Contents

Abstract

Let A be a, not necessarily closed, linear relation in a Hilbert space $\mathfrak{H}$ with a multivalued part $\operatorname{mul} A$. An operator B in $\mathfrak{H}$ with $\operatorname{ran} B \perp \operatorname{mul} A^{**}$ is said to be an operator part of A when $A = B \,\widehat{+}\, (\{0\} \times \operatorname{mul} A)$, where the sum is componentwise (i.e. span of the graphs). This decomposition provides a counterpart and an extension for the notion of closability of (unbounded) operators to the setting of linear relations. Existence and uniqueness criteria for an operator part are established via the so-called canonical decomposition of A. In addition, conditions are developed for the above decomposition to be orthogonal (components defined in orthogonal subspaces of the underlying space). Such orthogonal decompositions are shown to be valid for several classes of relations. The relation A is said to have a Cartesian decomposition if $A = U + \mathrm{i}\,V$, where U and V are symmetric relations and the sum is operatorwise. The connection between a Cartesian decomposition of A and the real and imaginary parts of A is investigated.

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Dedicated to Schôichi Ôta
on the occasion of his sixtieth birthday

1. Introduction

1.1. Some background. A linear *relation* A in a Hilbert space $\mathfrak{H}$ is by definition a linear subspace of the product space $\mathfrak{H} \times \mathfrak{H}$. A linear relation A is (the graph of) a linear operator if and only if $\operatorname{mul} A = \{0\}$, where the *multivalued part* $\operatorname{mul} A$ of A is defined as $\{g \in \mathfrak{H};\ \{0, g\} \in A\}$. The formal *inverse* A^{-1} of a linear relation A is given by $A^{-1} = \{\{k, h\};\ \{h, k\} \in A\}$, so that $\operatorname{dom} A^{-1} = \operatorname{ran} A$, $\operatorname{ran} A^{-1} = \operatorname{dom} A$, $\ker A^{-1} = \operatorname{mul} A$, and $\operatorname{mul} A^{-1} = \ker A$. The *closure* of a linear relation is a linear relation which is obtained by taking the closure of the corresponding subspace in $\mathfrak{H} \times \mathfrak{H}$. The linear relation A is called *closed* as a relation in $\mathfrak{H}$ if the subspace is closed in $\mathfrak{H} \times \mathfrak{H}$. If A is (the graph of) a linear operator, then A is said to be closable if the closure of A is (the graph of) a linear operator. The *adjoint* $A^* = JA^{\perp} = (JA)^{\perp}$, with the operator J defined by $J\{f, f'\} = \{f', -f\}$, $\{f, f'\} \in \mathfrak{H} \times \mathfrak{H}$, is automatically a closed linear relation in $\mathfrak{H}$. Then the second adjoint A^{**} is equal to the closure $\overline{A}$ of A. A relation is said to be *symmetric* if $A \subset A^*$ and *selfadjoint* if $A = A^*$. The study of general relations was initiated by R. Arens [2]. Further work has been concerned with symmetric and selfadjoint relations and, more generally, with normal, accretive, dissipative, and sectorial relations; see for instance [3], [4], [8], [11], [14], [34].

Linear relations can be viewed as multivalued linear operators. They show up in a natural way in a variety of problems. Some of these will be presented for the convenience of the reader.

The first example shows the usefulness of relations by relating results for A to those for the formal inverse A^{-1}.

Example 1.1. Let A be a linear operator or a linear relation in a Hilbert space $\mathfrak{H}$, which is not necessarily closed or densely defined. An element $h \in \mathfrak{H}$ belongs to $\operatorname{dom} A^*$ if and only if

$$\sup\{(h, g) + (g, h) - (f, f);\ \{f, g\} \in A\} < \infty, \tag{1.1}$$

and an element $k \in \mathfrak{H}$ belongs to $\operatorname{ran} A^*$ if and only if

$$\sup\{(f, k) + (k, f) - (g, g);\ \{f, g\} \in A\} < \infty. \tag{1.2}$$

The formulas (1.1) and (1.2) show the advantage of the language of relations: the formula (1.2) is in fact the same as the formula (1.1) when the relation A is replaced by its formal inverse A^{-1}. Moreover, (1.1) is equivalent to

$$\sup\{|(g, h)|^2;\ \{f, g\} \in A,\ (f, f) \le 1\} < \infty, \tag{1.3}$$

and (1.2) is equivalent to

$$\sup\{|(f,k)|^2;\ \{f,g\}\in A,\ (g,g)\leq 1\}<\infty. \tag{1.4}$$

Again the relation between (1.3) and (1.4) via the formal inverse of A is evident. The last two characterizations are versions of results which go back to Shmul'yan for bounded operators A; for more details see [20].

As a second example it is shown that under very general conditions a densely defined closable operator can be decomposed as the sum of a closable operator and a singular operator (whose closure is a Cartesian product).

Example 1.2. Let A be a densely defined closable operator in a Hilbert space $\mathfrak{H}$; i.e., the closure $\overline{A}$ of A in $\mathfrak{H}\times\mathfrak{H}$ is the graph of a linear operator. Let $\varphi\in\mathfrak{H}$ and let P_φ be the orthogonal projection from $\mathfrak{H}$ onto the linear space spanned by φ. Then the operator A admits the decomposition

$$A=B+C, \tag{1.5}$$

with the densely defined operators B and C defined by

$$B=(I-P_\varphi)A, \qquad C=P_\varphi A. \tag{1.6}$$

Then the operator B is closable for any choice of $\varphi\in\mathfrak{H}$, but the behaviour of the operator C depends on the choice of $\varphi\in\mathfrak{H}$. If $\varphi\in\operatorname{dom}A^*$, then $\overline{C}\in\boldsymbol{B}(\mathfrak{H})$ (bounded linear operators on $\mathfrak{H}$) and $\overline{C}h=(h,A^*\varphi)\varphi$ for $h\in\mathfrak{H}$. However, if $\varphi\in\mathfrak{H}\setminus\operatorname{dom}A^*$, then C is a so-called singular operator, i.e., $\operatorname{ran}C\subset\operatorname{mul}\overline{C}$ and $\overline{C}=\mathfrak{H}\times\operatorname{span}\{\varphi\}$. For more details and the connection with Lebesgue type decompositions, see [19].

As a third example consider the case of a monotonically increasing sequence of bounded linear operators in the absence of a uniform upper bound.

Example 1.3. Let $\mathfrak{H}$ be a Hilbert space and let $A_n\in\boldsymbol{B}(\mathfrak{H})$ be a nondecreasing sequence of nonnegative operators, i.e., $0\leq(A_mh,h)\leq(A_nh,h)$, $h\in\mathfrak{H}$, for $n\geq m$. If the sequence A_n is bounded from above, i.e., $(A_nh,h)\leq M(h,h)$, $h\in\mathfrak{H}$, for some $M\geq 0$, then it is known that there exists a strong limit $A_\infty\in\boldsymbol{B}(\mathfrak{H})$, i.e., $\|A_nh-A_\infty h\|\to 0$, $h\in\mathfrak{H}$, and A_∞ has the same upper bound. The situation is different when the family A_n does not have an upper bound. The absence of a uniform bound leads to phenomena which involve unbounded operators and relations. In fact, there exists a selfadjoint relation A_∞ which is nonnegative, i.e., $(f',f)\geq 0$, $\{f,f'\}\in A_\infty$, such that A_n converges to A_∞ in the strong resolvent sense, i.e.,

$$(A_n-\lambda)^{-1}h\to(A_\infty-\lambda)^{-1}h, \qquad h\in\mathfrak{H},\ \lambda\in\mathbb{C}\setminus\mathbb{R}.$$

Moreover, the domain of the square root of A_∞ is given by

$$\operatorname{dom}A_\infty^{1/2}=\{h\in\mathfrak{H};\ \sup_{n\in\mathbb{N}}(A_nh,h)<\infty\}.$$

For more details and the connection with monotone sequences of semibounded closed forms, see [5].

Often multivalued operators appear as extensions of symmetric operators, like in boundary value problems for differential operators. Boundary conditions impose restrictions in such a way that an underlying symmetric operator becomes nondensely defined; cf. [9]. The situation can often be formalized in simple terms as follows.

Example 1.4. Let A be a selfadjoint operator in a Hilbert space $\mathfrak{H}$ and let Z be, for simplicity, a finite-dimensional subspace of $\mathfrak{H} \times \mathfrak{H}$. Then the intersection $A \cap Z^*$ is a symmetric restriction of A, which may be nondensely defined, so that its adjoint $A \mathbin{\widehat{+}} Z$, a componentwise sum, may be multivalued. In this case, among the selfadjoint extensions of $A \cap Z^*$, there also occur multivalued operators. In connection with differential operators this construction gives rise to nonstandard boundary conditions. If, for instance, A is a selfadjoint Sturm–Liouville operator, then integral boundary conditions or perturbations via delta functions or their derivatives fit into this framework with a proper choice of the subspace Z; see [24] for more details.

In general, the spectral theory of differential equations offers many examples of multivalued operators. Linear relations provide the natural context for the study of general selfadjoint boundary value problems involving systems of differential equations; cf. [6]. In fact, the theory of boundary triplets and boundary relations has been formulated to discuss all extensions (singlevalued and multivalued) of symmetric relations; see [12], [13]. For instance, the description of selfadjoint extensions of a symmetric operator or relation is always in terms of selfadjoint relations in a parameter space; such selfadjoint relations also appear in Kreĭn's formula.

1.2. Decomposition of relations. There are many kinds of decompositions of linear relations. For instance, for semi-Fredholm relations and for quasi-Fredholm relations there is a so-called Kato decomposition (see [27]) or for closed linear relations a Stone decomposition (see [21], [28]). The decompositions appearing in the present paper are concerned with splitting linear operators and relations via components that are closable, nonclosable, or purely multivalued, and components involving the real and imaginary parts of relations.

It is necessary to begin by explaining the so-called *canonical decomposition* of linear relations which has been studied recently in [21]. Let A be a relation in a Hilbert space and let A^{**} be its closure. Let P the orthogonal projection from $\mathfrak{H}$ onto $\operatorname{mul} A^{**}$ and define the relations $A_{\rm reg}$ and $A_{\rm sing}$, the *regular* part and the *singular* part of A respectively, by

$$A_{\rm reg} = \{\{f, (I-P)f'\};\, \{f, f'\} \in A\}, \qquad A_{\rm sing} = \{\{f, Pf'\};\, \{f, f'\} \in A\}.$$

Then A admits the decomposition

$$A = A_{\rm reg} + A_{\rm sing} = \{\{f, h+k\};\, \{f, h\} \in A_{\rm reg},\, \{f, k\} \in A_{\rm sing}\}.$$

The regular part $A_{\rm reg}$ is actually a closable operator, whereas the singular part $A_{\rm sing}$ is a singular relation, i.e., its closure is a Cartesian product; cf. [21]. The canonical decomposition of A above is strongly related to the Lebesgue decompositions of forms; see [21]. The canonical decomposition of a relation is an example of a decomposition as an operatorwise sum. However, relations also admit componentwise decompositions. The aim of this paper is to present several decompositions of linear relations as operatorwise sums and as componentwise sums.

The second type of decomposition introduced in the present paper for general, not necessarily closed, linear relations is a *componentwise decomposition* of a relation A in an operator part and a multivalued part of the form

$$A = B \,\widehat{+}\, A_{\rm mul}, \tag{1.7}$$

where the operator part B is (the graph of) an operator in $\mathfrak{H}$, $A_{\rm mul} = \{0\} \times {\rm mul}\, A$, and the sum in (1.7) is componentwise (as indicated by $\widehat{+}$). To make the decomposition somewhat reasonable or unique it is necessary to impose some additional assumptions on (1.7). Assume for the moment that the relation A is closed. Then one possible choice is $B = A_{\rm op}$ where $A_{\rm op} = \{\{f, f'\} \in A;\ f' \perp {\rm mul}\, A\}$, so that B is a closed operator. Since ${\rm mul}\, A$ is closed and $\mathfrak{H} = \overline{{\rm dom}}\, A^* \oplus {\rm mul}\, A$, the identity (1.7) follows. This motivates the construction in the general case. The extra assumption that ${\rm ran}\, B \subset \overline{{\rm dom}}\, A^* = ({\rm mul}\, A^{**})^\perp$ makes B unique, namely $B = A_{\rm op}$, where now

$$A_{\rm op} = \{\{f, f'\} \in A;\ f' \perp {\rm mul}\, A^{**}\}$$

is a closable operator. Observe that $A_{\rm op} \subset A_{\rm reg}$. It will be shown that $B = A_{\rm op}$ satisfies (1.7) precisely when $A_{\rm op} = A_{\rm reg}$. A relation A which allows a decomposition (1.7) with ${\rm ran}\, B \subset \overline{{\rm dom}}\, A^*$ will be called *decomposable*.

The third decomposition is related to the second type of decomposition, so it is again componentwise. Assuming that A is decomposable the question is when the decomposition (1.7) is orthogonal with regard to the *orthogonal splitting* of the Hilbert space

$$\mathfrak{H} = \overline{{\rm dom}}\, A^* \oplus {\rm mul}\, A^{**}.$$

A necessary and sufficient additional condition that appears now for A is

$${\rm dom}\, A \subset \overline{{\rm dom}}\, A^* \quad \text{or, equivalently,} \quad {\rm mul}\, A^{**} \subset {\rm mul}\, A^*.$$

Particular cases are studied for decomposable relations A which are in addition *formally domain tight* and *domain tight*, i.e., satisfy

$${\rm dom}\, A \subset {\rm dom}\, A^* \quad \text{and} \quad {\rm dom}\, A = {\rm dom}\, A^*,$$

respectively. Furthermore, decomposable relations are studied under the condition that their numerical range is a proper subset of $\mathbb{C}$. Orthogonal decompositions for normal, selfadjoint, and, for instance, maximal sectorial relations are obtained as byproducts.

The fourth type of decomposition to be studied in the present paper is the Cartesian decomposition of a relation. By definition a *Cartesian decomposition* of a relation A is of the form

$$A = A_1 + {\rm i}\, A_2, \tag{1.8}$$

where A_1 and A_2 are symmetric relations in $\mathfrak{H}$, i.e.,

$$A_1 \subset (A_1)^*, \qquad A_2 \subset (A_2)^*,$$

and where the sum in (1.8) is now again operatorwise; see [39] for the operator case. It is a consequence of the Cartesian decomposition (1.8) that A satisfies the condition ${\rm dom}\, A \subset {\rm dom}\, A^*$, and it will be shown that this is also a sufficient condition for the existence of a Cartesian decomposition. The connection between the components A_1 and A_2 of a Cartesian decomposition (1.8) of A and the real and imaginary parts of A is clear if A is a densely defined normal operator; cf. [39]. In the general case, the connection is vague, but the situation becomes clear when the following extension of A is introduced:

$$A_\infty = A \,\widehat{+}\, (\{0\} \times \operatorname{mul} A^*).$$

The special situation of Cartesian decompositions for normal relations will be treated in [23].

1.3. Brief description. Here is a brief review of the contents of the paper. Section 2 contains a number of preliminary definitions and facts concerning linear relations. A number of results which are known for linear operators are stated for the case of linear relations; for completeness proofs are included. The notions of formally domain tight and domain tight relations are introduced. Canonical decompositions and decompositions of linear relations of the form (1.7) are taken up in Section 3. The notion of decomposable relation is characterized in various ways. A number of examples is included illustrating relations which are not decomposable. The question of the orthogonality of such decompositions is taken up in Section 4. In particular, relations whose numerical range is a proper subset of $\mathbb{C}$ are treated. Cartesian decompositions of the form (1.8) are treated in Section 5. This section also contains a treatment of the real and imaginary parts of a linear relation.

2. Preliminaries

This section contains a number of basic definitions and results concerning linear relations in a Hilbert space. These results are analogs or natural extensions of results which are better known in the case of operators. It should be mentioned that many of the stated results have their analogs also for linear relations acting from one Hilbert space to another Hilbert space. However, for simplicity all the statements are formulated here for the case of linear relations from a given Hilbert space back to itself.

2.1. Linear relations in a Hilbert space. Let $\mathfrak{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$. The Cartesian product $\mathfrak{H}\times\mathfrak{H}$ will be provided with the usual inner product. A linear *relation* (or relation, for short) A in $\mathfrak{H}$ is by definition a linear subspace of the Hilbert space $\mathfrak{H}\times\mathfrak{H}$. The *domain*, *range*, *kernel*, and *multivalued part* of A are denoted by $\operatorname{dom} A$, $\operatorname{ran} A$, $\ker A$, and $\operatorname{mul} A$:

$$\begin{aligned} \operatorname{dom} A &\stackrel{\text{def}}{=} \{f;\ \{f,f'\}\in A\}, & \ker A &\stackrel{\text{def}}{=} \{f;\ \{f,0\}\in A\},\\ \operatorname{ran} A &\stackrel{\text{def}}{=} \{f';\ \{f,f'\}\in A\}, & \operatorname{mul} A &\stackrel{\text{def}}{=} \{f';\ \{0,f'\}\in A\}; \end{aligned}$$

they are linear subspaces of $\mathfrak{H}$. An operator is a relation when its is identified with its graph. Clearly in this sense a relation A is an operator precisely when $\operatorname{mul} A = \{0\}$. Define the *inverse* of A by

$$A^{-1} \stackrel{\text{def}}{=} \{\{f',f\};\ \{f,f'\}\in A\};$$

then, by complete symmetry,

$$\operatorname{dom} A^{-1} = \operatorname{ran} A, \qquad \ker A^{-1} = \operatorname{mul} A,$$
$$\operatorname{ran} A^{-1} = \operatorname{dom} A, \qquad \operatorname{mul} A^{-1} = \ker A.$$

A relation A is *closed* if it is closed as a subspace of $\mathfrak{H} \times \mathfrak{H}$, in which case $\ker A$ and $\operatorname{mul} A$ are closed subspaces of $\mathfrak{H}$. The closure of a relation A in $\mathfrak{H} \times \mathfrak{H}$ is denoted by $\operatorname{clos} A$; the notations $\overline{\operatorname{dom}}\, A$ and $\overline{\operatorname{ran}}\, A$ indicate the closures of $\operatorname{dom} A$ and $\operatorname{ran} A$ in $\mathfrak{H}$, respectively. The closure of (the graph of) an operator is a closed relation which is not necessarily (the graph of) an operator. An operator is said to be *closable* if the closure of its graph is (the graph of an) operator. In what follows, the class of bounded everywhere defined operators on $\mathfrak{H}$ is denoted by $\boldsymbol{B}(\mathfrak{H})$.

Observe that for any relation A one has

$$\overline{\operatorname{dom}}(\operatorname{clos} A) = \overline{\operatorname{dom}}\, A, \qquad \overline{\operatorname{ran}}(\operatorname{clos} A) = \overline{\operatorname{ran}}\, A. \tag{2.1}$$

Sometimes these identities can be improved. The following result for bounded operators is standard; an extension for linear relations will appear later in Corollary 3.22 (see also Proposition 2.12). A proof is given here for completeness.

LEMMA 2.1. *Let A be a bounded, not necessarily densely defined, operator in a Hilbert space $\mathfrak{H}$. Then*

(i) *A is closed if and only if $\operatorname{dom} A$ is closed;*
(ii) *A is closable and $\operatorname{clos} A$ is bounded with $\|\operatorname{clos} A\| = \|A\|$;*
(iii) $\operatorname{dom}(\operatorname{clos} A) = \overline{\operatorname{dom}}\, A$.

Proof. (i) Assume that A is closed. If the sequence $f_n \in \operatorname{dom} A$ tends to $f \in \mathfrak{H}$, then the inequality $\|A(f_n - f_m)\| \le \|A\|\,\|f_n - f_m\|$ shows that Af_n is a Cauchy sequence, so that $Af_n \to g$ for some $g \in \mathfrak{H}$. Therefore $\{f_n, Af_n\} \to \{f, g\}$, which implies that $f \in \operatorname{dom} A$ and $g = Af$, since A is closed. In particular, $\operatorname{dom} A$ is closed.

Conversely, assume that $\operatorname{dom} A$ is closed. Let the sequence $\{f_n, Af_n\} \in A$ converge to $\{f, g\}$. Then $f \in \operatorname{dom} A$ since $\operatorname{dom} A$ is closed. It follows from the inequality $\|Af_n - Af\| \le \|A\|\,\|f_n - f\|$ that $Af_n \to Af$, in other words, $g = Af$ or, equivalently, $\{f, g\} \in A$. Hence, A is closed.

(ii) In order to show that A is closable, assume that $\{0, g\} \in \operatorname{clos} A$. Then there is a sequence $\{f_n, Af_n\} \in A$ such that $\{f_n, Af_n\} \to \{0, g\}$, i.e., $f_n \to 0$ and $Af_n \to g$. However, $f_n \to 0$ implies that $Af_n \to 0$, so that $g = 0$. Thus, A is closable.

As to boundedness, recall that by definition

$$\|A\| = \sup\{\|Af\| :\ f \in \operatorname{dom} A,\ \|f\| \le 1\}.$$

Since $\operatorname{clos} A$ is an operator and every $f \in \operatorname{dom}(\operatorname{clos} A)$ can be approximated by a sequence $f_n \in \operatorname{dom} A$ with $f_n \to f$ and $Af_n \to (\operatorname{clos} A)f$, the equality $\|\operatorname{clos} A\| = \|A\|$ follows easily from the above definition of the operator norm.

(iii) It follows from (2.1) that $\operatorname{dom}(\operatorname{clos} A) \subset \overline{\operatorname{dom}}(\operatorname{clos} A) = \overline{\operatorname{dom}}\, A$.

Conversely, assume that $f \in \overline{\operatorname{dom}}\, A$. Then there exists a sequence $f_n \in \operatorname{dom} A$ such that $f_n \to f$. Since Af_n is a Cauchy sequence there exists an element g such that $Af_n \to g$.

Observe that $\{f,g\} \in \operatorname{clos} A$. By (ii), $\operatorname{clos} A$ is an operator, and hence $f \in \operatorname{dom}(\operatorname{clos} A)$ and $g = (\operatorname{clos} A)f$. ■

The following statement concerning closable extensions of bounded densely defined operators is an immediate consequence of Lemma 2.1.

COROLLARY 2.2. *If $A \subset B$, B is closable, and A is bounded and densely defined, then B is bounded and, moreover, $B^{**} = A^{**}$.*

The assumption that B is closable is essential in Corollary 2.2; cf. Example 3.24.

2.2. Adjoint relations. Let A be a relation in a Hilbert space $\mathfrak{H}$. The *adjoint* A^* of A is the closed (automatically linear) relation defined by

$$A^* \stackrel{\text{def}}{=} \{\{f,f'\} \in \mathfrak{H} \times \mathfrak{H};\ \langle \{f,f'\},\{h,h'\}\rangle = 0 \text{ for all } \{h,h'\} \in A\},$$

where the form $\langle \cdot,\cdot \rangle$ is defined by

$$\langle \{f,f'\},\{h,h'\}\rangle = (f',h) - (f,h'), \qquad \{f,f'\},\{h,h'\} \in \mathfrak{H} \times \mathfrak{H}.$$

Note that the adjoint A^* is given by

$$A^* = JA^{\perp} = (JA)^{\perp}, \tag{2.2}$$

where the operator J, defined by

$$J\{f,f'\} = \{f',-f\}, \qquad \{f,f'\} \in \mathfrak{H} \times \mathfrak{H}, \tag{2.3}$$

is unitary in $\mathfrak{H} \times \mathfrak{H}$. If A is a relation, then $A^{**} = (A^*)^*$ gives the closure of A, i.e., $A^{**} = \operatorname{clos} A$, due to (2.2). Note that for two relations A and B one has

$$A \subset B \ \Rightarrow\ B^* \subset A^*. \tag{2.4}$$

Furthermore, it follows directly from the definition that

$$(A^{-1})^* = (A^*)^{-1}.$$

LEMMA 2.3. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then*

$$(\operatorname{dom} A)^{\perp} = \operatorname{mul} A^*, \qquad (\operatorname{ran} A)^{\perp} = \ker A^*, \tag{2.5}$$

and, likewise,

$$(\operatorname{dom} A^*)^{\perp} = \operatorname{mul} A^{**}, \qquad (\operatorname{ran} A^*)^{\perp} = \ker A^{**}. \tag{2.6}$$

Proof. The first identity in (2.5) follows from

$$\{0,g\} \in A^* \ \Leftrightarrow\ \{0,g\} \in J(A^{\perp}) \ \Leftrightarrow\ \{g,0\} \in A^{\perp} \ \Leftrightarrow\ g \in (\operatorname{dom} A)^{\perp}.$$

The second identity is obtained by going over to the inverse. The identities in (2.6) follow from those in (2.5) by going over to the adjoint. ■

In particular, observe that

$$\operatorname{mul} A^{**} = \{0\} \ \Leftrightarrow\ \operatorname{dom} A^* \text{ dense in } \mathfrak{H}. \tag{2.7}$$

LEMMA 2.4. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then the following equivalences are valid:*

$$\operatorname{dom} A \subset \overline{\operatorname{dom}}\, A^* \ \Leftrightarrow\ \overline{\operatorname{dom}}\, A \subset \overline{\operatorname{dom}}\, A^* \ \Leftrightarrow\ \operatorname{mul} A^{**} \subset \operatorname{mul} A^*, \tag{2.8}$$

and, likewise,

$$\operatorname{dom} A^* \subset \overline{\operatorname{dom}}\, A \;\Leftrightarrow\; \overline{\operatorname{dom}}\, A^* \subset \overline{\operatorname{dom}}\, A \;\Leftrightarrow\; \operatorname{mul} A^* \subset \operatorname{mul} A^{**}. \tag{2.9}$$

In particular,

$$\overline{\operatorname{dom}}\, A = \overline{\operatorname{dom}}\, A^* \;\Leftrightarrow\; \operatorname{mul} A^{**} = \operatorname{mul} A^*, \tag{2.10}$$

Proof. The first equivalence in (2.8) is valid since the subspace $\overline{\operatorname{dom}}\, A^*$ of $\mathfrak{H}$ is closed. The second equivalence in (2.8) is based on the identity $\operatorname{mul} A^* = (\operatorname{dom} A)^{\perp}$. The equivalences in (2.9) follow if in (2.8) the relation A is replaced by the relation A^* and the identity (2.1) is used. The identity (2.10) is now obvious. ■

It is a consequence of Lemma 2.3 that the Hilbert space $\mathfrak{H}$ has the following orthogonal decompositions:

$$\mathfrak{H} = \overline{\operatorname{dom}}\, A^{**} \oplus \operatorname{mul} A^*, \qquad \mathfrak{H} = \overline{\operatorname{ran}}\, A^{**} \oplus \ker A^*.$$

However, there are also similar, nonorthogonal, decompositions of $\mathfrak{H}$.

Lemma 2.5. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then*

$$\mathfrak{H} = \operatorname{dom} A^{**} + \operatorname{ran} A^*, \qquad \mathfrak{H} = \operatorname{ran} A^{**} + \operatorname{dom} A^*. \tag{2.11}$$

Proof. Recall from (2.2) that $JA^* = A^{\perp}$. This implies that $\mathfrak{H} \times \mathfrak{H} = A^{**} \oplus JA^*$, which leads to (2.11). ■

2.3. Special relations. A relation A is said to be *symmetric* if $A \subset A^*$; a relation is symmetric if and only if $(g, f) \in \mathbb{R}$ for all $\{f, g\} \in A$. A relation A is said to be *essentially selfadjoint* if $A^{**} = A^*$ and it is said to be *selfadjoint* if $A = A^*$. A relation A in a Hilbert space $\mathfrak{H}$ is said to be *formally normal* if there exists an isometry V from A into A^* of the form

$$V\{f, g\} = \{f, h\}, \qquad \{f, g\} \in A,\ \{f, h\} \in A^*,$$

i.e., V leaves the first component f invariant and $\|g\| = \|h\|$. A formally normal relation A in a Hilbert space $\mathfrak{H}$ is said to be *normal* if the isometry V is from A onto A^*. Normal relations and consequently selfadjoint relations are automatically closed. Finally, a relation A in $\mathfrak{H}$ is said to be *subnormal* if there exists a Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$ isometrically and a normal relation B in $\mathfrak{K}$ such that $A \subset B$.

2.4. Sums and products. Let A_1 and A_2 be relations in $\mathfrak{H}$. The notation $A_1 \,\widehat{+}\, A_2$ denotes the *componentwise sum* of A_1 and A_2:

$$A_1 \,\widehat{+}\, A_2 \stackrel{\text{def}}{=} \{\{f_1 + f_2, f_1' + f_2'\};\ \{f_1, f_1'\} \in A_1,\ \{f_2, f_2'\} \in A_2\}. \tag{2.12}$$

In particular,

$$\operatorname{dom}(A_1 \,\widehat{+}\, A_2) = \operatorname{dom} A_1 + \operatorname{dom} A_2, \qquad \operatorname{mul}(A_1 \,\widehat{+}\, A_2) = \operatorname{mul} A_1 + \operatorname{mul} A_2.$$

Lemma 2.6. *The componentwise sum satisfies the identities*

$$(A_1 \,\widehat{+}\, A_2)^* = A_1^* \cap A_2^*, \qquad \operatorname{clos}(A_1 \,\widehat{+}\, A_2) = (A_1^* \cap A_2^*)^*.$$

Proof. Observe that

$$(A_1 \,\widehat{+}\, A_2)^* = J(A_1 \,\widehat{+}\, A_2)^{\perp} = J(A_1^{\perp} \cap A_2^{\perp}) = JA_1^{\perp} \cap JA_2^{\perp} = A_1^* \cap A_2^*,$$

according to the definition of the adjoint operation. This gives the first identity, and the second identity is obtained by taking adjoints in the first one. ■

The following identities are also clear:

$$\operatorname{clos}(A_1 \mathbin{\widehat{+}} A_2) = \operatorname{clos}(A_1 \mathbin{\widehat{+}} \operatorname{clos} A_2) = \operatorname{clos}(\operatorname{clos} A_1 \mathbin{\widehat{+}} \operatorname{clos} A_2).$$

The notation $A_1 + A_2$ is reserved for the *operatorwise sum* of A_1 and A_2:

$$A_1 + A_2 \stackrel{\text{def}}{=} \{\{f, f' + f''\};\ \{f, f'\} \in A_1,\ \{f, f''\} \in A_2\}. \tag{2.13}$$

In particular, it follows from the definition in (2.13) that

$$\operatorname{dom}(A_1 + A_2) = \operatorname{dom} A_1 \cap \operatorname{dom} A_2, \qquad \operatorname{mul}(A_1 + A_2) = \operatorname{mul} A_1 + \operatorname{mul} A_2. \tag{2.14}$$

In the case when A_1 and A_2 are operators this sum is the (graph of the) usual operator sum.

Lemma 2.7. *The operatorwise sum satisfies*

$$A_1^* + A_2^* \subset (A_1 + A_2)^*. \tag{2.15}$$

If A_1 or A_2 belongs to $\boldsymbol{B}(\mathfrak{H})$, then

$$A_1^* + A_2^* = (A_1 + A_2)^*. \tag{2.16}$$

Proof. Let $\{f, f_1' + f_2'\} \in A_1^* + A_2^*$ with $\{f, f_1'\} \in A_1^*$ and $\{f, f_2'\} \in A_2^*$. Now assume that $\{h, h_1 + h_2\} \in A_1 + A_2$ with $\{h, h_1\} \in A_1$ and $\{h, h_2\} \in A_2$. Then

$$\langle \{f, f_1' + f_2'\}, \{h, h_1 + h_2\} \rangle = (f_1', h) - (f, h_1) + (f_2', h) - (f, h_2) = 0,$$

which implies that $\{f, f_1' + f_2'\} \in (A_1 + A_2)^*$. This shows (2.15).

For the converse, let $\{f, f'\} \in (A_1 + A_2)^*$, so that for all $\{h, h_1\} \in A_1$ and $\{h, h_2\} \in A_2$,

$$0 = \langle \{f, f'\}, \{h, h_1 + h_2\} \rangle = (f', h) - (f, h_1 + h_2) = (f', h) - (f, h_1) - (f, h_2).$$

Suppose that, for instance, $A_2 \in \boldsymbol{B}(\mathfrak{H})$; then $h_2 = A_2 h$ and the above identity implies that

$$(f' - A_2^* f, h) = (f, h_1)$$

for all $\{h, h_1\} \in A_1$, so that $\{f, f' - A_2^* f\} \in A_1^*$. Together with $\{f, A_2^* f\} \in A_2^*$, this means that $\{f, f'\} \in A_1^* + A_2^*$. This shows (2.16). ■

The notation $A_1 A_2$ indicates the *product* of A_1 and A_2:

$$A_1 A_2 \stackrel{\text{def}}{=} \{\{f, f'\};\ \{f, h\} \in A_2,\ \{h, f'\} \in A_1\}. \tag{2.17}$$

In particular, $\operatorname{mul} A_1 \subset \operatorname{mul}(A_1 A_2)$. Moreover, if A_2 is an operator, then $\operatorname{mul} A_1 = \operatorname{mul}(A_1 A_2)$. In the case when A_1 and A_2 are both operators the product in (2.17) is the (graph of the) usual operator product. The product of relations is clearly associative. Observe that

$$AA^{-1} = I_{\operatorname{ran} A} \mathbin{\widehat{+}} (\{0\} \times \operatorname{mul} A), \qquad A^{-1}A = I_{\operatorname{dom} A} \mathbin{\widehat{+}} (\{0\} \times \ker A),$$

which shows that products of relations require some care. For $\lambda \in \mathbb{C}$ the notation λA agrees in this sense with $(\lambda I)A$.

LEMMA 2.8. *The product satisfies*

$$A_2^*A_1^* \subset (A_1A_2)^*. \tag{2.18}$$

If A_1 belongs to $\boldsymbol{B}(\mathfrak{H})$, then

$$(A_1A_2)^* = A_2^*A_1^*. \tag{2.19}$$

Proof. Let $\{f, f'\} \in A_2^*A_1^*$, so that $\{f, g\} \in A_1^*$ and $\{g, f'\} \in A_2^*$. Now assume that $\{h, h'\} \in A_1A_2$, so that $\{h, k\} \in A_2$ and $\{k, h'\} \in A_1$. Then

$$\langle\{f, f'\}, \{h, h'\}\rangle = (f', h) - (f, h') = (g, k) - (g, k) = 0,$$

which yields $\{f, f'\} \in (A_1A_2)^*$. This shows (2.18).

Conversely, let $\{f, f'\} \in (A_1A_2)^*$, so that for all $\{h, h'\} \in A_1A_2$ one has

$$0 = \langle\{f, f'\}, \{h, h'\}\rangle = (f', h) - (f, h').$$

However, since $A_1 \in \boldsymbol{B}(\mathfrak{H})$ it is easily seen that $\{h, h'\} \in A_1A_2$ if and only if $\{h, k\} \in A_2$ and $h' = A_1k$. Hence, $\{f, f'\} \in (A_1A_2)^*$ if and only if for all $\{h, k\} \in A_2$,

$$0 = (f', h) - (f, A_1k) = (f', h) - (A_1^*f, k).$$

Therefore $\{f', A_1^*f\} \in A_2^*$, and $\{f, f'\} \in A_2^*A_1^*$. This shows (2.19). ■

Now let A_1 and A_2 be relations in the Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. The notation $A_1 \,\widehat{\oplus}\, A_2$ stands for the *componentwise orthogonal sum* of A_1 and A_2 in the Hilbert space $(\mathfrak{H}_1 \oplus \mathfrak{H}_2) \times (\mathfrak{H}_1 \oplus \mathfrak{H}_2)$:

$$A_1 \,\widehat{\oplus}\, A_2 \stackrel{\text{def}}{=} \{\{f_1 \oplus f_2, f_1' \oplus f_2'\}; \{f_1, f_1'\} \in A_1,\ \{f_2, f_2'\} \in A_2\}.$$

Hence $(A_1 \,\widehat{\oplus}\, A_2)^* = A_1^* \,\widehat{\oplus}\, A_2^*$, where the adjoints are taken in the corresponding Hilbert spaces.

It follows from the definition (2.17) that for any $R \in \boldsymbol{B}(\mathfrak{H})$ the product AR is given by

$$AR = \{\{f, f'\};\ \{Rf, f'\} \in A\}.$$

This product can be made more explicit if R or $I - R$ is an orthogonal projection onto a closed subspace containing $\overline{\operatorname{dom}}\, A$.

LEMMA 2.9. *Let A be a relation in a Hilbert space $\mathfrak{H}$, let $\mathfrak{X}$ and $\mathfrak{Y}$ be closed subspaces of $\mathfrak{H}$ such that* $\operatorname{mul} A^* = \mathfrak{X} \oplus \mathfrak{Y}$, *and let R be the orthogonal projection onto* $\overline{\operatorname{dom}}\, A \oplus \mathfrak{X}$. *Then*

$$AR = A \,\widehat{\oplus}\, (\mathfrak{Y} \times \{0\}), \qquad A(I - R) = (\overline{\operatorname{dom}}\, A \oplus \mathfrak{X}) \times \operatorname{mul} A.$$

In particular,

$$\operatorname{dom} AR = \operatorname{dom} A \oplus \mathfrak{Y}, \qquad \operatorname{dom} A(I - R) = \overline{\operatorname{dom}}\, A \oplus \mathfrak{X}.$$

Proof. Since $\operatorname{dom} A \subset \operatorname{ran} R$ the definition of the product AR shows that $A \subset AR$, and since $\mathfrak{Y} = \ker R \subset \ker AR$ it is also clear that $\mathfrak{Y} \times \{0\} \subset AR$. Hence

$$A \,\widehat{\oplus}\, (\mathfrak{Y} \times \{0\}) \subset AR.$$

For the converse inclusion, let $\{f, f'\} \in AR$. Then

$$\{f, f'\} = \{Rf, f'\} + \{(I - R)f, 0\} \in A \,\widehat{\oplus}\, (\mathfrak{Y} \times \{0\}.$$

This shows the first identity.

On the other hand, $\operatorname{ran}(I-R) \cap \operatorname{dom} A = \mathfrak{Y} \cap \operatorname{dom} A = \{0\}$. Hence, the definition of the product gives $A(I-R) = \ker(I-R) \times \operatorname{mul} A$, which yields the second identity. ■

2.5. Some auxiliary results. Let $\mathfrak{H}$ be a Hilbert space and let $\mathfrak{M}$ and $\mathfrak{N}$ be closed subspaces of $\mathfrak{H}$. Then $\mathfrak{M}+\mathfrak{N}$ is closed if and only if $\mathfrak{M}^\perp + \mathfrak{N}^\perp$ is closed; see, for instance, [26, IV, Theorem 4.8].

LEMMA 2.10. *Let A and B be closed relations in a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) *$A \mathbin{\widehat{+}} B$ is closed;*
(ii) *$A^* \mathbin{\widehat{+}} B^*$ is closed.*

Proof. (i)⇒(ii). The graphs of A and B are closed linear subspaces of the Hilbert space $\mathfrak{H} \times \mathfrak{H}$. Hence, the sum $A \mathbin{\widehat{+}} B$ is a closed linear subspace of $\mathfrak{H} \times \mathfrak{H}$ if and only if the sum of the orthogonal complements

$$A^\perp \mathbin{\widehat{+}} B^\perp \tag{2.20}$$

in $\mathfrak{H} \oplus \mathfrak{H}$ is also closed. Recall that the adjoints of A and B are given by $A^* = JA^\perp$ and $B^* = JB^\perp$, where the operator J is defined in (2.3). Hence the sum in (2.20) is closed in $\mathfrak{H} \times \mathfrak{H}$ if and only if

$$J(A^\perp \mathbin{\widehat{+}} B^\perp) = JA^\perp \mathbin{\widehat{+}} JB^\perp = A^* \mathbin{\widehat{+}} B^*$$

is closed in $\mathfrak{H} \times \mathfrak{H}$.

(ii)⇒(i). Since A and B are closed one has $A^{**} = A$ and $B^{**} = B$. Hence this implication follows by symmetry. ■

The following observation, based on Lemma 2.10, goes back to Yu. L. Shmul'yan [37]. A weaker version for so-called range space relations can be found in [27].

THEOREM 2.11. *Let A be a closed relation in a Hilbert space $\mathfrak{H}$. Then*

(i) $\operatorname{dom} A$ *closed* ⇔ $\operatorname{dom} A^*$ *closed;*
(ii) $\operatorname{ran} A$ *closed* ⇔ $\operatorname{ran} A^*$ *closed.*

Proof. (i) First observe that $A = A^{**}$, since A is assumed to be closed. Hence,

$$(A^* \mathbin{\widehat{+}} (\{0\} \times \mathfrak{H}))^* = A \cap (\{0\} \times \mathfrak{H}) = \{0\} \times \operatorname{mul} A. \tag{2.21}$$

In particular, (2.21) leads to

$$(A^* \mathbin{\widehat{+}} (\{0\} \times \mathfrak{H}))^{**} = (\operatorname{mul} A)^\perp \times \mathfrak{H}. \tag{2.22}$$

Assume that $\operatorname{dom} A$ is closed, so that $A \mathbin{\widehat{+}} (\{0\} \times \mathfrak{H})$ is a closed subspace in $\mathfrak{H} \times \mathfrak{H}$. By Lemma 2.10 this implies that $A^* \mathbin{\widehat{+}} (\{0\} \times \mathfrak{H})$ is a closed subspace of $\mathfrak{H} \times \mathfrak{H}$, so that with (2.22) it follows that

$$A^* \mathbin{\widehat{+}} (\{0\} \times \mathfrak{H}) = (\operatorname{mul} A)^\perp \times \mathfrak{H}, \tag{2.23}$$

or, equivalently, $\operatorname{dom} A^* = (\operatorname{mul} A)^\perp$. Hence, $\operatorname{dom} A^*$ is closed.

Now assume that $\operatorname{dom} A^*$ is closed, so that $A^* \mathbin{\widehat{+}} (\{0\} \times \mathfrak{H})$ is closed. By Lemma 2.10 this implies that $A \mathbin{\widehat{+}} (\{0\} \times \mathfrak{H})$ is closed, i.e., $\operatorname{dom} A$ is closed.

(ii) This can be seen by going over to the inverse of A. ■

The next proposition enhances the previous theorem by giving necessary and sufficient conditions for $\operatorname{dom} A^*$ and $\operatorname{ran} A^*$ to be closed, respectively.

Proposition 2.12. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $\operatorname{dom} A^*$ *is closed;*
(ii) $\operatorname{ran} PA^{**} \subset \operatorname{dom} A^*$, *where P is the orthogonal projection onto* $\overline{\operatorname{dom}}\, A^*$*;*
(iii) $\operatorname{ran} QA^* \subset \operatorname{dom} A^{**}$, *where Q is the orthogonal projection onto* $\overline{\operatorname{dom}}\, A$.

Similarly the following statements are equivalent:

(iv) $\operatorname{ran} A^*$ *is closed;*
(v) $P'(\operatorname{dom} A^{**}) \subset \operatorname{ran} A^*$, *where P' is the orthogonal projection onto* $\overline{\operatorname{ran}}\, A^*$*;*
(vi) $Q'(\operatorname{dom} A^*) \subset \operatorname{ran} A^{**}$, *where Q' is the orthogonal projection onto* $\overline{\operatorname{ran}}\, A$.

Proof. By Lemma (2.5) A satisfies the identities (2.11). The implications (ii)⇒(i) and (v)⇒(iv) are obtained by applying P to the second identity in (2.11) and P' to the first identity in (2.11). The implications (iii)⇒(i) and (vi)⇒(iv) follow by first applying Q to the first identity in (2.11) and Q' to the second to see that $\operatorname{dom} A^{**}$ and $\operatorname{ran} A^{**}$, respectively, are closed; then apply Theorem 2.11.

The implications (i)⇒(ii) and (iv)⇒(v) are clear, while (i)⇒(iii) and (iv)⇒(vi) follow from Theorem 2.11, because then equivalently $\operatorname{dom} A^{**}$ ($\operatorname{ran} A^{**}$, respectively) is closed. ∎

Observe that in Proposition 2.12 the statements (i)–(iii) are actually connected with the statements (iv)–(vi) via the formal inverse A^{-1} of A.

The descriptions of $\operatorname{dom} A^*$ and $\operatorname{ran} A^*$ can be given by means of certain functionals; cf. Example 1.1 (see [20] for further details).

The following result (cf. [19, Lemma 4.1]) follows easily from Proposition 2.12.

Corollary 2.13. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $\operatorname{dom} A^* = \mathfrak{H}$*;*
(ii) $\operatorname{ran} A^{**} \subset \operatorname{dom} A^*$*;*
(iii) *A (and thus also A^{**}) is the graph of a bounded operator.*

Proof. The equivalence of (i) and (ii) is obtained directly from Proposition 2.12.

(i)⇒(iii). If $\operatorname{dom} A^* = \mathfrak{H}$, then $\operatorname{dom} A^{**}$ is closed by Theorem 2.11 and $\operatorname{mul} A^{**} = \{0\}$. Now apply the closed graph theorem.

(iii)⇒(i). The boundedness of A^{**} implies that $\operatorname{dom} A^{**}$ is closed; see Lemma 2.1. Hence, also $\operatorname{dom} A^*$ is closed by Theorem 2.11. It follows from $\operatorname{mul} A^{**} = \{0\}$ that $\operatorname{dom} A^*$ is dense. Therefore $\operatorname{dom} A^* = \mathfrak{H}$. ∎

Remark 2.14. Note that the decomposition $A = B + C$ in Example 1.2 with a nontrivial singular part C is possible if and only if $\operatorname{dom} A^* \neq \mathfrak{H}$; according to Corollary 2.13 this is equivalent to the operator A in Example 1.2 being unbounded.

There are similar corollaries characterizing A^{-1}, A^*, or A^{-*} being a bounded (singlevalued) operator. It can also be noted that PA^{**} appearing in Proposition 2.12 is in fact the regular part of the closure A; see Section 3. The connection between Proposition 2.12 and Corollary 2.13 can be strengthened by means of decompositions in Section 3.

2.6. Points of regular type and the resolvent set. Let A be a relation in a Hilbert space $\mathfrak{H}$. Then $\lambda \in \mathbb{C}$ is said to be an *eigenvalue* of A if $\{f, \lambda f\} \in A$ for some nonzero $f \in \mathfrak{H}$. The set of points of *regular type* of A is denoted by $\gamma(A)$; it consists of those $\lambda \in \mathbb{C}$ for which there exists a positive constant $c(\lambda) > 0$ such that

$$\|f' - \lambda f\| \geq c(\lambda)\|f\|, \qquad \{f, f'\} \in A. \tag{2.24}$$

In other words, $\lambda \in \mathbb{C}$ is a point of *regular type* of A if and only if $(A-\lambda)^{-1}$ is (the graph of) a bounded linear operator, defined on $\operatorname{ran}(A-\lambda)$. In particular, the relation A is closed if and only if $\operatorname{ran}(A - \lambda)$ is closed in $\mathfrak{H}$ for some $\lambda \in \mathbb{C}$ of regular type. Furthermore, $\gamma(\operatorname{clos} A) = \gamma(A)$. It is clear that $\gamma(A) \subset \gamma(\operatorname{clos} A)$. To see the other inclusion, let $\lambda \in \gamma(A)$, so that $(A-\lambda)^{-1}$ is a bounded linear operator. From $\operatorname{clos}(A-\lambda)^{-1} = (\operatorname{clos} A-\lambda)^{-1}$ it follows that $\lambda \in \gamma(\operatorname{clos} A)$.

It is well known that $\gamma(A)$ is an open set for operators, and this remains true also for relations; see [43], [44]; cf. also [15, 16].

THEOREM 2.15. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then $\gamma(A)$ is an open set. In particular, if $\mu \in \gamma(A)$ and $|\lambda - \mu| \, \|(A - \mu)^{-1}\| < 1$, then $\lambda \in \gamma(A)$ and*

$$\|(A-\lambda)^{-1}\| \leq \frac{\|(A-\mu)^{-1}\|}{1 - |\lambda - \mu| \, \|(A-\mu)^{-1}\|}. \tag{2.25}$$

Moreover, if $\mu \in \gamma(A)$ and $|\lambda-\mu| \, \|(A-\mu)^{-1}\| < 1$, then $\overline{\operatorname{ran}}(A-\lambda)$ is not a proper subset of $\overline{\operatorname{ran}}(A - \mu)$.

Proof. Let $\mu \in \gamma(A)$ and $\{f, g\} \in A$. Since $(A - \mu)^{-1}$ is a bounded linear operator, it follows from $(A-\mu)^{-1}(g-\mu f) = f$ that

$$\|f\| \leq \|(A-\mu)^{-1}\| \, \|(g - \mu f)\|.$$

For each $\lambda \in \mathbb{C}$ one has

$$g - \lambda f = g - \mu f - (\lambda - \mu) f,$$

which implies that

$$\|g - \lambda f\| \geq \|g - \mu f\| - |\lambda - \mu| \, \|f\|.$$

Hence,

$$\begin{aligned}
\|(A-\mu)^{-1}\| \, \|g - \lambda f\| &\geq \|(A-\mu)^{-1}\| \, \|g - \mu f\| - |\lambda - \mu| \, \|(A-\mu)^{-1}\| \, \|f\| \\
&\geq \|f\| - |\lambda - \mu| \, \|(A-\mu)^{-1}\| \, \|f\| \\
&= (I - |\lambda - \mu| \, \|(A-\mu)^{-1}\|)\|f\|.
\end{aligned}$$

With the inclusion $\{g-\lambda f, f\} \in (A-\lambda)^{-1}$ and the assumption $|\lambda-\mu| \, \|(A-\mu)^{-1}\| < 1$ this inequality shows that $(A - \lambda)^{-1}$ is a bounded linear operator, whose norm is estimated by (2.25).

Assume that $\overline{\operatorname{ran}}(A - \lambda)$ is a proper subset of $\overline{\operatorname{ran}}(A - \mu)$. Choose $k \in \overline{\operatorname{ran}}(A - \mu) \ominus \overline{\operatorname{ran}}(A-\lambda)$ with $\|k\| = 1$. Then $\|k - g\| \geq 1$ for all $g \in \overline{\operatorname{ran}}(A-\lambda)$. Let $k_n \in \operatorname{ran}(A-\mu)$

be such that $k_n \to k$. Then there exist h_n such that $\{h_n, k_n\} \in A - \mu$, so that also $\{h_n, k_n + (\mu - \lambda)h_n\} \in A - \lambda$. In particular,

$$\begin{aligned} 1 &\le \|k - (k_n + (\mu - \lambda)h_n)\| \le \|k - k_n\| + |\mu - \lambda|\,\|h_n\| \\ &\le \|k - k_n\| + |\mu - \lambda|\,\|(A - \mu)^{-1}\|\,\|k_n\|. \end{aligned}$$

Letting $n \to \infty$ leads to

$$1 \le |\mu - \lambda|\,\|(A - \mu)^{-1}\|,$$

a contradiction. Hence $\overline{\operatorname{ran}}(A - \lambda)$ is not a proper subset of $\overline{\operatorname{ran}}(A - \mu)$. ■

The *resolvent set* $\rho(A)$ of A is the set of all $\lambda \in \mathbb{C}$ such that $\lambda \in \gamma(A)$ and $\operatorname{ran}(A - \lambda)$ is dense in $\mathfrak{H}$. Observe that $\rho(\operatorname{clos} A) = \rho(A)$.

THEOREM 2.16. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then $\rho(A)$ is open. In particular, if $\mu \in \rho(A)$ and $|\lambda - \mu|\,\|(A - \mu)^{-1}\| < 1$, then $\lambda \in \rho(A)$.*

Proof. Since $\mu \in \rho(A)$ one has $\mu \in \gamma(A)$ and $\overline{\operatorname{ran}}(A - \mu) = \mathfrak{H}$. Now by Theorem 2.15 $\lambda \in \gamma(A)$ and $\overline{\operatorname{ran}}(A - \lambda)$ is not a proper subset of $\overline{\operatorname{ran}}(A - \mu)$. Therefore $\overline{\operatorname{ran}}(A - \lambda) = \mathfrak{H}$, so that $\lambda \in \rho(A)$. ■

If A is closed, then $\lambda \in \rho(A)$ if and only if $(A - \lambda)^{-1} \in \boldsymbol{B}(\mathfrak{H})$.

2.7. Defect numbers. It is useful to recall the notion of opening between subspaces. Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be linear (not necessarily closed) subspaces of a Hilbert space $\mathfrak{H}$. Let P_1 and P_2 be the orthogonal projections onto the closures $\overline{\mathfrak{L}_1}$ and $\overline{\mathfrak{L}_2}$ of $\mathfrak{L}_1$ and $\mathfrak{L}_2$, respectively. The *opening* $\theta(\mathfrak{L}_1, \mathfrak{L}_2)$ is defined by $\theta(\mathfrak{L}_1, \mathfrak{L}_2) = \|P_1 - P_2\|$. It is clear that $\theta(\mathfrak{L}_1, \mathfrak{L}_2) = \theta(\overline{\mathfrak{L}_1}, \overline{\mathfrak{L}_2}) = \theta(\mathfrak{L}_1^{\perp}, \mathfrak{L}_2^{\perp})$. Moreover, $\theta(\mathfrak{L}_1, \mathfrak{L}_2) \le 1$, and if $\theta(\mathfrak{L}_1, \mathfrak{L}_2) < 1$, then $\dim \mathfrak{L}_1 = \dim \mathfrak{L}_2$. In order to use the opening the following formula is useful:

$$\theta(\mathfrak{L}_1, \mathfrak{L}_2) = \max\left(\sup_{f \in \mathfrak{L}_1} \frac{\|(I - P_2)f\|}{\|f\|}, \sup_{f \in \mathfrak{L}_2} \frac{\|(I - P_1)f\|}{\|f\|} \right).$$

The following result is a standard fact for operators; for relations it appears precisely in the same form.

THEOREM 2.17. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then the defect*

$$\dim \operatorname{ran}(A - \lambda)^{\perp}$$

is constant for λ in connected components of $\gamma(A)$.

Proof. Let $\lambda, \mu \in \gamma(A)$ and let P_λ and P_μ be the orthogonal projections onto the subspaces $\operatorname{ran}(A - \lambda)^{\perp}$ and $\operatorname{ran}(A - \mu)^{\perp}$, respectively.

Step 1. For each $h \in \mathfrak{H}$,

$$\|(I - P_\mu)h\| = \sup_{\{f,g\} \in A} \frac{|(h, g - \mu f)|}{\|g - \mu f\|} = \sup_{\{f,g\} \in A} \frac{|(h, g - \lambda f + (\lambda - \mu) f)|}{\|g - \mu f\|}.$$

In particular, if $h \in \operatorname{ran}(A - \lambda)^{\perp}$, then

$$\|(I - P_\mu)h\| = |\lambda - \mu| \sup_{\{f,g\} \in A} \frac{|(h, f)|}{\|g - \mu f\|}.$$

Since $\|f\| \leq \|(A-\mu)^{-1}\| \, \|g-\mu f\|$, $\{f,g\} \in A$, it follows that

$$\|(I-P_\mu)h\| \leq |\lambda-\mu| \, \|(A-\mu)^{-1}\| \, \|h\|.$$

Step 2. Completely similar, it follows for $k \in \operatorname{ran}(A-\mu)^\perp$ that

$$\|(I-P_\lambda)k\| = |\lambda-\mu| \sup_{\{f,g\}\in A} \frac{|(k,f)|}{\|g-\lambda f\|} \leq |\lambda-\mu| \, \|(A-\lambda)^{-1}\| \, \|k\|.$$

Hence, if $|\lambda-\mu| \, \|(A-\mu)^{-1}\| < 1$, then

$$\|(I-P_\lambda)k\| \leq \frac{|\lambda-\mu| \, \|(A-\mu)^{-1}\|}{1-|\lambda-\mu| \, \|(A-\mu)^{-1}\|} \, \|k\|.$$

Step 3. Now let $|\lambda-\mu| \, \|(A-\mu)^{-1}\| < 1/2$. Then it follows from Steps 1 and 2 that

$$\theta(\operatorname{ran}(A-\lambda)^\perp, \operatorname{ran}(A-\mu)^\perp) < 1,$$

which implies the equality

$$\dim \operatorname{ran}(A-\lambda)^\perp = \dim \operatorname{ran}(A-\mu)^\perp;$$

see [1], [26].

Step 4. For each $\mu \in \gamma(A)$ there exists a positive number $\delta = \frac{1}{2}\|(A-\mu)^{-1}\|^{-1}$ such that $|\lambda-\mu| < \delta$ implies that $\lambda \in \gamma(A)$ and that at λ there is the same defect as at μ. Now let Γ be a connected open component of $\gamma(A)$. Then Γ is arcwise connected and each pair of points in Γ can be connected by a (piecewise) connected curve with compact image. It remains to use compactness to divide the curve into pieces of length $\delta/2$ to conclude that $\dim\ker(A^*-\bar{\lambda})$ is constant in Γ. ∎

2.8. The numerical range. Let A be a relation in a Hilbert space $\mathfrak{H}$. The *numerical range* $\mathcal{W}(A)$ of A is defined by

$$\mathcal{W}(A) = \{(f',f);\ \{f,f'\} \in A,\ \|f\| = 1\} \subset \mathbb{C},$$

and by $\{0\} \subset \mathbb{C}$ if A is purely multivalued, i.e. if $\operatorname{dom} A = \{0\}$. Clearly, all eigenvalues of A belong to the numerical range $\mathcal{W}(A)$ of A. Observe that the numerical range of the inverse of A is given by

$$\mathcal{W}(A^{-1}) = \{\lambda \in \mathbb{C};\ \bar{\lambda} \in \mathcal{W}(A)\}.$$

The following result will be proved along the lines of [40]; cf. [26], [36].

PROPOSITION 2.18. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then the numerical range $\mathcal{W}(A)$ is a convex set in $\mathbb{C}$.*

Proof. Let $\lambda_1, \lambda_2 \in \mathcal{W}(A)$ and assume that $\lambda_1 \neq \lambda_2$. It will be shown that each point on the segment between λ_1 and λ_2 belongs to $\mathcal{W}(A)$, i.e., it will be shown that for each $u \in [0,1]$,

$$u\lambda_1 + (1-u)\lambda_2 \in \mathcal{W}(A).$$

For this purpose write $\lambda_i = (g_i, f_i)$, where $\{f_i, g_i\} \in A$, $\|f_i\| = 1$, $i = 1,2$, and define, for $x_1, x_2 \in \mathbb{C}$,

$$F(x_1,x_2) = (x_1g_1 + x_2g_2, x_1f_1 + x_2f_2), \qquad G(x_1,x_2) = \|x_1f_1 + x_2f_2\|^2,$$

and

$$H(x_1,x_2)=\frac{F(x_1,x_2)-\lambda_2 G(x_1,x_2)}{\lambda_1-\lambda_2}.$$

Note that if $G(x_1,x_2)=1$, then $F(x_1,x_2)\in\mathcal{W}(A)$, or, in other words,

$$H(x_1,x_2)\lambda_1+(1-H(x_1,x_2))\lambda_2=\lambda_2+H(x_1,x_2)(\lambda_1-\lambda_2)\in\mathcal{W}(A).$$

Hence, the proof will be complete if for each $u\in[0,1]$ there exist numbers $x_1,x_2\in\mathbb{C}$ for which

$$G(x_1,x_2)=1,\qquad H(x_1,x_2)=u.$$

Observe that $H(x_1,x_2)=x_1\bar{x}_1+c_1\bar{x}_1x_2+c_2x_1\bar{x}_2$ for some $c_1,c_2\in\mathbb{C}$. Define

$$\delta=1\quad\text{if } \bar{c}_1=c_2,\qquad \delta=\frac{\bar{c}_1-c_2}{|\bar{c}_1-c_2|}\quad\text{if } \bar{c}_1\neq c_2$$

so that $|\delta|=1$. When $t_1,t_2\in\mathbb{R}$ it follows that

$$G(t_1,\delta t_2)=t_1^2+2\beta t_1t_2+t_2^2,\qquad H(t_1,\delta t_2)=t_1^2+\gamma t_1t_2,$$

where $\beta=\mathfrak{re}(\delta(f_2,f_1))$ and $\gamma=\delta c_1+\bar{\delta}c_2$. Hence $-1\leq\beta\leq 1$ and $\gamma\in\mathbb{R}$. For $t_1\in[-1,1]$ note that $(1-\beta^2)t_1^2\leq 1$ and choose

$$t_2=-\beta t_1\pm\sqrt{1-(1-\beta^2)t_1^2},$$

with the $+$ sign when $\beta\geq 0$ and the $-$ sign when $\beta<0$. Then

$$G(t_1,\delta(-\beta t_1\pm\sqrt{1-(1-\beta^2)t_1^2}))=1,$$

and

$$H(t_1,\delta(-\beta t_1\pm\sqrt{1-(1-\beta^2)t_1^2}))=(1-\beta\gamma)t_1^2\pm\gamma t_1\sqrt{1-(1-\beta^2)t_1^2}.$$

The last expression is a real continuous function in t_1 which takes the value 0 at $t_1=0$ and the value 1 at $t_1=1$. Hence the segment $[0,1]$ is in the range of values of this function. ∎

Hence either $\mathcal{W}(A)=\mathbb{C}$ or $\mathcal{W}(A)\neq\mathbb{C}$, in which case $\mathcal{W}(A)$ lies in some halfplane. The first case may actually occur if, for instance, $\ker A\cap\operatorname{mul}A\neq\{0\}$, so that A contains nontrivial elements $\{0,h\}$ and $\{h,0\}$. If the relation A' is an extension of A, i.e., $A\subset A'$, then $\mathcal{W}(A)\subset\mathcal{W}(A')$. In particular,

$$\mathcal{W}(A)\subset\mathcal{W}(\operatorname{clos}A)\subset\operatorname{clos}\mathcal{W}(A), \tag{2.26}$$

where the last inclusion is straightforward to verify. All sets in (2.26) are convex.

2.9. An extension preserving the numerical range. Let A be a relation in a Hilbert space $\mathfrak{H}$ and associate with it the relation A_∞ defined by

$$A_\infty\stackrel{\text{def}}{=}A\,\widehat{+}\,(\{0\}\times\operatorname{mul}A^*); \tag{2.27}$$

the sum in (2.27) is direct if and only if $\operatorname{mul}A\cap\operatorname{mul}A^*=\{0\}$. The relation A_∞ is an extension of A and

$$\operatorname{dom}A_\infty=\operatorname{dom}A,\qquad \operatorname{mul}A_\infty=\operatorname{mul}A+\operatorname{mul}A^*. \tag{2.28}$$

Clearly, if $\operatorname{mul} A \subset \operatorname{mul} A^*$ then $\operatorname{mul} A_\infty = \operatorname{mul} A^*$. Moreover, $A_\infty = A$ if and only $\operatorname{mul} A^* \subset \operatorname{mul} A$ (which is the case when, for instance, A is densely defined). Due to $\operatorname{mul} A^* = (\operatorname{dom} A)^\perp$ it follows from (2.27) that

$$\mathcal{W}(A_\infty) = \mathcal{W}(A). \tag{2.29}$$

Constructions in terms of the extension A_∞ can be found in [10] and [22]. A key observation is given in the following lemma.

LEMMA 2.19. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then $(A_\infty)^*$ can be expressed as a restriction of A^*:*

$$(A_\infty)^* = \{\{f, f'\} \in A^*;\ f \in \overline{\operatorname{dom}}\, A\}. \tag{2.30}$$

In particular,

$$\operatorname{dom}(A_\infty)^* = \overline{\operatorname{dom}}\, A \cap \operatorname{dom} A^*, \qquad \operatorname{mul}(A_\infty)^* = \operatorname{mul} A^*. \tag{2.31}$$

Proof. It follows from (2.27) and Lemma 2.6 that

$$(A_\infty)^* = A^* \cap (\overline{\operatorname{dom}}\, A \times \mathfrak{H}),$$

which leads to the description (2.30) and the identities in (2.31). ∎

2.10. Formally domain tight and domain tight relations. A relation A in a Hilbert space $\mathfrak{H}$ is said to be *formally domain tight* if

$$\operatorname{dom} A \subset \operatorname{dom} A^*. \tag{2.32}$$

Formally normal and symmetric relations are formally domain tight. If a relation A is formally domain tight, then (2.32) shows that

$$(\operatorname{mul} A \subset)\ \operatorname{mul} A^{**} \subset \operatorname{mul} A^*. \tag{2.33}$$

A densely defined formally domain tight relation A is (the graph of) a closable operator, i.e., $\operatorname{mul} A^{**} = \{0\}$. Furthermore, for a formally domain tight relation A it follows that

$$\operatorname{mul} A^* \subset \operatorname{mul} A \ \Rightarrow\ \operatorname{mul} A = \operatorname{mul} A^* = \operatorname{mul} A^{**}. \tag{2.34}$$

A relation A in a Hilbert space $\mathfrak{H}$ is said to be *domain tight* if

$$\operatorname{dom} A = \operatorname{dom} A^*. \tag{2.35}$$

Normal and selfadjoint relations are domain tight. If a relation A is domain tight, then

$$\operatorname{mul} A^{**} = \operatorname{mul} A^*. \tag{2.36}$$

A domain tight relation A is densely defined if and only if A is (the graph of) a closable operator, i.e., $\operatorname{mul} A^{**} = \{0\}$.

REMARK 2.20. The notions of formally domain tight and domain tight relations seem to be new. It is clear that symmetric, formally normal, and subnormal relations may be viewed as prototypes of formally domain tight relations and that selfadjoint and normal relations may be viewed as prototypes of domain tight relations. Densely defined domain tight symmetric or formally normal operators must necessarily be selfadjoint or normal, respectively; on the other hand, domain tight symmetric or domain tight formally normal relations are selfadjoint or normal when extra information about the multivalued parts is provided; cf. Corollary 2.27. For subnormal operators the situation is different: in principle they are not domain tight (see [38] for some discussion) but even if they are, they may not be normal as their normal extensions in most cases go beyond the initial space; this is less visible in the case of relations and Section 2.12 sheds some more light on that.

Remark 2.21. Further examples of formally domain tight and domain tight relations or operators come from the q-deformation of the above mentioned classes. This is motivated by the theory of quantum groups; the relevant Hilbert space operators were introduced by S. Ôta [30], [31]. The balanced operators proposed by S. L. Woronowicz [45] appear to be in the same spirit.

Lemma 2.22. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then*

(i) *A is formally domain tight if and only if*

$$\operatorname{dom} A \subset \overline{\operatorname{dom}}\, A \cap \operatorname{dom} A^*; \tag{2.37}$$

(ii) *if A is domain tight then*

$$\operatorname{dom} A = \overline{\operatorname{dom}}\, A \cap \operatorname{dom} A^*. \tag{2.38}$$

Proof. (i) The inclusion $\operatorname{dom} A \subset \operatorname{dom} A^*$ is equivalent to the inclusion in (2.37).

(ii) If A is formally domain tight, then (2.37) gives

$$\operatorname{dom} A \subset \overline{\operatorname{dom}}\, A \cap \operatorname{dom} A^* \subset \operatorname{dom} A^*.$$

Hence, if A is domain tight, then (2.38) follows. ∎

If B is a formally domain tight relation, then any restriction A of B, i.e., $A \subset B$, is also formally domain tight; see (2.4). The following lemma contains a kind of converse statement.

Lemma 2.23. *Let A and B be relations in a Hilbert space $\mathfrak{H}$ which satisfy $A \subset B$. If A is domain tight and B is formally domain tight, then B is domain tight.*

Proof. The inclusion $A \subset B$ implies that $B^* \subset A^*$. Therefore, it follows that

$$\operatorname{dom} A \subset \operatorname{dom} B, \qquad \operatorname{dom} B^* \subset \operatorname{dom} A^*. \tag{2.39}$$

The assumptions on A and B are

$$\operatorname{dom} A = \operatorname{dom} A^*, \qquad \operatorname{dom} B \subset \operatorname{dom} B^*.$$

Combining these assumptions with the inclusions in (2.39) gives

$$\operatorname{dom} A \subset \operatorname{dom} B \subset \operatorname{dom} B^* \subset \operatorname{dom} A^* = \operatorname{dom} A,$$

which leads to $\operatorname{dom} B = \operatorname{dom} B^*$, i.e., B is domain tight. ∎

Remark 2.24. Let A be a relation in a Hilbert space. Then clearly

$$A \text{ domain tight} \;\Rightarrow\; A \text{ and } A^* \text{ formally domain tight.}$$

Moreover, if $\operatorname{dom} A^{**} = \operatorname{dom} A$, then

$$A \text{ and } A^* \text{ formally domain tight} \;\Rightarrow\; A \text{ domain tight.}$$

If A^{**} is formally domain tight, then A is formally domain tight.

The relation A_∞ introduced in (2.27) can be used to obtain a characterization for A to be domain tight or formally domain tight.

Proposition 2.25. *Let A be a relation in a Hilbert space $\mathfrak{H}$ and let the extension A_∞ of A be defined by* (2.27). *Then*

(i) *A is formally domain tight if and only if A_∞ is formally domain tight;*
(ii) *A_∞ is domain tight if and only if* $\operatorname{dom} A = \overline{\operatorname{dom}}\, A \cap \operatorname{dom} A^*$*;*
(iii) *A is domain tight if and only if A_∞ is domain tight and* $\operatorname{dom} A^* \subset \overline{\operatorname{dom}}\, A$*. Furthermore, in this case $(A_\infty)^* = A^* = (A^*)_\infty$.*

Proof. (i) According to (2.28) and (2.31) the relation A_∞ is formally domain tight (i.e., $\operatorname{dom} A_\infty \subset \operatorname{dom}(A_\infty)^*$) if and only if

$$\operatorname{dom} A \subset \overline{\operatorname{dom}}\, A \cap \operatorname{dom} A^*.$$

Hence, the statement follows from Lemma 2.22.

(ii) The assertion follows from (2.28) and (2.31).

(iii) Let A be domain tight. Then $\operatorname{dom} A = \overline{\operatorname{dom}}\, A \cap \operatorname{dom} A^*$ by Lemma 2.22. Hence, A_∞ is domain tight by (ii). Moreover, $\operatorname{dom} A^* = \operatorname{dom} A \subset \overline{\operatorname{dom}}\, A$.

Conversely, if A_∞ is domain tight and $\operatorname{dom} A^* \subset \overline{\operatorname{dom}}\, A$, then part (ii) implies that $\operatorname{dom} A = \overline{\operatorname{dom}}\, A \cap \operatorname{dom} A^* = \operatorname{dom} A^*$. Thus, A is domain tight.

It is clear for a domain tight relation A that

$$\{\{f, f'\} \in A^*;\ f \in \overline{\operatorname{dom}}\, A\} = A^*.$$

Hence, the identity (2.30) implies that $(A_\infty)^* = A^*$. In general, $(A^*)_\infty$ is an extension of A^*, and $A^* = (A^*)_\infty$ if and only if $\operatorname{mul} A^{**} \subset \operatorname{mul} A^*$. Therefore, if A is domain tight, the identity (2.36) implies that $A^* = (A^*)_\infty$. ■

Lemma 2.26. *Let A be a relation in a Hilbert space $\mathfrak{H}$ and let the extension A_∞ of A be defined by* (2.27). *If A is formally domain tight, then*

(i) $\operatorname{mul} A_\infty = \operatorname{mul} A^*$*;*
(ii) *$A_\infty = A$ if and only* $\operatorname{mul} A^* = \operatorname{mul} A$*;*
(iii) $A \cap (\{0\} \times \operatorname{mul} A^*) = \{0\} \times \operatorname{mul} A$*, and the sum in* (2.27) *is direct if and only if A is an operator;*
(iv) *A_∞ is an operator if and only if A is densely defined.*

Moreover, if A is domain tight and $\operatorname{mul} A^{**} = \operatorname{mul} A$*, then $A_\infty = A$. In particular, if A is domain tight and closed, then $A_\infty = A$.*

Proof. (i) Since A is formally domain tight, (2.33) shows that $\operatorname{mul} A \subset \operatorname{mul} A^*$. This shows the assertion.

(ii) Note that $A = A_\infty$ if and only if $\operatorname{mul} A^* \subset \operatorname{mul} A$. If A is formally domain tight, then (2.34) implies that the inclusion $\operatorname{mul} A^* \subset \operatorname{mul} A$ is equivalent to the identity $\operatorname{mul} A^* = \operatorname{mul} A$.

(iii) Since A is formally domain tight, the inclusion $\operatorname{mul} A \subset \operatorname{mul} A^*$ in (2.33) leads to the assertions.

(iv) If A is densely defined, then $\operatorname{mul} A^* = \{0\}$, so that $\operatorname{mul} A^{**} = \{0\}$ by (2.33), and A is a closable operator. Hence, A_∞ is an operator. Conversely, if A_∞ is an operator, then necessarily $\operatorname{mul} A^* = \{0\}$, so that A is densely defined.

For the last statement, observe that (2.35) implies (2.36). The assumption $\operatorname{mul} A^{**} = \operatorname{mul} A$ implies that $\operatorname{mul} A^* = \operatorname{mul} A$. The assertion now follows from (ii). ∎

2.11. Selfadjointness of symmetric relations. Let A be a symmetric relation in a Hilbert space $\mathfrak{H}$. Then its closure A^{**} is formally domain tight, as the closure is symmetric. If A is densely defined, then $\operatorname{mul} A^{**} = \{0\}$, so that, in fact, A is a closable operator.

If A is a selfadjoint relation, then, in particular, A is symmetric, domain tight, and $\operatorname{mul} A^* \subset \operatorname{mul} A$.

LEMMA 2.27. *Let A be a symmetric domain tight relation in a Hilbert space $\mathfrak{H}$, such that* $\operatorname{mul} A^* \subset \operatorname{mul} A$. *Then A is selfadjoint. In particular, a closed domain tight symmetric relation is selfadjoint.*

Proof. It suffices to show that $A^* \subset A$. Let $\{f, g\} \in A^*$, so that $f \in \operatorname{dom} A^* = \operatorname{dom} A$, which implies that there is an element h such that $\{f, h\} \in A$ $(\subset A^*)$. Hence, $g - h \in \operatorname{mul} A^* \subset \operatorname{mul} A$. Therefore

$$\{f, g\} = \{f, h\} + \{0, g - h\} \in A.$$

Hence, $A^* \subset A$, and thus A is selfadjoint.

When A is closed and domain tight, it follows from (2.36) that $\operatorname{mul} A^* = \operatorname{mul} A$. Hence, the last observation is clear. ∎

If A is a symmetric relation, then, clearly, also the extension A_∞ is symmetric (for instance, see (2.29)). The following result goes back to [10].

LEMMA 2.28. *Let A be a relation in a Hilbert space $\mathfrak{H}$ and let the extension A_∞ of A be defined by* (2.27). *Then A_∞ is selfadjoint if and only if A is symmetric and* $\operatorname{dom} A = \overline{\operatorname{dom}} A \cap \operatorname{dom} A^*$.

Proof. ($\Rightarrow$) If A_∞ is selfadjoint, then A is symmetric and A_∞ is domain tight, so that $\operatorname{dom} A = \overline{\operatorname{dom}} A \cap \operatorname{dom} A^*$; cf. Proposition 2.25.

($\Leftarrow$) If A is symmetric and $\operatorname{dom} A = \overline{\operatorname{dom}} A \cap \operatorname{dom} A^*$, then A_∞ is symmetric and domain tight. Moreover $\operatorname{mul} (A_\infty)^* = \operatorname{mul} A^*$ by Lemma 2.19 and $\operatorname{mul} A_\infty = \operatorname{mul} A^*$ by Lemma 2.26, so that A_∞ is selfadjoint; cf. Lemma 2.27. ∎

2.12. Extensions in larger Hilbert spaces. Let $\mathfrak{H}$ and $\mathfrak{K}$ be two Hilbert spaces with the inclusion $\mathfrak{H} \subset \mathfrak{K}$ being isometric. Let A be a relation in $\mathfrak{H}$ and let B be a relation in $\mathfrak{K}$. Assume that B is an extension of A, i.e.,

(2.40) $$A \subset B.$$

Then it is clear that

(2.41) $$\operatorname{dom} A \subset \operatorname{dom} B \cap \mathfrak{H}, \qquad P(\operatorname{dom} B^*) \subset \operatorname{dom} A^*,$$

where P is the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$. The assumption $A \subset B$ implies the first inclusion in (2.41) trivially and it implies the second inclusion in (2.41) since

$\{Pf, Pg\} \in A^*$ for all $\{f, g\} \in B^*$. The relation B is said to be a *tight extension* of A if

$$\operatorname{dom} A = \operatorname{dom} B \cap \mathfrak{H},$$

and, likewise, B is said to be a $*$-*tight extension* of A if

$$P(\operatorname{dom} B^*) = \operatorname{dom} A^*.$$

Tight and $*$-tight extensions will be discussed only in this subsection. If the relation B is formally domain tight in $\mathfrak{K}$, then

$$\operatorname{dom} B \cap \mathfrak{H} \subset P(\operatorname{dom} B^*). \tag{2.42}$$

Hence, if B is a tight and $*$-tight extension of A, and if B is formally domain tight in $\mathfrak{K}$, then (2.42) shows that A is formally domain tight in $\mathfrak{H}$. The next result is a counterpart to Lemma 2.23.

LEMMA 2.29. *Let A be a relation in the Hilbert space $\mathfrak{H}$ and let B be a relation in the Hilbert space $\mathfrak{K}$ which satisfy* (2.40).

(i) *If A is domain tight in $\mathfrak{H}$ and B is formally domain tight in $\mathfrak{K}$, then*

$$\operatorname{dom} B \cap \mathfrak{H} = P(\operatorname{dom} B^*), \tag{2.43}$$

and B is a tight and $*$-tight extension of A.

(ii) *If the identity* (2.43) *holds and if B is a tight and $*$-tight extension of A, then A is domain tight in $\mathfrak{H}$.*

Proof. (i) If the extension B of A is formally domain tight in $\mathfrak{K}$, then

$$\operatorname{dom} A \subset \operatorname{dom} B \cap \mathfrak{H} \subset P(\operatorname{dom} B^*) \subset \operatorname{dom} A^*. \tag{2.44}$$

The second inclusion follows from $\operatorname{dom} B \subset \operatorname{dom} B^*$. The other inclusions follow from (2.41). The assumption that A is domain tight in $\mathfrak{H}$ and the inclusions in (2.44) imply the identity in (2.43). In particular, B is a tight and $*$-tight extension of A.

(ii) Assume that the identity (2.43) holds and that B is a tight and $*$-tight extension of A. By the definitions of tight and $*$-tight extensions it follows that $\operatorname{dom} A = \operatorname{dom} A^*$. ■

If B is a tight extension of A, then any tight extension of B is again a tight extension of A. There is a similar statement for $*$-tight extensions of A.

A densely defined symmetric operator always has a tight selfadjoint extension; a detailed argument is given in [38], which in turn implements the suggestion made in [1], where a tight extension is called an extension of the second kind. A densely defined subnormal operator need not have any tight normal extensions; an example of Ôta [32] gives a negative answer to the question in [38].

Tight and $*$-tight extensions as discussed in [41] are essential in identifying solutions of the commutation relation of the q-harmonic oscillator as q-creation operators when $q > 1$, in which case nonuniqueness of normal extensions occurs; see [42, Theorem 21].

2.13. Range tight relations. Let A be a relation in a Hilbert space $\mathfrak{H}$. The notions of formally domain tight and domain tight refer to properties relative to the domains dom A and dom A^*. Similar notions exist relative to the ranges $\operatorname{ran} A$ and $\operatorname{ran} A^*$. A relation A

in a Hilbert space $\mathfrak{H}$ is said to be *formally range tight* if

$$\operatorname{ran} A \subset \operatorname{ran} A^*,$$

and it is said to be *range tight* if

$$\operatorname{ran} A = \operatorname{ran} A^*.$$

Clearly, a relation A is (formally) range tight if and only if the relation A^{-1} is (formally) domain tight. Hence, all earlier statements for (formally) domain tight relations have their counterparts for (formally) range tight relations. As an example consider the following consequence of Lemma 2.27.

Let A be a symmetric range tight relation in a Hilbert space $\mathfrak{H}$, such that $\ker A^* \subset \ker A$. Then A is selfadjoint. In particular, a closed range tight symmetric relation is selfadjoint. The same result for densely defined closed range tight symmetric operators was obtained independently by Z. Sebestyén and Z. Tarcsay (personal communication).

2.14. Maximality with respect to the numerical range. The following results are included for completeness. In some form or other they go back to R. McKelvey (unpublished lecture notes) and F. S. Rofe-Beketov [34]; see also [18].

LEMMA 2.30. *Let A be a relation in a Hilbert space $\mathfrak{H}$ with $\mathcal{W}(A) \neq \mathbb{C}$. Let $\lambda \notin \operatorname{clos} \mathcal{W}(A)$, i.e., $d(\lambda) = \operatorname{dist}(\lambda, \operatorname{clos} \mathcal{W}(A)) > 0$. Then*

(i) *$(A-\lambda)^{-1}$ is a bounded linear operator with*

$$\|(A-\lambda)^{-1}\| \leq 1/d(\lambda); \tag{2.45}$$

(ii) *$\operatorname{mul} A \subset \operatorname{mul} A^*$.*

Proof. (i) Let $\lambda \notin \operatorname{clos} \mathcal{W}(A)$ and let $\{f, f'\} \in A$ with $\|f\| = 1$. Then

$$(f', f) - \lambda = (f', f) - \lambda(f, f) = (f' - \lambda f, f),$$

so that

$$d(\lambda) \leq |(f', f) - \lambda| \leq \|f' - \lambda f\|, \quad \{f, f' - \lambda f\} \in A - \lambda.$$

Since λ is not an eigenvalue of A, the inequality in (2.45) follows from the above inequality.

(ii) Let $\varphi \in \operatorname{mul} A$, so that $\{f, f' + c\varphi\} \in A$ for all $\{f, f'\} \in A$ and all $c \in \mathbb{C}$. Since $\mathcal{W}(A) \neq \mathbb{C}$, the identity

$$(f' + c\varphi, f) = (f', f) + c(\varphi, f)$$

shows that $(\varphi, f) = 0$. Hence $\operatorname{mul} A \subset (\operatorname{dom} A)^\perp = \operatorname{mul} A^*$. ∎

Let A be a relation in a Hilbert space $\mathfrak{H}$ with $\mathcal{W}(A) \neq \mathbb{C}$. According to Lemma 2.30 the complement $\Delta(A) = \mathbb{C} \setminus \operatorname{clos} \mathcal{W}(A)$ is a subset of the set of regular points of A. Hence $\operatorname{ran}(A-\lambda)$ is closed for some $\lambda \notin \operatorname{clos} \mathcal{W}(A)$ if and only if A is closed. Since $\operatorname{clos} \mathcal{W}(A)$ is a closed convex set (see Proposition 2.18 and (2.26)), it follows that either $\Delta(A)$ is an open connected set, or $\Delta(A)$ consists of two open connected components (if $\mathcal{W}(A)$ is a strip bounded by two parallel straight lines). Furthermore, by Theorem 2.17, $\dim \ker(A^* - \bar{\lambda})$ is constant for $\lambda \in \Delta(A)$ or for λ in each of the connected components of $\Delta(A)$. If $\ker(A^* - \bar{\lambda}) = \{0\}$ for some $\lambda \in \mathbb{C} \setminus \operatorname{clos} \mathcal{W}(A)$ then $\Delta(A)$ or the corresponding component (to which λ belongs) is a subset of $\rho(A)$.

Note that in the statements (i) and (ii) of Lemma 2.30 the relation A may be replaced by the closure A^{**}. In particular, this shows that a densely defined relation A with $\mathcal{W}(A) \neq \mathbb{C}$ satisfies $\operatorname{mul} A^{**} = \{0\}$; in other words, A is a closable operator. Furthermore, it follows that $\operatorname{ran}(A^{**}-\lambda)$ is closed. These observations lead to the following useful result.

Corollary 2.31. *Let A be a relation in a Hilbert space $\mathfrak{H}$ with $\mathcal{W}(A) \neq \mathbb{C}$. Let $\lambda \notin \operatorname{clos} \mathcal{W}(A)$. Then*

$$\operatorname{ran}(A^* - \bar{\lambda}) = \mathfrak{H}.$$

Proof. In general $\mathfrak{H} = \overline{\operatorname{ran}}(A^* - \bar{\lambda}) \oplus \ker(A^{**} - \lambda)$. By Lemma 2.30 and the above remarks, it follows that $\mathfrak{H} = \overline{\operatorname{ran}}(A^* - \bar{\lambda})$ and that $\operatorname{ran}(A^{**} - \lambda)$ is closed. Then also $\operatorname{ran}(A^* - \bar{\lambda})$ is closed by Theorem 2.11, so that $\operatorname{ran}(A^* - \bar{\lambda}) = \mathfrak{H}$. ∎

A relation A in a Hilbert space $\mathfrak{H}$ with $\mathcal{W}(A) \neq \mathbb{C}$ is said to be *maximal* with respect to the numerical range $\mathcal{W}(A)$ if $\operatorname{ran}(A - \lambda) = \mathfrak{H}$ for some $\lambda \notin \operatorname{clos} \mathcal{W}(A)$. Then, clearly, $\lambda \in \rho(A)$ and A is closed. In fact, A is maximal if and only if some open connected component of $\Delta(A)$ is contained in the resolvent set of A.

Lemma 2.32. *Let A be a relation in a Hilbert space $\mathfrak{H}$ with $\mathcal{W}(A) \neq \mathbb{C}$. Assume that A is maximal with respect to $\mathcal{W}(A)$. Then*

$$\operatorname{mul} A = \operatorname{mul} A^*. \tag{2.46}$$

Proof. It suffices to show that $\operatorname{mul} A^* \subset \operatorname{mul} A$; cf. Lemma 2.30. Let A_∞ be the extension of A defined in (2.27). Then $\mathcal{W}(A_\infty) = \mathcal{W}(A)$ according to (2.29). Hence, if $\lambda \notin \operatorname{clos} \mathcal{W}(A)$, then λ is not an eigenvalue of A_∞. Moreover, since A_∞ is an extension of A it follows that $\operatorname{ran}(A - \lambda) \subset \operatorname{ran}(A_\infty - \lambda)$. It follows from $\mathcal{W}(A_\infty) = \mathcal{W}(A)$ and $\operatorname{ran}(A_\infty - \lambda) = \mathfrak{H}$ that A_∞ is closed. Therefore $(A_\infty - \lambda)^{-1} \in \boldsymbol{B}(\mathfrak{H})$, so that $\lambda \in \rho(A_\infty)$. It follows from $(A - \lambda)^{-1} \subset (A_\infty - \lambda)^{-1}$ that $(A - \lambda)^{-1} = (A_\infty - \lambda)^{-1}$, in other words $A_\infty = A$. This shows that $\operatorname{mul} A^* \subset \operatorname{mul} A$. ∎

3. Componentwise decompositions of relations

In this section the canonical operatorwise decomposition of a relation in a Hilbert space is used to characterize componentwise decompositions by means of an operator part. Again, for simplicity, the results are formulated for linear relations in a Hilbert space, instead of linear relations acting from one Hilbert space to another Hilbert space.

3.1. Canonical decompositions of relations. A relation A in a Hilbert space $\mathfrak{H}$ (or a relation from a Hilbert space $\mathfrak{H}$ to another Hilbert space $\mathfrak{K}$) is said to be *singular* if

$$\operatorname{ran} A \subset \operatorname{mul} A^{**} \quad \text{or equivalently} \quad \overline{\operatorname{ran}}\, A \subset \operatorname{mul} A^{**}. \tag{3.1}$$

The equivalence here is due to the closedness of $\operatorname{mul} A^{**}$. Furthermore, the inclusion

$$\operatorname{mul} A^{**} \subset \overline{\operatorname{ran}}\, A \tag{3.2}$$

follows from (2.1) as $\operatorname{mul} A^{**} \subset \operatorname{ran} A^{**}$. Therefore, a linear relation A is singular if and only if

$$\overline{\operatorname{ran}}\, A = \operatorname{mul} A^{**}, \tag{3.3}$$

which follows from (3.1) and (3.2). There is also an alternative characterization in terms of sequences which goes back to Ôta [29] in the case of densely defined operators; cf. [19].

PROPOSITION 3.1. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) *A is singular;*
(ii) *for each $\varphi' \in \operatorname{ran} A$ there exists a sequence $\{h_n, h_n'\} \in A$ such that $h_n \to 0$ and $h_n' \to \varphi'$.*

Proof. The equivalence is obtained by rewriting the condition $\operatorname{ran} A \subset \operatorname{mul} A^{**}$ elementwise using the definition of the closure A^{**} of A. ■

In what follows, a relation A in a Hilbert space $\mathfrak{H}$ (or a relation from a Hilbert space $\mathfrak{H}$ to another Hilbert space $\mathfrak{K}$) is said to be *regular* if its closure A^{**} is an operator. Thus a regular relation is automatically an operator.

Let A be a not necessarily closed relation in the Hilbert $\mathfrak{H}$ and define the subspace $\mathfrak{H}_A$ by

$$\mathfrak{H}_A \stackrel{\text{def}}{=} \overline{\operatorname{dom}}\, A^* = \mathfrak{H} \ominus \operatorname{mul} A^{**}. \tag{3.4}$$

Since $\operatorname{mul} A \subset \operatorname{mul} A^{**}$, it follows that

$$\mathfrak{H}_A \subset \mathfrak{H} \ominus \operatorname{mul} A. \tag{3.5}$$

Let P be the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{H}_A$. Introduce the following relations:

$$A_{\text{reg}} \stackrel{\text{def}}{=} PA = \{\{f, Pg\};\ \{f, g\} \in A\}, \tag{3.6}$$

called the *regular part* of A, and

$$A_{\text{sing}} \stackrel{\text{def}}{=} (I-P)A = \{\{f, (I-P)g\};\ \{f, g\} \in A\}, \tag{3.7}$$

called the *singular part* of A. Observe that $\operatorname{dom} A_{\text{reg}} = \operatorname{dom} A_{\text{sing}} = \operatorname{dom} A$. The following operatorwise sum decomposition for linear relations acting from one Hilbert space to another was proved in [21, Theorem 4.1]; in the case that A is an operator it can be found in [29], [25]. A short proof can be given by means of Lemmas 2.8 and 2.9.

THEOREM 3.2. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then A admits a canonical operatorwise sum decomposition*

$$A = A_{\text{reg}} + A_{\text{sing}}, \tag{3.8}$$

where A_{reg} is a regular operator in $\mathfrak{H}$ and A_{sing} is a singular relation in $\mathfrak{H}$ with

$$(A_{\text{reg}})^{**} = (A^{**})_{\text{reg}}, \quad (A_{\text{sing}})^{**} = ((A^{**})_{\text{sing}})^{**}, \quad \operatorname{mul} A_{\text{sing}} = \operatorname{mul} A. \tag{3.9}$$

Proof. Let P be the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{H}_A = \overline{\operatorname{dom}}\, A^*$. The decomposition (3.8) is clear.

By definition $A_{\text{reg}} = PA$ and hence by Lemmas 2.8 and 2.9,

$$(A_{\text{reg}})^* = (PA)^* = A^*P = A^* \mathbin{\widehat{\oplus}} (\operatorname{mul} A^{**} \times \{0\}).$$

In particular, $\operatorname{dom}(A_{\rm reg})^* = \operatorname{dom} A^* \oplus \operatorname{mul} A^{**}$, so that $\overline{\operatorname{dom}}\,(A_{\rm reg})^* = \mathfrak{H}$, which is equivalent to $\operatorname{mul}(A_{\rm reg})^{**} = \{0\}$; cf. Lemma 2.3. Thus, the relation $A_{\rm reg}$ in (3.6) is regular.

Again, by definition $A_{\rm sing} = (I-P)A$ and hence by Lemmas 2.8 and 2.9,

$$(A_{\rm sing})^* = ((I-P)A)^* = A^*(I-P) = \overline{\operatorname{dom}}\,A^* \times \operatorname{mul} A^*.$$

Since $\operatorname{dom}(A_{\rm sing})^* = \overline{\operatorname{dom}}\,A^*$, it follows that $\operatorname{mul}(A_{\rm sing})^{**} = \operatorname{mul} A^{**}$; cf. Lemma 2.3. Therefore, $\operatorname{ran} A_{\rm sing} \subset \operatorname{mul} A^{**} = \operatorname{mul}(A_{\rm sing})^{**}$ and $A_{\rm sing}$ is singular.

It remains to prove the identities in (3.9). The identities $(PA)^* = A^*P = (PA^{**})^*$ show that

$$(A_{\rm reg})^* = A^*P = ((A^{**})_{\rm reg})^*$$

and hence $(A_{\rm reg})^{**} = ((A^{**})_{\rm reg})^{**}$. Since $\operatorname{ran}(I-P) = \operatorname{mul} A^{**}$ it follows that $(A^{**})_{\rm reg} \subset A^{**}$. This implies that $(A^{**})_{\rm reg}$ is closed: indeed, if $\{f_n, f_n'\} \in A^{**}$ and $\{f_n, Pf_n'\} \to \{f, f'\}$, then $\{f, f'\} \in A^{**}$ and $f' = Pf'$, so that $\{f, f'\} = \{f, Pf'\} \in (A^{**})_{\rm reg}$. Therefore, $((A^{**})_{\rm reg})^{**} = (A^{**})_{\rm reg}$, yielding the first identity in (3.9).

Likewise, the equalities $((I-P)A)^* = A^*(I-P) = ((I-P)A^{**})^*$ imply that

$$(A_{\rm sing})^* = A^*(I-P) = ((A^{**})_{\rm sing})^*.$$

Hence $(A_{\rm sing})^{**} = ((A^{**})_{\rm sing})^{**}$, and the second identity in (3.9) is proved.

Finally, since $\operatorname{mul} A \subset \operatorname{mul} A^{**}$, one obtains

$$\operatorname{mul} A_{\rm sing} = \{(I-P)f' : \{0, f'\} \in A\} = \{f' : \{0, f'\} \in A\} = \operatorname{mul} A.$$

This completes the proof. ■

Several illustrations of Theorem 3.2 can be found in [19], [21]. Canonical decompositions of relations have their counterparts in the canonical decomposition of pairs of nonnegative sesquilinear forms (see [19]).

It is clear from the definitions that A is regular if and only if $A_{\rm sing}$ in (3.8) is the zero operator on $\operatorname{dom} A$, and similarly, A is singular if and only if $A_{\rm reg}$ in (3.8) is the zero operator on $\operatorname{dom} A$. The condition that A is singular can also be characterized as follows; cf. [21].

Proposition 3.3. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) *A is singular;*
(ii) $\operatorname{dom} A^* \subset \ker A^*$ *or, equivalently,* $\operatorname{dom} A^* = \ker A^*$;
(iii) $A^* = \operatorname{dom} A^* \times \operatorname{mul} A^*$;
(iv) $A^{**} = \overline{\operatorname{dom}}\,A \times \operatorname{mul} A^{**}$.

In particular, if one of the relations A, A^{-1}, A^*, or A^{**} is singular, then all of them are singular.

Proof. (i)⇒(ii). The identity in (3.3) implies that $(\overline{\operatorname{ran}}\,A)^\perp = (\operatorname{mul} A^{**})^\perp$, which is equivalent to $\ker A^* = \overline{\operatorname{dom}}\,A^*$ by Lemma 2.3. In particular, $\operatorname{dom} A^* \subset \ker A^*$.

(ii)⇒(iii). Let $\{f, g\} \in A^*$. Now $f \in \operatorname{dom} A^*$ implies that $f \in \ker A^*$. Therefore $\{f, 0\} \in A^*$ and then also $\{0, g\} \in A^*$, or $g \in \operatorname{mul} A^*$. This shows that $\{f, g\} \in \operatorname{dom} A^* \times$

$\operatorname{mul} A^*$. Conversely, let $\{f,g\} \in \operatorname{dom} A^* \times \operatorname{mul} A^*$. Then $\{0,g\} \in A^*$. Moreover, $f \in \operatorname{dom} A^*$ and by (ii), $f \in \ker A^*$, i.e., $\{f,0\} \in A^*$. Thus $\{f,g\} \in A^*$.

(iii)$\Rightarrow$(iv). Taking adjoints in (iii) yields $A^{**} = (\operatorname{mul} A^*)^\perp \times (\operatorname{dom} A^*)^\perp$, which gives (iv) by means of Lemma 2.3.

(iv)$\Rightarrow$(i). Now $\operatorname{ran} A^{**} = \operatorname{mul} A^{**}$ gives $\operatorname{ran} A \subset \operatorname{mul} A^{**}$. Thus A is singular.

The last statement is clear from the equivalence of (i)–(iv). ■

The following characterizations for regularity of A are immediate from the definitions. Further characterizations of regularity are given after componentwise decompositions have been introduced; see Proposition 3.11.

PROPOSITION 3.4. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) *A is regular, i.e., a closable operator;*
(ii) $\operatorname{mul} A^{**} = \{0\}$*;*
(iii) *A^* is densely defined.*

Proof. The equivalence of (i) and (ii) holds by definition of closability. The equivalence of (ii) and (iii) is obtained from Lemma 2.3. ■

Boundedness of the regular and singular parts of A in Theorem 3.2 can be characterized as follows.

PROPOSITION 3.5. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then:*

(i) *A_{reg} is a bounded operator if and only if* $\operatorname{dom} A^*$ *is closed;*
(ii) *A_{sing} is a bounded operator if and only if it is the zero operator on* $\operatorname{dom} A$*, i.e.,* $A_{\mathrm{sing}} = \operatorname{dom} A \times \{0\}$.

In particular, if $\operatorname{ran} A_{\mathrm{sing}} \neq \{0\}$ *then A_{sing} is either an unbounded operator or a multivalued relation with* $\operatorname{mul} A_{\mathrm{sing}} = \operatorname{mul} A$.

Proof. (i) According to Theorem 3.2, A_{reg} is regular (i.e. closable) and $(A_{\mathrm{reg}})^{**} = (A^{**})_{\mathrm{reg}}$. Hence by Lemma 2.1, A_{reg} is bounded if and only if $(A^{**})_{\mathrm{reg}}$ is bounded, or equivalently, $\operatorname{dom}(A^{**})_{\mathrm{reg}} = \operatorname{dom} A^{**}$ is closed. Then, equivalently, $\operatorname{dom} A^*$ is closed by Theorem 2.11.

(ii) Assume that A_{sing} is a bounded operator, so that also $(A_{\mathrm{sing}})^{**}$ is a bounded operator. According to Theorem 3.2, A_{sing} is singular, so that

$$\operatorname{ran} A_{\mathrm{sing}} \subset \operatorname{mul}(A_{\mathrm{sing}})^{**} = \{0\}.$$

Therefore, $A_{\mathrm{sing}} = \operatorname{dom} A \times \{0\}$. Conversely, if $A_{\mathrm{sing}} = \operatorname{dom} A \times \{0\}$ then $\operatorname{ran} A_{\mathrm{sing}} = \{0\}$, and A_{sing} is bounded and singular.

The last statement is immediate from (ii) and (3.9) in Theorem 3.2. ■

Note that by Proposition 3.5, $\operatorname{dom} A^*$ is closed if and only if A_{reg} is bounded, which by Corollary 2.13 is equivalent to $\operatorname{dom}(A_{\mathrm{reg}})^* = \mathfrak{H}$. Thus, $\operatorname{dom} A^*$ is closed if and only if $\operatorname{dom}(A_{\mathrm{reg}})^* = \mathfrak{H}$, which is also clear from the identity

$$\operatorname{dom}(A_{\mathrm{reg}})^* = \operatorname{dom} A^* \oplus \operatorname{mul} A^{**}.$$

From Proposition 2.12 one obtains, in place of part (i) in Proposition 3.5, the following formally weaker, but equivalent, criterion for boundedness of $A_{\rm reg}$.

COROLLARY 3.6. *$A_{\rm reg}$ is a bounded operator if and only if* $\operatorname{ran}(A^{**})_{\rm reg} \subset \operatorname{dom} A^*$, *or equivalently,* $\operatorname{ran}(A_{\rm reg})^{**} \subset \operatorname{dom} A^*$.

Proof. By Theorem 3.2, $(A_{\rm reg})^{**} = (A^{**})_{\rm reg}$ and hence the assertion follows from Proposition 3.5(i) and the equivalence of items (i) and (ii) in Proposition 2.12. ∎

COROLLARY 3.7. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then*

$$A \subset A_{\rm reg} \,\widehat{+}\, (A^{**})_{\rm mul} \subset (A^{**})_{\rm reg} \,\widehat{+}\, (A^{**})_{\rm mul}.$$

Proof. Let $\{f, f'\} \in A$ and consider $f' = Pf' + (I-P)f'$. This leads to

$$\{f, f'\} = \{f, Pf'\} + \{0, (I-P)f'\}.$$

Hence, the first inclusion is clear. Furthermore, the second inclusion follows from

$$A_{\rm reg} \subset (A_{\rm reg})^{**} = (A^{**})_{\rm reg},$$

where the identity holds by Theorem 3.2. ∎

REMARK 3.8. Let A be a relation in a Hilbert space $\mathfrak{H}$, which satisfies $\operatorname{mul} A^{**} \subset \operatorname{mul} A^*$. Then $\mathcal{W}(A) = \mathcal{W}(A_{\rm reg})$. To see this, observe that

$$(A_{\rm reg} f, f) = (Pf', f) = (f', f), \qquad \{f, f'\} \in A;$$

cf. Lemma 2.4.

3.2. Componentwise decompositions of relations via the operator part. By means of the Hilbert space $\mathfrak{H}_A$ the restriction $A_{\rm op}$ of A is defined by

$$A_{\rm op} \stackrel{\rm def}{=} \{\{f, g\} \in A;\ g \in \mathfrak{H}_A\}. \tag{3.10}$$

Equivalently, $A_{\rm op}$ can be written in the following way:

$$A_{\rm op} = A \cap (\mathfrak{H} \times \mathfrak{H}_A). \tag{3.11}$$

By definition $A_{\rm op}$ is (the graph of) an operator in $\mathfrak{H}$ (see (2.1)) and clearly

$$A_{\rm op} \subset A_{\rm reg}, \tag{3.12}$$

where $A_{\rm reg}$ is as in (3.6). Since $A_{\rm reg}$ is closable in $\mathfrak{H}$, the operator $A_{\rm op}$ is also closable in $\mathfrak{H}$. By means of the multivalued part of A the restriction $A_{\rm mul}$ of A is defined by

$$A_{\rm mul} \stackrel{\rm def}{=} \{0\} \times \operatorname{mul} A. \tag{3.13}$$

In particular, the relation $A_{\rm mul}$ is closed in $\mathfrak{H} \times \mathfrak{H}$ if and only if the subspace $\operatorname{mul} A$ is closed in $\mathfrak{H}$. By taking adjoints in (3.13) one gets

$$(A_{\rm mul})^* = (\operatorname{mul} A)^\perp \times \mathfrak{H}, \tag{3.14}$$

so that $A_{\rm mul}$ is a symmetric relation in $\mathfrak{H}$. By taking adjoints in (3.14) one gets

$$(A_{\rm mul})^{**} = \{0\} \times \overline{\operatorname{mul}} A. \tag{3.15}$$

The following theorem is concerned with the decomposition of a, not necessarily closed, relation A in the graph sense via its multivalued part.

THEOREM 3.9. *Let A be a relation in a Hilbert space $\mathfrak{H}$. If there exists a relation B in $\mathfrak{H}$ such that*

$$A = B \mathbin{\widehat{+}} A_{\rm mul}, \qquad \operatorname{ran} B \subset \mathfrak{H}_A, \tag{3.16}$$

then the sum in (3.16) *is direct and B is a closable operator which coincides with $A_{\rm op}$. In particular, the decomposition of A in* (3.16) *is unique.*

Proof. It follows from (2.1) that the sum in (3.16) is direct. The equality in (3.16) implies that $B \subset A$ and $\operatorname{dom} B = \operatorname{dom} A$. Since $\operatorname{ran} B \subset \mathfrak{H}_A$, it follows from (3.10) that $B \subset A_{\rm op}$; in particular, B is a closable operator in $\mathfrak{H}$. Furthermore, the inclusion $B \subset A_{\rm op}$ implies that $\operatorname{dom} A = \operatorname{dom} B \subset \operatorname{dom} A_{\rm op}$, and thus $\operatorname{dom} A_{\rm op} = \operatorname{dom} A$. Since $A_{\rm op}$ and $B \subset A_{\rm op}$ are (closable) operators with $\operatorname{dom} B = \operatorname{dom} A_{\rm op}$, the equality $B = A_{\rm op}$ follows. ∎

Hence if A admits a componentwise sum decomposition of the form (3.16), then

$$A = A_{\rm op} \mathbin{\widehat{+}} A_{\rm mul}, \tag{3.17}$$

and $A_{\rm op}$ in (3.10) can be viewed as the *minimal operator part* of A which together with $A_{\rm mul}$ decomposes A as a componentwise sum; cf. (2.12). Clearly, by (3.5) the condition $\operatorname{ran} B \subset \mathfrak{H}_A = \mathfrak{H} \ominus \operatorname{mul} A^{**}$ implies that $\operatorname{ran} B \subset \mathfrak{H} \ominus \operatorname{mul} A$. It is precisely in the case where $\operatorname{mul} A$ is dense in $\operatorname{mul} A^{**}$ (recall that A is not necessarily closed) that the condition $\operatorname{ran} B \subset \mathfrak{H}_A$ in (3.16) is equivalent to the condition $\operatorname{ran} B \subset \mathfrak{H} \ominus \operatorname{mul} A$.

A relation A in a Hilbert space $\mathfrak{H}$ is said to be *decomposable* if the componentwise decomposition (3.16), or equivalently, (3.17) is valid; cf. Subsection 1.2. The next theorem gives necessary and sufficient conditions for A to be decomposable and, furthermore, relates the decomposition of the relation A in (3.17) to the operatorwise sum decomposition of A in (3.8).

THEOREM 3.10. *Let A be a relation in a Hilbert space $\mathfrak{H}$, let P be the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{H}_A = \overline{\operatorname{dom}}\, A^*$, and let the relations $A_{\rm reg}$, $A_{\rm mul}$, and $A_{\rm op}$ be defined as above. Then the following statements are equivalent:*

(i) *A is decomposable;*
(ii) $\operatorname{dom} A_{\rm op} = \operatorname{dom} A$*;*
(iii) $A_{\rm reg} = A_{\rm op}$*;*
(iv) $A_{\rm reg} \subset A$*;*
(v) $\operatorname{ran} (I - P)A \subset \operatorname{mul} A$*;*
(vi) $A = A_{\rm reg} \mathbin{\widehat{+}} A_{\rm mul}$.

Proof. (i)⇒(ii). This implication is clear, since $\operatorname{dom} A_{\rm mul} = \{0\}$.

(ii)⇒(iii). The assumption gives $\operatorname{dom} A_{\rm op} = \operatorname{dom} A = \operatorname{dom} A_{\rm reg}$. Now (3.12) implies that $A_{\rm op} = A_{\rm reg}$, since $A_{\rm op}$ and $A_{\rm reg}$ are operators.

(iii)⇒(iv). This implication is clear, since $A_{\rm op} \subset A$ by definition.

(iv)⇔(v). Let $\{f, g\} \in A$ and write $\{f, g\} = \{f, Pg\} \mathbin{\widehat{+}} \{0, (I - P)g\}$. Here $\{f, Pg\} \in A_{\rm reg}$ and the condition $\{f, Pg\} \in A$ is equivalent to $\{0, (I - P)g\} \in A$. This shows that $A_{\rm reg} \subset A$ if and only if $(I - P)(\operatorname{ran} A) \subset \operatorname{mul} A$, which proves the claim.

(iv)&(v)⇒(vi). By decomposing $\{f, g\} \in A$ as $\{f, g\} = \{f, Pg\} \mathbin{\widehat{+}} \{0, (I - P)g\}$ one concludes that $A \subset A_{\rm reg} \mathbin{\widehat{+}} A_{\rm mul}$. The reverse inclusion is clear, and thus (vi) follows.

(vi)$\Rightarrow$(i). It suffices to prove that $A_{\text{reg}} = A_{\text{op}}$. The equality in (vi) implies that $A_{\text{reg}} \subset A$. Hence, if $\{f, g\} \in A_{\text{reg}}$ then $\{f, g\} \in A$, $g \in \mathfrak{H}_A$, and thus $\{f, g\} \in A_{\text{op}}$. Therefore, $A_{\text{reg}} \subset A_{\text{op}}$, while the reverse inclusion is always true; cf. (3.12).

This completes the proof. ∎

Recall that A is a bounded operator if and only if $\operatorname{ran} A^{**} \subset \operatorname{dom} A^*$; see Corollary 2.13. From Theorem 3.10 one gets the following characterization for the essentially weaker condition $\operatorname{ran} A \subset \overline{\operatorname{dom}} A^*$.

Proposition 3.11. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $\operatorname{ran} A \subset \overline{\operatorname{dom}} A^*$ $(= \mathfrak{H}_A)$;
(ii) $A_{\text{op}} = A$;
(iii) *A is regular, i.e., a closable operator;*
(iv) $\mathfrak{H}_A = \mathfrak{H}$;
(v) *A is a decomposable operator.*

Proof. (i)$\Leftrightarrow$(ii). This is clear from the definition of A_{op} in (3.10).

(ii)$\Rightarrow$(iii). If $A_{\text{op}} = A$ then, together with A_{op}, A is closable.

(iii)$\Rightarrow$(iv). If A is closable, then $\operatorname{mul} A^{**} = \{0\}$ and hence $\mathfrak{H}_A = \mathfrak{H}$.

(iv)$\Rightarrow$(v). If $\mathfrak{H}_A = \mathfrak{H}$ then $A_{\text{reg}} = A$ and hence A is decomposable by Theorem 3.10(iv).

(v)$\Rightarrow$(i). If A is a decomposable operator, then $A_{\text{mul}} = \{0\} \times \{0\}$ and hence $(I - P)A = 0$ by Theorem 3.10(v). This means that $\operatorname{ran} A \subset \ker(I - P) = \mathfrak{H}_A$. ∎

The next result is clear from Proposition 3.11.

Corollary 3.12. *An operator A in a Hilbert space $\mathfrak{H}$ is decomposable if and only if it is regular, i.e., $A_{\text{sing}} = 0$.*

Hence, an operator A is decomposable in the sense of Theorem 3.9 if and only if it is closable; in this case $A_{\text{mul}} = \{0\} \times \{0\}$ and $A = A_{\text{op}}$. In this sense the decomposability property introduced via Theorem 3.9 can be seen as an extension to linear relations of the notion of closability of operators.

Singular operators and relations are not in general decomposable; for them the following result holds.

Proposition 3.13. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then*

(i) *A is singular and decomposable if and only if $A = \operatorname{dom} A \times \operatorname{mul} A$, or equivalently, $\operatorname{dom} A = \ker A$.*
(ii) *A singular operator A is decomposable if and only if it is bounded, or equivalently, A is the zero operator on its domain, i.e., $A = \operatorname{dom} A \times \{0\}$.*

Proof. (i) The relation A is singular if $\operatorname{ran} A \subset \operatorname{mul} A^{**}$. This is equivalent to $A_{\text{reg}} = \operatorname{dom} A \times \{0\}$. By Theorem 3.10, A is decomposable if and only if $A = A_{\text{reg}} \widehat{+} A_{\text{mul}}$. Hence, if A is singular and decomposable, then $A = \operatorname{dom} A \times \operatorname{mul} A$. Conversely, if A is of the form $A = \operatorname{dom} A \times \operatorname{mul} A$, then clearly A is singular and decomposable. Furthermore, it is easy to check that $A = \operatorname{dom} A \times \operatorname{mul} A$ is equivalent to $\operatorname{dom} A = \ker A$.

(ii) This is clear from part (i) and Proposition 3.5(ii). ∎

Next some sufficient conditions for decomposability of relations are given.

COROLLARY 3.14. *If the relation A satisfies* $\operatorname{mul} A = \operatorname{mul} A^{**}$*, then A is decomposable and the relation A_{mul} is closed.*

Proof. Note that $I - P$ is the orthogonal projection onto $\operatorname{mul} A^{**}$. Therefore, in this case $\operatorname{ran}(I-P)A \subset \operatorname{mul} A^{**} = \operatorname{mul} A$, and hence A is decomposable by Theorem 3.10(v). Since A^{**} is closed, also $\operatorname{mul} A = \operatorname{mul} A^{**}$ and A_{mul} are closed. ∎

COROLLARY 3.15. *If the relation A is a closed, then A is decomposable and the relations $A_{\mathrm{op}} = A_{\mathrm{reg}}$ and A_{mul} are closed.*

Proof. Since A is closed, $\operatorname{mul} A = \operatorname{mul} A^{**}$ and the first statement is obtained from Corollary 3.14. Moreover, it is clear from (3.11) that A_{op} is closed. ∎

Later, in Proposition 3.21, it is shown that if $\operatorname{mul} A$ is closed then the sufficient condition $\operatorname{mul} A = \operatorname{mul} A^{**}$ for decomposability becomes also necessary.

Let A be a relation in the Hilbert space $\mathfrak{H}$ which is not necessarily closed. Then the closure of A is given by A^{**}; recall that $(A^{**})_{\mathrm{mul}} = \{0\} \times \operatorname{mul} A^{**}$. It is useful to observe that

$$\operatorname{mul} A \subset \overline{\operatorname{mul}}\, A \subset \operatorname{mul} A^{**},$$

and, furthermore, that

$$(A_{\mathrm{mul}})^{**} = (A^{**})_{\mathrm{mul}} \;\Leftrightarrow\; \overline{\operatorname{mul}}\, A = \operatorname{mul} A^{**}; \tag{3.18}$$

cf. (3.15). Observe that $\mathfrak{H}_{A^{**}} = \mathfrak{H}_A$. Therefore the operator $(A^{**})_{\mathrm{op}}$ is given by

$$(A^{**})_{\mathrm{op}} = A^{**} \cap (\mathfrak{H} \times \mathfrak{H}_A). \tag{3.19}$$

It is clear from (3.11), (3.13), and (3.19) that $A_{\mathrm{op}} \subset (A^{**})_{\mathrm{op}}$ and $A_{\mathrm{mul}} \subset (A^{**})_{\mathrm{mul}}$. Therefore, Corollary 3.15, applied to A^{**}, implies that

$$(A_{\mathrm{op}})^{**} \subset (A^{**})_{\mathrm{op}}, \qquad (A_{\mathrm{mul}})^{**} \subset (A^{**})_{\mathrm{mul}}.$$

The following result is a direct consequence of Theorem 3.10.

PROPOSITION 3.16. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then A^{**} is decomposable and has the following componentwise sum decomposition:*

$$A^{**} = (A^{**})_{\mathrm{op}} \,\widehat{+}\, (A^{**})_{\mathrm{mul}}. \tag{3.20}$$

Moreover, if the relation A is decomposable, then

$$(A_{\mathrm{op}})^{**} = (A^{**})_{\mathrm{op}}, \qquad (A_{\mathrm{mul}})^{**} = (A^{**})_{\mathrm{mul}}. \tag{3.21}$$

Proof. Let A be any relation in $\mathfrak{H}$. Then A^{**} is closed, and by Corollary 3.15, A^{**} is decomposable, which leads to the decomposition (3.20).

Now assume that A is decomposable as $A = A_{\mathrm{op}} \,\widehat{+}\, A_{\mathrm{mul}}$. Then it follows from $\operatorname{ran} A_{\mathrm{op}} \subset \mathfrak{H}_A$ that

$$A^{**} = (A_{\mathrm{op}})^{**} \,\widehat{+}\, (A_{\mathrm{mul}})^{**}. \tag{3.22}$$

The identities (3.22) and (3.15) lead to the decomposition

$$A^{**} = (A_{\mathrm{op}})^{**} \,\widehat{+}\, (\{0\} \times \overline{\operatorname{mul}}\, A). \tag{3.23}$$

The operator $A_{\rm op}$ is closable and $\operatorname{ran} A_{\rm op} \subset \mathfrak{H}_A$. Hence, $(A_{\rm op})^{**}$ is an operator and $\operatorname{ran} (A_{\rm op})^{**} \subset \mathfrak{H}_A$. Because $(A_{\rm op})^{**}$ is an operator, it follows from (3.23) that $\operatorname{mul} A^{**} = \overline{\operatorname{mul} A}$; thus (3.23) reads

$$A^{**} = (A_{\rm op})^{**} \mathbin{\widehat{+}} (A^{**})_{\rm mul}.$$

An application of Theorem 3.9 to A^{**} shows that $(A_{\rm op})^{**} = (A^{**})_{\rm op}$. This completes the proof. ■

If a relation A is closed, then it is decomposable by Corollary 3.15, and Proposition 3.16 is a refinement of earlier results. Observe that in Proposition 3.16 one has

$$(A^{**})_{\rm op} = (A^{**})_{\rm reg} = (A_{\rm reg})^{**} \tag{3.24}$$

by Theorems 3.10 and 3.2. For a relation A which is not necessarily decomposable, it follows from $A \subset A^{**}$ and (3.20) that

$$A \subset (A^{**})_{\rm op} \mathbin{\widehat{+}} (A^{**})_{\rm mul}.$$

This inclusion can also be seen from Corollary 3.7. If A is a relation and one of the identities in (3.21) is not satisfied, then A is not decomposable. Although the conditions in (3.21) are necessary for A to be decomposable, they are not sufficient. In fact, it is possible that both identities in (3.21) are satisfied, while A is not decomposable; see Example 3.25.

A relation A whose regular part is bounded need not be decomposable; see e.g. Example 3.24. Decomposability of such relations is characterized in the next result.

Proposition 3.17. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) *A is decomposable with a bounded operator part $A_{\rm op}$;*
(ii) *$A_{\rm reg} = A_{\rm op}$ is bounded;*
(iii) *$\operatorname{dom} A^*$ is closed and $A_{\rm reg} = A_{\rm op}$.*

Furthermore, the following weaker statements are equivalent:

(iv) *$A_{\rm op}$ is bounded, densely defined in $\overline{\operatorname{dom}} A$, and $(A_{\rm mul})^{**} = (A^{**})_{\rm mul}$;*
(v) *$A_{\rm reg}$ is bounded and the conditions in (3.21) are satisfied;*
(vi) *$\operatorname{dom} A^*$ is closed and the conditions in (3.21) are satisfied.*

If, in addition, $\operatorname{ran} (I - P)A \subset \operatorname{mul} A$ or $\operatorname{mul} A$ is closed, then the conditions (iv)–(vi) are also equivalent to the conditions (i)–(iii).

Proof. (i)⇔(ii). This is clear from Theorem 3.10; see items (i) and (iii).

(ii)⇔(iii). This is an immediate consequence of Proposition 3.5.

(iv)⇒(v). Since $A_{\rm op} \subset A_{\rm reg}$ and the operator $A_{\rm reg}$ is closable, the assumption that $A_{\rm op}$ is densely defined and bounded in $\overline{\operatorname{dom}} A$ leads to the equality

$$(A_{\rm op})^{**} = (A_{\rm reg})^{**};$$

cf. Corollary 2.2. Hence $(A_{\rm reg})^{**}$ and, in particular, $A_{\rm reg}$ is bounded. Moreover, $(A_{\rm op})^{**} = (A^{**})_{\rm op}$ is now obtained from (3.24).

(v)⇒(iv). If $A_{\rm reg}$ is bounded then $A_{\rm op} \subset A_{\rm reg}$ is bounded, too. By Proposition 3.16, $(A^{**})_{\rm reg} = (A^{**})_{\rm op}$. Hence, if $(A_{\rm op})^{**} = (A^{**})_{\rm op}$ then $\overline{\rm dom}\, A_{\rm op} = {\rm dom}\,(A^{**})_{\rm op} = {\rm dom}\,(A^{**})_{\rm reg} = \overline{\rm dom}\, A$ (cf. Lemma 2.1), i.e., $A_{\rm op}$ is densely defined in $\overline{\rm dom}\, A$.

(v)⇔(vi). Again this holds by Proposition 3.5.

To prove the last statement note that (i) implies (v) by Proposition 3.16. On the other hand, if ${\rm ran}\,(I-P)A \subset {\rm mul}\, A$ then A is decomposable by Theorem 3.10 and thus (iv) implies (i). Similarly, the assumption that ${\rm mul}\, A$ is closed together with $(A_{\rm mul})^{**} = (A^{**})_{\rm mul}$ implies that ${\rm mul}\, A = {\rm mul}\, A^{**}$, so that A is decomposable by Corollary 3.14. Hence, again (iv) implies (i). ■

Proposition 3.17 indicates that even in the case where $A_{\rm reg}$ is a bounded operator, the equalities in (3.21) are not sufficient for the decomposability of A. In fact, this may happen also in the case where $A_{\rm reg}$ is closed and bounded; see Example 3.28. However, if ${\rm mul}\, A$ is closed then the situation is different; see Corollary 3.22.

3.3. Componentwise decompositions for relations via the multivalued part. Theorem 3.10 shows that a relation A in a Hilbert $\mathfrak{H}$ is decomposable in the sense of Theorem 3.9 if and only if $A_{\rm op} = A_{\rm reg}$, where $A_{\rm reg} = PA$ with P the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{H}_A = \mathfrak{H} \ominus {\rm mul}\, A^{**}$. Closely related to the regular part $A_{\rm reg}$ is the relation

$$A_{\rm m} \stackrel{\rm def}{=} P_{\rm m} A = \{\{f, P_{\rm m} f'\};\, \{f, f'\} \in A\}, \tag{3.25}$$

where $P_{\rm m}$ is the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{H} \ominus \overline{\rm mul}\, A$. $A_{\rm m}$ can be thought of as the *maximal operator part* of A; cf. Theorem 3.18 below. Observe that

$${\rm mul}\, A_{\rm m} = \{P_{\rm m} f';\, \{0, f'\} \in A\} = \{0\},$$

i.e., $A_{\rm m}$ is an operator. Note that

$$\mathfrak{H} = \overline{\rm dom}\, A \oplus {\rm mul}\, A^{**} = \overline{\rm dom}\, A \oplus ({\rm mul}\, A^{**} \ominus \overline{\rm mul}\, A) \oplus \overline{\rm mul}\, A, \tag{3.26}$$

so that ${\rm ran}\, P \subset {\rm ran}\, P_{\rm m}$. Therefore $A_{\rm reg} = PA_m$ and, in addition,

$$A_{\rm op} \subset A_m. \tag{3.27}$$

The operator $A_{\rm m}$ can be used to give a further equivalent condition for A to be decomposable, which is stated as item (ii) in the next theorem.

Theorem 3.18. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then $A_{\rm m}$ is an operator and the following statements are equivalent:*

(i) *A is decomposable;*
(ii) *$A_{\rm m} = A_{\rm op}$.*

Furthermore, the following weaker statements are equivalent:

(iii) *$A_{\rm m} = A_{\rm reg}$;*
(iv) *${\rm ran}\, A_{\rm m} \subset \mathfrak{H}_A$;*
(v) *${\rm mul}\, A^{**} = \overline{\rm mul}\, A$;*
(vi) *$A_{\rm m}$ is a closable operator.*

If, in addition, ${\rm mul}\, A$ *is closed, then the conditions* (iii)–(vi) *are also equivalent to the conditions* (i)–(ii).

Proof. (i)⇒(ii). It suffices to show that $A_{\mathrm{m}} \subset A_s$. Let A be decomposed as in (3.17). If $\{f, f'\} \in A$, then $\{f, f'\} = \{f, A_{\mathrm{op}} f\} + \{0, \varphi\}$ with $\varphi \in \operatorname{mul} A$. In particular, $P_{\mathrm{m}} f' = A_{\mathrm{op}} f$ for $\{f, f'\} \in A$, i.e. $A_{\mathrm{m}} \subset A_{\mathrm{op}}$.

(ii)⇒(i). If $A_{\mathrm{m}} = A_{\mathrm{op}}$, then $\operatorname{dom} A_{\mathrm{op}} = \operatorname{dom} A$ and by Theorem 3.10 this is equivalent to A being decomposable.

Next the equivalence of (iii)–(vi) will be proved.

(iii)⇒(iv). This is clear since $\operatorname{ran} A_{\mathrm{reg}} \subset \mathfrak{H}_A$.

(iv)⇒(v). Observe that $(A_{\mathrm{m}})^* = (P_{\mathrm{m}} A)^* = A^* P_{\mathrm{m}}$ and $(A_{\mathrm{m}})^{**} = (A^* P_{\mathrm{m}})^* \supset P_{\mathrm{m}} A^{**}$ by Lemma 2.8. The assumption in (iv) implies that $\operatorname{ran}(A_{\mathrm{m}})^{**} \subset \mathfrak{H}_A$; cf. (2.1). Then also $\operatorname{ran} P_{\mathrm{m}} A^{**} \subset \mathfrak{H}_A$ and, in particular, $P_{\mathrm{m}}(\operatorname{mul} A^{**}) \subset \mathfrak{H}_A$, which means that $\operatorname{mul} A^{**} = \overline{\operatorname{mul}} A$; cf. (3.26).

(v)⇔(vi). It follows from Lemma 2.9 that $(A_{\mathrm{m}})^* = A^* P_{\mathrm{m}} = A^* \mathbin{\widehat{\oplus}} (\overline{\operatorname{mul}} A \times \{0\})$, so that

$$\operatorname{dom}(A_{\mathrm{m}})^* = \operatorname{dom} A^* \oplus \overline{\operatorname{mul}} A.$$

Now the operator A_{m} is closable if and only if $(A_{\mathrm{m}})^*$ is densely defined (cf. Proposition 3.4), which is equivalent to $\operatorname{mul} A^{**} = \overline{\operatorname{mul}} A$.

(v)⇒(iii). If $\operatorname{mul} A^{**} = \overline{\operatorname{mul}} A$ then $P_{\mathrm{m}} = P$ and, therefore, $A_{\mathrm{m}} = PA = A_{\mathrm{reg}}$.

Next it is shown that the conditions (iii)–(vi) follow from the conditions (i) and (ii). Namely, if $A_{\mathrm{m}} = A_{\mathrm{op}}$ or, equivalently, A is decomposable, then $A_{\mathrm{op}} = A_{\mathrm{reg}}$ by Theorem 3.10, and hence $A_{\mathrm{m}} = A_{\mathrm{op}} = A_{\mathrm{reg}}$.

As to the last statement of the theorem, observe that if $\operatorname{mul} A$ is closed then the condition (v) implies (i) by Corollary 3.14. ■

Observe that the conditions (iii)–(vi) in Theorem 3.18 do not in general imply decomposability of A; see for instance Example 3.25.

Next, decompositions of linear relations A whose multivalued part $\operatorname{mul} A$ is closed in the Hilbert space $\mathfrak{H}$ will be briefly treated.

Lemma 3.19. *Let A be a relation in a Hilbert space $\mathfrak{H}$ with* $\operatorname{mul} A$ *closed. Then A admits the decomposition*

$$A = A_{\mathrm{m}} \mathbin{\widehat{+}} A_{\mathrm{mul}}, \tag{3.28}$$

where A_{m} is an operator with $\operatorname{dom} A_{\mathrm{m}} = \operatorname{dom} A$.

Proof. It has been shown that A_{m} is an operator and that $\operatorname{dom} A_{\mathrm{m}} = \operatorname{dom} A$. Now, rewrite A as follows:

$$A = P_{\mathrm{m}} A + (I - P_{\mathrm{m}}) A = \{\{f, P_{\mathrm{m}} g\} \mathbin{\widehat{+}} \{0, (I - P_{\mathrm{m}}) g\};\ \{f, g\} \in A\}.$$

This implies that $A \subset A_{\mathrm{m}} \mathbin{\widehat{+}} A_{\mathrm{mul}}$.

Conversely, since $\operatorname{mul} A$ is closed, one has $A_{\mathrm{mul}} \subset A$ and thus also $P_{\mathrm{m}} A \subset A$. Therefore, $P_{\mathrm{m}} A + (I - P_{\mathrm{m}}) A \subset A$. ■

The decomposition of A in Lemma 3.19 for relations A with $\operatorname{mul} A$ closed is not of the type as introduced via Theorem 3.9, since the condition $\operatorname{ran} A_{\mathrm{m}} \subset \mathfrak{H}_A$ $(= \overline{\operatorname{dom} A^*})$ need not be satisfied. This implies that the decomposition given in Lemma 3.19 does not behave well for instance under closures: in particular, the operator A_{m} in (3.28) is not in general closable. In fact, when $\operatorname{mul} A$ is closed, Theorem 3.18 shows that the operator A_{m} is closable precisely when A is decomposable in the sense of Theorem 3.9.

One can reformulate the situation also by means of the decompositions of the form (3.17) alone.

Corollary 3.20. *Let A be a relation in a Hilbert space $\mathfrak{H}$ with* $\operatorname{mul} A$ *closed. Then*

$$A \text{ is decomposable} \;\Leftrightarrow\; A_{\mathrm{m}} \text{ is a decomposable operator.}$$

In this case, the decomposition of A_{m} is trivial, i.e., $A_{\mathrm{m}} = (A_{\mathrm{m}})_{\mathrm{op}}$, and the decompositions in (3.17) *and* (3.28) *coincide:*

$$A = A_{\mathrm{m}} \mathbin{\widehat{+}} A_{\mathrm{mul}} = A_{\mathrm{op}} \mathbin{\widehat{+}} A_{\mathrm{mul}}. \tag{3.29}$$

Proof. According to Proposition 3.11 the operator A_{m} is decomposable if and only if $(A_{\mathrm{m}})_{\mathrm{op}} = A_{\mathrm{m}}$, or equivalently, A_{m} is closable. This means that $\operatorname{mul} A = \operatorname{mul} A^{**}$. By Theorem 3.18 this last condition is equivalent to A being decomposable. In this case $A_{\mathrm{m}} = A_{\mathrm{op}}$ and (3.29) follows. ∎

The characterizations of decomposability of A in the case where $\operatorname{mul} A$ is closed are collected in the next result. It shows that decomposability of A (with $\operatorname{mul} A$ closed) is a natural counterpart and extension of the notion of closability of operators; see also Proposition 3.11.

Proposition 3.21. *Let A be a relation in a Hilbert space $\mathfrak{H}$ with* $\operatorname{mul} A$ *closed. Then the following statements are equivalent:*

(i) *A is decomposable;*
(ii) $A_{\mathrm{m}} = A_{\mathrm{op}}$*;*
(iii) $A_{\mathrm{m}} = A_{\mathrm{reg}}$*;*
(iv) $\operatorname{ran} A_{\mathrm{m}} \subset \mathfrak{H}_A$*;*
(v) $\operatorname{mul} A^{**} = \operatorname{mul} A$*;*
(vi) *A_{m} is a closable operator.*

Proof. Since $\operatorname{mul} A$ is closed, the result is obtained from Theorem 3.18. ∎

The next result enhances Proposition 3.17.

Corollary 3.22. *Let A be a relation in a Hilbert space $\mathfrak{H}$ with* $\operatorname{mul} A$ *closed. Then the following statements are equivalent:*

(i) *A is decomposable with a bounded operator part A_{op};*
(ii) *the operator A_{m} in* (3.25) *is bounded.*

Proof. (i)⇒(ii). Since A is decomposable, Proposition 3.21 shows that $A_{\mathrm{m}} = A_{\mathrm{op}}$, and hence (ii) follows.

(ii)⇒(i). If A_{m} is bounded, then it is closable (see Lemma 2.1). Hence, by Proposition 3.21, A is decomposable and $A_{\mathrm{op}} = A_{\mathrm{m}}$ is bounded. ∎

Observe that the condition (ii) in Corollary 3.22 is essentially weaker than, for instance, the condition (ii) (or (v)) in Proposition 3.17; in particular, no equality $A_{\mathrm{m}} = A_{\mathrm{op}}$ is assumed in part (ii) of Corollary 3.22. In fact, Corollary 3.22 is a natural extension of the basic Lemma 2.1 stating that a bounded operator is closable.

3.4. Componentwise decompositions of adjoint relations. Let A be a relation in a Hilbert space $\mathfrak{H}$, then its adjoint A^* is automatically a closed (linear) relation. Let $\mathfrak{H}_{A^*} = \mathfrak{H} \ominus \operatorname{mul} A^*$ and let P_* be the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{H}_{A^*}$. Recall the definitions of the regular part of A^*:

$$(A^*)_{\mathrm{reg}} \stackrel{\mathrm{def}}{=} \{\{f, P_* g\};\ \{f, g\} \in A^*\},$$

and of the singular part of A^*:

$$(A^*)_{\mathrm{sing}} \stackrel{\mathrm{def}}{=} \{\{f, (I - P_*)g\};\ \{f, g\} \in A^*\}.$$

Observe that $\operatorname{dom}(A^*)_{\mathrm{reg}} = \operatorname{dom}(A^*)_{\mathrm{sing}} = \operatorname{dom} A^*$. The relation A^* admits the canonical operatorwise sum decomposition:

$$A^* = (A^*)_{\mathrm{reg}} + (A^*)_{\mathrm{sing}},$$

where $(A^*)_{\mathrm{reg}}$ is a regular operator in $\mathfrak{H}$ and $(A^*)_{\mathrm{sing}}$ is a singular relation in $\mathfrak{H}$; cf. Theorem 3.2. By means of the Hilbert space $\mathfrak{H}_{A^*}$ the restriction $(A^*)_{\mathrm{op}}$ of A^* is defined by

$$(A^*)_{\mathrm{op}} \stackrel{\mathrm{def}}{=} \{\{f, g\} \in A^*;\ g \in \mathfrak{H}_{A^*}\}.$$

Observe that $(A^*)_{\mathrm{op}}$ can be rewritten in the following way:

$$(A^*)_{\mathrm{op}} = A^* \cap (\mathfrak{H} \times \mathfrak{H}_{A^*}).$$

The next decomposition result follows from Theorem 3.9, Theorem 3.10, and Corollary 3.15.

THEOREM 3.23. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then $(A^*)_{\mathrm{op}} = (A^*)_{\mathrm{reg}}$ is a closed operator and A^* has the following componentwise decomposition:*

$$A^* = (A^*)_{\mathrm{op}} \mathbin{\widehat{+}} (A^*)_{\mathrm{mul}}. \tag{3.30}$$

If there exists a relation B in $\mathfrak{H}$ such that

$$A^* = B \mathbin{\widehat{+}} (A^*)_{\mathrm{mul}}, \qquad \operatorname{ran} B \subset \mathfrak{H}_{A^*}, \tag{3.31}$$

then the sum in (3.31) is direct and $B = (A^*)_{\mathrm{op}}$ is a closed operator. In particular, the decomposition of A^* in (3.31) is unique.

3.5. Some examples of operators or relations which are not decomposable. Let A be a relation in a Hilbert space $\mathfrak{H}$. If A is decomposable then Proposition 3.16 shows that both identities in (3.21) are satisfied. By Corollary 3.12 any operator which is not regular or, equivalently, not closable is not decomposable, as it violates the second identity in (3.21) in Proposition 3.16. This subsection contains examples which illustrate the absence of decomposability.

The first example provides a singular operator which is not regular, but for which the first identity in (3.21) holds. The second example shows a relation which is not decomposable as it violates the first identity in (3.21), while the second identity in (3.21) is satisfied. In the second example there is also a relation for which both identities (3.21) are satisfied, while the relation is not decomposable. The third example gives a relation A which is not decomposable and which satisfies neither of the identities (3.21). Finally, the fourth example shows that a decomposable relation A whose operator part is bounded, can become nondecomposable after a one-dimensional perturbation of its operator part.

Example 3.24. Let $T = T^*$ be an unbounded selfadjoint operator in a Hilbert space $\mathfrak{H}$ and let $\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_-$ be the rigged Hilbert spaces associated with $|T|^{1/2}$; cf. [7]. Denote the duality between $\mathfrak{H}_+$ and $\mathfrak{H}_-$ by (f, φ), $f \in \mathfrak{H}_+$ and $\varphi \in \mathfrak{H}_-$. With elements $\varphi \in \mathfrak{H}_- \backslash \mathfrak{H}$ and $y_0 \in \mathfrak{H}$ define the following unbounded operator A in $\mathfrak{H}$:

$$Af \stackrel{\text{def}}{=} (f, \varphi) y_0, \qquad f \in \operatorname{dom} A \stackrel{\text{def}}{=} \mathfrak{H}_+ = \operatorname{dom} |T|^{1/2}. \tag{3.32}$$

Clearly, the operator A is densely defined. To determine A^* assume that $\{h, k\} \in \mathfrak{H} \times \mathfrak{H}$ satisfies

$$0 = (k, f) - (h, Af) = (k, f) - (h, (f, \varphi) y_0) = (k - (h, y_0)\varphi, f)$$

for all $f \in \operatorname{dom} A$; here (k, f) is meant in the sense of the duality. Since $\operatorname{dom} A = \operatorname{dom} |T|^{1/2} = \mathfrak{H}_+$, the previous identities imply that $k - (h, y_0)\varphi = 0$. Now $k \in \mathfrak{H}$ and $\varphi \in \mathfrak{H}_- \backslash \mathfrak{H}$, thus $k = 0$ and $(h, y_0) = 0$. Conversely, if $\{h, k\} \in \mathfrak{H} \times \mathfrak{H}$, and $(h, y_0) = 0$ and $k = 0$, then $\{h, k\} \in A^*$. Therefore, A^* is given by

$$A^* = \{\{h, 0\} \in \mathfrak{H} \times \mathfrak{H};\ (h, y_0) = 0\}.$$

Note that A^* is (the graph of) an operator (since A is densely defined) and that $\operatorname{dom} A^*$ is not dense. Clearly,

$$A^{**} = \{\{f, g\} \in \mathfrak{H} \times \mathfrak{H};\ g \in \operatorname{span}\{y_0\}\} = \mathfrak{H} \times \operatorname{span}\{y_0\}, \tag{3.33}$$

so that

$$\operatorname{mul} A^{**} = \operatorname{span}\{y_0\}. \tag{3.34}$$

The orthogonal projection P onto $\mathfrak{H}_A = (\operatorname{span}\{y_0\})^\perp$ satisfies $Py_0 = 0$. Therefore the canonical decomposition (3.8) of A is trivial:

$$A_{\text{reg}} = \{\{f, 0\};\ f \in \operatorname{dom} A\}, \qquad A = A_{\text{sing}}. \tag{3.35}$$

Next, observe that the operator A_{op} in (3.10) is given by

$$A_{\text{op}} = \{\{f, 0\};\ f \in \operatorname{dom} A,\ (f, \varphi) = 0\}. \tag{3.36}$$

It follows from the identity (3.34) that the operator A is not decomposable (cf. Corollary 3.12); of course, this also follows by comparing (3.35) and (3.36). Since A_{op} is densely defined it follows that

$$(A_{\text{op}})^{**} = \mathfrak{H} \times \{0\},$$

and it follows from (3.33) that

$$(A^{**})_{\text{op}} = \{\{f, g\} \in A^{**};\ g \in \mathfrak{H}_{\text{op}}\} = \mathfrak{H} \times \{0\}.$$

Hence, the first equality in (3.21) is satisfied, but the second is not. Finally, observe that while the operator A in (3.32) is singular and not decomposable, its closure A^{**} is singular and decomposable (cf. (3.33) and Proposition 3.13).

Example 3.25. Let $\mathfrak{M}$ be a dense subspace of the Hilbert space $\mathfrak{H}$ and let B be a relation in $\mathfrak{H}$. Define the relation A in $\mathfrak{H}$ by

$$A \stackrel{\text{def}}{=} B \,\widehat{+}\, (\{0\} \times \mathfrak{M}), \tag{3.37}$$

so that $\operatorname{dom} A = \operatorname{dom} B$ and $\operatorname{mul} A = \operatorname{mul} B + \mathfrak{M}$. It follows from (3.37) that

$$A^* = B^* \cap (\mathfrak{M}^\perp \times \mathfrak{H}),$$

and, since $\mathfrak{M}$ is dense, one obtains

$$A^{**} = \operatorname{clos}(B^{**} \,\widehat{+}\, (\{0\} \times \mathfrak{H})). \tag{3.38}$$

Observe that $B^{**} \,\widehat{+}\, (\{0\} \times \mathfrak{H}) = \operatorname{dom} B^{**} \times \mathfrak{H}$. Hence, by means of (2.1) it follows from (3.38) that

$$A^{**} = \overline{\operatorname{dom}}\, B^{**} \times \mathfrak{H} = \overline{\operatorname{dom}}\, B \times \mathfrak{H}. \tag{3.39}$$

It is clear that

$$\overline{\operatorname{mul}}\, A = \mathfrak{H}, \qquad \operatorname{mul} A^{**} = \mathfrak{H}. \tag{3.40}$$

In particular, $\mathfrak{H}_A = \{0\}$ (see (3.4)), so that the orthogonal projection P is trivial: $P = 0$. Therefore the canonical decomposition (3.8) of A is trivial:

$$A_{\rm reg} = \operatorname{dom} B \times \{0\}, \qquad A = A_{\rm sing}. \tag{3.41}$$

Next, observe that $A_{\rm op}$ in (3.10) is given by

$$A_{\rm op} = A \cap (\mathfrak{H} \times \{0\}) = \ker A \times \{0\}. \tag{3.42}$$

It follows from (3.41) and (3.42) that

$$A \text{ decomposable } \Leftrightarrow \ker A = \operatorname{dom} B; \tag{3.43}$$

cf. Proposition 3.13. The identities (3.42) and (3.39) give

$$(A_{\rm op})^{**} = \overline{\ker}\, A \times \{0\}, \tag{3.44}$$

and

$$(A^{**})_{\rm op} = \overline{\operatorname{dom}}\, B \times \{0\}. \tag{3.45}$$

Hence, as to the first equality in (3.21) of Proposition 3.16, a comparison of (3.44) and (3.45) leads to

$$(A_{\rm op})^{**} = (A^{**})_{\rm op} \;\Leftrightarrow\; \overline{\ker}\, A = \overline{\operatorname{dom}}\, B. \tag{3.46}$$

It follows from (3.40) that the second equality $(A_{\rm mul})^{**} = (A^{**})_{\rm mul}$ in (3.21) is satisfied. The conditions (3.43) and (3.46) will now be reformulated in a special case.

Lemma 3.26. *Let $\mathfrak{M}$ be a dense subspace of the Hilbert space $\mathfrak{H}$ and let B be a relation in $\mathfrak{H}$. Define the relation A by* (3.37) *and assume that* $\operatorname{ran} B \cap \mathfrak{M} = \{0\}$. *Then*

$$A \textit{ decomposable } \Leftrightarrow \ker B = \operatorname{dom} B \;\Leftrightarrow\; B \textit{ singular and decomposable}, \tag{3.47}$$

and

$$(A_{\rm op})^{**} = (A^{**})_{\rm op} \;\Leftrightarrow\; \overline{\ker}\, B = \overline{\operatorname{dom}}\, B \;\Rightarrow\; B \textit{ singular}. \tag{3.48}$$

Proof. It follows from the definition (3.37) that $\ker B \subset \ker A$. To show the converse inclusion, let $\{f, 0\} \in A$, so that $\{f, 0\} = \{f, g\} + \{0, \varphi\}$ with $\{f, g\} \in B$ and $\varphi \in \mathfrak{M}$. The condition $\operatorname{ran} B \cap \mathfrak{M} = \{0\}$ implies that $g = 0$ and $\varphi = 0$. In particular, $\{f, 0\} \in B$. Hence $\ker B = \ker A$. The first equivalences in (3.47) and (3.48) now follow from (3.43), (3.46). The second equivalence in (3.47) holds by Proposition 3.13. Finally, to see the implication in (3.48) observe that $\overline{\operatorname{dom}} B = \overline{\ker} B \subset \ker B^{**}$, so that B^{-1}, and thus also B, is singular by Proposition 3.3. ■

Let B be a nontrivial injective operator which satisfies $\operatorname{ran} B \cap \mathfrak{M} = \{0\}$. Then $\overline{\operatorname{dom}} B \neq \overline{\ker} B = \{0\}$ and the first equality in (3.21) is not satisfied (and A is not decomposable). For instance, take $B = \operatorname{span}\{h, h\}$ where $h \in \mathfrak{H}$ is nontrivial and $h \notin \mathfrak{M}$.

Let B be the densely defined operator in Example 3.24 (see (3.32)), where $y_0 \notin \mathfrak{M}$ is a nontrivial vector, so that $\operatorname{ran} B \cap \mathfrak{M} = \{0\}$. Then B satisfies $\overline{\operatorname{dom}} B = \overline{\ker} B$ and the first equality in (3.21) is satisfied. Clearly, B does not satisfy $\ker B = \operatorname{dom} B$, so that A is not decomposable.

EXAMPLE 3.27. Let $\mathfrak{M}$ be a nonclosed subspace and let $y_0 \in \mathfrak{H}$, and assume that $\mathfrak{H} = \operatorname{clos} \mathfrak{M} \oplus \operatorname{span}\{y_0\}$. Let B be a densely defined singular operator with $\operatorname{mul} B^{**} = \operatorname{span}\{y_0\}$; cf. e.g. Example 3.24. Let the bounded operator $C \in \boldsymbol{B}(\mathfrak{H})$, $C \neq 0$, have the property that $\operatorname{ran} C \subset \operatorname{clos} \mathfrak{M} \setminus \mathfrak{M}$. The operator $B + C$ is densely defined with $\operatorname{dom}(B + C) = \operatorname{dom} B$, and according to (2.15),

$$(B + C)^* = B^* + C^*, \qquad (B + C)^{**} = B^{**} + C,$$

so that

$$\operatorname{dom}(B + C)^{**} = \operatorname{dom} B^{**}, \qquad \operatorname{mul}(B + C)^{**} = \operatorname{span}\{y_0\},$$

cf. (2.14). Define the relation A in $\mathfrak{H}$ by

$$A \stackrel{\text{def}}{=} (B + C) \mathbin{\widehat{+}} (\{0\} \times \mathfrak{M}),$$

so that $A^* = (B + C)^* \cap (\mathfrak{M}^\perp \times \mathfrak{H})$, which leads to

$$A^{**} = \overline{\operatorname{span}}\{(B + C)^{**} \mathbin{\widehat{+}} (\{0\} \times \operatorname{clos} \mathfrak{M})\}. \tag{3.49}$$

Observe that $(B+C)^{**} = (B+C)^{**} \mathbin{\widehat{+}} (\{0\} \times \operatorname{span}\{y_0\})$ and since $\operatorname{clos} \mathfrak{M} \oplus \operatorname{span}\{y_0\} = \mathfrak{H}$, one concludes that

$$(B + C)^{**} \mathbin{\widehat{+}} (\{0\} \times \operatorname{clos} \mathfrak{M}) = (B + C)^{**} \mathbin{\widehat{+}} (\{0\} \times \mathfrak{H}) = \operatorname{dom} B^{**} \times \mathfrak{H}. \tag{3.50}$$

A combination of (3.49) and (3.50) leads to

$$A^{**} = \overline{\operatorname{dom}} B^{**} \times \mathfrak{H} = \mathfrak{H} \times \mathfrak{H}, \tag{3.51}$$

since $\operatorname{dom} B$ is dense in $\mathfrak{H}$. In particular, $\operatorname{mul} A^{**} = \mathfrak{H}$, so that $\mathfrak{H}_A = \{0\}$ (see (3.4)) and the orthogonal projection P is trivial: $P = 0$. Therefore the canonical decomposition (3.8) of A is trivial:

$$A_{\mathrm{reg}} = \operatorname{dom} B \times \{0\}, \qquad A = A_{\mathrm{sing}}. \tag{3.52}$$

Next, observe that A_{op} in (3.10) is given by $A_{\mathrm{op}} = A \cap (\mathfrak{H} \times \{0\})$, so that

$$A_{\mathrm{op}} = (\ker B \cap \ker C) \times \{0\}, \tag{3.53}$$

since $\operatorname{ran}(B + C) \cap \mathfrak{M} = \{0\}$ and $\operatorname{ran} B \cap \operatorname{ran} C = \{0\}$. Therefore, a comparison of (3.52) and (3.53) shows that the relation A is not decomposable; already $\operatorname{dom} B \neq \ker B$ since by construction B is an operator with $\operatorname{ran} B = \operatorname{span}\{y_0\}$. Furthermore, note that (3.53) implies that

$$(A_{\mathrm{op}})^{**} = \operatorname{clos}(\ker B \cap \ker C) \times \{0\}, \tag{3.54}$$

while it follows from (3.51) that

$$(A^{**})_{\rm op} = \mathfrak{H} \times \{0\}. \tag{3.55}$$

A comparison of (3.54) and (3.55) shows that the first identity of (3.21) is not satisfied, since $\ker C \neq \mathfrak{H}$ by the assumption $C \neq 0$. Finally, the identities $\operatorname{mul} A^{**} = \mathfrak{H}$ and $\operatorname{mul} A = \mathfrak{M}$ and $\overline{\operatorname{mul}}\, A = \operatorname{clos} \mathfrak{M}$ imply that the second identity of (3.21) is not satisfied; cf. (3.18).

Hence, the relation A in this example is not decomposable and, moreover, the two identities (3.21) in Proposition 3.16 are not satisfed. Another way to construct such an example is to take the orthogonal sum of the relations in Examples 3.24 and 3.25.

The next example shows that a decomposable relation A whose operator part is bounded, can become nondecomposable after a one-dimensional perturbation of its operator part.

Example 3.28. Let B be a bounded operator in $\mathfrak{H}$ and let $\mathfrak{M} \subset \mathfrak{H} \ominus \overline{\operatorname{ran}}\, B$ be a nonclosed subspace. Define the relation A by $A = B \,\widehat{+}\, (\{0\} \times \mathfrak{M})$, so that

$$A^{**} = B^{**} \,\widehat{+}\, (\{0\} \times \operatorname{clos} \mathfrak{M}).$$

The relation A is decomposable with $A_{\rm reg} = B$ and $A_{\rm mul} = \{0\} \times \mathfrak{M}$. Let $f_0 \in \overline{\operatorname{dom}}\, B$ and let $e \in (\operatorname{clos} \mathfrak{M}) \setminus \mathfrak{M}$. Define $B_e f = Bf + (f, f_0)e$, $f \in \operatorname{dom} B$, and define the relation A_e by $A_e = B_e \,\widehat{+}\, (\{0\} \times \mathfrak{M})$, so that

$$A_e^{**} = B_e^{**} \,\widehat{+}\, (\{0\} \times \operatorname{clos} \mathfrak{M}).$$

Observe that $\operatorname{mul} A_e = \operatorname{mul} A = \mathfrak{M}$ and $\operatorname{mul} A_e^{**} = \operatorname{mul} A^{**} = \operatorname{clos} \mathfrak{M}$. However, $\operatorname{ran}(I-P)A_e = \operatorname{span}\{e\} + \mathfrak{M}$ so that $\operatorname{ran}(I-P)A_e \not\subset \operatorname{mul} A_e$, and thus A_e is not decomposable by Theorem 3.10. In this case A_e still satisfies the equalities in (3.21):

$$((A_e)_{\rm op})^{**} = (B \restriction_{f_0^\perp})^{**} = (B_e^{**} \,\widehat{+}\, (\{0\} \times \operatorname{clos} \mathfrak{M}))_{\rm op} = ((A_e)^{**})_{\rm op}$$

and

$$((A_e)_{\rm mul})^{**} = \{0\} \times \operatorname{clos} \mathfrak{M} = ((A_e)^{**})_{\rm mul}.$$

4. Orthogonal componentwise decompositions of relations

Let A be a decomposable relation in a Hilbert space $\mathfrak{H}$, so that it has a componentwise sum decomposition as in (3.17). Furthermore, the adjoint A^*, being closed, has a componentwise decomposition as in (3.30). Necessary and sufficient conditions for these componentwise decompositions to be orthogonal will be given.

4.1. Orthogonality for componentwise sum decompositions of relations. For any relation A in a Hilbert space $\mathfrak{H}$ the identities

$$(A_{\rm mul})^* = (\operatorname{mul} A)^\perp \times \mathfrak{H}, \qquad (A_{\rm mul})^{**} = \{0\} \times \overline{\operatorname{mul}}\, A$$

are valid, where the adjoint is, as usual, with respect to the Hilbert space $\mathfrak{H}$. The last identity is concerned with taking closures, which are automatically with respect to the

Hilbert space $\operatorname{mul} A^{**}$. It is also useful to consider the adjoint of A_{mul} as a relation in the Hilbert space $\operatorname{mul} A^{**}$. The proof of the following lemma is straightforward.

LEMMA 4.1. *Let A be a relation in a Hilbert space $\mathfrak{H}$. The adjoint of the relation $A_{\mathrm{mul}} = \{0\} \times \operatorname{mul} A$ in the Hilbert space $\operatorname{mul} A^{**}$ is given by*

$$(A_{\mathrm{mul}})^* = (\operatorname{mul} A^{**} \ominus \operatorname{mul} A) \times \operatorname{mul} A^{**}.$$

In particular,

$$\begin{aligned} \overline{\operatorname{mul}} A^* = \operatorname{mul} A^{**} \ &\Leftrightarrow\ (A_{\mathrm{mul}})^* = (A_{\mathrm{mul}})^{**}, \\ \operatorname{mul} A = \operatorname{mul} A^{**} \ &\Leftrightarrow\ (A_{\mathrm{mul}})^* = A_{\mathrm{mul}}. \end{aligned}$$

Hence, the relation A_{mul} is essentially selfadjoint in the Hilbert space $\operatorname{mul} A^{**}$ if and only if $\overline{\operatorname{mul}} A = \operatorname{mul} A^{**}$, and the relation A_{mul} is selfadjoint in the Hilbert space $\operatorname{mul} A$ if and only if $\operatorname{mul} A = \operatorname{mul} A^{**}$. The following proposition is a further specification of the results in Proposition 3.16. Recall that $\mathfrak{H}_A = \mathfrak{H}_{A^{**}}$; it will be shown that the decompositions (3.17) and (3.20) are orthogonal with respect to the splitting $\mathfrak{H} = \mathfrak{H}_A \oplus \operatorname{mul} A^{**}$, simultaneously.

PROPOSITION 4.2. *Let A be a decomposable relation in a Hilbert space $\mathfrak{H}$. Then the componentwise sum decomposition* (3.17) *of A is orthogonal,*

$$A = A_{\mathrm{op}} \mathbin{\widehat{\oplus}} A_{\mathrm{mul}}, \tag{4.1}$$

if and only if

$$\operatorname{dom} A \subset \overline{\operatorname{dom}} A^* \quad \textit{or, equivalently,} \quad \operatorname{mul} A^{**} \subset \operatorname{mul} A^*. \tag{4.2}$$

In this case A_{mul} is essentially selfadjoint in $\operatorname{mul} A^{**}$. *Moreover, in this case the componentwise sum decomposition* (3.20) *of A^{**} is automatically orthogonal,*

$$A^{**} = (A^{**})_{\mathrm{op}} \mathbin{\widehat{\oplus}} (A^{**})_{\mathrm{mul}}, \tag{4.3}$$

and $(A^{**})_{\mathrm{mul}}$ is selfadjoint in $\operatorname{mul} A^{**}$.

Proof. It is assumed that A is decomposable, i.e., $A = A_{\mathrm{op}} \mathbin{\widehat{+}} A_{\mathrm{mul}}$. Clearly, the subspaces $\operatorname{ran} A_{\mathrm{op}}$ and $\operatorname{mul} A$ are orthogonal; cf. (3.11). Hence, the componentwise sum decomposition is orthogonal if and only if the condition $\operatorname{dom} A_{\mathrm{op}} \subset \mathfrak{H} \ominus \operatorname{mul} A^{**}$ is satisfied. Note that by Theorem 3.10 this last condition is equivalent to (4.2); cf. Lemma 2.4. Furthermore, the decomposability of A implies that $\overline{\operatorname{mul}} A = \operatorname{mul} A^{**}$; cf. Proposition 3.16. Lemma 4.1 now guarantees that A_{mul} is essentially selfadjoint in $\operatorname{mul} A^{**}$. It is clear that $(A^{**})_{\mathrm{mul}}$ is selfadjoint in $\operatorname{mul} A^{**}$; cf. Lemma 4.1. ■

COROLLARY 4.3. *Let A be a decomposable relation in a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $\operatorname{mul} A^{**} \subset \operatorname{mul} A^*$ *and* $(A^{**})_{\mathrm{op}} \in \boldsymbol{B}(\overline{\operatorname{dom}} A^*)$*;*
(ii) $\operatorname{dom} A^{**} = \overline{\operatorname{dom}} A^*$.

Proof. (i)⇒(ii). If $\operatorname{mul} A^{**} \subset \operatorname{mul} A^*$, then (4.3) holds by Proposition 4.2. Moreover, if $(A^{**})_{\mathrm{op}} \in \boldsymbol{B}(\overline{\operatorname{dom}} A^*)$, then

$$\operatorname{dom} A^{**} = \operatorname{dom} (A^{**})_{\mathrm{op}} = \overline{\operatorname{dom}} A^*.$$

(ii)⇒(i). If $\operatorname{dom} A^{**} = \overline{\operatorname{dom}}\, A^*$, then $\operatorname{mul} A^{**} = \operatorname{mul} A^*$; cf. Lemma 2.4. Hence (4.3) holds by Proposition 4.2. Furthermore,

$$\operatorname{dom}(A^{**})_{\mathrm{op}} = \operatorname{dom} A^{**} = \overline{\operatorname{dom}}\, A^*.$$

Hence the closed operator $(A^{**})_{\mathrm{op}}$ is defined on all of $\operatorname{dom} A^*$, so that it is bounded by the closed graph theorem. ■

Let A be a decomposable relation. Then it has already been shown in Proposition 3.16 that

$$(A_{\mathrm{op}})^{**} = (A^{**})_{\mathrm{op}}, \qquad (A_{\mathrm{mul}})^{**} = (A^{**})_{\mathrm{mul}}.$$

When A is decomposable and satisfies (4.2), these equalities also follow from a comparison between (4.1) and (4.3). Under the same circumstances, A is closed if and only if A_{op} and A_{mul} are closed; and A_{op} is densely defined if and only if $\overline{\operatorname{dom}}\, A = \overline{\operatorname{dom}}\, A^*$, which is equivalent to $\operatorname{mul} A^{**} = \operatorname{mul} A^*$.

PROPOSITION 4.4. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Assume that there is a closable operator B (in $\mathfrak{H}_A$) such that*

$$A = B \,\widehat{\oplus}\, A_{\mathrm{mul}}. \tag{4.4}$$

Then B coincides with A_{op}. In particular, the relation A is decomposable and satisfies the condition (4.2).

Proof. The assumption (4.4) implies that the condition in Theorem 3.9 is satisfied, so that $B = A_{\mathrm{op}}$. In particular, it follows that $\operatorname{dom} A_{\mathrm{op}} = \operatorname{dom} A$, so that A is decomposable by Theorem 3.10. Since B is an operator in $\mathfrak{H}_{\mathrm{op}}$ it is clear that $\operatorname{dom} A = \operatorname{dom} A_{\mathrm{op}} = \operatorname{dom} B \subset \mathfrak{H}_{\mathrm{A}} = \overline{\operatorname{dom}}\, A^*$, which leads to (4.2). ■

A combination of Propositions 4.2 and 4.4 leads to the following corollary.

COROLLARY 4.5. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then A has an orthogonal decomposition of the form* (4.1) *if and only if A is decomposable and satisfies* (4.2).

COROLLARY 4.6. *Let A be a relation in a Hilbert space $\mathfrak{H}$ which satisfies* $\operatorname{mul} A = \operatorname{mul} A^{**}$*, so that A is decomposable and A_{mul} is selfadjoint in* $\operatorname{mul} A^{**}$*. Then A admits the orthogonal composition* (4.1) *if and only if* $\operatorname{mul} A \subset \operatorname{mul} A^*$.

Proof. The condition $\operatorname{mul} A = \operatorname{mul} A^{**}$ implies that A is decomposable; cf. Corollary 3.14 and Lemma 4.1. Furthermore, the condition (4.2) in Proposition 4.2 is now equivalent to $\operatorname{mul} A \subset \operatorname{mul} A^*$. ■

If A is a closed relation in a Hilbert space $\mathfrak{H}$, then A is decomposable and A_{mul} is selfadjoint in $\operatorname{mul} A^{**}$; cf. Corollary 3.15 and Lemma 4.1. Hence A admits the orthogonal composition (4.1) if and only if $\operatorname{mul} A \subset \operatorname{mul} A^*$ (see Corollary 4.6).

4.2. Orthogonality for componentwise sum decompositions of adjoint relations. Let A be a relation in a Hilbert space $\mathfrak{H}$. Since the relation A^* is closed it is decomposable and has the componentwise decomposition (3.30); cf. Theorem 3.23. The adjoint of the relation $(A^*)_{\mathrm{mul}}$ in the Hilbert space $\operatorname{mul} A^*$ is given by

$$((A^*)_{\mathrm{mul}})^* = \{0\} \times \operatorname{mul} A^* = (A^*)_{\mathrm{mul}},$$

and the relation $(A^*)_{\rm mul}$ is selfadjoint in the Hilbert space $\operatorname{mul} A^*$; cf. Lemma 4.1. The following result is obtained by combining Lemma 2.4, Corollary 3.15, Proposition 4.2, and Proposition 4.4. The orthogonal componentwise decomposition is with respect to the orthogonal splitting $\mathfrak{H} = \mathfrak{H}_{A^*} \oplus \operatorname{mul} A^*$.

Proposition 4.7. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then the componentwise sum decomposition of A^* in* (3.30) *is orthogonal,*

$$A^* = (A^*)_{\rm op} \mathbin{\widehat{\oplus}} (A^*)_{\rm mul}, \tag{4.5}$$

if and only if

$$\operatorname{mul} A^* \subset \operatorname{mul} A^{**}. \tag{4.6}$$

Corollary 4.8. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $\operatorname{mul} A^* \subset \operatorname{mul} A^{**}$ *and* $(A^*)_{\rm op} \in \boldsymbol{B}(\overline{\operatorname{dom}}\, A)$*;*
(ii) $\overline{\operatorname{dom}}\, A = \operatorname{dom} A^*$.

Proof. (i)⇒(ii). If $\operatorname{mul} A^* \subset A^{**}$, then (4.5) holds by Proposition 4.7. Moreover, if $(A^*)_{\rm op} \in \boldsymbol{B}(\overline{\operatorname{dom}}\, A)$, then

$$\operatorname{dom} A^* = \operatorname{dom} (A^*)_{\rm op} = \overline{\operatorname{dom}}\, A.$$

(ii)⇒(i). If $\overline{\operatorname{dom}}\, A = \operatorname{dom} A^*$, then $\operatorname{mul} A^* = \operatorname{mul} A^{**}$; cf. Lemma 2.4. Hence (4.5) holds by Proposition 4.7. Furthermore, the closed operator $(A^*)_{\rm op}$ is defined on all of $\overline{\operatorname{dom}}\, A$, so that it is bounded by the closed graph theorem. ■

Another way to decompose A^* is to assume that A has an orthogonal componentwise decomposition as in (4.1). Hence, the following orthogonal componentwise decomposition is with respect to the orthogonal splitting $\mathfrak{H} = \mathfrak{H}_A \oplus \operatorname{mul} A^{**}$.

Proposition 4.9. *Let A be a decomposable relation in a Hilbert space $\mathfrak{H}$, which satisfies* (4.2)*. Then A^* has the orthogonal componentwise decomposition*

$$A^* = (A_{\rm op})^* \mathbin{\widehat{\oplus}} (A_{\rm mul})^*, \tag{4.7}$$

where $(A_{\rm op})^*$ and $(A_{\rm mul})^*$ stand for the adjoints of $A_{\rm op}$ and $A_{\rm mul}$ in $\mathfrak{H}_{\rm op}$ and $\operatorname{mul} A^{**}$*, respectively. Moreover, $(A_{\rm mul})^* = \{0\} \times \operatorname{mul} A^{**}$ is selfadjoint in* $\operatorname{mul} A^{**}$.

Proof. Taking adjoints in (4.1) gives (4.7). It follows from Proposition 4.2 that $A_{\rm mul}$ is essentially selfadjoint in $\operatorname{mul} A^{**}$ or, equivalently, that $(A_{\rm mul})^*$ is selfadjoint in $\operatorname{mul} A^{**}$; cf. Lemma 4.1. ■

Since the closable operator $A_{\rm op}$ need not be densely defined in $\mathfrak{H}_A$ its adjoint $(A_{\rm op})^*$ is a relation with multivalued part $\operatorname{mul} (A_{\rm op})^*$. The following result is a direct consequence of (4.7).

Corollary 4.10. *Let A be a decomposable relation in a Hilbert space $\mathfrak{H}$, which satisfies* (4.2)*. Then*

$$\operatorname{mul} A^* \ominus \operatorname{mul} A^{**} = \operatorname{mul} (A_{\rm op})^*,$$

so that

$$(A^*)_{\rm mul} = \{0\} \times (\operatorname{mul} (A_{\rm op})^* \oplus \overline{\operatorname{mul}}\, A).$$

A combination of Propositions 4.2 and 4.10 leads to a decomposition result for formally domain tight relations.

PROPOSITION 4.11. *Let A be a decomposable relation which is formally domain tight. Then A admits the orthogonal decomposition* (4.1)*, where $A_{\rm op}$ is a formally domain tight operator in $\mathfrak{H}_{\rm op}$ and $A_{\rm mul}$ is essentially selfadjoint in* $\operatorname{mul} A^{**}$*.*

Proof. Since A is formally domain tight, it follows that $\operatorname{mul} A^{**} \subset \operatorname{mul} A^*$. Since A is assumed to be also decomposable, the conditions of Proposition 4.2 are satisfied. Hence the orthogonal decomposition of A in (4.1) and the orthogonal decomposition of A^* in (4.7) are valid. Recall that $A_{\rm mul} = \{0\} \times \operatorname{mul} A$ and $(A_{\rm mul})^* = \{0\} \times \operatorname{mul} A^{**}$; cf. Proposition 4.9. Hence, it follows from (4.1) and (4.7) that

$$\operatorname{dom} A_{\rm op} = \operatorname{dom} A \subset \operatorname{dom} A^* = \operatorname{dom} (A_{\rm op})^*.$$

In other words, the operator $A_{\rm op}$ is formally domain tight in $\mathfrak{H}_{\rm op}$. ■

Let A be a relation in a Hilbert space $\mathfrak{H}$, which satisfies $\operatorname{mul} A^{**} = \operatorname{mul} A^*$. Then the orthogonal splitting $\mathfrak{H} = \mathfrak{H}_A \oplus \operatorname{mul} A^{**}$ generated by $\operatorname{mul} A^{**}$ coincides with the orthogonal splitting $\mathfrak{H} = \mathfrak{H}_{A^*} \oplus \operatorname{mul} A^*$ generated by $\operatorname{mul} A^*$. Hence, in this case the orthogonal decompositions (4.1), (4.7), and (4.5) (cf. (4.6)) are with respect to the same splitting.

PROPOSITION 4.12. *Let A be a decomposable relation in a Hilbert space $\mathfrak{H}$, which satisfies* $\operatorname{mul} A^{**} = \operatorname{mul} A^*$*. Then A admits the orthogonal decomposition* (4.1) *where $A_{\rm op}$ is a densely defined operator in $\mathfrak{H}_{\rm op}$ and $A_{\rm mul}$ is an essentially selfadjoint relation in* $\operatorname{mul} A^{**}$*. Moreover,*

$$(A_{\rm op})^* = (A^*)_{\rm op}. \tag{4.8}$$

Proof. It follows from the condition $\operatorname{mul} A^{**} = \operatorname{mul} A^*$ that the identity (4.5) is valid. Since A is assumed to be decomposable, the condition $\operatorname{mul} A^{**} = \operatorname{mul} A^*$ also implies that the identity (4.1) holds. It follows from Corollary 4.10 that $A_{\rm op}$ is a densely defined operator in $\mathfrak{H}_{\rm op}$. The identity (4.1) itself shows that the identity (4.7) holds. Furthermore, the condition $\operatorname{mul} A^{**} = \operatorname{mul} A^*$ implies that both decompositions (4.5) and (4.7) are relative to the same orthogonal splitting of the Hilbert space $\mathfrak{H}$. Therefore, the identity (4.8) is immediate. ■

A combination of Propositions 4.2 and 4.12 leads to a decomposition result for domain tight relations.

PROPOSITION 4.13. *Let A be a decomposable relation in a Hilbert space $\mathfrak{H}$, which is domain tight. Then A admits the orthogonal decomposition* (4.1) *where $A_{\rm op}$ is a densely defined domain tight operator in $\mathfrak{H}_A$ and $A_{\rm mul}$ is essentially selfadjoint in* $\operatorname{mul} A^{**}$*.*

Proof. If A is a domain tight relation, so that $\operatorname{dom} A = \operatorname{dom} A^*$, then $\operatorname{mul} A^{**} = \operatorname{mul} A^*$ and Proposition 4.12 applies. It follows from the decompositions (4.1) and (4.7), that

$$\operatorname{dom} (A_{\rm op})^* = \operatorname{dom} (A^*)_{\rm op} = \operatorname{dom} A^* = \operatorname{dom} A = \operatorname{dom} A_{\rm op},$$

which shows that $A_{\rm op}$ is domain tight in $\mathfrak{H}_A$. ■

The relations A which are domain tight, i.e., $\operatorname{dom} A = \operatorname{dom} A^*$, and which satisfy the additional condition $\operatorname{mul} A = \operatorname{mul} A^*$, can be characterized in terms of orthogonal decompositions.

PROPOSITION 4.14. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then A is domain tight and* $\operatorname{mul} A = \operatorname{mul} A^*$ *if and only if $A = B \,\widehat{\oplus}\, A_{\mathrm{mul}}$ where B is a densely defined domain tight (closable) operator in $\mathfrak{H}_A$ and A_{mul} is selfadjoint in* $\operatorname{mul} A^{**}$. *In this case $B = A_{\mathrm{op}}$.*

Proof. ($\Rightarrow$) Assume that A is domain tight and that $\operatorname{mul} A = \operatorname{mul} A^*$. Then it follows that $\operatorname{mul} A^{**} = \operatorname{mul} A$. Hence, A is decomposable by Corollary 3.14 and A_{mul} is selfadjoint in $\operatorname{mul} A^{**}$ by Lemma 4.1. By Proposition 4.13 it follows that A_{op} is a densely defined domain tight operator in $\mathfrak{H}_A$. Furthermore, A_{op} is closable, which is clear from the fact that A is decomposable, but also from the fact that A_{op} is domain tight and densely defined. According to Proposition 4.13 the relation A decomposes as $A = A_{\mathrm{op}} \,\widehat{\oplus}\, A_{\mathrm{mul}}$.

($\Leftarrow$) Assume that $A = B \,\widehat{\oplus}\, A_{\mathrm{mul}}$ where B is a densely defied domain tight operator in $\mathfrak{H}_A$ and A_{mul} is selfadjoint in $\operatorname{mul} A^{**}$. Then $A^* = B^* \,\widehat{\oplus}\, A_{\mathrm{mul}}$, so that $\operatorname{dom} A = \operatorname{dom} B = \operatorname{dom} B^* = \operatorname{dom} A^*$, and A is domain tight. The condition that A_{mul} is selfadjoint in $\operatorname{mul} A^{**}$ implies that $\operatorname{mul} A = \operatorname{mul} A^{**}$; cf. Lemma 4.1. Since B is densely defined and domain tight, it follows that B is a closable operator. Hence, by Proposition 4.4, the identity $B = A_{\mathrm{op}}$ is established. ∎

4.3. Some classes of relations with orthogonal componentwise decompositions. This subsection describes orthogonal componentwise decompositions for some classes of relations described via the numerical range and for some subclasses of domain tight relations.

Let A be a decomposable relation in a Hilbert space $\mathfrak{H}$ and assume that $\operatorname{mul} A^{**} \subset \operatorname{mul} A^*$. Then

$$\mathcal{W}(A) = \mathcal{W}(A_{\mathrm{op}}). \tag{4.9}$$

To see this, note that Theorem 3.10 shows that $A_{\mathrm{reg}} = A_{\mathrm{op}}$, and then apply Remark 3.8. Now some consequences of the assumption $\mathcal{W}(A) \neq \mathbb{C}$ are listed.

PROPOSITION 4.15. *Let A be a decomposable relation in a Hilbert space $\mathfrak{H}$ such that $\mathcal{W}(A) \neq \mathbb{C}$. Then the relation A admits the orthogonal decomposition* (4.1), *A_{mul} is essentially selfadjoint in* $\operatorname{mul} A^{**}$, *and $\mathcal{W}(A_{\mathrm{op}}) = \mathcal{W}(A)$. Moreover, if $\rho(A) \neq \emptyset$, then A_{op} is a closed densely defined operator in $\mathfrak{H}_{\mathrm{op}}$, A_{mul} is selfadjoint in* $\operatorname{mul} A^{**}$, *and $\rho(A_{\mathrm{op}}) \neq \emptyset$.*

Proof. By Lemma 2.30 the condition $\mathcal{W}(A) \neq \mathbb{C}$ implies that $\operatorname{mul} A \subset \operatorname{mul} A^*$, and thus also $\overline{\operatorname{mul}}\, A \subset \operatorname{mul} A^*$. By Proposition 3.16 the condition that A is decomposable implies that $\overline{\operatorname{mul}}\, A = \operatorname{mul} A^{**}$. Therefore the inclusion $\operatorname{mul} A^{**} \subset \operatorname{mul} A^*$ is valid. Since A is assumed to be decomposable, Proposition 4.2 may be applied. The identity $\mathcal{W}(A_{\mathrm{op}}) = \mathcal{W}(A)$ follows from (4.9).

If $\rho(A) \neq \emptyset$, then Lemma 2.32 shows that A is closed and $\operatorname{mul} A^* = \operatorname{mul} A$. Hence, Proposition 4.12 applies, so that A_{op} is densely defined closed operator in $\mathfrak{H}_A$ and $\operatorname{mul} A$ is closed. The decomposition $A = A_{\mathrm{op}} \,\widehat{\oplus}\, A_{\mathrm{mul}}$, where A_{mul} is selfadjoint in $\operatorname{mul} A^{**}$, shows that A and A_{op} have the same resolvent set. ∎

Let A be a relation in a Hilbert space $\mathfrak{H}$. Then A is symmetric if and only if $\mathcal{W}(A) \subset \mathbb{R}$. A relation A is said to be *dissipative* if $\mathcal{W}(A)$ is a subset of the upper halfplane:

$$\mathfrak{im}\,(f', f) \geq 0, \qquad \{f, f'\} \in A,$$

and a relation A is said to be *accretive* if $\mathcal{W}(A)$ is a subset of the right halfplane:

$$\mathfrak{re}\,(f', f) \geq 0, \qquad \{f, f'\} \in A.$$

A relation A is said to be *sectorial with vertex at the origin and semiangle* α, $\alpha \in (0, \pi/2)$, if $\mathcal{W}(A)$ is a subset of the corresponding sector in the right halfplane:

$$(\tan\alpha)\,\mathfrak{re}\,(f', f) \geq |\mathfrak{im}\,(f', f)|, \qquad \{f, f'\} \in A; \tag{4.10}$$

cf. [3], [4], [18], [35]. A relation A is said to be *nonnegative* if $\mathcal{W}(A)$ is a subset of $[0, \infty)$. In each of these cases the closure gives rise to a similar inequality. Hence, if the relation A belongs to one of the above classes, it may be assumed in addition that A is closed. Therefore Proposition 4.15 may be applied and the operator part A_{op} in the orthogonal decomposition (4.1) belongs to the same class as the original relation A.

In each of these cases the relation A is said to be maximal with respect to the indicated property if the complement of $\operatorname{clos}\mathcal{W}(A)$ (or one of its components) belongs to the resolvent set so that $\rho(A)$ is not empty. It can be shown that maximality is equivalent to the absence of nontrivial (relation) extensions with the same property; cf. [26], [33], [14], [18], [17].

Corollary 4.16. *Let A be a maximal symmetric (dissipative, accretive, sectorial, nonnegative) relation in a Hilbert space $\mathfrak{H}$. Then A admits an orthogonal decomposition of the form $A = A_{\mathrm{op}} \mathbin{\widehat{\oplus}} A_{\mathrm{mul}}$, where A_{op} is a closed, densely defined, maximal symmetric (dissipative, accretive, sectorial, nonnegative) operator in the Hilbert space $\mathfrak{H}_A$ and A_{mul} is a selfadjoint relation in* $\operatorname{mul} A^{**}$.

The result for maximal symmetric relations can also be seen as a consequence of Proposition 4.11, since symmetric relations are formally domain tight. Selfadjoint and normal relations are domain tight and there is a decomposition result for them corresponding to Corollary 4.16, as an application of Proposition 4.12; see [8] and [23] for further details.

Corollary 4.17. *Let A be a selfadjoint (normal) relation in a Hilbert space $\mathfrak{H}$. Then A admits an orthogonal decomposition of the form $A = A_{\mathrm{op}} \mathbin{\widehat{\oplus}} A_{\mathrm{mul}}$, where A_{op} is a selfadjoint (normal) operator in the Hilbert space $\mathfrak{H}_A$ and A_{mul} is a selfadjoint relation in* $\operatorname{mul} A^{**}$.

Recall that selfadjoint and normal operators are automatically densely defined; cf. (2.36).

5. Cartesian decompositions of relations

In this section the notions of real and imaginary parts of a relation in a Hilbert space are confronted with the notion of a Cartesian decomposition.

5.1. Real and imaginary parts of relations. Let A be a relation in a Hilbert space $\mathfrak{H}$. The *real part* $\mathfrak{re}\, A$ and the *imaginary part* $\mathfrak{im}\, A$ of A are defined by

$$\mathfrak{re}\, A \stackrel{\text{def}}{=} \frac{1}{2}(A + A^*) = \left\{ \left\{ f, \frac{f' + f''}{2} \right\};\ \{f, f'\} \in A,\ \{f, f''\} \in A^* \right\}, \tag{5.1}$$

and

$$\mathfrak{im}\, A \stackrel{\text{def}}{=} \frac{1}{2\,\mathrm{i}}(A - A^*) = \left\{ \left\{ f, \frac{f' - f''}{2\,\mathrm{i}} \right\};\ \{f, f'\} \in A,\ \{f, f''\} \in A^* \right\}, \tag{5.2}$$

with the operatorwise sums defined as in (2.13). It is clear from the definitions that

$$\begin{cases} \operatorname{dom} \mathfrak{re}\, A = \operatorname{dom} \mathfrak{im}\, A = \operatorname{dom} A \cap \operatorname{dom} A^*, \\ \operatorname{dom} \mathfrak{re}\, A^* = \operatorname{dom} \mathfrak{im}\, A^* = \operatorname{dom} A^{**} \cap \operatorname{dom} A^*. \end{cases} \tag{5.3}$$

The real and imaginary parts of A are connected by

$$\mathfrak{re}(\mathrm{i}\, A) = -\,\mathfrak{im}\, A, \qquad \mathfrak{im}(\mathrm{i}\, A) = \mathfrak{re}\, A. \tag{5.4}$$

In what follows the relations $\mathfrak{re}\, A \pm \mathrm{i}\,\mathfrak{im}\, A$ and their connections with the original relation A will be studied.

PROPOSITION 5.1. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then*

(i) $\mathfrak{re}\, A \subset \mathfrak{re}\, A^* = \mathfrak{re}\, A^{**} \subset (\mathfrak{re}\, A)^*$ *and* $\mathfrak{im}\, A \subset -\,\mathfrak{im}\, A^* = \mathfrak{im}\, A^{**} \subset (\mathfrak{im}\, A)^*$*;*
(ii) *if A is closed, then* $\mathfrak{re}\, A = \mathfrak{re}\, A^*$ *and* $\mathfrak{im}\, A = -\,\mathfrak{im}\, A^*$*;*
(iii) $\operatorname{mul} \mathfrak{re}\, A = \operatorname{mul} \mathfrak{im}\, A = \operatorname{mul} A + \operatorname{mul} A^*$ *and if, in addition, A is formally domain tight, then* $\operatorname{mul} \mathfrak{re}\, A = \operatorname{mul} \mathfrak{im}\, A = \operatorname{mul} A^*$*.*

Proof. (i) Since $A \subset A^{**}$, it follows from (2.15) that

$$\frac{1}{2}(A + A^*) \subset \frac{1}{2}(A^{**} + A^*) \subset \left(\frac{1}{2}(A + A^*) \right)^*,$$

$$\frac{1}{2\,\mathrm{i}}(A - A^*) \subset -\frac{1}{2\,\mathrm{i}}(A^* - A^{**}) \subset \left(\frac{1}{2\,\mathrm{i}}(A - A^*) \right)^*.$$

The assertions concerning $\mathfrak{re}\, A$ and $\mathfrak{im}\, A$ are now clear.

(ii) Here $A = A^{**}$ and thus the stated equalities are clear from (5.1) and (5.2).

(iii) The first assertion is immediate from (5.1) and (5.2). If A is formally domain tight, then it follows from (2.33) that $\operatorname{mul} A \subset \operatorname{mul} A^*$ and thus $\operatorname{mul} A + \operatorname{mul} A^* = \operatorname{mul} A^*$, which implies the second assertion. ∎

The real and imaginary parts $\mathfrak{re}\, A$ and $\mathfrak{im}\, A$ of a relation A are symmetric relations, due to Proposition 5.1. They are defined in terms of operatorwise sums involving A and A^*. There are also connections with the componentwise sum $A \,\widehat{+}\, A^*$.

PROPOSITION 5.2. *Let A be a linear relation in a Hilbert space $\mathfrak{H}$. Then*

(i) $\mathfrak{re}\, A \subset A \,\widehat{+}\, A^*$ *and* $\mathfrak{im}\, A \subset A \,\widehat{+}\, A^*$*;*
(ii) $\operatorname{ran}(\mathfrak{re}\, A) = \operatorname{mul}(A \,\widehat{+}\, A^*)$ *and* $\operatorname{ran}(\mathfrak{im}\, A) = \operatorname{mul}(A \,\widehat{+}\, A^*)$*;*
(iii) $\mathfrak{re}\, A \pm \mathrm{i}\,\mathfrak{im}\, A \subset \mathfrak{re}\, A \,\widehat{+}\, (\{0\} \times \operatorname{ran} \mathfrak{im}\, A) \subset A \,\widehat{+}\, A^*$*;*
(iv) $\mathfrak{im}\, A \pm \mathrm{i}\,\mathfrak{re}\, A \subset \mathfrak{im}\, A \,\widehat{+}\, (\{0\} \times \operatorname{ran} \mathfrak{re}\, A) \subset A \,\widehat{+}\, A^*$*.*

Proof. (i) The first inclusion follows from (5.1) and

$$2\left\{f, \frac{f'+f''}{2}\right\} = \{f, f'\} + \{f, f''\} \in A \,\widehat{+}\, A^*, \qquad \{f, f'\} \in A, \{f, f''\} \in A^*.$$

The second inclusion can be shown similarly.

(ii) The second inclusion will be shown. Let $\{0, g\} \in A \,\widehat{+}\, A^*$. Then

$$\{0, g\} = \{f, f'\} - \{f, f''\}, \qquad \{f, f'\} \in A, \{f, f''\} \in A^*,$$

so that

$$\left\{f, \frac{g}{2\,\mathrm{i}}\right\} = \left\{f, \frac{f'-f''}{2\,\mathrm{i}}\right\} \in \mathfrak{im}\, A.$$

Hence $\operatorname{mul}(A \,\widehat{+}\, A^*) \subset \operatorname{ran}(\mathfrak{im}\, A)$. The reverse inclusion follows immediately from (5.2). This proves the second identity. The first identity is now obtained as follows:

$$\operatorname{ran}(\mathfrak{re}\, A) = \operatorname{ran}(\mathfrak{im}\, iA) = \operatorname{mul}(iA \,\widehat{+}\, (iA)^*) = \operatorname{mul}(-A \,\widehat{+}\, A^*).$$

(iii) Let $\{f, \varphi \pm \mathrm{i}\,\psi\} \in \mathfrak{re}\, A \pm \mathrm{i}\, \mathfrak{im}\, A$ with $\{f, \varphi\} \in \mathfrak{re}\, A$ and $\{f, \psi\} \in \mathfrak{im}\, A$. Then clearly

$$\{f, \varphi \pm \mathrm{i}\,\psi\} = \{f, \varphi\} \,\widehat{+}\, \{0, \pm \mathrm{i}\,\psi\} \in \mathfrak{re}\, A \,\widehat{+}\, (\{0\} \times \operatorname{ran} \mathfrak{im}\, A),$$

which shows the first inclusion in (iii). The second inclusion in (iii) follows from (i) and (ii).

(iv) This is obtained from (iii) by means of (5.4). ■

The next result gives necessary and sufficient conditions for a relation A to be formally domain tight.

THEOREM 5.3. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) *A is formally domain tight;*
(ii) $A \subset \mathfrak{re}\, A + \mathrm{i}\, \mathfrak{im}\, A$*;*
(iii) $(\mathfrak{re}\, A) \,\widehat{+}\, A^* = A \,\widehat{+}\, A^*$*;*
(iv) *there exists a relation B in $\mathfrak{H}$ such that* $\operatorname{dom} A = \operatorname{dom} B$ *and* $A \subset B^*$*;*
(v) *there exists a relation C in $\mathfrak{H}$ such that* $A \subset \mathfrak{re}\, C + \mathrm{i}\, \mathfrak{im}\, C$.

Proof. (i)⇒(ii). Let $\{f, g\} \in A$. Since $\operatorname{dom} A \subset \operatorname{dom} A^*$, there exists $h \in \mathfrak{H}$ such that $\{f, h\} \in A^*$. Then clearly

$$\{f, g\} = \left\{f, \frac{g+h}{2} + \mathrm{i}\, \frac{g-h}{2\,\mathrm{i}}\right\} \in \mathfrak{re}\, A + \mathrm{i}\, \mathfrak{im}\, A.$$

Hence $A \subset \mathfrak{re}\, A + \mathrm{i}\, \mathfrak{im}\, A$.

(ii)⇒(iii). By Proposition 5.2, $\mathfrak{re}\, A \subset A \,\widehat{+}\, A^*$ and hence

$$(\mathfrak{re}\, A) \,\widehat{+}\, A^* \subset A \,\widehat{+}\, A^*.$$

Thus it is enough to prove the reverse inclusion: $A \,\widehat{+}\, A^* \subset (\mathfrak{re}\, A) \,\widehat{+}\, A^*$. It suffices to prove that $A \subset (\mathfrak{re}\, A) \,\widehat{+}\, A^*$. Therefore, let $\{f, f'\} \in A$. Then by (ii), $\{f, f'\} \in \mathfrak{re}\, A + \mathrm{i}\, \mathfrak{im}\, A$, so that $f \in \operatorname{dom} A \cap \operatorname{dom} A^*$ by (5.3) and, in particular, $f \in \operatorname{dom} A^*$. Hence, there exists an element f'' such that $\{f, f''\} \in A^*$. Then

$$\{f, f'\} = \{2f, f' + f''\} - \{f, f''\} \in (\mathfrak{re}\, A) \mathbin{\widehat{+}} A^*.$$

This completes the proof of the equality in (iii).

(iii)⇒(i). Let $f \in \operatorname{dom} A$. Then $\{f, f'\} \in \mathrm{i}\, A$ for some $f' \in \mathfrak{H}$. By (iii), $\{f, f'\} \in (\mathfrak{re}\, A) \mathbin{\widehat{+}} A^*$, so that $f = f_1 + f_2$ with $f_1 \in \operatorname{dom} \mathfrak{re}\, A$ and $f_2 \in \operatorname{dom} A^*$. It follows from (5.3) that $f_1 \in \operatorname{dom} A^*$. Hence, $f = f_1 + f_2 \in \operatorname{dom} A^*$. Thus (i) has been shown.

(i)&(ii)⇒(iv). Define $B \overset{\text{def}}{=} \mathfrak{re}\, A - \mathrm{i}\, \mathfrak{im}\, A$. Then by Proposition 5.1 and (2.15),

$$B^* \supset (\mathfrak{re}\, A)^* + \mathrm{i}(\mathfrak{im}\, A)^* \supset \mathfrak{re}\, A + \mathrm{i}\, \mathfrak{im}\, A \supset A.$$

Furthermore, it follows from $\operatorname{dom} A \subset \operatorname{dom} A^*$ and (5.3) that

$$\operatorname{dom} B = \operatorname{dom} \mathfrak{re}\, A = \operatorname{dom} \mathfrak{im}\, A = \operatorname{dom} A \cap \operatorname{dom} A^* = \operatorname{dom} A.$$

Hence (iv) has been shown.

(iv)⇒(i). By taking adjoints in $A \subset B^*$ one gets $B \subset B^{**} \subset A^*$, so that $\operatorname{dom} A = \operatorname{dom} B \subset \operatorname{dom} A^*$. Hence A is formally domain tight.

(v)⇒(i). Taking adjoints in $A \subset \mathfrak{re}\, C + \mathrm{i}\, \mathfrak{im}\, C$ one obtains, by Proposition 5.1 and (2.15),

$$A^* \supset (\mathfrak{re}\, C + \mathrm{i}\, \mathfrak{im}\, C)^* \supset (\mathfrak{re}\, C)^* - \mathrm{i}(\mathfrak{im}\, C)^* \supset \mathfrak{re}\, C - \mathrm{i}\, \mathfrak{im}\, C.$$

Since $\operatorname{dom} A \subset \operatorname{dom} \mathfrak{re}\, C = \operatorname{dom} \mathfrak{im}\, C \subset \operatorname{dom} A^*$, this shows that A is formally domain tight.

(ii)⇒(v). This implication is trivial. ■

The following lemma contains a result analogous to the equivalence of (i) and (iii) in Theorem 5.3. Moreover, the identities $\mathfrak{re}\, A = \mathfrak{re}\, A^*$ and $\mathfrak{im}\, A = -\, \mathfrak{im}\, A^*$ will be shown under different conditions than in Proposition 5.1.

Lemma 5.4. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then*

(i) $\operatorname{dom} A^* \subset \operatorname{dom} A$ *if and only if*

$$(\mathfrak{re}\, A) \mathbin{\widehat{+}} A = A \mathbin{\widehat{+}} A^*; \tag{5.5}$$

(ii) *if* $\operatorname{dom} A^* \subset \operatorname{dom} A \subset \overline{\operatorname{dom}}\, A^*$, *then*

$$\mathfrak{re}\, A = \mathfrak{re}\, A^*, \qquad \mathfrak{im}\, A = -\, \mathfrak{im}\, A^*. \tag{5.6}$$

Proof. (i) Assume that $A \mathbin{\widehat{+}} A^* = (\mathfrak{re}\, A) \mathbin{\widehat{+}} A$, which, in particular, leads to $A^* \subset (\mathfrak{re}\, A) \mathbin{\widehat{+}} A$. Since $\operatorname{dom} \mathfrak{re}\, A = \operatorname{dom} A \cap \operatorname{dom} A^*$ (see (5.3)), it follows that $\operatorname{dom} A^* \subset \operatorname{dom} A$.

Now assume $\operatorname{dom} A^* \subset \operatorname{dom} A$. It suffices to show that $A \mathbin{\widehat{+}} A^* \subset (\mathfrak{re}\, A) \mathbin{\widehat{+}} A$, as the reverse inclusion is always true by Proposition 5.2. Let $\{f, f''\} \in A^*$, then there exists $\{f, f'\} \in A$. Hence,

$$\{f, f''\} = \{2f, f' + f''\} - \{f, f'\} \in (\mathfrak{re}\, A) \mathbin{\widehat{+}} A.$$

It follows that $A^* \subset (\mathfrak{re}\, A) \mathbin{\widehat{+}} A$, but then also $A \mathbin{\widehat{+}} A^* \subset (\mathfrak{re}\, A) \mathbin{\widehat{+}} A$. Therefore, (5.5) has been proved.

(ii) By Lemma 2.4, it follows from $\operatorname{dom} A^* \subset \operatorname{dom} A \subset \overline{\operatorname{dom}}\, A^*$ that $\operatorname{mul} A^{**} = \operatorname{mul} A^*$. According to Proposition 5.1, $\mathfrak{re}\, A \subset \mathfrak{re}\, A^*$. To prove the reverse inclusion assume that $\{f, g\} \in \mathfrak{re}\, A^*$. Then for some $\{f, g'\} \in A^*$ and $\{f, g''\} \in A^{**}$ one has $2g = g' + g''$. Here

$f \in \operatorname{dom} A^* \cap \operatorname{dom} A^{**}$ and since $\operatorname{dom} A^* \subset \operatorname{dom} A$, one has $\{f, f'\} \in A$ for some f'. Consequently, $\{f, g''\} - \{f, f'\} \in A^{**}$ and

$$g'' - f' \in \operatorname{mul} A^{**} = \operatorname{mul} A^* \subset \operatorname{mul} \mathfrak{re}\, A,$$

where the last inclusion is due to Proposition 5.1(iii). Therefore,

$$\{f, g\} = \left\{f, \frac{f' + g'}{2}\right\} + \left\{0, \frac{g'' - f'}{2}\right\} \in \mathfrak{re}\, A,$$

and hence $\mathfrak{re}\, A^* \subset \mathfrak{re}\, A$. This proves the identity $\mathfrak{re}\, A = \mathfrak{re}\, A^*$. The second identity in (5.6) is obtained from the first one by means of the equalities $\mathfrak{re}(iA) = -\mathfrak{im}\, A$ and $\mathfrak{re}\,(iA)^* = \mathfrak{im}\, A^*$; cf. (5.4). ■

The following characterizations for a relation to be domain tight are consequences of Lemma 5.4; cf. Theorem 5.3.

PROPOSITION 5.5. *Let A be a relation in a Hilbert space $\mathfrak{H}$. The following conditions are equivalent:*

(i) *A is domain tight;*
(ii) *$(\mathfrak{re}\, A) \,\widehat{+}\, A = (\mathfrak{re}\, A) \,\widehat{+}\, A^*$;*
(iii) *$\mathfrak{re}\, A \,\widehat{+}\, (\{0\} \times \operatorname{ran} \mathfrak{im}\, A) = A \,\widehat{+}\, A^*$.*

In this case,

$$\mathfrak{re}\, A \,\widehat{+}\, (\{0\} \times \operatorname{ran} \mathfrak{im}\, A) = (\mathfrak{re}\, A) \,\widehat{+}\, A = (\mathfrak{re}\, A) \,\widehat{+}\, A^* = A \,\widehat{+}\, A^*. \tag{5.7}$$

Proof. (i)⇒(ii). If $\operatorname{dom} A = \operatorname{dom} A^*$ then $(\mathfrak{re}\, A) \,\widehat{+}\, A^* = A \,\widehat{+}\, A^*$ by Theorem 5.3(iii), while $(\mathfrak{re}\, A) \,\widehat{+}\, A = A \,\widehat{+}\, A^*$ due to (5.5) in Lemma 5.4. This gives the identity in (ii).

(ii)⇐(i). If $(\mathfrak{re}\, A) \,\widehat{+}\, A = (\mathfrak{re}\, A) \,\widehat{+}\, A^*$, then, in particular, $A \subset (\mathfrak{re}\, A) \,\widehat{+}\, A^*$. Since, by (5.3), $\operatorname{dom} \mathfrak{re}\, A = \operatorname{dom} A \cap \operatorname{dom} A^*$, it follows that $\operatorname{dom} A \subset \operatorname{dom} A^*$. The inclusion $\operatorname{dom} A^* \subset \operatorname{dom} A$ follows in a similar way. Hence, A is domain tight.

(i)⇒(iii). In view of the second inclusion in Proposition 5.2(iii) it suffices to show the inclusion $A \,\widehat{+}\, A^* \subset \mathfrak{re}\, A \,\widehat{+}\, (\{0\} \times \operatorname{ran} \mathfrak{im}\, A)$ when A is domain tight. Since A is domain tight, A^* is formally domain tight; cf. Remark 2.24. Hence, Theorem 5.3 implies

$$A \subset \mathfrak{re}\, A + \mathrm{i}\, \mathfrak{im}\, A, \qquad A^* \subset \mathfrak{re}\, A^* + \mathrm{i}\, \mathfrak{im}\, A^* = \mathfrak{re}\, A - \mathrm{i}\, \mathfrak{im}\, A,$$

where the last identity is obtained from Lemma 5.4. It remains to use Proposition 5.2(i) to get the claimed inclusion.

(iii)⇒(i). The equality in (iii) implies that $\operatorname{dom} A \cap \operatorname{dom} A^* = \operatorname{dom} A + \operatorname{dom} A^*$; cf. (5.3). This last identity is clearly equivalent to $\operatorname{dom} A = \operatorname{dom} A^*$.

Finally, the equalities stated in (5.7) are clear from the above arguments. ■

5.2. Cartesian decompositions of relations. A relation A in a Hilbert space $\mathfrak{H}$ is said to have a *Cartesian decomposition* if there are two symmetric relations A_1 and A_2 in $\mathfrak{H}$ such that

$$A = A_1 + \mathrm{i}\, A_2, \tag{5.8}$$

with the operatorwise sum defined as in (2.13), so that $\operatorname{dom} A = \operatorname{dom} A_1 \cap \operatorname{dom} A_2$ and $\operatorname{mul}(A_1 + A_2) = \operatorname{mul} A_1 + \operatorname{mul} A_2$; cf. (2.14). In particular, if A is an operator, then A_1

and A_2 in (5.8) are operators. The Cartesian decomposition for operators is extensively considered in [39].

EXAMPLE 5.6. Let A be a maximal sectorial relation in $\mathfrak{H}$ with vertex at the origin and semiangle α; cf. (4.10). Then there exist a nonnegative selfadjoint relation H in $\mathfrak{H}$ and a selfadjoint operator $B \in \boldsymbol{B}(\mathfrak{H})$ with $\overline{\operatorname{ran}}\, B \subset (\operatorname{mul} A)^\perp$ and $\|B\| \leq \tan \alpha$ such that

$$A = H^{1/2}(I + \mathrm{i}\, B)H^{1/2};$$

cf. [18], [26], [35]. Clearly, H and $H^{1/2}BH^{1/2}$ are symmetric relations, but $H + \mathrm{i}\, H^{1/2}BH^{1/2}$, their operatorwise sum, need not be equal to A. In general, the inclusion

$$H + \mathrm{i}\, H^{1/2}BH^{1/2} \subset A$$

holds. There is equality if, for instance, $\operatorname{ran} B \subset \operatorname{dom} H^{1/2}$.

PROPOSITION 5.7. *Let A be a relation in a Hilbert space $\mathfrak{H}$, let A have a Cartesian decomposition* (5.8)*, and define the relation B by $B = A_1 - \mathrm{i}\, A_2$. Then A and B have the same domain* $\operatorname{dom} B = \operatorname{dom} A$*, they are formally domain tight, and form a dual pair:*

$$B \subset A^*, \qquad A \subset B^*.$$

Moreover, the symmetric components A_1 and A_2 satisfy

$$A_1 \cap (\operatorname{dom} A \times \mathfrak{H}) \subset \mathfrak{re}\, A, \qquad A_2 \cap (\operatorname{dom} A \times \mathfrak{H}) \subset \mathfrak{im}\, A, \tag{5.9}$$

and $A_1 \pm \mathrm{i}\, A_2 \subset \mathfrak{re}\, A \pm \mathrm{i}\, \mathfrak{im}\, A$.

Proof. If A has a Cartesian decomposition of the form (5.8), then clearly A and B have the same domain. By (2.15) and the symmetry of A_1 and A_2 it follows that

$$A^* = (A_1 + \mathrm{i}\, A_2)^* \supset A_1^* - \mathrm{i}\, A_2^* \supset A_1 - \mathrm{i}\, A_2 = B. \tag{5.10}$$

Hence, $\operatorname{dom} A = \operatorname{dom} B \subset \operatorname{dom} A^*$, so that A is formally domain tight. A similar argument shows that B is formally domain tight. Moreover, (5.10) shows that $B \subset A^*$, which also leads to $A \subset A^{**} \subset B^*$; hence A and B form a dual pair.

To show the first inclusion in (5.9), let $\{f, f_1'\} \in A_1$ with $f \in \operatorname{dom} A$. Then there exists $f_2' \in \mathfrak{H}$ such that $\{f, f_2'\} \in A_2$. Hence, $\{f, f_1' + \mathrm{i}\, f_2'\} \in A$ due to (5.8) and $\{f, f_1' - \mathrm{i}\, f_2'\} \in A^*$ due to (5.10), so that $\{f, f_1'\} \in \mathfrak{re}\, A$. Thus, $A_1 \cap (\operatorname{dom} A \times \mathfrak{H}) \subset \mathfrak{re}\, A$ and then in view of (5.4) the second inclusion in (5.9) follows as well.

The inclusions $A_1 \pm \mathrm{i}\, A_2 \subset \mathfrak{re}\, A \pm \mathrm{i}\, \mathfrak{im}\, A$ follow directly from (5.9). ∎

A formally domain tight relation A satisfies $A \subset \mathfrak{re}\, A + \mathrm{i}\, \mathfrak{im}\, A$; cf. Theorem 5.3. By means of Cartesian decompositions this inclusion can be made more precise, yielding some characterizations of a relation A being formally domain tight.

THEOREM 5.8. *Let A be a relation in a Hilbert space $\mathfrak{H}$ and let the extension A_∞ of A be as defined in* (2.27)*. Then the following conditions are equivalent:*

(i) *A is formally domain tight;*
(ii) *A admits a Cartesian decomposition $A = A_1 + \mathrm{i}\, A_2$ for some symmetric relations A_1 and A_2 in $\mathfrak{H}$;*
(iii) *A_∞ admits the Cartesian decomposition*

$$A_\infty = \mathfrak{re}\, A + \mathrm{i}\, \mathfrak{im}\, A. \tag{5.11}$$

Proof. (ii)⇒(i). This implication follows from Proposition 5.7.

(iii)⇒(i). Since $A \subset A_\infty$ this implication follows from Theorem 5.3. Another approach is that A_∞ is formally domain tight by Proposition 5.7, but then A is formally domain tight by Proposition 2.25.

(i)⇒(iii). Let A be formally domain tight. Then $A \subset \mathfrak{re}\, A + \mathrm{i}\, \mathfrak{im}\, A$ by Theorem 5.3. Furthermore, Proposition 5.1(iii) shows that $\{0\} \times \operatorname{mul} A^* \subset \mathfrak{re}\, A + \mathrm{i}\, \mathfrak{im}\, A$. This yields the inclusion $A_\infty = A \,\widehat{+}\, (\{0\} \times \operatorname{mul} A^*) \subset \mathfrak{re}\, A + \mathrm{i}\, \mathfrak{im}\, A$. Conversely, each element $\{f, g\} \in \mathfrak{re}\, A + \mathrm{i}\, \mathfrak{im}\, A$ is given by

$$\{f, g\} = \left\{ f, \frac{f' + f''}{2} + \mathrm{i}\, \frac{h' - h''}{2\,\mathrm{i}} \right\}, \tag{5.12}$$

where $\{f, f'\}, \{f, h'\} \in A$ and $\{f, f''\}, \{f, h''\} \in A^*$. Then

$$\begin{aligned} 2\{f, g\} &= \{2f, f' + f'' + h' - h''\} = \{f, f'\} + \{f, h'\} + \{0, f'' - h''\} \\ &\in A \,\widehat{+}\, (\{0\} \times \operatorname{mul} A^*). \end{aligned} \tag{5.13}$$

This proves the inclusion $\mathfrak{re}\, A + \mathrm{i}\, \mathfrak{im}\, A \subset A_\infty$.

(i)⇒(ii). By Theorem 3.23, A^* can be decomposed as $A^* = (A^*)_{\mathrm{op}} \,\widehat{+}\, (A^*)_{\mathrm{mul}}$; see also Corollary 3.15. Here $(A^*)_{\mathrm{op}}$ is an operator with $\operatorname{dom} (A^*)_{\mathrm{op}} = \operatorname{dom} A^*$. Now define

$$A_1 \overset{\mathrm{def}}{=} \frac{1}{2}\,(A + (A^*)_{\mathrm{op}}), \qquad A_2 \overset{\mathrm{def}}{=} \frac{1}{2\,\mathrm{i}}\,(A - (A^*)_{\mathrm{op}});$$

cf. (5.1), (5.2). The assumption $\operatorname{dom} A \subset \operatorname{dom} A^*$ shows that $\operatorname{dom} A_1 = \operatorname{dom} A_2 = \operatorname{dom} A$; therefore $A_1 \subset \mathfrak{re}\, A$ and $A_2 \subset \mathfrak{im}\, A$, so that A_1 and A_2 are symmetric relations. The inclusion $A \subset A_1 + \mathrm{i}\, A_2$ can be proved in the same way as the implication (i) ⇒ (ii) in Theorem 5.3, when $\operatorname{dom} A \subset \operatorname{dom} A^* = \operatorname{dom}(A^*)_{\mathrm{op}}$ is used. The reverse inclusion $A_1 + \mathrm{i}\, A_2 \subset A$ can be seen with a similar, but simpler, calculation as used in (5.12), (5.13). Therefore, $A = A_1 + \mathrm{i}\, A_2$. ∎

Domain tight relations can now be characterized via Cartesian decompositions as follows.

THEOREM 5.9. *Let A be a relation in a Hilbert space $\mathfrak{H}$ and let the extension A_∞ be as defined in* (2.27). *Then the following conditions are equivalent:*

(i) *A is domain tight;*

(ii) *A_∞ and $(A^*)_\infty$ admit the Cartesian decompositions*

$$A_\infty = \mathfrak{re}\, A + \mathrm{i}\, \mathfrak{im}\, A, \qquad (A^*)_\infty = \mathfrak{re}\, A - \mathrm{i}\, \mathfrak{im}\, A;$$

(iii) *A and A^* satisfy*

$$A \subset \mathfrak{re}\, A + \mathrm{i}\, \mathfrak{im}\, A, \qquad A^* = \mathfrak{re}\, A - \mathrm{i}\, \mathfrak{im}\, A; \tag{5.14}$$

(iv) *for some symmetric relations A_1 and A_2 in $\mathfrak{H}$ one has*

$$A = A_1 + \mathrm{i}\, A_2, \qquad A^* = \mathfrak{re}\, A - \mathrm{i}\, \mathfrak{im}\, A. \tag{5.15}$$

Proof. (i)⇒(ii). Assume that A is domain tight. Then A and A^* are formally domain tight; cf. Remark 2.24. The first identity in (5.9) holds by (5.11). Since A is domain tight, Lemma 5.4 shows that $\mathfrak{re}\, A^* = \mathfrak{re}\, A$ and $\mathfrak{im}\, A^* = -\,\mathfrak{im}\, A$. Now Theorem 5.8 (applied to A^*) gives the second identity in (5.9):

$$(A^*)_\infty = \mathfrak{re}\, A^* + \mathrm{i}\, \mathfrak{im}\, A^* = \mathfrak{re}\, A - \mathrm{i}\, \mathfrak{im}\, A\cdot$$

(ii)⇒(i). It follows from (5.3) and the Cartesian decompositions in (5.9) that

$$\mathrm{dom}\, A = \mathrm{dom}\, A_\infty = \mathrm{dom}\, A \cap \mathrm{dom}\, A^* = \mathrm{dom}\,(A^*)_\infty = \mathrm{dom}\; A^*,$$

which shows that A is domain tight.

(i)&(ii)⇒(iii). The inclusion in (5.14) is clear from (5.9) as $A \subset A_\infty$. Since A is domain tight, $\mathrm{mul}\, A^{**} = \mathrm{mul}\, A^*$ and therefore $(A^*)_\infty = A^* \mathbin{\widehat{+}} (\{0\} \times \mathrm{mul}\, A^{**}) = A^*$. Thus the second identity in (5.14) is also immediate from (5.9).

(iii)⇒(iv). The inclusion in (5.14) implies that A is formally domain tight. Hence, the first identity in (5.15) is obtained from Theorem 5.8(ii).

(iv)⇒(i). The first identity in (5.15) shows that A is formally domain tight by Theorem 5.8, while the second identity in (5.15) implies that $\mathrm{dom}\, A^* \subset \mathrm{dom}\, \mathfrak{re}\, A = \mathrm{dom}\, \mathfrak{im}\, A = \mathrm{dom}\, A \cap \mathrm{dom}\, A^*$; cf. (5.3). Hence, A is domain tight. ■

In the above characterization some of the conditions do not look symmetric. By turning to a more special class of domain tight relations the description will be more symmetric.

Theorem 5.10. *Let A be a relation in a Hilbert space $\mathfrak{H}$. Then the following conditions are equivalent:*

(i) *A is domain tight and* $\mathrm{mul}\, A = \mathrm{mul}\, A^*$*;*

(ii) *A and A^* admit the Cartesian decompositions*

$$A = \mathfrak{re}\, A + \mathrm{i}\, \mathfrak{im}\, A, \qquad A^* = \mathfrak{re}\, A - \mathrm{i}\, \mathfrak{im}\, A; \tag{5.16}$$

(iii) *A and A^* admit the Cartesian decompositions*

$$A = A_1 + \mathrm{i}\, A_2, \qquad A^* = A_1 - \mathrm{i}\, A_2 \tag{5.17}$$

for some symmetric relations A_1 and A_2 in $\mathfrak{H}$.

Proof. (i)⇒(ii). The assumption $\mathrm{mul}\, A = \mathrm{mul}\, A^*$ implies that $A_\infty = A$. Therefore, the statement follows from (5.9) and (5.15).

(ii)⇒(iii). In (5.16) the relations $\mathfrak{re}\, A$ and $\mathfrak{im}\, A$ are symmetric. Hence, this implication is trivial.

(iii)⇒(i). It is clear from (5.17) that $\mathrm{dom}\, A = \mathrm{dom}\, A_1 \cap \mathrm{dom}\, A_2 = \mathrm{dom}\, A^*$ and $\mathrm{mul}\, A = \mathrm{mul}\, A_1 + \mathrm{mul}\, A_2 = \mathrm{mul}\, A^*$. ■

The special domain tight relations in Theorem 5.10 can also be characterized by means of decomposable domain tight relations; cf. Proposition 4.14.

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