in G, and the *center* of G is a vertex with the minimum eccentricity, where the *eccentricity* of a vertex v is the maximum distance to other vertices.

LEMMA 6.16 ([62]). *Let $G = (V, E)$ be a graph and let $E_{\min} \subset E$ be a minimum subset of edges satisfying* (1)–(2). *Let $G_S = (V_S, E_S)$ be any connected component of the graph $G[E_{min}]$. Then G_S has the following properties*:

(a) *G_S is acyclic;*
(b) $2 \leq \text{diam}(G_S) \leq 4$*;*
(c) *there is at most one vertex of degree at least* 3 *in G_S, and it is the center of G_S.*

Proof. (a) Conversely, suppose that in G_S there is a cycle $C = e_1 \dots e_k$, where $e_i = \{v_i, v_{i+1}\}$ for $i = 1, \dots, k-1$, and $e_k = \{v_k, v_1\}$. Consider the edge e_k and set $E^*_{\min} = E_{\min} \setminus \{e_k\}$. We will show that $E^*_{\min}$ satisfies (1)–(2), thus contradicting the minimality of $E_{\min}$.

As all vertices still remain covered and the number of connected components does not change, all we need is to show that the property (2) remains valid. Clearly, (2) can be disturbed only in G_S, and moreover, only for an edge e in $E(v_1) \cup E(v_k)$, where $E(v)$ denotes the set of edges incident to the vertex v. We give the reasoning only for $e \in E(v_1)$; the other case can be solved in a similar way.

If $e = e_1$, then $|e_1 \cap e_2| = 1$. Otherwise, $|e \cap e_1| = 1$. Hence $E^*_{\min}$ satisfies both (1) and (2), and this contradicts the minimality of $E_{\min}$. Therefore G_S is acyclic.

(b) The inequality $2 \leq \text{diam}(G_S)$ follows from (2), so we only have to show that $\text{diam}(G_S) \leq 4$. Suppose, on the contrary, that $\text{diam}(G_S) > 4$. This implies that there is a path P_6 of order 6 in G_S. Let $P_6 = e_1e_2e_3e_4e_5$, where $e_i = \{v_i, v_{i+1}\}$, $i = 1, \dots, 5$. Then applying a similar reasoning to that in (a), we find that the edge e_3 is needless, thus contradicting the minimality of $E_{\min}$.

(c) We have to consider two cases. First, suppose that there are at least two vertices of degree at least 3 in G_S, say v_1 and v_2. Then by removing any edge from a path joining v_1 and v_2 in G_S, we get a smaller subset of edges satisfying (1)–(2).

If there is only one vertex v of degree at least 3 and v is not the center of G_S, then by (b) there is a path $P_5 = v_3v_2v_1vv_4$ with v_3 and v_4 as leaves in G_S. Clearly, v_1 is the center of G_S. But then the edge $\{v_1, v\}$ is redundant. ■

From now on, a graph G_S satisfying conditions (a)–(c) will be called a *spider*. Note that in the case $\mathrm{diam}(G_S) = 2$, a spider is just a star-graph, a tree with at most one non-leaf.

6.2.3. Maximum P_3-matching and spider cover. Let us define the *maximum spider cover problem* (MaxSC problem for short) in a graph $G = (V, E)$ as the problem of finding the maximum number of vertex-disjoint spiders that cover V (that is, each vertex of G is a vertex of a spider).

LEMMA 6.17 ([62]). *A family of spiders $S_1, \ldots, S_p$ is a solution to the* MaxSC *problem in the intersection graph of a grid P if and only if $\bigcup_{i=1,\ldots,p} E(S_i)$ is a solution to the* MinWCG *problem in P.*

Proof. Because the number of edges of the spiders $S_1, \ldots, S_p$ is $|V| - p$, by Lemma 6.16 it is easy to observe that maximizing the number of spiders is equivalent to minimizing the number of guards. ∎

LEMMA 6.18 ([62]). *Let p be the number of spiders in a maximum spider cover of a graph G and let p_3 be the cardinality of a maximum P_3-matching in G. Then $p = p_3$ and solutions of these problems are equivalent in the sense that one can be constructed from the other.*

Proof. ($p \le p_3$) This inequality is obvious because all spiders are vertex-disjoint and each spider has a 3-vertex path as its subgraph.

($p \ge p_3$) Let V' be a set of vertices covered by a maximum P_3-matching P in $G = (V, E)$, and let $E' \subseteq E$ be the set of edges of all paths in P. By the definition, E' satisfies (2).

CLAIM. There is no vertex in G whose shortest distance to a vertex of P is 3 or more, otherwise P is not maximum.

Now, let $V_1, V_2 \subseteq V$ be subsets of vertices whose distance from V' is 1 and 2, respectively. Let us connect vertices from V_1 to V' by adding edges to E', one edge per each vertex in V_1. The resulting graph $G[E']$ is acyclic, and the diameter of any of its connected components is at most 4; the new set E' still satisfies (2). Next, let us connect vertices from V_2 to the new V' by adding edges to the new E', one edge per each vertex in V_2. It is easy to see that:

- the number of connected components in the new $G[E']$ remains p_3;
- $G[E']$ remains acyclic;
- the diameter of any connected component of $G[E']$ is at most 4, otherwise the matching P is not maximum;
- in any connected component of $G[E']$ there is at most one vertex of degree three which is the center, otherwise P is not maximum;
- E' still satisfies (2).

By all these observations, the final E' is a set of spiders. Moreover, $G[E']$ covers all vertices of G by the above claim. ∎

The above reductions can be done in $O(|V| + |E|)$ time by checking the neighbours of all vertices from the maximum P_3-matching,

Recall that a minimum (arbitrary) guard cover and a minimum cooperative guard cover of an n-segment grid have $n - m$ and $n - 1$ guards, respectively, where m is the cardinality of the maximum matching in the intersection graph of the grid. We have an analogous formula for weakly cooperative guards.

THEOREM 6.19 ([62]). *A minimum weakly cooperative guard cover for a grid P of n segments has $n - p_3$ guards, where p_3 is the cardinality of a maximum P_3-matching in the intersection graph of P.*

Proof. By Lemma 6.18, the number of edges in a maximum spider cover is equal to $2p_3 + (n - 3p_3) = n - p_3$. By Lemma 6.17, the assertion follows. ∎

We have just proven that the minimum weakly cooperative guards problem in a grid is equivalent to the maximum P_3-matching problem in the intersection graph of the grid. Recall that any intersection graph is bipartite, but not every bipartite graph is the intersection graph of a grid. Nevertheless, any bipartite planar graph is the intersection graph of a grid [21], and hence by Theorems 6.15 and 6.19, we get the following theorem.

THEOREM 6.20 ([62]). *The minimum weakly cooperative guards problem is NP-hard even for grids in which any segment crosses at most three other segments.*

6.2.4. Final remarks. Masuyama and Ibaraki [63] showed that the maximum P_i-matching problem in trees can be solved in linear time, for any $i \geq 3$. The idea of their algorithm is to treat a tree T as a rooted tree (T, r) (with an arbitrary vertex r as the root) and to pack i-vertex paths while traversing (T, r) in the bottom-up manner. Hence by Theorem 6.19, we get the following corollary.

COROLLARY 6.21 ([62]). *The minimum weakly cooperative guards problem for grids with trees as intersection graphs can be solved in linear time.*

Finally, recall that the guards problem was also stated for three-dimensional grids. In the case of arbitrary guards, the minimum guard problem is NP-complete (reduction from the vertex cover problem), whereas for cooperative guards, the minimum cooperative guards problem can be solved in polynomial time, and a solution is obtained from a spanning set of a 2-polymatroid constructed from the intersection graph of a grid. Of course, we can ask about the minimum weakly cooperative guards problem in three-dimensional grids, but the MinWCG problem in this class of grids is NP-complete by Theorem 6.20.

6.3. Mobile guards

In 1981, Toussaint introduced the idea of mobile guards in art galleries [6]: a *mobile guard* was constrained to patrol either along an edge of a polygon or along a straight line wholly contained within the polygon. Note that in the mobile guards problem, we do not require that every point of a polygon is permanently covered, but we only need every point to be seen by at least one guard during his walk. Toussaint conjectured that except for a small number of polygons, $\lfloor n/4 \rfloor$ edge guards are sufficient to guard a polygon, and this problem still remains open.

In this section, we explore the problem of mobile guards in grids. Specifically, each mobile guard is allowed to move along a grid segment, and then a point x in a grid P is said to be seen by a guard g if there is a point $y \in g$ such that the segment $xy \subseteq P$. Thus x is covered by the guard g if either $x \in g$ or x belongs to a grid segment crossing g. Now by the definition, a mobile guard corresponds to a vertex in the intersection graph $G = (V, E)$ of P, and P is covered by a set S of mobile guards if and only if S *dominates* all vertices in G, that is, every vertex in $V \setminus S$ is adjacent to at least one vertex in S. Thus there is a one-to-one correspondence between a minimum mobile guard set in P and a minimum dominating set in G. Consequently, if the *domination number* of G, denoted by $\gamma(G)$, is defined to be the cardinality of a minimum dominating set in G, then a minimum mobile guard set of P has $\gamma(G)$ mobile guards. This crucial fact was used by Katz *et al.* [51], who proved that the problem of finding the minimum number of mobile guards covering a grid is NP-hard.

THEOREM 6.22 ([51]). *The problem of finding the minimum number of mobile guards covering a grid is NP-hard.*

Before proceeding further, note that a grid is a geometrical object, and not a combinatorial one. In the previous sections, we did not focus on this fact, as no confusion could arise. However, we have to be careful when we say a 'segment' of a grid, as geometrically speaking, there are infinitely many (sub)segments of a grid. Thus a segment of a grid which is not strictly contained in any other segment of the grid will be called a *grid segment*, and consequently, by the number n of segments of a grid we shall mean the number of grid segments.

In order to discuss cooperative mobile guards in grids, we first have to modify the definition of cooperation. More precisely, a set S of mobile guards is *cooperative* if the subgraph $G[S]$ in the intersection graph G of the grid induced by S is connected. Moreover, a set S of mobile guards is *weakly cooperative* if $G[S]$ has no isolated vertices. Then the cooperative mobile guards problem also has its counterparts in the theory of domination. Recall that a *connected dominating set* is a dominating set which induces a connected subgraph of G. The minimum cardinality of a connected dominating set is called the *connected domination number*, and denoted by $\gamma_c(G)$. A dominating set is a *total dominating set* if the subgraph induced by the set has no isolated vertices. The minimum cardinality of a total dominating set, denoted by $\gamma_t(G)$, is called the *total domination number*. Following [51], it is easy to see that the following properties hold.

REMARK 6.23 ([54]). *A minimum cooperative mobile guard set of an n-segment grid has $\gamma_c(G)$ guards, where G is the intersection graph of the grid.*

REMARK 6.24 ([54]). *A minimum weakly cooperative mobile guard set of an n-segment grid has $\gamma_t(G)$ guards, where G is the intersection graph of the grid.*

In the next section, we establish that the problem of determining γ_c for bipartite planar graphs is NP-hard [90], which makes the problem of determining the minimum number of cooperative mobile guards intractable. Next, in Section 6.3.2, we show that the problem of determining γ_t for subcubic bipartite planar graphs is NP-hard as well, thus getting the NP-hardness of the problem of determining the minimum number of weakly cooperative mobile guards (MinWCMG problem for short). Next, in Section 6.3.3, we discuss a restricted class of grids, so-called *polygon-bounded* grids, for which we propose a quadratic time algorithm for solving the MinWCMG problem. The algorithm is based upon the property that horizontal and vertical grid segments may be covered independently, whereas the constructed guard set satisfies the condition of weak cooperation. We then explore *horizontally and vertically unobstructed grids* for which we propose an $O(n \log n)$ time algorithm for the MinWCMG problem. Finally, we investigate *complete rectangular grids with obstacles.* We show that as long as both dimensions of a grid are larger than k, $k+2$ weakly cooperative mobile guards are always sufficient to cover the grid with k obstacles.

6.3.1. Cooperative mobile guards. Let us define the *connected dominating set problem* (CDS problem for short) as follows:

Instance: A bipartite planar graph $G = (V, E)$ and a positive integer d.

Question: Is there a connected dominating set in G of cardinality at most d?

In 1985, White *et al.* [90] proved the following theorem.

THEOREM 6.25 ([90]). *The CDS problem in bipartite planar graphs is NP-complete.*

Proof. Let $G = (V, E)$ be a planar graph, with $E \neq \emptyset$. Let $\widetilde{G}$ be a fixed planar embedding of G, and let F be the set of faces of $\widetilde{G}$. Define the graph $H = (V', E')$, where $V' = V \cup E \cup (F \times \{1, 2\})$, and $\{x, y\} \in E'$ if one of the following conditions holds:

(i) $x \in V$, $y \in E$, and x is incident with y.
(ii) $x \in V$, $y = (f, 1)$ for some $f \in F$, and x is incident with f.
(iii) $x = (f, 1)$ and $y = (f, 2)$ for some $f \in F$.

In other words, H is obtained from $\widetilde{G}$ by inserting an edge in each face, joining one end of each such edge to every vertex on the face, and then subdividing each original edge of $\widetilde{G}$. Thus H is a bipartite planar graph with the bipartition

$$(V \cup (F \times \{2\}), E \cup (F \times \{1\})).$$

Now, let $R = E \cup (F \times \{2\})$. Then $R \geq 2$, and R is independent. It is straightforward to verify that a set D is a connected domination set in H if and only if $D \backslash R$ is a connected dominating set in H. Moreover, if $S \subseteq V \cup (F \times \{1\})$, then the following conditions are equivalent:

(i) $S = S' \cup (F \times \{1\})$, where S' is a vertex cover of G.
(ii) S is a connected dominating set in H.
(iii) $S \cup R$ is a connected, i.e., $S \cup R$ is the vertex set of a Steiner tree of R in H.

Since a non-deterministic polynomial algorithm for the CDS problem just guesses the subset S of vertices and checks whether the subgraph induced by set S is connected, and the vertex cover problem is NP-complete for planar graphs [41], the assertion follows. ∎

Thus by the theorem above, Remark 6.23, and the fact that any bipartite planar graph is the intersection graph of a grid [21], we get the following theorem.

THEOREM 6.26 ([54]). *The problem of finding the minimum number of cooperative mobile guards covering a grid is NP-hard.*

6.3.2. Weakly cooperative mobile guards. In this section, we will prove that the problem of finding the minimum number of weakly cooperative mobile guards covering a grid is NP-hard. The idea of the proof is based upon Remark 6.24 and proceeds by reduction from the 3DM problem to the problem of determining the total domination number in subcubic bipartite planar graphs. Let us define the *total dominating set problem* (TDS problem for short) as follows:

Instance: A bipartite planar graph G and an integer d.

Question: Is there a total dominating set in G of cardinality at most d?

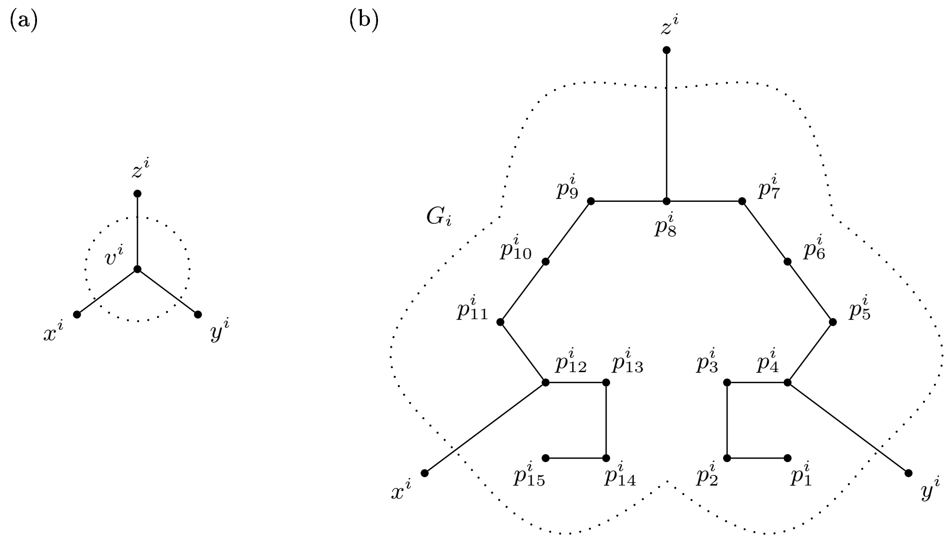

Fig. 6.6. The vertex v^i is replaced with the graph G_i.

Now, using the result of Theorem 6.12, we will show that the TDS problem on subcubic bipartite planar graphs is NP-complete (cf. the proof of Lemma 6.13). Let $G = (V \cup M, E)$ be a subcubic bipartite planar graph, where $V = X \cup Y \cup Z$, $|X| = |Y| = |Z| = q$, every vertex $m \in M$ has degree 3, m is adjacent to exactly one vertex from each of sets X, Y and Z. Let $G^* = (V^*, E^*)$ be the graph obtained from G by replacing each vertex $v^i \in M$, $i = 1, \ldots, |M|$, (and all edges incident to it) with the graph $G_i = (V_i, E_i)$ of Fig. 6.6. Formally:

- $V_i = \{p_j^i\}_{j=1}^{15}$;
- $V^* = V \cup \bigcup_{i=1}^{|M|} V_i$;

- $E^* = E \setminus E^- \cup E^+$, where
$$E^- = \bigcup_{i=1}^{|M|} \{\{x^i, v^i\}, \{y^i, v^i\}, \{z^i, v^i\}\},$$
$$E^+ = \bigcup_{i=1}^{|M|} (E_i \cup \{\{x^i, p_{12}^i\}, \{y^i, p_4^i\}, \{z^i, p_8^i\}\}),$$
and x^i, y^i and z^i are the neighbours of the vertex v^i in G.

Clearly, G^* has $|V| + 15|M|$ vertices and $|E| + 14|M|$ edges, and $\Delta(G^*) = 3$.

LEMMA 6.27 ([54]). *There exists a solution of the* 3DM *problem in G if and only if there exists a total domination set of cardinality $q + 8|M|$ in G^*.*

Proof. ($\Rightarrow$) Let M' be a solution to the 3DM problem in G, $|M'| = q$. A total dominating set in G^* consists of the following vertices:

- if the vertex v^i corresponding to the graph G_i is in M', then in G_i with attached vertices x^i, y^i and z^i (see Fig. 6.6) we choose the following vertices:
$$\{p_2^i, p_3^i, p_4^i\} \cup \{p_7^i, p_8^i, p_9^i\} \cup \{p_{12}^i, p_{13}^i, p_{14}^i\};$$
- otherwise, if v^j corresponding to G_j is not in M', then in G_j with attached vertices x^j, y^j and z^j we choose the vertices
$$\{p_2^i, p_3^i\} \cup \{p_6^i, p_7^i\} \cup \{p_{10}^i, p_{11}^i\} \cup \{p_{13}^i, p_{14}^i\}.$$

($\Leftarrow$) First, note that the following properties are consequences of the structure of graph G_i.

(i) Every graph G_i is dominated by at least eight vertices from V_i.

(ii) If at most eight vertices from $p_1^i, \ldots, p_{15}^i$ form a solution that dominates the graph G_i, then none of x_i, y_i, z_i is dominated by this solution.

Now, suppose that $D \subseteq V^*$ dominates V^*, i.e., $G^*[D]$ has no isolated vertices, and $|D| \leq 8|M| + q$. Denote by p the number of graphs G_i such that more than eight vertices dominate V_i. By (i), we have $|D| \geq 8(|M| - p) + 9p$, hence $p \leq q$.

Hence, by (ii), there are $p \leq q$ vertices which dominate all vertices from $X \cup Y \cup Z$. Because every vertex from X needs at least one adjacent vertex from any graph G_i that is in the domination set D, and no two different vertices from X have a common vertex from any G_i, exactly $p = q$ vertices from the graphs G_i dominate set X. Analogously, the same set of q vertices must dominate Y and Z. Thus we have constructed a solution
$$M' = \{v^i : D \cap V_i \text{ dominates three vertices from } X \cup Y \cup Z\}$$
to the 3DM problem in polynomial time. ∎

A non-deterministic polynomial algorithm for the TDS problem just guesses a subset S of vertices and checks whether the subgraph induced by S has no isolated vertices. Therefore, by the above lemma and Theorem 6.12, we have the following theorem.

THEOREM 6.28 ([54]). *The* TDS *problem in subcubic bipartite planar graphs is NP-complete.*

And, as a consequence, we get

THEOREM 6.29 ([54]). *The problem of determining the total domination number* γ_t *in subcubic bipartite planar graphs is NP-hard.*

Thus by Remark 6.24 and the theorem above, we have

THEOREM 6.30 ([54]). *The problem of finding the minimum number of weakly cooperative mobile guards is NP-hard even for grids in which any segment crosses at most three other segments.*

6.3.3. Polynomially solvable cases of the MinWCMG **problem.** Recall that the minimum connected dominating set problem and the minimum total dominating set problem can be solved in linear time for trees [55, 81], more precisely:

THEOREM 6.31 ([81]). *If* T *is a tree with* $n > 3$ *vertices, then* $\gamma_c(T) = n - l(T)$*, where* $l(T)$ *denotes the number of leaves in* T.

THEOREM 6.32 ([55]). *If* T *is a tree, then* $\gamma_t(T)$ *can be determined in linear time.*

Thus we have the following corollary.

COROLLARY 6.33 ([54]). *The* MinCMG *problem and the* MinWCMG *problem for grids with trees as intersection graphs can be solved in linear time.*

In this section, we will show that there exist a wider class of grids for which the optimum placement of weakly cooperative mobile guards (and consequently, the total domination number of the intersection graph) can also be computed in polynomial time. Before we characterize some of these classes, let us establish an elementary property of a total dominating set in a bipartite graph.

LEMMA 6.34 ([54]). *Let* $G = (V, E)$ *be a bipartite graph with vertex partitions* V_1 *and* V_2. *The problem of determining a minimum total dominating set* $TD(G) \subset V$ *is equivalent to finding a solution to the following two independent problems:*

1. *Find a minimum vertex set* $TD_1(G) \subset V_1$ *which dominates* V_2.
2. *Find a minimum vertex set* $TD_2(G) \subset V_2$ *which dominates* V_1.

In particular, for any pair of minimum sets $TD_1(G)$ *dominating* V_2 *and* $TD_2(G)$ *dominating* V_1*, the set* $TD_1(G) \cup TD_2(G)$ *is a minimum total dominating set for* G. *On the other hand, if* $TD(G)$ *is a minimum total dominating set for* G*, then* $TD(G) \cap V_1$ *is a minimum set dominating* V_2*, and* $TD(G) \cap V_2$ *is a minimum set dominating* V_1.

Proof. Consider any vertex set $T_1 \subseteq V_1$ whose set of neighbours is V_2, and any vertex set $T_2 \subseteq V_2$ whose set of neighbours is V_1. The set $T_1 \cup T_2$ is obviously a total dominating set for G.

Now, take any total dominating set $T \subseteq V$. We will show that the set of neighbours of $T \cap V_1$ is V_2. Conversely, assume that there exists $v \in V_2$ with no neighbours in $T \cap V_1$. It is easy to observe that v must belong to T and since the dominating set T is total, there must exist a vertex in $T \cap V_1$ which is a neighbour of v, a contradiction. Similarly, it is possible to show that the set of neighbours of $T \cap V_2$ is V_1.

We have shown a natural one-to-one correspondence between total dominating sets in bipartite graphs and pairs of sets covering the graph's partitions. The nature of this relation is such that it preserves the minimality of the discussed sets, which completes the proof. ■

Clearly, the above lemma has an immediate application to the minimum weakly cooperative guards problem in grids—recall that intersection graphs of grids are bipartite, as shown in [21]. More precisely, to solve the MinWCMG problem, all we need is to find a minimum set of vertical segments covering all horizontal segments and *vice versa*; that is, horizontal and vertical grid segments may be covered independently, whereas the constructed guard set satisfies the condition of weak cooperation, and we shall use this observation to construct exact algorithms in polygon-bounded grids, simple grids and vertically (horizontally) unobstructed grids.

COROLLARY 6.35 ([54]). *To solve the* MinWCMG *problem in a grid P, all we need is to find a minimum set of vertical segments covering all horizontal segments of P and a minimum set of horizontal segments covering all vertical segments of P.*

Polygon-bounded grids. A *complete rectangular grid* is a grid in which all endpoints of the grid segments are located on the boundary of a rectangle formed by four extremal grid segments (northernmost, westernmost, southernmost, easternmost). We assume that the set of intersections of segments of a rectangular grid is a subset of the integer point grid $\mathbb{Z}^2$; an example of a complete rectangular grid is shown in Fig. 6.7(a).

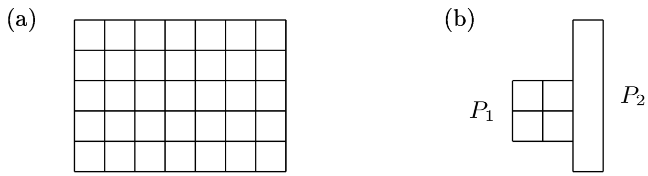

Fig. 6.7. (a) A complete rectangular grid. (b) The grid $P = P_1 \cup P_2$ is not polygon-bounded as P_2 is not complete rectangular.

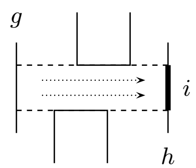

Fig. 6.8. The orthogonal projection i of g onto h with respect to the orthogonal hull.

The class of *polygon-bounded* grids is constructed as follows:

(1) A grid consisting of a single segment is polygon-bounded.
(2) A complete rectangular grid is polygon-bounded.
(3) Any other grid P is polygon-bounded if it has induced polygon-bounded subgrids P_1 and P_2 such that $P = P_1 \cup P_2$ and $P_1 \cap P_2$ is a segment or a point.

Examples of polygon-bounded grids are shows in Figs. 6.9–6.10; the grid in Fig. 6.7(b) is not polygon-bounded as P_2 is not a complete rectangular grid. Thus a polygon-bounded grid can be thought of as a grid which consists of all segments cut off by an orthogonal polygon without holes from grid paper. Note that the class of polygon-bounded grids is a subclass of *simple grids* introduced by Gewali and Ntafos [34]; we shall discuss this later.

The algorithm. Let H and S be arbitrary sets of segments. We say that H *covers* S if for any segment $p \in S$ there exists a segment $h \in H$ such that $p \cap h \neq \emptyset$. We will now construct an efficient algorithm for solving the MinWCMG problem in polygon-bounded grids. Let a polygon-bounded grid P consist of a set S_H of horizontal grid segments and a set S_V of vertical grid segments. The algorithm finds a minimum set of guards in P by determining the minimum set of horizontal grid segments of P covering S_V and the minimum set of vertical grid segments of P covering S_H (Corollary 6.35). We assume that the input is in the form of an n-vertex sequence describing the orthogonal hull of a polygon-bounded grid—this is important when speaking about complexity, as for example a complete rectangular grid can be described by four points, but can have arbitrarily many (an exponential number of) grid segments.

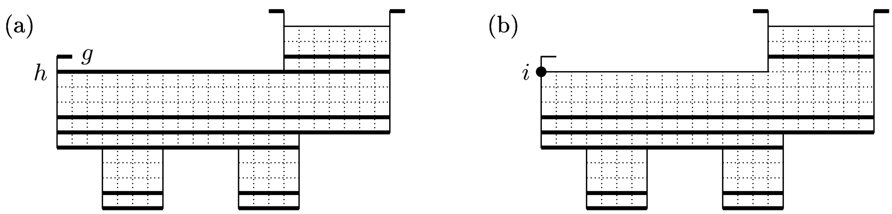

Fig. 6.9. (a) The initial set S in Step 1 of the algorithm (bold lines). (b) The segments g and h are replaced with the segment (point) i.

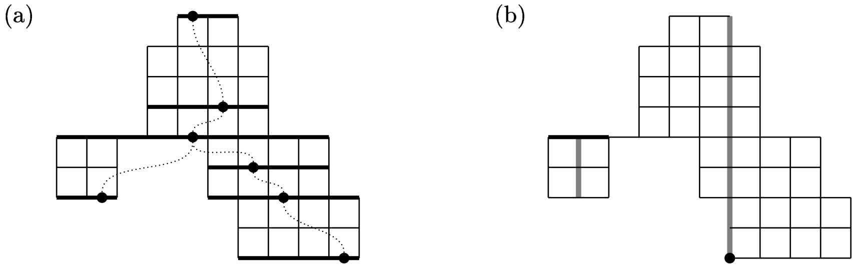

Fig. 6.10. (a) An illustration of the algorithm: a polygon-bounded grid P, the corresponding set S (bold line) and the neighbourhood graph $G(S)$ (edges drawn with a dotted line). (b) The resulting minimum set of vertical grid segments (gray lines) covering all horizontal line segments after Step 3.

THEOREM 6.36 ([54]). *There exists a quadratic time algorithm solving the* MinWCMG *problem in polygon-bounded grids.*

Proof. To simplify the considerations, we confine ourselves to the description of the algorithm for determining a minimum cover of horizontal grid segments with vertical grid segments (Corollary 6.35). Let P be a polygon-bounded grid. Segments g and h belonging to a set of segments S are regarded as *neighbouring in* S (with respect to P) if they are parallel, both intersect a grid segment s, and there is no segment in S that intersects s between g and h. Now, let S_H be a set of horizontal segments, and suppose that S_H contains a segment g with exactly one neighbouring segment h in S_H with respect to P. Define the segment set $S'_H = (S_H \setminus \{g, h\}) \cup \{i\}$, where i is the orthogonal projection of g onto h with respect to the orthogonal hull of P (see Fig. 6.8; note that i is connected by the definition of a polygon-bounded grid). Then S'_H has the following property.

REMARK 6.37. *Every minimum subset M of S_V covering S'_H also covers set S_H.*

Indeed, since all segments of S'_H are covered by $M \subseteq S_V$, some segment s of M intersects i, and thus it intersects h as well. Next, it is easy to see that s also intersects g by the definition of a polygon-bounded grid. Hence s covers both g and h. The minimality of M as a cover of S_H is a direct conclusion of M being a cover of S'_H, which is geometrically contained in S_H.

The idea of the algorithm is to construct a set S of horizontal segments whose minimum cover M with grid segments from S_V can be easily determined. By Remark 6.37, M will cover S_H as well.

1. Given a polygon-bounded grid P in the form of its orthogonal hull, create S as shown in Fig. 6.9, by dissecting P into the minimum possible number of complete rectangular grids whose pairwise intersections are horizontal segments and selecting at most two segments from all such rectangular grids as elements of S.
2. Take any segment $g \in S$ with only one neighbour $h \in S$, and replace S with S' by applying the construction used when discussing Remark 6.37. Repeat Step 2 as long as S contains a segment with only one neighbour in S.
3. Construct the solution M by selecting one intersecting grid segment of S_V for every segment of S.

It remains to show that M is a minimum set of segments covering S. Consider the graph $G(S)$ whose vertex set is S, in which two vertices are neighbours if and only if the corresponding segments of S are neighbours (with respect to P). At the end of Step 1 the graph $G(S)$ is a tree (see Fig. 6.10(a)). Throughout Step 2, the modifications of S result in the iterated removal of leaves and edges from $G(S)$, thus $G(S)$ remains a forest until the end of the algorithm. The algorithm proceeds to Step 3 when $G(S)$ has no leaves left, or equivalently, when $G(S)$ is a graph with no edges. Thus during Step 3 no two segments of S are neighbouring (with respect to P) (see Fig. 6.10(b)). It transpires that a separate vertical guard is required to cover every segment in S.

The operating time of Step 1 of the algorithm is linear with respect to the number of sides n of the orthogonal hull of the grid P, since the hull of P can always be decomposed into no more rectangles (intersecting only at horizontal segments) than there are horizontal sides in P. The set S and the neighbouring relation between segments can be represented in the form of the previously defined forest $G(S)$, whose size and order are bounded by $2n$. Step 2 consists of $O(n)$ iterations, each of which requires at most $O(n)$ time. Step 3 can be done in linear time. ■

Simple grids. A grid is called *simple* if all the endpoints of its segments lie on the outer face of the planar subdivision formed by the grid and if there exists $\varepsilon > 0$ such that each of the grid segments can be extended by ε in both directions provided that its new endpoints still lie on the outer face (see [34] for more details). For example, the grid shown in Fig. 6.11(a) is simple, whereas the grid shown in Fig. 6.11(b) is not. Of course, by the definition, a polygon-bounded grid is a simple grid.

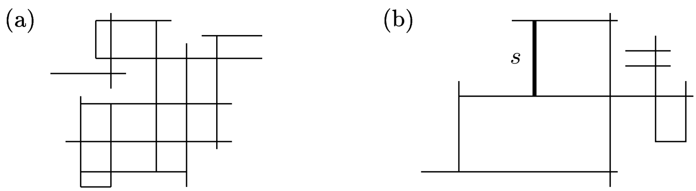

Fig. 6.11. (a) An example of a simple grid. (b) A grid which is not simple, as the segment s cannot be extended without losing the property that all segment endpoints lie on the outer face.

It is easy to see that the algorithm discussed above can be directly applied to the MinWCMG problem in simple grids with the only difference that now the initial set S during Step 1 consists of all horizontal segments (respectively, vertical segments), and the complexity of the algorithm is quadratic in the number of grid segments. Thus we have the following theorem.

THEOREM 6.38 ([54]). *There exists a quadratic time algorithm solving the* MinWCMG *problem for simple grids.*

Horizontally and vertically unobstructed grids. A grid is called *vertically* (*horizontally*) *unobstructed* if it can be constructed by removing some set of horizontal (vertical) segments of the plane from a complete rectangular grid. An example of a vertically unobstructed grid is shown in Fig. 6.12(a). The problem of determining a minimum set of weakly cooperative mobile guards covering a vertically unobstructed grid or a horizontally unobstructed grid can also be solved in polynomial time. In the following, we shall give an $O(n \log n)$ algorithm which solves the problem for vertically unobstructed grids consisting of n segments.

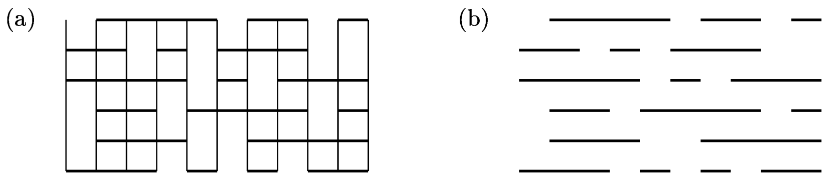

Fig. 6.12. (a) A vertically unobstructed grid. (b) Its set of horizontal segments.

The algorithm. For a given grid $P = S_H \cup S_V$, the algorithm returns a set of grid segments of P representing the positions of guards in some solution of MinWCMG, expressed as the union of the following sets:

1. A minimum set of horizontal segments $T \subseteq S_H$ such that the following inclusion of segments of the real line holds:
$$\bigcup_{s \in S_H} [\lceil l(s) \rceil, \lfloor r(s) \rfloor] \subseteq \bigcup_{t \in T} [\lceil l(t) \rceil - 1/2, \lfloor r(t) \rfloor + 1/2]$$
($l(h)$ and $r(h)$ denote the horizontal coordinates of the left-hand and right-hand endpoints of the segment h, respectively).
2. A set of vertical segments $W \subseteq S_V$ such that W covers all segments of S_H and the set of segments obtained from W by replacing any vertical segment of W with its nearest neighbour to the right, or by removing a segment from W, does not cover S_H.

By Corollary 6.35, both stages of the algorithm may be analysed separately. Stage 1 is equivalent to the solution of the problem of covering a sequence of points representing consecutive integers with a minimum subset of a given set of segments of the real line (see Fig. 6.12(b)), and can be solved with a simple $O(n \log n)$ plane sweep algorithm. Stage 2 describes an $O(n \log n)$ greedy left-to-right sweep approach to the problem of covering S_H with a minimum number of segments from S_V. The correctness of this approach is intuitively obvious and can be proven by induction on the number of vertical segments of the grid.

6.3.4. Grids with obstacles. Consider a complete rectangular grid P. If we put an *obstacle* b on a grid segment s of P, then b blocks the visibility on s, that is, the segment s is divided into two grid segments (we assume that an obstacle is never placed at a crossing). Consequently, a *grid P with k obstacles* is a complete rectangular grid in which we put k obstacles. Note that if the obstacles are put only on horizontal (or vertical) grid segments, then the resulting grid is vertically unobstructed (or resp., horizontally unobstructed). Let $\mathrm{wcmg}(n,k)$ denote the maximum number of weakly cooperative mobile guards that are ever needed for an n-segment grid with k obstacles. Fig. 6.13 shows a

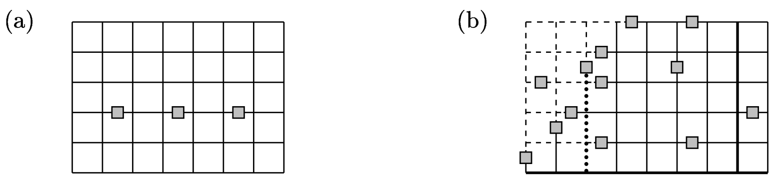

Fig. 6.13. (a) A grid with k obstacles may require as many as $k+2$ weakly cooperative mobile guards. (b) $\mathrm{wcmg}(n,k) \le k+2$.

class of grids with k obstacles that requires as many as $k+2$ weakly cooperative mobile guards. Note that the exemplary grid requires $k+2$ arbitrary and cooperative mobile guards as well. Thus we have $\mathrm{wcmg}(n,k) \ge k+2$.

Proposition 6.39 ([54]). *As long as both the dimensions of the grid are larger than k,*

$$\mathrm{wcmg}(n,k) \leq k+2.$$

Proof. Since the dimensions of the grid are sufficiently large, it is possible to find a vertical grid segment and a horizontal grid segment in P which span from one of the sides of the rectangle bounding P to the opposite side (these segments are marked with bold lines in Fig. 6.13(b), and will be referred to as the *backbone* of the grid). We place guards in both grid segments of the backbone of the grid, leaving k guards still to be placed. The number of uncovered segments in P is at this point at most k. Consider any connected partition D of the disjoint subgrid U of P consisting of all uncovered segments (one such partition is marked with dashed lines in Fig. 6.13(b)). The grid D must be connected to the backbone of P by some segment d (denoted by a dotted line in Fig. 6.13(b)). By placing guards in d and in all segments of D but one (leaving out a segment corresponding to one of the leaves in the spanning tree of the intersection graph of $D \cup \{d\}$), we obtain a cover of D with $|D|$ guards, and the set of guards is always connected to the backbone of P. By repeating this procedure for all connected partitions of U, we finally obtain a cooperative guard cover of P using two guards along the backbone and $|U|$ guards to cover U, and $|U|+2 \leq k+2$. ■

7. FINAL REMARKS

We are by no means able to mention all open art gallery problems ([73], [85], [88]), and thus the cooperative version of them. A part of them was mentioned in the previous chapters, now we shall present some others.

1. *Restricted visibility.* Let P be a polygon. A guard of a range of vision α (or α-*guard* for short) is a point $g \in P$ and an angular domain H_g of angle α with apex at g. An α-guard (g, H_g) sees a point x if the line segment $gx \subset P \cap H_g$. Denote by $\mathrm{cg}(n, \alpha)$ the maximal number of α-guards required to cover a polygon with n vertices for fixed $\alpha \in [0°, 360°)$.
   Problem: Determine $\mathrm{cg}(n, \alpha)$.

2. *Exterior visibility—other obstacles.* In general, throughout this dissertation we have concentrated on polygons, but the cooperative guards problem for the exterior visibility may also be posed for other obstacles:
   2.1. *Line segment obstacles.* Let L be a set of n non-intersecting line segments in the plane. Visibility is defined as follows: a guard at a point x sees a point y if the line segment xy does not cross the interior of any line segment obstacle; xy may be collinear with a segment or touch one if its endpoints. Provide necessity and sufficiency bounds for a complete cooperative coverage of all points of the plane.
   2.2. *Convex polygon obstacles.* Let C be a set of n disjoint convex polygons. Similarly to the case above, a guard at x sees y if the line segment xy does not cross the interior of any polygon. Provide necessity and sufficiency bounds for a complete cooperative coverage of all points of the plane.

3. *Protecting convex sets.* Another variation of the original art gallery problem was proposed by Czyzowicz *et al.* [22]. We say that a set S is *protected* by a guard g if at least one point on the boundary of S is visible from g. A guard g sees a point p if the line segment gp does not cross the interior of any other set. The idea behind this concept is that as long as we can see a part of an object, we know it has not been stolen. In [22] the authors established that $\lfloor 2(n-2)/3 \rfloor$ guards are always sufficient and occasionally necessary to protect any family of n disjoint convex sets, $n > 2$.
   Problem: Provide necessity and sufficiency bounds for complete cooperative protection of a family of n disjoint convex sets.

7.1. Third dimension

Little is known about guarding of polyhedra in 3D, even for arbitrary guards. The reason is that the main tool used for two-dimensional problems—triangulation—does not generalize: Lennes [57] proved that there exist polyhedra whose interior cannot be partitioned into tetrahedra whose vertices are selected from the polyhedra vertices. The smallest example of an untetrahedralizable polyhedron, due to Schönhardt, is shown in Fig. 7.1. Moreover, given a polyhedron P, the problem of determining whether P is tetrahedralizable is NP-complete [79].

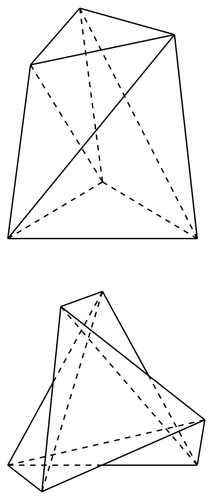

Fig. 7.1. Schönhardt's untetrahedralizable polyhedron.

Another surprising fact: one can expect that placing guards at every vertex of a polyhedron will cover the entire interior. But this would only be true if every polyhedron were tetrahedralizable. In the absence of tetrahedralization, however, this method does not provide a complete coverage. In fact, R. Seidel constructed a polyhedron with two properties:

- Guards placed at every vertex do not cover the interior.
- $\Omega(n^{3/2})$ guards are necessary, where n is the number of vertices.

Moreover, the polyhedron that realized these properties was orthogonal and of genus zero (the complete construction can be found for example in [73]). Thus $\Omega(n^{3/2})$ is the lower bound. Up to now, the best upper bound for arbitrary guards has been $O(n^2)$, and it is provided by the following theorem.

Theorem 7.1 ([15]). *If Steiner points (that is, vertices that are not vertices of the original polyhedron) are allowed in the decomposition, then a polyhedron with n vertices can always be partitioned into $O(n^2)$ convex pieces, and this bound is tight in the worst case.*

In the orthogonal case, the $\Theta(n^{3/2})$ bound is tight, as Paterson and Yao [75] established that an n-vertex orthogonal polyhedron can be decomposed into $O(n^{3/2})$ convex pieces.

To the best of our knowledge, apart from the results discussed in Chapter 6, nothing is known for the cooperative guards problem in the case of more than two dimensions. Of course, all lower bounds for arbitrary guards become bounds for cooperative guards, and it would be desirable to investigate the cooperative guards problem in the 3D case.

7.2. Précis

We conclude with Tables 7.1–7.3 of the major art gallery theorems for cooperative guards discussed in this dissertation. The bounds in the case of the fortress problem are not given (see Chapter 5).

Table 7.1. The cooperative guards problem.

<table>
<tr><th>Polygon shape</th><th>Holes</th><th>Guard type</th><th>Lower bound</th><th>Upper bound</th><th>Section</th></tr>
<tr><td>arbitrary</td><td>0</td><td>v/p</td><td colspan="2">$\lfloor n/2 \rfloor - 1$</td><td>2.1</td></tr>
<tr><td></td><td>1</td><td></td><td colspan="2">$\lfloor (n-1)/2 \rfloor$</td><td>2.4.1</td></tr>
<tr><td></td><td>≥ 2</td><td>point</td><td>$\lfloor n/2 \rfloor - h$</td><td>$\lfloor (n+h-2)/2 \rfloor$</td><td>2.4.2</td></tr>
<tr><td></td><td></td><td>vertex</td><td>$\lfloor n/2 \rfloor$</td><td>$\lfloor (n+2h-2)/2 \rfloor$</td><td>2.4.2</td></tr>
<tr><td>orthogonal</td><td>0</td><td>v/p</td><td colspan="2">$\lfloor n/2 \rfloor - 2$</td><td>2.2.1</td></tr>
<tr><td>orthogonal</td><td>1</td><td></td><td colspan="2">$\lfloor (n - (n \bmod 4))/2 \rfloor - 1$</td><td>2.4.3</td></tr>
<tr><td></td><td>≥ 2</td><td>point</td><td>$\lfloor n/2 \rfloor - h + \lfloor (h-1)/4 \rfloor$</td><td>$\lfloor n/2 \rfloor - 1$</td><td>2.4.3</td></tr>
<tr><td></td><td></td><td>vertex</td><td>$\lfloor n/2 \rfloor - h + \lfloor (h-1)/4 \rfloor$</td><td>$\lfloor (n+2h)/2 \rfloor - 2$</td><td>2.4.3</td></tr>
<tr><td>monotone</td><td>0</td><td>v/p</td><td colspan="2">$\lfloor n/2 \rfloor - 1$</td><td>2.2.2</td></tr>
<tr><td>spiral</td><td></td><td></td><td colspan="2">$\lfloor n/2 \rfloor - 1$</td><td>2.2.3</td></tr>
<tr><td>star</td><td></td><td>vertex</td><td colspan="2">$\lfloor n/2 \rfloor - 1$</td><td>2.2.4</td></tr>
</table>

Table 7.2. The k-guarded guards problem.

<table>
<tr><th>Polygon shape</th><th>k</th><th>Guard type</th><th>Lower bound</th><th>Upper bound</th><th>Section</th></tr>
<tr><td>arbitrary</td><td>1</td><td>v/p</td><td colspan="2">$\lfloor (3n-1)/7 \rfloor$</td><td>3.1</td></tr>
<tr><td>orthogonal</td><td></td><td></td><td colspan="2">$\lfloor n/3 \rfloor$</td><td>3.2</td></tr>
<tr><td>monotone</td><td></td><td>point</td><td colspan="2">$\lfloor 2n/5 \rfloor$</td><td>3.3</td></tr>
<tr><td></td><td></td><td>vertex</td><td colspan="2">$\lfloor (3n-1)/7 \rfloor$</td><td>3.3</td></tr>
<tr><td>spiral</td><td></td><td>v/p</td><td colspan="2">$\lfloor 2n/5 \rfloor$</td><td>3.3.3</td></tr>
<tr><td>star</td><td></td><td>vertex</td><td colspan="2">$\lfloor (3n-1)/7 \rfloor$</td><td>3.4</td></tr>
<tr><td>arbitrary</td><td>≥ 2</td><td>v/p</td><td colspan="2">$k\lfloor n/5 \rfloor + \lfloor (n+2)/5 \rfloor$</td><td>4.1</td></tr>
<tr><td>orthogonal</td><td></td><td></td><td colspan="2">$k\lfloor n/6 \rfloor + \lfloor (n+2)/6 \rfloor$</td><td>4.3.1</td></tr>
<tr><td>star</td><td></td><td>vertex</td><td colspan="2">$k\lfloor n/5 \rfloor + \lfloor (n+2)/5 \rfloor$</td><td>4.3.2</td></tr>
<tr><td>monotone</td><td></td><td></td><td colspan="2">$k\lfloor n/5 \rfloor + \lfloor (n+2)/5 \rfloor$</td><td>4.3.3</td></tr>
</table>

Table 7.3. Cooperative guards in grids.

Guard type	Dimension	Complexity	Section
cooperative	2	$O(n+m)$	6
	3	$O(mn^{2.5})$	6.1
weakly cooperative	2	NPH	6.2
mobile cooperative	2	NPH	6.3.1
mobile weakly cooperative	2	NPH	6.3.2
in polygon-bounded grids		$O(n^2)$	6.3.3
in horizontally/vertically unobstructed grids		$O(n \log n)$	6.3.3

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