# Contents

## Abstract

We investigate Hartman functions on a topological group $G$. Recall that $(\iota, C)$ is a group compactification of $G$ if $C$ is a compact group, $\iota : G \to C$ is a continuous group homomorphism and $\iota(G) \subseteq C$ is dense. A bounded function $f : G \to \mathbb{C}$ is a Hartman function if there exists a group compactification $(\iota, C)$ and $F : C \to \mathbb{C}$ such that $f = F \circ \iota$ and $F$ is Riemann integrable, i.e. the set of discontinuities of $F$ is a null set with respect to the Haar measure. In particular, we determine how large a compactification for a given group $G$ and a Hartman function $f : G \to \mathbb{C}$ must be to admit a Riemann integrable representation of $f$. The connection to (weakly) almost periodic functions is investigated.

In order to give a systematic presentation which is self-contained to a reasonable extent, we include several separate sections on the underlying concepts such as finitely additive measures on Boolean set algebras, means on algebras of functions, integration on compact spaces, compactifications of groups and semigroups, the Riemann integral on abstract spaces, invariance of measures and means, continuous extensions of transformations and operations to compactifications, etc.

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# 1. Introduction

**1.1. Motivation.** By a topological dynamical system $(X, T)$ we mean a continuous transformation $T : X \to X$ acting on a compact space $X$ (which in many cases is supposed to be metrizable). Symbolic dynamics is concerned with the special case $X = A^{\mathbb{N}}$ or $X = A^{\mathbb{Z}}$ with a finite set $A$, called the alphabet. Here the transformation is the shift $T = \sigma : (a_n) \in X \mapsto (a_{n+1}) \in X$. The importance of this special case is due to the fact that, for a suitable finite partition (Markov partition) $X = X_1 \cup \dots \cup X_n$ of a metrizable space $X$ and the alphabet $A = \{1, \dots, n\}$, most information on the original system $(X, T)$ is contained in the associated symbolic system which is defined below.

Consider the coding $F : X \to A$, $F(x) = i$ if $x \in X_i$. Let $\varphi : X \to A^{\mathbb{N}}$, $x \mapsto (T^n x)_{n\in\mathbb{N}}$ or, if $T$ is bijective, $\varphi : X \to A^{\mathbb{Z}}$, $x \mapsto (T^n x)_{n\in\mathbb{Z}}$. The case of bijective $T$ applies for the major part of the exposition. The associated dynamical system $(Y, \sigma)$ with $Y = \overline{\varphi(X)}$ is a subshift, i.e. $Y$ is a closed and $\sigma$-invariant subset of $A^{\mathbb{Z}}$. The connection between $(X, T)$ and $(Y, \sigma)$ is expressed by the commuting diagram

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ Y & \xrightarrow{\sigma} & Y \end{array}$$

If $\varphi$ is continuous this means that $(Y, \sigma)$ is a factor of $(X, T)$. However, this can be guaranteed only if the $X_i$ are clopen subsets of $X$, which, for instance for connected $X$, is impossible. The classical way of avoiding this disadvantage is to choose the partition in such a way that $\varphi$ is injective and $\varphi^{-1}$ has a continuous extension $\psi$ such that $(X, T)$ is a factor of $(Y, T)$:

$$\begin{array}{ccc} Y & \xrightarrow{\sigma} & Y \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} \\ X & \xrightarrow{T} & X \end{array}$$

In order to apply results from ergodic theory (such as Birkhoff's Theorem) one looks for invariant measures. Assume that $\mu$ is a $\sigma$-invariant measure on $Y$, i.e. $\mu(\sigma^{-1}[B]) = \mu(B)$ for all Borel sets $B \subseteq Y$. Then $\mu_T(M) := \mu(T^{-1}[M])$ defines a $T$-invariant measure $\mu_T$ on $X$.

The situation is particularly nice if $T$ is uniquely ergodic, i.e. if there is a unique $T$-invariant Borel measure. In this case the limit relation

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{n-1} f(T^n x) = \int_X f \, d\mu_T \tag{1.1}$$

does hold not only up to a set of zero $\mu_T$-measure, but even uniformly for all $x \in X$ whenever $f : X \to \mathbb{R}$ is continuous and bounded. By obvious approximation this statement extends to all bounded $f : X \to \mathbb{R}$ with

$$\forall \varepsilon > 0\ \exists f_1, f_2 : X \to \mathbb{R} \text{ continuous, } f_1 \leq f \leq f_2, \qquad \int_X (f_2 - f_1)\, d\mu_T < \varepsilon. \tag{1.2}$$

In the case $X = [0,1]$, equipped with the Lebesgue measure, (1.2) is equivalent to the requirement $\mu_T(\mathrm{disc}) = 0$, i.e. that the set $\mathrm{disc}(f)$ of discontinuity points of $f$ is a null set. In other words, $f$ is Riemann integrable. If $f$ takes only finitely many values $r_1, \ldots, r_s$ this condition is equivalent to $\mu_T(\partial X_i) = 0$ for the topological boundary of $X_i := f^{-1}[\{r_i\}]$, $i = 1, \ldots, s$. Indeed, this condition is usually assumed for partitions in the context of symbolic dynamics. In this paper we allow $F : X \to \mathbb{C}$ to have infinitely many values, but, motivated by the above considerations, we assume that $F$ is Riemann integrable.

A very important class of uniquely ergodic systems are group rotations, i.e. $T : C \to C$, $x \mapsto x + g$, where $g \in C$ is a topological generator of the compact (abelian) group $C$, meaning that the cyclic group generated by $g$ is dense in $C$. The unique invariant measure for the transformation $T$ is given by the Haar measure $\mu_C$ on $C$. The induced coding sequences $(a_n)_{n\in\mathbb{Z}}$ are given by $a_n = F(x + ng)$ and may be used to form a factor of $(\iota, C)$. Indeed, if we consider the mapping $\iota : \mathbb{Z} \to C$, $n \mapsto ng$, we have $a = F \circ \iota$. $(\iota, C)$ is a group compactification of $\mathbb{Z}$ since $\iota$ is a (trivial) continuous group homomorphism with image $\iota(\mathbb{Z})$ dense in $C$. Allowing $\mathbb{Z}$ to be replaced by an arbitrary topological group, we finally arrive at the definition of Hartman functions, the main objects of our paper:

A function $f : G \to \mathbb{C}$ on a topological group $G$ is called a Hartman function if there is a group compactification $(\iota, C)$ of $G$ and a function $F : C \to \mathbb{C}$ which is Riemann integrable with respect to the Haar measure and satisfies $f = F \circ \iota$; $F$ is called a representation of $(\iota, C)$.

In particular, almost periodic functions (defined by continuous $F$) are Hartman functions. The name *Hartman function* refers to the Polish mathematician Stanisław Hartman who was, up to our knowledge, the first to consider these objects in the 1960s in his work in harmonic analysis [19, 20, 21]. He focused on the Bohr compactification $(\iota_b, bG)$ of the group $G$. It is not difficult to see that our definition is equivalent to the analogous requirement for $(\iota, C) = (\iota_b, bG)$. The question whether for a given Hartman function $f$, there are small compactifications with a representation $f = F \circ \iota$ is one of our major topics.

Additionally we investigate the connection of Hartman functions and weak almost periodicity. Recall that a function is weakly almost periodic if it has a continuous representation in a semitopological semigroup compactification, or, equivalently, in the weak almost periodic compactification $(\iota_w, wG)$. While every almost periodic function is Hartman, this is not true in the weak case. A more systematic overview of the content of this paper is given at the end of this section.

**1.2. Recent results on Hartman sets, sequences and functions.** For an extended survey on recent research on Hartman sets, Hartman sequences and Hartman functions we refer to [57]. Here we only give a very brief summary.

The series of papers we report on was initiated by investigations of M. Paštéka and R. F. Tichy [31, 32, 33] on the distribution of sequences induced by the algebraic structure in commutative rings $R$. The authors used the completion $\overline{R}$ with respect to a natural metric structure such that $R$ is compact and thus carries a Haar measure $\mu$. The restriction of $\mu$ to the $\mu$-continuity sets $M$, i.e. to those sets with $\mu(\partial M) = 0$ has been pulled back in order to obtain a natural concept of uniform distribution in the original structure $R$.

One easily observes that the measure-theoretic part of the construction depends only on the additive group structure of $R$. Thus the natural framework for a systematic investigation is that of group compactifications $(\iota, C)$ of a topological group $G$ and of the finitely additive measure $\mu_{(\iota,C)}$ on $G$ defined for $\iota$-preimages of $\mu$-continuity sets as follows:

$$\mu_{(\iota,C)}(\iota^{-1}[M]) := \mu(M), \quad M \subseteq C. \tag{1.3}$$

This has been studied in [12]. Results for the special case $G = \mathbb{Z}$ are presented in [44, 45]: Hartman sets $\iota^{-1}[M] \subseteq \mathbb{Z}$ are identified with the functions $\mathbb{1}_{\iota^{-1}[M]} : \mathbb{Z} \to \{0, 1\}$ and called Hartman sequences. The relation to Beatty resp. Sturmian sequences and continued fractions expansion is described. It is shown that the system of Hartman sequences is generated by the system of Beatty sequences by means of Boolean combinations and approximation in measure.

The connection to ergodic theory already mentioned in [45] is stressed further in [56]: Hartman sequences can be considered as symbolic coding sequences of group rotations (as described in the previous section). The problem to identify the underlying dynamical system turns out to be equivalent to the identification of the group compactification $(\iota, C)$ of $\mathbb{Z}$ inducing the Hartman set $\iota^{-1}[M] \subseteq \mathbb{Z}$. As an alternative to classical methods such as spectral analysis of the dynamical system, a purely topological method has been presented. Each Hartman set $\iota^{-1}[M] \subseteq \mathbb{Z}$ defines in a natural way a filter on $\mathbb{Z}$. Under rather mild assumptions this filter is the $\iota$-preimage of the neighborhood filter $\mathfrak{U}(0_C)$ of the identity in $C$ and contains all necessary information about $(\iota, C)$.

These methods have been applied to questions from number theory in [2] and generalized to the setting of topological groups in [3].

The aspect of symbolic dynamics has been studied further in [47] by investigation of subword complexity of Hartman sequences. Recall that the subword complexity $p_a : \mathbb{N} \to \mathbb{N}$ induced by the sequence $a \in \{0, 1\}^{\mathbb{Z}}$ is a function associating to each $n \in \mathbb{N}$ the number of different 0-1 blocks of length $n$ occurring in $a$. Clearly $1 \leq p_a(n) \leq 2^n$. The main facts in this context are:

1. $\lim_{n\to\infty} n^{-1} \log p_a(n) = 0$, corresponding to the fact that group rotations have entropy 0.
2. Whenever $\lim_{n\to\infty} n^{-1} \log p_n = 0$ for a sequence $p_n$ with $1 \leq p_n \leq 2^n$, then there is a Hartman sequence $a$ with $p_n(a) \geq p_n$ for every $n \in \mathbb{N}$.
3. The Hartman sequence $a = \mathbb{1}_{\iota^{-1}[M]}$, where $M \subseteq \mathbb{T}^s$, an $s$-dimensional cube, satisfies $p_a(n) \sim c_M \cdot n^s$ with an explicit constant $c_M > 0$ (we omit the number-theoretic assumptions).

An amazing geometric interpretation of the constant $c_M$ was recently given in [46], where statement 3 has been generalized to convex polygons $M$ and $c_M$ corresponds to the volume of the projection body of $M$.

The investigation of Hartman functions has been started in [27] where, for instance, results from [56] on Hartman sequences have been generalized. In the present paper we continue these investigations and include a systematic and considerably self-contained treatment of the topological and measure-theoretic background.

**1.3. Content of the paper.** Chapter 2 presents measure-theoretic and topological preliminaries. Section 2.1 fixes notation concerning (Boolean) set algebras and related algebras of functions. In Section 2.2 we investigate finitely additive measures on set algebras and the integration of functions from corresponding function algebras. Then we present the connection between measures and means. One of the most fundamental phenomena in analysis is that compactness is used to obtain $\sigma$-additivity of measures and thus makes Lebesgue's integration theory work. Riesz' Representation Theorem plays a crucial rôle in this context; we recall it in Section 2.3. If compactness is absent one can try to force it by considering compactifications. In Section 2.4 we construct compactifications in such a way that a given set of bounded functions admits continuous extensions. We touch the classical representation theorems of Gelfand and Stone. Among all compactifications of a given (completely regular) topological space $X$ there is a, in a natural sense, maximal compactification, the Stone–Čech compactification $(\iota_\beta, \beta X)$. In Section 2.5 we collect its important properties. Having presented the basics concerning compactifications, measures, means and the Riemann integral, we put these concepts together in Section 2.6. Section 2.7, the last one in Chapter 2, presents the interpretation of the Stone–Čech compactification $(\iota_\beta, \beta X)$ of a discrete space $X$ as the set of all multiplicative means on $X$. This motivates us to investigate means with more restrictive properties, such as invariance.

Chapter 3 is concerned with invariance of measures and means under transformations and operations. In particular, we investigate in Section 3.1 questions of existence and uniqueness. For a transformation $T : X \to X$ invariance is closely related to the behavior of Cesàro means along $T$-orbits, a concept which leads to the notion of Banach density. In Section 3.2 we treat several examples and applications: finite $X$, $X = \mathbb{Z}$ and $T : x \mapsto x + 1$, compact $X$ and continuous $T : X \to X$, shift spaces and symbolic dynamics, the free group generated by two elements. In Section 3.3 we consider compactifications under the additional aspect of extending transformations and (semi)group actions in a continuous way. For binary or, more generally, $n$-ary, operations continuous extensions do not always exist. The arising problems are treated in Section 3.4. In particular, $n$-ary operations on $X$, $n \geq 2$, can be continuously extended to $(\iota_\beta, \beta X)$ only in very special cases. Nevertheless, it is useful to formulate a general framework in order to unify the most interesting classical situations: topological and semitopological group and semigroup compactifications. This is done in Section 3.5. In Section 3.6 these constructions are discussed in the context of invariant means and measures. We mention the notion of weak almost periodicity and touch amenable groups and semigroups very briefly.

Chapter 4 develops the basic theory of Hartman functions. Section 4.1 presents several equivalent conditions describing the connection with almost periodicity and the Bohr compactification, i.e. the maximal group compactification. Replacing group compactifications by semitopological semigroup compactifications one obtains the weak almost periodic compactification, weak almost periodic functions and weak Hartman functions. This is presented in Section 4.2. The category of all group compactifications of a topological group $G$ is particularly well understood if $G$ is abelian and carries a locally compact group topology. The key ingredient is Pontryagin's Duality Theorem. We recall this situation in Section 4.3. One of the most interesting questions concerning a Hartman function $f : G \to \mathbb{C}$ is how small a group compactification $(\iota, C)$ can be taken if one asks for a Riemann integrable representation of $f$. This question is treated in Section 4.4. We give an answer for LCA groups in terms of the minimal cardinality of a dense subgroup in the Pontryagin dual $\hat{G}$ of $G$.

Chapter 5 is devoted to the comparison of Hartman functions and weakly almost periodic functions. It turns out that a generalization of what is called a jump discontinuity in basic analysis plays an important rôle. Generalized jump discontinuities are established in Section 5.1 and used in Section 5.2 to give necessary conditions of weak almost periodicity of Hartman functions. This leads to the investigation of Hartman functions without such generalized jumps in Section 5.3. Hartman functions with small support are treated in 5.4. Finally, Section 5.5 discusses particular examples of Hartman functions on the integers which are neither almost periodic nor converge to 0. The results use the Fourier–Stieltjes transform of measures.

Finally, a short summary is given, including a diagram which illustrates the relation between several spaces of functions which are interesting in our context.

## 2. Measure-theoretic and topological preliminaries

**2.1. Set algebras $\mathfrak{A}$ and $\mathfrak{A}$-functions.** We start by fixing notation which is suitable to imitate the construction of the Riemann integral in the slightly more general context which will be ours.

Definition 2.1.1. A (boolean) *set algebra* $\mathfrak{A}$ (on a set $X$) is a system of subsets of $X$ with $\emptyset, X \in \mathfrak{A}$ for which $A, B \in \mathfrak{A}$ implies $A \cup B, A \cap B, X \setminus A \in \mathfrak{A}$.

Example 2.1.2. Let $X = [0, 1] \subseteq \mathbb{R}$ be the unit interval and $\mathfrak{A} = \mathfrak{A}([0, 1])$ the system of all finite unions of subintervals $I \subseteq [0, 1]$ (open, closed and one-sided closed, also including singletons and the empty set). This is the most classical situation. But it is worth noting that we might replace $[0, 1]$ by any totally ordered $X$, for instance by any $D \subseteq [0, 1]$ dense in $[0, 1]$ (as $D = \mathbb{Q} \cap [0, 1]$).

We are interested in integration of complex-valued functions on $X$:

Definition 2.1.3. Let $\mathcal{A}$ be a set of functions $f : X \to \mathbb{C}$. We call the subset $\mathcal{A}_{\mathbb{R}}$ of all $f \in \mathcal{A}$ with $f(X) \subseteq \mathbb{R}$ the *real part* of $\mathcal{A}$. $\mathcal{A}$ is called *real* if $\mathcal{A}_{\mathbb{R}} = \mathcal{A}$. If $\mathcal{A}$ is a vector space or an algebra over $\mathbb{R}$ (or $\mathbb{C}$) we call $\mathcal{A}$ a *real* (or *complex*) *space* resp. a *real* (or *complex*) *algebra of functions*. For any $A \subseteq X$ let $\mathbb{1}_A(x) = 1$ for $x \in A$ and $\mathbb{1}_A(x) = 0$ for $x \in X \setminus A$. For an algebra $\mathcal{A}$ we always assume $\mathbb{1}_X \in \mathcal{A}$. A complex space or algebra $\mathcal{A}$ of functions is called a $*$-*space* resp. a $*$-*algebra* if $f \in \mathcal{A}$ implies $\bar{f} \in \mathcal{A}$ for the complex conjugate $\bar{f}$ of the function $f$. We write $B(X)$ for the set of all bounded $f : X \to \mathbb{C}$, $B_{\mathbb{R}}(X) := B(X)_{\mathbb{R}}$ for its real part. (Later we will also use the notation $\mathcal{B}$ for the Fourier–Stieltjes algebra.) A $*$-algebra $\mathcal{A}$ on $X$ which is complete with respect to the topology of uniform convergence on $X$ is called a *$C^*$-algebra*.

Note that whenever $\mathcal{A}$ is a real space we can form the complexification $\mathcal{A}_{\mathbb{C}} = \{f_1+if_2 : f_1, f_2 \in \mathcal{A}\}$ which is a complex vector space, and a $*$-algebra whenever $\mathcal{A}$ is a real algebra. For any complex linear space or algebra $\mathcal{A}$, to be a $*$-space resp. a $*$-algebra is equivalent to the following property: Whenever $f = f_1 + if_2$ is the decomposition of $f$ into the real part $f_1$ and imaginary part $f_2$, then $f \in \mathcal{A}$ if and only if $f_1, f_2 \in \mathcal{A}_{\mathbb{R}}$. Thus for the investigation of $*$-algebras $\mathcal{A}$ it suffices to investigate the real part $\mathcal{A}_{\mathbb{R}}$ whenever convenient. Furthermore, any $C^*$-algebra of bounded functions is closed under taking absolute values: $|f| = \sqrt{f\bar{f}}$, a fact which can be seen by approximating the square root by polynomials.

Definition 2.1.4. Let $\mathfrak{A}$ be a set algebra on $X$. A function $f : X \to \mathbb{C}$ is called $\mathfrak{A}$-*simple* if it has a representation

$$f = \sum_{i=1}^{n} c_i \mathbb{1}_{A_i}$$

with $A_i \in \mathfrak{A}$ and $c_i \in \mathbb{C}$. The set of all $\mathcal{A}$-simple $f$ is denoted by $\mathcal{S}_{\mathfrak{A}}$. We denote the uniform closure $\overline{\mathcal{S}_{\mathfrak{A}}}$ of $\mathcal{S}_{\mathfrak{A}}$ by $B(\mathfrak{A})$. Members of $B(\mathfrak{A})$ are also called $\mathfrak{A}$-*functions*.

More explicitly, for a set algebra $\mathfrak{A}$ on $X$ the function $f : X \to \mathbb{C}$ lies in $B(\mathfrak{A})$ if and only if for all $\varepsilon > 0$ there is an $f' \in \mathcal{S}_{\mathfrak{A}}$ with $|f(x) - f'(x)| < \varepsilon$ for all $x \in X$.

Proposition 2.1.5. *All the sets $\mathcal{S}_{\mathfrak{A}} \subseteq B(\mathfrak{A}) \subseteq B(X)$ are $*$-algebras. In general, the inclusions are strict.*

*Proof.* It is clear that $\mathcal{S}_{\mathfrak{A}}$, $B(\mathfrak{A})$ and $B(X)$ are $*$-algebras satisfying the stated inclusions. Thus it suffices to show that $\mathcal{S}_{\mathfrak{A}} \neq B(\mathfrak{A}) \neq B(X)$ if one takes $X = [0,1]$ and $\mathfrak{A} = \mathfrak{A}([0,1])$, the set algebra of all finite unions of subintervals of $[0,1]$. Then $f \in C(X) \subseteq B(\mathfrak{A})$ but $f \notin \mathcal{S}_{\mathfrak{A}}$ if we take $f(x) = x$, hence $\mathcal{S}_{\mathfrak{A}} \neq B(\mathfrak{A})$. On the other hand, all $f \in B(\mathfrak{A})$ are Riemann integrable in the classical sense, which is not the case for arbitrary $f \in B(X)$. ∎

For every set algebra $\mathfrak{A}$, $B(\mathfrak{A})$ is a $C^*$-algebra. But not every $C^*$-algebra $\mathcal{A}$ can be written as $\mathcal{A} = B(\mathfrak{A})$ for an appropriate $\mathfrak{A}$. The situation is explained by the following facts.

Proposition 2.1.6. *For a set $\mathcal{A}$ of complex-valued functions $f : X \to \mathbb{C}$ define $\mathfrak{A}_{\mathcal{A}} := \{A \subseteq X : \mathbb{1}_A \in \mathcal{A}\}$. Then:*

(i) *$\mathfrak{A}_{\mathcal{A}}$ is a set algebra whenever $\mathcal{A}$ is an algebra.*
(ii) *Every set algebra $\mathfrak{A}$ on $X$ satisfies $\mathfrak{A} = \mathfrak{A}_{B(\mathfrak{A})}$.*
(iii) *For every uniformly closed algebra $\mathcal{A}$ one has $B(\mathfrak{A}_{\mathcal{A}}) \subseteq \mathcal{A}$, while the converse inclusion does not hold in general.*

*Proof.* (i) Follows from $\mathbb{1}_X \in \mathcal{A}$, $\mathbb{1}_{A_1 \cap A_2} = \mathbb{1}_{A_1} \cdot \mathbb{1}_{A_2}$, $\mathbb{1}_{X \setminus A} = \mathbb{1}_X - \mathbb{1}_A$ and the identity $A_1 \cup A_2 = X \setminus ((X \setminus A_1) \cap (X \setminus A_2))$.

(ii) The inclusion $\mathfrak{A} \subseteq \mathfrak{A}_{B(\mathfrak{A})}$ is obvious. For the converse assume $A \in \mathfrak{A}_{B(\mathfrak{A})}$, i.e. $\mathbb{1}_A \in B(\mathfrak{A})$. Then there are $f_n \in \mathcal{S}_{\mathfrak{A}}$ uniformly converging to $\mathbb{1}_A$. There are representations $f_n = \sum_{i=1}^{k_n} \alpha_{i,n} \mathbb{1}_{A_{n,i}}$ such that for each $n$ the $A_{n,i} \in \mathfrak{A}$, $i = 1, \ldots, k_n$, are pairwise disjoint. For sufficiently large fixed $n$, each $x \in X$ satisfies either $|f_n(x) - 1| < 1/2$ (if $x \in A$) or $|f_n(x)| < 1/2$ (if $x \notin A$). This shows that $A_{n,i} \subseteq A$ or $A_{n,i} \subseteq X \setminus A$ for any such fixed $n$ and all $i = 1, \ldots, k_n$, hence $A = \bigcup_{i:A_{n,i} \subseteq A} A_{n,i} \in \mathfrak{A}$.

(iii) The stated inclusion is obvious. The example $\mathcal{A} = C([0,1])$, $\mathfrak{A}_{\mathcal{A}} = \{\emptyset, X\}$, $B(\mathfrak{A}_{\mathcal{A}}) = \{c\mathbb{1}_X : c \in \mathbb{C}\}$ shows that the inclusion may be strict. ■

## 2.2. Finitely additive measures and means

Definition 2.2.1. Let $\mathfrak{A}$ be a set algebra on $X$. A function $p : \mathfrak{A} \to [0, \infty]$ with $p(\emptyset) = 0$ is called a *finitely additive measure*, briefly *fam* (on $X$ or, more precisely, on $\mathfrak{A}$) if it is finitely additive, i.e. if $p(A_1 \cup A_2) = p(A_1) + p(A_2)$ whenever $A_1 \cap A_2 = \emptyset$. $p$ is called a *finitely additive probability measure*, briefly *fapm*, if furthermore $p(X) = 1$.

Example 2.2.2. Continuing Example 2.1.2, for $X = [0,1]$ and $\mathfrak{A} = \mathfrak{A}([0,1])$, the system of all finite unions of intervals, one takes $p(I) = b - a$ for $I = [a,b]$ with $0 \le a \le b \le 1$. This definition uniquely extends to a fapm on the set algebra $\mathfrak{A}([0,1])$ of all finite unions of intervals. We will refer to this $p$ as the *natural measure*. The construction does not depend on the completeness (compactness) of $[0,1]$ and hence can be done as well for dense subsets $D \subset X$. For instance, one could consider (finite unions of) intervals of rationals.

Definition 2.2.3. Let $\mathcal{A}$ be a linear space of functions on a set $X$. Then a *mean* $m$ on $\mathcal{A}$ is a linear functional $m : \mathcal{A} \to \mathbb{C}$ which is *positive*, i.e. $f \ge 0$ implies $m(f) \ge 0$, and satisfies $m(\mathbb{1}_X) = 1$.

Note that whenever $\mathcal{A}$ is real and $m$ is a mean on $\mathcal{A}$ then $\overline{m}(f_1 + if_2) := m(f_1) + im(f_2)$ for $f_1, f_2 \in \mathcal{A}$ is the unique extension of $m$ to the complexification $\mathcal{A}_{\mathbb{C}}$ of $\mathcal{A}$. Very often we simply write $m$ for $\overline{m}$.

For real functions $f$ every mean $m$, by positivity, satisfies $\inf f \le m(f) \le \sup f$. As a consequence we have:

Proposition 2.2.4. *Every mean $m$ on $\mathcal{A}$ is continuous with respect to the norm $\|f\|_\infty := \sup_{x \in X} |f(x)|$ and thus has a unique extension to the uniform closure $\bar{\mathcal{A}}$ of $\mathcal{A}$.*

Every mean induces a further notion of closure:

Definition 2.2.5. Let $m$ be a mean on a linear space $\mathcal{A}$ of functions on $X$. Then the *real $m$-closure* $\overline{\mathcal{A}}_{\mathbb{R}}^{(m)}$ of $\mathcal{A}$ is the set of all $f : X \to \mathbb{R}$ such that for all $\varepsilon > 0$ there are $f_1, f_2 \in \mathcal{A}_{\mathbb{R}}$ with $f_1 \le f \le f_2$ and $m(f_2 - f_1) < \varepsilon$. For $f \in \overline{\mathcal{A}}_{\mathbb{R}}^{(m)}$, $\overline{m}(f)$ is defined to be the *unique* value $\alpha \in \mathbb{R}$ with $m(f_1) \le \alpha \le m(f_2)$ for all $f_1, f_2 \in \mathcal{A}$ with $f_1 \le f \le f_2$. The (complex) *$m$-closure* $\overline{\mathcal{A}}^{(m)}$ is the set of all $f = f_1 + if_2$ with $f_1, f_2 \in \overline{\mathcal{A}}_{\mathbb{R}}^{(m)}$. Furthermore, we define $\overline{m}(f) := \overline{m}(f_1) + i\overline{m}(f_2)$ for such $f = f_1 + if_2$. $\overline{m}$ is called the *completion* of $m$, sometimes also simply denoted by $m$. In the case $\overline{\mathcal{A}}^{(m)} = \mathcal{A}$ we call $m$ *complete* and $\mathcal{A}$ *$m$-closed*.

REMARK 2.2.6. Distinguish the $m$-closure from the completion with respect to the pseudo-metric $d_m(f,g) := m(|f-g|)$. By definition ($m$ is continuous with respect to $d_m$) the $m$-closure is always contained in the $d_m$-completion: $\overline{\mathcal{A}}^{(m)} \subseteq \overline{\mathcal{A}}^{(d_m)}$. The closure with respect to $m$ corresponds to the integral in the sense of Riemann, the completion with respect to $d_m$ to that of Lebesgue (modulo null-sets).

Every fapm $p$ defined on a set algebra $\mathfrak{A}$ on a set $X$ induces a linear functional $m_p$ in the natural way. Standard arguments (using the fact that $\mathfrak{A}$ is closed under intersections and that $p$ is finitely additive) show that for an $\mathfrak{A}$-simple $f = \sum_{i=1}^n c_i \mathbb{1}_{A_i} \in \mathcal{S}_{\mathfrak{A}}$ the value

$$m_p(f) = m_p\Big(\sum_{i=1}^n c_i \mathbb{1}_{A_i}\Big) := \sum_{i=1}^n c_i p(A_i)$$

does not depend on this particular representation of $f$ as a linear combination. Obviously this $m_p$ is a mean on $\mathcal{S}_{\mathfrak{A}}$ and thus, by Proposition 2.2.4, has a unique extension to the algebra $B(\mathfrak{A}) = \overline{\mathcal{S}}_{\mathfrak{A}}$ as well as to $\overline{\mathcal{S}}_{\mathfrak{A}}^{(m_p)}$.

We want to extend the domain of $m_p$ from $\mathcal{S}_{\mathfrak{A}}$ to the space $\mathcal{I}_p$ defined as follows.

DEFINITION 2.2.7. For a given fapm $p$ on $\mathfrak{A}$ let $\mathcal{I}_p := \overline{\mathcal{S}}_{\mathfrak{A}}^{(m_p)}$. The members $f \in \mathcal{I}_p$ are called *integrable* (with respect to $p$). The extension of $m_p$ to $\mathcal{I}_p$, usually also denoted by $m_p$, is called the *mean induced by p*.

We leave the proof of the following easy properties to the reader:

PROPOSITION 2.2.8. *Let $\mathfrak{A}$ be a set algebra on the set $X$, and $p$ a fapm defined on $\mathfrak{A}$. Then $B(\mathfrak{A}) \subseteq \mathcal{I}_p \subseteq B(X)$, $\mathcal{I}_p$ is $m_p$-closed and $m_p$ is a mean on $\mathcal{I}_p$.*

REMARK 2.2.9. $\mathcal{I}_p$ is uniformly closed. In particular, $\mathcal{I}_p$ is a $C^*$-algebra. Indeed, let $f_n \to f$ uniformly where $f_n \in \mathcal{I}_p$. For given $\varepsilon > 0$ there exists $f_n$ such that $\|f - f_n\|_\infty \leq \varepsilon/4$ and $f_{n,1}, f_{n,2} \in B(\mathfrak{A})$ such that $f_{n,1} \leq f_n \leq f_{n,2}$ and $m_p(f_{n,2} - f_{n,1}) \leq \varepsilon/2$. Observe that

$$f_{n,1} - \varepsilon/4 \leq f_n - \varepsilon/4 \leq f \leq f_n + \varepsilon/4 \leq f_{n,2} + \varepsilon/4$$

and thus $m_p((f_{n,2} + \varepsilon/4) - (f_{n,1} - \varepsilon/4)) \leq \varepsilon$ shows $f \in \mathcal{I}_p$.

The inclusions stated in Proposition 2.2.8 are in general strict as the following example shows.

EXAMPLE 2.2.10. Let again $\mathfrak{A} = \mathfrak{A}([0,1])$ be the set algebra of all finite unions of subintervals of $X = [0,1]$, $p$ the natural measure on $\mathfrak{A}$. Then $\mathcal{I}_p$ is the set of all $f : [0,1] \to \mathbb{C}$ which are integrable in the classical Riemann sense, thus a proper subset of $B(X)$. Consider $f := \mathbb{1}_C$ where $C = \{\sum_{n=1}^\infty a_n/3^n : a_n \in \{0,2\}\}$ is Cantor's middle third set. Then $f \in \mathcal{I}_p$, but $f \notin B(\mathfrak{A})$: $f \in B(\mathfrak{A})$ would yield the existence of $f_1 = \sum_{i=1}^n c_i \mathbb{1}_{A_i} \in \overline{\mathcal{S}_{\mathfrak{A}}}$ with $A_i \in \mathfrak{A}$, $c_i \in \mathbb{C}$ and $\|f - f_1\|_\infty < 1/2$. We may assume that the $A_i$ are pairwise disjoint. Consider $A := \bigcup_{i:|c_i - 1|<1/2} A_i \in \mathfrak{A}$ and $f_2 := \mathbb{1}_A \in \mathcal{S}_{\mathfrak{A}}$. Then $\|f - f_2\|_\infty < 1/2$, which, since $f$ and $f_2$ only take the values 0 and 1, implies $f = f_2$ and $C = A$, a finite union of intervals, a contradiction.

We have seen that each fapm $p$ on a set algebra in a natural way induces a mean $m$ on the $C^*$-algebra $\mathcal{I}_p$. Recall from the first statement in Proposition 2.1.6 that $\mathfrak{A}_{\mathcal{A}} := \{A \subset X : \mathbb{1}_A \in \mathcal{A}\}$ is a set algebra whenever $\mathcal{A}$ is an algebra of functions. Given a mean $m$ on $\mathcal{A}$, $p_m(A) := m(\mathbb{1}_A)$ clearly defines a fapm on $\mathfrak{A}_m := \mathfrak{A}_{\mathcal{A}}$. We ask whether the constructions $\varphi : (\mathfrak{A}, p) \mapsto (\mathcal{I}_p, m_p)$ and $\psi : (\mathcal{A}, m) \mapsto (\mathfrak{A}_m, p_m)$ are inverse to each other. In general this is not the case.

EXAMPLE 2.2.11. Consider any algebra $\mathcal{A}$ of continuous functions on a nontrivial connected space $X$ (for instance $X = [0, 1]$) containing functions which are not constant, and any nontrivial mean $m$ on $\mathcal{A}$. Then $\mathfrak{A}_{\mathcal{A}} = \{\emptyset, X\}$ and hence $\mathcal{I}_{p_m}$ only contains the constant functions and does not coincide with $\mathcal{A}$.

However, this is not surprising if we note that $\mathcal{A}$ in the above example is not $m$-closed, while $\mathcal{I}_p$ is $m_p$-closed. Thus we have to assume this property for all function algebras and means, and to use the analogous property for fapm's.

DEFINITION 2.2.12. Consider a fapm $p$ on a set algebra $\mathfrak{A}$ on the set $X$. Then the *$p$-completion* $\overline{\mathfrak{A}}^{(p)}$ of $\mathfrak{A}$ is defined as the set of all $A \subseteq X$ with the following property: For each $\varepsilon > 0$ there are $A_1, A_2 \in \mathfrak{A}$ with $A_1 \subseteq A \subseteq A_2$ and $p(A_2 \setminus A_1) < \varepsilon$. For $A \in \overline{\mathfrak{A}}^{(p)}$ we define $p(A)$ to be the unique $\alpha$ with $p(A_1) \leq \alpha \leq p(A_2)$ for all $A_1, A_2 \in \mathfrak{A}$ with $A_1 \subseteq A \subseteq A_2$. In this way we canonically extend $p$ to all of $\overline{\mathfrak{A}}^{(p)}$. In the case $\overline{\mathfrak{A}}^{(p)} = \mathfrak{A}$ we call $p$ *complete* and $\mathfrak{A}$ *$p$-closed.*

It is clear that the $p$-completion of a set algebra is again a set algebra. Note furthermore that for $p$ $\sigma$-additive the notion coincides with the usual concept of a complete measure.

PROPOSITION 2.2.13. *Let $\mathfrak{A}$ be a set algebra on $X$ and $p$ a fapm on $\mathfrak{A}$.*

(i) $\mathfrak{A} \subseteq \mathfrak{A}_{m_p}$ *and* $p_{m_p}(A) = p(A)$ *whenever* $A \in \mathfrak{A}$.

(ii) $\overline{\mathfrak{A}}^{(p)} = \mathfrak{A}_{m_p}$. *In particular, the equality $\mathfrak{A} = \mathfrak{A}_{m_p}$ holds if and only if $\mathfrak{A}$ is $p$-closed.*

*Proof.* (i) is obvious. To prove (ii) assume first that $A \in \mathfrak{A}_{m_p}$ and pick any $\varepsilon > 0$. Then $\mathbb{1}_A \in \mathcal{I}_p$ by definition of $\mathfrak{A}_{m_p}$. By definition of $\mathcal{I}_p$ this means that there are $f_1, f_2 \in \mathcal{S}_{\mathfrak{A}}$ such that $f_1 \leq \mathbb{1}_A \leq f_2$ and $m_p(f_2 - f_1) < \varepsilon$. There is a representation $f_2 - f_1 = \sum_{i=1}^n c_i \mathbb{1}_{A_i}$ such that the $A_i$ are nonempty, pairwise disjoint and both $f_1$ and $f_2$ are constant on each $A_i$. $f_2 - f_1 \geq 0$ implies $c_i \geq 0$ for all $i$. Consider the partition of $\{1, \dots, n\}$ into three sets $I_1, I_2, I_3$ in such a way that $A_i \subseteq A$ for $i \in I_1$ and $A_i \cap A = \emptyset$ for $i \in I_2$. For $i \in I_3$ we require that $A_i$ intersects $A$ as well as $X \setminus A$. We define $B_1 := \bigcup_{i \in I_1} A_i$ and $B_2 := B_1 \cup \bigcup_{i \in I_3} A_i$, hence $B_1 \subseteq A \subseteq B_2$ and $B_1, B_2 \in \mathfrak{A}$. Note that $f_1 \leq \mathbb{1}_A \leq f_2$ together with the fact that the $f_1$ and $f_2$ are constant on each $A_i$ implies that for $i \in I_3$ we have $f_1 \leq 0$ and $f_2 \geq 1$, therefore $c_i \geq 1$. We conclude

$$\begin{aligned} p(B_2 \setminus B_1) = \sum_{i \in I_3} p(A_i) \leq \sum_{i \in I_3} c_i p(A_i) = m_p\Big(\sum_{i \in I_3} c_i \mathbb{1}_{A_i}\Big) \\ \leq m_p\Big(\sum_{i=1}^n c_i \mathbb{1}_{A_i}\Big) = m_p(f_2 - f_1) < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary this implies $A \in \overline{\mathfrak{A}}^{(p)}$.

If on the other hand we are given a set $A \in \overline{\mathfrak{A}}^{(p)}$ and $\varepsilon > 0$, then there exist $B_1, B_2 \in \mathfrak{A}$ such that $B_1 \subseteq A \subseteq B_2$ and $p(B_2 \setminus B_1) < \varepsilon$. Passing to the indicator functions $\mathbb{1}_{B_1} \leq \mathbb{1}_A \leq \mathbb{1}_{B_2}$ and noting $\mathbb{1}_{B_1}, \mathbb{1}_{B_2} \in \mathcal{S}_{\mathfrak{A}}$ we see that $A \in \mathfrak{A}_{m_p}$. ∎

The analogous statement for the converse construction says that, given a mean $m$ on a $C^*$-algebra $\mathcal{A}$, $\mathcal{A} = \mathcal{I}_{p_m}$ if and only if $\mathcal{A}$ is $m$-closed. Later we will use topological constructions for a proof of this fact (see Proposition 2.6.6).

**2.3. Integration on compact spaces.** Throughout this text the notion of compactness always includes the Hausdorff separation axiom. In this section we assume that $X$ is a compact space. If $\mu$ is a Borel probability measure on $X$ then $m = m_\mu : f \mapsto \int_X f d\mu$ defines a mean on $\mathcal{A} = C(X)$, the $C^*$-algebra of all continuous $f : X \to \mathbb{C}$. One of the main reasons that integration theory is particularly successful on (locally) compact spaces is that also a converse is true: Positive functionals induce $\sigma$-additive measures. This is the content of the celebrated Riesz' Representation Theorem, which we use in the following version:

Proposition 2.3.1 (Riesz). *Let $X$ be compact and $m$ a mean on $C(X)$. Then there is a unique regular probability measure $\mu = \mu_m$ which is the completion of its restriction to the $\sigma$-algebra of Borel sets on $X$ and such that $m(f) = \int_X f\, d\mu$ for all $f \in C(X)$. (Recall that **regular** means that for every $\mu$-measurable $A$ and all $\varepsilon > 0$ there are closed $F$ and open $G$ with $F \subseteq A \subseteq G$ and $\mu(G \setminus F) < \varepsilon$.)*

A proof can be found for instance in Rudin's book [38].

On the compact unit interval $X = [0, 1]$ the classical Riemann integral can be taken as a mean $m$ on $C(X)$. Then the measure $\mu_m$ according to Riesz' Representation Theorem is the Lebesgue measure on $[0, 1]$. Note that in this case $\mathcal{A} = C(X)$ is not $m_\mu$-closed, since all Riemann integrable functions (essentially by the very definition of the Riemann integral) are members of the $m$-closure of $\mathcal{A}$ but not necessarily continuous. Sets $A$ with topological boundary $\partial A$ of measure 0 play an important rôle.

Definition 2.3.2. Let $\mu$ be a complete Borel measure on $X$. A set $A \subseteq X$ is called *$\mu$-Jordan measurable* or a *$\mu$-continuity set* if the topological boundary $\partial A$ of $A$ satisfies $\mu(\partial A) = 0$. The system of all $\mu$-continuity sets (which forms a set algebra on $X$) is denoted by $\mathfrak{C}_\mu(X)$.

In the classical case $X = [0, 1]$, $\mu$ the Lebesgue measure, the continuity sets $A$ are exactly those $A \subseteq [0, 1]$ for which $\mathbb{1}_A$ is integrable in the Riemann sense. The uniform closure of the linear span of such $\mathbb{1}_A$ coincides with the Riemann integrable functions. In order to treat the Riemann integral in the context of compactifications we fix well-known characterizations of classical Riemann integrability in our somewhat more general context.

For a function $f$, defined on the topological space $X$, we will denote by $\mathrm{disc}(f)$ the set of discontinuity points of $f$.

Proposition 2.3.3. *Let $X$ be compact, $\mu$ a finite complete regular Borel measure on $X$ and $f : X \to \mathbb{R}$ bounded. Then the following conditions are equivalent:*

(i) $\mathrm{disc}(f)$ *is $\mu$-measurable and a $\mu$-null set.*
(ii) $f \in \overline{S}_{\mathfrak{C}_\mu} = B(\mathfrak{C}_\mu)$, *i.e. $f$ can be approximated by simple $\mathfrak{C}_\mu$-functions with respect to uniform convergence.*

(iii) $f \in \overline{C(X)}^{m_\mu}$, *i.e. for every* $\varepsilon > 0$ *there exist* $f_1, f_2 \in C(X)$ *such that* $f_1 \le f \le f_2$ *and* $\int_X (f_2 - f_1)\, d\mu < \varepsilon$.

*If one (and hence all) of these conditions are satisfied, then* $f$ *is* $\mu$*-measurable.*

*Proof.* First we prove that (i) implies that $f$ is measurable. By regularity there is a decreasing sequence of open sets $O_n$, $n \in \mathbb{N}$, of measure $\mu(O_n) < 1/n$ with $\operatorname{disc}(f) \subseteq O_n$. Let $f_n$ be the restriction of $f$ to $X \setminus O_n$. For any Borel set $B \subseteq \mathbb{R}$ we have $f^{-1}[B] = \bigcup_{n\in\mathbb{N}} f_n^{-1}[B] \cup N$ with $N \subseteq D := \bigcap_{n\in\mathbb{N}} O_n$, $\mu(D) = 0$. By the completeness of $\mu$ we conclude that $N$ and thus $f^{-1}[B]$ and finally $f$ is measurable. Now we start with the cyclic proof of the equivalences.

(i)$\Rightarrow$(ii): Assume that $\mu(\operatorname{disc}(f)) = 0$ and (without loss of generality) $f(X) \subseteq [0,1]$. We introduce the level sets $M_t := [0 \le f < t]$, which are measurable by the first part of the proof, and the function

$$\varphi_f(t) := \mu(M_t).$$

Since $\varphi_f$ is increasing, it has at most countably many points of discontinuity. Consider $\mu(\{x : f(x) = t\}) \le \varphi_f(r) - \varphi_f(s)$ for $s < t < r$. If $\varphi_f$ is continuous at $t$ this implies

$$\sup_{s<t} \varphi_f(s) = f(t) = \inf_{r>t} \varphi_f(r),$$

and so $\{x : f(x) = t\}$ is a $\mu$-null set for $t \notin \operatorname{disc}(\varphi_f)$. Now let $x \in \partial M_t$. If $f$ is continuous at $x$ we clearly have $f(x) = t$. So

$$\partial M_t \subseteq \operatorname{disc}(f) \cup \{x : f(x) = t\}.$$

The first set on the right-hand side is a $\mu$-null set by our assumption and the second one is a $\mu$-null set at least for each continuity point $t$ of $\varphi_f$. So for all but at most countably many $t$ the set $M_t$ is a $\mu$-continuity set. In particular, the set $N_f := \{t : \mu(\partial M_t) = 0\} \subseteq [0,1]$ is dense.

Now we approximate $f$ uniformly by members of $S_{\mathfrak{C}_\mu}$: Given $\varepsilon > 0$, pick $n \in \mathbb{N}$ such that $n > 1/\varepsilon$ and pick real numbers $\{t_i\}_{i=0}^n \subset N_f$ with

$$t_0 = 0 < t_1 < \frac{1}{n} < \cdots < t_i < \frac{i}{n} < t_{i+1} < \ldots < \frac{n-1}{n} < t_n = \|f\|_\infty \le 1.$$

Let $A_i := M_{t_i} \setminus M_{t_{i-1}}$. Then $\left|f(x) - \frac{i-1}{n}\right| < \varepsilon$ on $A_i$, $i = 1, \ldots, n$. Since $X = M_1 \setminus M_0 = \bigcup_{i=1}^n A_i$ we conclude

$$\left| \sum_{i=1}^n \frac{i}{n} \mathbb{1}_{A_i}(x) - f(x) \right| < \varepsilon.$$

(ii)$\Rightarrow$(iii): Let $\mathcal{A}_0$ denote the set of all bounded $g : X \to \mathbb{R}$ satisfying (iii), i.e. such that for each $\varepsilon > 0$ there are $g_1, g_2 \in C(X)$ with $g_1 \le g \le g_2$ and $\int_X (g_2 - g_1)\, d\mu < \varepsilon$. It is a routine check that $\mathcal{A}_0$ is a linear space and uniformly closed. Thus it suffices to show that $\mathbb{1}_A \in \mathcal{A}_0$ whenever $A \in \mathfrak{C}_\mu$. For such an $A$ and any given $\varepsilon > 0$ we use the regularity of $\mu$ to get an open set $O$ with $\partial A \subseteq O$ and $\mu(O) < \varepsilon$. Since compact spaces are normal we can find closed sets $A_1, A_2$ and open sets $O_1, O_2$ with

$$A \setminus O \subseteq O_1 \subseteq A_1 \subseteq A^o \subseteq \overline{A} \subseteq O_2 \subseteq A_2 \subseteq A \cup O.$$

Take continuous Urysohn functions $f_1$ for $A \setminus O$ and $X \setminus O_1$, $f_2$ for $A_2$ and $X \setminus (A \cup O)$, i.e.

$$\mathbb{1}_{A\setminus O} \le f_1 \le f \le f_2 \le \mathbb{1}_{A\cup O}.$$

Then $\int_X (f_2 - f_1)\, d\mu \le \mu(O) < \varepsilon$.

(iii)⇒(i): Define the oscillation $\mathrm{Os}_f(x)$ of $f$ at a point $x$ by

$$\mathrm{Os}_f(x) := \limsup_{y\to x} f(y) - \liminf_{y\to x} f(y).$$

Let $A_k := [\mathrm{Os}_f(x) \ge 1/k]$ be the set of all $x \in X$ where the oscillation of $f$ is at least $1/k$. Pick any $\varepsilon > 0$ and $k \in \mathbb{N}$. By (iii) there are continuous $f_1^\varepsilon, f_2^\varepsilon$ with $f_1^\varepsilon \le f \le f_2^\varepsilon$ and $\int_X (f_2^\varepsilon - f_1^\varepsilon)\, d\mu < \varepsilon/k$. Note that $A_k \subseteq B_k^\varepsilon := \{x \in X : f_2^\varepsilon(x) - f_1^\varepsilon(x) \ge 1/k\}$ and $\mu(B_k^\varepsilon) < 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary we have $\mu(A_k) = 0$. Since $\mathrm{disc}(f) = \bigcup_{k\in\mathbb{N}} A_k$ this proves (i). ∎

The equivalence of (i) and (iii) can also be found in [49].

Definition 2.3.4. Let $\mu$ be a finite, complete and regular Borel measure on the compact space $X$ and $f : X \to \mathbb{C}$ be a bounded function with decomposition $f = f_1 + if_2$ into real and imaginary parts. Then $f$ is called *$\mu$-Riemann integrable* if both $f_1$ and $f_2$ satisfy the equivalent conditions of Proposition 2.3.3. We denote the set of all $\mu$-Riemann integrable $f$ by $\mathcal{R}_\mu(X)$ or $\mathcal{R}_\mu$.

The three conditions in Proposition 2.3.3 immediately transfer to complex-valued functions.

Corollary 2.3.5. *Let $\mu$ be a finite, complete and regular Borel measure on the compact space $X$. For a bounded $f : X \to \mathbb{C}$ the following conditions are equivalent.*

(i) *$f \in \mathcal{R}_\mu$, i.e. $f$ is $\mu$-Riemann integrable.*
(ii) *$\mu(\mathrm{disc}(f)) = 0$.*
(iii) *$f \in B(\mathcal{S}_{\mathfrak{C}_\mu})$.*

*In particular, $\mathcal{R}_\mu = \mathcal{I}_p$ if $p(A) := \mu(A)$ for $A \in \mathfrak{C}_\mu$.*

Every $f \in \mathcal{R}_\mu$ is $\mu$-measurable and the set $\mathrm{disc}(f)$ of discontinuities of a Riemann integrable $f$ is small not only in the measure-theoretic but also in the topological sense.

Proposition 2.3.6. *Let $X$ be compact and $\mu$ a finite regular Borel measure with* $\mathrm{supp}(\mu) = X$. *Let $f \in \mathcal{R}_\mu(X)$ be Riemann integrable. Then* $\mathrm{disc}(f)$ *is a meager $\mu$-null set, in particular the set of continuity points of $f$ is dense in $X$.*

*Proof.* We may assume that $f \in R_\mu(X)$ is real-valued. It suffices to show that $\mathrm{disc}(f)$ is meager. As in the proof of Proposition 2.3.3 let us denote the oscillation of $f$ at $x$ by $\mathrm{Os}_f(x)$. A standard argument shows that the sets $A_n := [\mathrm{Os}_f \ge 1/n]$, $n > 0$, are closed. The sets $A_n$ are all $\mu$-null sets since $A_n \subseteq \mathrm{disc}(f)$. Since $\mu$ has full support, this implies that all sets $A_n$ are nowhere dense, i.e. $\mathrm{disc}(f) = \bigcup_{n>0} A_n$ is a meager $F_\sigma$-set of zero $\mu$-measure. ∎

We want to illustrate the rôle of the regularity assumption on $\mu$ in Proposition 2.3.3. For this we use the example of a nonregular Borel measure occurring in Rudin's book [38, Exercise 2.18].

Example 2.3.7. Let $X = [0, \omega_1]$ be the set of all ordinals up to the first uncountable one equipped with the order topology. Thus $X$ is a compact space.

We need the fact that every (at most) countable family of uncountable compact subsets $K_n \subseteq X$ has an uncountable intersection $K$. To see this consider any increasing sequence $x_0 < x_1 < x_2 < \cdots \in X$ which meets every $K_n$ infinitely many times. It follows that $\alpha_0 := \sup_n x_n < \omega_1$ is in the closure of all $K_n$, hence in $K$. Since we may require $x_0 > x$ for any given $x < \omega_1$ the same construction can be repeated in order to obtain an $\alpha_1 \in K$ with $\alpha_1 > \alpha_0$. Transfinite induction with the limit step $\alpha_\lambda := \sup_{\nu < \lambda} \alpha_\nu$ generates the closed and thus compact subset of all $\alpha_\nu$, $\nu < \omega_1$, which is contained in $K$.

Easy consequences: We call a set $S \subseteq X$ *of type 1* if $S \cup \{\omega_1\}$ contains an uncountable compact $K$. If $S$ is of type 1 the complement of $S$ cannot have the same property. Call $S \subseteq X$ *of type 0* if $(X \setminus S) \cup \{\omega_1\}$ contains an uncountable compact $K$. The system of all sets of either type 0 or type 1 forms a $\sigma$-algebra $\mathfrak{A}$ containing all Borel sets.

Let $\mu(S) = i$ if $S$ is of type $i = 0, 1$; then $\mu$ is a complete measure defined on $\mathfrak{A}$. Note that every countable set is a $\mu$-null set. The set $\{\omega_1\}$ has measure 0 and is a counterexample for outer regularity: The function $\mathbb{1}_{\{\omega_1\}}$ obviously satisfies conditions (i) and (ii) in Proposition 2.3.3, but not (iii). To see this last assertion consider any continuous $f : X \to \mathbb{C}$ and take $\beta_n$ such that $|f(x) - f(\omega_1)| < 1/n$ for all $x \geq \beta_n$. Then $\beta := \sup_n \beta_n < \omega_1$ has the property that $f(x) = f(\omega_1)$ for all $x \geq \beta$. It follows that $\int_X f \, d\mu = f(\omega_1)$ for all $f \in C(X)$. In particular, $g \leq \mathbb{1}_{\omega_1} \leq h$, $g, h \in C(X)$ implies $\int_X (h - g) \, d\mu \geq 1$, contradicting (iii).

Nevertheless, we can apply Riesz' Representation Theorem 2.3.1 to the functional $m(f) := \int_X f \, d\mu$. A quick inspection shows that $\mu_m = \delta_{\omega_1}$, i.e. the associated unique regular Borel measure is the point measure concentrated at the point $\omega_1$. As a complete measure, this $\mu_m$ is defined on the whole power set of $X$. Finally, we observe that $\mathbb{1}_{\omega_1} \notin \mathcal{I}_{\mu_m}$.

**2.4. Compactifications and continuity.** The previous section has illustrated that compactness plays an important rôle in integration theory. This motivates us to investigate compactifications, the topic of this purely topological section. Let $X$ be a, possibly discrete, topological space.

We will interpret functions $f : X \to \mathbb{C}$ as restrictions of functions $F : K \to \mathbb{C}$ on compact spaces $K$. For our needs the following setting is appropriate.

Definition 2.4.1. A pair $(\iota, K)$, $K$ compact, $\iota : X \to K$ a continuous mapping, is called a *compactification* of $X$ whenever $\overline{\iota(X)} = K$, i.e. whenever the image of $X$ under $\iota$ is dense in $K$. The function $F : K \to \mathbb{C}$ is called a *representation* of $f : X \to \mathbb{C}$ whenever $f = F \circ \iota$, i.e. whenever the diagram

$$\begin{tikzcd} & K \arrow[d, "F"] \\ X \arrow[ur, "\iota"] \arrow[r, "f"] & \mathbb{C} \end{tikzcd}$$

commutes. In this case we also say that $f$ can be represented in $(\iota, K)$. If $F \in C(K)$ we say that $F$ is a continuous representation.

Note that in the definition of a compactification $\iota$ is not required to be a homeomorphic embedding nor to be injective. If there is a continuous representation $F$ of $f$ in $(\iota, K)$, then this $F$ is uniquely determined by continuity and the fact that $\iota(X)$ is dense in $K$. Furthermore, $f = F \circ \iota$ is continuous as well. In this section we are therefore mainly interested in continuous $f$. Let us consider first a rather trivial example.

Example 2.4.2.

- Let $f : X \to \mathbb{C}$ be bounded and continuous. Surely $K_f := \overline{f(X)}$ is compact. Define $\iota_f : x \mapsto f(x)$ and let $F_f : K_f \to \mathbb{C}$ be the inclusion mapping. Then $(\iota_f, K_f)$ is a compactification of $X$ and $F_f$ is a continuous representation of $f$ in $(\iota_f, K_f)$. We call $F_f$ the *natural continuous representation* of $f$.
- Let $f : X \to \mathbb{C}$ be merely bounded. If we impose the discrete topology on $X$, then $f$ is continuous and the associated compactification $(\iota_f, K_f)$ is a compactification of the discrete space $X_{\text{dis}}$.

One observes the following minimality property of the natural continuous representation: If $F : K \to \mathbb{C}$ is any continuous representation of $f$ in any compactification $(K, \iota)$ of $X$, then $\pi : K \to K_f = \overline{f(X)}$, $\pi(k) := F(k)$, is continuous, onto and satisfies $\pi \circ \iota = \iota_f$. This motivates the following definition.

Definition 2.4.3. Let $(\iota_1, K_1)$ and $(\iota_2, K_2)$ be two compactifications of $X$. Then we write $(\iota_1, K_1) \le (\iota_2, K_2)$ (via $\pi$) and say that $(\iota_1, K_1)$ is smaller than $(\iota_2, K_2)$ or, equivalently, $(\iota_2, K_2)$ is bigger than $(\iota_1, K_1)$, if $\pi : K_2 \to K_1$ is continuous satisfying $\iota_1 = \pi \circ \iota_2$, i.e. making the diagram

$$\begin{tikzcd} & K_2 \arrow[d, "\pi"] \\ X \arrow[ur, "\iota_2"] \arrow[r, "\iota_1"'] & K_1 \end{tikzcd}$$

commutative. If $\pi$ is a homeomorphism we say that $(\iota_1, K_1)$ and $(\iota_2, K_2)$ are equivalent via $\pi$ and write $(\iota_1, K_1) \cong (\iota_2, K_2)$.

A consequence of the continuity of the maps involved and of the fact that the images $\iota_i(X)$ are dense is that $\pi$ as in Definition 2.4.3 is unique. By compactness, $\pi$ is onto as well. If $\pi$ happens to be injective it is a homeomorphism, i.e. $(\iota_1, K_1)$ and $(\iota_2, K_2)$ are equivalent. Furthermore, one easily sees that, whenever $(\iota_1, K_1) \le (\iota_2, K_2)$ via $\pi_1$ and $(\iota_2, K_2) \le (\iota_1, K_1)$ via $\pi_2$ then $\pi_2 \circ \pi_1$ is the identity on $K_1$ and $\pi_1 \circ \pi_2$ is the identity on $K_2$, hence $\pi_2 = \pi_1^{-1}$, $\pi_1$ and $\pi_2$ are isomorphisms and both compactifications are equivalent.

Proposition 2.4.4. *$(\iota_1, K_1) \cong (\iota_2, K_2)$ if and only if both $(\iota_1, K_1) \le (\iota_2, K_2)$ and $(\iota_2, K_2) \le (\iota_1, K_1)$.*

Note that maps $\pi_1, \pi_2$ as in Definition 2.4.3 may as well be considered to be the morphisms in a category whose objects are all compactifications of $X$. Other related categories arise if one allows only continuous representations of one fixed $f : X \to \mathbb{C}$. In these terms the minimality property of the natural compactification asserts that $(\iota_f, K_f)$ is a universal object and thus unique up to equivalence.

PROPOSITION 2.4.5. *Let $F_1$ be a representation of $f : X \to \mathbb{C}$ in a compactification $(\iota_1, K_1)$ of $X$, and suppose $(\iota_1, K_1) \le (\iota_2, K_2)$ via $\pi$. Then $F_2 := F_1 \circ \pi$ is a representation of $f$ in $(\iota_2, K_2)$ which is continuous whenever $F_1$ is continuous.*

Given a family of compactifications $(\iota_i, K_i)$, $i \in I$, of $X$, we get a common upper bound by taking products: Let $\iota(x) := (\iota_i(x))_{i\in I} \in P := \prod_{i\in I} K_i$ and $K := \overline{\iota(X)} \subseteq P$. Then one obtains a compactification $(\iota, K)$ which, via the projections $\pi_{i_0} : K \to K_{i_0}$, $i_0 \in I$, $(k_i)_{i\in I} \mapsto k_{i_0}$, indeed satisfies $(\iota_i, K_i) \le (\iota, K)$ for all $i \in I$. Sometimes we use the notation $\bigvee_{i\in I}(\iota_i, K_i)$ for $(\iota, K)$.

DEFINITION 2.4.6. For compactifications $(\iota_i, K_i)$ of $X$, $i \in I$, the compactification $(\iota, K)$, $\iota : x \mapsto (\iota_i(x))_{i\in I}$, $K := \overline{\iota(X)} \subseteq \prod_{i\in I} K_i$, is called the *product compactification* of all $(\iota_i, K_i)$, $i \in I$.

PROPOSITION 2.4.7. *For any compactifications $(\iota_i, K_i)$ of $X$, $i \in I$, the supremum $\sup_{i\in I}(\iota_i, K_i)$ is equivalent to the product compactification $(\iota, K)$ of all $(\iota_i, K_i)$, $i \in I$.*

*Proof.* We have already seen that $\sup_{i\in I}(\iota_i, K_i) \le (\iota, K)$. Let $(\iota', K')$ be another compactification of $X$ such that $(\iota_i, K_i) \le (\iota', K')$, $i \in I$. Denote by $\pi_i : K' \to K_i$ the $i$th projection. Define a mapping $\pi : K' \to K$ via $k' \mapsto (\pi_i(k'))_{i\in I}$. Note that $\pi \circ \iota' = \iota$, hence $\pi(\iota'(X)) \subseteq K$ and

$$\pi(K') = \pi(\overline{\iota'(X)}) \subseteq \overline{\pi(\iota'(X))} \subseteq K.$$

It is immediate to check that $\pi$ is continuous; thus $(\iota, K) \le (\iota', K')$. ∎

Analogously the product compactification can be used to obtain a minimal compactification where all functions from an arbitrary given family have a continuous representation: Let $f_i : X \to \mathbb{C}$, $i \in I$, be bounded and continuous functions on $X$. We consider the natural continuous representations of the $f_i$, i.e. $(\iota_i, K_i) := (\iota_{f_i}, K_{f_i})$, and $F_i : K_i \to \mathbb{C}$ the inclusion mappings. Let $(\iota, K)$ be the product of all $(\iota_i, K_i)$, $i \in I$.

DEFINITION 2.4.8. Let us denote the $C^*$-algebras of bounded resp. continuous resp. bounded and continuous $f : X \to \mathbb{C}$ by $B(X)$, $C(X)$ resp. $C_b(X)$. For a given family of $f_i \in C_b(X)$, $i \in I$, the compactification $(\iota, K)$ constructed as above is called the *natural compactification* for the family of all $f_i$, $i \in I$. If $\mathcal{A} = \{f_i : i \in I\}$ we also write $(\iota_{\mathcal{A}}, K_{\mathcal{A}})$ for $(\iota, K)$.

PROPOSITION 2.4.9. *Let $\mathcal{A} \subseteq C_b(X)$. Then*:

(i) *Every $f \in \mathcal{A}$ has a continuous representation in the natural compactification $(\iota_{\mathcal{A}}, K_{\mathcal{A}})$ of $\mathcal{A}$.*
(ii) *Suppose that $(\iota, K)$ is any compactification of $X$ where every $f \in \mathcal{A}$ has a continuous representation. Then $(\iota_{\mathcal{A}}, K_{\mathcal{A}}) \le (\iota, K)$, i.e. $(\iota_{\mathcal{A}}, K_{\mathcal{A}})$ is minimal among the compactifications with this property.*
(iii) *$\{F \circ \iota : F \in C(K_{\mathcal{A}})\}$ is a $C^*$-algebra and the $*$-algebra generated by $\mathcal{A}$ is dense in this $C^*$-algebra. In particular, if $\mathcal{A}$ is $C^*$-algebra, then $\mathcal{A}$ contains exactly those $f$ which have a continuous representation in $(\iota_{\mathcal{A}}, K_{\mathcal{A}})$.*

*Proof.* (i) For each $i_0 \in I$, $G_{i_0} : K \to \mathbb{C}$, $(c_i)_{i\in I} \mapsto c_{i_0}$, is continuous and satisfies $f_{i_0} = G_{i_0} \circ \iota$ for each $i_0 \in I$. Thus all $f_i$ can be continuously represented in $(\iota_{\mathcal{A}}, K_{\mathcal{A}})$.

(ii) Let $(\iota', K')$ be an arbitrary compactification of $X$ where continuous representations $G_i' : K' \to \mathbb{C}$ of $f_i = G_i' \circ \iota'$ exist. As in Proposition 2.4.7, we define $\pi : k' \mapsto (G_i'(k'))_{i \in I} \in \prod_{i\in I} \overline{f_i(X)}$. Then $\pi$ is continuous because all components are. Again we have $\pi(K') \subseteq K$, $\pi : K' \to K$ and $(\iota, K) \leq (\iota', K')$. Furthermore, $G_i' = G_i \circ \pi$ for all $i \in I$, since the mappings on both sides are continuous and coincide on the dense set $\iota'(X)$.

(iii) It is clear that the mapping $F \mapsto F \circ \iota$ maps the $C^*$-algebra $C(K_{\mathcal{A}})$ again onto a $C^*$-algebra and that this map is a continuous homomorphism between $C^*$-algebras.

For the rest of the proof we can assume that $\mathcal{A}$ is a $*$-algebra. It remains to prove that the $*$-algebra $\mathcal{A}' := \{F \in C(K_{\mathcal{A}}) : F \circ \iota \in \mathcal{A}\}$ is dense in $C(K_{\mathcal{A}})$. We employ the Stone–Weierstraß theorem. Obviously $\mathcal{A}'$ is a $*$-algebra containing all constant functions. We are done if $\mathcal{A}'$ is point separating. Pick $c \neq c' \in K_{\mathcal{A}}$. Recall that the points in $K_{\mathcal{A}}$ are of the form $c = (c_f)_{f\in\mathcal{A}}$ and $c' = (c_f')_{f\in\mathcal{A}}$ with $c_f, c_f' \in \mathbb{C}$. Hence there is some $f_0 \in \mathcal{A}$ such that $c_{f_0} \neq c'_{f_0}$. By definition, $K_{\mathcal{A}}$ is the closure of the set of all $(f(x))_{f\in\mathcal{A}}$, $x \in X$. It follows that there are $x, x' \in X$ with $f_0(x)$ arbitrary close to $c_{f_0}$, $f_0(x')$ to $c'_{f_0}$, hence $f_0(x) \neq f_0(x')$. Let $F_0 = \pi_{f_0} \in C_b(X)$, implying $f_0 = F_0 \circ \iota$ and $F_0 \in \mathcal{A}'$ with $F_0(c) = f_0(x) \neq f_0(x') = F_0(c')$. Thus $\mathcal{A}'$ is indeed point separating, which completes the proof. ∎

Proposition 2.4.10. *Let $\mathcal{A}$ be a $C^*$-algebra on $X$. Then $\mathcal{A}$ separates points of $X$ if and only if in the natural compactification $(\iota_{\mathcal{A}}, K_{\mathcal{A}})$ the map $\iota_{\mathcal{A}} : X \to K_{\mathcal{A}}$ is one-one.*

*Proof.* Recall that $\iota_{\mathcal{A}}(x) := (f(x))_{x\in\mathcal{A}}$. Now, $\mathcal{A}$ separates points of $X$ if and only if for all $x_1, x_2 \in X$ with $x_1 \neq x_2$ there exists $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$, i.e. $\iota_{\mathcal{A}}(x_1) \neq \iota_{\mathcal{A}}(x_2)$. ∎

Corollary 2.4.11 (Gelfand). *The mapping $\mathcal{A} \mapsto (\iota_{\mathcal{A}}, K_{\mathcal{A}})$ is (modulo equivalence of compactifications) a bijective and order-preserving correspondence between compactifications of $X$ and $C^*$-subalgebras of $C_b(X)$ which contain $\mathbb{1}_X$. In particular, $\mathcal{A}$ and $C(K_{\mathcal{A}})$ are isomorphic as $C^*$-algebras.*

Remark 2.4.12. Note that Corollary 2.4.11 applies to $C^*$-subalgebras of $B(X)$ as well. All one has to do is to identify $B(X)$ with $C_b(X_{\mathrm{dis}})$. Thus $B(X)$ is a $C^*$-algebra of continuous functions.

Example 2.4.13. Let us consider the special case where $\mathcal{A} = B(\mathfrak{A}_{\mathcal{A}}) \subseteq C_b(X_{\mathrm{dis}})$ and write $\mathfrak{A} = \mathfrak{A}_{\mathcal{A}}$. We consider the set $\mathcal{A}_1 := \{\mathbb{1}_A : A \in \mathfrak{A}\}$, the corresponding compactification $(\iota_1, K_1) := (\iota_{\mathcal{A}_1}, K_{\mathcal{A}_1})$ and the commutative diagram

$$\begin{tikzcd} & K_{\mathcal{A}} \arrow[d, "\pi"] \\ X \arrow[ur, "\iota_{\mathcal{A}}"] \arrow[r, "\iota_{\mathcal{A}_1}"] & K_1 \end{tikzcd}$$

with $\pi : (c_f)_{f\in\mathcal{A}} \mapsto (c_f)_{f\in\mathcal{A}_1} \in \{0,1\}^{\mathcal{A}_1}$. We claim that $\pi$ is injective. Suppose first $\mathbb{1}_A(x) = \mathbb{1}_A(y)$ for all $A \in \mathfrak{A}$. Then $f(x) = f(y)$ for all $f \in \mathcal{S}_{\mathfrak{A}}$ and hence for all $f$ from the closure $B(\mathfrak{A}) = \mathcal{A}$. Suppose now that $c = (c_f)_{f\in\mathcal{A}_1} = \pi(a) = \pi(b) \in \overline{\iota_{\mathcal{A}_1}(X)}$ with $a = (a_f)_{f\in\mathcal{A}}$ and $b = (b_f)_{f\in\mathcal{A}}$. Then $a_f = b_f = c_f$ for all $f \in \mathcal{A}_1$. There is a net $(x_\nu)_{\nu\in N}$, $N$ a directed set, such that $\iota_{\mathcal{A}_1}(x_\nu) \to c$. Define $\iota_{\mathcal{A}_1}(x_\nu) = (c_f^\nu)_{f\in\mathcal{A}}$. Note that $c_f^\nu = f(x_\nu)$. Thus we have $f(x_\nu) \to c_f = a_f = b_f$ for all $f \in \mathcal{A}_1$, hence, by linearity, for all $f \in \mathcal{S}_{\mathfrak{A}}$ and, by uniform closure, for all $f \in B(\mathfrak{A}) = \mathcal{A}$. Therefore we conclude that $\iota_{\mathcal{A}}(x_\nu) = (f(x_\nu))_{f\in\mathcal{A}} \to a = (a_f)_{f\in\mathcal{A}} = (b_f)_{f\in\mathcal{A}} = b$, proving that $\pi$ is injective. Thus $(\iota_{\mathcal{A}}, K_{\mathcal{A}}) \cong (\iota_{\mathcal{A}_1}, K_{\mathcal{A}_1})$. A clopen subbasis of $K_1$ is given by all sets $A_0' := \{(c_A)_{A\in\mathfrak{A}} : c_{A_0} = 1\}$, $A_0 \in \mathfrak{A}$.

Corollary 2.4.14 (Stone). *If $\mathcal{A} = B(\mathfrak{A}_{\mathcal{A}})$ then the compact space $K_{\mathcal{A}}$ is totally disconnected.*

Note that the natural context of our discussion are classical theorems due to Gelfand, Banach and Stone. Without going into formal details these results are as follows. Gelfand's representation theorem states that every (abstract) commutative unital $C^*$-algebra $\mathcal{A}$ (meaning that complex conjugation is replaced by an abstract operation with corresponding properties) is isometrically isomorphic to some $C(K)$ where $K$ is a suitable compact space. In this context $K$ is also called the structure space or Gelfand compactum for $\mathcal{A}$. By the Banach–Stone theorem, two compact spaces $K_1$ and $K_2$ are homeomorphic if and only if $C(K_1) \cong C(K_2)$ as unital Banach algebras. Furthermore, by Stone's theorem, for every Boolean algebra $B$ there is a totally disconnected compact space $K$, the so called Stone space associated to $B$, such that for the systems $\mathrm{Cl}(K)$ of all clopen subsets of $K$ we have $B \cong \mathrm{Cl}(K)$ as Boolean algebras. Two such spaces $K_1$ and $K_2$ are homeomorphic if and only if $\mathrm{Cl}(K_1) \cong \mathrm{Cl}(K_2)$. Finally, the Stone space of a Boolean set algebra $\mathfrak{A}$ is homeomorphic to the Gelfand compactum for $B(\mathfrak{A})$. The interested reader is referred to [8] and [9].

**2.5. The Stone–Čech compactification $\beta X$.** We now apply the construction of the natural compactification for an algebra $\mathcal{A}$ to the case $\mathcal{A} = C_b(X)$, i.e. to the algebra of all bounded and continuous $f : X \to \mathbb{C}$.

Definition 2.5.1. The maximal compactification $(\iota_\beta, \beta X)$ of a topological space $X$, corresponding to the algebra $C_b(X)$ in the sense of Corollary 2.4.11, is denoted by $(\iota_\beta, \beta X)$ and is called the *Stone–Čech compactification* of $X$.

$(\iota_\beta, \beta X)$ is characterized uniquely up to equivalence by the universal property that for every continuous $\varphi : X \to K$, $K$ compact, there is a (unique) continuous $\psi : \beta X \to K$ with $\varphi = \psi\circ\iota_\beta$. To see this we may assume $K = \overline{\varphi(X)}$, so that $(\varphi, K)$ is a compactification of $X$. By the maximality of $(\iota_\beta, \beta X)$ and Corollary 2.4.11 this just means that there is a $\psi$ as claimed. For uniqueness assume that $(\iota, K)$ is another compactification of $X$ with this universal property. Every $f \in C_b(X)$ has a range contained in a compact set $K_0 \subseteq \mathbb{C}$. By the universal property there is a continuous $\psi : K \to K_0$ with $\psi \circ \iota = f$. Hence, again by Corollary 2.4.11, the algebra corresponding to $(\iota, K)$ contains $C_b(X)$. Thus $(\iota, K)$ has to be maximal, i.e. equivalent to $(\iota_\beta, \beta X)$.

Nevertheless, in order to obtain an interesting and rich structure one needs sufficiently many bounded and continuous functions.

Definition 2.5.2. $X$ is called *completely regular* if it has the following separation property: For every closed $A \subseteq X$ and $x \in X \setminus A$ there is a continuous $f : X \to [0, 1]$ with $f(x) = 1$ and $f(a) = 0$ for all $a \in A$. Such an $f$ is called an *Urysohn function* for $A$ and $x$.

Under this assumption every Urysohn function gives rise to a compactification separating two points $x \neq y \in X$, implying that $\iota_\beta$ is injective. $\iota_\beta$ is even a homeomorphic embedding of $X$ into $\beta X$. To see this, it suffices to show that for $x \in O \subseteq X$, $O$ open, there is an open set $O_\beta \subseteq \beta X$ containing $\iota_\beta(x)$ such that $\iota_\beta(O) \supseteq O_\beta \cap \iota_\beta(X)$. Take an Urysohn function $f_0$ for $x$ and $A := X \setminus O$, recall that $\iota_\beta : x \mapsto (f(x))_{f \in C_b(X)}$ and observe that $O_\beta := \{(c_f)_{f \in C_b(X)} : c_{f_0} > 0\}$ has the required properties.

Let us now consider the case of discrete $X$, i.e. $C_b(X) = B(X)$. Then each $\mathbb{1}_A \in B(X)$, $A \subseteq X$, has a continuous representation in $(\iota_\beta, \beta X)$ which must be of the form $\mathbb{1}_{A^*}$ with some clopen $A^* = \overline{\iota_\beta(A)} \subseteq \beta X$. (Therefore the usual notation $A^* = \bar{A}$ as a closure, though not rigorously correct in our setting, does not lead to contradictions.)

Conversely, every clopen set $B \subseteq \beta X$ can be written as $B = A^*$ with $A := \iota_\beta^{-1}[B]$. Furthermore, such sets form a basis for the topology in $\beta X$: Let $O \subseteq \beta X$ be open and $x \in O$. Then, by the separation properties of compact spaces, there is an open set $O_x$ such that $x \in O_x \subseteq \bar{O}_x \subseteq O$. For $A_x := \iota_\beta^{-1}[O_x]$ we obtain $x \in A_x^* \subseteq O$. This shows that $O = \bigcup_{x \in O} A_x^*$ can be written as a union of clopen sets.

Let $A = \{a\}$, $a \in X$, be a singleton and $x \neq \iota_\beta(a)$. There is an open neighborhood $O$ of $x$ not containing $\iota_\beta(a)$. Thus the continuous representation of $\mathbb{1}_A$ in $(\iota_\beta, \beta X)$ has to take the constant value 0 on $O$, hence $\mathbb{1}_{A^*} = \mathbb{1}_{\{\iota_\beta(a)\}}$. By continuity this shows that $\{\iota_\beta(a)\}$ is open, i.e. $\iota_\beta(a)$ is an isolated point in $\beta X$. A further consequence is that $A^* \cap \iota_\beta(X \setminus A) = \emptyset$ and $A^* \cap (X \setminus A)^* = \emptyset$. Since

$$\beta X = \overline{\iota_\beta(X)} = \overline{\iota_\beta(A) \cup \iota_\beta(X \setminus A)} = \overline{\iota_\beta(A)} \cup \overline{\iota_\beta(X \setminus A)} = A^* \cup (X \setminus A)^*$$

we conclude that $\Phi : A \mapsto A^*$ is an isomorphism of Boolean set algebras between $\mathfrak{P}(X)$, the powerset of $X$, and $\mathrm{Cl}(\beta X)$, the system of all clopen sets in $\beta X$.

Consider $\mathcal{F}_x := \{\iota_\beta^{-1}[O] : x \in O \subseteq \beta X, O \text{ open}\}$. Obviously $\mathcal{F}_x$ is a filter on $X$. For arbitrary $A \subseteq X$, as $A^* \cup (X \setminus A)^* = \beta X$, we have either $x \in A^*$ or $x \in (X \setminus A)^*$. In the first case this means $A = \iota_\beta^{-1}[A^*] \in \mathcal{F}_x$, in the second case $X \setminus A \in \mathcal{F}_x$. Thus $\mathcal{F}_x$ is an ultrafilter. Conversely, every ultrafilter $\mathcal{F}$ on $X$ induces an ultrafilter $\mathcal{F}_\beta$ on $\beta X$ consisting of all $F_\beta \subseteq \beta X$ which contain $\iota_\beta(F)$ for at least one $F \in \mathcal{F}$. The compactness of $\beta X$ guarantees that $\mathcal{F}_\beta$ converges to some $x \in \beta X$, which is possible only if $\mathcal{F} = \mathcal{F}_x$. This shows that the points in $\beta X$ are in a natural bijective correspondence with the ultrafilters on $X$.

We summarize the collected facts about $\beta X$.

Proposition 2.5.3. *Let $X$ be a completely regular topological space. Then the Stone–Čech compactification $(\iota_\beta, \beta X)$ of $X$ has the following properties.*

(i) *For every continuous $f : X \to K$, $K$ compact, there is a continuous $\varphi : \beta X \to K$ with $f = \varphi \circ \iota_\beta$, i.e. making the diagram*

$$\begin{tikzcd} & \beta X \arrow[d, "\varphi"] \\ X \arrow[ur, "\iota_\beta"] \arrow[r, "f"] & K \end{tikzcd}$$

*commutative.*

(ii) *$\iota_\beta : X \to \iota_\beta(X) \subseteq \beta X$ is a homeomorphism.*

(iii) *Assume that $X$ is discrete.*

(a) *The mapping $A \mapsto A^* := \overline{\iota_\beta(A)}$ is an isomorphism of Boolean set algebras between $\mathfrak{P}(X)$, the powerset of $X$, and $\mathrm{Cl}(\beta X)$, the system of all clopen sets in $\beta X$.*

(b) *The clopen subsets of $\beta X$ form a topological basis in $\beta X$.*

(c) *The isolated points in $\beta X$ are exactly those of the form $\iota_\beta(x)$, $x \in X$.*

(d) *$\beta X$ can be represented as the set of all ultrafilters on $X$ where $\iota_\beta(x) = \mathcal{F}_x := \{F \subseteq X : x \in F\}$ for all $x \in X$. Then $A^*$ consists of those ultrafilters $\mathcal{F}$ on $X$ with $A \in \mathcal{F}$.*

**2.6. Compactifications, measures, means and Riemann integral.** We are now going to consider compactifications $(\iota, K)$ of a set (or a topological space) $X$ in connection with complete Borel probability measures $\mu$ on $K$.

Definition 2.6.1. Let $(\iota, K)$ be a compactification of $X$, $\mu$ a complete and regular Borel probability measure on $K$, and $\mathcal{A}$ a set of complex-valued $\mu$-measurable functions on $K$. Then we call the quadruple $(\iota, K, \mu, \mathcal{A})$ *admissible* if the following condition is satisfied: Whenever $F_1 \circ \iota = F_2 \circ \iota$ for $F_1, F_2 \in \mathcal{A}$ then

$$\int_K F_1 \, d\mu = \int_K F_2 \, d\mu.$$

For arbitrary $\mathcal{A}$ define $\mathcal{A}^* := \iota^*(\mathcal{A}) = \{F \circ \iota : F \in \mathcal{A}\}$. For admissible $(\iota, K, \mu, \mathcal{A})$ we define

$$m_\mu : \ f = F \circ \iota \mapsto \int_K F \, d\mu, \quad F \in \mathcal{A}.$$

Note that $m_\mu$ is well defined on $\mathcal{A}^*$ and a bounded linear functional whenever $\mathcal{A}$ is a linear space, called the *mean induced by* $(\iota, K, \mu, \mathcal{A})$.

It is clear that for all compactifications $(\iota, K)$ of $X$ and all $\mu$ we get an admissible quadruple if we take $\mathcal{A} := C(K)$. In this case $F_1 \circ \iota = F_2 \circ \iota$ with $F_1, F_2 \in \mathcal{A}$ is possible only for $F_1 = F_2$ (recall that $\iota(X)$ is dense in $K$). For our subsequent investigations the following similar statement for $\mathcal{A} = \mathcal{R}_\mu$ is fundamental.

Proposition 2.6.2. *For every compactification $(\iota, K)$ of $X$ and every complete and regular Borel probability measure on $\mu$ on $K$ the quadruple $(\iota, K, \mu, \mathcal{R}_\mu)$ is admissible. Hence*

$$m(F \circ \iota) := \int_K F \, d\mu$$

*is a well defined mean on the algebra $\mathcal{R}_\mu^*$.*

*Proof.* We may assume that $\mu$ has full support, i.e. all nonempty open sets in $K$ have positive measure. Let $f = F_1 \circ \iota = F_2 \circ \iota$ with $F_i \in \mathcal{R}_\mu$. First we assume that $F_i = \mathbb{1}_{A_i}$ for certain $\mu$-continuity sets $A_i \in \mathfrak{C}_\mu$. The symmetric difference $A := A_1 \triangle A_2$ is a $\mu$-continuity set with empty interior. We conclude that $\partial A$ has zero measure and hence $\int \mathbb{1}_{A_1} \, d\mu = \int \mathbb{1}_{A_2} \, d\mu$. By linearity this property extends to functions $F_i \in \mathcal{S}_{\mathfrak{C}_\mu}$ and, using a standard approximation argument, to arbitrary $F_i \in \mathcal{R}_\mu$. ∎

We have to compare compactifications also in a measure-theoretic sense. For this reason we fix further notation.

DEFINITION 2.6.3. Suppose that $\mu_i$ is a complete Borel probability measure on $K_i$, where $(\iota_i, K_i)$ is a compactification of $X$, $i = 1, 2$. Then we write $(\iota_1, K_1, \mu_1) \leq (\iota_2, K_2, \mu_2)$ if $(\iota_1, K_1) \leq (\iota_2, K_2)$ via $\pi : K_2 \to K_1$ which, in addition to being continuous is also measure preserving, i.e. whenever $A_1 \subseteq K_1$ is $\mu_1$-measurable then its pre-image $A_2 := \pi^{-1}[A_1]$ is $\mu_2$-measurable with $\mu_2(A_2) = \mu_1(A_1)$.

REMARK 2.6.4. In the above situation we also could have *defined* the measure on $K_1$ via $\mu_1(A_1) := \mu_2(\pi^{-1}[A_1])$. This construction is called pullback, and $\mu_1$ is often denoted by $\pi \circ \mu_2$.

We know by Proposition 2.4.5 that every $f : X \to \mathbb{C}$ which has a continuous representation $F_1 : K_1 \to \mathbb{C}$ in the compactification $(\iota_1, K_1)$ has a continuous representation $F_2 := F_1 \circ \pi$ in $(\iota_2, K_2)$ whenever $(\iota_1, K_1) \leq (\iota_2, K_2)$ via $\pi : K_2 \to K_1$. We get a similar assertion if we replace continuity by Riemann integrability.

PROPOSITION 2.6.5. *Suppose that $f : X \to \mathbb{C}$ has a $\mu_1$-Riemann integrable representation $F_1 : K_1 \to \mathbb{C}$ in the compactification $(\iota_1, K_1, \mu_1)$. Whenever $(\iota_1, K_1, \mu_1) \leq (\iota_2, K_2, \mu_2)$ via $\pi$ then $F_2 := F_1 \circ \pi$ is a $\mu_2$-Riemann integrable representation of $f$ in $(\iota_2, K_2, \mu_2)$.*

*Proof.* It is clear that $F_2 := F_1 \circ \pi$ is a realization of $f$ whenever $F_1$ is. It is immediate to check that $\mathrm{disc}(F_2 \circ \pi) \subseteq \pi^{-1}[\mathrm{disc}(F_1)]$. Thus one obtains

$$\mu_2(\mathrm{disc}(F_2)) \leq \mu_2(\pi^{-1}[\mathrm{disc}(F_1)]) = \mu_1(\mathrm{disc}(F_1)) = 0.$$

Thus $F_2$ is $\mu_2$-Riemann integrable whenever $F_1$ is $\mu_1$-Riemann integrable. ∎

Proposition 2.6.5 shows that $(\iota_1, K_1, \mu_1) \leq (\iota_2, K_2, \mu_2)$ implies $\{F_1 \circ \iota : F_1 \in \mathcal{R}_{\mu_1}\} \subseteq \{F_2 \circ \iota : F_2 \in \mathcal{R}_{\mu_2}\}$. This observation is of particular interest if there exists a maximal $(\iota, K, \mu)$.

Conversely, assume that a $C^*$-algebra $\mathcal{A}$ of bounded functions $f : X \to \mathbb{C}$ and a mean $m$ on $\mathcal{A}$ are given. Let $(\iota_\mathcal{A}, K_\mathcal{A})$ be the natural compactification for $\mathcal{A}$. By Proposition 2.4.9 the mapping $\iota^* : F \to F \circ \iota$ is a bijection between $C(K_\mathcal{A})$ and $\mathcal{A}$. Thus $m'(F) := m(F \circ \iota)$ is well defined and a mean on $C(K_\mathcal{A})$. By Riesz' Representation Theorem 2.3.1, $m'$ induces a Borel probability measure $\mu$ on $K_\mathcal{A}$ with $m'(F) = \int F \, d\mu$ for all $F \in C(K_\mathcal{A})$ which is unique on the $\sigma$-algebra of Borel sets and its $\mu$-completion. So it is not surprising that the $m$-closure of $\mathcal{A}$ contains all $f = F \circ \iota$ with $F \in \mathcal{R}_\mu$.

PROPOSITION 2.6.6. *Let $\mathcal{A}$ be a $C^*$-algebra on $X$ and $m$ a mean on $\mathcal{A}$. Let $(\iota_\mathcal{A}, K_\mathcal{A}, \mu)$ be the compactification where $(\iota_\mathcal{A}, K_\mathcal{A})$ is the natural compactification for $\mathcal{A}$ and $\mu$ is the complete and regular Borel measure on $K_\mathcal{A}$ which satisfies $\int F \, d\mu = m(F \circ \iota_\mathcal{A})$ for all $F \in C(K_\mathcal{A})$. Then*

$$\mathcal{R}_\mu^* := \{F \circ \iota_\mathcal{A} : F \in \mathcal{R}_\mu\} \subseteq \overline{\mathcal{A}}^{(m)}$$

*for the $m$-completion $\overline{\mathcal{A}}^{(m)}$ of $\mathcal{A}$. Furthermore, if $\mathcal{A}$ separates points of $X$, then $\mathcal{R}_\mu^* = \overline{\mathcal{A}}^{(m)}$.*

*Proof.* Let $F \in \mathcal{R}_\mu$. Then it is straightforward to check that $F \circ \iota_\mathcal{A}$ is in the $m$-closure of $\mathcal{A}$, i.e. $\mathcal{R}_\mu^* \subseteq \bar{\mathcal{A}}^{(m)}$. Assume now that $\mathcal{A}$ separates points and take the real-valued function $f \in \bar{\mathcal{A}}^{(m)}$. By definition, for every $\varepsilon > 0$ there are real-valued $F_1, F_2 \in C(K_\mathcal{A})$ such that $F_1 \circ \iota_\mathcal{A} \leq f \leq F_2 \circ \iota_\mathcal{A}$ with $\int (F_2 - F_1)\, d\mu \leq \varepsilon$. Observe

$$F_\flat := \sup_{F_1 \circ \iota_\mathcal{A} \leq f} F_1 \leq F \leq \inf_{f \leq F_2 \circ \iota_\mathcal{A}} F_2 =: F^\#, \quad F_1, F_2 \in C(K_\mathcal{A}).$$

The fact that $f$ is in the $m$-closure of $\mathcal{A}$ implies that every $F$ with $F_\flat \leq F \leq F^\#$ is $\mu$-Riemann integrable. Since $\mathcal{A}$ separates points of $X$ the map $\iota_\mathcal{A} : X \to K_\mathcal{A}$ is one-one (cf. Proposition 2.4.10). Thus we can define a function $F^\circ : K_\mathcal{A} \to \mathbb{R}$ via

$$F^\circ(k) = \begin{cases} f(x) & \text{if } k = \iota_\mathcal{A}(x), \\ 0 & \text{otherwise.} \end{cases}$$

Then $F := \max\{F^\circ, F^\flat\}$ is $\mu$-Riemann integrable and satisfies $F \circ \iota_\mathcal{A} = f$. Hence $\mathcal{R}_\mu^* \supseteq \bar{\mathcal{A}}^{(m)}$. ∎

Remark 2.6.7. In any case $\mathcal{R}_\mu^*$ is a $C^*$-algebra. $\iota_\mathcal{A}^* : F \mapsto F \circ \iota_\mathcal{A}$ is a bounded $*$-homomorphism which maps $\mathcal{R}_\mu$ into $B(X)$, and the image of every bounded $*$-homomorphism is again a $C^*$-algebra (cf. [8, Theorem I.5.5]).

In the general case of a non-point-separating algebra $\mathcal{A} \subseteq B(X)$ we can do a general construction: Consider the equivalence relation on $X$ defined by $x \approx y$ if for every $f \in \mathcal{A}$ we have $f(x) = f(y)$. Then $\mathcal{A}$ induces an algebra $\mathcal{A}/_\approx \subseteq B(X/_\approx)$ which is isomorphic to $\mathcal{A}$ but point separating.

Example 2.6.8. Let $X = \{a, b, c\}$, $\mathcal{A} := \{f : X \to \mathbb{C} : f(a) = f(b)\}$ and consider the fapm $\delta_c$. Then $K_\mathcal{A} := \{\alpha, \beta\}$ is a two-element set and $\iota_\mathcal{A}(a) = \iota_\mathcal{A}(b) =: \alpha$. The inclusion $\mathcal{A} = \mathcal{R}_{\delta_\beta}^* \subset \bar{\mathcal{A}}^{m_{\delta_c}} = B(X)$ is strict, showing that in the last statement of Proposition 2.6.6 the point-separation property cannot be omitted.

However, the identity $\mathcal{R}_\mu^* = \bar{\mathcal{A}}^{(m)}$ may hold even for certain non-point-separating algebras $\mathcal{A}$, e.g. if $\mathcal{A}$ consists of constant functions.

**2.7. The set of all means.** For an infinite discrete set $X$ there is an abundance of means on the algebra $B(X)$ of bounded $f : X \to \mathbb{C}$. For a better understanding of the structure of the set of all means, compactifications turn out very useful. We start by restricting to very special means, namely multiplicative ones. (A mean $m$ defined on an algebra $\mathcal{A}$ of functions is called *multiplicative* if $m(f_1 \cdot f_2) = m(f_1)m(f_2)$ for all $f_1, f_2 \in \mathcal{A}$.) As a standard reference we mention [17].

Given a multiplicative mean $m$ on $B(X)$, let $p = p_m$ be the corresponding fapm, defined on the whole power set $\mathfrak{A} = \mathfrak{P}(X)$ of $X$. For every $A \subseteq X$ multiplicativity of $m$ yields $p_m(A) = m(\mathbb{1}_A) = m(\mathbb{1}_A \cdot \mathbb{1}_A) = m(\mathbb{1}_A)m(\mathbb{1}_A) = p_m^2(A)$, hence $p_m(A) \in \{0, 1\}$.

Conversely, every fapm $p$ defined for all $A \subseteq X$ and taking only the values 0 and 1 induces a multiplicative mean on $B(X)$: First, check that in all four possible cases for

$p(A_1), p(A_2) \in \{0,1\}$ one gets $p(A_1 \cap A_2) = p(A_1)p(A_2)$. This implies $m_p(f_1 \cdot f_2) = m_p(f_1)m_p(f_2)$ whenever $f_i = \mathbb{1}_{A_i}$. By multiplicativity and distributivity this transfers to $f_i \in \mathcal{S}_{\mathfrak{A}}$. Finally, observe that $B(X)$ is the uniform closure of $\mathcal{S}_{\mathfrak{A}}$ to conclude by standard approximation arguments that $m_p$ is indeed a multiplicative mean on $B(X)$.

Thus multiplicative means are in a one-one correspondence with fapm's on the power set taking only the values 0 and 1. For an arbitrary such $p$ the system $\mathcal{F}_p$ of all $A \subseteq X$ with $p(A) = 1$ is closed under finite intersections, supersets and contains $X$, i.e. $\mathcal{F}_p$ is a filter. Since for each $A$ either $p(A) = 1$ or $p(X \setminus A) = 1$, $\mathcal{F}_p$ is an ultrafilter. Obviously also this argument is reversible: Every ultrafilter $\mathcal{F}$ on $X$ induces a fapm $p_{\mathcal{F}}$ by $p_{\mathcal{F}}(A) = \mathbb{1}_{\mathcal{F}}(A)$ which takes only the values 0 and 1. Consider the Stone–Čech compactification $\beta X$ as the space of ultrafilters on $X$, according to Proposition 2.5.3. Then the means $m$ on $B(X)$ transfer to positive linear functionals on $C(\beta X)$ and thus, by Riesz' Representation Theorem, to Borel probability measures on $\beta X$. The functionals, taking only the values 0 and 1, are point evaluations $F \mapsto F(y)$, $F \in C(\beta X)$, corresponding to Dirac measures $\delta_y$ concentrated at the point $y \in \beta X$. As an ultrafilter, $y$ contains exactly those $A \subseteq X$ with $p(A) = 1$. Note that, in the set of all sub-probability Borel measures, normalized point measures are exactly the extreme ones, i.e. they can be represented as a convex combination only in the trivial way. In the weak-$^*$-topology the set of all sub-probability measures is compact. Thus, by the Krein–Milman Theorem (cf. for instance [40]), an arbitrary Borel measure on $\beta X$ is in the weak-$^*$-closure of the convex hull of certain point measures. Going back to $X$ and means on $X$ we thus have:

Proposition 2.7.1. *The set of all means on $B(X)$, $X$ discrete, is given by the convex hull of all multiplicative means on $X$. The multiplicative means on $X$ are in a natural bijective correspondence with the points of the Stone–Čech compactification $\beta X$.*

Indlekofer has systematically used the relation between means and fapm's on $\mathbb{N}$ or $\mathbb{Z}$ with probability measures on the Stone–Čech compactification in probabilistic number theory (cf. for instance [24]).

In Section 2.4 we have seen that for $f : X \to \mathbb{C}$, $X$ discrete, there is a smallest continuous representation which we called the natural one and which is unique up to equivalence. Looking for Riemann integrable representations, also a measure has to be involved and thus the situation is more complicated. This has the consequence that there is not one unique (up to equivalence) smallest Riemann integrable representation. Nevertheless, at least for discrete $X$, one can easily find many minimal compactifications:

Example 2.7.2. Let $X$ be discrete. For given bounded $f : X \to \mathbb{C}$ equip $K := f(X)$ with a compact topology and let $\iota : x \mapsto f(x)$. Then $(\iota, K)$ is a compactification and $F : K \to \mathbb{C}$, $k \mapsto k$, is the only representation of $f$ in $(\iota, K)$. This representation is clearly minimal, provided $K$ carries an appropriate Borel probability measure $\mu$. If $K$ is finite, the discrete topology is compact and does the job together with any probability measure $\mu$ defined on $\mathfrak{P}(K)$. In the infinite case we define a compact topology on $K$ by fixing any $k_0 \in K$ and taking as open sets all subsets of $K$ not containing $k_0$ and all cofinite sets which contain $k_0$. Note that all $k \in K \setminus \{k_0\}$ are isolated points, hence every function is continuous at such $k$. The only possible discontinuity point is $k_0$. Thus, provided $\mu(\{k_0\}) = 0$, we have $\mathcal{R}_\mu = B(K)$. In particular, $F$ is $\mu$-Riemann integrable.

For many reasons this construction is not very satisfactory. One of them is that there is no canonical choice of $\mu$. The most natural way to find canonical measures is by invariance requirements. In the forthcoming chapters we will be concerned with invariance mainly with respect to group or semigroup operations, to some extent also with respect to a single transformation in the sense of topological and symbolic dynamics.

## 3. Invariance under transformations and operations

**3.1. Invariant means for a single transformation.** At the end of the previous chapter we have seen that there are an abundance of means on an infinite discrete set $X$. If $X$ carries further structure one asks for means and measures with certain interesting additional, namely invariance properties.

Definition 3.1.1. Let $X$ be any nonempty set and $T : X \to X$. Then $U_T : B(X) \to B(X)$ is defined by $f \mapsto f \circ T$. A set $\mathcal{A} \subseteq B(X)$ is called *$T$-invariant* if $U_T(\mathcal{A}) \subseteq \mathcal{A}$. Assume that $\mathcal{A}$ is a *$T$-invariant* vector space and $m$ is a mean on $\mathcal{A}$. Then $m$ is called *$T$-invariant* if $U_T^*(m) = m \circ U_T = m$, i.e. if

$$m(f \circ T) = m(f)$$

for all $f \in \mathcal{A}$. By $M(\mathcal{A})$ we denote the set of all means on $\mathcal{A}$, $M(X) := M(B(X))$, and by $M(\mathcal{A}, T)$ the set of all $T$-invariant $m \in M(\mathcal{A})$. For bijective $T$ we call $\mathcal{A}$ resp. $m$ *two-sided $T$-invariant* if it is both $T$- and $T^{-1}$-invariant.

Note that in the two-sided invariant case one has $M(\mathcal{A}, T) = M(\mathcal{A}, T^{-1})$. It is easy to check that $M(\mathcal{A})$ and $M(\mathcal{A}, T)$ are weak-$^*$-closed subsets of the unit ball in $B(X)^*$, the dual space of the Banach space $B(X)$. Thus, since by the Banach–Alaoglu Theorem the dual unit ball is compact in this topology, $M(\mathcal{A})$ and $M(\mathcal{A}, T)$ are compact as well. More directly, compactness becomes clear from applying Tikhonov's Theorem to

$$M(\mathcal{A}) \subseteq \prod_{f \in \mathcal{A}} \{z \in \mathbb{C} : |z| \leq \|f\|_\infty\}.$$

As a consequence, any sequence $m_n \in M(\mathcal{A})$ has at least one accumulation point (accumulation measure) $m \in M(\mathcal{A})$. In particular, the set $M_{T,(m_n)}$ of accumulation means of the sequence $(\overline{m}_n)$ is nonempty:

$$\overline{m}_n := \frac{1}{n} \sum_{k=0}^{n-1} m_n(f \circ T^k) = \frac{1}{n} \sum_{k=0}^{n-1} m_n(U_T^k(f)) = \frac{1}{n} \sum_{k=0}^{n-1} {U_T^*}^k(m_n)(f).$$

Proposition 3.1.2. *$M_{T,(m_n)} \subseteq M(\mathcal{A}, T)$. In particular, there are $T$-invariant means. We can take for instance the point evaluation $m_n := m_{\delta_x} : f \mapsto f(x)$ for any $x \in X$.*

*Proof.* Let $m \in M_{T,(m_n)}$. As $m$ is an accumulation mean, for every $\varepsilon > 0$ and bounded $f : X \to \mathbb{C}$ there is a sequence $n_1 < n_2 < \cdots$ such that both $|m(f) - \overline{m}_{n_k}(f)| \leq \varepsilon$ and

$|m(f\circ T)-\overline{m}_{n_k}(f\circ T)|\le\varepsilon$ for all $k\in\mathbb{N}$. From the defining properties of $\overline{m}_{n_k}$ we obtain

$$\begin{aligned}|\overline{m}_{n_k}(f\circ T)-\overline{m}_{n_k}(f)| &= \frac{1}{n_k}\Big|\sum_{j=0}^{n_k-1} m_{n_k}(f\circ T^{j+1}-f\circ T^j)\Big| \\ &= \frac{1}{n_k}|m_{n_k}(f\circ T^{n_k})-m_{n_k}(f)|\le\frac{2}{n_k}\|f\|_\infty, \\ |m(f)-m(f\circ T)| &\le |m(f)-\overline{m}_{n_k}(f)|+|\overline{m}_{n_k}(f)-\overline{m}_{n_k}(f\circ T)| \\ &\quad+|\overline{m}_{n_k}(f\circ T)-m(f\circ T)| \\ &\le 2\varepsilon+\frac{2}{n_k}\|f\|_\infty.\end{aligned}$$

As this is true for all $k\in\mathbb{N}$ and $\varepsilon>0$ we obtain $T$-invariance of $m$. ■

We now study which values of $m(f)$ are possible for $m\in M(T,\mathcal{A})$ and $f\in\mathcal{A}$.

PROPOSITION 3.1.3. *Let $T:X\to X$, $\mathcal{A}\subseteq B(X)$ a $T$-invariant vector space, and $f\in\mathcal{A}_{\mathbb{R}}$. Then the set $\{m(f):m\in M(\mathcal{A},T)\}$ coincides with the interval $[a,b]$ where*

$$a=\lim_{n\to\infty}\inf_{x\in X}s_n(x)\quad\text{and}\quad b=\lim_{n\to\infty}\sup_{x\in X}s_n(x),\qquad s_n=s_{n,T,f}:=\frac{1}{n}\sum_{k=0}^{n-1}f\circ T^k. \tag{3.1}$$

*In particular, this set does not depend on $\mathcal{A}$.*

*Proof.* Note first that $m(s_n)=m(f)$ for every $m\in M(\mathcal{A},T)$. For the proof it suffices to show that, for every $\alpha\in\mathbb{R}$, there is an $m\in M(\mathcal{A},T)$ with $m(f)=\alpha$ if and only if the following condition is satisfied:

Condition (C): *For all $\varepsilon>0$ and $n\in\mathbb{N}$ there are $x=x(\varepsilon,n),y=y(\varepsilon,n)\in X$ such that $s_n(x)>\alpha-\varepsilon$ and $s_n(y)<\alpha+\varepsilon$.*

Necessity of (C): Let $m(f)=\alpha$ with $m\in M(\mathcal{A},T)$ and suppose, by contradiction, that (C) fails. Then there is an $\varepsilon>0$ and an $n\in\mathbb{N}$ such that $s_n(x)\le\alpha-\varepsilon$ for all $x\in X$ (the case $s_n(x)\ge\alpha+\varepsilon$ can be treated similarly), hence $m(f)=m(s_n)\le\|s_n\|_\infty\le\alpha-\varepsilon$, a contradiction.

Sufficiency of (C): Assume that (C) holds and consider the point measures $m_n:=\delta_{x(1/n,n)}$. With the notation of Proposition 3.1.2 this means $\overline{m}_n(f)>\alpha-1/n$ for all $n$. Use Proposition 3.1.2 to find an $m'\in M_{T,(m_n)}\subseteq M(\mathcal{A},T)$. Then $m'(f)\ge\alpha$. Similarly one finds an $m''\in M(\mathcal{A},T)$ with $m''(f)\le\alpha$. It follows that there is a $\lambda\in[0,1]$ such that

$$\lambda m'(f)+(1-\lambda)m''(f)=\alpha.$$

Since $M(\mathcal{A},T)$ is convex, $m:=\lambda m'+(1-\lambda)m''$ has the required properties. ■

Of particular interest are the functions $f$ with a unique mean value.

DEFINITION 3.1.4. Let $T:X\to X$ and $\mathcal{A}\subseteq B(X)$ be a $T$-invariant linear space. A function $f\in\mathcal{A}$ is called $\mathcal{A}$-*almost convergent* if $m(f)$ has the same value for all $m\in M(\mathcal{A},T)$. The set of all $\mathcal{A}$-almost convergent $f\in\mathcal{A}$ is denoted by $AC(\mathcal{A},T)$; for $\mathcal{A}=B(X)$ we also write $AC(B(X),T)=AC(T)$. We write $m_{\mathcal{A}}$ for the restriction of $m\in M(\mathcal{A},T)$ to $AC(\mathcal{A},T)$. If $AC(\mathcal{A},T)=\mathcal{A}$ we call $T$ *uniquely ergodic* (with respect to $\mathcal{A}$).

By definition, $m_{\mathcal{A}}$ does not depend on $m$. It is clear that $AC(\mathcal{A},T)$ is a $T$-invariant uniformly closed linear space containing all constant functions. Furthermore, $AC(\mathcal{A},T) = AC(T) \cap \mathcal{A}$. Finally, $f \in AC(\mathcal{A},T)$ with $m_{\mathcal{A}}(f) = \alpha$ if and only if $f \in \mathcal{A}$ and, for all $x_n \in X$,

$$\lim_{n\to\infty} s_{n,T,f}(x_n) = \frac{1}{n}\sum_{k=0}^{n-1} f(T^k(x_n)) = \alpha.$$

The obvious way to define $T$-invariance of a set algebra $\mathfrak{A}$ on $X$ or a finitely additive measure $p$ defined on $\mathfrak{A}$ is to require that $\{\mathbb{1}_A : A \in \mathfrak{A}\} \subseteq B(X)$ resp. $m_p$ as defined in Section 2.2 is $T$-invariant. Since $\mathbb{1}_A \circ T = \mathbb{1}_{T^{-1}[A]}$ this is the case if and only if $T^{-1}[A] \in \mathfrak{A}$ resp. $p(T^{-1}[A]) = p(A)$ for all $A \in \mathfrak{A}$. From Proposition 3.1.3 we get:

PROPOSITION 3.1.5. *The possible values $p(A)$ for $T$-invariant fapm $p$ are given by the interval $[a,b]$ where*

$$a = \lim_{n\to\infty} \inf_{x\in X} d_n(x), \qquad b = \lim_{n\to\infty} \sup_{x\in X} d_n(x), \qquad d_n(x) = \frac{1}{n}|\{k : T^k(x) \in A,\ 0 \le k < n\}|.$$

*In particular, $p(A)$ takes the same value for all $T$-invariant $p$ if and only if $a = b$.*

## 3.2. Applications

**3.2.1. Finite $X$.** Let $X$ be finite, $T : X \to X$ and $\mathcal{A} = B(X) = \mathbb{C}^X$. For every $x \in X$ there is a minimal $m \ge 0$ and a minimal $k > m$ such that $T^k(x) = T^m(x)$. We call $C_x := \{T^m(x), T^{m+1}(x), \dots, T^{k-1}(x)\}$ the *cycle (cyclic attractor)* induced by $x$ and $B_x = B(C_x) := \{y \in X : C_y = C_x\}$ the *basin* of $C_x$. It is clear that $C_x \subseteq B_x$, the $B_x$ forming a partition. $C_x = B_x$ if and only if the restriction of $T$ to this set is bijective. Furthermore, the $s_n = s_{n,T,f}$ defined by

$$s_n(x) = \frac{1}{n}\sum_{k=1}^{n-1} f(T^k(x))$$

converge to a function $\bar{f}$ which, on each $C = C_x$, takes the constant value

$$m_C(f) := \frac{1}{|C|}\sum_{y\in C} f(y).$$

It is clear that $m_C \in M(B(X),T)$ for each cycle $C$. The same holds for all convex combinations. We claim that, conversely, every $m \in M(B(X),T)$ is of this type, i.e. $m = \sum_C \lambda_C m_C$ with $0 \le \lambda_C \le 1$ for all $C$ and $\sum_C \lambda_C = 1$. To see this, define $\lambda_C := m(\mathbb{1}_{B(C)})$ and observe that

$$\bar{f} = \sum_C m_C(f)\mathbb{1}_{B(C)}.$$

This implies

$$m(f) = m(s_n) = m(\bar{f}) = \sum_C m_C(f) m(\mathbb{1}_{B(C)}) = \sum_C \lambda_C m_C(f).$$

The uniqueness of the $\lambda_C$ follows since the $m_C$ are linearly independent. This gives an obvious description of almost convergent functions: $f \in AC(B(X),T)$ if and only if $m_C(f)$ takes the same value for all cycles $C$.

In terms of measures this means that every $T$-invariant $p$ is a convex combination of the ergodic measures $p_C$ defined by $p_C(A) := |A \cap C|/|C|$. This is the finite version of the ergodic decomposition given by Birkhoff's Ergodic Theorem (cf. for instance [55]). Infinite $X$ would have to be treated in this context, but we do not go further into this direction.

**3.2.2. $X = \mathbb{Z}$, $T : x \mapsto x + 1$.** First note that whenever $\mathcal{A} \subseteq B(\mathbb{Z})$ is two-sided $T$-invariant then $T$-invariance of a mean $m$ or a fam $p$ implies invariance with respect to all translations on the additive group $\mathbb{Z}$. In Section 3.6 we will focus on this aspect. Here we want to apply our analysis from Section 3.1. For this reason we have to consider the quantities

$$s_{N,f}(n) := \frac{1}{N} \sum_{k=n}^{n+N-1} f(k)$$

and, for real-valued $f$, the corresponding lower and upper limits

$$m_*(f) := \lim_{N\to\infty} \inf_{n\in\mathbb{Z}} s_{N,f}(n) \quad \text{and} \quad m^*(f) := \lim_{N\to\infty} \sup_{n\in\mathbb{Z}} s_{N,f}(n).$$

For $f = \mathbb{1}_A$ these values $m_*(A) := m_*(\mathbb{1}_A)$ and $m^*(A) := m^*(\mathbb{1}_A)$ are known as *lower Banach density* resp. *upper Banach density* of $A$. The possible values of $T$-invariant measures are given by

$$\{m(f) : m \in M(T, \mathcal{A})\} = [m_*(f), m^*(f)],$$

hence

$$AC(\mathcal{A}, T) = \{f \in \mathcal{A} : m_*(f) = m^*(f)\}.$$

The restriction of $m_*$ and $m^*$ to $AC(\mathbb{Z}) := AC(B(X), T)$ is known as *Banach density* and denoted by $m_B$.

We have already mentioned that the set $AC(\mathbb{Z})$ of almost convergent $f$ on $\mathbb{Z}$ is a linear space and, furthermore, uniformly closed. Having the results about compactifications and complex-valued functions in mind, we ask whether $AC(\mathbb{Z})$ is an algebra as well. But this is not the case as the following example shows.

Example 3.2.1. Consider the sets $A = 2\mathbb{Z}$ of even numbers and $B = (B_1 \cup B_2) \cup (-B_1 \cup -B_2)$ with

$$B_1 = \bigcup_{n=1}^{\infty} ((2n-1)!, (2n)!] \cap 2\mathbb{Z}, \qquad B_2 = \bigcup_{n=1}^{\infty} ((2n)!, (2n+1)!] \cap (2\mathbb{Z}+1).$$

In $B$ one has very long blocks of even numbers alternating with very long blocks of odd numbers. It is clear that both $A$ and $B$ have Banach density $1/2$ while $B_1 = A \cap B$ has lower Banach density 0 not coinciding with its upper Banach density $1/2$. It follows that $\mathbb{1}_{B_1} = \mathbb{1}_A \cdot \mathbb{1}_B \notin AC(\mathbb{Z})$ although $\mathbb{1}_A, \mathbb{1}_B \in AC(\mathbb{Z})$. Thus $AC(\mathbb{Z})$ is not an algebra. In particular, there is no compactification $(\iota, K)$ of $\mathbb{Z}$ such that $AC(\mathbb{Z})$ is the set of all $f$ having a continuous (or Riemann integrable) representation in $(\iota, K)$.

**3.2.3. $X$ compact, $\mathcal{A} = C(X)$, $T$ continuous.** Let $X$ be a compact space, $\mathcal{A} = C(X)$ the algebra of complex-valued continuous functions on $X$ and $T : X \to X$ continuous. Then $C(X)$ is $T$-invariant since $f \in C(X)$ implies $T^*f = f \circ T \in C(X)$. (If $T$ is bijective then it is a homeomorphism, hence $C(X)$ is even two-sided $T$-invariant.) This is the framework of classical topological dynamics.

Proposition 3.1.2 guarantees that there are $T$-invariant means on $C(X)$. By Riesz' Representation Theorem 2.3.1 every $m \in M(T, C(X))$ induces a unique regular Borel probability measure $\mu_m$ with

$$m(f) = \int_X f \, d\mu_m \quad \text{for all } f \in C(X).$$

Definition 3.2.2. Let $\mu$ be a measure defined on a $\sigma$-algebra $\mathfrak{A}$ on $X$ and $T : X \to X$ measurable. Then $\mu$ is called *$T$-invariant* if $\mu(T^{-1}[A]) = \mu(A)$ for all $A \in \mathfrak{A}$.

In the context of compact $X$ we are particularly interested in the case where $\mathfrak{A}$ contains all Borel sets and that $\mu$ is complete. In the case of regular Borel measures the invariance of a mean $m$ is equivalent to invariance of the corresponding $\mu_m$:

Proposition 3.2.3. *Let $X$ be a compact space, $T : X \to X$ continuous, $m$ a mean on $C(X)$ and $\mu$ a Borel measure, i.e. defined on a $\sigma$-algebra $\mathfrak{A}$ containing all Borel sets, such that $m(f) = \int_X f \, d\mu$ for all $f \in C(X)$. Then:*

(i) *If $\mu$ is $T$-invariant then $m$ is $T$-invariant.*
(ii) *Assume that $\mu$ is regular. If $m$ is $T$-invariant then $\mu$ is $T$-invariant.*
(iii) *The implication in the second statement does not hold if one drops the regularity assumption on $\mu$.*

*Sketch.* (i) In order to show that $m(f \circ T) = m(f)$ for all $f \in C(X)$ one first considers $f = \mathbb{1}_A$ with $A \in \mathcal{A}$, then linear combinations of such $f$, and finally one uses the fact that any $f \in C(X)$ can be uniformly approximated by such linear combinations.

(ii) Let $A \in \mathfrak{A}$ and $\varepsilon > 0$. By regularity of $\mu$ there are a closed set $C$ and an open set $O$ such that $C \subseteq A \subseteq O \subseteq X$ and $\mu(O \setminus C) < \varepsilon$, and corresponding Urysohn functions, i.e. continuous $f, g : X \to [0, 1]$ with $\mathbb{1}_C \leq f \leq \mathbb{1}_A \leq g \leq \mathbb{1}_O$. By invariance of $m$ we obtain

$$\mu(T^{-1}[A]) \leq m(g \circ T) = m(g) \leq \mu(O) \leq \mu(A) + \varepsilon$$

and similarly $\mu(T^{-1}[A]) \geq \mu(A) - \varepsilon$, hence $\mu(T^{-1}[A]) = \mu(A)$.

(iii) We use the example $X = [0, \omega_1]$ from the end of Section 2.3 and the constant mapping $T : X \to X$, $x \mapsto \omega_1$. Then the point evaluation mean $m : f \mapsto f(\omega_1)$ is $T$-invariant (as also is the corresponding point measure $\delta_{\omega_1}$ concentrated at $\omega_1$). The measure $\mu$ from the end of Section 2.3 satisfies $m(f) = \int_X f(x) \, d\mu$ for all $f \in C(X)$. Nevertheless, for $A = \{\omega_1\}$ we have $\mu(A) = 0 \neq 1 = \mu(X) = \mu(T^{-1}[A])$. ∎

If $M(C(X), T)$ consists of only one measure, we say that $T$ is *uniquely ergodic*.

Corollary 3.2.4. *Let $M(C(X), T) = \{m\}$. For every $f \in C(X)$,*

$$s_{n,T,f}(x) := \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) \to m(f) \qquad \textit{uniformly in } x \in X,$$

*in particular, the uniform closure of the convex hull of the $T$-orbit contains the constant function $m(f) \cdot \mathbb{1}_X$.*

*Proof.* By Proposition 3.1.3 we know that

$$\lim_{n\to\infty}\Big(\inf_{x\in X} s_{n,T,f}(x)\Big) = m(f) = \lim_{n\to\infty}\Big(\sup_{x\in X} s_{n,T,f}(x)\Big),$$

i.e. for every $\varepsilon > 0$ and large enough $n$ we have $\sup_{x\in X} |s_{n,T,f}(x) - m(f)| < \varepsilon$. ∎

If $T$ is uniquely ergodic, $m(f)\mathbb{1}_X$ is the only constant function in the uniform closure of the convex hull of the $T$-orbit. For an arbitrary constant $\lambda \in \mathbb{C}$ in this closure we have

$$\lambda = m(\lambda\mathbb{1}_X) = \lim_{k\to\infty} m(f\circ T^{n_k}) = m(f) \quad \text{whenever } f\circ T^{n_k} \to \lambda\mathbb{1}_X.$$

**3.2.4. Shift spaces and symbolic dynamics.** We now consider a special case of the situation treated in Section 3.2.3. Let $A$ be a finite set. In this context $A$ is called an *alphabet* and its members are called *symbols*. Furthermore, let $X$ be a closed subset of the compact space $A^{\mathbb{Z}}$ which is *shift invariant*, i.e. $\sigma(X) = X$ for the shift $\sigma : (a_k)_{k\in\mathbb{Z}} \mapsto (a_{k+1})_{k\in\mathbb{Z}}$. Such dynamical systems $(X,\sigma)$ are also called *subshifts* and are the main objects of symbolic dynamics.

The importance of these apparently simple objects is due to the abundance of $\sigma$-invariant closed subsets $X$ of $A^{\mathbb{Z}}$ by means of which a quite big class of dynamical systems can be represented in a reasonable way. Assume that $Y$ is a compact space, $T : Y \to Y$ a continuous transformation, $Y = Y_0 \cup Y_1 \cup \cdots \cup Y_{s-1}$ (pairwise disjoint union), $s \in \mathbb{N}$, a finite partition of $Y$ and $F : Y \to \{0,1,\ldots,s-1\}$, $y \mapsto i$, if $y \in Y_i$.

Note that $F$ is in general (in particular for $Y$ connected) not continuous, but if the $Y_i$ are continuity sets with respect to an appropriate measure on $Y$, Riemann integrable. Therefore we should not expect that the induced mapping $y \mapsto x(y) := (F(T^k y))_{k\in\mathbb{Z}}$ is continuous. But we may consider the closure $X$ of its image which is shift invariant. In many interesting cases with $Y$ metrizable there is a continuous surjection $\varphi : X \to Y$ such that $y = \varphi(x)$ whenever $x = x(y)$. One says that $(Y,T)$ is a *factor* of $(X,\sigma)$ and we have the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\sigma"] & Y \arrow[d, "T"] \\ X \arrow[r, "\varphi"] & Y \end{tikzcd}$$

Example 3.2.5 (Sturmian sequence). Let $Y = \mathbb{R}/\mathbb{Z}$, $\alpha \in Y$ and $T$ be the homeomorphism $y \mapsto y+\alpha$ (rotation). Consider the partition of $Y$ into two segments $Y_0 = [0,\alpha)+\mathbb{Z}$ and $Y_1 = [\alpha,1)+\mathbb{Z}$. Hence $A = \{0,1\}$. For the $T$-orbit $x(0)$ of $y = 0$, i.e. $x(0) = (a_k)_{k\in\mathbb{Z}}$ with $a_k = 1$ if $k\alpha \in Y_1$ and $a_k = 0$ if $k\alpha \in Y_0$, it turns out that $X$ is the closure of the shift orbit $\{\sigma^k(x(0)) : k \in \mathbb{Z}\}$ of $x(0)$ in $\{0,1\}^{\mathbb{Z}}$. The two-sided sequence $x(0)$ is an example of a so called Sturmian sequence, a class providing some of the simplest but typical examples of Hartman functions, the main topic of the forthcoming chapters.

In order to understand invariant means, or equivalently by Riesz' Representation Theorem 2.3.1 and Proposition 3.2.3, invariant measures on shift spaces $X \subseteq A^{\mathbb{Z}}$, note that every invariant Borel probability measure on $X$ is uniquely determined by its values on cylinder sets $S = [b_0,\ldots,b_{n-1}] = \{(a_k)_{k\in\mathbb{Z}} : a_k = b_k,\ k = 0,\ldots,n-1\}$. Thus it suffices to consider functions $f = \mathbb{1}_S$ for such $S$ and apply the arguments from Section 3.1. For instance, the numbers $a, b$ in Proposition 3.1.3 can be described in terms of relative frequencies of blocks $(b_0, \dots, b_{n-1})$ in symbolic sequences $x \in X$.

**3.2.5. The free group $F(x, y)$.** Let $X = F(x, y)$ denote the group with two free generators $x$ and $y$. As usual we assume each member $w \in F(x, y)$ to be a reduced word built up from the four allowed symbols $x, y, x^{-1}, y^{-1}$, including the empty word $w = \emptyset$. Denote by $W_x$ all reduced words ending with the symbol $x$; $W_{x^{-1}}, W_y$ and $W_{y^{-1}}$ are defined analogously. Consider the transformation $T_1 : w \mapsto wx^{-1}$. In particular, we have $T_1(W_x) = W_x \cup W_y \cup W_{y^{-1}} \cup \{\emptyset\}$. Assume $m_1 \in M(T_1, B(X))$. Then, for the associated fapm $p_1 = m_{p_1}$ we compute

$$0 = p_1(T_1(W_x)) - p_1(W_x) = p_1(W_y \cup W_{y^{-1}} \cup \{\emptyset\}). \tag{3.2}$$

By symmetry, for $T_2 : w \mapsto wy^{-1}$ we have $T_2(W_y) = W_y \cup W_x \cup W_{x^{-1}} \cup \{\emptyset\}$ and every $m_2 \in M(T_2, B(X))$ satisfies

$$0 = p_2(T_2(W_y)) - p_2(W_y) = p_2(W_x \cup W_{x^{-1}} \cup \{\emptyset\}). \tag{3.3}$$

In particular, we have $T_1(W_x) \setminus W_x \cup T_2(W_y) \setminus W_y = X$. Assume $m \in M(T_1, B(X)) \cap M(T_2, B(X))$. The associated fapm $p = m_p$ satisfies

$$1 = p(X) = p(T_1(W_x) \setminus W_x \cup T_2(W_y) \setminus W_y) \tag{3.4}$$

$$\leq p(T_1(W_x)) - p(W_x) + p(T_2(W_y)) - p(W_y) = 0, \tag{3.5}$$

a contradiction. Thus there is no mean on $B(X)$ which is both $T_1$- and $T_2$-invariant.

Since $T_1$ and $T_2$ are group translations this shows that $X = F(x, y)$ is not an amenable group (see also Section 3.6.4). Together with the fact that $F(x, y)$ can be realized as a group of orthogonal transformations of $\mathbb{R}^3$ this is the core of the celebrated Banach–Tarski paradox (see [54]). We will focus on (semi-)group structures systematically in Section 3.6.

**3.3. Compactifications for transformations and actions.** We have seen the importance of compactifications for means and measures already in Chapter 2. The rôle of transformations for identifying interesting measures in terms of invariance properties was pointed out in Section 3.1. We now combine both points of view by investigating the following setting.

DEFINITION 3.3.1. Let $X$ be a topological space, $T : X \to X$ continuous and $(\iota, K)$ a compactification of $X$. Then $(\iota, K)$ is called a compactification *compatible* with $T$ and, vice versa, $T$ a transformation *compatible* with $(\iota, K)$ if there is a continuous $T_K : K \to K$ such that $\iota \circ T = T_K \circ \iota$, i.e. making the diagram

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[d, "T"] & K \arrow[d, "T_K"] \\ X \arrow[r, "\iota"] & K \end{tikzcd}$$

commutative. $T_K$ is called a *continuous extension* of $T$ in $(\iota, K)$.

Continuous extensions are unique and compatible with composition:

PROPOSITION 3.3.2. *Let $T_K$ and $T'_K$ be continuous extensions of $T : X \to X$ in the compactification $(\iota, K)$ of $X$. Then $T_K = T'_K$. Furthermore, if $S_K$ is a continuous extension of $S : X \to X$ in $(\iota, K)$ then $(S \circ T)_K = S_K \circ T_K : K \to K$ is a continuous extension of $S \circ T : X \to X$.*

*Proof.* The first statement follows since $T_K$ and $T'_K$ are continuous and coincide on the dense set $\iota(X) \subseteq K$. For the second statement observe

$$(S_K \circ T_K) \circ \iota = S_K \circ (T_K \circ \iota) = S_K \circ (\iota \circ T) = (S_K \circ \iota) \circ T = (\iota \circ S) \circ T = \iota \circ (S \circ T)$$

and use the uniqueness of continuous extensions to obtain $S_K \circ T_K = (S \circ T)_K$. ∎

Note that from a certain point of view, for continuous extensions of transformations $T : X \to X$ the situation is more complicated than for representations of complex-valued $f : X \to \mathbb{C}$ in the sense of Section 3.5. This is due to the fact that there is no obvious analogue of Proposition 2.4.9 which yields a natural compactification for $f$ or even for a unital $C^*$-algebra $\mathcal{A}$ which is minimal. Proposition 2.4.9 was based on Proposition 2.4.5 implying that for each compactification allowing a continuous representation every bigger compactification has the same property. The following example shows that this is not true for $T : X \to X$.

EXAMPLE 3.3.3. Let $X = \mathbb{Z}$, $T : k \mapsto k+1$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and consider the compactifications $(\iota_i, K_i)$, $i = 1, 2$, given by $K_1 := \mathbb{T} = \mathbb{R}/\mathbb{Z}$, $\iota_1 : k \mapsto k\alpha + \mathbb{Z}$, and $K_2 := [0, 1]$, $\iota_2 : k \mapsto \{k\alpha\} := k\alpha - \max\{m \in \mathbb{Z} : m \leq k\alpha\}$. Obviously $(\iota_1, K_1) \leq (\iota_2, K_2)$ via $\pi : K_2 = [0, 1] \to \mathbb{R}/\mathbb{Z} = K_2$, $x \mapsto x + \mathbb{Z}$. The compactification $(\iota_1, K_1)$ is compatible with $T$ since $T_1 : x \mapsto x + \alpha$ is a continuous extension of $T$ in $(\iota_1, K_1)$. There is no continuous extension $T_2$ of $T$ in $(\iota_2, K_2)$. Indeed, suppose that such a $T_2 : [0, 1] \to [0, 1]$ exists. Since $\iota_2(\mathbb{Z})$ is dense in $[0, 1]$ we can find a sequence $x_n = \{k_n\alpha\} = \iota_2(k_n)$ such that $x_n \to 1 - \alpha$. Furthermore, we can arrange $0 \leq x_n < 1 - \alpha$ for all $n \in \mathbb{N}$. Using $T_2 \circ \iota_2 = \iota_2 \circ T$ we get

$$\begin{aligned} T_2(1-\alpha) &= \lim_{n\to\infty} T_2(x_n) = \lim_{n\to\infty} T_2 \circ \iota_2(k_n) = \lim_{n\to\infty} \iota_2 \circ T(k_n) \\ &= \lim_{n\to\infty} \iota_2(k_n + 1) = \lim_{n\to\infty} \{k_n\alpha + \alpha\} = \lim_{n\to\infty} x_n + \alpha = 1. \end{aligned}$$

We now pick another sequence $y_n = \{l_n\alpha\} = \iota_2(l_n)$ such that $y_n \to 1 - \alpha$ but now with the requirement $1 - \alpha < y_n \leq 1$ for all $n \in \mathbb{N}$. Similarly, we get

$$\begin{aligned} T_2(1-\alpha) &= \lim_{n\to\infty} T_2(y_n) = \lim_{n\to\infty} T_2 \circ \iota_2(l_n) = \lim_{n\to\infty} \iota_2 \circ T(l_n) \\ &= \lim_{n\to\infty} \iota_2(l_n + 1) = \lim_{n\to\infty} \{l_n\alpha + \alpha\} = \lim_{n\to\infty} y_n + \alpha - 1 = 0, \end{aligned}$$

a contradiction.

We see that taking bigger compactifications does not guarantee that we find continuous extensions. Nevertheless, for the Stone–Čech compactification $(\iota_\beta, \beta X)$ (cf. Section 2.5), everything works. In particular, the first statement of Proposition 2.5.3 applies to $K = \beta X$: For every continuous $T : X \to X$ the map $\iota_\beta \circ T : X \to \beta X$ is continuous. Therefore there is a continuous $T_\beta : \beta X \to \beta X$ such that $T_\beta \circ \iota_\beta = \iota_\beta \circ T$.

DEFINITION 3.3.4. Let $X$ be a completely regular space and $T : X \to X$ be continuous. Then $T_\beta : \beta X \to \beta X$ denotes the (unique) continuous extension of $T$ in the Stone–Čech compactification $(\iota_\beta, \beta X)$ of $X$.

There is no obstacle to considering families of transformations instead of a single $T$. In order to proceed in this direction recall the notion of (semi)group actions.

Definition 3.3.5. Let $S$ be a semigroup and $X$ a set. A mapping $\alpha : S \times X \to X$, $(s, x) \mapsto \alpha(s, x)$, is called a *semigroup action* of $S$ on $X$ if $\alpha(s_1, s_2(x)) = \alpha(s_1 s_2, x)$ for all $s_1, s_2 \in S$ and all $x \in X$.

This construction carries over to groups in the obvious way:

Definition 3.3.6. Let $G$ be a group and $X$ a set. A mapping $\alpha : G \times X \to X$, $(g, x) \mapsto \alpha(g, x)$, is called a *group action* of $G$ on $X$ if it is a semigroup action of $G$ considered as a semigroup and $\alpha(e_G, x) = x$ for the unit element $e_G \in G$ and all $x \in X$.

For a semigroup action the maps $s^\alpha : x \mapsto \alpha(s, x)$ are self-maps of $X$. If we impose a semigroup structure on $X^X$, the set of all maps $f : X \to X$, by using the composition of maps as semigroup operation, a semigroup action of $S$ on $X$ is nothing other than a homomorphism $\alpha : S \to X^X$, $s \mapsto s^\alpha$. Similarly we can impose a group structure on $\mathrm{Sym}(X) = \{f \in X^X : f \text{ bijective}\}$. In the group case we have

$$s^\alpha (s^{-1})^\alpha = (ss^{-1})^\alpha = \mathrm{Id}_X = (s^{-1})^\alpha s^\alpha,$$

thus a group action is a homomorphism $G \to \mathrm{Sym}(X)$. So far the set $X$ on which $\alpha$ acts carries no structure itself.

Definition 3.3.7. Let $X$ be a topological space and $\alpha : S \times X \to X$ a semigroup action. If $s^\alpha$ is continuous for every $s \in S$ we say that $S$ *acts by continuous maps* on $X$.

Let $\alpha$ be a semigroup action of $S$ on $X$ by continuous maps. Suppose that all $s^\alpha : X \to X$ have continuous extensions $s^\alpha_K : K \to K$ in the compactification $(\iota, K)$ of $X$. As a consequence of Proposition 2.5.3 we have $(st)^\alpha_K = s^\alpha_K \circ t^\alpha_K$ for all $s, t \in S$, hence $\alpha_K : S \times K \to K$, $(s, c) \mapsto s^\alpha_K(c)$, defines a semigroup action of $S$ on $K$ by continuous maps.

Definition 3.3.8. The action $\alpha_K$ defined as above is called the *extension* of the action $\alpha$ in the compactification $(\iota, K)$ of $X$. For $(\iota, K) = (\iota_\beta, \beta X)$ all $s^\alpha$ have continuous extensions and we write $\alpha_\beta$ for $\alpha_K$.

Definition 3.3.9. Assume now that $S$ is equipped with a topology for which the semigroup operation $S \times S \to S$, $(s_1, s_2) \mapsto s_1 s_2$, is jointly continuous on $S \times S$. Then $S$ is called a *topological semigroup* (see also Section 3.5).

Definition 3.3.10. Let $S$ be a topological semigroup which acts by continuous maps on $X$. The semigroup action $\alpha : S \times X \to X$ is called a *jointly continuous semigroup action* if $\alpha$ is jointly continuous on $S \times X$.

In the next section we will analyze when the extension $\alpha_\beta$ of a jointly continuous semigroup action $\alpha$ is again a jointly continuous semigroup action.

**3.4. Separate and joint continuity of operations.** Let us now focus on the case of a discrete semigroup $S$. Then the semigroup operation $\alpha : S \times S \to S$, $(s, t) \mapsto st$, is an action of $S$ on itself which has a continuous extension $\alpha_\beta : S \times \beta S \to \beta S$ to its Stone–Čech compactification $(\iota_\beta, \beta S)$. All $r_a : S \to \beta S$, $r_a(s) := \alpha_\beta(s, a)$, $a \in \beta S$, have continuous extensions $\rho_a : \beta S \to \beta S$ fulfilling $\rho_a(\iota_\beta(s)) = r_a(s) = \alpha_\beta(s,a)$. Consider now the operation $* : \beta S \times \beta S \to \beta S$, $(a,b) \mapsto a * b := \rho_b(a)$. This operation is described in the following statement.

Proposition 3.4.1. *Let $S$ be a discrete semigroup. Then there is a unique semigroup operation $* : \beta S \times \beta S \to \beta S$ on the Stone–Čech compactification $(\iota_\beta, \beta S)$ of $S$ such that:*

(i) *$*$ extends the semigroup operation on $S$, i.e. $\iota_\beta(s) * \iota_\beta(t) = \iota_\beta(st)$ for all $s,t \in S$.*
(ii) *The right translations $\rho_a : \beta S \to \beta S$, $x \mapsto x * a$, are continuous for all $a \in \beta S$.*
(iii) *The left translations $\lambda_a : \beta S \to \beta S$, $x \mapsto a * x$, are continuous for all $a \in \iota_\beta(S)$.*

*Proof.* It suffices to prove that the operation $*$ defined before the proposition is associative. For all $s,t,u \in \iota_\beta(S)$ we have

$$\lambda_{s*t}(u) = (s*t)*u = s*(t*u) = \lambda_s \circ \lambda_t(u).$$

Since $\lambda_{s*t}$ and $\lambda_s \circ \lambda_t$ are continuous and $\iota_\beta(S)$ is dense in $S$ this equation extends to $\lambda_{s*t}(z) = \lambda_s \circ \lambda_t(z)$ for all $z \in \beta S$. Hence

$$\rho_z \circ \lambda_s(t) = (s*t)*z = \lambda_{s*t}(z) = \lambda_s \circ \lambda_t(z) = s*(t*z) = \lambda_s \circ \rho_z(t).$$

Since $\rho_z \circ \lambda_s$ and $\lambda_s \circ \rho_z$ are continuous this equation similarly extends to $\rho_z \circ \lambda_s(y) = \lambda_s \circ \rho_z(y)$ for all $y \in \beta S$. Hence

$$\rho_z \circ \rho_y(s) = (s*y)*z = \rho_z \circ \lambda_s(y) = \lambda_s \circ \rho_z(y) = s*(y*z) = \rho_{y*z}(s).$$

Once again, since $\rho_z \circ \rho_y$ and $\rho_{y*z}$ are continuous this equation extends to $\rho_z \circ \rho_y(x) = \rho_{y*z}(x)$ for all $x \in \beta S$, hence

$$(x*y)*z = \rho_z \circ \rho_y(x) = \rho_{y*z}(x) = x*(y*z)$$

for all $x,y,z \in \beta S$. ∎

For much more information about the algebraic structure of $\beta S$ we refer to [23]. There one can also find information about related constructions as the enveloping semigroup of a semigroup of continuous transformations etc. We are now going to show that $*$ is jointly continuous only in very special cases (which are not particularly interesting). Not for maximizing generality but in order to identify the natural context we use terminology from General Algebra.

Definition 3.4.2. For any set $X$, a function $\omega : X^n \to X$, $n \in \mathbb{N}$, is called an *$n$-ary operation on $X$*. If $(\iota, K)$ is a compactification of $X$, $\omega_K : K^n \to K$ is called an *extension* of $\omega$ if $\iota \circ \omega = \omega_K \circ \iota^n$ with $\iota^n : X^n \to X^n$, $(x_1, \ldots, x_n) \mapsto (\iota(x_1), \ldots, \iota(x_n))$. $(\iota, K)$ is called *compatible* with $\omega$ and vice versa if a continuous extension $\omega_K$ of $\omega$ exists.

Remark 3.4.3. If $\omega_0$ is an $m$-ary operation on $X$ and $\omega_1, \ldots, \omega_m$ are $n$-ary operations on $X$ then $\omega(x_1, \ldots, x_n) := \omega_0(\omega_1(x_1, \ldots, x_n), \ldots, \omega_m(x_1, \ldots, x_n))$ defines an $n$-ary operation $\omega$ on $X$, called the *composition* of $\omega_0$ and the $\omega_i$, $i = 1, \ldots, m$. If all the operations involved are continuous then so is $\omega$. Other (trivial) examples of continuous $n$-ary operations are the projections $\pi_i^n : (x_1, \ldots, x_n) \mapsto x_i$, $1 \le i \le n$. A set $\Omega$ of operations on $X$ which contains all projections and is closed under composition is called a *clone* on $X$. Thus, for every family of continuous $n_i$-ary operations $\omega_i$ on $X$, $i \in I$, all operations in the clone generated by the $\omega_i$ are continuous as well. A standard reference on the clone of continuous functions is [50]. From this point of view the traditional approach in General Algebra, namely to define a universal algebra as an object of the type $(X, (\omega_i)_{i\in I})$, is intimately connected with the investigation of clones. In particular, for $X$ infinite there is indeed much current research on clones on $X$ (cf. [16]). But here we proceed in a different direction.

Recall that, by definition, a topological space is 0-*dimensional* if there exists a topological basis of clopen sets.

LEMMA 3.4.4. *Let $X$ be a 0-dimensional compact Hausdorff space and $R \subseteq X^n$ a clopen subset. Then $R = \bigcup_{i=1}^k R_i$ is a finite union of generalized rectangles $R_i = A_{i,1} \times \cdots \times A_{i,n}$ with clopen $A_{i,j} \subseteq X$, $i = 1, \ldots, k$, $j = 1, \ldots, n$. The $R_i$ can be taken pairwise disjoint and such that all the $A_{i,j}$ are from a fixed finite partition $P = \{A_1, \ldots, A_k\}$.*

*Proof.* Pick $x = (x_1, \ldots, x_n) \in R$. Since $R$ is open and $X$ has a clopen basis there are clopen neighborhoods $A_{x,i}$ of $x_i$ such that $A_x := A_{x,1} \times \cdots \times A_{x,n} \subseteq R$. Hence $R = \bigcup_{x\in R} A_x$. This covering is open. Since $R$, being a closed subset of $X$, is compact, finitely many $R_i := A_{x_i}$, $i = 1, \ldots, k$, form a covering as well. It is clear that by finite refinements, the $R_i$ can be taken pairwise disjoint and all the resulting $A_{i,j}$ form one finite partition. ∎

This lemma yields a characterization of operations having a continuous extension in the Stone–Čech compactification.

THEOREM 1. *Let $X$ be discrete and $\omega : X^n \to X$ an $n$-ary operation on $X$. Then $\omega$ has a continuous extension in the Stone–Čech compactification $(\iota_\beta, \beta X)$ if and only if for every $S \subseteq X$ the preimage is a finite union of rectangles, i.e.*

$$\omega^{-1}[S] = \bigcup_{i=1}^{k} R_i \quad \textit{with} \quad R_i = A_{i,1} \times \cdots \times A_{i,n}.$$

*Proof.* Necessity: Assume that $\omega_\beta$ is the continuous extension of $\omega$ in $(\iota_\beta, \beta X)$. $S^* = \overline{\iota_\beta(S)}$ (notation as in Proposition 2.5.3) is clopen, hence, by continuity of $\omega_\beta$, $\omega_\beta^{-1}[S^*]$ is clopen as well. So Lemma 3.4.4 applies, showing that this set is a finite union of rectangles. This immediately translates to the same property of $\omega^{-1}[S]$.

Sufficiency: Assume that for $\omega : X^n \to X$ all preimages $\omega^{-1}[S]$, $S \subseteq X$, are finite unions of rectangles. We have to construct a continuous extension $\omega_\beta$ of $\omega$ in $(\iota_\beta, \beta X)$. We use the ultrafilter description from Proposition 2.5.3. So let $p_1, \ldots, p_n$ be ultrafilters on $X$. We define an ultrafilter $p := \omega_\beta(p_1, \ldots, p_n)$ on $X$ by letting $F \subseteq X$ be a member of $p$ if and only if $\omega(F_1 \times \cdots \times F_n) \subseteq F$ for some sets $F_i \in p_i$. It is straightforward to check that $\emptyset \notin p$, that $F \in p$ and $F \subseteq F' \subseteq X$ implies $F' \in p$, and that $F, F' \in p$ implies $F \cap F' \in p$. But $p$ is even maximal: For arbitrary $F \subseteq X$ our assumption shows that $R := \omega^{-1}[F]$ can be taken as stated in Lemma 3.4.4. Since for each $j = 1, \ldots, n$, $p_j$ is an ultrafilter on $X$, there is exactly one $A_{k_j} \in p$ such that $A_{k_j} \in p_j$. For the rectangle $R' := A_{k_1} \times \cdots \times A_{k_n}$ we have either $R' \subseteq R$ or $R' \subseteq X \setminus R$. In the first case this implies $F \in p$, in the second case $X \setminus F \in p$, showing that $p$ is an ultrafilter.

Finally, we have to prove that $\omega_\beta$ is continuous on $(\beta X)^n$. We use Proposition 2.5.3 several times. Take arbitrary ultrafilters $p_1, \ldots, p_n \in \beta X$ and any neighborhood $U$ of

$p := \omega_\beta(p_1, \dots, p_n)$. By the definition of the topology on $\beta X$ there is a set $F \subseteq X$ such that $F \in p_1$ and $U$ contains all ultrafilters $p$ with $F \in p$. By the definition of $\omega_\beta$ there are $F_i \in p_i$ such that $\omega(F_1 \times \dots \times F_n) \subseteq F$. Each $F_i$ defines a neighborhood $U_i$ of $p_i$ consisting of all ultrafilters which contain $F_i$. It is clear that $\omega_\beta(U_1 \times \dots \times U_n) \subseteq U$, showing that $\omega_\beta$ is continuous. ∎

Corollary 3.4.5. *Let $S$ be an infinite discrete group. Then there is no continuous extension of the group operation on $S$ to $(\iota_\beta, \beta S)$.*

*Proof.* Preimages of singletons are infinite but contain only singleton rectangles, thus cannot be finite unions of rectangles. ∎

Similar arguments apply for many semigroups as $\mathbb{N}$ with addition or with multiplication, and for infinite totally ordered sets with min or max as semigroup operation.

Definition 3.4.6. An $n$-ary operation $\omega : X^n \to X$ is called *essentially unary* (depending on the $i$th component) if there is an $f : X \to X$ such that $\omega(x_1, \dots, x_n) = f(x_i)$ for all $x_1, \dots, x_n \in X^n$. $\omega$ is called *locally essentially unary* if there is a finite partition of $X$ into sets $A_i$, $i = 1, \dots, k$, such that the restriction of $f$ to each rectangle $R = A_{i_1} \times \dots \times A_{i_n}$, $i_j \in \{1, \dots, k\}$ is essentially unary.

Proposition 3.4.7. *Let $\omega : X^n \to X$ be locally essentially unary. Then there is a continuous extension $\omega_\beta$ of $\omega$ to $(\iota_\beta, \beta X)$.*

*Proof.* As the reader checks easily, every locally unary operation $\omega$ satisfies the condition of Theorem 1. ∎

Continuing work of van Douwen [52], Farah was able to show in [10, 11] that the converse of Proposition 3.4.7 also holds true.

Proposition 3.4.8 (Farah). *Let $X$ be an infinite discrete set and assume that $\omega : X^n \to X$ has a continuous extension to $(\iota_\beta, \beta X)$. Then $\omega$ is locally essentially unary.*

**3.5. Compactifications for operations.** We have seen in the previous section that many interesting binary operations cannot be continuously extended to the Stone–Čech compactification. Nevertheless, some ideas presented in Section 2.4 can be adapted. In order to be more flexible it is useful to consider the following setting.

Definition 3.5.1. Let $I$ be an index set, $n_i \in \mathbb{N}$ and $\gamma_i \subseteq \mathfrak{P}(\{1, \dots, n_i\})$ for all $i \in I$. A *semitopological* (general) *algebra* of type $\tau = ((n_i)_{i \in I}, (\gamma_i)_{i \in I})$ is a topological space $X$ together with a family of $n_i$-ary operations $\omega_i : X^{n_i} \to X$ for which $(x_{j_1}, \dots, x_{j_s}) \mapsto \omega_i(x_1, \dots, x_{n_i})$ is continuous for all $\{j_1, \dots, j_s\} \in \gamma_i$ and all fixed $x_i \in X$, $i \notin \{j_1, \dots, j_s\}$. This semitopological algebra is called a *topological algebra* if furthermore $\{1, \dots, n_i\} \in \gamma_i$ for all $i \in I$. In this case one might omit the information contained in the $\gamma_i$ and consider $\tau$ to be given by the $\tau = (n_i)_{i \in I}$.

Example 3.5.2 (Semitopological algebras).

- Topological groups are groups considered as topological algebras of type $\tau = (2, 1)$, requiring joint continuity of the binary operation as well as continuity of the operation $x \mapsto x^{-1}$.

- Topological groups can also be seen as topological algebras of type $\tau = (2, 1, 0)$ if one wishes to emphasize that the neutral element may be considered as a 0-ary operation.
- Topological semigroups are semigroups which are topological algebras of type $\tau = (2)$.
- Semitopological semigroups are semigroups considered as semitopological algebras of type $(2, \{\{1\}, \{2\}\})$, i.e. the semigroup operation is continuous in each component but not necessarily jointly continuous. Similarly the type of left and right topological semigroups is $\tau = (2, \gamma)$ with $\gamma = \{\{i\}\}$ with $i = 1$ resp. $i = 2$.

The value of the rather technical concept of a semitopological algebra gets clear by considering compactifications of general algebras.

Definition 3.5.3. Let $X$ and $K$ be (semi)topological algebras of type $\tau$. If $(\iota, K)$ is a compactification of the set $X$ such that each operation on $K$ extends the corresponding operation on $X$, we call $(\iota, K)$ a *$\tau$-compactification* of $X$. In the case of topological groups, (semi)topological semigroups etc. these compactifications are also called group, (semi)topological (semi)group etc. compactifications in the obvious way.

Later we will discuss the special cases of group, semigroup and semitopological semigroup compactifications in more detail. In the general context the following observations hold.

Proposition 3.5.4.

(i) *The direct product of a family of (semi)topological algebras of type $\tau$ is again a (semi)topological algebra of type $\tau$.*
(ii) *Every (semi)topological algebra of type $\tau$ has a maximal $\tau$-compactification.*

*Proof.* (ii) is obvious. For (iii), the product compactification (cf. Definition 2.4.6) of all (semi)topological compactifications of type $\tau$ has the required properties. To justify this construction it suffices to show that there is a set $\mathcal{S}$ of compactifications of $X$ such that for every $\tau$-compactification $(\iota, K)$ of $X$ there is an equivalent compactification in $\mathcal{S}$. Since $|K| \leq |\beta X|$ one can take for $\mathcal{S}$ the set of all compactifications $(\iota, K)$ of $X$ with $K \subseteq \beta X$ (as a set, not necessarily as a topological subspace or subalgebra). ■

Example 3.5.5.

(i) For a topological group $G$ the maximal group compactification is called the almost periodic or *Bohr compactification* and denoted by $(\iota_b, bG)$.
(ii) For a semitopological semigroup $S$ the maximal semitopological semigroup compactification is called the *weak almost periodic compactification* and denoted by $(\iota_w, wS)$ (see also Section 4.2). For the realization of $wS$ as a space of filters in the spirit of Proposition 2.5.3 we refer to [4].

## 3.6. Invariance on groups and semigroups

**3.6.1. The action of a semigroup by translations.** With every (semi)group $S$ comes a natural action by right translations.

Definition 3.6.1. Let $S$ be a semitopological semigroup. Then $S$ acts on $B(S)$ by right translations in the following way (notation as in Proposition 3.4.1):

$$R : S \times B(S) \to B(S), \quad R_s(f)(t) := f(\rho_s(t)) = f(ts).$$

For every $s \in S$ the map $R_s$ is a bounded linear operator. Since for the left translations

$$L : S \times B(S) \to B(S), \quad L_s(f)(t) := f(\lambda_s(t)) = f(st),$$

we have $L_s L_t = L_{ts}$ the map $(s, f) \mapsto L_f$ is not a semigroup action, but merely an "anti"-action of $S$ on $B(S)$. However, in the group case we can define an action $\Lambda : G \times B(G) \to B(G)$ via $\Lambda_s(f)(t) = f(\lambda_{s^{-1}}t)$.

In the following we use compatibility with respect to these translations to single out a unique measure or mean on certain algebras $\mathcal{A} \subseteq B(S)$. We will mainly focus on the group case. In contrast to the previous sections this section will be less self-contained. As standard references (which also extensively treat the semigroup case) we mention [5, 6, 18, 34, 39, 40, 41].

### 3.6.2. Means

Definition 3.6.2. Let $\mathcal{A} \subseteq B(S)$ be a $*$-algebra which is invariant under translations. A mean $m \in M(\mathcal{A})$ is *left* (resp. *right*) *invariant* if $m(f) = m(L_s f)$ (resp. $m(f) = m(R_s f)$) for all $s \in S$ and $f \in \mathcal{A}$. A mean which is both left and right invariant is called *bi-invariant*, or simply *invariant*.

It is a nontrivial task to find conditions on $\mathcal{A}$ which ensure the existence of an invariant mean. It turns out that the closure of the convex hull of the orbits with respect to various topologies plays an important rôle.

Proposition 3.6.3. *Let $S$ be a semitopological semigroup and $\mathcal{A} \subseteq B(S)$ a $C^*$-algebra such that*

(i) *for each $f \in \mathcal{A}$, $\overline{\mathrm{co}}(\{L_s f : s \in S\})$ contains a constant,*
(ii) *for each $f \in \mathcal{A}$, $\overline{\mathrm{co}}(\{R_s f : s \in S\})$ contains a constant,*

*where $\overline{\mathrm{co}}$ indicates the closure of the convex hull with respect to uniform convergence. Then there exists a mean $m \in M(\mathcal{A})$ which is bi-invariant. Furthermore, $m$ is unique.*

The proof of this assertion can be found for example in [6]. It is a general principle in the theory of function spaces on semigroups that constants in convex closures are intimately linked to invariant means (see [5, Chapter 2]). Proposition 3.6.3 states that if we can find constants in the uniform closures of the convex hull, there exists already a corresponding bi-invariant mean which is unique.

Having Proposition 3.6.2 at hand, we can establish the existence and uniqueness of an invariant mean for (weak) almost periodic functions. Recall the notion of almost periodicity:

Definition 3.6.4. A bounded function $f : S \to \mathbb{C}$ on a semitopological semigroup $S$ is called *(weakly) almost periodic* if the set $\{L_s f : s \in S\}$ of left translations is relatively compact in the norm (weak) topology. Let us denote the algebra of almost periodic functions by $AP(S)$ and the algebra of weakly almost periodic functions by $\mathcal{W}(S)$.

Evidently $AP(S) \subseteq \mathcal{W}(S)$. Weakly almost periodic functions may be characterized using the following double limit criterion.

Proposition 3.6.5 (Grothendieck). *Let $S$ be a semitopological semigroup. A bounded function $f : S \to \mathbb{C}$ is weakly almost periodic if and only if*

$$\lim_{n\to\infty}\lim_{m\to\infty} f(t_n s_m) = \lim_{m\to\infty}\lim_{n\to\infty} f(t_n s_m)$$

*whenever $(t_n)_{n=1}^\infty, (s_m)_{m=1}^\infty \subseteq S$ are sequences such that the limits involved exist.*

Using weak compactness of the translation orbit it is easy to check that weakly almost periodic functions satisfy the double limit condition. The complete proof can be found for instance in [25].

Remark 3.6.6 (Weak almost periodicity).

- This definition of (weakly) almost periodicity does not depend on the given topology on $S$ since the norm (weak) topology on $C_b(S)$ coincides with the relative topology inherited from the norm (weak) topology on $B(S) = C_b(S_{\mathrm{dis}})$. The statement is obvious for the norm topology; for the weak topology it follows from the Hahn–Banach Theorem.
- For any (weakly) almost periodic function the right orbit $\{R_s f : s \in S\}$ is relatively (weakly) compact as well. However, in the weak case left and right orbit closures will not in general coincide, while for almost periodic functions they always do.
- One can show that the set of (weakly) almost periodic functions is a $C^*$-algebra. Its structure space (see also Section 2.4) is a topological group (semitopological semigroup). The structure space coincides with the *Bohr compactification* in the almost periodic case and with the weakly almost periodic compactification in the weak case.

Before we go on, we quote the celebrated fixed point theorem of C. Ryll-Nardzewski which is vital to the theory of almost periodicity.

Proposition 3.6.7 (Ryll-Nardzewski). *Let $X$ be a Banach space and $K \subseteq X$ a weakly compact convex set. Let $S$ be a semigroup which acts on $K$ by continuous affine mappings $T_s$, i.e. $T_{st} = T_s T_t$ for all $s, t \in S$ and for each $s \in S$ there exists a continuous linear map $A_s$ and a vector $\tau_s$ such that $T_s x = A_s x + \tau_s$. If furthermore the action of $S$ is distal, i.e. for all $x \neq y$ we have $\inf\{\|T_s x - T_s y\| : s \in S\} > 0$, then there exists a common fixed point of the action.*

*Proof.* See the original paper [43] for a probabilistic or [18] for a geometric proof. ∎

Proposition 3.6.8. *Let $S$ be a semitopological semigroup. There exists a unique invariant mean on $\mathcal{W}(S)$ and hence also on $AP(S)$.*

We only give a sketch of the argument for the case where $S = G$ is a group. We will employ the Ryll-Nardzewski Theorem to show that $\mathcal{W}(S)$ meets the requirements of Proposition 3.6.3. Observe that a function $f : S \to S$ satisfying $L_s f = f$ for all $s \in S$ has to be constant. The weak closure of the convex hull of a weakly compact set is again weakly compact (Krein–Shmul'yan Theorem). So for $f \in \mathcal{W}(S)$ the set $K := \overline{\mathrm{co}}^{(w)}(\{L_s f : s \in S\})$ is again weakly compact. As $K$ is convex, the norm closure and the weak closure coincide (Hahn–Banach Theorem). The action of $S$ by the translations

$L_s$ leaves the set $K$ invariant, in fact $S$ acts by linear isometries. Thus we can use the Ryll-Nardzewski Theorem to conclude that there exists a common fixed point, i.e. a constant.

EXAMPLE 3.6.9.

- Let $\chi : S \to \mathbb{C}$ be a (semi)character, i.e. a continuous (semi)group homomorphism which satisfies $|\chi(s)| = 1$. Then $\chi$ is almost periodic. For the invariant mean on $AP(S)$ we have
$$m(\chi) = m(L_s\chi) = \chi(a)m(\chi).$$
Thus, if $\chi$ is not the constant character $\mathbb{1}_S$, then $m(\chi) = 0$.
- Let $S$ be locally compact. Every $f \in C_0(S)$ is weakly almost periodic and $m(f) = 0$ for the unique mean $m$ on $\mathcal{W}(S)$.
- Let $S = G$ be a locally compact abelian (LCA) group. Then the Fourier transform $\hat{\mu} : \hat{G} \to \mathbb{C}$ of a Borel measure $\mu$ on $G$,
$$\hat{\mu}(\chi) := \int_G \chi \, d\mu,$$
is weakly almost periodic and $m(\hat{\mu}) = \mu(\{e_G\})$ for the unique mean $m$ on $\mathcal{W}(\hat{G})$ and $e_G$ the neutral element of $G$. By Bochner's Theorem every positive definite function is weakly almost periodic. Recall that a function $f : \hat{G} \to \mathbb{C}$ is positive definite if for all $\chi_1, \dots, \chi_n \in \hat{G}$ the matrix $(f(\chi_i\overline{\chi}_j))_{i,j=1}^n \in \mathbb{C}^{n\times n}$ is positive definite.

**3.6.3. Measures.** Proposition 3.6.8 takes a particularly nice form if $S = G$ is a compact topological group.

PROPOSITION 3.6.10 (Haar measure). *Let $G$ be a compact topological group. Then there exists a regular Borel probability measure $\mu$ on $G$ which is invariant under left and right translations, i.e. $\mu(A) = \mu(gA) = \mu(Ag)$ for every Borel set $A \subseteq G$ and $g \in G$. This measure is unique and called the **Haar measure**.*

*Proof.* For $G$ compact and $f : G \to \mathbb{C}$ continuous the map $L^f : G \to B(G)$, $g \mapsto L_g f$, is continuous in the norm (weak) topology on $G$. Thus the norm (weak) closure of $\mathrm{co}(\{L_g f : g \in G\})$ is compact, i.e. $f$ is (weakly) almost periodic; $AP(G) = \mathcal{W}(G) = C_b(G)$. Proposition 3.6.8 together with Riesz' Representation Theorem 2.3.1 yields the existence of a unique left invariant measure $\mu_m$. Note that $\mu_m(G) = m(\mathbb{1}_G) = 1$, so $\mu$ is a probability measure.

Inversion $g \mapsto g^{-1}$ turns any left invariant measure (or mean) on $G$ into a right invariant measure (mean). Since the unique mean on $AP(G)$ is bi-invariant the left invariant Haar measure on a compact group is also right invariant. ■

REMARK 3.6.11. If $G$ is only *locally compact* we can still construct a left invariant measure. In this setting, however, uniqueness holds only up to a multiplicative constant and left invariance does not in general imply right invariance. For a rigorous treatment of the Haar measure we refer to [22, 48].

We can use the Haar measure to give an alternative approach to the unique invariant mean on the (weakly) almost periodic functions. As the Bohr compactification $(\iota_b, bG)$ of a topological group $G$ is compact, there exists the Haar measure $\mu_b$ on $bG$. As $bG$ is the structure space of the $C^*$-algebra $AP(G)$, there is an isomorphism $C(bG) \cong AP(G)$ given by $F \mapsto F \circ \iota$ (see Section 2.4). Consequently, $m(F \circ \iota) := \int_{bG} F \, d\mu_b$ defines an invariant mean on $AP(G)$, which, by uniqueness of the Haar measure, is the unique invariant mean on $AP(G)$.

COROLLARY 3.6.12. *Let $G$ be a topological group and $f \in AP(G)$. If $m(|f|) = 0$ for the unique invariant mean $m$ on $AP(G)$, then $f = 0$.*

*Proof.* As the Haar measure gives positive measure to open sets in $bG$, the equality

$$0 = m(|F \circ \iota|) = \int_{bG} |F| \, d\mu_b$$

with $F \in C(bG)$ can only hold if $F = 0$. ∎

COROLLARY 3.6.13. *Let $G$ be a topological group and let $(C_1, \iota_1)$, $(C_2, \iota_2)$ be group compactifications of $G$. If $(C_1, \iota_1) \leq (C_2, \iota_2)$ via $\pi : C_2 \to C_1$ then $\mu_1 = \pi \circ \mu_2$, where $\mu_1$ resp. $\mu_2$ is the Haar measure on $C_1$ resp. $C_2$. In particular, Proposition 2.6.5 applies.*

*Proof.* Using continuity of $\pi$ and density of $\iota_i(C) \in C_i$ it is straightforward to check that $\pi \circ \mu_2$ is an invariant Borel measure on $C_1$, hence it must be the Haar measure. ∎

**3.6.4. Amenability.** Finally, we will drop any uniqueness assumptions and focus on (semi)groups such that there exists at least one invariant mean.

DEFINITION 3.6.14. A discrete (semi)group $S$ is called *amenable* if there exists a bi-invariant mean on $B(S)$.

In the group case one can use the inversion $g \mapsto g^{-1}$ to show that existence of a one-sided invariant mean is equivalent to the existence of a bi-invariant mean. Indeed, let $m_l, m_r : B(X) \to \mathbb{C}$ be left resp. right invariant. For bounded $f : G \to G$ the function $M_l f(g) := m_l(R_g f)$ is again bounded. Then $m(f) := m_r(M_l f)$ defines a bi-invariant mean. Furthermore, if $G$ carries a locally compact topology then the existence of an invariant mean on $UCB(G)$, the algebra of uniformly continuous functions, implies the existence of an invariant mean on $B(G)$ and vice versa. For details we refer to [18, 34].

As we have seen in the previous section, every group which admits a compact group topology is amenable, as the Haar measure defines an invariant mean on $C_b(G)$. Among many other more or less remarkable properties, amenable groups have interesting dynamical behavior.

PROPOSITION 3.6.15 (Markov–Kakutani). *Let $S$ be amenable. If $S$ acts on a compact convex subset $K$ of a topological vector space $X$ by affine mappings, then there exists a common fixed point.*

The proof of this statement can be found in [18]. Note that the classical Markov–Kakutani fixed-point theorem is stated for the case $S = \mathbb{Z}$, or, slightly more generally, for abelian $S$.

The class **Am** of amenable *groups* is closed under elementary group-theoretical constructions, i.e. if $G$ is amenable, then so is every subgroup and homomorphic image of $G$.

Similarly, if $G_1$ and $G_2$ are amenable, then $G_1 \times G_2$ is amenable, and also the extension of an amenable group by an amenable group is amenable. Finally, directed unions of amenable groups are amenable.

It is trivial that every finite group is amenable and it is well-known that every abelian group is amenable. The class **ElAm** of elementary amenable groups is defined as follows: all finite groups and all abelian groups belong to **ElAm**, and **ElAm** is closed under taking subgroups, homomorphic images, finite direct products, extensions and direct unions, and is minimal with this property. One might ask whether **ElAm** = **Am**. The answer to this question is negative. For details see [34].

In Section 3.2.5 we have seen that the free group $F_2 := F(x, y)$ is not amenable. Consequently, no group containing $F_2$ as a subgroup (such as $SO(n)$ for $n \geq 3$) can be amenable. Define the class **NFree** of groups which do not contain $F_2$ as a subgroup. The so called *von Neumann conjecture* is that **AM** = **NFree**. However, this long standing conjecture was disproved by Ol'shanskii; for details see again [34].

We conclude this detour to amenable groups with the notion of extreme amenability. A group is called *extremely amenable* if there exists a *multiplicative* invariant mean. Extremely amenable groups arise as transformation groups of infinite-dimensional Hilbert spaces. They are intimately linked to concentration of measure phenomena; compact groups which are extremely amenable must be trivial (uniqueness of the Haar measure), but also locally compact groups can never be extremely amenable (see [13, 51]).

# 4. Hartman measurability

**4.1. Definition of Hartman functions.** The following definition fixes the main objects for the rest of the paper.

Definition 4.1.1. Let $G$ be a topological group. We call a bounded function $f : G \to \mathbb{C}$ *Hartman measurable* or a *Hartman function* if $f$ can be extended to a Riemann integrable function on some group compactification. According to Proposition 2.6.5 such a compactification can always be taken to be the maximal one, i.e. the Bohr compactification $(\iota_b, bG)$. The Haar measure on $bG$ is denoted by $\mu_b$. Let $\mathfrak{C}_{\mu_b}(bG)$ denote the system of all $\mu_b$-continuity sets on the Bohr compactification (see Definition 2.3.2). We denote by $\mathcal{H}(G)$ the set of all Hartman measurable functions and by $\mathfrak{H}(G)$ the system of all *Hartman sets* $\{{\iota_b^*}^{-1}[A] : A \in \mathfrak{C}_{\mu_b}(bG)\}$. It is easy to verify that $\mathfrak{H}(G)$ is a set algebra on $G$. We define a fapm $p$ on $\mathfrak{H}(G)$ via $p({\iota_b^*}^{-1}[A]) := \mu_b(A)$. The definition is correct by Proposition 2.6.2.

Proposition 4.1.2. *The following assertions are equivalent:*

(i) *$f \in \mathcal{H}(G)$, i.e. by definition $f = F \circ \iota$ with $F \in \mathcal{R}_{\mu_K}(K)$, $\mathcal{R}_{\mu_K}(K)$ denoting the set of all $F : K \to \mathbb{C}$ which are Riemann integrable with respect to the Haar measure $\mu_K$ on $K$, for some group compactification $(\iota, K)$ of $G$.*

(ii) *$f = F \circ \iota_b$ with $F \in \mathcal{R}_{\mu_b}(bG)$.*

(iii) $f \in B(\mathfrak{H}(G))$.

*Furthermore, if* $\iota_b : G \to bG$ *is one-one,* (i), (ii) *and* (iii) *are equivalent to*

(iv) $f \in \overline{AP(G)}^{(m)}$, *the $m$-completion of the almost periodic functions with respect to the unique invariant mean $m$.*

*Proof.* (i)⇔(ii): Apply Proposition 2.6.5.

(ii)⇔(iii): Consider the map $\iota_b^* : F \mapsto F \circ \iota_b$ which sends a function defined on the Bohr compactification $bG$ to a function defined on the group $G$. Then $\iota_b^*$ maps $\mathcal{R}_{\mu_b}(bG)$, the set of Riemann integrable functions on $bG$ (Definition 2.3.4), onto $\mathcal{H}(G)$. Thus $\mathcal{H}(G) = \iota_b^* \mathcal{R}_{\mu_b}(bG)$. The map $\iota_b^* : \mathcal{R}_{\mu_b}(bG) \to B(G)$ is a bounded homomorphism of $*$-algebras as the reader may quickly verify. Consequently, its image, $\mathcal{H}(G)$, is a $C^*$-algebra (see [8, Theorem I.5.5]). In particular, $\mathcal{H}(G)$ is closed.

Recall that $\mathfrak{C}_{\mu_b}(bG)$ denotes the set algebra of $\mu_b$-continuity sets on the Bohr compactification (cf. Definition 2.3.2). We then have the inclusions $\mathcal{S}_{\mathfrak{H}} \subseteq \mathcal{H}(G) \subseteq B(\mathfrak{H}(G))$. The first inclusion is valid since due to linearity of $\iota_b^*$ every $f \in \mathcal{S}_{\mathfrak{H}}$ is of the form $F \circ \iota_b$ for some $\mathfrak{C}_{\mu_b}$-simple function $F$. The second inclusion is true by the following argument: $f \in \mathcal{H}(G)$ if there are $\mathfrak{C}_{\mu_b}$-simple functions $F_n$ such that $\lim_{n\to\infty} \|F_n \circ \iota_b^* - f\|_\infty = 0$. Every function $F_n \circ \iota_b^*$ is $\mathfrak{H}$-simple, thus $f$ is in the uniform closure $B(\mathfrak{H}(G))$. Since $\mathcal{H}(G)$ is closed we have $\mathcal{H}(G) = \overline{\mathcal{S}}_{\mathfrak{H}} = B(\mathfrak{H}(G))$ in the notation of Section 2.1.

(iii)⇔(iv): Apply Proposition 2.6.6. ■

In [19] Hartman has used the $m$-closure of almost periodic functions to define a class of functions called “R-fastperiodisch” (“R-almost periodic”). According to [19] this term was suggested by C. Ryll-Nardzewski. In our terminology the R-almost periodic functions coincide with $\mathcal{H}(\mathbb{R})$, the Hartman functions on the reals.

The equivalence of (i) and (iv) in Proposition 4.1.2 for $G = \mathbb{Z}^n, \mathbb{R}^n$ has independently been obtained by J.-L. Mauclaire (oral communication). In [28, 29] J.-L. Mauclaire used extensions of arithmetic functions to (semi)group compactifications to prove number-theoretic results.

While the inclusion $\mathcal{H}(G) \subseteq \overline{AP(G)}^{(m)}$ is always valid, the converse does not hold true. The crucial property is injectivity of the map $\iota_b : G \to bG$. Topological groups where $\iota_b$ is one-one are called *maximally almost periodic*.

Example 4.1.3. Let $G_1$ be a topological group such that $bG_1 = \{e\}$ is trivial (such groups exist in abundance (cf. [15, 35, 37]; they are called *minimally almost periodic*) and $G_2 = \mathbb{T}$, the torus. Denote by $\mu$ the Haar measure on $\mathbb{T}$. Consider $G := G_1 \times G_2$. Then $\iota_b : G \to bG$ is the projection onto the second factor $\iota_b : (x, y) \mapsto y$. Consider $\mathcal{A} = AP(G)$ and $m$ the unique invariant mean on $AP(G)$. A function $f : G_1 \times G_2 \to \mathbb{C}$ belongs to $\mathcal{H}(G)$ if and only if $f(x, y) = F(y)$ for a function $F \in \mathcal{H}(\mathbb{T})$. Let $F : \mathbb{T} \to \mathbb{R}$ be Riemann integrable such that

$$F_\flat := \sup_{\substack{F_1 \le F \\ F_1 \in C(\mathbb{T})}} F_1 \ne \inf_{\substack{F_2 \ge F \\ F_2 \in C(\mathbb{T})}} F_2 =: F^\sharp.$$

We can take for instance $F = \mathbb{1}_A$, where $A$ is the Cantor middle-third set (in this case $F_\flat = 0$ and $F^\sharp = F$). Pick any $y_0 \in \mathbb{T}$ such that $\alpha := F_\flat(y_0) < F^\sharp(y_0) =: \beta$ and pick any nonconstant function $F_0 : G_1 \to [\alpha, \beta]$. Define

$$f(x,y) = \begin{cases} F(y) & \text{for } y \neq y_0, \\ F_0(x) & \text{for } y = y_0. \end{cases}$$

Then $F \in \overline{AP(G)}^m$ (since $F_\flat(y) \le f(x,y) \le F^\sharp(y)$), but $F \notin \mathcal{H}(G)$.

PROPOSITION 4.1.4. *$\mathcal{H}(G)$ is a translation invariant $C^*$-subalgebra of $B(G)$ and there exists a unique invariant mean on $\mathcal{H}(G)$.*

*Proof.* Translation invariance is a consequence of the fact that $\iota_b$ is a group homomorphism. In Proposition 4.1.2 we have already seen that $\mathcal{H}(G)$ is a $C^*$-algebra.

Every mean $m$ on $\mathcal{H}(G)$ lifts to a mean $m_b$ on $\mathcal{R}_{\mu_b}(bG)$ via the definition $m_b(F) := m(F \circ \iota_b^*)$ for $F \in \mathcal{R}_{\mu_b}(bG)$. For $m$ invariant one has $m_b(F) = \int_{bG} F\, d\mu_b$ for all *continuous* $F : bG \to bG$ (Riesz' Representation Theorem 2.3.1 and uniqueness of the Haar measure). Since $\mathcal{R}_{\mu_b}(bG)$ is the $\mu_b$-closure of $C(bG)$, $m_b$ is not only unique on $C(bG)$ but also on $\mathcal{R}_{\mu_b}(bG)$. This settles the uniqueness of $m$. On the other hand,

$$m(F \circ \iota_b) := \int_{bG} F\, d\mu_b, \quad F \in \mathcal{R}_{\mu_b}(bG),$$

defines such an invariant mean on $\mathcal{H}(G)$. ∎

In light of Proposition 4.1.2 the fapm $p$ on $\mathfrak{H}(G)$ resp. the mean $m$ on $\mathcal{H}(G)$ has a nice completeness property.

COROLLARY 4.1.5. *Let $G$ be a topological group such that $\iota_b : G \to bG$ is one-one.*

(i) *Let $A \in \mathfrak{H}(G)$ be a null-set, i.e. $p(A) = 0$. If $B \subseteq A$ then $B \in \mathfrak{H}(G)$.*
(ii) *Let $f \in \mathcal{H}(G)$ be a function with zero absolute mean value, i.e. $m(|f|) = 0$. If $f : G \to \mathbb{C}$ is such that $|g| \le |f|$ then $g \in \mathcal{H}(G)$.*

**4.2. Definition of weak Hartman functions.** We need some results concerning the weakly almost periodic compactification $(\iota_w, wS)$ of a semitopological semigroup $S$. Recall from Section 3.5 that a semitopological semigroup $S$ is a semigroup where all left translations $\lambda_s : S \to S$ and all right translations $\rho_s : S \to S$ are continuous.

DEFINITION 4.2.1. Let $S$ be a semitopological semigroup. By Proposition 3.5.4 there exists a maximal compactification $(\iota_w, wS)$ which is a semitopological semigroup. $(\iota_w, wS)$ is called the *weakly almost periodic compactification* of $S$.

COROLLARY 4.2.2. *Let $S$ be an abelian semitopological semigroup. Then $wS$, the weakly almost periodic compactification of $S$, is also abelian.*

*Proof.* For every $s \in S$ the continuous maps $\lambda_{\iota_w(s)}, \rho_{\iota_w(s)} : wS \to wS$ coincide on the dense set $\iota_w(S)$. Therefore $\lambda_{\iota_w(s)} = \rho_{\iota_w(s)}$. For arbitrary $x \in wS$ and $s \in S$ we have

$$\lambda_x(\iota_w(s)) = \rho_{\iota_w(s)}(x) \stackrel{!}{=} \lambda_{\iota_w(s)}(x) = \rho_x(\iota_w(s)),$$

thus also $\lambda_x$ and $\rho_x$ coincide on a dense set and therefore are equal. ∎

Definition 4.2.3. Let $S$ be a semigroup. A subset $I \subseteq S$ is called a (two-sided) *ideal* if $\lambda_s(I) \subseteq I$ and $\rho_s(I) \subseteq I$ for every $s \in S$. We denote by $K(S)$ the *kernel* of $S$, i.e. the intersection of all ideals in $S$.

From now on we will stick to the special case where $S = G$ is algebraically an abelian group. Here the kernel $K(G)$ has particularly nice properties.

Proposition 4.2.4. *Let $G$ be a semitopological abelian group. Then the kernel $K(G)$ is a compact topological group.*

The proof of this assertion can be found in [5, 41].

Let $e \in G$ denote the neutral element of the group $K(wG)$. Then $K(wG) = e + wG$ and the mapping $\rho : wG \to K(wG)$ defined via $x \mapsto e + x$ is a continuous retraction, i.e. $\rho(x) = x$ for all $x \in K(wG)$.

Proposition 4.2.5. *Let $G$ be an abelian topological group and $(\iota_w, wG)$ the weakly almost periodic compactification of $G$. Then the compactification $(\rho \circ \iota_w, K(wG))$ is equivalent to the Bohr compactification of $G$.*

*Proof.* Note that $(\rho \circ \iota_w, K(wG))$ is a group compactification. We show that it has the universal property of the Bohr compactification. Each almost periodic function $f$ on $G$ may be extended to a continuous function $F$ on $wG$. Consider the function $F - F \circ \rho$. Since $F \circ \rho$ may be regarded as a continuous function on the group compactification $(\rho \circ \iota_w, K(wG))$ the function $|F - F \circ \rho|$ induces a nonnegative almost periodic function on $G$. Since this function vanishes on $K(wG)$, the induced almost periodic function has zero mean value (note that the mean value is given by integration over $K(wG)$ with respect to $\mu_b$). By Corollary 4.1.5 and continuity this implies $F = F \circ \rho$. Thus we have $f = F \circ \iota_w = (F \circ \rho) \circ \iota_w$. So $F \circ \rho$ is a continuous extension of $f$ on $(\rho \circ \iota_w, K(wG))$. ∎

Similarly one proves that for an arbitrary semitopological semigroup compactification $(\iota, C)$ of $G$ the kernel $K(C)$ is a compact topological group and coincides with $\rho(C) = e + C$, where $e$ is the neutral element of $K(C)$ and $\rho$ the retraction defined as above. In this setting, $(\rho \circ \iota, K(C))$ constitutes a group compactification of $G$.

Proposition 4.2.6. *Let $G$ be an abelian topological group and $(\iota, C)$ a semitopological semigroup compactification. Then there exists a unique translation invariant Borel measure on $C$.*

*Proof.* Suppose $\mu$ is an invariant measure on $C$. Then $\mu(C) = \mu(e + C) = \mu(K(C))$ implies that $\mu$ is supported on the compact group $K(C)$. Consequently, $\mu_{|K(C)}$ coincides with the Haar measure on $K(C)$. Thus

$$\mu(A) = \mu_b(A \cap K(C)), \tag{4.1}$$

where $\mu_b$ denotes the Haar measure on $K(C)$. This settles the uniqueness of $\mu$. Since (4.1) indeed defines a translation invariant Borel measure, also the existence is guaranteed. ∎

Let us use the framework of semigroup compactifications to define weak Hartman functions:

Definition 4.2.7. Let $G$ be an abelian topological group and $f : G \to \mathbb{C}$ a bounded function. Then $f$ is called *weak Hartman measurable* or a *weak Hartman function* if there exists a semitopological semigroup compactification $(\iota, C)$ of $G$ and an $F \in \mathcal{R}_{\mu_C}(C)$ such that $f = F \circ \iota$ for the unique translation invariant measure $\mu_C$ on $C$. The set of all weak Hartman functions on $G$ is denoted by $\mathcal{H}^w(G)$.

It is almost, but not quite, entirely analogous to the strong case to check that $\mathcal{H}^w(G)$ is a translation invariant $C^*$-subalgebra of $B(G)$ on which a unique invariant mean exists. Furthermore, the universal property of the weakly almost periodic compactification $(\iota_w, wG)$ implies that a bounded function $f$ is weak Hartman if there exists a $\mu_w$-Riemann integrable function $F \in \mathcal{R}(wG) = \mathcal{R}_{\mu_w}(wG)$, $\mu_w$ denoting the unique translation invariant measure on $wG$, such that $f = F \circ \iota$. From Definition 4.2.7 it is also obvious that $\mathcal{H}^w(G) \supseteq \mathcal{H}(G) \cup \mathcal{W}(G)$.

Definition 4.2.8. Let $G$ be an abelian topological group. We denote by $\mathcal{H}_0^w(G)$ the set of all weak Hartman functions $f$ such that $|f|$ has zero mean value.

$\mathcal{H}_0^w(G)$ is a closed ideal of $\mathcal{H}(G)$. We will now identify the corresponding quotient space. Given any $\mu_w$-Riemann integrable function $F : wG \to \mathbb{C}$ we can write

$$F = \underbrace{(F - F \circ \rho)}_{=:F_0} + \underbrace{F \circ \rho}_{=:F_h}.$$

Note that the retraction $\rho : wG \to wG$ is measure preserving:

$$\begin{aligned}\rho \circ \mu_w(A) &= \mu_w(\rho^{-1}[A]) = \mu_w(\{x \in wG : e + x \in A\})\\ &= \mu_b(\{x \in wG : x = e + x \text{ and } e + x \in A)\\ &= \mu_b(A \cap K(wG)) = \mu_w(A).\end{aligned}$$

Thus both $F_0$ and $F_h$ are $\mu_w$-Riemann integrable. The induced weak Hartman function $f := F \circ \iota_w$ can be written as the sum $f = f_0 + f_h$ where $f_0 := F_0 \circ \iota_w$ is a weak Hartman function with zero mean value and $f_h := F_h \circ \iota_w$ is an ordinary Hartman function. This decomposition is unique. So we have proved:

Theorem 2. *Let $G$ be an abelian topological group and denote by $\mathcal{H}^w(G)$ and $\mathcal{H}_0^w(G)$ the space of weak Hartman functions resp. the space of weak Hartman functions with zero mean value. Then $\mathcal{H}^w(G) = \mathcal{H}_0^w(G) \oplus \mathcal{H}(G)$.*

**4.3. Compactifications of LCA groups.** In the following sections we will deal with *locally compact abelian (LCA)* groups. If $H$ is a subgroup of the topological group $G$, we will denote this by $H \leq G$. By $G_d$ we mean the group $G$ equipped with the discrete topology. We will use standard notation such as $\hat{G}$ for the Pontryagin dual, $\chi$ for characters, $H^\perp$ for the annihilator of a subgroup and $\varphi^*$ for the adjoint of a homomorphism without further mention, and refer the reader instead to standard textbooks on this topic such as [1, 22, 48].

Similarly to Proposition 2.4.9 where we used a function algebra $\mathcal{A}$ to construct a compactification we will now use a group of characters. Let $H \leq \hat{G}_d$ be an (algebraic) subgroup of the dual of $G$. Then $H$ induces a group compactification $(\iota_H, K_H)$ of $G$ in the following way:

$$\iota_H : g \mapsto (\chi(g))_{\chi \in H}, \qquad K_H := \overline{\iota_H(G)} \le \mathbb{T}^H,$$

and for every such $H \le \hat{G}_d$ the kernel of $\iota_H$ coincides with the annihilator $H^\perp \le G$. Remarkably, also the converse is true.

Proposition 4.3.1. *Let $G$ be an LCA group and let $(\iota, C)$ be a group compactification of $G$. Then there exists a unique subgroup $H \le \hat{G}_d$ such that $(\iota, C)$ and $(\iota_H, K_H)$ are equivalent, namely $H = \iota^*(\hat{C})$.*

*Proof.* As $\iota : G \to C$ has dense image, the adjoint homomorphism $\iota^* : \hat{C} \to \hat{G}$ is one-one. Let $H := \iota^*(\hat{C}) \le \hat{G}$ and consider the group compactification $(\iota_H, K_H)$. Note that for $g \in G$ we have, due to injectivity of $\iota^*$,

$$(\chi(\iota(g)))_{\chi \in \hat{C}} = (\iota^*(\chi)(g))_{\chi \in \hat{C}} = (\eta(g))_{\eta \in H} = \iota_H(g).$$

Define $\pi : C \to \mathbb{T}^{\hat{C}}$ via $c \mapsto (\chi(c))_{\chi \in \hat{C}}$. Then $\pi$ is a continuous homomorphism and maps the dense subgroup $\iota(G) \le C$ onto the dense subgroup $\iota_H(G) \le K_H$. As $C$ is compact, $\pi(C)$ is closed and thus contains $K_H$. On the other hand, $\pi^{-1}[K_H]$ is closed since $\pi$ is continuous and so contains $C$. Thus $\pi$ maps $C$ onto $K_H$. Since $\iota_H = \pi \circ \iota$ this implies $(\iota, C) \ge (\iota_H, K_H)$. If $\pi(c) = 0$ then $\chi(c) = 0$ for all $\chi \in \hat{C}$. Thus $c = 0$ and $\pi$ is one-one. So $(\iota, C) \cong (\iota_H, K_H)$ via $\pi$ (cf. Definition 2.4.3). ∎

**4.4. Realizability on LCA groups.** Let us now turn to the *realizability* of Hartman functions by Riemann integrable functions (cf. Definition 2.4.1). It follows from Theorem 4 in [56] that every $f \in \mathcal{H}(\mathbb{Z})$ which is a characteristic function can be realized in a metrizable compactification. We are going to generalize this result. As a corollary we prove that metric realizability of every $f \in \mathcal{H}(G)$ is possible precisely for LCA groups with separable dual. First we have to introduce some useful concepts.

### 4.4.1. Preparation

Definition 4.4.1. Let $G$ be an LCA-group. The *topological weight* $\kappa(G)$ is defined as the cardinal number

$$\kappa(G) = \min\{|I| : (O_i)_{i \in I} \text{ is an open basis of } G\}.$$

Note that this minimum exists (and is not merely an infimum) since cardinal numbers are well-ordered. The topological weight behaves well with respect to products, i.e. for infinite $G$ we have $\kappa(G \times H) = \max\{\kappa(G), \kappa(H)\}$ and $\kappa(G^I) = \kappa(G) \cdot |I|$. For $H \subseteq G$ we clearly have $\kappa(H) \le \kappa(G)$.

From the theory of LCA-groups it is known that $\kappa(G) = \kappa(\hat{G})$ (see [22, §24.14]). The topological weight of the group compactification $K_H$ can thus be computed very easily via $\kappa(K_H) = \kappa(\hat{K}_H) = \kappa(H_d) = |H|$.

Definition 4.4.2. Let $G$ be an LCA-group. The *co-weight* $c(G)$ is defined as $\min\{|H| : H \le \hat{G} \text{ and } \overline{H} = \hat{G}\}$.

We collect some facts concerning the co-weight:

(i) $c(G) \le \aleph_0 \Leftrightarrow G$ has separable dual.

(ii) $c(G) < \infty \Leftrightarrow G$ is finite.
(iii) If $H$ is a closed subgroup of $G$, then $c(H) \leq c(G)$.

The statements (i) and (ii) are obvious. For the sake of completeness we give the argument for (iii): By duality $\hat{H}$ and $\hat{G}/H^{\perp}$ are isomorphic LCA groups. Let $G_0 \leq \hat{G}$ be a subgroup of $\hat{G}$ with $|G_0| = c(G)$. Then $H_0 := G_0 + H^{\perp}$ is a subgroup of $\hat{G}/H^{\perp}$ and $|H_0| \leq |G_0|$. As the canonical projection $\pi_{H^{\perp}} : \hat{G} \to \hat{G}/H^{\perp}$ is continuous and onto, dense sets are mapped onto dense sets. So pick $G_0 \leq \hat{G}$ which is dense with $|G_0| = c(G)$ to conclude $c(H) \leq |\pi_{H^{\perp}}(G_0)| \leq |G_0| = c(G)$.

Definition 4.4.3. Let $G$ be an LCA-group and $f \in \mathcal{H}(G)$ a Hartman function. The *weight* of $f$, $\kappa(f)$, is defined by

$$\kappa(f) := \min\{\kappa(K) : f \text{ can be realized on } (\iota, K)\}.$$

By virtue of Proposition 4.3.1 we can compute $\kappa(f)$ as the minimum of all $|H|$ such that $f$ can be realized on $(\iota_H, K_H)$. We want to prove the following:

Theorem 3. *Let $G$ be an LCA group. Then*

$$\max\{\kappa(f) : f \in \mathcal{H}(G)\} = c(G),$$

*i.e. every Hartman measurable function on $G$ can be realized on a compactification whose topological weight is at most $c(G)$, and this is best possible.*

Corollary 4.4.4. *Let $G$ be an LCA group. Then the following are equivalent:*

(i) *$\hat{G}$ is separable,*
(ii) *Every $f \in \mathcal{H}(G)$ can be realized on a metrizable compactification.*

*Proof.* Separability of $\hat{G}$ is equivalent to $c(G) \leq \aleph_0$. ■

The rest of this section is devoted to the proof of Theorem 3.

### 4.4.2. Estimate from above

Lemma 4.4.5. *Let $G$ be an LCA group. Then there exists an injective group compactification of $G$, i.e. a group compactification $(\iota, C)$ such that $\iota : G \to C$ is one-one. Furthermore,*

$$\min\{\kappa(C) : (\iota, C) \text{ is injective}\} = c(G).$$

*Proof.* The result follows from the fact that any group compactification is equivalent to some $(\iota_H, K_H)$ with $H \leq \hat{G}$ and that $\ker \iota_H = H^{\perp}$. Thus $\iota_H$ is injective if $H^{\perp} = \{0\}$ and this is equivalent to $H$ being dense in $\hat{G}$. Thus $c(G) \geq |H| = \kappa(K_H) = \kappa(C)$ and equality is obtained if $|H| = c(G)$ and $(\iota, C) = (\iota_H, K_H)$. ■

In the following let us call a group compactification $(\iota, C)$ of an LCA group a *finite-dimensional compactification* if $C \leq \mathbb{T}^s$ for some $s \in \mathbb{N}$.

Lemma 4.4.6. *Let $G$ be an LCA group and $T \subseteq G$ a Hartman set. For every $\varepsilon > 0$ there are Hartman sets $T_\varepsilon$ and $T^\varepsilon$, realized on a finite-dimensional compactification $(\iota, C)$ such that $T_\varepsilon \subseteq T \subseteq T^\varepsilon$ and $m(T^\varepsilon \setminus T_\varepsilon) < \varepsilon$.*

*Proof.* We proceed similarly to [56, Theorem 2]. Let $M \subseteq bG$ be a $\mu_b$-continuity set realizing $T$, i.e. $T = \iota_b^{-1}[M]$. Use the inner regularity of the Haar measure on $bG$ to find a compact inner approximation $K \subset M^\circ$ with $\mu_b(M \setminus K) = \mu_b(M^\circ \setminus K) < \varepsilon/2$.

Recall that one can construct the Bohr compactification as $(\iota_{\hat{G}_d}, K_{\hat{G}_d})$ (see Proposition 4.3.1). As $bG = K_{\hat{G}_d} \subseteq \prod_{\chi\in\hat{G}} \overline{\chi(G)}$ one can obtain a basis $(B_i)_{i\in I}$ of open sets in $bG$ by restricting the standard basis of the product space to the subspace $bG$. The sets $B_i$ can be chosen to be finite intersections of sets of the form

$$D_{\chi_0;a,b} := \{(\alpha_\chi)_{\chi\in\hat{G}} \in bG : \alpha_{\chi_0} \in (a, b)\}$$

where $(a, b)$ denotes an open segment in $\mathbb{T}$ and such that the basis $(B_i)_{i\in I}$ consists of $\mu_b$-continuity sets.

We can cover $K$ by finitely many sets of the form $O_j = B_{i_j} \cap M^o$, $j = 1, \ldots n$, with $i_j \in I$. Each $O_j$ is a $\mu_b$-continuity set and induces a Hartman set $T_j = \iota_b^{-1}[O_j]$ on $G$ that may be realized on a finite-dimensional group compactification $(\iota_j, C_j)$, i.e. $C_j \leq \mathbb{T}^{s_j}$. Let $(\iota_0, C_0)$ denote the supremum of all $(\iota_j, C_j)$, $j = 1, \ldots, n$. It is easy to check that $C_0 \leq \mathbb{T}^{s_0}$ with $s_0 = \sum_{j=1}^n s_j$ and that $T_\varepsilon = \iota_b^{-1}(\bigcup_{j=1}^n O_j)$ is a Hartman set which can be realized in $(\iota_0, C_0)$ (see Definition 2.4.6).

In a similar way one finds an outer approximation $T^\varepsilon$ which can be realized in some compactification $(\iota^0, C^0)$ with $C^0 \leq T^{s^0}$. Then we can take the supremum $(\iota, C)$ of $(\iota_0, C_0)$ and $(\iota^0, C^0)$ and $s = s_0 + s^0$. ■

LEMMA 4.4.7. *Every Hartman set $T$ on an infinite LCA group $G$ can be realized on a group compactification with topological weight $c(G)$.*

*Proof.* We follow the lines of [56, Theorem 4]. Let $T$ be a Hartman set and $(T_{1/n})_{n=1}^\infty$, $(T^{1/n})_{n=1}^\infty$ sequences of Hartman sets as in Lemma 4.4.6, approximating $T$ from inside resp. outside. Let $(\iota, C)$ be the supremum of all at most countably many finite-dimensional compactifications involved. As $\kappa(\mathbb{T}^s) = \aleph_0$ for every $s \in \mathbb{N}$, the topological weight of $C$ cannot exceed $\aleph_0 \cdot \aleph_0 = \aleph_0$. By Lemma 4.4.5 we can find an injective group compactification, covering $(\iota, C)$ and having topological weight $\max\{c(G), \aleph_0\} = c(G)$. For notational convenience we call this compactification again $(\iota, C)$.

Denote by $M_n$ resp. $M^n$ the $\mu_C$-continuity sets in $C$ that realize the Hartman sets $T_{1/n}$ resp. $T^{1/n}$. Thus $M_\infty := \bigcup_{n=1}^\infty M_n^\circ$ is open, $M^\infty := \bigcap_{n=1}^\infty \overline{M^n}$ is closed and $\iota^{-1}[M_\infty] \subseteq T \subseteq \iota^{-1}[M^\infty]$. Let $M := M_\infty \cup \iota(T)$. Since $\iota$ is one-one the preimage of $M$ under $\iota$ coincides with the given Hartman set $T$. Furthermore,

$$\mu_C(\partial M) \leq \mu(M^\infty \setminus M_\infty) = \lim_{n\to\infty} \mu(M^n \setminus M_n) = 0$$

shows that $M$ is a $\mu_C$-continuity set. ■

COROLLARY 4.4.8. *Let $G$ be an infinite LCA group and $f \in \mathcal{H}(G)$ with $f(G)$ finite. Then $f$ can be realized in a group compactification $(\iota, C)$ with topological weight $c(G)$ by a simple $\mu_C$-continuity function.*

*Proof.* By assumption $f = \sum_{i=1}^n \alpha_i \mathbb{1}_{T_i}$. It is clear that the $T_i$ can be taken to be Hartman sets. By Lemma 4.4.7, $T_i$, $i = 1, \ldots, n$, can be realized on a compactification $(\iota_i, C_i)$ with

$\kappa(C_i) = c(G)$. The supremum $(\iota, C)$ of the $(\iota_i, C_i)$ has again topological weight $c(G)$ and, as a consequence of Proposition 2.4.5, each $T_i$ and hence $f$ can be realized in $(\iota, C)$. ∎

Now we can prove the first part of Theorem 3, namely $\kappa(f) \le c(G)$ for every $f \in \mathcal{H}(G)$:

*Proof.* First consider the finite (compact) case: If $G$ is compact then every $f \in \mathcal{H}(G)$ can be realized on $(\mathrm{id}_G, G)$. Hence $\kappa(f) \le \kappa(G) = \kappa(\hat{G}) = |\hat{G}| = c(G)$.

Now we show that on an infinite LCA group $G$ every $f \in \mathcal{H}(G)$ can be realized on a group compactification with topological weight not exceeding $c(G)$. We may assume that $f$ is real-valued. By Proposition 4.1.2, $f$ can be realized in the maximal compactification $(\iota_b, bG)$ by some $F^b \in \mathcal{R}_{\mu_b}(bG)$, i.e. $f = F^b \circ \iota_b$. By Lemma 2.3.3 there is a sequence of simple $\mu_b$-continuity functions $F_n^b$ on $bG$ converging to $F^b$ uniformly. Consider the Hartman functions $f_n = F_n^b \circ \iota_b$ and note that each $f_n$ takes only finitely many values. Corollary 4.4.8 guarantees that each $f_n$ can be realized on a group compactification $(\iota_n, C_n)$ with $\kappa(C_n) = c(G)$ by simple $\mu_{C_n}$-continuity functions $F_n^0$, i.e. $f_n = F_n^0 \circ \iota_n$. The supremum of countably many group compactifications of topological weight $c(G)$ has a topological weight not exceeding $\aleph_0 \cdot c(G) = c(G)$. By technical convenience we use Lemma 4.4.5 to get an injective group compactification $(\iota, C)$ with $\kappa(C) = c(G)$ covering all $(\iota_n, C_n)$. For each $n$ let $\pi_n : C \to C_n$ denote the canonical projection, i.e. $\iota_n = \pi_n \circ \iota$. Consider the functions $F_n = F_n^0 \circ \pi_n$ which are in $\mathcal{R}_{\mu_C}(C)$ by Proposition 2.4.5 and in fact simple $\mu_C$-continuity functions.

$$\begin{tikzcd} & C \arrow[d, "\pi_n"] \arrow[dr, "F_n"] & \\ G \arrow[ur, "\iota"] \arrow[r, "\iota_n"] & C_n \arrow[r, "F_n^0"] & \mathbb{C} \end{tikzcd}$$

In order to realize $f$ in $(\iota, C)$ by $F \in \mathcal{R}_{\mu_C}(C)$ we have to define $F(x) = f(g)$ whenever $x = \iota(g)$ for some $g \in G$. Since $\iota$ is one-one, $F$ is well-defined on $\iota(G)$. For $x \in C \setminus \iota(G)$ we define

$$F(x) = \limsup_{\iota(g) \to x} F(\iota(g)).$$

It remains to show that $F$ is $\mu_C$-Riemann integrable. For each $n \in \mathbb{N}$ let $F_n = \sum_{i=1}^{k_n} \alpha_i \mathbb{1}_{A_{n,i}}$ be a representation of $F_n$ with pairwise disjoint continuity sets $A_{n,i}$, $i = 1, \ldots, k_n$. The open sets $U_n = \bigcup_{i=1}^{k_n} A_{n,i}^\circ$ have full $\mu_C$-measure. Thus the dense $G_\delta$-set $U = \bigcap_{n \in \mathbb{N}} U_n$ has full $\mu_C$-measure as well. If we can prove the following claim, we are done.

Claim. *Each $x \in U$ is a point of continuity for $F$.*

Fix $x \in U$ and $\varepsilon > 0$. We are looking for an open neighborhood $V \in \mathfrak{U}(x)$ ($\mathfrak{U}(x)$ denoting the filter of neighborhoods of $x$) such that $\iota(g_1), \iota(g_2) \in V$ implies $|f(g_1) - f(g_2)| < \varepsilon$. This suffices to guarantee $|F(x_1) - F(x_2)| \le \varepsilon$ for all $x_1, x_2 \in V$, in particular $|F(x_1) - F(x)| \le \varepsilon$, yielding continuity of $F$ in $x$. To find such a $V$ note that, by construction, the $F_n$ converge uniformly to $F$ on the dense set $\iota(G)$. Choose $n \in \mathbb{N}$ in such a way that $|F_n(\iota(g)) - F(\iota(g))| < \varepsilon/2$ for all $g \in G$. There is a unique $i \in \{1, \ldots, k_n\}$ such that $x \in A_{n,i}^\circ$. The set $V := A_{n,i}^\circ$ has the desired property: For $\iota(g_1), \iota(g_2) \in V$ we have $F_n(\iota(g_1)) = F_n(\iota(g_2))$ and

$$|f(g_1) - f(g_2)| = |F(\iota(g_1)) - F_n(\iota(g_1))| + |F_n(\iota(g_2)) - F(\iota(g_2))| < \varepsilon. \ ∎$$

**4.4.3. Estimate from below.** In this section we are concerned with the construction of a Hartman function with $\kappa(f) = c(G)$ for a given infinite group $G$.

LEMMA 4.4.9. *Let $G$ be an uncountable LCA-group. Then there exists a subset $A \subseteq G$ such that $\mathbb{1}_A$ can be realized only in injective group compactifications of $G$.*

*Proof.* Let $\alpha \mapsto g_\alpha$ be a bijection between the set of all ordinals $\alpha < |G|$ and $G \setminus \{0\}$. By choosing for each $\alpha$ elements of the co-sets of the subgroup $\langle g_\alpha \rangle$ generated by $g_\alpha$ we find (for each $\alpha$) $x_\alpha^{(i)} \in G$ with $|I| \leq |G|$ such that

$$G = \bigcup_{i \in I} (\langle g_\alpha \rangle + x_\alpha^{(i)}), \quad \alpha < |G|,$$

is a disjoint union. As $G$ is uncountable and $\langle g_\alpha \rangle$ is at most countable we must have $|I| = |G|$. We start with the construction of $A$: Assume, by transfinite induction, that for given $\alpha_0 < |G|$ we have already constructed elements $x_\alpha, y_\alpha$, $\alpha < \alpha_0$, such that $y_\alpha \in \langle g_\alpha \rangle + x_\alpha$ with $x_\alpha = x_\alpha^{(i)}$ for some $i = i(\alpha) \in I$ and all the $x_\alpha, y_\alpha$ with $\alpha < \alpha_0$ are pairwise distinct. To find $y_{\alpha_0}$ we first observe that

$$N_1 := \{y_\alpha, x_\alpha : \alpha < \alpha_0\} \subseteq \bigcup_{\alpha < \alpha_0} (\langle g_\alpha \rangle + x_\alpha),$$

hence we have $|N_1| \leq |\alpha_0| \cdot \aleph_0 < |G|$. Therefore, by the cardinality of

$$G = \bigcup_{i \in I} (\langle g_{\alpha_0} \rangle + x_{\alpha_0}^{(i)}),$$

there are $|G|$ indices $i \in I$ with $\langle g_{\alpha_0} \rangle + x_{\alpha_0}^{(i)}$ disjoint from $\{y_\alpha, x_\alpha : \alpha < \alpha_0\}$. Pick such an $i = i(\alpha_0)$ and $x_{\alpha_0} = x_{\alpha_0}^{(i)}, y_{\alpha_0} = y_{\alpha_0}^{(i)} \in \langle g_{\alpha_0} \rangle + x_{\alpha_0}^{(i)}$ with $x_{\alpha_0} \neq y_{\alpha_0}$. Let $A = \{x_\alpha : \alpha < |G|\}$.

Note that by its very definition, $y_\alpha \notin A$ for all $\alpha < |G|$. Suppose $A$ can be realized in some noninjective group compactification $(\iota, C)$, i.e. there exists a set $M \subseteq C$ such that $\iota^{-1}[M] = A$. As $\iota$ is not injective, there exists some $g_\alpha \in \ker \iota$, $g_\alpha \neq 0$. Then $\langle g_\alpha \rangle \subseteq \ker \iota$. As $y_\alpha \in \langle g_\alpha \rangle + x_\alpha$ we have $\iota(y_\alpha) = \iota(x_\alpha) \in M$ , i.e. $y_\alpha \in \iota^{-1}[M] = A$, a contradiction. ∎

REMARK 4.4.10. For countable $G$ we could use a similar and even simpler construction. However, we will use a different approach for this case.

LEMMA 4.4.11. *Let $G$ be an infinite LCA group. Then there exists a closed subgroup $G_0 \leq G$ such that $c(G_0) = c(G)$ and $G_0$ is a Hartman null-set.*

*Proof.* First we distinguish two cases and employ in each of them the existence of a nontrivial closed subgroup with zero Hartman measure.

(i) There exists a $\chi \in \hat{G}$ such that $\chi(G)$ is infinite. The annihilator $G_0 := \{\chi\}^\perp$ is a Hartman null-set. To see this consider the preimage of the singleton $\{0\}$ in the group compactification $(\chi, \mathbb{T})$.
(ii) All characters are torsion elements, i.e. $\chi(G)$ is finite for every $\chi \in \hat{G}$. Pick a sequence of pairwise (algebraically) independent characters $\chi_n$ and consider $G_0 := \bigcap_{i=1}^\infty \{\chi_i\}^\perp$ and the group compactification $(\iota, C) := \bigvee_{i=1}^\infty (\chi_i, \mathbb{Z}_{m_i})$ with $m_i = |\chi_i(G)|$. As $C$ is infinite, every singleton in $C$ has zero $\mu_C$-measure. Therefore $G_0 = \iota^{-1}[\{0\}]$ is a Hartman null-set.

As $G_0 \leq G$ is a closed subgroup, $c(G_0) \leq c(G)$. Let $H$ be a dense subgroup of $\hat{G_0}$ with $|H| = c(G_0)$. Any $\eta \in H$ can be extended from a character on $G_0$ to a character $\tilde{\eta}$ on the whole of $G$. Let us denote the set of those extended characters by $\tilde{H}$. Note that as $H$ is infinite the subgroup

$$H_0 := \begin{cases} \tilde{H} & \text{for } G \text{ as in case (i)}, \\ \langle \tilde{H} \cup \{\chi_i : i \in \mathbb{N}\} \rangle & \text{for } G \text{ as in case (ii)}, \end{cases}$$

has the same cardinality as $H$. We show that $H_0$ is dense in $\hat{G}$ by computing its annihilator. Pick any $g \in G$ such that $g \in H_0^{\perp}$. If $g \in G_0$ we have in particular $\chi_{|G_0}(g) = 0$ for every $\chi \in \tilde{H}$ and thus $\eta(g) = 0$ for every $\eta \in H$. As $H$ is dense in $\hat{G_0}$ we have $g = 0$. Thus $H_0^{\perp} \cap G_0 = \{0\}$. On the other hand, by its very definition, $H_0^{\perp} \leq G_0$. So $H_0^{\perp} = \{0\}$ and thus $\overline{H}_0 = G$. This gives the reverse inequality $c(G) \leq |H_0| = |H| = c(G_0)$. ■

By proving the next statement we conclude the remaining part of Theorem 3.

Proposition 4.4.12. *Let $G$ be an LCA group. Then there exists a Hartman function $f \in \mathcal{H}(G)$ such that $\kappa(f) = c(G)$.*

*Proof.*

- $G$ finite: Let $\hat{G} = \{\chi_1, \dots \chi_n\}$ and take $f = \chi_1 + \ldots + \chi_n$. Then $\kappa(f) = n = c(G)$.
- $G$ countable: $|G| = \aleph_0$ implies $c(G) = \aleph_0$. Note that every countable LCA group is discrete. Thus $\hat{G}$ is isomorphic to a closed subgroup of $\mathbb{T}^{\aleph_0}$. In particular, $\hat{G}$ is uncountable. Consider any sequence $(\chi_i)_{i=1}^{\infty}$ of pairwise (algebraically) independent characters on $G$ and define the almost periodic function

$$f = \sum_{i=1}^{\infty} \frac{\chi_i}{2^n}$$

  on $G$. Denoting by $m_G$ the unique invariant mean on $AP(G)$, we see that

$$\Gamma(f) := \langle \{\chi \in \hat{G} : m_G(f \cdot \chi) \neq 0\} \rangle = \langle \{\chi_i : i \in \mathbb{N}\} \rangle.$$

  Then, by [27, Corollary 13], we have $\Gamma(f) \subseteq H$ for any compactification $(\iota_H, K_H)$ on which $f$ can be realized. As $f$ can only be realized on compactifications $(\iota_H, K_H)$ with $H$ infinite we conclude $\kappa(f) \geq |H| = \aleph_0 = c(G)$.
- $G$ uncountable: Take $G_0 \leq G$ as in Lemma 4.4.11 and let $f := \mathbb{1}_A$ for the set $A \subsetneq G_0$ from Lemma 4.4.9. As $G_0$ has zero Hartman measure Corollary 4.1.5 implies $f \in \mathcal{H}(G)$. By construction for every compactification $(\iota, C)$ where $f$ can be realized $\iota$ must be injective on $G_0$, hence $\kappa(f) \geq c(G_0) = c(G)$. ■

# 5. Classes of Hartman functions

**5.1. Generalized jump discontinuities.** The concept of generalized jump discontinuities is useful for comparing Hartman functions and weakly almost periodic functions. In the present section we do not need the group setting.

Definition 5.1.1. Let $X, Y$ be topological spaces. A function $f : X \to Y$ has a *generalized jump discontinuity (g.j.d.)* at $x \in X$ if there are (disjoint) open sets $O_1$ and $O_2$ such that $x \in \bar{O}_1 \cap \bar{O}_2$ but $\overline{f(O_1)} \cap \overline{f(O_2)} = \emptyset$.

Example 5.1.2.

- The function $f_1(x) = \mathbb{1}_{[0,1/2)}(x)$ on $X = [0,1]$ has a g.j.d. at $x = 1/2$. The function $f_2(x) = \mathbb{1}_{\{1/2\}}(x)$ has no g.j.d.
- The function $f : [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} \sin(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

  has a g.j.d. at 0. To see this, consider the open set $O_1 := f^{-1}[(1/2, 1]]$ and the open set $O_2 := f^{-1}[[-1, -1/2)]$.
- Generalizing the first example, let $X$ be compact, $\mu$ be a finite complete regular Borel measure on $X$ and $A$ a $\mu$-continuity set. Then the characteristic function $\mathbb{1}_A$ has g.j.d.s on $\partial(A^\circ) \cap \partial(A^{\circ c})$, the common boundary of $A^\circ$ and its complement.
- Let $X$ be compact and $\mu$ a finite complete regular Borel measure with $\operatorname{supp}(\mu) = X$. If $f : X \to \mathbb{R}$ is constant $\mu$-a.e. then $f$ has no g.j.d.

Proposition 5.1.3. *Let $X$ be a topological space. Denote by $J(X)$ the set of all bounded functions $f : X \to \mathbb{R}$ having a g.j.d. Then $J(X) \subseteq B(X)$ is open in the topology of uniform convergence.*

*Proof.* Let $f \in J(X)$. Then there exist disjoint open sets $O_1$ and $O_2$ with $\partial O_1 \cap \partial O_2 \neq \emptyset$ but $\overline{f(O_1)} \cap \overline{f(O_2)} = \emptyset$. Let $\varepsilon := d(f(O_1), f(O_2)) > 0$ and suppose $\|f - g\|_\infty < \varepsilon/8$. Then $d(f(Y), g(Y)) < \varepsilon/4$ for any set $Y \subseteq X$, hence

$$|d(g(O_1), g(O_2)) - d(f(O_1), f(O_2))| < \varepsilon/2.$$

In particular, $d(g(O_1), g(O_2)) > 0$, i.e. $g$ has a g.j.d. ∎

Lemma 5.1.4. *Let $X$ be compact and $\mu$ a finite complete regular Borel measure with $\operatorname{supp}(\mu) = X$. Let $f, g \in \mathcal{R}_\mu(X)$ be Riemann integrable functions.*

(i) *If $f$ and $g$ coincide on a dense set, then they coincide on a co-meager set of full $\mu$-measure.*
(ii) *If $f$ and $g$ coincide on a dense set and $f$ has a g.j.d. at $x \in X$, then also $g$ has a g.j.d. at $x \in X$.*

*Proof.* (i) By Proposition 2.3.6 it suffices to show $[f = g] := \{x \in X : f(x) = g(x)\} \supseteq X \setminus (\operatorname{disc}(f) \cup \operatorname{disc}(g))$. Let $x \in X$ be a point of continuity for both $f$ and $g$, and $U \subseteq X$ a neighborhood of $x$ such that $y \in U$ implies $|f(y) - f(x)| < \varepsilon/2$ and $|g(y) - g(x)| < \varepsilon/2$. As $[f = g]$ is dense in $X$ we can pick $y_\varepsilon \in U \cap [f = g]$. Thus

$$|f(x) - g(x)| \leq |f(x) - f(y_\varepsilon)| + |g(y_\varepsilon) - g(x)| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary this implies $f(x) = g(x)$.

(ii) Choose $O_1$ and $O_2$ according to the definition of a g.j.d. of $f$ at $x$. By (i), $f$ and $g$ coincide on a dense set of common continuity points. Thus for every $x \in X$ and $U \in \mathfrak{U}(x)$ we can pick $x_i^U \in U \cap O_i$, $i = 1, 2$, which are both points of continuity for $f$

and $g$ and such that $f(x_i^U) = g(x_i^U)$, $i = 1, 2$. Pick open neighborhoods $O_i^U$ of $x_i^U$ such that $O_i^U \subseteq U \cap O_i$ and

$$\operatorname{diam}(g(O_i^U)) < \tfrac{1}{3} \operatorname{dist}(f(O_1), f(O_2)), \quad i = 1, 2.$$

Consider the open sets $U_i := \bigcup_{U \in \mathfrak{U}(x)} O_i^U$, $i = 1, 2$. Then $g(U_1)$ is separated from $g(U_2)$ and $x_i^U \in U_i$ for all $U \in \mathfrak{U}(x)$. This implies $x \in \overline{U}_i$, $i = 1, 2$, proving that $x$ is a g.j.d. for $g$. ■

Corollary 5.1.5. *Let $X$ be compact and $\mu$ a finite complete regular Borel measure on $X$ with* $\operatorname{supp}(\mu) = X$. *Let $f, g$ be simple $\mathfrak{C}_\mu$-functions* (*see Definition* 2.3.2). *If $f$ and $g$ coincide on a dense set, then they coincide on an open set of full $\mu$-measure.*

*Proof.* Lemma 5.1.4 implies that $[f = g]$ has full $\mu$-measure. Since $f$ and $g$ are simple $\mathfrak{C}_\mu$-functions, $[f = g]$ is a $\mu$-continuity set. Thus $[f = g]^\circ$ and $[f = g]$ have the same $\mu$-measure $\mu(X)$. ■

Proposition 5.1.6. *Let $X$ be compact and $\mu$ a finite complete regular Borel measure with* $\operatorname{supp}(\mu) = X$. *Let $f \in \mathcal{R}_\mu(X) \setminus J(X)$, i.e. $f$ is Riemann integrable without a g.j.d. Then there exists a* unique *continuous function $f_r \in C(X)$, the* regularization *of $f$, such that $f$ and $f_r$ coincide on $X \setminus \operatorname{disc}(f)$.*

*Proof.* For $f$ Riemann integrable $f$ the set $X \setminus \operatorname{disc}(f)$ is dense in $X$ by Proposition 2.3.6. Hence there is at most one continuous $f_r$ with $f_r(x) = f(x)$ for $x \notin \operatorname{disc}(f)$.

Let $x \in \operatorname{disc}(f)$. For each $U \in \mathfrak{U}(x)$ (the neighborhood system of $x$), $y \in U \setminus \operatorname{disc}(f)$ and $\varepsilon > 0$ pick an open neighborhood $O = O(U, \varepsilon, y) \in \mathfrak{U}(y)$ such that $O \subseteq U$ and $\operatorname{diam}(\overline{f(O)}) < \varepsilon$. Let

$$O(U, \varepsilon) := \bigcup_{y \in U \setminus \operatorname{disc}(f)} O(U, \varepsilon, y).$$

Claim. *The set $\Lambda(x)$ consists of exactly one point $\lambda_x$, where*

$$\Lambda(x) := \bigcap_{\varepsilon > 0} \bigcap_{U \in \mathfrak{U}(x)} \overline{f(O(U, \varepsilon))}.$$

$\Lambda(x) \neq \emptyset$ by the finite-intersection property of the compact sets $\overline{f(O(U, \varepsilon))}$, $U \in \mathfrak{U}(x)$, $\varepsilon > 0$. Suppose by contradiction that $\lambda_1, \lambda_2 \in \Lambda(x)$ and $\lambda_1 \neq \lambda_2$. Consider the open sets $O_i := \bigcup O(U, \varepsilon, y)$, $i = 1, 2$, where the union is taken over all triples $(U, \varepsilon, y)$ with $U \in \mathfrak{U}(x)$, $\varepsilon < |\lambda_1 - \lambda_2|/4$ and $y \in U \setminus \operatorname{disc}(f)$ such that $|f(y) - \lambda_i| < \varepsilon$. By construction we have $x \in \bar{O}_1 \cap \bar{O}_2$ and $\overline{f(O_2)} \cap \overline{f(O_2)} = \emptyset$. Hence $x$ is a g.j.d. of $f$, contradiction.

Claim. $f_r : X \to \mathbb{R}$,

$$f_r(x) = \begin{cases} f(x) & \text{for } x \notin \operatorname{disc}(f), \\ \lambda_x & \text{for } x \in \operatorname{disc}(f), \end{cases}$$

*is continuous.*

It is immediate to check that $\operatorname{disc}(f_r) \subseteq \operatorname{disc}(f)$. Suppose for contradiction that there exists $x \in \operatorname{disc}(f_r) \subseteq \operatorname{disc}(f)$. Then an inspection of the argument above shows that $x$ would be a g.j.d. for $f$. ■

Note that for $f : X \to \mathbb{R}$ meeting the requirements of Proposition 5.1.6 we have

$$\|f_r\|_\infty = \sup_{x \in X} |f_r(x)| = \sup_{x \in X \setminus \operatorname{disc}} |f_r(x)| = \sup_{x \in X \setminus \operatorname{disc}(f)} |f(x)| \leq \sup_{x \in X} |f(x)| = \|f\|_\infty.$$

Thus the mapping $f \mapsto f_r$ is continuous with respect to the topology of uniform convergence on its domain of definition, i.e. on $\mathcal{R}_\mu(X) \setminus J(X) \subseteq B(X)$.

COROLLARY 5.1.7. *Let $X$ be compact and $\mu$ a finite complete regular Borel measure with* $\operatorname{supp}(\mu) = X$. *For $f \in \mathcal{R}_\mu(X)$ the following assertions are equivalent:*

(i) *There exists $g \in C(X)$ such that $f$ and $g$ coincide on a co-meager set of full $\mu$-measure.*
(ii) *$f$ has no g.j.d.*

*Proof.* (i)⇒(ii): Suppose $f$ has a g.j.d. at $x \in X$. Pick open sets $O_1$ and $O_2$ according to the definition of a g.j.d. at $x$. Next, pick nets $\{x_\nu^{(i)}\}_{\nu \in \mathcal{N}_i}$, where $(\mathcal{N}_i, \leq)$ are directed sets, such that

$$x_\nu^{(i)} \in O_i \cap [f = g], \quad \lim_{\nu \in \mathcal{N}_i} = x, \quad i = 1, 2.$$

This gives the desired contradiction

$$\overline{f(O_1)} \ni \lim_{\nu \in \mathcal{N}_1} f(x_\nu^{(1)}) = \lim_{\nu \in \mathcal{N}_1} g(x_\nu^{(1)}) = \lim_{\nu \in \mathcal{N}_2} g(x_\nu^{(2)}) = \lim_{\nu \in \mathcal{N}_2} f(x_\nu^{(2)}) \in \overline{f(O_2)}.$$

(ii)⇒(i): The statement follows from Proposition 5.1.6 and Lemma 5.1.4. ∎

**5.2. Hartman functions that are weakly almost periodic.** Recall the notion of weak almost periodicity from Sections 2.7 and 4.2.

THEOREM 4. *Let $G$ be a topological group and $f \in \mathcal{H}(G) \cap \mathcal{W}(G)$ a weakly almost periodic Hartman function. Let $(\iota, C)$ be a group compactification on which $f$ can be realized by $F \in \mathcal{R}_{\mu_C}(C)$. Then $F : C \to \mathbb{C}$ has no g.j.d.*

*Proof.* Assume, for contradiction, that $f \in \mathcal{H}(G) \cap \mathcal{W}(G)$ can be realized on the group compactification $(\iota, C)$ by $F \in \mathcal{R}_{\mu_C}(C)$, where $F$ has a g.j.d. at $x_0 \in C$. Pick $O_1, O_2 \subseteq C$ as in the definition of a g.j.d. at $x_0$.

Pick a net $(g_\nu)_{\nu \in \mathcal{N}}$ in $G$, where $(\mathcal{N}, \leq)$ is a directed set, in such a way that $\iota(g_\nu) \in O_1$ and $\lim_{\nu \in \mathcal{N}} \iota(g_\nu) = x_0$. We can take $(g_\nu)_{\nu \in \mathcal{N}}$ to be a universal net, i.e. for every $A \subseteq G$, $(g_\nu)_{\nu \in \mathcal{N}}$ stays eventually in $A$ or $G \setminus A$. Furthermore, we define

$$\varphi_{\mathcal{N}} : \tilde{f} \mapsto \lim_{\nu \in \mathcal{N}} \tilde{f}(g_\nu) \quad (= \lim_{\nu \in \mathcal{N}} \delta_{g_\nu}(\tilde{f})),$$

where $\delta_{g_\nu}$ denotes the evaluation functional at the point $g_\nu$. By universality of $(g_\nu)_{\nu \in \mathcal{N}}$, $\varphi_{\mathcal{N}}$ is well-defined and a bounded linear functional on $B(G)$. Since $x_0 \in \overline{O_2}$, for every neighborhood $V \subseteq C$ of $e$, the neutral element of the group $C$, we can find a neighborhood $U \subseteq C$ of $x_0 \in C$ and a $g = g_{U,V} \in G$ such that $\iota(g_{U,V}) \in V$ and $\iota(g_{U,V}) \cdot U \subseteq O_2$. All such pairs $(U, V)$ form a directed set $\mathcal{M}'$ equipped with the order $(U_1, V_1) \leq (U_2, V_2) :\Leftrightarrow U_1 \supseteq U_2$ and $V_1 \supseteq V_2$. The net $(g_{\mu'})_{\mu' \in \mathcal{M}'}$ has the property that for every $\mu' \in \mathcal{M}'$ the net $(\iota(g_\nu g_{\mu'}))_{\nu \in \mathcal{N}}$ stays eventually in $O_2$.

Pick a directed set $(\mathcal{M}, \preceq)$ such that $(g_\mu)_{\mu \in \mathcal{M}}$ is a universal refinement of the net $(g_{\mu'})_{\mu' \in \mathcal{M}'}$. Then $\lim_{\mu \in \mathcal{M}} \iota(g_\mu) = e \in C$. As $f$ is weakly almost periodic, the closure of the left-translation orbit

$$O_L(f) = (L_g f : g \in G)$$

is weakly compact in $B(G)$. This implies that there exists $f_0 \in B(G)$ in the weak closure of $O_L(f)$ such that weak-$\lim_{\mu\in\mathcal{M}} L_{g_\mu} f = f_0$. Consider the evaluation functionals $\delta_{g_\nu} \in B(G)^*$:

$$\delta_{g_\nu}(f_0) = f_0(g_\nu) = \lim_{\mu\in\mathcal{M}} L_{g_\mu} f(g_\nu) = \lim_{\mu\in\mathcal{M}} f(g_\mu g_\nu) = \lim_{\mu\in\mathcal{M}} F(\iota(g_\mu g_\nu)).$$

As $\lim_{\mu\in\mathcal{M}} \iota(g_\mu) = e \in C$ for fixed $\nu \in \mathcal{N}$ the net $(\iota(g_\mu g_\nu))_{\mu\in\mathcal{M}}$ stays in $O_1$ eventually. Hence $\delta_{g_\nu}(f_0) \in \overline{F(O_1)}$ and thus $\varphi_{\mathcal{N}}(f_0) = \lim_{\nu\in\mathcal{N}} f_0(g_\nu) \in \overline{F(O_1)}$. Let us now compute the value of the functional $\varphi_{\mathcal{N}}$ at $f_0$ directly:

$$\varphi_{\mathcal{N}}(f_0) = \lim_{\mu\in\mathcal{M}} \varphi_{\mathcal{N}}(L_{g_\mu} f) = \lim_{\mu\in\mathcal{M}} \lim_{\nu\in\mathcal{N}} L_{g_\mu} f(g_\nu) = \lim_{\mu\in\mathcal{M}} \lim_{\nu\in\mathcal{N}} f(g_\mu g_\nu) = \lim_{\mu\in\mathcal{M}} \lim_{\nu\in\mathcal{N}} F(\iota(g_\mu g_\nu)).$$

Since $\iota(g_\mu g_\nu) = \iota(g_\mu)\iota(g_\nu) \in O_2$ eventually, this yields $\varphi_{\mathcal{N}}(f_0) \in \overline{F(O_1)} \cap \overline{F(O_1)}$, a contradiction. ∎

Remark 5.2.1. The proof of Theorem 4 employs the same argument (but regarding nets instead of sequences) that may be used to establish the easy direction of Grothendieck's Double Limit Theorem 3.6.5.

Corollary 5.2.2. *Let $G$ be an infinite LCA group. Then there exists a Hartman function which is not weakly almost periodic. In particular, $\mathcal{H}(G) \neq \mathcal{W}(G)$.*

*Proof.* Let $(\iota, C)$ be any infinite metrizable compactification of $G$. (This can be obtained by taking pairwise distinct characters $\chi_n$, $n \in \mathbb{N}$, and $\iota : g \mapsto (\chi_n(g))_{n\in\mathbb{N}}$, $C := \overline{\iota(G)} \leq \mathbb{T}^{\mathbb{N}}$.) It suffices to find two disjoint open $\mu_C$-continuity sets $O_1$ and $O_2$ in $C$ with a common boundary point $x \in \partial O_1 \cap \partial O_2$. Then $x$ is a g.j.d. for $F = \mathbb{1}_{O_1}$ and, by Theorem 4, $f = F \circ \iota \in \mathcal{H}(G) \setminus \mathcal{W}(G)$.

Let $d : C \times C \to [0, 1)$ be a bounded metric which generates the topology of $C$. We use the fact that for every $x \in C$ there are open balls $B(r, x) := \{y \in C : d(x, y) < r\}$ with center $x$ and arbitrarily small radius $r > 0$ which are $\mu_C$-continuity sets (see [26, Example 1.3], or an argument similar to the proof of our Proposition 4.4.6).

Construction of $O_1, O_2$: Pick any $x \in C$. We define two sequences of disjoint open $\mu_C$-continuity sets $(O_j^{(1)})_{j=0}^\infty$ and $(O_j^{(2)})_{j=0}^\infty$. Let $O_0^{(1)}$ and $O_0^{(2)}$ be any two disjoint open balls which are $\mu_C$-continuity sets, separated from $x$ and have $\mu_C$-measure smaller than $1/2$. We proceed by induction: Suppose we have already defined $O_0^{(1)}, \dots, O_n^{(1)}$ and $O_0^{(2)}, \dots, O_n^{(2)}$ such that

$$\mu\Big(\bigcup_{j=0}^n O_j^{(i)}\Big) < \frac{1}{2}\Big(1 - \frac{1}{2^n}\Big) \quad \text{and} \quad 0 < \operatorname{dist}\Big(\bigcup_{j=0}^n O_j^{(i)}, x\Big) < \frac{1}{2^n}, \qquad i = 1, 2.$$

Let

$$r < \min\{\operatorname{dist}(O_j^{(i)}, x) : j = 0, \dots, n \text{ and } i = 1, 2\}$$

and pick distinct $x_1, x_2 \in B(r, x)$ and $\rho < \min\{r/2, 1/2^{n+1}\}$ such that $O_{n+1}^{(i)} := B(\rho, x_i)$, $i = 1, 2$, are $\mu_C$-continuity sets of $\mu_C$-measure less than $1/2^{n+1}$. Choosing $O_i := \bigcup_{j=0}^\infty O_j^{(i)}$, $i = 1, 2$, we obtain two disjoint open sets $O_1, O_2 \subseteq C$ with the required properties. ∎

The converse problem, namely to find weakly almost periodic functions that are not Hartman measurable appears to be harder. We content ourselves with the special case $G = \mathbb{Z}$. The key ingredient for our example are ergodic sequences. These sequences were extensively studied by Rosenblatt and Wierdl [36]. Also in the context of Hartman measurability ergodic sequences were already mentioned in [45].

Example 5.2.3 (Ergodic sequences). A sequence $n_k$ of nonnegative integers is called *ergodic* if for every measure preserving system $(X, T, \mu)$ with ergodic transformation $T : X \to X$ and every $\mu$-integrable $f$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f \circ T^{n_k}(x) = \int_X f \, d\mu$$

for $\mu$-almost every $x \in X$. Birkhoff's ergodic theorem (cf. [55]) states that $n_k = k$ is an ergodic sequence. It is known ([45, Theorem 11] and the examples therein) that there are other ergodic sequences, such as $(k \log k)_{k\in\mathbb{N}}$, which cannot be Hartman measurable. On the other hand, 0-1 sequences with the property that the length between consecutive 1s tends to infinity while the length of consecutive 0s stays bounded are weakly almost periodic [5, Theorem 4.2]. Thus $\mathcal{E} \subseteq \mathcal{W}(\mathbb{Z}) \setminus \mathcal{H}(\mathbb{Z})$, where $\mathcal{E}$ is the set of all ergodic sequences on $\mathbb{Z}$.

Problem 5.2.4. *Construct $f \in \mathcal{W}(G) \setminus \mathcal{H}(G)$ on more general LCA, or even arbitrary topological groups.*

**5.3. Hartman functions without generalized jumps.** Theorem 4 motivates us to further investigate Hartman functions having no g.j.d. First we show that the property of having a g.j.d. does not depend on the particular compactification.

Proposition 5.3.1. *Let $G$ be a topological group and $f \in \mathcal{H}(G)$ a Hartman function. Let $F_1 \in \mathcal{R}_{\mu_1}(C_1)$ and $F_2 \in \mathcal{R}_{\mu_2}(C_2)$ be realizations of $f$ on the group compactifications $(\iota_1, C_1)$ resp. $(\iota_2, C_2)$. If $F_1$ has a g.j.d., then $F_2$ also has a g.j.d.*

*Proof.* Let $x \in G$ be a g.j.d. for $F_1$. Suppose $(\iota_1, C_1) \geq (\iota_2, C_2)$, i.e. that there is a continuous surjection $\pi : C_1 \to C_2$ with $\iota_2 = \pi \circ \iota_1$ and $f = F_1 \circ \iota_1 = F_2 \circ \iota_2$. Thus $F_1$ and $F_2 \circ \pi$ coincide on $\iota_1(G)$. (Note that the right triangle in the diagram

$$\begin{tikzcd}
& C_1 \arrow[d, "\pi"] \arrow[dr, "F_1"] & \\
G \arrow[ur, "\iota_1"] \arrow[r, "\iota_2"] & C_2 \arrow[r, "F_2"] & \mathbb{C}
\end{tikzcd}$$

does not necessarily commute on the whole set $C_1$.) Hence Lemma 5.1.4(i) implies that $F_1 = F_2 \circ \pi$ $\mu_1$-a.e., and Lemma 5.1.4(ii) implies that $F_2 \circ \pi$ has a g.j.d. at $x \in C_1$ whenever $F_1$ has a g.j.d. at $x \in C_1$. Pick disjoint open sets $O_1, O_2 \subseteq C_1$ according to the definition of a g.j.d. for $F_2 \circ \pi$ at $x \in C_2$, i.e. $x \in \overline{O}_1 \cap \overline{O}_2$ but $\overline{F_2 \circ \pi(O_1)} \cap \overline{F_2 \circ \pi(O_2)} = \emptyset$. Thus $\pi(O_1)$ and $\pi(O_2)$ are disjoint. Since $\pi$ is an open mapping, $\pi(O_1)$ and $\pi(O_2)$ are open sets with $\pi(x) \in \overline{\pi(O_1)} \cap \overline{\pi(O_2)}$. Thus $\pi(x)$ is a g.j.d. for $F_2$.

In the general case let $\pi$ be the canonical projection $bG \to C_1$ and define $F^b := F_1 \circ \pi$. It is easy to check that if $F_1$ has a g.j.d. at $x \in C_1$, then $F^b$ has a g.j.d. at every point of $\pi^{-1}[\{x\}]$. Moreover, $F^b, F_1$ and $F_2$ induce the same Hartman function $f$ on $G$. Now apply the first part of this proof to the two functions $F^b$ and $F_2$. ∎

This result shows that being realized by a function with a g.j.d. is an intrinsic property of a Hartman function and does not depend on the particular realization. In virtue of this result we can consider the set of all Hartman functions such that one (and hence all) realizations lack a g.j.d.

Definition 5.3.2. Let $G$ be a topological group. Let

$$\begin{aligned}\mathcal{H}_c(G) &:= \{f \in \mathcal{H}(G) : \forall(\iota, C) \;\; f = F \circ \iota \text{ with } F \in R_{\mu_C}(C) \\ &\qquad\qquad \text{implies that } F \text{ has no g.j.d.}\} \\ &= \{f \in \mathcal{H}(G) : \exists(\iota, C) \;\; f = F \circ \iota \text{ with some } F \in R_{\mu_C}(C) \\ &\qquad\qquad \text{without any g.j.d.}\}.\end{aligned}$$

In the next section we will see that $\mathcal{H}_c(G)$ enjoys nice algebraic and topological properties.

**5.4. Hartman functions with small support.** Similar to the situation of g.j.d.s, for different realizations of a Hartman function also the property of vanishing outside a meager null set does not depend on the special choice of the realization.

Proposition 5.4.1. *Let $G$ be a topological group and $f \in \mathcal{H}(G)$ a Hartman function. Let $F_1 \in \mathcal{R}_{\mu_1}(C_1)$ and $F_2 \in \mathcal{R}_{\mu_2}(C_2)$ be realizations of $f$ on the group compactifications $(\iota_1, C_1)$ resp. $(\iota_2, C_2)$. If $[F_1 \neq 0]$ is a meager $\mu_1$-null set, then $[F_2 \neq 0]$ is a meager $\mu_2$-null set.*

*Proof.*

(i) First consider the case where $(\iota_1, C_1) \leq (\iota_2, C_2)$ via $\pi : C_2 \to C_1$. By assumption $[F_1 \neq 0]$ is a meager $\mu_1$-null set. Use Lemma 5.1.4(i) to see that $[F_2 = F_1 \circ \pi]$ is a co-meager set of full $\mu_2$-measure. Thus $\pi^{-1}[[F_1 \neq 0]] \triangle [F_2 \neq 0]$ is a meager $\mu_2$-null set. This implies

$$\mu_2([F_2 \neq 0]) = \mu_2(\pi^{-1}[[F_1 \neq 0]]) = \mu_1([F_1 \neq 0]) = 0.$$

Next we show that $[F_2 \neq 0]$ is meager. Indeed, $\pi : C_2 \to C_1$ is open, closed, continuous and surjective. Thus one easily verifies that preimages of meager sets are meager, in particular if $[F_1 \neq 0]$ is meager in $C_1$, then $\pi^{-1}[[F_1 \neq 0]]$ is meager in $C_2$. Since $\pi^{-1}[[F_1 \neq 0]]$ and $[F_2 \neq 0]$ differ at most on a meager $\mu_2$-null set, $[F_2 \neq 0]$ is meager, proving the claim.

(ii) Suppose $(\iota_1, C_1) \geq (\iota_2, C_2)$. We use again the fact that $\pi : C_1 \to C_2$ is an open and continuous surjection of compact spaces to conclude that $\pi[[F_1 \neq 0]]$ is meager in $C_2$ whenever $[F_1 \neq 0]$ is meager in $C_1$. The rest of the proof is analogous to the first case.

In the general case the property of vanishing outside a meager null-set transfers first by (i) from $(\iota_1, C_1)$ to $(\iota_b, bG)$ and then by (ii) from $(\iota_b, bC)$ to $(\iota_2, C_2)$. ■

We will consider the set of those Hartman functions all realizations of which vanish outside a meager null set.

Definition 5.4.2. Let $G$ be a topological group. Let

$$\begin{aligned}\mathcal{H}_0(G) &:= \{f \in \mathcal{H}(G) : \forall(\iota, C)\ \ f = F \circ \iota \text{ with } F \in R_{\mu_C}(C)\\ &\qquad\qquad\qquad\text{implies that } [F \neq 0] \text{ is a meager } \mu_C\text{-null set}\}\\ &= \{f \in \mathcal{H}(G) : \exists(\iota, C)\ \ f = F \circ \iota \text{ with some } F \in R_{\mu_C}(C)\\ &\qquad\qquad\qquad\text{such that } [F \neq 0] \text{ is a meager } \mu_C\text{-null set}\}.\end{aligned}$$

Proposition 5.4.3. *Let $G$ be a topological group. Then $\mathcal{H}_0(G)$ and $\mathcal{H}_c(G)$ are translation invariant $C^*$-subalgebras of $B(G)$. Furthermore, $\mathcal{H}_c(G)$ contains all constant functions.*

*Proof.* By their definition it is clear that $\mathcal{H}_0(G)$ and $\mathcal{H}_c(G)$ are subalgebras of $B(G)$, invariant under translations, and that $\mathcal{H}_c(G)$ contains all constants. It remains to prove that $\mathcal{H}_0(G)$ and $\mathcal{H}_c(G)$ are closed in the topology of uniform convergence.

Let $\mathcal{R}_0(bG) := \{f \in \mathcal{R}_{\mu_b}(bG) : [f \neq 0] \text{ is a meager } \mu_b\text{-null set}\}$. Note that $\mathcal{R}_0(bG)$ is a closed subalgebra of $\mathcal{R}_\mu(bG)$ (as a countable union of meager null sets is again a meager null set). Since $\iota_b^* : \mathcal{R}_{\mu_b}(bG) \to B(G)$ is a continuous homomorphism of $C^*$-algebras and $\iota_b^*(\mathcal{R}_0(bG)) = \mathcal{H}_0(G)$ (Definition 5.4.2), $\mathcal{H}_0(G)$ is closed ([8, Theorem I.5.5]).

Now, $J(bG)$, the set of all bounded functions on $bG$ having a g.j.d., is open in the topology of uniform convergence (Proposition 5.1.3). Thus $C(bG) \oplus \mathcal{R}_0(bG)$, the set of all bounded functions on $bG$ without a g.j.d. (Corollary 5.1.7), is closed (Proposition 5.1.3). Since $\mathcal{H}_c(G) = \iota_b^*(C(bG) \oplus R_0(bG))$ by [8, Theorem I.5.5], it follows that $\mathcal{H}_c(G)$ is closed. ∎

The last part of this section is devoted to the relations of the algebras $\mathcal{H}_0$, $\mathcal{H}_c$ and $AP$. Note that $AP(G) \cap \mathcal{H}_0(G) = \{0\}$. This is due to the fact that $f \in \mathcal{H}_0(G)$ implies $m(|f|) = 0$, which is impossible for a nonzero almost periodic function.

Lemma 5.4.4. *Let $G$ be a topological group. Then $\mathcal{H}_0(G) \subseteq \mathcal{H}_c(G)$.*

*Proof.* It suffices to show that for every $F \in \mathcal{R}_{\mu_C}(C)$ on a group compactification $(\iota, C)$ such that $F \circ \iota \in \mathcal{H}_0(G)$ there are no two distinct open sets $O_1, O_2 \subseteq C$ with $\overline{F(O_1)} \cap \overline{F(O_2)} = \emptyset$. As $[F \neq 0]$ is a $\mu_C$-null set (Proposition 5.4.1) the set $[F = 0]$ is dense in $C$, i.e. $0 \in \overline{F(O_1)} \cap \overline{F(O_2)}$. ∎

Proposition 5.4.5. *Let $G$ be a topological group. For every $f \in \mathcal{H}_c(G)$ there exists a unique almost periodic function $f_a \in AP(G)$ and a unique function $f_0 \in \mathcal{H}_0(G)$ such that $f := f_a + f_0$. Furthermore, if $f \geq 0$, then $f_a \geq 0$.*

*Proof.* Let $F$ be a realization of $f$ on a group compactification $(\iota, C)$. Using Proposition 5.1.6 we can decompose $F = F^r + (F - F^r)$, the first summand being continuous and the second one having support on a meager $\mu_C$-null set.

Existence: Let $f_a := F \circ \iota \in AP(G)$ and $f_0 := (F - F^r) \circ \iota \in H_0(G)$. By construction $f = f_a + f_0$.

Uniqueness: Suppose $f = f_a^{(1)} + f_0^{(1)} = f_a^{(2)} + f_0^{(2)}$ with $f_a^{(1)}, f_a^{(2)} \in AP(G)$ and $f_0^{(1)}, f_0^{(2)} \in \mathcal{H}_0(G)$. This implies $f_a^{(1)} - f_a^{(2)} = f_0^{(1)} - f_0^{(2)} \in AP(G) \cap \mathcal{H}_0(G) = \{0\}$, i.e. $f_a^{(1)} = f_a^{(2)}$ and $f_0^{(1)} = f_0^{(2)}$.

Positivity: Let $f \geq 0$. We claim that $F \geq 0$ outside $\operatorname{disc}(F)$. To see this, let $x \in C$ be a point of continuity for $F$ and suppose for contradiction that $F(x) < 0$. Pick an open neighborhood $V$ of $x$ such that $F(y) < 0$ for any $y \in V$. As $\iota(G)$ is dense in $C$ there exists an element $\iota(g) \in V$ with $F(\iota(g)) = f(g) \geq 0$, a contradiction. $F$ and $F^r$ coincide on the dense set $C \setminus \mathrm{disc}(F)$. By continuity of $F^r$ we have $F^r \geq 0$, implying $f_a \geq 0$. ∎

An immediate consequence is:

THEOREM 5. *Let $G$ be a topological group. Then $\mathcal{H}_c(G) = AP(G) \oplus \mathcal{H}_0(G)$. Furthermore, the mapping $P : \mathcal{H}_c \to AP(G)$ defined via $f \mapsto f_a$, where $f = f_a + f_0$ is the decomposition from Proposition* 5.4.5, *is a bounded positive projection with $\|P\| = 1$ and $m(Pf) = m(f)$ for the unique invariant mean $m$ on $\mathcal{H}(G)$.*

Recall from Example 4.1.3 that a topological group $G$ is called *minimally almost periodic* (map) if $AP(G)$ consists only of the constant functions. $G$ is called *maximally almost periodic* (MAP) if $AP(G)$ separates the points of $G$ (cf. Proposition 4.1.2). Every LCA group is maximally almost periodic.

COROLLARY 5.4.6. *Let $G$ be a topological group. The following assertions are equivalent:*

(i) $\mathcal{H}_c(G) = \mathcal{H}_0(G)$,
(ii) *$G$ is minimally almost periodic.*

*Proof.* Both (i) and (ii) are equivalent to $bG = \{0\}$. ∎

PROBLEM 5.4.7. *For which topological groups is the inclusion $\mathcal{H}_c(G) \supseteq \mathcal{H}(G) \cap \mathcal{W}(G)$ strict? Construct $f \in \mathcal{H}_c(G) \setminus (\mathcal{H}(G) \cap \mathcal{W}(G))$.*

LEMMA 5.4.8. *Let $G$ be a noncompact topological group and let $(\iota_b, bG)$ be the Bohr compactification of $G$.*

(i) *If $G$ is MAP then $\mu_b(\iota_b(K)) = 0$ for every $\sigma$-compact $K \subseteq G$.*
(ii) *If $G$ is an LCA group and $\iota_b(G)$ is $\mu_C$-measurable then $\mu_b(\iota_b(G)) = 0$.*

*Proof.* (i) First suppose that $K$ is compact. We inductively construct a sequence $(g_i)_{i=1}^{\infty} \subseteq G$ such that $g_i K \cap g_j K = \emptyset$ for $i \neq j$: Suppose that $(g_i K)_{i=1}^{n}$ is a family of pairwise disjoint sets; we prove that there exists $g_{n+1} \in G$ such that $(g_i K)_{i=1}^{n+1}$ is also a family of pairwise disjoint sets. Suppose by contradiction that for every $g \in G$ there is a $j$ such that $g_j K \cap gK \neq \emptyset$. Then $g \in g_j K K^{-1}$. So $G = \bigcup_{j=1}^{n} g_j K K^{-1}$ would be compact, a contradiction.

Since $G$ is MAP, $\iota_b$ is one-one. The sets $(\iota_b(g_i K))_{i=1}^{\infty}$ form an infinite sequence of pairwise disjoint translates of the compact (and thus measurable) set $\iota_b(K) \subseteq bG$. If $\mu_b(\iota_b(K)) > 0$ then

$$1 = \mu_b(bG) \geq \sum_{i=1}^{\infty} \mu_b(\iota_b(g_i K)) = \sum_{i=1}^{\infty} \mu_b(\iota_b(K)) = \infty,$$

a contradiction. Consequently, $\mu_b(\iota_b(K)) = 0$. If $K$ is $\sigma$-compact the assertion follows from the $\sigma$-additivity of $\mu_b$.

(ii) Follows from the fact that $\iota_b(G)$ has zero outer $\mu_b$-measure (see [14, 53]). ∎

If we replace in Lemma 5.4.8 the Bohr compactification by an arbitrary compactification $(\iota, C)$ the measurability condition on the set $\iota(G)$ becomes crucial.

EXAMPLE 5.4.9. Consider the compact group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and any fixed irrational $\alpha \in \mathbb{T}$. By Zorn's Lemma there is a maximal subgroup $G$ of $\mathbb{T}$ with $\alpha \notin G$. The subgroup $G$

equipped with the discrete topology is an LCA group. Let $\iota : G \to \mathbb{T}$ be the inclusion mapping and $C = \mathbb{T}$. Then $(\iota, C)$ is a group compactification of $G$, distinct from the Bohr compactification. Let $\mu_C$ be the Haar measure on $C$. Assume, for contradiction, that $G$ is a $\mu_C$-measurable null set in $\mathbb{T}$. Consider the measure preserving mappings $\varphi_k : \mathbb{T} \to \mathbb{T}$, $x \mapsto kx$, $k \in \mathbb{Z}$. Then all sets $\varphi_k^{-1}[\alpha + G]$ are measurable $\mu_C$-null sets. Pick any $x \in \mathbb{T}$. If $x \notin G$ then, by the maximality property of $G$, $\alpha = kx + g$ for some $k \in \mathbb{Z} \setminus \{0\}$ and $g \in G$. This implies $\varphi_k(x) \in \alpha + G$, i.e. $x \in \varphi_k^{-1}[\alpha + G]$. We conclude that

$$\mathbb{T} = G \cup \bigcup_{k \in \mathbb{Z} \setminus \{0\}} \varphi_k^{-1}[\alpha + G]$$

is a countable union of $\mu_C$-null sets, hence $1 = \mu_C(C) = 0$, a contradiction.

Let us denote by $\mathcal{F}_0(G)$ the set of all bounded (not necessarily continuous or even measurable) complex-valued functions $f : G \to \mathbb{C}$ vanishing at infinity, i.e. $f \in \mathcal{F}_0(G)$ if for every $\varepsilon > 0$ there is a compact set $K \subseteq G$ with $|f(x)| < \varepsilon$ for all $x \in G \setminus K$. As usual, $C_0(G)$ denotes the set of all *continuous* $f \in \mathcal{F}_0(G)$.

THEOREM 6. *Let $G$ be a MAP group. Then $C_0(G) \subseteq \mathcal{H}(G)$. If $G$ is not compact then even $\mathcal{F}_0(G) \subseteq \mathcal{H}_0(G)$.*

*Proof.* In the first step we show $\mathcal{F}_0(G) \subseteq \mathcal{H}(G)$. If $G$ is compact there is nothing to prove. Suppose $G$ is not compact. Let $f \in \mathcal{F}_0(G)$ and define $F : bG \to \mathbb{C}$ by

$$F(x) := \begin{cases} f(g) & \text{if } x = \iota_b(g),\ g \in G, \\ 0 & \text{else.} \end{cases}$$

Then $f = F \circ \iota_b$. It suffices to consider $f$ such that $0 \le f \le 1$. For every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq G$ such that $f(x) < \varepsilon$ for $x \in G \setminus K_\varepsilon$. By Lemma 5.4.8, we have $\mu_b(A) = 0$, where $A = \iota_b(K_\varepsilon)$. Regularity of the Haar measure implies that we can find an open set $O \supset A$ such that $\mu_b(O) < \varepsilon$. Let $h$ be an Urysohn function for $A$ and $bG \setminus O$, i.e. $h : bG \to [0,1]$ is continuous with $h = 1$ on $A$ and $h = 0$ on $bG \setminus O$. Consider the continuous function $g_\varepsilon := h + \varepsilon \mathbb{1}_{bG}$. Since $0 \le F \le g_\varepsilon$ and

$$\int_{bG} g_\varepsilon \, d\mu_b \le \mu_b([h > 0]) + \varepsilon \le 2\varepsilon,$$

Proposition 2.3.3 implies $F \in \mathcal{R}_\mu(bG)$. Hence $f$ is a Hartman function. It remains to show that $[F \ne 0]$ is a meager $\mu_b$-null set. For each $n \in \mathbb{N}$ the set $[f \ge 1/n]$ is compact. Hence $\iota_b([f \ge 1/n]) = [F \ge 1/n]$ is a compact $\mu_b$-null set and therefore nowhere dense. Thus $[F \ne 0] = \bigcup_{n=1}^{\infty} [|F| \ge 1/n]$ is a meager $\mu_b$-null set and $f \in \mathcal{H}_0(G)$. ∎

COROLLARY 5.4.10. *Hartman functions $f \in \mathcal{H}(G)$ need not be measurable with respect to the Haar measure on $G$.*

*Proof.* As a counterexample take $G = \mathbb{R}$ with the Lebesgue measure and any set $A \subset [0,1]$ which is not Lebesgue measurable. Then $f = \mathbb{1}_A$ is a Hartman function by Theorem 6 but not Lebesgue measurable. ∎

As a further consequence of Theorem 6 we get the following supplement to Corollary 5.2.2.

COROLLARY 5.4.11. *Let $G$ be a nondiscrete MAP group. Then $\mathcal{H}_0(G) \setminus \mathcal{W}(G) \neq \emptyset$. In particular, the inclusion $C_0(G) \subset \mathcal{H}_0(G)$ is strict.*

*Proof.* Let $f := \mathbb{1}_{\{0\}}$. Then $f \in \mathcal{H}_0(G)$ (trivially for compact $G$, otherwise by Theorem 6). Since $f$ is not continuous $f \notin \mathcal{W}(G)$. (Recall that every weakly almost periodic $f$ has a representation $f = F \circ \iota_w$ with $F : wG \to \mathbb{C}$ continuous on the weakly almost periodic compactification $(\iota_w, wG)$ and thus is continuous.) ∎

The following example shows that also for the integers the space $C_0(\mathbb{Z})$ of functions vanishing at infinity is a proper subspace of $\mathcal{H}_0(\mathbb{Z})$.

EXAMPLE 5.4.12. Let $T = \{t_n : n \in \mathbb{N}\}$ be a *lacunary* set of positive integers, i.e. $t_1 < t_2 < \cdots$ with $\limsup_{n\to\infty} t_n/t_{n+1} = \varepsilon < 1$. Then $\mathbb{1}_T \in \mathcal{H}_0(\mathbb{Z}) \setminus C_0(\mathbb{Z})$.

*Proof.* By [45, Theorem 9] for each $n \in \mathbb{N}$ there exists an $n$-dimensional compactification $(\iota_n, C_n)$ and a compact $\mu_{C_n}$-continuity set $K_n \subseteq C_n$ with $\mu_n(K_n) \leq 4n\varepsilon^n$ such that $\iota_n^{-1}[K_n] \supseteq T$. Furthermore, we can arrange $(\iota_n, C_n) \leq (\iota_{n+1}, C_{n+1})$ and $\pi_{n+1,n}^{-1}[K_n] \supseteq K_{n+1}$, where $\pi_{n+1,n} : C_{n+1} \to C_n$ is the canonical projection, i.e. $\iota_n = \pi_{n+1,n} \circ \iota_{n+1}$. Let $(\iota, C) := \bigvee_{n=1}^{\infty} (\iota_n, C_n)$ and let $\pi_n : C \to C_n$ be the canonical projection onto $C_n$. Thus $K := \bigcap_{n=1}^{\infty} \pi_n^{-1}[K_n]$ is a compact $\mu_C$-null set (hence a $\mu_C$-continuity set) with $\iota^{-1}[K] \supseteq T$. This shows $\mathbb{1}_T \in \mathcal{H}_0(\mathbb{Z})$. Since $T \subseteq \mathbb{Z}$ is infinite, we have $\mathbb{1}_T \notin C_0(\mathbb{Z})$. ∎

**5.5. Hartman functions on $\mathbb{Z}$.** For locally compact groups $G$ it is very easy to see that $C_0(G) \subseteq \mathcal{W}(G)$. A much harder problem is to find functions in $\mathcal{W}(G) \setminus (AP(G) \oplus C_0(G))$ (see for instance [42]).

Topological groups with $\mathcal{W}(G) = AP(G) \oplus C_0(G)$ are called *minimally w.a.p.* A famous example, due to M. Megrelishvili, is $H_+[0,1]$, the group of all orientation preserving self-homeomorphisms of the closed unit interval $[0,1]$ endowed with the compact-open topology (see [30]). For minimally w.a.p. groups our Theorem 6 implies $\mathcal{W}(G) \subseteq \mathcal{H}(G)$. However, it is known that noncompact LCA groups are never minimally w.a.p. (see [7]).

PROBLEM 5.5.1. *Find a nontrivial topological group $G$ (necessarily not minimally w.a.p.) such that $\mathcal{W}(G) \subseteq \mathcal{H}(G)$.*

Throughout the rest of this section all results are stated for the case $G = \mathbb{Z}$. A quick way to obtain $f \in (\mathcal{W} \cap \mathcal{H}_0) \setminus (AP \oplus C_0)$ is through the following result.

PROPOSITION 5.5.2. *Let $(t_n)_{n=1}^{\infty} \subseteq \mathbb{Z}$ be a lacunary sequence of positive integers, i.e.*

$$\limsup_{n\to\infty} \frac{t_n}{t_{n+1}} = \varepsilon < 1.$$

*Let $T := \{t_n : n \in \mathbb{N}\} \subseteq \mathbb{Z}$. Then $\mathbb{1}_T \in (\mathcal{W} \cap \mathcal{H}_0) \setminus (AP \oplus C_0)$.*

*Proof.* According to our Example 5.4.12, $f = \mathbb{1}_T$ is a member of $\mathcal{H}_0$. Since $T$ is a lacunary set, [5, Theorem 4.2] implies $f \in \mathcal{W}$. Furthermore, $\liminf_{k\to\pm\infty} = 1$ implies $f \notin C_0$ and $f \notin AP$. Suppose for contradiction that $f = f_a + f_0$ where $f_a \in AP$ and $f_0 \in C_0$. Then

$$0 = \mathrm{dens}(T) = m(f) = m(f_a) + m(f_0)$$

implies $m(f_a) = 0$. As $f_a(k) \geq 0$ for all but finitely many $k$, this implies $f_a = 0$, a contradiction. ∎

The main objective of this section is to illustrate a further method to construct functions $f \in (\mathcal{W} \cap \mathcal{H}_0) \setminus (AP \oplus C_0)$.

**5.5.1. Fourier–Stieltjes transformation.** Let us recall some facts about the Fourier transformation of measures on LCA groups. Let $G$ be an LCA group. We denote by $\mathcal{M}(G)$ the set of all finite complex Borel measures on $G$. Recall that $\mathcal{M}(G)$ can be regarded as the dual $C_0(G)^*$ of the Banach space $C_0(G)$. The canonical pairing $C_0(G) \times C_0(G)^* \to \mathbb{C}$ is given by

$$\langle f, \mu \rangle := \int_G f(x)\, d\mu(x).$$

Also recall that we convolute two measures $\mu, \nu \in \mathcal{M}(G)$ according to the formula

$$\langle f, \mu * \nu \rangle = \int_{G \times G} f(x+y)\, d(\mu \otimes \nu)(x, y).$$

The Fourier–Stieltjes transform $\mu \mapsto \hat{\mu}$ assigns to a measure $\mu \in \mathcal{M}(G)$ the uniformly continuous function

$$\hat{\mu}(\chi) := \int_G \chi(x)\, d\mu(x)$$

defined on the dual group $\hat{G}$. The map $\mu \mapsto \hat{\mu}$ is a continuous homomorphism of the convolution algebra $(\mathcal{M}(G), *)$ into the function algebra $(UCB(\hat{G}), \cdot)$ of uniformly continuous functions on $\hat{G}$. The set $\{\hat{\mu} : \mu \in \mathcal{M}(G)\}$ of all Fourier–Stieltjes transforms is called the *Fourier–Stieltjes algebra* and denoted by $\mathcal{B}(\hat{G})$. It is well known (see [5, 39]) that for noncompact LCA groups $G$ the inclusions

$$AP(G) \subset \overline{\mathcal{B}(G)} \subset \mathcal{W}(G)$$

hold and are strict.

PROPOSITION 5.5.3. *Let $G$ be a locally compact group. The following assertions hold*:

(i) *If $\mu$ is discrete, then $\hat{\mu}$ is almost periodic.*
(ii) *If $\mu$ is absolutely continuous with respect to the Haar measure on $G$, then $\hat{\mu} \in C_0(\hat{G}) \subseteq \mathcal{W}_0(\hat{G})$ (for $G = \mathbb{T}$ this is the Riemann–Lebesgue Lemma).*
(iii) *$m_{\hat{G}}(\hat{\mu}) = \mu(\{0_G\})$ for the unique invariant mean $m_{\hat{G}}$ on $\hat{G}$. In particular, $\hat{\mu}$ has zero mean value whenever $\mu$ is atomless.*

*Proof.* See [39, Section 1.3]. ∎

Recall that an LCA group, by Pontryagin's duality theorem, is algebraically and topologically isomorphic to its bi-dual.

LEMMA 5.5.4. *Let $G$ be a discrete LCA group and $(\nu_n)_{n=1}^{\infty} \subseteq \mathcal{M}(\hat{G})$ a bounded sequence of discrete measures. Then the following assertions are equivalent:*

(i) *The sequence $(\hat{\nu}_n)_{n=1}^{\infty} \subseteq AP(G)$ converges pointwise to a bounded function $f : G \to C$.*
(ii) *The sequence $(\nu_n)_{n=1}^{\infty}$ of discrete measures converges weak-$^*$ to a measure $\mu \in \mathcal{M}(\hat{G})$.*

*In this case $f = \hat{\mu}$, the Fourier–Stieltjes transform of the measure $\mu$.*

*Proof.* (i)$\Rightarrow$(ii): Let $f_n := \hat{\nu}_n$. By weak-*-compactness of the unit ball in $C_0(\hat{G})^* = \mathcal{M}(\hat{G})$ we can find a weak-*-limit point $\mu$ of the set $\{\nu_n : n \geq 0\}$. Due to compactness of $\hat{G}$, for every $x \in G$ the map $\mu \mapsto \int_{\hat{G}} \chi(x)\, d\mu(\chi)$ is a weak-*-continuous functional defined on $\mathcal{M}(\hat{G})$. Thus for every $x \in G$ and $\varepsilon > 0$ there exist infinitely many $n_k$, $k \in \mathbb{N}$, (depending of course on $x$) such that

$$|\hat{\mu}(x) - f_{n_k}(x)| = \left| \int_{\hat{G}} \chi(x)\, d\mu(\chi) - \int_{\hat{G}} \chi(x)\, d\nu_{n_k}(\chi) \right| < \varepsilon.$$

Since $f_n(x) \to f(x)$ pointwise, we obtain

$$|\hat{\mu}(x) - f(x)| \leq |\hat{\mu}(x) - f_{n_k}(x)| + |f_{n_k}(x) - f(x)| < 2\varepsilon.$$

Thus $\lim_{n\to\infty} f_n(x) = \hat{\mu}(x)$. Let $\tilde{\mu}$ be another weak-*-limit point of the set $\{\nu_n : n \in \mathbb{N}\}$. On a compact space, weak-*-convergence of measures implies pointwise convergence of their Fourier–Stieltjes transforms. Thus $\hat{\tilde{\mu}}$ and $\hat{\mu}$ coincide. Hence $\mu = \text{weak-*-}\lim_{n\to\infty} \nu_n = \tilde{\mu}$.

(ii)$\Rightarrow$(i): By compactness of $\hat{G}$, for every $x \in G$ the mapping $\mu \mapsto \hat{\mu}(x) = \int_{\hat{G}} \chi(x)\, d\mu(\chi)$ is a weak-*-continuous functional. Thus $f_n := \hat{\nu}_n$ converges pointwise. ■

**5.5.2. Example.** In the following we will investigate the function

$$f(k) = \prod_{j=1}^{\infty} \cos^2\left(2\pi \frac{k}{3^j}\right)$$

defined on the group $G = \mathbb{Z}$ of integers, and we will prove Theorem 7 below. For its formulation we use the singular measure $\mu_3$ concentrated on the ternary Cantor (middle-third) set in the natural way. To be more precise: Let $\lambda$ be the Lebesgue measure on $[0,1)$. Consider the $\lambda$-almost everywhere uniquely defined mapping $\varphi : [0,1) \to [0,1)$ with

$$\varphi : \sum_{i=1}^{\infty} \frac{a_i}{2^i} \mapsto \sum_{i=1}^{\infty} \frac{2a_i}{3^i},$$

$a_i \in \{0,1\}$, and the canonical inclusion $\iota : [0,1) \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$, $x \mapsto x + \mathbb{Z}$. Then $\mu_3 = (\iota \circ \varphi) \circ \lambda$ (notation as in Proposition 2.4.5).

THEOREM 7. *Let $f : \mathbb{Z} \to [0,1]$ be given by*

$$f(k) = \prod_{j=1}^{\infty} \cos^2\left(2\pi \frac{k}{3^j}\right).$$

*Then the following statements hold:*

(i) $f \in (\mathcal{H}_0 \cap \mathcal{W}) \setminus (AP \oplus C_0)$.
(ii) $m_{\mathbb{Z}}(f) = 0$ *for $m_{\mathbb{Z}}$ the unique invariant mean on $\mathcal{H}(\mathbb{Z})$.*
(iii) *$f$ can be realized by a Riemann integrable function on the 3-adic compactification $\overline{\mathbb{Z}}^{(3)}$.*
(iv) *$f$ is the Fourier–Stieltjes transform of the singular measure $\mu_3$ corresponding to the ternary Cantor set canonically embedded into $\mathbb{T}$.*

*Proof.* Everything will follow from Lemmas 5.5.5, 5.5.7 and 5.5.8. ∎

We have to fix some notation and then prove the auxiliary statements. We will construct a function on $\mathbb{Z}$ using discrete measures on $\hat{\mathbb{Z}} = \mathbb{T}$. Note that $\mathbb{T}$ is algebraically and topologically isomorphic to the interval $[0,1)$ equipped with addition modulo 1.

For $\alpha \in [0,1)$ let us denote by $\delta_{\chi_\alpha} \in \mathcal{M}(\hat{\mathbb{Z}})$ the probability measure concentrated on the character $\chi_\alpha : k \mapsto \exp(2\pi i k\alpha)$. We define recursively discrete probability measures $\nu_n \in \mathcal{M}(\mathbb{Z})$ by $\nu_0 := \delta_{\chi_{1/2}}$ and

$$\nu_n := \nu_{n-1} * \tfrac{1}{2}(\delta_{\chi_{-1/3^n}} + \delta_{\chi_{1/3^n}}).$$

Note that $\nu_n \to \mu_3$ in the weak-*-topology of $\mathcal{M}(\mathbb{Z})$. Using the fact that $(\nu_n * \nu_{n-1})\hat{} = \hat{\nu}_n \hat{\nu}_{n-1}$ and $\hat{\delta}_{\chi_\alpha}(k) = \chi_\alpha(k) = \exp(2\pi i k\alpha)$ one easily computes $\hat{\nu}_n(k) = \prod_{j=1}^n \cos\big(2\pi \frac{k}{3^j}\big)$.

Lemma 5.5.5. *Let*

$$\tilde{f}_n := \hat{\nu}_n(k) = \prod_{j=1}^{n} \cos\left(2\pi \frac{k}{3^j}\right).$$

*Then $\tilde{f}_n$ converges pointwise to $\hat{\mu}_3$, the Fourier–Stieltjes transform of the singular measure $\mu_3$ concentrated on the ternary Cantor set. In particular, $\lim_{n\to\infty} \tilde{f}_n$ is weakly almost periodic and has zero mean value.*

*Proof.* Each $\tilde{f}_n$ is a product of finitely many periodic factors with rational periods, so $\tilde{f}_n$ is periodic. We show that the functions $\tilde{f}_n$ converge pointwise. Observe that $\lim_{j\to\infty} \cos(2\pi k/3^j) = 1$ for fixed $k \in \mathbb{N}$. All terms of this sequence are nonnegative provided $j \geq \log_3(2k) =: j(k)$. Thus $(\tilde{f}_{j(k)+n}(k)/\tilde{f}_{j(k)}(k))_{n=1}^\infty$ is a decreasing sequence of nonnegative real numbers, hence its limit exists. By Lemma 5.5.4 we know that

$$\tilde{f}(k) := \lim_{n\to\infty} \tilde{f}_n(k) = \prod_{j=1}^{\infty} \cos\left(2\pi \frac{k}{3^j}\right)$$

is the Fourier–Stieltjes transform of the measure $\mu = \mu_3 \in \mathcal{M}(\mathbb{Z})$. Since $\mu_3$ has no atoms, Proposition 5.5.3 implies that $\tilde{f} = \hat{\mu}_3 \in \mathcal{W}(\mathbb{Z})$ and that $m_{\mathbb{Z}}(\tilde{f}) = 0$ for the unique invariant mean $m_{\mathbb{Z}}$ on $\mathcal{W}(\mathbb{Z})$.

The same considerations apply to the discrete measures $\nu_n * \nu_n$, the nonnegative periodic functions

$$f_n(k) := \tilde{f}_n^2(k) = \prod_{j=1}^{n} \cos^2\left(2\pi \frac{k}{3^j}\right)$$

and the limit $f = \tilde{f}^2 = (\mu_3 * \mu_3)\hat{}$, which is weakly almost periodic with zero mean value. ∎

Lemma 5.5.6. *The periodic functions $f_n : \mathbb{Z} \to [0,1]$ defined via*

$$f_n(k) := \tilde{f}_n^2(k) = \prod_{j=1}^{n} \cos^2\left(2\pi \frac{k}{3^j}\right)$$

*have the mean value* $m_{\mathbb{Z}}(f_n) = 1/2^n$*, where* $m_{\mathbb{Z}}$ *is the unique invariant mean on* $AP(\mathbb{Z})$.

*Proof.* By Proposition 5.5.3 it suffices to compute $(\nu_n * \nu_n)(\{0\})$. We leave the elementary calculation to the reader. ■

LEMMA 5.5.7. *Let* $f = \lim_{n\to\infty} f_n$ *be as above. Then* $f \notin AP \oplus C_0$.

*Proof.* $\tilde{f}$ satisfies the functional equation $\tilde{f}(3k) = \tilde{f}(k)$, $k \in \mathbb{Z}$. This implies

$$\tilde{f}(3^k) = \tilde{f}(0) = 1.$$

Thus both $\tilde{f}, f \notin C_0$. As $f \geq 0$ but $m_{\mathbb{Z}}(f) = 0$ we have $f \notin AP$ by Corollary 3.6.12.

Suppose there exists a representation $f = f_a + f_0 \geq 0$ with nontrivial $f_a \in AP$ and $f_0 \in C_0$. Furthermore, let

$$f_a = \underbrace{\max\{f_a, 0\}}_{:=f_a^+ \geq 0} + \underbrace{\min\{f_a, 0\}}_{:=f_a^- \leq 0}.$$

Note that $f_a^+, f_a^- \in AP$ as $AP$ is a lattice. $m_{\mathbb{Z}}(f_a) = m_{\mathbb{Z}}(f) = 0$ implies $m_{\mathbb{Z}}(f_a^+) = -m_{\mathbb{Z}}(f_a^-)$. As $f_a^-$ is a nonpositive almost periodic function, $m_{\mathbb{Z}}(f_a^-) < 0$. Thus there exists $\varepsilon > 0$ such that for all $N \in \mathbb{N}$,

$$\inf_{|k|\geq N} f_a(k) = \inf_{|k|\geq N} f_a^-(k) \leq -\varepsilon < 0.$$

Note that $f_a^-(k_0) \neq 0$ implies $f_a^+(k_0) = 0$. Let $N_0$ be such that $|f_0(k)| < \varepsilon/2$ for $|k| \geq N_0$. Thus there exists $k_0 \geq N_0$ such that

$$f(k_0) = f_a(k_0) + f_0(k_0) = f_a^-(k_0) + f_0(k_0) \leq -\varepsilon + \varepsilon/2 = -\varepsilon/2 < 0.$$

This contradicts $f \geq 0$. ■

Consider the compact group of 3-adic integers $\overline{\mathbb{Z}}^{(3)}$ realized as projective limit of the projective system of cyclic groups $C_n := \mathbb{Z}/3^n\mathbb{Z}$ and mappings $\pi_n$ (reducing $k$ mod $3^{n+1}$ to $k$ mod $3^n$):

$$\begin{tikzcd}
& & & & \overline{\mathbb{Z}}^{(3)} \arrow[dlll, "\kappa_{n-1}"'] \arrow[dl, "\kappa_n"] \arrow[d, dotted, no head] \\
\{0\} & \mathbb{Z}/3\mathbb{Z} \arrow[l, "\pi_0"] & \ldots \arrow[l, "\pi_1"] & \mathbb{Z}/3^n\mathbb{Z} \arrow[l, "\pi_{n-1}"] & {} \arrow[l, "\pi_n"]
\end{tikzcd}$$

The projective limit $\overline{\mathbb{Z}}^{(3)} := \lim_{\leftarrow} C_n$ of this system can be identified with a certain closed subgroup of the compact group $\prod_{n=1}^{\infty} C_n$. Regarding $C_n$ as the set $\{0, 1/3^n, \ldots, 1-1/3^n\}$ with addition modulo 1, one easily checks that for each integer $k \in \mathbb{Z}$ the sequence $\iota(k) := (k/3^n)_{n=1}^{\infty}$ defines an element of the projective limit $\overline{\mathbb{Z}}^{(3)}$. The mapping $\iota : \mathbb{Z} \to \overline{\mathbb{Z}}^{(3)}$ is a (continuous) homomorphism. Hence $(\iota, \overline{\mathbb{Z}}^{(3)})$ is a group compactification of $\mathbb{Z}$, the so-called 3-*adic compactification*. Note that each $(C_n, \iota_n)$ is a group compactification of $\mathbb{Z}$, where $\iota_n$ is reduction modulo $3^n$. Furthermore, $(C_n, \iota_n) \leq (C_{n+1}, \iota_{n+1})$ via $\pi_n$ and $(C_n, \iota_n) \leq (\iota, \overline{\mathbb{Z}}^{(3)})$ via $\kappa_n$ for each $n \in \mathbb{N}$. By construction every $3^n$-periodic function $f : \mathbb{Z} \to \mathbb{C}$ can be realized by a continuous function $F : C_n \to \mathbb{C}$.

Lemma 5.5.8. *Let $f = \lim_{n\to\infty} f_n$ be as above. Then $f \in \mathcal{H}_0$ and $f$ can be realized in the 3-adic integers.*

*Proof.* Since every $3^n$-periodic function can be realized by a continuous function on $C_n$, we can in particular realize $f_n := \prod_{j=1}^n \cos^2(2\pi k/3^j)$. Consequently, there exists a unique continuous function $F_n$ on the 3-adic integers $\overline{\mathbb{Z}}^{(3)}$ such that $f_n = F_n \circ \iota$.

Since for $x \in \iota(\mathbb{Z})$ the sequence of $(F_n(x))_{n=1}^\infty$ is decreasing (note that $0 \le \cos^2(2\pi k/3^j) \le 1$), $(F_n(x))_{n=1}^\infty$ is decreasing for every $x \in \overline{\mathbb{Z}}^{(3)}$ by continuity of $F_n$. In particular, the limit $F(x) := \lim_{n\to\infty} F_n(x)$ exists and $F \circ \iota = f$. We show that $F$ is Riemann integrable on $\overline{\mathbb{Z}}^{(3)}$: For each $n \in \mathbb{N}$ we have $0 \le F \le F_n$. Lemma 5.5.6 and uniqueness of the invariant mean $m_{\mathbb{Z}}$ on $AP$ yield

$$\int_{\overline{\mathbb{Z}}^{(3)}} F_n \, d\lambda = m_{\mathbb{Z}}(f_n) = \frac{1}{2^n}$$

for the normalized Haar measure $\lambda$ on $\overline{\mathbb{Z}}^{(3)}$. Thus Proposition 2.3.3 implies that $F$ is Riemann integrable on $\overline{\mathbb{Z}}^{(3)}$.

Finally, suppose $F$ has a g.j.d. Then Theorem 4 implies $f \notin \mathcal{W}$, contradicting Lemma 5.5.5. Thus $f \in \mathcal{H}_c$. By Proposition 5.1.6 we can find unique functions $f_a \in AP$ and $f_0 \in \mathcal{H}_0$ such that $f = f_a + f_0$. As $f \ge 0$ we have $f_a \ge 0$. $m_{\mathbb{Z}}(f_a) = m_{\mathbb{Z}}(f) = 0$ implies $f_a = 0$. So, indeed $f = f_0 \in \mathcal{H}_0$. ■

Problem 5.5.9. *Construct functions $f_1 \in \overline{\mathcal{B}} \setminus \mathcal{H}$ and $f_2 \in \mathcal{H} \setminus \overline{\mathcal{B}}$.*

Problem 5.5.10. *How are $\overline{\mathcal{B}}$ and $\mathcal{W} \cap \mathcal{H}$ related? Is there a reasonable condition on functions in $\overline{\mathcal{B}}$ that implies Hartman measurability?*

# 6. Summary

The following diagram summarizes some of our results concerning the space $\mathcal{H} = \mathcal{H}(G)$ of Hartman functions on a topological group $G$. Recall the following function spaces:

$AC$ — almost convergent functions (Definition 3.1.4)
$AP$ — almost periodic functions (Definition 3.6.4)
$\mathcal{W}$ — weakly almost periodic functions (Definition 3.6.4)
$\overline{\mathcal{B}}$ — Fourier–Stieltjes algebra (Section 5.5.1)
$\mathcal{H}$ — Hartman functions (Definition 4.1.1)
$\mathcal{H}^w$ — weak Hartman functions (Definition 4.2.7)
$\mathcal{H}_c$ — Hartman functions realized without g.j.d. (Definition 5.3.2)
$\mathcal{H}_0$ — Hartman functions realized by functions supported on a meager null set (Definition 5.4.2)
$C_0$ — continuous functions vanishing at infinity (p. 63)

Inclusions indicated by | are proper (at least for certain groups $G$, e.g. for $G = \mathbb{Z}$). For spaces connected by : we did not prove strict inclusions.

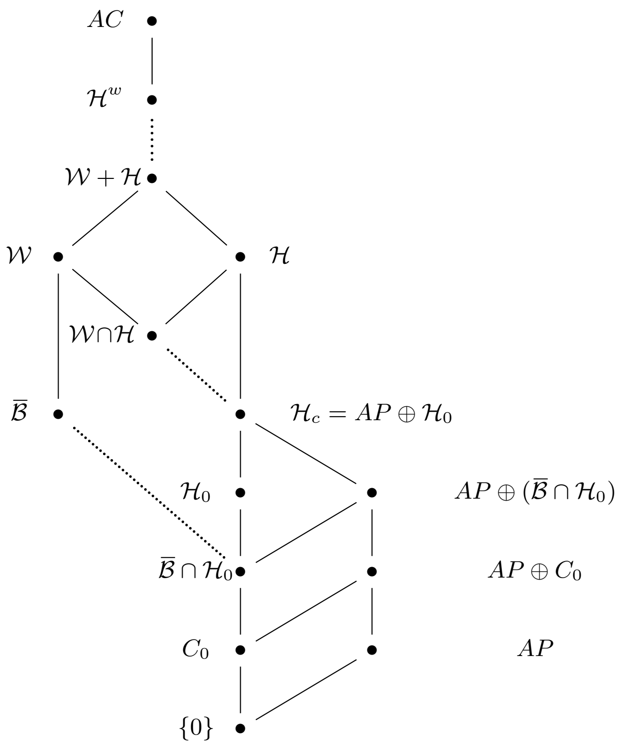


Fig. 1. Spaces of Hartman measurable functions

# References


[1] D. L. Armacost, *The Structure of Locally Compact Abelian Groups*, Monogr. Textbooks Pure Appl. Math. 68, Dekker, New York, 1981.

[2] M. Beiglböck, *Strong characterizing sequences of countable groups*, J. Number Theory 127 (2007), 145–152.

[3] M. Beiglböck, C. Steineder, and R. Winkler, *Sequences and filters of characters characterizing subgroups of compact abelian groups*, Topology Appl. 153 (2006), 1682–1695.

[4] J. F. Berglund and N. Hindman, *Filters and the weak almost periodic compactification of a discrete semigroup*, Trans. Amer. Math. Soc. 284 (1984), 1–38.

[5] J. F. Berglund, H. D. Junghenn, and P. Milnes, *Analysis on Semigroups*, Canad. Math. Soc. Ser. Monogr. Adv. Texts, Wiley, New York, 1989.

[6] R. B. Burckel, *Weakly Almost Periodic Functions on Semigroups*, Gordon and Breach, New York, 1970.

[7] C. Chou, *Minimally weakly almost periodic groups*, J. Funct. Anal. 36 (1980), 1–17.

[8] K. R. Davidson, *$C^*$-Algebras by Example*, Fields Inst. Monogr., Amer. Math. Soc., Providence, RI, 1996.

[9] N. Dunford and J. T. Schwartz, *Linear Operators. Part I*, Wiley, New York, 1988 (reprint of the 1958 original).

[10] I. Farah, *Functions essentially depending on at most one variable*, J. Combin. Theory Ser. A 94 (2001), 169–177.

[11] —, *Dimension phenomena associated with $\beta\mathbb{N}$-spaces*, Topology Appl. 125 (2002), 279–297.

[12] S. Frisch, M. Paštéka, R. F. Tichy, and R. Winkler, *Finitely additive measures on groups and rings*, Rend. Circ. Mat. Palermo (2) 48 (1999), 323–340.

[13] A. A. Giannopoulos and V. D. Milman, *Concentration property on probability spaces*, Adv. Math. 156 (2000), 77–106.

[14] I. Glicksberg, *Uniform boundedness for groups*, Canad. J. Math. 14 (1962), 269–276.

[15] H. Glöckner and K.-H. Neeb, *Minimally almost periodic abelian groups and commutative $W^*$-algebras*, in: Nuclear Groups and Lie Groups (Madrid, 1999), Res. Exp. Math. 24, Heldermann, Lemgo, 2001, 163–185.

[16] M. Goldstern and M. Pinsker, *A survey of clones on infinite sets*, Algebra Universalis (2008), http://www.arxiv.org/math.RA/0701030.

[17] E. E. Granirer, *Exposed points of convex sets and weak sequential convergence*, Mem. Amer. Math. Soc. 123 (1972).

[18] F. P. Greenleaf, *Invariant Means on Topological Groups and Their Applications*, Van Nostrand Math. Stud. 16, Van Nostrand Reinhold, New York, 1969.

[19] S. Hartman, *Über Niveaulinien fastperiodischer Funktionen*, Studia Math. 20 (1961), 313–325.

[20] —, *R-fastperiodische Funktionen*, Studia Math. (Ser. Specjalna) 1963, no. 1, 39–40.

[21] —, *Remarks on equidistribution on non-compact groups*, Compos. Math. 16 (1964), 66–71 (1964).

[22] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis. Vol. I*, 2nd ed., Grundlehren Math. Wiss. 115, Springer, Berlin, 1979.

[23] N. Hindman and D. Strauss, *Algebra in the Stone-Čech Compactification*, de Gruyter Exp. Math. 27, de Gruyter, Berlin, 1998.

[24] K.-H. Indlekofer, *Number theory—probabilistic, heuristic, and computational approaches*, Comput. Math. Appl. 43 (2002), 1035–1061.

[25] S. Kremp, *An elementary proof of the Eberlein–Šmulian theorem and the double limit criterion*, Arch. Math. (Basel) 47 (1986), 66–69.

[26] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Wiley-Interscience, New York, 1974.

[27] G. Maresch, *Filters and subgroups associated with Hartman measurable functions*, Integers 5 (2005), no. 3, A12, 17 pp.

[28] J.-L. Mauclaire, *Intégration et théorie des nombres*, Travaux en Cours, Hermann, Paris, 1986.

[29] —, *Integration and number theory*, in: Prospects of Mathematical Science (Tokyo, 1986), World Sci., Singapore, 1988, 197–125.

[30] M. G. Megrelishvili, *Every semitopological semigroup compactification of the group $H_+[0,1]$ is trivial*, Semigroup Forum 63 (2001), 357–370.

[31] M. Paštéka, *The measure density on Dedekind domains*, Ricerche Mat. 45 (1996), 21–36.

[32] —, *Submeasures and uniform distribution on Dedekind domains*, Atti Sem. Mat. Fis. Univ. Modena 45 (1997), 357–372.

[33] M. Paštéka and R. F. Tichy, *Distribution problems in Dedekind domains and submeasures*, Ann. Univ. Ferrara Sez. VII (N.S.) 40 (1994), 191–206 (1996).

[34] A. L. T. Paterson, *Amenability*, Math. Surveys Monogr. 29, Amer. Math. Soc., Providence, RI, 1988.

[35] D. Remus, *Topological groups without nontrivial characters*, in: General Topology and its Relations to Modern Analysis and Algebra, VI (Prague, 1986), Res. Exp. Math. 16, Heldermann, Berlin, 1988, 477–484.

[36] J. M. Rosenblatt and M. Wierdl, *Pointwise ergodic theorems via harmonic analysis*, in: Ergodic Theory and its Connections with Harmonic Analysis (Alexandria, 1993), London Math. Soc. Lecture Note Ser. 205, Cambridge Univ. Press, Cambridge, 1995, 3–151.

[37] S. Rothman, *The von Neumann kernel and minimally almost periodic groups*, Trans. Amer. Math. Soc. 259 (1980), 401–421.

[38] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1987.

[39] —, *Fourier Analysis on Groups*, Wiley, New York, 1990 (reprint of the 1962 original).

[40] —, *Functional Analysis*, 2nd ed., Int. Ser. Pure Appl. Math., McGraw-Hill, New York, 1991.

[41] W. Ruppert, *Compact Semitopological Semigroups: an Intrinsic Theory*, Lecture Notes in Math. 1079, Springer, Berlin, 1984.

[42] —, *On signed* $\mathbf{a}$*-adic expansions and weakly almost periodic functions*, Proc. London Math. Soc. (3) 63 (1991), 620–656.

[43] C. Ryll-Nardzewski, *On fixed points of semigroups of endomorphisms of linear spaces*, in: Proc 5th Berkeley Symposium on Mathematical Statistics and Probability, Vol. II, Part 1, Univ. California Press, Berkeley, CA, 1967, 55–61.

[44] J. W. Sander, *On the independence of Hartman sequences*, Monatsh. Math. 135 (2002), 327–332.

[45] J. Schmeling, E. Szabó, and R. Winkler, *Hartman and Beatty bisequences*, in: Algebraic Number Theory and Diophantine Analysis (Graz, 1998), de Gruyter, Berlin, 2000, 405–421.

[46] C. Steineder, *Subword complexity and projection bodies*, Adv. Math. 217 (2008), 2377–2400.

[47] C. Steineder and R. Winkler, *Complexity of Hartman sequences*, J. Théor. Nombres Bordeaux 17 (2005), 347–357.

[48] M. Stroppel, *Locally Compact Groups*, EMS Textbooks in Math., Eur. Math. Soc., Zürich, 2006.

[49] M. Talagrand, *Closed convex hull of set of measurable functions, Riemann-measurable functions and measurability of translations*, Ann. Inst. Fourier (Grenoble) 32 (1982), no. 1, vi, 39–69.

[50] W. Taylor, *The Clone of a Topological Space*, Res. Exp. Math. 13, Heldermann, Berlin, 1986.

[51] V. Uspenskij, *Compactifications of topological groups*, in: Proc. Ninth Prague Topological Symposium (2001), Topol. Atlas, North Bay, ON, 2002, 331–346.

[52] E. K. van Douwen, *Prime mappings, number of factors and binary operations*, Dissertationes Math. (Rozprawy Mat.) 199 (1981).

[53] N. Th. Varopoulos, *A theorem on the Bohr compactification of a locally compact Abelian group*, Proc. Cambridge Philos. Soc. 61 (1965), 65–68.

[54] S. Wagon, *The Banach–Tarski Paradox*, Encyclopedia Math. Appl. 24, Cambridge Univ. Press, Cambridge, 1985.

[55] P. Walters, *An Introduction to Ergodic Theory*, Grad. Texts in Math. 79, Springer, New York, 1982.

[56] R. Winkler, *Ergodic group rotations, Hartman sets and Kronecker sequences*, Monatsh. Math. 135 (2002), 333–343.

[57] —, *Hartman sets, functions and sequences—a survey*, Adv. Stud. Pure Math. 49 (2007), 517–543.