1. Introduction

C- and C^*-embedded subspaces play a central role in general topology, and the corresponding frame quotients, here termed C- and C^*-quotients, are no less important to pointfree topology. Indeed, the characterizations of these quotients draw together many of the central strands of frame theory. But what the pointfree formulation adds to the classical theory is a remarkable combination of elegance of statement, simplicity of proof, and increase of extent.

The central results are the frame characterizations of C- and C^*-quotients, Theorems 7.1.1 and 7.2.7. Sections 2–6 are ground-clearing and machinery-building for these results, and Section 8 consists of applications of them. Crucial to both theorems is the preservation of certain features of the cozero parts of the underlying frames. Covering properties also play an important role, and such properties naturally tie in closely with uniformities. In particular, the characterization of C-quotients leads to the study of what might be called the geometry of cozero covers. Because a good deal of machinery is required to tie together the long list of concepts upon which these embeddings impinge, we mention here only one particularly novel and interesting completeness feature of CL which emerged; we call it $*$-completeness.

In order to close the circles of ideas we found it necessary to employ the algebraic properties of CL as an archimedean f-ring. Roughly sixty percent of the arguments hinge on the algebra, even though a much smaller percentage of the results themselves mention it. This, of course, is squarely in the tradition of classical general topology, but it is a heretofore underused technique for the study of frames. And this may be the central thrust of our work: archimedean f-rings are just as important for understanding frames as they are for understanding general topology.

Finally, it is a pleasure to record our gratitude to a generous and erudite referee, whose suggestions significantly improved the exposition of these results.

2. Background

In this section we briefly record the definitions and standard results necessary to read this article. Although all the results are well known, we include a few proofs to spare the uninitiated reader the trouble of recovering information which is sometimes difficult to dig out of the literature.

2.1. Frames. A *frame* is a complete lattice L with top element $\top$ and bottom element $\bot$ satisfying the *frame distributivity law*

$$a \wedge \bigvee_I b_i = \bigvee_I (a \wedge b_i)$$

for all $a, b_i \in L$. A *frame map* is a lattice homomorphism $m : L \to M$ which preserves top, bottom, binary meets and arbitrary joins. We work in the category of frames with frame maps; the standard reference is [23]. The prototypical frame is the topology $\mathcal{O}X$ of a space X, and the most frequently encountered space is $\mathbb{R}$, the space of real numbers. Because $\mathcal{O}\mathbb{R}$ plays a special role in what follows, we reserve the letters U, V, and W, sometimes decorated with primes or subscripts, for its elements, and we write the join and meet operations as $U \cup V$ and $U \cap V$, respectively. Furthermore, we abbreviate oft-mentioned members of $\mathcal{O}\mathbb{R}$ as follows:

$$\mathbb{R}^+ \equiv (0, \infty), \quad \mathbb{R}^- \equiv (-\infty, 0), \quad \mathbb{R}_0 \equiv \mathbb{R}^+ \cup \mathbb{R}^- = \mathbb{R} \setminus \{0\}, \quad \mathbb{R}_1 \equiv \mathbb{R} \setminus \{1\}.$$

Any continuous function $f : X \to Y$ between spaces gives rise to a frame map $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ with $\mathcal{O}f(U) = f^{-1}(U)$ for U an open of Y, and this results in a contravariant functor $\mathcal{O}$ from the category **Top** of topological spaces and continuous functions to **Frm**. There is another contravariant functor in the opposite direction, the *spectrum functor* Σ : **Frm** $\to$ **Top**, and these two functors are adjoint on the right.

The topological notion of subspace is captured by the concept of the quotient frame [23]. A *quotient* of a frame L is the codomain M of a frame surjection $m : L \twoheadrightarrow M$, and we refer to m as the *quotient map*. The quotients of L are naturally preordered: quotient M is *larger* than quotient N if the quotient map $n : L \twoheadrightarrow N$ factors through the quotient map $m : L \twoheadrightarrow M$, i.e., if $km = n$ for some frame surjection $k : M \twoheadrightarrow N$. This is equivalent to the condition that $m(a_1) = m(a_2)$ implies $n(a_1) = n(a_2)$ for all $a_i \in L$. The equivalence relation induced by this preorder identifies two quotients M and N if each is larger than the other, and this happens if and only if there is a frame isomorphism $k : M \to N$ such that $km = n$. The collection of all equivalence classes of quotients of a given frame L forms a frame in this order [23], a fact we use without comment in what follows when we discuss the infima of quotients. We suppress the distinction between quotients and their equivalence classes; the top element of the frame of quotients is L itself and the bottom is the trivial one-element frame.

With each element $b \in L$ are associated two special quotients. The *open quotient* of b is the frame $\downarrow b \equiv \{c \in L : c \leq b\}$ with quotient map $a \mapsto a \wedge b$, where the former is regarded as a frame in the order inherited from L with $\top_{\downarrow b} = b$ and $\bot_{\downarrow b} = \bot_L$. The *closed quotient* of b is the frame $\uparrow b \equiv \{c \in L : c \geq b\}$ with quotient map $a \mapsto a \vee b$, where the former is regarded as a frame in the order inherited from L with $\top_{\uparrow b} = \top_L$ and $\bot_{\uparrow b} = b$.

The *pseudocomplement* of a frame element $a \in L$ is

$$a^* \equiv \bigvee \{b : a \wedge b = \bot\}.$$

An element $a \in L$ is *dense* if $a^* = \bot$. Thus a dense frame element of $\mathcal{O}X$ is a dense open subset of X. A frame map $m : L \to M$ is *dense* if $m(a) = \bot$ implies $a = \bot$. Thus for a continuous function $f : Y \to X$, the corresponding frame map $f^{-1} : \mathcal{O}X \to \mathcal{O}Y$ is dense if and only if $f(Y)$ is a dense subspace of X. Note that the open quotient map of b is dense if and only if b is a dense element, and the closed quotient map of b is dense if and only if $b = \bot$. A frame map $m : L \to M$ is *codense* if $m(a) = \top$ implies $a = \top$.

LEMMA 2.1.1. *A frame map m factors through the open quotient map $a \mapsto a \wedge b$ if and only if $m(b) = \top$. And m factors through the closed quotient map $a \mapsto a \vee b$ if and only if $m(b) = \bot$.*

$$\begin{tikzcd} L \arrow[r, "m"] \arrow[d, "q"'] & M \\ \downarrow b \arrow[ur, "g"'] & \end{tikzcd}$$

Proof. Suppose that $m = gq$ for some g, where q names the map $a \mapsto a \wedge b$. Then

$$m(b) = gq(b) = g(\top_{\downarrow b}) = \top_L.$$

Conversely, if $m : L \to M$ satisfies $m(b) = \top$ then define $g : \downarrow b \to M$ by the rule $g(a) \equiv f(a)$ for $a \in \downarrow b$. Clearly g preserves binary infima, arbitrary suprema, and $\bot$; it preserves $\top$ because it takes b to $\top$. And for all $a \in L$ we have

$$m(a) = m(a \wedge \top) = m(a) \wedge \top = m(a) \wedge m(b) = f(a \wedge b) = gq(a).$$

The argument for closed quotient maps is similar. ■

COROLLARY 2.1.2. *Let $m : L \to M$ be a frame surjection and let $b \in L$. Then the quotient M is smaller than the open quotient $\downarrow b$ if and only if $m(b) = \top$, and M is smaller than the closed quotient $\uparrow b$ if and only if $m(b) = \bot$. Therefore there is a unique smallest closed quotient $\uparrow b$ larger than M, and it is given by $b \equiv \bigvee_{m(a)=\bot} a$. We refer to this closed quotient as the* closure *of m.*

For elements a and b of a frame L we say that a is *rather below* b, and write $a \prec b$, if there is a *separating element* c such that $a \wedge c = \bot$ and $b \vee c = \top$, i.e., if $a^* \vee b = \top$. A frame L is *regular* if $a = \bigvee_{b \prec a} b$ for all $a \in L$.

PROPOSITION 2.1.3. *A quotient of a regular frame is the infimum of the open quotients above it.*

Proof. Let $m : L \twoheadrightarrow M$ be a quotient map out of the regular frame L, and let F be the filter of elements $b \in L$ such that $\downarrow b$ is larger than M, i.e.,

$$F = \{b \in L : m(b) = \top\}.$$

Consider a quotient N of L which is smaller than $\downarrow b$ for all $b \in F$, and let $n : L \twoheadrightarrow N$ be the quotient map. This means that $n(b) = \top$ for all $b \in F$. Suppose $m(a_1) = m(a_2)$; consider $x_1 \prec a_1$ and set $b \equiv x_1^* \vee a_2$. Note that $b \in F$ because

$$\top = m(\top) = m(x_1^* \vee a_1) = m(x_1^*) \vee m(a_1) = m(x_1^*) \vee m(a_2) = m(x_1^* \vee a_2) = m(b).$$

It follows that $g(b) = \top$. Therefore

$$g(x_1) = g(x_1) \wedge \top = g(x_1) \wedge g(b) = g(x_1 \wedge b) = g(x_1 \wedge (x_1^* \vee a_2)) = g(x_1 \wedge a_2) \leq g(a_2),$$

so that $g(a_1) = \bigvee_{x_1 \prec a_1} g(x_1) \leq g(a_2)$. By interchanging a_1 with a_2 we get $g(a_2) \leq g(a_1)$, i.e., $g(a_1) = g(a_2)$, from which we conclude that N is smaller than M. ∎

In a frame L, a *scale from a_0 to a_1* is a subset

$$\{a_q : q \in [0,1]_{\mathbb{Q}}\} \subseteq L,$$

indexed by the rational interval $[0,1]_{\mathbb{Q}}$, such that $a_p \prec a_q$ whenever $p < q$ in $[0,1]_{\mathbb{Q}}$. We say of elements $a, b \in L$ that a is *completely below* b, and write $a \prec\prec b$, if there is a scale from a to b. A frame L is *completely regular* if $a = \bigvee_{b \prec\prec a} b$ for all $a \in L$. A space is completely regular if every open set is the union of the cozero sets it contains. Thus we see from the equivalence of (1) and (3) in Proposition 2.1.4 that a space X is completely regular if and only if $\mathcal{O}X$ is a completely regular frame.

Of central importance for our purposes are the cozero elements of a frame. A *cozero element* of L, or simply a *cozero* of L, is an element of the form $h(\mathbb{R}_0)$ for some frame map $h : \mathcal{O}\mathbb{R} \to L$, and for any such map we refer to $h(\mathbb{R}_0)$ as $\operatorname{coz} h$. The collection of all cozeros of L, denoted by $\operatorname{Coz} L$, forms a sub-σ-frame of L, i.e., a sublattice which is closed under countable joins and satisfies the frame distributivity law for countable joins. Subsection 3.2 develops the basic facts about cozero elements which will be needed in what follows.

Proposition 2.1.4. *The following are equivalent for $a, b \in L$:*

(1) *$a \prec\prec b$.*

(2) *There is a frame map $f : \mathcal{O}\mathbb{R} \to L$ such that $f(\mathbb{R}_0) \wedge a = \bot$ and $f(\mathbb{R}_1) \leq b$. Such a map can be chosen to satisfy $0 \leq f \leq 1$ when it exists.*

(3) *There is some $c \in \operatorname{Coz} L$ such that $a \prec c \prec b$.*

Proof. We outline the proof of the equivalence of (1) and (2); the details may be found in [23, IV, 1.4]. If f satisfies (2) then we may form a scale $\{a_q : q \in [0,1]_{\mathbb{Q}}\}$ from a to b by setting $a_0 = a$, $a_1 = b$, and $a_q = f(-\infty, q)$ for $q \in (0,1)_{\mathbb{Q}}$. That is because $f(p, \infty)$ is a separating element witnessing $a_p \prec a_q$ for $p < q$ in $(0,1)_{\mathbb{Q}}$. On the other hand, if $\{a_q : q \in [0,1]_{\mathbb{Q}}\}$ is a scale from a to b then, after extending the index set to $\mathbb{Q}$ by setting $a_q = \bot$ for $q < 0$ and $a_q = \top$ for $q > 1$, the formula

$$f(U) \equiv \bigvee\{a_j \wedge a_i^* : i < j,\ (i,j) \prec U\}, \qquad U \in \mathcal{O}\mathbb{R},$$

produces a frame map satisfying (2). The equivalence of (1) and (3) is due to Banaschewski [6]. ∎

Proposition 2.1.5. *A quotient of a (completely) regular frame is itself (completely) regular.*

Proof. If M is a quotient of L with quotient map m and if $a = \bigvee\{b : b \prec a\}$ then $m(a) = \bigvee\{m(b) : b \prec a\}$ in M, and $b \prec a$ implies $m(b) \prec m(a)$. The reason for the latter is that if $b \prec a$ then $b^* \vee a = \top$, hence $m(b^*) \vee m(a) = \top$ and $m(b^*) \wedge m(b) = m(b^* \wedge b) = m(\bot) = \bot$, with the consequence that $m(b)^* \vee m(a) = \top$. The proof for complete regularity is similar. ∎

Frame maps preserve cozero elements: for any frame map $m : L \to M$ and any $a \in \operatorname{Coz} L$, $m(a)$ lies in $\operatorname{Coz} M$. Attributes of the induced cozero map are ascribed to a frame map with the prefix coz. Thus m is said to be *coz-codense* if $m(a) = \top$ implies $a = \top$ for all $a \in \operatorname{Coz} L$, m is said to be *coz-onto* if for every $b \in \operatorname{Coz} M$ there is some $a \in \operatorname{Coz} L$ such that $m(a) = b$, and m is said to be *coz-iso* if for every $b \in \operatorname{Coz} M$ there is a unique $a \in \operatorname{Coz} L$ such that $m(a) = b$. Coz-ontoness is the frame counterpart of the important notion of z-embedding. A subspace Y of a space X is z-embedded if for every zero set Z of Y there is some zero set W of X such that $W \cap Y = Z$, and this will be true if and only if the frame map $f^{-1} : \mathcal{O}X \to \mathcal{O}Y$ of the subspace embedding $f : Y \to X$ is coz-codense.

A frame L is said to be *compact*, respectively *Lindelöf*, if every cover has a finite, respectively countable, subcover. The compact regular frames form a coreflective subcategory, **KRegFrm**, of the completely regular frames, **CRegFrm**, and we will denote this by $\beta L \twoheadrightarrow L$. Thus βL is the *Stone–Čech compactification* of the frame L. There are various descriptions for βL; we choose to consider βL as the frame of all completely regular ideals of L, where an ideal I is *completely regular* provided that for every $x \in I$ there exists $y \in I$ with $x \prec\!\prec y$. Note that βL is closed under arbitrary joins (of ideals) and finite meets (intersections). Then the coreflection map itself is the join operation $I \mapsto \bigvee I$, and it has as right adjoint the map k given by the rule $k(x) \equiv \{a \in L : a \prec\!\prec x\}$.

βL is determined by the completely below relation. In fact, it is well known that any compactification determines, and is determined by, a strong inclusion (proximity relation) [6]. The completely below relation is the finest such relation. Specifically, if $h : K \twoheadrightarrow L$ is a compactification then the associated strong inclusion is defined by $x \lhd y$ if and only if there exist $a, b \in K$ with $a \prec b$ and $h(a) = x$, $h(b) = y$. Conversely, if $\lhd$ is a strong inclusion on L then the frame of all strongly regular ideals forms a compactification, and the right adjoint to the join map is given by $r(x) = \{a \in L : a \lhd x\}$. Therefore to show a given compactification is the Stone–Čech compactification it suffices to show that the associated strong inclusion is precisely the completely below relation.

It is also well known that the regular Lindelöf frames form a coreflective subcategory, **RegLindFrm**, of **CRegFrm**. We think of the "Lindelöfication" of a completely regular frame L as given by $\mathcal{H} \operatorname{Coz} L$, where $\mathcal{H}$ is the functor that takes a σ-frame to its frame of σ-ideals (that is, ideals closed under countable join), and the dense map $\varepsilon_L : \mathcal{H} \operatorname{Coz} L \twoheadrightarrow L$ is given by join. Its right adjoint is defined by $\downarrow$ on the cozero part of L, i.e., for $c \in L$,

$$(\varepsilon_L)_*(c) = {\downarrow_{\operatorname{Coz} L}}(c) = \{a \in \operatorname{Coz} L : a \leq c\}.$$

2.2. Uniform frames. A *cover* of a frame L is a subset $A \subseteq L$ with $\bigvee A = \top$. For covers A, B of a frame L, A is called a *refinement* of B if $A \subseteq {\downarrow} B$, i.e., if for each $a \in A$ there exists $b \in B$ with $a \leq b$. For any $x \in L$, the *star* of x relative to a cover A is the element $Ax = \bigvee\{a \in A : a \wedge x \neq \bot\}$. If $\mathcal{A}$ is a collection of covers, then $x \lhd_{\mathcal{A}} y$ in L if there exists $A \in \mathcal{A}$ with $Ax \leq y$. It is immediate that $x \lhd_{\mathcal{A}} y$ implies $x \prec y$. A *star refinement* of a cover B of L is a cover A of L such that the cover $\{Ax : x \in A\}$ refines B, written as $A \leq^* B$. A cover A is said to be *normal* whenever there exists a sequence

$(A_n)_{n\in\mathbb{N}}$ of covers such that $A = A_0$ and $A_{n+1} \leq^* A_n$ for all $n \in \mathbb{N}$. A *uniformity* on L is a filter μ of covers such that each $A \in \mu$ is star refined by some $B \in \mu$, and which satisfies the compatibility condition that each $a \in L$ is the join of all $x \in L$ such that $Ax \leq a$ for some $A \in \mu$. When $Ax \leq a$ for some $A \in \mu$, then x is said to be *uniformly below* a, written $x \lhd a$. A *uniform frame* is a frame L together with a specified uniformity μ and is denoted by (L, μ).

A *uniform map* is a frame map which preserves uniform covers. The resulting category will be denoted by **UniFrm**. It is easy to verify that the uniformly below relation implies the rather below relation, and in fact, the completely below relation. Thus every uniform frame is completely regular. Moreover, every completely regular frame admits a uniformity. In fact, every completely regular frame L admits a finest uniformity which consists of all the normal covers. This uniformity will be called the *fine uniformity* of L and will be denoted by α_L.

A uniform map $h : (L, \mu) \twoheadrightarrow (M, \nu)$ is a *surjection*, or (uniform) quotient map, if it is onto, and ν is generated by the image covers $h[A]$, $A \in \mu$. A uniform frame (L, μ) is *precompact* if μ has a basis of finite covers. For any uniform frame (L, μ), the uniformity generated by all finite uniform covers, denoted by $p\mu$, is compatible with L and gives the coreflection to the subcategory of precompact uniform frames. This uniformity may also be described as being generated by uniform maps from $\mathcal{O}[0, 1]$ to (L, μ) and thus may be thought of as $c^*\mu$ (see [28]). In our context, it is clearly the latter description that will be of interest. In particular, $c^*\alpha_L$ will be referred to as the *Stone–Čech uniformity* on L. (So called because $c^*\alpha_L$ is clearly the uniformity generated by the image of the unique uniformity on βL.) In an analogous way using countable covers, we may define a *separable* uniform frame and the associated coreflection is denoted by $e\mu$. In particular we are interested in the separable coreflection of the fine uniformity, $e\alpha_L$, called the *Shirota uniformity* [29], which may be thought of as the uniformity generated by all countable covers of cozero elements. Also we need to consider the uniformity $c\mu$ generated by all uniform maps from $\mathcal{O}\mathbb{R}$ to (L, μ). We call $c\alpha_L$ the *real uniformity* on L. It is obvious that this uniformity is always coarser than the Shirota uniformity.

2.3. Archimedean f-rings. The major tool employed in our investigation is the adjunction between frames and certain kinds of lattice-ordered rings. A *lattice-ordered ring* (*ℓ-ring*) is a commutative ring G with identity 1 whose underlying set is endowed with a lattice ordering such that $f, g \geq 0$ implies $f + g \geq 0$ and $fg \geq 0$ for all $f, g, h \in G$. In any ℓ-ring G, the *positive part*, *negative part*, *and absolute value of* $g \in G$ are

$$g^+ \equiv g \vee 0, \qquad g^- \equiv (-g) \vee 0, \qquad |g| \equiv g \vee (-g),$$

and these satisfy

$$g = g^+ - g^-, \qquad |g| = g^+ + g^- = g^+ \vee g^-, \qquad g^+ \wedge g^- = 0,$$
$$|f + g| \leq |f| + |g|, \qquad |fg| \leq |f||g|.$$

The *positive* and *negative cones* of G are

$$G^+ \equiv \{g \in G : g \geq 0\}, \qquad G^- \equiv \{g \in G : g \leq 0\}.$$

The unique extraction of roots is a property enjoyed by any ℓ-ring G, i.e., $nf = ng$ implies $f = g$ for all $f, g \in G$ and $n \in \mathbb{N}$. That means that the notation $\frac{p}{q}g$, standing for that element f which satisfies $qf = pg$ if such exists in G, makes sense for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. We term G *divisible* if $\frac{p}{q}g$ exists for all $g \in G$, $p \in \mathbb{Z}$, and $q \in \mathbb{N}$. An *ℓ-ring homomorphism* is a mapping between ℓ-rings which is a ring and lattice homomorphism.

An *f-ring* is an ℓ-ring G which satisfies

$$(f \wedge g)h = fh \wedge gh$$

for all $f, g \in G$ and $h \in G^+$. Subdirect products of totally ordered rings are clearly f-rings, and in fact, every f-ring is of this form [15]. This makes two important special properties of f-rings clear, namely that for all $f, g \in G$,

$$f^2 \geq 0, \quad \text{and} \quad |fg| = |f| \cdot |g|.$$

Both ℓ-rings and f-rings are equationally defined, even though we have not chosen an equational definition here. Consequently, these classes are closed under products, subalgebras, and homomorphic images. In particular, an *f-subring* G of an f-ring H is a subring and sublattice; we write $G \leq H$.

Let G be an f-ring. G is said to be *archimedean* if for all $f, g \in G^+$,

$$(\forall n \in \mathbb{N}\ (nf \leq g)) \ \Rightarrow\ f = 0.$$

Now it is always true that $g \wedge 1 = 0$ implies $g = 0$ for any $g \in G$. However, not every f-ring enjoys the stronger property of being *bounded*, i.e., for all $g \in G$ there is some $n \in \mathbb{N}$ such that

$$|g| \leq n \equiv n1.$$

The category **AfR** (**bAfR**) has objects which are (bounded) divisible archimedean f-rings, and morphisms which are ℓ-ring homomorphisms. Since we will be working exclusively in the categories **AfR** and **bAfR**, we shall refer to these morphisms as *f-homomorphisms*. Each **Afr**-object G contains a copy of the rational numbers Q: $\frac{p}{q}$ denotes the unique element $g \in G$ such that $qg = p1$ for $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. **bAfR** is coreflective in **AfR**, and the coreflection of $G \in$ **AfR** is just the insertion into G of its *bounded part*

$$G^* \equiv \{g \in G : \exists n \in \mathbb{N}\ (|g| \leq n)\}.$$

[14], [17], and [1] are fine general references on f-rings, among other topics, but the reader interested in additional background should overlook neither Henriksen's masterful survey [21] nor Banaschewski's elegant expository paper [8]. The latter is particularly close to the spirit of this article.

We remark in passing that we make very little use of the multiplicative structure of **AfR**-objects. (See the remark following Theorem 7.2.3.) Thus for most purposes the reader may replace **AfR** with **W**, the category of archimedean lattice-ordered groups with weak order units.

The **AfR**-objects of central interest here are those of the form

$$CL \equiv \{f \in \mathbf{Frm} : f : \mathcal{O}\mathbb{R} \to L\}.$$

The fundamental fact is that CL is universal for **AfR** in the sense that any **AfR**-object is isomorphic to a subobject of some CL. This means that archimedean f-rings constitute a satisfactory abstraction of the algebraic structure of CL. (Section 3 is a development of this algebraic structure.) The universality of CL follows from the localic Yosida representation, an achievement of Madden and Vermeer [25]; see [5] for another development.

Important as it is, we shall have little need here for the representation of general **AfR**-objects. Instead, it is the localic Yosida representation of **bAfR**-objects which will play a significant role in the main Theorem 7.1.1. This representation can be developed either by turning the arrows around in the classical (pointed) Yosida representation [30], or by specializing the localic Yosida representation for **AfR** to bounded objects. It is the latter approach which leads to insight in our work, and we proceed to outline it now. Proofs are available in [25] or [5].

A **bAfR**-*kernel* of G is a subset of the form $\theta^{-1}(0) \subseteq G$ for some surjection $\theta : G \to H$ in **bAfR**. The fact that **bAfR** is closed under products ([1]) implies that the set $\ker G$ of **bAfR**-kernels of G is closed under intersection, so that every subset $K \subseteq G$ is contained in a smallest such kernel, denoted by $\langle K \rangle_G$ or simply $\langle K \rangle$. A subset K is a **bAfR**-kernel of G if and only if it has the following features:

(1) K is a sublattice and subgroup (ℓ-subgroup) of G which is *convex*, i.e., for all $k, g \in G$,

$$0 \leq g \leq k \in K \;\Rightarrow\; g \in K.$$

(2) K is *uniformly closed*. Recall that a sequence $\{g_n\} \subseteq G \in \mathbf{AfR}$ *converges uniformly* to $g \in G$, written $g_n \to g$, provided that

$$\forall k \in \mathbb{N}\ \exists m \in \mathbb{N}\ \forall n \geq m \left(|g_n - g| \leq \frac{1}{k} \right).$$

We often use the more traditional symbol ε for $\frac{1}{k}$. For any subset $K \subseteq G$, the *uniform closure* of K consists of all $g \in G$ for which $k_n \to g$ for some sequence $\{k_n\} \subseteq K$. We say that K is *uniformly closed* if it is equal to its uniform closure, and *uniformly dense in* G if its uniform closure is G.

Condition (1) ensures that G/K is an ℓ-group, and it is an ℓ-ring because a convex ℓ-subgroup of a bounded f-ring is a ring ideal. (2) ensures that G/K is archimedean [24]. The property of being bounded is inherited by quotients.

The *Yosida frame* of G, designated by YG, is $\ker G$ in the inclusion order. This frame is compact and regular and therefore is the topology of a unique compact Hausdorff space ΣYG, called the *Yosida space* of G. Y is a functor from **bAfR** into the category **KRegFrm** of compact regular frames: for a **bAfR**-morphism $\theta : G \to H$, $Y\theta : YG \to YH$ is defined by the rule

$$(Y\theta)(K) \equiv \langle \theta(K) \rangle_H.$$

([1]) One must exercise caution here. The product of a family $\{G_i : i \in I\}$ in **bAfR** is the set of maps $g : I \to \bigcup_I G_i$ such that $g(i) \in G_i$ for all $i \in I$, and such that there is some $n \in \mathbb{N}$ for which $|g(i)| \leq n$ for all $i \in I$.

This functor is the left adjoint to the functor C^* which takes any frame L to the **bAfR**-object $C^*L \equiv (CL)^*$ of bounded frame maps from $\mathcal{O}\mathbb{R}$ into L. The elements of C^*L are precisely those frame maps $f \in CL$ which factor through some closed quotient map $a \mapsto a \vee a_r$, where $a_r \equiv \mathbb{R} \setminus [-r, r]$ for $r \in \mathbb{Q}^+$; see Lemma 4.1.1. For a frame map $m : L \to M$, $Cm : CL \to CM$ is defined by the rule

$$(Cm)(k) \equiv mk.$$

Of particular interest for our purposes is the unit of this adjunction. For each $G \in$ **bAfR** there is an injection $\mu_G : G \to C^*YG$ in **bAfR** which, for a given $g \in G$, is given by the formulas

$$\begin{aligned} \mu_G(g)(p,q) &\equiv \langle (g-p)^+ \wedge (q-g)^+ \rangle, && p, q \in \mathbb{Q}, \\ \mu_G(g)(U) &\equiv \bigvee_{(p,q) \subseteq U} (\mu_G g)(p,q), && U \in O\mathbb{R}. \end{aligned}$$

We abbreviate this by writing $\widehat{g}$ for $\mu_G(g)$ and $\widehat{G}$ for $\mu_G(G)$, and we refer to this map, or its image $\widehat{G}$, as the *Yosida representation* of G. We will have need for one additional fact which follows readily from the definitions above, namely that for any $g \in G$,

$$\operatorname{coz} \widehat{g} \equiv \widehat{g}(\mathbb{R}_0) = \langle g \rangle.$$

What distinguishes this representation from others is that $\widehat{G}$ *separates the elements of* YG, i.e., for all $a \in YG$,

$$a = \bigvee_{\widehat{g}(\mathbb{R}_0) \leq a} \widehat{g}(\mathbb{R}_0).$$

Furthermore YG is unique with respect to this property: if L is any compact regular frame which admits a **bAfR**-injection $\mu : G \to C^*L$ such that $\mu(G)$ separates the elements of L then there is a unique frame isomorphism $m : YG \to L$ such that $(Cm)\mu_G = \mu$. The counit of the adjunction for a frame L is the frame homomorphism $YC^*L \to L$ taking K to $\bigvee_K \operatorname{coz} k$, and this is the coreflection map from compact regular frames, i.e., the Stone–Čech compactification $\beta L \twoheadrightarrow L$.

3. Calculating in CL

In this section we do the basic calculations underlying the results that follow. We begin by exhibiting in Proposition 3.1.1 the fundamental formula which reduces the arithmetic of CL to the arithmetic of $\mathbb{R}$. We then apply this formula to establish in Propositions 3.2.3 and 3.2.6 the basic relationships between the members of CL and their cozero elements in L. A detailed account of this material can be found in [9].

We remind the reader of some notation, and introduce a little more. $\mathbb{R}^+$, $\mathbb{R}^-$, $\mathbb{R}_0$, $\mathbb{R}_1$ designate $(0, \infty)$, $(-\infty, 0)$, $\mathbb{R} \setminus \{0\}$, and $\mathbb{R} \setminus \{1\}$, respectively. For any $U \in \mathcal{O}\mathbb{R}$ we use U^+ and U^- to abbreviate $U \cap \mathbb{R}^+$ and $U \cap \mathbb{R}^-$, respectively. Note that $U^+ \cap U^- = \emptyset = \bot$, and that $U = U^+ \cup U^-$ if and only if $U \subseteq \mathbb{R}_0$. We use $\operatorname{coz} f$ to denote $f(\mathbb{R}_0)$ for $f \in CL$, and $b(r, \varepsilon)$ for the ε-ball $(r - \varepsilon, r + \varepsilon)$ about r. Somewhat more generally, we also use $b(U, \varepsilon)$ to designate the ε-ball $\bigvee_{r \in U} b(r, \varepsilon)$ about $U \in \mathcal{O}\mathbb{R}$.

3.1. The fundamental formula. We briefly review here the process of canonically lifting operations on $\mathbb{R}$ to operations on CL. Consider an n-ary operation w on $\mathbb{R}$, by which we mean a continuous function $w : \mathbb{R}^n \to \mathbb{R}$. Because $\mathbb{R}$ is locally compact we can, and do, regard $\mathcal{O}(\mathbb{R}^n)$ as the frame coproduct $\bigoplus_n \mathcal{O}\mathbb{R}$ [23, II 2.13], with the ith coproduct map j_i being the frame map of the ith projection on $\mathbb{R}^n$, $1 \le i \le n$. Thus a basic open subset of $\mathbb{R}^n$ has the form

$$U_1 \times \ldots \times U_n = j_1(U_1) \cap \ldots \cap j_n(U_n), \qquad U_i \in \mathcal{O}\mathbb{R}.$$

Now the frame map of w is $\mathcal{O}w : \mathcal{O}\mathbb{R} \to \mathcal{O}(\mathbb{R}^n)$, given by $\mathcal{O}w(U) = w^{-1}(U)$ for $U \in \mathcal{O}\mathbb{R}$. Thus for $U \in \mathcal{O}\mathbb{R}$,

$$\begin{aligned}\mathcal{O}w(U) = w^{-1}(U) &= \bigcup_{U_1 \times \ldots \times U_n \subseteq w^{-1}(U)} (U_1 \times \ldots \times U_n) \\ &= \bigcup_{U_1 \times \ldots \times U_n \subseteq w^{-1}(U)} \bigcap_{1 \le i \le n} j_i(U_i) = \bigcup_{w(U_1, \ldots, U_n) \subseteq U} \bigcap_{1 \le i \le n} j_i(U_i),\end{aligned}$$

where we have expressed the condition $U_1 \times \ldots \times U_n \subseteq w^{-1}(U)$ by writing $w(U_1, \ldots, U_n) \subseteq U$.

Now for a given frame L, w lifts canonically to $w' : (CL)^n \to CL$ as follows. Given $f_i \in CL$, $1 \le i \le n$, let f be the coproduct map satisfying $f j_i = f_i$, $1 \le i \le n$. Then $w'(f_1, \ldots, f_n)$ is simply $f(\mathcal{O}w)$, as in the diagram

$$\begin{tikzcd}
& \mathcal{O}\mathbb{R} \arrow[d, "j_i"'] \arrow[dr, "f_i"] & \\
\mathcal{O}\mathbb{R} \arrow[r, "\mathcal{O}w"] & \mathcal{O}(\mathbb{R}^n) \arrow[r, "f"] & L
\end{tikzcd}$$

Combining these facts gives the fundamental formula for our calculations.

PROPOSITION 3.1.1. *The canonical lifting $w' : (CL)^n \to CL$ of a continuous function $w : \mathbb{R}^n \to \mathbb{R}$ is given by the formula*

$$w'(f_1 \ldots, f_n)(U) = \bigvee_{w(U_1, \ldots, U_n) \subseteq U} \bigwedge_{1 \le i \le n} f_i(U_i), \qquad f_i \in CL,\ U \in \mathcal{O}\mathbb{R}.$$

Furthermore, the U_i's in the supremum may be taken to be rational intervals.

We mention for emphasis that Proposition 3.1.1 applies to nullary operations. Such an operation corresponds to the choice of a constant from $\mathbb{R}$, i.e., a continuous function $w : \mathbf{1} \to \mathbb{R}$, where $\mathbf{1}$ designates the one-element space. In this case $\mathcal{O}w$ maps $\mathcal{O}\mathbb{R}$ into $\mathcal{O}\mathbf{1} = 2$, the two-element frame, f is the unique frame map from $\mathbf{2}$ into L, and if $\mathbf{1} = \{x\}$ and $w(x) = r$ then w' satisfies

$$w'(U) = \begin{cases} \top & \text{if } r \in U, \\ \bot & \text{if } r \notin U. \end{cases}$$

We generally abuse the notation to the extent of writing w' as simply w, and in particular we write the aforementioned constant frame map as r.

Most often the continuous function w will be associated with an *ℓ-ring term*, i.e., an expression τ built up from free variables and constants using the ℓ-ring operations. In this case the continuous function w associated with τ is obtained by interpreting each n-ary operation and constant of τ to be the corresponding operation from $\mathbb{R}^n$ into $\mathbb{R}$. Thus we can interpret any ℓ-ring term in CL. For example, for $f, g \in CL$ the expression $(f-g)^+$ means $w(f, g, 0)$ for the ℓ-ring term

$$w(v_1, v_2, v_3) \equiv (v_1 - v_2) \vee v_3,$$

and as such designates that member $k \in CL$ which satisfies

$$k(U) = \bigvee_{(U_1 - U_2) \vee U_3 \subseteq U} (f(U_1) \wedge g(U_2) \wedge 0(U_3)), \quad U \in \mathcal{O}\mathbb{R},$$

where the U_i's can be assumed to range over rational intervals in $\mathcal{O}\mathbb{R}$.

We show that CL is an **AfR**-object in two steps, Corollaries 3.1.3 and 3.2.8.

Corollary 3.1.2. *If τ_1 and τ_2 are ℓ-ring terms such that the identity $\tau_1 = \tau_2$ holds in $\mathbb{R}$ then this identity also holds in CL. More generally, if τ_1 and τ_2 are ℓ-ring terms such that the inequality $\tau_1 \leq \tau_2$ holds in $\mathbb{R}$ then this inequality also holds in CL.*

Proof. Let w_1 and w_2 be the continuous functions determined by n-ary ℓ-ring terms τ_1 and τ_2. To say that the identity $\tau_1 = \tau_2$ holds in $\mathbb{R}$ is to say that $w_1 = w_2$, and in this case it easily follows from Proposition 3.1.1 that $w_1(f_1, \ldots, f_n) = w_2(f_1, \ldots, f_n)$ for $f_i \in CL$, $1 \leq i \leq n$, i.e., that $\tau_1 = \tau_2$ holds in CL. And the inequality $\tau_1 \leq \tau_2$ is equivalent to the equality $\tau_1 \vee \tau_2 = \tau_2$. ■

Corollary 3.1.3. *CL is an ℓ-ring.*

Proof. The identities which define ℓ-rings hold in $\mathbb{R}$ and therefore also in CL by Corollary 3.1.2. ■

3.2. Cozero facts. We collect the basic facts about cozero elements which will be necessary for what follows. Proposition 3.2.6 completes the list of cozero facts begun in Proposition 3.2.3, but its proof requires the intermediate Lemma 3.2.4. Corollary 3.2.11 establishes that the open quotient of a cozero frame element is always coz-onto. All of these facts will be heavily used in what follows.

We begin by characterizing cozero elements in a fashion which is intrinsic to the frame L in which they occur, without reference to a map from $\mathcal{O}\mathbb{R}$ into L. This is Proposition 3.2.2.

Lemma 3.2.1. *For any $f \in CL$ and $q \in \mathbb{Q}$, $\mathrm{coz}(f - q) = f(\mathbb{R} \setminus \{q\})$.*

Proof. By Proposition 3.1.1,

$$\mathrm{coz}(f-q) = (f-q)(\mathbb{R}_0) = \bigvee_{U - V \subseteq \mathbb{R}_0} (f(U) \wedge q(V)) = \bigvee_{U \cap V = \emptyset} (f(U) \wedge q(V)).$$

The only nontrivial contributions to this supremum occur when $1 \in V$. Thus we may take V to be of the form $(q - \varepsilon, q + \varepsilon)$ for $\varepsilon > 0$, and in this case U may be assumed to have the form $(-\infty, q-\varepsilon) \cup (q+\varepsilon, \infty)$. Therefore the supremum reduces to $\bigvee_{\varepsilon > 0}(f(-\infty, q-\varepsilon) \vee f(q+\varepsilon, \infty)) = f(\mathbb{R} \setminus \{q\})$. ■

Proposition 3.2.2. *An element a of a frame L is a cozero if and only if $a = \bigvee_{q<1} a_q$ for some scale $\{a_q : q \in [0,1]_{\mathbb{Q}}\}$.*

Proof. If $a = \bigvee_{q<1} a_q$ for some scale $\{a_q : q \in [0,1]_{\mathbb{Q}}\}$ then, by changing a_1 to a if necessary, we may assume that the scale is from a_0 to a. Proposition 2.1.4 then produces a function $f \in CL$ such that $f(\mathbb{R}_1) \leq a$, and an examination of the proof shows that $f(\mathbb{R}_1) = \bigvee_{q<1} a_q = a$. Thus $\operatorname{coz}(f-1) = a$ by Lemma 3.2.1. On the other hand, if $\operatorname{coz} f = a$ for some $f \in CL$ then $a = \bigvee_{q<1} a_q$ for the scale defined by

$$a_q \equiv f((-\infty, q-1) \cup (1-q, \infty)), \qquad q \in [0,1]_{\mathbb{Q}}.$$

The point is that $f(-p,p)$ is a separating element witnessing $a_p \prec a_q$ for $p < q$ in $[0,1]_{\mathbb{Q}}$. ■

Proposition 3.2.3. *The following hold for* $f, g \in CL$:

(1) $\operatorname{coz} f = \operatorname{coz}|f| = |f|(\mathbb{R}^+)$.
(2) $\operatorname{coz} f \wedge \operatorname{coz} g = \operatorname{coz}(|f| \wedge |g|)$.
(3) $\operatorname{coz}(f+g) \leq \operatorname{coz} f \vee \operatorname{coz} g$.
(4) $\operatorname{coz} f = \perp$ *if and only if* $f = 0$.

Proof. To prove (1) first note that for any $V \in \mathcal{O}\mathbb{R}$ we have

$$|f|(V) = (f \vee (-f))(V) = \bigvee_{U \vee (-U) \subseteq V} f(U).$$

In particular, since an interval $U \in \mathcal{O}\mathbb{R}$ satisfies $U \vee (-U) \subseteq \mathbb{R}_0$ if and only if $U \vee (-U) \subseteq \mathbb{R}^+$, we get

$$\operatorname{coz}|f| = |f|(\mathbb{R}_0) = \bigvee_{U \vee (-U) \subseteq \mathbb{R}_0} f(U) = \bigvee_{U \vee (-U) \subseteq \mathbb{R}^+} f(U) = |f|(\mathbb{R}^+).$$

And since any $U \in \mathcal{O}\mathbb{R}$ satisfies $U \vee (-U) \subseteq \mathbb{R}_0$ if and only if $U \subseteq \mathbb{R}_0$, we get

$$\operatorname{coz}|f| = \bigvee_{U \vee (-U) \subseteq \mathbb{R}_0} f(U) = \bigvee_{U \subseteq \mathbb{R}_0} f(U) = f\Big(\bigvee_{U \subseteq \mathbb{R}_0} U\Big) = f(\mathbb{R}_0) = \operatorname{coz} f.$$

To prove (2) we let w designate the ℓ-ring term

$$w(v_1, v_2) \equiv (v_1 \vee (-v_1)) \wedge (v_2 \vee (-v_2)).$$

Then since for $U_i \in \mathcal{O}\mathbb{R}$ we have $w(U_1, U_2) \subseteq \mathbb{R}_0$ if and only if $U_1, U_2 \subseteq \mathbb{R}_0$,

$$\begin{aligned}\operatorname{coz}(|f| \wedge |g|) = (|f| \wedge |g|)(\mathbb{R}_0) &= \bigvee_{w(U_1,U_2) \subseteq \mathbb{R}_0} (f(U_1) \wedge g(U_2)) \\ &= \bigvee_{U_1, U_2 \subseteq \mathbb{R}_0} (f(U_1) \wedge g(U_2)) = \bigvee_{U_1 \subseteq \mathbb{R}_0} f(U_1) \wedge \bigvee_{U_2 \subseteq \mathbb{R}_0} g(U_2) \\ &= f(\mathbb{R}_0) \wedge g(\mathbb{R}_0) = \operatorname{coz} f \wedge \operatorname{coz} g.\end{aligned}$$

To prove (3) observe that

$$\begin{aligned}\operatorname{coz}(f+g) = (f+g)(\mathbb{R}_0) &= \bigvee_{U_1 + U_2 \subseteq \mathbb{R}_0} (f(U_1) \wedge g(U_2)) \\ &\leq \bigvee_{U_1 \subseteq \mathbb{R}_0} f(U_1) \vee \bigvee_{U_2 \subseteq \mathbb{R}_0} g(U_2) = \operatorname{coz} f \vee \operatorname{coz} g\end{aligned}$$

because $U_1 + U_2 \subseteq \mathbb{R}_0$ only if $U_1 \subseteq \mathbb{R}_0$ or $U_2 \subseteq \mathbb{R}_0$. Finally, (4) holds because it is clear that $\operatorname{coz} f = \perp$ if and only if f is the constant 0 frame map, and this is the additive identity of CL by Corollary 3.1.2 because the identity $v + 0 = v$ holds in $\mathbb{R}$. ■

Lemma 3.2.4. *The following are equivalent for $f, g \in CL$:*

(1) $f \leq g$.
(2) $\operatorname{coz}(g-f) = (g-f)(\mathbb{R}^+)$.
(3) $(g-f)(\mathbb{R}^-) = \perp$.
(4) $(g-f)(r,\infty) = \top$ *for all* $r \in \mathbb{R}^-$.
(5) $f(r,\infty) \leq g(r,\infty)$ *for all* $r \in \mathbb{R}$.
(6) $f(-\infty,r) \geq g(-\infty,r)$ *for all* $r \in \mathbb{R}$.

Proof. If (1) holds then (2) follows from Proposition 3.2.3(1) because $g - f = |g-f|$. Suppose that (2) holds and abbreviate $g-f$ to k. Then $k(\mathbb{R}^-) \leq k(\mathbb{R}^+)$ because

$$k(\mathbb{R}^+) = \operatorname{coz} k = k(\mathbb{R}_0) = k(\mathbb{R}^+ \vee \mathbb{R}^-) = k(\mathbb{R}^+) \vee k(\mathbb{R}^-).$$

Therefore (3) holds because

$$k(\mathbb{R}^-) = k(\mathbb{R}^+) \wedge k(\mathbb{R}^-) = k(\mathbb{R}^+ \wedge \mathbb{R}^-) = k(\perp) = \perp.$$

If (3) holds and $r < 0$ then (4) holds because

$$\top = k(\mathbb{R}^- \vee (r,\infty)) = k(\mathbb{R}^-) \vee k(r,\infty) = k(r,\infty).$$

If (4) holds then (5) follows because for any $r \in \mathbb{R}$ we have

$$\begin{aligned} g(r,\infty) = (f + (g-f))(r,\infty) &= \bigvee_{U_1+U_2 \subseteq (r,\infty)} (f(U_1) \wedge (g-f)(U_2)) \\ &\geq \bigvee_{\varepsilon>0} (f(r+\varepsilon,\infty) \wedge (g-f)(-\varepsilon,\infty)) = \bigvee_{\varepsilon>0} f(r+\varepsilon,\infty) = f(r,\infty). \end{aligned}$$

Assume (5), and in order to prove (6) consider $s < r$ in $\mathbb{R}$. Then since

$$g(-\infty,s) \wedge f(s,\infty) \leq g(-\infty,s) \wedge g(s,\infty) = g((-\infty,s) \wedge (s,\infty)) = g(\perp) = \perp,$$

it follows that $g(-\infty,s) \leq f(-\infty,r)$ because

$$\begin{aligned} g(-\infty,s) = g(-\infty,s) \wedge \top &= g(-\infty,s) \wedge f((-\infty,r) \vee (s,\infty)) \\ &= (g(-\infty,s) \wedge f(-\infty,r)) \vee (g(-\infty,s) \wedge f(s,\infty)) \\ &= (g(-\infty,s) \wedge f(-\infty,r)) \vee \perp = g(-\infty,s) \wedge f(-\infty,r). \end{aligned}$$

Therefore (6) holds because $g(-\infty,r) = \bigvee_{s<r} g(-\infty,s) \leq f(-\infty,r)$. Assume (6) to prove (3). Observe that $U_i \in \mathcal{O}\mathbb{R}$ satisfy $U_1 - U_2 \subseteq \mathbb{R}^-$ if and only if there is some $r \in \mathbb{R}$ for which $U_1 \subseteq (-\infty,r)$ and $U_2 \subseteq (r,\infty)$. Therefore

$$\begin{aligned} (g-f)(\mathbb{R}^-) &= \bigvee_{U_1-U_2 \subseteq \mathbb{R}^-} (g(U_1) \wedge f(U_2)) = \bigvee_{r \in \mathbb{R}} (g(-\infty,r) \wedge f(r,\infty)) \\ &\leq \bigvee_{r \in \mathbb{R}} (f(-\infty,r) \wedge f(r,\infty)) = \perp. \end{aligned}$$

It only remains to show that (1) follows from the conjunction of (3) and (4). For that purpose abbreviate $g - f$ to k; we aim to show that for any $U \in \mathcal{O}\mathbb{R}$,

$$(k \wedge 0)(U) = \bigvee_{U_1 \wedge U_2 \subseteq U} (k(U_1) \wedge 0(U_2)) = 0(U) = \begin{cases} \top & \text{if } 0 \in a, \\ \perp & \text{if } 0 \notin a, \end{cases}$$

If $0 \notin U$ then the nontrivial contributions to the supremum above have $0 \in U_2$, which forces $U_1 \subseteq \mathbb{R}^-$ and in light of (3) that fact implies that the term is trivial after all. That is, $(k \wedge 0)(U) = 0(U) = \perp$ in this case. If $0 \in U$ then $(-\varepsilon, \varepsilon) \subseteq U$ for some $\varepsilon > 0$ and so by taking $U_1 \equiv (-\varepsilon, \infty)$ and $U_2 \equiv (-\varepsilon, \varepsilon)$ in the supremum above we deduce by (4) that

$$(k \wedge 0)(U) \geq k(-\varepsilon, \infty) \wedge 0(-\varepsilon, \varepsilon) = \top \wedge \top = \top = 0(U). \blacksquare$$

Corollary 3.2.5. *The following are equivalent for $k \in CL$:*

(1) $k \geq 0$.
(2) $\operatorname{coz} k = k(\mathbb{R}^+)$.
(3) $k(\mathbb{R}^-) = \perp$.
(4) $k(r, \infty) = \top$ *for all* $r \in \mathbb{R}^-$.

Proposition 3.2.6. *For $f, g \in CL$,*

$$\operatorname{coz} f \vee \operatorname{coz} g = \operatorname{coz}(|f| \vee |g|) = \operatorname{coz}(|f| + |g|).$$

Proof. Since $\operatorname{coz} f = \operatorname{coz} |f|$ by Proposition 3.2.3(1), we may take $f, g \geq 0$ here. Then by Lemma 3.2.4(5) we have

$$f(\mathbb{R}^+) \vee g(\mathbb{R}^+) \leq (f \vee g)(\mathbb{R}^+) \leq (f + g)(\mathbb{R}^+),$$

i.e.,

$$\operatorname{coz} f \vee \operatorname{coz} g \leq \operatorname{coz}(f \vee g) \leq \operatorname{coz}(f + g).$$

Since $\operatorname{coz}(f + g) \leq \operatorname{coz} f \vee \operatorname{coz} g$ by Proposition 3.2.3(3), the proof is complete. $\blacksquare$

Corollary 3.2.7. *If $0 \leq f \leq g$ in CL then $\operatorname{coz} f \leq \operatorname{coz} g = \operatorname{coz}(f \vee g) = \operatorname{coz}(f + g)$.*

Corollary 3.2.8. *CL is archimedean.*

Proof. Consider $f, g \in CL$ such that $0 \leq nf \leq g$ for all $n \in \mathbb{N}$. First observe that for any $r \in \mathbb{R}$ and $n \in \mathbb{N}$, $nf(r, \infty) = \bigvee_{nU \subseteq (r,\infty)} f(U) = f\big(\frac{r}{n}, \infty\big)$ because $nU \subseteq (r, \infty)$ if and only if $U \subseteq \big(\frac{r}{n}, \infty\big)$. It follows from Lemma 3.2.4 that for any $r \in \mathbb{R}^+$ we have

$$\operatorname{coz} f = f(\mathbb{R}^+) = \bigvee_{\mathbb{N}} f\left(\frac{r}{n}, \infty\right) = \bigvee_{\mathbb{N}} nf(r, \infty) \leq g(r, \infty).$$

Consequently,

$$\begin{aligned} \operatorname{coz} f = \operatorname{coz} f \wedge \top = \operatorname{coz} f \wedge \bigvee_{\mathbb{R}^+} g(-\infty, r) &= \bigvee_{\mathbb{R}^+} (\operatorname{coz} f \wedge g(-\infty, r)) \\ &\leq \bigvee_{\mathbb{R}^+} (g(r, \infty) \wedge g(-\infty, r)) = \perp. \end{aligned}$$

We conclude from Proposition 3.2.3(4) that $f = 0$. $\blacksquare$

Corollary 3.2.9. *Any equality or inequality of ℓ-ring terms which holds in $\mathbb{R}$ must hold in every **AfR**-object.*

Proof. By virtue of the localic Yosida representation discussed in Subsection 2.3, every **AfR**-object is isomorphic to a subobject of CL for some frame L. $\blacksquare$

We close this subsection by showing that a cozero element of (the frame of the open quotient of) a cozero element is itself a cozero element. More precisely, $\operatorname{Coz}(\downarrow a) =$

$\operatorname{Coz} L \cap ({\downarrow}a)$ for all $a \in \operatorname{Coz} L$. The significance of this fact for our purposes is captured by Corollary 3.2.11.

PROPOSITION 3.2.10. *If $a \in \operatorname{Coz} L$ and $b \in \operatorname{Coz}({\downarrow}a)$ then $b \in \operatorname{Coz} L$.*

Proof. By Proposition 3.2.2, we need only find a scale $\{c_i\}$ in L such that $\bigvee_{i<1} c_i = b$. Let $\{a_i : i \in [0,1]_{\mathbb{Q}}\}$ be a scale in L such that $\bigvee_{i<1} a_i = a$ and let $\{b_i : i \in [0,1]_{\mathbb{Q}}\}$ be a scale in ${\downarrow}a$ such that $\bigvee_{i<1} b_i = b$. Consider indices $i < j$ in $[0,1]_{\mathbb{Q}}$. The fact that $b_i \prec b_j$ in ${\downarrow}a$ means that there is some $c_i \in {\downarrow}a$ such that $b_i \wedge c_i = \bot$ and $b_j \vee c_i = a$, while the fact that $a_i \prec a_j$ in L means that there is some $d_i \in L$ such that $a_i \wedge d_i = \bot$ and $a_j \vee d_i = \top$. But then $e \equiv c_i \vee d_i$ satisfies

$$\begin{aligned}
&(a_i \wedge b_i) \wedge e = (a_i \wedge b_i \wedge c_i) \vee (a_i \wedge b_i \wedge d_i) = \bot, \\
&(a_j \wedge b_j) \vee e = (a_j \vee c_i \vee d_i) \wedge (b_j \vee c_i \vee d_i) = \top \wedge (a \vee d_i) \geq a_j \vee d_i = \top,
\end{aligned}$$

which is to say that $a_i \wedge b_i \prec a_j \wedge b_j$ in L. Therefore $\{a_i \wedge b_i : i \in [0,1]_{\mathbb{Q}}\}$ is a scale in L which joins to b, meaning $b \in \operatorname{Coz} L$. ∎

COROLLARY 3.2.11. *The open quotient of a cozero element is coz-onto.*

3.3. Division in CL. The fundamental formula of Proposition 3.1.1 applies to the operation of division as well, but with the prohibition of division by zero. We introduce the notation

$$\widetilde{U} \equiv \left\{ x \in \mathbb{R}_0 : \frac{1}{x} \in U \right\}, \qquad U \in \mathcal{O}\mathbb{R}.$$

Here $\widetilde{U}$ is regarded as an element of $\mathcal{O}\mathbb{R}$.

PROPOSITION 3.3.1. *An element f has an inverse in CL if and only if $\operatorname{coz} f = \top$, and in that case it is given by the formula*

$$f^{-1}(U) = f(\widetilde{U}), \qquad U \in \mathcal{O}\mathbb{R}.$$

Proof. If for any $g \in CL$ we have $fg = 1$ then by Proposition 3.2.3,

$$\operatorname{coz} f \wedge \operatorname{coz} g = \operatorname{coz}(fg) = \operatorname{coz} 1 = \top,$$

so that $\operatorname{coz} f = \operatorname{coz} g = \top$. Conversely suppose that $\operatorname{coz} f = \top$, let $q : \mathcal{O}\mathbb{R} \to {\downarrow}\mathbb{R}_0$ be the open quotient map given by $q(U) \equiv U \cap \mathbb{R}_0$ for $U \in \mathcal{O}\mathbb{R}$, and let $d : \mathcal{O}\mathbb{R} \to {\downarrow}\mathbb{R}_0$ be the frame map of the continuous function $x \mapsto \frac{1}{x}$, so that $d(U) = q(\widetilde{U})$ for all $U \in \mathcal{O}\mathbb{R}$.

$$\begin{tikzcd}
\mathcal{O}\mathbb{R} \arrow[r, "f"] \arrow[d, "q"'] & L \\
{\downarrow}\mathbb{R}_0 \arrow[ur, "f'"] & \mathcal{O}\mathbb{R} \arrow[u, "g"'] \arrow[l, "d"]
\end{tikzcd}$$

Now f factors through q by Lemma 2.1.1, say $f = f'q$. Put $g \equiv f'd$. Then clearly $g(U) = f'd(U) = f'q(\widetilde{U}) = f(\widetilde{U})$ for all $U \in \mathcal{O}\mathbb{R}$.

It remains to show that $fg = 1$. By Proposition 3.1.1, for $W \in \mathcal{O}\mathbb{R}$ we have

$$fg(W) = \bigvee_{UV \subseteq W} (f(U) \wedge g(V)) = \bigvee_{UV \subseteq W} (f(U) \wedge f(\widetilde{V})) = f\Big(\bigcup_{UV \subseteq W} (U \cap \widetilde{V})\Big).$$

If there exist U and V such that $U \cap \widetilde{V} \neq \emptyset$ and $UV \subseteq W$ then there exists $x \in U$ such that $x \in \mathbb{R}_0$ and $\frac{1}{x} \in V$, hence $1 \in UV \subseteq W$, showing that $fg(W) = \perp$ whenever $1 \notin W$. On the other hand, if $1 \in W$ then for any $x \in \mathbb{R}_0$ there exist U and V such that $x \in U$, $\frac{1}{x} \in V$, and $UV \subseteq W$ by the continuity of the multiplication in $\mathbb{R}$. Thus $\bigcup_{UV \subseteq W}(U \cap \widetilde{V}) \supseteq \mathbb{R}_0$, and hence $f(W) = \top$. In all $fg = 1$. ■

An **AfR**-object G is said to be *closed under bounded inversion* if every $g \in G$ such that $g \geq 1$ has an inverse.

Corollary 3.3.2. *CL is closed under bounded inversion.*

Proof. If $g \geq 1$ then $\operatorname{coz} g = g(\mathbb{R}^+) \geq 1(\mathbb{R}^+) = \top$ by Lemma 3.2.4. ■

4. Completeness properties of CL

4.1. The uniform completeness of CL. In an **AfR**-object G, a sequence $\{g_n\}$ is *Cauchy* if for any $k \in \mathbb{N}$ there is an index m such that $|g_i - g_j| \leq \frac{1}{k}$ for all $i, j \geq m$. We say that G is *uniformly complete* if every Cauchy sequence in G converges uniformly to a limit in G. Because CL is uniformly complete whenever L is the topology of a space, it may not be surprising that CL is uniformly complete for any frame L, a fact we establish in Theorem 4.1.5. Nevertheless the general result does not follow from the spatial version, and it is needed in the proof of our main Theorem 7.1.1.

We remark in passing that an **AfR**-object is uniformly complete and bounded if and only if it is isomorphic to $C(\mathcal{O}X)$ for some compact Hausdorff space X; this is the content of the famous Stone–Weierstrass Theorem. However, a uniformly complete but unbounded **AfR**-object need not be of the form CL for any frame L. Nevertheless there is another convergence, closely related to uniform convergence and defined intrinsically from the f-ring operations, with respect to which the completeness of an **AfR**-object G is equivalent to the existence of a locale L for which G is isomorphic to CL; see [4]. (In fact, this development makes no reference to the multiplication on G and therefore takes place in the broader category **W** of archimedean ℓ-groups with weak order unit.)

We say that two intervals $U_1, U_2 \in \mathcal{O}\mathbb{R}$ are *at least ε units apart* provided that $b(U_1, \varepsilon) \cap U_2 = \emptyset$, i.e., that $|x_1 - x_2| > \varepsilon$ for all $x_i \in U_i$. And in $\mathcal{O}\mathbb{R}$ we use the notation $U \subseteq_\varepsilon V$ to mean that $b(U, \varepsilon) \subseteq V$. Note that every $U \in \mathcal{O}\mathbb{R}$ satisfies

$$U = \bigcup_{\varepsilon > 0} \bigcup_{V \subseteq_\varepsilon U} V.$$

Lemma 4.1.1. *The following are equivalent for $f \in CL$:*

(1) *f is bounded by $r \in \mathbb{R}^+$, i.e., $|f| \leq r$.*
(2) *$f(\mathbb{R} \setminus [-r, r]) = \perp$.*
(3) *f factors through the closed quotient $U \mapsto U \cup (\mathbb{R} \setminus [-r, r])$.*

Proof. By parts (5) and (6) of Lemma 3.2.4,

$$\begin{aligned} |f| \le r \;\Leftrightarrow\; -r \le f \le r \;\Leftrightarrow\; f(-\infty,-r) = f(r,\infty) = \bot \\ \Leftrightarrow\; f(-\infty,-r) \vee f(r,\infty) = f(\mathbb{R} \setminus [-r,r]) = \bot. \end{aligned}$$

The equivalence of (2) and (3) is an instance of Lemma 2.1.1. ■

PROPOSITION 4.1.2. *For $f, g \in CL$ and $\varepsilon > 0$, $|f - g| \le \varepsilon$ if and only if $f(U_1) \wedge g(U_2) = \bot$ for all pairs of intervals U_i which are at least ε units apart.*

Proof. By Lemma 4.1.1, $|f - g| \le \varepsilon$ if and only if

$$|f-g|(\mathbb{R} \setminus [-\varepsilon,\varepsilon]) = |f-g|((-\infty,-\varepsilon) \vee (\varepsilon,\infty)) = |f-g|(\varepsilon,\infty) = \bot.$$

If we express the last equality by means of the fundamental formula of Proposition 3.1.1 we get

$$|f-g|(\varepsilon,\infty) = \bigvee_{|U_1 - U_2| \subseteq (\varepsilon,\infty)} (f(U_1) \wedge g(U_2)) = \bot.$$

This is evidently equivalent to the assertion that $f(U_1) \wedge g(U_2) = \bot$ for all pairs of intervals U_i which are at least ε units apart. ■

COROLLARY 4.1.3. *A sequence $\{g_n\}$ in CL is Cauchy if and only if for all $\varepsilon > 0$ there is some index m such that for all $k, l \ge m$ we have $g_k(U_1) \wedge g_l(U_2) = \bot$ for all pairs of intervals U_i which are at least ε units apart.*

LEMMA 4.1.4. *If $f, g \in CL$ and $\varepsilon > 0$ satisfy $|f - g| \le \varepsilon$ then $f(U) \le g(V)$ for all $U \subseteq_\varepsilon V$ in $\mathcal{O}\mathbb{R}$.*

Proof. Consider intervals $W \subseteq_\delta U \subseteq_\varepsilon V$ in $\mathcal{O}\mathbb{R}$. Set

$$W_1 \equiv \{s \in \mathbb{R} : \forall r \in W\ (|s - r| > \varepsilon)\} \in \mathcal{O}\mathbb{R}.$$

Since W_1 is the union of at most two intervals, both of which are at least ε units apart from W, it follows from Proposition 4.1.2 that $f(W) \wedge g(W_1) = \bot$. And since $g(W_1) \vee g(V) = g(W_1 \cup V) = \top$, it follows that $f(W) \le g(V)$. Therefore

$$f(U) = f\Big(\bigcup_{\delta>0} \bigcup_{W \subseteq_\delta U} W\Big) = \bigvee_{\delta>0} \bigvee_{W \subseteq_\delta U} f(W) \le g(V). \ ■$$

THEOREM 4.1.5. *CL is uniformly complete.*

Proof. For a Cauchy sequence $\{g_n\}$ in CL define $g : \mathcal{O}\mathbb{R} \to L$ by the formula

$$g(U) \equiv \bigvee_{\varepsilon>0} \{g_n(V) : V \subseteq_{2\varepsilon} U,\ \forall k, l \ge n\ (|g_k - g_l| \le \varepsilon)\}, \qquad U \in \mathcal{O}\mathbb{R}.$$

To show that g is a frame map first observe that $g(\emptyset) = \bot$ and $g(\mathbb{R}) = \top$ clearly hold. To check that g preserves binary meets note that $g(U_1) \wedge g(U_2)$ can be written as

$$\begin{aligned} &\bigvee_{\varepsilon_1>0} \{g_{n_1}(V_1) : V_1 \subseteq_{2\varepsilon_1} U_1,\ \forall k, l \ge n_1\ (|g_k - g_l| \le \varepsilon_1)\} \\ &\qquad \wedge \bigvee_{\varepsilon_2>0} \{g_{n_2}(V_2) : V_2 \subseteq_{2\varepsilon_2} U_2,\ \forall k, l \ge n_2\ (|g_k - g_l| \le \varepsilon_2)\} \\ &\quad = \bigvee_{\varepsilon_i>0} \{g_{n_1}(V_1) \wedge g_{n_2}(V_2) : V_i \subseteq_{2\varepsilon_i} U_i,\ \forall k, l \ge n_i\ (|g_k - g_l| \le \varepsilon_i)\}. \end{aligned}$$

Consider a term $g_{n_1}(V_1)\wedge g_{n_2}(V_2)$ that appears in the last supremum above, set $\varepsilon \equiv \varepsilon_1 \wedge \varepsilon_2$, and find an integer n large enough that $|g_k - g_l| \leq \varepsilon/2$ for all $k, l \geq n$, and large enough that $n \geq n_1 \vee n_2$. Set $W_i \equiv b(V_i, \varepsilon)$ and $W \equiv W_1 \cap W_2$. Then because $V_i \subseteq_{\varepsilon_i} W_i$ and $|g_n - g_{n_i}| \leq \varepsilon_i$, Lemma 4.1.4 gives the inequality

$$g_{n_1}(V_1) \wedge g_{n_2}(V_2) \leq g_n(W_1) \wedge g_n(W_2) = g_n(W).$$

But

$$W_i \subseteq_{\varepsilon_i} U_i \ \Rightarrow \ W_i \subseteq_{\varepsilon} U_i \ \Rightarrow \ W = W_1 \cap W_2 \subseteq_{\varepsilon} U_1 \cap U_2,$$

and since $|g_k - g_l| \leq \varepsilon/2$ for all $k, l \geq n$, the term $g_n(W)$ appears in the supremum which defines $g(U_1 \cap U_2)$, namely

$$g(U_1 \cap U_2) \equiv \bigvee_{\varepsilon>0} \{g_n(W) : W \subseteq_{2\varepsilon} U_1 \cap U_2,\ \forall k, l \geq n\ (|g_k - g_l| \leq \varepsilon)\}.$$

Thus $g(U_1)\wedge g(U_2) \leq g(U_1 \cap U_2)$. But the opposite inequality is clear because g is obviously order-preserving, so we can conclude that g preserves binary meets.

Now consider a supremum $\bigcup_I U_i = U$ in $\mathcal{O}\mathbb{R}$. It is clear that $\bigvee_I g(U_i) \leq g(U)$ simply because g is order-preserving. To verify the opposite inequality it is enough to show that for any $\varepsilon > 0$, any bounded interval $V \subseteq_{2\varepsilon} U$, and any index n such that $|g_k - g_l| \leq \varepsilon$ for $k, l \geq n$, we have $\bigvee_I g(U_i) \geq g_n(V)$. Set $W \equiv b(V, \varepsilon)$. Since the U_i's cover the compact set $\operatorname{cl} W$, there is a finite subset $I_0 \subseteq I$ for which $\bigcup_{I_0} U_{i_0} \supseteq \operatorname{cl} W$. Therefore there exist open sets V_{i_0}, $i_0 \in I_0$, and a real number δ, $\varepsilon > \delta > 0$, such that $\bigcup_{I_0} V_{i_0} \supseteq W$ and $V_{i_0} \subseteq_{2\delta} U_{i_0}$, $i_0 \in I_0$. Let $m \geq n$ be an integer large enough that $|g_k - g_l| \leq \delta$ for $k, l \geq m$. Then by definition we have $g(U_{i_0}) \geq g_m(V_{i_0})$ for all $i \in I_0$, hence

$$\bigvee_I g(U_i) \geq \bigvee_{I_0} g(U_{i_0}) \geq \bigvee_{I_0} g_m(V_{i_0}) = g_m\Big(\bigcup_{I_0} V_{i_0}\Big) \geq g_m(W) \geq g_n(V).$$

The last inequality holds by Lemma 4.1.4 since $m \geq n$ implies $|g_m - g_n| \leq \varepsilon$ and since $V \subseteq_{\varepsilon} W$.

It remains to show that g is the uniform limit of the g_n's. For a given $\varepsilon > 0$ find an index n such that $|g_k - g_l| \leq \varepsilon$ for all $k, l \geq n$. We claim that $|g - g_j| \leq \varepsilon$ for all $j \geq n$. To verify the claim fix $j \geq n$ and consider $U_i \in \mathcal{O}\mathbb{R}$ which are at least ε units apart; we seek to show that $g(U_1) \wedge g_j(U_2) = \perp$. For that purpose consider an arbitrary term $g_m(V_1)$ of the supremum which defines $g(U_1)$, i.e., there is some $\delta > 0$ such that $V_1 \subseteq_{2\delta} U_1$ and $|g_k - g_l| \leq \delta$ for all $k, l \geq m$. Fix a particular $k \geq m \vee n$, and set $W_1 \equiv b(V_1, \delta)$. Then Lemma 4.1.4 gives $g_m(V_1) \leq g_k(W_1)$, and Lemma 4.1.2 gives $g_k(W_1) \wedge g_j(U_2) = \perp$ because $|g_k - g_j| \leq \varepsilon$ and because W_1 and U_2 are at least ε units apart. It follows that $g_m(V_1) \wedge g_j(U_2) = \perp$, and since $g_m(V_1)$ is an arbitrary term in the supremum which defines $g(U_1)$, we can conclude that $g(U_1) \wedge g_j(U_2) = \perp$. This concludes the proof of the claim and of the theorem. ∎

Corollary 4.1.6. *C^*L is uniformly complete.*

Proof. A Cauchy sequence $\{g_n\}$ in C^*L is Cauchy in CL, and its limit $g \in CL$ must lie in C^*L because the limit is bounded by $g_m + 1$, where m is an index for which $|g_i - g_j| \leq 1$ for all $i, j \geq m$. ∎

The proof of the following lemma makes use of the fact that constant functions pass through **AfR**-morphisms, i.e., for any **AfR**-morphism $\theta : G \to H$ and any rational number p, $\theta(p) = p$.

LEMMA 4.1.7. *A quotient of a uniformly complete* **AfR***-object is uniformly complete.*

Proof. Let $\theta : G \to H$ be a surjection in **AfR**, and let $\{h_n\}$ be a Cauchy sequence in H. By passing to a subsequence if necessary, we may assume that $|h_i - h_j| \leq 1/2^m$ for all $i, j \geq m$. For each n choose g_n such that $\theta(g_n) = h_n$; we will inductively define a sequence $\{g'_n\}$ which is Cauchy in G and which satisfies $\theta(g'_n) = h_n$ for all n. Put $g'_1 \equiv g_1$. Suppose g'_n has been defined so that $\theta(g'_n) = h_n$. Since

$$\theta\left(g'_n - \frac{1}{2^n}\right) = h_n - \frac{1}{2^n} \leq h_{n+1} \leq h_n + \frac{1}{2^n} = \theta\left(g'_n + \frac{1}{2^n}\right),$$

it follows that if we set

$$g'_{n+1} \equiv \left(g_{n+1} \vee \left(g'_n - \frac{1}{2^n}\right)\right) \wedge \left(g'_n + \frac{1}{2^n}\right)$$

then we will get $\theta(g'_{n+1}) = h_{n+1}$. The sequence $\{g'_n\}$ is Cauchy by construction, and it converges uniformly to a unique element $g \in G$. And $\theta(g)$ is clearly the limit of the given sequence $\{h_n\}$ in H: for given $\varepsilon > 0$ there is an index m for which $|g'_n - g| \leq \varepsilon$ for all $n \geq m$, and if we apply θ to this inequality we get $|h_n - \theta(g)| \leq \varepsilon$ for all $n \geq m$. ∎

COROLLARY 4.1.8. *Any quotient of CL or C^*L is uniformly complete.*

4.2. The $*$-completeness of CL. We introduce a convergence notion, termed $*$-convergence, with respect to which CL is complete. In contrast to uniform convergence, however, this notion cannot be readily defined in an abstract f-ring, but makes sense only in CL for some frame L. Nevertheless this completeness property will be important for our purposes.

DEFINITION 4.2.1. In CL, we say that the sequence $\{g_n\}$ *$*$-converges to* 0, and write $g_n \xrightarrow{*} 0$, provided that

$$\bigvee_m \bigwedge_{n \geq m} (\operatorname{coz} g_n)^* = \top.$$

We say that $\{g_n\}$ *$*$-converges to* g, and write $g_n \xrightarrow{*} g$, if $(g_n - g) \xrightarrow{*} 0$.

We show in Proposition 4.2.3 that $*$-convergence is well behaved insofar as it is Hausdorff and renders all of the operations on CL continuous. Thus $*$-convergence is an ℓ-convergence in the sense of [2]. Corollary 4.2.6 provides further evidence that $*$-convergence is important for the study of CL; in it we show that C^*L is dense in CL with respect to $*$-convergence. Finally, we point out that $*$-convergence is trivial if L is compact, for in that case it is the convergence of the discrete topology on CL.

LEMMA 4.2.2. *The following hold in CL:*

(1) *If $g_n \xrightarrow{*} 0$ then $(-g_n) \xrightarrow{*} 0$.*
(2) *If $g_n \xrightarrow{*} 0$ and $h_n \xrightarrow{*} 0$ then $(g_n + h_n) \xrightarrow{*} 0$.*
(3) *If $h_n \xrightarrow{*} 0$ then for any sequence (g_n) we have $g_n h_n \xrightarrow{*} 0$.*

Proof. Part (1) is a consequence of the fact that $\operatorname{coz} g_n = \operatorname{coz}(-g_n)$. Part (2) holds because

$$\begin{aligned}\top &= \bigvee_{m_1}\bigwedge_{n_1\geq m_1}(\operatorname{coz} g_{n_1})^* \wedge \bigvee_{m_2}\bigwedge_{n_2\geq m_2}(\operatorname{coz} h_{n_2})^* = \bigvee_{m_i}\bigwedge_{n_i\geq m_i}((\operatorname{coz} g_{n_1})^* \wedge (\operatorname{coz} h_{n_2})^*)\\ &= \bigvee_{m}\bigwedge_{n\geq m}((\operatorname{coz} g_n)^* \wedge (\operatorname{coz} h_n)^*) = \bigvee_{m}\bigwedge_{n\geq m}(\operatorname{coz} g_n \vee \operatorname{coz} h_n)^*\\ &\leq \bigvee_{m}\bigwedge_{n\geq m}(\operatorname{coz}(g_n + h_n))^*.\end{aligned}$$

The inequality is a consequence of Proposition 3.2.3(3). Part (3) holds because by Proposition 3.2.3(2) we have

$$\bigvee_{m}\bigwedge_{n\geq m}(\operatorname{coz}(h_n g_n))^* = \bigvee_{m}\bigwedge_{n\geq m}(\operatorname{coz} h_n \wedge \operatorname{coz} g_n)^* \geq \bigvee_{m}\bigwedge_{n\geq m}(\operatorname{coz} h_n)^* = \top. \blacksquare$$

PROPOSITION 4.2.3. *Suppose that* $g_n \xrightarrow{*} g$ *and* $h_n \xrightarrow{*} h$ *in* CL. *Then*:

(1) $(-g_n) \xrightarrow{*} (-g)$.
(2) $(g_n + h_n) \xrightarrow{*} (g + h)$.
(3) $g_n^+ \xrightarrow{*} g^+$.
(4) $(g_n \vee h_n) \xrightarrow{*} (g \vee h)$.
(5) $(g_n \wedge h_n) \xrightarrow{*} (g \wedge h)$.
(6) $(g_n h_n) \xrightarrow{*} (gh)$.
(7) *If* $g_n = h_n$ *for all* n *then* $g = h$.

Proof. (1) follows from Lemma 4.2.2(1) and Proposition 3.2.3(1), since the latter implies that $\operatorname{coz} k = \operatorname{coz}(-k)$ for all $k \in CL$. (2) follows from Lemma 4.2.2(2), and (6) follows from Lemma 4.2.2(3) because

$$g_n h_n - gh = g_n(h_n - h) + h(g_n - g).$$

To establish (3), (4), and (5), first note these relations which hold in $\mathbb{R}$ and therefore in any **W**-object by Corollary 3.2.9:

$$f \vee k = (f - k)^+ + k, \qquad f \wedge k = -((-f) \vee (-k)), \qquad |f - k| \geq |f^+ - k^+|.$$

From the displayed equalities we deduce that both (4) and (5) follow from (3). To prove (3) we use the displayed inequality together with Corollary 3.2.7, which together yield

$$\operatorname{coz}(g_n - g) = \operatorname{coz}|g_n - g| \geq \operatorname{coz}|g_n^+ - g^+| = \operatorname{coz}(g_n^+ - g^+), \qquad n \in \mathbb{N},$$

from which follows

$$\top = \bigvee_{m}\bigwedge_{n\geq m}(\operatorname{coz}(g_n - g))^* \leq \bigvee_{m}\bigwedge_{n\geq m}(\operatorname{coz}(g_n^+ - g^+))^*.$$

To show (7) assume that $g_n = h_n$ for all $n \in \mathbb{N}$. Then $(g_n - h_n) \xrightarrow{*} (g - h)$ implies

$$\top = \bigvee_{m}\bigwedge_{n\geq m}(\operatorname{coz}(g_n - h_n + h - g))^* = \bigvee_{m}\bigwedge_{n\geq m}(\operatorname{coz}(h - g))^* = (\operatorname{coz}(h - g))^*,$$

from which we conclude that $\operatorname{coz}(h - g) = \bot$ and $h = g$ by Proposition 3.2.3(4). $\blacksquare$

Along with any suitable convergence comes the corresponding completeness notion [2].

Definition 4.2.4. A sequence $\{g_n\}$ in CL is $*$-*Cauchy* provided that

$$\bigvee_m \bigwedge_{i,j\geq m} (\mathrm{coz}(g_i - g_j))^* = \top.$$

An **AfR**-subobject $G \leq CL$ is said to be $*$-*complete* if every $*$-Cauchy sequence in G $*$-converges to a limit in G.

Proposition 4.2.5. *CL is $*$-complete.*

Proof. Suppose $\{g_n\}$ is a $*$-Cauchy sequence in CL, and $d_m \equiv \bigwedge_{i,j\geq m}(\mathrm{coz}(g_i - g_j))^*$. Define $g : \mathcal{O}\mathbb{R} \to L$ by the rule

$$g(U) \equiv \bigvee_m (g_m(U) \wedge d_m).$$

We first claim that for all $i, j \geq m$ and $U \in \mathcal{O}\mathbb{R}$,

$$g_i(U) \wedge d_m = g_j(U) \wedge d_m.$$

To verify this claim, first use Proposition 3.1.1 to write $g_i(U) \wedge d_m$ as

$$((g_i - g_j) + g_j)(U) \wedge d_m = \bigvee_{U_1+U_2\subseteq U} ((g_i - g_j)(U_1) \wedge g_j(U_2) \wedge d_m)$$

Now since $\mathrm{coz}(g_i - g_j) \wedge d_m = \bot$, the only nontrivial contributions to this supremum occur when $0 \in U_1$, in which case there is some $\delta > 0$ such that $(-\delta, \delta) \subseteq U_1$ and $b(U_2, \delta) \subseteq U$. In such a case, furthermore,

$$(g_i - g_j)(U_1) \wedge d_m = ((g_i - g_j)(-\delta, \delta) \vee (g_i - g_j)(U_1 \cap \mathbb{R}_0)) \wedge d_m = (g_i - g_j)(-\delta, \delta) \wedge d_m.$$

Therefore

$$\begin{aligned} g_i(U) \wedge d_m &= \bigvee_{\delta>0} \bigvee_{b(U_2,\delta)\subseteq U} ((g_i - g_j)(-\delta, \delta) \wedge g_j(U_2) \wedge d_m) \\ &\leq \bigvee_{\delta>0} \bigvee_{b(U_2,\delta)\subseteq U} (g_j(U_2) \wedge d_m) = \bigvee_{\delta>0} \bigvee_{b(U_2,\delta)\subseteq U} g_j(U_2) \wedge d_m \\ &= g_j(U) \wedge d_m. \end{aligned}$$

A symmetrical argument establishes the opposite inequality and the claim.

Let us now show that g is a frame map. Clearly $g(\emptyset) = \bot$, and $g(\mathbb{R}) = \top$ because $\bigvee_m d_m = \top$. To show that g preserves binary meets, consider arbitrary $U_i \in \mathcal{O}\mathbb{R}$. Then by the claim

$$\begin{aligned} g(U_1) \wedge g(U_2) &= \bigvee_{m_1} (g_{m_1}(U_1) \wedge d_{m_1}) \wedge \bigvee_{m_2} (g_{m_2}(U_2) \wedge d_{m_2}) \\ &= \bigvee_{m_i} (g_{m_1}(U_1) \wedge g_{m_2}(U_2) \wedge d_{m_1} \wedge d_{m_2}) \\ &= \bigvee_m (g_m(U_1) \wedge g_m(U_2) \wedge d_m) = \bigvee_m (g_m(U_1 \cap U_2) \wedge d_m) \\ &= g(U_1 \cap U_2). \end{aligned}$$

To show that g preserves arbitrary joins, consider $\bigcup_I U_i = U$ in $\mathcal{O}\mathbb{R}$. Then

$$\begin{aligned}\bigvee_I g(U_i) &= \bigvee_I \bigvee_{m_i} (g_{m_i}(U_i) \wedge d_{m_i}) = \bigvee_m \bigvee_I (g_m(U_i) \wedge d_m) \\ &= \bigvee_m \Big(\bigvee_I g_m(U_i) \wedge d_m \Big) = \bigvee_m (g_m(U) \wedge d_m) = g(U).\end{aligned}$$

This completes the proof that g is a frame map.

We next claim that $g(U) \wedge d_n = g_n(U) \wedge d_n$ for all $n \in \mathbb{N}$ and $U \in \mathcal{O}\mathbb{R}$. Indeed

$$g(U) \wedge d_n = \bigvee_m (g_m(U) \wedge d_m) \wedge d_n = \bigvee_{m \in \mathbb{N}} (g_m(U) \wedge d_m \wedge d_n),$$

and by the first claim the terms in the last supremum reduce to $g_n(U) \wedge d_m$ for $m \leq n$, and to $g_n(U) \wedge d_n$ for $m \geq n$. Thus the supremum itself comes to $g_n(U) \wedge d_n$, proving the second claim.

It remains to show that $g_n \xrightarrow{*} g$. For that purpose it is enough to show that $\operatorname{coz}(g_n - g) \wedge d_m = \bot$ for $n \geq m$, for that will establish that $d_m \leq \bigwedge_{n \geq m} (\operatorname{coz}(g_n - g))^*$ and hence that

$$\bigvee_m \bigwedge_{n \geq m} (\operatorname{coz}(g_n - g))^* \geq \bigvee_m d_m = \top.$$

The calculation depends on both claims:

$$\begin{aligned}\operatorname{coz}(g_n - g) \wedge d_m &= \bigvee_{U_1 - U_2 \subseteq \mathbb{R}_0} (g_n(U_1) \wedge g(U_2)) \wedge d_m = \bigvee_{U_1 - U_2 \subseteq \mathbb{R}_0} (g_n(U_1) \wedge g(U_2) \wedge d_m) \\ &= \bigvee_{U_1 - U_2 \subseteq \mathbb{R}_0} (g_n(U_1) \wedge g_m(U_2) \wedge d_m) = \bigvee_{U_1 \cap U_2 = \emptyset} (g_m(U_1) \wedge g_m(U_2) \wedge d_m) \\ &= \bigvee_{U_1 \cap U_2 = \emptyset} (g_m(U_1 \cap U_2) \wedge d_m) = \bot. \ \blacksquare\end{aligned}$$

Corollary 4.2.6. *C^*L is dense in CL with respect to $*$-convergence.*

Proof. Consider $h \in (CL)^+$ and set $h_n \equiv h \wedge n$ for all $n \in \mathbb{N}$. Then $h_n \in C^*L$ for all n, and the proof is completed by establishing that $h_n \xrightarrow{*} h$. Now the reader will have no trouble verifying from the fundamental formula of Proposition 3.1.1 that $(h-n)^+(\mathbb{R}_0) = h(n, \infty)$, so that

$$\operatorname{coz}(h - h_n) = \operatorname{coz}(h - n)^+ = h(n, \infty).$$

Therefore for $n \geq m$ we get

$$h(-\infty, m) \wedge \operatorname{coz}(h - h_n) = h(-\infty, m) \wedge h(n, \infty) = \bot.$$

That is, $h(-\infty, m) \leq \bigwedge_{n \geq m} (\operatorname{coz}(h - h_n))^*$, and since $\bigvee_m h(-\infty, m) = \top$, we have $h_n \xrightarrow{*} h$. $\blacksquare$

Corollary 4.2.7. *Let G be an* **AfR**-*subobject of CL. Then $G = CL$ if and only if $G^* = C^*L$ and G is $*$-complete.*

We close this subsection by pointing out a consequence of the proof of Proposition 4.2.5 which will be needed in what follows.

Corollary 4.2.8. *Suppose* $g_n \xrightarrow{*} g$ *in* CL, *and set* $d_m \equiv \bigwedge_{i,j\geq m}(\operatorname{coz}(g_i - g_j))^*$ *for* $m \in \mathbb{N}$. *Then for all* $i, j \geq m$ *and all* $U \in \mathcal{O}\mathbb{R}$,

(1) $g_i(U) \wedge d_m = g_j(U) \wedge d_m$, *and*
(2) $g(U) \wedge d_m = g_m(U) \wedge d_m$.

Proof. Under the hypotheses, $\{g_n\}$ is $*$-Cauchy and g is the limit constructed in the proof of Proposition 4.2.5. That proof establishes the properties claimed. ■

5. Cozero covers

Our development will require a detailed understanding of cozero covers. We begin with an analysis of principal covers.

5.1. Principal covers. A principal cover is generated by a single frame map.

Definition 5.1.1. A cover of a frame L is said to be *principal* if it is of the form $f(B)$ for a cover B of $\mathcal{O}\mathbb{R}$ and a single $f \in CL$.

Note that a principal cover is a cozero cover, i.e., consists of cozero elements. Furthermore, every principal cover has a countable subcover because every cover of $\mathcal{O}\mathbb{R}$ has a countable subcover, and likewise every principal cover of the form $f(B)$ for $f \in C^*L$ has a finite subcover. And if $m : L \to M$ is a frame map and C is a principal cover of L then $m(C)$ is a principal cover of M. It follows that the real uniformity on a completely regular frame may be thought of as the uniformity generated by all principal covers, and the Stone–Čech uniformity as the one generated by all finite principal covers.

Although we prove in Proposition 5.1.2 that every binary cozero cover is principal, we give in Example 5.1.4 a ternary cozero cover which cannot be refined by a single principal cover. Nevertheless, we prove in Proposition 5.1.6 that every finite cozero cover is refined by a finite meet of principal covers, and this result is the primary objective of this subsection. We remind the reader that $\mathbb{R}^+$, $\mathbb{R}^-$, $\mathbb{R}_0$, and $\mathbb{R}_1$ designate respectively $(0,\infty)$, $(-\infty, 0)$, $\mathbb{R} \setminus \{0\}$, and $\mathbb{R} \setminus \{1\}$, and that for $U \in \mathcal{O}\mathbb{R}$, U^+ and U^- designate respectively $U \cap \mathbb{R}^+$ and $U \cap \mathbb{R}^-$.

Proposition 5.1.2. *Every binary cozero cover is principal. More specifically, for elements* $a_i \in L$ *there is some* $f \in CL$ *such that*

$$a_0 = f(\mathbb{R}_0) \quad \textit{and} \quad a_1 = f(\mathbb{R}_1)$$

if and only if $a_i \in \operatorname{Coz} L$ *and* $a_0 \vee a_1 = \top$. *Moreover, f may be chosen to satisfy* $0 \leq f \leq 1$.

Proof. If such an f exists then clearly $a_i \in \operatorname{Coz} L$ and $a_0 \vee a_1 = \top$. Now assume $a_i \in \operatorname{Coz} L$ satisfy $a_0 \vee a_1 = \top$. Because $\operatorname{Coz} L$ is normal we may find $b_i \in \operatorname{Coz} L$ satisfying $a_0 \vee b_1 = b_0 \vee a_1 = \top$ and $b_0 \wedge b_1 = \bot$. The element b_0 witnesses $b_1 \prec a_1$, so by Proposition 2.1.4 there is some $g \in CL$ satisfying $g(\mathbb{R}_0) \wedge b_1 = \bot$, $g(\mathbb{R}_1) \leq a_1$, and $0 \leq g \leq 1$. It follows that $\operatorname{coz} g \leq a_0$ and, by Lemma 3.2.1, that $\operatorname{coz}(1-g) \leq a_1$. Next find elements $g_i \in \operatorname{Coz} L$ for which $\operatorname{coz} g_i = a_i$. Since $\operatorname{coz} g_i = \operatorname{coz}\big(|g_i| \wedge \frac{1}{2}\big)$ by Proposition 3.2.3, we may assume

$0 \le g_i \le \frac{1}{2}$. Put

$$f \equiv (g \vee g_0) \wedge (1 - g_1).$$

Then because $1 - g_1 \ge \frac{1}{2}$ it follows that $\operatorname{coz}(1 - g_1) = \top$, so that from Propositions 3.2.3 and 3.2.6 we get

$$\operatorname{coz} f = \operatorname{coz}(g \vee g_0) \wedge \operatorname{coz}(1 - g_1) = \operatorname{coz}(g \vee g_0) = \operatorname{coz} g \vee \operatorname{coz} g_0 = a_0.$$

And $1 - f = ((1 - g) \wedge (1 - g_0)) \vee g_1$, hence

$$\operatorname{coz}(1 - f) = (\operatorname{coz}(1 - g) \wedge \operatorname{coz}(1 - g_0)) \vee \operatorname{coz} g_1 = \operatorname{coz}(1 - g) \vee \operatorname{coz} g_1 = a_1.$$

We have shown that $f(\mathbb{R}_0) = a_0$ and $f(\mathbb{R}_1) = a_1$, and f clearly satisfies $0 \le f \le 1$. ∎

Corollary 5.1.3. *If $a_0 \vee a_1 = \top$ in $\operatorname{Coz} L$ then there exist $b_i \in \operatorname{Coz} L$ such that $b_i \prec\!\prec a_i$ and $b_0 \vee b_1 = \top$.*

Proof. Find $f \in CL$ for which $a_i = f(\mathbb{R}_i)$ and $0 \le f \le 1$. Set $b_0 \equiv f\left(\frac{1}{4}, \infty\right)$ and $b_1 \equiv f\left(-\infty, \frac{3}{4}\right)$. ∎

Here is a simple example of a ternary cozero cover which is not principal. We encourage the reader to draw a picture, and we do so by using spatial terminology.

Example 5.1.4. The space X is the union of three overlapping open disks in the Euclidean plane, labeled d_1, d_2, and d_3, with centers labeled x_1, x_2, and x_3. The disks are arranged so that x_i lies only in d_i. Let $P(x_j, x_k)$ be a path from x_j to x_k which does not pass through d_i for i not j or k. We claim that there is no continuous function $g : X \to [0, 1]$ for which there is an open cover $\{a_1, a_2, a_3\}$ of $[0, 1]$ such that $g^{-1}(a_i) \subseteq d_i$ for $i = 1, 2, 3$.

Suppose for contradiction that g is such a function and $\{a_1, a_2, a_3\}$ such a cover. Label $g(x_i)$ as r_i. It follows that r_i lies in a_i but not a_j or a_k for i not j or k. Thus the r_i's are distinct points in $[0, 1]$, say $r_1 < r_2 < r_3$. But $g(P(x_1, x_3))$ is a connected subset of $[0, 1]$ which contains r_1 and r_3 and so must also contain r_2 and thus meet a_2. This contradicts the assumption that $g^{-1}(a_2) \subseteq d_2$.

Even though not every finite cozero cover is refined by a single principal cover, it is at least refined by the finite meet of principal, in fact binary, covers. This is the content of Proposition 5.1.6.

Lemma 5.1.5. *The system of inequalities*

$$\begin{array}{rcrcr}
(1-n)x_1 + & & x_2 + \ldots + & & x_n > 0 \\
x_1 + & (1-n) & x_2 + \ldots + & & x_n > 0 \\
\vdots\ \vdots & & \vdots\ \vdots\quad \vdots & & \vdots\ \vdots\ \vdots \\
x_1 + & & x_2 + \ldots + & (1-n) & x_n > 0
\end{array}$$

has no solution in real numbers x_i.

Proof. The inequalities add to the contradiction $0 > 0$. ∎

Proposition 5.1.6. *Every finite cozero cover is refined by the meet of finitely many binary covers.*

Proof. Let $C = \{c_i : 1 \leq i \leq n\}$, $n > 2$, be a finite cozero cover, and for each i let $f_i \in (CL)^+$ be such that $\operatorname{coz} f_i = c_i$. For each i let w_i designate the n-ary operation on CL associated with the term $\tau_i \equiv \sum_{i \neq j} x_j - (n-1)x_i$, and let $g_i \equiv w_i(f_1, \ldots, f_n)$. We aim to show that $\{\{c_i, \operatorname{coz} g_i^+\} : 1 \leq i \leq n\}$ is a family of binary cozero covers whose common refinement refines C.

The first step is to demonstrate that each member of this family is a cover, i.e., for each i,

$$c_i \vee \operatorname{coz} g_i^+ = c_i \vee g_i(\mathbb{R}^+) = c_i \vee \bigvee_{w_i(U_1,\ldots,U_n) \subseteq \mathbb{R}^+} \bigwedge_{1 \leq j \leq n} f_j(U_j) = \top.$$

For that purpose fix $k \neq i$, $1 \leq k \leq n$, fix $\varepsilon > 0$, and put

$$U_j \equiv \begin{cases} (-\varepsilon, \infty) & \text{if } j \neq i, k, \\ \left(-\infty, \dfrac{\varepsilon}{n-1}\right) & \text{if } j = i, \\ ((n-1)\varepsilon, \infty) & \text{if } j = k. \end{cases}$$

Clearly $w_i(U_1, \ldots, U_n) \subseteq \mathbb{R}^+$, and since $f_j(U_j) = \top$ for $j \neq i, k$, we get

$$\begin{aligned} c_i \vee \bigwedge_{1 \leq j \leq n} f_j(U_j) &= c_i \vee \left(f_k((n-1)\varepsilon, \infty) \wedge f_i\left(-\infty, \frac{\varepsilon}{n-1}\right)\right) \\ &= (c_i \vee f_k((n-1)\varepsilon, \infty)) \wedge \left(c_i \vee f_i\left(-\infty, \frac{\varepsilon}{n-1}\right)\right) \\ &= (c_i \vee f_k((n-1)\varepsilon, \infty)) \wedge \left(f_i(\mathbb{R}^+) \vee f_i\left(-\infty, \frac{\varepsilon}{n-1}\right)\right) \\ &= c_i \vee f_k((n-1)\varepsilon, \infty). \end{aligned}$$

Therefore

$$c_i \vee \operatorname{coz} g_i^+ \geq \bigvee_{\varepsilon > 0} (c_i \vee f_k((n-1)\varepsilon, \infty)) = c_i \vee \bigvee_{\varepsilon > 0} f_k((n-1)\varepsilon, \infty) = c_i \vee c_k.$$

Since k could be any index other than i, we conclude that $c_i \vee \operatorname{coz} g_i^+ \geq \bigvee_{1 \leq j \leq n} c_j = \top$.

What remains is to show that the common refinement of these binary covers refines C. For that purpose we need only show that $\bigwedge_{1 \leq i \leq n} \operatorname{coz} g_i^+ = \bot$. But

$$\begin{aligned} \bigwedge_{1 \leq i \leq n} \operatorname{coz} g_i^+ &= \operatorname{coz}\Big(\bigwedge_{1 \leq i \leq n} g_i^+\Big) = \operatorname{coz}\Big(\bigwedge_{1 \leq i \leq n} g_i\Big)^+ = \Big(\bigwedge_{1 \leq i \leq n} g_i\Big)(\mathbb{R}^+) \\ &= \bigvee_{w(U_1,\ldots,U_n) \subseteq \mathbb{R}^+} \bigwedge_{1 \leq i \leq n} f_i(U_i), \end{aligned}$$

where w is the n-ary operation on CL associated with the term $\tau \equiv \bigwedge_{1 \leq i \leq n} \tau_i$. But it is precisely the purpose of Lemma 5.1.5 to point out that $w(U_1, \ldots, U_n) \subseteq \mathbb{R}^+$ implies $U_i = \bot$ for all i. ∎

COROLLARY 5.1.7. *The finite cozero covers of a frame L are generated by the finite principal covers, by the finite linked cozero covers, and by the binary covers.*

We remark that not every countable cozero cover can be refined by a finite meet of principal covers. Corson and Isbell [16] show that a space X has the feature that every cover is refined by a finite meet of principal covers if and only if X is Lindelöf and has a compact subset C such that every closed subset disjoint from C has finite dimension. The space $\mathbb{R}^\omega$ is Lindelöf but lacks any such compact subset. Therefore $\mathbb{R}^\omega$ has a countable cover which cannot be refined by the finite meet of finitely many principal covers. The authors would like to thank A. W. Hager for bringing this paper to their attention.

5.2. Linked covers and towers. Principal covers are closely related to two particularly simple kinds of covers, linked covers and towers. To define these types of covers requires a little terminology. A subset $I \subseteq \mathbb{Z}$ is said to be *convex* if for all $i, j \in I$ and $n \in \mathbb{Z}$, $i \leq n \leq j$ implies $n \in I$. Note that convex subsets of $\mathbb{Z}$ are of essentially only four order types: finite, infinite and left bounded, infinite and right bounded, and unbounded.

Definition 5.2.1. A cover $\{c_i : i \in I\}$ of L is *linked* if it is indexed by a convex subset $I \subseteq \mathbb{Z}$ such that for all $i, j \in I$,

$$c_i \wedge c_j > \bot \;\Rightarrow\; |i - j| \leq 1.$$

That is, each c_i is allowed to meet at most c_{i-1} and c_i nontrivially.

Hager investigated linked covers, under the name linear covers, in uniform spaces [20]. He showed that they form a subbase for the principal covers generated by uniformly continuous functions. Thus his work prefigures Proposition 5.2.2.

Proposition 5.2.2. *Every principal cover of a frame L is refined by a linked cozero cover. And every linked cozero cover is principal.*

Proof. The first sentence of the proposition follows from the fact that every cover of $\mathcal{O}\mathbb{R}$ is refined by a linked cover. Consider now a linked cozero cover; we consider first the case of a finite index set, say $\{c_i : 1 \leq i \leq n\}$, $n > 2$. For each i, $1 \leq i < n$, let $f_i \in CL$ be given by Proposition 5.1.2 and such that

$$f_i(\mathbb{R}_1) = \bigvee_{1 \leq j \leq i} c_j, \quad f_i(\mathbb{R}_0) = \bigvee_{i < j \leq n} c_j, \qquad 0 \leq f_i \leq 1.$$

In view of the fact that $0 \leq f_i \leq 1$, we have $f_i(\mathbb{R}_1) = f_i(-\infty, 1)$ and $f_i(\mathbb{R}_0) = f_i(\mathbb{R}^+)$ for all i. Observe that for $1 < i < n$ and $\varepsilon > 0$ we have $f_i(-\infty, \varepsilon) \geq \bigvee_{1 \leq j < i} c_j$. This follows from the facts that

$$f_i(-\infty, \varepsilon) \vee \bigvee_{i < j \leq n} c_j = f_i(-\infty, \varepsilon) \vee f_i(0, \infty) = f_i(\mathbb{R}) = \top,$$

$$\bigvee_{1 \leq j < i} c_j \wedge \bigvee_{i < j \leq n} c_j = \bot.$$

Set $g \equiv \sum_{1 \leq i < n} f_i \in CL$. We first claim that $g(-\infty, k) = \bigvee_{1 \leq j \leq k} c_j$ for any integer k, $1 \leq k < n$, i.e., that

$$g(-\infty, k) = \bigvee_{\sum_{1 \leq i < n} U_i \subseteq (-\infty, k)} \bigwedge_{1 \leq i < n} f_i(U_i) = \bigvee_{1 \leq j \leq k} c_j.$$

To establish this equality first consider a family $\{U_i\} \subseteq \mathcal{O}\mathbb{R}$ such that $\sum_{1 \leq i < n} U_i \subseteq (-\infty, k)$. Since each U_i must be contained in an interval bounded on the right, we may assume it to be of the form $(-\infty, y_i)$ for $y_i \in \mathbb{R}$. If $\bigwedge_{1 \leq i < n} f_i(U_i) > \bot$ it follows that

$y_i > 0$ for $1 \le i < n$, and since $\sum_{1\le i<n} y_i \le k$ there must be an index $l \le k$ for which $y_l \le 1$. But then

$$\bigwedge_{1\le i<n} f_i(U_i) \le f_l(U_l) \le f_l(-\infty,1) = \bigvee_{1\le j\le l} c_j \le \bigvee_{1\le j\le k} c_j.$$

This shows that $g(-\infty,k) \le \bigvee_{1\le j\le k} c_j$. Next fix $\varepsilon > 0$, and put

$$U_i \equiv \begin{cases} (-\infty, 1+\varepsilon) & \text{if } 1 \le i < k,\\ (-\infty, 1-(n-2)\varepsilon) & \text{if } i = k,\\ (-\infty, \varepsilon) & \text{if } k < i < n.\end{cases}$$

(If $k = 1$ the first clause of the definition applies to no U_i, and if $k = n-1$ the last clause applies to no U_i.) Observe that $\sum_{1\le i<n} U_i \subseteq (-\infty, k)$. Since $f_i(U_i) = \top$ for $1 \le i < k$, and since for $k < i < n$,

$$f_i(U_i) \ge \bigvee_{1\le j\le k} c_j = f_k(-\infty,1) \ge f_k(U_k)$$

by the observation at the end of the first paragraph above, it follows that

$$\bigwedge_{1\le i<n} f_i(U_i) = f_k(U_k) = f_k(-\infty, 1-(n-2)\varepsilon).$$

Therefore

$$g(-\infty,k) \ge \bigvee_{\varepsilon>0} f_k(-\infty,1-(n-2)\varepsilon) = f_k(-\infty,1) = \bigvee_{1\le j\le k} c_j,$$

which completes the proof of the first claim. A similar argument can be used to establish that $g(k-1,\infty) = \bigvee_{k+1\le l\le n} c_l$ for $0 \le k < n$.

To complete the proof of the case, simply note that for $0 \le k < n$ we get

$$\begin{aligned} g(k-1,k+1) &= g(-\infty,k+1) \wedge g(k-1,\infty) = \bigvee_{1\le j\le k+1} c_j \wedge \bigvee_{k+1\le l\le n} c_l \\ &= \bigvee\{(c_j \wedge c_l) : 1 \le j \le k+1 \le l \le n\} = c_{k+1}, \end{aligned}$$

the last equality holding because $\{c_j\}$ is a linked cozero cover. The desired cover of $\mathbb{R}$ is obtained by setting

$$U_1 \equiv (-\infty,1), \qquad U_{k+1} \equiv (k-1,k+1), \quad 1 \le k < n-1, \qquad U_n \equiv (n-2,\infty).$$

We have arranged for $g(U_j)$ to be c_j for $1 \le j \le n$.

The second case is that of a linked cozero cover $\{c_i : i \in \mathbb{N}\}$ indexed by the natural numbers. As in the first case, find for each $i \in \mathbb{N}$ some $f_i \in CL$ such that

$$f_i(\mathbb{R}_1) = \bigvee_{1\le j\le i} c_j, \quad f_i(\mathbb{R}_0) = \bigvee_{i<j} c_j, \qquad 0 \le f_i \le 1.$$

(The definition of $f_i(\mathbb{R}_0)$ uses the fact that the $\operatorname{Coz} L$ is a σ-frame, and is therefore closed under countable joins. See Subsection 2.1.) Set $g_n \equiv \sum_{1\le i\le n} f_i$ for each $n \in \mathbb{N}$. We claim that $\{g_n\}$ is a $*$-Cauchy sequence in CL. For if $m < n$ then

$$\operatorname{coz}(g_n - g_m) = \operatorname{coz} \sum_{m<i\le n} f_i = \bigvee_{m<i\le n} \operatorname{coz} f_i = \bigvee_{m<i\le n} f_i(\mathbb{R}_0) = \bigvee_{m+1<j} c_j.$$

Therefore $(\bigvee_{1\le j\le m} c_j)\wedge \mathrm{coz}(g_n-g_m)=\bot$, meaning that

$$\bigvee_{1\le j\le m} c_j \le d_m \equiv \bigwedge_{k,l\ge m} (\mathrm{coz}(g_k-g_l))^*.$$

Since

$$\bigvee_m \bigwedge_{k,l\ge m} (\mathrm{coz}(g_k-g_l))^* = \bigvee_m d_m \ge \bigvee_j c_j = \top,$$

it follows that $\{g_n\}$ is $*$-Cauchy and therefore has a limit g in CL by Proposition 4.2.5. Set

$$U_1 \equiv (-\infty,1), \quad U_{k+1}\equiv (k-1,k+1), \quad k\ge 1.$$

Then for any $k\in\mathbb{N}$ we get

$$\begin{aligned} g(U_k) &= g(U_k)\wedge\top = g(U_k)\wedge\bigvee_m d_m = \bigvee_m (g(U_k)\wedge d_m) = \bigvee_{m\ge k} (g(U_k)\wedge d_m)\\ &= \bigvee_{m\ge k} (g_m(U_k)\wedge d_m). \end{aligned}$$

(The penultimate equality is justified by the fact that $\{d_m\}$ is increasing, while the last equality holds by Corollary 4.2.8.) But g_m is a finite sum as in the first case, so for $m\ge k$ we get

$$g_m(U_k) = c_k \le \bigvee_{1\le j\le m} c_j \le d_m,$$

with the result that $g(U_k)=c_k$ for all k.

In the only remaining case we have a linked cozero cover $\{c_j : j\in\mathbb{Z}\}$ indexed by the integers. By two appeals to the second case above we can construct elements $g_+, g_-\in (CL)^+$ such that

$$\begin{aligned} g_+(-\infty,1) &= \bigvee_{j\le 0} c_j, \quad g_+(k-1,k+1)=c_k, \quad k\ge 1,\\ g_-(-\infty,1) &= \bigvee_{j\ge 0} c_j, \quad g_-(k-1,k+1)=c_k, \quad k\le 1. \end{aligned}$$

Then it is light work to verify that $g\equiv g_+-g_-\in CL$ has the feature that $g(k-1,k+1) = c_k$ for all $k\in\mathbb{Z}$. ∎

Linked covers are closely related to another kind of cover called a tower. The relationship between them is the content of Proposition 5.2.4.

Definition 5.2.3. A *tower* in L is a subset $\{c_i : i\in I\}\subseteq L$ indexed by a convex subset $I\subseteq\mathbb{Z}$ such that $c_i\le c_j$ for all $i\le j$ in I, and such that $\bigvee_I c_i=\top$. A *cozero tower* is a tower of cozero elements. A tower is said to be *regular* if $c_i\prec c_j$ for all $i<j$ in I.

Observe that a regular cozero tower $\{c_i : i\in I\}$ is actually completely regular in the sense that $c_i \prec\!\prec c_j$ for $i<j$ in I.

Proposition 5.2.4. *If $\{c_i : i\in I\}$ is a linked cover of L then $\{\bigvee_{j\le i} c_j : i\in I\}$ is a regular tower. On the other hand, for every regular tower $\{a_i : i\in I\}$ there is a linked cover $\{c_i : i\in I\}$ for which $a_i=\bigvee_{j\le i} c_j$ for all $i\in I$. And if either cover consists of cozero elements, so does the other.*

Proof. Suppose that $\{c_i : i \in I\}$ is a linked cover of L, and set $a_i = \bigvee_{j \le i} c_j$ for all $i \in I$. If $|I| \le 2$ then $\{a_i\}$ is clearly a regular tower, and if $|I| \ge 3$ then $\{a_i\}$ is also a regular tower because, for each $i \in I$ such that $i + 1 \in I$, the element $\bigvee_{i+1<j} c_j$ serves as a separating element for $a_i \prec a_{i+1}$. Now suppose that $\{a_i : i \in I\}$ is a regular tower. If $|I| \le 2$ we may take this same cover to be the desired linked cover. If $|I| \ge 3$ then for each $i \in I$ such that $i + 1 \in I$ choose a separating element b_i witnessing $a_i \prec a_{i+1}$. Note that $b_i \ge b_{i+1}$ for all $i \in I$ such that $i + 2 \in I$. Put

$$c_1 \equiv a_1, \quad c_2 \equiv a_2, \quad c_i \equiv a_i \wedge b_{i-2}, \quad 3 \le i \in I.$$

Then a simple induction shows that $\bigvee_{j \le i} c_j = a_i$ for all $i \in I$. And if $i + 2 \le j$ in I then the fact that $c_j \le b_i$ implies that $c_i \wedge c_j = \perp$, which is to say that $\{c_i\}$ is a linked cover. ∎

COROLLARY 5.2.5. *Every regular cozero tower is a principal cover.*

Proof. For a given regular cozero tower $\{a_i : i \in I\}$ in L, first use Proposition 5.2.4 to get a linked cozero cover $\{c_i : i \in I\}$ such that $a_i = \bigvee_{j \le i} c_j$ for all $i \in I$. Then use Proposition 5.2.2 to find $f \in CL$ and cover $\{U_i\}$ of $\mathcal{O}\mathbb{R}$ such that $f(U_i) = c_i$ for all $i \in I$. Set $V_i \equiv \bigvee_{j \le i} U_j$ for all $i \in I$. Then $\{V_i\}$ is a cover of $\mathcal{O}\mathbb{R}$, and $f(V_i) = a_i$ for all $i \in I$. That is, $\{a_i\}$ is principal. ∎

We conclude this subsection with two results which will be needed in the proof of Theorem 7.2.7.

PROPOSITION 5.2.6. *Let $m : L \twoheadrightarrow M$ be a surjective homomorphism such that every principal cover of M is refined by the image of the meet of finitely many principal covers of L. Then every regular tower in* $\operatorname{Coz} M$ *is refined by the image of a regular tower in* $\operatorname{Coz} L$.

Proof. Let A be a regular tower in $\operatorname{Coz} M$. Since this cover is principal by Corollary 5.2.5, there must be principal covers $C_i \subseteq L$, $1 \le i \le p$, such that the image of their meet C refines A. Without loss of generality we may assume each C_i to be linked by Proposition 5.2.2, say $C_i = \{c_{ij} : j \in I_i\}$. Of course, the index sets I_i need not be the same for all i, but by setting $c_{ij} = \perp$ for all j for which c_{ij} is not initially defined, we may assume that $I_i = \mathbb{Z}$ for all i. Set

$$b_n \equiv \bigwedge_{1 \le i \le p} \bigvee_{|j| \le n} c_{ij}, \qquad n \in \mathbb{N}.$$

Observe that $B \equiv \{b_n\}$ constitutes a regular tower in $\operatorname{Coz} L$ because for any $n \in \mathbb{N}$, $\bigvee_{|j| \le n} c_{ij} \prec \bigvee_{|j| \le n+1} c_{ij}$ for $1 \le i \le p$ implies $b_n \prec b_{n+1}$. Now fix $n \in \mathbb{N}$, and write

$$m(b_n) = \bigwedge_{1 \le i \le p} \bigvee_{|j| \le n} m(c_{ij}) = \bigvee_{\theta \in \Theta} \bigwedge_{1 \le i \le p} m(c_{i\theta(i)}),$$

where Θ designates the set of choice functions $\theta : \{1, \ldots, p\} \to \{j : |j| \le n\}$. Now $\bigwedge_{1 \le i \le p} c_{i\theta(i)}$ lies in C for each such θ, and since $m(C)$ refines A, there must be some $a \in A$ for which $\bigwedge_{1 \le i \le p} m(c_{i\theta(i)}) \le a$. Since Θ is finite and A is totally ordered, this implies that there is some $a_n \in A$ such that $m(b_n) \le a_n$. We have shown that $m(B)$ refines A. ∎

The last result of this subsection shows how to construct an element of CL with any prescribed rate of growth on a regular cozero tower.

Proposition 5.2.7. *Let $\{b_n : n \in \mathbb{N}\}$ be a regular tower in $\operatorname{Coz} L$, and let $\{r_n\}$ be a strictly increasing sequence of positive real numbers. Then there is some $g \in (CL)^+$ such that $g(-\infty, r_n) = b_n$ for all $n \in \mathbb{N}$.*

Proof. Choose separating elements c_n for $b_n \prec b_{n+1}$, i.e., $c_n \wedge b_n = \bot$ and $c_n \vee b_{n+1} = \top$ for all n. Note that necessarily $c_n \succ c_{n+1}$ for all n. As a preliminary step in the construction, we choose a sequence $\{h_n\} \subseteq CL$ as follows. Set $h_1 \equiv r_1 - f$, where f is some element of CL such that $0 \leq f \leq r_1$ and $\operatorname{coz} f = b_1$. Then $\operatorname{coz}(r_1 - h_1) = \operatorname{coz} f = b_1$, and an easy application of the fundamental formula of Proposition 3.1.1 gives $h_1(-\infty, r_1) = b_1$. For $n \geq 2$ use Proposition 5.1.2 to find $h'_n \in (CL)^+$ such that $0 \leq h'_n \leq 1$, $h'_n(-\infty, 1) = b_n$, and $h'_n(0, \infty) = \operatorname{coz} h'_n = c_{n-1}$. Then put $h_n \equiv r_n h'_n$, so that $0 \leq h_n \leq r_n$, $h_n(-\infty, r_n) = b_n$, and $h_n(0, \infty) = \operatorname{coz} h_n = c_{n-1}$. Finally, set

$$g_n \equiv \bigvee_{1 \leq i \leq n} h_n.$$

We claim that

$$g_m(-\infty, r_n) = b_n \qquad \text{for all } m \geq n.$$

Indeed,

$$\begin{aligned} g_m(-\infty, r_n) &= \bigvee_{\bigvee_{1 \leq i \leq m} U_i \subseteq (-\infty, r_n)} \bigwedge_{1 \leq i \leq m} h_i(U_i) = \bigwedge_{1 \leq i \leq m} h_i(-\infty, r_n) \\ &= \bigwedge_{1 \leq i < n} h_i(-\infty, r_n) \wedge h_n(-\infty, r_n) \wedge \bigwedge_{n < i \leq m} h_i(-\infty, r_n). \end{aligned}$$

Now for $i < n$ we have $r_i < r_n$, and because $h_i \leq r_i$ it follows that $h_i(-\infty, r_n) = \top$, so that $\bigwedge_{1 \leq i < n} h_i(-\infty, r_n) = \top$. For $i > n$ we have

$$h_i(-\infty, r_n) \vee c_n \geq h_i(-\infty, r_n) \vee c_{i-1} = h_i(-\infty, r_n) \vee h_i(0, \infty) = \top,$$

and because $c_n \wedge b_n = \bot$, it follows that $b_n \leq h_i(-\infty, r_n)$, so that $b_n \leq \bigwedge_{n < i \leq m} h_i(-\infty, r_n)$. Since $h_n(-\infty, r_n) = b_n$, the claim that $g_m(-\infty, r_n) = b_n$ has been established.

We next prove that $\{g_n\}$ is a $*$-Cauchy sequence. This will follow from the claim that $\operatorname{coz}(g_l - g_k) \leq c_k$ for all $k < l$, for this claim implies that

$$d_m \equiv \bigwedge_{k, l \geq m} (\operatorname{coz}(g_l - g_k))^* \geq b_m$$

for all $m \in \mathbb{N}$, with the result that $\bigvee_{\mathbb{N}} d_m \geq \bigvee_{\mathbb{N}} b_m = \top$. To prove the claim we fix $k < l$ and examine

$$\begin{aligned} g_l - g_k &= \bigvee_{1 \leq i \leq l} h_i - \bigvee_{1 \leq j \leq k} h_j = \Big(\bigvee_{1 \leq j \leq k} h_j \vee \bigvee_{k < i \leq l} h_i \Big) - \bigvee_{1 \leq j \leq k} h_j \\ &= 0 \vee \Big(\bigvee_{k < i \leq l} h_i - \bigvee_{1 \leq j \leq k} h_j \Big) = \bigvee_{k < i \leq l} \Big(h_i - \bigvee_{1 \leq j \leq k} h_j \Big)^+ \end{aligned}$$

Writing $g_l - g_k$ in this way gives

$$\operatorname{coz}(g_l - g_k) = \operatorname{coz}\Big(\bigvee_{k<i\le l}\Big(h_i - \bigvee_{1\le j\le k} h_j\Big)^+\Big) = \bigvee_{k<i\le l}\operatorname{coz}(h_i - \bigvee_{1\le j\le k} h_j)^+$$
$$\le \bigvee_{k<i\le l}\operatorname{coz} h_i = \bigvee_{k<i\le l} c_{i-1} = c_k.$$

This proves the claim that $\{g_n\}$ is $*$-Cauchy.

By Proposition 4.2.5 there is a unique $g \in CL$ such that $g_n \xrightarrow{*} g$. It remains to show that $g(-\infty, r_n) = b_n$ for all n. We have

$$g(-\infty, r_n) = g(-\infty, r_n) \wedge \top = g(-\infty, r_n) \wedge \bigvee_m b_m = \bigvee_m (g(-\infty, r_n) \wedge b_m)$$
$$= \bigvee_m (g_m(-\infty, r_n) \wedge b_m).$$

The last equality is justified by Corollary 4.2.8, for

$$g(-\infty, r_n) \wedge b_m = g(-\infty, r_n) \wedge d_m \wedge b_m = g_m(-\infty, r_n) \wedge d_m \wedge b_m = g_m(-\infty, r_n) \wedge b_m.$$

Now for $m < n$ we have $g_m(-\infty, r_n) = \top$ as a result of the fact that $g_m \le r_m < r_n$. Therefore $\bigvee_{m<n}(g_m(-\infty, r_n) \wedge b_m) = b_{n-1}$. And for $m \ge n$ we have remarked above that $g_m(-\infty, r_n) = b_n$. It follows that $g(-\infty, r_n) = b_n$. ∎

We remark that Proposition 5.2.7 provides a second proof of Corollary 5.2.5, at least in the case of regular cozero towers indexed by $\mathbb{N}$.

6. Complete separation

Complete separation is of central importance when dealing with the problem of extending continuous functions. In this section we will consider complete separation in two different contexts, namely f-rings and frames, and then show in Proposition 6.2.10 how the two versions are simply different views of the same idea. Both views are required for our proof of the main Theorem 7.1.1.

6.1. Complete separation in archimedean f-rings. We first formulate the notion of complete separation in **AfR**. We assume the terminology of the localic Yosida representation outlined in Subsection 2.3. In particular, if $K \subseteq G \in$ **bAfR** then $\langle K\rangle$ denotes the **bAfR**-kernel generated by K.

Definition 6.1.1. In an **AfR**-object H, g *completely separates* h_1 *from* h_2 provided that $0 \le g \le 1$ and

$$g \wedge |h_1| = 0 \quad \text{and} \quad (1-g) \wedge |h_2| = 0.$$

For $G \le H$, we say that elements $h_i \in H$ are *completely separated in* G provided that there exists some $g \in G$ which completely separates h_1 from h_2. Elements $h_i \in H$ are *completely separated* if they are completely separated in H.

A key concept needed to work with complete separation is lifting. Let us agree to say that two continuous real-valued functions on a space *lift one another* if their join is bounded away from zero. We formulate this idea more generally in **AfR** as follows.

Definition 6.1.2. For $0 \le f, g \in G \in \mathbf{AfR}$, we say that f *lifts* g whenever $n(f \vee g) \ge 1$ for some $n \in \mathbb{N}$.

Note that f lifts g if and only if g lifts f if and only if f lifts ng for some $n \in \mathbb{N}$ if and only if f lifts ng for all $n \in \mathbb{N}$. More important, note that lifting really is an attribute determined by bounded elements: f lifts g if and only if $f \wedge 1$ lifts $g \wedge 1$.

Proposition 6.1.3. *The following are equivalent for* $0 \le f, g \in G \in \mathbf{AfR}$:

(1) f *lifts* g.
(2) *No nonzero* **bAfR**-*morphism out of* G^* *takes both* f *and* g *to* 0, *i.e.*, $1 \in \langle f, g\rangle$.
(3) *There is some* $h \in G$, $0 \le h \le 1$, *such that* $\langle h\rangle = \langle f\rangle$ *and* $\langle 1-h\rangle = \langle g\rangle$.
(4) *There is some* $h \in G$, $0 \le h \le 1$, *such that* $\langle h\rangle \subseteq \langle f\rangle$ *and* $\langle 1-h\rangle \subseteq \langle g\rangle$.
(5) $\operatorname{coz}\widehat{f} \vee \operatorname{coz}\widehat{g} = \top$.
(6) $\operatorname{coz}\theta(f) \vee \operatorname{coz}\theta(g) = \top$ *for any frame* L *and any* **bAfR**-*morphism* $\theta : G^* \to C^*L$.

Proof. If (1) holds then, because $\langle f, g\rangle$ contains $n(f \vee g)$ and is convex, (2) clearly also holds. If (2) holds then 1 must lie in the **bAfR**-kernel generated by $f \vee g$, which is the uniform closure of the convex ℓ-subgroup generated by $f \vee g$. Since this convex ℓ-subgroup is

$$\{h \in G : |h| \le n(f \vee g) \text{ for some } n \in \mathbb{N}\},$$

it follows that the sequence $\{n(f \vee g) \wedge 1\}$ converges uniformly to 1. Thus $n(f \vee g) \wedge 1 \ge \frac{1}{2}$ for some n, i.e., (1) holds.

If (1) holds for f and g then, by exchanging them for $nf \wedge 1$ and $ng \wedge 1$ for sufficiently large $n \in \mathbb{N}$, we may assume that $f \vee g = 1$. Set

$$h \equiv \frac{1}{2}(1 + f - g) = \frac{1}{2}(f \vee g + f - g) = \frac{1}{2}((2f - g) \vee f).$$

Then $\langle h\rangle = \langle f\rangle$ because

$$\frac{1}{2}f \le \frac{1}{2}((2f - g) \vee f) \le \frac{1}{2}(2f \vee f) = f,$$

and since $1 - h = \frac{1}{2}(1 - f + g)$, a symmetrical argument shows that $\langle 1-h\rangle = \langle g\rangle$. Thus (3) holds. Finally, (3) clearly implies (4), and if (4) holds then (2) holds as well because

$$1 \in \langle h\rangle \vee \langle 1-h\rangle \subseteq \langle f\rangle \vee \langle g\rangle = \langle f \vee g\rangle.$$

Since $\operatorname{coz}\widehat{f}$ is literally $\langle f\rangle$ in the Yosida representation of G, the equivalence of (2) and (5) is obvious. To show that (5) implies (6), consider a frame L and a morphism $\theta : G^* \to C^*L$ in **bAfR**. Let $k : YG \to L$ be the unique frame map for which $(Ck)\mu_G = \theta$. Then because

$$\operatorname{coz}\theta(f) = \theta(f)(\mathbb{R}_0) = (Ck)\widehat{f}(\mathbb{R}_0) = k\widehat{f}(\mathbb{R}_0) = k \operatorname{coz}\widehat{f},$$

and because $\operatorname{coz}\theta(g) = k \operatorname{coz}\widehat{g}$ for similar reasons, it is clear that (5) must imply (6) and is therefore equivalent to it. ∎

We now use lifting to characterize complete separation.

LEMMA 6.1.4. *Elements h_1 and h_2 in some* **AfR**-*object H are completely separated in G if and only if there are elements $g_i \in G^+$ such that $g_i \wedge |h_i| = 0$ and g_1 lifts g_2.*

Proof. If $g \in G$ completely separates h_1 from h_2 then we set $g_1 \equiv g$ and $g_2 \equiv 1 - g$. These elements lift one another because

$$g_1 \vee g_2 = g \vee (1-g) \geq \frac{1}{2}$$

is an application of the ℓ-ring identity $2(h \vee k) \geq h + k$. On the other hand, if $g_i \in G^+$ satisfy $g_i \wedge |h_i| = 0$ and if g_1 lifts g_2 then by Proposition 6.1.3 there is some $g \in G$, $0 \leq g \leq 1$, such that $\langle g \rangle = \langle g_1 \rangle$ and $\langle 1-g \rangle = \langle g_2 \rangle$. Now from the fact that

$$\langle g \wedge |h_1| \rangle = \langle g \rangle \cap \langle |h_1| \rangle = \langle g_1 \rangle \cap \langle |h_1| \rangle = \langle g_1 \wedge |h_1| \rangle = 0$$

it follows that $g \wedge |h_1| = 0$, and we likewise deduce that $(1-g) \wedge |h_2| = 0$. Thus g completely separates h_1 from h_2. ∎

LEMMA 6.1.5. *For any $h \in H \in$* **AfR** *and $p < q$ in Q, $(h-q)^+$ is completely separated from $(p-h)^+$.*

Proof. Let $h_1 \equiv (q-h)^+$ and $h_2 \equiv (h-p)^+$. Clearly

$$h_1 \wedge (h-q)^+ = h_2 \wedge (p-h)^+ = 0,$$

and

$$h_1 \vee h_2 = ((q-h) \vee (h-p)) \vee 0 \geq \frac{1}{2}(q-p).$$

Thus h_1 lifts h_2, which is to say that there is some $h \in H$ which completely separates $(h-q)^+$ from $(p-h)^+$ by Lemma 6.1.4. ∎

LEMMA 6.1.6. *Suppose $G \leq H \in$* **bAfR**. *Then G is uniformly dense in H if and only if there is some rational number q, $0 \leq q < \frac{1}{2}$, such that for every $h \in H$, $0 \leq h \leq 1$, there exists some $g \in G$ with $|h-g| \leq q$.*

Proof. The condition is certainly necessary, so suppose it holds and fix $h \in H$. Now h is bounded, say $|h| \leq r$ for some rational number $r > 0$. We claim that for any $n \in \mathbb{N}$ there is some $g_n \in G$ for which

$$|h - g_n| \leq r(2q)^n.$$

Now $h_1 \equiv \frac{1}{2r}(h+r)$ satisfies $0 \leq h_1 \leq 1$, so there is some $g \in G$ for which $|h_1 - g| \leq q$. Then $g_1 \equiv r(2g-1)$ satisfies

$$|h - g_1| = |h + r - 2rg| = 2r\left|\frac{1}{2r}(h+r) - g\right| = 2r|h_1 - g| \leq 2rq.$$

Now suppose the claim holds for n, say $|h - g_n| \leq r(2q)^n$ for some $g_n \in G$. Repeating the foregoing with $h_{n+1} \equiv h - g_n$ in place of h and $r(2q)^n$ in place of r yields a $g \in G$ for which $|h_{n+1} - g| \leq r(2q)^{n+1}$. Setting $g_{n+1} \equiv g_n + g$ completes the induction and the proof. ∎

THEOREM 6.1.7. *The following are equivalent for $G \leq H$ in* **AfR**:

(1) *Two elements of H which are completely separated in H are completely separated in G.*

(2) *G^* is uniformly dense in H^*.*

(3) *G^* and H^* have the same* **bAfR**-*kernels. That is, the map Y^*i which realizes (the restriction to G^* of) the inclusion $i : G \to H$ is a frame isomorphism from* $\ker G^* \equiv Y^*G$ *onto* $\ker H^* \equiv Y^*H$.

Proof. Suppose that (2) holds and consider elements $h_i \in H$ which are completely separated in H. By Lemma 6.1.4 we may find elements $k_i \in H^+$ such that $k_i \wedge |h_i| = 0$ and k_1 lifts k_2. By replacing each k_i by $nk_i \wedge 1$ for sufficiently large n, we may assume that $k_1 \vee k_2 = 1$. Now use the uniform density of G^* in H^* to find $f_i \in G$ satisfying

$$\left|\left(k_i - \frac{1}{4}\right) - f_i\right| \leq \frac{1}{4},$$

and set $g_i \equiv f_i^+$. Then $g_i \wedge |h_i| = 0$ because $g_i \leq k_i$ since $f_i \leq k_i$, and

$$2g_1 \vee 2g_2 = 2(f_1 \vee f_2)^+ \geq 2\left(\left(k_1 - \frac{1}{2}\right) \vee \left(k_2 - \frac{1}{2}\right)\right)^+ = 2\left(k_1 \vee k_2 - \frac{1}{2}\right)^+ = 1.$$

Thus by Lemma 6.1.4 the h_i's are completely separated in G, i.e., (1) holds.

Now suppose (1) holds. To show that (2) holds it is enough by Lemma 6.1.6 to find for given $h \in H$, $0 \leq h \leq 1$, some $g \in G$ satisfying $|h - g| \leq \frac{1}{3}$. First note that by Lemma 6.1.5, $\left(\frac{1}{3} - h\right)^+$ is completely separated from $\left(h - \frac{2}{3}\right)^+$ in H and therefore also by some $f \in G$. We claim that $g \equiv \frac{1}{3}(f + 1)$ is the desired element of G, and we aim to establish this claim in two steps.

The first step is to observe that since f is disjoint from $\left(\frac{1}{3} - h\right)^+$ it is also disjoint from $(1 - 3h)^+$. But

$$f \wedge ((1 - 3h) \vee 0) = 0 \ \Rightarrow \ (f + 3h) \wedge (1 \vee 3h) = 3h,$$

and since f is a lower bound for each term in the meet on the right, it follows that $f \leq 3h$. The second step is to observe that since $1 - f$ is disjoint from $\left(h - \frac{2}{3}\right)^+$ it is also disjoint from $(3h - 2)^+$. But

$$(1 - f) \wedge ((3h - 2) \vee 0) = 0 \ \Rightarrow \ 3 \wedge ((3h + f) \vee (f + 2)) = f + 2,$$

and since $3h$ is a lower bound for each term in the meet on the right, it follows that $3h \leq f + 2$. Combining the steps yields $-1 \leq 3h - (f + 1) \leq 1$, i.e., $|h - g| \leq \frac{1}{3}$.

Finally, the equivalence of (2) and (3) is a consequence of the fact that the uniform density of G in H is equivalent to the two having the same classical Yosida space, whose frames Y^*G and Y^*H are therefore also isomorphic. ■

6.2. Complete separation in frames. We develop the concepts surrounding complete separation in frames. The first step is to formulate the frame counterpart of the notion of completely separated subspaces. Throughout this section we consider a fixed frame surjection $m : L \twoheadrightarrow M$.

$$\begin{tikzcd}
M \arrow[r, two heads, "m_0"] \arrow[d, two heads, "m_1"'] & M_0 & \\
M_1 & L \arrow[ul, two heads, "m"'] & \mathcal{O}\mathbb{R} \arrow[l, "f"']
\end{tikzcd}$$

Definition 6.2.1. Let M_0 and M_1 be quotients of M with quotient maps m_0 and m_1. We say that M_0 and M_1 are *m-completely separated by $f \in CL$* if

$$m_0 m f(\mathbb{R}_0) = m_1 m f(\mathbb{R}_1) = \bot.$$

We say that M_0 and M_1 are *m-completely separated* if they are m-completely separated by some $f \in CL$. When m is the identity map we simply say that M_0 and M_1 are *completely separated*. Elements $b_i \in M$ are m-completely separated if their open quotients $\downarrow b_i$ are m-completely separated.

Observe that for any subset A of a topological space X, $f(x) = 0$ for all $x \in \operatorname{cl} A$ if and only if $f^{-1}(\mathbb{R}_0) \subseteq X \setminus \operatorname{cl} A$. Thus it is clear that the foregoing definition is the frame equivalent of the topological notion of completely separated subspaces. Notice that if a and b are completely separated then $a^* \vee b^* = \top$, $a \wedge b = \bot$ and hence $a \prec b^*$ and $b \prec a^*$. In fact, $a^{**} \prec b^*$ and $b^{**} \prec a^*$. The following result shows that frame maps preserve complete separation, as one would expect.

Proposition 6.2.2. *Let k be a frame surjection for which surjections m_i, k_i, and l_i exist making the following diagram commute*:

$$\begin{tikzcd}
K \arrow[r, two heads, "k_0"] \arrow[d, two heads, "k_1"'] & K_0 & & \\
K_1 & M \arrow[ul, two heads, "k"'] \arrow[r, two heads, "m_0"] \arrow[d, two heads, "m_1"] & M_0 \arrow[ul, two heads, "l_0"'] & \\
& M_1 \arrow[ul, two heads, "l_1"] & L \arrow[ul, two heads, "m"'] & \mathcal{O}\mathbb{R} \arrow[l, "f"']
\end{tikzcd}$$

Then K_0 is km-completely separated from K_1 whenever M_0 is m-completely separated from M_1.

Proposition 6.2.3. *Let M_0 and M_1 be quotients of M with quotient maps m_0 and m_1. Then M_0 and M_1 are m-completely separated if and only if there are $a_i \in \operatorname{Coz} L$ such that $m_i m(a_i) = \bot$ and $a_0 \vee a_1 = \top$. In particular, elements $b_i \in M$ are m-completely separated if and only if there are $a_i \in \operatorname{Coz} L$ such that $m(a_i) \leq b_i^*$ and $a_0 \vee a_1 = \top$.*

Proof. This follows from Proposition 5.1.2 by setting $a_i \equiv f(\mathbb{R}_i)$. ∎

Definition 6.2.4. We say of elements $b_i \in M$ that b_0 *is m-completely below* b_1, and write $b_0 \prec\!\prec_m b_1$, if there exist $c_0 \prec\!\prec c_1$ in L such that $b_0 \leq m(c_0)$ and $m(c_1) \leq b_1$.

Note that when m is the identity map, the m-completely below relation is simply the completely below relation.

Lemma 6.2.5. *Suppose that for $b_0, b_1 \in M$ there are rational numbers $p < q$ and some $f \in CL$ with*

$$b_0 \leq mf(-\infty, p) \leq mf(-\infty, q) \leq b_1.$$

Then $b_0 \prec\!\prec_m b_1$.

Proof. For $g \equiv (f-p)/(q-p) \in CL$, one can readily show by use of the fundamental formula of Proposition 3.1.1 that $f(-\infty,p) = g(-\infty,0)$ and $f(-\infty,q) = g(-\infty,1)$. ∎

LEMMA 6.2.6. *$b_0 \prec\!\prec_m b_1$ in M if and only if there exists a scale $\{c_i\}$ in L with $b_0 \leq m(c_0) \leq m(c_1) \leq b_1$.*

PROPOSITION 6.2.7. *The following are equivalent for elements $b_i \in M$:*

(1) b_0 and b_1 are m-completely separated, i.e., there is some $f \in CL$ for which $b_i \wedge mf(\mathbb{R}_i) = \perp$.

(2) There exist $a_i \in \operatorname{Coz} L$ such that $a_0 \vee a_1 = \top$ and $m(a_i) \wedge b_i = \perp$.

(3) $b_i \prec\!\prec_m b_j^*$ for $i \neq j$.

(4) There is a scale $\{a_i : i \in [0,1]_{\mathbb{Q}}\} \subseteq L$ for which $m(a_0) = b_0$ and $m(a_1) = b_1^*$.

Proof. The equivalence of conditions (1) and (2) is an application of Proposition 6.2.3. To verify that (1) implies (3) observe that if f is a function which m-completely separates m_0 from m_1 then

$$b_0 \leq mf\left(-\infty, \frac{1}{4}\right) \leq mf\left(-\infty, \frac{3}{4}\right) \leq b_1^*.$$

(The first inequality follows from the facts that $b_0 \wedge mf(\mathbb{R}_0) = \perp$ and $mf\left(-\infty, \frac{1}{4}\right) \vee mf(\mathbb{R}_0) = \top$, while the third follows from the fact that $b_1 \wedge mf\left(-\infty, \frac{3}{4}\right) \leq b_1 \wedge mf(\mathbb{R}_1)$ $= \perp$.) This clearly yields (3) because $f\left(-\infty, \frac{1}{4}\right) \prec\!\prec f\left(-\infty, \frac{3}{4}\right)$ in L. To show that (3) implies (1), let $c_0 \prec\!\prec c_1$ satisfy $b_0 \leq m(c_0)$ and $m(c_1) \leq b_1$. By Proposition 2.1.4 there is some $g \in CL$ such that $g(\mathbb{R}_0) \wedge c_0 = \perp$ and $g(\mathbb{R}_1) \leq c_1$. Notice that

$$b_0 \leq mf(-\infty,0) \leq mf(-\infty,1) \leq b_1^*$$

and set $g \equiv (f \vee 0) \wedge 1$. Then use the fundamental formula of Proposition 3.1.1 to show that g m-completely separates b_0 from b_1, i.e., $b_i \wedge mg(\mathbb{R}_i) = \perp$. The proof of the equivalence of (3) and (4) is a minor modification of the proof of Proposition 2.1.4, which can be found in [23, 1.4, IV]. ∎

COROLLARY 6.2.8. *Suppose elements $b_0, b_1 \in M$ are m-completely separated. Then they are completely separated, $b_0 \wedge b_1 = \perp$, and any $a_i \in M$ such that $a_i \leq b_i$ are also m-completely separated.*

COROLLARY 6.2.9. *Elements $b_0, b_1 \in M$ are m-completely separated if and only if there are elements $a_i \in \operatorname{Coz} M$ which are m-completely separated and which satisfy $b_i \leq a_i$.*

Proof. Suppose the b_i's are m-completely separated, so that $b_0 \prec\!\prec_m b_1^*$ by Proposition 6.2.7, say $b_0 \leq mf(-\infty,0) \leq mf(-\infty,1) \leq b_1^*$ for some $f \in CL$. Then

$$b_0 \leq mf(-\infty,0) \leq mf\left(-\infty, \frac{3}{4}\right) \leq mf\left(\frac{3}{4}, \infty\right)^*$$

because $\left(-\infty, \frac{3}{4}\right) \wedge \left(\frac{3}{4}, \infty\right) = \perp$, and $b_1 \leq mf\left(\frac{3}{4}, \infty\right)$ because $(-\infty,1) \vee \left(\frac{3}{4}, \infty\right) = \top$ and $b_1 \wedge mf(-\infty,1) = \perp$. Therefore $mf(-\infty,0)$ and $mf\left(\frac{3}{4}, \infty\right)$ are the cozero elements we seek. ∎

Finally, we demonstrate the relationship between the notions of complete separation in f-rings and in frames, considering only the case when the quotient map is the identity. This may be generalized to include any quotient.

PROPOSITION 6.2.10. *For any frame L, elements $h_0, h_1 \in CL$ are completely separated in CL if and only if $\operatorname{coz} h_0$ and $\operatorname{coz} h_1$ are completely separated in L.*

Proof. If h_0 and h_1 are completely separated by g in CL then $g \wedge |h_0| = 0$ and $(1-g) \wedge |h_1| = 0$. Let $a_0 = \operatorname{coz} g$ and $a_1 = \operatorname{coz}(1-g)$. Then $a_0 \vee a_1 = \top$ since $g \vee (1-g) \geq \frac{1}{2}$, and since $a_i \wedge \operatorname{coz} h_i = \bot$, it follows from Proposition 6.2.7 that $\operatorname{coz} h_0$ and $\operatorname{coz} h_1$ are completely separated in L. Conversely, if $\operatorname{coz} h_0$ and $\operatorname{coz} h_1$ are completely separated in L then there exists $f \in CL$ with $\operatorname{coz} h_i \wedge f(\mathbb{R}_i) = \bot$. But then $f \wedge |h_0| = 0$ and $(1-f) \wedge |h_1| = 0$ and so h_0 and h_1 are completely separated in CL. ∎

7. C-quotients and C^*-quotients

A subspace is C-embedded if every continuous real-valued function out of the subspace can be continuously extended to the space. The corresponding notion in frames is that of a C-quotient.

$$
\begin{tikzcd}
L \arrow[r, two heads, "m"] & M \\
 & \mathcal{O}\mathbb{R} \arrow[ul, "g"] \arrow[u, "h"']
\end{tikzcd}
$$

DEFINITION 7.0.1. Let $m : L \twoheadrightarrow M$ be a frame surjection. We say of frame maps $h \in CM$ and $g \in CL$ that *g is an extension of h over m* if $mg = h$. We say that M is a *C-quotient* and that m is a *C-quotient map* if every $h \in CM$ has an extension over m. We say that M is a *C^*-quotient* and that m is a *C^*-quotient map* if every $h \in C^*M$ has an extension over m.

A C^*-quotient map m is always coz-onto, for any $a \in \operatorname{Coz} L$ is of the form $a = h(\mathbb{R}_0)$ for some $h \in CM$, and by replacing h by $(h \vee 0) \wedge 1$ if necessary, we may assume that $h \in C^*M$. If g is an extension of h over m then $b \equiv g(\mathbb{R}_0)$ is an element of $\operatorname{Coz} L$ such that $m(b) = a$.

7.1. C^*-quotients. We finally arrive at the characterization of C^*-quotients, our main Theorem 7.1.1. Among its equivalent conditions is the surjectivity of the induced map in uniform frames, for which terminology we refer the reader to Subsection 2.2. The equivalence of (1) and (3) is the frame formulation of Urysohn's Theorem.

THEOREM 7.1.1. *The following are equivalent for a quotient M of a frame L with quotient map m:*

(1) *M is a C^*-quotient.*
(2) *In M, $b_0 \prec\!\prec b_1$ if and only if $b_0 \prec\!\prec_m b_1$.*
(3) *Two completely separated elements of M are m-completely separated.*
(4) *Two completely separated quotients of M are m-completely separated.*
(5) *Every binary cozero cover of M is refined by the image of a binary cozero cover of L.*

(6) *Every finite cozero cover of M is refined by the image of a finite cozero cover of L.*

(7) *Every binary cozero cover of M is the image of a binary cozero cover of L.*

(8) *$(C^*m)(C^*L) = \{mg : g \in C^*L\}$ is uniformly dense in C^*M.*

(9) *The map $C^*m : C^*L \to C^*M$ is a surjection in* **bAfR**.

When L is completely regular these conditions are equivalent to the following:

(10) *The map induced between the frames with Stone–Čech uniformities is a uniform surjection, i.e., $m : (L, c^*\alpha_L) \to (M, c^*\alpha_M)$ is a surjection in* **UniFrm**.

Proof. Suppose that (1) holds. If $b_0 \prec\!\prec b_1$ in M by means of an $h \in CM$ such that

$$b_0 \le h(-\infty, 0) \le h(-\infty, 1) \le b_1,$$

then $b_0 \prec\!\prec_m b_1$ by means of an extension of h over m, which is to say that (2) holds. The equivalence of (2) and (3) here is a consequence of the equivalence of (1) and (3) in Proposition 6.2.7. Assume (3), and to prove (4) consider completely separated quotients M_i of M with quotient maps m_i. By Proposition 6.2.3 there exist $a_i \in \operatorname{Coz} M$ such that $m_i(a_i) = \bot$ and $a_0 \vee a_1 = \top$. Corollary 5.1.3 provides $c_i \in \operatorname{Coz} M$ such that $c_i \prec\!\prec a_i$ and $c_0 \vee c_1 = \top$. Since c_0^* and c_1^* are completely separated elements of M, they are m-completely separated by (3). Applying Proposition 6.2.3 again provides $b_i \in \operatorname{Coz} L$ such that $m(b_i) \le c_i^{**}$. But $c_i^{**} \le a_i$ because $c_i \prec a_i$, so we get $m_i m(b_i) = \bot$, i.e., m_0 and m_1 are m-completely separated. Thus (3) implies (4), and of course, (3) is a special case of (4).

Now assume (3) holds and, to establish (5), consider $b_i \in \operatorname{Coz} M$ such that $b_0 \vee b_1 = \top$. Applying Corollary 5.1.3 gives $c_0, c_1 \in \operatorname{Coz} M$ with $c_0 \vee c_1 = \top$ and $c_i \prec b_i$. Then c_0^* and c_1^* are completely separated by the implication from (2) to (1) in Proposition 6.2.7, hence m-completely separated, whereupon the implication from (1) to (2) in Proposition 6.2.7 provides elements $a_i \in \operatorname{Coz} L$ with $m(a_i) \wedge c_j^* = \bot$ for $i \ne j$. It follows that $m(a_i) \le c_i^{**} \le b_i$, and so we have established (5).

Assume (5), and to prove (6) consider a finite cozero cover $B \subseteq M$. By Proposition 5.1.6 there are binary cozero covers $B_i \subseteq M$, $1 \le i \le n$, whose meet refines B. For each i let A_i be a binary cozero cover of L whose image refines B_i. Then the meet of the A_i's lies in $c^*\alpha_L$, and its image refines B. Thus (6) holds, and since (5) is just a special case of it, the two are equivalent. Likewise (7) implies (5), and (7) is implied by the conjunction of (1) and (5) in light of the fact that C^*-quotient maps are coz-onto.

Assuming (5), we aim to prove (8) by verifying condition (1) of Theorem 6.1.7. Consider elements $h_0, h_1 \in C^*M$ completely separated by $k \in C^*M$. Then in M,

$$\operatorname{coz} k \vee \operatorname{coz}(1-k) = k(0,\infty) \vee k(-\infty, 1) = \top,$$

so by (5) above there must be elements $a_i \in \operatorname{Coz} L$ which satisfy

$$m(a_0) \le \operatorname{coz} k, \quad m(a_1) \le \operatorname{coz}(1-k), \quad a_0 \vee a_1 = \top.$$

Now by Proposition 5.1.2 there is some $g \in C^*L$ such that $g(\mathbb{R}_i) = a_i$ and $0 \le g \le 1$. Clearly mg completely separates h_0 from h_1, for

$$\operatorname{coz} mg = mg(\mathbb{R}_0) = m(a_0) \le \operatorname{coz} k \;\Rightarrow\; mg \wedge |h_0| = 0,$$

and $(1 - mg) \wedge |h_1|$ likewise. Thus (8) holds.

The implication from (8) to (9) is immediate from Corollary 4.1.8, and (9) is clearly a reformulation of (1). Thus the equivalence of the first nine conditions has been proven. So assume now that L, and therefore also M, is completely regular, a necessary condition for uniform frames. Since the uniformities $c^*\alpha_L$ and $c^*\alpha_M$ are generated by finite cozero covers, we see that (10) is a reformulation of (6); see Subsection 2.2. ■

The following corollary and its equivalent after Theorem 7.2.3 were pointed out to us by the referee and are noteworthy because they include frames which are very much nonspatial. It also nicely generalizes the obvious fact that any subspace of a discrete space is C^*-embedded.

Corollary 7.1.2. *All quotients of a Boolean frame are C^*-quotients.*

We can now give the frame counterpart of the spatial result that a zero set Z is C^*-embedded in a space X if and only if it is z-embedded [18], i.e., if and only if every zero set of Z is the intersection with Z of a zero set of X.

Corollary 7.1.3. *For $x \in \operatorname{Coz} L$, the closed quotient $\uparrow x$ is a C^*-quotient if and only if the quotient map is coz-onto.*

Proof. As we pointed out at the beginning of this subsection, every C^*-quotient map is coz-onto. Assuming that $x \in \operatorname{Coz} L$ and that the closed quotient map $a \mapsto a \vee x$ is coz-onto, we show that $\uparrow x$ satisfies Theorem 7.1.1(6). Consider $a, b \in \operatorname{Coz}(\uparrow x)$ with $a \vee b = \top_{\uparrow x}$; since $a, b \in \uparrow x$ it follows that $a \vee b = \top_L$. Then there exist $a', b' \in \operatorname{Coz} L$ with $a' \vee x = a$ and $b' \vee x = b$. But because $x \in \operatorname{Coz} L$, both a and b lie in $\operatorname{Coz} L$ as well. ■

A reformulation of Corollary 7.1.3 making use of the adjoint map $m_* : M \to L$, where $m_*(x) \equiv \bigvee\{a : m(a) \le x\}$, $x \in M$, deserves emphasis:

Corollary 7.1.4. *Suppose we have $m : L \twoheadrightarrow M \cong \uparrow c$ for some $c \in \operatorname{Coz} L$. If m_* maps $\operatorname{Coz} M$ onto $\operatorname{Coz} L$ then M is a C^*-quotient.*

7.2. C-quotients. A subspace is C-embedded if and only if it is C^*-embedded and completely separated from every zero set disjoint from it [18, 1.18]. We will show that the corresponding frame result holds. The reader familiar with the classical proof will recognize its shadow in our demonstration.

Recall that a frame map $m : L \to M$ is coz-codense if $c = \top$ whenever $m(c) = \top$ for $c \in \operatorname{Coz} L$. We now define the frame map m to be *almost coz-codense* if for every $c \in \operatorname{Coz} L$ such that $m(c) = \top$ there is a $d \in \operatorname{Coz} L$ such that $m(d) = \bot$ and $c \vee d = \top$. Observe that a coz-codense frame map is almost coz-codense, and that a dense frame map which is almost coz-codense is in fact coz-codense. Lemma 7.2.1 points out that almost coz-codensity is the frame version of the condition that a subspace be completely separated from every zero set disjoint from it.

Lemma 7.2.1. *A frame map $m : L \to M$ is almost coz-codense if and only if for every $c \in \operatorname{Coz} L$ such that $m(c) = \top$ there is an $f \in CL$, $0 \le f \le 1$, such that $f(\mathbb{R}_0) \le c$ and $mf(\mathbb{R}_1) = \bot$.*

Proof. The equivalence of the two conditions follows from Proposition 5.1.2 by taking $d = f(\mathbb{R}_1)$. ■

For frame surjections, the conjunction of the properties of coz-codensity with coz-ontoness implies the property of being a C^*-quotient map. In fact more is true: this conjunction is equivalent to being a C-quotient map, but we need the weaker result to prove the stronger.

PROPOSITION 7.2.2. *A frame surjection which is coz-onto and almost coz-codense is a C^*-quotient map.*

Proof. Let $m : L \twoheadrightarrow M$ be coz-onto and almost coz-codense. By Theorem 7.1.1, we need only show that binary cozero covers of M are refined by the image of binary cozero covers of L . If $x \vee x' = \top$ in $\operatorname{Coz} M$ then by coz-ontoness there exist $a, a' \in \operatorname{Coz} L$ with $m(a) = x$ and $m(a') = x'$. Thus $m(a \vee a') = \top$, and so using the almost coz-codensity, we can find $d \in \operatorname{Coz} L$ with $m(d) = \perp$ and $a \vee a' \vee d = \top$. Now let $c = a \vee d$; then $m(c) = x$ and $c \vee a' = \top$. ∎

THEOREM 7.2.3. *A frame surjection is a C-quotient map if and only if it is coz-onto and almost coz-codense*

Proof. Assume that $m : L \twoheadrightarrow M$ is a C-quotient map. To prove that m is almost coz-codense, we show that the equivalent condition of Lemma 7.2.1 holds. Take $c \in \operatorname{Coz} L$ with $m(c) = \top$, say $c = \operatorname{coz} k$ for $k \in CL^+$. By Proposition 3.3.1 we have $h \in CM$ satisfying $h \cdot mk = 1$. Let g be an extension of h over m, and set $f \equiv g \cdot k$ in CL. We verify that

$$f(\mathbb{R}_0) = (g \cdot k)(\mathbb{R}_0) = \bigvee_{a_1 \cdot a_2 \subseteq \mathbb{R}_0} [g(a_1) \wedge k(a_2)] = g(\mathbb{R}_0) \wedge k(\mathbb{R}_0) \leq k(\mathbb{R}_0) = c.$$

We likewise verify that

$$\begin{aligned} mf(\mathbb{R}_1) = m(g \cdot k)(\mathbb{R}_1) = m\Big(\bigvee_{a_1 \cdot a_2 \subseteq \mathbb{R}_1} [g(a_1) \wedge k(a_2)] \Big) = \bigvee_{a_1 \cdot a_2 \subseteq \mathbb{R}_1} [h(a_1) \wedge mk(a_2)] \\ = (h \cdot mk)(\mathbb{R}_1) = 1(\mathbb{R}_1) = \perp. \end{aligned}$$

This completes the proof of the necessity of almost coz-codensity.

Now suppose that $m : L \twoheadrightarrow M$ is a frame surjection which is coz-onto and almost coz-codense. Then by Proposition 7.2.2 it is a C^*-quotient map. To show m to be a C-quotient map, consider a given $h \in CM$ and let p and q be the frame maps of the inclusions of the subspaces indicated in the following diagram:

$$\begin{tikzcd}
 & L \arrow[r, "m"] & M \\
\mathcal{O}(-1,1) \arrow[ur, "k"] & \mathcal{O}\mathbb{R} \arrow[l, "t"'] \arrow[r, "q"] \arrow[u, "i"] \arrow[ur, "h"] & \mathcal{O}[-1,1] \arrow[ul, "g"'] \arrow[u, "l"'] \arrow[ll, bend left, "p"]
\end{tikzcd}$$

Let t be the frame map of any homeomorphism from $\mathcal{O}(-1,1)$ onto $\mathcal{O}\mathbb{R}$, let l abbreviate $ht^{-1}p$, and let g be an extension of l over m, so that $mg = l$. Set $c \equiv g(-1,1) \in \operatorname{Coz} L$, and observe that

$$m(c) = mg(-1,1) = l(-1,1) = ht^{-1}p(-1,1) = ht^{-1}(\top) = \top,$$

so that by Lemma 7.2.1 there is some $f \in CL$, $0 \le f \le 1$, for which $f(\mathbb{R}_0) \le c$ and $mf(\mathbb{R}_1) = \bot$. Set $i \equiv f \cdot (gq)$. We first claim that $mi = lq$, i.e., that the outer square of the diagram commutes. To verify this claim recall that for any $U \in \mathcal{O}\mathbb{R}$,

$$mi(U) = m\Big(\bigvee_{V\cdot W\subseteq U} [f(V) \wedge gq(W)]\Big) = \bigvee_{V\cdot W\subseteq U} [mf(V) \wedge lq(W)].$$

But if $1 \notin V$ then $mf(V) \le mf(\mathbb{R}_1) = \bot$, meaning that all terms in which $1 \notin V$ drop out of the supremum displayed above. For those that remain, namely those for which $1 \in V$, we have $mf(V) = \top$ because

$$\top = mf(\top) = mf(V \vee \mathbb{R}_1) = mf(V) \vee mf(\mathbb{R}_1) = mf(V).$$

Therefore

$$\begin{aligned} mi(U) &= \bigvee\{lq(W) : \exists V\ (1 \in V \text{ and } V \cdot W \subseteq U)\} \\ &= lq\Big(\bigvee\{W : \exists V\ (1 \in V \text{ and } V \cdot W \subseteq U)\}\Big) = lq(U). \end{aligned}$$

This proves the claim that the outer square commutes.

We next claim that i factors through pq. To that end it suffices by Lemma 2.1.1 to demonstrate that $i(-1,1) = \top$. First observe that

$$\begin{aligned} i(-1,1) &= \bigvee_{V\cdot W\subseteq(-1,1)} [f(V) \wedge gq(W)] \ge \bigvee_{y>1} \left[f(-y,y) \wedge g\Big(-\frac{1}{y}, \frac{1}{y}\Big)\right] \\ &= \bigvee_{y>1} g\Big(-\frac{1}{y}, \frac{1}{y}\Big) = g(-1,1) = c. \end{aligned}$$

(The second equality is justified by the fact that $f(-y,y) = \top$ for $y > 1$ because $0 \le f \le 1$.) Therefore

$$\begin{aligned} i(-1,1) = i(-1,1) \vee c &= \bigvee_{V\cdot W\subseteq(-1,1)} [f(V) \wedge gq(W)] \vee c \\ &= \bigvee_{V\cdot W\subseteq(-1,1)} [(f(V) \wedge gq(W)) \vee c] \ge \bigvee_{V\cdot W\subseteq(-1,1)} [(f(V) \wedge gq(W)) \vee f(\mathbb{R}_0)] \\ &\ge \bigvee_{1>y>0} \left[\Big(f(-y,y) \wedge gq\Big(-\frac{1}{y}, \frac{1}{y}\Big)\Big) \vee f(\mathbb{R}_0)\right] \\ &= \bigvee_{1>y>0} [f(-y,y) \vee f(\mathbb{R}_0)] = \bigvee_{1>y>0} f((-y,y) \vee \mathbb{R}_0) = \top. \end{aligned}$$

(The fourth equality follows from the fact that $q\big(-\frac{1}{y}, \frac{1}{y}\big) = \top$ for $0 < y < 1$.) This proves the claim that i factors through pq, say $i = kpq$.

The desired extension of h is kt because

$$mktt^{-1}pq = mkpq = mi = lq = ht^{-1}pq,$$

and since $t^{-1}pq$ is surjective and hence an epimorphism, it follows that $mkt = h$. ■

We remark in passing that the proof of Theorem 7.2.3 is the only occasion on which we could not avoid using the multiplicative structure of CL. (Use of the property of closure under bounded inversion, which figures into Corollary 3.3.2, Lemma 7.2.5, and Theorem 7.2.7, can be eliminated at the expense of more complicated arguments.) It should be possible to avoid the use of multiplication altogether because the Yosida adjunction exists between **Frm** and the category **W** of archimedean lattice ordered groups with weak order unit. That is, the weaker structure still carries all the information. We challenge the reader to prove Theorem 7.2.3 in **W**.

Corollary 7.2.4. *All quotients of a Boolean frame are C-quotients.*

Proof. Let M be a quotient of the Boolean frame L with quotient map m. Since every element of L is a cozero element, it is clear that m is coz-onto. Now, if $a \in L$ satisfies $m(a) = \top$ then $m(a^*) = \bot$ and $a \vee a^* = \top$. ∎

We aim to prove Theorem 7.2.7, which characterizes C-quotients much as Theorem 7.1.1 characterizes C^*-quotients. For that purpose the following two results are preparatory, and of these two results, the first is folklore. An extension $G \leq H$ in **AfR** is said to be *majorizing* if for every $h \in H^+$ there is some $g \in G^+$ such that $h \leq g$. One says that G *majorizes* H.

Lemma 7.2.5. *Suppose that G is closed under bounded inversion, and suppose that $G \leq H$ is a majorizing extension in* **AfR** *such that $G^* = H^*$. Then $G = H$.*

Proof. Given $h \in H^+$, find $g \in G$ satisfying $h' \equiv h + 1 \leq g$. Since $g \geq 1$, g^{-1} exists in G and therefore $h'g^{-1}$ exists in H. In fact, $h'g^{-1}$ lies in G, for it is bounded by 1. But then $h' = (h'g^{-1})g$ must lie in G as well, so that $h = h' - 1 \in G$. ∎

Proposition 7.2.6. *The following are equivalent for a frame surjection $m : L \twoheadrightarrow M$.*

(1) *Every regular cozero tower in M is refined by the image of a regular cozero tower in L.*

(2) $(Cm)(CL) = \{mg : g \in CL\}$ *majorizes CM. That is, for every $h \in (CM)^+$ there is some $g \in (CL)^+$ such that $h \leq mg$.*

Proof. To prove that (1) implies (2), consider $h \in (CM)^+$ and set $a_n \equiv h(-\infty, n)$ for all $n \in \mathbb{N}$. If h is bounded there is nothing to prove, so assume that $h \notin C^*M$, i.e., $a_n < \top$ for all n. Because $\{a_n\}$ is a regular tower in $\operatorname{Coz} M$, there is by (1) a regular tower $\{b_n\}$ in $\operatorname{Coz} L$ such that $\{m(b_n)\}$ refines $\{a_n\}$; for each positive integer n let $k(n)$ designate the least integer k such that $m(b_n) \leq a_k$. The sequence $\{k(n)\}$ is nondecreasing, and it is unbounded for otherwise some a_k would exceed all the $m(b_n)$'s, giving the contradiction

$$a_k \geq \bigvee_{n\in\mathbb{N}} m(b_n) = m\Big(\bigvee_{n\in\mathbb{N}} b_n\Big) = m(\top) = \top.$$

By passing to a subsequence of $\{b_n\}$ if necessary, we may assume that $k(n) < k(n+1)$ for all n.

Use Proposition 5.2.7 to construct $g \in (CL)^+$ such that $g(-\infty, k(n+1)) = b_n$ for all n. By replacing g with $g \vee k(1)$ if necessary, we may assume that $g \geq k(1)$. (Here $k(1)$ is the constant frame map.) We claim that $mg \geq h$, and we show this with the aid of

Lemma 3.2.4(6) by verifying that $h(-\infty, r) \geq mg(-\infty, r)$ for all real numbers r. So fix r and let n be the least integer for which $r < k(n)$. If $n = 1$ then

$$mg(-\infty, r) \leq mg(-\infty, k(1)) = \perp \leq h(-\infty, r)$$

because $g \geq k(1)$. If $n > 1$ then $k(n-1) \leq r < k(n)$ implies

$$mg(-\infty, r) \leq mg(-\infty, k(n)) = m(b_{n-1}) \leq a_{k(n-1)} = h(-\infty, k(n-1)) \leq h(-\infty, r).$$

This proves the claim and shows that $(Cm)(CL)$ majorizes CM.

Assume (2), and to prove (1) consider a regular tower $\{a_n : n \in I\}$ in $\operatorname{Coz} M$. We seek a tower $\{b_n : n \in I\}$ in $\operatorname{Coz} L$ such that $\{m(b_n)\}$ refines $\{a_n\}$. If I is bounded above in $\mathbb{Z}$ then we may simply take $b_n = \top$ for all $n \in I$. If I is $\mathbb{Z}$ we simply truncate the index set by omitting all negative integers. The reason we can do this is that if $\{b_n : n \in \mathbb{N}\}$ is a regular tower in $\operatorname{Coz} L$ such that $\{m(b_n)\}$ refines $\{a_n : n \in \mathbb{N}\}$ then, by setting

$$b'_n \equiv \begin{cases} \perp & \text{if } n \leq 0, \\ b_n & \text{if } n \geq 1, \end{cases}$$

we get a regular tower $\{b'_n : n \in \mathbb{Z}\}$ in $\operatorname{Coz} L$ such that $\{m(b'_n)\}$ refines $\{a_n\}$. So assume that the index set I is $\mathbb{N}$. Now use Proposition 5.2.7 to construct $h \in (CM)^+$ such that $h(-\infty, n) = a_n$ for all n, and then use (2) to find $g \in (CL)^+$ such that $mg \geq h$. If we set $b_n \equiv g(-\infty, n)$ then we deduce by Lemma 3.2.4(6) that for all n,

$$m(b_n) = mg(-\infty, n) \leq h(-\infty, n) = a_n. \blacksquare$$

Finally we are equipped to prove our main result on C-quotients.

THEOREM 7.2.7. *The following are equivalent for a quotient M of a frame L with quotient map m*:

(1) *M is a C-quotient.*

(2) *M is a C^*-quotient and m is almost coz-codense.*

(3) *m is coz-onto and almost coz-codense.*

(4) *Every principal cover of M is refined by the image of a principal cover of L.*

(5) *Every linked cozero cover of M is refined by the image of a linked cozero cover of L.*

(6) *M is a C^*-quotient and every regular cozero tower of M is refined by the image of a regular cozero tower of L.*

(7) *M is a C^*-quotient and $(Cm)(CL) = \{mg : g \in G\}$ majorizes CM.*

(8) *The map $Cm : CL \to CM$ is a surjection in* **AfR**.

If L is completely regular, *these conditions are equivalent to the following*:

(9) *The map induced between the frames with real uniformities is a uniform surjection*, *that is*, *$m : (L, c\alpha_L) \to (M, c\alpha_M)$ is a surjection in* **UniFrm**.

Proof. The equivalences of (1) and (2) and (3) follow from Theorem 7.2.3. And (1) certainly implies (4), for if A is a cover of M generated by some $f \in CM$ then A is the image of the cover of L generated by any extension of f over m. The equivalence of (4) and (5) is a consequence of Proposition 5.2.2.

To show that (4) implies (6), first observe that a binary cover $A = \{a_0, a_1\}$ of M, being principal by Proposition 5.1.2, must be refined by the image of a principal cover C of L by (4). We may without loss of generality take C to be countable. Then $\{c_0, c_1\}$ is also a principal cover of L, where $c_i \equiv \bigvee\{c \in C : m(c) \leq a_i\}$, and its image refines A as well. This shows that m is a C^*-quotient map by Theorem 7.1.1. Thanks to Proposition 5.2.6, (4) also implies that every regular tower in $\operatorname{Coz} M$ is refined by the image of a regular tower in $\operatorname{Coz} L$. That is, (6) holds. And (6) is clearly equivalent to (7) by Proposition 7.2.6.

If (7) holds then (8) follows by Lemma 7.2.5, whose hypotheses are established in Proposition 3.3.2 and Theorem 7.1.1. And since (8) is just a reformulation of (1), we have established the equivalence of the first eight conditions. Finally, if L (and hence M) is completely regular, so that we may speak in terms of uniform frames, the surjectivity condition in (9) is a reformulation of (4). ∎

8. Applications

Theorems 7.1.1 and 7.2.7 have broad application in the theory of topological spaces, and so it should come as no surprise that they also have broad application for frames. We outline a few of these applications in this section. Some of them involve sharpening Theorems 7.1.1 and 7.2.7 under additional hypotheses. Of the various additional hypotheses we consider, the most powerful is the density of the quotient.

8.1. Maximal extensions of frame maps into dense quotients. Suppose we have a bounded continuous real-valued function g on a dense subspace M of a space L. Then there is a unique largest subspace S to which g can be continuously extended, namely

$$S = \Big\{x \in X : \bigwedge_{U \in \mathcal{N}(x)} \Big(\bigvee_{M \cap U} g(y) - \bigwedge_{M \cap U} g(y)\Big) = 0\Big\}.$$

(Here $\mathcal{N}(x)$ is the filter of neighborhoods of x.) S is often called the *maximum domain* of g. In Theorem 8.1.2 we generalize this result in two directions: to frames, and to functions g out of an arbitrary regular frame, not just $\mathcal{O}[0,1]$. Consequently, for every dense quotient M of a frame L there is a unique largest quotient having M as a C^*-quotient (C-quotient); this is Corollary 8.1.5. Other consequences of these results augment Theorems 7.1.1 and 7.2.7.

We begin by reformulating the preceding arrangement for frames, using the same letters for the objects. We are given dense quotient maps m and q such that m factors through q, say $m = nq$. We consider a frame map g, whose domain B plays the role of $\mathcal{O}[0,1]$ in the preceding paragraph, but which is here assumed only to be a regular frame. We seek necessary and sufficient conditions for the existence of a frame map g' such that $ng' = g$.

We remind the reader that to say that m is dense means that $m_*(\bot) = \bot$, where $m_* : M \to L$ is the adjoint map given by the rule $m_*(b) \equiv \bigvee\{a : m(a) \leq b\}$. We abbreviate $m_* g$ to $\overline{g}$. Both m_* and $\overline{g}$ preserve binary meets, but neither preserves arbitrary joins in general. We fix this terminology for the remainder of the subsection.

Lemma 8.1.1. *$qm_* = n_*$, so that $n_*g = q\bar{g}$.*

Proof. Since $m = nq$, we have $m_* = q_*n_*$. But q is surjective so qq_* is the identity map on S, and so the result follows by applying q to both sides. ■

For each cover C of B we use a_C to designate $\bigvee_C \bar{g}(c)$. Observe that because m is dense, each a_C is a dense element of L, i.e., $a_C^* = \bot$.

Theorem 8.1.2. *There is a frame map g' which makes the diagram*

$$\begin{tikzcd} L \arrow[r, two heads, "m"] \arrow[d, two heads, "q"'] & M \\ S \arrow[ur, two heads, "n"] & B \arrow[l, "g'"] \arrow[u, "g"'] \end{tikzcd}$$

commute if and only if S is smaller than all open quotients of the form $\downarrow a_C$ for covers $C \subseteq B$, i.e., if and only if $q(a_C) = \top$ for every cover C of B.

Proof. Suppose g' exists. To understand why q must take each a_C to $\top$, first note that for any $b \in B$,

$$ng'(b) = g(b) \;\Rightarrow\; g'(b) \leq n_*g(b) = q\bar{g}(b).$$

Thus for a cover C of B,

$$\top = g'\Big(\bigvee C\Big) = \bigvee_C g'(c) \leq \bigvee_C q\bar{g}(c) = q\Big(\bigvee_C \bar{g}(c)\Big) = q(a_C).$$

Now suppose that $q(a_C) = \top$ for every cover C of B. Define $g' : B \to S$ by the rule

$$g'(b) \equiv \bigvee_{z \prec b} q\bar{g}(z) = \bigvee_{z \prec b} n_*g(z).$$

We must first verify that g' is a frame map. It can readily be seen to preserve $\bot$ and $\top$. It preserves binary meets because

$$\begin{aligned} g'(b_1) \wedge g'(b_2) &= \bigvee_{z_1 \prec b_1} q\bar{g}(z_1) \wedge \bigvee_{z_2 \prec b_2} q\bar{g}(z_2) = q\Big(\bigvee_{z_1 \prec b_1} \bar{g}(z_1) \wedge \bigvee_{z_2 \prec b_2} \bar{g}(z_2)\Big) \\ &= q\Big(\bigvee_{z_i \prec b_i} (\bar{g}(z_1) \wedge \bar{g}(z_2))\Big) = q(\bigvee_{z_i \prec b_i} \bar{g}(z_1 \wedge z_2)) \\ &= q\Big(\bigvee_{z \prec b_1 \wedge b_2} \bar{g}(z)\Big) = \bigvee_{z \prec b_1 \wedge b_2} q\bar{g}(z) = g'(b_1 \wedge b_2). \end{aligned}$$

It remains to show that g' preserves arbitrary joins. For that purpose consider $\bigvee_I b_i = b$ in B; we need to prove that $g'(b) \leq \bigvee_I g'(b_i)$, i.e., that

$$g'(b) \equiv \bigvee_{z \prec b} q\bar{g}(z) \leq \bigvee_I g'(b_i) = \bigvee_I \bigvee_{z_i \prec b_i} q\bar{g}(z_i) = q\Big(\bigvee_{z_i \prec b_i} \bar{g}(z_i)\Big).$$

For this, in turn, it is enough to show that $q\bar{g}(z) \leq q(\bigvee_{z_i \prec b_i} \bar{g}(z_i))$ for any $z \prec b$. Here we have some good luck. It turns out that there is a single cover $C \subseteq B$ such that

$$\bar{g}(z) \wedge a_C = \bar{g}(z) \wedge \bigvee_C \bar{g}(c) = \bigvee_C \bar{g}(z \wedge c) \leq \bigvee_{z_i \prec b_i} \bar{g}(z_i).$$

This does it, because an application of q to this inequality gives the conclusion we seek. The desired cover is

$$C \equiv \{z_i : z_i \prec b_i \text{ for some } i \in I\} \cup \{z^*\}.$$

That $\bigvee C = \top$ follows from the fact that $b = \bigvee_{z_i \prec b_i} z_i$ by virtue of the regularity of B, and from the fact that $z^* \vee b = \top$ because $z \prec b$. And C clearly has the property claimed for it: if $c \in C$ then either $c = z_i \prec b_i$ for some i, in which case $\bar{g}(z \wedge c) \leq \bar{g}(z_i) \leq \bigvee_{z_i \prec b_i} \bar{g}(z_i)$, or $c = z^*$, in which case $\bar{g}(z \wedge c) = \bar{g}(\perp) = \perp \leq \bigvee_{z_i \prec b_i} \bar{g}(z_i)$. This completes the proof that g' is a frame map. To verify that $ng' = g$ simply observe that for any $b \in B$,

$$ng'(b) = n\Big(\bigvee_{z \prec b} n_* g(z)\Big) = \bigvee_{z \prec b} nn_* g(z) = \bigvee_{z \prec b} g(z) = g\Big(\bigvee_{z \prec b} z\Big) = g(b). \;\blacksquare$$

Corollary 8.1.3. *There is a unique largest quotient S through whose quotient map m factors as $m = nq$ and into which a frame map g' exists such that $g = ng'$. It is the infimum of the open quotients $\downarrow a_C$ for covers $C \subseteq B$.*

The quotient S which arises in Corollary 8.1.3 plays the frame role of the maximum domain mentioned at the beginning of this section. We therefore refer to S as the *maximum codomain* of g.

Corollary 8.1.4. *A dense quotient M of a frame L is not C^*-embedded (C-embedded) if and only if it lies below the maximum codomain of some $g \in C^*(M)$ ($g \in C(M)$).*

We now allow the function g to vary over all of $C^*(M)$ ($C(M)$).

Corollary 8.1.5. *There is a unique largest quotient S of L having M as a C^*-quotient (C-quotient).*

Proof. To get the C^* version take $B \equiv \mathcal{O}[0,1]$ and take S to be the infimum of the open quotients $\downarrow(\bigvee_C \bar{g}(c))$ for all covers $C \subseteq B$ and all $g \in C^*M$. To get the C version take $B \equiv \mathcal{O}\mathbb{R}$ and let g range over CM. $\blacksquare$

We would like to thank the referee for pointing out that parts of the foregoing results are known and may be used to give an alternate proof. We outline it here. The lifting of g does not depend on S and M being quotients of L. Requiring that $q\bar{g}$ preserves covers is equivalent to requiring that $n_* g$ does (see Lemma 8.1.1), and this, by the Extension Theorem in [12], is exactly the necessary and sufficient condition for the existence of $g' : B \to S$ such that $ng' = g$. Now in the context of quotients, given $m : L \twoheadrightarrow M$ one can define S as the quotient of L which makes all $m_* g[C]$, for C any cover of B, into covers.

We can now interpret the classical result about maximum domains in light of Theorem 8.1.2. Given a continuous function f from a dense subspace M of L into a regular space, there is a unique largest *sublocale* S containing M to which f can be continuously extended. The spatial part of S is the maximum domain of f.

Using Theorem 8.1.2 one may obtain the following refinements of Theorems 7.1.1 and 7.2.7; for this we require that $S = L$. The appearance of the adjoint map in these results prefigures Theorems 8.2.6 and 8.2.12.

Corollary 8.1.6. *A dense quotient map m is a C^*-quotient if and only if $\bigvee_A m_*(a) = \top$ for all finite (or binary) cozero covers $A \subseteq M$. A dense quotient map m is a C-quotient if and only if $\bigvee_A m_*(a) = \top$ for all principal (or linked cozero) covers $A \subseteq M$.*

8.2. Dense quotients of completely regular frames. With the added condition of density in the setting of complete regularity, it is possible to greatly simplify the proofs of Theorems 7.1.1 and 7.2.7. Bernhard Banaschewski kindly pointed out to us the following two elegant demonstrations, which take advantage of the beautiful properties of the uniform completions in the context of dense quotients.

Theorem 8.2.1. *A dense surjection $m : L \twoheadrightarrow M$ is a C^*-quotient map if and only if the m-completely below relation coincides with the completely below relation.*

Proof. The forward implication is always true and easy to prove. For the converse we recall that the Stone–Čech uniformities on L and M are denoted by $c^*\alpha_L$ and $c^*\alpha_M$ respectively. Since $\prec\!\prec$ coincides with $\prec\!\prec_m$, it is easy to verify that the map $m : (L, c^*\alpha_L) \twoheadrightarrow (M, c^*\alpha_M)$ is a uniform surjection. Now consider the following diagram:

$$\begin{tikzcd} C(L, c^*\alpha_L) \arrow[r, two heads] \arrow[d] & C(M, c^*\alpha_M) \arrow[d] \\ (L, c^*\alpha_L) \arrow[r, two heads] & (M, c^*\alpha_M) \end{tikzcd}$$

Since m is a dense surjection, the top arrow is an isomorphism by [6]. Thus any $f : \mathcal{O}[0,1] \to M$, which is then uniform $f : \mathcal{O}[0,1] \to (M, c^*\alpha_M)$, extends to $C(M, c^*\alpha_M) \simeq C(L, c^*\alpha_L)$ since $\mathcal{O}[0,1]$ is complete, and thus to L. ■

Theorem 8.2.2. *A dense surjection $m : L \twoheadrightarrow M$ is a C-quotient map if and only if every principal cover of M is refined by the image of a principal cover of L.*

Proof. One uses the same proof as above. Note that the condition on the principal covers is precisely what is required to show that the map $m : (L, c\alpha_L) \twoheadrightarrow (M, c\alpha_M)$ is a uniform surjection. ■

In the remainder of this subsection we obtain various characterizations of dense C and C^*-quotients in terms of the adjoint map. We also investigate interactions with some well known frame coreflections resulting in some localic versions of classical results in [18] and [27]. We first need some technical definitions and lemmas.

Definition 8.2.3. For $b \in L$ we say that *a is rather below b in* $\operatorname{Coz} L$, and write $a \prec b$ *in* $\operatorname{Coz} L$, if $a \in \operatorname{Coz} L$ and there exists a cozero separating element.

Every element in a completely regular frame L is a join of cozeros rather below it in $\operatorname{Coz} L$.

Lemma 8.2.4. *For any quotient map $m : L \twoheadrightarrow M$, any $x \in L$ and any $a \in M$,*

$$x^* \wedge m_*(m(x) \vee a) \leq m_*(a).$$

Proof. We show that $m(x^* \wedge m_*(m(x) \vee a)) \leq a$:

$$m(x^* \wedge m_*(m(x) \vee a)) = m(x^*) \wedge (m(x) \vee a) = (m(x^*) \wedge m(x)) \vee (m(x^*) \wedge a)$$
$$= \perp \vee (m(x^*) \wedge a) \leq a. \ \blacksquare$$

LEMMA 8.2.5. *For dense $m : L \twoheadrightarrow M$ and $a \in M$, $m_*(a^*) = m_*(a)^*$.*

Proof. Because

$$m(m_*(a)^*) \wedge a = m(m_*(a)^*) \wedge mm_*(a) = m(m_*(a)^* \wedge m_*(a)) = m(\perp) = \perp,$$

it follows that $m(m_*(a)^*) \leq a^*$ and $m_*(a)^* \leq m_*(a^*)$. Because m is dense we also get

$$m_*(a^*) \wedge m_*(a) = m_*(a^* \wedge a) = m_*(\perp) = \perp,$$

and thus $m_*(a^*) \leq m_*(a)^*$. $\blacksquare$

THEOREM 8.2.6. *The following are equivalent for a dense quotient map $m : L \twoheadrightarrow M$ in* **CRegFrm**:

(1) *Every frame map from a compact domain to M extends to L. That is, for every compact frame B and frame map $g : B \to M$ there exists $g' : B \to L$ with $mg' = g$.*

(2) *m is a C^*-quotient map.*

(3) *If $b_0 \vee b_1 = \top$ in $\operatorname{Coz} M$ then $m_*(b_0) \vee m_*(b_1) = \top$ in L. Moreover this is true for any finite cozero cover.*

(4) *For any $b_i \in \operatorname{Coz} M$, $m_*(b_0) \vee m_*(b_1) = m_*(b_0 \vee b_1)$.*

(5) *If b_0 and b_1 are completely separated in M then $m_*(b_0)$ and $m_*(b_1)$ are completely separated in L.*

(6) *If b_0 and b_1 are completely separated in M then $m_*({b_0}^*) \vee m_*(b_1^*) = \top$.*

(7) *If M_0 and M_1 are completely separated quotients of M with quotient maps m_0 and m_1 then*

$$(m_0m)_*(\perp) \vee (m_1m)_*(\perp) = \top.$$

(8) *For any $I \subseteq \operatorname{Coz} M$, $m_*(I)$ generates a proper ideal in L whenever I generates a proper ideal in M.*

Proof. Since $\mathcal{O}[0, 1]$ is compact, (1) implies (2). That (2) implies (3) follows directly from Theorem 7.1.1. Assume (3) is true and take $a \in \operatorname{Coz} L$ with $a \prec m_*(b_0 \vee b_1)$ in $\operatorname{Coz} L$ and separating element $a' \in \operatorname{Coz} L$. Then $a' \vee m_*(b_0 \vee b_1) = \top$ so $m(a') \vee b_0 \vee b_1 = \top$ and hence by (3), $m_*(m(a') \vee b_0) \vee m_*(b_1) = \top$. Therefore

$$\begin{aligned} a &= [a \wedge m_*(m(a') \vee b_0)] \vee [a \wedge m_*(b_1)] \\ &\leq [(a')^* \wedge m_*(m(a') \vee b_0)] \vee [a \wedge m_*(b_1)] \leq m_*(b_0) \vee m_*(b_1), \end{aligned}$$

and so $m_*(b_0 \vee b_1) \leq m_*(b_0) \vee m_*(b_1)$. The opposite inequality is always true since m_* preserves order, and thus we have shown that (4) is true. And (4) implies (1) by Corollary 8.1.6, so that we have shown the first four condition equivalent.

Assume (2), and to prove (5) consider elements $b_i \in L$ which are completely separated by $f \in C^*M$, i.e., $f(\mathbb{R}_i) \leq b_i^*$. Since we may assume without loss of generality that $0 \leq f \leq 1$, f has an extension f' over m. Now $mf'(\mathbb{R}_i) = f(\mathbb{R}_i) \leq {b_i}^*$ and so $f'(\mathbb{R}_i) \leq m_*(b_i^*) = m_*(b_i)^*$ and thus f' completely separates $m_*(b_0)$ from $m_*(b_1)$. The implication from (5) to (6) is clear, since the complete separation of $m_*(b_0)$ from $m_*(b_1)$ implies that

$$\top = m_*(b_0)^* \vee m_*(b_1)^* = m_*(b_0^*) \vee m_*(b_1^*).$$

Assume (6), and to prove (7) consider completely separated quotients M_0 and M_1 of M. This means that there exist $b_i \in \operatorname{Coz} M$ such that $m_i(b_i) = \bot$ and $b_0 \vee b_1 = \top$. But since $m_i m m_*(b_i) = \bot$ it follows that $m_*(b_i) \le (m_i m)_*(\bot)$, so that from (6) we get (7) like this:

$$\top = m_*(b_0) \vee m_*(b_1) \le (m_0 m)_*(\bot) \vee (m_1 m)_*(\bot).$$

(6) follows from (7) as the special case in which each M_i is the open quotient $\downarrow b_i$. Assume (6), and to prove (3) consider $b_i \in \operatorname{Coz} M$ such that $b_0 \vee b_1 = \top$. Then use Corollary 5.1.3 to get $c_i \in \operatorname{Coz} M$ with $c_i \prec b_i$ and $c_0 \vee c_1 = \top$. Thus c_0^* and c_1^* are completely separated and so $m_*(c_0^{**}) \vee m_*(c_1^{**}) = \top$ by (6). Now $c_i^{**} \le b_i$ because $c_i \prec b_i$, and hence $m_*(b_0) \vee m_*(b_1) = \top$ and (3) holds. Since (8) is clearly a reformulation of (3), the proof of the theorem is complete. ∎

Recall that we use $\beta L \twoheadrightarrow L$ to denote the Stone–Čech compactification of a completely regular frame L; see Subsection 2.1. Since this map is dense, Theorem 8.2.6 applies to it. The following result summarizes the characteristic properties of the Stone–Čech compactification.

Corollary 8.2.7. *The compact regular coreflection $\beta L \twoheadrightarrow L$ of a completely regular frame L is the unique compactification $m : L \twoheadrightarrow M$ of L which has the following equivalent properties*:

(1) *L is a C^*-quotient of βL.*
(2) *If $b_0 \vee b_1 = \top$ in $\operatorname{Coz} L$ then $k(b_0) \vee k(b_1) = \top$ $(= L)$.*
(3) *For any $b_i \in \operatorname{Coz} L$, $k(b_0) \vee k(b_1) = k(b_0 \vee b_1)$.*
(4) *Completely separated elements in L have "covering pseudocomplements" in βL. That is, if b_0 is completely separated from b_1 in L then $k(b_0)^* \vee k(b_1)^* = \top$ $(= L)$.*
(5) *$\beta L \twoheadrightarrow L$ is maximal in the partially ordered set of compactifications of L.*

We close this subsection by giving the analogs of Theorem 8.2.6 and Corollary 8.2.7 for C-quotients, Theorem 8.2.12 and Corollary 8.2.13. These results are expedited by a couple of preliminary observations. The first follows from the fact that, as we remarked at the beginning of Subsection 7.2, there is no distinction between coz-codensity and almost coz-codensity for dense quotients.

Proposition 8.2.8. *A dense quotient is a C-quotient if and only if it is a C^*-quotient with a coz-codense quotient map.*

The second remark is that the coz-codensity of a dense quotient implies that it is one-one on cozero parts.

Lemma 8.2.9. *A dense quotient map is coz-iso if and only if it is both coz-onto and coz-codense.*

Proof. If $m : L \twoheadrightarrow M$ is dense and coz-codense then $a_0 \prec a_1$ in $\operatorname{Coz} L$ whenever $m(a_0) \prec m(a_1)$ in $\operatorname{Coz} M$. We claim that such a map is one-one on cozero parts. For if $a_i \in \operatorname{Coz} L$ satisfy $m(a_0) = m(a_1)$ then for any b such that $b \prec a_0$ in $\operatorname{Coz} L$ we would have $m(b) \prec$

$m(a_1)$ in $\operatorname{Coz} M$, hence $b \prec a_1$ in $\operatorname{Coz} L$. Since any element of $\operatorname{Coz} L$ is the (countable) join of elements rather below it in $\operatorname{Coz} L$, the result follows. ∎

COROLLARY 8.2.10. *If M is a dense C-quotient of L then $\operatorname{Coz} L$ is isomorphic to $\operatorname{Coz} M$.*

The converse of Corollary 8.2.10 is in fact also true; see Theorem 8.2.12(6). Recall that $\mathcal{H}\operatorname{Coz} L \twoheadrightarrow L$ denotes the Lindelöfication of a completely regular frame L; see Subsection 2.1. We refer the reader to [25] for basic results on Lindelöf frames.

COROLLARY 8.2.11. *If L is a dense C-quotient of K and if K is Lindelöf then $\mathcal{H}\operatorname{Coz} L \cong K$.*

THEOREM 8.2.12. *The following are equivalent for a dense quotient map $m : L \twoheadrightarrow M$ in* **CRegFrm**:

(1) *Every frame map from a Lindelöf domain to M extends to L. That is, whenever $g : B \to M$ with B Lindelöf there exists $g' : B \to L$ with $mg' = g$.*

(2) *M is a C-quotient.*

(3) *If $\bigvee_{n\in\mathbb{N}} c_n = \top$ in $\operatorname{Coz} M$ then there exist $\{d_n\} \subseteq \operatorname{Coz} L$ with $m(d_n) = c_n$ for all n and $\bigvee_{n\in\mathbb{N}} d_n = \top$. In particular, $\bigvee_{n\in\mathbb{N}} m_*(c_n) = \top$.*

(4) *For $\{c_n : n \in \mathbb{N}\} \subseteq \operatorname{Coz} M$, $m_*(\bigvee_{n\in\mathbb{N}} c_n) = \bigvee_{n\in\mathbb{N}} m_*(c_n)$.*

(5) *M is a C^*-quotient and m is coz-codense.*

(6) *m is coz-iso, i.e., $\operatorname{Coz} L \cong \operatorname{Coz} M$.*

(7) *m induces a* **UniFrm** *surjection from $(L, e\alpha_L)$ onto $(M, e\alpha_M)$.*

Proof. One may use arguments similar to those used in the proof of Theorem 8.2.6. ∎

Theorem 8.2.12 applies to the Lindelöfication $\mathcal{H}\operatorname{Coz} L \twoheadrightarrow L$ of a completely regular frame L. The following corollary summarizes the characteristic properties of the Lindelöfication of a completely regular frame.

COROLLARY 8.2.13. *The regular Lindelöf coreflection $\mathcal{H}\operatorname{Coz} L \twoheadrightarrow L$ of a completely regular frame L is the unique Lindelöfication $m : L \twoheadrightarrow M$ of L which has the following equivalent properties*:

(1) *L is a C-quotient of $\mathcal{H}\operatorname{Coz} L$.*

(2) *m_* takes countable cozero covers of L to covers. That is, if $\bigvee_{n\in\mathbb{N}} c_n = \top$ in $\operatorname{Coz} L$ then $\bigvee_{n\in\mathbb{N}} {\downarrow_{\operatorname{Coz} L}}(c_n) = \top$ $(= \operatorname{Coz} L)$.*

(3) *m_* preserves countable joins of cozero elements of L. That is, for $\{c_n : n \in \mathbb{N}\} \subseteq \operatorname{Coz} L$, $\bigvee_{n\in\mathbb{N}} {\downarrow_{\operatorname{Coz} L}}(c_n) = {\downarrow_{\operatorname{Coz} L}}(\bigvee_{n\in\mathbb{N}} c_n)$.*

Using the above results we may deduce the frame version of the result in [18, 8A.1] which characterizes realcompact spaces as those which have no proper dense C-extensions, and thus any C-embedded realcompact subset is closed.

PROPOSITION 8.2.14. *In* **CRegFrm**, *a frame M is Lindelöf if and only if every frame having M as a dense C-quotient is isomorphic to M.*

Proof. If $m : L \twoheadrightarrow M$ is a dense C-quotient map then $\operatorname{Coz} L \cong \operatorname{Coz} M$ by Corollary 8.2.10, and so $\mathcal{H}\operatorname{Coz} L \cong \mathcal{H}\operatorname{Coz} M$. If in addition M is Lindelöf then $\mathcal{H}\operatorname{Coz} L \cong M$ and so it follows that $L \cong M$. Conversely, if the only dense C-quotient maps onto M are isomorphisms then in particular $\mathcal{H}\operatorname{Coz} M \twoheadrightarrow M$ is an isomorphism and so M is Lindelöf. ■

COROLLARY 8.2.15. *In* **CRegFrm**, *any Lindelöf C-quotient is closed. That is, if $m : L \twoheadrightarrow M$ is a C-quotient map and M is Lindelöf then m is equivalent to the closed quotient map $a \mapsto a \vee b$, where $b \equiv m_*(\top)$.*

Proof. m factors through the closed quotient map by Lemma 2.1.1, say $m = pq$ with q the closed quotient map. Since p is clearly dense, it is an isomorphism by Proposition 8.2.14. ■

8.3. Normality. We recall that a frame L is *normal* if whenever $a \vee b = \top$, there exist $x, y \in L$ with $x \wedge y = \bot$ and $x \vee a = \top = y \vee b$. As with complete regularity, normality may be characterized in terms of real-valued functions. We recall that in normal frames the rather below relation and completely below relation coincide [23], and so one can deduce a localic version of Urysohn's Lemma.

PROPOSITION 8.3.1. *L is normal if and only if whenever $a \vee b = \top$ there exists $f \in CL$ with $f(\mathbb{R}_0) \le a$ and $f(\mathbb{R}_1) \le b$.*

Proof. Suppose $a \vee b = \top$ in L. By normality there exist $x, y \in L$ with $x \wedge y = \bot$ and $x \vee a = \top = y \vee b$. Then $x \prec b$ and so $x \prec\!\prec b$. Hence by Proposition 2.1.4 there exists $f \in CL$ with $f(\mathbb{R}_0) \le x^*$ and $f(\mathbb{R}_1) \le b$. But $x^* \le a$ since $x \vee a = \top$, and so we get the required result. Conversely, given $a \vee b = \top$ and such an f, let $x = f\big(-\infty, \frac{1}{3}\big)$ and $y = f\big(\frac{2}{3}, \infty\big)$. ■

COROLLARY 8.3.2. *L is normal if and only if whenever $a \vee b = \top$ there exist $c, d \in \operatorname{Coz} L$ with $c \le a$, $d \le b$, and $c \vee d = \top$.*

Tietze's Extension Theorem characterizes normal spaces as precisely those in which every closed subspace is C-embedded. We use results obtained in Section 6 to get the following simple and elegant proof of the localic version of that theorem (cf. [28]).

THEOREM 8.3.3. *The following are equivalent for a frame L:*

(1) *L is normal.*
(2) *Every closed quotient of L is a C-quotient.*
(3) *Every closed quotient of L is a C^*-quotient.*

Proof. Suppose L is normal, and consider the closed quotient map $m : L \twoheadrightarrow {\uparrow}c$ for $c \in L$, i.e., $m(a) = a \vee c$ for all $a \in L$. We first show that $a \prec\!\prec_m b$ whenever $a \prec\!\prec b$ in ${\uparrow}c$, and this will establish that m is a C^*-quotient map by Theorem 7.1.1. Now the fact that $a \prec b$ in ${\uparrow}c$ implies that there exists $d \in L$ with $d \ge c$, $d \wedge a = c$, and $d \vee b = \top$. Applying the normality of L to b and d produces $x, y \in L$ with $x \wedge y = \bot$ and $x \vee d = \top = y \vee b$. Since $(x \wedge a) \wedge y = \bot$, $x \wedge a \prec b$ in L and hence $x \wedge a \prec\!\prec b$. Now

$$\begin{aligned}(x \wedge a) \vee c &= (x \vee c) \wedge (a \vee c) = (x \vee c) \wedge a = (x \vee (d \wedge a)) \wedge a\\ &= (x \vee d) \wedge (x \vee a) \wedge a = \top \wedge a = a.\end{aligned}$$

Thus $m(x \wedge a) = a$ and obviously $m(b) = b$, so $a \prec\!\prec_m b$.

We now show that m is almost coz-codense, and this will establish that m is a C-quotient map by Theorem 7.2.7(2). Consider $a \in \operatorname{Coz} L$ with $m(a) = a \vee c = \top$. By the normality of L we get $x, y \in \operatorname{Coz} L$ with $x \wedge y = \perp$ and $x \vee c = y \vee a = \top$. Then $y \prec c$ with separating element x, and since L is normal it follows that $y \prec\!\prec c$. Therefore there is some $b \in \operatorname{Coz} L$ such that $y \leq b \leq c$ by Proposition 2.1.4. Since $b \vee a = \top$ and $m(b) = \perp$, we have established that m is almost coz-codense.

Assume (3), and to prove (1) suppose $a \vee b = \top$ in L and consider the closed quotient map $m : L \twoheadrightarrow \uparrow(a \wedge b)$. Define $f : \mathcal{O}\mathbb{R} \to \uparrow(a \wedge b)$ for $U \in \mathcal{O}\mathbb{R}$ as follows:

$$f(c) \equiv \begin{cases} \top & \text{if } 0 \in U \text{ and } 1 \in U, \\ b & \text{if } 0 \in U \text{ and } 1 \notin U, \\ a & \text{if } 0 \notin U \text{ and } 1 \in U, \\ a \wedge b & \text{if } 0 \notin U \text{ and } 1 \notin U. \end{cases}$$

Then f is a frame map and if m is a C-quotient map then f extends over m to some $f' \in CL$. We have $mf'(\mathbb{R}_0) = f(\mathbb{R}_0) = a$ and $mf'(\mathbb{R}_1) = f(\mathbb{R}_1) = b$, from which it follows that $f'(\mathbb{R}_0) \leq a$ and $f'(\mathbb{R}_1) \leq b$. Therefore L is normal by Proposition 8.3.1. ■

8.4. Disconnectivity. Elements a and b of a frame L are *complements* of one another if $a \wedge b = \perp$ and $a \vee b = \top$, and in this case we say that either one is *complemented*. The complemented elements of L form a boolean algebra in the order inherited from L, and in fact all such elements are cozeros. For if a and b are complements then $a \prec a$ with separating element b, and by putting $a_i \equiv a$ for all $i \in [0,1]_{\mathbb{Q}}$ we get a scale whose join is a. Put another way, the frame map $f \in C^*L$ defined by the rule

$$f(c) \equiv \begin{cases} \top & \text{if } 0 \in U \text{ and } 1 \in U, \\ a & \text{if } 0 \notin U \text{ and } 1 \in U, \\ b & \text{if } 0 \in U \text{ and } 1 \notin U, \\ \perp & \text{if } 0 \notin U \text{ and } 1 \notin U, \end{cases} \qquad U \in \mathcal{O}[0,1],$$

satisfies $\operatorname{coz} f = f(\mathbb{R}_0) = a$ and $f(\mathbb{R}_1) = b$.

A frame is *disconnected* if there is at least one nontrivial complemented element, i.e., if there exists a binary cover of disjoint non-$\perp$ elements. A frame is *connected* if it is not disconnected, or equivalently, if $a \wedge b = \perp$ and $a \vee b = \top$ imply $a = \top$ or $b = \top$. Baboolal and Banaschewski show that a frame is connected if and only if its Stone–Čech compactification is connected. A frame is *zero-dimensional* if it has a base of complemented elements, i.e., if for every $a \in L$ such that $a > \perp$ there is a complemented element b such that $a > b > \perp$. Clearly every zero-dimensional frame is disconnected. We consider weaker variations of disconnectivity and see how some of these may be characterized using complete separation, C- and C^*-quotients. We recall from 6.2.1 that elements $a, b \in L$ are said to be *completely separated* if $a \prec\!\prec b^*$.

8.4.1. *Extremally disconnected frames.* A frame is *extremally disconnected* if it has the attributes identified in Proposition 8.4.1. Each of these has a spatial counterpart characterizing extremally disconnected spaces: (1) says that disjoint open sets have disjoint closures, (2) asserts that regular open sets are clopen, (3) can be interpreted as saying that the closure of an open subset is open, (4) is the statement that disjoint open sets are completely separated, (5) says that dense subspaces are C^*-embedded, and (6) says that open subsets are C^*-embedded. Johnstone [23] defines extremal disconnectivity using condition (2). The identity $a^{**} \vee a^* = \top$ is equivalent to the De Morgan law $(a \wedge b)^* = (a^* \vee b^*)$, and hence these frames are often called De Morgan frames [7].

PROPOSITION 8.4.1. *The following are equivalent for a frame L:*

(1) *For all $a, b \in L$, $a \wedge b = \perp$ implies $a^* \vee b^* = \top$.*
(2) *For all $a \in L$, $a^* \vee a^{**} = \top$.*
(3) *For all $a \in L$, the map $x \mapsto x \vee a^*$ effects a frame isomorphism from $\downarrow a^{**}$ onto $\uparrow a^*$.*
(4) *For all $a, b \in L$, $a \wedge b = \perp$ implies that a and b are completely separated.*
(5) *Every dense quotient of L is a C^*-quotient.*
(6) *Every open quotient of L is a C^*-quotient.*
(7) *Every dense open quotient of L is a C^*-quotient.*

Proof. (2) is a special case of (1), and if (2) holds then (1) follows, for $a \wedge b = \perp$ implies

$$\top = (a^* \vee a^{**}) \vee (b^* \vee b^{**}) = (a^* \wedge b^*) \vee (a^* \wedge b^{**}) \vee (a^{**} \wedge b^*) \vee (a^{**} \wedge b^{**}) \leq a^* \vee b^*.$$

The equivalence of (2) and (3) is clear, since $x \mapsto x \vee a^*$ is a frame map if and only it carries the top of $\downarrow a^{**}$, namely a^{**}, to the top of $\uparrow a^*$, namely $\top$. And when this condition obtains, it is easy to see that this map is bijective. If (2) holds then (4) follows, for if $a \wedge b = \perp$ then a^* and b^* are cozero elements which witness their complete separation by Proposition 6.2.3. Likewise (2) follows from (4), for any two completely separated elements a and b must satisfy $a^* \vee b^* = \top$, again by Proposition 6.2.3. We have shown the equivalence of the first four conditions.

Assume (4), and to prove (5) consider a dense quotient map $m : L \twoheadrightarrow M$ and completely separated elements $x, y \in M$. Locate $a, b \in L$ such that $m(a) = x$ and $m(b) = y$. Then $a \wedge b = \perp$ because m is dense, and so a and b are completely separated in L by (4). That means that there exist $c, d \in \operatorname{Coz} L$ such that $c \leq b^*$, $d \leq a^*$, and $c \vee d = \top$. But then $m(c) \wedge y = m(d) \wedge x = \perp$, so x and y are m-completely separated. It follows from Theorem 7.1.1 that m is a C^*-quotient map, i.e., that (5) holds. The proof of (6) from (4) goes along the same lines, and, of course, (7) is a special case of either (5) or (6).

Assume (7), and to prove (2) consider $a \in L$ for the purpose of showing that $a^* \vee a^{**} = \top$. The open quotient map $m : L \twoheadrightarrow \downarrow(a \vee a^*)$ is dense because $a \vee a^*$ is a dense element, and so m is a C^*-quotient map. Since a and a^* are complements in $\downarrow(a \vee a^*)$, they are also cozero elements which join to the top of their frame, and so by Theorem 7.1.1 there exist $c, d \in \operatorname{Coz} L$ such that $m(c) = a$ and $m(d) = a^*$ and $c \vee d = \top$. Now

$$a = m(c) = c \wedge (a \vee a^*) = (c \wedge a) \vee (c \wedge a^*) \;\Rightarrow\; a \leq c \leq a^{**},$$

and similarly $a^* \leq d \leq a^{***} = a^*$. But it follows from the density of m that $c \wedge d = \perp$, and so c and d are complements in L. From this in turn it follows that $c = a^{**}$, i.e., $a^* \vee a^{**} = \top$, meaning that (2) holds. ∎

PROPOSITION 8.4.2. *L is extremally disconnected if and only if βL is extremally disconnected.*

Proof. Suppose βL is extremally disconnected and take $a \in L$. Then $k(a) \in \beta L$, where $k(a) = \{x \in L : x \prec\!\prec a\}$, so $k(a)^* \vee k(a)^{**} = L$. Recall that $k(a^*) = k(a)^*$ so there exist $x \prec\!\prec a^*$ and $y \prec\!\prec a^{**}$ with $x \vee y = \top$ and hence $a^{**} \vee a^* = \top$. Conversely, suppose that

L is extremally disconnected and take any $I \in \beta L$. Let $x = \bigvee I^{**}$ and $y = \bigvee I^*$. Then $x \wedge y \in I^* \wedge I^{**} = {\downarrow}\bot$ implies $x \wedge y = \bot$, so $x^* \vee y^* = \top$. Since $x^* \vee x^{**} = \top$, $x^* \prec x^*$, and so $x^* \prec\!\prec x^*$. Thus $x^* \in k(x^*)$. Moreover, $k(x^*) \leq I^*$ because if $a \in k(x^*) \wedge I$ then $a \prec\!\prec x^*$ and $a \leq x$ and so $a = \bot$. Thus $x^* \in I^*$. Similarly we can show that $y^* \in I^{**}$ and hence $I^* \vee I^{**} = L = \top$. ∎

Banaschewski obtained the same result for zero-dimensional frames using biframes [10].

8.4.2. *Basically disconnected frames.* A frame is *basically disconnected* if it has the attributes identified in Proposition 8.4.3. Each of these has a spatial counterpart characterizing basically disconnected spaces: (1) says that disjoint open sets, one of which is a cozero, have disjoint closures, (2) asserts that the regular open set generated by a cozero is clopen, (3) can be interpreted as saying that the closure of a cozero subset is open, and (4) is the statement that disjoint open sets, one of which is a cozero, are completely separated. To avoid trivialities arising from a paucity of cozero elements, *we assume that all frames are completely regular for the rest of the article.* Proposition 8.4.3 can be established on the basis of arguments similar to those used in the proof of Proposition 8.4.1.

PROPOSITION 8.4.3. *The following are equivalent for a frame L:*

(1) *For all $a \in \operatorname{Coz} L$ and $b \in L$, $a \wedge b = \bot$ implies $a^* \vee b^* = \top$.*
(2) *For all $a \in \operatorname{Coz} L$, $a^* \vee a^{**} = \top$.*
(3) *For all $a \in L$, the map $x \mapsto x \vee a^*$ effects a frame isomorphism from ${\downarrow}a^{**}$ onto ${\uparrow}a^*$.*
(4) *For all $a \in \operatorname{Coz} L$ and $b \in L$, $a \wedge b = \bot$ implies that a and b are completely separated.*

PROPOSITION 8.4.4. *Every basically disconnected completely regular frame is zero-dimensional, i.e., has a base of complemented elements.*

Proof. Recall that $x \prec a$ implies $x^{**} \leq a$. Therefore

$$a = \bigvee \{x \in \operatorname{Coz} L : x \prec a\} = \bigvee \{x^{**} : x \in \operatorname{Coz} L,\ x \prec a\},$$

and the result follows from Proposition 8.4.3(2). ∎

PROPOSITION 8.4.5. *L is basically disconnected if and only if βL is basically disconnected.*

Proof. Suppose that βL is basically disconnected and, in order to show that L is basically disconnected, consider $a \in \operatorname{Coz} L$. Since $\beta L \twoheadrightarrow L$ is a C^*-quotient map, it is coz-onto, so there exists some $I \in \operatorname{Coz} \beta L$ such that $\bigvee I = a$. Since I^* and I^{**} are complements in βL, $c \equiv \bigvee I^*$ and $d \equiv \bigvee I^{**}$ are complements in L. Now $k(a^*) \wedge I = {\downarrow}\bot$ implies that $k(a^*) \leq I^*$ and hence $a^* \leq c$. And because $I \wedge I^* = {\downarrow}\bot$ implies $\bot = \bigvee I \wedge \bigvee I^* = a \wedge c$, we get $c \leq a^*$ and therefore $c = a^*$. It follows that

$$a^* \vee a^{**} = c \vee c^* = c \vee d = \top.$$

Suppose conversely that L is basically disconnected and, in order to show that βL is basically disconnected, take $I \in \operatorname{Coz} \beta L$. Then $\bigvee I \equiv x \in \operatorname{Coz} L$ so $x^* \vee x^{**} = \top$. By the same argument used in the proof of Proposition 8.4.2, it can be shown that $x^* \in I^*$ and $x^{**} \in I^{**}$ and so $I^* \vee I^{**} = L$. ■

8.4.3. *P-frames and almost P-frames.* A *P-space* (or *pseudo-discrete space*) is a space in which each cozero set is closed. These spaces are defined and variously characterized in [18]; see also [19]. An *almost P-space* is a space in which every cozero set is the interior of its closure, or equivalently, every zero set is the closure of its interior.

DEFINITION 8.4.6. A frame L is a *P-frame* if $\operatorname{Coz} L$ is complemented, i.e., $a \in \operatorname{Coz} L$ implies $a^* \in \operatorname{Coz} L$. A frame L is an *almost P-frame* if $a = a^{**}$ for all $a \in \operatorname{Coz} L$, i.e., $a^* = \bot$ implies $a = \top$ for $a \in \operatorname{Coz} L$.

PROPOSITION 8.4.7. *The following are equivalent for a frame L.*

(1) *L is a P-frame.*
(2) *L is a basically disconnected almost P-frame.*
(3) *The open quotient of any cozero element of L is a C-quotient.*

Proof. A little reflection on the definitions makes it clear that (1) and (2) are equivalent. Suppose that L is a P-frame, and consider the open quotient map $m : L \twoheadrightarrow {\downarrow}a$ for some $a \in \operatorname{Coz} L$. Then m is coz-onto by Proposition 3.2.10, and is almost coz-codense because $a^* \vee b = \top$ for any $b \in \operatorname{Coz} L$ such that $m(b) = b \wedge a = \top = a$. Therefore m is a C-quotient map by Theorem 7.2.7, and (3) holds. On the other hand, if (3) holds and $a \in \operatorname{Coz} L$ then the fact that the open quotient map $m : L \twoheadrightarrow {\downarrow}a$ is almost coz-codense requires the existence of some $b \in \operatorname{Coz} L$ such that $m(b) = b \wedge a = \bot$ and $b \vee a = \top$. That is, b is the complement of a, and (1) holds. ■

See [18, 4M] for an example of a space which is basically disconnected but not a P-space.

PROPOSITION 8.4.8. *A frame is a P-frame if and only if its Lindelöfication is a P-frame if and only if its realcompactification is a P-frame.*

Proof. The Lindelöfication of L is given by $\mathcal{H} \operatorname{Coz} L$ and the realcompactification is a quotient of $\mathcal{H} \operatorname{Coz} L$ denoted by νL. The result follows easily since $\operatorname{Coz} L \cong \operatorname{Coz} \mathcal{H} \operatorname{Coz} L \cong \operatorname{Coz} \nu L$. ■

8.4.4. *F-frames and F'-frames.* A topological space is called an *F-space* if every cozero set is C^*-embedded [18], [27], a *quasi-F-space* if every dense cozero set is C^*-embedded, and an *F'-space* if disjoint cozero sets have disjoint closures.

DEFINITION 8.4.9. L is an *F-frame* (a *quasi-F-frame*) if the open quotient of each (dense) cozero element is a C^*-quotient. L is an *F'-frame* if $a \wedge b = \bot$ for $a, b \in \operatorname{Coz} L$ implies $a^* \vee b^* = \top$.

Every P-frame is an F-frame by Proposition 8.4.7, every basically disconnected frame is an F-frame by Proposition 8.4.3, and every F-frame is an F'-frame and a quasi-F-frame. See [25] for a treatment of quasi-F-frames.

PROPOSITION 8.4.10. *L is an F-frame if and only if disjoint cozero elements are completely separated*, *i.e.*, *if and only if for all $a, b \in \operatorname{Coz} L$ such that $a \wedge b = \bot$ there exist*

$c, d \in \operatorname{Coz} L$ such that $c \wedge b = d \wedge a = \perp$ and $c \vee d = \top$. Further, L is a quasi-F-frame if and only if disjoint cozero elements with dense join are completely separated, i.e., if and only if for all $a, b \in \operatorname{Coz} L$ such that $a \wedge b = \perp$ and $a \vee b$ dense there exist $c, d \in \operatorname{Coz} L$ such that $c \wedge b = d \wedge a = \perp$ and $c \vee d = \top$.

Proof. We prove the assertion for F-frames; the proof for quasi-F-frames is almost identical. Suppose L is an F-frame and take $a \wedge b = \perp$ in $\operatorname{Coz} L$. Then since $a \vee b$ lies in $\operatorname{Coz} L$, the open quotient map $m : L \twoheadrightarrow \downarrow(a \vee b)$ is a C^*-quotient map. Since a and b are complements they are also cozero elements in $\downarrow(a \vee b)$, and because $a \vee b$ is the top of $\downarrow(a \vee b)$, Theorem 7.1.1 gives $c, d \in \operatorname{Coz} L$ such that $m(c) = a$, $m(d) = b$, and $c \vee d = \top$. But

$$a = m(c) = c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b) \ \Rightarrow\ c \wedge b = \perp,$$

and likewise $d \wedge a = \perp$. That is, a and b are completely separated in L.

Conversely, suppose that disjoint cozero elements of L are completely separated, and consider the open quotient map $m : L \twoheadrightarrow \downarrow x$ for some $x \in \operatorname{Coz} L$. If a and b are completely separated in $\downarrow x$ then by Corollary 6.2.9 we can assume that a and b lie in $\operatorname{Coz}(\downarrow x)$, and so $a, b \in \operatorname{Coz} L$ by Proposition 3.2.10. Because $a \wedge b = \perp$ in $\operatorname{Coz} L$, a and b are completely separated in L. It follows that a and b are m-completely separated, and that m is a C^*-quotient map by Theorem 7.1.1. ■

PROPOSITION 8.4.11. *L is an F-frame if and only if βL is an F-frame.*

Proof. Assume that L is an F-frame. Take $J \in \operatorname{Coz} \beta L$, and put $a \equiv \bigvee J \in \operatorname{Coz} L$. Now take $g : \mathcal{O}[0,1] \to \downarrow J$ and consider the following diagram:

$$\begin{tikzcd}
\beta L \arrow[rr, "l"] \arrow[dd, "m"'] & & \downarrow J \arrow[dd, "k"] \\
& \mathcal{O}[0,1] \arrow[ul, "h"] \arrow[ur, "g"'] \arrow[dl, "g'"'] \arrow[dr, "f"] & \\
L \arrow[rr, "n"'] & & \downarrow a
\end{tikzcd}$$

The maps in the diagram arise as follows: m is the canonical join map $I \mapsto \bigvee I$, and k is its restriction to $\downarrow J$; f is kg; n is the open quotient map of a; g' is the extension of f over n whose existence is guaranteed by the fact that n is a C^*-quotient map; h is the result of factoring g' through m, which can be done because m is the coreflection of L in compact regular frames; l is the open quotient map of J.

We claim that h is an extension of g over l, i.e., that $lh = g$. To establish this claim first note that the outer square commutes, i.e., that $kl = nm$, since

$$\bigvee (J \wedge I) = \bigvee J \wedge \bigvee I = a \wedge \bigvee I.$$

Therefore

$$kg = f = ng' = nmh = klh,$$

and since k is monic in **CRegFrm** by virtue of being dense, it follows that $g = lh$.

Conversely, assume βL is an F-frame, and take $a \wedge b = \perp$ in $\operatorname{Coz} L$. Because $\beta L \twoheadrightarrow L$ is coz-onto, there exist $I_a, I_b \in \operatorname{Coz} \beta L$ such that $\bigvee I_a = a$ and $\bigvee I_b = b$. Since $I_a \wedge I_b = {\downarrow}\perp$ in the F-frame βL, there exist $J_a, J_b \in \operatorname{Coz} \beta L$ such that $J_a \vee J_b = L$ and

$$J_a \wedge I_b = I_a \wedge J_b = {\downarrow}\perp.$$

Since $J_a \vee J_b = L$ it follows that there exist $c \in J_a$ and $d \in J_b$ such that $c \vee d = \top$, and we may without loss of generality assume that $c, d \in \operatorname{Coz} L$. But then $J_a \wedge I_b = {\downarrow}\perp$ implies that $c \wedge b = c \wedge \bigvee I_b = \perp$, and likewise $a \wedge d = \perp$. That is, c and d witness the complete separation of a from b. ∎

COROLLARY 8.4.12. *If L is an F-frame then ${\downarrow}a$ is an F-frame for each $a \in \operatorname{Coz} L$.*

Proof. If $x \in \operatorname{Coz}({\downarrow}a)$ then $x \in \operatorname{Coz} L$ by Proposition 3.2.10. Consider the diagram

$$\begin{tikzcd}
\mathcal{O}[0,1] \arrow[r, "f'"] \arrow[rr, bend left, "f"] \arrow[dr] & L \arrow[r, two heads, "m"] \arrow[d, two heads, "n"] & {\downarrow}x \\
& {\downarrow}a \arrow[ur, "p"'] &
\end{tikzcd}$$

in which f is an arbitrary frame map, m and n are the open quotient maps, p is the result of factoring m through n, and f' is the extension of f over m guaranteed by the fact that m is a C^*-quotient map. Then nf' is an extension of f over p. ∎

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