1. Introduction

In this work we deal with several joint spectra defined for representations of complex solvable finite-dimensional Lie algebras in complex Banach spaces. Our main concern is to study the behavior of some joint spectra with respect to the procedure of passing from two given such representations, $\varrho_1\colon L_1 \to \mathrm{L}(X_1)$ and $\varrho_2\colon L_2 \to \mathrm{L}(X_2)$, to the tensor product representation of the direct sum of the algebras, $\varrho\colon L_1 \times L_2 \to \mathrm{L}(X_1 \mathbin{\widetilde{\otimes}} X_2)$, $\varrho = \varrho_1 \otimes I + I \otimes \varrho_2$, where $X_1 \mathbin{\widetilde{\otimes}} X_2$ is a tensor product of the Banach spaces X_1 and X_2 in the sense of [14], and I denotes the identity operator of both X_1 and X_2. In addition, we describe the spectral contributions of ϱ_1 and ϱ_2 to some joint spectra of the multiplication representation $\widetilde{\varrho}\colon L_1 \times L_2^{\mathrm{op}} \to \mathrm{L}(J)$, $\widetilde{\varrho}(T) = \varrho_1(l_1)T + T\varrho_2(l_2)$, where $J \subseteq \mathrm{L}(X_2, X_1)$ is an operator ideal between the Banach spaces X_1 and X_2 in the sense of [14], and L_2^{op} is the opposite algebra of L_2. In order to accurately present the problems we are concerned with, we review how the theory of tensor products is placed within the general theory of joint spectra. We first recall some of the best known joint spectra in the commutative and non-commutative setting and their relation with tensor products.

Given a commutative complex Banach algebra $\mathcal{A}$ with unit element I, if $a = (a_1, \ldots, a_n) \in \mathcal{A}^n$, $n \geq 1$, then the joint spectrum of a is defined by

$$\sigma_{\mathcal{A}}(a) = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : \text{the elements } a_i - \lambda_i I,\ i = 1, \ldots, n, \text{ generate a proper ideal in } \mathcal{A}\}.$$

Another well known formula giving the same set is

$$\sigma_{\mathcal{A}}(a) = \{(f(a_1), \ldots, f(a_n)) \in \mathbb{C}^n : f \in \mathcal{M}(\mathcal{A})\},$$

where $\mathcal{M}(\mathcal{A})$ is the set of all non-zero multiplicative linear functionals on $\mathcal{A}$.

The joint spectrum $\sigma_{\mathcal{A}}(a)$ is always a non-void compact subset of $\mathbb{C}^n$. Moreover, the joint spectrum is a fundamental concept in the theory of commutative Banach algebras, for it provides an analytic functional calculus for several elements in such an algebra; see [22], [3], [26] and [2]. For a general account of the joint spectrum see [11] and [17].

When $\mathcal{A}$ is a non-commutative unital Banach algebra, say $\mathcal{A} = \mathrm{L}(X)$, where X is a Banach space, one could define the joint spectrum of a commutative n-tuple $a = (a_1, \ldots, a_n)$ in $\mathcal{A}$ as the joint spectrum $\sigma_{\mathcal{B}}(a)$ of a relative to a maximal abelian subalgebra $\mathcal{B}$ containing $a_1, \ldots, a_n$. Unfortunately, the joint spectrum so defined depends very strongly on the choice of $\mathcal{B}$. Indeed, if we consider two maximal abelian subalgebras $\mathcal{B}_1$ and $\mathcal{B}_2$ containing a_i, $i = 1, \ldots n$, unlike the case $n = 1$ it is not generally true that $\sigma_{\mathcal{B}_1}(a) = \sigma_{\mathcal{B}_2}(a)$; see [1].

So far we have considered a Banach algebra convention, i.e., all concepts are related to a Banach algebra $\mathcal{A}$. However, there is another way to introduce joint spectra, the

so-called spatial convention, i.e., the joint spectra are defined for tuples of commuting operators in the algebras $\mathrm{L}(X)$, with X a Banach space, and in the definitions elements of X are involved. For a given Banach algebra $\mathcal{A}$, we put $X = \mathcal{A}$ and interpret the elements of $\mathcal{A}$ as operators of left multiplication, i.e., to $a \in \mathcal{A}$ we associate the map $L_a \in L(A)$, where $L_a(b) = a.b$, $b \in \mathcal{A}$. Thus, a joint spectrum defined for commutative tuples of Banach space operators, $\sigma(\cdot)$, gives rise to a joint spectrum on $\mathcal{A}$, $\sigma(a, \mathcal{A}) = \sigma(L_a)$, where $L_a = (L_{a_1}, \ldots, L_{a_n})$ and $a = (a_1, \ldots, a_n)$ is a commutative tuple in $\mathcal{A}$.

Among the most important joint spectra defined in the spatial convention, we have the Taylor joint spectrum; see [24] and [11]. This joint spectrum is defined for commuting systems of Banach space operators $T = (T_1, \ldots, T_n)$, and it has the advantage that its definition depends on the action of the maps $T_1, \ldots, T_n$. The Taylor joint spectrum, $\sigma_{\mathrm{T}}(T)$, is a compact non-void subset of $\mathbb{C}^n$ and it has several additional important properties, such as an analytic functional calculus and the so-called projection property. When $\mathcal{A}$ is a commutative Banach algebra, if $a = (a_1, \ldots, a_n) \in \mathcal{A}^n$, then $\sigma_{\mathrm{T}}(a, \mathcal{A}) = \sigma_{\mathcal{A}}(a)$; see [24] and [11]. Therefore, the joint spectrum $\sigma_{\mathcal{A}}(a)$ can be thought of as the Taylor joint spectrum $\sigma_{\mathrm{T}}(a, \mathcal{A})$.

There are many other interesting joint spectra defined in the spatial convention, for example, the Słodkowski joint spectra, [23], the Fredholm or essential joint spectra, [15] and [19], and the split and essential split joint spectra, [13]. All these joint spectra are related to the Taylor spectrum and have similar properties.

On the other hand, over the last years some of the joint spectra originally introduced for commuting systems of operators have been extended to the non-commutative case. Indeed, the Taylor, Słodkowski and split joint spectra have been extended to representations of complex solvable finite-dimensional Lie algebras in complex Banach spaces and their main properties have been proved; see [5], [7], [16], [20] and [21].

One of the most deeply studied problems within the theory of joint spectra has been the determination of the spectral contributions that two commuting systems of operators $S = (S_1, \ldots, S_n)$ and $T = (T_1, \ldots, T_m)$, defined in the Banach spaces X_1 and X_2 respectively, make to the joint spectra of the system $(S \otimes I, I \otimes T) = (S_1 \otimes I, \ldots, S_n \otimes I, I \otimes T_1, \ldots, I \otimes T_m)$ defined in $X_1 \widetilde{\otimes}_\alpha X_2$, i.e., the completion of the algebraic tensor product $X_1 \otimes X_2$ with respect to a quasi-uniform crossnorm α, and where the symbol I stands for the identity map both in X_1 and X_2. For example, if X_1 and X_2 are Hilbert spaces and $X_1 \overline{\otimes} X_2$ is the canonical completion of $X_1 \otimes X_2$, then in [10] the Taylor joint spectrum of $(S \otimes I, I \otimes T)$ in $X_1 \overline{\otimes} X_2$ was characterized. Indeed, it was proved that

$$\sigma_{\mathrm{T}}(S \otimes I, I \otimes T) = \sigma_{\mathrm{T}}(S) \times \sigma_{\mathrm{T}}(T);$$

see the related work [9]. In addition, the results of [9] and [10] were extended in [27] and [28] to Banach spaces and quasi-uniform crossnorms.

Furthermore, in an operator ideal $J \subseteq \mathrm{L}(X_2, X_1)$ between the Banach spaces X_1 and X_2, it is possible to consider tuples of left and right multiplication: $L_S = (L_{S_1}, \ldots, L_{S_n})$ and $R_T = (R_{T_1}, \ldots, R_{T_m})$ respectively, induced by commuting systems of operators $S = (S_1, \ldots, S_n)$ and $T = (T_1, \ldots, T_m)$ defined in X_1 and X_2 respectively, where $L_U(A) = UA$ and $R_V(B) = BV$, $U \in \mathrm{L}(X_1)$, $V \in \mathrm{L}(X_2)$ and $A, B \in J$. However, the tuple (L_S, R_T) is closely related to the system $(S \otimes I, I \otimes T')$; see [12], [14]. Indeed,

the completion $H \widetilde{\otimes}_\alpha H'$ of the algebraic tensor product of a Hilbert space H and its dual relative to a uniform crossnorm α can be regarded as an operator ideal in $\mathrm{L}(H)$; see [14]. As regards this identification the operators $S_i \otimes I$ and $I \otimes T'_j$ correspond to the operators L_{S_i} and R_{T_j} respectively, for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. In particular, the joint spectra of (L_S, R_T) are closely related to the corresponding joint spectra of $(S \otimes I, I \otimes T')$. The Taylor joint spectrum and the essential joint spectrum of (L_S, R_T) were studied in [12] and [14] in the Hilbert and Banach space setting respectively.

In addition, an axiomatic tensor product was introduced in [14]. This tensor product is general and rich enough to allow, on the one hand, the description of the Taylor, split, essential Taylor and essential split joint spectra of a system $(S \otimes I, I \otimes T)$ defined in the tensor product of two Banach spaces, and on the other hand, the description of all the above-mentioned joint spectra of tuples of left and right multiplications (L_S, R_T) defined in a class of operator ideals between Banach spaces introduced in [14].

Some of the main results in [9], [10], [12], [14], [27] and [28] were extended to the non-commutative setting. In fact, the main result of [10] was extended in [6] to solvable Lie algebras of operators defined in Hilbert spaces, and in [21] the descriptions from [14] in connection with the Taylor and the split joint spectra of a system $(S \otimes I, I \otimes T)$ and of a tuple of left and right multiplications (L_S, R_T) were extended to the tensor product representation of the direct sum of two solvable Lie algebras, and to the multiplication representation respectively; see [21; 3]. This work aims at extending the central results in [14] and [21; 3] to other joint spectra in the commutative and non-commutative settings.

Indeed, one of our main objectives is to describe, by means of the tensor product introduced in [14], the Słodkowski and split joint spectra of the tensor product representation of the direct sum of two solvable Lie algebras, and of the multiplication representation in an operator ideal between Banach spaces in the sense of [14]; see Sections 5 and 7. These descriptions provide an extension, from the Taylor joint spectrum and the usual split joint spectrum to the Słodkowski and split joint spectra, of two of the main results in [21; 3] for the tensor product introduced in [14]. Moreover, we consider nilpotent systems of operators, in particular commutative, and we describe the Słodkowski and split joint spectra of a system $(S \otimes I, I \otimes T)$, and of a tuple of left and right multiplications (L_S, R_T) in operator ideals between Banach spaces in the sense of [14]; see Sections 5 and 7.

Our second main objective is to describe the essential Słodkowski and essential split joint spectra of the tensor product representation of the direct sum of two solvable Lie algebras and of the multiplication representation in an operator ideal between Banach spaces in the sense of [14]; see Sections 6 and 7. These results are an extension of the description proved in [14], from the essential Taylor and essential split joint spectra to the essential Słodkowski and essential split joint spectra, and from commuting tuples of operators to representations of solvable Lie algebras. Furthermore, we consider nilpotent systems of operators and we describe the essential Słodkowski and essential split joint spectra of the systems mentioned in the last paragraph.

To prove our second main result, we need to introduce the essential Słodkowski and essential split joint spectra of a representation of a complex solvable finite-dimensional

Lie algebra in a complex Banach space, and to prove the main properties of these joint spectra; see Section 3.

In addition, as an application, in Section 8 we describe all the above-mentioned joint spectra of two particular representations of a nilpotent Lie algebra, one in a tensor product of Banach spaces, where the tensor product is that introduced in [14], and the other in an operator ideal between Banach spaces in the sense of [14].

This work is organized as follows. In Section 2 we recall the definitions and main properties of the Słodkowski and split joint spectra; we also include a brief review of Lie algebras. In Section 3 we introduce the essential Słodkowski and essential split joint spectra, and we prove their main properties. In Section 4 we recall the axiomatic tensor product introduced in [14] and we prove some results needed for our main theorems. In Section 5 we describe the Słodkowski and split joint spectra of the tensor product representation of the direct sum of two solvable Lie algebras. In Section 6 we describe the essential Słodkowski and essential split joint spectra of the tensor product representation of the direct sum of two solvable Lie algebras. In Section 7 we describe the Słodkowski, split, essential Słodkowski and essential split joint spectra of the multiplication representation in an operator ideal between Banach spaces in the sense of [14]. In addition, in Sections 5, 6 and 7 we consider nilpotent systems of operators and we obtain descriptions of the corresponding joint spectra. Finally, in Section 8, we apply our main results to some representations of nilpotent Lie algebras.

2. The Taylor, Słodkowski and split joint spectra

In this section we review the definitions and main properties of the Taylor, Słodkowski and split joint spectra of a representation of a Lie algebra in a Banach space; see [24], [23], [13], [14], [16], [7], [5], [20] and [21]. In order to make the exposition reasonably self-contained, we first review some well known facts on Lie algebras used in this work. Since we are interested in solvable Lie algebras acting on complex Banach spaces, we limit our review to this case; for a complete exposition see [8].

A *complex Lie algebra* is a vector space over the complex field $\mathbb{C}$ provided with a bilinear bracket, named the *Lie product*, $[\cdot,\cdot]\colon L\times L\to L$, which satisfies the Lie conditions

$$[x,x]=0, \qquad [[x,y],z]+[[y,z],x]+[[z,x],y]=0$$

for every $x,y,z\in L$. The second of these equations is called the *Jacobi identity*. By L^{op} we denote the *opposite Lie algebra* of L, i.e., the algebra that coincides with L as a vector space and has the bracket $[x,y]^{\mathrm{op}}=-[x,y]=[y,x]$ for $x,y\in L$.

An example of a Lie algebra structure is given by the algebra $\mathrm{L}(X)$ of all bounded linear maps defined in a Banach space X, and the bracket $[\cdot,\cdot]\colon \mathrm{L}(X)\times\mathrm{L}(X)\to\mathrm{L}(X)$, $[S,T]=ST-TS$ for $S,T\in\mathrm{L}(X)$.

Given two Lie algebras L_1 and L_2 with Lie brackets $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$ respectively, a *morphism of Lie algebras* $H\colon L_1\to L_2$ is a linear map such that $H([x,y]_1)=[H(x),H(y)]_2$ for $x,y\in L_1$. In particular, when $L_2=\mathrm{L}(X)$, X a Banach space, we say that $H\colon L_1\to\mathrm{L}(X)$ is a *representation* of L_1.

We say that a subspace I of L is a *subalgebra* when $[I,I] \subseteq I$, and an *ideal* when $[I,L] \subseteq I$, where if M and N are two subsets of L, then $[M,N]$ denotes the set $\{[m,n] : m \in M,\ n \in N\}$. In particular, $L^2 = [L,L] = \{[x,y] : x,y \in L\}$ is an ideal of L. In addition, we say that a linear map $f\colon L \to \mathbb{C}$ is a *character* when $f(L^2) = 0$, i.e., when $f\colon L \to \mathbb{C}$ is a Lie morphism.

For any Lie algebra L we can consider the following two series of ideals: the *derived series*

$$L = L^{(1)} \supseteq L^{(2)} = [L,L] \supseteq L^{(3)} = [L^{(2)}, L^{(2)}] \supseteq \ldots \supseteq L^{(k)} = [L^{(k-1)}, L^{(k-1)}],$$

and the *descending central series*

$$L = L^1 \supseteq L^2 = L^{(2)} = [L,L] \supseteq L^3 = [L,L^2] \supseteq \ldots \supseteq L^k = [L,L^{k-1}] \supseteq \ldots.$$

A Lie algebra L is *solvable* or *nilpotent* if there is some positive integer k such that $L^{(k)} = 0$ or $L^k = 0$ respectively. Obviously all nilpotent Lie algebras are solvable.

One of the most useful properties of a complex solvable finite-dimensional Lie algebra L is the existence of a *Jordan–Hölder sequence*, i.e., a sequence $(L_k)_{0\le k\le n}$ of ideals such that

(i) $L_0 = 0$, $L_n = L$,
(ii) $L_i \subseteq L_{i+1}$ for $i = 0,\ldots,n-1$,
(iii) $\dim L_i = i$, where $n = \dim L$; see [8; 5.3].

Another important property of these algebras is the existence of polarizations. A *polarization* of a character f of L is a subalgebra $P(f)$ of L maximal with respect to the property $f([I,I]) = 0$, where I is a subalgebra of L. In fact, if $(L_k)_{0\le k\le n}$ is a Jordan–Hölder sequence of ideals of L, then $P(f;(L_k)_{0\le k\le n}) = \sum_{i=0}^n N_i(f_i)$ is a polarization of f, where $N_i(f_i) = \{x \in L_i : f([x,L_i]) = 0\}$; see [4; IV.4].

Next we review the definitions of the Taylor, Słodkowski and split joint spectra. From now on L denotes a complex solvable finite-dimensional Lie algebra, X a complex Banach space and $\varrho\colon L \to \mathrm{L}(X)$ a representation of L in X. We consider the *Koszul complex* of the representation ϱ, i.e., $(X \otimes \bigwedge L, d(\varrho))$, where $\bigwedge L$ denotes the exterior algebra of L, and $d_p(\varrho)\colon X \otimes \bigwedge^p L \to X \otimes \bigwedge^{p-1} L$ is the map defined by

$$\begin{aligned} d_p(\varrho)&(x \otimes \langle l_1 \wedge \ldots \wedge l_p\rangle) \\ &= \sum_{k=1}^p (-1)^{k+1} \varrho(l_k)(x) \otimes \langle l_1 \wedge \ldots \wedge \widehat{l_k} \wedge \ldots \wedge l_p\rangle \\ &\quad + \sum_{1\le i<j\le p} (-1)^{i+j} x \otimes \langle [l_i,l_j] \wedge l_1 \wedge \ldots \wedge \widehat{l_i} \wedge \ldots \wedge \widehat{l_j} \wedge \ldots \wedge l_p\rangle, \end{aligned}$$

where $\widehat{\ }$ means deletion. For p such that $p \le 0$ or $p \ge n+1$, where $n = \dim L$, we define $d_p(\varrho) = 0$.

If f is a character of L, then we consider the representation $\varrho - f \equiv \varrho - f \cdot I$ of L in X, where I denotes the identity map of X. Now, if $H_*(X \otimes \bigwedge L, d(\varrho - f))$ denotes the homology of the Koszul complex of the representation $\varrho - f$, then we consider the set

$$\sigma_p(\varrho) = \{f \in L^* : f(L^2) = 0,\ H_p(X \otimes \textstyle\bigwedge L, d(\varrho - f)) \ne 0\}.$$

Now we state the definition of the Taylor and Słodkowski joint spectra; see [5], [7], [16], [20] and [21]. We follow the notation of [21; 2.11].

Definition 1. Let X be a complex Banach space, L a complex solvable finite-dimensional Lie algebra, and $\varrho\colon L \to \mathrm{L}(X)$ a representation of L in X. Then the *Taylor joint spectrum* of ϱ is the set

$$\sigma(\varrho) = \bigcup_{p=0}^{n} \sigma_p(\varrho) = \{f \in L^* : f(L^2) = 0,\ H_*(X \otimes \textstyle\bigwedge L, d(\varrho - f)) \neq 0\}.$$

Moreover, the *kth δ-Słodkowski joint spectrum* of ϱ is the set

$$\sigma_{\delta,k}(\varrho) = \bigcup_{p=0}^{k} \sigma_p(\varrho),$$

and the *kth π-Słodkowski joint spectrum* of ϱ is the set

$$\sigma_{\pi,k}(\varrho) = \bigcup_{p=n-k}^{n} \sigma_p(\varrho) \cup \{f \in L^* : f(L^2) = 0,\ R(d_{n-k}(\varrho - f)) \text{ is not closed}\}$$

for $0 \leq k \leq n = \dim L$.

We observe that $\sigma_{\delta,n}(\varrho) = \sigma_{\pi,n}(\varrho) = \sigma(\varrho)$.

The Taylor and Słodkowski joint spectra are compact non-void subsets of L^*. If $L \subseteq \mathrm{L}(X)$ is a commutative subalgebra of operators and the representation is the inclusion $\iota\colon L \to \mathrm{L}(X)$, $\iota(T) = T$, $T \in L$, then $\sigma(\iota)$, $\sigma_{\delta,k}(\iota)$ and $\sigma_{\pi,k}(\iota)$ are reduced to the usual Taylor and Słodkowski joint spectra respectively in the following sense. If $l = (l_1, \ldots, l_n)$ is a basis of L and σ denotes either the Taylor joint spectrum or one of the Słodkowski joint spectra of ι, then $\{(f(l_1), \ldots, f(l_n)) : f \in \sigma\} = \sigma(l_1, \ldots, l_n)$, i.e., the joint spectrum σ in terms of the basis $l = (l_1, \ldots, l_n)$ coincides with the spectrum of the n-tuple l. In addition, these joint spectra have the so-called projection property. Since this property is one of the most important results that all the joint spectra considered in this work have in common, we give the explicit definition.

Definition 2. Let X be a complex Banach space and σ a function which assigns a compact non-void subset of the characters of L to each representation $\varrho\colon L \to \mathrm{L}(X)$ of a complex solvable finite-dimensional Lie algebra L in X. In addition, let I be an ideal or a subalgebra of L, in the solvable or nilpotent case respectively, and consider the representation $\varrho|I\colon I \to \mathrm{L}(X)$, i.e., the restriction of ϱ to I. Then we say that σ has the *projection property* if for each ideal or subalgebra I, in the solvable or nilpotent case respectively, we have

$$\pi(\sigma(\varrho)) = \sigma(\varrho|I),$$

where $\pi\colon L^* \to I^*$ is the restriction map.

Next we review the definition of the split joint spectra, and we prove their most important properties, the projection property among them.

A finite *complex* (X, d) of Banach spaces and bounded linear operators is a sequence

$$0 \to X_n \xrightarrow{d_n} X_{n-1} \to \ldots \to X_1 \xrightarrow{d_1} X_0 \to 0,$$

where $n \in \mathbb{N}$, each X_p is a Banach space, and the maps $d_p \in \mathrm{L}(X_p, X_{p-1})$ are such that $d_p \circ d_{p-1} = 0$ for $p = 0, \ldots, n$.

For a fixed integer p, $0 \le p \le n$, we say that (X, d) is *split in degree* p if there are continuous linear operators

$$X_{p+1} \overset{h_p}{\leftarrow} X_p \overset{h_{p-1}}{\longleftarrow} X_{p-1}$$

such that $d_{p+1}h_p + h_{p-1}d_p = I_p$, where I_p denotes the identity operator of X_p.

In addition, if L, X and ϱ are as above, then for each p we consider the set

$$\mathrm{sp}_p(\varrho) = \{f \in L^* : f(L^2) = 0,\ (X \otimes \textstyle\bigwedge L, d(\varrho - f)) \text{ is not split in degree } p\}.$$

Now we give the definition of split joint spectra; see [13] and [21].

Definition 3. Let X be a complex Banach space, L a complex solvable finite-dimensional Lie algebra, and $\varrho\colon L \to \mathrm{L}(X)$ a representation of L in X. Then the *split joint spectrum* of ϱ is the set

$$\mathrm{sp}(\varrho) = \bigcup_{p=0}^{n} \mathrm{sp}_p(\varrho).$$

Moreover, the *kth δ-split joint spectrum* of ϱ is the set

$$\mathrm{sp}_{\delta,k}(\varrho) = \bigcup_{p=0}^{k} \mathrm{sp}_p(\varrho),$$

and the *kth π-split joint spectrum* of ϱ is the set

$$\mathrm{sp}_{\pi,k}(\varrho) = \bigcup_{p=n-k}^{n} \mathrm{sp}_p(\varrho),$$

for $0 \le k \le n = \dim L$.

We observe that $\mathrm{sp}_{\delta,n}(\varrho) = \mathrm{sp}_{\pi,n}(\varrho) = \mathrm{sp}(\varrho)$.

It is clear that $\sigma_{\delta,k}(\varrho) \subseteq \mathrm{sp}_{\delta,k}(\varrho)$, $\sigma_{\pi,k}(\varrho) \subseteq \mathrm{sp}_{\pi,k}(\varrho)$, and $\sigma(\varrho) \subseteq \mathrm{sp}(\varrho)$. Moreover, if X is a Hilbert space, the above inclusions are equalities. In addition, when $L \subseteq \mathrm{L}(X)$ is a commutative subalgebra of operators and the representation is the inclusion $\iota\colon L \to \mathrm{L}(X)$, these joint spectra coincide with the ones introduced by J. Eschmeier in [13] for commuting tuples of operators in the same sense explained for the Taylor and Słodkowski joint spectra.

In the following theorem we consider the main properties of split joint spectra; for a complete exposition see [21; 3].

Theorem 1. *Let X be a complex Banach space, L a complex solvable finite-dimensional Lie algebra, and $\varrho\colon L \to \mathrm{L}(X)$ a representation of L in X. Then the sets* $\mathrm{sp}(\varrho)$, $\mathrm{sp}_{\delta,k}(\varrho)$, *and* $\mathrm{sp}_{\pi,k}(\varrho)$ *are compact non-void subsets of L^* that have the projection property.*

Proof. First of all, in [21; 3.1.9] it was proved that $\mathrm{sp}(\varrho)$ is a compact non-void subset of L^* that has the projection property.

Furthermore, by [21; 3.1.5], [21; 3.1.7] and an argument similar to the one in [21; 3.1.9], it is easy to prove that $\mathrm{sp}_{\delta,k}(\varrho)$ and $\mathrm{sp}_{\pi,k}(\varrho)$ are compact non-void subsets of L^* that have the projection property. ∎

3. The Fredholm joint spectra

In order to prove the main results in Sections 6 and 7 we need to study several essential joint spectra. We first consider the essential joint spectra introduced by A. S. Faĭnshteĭn [15] and by V. Müller [19] for commuting tuples of operators and we extend them to representations of solvable Lie algebras in Banach spaces. In addition, we extend the essential split joint spectra introduced by J. Eschmeier [13] to such representations. We begin with the essential Taylor and Słodkowski joint spectra.

Let X, L, and $\varrho\colon L \to \mathrm{L}(X)$ be as in Section 2, and for each p consider the set

$$\sigma_{p,\mathrm{e}}(\varrho) = \{f \in L^* : f(L^2) = 0,\ \dim H_p(X \otimes \textstyle\bigwedge L, d(\varrho - f)) = \infty\}.$$

Definition 4. Let X be a complex Banach space, L a complex solvable finite-dimensional Lie algebra, and $\varrho\colon L \to \mathrm{L}(X)$ a representation of L in X. Then the *Fredholm* or *essential Taylor joint spectrum* of ϱ is the set

$$\sigma_{\mathrm{e}}(\varrho) = \bigcup_{p=0}^{n} \sigma_{p,\mathrm{e}}(\varrho).$$

Moreover, the *kth Fredholm* or *essential δ-Słodkowski joint spectrum* of ϱ is the set

$$\sigma_{\delta,k,\mathrm{e}}(\varrho) = \bigcup_{p=0}^{k} \sigma_{p,\mathrm{e}}(\varrho),$$

and the *kth Fredholm* or *essential π-Słodkowski joint spectrum* of ϱ is the set

$$\sigma_{\pi,k,\mathrm{e}}(\varrho) = \bigcup_{p=n-k}^{n} \sigma_{p,\mathrm{e}}(\varrho) \cup \{f \in L^* : f(L^2) = 0,\ R(d_{n-k}(\varrho)) \text{ is not closed}\}$$

for $0 \le k \le n = \dim L$.

We observe that $\sigma_{\mathrm{e}}(\varrho) = \sigma_{\delta,n,\mathrm{e}}(\varrho) = \sigma_{\pi,n,\mathrm{e}}(\varrho)$.

Now we prove that these sets are really joint spectra. In fact, we first show the properties of the sets $\sigma_{\delta,k,\mathrm{e}}(\varrho)$ and then by a duality argument we obtain the properties of the sets $\sigma_{\pi,k,\mathrm{e}}(\varrho)$. Our proof of the properties of $\sigma_{\delta,k,\mathrm{e}}(\varrho)$ is a direct generalization of the one in [15].

Theorem 2. *Let X be a complex Banach space, L a complex solvable finite-dimensional Lie algebra, and $\varrho\colon L \to \mathrm{L}(X)$ a representation of L in X. Then the sets $\sigma_{\delta,k,\mathrm{e}}(\varrho)$ are compact non-void subsets of L^* that have the projection property. In particular, $\sigma_e(\varrho)$ is a compact non-void subset of L^* that has the projection property.*

Proof. It is clear that $\sigma_{\delta,k,\mathrm{e}}(\varrho) \subseteq \sigma_{\delta,k}(\varrho)$. Moreover, by [25; 2.11], $\sigma_{\delta,k,\mathrm{e}}(\varrho)$ is a closed set. Thus, $\sigma_{\delta,k,\mathrm{e}}(\varrho)$ is a compact subset of L^*.

In order to prove the projection property for ideals of a solvable Lie algebra, by [8; 5.3] it is enough to consider an ideal I of L of codimension 1. Then, if we consider the usual decomposition of the chain complex $(X \otimes \bigwedge L, d(\varrho))$ associated to the ideal I and the short exact sequence defined by this decomposition (see [7], [5] and [20]), an easy calculation shows that

$$\pi(\sigma_{\delta,k,\mathrm{e}}(\varrho)) \subseteq \sigma_{\delta,k,\mathrm{e}}(\varrho|I).$$

To prove the reverse inclusion we may apply A. S. Faĭnshteĭn's argument in [15; 1], i.e., the essential version of [23; 1.6]; see also [19]. However, we have to verify the following fact: if $\widetilde{f} \in \sigma_{\delta,k,\mathrm{e}}(\varrho|I)$, then for each $f \in L^*$ such that $f|I = \widetilde{f}$, f is a character of L, i.e., $f(L^2) = 0$.

Indeed, since $\widetilde{f} \in \sigma_{\delta,k,\mathrm{e}}(\varrho|I) \subseteq \sigma_{\delta,k}(\varrho|I)$, $\widetilde{f}$ is a character of I, i.e., $\widetilde{f}(I^2) = 0$. Moreover, since I is an ideal of L, by the projection property of the joint spectrum $\sigma_{\delta,k}(\varrho)$ (see [5; 4.5], [20; 3.4] and [21; 2.11]), there is $f \in \sigma_{\delta,k,\mathrm{e}}(\varrho)$ such that $f|I = \widetilde{f}$.

Now, since f is a character of L, L is a polarization for f (see [4; IV] or Section 2). Moreover, as I is an ideal of codimension 1 in L and $\widetilde{f}$ is a character of I, if there were $f' \in L^*$ such that $f'|I = \widetilde{f}$ and such that f' was not a character of L, then I would be a polarization of f' (see [4; IV]). However, if we considered $f - f'$, by [8; 5.3] and [4; IV.4.1.1.4] we would have $I = L$, which is impossible according to our assumption. Thus, every extension of $\widetilde{f}$ to L^* is a character of L. So, we have shown the projection property for ideals of a solvable Lie algebra.

Suppose that L is a nilpotent Lie algebra and that I is a subalgebra of L. As in [21; 0.3.7] we consider the sequence $(L^k + I)_{k\in\mathbb{N}}$ of subalgebras of L, where $(L^k)_{k\in\mathbb{N}}$ is the descending central series of L. In particular, we have $L^1 + I = L + I = L$. Moreover, since L is a nilpotent Lie algebra, there is $k_0 \in \mathbb{N}$ such that $L^k = 0$ for all $k \geq k_0$, which implies that $L^k + I = I$ for all $k \geq k_0$. In addition, since $[L, L^k] = L^{k+1}$ for all $k \in \mathbb{N}$, we have $[L^k + I, L^{k+1} + I] \subseteq [L, L^{k+1}] + [L^k, L] + [I, I] \subseteq L^{k+1} + I$, i.e., for each $k \in \mathbb{N}$, $L^{k+1} + I$ is an ideal of $L^k + I$. Thus, in view of the projection property for ideals, we get the projection property for subalgebras of nilpotent Lie algebras. ∎

We have proved the main properties of the joint spectra $\sigma_{\delta,k,\mathrm{e}}(\varrho)$. For $\sigma_{\pi,k,\mathrm{e}}(\varrho)$ we proceed by a duality argument. We begin with the following proposition, which extends a result of Z. Słodkowski (see [23; 2.1]).

Proposition 3. *Let $X \xrightarrow{A} Y \xrightarrow{B} Z$ be a chain complex of Banach spaces and bounded linear operators. Then the following conditions are equivalent*:

(i) $\dim \mathrm{Ker}(B)/R(A) < \infty$ *and* $R(B)$ *is closed*,

(ii) $\dim \mathrm{Ker}(A^*)/R(B^*) < \infty$ *and* $R(A^*)$ *is closed.*

Proof. First of all, if $\dim \mathrm{Ker}(B)/R(A) < \infty$, then $R(A)$ is closed, and then $R(A^*)$ is closed.

Now, if N is a finite-dimensional subspace of Y such that $R(A) \oplus N = \mathrm{Ker}(B)$ and if $i\colon N \to Y$ is the inclusion map, then we consider the chain complex

$$X \oplus N \xrightarrow{A'} Y \xrightarrow{B} Z,$$

where $A' = A \oplus i$, i.e., for $x \in X$ and $n \in N$,

$$A'(x, n) = A(x) + i(n).$$

Since $R(A') = \mathrm{Ker}(B)$ and $R(B)$ is closed, by [23; 2.1] we have

$$R(B^*) = \mathrm{Ker}(A'^*) = \mathrm{Ker}(A^*) \cap \mathrm{Ker}(i^*) = \mathrm{Ker}(A^*) \cap N^\perp \subseteq \mathrm{Ker}(A^*).$$

Now we consider the inclusion map

$$\iota_1\colon \operatorname{Ker}(A^*) \to Y',$$

where Y' denotes the dual Banach space of Y. Since $\iota_1(R(B^*)) \subseteq N^\perp$, we may consider the quotient map

$$\widetilde{\iota}_1\colon \operatorname{Ker}(A^*)/R(B^*) \to Y'/N^\perp.$$

But if M is a closed subspace of Y such that $N \oplus M = Y$, then

$$Y'/N^\perp \cong M^\perp \cong N'.$$

In particular, $\dim Y'/N^\perp < \infty$, and since $\iota_1^{-1}(N^\perp) = R(B^*)$, we see that $\widetilde{\iota}_1$ is an injection, which implies that $\dim \operatorname{Ker}(A^*)/R(B^*) < \infty$.

Conversely, if $\dim \operatorname{Ker}(A^*)/R(B^*) < \infty$, then $R(B^*)$ is closed and then $R(B)$ is closed.

Now, if we identify Y and Z in the canonical way with subspaces of Y'' and Z'' respectively, then

$$R(A^{**}) \cap Y = R(A), \qquad \operatorname{Ker}(B^{**}) \cap Y = N(B).$$

Thus,

$$\dim \operatorname{Ker}(B)/R(A) = \dim \operatorname{Ker}(B^{**}) \cap Y/R(A^{**}) \cap Y.$$

In addition, if we consider the inclusion map

$$\iota_2\colon \operatorname{Ker}(B^{**}) \cap Y \to \operatorname{Ker}(B^{**}),$$

since $\iota_2(R(A^{**}) \cap Y) \subseteq R(A^{**})$ and $\iota_2^{-1}(R(A^{**})) = R(A^{**}) \cap Y$, the quotient map

$$\widetilde{\iota}_2\colon \operatorname{Ker}(B^{**}) \cap Y/R(A^{**}) \cap Y \to \operatorname{Ker}(B^{**})/R(A^{**})$$

is an injection. In particular, $\dim \operatorname{Ker}(B)/R(A) \le \dim N(B^{**})/R(A^{**})$.

Finally, from the first part of the proposition, which has just been proved, we know that $\dim N(B^{**})/R(A^{**}) < \infty$. ∎

When $\varrho\colon L \to \mathrm{L}(X)$ is a representation of the Lie algebra L in the Banach space X, we may consider the *adjoint representation* of ϱ, i.e., $\varrho^*\colon L^{\mathrm{op}} \to L(X')$, $\varrho^*(l) = (\varrho(l))^*$, where X' denotes the dual space of X. Now we relate the joint spectra $\sigma_{\delta,k,\mathrm{e}}(\varrho)$ and $\sigma_{\pi,k,\mathrm{e}}(\varrho)$.

THEOREM 4. *Let X be a complex Banach space, L a complex solvable finite-dimensional Lie algebra, and $\varrho\colon L \to \mathrm{L}(X)$ a representation of L in X. If $\varrho^*\colon L^{\mathrm{op}} \to \mathrm{L}(X')$ is the adjoint representation of ϱ, then there is a character of L, h, depending only on the Lie structure of L, such that, for $0 \le k \le n$.*

(i) $\sigma_{\delta,k,\mathrm{e}}(\varrho) + h = \sigma_{\pi,k,\mathrm{e}}(\varrho^*)$,
(ii) $\sigma_{\pi,k,\mathrm{e}}(\varrho) + h = \sigma_{\delta,k,\mathrm{e}}(\varrho^*)$.

Proof. This is a consequence of Proposition 3, [5; 1] and [21; 2.4.5]. ∎

Now we state the main properties of the joint spectra $\sigma_{\pi,k,\mathrm{e}}(\varrho)$.

THEOREM 5. *Let X be a complex Banach space, L a complex solvable finite-dimensional Lie algebra, and $\varrho\colon L \to \mathrm{L}(X)$ a representation of L in X. Then the sets $\sigma_{\pi,k,\mathrm{e}}(\varrho)$ are compact non-void subsets of L^* that have the projection property.*

Proof. According to Theorems 2 and 4, $\sigma_{\pi,k,\mathrm{e}}(\varrho)$ are compact non-void subsets of L^*, $0 \le k \le n = \dim L$.

Furthermore, if L is a solvable Lie algebra and I an ideal of L, then by [8; 5.3] there is a Jordan–Hölder sequence of ideals of L such that I is one of its terms. Thus we may suppose that $\dim I = n-1$. In addition, if h and h_I are the characters of L and I involved in formulas (i) and (ii) of Theorem 4 that correspond to the Lie algebra L and the ideal I respectively, then by [5; 1] and [8; 5.3], or by [21; 2.4], $h|I = h_I$. In particular,

$$\sigma_{\pi,k,\mathrm{e}}(\varrho|I) + h|I = \sigma_{\delta,k,\mathrm{e}}(\varrho^*|I).$$

Thus, according to Theorem 4 we have

$$\pi(\sigma_{\pi,k,\mathrm{e}}(\varrho)) = \pi(\sigma_{\delta,k,\mathrm{e}}(\varrho^*) - h) = \sigma_{\delta,k,\mathrm{e}}(\varrho^*) - h|I = \sigma_{\pi,k,\mathrm{e}}(\varrho|I).$$

So, we have proved the projection property for ideals of a solvable Lie algebra. To prove it for subalgebras of a nilpotent Lie algebra, it is enough to apply the corresponding proof of Theorem 2. ∎

Now we study the essential split joint spectra. These are extensions to representations of solvable Lie algebras in a Banach space of the corresponding joint spectra introduced by J. Eschmeier in [13] for finite tuples of commuting Banach space operators. In order to show their main properties, we use a characterization proved in [13].

As in Section 2, we now consider a finite complex (X, d) of Banach spaces and bounded linear operators,

$$0 \to X_n \xrightarrow{d_n} X_{n-1} \to \ldots \to X_1 \xrightarrow{d_1} X_0 \to 0.$$

Given a fixed integer p, $0 \le p \le n$, we say that (X, d) is *Fredholm split in degree* p if there are continuous linear operators

$$X_{p+1} \xleftarrow{h_p} X_p \xleftarrow{h_{p-1}} X_{p-1}$$

and a compact operator k_p defined in X_p such that $d_{p+1}h_p + h_{p-1}d_p = I_p - k_p$, where I_p denotes the identity operator of X_p.

Let X, L, and ϱ: $L \to \mathrm{L}(X)$ be as in Section 2, and for each p consider the set

$$\mathrm{sp}_{p,e}(\varrho) = \{f \in L^* : f(L^2) = 0,\ (X \otimes \textstyle\bigwedge L, d(\varrho - f)) \text{ is not Fredholm split in degree } p\}.$$

Now we state the definition of essential split joint spectra; see [13].

Definition 5. Let X be a complex Banach space, L a complex solvable finite-dimensional Lie algebra, and ϱ: $L \to \mathrm{L}(X)$ a representation of L in X. Then the *Fredholm* or *essential split joint spectrum* of ϱ is the set

$$\mathrm{sp}_{\mathrm{e}}(\varrho) = \bigcup_{p=0}^{n} \mathrm{sp}_{p,\mathrm{e}}(\varrho).$$

Moreover, the *kth Fredholm* or *essential δ-split joint spectrum* of ϱ is the set

$$\mathrm{sp}_{\delta,k,\mathrm{e}}(\varrho) = \bigcup_{p=0}^{k} \mathrm{sp}_{p,\mathrm{e}}(\varrho),$$

and the *kth Fredholm* or *essential π-split joint spectrum* of ϱ is the set

$$\mathrm{sp}_{\pi,k,\mathrm{e}}(\varrho) = \bigcup_{p=n-k}^{n} \mathrm{sp}_{p,\mathrm{e}}(\varrho),$$

for $0 \leq k \leq n$.

We observe that $\mathrm{sp}_{\delta,n,\mathrm{e}}(\varrho) = \mathrm{sp}_{\pi,n,\mathrm{e}}(\varrho) = \mathrm{sp}_{e}(\varrho)$.

In order to show the main properties of these joint spectra, we need to prove some technical results. We first review several facts related to complexes of Banach space operators.

Given a finite complex (X, d) of Banach spaces and bounded linear operators and a Banach space Y, we denote the complex

$$0 \to \mathrm{L}(Y, X_n) \xrightarrow{L_{d_n}} \mathrm{L}(Y, X_{n-1}) \to \ldots \to \mathrm{L}(Y, X_1) \xrightarrow{L_{d_1}} \mathrm{L}(Y, X_0) \to 0$$

by $\mathrm{L}(Y, X.)$, where L_{d_p} denotes the induced operator of left multiplication with d_p, i.e., for $T \in \mathrm{L}(Y, X_p)$, $L_{d_p}(T) = d_p \circ T \in \mathrm{L}(Y, X_{p-1})$, $0 \leq p \leq n$; see [13].

In addition, if X_1 and X_2 are two complex Banach spaces, and if $\mathrm{K}(X_1, X_2)$ denotes the ideal of all compact operators in $\mathrm{L}(X_1, X_2)$, then it is clear that $L_{d_p}(\mathrm{K}(Y, X_p)) \subseteq \mathrm{K}(Y, X_{p-1})$. Thus, we may consider the complex $\mathrm{C}(Y, X.) = (\mathrm{C}(Y, X_p), \widetilde{L}_{d_p})$, where $\mathrm{C}(Y, X_p) = \mathrm{L}(Y, X_p)/\mathrm{K}(Y, X_p)$ and $\widetilde{L}_{d_p}$ is the quotient operator associated to L_{d_p}; see [13].

On the other hand, if L, X, and $\varrho\colon L \to \mathrm{L}(X)$ are as in Section 2, then we consider the representation

$$L_\varrho\colon L \to \mathrm{L}(\mathrm{L}(X)), \qquad l \mapsto L_{\varrho(l)},$$

where $L_{\varrho(l)}$ denotes the left multiplication operator associated to $\varrho(l)$, $l \in L$; see [21; 3.1].

In addition, since $L_{\varrho(l)}(\mathrm{K}(X)) \subseteq \mathrm{K}(X)$, it is possible to consider the representation

$$\widetilde{L}_\varrho\colon L \to \mathrm{L}(\mathrm{C}(X)),$$

where $\mathrm{C}(X) = \mathrm{L}(X)/\mathrm{K}(\mathrm{X})$ and $\widetilde{L}_\varrho(l)$ is the quotient operator defined in $\mathrm{C}(X)$ associated to $L_{\varrho(l)}$.

In the following proposition we relate the complexes $(\mathrm{C}(X) \otimes \bigwedge L, d(\widetilde{L}_\varrho))$ and $\mathrm{C}(X, (X \otimes \bigwedge L, d(\varrho)).)$.

Proposition 6. *Let X be a complex Banach space, L a complex solvable finite-dimensional Lie algebra, and $\varrho\colon L \to \mathrm{L}(X)$ a representation of L in X. Then the complexes $(\mathrm{C}(X) \otimes \bigwedge L, d(\widetilde{L}_\varrho))$ and $\mathrm{C}(X, (X \otimes \bigwedge L, d(\varrho)).)$ are naturally isomorphic.*

Proof. First, we consider the complexes $(\mathrm{L}(X) \otimes \bigwedge L, d(L_\varrho))$ and $\mathrm{L}(X, (X \otimes \bigwedge L, d(\varrho)).)$. In [21; 3.1.4] it was proved that these two complexes are naturally isomorphic. Indeed, if $\Phi_p\colon \mathrm{L}(X) \otimes \bigwedge^p L \to \mathrm{L}(X, X \otimes \bigwedge^p L)$ is the map

$$\Phi_p(T \otimes \xi)(x) = T(x) \otimes \xi,$$

$T \in \mathrm{L}(X)$, $\xi \in \bigwedge^p L$ and $x \in X$, then $\Phi : (\mathrm{L}(X) \otimes \bigwedge L, d(L_\varrho)) \to \mathrm{L}(X, (X \otimes \bigwedge L, d(\varrho)).)$ is

an isomorphism of chain complexes. In particular, the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{L}(X)\otimes\bigwedge^p L & \xrightarrow{d_p(L_\varrho)} & \mathrm{L}(X)\otimes\bigwedge^{p-1} L \\ \Big\downarrow{\scriptstyle \Phi_p} & & \Big\downarrow{\scriptstyle \Phi_{p-1}} \\ \mathrm{L}(X, X\otimes\bigwedge^p L) & \xrightarrow{L_{d_p}} & \mathrm{L}(X, X\otimes\bigwedge^{p-1} L) \end{array}$$

Moreover, since Φ_p is an isomorphism, an easy calculation shows that $\Phi_p(\mathrm{K}(X)\otimes\bigwedge^p L) = \mathrm{K}(X, X\otimes\bigwedge^p L)$. Thus, we may consider the associated quotient map $\widetilde{\Phi}_p\colon \mathrm{C}(X)\otimes\bigwedge^p L \to \mathrm{C}(X, X\otimes\bigwedge^p L)$, which is an isomorphism.

In addition, it is clear that $d_p(L_\varrho)(\mathrm{K}(X)\otimes\bigwedge^p L) \subseteq \mathrm{K}(X)\otimes\bigwedge^{p-1} L$. Furthermore, if $\pi_p\colon \mathrm{L}(X)\otimes\bigwedge^p L \to \mathrm{C}(X)\otimes\bigwedge^p L$ denotes the projection map, it is easy to prove that the quotient map associated to $d_p(L_\varrho)$ coincides with $d_p(\widetilde{L}_\varrho)$, i.e., we have the commutative diagram

$$\begin{array}{ccc} \mathrm{L}(X)\otimes\bigwedge^p L & \xrightarrow{d_p(L_\varrho)} & \mathrm{L}(X)\otimes\bigwedge^{p-1} L \\ \Big\downarrow{\scriptstyle \pi_p} & & \Big\downarrow{\scriptstyle \pi_{p-1}} \\ \mathrm{C}(X)\otimes\bigwedge^p L & \xrightarrow{d_p(\widetilde{L}_\varrho)} & \mathrm{C}(X)\otimes\bigwedge^{p-1} L \end{array}$$

In particular, the family $(\pi_p)_{0\le p\le n}\colon (\mathrm{L}(X)\otimes\bigwedge L, d(L_\varrho)) \to (\mathrm{C}(X)\otimes\bigwedge L, d(\widetilde{L}_\varrho))$ is a morphism of chain complexes.

Thus, we obtain the commutative diagram

$$\begin{array}{ccc} \mathrm{C}(X)\otimes\bigwedge^p L & \xrightarrow{d_p(\widetilde{L}_\varrho)} & \mathrm{C}(X)\otimes\bigwedge^{p-1} L \\ \Big\downarrow{\scriptstyle \widetilde{\Phi}_p} & & \Big\downarrow{\scriptstyle \widetilde{\Phi}_{p-1}} \\ \mathrm{C}(X, X\otimes\bigwedge^p L) & \xrightarrow{\widetilde{L}_{d_p}} & \mathrm{C}(X, X\otimes\bigwedge^{p-1} L) \end{array}$$

Finally, since for each p, $0 \le p \le n$, the map $\widetilde{\Phi}_p$ is an isomorphism, the family $\widetilde{\Phi} = (\widetilde{\Phi}_p)_{0\le p\le n}\colon (\mathrm{C}(X)\otimes\bigwedge L, d(\widetilde{L}_\varrho)) \to \mathrm{C}(X, (X\otimes\bigwedge L, d(\varrho)).)$ is an isomorphism of chain complexes. ∎

In order to show that the sets introduced in Definition 5 are really joint spectra, we need to prove an isomorphism similar to the one in Proposition 6, but related to right multiplication instead of left multiplication. We first review some results necessary for our objective.

Let (X, d) be a finite complex of Banach spaces and bounded linear operators and Y a complex Banach space. We denote the complex

$$0 \to \mathrm{L}(X_0, Y) \xrightarrow{R_{d_1}} \mathrm{L}(X_1, Y) \to \ldots \to \mathrm{L}(Y, X_{n-1}) \xrightarrow{R_{d_n}} \mathrm{L}(X_n, Y) \to 0$$

by $\mathrm{L}(X., Y)$, where R_{d_p} denotes the induced operator of right multiplication with d_p, i.e., for $T \in \mathrm{L}(X_{p-1}, Y)$, $R_{d_p}(T) = T \circ d_p \in \mathrm{L}(X_p, Y)$, $0 \le p \le n$; see [13].

It is clear that $R_{d_p}(\mathrm{K}(X_{p-1}, Y)) \subseteq \mathrm{K}(X_p, Y)$. Thus, we may consider the complex $\mathrm{C}(X., Y) = (\mathrm{C}(X_p, Y), \widetilde{R}_{d_p})$, where $\mathrm{C}(X_p, Y) = \mathrm{L}(X_p, Y)/\mathrm{K}(X_p, Y)$ and $\widetilde{R}_{d_p}$ is the quotient operator associated to R_{d_p}; see [13].

On the other hand, if L, X, and $\varrho\colon L \to \mathrm{L}(X)$ are as in Section 2, then we consider the representation

$$R_\varrho\colon L^{\mathrm{op}} \to \mathrm{L}(\mathrm{L}(X)), \qquad l \mapsto R_{\varrho(l)},$$

where $R_{\varrho(l)}$ denotes the right multiplication operator associated to $\varrho(l)$, $l \in L^{\mathrm{op}}$; see [21; 3.1].

Furthermore, since $R_{\varrho(l)}(\mathrm{K}(X)) \subseteq \mathrm{K}(X)$, it is possible to consider the representation

$$\widetilde{R}_\varrho\colon L^{\mathrm{op}} \to \mathrm{L}(\mathrm{C}(X)),$$

where $\widetilde{R}_\varrho(l)$ is the quotient operator associated to $R_{\varrho(l)}$.

Now we consider the Chevalley–Eilenberg cochain complex associated to the representation $\widetilde{R}_\varrho\colon L^{\mathrm{op}} \to \mathrm{L}(\mathrm{C}(X))$, i.e., $\mathrm{ChE}(\widetilde{R}_\varrho) = (\mathrm{Hom}(\bigwedge L, \mathrm{C}(X)), \delta(\widetilde{R}_\varrho))$, where $\delta_p(\widetilde{R}_\varrho)\colon \mathrm{Hom}(\bigwedge^p L, \mathrm{C}(X)) \to \mathrm{Hom}(\bigwedge^{p+1} L, \mathrm{C}(X))$ is the map defined by

$$\begin{aligned}(\delta_p(\widetilde{R}_\varrho)f)(x_1 \dots x_{p+1}) = {} & \sum_{i=1}^{p+1} (-1)^{i-1} \widetilde{R}_\varrho(x_i) f(x_1 \dots \widehat{x_i} \dots x_{p+1}) \\ & + \sum_{1 \le i < k \le p+1} (-1)^{i+k} f([x_i, x_k].x_1 \dots \widehat{x_i} \dots \widehat{x_k} \dots x_{p+1})\end{aligned}$$

for $f \in \mathrm{Hom}(\bigwedge^p L, \mathrm{C}(X))$ and $x_i \in L^{\mathrm{op}}$, $1 \le i \le p+1$; see [21; 2.1.9].

In the next proposition we relate the complexes $\mathrm{ChE}(\widetilde{R}_\varrho)$ and $\mathrm{C}((X \otimes \bigwedge L, d(\varrho))., X)$.

Proposition 7. *The complexes* $\mathrm{ChE}(\widetilde{R}_\varrho)$ *and* $\mathrm{C}((X \otimes \bigwedge L, d(\varrho))., X)$ *are naturally isomorphic.*

Proof. First of all, we consider the representation $R_\varrho\colon L^{\mathrm{op}} \to \mathrm{L}(\mathrm{L}(X))$ and the Chevalley–Eilenberg cochain complex associated to it, i.e., $\mathrm{ChE}(R_\varrho) = (\mathrm{Hom}(\bigwedge L, \mathrm{L}(X)), \delta(R_\varrho))$, where $\delta_p(R_\varrho)\colon \mathrm{Hom}(\bigwedge^p L, \mathrm{L}(X)) \to \mathrm{Hom}(\bigwedge^{p+1} L, \mathrm{L}(X))$ is the map defined by

$$\begin{aligned}(\delta_p(R_\varrho)f)(x_1 \dots x_{p+1}) = {} & \sum_{i=1}^{p+1} (-1)^{i-1} R_\varrho(x_i) f(x_1 \dots \widehat{x_i} \dots x_{p+1}) \\ & + \sum_{1 \le i < k \le p+1} (-1)^{i+k} f([x_i, x_k].x_1 \dots \widehat{x_i} \dots \widehat{x_k} \dots x_{p+1}),\end{aligned}$$

for $f \in \mathrm{Hom}(\bigwedge^p L, \mathrm{L}(X))$ and $x_i \in L^{\mathrm{op}}$, $1 \le i \le p+1$; see [21; 2.1.9].

Now, in [21; 3.1.6] it was proved that $\mathrm{ChE}(R_\varrho)$ and $\mathrm{L}((X \otimes \bigwedge L, d(\varrho))., X)$ are naturally isomorphic. Indeed, if $\Psi_p\colon \mathrm{Hom}(\bigwedge^p L, \mathrm{L}(X)) \to \mathrm{L}(X \otimes \bigwedge^p L, X)$ is the map

$$(\Psi_p(f))(x \otimes \xi) = f(\xi)(x),$$

$f \in \mathrm{Hom}(\bigwedge^p L, \mathrm{L}(X))$, $\xi \in \bigwedge^p L$ and $x \in X$, then $\Psi : \mathrm{ChE}(R_\varrho) \to \mathrm{L}((X \otimes \bigwedge L, d(\varrho))., X)$ is an isomorphism of chain complexes. In particular, the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Hom}(\bigwedge^p L, \mathrm{L}(X)) \arrow[r, "\delta_p(R_\varrho)"] \arrow[d, "\Psi_p"'] & \mathrm{Hom}(\bigwedge^{p+1} L, \mathrm{L}(X)) \arrow[d, "\Psi_{p+1}"] \\
\mathrm{L}(X \otimes \bigwedge^p L, X) \arrow[r, "R_{d_{p+1}}"] & \mathrm{L}(X \otimes \bigwedge^{p+1} L, X)
\end{tikzcd}$$

Since Ψ_p is an isomorphism, an easy calculation shows $\Psi_p(\mathrm{Hom}(\bigwedge^p L, \mathrm{K}(X)) = \mathrm{K}(X \otimes \bigwedge^p L, X)$. Thus, we may consider the associated quotient map $\widetilde{\Psi}_p\colon \mathrm{Hom}(\bigwedge^p L, \mathrm{C}(X)) \to \mathrm{C}(X \otimes \bigwedge^p L, X)$, which is an isomorphism.

In addition, it is clear that $\delta_p(R_\varrho)(\mathrm{Hom}(\bigwedge^p L, \mathrm{K}(X)) \subseteq \mathrm{Hom}(\bigwedge^{p+1} L, \mathrm{K}(X))$. Furthermore, if $\pi_p\colon \mathrm{Hom}(\bigwedge^p L, \mathrm{L}(X)) \to \mathrm{Hom}(\bigwedge^p L, \mathrm{C}(X))$ denotes the projection map, it is easy to prove that the quotient map associated to $\delta_p(R_\varrho)$ coincides with $\delta_p(\widetilde{R}_\varrho)$, i.e., we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(\bigwedge^p L, \mathrm{L}(X)) \arrow[r, "\delta_p(R_\varrho)"] \arrow[d, "\pi_p"'] & \mathrm{Hom}(\bigwedge^{p+1} L, \mathrm{L}(X)) \arrow[d, "\pi_{p+1}"] \\
\mathrm{Hom}(\bigwedge^p L, \mathrm{C}(X)) \arrow[r, "\delta_p(\widetilde{R}_\varrho)"] & \mathrm{Hom}(\bigwedge^{p+1} L, \mathrm{C}(X))
\end{tikzcd}$$

In particular, the family $(\pi_p)_{0\le p\le n}\colon \mathrm{ChE}(R_\varrho) \to \mathrm{ChE}(\widetilde{R}_\varrho)$ is a morphism of chain complexes.

Thus, we obtain the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(\bigwedge^p L, \mathrm{C}(X)) \arrow[r, "\delta_p(\widetilde{R}_\varrho)"] \arrow[d, "\widetilde{\Psi}_p"'] & \mathrm{Hom}(\bigwedge^{p+1} L, \mathrm{C}(X)) \arrow[d, "\widetilde{\Psi}_{p+1}"] \\
\mathrm{C}(X \otimes \bigwedge^p L, X) \arrow[r, "\widetilde{R}_{d_p}"] & \mathrm{C}(X \otimes \bigwedge^{p+1} L, X)
\end{tikzcd}$$

Finally, since for each p, $0 \le p \le n$, the map $\widetilde{\Psi}_p$ is an isomorphism, the family $\widetilde{\Psi} = (\widetilde{\Psi}_p)_{0\le p\le n}\colon \mathrm{ChE}(\widetilde{R}_\varrho) \to \mathrm{C}((X \otimes \bigwedge L, d(\varrho))., X)$ is an isomorphism of chain complexes. ∎

Now we state the main spectral properties of essential split joint spectra.

THEOREM 8. *Let X be a complex Banach space, L a complex solvable finite-dimensional Lie algebra, and $\varrho\colon L \to \mathrm{L}(X)$ a representation of L in X. Then*

(i) $\mathrm{sp}_{\delta,k,\mathrm{e}}(\varrho) = \sigma_{\delta,k}(\widetilde{L}_\varrho)$,
(ii) $\mathrm{sp}_{\pi,k,\mathrm{e}}(\varrho) = \sigma_{\delta,k}(\widetilde{R}_\varrho) + h$,
(iii) $\mathrm{sp}_{\mathrm{e}}(\varrho) = \sigma_{\mathrm{e}}(\widetilde{L}_\varrho) = \sigma_{\mathrm{e}}(\widetilde{R}_\varrho) + h$,

where h is the character of L considered in Theorem 4 and $0 \le k \le n$.

Proof. Since the argument in [13; 2.4(a-iii)] applies in the non-commutative case, according to Proposition 6 we have

$$\mathrm{sp}_{\delta,k,\mathrm{e}}(\varrho) = \sigma_{\delta,k}(\widetilde{L}_\varrho).$$

Next, since the argument in [13; 2.4(b-iii)] applies in the non-commutative case, if h is the character of L considered in Theorem 4 (see [5; 1] and [21; 2.4.5]), then according to Proposition 7 and [21; 2.4.4] we have

$$\mathrm{sp}_{\pi,k,\mathrm{e}}(\varrho) = \sigma_{\delta,k}(\widetilde{R}_\varrho) + h.$$

The third statement is clear. ∎

THEOREM 9. *Let X be a complex Banach space, L a complex solvable finite-dimensional Lie algebra, and $\varrho\colon L \to \mathrm{L}(X)$ a representation of L in X. Then the sets* $\mathrm{sp}_{\mathrm{e}}(\varrho)$,

$\mathrm{sp}_{\delta,k,\mathrm{e}}(\varrho)$, *and* $\mathrm{sp}_{\pi,k,\mathrm{e}}(\varrho)$ *are compact non-void subsets of* L^* *that have the projection property.*

Proof. The main properties of the essential split joint spectra may be deduced from the corresponding ones of the Słodkowski and Taylor joint spectra, and from the particular behavior of the character h with respect to Lie ideals of L; see the proof of Theorem 5. ∎

Finally, in the following proposition we consider two nilpotent Lie algebras and two representations of the algebras in a complex Banach space related by an epimorphism, and we describe the connection between the joint spectra of the representations. We need this result for nilpotent and commutative systems of operators. In addition, these results provide an extension of [21; 2.7.4] and [21; 3.1.10] for representations of nilpotent Lie algebras, from the Taylor to Słodkowski joint spectra and from the usual split spectrum to the split joint spectra. Moreover, we consider the corresponding essential joint spectra and prove similar characterizations.

PROPOSITION 10. *Let* X *be a complex Banach space*, L_1 *and* L_2 *two complex nilpotent finite-dimensional Lie algebras*, $\varrho_1\colon L_1 \to \mathrm{L}(X)$ *a representation of* L_1, *and* $f\colon L_2 \to L_1$ *a Lie algebra epimorphism. Then*, *if we consider the representation* $\varrho_2 = \varrho_1 \circ f\colon L_2 \to \mathrm{L}(X)$, *we have*

(i) $\sigma_{\delta,k}(\varrho_2) = \sigma_{\delta,k}(\varrho_1) \circ f$, $\sigma_{\pi,k}(\varrho_2) = \sigma_{\pi,k}(\varrho_1) \circ f$,
(ii) $\sigma_{\delta,k,\mathrm{e}}(\varrho_2) = \sigma_{\delta,k,\mathrm{e}}(\varrho_1) \circ f$, $\sigma_{\pi,k,\mathrm{e}}(\varrho_2) = \sigma_{\pi,k,\mathrm{e}}(\varrho_1) \circ f$,
(iii) $\mathrm{sp}_{\delta,k}(\varrho_2) = \mathrm{sp}_{\delta,k}(\varrho_1) \circ f$, $\mathrm{sp}_{\pi,k}(\varrho_2) = \mathrm{sp}_{\pi,k}(\varrho_1) \circ f$,
(iv) $\mathrm{sp}_{\delta,k,\mathrm{e}}(\varrho_2) = \mathrm{sp}_{\delta,k,\mathrm{e}}(\varrho_1) \circ f$, $\mathrm{sp}_{\pi,k,\mathrm{e}}(\varrho_2) = \sigma_{\pi,k,\mathrm{e}}(\varrho_1) \circ f$,

where $\sigma_*(\varrho_1) \circ f = \{\alpha \circ f : \alpha \in \sigma_*(\varrho_1)\}$ *and* $\mathrm{sp}_*(\varrho_1) \circ f = \{\alpha \circ f : \alpha \in \mathrm{sp}_*(\varrho_1)\}$.

Proof. A careful inspection of [16; 2.5] and [16; 2.6] shows that it is possible to refine the arguments of these results in order to prove that the Koszul complex of ϱ_1 is exact for $p = 0, \ldots, k$ if and only if the Koszul complex of ϱ_2 is exact for $p = 0, \ldots, k$. In particular, if $\alpha \in \sigma_{\delta,k}(\varrho_1)$, then $\varrho_2 - \alpha \circ f = (\varrho_1 - \alpha) \circ f$, which implies that $\sigma_{\delta,k}(\varrho_1) \circ f \subseteq \sigma_{\delta,k}(\varrho_2)$. On the other hand, since $\sigma_{\delta,k}(\varrho_2) \subseteq \sigma(\varrho_2)$, by [21; 2.7.4], if $\beta \in \sigma_{\delta,k}(\varrho_2)$, then there is $\alpha \in \sigma(\varrho_1)$ such that $\beta = \alpha \circ f$. But by the above observation, since $\varrho_2 - \beta = (\varrho_1 - \alpha) \circ f$, we have $\alpha \in \sigma_{\delta,k}(\varrho_1)$. Thus, $\sigma_{\delta,k}(\varrho_2) = \sigma_{\delta,k}(\varrho_1) \circ f$.

In addition, a careful inspection of [16; 2.5] and [16; 2.6] shows that it is possible to extend the arguments developed in these results to the essential δ-Słodkowski joint spectra, i.e., it is possible to prove that the Koszul complex of ϱ_1 is Fredholm for $p = 0, \ldots, k$ if and only if the Koszul complex of ϱ_2 is Fredholm for $p = 0, \ldots, k$. In particular, we may apply the same argument that we developed for the joint spectra $\sigma_{\delta,k}$ to the joint spectra $\sigma_{\delta,k,\mathrm{e}}$. Thus, $\sigma_{\delta,k,\mathrm{e}}(\varrho_2) = \sigma_{\delta,k,\mathrm{e}}(\varrho_1) \circ f$.

Now if we consider the representations $\varrho_1^*\colon L_1^{\mathrm{op}} \to \mathrm{L}(X')$ and $\varrho_2^*\colon L_1^{\mathrm{op}} \to \mathrm{L}(X')$ defined in Theorem 4, then $\varrho_2^* = \varrho_1^* \circ f$. Moreover, by [5; 7], [21; 2.11.4] and Theorem 4 we have $\sigma_{\pi,k}(\varrho_1) \circ f = \sigma_{\pi,k}(\varrho_2)$ and $\sigma_{\pi,k,\mathrm{e}}(\varrho_2) = \sigma_{\pi,k,\mathrm{e}}(\varrho_1) \circ f$.

Furthermore, if we consider the representations $L_{\varrho_i}\colon L_i \to \mathrm{L}(\mathrm{L}(X))$ and $R_{\varrho_i}\colon L_i^{\mathrm{op}} \to \mathrm{L}(\mathrm{L}(X))$, for $i = 1, 2$, then $L_{\varrho_2} = L_{\varrho_1} \circ f$ and $R_{\varrho_2} = R_{\varrho_1} \circ f$. Then, by [21; 3.1.5] and [21; 3.1.7] we have $\mathrm{sp}_{\delta,k}(\varrho_2) = \mathrm{sp}_{\delta,k}(\varrho_1) \circ f$ and $\mathrm{sp}_{\pi,k}(\varrho_2) = \mathrm{sp}_{\pi,k}(\varrho_1) \circ f$.

Finally, if we consider the representations $\widetilde{L}_{\varrho_i}\colon L_i \to \mathrm{L}(\mathrm{C}(X))$ and $\widetilde{R}_{\varrho_i}\colon L_i^{\mathrm{op}} \to \mathrm{L}(\mathrm{C}(X))$, for $i = 1, 2$, then $\widetilde{L}_{\varrho_2} = \widetilde{L}_{\varrho_1} \circ f$ and $\widetilde{R}_{\varrho_2} = \widetilde{R}_{\varrho_1} \circ f$. Then, according to Theorem 8 we have $\mathrm{sp}_{\delta,k,\mathrm{e}}(\varrho_2) = \mathrm{sp}_{\delta,k,\mathrm{e}}(\varrho_1) \circ f$ and $\mathrm{sp}_{\pi,k,\mathrm{e}}(\varrho_2) = \mathrm{sp}_{\pi,k,\mathrm{e}}(\varrho_1) \circ f$. ■

4. Tensor products of Banach spaces

In this section we review the definition and main properties of the tensor product of complex Banach spaces introduced by J. Eschmeier [14]. In addition, we prove some propositions necessary for our main results.

A pair $\langle X, \widetilde{X}\rangle$ of Banach spaces will be called a *dual pairing* if

$$\text{(A) } \widetilde{X} = X' \quad \text{or} \quad \text{(B) } X = \widetilde{X}'.$$

In both cases, the canonical bilinear mapping is denoted by

$$X \times \widetilde{X} \to \mathbb{C}, \qquad (x, u) \mapsto \langle x, u\rangle.$$

If $\langle X, \widetilde{X}\rangle$ is a dual pairing, we consider the subalgebra $\mathcal{L}(X)$ of $\mathrm{L}(X)$ consisting of all operators $T \in \mathrm{L}(X)$ for which there is an operator $T' \in \mathrm{L}(\widetilde{X})$ with

$$\langle Tx, u\rangle = \langle x, T'u\rangle$$

for all $x \in X$ and $u \in \widetilde{X}$. It is clear that if the dual pairing is $\langle X, X'\rangle$, then $\mathcal{L}(X) = \mathrm{L}(X)$, and that if the dual pairing is $\langle X', X\rangle$, then $\mathcal{L}(X) = \{T^* : T \in \mathrm{L}(\widetilde{X})\}$. In particular, each operator of the form

$$f_{y,v}\colon X \to X, \qquad x \mapsto \langle x, v\rangle y,$$

is contained in $\mathcal{L}(X)$, for $y \in X$ and $v \in \widetilde{X}$.

Now we recall the definition of the tensor product introduced by J. Eschmeier [14].

Definition 6. Given two dual pairings $\langle X, \widetilde{X}\rangle$ and $\langle Y, \widetilde{Y}\rangle$, a *tensor product* of the Banach spaces X and Y relative to the dual pairings $\langle X, \widetilde{X}\rangle$ and $\langle Y, \widetilde{Y}\rangle$ is a Banach space Z together with continuous bilinear mappings

$$X \times Y \to Z, \quad (x, y) \mapsto x \otimes y; \qquad \mathcal{L}(X) \times \mathcal{L}(Y) \to \mathrm{L}(Z), \quad (T, S) \mapsto T \otimes S,$$

which satisfy the following conditions:

(T1) $||x \otimes y|| = ||x|| \cdot ||y||$,
(T2) $T \otimes S(x \otimes y) = (Tx) \otimes (Sy)$,
(T3) $(T_1 \otimes S_1) \circ (T_2 \otimes S_2) = (T_1T_2) \otimes (S_1S_2)$, $I \otimes I = I$,
(T4) $\mathrm{Im}(f_{x,u} \otimes I) \subseteq \{x \otimes y : y \in Y\}, \mathrm{Im}(f_{y,v} \otimes I) \subseteq \{x \otimes y : x \in X\}$.

As in [14], we write $X \widetilde{\otimes} Y$ instead of Z. In addition, as in [14] we have two applications of Definition 6, namely, the completion $X \widetilde{\otimes}_\alpha Y$ of the algebraic tensor product of the Banach spaces X and Y with respect to a quasi-uniform crossnorm α (see [18]), and an operator ideal between Banach spaces (see [14] and Section 7).

In order to prove our main results we need to study the behavior of a split and Fredholm split complex of Banach spaces with respect to the procedure of tensoring

it with a fixed Banach space. We begin with some preparation and then we prove our characterization.

Let (X, d) be, as in Section 2, a complex of Banach spaces and bounded linear operators, and suppose that (X, d) is Fredholm split for $p = 0, \ldots, k$. Then, by [13; 2.7] and its proof, the complex (X, d) is Fredholm for $p = 0, \ldots, k$, and $\mathrm{Ker}(d_p)$ is a complemented subspace of X_p for $p = 1, \ldots, k+1$. In addition, if for $p = 1, \ldots, k+1$ we decompose $X_p = \mathrm{Ker}(d_p) \oplus L_p$, then for $p = 1, \ldots, k$ we have $X_p = R(d_{p+1}) \oplus N_p \oplus L_p$, where N_p is a finite-dimensional subspace of X_p such that $R(d_{p+1}) \oplus N_p = \mathrm{Ker}(d_p)$. Moreover, $X_0 = R(d_1) \oplus N_0$, where N_0 is a finite-dimensional subspace of X_0; in particular, we may define $L_0 = 0$. Thanks to these decompositions, for $p = 0, \ldots, k$ there are well defined operators $h_p\colon X_p \to X_{p+1}$ such that

(i) $h_p|L_p = 0$, $h_p|N_p = 0$, $h_p \circ d_{p+1} = I_p|L_{p+1}$, where I_p denotes the identity operator of X_p,

(ii) $d_{p+1}h_p + h_{p-1}d_p = I_p - k_p$, where k_p is the projector of X_p with range N_p and null space $R(d_{p+1}) \oplus L_p$,

(iii) $h_p h_{p-1} = 0$ for $p = 1, \ldots, k$.

In addition, if the complex (X, d) is split for $p = 0, \ldots, k$, then it is exact for $p = 0, \ldots, k$, and in the above decompositions $N_p = 0$ for $p = 0, \ldots, k$. In particular, $k_p = 0$ for $p = 0, \ldots, k$.

If there is a Banach space Z such that for each $p \in \mathbb{Z}$ there is an $n_p \in \mathbb{N}_0$ with $X_p = Z^{n_p}$, and a Banach space Y such that there is a tensor product $Y \widetilde{\otimes} Z$ relative to $\langle Y, Y' \rangle$ and $\langle Z, Z' \rangle$, then we may consider the chain complex

$$Y \widetilde{\otimes} X_{k+1} \xrightarrow{I \otimes d_{k+1}} Y \widetilde{\otimes} X_k \xrightarrow{I \otimes d_k} Y \widetilde{\otimes} X_{k-1} \to \ldots \to Y \widetilde{\otimes} X_1 \xrightarrow{I \otimes d_1} Y \widetilde{\otimes} X_0 \to 0,$$

where I denotes the identity of Y. Moreover, if for $p = 0, \ldots, k$ we consider the maps $I \otimes h_p\colon Y \widetilde{\otimes} X_p \to Y \widetilde{\otimes} X_{p+1}$, then

(i) $I \otimes d_{p+1} \circ I \otimes h_p + I \otimes h_{p-1} \circ I \otimes d_p = I - I \otimes k_p$,

(ii) $I \otimes h_p \circ I \otimes h_{p-1} = 0$.

It is worth noticing that the properties of the tensor product and the fact that $X_p = Z^{n_p}$ imply that the maps $I \otimes d_p$, $p = 0, \ldots, k+1$, and $I \otimes h_p$, $p = 0, \ldots, k$, are well defined and the compositions behave as usual.

Similarly, we consider a chain complex that is split or Fredholm split for $p = k, \ldots, n$.

Let (X, d) be, as in Section 2, a complex of Banach spaces and bounded linear operators, and suppose that (X, d) is Fredholm split for $p = k, \ldots, n$. Then, by [13; 2.7] and its proof, the complex (X, d) is Fredholm for $p = k, \ldots, n$ and $R(d_{p+1})$ is a closed complemented subspace of X_p for $p = k-1, \ldots, n-1$. In addition, for $p = k, \ldots, n-1$ we may decompose $X_p = R(d_{p+1}) \oplus N_p \oplus L_p$, where N_p is a finite-dimensional subspace of $\mathrm{Ker}(d_p)$ such that $\mathrm{Ker}(d_p) = R(d_{p+1}) \oplus N_p$. Moreover, for $p = n$ we know that $X_n = N_n \oplus L_n$, where $N_n = \mathrm{Ker}(d_n)$, and for $p = k-1$ we define $N_{k-1} = 0$ and L_{k-1} such that $X_{k-1} = R(d_k) \oplus L_{k-1}$. Thanks to these decompositions, for $p = k-1, \ldots, n$ there are well defined operators $h_p\colon X_p \to X_{p+1}$ such that

(i) $h_p|L_p = 0$, $h_p \circ d_{p+1} = I_p|L_{p+1}$, $h_p|N_p = 0$, where I_p denotes the identity operator of X_p,

(ii) $d_{p+1}h_p + h_{p-1}d_p = I_p - k_p$ for $p = k, \dots, n$, where k_p is the projector of X_p with range N_p and null space $L_p \oplus R(d_{p+1})$,

(iii) $h_p h_{p-1} = 0$ for $p = k, \dots, n$.

In addition, if the complex (X, d) is split for $p = k, \dots, n$, it is exact for $p = k, \dots, n$, and in the above decompositions $N_p = 0$ for $p = k, \dots, n$. In particular, $k_p = 0$ for $p = k, \dots, n$.

If there is a Banach space Z such that for each $p \in \mathbb{Z}$ there is an $n_p \in \mathbb{N}_0$ with $X_p = Z^{n_p}$, and a Banach space Y such that there is a tensor product $Y \widetilde{\otimes} Z$ relative to $\langle Y, Y' \rangle$ and $\langle Z, Z' \rangle$, then we may consider the chain complex

$$0 \to Y \widetilde{\otimes} X_n \xrightarrow{I \otimes d_n} Y \widetilde{\otimes} X_{n-1} \to \dots \to Y \widetilde{\otimes} X_k \xrightarrow{I \otimes d_k} Y \widetilde{\otimes} X_{k-1} \to \dots$$

where I denotes the identity of Y. Moreover, if for $p = k-1, \dots, n-1$ we consider the maps $I \otimes h_p \colon Y \widetilde{\otimes} X_p \to Y \widetilde{\otimes} X_{p+1}$, for $p = k, \dots, n$ we have

(i) $I \otimes d_{p+1} \circ I \otimes h_p + I \otimes h_{p-1} \circ I \otimes d_p = I - I \otimes k_p$,

(ii) $I \otimes h_p \circ I \otimes h_{p-1} = 0$.

As before, the maps $I \otimes d_p$, $p = n, \dots k$, and $I \otimes h_p$, $p = k-1, \dots n-1$, are well defined and the compositions behave as usual.

Proposition 11. *In the above conditions, for $p = 0, \dots, k$ we have*

(i) $I \otimes h_p \circ I \otimes d_{p+1} = I \otimes h_p d_{p+1}$ *is a projector defined in* $Y \widetilde{\otimes} X_{p+1}$. *In particular,* $Y \widetilde{\otimes} X_{p+1} = \operatorname{Ker}(I \otimes h_p d_{p+1}) \oplus R(I \otimes h_p d_{p+1})$.

(ii) $\operatorname{Ker}(I \otimes h_p d_{p+1}) = \operatorname{Ker}(I \otimes d_{p+1})$, $R(I \otimes h_p d_{p+1}) = R(I \otimes h_p)$, *and* $\operatorname{Ker}(I \otimes h_p) = R(I \otimes h_{p-1}) \oplus R(I \otimes k_p)$.

Similarly, for $p = k, \dots, n$ we have

(i) $I \otimes d_p \circ I \otimes h_{p-1} = I \otimes d_p h_{p-1}$ *is a projector defined in* $Y \widetilde{\otimes} X_{p-1}$. *In particular,* $Y \widetilde{\otimes} X_{p-1} = \operatorname{Ker}(I \otimes d_p h_{p-1}) \oplus R(I \otimes d_p h_{p-1})$.

(ii) $\operatorname{Ker}(I \otimes d_p h_{p-1}) = \operatorname{Ker}(I \otimes h_{p-1})$, $R(I \otimes d_p h_{p-1}) = R(I \otimes d_p)$, *and* $\operatorname{Ker}(I \otimes h_p) = R(I \otimes h_{p-1}) \oplus R(I \otimes k_p)$.

Proof. We only prove the first part of the proposition; the proof of the second one is similar.

It is easy to prove that $h_p d_{p+1} \colon X_{p+1} \to X_{p+1}$ is a projector. Thus, according to the properties of the tensor product we obtain the first assertion.

Since $I \otimes h_p d_{p+1} = I \otimes h_p \circ I \otimes d_{p+1}$, it is clear that $R(I \otimes h_p d_{p+1}) \subseteq R(I \otimes h_p)$.

On the other hand, since

$$I \otimes h_p d_{p+1} \circ I \otimes h_p = I \otimes h_p d_{p+1} h_p = I \otimes h_p (I_p - k_p - h_{p-1} d_p) = I \otimes h_p,$$

we have $R(I \otimes h_p) \subseteq R(I \otimes h_p d_{p+1})$. Thus, the equality is proved.

Since $I \otimes h_p d_{p+1} = I \otimes h_p \circ I \otimes d_{p+1}$, it is clear that $\operatorname{Ker}(I \otimes d_{p+1}) \subseteq \operatorname{Ker}(I \otimes h_p d_{p+1})$. On the other hand,

$$I \otimes d_{p+1} \circ I \otimes h_p d_{p+1} = I \otimes d_{p+1} h_p d_{p+1} = I \otimes (I_p - k_p - h_{p-1} d_p) d_{p+1} = I \otimes d_{p+1}.$$

Thus $\mathrm{Ker}(I \otimes h_p d_{p+1}) \subseteq \mathrm{Ker}(I \otimes d_{p+1})$, and we have the equality.

In order to prove the decomposition of $\mathrm{Ker}(I \otimes h_p)$, we first suppose that $p = 1, \ldots, k$. We observe that $I \otimes h_p \circ I \otimes h_{p-1} = I \otimes h_p h_{p-1} = 0$ and $I \otimes h_p \circ I \otimes k_p = I \otimes h_p k_p = 0$. Thus, $R(I \otimes k_p) + R(I \otimes h_{p-1}) \subseteq \mathrm{Ker}(I \otimes h_p)$. Moreover, $R(I \otimes k_p) \cap R(I \otimes h_{p-1}) = 0$.

In fact, since k_p is a projector, $I \otimes k_p$ is a projector. In particular, we may suppose that if $z \in R(I \otimes k_p)$, then $z = I \otimes k_p(z)$. In addition, if $z = I \otimes h_{p-1}(w)$, then we have

$$z = I \otimes k_p(z) = I \otimes k_p(I \otimes h_{p-1}(w)) = I \otimes k_p h_{p-1}(w) = 0.$$

Therefore $R(I \otimes k_p) \oplus R(I \otimes h_{p-1}) \subseteq \mathrm{Ker}(I \otimes h_p)$.

On the other hand, if $z \in \mathrm{Ker}(I \otimes h_p)$, then we have $z = I \otimes k_p(z) + I \otimes h_{p-1} d_p(z)$. Thus, $z \in R(I \otimes h_{p-1}) \oplus R(I \otimes k_p)$, and we have the equality.

Now, if $p = 0$, it is clear that $R(I \otimes k_0) \subseteq \mathrm{Ker}(I \otimes h_0)$. On the other hand, $I - I \otimes k_0 = I \otimes d_1 \circ I \otimes h_0$. In particular, if $z \in \mathrm{Ker}(I \otimes h_0)$, then $z \in R(I \otimes k_0)$. Thus, $\mathrm{Ker}(I \otimes h_0) = R(I \otimes k_0)$. ∎

Remark 12. In the above conditions, if there is a Banach space Y and a tensor product $Z \widetilde{\otimes} Y$ relative to $\langle Z, Z' \rangle$ and $\langle Y, Y' \rangle$, then by similar arguments it is possible to obtain similar results to the ones of Proposition 11, but in which the order of the spaces and maps in the tensor products are interchanged.

Now we review the relation between the tensor product of J. Eschmeier and complexes of Banach spaces; see [14; 3].

Let $(\langle X_i, \widetilde{X}_i \rangle)_{0 \le i \le n}$ be a system of dual pairings of Banach spaces such that $\widetilde{X}_i = X_i'$ for all $i = 0, \ldots, n$, or $X_i = \widetilde{X}_i'$ for all $i = 0, \ldots, n$. Then, if $\mathcal{X} = \bigoplus_{p=0}^n X_p$ and if $\widetilde{\mathcal{X}} = \bigoplus_{p=0}^n \widetilde{X}_p$, according to the observations in [14; 3], $\langle \mathcal{X}, \widetilde{\mathcal{X}} \rangle$ is a dual pairing. Moreover, if for all $i = 1, \ldots, n$ there is an operator $d_i' \in \mathcal{L}(X_i, X_{i-1})$ such that $d_{i-1}' \circ d_i' = 0$, then

$$0 \to X_n \xrightarrow{d_n'} X_{n-1} \to \ldots \to X_1 \xrightarrow{d_1'} X_0 \to 0$$

is a complex of Banach spaces and bounded linear operators; we denote it by (X, d'). In addition, if $\partial' = \bigoplus_{p=1}^n d_p'$, then $(\mathcal{X}, \partial')$ is the differential space associated to the complex (X, d') and $\partial' \in \mathcal{L}(\mathcal{X})$.

Now we consider another system of dual pairings $(\langle Y_j, \widetilde{Y}_j \rangle)_{0 \le j \le m}$ with the property stated above, i.e., $\widetilde{Y}_j = Y_j'$ for all $j = 0, \ldots, m$, or $Y_j = \widetilde{Y}_j'$ for all $j = 0, \ldots, m$. As above, we suppose that for all $j = 1, \ldots, m$ there is an operator $d_j'' \in \mathcal{L}(Y_j, Y_{j-1})$ such that $d_{j-1}'' \circ d_j'' = 0$. Thus, we have a differential complex

$$0 \to Y_m \xrightarrow{d_m''} Y_{m-1} \to \ldots \to Y_1 \xrightarrow{d_1''} Y_0 \to 0;$$

we denote it by (Y, d''). In addition, if $\partial'' = \bigoplus_{q=1}^m d_q''$, then $(\mathcal{Y}, \partial'')$ is the differential space associated to the complex (Y, d'') and $\partial'' \in \mathcal{L}(\mathcal{Y})$.

We suppose that for each $i = 0, \ldots, n$ and for each $j = 0, \ldots, m$ there is a tensor product $X_i \widetilde{\otimes} Y_j$ relative to $\langle X_i, \widetilde{X}_i \rangle$ and $\langle Y_j, \widetilde{Y}_j \rangle$, in such a way that all these tensor products are compatible in the sense described at the end of Section 1 in [14]. In particular, it is possible to consider the tensor product $\mathcal{X} \widetilde{\otimes} \mathcal{Y}$ relative to $\langle \mathcal{X}, \widetilde{\mathcal{X}} \rangle$ and $\langle \mathcal{Y}, \widetilde{\mathcal{Y}} \rangle$; see

[14; 1]. Moreover, if $\eta \in \mathcal{L}(\mathcal{X})$ is the map defined by $\eta|X_p = (-1)^p I_p$, where I_p denotes the identity of X_p, then the map $\partial\colon \mathcal{X} \mathbin{\widetilde{\otimes}} \mathcal{Y} \to \mathcal{X} \mathbin{\widetilde{\otimes}} \mathcal{Y}$ defined by

$$\partial = \partial' \otimes I_q + \eta \otimes \partial''$$

is such that $\partial \circ \partial = 0$ and $\partial \in \mathcal{L}(\mathcal{X} \mathbin{\widetilde{\otimes}} \mathcal{Y})$, where I_q denotes the identity of Y_q.

Furthermore, if we consider the double complex

$$\begin{tikzcd}
X_{p-1} \mathbin{\widetilde{\otimes}} Y_q \arrow[d, "(-1)^{p-1} I_{p-1} \otimes d''_q"'] & X_p \mathbin{\widetilde{\otimes}} Y_q \arrow[l, "d'_p \otimes I_q"'] \arrow[d, "(-1)^p I_p \otimes d''_q"] \\
X_{p-1} \mathbin{\widetilde{\otimes}} Y_{q-1} & X_p \mathbin{\widetilde{\otimes}} Y_{q-1} \arrow[l, "d'_p \otimes I_{q-1}"']
\end{tikzcd}$$

then the differential space associated to the total complex of this double complex is $(\mathcal{X} \mathbin{\widetilde{\otimes}} \mathcal{Y}, \partial)$.

Now, if L_1 and L_2 are two complex solvable finite-dimensional Lie algebras of dimensions n and m respectively, X_1 and X_2 two complex Banach spaces, and $\varrho_i\colon L_i \to \mathrm{L}(X_i)$, $i = 1, 2$, two representations of the Lie algebras, then we may consider the Koszul complexes associated to the representations ϱ_1 and ϱ_2, i.e., $(X_1 \otimes \bigwedge L_1, d(\varrho_1))$ and $(X_2 \otimes \bigwedge L_2, d(\varrho_2))$ respectively.

It is clear that for $p = 0, \dots, n$ and for $q = 0, \dots, m$, $\langle X_1 \otimes \bigwedge^p L_1, X_1 \otimes \bigwedge^p L'_1 \rangle$ and $\langle X_2 \otimes \bigwedge^q L_2, X_2 \otimes \bigwedge^q L'_2 \rangle$ are dual pairings. Moreover, $d_p(\varrho_1) \in \mathcal{L}(X_1 \otimes \bigwedge^p L_1, X_1 \otimes \bigwedge^{p-1} L_1)$ and $d_q(\varrho_2) \in \mathcal{L}(X_2 \otimes \bigwedge^q L_2, X_2 \otimes \bigwedge^{q-1} L_2)$ for $p = 0, \dots, n$ and $q = 0, \dots, m$. Thus, we may consider the differential spaces $(\mathcal{X}_1, \partial_1)$ and $(\mathcal{X}_2, \partial_2)$, where $\mathcal{X}_1 = X_1 \otimes \bigwedge L_1$, $\mathcal{X}_2 = X_2 \otimes \bigwedge L_2$, $\partial_1 = \bigoplus_{p=1}^n d_p(\varrho_1)$ and $\partial_2 = \bigoplus_{q=1}^m d_q(\varrho_2)$.

We suppose that there is a tensor product $X_1 \mathbin{\widetilde{\otimes}} X_2$ of X_1 and X_2 with respect to $\langle X_1, X'_1 \rangle$ and $\langle X_2, X'_2 \rangle$. Then, according to the considerations at the end of Section 1 in [14], for all $p = 0, \dots, n$ and $q = 0, \dots, m$ there is a well defined tensor product $X_1 \otimes \bigwedge^p L_1 \mathbin{\widetilde{\otimes}} X_2 \otimes \bigwedge^q L_2$ of $X_1 \otimes \bigwedge^p L_1$ and $X_2 \otimes \bigwedge^q L_2$ relative to $\langle X_1 \otimes \bigwedge^p L_1, X_1 \otimes \bigwedge^p L'_1 \rangle$ and $\langle X_2 \otimes \bigwedge^q L_2, X_2 \otimes \bigwedge^q L'_2 \rangle$. Furthermore, since for all p and q such that $p = 0, \dots, n$ and $q = 0, \dots, m$, these tensor products are compatible in the sense described at the end of Section 1 in [14], as above, we may consider the tensor product $\mathcal{X}_1 \mathbin{\widetilde{\otimes}} \mathcal{X}_2$ of $\mathcal{X}_1$ and $\mathcal{X}_2$, which is a differential space with differential $\partial \in \mathcal{L}(\mathcal{X}_1 \mathbin{\widetilde{\otimes}} \mathcal{X}_2)$, $\partial = \partial_1 \otimes I + \eta \otimes \partial_2$. Then $(\mathcal{X}_1 \mathbin{\widetilde{\otimes}} \mathcal{X}_2, \partial)$ is the differential space associated to the total complex of the double complex

$$\begin{tikzcd}
X_1 \otimes \bigwedge^{p-1} L_1 \mathbin{\widetilde{\otimes}} X_2 \otimes \bigwedge^q L_2 \arrow[d, "(-1)^{p-1} I_{p-1} \otimes d_q(\varrho_2)"'] & X_1 \otimes \bigwedge^p L_1 \mathbin{\widetilde{\otimes}} X_2 \otimes \bigwedge^q L_2 \arrow[l, "d_p(\varrho_1) \otimes I_q"'] \arrow[d, "(-1)^p I_p \otimes d_q(\varrho_2)"] \\
X_1 \otimes \bigwedge^{p-1} L_1 \mathbin{\widetilde{\otimes}} X_2 \otimes \bigwedge^{q-1} L_2 & X_1 \otimes \bigwedge^p L_1 \mathbin{\widetilde{\otimes}} X_2 \otimes \bigwedge^{q-1} L_2 \arrow[l, "d_p(\varrho_1) \otimes I_{q-1}"']
\end{tikzcd}$$

We recall that given the Koszul complexes $(X_1 \otimes \bigwedge L_1, d(\varrho_1))$ and $(X_2 \otimes \bigwedge L_2, d(\varrho_2))$, according to the properties of the tensor product introduced in [14] and the considerations of Sections 1 and 3 in [14], it is possible to consider the complex of Banach spaces defined by the tensor product of $(X_1 \otimes \bigwedge L_1, d(\varrho_1))$ and $(X_2 \otimes \bigwedge L_2, d(\varrho_2))$, denoted by $(X_1 \otimes \bigwedge L_1, d(\varrho_1)) \mathbin{\widetilde{\otimes}} (X_2 \otimes \bigwedge L_2, d(\varrho_2))$. This complex is the total complex of the above

double complex, i.e., for $0 \le k \le n+m$, the kth space is $\bigoplus_{p+q=k} X_1 \otimes \bigwedge^p L_1 \widetilde{\otimes} X_2 \otimes \bigwedge^q L_2$, and the boundary map, d_k, restricted to $X_1 \otimes \bigwedge^p L_1 \widetilde{\otimes} X_2 \otimes \bigwedge^q L_2$ is $d_k = d_p(\varrho_1) \otimes I_q + (-1)^p \otimes d_q(\varrho_2)$. In particular, $(\mathcal{X}_1 \widetilde{\otimes} \mathcal{X}_2, \partial)$ is the differential space of the complex $(X_1 \otimes \bigwedge L_1, d(\varrho_1)) \widetilde{\otimes} (X_2 \otimes \bigwedge L_2, d(\varrho_2))$.

On the other hand, we may consider the direct sum $L = L_1 \times L_2$ of the Lie algebras L_1 and L_2, which is a complex solvable finite-dimensional Lie algebra, and the tensor product representation of L in $X_1 \widetilde{\otimes} X_2$, i.e.,

$$\varrho = \varrho_1 \times \varrho_2 \colon L \to \mathrm{L}(X_1 \widetilde{\otimes} X_2), \qquad \varrho_1 \times \varrho_2(l_1, l_2) = \varrho_1(l_1) \otimes I + I \otimes \varrho_2(l_2),$$

where I denotes the identity operator of both X_2 and X_1. In particular, we may consider the Koszul complex of the representation $\varrho \colon L \to \mathrm{L}(X_1 \widetilde{\otimes} X_2)$, i.e., $(X_1 \widetilde{\otimes} X_2 \otimes \bigwedge L, d(\varrho))$, and the differential space associated to it, i.e., $(X_1 \widetilde{\otimes} X_2 \otimes \bigwedge L, \widetilde{\partial})$, where $\widetilde{\partial} = \bigoplus_{k=1}^{n+m} d_k(\varrho)$.

In the following proposition we relate $(X_1 \otimes \bigwedge L_1, d(\varrho_1)) \widetilde{\otimes} (X_2 \otimes \bigwedge L_2, d(\varrho_2))$ and $(X_1 \widetilde{\otimes} X_2 \otimes \bigwedge L, d(\varrho))$.

Proposition 13. *Let X_1 and X_2 be two complex Banach spaces, L_1 and L_2 two complex solvable finite-dimensional Lie algebras, and $\varrho_i \colon L_i \to \mathrm{L}(X_i)$, $i = 1, 2$, two representations of the algebras. Then the complexes $(X_1 \widetilde{\otimes} X_2 \otimes \bigwedge L, d(\varrho))$ and $(X_1 \otimes \bigwedge L_1, d(\varrho_1)) \widetilde{\otimes} (X_2 \otimes \bigwedge L_2, d(\varrho_2))$ are isomorphic. In particular, the differential spaces $(\mathcal{X}_1 \widetilde{\otimes} \mathcal{X}_2, \partial)$ and $(X_1 \widetilde{\otimes} X_2 \otimes \bigwedge L, \widetilde{\partial})$ are isomorphic.*

Proof. First of all we consider the identification

$$\Phi \colon \bigwedge L_1 \otimes \bigwedge L_2 \to \bigwedge L, \qquad \Phi(w_1 \otimes w_2) = w_1 \wedge w_2,$$

for $w_1 \in L_1$, $w_2 \in L_2$. Now an easy calculation shows that for $k = 0, \ldots, n+m$ the map

$$\widetilde{\Phi}_k \colon \bigoplus_{p+q=k} X_1 \otimes \bigwedge^p L_1 \widetilde{\otimes} X_2 \otimes \bigwedge^q L_2 \to X_1 \widetilde{\otimes} X_2 \otimes \bigwedge^k L,$$
$$\widetilde{\Phi}(x_1 \otimes w_1 \widetilde{\otimes} x_2 \otimes w_2) = x_1 \widetilde{\otimes} x_2 \otimes w_1 \wedge w_2,$$

is a well defined isomorphism. Moreover, since L is the direct sum of L_1 and L_2, it is easy to prove that $\widetilde{\Phi} = (\widetilde{\Phi}_k)_{0 \le k \le n+m}$ is a chain map, i.e., $\widetilde{\Phi}(d) = d(\varrho)\widetilde{\Phi}$. ∎

5. Joint spectra of the tensor product representation

In this section we consider two representations of Lie algebras in two Banach spaces and a tensor product of the Banach spaces in the sense of [14], and we describe the Słodkowski and split joint spectra of the tensor product representation of the direct sum of the algebras; see Section 4. Moreover, for Hilbert spaces, the joint spectra are characterized in a precise manner. In addition, we apply our results to nilpotent systems of operators. We start by recalling the objects we shall work with.

Let L_1 and L_2 be two complex solvable finite-dimensional Lie algebras, X_1 and X_2 two complex Banach spaces, and $\varrho_i \colon L_i \to \mathrm{L}(X_i)$, $i = 1, 2$, two representations of Lie algebras. We suppose that there is a tensor product $X_1 \widetilde{\otimes} X_2$ of X_1 and X_2 relative to $\langle X_1, X_1' \rangle$ and $\langle X_2, X_2' \rangle$. Thus, as in Section 4, we may consider the direct sum $L = L_1 \times L_2$

of the Lie algebras L_1 and L_2, which is a complex solvable finite-dimensional Lie algebra, and the tensor product representation of L in $X_1 \widetilde{\otimes} X_2$, i.e.

$$\varrho = \varrho_1 \times \varrho_2 \colon L \to \mathrm{L}(X_1 \widetilde{\otimes} X_2), \qquad \varrho_1 \times \varrho_2(l_1, l_2) = \varrho_1(l_1) \otimes I + I \otimes \varrho_2(l_2),$$

where I denotes the identity of X_2 and X_1 respectively. In particular, we may consider the Koszul complex $(X_1 \widetilde{\otimes} X_2 \otimes \bigwedge L, d(\varrho))$ of the representation $\varrho \colon L \to \mathrm{L}(X_1 \widetilde{\otimes} X_2)$.

Now we state the most important result of this section. However, we first observe that the set of characters of L may be naturally identified with the cartesian product of the sets of characters of L_1 and L_2. Indeed, it is clear that $L^* \cong L_1^* \times L_2^*$. Moreover, since as Lie algebra, L is the direct sum of L_1 and L_2, if $[\cdot,\cdot]$ denotes the Lie bracket of L, then the restriction of $[\cdot,\cdot]$ to L_1 or L_2 coincides with the bracket of L_1 or L_2 respectively, and for $l_1 \in L_1$ and $l_2 \in L_2$, $[l_1, l_2] = 0$. Thus, the map

$$H \colon L^* \to L_1^* \times L_2^*, \qquad f \mapsto (f \circ \iota_1, f \circ \iota_2),$$

defines an identification of the characters of L and the cartesian product of the characters of L_1 and L_2, where $\iota_j \colon L_j \to L$ denotes the inclusion map, $j = 1, 2$. In the following theorem we use this identification.

THEOREM 14. *Let L_1 and L_2 be two complex solvable finite-dimensional Lie algebras, X_1 and X_2 two complex Banach spaces, and $\varrho_i \colon L_i \to \mathrm{L}(X_i)$, $i = 1, 2$, two representations of Lie algebras. Suppose that there is a tensor product $X_1 \widetilde{\otimes} X_2$ of X_1 and X_2 relative to $\langle X_1, X_1' \rangle$ and $\langle X_2, X_2' \rangle$. Consider the tensor product representation $\varrho = \varrho_1 \times \varrho_2 \colon L \to \mathrm{L}(X_1 \widetilde{\otimes} X_2)$ of $L = L_1 \times L_2$. Then*

$$\text{(i)} \quad \bigcup_{p+q=k} \sigma_{\delta,p}(\varrho_1) \times \sigma_{\delta,q}(\varrho_2) \subseteq \sigma_{\delta,k}(\varrho) \subseteq \mathrm{sp}_{\delta,k}(\varrho) \subseteq \bigcup_{p+q=k} \mathrm{sp}_{\delta,p}(\varrho_1) \times \mathrm{sp}_{\delta,q}(\varrho_2),$$

$$\text{(ii)} \quad \bigcup_{p+q=k} \sigma_{\pi,p}(\varrho_1) \times \sigma_{\pi,q}(\varrho_2) \subseteq \sigma_{\pi,k}(\varrho) \subseteq \mathrm{sp}_{\pi,k}(\varrho) \subseteq \bigcup_{p+q=k} \mathrm{sp}_{\pi,p}(\varrho_1) \times \mathrm{sp}_{\pi,q}(\varrho_2).$$

In particular, if X_1 and X_2 are Hilbert spaces, the above inclusions are equalities.

Proof. We begin with the first statement.

We consider $\alpha \in \sigma_{\delta,p}(\varrho_1)$, $\beta \in \sigma_{\delta,q}(\varrho_2)$, $p+q = k$, and the Koszul complexes associated to the representations $\varrho_1 - \alpha \colon L_1 \to \mathrm{L}(X_1)$ and $\varrho_2 - \beta \colon L_2 \to \mathrm{L}(X_2)$, $(X_1 \otimes \bigwedge L_1, d(\varrho_1 - \alpha))$ and $(X_2 \otimes \bigwedge L_2, d(\varrho_2 - \beta))$ respectively. Then there is p_1, $0 \le p_1 \le p$, and q_2, $0 \le q_2 \le q$, such that $H_{p_1}(X_1 \otimes \bigwedge L_1, d(\varrho_1 - \alpha)) \neq 0$ and $H_{q_2}(X_2 \otimes \bigwedge L_2, d(\varrho_2 - \beta)) \neq 0$.

In addition, if we consider the differential spaces associated to the Koszul complexes of $\varrho_1 - \alpha$ and $\varrho_2 - \beta$, $(\mathcal{X}_1, \partial_1)$ and $(\mathcal{X}_2, \partial_2)$ respectively, then by [14; 2.2] we have $H_*(\mathcal{X}_1 \widetilde{\otimes} \mathcal{X}_2) \neq 0$. Moreover, since $(\mathcal{X}_1 \widetilde{\otimes} \mathcal{X}_2, \partial)$ is the differential space of $(X_1 \otimes \bigwedge L_1, d(\varrho_1 - \alpha)) \widetilde{\otimes} (X_2 \otimes \bigwedge L_2, d(\varrho_2 - \beta))$, according to the structure of the map φ in [14; 2.2], we have $H_{p_1+q_2}((X_1 \otimes \bigwedge L_1, d(\varrho_1 - \alpha)) \widetilde{\otimes} (X_2 \otimes \bigwedge L_2, d(\varrho_2 - \beta))) \neq 0$. Furthermore, according to Proposition 13, since $(\varrho_1 - \alpha) \times (\varrho_2 - \beta) = \varrho - (\alpha, \beta)$, we have $H_{p_1+q_2}(X_1 \widetilde{\otimes} X_2 \otimes \bigwedge L, d(\varrho - (\alpha, \beta))) \neq 0$. In particular, since $0 \le p_1 + q_2 \le p + q = k$, $(\alpha, \beta) \in \sigma_{\delta,k}(\varrho)$.

The middle inclusion is clear.

For the rightmost inclusion, we prove that if (α, β) does not belong to $\bigcup_{p+q=k} \mathrm{sp}_{\delta,p}(\varrho_1) \times \mathrm{sp}_{\delta,q}(\varrho_2)$, then (α, β) does not belong to $\mathrm{sp}_{\delta,k}(\varrho)$. To this end, we shall construct a homotopy operator. There are several cases to be considered.

We first suppose that $\alpha \notin \mathrm{sp}_{\delta,k}(\varrho_1)$. Thus, the complex $(X_1 \otimes \bigwedge L_1, d(\varrho_1 - \alpha))$ is split for $p = 0, \dots, k$, i.e., for $p = 0, \dots, k$ there are bounded linear operators $h_p\colon X_1 \otimes \bigwedge^p L_1 \to X_1 \otimes \bigwedge^{p+1} L_1$ such that $h_{p-1} d_p(\varrho_1 - \alpha) + d_{p+1}(\varrho_1 - \alpha) h_p = I_p$, where I_p denotes the identity of $X_1 \otimes \bigwedge^p L_1$. Then, if p and q are such that $0 \le p + q \le k$, we define

$$H_{p,q}\colon X_1 \otimes \bigwedge^p L_1 \mathbin{\widetilde{\otimes}} X_2 \otimes \bigwedge^q L_2 \to X_1 \otimes \bigwedge^{p+1} L_1 \mathbin{\widetilde{\otimes}} X_2 \otimes \bigwedge^q L_2, \qquad H_{p,q} = h_p \otimes I_q,$$

where I_q denotes the identity map of $X_2 \otimes \bigwedge^q L_2$. We observe that since $\mathcal{L}(X_1 \otimes \bigwedge^p L_1, X_1 \otimes \bigwedge^{p+1} L_1) = \mathrm{L}(X_1 \otimes \bigwedge^p L_1, X_1 \otimes \bigwedge^{p+1} L_1)$, $H_{p,q}$ is a well defined map. Moreover, a direct calculation shows that the maps $H_r = \bigoplus_{p+q=r} H_{p,q}$, $r = 0, \dots, k$, define a homotopy operator for the complex $(X_1 \otimes \bigwedge L_1) \mathbin{\widetilde{\otimes}} (X_2 \otimes \bigwedge L_2)$. Thus, according to Proposition 13 the complex $(X_1 \mathbin{\widetilde{\otimes}} X_2 \otimes \bigwedge L, d(\varrho - (\alpha, \beta)))$ is split for $r = 0, \dots, k$, i.e., (α, β) does not belong to $\mathrm{sp}_{\delta,k}(\varrho)$.

By a similar argument, it is possible to prove that if $\beta \notin \mathrm{sp}_{\delta,k}(\varrho_2)$, then (α, β) does not belong to $\mathrm{sp}_{\delta,k}(\varrho)$. Thus, we may suppose that $\alpha \in \mathrm{sp}_{\delta,k}(\varrho_1)$ and $\beta \in \mathrm{sp}_{\delta,k}(\varrho_2)$.

Now, since (α, β) does not belong to $\bigcup_{p+q=k} \mathrm{sp}_{\delta,p}(\varrho_1) \times \mathrm{sp}_{\delta,q}(\varrho_2)$ and $\alpha \in \mathrm{sp}_{\delta,k}(\varrho_1)$, we have $\beta \notin \mathrm{sp}_{\delta,0}(\varrho_2)$. Similarly, since $\beta \in \mathrm{sp}_{\delta,k}(\varrho_2)$ we have $\alpha \notin \mathrm{sp}_{\delta,0}(\varrho_1)$. Thus, there is p_1, $1 \le p_1 \le k$, such that $\alpha \notin \mathrm{sp}_{\delta,p_1-1}(\varrho_1)$, $\alpha \in \mathrm{sp}_{\delta,p_1}$, and $\beta \notin \mathrm{sp}_{\delta,k-p_1}(\varrho_2)$.

In order to construct a homotopy operator for the Koszul complex associated to $\varrho - (\alpha, \beta)$, for (α, β) as in the last paragraph, it is necessary to consider several cases. In fact, we shall define the operator according to the relation of p and q with p_1 and $k - p_1$ respectively, and for each particular case, we shall prove that it is a homotopy. At the end of the proof, it is clear that this map is a well defined homotopy for the Koszul complex of ϱ at $r = 0, \dots, k$.

Moreover, according to Proposition 13, it is enough to prove that the complex $(X_1 \otimes \bigwedge L_1, d(\varrho_1 - \alpha)) \mathbin{\widetilde{\otimes}} (X_2 \otimes \bigwedge L_2, d(\varrho_2 - \beta))$ is split in dimension $r = 0, \dots, k$. Now, the rth space of this complex is $\bigoplus_{p+q=k} X_1 \otimes \bigwedge^p L_1 \mathbin{\widetilde{\otimes}} X_2 \otimes \bigwedge^q L_2$. We construct an operator $H_{p,q}$ satisfying the homotopy identity for p and q such that $p + q = r$, and then we verify that $(H_r)_{0 \le r \le k}$ is a homotopy operator for the complex, where $H_r = \bigoplus_{p+q=r} H_{p,q}$. The construction of the maps $H_{p,q}$ is divided into five cases.

We first suppose that $0 \le p \le p_1 - 1$ and $0 \le q \le k - p_1$. Then we have well defined maps

$$X_1 \otimes \bigwedge^{p-1} L_1 \xrightarrow{h_{p-1}} X_1 \otimes \bigwedge^p L_1 \xrightarrow{h_p} X_1 \otimes \bigwedge^{p+1} L_1$$

such that $d_{p+1}(\varrho_1 - \alpha) h_p + h_{p-1} d_p(\varrho_1 - \alpha) = I_p$, where I_p denotes the identity of $X_1 \otimes \bigwedge^p L_1$, and

$$X_2 \otimes \bigwedge^{q-1} L_2 \xrightarrow{g_{q-1}} X_2 \otimes \bigwedge^q L_2 \xrightarrow{g_q} X_2 \otimes \bigwedge^{q+1} L_2$$

such that $d_{q+1}(\varrho_2 - \beta) g_q + g_{q-1} d_q(\varrho_2 - \beta) = I_q$, where I_q denotes the identity of $X_2 \otimes \bigwedge^q L_2$. Thus, we may define the map

$$H_{p,q}\colon X_1 \otimes \bigwedge^p L_1 \mathbin{\widetilde{\otimes}} X_2 \otimes \bigwedge^q L_2 \to X_1 \otimes \bigwedge^{p+1} L_1 \mathbin{\widetilde{\otimes}} X_2 \otimes \bigwedge^q L_2 \oplus X_1 \otimes \bigwedge^p L_1 \mathbin{\widetilde{\otimes}} X_2 \otimes \bigwedge^{q+1} L_2,$$
$$H_{p,q} = \tfrac{1}{2}(h_p \otimes I_q \oplus (-1)^p I_p \otimes g_q).$$

We observe that according to the properties of the tensor product, $H_{p,q}$ is a well defined map.

In addition, since $p-1<p\leq p_1-1$ and $q-1<q\leq k-p_1$, we may define the maps $H_{p-1,q}$ and $H_{p,q-1}$. A direct calculation shows that in $X_1\otimes\bigwedge^p L_1\widetilde{\otimes}X_2\otimes\bigwedge^q L_2$, we have

$$d_{r+1}H_{p,q}+(H_{p-1,q}\oplus H_{p,q-1})d_r=I,$$

where d and I are the boundary and the identity of the complex $(X_1\otimes\bigwedge L_1,d(\varrho_1-\alpha))\widetilde{\otimes}(X_2\otimes\bigwedge L_2,d(\varrho_2-\beta))$ respectively.

In the second case we suppose that p and q are such that $p\leq p_1-1$ and $q=k-p_1+1$. Then we know that for $q=0,\dots,k-p_1$ there are bounded maps

$$X_2\otimes\bigwedge^{q-1}L_2\xrightarrow{g_{q-1}}X_2\otimes\bigwedge^q L_2\xrightarrow{g_q}X_2\otimes\bigwedge^{q+1}L_2$$

such that $d_{q+1}(\varrho_2-\beta)g_q+g_{q-1}d_q(\varrho_2-\beta)=I_q$.

In addition, we may suppose that the maps g_q satisfy the preliminary facts recalled before Proposition 11, for $q=0,\dots,k-p_1$. Moreover, according to Proposition 11, we have $X_1\otimes\bigwedge^p L_1\widetilde{\otimes}X_2\otimes\bigwedge^q L_2=\operatorname{Ker}(I_p\otimes d_q(\varrho_2-\beta))\oplus R(I_p\otimes g_{q-1})$ for $p=0,\dots,n$ and $q=0,\dots,k-p_1+1$.

It is easy to prove that

(i) $d_p(\varrho_1-\alpha)\otimes I_q(R(I_p\otimes g_{q-1}))\subseteq R(I_{p-1}\otimes g_{q-1})$ and $d_p(\varrho_1-\alpha)\otimes I_q(\operatorname{Ker}(I_p\otimes d_q(\varrho_2-\beta)))\subseteq\operatorname{Ker}(I_{p-1}\otimes d_q(\varrho_2-\beta))$,

(ii) $I_p\otimes d_q(\varrho_2-\beta)(\operatorname{Ker}(I_p\otimes d_q(\varrho_2-\beta)))=0$ and $I_p\otimes d_q(\varrho_2-\beta)(R(I_p\otimes g_{q-1}))=R(I_p\otimes d_q(\varrho_2-\beta))=\operatorname{Ker}(I_p\otimes d_{q-1}(\varrho_2-\beta))$.

Furthermore, as in the first case, we have well defined maps $(h_p)_{0\leq p\leq p_1-1}$ such that $h_p\colon X_1\otimes\bigwedge^p L_1\to X_1\otimes\bigwedge^{p+1}L_1$ and $d_{p+1}(\varrho_1-\alpha)h_p+h_pd_{p-1}(\varrho_1-\alpha)=I_p$ for $p=0,\dots,p_1-1$. A straightforward calculation shows that

(iii) $h_p\otimes I_q(R(I_p\otimes g_{q-1}))\subseteq R(I_{p+1}\otimes g_{q-1})$ and $h_p\otimes I_q(\operatorname{Ker}(I_p\otimes d_q(\varrho_2-\beta)))\subseteq\operatorname{Ker}(I_{p+1}\otimes d_q(\varrho_2-\beta))$.

Now, for $p=0,\dots,p_1-1$ and $q=k-p_1+1$ we define $H_{p,q}$ as follows:

$$H_{p,q}|R(I_p\otimes g_{q-1})=\tfrac{1}{2}(h_p\otimes I_q),\qquad H_{p,q}|\operatorname{Ker}(I_p\otimes d_q(\varrho_2-\beta))=h_p\otimes I_q.$$

According to the properties of the tensor product, the map $H_{p,q}$ is well defined.

In addition, for $p=0,\dots,p_1-1$ and $q-1=k-p_1$, according to the first case, we have the well defined map $H_{p,q-1}$. On the other hand, for $p-1$, $p=0,\dots,p_1-1$, and $q=k-p_1+1$, we may define $H_{p-1,q}$ in a similar way to $H_{p,q}$.

Now, using (i)–(iii) it is easy to prove that

$$d_{r+1}H_{p,q}+(H_{p,q-1}\oplus H_{p-1,q})d_r=I,$$

where d and I are as above.

In the third case $p=0,\dots,p_1-1$ and $q>k-p_1+1$. There are two subcases: $q-1>k-p_1+1$ and $q-1=k-p_1+1$. We begin with the first subcase.

For $p=0,\dots,p_1-1$ and $q>q-1>k-p_1+1$ we define

$$H_{p,q}|X_1\otimes\bigwedge^p L_1\widetilde{\otimes}X_2\otimes\bigwedge^q L_2\to X_1\otimes\bigwedge^{p+1}L_1\widetilde{\otimes}X_2\otimes\bigwedge^q L_2,\qquad H_{p,q}=h_p\otimes I_q.$$

According to the properties of the tensor product, $H_{p,q}$ is a well defined map.

Moreover, since $q-1>q>k-p_1+1$, we may define $H_{p-1,q}$ and $H_{p,q-1}$ in a similar way. Then an easy calculation shows that

$$d_{r+1}H_{p,q}+(H_{p,q-1}\oplus H_{p-1,q})d_r=I.$$

On the other hand, for $p=0,\dots,p_1-1$ and $q=k-p_1+1$, we define $H_{p,q}=h_p\otimes I_q$. Furthermore, for $p-1$ and $q=k-p_1+1$, we may define $H_{p-1,q}=h_{p-1}\otimes I_q$, and for $p=0,\dots,p_1-1$ and $q-1=k-p_1$, $H_{p,q-1}$ was defined in the second case. A direct calculation shows that

$$d_{r+1}H_{p,q}+(H_{p,q-1}\oplus H_{p-1,q})d_r=I.$$

In the fourth case $p=p_1$ and $q\le k-p_1$. This case is similar to the second one.

We consider the complex associated to the representation $\varrho_1-\alpha$, i.e., $(X_1\otimes\bigwedge L_1, d(\varrho_1-\alpha))$. We know that for $p=0,\dots,p_1-1$ there are bounded maps

$$X_1\otimes\textstyle\bigwedge^{p-1}L_1\xrightarrow{h_{p-1}}X_1\otimes\bigwedge^{p}L_1\xrightarrow{h_p}X_1\otimes\bigwedge^{p+1}L_1$$

such that $d_{p+1}(\varrho_1-\alpha)h_p+h_{p-1}d_p(\varrho_1-\alpha)=I_p$.

Moreover, as in the second case, we may suppose that the maps h_p satisfy the preliminary facts recalled before Proposition 11, for $p=0,\dots,p_1-1$. Furthermore, according to Proposition 11 and Remark 12, $X_1\otimes\bigwedge^p L_1\mathbin{\widetilde{\otimes}}X_2\otimes\bigwedge^q L_2=\mathrm{Ker}(d_p(\varrho_1-\alpha)\otimes I_q)\oplus R(h_{p-1}\otimes I_q)$ for $p=0,\dots,p_1$ and $q=0,\dots,m$.

As in the second case, it is easy to prove that

(i) $I_p\otimes d_q(\varrho_2-\beta)(R(h_{p-1}\otimes I_q))\subseteq R(h_{p-1}\otimes I_{q-1})$ and $I_p\otimes d_q(\varrho_2-\beta)(\mathrm{Ker}(d_p(\varrho_1-\alpha)\otimes I_q))\subseteq\mathrm{Ker}(d_p(\varrho_1-\alpha)\otimes I_{q-1})$,

(ii) $d_p(\varrho_1-\alpha)\otimes I_q(\mathrm{Ker}(d_p(\varrho_1-\alpha)\otimes I_q))=0$ and $d_p(\varrho_1-\alpha)\otimes I_q(R(h_{p-1}\otimes I_q))=R(d_p(\varrho_1-\alpha)\otimes I_q)=\mathrm{Ker}(d_{p-1}(\varrho_1-\alpha)\otimes I_q)$.

In addition, for $q=0,\dots,k-p_1$, we have well defined maps $(g_q)_{0\le q\le k-p_1}$ such that $g_q\colon X_2\otimes\bigwedge^q L_2\to X_2\otimes\bigwedge^{q+1}L_2$ and $d_{q+1}(\varrho_2-\beta)g_p+g_{q-1}d_{q-1}(\varrho_2-\beta)=I_q$. A straightforward calculation shows

(iii) $I_p\otimes g_q(R(h_{p-1}\otimes I_q))\subseteq R(h_{p-1}\otimes I_{q+1})$ and $I_p\otimes g_q(\mathrm{Ker}(d_p(\varrho_1-\alpha)\otimes I_q))\subseteq\mathrm{Ker}(d_p(\varrho_1-\alpha)\otimes I_{q+1})$.

Now for $p=p_1$ and $0\le q\le k-p_1$, we define $H_{p_1,q}$ as follows:

$$H_{p_1,q}|R(h_{p_1-1}\otimes I_q)=(-1)^p1/2(I_p\otimes g_q),\qquad H_{p_1,q}|\mathrm{Ker}(d_{p_1}(\varrho_1-\alpha)\otimes I_q)=(-1)^pI_p\otimes g_q.$$

According to the properties of the tensor product, $H_{p_1,q}$ is a well defined map.

In addition, according to the first case, we have the well defined map $H_{p_1-1,q}$, $p=p_1-1$ and $q=0,\dots,k-p_1$. On the other hand, we may define $H_{p_1,q-1}$ like $H_{p_1,q}$, $p=p_1$ and $q-1=0,\dots,k-p_1$.

Now, as in the second case, using (i)–(iii) it is easy to prove that

$$d_{r+1}H_{p_1,q}+(H_{p_1-1,q}\oplus H_{p_1,q})d_r=I.$$

In the last case, we have $p\ge p_1+1$ and $q=0,\dots,k-p_1$. Moreover, as in the third case, there are two subcases: $p-1\ge p_1+1$ and $p-1=p_1$. We begin with the first subcase.

For $p > p-1 \geq p_1+1$ and $q = 0, \ldots, k-p_1$, we define

$$H_{p,q} : X_1 \otimes \bigwedge^p L_1 \widetilde{\otimes} X_2 \otimes \bigwedge^q L_2 \to X_1 \otimes \bigwedge^p L_1 \widetilde{\otimes} X_2 \otimes \bigwedge^{q+1} L_2, \qquad H_{p,q} = I_p \otimes g_q.$$

According to the properties of the tensor product, the map $H_{p,q}$ is well defined.

Since $p-1 > p_1+1$, we may define $H_{p-1,q}$ and $H_{p,q-1}$. Then an easy calculation shows that

$$d_{r+1}H_{p,q} + (H_{p,q-1} \oplus H_{p-1,q})d_r = I.$$

On the other hand, for $p-1 = p_1$ and $q = 0, \ldots, k-p_1$, we define $H_{p,q} = I_p \otimes g_q$. Moreover, for $p-1 = p_1$ and q, $H_{p-1,q}$ was defined in the fourth case, and for p and $q-1$, we may define $H_{p,q-1} = I_p \otimes g_{q-1}$. A direct calculation shows

$$d_{r+1}H_{p,q} + (H_{p-1,q} \oplus H_{p,q-1})d_r = I.$$

Since we considered all the possible cases for p and q, $0 \leq p+q \leq k$, if for $r = 0, \ldots, k$ we consider the map $H_r = \bigoplus_{p+q=r} H_{p,q}$, then the above computations show that $(H_r)_{0\leq r\leq k}$ is a homotopy for the complex $(X_1 \otimes \bigwedge L_1, d(\varrho_1 - \alpha)) \widetilde{\otimes} (X_2 \otimes \bigwedge L_2, d(\varrho_2 - \beta))$. Thus, according to Proposition 13, (α, β) does not belong to $\mathrm{sp}_{\delta,k}(\varrho)$.

The second part of the theorem may be proved by a similar argument, using the second half of Proposition 11 for the rightmost inclusion. ∎

We recall that in [21; 3.3], the axiomatic tensor product of [14] was generalized. However, as explained in [21; 3.3], the objective was to simplify the form of the axioms rather than to generalize the definition of [14]; in addition, the known applications of both tensor products coincide. Since for our proofs of the main results in this work, the definition of [14] is more useful than the one of [21], we proved Theorem 14 and shall prove the other results for the tensor product introduced in [14]. In particular, Theorem 14 may be seen as an extension of [21; 3.6.8] for the tensor product of [14]. However, we believe that with the axiomatic tensor product introduced in [21; 3.3], it would be possible to obtain results similar to ours.

Now we consider nilpotent systems of operators and we prove a variant of Theorem 14 for this case. This result extends [21; 3.7.2] for the tensor product of [14]. Moreover, the following theorem is an extension of well known results for commuting tuples of operators; see [9], [10], [28] and [14]. First we give a definition.

Let X be a complex Banach space and $T = (T_1, \ldots, T_n)$ an n-tuple of operators defined in X such that the linear subspace of $\mathrm{L}(X)$ generated by them, $\langle T_i \rangle_{1\leq i\leq n} = L$, is a nilpotent Lie subalgebra of $\mathrm{L}(X)$. We consider the representation defined by the inclusion $\iota_L \colon L \to \mathrm{L}(X)$. Then, if σ denotes a subset of a joint spectrum defined for representations of complex solvable finite-dimensional Lie algebras, we denote the set $\{(\alpha(T_1), \ldots, \alpha(T_n)) : \alpha \in \sigma(\iota_L)\}$ by $\sigma(T)$.

THEOREM 15. *Let X_1 and X_2 be two complex Banach spaces. Suppose that there is a tensor product $X_1 \widetilde{\otimes} X_2$ of X_1 and X_2 with respect to $\langle X_1, X_1' \rangle$ and $\langle X_2, X_2' \rangle$. Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_m)$ be two tuples of operators, $a_i \in \mathrm{L}(X_1)$, $1 \leq i \leq n$, and $b_j \in \mathrm{L}(X_2)$, $1 \leq j \leq m$, such that the vector subspaces generated by them, $\langle a_i \rangle_{1\leq i\leq n}$ and $\langle b_j \rangle_{1\leq j\leq m}$, are nilpotent Lie subalgebras of $\mathrm{L}(X_1)$ and $\mathrm{L}(X_2)$ respectively. Consider the*

$(n+m)$-tuple of operators defined in $X_1 \widetilde{\otimes} X_2$, $c = (a_1 \otimes I, \ldots, a_n \otimes I, I \otimes b_1, \ldots, I \otimes b_m)$, where I denotes the identity of X_2 and X_1 respectively. Then

(i) $\displaystyle\bigcup_{p+q=k} \sigma_{\delta,p}(a) \times \sigma_{\delta,q}(b) \subseteq \sigma_{\delta,k}(c) \subseteq \mathrm{sp}_{\delta,k}(c) \subseteq \bigcup_{p+q=k} \mathrm{sp}_{\delta,p}(a) \times \mathrm{sp}_{\delta,q}(b)$,

(ii) $\displaystyle\bigcup_{p+q=k} \sigma_{\pi,p}(a) \times \sigma_{\pi,q}(b) \subseteq \sigma_{\pi,k}(c) \subseteq \mathrm{sp}_{\pi,k}(c) \subseteq \bigcup_{p+q=k} \mathrm{sp}_{\pi,p}(a) \times \mathrm{sp}_{\pi,q}(b)$.

In particular, if X_1 and X_2 are Hilbert spaces, the above inclusions are equalities.

Proof. We consider the nilpotent Lie algebras $L_1 = \langle a_i \rangle_{1 \le i \le n}$ and $L_2 = \langle b_j \rangle_{1 \le j \le m}$, and the representations of the above algebras defined by inclusion, i.e.,

$$\iota_1 \colon L_1 \to \mathrm{L}(X_1), \qquad \iota_2 \colon L_2 \to \mathrm{L}(X_2).$$

Then, if we consider the representation $\iota = \iota_1 \times \iota_2 \colon L_1 \times L_2 \to \mathrm{L}(X_1 \widetilde{\otimes} X_2)$, according to Theorem 14 we have

$$\bigcup_{p+q=k} \sigma_{\delta,p}(\iota_1) \times \sigma_{\delta,q}(\iota_2) \subseteq \sigma_{\delta,k}(\iota) \subseteq \mathrm{sp}_{\delta,k}(\iota) \subseteq \bigcup_{p+q=k} \mathrm{sp}_{\delta,p}(\iota_1) \times \mathrm{sp}_{\delta,q}(\iota_2).$$

Now, if we consider the identification of the characters of $L_1 \times L_2$ with the cartesian product of the characters of L_1 and L_2, it is clear that $\sigma_{\delta,p}(a) \times \sigma_{\delta,q}(b)$ coincides with the set

$$\{(\alpha(a_1), \ldots, \alpha(a_n), \beta(b_1), \ldots, \beta(b_m)) : (\alpha, \beta) \in \sigma_{\delta,p}(\iota_1) \times \sigma_{\delta,q}(\iota_2)\}.$$

Similarly, $\mathrm{sp}_{\delta,p}(a) \times \mathrm{sp}_{\delta,q}(b)$ coincides with

$$\{(\alpha(a_1), \ldots, \alpha(a_n), \beta(b_1), \ldots, \beta(b_m)) : (\alpha, \beta) \in \mathrm{sp}_{\delta,p}(\iota_1) \times \mathrm{sp}_{\delta,q}(\iota_2)\}.$$

On the other hand, we consider the nilpotent Lie subalgebra of $\mathrm{L}(X_1 \widetilde{\otimes} X_2)$ generated by the elements of the $(n+m)$-tuple c; we denote it by L. Then, if $\iota \colon L \to \mathrm{L}(X_1 \widetilde{\otimes} X_2)$ is the representation defined by the inclusion, we have $\iota_1 \times \iota_2 = \iota \circ h$, where $h \colon L_1 \times L_2 \to L$ is the epimorphism of Lie algebras that satisfies $h(a_i) = a_i \otimes I$ and $h(b_j) = I \otimes b_j$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. In particular, according to Proposition 10 we have

$$\sigma_{\delta,k}(\iota_1 \times \iota_2) = \sigma_{\delta,k}(\iota) \circ h, \qquad \mathrm{sp}_{\delta,k}(\iota_1 \times \iota_2) = \mathrm{sp}_{\delta,k}(\iota) \circ h.$$

Furthermore,

$$\sigma_{\delta,k}(c) = \{(\gamma \circ h(a_1), \ldots, \gamma \circ h(a_n), \gamma \circ h(b_1), \ldots, \gamma \circ h(b_m)) : \gamma \in \sigma_{\delta,k}(\iota)\}.$$

Moreover, according to Proposition 10, $\sigma_{\delta,k}(c)$ coincides with

$$\{(\alpha(a_1), \ldots, \alpha(a_n), \beta(b_1), \ldots, \beta(b_m)) : (\alpha, \beta) \in \sigma_{\delta,k}(\iota_1 \times \iota_2)\}.$$

Similarly,

$$\mathrm{sp}_{\delta,k}(c) = \{(\alpha(a_1), \ldots, \alpha(a_n), \beta(b_1), \ldots, \beta(b_m)) : (\alpha, \beta) \in \mathrm{sp}_{\delta,k}(\iota_1 \times \iota_2)\}.$$

Thus, the above equalities prove the first part of the theorem. The second statement may be proved by a similar argument. ∎

6. Fredholm joint spectra of the tensor product representation

In this section we consider two representations of Lie algebras in two Banach spaces and a tensor product of the Banach spaces in the sense of [14], and we describe the essential Słodkowski and essential split joint spectra of the tensor product representation of the direct sum of the algebras; see Section 4. In addition, we apply our results to nilpotent systems of operators. We first prove a result needed for the main theorem in this section.

Proposition 16. *Let X_1 and X_2 be two Banach spaces. Suppose that there is a tensor product $X_1 \widetilde{\otimes} X_2$ of X_1 and X_2 relative to $\langle X_1, X_1'\rangle$ and $\langle X_2, X_2'\rangle$. Consider in X_1 and X_2 two projectors with finite-dimensional range, k_1 and k_2 respectively. Then $k_1 \otimes k_2 \in \mathrm{L}(X_1 \widetilde{\otimes} X_2)$ is a projector with finite-dimensional range. In fact, $R(k_1 \otimes k_2) = R(k_1) \otimes R(k_2)$.*

Proof. According to the properties of the tensor product, it is clear that $k_1 \otimes k_2$ is a projector and that $R(k_1 \otimes k_2) \supseteq R(k_1) \otimes R(k_2)$.

In order to prove the other inclusion, we consider a base $(v_i)_{1\le i\le n}$ of $R(k_1)$, i.e., $R(k_1) = \langle v_i\rangle_{1\le i\le n}$. Then we have $X_1 = \mathrm{Ker}(k_1) \oplus \bigoplus_{i=1}^n \langle v_i\rangle$. Moreover, if for each $s = 1,\dots,n$ we consider the map $l_s\colon X_1 \to \mathbb{C}$, $l_s|\mathrm{Ker}(k_1) \equiv 0$, $l_s|\langle v_i\rangle \equiv 0$, $i = 1,\dots,n$, $i \ne s$, and $l_s(v_s) = 1$, then we may define the maps $f_{v_i l_i}\colon X_1 \to X_1$, $f_{v_i l_i}(x_1) = l_i(x_1)v_i$ for $x_1 \in X_1$. Now, an easy calculation shows that $k_1 = \sum_{i=1}^n f_{v_i l_i}$.

Similarly, we may consider a base $(v_j')_{1\le j\le m}$ of $R(k_2)$, and then we have $X_2 = \mathrm{Ker}(k_2) \oplus \bigoplus_{j=1}^m \langle v_j'\rangle$. Moreover, if for $j = 1,\dots,m$ we consider the maps $h_j\colon X_2 \to \mathbb{C}$, $h_j|\mathrm{Ker}(k_2) \equiv 0$, $h_j(v_t') = 0$, $t = 1,\dots,m$, $t \ne j$, and $h_j(v_j') = 1$, then we may define the maps $f_{v_j' h_j}\colon X_2 \to X_2$, $f_{v_j' h_j}(x_2) = h_j(x_2)v_j'$ for $x_2 \in X_2$. As above, an easy calculation shows that $k_2 = \sum_{j=1}^m f_{v_j' h_j}$.

Now, according to the properties of the tensor product, we have

$$k_1 \otimes k_2 = \sum_{i,j} f_{v_i l_i} \otimes f_{v_j' h_j} = \sum_{i,j} f_{v_i l_i} \otimes I \circ I \otimes f_{v_j' h_j}.$$

Moreover, by [14; 1.1], for each l_i, $i = 1,\dots,n$, there is a map $f_{l_i}\colon X_1 \widetilde{\otimes} X_2 \to X_2$ such that $f_{x_1 l_i} \otimes I(z) = x_1 \otimes f_{l_i}(z)$ for $x_1 \in X_1$ and $z \in X_1 \widetilde{\otimes} X_2$, where $f_{x_1 l_i}\colon X_1 \to X_1$ is the map $f_{x_1 l_i}(x) = l_i(x)x_1$. In addition, for each h_j, $j = 1,\dots,m$, there is a map $g_{h_j}\colon X_1 \widetilde{\otimes} X_2 \to X_1$ such that $I \otimes f_{x_2 h_j}(z) = g_{h_j}(z) \otimes x_2$ for $x_2 \in X_2$ and $z \in X_1 \widetilde{\otimes} X_2$, where $f_{x_2 h_j}$ has a definition similar to that of $f_{x_1 l_i}$. In particular, for $z \in X_1 \widetilde{\otimes} X_2$ we have

$$\begin{aligned} k_1 \otimes k_2(z) &= \sum_{i,j} f_{v_i l_i} \otimes I \circ I \otimes f_{v_j' h_j}(z) = \sum_{i,j} f_{v_i l_i} \otimes I(g_{h_j}(z) \otimes v_j') \\ &= \sum_{i,j} v_i \otimes f_{l_i}(g_{h_j}(z) \otimes v_j'). \end{aligned}$$

Thus, $R(k_1 \otimes k_2) \subseteq R(k_1) \otimes X_2$.

Moreover, since k_2 is a projection, if for $z \in X_1 \widetilde{\otimes} X_2$ we define $z_{ij} = f_{l_i}(g_{h_j}(z) \otimes v_j')$, then we have $z_{ij} = k_2(z_{ij}) + (I - k_2)(z_{ij})$. In particular

$$k_1 \otimes k_2(z) = \sum_{i,j} v_i \otimes z_{ij} = \sum_{i,j} v_i \otimes k_2(z_{ij}) + \sum_{i,j} v_i \otimes (I - k_2)(z_{ij}).$$

But since $k_1 \otimes k_2$ is a projector in $X_1 \widetilde{\otimes} X_2$, we have

$$k_1 \otimes k_2(z) = (k_1 \otimes k_2)^2(z) = \sum_{i,j} v_i \otimes k_2(z_{ij}).$$

In particular, $R(k_1 \otimes k_2) \subseteq R(k_1) \otimes R(k_2)$. ∎

Now we state the main result of this section. The following theorem is an extension of [14; 3.2].

THEOREM 17. *Let X_1 and X_2 be two complex Banach spaces, L_1 and L_2 two complex solvable finite-dimensional Lie algebras, and $\varrho_i\colon L_i \to \mathrm{L}(X_i)$, $i = 1, 2$, two representations of Lie algebras. Suppose that there is a tensor product $X_1 \widetilde{\otimes} X_2$ of X_1 and X_2 relative to $\langle X_1, X_1'\rangle$ and $\langle X_2, X_2'\rangle$. Consider the tensor product representation $\varrho = \varrho_1 \times \varrho_2\colon L_1 \times L_2 \to \mathrm{L}(X_1 \widetilde{\otimes} X_2)$ of the direct sum of L_1 and L_2. Then*

(i) $$\bigcup_{p+q=k} \sigma_{\delta,p,\mathrm{e}}(\varrho_1) \times \sigma_{\delta,q}(\varrho_2) \cup \bigcup_{p+q=k} \sigma_{\delta,p}(\varrho_1) \times \sigma_{\delta,q,\mathrm{e}}(\varrho_2) \subseteq \sigma_{\delta,k,\mathrm{e}}(\varrho)$$
$$\subseteq \mathrm{sp}_{\delta,k,\mathrm{e}}(\varrho) \subseteq \bigcup_{p+q=k} \mathrm{sp}_{\delta,p,\mathrm{e}}(\varrho_1) \times \mathrm{sp}_{\delta,q}(\varrho_2) \cup \bigcup_{p+q=k} \mathrm{sp}_{\delta,p}(\varrho_1) \times \mathrm{sp}_{\delta,q,\mathrm{e}}(\varrho_2),$$

(ii) $$\bigcup_{p+q=k} \sigma_{\pi,p,\mathrm{e}}(\varrho_1) \times \sigma_{\pi,q}(\varrho_2) \cup \bigcup_{p+q=k} \sigma_{\pi,p}(\varrho_1) \times \sigma_{\pi,q,\mathrm{e}}(\varrho_2) \subseteq \sigma_{\pi,k,\mathrm{e}}(\varrho)$$
$$\subseteq \mathrm{sp}_{\pi,k,\mathrm{e}}(\varrho) \subseteq \bigcup_{p+q=k} \mathrm{sp}_{\pi,p,\mathrm{e}}(\varrho_1) \times \mathrm{sp}_{\pi,q}(\varrho_2) \bigcup \bigcup_{p+q=k} \mathrm{sp}_{\pi,p}(\varrho_1) \times \mathrm{sp}_{\pi,q,\mathrm{e}}(\varrho_2).$$

In particular, if X_1 and X_2 are Hilbert spaces, the above inclusions are equalities.

Proof. First of all, in the proof of this theorem we use the notations and identifications of Theorem 14. In particular, if α is a character of L_1 and β is a character of L_2 we work with the complex $(X_1 \otimes \bigwedge L_1, d(\varrho_1 - \alpha)) \widetilde{\otimes} (X_2 \otimes \bigwedge L_2, d(\varrho_2 - \beta))$ instead of the Koszul complex associated to the representation $\varrho - (\alpha, \beta)\colon L_1 \times L_2 \to \mathrm{L}(X_1 \widetilde{\otimes} X_2)$. We begin with the first statement.

In order to prove the leftmost inclusion, the same argument used in Theorem 14 for the $\sigma_{\delta,k}$ joint spectra may be applied to the essential δ-Słodkowski joint spectra. In fact, the argument still works when we consider two homology spaces, one of which is non-null and the other is infinite-dimensional, instead of considering two non-null homology spaces.

As in Theorem 14, the middle inclusion is clear.

For the rightmost inclusion, we shall use an induction argument.

First of all we study the case $k = 0$.

We consider a pair $(\alpha, \beta) \in \mathrm{sp}_{\delta,0,\mathrm{e}}(\varrho) \setminus (\mathrm{sp}_{\delta,0,\mathrm{e}}(\varrho_1) \times \mathrm{sp}_{\delta,0}(\varrho_2) \cup \mathrm{sp}_{\delta,0}(\varrho_1) \times \mathrm{sp}_{\delta,0,\mathrm{e}}(\varrho_2))$. Since by Theorem 14, $(\alpha, \beta) \in \mathrm{sp}_{\delta,0}(\varrho_1) \times \mathrm{sp}_{\delta,0}(\varrho_2)$, we have $\alpha \in \mathrm{sp}_{\delta,0}(\varrho_1) \setminus \mathrm{sp}_{\delta,0,\mathrm{e}}(\varrho_2)$ and $\beta \in \mathrm{sp}_{\delta,0}(\varrho_2) \setminus \mathrm{sp}_{\delta,0,\mathrm{e}}(\varrho_2)$. In particular, there are bounded linear maps

$$h_0\colon X_1 \to X_1 \otimes \textstyle\bigwedge^1 L_1, \qquad g_0\colon X_2 \to X_2 \otimes \bigwedge^1 L_2,$$

and finite range projectors

$$k_0\colon X_1 \to X_1, \qquad k_0'\colon X_2 \to X_2,$$

such that

$$d_1(\varrho_1 - \alpha)h_0 = I_0 - k_0, \qquad d_1(\varrho_2 - \beta)g_0 = I_0 - k_0'.$$

Now, if we consider the map

$$H_0\colon X_1 \mathbin{\widetilde{\otimes}} X_2 \to X_1 \mathbin{\widetilde{\otimes}} X_2 \otimes \textstyle\bigwedge^1 L_2 \oplus X_1 \otimes \bigwedge^1 L_1 \mathbin{\widetilde{\otimes}} X_2, \qquad H_0 = (I_0 \otimes g_0, h_0 \otimes I_0),$$

then it is easy to prove that

$$d_1 H_0 = I - k_0 \otimes k_0',$$

where d and I are the boundary and the identity of the complex $(X_1 \otimes \bigwedge L_1, d(\varrho_1 - \alpha)) \mathbin{\widetilde{\otimes}} (X_2 \otimes \bigwedge L_2, d(\varrho_2 - \beta))$ respectively.

By Proposition 16, the map $k_0 \otimes k_0'$ is a projector with finite-dimensional range. In particular, according to Proposition 13, (α, β) does not belong to $\mathrm{sp}_{\delta,k,\mathrm{e}}(\varrho)$, which is impossible according to our assumption.

Now we suppose that the rightmost inclusion is true for 0 and for all natural numbers less than k, and we prove it for k. We proceed as in the case $k = 0$.

We consider a pair $(\alpha, \beta) \in \mathrm{sp}_{\delta,k,\mathrm{e}}(\varrho)$ which does not belong to $\bigcup_{p+q=k} \mathrm{sp}_{\delta,p,\mathrm{e}}(\varrho_1) \times \mathrm{sp}_{\delta,q}(\varrho_2) \cup \bigcup_{p+q=k} \mathrm{sp}_{\delta,p}(\varrho_1) \times \mathrm{sp}_{\delta,q,\mathrm{e}}(\varrho_2)$. In particular, $(\alpha, \beta) \notin \bigcup_{p+q=k-1} \mathrm{sp}_{\delta,p,\mathrm{e}}(\varrho_1) \times \mathrm{sp}_{\delta,q}(\varrho_2) \cup \bigcup_{p+q=k-1} \mathrm{sp}_{\delta,p}(\varrho_1) \times \mathrm{sp}_{\delta,q,\mathrm{e}}(\varrho_2)$. Thus, by the inductive hypothesis $(\alpha, \beta) \notin \mathrm{sp}_{\delta,k-1,\mathrm{e}}(\varrho)$.

In addition, since according to Theorem 14, $\mathrm{sp}_{\delta,k,\mathrm{e}}(\varrho) \subseteq \mathrm{sp}_{\delta,k}(\varrho) \subseteq \bigcup_{p+q=k} \mathrm{sp}_{\delta,p}(\varrho_1) \times \mathrm{sp}_{\delta,q}(\varrho_2)$, there are p_0 and q_0, $p_0 + q_0 = k$, such that $\alpha \in \mathrm{sp}_{\delta,p_0}(\varrho_1)$ and $\beta \in \mathrm{sp}_{\delta,q_0}(\varrho_2)$. Moreover, we may suppose that $p_0 = \min\{p,\ 0 \le p \le k : \alpha \in \mathrm{sp}_{\delta,p}(\varrho_1)\}$. It is easy to prove that the following assertions are true:

(i) $\alpha \in \mathrm{sp}_{\delta,p_0}(\varrho_1)$, $\alpha \notin \mathrm{sp}_{\delta,p}(\varrho_1)$, $p = 0, \dots, p_0 - 1$, and $\beta \notin \mathrm{sp}_{\delta,q_0,\mathrm{e}}(\varrho_2)$,

(ii) $\beta \in \mathrm{sp}_{\delta,q_0}(\varrho_2)$, and either $\alpha \notin \mathrm{sp}_{\delta,k,\mathrm{e}}(\varrho_1)$ and $\beta \in \mathrm{sp}_{\delta,0}(\varrho_2)$, or there is p_1, $p_0 \le p_1 \le k - 1$, such that $\alpha \notin \mathrm{sp}_{\delta,p_1,\mathrm{e}}(\varrho_1)$, $\alpha \in \mathrm{sp}_{\delta,p_1+1,\mathrm{e}}(\varrho_1)$, and $\beta \notin \mathrm{sp}_{\delta,k-p_1-1}(\varrho_2)$.

By means of assertions (i) and (ii), we prove that $\dim \mathrm{Ker}(d_k)/R(d_{k+1})$ is finite, and that $\mathrm{Ker}(d_{k+1})$ is a complemented subspace. Since $(\alpha, \beta) \notin \mathrm{sp}_{\delta,k-1,\mathrm{e}}(\varrho)$, by [13; 2.7], we have $(\alpha, \beta) \notin \mathrm{sp}_{\delta,k,\mathrm{e}}(\varrho)$, which is impossible according to our assumption.

Before we begin the proof we observe the following point. Assertions (i) and (ii) in fact consist of two different pairs of assertions, therefore, we should give two proofs. However, we work only with assertion (i) and the second part of assertion (ii). The proof that needs assertion (i) and the first part of assertion (ii) is easier and it is left to the reader.

By (i) and (ii) there are bounded linear operators $h_p\colon X_1 \otimes \bigwedge^p L_1 \to X_1 \otimes \bigwedge^{p+1} L_1$, $p = 0, \dots, p_1$, and there are projectors with finite-dimensional range, $k_p\colon X_1 \otimes \bigwedge^p L_1 \to X_1 \otimes \bigwedge^p L_1$, $p = p_0, \dots, p_1$, such that for $p = 0, \dots, p_0 - 1$,

$$h_{p-1} d_p(\varrho_1 - \alpha) + d_{p+1}(\varrho_1 - \alpha) h_p = I_p,$$

and for $p = p_0, \dots, p_1$,

$$h_{p-1} d_p(\varrho_1 - \alpha) + d_{p+1}(\varrho - \alpha) h_p = I_p - k_p.$$

In addition, by (i) and (ii) there are bounded linear maps $g_q\colon X_2 \otimes \bigwedge^q L_2 \to X_2 \otimes \bigwedge^{q+1} L_2$, $q = 0, \dots, q_0 = k - p_0$, and there are projectors with finite-dimensional range, $k_q'\colon X_2 \otimes \bigwedge^q L_2 \to X_2 \otimes \bigwedge^q L_2$, $q = k - p_1, \dots, q_0$, such that for $q = 0, \dots, k - p_1 - 1$,

$$g_{q-1} d_q(\varrho_2 - \beta) + d_{q+1}(\varrho_2 - \beta) g_q = I_q,$$

and for $q = k - p_1, \dots, q_0$,

$$g_{q-1} d_q(\varrho_2 - \beta) + d_{q+1}(\varrho_2 - \beta) g_q = I_q - k'_q.$$

In order to prove that $\mathrm{Ker}(d_{k+1})$ is a complemented subspace of $\bigoplus_{p+q=k+1} X_1 \otimes \bigwedge^p L_1 \widetilde{\otimes} X_2 \otimes \bigwedge^q L_2$, we first characterize it and then exhibit a complement.

It is easy to prove that $\mathrm{Ker}(d_{k+1})$ is the set of all $(x_{p,q})$, $p + q = k + 1$, $x_{p,q} \in X_1 \otimes \bigwedge^p L_1 \widetilde{\otimes} X_2 \otimes \bigwedge^q L_2$, such that in $X_1 \otimes \bigwedge^{p-1} L_1 \widetilde{\otimes} X_2 \otimes \bigwedge^q L_2$,

$$d_p(\varrho_1 - \alpha) \otimes I_q(x_{p,q}) + (-1)^{p-1} I_{p-1} \otimes d_{q+1}(\varrho_2 - \beta)(x_{p-1,q+1}) = 0.$$

According to Proposition 11, we know that for $q = 0, \dots, k - p_1$ and $p + q = k + 1$,

$$X_1 \otimes \bigwedge^p L_1 \widetilde{\otimes} X_2 \otimes \bigwedge^q L_2 = R(I_p \otimes g_{q-1}) \oplus \mathrm{Ker}(I_p \otimes d_q(\varrho_2 - \beta)),$$

$$X_1 \otimes \bigwedge^{p-1} L_1 \widetilde{\otimes} X_2 \otimes \bigwedge^q L_2 = R(I_{p-1} \otimes g_{q-1}) \oplus \mathrm{Ker}(I_{p-1} \otimes d_q(\varrho_2 - \beta)),$$

and for $q = 0, \dots, k - p_1 - 1$,

$$X_1 \otimes \bigwedge^{p-1} L_1 \widetilde{\otimes} X_2 \otimes \bigwedge^{q+1} L_2 = R(I_{p-1} \otimes g_q) \oplus \mathrm{Ker}(I_{p-1} \otimes d_{q+1}(\varrho_2 - \beta)).$$

In particular, we may represent each $x_{p,q} \in X_1 \otimes \bigwedge^p L_1 \widetilde{\otimes} X_2 \otimes \bigwedge^q L_2$, $p + q = k + 1$, $q = 0, \dots, k - p_1$, as $x_{p,q} = (a_{p,q}, b_{p,q})$, where $a_{p,q} \in \mathrm{Ker}(I_p \otimes d_q(\varrho_2 - \beta))$ and $b_{p,q} \in R(I_p \otimes g_{q-1})$.

On the other hand, according to Proposition 11,

$$I_{p-1} \otimes g_q \colon \mathrm{Ker}(I_{p-1} \otimes d_q(\varrho_2 - \beta)) \to R(I_{p-1} \otimes g_q)$$

is a topological isomorphism for $q = 0, \dots, k - p_1 - 1$. Then an easy calculation shows that $x_{k+1,0} = a_{k+1,0}$, and that $b_{p,q} = (-1)^{p+1} d_{p+1}(\varrho_1 - \alpha) \otimes g_{q-1}(a_{p+1,q-1})$ for $q = 1, \dots, k - p_1$.

Thus, $x_{p,q}$ is described for $q = 0, \dots, k - p_1 - 1$ and p such that $p + q = k + 1$. However, we may continue this procedure till $q = k - p_0$.

In fact, according to Proposition 11 the above decompositions of the spaces $X_1 \otimes \bigwedge^p L_1 \widetilde{\otimes} X_2 \otimes \bigwedge^q L_2$, $X_1 \otimes \bigwedge^{p-1} L_1 \widetilde{\otimes} X_2 \otimes \bigwedge^q L_2$ and $X_1 \otimes \bigwedge^{p-1} L_1 \widetilde{\otimes} X_2 \otimes \bigwedge^{q+1} L_2$ remain true for $q = k - p_1, \dots, q_0 + 1 = k - p_0 + 1$. Moreover, according to Proposition 11, it is easy to prove that $\mathrm{Ker}(I_{p-1} \otimes d_q(\varrho_2 - \beta)) = R(I_{p-1} \otimes k'_q) \oplus R(I_{p-1} \otimes d_{q+1}(\varrho_2 - \beta))$, $q = k - p_1, \dots, k - p_0$, and

$$I_{p-1} \otimes g_q \colon R(I_{p-1} \otimes d_{q+1}(\varrho_2 - \beta)) \to R(I_{p-1} \otimes g_q)$$

is a topological isomorphism. Then, if for $q = k - p_1, \dots, k - p_0 + 1$ we decompose $x_{p,q} = ((a^1_{p,q}, a^2_{p,q}), b_{p,q})$, where $a^1_{p,q} \in R(I_p \otimes d_{q+1}(\varrho_2 - \beta))$, $a^2_{p,q} \in R(I_p \otimes k'_q)$ and $b_{p,q} \in R(I_p \otimes g_{q-1})$, an easy calculation shows that $a^2_{p,q} \in R(I_p \otimes k'_q) \cap \mathrm{Ker}(d_p(\varrho_1 - \alpha) \otimes I_q)$, $q = k - p_1, \dots, k - p_0$, and $b_{p,q} = (-1)^{p+1} d_{p+1}(\varrho_1 - \alpha) \otimes g_{q-1}(a^1_{p+1,q-1})$, $q = k - p_1, \dots, k - p_0 + 1$.

On the other hand, by a similar argument, it is possible to prove the following fact. If we consider for $p = 0, \dots, p_0$ the decomposition

$$X_1 \otimes \bigwedge^p L_1 \widetilde{\otimes} X_2 \otimes \bigwedge^q L_2 = R(h_{p-1} \otimes I_q) \oplus \mathrm{Ker}(d_p(\varrho_1 - \alpha) \otimes I_q),$$

and we represent $x_{p,q}$ as $x_{p,q} = (c_{p,q}, d_{p,q})$, where $c_{p,q} \in \mathrm{Ker}(d_p(\varrho_1 - \alpha) \otimes I_q)$ and $d_{p,q} \in R(h_{p-1} \otimes I_q)$, then $x_{0,k+1} = c_{0,k+1}$ and $d_{p,q} = (-1)^p h_{p-1} \otimes d_{q+1}(\varrho_2 - \beta)(c_{p-1,q+1})$ for $p = 1, \dots, p_0$.

Thus, if $(x_{p,q})$, $p+q=k+1$, belongs to $\mathrm{Ker}(d_{k+1})$, then $x_{p,q}$ is described for $p=0,\dots,p_0-1$ and $q=0,\dots,k-p_0$. In order to characterize $\mathrm{Ker}(d_{k+1})$ in a complete way, we have to consider $X_1\otimes\bigwedge^{p_0}L_1\,\widetilde{\otimes}\,X_2\otimes\bigwedge^{k+1-p_0}L_2$.

In $X_1\otimes\bigwedge^{p_0}L_1\,\widetilde{\otimes}\,X_2\otimes\bigwedge^{k+1-p_0}L_2$, we have two well defined projectors,

$$S=I_{p_0}\otimes g_{k-p_0}d_{k-p_0+1}(\varrho_2-\beta),\qquad T=h_{p_0-1}d_{p_0}(\varrho_1-\alpha)\otimes I_{k-p_0+1}.$$

Moreover, since S commutes with T, $X_1\otimes\bigwedge^{p_0}L_1\,\widetilde{\otimes}\,X_2\otimes\bigwedge^{k+1-p_0}L_2$ may be decomposed as the direct sum of the ranges of the operators ST, $S(I-T)$, $(I-S)T$ and $(I-S)(I-T)$, and each x that belongs to this space may be decomposed as $x=(x_{ST},x_{S(I-T)},x_{(I-S)T},x_{(I-S)(I-T)})$.

Now, if $(x_{p,q})$, $p+q=k+1$, belongs to $\mathrm{Ker}(d_{k+1})$, in order to determine $x_{p_0,k-p_0+1}$ it is enough to consider the equations in which it takes part, i.e.,

$$d_{p_0+1}(\varrho_1-\alpha)\otimes I_{k-p_0}(x_{p_0+1,k-p_0})+(-1)^{p_0}I_{p_0}\otimes d_{k+1-p_0}(\varrho_2-\beta))(x_{p_0,k+1-p_0})=0,$$
$$d_{p_0}(\varrho_1-\alpha)\otimes I_{k+1-p_0}(x_{p_0,k+1-p_0})+(-1)^{p_0-1}I_{p_0-1}\otimes d_{k+2-p_0}(\varrho_2-\beta)(x_{p_0-1,k+2-p_0})=0.$$

In addition, an easy calculation shows that if we decompose $x_{p_0,k-p_0+1}=x$ as above, then $x_{ST}=0$, $x_{(I-S)T}=d_{p_0,k+1-p_0}$, $x_{S(I-T)}=b_{p_0,k+1-p_0}$, and $x_{(I-S)(I-T)}$ is an arbitrary element in the range of $(I-S)(I-T)$.

Thus, $\mathrm{Ker}(d_{k+1})$ may be represented as the direct sum of the following spaces:

(i) for $q=0,\dots,k-p_0$, the graph of $(-1)^pd_p(\varrho_1-\alpha)\otimes g_q$: $R(I_p\otimes d_{q+1}(\varrho_2-\beta))\to R(I_{p-1}\otimes g_q)$, $p+q=k+1$;

(ii) for $q=k-p_1,\dots,k-p_0$, $R(I_p\otimes k'_q)\cap\mathrm{Ker}(d_p(\varrho_1-\alpha)\otimes I_q)$, $p+q=k+1$;

(iii) for $p=0,\dots,p_0-1$, the graph of $(-1)^ph_p\otimes d_q(\varrho_2-\beta)$: $R(d_{p+1}(\varrho_1-\alpha)\otimes I_q)\to R(h_p\otimes I_{q-1})$, $p+q=k+1$;

(iv) the range of the projector $(I-S)(I-T)$.

In order to construct a direct complement of $\mathrm{Ker}(d_{k+1})$ we need the following observations.

First, if X and Y are Banach spaces and $T\in\mathrm{L}(X,Y)$, then $X\oplus Y=\mathrm{Graph}(T)\oplus Y$. Second, an easy calculation shows $R(I_p\otimes k'_q)\cap\mathrm{Ker}(d_p(\varrho_1-\alpha)\otimes I_q)\oplus R(h_{p-1}d_p(\varrho_1-\alpha)\otimes k'_q)=R(I_p\otimes k'_q)$, for $q=k-p_1,\dots,k-p_0$.

Now, depending on p and q, $p+q=k+1$, the space $X_1\otimes\bigwedge^pL_1\,\widetilde{\otimes}\,X_2\otimes\bigwedge^qL_2$ is equal to the direct sum of the following spaces:

(i) for $p=0,\dots,p_0-1$, $R(d_{p+1}(\varrho_1-\alpha)\otimes I_q)$ and $R(h_{p-1}\otimes I_q)$;

(ii) for $q=0,\dots,k-p_0$, $R(I_p\otimes d_{q+1}(\varrho_2-\beta))$, $R(I_p\otimes g_{q-1})$ and $R(I_p\otimes k'_q)$; when $q=0,\dots,k-p_1-1$, we have $k'_q=0$;

(iii) for $p=p_0$ and $q=k-p_0+1$, the ranges of the operators ST, $S(I-T)$, $(I-S)T$ and $(I-S)(I-T)$.

Then, if we define V to be the direct sum of the spaces $R(h_{p-1}\otimes I_q)$, $p=0,\dots,p_0$, $R(I_p\otimes g_{q-1})$, $q=0,\dots,k-p_0+1$, $R(h_{p-1}d_p(\varrho_1-\alpha)\otimes k'_q)$, $q=k-p_1,\dots,k-p_0$, and $R(ST)$ for $p=p_0$ and $q=k-p_0+1$, we have $\bigoplus_{p+q=k+1}X_1\otimes\bigwedge^pL_1\,\widetilde{\otimes}\,X_2\otimes\bigwedge^qL_2=\mathrm{Ker}(d_{k+1})\oplus V$.

We now prove that $\dim\mathrm{Ker}(d_k)/R(d_{k+1})$ is finite.

As with $\mathrm{Ker}(d_{k+1})$, we may represent $\mathrm{Ker}(d_k)$ as the direct sum of the following spaces:

(i) for $q = 0, \ldots, k-p_0-1$, the graph of $(-1)^p d_p(\varrho_1-\alpha)\otimes g_q$: $R(I_p\otimes d_{q+1}(\varrho_2-\beta)) \to R(I_{p-1}\otimes g_q)$, $p+q=k$;

(ii) for $q = k-p_1, \ldots, k-p_0-1$, $R(I_p\otimes k'_q)\cap \mathrm{Ker}(d_p(\varrho_1-\alpha)\otimes I_q)$, $p+q=k$;

(iii) for $p = 0, \ldots, p_0-1$, the graph of $(-1)^p h_p\otimes d_q(\varrho_2-\beta)$: $R(d_{p+1}(\varrho_1-\alpha)\otimes I_q) \to R(h_p\otimes I_{q-1})$, $p+q=k$;

(iv) for $p=p_0$ and $q=k-p_0$, the range of the projector $(I-S)(I-T)$, where

$$S = I_{p_0}\otimes g_{k-p_0-1}d_{k-p_0}(\varrho_2-\beta), \qquad T = h_{p_0-1}d_{p_0}(\varrho_1-\alpha)\otimes I_{k-p_0}.$$

Now we consider p and q such that $p+q=k$ and $q=0,\ldots,k-p_0-1$. Then, if we consider $(-1)^p I_p\otimes g_q(a)$, $a\in R(I_p\otimes d_{q+1}(\varrho_2-\beta))$, it is easy to prove that $(a,(-1)^p d_p(\varrho_1-\alpha)\otimes g_q(a))\in R(d_{k+1})$. Thus, the graph of $(-1)^p d_p(\varrho_1-\alpha)\otimes g_q$: $R(I_p\otimes d_{q+1}(\varrho_2-\beta)) \to R(I_{p-1}\otimes g_q)$ is contained in $R(d_{k+1})$.

In a similar way, we may prove that the graph of $(-1)^p h_p\otimes d_q(\varrho_2-\beta)$: $R(d_{p+1}(\varrho_1-\alpha)\otimes I_q)\to R(h_p\otimes I_{q-1})$, $p+q=k$, $p=0,\ldots,p_0-1$, is contained in $R(d_{k+1})$.

We denote the following spaces by $S_{p,q}$, $p+q=k$:

(i) for $q=k-p_1,\ldots,k-p_0-1$, $S_{p,q}=R(I_p\otimes k'_q)\cap\mathrm{Ker}(d_p(\varrho_1-\alpha)\otimes I_q)$,

(ii) for $p=p_0$ and $q=k-p_0$, $S_{p,q}=R(I-S)(I-T)$.

Since $k-p_1\le q\le k-p_0$ and $p_0\le p\le p_1$, we may consider the well defined map

$$H_{p,q}\colon X_1\otimes\bigwedge^p L_1\widetilde{\otimes}X_2\otimes\bigwedge^q L_2\to X_1\otimes\bigwedge^{p+1} L_1\widetilde{\otimes}X_2\otimes\bigwedge^q L_2\oplus X_1\otimes\bigwedge^p L_1\widetilde{\otimes}X_2\otimes\bigwedge^{q+1} L_2,$$
$$H_{p,q}=h_p\otimes I_q+k_p\otimes g_q.$$

Moreover, if we define $k_{p_0-1}=0$ and $k'_{k-p_1-1}=0$, then we may define the corresponding maps $H_{p-1,q}$ and $H_{p,q-1}$, and an easy calculation shows that

$$(H_{p-1,q}\oplus H_{p,q-1})d_k+d_{k+1}H_{p,q}=I-k_p\otimes k'_q.$$

Since $S_{p,q}$ is contained in $\mathrm{Ker}(d_k)$,

$$d_{k+1}(H_{p,q}(S_{p,q}))+k_p\otimes k'_q(S_{p,q})=S_{p,q}.$$

Thus, according to Proposition 16, the codimension of $R(d_{k+1})$ in $\mathrm{Ker}(d_k)$ is finite.

The second statement of the theorem may be proved by a similar argument, using the second part of Proposition 11. ∎

As in the last section, we consider two nilpotent systems of operators and prove a variant of Theorem 17 for this case. In particular, in the commuting case we obtain an extension of [14; 3.2].

THEOREM 18. *Let X_1 and X_2 be two complex Banach spaces. Suppose that there is a tensor product $X_1\widetilde{\otimes}X_2$ of X_1 and X_2 with respect to $\langle X_1,X_1'\rangle$ and $\langle X_2,X_2'\rangle$. Let $a=(a_1,\ldots,a_n)$ and $b=(b_1,\ldots,b_m)$ be two tuples of operators, $a_i\in \mathrm{L}(X_1)$, $1\le i\le n$, and $b_j\in\mathrm{L}(X_2)$, $1\le j\le m$, such that the vector subspaces generated by them, $\langle a_i\rangle_{1\le i\le n}$ and $\langle b_j\rangle_{1\le j\le m}$, are nilpotent Lie subalgebras of $\mathrm{L}(X_1)$ and $\mathrm{L}(X_2)$ respectively. Consider the $(n+m)$-tuple of operators defined in $X_1\widetilde{\otimes}X_2$, $c=(a_1\otimes I,\ldots,a_n\otimes I, I\otimes b_1,\ldots,I\otimes b_m)$, where I denotes the identity of X_2 and X_1 respectively. Then*

$$\text{(i)}\quad \bigcup_{p+q=k} \sigma_{\delta,p,\mathrm{e}}(a)\times\sigma_{\delta,q}(b)\cup \bigcup_{p+q=k} \sigma_{\delta,p}(a)\times\sigma_{\delta,q,\mathrm{e}}(b)\subseteq \sigma_{\delta,k,\mathrm{e}}(c)$$
$$\subseteq \mathrm{sp}_{\delta,k,\mathrm{e}}(c)\subseteq \bigcup_{p+q=k} \mathrm{sp}_{\delta,p,\mathrm{e}}(a)\times\mathrm{sp}_{\delta,q}(b)\cup \bigcup_{p+q=k} \mathrm{sp}_{\delta,p}(a)\times\mathrm{sp}_{\delta,q,\mathrm{e}}(b),$$
$$\text{(ii)}\quad \bigcup_{p+q=k} \sigma_{\pi,p,\mathrm{e}}(a)\times\sigma_{\pi,q}(b)\cup \bigcup_{p+q=k} \sigma_{\pi,p}(a)\times\sigma_{\pi,q,\mathrm{e}}(b)\subseteq \sigma_{\pi,k,\mathrm{e}}(c)$$
$$\subseteq \mathrm{sp}_{\pi,k,\mathrm{e}}(c)\subseteq \bigcup_{p+q=k} \mathrm{sp}_{\pi,p,\mathrm{e}}(a)\times\mathrm{sp}_{\pi,q}(b)\cup \bigcup_{p+q=k} \mathrm{sp}_{\pi,p}(a)\times\mathrm{sp}_{\pi,q,\mathrm{e}}(b).$$

Proof. Adapt the argument in Theorem 15. ∎

7. Joint spectra of the multiplication representation

In this section we deal with an operator ideal in the sense of J. Eschmaier (see [14] or below). These operator ideals are naturally a tensor product of two Banach spaces, and since the multiplication representation may be seen as a tensor product representation, we shall extend the results of Sections 5 and 6 to the multiplication representation. We begin with the definition of an operator ideal in the sense of J. Eschmeier.

Definition 7. An *operator ideal* J between Banach spaces X_2 and X_1 is a linear subspace of $\mathrm{L}(X_2,X_1)$ equipped with a space norm α such that

(i) $x_1\otimes x_2'\in J$ and $\alpha(x_1\otimes x_2')=||x_1||\cdot||x_2'||$,
(ii) $SAT\in J$ and $\alpha(SAT)\le ||S||\alpha(A)||T||$,

where $x_1\in X_1$, $x_2'\in X_2'$, $A\in J$, $S\in\mathrm{L}(X_1)$, $T\in\mathrm{L}(X_2)$, and $x_1\otimes x_2'$ is the usual rank one operator $X_2\to X_1$, $x_2\mapsto\langle x_2,x_2'\rangle x_1$.

Examples of this kind of ideals are given in [14; 1].

We recall that such an operator ideal J is naturally a tensor product relative to $\langle X_1,X_1'\rangle$ and $\langle X_2',X_2\rangle$, with the bilinear mappings

$$X_1\times X_2'\to J,\quad (x_1,x_2')\mapsto x_1\otimes x_2',\qquad \mathcal{L}(X_1)\times\mathcal{L}(X_2')\to\mathrm{L}(J),\quad (S,T')\mapsto S\otimes T',$$

where $S\otimes T'(A)=SAT$.

Now, let L_1 and L_2 be two complex solvable finite-dimensional Lie algebras, X_1 and X_2 two complex Banach spaces, and $\varrho_i\colon L_i\to\mathrm{L}(X_i)$, $i=1,2$, two representations of Lie algebras. We consider the Lie algebra L_2^{op} and the adjoint representation $\varrho_2^*\colon L_2^{\mathrm{op}}\to\mathrm{L}(X_2')$. Now, if L is the direct sum of L_1 and L_2^{op}, $L=L_1\times L_2^{\mathrm{op}}$, then the multiplication representation of L in J considered in [21; 3.6] is

$$\widetilde{\varrho}\colon L\to\mathrm{L}(J),\qquad \widetilde{\varrho}(l_1,l_2)(T)=\varrho_1(l_1)T+T\varrho_2(l_2).$$

According to [21; 3.6.10], $\widetilde{\varrho}$ is a representation of L in $\mathrm{L}(J)$, and when J is viewed as a tensor product of X_1 and X_2' relative to $\langle X_1,X_1'\rangle$ and $\langle X_2',X_2\rangle$, $\widetilde{\varrho}$ coincides with the representation

$$\varrho_1\times\varrho_2^*\colon L\to\mathrm{L}(X_1\mathbin{\widetilde{\otimes}}X_2'),\qquad \varrho_1\times\varrho_2^*(l_1,l_2)=\varrho_1(l_1)\otimes I+I\otimes\varrho_2^*(l_2).$$

Moreover, by a similar argument to the one in Proposition 13, it is easy to prove that the complex $(X_1 \otimes \bigwedge L_1, d(\varrho_1)) \widetilde{\otimes} (X_2 \otimes \bigwedge L_2^{\mathrm{op}}, d(\varrho_2^*))$ is well defined, and that it is isomorphic to the complex $((X_1 \widetilde{\otimes} X_2') \otimes \bigwedge L, d(\varrho_1 \times \varrho_2^*))$, which may be identified with the complex $(J \otimes \bigwedge L, d(\widetilde{\varrho}))$ when J is viewed as a tensor product of X_1 and X_2' relative to $\langle X_1, X_1' \rangle$ and $\langle X_2', X_2 \rangle$.

In the following theorems we describe the joint spectra of the representation $\widetilde{\varrho}$.

THEOREM 19. *Let L_1 and L_2 be two complex solvable finite-dimensional Lie algebras, X_1 and X_2 two complex Banach spaces, and $\varrho_i\colon L_i \to \mathrm{L}(X_i)$, $i = 1, 2$, two representations of Lie algebras. Suppose that there is an operator ideal J between X_2 and X_1 in the sense of Definition 7, and represent it as the tensor product of X_1 and X_2' relative to $\langle X_1, X_1' \rangle$ and $\langle X_2', X_2 \rangle$. Consider the multiplication representation $\widetilde{\varrho}\colon L_1 \times L_2^{\mathrm{op}} \to \mathrm{L}(J)$. Then*

(i) $$\bigcup_{p+q=k} \sigma_{\delta,p}(\varrho_1) \times (\sigma_{\pi,m-q}(\varrho_2) - h_2) \subseteq \sigma_{\delta,k}(\widetilde{\varrho}) \subseteq \mathrm{sp}_{\delta,k}(\widetilde{\varrho}) \subseteq \bigcup_{p+q=k} \mathrm{sp}_{\delta,p}(\varrho_1) \times (\mathrm{sp}_{\pi,m-q}(\varrho_2) - h_2),$$

(ii) $$\bigcup_{p+q=k} \sigma_{\pi,p}(\varrho_1) \times (\sigma_{\delta,m-q}(\varrho_2) - h_2) \subseteq \sigma_{\pi,k}(\widetilde{\varrho}) \subseteq \mathrm{sp}_{\pi,k}(\widetilde{\varrho}) \subseteq \bigcup_{p+q=k} \mathrm{sp}_{\pi,p}(\varrho_1) \times (\mathrm{sp}_{\delta,m-q}(\varrho_2) - h_2),$$

where h_2 is the character of L_2 considered in Theorem 4.

In particular, if X_1 and X_2 are Hilbert spaces, the above inclusions are equalities.

Proof. We begin with the first statement.

We consider the complexes $(X_1 \otimes \bigwedge L_1, d(\varrho_1))$ and $(X_2' \otimes \bigwedge L_2^{\mathrm{op}}, d(\varrho_2^*))$. Since the complex $(J \otimes \bigwedge L, d(\widetilde{\varrho}))$ is isomorphic to $(X_1 \otimes \bigwedge L_1, d(\varrho_1)) \widetilde{\otimes} (X_2' \otimes \bigwedge L_2^{\mathrm{op}}, d(\varrho_2^*))$, we work with the latter.

In addition, if we consider the differentiable spaces associated to the Koszul complexes defined by the representations ϱ_1 and ϱ_2^*, $(\mathcal{X}_1, \partial_1)$ and $(\mathcal{X}_2', \partial_2^*)$ respectively, since $\partial_1 \in \mathcal{L}(\mathcal{X}_1)$ and $\partial_2^* \in \mathcal{L}(\mathcal{X}_2')$, we may consider the tensor product $(\mathcal{X}_1, \partial_1) \widetilde{\otimes} (\mathcal{X}_2', \partial_2^*)$ of $(\mathcal{X}_1, \partial_1)$ and $(\mathcal{X}_2', \partial_2^*)$ relative to $\langle \mathcal{X}_1, \mathcal{X}_1' \rangle$ and $\langle \mathcal{X}_2', \mathcal{X}_2 \rangle$, which has the boundary $\widetilde{\partial} = \partial_1 \otimes I + \eta \otimes \partial_2^*$; see [14] or Section 4. Moreover, $(\mathcal{X}_1, \partial_1) \widetilde{\otimes} (\mathcal{X}_2', \partial_2^*)$ is the differentiable space associated to the complex $(X_1 \otimes \bigwedge L_1, d(\varrho_1)) \widetilde{\otimes} (X_2' \otimes \bigwedge L_2^{\mathrm{op}}, d(\varrho_2^*))$; see Section 4 or [14].

Now we consider $\alpha \in \sigma_{\delta,p}(\varrho_1)$ and $\beta \in \sigma_{\pi,m-q}(\varrho_2) - h_2$, $p+q = k$. Then, by the duality property of the Słodkowski joint spectra, [5; 7] and [21; 2.11.4], $\beta \in \sigma_{\delta,q}(\varrho_2^*)$. Now, if we consider the Koszul complexes associated to the representations $\varrho_1 - \alpha\colon L_1 \to \mathrm{L}(X_1)$ and $\varrho_2^* - \beta\colon L_2^{\mathrm{op}} \to \mathrm{L}(X_2')$, the differentiable spaces associated to them, $(\mathcal{X}_1, \partial_1)$ and $(\mathcal{X}_2, \partial_2^*)$ respectively, and the tensor product $(\mathcal{X}_1, \partial_1) \widetilde{\otimes} (\mathcal{X}_2', \partial_2^*)$, then we may apply [14; 2.2], and a similar argument to the one in Theorem 14 shows the leftmost inclusion.

The middle inclusion is clear.

For the rightmost inclusion, we adapt the corresponding argument in Theorem 14 to the present situation.

We consider the complex $(X_2 \otimes \bigwedge L_2, d(\varrho_2))$. By the duality property of the Koszul complex associated to ϱ_2 (see [5; 1] and [21; 2.4.5]), if $\beta \notin \mathrm{sp}_{\pi,m-q}(\varrho_2) - h_2$, then $\beta \notin$

$\mathrm{sp}_{\delta,q}(\varrho_2^*)$. In particular, if $(\alpha,\beta)\notin\bigcup_{p+q=k}\mathrm{sp}_{\delta,p}(\varrho_1)\times(\mathrm{sp}_{\pi,m-q}(\varrho_2)-h_2)$, then $(\alpha,\beta)\notin\bigcup_{p+q=k}\mathrm{sp}_{\delta,p}(\varrho_1)\times\mathrm{sp}_{\delta,q}(\varrho_2^*)$.

In addition, by the duality property of the Koszul complex of the representation ϱ_2 and by elementary properties of the adjoint of an operator, it is easy to prove that if $\beta\notin\mathrm{sp}_{\pi,m-t}(\varrho_2)-h_2$, then there is a homotopy for the complex $(X_2'\otimes L_2^{\mathrm{op}},d(\varrho_2^*-\beta))$, $(g_s)_{0\le s\le t}$, which satisfies the preliminary facts recalled before Proposition 11. Besides, if for each $s=0,\dots,t$ we think about each map g_s as a matrix of operators, then each component of this matrix is an adjoint operator.

Now, according to the properties of the axiomatic tensor product introduced in [14], if there is a tensor product $Y\mathbin{\widetilde{\otimes}}X'$ of a Banach space Y and X' relative to $\langle Y,Y'\rangle$ and $\langle X',X\rangle$ then it is possible to prove similar results to those in Proposition 11. In particular, it is possible to adapt the proof in Theorem 14 to the present case in order to prove the rightmost inclusion.

The second statement may be proved by a similar argument. ∎

Now we describe the essential Słodkowski and essential split joint spectra of the multiplication representation $\widetilde{\varrho}$.

THEOREM 20. *Let L_1 and L_2 be two complex solvable finite-dimensional Lie algebras, X_1 and X_2 two complex Banach spaces, and $\varrho_i\colon L_i\to\mathrm{L}(X)$, $i=1,2$, two representations of Lie algebras. Suppose that there is an operator ideal J between X_2 and X_1 in the sense of Definition* 7, *and represent it as the tensor product of X_1 and X_2' relative to $\langle X_1,X_1'\rangle$ and $\langle X_2',X_2\rangle$. Consider the multiplication representation $\widetilde{\varrho}\colon L_1\times L_2^{\mathrm{op}}\to\mathrm{L}(J)$. Then*

$$\text{(i)}\quad \bigcup_{p+q=k}\sigma_{\delta,p,\mathrm{e}}(\varrho_1)\times(\sigma_{\pi,m-q}(\varrho_2)-h_2)\cup\bigcup_{p+q=k}\sigma_{\delta,p}(\varrho_1)\times(\sigma_{\pi,m-q,\mathrm{e}}(\varrho_2)-h_2)$$
$$\subseteq\sigma_{\delta,k,\mathrm{e}}(\widetilde{\varrho})\subseteq\mathrm{sp}_{\delta,k,\mathrm{e}}(\widetilde{\varrho})$$
$$\subseteq\bigcup_{p+q=k}\mathrm{sp}_{\delta,p,\mathrm{e}}(\varrho_1)\times(\mathrm{sp}_{\pi,m-q}(\varrho_2)-h_2)\cup\bigcup_{p+q=k}\mathrm{sp}_{\delta,p}(\varrho_1)\times(\mathrm{sp}_{\pi,m-q,\mathrm{e}}(\varrho_2)-h_2),$$

$$\text{(ii)}\quad \bigcup_{p+q=k}\sigma_{\pi,p,\mathrm{e}}(\varrho_1)\times(\sigma_{\delta,m-q}(\varrho_2)-h_2)\cup\bigcup_{p+q=k}\sigma_{\pi,p}(\varrho_1)\times(\sigma_{\delta,m-q,\mathrm{e}}(\varrho_2)-h_2)$$
$$\subseteq\sigma_{\pi,k,\mathrm{e}}(\widetilde{\varrho})\subseteq\mathrm{sp}_{\delta,k,\mathrm{e}}(\widetilde{\varrho})$$
$$\subseteq\bigcup_{p+q=k}\mathrm{sp}_{\pi,p,\mathrm{e}}(\varrho_1)\times(\mathrm{sp}_{\delta,m-q}(\varrho_2)-h_2)\cup\bigcup_{p+q=k}\mathrm{sp}_{\pi,p}(\varrho_1)\times(\mathrm{sp}_{\delta,m-q,\mathrm{e}}(\varrho_2)-h_2),$$

where h_2 is the character of L_2 considered in Theorem 4.

In particular, if X_1 and X_2 are Hilbert spaces, the above inclusions are equalities.

Proof. Adapt the proof of Theorem 17. ∎

As in Sections 5 and 6, we consider nilpotent systems of operators, and we obtain variants of Theorems 19 and 20 for this case.

THEOREM 21. *Let X_1 and X_2 be two complex Banach spaces, and $a=(a_1,\dots,a_n)$ and $b=(b_1,\dots,b_m)$ two tuples of operators, $a_i\in\mathrm{L}(X_1)$, $1\le i\le n$, and $b_j\in\mathrm{L}(X_2)$, $1\le j\le m$, such that the vector subspaces generated by them, $\langle a_i\rangle_{1\le i\le n}$ and $\langle b_j\rangle_{1\le j\le m}$, are nilpotent Lie subalgebras of $\mathrm{L}(X_1)$ and $\mathrm{L}(X_2)$ respectively. Consider an operator ideal*

$J \subseteq \mathrm{L}(X_2, X_1)$ in the sense of Definition 7, and the $(n+m)$-tuple of operators defined in $\mathrm{L}(J)$, $c = (L_{a_1}, \ldots, L_{a_n}, R_{b_1}, \ldots, R_{b_m})$, where if $S \in \mathrm{L}(X_1)$ and if $T \in \mathrm{L}(X_2)$, the maps $L_S, R_T\colon J \to J$ are defined by

$$L_S(U) = SU, \qquad R_T(U) = UT.$$

Then

$$\text{(i)} \quad \bigcup_{p+q=k} \sigma_{\delta,p}(a) \times \sigma_{\pi,m-q}(b) \subseteq \sigma_{\delta,k}(c) \subseteq \mathrm{sp}_{\delta,k}(c) \subseteq \bigcup_{p+q=k} \mathrm{sp}_{\delta,p}(a) \times \mathrm{sp}_{\pi,m-q}(b),$$

$$\text{(ii)} \quad \bigcup_{p+q=k} \sigma_{\pi,p}(a) \times \sigma_{\delta,m-q}(b) \subseteq \sigma_{\pi,k}(c) \subseteq \mathrm{sp}_{\pi,k}(c) \subseteq \bigcup_{p+q=k} \mathrm{sp}_{\pi,p}(a) \times \mathrm{sp}_{\delta,m-q}(b).$$

Proof. As in Theorem 15, we consider the Lie algebras $L_1 = \langle a_i \rangle_{1 \le i \le n}$ and $L_2 = \langle b_j \rangle_{1 \le j \le m}$, the representations of the above algebras defined by inclusion, i.e., $\iota_1\colon L_1 \to \mathrm{L}(X_1)$ and $\iota_2\colon L_2 \to \mathrm{L}(X_2)$, and the representation $\iota = \iota_1 \times \iota_2^*\colon L_1 \times L_2^{\mathrm{op}} \to \mathrm{L}(X_1 \mathbin{\widetilde{\otimes}} X_2')$. Then, if J is viewed as a tensor product of X_1 and X_2 relative to $\langle X_1, X_1' \rangle$ and $\langle X_2', X_2 \rangle$, ι coincides with the representation $\varrho\colon L_1 \times L_2^{\mathrm{op}} \to \mathrm{L}(J)$, $\varrho(A, B)(T) = AT + TB$.

Now, the argument in Theorem 15 may be adapted to the present situation using Proposition 10 and Theorem 19 instead of Theorem 14. ∎

THEOREM 22. *Under the assumptions of Theorem 21,*

$$\text{(i)} \quad \bigcup_{p+q=k} \sigma_{\delta,p,\mathrm{e}}(a) \times \sigma_{\pi,m-q}(b) \cup \bigcup_{p+q=k} \sigma_{\delta,p}(a) \times \sigma_{\pi,m-q,\mathrm{e}}(b) \subseteq \sigma_{\delta,k,\mathrm{e}}(c)$$
$$\subseteq \mathrm{sp}_{\delta,k,\mathrm{e}}(c) \subseteq \bigcup_{p+q=k} \mathrm{sp}_{\delta,p,\mathrm{e}}(a) \times \mathrm{sp}_{\pi,m-q}(b) \cup \bigcup_{p+q=k} \mathrm{sp}_{\delta,p}(a) \times \mathrm{sp}_{\pi,m-q,\mathrm{e}}(b),$$

$$\text{(ii)} \quad \bigcup_{p+q=k} \sigma_{\pi,p,\mathrm{e}}(a) \times \sigma_{\delta,m-q}(b) \cup \bigcup_{p+q=k} \sigma_{\pi,p}(a) \times \sigma_{\delta,m-q,\mathrm{e}}(b) \subseteq \sigma_{\pi,k,\mathrm{e}}(c)$$
$$\subseteq \mathrm{sp}_{\pi,k,\mathrm{e}}(c) \subseteq \bigcup_{p+q=k} \mathrm{sp}_{\pi,p,\mathrm{e}}(a) \times \mathrm{sp}_{\delta,m-q}(b) \cup \bigcup_{p+q=k} \mathrm{sp}_{\pi,p}(a) \times \mathrm{sp}_{\delta,m-q,\mathrm{e}}(b).$$

Proof. Adapt the argument in Theorem 18, using Proposition 10 and Theorem 20 instead of Theorem 17. ∎

We observe that similar remarks to the ones in Sections 5 and 6 may be made for the theorems in this section. In particular, Theorems 19 and 21 are extensions of [21; 3.6.10] and [21; 3.7.4] respectively for the tensor product introduced in [14]. In addition, Theorems 20 and 22 extend [14; 3.1] and [14; 3.2] respectively for the essential joint spectra.

8. Applications

In this section we apply the results that we obtained in Sections 5–7 to particular representations of nilpotent Lie algebras.

We consider two complex Banach spaces X_1 and X_2, a complex nilpotent finite-dimensional Lie algebra L, and two representations of L, $\varrho_1\colon L \to \mathrm{L}(X_1)$ and $\varrho_2\colon L \to \mathrm{L}(X_2)$.

We suppose that there is a tensor product $X_1 \widetilde{\otimes} X_2$ of X_1 and X_2 relative to $\langle X_1, X_1' \rangle$ and $\langle X_2, X_2' \rangle$. Thus, we may consider the tensor product representation

$$\varrho = \varrho_1 \times \varrho_2 \colon L \times L \to \mathrm{L}(X_1 \widetilde{\otimes} X_2), \qquad \varrho = \varrho_1 \otimes I + I \otimes \varrho_2.$$

Now we consider the diagonal map

$$\Delta \colon L \to L \times L, \qquad \Delta(l) = (l, l),$$

and we identify L with $\Delta(L)$. In addition, we may consider the representation

$$\theta = \varrho \circ \Delta \colon L \to \mathrm{L}(X_1 \widetilde{\otimes} X_2), \qquad \theta(l) = \varrho_1(l) \otimes I + I \otimes \varrho_2(l).$$

In the following theorem we describe the Słodkowski, split, essential Słodkowski and essential split joint spectra of the representation θ.

THEOREM 23. *Let L be a complex nilpotent finite-dimensional Lie algebra, X_1 and X_2 two complex Banach spaces, and $\varrho_i \colon L \to \mathrm{L}(X_i)$, $i = 1, 2$, two representations of L. Suppose that there is a tensor product $X_1 \widetilde{\otimes} X_2$ of X_1 and X_2 relative to $\langle X_1, X_1' \rangle$ and $\langle X_2, X_2' \rangle$. Consider the representation $\theta \colon L \to \mathrm{L}(X_1 \widetilde{\otimes} X_2)$. Then*

(i) $$\bigcup_{p+q=k} (\sigma_{\delta,p}(\varrho_1) + \sigma_{\delta,q}(\varrho_2)) \subseteq \sigma_{\delta,k}(\theta) \subseteq \mathrm{sp}_{\delta,k}(\theta) \subseteq \bigcup_{p+q=k} (\mathrm{sp}_{\delta,p}(\varrho_1) + \mathrm{sp}_{\delta,q}(\varrho_2)),$$

(ii) $$\bigcup_{p+q=k} (\sigma_{\pi,p}(\varrho_1) + \sigma_{\pi,q}(\varrho_2)) \subseteq \sigma_{\pi,k}(\theta) \subseteq \mathrm{sp}_{\pi,k}(\theta) \subseteq \bigcup_{p+q=k} (\mathrm{sp}_{\pi,p}(\varrho_1) + \mathrm{sp}_{\pi,q}(\varrho_2)),$$

(iii) $$\bigcup_{p+q=k} (\sigma_{\delta,p,\mathrm{e}}(\varrho_1) + \sigma_{\delta,q}(\varrho_2)) \cup \bigcup_{p+q=k} (\sigma_{\delta,p}(\varrho_1) + \sigma_{\delta,q,\mathrm{e}}(\varrho_2)) \subseteq \sigma_{\delta,k,\mathrm{e}}(\theta)$$
$$\subseteq \mathrm{sp}_{\delta,k,\mathrm{e}}(\theta) \subseteq \bigcup_{p+q=k} (\mathrm{sp}_{\delta,p,\mathrm{e}}(\varrho_1) + \mathrm{sp}_{\delta,q}(\varrho_2)) \cup \bigcup_{p+q=k} (\mathrm{sp}_{\delta,p}(\varrho_1) + \mathrm{sp}_{\delta,q,\mathrm{e}}(\varrho_2)),$$

(iv) $$\bigcup_{p+q=k} (\sigma_{\pi,p,\mathrm{e}}(\varrho_1) + \sigma_{\pi,q}(\varrho_2)) \cup \bigcup_{p+q=k} (\sigma_{\pi,p}(\varrho_1) + \sigma_{\pi,q,\mathrm{e}}(\varrho_2)) \subseteq \sigma_{\pi,k,\mathrm{e}}(\theta)$$
$$\subseteq \mathrm{sp}_{\pi,k,\mathrm{e}}(\theta) \subseteq \bigcup_{p+q=k} (\mathrm{sp}_{\pi,p,\mathrm{e}}(\varrho_1) + \mathrm{sp}_{\pi,q}(\varrho_2)) \bigcup \bigcup_{p+q=k} (\mathrm{sp}_{\pi,p}(\varrho_1) + \mathrm{sp}_{\pi,q,\mathrm{e}}(\varrho_2)).$$

In particular, if X_1 and X_2 are Hilbert spaces, the above inclusions are equalities.

Proof. In order to prove the first statement we recall that according to Theorem 14 we have

$$\bigcup_{p+q=k} \sigma_{\delta,p}(\varrho_1) \times \sigma_{\delta,q}(\varrho_2) \subseteq \sigma_{\delta,k}(\varrho) \subseteq \mathrm{sp}_{\delta,k}(\varrho) \subseteq \bigcup_{p+q=k} \mathrm{sp}_{\delta,p}(\varrho_1) \times \mathrm{sp}_{\delta,q}(\varrho_2).$$

Now, the map $\Delta \colon L \to L \times L$ is an identification between L and $\Delta(L)$, which is a subalgebra of the nilpotent Lie algebra $L \times L$. Then, if we consider the representation $\varrho|\Delta(L) \colon \Delta(L) \to \mathrm{L}(X_1 \widetilde{\otimes} X_2)$, since $\theta = \varrho|\Delta(L) \circ \Delta$, according to Proposition 10 we have

$$\sigma_{\delta,k}(\theta) = \sigma_{\delta,k}(\varrho|\Delta(L)) \circ \Delta = \{\alpha \circ \Delta : \alpha \in \sigma_{\delta,k}(\varrho|\Delta(L))\},$$

and

$$\mathrm{sp}_{\delta,k}(\theta) = \mathrm{sp}_{\delta,k}(\varrho|\Delta(L)) \circ \Delta = \{\alpha \circ \Delta : \alpha \in \mathrm{sp}_{\delta,k}(\varrho|\Delta(L))\}.$$

In addition, since $\Delta(L)$ is a subalgebra of the nilpotent Lie algebra $L \times L$, by the projection property for the Słodkowski and split joint spectra, [21; 2.11.5], [21; 3.1.5] and

Theorem 1, we have

$$\pi(\sigma_{\delta,k}(\varrho)) = \sigma_{\delta,k}(\varrho|\Delta(L)), \qquad \pi(\mathrm{sp}_{\delta,k}(\varrho)) = \mathrm{sp}_{\delta,k}(\varrho|\Delta(L)),$$

where $\pi\colon (L\times L)^* \to \Delta(L)^*$ denotes the restrictiton map.

Now, it is easy to prove that

$$\pi\Big(\bigcup_{p+q=k} \sigma_{\delta,p}(\varrho_1)\times\sigma_{\delta,q}(\varrho_2)\Big)\circ\Delta = \bigcup_{p+q=k}(\sigma_{\delta,p}(\varrho_1)+\sigma_{\delta,q}(\varrho_2)),$$

and that

$$\pi\Big(\bigcup_{p+q=k} \mathrm{sp}_{\delta,p}(\varrho_1)\times\mathrm{sp}_{\delta,q}(\varrho_2)\Big)\circ\Delta = \bigcup_{p+q=k}(\mathrm{sp}_{\delta,p}(\varrho_1)+\mathrm{sp}_{\delta,q}(\varrho_2)).$$

Thus, we proved the first part of the theorem.

The other statements may be proved by similar arguments, using for (ii) Theorem 14 and the projection property for the Słodkowski and split joint spectra, and for (iii) and (iv) Theorem 17 and the projection property for the essential Słodkowski and essential split joint spectra, Theorems 2, 5 and 9. ∎

Now we consider two complex Banach spaces X_1 and X_2, an operator ideal between X_2 and X_1 in the sense of [14], a complex nilpotent Lie algebra L, two representations of L, $\varrho_1\colon L\to \mathrm{L}(X_1)$ and $\varrho_2\colon L\to\mathrm{L}(X_2)$, and the representation of L^{op}, $\nu=-\varrho_2\colon L^{\mathrm{op}}\to \mathrm{L}(X_2)$. As in Section 7, we may consider the multiplication representation

$$\widetilde{\varrho}\colon L\times L\to \mathrm{L}(J), \qquad \widetilde{\varrho}(l_1,l_2)(T)=\varrho_1(l_1)T-T\varrho_2(l_2).$$

As above, we may consider the representation

$$\widetilde{\theta}=\widetilde{\varrho}\circ\Delta\colon L\to\mathrm{L}(J).$$

In the following theorem we describe the Słodkowski, split, essential Słodkowski and essential split joint spectra of the representation $\widetilde{\theta}\colon L\to\mathrm{L}(J)$.

THEOREM 24. *Let L be a complex nilpotent finite-dimensional Lie algebra, X_1 and X_2 two complex Banach spaces, and $\varrho_i\colon L\to\mathrm{L}(X_i)$, $i=1,2$, two representations of the Lie algebra L. Suppose that there is an operator ideal J between X_2 and X_1 in the sense of Definition 7. Consider the representation $\widetilde{\theta}\colon L\to\mathrm{L}(J)$. Then*

$$\text{(i)}\quad \bigcup_{p+q=k}(\sigma_{\delta,p}(\varrho_1)-\sigma_{\pi,m-q}(\varrho_2)+h_2)\subseteq\sigma_{\delta,k}(\widetilde{\theta})\subseteq\mathrm{sp}_{\delta,k}(\widetilde{\theta})$$
$$\subseteq\bigcup_{p+q=k}(\mathrm{sp}_{\delta,p}(\varrho_1)-\mathrm{sp}_{\pi,m-q}(\varrho_2)+h_2),$$

$$\text{(ii)}\quad \bigcup_{p+q=k}(\sigma_{\pi,p}(\varrho_1)-\sigma_{\delta,m-q}(\varrho_2)+h_2)\subseteq\sigma_{\pi,k}(\widetilde{\theta})\subseteq\mathrm{sp}_{\pi,k}(\widetilde{\theta})$$
$$\subseteq\bigcup_{p+q=k}(\mathrm{sp}_{\pi,p}(\varrho_1)-\mathrm{sp}_{\delta,m-q}(\varrho_2)+h_2),$$

(iii) $$\bigcup_{p+q=k}(\sigma_{\delta,p,\mathrm{e}}(\varrho_1)-\sigma_{\pi,m-q}(\varrho_2)+h_2)\cup\bigcup_{p+q=k}(\sigma_{\delta,p}(\varrho_1)-\sigma_{\pi,m-q,\mathrm{e}}(\varrho_2)+h_2)$$
$$\subseteq\sigma_{\delta,k,\mathrm{e}}(\widetilde{\theta})\subseteq\mathrm{sp}_{\delta,k,\mathrm{e}}(\widetilde{\theta})$$
$$\subseteq\bigcup_{p+q=k}(\mathrm{sp}_{\delta,p,\mathrm{e}}(\varrho_1)-\mathrm{sp}_{\pi,m-q}(\varrho_2)+h_2)\cup\bigcup_{p+q=k}(\mathrm{sp}_{\delta,p}(\varrho_1)-\mathrm{sp}_{\pi,m-q,\mathrm{e}}(\varrho_2)+h_2),$$

(iv) $$\bigcup_{p+q=k}(\sigma_{\pi,p,\mathrm{e}}(\varrho_1)-\sigma_{\delta,m-q}(\varrho_2)+h_2)\cup\bigcup_{p+q=k}(\sigma_{\pi,p}(\varrho_1)-\sigma_{\delta,m-q,\mathrm{e}}(\varrho_2)+h_2)$$
$$\subseteq\sigma_{\pi,k,\mathrm{e}}(\widetilde{\theta})\subseteq\mathrm{sp}_{\delta,k,\mathrm{e}}(\widetilde{\theta})$$
$$\subseteq\bigcup_{p+q=k}(\mathrm{sp}_{\pi,p,\mathrm{e}}(\varrho_1)-\mathrm{sp}_{\delta,m-q}(\varrho_2)+h_2)\cup\bigcup_{p+q=k}(\mathrm{sp}_{\pi,p}(\varrho_1)-\mathrm{sp}_{\delta,m-q,\mathrm{e}}(\varrho_2)+h_2),$$

where h_2 is the character of L_2 considered in Theorem 4.

In particular, if X_1 and X_2 are Hilbert spaces, the above inclusions are equalities.

Proof. The theorem may be proved by a similar argument to the one in Theorem 23, using Theorems 19 and 20 instead of Theorems 14 and 17. ∎

Finally, Theorems 23 and 24 provide an extension of two of the main results in [21; 3.8] for the tensor product introduced in [14].

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