1. Introduction

The main object of this paper is to construct a calculus of anisotropic pseudodifferential operators (ψDOs) on a manifold and apply it to semi-elliptic operators generated by vector fields.

A simple example of a semi-elliptic differential operator on $\mathbb{R}^n$ is the operator

$$\sum_{k=1}^{n}(-1)^{m_k}\partial_{x_k}^{2m_k}, \qquad m_k \in \mathbb{N}. \tag{1.1}$$

Its symbol $\sigma(\xi) = \sum_{k=1}^{n} \xi_k^{2m_k}$ does not vanish in $\mathbb{R}^n \backslash \{0\}$, i.e. is of the "elliptic" nature. On the other hand the usual isotropic homogeneity of principal symbols of elliptic operators is now replaced by its anisotropic analogue

$$\sigma(t^{1/m_1}\xi_1, \dots, t^{1/m_n}\xi_n) = t^2\sigma(\xi), \qquad \forall t > 0.$$

Therefore, the corresponding calculus of ψDOs should include operators with symbols which satisfy anisotropic estimates. Such symbols $a(x,\xi)$ have different growth (decay) rates as $\xi \to \infty$ in different directions.

Let M be a C^∞-smooth n-dimensional manifold and $\nu_1, \dots, \nu_n$ be C^∞-smooth vector fields on M which span the tangent space T_xM at each point $x \in M$. The generalization of (1.1) to this situation is the semi-elliptic operator

$$\sum_{k=1}^{n}(-1)^{m_k}\partial_{\nu_k}^{2m_k}. \tag{1.2}$$

The vector fields $\nu_1, \dots, \nu_n$ may have nonzero commutators and the properties of operators like (1.2) depend on the structure of the Lie algebra generated by $\nu_1, \dots, \nu_n$. Correspondingly the theory of such operators is more geometric in spirit than the standard "elliptic" one. But still this is an "elliptic" theory and operators like (1.2) are much more "elliptic" than Hörmander's sums of squares of vector fields and their generalizations (see [Ho2], [Ho3, Vol. III], [RS], [Goo], [HN], [Ta1], [VSC], [Nua], [Mal] and the references therein). In a sense semi-elliptic operators generated by vector fields may be viewed as a bridge between elliptic operators and hypoelliptic operators of the type of Hörmander's sums of squares of vector fields.

In this paper we construct a calculus of anisotropic ψDOs on M which allows one to handle semi-elliptic operators generated by $\nu_1, \dots, \nu_n$. The symbols of these ψDOs belong to anisotropic analogues of the Hörmander classes $S^r_{\varrho,\delta}$. The coordinate directions defining the anisotropy correspond to $\nu_1(x)$, ..., $\nu_n(x)$ at each point $x \in M$. Since in this situation invariance under diffeomorphisms is clearly out of the question, we cannot follow the usual way of defining ψDOs on manifolds: local coordinates plus partitions of unity. We have to use invariant tools only. In the case when M is a nilpotent Lie group one can use the group convolution and Fourier transform to construct a calculus of anisotropic ψDOs on M (see, e.g., [Dy1], [Dy2], [NS], [How], [Mil], [Mel], [Ta2], [BG], [Cum], [CGGP]). We do not suppose that M is a Lie group and thus have to choose a different approach.

An invariant (= intrinsic = covariant) calculus of ψDOs on manifolds equipped with linear connections was developed in [Bok], [Wi1], [Wi2], [Dra], [FK] and [Saf]. It is related to quantization on manifolds and has been used to explicitly compute coefficients of the short time on-diagonal asymptotics for the heat kernels corresponding to elliptic operators (see [Und], [LQ], [GK1], [GK2], [Fu] and the references therein). All these papers except [Saf] dealt with the standard ψDOs, i.e. with ψDOs which can be defined via local coordinates. For those operators invariant complete symbols were defined in a coordinate-free way with the help of linear connections. The approach of [Saf] is more radical: the invariant coordinate-free definition of ψDOs allows one to cover new classes of operators where the coordinate approach is not applicable.

We follow the method of [Saf] to construct a calculus of anisotropic ψDOs generated by $\nu_1, \dots, \nu_n$. We deal with operators which act on sections of vector bundles over M. In such a situation one needs connections of two kinds: connections on the above-mentioned vector bundles and a connection on the underlying manifold M, more precisely on the cotangent bundle T^*M. In our case the latter is defined by the vector fields $\nu_1, \dots, \nu_n$. All necessary definitions and results from differential geometry are collected in Section 2. Some textbook material has been included in order to fix the notation and make the presentation reasonably self-contained. The words "reasonably self-contained" are understood in a purely pseudodifferential fashion: only those notions and facts of differential geometry which can be found in [Ho3, Vol. I] and [Tre, Vol. I] are included in the pre-requisites.

There are two polynomials naturally associated with the operator (1.2). One is its (principal) symbol $\sum_{k=1}^n (\sum_{l=1}^n \nu_k^l(x)\xi_l)^{2m_k}$ defined on T^*M. The other is the (principal) presymbol $\sum_{k=1}^n \eta_k^{2m_k}$ defined on $M \times \mathbb{R}^n$. Not surprisingly presymbols of ψDOs are more convenient to work with than symbols. We formulate the results in this paper in terms of presymbols. The corresponding symbols are obtained by substituting $\sum_{l=1}^n \nu_k^l(x)\xi_l$ for η_k into the presymbols.

We introduce the spaces of (pre)symbols in Section 3 and define anisotropic ψDOs generated by $\nu_1, \dots, \nu_n$ in Section 4. The latter also contains a formula expressing the presymbol of a ψDO in terms of its amplitude and a theorem on the change of the presymbol under a change of connections on the vector bundles. Section 5 deals with the adjoints of ψDOs.

The key result of the paper is Theorem 6.7 on composition of ψDOs with symbols from the anisotropic classes $S^r_{\varrho,\delta}$. This theorem and the boundedness results of Section 7 are obtained under the restriction (6.1) which connects ϱ and the orders of anisotropy to the geometry of the family $\nu_1, \dots, \nu_n$. In the case $\varrho = 1$ this restriction means that the commutator $[\partial_{\nu_j}, \partial_{\nu_k}]$ is an operator of a strictly lower order than $\partial_{\nu_j}\partial_{\nu_k}$ and $\partial_{\nu_k}\partial_{\nu_j}$.

In Section 8 we show how our calculus relates to the existing works on anisotropic (pseudo)differential operators on nilpotent Lie groups with dilations and in particular discuss the condition (6.1) which is supposed to be satisfied in the remaining part of the paper.

Section 9 is devoted to compact ψDOs. We prove in Section 10 that a semi-elliptic ψDO is Fredholm in anisotropic analogues of Bessel-potential spaces H_p^s if M is compact and establish some basic properties of these spaces. More detailed results on anisotropic function spaces on manifolds will be published in a forthcoming paper. It is shown in Section 11 that under reasonable conditions on a semi-elliptic ψDO A the resolvent $(A-\lambda I)^{-1}$ is a ψDO with a "well behaved" symbol for sufficiently large λ's lying outside a "parabolic" neighbourhood of the spectrum of the principal presymbol of A (see Definition 11.4). The results of this section are applied in Section 12 to construct complex powers of A and study meromorphic continuations of their kernels. The information obtained there is used in Section 13 where we deal with the exponential e^{-tA} and establish the full on-diagonal short time asymptotics of the heat kernel corresponding to A.

We return to the resolvent $(A-\lambda I)^{-1}$ in Section 14 and prove asymptotic formulae for its kernel. These formulae and a generalization of the Pleijel–Malliavin Tauberian theorem (see Lemmas 15.1, 15.2) allow us to obtain an asymptotic formula for the spectral function of A with a remainder estimate and more precise, in particular two-term, asymptotic formulae for the Riesz means of the spectral function. The latter show that the remainder estimate in the former is not optimal and that our asymptotic formula for the spectral function is an analogue of that from [Ag]. It is unclear (to me!) whether one can obtain the optimal remainder estimate using the wave equation method, because the standard reduction to the first order operators does not seem to be working in the anisotropic case. The method due to D. Robert and G. Métivier (see [Ro1] and [Me2]) is probably more promising here. (For asymptotic properties of the spectra of anisotropic elliptic ψDOs on $\mathbb{R}^n$ see [Ar], [BS], [BB], [BBR], [HR1], [Ro2] and the references therein.)

It is worth mentioning that due to the anisotropy the second terms in the asymptotic formulae for the Riesz means of the spectral function and of the distribution function of eigenvalues may be nonzero even if A is a differential operator (see Remark 15.7), which does not happen in the standard elliptic case (see [DG] and Remark 15.6).

2. Auxiliary geometric results

Vector fields. Let M be a C^∞-smooth n-dimensional manifold and $\nu_1,\dots,\nu_n$ be C^∞-smooth vector fields on M. Suppose that *$\nu_1(x),\dots,\nu_n(x)$ span the tangent space T_xM at each point $x\in M$*. Then there exist functions $C^m_{j,k}\in C^\infty(M)$ such that

$$[\partial_{\nu_j},\partial_{\nu_k}]=\sum_{m=1}^{n} C^m_{j,k}\partial_{\nu_m}, \tag{2.1}$$

where $[\cdot,\cdot]$ denotes the commutator: $[A,B]=AB-BA$, while ∂_{ν_j} is the derivative in the direction of ν_j, i.e.

$$\partial_{\nu_j} = \sum_{l=1}^{n} \nu_j^l(x)\partial_{x^l} \tag{2.2}$$

in any local coordinate system.

Let $\widetilde{\nu}^1(x), \dots, \widetilde{\nu}^n(x)$ be the basis of the cotangent space T_x^*M dual to $\nu_j(x)$, $j = 1, \dots, n$:

$$\langle \widetilde{\nu}^k(x), \nu_j(x) \rangle = \delta_j^k = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases} \tag{2.3}$$

It is clear that $\widetilde{\nu}^1, \dots, \widetilde{\nu}^n$ are C^∞-smooth 1-forms on M.

We will call a curve $\gamma : [a, b] \to M$ a *geodesic* if

$$\dot{\gamma}(t) = \sum_{j=1}^{n} c_j \nu_j(\gamma(t)), \qquad \forall t \in [a, b] \subset \mathbb{R}, \tag{2.4}$$

for some constants $c_1, \dots, c_n \in \mathbb{R}$. As usual, dot denotes the derivative with respect to the "time" t.

2.1. Remark. It is not difficult to check that our vector fields $\nu_j(x)$, $j = 1, \dots, n$, generate a linear connection on M with Christoffel symbols

$$\Gamma_{j,k}^m(x) = -\sum_{l=1}^{n} \widetilde{\nu}_k^l(x)\partial_{x^j}\nu_l^m(x)$$

(see (2.3)), such that the classical definition of a geodesic corresponding to this connection coincides with (2.4).

Let us take an arbitrary point $x \in M$. It follows from the standard results on ordinary differential equations that if $\widetilde{y} = (\widetilde{y}^1, \dots, \widetilde{y}^n) \in \mathbb{R}^n$ is sufficiently small, then there exists a unique geodesic $\gamma = \gamma_x^{\widetilde{y}} : [0, 1] \to M$ such that $\gamma(0) = x$ and

$$\dot{\gamma}(0) = \sum_{j=1}^{n} \widetilde{y}^j \nu_j(x),$$

i.e.

$$\dot{\gamma}(t) = \sum_{j=1}^{n} \widetilde{y}^j \nu_j(\gamma(t)), \qquad \forall t \in [0, 1]. \tag{2.5}$$

So, we have a well defined mapping of a neighbourhood $U \subset \mathbb{R}^n$ of $0 \in \mathbb{R}^n$ into M:

$$U \ni \widetilde{y} \mapsto \exp_x(\widetilde{y}) := \gamma_x^{\widetilde{y}}(1) \in M. \tag{2.6}$$

It is clear that

$$x = \exp_x(0). \tag{2.7}$$

Applying the theorem on the smoothness of solutions of ordinary differential equations with respect to parameters we deduce that the mapping (2.6) is C^∞-smooth. Differentiating the equality

$$\gamma_x^{\tau\widetilde{y}}(t) = \gamma_x^{\widetilde{y}}(\tau t)$$

with respect to τ and taking $\tau = 0$, $t = 1$ one can easily prove that the differential of (2.6) at $0 \in \mathbb{R}^n$ corresponds to the identity matrix in the coordinate system in T_xM defined by the basis $\nu_1(x), \dots, \nu_n(x)$. Hence (2.6) is a C^∞-diffeomorphism of a sufficiently small neighbourhood $U \subset \mathbb{R}^n$ of the origin onto a neighbourhood $W_x \subset M$ of $x \in M$. This defines a coordinate system on W_x: we say that the coordinates of a point $y \in W_x$ are $\widetilde{y} = (\widetilde{y}^1, \dots, \widetilde{y}^n) \in U \subset \mathbb{R}^n$ if

$$y = \exp_x(\widetilde{y}).$$

We will call this coordinate system the *canonical coordinate system with the origin at* $x \in M$.

In this coordinate system any geodesic $\gamma_x^{\widetilde{y}}(t)$ has the form $t\widetilde{y}$:

$$\gamma_x^{\widetilde{y}}(t) = \gamma_x^{t\widetilde{y}}(1) = \exp_x(t\widetilde{y}). \tag{2.8}$$

It follows from the above that if x and y are sufficiently close to each other, then there exists a unique geodesic $\gamma_{y,x} : [0,1] \to M$ such that $\gamma_{y,x}(0) = x$, $\gamma_{y,x}(1) = y$. So,

$$\dot{\gamma}_{y,x}(t) = \sum_{j=1}^{n} c_j \nu_j(\gamma_{y,x}(t)), \qquad \forall t \in [0,1],$$

for some constants $c_j = c_j(x,y) \in \mathbb{R}$ depending on x and y (see (2.5)). It is easy to see that c_j are C^∞-smooth in a neighbourhood of the diagonal of $M \times M$ and

$$c_j(x,y) = \widetilde{y}^j, \qquad j = 1, \dots, n, \tag{2.9}$$

where $\widetilde{y} = (\widetilde{y}^1, \dots, \widetilde{y}^n)$ are the coordinates of the point y in the canonical coordinate system with the origin at x (cf. (2.5)). The last equality is equivalent to

$$\gamma_{y,x} = \gamma_x^{\widetilde{y}} \quad \text{if } y = \exp_x(\widetilde{y}). \tag{2.10}$$

It is clear that

$$c_j(x,x) = 0. \tag{2.11}$$

Using the change of variable $t \mapsto 1 - t$ we obtain

$$c_j(x,y) = -c_j(y,x). \tag{2.12}$$

Let ν_0 be a C^∞-smooth vector field on M. For any $w \in M$ there exists a unique integral curve $E(\cdot, w) : [0, \delta] \to M$ of ν_0 starting at w:

$$\frac{\partial E(t,w)}{\partial t} = \nu_0(E(t,w)), \qquad E(0,w) = w.$$

Here $\delta = \delta(w, \nu_0) > 0$ is a sufficiently small number. If ν_0 is sufficiently small, i.e.

$$\nu_0(t) = \sum_{j=1}^{n} a_j(u)\nu_j(u), \qquad u \in M,$$

where $\sum_{j=1}^{n} |a_j(u)|$ is sufficiently small, then we can take $\delta(w, \nu_0) = 1$.

We define

$$\operatorname{Exp}(\nu_0)(w) = E(1,w) \tag{2.13}$$

whenever the right hand side is defined. It is obvious that

$$\exp_x(\widetilde{y}) = \operatorname{Exp}\Big(\sum_{j=1}^{n} \widetilde{y}^j \nu_j\Big)(x) \tag{2.14}$$

Throughout the paper ∂_ν^α *will denote the symmetrization of* $\partial_{\nu_1}^{\alpha_1} \dots \partial_{\nu_n}^{\alpha_n}$.

2.2. Lemma. *For any multi-index $\alpha \in \mathbb{Z}_+^n$ and any smooth function f we have*

$$\partial_\nu^\alpha f(x) = \partial_{\widetilde{y}}^\alpha f(\exp_x(\widetilde{y}))|_{\widetilde{y}=0}. \tag{2.15}$$

Proof. It is well known (see, e.g., [NSW, Proposition 4.2]) that the formal Taylor series

$$\sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!}(\partial_{\widetilde{y}}^\alpha f(\exp_x(\widetilde{y}))|_{\widetilde{y}=0})\widetilde{y}^\alpha$$

of $f(\exp_x(\widetilde{y})) = f(\mathrm{Exp}(\sum_{j=1}^n \widetilde{y}^j \nu_j)(x))$ at $0 \in \mathbb{R}^n$ equals

$$\sum_{l=0}^{\infty} \frac{1}{l!}\Big(\sum_{j=1}^n \widetilde{y}^j \partial_{\nu_j}\Big)^l f(x) = \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!}(\partial_\nu^\alpha f(x))\widetilde{y}^\alpha.$$

Comparing the coefficients we obtain (2.15). ∎

2.3. Lemma. *For any multi-indices $\alpha, \beta \in \mathbb{Z}_+^n$ such that $|\alpha+\beta| \geq 2$, the derivative*

$$\partial_{\nu(y)}^\alpha \partial_{\nu(z)}^\beta c_j(y,z)|_{y=z=x} = \partial_{\widetilde{y}}^\alpha \partial_{\widetilde{z}}^\beta c_j(\exp_x(\widetilde{y}), \exp_x(\widetilde{z}))|_{\widetilde{y}=\widetilde{z}=0}$$

is a linear combination of terms of the form

$$\sum \partial_\nu^{\mu^{(1)}} C_{k_1,m_1}^{j}(x) \partial_\nu^{\mu^{(2)}} C_{k_2,m_2}^{j_2}(x) \dots \partial_\nu^{\mu^{(q)}} C_{k_q,m_q}^{j_q}(x). \tag{2.16}$$

Here each of the "upper" indices $j_2, \dots, j_q$ coincides with one of the "lower" indices, which are k_l, m_l and those corresponding to the multi-indices $\mu^{(l)}$, and $\sum$ denotes the contraction, i.e. the summation from 1 to n over these repeated indices. The remaining

$$|\mu^{(1)}| + \dots + |\mu^{(q)}| + q + 1 = |\alpha+\beta|$$

"lower" indices correspond to those of $\alpha+\beta$. Moreover,

$$\frac{\partial}{\partial \widetilde{y}^k} c_j(\exp_x(\widetilde{y}), x) = -\delta_k^j, \tag{2.17}$$

$$\frac{\partial}{\partial \widetilde{z}^k} c_j(x, \exp_x(\widetilde{z})) = \delta_k^j, \tag{2.18}$$

$$\partial_{\widetilde{y}}^\alpha c_j(\exp_x(\widetilde{y}), x) = 0 \quad \text{if } |\alpha| \geq 2, \tag{2.19}$$

$$\partial_{\widetilde{z}}^\beta c_j(x, \exp_x(\widetilde{z})) = 0 \quad \text{if } |\beta| \geq 2, \tag{2.20}$$

$$\frac{\partial}{\partial \widetilde{y}^k}\frac{\partial}{\partial \widetilde{z}^m} c_j(\exp_x(\widetilde{y}), \exp_x(\widetilde{z}))|_{\widetilde{y}=\widetilde{z}=0} = \frac{1}{2} C_{k,m}^j(x). \tag{2.21}$$

Proof. The equalities (2.17)–(2.20) follow directly from (2.9) and (2.12).

Let $y = \exp_x(\widetilde{y})$, $z = \exp_x(\widetilde{z})$. It follows from (2.9), (2.12) and (2.14) that

$$\mathrm{Exp}\Big(-\sum_{l=1}^n \widetilde{y}^l \nu_l\Big)(y) = x.$$

Consequently

$$H(\widetilde{y},\widetilde{z})(y) = z, \quad \text{where} \quad H(\widetilde{y},\widetilde{z})(y) = \mathrm{Exp}\Big(\sum_{l=1}^n \widetilde{z}^l \nu_l\Big)\mathrm{Exp}\Big(-\sum_{l=1}^n \widetilde{y}^l \nu_l\Big).$$

Further,

$$\begin{aligned}\partial_{\widetilde{y}}^\alpha \partial_{\widetilde{z}}^\beta c_j(\exp_x(\widetilde{y}), \exp_x(\widetilde{z}))|_{\widetilde{y}=\widetilde{z}=0} &= \partial_{\widetilde{y}}^\alpha \partial_{\widetilde{z}}^\beta c_j(y, H(\widetilde{y},\widetilde{z})(y))|_{\widetilde{y}=\widetilde{z}=0} \\ &= (\partial_{\widetilde{y}} + \partial_{\widetilde{w}})^\alpha \partial_{\widetilde{z}}^\beta c_j(w, H(\widetilde{y},\widetilde{z})(w))|_{\widetilde{y}=\widetilde{z}=\widetilde{w}=0},\end{aligned}$$

where $w = \exp_x(\widetilde{w})$. Hence, it is sufficient to prove that

$$\partial_{\widetilde{w}}^{\alpha''} \partial_{\widetilde{y}}^{\alpha'} \partial_{\widetilde{z}}^{\beta} c_j(w, H(\widetilde{y}, \widetilde{z})(w))|_{\widetilde{y}=\widetilde{z}=\widetilde{w}=0} \tag{2.22}$$

is a linear combination of terms of the form (2.16) with $\alpha = \alpha' + \alpha''$. Let us evaluate

$$\partial_{\widetilde{y}}^{\alpha'} \partial_{\widetilde{z}}^{\beta} c_j(w, H(\widetilde{y}, \widetilde{z})(w))|_{\widetilde{y}=\widetilde{z}=0}. \tag{2.23}$$

We have to find the formal Taylor $(\widetilde{y}, \widetilde{z})$-series of the function

$$c_j\Big(w, \operatorname{Exp}\Big(\sum_{l=1}^{n} \widetilde{z}^l \nu_l\Big) \operatorname{Exp}\Big(-\sum_{l=1}^{n} \widetilde{y}^l \nu_l\Big)(w)\Big)$$

at $0 \in \mathbb{R}^n \times \mathbb{R}^n$. Using the Campbell–Hausdorff formula (see, e.g., [Ser, Part I, IV.7, IV.8]) one can prove (see [NSW, Appendix]) that this formal series equals

$$\sum_{p=0}^{\infty} \frac{1}{p!} \partial_{\nu_0(v)}^{p} c_j(w, v)|_{v=w}, \tag{2.24}$$

where $\partial_{\nu_0(v)}$ is a formal series of linear combinations of iterated commutators of $\sum_{l=1}^{n} \widetilde{z}^l \partial_{\nu_l}$ and $-\sum_{l=1}^{n} \widetilde{y}^l \partial_{\nu_l}$:

$$\begin{aligned}\partial_{\nu_0} &= \sum_{l=1}^{n} \widetilde{z}^l \partial_{\nu_l} - \sum_{l=1}^{n} \widetilde{y}^l \partial_{\nu_l} - \frac{1}{2}\Big[\sum_{l=1}^{n} \widetilde{z}^l \partial_{\nu_l}, \sum_{l=1}^{n} \widetilde{y}^l \partial_{\nu_l}\Big] \\ &\quad - \frac{1}{12}\Big[\sum_{l=1}^{n} \widetilde{z}^l \partial_{\nu_l}, \Big[\sum_{l=1}^{n} \widetilde{z}^l \partial_{\nu_l}, \sum_{l=1}^{n} \widetilde{y}^l \partial_{\nu_l}\Big]\Big] + \frac{1}{12}\Big[\sum_{l=1}^{n} \widetilde{y}^l \partial_{\nu_l} \Big[\sum_{l=1}^{n} \widetilde{y}^l \partial_{\nu_l}, \sum_{l=1}^{n} \widetilde{z}^l \partial_{\nu_l}\Big]\Big] + \ldots\end{aligned}$$

Using (2.1) we obtain

$$\partial_{\nu_0(v)} = \sum_{l=1}^{n} (\widetilde{z}^l - \widetilde{y}^l) \partial_{\nu_l(v)} + \frac{1}{2} \sum_{k,m}^{n} \sum_{l=1}^{n} \widetilde{y}^k \widetilde{z}^m C_{k,m}^l(v) \partial_{\nu_l(v)} + \sum_{|\alpha|+|\beta| \geq 3} \sum_{l=1}^{n} \widetilde{y}^\alpha \widetilde{z}^\beta \mathcal{P}_{\alpha,\beta}^l(v) \partial_{\nu_l(v)},$$

where $\mathcal{P}_{\alpha,\beta}^l(v)$ is a linear combination of terms of the form (2.16) with l and v instead of j and x respectively. Now it follows from (2.15), (2.18) and (2.20) that (2.24) equals

$$\widetilde{z}^j - \widetilde{y}^j + \frac{1}{2} \sum_{k,m}^{n} C_{k,m}^j(w) \widetilde{y}^k \widetilde{z}^m + \sum_{|\alpha|+|\beta| \geq 3} \mathcal{Q}_{\alpha,\beta}^j(w) \widetilde{y}^\alpha \widetilde{z}^\beta,$$

where $\mathcal{Q}_{\alpha,\beta}^j(w)$ is a linear combination of terms of the form (2.16) with $x = w$. It is clear that (2.23) equals $\mathcal{Q}_{\alpha',\beta}^j(w)$ if $|\alpha'| + |\beta| \geq 3$ and

$$\frac{\partial}{\partial \widetilde{y}^k} \frac{\partial}{\partial \widetilde{z}^m} c_j(w, H(\widetilde{y}, \widetilde{z})(w))|_{\widetilde{y}=\widetilde{z}=0} = \frac{1}{2} C_{k,m}^j(w).$$

The cases when $\alpha' = 0$ or $\beta = 0$ are covered by (2.17)–(2.20). This enables us to find (2.22) and completes the proof (see also (2.15)). ∎

2.4. Lemma. *We have*

$$\sum_{k=1}^{n} c_k(y, z) \partial_{\nu_k(z)} c_j(y, z) = c_j(y, z), \qquad \sum_{k=1}^{n} c_k(y, z) \partial_{\nu_k(y)} c_j(y, z) = -c_j(y, z).$$

Proof. Suppose $z = \exp_y(\widetilde{z})$. Then using (2.8)–(2.10) we obtain

$$\sum_{k=1}^{n} c_k(y,z)\partial_{\nu_k(z)} c_j(y,z) = \frac{d}{dt} c_j(y, \gamma_{z,y}(t))\Big|_{t=1}$$
$$= \frac{d}{dt} c_j(y, \exp_y(t\widetilde{z}))\Big|_{t=1} = \frac{d}{dt}(t\widetilde{z}^j)\Big|_{t=1} = \widetilde{z}^j = c_j(y,z).$$

The second equality of the lemma follows from the first one and (2.12). ∎

Connections on vector bundles. Let $\mathcal{E}$ be a C^∞-smooth vector bundle over M. We denote by $C^\infty(\mathcal{E})$ the space of C^∞-sections of $\mathcal{E}$. In particular $C^\infty(TM)$ is the space of C^∞-smooth vector fields on M.

A *connection* on $\mathcal{E}$ is a continuous mapping

$$\nabla : C^\infty(TM) \times C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E}), \quad C^\infty(TM) \times C^\infty(\mathcal{E}) \ni (\nu_0, \omega) \mapsto \nabla_{\nu_0}\omega \in C^\infty(\mathcal{E}),$$

satisfying the following linearity conditions:

$$\nabla_{\nu_0+\nu_1}\omega = \nabla_{\nu_0}\omega + \nabla_{\nu_1}\omega, \quad \nabla_{\nu_0}(\omega + \omega_1) = \nabla_{\nu_0}\omega + \nabla_{\nu_0}\omega_1, \quad \nabla_{\varphi\nu_0}\omega = \varphi\nabla_{\nu_0}\omega,$$

and the Leibniz rule

$$\nabla_{\nu_0}(\varphi\omega) = \varphi\nabla_{\nu_0}\omega + (\partial_{\nu_0}\varphi)\omega,$$

for any $\nu_0, \nu_1 \in C^\infty(TM)$, $\omega, \omega_1 \in C^\infty(\mathcal{E})$ and $\varphi \in C^\infty(M)$.

The mapping $\nabla_{\nu_0} : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$ is a first order linear differential operator: it can be expressed locally as a sum of ∂_{ν_0} and a zero order term with a C^∞-smooth matrix-valued coefficient. This operator is called the *covariant derivative in the direction of* ν_0. If ∇' is another connection on $\mathcal{E}$, then $\nabla_{\nu_0} - \nabla'_{\nu_0}$ is a zero order operator, i.e. can be viewed as an automorphism of $\mathcal{E}$.

Every C^∞-smooth vector bundle over a paracompact manifold M does have a connection (see, e.g., [MS, Appendix C, Lemma 2]).

Let Z be a vector space and $\mathcal{H}om(Z,\mathcal{E})$ and $\mathcal{H}om(\mathcal{E},Z)$ be the C^∞-smooth vector bundles with the fibres

$$\mathcal{H}om(Z,\mathcal{E})_x = \mathrm{Hom}(Z,\mathcal{E}_x), \quad \mathcal{H}om(\mathcal{E},Z)_x = \mathrm{Hom}(\mathcal{E}_x,Z)$$

constructed in the standard way (see [MS, §3]). Here $\mathrm{Hom}(X,Y)$ denotes the vector space of all linear mappings from the vector space X to the vector space Y. For any $F \in C^\infty(\mathcal{H}om(Z,\mathcal{E}))$ and $\zeta \in Z$ we have $F\zeta \in C^\infty(\mathcal{E})$. So, we can define $\nabla_{\nu_0}F \in C^\infty(\mathcal{H}om(Z,\mathcal{E}))$ by the equality

$$(\nabla_{\nu_0}F)\zeta = \nabla_{\nu_0}(F\zeta). \tag{2.25}$$

Further, any $\Phi \in C^\infty(\mathcal{H}om(\mathcal{E},Z))$ and $\omega \in C^\infty(\mathcal{E})$ give rise to a C^∞-smooth vector-valued function $\Phi\omega : M \to Z$. Hence we can define $\nabla_{\nu_0}\Phi \in C^\infty(\mathcal{H}om(\mathcal{E},Z))$ by the equality

$$(\nabla_{\nu_0}\Phi)\omega = \partial_{\nu_0}(\Phi\omega) - \Phi\nabla_{\nu_0}\omega. \tag{2.26}$$

Throughout the paper ∇_ν^α *will denote the symmetrization of* $\nabla_{\nu_1}^{\alpha_1} \dots \nabla_{\nu_n}^{\alpha_n}$, *where* $\nu_1, \dots, \nu_n$ *are the given vector fields.*

Let $x, y \in M$ and $\gamma : [0,1] \to M$ be a C^∞-smooth curve such that $\gamma(0) = x$, $\gamma(1) = y$. Let $\mathcal{E}_0$ be the restriction of $\mathcal{E}$ to $\gamma([0,1]) \subset M$ and $\omega_0 \in C^\infty(\mathcal{E}_0)$. Suppose $\omega \in C^\infty(\mathcal{E})$ is an extension of ω_0, i.e. $\omega|_{\gamma([0,1])} = \omega_0$, and $\nu_0 \in C^\infty(TM)$ is such that

$$\nu_0(\gamma(t)) = \dot\gamma(t), \qquad \forall t \in [0,1].$$

Then it is not difficult to prove that $\nabla_{\nu_0}\omega|_{\gamma([0,1])}$ does not depend on the choice of ω and ν_0. Thus for any $\omega_0 \in C^\infty(\mathcal{E}_0)$,

$$\nabla_{\dot\gamma}\omega_0 \in C^\infty(\mathcal{E}_0)$$

is a well defined object. This construction generates a connection on $\mathcal{E}_0$.

We will say that a vector $\omega(y) \in \mathcal{E}_y$ is a *parallel displacement of a vector* $\omega(x) \in \mathcal{E}_x$ *along the curve* γ if there exists $\omega_0 \in C^\infty(\mathcal{E}_0)$ such that $\omega_0(x) = \omega(x)$, $\omega_0(y) = \omega(y)$ and

$$\nabla_{\dot\gamma}\omega_0 = 0.$$

Using standard results on linear ordinary differential equations one can prove that for any vector $\omega(x) \in \mathcal{E}_x$ there exists its unique parallel displacement $\omega(y) \in \mathcal{E}_y$ along the curve γ and that the mapping $\mathcal{E}_x \ni \omega(x) \mapsto \omega(y) \in \mathcal{E}_y$ is linear.

Let $x, y \in M$ be sufficiently close to each other and let $\Phi_{y,x} : \mathcal{E}_x \to \mathcal{E}_y$ be the parallel displacement along the geodesic $\gamma_{y,x}$. It is not difficult to see that $\Phi_{y,y}$ and $\Phi_{y,x}\Phi_{x,y}$ equal the identity automorphism of $\mathcal{E}_y$:

$$\Phi_{y,y} = I_{\mathcal{E}_y}, \qquad \Phi_{y,x}\Phi_{x,y} = I_{\mathcal{E}_y}. \tag{2.27}$$

2.5. Lemma. *For any multi-index $\gamma \in \mathbb{Z}_+^n \setminus \{0\}$ and any $p, q \in \mathbb{Z}_+$ such that $p + q = |\gamma|$ we have*

$$\sum_{\substack{\alpha+\beta=\gamma \\ |\alpha|=p,\ |\beta|=q}} \frac{1}{\alpha!\beta!} \nabla^\alpha_{\nu(y)} \nabla^\beta_{\nu(x)} \Phi_{y,x}|_{y=x} = 0.$$

In particular

$$\nabla^\gamma_{\nu(y)}\Phi_{y,x}|_{y=x} = 0, \qquad \nabla^\gamma_{\nu(x)}\Phi_{y,x}|_{y=x} = 0.$$

Proof. Let us fix arbitrary points $x, y \in M$ which are sufficiently close to each other. It follows from the definition of the parallel displacement that

$$\nabla_{\dot\gamma_{y,x}(s)}\Phi_{\gamma_{y,x}(s),\gamma_{y,x}(t)}\omega_{(t)} = 0, \qquad \forall \omega_{(t)} \in \mathcal{E}_{\gamma_{y,x}(t)}, \qquad \forall s, t \in [0,1].$$

According to (2.25)–(2.27) and the last equality,

$$\begin{aligned}
&(\nabla_{\dot\gamma_{y,x}(t)}\Phi_{\gamma_{y,x}(s),\gamma_{y,x}(t)})\Phi_{\gamma_{y,x}(t),\gamma_{y,x}(s)}\omega_{(s)} \\
&\qquad = \partial_{\dot\gamma_{y,x}(t)}\omega_{(s)} - \Phi_{\gamma_{y,x}(s),\gamma_{y,x}(t)}\nabla_{\dot\gamma_{y,x}(t)}\Phi_{\gamma_{y,x}(t),\gamma_{y,x}(s)}\omega_{(s)} \\
&\qquad = -\Phi_{\gamma_{y,x}(s),\gamma_{y,x}(t)}(\nabla_{\dot\gamma_{y,x}(t)}\Phi_{\gamma_{y,x}(t),\gamma_{y,x}(s)}\omega_{(s)}) = 0, \qquad \forall \omega_{(s)} \in \mathcal{E}_{\gamma_{y,x}(s)}.
\end{aligned}$$

Therefore

$$\nabla_{\dot\gamma_{y,x}(s)}\Phi_{\gamma_{y,x}(s),\gamma_{y,x}(t)} = 0, \quad \nabla_{\dot\gamma_{y,x}(t)}\Phi_{\gamma_{y,x}(s),\gamma_{y,x}(t)} = 0, \qquad \forall s, t \in [0,1].$$

Hence,

$$\nabla^p_{\dot\gamma_{y,x}(s)}\nabla^q_{\dot\gamma_{y,x}(t)}\Phi_{\gamma_{y,x}(s),\gamma_{y,x}(t)} = 0, \qquad \forall p, q \in \mathbb{Z}_+,\ p + q > 0. \tag{2.28}$$

Since $\dot\gamma_{y,x} = \sum_{j=1}^n c_j(x,y)\nu_j$, we have

$$\sum_{|\alpha|=p,\,|\beta|=q} \frac{p!q!}{\alpha!\beta!} c^{\alpha+\beta}(x,y)\nabla^\alpha_{\nu(\gamma_{y,x}(s))}\nabla^\beta_{\nu(\gamma_{y,x}(t))}\Phi_{\gamma_{y,x}(s),\gamma_{y,x}(t)} = 0, \qquad \forall s,t\in[0,1],$$

where $c^\mu(x,y) = c_1^{\mu_1}(x,y)\dots c_n^{\mu_n}(x,y)$. Taking $s=1$, $t=0$ we obtain

$$\sum_{|\alpha|=p,\,|\beta|=q} \frac{1}{\alpha!\beta!} c^{\alpha+\beta}(x,y)\nabla^\alpha_{\nu(y)}\nabla^\beta_{\nu(x)}\Phi_{y,x} = 0, \quad \forall p,q\in\mathbb{Z}_+,\ p+q>0. \tag{2.29}$$

Let $b=(b^1,\dots,b^n)\in\mathbb{R}^n$ be an arbitrary vector. Then the last equality implies, for sufficiently small $t>0$,

$$\sum_{|\alpha|=p,\,|\beta|=q} \frac{1}{\alpha!\beta!} t^{p+q} b^{\alpha+\beta}\nabla^\alpha_{\nu(y)}\nabla^\beta_{\nu(x)}\Phi_{y,x}|_{y=\exp_x(tb)} = 0$$

(see (2.9)). Dividing by t^{p+q} and taking in the resulting equality $t=0$ we obtain

$$\sum_{|\alpha|=p,\,|\beta|=q} \frac{1}{\alpha!\beta!} b^{\alpha+\beta}\nabla^\alpha_{\nu(y)}\nabla^\beta_{\nu(x)}\Phi_{y,x}|_{y=x} = 0, \quad \forall b\in\mathbb{R}^n. \ \blacksquare$$

2.6. Lemma. *For any multi-index $\alpha\in\mathbb{Z}^n_+$ and any $\omega\in C^\infty(\mathcal{E})$ we have*

$$\partial^\alpha_{\nu(y)}\Phi_{x,y}\omega(y)|_{y=x} = \nabla^\alpha_\nu\omega(x).$$

Proof. Suppose $|\alpha|=r$. Let $b=(b^1,\dots,b^n)\in\mathbb{R}^n$ be an arbitrary vector and $\nu_0 = \sum_{k=1}^n b^k\nu_k$. It follows from (2.26) that

$$r!\sum_{|\beta|=r}\frac{1}{\beta!}b^\beta\partial^\beta_{\nu(y)}\Phi_{x,y}\omega(y) = \partial^r_{\nu_0(y)}\Phi_{x,y}\omega(y) = \sum_{p+q=r}\frac{r!}{p!q!}(\nabla^p_{\nu_0(y)}\Phi_{x,y})\nabla^q_{\nu_0(y)}\omega(y)$$

$$= r!\sum_{|\beta'|+|\beta''|=r}\frac{1}{\beta'!\beta''!}b^{\beta'+\beta''}(\nabla^{\beta'}_{\nu(y)}\Phi_{x,y})\nabla^{\beta''}_{\nu(y)}\omega(y).$$

Comparing the coefficients we obtain

$$\partial^\alpha_{\nu(y)}\Phi_{x,y}\omega(y) = \sum_{\alpha'+\alpha''=\alpha}\frac{\alpha!}{\alpha'!\alpha''!}(\nabla^{\alpha'}_{\nu(y)}\Phi_{x,y})\nabla^{\alpha''}_{\nu(y)}\omega(y). \tag{2.30}$$

Applying Lemma 2.5 we prove that this sum equals $\nabla^\alpha_\nu\omega(x)$ when $y=x$. $\blacksquare$

2.7. Lemma. *For any multi-index $\alpha\in\mathbb{Z}^n_+$ and any smooth function φ,*

$$\nabla^\alpha_{\nu(y)}\varphi(y)\Phi_{y,x}|_{y=x} = \partial^\alpha_\nu\varphi(x).$$

Proof. We have

$$\nabla^\alpha_{\nu(y)}\varphi(y)\Phi_{y,x} = \sum_{\alpha'+\alpha''=\alpha}\frac{\alpha!}{\alpha'!\alpha''!}(\partial^{\alpha'}_{\nu(y)}\varphi(y))\nabla^{\alpha''}_{\nu(y)}\Phi_{y,x}.$$

It follows from Lemma 2.5 that this sum equals $\partial^\alpha_\nu\varphi(x)$ when $y=x$. $\blacksquare$

Let $\mathcal{E}$ and $\mathcal{F}$ be C^∞-smooth vector bundles over M with connections $\nabla^{\mathcal{E}}$ and $\nabla^{\mathcal{F}}$ respectively and let $\sigma:\mathcal{E}\to\mathcal{F}$ be a C^∞-smooth vector bundle morphism. σ can be viewed as an element of $C^\infty(\mathcal{H}om(\mathcal{E},\mathcal{F}))$, where $\mathcal{H}om(\mathcal{E},\mathcal{F})$ is the C^∞-smooth vector bundle with the fibres

$$\mathcal{H}om(\mathcal{E},\mathcal{F})_x = \mathrm{Hom}(\mathcal{E}_x,\mathcal{F}_x)$$

constructed in the standard way (see, e.g., [MS, §3]). Any $\omega \in C^\infty(\mathcal{E})$ gives rise to a section $\sigma\omega \in C^\infty(\mathcal{F})$. So, we can define $\nabla_{\nu_0}^{\mathcal{F},\mathcal{E}}\sigma \in C^\infty(\mathcal{H}om(\mathcal{E},\mathcal{F}))$ by the equality

$$(\nabla_{\nu_0}^{\mathcal{F},\mathcal{E}}\sigma)\omega = \nabla_{\nu_0}^{\mathcal{F}}(\sigma\omega) - \sigma\nabla_{\nu_0}^{\mathcal{E}}\omega. \tag{2.31}$$

As usual, $(\nabla_\nu^{\mathcal{F},\mathcal{E}})^\alpha$ *will denote the symmetrization of* $(\nabla_{\nu_1}^{\mathcal{F},\mathcal{E}})^{\alpha_1}\dots(\nabla_{\nu_n}^{\mathcal{F},\mathcal{E}})^{\alpha_n}$*, where* $\nu_1,\dots,\nu_n$ *are the given vector fields.*

3. Classes of symbols

Let $\mathbf{d} = (d_1,\dots,d_n) \in \mathbb{R}^n$ be a vector such that $d_k > 0$, $k = 1,\dots,n$, and

$$\sum_{k=1}^n \frac{1}{d_k} = n. \tag{3.1}$$

For any multi-index $\alpha \in \mathbb{Z}_+^n$ we put

$$|\alpha : \mathbf{d}| = \sum_{k=1}^n \frac{\alpha_k}{d_k}. \tag{3.2}$$

For any vector $\eta \in \mathbb{R}^n \setminus \{0\}$ we denote by $|\eta|_{\mathbf{d}}$ the unique solution $\tau = \tau(\eta) = |\eta|_{\mathbf{d}}$ of the equation

$$\sum_{k=1}^n \tau^{-2/d_k}\eta_k^2 = 1.$$

It follows from the implicit function theorem that $\tau(\cdot) \in C^\infty(\mathbb{R}^n \setminus \{0\})$. Further, we put $|0|_{\mathbf{d}} = 0$. It is not difficult to prove that there exists $C_{\mathbf{d}} \geq 1$ such that

$$|\eta + \eta'|_{\mathbf{d}} \leq C_{\mathbf{d}}(|\eta|_{\mathbf{d}} + |\eta'|_{\mathbf{d}}), \qquad \forall \eta, \eta' \in \mathbb{R}^n. \tag{3.3}$$

One can take, e.g., $C_{\mathbf{d}} = 2^{\max\{d_k\}}$. It is clear that

$$\max\{|\eta_k|^{d_k}\} \leq |\eta|_{\mathbf{d}} \leq n^{\max\{d_k\}/2}\max\{|\eta_k|^{d_k}\}, \tag{3.4}$$

$$\begin{aligned} |\eta|^{\min\{d_k\}} \leq |\eta|_{\mathbf{d}} \leq |\eta|^{\max\{d_k\}} \quad &\text{if } |\eta| \geq 1, \\ |\eta|^{\max\{d_k\}} \leq |\eta|_{\mathbf{d}} \leq |\eta|^{\min\{d_k\}} \quad &\text{if } |\eta| \leq 1. \end{aligned} \tag{3.5}$$

Let $r \in \mathbb{R}$, $0 \leq \delta, \varrho \leq 1$. We denote by $S_{\varrho,\delta}^{r,\mathbf{d}}(M \times \mathbb{R}^n)$ the class of all functions $\widetilde{a} \in C^\infty(M \times \mathbb{R}^n)$ such that for any compact set $K \subset M$,

$$\begin{aligned} &|\partial_\eta^\alpha \partial_{\nu_{j_1}(x)} \dots \partial_{\nu_{j_q}(x)} \widetilde{a}(x,\eta)| \leq \mathrm{const}_{K,\alpha,j_1,\dots,j_q}(1 + |\eta|_{\mathbf{d}})^{r - \varrho|\alpha:\mathbf{d}| + \delta|\beta:\mathbf{d}|}, \\ &\qquad \forall \alpha \in \mathbb{Z}_+^n,\ \forall j_1,\dots,j_q \in \{1,\dots,n\},\ \forall q \in \mathbb{Z}_+,\ \forall \eta \in \mathbb{R}^n,\ \forall x \in K, \end{aligned} \tag{3.6}$$

where β is the multi-index corresponding to the set of indices $\{j_1,\dots,j_q\}$.

Let $\mathcal{E}$ and $\mathcal{F}$ be C^∞-smooth complex vector bundles over M and let $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{F}}$ be the corresponding induced vector bundles over $M \times \mathbb{R}^n$ defined by the projection $M \times \mathbb{R}^n \to M$ (see, e.g., [MS, §3]). We say that $\widetilde{a} \in C^\infty(\mathcal{H}om(\widetilde{\mathcal{E}},\widetilde{\mathcal{F}}))$ belongs to the class $\widetilde{S}_{\varrho,\delta}^{r,\mathbf{d}}(\mathcal{E},\mathcal{F})$ if it can be expressed locally as a matrix-valued function with components satisfying (3.6).

In order to give an invariant definition of $\widetilde{S}_{\varrho,\delta}^{r,\mathbf{d}}(\mathcal{E},\mathcal{F})$ we need connections $\nabla^{\mathcal{E}}$ and $\nabla^{\mathcal{F}}$ on the bundles $\mathcal{E}$ and $\mathcal{F}$ respectively. The class $\widetilde{S}_{\varrho,\delta}^{r,\mathbf{d}}(\mathcal{E},\mathcal{F})$ consists of all morphisms

$\widetilde{a} \in C^\infty(\mathcal{H}om(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}))$ such that for any compact set $K \subset M$,

$$\|\partial_\eta^\alpha \nabla^{\mathcal{F},\mathcal{E}}_{\nu_{j_1}(x)} \dots \nabla^{\mathcal{F},\mathcal{E}}_{\nu_{j_q}(x)} \widetilde{a}(x, \eta)\| \le \mathrm{const}_{K,\alpha,j_1,\dots,j_q} (1 + |\eta|_{\mathbf{d}})^{r - \varrho|\alpha:\mathbf{d}| + \delta|\beta:\mathbf{d}|}, \quad \forall \alpha \in \mathbb{Z}_+^n,\ \forall j_1, \dots, j_q \in \{1, \dots, n\},\ \forall q \in \mathbb{Z}_+,\ \forall \eta \in \mathbb{R}^n,\ \forall x \in K, \tag{3.7}$$

where β is the multi-index corresponding to the set of indices $\{j_1, \dots, j_q\}$ (see (2.31)). Here $\|\cdot\| : \mathcal{H}om(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}) \to \mathbb{R}$ is a continuous function such that its restriction to each fibre

$$\mathcal{H}om(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})_{(x,\eta)} = \mathrm{Hom}(\widetilde{\mathcal{E}}_{(x,\eta)}, \widetilde{\mathcal{F}}_{(x,\eta)})$$

is a norm on $\mathrm{Hom}(\mathcal{E}_x, \mathcal{F}_x)$. This definition of $\widetilde{S}^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E}, \mathcal{F})$ does not depend on the choice of connections $\nabla^{\mathcal{E}}$ and $\nabla^{\mathcal{F}}$ due to the fact that if ∇ and ∇' are connections on the same vector bundle, then $\nabla_{\nu_0} - \nabla'_{\nu_0}$ is a zero order operator for any vector field ν_0 on M.

Finally, let $\mathcal{E}_*$ and $\mathcal{F}_*$ be the vector bundles over the cotangent bundle T^*M induced by $\mathcal{E}$ and $\mathcal{F}$ with the help of the projection $T^*M \to M$. We say that a vector bundle morphism $a \in C^\infty(\mathcal{H}om(\mathcal{E}_*, \mathcal{F}_*))$ belongs to the class $S^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E}, \mathcal{F})$ if there exists $\widetilde{a} \in \widetilde{S}^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E}, \mathcal{F})$ such that

$$a(x, \xi) = \widetilde{a}(x, \sigma(x, \xi)), \tag{3.8}$$

where

$$\sigma(x, \xi) = (\sigma_1(x, \xi), \dots, \sigma_n(x, \xi)), \quad \sigma_k(x, \xi) = \langle \nu_k(x), \xi \rangle, \quad \forall \xi \in T^*_x M. \tag{3.9}$$

Elements of $S^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E}, \mathcal{F})$ will play the role of symbols of pseudodifferential operators. Pseudodifferential operators can also be defined with the help of amplitudes. So, we will need the corresponding classes of amplitudes. Let $\mathcal{E}^M_*$ and $\mathcal{F}^M_*$ be the vector bundles over $M \times T^*M$ induced by $\mathcal{E}_*$ and $\mathcal{F}_*$ with the help of the projection $M \times T^*M \to T^*M$. We will say that a vector bundle morphism $a \in C^\infty(\mathcal{H}om(\mathcal{E}^M_*, \mathcal{F}^M_*))$ belongs to the class $S^{r,\mathbf{d}}_{\varrho,\delta}(M; \mathcal{E}, \mathcal{F})$ if there exists $\widetilde{a} \in \widetilde{S}^{r,\mathbf{d}}_{\varrho,\delta}(M; \mathcal{E}, \mathcal{F})$ such that

$$a(y; x, \xi) = \widetilde{a}(y; x, \sigma(x, \xi)), \quad \forall x, y \in M,\ \forall \xi \in T^*_x M. \tag{3.10}$$

Here $\widetilde{S}^{r,\mathbf{d}}_{\varrho,\delta}(M; \mathcal{E}, \mathcal{F})$ denotes the class of all morphisms $\widetilde{a} \in C^\infty(\mathcal{H}om(\widetilde{\mathcal{E}}^M, \widetilde{\mathcal{F}}^M))$ such that for any compact set $K \subset M \times M$,

$$\begin{aligned} &\|\partial_\eta^\alpha \partial_{\nu_{k_1}(y)} \dots \partial_{\nu_{k_p}(y)} \nabla^{\mathcal{F},\mathcal{E}}_{\nu_{j_1}(x)} \dots \nabla^{\mathcal{F},\mathcal{E}}_{\nu_{j_q}(x)} \widetilde{a}(y; x, \eta)\| \\ &\qquad \le \mathrm{const}_{K,\alpha,k_1,\dots,k_p,j_1,\dots,j_q} (1 + |\eta|_{\mathbf{d}})^{r - \varrho|\alpha:\mathbf{d}| + \delta|(\beta+\mu):\mathbf{d}|}, \\ &\forall \alpha \in \mathbb{Z}_+^n,\ \forall k_1, \dots, k_p, j_1, \dots, j_q \in \{1, \dots, n\},\ \forall p, q \in \mathbb{Z}_+,\ \forall \eta \in \mathbb{R}^n,\ \forall (y, x) \in K, \end{aligned} \tag{3.11}$$

where β, μ are the multi-indices corresponding to the sets of indices $\{j_1, \dots, j_q\}$, $\{k_1, \dots, k_p\}$ and $\widetilde{\mathcal{E}}^M$, $\widetilde{\mathcal{F}}^M$ are the vector bundles over $M \times (M \times \mathbb{R}^n)$ induced by $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{F}}$ with the help of the projection $M \times (M \times \mathbb{R}^n) \to M \times \mathbb{R}^n$.

For all ϱ and δ the intersection

$$S^{-\infty} = \bigcap_{r \in \mathbb{R}} S^{r,\mathbf{d}}_{\varrho,\delta} = \bigcap_{r \in \mathbb{R}} S^{r,\mathbf{d}}_{1,0} \tag{3.12}$$

consists of morphisms which vanish with all their derivatives faster than any power of $|\xi|$ as $|\xi| \to \infty$. Note that the mapping $\xi \mapsto \eta = \sigma(x, \xi)$ is invertible since the vectors

$\nu_1(x), \dots, \nu_n(x)$ span the tangent space T_xM (see also (3.5)). We will also use the notation

$$\widetilde{S}^{-\infty} = \bigcap_{r\in\mathbb{R}} \widetilde{S}^{r,\mathbf{d}}_{\varrho,\delta} = \bigcap_{r\in\mathbb{R}} \widetilde{S}^{r,\mathbf{d}}_{1,0} \tag{3.13}$$

and the standard notation for asymptotic expansions of symbols and amplitudes. Namely, let $\widetilde{a}_j \in \widetilde{S}^{r_j,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})$, $j \in \mathbb{N}$, $r_j \to -\infty$ as $j \to \infty$, and let $\widetilde{a} \in C^\infty(\mathcal{H}om(\widetilde{\mathcal{E}},\widetilde{\mathcal{F}}))$. We will write

$$\widetilde{a}(x,\eta) \sim \sum_{j=1}^{\infty} \widetilde{a}_j(x,\eta) \tag{3.14}$$

if

$$\widetilde{a}(x,\eta) - \sum_{j=1}^{l} \widetilde{a}_j(x,\eta) \in \widetilde{S}^{\overline{r}_{l+1},\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F}), \qquad \forall l \in \mathbb{N},$$

where $\overline{r}_{l+1} = \max\{r_j : j \geq l+1\}$. The asymptotic expansion

$$\widetilde{a}(y;x,\eta) \sim \sum_{j=1}^{\infty} \widetilde{a}_j(y;x,\eta)$$

will be understood analogously.

Exactly as in the standard calculus of pseudodifferential operators (see, e.g., [Shu, 3.3] or [Ta1, Ch. II, §3]) one can prove that for any sequence $\widetilde{a}_j \in \widetilde{S}^{r_j,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})$, $j \in \mathbb{N}$, such that $r_j \to -\infty$ as $j \to \infty$, there exists a unique modulo $\widetilde{S}^{-\infty}$ symbol $\widetilde{a} \in \widetilde{S}^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})$, $r = \max\{r_j\}$, satisfying (3.14).

4. Anisotropic pseudodifferential operators on a manifold

We will need measures on M and the cotangent spaces T^*_yM, $y \in M$, in order to give a coordinate free definition of pseudodifferential operators. Let us consider the determinant

$$\det(\nu_1(y), \dots, \nu_n(y)) = \det(\nu^j_k(y))^n_{j,k=1}. \tag{4.1}$$

It depends on the choice of a coordinate system, but it is easy to see that

$$d\mathcal{M}(y) := |\det(\nu^j_k(y))|^{-1} dy, \qquad d\mathcal{N}_y(\zeta) := |\det(\nu^j_k(y))| d\zeta, \quad \zeta \in T^*_yM, \tag{4.2}$$

are invariant, i.e. independent of the choice of a coordinate system. The determinant (4.1) always differs from 0 because the vectors $\nu_1(y), \dots, \nu_n(y)$ span the tangent space T_yM. Note that since

$$(\det(\nu^j_k(y)))^{-1} = \det(\widetilde{\nu}^k_j(y)),$$

where $\widetilde{\nu}^k(y) = \sum_{j=1}^n \widetilde{\nu}^k_j(y) dy^j$ (see (2.3)), we have the following equality for differential forms:

$$(\det(\nu^j_k(y)))^{-1} dy^1 \wedge \dots \wedge dy^n = \widetilde{\nu}^1(y) \wedge \dots \wedge \widetilde{\nu}^n(y).$$

Let us evaluate the determinant (4.1) in the canonical coordinate system with the origin at $x \in M$. Since

$$\partial_{\nu_k(y)} \widetilde{y}^j = \sum_{m=1}^n \nu^m_k(y) \partial_{\widetilde{y}^m} \widetilde{y}^j = \sum_{m=1}^n \nu^m_k(y) \delta^j_m = \nu^j_k(y),$$

(2.9) implies that our determinant equals

$$\Upsilon(x,y) := \det(\partial_{\nu_k(y)} c_j(x,y)). \tag{4.3}$$

It is not difficult to prove that for any multi-index $\alpha \in \mathbb{Z}_+^n$,

$$\partial_{\nu(y)}^{\alpha} \Upsilon(x,y)|_{y=x} \tag{4.4}$$

is a universal polynomial in $\partial_\nu^\beta C_{j,k}^m(x)$, $|\beta| \le |\alpha| - 1$, whose coefficients do not depend on M or ν_k, $k = 1, \dots, n$. Indeed, $\partial_{\nu(y)}^{\alpha} \Upsilon(x,y)$ is a linear combination of determinants whose components have the form

$$\partial_{\nu_{k_r}(y)} \dots \partial_{\nu_{k_1}(y)} c_j(x,y).$$

Using (2.1) we can express $\partial_{\nu_{k_r}} \dots \partial_{\nu_{k_1}}$ in terms of symmetrized derivatives ∂_ν^μ and operators of multiplication by ∂_ν-derivatives of the functions $C_{j,k}^m$. According to (2.15) and (2.20) all symmetrized derivatives of order greater than 1 vanish at $y = x$, while (2.18) implies that

$$\partial_{\nu_k(y)} c_j(x,y)|_{y=x} = \delta_k^j.$$

Consequently, (4.4) is a linear combination of determinants whose components are polynomials in $\partial_\nu^\beta C_{j,k}^m(x)$.

It follows from (2.15), (2.18) that

$$\Upsilon(x,x) = 1. \tag{4.5}$$

Since $\Upsilon(x,y) \ne 0$ if y is sufficiently close to x, the last equality implies that $\Upsilon(x,y) > 0$. Let us give an explicit formula for (4.4) when $|\alpha| = 1$. The columns of the determinant $\Upsilon(x,y)$ are equal to $\partial_{\nu_k(y)} c(x,y)$, $k = 1, \dots, n$, where

$$c(x,y) = (c_1(x,y), \dots, c_n(x,y)). \tag{4.6}$$

Using (2.1), (2.15) and (2.20) we obtain

$$\begin{aligned}
\partial_{\nu_j(y)} \partial_{\nu_k(y)} c(x,y)|_{y=x} &= \frac{1}{2}(\partial_{\nu_j(y)}\partial_{\nu_k(y)} + \partial_{\nu_k(y)}\partial_{\nu_j(y)}) c(x,y)|_{y=x} \\
&\quad + \frac{1}{2} \sum_{m=1}^n C_{j,k}^m(y) \partial_{\nu_m(y)} c(x,y)|_{y=x} \\
&= \frac{1}{2} \sum_{m=1}^n C_{j,k}^m(y) \partial_{\nu_m(y)} c(x,y)|_{y=x}.
\end{aligned}$$

Hence

$$\begin{aligned}
\partial_{\nu_j(y)} \Upsilon(x,y)|_{y=x} &= \partial_{\nu_j(y)} \det(\partial_{\nu_1(y)} c(x,y), \dots, \partial_{\nu_n(y)} c(x,y))|_{y=x} \\
&= \sum_{k=1}^n \det(\partial_{\nu_1(y)} c(x,y), \dots, \partial_{\nu_{k-1}(y)} c(x,y), \partial_{\nu_j(y)}\partial_{\nu_k(y)} c(x,y), \\
&\qquad \partial_{\nu_{k+1}(y)} c(x,y), \dots, \partial_{\nu_n(y)} c(x,y))|_{y=x}
\end{aligned} \tag{4.7}$$

$$= \sum_{k=1}^{n} \det \Big(\partial_{\nu_1(y)} c(x,y), \ldots, \partial_{\nu_{k-1}(y)} c(x,y), \frac{1}{2} \sum_{m=1}^{n} C_{j,k}^{m}(y) \partial_{\nu_m(y)} c(x,y), \partial_{\nu_{k+1}(y)} c(x,y), \ldots, \partial_{\nu_n(y)} c(x,y) \Big) \Big|_{y=x}$$

$$= \frac{1}{2} \sum_{k=1}^{n} C_{j,k}^{k}(x) \Upsilon(x,x) = \frac{1}{2} \sum_{k=1}^{n} C_{j,k}^{k}(x).$$

It was noticed in [Saf] that $\Upsilon(x,y) \equiv 1$ if

$$C_{j,k}^{m} \equiv 0, \qquad \forall m \geq k. \tag{4.8}$$

Indeed, using (2.1) and Lemma 2.4 we obtain

$$\sum_{l=1}^{n} c_l(x,z) \partial_{\nu_l(z)} \partial_{\nu_k(z)} c_j(x,z)$$

$$= \sum_{l=1}^{n} \partial_{\nu_k(z)} c_l(x,z) \partial_{\nu_l(z)} c_j(x,z) - \sum_{l=1}^{n} (\partial_{\nu_k(z)} c_l(x,z)) \partial_{\nu_l(z)} c_j(x,z)$$
$$+ \sum_{l,m=1}^{n} c_l(x,z) C_{l,k}^{m}(z) \partial_{\nu_m(z)} c_j(x,z)$$

$$= \partial_{\nu_k(z)} c_j(x,z) - \sum_{l=1}^{n} (\partial_{\nu_k(z)} c_l(x,z)) \partial_{\nu_l(z)} c_j(x,z) + \sum_{l,m=1}^{n} c_l(x,z) C_{l,k}^{m}(z) \partial_{\nu_m(z)} c_j(x,z).$$

Now it follows from the equality

$$c_j(x, \gamma_{y,x}(t)) = c_j(x, \exp_x(t\widetilde{y})) = t\widetilde{y} = t c_j(x,y), \qquad y = \exp_x(\widetilde{y}), \tag{4.9}$$

(see (2.8)–(2.10)) that the matrix function

$$F(t) := (\partial_{\nu_k(z)} c_j(x,z))|_{z=\gamma_{y,x}(t)}$$

satisfies the equation

$$tF'(t) = F(t) - F^2(t) + t\Omega(t) F(t),$$

where

$$\Omega(t) := \Big(\sum_{l=1}^{n} c_l(x,y) C_{l,k}^{m}(\gamma_{y,x}(t)) \Big)$$

is a triangular matrix function with 0 on the diagonal. Consequently, the matrix function $G = F^{-1}$ satisfies the equation

$$tG'(t) = -G(t) + I - tG(t)\Omega(t),$$

where I is the identity matrix. Thus the matrix function $H(t) = t(G(t) - I)$ solves the following problem:

$$H'(t) = -H(t)\Omega(t) - t\Omega(t), \qquad H(0) = 0,$$

which can be regarded as a Cauchy problem in the space of triangular matrix functions with 0 on the diagonal. Hence H is also such a matrix function. So, G and $F = G^{-1}$ are triangular matrix functions with 1 on the diagonal. This implies that $\Upsilon(x,y) = \det F(1) = 1$.

The determinant $\Upsilon(x,y)$ is an analogue of the so-called Van Vleck–Morette determinant (cf. [DW, (17.28)], [Fri, (4.2.19)] and [Fu]; see also [FK, Remarks 2.4 and 2.5]).

Let V be a sufficiently small neighbourhood of the diagonal of $M \times M$ such that $\gamma_{y,x}$ is defined and $\Upsilon(x,y) \neq 0$ for any $(x,y) \in V$. Let us fix a function $\chi \in C^\infty(M \times M)$ which equals 1 in some neighbourhood of the diagonal of $M \times M$ and vanishes on $(M \times M) \setminus V$.

4.1. Definition. Let $\mathcal{E}$ and $\mathcal{F}$ be C^∞-smooth complex vector bundles over M with connections $\nabla^{\mathcal{E}}$ and $\nabla^{\mathcal{F}}$ respectively and let θ be a complex number, $\tau \in [0,1]$, $r \in \mathbb{R}$, $0 \le \delta < \varrho \le 1$. We will say that a continuous linear operator

$$A : C_0^\infty(\mathcal{E}) \to C^\infty(\mathcal{F})$$

is a *pseudodifferential operator* (ψDO) *with a* τ*-symbol*

$$\sigma_{A,\tau} = a \in S^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})$$

if

$$(4.10)\quad (A\omega)(x) = \frac{1}{(2\pi)^n} \int_M \int_{T^*_{z_\tau}} e^{-i\langle \dot\gamma_{y,x}(\tau),\zeta\rangle} \Phi^{\mathcal{F}}_{x,z_\tau} a(z_\tau,\zeta) \Phi^{\mathcal{E}}_{z_\tau,y} \omega(y) \Upsilon^\theta(x,y) \times \chi(x,y)\, d\mathcal{N}_{z_\tau}(\zeta)\, d\mathcal{M}(y) + \int_M K(x,y)\omega(y)\, d\mathcal{M}(y), \qquad \forall \omega \in C_0^\infty(\mathcal{E}),$$

where $z_\tau = \gamma_{y,x}(\tau)$, $d\mathcal{M}(y)$ and $d\mathcal{N}_{z_\tau}(\zeta)$ are defined by (4.2), the superscripts $\mathcal{F}$ and $\mathcal{E}$ indicate that the parallel displacements along geodesics $\Phi^{\mathcal{F}}$ and $\Phi^{\mathcal{E}}$ correspond to the connections $\nabla^{\mathcal{F}}$ and $\nabla^{\mathcal{E}}$ (see Section 2),

$$K \in C^\infty(\mathcal{H}om_{M\times M}(\mathcal{E},\mathcal{F}))$$

and $\mathcal{H}om_{M\times M}(\mathcal{E},\mathcal{F})$ is the C^∞-smooth vector bundle over $M \times M$ with the fibres

$$(\mathcal{H}om_{M\times M}(\mathcal{E},\mathcal{F}))_{(x,y)} = \mathrm{Hom}(\mathcal{E}_y, \mathcal{F}_x)$$

constructed in the standard way (see, e.g., [Tre, Vol. I, Ch. I, §7]). The first term on the right hand side of (4.10) is understood as an oscillatory integral (see below). The class of such operators will be denoted by $\Psi^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})$. The functions $\sigma_A = \sigma_{A,0}$ and $\sigma_A^{\mathrm{W}} = \sigma_{A,1/2}$ are said to be the *symbol* and the *Weyl symbol* of the ψDO A. The morphism $\widetilde{a}$ from (3.8) corresponding to the τ-symbol (symbol, Weyl symbol) a of the ψDO A is called its τ*-presymbol* (*presymbol*, *Weyl presymbol*) and is denoted by $\widetilde{\sigma}_{A,\tau}$ ($\widetilde{\sigma}_A$, $\widetilde{\sigma}_A^{\mathrm{W}}$).

The strange factor $\Upsilon^\theta(x,y)$ appears in (4.10) by the following reason. If $\theta = 1$ then the connection between partial differential operators and their symbols becomes very simple (see (4.15), (4.16) and (4.21) below). On the other hand the formula (4.10) itself and the formula for the adjoint operator are simpler if $\theta = 0$ (see Theorem 5.1 below). The case $\theta = 1/2$ is also of interest due to its applications in quantum mechanics. All three definitions of ψDOs have been used in the mathematics literature ($\theta = 1$ in [FK], [Wi1], [Wi2], $\theta = 1/2$ in [LQ], $\theta = 0$ in [Saf]). We follow the suggestion made in [Fu] to define ψDOs for an arbitrary θ in order to treat all the above cases simultaneously. Of course one can use factors much more general than $\Upsilon^\theta(x,y)$. However, this does not change the class $\Psi^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})$ (see Theorem 4.4 below).

The phase function $\langle \dot\gamma_{y,x}(\tau), \zeta\rangle$ seems to be a reasonable generalization of the standard Euclidean one $\langle y - x, \zeta\rangle$, because in any local coordinate system we have $\dot\gamma_{y,x}(\tau) = y' - x' + O(\|y' - x'\|^2)$, where $x', y' \in \mathbb{R}^n$ are the coordinates of x and y respectively (see [Saf]). Indeed, since the coordinates of x and y in the canonical coordinate system with the origin at x are 0 and $\widetilde{y}$, we have, due to (2.9),

$$\|c(x, y)\| = O(\|y' - x'\|) \quad \text{as } y \to x$$

(see (4.6)). Now using the equality

$$\dot\gamma_{y,x}(t) = \sum_{j=1}^{n} c_j(x, y)\nu_j(\gamma_{y,x}(t)), \qquad \forall t \in [0, 1], \tag{4.11}$$

we obtain in the chosen coordinate system

$$\begin{aligned}
\gamma_{y,x}(t) - \gamma_{y,x}(\tau) &= \int_\tau^t \dot\gamma_{y,x}(s)\, ds = O(\|y' - x'\|),\\
y' - x' &= \gamma_{y,x}(1) - \gamma_{y,x}(0)\\
&= \int_0^1 \dot\gamma_{y,x}(t)\, dt = \dot\gamma_{y,x}(\tau) + O(\max_{0\le t\le 1} \|\dot\gamma_{y,x}(t) - \dot\gamma_{y,x}(\tau)\|)\\
&= \dot\gamma_{y,x}(\tau) + O(\|c(x, y)\| \max_{0\le t\le 1} \|\gamma_{y,x}(t) - \gamma_{y,x}(\tau)\|)\\
&= \dot\gamma_{y,x}(\tau) + O(\|y' - x'\|^2) \quad \text{as } y' \to x'.
\end{aligned}$$

If $\tau = 0$ or $\delta < \min\{d_k\} \min\{d_k^{-1}\}$, the oscillatory integral on the right hand side of (4.10) can be regularized in the standard way (see, e.g., [Shu, §1]). We can rewrite this integral in a more convenient form which will be used throughout the paper. According to (4.11) we have

$$\langle \dot\gamma_{y,x}(\tau), \zeta\rangle = \sum_{j=1}^{n} c_j(x, y)\sigma_j(z_\tau, \zeta) \tag{4.12}$$

(see (3.9)). Using the change of variables

$$\zeta \mapsto \eta = \sigma(z_\tau, \zeta)$$

and the definition (3.8) we can show that our oscillatory integral equals

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_M e^{-i\langle c(x,y),\eta\rangle} \Phi^{\mathcal{F}}_{x,z_\tau} \widetilde{a}(z_\tau, \eta) \Phi^{\mathcal{E}}_{z_\tau,y} \omega(y) \Upsilon^\theta(x, y)\chi(x, y)\, d\mathcal{M}(y)\, d\eta, \tag{4.13}$$

where

$$\langle c(x, y), \eta\rangle = \sum_{j=1}^{n} c_j(x, y)\eta_j.$$

Choosing in (4.13) the canonical coordinate system with the origin at x we obtain

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle \widetilde{y},\eta\rangle} \Phi^{\mathcal{F}}_{x,z_\tau} \widetilde{a}(z_\tau, \eta) \Phi^{\mathcal{E}}_{z_\tau,y} \omega(y)\chi(x, y) \Upsilon^{\theta-1}(x, y)\, d\widetilde{y}\, d\eta, \tag{4.14}$$

$$y = \exp_x(\widetilde{y}),$$

(see (2.9) and (4.3)). Note that in (4.14) we integrate with respect to $\widetilde{y}$ over a small neighbourhood of $0 \in \mathbb{R}^n$, because the function $\chi(x,\cdot)$ vanishes outside a small neighbourhood of x.

Let us show how to regularize the oscillatory integral on the right hand side of (4.10) in the general case. We will regularize (4.13) and consider the result as a regularization of our oscillatory integral. We use the following Taylor formula:

$$\begin{aligned}
&\Phi^{\mathcal{F}}_{x,z_\tau}\widetilde{a}(z_\tau,\eta)\Phi^{\mathcal{E}}_{z_\tau,y} \\
&\quad= \sum_{|\alpha|=0}^{N} \frac{1}{\alpha!} c^\alpha(x,y)\tau^{|\alpha|}(\nabla^{\mathcal{F},\mathcal{E}}_{\nu(x)})^\alpha \widetilde{a}(x,\eta)\Phi^{\mathcal{E}}_{x,y} \\
&\qquad + (N+1)\sum_{|\alpha|=N+1}\frac{1}{\alpha!}c^\alpha(x,y)\int_0^\tau \Phi^{\mathcal{F}}_{x,z_t}(\nabla^{\mathcal{F},\mathcal{E}}_{\nu(z)})^\alpha \widetilde{a}(z,\eta)|_{z=z_t}\Phi^{\mathcal{E}}_{z_t,y}(\tau-t)^N\,dt
\end{aligned}$$

(see (4.23) below). The oscillatory integral corresponding to the first sum on the right hand side can be regularized in the standard way (see [Shu, §1] and also (4.24) below). It is left to consider the integral corresponding to the second sum. If $(N+1)\min\{d_k^{-1}\}(\varrho-\delta) > r+n$, then integration by parts in η gives an absolutely convergent integral (see (4.24)).

It is not difficult to prove that for any

$$a \in S^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F}) \quad \text{and} \quad K \in C^\infty(\mathcal{H}om_{M\times M}(\mathcal{E},\mathcal{F})),$$

(4.10) defines a continuous operator $A : C_0^\infty(\mathcal{E}) \to C^\infty(\mathcal{F})$ (cf. [Shu, §2]).

Let us take arbitrary $\eta' \in \mathbb{R}^n$, $\omega_0 \in C^\infty(\mathcal{E})$, and in (4.14) put $\tau = 0$,

$$\omega(y) = e^{i\langle \widetilde{y},\eta'\rangle}\chi(x,y)\Upsilon^{1-\theta}(x,y)\Phi^{\mathcal{E}}_{y,x}\omega_0(x), \qquad y = \exp_x(\widetilde{y}).$$

Then due to (2.27) and the equality $z_0 = \gamma_{y,x}(0) = x$ the result equals

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle \widetilde{y},\eta-\eta'\rangle}\widetilde{a}(x,\eta)\omega_0(x)\chi^2(x,\exp_x(\widetilde{y}))\,d\widetilde{y}\,d\eta.$$

Since the function $\chi^2(x,\exp_x(\cdot))$ equals 1 in some neighbourhood of $0 \in \mathbb{R}^n$, it is not difficult to prove that the last integral equals

$$\widetilde{a}(x,\eta')\omega_0(x) + \widetilde{q}_0(x,\eta')\omega_0(x), \qquad \widetilde{q}_0 \in \widetilde{S}^{-\infty}(\mathcal{E},\mathcal{F})$$

(see, e.g., the proof of [Ho1, Lemma 3.2]). Simple integration by parts shows that for the chosen ω the second term on the right hand side of (4.10) equals

$$\widetilde{q}_1(x,\eta')\omega_0(x), \qquad \widetilde{q}_1 \in \widetilde{S}^{-\infty}(\mathcal{E},\mathcal{F}).$$

Thus we have proved the following result.

4.2. Lemma. *Let A be a ψDO with a symbol $\sigma_A \in S^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})$. Then*

$$\sigma_A(x,\xi)\omega_0(x) = (A\omega)(x) + q(x,\xi)\omega_0(x), \qquad \forall \omega_0 \in C^\infty(\mathcal{E}),$$

where

$$\omega(y) = e^{i\langle \dot{\gamma}_{y,x}(0),\xi\rangle}\chi(x,y)\Upsilon^{1-\theta}(x,y)\Phi^{\mathcal{E}}_{y,x}\omega_0(x), \qquad q \in S^{-\infty}(\mathcal{E},\mathcal{F}).$$

(*Here we fix an arbitrary $(x,\xi) \in T^*M$, apply the operator A to the section ω, regarding y as the variable, and then evaluate the result $A\omega$ at the point x.*)

Let us consider the case when in (4.10), $\tau = 0$, $K = 0$ and $a(x,\xi)$ is a polynomial in ξ:

$$a(x,\xi) = \sum_{|\alpha|\le N} a_\alpha(x)\sigma^\alpha(x,\xi), \qquad a_\alpha \in C^\infty(\mathcal{H}om(\mathcal{E},\mathcal{F})) \tag{4.15}$$

(see (3.9)). Using the representation (4.14), (2.15), Lemma 2.6 and the fact that the function $\chi(x,\cdot)$ equals 1 in some neighbourhood of x we obtain

$$\begin{aligned}
(A\omega)(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle\widetilde{y},\eta\rangle}\Big(\sum_{|\alpha|\le N} a_\alpha(x)\eta^\alpha\Big)\varPhi^{\mathcal{E}}_{x,y}\omega(y)\chi(x,y)\varUpsilon^{\theta-1}(x,y)\,d\widetilde{y}\,d\eta\\
&= \sum_{|\alpha|\le N} a_\alpha(x)(-i)^{|\alpha|}\partial^\alpha_{\widetilde{y}}(\varPhi^{\mathcal{E}}_{x,\exp_x(\widetilde{y})}\omega(\exp_x(\widetilde{y}))\varUpsilon^{\theta-1}(x,\exp_x(\widetilde{y})))|_{\widetilde{y}=0}\\
&= \sum_{|\alpha|\le N} a_\alpha(x)(-i)^{|\alpha|}\partial^\alpha_{\nu(y)}(\varPhi^{\mathcal{E}}_{x,y}\omega(y)\varUpsilon^{\theta-1}(x,y))|_{y=x}\\
&= \sum_{|\alpha|\le N} a_\alpha(x)(-i)^{|\alpha|}(\nabla^{\mathcal{E}}_{\nu(y)})^\alpha(\omega(y)\varUpsilon^{\theta-1}(x,y))|_{y=x}\\
&= \sum_{|\alpha|\le N} a_\alpha(x)(-i)^{|\alpha|}\alpha!\sum_{\alpha'+\alpha''=\alpha}\frac{1}{\alpha'!\alpha''!}(\partial^{\alpha''}_{\nu(y)}\varUpsilon^{\theta-1}(x,y))|_{y=x}(\nabla^{\mathcal{E}}_\nu)^{\alpha'}\omega(x)\\
&= \sum_{|\beta|\le N}\Bigg(\sum_{|\beta'|\le N-|\beta|}\frac{(\beta+\beta')!}{\beta!\beta'!}(-i)^{|\beta+\beta'|}\partial^{\beta'}_{\nu(y)}\varUpsilon^{\theta-1}(x,y)|_{y=x}a_{\beta+\beta'}(x)\Bigg)(\nabla^{\mathcal{E}}_\nu)^\beta\omega(x).
\end{aligned}$$

Hence

$$A = \sum_{|\beta|\le N} b_\beta(x)(\nabla^{\mathcal{E}}_\nu)^\beta, \qquad b_\beta \in C^\infty(\mathcal{H}om(\mathcal{E},\mathcal{F})), \tag{4.16}$$

where the coefficients are given by the formula

$$b_\beta(x) = \sum_{|\beta'|\le N-|\beta|}\frac{(\beta+\beta')!}{\beta!\beta'!}(-i)^{|\beta+\beta'|}\partial^{\beta'}_{\nu(y)}\varUpsilon^{\theta-1}(x,y)|_{y=x}a_{\beta+\beta'}(x). \tag{4.17}$$

Conversely, any differential operator (4.16) can be represented in the form (4.10), where $\tau = 0$, $K = 0$ and the coefficients of the symbol (4.15) are given by the following recurrent formulae:

$$a_\alpha(x) = \begin{cases} i^N b_\alpha(x) & \text{if } |\alpha| = N,\\ i^{|\alpha|}b_\alpha(x) \\ \qquad - \displaystyle\sum_{1\le|\alpha'|\le N-|\alpha|}\frac{(\alpha+\alpha')!}{\alpha!\alpha'!}(-i)^{|\alpha'|}\partial^{\alpha'}_{\nu(y)}\varUpsilon^{\theta-1}(x,y)|_{y=x}a_{\alpha+\alpha'}(x) \\ & \text{if } |\alpha| < N \end{cases} \tag{4.18}$$

(see (4.5)). However, it is easier to find the coefficients a_α with the help of Lemma 4.2:

$$\begin{aligned}
a(x,\xi) &= \sum_{|\beta|\le N} b_\beta(x)(\nabla^{\mathcal{E}}_{\nu(y)})^\beta(e^{i\langle c(x,y),\sigma(x,\xi)\rangle}\varUpsilon^{1-\theta}(x,y)\varPhi^{\mathcal{E}}_{y,x})|_{y=x}\\
&= \sum_{|\beta|\le N} b_\beta(x)\partial^\beta_{\nu(y)}(e^{i\langle c(x,y),\sigma(x,\xi)\rangle}\varUpsilon^{1-\theta}(x,y))|_{y=x}
\end{aligned}$$

$$= \sum_{|\beta|\le N} b_\beta(x)\beta! \sum_{\beta'+\beta''=\beta} \frac{1}{\beta'!\beta''!} \partial^{\beta''}_{\nu(y)} \Upsilon^{1-\theta}(x,y)|_{y=x} i^{|\beta'|}\sigma^{\beta'}(x,\xi)$$

$$= \sum_{|\alpha|\le N} i^{|\alpha|}\left(\sum_{|\alpha'|\le N-|\alpha|} \frac{(\alpha+\alpha')!}{\alpha!\alpha'!} \partial^{\alpha'}_{\nu(y)} \Upsilon^{1-\theta}(x,y)|_{y=x} b_{\alpha+\alpha'}(x)\right)\sigma^\alpha(x,\xi)$$

(see Lemma 2.7, (2.15), (2.18), (2.20) and (4.12)). Consequently,

$$a_\alpha(x) = i^{|\alpha|} \sum_{|\alpha'|\le N-|\alpha|} \frac{(\alpha+\alpha')!}{\alpha!\alpha'!} \partial^{\alpha'}_{\nu(y)} \Upsilon^{1-\theta}(x,y)|_{y=x} b_{\alpha+\alpha'}(x). \tag{4.19}$$

It is worth repeating that $\partial^{\alpha'}_{\nu(y)}\Upsilon(x,y)|_{y=x}$ is a universal polynomial in $\partial^\mu_\nu C^m_{j,k}(x)$, $|\mu| \le |\alpha'|-1$. In particular, using (4.7) we obtain

$$a_\alpha(x) = i^{N-1}\left(b_\alpha(x) + \frac{1-\theta}{2}\sum_{k,j=1}^n (\alpha_j+1)C^k_{j,k}(x) b_{\alpha+e_j}(x)\right) \quad \text{if } |\alpha| = N-1, \tag{4.20}$$

where $e_j = (0,\dots,0,1,0,\dots,0)$ with 1 at the jth place.

The above formulae imply that the symbol of the operator $\nabla^{\mathcal{E}}_{\nu_j}$ equals

$$i\sigma_j(x,\xi) + \frac{1-\theta}{2}\sum_{k=1}^n C^k_{j,k}(x).$$

In the case $\theta = 1$ the formulae (4.17), (4.19) become very simple:

$$a_\alpha(x) = i^{|\alpha|} b_\alpha(x). \tag{4.21}$$

The same is true for an arbitrary θ if (4.8) holds.

4.3. Theorem. *Let A be a ψDO with a τ-presymbol $\widetilde{\sigma}_{A,\tau} \in \widetilde{S}^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})$. Then A is a ψDO with an s-presymbol $\widetilde{\sigma}_{A,s} \in \widetilde{S}^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})$,*

$$\widetilde{\sigma}_{A,s}(x,\eta) \sim \sum_{|\alpha|=0}^\infty \frac{i^{|\alpha|}}{\alpha!}(s-\tau)^{|\alpha|}\partial^\alpha_\eta (\nabla^{\mathcal{F},\mathcal{E}}_\nu)^\alpha \widetilde{\sigma}_{A,\tau}(x,\eta). \tag{4.22}$$

Proof. Let us consider the Taylor expansion of the $\mathrm{Hom}(\mathcal{E}_y,\mathcal{F}_x)$-valued function

$$\psi(t) = \Psi(z_t) := \Phi^{\mathcal{F}}_{x,z_t}\widetilde{\sigma}_{A,\tau}(z_t,\eta)\Phi^{\mathcal{E}}_{z_t,y}.$$

Using (2.26), (2.28) and (2.31) we obtain

$$\frac{d^r\Psi(z_t)}{dt^r} = \partial^r_{\dot\gamma_{y,x}(t)} \Phi^{\mathcal{F}}_{x,z_t}\widetilde{\sigma}_{A,\tau}(z_t,\eta)\Phi^{\mathcal{E}}_{z_t,y}$$
$$= \sum_{m+p+q=r}\frac{r!}{m!p!q!}(\nabla^{\mathcal{F}}_{\dot\gamma_{y,x}(t)})^m \Phi^{\mathcal{F}}_{x,\gamma_{y,x}(t)} (\nabla^{\mathcal{F},\mathcal{E}}_{\dot\gamma_{y,x}(t)})^p \widetilde{\sigma}_{A,\tau}(z_t,\eta) (\nabla^{\mathcal{E}}_{\dot\gamma_{y,x}(t)})^q \Phi^{\mathcal{E}}_{\gamma_{y,x}(t),y}$$
$$= \Phi^{\mathcal{F}}_{x,z_t}(\nabla^{\mathcal{F},\mathcal{E}}_{\dot\gamma_{y,x}(t)})^r \widetilde{\sigma}_{A,\tau}(z_t,\eta)\Phi^{\mathcal{E}}_{z_t,y}.$$

Since $\dot\gamma_{y,x} = \sum_{j=1}^n c_j(x,y)\nu_j$, we have

$$\frac{d^r\Psi(z_t)}{dt^r} = \Phi^{\mathcal{F}}_{x,z_t}\sum_{|\alpha|=r}\frac{r!}{\alpha!}c^\alpha(x,y)(\nabla^{\mathcal{F},\mathcal{E}}_{\nu(z)})^\alpha \widetilde{\sigma}_{A,\tau}(z,\eta)|_{z=z_t}\Phi^{\mathcal{E}}_{z_t,y}.$$

Hence

(4.23)
$$\begin{aligned}\Psi(z_\tau) &= \sum_{r=0}^{N} \frac{1}{r!}\frac{d^r\Psi(z_t)}{dt^r}\bigg|_{t=s}(\tau-s)^r + \frac{1}{N!}\int_s^\tau \frac{d^{N+1}\Psi(z_t)}{dt^{N+1}}(\tau-t)^N\,dt \\ &= \sum_{|\alpha|=0}^{N} \frac{1}{\alpha!}c^\alpha(x,y)(\tau-s)^{|\alpha|}\Phi^{\mathcal{F}}_{x,z_s}(\nabla^{\mathcal{F},\mathcal{E}}_{\nu(z)})^\alpha\widetilde{\sigma}_{A,\tau}(z,\eta)|_{z=z_s}\Phi^{\mathcal{E}}_{z_s,y} \\ &\quad + (N+1)\sum_{|\alpha|=N+1}\frac{1}{\alpha!}c^\alpha(x,y)\int_s^\tau \Phi^{\mathcal{F}}_{x,z_t}(\nabla^{\mathcal{F},\mathcal{E}}_{\nu(z)})^\alpha\widetilde{\sigma}_{A,\tau}(z,\eta)|_{z=z_t}\Phi^{\mathcal{E}}_{z_t,y}(\tau-t)^N\,dt.\end{aligned}$$

Now using the equality

(4.24)
$$c^\alpha(x,y)e^{-i\langle c(x,y),\eta\rangle} = i^{|\alpha|}\partial_\eta^\alpha e^{-i\langle c(x,y),\eta\rangle}$$

and integrating by parts we can derive (4.22) from (4.13). We skip the details which are similar to those from the standard calculus of ψDOs. ■

In the same way one can prove the following proposition.

4.4. Theorem. *Let $\tau, s \in [0,1]$, $a \in S^{r,\mathbf{d}}_{\varrho,\delta}(M;\mathcal{E},\mathcal{F})$, $0 \le \delta < \varrho \le 1$, and let an operator $A : C_0^\infty(\mathcal{E}) \to C^\infty(\mathcal{F})$ be defined by the formula*

$$(A\omega)(x) = \frac{1}{(2\pi)^n}\int_M\int_{T^*_{z_\tau}} e^{-i\langle\dot\gamma_{y,x}(\tau),\zeta\rangle}\Phi^{\mathcal{F}}_{x,z_\tau}a(z_s;z_\tau,\zeta)\Phi^{\mathcal{E}}_{z_\tau,y}\omega(y)\Upsilon^\theta(x,y) \times \chi(x,y)\,d\mathcal{N}_{z_\tau}(\zeta)\,d\mathcal{M}(y), \qquad \forall\omega\in C_0^\infty(\mathcal{E}).$$

Then A is a ψDO of the class $\Psi^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})$ and its τ- and s-presymbols are given by the following formulae:

$$\widetilde{\sigma}_{A,\tau}(x,\eta) \sim \sum_{|\alpha|=0}^{\infty}\frac{i^{|\alpha|}}{\alpha!}(\tau-s)^{|\alpha|}\partial_\eta^\alpha\partial^\alpha_{\nu(y)}\widetilde{a}(y;x,\eta)|_{y=x},$$
$$\widetilde{\sigma}_{A,s}(x,\eta) \sim \sum_{|\alpha|=0}^{\infty}\frac{i^{|\alpha|}}{\alpha!}(s-\tau)^{|\alpha|}\partial_\eta^\alpha(\nabla^{\mathcal{F},\mathcal{E}}_{\nu(x)})^\alpha\widetilde{a}(y;x,\eta)|_{y=x},$$

where $a(y;x,\xi) = \widetilde{a}(y;x,\sigma(x,\xi))$ (cf. (3.10)).

It was mentioned in Section 3 that the class $S^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})$ of symbols does not depend on the choice of the connections $\nabla^{\mathcal{E}}$ and $\nabla^{\mathcal{F}}$. It turns out that the class $\Psi^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})$ of ψDOs also does not depend on the choice of $\nabla^{\mathcal{E}}$ and $\nabla^{\mathcal{F}}$. Let $\widehat{\nabla}^{\mathcal{E}}$ and $\widehat{\nabla}^{\mathcal{F}}$ be another pair of connections on $\mathcal{E}$ and $\mathcal{F}$ and let $\widehat{\Phi}^{\mathcal{E}}$ and $\widehat{\Phi}^{\mathcal{F}}$ be the corresponding parallel displacements along geodesics.

4.5. Theorem. *Let A be a ψDO with a τ-presymbol $\widetilde{a} \in \widetilde{S}^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})$ with respect to the pair $\nabla^{\mathcal{E}}$, $\nabla^{\mathcal{F}}$. Then A is a ψDO with a τ-presymbol $\widetilde{b} \in \widetilde{S}^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})$ with respect to the* *pair* $\widehat{\nabla}^{\mathcal{E}}$, $\widehat{\nabla}^{\mathcal{F}}$ *and*

$$\widetilde{b}(x,\eta) \sim \sum_{|\alpha|,|\beta|=0}^{\infty} \frac{i^{|\alpha|+|\beta|}\tau^{|\alpha|}(\tau-1)^{|\beta|}}{\alpha!\beta!} (\nabla^{\mathcal{F}}_{\nu(z)})^{\alpha} \widehat{\Phi}^{\mathcal{F}}_{x,z}|_{z=x} \partial^{\alpha+\beta}_{\eta} \widetilde{a}(x,\eta) (\nabla^{\mathcal{E}}_{\nu(z)})^{\beta} \widehat{\Phi}^{\mathcal{E}}_{z,x}|_{z=x}$$

$$\sim \sum_{|\alpha|,|\beta|=0}^{\infty} \frac{i^{|\alpha|+|\beta|}\tau^{|\alpha|}(\tau-1)^{|\beta|}}{\alpha!\beta!} (\widehat{\nabla}^{\mathcal{F}}_{\nu(z)})^{\alpha} \Phi^{\mathcal{F}}_{z,x}|_{z=x} \partial^{\alpha+\beta}_{\eta} \widetilde{a}(x,\eta) (\widehat{\nabla}^{\mathcal{E}}_{\nu(z)})^{\beta} \Phi^{\mathcal{E}}_{x,z}|_{z=x}$$

Proof. The equality (2.27) implies

$$\Phi^{\mathcal{F}}_{x,z_\tau} \widetilde{a}(z_\tau,\eta) \Phi^{\mathcal{E}}_{z_\tau,y} = \widehat{\Phi}^{\mathcal{F}}_{x,z_\tau} \widehat{b}(x,y;z_\tau,\eta) \widehat{\Phi}^{\mathcal{E}}_{z_\tau,y},$$

where

$$\widehat{b}(x,y;z_\tau,\eta) = \widehat{\Phi}^{\mathcal{F}}_{z_\tau,x} \Phi^{\mathcal{F}}_{x,z_\tau} \widetilde{a}(z_\tau,\eta) \Phi^{\mathcal{E}}_{z_\tau,y} \widehat{\Phi}^{\mathcal{E}}_{y,z_\tau}.$$

Similarly to (4.23) we obtain

$$\widehat{b}(x,y;z_\tau,\eta) \sim \left(\sum_{|\alpha|=0}^{\infty} \frac{(-\tau)^{|\alpha|}}{\alpha!} c^{\alpha}(x,y) (\nabla^{\mathcal{F}}_{\nu(z)})^{\alpha} \widehat{\Phi}^{\mathcal{F}}_{z_\tau,z}|_{z=z_\tau} \right) \widetilde{a}(z_\tau,\eta)$$
$$\times \left(\sum_{|\beta|=0}^{\infty} \frac{(1-\tau)^{|\beta|}}{\beta!} c^{\beta}(x,y) (\nabla^{\mathcal{E}}_{\nu(z)})^{\beta} \widehat{\Phi}^{\mathcal{E}}_{z,z_\tau}|_{z=z_\tau} \right),$$

$$\widehat{b}(x,y;z_\tau,\eta) \sim \left(\sum_{|\alpha|=0}^{\infty} \frac{(-\tau)^{|\alpha|}}{\alpha!} c^{\alpha}(x,y) (\widehat{\nabla}^{\mathcal{F}}_{\nu(z)})^{\alpha} \Phi^{\mathcal{F}}_{z,z_\tau}|_{z=z_\tau} \right) \widetilde{a}(z_\tau,\eta)$$
$$\times \left(\sum_{|\beta|=0}^{\infty} \frac{(1-\tau)^{|\beta|}}{\beta!} c^{\beta}(x,y) (\widehat{\nabla}^{\mathcal{E}}_{\nu(z)})^{\beta} \Phi^{\mathcal{E}}_{z_\tau,z}|_{z=z_\tau} \right).$$

The remaining part of the proof uses (4.24) and integration by parts and is similar to that of Theorem 4.3. ∎

4.6. Remark. Later on we will need the following class of operators:

$$\Psi^{-\infty}(\mathcal{E},\mathcal{F}) = \bigcap_{r\in\mathbb{R}} \Psi^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F}) = \bigcap_{r\in\mathbb{R}} \Psi^{r,\mathbf{d}}_{1,0}(\mathcal{E},\mathcal{F}).$$

It is easy to see that any element of $\Psi^{-\infty}(\mathcal{E},\mathcal{F})$ is an integral operator with a C^∞-smooth kernel.

5. Dual and adjoint operators

Let $\mathcal{E}'$ be the vector bundle dual to $\mathcal{E}$: $\mathcal{E}' = \mathcal{H}om(\mathcal{E},\mathbb{C})$. The transition matrices $c_{J,K}$ corresponding to $\mathcal{E}$ are replaced by ${}^t c^{-1}_{J,K}$ in the case of $\mathcal{E}'$. Let $\mathcal{E}^*$ be the vector bundle adjoint to $\mathcal{E}$, i.e. let $\mathcal{E}^*$ be the complex conjugate of $\mathcal{E}'$ (see, e.g., [MS, §14]). The transition matrices corresponding to $\mathcal{E}^*$ are $\overline{{}^t c^{-1}_{J,K}}$. In this case one has well defined bilinear and sesquilinear mappings

(5.1) $$\langle\cdot,\cdot\rangle_{\mathcal{E}} : \mathcal{E}_x \times \mathcal{E}'_x \to \mathbb{C},$$

(5.2) $$(\cdot,\cdot)_{\mathcal{E}} : \mathcal{E}_x \times \mathcal{E}^*_x \to \mathbb{C}, \quad x \in M.$$

Using them we can define the connections $\nabla^{\mathcal{E}'}$ and $\nabla^{\mathcal{E}^*}$ generated by $\nabla^{\mathcal{E}}$. For any sections $v \in C^\infty(\mathcal{E}')$, $w \in C^\infty(\mathcal{E}^*)$ and any vector field $\nu_0 \in C^\infty(TM)$ we define $\nabla^{\mathcal{E}'}_{\nu_0} v$ and $\nabla^{\mathcal{E}^*}_{\nu_0} w$ by the equalities

$$\langle \omega, \nabla^{\mathcal{E}'}_{\nu_0} v\rangle_{\mathcal{E}} = \partial_{\nu_0}\langle \omega, v\rangle_{\mathcal{E}} - \langle \nabla^{\mathcal{E}}_{\nu_0}\omega, v\rangle_{\mathcal{E}}, \tag{5.3}$$

$$(\omega, \nabla^{\mathcal{E}^*}_{\nu_0} w)_{\mathcal{E}} = \partial_{\nu_0}(\omega, w)_{\mathcal{E}} - (\nabla^{\mathcal{E}}_{\nu_0}\omega, w)_{\mathcal{E}}, \qquad \forall \omega \in C^\infty(\mathcal{E}). \tag{5.4}$$

The first equality is in fact a special case of (2.26) with $Z = \mathbb{C}$.

The connections $\nabla^{\mathcal{E}'}$ and $\nabla^{\mathcal{E}^*}$ generate the corresponding parallel displacements along geodesics

$$\Phi^{\mathcal{E}'}_{y,x} : \mathcal{E}'_x \to \mathcal{E}'_y, \qquad \Phi^{\mathcal{E}^*}_{y,x} : \mathcal{E}^*_x \to \mathcal{E}^*_y.$$

It follows from (5.3), (5.4) that for any $x \in M$, $\epsilon \in \mathcal{E}_x$, $\epsilon' \in \mathcal{E}'_x$ and $\epsilon^* \in \mathcal{E}^*_x$ the functions

$$y \mapsto \langle \Phi^{\mathcal{E}}_{y,x}\epsilon, \Phi^{\mathcal{E}'}_{y,x}\epsilon'\rangle_{\mathcal{E}}, \qquad y \mapsto (\Phi^{\mathcal{E}}_{y,x}\epsilon, \Phi^{\mathcal{E}^*}_{y,x}\epsilon^*)_{\mathcal{E}}$$

are constant along geodesics starting at x. Taking $\epsilon' = \Phi^{\mathcal{E}'}_{x,y}\epsilon'_y$, $\epsilon^* = \Phi^{\mathcal{E}^*}_{x,y}\epsilon^*_y$ with arbitrary $\epsilon'_y \in \mathcal{E}'_y$, $\epsilon^*_y \in \mathcal{E}^*_y$ we obtain

$$\langle \Phi^{\mathcal{E}}_{y,x}\epsilon, \epsilon'_y\rangle_{\mathcal{E}} = \langle \epsilon, \Phi^{\mathcal{E}'}_{x,y}\epsilon'_y\rangle_{\mathcal{E}}, \qquad \forall \epsilon \in \mathcal{E}_x,\ \forall \epsilon'_y \in \mathcal{E}'_y, \tag{5.5}$$

$$(\Phi^{\mathcal{E}}_{y,x}\epsilon, \epsilon^*_y)_{\mathcal{E}} = (\epsilon, \Phi^{\mathcal{E}^*}_{y,x}\epsilon^*_y)_{\mathcal{E}}, \qquad \forall \epsilon \in \mathcal{E}_x,\ \forall \epsilon^*_y \in \mathcal{E}^*_y. \tag{5.6}$$

Let $\omega \in C^\infty(\mathcal{E})$, $v \in C^\infty(\mathcal{E}')$ and suppose that at least one of these sections has a compact support. Similarly, let $w \in C^\infty(\mathcal{E}^*)$ and suppose that either w or ω has compact support. Then using (5.1), (5.2) we can define the following bilinear and sesquilinear forms:

$$\langle \omega, v\rangle_{\mathcal{E},\mathcal{M}} := \int_M \langle \omega(x), v(x)\rangle_{\mathcal{E}}\, d\mathcal{M}(x), \tag{5.7}$$

$$(\omega, w)_{\mathcal{E},\mathcal{M}} := \int_M (\omega(x), w(x))_{\mathcal{E}}\, d\mathcal{M}(x) \tag{5.8}$$

(see (4.2)).

All the above constructions can be carried out for the vector bundle $\mathcal{F}$ without any changes.

5.1. Theorem. *Let $\tau, s \in [0,1]$ and $A \in \Psi^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})$. Then there exist ψDOs $A' \in \Psi^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{F}',\mathcal{E}')$ and $A^* \in \Psi^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{F}^*,\mathcal{E}^*)$ such that*

$$\langle A\omega, v\rangle_{\mathcal{F},\mathcal{M}} = \langle \omega, A'v\rangle_{\mathcal{E},\mathcal{M}}, \qquad (A\omega, w)_{\mathcal{F},\mathcal{M}} = (\omega, A^*w)_{\mathcal{E},\mathcal{M}}$$

for any $\omega \in C^\infty_0(\mathcal{E})$, $v \in C^\infty_0(\mathcal{F}')$, $w \in C^\infty_0(\mathcal{F}^*)$, and

$$\widetilde{\sigma}_{A',s}(x,\eta) \sim \sum_{|\alpha|,|\beta|,|\mu|=0}^{\infty} \frac{i^{|\alpha|+|\beta|+|\mu|}(s+\tau-1)^{|\alpha|}s^{|\beta|}(s-1)^{|\mu|}}{\alpha!\beta!\mu!} \times \partial^\beta_{\nu(z)}\partial^\mu_{\nu(z')}\Delta^\theta(z,z')|_{z=z'=x}\partial^{\alpha+\beta+\mu}_\eta(\nabla^{\mathcal{E}',\mathcal{F}'}_\nu)^\alpha\widetilde{\sigma}'_{A,\tau}(x,-\eta), \tag{5.9}$$

$$\widetilde{\sigma}_{A^*,s}(x,\eta) \sim \sum_{|\alpha|,|\beta|,|\mu|=0}^{\infty} \frac{i^{|\alpha|+|\beta|+|\mu|}(s+\tau-1)^{|\alpha|}s^{|\beta|}(s-1)^{|\mu|}}{\alpha!\beta!\mu!} \times \partial^\beta_{\nu(z)}\partial^\mu_{\nu(z')}\Delta^\theta(z,z')|_{z=z'=x}\partial^{\alpha+\beta+\mu}_\eta(\nabla^{\mathcal{E}^*,\mathcal{F}^*}_\nu)^\alpha\widetilde{\sigma}^*_{A,\tau}(x,\eta), \tag{5.10}$$

where $\widetilde{\sigma}'_{A,\tau} \in C^\infty(\mathcal{H}om(\widetilde{\mathcal{F}}', \widetilde{\mathcal{E}}'))$ *and* $\widetilde{\sigma}^*_{A,\tau} \in C^\infty(\mathcal{H}om(\widetilde{\mathcal{F}}^*, \widetilde{\mathcal{E}}^*))$ *are the morphisms* (see Section 3) *dual and adjoint to* $\widetilde{\sigma}_{A,\tau}$:

$$\langle \widetilde{\sigma}_{A,\tau}(x,\eta)\epsilon, \psi' \rangle_{\mathcal{F}} = \langle \epsilon, \widetilde{\sigma}'_{A,\tau}(x,\eta)\psi' \rangle_{\mathcal{E}}, \qquad \forall \epsilon \in \mathcal{E}_x,\ \forall \psi' \in \mathcal{F}'_x,$$
$$(\widetilde{\sigma}_{A,\tau}(x,\eta)\epsilon, \psi^*)_{\mathcal{F}} = (\epsilon, \widetilde{\sigma}^*_{A,\tau}(x,\eta)\psi^*)_{\mathcal{E}}, \qquad \forall \epsilon \in \mathcal{E}_x,\ \forall \psi^* \in \mathcal{F}^*_x,$$

and

$$\Delta(x,y) = \frac{\Upsilon(y,x)}{\Upsilon(x,y)}.$$

If $\theta = 0$ *or* (4.8) *holds, then*

$$\widetilde{\sigma}_{A',s}(x,\eta) \sim \sum_{|\alpha|=0}^{\infty} \frac{i^{|\alpha|}(s+\tau-1)^{|\alpha|}}{\alpha!} \partial_\eta^\alpha (\nabla_\nu^{\mathcal{E}',\mathcal{F}'})^\alpha \widetilde{\sigma}'_{A,\tau}(x,-\eta), \tag{5.11}$$

$$\widetilde{\sigma}_{A^*,s}(x,\eta) \sim \sum_{|\alpha|=0}^{\infty} \frac{i^{|\alpha|}(s+\tau-1)^{|\alpha|}}{\alpha!} \partial_\eta^\alpha (\nabla_\nu^{\mathcal{E}^*,\mathcal{F}^*})^\alpha \widetilde{\sigma}^*_{A,\tau}(x,\eta), \tag{5.12}$$

and in particular the Weyl presymbols satisfy the equalities

$$\widetilde{\sigma}^{\mathrm{W}}_{A'}(x,\eta) = (\widetilde{\sigma}^{\mathrm{W}}_{A}(x,-\eta))', \qquad \widetilde{\sigma}^{\mathrm{W}}_{A^*}(x,\eta) = (\widetilde{\sigma}^{\mathrm{W}}_{A}(x,\eta))^*. \tag{5.13}$$

Proof. Using (5.5) and (5.6), from (4.10), (4.13) we obtain

$$\begin{aligned}(A'v)(y) &= \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \int\limits_M e^{-i\langle c(x,y),\eta\rangle} \varPhi^{\mathcal{E}'}_{y,z_\tau} \widetilde{\sigma}'_{A,\tau}(z_\tau,\eta) \varPhi^{\mathcal{F}'}_{z_\tau,x} v(x) \Upsilon^\theta(x,y)\chi(x,y)\, d\mathcal{M}(x)\, d\eta \\ &\quad + \int\limits_M K'(x,y)v(x)\, d\mathcal{M}(x), \\ (A^*w)(y) &= \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \int\limits_M e^{i\langle c(x,y),\eta\rangle} \varPhi^{\mathcal{E}^*}_{y,z_\tau} \widetilde{\sigma}^*_{A,\tau}(z_\tau,\eta) \varPhi^{\mathcal{F}^*}_{z_\tau,x} w(x) \Upsilon^\theta(x,y)\chi(x,y)\, d\mathcal{M}(x)\, d\eta \\ &\quad + \int\limits_M K^*(x,y)w(x)\, d\mathcal{M}(x),\end{aligned}$$

since (5.1) and (5.2) are bilinear and sesquilinear respectively. We will sketch the remaining part of the proof for the operator A'. For A^* it is quite similar.

Taking into account (2.12) and the equality $z_\tau(x,y) = z_{1-\tau}(y,x)$ and making the change of variables $\eta \mapsto -\eta$ we arrive at

$$\begin{aligned}(A'v)(x) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \int\limits_M & e^{-i\langle c(x,y),\eta\rangle} \varPhi^{\mathcal{E}'}_{x,z_{1-\tau}} \widetilde{\sigma}'_{A,\tau}(z_{1-\tau},-\eta) \varPhi^{\mathcal{F}'}_{z_{1-\tau},y} v(y) \\ & \times \Delta^\theta(x,y)\Upsilon^\theta(x,y)\chi(y,x)\, d\mathcal{M}(y)\, d\eta + \int\limits_M K'(y,x)v(y)\, d\mathcal{M}(y).\end{aligned}$$

Now (5.9) can be proved similarly to Theorems 4.3–4.5.

If $\theta = 0$ or (4.8) holds, then $\Delta^\theta(x,y) \equiv 1$ and (5.11), (5.12) follow from (5.9) and (5.10). Taking $s = \tau = 1/2$ in (5.11), (5.12) we obtain (5.13). ∎

5.2. Remark. Using Lemma 2.3 one can prove that $\partial^\beta_{\nu(z)} \partial^\mu_{\nu(z')} \Delta^\theta(z,z')|_{z=z'=x}$ is a polynomial in $\partial^\alpha_\nu C^m_{j,k}(x)$ (cf. (4.4)).

The operators A' and A^* were defined above with the help of the forms (5.7) and (5.8), which depended on the choice of the measure $d\mathcal{M}$ (see (4.2)). In order to define these operators in a more invariant way one has to deal with densities.

Let $\kappa \in \mathbb{R}$ and let Ω^κ be the bundle of κ-densities over M (see, e.g., [Ho3, Vol. 3, §18.1] or [Tre, Vol. 2, Ch. VII, §2.5]). Since this bundle has one-dimensional fibres and the corresponding transition functions are positive, it is clear that (5.1) and (5.2) define the following mappings:

$$\langle\cdot,\cdot\rangle_{\mathcal{E}} : (\mathcal{E}\otimes\Omega^{\kappa_1})_x\times(\mathcal{E}'\otimes\Omega^{\kappa_2})_x \to \Omega_x^{\kappa_1+\kappa_2}, \qquad (\cdot,\cdot)_{\mathcal{E}} : (\mathcal{E}\otimes\Omega^{\kappa_1})_x\times(\mathcal{E}^*\otimes\Omega^{\kappa_2})_x \to \Omega_x^{\kappa_1+\kappa_2}.$$

Therefore for any $\omega \in C^\infty(\mathcal{E}\otimes\Omega^\kappa)$, $v \in C^\infty(\mathcal{E}'\otimes\Omega^{1-\kappa})$ and $w \in C^\infty(\mathcal{E}^*\otimes\Omega^{1-\kappa})$ we obtain densities $\langle\omega(x), v(x)\rangle_{\mathcal{E}}$ and $(\omega(x), w(x))_{\mathcal{E}}$. Hence, the following objects are well defined:

$$\langle\omega, v\rangle_{\mathcal{E},\Omega} := \int_M \langle\omega(x), v(x)\rangle_{\mathcal{E}}\, dx, \tag{5.14}$$

$$(\omega, w)_{\mathcal{E},\Omega} := \int_M (\omega(x), w(x))_{\mathcal{E}}\, dx, \tag{5.15}$$

if at least one of the sections in each equality has a compact support.

For any $\kappa \in \mathbb{R}$ we can equip the tensor product $\mathcal{E}\otimes\Omega^\kappa$ with a connection taking the following natural parallel displacements along geodesics:

$$\Phi_{y,x}^{\mathcal{E}\otimes\Omega^\kappa} = |\det(\nu_k^j(y))|^{-\kappa}\Phi_{y,x}^{\mathcal{E}}|\det(\nu_k^j(x))|^{\kappa} : (\mathcal{E}\otimes\Omega^\kappa)_x \to (\mathcal{E}\otimes\Omega^\kappa)_y \tag{5.16}$$

(see [KN, Ch. III, Section 1]).

It is clear that the operator

$$N_\kappa : C^\infty(\mathcal{E}\otimes\Omega^\kappa) \to C^\infty(\mathcal{E}), \qquad (N_\kappa\omega)(x) = |\det(\nu_k^j(x))|^{\kappa}\omega(x),$$

is an isomorphism with the inverse

$$N_{-\kappa} : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E}\otimes\Omega^\kappa), \qquad (N_{-\kappa}v)(x) = |\det(\nu_k^j(x))|^{-\kappa}v(x),$$

and that any morphism $a \in C^\infty(\mathcal{H}om(\mathcal{E}_*, \mathcal{F}_*))$ can be regarded as an element of the space $C^\infty(\mathcal{H}om((\mathcal{E}\otimes\Omega^\kappa)_*, (\mathcal{F}\otimes\Omega^\kappa)_*))$ (see Section 3).

Let $A_\kappa \in \Psi_{\varrho,\delta}^{r,\mathbf{d}}(\mathcal{E}\otimes\Omega^\kappa, \mathcal{F}\otimes\Omega^\kappa)$ be a ψDO with a τ-symbol $a \in S_{\varrho,\delta}^{r,\mathbf{d}}(\mathcal{E},\mathcal{F})$ defined with the help of (5.16). Then it is easy to see that the operator $A := N_\kappa A_\kappa N_{-\kappa}$ is a ψDO from $\Psi_{\varrho,\delta}^{r,\mathbf{d}}(\mathcal{E},\mathcal{F})$ with the same τ-symbol a.

Using the forms (5.14), (5.15) we can define the dual operator $A'_\kappa : C_0^\infty(\mathcal{F}'\otimes\Omega^{1-\kappa}) \to C^\infty(\mathcal{E}'\otimes\Omega^{1-\kappa})$ and the adjoint operator $A^*_\kappa : C_0^\infty(\mathcal{F}^*\otimes\Omega^{1-\kappa}) \to C^\infty(\mathcal{E}^*\otimes\Omega^{1-\kappa})$ by the formulae

$$\langle A_\kappa\omega, v\rangle_{\mathcal{F},\Omega} = \langle\omega, A'_\kappa v\rangle_{\mathcal{E},\Omega}, \qquad (A_\kappa\omega, w)_{\mathcal{F},\Omega} = (\omega, A^*_\kappa w)_{\mathcal{E},\Omega}$$

for any $\omega \in C_0^\infty(\mathcal{E}\otimes\Omega^\kappa)$, $v \in C_0^\infty(\mathcal{F}'\otimes\Omega^{1-\kappa})$, $w \in C_0^\infty(\mathcal{F}^*\otimes\Omega^{1-\kappa})$. Since

$$\begin{aligned}\langle A_\kappa\omega, v\rangle_{\mathcal{F},\Omega} &= \langle N_{-\kappa}AN_\kappa\omega, v\rangle_{\mathcal{F},\Omega} = \langle AN_\kappa\omega, N_{1-\kappa}v\rangle_{\mathcal{F},\mathcal{M}}\\ &= \langle N_\kappa\omega, A'N_{1-\kappa}v\rangle_{\mathcal{F},\mathcal{M}} = \langle\omega, N_{\kappa-1}A'N_{1-\kappa}v\rangle_{\mathcal{F},\Omega},\end{aligned}$$

we have $A'_\kappa = N_{\kappa-1}A'N_{1-\kappa}$. Similarly we obtain $A^*_\kappa = N_{\kappa-1}A^*N_{1-\kappa}$. Therefore

$$A'_\kappa \in \Psi^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{F}' \otimes \Omega^{1-\kappa}, \mathcal{E}' \otimes \Omega^{1-\kappa}), \qquad A^*_\kappa \in \Psi^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{F}^* \otimes \Omega^{1-\kappa}, \mathcal{E}^* \otimes \Omega^{1-\kappa})$$

and the s-presymbols of these ψDOs are given by (5.9), (5.10) (see also (5.11)–(5.13)).

5.3. Remark. Using the results of this section one can easily prove that any $A \in \Psi^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})$ admits an extension to a continuous operator from $E'(\mathcal{E}) = (C^\infty(\mathcal{E}^* \otimes \Omega))'$ to $\mathcal{D}'(\mathcal{F}) = (C_0^\infty(\mathcal{F}^* \otimes \Omega))'$. Suppose A is *properly supported*, i.e. for any $\varphi \in C_0^\infty(M)$ the Schwartz kernels of the ψDOs φA and $A\varphi I$ have compact supports. Here I is the identity operator. Then A maps $C_0^\infty(\mathcal{E})$ into $C_0^\infty(\mathcal{F})$ and can be extended to continuous operators from $C^\infty(\mathcal{E})$ to $C^\infty(\mathcal{F})$, from $E'(\mathcal{E})$ to $E'(\mathcal{F})$ and from $\mathcal{D}'(\mathcal{E})$ to $\mathcal{D}'(\mathcal{F})$ (cf. [Shu, Proposition 3.1]).

6. Composition of ψDOs

In the remaining part of the paper except Section 8, *we will always suppose that*

$$\varrho\left(\frac{1}{d_j} + \frac{1}{d_k}\right) > \frac{1}{d_m} \qquad \text{if } C^m_{j,k} \not\equiv 0 \tag{6.1}$$

(see (2.1)). For the applications we have in mind the most important class of symbols is $S^{r,\mathbf{d}}_{1,0}$. In this case (6.1) takes the form

$$\frac{1}{d_j} + \frac{1}{d_k} > \frac{1}{d_m} \qquad \text{if } C^m_{j,k} \not\equiv 0. \tag{6.2}$$

Since we regard ∂_{ν_j} as an operator of order d_j^{-1}, $\partial_{\nu_j}\partial_{\nu_k}$ and $\partial_{\nu_k}\partial_{\nu_j}$ are operators of order $d_j^{-1} + d_k^{-1}$ and (6.2) means that the commutator $[\partial_{\nu_j}, \partial_{\nu_k}]$ has a strictly lower order (see (2.1)).

Let

$$\varepsilon := \min\{d_l^{-1}, \varrho(d_j^{-1} + d_k^{-1}) - d_m^{-1} : l = 1, \ldots, n,\ C^m_{j,k} \not\equiv 0\}. \tag{6.3}$$

6.1. Lemma. *If* $|\alpha + \beta| \geq 2$ *and*

$$\partial^\alpha_{\nu(y)} \partial^\beta_{\nu(z)} c_j(y,z)|_{y=z=x} \neq 0, \tag{6.4}$$

then

$$\frac{1}{d_j} \leq \varrho(|(\alpha+\beta) : \mathbf{d}| - \varepsilon(|\alpha+\beta| - 1))$$
$$\leq \varrho(|(\alpha+\beta) : \mathbf{d}| - \varepsilon \max\{|\alpha|, |\beta|\}) \leq \varrho\left(|(\alpha+\beta) : \mathbf{d}| - \frac{\varepsilon}{2}|\alpha+\beta|\right).$$

Proof. It follows from (6.4) that at least one term

$$\partial^{\mu^{(1)}}_\nu C^j_{k_1,m_1}(x) \partial^{\mu^{(2)}}_\nu C^{j_2}_{k_2,m_2}(x) \ldots \partial^{\mu^{(q)}}_\nu C^{j_q}_{k_q,m_q}(x)$$

of at least one of the scalars (2.16) does not equal 0. Here each of the “upper” indices $j_2, \ldots, j_q$ coincides with one of the “lower” indices, which are k_l, m_l and those corresponding to the multi-indices $\mu^{(l)}$. The remaining

$$|\mu^{(1)}| + \ldots + |\mu^{(q)}| + q + 1 = |\alpha + \beta| \tag{6.5}$$

"lower" indices correspond to those of $\alpha+\beta$. Therefore

$$|(\alpha+\beta):\mathbf{d}| + \frac{1}{d_{j_2}} + \ldots + \frac{1}{d_{j_q}} = |\mu^{(1)}:\mathbf{d}| + \ldots + |\mu^{(q)}:\mathbf{d}| + \left(\frac{1}{d_{k_1}} + \frac{1}{d_{m_1}}\right) + \ldots + \left(\frac{1}{d_{k_q}} + \frac{1}{d_{m_q}}\right).$$

Using (6.1) we obtain

$$\begin{aligned}\varrho|(\alpha+\beta):\mathbf{d}| - \frac{1}{d_j} &= \varrho|(\alpha+\beta):\mathbf{d}| - d_j^{-1} + \varrho(d_{j_2}^{-1} - d_{j_2}^{-1}) + \ldots + \varrho(d_{j_q}^{-1} - d_{j_q}^{-1}) \\ &= \varrho|\mu^{(1)}:\mathbf{d}| + \ldots + \varrho|\mu^{(q)}:\mathbf{d}| + (\varrho(d_{k_1}^{-1} + d_{m_1}^{-1}) - d_j^{-1}) \\ &\quad + (\varrho(d_{k_2}^{-1} + d_{m_2}^{-1}) - \varrho d_{j_2}^{-1}) + \ldots + (\varrho(d_{k_q}^{-1} + d_{m_q}^{-1}) - \varrho d_{j_q}^{-1}) \\ &\geq \varrho\varepsilon(|\mu^{(1)}| + \ldots + |\mu^{(q)}|) + \varepsilon q \geq \varrho\varepsilon(|\alpha+\beta| - 1)\end{aligned}$$

(see (6.5)). It follows from (2.19), (2.20) that if (6.4) is satisfied and $|\alpha+\beta|\geq 2$, then $|\alpha|,|\beta|\geq 1$ and $|\alpha+\beta|-1\geq\max\{|\alpha|,|\beta|\}\geq|\alpha+\beta|/2$. ∎

Let

$$h_j(x,y,z) := c_j(z,x) + c_j(x,y) - c_j(z,y). \tag{6.6}$$

It follows from (2.11) and (2.12) that

$$\begin{aligned} h_j(x,y,x) = h_j(x,x,z) &= h_j(x,y,y) = 0, \\ \partial^\alpha_{\nu(y)} h_j(x,y,z)|_{z=x} &= \partial^\alpha_{\nu(y)} h_j(x,y,x) = 0, \\ \partial^\beta_{\nu(z)} h_j(x,y,z)|_{y=x} &= \partial^\beta_{\nu(z)} h_j(x,x,z) = 0. \end{aligned} \tag{6.7}$$

These equalities and Lemma 6.1 imply the following result.

6.2. Corollary. *If*

$$\partial^\alpha_{\nu(y)}\partial^\beta_{\nu(z)} h_j(x,y,z)|_{y=z=x} \neq 0,$$

then

$$\begin{aligned}\frac{1}{d_j} &\leq \varrho(|(\alpha+\beta):\mathbf{d}| - \varepsilon(|\alpha+\beta|-1)) \\ &\leq \varrho(|(\alpha+\beta):\mathbf{d}| - \varepsilon\max\{|\alpha|,|\beta|\}) \leq \varrho\left(|(\alpha+\beta):\mathbf{d}| - \frac{\varepsilon}{2}|\alpha+\beta|\right).\end{aligned} \tag{6.8}$$

Let

$$\psi(x,y,z) := c(x,y) - h(x,y,z) = c(z,y) - c(z,x), \tag{6.9}$$

i.e. $\psi_j(x,y,z) = c_j(x,y) - h_j(x,y,z)$, $j=1,\ldots,n$.

6.3. Lemma. *Suppose y and z are sufficiently close to x. Then for any $\alpha\in\mathbb{Z}^n_+$, $q\in\mathbb{Z}_+$ and $j=1,\ldots,n$ we have*

$$\begin{aligned}\partial^\alpha_{\tilde z} h_j(x,y,z) &= \sum_{|\beta'|+|\beta''|\leq q} H_{j,\alpha,\beta',\beta''}(x)\,\psi^{\beta'}(x,y,z)\,c^{\beta''}(x,z) \\ &\quad + \sum_{|\beta'|+|\beta''|=q+1} \widehat{H}_{j,\alpha,\beta',\beta''}(x,y,z)\,\psi^{\beta'}(x,y,z)\,c^{\beta''}(x,z), \qquad z=\exp_x(\tilde z),\end{aligned}$$

where $\widehat{H}_{j,\alpha,\beta',\beta''}$ *are* C^∞*-smooth functions,* $H_{j,\alpha,\beta',\beta''}(x)$ *are polynomials in* $\partial_\nu^\alpha C_{j,k}^m(x)$ *and* $H_{j,\alpha,\beta',\beta''}(x) \neq 0$ *implies*

$$\frac{1}{d_j} \le \varrho\Big(|(\alpha+\beta'+\beta''):\mathbf{d}| - \frac{\varepsilon}{2}|\alpha+\beta'+\beta''|\Big).$$

Proof. Let us use the Taylor expansion

$$\partial_{\tilde z}^\alpha h_j(x,y,z) = \sum_{|\alpha'|+|\alpha''|\le q} \frac{1}{\alpha'!\alpha''!}\partial_{\tilde y}^{\alpha'}\partial_{\tilde z}^{\alpha''+\alpha} h_j(x,y,z)|_{y=z=x} c^{\alpha'}(x,y)c^{\alpha''}(x,z)$$
$$+ \sum_{|\alpha'|+|\alpha''|=q+1} \widehat{G}_{j,\alpha,\alpha',\alpha''}(x,y,z)c^{\alpha'}(x,y)c^{\alpha''}(x,z),$$
$$y = \exp_x(\tilde y),\ z = \exp_x(\tilde z).$$

It follows from Corollary 6.2 and Lemma 2.2 that for all nonzero terms of the first sum we have

$$\frac{1}{d_j} \le \varrho\Big(|(\alpha+\alpha'+\alpha''):\mathbf{d}| - \frac{\varepsilon}{2}|\alpha+\alpha'+\alpha''|\Big). \tag{6.10}$$

After the substitution $c(x,y) = \psi(x,y,z) + h(x,y,z)$ each of these terms takes the form

$$\frac{1}{\alpha'!\alpha''!}\partial_{\tilde y}^{\alpha'}\partial_{\tilde z}^{\alpha''+\alpha} h_j(x,y,z)|_{y=z=x}\psi^{\alpha'}(x,y,z)c^{\alpha''}(x,z)$$
$$\sum_{\substack{\gamma'+\gamma''=\alpha' \\ \gamma''\neq 0}} \frac{1}{\gamma'!\gamma''!\alpha''!}\partial_{\tilde y}^{\alpha'}\partial_{\tilde z}^{\alpha''+\alpha} h_j(x,y,z)|_{y=z=x}\psi^{\gamma'}(x,y,z)h^{\gamma''}(x,y,z)c^{\alpha''}(x,z).$$

Using the Taylor expansions of $h_k(x,y,z)$ and Corollary 6.2 we can rewrite the last sum in the form

$$\sum_{|\gamma'|+|\mu'|+|\mu''|+|\alpha''|\le q} G_{j,\alpha,\alpha'',\gamma',\mu',\mu''}(x)\psi^{\gamma'}(x,y,z)c^{\mu'}(x,y)c^{\mu''+\alpha''}(x,z)$$
$$+ \sum_{|\gamma'|+|\mu'|+|\mu''|+|\alpha''|=q+1} \widehat{G}_{j,\alpha,\alpha'',\gamma',\mu',\mu''}(x,y,z)\psi^{\gamma'}(x,y,z)c^{\mu'}(x,y)c^{\mu''+\alpha''}(x,z),$$

where $\widehat{G}_{j,\alpha,\alpha'',\gamma',\mu',\mu''}$ are C^∞-smooth functions and $G_{j,\alpha,\alpha'',\gamma',\mu',\mu''}(x)$ are polynomials in $\partial_\nu^\alpha C_{j,k}^m(x)$ (see Lemmas 2.2, 2.3 and (6.6)). Corollary 6.2 implies that if $G_{j,\alpha,\alpha'',\gamma',\mu',\mu''}(x) \neq 0$ then

$$|\gamma'':\mathbf{d}| \le \varrho\Big(|(\mu'+\mu''):\mathbf{d}| - \frac{\varepsilon}{2}|\mu'+\mu''|\Big).$$

From this inequality, (6.10) and the equality $\gamma'+\gamma''=\alpha'$ we obtain

$$\frac{1}{d_j} \le \varrho\Big(|(\alpha+\gamma'+\alpha''):\mathbf{d}| - \frac{\varepsilon}{2}|\alpha+\gamma'+\alpha''|\Big) + |\gamma'':\mathbf{d}|$$
$$\le \varrho\Big(|(\alpha+\gamma'+\mu'+\mu''+\alpha''):\mathbf{d}| - \frac{\varepsilon}{2}|\alpha+\gamma'+\mu'+\mu''+\alpha''|\Big).$$

Since due to (6.7) the Taylor expansion of $h_k(x,y,z)$ does not contain first order terms, we have $|\mu'+\mu''| > |\gamma''|$ if $\gamma''\neq 0$, i.e.

$$|\gamma'+\mu'+\mu''+\alpha''| > |\alpha'+\alpha''|.$$

Using the equality $c(x,y) = \psi(x,y,z) + h(x,y,z)$ again and repeating the above procedure several times we end up with

$$\partial_z^\alpha h_j(x,y,z) = \sum_{|\beta'|+|\beta''|\le q} H_{j,\alpha,\beta',\beta''}(x)\psi^{\beta'}(x,y,z)c^{\beta''}(x,z) + \sum_{|\gamma|+|\mu|+|\beta''|=q+1} \widehat{H}_{j,\alpha,\gamma,\mu,\beta''}(x,y,z)\psi^\gamma(x,y,z)c^\mu(x,y)c^{\beta''}(x,z), \tag{6.11}$$

where $H_{j,\alpha,\beta',\beta''}(x)$ have the required properties and $\widehat{H}_{j,\alpha,\gamma,\mu,\beta''}$ are C^∞-smooth.

The equalities (6.7) imply

$$h(x,y,z) = \varGamma(x,y,z)c(x,y),$$

where $\varGamma$ is a C^∞-smooth matrix function and $\|\varGamma(x,y,z)\| \le 1/2$ if y and z are sufficiently close to x. Hence

$$\psi(x,y,z) = c(x,y) - h(x,y,z) \;\Rightarrow\; c(x,y) = (I - \varGamma(x,y,z))^{-1}\psi(x,y,z).$$

Plugging the last equality into (6.11) and taking into account that $(I-\varGamma)^{-1}$ is a C^∞-smooth matrix function we conclude the proof. ■

We define the $\mathbf{d}$-degree of a polynomial $p(\eta) = \sum_{|\gamma|\le N} c_\gamma \eta^\gamma$ by the equalities

$$\mathbf{d}(p) := \max_{c_\gamma \ne 0} |\gamma : \mathbf{d}|, \quad p \not\equiv 0, \qquad \mathbf{d}(0) := -\infty. \tag{6.12}$$

It follows from the obvious inequality

$$|\eta^\gamma| = |\eta_1|^{\gamma_1} \dots |\eta_n|^{\gamma_n} \le |\eta|_{\mathbf{d}}^{\gamma_1/d_1} \dots |\eta|_{\mathbf{d}}^{\gamma_n/d_n} = |\eta|_{\mathbf{d}}^{|\gamma:\mathbf{d}|}$$

(see (3.4)) that

$$|p(\eta)| \le \mathrm{const}\,(1 + |\eta|_{\mathbf{d}})^{\mathbf{d}(p)}. \tag{6.13}$$

It is clear that

$$\mathbf{d}(p_1 + p_2) \le \max\{\mathbf{d}(p_1), \mathbf{d}(p_2)\}, \qquad \mathbf{d}(\partial_\eta^\mu p) \le \mathbf{d}(p) - |\mu : \mathbf{d}|. \tag{6.14}$$

Suppose $\mathcal{J}$ is a C^∞-smooth complex vector bundle over M with a connection $\nabla^{\mathcal{J}}$ and $\varPhi^{\mathcal{J}}$ is the corresponding parallel displacement along geodesics. Let

$$\varUpsilon_\theta(x,y,z) := \varUpsilon^{-\theta}(x,y)\varUpsilon^{\theta-1}(x,z)\varUpsilon^\theta(z,y),$$

$$\begin{aligned} \mathcal{P}^{\mathcal{J}}_{\beta,\gamma}(x,\eta) :=& \sum_{\beta'+\beta''=\beta} \sum_{|\mu|\le|\beta|} \frac{\beta!}{\beta'!\beta''!} \frac{i^{-|\mu|}}{\mu!} \partial_\eta^\mu \partial_{\nu(z)}^{\beta''} \partial_{\nu(y)}^{\beta'+\gamma+\mu} \\ & \times \Big(\varUpsilon_\theta(x,y,z) \exp\Big(i\sum_{j=1}^n h_j(x,y,z)\eta_j\Big) \varPhi^{\mathcal{J}}_{x,z}\varPhi^{\mathcal{J}}_{z,y}\varPhi^{\mathcal{J}}_{y,x}\Big)\Big|_{y=z=x} \\ =& \sum_{\beta'+\beta''=\beta} \sum_{|\mu|\le|\beta|} \frac{\beta!}{\beta'!\beta''!} \frac{i^{-|\mu|}}{\mu!} \partial_\eta^\mu (\nabla^{\mathcal{J}}_{\nu(z)})^{\beta''} (\nabla^{\mathcal{J}}_{\nu(y)})^{\beta'+\gamma+\mu} \\ & \times \Big(\varUpsilon_\theta(x,y,z) \exp\Big(i\sum_{j=1}^n h_j(x,y,z)\eta_j\Big) \varPhi^{\mathcal{J}}_{z,y}\Big)\Big|_{y=z=x}, \qquad \beta,\gamma \in \mathbb{Z}^n_+ \end{aligned} \tag{6.15}$$

(see Lemma 2.5 and (2.27), (2.30)).

6.4. Remark. If $|\mu| > |\beta|$, $\beta' + \beta'' = \beta$, then

$$\partial_\eta^\mu \partial_{\nu(z)}^{\beta''} \partial_{\nu(y)}^{\beta'+\gamma+\mu} \Big(\Upsilon_\theta \exp\Big(i \sum_{j=1}^n h_j \eta_j \Big) \Phi_{x,z}^{\mathcal{J}} \Phi_{z,y}^{\mathcal{J}} \Phi_{y,x}^{\mathcal{J}} \Big) \Big|_{y=z=x} = 0. \tag{6.16}$$

Indeed,

$$\partial_\eta^\mu \exp\Big(i \sum_{j=1}^n h_j \eta_j \Big) = i^{|\mu|} h^\mu(x,y,z) \exp\Big(i \sum_{j=1}^n h_j \eta_j \Big)$$

and (6.7) implies (6.16), since $|\mu| > |\beta''|$.

The morphisms $\mathcal{P}_{\beta,\gamma}^{\mathcal{J}} \in C^\infty(\mathcal{H}om(\widetilde{\mathcal{J}}, \widetilde{\mathcal{J}}))$ (see Section 3 for the notation) are polynomials in η (see (6.7)). Their coefficients are linear forms in $(\nabla_{\nu(z)}^{\mathcal{J}})^{\alpha''} (\nabla_{\nu(y)}^{\mathcal{J}})^{\alpha'} \Phi_{z,y}^{\mathcal{J}}|_{y=z=x}$. Due to Lemma 2.3, (4.3) and (6.6) the coefficients of these forms are polynomials in $\partial_\nu^\alpha C_{j,k}^m(x)$.

It is clear that the definition (6.12) of the **d**-degree of a polynomial can be extended to polynomials with coefficients from an arbitrary algebra (or a ring). For the polynomials $\mathcal{P}_{\beta,\gamma}^{\mathcal{J}}$ we have the following result.

6.5. Lemma. *For any* $\beta, \gamma \in \mathbb{Z}_+^n$,

$$\mathbf{d}(\mathcal{P}_{\beta,\gamma}^{\mathcal{J}}) \le \varrho\Big(|(\beta+\gamma) : \mathbf{d}| - \frac{\varepsilon}{2} |\beta+\gamma| \Big),$$
$$\mathbf{d}(\mathcal{P}_{\beta,\gamma}^{\mathcal{J}}) \le \varrho(|(\beta+\gamma) : \mathbf{d}| - \varepsilon|\gamma|),$$
$$\mathbf{d}(\mathcal{P}_{\beta,\gamma}^{\mathcal{J}}) \le |\beta| \max\{d_k^{-1}\}.$$

If $|\beta+\gamma| = 3$, *then*

$$\mathbf{d}(\mathcal{P}_{\beta,\gamma}^{\mathcal{J}}) \le \varrho(|(\beta+\gamma) : \mathbf{d}| - 2\varepsilon).$$

Proof. A factor η_j can appear in a monomial of the polynomial

$$P_{\beta',\beta'',\gamma,\mu}(x,\eta) := \partial_{\nu(z)}^{\beta''} \partial_{\nu(y)}^{\beta'+\gamma+\mu} \Big(\Upsilon_\theta \exp\Big(i \sum_{j=1}^n h_j \eta_j \Big) \Phi_{x,z}^{\mathcal{J}} \Phi_{z,y}^{\mathcal{J}} \Phi_{y,x}^{\mathcal{J}} \Big) \Big|_{y=z=x}$$

only together with some partial derivative of h_j evaluated at $y = z = x$. The sum of multi-indices corresponding to these derivatives is $\le \beta' + \beta'' + \gamma + \mu = \beta + \gamma + \mu$ for every monomial. Using Corollary 6.2 one can easily prove that

$$\mathbf{d}(P_{\beta',\beta'',\gamma,\mu}) \le \varrho(|(\beta+\gamma+\mu) : \mathbf{d}| - \frac{\varepsilon}{2}|\beta+\gamma+\mu|) \le \varrho(|(\beta+\gamma) : \mathbf{d}| - \frac{\varepsilon}{2}|\beta+\gamma|) + |\mu : \mathbf{d}|,$$
$$\mathbf{d}(P_{\beta',\beta'',\gamma,\mu}) \le \varrho(|(\beta+\gamma+\mu) : \mathbf{d}| - \varepsilon|\beta'+\gamma+\mu|) \le \varrho(|(\beta+\gamma) : \mathbf{d}| - \varepsilon|\gamma|) + |\mu : \mathbf{d}|.$$

Now the first and the second inequalities of the lemma follow from (6.14).

Let $p_\alpha(x)\eta^\alpha$, $\alpha \in \mathbb{Z}_+^n$, be a monomial of $P_{\beta',\beta'',\gamma,\mu}$. Then (6.7) implies that $|\alpha| \le |\beta''| \le |\beta|$. Therefore $|\alpha : \mathbf{d}| \le |\beta| \max\{d_k^{-1}\}$ and we obtain the third inequality of the lemma.

Let $|\beta+\gamma| = 3$. If $\mu \ne 0$, we obtain as above

$$\mathbf{d}(P_{\beta',\beta'',\gamma,\mu}) \le \varrho(|(\beta+\gamma) : \mathbf{d}| - 2\varepsilon) + |\mu : \mathbf{d}|.$$

It follows from (6.7) that $P_{\beta',\beta'',\gamma,0}$ does not contain powers of η higher than 1. Therefore, we can apply the first inequality in (6.8), which gives us

$$\mathbf{d}(P_{\beta',\beta'',\gamma,0}) \le \varrho(|(\beta+\gamma):\mathbf{d}| - \varepsilon(|\beta+\gamma|-1)) = \varrho(|(\beta+\gamma):\mathbf{d}| - 2\varepsilon).$$

It is now left to apply (6.14). ∎

6.6. Proposition. *We have*

$$\mathcal{P}^{\mathcal{J}}_{0,0} \equiv 1, \quad \mathcal{P}^{\mathcal{J}}_{0,\gamma} \equiv 0, \quad \forall\gamma \ne 0, \quad \mathcal{P}^{\mathcal{J}}_{\beta,0} \equiv 0, \quad \forall\beta \ne 0. \tag{6.17}$$

For any $k, m = 1, \dots, n$,

$$\mathcal{P}^{\mathcal{J}}_{k,m}(x,\eta) = \frac{1}{2i}\sum_{j=1}^{n} C^{j}_{k,m}(x)\eta_j \tag{6.18}$$

modulo a "function" of x. (*Here and below we call elements of* $C^\infty(\mathcal{H}om(\mathcal{J},\mathcal{J}))$ *"functions" of* x *in order to emphasize that they do not depend on* η. *We also denote the identity automorphism by* 1.)

Proof. The first equality in (6.17) follows from (2.27), (4.5) and (6.7). The second one is a consequence of (2.27), (6.7) and the equality $\varUpsilon_\theta(x,y,x) = 1$. Let us prove the third equality in (6.17) [1].

Using Lemma 2.2 we obtain

$$\begin{aligned} &\mathcal{P}^{\mathcal{J}}_{\beta,\gamma}(x,\eta) \\ &= \sum_{\beta'+\beta''=\beta}\sum_{|\mu|\le|\beta|}\frac{\beta!}{\beta'!\beta''!}\frac{i^{-|\mu|}}{\mu!}\partial^{\mu}_{\eta}\partial^{\beta''}_{\widetilde z}\partial^{\beta'+\gamma+\mu}_{\widetilde y} \\ &\qquad\times\Big(\varUpsilon_\theta \exp\Big(i\sum_{j=1}^{n}h_j\eta_j\Big)\varPhi^{\mathcal{J}}_{x,z}\varPhi^{\mathcal{J}}_{z,y}\varPhi^{\mathcal{J}}_{y,x}\Big)\Big|_{y=z=x} \\ &= (\partial_{\widetilde z}+\partial_{\widetilde y})^{\beta}\sum_{|\mu|\le|\beta|}\frac{i^{-|\mu|}}{\mu!}\partial^{\mu}_{\eta}\partial^{\gamma+\mu}_{\widetilde y}\Big(\varUpsilon_\theta \exp\Big(i\sum_{j=1}^{n}h_j\eta_j\Big)\varPhi^{\mathcal{J}}_{x,z}\varPhi^{\mathcal{J}}_{z,y}\varPhi^{\mathcal{J}}_{y,x}\Big)\Big|_{y=z=x} \\ &= \partial^{\beta}_{\widetilde z}\Big(\sum_{|\mu|\le|\beta|}\frac{i^{-|\mu|}}{\mu!}\partial^{\gamma+\mu}_{\widetilde y}\partial^{\mu}_{\eta}\Big(\varUpsilon_\theta \exp\Big(i\sum_{j=1}^{n}h_j\eta_j\Big)\varPhi^{\mathcal{J}}_{x,z}\varPhi^{\mathcal{J}}_{z,y}\varPhi^{\mathcal{J}}_{y,x}\Big)\Big|_{y=z}\Big)\Big|_{z=x}, \\ &\qquad\qquad y=\exp_x(\widetilde y),\ z=\exp_x(\widetilde z). \end{aligned} \tag{6.19}$$

In particular

$$\begin{aligned} &\mathcal{P}^{\mathcal{J}}_{\beta,0}(x,\eta) \\ &= \partial^{\beta}_{\widetilde z}\Big(\sum_{|\mu|\le|\beta|}\frac{i^{-|\mu|}}{\mu!}\partial^{\mu}_{\widetilde y}\partial^{\mu}_{\eta}\Big(\varUpsilon_\theta \exp\Big(i\sum_{j=1}^{n}h_j\eta_j\Big)\varPhi^{\mathcal{J}}_{x,z}\varPhi^{\mathcal{J}}_{z,y}\varPhi^{\mathcal{J}}_{y,x}\Big)\Big|_{y=z}\Big)\Big|_{z=x}, \\ &\qquad\qquad y=\exp_x(\widetilde y),\ z=\exp_x(\widetilde z). \end{aligned} \tag{6.20}$$

[1] The idea of the proof is due to Yu. Safarov.

It follows from (6.7) and the equality $\varUpsilon_\theta(x,z,z) = \varUpsilon^{-1}(x,z)$ that for $\beta \neq 0$,

$$\begin{aligned}
(6.21)\qquad & \sum_{|\mu|\le|\beta|} \frac{i^{-|\mu|}}{\mu!}\partial^\mu_{\tilde y}\partial^\mu_\eta\Big(\varUpsilon_\theta \exp\Big(i\sum_{j=1}^n h_j\eta_j\Big)\varPhi^{\mathcal J}_{x,z}\varPhi^{\mathcal J}_{z,y}\varPhi^{\mathcal J}_{y,x}\Big)\Big|_{y=z} \\
&= \sum_{|\mu|\le|\beta|} \frac{1}{\mu!}\partial^\mu_{\tilde y}\Big(h^\mu\varUpsilon_\theta \exp\Big(i\sum_{j=1}^n h_j\eta_j\Big)\varPhi^{\mathcal J}_{x,z}\varPhi^{\mathcal J}_{z,y}\varPhi^{\mathcal J}_{y,x}\Big)\Big|_{y=z} \\
&= \varUpsilon^{-1}(x,z)\sum_{|\mu|\le|\beta|}\frac{1}{\mu!}\partial^\mu_{\tilde y}h^\mu(x,y,z)|_{y=z} \\
&= \varUpsilon^{-1}(x,z)\sum_{l=0}^{|\beta|}\frac{1}{l!}\sum_{1\le j_1,\dots,j_l\le n}\frac{\partial^l}{\partial\tilde y^{j_1}\dots\partial\tilde y^{j_l}}h_{j_1}\dots h_{j_l}\Big|_{y=z} \\
&= \varUpsilon^{-1}(x,z)\sum_{l=0}^{|\beta|}\frac{1}{l!}\sum_{1\le j_1,\dots,j_l\le n}\sum_{\sigma\in S_l}\partial_{\tilde y^{j_{\sigma(1)}}}h_{j_1}|_{y=z}\dots\partial_{\tilde y^{j_{\sigma(l)}}}h_{j_l}|_{y=z} \\
&= \varUpsilon^{-1}(x,z)\mathcal R_\beta(x,z),
\end{aligned}$$

where S_l is the symmetric group of degree l, i.e. the group of all permutations on $\{1,\dots,l\}$. Let $\varPsi(x,z) := (\partial_{\tilde y^k}h_j|_{y=z})_{n\times n}$. Since any permutation is a composition of disjoint cyclic permutations (see, e.g., [MB, Ch. II, Theorem 14]) and

$$\sum_{1\le k_1,\dots,k_q\le n}\partial_{\tilde y^{k_1}}h_{k_2}|_{y=z}\partial_{\tilde y^{k_2}}h_{k_3}|_{y=z}\dots\partial_{\tilde y^{k_{q-1}}}h_{k_q}|_{y=z}\partial_{\tilde y^{k_q}}h_{k_1}|_{y=z} = \operatorname{Tr}\varPsi^q(x,z),$$

we have

$$\mathcal R_\beta(x,z) = \sum_{0\le q_1+\dots+q_m\le|\beta|} c_{q_1,\dots,q_m}\operatorname{Tr}\varPsi^{q_1}(x,z)\dots\operatorname{Tr}\varPsi^{q_m}(x,z),$$

where $c_{q_1,\dots,q_m}$ are some constants depending only on $q_1,\dots,q_m,n$ and $|\beta|$. Therefore $\mathcal R_\beta(x,z)$ is a polynomial in the eigenvalues $\lambda_1=\lambda_1(x,z),\dots,\lambda_n=\lambda_n(x,z)$ of the matrix $\varPsi(x,z)$ whose coefficients are independent of $\varPsi(x,z)$. In order to find these coefficients we may assume that $\varPsi(x,z)$ is diagonal. In this case we have

$$\mathcal R_\beta(x,z) = \sum_{|\mu|\le|\beta|}\frac{1}{\mu!}\partial^\mu_{\tilde y}h^\mu(x,y,z)|_{y=z} = \sum_{0\le p_1+\dots+p_n\le|\beta|}\lambda_1^{p_1}\dots\lambda_n^{p_n}.$$

It is clear that this equality holds for any matrix $\varPsi(x,z)$.

From (2.18) and (6.6) we obtain

$$\partial_{\tilde y^k}h_j(x,y,z)|_{y=z} = \partial_{\tilde y^k}c_j(x,y)|_{y=z} - \partial_{\tilde y^k}c_j(z,y)|_{y=z} = \delta^j_k - \partial_{\tilde y^k}c_j(z,y)|_{y=z}.$$

The chain rule, (2.9), (2.18) and Lemma 2.2 imply

$$\begin{aligned}
\sum_{k=1}^n(\partial_{\nu_l(z)}c_k(x,z))\partial_{\tilde y^k}c_j(z,y)|_{y=z} &= \sum_{k=1}^n(\partial_{\nu_l(z)}\tilde z^k)\partial_{\tilde y^k}c_j(z,y)|_{y=z} \\
&= \Big(\sum_{k=1}^n(\partial_{\nu_l(y)}\tilde y^k)\partial_{\tilde y^k}c_j(z,y)\Big)\Big|_{y=z} = (\partial_{\nu_l(y)}c_j(z,y))|_{y=z} = \delta^j_l,
\end{aligned}$$

i.e. $(\partial_{\nu_l(z)} c_k(x,z))_{n\times n} = (I - \Psi(x,z))^{-1}$. Therefore

$$\Upsilon^{-1}(x,z) = \det(I - \Psi(x,z)) = (1-\lambda_1)\dots(1-\lambda_n)$$

(see (4.3)). Thus the LHS of (6.21) equals

$$(1-\lambda_1)\dots(1-\lambda_n) \sum_{0\le p_1+\ldots+p_n\le|\beta|} \lambda_1^{p_1}\dots\lambda_n^{p_n}.$$

By induction on n one can easily prove that this polynomial equals 1 modulo a polynomial in $\lambda_1,\dots,\lambda_n$ which contains only terms of degree higher than $|\beta|$. Since $\Psi(x,x)=0$ due to (6.7), we have $\lambda_m = O(|\widetilde{z}|)$, $m=1,\dots,n$. Hence the LHS of (6.21) is a C^∞-smooth function which equals 1 modulo $O(|\widetilde{z}|^{|\beta|+1})$. Now it follows from (6.20) that $\mathcal{P}^{\mathcal{J}}_{\beta,0}\equiv 0$, $\forall\beta\ne 0$.

Let us prove (6.18). It follows from (6.7) that

$$\sum_{s=1}^n \frac{1}{i}\partial_{\eta_s}\partial_{\widetilde{y}^s}(\partial_{\widetilde{z}^k}+\partial_{\widetilde{y}^k})\partial_{\widetilde{y}^m}\Big(\Upsilon_\theta \exp\Big(i\sum_{j=1}^n h_j\eta_j\Big)\Phi^{\mathcal{J}}_{x,z}\Phi^{\mathcal{J}}_{z,y}\Phi^{\mathcal{J}}_{y,x}\Big)\Big|_{y=z=x}$$
$$+\,\partial_{\widetilde{y}^k}\partial_{\widetilde{y}^m}\Big(\Upsilon_\theta \exp\Big(i\sum_{j=1}^n h_j\eta_j\Big)\Phi^{\mathcal{J}}_{x,z}\Phi^{\mathcal{J}}_{z,y}\Phi^{\mathcal{J}}_{y,x}\Big)\Big|_{y=z=x}$$

does not depend on η, i.e. is a "function" of x. Using (2.21) we obtain

$$\partial_{\widetilde{z}^k}\partial_{\widetilde{y}^m}\Big(\Upsilon_\theta \exp\Big(i\sum_{j=1}^n h_j\eta_j\Big)\Phi^{\mathcal{J}}_{x,z}\Phi^{\mathcal{J}}_{z,y}\Phi^{\mathcal{J}}_{y,x}\Big)\Big|_{y=z=x} = -\frac{i}{2}\sum_{j=1}^n C^j_{k,m}(x)\eta_j$$

modulo a "function" of x. Now (6.18) follows from (6.19). ∎

Let $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{J}$ be C^∞-smooth complex vector bundles over M and let $A\in\Psi^{r_1,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})$, $B\in\Psi^{r_2,\mathbf{d}}_{\varrho,\delta}(\mathcal{J},\mathcal{E})$. In order to be able to define the composition $AB: C_0^\infty(\mathcal{J})\to C^\infty(\mathcal{F})$ we need at least one of the ψDOs A and B to be properly supported (see Remark 5.3).

6.7. Theorem. *Let $A\in\Psi^{r_1,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})$, $B\in\Psi^{r_2,\mathbf{d}}_{\varrho,\delta}(\mathcal{J},\mathcal{E})$, $r_1,r_2\in\mathbb{R}$, $0\le\delta<\varrho\le 1$. Suppose at least one of these ψDOs is properly supported and* (6.1) *is satisfied. Then $AB\in\Psi^{r_1+r_2,\mathbf{d}}_{\varrho,\delta}(\mathcal{J},\mathcal{F})$ and*

$$\widetilde{\sigma}_{AB}(x,\eta)\sim\sum_{|\alpha|,|\beta|,|\gamma|=0}^{\infty}\frac{i^{-(|\alpha|+|\beta|+|\gamma|)}}{\alpha!\beta!\gamma!}\partial_\eta^{\beta+\alpha}\widetilde{\sigma}_A(x,\eta)\partial_\eta^\gamma(\nabla^{\mathcal{E},\mathcal{J}}_{\nu(x)})^\alpha\widetilde{\sigma}_B(x,\eta)\mathcal{P}^{\mathcal{J}}_{\beta,\gamma}(x,\eta). \tag{6.22}$$

6.8. Remark. Lemma 6.5 implies that the terms on the RHS of (6.22) form an asymptotic series (see also (6.13)).

6.9. Remark. By the definition (6.15) the polynomials $\mathcal{P}^{\mathcal{J}}_{\beta,\gamma}$ do not depend on the vector bundles $\mathcal{E}$ and $\mathcal{F}$. Suppose $\mathcal{E}=\mathcal{F}$ or $\mathcal{E}=\mathcal{J}$. Substituting $A=I$ or $B=I$ (i.e. $\widetilde{\sigma}_A\equiv I$ or $\widetilde{\sigma}_B\equiv I$) in (6.22), we obtain (6.17).

Proof of Theorem 6.7. Let $\widetilde{\sigma}_{A,B}\in\widetilde{S}^{r_1+r_2,\mathbf{d}}_{\varrho,\delta}(\mathcal{J},\mathcal{F})$ be a morphism satisfying (6.22). The existence of such $\widetilde{\sigma}_{A,B}$ can be proved in the same way as in the standard calculus of ψDOs (see, e.g., [Shu, Proposition 3.5]). The only difference is that we have to use the mapping $\eta\mapsto(\tau^{-1/d_1}\eta_1,\dots,\tau^{-1/d_n}\eta_n)$ instead of $\eta\mapsto\tau^{-1}\eta$. Our aim is to prove that AB is a ψDO with the presymbol $\widetilde{\sigma}_{A,B}$.

Using (4.13) with $\tau = 0$ for A and $\tau = 1$ for B we obtain

$$(6.23)\qquad (AB\omega)(x) = \int_M S(x,y)\Phi^{\mathcal{J}}_{x,y}\omega(y)\Upsilon^{\theta}(x,y)\chi_0(x,y)d\mathcal{M}(y) + \int_M K_{A,B}(x,y)\omega(y)\,d\mathcal{M}(y), \qquad \forall\omega \in C_0^\infty(\mathcal{E}),$$

where $K_{A,B} \in C^\infty(\mathcal{H}om_{M\times M}(\mathcal{J},\mathcal{F}))$ and the first term is understood as an oscillatory integral with

$$S(x,y) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n}\int_M\int_{\mathbb{R}^n} e^{-i\langle c(x,z),\eta'\rangle}\widetilde{\sigma}_A(x,\eta')\Phi^{\mathcal{E}}_{x,z}e^{-i\langle c(z,y),\eta\rangle}\Phi^{\mathcal{E}}_{z,y} \times \widetilde{\sigma}_{B,1}(y,\eta)\Phi^{\mathcal{J}}_{y,x}\Upsilon^{-\theta}(x,y)\Upsilon^{\theta}(z,y)\Upsilon^{\theta}(x,z)\chi(z,y)\chi(x,z)\,d\eta\,d\mathcal{M}(z)\,d\eta'$$

(see (2.27)). The cut-off function $\chi_0 \in C^\infty(M\times M)$ equals 1 in some neighbourhood of the diagonal of $M\times M$ and satisfies the equality $\chi_0\chi = \chi_0$.

Under the change of variables $\eta' = \widetilde{\eta} + \eta$ the last integral takes the form

$$S(x,y) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n}\int_M\int_{\mathbb{R}^n} e^{-i\langle c(x,z),\widetilde{\eta}\rangle}e^{-i\langle c(x,y),\eta\rangle} \times \widetilde{\sigma}_A(x,\eta+\widetilde{\eta})\exp\Big(i\sum_{j=1}^n h_j(x,y,z)\eta_j\Big)\Phi^{\mathcal{E}}_{x,z}\Phi^{\mathcal{E}}_{z,y}\widetilde{\sigma}_{B,1}(y,\eta)\Phi^{\mathcal{J}}_{y,x} \times \Upsilon^{-\theta}(x,y)\Upsilon^{\theta}(z,y)\Upsilon^{\theta}(x,z)\chi(z,y)\chi(x,z)\,d\eta\,d\mathcal{M}(z)\,d\widetilde{\eta}$$

(see (2.12) and (6.6)). Let $z = \exp_x(\widetilde{z})$. Then due to (2.9), (4.2) and (4.3) the last integral equals

$$S(x,y) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle \widetilde{z},\widetilde{\eta}\rangle}e^{-i\langle c(x,y),\eta\rangle}\widetilde{\sigma}_A(x,\eta+\widetilde{\eta}) \times \Upsilon_\theta(x,y,z)\exp\Big(i\sum_{j=1}^n h_j\eta_j\Big)\Phi^{\mathcal{E}}_{x,z}\Phi^{\mathcal{E}}_{z,y}\widetilde{\sigma}_{B,1}(y,\eta)\Phi^{\mathcal{J}}_{y,x}\chi(z,y)\chi(x,z)\,d\eta\,d\widetilde{z}\,d\widetilde{\eta}.$$

Taking the Taylor expansion of $\widetilde{\sigma}_A(x,\eta+\widetilde{\eta})$ at the point $\widetilde{\eta} = 0$ we obtain

$$(6.24)\qquad S(x,y) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle \widetilde{z},\widetilde{\eta}\rangle}e^{-i\langle c(x,y),\eta\rangle} \times \sum_{|\alpha|\le N}\frac{1}{\alpha!}\widetilde{\eta}^\alpha\partial^\alpha_\eta\widetilde{\sigma}_A(x,\eta)\Upsilon_\theta(x,y,z)\exp\Big(i\sum_{j=1}^n h_j\eta_j\Big)\Phi^{\mathcal{E}}_{x,z}\Phi^{\mathcal{E}}_{z,y} \times \widetilde{\sigma}_{B,1}(y,\eta)\Phi^{\mathcal{J}}_{y,x}\chi(z,y)\chi(x,z)\,d\eta\,d\widetilde{z}\,d\widetilde{\eta} + \frac{N+1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle \widetilde{z},\widetilde{\eta}\rangle}e^{-i\langle c(x,y),\eta\rangle}\sum_{|\alpha|=N+1}\frac{1}{\alpha!}\widetilde{\eta}^\alpha \times \Big(\int_0^1(1-t)^N\partial^\alpha_\eta\widetilde{\sigma}_A(x,\eta+t\widetilde{\eta})\,dt\Big)\Upsilon_\theta(x,y,z)\exp\Big(i\sum_{j=1}^n h_j\eta_j\Big) \times \Phi^{\mathcal{E}}_{x,z}\Phi^{\mathcal{E}}_{z,y}\widetilde{\sigma}_{B,1}(y,\eta)\Phi^{\mathcal{J}}_{y,x}\chi(z,y)\chi(x,z)\,d\eta\,d\widetilde{z}\,d\widetilde{\eta} = S_N(x,y) + R_N(x,y).$$

Due to the presence of the factor $\chi_0(x,y)$ in the first term on the RHS of (6.23) we can suppose that y is sufficiently close to x. In this case integration with respect to $\widetilde{z}$ and $\widetilde{\eta}$ gives

$$S_N(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\langle c(x,y),\eta\rangle} \sum_{|\alpha|\le N} \frac{i^{-|\alpha|}}{\alpha!} Q_\alpha(y;x,\eta)d\eta,$$

where

$$Q_\alpha(y;x,\eta) = \partial_\eta^\alpha \widetilde{\sigma}_A(x,\eta) P_\alpha(y;x,\eta) \Phi^{\mathcal{E}}_{x,y} \widetilde{\sigma}_{B,1}(y,\eta) \Phi^{\mathcal{J}}_{y,x}$$

and

$$P_\alpha(y;x,\eta) := \partial_{\widetilde{z}}^\alpha \Big(\Upsilon_\theta(x,y,z) \exp\Big(i \sum_{j=1}^n h_j \eta_j \Big) \Phi^{\mathcal{E}}_{x,z} \Phi^{\mathcal{E}}_{z,y} \Phi^{\mathcal{E}}_{y,x} \Big)\Big|_{z=x}$$

is a polynomial in η (see (2.27) and (6.7)). Therefore according to Theorem 4.4 the operator defined by the oscillatory integral

$$\int_M S_N(x,y) \Phi^{\mathcal{J}}_{x,y} \omega(y) \Upsilon^\theta(x,y) \chi_0(x,y)\, d\mathcal{M}(y) \tag{6.25}$$

is a ψDO with the presymbol

$$\begin{aligned}
(6.26)\quad \widetilde{\sigma}_{(N)}(x,\eta) &\sim \sum_{|\mu|=0}^{\infty} \frac{i^{-|\mu|}}{\mu!} \\
&\quad \times \partial_\eta^\mu \partial_{\widetilde{y}}^\mu \sum_{|\alpha|\le N} \frac{i^{-|\alpha|}}{\alpha!} \partial_\eta^\alpha \widetilde{\sigma}_A(x,\eta) (P_\alpha(y;x,\eta) \Phi^{\mathcal{E}}_{x,y} \widetilde{\sigma}_{B,1}(y,\eta) \Phi^{\mathcal{J}}_{y,x})|_{y=x} \\
&= \sum_{|\alpha|\le N} \sum_{\mu'+\mu''=\beta+\gamma'+\gamma''} \frac{i^{-(|\alpha|+|\beta|+|\gamma'|+|\gamma''|)}}{\alpha!\beta!\gamma'!\gamma''!} \frac{(\mu'+\mu'')!}{\mu'!\mu''!} \partial_\eta^{\beta+\alpha} \widetilde{\sigma}_A(x,\eta) \\
&\quad \times \partial_\eta^{\gamma'} \partial_{\widetilde{y}}^{\mu'} P_\alpha(y;x,\eta)|_{y=x} \partial_\eta^{\gamma''} (\nabla^{\mathcal{E},\mathcal{J}}_{\nu(x)})^{\mu''} \widetilde{\sigma}_{B,1}(x,\eta) \\
&= \sum_{|\alpha|\le N} \sum_{|\beta|=0}^{\infty} \frac{i^{-(|\alpha|+|\beta|)}}{\alpha!\beta!} \partial_\eta^{\beta+\alpha} \widetilde{\sigma}_A(x,\eta) \\
&\quad \times \sum_{\mu'+\mu''-\gamma'-\gamma''=\beta} \frac{i^{-(|\gamma'|+|\gamma''|)}}{\gamma'!\gamma''!} \frac{(\mu'+\mu'')!}{\mu'!\mu''!} \partial_\eta^{\gamma'} \partial_{\widetilde{y}}^{\mu'} P_\alpha(y;x,\eta)|_{y=x} \\
&\quad \times \partial_\eta^{\gamma''} (\nabla^{\mathcal{E},\mathcal{J}}_{\nu(x)})^{\mu''} \widetilde{\sigma}_{B,1}(x,\eta)
\end{aligned}$$

(see Lemmas 2.2, 2.5, 2.6 and (2.31)). Similarly to Lemma 6.5 one can prove that

$$\mathbf{d}(\partial_\eta^{\gamma'} \partial_{\widetilde{y}}^{\mu'} P_\alpha(y;x,\eta)|_{y=x}) \le \varrho\Big(|(\alpha+\mu') : \mathbf{d}| - \frac{\varepsilon}{2}|\alpha+\mu'| \Big) - |\gamma' : \mathbf{d}|.$$

Hence, the terms in (6.26) belong to $\widetilde{S}^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{J},\mathcal{F})$, where

$$\begin{aligned}
(6.27)\qquad r &\le r_1 - \varrho|(\beta+\alpha) : \mathbf{d}| \\
&\quad + \varrho\Big(|(\alpha+\mu') : \mathbf{d}| - \frac{\varepsilon}{2}|\alpha+\mu'| \Big) - |\gamma' : \mathbf{d}| + r_2 - \varrho|\gamma'' : \mathbf{d}| + \delta|\mu'' : \mathbf{d}|
\end{aligned}$$

$$\le r_1 + r_2 - \frac{\varrho\varepsilon}{2}|\alpha + \mu'| - (\varrho - \delta)|\mu'' : \mathbf{d}|$$
$$- \varrho(|\beta : \mathbf{d}| - |\mu' : \mathbf{d}| + |\gamma' : \mathbf{d}| + |\gamma'' : \mathbf{d}| - |\mu'' : \mathbf{d}|)$$
$$\le r_1 + r_2 - \min\left\{\varrho - \delta, \frac{\varrho}{2}\right\}\varepsilon(|\alpha| + |\mu'| + |\mu''|),$$

since $\mu' + \mu'' - \gamma' - \gamma'' = \beta$. Thus,

$$\widetilde{\sigma}_{(N)}(x,\eta) \sim \sum_{|\alpha|,|\beta|=0}^{\infty} \frac{i^{-(|\alpha|+|\beta|)}}{\alpha!\beta!}\partial_\eta^{\beta+\alpha}\widetilde{\sigma}_A(x,\eta) \times \sum_{\mu'+\mu''-\gamma'-\gamma''=\beta} \frac{i^{-(|\gamma'|+|\gamma''|)}}{\gamma'!\gamma''!}\frac{(\mu'+\mu'')!}{\mu'!\mu''!}\partial_\eta^{\gamma'}\partial_{\widetilde{y}}^{\mu'}P_\alpha(y;x,\eta)|_{y=x} \times \partial_\eta^{\gamma''}(\nabla_{\nu(x)}^{\mathcal{E},\mathcal{J}})^{\mu''}\widetilde{\sigma}_{B,1}(x,\eta) + \widetilde{q}_N(x,\eta), \tag{6.28}$$

where

$$\widetilde{q}_N(x,\eta) \sim -\sum_{|\alpha|\ge N+1}\sum_{|\beta|=0}^{\infty} \frac{i^{-(|\alpha|+|\beta|)}}{\alpha!\beta!}\partial_\eta^{\beta+\alpha}\widetilde{\sigma}_A(x,\eta) \times \sum_{\mu'+\mu''-\gamma'-\gamma''=\beta} \frac{i^{-(|\gamma'|+|\gamma''|)}}{\gamma'!\gamma''!}\frac{(\mu'+\mu'')!}{\mu'!\mu''!}\partial_\eta^{\gamma'}\partial_{\widetilde{y}}^{\mu'}P_\alpha(y;x,\eta)|_{y=x} \times \partial_\eta^{\gamma''}(\nabla_{\nu(x)}^{\mathcal{E},\mathcal{J}})^{\mu''}\widetilde{\sigma}_{B,1}(x,\eta), \tag{6.29}$$

$$\widetilde{q}_N \in \widetilde{S}^{r_N,\mathbf{d}}_{\varrho,\delta}(\mathcal{J},\mathcal{F}),\ r_N \le r_1 + r_2 - \min\left\{\varrho - \delta, \frac{\varrho}{2}\right\}\varepsilon(N+1).$$

Let us prove that the first term on the RHS of (6.28) equals $\widetilde{\sigma}_{A,B}$ (i.e. the RHS of (6.22)) modulo $\widetilde{S}^{-\infty}$. Suppose A is a differential operator, i.e. $\widetilde{\sigma}_A(x,\eta)$ is a polynomial in η. We can represent AB in the form (6.23), where

$$S(x,y) = \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\int_M\int_{\mathbb{R}^n} e^{-i\langle c(x,z),\eta'\rangle}\widetilde{\sigma}_A(x,\eta')\Phi^{\mathcal{E}}_{x,z}e^{-i\langle c(z,y),\eta\rangle}\widetilde{\sigma}_B(z,\eta) \times \Phi^{\mathcal{J}}_{z,y}\Phi^{\mathcal{J}}_{y,x}\Upsilon^{-\theta}(x,y)\Upsilon^{\theta}(z,y)\Upsilon^{\theta}(x,z)\chi(z,y)\chi(x,z)\,d\eta\,d\mathcal{M}(z)\,d\eta'.$$

Acting as above we obtain

$$S(x,y) = \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i\langle\widetilde{z},\widetilde{\eta}\rangle}e^{-i\langle c(x,y),\eta\rangle}\sum_\alpha\frac{1}{\alpha!}\widetilde{\eta}^\alpha\partial_\eta^\alpha\widetilde{\sigma}_A(x,\eta)\Phi^{\mathcal{E}}_{x,z}\widetilde{\sigma}_B(z,\eta)\Phi^{\mathcal{J}}_{z,y}\Phi^{\mathcal{J}}_{y,x} \times \exp\left(i\sum_{j=1}^n h_j\eta_j\right)\Upsilon_\theta(x,y,z)\chi(z,y)\chi(x,z)\,d\eta\,d\widetilde{z}\,d\widetilde{\eta}.$$

Note that here we have a finite sum, since $\widetilde{\sigma}_A(x,\eta)$ is a polynomial in η. Further,

$$S(x,y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i\langle c(x,y),\eta\rangle}\sum_\alpha\frac{i^{-|\alpha|}}{\alpha!}\partial_\eta^\alpha\widetilde{\sigma}_A(x,\eta) \times \partial_{\widetilde{z}}^\alpha\left(\Phi^{\mathcal{E}}_{x,z}\widetilde{\sigma}_B(z,\eta)\Phi^{\mathcal{J}}_{z,y}\Phi^{\mathcal{J}}_{y,x}\Upsilon_\theta(x,y,z)\exp\left(i\sum_{j=1}^n h_j\eta_j\right)\right)\Big|_{z=x}d\eta.$$

Using Lemmas 2.2, 2.5, 2.6, Theorem 4.4, Remark 6.4 and (2.27), (2.31) we can prove as above that AB is a ψDO with the presymbol

$$\widetilde{\sigma}(x,\eta) \sim \sum_{|\mu|=0}^{\infty} \frac{i^{-|\mu|}}{\mu!} \partial_\eta^\mu \partial_{\widetilde{y}}^\mu \sum_{\alpha} \frac{i^{-|\alpha|}}{\alpha!} \sum_{\alpha'+\alpha''=\alpha} \frac{\alpha!}{\alpha'!\alpha''!} \partial_\eta^\alpha \widetilde{\sigma}_A(x,\eta)$$
$$\times (\nabla_{\nu(x)}^{\mathcal{E},\mathcal{J}})^{\alpha'} \widetilde{\sigma}_B(x,\eta) \Big(\partial_{\widetilde{z}}^{\alpha''} \Big(\Upsilon_\theta \exp\Big(i \sum_{j=1}^n h_j \eta_j\Big) \Phi_{x,z}^{\mathcal{J}} \Phi_{z,y}^{\mathcal{J}} \Phi_{y,x}^{\mathcal{J}} \Big)\Big|_{z=x} \Big)\Big|_{y=x}$$
$$\sim \sum_{|\mu|,|\alpha'|,|\alpha''|=0}^{\infty} \frac{i^{-(|\alpha'|+|\alpha''|+|\mu|)}}{\alpha'!\alpha''!\mu!} \partial_\eta^\mu \Big\{ \partial_\eta^{\alpha'+\alpha''} \widetilde{\sigma}_A(x,\eta) (\nabla_{\nu(x)}^{\mathcal{E},\mathcal{J}})^{\alpha'} \widetilde{\sigma}_B(x,\eta)$$
$$\times \partial_{\widetilde{y}}^\mu \partial_{\widetilde{z}}^{\alpha''} \Big(\Upsilon_\theta \exp\Big(i \sum_{j=1}^n h_j \eta_j\Big) \Phi_{x,z}^{\mathcal{J}} \Phi_{z,y}^{\mathcal{J}} \Phi_{y,x}^{\mathcal{J}} \Big)\Big|_{z=x} \Big\}\Big|_{y=x}$$
$$= \sum_{|\alpha'|,|\alpha''|,|\mu|,|\gamma'|,|\gamma''|=0}^{\infty} \frac{i^{-(|\alpha'|+|\alpha''|+|\mu|+|\gamma'|+|\gamma''|)}}{\alpha'!\alpha''!\mu!\gamma'!\gamma''!} \partial_\eta^{\alpha'+\alpha''+\gamma'} \widetilde{\sigma}_A(x,\eta)$$
$$\times \partial_\eta^{\gamma''} (\nabla_{\nu(x)}^{\mathcal{E},\mathcal{J}})^{\alpha'} \widetilde{\sigma}_B(x,\eta) \partial_\eta^\mu \partial_{\widetilde{y}}^{\mu+\gamma'+\gamma''} \partial_{\widetilde{z}}^{\alpha''} \Big(\Upsilon_\theta \exp\Big(i \sum_{j=1}^n h_j \eta_j\Big) \Phi_{x,z}^{\mathcal{J}} \Phi_{z,y}^{\mathcal{J}} \Phi_{y,x}^{\mathcal{J}} \Big)\Big|_{y=z=x}$$
$$= \sum_{|\alpha|,|\beta'|,|\beta''|,|\gamma|,|\mu|=0}^{\infty} \frac{i^{-(|\alpha|+|\beta'|+|\beta''|+|\gamma|+|\mu|)}}{\alpha!\beta'!\beta''!\gamma!\mu!} \partial_\eta^{\alpha+\beta'+\beta''} \widetilde{\sigma}_A(x,\eta)$$
$$\times \partial_\eta^\gamma (\nabla_{\nu(x)}^{\mathcal{E},\mathcal{J}})^{\alpha} \widetilde{\sigma}_B(x,\eta) \partial_\eta^\mu \partial_{\widetilde{z}}^{\beta''} \partial_{\widetilde{y}}^{\beta'+\gamma+\mu} \Big(\Upsilon_\theta \exp\Big(i \sum_{j=1}^n h_j \eta_j\Big) \Phi_{x,z}^{\mathcal{J}} \Phi_{z,y}^{\mathcal{J}} \Phi_{y,x}^{\mathcal{J}} \Big)\Big|_{y=z=x}$$
$$= \sum_{|\alpha|,|\beta|,|\gamma|=0}^{\infty} \frac{i^{-(|\alpha|+|\beta|+|\gamma|)}}{\alpha!\beta!\gamma!} \partial_\eta^{\beta+\alpha} \widetilde{\sigma}_A(x,\eta) \partial_\eta^\gamma (\nabla_{\nu(x)}^{\mathcal{E},\mathcal{J}})^{\alpha} \widetilde{\sigma}_B(x,\eta) \mathcal{P}_{\beta,\gamma}^{\mathcal{J}}(x,\eta).$$

On the other hand, we have in fact proved above that if A is a differential operator then AB is a ψDO with the presymbol

$$\widetilde{\sigma}(x,\eta) \sim \sum_{|\alpha|,|\beta|=0}^{\infty} \frac{i^{-(|\alpha|+|\beta|)}}{\alpha!\beta!} \partial_\eta^{\beta+\alpha} \widetilde{\sigma}_A(x,\eta)$$
$$\times \sum_{\mu'+\mu''-\gamma'-\gamma''=\beta} \frac{i^{-(|\gamma'|+|\gamma''|)}}{\gamma'!\gamma''!} \frac{(\mu'+\mu'')!}{\mu'!\mu''!} \partial_\eta^{\gamma'} \partial_{\widetilde{y}}^{\mu'} P_\alpha(y;x,\eta)|_{y=x}$$
$$\times \partial_\eta^{\gamma''} (\nabla_{\nu(x)}^{\mathcal{E},\mathcal{J}})^{\mu''} \widetilde{\sigma}_{B,1}(x,\eta).$$

Since $\widetilde{\sigma}_A(x,\eta)$ is an arbitrary polynomial, we conclude that for any $\varsigma \in \mathbb{Z}_+^n$,

$$\sum_{\alpha+\beta=\varsigma} \frac{1}{\alpha!\beta!} \sum_{\mu'+\mu''-\gamma'-\gamma''=\beta} \frac{i^{-(|\gamma'|+|\gamma''|)}}{\gamma'!\gamma''!} \frac{(\mu'+\mu'')!}{\mu'!\mu''!} \partial_\eta^{\gamma'} \partial_{\widetilde{y}}^{\mu'} P_\alpha(y;x,\eta)|_{y=x}$$
$$\times \partial_\eta^{\gamma''} (\nabla_{\nu(x)}^{\mathcal{E},\mathcal{J}})^{\mu''} \widetilde{\sigma}_{B,1}(x,\eta)$$
$$= \sum_{\alpha+\beta=\varsigma} \frac{1}{\alpha!\beta!} \sum_{|\gamma|=0}^{\infty} \frac{i^{-|\gamma|}}{\gamma!} \partial_\eta^\gamma (\nabla_{\nu(x)}^{\mathcal{E},\mathcal{J}})^{\alpha} \widetilde{\sigma}_B(x,\eta) \mathcal{P}_{\beta,\gamma}^{\mathcal{J}}(x,\eta) \tag{6.30}$$

modulo $\widetilde{S}^{-\infty}$.

Let us return to the case when A is a ψDO. It follows from (6.28), (6.30) that (6.25) is the sum of ψDOs with the presymbols $\widetilde{\sigma}_{A,B}$ and $\widetilde{q}_N$ and an operator with the kernel from $C^\infty(\mathcal{H}om_{M\times M}(\mathcal{J},\mathcal{F}))$. Consequently, AB is the sum of ψDOs with the presymbols $\widetilde{\sigma}_{A,B}$ and $\widetilde{q}_N$, an operator with the kernel from $C^\infty(\mathcal{H}om_{M\times M}(\mathcal{J},\mathcal{F}))$ and the operator defined by the oscillatory integral

$$\int_M R_N(x,y)\Phi^{\mathcal{J}}_{x,y}\omega(y)\Upsilon^\theta(x,y)\chi_0(x,y)\,d\mathcal{M}(y)$$

(see (6.23), (6.24)). Due to (6.29) it is now left to prove that for any given $L\in\mathbb{N}$ the kernel $R_N(x,y)$ is C^L-smooth if N is sufficiently large.

Let us take an arbitrary compact set $K\subset M$ and suppose $x\in K$. Using (6.9), the formula

$$\widetilde{\eta}^\alpha e^{-i\langle\widetilde{z},\widetilde{\eta}\rangle} = i^{|\alpha|}\partial^\alpha_{\widetilde{z}} e^{-i\langle\widetilde{z},\widetilde{\eta}\rangle}$$

and integrating by parts, from (6.24) we obtain

$$\begin{aligned} R_N(x,y) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_0^1 &(1-t)^N e^{-i\langle\widetilde{z},\widetilde{\eta}\rangle} e^{-i\langle\psi(x,y,z),\eta\rangle} \\ &\times\sum_{|\alpha|=N+1}\partial^\alpha_\eta\widetilde{\sigma}_A(x,\eta+t\widetilde{\eta})\Pi_\alpha(x,y,z,\eta)\Phi^{\mathcal{E}}_{x,y}\widetilde{\sigma}_{B,1}(y,\eta)\Phi^{\mathcal{J}}_{y,x}\,dt\,d\eta\,d\widetilde{z}\,d\widetilde{\eta}, \end{aligned}$$

where $\Pi_\alpha(x,y,z,\eta)$ is a linear combination of terms of the form

$$\Pi_{\alpha^{(1)},\dots,\alpha^{(l)}}(x,y,z)(\partial^{\alpha^{(1)}}_{\widetilde{z}}h_{j_1}(x,y,z)\eta_{j_1})\dots(\partial^{\alpha^{(l)}}_{\widetilde{z}}h_{j_l}(x,y,z)\eta_{j_l})$$

with C^∞-smooth $\Pi_{\alpha^{(1)},\dots,\alpha^{(l)}}$ and $\alpha^{(1)}+\dots+\alpha^{(l)}\le\alpha$. Here the inequality between multi-indices is understood component-wise and $\Pi_{\alpha^{(1)},\dots,\alpha^{(l)}}(x,y,z)\in\mathrm{Hom}(\mathcal{E}_x,\mathcal{E}_x)$. Since $\Pi_{\alpha^{(1)},\dots,\alpha^{(l)}}(x,y,z)$ contains factors $\chi(z,y)$ and $\chi(x,z)$ or their derivatives it can be nonzero only if y and z are sufficiently close to x. So, we can apply Lemma 6.3 with q which will be chosen later (see (6.42) below). Using the formulae

$$\begin{aligned} \psi^{\beta'}(x,y,z)e^{-i\langle\psi(x,y,z),\eta\rangle} &= i^{|\beta'|}\partial^{\beta'}_\eta e^{-i\langle\psi(x,y,z),\eta\rangle}, \\ c^{\beta''}(x,z)e^{-i\langle\widetilde{z},\widetilde{\eta}\rangle} = \widetilde{z}^{\beta''}e^{-i\langle\widetilde{z},\widetilde{\eta}\rangle} &= i^{|\beta''|}\partial^{\beta''}_{\widetilde{\eta}} e^{-i\langle\widetilde{z},\widetilde{\eta}\rangle} \end{aligned}$$

and integrating by parts we can represent $R_N(x,y)$ as a linear combination of terms of the form

$$\begin{aligned} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_0^1 &e^{-i\langle\widetilde{z},\widetilde{\eta}\rangle}e^{-i\langle\psi(x,y,z),\eta\rangle}\partial^{\alpha+\beta}_\eta\widetilde{\sigma}_A(x,\eta+t\widetilde{\eta}) \\ &\times\Pi_{\alpha,\beta,\gamma,\mu}(t,x,y,z)\eta^\mu\Phi^{\mathcal{E}}_{x,y}\partial^\gamma_\eta\widetilde{\sigma}_{B,1}(y,\eta)\Phi^{\mathcal{J}}_{y,x}\,dt\,d\eta\,d\widetilde{z}\,d\widetilde{\eta}, \end{aligned} \tag{6.31}$$

where

$$|\mu|\le|\alpha| \tag{6.32}$$

and either

$$|\mu:\mathbf{d}|\le\varrho\Big(|(\alpha+\beta+\gamma):\mathbf{d}|-\frac{\varepsilon}{2}|\alpha+\beta+\gamma|\Big) \tag{6.33}$$

or

$$|\beta+\gamma|\ge q.$$

In the last case we have

$$|(\beta+\gamma):\mathbf{d}| \ge \varepsilon q. \tag{6.34}$$

Let us consider the case (6.33). Suppose

$$|\alpha| = N+1 \ge \frac{4}{\varrho\varepsilon}\max\{d_k^{-1}\}, \quad \text{i.e.} \quad \frac{\varrho\varepsilon}{4}|\alpha| \ge \max\{d_k^{-1}\}. \tag{6.35}$$

It is not difficult to see that μ can be represented as a sum $\mu = \mu' + \mu''$, where

$$|\mu'':\mathbf{d}| \le \varrho\Big(|\gamma:\mathbf{d}| - \frac{\varepsilon}{2}|\gamma|\Big),$$
$$|\mu':\mathbf{d}| \le \varrho\Big(|(\alpha+\beta):\mathbf{d}| - \frac{\varepsilon}{2}|\alpha+\beta|\Big) + \max\{d_k^{-1}\} \le \varrho\Big(|(\alpha+\beta):\mathbf{d}| - \frac{\varepsilon}{4}|\alpha|\Big).$$

It is convenient to rewrite (6.31) in the form

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_0^1 e^{-i\langle\widetilde{z},\widetilde{\eta}\rangle} e^{-i\langle\psi(x,y,z),\eta\rangle}\eta^{\mu'}\partial_\eta^{\alpha+\beta}\widetilde{\sigma}_A(x,\eta+t\widetilde{\eta})$$
$$\times \varPi_{\alpha,\beta,\gamma,\mu}(t,x,y,z)\varPhi^{\mathcal{E}}_{x,y}\eta^{\mu''}\partial_\eta^\gamma\widetilde{\sigma}_{B,1}(y,\eta)\varPhi^{\mathcal{J}}_{y,x}\,dt\,d\eta\,d\widetilde{z}\,d\widetilde{\eta}.$$

Then we use the equality $\eta = (\eta+t\widetilde{\eta}) - t\widetilde{\eta}$ and represent (6.31) as a sum of terms of the form

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_0^1 e^{-i\langle\widetilde{z},\widetilde{\eta}\rangle} e^{-i\langle\psi(x,y,z),\eta\rangle}\widetilde{\eta}^{\alpha'}\widetilde{a}_{\alpha,\beta,\mu',\alpha'}(x,\eta+t\widetilde{\eta}) \tag{6.36}$$
$$\times \varPi_{\alpha,\beta,\gamma,\mu',\mu'',\alpha'}(t,x,y,z)\varPhi^{\mathcal{E}}_{x,y}\widetilde{b}_{\gamma,\mu''}(y,\eta)\varPhi^{\mathcal{J}}_{y,x}\,dt\,d\eta\,d\widetilde{z}\,d\widetilde{\eta},$$

where $\alpha' \le \mu'$ and

$$\widetilde{a}_{\alpha,\beta,\mu',\alpha'}(x,\eta) := \eta^{\mu'-\alpha'}\partial_\eta^{\alpha+\beta}\widetilde{\sigma}_A(x,\eta) \in \widetilde{S}^{r',\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F}), \qquad r' \le r_1 - |\alpha':\mathbf{d}| - \frac{\varrho\varepsilon}{4}|\alpha|,$$
$$\widetilde{b}_{\gamma,\mu''}(y,\eta) := \eta^{\mu''}\partial_\eta^\gamma\widetilde{\sigma}_{B,1}(y,\eta) \in \widetilde{S}^{r'',\mathbf{d}}_{\varrho,\delta}(\mathcal{J},\mathcal{E}),$$
$$r'' \le r_2,\ r'+r'' \le r_1+r_2 - |\alpha':\mathbf{d}| - \frac{\varrho\varepsilon}{2}|\alpha+\beta+\gamma|$$

(see (6.33)).

For each of the terms (6.36) such that

$$|\alpha'| \ge \frac{4}{\varrho\varepsilon}\max\{d_k^{-1}\}, \quad \text{i.e.} \quad \frac{\varrho\varepsilon}{4}|\alpha'| \ge \max\{d_k^{-1}\},$$

we repeat the above manipulations: integration by parts with respect to $\widetilde{z}$, Lemma 6.3, integration by parts with respect to η and $\widetilde{\eta}$, the equality $\eta = (\eta+t\widetilde{\eta}) - t\widetilde{\eta}$. Doing so we decrease r' and $r'+r''$ by at least

$$\frac{\varrho\varepsilon}{4}|\alpha'| \ge \max\{d_k^{-1}\} \quad \text{and} \quad \frac{\varrho\varepsilon}{2}|\alpha'| \ge 2\max\{d_k^{-1}\}$$

respectively. Repeating the entire cycle p times we prove that $R_N(x,y)$ is a sum of terms of the form

$$(6.37)\qquad \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_0^1 e^{-i\langle\widetilde{z},\widetilde{\eta}\rangle} e^{-i\langle\psi(x,y,z),\eta\rangle}\widetilde{\eta}^{\alpha'}\widetilde{a}_{\alpha'}(x,\eta+t\widetilde{\eta}) \times \widehat{\Pi}(t,x,y,z)\Phi^{\mathcal{E}}_{x,y}\widetilde{b}_{\alpha'}(y,\eta)\Phi^{\mathcal{J}}_{y,x}\,dt\,d\eta\,d\widetilde{z}\,d\widetilde{\eta}$$

and

$$(6.38)\qquad \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_0^1 e^{-i\langle\widetilde{z},\widetilde{\eta}\rangle} e^{-i\langle\psi(x,y,z),\eta\rangle}\eta^{\mu}\widetilde{a}'_{\mu}(x,\eta+t\widetilde{\eta}) \times \widehat{\Pi}'(t,x,y,z)\Phi^{\mathcal{E}}_{x,y}\widetilde{b}'_{\mu}(y,\eta)\Phi^{\mathcal{J}}_{y,x}\,dt\,d\eta\,d\widetilde{z}\,d\widetilde{\eta},$$

where

$$(6.39)\qquad \widetilde{a}_{\alpha'}\in\widetilde{S}^{r',\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F}),\quad r'\le r_1-\frac{\varrho\varepsilon}{4}|\alpha|,\quad \widetilde{b}_{\alpha'}\in\widetilde{S}^{r'',\mathbf{d}}_{\varrho,\delta}(\mathcal{J},\mathcal{E}),$$
$$r''\le r_2,\quad r'+r''\le r_1+r_2-|\alpha':\mathbf{d}|-\frac{\varrho\varepsilon}{2}|\alpha|,\quad |\alpha'|<\frac{4}{\varrho\varepsilon}\max\{d_k^{-1}\},$$

$|\mu|\le|\alpha|$ (see (6.32)) and either

$$(6.40)\qquad \begin{aligned} &\widetilde{a}'_{\mu}\in\widetilde{S}^{r',\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F}),\quad r'\le r_1-|\mu:\mathbf{d}|-\frac{\varrho\varepsilon}{4}|\alpha|-p\max\{d_k^{-1}\},\\ &\widetilde{b}'_{\mu}\in\widetilde{S}^{r'',\mathbf{d}}_{\varrho,\delta}(\mathcal{J},\mathcal{E}),\quad r''\le r_2,\\ &\qquad r'+r''\le r_1+r_2-|\mu:\mathbf{d}|-\frac{\varrho\varepsilon}{2}|\alpha|-2p\max\{d_k^{-1}\},\end{aligned}$$

or

$$(6.41)\qquad \widetilde{a}'_{\mu}\in\widetilde{S}^{r_1-\varrho|(\alpha+\beta):\mathbf{d}|,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F}),\quad \widetilde{b}'_{\mu}\in\widetilde{S}^{r_2-\varrho|\gamma:\mathbf{d}|,\mathbf{d}}_{\varrho,\delta}(\mathcal{J},\mathcal{E}),\quad |(\beta+\gamma):\mathbf{d}|\ge\varepsilon q$$

(see (6.34)).

Let us take q such that

$$(6.42)\qquad \varrho\varepsilon q\ge 2\max\{d_k^{-1}\}(N+1+p).$$

If (6.41) holds, then either $|\beta:\mathbf{d}|\ge 2^{-1}\varepsilon q$ or $|\gamma:\mathbf{d}|\ge 2^{-1}\varepsilon q$. In the first case we obtain (6.40), since

$$|\alpha:\mathbf{d}|\ge\varepsilon|\alpha|,\quad |\mu:\mathbf{d}|\le\max\{d_k^{-1}\}|\mu|\le\max\{d_k^{-1}\}|\alpha|=\max\{d_k^{-1}\}(N+1).$$

In the second case

$$\eta^{\mu}\widetilde{b}'_{\mu}(y,\eta)\in\widetilde{S}^{r'',\mathbf{d}}_{\varrho,\delta}(\mathcal{J},\mathcal{E}),\quad r''\le r_2,$$

and we have (6.37), (6.39) with $\alpha'=0$. So, it is sufficient to consider the integrals (6.37), (6.39) and (6.38), (6.40).

The equalities (6.7) and (6.9) imply that the matrix

$$\partial_{\widetilde{z}}\psi(x,y,z)=-\partial_{\widetilde{z}}h(x,y,z)=-(\partial_{\widetilde{z}_j}h_k(x,y,z))_{n\times n}$$

satisfies the inequality $\|\partial_{\widetilde{z}}\psi(x,y,z)\|\le 1/2$ if y and z are sufficiently close to x. Therefore

$$\begin{aligned}\eta&=(I-t\partial_{\widetilde{z}}\psi(x,y,z))^{-1}((\eta+t\widetilde{\eta})-t(\widetilde{\eta}+\partial_{\widetilde{z}}\psi(x,y,z)\eta)),\\ \widetilde{\eta}&=(I-t\partial_{\widetilde{z}}\psi(x,y,z))^{-1}((\widetilde{\eta}+\partial_{\widetilde{z}}\psi(x,y,z)\eta)-\partial_{\widetilde{z}}\psi(x,y,z)(\eta+t\widetilde{\eta}))\end{aligned}$$

and

$$(6.43)\qquad |\eta|+|\widetilde{\eta}|\le\mathrm{const}\,(|\widetilde{\eta}+\partial_{\widetilde{z}}\psi(x,y,z)\eta|+|\eta+t\widetilde{\eta}|).$$

Let us consider the integral (6.37), (6.39). We apply the equality

$$e^{-i\langle\widetilde{z},\widetilde{\eta}\rangle}e^{-i\langle\psi(x,y,z),\eta\rangle} = (1+|\widetilde{\eta}+\partial_{\widetilde{z}}\psi(x,y,z)\eta|^2)^{-1} \times\Big(1+i\sum_{j=1}^{n}(\widetilde{\eta}_j+\partial_{\widetilde{z}_j}\langle\psi(x,y,z),\eta\rangle)\partial_{\widetilde{z}_j}\Big)e^{-i\langle\widetilde{z},\widetilde{\eta}\rangle}e^{-i\langle\psi(x,y,z),\eta\rangle} \tag{6.44}$$

and integrate by parts with respect to $\widetilde{z}$. We repeat this p_1 times, where

$$p_1 := \left[\frac{1}{2}\min\{d_k\}\left(\frac{\varrho\varepsilon}{4}(N+1)-r_1\right)\right] \tag{6.45}$$

and $[\cdot]$ denotes the integer part. It follows from (3.5), (6.39), the inequality

$$\delta(d_{j_1}^{-1}+\ldots+d_{j_m}^{-1}) \le d_{j_1}^{-1}+\ldots+d_{j_m}^{-1} \le \max\{d_k^{-1}\}m \tag{6.46}$$

and the equalities $|\alpha|=N+1$, $\min\{d_k\}\max\{d_k^{-1}\}=1$ that

$$\|\nabla^{\mathcal{F},\mathcal{E}}_{\nu_{j_1}(x)}\ldots\nabla^{\mathcal{F},\mathcal{E}}_{\nu_{j_m}(x)}\widetilde{a}_{\alpha'}(x,\eta+t\widetilde{\eta})\| \le \mathrm{const}_{j_1,\ldots,j_m}(1+|\eta+t\widetilde{\eta}|)^{r(m)},$$
$$r(m) \le \min\{d_k\}\left(r_1-\frac{\varrho\varepsilon}{4}(N+1)\right)+m \le -2p_1+m,$$

for any $j_1,\ldots,j_m\in\{1,\ldots,n\}$ and any $m\in\mathbb{Z}_+$ such that

$$\max\{d_k^{-1}\}m < \frac{\varrho\varepsilon}{4}(N+1)-r_1.$$

Using (6.43), (6.46) and the inequality

$$(1+|\widetilde{\eta}+\partial_{\widetilde{z}}\psi(x,y,z)\eta|+|\eta+t\widetilde{\eta}|) \le (1+|\widetilde{\eta}+\partial_{\widetilde{z}}\psi(x,y,z)\eta|)(1+|\eta+t\widetilde{\eta}|),$$

we prove that the integral (6.37), (6.39) can be represented in the form

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\widetilde{g}(x,y,z,\eta,\widetilde{\eta})\,d\widetilde{\eta}\,d\widetilde{z}\,d\eta, \tag{6.47}$$

where $\widetilde{g}(x,y,z,\eta,\widetilde{\eta})$ can be nonzero only if y and z are sufficiently close to x, and satisfies the estimates

$$\|\nabla^{\mathcal{F},\mathcal{J}}_{\nu_{j_1}(x)}\ldots\nabla^{\mathcal{F},\mathcal{J}}_{\nu_{j_{m_1}}(x)}\partial_{\nu_{k_1}(y)}\ldots\partial_{\nu_{k_{m_2}}(y)}\widetilde{g}(x,y,z,\eta,\widetilde{\eta})\| \le \mathrm{const}_{j_1,\ldots,j_{m_1},k_1,\ldots,k_{m_2}}(1+|\eta|_{\mathbf{d}})^{r_2(m_2)}(1+|\eta|+|\widetilde{\eta}|)^{r_1(m_1)}, \tag{6.48}$$
$$r_1(m_1) \le -p_1+\frac{4}{\varrho\varepsilon}\max\{d_k^{-1}\}+m_1, \qquad r_2(m_2) \le r_2+\max\{d_k^{-1}\}m_2,$$

for any $j_1,\ldots,j_{m_1},k_1,\ldots,k_{m_2}\in\{1,\ldots,n\}$ and any $m_1,m_2\in\mathbb{Z}_+$ such that

$$\max\{d_k^{-1}\}m_1 < \frac{\varrho\varepsilon}{4}(N+1)-r_1.$$

Hence, due to (3.5) the integral (6.37), (6.39) can be represented in the form

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\widetilde{g}(x,y,z,\eta)\,d\widetilde{z}\,d\eta, \tag{6.49}$$

where $\widetilde{g}(x,y,z,\eta)$ satisfies the estimates

$$\|\nabla^{\mathcal{F},\mathcal{J}}_{\nu_{j_1}(x)}\ldots\nabla^{\mathcal{F},\mathcal{J}}_{\nu_{j_{m_1}}(x)}\partial_{\nu_{k_1}(y)}\ldots\partial_{\nu_{k_{m_2}}(y)}\widetilde{g}(x,y,z,\eta)\| \le \mathrm{const}_{j_1,\ldots,j_{m_1},k_1,\ldots,k_{m_2}}(1+|\eta|)^{r(m_1,m_2)}, \tag{6.50}$$

$$r(m_1, m_2) \le -p_1 + \frac{4}{\varrho\varepsilon}\max\{d_k^{-1}\} + m_1 + n + \max\{d_k\}(r_2 + \max\{d_k^{-1}\}m_2)_+,$$

for any $j_1, \dots, j_{m_1}, k_1, \dots, k_{m_2} \in \{1, \dots, n\}$ and any $m_1, m_2 \in \mathbb{Z}_+$ such that

$$\max\{d_k^{-1}\}m_1 < \frac{\varrho\varepsilon}{4}(N+1) - r_1.$$

Here $s_+ := \max\{s, 0\}$.

Let us take an arbitrary $L \in \mathbb{N}$. Choosing N such that

$$-p_1 + \frac{4}{\varrho\varepsilon}\max\{d_k^{-1}\} + L + n + \max\{d_k\}(r_2 + \max\{d_k^{-1}\}L)_+ < -(n+1) \tag{6.51}$$

(see (6.45)), we prove that (6.37), (6.39) is C^L-smooth. It is left to prove that for the given L and N we can choose p such that (6.38), (6.40) is C^L-smooth.

We apply (6.44) and integrate by parts p_2 times, where

$$p_2 := \left[\frac{1}{2}\left(p + \min\{d_k\}\left(\frac{\varrho\varepsilon}{4}(N+1) - r_1\right)\right)\right] = p + p_1. \tag{6.52}$$

We then prove as above that (6.38), (6.40) can be represented in the form (6.47), (6.48), where

$$r_1(m_1) \le -p_2 + (N+1) + m_1, \qquad \max\{d_k^{-1}\}m_1 < p\max\{d_k^{-1}\} + \frac{\varrho\varepsilon}{4}(N+1) - r_1.$$

Therefore (6.38), (6.40) admits a representation of the form (6.49), (6.50) with

$$r(m_1, m_2) \le -p_2 + (N+1) + m_1 + n + \max\{d_k\}(r_2 + \max\{d_k^{-1}\}m_2)_+.$$

Choosing p such that

$$-p_2 + (N+1) + L + n + \max\{d_k\}(r_2 + \max\{d_k^{-1}\}L)_+ < -(n+1) \tag{6.53}$$

(see (6.52)), we prove that (6.38), (6.40) is C^L-smooth. ∎

Lemma 6.5, Proposition 6.6 and Theorem 6.7 imply the following result.

6.10. Corollary. *Let the conditions of Theorem* 6.7 *be satisfied. Then*

$$\widetilde{\sigma}_{AB}(x,\eta) = \widetilde{\sigma}_A(x,\eta)\widetilde{\sigma}_B(x,\eta) - i\sum_{j=1}^{n}\partial_{\eta_j}\widetilde{\sigma}_A(x,\eta)\nabla^{\mathcal{E},\mathcal{J}}_{\nu_j(x)}\widetilde{\sigma}_B(x,\eta)$$
$$- \frac{1}{2i}\sum_{j,k,m=1}^{n} C^j_{k,m}(x)\eta_j\partial_{\eta_k}\widetilde{\sigma}_A(x,\eta)\partial_{\eta_m}\widetilde{\sigma}_B(x,\eta)$$

modulo $\widetilde{S}^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{J},\mathcal{F})$, $r \le r_1 + r_2 - 2\min\{(\varrho-\delta)\min\{d_k^{-1}\}, \varrho\varepsilon\}$.

7. L_p-estimates

This section is devoted to the L_p-boundedness of ψDOs. It is sufficient to consider ψDOs acting on scalar functions, i.e. the case $\mathcal{E} = \mathcal{F} = M \times \mathbb{C}$, when dealing with this problem. We will use the notation $\widetilde{S}^{r,\mathbf{d}}_{\varrho,\delta}(M) := \widetilde{S}^{r,\mathbf{d}}_{\varrho,\delta}(M\times\mathbb{C}, M\times\mathbb{C})$ and $\Psi^{r,\mathbf{d}}_{\varrho,\delta}(M) := \Psi^{r,\mathbf{d}}_{\varrho,\delta}(M\times\mathbb{C}, M\times\mathbb{C})$.

Theorem 6.7 allows one to prove L_2-continuity of ψDOs from $\Psi^{0,\mathbf{d}}_{\varrho,\delta}(M)$. The proof is based on the method due to L. Hörmander and is almost identical to the corresponding argument from the standard theory of ψDOs (see, e.g., [Shu, §6] or [Ta1, Ch. II, §6]).

7.1. Theorem. *Let* (6.1) *be satisfied and* $A \in \Psi^{0,\mathbf{d}}_{\varrho,\delta}(M)$, $0 \le \delta < \varrho \le 1$. *Then* $A : L_{2,\mathrm{comp}}(M) \to L_{2,\mathrm{loc}}(M)$ *is continuous, i.e. for any* $\varphi, \varphi_0 \in C_0^\infty(M)$ *the operator* $\varphi_0 A \varphi I : L_2(M) \to L_2(M)$ *is bounded.*

Now we are going to prove L_p-continuity of ψDOs from $\Psi^{0,\mathbf{d}}_{1,\delta}(M)$. The idea is to show that these ψDOs are Calderón–Zygmund operators on a suitable space of homogeneous type.

7.2. Lemma. *Let* (6.2) *be satisfied and let* $W \subset M$ *be a set such that* $c(x,y)$ *is defined for any* $x, y \in W$ *and* $|c(x,y)|_{\mathbf{d}} \le 1$. *Then there exists a constant* $C(W) < \infty$ *such that*

$$|c(y,z)|_{\mathbf{d}} \le C(W)(|c(y,x)|_{\mathbf{d}} + |c(x,z)|_{\mathbf{d}}), \qquad \forall x, y, z \in W,$$

i.e. $|c(x,y)|_{\mathbf{d}}$ *is a quasimetric on* W *(see* (2.12)*).*

Proof. Let us use the Taylor expansion

$$c_j(y,z) = c_j(y,x) + c_j(x,z) + \sum_{2 \le |\alpha+\beta| < N} \frac{(-1)^{|\alpha|}}{\alpha!\beta!} \partial^\alpha_{\nu(y)} \partial^\beta_{\nu(z)} c_j(y,z)|_{y=z=x} c^\alpha(y,x) c^\beta(x,z)$$
$$+ \sum_{|\alpha+\beta|=N} G_{\alpha,\beta}(x,y,z) c^\alpha(y,x) c^\beta(x,z),$$

where $N \ge \max\{d_l\} \max\{d_l^{-1}\}$ and the functions $G_{\alpha,\beta}$ are C^∞-smooth (see (2.9), (2.12), (2.17) and (2.18)). Now our statement follows from (3.4), Lemma 6.1 and the obvious inequality

$$\max\{|c_l(y,x)|, |c_l(x,z)|\} \le (|c(y,x)|_{\mathbf{d}} + |c(x,z)|_{\mathbf{d}})^{d_l^{-1}}, \qquad l = 1, \dots, n. \ \blacksquare$$

7.3. Remark. The last statement remains true if (6.2) is replaced by a weaker restriction:

$$\frac{1}{d_j} + \frac{1}{d_k} \ge \frac{1}{d_m} \qquad \text{if } C^m_{j,k} \not\equiv 0.$$

Indeed, in this case Lemma 6.1 holds with $\varepsilon = 0$, $\varrho = 1$. This generalization of Lemma 7.2 is a special case of the results obtained in [Nag], [NSW].

7.4. Lemma. *Let* $\widetilde{a} \in \widetilde{S}^{0,\mathbf{d}}_{1,1}(M)$. *Then*

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\langle c(x,y),\eta\rangle} \widetilde{a}(x,\eta) \chi(x,y)\, d\eta = \chi(x,y) k(x, c(x,y)),$$

where for any compact set $K \subset M$,

$$|\partial_{\nu_{j_1}(x)} \dots \partial_{\nu_{j_q}(x)} \partial^\alpha_w k(x,w)| \le \mathrm{const}_{K,\alpha,j_1,\dots,j_q} |w|_{\mathbf{d}}^{-(n+|\alpha:\mathbf{d}|+|\beta:\mathbf{d}|)},$$
$$\forall \alpha \in \mathbb{Z}^n_+,\ \forall j_1, \dots, j_q \in \{1, \dots, n\},\ \forall q \in \mathbb{Z}_+,\ \forall w \in \mathbb{R}^n \setminus \{0\},\ \forall x \in K,$$

and β *is the multi-index corresponding to the set of indices* $\{j_1, \dots, j_q\}$.

Proof. The proof follows closely the corresponding argument from the standard theory of ψDOs (see, e.g., [Ste, Ch. VI, §4 and Ch. VII, §1]). Let us take $u \in C_0^\infty(\mathbb{R}^n)$ with the properties that $u(\eta) = 1$ for $|\eta|_{\mathbf{d}} \le 1$, and $u(\eta) = 0$ for $|\eta|_{\mathbf{d}} \ge 2$. Let

$$v(\eta) := u(\eta) - u(2^{1/d_1}\eta_1, \dots, 2^{1/d_n}\eta_n),$$
$$v_0 \equiv u, \quad v_m(\eta) := v(2^{-m/d_1}\eta_1, \dots, 2^{-m/d_n}\eta_n), \quad m \in \mathbb{N}.$$

It is clear that v_m, $m \in \mathbb{N}$, is supported in the shell

$$\Omega_m := \{\eta \in \mathbb{R}^n : 2^{m-1} \le |\eta|_{\mathbf{d}} \le 2^{m+1}\}.$$

Since Ω_m is the image of Ω_0 under the map $\eta \mapsto (2^{m/d_1}\eta_1, \dots, 2^{m/d_n}\eta_n)$, (3.1) implies the following equality for the volume of Ω_m: $\operatorname{Vol}\Omega_m = 2^{mn}\operatorname{Vol}\Omega_0$. Hence, for

$$k_m(x, w) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\langle w,\eta\rangle} \widetilde{a}(x, \eta) v_m(\eta)\, d\eta$$

we have

$$\text{(7.1)} \quad |w^\gamma \partial_{\nu_{j_1}(x)} \dots \partial_{\nu_{j_q}(x)} \partial_w^\alpha k_m(x, w)| \le \operatorname{const}_{K,\alpha,\gamma,j_1,\dots,j_q} 2^{m(n-|\gamma:\mathbf{d}|+|\alpha:\mathbf{d}|+|\beta:\mathbf{d}|)},$$
$$\forall \alpha, \gamma \in \mathbb{Z}_+^n,\ \forall j_1, \dots, j_q \in \{1, \dots, n\},\ \forall q \in \mathbb{Z}_+,\ \forall w \in \mathbb{R}^n \setminus \{0\},\ \forall x \in K,$$

where β is the multi-index corresponding to the set of indices $\{j_1, \dots, j_q\}$.

The equality $1 = \sum_{m=0}^\infty v_m$ implies

$$\partial_{\nu_{j_1}(x)} \dots \partial_{\nu_{j_q}(x)} \partial_w^\alpha k(x, w) = \sum_{m=0}^\infty \partial_{\nu_{j_1}(x)} \dots \partial_{\nu_{j_q}(x)} \partial_w^\alpha k_m(x, w).$$

Therefore it will suffice to estimate the sum $\sum_{m=0}^\infty |\partial_{\nu_{j_1}(x)} \dots \partial_{\nu_{j_q}(x)} \partial_w^\alpha k_m(x, w)|$. We break this sum into two parts: the first where $2^m \le |w|_{\mathbf{d}}^{-1}$, the second where $2^m > |w|_{\mathbf{d}}^{-1}$. For the first one we use (7.1) with $\gamma = 0$. This gives the upper bound

$$O\Big(\sum_{2^m \le |w|_{\mathbf{d}}^{-1}} 2^{m(n+|\alpha:\mathbf{d}|+|\beta:\mathbf{d}|)}\Big) = O(|w|_{\mathbf{d}}^{-(n+|\alpha:\mathbf{d}|+|\beta:\mathbf{d}|)}).$$

Let us estimate the second sum. For a given w we can choose l such that $|w_l|^{d_l} \ge n^{-\max\{d_j\}/2}|w|_{\mathbf{d}}$ (see (3.4)). Then we apply (7.1), where $\gamma = (0, \dots, 0, N, 0, \dots, 0)$ with $N > d_l(n + |\alpha : \mathbf{d}| + |\beta : \mathbf{d}|)$ at the lth place. This gives the upper bound

$$O(|w|_{\mathbf{d}}^{-N/d_l}) \sum_{2^m > |w|_{\mathbf{d}}^{-1}} 2^{m(n-N/d_l+|\alpha:\mathbf{d}|+|\beta:\mathbf{d}|)} = O(|w|_{\mathbf{d}}^{-(n+|\alpha:\mathbf{d}|+|\beta:\mathbf{d}|)}). \ \blacksquare$$

7.5. Lemma. *Let* $\widetilde{k}(x, y) := \chi(x, y)k(x, c(x, y))$ *be the function from the previous lemma. Then for any compact set* $K \subset M$ *there exist constants* $C_K, C'_K < \infty$ *such that for any* $x \in K$, $y \neq x$ *and* $z \in M$ *with* $|c(x, z)|_{\mathbf{d}} < C_K|c(x, y)|_{\mathbf{d}}$ *we have*

$$|\widetilde{k}(x, y)| \le C'_K |c(x, y)|_{\mathbf{d}}^{-n},$$
$$|\widetilde{k}(x, y) - \widetilde{k}(z, y)| + |\widetilde{k}(y, x) - \widetilde{k}(y, z)| \le C'_K |c(x, y)|_{\mathbf{d}}^{-n} \left(\frac{|c(x, z)|_{\mathbf{d}}}{|c(x, y)|_{\mathbf{d}}}\right)^{\min\{d_j^{-1}\}}.$$

Proof. The first inequality follows directly from Lemma 7.4. Let us prove the second one. We can break K into a finite number of sets W satisfying the conditions of Lemma 7.2. So, it is sufficient to prove the inequality for such W instead of K.

Suppose $x, y, z \in W$ and $|c(x,z)|_{\mathbf{d}} < (2C(W))^{-1}|c(x,y)|_{\mathbf{d}}$ (see Lemma 7.2). It follows from (4.9) and the definition of $|\cdot|_{\mathbf{d}}$ that

$$|c(x,\gamma_{z,x}(t))|_{\mathbf{d}} = |tc(x,z)|_{\mathbf{d}} \le |c(x,z)|_{\mathbf{d}} < (2C(W))^{-1}|c(x,y)|_{\mathbf{d}}, \qquad \forall t \in [0,1].$$

Therefore, (2.12) and Lemma 7.2 imply

$$\begin{aligned}|c(\gamma_{z,x}(t),y)|_{\mathbf{d}} &\le C(W)(|c(\gamma_{z,x}(t),x)|_{\mathbf{d}} + |c(x,y)|_{\mathbf{d}}) \le (C(W)+1/2)|c(x,y)|_{\mathbf{d}},\\ |c(x,y)|_{\mathbf{d}} &\le C(W)(|c(x,\gamma_{z,x}(t))|_{\mathbf{d}} + |c(\gamma_{z,x}(t),y)|_{\mathbf{d}})\\ &\le \frac{1}{2}|c(x,y)|_{\mathbf{d}} + C(W)|c(\gamma_{z,x}(t),y)|_{\mathbf{d}}.\end{aligned}$$

Hence,

$$(2C(W))^{-1}|c(x,y)|_{\mathbf{d}} \le |c(\gamma_{z,x}(t),y)|_{\mathbf{d}} \le (C(W)+1/2)|c(x,y)|_{\mathbf{d}}. \tag{7.2}$$

Using the equality $\dot\gamma_{z,x} = \sum_{m=1}^{n} c_m(x,z)\nu_m$, we obtain

$$\begin{aligned}&k(x,c(x,y)) - k(z,c(z,y))\\ &\quad= -\int_0^1 \frac{d}{dt}k(\gamma_{z,x}(t),c(\gamma_{z,x}(t),y))\,dt\\ &\quad= -\sum_{m=1}^{n} c_m(x,z)\int_0^1 \partial_{\nu_m(z')}k(z',c(\gamma_{z,x}(t),y))|_{z'=\gamma_{z,x}(t)}\,dt\\ &\qquad - \sum_{m,l=1}^{n} c_m(x,z)\int_0^1 \partial_{w_l}k(\gamma_{z,x}(t),w)|_{w=c(\gamma_{z,x}(t),y)}\partial_{\nu_m(z')}c_l(z',y)|_{z'=\gamma_{z,x}(t)}\,dt.\end{aligned}$$

Let us substitute in this formula the Taylor expansions

$$\begin{aligned}\partial_{\nu_m(z')}c_l(z',y) = {}& -\delta_m^l + \sum_{1\le|\alpha|<N}\frac{1}{\alpha!}\partial^{\alpha}_{\nu(y)}\partial_{\nu_m(z')}c_l(z',y)|_{y=z'}c^{\alpha}(z',y)\\ &+\sum_{|\alpha|=N}G_\alpha(z',y)c^{\alpha}(z',y),\end{aligned}$$

where $N \ge \max\{d_j\}\max\{d_j^{-1}\}$ and the functions G_α are C^∞-smooth (see (2.9) and (2.17)). Then it follows from (3.4), (7.2) and Lemmas 6.1, 7.4 that

$$|\widetilde{k}(x,y) - \widetilde{k}(z,y)| \le \mathrm{const}\,|c(x,y)|_{\mathbf{d}}^{-n}\left(\frac{|c(x,z)|_{\mathbf{d}}}{|c(x,y)|_{\mathbf{d}}}\right)^{\min\{d_j^{-1}\}}.$$

The estimate for $|\widetilde{k}(y,x) - \widetilde{k}(y,z)|$ can be proved similarly. ■

7.6. Theorem. *Let* (6.2) *be satisfied and* $A \in \Psi^{0,\mathbf{d}}_{1,\delta}(M)$, $0 \le \delta < 1$. *Then* $A : L_{p,\mathrm{comp}}(M) \to L_{p,\mathrm{loc}}(M)$, $1 < p < \infty$, *is continuous, i.e. for any* $\varphi, \varphi_0 \in C_0^\infty(M)$ *the operator* $\varphi_0 A\varphi I : L_p(M) \to L_p(M)$, $1 < p < \infty$, *is bounded.*

Proof. It is sufficient to prove L_p-boundedness of $\varphi_0 A \varphi I$ in the case when $\varphi, \varphi_0 \in C_0^\infty(W)$ and the open set W satisfies the conditions of Lemma 7.2.

Let us consider the ball

$$B(x,\tau) := \{y \in M : |c(x,y)|_{\mathbf{d}} < \tau\}, \qquad x \in W,\ \tau < 1.$$

It follows from the definition of the measure $\mathcal{M}$ (see (4.2)) that

$$\mathcal{M}(B(x,\tau)) = \int\limits_{|\widetilde{y}|_{\mathbf{d}}<\tau} \Upsilon^{-1}(x, \exp_x(\widetilde{y}))\, d\widetilde{y}$$

(see (2.9) and (4.3)). Since, due to (3.1),

$$\mathrm{Vol}\{\widetilde{y} \in \mathbb{R}^n : |\widetilde{y}|_{\mathbf{d}} < \tau\} = \tau^n\, \mathrm{Vol}\{\widetilde{y} \in \mathbb{R}^n : |\widetilde{y}|_{\mathbf{d}} < 1\},$$

there exist positive constants $C_1, C_2 < \infty$ such that

$$C_1\tau^n \le \mathcal{M}(B(x,\tau)) \le C_2\tau^n, \quad \forall x \in W,\ \forall \tau \in (0,1).$$

This estimate and Lemma 7.5 allow one to deduce L_p-boundedness of $\varphi_0 A\varphi I$ from its L_2-boundedness (Theorem 7.1) with the help of the theory of Calderón–Zygmund singular integral operators on spaces of homogeneous type (see [CW, Ch. III], [Chr, Ch. VI]). ∎

7.7. Remark. In the same way one can prove that $\varphi_0 A\varphi I$ is of weak type $(1,1)$ and maps L_∞ boundedly to BMO.

8. Example: anisotropic ψDOs on Lie groups

Let G be an n-dimensional Lie group and $\mathcal{G}$ be the corresponding Lie algebra of right-invariant vector fields on G. Let $\nu_1, \dots, \nu_n$ be a basis of $\mathcal{G}$. Using it, we can identify $\mathcal{G}$ with $\mathbb{R}^n$. There exist neighbourhoods $U = \{\widetilde{z} \in \mathbb{R}^n : |\widetilde{z}|_{\mathbf{d}} < \mathrm{const}\}$ of $0 \in \mathbb{R}^n$ and W of the identity $e \in G$ such that the exponential mapping $\exp_e : \mathbb{R}^n \to G$ is a diffeomorphism of U onto W. We set

$$\delta_\tau \widetilde{z} := (\tau^{1/d_1}\widetilde{z}_1, \dots, \tau^{1/d_n}\widetilde{z}_n), \quad \widetilde{z} = (\widetilde{z}_1, \dots, \widetilde{z}_n) \in \mathbb{R}^n,\ \tau > 0,$$

$$\delta_\tau z := \exp_e(\delta_\tau \widetilde{z}), \quad z = \exp_e(\widetilde{z}) \in W,$$

$$\varphi_\tau(z) := \begin{cases} \tau^n \varphi(\delta_\tau z), & z \in W, \\ 0, & z \notin W, \end{cases} \quad \varphi \in C_0^\infty(W),\ \tau > 1.$$

It is clear that $\varphi_\tau \in C_0^\infty(W)$ for any $\varphi \in C_0^\infty(W)$.

We will say that a distribution $f \in \mathcal{D}'(G)$ is *locally* $\mathbf{d}$*-homogeneous of degree* $\mu \in \mathbb{C}$ if

$$\langle f, \varphi_{1/\tau}\rangle = \tau^\mu \langle f, \varphi\rangle, \quad \forall \tau \in (0,1),\ \forall \varphi \in C_0^\infty(W).$$

A distribution is called *regular* if it is smooth away from e, i.e. belongs to $C^\infty(G \setminus \{e\})$.

If there are no $m_1, \dots, m_n \in \mathbb{Z}_+$ such that $\mu = \sum_{j=1}^n m_j/d_j$, we denote by $\mathcal{RH}_\mu^{\mathbf{d}}(G)$ the class of all regular locally $\mathbf{d}$-homogeneous distributions of degree μ on G. If $\mu = \sum_{j=1}^n m_j/d_j$ for some $m_1, \dots, m_n \in \mathbb{Z}_+$, then $\mathcal{RH}_\mu^{\mathbf{d}}(G)$ will denote the class of all regular distributions of the form $f = f_0 + f_1$, where f_0 is a regular locally $\mathbf{d}$-homogeneous distribution of degree μ,

$$f_1(\exp_e(\widetilde{z})) = p(\widetilde{z}) \log |\widetilde{z}|_{\mathbf{d}}, \quad \forall \widetilde{z} \in U,$$

and p is a $\mathbf{d}$-homogeneous polynomial of $\mathbf{d}$-degree μ.

Suppose $k(x,\cdot) \in \mathcal{RH}_\mu^{\mathbf{d}}(G)$ for any $x \in G$. We will say that $k \in C^\infty(G, \mathcal{RH}_\mu^{\mathbf{d}})$ if $k \in C^\infty(G \times (G \setminus \{e\}))$ and the function $G \ni x \mapsto \langle k(x,\cdot), \varphi\rangle \in \mathbb{C}$ belongs to $C^\infty(G)$ for any $\varphi \in C_0^\infty(G)$.

It follows from the right-invariance of the vector fields $\nu_1, \dots, \nu_n$ that

$$d\mathcal{M}(y) := |\det(\nu_k^j(y))|^{-1} dy, \qquad y \in G,$$

(see (4.2)) is a right Haar measure on G (cf. [Hel, Ch. X, §1, Sect. 1]).

Let us consider an operator $A : C_0^\infty(G) \to C^\infty(G)$ defined by

$$(A\varphi)(x) := (k(x,\cdot) * \varphi)(x), \qquad \varphi \in C_0^\infty(G), \tag{8.1}$$

where $k \in C^\infty(G, \mathcal{RH}_\mu^{\mathbf{d}})$ and $*$ is the group convolution. So,

$$(k(x,\cdot) * \varphi)(x) = \langle k(x,\cdot), \varphi((\cdot)^{-1}x)\Delta\rangle$$

and Δ is the modular function of G, i.e. $d\mathcal{M}(z^{-1}) = \Delta(z) d\mathcal{M}(z)$. If $k(x,\cdot)$, $x \in G$, is locally integrable, then

$$(k(x,\cdot) * \varphi)(x) = \int_G k(x,z)\varphi(z^{-1}x)\Delta(z)\, d\mathcal{M}(z) = \int_G k(x, xy^{-1})\varphi(y)\, d\mathcal{M}(y).$$

Let us take $\chi \in C_0^\infty(W)$ which equals 1 in some neighbourhood of e. For every $x \in G$ we consider the following distribution on $\mathbb{R}^n$: $\widetilde{k}(x,\cdot) = \chi(\exp_e(\cdot))k(x, \exp_e(\cdot))$. Let $\widetilde{a}(x,\cdot)$ be the Fourier transform of $\widetilde{k}(x,\cdot)$, i.e. $\widetilde{a}(x,\eta) = F_{\widetilde{z}\to\eta}\widetilde{k}(x,\widetilde{z})$. It follows from the properties of k and the equality

$$p(\delta_\tau \widetilde{z}) \log|\delta_\tau \widetilde{z}|_{\mathbf{d}} = \tau^\mu p(\widetilde{z}) \log|\widetilde{z}|_{\mathbf{d}} + \tau^\mu \log\tau\, p(\widetilde{z}), \qquad \widetilde{z} \in \mathbb{R}^n \setminus \{0\},\ \tau > 0,$$

that $\widetilde{a} \in C^\infty(G \times \mathbb{R}^n)$ and $\widetilde{a} = \widetilde{a}_0 + \widetilde{a}_1$, $\widetilde{a}_0, \widetilde{a}_1 \in C^\infty(G \times \mathbb{R}^n)$,

$$\widetilde{a}_0(x, \delta_\tau\eta) = \tau^{-\mu-n}\widetilde{a}_0(x,\eta), \qquad |\eta|_{\mathbf{d}} \ge 1,\ \tau \ge 1,$$

$$\sup_{x\in\Xi,\, |\eta|_{\mathbf{d}}\ge 1} |\eta|_{\mathbf{d}}^N |\partial_\eta^\alpha \partial_{\nu_{k_1}(x)} \dots \partial_{\nu_{k_p}(x)} \widetilde{a}_1(x,\eta)| < \infty,$$

$$\forall N \in \mathbb{R},\ \forall \alpha \in \mathbb{Z}_+^n,\ \forall k_1, \dots, k_p \in \{1, \dots, n\},\ \forall p \in \mathbb{Z}_+,$$

for any compact set $\Xi \subset G$. Hence $\widetilde{a} \in \widetilde{S}_{1,0}^{-\operatorname{Re}\mu-n,\mathbf{d}}(G)$.

Now using the inverse Fourier transform we can represent the operator (8.1) in the form

$$(A\varphi)(x) = \frac{1}{(2\pi)^n} \int_G \int_{\mathbb{R}^n} e^{i\langle\widetilde{z},\eta\rangle} \widetilde{a}(x,\eta)\varphi(z^{-1}x)\chi(z)\Delta(z)\, d\eta\, d\mathcal{M}(z)$$
$$+ \int_G (1 - \chi^2(z))k(x,z)\varphi(z^{-1}x)\Delta(z)\, d\mathcal{M}(z),$$

where the first term on the right hand side is understood as an oscillatory integral (see, e.g., [Shu, §1]). It follows from the right-invariance of $\nu_1, \dots, \nu_n$ and (2.12) that $c_j(e, xy^{-1}) = c_j(y,x) = -c_j(x,y)$, $j = 1, \dots, n$. Since $\widetilde{z}_j = c_j(e,z)$, the change of variable yields

$$(A\varphi)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_G e^{-i\langle c(x,y),\eta\rangle} \widetilde{a}(x,\eta)\varphi(y)\chi(xy^{-1})\, d\mathcal{M}(y)\, d\eta$$
$$+ \int_G (1 - \chi^2(xy^{-1}))k(x, xy^{-1})\varphi(y)\, d\mathcal{M}(y).$$

Thus (8.1) is a ψDO: $A \in \Psi_{1,0}^{-\operatorname{Re}\mu-n,\mathbf{d}}(G)$ (cf. Definition 4.1 and (4.13)) and the results of Sections 4 and 5 apply to it. This is true for any Lie group G and any $\mathbf{d} = (d_1, \dots, d_n)$. However the results of Sections 6, 7 and of the following sections rely upon the restriction (6.2): $d_j^{-1} + d_k^{-1} > d_m^{-1}$ if $C_{j,k}^m \not\equiv 0$. So, the main part of our calculus does not cover the case considered in [Dy1], [Dy2], [NS], [How], [Mil], [Mel], [Ta2], [BG], [Cum] and [CGGP]. These works deal with the case where G is a homogeneous group (see [FS, Ch. I]) and δ_τ is an algebra automorphism of $\mathcal{G}$ (a group automorphism of G) for any $\tau > 0$. This means in particular that G is a nilpotent Lie group and $d_j^{-1} + d_k^{-1} = d_m^{-1}$ if $C_{j,k}^m \not\equiv 0$.

9. Compact ψDOs

In this section we deal with compactness of ψDOs. It is sufficient to consider ψDOs acting on scalar functions, i.e. the case $\widetilde{S}^{r,\mathbf{d}}_{\varrho,\delta}(M) := \widetilde{S}^{r,\mathbf{d}}_{\varrho,\delta}(M \times \mathbb{C}, M \times \mathbb{C})$, $\Psi^{r,\mathbf{d}}_{\varrho,\delta}(M) := \Psi^{r,\mathbf{d}}_{\varrho,\delta}(M \times \mathbb{C}, M \times \mathbb{C})$. Let us start with the following simple statement.

9.1. Lemma. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. If there exist a constant $C \geq 0$ and a compact linear operator $K : \mathcal{H}_1 \to \mathcal{H}_2$ such that*

$$\|Au\| \leq C\|u\| + \|Ku\|, \qquad \forall u \in \mathcal{H}_1,$$

then for any $\varepsilon > 0$ there exists a compact linear operator $K_\varepsilon : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\|A - K_\varepsilon\| \leq C + \varepsilon.$$

Proof. Let $P_\varepsilon : \mathcal{H}_1 \to \mathcal{H}_1$ be the orthogonal projection onto the linear span of a finite ε-net of the relatively compact set $\{K^*v : v \in \mathcal{H}_2,\ \|v\| \leq 1\}$ (see, e.g., [BN, Definitions 5.6, 5.7, Theorems 5.7 and 17.8]). Since P_ε is a finite rank operator, $K_\varepsilon := AP_\varepsilon$ is compact and

$$\|A - K_\varepsilon\| = \|A(I - P_\varepsilon)\| \leq \sup_{\|u\|\leq 1} \left(C\|(I - P_\varepsilon)u\| + \|K(I - P_\varepsilon)u\|\right)$$
$$\leq C + \|K(I - P_\varepsilon)\| = C + \|(I - P_\varepsilon)K^*\| \leq C + \varepsilon$$

(see [BN, Section 10.4 and Theorem 22.1]). ∎

9.2. Theorem. *Let* (6.1) *be satisfied*, $A \in \Psi^{0,\mathbf{d}}_{\varrho,\delta}(M)$, $0 \leq \delta < \varrho \leq 1$ *and*

$$\sup_{x\in\Xi} |\widetilde{\sigma}_A(x, \eta)| \to 0 \qquad \text{as } |\eta|_{\mathbf{d}} \to \infty \tag{9.1}$$

for any compact set $\Xi \subset M$. Then $\varphi_0 A\varphi I$ is compact on $L_2(M)$ for any $\varphi, \varphi_0 \in C_0^\infty(M)$.

Proof. Similarly to [Shu, Theorem 6.2] or [Ta1, Ch. II, Theorem 6.3] one can prove that for any $\delta > 0$ there exists an integral operator $\mathcal{K}_\delta$ with a C^∞-smooth kernel having compact support such that

$$\|\varphi_0 A\varphi u\| \leq \delta\|u\| + \|\mathcal{K}_\delta u\|, \qquad \forall u \in L_2(M)$$

(see also Theorem 6.7). Now the compactness of A follows from the previous lemma. ∎

9.3. Theorem. *Let* (6.2) *be satisfied and $A \in \Psi^{0,\mathbf{d}}_{1,\delta}(M)$, $0 \leq \delta < 1$. If* (9.1) *holds, then $\varphi_0 A\varphi I$ is compact on $L_p(M)$, $1 < p < \infty$, for any, $\varphi, \varphi_0 \in C_0^\infty(M)$.*

Proof. The compactness of A can be obtained from the previous theorem and Theorem 7.6 by interpolation (see [Kra]). ∎

9.4. Remark. It is clear that (9.1) is satisfied if $A \in \Psi^{-\tau,\mathbf{d}}_{\varrho,\delta}(M)$, $\tau > 0$.

10. Fredholm ψDOs

10.1. DEFINITION. A ψDO $A \in \Psi^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})$ with a presymbol $\widetilde{a} \in \widetilde{S}^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})$ is called *semi-elliptic* if for every compact set $\Xi \subset M$ there exist $C, N > 0$ such that

$$\|\widetilde{a}^{-1}(x,\eta)\| \leq C(1+|\eta|_{\mathbf{d}})^{-r}, \qquad |\eta|_{\mathbf{d}} \geq N,\ x \in \Xi,$$

where $\|\cdot\| : \mathcal{H}om(\widetilde{\mathcal{F}},\widetilde{\mathcal{E}}) \to \mathbb{R}$ is a continuous function such that its restriction to each fibre $\mathcal{H}om(\widetilde{\mathcal{F}},\widetilde{\mathcal{E}})_{(x,\eta)} = \mathrm{Hom}(\widetilde{\mathcal{F}}_{(x,\eta)},\widetilde{\mathcal{E}}_{(x,\eta)})$ is a norm on $\mathrm{Hom}(\mathcal{F}_x,\mathcal{E}_x)$ (cf. Section 3).

It is not difficult to see that if A is semi-elliptic then there exists $\widetilde{b} \in \widetilde{S}^{-r,\mathbf{d}}_{\varrho,\delta}(\mathcal{F},\mathcal{E})$ such that $\widetilde{b}\widetilde{a} - I \in \widetilde{S}^{-\infty}$, $\widetilde{a}\widetilde{b} - I \in \widetilde{S}^{-\infty}$, where I denotes the identity morphism of the corresponding vector bundle. Using Theorem 6.7 one can prove in the standard way (see, e.g., [Shu, §5] or [Ta1, Ch. III, §1]) the following result.

10.2. THEOREM. *Let* (6.1) *be satisfied and* $A \in \Psi^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})$ *be semi-elliptic. Then there exists a properly supported* ψ*DO* $B \in \Psi^{-r,\mathbf{d}}_{\varrho,\delta}(\mathcal{F},\mathcal{E})$ *such that* $BA - I \in \Psi^{-\infty}(\mathcal{E},\mathcal{E})$, $AB - I \in \Psi^{-\infty}(\mathcal{F},\mathcal{F})$ (*cf. Remark* 4.6).

The operator B from the last theorem is said to be a *parametrix* of A. A ψDO B_0 is called a *right* (resp. *left*) *parametrix* of A if $AB_0 - I \in \Psi^{-\infty}(\mathcal{F},\mathcal{F})$ (resp. $B_0A - I \in \Psi^{-\infty}(\mathcal{E},\mathcal{E})$). Suppose A has a left parametrix $B_0 \in \Psi^{-r,\mathbf{d}}_{\varrho,\delta}(\mathcal{F},\mathcal{E})$ and a right parametrix $B_1 \in \Psi^{-r,\mathbf{d}}_{\varrho,\delta}(\mathcal{F},\mathcal{E})$. Considering B_0AB_1 we obtain $B_0 - B_1 \in \Psi^{-\infty}(\mathcal{F},\mathcal{E})$. So, each of B_0 and B_1 is a parametrix of A.

In the remaining part of this section we will suppose that the manifold M is compact. This assumption together with the requirement that $\nu_1(x),\dots,\nu_n(x)$ span the tangent space T_xM at each point $x \in M$, i.e. that M is *parallelizable*, impose a strong restriction on the topology of M. For example, the unit n-dimensional sphere S^n is parallelizable if and only if $n = 1$, 3 or 7 (see, e.g., [Sch, Ch. X]). On the other hand any Lie group is parallelizable. We will also suppose that (6.2) is satisfied.

Now we are going to consider anisotropic analogues of Bessel-potential spaces on M. Our approach is similar to the treatment of isotropic Sobolev H^s_2-spaces in [Shu, §7].

Let $I_s \in \Psi^{s,\mathbf{d}}_{1,0}(M)$, $s > 0$, be a ψDO with the presymbol

$$(\psi(\eta) + (1-\psi(\eta))|\eta|_{\mathbf{d}})^s, \tag{10.1}$$

where the function $\psi \in C_0^\infty(\mathbb{R}^n)$ equals 1 in some neighbourhood of 0. Since I_s is semi-elliptic, there exists $I_{-s} \in \Psi^{-s,\mathbf{d}}_{1,0}(M)$ such that

$$R_{-s} := I_{-s}I_s - I, \qquad R_s := I_sI_{-s} - I \in \Psi^{-\infty}(M) \tag{10.2}$$

(see Theorem 10.2). It follows from Theorem 6.7 (see also Corollary 6.10) that the presymbol $\widetilde{a}_{-s}$ of I_{-s} satisfies the condition

$$\widetilde{a}_{-s}(x,\eta) - (\psi(\eta) + (1-\psi(\eta))|\eta|_{\mathbf{d}})^{-s} \in \widetilde{S}^{-s-\varepsilon,\mathbf{d}}_{1,0}(M),$$

where ε is defined by (6.3) with $\varrho = 1$.

Let $I_0 := I$ be the identity mapping. The ψDO I_s is now defined for all $s \in \mathbb{R}$.

Below we will use the notation $\|\cdot\|_p := \|\cdot \mid L_p(M)\|$.

10.3. Theorem. *Let (6.2) be satisfied, $s \in \mathbb{R}$ and $1 < p < \infty$. Then*

$$H_p^{s,\mathbf{d}}(M) := \{u \in \mathcal{D}'(M) : \|u\|_{s,p}^{(\mathbf{d})} := \|I_s u\|_p + \|R_{-s} u\|_p < \infty\}$$

is a Banach space with the norm $\|\cdot\|_{s,p}^{(\mathbf{d})}$. $H_2^{s,\mathbf{d}}(M)$ is a Hilbert space with the inner product

$$(u,v)_s^{(\mathbf{d})} := \int_M I_s u \overline{(I_s v)}\, d\mathcal{M} + \int_M R_{-s} u \overline{(R_{-s} v)}\, d\mathcal{M}.$$

Proof. Since $u = I_{-s} I_s u - R_{-s} u$ (see (10.2)), $\|u\|_{s,p}^{(\mathbf{d})} = 0$ implies $u \equiv 0$. Hence $\|\cdot\|_{s,p}^{(\mathbf{d})}$ is a norm and $(\cdot,\cdot)_s^{(\mathbf{d})}$ is an inner product. So, we only need to prove the completeness of $H_p^{s,\mathbf{d}}(M)$.

Suppose $u_n \in H_p^{s,\mathbf{d}}(M)$, $n \in \mathbb{N}$, is a Cauchy sequence. Then there exist $v, w \in L_p(M)$ such that

$$\lim_{n\to\infty} \|v - I_s u_n\|_p = 0, \qquad \lim_{n\to\infty} \|w - R_{-s} u_n\|_p = 0. \tag{10.3}$$

The operator $I_{-s} : \mathcal{D}'(M) \to \mathcal{D}'(M)$ is continuous (see Remark 5.3). Therefore the first equalities in (10.2) and (10.3) imply that u_n converges in $\mathcal{D}'(M)$ to some $u \in \mathcal{D}'(M)$. Since R_{-s} is an integral operator with a C^∞-smooth kernel (see Remark 4.6), it maps $\mathcal{D}'(M)$ continuously into $C^\infty(M)$. So, using (10.3) and the continuity of the operator $I_s : \mathcal{D}'(M) \to \mathcal{D}'(M)$, we obtain $I_s u = v \in L_p(M)$, $R_{-s} u = w \in L_p(M)$. Thus, $u \in H_p^{s,\mathbf{d}}(M)$ and it is left to prove that u_n converges to u in $H_p^{s,\mathbf{d}}(M)$-norm.

We have $u = I_{-s} I_s u - R_{-s} u = I_{-s} v - w$ and

$$u - u_n = I_{-s} v - w - I_{-s} I_s u_n + R_{-s} u_n = I_{-s}(v - I_s u_n) - (w - R_{-s} u_n).$$

Consequently,

$$\begin{aligned}\|u - u_n\|_{s,p}^{(\mathbf{d})} \le \|I_s I_{-s}(v - I_s u_n)\|_p + \|R_{-s} I_{-s}(v - I_s u_n)\|_p \\ + \|I_s(w - R_{-s} u_n)\|_p + \|R_{-s}(w - R_{-s} u_n)\|_p.\end{aligned}$$

The first, second and fourth terms on the right hand side converge to 0 due to (10.3) and the continuity of the operators $I_s I_{-s} = I + R_s$ and $R_{-s} I_{-s}$, $R_{-s} \in \Psi^{-\infty}(M)$ in $L_p(M)$. The third term tends to 0 because $R_{-s} u_n$ converges to w in $C^\infty(M)$ and I_s is continuous on $C^\infty(M)$. ∎

10.4. Remark. A distribution $u \in \mathcal{D}'(M)$ belongs to $H_p^{s,\mathbf{d}}(M)$ if and only if $I_s u \in L_p(M)$, since $R_{-s} u \in C^\infty(M) \subset L_p(M)$.

10.5. Theorem. *$C^\infty(M)$ is dense in $H_p^{s,\mathbf{d}}(M)$.*

Proof. Let us take an arbitrary $u \in H_p^{s,\mathbf{d}}(M)$. Then $v := I_s u \in L_p(M)$, $w := R_{-s} u \in C^\infty(M)$ and there exists a sequence $v_n \in C^\infty(M)$, $n \in \mathbb{N}$, which converges to v in $L_p(M)$. Let us show that $u_n := I_{-s} v_n - w \in C^\infty(M)$ converges to u in $H_p^{s,\mathbf{d}}(M)$. We have $u - u_n = I_{-s} v - w - I_{-s} v_n + w = I_{-s}(v - v_n)$ (see (10.2)) and

$$\|u - u_n\|_{s,p}^{(\mathbf{d})} \le \|I_s I_{-s}(v - v_n)\|_p + \|R_{-s} I_{-s}(v - v_n)\|_p.$$

The convergence of the right hand side to 0 can be proved as in the proof of Theorem 10.3. ∎

10.6. Theorem. *For any $\tau > 0$ the space $H_p^{s+\tau,\mathbf{d}}(M)$ is compactly embedded in $H_p^{s,\mathbf{d}}(M)$.*

Proof. Any operator $R \in \Psi^{-\infty}(M)$ maps $\mathcal{D}'(M)$ continuously into $C^\infty(M)$ (see Remark 4.6). Since $H_p^{r,\mathbf{d}}(M)$, $r \in \mathbb{R}$, is continuously embedded in $\mathcal{D}'(M)$ and $C^\infty(M)$ is compactly embedded in $L_p(M)$, the operator $R : H_p^{r,\mathbf{d}}(M) \to L_p(M)$ is compact. It follows from Theorems 6.7, 9.3 and Remark 9.4 that the ψDO $I_s I_{-s-\tau} \in \Psi_{1,0}^{-\tau,\mathbf{d}}(M)$ is compact in $L_p(M)$. Now the statement follows from the equality $I_s u = I_s I_{-s-\tau}(I_{s+\tau}u) - (I_s R_{-s-\tau})u$, $u \in H_p^{s+\tau,\mathbf{d}}(M)$ (see (10.2)). ∎

10.7. Theorem. *Let $s, r \in \mathbb{R}$, $\tau > 0$ and $1 < p < \infty$.*

(i) *If* (6.1) *is satisfied, then any $A \in \Psi_{\varrho,\delta}^{r,\mathbf{d}}(M)$, $0 \le \delta < \varrho \le 1$, is continuous from $H_2^{s,\mathbf{d}}(M)$ to $H_2^{s-r,\mathbf{d}}(M)$ and compact from $H_2^{s,\mathbf{d}}(M)$ to $H_2^{s-r-\tau,\mathbf{d}}(M)$.*

(ii) *If* (6.2) *is satisfied, then any $A \in \Psi_{1,\delta}^{r,\mathbf{d}}(M)$, $0 \le \delta < 1$, is continuous from $H_p^{s,\mathbf{d}}(M)$ to $H_p^{s-r,\mathbf{d}}(M)$ and compact from $H_p^{s,\mathbf{d}}(M)$ to $H_p^{s-r-\tau,\mathbf{d}}(M)$.*

Proof. Let us prove (i). In order to show that $A : H_2^{s,\mathbf{d}}(M) \to H_2^{s-r,\mathbf{d}}(M)$ is continuous it is sufficient to prove that $I_{s-r}A : H_2^{s,\mathbf{d}}(M) \to L_2(M)$ is continuous. The continuity of the last operator follows from the equality $I_{s-r}A = (I_{s-r}AI_{-s})I_s - I_{s-r}AR_{-s}$, since $I_{s-r}AI_{-s} \in \Psi_{\varrho,\delta}^{0,\mathbf{d}}(M)$ is bounded on $L_2(M)$ (see Theorems 6.7 and 7.1). Now the compactness of $A : H_2^{s,\mathbf{d}}(M) \to H_2^{s-r-\tau,\mathbf{d}}(M)$ follows from Theorem 10.6.

The proof of (ii) is almost identical. The only difference is that we use Theorem 7.6 instead of Theorem 7.1. ∎

10.8. Theorem. *Let* (6.2) *be satisfied, $s \in \mathbb{R}$, $1 < p < \infty$, and $1/p + 1/p' = 1$. Then the bilinear form*

$$\langle u, v\rangle = \int_M u(x)v(x)\, d\mathcal{M}(x), \qquad u, v \in C^\infty(M),$$

can be extended to a continuous bilinear form $\langle\cdot,\cdot\rangle : H_p^{s,\mathbf{d}}(M) \times H_{p'}^{-s,\mathbf{d}}(M) \to \mathbb{C}$. For any continuous linear functional $l : H_p^{s,\mathbf{d}}(M) \to \mathbb{C}$ there exists a unique $v \in H_{p'}^{-s,\mathbf{d}}(M)$ such that $l(u) = \langle u, v\rangle$, $\forall u \in H_p^{s,\mathbf{d}}(M)$. The mapping $l \mapsto v$ is an isomorphism of the spaces $(H_p^{s,\mathbf{d}}(M))'$ and $H_{p'}^{-s,\mathbf{d}}(M)$.

Proof. It follows from (10.2) and Theorem 5.1 that

$$\begin{aligned}\langle u, v\rangle &= \langle (I_{-s}I_s - R_{-s})u, v\rangle = \langle I_s u, I'_{-s}v\rangle - \langle R_{-s}u, v\rangle \\ &= \langle I_s u, I'_{-s}v\rangle - \langle R_{-s}u, (I_sI_{-s} - R_s)v\rangle \\ &= \langle I_s u, I'_{-s}v\rangle - \langle I'_s R_{-s}u, I_{-s}v\rangle + \langle R_{-s}u, R_s v\rangle, \qquad u, v \in C^\infty(M),\end{aligned}$$

where $I'_{-s} \in \Psi_{1,0}^{-s,\mathbf{d}}(M)$, $I'_s \in \Psi_{1,0}^{s,\mathbf{d}}(M)$. Applying the Hölder inequality $|\langle f, g\rangle| \le \|f\|_p \|g\|_{p'}$ and Theorem 10.7 we obtain

$$|\langle u, v\rangle| \le \text{const}\, \|u\|_{s,p}^{(\mathbf{d})} \|v\|_{-s,p'}^{(\mathbf{d})}, \qquad \forall u, v \in C^\infty(M).$$

Now our first statement follows from Theorem 10.5.

Let us take an arbitrary continuous linear functional $l : H_p^{s,\mathbf{d}}(M) \to \mathbb{C}$. Since $C^\infty(M)$ is continuously and densely embedded in $H_p^{s,\mathbf{d}}(M)$, there exists a unique $v \in \mathcal{D}'(M)$ such that $l(u) = \langle u, v\rangle$, $\forall u \in C^\infty(M)$. So, we only need to prove that $v \in H_{p'}^{-s,\mathbf{d}}(M)$, i.e. that

$I_{-s}v \in L_{p'}(M)$ (see Remark 10.4). We have

$$\langle u, I_{-s}v\rangle = \langle I'_{-s}u, v\rangle = l(I'_{-s}u).$$

Therefore

$$|\langle u, I_{-s}v\rangle| = |l(I'_{-s}u)| \le \text{const}\,\|I'_{-s}u\|^{(\mathbf{d})}_{s,p} \le \text{const}\,\|u\|_p, \qquad \forall u \in C^\infty(M)$$

(see Theorem 10.7), i.e. $I_{-s}v \in L_{p'}(M)$. ∎

10.9. Remark. It follows from Theorem 10.7(ii) that $\sum_{|\gamma|\le m} c_\gamma \partial_\nu^\gamma$ is continuous from $H^{s,\mathbf{d}}_p(M)$ to $L_p(M)$ if $s \ge m\max\{d_k^{-1}\}$ and $c_\gamma \in C^\infty(M)$. Therefore $H^{s,\mathbf{d}}_p(M)$ is continuously embedded in the standard isotropic Sobolev space $W^m_p(M)$, $m \in \mathbb{Z}_+$, if $s \ge m\max\{d_k^{-1}\}$. Consequently, $H^{s,\mathbf{d}}_p(M)$ is continuously embedded in $C^L(M)$ if $s \ge ([L + n/p] + 1)\max\{d_k^{-1}\}$, where $[\cdot]$ denotes the integer part (see, e.g., [Ad, Theorem 5.4]). On the other hand it is not difficult to show that $I_s u \in C(M) \subset L_p(M)$ for any $u \in C^l(M)$ if l is sufficiently large (see [Shu, §1]). Hence $C^l(M)$ is continuously embedded in $H^{s,\mathbf{d}}_p(M)$ if l is sufficiently large (see Remark 10.4).

The above results admit an obvious extension to the spaces $H^{s,\mathbf{d}}_p(\mathcal{E})$ of sections of a C^∞-smooth vector bundle $\mathcal{E}$ over M and ψDOs acting on such spaces. Using Theorems 5.1, 10.2, 10.7 and 10.8 one can prove in the standard way (see, e.g., [Shu, Theorem 8.1]) the following result.

10.10. Theorem. *Let* (6.1) *be satisfied and* $A \in \Psi^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})$ *be semi-elliptic. Then the operator* $A : H^{s,\mathbf{d}}_2(\mathcal{E}) \to H^{s-r,\mathbf{d}}_2(\mathcal{F})$ *is Fredholm*, $\operatorname{Ker}(A) \subset C^\infty(\mathcal{E})$ *and there exist a finite number of independent of* s *sections* $w_1, \dots, w_N \in C^\infty(\mathcal{F}')$ *such that*

$$\operatorname{Ran}(A) = \{v \in H^{s-r,\mathbf{d}}_2(\mathcal{F}) : \langle v, w_j\rangle_{\mathcal{F},\mathcal{M}} = 0,\ j = 1,\dots,N\}$$

(see (5.7)). *If* $Au \in H^{s-r,\mathbf{d}}_2(\mathcal{F})$ *for* $u \in \mathcal{D}'(M)$ *then* $u \in H^{s,\mathbf{d}}_2(\mathcal{E})$, *if* $Au \in C^\infty(\mathcal{F})$ *then* $u \in C^\infty(\mathcal{E})$. *For any* $B \in \Psi^{r-\tau,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})$, $\tau > 0$, *we have* $\operatorname{Ind}(A+B) = \operatorname{Ind}(A)$. (*Here* "Ker", "Ran" *and* "Ind" *stand for the kernel, range and index respectively.*) *In the case* $\varrho = 1$ *the same is true for the operator* $A : H^{s,\mathbf{d}}_p(\mathcal{E}) \to H^{s-r,\mathbf{d}}_p(\mathcal{F})$, $1 < p < \infty$, *and* $w_1, \dots, w_N \in C^\infty(\mathcal{F}')$ *can be chosen to be independent of* s *and* p.

The last theorem shows that the index of A does not depend on s (or p) and is defined by the equivalence class of the presymbol of A in the quotient space $\widetilde{S}^{r,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})/\widetilde{S}^{r-\tau,\mathbf{d}}_{\varrho,\delta}(\mathcal{E},\mathcal{F})$, $\tau > 0$. It would be interesting to obtain an Atiyah–Singer type formula for the index of A.

10.11. Remark. Theorem 10.10 remains true if we replace $\mathcal{F}'$ and $\langle\cdot,\cdot\rangle_{\mathcal{F},\mathcal{M}}$ by $\mathcal{F}^*$ and $(\cdot,\cdot)_{\mathcal{F},\mathcal{M}}$ respectively (see (5.8)). A C^∞-smooth vector bundle $\mathcal{E}$ is called *Hermitian* if there exists a *Hermitian metric* on $\mathcal{E}$, i.e. a C^∞-smooth function $G : \mathcal{E}\times\mathcal{E} \to \mathbb{C}$ such that its restriction to each fibre $\mathcal{E}_x \times \mathcal{E}_x$ is an inner product on $\mathcal{E}_x$. Every C^∞-smooth vector bundle over a paracompact manifold has a Hermitian metric (see, e.g., [Hus, Ch. 3, Theorems 5.5 and 9.5]). Using a Hermitian metric on $\mathcal{E}$ one can identify $\mathcal{E}^*$ with $\mathcal{E}$.

10.12. Theorem. *Let* (6.2) *be satisfied and* $r \in \mathbb{R}$. *Then there exists* $\Lambda_r \in \Psi^{r,\mathbf{d}}_{1,0}(\mathcal{E},\mathcal{E})$ *with the presymbol*

$$(\psi(\eta) + (1-\psi(\eta))|\eta|_{\mathbf{d}})^r I \in \widetilde{S}^{r,\mathbf{d}}_{1,0}(\mathcal{E},\mathcal{E}) \tag{10.4}$$

(cf. (10.1)) *such that* $\Lambda_r : H_p^{s,\mathbf{d}}(\mathcal{E}) \to H_p^{s-r,\mathbf{d}}(\mathcal{E})$ *is an isomorphism for every* $s \in \mathbb{R}$ *and* $1 < p < \infty$. *The* ψ*DO* Λ_r *also induces an isomorphism* $C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$.

Proof. According to the preceding remark we can suppose that $\mathcal{E}$ is Hermitian. Let $J_r \in \Psi_{1,0}^{r,\mathbf{d}}(\mathcal{E},\mathcal{E})$ be a ψDO with the presymbol (10.4). J_r is semi-elliptic and Theorem 10.10 applies to it. We have $\mathrm{Ind}(J_r) = -\mathrm{Ind}(J_r^*)$. On the other hand, Theorem 5.1 implies that $J_r - J_r^* \in \Psi_{1,0}^{r-\min\{d_k^{-1}\},\mathbf{d}}(\mathcal{E},\mathcal{E})$. Here we identify $\mathcal{E}^*$ with $\mathcal{E}$ (see Remark 10.11). So, $\mathrm{Ind}(J_r) = \mathrm{Ind}(J_r^*)$ due to Theorem 10.10. Therefore $\mathrm{Ind}(J_r) = 0$. Let $u_1, \ldots, u_N \in C^\infty(\mathcal{E})$ be a basis of $\mathrm{Ker}(J_r)$. There exist sections $w_1, \ldots, w_N \in C^\infty(\mathcal{E})$ independent of s and p such that

$$\mathrm{Ran}(J_r) = \{v \in H_p^{s-r,\mathbf{d}}(\mathcal{E}) : (v, w_j)_{G,\mathcal{M}} = 0,\ j = 1, \ldots, N\},$$

where $(v,w)_{G,\mathcal{M}} := \int_M G(v(x), w(x))\, d\mathcal{M}(x)$ and G is a Hermitian metric on $\mathcal{E}$. It is easy to see that $K := \sum_{j=1}^N (\cdot, u_j)_{G,\mathcal{M}} w_j$ is an integral operator with a C^∞-smooth kernel and $\Lambda_r := J_r + K$ induces isomorphisms $C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$ and $H_p^{s,\mathbf{d}}(\mathcal{E}) \to H_p^{s-r,\mathbf{d}}(\mathcal{E})$ for every $s \in \mathbb{R}$ and $1 < p < \infty$. ■

The following *Gårding inequality* can be proved exactly as in the standard calculus of ψDOs (see, e.g., [Ta1, Ch. II, §8]).

10.13. THEOREM. *Suppose* (6.1) *is satisfied*, $\mathcal{E}$ *is Hermitian*, $A \in \Psi_{\varrho,\delta}^{r,\mathbf{d}}(\mathcal{E},\mathcal{E})$ *and* $\mathrm{Re}\,\widetilde{\sigma}_A(x,\eta) := (\widetilde{\sigma}_A(x,\eta) + \widetilde{\sigma}_A(x,\eta)^*)/2 \geq C|\eta|_{\mathbf{d}}^r > 0$ *for large* $|\eta|_{\mathbf{d}}$. *Then for any* $C_0 < C$ *and any* $s \in \mathbb{R}$ *there exists* $C_1 > 0$ *such that*

$$\mathrm{Re}(Au,u)_{\mathcal{E},\mathcal{M}} \geq C_0 \|u\|_{r/2,2}^2 - C_1 \|u\|_{s,2}^2, \qquad \forall u \in H_2^{r/2,\mathbf{d}}(\mathcal{E}).$$

11. The resolvent of a semi-elliptic ψDO

In this section we deal with ψDOs from $\Psi_{1,0}^{r,\mathbf{d}}$. We suppose as usual that (6.2) is satisfied. We have

$$\varepsilon = \min\{d_l^{-1}, d_j^{-1} + d_k^{-1} - d_m^{-1} : l = 1, \ldots, n,\ C_{j,k}^m \not\equiv 0\} > 0$$

(cf. (6.3)). Let $\mathcal{N} := \{\nu_1, \ldots, \nu_n\}$ and

$$\mathbb{Z}_+(\mathcal{N},\mathbf{d}) := \Big\{ \sum_{l=1}^n \tau_l d_l^{-1} + \sum_{C_{j,k}^m \not\equiv 0} \tau_{j,k}^m (d_j^{-1} + d_k^{-1} - d_m^{-1}) : \tau_l, \tau_{j,k}^m \in \mathbb{Z}_+ \Big\}. \tag{11.1}$$

It is clear that

$$\varepsilon = \min \mathbb{Z}_+(\mathcal{N},\mathbf{d}) \setminus \{0\}. \tag{11.2}$$

11.1. LEMMA. *If* $|\alpha + \beta| \geq 2$ *and*

$$\partial_{\nu(y)}^\alpha \partial_{\nu(z)}^\beta c_j(y,z)|_{y=z=x} \neq 0,$$

then $|(\alpha+\beta) : \mathbf{d}| - d_j^{-1} \in \mathbb{Z}_+(\mathcal{N},\mathbf{d})$.

Proof. Acting as in the proof of Lemma 6.1 we find that at least one term

$$\partial_\nu^{\mu^{(1)}} C_{k_1,m_1}^j(x) \partial_\nu^{\mu^{(2)}} C_{k_2,m_2}^{j_2}(x) \ldots \partial_\nu^{\mu^{(q)}} C_{k_q,m_q}^{j_q}(x)$$

of at least one of the scalars (2.16) does not equal 0 and

$$\begin{aligned}|(\alpha+\beta):\mathbf{d}|-d_j^{-1} &= |(\alpha+\beta):\mathbf{d}|-d_j^{-1}+(d_{j_2}^{-1}-d_{j_2}^{-1})+\ldots+(d_{j_q}^{-1}-d_{j_q}^{-1})\\ &= |\mu^{(1)}:\mathbf{d}|+\ldots+|\mu^{(q)}:\mathbf{d}|+(d_{k_1}^{-1}+d_{m_1}^{-1}-d_j^{-1})\\ &\quad+(d_{k_2}^{-1}+d_{m_2}^{-1}-d_{j_2}^{-1})+\ldots+(d_{k_q}^{-1}+d_{m_q}^{-1}-d_{j_q}^{-1})\in\mathbb{Z}_+(\mathcal{N},\mathbf{d}).\ \blacksquare\end{aligned}$$

This lemma and the equalities (6.7) imply the following result.

11.2. Corollary. *If*

$$\partial^\alpha_{\nu(y)}\partial^\beta_{\nu(z)}h_j(x,y,z)|_{y=z=x}\neq 0,$$

then $|(\alpha+\beta):\mathbf{d}|-d_j^{-1}\in\mathbb{Z}_+(\mathcal{N},\mathbf{d})$.

Our next statement concerns the polynomials $\mathcal{P}^{\mathcal{J}}_{\beta,\gamma}$ defined by (6.15).

11.3. Lemma. $|(\beta+\gamma):\mathbf{d}|-\mathbf{d}(\mathcal{P}^{\mathcal{J}}_{\beta,\gamma})\in\mathbb{Z}_+(\mathcal{N},\mathbf{d})$ *for any* $\beta,\gamma\in\mathbb{Z}^n_+$ *such that* $\mathcal{P}^{\mathcal{J}}_{\beta,\gamma}\not\equiv 0$. *The same is true for the* $\mathbf{d}$*-degree of any monomial of the polynomial* $\mathcal{P}^{\mathcal{J}}_{\beta,\gamma}$.

Proof. This follows from the preceding corollary and the fact that a factor η_j can appear in a monomial of the polynomial $\mathcal{P}^{\mathcal{J}}_{\beta,\gamma}$ only together with some partial derivative of h_j evaluated at $y=z=x$ (cf. the proof of Lemma 6.5). ■

11.4. Definition. Let $\mu,\kappa\in\mathbb{C}$. A morphism $\widetilde{b}\in\widetilde{S}^{\operatorname{Re}\mu,\mathbf{d}}_{1,0}(\mathcal{E},\mathcal{F})$ will be called *almost* $\mathbf{d}$*-homogeneous of degree* $\mu\in\mathbb{C}$ if

$$\widetilde{b}(x,\tau^{1/d_1}\eta_1,\ldots,\tau^{1/d_n}\eta_n)=\tau^\mu\widetilde{b}(x,\eta),\qquad |\eta|_{\mathbf{d}}\geq 1,\ \tau\geq 1. \tag{11.3}$$

We will say that $\widetilde{a}\in\widetilde{S}^{\operatorname{Re}\kappa,\mathbf{d}}_{1,0}(\mathcal{E},\mathcal{F})$ belongs to $\mathcal{H}\widetilde{S}^{\kappa,\mathbf{d}}(\mathcal{E},\mathcal{F})$ if

$$\widetilde{a}(x,\eta)\sim\sum_{l\in\mathbb{Z}_+(\mathcal{N},\mathbf{d})}\widetilde{a}_l(x,\eta) \tag{11.4}$$

(cf. (3.14)), where each $\widetilde{a}_l\in\widetilde{S}^{\operatorname{Re}\kappa-l,\mathbf{d}}_{1,0}(\mathcal{E},\mathcal{F})$ is almost $\mathbf{d}$-homogeneous of degree $\kappa-l$. Correspondingly we say that $A\in\Psi^{\operatorname{Re}\kappa,\mathbf{d}}_{1,0}(\mathcal{E},\mathcal{F})$ belongs to $\mathcal{H}\Psi^{\kappa,\mathbf{d}}(\mathcal{E},\mathcal{F})$ if its presymbol $\widetilde{a}$ belongs to $\mathcal{H}\widetilde{S}^{\kappa,\mathbf{d}}(\mathcal{E},\mathcal{F})$. In this case $\widetilde{a}_0$ from the right hand side of (11.4) is called the *principal presymbol* of A.

We take $l\in\mathbb{Z}_+(\mathcal{N},\mathbf{d})$ in the asymptotic expansion (11.4) for the following reason. Let $A\in\mathcal{H}\Psi^{\kappa_1,\mathbf{d}}(\mathcal{E},\mathcal{F})$, $B\in\mathcal{H}\Psi^{\kappa_2,\mathbf{d}}(\mathcal{J},\mathcal{E})$ and suppose at least one of these ψDOs is properly supported. Then it follows from Theorem 6.7 and Lemma 11.3 that $AB\in\mathcal{H}\Psi^{\kappa_1+\kappa_2,\mathbf{d}}(\mathcal{J},\mathcal{F})$. On the other hand, Corollary 6.10 shows that even in the case where the presymbols of A and$B$ are almost $\mathbf{d}$-homogeneous of degrees κ_1 and κ_2 respectively, the last inclusion does not necessarily hold if we take $\{\sum_{l=1}^n\tau_l d_l^{-1}:\tau_l\in\mathbb{Z}_+\}$ instead of $\mathbb{Z}_+(\mathcal{N},\mathbf{d})$ in Definition 11.4.

Let $A\in\mathcal{H}\Psi^{\kappa,\mathbf{d}}(\mathcal{E},\mathcal{E})$ be semi-elliptic and $\kappa\in(0,\infty)$. We are going to construct a special parametrix of $A-\lambda I$ which depends on λ "in a nice way". Since the principal presymbol $\widetilde{a}_0$ of A is almost $\mathbf{d}$-homogeneous, the set of all eigenvalues of $\widetilde{a}_0(x,\eta)$, $x\in\Xi$, $\eta\in\mathbb{R}^n$, coincides with some cone with vertex at 0 outside a bounded subset of $\mathbb{C}$, for any compact $\Xi\subset M$. *We will suppose that the set of all eigenvalues of* $\widetilde{a}_0(x,\eta)$, $x\in M$, $\eta\in\mathbb{R}^n$, *lies in a closed cone* $\mathcal{C}\subset\mathbb{C}$ *with vertex at* 0, $\mathcal{C}\neq\mathbb{C}$. Since A is semi-elliptic, Definition 11.4 implies that $\widetilde{a}_0(x,\eta)$ is invertible for $|\eta|_{\mathbf{d}}\geq 1$. If A is a differential operator,

$\widetilde{a}_0(x,\eta)$ is a polynomial and hence is invertible for $\eta \neq 0$. *If A is not a differential operator we will suppose*, changing $\widetilde{a}_0(x,\eta)$ for $|\eta|_{\mathbf{d}} < 1$ if necessary, *that $\widetilde{a}_0(x,\eta)$ is invertible for all $\eta \in \mathbb{R}^n$.*

We are looking for a parametrix $B(\lambda) \in \Psi_{1,0}^{-\kappa,\mathbf{d}}(\mathcal{E},\mathcal{E})$, $\lambda \in \mathbb{C} \setminus \mathcal{C}$, of $A - \lambda I$ such that its presymbol $\widetilde{b}(\lambda)$ admits the following asymptotic expansion:

$$\widetilde{b}(\lambda; x, \eta) \sim \sum_{q \in \mathbb{Z}_+(\mathcal{N},\mathbf{d})} \widetilde{b}_q(\lambda; x, \eta), \tag{11.5}$$

where $\widetilde{b}_q(\lambda) \in \widetilde{S}_{1,0}^{-\kappa-q,\mathbf{d}}(\mathcal{E},\mathcal{E})$ are rational functions of λ and

$$\widetilde{b}_q(\tau^\kappa \lambda; x, \tau^{1/d_1}\eta_1, \ldots, \tau^{1/d_n}\eta_n) = \tau^{-\kappa-q}\widetilde{b}_q(\lambda; x, \eta), \qquad |\eta|_{\mathbf{d}} \geq 1, \ \tau \geq 1. \tag{11.6}$$

Applying Theorem 6.7 we obtain from $AB(\lambda) - I \in \Psi^{-\infty}(\mathcal{E},\mathcal{E})$ the system

$$(\widetilde{a}_0(x,\eta) - \lambda I)\widetilde{b}_0(\lambda; x, \eta) = I, \tag{11.7}$$

$$\begin{aligned}(\widetilde{a}_0(x,\eta) - \lambda I)\widetilde{b}_q(\lambda; x, \eta) + \sum_{\substack{l+p+|\alpha:\mathbf{d}|+|(\beta+\gamma):\mathbf{d}|-\mathbf{d}(\mathcal{P}^{\mathcal{E}}_{\beta,\gamma})+j=q \\ p<q}} \frac{i^{-(|\alpha|+|\beta|+|\gamma|)}}{\alpha!\beta!\gamma!} \\ \times \partial_\eta^{\beta+\alpha}\widetilde{a}_l(x,\eta)\partial_\eta^\gamma(\nabla^{\mathcal{E},\mathcal{E}}_{\nu(x)})^\alpha \widetilde{b}_p(\lambda; x, \eta)\mathcal{P}^{\mathcal{E}}_{\beta,\gamma,j}(x,\eta) = 0, \ q \in \mathbb{Z}_+(\mathcal{N},\mathbf{d}) \setminus \{0\},\end{aligned} \tag{11.8}$$

where $\mathcal{P}^{\mathcal{E}}_{\beta,\gamma,j}$ is the sum of all monomials of the polynomial $\mathcal{P}^{\mathcal{E}}_{\beta,\gamma}$ of $\mathbf{d}$-degree $\mathbf{d}(\mathcal{P}^{\mathcal{E}}_{\beta,\gamma}) - j$. From this system we can find successively $\widetilde{b}_0(\lambda), \widetilde{b}_\varepsilon(\lambda), \ldots$, $\lambda \in \mathbb{C} \setminus \mathcal{C}$:

$$\begin{aligned}\widetilde{b}_0(\lambda; x, \eta) &= (\widetilde{a}_0(x,\eta) - \lambda I)^{-1}, \\ \widetilde{b}_\varepsilon(\lambda; x, \eta) &= -(\widetilde{a}_0(x,\eta) - \lambda I)^{-1}\widetilde{a}_\varepsilon(x,\eta)(\widetilde{a}_0(x,\eta) - \lambda I)^{-1} \\ &\quad - i \sum_{d_k^{-1}=\varepsilon} (\widetilde{a}_0(x,\eta) - \lambda I)^{-1}\partial_{\eta_k}\widetilde{a}_0(x,\eta)(\widetilde{a}_0(x,\eta) - \lambda I)^{-1}\nabla^{\mathcal{E},\mathcal{E}}_{\nu_k(x)}\widetilde{a}_0(x,\eta) \\ &\quad \times (\widetilde{a}_0(x,\eta) - \lambda I)^{-1} - \frac{1}{2i}\sum_{d_k^{-1}+d_m^{-1}-d_s^{-1}=\varepsilon} C^s_{k,m}(x)\eta_s(\widetilde{a}_0(x,\eta) - \lambda I)^{-1} \\ &\quad \times \partial_{\eta_k}\widetilde{a}_0(x,\eta)(\widetilde{a}_0(x,\eta) - \lambda I)^{-1}\partial_{\eta_m}\widetilde{a}_0(x,\eta)(\widetilde{a}_0(x,\eta) - \lambda I)^{-1}\end{aligned} \tag{11.9}$$

(see (11.2) and Corollary 6.10).

11.5. Lemma. *The system* (11.7), (11.8) *has a unique solution and each $\widetilde{b}_q(\lambda; x, \eta)$ is a linear combination of products of factors $(\widetilde{a}_0(x,\eta) - \lambda I)^{-1}$ and $\widetilde{b}_{q,t}(x,\eta)$, $t = 1, \ldots, N$. For each of these products the number L of the factors $(\widetilde{a}_0(x,\eta) - \lambda I)^{-1}$ satisfies the inequality $L \leq 2\varepsilon^{-1}q + 1$, the left factor is always $(\widetilde{a}_0(x,\eta) - \lambda I)^{-1}$ and $\widetilde{b}_{q,t} \in \mathcal{H}\Psi^{\kappa_t,\mathbf{d}}(\mathcal{E},\mathcal{E})$ is an almost $\mathbf{d}$-homogeneous* (*of degree κ_t*) *product of $\widetilde{a}_l$, $C^s_{k,m}$, their* (*covariant*) *derivatives and powers of η, where $\sum_{t=1}^N \kappa_t = (L-1)\kappa - q$. If $q \neq 0$, then $L \geq 2$.*

Proof. The conditions

$$l + p + |\alpha : \mathbf{d}| + |(\beta+\gamma) : \mathbf{d}| - \mathbf{d}(\mathcal{P}^{\mathcal{E}}_{\beta,\gamma}) + j = q, \qquad p < q$$

from (11.8) imply $p + \varepsilon(|\alpha| + |\gamma|) \leq q$ and $q - p \geq \varepsilon$ (see Lemmas 6.5, 11.3 and (11.1), (11.2)). Therefore $1 + |\alpha| + |\gamma| \leq 2\varepsilon^{-1}(q - p)$. Now the inequality $L \leq 2\varepsilon^{-1}q + 1$ follows by induction. The remaining part of the lemma is also proved by induction. ∎

11.6. Remark. If A is a differential operator, then $\kappa_t \geq 0$. So, $(L-1)\kappa - q \geq 0$, i.e. $L \geq q/\kappa + 1$.

Let $\Xi \subset M$ be an arbitrary compact set. It is easy to see that if A is not a differential operator and $\widetilde{a}_0(x,\eta)$ is invertible for all $\eta \in \mathbb{R}^n$, then $\widetilde{b}_q(\lambda; x, \eta)$ is well defined for $\lambda \in (\mathbb{C} \setminus \mathcal{C}) \cup \{\mu \in \mathbb{C} : |\mu| \leq \delta_0\}$, $x \in \Xi$, where $\delta_0 = \delta_0(\Xi) > 0$ is sufficiently small. This property will be used in the construction of complex powers of A (see Section 12). If A is a differential operator, we can, of course, change $\widetilde{a}_0(x,\eta)$ for $|\eta|_{\mathbf{d}} < 1$ and make it invertible for all $\eta \in \mathbb{R}^n$, but in this case Remark 11.6 is false. It is convenient to introduce the following cut-off function. Since A is semi-elliptic and its presymbol is a polynomial in η, the numbers $l_k := \kappa d_k$ are integers (see (3.4)). The polynomial $p_{2\kappa}(\eta) := \sum_{k=1}^n \eta_k^{2l_k}$ is $\mathbf{d}$-homogeneous of degree 2κ and there exist $C_1, C_2 > 0$ such that $C_1|\eta|_{\mathbf{d}}^{2\kappa} \leq p_{2\kappa}(\eta) \leq C_2|\eta|_{\mathbf{d}}^{2\kappa}$, $\forall \eta \in \mathbb{R}^n$. Our cut-off function is $\widetilde{\chi}_\kappa(\lambda,\eta) = h(p_{2\kappa}(\eta) + |\lambda|^2)$, where $h \in C^\infty(\mathbb{R})$, $h(t) = 0$ for $t \leq 1/2$, $h(t) = 1$ for $t \geq 1$.

We will use the notation

$$\widetilde{b}_{0,q}(\lambda; x, \eta) := \widetilde{b}_q(\lambda; x, \eta), \tag{11.10}$$

where it is supposed that $\widetilde{a}_0(x,\eta)$ *is invertible for all* $\eta \in \mathbb{R}^n$. If A is a differential operator, we will also use the notation

$$\widetilde{b}_{\chi,q}(\lambda; x, \eta) := \widetilde{\chi}_\kappa(\lambda,\eta)\widetilde{b}_q(\lambda; x, \eta). \tag{11.11}$$

It is easy to see that $\widetilde{b}_{0,q}(\lambda; x, \eta)$ and $\widetilde{b}_{\chi,q}(\lambda; x, \eta)$ are well defined for $\lambda \in (\mathbb{C} \setminus \mathcal{C}) \cup \{\mu \in \mathbb{C} : |\mu| \leq \delta_0\}$, $x \in \Xi$, if $\delta_0 = \delta_0(\Xi) > 0$ is sufficiently small.

11.7. Lemma. *Let* n_0 *be the dimension of a fibre of* $\mathcal{E}$. *Then for any compact set* $\Xi \subset M$,

$$\begin{aligned} &\|\partial_\eta^\alpha \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_1}(x)} \cdots \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_m}(x)} \widetilde{b}_{0,q}(\lambda; x, \eta)\| \\ &\leq \mathrm{const}_{\Xi,q,\alpha,j_1,\dots,j_m} \sum_{L_0=l}^{[2\varepsilon^{-1}q]+1+|\alpha|+m} (1+|\eta|_{\mathbf{d}})^{(L_0-1)\kappa-q-|\alpha:\mathbf{d}|}(1+|\lambda|+|\eta|_{\mathbf{d}}^{\kappa})^{-L_0}\left(\frac{|\lambda|}{d(\lambda)}\right)^{n_0L_0}, \\ &\forall \lambda \in \mathbb{C}\setminus\mathcal{C},\ \forall \alpha \in \mathbb{Z}_+^n,\ \forall j_1,\dots,j_m \in \{1,\dots,n\},\ \forall m \in \mathbb{Z}_+,\ \forall \eta \in \mathbb{R}^n,\ \forall x \in \Xi, \end{aligned} \tag{11.12}$$

where $l = 1$ *if* $q + |\alpha| + m = 0$, $l = 2$ *if* $q + |\alpha| + m > 0$, $[\cdot]$ *denotes the integer part and* $d(\lambda) := \mathrm{dist}(\lambda, \mathcal{C})$. *If* $\mathcal{E}$ *is Hermitian and* $\widetilde{a}_0(x,\eta)$ *is normal for all* $(x,\eta) \in M \times \mathbb{R}^n$, *then one can take* $(|\lambda|/d(\lambda))^{L_0}$ *instead of* $(|\lambda|/d(\lambda))^{n_0L_0}$ *on the right hand side of* (11.12).

Proof. It can easily be derived from Lemma 11.5 that

$$\begin{aligned} &\|\partial_\eta^\alpha \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_1}(x)} \cdots \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_m}(x)} \widetilde{b}_{0,q}(\lambda; x, \eta)\| \\ &\quad\leq \mathrm{const} \sum_{L_0=l}^{[2\varepsilon^{-1}q]+1+|\alpha|+m} (1+|\eta|_{\mathbf{d}})^{(L_0-1)\kappa-q-|\alpha:\mathbf{d}|}\|(\widetilde{a}_0(x,\eta)-\lambda I)^{-1}\|^{L_0}. \end{aligned} \tag{11.13}$$

Let $\mu_1(x,\eta), \dots, \mu_{n_0}(x,\eta)$ be the eigenvalues of $\widetilde{a}_0(x,\eta)$. The simple inequality

$$\|(\widetilde{a}_0(x,\eta)-\lambda I)^{-1}\| \leq \mathrm{const}\, \|\widetilde{a}_0(x,\eta)-\lambda I\|^{n_0-1}|\det(\widetilde{a}_0(x,\eta)-\lambda I)|^{-1}$$

(see, e.g., [Kat, Ch. I, (4.12)]) implies

$$\|(\widetilde{a}_0(x,\eta)-\lambda I)^{-1}\| \le \mathrm{const}\,(1+|\lambda|+|\eta|_{\mathbf{d}}^{\kappa})^{n_0-1}\prod_{l=1}^{n_0}|\mu_l(x,\eta)-\lambda|^{-1}. \tag{11.14}$$

An elementary geometric argument shows that

$$\frac{d(\lambda)}{|\lambda|} \le \frac{|\mu_l(x,\eta)-\lambda|}{|\mu_l(x,\eta)|}, \quad \text{i.e.} \quad |\mu_l(x,\eta)-\lambda| \ge d(\lambda)\frac{|\mu_l(x,\eta)|}{|\lambda|}.$$

Since $|\mu_l(x,\eta)-\lambda| \ge d(\lambda)$, we have

$$|\mu_l(x,\eta)-\lambda| \ge \frac{1}{2}d(\lambda)\frac{|\lambda|+|\mu_l(x,\eta)|}{|\lambda|}.$$

Our assumptions imply that there exists $\delta_0>0$ such that $|\mu_l(x,\eta)|\ge\delta_0$, $l=1,\dots,n_0$, for any $\eta\in\mathbb{R}^n$ and $x\in\Xi$. Using the equality

$$\mu_l(x,\tau^{1/d_1}\eta_1,\dots,\tau^{1/d_n}\eta_n)=\tau^{\kappa}\mu_l(x,\eta), \qquad |\eta|_{\mathbf{d}}\ge 1,\ \tau\ge 1,$$

we obtain $|\mu_l(x,\eta)|\ge \mathrm{const}\,(1+|\eta|_{\mathbf{d}}^{\kappa})$ for any $\eta\in\mathbb{R}^n$ and $x\in\Xi$. Therefore

$$|\mu_l(x,\eta)-\lambda|^{-1} \le \mathrm{const}\,\frac{|\lambda|}{d(\lambda)}(1+|\lambda|+|\eta|_{\mathbf{d}}^{\kappa})^{-1}. \tag{11.15}$$

Substituting this estimate into (11.14) we get

$$\|(\widetilde{a}_0(x,\eta)-\lambda I)^{-1}\| \le \mathrm{const}\,(1+|\lambda|+|\eta|_{\mathbf{d}}^{\kappa})^{-1}(|\lambda|/d(\lambda))^{n_0}.$$

Now (11.12) follows from (11.13).

If $\mathcal{E}$ is Hermitian and $\widetilde{a}_0(x,\eta)$ is normal we can use

$$\|(\widetilde{a}_0(x,\eta)-\lambda I)^{-1}\| \le \mathrm{const}\max_{l=1,\dots,n_0}|\mu_l(x,\eta)-\lambda|^{-1}$$

instead of (11.14). This proves the last statement of the lemma. ■

Let us introduce the following "parabolic neighbourhood" of the cone $\mathcal{C}$:

$$\Sigma(\mathcal{C},\theta) := \{\lambda\in\mathbb{C} : d(\lambda) < \mathrm{const}\,|\lambda|^{1-\theta}\}, \qquad 0\le\theta\le 1. \tag{11.16}$$

11.8. Corollary. *For any compact set $\Xi\subset M$,*

$$\|\partial_\eta^\alpha \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_1}(x)}\cdots\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_m}(x)}\widetilde{b}_{0,q}(\lambda;x,\eta)\| \tag{11.17}$$

$$\le \mathrm{const}_{\Xi,q,\alpha,j_1,\dots,j_m}(1+|\eta|_{\mathbf{d}})^{(l-1)\kappa-q-|\alpha:\mathbf{d}|}(1+|\lambda|+|\eta|_{\mathbf{d}}^{\kappa})^{-l}\left(\frac{|\lambda|}{d(\lambda)}\right)^{n_0([2\varepsilon^{-1}q]+1+|\alpha|+m)}, \quad \forall\lambda\in\mathbb{C}\setminus\mathcal{C},$$

$$\|\partial_\eta^\alpha \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_1}(x)}\cdots\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_m}(x)}\widetilde{b}_{0,q}(\lambda;x,\eta)\| \tag{11.18}$$

$$\le \mathrm{const}_{\Xi,q,\alpha,j_1,\dots,j_m}(1+|\eta|_{\mathbf{d}})^{\theta n_0([2\varepsilon^{-1}q]+|\alpha|+m)\kappa+(l-1)(1-\theta n_0)\kappa-q-|\alpha:\mathbf{d}|}(1+|\lambda|+|\eta|_{\mathbf{d}}^{\kappa})^{l(\theta n_0-1)},$$

$$\forall\lambda\in\mathbb{C}\setminus\Sigma(\mathcal{C},\theta),\ \forall\theta\in[0,1/n_0],\ \forall\alpha\in\mathbb{Z}_+^n,$$

$$\forall j_1,\dots,j_m\in\{1,\dots,n\},\ \forall m\in\mathbb{Z}_+,\ \forall\eta\in\mathbb{R}^n,\ \forall x\in\Xi,$$

where $l=1$ if $q+|\alpha|+m=0$, $l=2$ if $q+|\alpha|+m>0$. If $\mathcal{E}$ is Hermitian and $\widetilde{a}_0(x,\eta)$ is normal for all $(x,\eta)\in M\times\mathbb{R}^n$, one can take 1 *instead of n_0 on the right hand sides of* (11.17) *and* (11.18).

Proof. (11.17) is a direct consequence of (11.12). Let us prove (11.18). Since $\lambda \in \mathbb{C} \setminus \Sigma(\mathcal{C}, \theta)$, we have $|\lambda|/d(\lambda) \le \text{const}\, |\lambda|^\theta$. It is clear that $\theta n_0 \le 1$ implies

$$(1 + |\lambda| + |\eta|_{\mathbf{d}}^\kappa)^{-1} |\lambda|^{\theta n_0} \le (1 + |\lambda| + |\eta|_{\mathbf{d}}^\kappa)^{\theta n_0 - 1} \le (1 + |\eta|_{\mathbf{d}}^\kappa)^{\theta n_0 - 1}.$$

Therefore the RHS of (11.12) can be estimated by

$$\text{const} \sum_{L_0 = l}^{[2\varepsilon^{-1} q] + 1 + |\alpha| + m} (1 + |\eta|_{\mathbf{d}})^{(L_0 - 1)\kappa - q - |\alpha : \mathbf{d}| + (\theta n_0 - 1)(L_0 - l)\kappa} (1 + |\lambda| + |\eta|_{\mathbf{d}}^\kappa)^{-l} |\lambda|^{l \theta n_0}$$
$$\le \text{const}\, (1 + |\eta|_{\mathbf{d}})^{\theta n_0 ([2\varepsilon^{-1} q] + |\alpha| + m)\kappa + (l-1)(1 - \theta n_0)\kappa - q - |\alpha : \mathbf{d}|} (1 + |\lambda| + |\eta|_{\mathbf{d}}^\kappa)^{l(\theta n_0 - 1)}. \quad \blacksquare$$

11.9. Remark. If A is a differential operator, (11.12), (11.17) and (11.18) remain valid if we replace $\widetilde{b}_{0,q}(\lambda; x, \eta)$ by $\widetilde{b}_{\chi,q}(\lambda; x, \eta)$ and l by the smallest integer which is greater than or equal to $(q + |\alpha : \mathbf{d}|)/\kappa + 1$. Indeed, similarly to Remark 11.6, in the analogue of (11.13) we have $(L_0 - 1)\kappa - q - |\alpha : \mathbf{d}| \ge 0$, i.e. $L_0 \ge (q + |\alpha : \mathbf{d}|)/\kappa + 1$. Note also that we can suppose $|\eta|_{\mathbf{d}}^\kappa + |\lambda| > \text{const}$ with a positive constant, because $b_{\chi,q}(\lambda; x, \eta) = 0$ otherwise (see (11.11)). Therefore the inequality $|\mu_l(x, \eta)| \ge \text{const}\, |\eta|_{\mathbf{d}}^\kappa$, $\eta \in \mathbb{R}^n$, $x \in \Xi$, implies

$$|\lambda| + |\mu_l(x, \eta)| \ge \text{const}\, (|\lambda| + |\eta|_{\mathbf{d}}^\kappa) \ge \text{const}\, (1 + |\lambda| + |\eta|_{\mathbf{d}}^\kappa).$$

Hence (11.15) holds and (11.12) follows as above.

Let us define a ψDO $B^{(N)}(\lambda)$, $N \in \mathbb{Z}_+(\mathcal{N}, \mathbf{d})$, by the formula

$$(11.19)\qquad (B^{(N)}(\lambda)\omega)(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_M e^{-i\langle c(x,y), \eta\rangle} \sum_{q \le N} \widetilde{b}_{0,q}(\lambda; x, \eta) \Phi_{x,y}^{\mathcal{E}} \omega(y) \Upsilon^\theta(x, y) \chi(x, y)\, d\mathcal{M}(y)\, d\eta,$$
$$\forall \omega \in C_0^\infty(\mathcal{E})$$

(see (4.10), (4.13)). We can suppose that χ is properly supported, i.e. for any compact set $\Xi \subset M$ the intersections of the support of χ with $\Xi \times M$ and $M \times \Xi$ are compact in $M \times M$. Then $B^{(N)}(\lambda)$ is properly supported. Let us consider the operator

$$(11.20)\qquad T^{(N)}(\lambda) := (A - \lambda I) B^{(N)}(\lambda) - I.$$

11.10. Lemma. *For any $L \in \mathbb{Z}_+$ and any $N \ge n + \kappa + L \max\{d_k^{-1}\}$, $N \in \mathbb{Z}_+(\mathcal{N}, \mathbf{d})$, there exists $\widetilde{N} \in \mathbb{R}$ such that $T^{(N)}(\lambda)$ is an integral operator with a kernel $r^{(N)}(\lambda; x, y)$ satisfying the estimate*

$$(11.21)\qquad \|\nabla_{\nu_{j_1}(x)}^{\mathcal{E},\mathcal{E}} \dots \nabla_{\nu_{j_{m_1}}(x)}^{\mathcal{E},\mathcal{E}} \nabla_{\nu_{k_1}(y)}^{\mathcal{E},\mathcal{E}} \dots \nabla_{\nu_{k_{m_2}}(y)}^{\mathcal{E},\mathcal{E}} r^{(N)}(\lambda; x, y)\|$$
$$\le \text{const}_{\Xi, N, j_1, \dots, j_{m_1}, k_1, \dots, k_{m_2}} (1 + |\lambda|)^{-1} \left(\frac{|\lambda|}{d(\lambda)}\right)^{\widetilde{N}},$$

$j_1, \dots, j_{m_1}, k_1, \dots, k_{m_2} \in \{1, \dots, n\}$, $m_1, m_2 \in \mathbb{Z}_+$, $m_1 + m_2 \le L$, $\forall \lambda \in \mathbb{C} \setminus \mathcal{C}$, $\forall x, y \in \Xi$, for any compact set $\Xi \subset M$.

Proof. The lemma is proved by using the arguments from the proof of Theorem 6.7, (11.7), (11.8), (11.10) and (11.17). Note that the function $(1 + |\eta|_{\mathbf{d}})^{-s}$ is integrable on $\mathbb{R}^n$

if $s > n$. This follows from the equality

$$\mathrm{Vol}\{\eta \in \mathbb{R}^n : \tau \le |\eta|_{\mathbf{d}} \le 2\tau\} = \tau^n \,\mathrm{Vol}\{\eta \in \mathbb{R}^n : 1 \le |\eta|_{\mathbf{d}} \le 2\}, \qquad \tau > 0$$

(cf. the proof of Lemma 7.4). ∎

11.11. Remark. It is easy to see that (11.12), (11.17) and (11.21) remain valid for any λ from a sufficiently small disk $\{\mu \in \mathbb{C} : |\mu| \le \delta_0\}$, if we replace on the right hand sides $|\lambda|$ and $|\lambda|/d(\lambda)$ by 0 and 1 respectively.

11.12. Lemma. *Suppose* $\theta < \varepsilon/(2n_0\kappa)$, $\theta \le 1/n_0$. *For any* $L \in \mathbb{Z}_+$ *there exists* $N_0 > 0$ *such that* $T^{(N)}(\lambda)$, $N \ge N_0$, *is an integral operator with a kernel* $r^{(N)}(\lambda; x, y)$ *satisfying the estimate*

$$(11.22) \qquad \|\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_1}(x)} \dots \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_{m_1}}(x)} \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{k_1}(y)} \dots \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{k_{m_2}}(y)} r^{(N)}(\lambda; x, y)\| \le \mathrm{const}_{\Xi,N,j_1,\dots,j_{m_1},k_1,\dots,k_{m_2}} (1 + |\lambda|)^{\theta n_0 - 1},$$
$$j_1, \dots, j_{m_1}, k_1, \dots, k_{m_2} \in \{1, \dots, n\},\ m_1, m_2 \in \mathbb{Z}_+,\ m_1 + m_2 \le L,$$
$$\forall \lambda \in \mathbb{C} \setminus \Sigma(\mathcal{C}, \theta),\ \forall x, y \in \Xi,$$

for any compact set $\Xi \subset M$. *If* $\mathcal{E}$ *is Hermitian and* $\widetilde{a}_0(x, \eta)$ *is normal for all* $(x, \eta) \in M \times \mathbb{R}^n$, *we can take* $\theta < \varepsilon/(2\kappa)$, $\theta \le 1$, *and replace* n_0 *by* 1 *on the right hand side of* (11.22).

Proof. It follows from (11.18) that

$$\|\partial^\alpha_\eta \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_1}(x)} \dots \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_m}(x)} \widetilde{b}_{0,q}(\lambda; x, \eta)\| \le \mathrm{const}\, (1 + |\eta|_{\mathbf{d}})^{-\varepsilon_1 q + \varepsilon_0(|\alpha| + m) - |\alpha : \mathbf{d}|} (1 + |\lambda|)^{\theta n_0 - 1},$$

where $0 < \varepsilon_0 := \theta n_0 \kappa < \varepsilon/2$ and $\varepsilon_1 := 1 - 2\varepsilon^{-1}\theta n_0 \kappa > 0$. A straightforward inspection of the proof of Theorem 6.7 shows that it remains almost unchanged if the restrictions on the ψDO B are slightly weakened, namely if

$$\|\partial^\alpha_\eta \nabla^{\mathcal{F},\mathcal{E}}_{\nu_{j_1}(x)} \dots \nabla^{\mathcal{F},\mathcal{E}}_{\nu_{j_m}(x)} \widetilde{\sigma}_B(x, \eta)\| \le \mathrm{const}_{K,\alpha,j_1,\dots,j_m} (1 + |\eta|_{\mathbf{d}})^{r + \varepsilon_0(|\alpha| + m) - \varrho|\alpha : \mathbf{d}| + \delta|\beta : \mathbf{d}|},$$
$$\forall \alpha \in \mathbb{Z}^n_+,\ \forall j_1, \dots, j_m \in \{1, \dots, n\},\ \forall m \in \mathbb{Z}_+,\ \forall \eta \in \mathbb{R}^n,\ \forall x \in K,$$

where β is the multi-index corresponding to the set of indices $\{j_1, \dots, j_m\}$ and $\varepsilon_0 < \min\{(\varrho - \delta)\min\{d_k^{-1}\}, \varrho\varepsilon\}/2$ (cf. (3.7)). The only difference is that (6.27) should be rewritten as follows:

$$\begin{aligned} r &\le \dots \le r_1 + r_2 - \frac{\varrho\varepsilon}{2}|\alpha + \mu'| - (\varrho - \delta)|\mu'' : \mathbf{d}| + \varepsilon_0(|\gamma''| + |\mu''|) \\ &\le r_1 + r_2 - \frac{\min\{\varrho\varepsilon, (\varrho - \delta)\min\{d_k^{-1}\}\}}{2}(|\alpha| + |\mu'| + 2|\mu''|) + \varepsilon_0(|\gamma''| + |\mu''|) \\ &\le r_1 + r_2 - \widetilde{\varepsilon}(|\alpha| + |\mu'| + 2|\mu''|), \end{aligned}$$

where $\widetilde{\varepsilon} := \min\{(\varrho - \delta)\min\{d_k^{-1}\}, \varrho\varepsilon\}/2 - \varepsilon_0 > 0$.

In our case $\varrho = 1$, $\delta = 0$. So, $\widetilde{b}_{0,q}$ have the necessary properties and we can obtain (11.22) using (11.7), (11.8) and (11.10). The last statement of the lemma is proved similarly. ∎

It is clear that the ψDO $B(\lambda)$ with the presymbol

$$\widetilde{b}(\lambda; x, \eta) \sim \sum_{q \in \mathbb{Z}_+(\mathcal{N}, \mathbf{d})} \widetilde{b}_{0,q}(\lambda; x, \eta)$$

is a right parametrix of $A - \lambda I$, i.e. $(A - \lambda I)B(\lambda) \in \Psi^{-\infty}(\mathcal{E}, \mathcal{E})$. Since A is semi-elliptic, $B(\lambda)$ is a parametrix of $A - \lambda I$ (cf. Theorem 10.2). So, $B^{(N)}(\lambda)$ is an "approximate parametrix" of $A - \lambda I$. We are going to show that $B^{(N)}(\lambda)$ is a good approximation of the resolvent of $A - \lambda I$.

Suppose that the manifold M is compact. If $A - \lambda I$ is invertible on $H_p^{s,\mathbf{d}}(\mathcal{E})$ for some λ, then A is an operator with a compact resolvent (see Theorems 10.6 and 10.10) and its spectrum $\operatorname{Spec}(A)$ consists of isolated eigenvalues with finite multiplicities (see, e.g., [Kat, Ch. III, Theorem 6.29]). Note that $\operatorname{Spec}(A)$ does not depend on s or p (see Theorem 10.10).

11.13. THEOREM. *Let* (6.2) *be satisfied and M be compact. Suppose $A \in \mathcal{H}\Psi^{\kappa,\mathbf{d}}(\mathcal{E}, \mathcal{E})$ is semi-elliptic,* $\kappa \in (0, \infty)$, $\operatorname{Ind}(A) = 0$ (*see Theorem* 10.10), *the set of all eigenvalues of* $\widetilde{a}_0(x, \eta)$, $x \in M$, $\eta \in \mathbb{R}^n$, *lies in a closed cone* $\mathcal{C} \subset \mathbb{C}$ *with vertex at* 0, $\mathcal{C} \neq \mathbb{C}$. *Then for any* $\theta < \min\{\varepsilon/(2n_0\kappa), 1/n_0\}$ *there exists* $R > 0$ *such that* $\operatorname{Spec}(A) \subset \Sigma(\mathcal{C}, \theta) \cup \{\mu \in \mathbb{C} : |\mu| \leq R\}$ (*see* (11.16)). *For any* $\theta \in [0, \varepsilon/(2n_0\kappa)) \cap [0, 1/n_0]$, $s \in \mathbb{R}$, $1 < p < \infty$, $L \in \mathbb{Z}_+$ *and sufficiently large* $N \in \mathbb{Z}_+(\mathcal{N}, \mathbf{d})$ *there exists* $\widetilde{N} \in \mathbb{R}$ *such that* $(A - \lambda I)^{-1} - B^{(N)}(\lambda)$, $\lambda \in \mathbb{C} \setminus (\mathcal{C} \cup \operatorname{Spec}(A))$, *is an integral operator with a kernel* $\mathcal{R}^{(N)}(\lambda; x, y)$ *satisfying the estimates*

$$\begin{aligned}(11.23)\qquad &\|\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_1}(x)} \cdots \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_{m_1}}(x)} \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{k_1}(y)} \cdots \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{k_{m_2}}(y)} \mathcal{R}^{(N)}(\lambda; x, y)\| \\ &\leq \text{const}\, \|(A - \lambda I)^{-1}\|_{H_p^{s,\mathbf{d}}(\mathcal{E}) \to H_p^{s,\mathbf{d}}(\mathcal{E})} (1 + |\lambda|)^{-1} (|\lambda|/d(\lambda))^{\widetilde{N}}, \\ &j_1, \ldots, j_{m_1}, k_1, \ldots, k_{m_2} \in \{1, \ldots, n\},\ m_1, m_2 \in \mathbb{Z}_+,\ m_1 + m_2 \leq L, \\ &\forall \lambda \in \mathbb{C} \setminus (\mathcal{C} \cup \operatorname{Spec}(A)),\ \forall x, y \in M,\end{aligned}$$

$$\begin{aligned}(11.24)\qquad &\|\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_1}(x)} \cdots \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_{m_1}}(x)} \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{k_1}(y)} \cdots \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{k_{m_2}}(y)} \mathcal{R}^{(N)}(\lambda; x, y)\| \\ &\leq \text{const}\, \|(A - \lambda I)^{-1}\|_{H_p^{s,\mathbf{d}}(\mathcal{E}) \to H_p^{s,\mathbf{d}}(\mathcal{E})} (1 + |\lambda|)^{\theta n_0 - 1}, \\ &j_1, \ldots, j_{m_1}, k_1, \ldots, k_{m_2} \in \{1, \ldots, n\},\ m_1, m_2 \in \mathbb{Z}_+,\ m_1 + m_2 \leq L, \\ &\forall \lambda \in \mathbb{C} \setminus (\Sigma(\mathcal{C}, \theta) \cup \operatorname{Spec}(A)),\ \forall x, y \in M.\end{aligned}$$

If $\mathcal{E}$ is Hermitian and $\widetilde{a}_0(x, \eta)$ is normal for all $(x, \eta) \in M \times \mathbb{R}^n$, we can replace n_0 by 1 *in the restrictions for θ and on the right hand side of* (11.24).

Proof. It follows from Lemma 11.12 that $\|T^{(N)}(\lambda)\|_{L_p(\mathcal{E}) \to L_p(\mathcal{E})} < 1$ for all $\lambda \in \mathbb{C} \setminus \Sigma(\mathcal{C}, \theta)$ with sufficiently large $|\lambda|$ if $\theta < \min\{\varepsilon/(2n_0\kappa), 1/n_0\}$. Since $(A - \lambda I)B^{(N)}(\lambda) = I + T^{(N)}(\lambda)$ (see (11.20)), $A - \lambda I$ is right-invertible for such λ's. Taking into account the equality $\operatorname{Ind}(A - \lambda I) = 0$ (see Theorem 10.10), we deduce that $A - \lambda I$ is invertible and $(A - \lambda I)^{-1} = B^{(N)}(\lambda)(I + T^{(N)})^{-1}$ if $\lambda \in \mathbb{C} \setminus (\Sigma(\mathcal{C}, \theta) \cup \{\mu \in \mathbb{C} : |\mu| \leq R\})$ for sufficiently large $R > 0$. This proves the first statement of the theorem.

The equalities $B^{(N)}(\lambda) = (A - \lambda I)^{-1}(A - \lambda I)B^{(N)}(\lambda) = (A - \lambda I)^{-1} + (A - \lambda I)^{-1}T^{(N)}(\lambda)$ imply $(A - \lambda I)^{-1} - B^{(N)}(\lambda) = -(A - \lambda I)^{-1}T^{(N)}(\lambda)$.

Let us show that for every $L \in \mathbb{Z}_+$ there exists $l \in \mathbb{Z}_+$ such that

$$\|(A - \lambda I)^{-1}\|_{C^l(\mathcal{E}) \to C^L(\mathcal{E})} \leq \text{const}\, \|(A - \lambda I)^{-1}\|_{H_p^{s,\mathbf{d}}(\mathcal{E}) \to H_p^{s,\mathbf{d}}(\mathcal{E})}, \qquad \forall \lambda \in \mathbb{C} \setminus \operatorname{Spec}(A).$$

According to Remark 10.9 for a given $L \in \mathbb{Z}_+$ there exist $k \in \mathbb{Z}_+$ and $l \in \mathbb{Z}_+$ such that the following continuous embeddings hold: $C^l(\mathcal{E}) \hookrightarrow H_p^{s+k\kappa,\mathbf{d}}(\mathcal{E}) \hookrightarrow C^L(\mathcal{E})$. Hence for an arbitrarily fixed $\lambda_0 \in \mathbb{C} \setminus \operatorname{Spec}(A)$ we have

$$\begin{aligned}
&\|(A-\lambda I)^{-1}\|_{C^l(\mathcal{E})\to C^L(\mathcal{E})} \\
&\qquad \le \text{const}\, \|(A-\lambda I)^{-1}\|_{H_p^{s+k\kappa,\mathbf{d}}(\mathcal{E})\to H_p^{s+k\kappa,\mathbf{d}}(\mathcal{E})} \\
&\qquad \le \text{const}\, \|(A-\lambda_0 I)^{-k}(A-\lambda I)^{-1}(A-\lambda_0 I)^{k}\|_{H_p^{s+k\kappa,\mathbf{d}}(\mathcal{E})\to H_p^{s+k\kappa,\mathbf{d}}(\mathcal{E})} \\
&\qquad \le \text{const}\, \|(A-\lambda_0 I)^{-k}\|_{H_p^{s,\mathbf{d}}(\mathcal{E})\to H_p^{s+k\kappa,\mathbf{d}}(\mathcal{E})} \\
&\qquad\quad \times \|(A-\lambda I)^{-1}\|_{H_p^{s,\mathbf{d}}(\mathcal{E})\to H_p^{s,\mathbf{d}}(\mathcal{E})} \|(A-\lambda_0 I)^{k}\|_{H_p^{s+k\kappa,\mathbf{d}}(\mathcal{E})\to H_p^{s,\mathbf{d}}(\mathcal{E})} \\
&\qquad \le \text{const}\, \|(A-\lambda I)^{-1}\|_{H_p^{s,\mathbf{d}}(\mathcal{E})\to H_p^{s,\mathbf{d}}(\mathcal{E})}, \qquad \forall \lambda \in \mathbb{C}\setminus \operatorname{Spec}(A)
\end{aligned}$$

(see Theorems 10.7 and 10.10).

Now (11.23) and (11.24) follow from Lemmas 11.10 and 11.12. ∎

11.14. Corollary. *Let the conditions of the last theorem be satisfied. For any $s \in \mathbb{R}$ and $1 < p < \infty$ there exists $R > 0$ such that*

$$\|(A-\lambda I)^{-1}\|_{H_p^{s,\mathbf{d}}(\mathcal{E})\to H_p^{s,\mathbf{d}}(\mathcal{E})} \le \text{const}\,(1+|\lambda|)^{-1}, \quad \forall \lambda \in \mathbb{C}\setminus(\Sigma(\mathcal{C},0)\cup\{\mu\in\mathbb{C} : |\mu|\le R\}). \tag{11.25}$$

For any $L \in \mathbb{Z}_+$ and sufficiently large $N \in \mathbb{Z}_+(\mathcal{N},\mathbf{d})$ there exist $R > 0$ and $\widetilde{N} \in \mathbb{R}$ such that $(A-\lambda I)^{-1} - B^{(N)}(\lambda)$, $\lambda \in \mathbb{C}\setminus(\mathcal{C}\cup\operatorname{Spec}(A))$, is an integral operator with a kernel $\mathcal{R}^{(N)}(\lambda;x,y)$ satisfying the estimates

$$\begin{gathered}
\|\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_1}(x)}\cdots\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_{m_1}}(x)}\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{k_1}(y)}\cdots\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{k_{m_2}}(y)}\mathcal{R}^{(N)}(\lambda;x,y)\| \le \text{const}\,(1+|\lambda|)^{-2}, \\
j_1,\dots,j_{m_1},k_1,\dots,k_{m_2}\in\{1,\dots,n\},\ m_1,m_2\in\mathbb{Z}_+,\ m_1+m_2\le L, \\
\forall\lambda\in\mathbb{C}\setminus(\Sigma(\mathcal{C},0)\cup\{\mu\in\mathbb{C}:|\mu|\le R\}),\ \forall x,y\in M.
\end{gathered} \tag{11.26}$$

Proof. (11.26) follows from (11.24) and (11.25). (Since the ratio $|\lambda|/d(\lambda)$ is bounded on $\mathbb{C}\setminus\Sigma(\mathcal{C},0)$ (see (11.16)), one can use (11.23) instead of (11.24).) Let us prove (11.25). Using Lemma 11.10 or Lemma 11.12, Theorem 10.8 and Remark 10.9 we can show that $\|T^{(N)}(\lambda)\|_{H_p^{s,\mathbf{d}}(\mathcal{E})\to H_p^{s,\mathbf{d}}(\mathcal{E})} \le \text{const} < 1$ for all $\lambda \in \mathbb{C}\setminus\Sigma(\mathcal{C},0)$ with sufficiently large $|\lambda|$. Now (11.25) follows from the equality $(A-\lambda I)^{-1} = B^{(N)}(\lambda)(I+T^{(N)})^{-1}$ (see the proof of Theorem 11.13), Theorem 10.7(ii) and Corollary 11.8. ∎

11.15. Corollary. *Let* (6.2) *be satisfied, M be compact and $\mathcal{E}$ be Hermitian. Suppose $A \in \mathcal{H}\Psi^{\kappa,\mathbf{d}}(\mathcal{E},\mathcal{E})$ is semi-elliptic, $\kappa\in(0,\infty)$, A is formally self-adjoint, i.e. $A^* = A$ (see Theorem* 5.1 *and Remark* 10.11*). Then A is a self-adjoint operator on $L_2(\mathcal{E},d\mathcal{M})$ with the domain $H_2^{\kappa,\mathbf{d}}(\mathcal{E})$ and for any $\theta\in[0,\varepsilon/(2\kappa))\cap[0,1]$, $L\in\mathbb{Z}_+$ and sufficiently large $N\in\mathbb{Z}_+(\mathcal{N},\mathbf{d})$ there exists $\widetilde{N}\in\mathbb{R}$ such that $(A-\lambda I)^{-1}-B^{(N)}(\lambda)$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$, is an integral operator with a kernel $\mathcal{R}^{(N)}(\lambda;x,y)$ satisfying the estimates*

$$\begin{aligned}
&\|\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_1}(x)}\cdots\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_{m_1}}(x)}\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{k_1}(y)}\cdots\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{k_{m_2}}(y)}\mathcal{R}^{(N)}(\lambda;x,y)\| \\
&\qquad \le \text{const}\,(1+|\lambda|)^{-1}\left(\frac{|\lambda|}{|\operatorname{Im}\lambda|}\right)^{\widetilde{N}}\operatorname{dist}(\lambda,\operatorname{Spec}(A))^{-1}, \qquad \forall\lambda\in\mathbb{C}\setminus\mathbb{R},\ \forall x,y\in M,
\end{aligned}$$

$$\|\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_1}(x)} \dots \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_{m_1}}(x)} \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{k_1}(y)} \dots \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{k_{m_2}}(y)} \mathcal{R}^{(N)}(\lambda; x, y)\|$$
$$\le \text{const}\,(1+|\lambda|)^{\theta-1}\text{dist}(\lambda, \text{Spec}(A))^{-1}, \qquad \forall\lambda \in \mathbb{C} \setminus \Sigma(\mathbb{R}, \theta),\ \forall x, y \in M,$$
$$j_1, \dots, j_{m_1}, k_1, \dots, k_{m_2} \in \{1, \dots, n\},\ m_1, m_2 \in \mathbb{Z}_+,\ m_1 + m_2 \le L,$$

where $\Sigma(\mathbb{R}, \theta) := \{\mu \in \mathbb{C} : |\text{Im}\,\mu| < \text{const}\,|\mu|^{1-\theta}\}$.

Proof. The first statement can easily be derived from Theorem 10.10 in the standard way (cf. [Shu, §8]). The equality $A^* = A$ implies that the principal presymbol $\widetilde{a}_0(x, \eta)$ is self-adjoint for $|\eta|_{\mathbf{d}} > 1$ (see Theorem 5.1, Remark 10.11 and Definition 11.4). We can suppose that $\widetilde{a}_0(x, \eta)$ is self-adjoint for any $\eta \in \mathbb{R}^n$. Therefore its eigenvalues are real and we can take $\mathcal{C} = \mathbb{R}$. Since A is self-adjoint on $L_2(\mathcal{E})$, we have $\text{Spec}(A) \subset \mathbb{R}$ and $\|(A - \lambda I)^{-1}\|_{L_2(\mathcal{E}) \to L_2(\mathcal{E})} = \text{dist}(\lambda, \text{Spec}(A))^{-1}$. Now the estimates for $\mathcal{R}^{(N)}(\lambda; x, y)$ follow from (11.23) and (11.24). ∎

11.16. Remark. Suppose $0 \notin \text{Spec}(A)$. Then it is easy to see that

$$\|\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_1}(x)} \dots \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_{m_1}}(x)} \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{k_1}(y)} \dots \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{k_{m_2}}(y)} \mathcal{R}^{(N)}(\lambda; x, y)\| \le \text{const}$$

for any λ from a sufficiently small disk $\{\mu \in \mathbb{C} : |\mu| \le \delta_0\}$ if N is sufficiently large (cf. Remark 11.11).

If A is a differential operator, we can construct a more accurate approximation of the resolvent $(A - \lambda I)^{-1}$. Let $B^{(N)}_\chi(\lambda)$ be the ψDO defined by (11.19) with $\widetilde{b}_{\chi,q}(\lambda; x, \eta)$ instead of $\widetilde{b}_{0,q}(\lambda; x, \eta)$ and let $T^{(N)}_\chi(\lambda) := (A - \lambda I)B^{(N)}_\chi(\lambda) - I$.

11.17. Lemma. *Suppose* $\theta < \min\{\varepsilon/(2n_0\kappa), 1/n_0\}$ *and* $\Xi \subset M$ *is a compact set. For any* $L, J \in \mathbb{Z}_+$ *there exists* $N_0 > 0$ *such that* $T^{(N)}_\chi(\lambda)$, $N \ge N_0$, *is an integral operator with a kernel* $r^{(N)}_\chi(\lambda; x, y)$ *satisfying the estimates*

$$\|\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_1}(x)} \dots \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_{m_1}}(x)} \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{k_1}(y)} \dots \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{k_{m_2}}(y)} r^{(N)}_\chi(\lambda; x, y)\| \tag{11.27}$$
$$\le \text{const}_{\Xi, N, j_1, \dots, j_{m_1}, k_1, \dots, k_{m_2}} (1+|\lambda|)^{-J}(|\lambda|/d(\lambda))^{\widetilde{N}},$$
$$j_1, \dots, j_{m_1}, k_1, \dots, k_{m_2} \in \{1, \dots, n\},\ m_1, m_2 \in \mathbb{Z}_+,\ m_1 + m_2 \le L,$$
$$\forall\lambda \in \mathbb{C} \setminus \mathcal{C},\ \forall x, y \in \Xi,$$

$$\|\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_1}(x)} \dots \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_{m_1}}(x)} \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{k_1}(y)} \dots \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{k_{m_2}}(y)} r^{(N)}_\chi(\lambda; x, y)\| \tag{11.28}$$
$$\le \text{const}_{\Xi, N, j_1, \dots, j_{m_1}, k_1, \dots, k_{m_2}} (1+|\lambda|)^{-J},$$
$$j_1, \dots, j_{m_1}, k_1, \dots, k_{m_2} \in \{1, \dots, n\},\ m_1, m_2 \in \mathbb{Z}_+,\ m_1 + m_2 \le L,$$
$$\forall\lambda \in \mathbb{C} \setminus \Sigma(\mathcal{C}, \theta),\ \forall x, y \in \Xi,$$

where $\widetilde{N} > 0$ *depends on* N. *If* $\mathcal{E}$ *is Hermitian and* $\widetilde{a}_0(x, \eta)$ *is normal for all* $(x, \eta) \in M \times \mathbb{R}^n$, *we can replace* n_0 *by* 1 *in the restrictions for* θ.

Proof. According to Remark 11.9 estimates (11.17) and (11.18) hold for $\widetilde{b}_{\chi,q}(\lambda; x, \eta)$ with $l \ge (q + |\alpha : \mathbf{d}|)/\kappa + 1$. It is easy to see that the right hand sides of these inequalities are estimated from above by

$$\text{const}_{\Xi, q, \alpha, j_1, \dots, j_m} (1 + |\lambda| + |\eta|^\kappa_{\mathbf{d}})^{-1-(q+|\alpha:\mathbf{d}|)/\kappa} (|\lambda|/d(\lambda))^{n_0([2\varepsilon^{-1}q]+1+|\alpha|+m)}$$

and

$$\mathrm{const}_{\Xi,q,\alpha,j_1,\dots,j_m}(1+|\eta|_{\mathbf{d}})^{\theta n_0\kappa m-\theta n_0((1-2\varepsilon^{-1}\theta n_0\kappa)q+(1-\varepsilon^{-1}\theta n_0\kappa)|\alpha:\mathbf{d}|)}$$
$$\times(1+|\lambda|+|\eta|_{\mathbf{d}}^{\kappa})^{-(1-\theta n_0)(\kappa+(1-2\varepsilon^{-1}\theta n_0\kappa)q+(1-\varepsilon^{-1}\theta n_0\kappa)|\alpha:\mathbf{d}|)/\kappa}$$

respectively. The remaining part of the proof is simpler than the proofs of Lemmas 11.10 and 11.12, because A is a differential operator now and we do not have difficulties with estimating the remainder in the composition formula (see also the third inequality in Lemma 6.5). ■

Similarly to Theorem 11.13 one can prove the following result.

11.18. THEOREM. *Let A be a differential operator and the conditions of Theorem 11.13 be satisfied. Then for any $\theta \in [0,\varepsilon/(2n_0\kappa)) \cap [0,1/n_0]$, $s\in\mathbb{R}$, $1<p<\infty$, $L,J\in\mathbb{Z}_+$ and sufficiently large $N\in\mathbb{Z}_+(\mathcal{N},\mathbf{d})$ there exists $\widetilde{N}\in\mathbb{R}$ such that $(A-\lambda I)^{-1}-B_\chi^{(N)}(\lambda)$, $\lambda\in\mathbb{C}\setminus(\mathcal{C}\cup\operatorname{Spec}(A))$, is an integral operator with a kernel $\mathcal{R}_\chi^{(N)}(\lambda;x,y)$ satisfying the estimates*

$$\|\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_1}(x)}\dots\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_{m_1}}(x)}\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{k_1}(y)}\dots\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{k_{m_2}}(y)}\mathcal{R}_\chi^{(N)}(\lambda;x,y)\|$$
$$\le \mathrm{const}\,\|(A-\lambda I)^{-1}\|_{H_p^{s,\mathbf{d}}(\mathcal{E})\to H_p^{s,\mathbf{d}}(\mathcal{E})}(1+|\lambda|)^{-J}(|\lambda|/d(\lambda))^{\widetilde{N}},$$
$$j_1,\dots,j_{m_1},k_1,\dots,k_{m_2}\in\{1,\dots,n\},\ m_1,m_2\in\mathbb{Z}_+,\ m_1+m_2\le L,$$
$$\forall\lambda\in\mathbb{C}\setminus(\mathcal{C}\cup\operatorname{Spec}(A)),\ \forall x,y\in M,$$
$$\|\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_1}(x)}\dots\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_{m_1}}(x)}\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{k_1}(y)}\dots\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{k_{m_2}}(y)}\mathcal{R}_\chi^{(N)}(\lambda;x,y)\|$$
$$\le \mathrm{const}\,\|(A-\lambda I)^{-1}\|_{H_p^{s,\mathbf{d}}(\mathcal{E})\to H_p^{s,\mathbf{d}}(\mathcal{E})}(1+|\lambda|)^{-J},$$
$$j_1,\dots,j_{m_1},k_1,\dots,k_{m_2}\in\{1,\dots,n\},\ m_1,m_2\in\mathbb{Z}_+,\ m_1+m_2\le L,$$
$$\forall\lambda\in\mathbb{C}\setminus(\Sigma(\mathcal{C},\theta)\cup\operatorname{Spec}(A)),\ \forall x,y\in M.$$

Remark 11.16 remains valid for $\mathcal{R}_\chi^{(N)}(\lambda;x,y)$, while the estimates analogous to those from Corollaries 11.14, 11.15 hold with the factor $(1+|\lambda|)^{-J}$ on the right hand sides.

12. Complex powers of a semi-elliptic ψDO

We will suppose in this section that (6.2) is satisfied, M is compact, $A\in\mathcal{H}\Psi^{\kappa,\mathbf{d}}(\mathcal{E},\mathcal{E})$ is semi-elliptic, $\kappa\in(0,\infty)$, the set of all eigenvalues of $\widetilde{a}_0(x,\eta)$, $x\in M$, $\eta\in\mathbb{R}^n$, lies in $\mathcal{C}_\upsilon:=\{\mu\in\mathbb{C}:-\upsilon\le\arg\mu\le\upsilon\}$, $0\le\upsilon<\pi$ and $\operatorname{Spec}(A)\subset\mathcal{C}_\upsilon\setminus\{\mu\in\mathbb{C}:|\mu|\le\delta_0\}$ for sufficiently small $\delta_0>0$.

For $z\in\mathbb{C}$, $\operatorname{Re}z<0$, we set

$$A^z:=\frac{i}{2\pi}\int_\Gamma \lambda^z(A-\lambda I)^{-1}\,d\lambda, \tag{12.1}$$

where Γ is a curve beginning at infinity, passing along the negative real line to a circle $|\lambda|=\delta$, $\delta\le\delta_0$, then clockwise around the circle, and back to infinity along the negative real line. The function λ^z is analytic in $\mathbb{C}\setminus\{\lambda\in\mathbb{R}:\lambda\le0\}$ and equals 1 at $\lambda=1$. In particular, $\lambda^z=|\lambda|^z e^{i\pi z}$ on the first part of Γ and $\lambda^z=|\lambda|^z e^{-i\pi z}$ on the third one. Note that the integral is absolutely convergent due to (11.25).

If $\operatorname{Re} z \geq 0$ we take an arbitrary $m \in \mathbb{N}$, $m > \operatorname{Re} z$, and set $A^z := A^m A^{z-m}$. It is not difficult to show that this definition does not depend on the choice of m and A^z has the standard properties of powers. In particular $A^0 = I$, $A^1 = A$ and $A^z A^w = A^{z+w}$, $\forall z, w \in \mathbb{C}$ (see, e.g., [Shu, §10]).

It follows from Theorem 6.7 and the results of Section 11 that A^z is a ψDO of class $\mathcal{H}\Psi^{\kappa z, \mathbf{d}}(\mathcal{E}, \mathcal{E})$ with a presymbol

$$\widetilde{a}^{(z)}(x,\eta) \sim \sum_{q \in \mathbb{Z}_+(\mathcal{N},\mathbf{d})} \widetilde{a}_q^{(z)}(x,\eta),$$

where each $\widetilde{a}_q \in \widetilde{S}_{1,0}^{\kappa \operatorname{Re} z - q, \mathbf{d}}(\mathcal{E},\mathcal{E})$ is almost $\mathbf{d}$-homogeneous of degree $\kappa z - q$. If $\operatorname{Re} z < 0$, then

$$\widetilde{a}_q^{(z)}(x,\eta) = \frac{i}{2\pi} \int_{\Gamma} \lambda^z \widetilde{b}_{0,q}(\lambda; x, \eta)\, d\lambda \tag{12.2}$$

(see (11.10)). $\widetilde{a}_q^{(m)}$ is the almost $\mathbf{d}$-homogeneous part of degree $\kappa m - q$ of the presymbol of the ψDO A^m, $m \in \mathbb{N}$, which can be found with the help of Theorem 6.7. If $\operatorname{Re} z \geq 0$, $m \in \mathbb{N}$ and $m > \operatorname{Re} z$ then we have

$$\begin{aligned} \widetilde{a}_q^{(z)}(x,\eta) = & \sum_{l+p+|\alpha:\mathbf{d}|+|(\beta+\gamma):\mathbf{d}|-\mathbf{d}(\mathcal{P}^{\mathcal{E}}_{\beta,\gamma})+j=q} \frac{i^{-(|\alpha|+|\beta|+|\gamma|)}}{\alpha!\beta!\gamma!} \\ & \times \partial_\eta^{\beta+\alpha} \widetilde{a}_l^{(m)}(x,\eta) \partial_\eta^\gamma (\nabla^{\mathcal{E},\mathcal{E}}_{\nu(x)})^\alpha \widetilde{a}_p^{(z-m)}(x,\eta) \mathcal{P}^{\mathcal{E}}_{\beta,\gamma,j}(x,\eta), \quad |\eta|_{\mathbf{d}} \geq 1, \end{aligned} \tag{12.3}$$

where $\mathcal{P}^{\mathcal{E}}_{\beta,\gamma,j}$ is the sum of all monomials of the polynomial $\mathcal{P}^{\mathcal{E}}_{\beta,\gamma}$ of $\mathbf{d}$-degree $\mathbf{d}(\mathcal{P}^{\mathcal{E}}_{\beta,\gamma}) - j$ (see (6.15)). It follows from the properties of A^z that (12.3) does not depend on the choice of m and

$$\begin{aligned} \widetilde{a}_q^{(z+w)}(x,\eta) = & \sum_{l+p+|\alpha:\mathbf{d}|+|(\beta+\gamma):\mathbf{d}|-\mathbf{d}(\mathcal{P}^{\mathcal{E}}_{\beta,\gamma})+j=q} \frac{i^{-(|\alpha|+|\beta|+|\gamma|)}}{\alpha!\beta!\gamma!} \\ & \times \partial_\eta^{\beta+\alpha} \widetilde{a}_l^{(z)}(x,\eta) \partial_\eta^\gamma (\nabla^{\mathcal{E},\mathcal{E}}_{\nu(x)})^\alpha \widetilde{a}_p^{(w)}(x,\eta) \mathcal{P}^{\mathcal{E}}_{\beta,\gamma,j}(x,\eta), \quad |\eta|_{\mathbf{d}} \geq 1,\ \forall z, w \in \mathbb{C}. \end{aligned}$$

It is easy to see that $\partial_\eta^\beta (\nabla^{\mathcal{E},\mathcal{E}}_{\nu(x)})^\alpha \widetilde{a}_q^{(z)}(x,\eta)$ is an entire analytic function of z for any $q \in \mathbb{Z}_+(\mathcal{N},\mathbf{d})$, $\alpha, \beta \in \mathbb{Z}^n_+$, $(x,\eta) \in M \times \mathbb{R}^n$ and

$$\widetilde{a}_0^{(z)}(x,\eta) = (\widetilde{a}_0(x,\eta))^z \tag{12.4}$$

for $|\eta|_{\mathbf{d}} \geq 1$.

Suppose that $\widetilde{a}_\varepsilon(x,\eta)$, $\partial_{\eta_k} \widetilde{a}_0(x,\eta)$ and $\nabla^{\mathcal{E},\mathcal{E}}_{\nu_k(x)} \widetilde{a}_0(x,\eta)$ commute with $\widetilde{a}_0(x,\eta)$. Then $\partial_{\eta_k} \widetilde{a}_0(x,\eta)$ and $\partial_{\eta_m} \widetilde{a}_0(x,\eta)$ commute (see Lemma 14.6 below). Using the equality $C^s_{k,m} = -C^s_{m,k}$ (see (2.1)) we obtain

$$\begin{aligned} \widetilde{b}_\varepsilon(\lambda; x, \eta) = & -\widetilde{a}_\varepsilon(x,\eta)(\widetilde{a}_0(x,\eta) - \lambda I)^{-2} \\ & - i \sum_{d_k^{-1}=\varepsilon} \partial_{\eta_k} \widetilde{a}_0(x,\eta) \nabla^{\mathcal{E},\mathcal{E}}_{\nu_k(x)} \widetilde{a}_0(x,\eta)(\widetilde{a}_0(x,\eta) - \lambda I)^{-3} \end{aligned} \tag{12.5}$$

(see (11.9)) and consequently

$$\widetilde{a}_{\varepsilon}^{(z)}(x,\eta) = z\widetilde{a}_{\varepsilon}(x,\eta)(\widetilde{a}_0(x,\eta))^{z-1} + \frac{z(z-1)}{2i} \sum_{d_k^{-1}=\varepsilon} \partial_{\eta_k}\widetilde{a}_0(x,\eta)\nabla^{\mathcal{E},\mathcal{E}}_{\nu_k(x)}\widetilde{a}_0(x,\eta)(\widetilde{a}_0(x,\eta))^{z-2}, \tag{12.6}$$

if $|\eta|_{\mathbf{d}} \geq 1$ and $\operatorname{Re} z < 0$. The last equality holds for all $z \in \mathbb{C}$ due to analyticity.

Since $\widetilde{b}_{0,q}(\lambda; x, \eta)$ is analytic with respect to λ everywhere in $\mathbb{C}$ except the eigenvalues $\mu_1(x,\eta), \dots, \mu_{n_0}(x,\eta)$ of $\widetilde{a}_0(x,\eta)$, (12.2) can be rewritten in terms of residues:

$$\widetilde{a}_q^{(z)}(x,\eta) = -\sum_{m=1}^{n_0} \operatorname{Res}_{\lambda=\mu_m(x,\eta)}(\lambda^z \widetilde{b}_{0,q}(\lambda; x,\eta)) \tag{12.7}$$

for $\operatorname{Re} z < 0$. This equality extends analytically to an arbitrary $z \in \mathbb{C}$.

Suppose $\kappa \operatorname{Re} z < -n$. Since $\widetilde{a}^{(z)} \in \mathcal{H}\widetilde{S}^{\kappa z, \mathbf{d}}(\mathcal{E},\mathcal{E}) \subset \widetilde{S}^{\kappa \operatorname{Re} z, \mathbf{d}}_{1,0}(\mathcal{E},\mathcal{E})$, A^z is an integral operator with a kernel $A_z(x,y)$ which is continuous in (x,y,z) and analytic in z. Moreover

$$A_z(x,y) - \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\langle c(x,y),\eta\rangle} \widetilde{a}^{(z)}(x,\eta)\, d\eta\right) \Phi^{\mathcal{E}}_{x,y} \Upsilon^{\theta}(x,y)\chi(x,y)$$

admits an analytic continuation to an entire $C^\infty(\mathcal{H}om_{M\times M}(\mathcal{E},\mathcal{E}))$-valued function of z.

12.1. Theorem (cf. [See]). (i) *The restriction of $A_z(x,y)$ to the complement of the diagonal in $M \times M$ can be continued to an entire analytic function of z which is C^∞-smooth in (x,y,z).*

(ii) *$A_z(x,x)$ can be continued to a meromorphic function of $z \in \mathbb{C}$ which can have poles only at the points $z_q = (q-n)/\kappa$ ($q \in \mathbb{Z}_+(\mathcal{N},\mathbf{d})$), and the poles are simple. The residue at z_q equals*

$$\begin{aligned} \varrho_q(x) &= -\frac{1}{(2\pi)^n\kappa} \int_{|\eta|=1} \widetilde{a}_q^{(z_q)}(x,\eta) \sum_{k=1}^n d_k^{-1}\eta_k^2\, dS \\ &= \frac{1}{(2\pi)^n\kappa} \int_{|\eta|=1} \sum_{m=1}^{n_0} \operatorname{Res}_{\lambda=\mu_m(x,\eta)}(\lambda^{(q-n)/\kappa}\widetilde{b}_{0,q}(\lambda;x,\eta)) \sum_{k=1}^n d_k^{-1}\eta_k^2\, dS, \end{aligned}$$

where $\mu_1(x,\eta), \dots, \mu_{n_0}(x,\eta)$ are the eigenvalues of $\widetilde{a}_0(x,\eta)$. In particular

$$\varrho_0(x) = -\frac{1}{(2\pi)^n\kappa} \int_{|\eta|=1} (\widetilde{a}_0(x,\eta))^{-n/\kappa} \sum_{k=1}^n d_k^{-1}\eta_k^2\, dS.$$

$A_z(x,x)$ is C^∞-smooth in (x,z) for z different from the poles. $(z-z_q)A_z(x,x)$ is C^∞-smooth in (x,z) for z close to z_q.

(iii) *$A_z(x,x)$ does not have a pole at $z=0$ and*

$$A_0(x,x) = \frac{1}{(2\pi)^n\kappa} \int_{|\eta|=1} \sum_{k=1}^n d_k^{-1}\eta_k^2\, dS \int_0^\infty \widetilde{b}_{0,n}(-t;x,\eta)\, dt.$$

(Note that $n = \sum_{k=1}^n d_k^{-1} \in \mathbb{Z}_+(\mathcal{N},\mathbf{d})$.)

(iv) *Suppose A is a differential operator. Then $A_z(x,y) = 0$ if $x \neq y$ and $z \in \mathbb{Z}_+$. $A_z(x,x)$ does not have a pole at $z = j \in \mathbb{N}$ and*

$$A_j(x,x) = \frac{(-1)^j}{(2\pi)^n \kappa} \int\limits_{|\eta|=1} \sum_{k=1}^{n} d_k^{-1} \eta_k^2 \, dS \int\limits_0^\infty t^j \widetilde{b}_{0,\kappa j+n}(-t; x, \eta)\, dt.$$

(*Note that due to* (3.4) *the numbers $l_k := \kappa d_k$ are integers. Therefore $\kappa j \in \mathbb{Z}_+(\mathcal{N}, \mathbf{d})$. Since $n \in \mathbb{Z}_+(\mathcal{N}, \mathbf{d})$, we have $\kappa j + n \in \mathbb{Z}_+(\mathcal{N}, \mathbf{d})$. The last integral is absolutely convergent due to Remark* 11.9.)

Proof. The theorem can be proved in the standard way (see, e.g., [Shu, §12]). The integrals of almost **d**-homogeneous functions are evaluated with the help of Lemma 12.2 and Corollary 12.3 (see below). ■

12.2. Lemma. *Suppose f is a function continuous outside the open unit ball and*

(12.8) $$f(\tau^{1/d_1}\eta_1, \dots, \tau^{1/d_n}\eta_n) = \tau^\mu f(\eta), \qquad |\eta|_{\mathbf{d}} \geq 1,\ \tau \geq 1,\ \operatorname{Re}\mu < -n.$$

Then

(12.9) $$\int\limits_{|\eta|\geq 1} f(\eta)\, d\eta = -\frac{1}{\mu+n} \int\limits_{|\eta|=1} f(\eta) \sum_{k=1}^{n} d_k^{-1}\eta_k^2 \, dS.$$

Proof. Since $\operatorname{Re}\mu < -n$, the integral on the left hand side of (12.9) is absolutely convergent (cf. the proof of Lemma 11.10). Approximating f by smooth functions on the unit sphere we can reduce everything to the case when f is smooth. Differentiating (12.8) in τ and taking $\tau = 1$ afterwards we arrive at the following analogue of Euler's formula: $\sum_{k=1}^n d_k^{-1}\eta_k \partial_{\eta_k} f(\eta) = \mu f(\eta)$. Then integrating by parts and using the equality (3.1) we obtain

$$\begin{aligned}
\int\limits_{1\leq|\eta|_{\mathbf{d}}\leq R} f(\eta) d\eta &= \frac{1}{\mu} \int\limits_{1\leq|\eta|_{\mathbf{d}}\leq R} \sum_{k=1}^n d_k^{-1}\eta_k \partial_{\eta_k} f(\eta)\, d\eta \\
&= -\frac{n}{\mu} \int\limits_{1\leq|\eta|_{\mathbf{d}}\leq R} f(\eta)\, d\eta - \frac{1}{\mu} \int\limits_{|\eta|=1} f(\eta) \sum_{k=1}^n d_k^{-1}\eta_k^2\, dS \\
&\quad + \frac{1}{\mu} \int\limits_{|\eta|_{\mathbf{d}}=R} f(\eta) \sum_{k=1}^n d_k^{-1}\eta_k e_k(\eta)\, dS,
\end{aligned}$$

i.e.

$$\int\limits_{1\leq|\eta|_{\mathbf{d}}\leq R} f(\eta)\, d\eta = -\frac{1}{\mu+n} \int\limits_{|\eta|=1} f(\eta) \sum_{k=1}^n d_k^{-1}\eta_k^2\, dS + \frac{1}{\mu+n} \int\limits_{|\eta|_{\mathbf{d}}=R} f(\eta) \sum_{k=1}^n d_k^{-1}\eta_k e_k(\eta)\, dS,$$

where $(e_1(\eta), \dots, e_n(\eta))$ is the unit outward normal. So, it is left to prove that the last integral vanishes as $R \to \infty$. The equality $|\eta|_{\mathbf{d}} = R$ is equivalent to $\sum_{k=1}^n R^{-2/d_k}\eta_k^2 - 1 = 0$. Therefore $(e_1(\eta), \dots, e_n(\eta)) = \mathrm{const}\,(R^{-2/d_1}\eta_1, \dots,\ R^{-2/d_n}\eta_n)$ with a positive constant. Consequently, $\eta_k e_k(\eta) \geq 0$. Hence

$$\Big|\int\limits_{|\eta|_{\mathbf d}=R} f(\eta)\sum_{k=1}^{n} d_k^{-1}\eta_k e_k(\eta)\,dS\Big| \le \max_{|\eta|_{\mathbf d}=R}|f(\eta)|\sum_{k=1}^{n} d_k^{-1}\int\limits_{|\eta|_{\mathbf d}=R}\eta_k e_k(\eta)\,dS$$
$$= \mathrm{const}\,R^{\operatorname{Re}\mu}\sum_{k=1}^{n} d_k^{-1}\int\limits_{|\eta|_{\mathbf d}\le R}\partial_{\eta_k}\eta_k\,d\eta$$
$$= \mathrm{const}\,R^{\operatorname{Re}\mu}\,\mathrm{Vol}\{\eta\in\mathbb R^n : |\eta|_{\mathbf d}\le R\}\to 0 \quad \text{as } R\to\infty. \ \blacksquare$$

For an arbitrary open subset U of the unit sphere $|\eta| = 1$ we denote by $\mathcal H(U,\mathbf d)$ the following "truncated horn":

$$\mathcal H(U,\mathbf d) := \{(\tau^{1/d_1}\eta_1,\dots,\tau^{1/d_n}\eta_n) : \tau\ge 1,\ \eta\in U\}.$$

12.3. Corollary. *Let f satisfy the conditions of the last lemma. Then*

$$\int\limits_{\mathcal H(U,\mathbf d)} f(\eta)\,d\eta = -\frac{1}{\mu+n}\int\limits_{U} f(\eta)\sum_{k=1}^{n} d_k^{-1}\eta_k^2\,dS$$

for any open subset U of the unit sphere.

Proof. We need to prove (12.9) for the function $\chi_{\mathcal H(U,\mathbf d)}f$, where $\chi_{\mathcal H(U,\mathbf d)}$ is the characteristic (indicator) function of $\mathcal H(U,\mathbf d)$. It is sufficient to approximate $\chi_{\mathcal H(U,\mathbf d)}$ pointwise by continuous functions on the unit sphere and apply Lebesgue's dominated convergence theorem. ■

For any $w\in(0,\pi-\upsilon]$ we set

$$\mathit\Pi(w) := \sup_{r\ge 0}\|(A - re^{\pm i(\upsilon+w)}I)^{-1}\|_{L_2(\mathcal E)\to L_2(\mathcal E)}.$$

For $w > \pi-\upsilon$ we take $\mathit\Pi(w) = 1$. Here υ is the number from the restrictions on A formulated at the beginning of this section. $\mathit\Pi(w)$ is finite due to (11.25). If $\upsilon = 0$ and A is self-adjoint, then $\mathit\Pi(w)\le \mathrm{const}\,w^{-1}$, $\forall w\in(0,\pi]$.

12.4. Theorem. *For any $s_1,s_2\in\mathbb R$ $(s_1<s_2)$ and $c>0$ there exist $\widetilde N\in\mathbb R$ and $C>0$ such that $\widetilde N$ does not depend on s_1 and c, and*

$$\|A_z(x,x)\| \le C(1+|\operatorname{Im} z|)^{\widetilde N}e^{\upsilon|\operatorname{Im} z|}\min\{\mathit\Pi(|\operatorname{Im} z|^{-1}), e^{c|\operatorname{Im} z|}\} \tag{12.10}$$

for all z in the vertical strip $s_1\le\operatorname{Re} z\le s_2$ excluding neighbourhoods of the poles z_q.

Proof. The proof is based on the method used in the proof of [Ar, Proposition 3.4] (see also [HR2]).

Let us start with the case $s_2<0$. It follows from (12.1) and the results of Section 11 that $A^z = A^{(z)}_{(N)} + E^{(z)}_{(N)}$, where $A^{(z)}_{(N)}$ is the ψDO with the presymbol $\sum_{q\le N}\widetilde a^{(z)}_q(x,\eta)$ defined by a formula similar to (11.19), $E^{(z)}_{(N)}$ is the integral operator with the kernel

$$\mathrm E^{(z)}_{(N)}(x,y) = \frac{i}{2\pi}\int\limits_{\mathit\Gamma}\lambda^z\mathcal R^{(N)}(\lambda;x,y)\,d\lambda$$

and N is sufficiently large.

Since $\mathcal R^{(N)}(\lambda;x,y)$ is analytic with respect to λ in $(\mathbb C\setminus\mathcal C_\upsilon)\cup\{\mu\in\mathbb C : |\mu|<\delta_0\}$, we can replace the contour of integration $\mathit\Gamma$ by $\mathit\Gamma_\psi = \mathit\Gamma^+_\psi\cup\mathit\Gamma^0_\psi\cup\mathit\Gamma^-_\psi$, where $\mathit\Gamma^\pm_\psi = \{re^{\pm i\psi} : r\in[\delta,\infty)\}$, $\mathit\Gamma^0_\psi = \{\delta e^{-i\varphi} : \varphi\in[-\psi,\psi]\}$, $\delta<\delta_0$ and $\psi\in(\upsilon,\pi]$ will be chosen later.

Using (11.23) and Remark 11.16 we obtain

$$\left\| \frac{i}{2\pi} \int_{\Gamma_\psi^0} \lambda^z \mathcal{R}^{(N)}(\lambda; x, y)\, d\lambda \right\| \le \mathrm{const} \int_{-\psi}^{\psi} \delta^{\operatorname{Re} z} e^{|\varphi|\cdot|\operatorname{Im} z|}\, d\varphi \le \mathrm{const}\, \delta^{\operatorname{Re} z} e^{\psi|\operatorname{Im} z|},$$

$$\left\| \frac{i}{2\pi} \int_{\Gamma_\psi^\pm} \lambda^z \mathcal{R}^{(N)}(\lambda; x, y)\, d\lambda \right\| \le \mathrm{const}\, e^{\psi|\operatorname{Im} z|} \varPi(\psi - \upsilon)(\sin(\psi - \upsilon))^{-\widetilde{N}} \int_{\delta}^{\infty} r^{\operatorname{Re} z - 1}\, dr$$

$$= \mathrm{const}\, |\operatorname{Re} z|^{-1} \delta^{\operatorname{Re} z} (\sin(\psi - \upsilon))^{-\widetilde{N}} e^{\upsilon|\operatorname{Im} z|} e^{(\psi-\upsilon)|\operatorname{Im} z|} \varPi(\psi - \upsilon)$$

with constants independent of (x, y, z) and ψ. For $\psi = \upsilon + c$ we get

$$\|\mathrm{E}_{(N)}^{(z)}(x, y)\| \le \mathrm{const}\, e^{(\upsilon+c)|\operatorname{Im} z|}.$$

Let us make a different choice of ψ. If $|\operatorname{Im} z| < \max\{2/\pi, (\pi - \upsilon)^{-1}\}$ we put $\psi = \pi$. If $|\operatorname{Im} z| \ge \max\{2/\pi, (\pi - \upsilon)^{-1}\}$ we take $\psi = \upsilon + |\operatorname{Im} z|^{-1}$. In the latter case $|\sin(\psi - \upsilon)| = \sin(|\operatorname{Im} z|^{-1}) \ge 2\pi^{-1}|\operatorname{Im} z|^{-1}$. Therefore

$$\|\mathrm{E}_{(N)}^{(z)}(x, y)\| \le \mathrm{const}\, (1 + |\operatorname{Im} z|)^{\widetilde{N}} e^{\upsilon|\operatorname{Im} z|} \varPi(|\operatorname{Im} z|^{-1}).$$

Derivatives of $\mathrm{E}_{(N)}^{(z)}(x, y)$ can be estimated in exactly the same way. So, for any $L \in \mathbb{Z}_+$ and sufficiently large $N \in \mathbb{Z}_+(\mathcal{N}, \mathbf{d})$ there exist $C_0 > 0$ and $\widetilde{N} \in \mathbb{R}$ such that

(12.11) $$\|\nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_1}(x)} \dots \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_{m_1}}(x)} \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{k_1}(y)} \dots \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{k_{m_2}}(y)} \mathrm{E}_{(N)}^{(z)}(x, y)\|$$
$$\le C_0 (1 + |\operatorname{Im} z|)^{\widetilde{N}} e^{\upsilon|\operatorname{Im} z|} \min\{\varPi(|\operatorname{Im} z|^{-1}), e^{c|\operatorname{Im} z|}\}, \qquad m_1 + m_2 \le L,$$

for all z in the strip $s_1 \le \operatorname{Re} z \le s_2 < 0$.

If $\operatorname{Re} z < -n/\kappa$ then the ψDO $A_{(N)}^{(z)}$ is an integral operator with a kernel $\mathrm{A}_{(N)}^{(z)}(x, y)$, and we need to estimate the meromorphic continuation of

$$\mathrm{A}_{(N)}^{(z)}(x, x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \sum_{q \le N} \widetilde{a}_q^{(z)}(x, \eta)\, d\eta.$$

It follows from (12.2) that

$$\mathrm{A}_{(N)}^{(z)}(x, x) = I_0^{(z)}(x) + I_1^{(z)}(x),$$

where

$$I_0^{(z)}(x) := \frac{1}{(2\pi)^n} \int_{|\eta|<1} \sum_{q \le N} \widetilde{a}_q^{(z)}(x, \eta)\, d\eta = \frac{i}{(2\pi)^{n+1}} \int_{|\eta|<1} \sum_{q \le N} \int_{\Gamma} \lambda^z \widetilde{b}_{0,q}(\lambda; x, \eta)\, d\lambda\, d\eta,$$

$$I_1^{(z)}(x) := \frac{1}{(2\pi)^n} \int_{|\eta|\ge 1} \sum_{q \le N} \widetilde{a}_q^{(z)}(x, \eta)\, d\eta$$

$$= -\frac{1}{(2\pi)^n} \sum_{q \le N} \frac{1}{\kappa z - q + n} \int_{|\eta|=1} \widetilde{a}_q^{(z)}(x, \eta) \sum_{k=1}^{n} d_k^{-1} \eta_k^2\, dS$$

$$= -\frac{i}{(2\pi)^{n+1}} \sum_{q \le N} \frac{1}{\kappa z - q + n} \int_{|\eta|=1} \int_{\Gamma} \lambda^z \widetilde{b}_{0,q}(\lambda; x, \eta)\, d\lambda \sum_{k=1}^{n} d_k^{-1} \eta_k^2\, dS$$

(see Lemma 12.2). It is clear that $I_0^{(z)}(x)$ is analytic with respect to z in the left half-plane $\operatorname{Re} z < 0$ and $I_1^{(z)}(x)$ is meromorphic there with possible simple poles at $z_q = (q-n)/\kappa$. We need to change the contour of integration in order to estimate these integrals as above.

Since $\widetilde{b}_{0,q}(\lambda; x, \eta)$ is analytic with respect to λ in $(\mathbb{C} \setminus \mathcal{C}_\upsilon) \cup \{\mu \in \mathbb{C} : |\mu| < \delta_0\}$, we have

$$\int_{\Gamma} \lambda^z \widetilde{b}_{0,q}(\lambda; x, \eta)\, d\lambda = \int_{\Gamma_\psi} \lambda^z \widetilde{b}_{0,q}(\lambda; x, \eta)\, d\lambda.$$

Using (11.12) and Remark 11.11 we obtain as above

$$\Big\| \int_{\Gamma_\psi^0} \lambda^z \widetilde{b}_{0,q}(\lambda; x, \eta) d\lambda \Big\| \le \operatorname{const} \delta^{\operatorname{Re} z} e^{\psi |\operatorname{Im} z|},$$

$$\Big\| \int_{\Gamma_\psi^\pm} \lambda^z \widetilde{b}_{0,q}(\lambda; x, \eta) d\lambda \Big\| \le \operatorname{const} |\operatorname{Re} z|^{-1} \delta^{\operatorname{Re} z} (\sin(\psi - \upsilon))^{-n_0([2\varepsilon^{-1} q]+1)} e^{\psi |\operatorname{Im} z|}$$

with constants independent of ψ, $z, x \in M$ and η, $|\eta| \le 1$.

Taking $\psi = \upsilon + |\operatorname{Im} z|^{-1}$ for $|\operatorname{Im} z| \ge \max\{2/\pi, (\pi - \upsilon)^{-1}\}$, we can show that

$$\|\widetilde{a}_q^{(z)}(x, \eta)\| \le \operatorname{const} (1 + |\operatorname{Im} z|)^{n_0([2\varepsilon^{-1} q]+1)} e^{\upsilon |\operatorname{Im} z|}, \qquad |\eta| \le 1,\ x \in M,$$

for all z in the strip $s_1 \le \operatorname{Re} z \le s_2$. Similarly we proceed for the derivatives

$$\|\partial_\eta^\alpha \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_1}(x)} \cdots \nabla^{\mathcal{E},\mathcal{E}}_{\nu_{j_m}(x)} \widetilde{a}_q^{(z)}(x, \eta)\| \le \operatorname{const} (1 + |\operatorname{Im} z|)^{n_0([2\varepsilon^{-1} q]+1+|\alpha|+m)} e^{\upsilon |\operatorname{Im} z|}, \tag{12.12}$$
$$\forall \alpha \in \mathbb{Z}_+^n,\ \forall j_1, \dots, j_m \in \{1, \dots, n\},\ \forall m \in \mathbb{Z}_+,\ |\eta| \le 1,\ x \in M.$$

The above inequalities imply that there exists $C' > 0$ such that

$$\|\mathrm{A}_{(N)}^{(z)}(x, x)\| \le C' (1 + |\operatorname{Im} z|)^{n_0([2\varepsilon^{-1} N]+1)} e^{\upsilon |\operatorname{Im} z|}$$

for all z in the strip $s_1 \le \operatorname{Re} z \le s_2$ excluding neighbourhoods of the poles z_q. This completes the proof in the case $s_2 < 0$.

Suppose $s_2 \ge 0$ and take $k > s_2$, $k \in \mathbb{N}$. Let us use the representation $A^z = A^k A^{z-k} = A^k A_{(N)}^{(z-k)} + A^k E_{(N)}^{(z-k)}$. Acting as in the proof of Theorem 11.13 we can show with the help of Theorem 10.7 and Remark 10.9 that A^k is bounded from $C^L(M)$ to $C(M)$ if L is sufficiently large. Hence (12.11) implies that if N is sufficiently large, then $A^k E_{(N)}^{(z-k)}$ is an integral operator with a kernel $\widehat{\mathrm{E}}_{(N)}^{(z)}(x, y)$ satisfying the estimate

$$\|\widehat{\mathrm{E}}_{(N)}^{(z)}(x, y)\| \le \operatorname{const} (1 + |\operatorname{Im} z|)^{\widetilde{N}} e^{\upsilon |\operatorname{Im} z|} \min\{\Pi(|\operatorname{Im} z|^{-1}), e^{c |\operatorname{Im} z|}\}$$

for all z in the strip $s_1 \le \operatorname{Re} z \le s_2 < k$.

Further,

$$A^k A_{(N)}^{(z-k)} = \frac{i}{2\pi} \int_{\Gamma} \lambda^{z-k} A^k B^{(N)}(\lambda)\, d\lambda$$

(see (11.19) and (12.2)). It follows from Theorem 6.7 and (12.3) that for sufficiently large N,

$$A^k A_{(N)}^{(z-k)} = A_{(N)}^{(z)} + \frac{i}{2\pi} \int_{\Gamma} \lambda^{z-k} G_k^{(N)}(\lambda)\, d\lambda,$$

where $G_k^{(N)}(\lambda)$ is an integral operator with a kernel $\mathrm{G}_k^{(N)}(\lambda)(x,y)$ satisfying the estimate

$$\|\mathrm{G}_k^{(N)}(\lambda)(x,y)\| \le \mathrm{const}\,(1+|\lambda|)^{-1}(|\lambda|/d(\lambda))^{\widetilde{N}}$$

(cf. Lemma 11.10). Therefore the kernel of $i(2\pi)^{-1}\int_\Gamma \lambda^{z-k}G_k^{(N)}(\lambda)\,d\lambda$ is estimated by $\mathrm{const}\,(1+|\mathrm{Im}\,z|)^{\widetilde{N}}e^{\upsilon|\mathrm{Im}\,z|}$ for all z in the strip $s_1 \le \mathrm{Re}\,z \le s_2$. This is easier to prove than (12.11), since we do not need to deal with the norm of $(A-\lambda I)^{-1}$ here. The kernel of $A_{(N)}^{(z)}$ is estimated as above (see (12.3) and (12.12)). ∎

12.5. Corollary. *For any $s_1, s_2 \in \mathbb{R}$ $(s_1 < s_2)$ and $c > 0$ there exists $C > 0$ such that*

$$\|A_z(x,x)\| \le Ce^{(\upsilon+c)|\mathrm{Im}\,z|}$$

for all z in the vertical strip $s_1 \le \mathrm{Re}\,z \le s_2$ excluding neighbourhoods of the poles z_q. If $\upsilon = 0$ and A is self-adjoint, there exist $\widetilde{N} \in \mathbb{R}$ and $C > 0$ such that $\widetilde{N}$ does not depend on s_1 and

$$\|A_z(x,x)\| \le C(1+|\mathrm{Im}\,z|)^{\widetilde{N}}$$

for z as above.

Let $\Pi_{s,p}(w)$ be the quantity which is obtained from $\Pi(w)$ when we replace, in the definition, the norm $\|\cdot\,|\,L_2(\mathcal{E}) \to L_2(\mathcal{E})\|$ by $\|\cdot\,|\,H_p^{s,\mathbf{d}}(\mathcal{E}) \to H_p^{s,\mathbf{d}}(\mathcal{E})\|$, $s \in \mathbb{R}$, $1 < p < \infty$. Then $\Pi_{s,p}(w)$ is finite due to (11.25). It is clear that Theorem 12.4 remains true with $\Pi_{s,p}(w)$ in place of $\Pi(w)$. Acting as in its proof one can easily show that for any $s_0 < 0$ and $c > 0$ there exists a constant $C > 0$ such that

$$\begin{aligned}(12.13)\qquad &\|A^z\|_{H_p^{s,\mathbf{d}}(\mathcal{E})\to H_p^{s,\mathbf{d}}(\mathcal{E})} \\ &\le C(1+\|A^{-1}\|_{H_p^{s,\mathbf{d}}(\mathcal{E})\to H_p^{s,\mathbf{d}}(\mathcal{E})})^k e^{\upsilon|\mathrm{Im}\,z|}\min\{\Pi_{s,p}(|\mathrm{Im}\,z|^{-1}), e^{c|\mathrm{Im}\,z|}\}\end{aligned}$$

for all z in the strip $-k \le \mathrm{Re}\,z \le s_0$.

Suppose $-1 < \mathrm{Re}\,z < 0$. Then we can shrink the circular part of Γ to 0 and from (12.1) we obtain

$$(12.14)\qquad A^z = -\frac{\sin \pi z}{\pi}\int_0^\infty r^z(A+rI)^{-1}\,dr.$$

Now the inverse Mellin transformation gives

$$(12.15)\qquad (A-\lambda I)^{-1} = \frac{i}{2}\int_{\mathrm{Re}\,z=s}\frac{(-\lambda)^{-z-1}}{\sin \pi z}A^z\,dz, \qquad -1 < s < 0,$$

for $\lambda \in \mathbb{R}_-$. The above estimate of the norm of A^z allows one to extend (12.15) analytically to all $\lambda \in \mathbb{C} \setminus \mathcal{C}_\upsilon$.

13. The heat kernel of a semi-elliptic ψDO

We will suppose in this section that the restrictions formulated at the beginning of the last section are satisfied and $0 \le \upsilon < \pi/2$. Then A generates the semigroup

$$(13.1)\qquad e^{-tA} := \frac{i}{2\pi}\int_{\Gamma'_\psi} e^{-t\lambda}(A-\lambda I)^{-1}\,d\lambda, \qquad |\arg t| < \pi/2-\upsilon,$$

where $\upsilon < \psi < \pi/2 - |\arg t|$ and Γ'_ψ is the curve beginning at infinity, passing to 0 along the ray $\lambda = re^{i\psi}$, $r > 0$, and back to infinity along $\lambda = re^{-i\psi}$, $r > 0$. Let us substitute (12.15) into (13.1), interchange the integrals and send ψ to 0. This gives

$$e^{-tA} = \frac{1}{2\pi i} \int\limits_{\operatorname{Re} z = s} t^z \Gamma(-z) A^z \, dz, \tag{13.2}$$

where $\Gamma(\cdot)$ is the gamma-function (see [WW, 12.22]), $-1 < s < 0$ and $t > 0$. Since the operator-valued function under the integral sign is analytic for $\operatorname{Re} z < 0$, the estimate (12.13) allows one to show that (13.2) holds for any $s < 0$. The last equality extends analytically in t to $|\arg t| < \pi/2 - \upsilon$.

One can easily prove by induction on l that

$$A^l e^{-tA} = \frac{i}{2\pi} \int\limits_{\Gamma'_\psi} \lambda^l e^{-t\lambda} (A - \lambda I)^{-1} d\lambda, \qquad |\arg t| < \pi/2 - \upsilon, \ \forall l \in \mathbb{Z}_+.$$

Therefore $A^l e^{-tA}$ is bounded on $H_p^{s,\mathbf{d}}(\mathcal{E})$ for any $s \in \mathbb{R}$ and $p \in (1, \infty)$. This also follows from (12.13) and (13.2). Theorems 10.7 and 10.10 imply that e^{-tA} is bounded from $H_p^{s,\mathbf{d}}(\mathcal{E})$ to $H_p^{s',\mathbf{d}}(\mathcal{E})$ for any $s' \in \mathbb{R}$. Hence we deduce from Theorem 10.8 and Remark 10.9 that e^{-tA} is an integral operator with a C^∞-smooth kernel $\Theta(t, x, y)$.

13.1. THEOREM. *$\Theta(t, x, x)$ admits the following asymptotic expansion*:

$$\Theta(t, x, x) \sim \sum_{q \in \mathbb{Z}_+(\mathcal{N}, \mathbf{d}) \cup J} \Theta_q(x) t^{(q-n)/\kappa} + \sum_{j=1}^{\infty} \widehat{\Theta}_j(x) t^j \log t \qquad \text{as } t \to 0, \quad |\arg t| < \pi/2 - \upsilon, \tag{13.3}$$

where $J = \{q = n + j\kappa : j \in \mathbb{Z}_+\}$;

$$\Theta_q(x) = -\Gamma((n-q)/\kappa) \varrho_q(x) \quad \text{if } q \in \mathbb{Z}_+(\mathcal{N}, \mathbf{d}) \setminus J;$$

$$\widehat{\Theta}_j(x) = 0, \qquad \Theta_q(x) = \frac{(-1)^j}{j!} A_j(x, x)$$

if $q = n + j\kappa$, $j \in \mathbb{Z}_+$ *and either* $q \notin \mathbb{Z}_+(\mathcal{N}, \mathbf{d})$ *or* $q \in \mathbb{Z}_+(\mathcal{N}, \mathbf{d})$ *but* $A_z(x, x)$ *does not have a pole at* $z = j$; *and*

$$\widehat{\Theta}_j(x) = \frac{(-1)^j}{j!} \varrho_q(x),$$

$$\Theta_q(x) = \frac{(-1)^j}{j!} \left(A_z(x, x) - \frac{\varrho_q(x)}{z - j} \right) \bigg|_{z=j} - \varrho_q(x) \left(\Gamma(-z) + \frac{(-1)^j}{j!(z-j)} \right) \bigg|_{z=j}$$

if $q = n + j\kappa \in \mathbb{Z}_+(\mathcal{N}, \mathbf{d})$, $j \in \mathbb{N}$ *and* $A_z(x, x)$ *has a pole at* $z = j$ (*see Theorem* 12.1). *The above asymptotic expansion means that for any* $N \in \mathbb{R}$, $L \in \mathbb{N}$ *and* $c \in (0, \pi/2 - \upsilon)$ *there exists a constant* $C > 0$ *such that*

$$\left\| \partial_t^l \Big(\Theta(t, x, x) - \sum_{(q-n)/\kappa < N} \Theta_q(x) t^{(q-n)/\kappa} - \sum_{j \le N} \widehat{\Theta}_j(x) t^j \log t \Big) \right\| \le C t^{N-l}$$

$$\text{for } 0 < |t| < 1, \ |\arg t| < \pi/2 - \upsilon - c, \ l = 1, \ldots, L.$$

Proof. It follows from (13.2) that

$$\Theta(t,x,x) = \frac{1}{2\pi i} \int\limits_{\operatorname{Re} z = s} t^z \Gamma(-z) A_z(x,x)\, dz, \qquad s < -n/\kappa.$$

Since

$$|\Gamma(-z)| \le \operatorname{const}_{\operatorname{Re} z} (1 + |\operatorname{Im} z|)^{-[\operatorname{Re} z]} e^{-\pi |\operatorname{Im} z|/2}, \qquad \operatorname{Re} z \in \mathbb{R} \setminus \mathbb{Z},$$

with a constant independent of $\operatorname{Im} z$ (see [Ar, Lemma 4.1]), Corollary 12.5 implies that we can shift the path of integration letting $s \to \infty$ and "jumping" over the poles. Using the residue theorem we can obtain the above asymptotic expansion in the standard way (see [DG], [Agr2]). ■

13.2. Remark. If A is a differential operator, then $A_z(x,x)$ does not have a pole at $z = j \in \mathbb{N}$ (see Theorem 12.1(iv)) and we do not have the second sum on the right hand side of (13.3).

14. The asymptotics of the resolvent kernel of a semi-elliptic ψDO

Let $\widehat{b}_q$, $q \in \mathbb{Z}_+(\mathcal{N}, \mathbf{d})$, be the solution of the system which is obtained from (11.7), (11.8) if we replace $\widetilde{a}_l$, $l \in \mathbb{Z}_+(\mathcal{N}, \mathbf{d})$, by their $\mathbf{d}$-homogeneous extensions $\widehat{a}_l$:

$$\widehat{a}_l(x,\eta) = \widetilde{a}_l(x,\eta), \qquad |\eta|_{\mathbf{d}} \ge 1,$$
$$\widehat{a}_l(x, \tau^{1/d_1}\eta_1, \dots, \tau^{1/d_n}\eta_n) = \tau^{\kappa - l} \widehat{a}_l(x,\eta), \qquad \eta \ne 0,\ \tau > 0.$$

Then

$$\widehat{b}_q(\lambda; x, \eta) = \widetilde{b}_q(\lambda; x, \eta), \qquad |\eta|_{\mathbf{d}} \ge 1,$$
$$\widehat{b}_q(\tau^\kappa \lambda; x, \tau^{1/d_1}\eta_1, \dots, \tau^{1/d_n}\eta_n) = \tau^{-\kappa - q} \widehat{b}_q(\lambda; x, \eta), \qquad \eta \ne 0,\ \tau > 0.$$

If A is a differential operator, then $\widehat{a}_l \equiv \widetilde{a}_l$, $\widehat{b}_q \equiv \widetilde{b}_q$.

14.1. Theorem. *Let* (6.2) *be satisfied*, M *be compact and* $\mathcal{E}$ *be Hermitian. Suppose* $A \in \mathcal{H}\Psi^{\kappa,\mathbf{d}}(\mathcal{E},\mathcal{E})$ *is semi-elliptic and self-adjoint* (*see Corollary* 11.15) *and* $\kappa > n$. *Then the resolvent* $(A - \lambda I)^{-1}$, $\lambda \in \mathbb{C} \setminus \operatorname{Spec}(A)$, *is an integral operator with a kernel* $\mathcal{R}(\lambda; x, y)$ *which admits for any* $\theta \in (0, \varepsilon/(2\kappa))$ *the following on-diagonal asymptotic expansion*:

$$\mathcal{R}(\lambda; x, x) \sim \sum_{\substack{q \in \mathbb{Z}_+(\mathcal{N},\mathbf{d}) \\ q < n + \kappa(1 - 2\theta)}} \mathcal{R}_q^{\pm}(x) (\mp i\lambda)^{(n-q)/\kappa - 1} + O(|\lambda|^{2(\theta - 1)}) \quad \text{as } |\lambda| \to \infty, \qquad \lambda \in \mathbb{C} \setminus \Sigma(\mathbb{R}, \theta),\ \pm \operatorname{Im} \lambda > 0, \tag{14.1}$$

where $\Sigma(\mathbb{R}, \theta) := \{\mu \in \mathbb{C} : |\operatorname{Im} \mu| < \operatorname{const} |\mu|^{1-\theta}\}$,

$$\mathcal{R}_q^{\pm}(x) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \widehat{b}_q(\pm i; x, \eta)\, d\eta. \tag{14.2}$$

If A *is a differential operator*, *then*

$$\mathcal{R}(\lambda; x, x) \sim \sum_{q \in \mathbb{Z}_+(\mathcal{N},\mathbf{d})} \mathcal{R}_q^{\pm}(x) (\mp i\lambda)^{(n-q)/\kappa - 1} \quad \text{as } |\lambda| \to \infty, \qquad \lambda \in \mathbb{C} \setminus \Sigma(\mathbb{R}, \theta),\ \pm \operatorname{Im} \lambda > 0, \tag{14.3}$$

where $\mathcal{R}_q^{\pm}$ *are defined by* (14.2). *The above asymptotic expansions are uniform in* $x \in M$.

Proof. Since

$$\operatorname{dist}(\lambda, \operatorname{Spec}(A)) \geq |\operatorname{Im}\lambda| \geq \operatorname{const}|\lambda|^{1-\theta}, \qquad \lambda \in \mathbb{C} \setminus \Sigma(\mathbb{R}, \theta),$$

Corollary 11.15 and Theorem 11.18 reduce the proof to the study of the integrals

$$I_q(\lambda; x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widetilde{b}_q(\lambda; x, \eta)\, d\eta$$

(see (11.10), (11.11) and note that $\widetilde{\chi}_\kappa(\lambda, \eta) = 1$ if $|\lambda| \geq 1$).

Let us start with the case when A is a differential operator. Suppose λ is purely imaginary. Then $|\lambda| = \mp i\lambda$ if $\pm \operatorname{Im}\lambda > 0$. Using the change of variables $\eta' = (|\lambda|^{-1/(\kappa d_1)}\eta_1, \ldots, |\lambda|^{-1/(\kappa d_n)}\eta_n)$ and taking into account the equality $\widehat{b}_q \equiv \widetilde{b}_q$ we obtain

$$I_q(\lambda; x) = (\mp i\lambda)^{(n-q)/\kappa - 1} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{b}_q(\pm i; x, \eta')\, d\eta'.$$

The above equality extends analytically to all λ such that $\pm \operatorname{Im}\lambda > 0$. This completes the proof for differential operators.

Let us consider the general case. It follows from Lemma 11.5 that if $q < n + \kappa$ then the integral

$$\widehat{I}_q(\lambda; x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{b}_q(\lambda; x, \eta)\, d\eta$$

exists. We obtain as above

$$\widehat{I}_q(\lambda; x) = (\mp i\lambda)^{(n-q)/\kappa - 1} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{b}_q(\pm i; x, \eta)\, d\eta, \qquad \pm \operatorname{Im}\lambda > 0.$$

Further,

$$I_q(\lambda; x) - \widehat{I}_q(\lambda; x) = \frac{1}{(2\pi)^n} \int_{|\eta|_{\mathbf{d}} \leq 1} (\widetilde{b}_q(\lambda; x, \eta) - \widehat{b}_q(\lambda; x, \eta))\, d\eta.$$

We have

$$\begin{aligned} \widetilde{b}_0(\lambda; x, \eta) - \widehat{b}_0(\lambda; x, \eta) &= (\widetilde{a}_0(x, \eta) - \lambda I)^{-1}(\widehat{a}_0(x, \eta) - \widetilde{a}_0(x, \eta))(\widehat{a}_0(x, \eta) - \lambda I)^{-1} \\ &= O(|\lambda|^{-2}), \\ \widetilde{b}_q(\lambda; x, \eta) - \widehat{b}_q(\lambda; x, \eta) &= (1 + |\eta|_{\mathbf{d}}^{\kappa - q}) O(|\lambda|^{-2}), \qquad q > 0,\ |\eta|_{\mathbf{d}} \leq 1, \end{aligned}$$

as $|\lambda| \to \infty$ (see Lemma 11.5). Hence

$$I_q(\lambda; x) - \widehat{I}_q(\lambda; x) = O(|\lambda|^{-2}),$$

i.e.

$$I_q(\lambda; x) = \mathcal{R}_q^{\pm}(x)(\mp i\lambda)^{(n-q)/\kappa - 1} + O(|\lambda|^{-2}), \qquad q < n + \kappa.$$

Suppose now $q \geq n + \kappa$. Using Lemma 11.5 again we obtain

$$I_q(\lambda; x) = \frac{1}{(2\pi)^n} \int_{|\eta|_{\mathbf{d}} < 1} + \int_{|\eta|_{\mathbf{d}} \geq 1} \widetilde{b}_q(\lambda; x, \eta)\, d\eta = O(|\lambda|^{-2}) + \int_{|\eta|_{\mathbf{d}} \geq 1} \widehat{b}_q(\lambda; x, \eta)\, d\eta$$

$$= O(|\lambda|^{-2}) + |\lambda|^{(n-q)/\kappa-1}\frac{1}{(2\pi)^n}\Big(\int\limits_{|\lambda|^{-1/\kappa}\le|\eta'|_{\mathbf{d}}\le 1} + \int\limits_{|\eta'|_{\mathbf{d}}>1}\Big)\widehat{b}_q(\lambda/|\lambda|; x, \eta')\,d\eta'$$

$$= O(|\lambda|^{-2}) + O(|\lambda|^{(n-q)/\kappa-1}) + |\lambda|^{(n-q)/\kappa-1}\frac{1}{(2\pi)^n}\int\limits_{|\lambda|^{-1/\kappa}\le|\eta|_{\mathbf{d}}\le 1}\widehat{b}_q(\lambda/|\lambda|; x, \eta)\,d\eta$$

$$= O(|\lambda|^{-2}) + |\lambda|^{(n-q)/\kappa-1}O\Big(\int\limits_{|\lambda|^{-1/\kappa}\le|\eta|_{\mathbf{d}}\le 1}|\eta|_{\mathbf{d}}^{\kappa-q}\,d\eta\Big).$$

Now,

$$\int\limits_{|\lambda|^{-1/\kappa}\le|\eta|_{\mathbf{d}}\le 1}|\eta|_{\mathbf{d}}^{\kappa-q}d\eta = \int\limits_{|\lambda|^{-1/\kappa}}^{1}\tau^{\kappa-q}d\,\mathrm{Vol}\{\eta\in\mathbb{R}^n : |\eta|_{\mathbf{d}}\le\tau\}$$

$$= n\,\mathrm{Vol}\{\eta\in\mathbb{R}^n : |\eta|_{\mathbf{d}}\le 1\}\int\limits_{|\lambda|^{-1/\kappa}}^{1}\tau^{\kappa-q+n-1}d\tau$$

$$= \begin{cases}\text{const}\ \log|\lambda|, & q = n+\kappa,\\ \text{const}\ (|\lambda|^{(q-n)/\kappa-1}-1), & q > n+\kappa.\end{cases}$$

Therefore

$$I_q(\lambda; x) = \begin{cases}O(|\lambda|^{-2}\log|\lambda|), & q = n+\kappa,\\ O(|\lambda|^{-2}), & q > n+\kappa.\ \blacksquare\end{cases}$$

14.2. Theorem. *Let the conditions of Theorem* 14.1 *be satisfied and the principal presymbol* $\widetilde{a}_0$ *of* A *be positive. Then*

(14.4) $$\mathcal{R}(\lambda; x, x) \sim \sum_{\substack{q\in\mathbb{Z}_+(\mathcal{N},\mathbf{d})\\ q<n+\kappa(1-2\theta)}}\mathcal{R}_q(x)(-\lambda)^{(n-q)/\kappa-1} + O(|\lambda|^{2(\theta-1)}) \quad \textit{as } |\lambda|\to\infty, \quad \lambda\in\mathbb{C}\setminus\Sigma(\mathbb{R}_+,\theta),$$

where $\Sigma(\mathbb{R}_+,\theta) := \{\mu\in\mathbb{C} : \mathrm{Re}\,\mu > 0,\ |\mathrm{Im}\,\mu| < \text{const}\,|\mu|^{1-\theta}\}$,

(14.5) $$\mathcal{R}_q(x) = \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}\widehat{b}_q(-1; x, \eta)\,d\eta.$$

If A *is a differential operator, then*

(14.6) $$\mathcal{R}(\lambda; x, x) \sim \sum_{q\in\mathbb{Z}_+(\mathcal{N},\mathbf{d})}\mathcal{R}_q(x)(-\lambda)^{(n-q)/\kappa-1} \quad \textit{as } |\lambda|\to\infty,\ \lambda\in\mathbb{C}\setminus\Sigma(\mathbb{R}_+,\theta),$$

where $\mathcal{R}_q$ *are defined by* (14.5)*. The above asymptotic expansions are uniform in* $x\in M$*.*

Proof. Our statement follows from Theorem 14.1. We only need to observe that now negative values of λ are allowed and we can use the formula

$$\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}\widehat{b}_q(\lambda; x, \eta)\,d\eta = (-\lambda)^{(n-q)/\kappa-1}\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}\widehat{b}_q(-1; x, \eta')\,d\eta', \qquad \lambda\notin[0,\infty).\ \blacksquare$$

14.3. Theorem. *Let the conditions formulated at the beginning of Section* 12 *be satisfied and* $\kappa > n$*. Then the resolvent* $(A-\lambda I)^{-1}$, $\lambda\in\mathbb{C}\setminus\mathrm{Spec}(A)$, *is an integral operator with a kernel* $\mathcal{R}(\lambda; x, y)$ *which admits for any* $w\in(0,\pi-v]$ *the following on-diagonal* *asymptotic expansion*:

$$\mathcal{R}(\lambda;x,x)\sim\sum_{q\in\mathbb{Z}_+(\mathcal{N},\mathbf{d})\cup J}\widetilde{\mathcal{R}}_q(x)(-\lambda)^{(n-q)/\kappa-1}+\sum_{j=1}^{\infty}\widehat{\mathcal{R}}_j(x)(-\lambda)^{-j-1}\log(-\lambda), \tag{14.7}$$
$$\text{as } |\lambda|\to\infty,\ v+w\le\pm\arg\lambda\le\pi,$$

where $J=\{q=n+j\kappa : j\in\mathbb{Z}_+\}$;

$$\widetilde{\mathcal{R}}_q(x)=\pi\left(\sin\pi\frac{q-n}{\kappa}\right)^{-1}\varrho_q(x)\qquad \text{if } q\in\mathbb{Z}_+(\mathcal{N},\mathbf{d})\setminus J; \tag{14.8}$$

$$\widehat{\mathcal{R}}_j(x)=0,\qquad \widetilde{\mathcal{R}}_q(x)=(-1)^jA_j(x,x) \tag{14.9}$$

if $q=n+j\kappa$, $j\in\mathbb{Z}_+$ *and either* $q\notin\mathbb{Z}_+(\mathcal{N},\mathbf{d})$ *or* $q\in\mathbb{Z}_+(\mathcal{N},\mathbf{d})$ *but* $A_z(x,x)$ *does not have a pole at* $z=j$; *and*

$$\widehat{\mathcal{R}}_j(x)=(-1)^{j+1}\varrho_q(x),\qquad \widetilde{\mathcal{R}}_q(x)=(-1)^j\left(A_z(x,x)-\frac{\varrho_q(x)}{z-j}\right)\bigg|_{z=j}$$

if $q=n+j\kappa\in\mathbb{Z}_+(\mathcal{N},\mathbf{d})$, $j\in\mathbb{N}$ *and* $A_z(x,x)$ *has a pole at* $z=j$ (see Theorem 12.1). *The above asymptotic expansion is uniform in* $x\in M$.

Proof. It follows from (12.15) that

$$\mathcal{R}(\lambda;x,x)=\frac{i}{2}\int_{\operatorname{Re}z=s}\frac{(-\lambda)^{-z-1}}{\sin\pi z}A_z(x,x)\,dz,\qquad -1<s<-n/\kappa.$$

Since

$$|\sin\pi z|^{-1}\le\text{const}\,e^{-\pi|\operatorname{Im}z|},\qquad |\operatorname{Im}z|\ge1,$$

with a constant independent of z, Corollary 12.5 implies that we can shift the path of integration letting $s\to\infty$ and "jumping" over the poles. Using the residue theorem we can obtain the above asymptotic expansion in the standard way (see [Agr1], [Agr2]). ∎

14.4. Remark. If A is a differential operator, $A_z(x,x)$ does not have a pole at $z=j\in\mathbb{N}$ (see Theorem 12.1(iv)) and we do not have the second sum on the right hand side of (14.7).

14.5. Remark. Suppose A satisfies the conditions of Theorem 14.2 and is positive definite. Then it also satisfies the conditions of Theorem 14.3 with $v=0$, and (14.4), (14.6), (14.7) imply

$$\mathcal{R}_q(x)=\widetilde{\mathcal{R}}_q(x) \tag{14.10}$$

if $q<n+\kappa$ or A is a differential operator. Let us give a direct proof of this equality. We start with the case $q\in\mathbb{Z}_+(\mathcal{N},\mathbf{d})\setminus J$. If $q<n+\kappa$, similarly to (12.14) we obtain

$$\widetilde{a}_q^{(z_q)}(x,\eta)=-\frac{\sin\pi z_q}{\pi}\int_0^{\infty}t^{z_q}\widetilde{b}_q(-t;x,\eta)\,dt,\qquad |\eta|_{\mathbf{d}}\ge1,\ z_q=(q-n)/\kappa.$$

It follows from Remark 11.6 that this formula remains true if $q\ge n+\kappa$ and A is a differential operator. Taking into account that $\widetilde{a}_q^{(z_q)}$ is almost $\mathbf{d}$-homogeneous of degree $-n$

and using Lemma 12.2, from (14.8) and Theorem 12.1 we obtain

$$\begin{aligned}
\widetilde{\mathcal{R}}_q(x) &= -\frac{\pi}{(2\pi)^n \kappa \sin \pi z_q} \int\limits_{|\eta|=1} \widetilde{a}_q^{(z_q)}(x,\eta) \sum_{k=1}^n d_k^{-1} \eta_k^2 \, dS \\
&= -\frac{\pi}{(2\pi)^n \kappa \sin \pi z_q} \int\limits_{|\eta|_{\mathbf{d}} \geq 1} |\eta|_{\mathbf{d}}^{-1} \widetilde{a}_q^{(z_q)}(x,\eta) \, d\eta \\
&= \frac{1}{(2\pi)^n \kappa} \int\limits_{|\eta|_{\mathbf{d}} \geq 1} \int\limits_0^\infty t^{z_q} |\eta|_{\mathbf{d}}^{-1} \widetilde{b}_q(-t; x, \eta) \, dt \, d\eta \\
&= \frac{1}{(2\pi)^n \kappa} \int\limits_0^\infty \int\limits_{|\eta'|_{\mathbf{d}} \geq t^{-1/\kappa}} t^{z_q} t^{-1/\kappa} t^{-1-q/\kappa} t^{n/\kappa} |\eta'|_{\mathbf{d}}^{-1} \widehat{b}_q(-1; x, \eta') \, d\eta' \, dt \\
&= \frac{1}{(2\pi)^n \kappa} \int\limits_{\mathbb{R}^n} |\eta|_{\mathbf{d}}^{-1} \widehat{b}_q(-1; x, \eta) \int\limits_{t \geq |\eta|_{\mathbf{d}}^{-\kappa}} t^{-1-1/\kappa} \, dt \, d\eta \\
&= \frac{1}{(2\pi)^n \kappa} \int\limits_{\mathbb{R}^n} |\eta|_{\mathbf{d}}^{-1} \widehat{b}_q(-1; x, \eta) \kappa |\eta|_{\mathbf{d}} \, d\eta = \mathcal{R}_q(x).
\end{aligned}$$

Suppose now $q = n + j\kappa$, $j \in \mathbb{Z}_+$, $q \in \mathbb{Z}_+(\mathcal{N}, \mathbf{d})$. Then due to our restrictions either A is a differential operator or $j = 0$. In both cases $A_z(x,x)$ does not have a pole at $z = j$ and

$$\widetilde{\mathcal{R}}_q(x) = (-1)^j A_j(x,x) = \frac{1}{(2\pi)^n \kappa} \int\limits_{|\eta|=1} \sum_{k=1}^n d_k^{-1} \eta_k^2 \, dS \int\limits_0^\infty t^j \widetilde{b}_q(-t; x, \eta) \, dt$$

(see Theorem 12.1(iii) and (iv)). Acting as above we obtain (14.10).

Let A be the operator from Theorem 14.3. It follows from (14.5), (14.8), (14.10) and Theorem 12.1(ii) that

$$\begin{aligned}
\mathcal{R}_0(x) &= \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} (\widehat{a}_0(x,\eta) + I)^{-1} \, d\eta \\
&= \pi \left(\sin \frac{\pi n}{\kappa} \right)^{-1} \frac{1}{(2\pi)^n \kappa} \int\limits_{|\eta|=1} (\widetilde{a}_0(x,\eta))^{-n/\kappa} \sum_{k=1}^n d_k^{-1} \eta_k^2 \, dS.
\end{aligned} \tag{14.11}$$

Suppose in addition that A is self-adjoint. Then $\widehat{a}_0(x,\eta)$ is positive self-adjoint (see the proof of Corollary 11.15) and

$$(\widehat{a}_0(x,\eta) + I)^{-1} = \int\limits_0^\infty (\tau + 1)^{-1} \, dE_\tau(x,\eta),$$

where $E_\tau(x,\eta)$ is the orthogonal projection onto the subspace spanned by the eigenvectors of $\widehat{a}_0(x,\eta)$ corresponding to eigenvalues less than or equal to τ. Therefore

$$\begin{aligned}
\mathcal{R}_0(x) &= \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \int\limits_0^\infty (\tau + 1)^{-1} \, dE_\tau(x,\eta) \, d\eta \\
&= \frac{1}{(2\pi)^n} \int\limits_0^\infty (\tau + 1)^{-1} \, d\Big(\int\limits_{\mathbb{R}^n} E_\tau(x,\eta) \, d\eta \Big) \\
&= \frac{1}{(2\pi)^n} \int\limits_0^\infty (\tau + 1)^{-1} \, d\tau^{n/\kappa} \int\limits_{\mathbb{R}^n} E_1(x,\eta) \, d\eta.
\end{aligned}$$

Since

$$\int_0^\infty (\tau+1)^{-1}\tau^{n/\kappa-1}\,d\tau = B\Big(\frac{n}{\kappa}, 1-\frac{n}{\kappa}\Big) = \frac{\Gamma(n/\kappa)\Gamma(1-n/\kappa)}{\Gamma(1)} = \pi\Big(\sin\frac{\pi n}{\kappa}\Big)^{-1}$$

(see, e.g., [WW, §§12.14, 12.4 and 12.41]), we obtain

$$R_0(x) = \frac{\pi n}{\kappa}\Big(\sin\frac{\pi n}{\kappa}\Big)^{-1}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} E_1(x,\eta)\,d\eta. \tag{14.12}$$

In the scalar case this equality takes the form

$$R_0(x) = \frac{\pi n}{\kappa}\Big(\sin\frac{\pi n}{\kappa}\Big)^{-1}\frac{1}{(2\pi)^n}\int_{\widehat{a}_0(x,\eta)\le 1} d\eta. \tag{14.13}$$

If $\widetilde{a}_\varepsilon(x,\eta)$, $\partial_{\eta_k}\widetilde{a}_0(x,\eta)$ and $\nabla^{\mathcal{E},\mathcal{E}}_{\nu_k(x)}\widetilde{a}_0(x,\eta)$ commute with $\widetilde{a}_0(x,\eta)$, then $\partial_{\eta_k}\widetilde{a}_0(x,\eta)$ and $\nabla^{\mathcal{E},\mathcal{E}}_{\nu_k(x)}\widetilde{a}_0(x,\eta)$ commute (see Lemma 14.6 below) and using (12.5) and integration by parts we deduce from (14.5) that

$$R_\varepsilon(x) = -\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{a}_{\rm sub}(x,\eta)(\widehat{a}_0(x,\eta)+I)^{-2}\,d\eta, \tag{14.14}$$

where

$$\widehat{a}_{\rm sub}(x,\eta) := \widehat{a}_\varepsilon(x,\eta) - \frac{1}{2i}\sum_{d_k^{-1}=\varepsilon}\partial_{\eta_k}\nabla^{\mathcal{E},\mathcal{E}}_{\nu_k(x)}\widehat{a}_0(x,\eta). \tag{14.15}$$

Note that $\widehat{a}_{\rm sub}(x,\eta)$ and $\widehat{a}_0(x,\eta)$ commute (see Lemma 14.6 below). Suppose in addition that A is self-adjoint. Then

$$\begin{aligned}
R_\varepsilon(x) &= -\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{a}_{\rm sub}(x,\eta)\int_0^\infty(\tau+1)^{-2}dE_\tau(x,\eta)\,d\eta\\
&= -\frac{1}{(2\pi)^n}\int_0^\infty(\tau+1)^{-2}\,d\Big(\int_{\mathbb{R}^n}\widehat{a}_{\rm sub}(x,\eta)E_\tau(x,\eta)\,d\eta\Big)\\
&= -\frac{1}{(2\pi)^n}\int_0^\infty(\tau+1)^{-2}\,d\tau^{1+(n-\varepsilon)/\kappa}\int_{\mathbb{R}^n}\widehat{a}_{\rm sub}(x,\eta)E_1(x,\eta)\,d\eta.
\end{aligned}$$

Since

$$\begin{aligned}
\int_0^\infty(\tau+1)^{-2}\tau^{(n-\varepsilon)/\kappa}d\tau &= \frac{\Gamma((n-\varepsilon)/\kappa+1)\Gamma(1-(n-\varepsilon)/\kappa)}{\Gamma(2)}\\
&= \frac{n-\varepsilon}{\kappa}\Gamma\Big(\frac{n-\varepsilon}{\kappa}\Big)\Gamma\Big(1-\frac{n-\varepsilon}{\kappa}\Big)\\
&= \frac{n-\varepsilon}{\kappa}\pi\Big(\sin\pi\frac{n-\varepsilon}{\kappa}\Big)^{-1}
\end{aligned}$$

(see [WW, §§12.12, 12.14, 12.4 and 12.41]), we have

$$R_\varepsilon(x) = -\frac{\pi(n-\varepsilon)(\kappa+n-\varepsilon)}{\kappa^2}\Big(\sin\pi\frac{n-\varepsilon}{\kappa}\Big)^{-1}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{a}_{\rm sub}(x,\eta)E_1(x,\eta)\,d\eta. \tag{14.16}$$

In the scalar case this equality takes the form

$$R_\varepsilon(x) = -\frac{\pi(n-\varepsilon)(\kappa+n-\varepsilon)}{\kappa^2}\left(\sin\pi\frac{n-\varepsilon}{\kappa}\right)^{-1}\frac{1}{(2\pi)^n}\int\limits_{\widehat{a}_0(x,\eta)\le 1}\widehat{a}_{\mathrm{sub}}(x,\eta)\,d\eta. \tag{14.17}$$

14.6. Lemma. *Let $d_1, d_2 : \mathbf{R} \to \mathbf{R}$ be derivations of a ring $\mathbf{R}$, i.e.*

$$d_j(x+y) = d_jx + d_jy, \quad d_j(xy) = (d_jx)y + xd_jy, \qquad \forall x, y \in \mathbf{R},\ j = 1, 2.$$

Suppose d_1 and d_2 commute, i.e. $[d_1, d_2] = 0$. If $a \in \mathbf{R}$ is such that $[d_ja, a] = 0$, $j = 1, 2$, then

$$[d_1a, d_2a] = 0, \qquad [d_1d_2a, a] = 0.$$

Proof. Since $[d_ja, a] = 0$, we have

$$\begin{aligned}[d_1, d_2]a^2 = 0 \ &\Rightarrow\ 2d_1((d_2a)a) - 2d_2((d_1a)a) = 0\\ &\Rightarrow\ (d_1d_2a)a + (d_2a)d_1a - (d_2d_1a)a - (d_1a)d_2a = 0\\ &\Rightarrow\ ([d_1, d_2]a)a - [d_1a, d_2a] = 0 \ \Rightarrow\ [d_1a, d_2a] = 0.\end{aligned}$$

Using the last equality we obtain

$$d_1[d_2a, a] = 0 \ \Rightarrow\ [d_1d_2a, a] + [d_2a, d_1a] = 0 \ \Rightarrow\ [d_1d_2a, a] = 0. \ \blacksquare$$

14.7. Remark. Let $A \in \mathcal{H}\Psi^{\kappa,\mathbf{d}}(\mathcal{E}, \mathcal{F})$ and let $\widehat{\sigma}_{A,\mathrm{sub}} = \widehat{a}_{\mathrm{sub}}$ be defined by (14.15) with $\nabla^{\mathcal{F},\mathcal{E}}_{\nu_k}$ instead of $\nabla^{\mathcal{E},\mathcal{E}}_{\nu_k}$. Suppose $A' \in \mathcal{H}\Psi^{\kappa',\mathbf{d}}(\mathcal{J}, \mathcal{E})$ is a ψDO with a presymbol $\widetilde{a}'(x,\eta) \sim \sum_{l\in\mathbb{Z}_+(\mathcal{N},\mathbf{d})} \widetilde{a}'_l(x,\eta)$ (see Definition 11.4). Using Corollary 6.10 and the equality

$$\nabla^{\mathcal{F},\mathcal{J}}_{\nu_k}(\widehat{a}_0\widehat{a}'_0) = (\nabla^{\mathcal{F},\mathcal{E}}_{\nu_k}\widehat{a}_0)\widehat{a}'_0 + \widehat{a}_0(\nabla^{\mathcal{E},\mathcal{J}}_{\nu_k}\widehat{a}'_0)$$

(see (2.31)) we obtain by a straightforward calculation

$$\begin{aligned}\widehat{\sigma}_{AA',\mathrm{sub}} &= \widehat{\sigma}_{A,\mathrm{sub}}\widehat{a}'_0 + \widehat{a}_0\widehat{\sigma}_{A',\mathrm{sub}}\\ &\quad + \frac{1}{2i}\{\widehat{a}_0, \widehat{a}'_0\}_\varepsilon - \frac{1}{2i}\sum_{d_k^{-1}+d_m^{-1}-d_s^{-1}=\varepsilon} C^s_{k,m}(x)\eta_s\partial_{\eta_k}\widehat{a}_0\partial_{\eta_m}\widehat{a}'_0,\end{aligned} \tag{14.18}$$

where

$$\{\widehat{a}_0, \widehat{a}'_0\}_\varepsilon := \sum_{d_k^{-1}=\varepsilon}(\partial_{\eta_k}\widehat{a}_0(x,\eta)\nabla^{\mathcal{E},\mathcal{J}}_{\nu_k}\widehat{a}'_0 - \nabla^{\mathcal{F},\mathcal{E}}_{\nu_k(x)}\widehat{a}_0(x,\eta)\partial_{\eta_k}\widehat{a}'_0)$$

is an analogue of the Poisson bracket.

Suppose now A satisfies the conditions formulated at the beginning of Section 12 and $\widetilde{a}_\varepsilon(x,\eta)$, $\partial_{\eta_k}\widetilde{a}_0(x,\eta)$ and $\nabla^{\mathcal{E},\mathcal{E}}_{\nu_k(x)}\widetilde{a}_0(x,\eta)$ commute with $\widetilde{a}_0(x,\eta)$. Then using (12.4) and (12.6) we obtain by a routine calculation

$$\widehat{\sigma}_{A^z,\mathrm{sub}} = z\widehat{\sigma}_{A,\mathrm{sub}}(\widehat{a}_0)^{z-1} = z\widehat{a}_{\mathrm{sub}}(\widehat{a}_0)^{z-1}, \qquad z \in \mathbb{C}. \tag{14.19}$$

14.8. Remark. Let A be the operator from Theorem 14.2. It follows from (14.8), (14.10), (14.12) and (14.13) that

$$\varrho_0(x) = -\frac{n}{\kappa}\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} E_1(x,\eta)\,d\eta \tag{14.20}$$

and in the scalar case

$$\varrho_0(x) = -\frac{n}{\kappa}\frac{1}{(2\pi)^n}\int\limits_{\widehat{a}_0(x,\eta)\le 1} d\eta. \tag{14.21}$$

If $\widetilde{a}_\varepsilon(x,\eta)$, $\partial_{\eta_k}\widetilde{a}_0(x,\eta)$ and $\nabla^{\mathcal{E},\mathcal{E}}_{\nu_k(x)}\widetilde{a}_0(x,\eta)$ commute with $\widetilde{a}_0(x,\eta)$, then (14.16) and (14.17) imply

$$\varrho_\varepsilon(x) = \frac{(n-\varepsilon)(\kappa+n-\varepsilon)}{\kappa^2}\,\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}\widehat{a}_{\rm sub}(x,\eta)E_1(x,\eta)\,d\eta \tag{14.22}$$

and in the scalar case

$$\varrho_\varepsilon(x) = \frac{(n-\varepsilon)(\kappa+n-\varepsilon)}{\kappa^2}\,\frac{1}{(2\pi)^n}\int\limits_{\widehat{a}_0(x,\eta)\le 1}\widehat{a}_{\rm sub}(x,\eta)\,d\eta. \tag{14.23}$$

Let us show that (14.20)–(14.23) remain true if $0<\kappa\le n$. Let $r\in\mathbb{N}$ be such that $r\kappa>n$. We will use the superscript "(r)" to denote objects corresponding to the operator A^r. The equality $A^z=(A^r)^{z/r}$ implies $\varrho_q(x)=r\varrho_q^{(r)}(x)$. It follows from (12.4) that $E^{(r)}_{\tau^r}(x,\eta)=E_\tau(x,\eta)$. Using the analogue of (14.20) for A^r we obtain

$$\varrho_0(x)=r\varrho_0^{(r)}(x)=-r\frac{n}{r\kappa}\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}E_1^{(r)}(x,\eta)\,d\eta=-\frac{n}{\kappa}\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}E_1(x,\eta)\,d\eta,$$

i.e. (14.20), (14.21) hold. Let us prove (14.22), (14.23). Note that under our conditions $\widetilde{a}^{(r)}_\varepsilon(x,\eta)$, $\partial_{\eta_k}\widetilde{a}^{(r)}_0(x,\eta)$ and $\nabla^{\mathcal{E},\mathcal{E}}_{\nu_k(x)}\widetilde{a}^{(r)}_0(x,\eta)$ commute with $\widetilde{a}^{(r)}_0(x,\eta)$ (see (12.4) and (12.6)). The analogues of (14.8), (14.10), (14.14) for A^r and (12.4), (14.19) imply

$$\begin{aligned}
\pi\left(\sin\pi\frac{n-\varepsilon}{r\kappa}\right)^{-1}\varrho_\varepsilon(x) &= \pi\left(\sin\pi\frac{n-\varepsilon}{r\kappa}\right)^{-1} r\varrho_\varepsilon^{(r)}(x) = -rR_\varepsilon^{(r)}(x)\\
&= \frac{r}{(2\pi)^n}\int\limits_{\mathbb{R}^n}\widehat{a}^{(r)}_{\rm sub}(x,\eta)(\widehat{a}^{(r)}_0(x,\eta)+I)^{-2}\,d\eta\\
&= \frac{r^2}{(2\pi)^n}\int\limits_{\mathbb{R}^n}\widehat{a}_{\rm sub}(x,\eta)(\widehat{a}_0(x,\eta))^{r-1}((\widehat{a}_0(x,\eta))^r+I)^{-2}\,d\eta\\
&= \frac{r^2}{(2\pi)^n}\int\limits_{\mathbb{R}^n}\widehat{a}_{\rm sub}(x,\eta)\int\limits_0^\infty\tau^{r-1}(\tau^r+1)^{-2}\,dE_\tau(x,\eta)\,d\eta\\
&= \frac{r^2}{(2\pi)^n}\int\limits_0^\infty\tau^{r-1}(\tau^r+1)^{-2}\,d\Big(\int\limits_{\mathbb{R}^n}\widehat{a}_{\rm sub}(x,\eta)E_\tau(x,\eta)\,d\eta\Big)\\
&= \frac{r^2}{(2\pi)^n}\int\limits_0^\infty\tau^{r-1}(\tau^r+1)^{-2}\,d\tau^{1+(n-\varepsilon)/\kappa}\int\limits_{\mathbb{R}^n}\widehat{a}_{\rm sub}(x,\eta)E_1(x,\eta)\,d\eta.
\end{aligned}$$

Since

$$\begin{aligned}
\int\limits_0^\infty\tau^{r-1}(\tau^r+1)^{-2}\,d\tau^{1+(n-\varepsilon)/\kappa} &= \int\limits_0^\infty t^{1-1/r}(t+1)^{-2}\,dt^{(1+(n-\varepsilon)/\kappa)/r}\\
&= \frac{1}{r}\left(1+\frac{n-\varepsilon}{\kappa}\right)\int\limits_0^\infty t^{(n-\varepsilon)/(r\kappa)}(t+1)^{-2}\,dt\\
&= \frac{1}{r}\left(1+\frac{n-\varepsilon}{\kappa}\right)\frac{n-\varepsilon}{r\kappa}\pi\left(\sin\pi\frac{n-\varepsilon}{r\kappa}\right)^{-1}
\end{aligned}$$

(see the proof of (14.16)), we arrive at (14.22), (14.23).

15. Spectral asymptotics

We will suppose in this section that (6.2) is satisfied, M is compact, $\mathcal{E}$ is Hermitian and $A \in \mathcal{H}\Psi^{\kappa,\mathbf{d}}(\mathcal{E},\mathcal{E})$, $\kappa > 0$, is semi-elliptic and self-adjoint (see Corollary 11.15). The spectrum $\mathrm{Spec}(A)$ of A on $L_2(\mathcal{E})$ consists of isolated eigenvalues with finite multiplicities (see the paragraph preceding Theorem 11.13). The inner product on $L_2(\mathcal{E})$ is given by the formula

$$(v,w)_{G,\mathcal{M}} := \int_M G(v(x), w(x))\, d\mathcal{M}(x),$$

where G is the Hermitian metric on $\mathcal{E}$ and the measure $\mathcal{M}$ is defined by (4.2). Let E^A be the spectral measure corresponding to A and let $\Lambda \subset \mathbb{R}$ be such that $\Lambda \cap \mathrm{Spec}(A)$ is finite. We will denote by $e^A(\Lambda; x, y)$ the kernel of the spectral projection $E^A(\Lambda)$, i.e.

$$e^A(\Lambda; x, y) = \sum_{\lambda_k \in \Lambda} G(\cdot, u_k(y)) u_k(x), \qquad x, y \in M, \tag{15.1}$$

where $\{u_k\}_{k\in\mathbb{Z}}$ is a complete orthonormal system of the eigenvectors of A corresponding to the eigenvalues λ_k. We number the eigenvalues λ_k in nondecreasing order taking into account their multiplicities. We will also consider the following functions:

$$N^A(\Lambda) := \dim E^A(\Lambda) L_2(\mathcal{E}) = \sum_{\lambda_k \in \Lambda} 1 = \mathrm{Tr}\, E^A(\Lambda). \tag{15.2}$$

We denote by "Tr" (resp. by "tr") traces of operators acting on $L_2(\mathcal{E})$ (resp. on fibres of $\mathcal{E}$). Let $\epsilon_1, \dots, \epsilon_{n_0}$ be an orthonormal basis of $\mathcal{E}_x$. Then

$$\begin{aligned}\mathrm{tr}\, e^A(\Lambda; x, x) &= \sum_{j=1}^{n_0} G(e^A(\Lambda; x, x)\epsilon_j, \epsilon_j) = \sum_{j=1}^{n_0} \sum_{\lambda_k \in \Lambda} G(\epsilon_j, u_k(x)) G(u_k(x), \epsilon_j) \\ &= \sum_{\lambda_k \in \Lambda} \sum_{j=1}^{n_0} |G(u_k(x), \epsilon_j)|^2 = \sum_{\lambda_k \in \Lambda} G(u_k(x), u_k(x)).\end{aligned}$$

Since $\{u_k\}_{k\in\mathbb{Z}}$ is an orthonormal system, we obtain

$$N^A(\Lambda) = \int_M \mathrm{tr}\, e^A(\Lambda; x, x)\, d\mathcal{M}(x). \tag{15.3}$$

We will use the notation

$$e^A_\pm(\lambda; x, y) := e^A(\pm(0,\lambda); x, y), \quad N^A_\pm(\lambda) := N^A(\pm(0,\lambda)), \qquad \lambda > 0. \tag{15.4}$$

Suppose the principal presymbol $\widetilde{a}_0$ of A is positive. Then A may have only a finite number of negative eigenvalues (see Theorem 11.13) and we can consider the *spectral function* of A

$$e^A(\lambda; x, y) := e^A((-\infty, \lambda); x, y) \tag{15.5}$$

and the *distribution function of eigenvalues*

$$N^A(\lambda) := N^A((-\infty, \lambda)). \tag{15.6}$$

If $\kappa > n$, then the equality

$$(A - \lambda I)^{-1} = \int_{\mathbb{R}} (\mu - \lambda)^{-1}\, dE^A(\mu)$$

and Mercer's theorem (see, e.g., [Smi, Theorem 7.7.2]) imply

$$\mathcal{R}(\lambda; x, x) = \int_{\mathbb{R}} (\mu - \lambda)^{-1} \, de^A(\mu; x, x), \tag{15.7}$$

$$\int_M \operatorname{tr} \mathcal{R}(\lambda; x, x) \, d\mathcal{M}(x) = \int_{\mathbb{R}} (\mu - \lambda)^{-1} \, dN^A(\mu) \tag{15.8}$$

for sufficiently large negative $\lambda \in \mathbb{R}$. By analyticity the above equalities hold for any $\lambda \in \mathbb{C} \setminus \operatorname{Spec}(A)$.

Let h be a (vector-valued) function defined on $[0, \infty)$. Its *Riesz means* are given by the formula

$$h_{(l)}(\lambda) := \int_0^\lambda \left(1 - \frac{\mu}{\lambda}\right)^l dh(\mu), \qquad l \in \mathbb{N}, \ \lambda \in [0, \infty). \tag{15.9}$$

We will consider the Riesz means $e^A_{\pm,(l)}(\lambda; x, x)$, $e^A_{(l)}(\lambda; x, x)$, $N^A_{\pm,(l)}(\lambda)$ and $N^A_{(l)}(\lambda)$. Our argument will rely upon the following result.

15.1. Lemma ([Sad], see also [Agr1] and [Agr2, §6.1c]). *Let h be a piecewise constant nondecreasing function on $[0, \infty)$ and let*

$$S(\zeta) := \int_0^\infty (\mu - \zeta)^{-1} \, dh(\mu), \qquad \zeta \notin [0, \infty).$$

Suppose $\zeta = \lambda + i\mu$, $\lambda > 0$, $\mu > 0$ and $\Gamma(\zeta)$ is a rectifiable curve which starts at $\bar{\zeta}$, terminates at ζ and has no common points with $[0, \infty)$. Then

$$\left| h_{(l)}(\lambda) - \frac{1}{2\pi i} \int_{\Gamma(\zeta)} S(z) \left(1 - \frac{z}{\lambda}\right)^l dz \right| \leq \frac{1}{\pi l} \frac{\mu^{l+1}}{\lambda^l} \, |S(\zeta)|, \qquad l \in \mathbb{N}.$$

If $l = 0$, then the above inequality holds for $h_{(0)} = h$ with the coefficient $(\pi^{-2} + 2^{-2})^{1/2}$ instead of $(\pi l)^{-1}$.

This lemma is a generalization of the Pleijel–Malliavin Tauberian theorem, which deals with the case $l = 0$ (see [Ple]). It admits the following obvious generalization.

15.2. Lemma. *Let $\mathcal{H}$ be a complex Hilbert space and h be a piecewise constant function on $[0, \infty)$ taking values in the set of bounded self-adjoint operators on $\mathcal{H}$. Suppose h is nondecreasing, i.e.*

$$(h(\mu)\epsilon, \epsilon) \leq (h(\lambda)\epsilon, \epsilon), \qquad \forall \epsilon \in \mathcal{H}, \qquad \text{if } \mu \leq \lambda.$$

Then

$$\left\| h_{(l)}(\lambda) - \frac{1}{2\pi i} \int_{\Gamma(\zeta)} S(z) \left(1 - \frac{z}{\lambda}\right)^l dz \right\| \leq \frac{2}{\pi l} \frac{\mu^{l+1}}{\lambda^l} \, \|S(\zeta)\|, \qquad l \in \mathbb{N},$$

where $S(\zeta)$ and $\Gamma(\zeta)$ are defined as in Lemma 15.1. If $l = 0$, then the above inequality holds for $h_{(0)} = h$ with the coefficient $2(\pi^{-2} + 2^{-2})^{1/2}$ instead of $2(\pi l)^{-1}$.

Proof. It is sufficient to apply Lemma 15.1 to the scalar-valued functions $(h\epsilon, \epsilon)$, $\epsilon \in \mathcal{H}$, and use the inequality

$$\|B\| \leq 2 \sup_{\|\epsilon\|=1} |(B\epsilon, \epsilon)|,$$

where B is an arbitrary bounded linear operator on $\mathcal{H}$. To prove the last inequality we represent B in the form $B = B_0 + iB_1$, where $B_0 = 2^{-1}(B+B^*)$ and $B_1 = (2i)^{-1}(B-B^*)$ are self-adjoint. Then

$$\begin{aligned}\|B\| \le \|B_0\| + \|B_1\| &= \sup_{\|\epsilon\|=1} |(B_0\epsilon,\epsilon)| + \sup_{\|\epsilon\|=1} |(B_1\epsilon,\epsilon)| \\ &\le 2 \sup_{\|\epsilon\|=1} |(B_0\epsilon,\epsilon) + i(B_1\epsilon,\epsilon)| = 2 \sup_{\|\epsilon\|=1} |(B\epsilon,\epsilon)|.\end{aligned}$$

(The example $\mathcal{H} = \mathbb{C}^2$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ shows that the constant 2 is optimal in the above inequality.) ∎

15.3. Theorem. *Let the principal presymbol $\widetilde{a}_0$ of A be positive. Then*

$$e^A(\lambda; x, x) = s_0(x)\lambda^{n/\kappa} + o(\lambda^{(n-\tau)/\kappa}), \quad \forall \tau < \varepsilon/2, \quad \text{as } \lambda \to \infty, \tag{15.10}$$

uniformly in $x \in M$ and

$$N^A(\lambda) = s_0\lambda^{n/\kappa} + o(\lambda^{(n-\tau)/\kappa}), \quad \forall \tau < \varepsilon/2, \quad \text{as } \lambda \to \infty, \tag{15.11}$$

where

$$s_0(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} E_1(x,\eta)\, d\eta, \qquad s_0 = \frac{1}{(2\pi)^n} \int_{M\times\mathbb{R}^n} N(1; \widehat{a}_0(x,\eta))\, d\eta\, d\mathcal{M}(x) \tag{15.12}$$

and $N(1; \widehat{a}_0(x,\eta)) = \operatorname{tr} E_1(x,\eta)$ is the number of eigenvalues of $\widehat{a}_0(x,\eta)$ which are less than or equal to 1. In the scalar case

$$s_0(x) = \frac{1}{(2\pi)^n} \int_{\widehat{a}_0(x,\eta)\le 1} d\eta, \qquad s_0 = \frac{1}{(2\pi)^n} \int_{\widehat{a}_0(x,\eta)\le 1} d\eta\, d\mathcal{M}(x). \tag{15.13}$$

Proof. Let us consider the operator

$$A' := A - \sum_{\lambda_k<0} \lambda_k (\cdot, u_k)_{G,\mathcal{M}} u_k.$$

Since $u_k \in C^\infty(\mathcal{E})$ (see Theorem 10.10), $A' \in \mathcal{H}\Psi^{\kappa,\mathbf{d}}(\mathcal{E},\mathcal{E})$ has the same presymbol as A, is nonnegative and

$$e^{A'}(\lambda; x, y) = e^A(\lambda; x, y), \qquad \forall \lambda \ge 0.$$

Therefore we can suppose without loss of generality that A is nonnegative.

Let us start with the case $\kappa > n$. From (15.7) we have

$$\mathcal{R}(\lambda; x, x) = \int_0^\infty (\mu - \lambda)^{-1}\, de^A(\mu; x, x). \tag{15.14}$$

On the other hand

$$\mathcal{R}(\zeta; x, x) = \mathcal{R}_0(x)(-\zeta)^{n/\kappa-1} + O(|\zeta|^{(n-\varepsilon)/\kappa-1}) \quad \text{as } |\zeta| \to \infty,\ \zeta \in \mathbb{C} \setminus \Sigma(\mathbb{R}_+, \theta),$$

for arbitrary $\theta := \tau/\kappa \in (0, \varepsilon/(2\kappa))$, due to Theorem 14.2. Now Lemma 15.2 implies

$$e^A(\lambda; x, x) = \frac{1}{2\pi i} \int_{\Gamma(\zeta)} \mathcal{R}_0(x)(-z)^{n/\kappa-1}\, dz + O(\lambda^{(n-\tau)/\kappa}), \qquad \zeta = \lambda + i\lambda^{1-\tau/\kappa},$$

uniformly in $x \in M$. Here $\Gamma(\zeta)$ is the arc of the circle centered at 0. Consequently,

$$e^A(\lambda; x, x) = \mathcal{R}_0(x) \frac{1}{2\pi i} \int_{\Gamma_0(\zeta)} (-z)^{n/\kappa - 1}\, dz + O(\lambda^{(n-\tau)/\kappa}),$$

where $\Gamma_0(\zeta)$ consists of $\Gamma(\zeta)$ and the vertical interval $[\zeta, \bar{\zeta}]$ and is oriented clockwise. Since

$$\frac{1}{2\pi i} \int_{\Gamma_0(\zeta)} (-z)^{n/\kappa - 1}\, dz = -\frac{1}{2\pi i} \frac{\kappa}{n} \lambda^{n/\kappa} (e^{-i\pi n/\kappa} - e^{i\pi n/\kappa}) = \frac{\kappa}{\pi n} \lambda^{n/\kappa} \sin \frac{\pi n}{\kappa},$$

we obtain

$$e^A(\lambda; x, x) = \frac{\kappa}{\pi n} \sin \frac{\pi n}{\kappa}\, \mathcal{R}_0(x) \lambda^{n/\kappa} + O(\lambda^{(n-\tau)/\kappa}) = s_0(x) \lambda^{n/\kappa} + O(\lambda^{(n-\tau)/\kappa})$$

(see (14.12)). This completes the proof in the case $\kappa > n$.

Suppose now $0 < \kappa \le n$. Let $r \in \mathbb{N}$ be such that $r\kappa > n$. We have

$$\begin{aligned} e^A(\lambda; x, x) = e^{A^r}(\lambda^r; x, x) &= (\lambda^r)^{n/(r\kappa)} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} E_1(x, \eta)\, d\eta + o((\lambda^r)^{(n-\tau)/(r\kappa)}) \\ &= s_0(x) \lambda^{n/\kappa} + o(\lambda^{(n-\tau)/\kappa}) \end{aligned}$$

(cf. Remark 14.8). ∎

15.4. THEOREM. *Let the principal presymbol $\widetilde{a}_0$ of A be positive, $\kappa > n$ and $l \in \mathbb{N}$. Then*

$$e^A_{(l)}(\lambda; x, x) = \sum_{\substack{q \in \mathbb{Z}_+(\mathcal{N}, \mathbf{d}) \\ q < (l+1)\varepsilon/2}} s^{(l)}_q(x) \lambda^{(n-q)/\kappa} + o(\lambda^{(n-(l+1)\tau)/\kappa}), \tag{15.15}$$

$$N^A_{(l)}(\lambda) = \sum_{\substack{q \in \mathbb{Z}_+(\mathcal{N}, \mathbf{d}) \\ q < (l+1)\varepsilon/2}} s^{(l)}_q \lambda^{(n-q)/\kappa} + o(\lambda^{(n-(l+1)\tau)/\kappa}) \quad \text{as } \lambda \to \infty, \tag{15.16}$$

for every $\tau < \varepsilon/2$ if $l \le 2(n+\kappa)/\varepsilon - 3$, and

$$e^A_{(l)}(\lambda; x, x) = \sum_{\substack{q \in \mathbb{Z}_+(\mathcal{N}, \mathbf{d}) \\ q < (n+\kappa)(1 - 2/(l+3))}} s^{(l)}_q(x) \lambda^{(n-q)/\kappa} + O(\lambda^{2(n+\kappa)/(\kappa(l+3)) - 1}), \tag{15.17}$$

$$N^A_{(l)}(\lambda) = \sum_{\substack{q \in \mathbb{Z}_+(\mathcal{N}, \mathbf{d}) \\ q < (n+\kappa)(1 - 2/(l+3))}} s^{(l)}_q \lambda^{(n-q)/\kappa} + O(\lambda^{2(n+\kappa)/(\kappa(l+3)) - 1}) \quad \text{as } \lambda \to \infty \tag{15.18}$$

if $l > 2(n+\kappa)/\varepsilon - 3$. Here

$$\begin{aligned} s^{(l)}_q(x) &= -B(l+1, (n-q)/\kappa) \varrho_q(x), \\ s^{(l)}_q &= -B(l+1, (n-q)/\kappa) \int_M \operatorname{tr} \varrho_q(x)\, d\mathcal{M}(x), \qquad q \ne n, \end{aligned} \tag{15.19}$$

and

$$s^{(l)}_n(x) = A_0(x, x), \qquad s^{(l)}_n = \int_M \operatorname{tr} A_0(x, x)\, d\mathcal{M}(x), \tag{15.20}$$

and

$$B(p, r) = \frac{\Gamma(p)\Gamma(r)}{\Gamma(p+r)}$$

is the beta-function.

Suppose A is a nonnegative differential operator. Then (15.15) *and* (15.16) *hold for all $l \in \mathbb{N}$, where $s_q^{(l)}(x)$, $s_q^{(l)}$, $q \in \mathbb{Z}_+(\mathcal{N}, \mathbf{d}) \setminus \{n + j\kappa : j \in \mathbb{Z}_+\}$ are given by* (15.19) *and*

$$\begin{aligned} s_q^{(l)}(x) &= \frac{(-1)^j l!}{j!(l-j)!} A_j(x,x), \\ s_q^{(l)} &= \frac{(-1)^j l!}{j!(l-j)!} \int_M \operatorname{tr} A_j(x,x)\, d\mathcal{M}(x), \qquad q = n + j\kappa. \end{aligned} \tag{15.21}$$

The above asymptotic expansions are uniform in $x \in M$.

Proof. We can assume without loss of generality that A is nonnegative (see the proof of Theorem 15.3). Using Theorem 14.2, Lemma 15.2 and (15.14) we obtain

$$\begin{aligned} e_{(l)}^A(\lambda; x, x) = &\sum_{\substack{q \in \mathbb{Z}_+(\mathcal{N}, \mathbf{d}) \\ q < n + \kappa - 2\tau}} \mathcal{R}_q(x) \frac{1}{2\pi i} \int_{\Gamma(\zeta)} (-z)^{(n-q)/\kappa - 1} \left(1 - \frac{z}{\lambda}\right)^l dz \\ &+ O(\lambda^{2\tau/\kappa - 1}) + O(\lambda^{(n-(l+1)\tau)/\kappa}) \\ = &\sum_{\substack{q \in \mathbb{Z}_+(\mathcal{N}, \mathbf{d}) \\ q < n + \kappa - 2\tau}} \mathcal{R}_q(x) \frac{1}{2\pi i} \int_{\Gamma_0(\zeta)} (-z)^{(n-q)/\kappa - 1} \left(1 - \frac{z}{\lambda}\right)^l dz \\ &+ O(\lambda^{\max\{2\tau/\kappa - 1, (n-(l+1)\tau)/\kappa\}}), \qquad \zeta = \lambda + i\lambda^{1-\tau/\kappa}, \end{aligned}$$

uniformly in $x \in M$. Here $\Gamma(\zeta)$ and $\Gamma_0(\zeta)$ are the same as in the proof of Theorem 15.3. Since the integrands are analytic in $\mathbb{C} \setminus [0, \infty)$, we have

$$\begin{aligned} \frac{1}{2\pi i} \int_{\Gamma_0(\zeta)} (-z)^{(n-q)/\kappa - 1} \left(1 - \frac{z}{\lambda}\right)^l dz &= -\frac{1}{2\pi i} \int_{|z| = \lambda} (-z)^{(n-q)/\kappa - 1} \left(1 - \frac{z}{\lambda}\right)^l dz \\ &= \lambda^{(n-q)/\kappa} \frac{1}{2\pi i} \int_{|z|=1} z^{(n-q)/\kappa - 1} (1+z)^l\, dz, \end{aligned}$$

where the circles $|z| = \lambda$ and $|z| = 1$ are oriented anticlockwise.

Let us evaluate the integral

$$I(\beta, l) := \frac{1}{2\pi i} \int_{|z|=1} z^{\beta - 1} (1+z)^l\, dz, \qquad \beta \in \mathbb{R},\ l \in \mathbb{Z}_+.$$

If $\beta \notin \mathbb{Z}$, successive integration by parts gives

$$\begin{aligned} I(\beta, l) &= -\frac{l}{\beta} I(\beta + 1, l - 1) = \ldots = (-1)^l \frac{l!}{\beta(\beta+1)\ldots(\beta+l-1)} I(\beta + l, 0) \\ &= (-1)^l \frac{l!\Gamma(\beta)}{\Gamma(\beta + l)} \frac{1}{2\pi i} \int_{|z|=1} z^{\beta + l - 1}\, dz = (-1)^l \frac{\Gamma(l+1)\Gamma(\beta)}{\Gamma(\beta + l + 1)} \pi^{-1} \sin \pi(\beta + l) \\ &= B(l+1, \beta) \pi^{-1} \sin \pi\beta. \end{aligned}$$

Suppose $\beta \in \mathbb{Z}$. Then the integrand is analytic in $\mathbb{C} \setminus \{0\}$ and the residue theorem implies

$$I(\beta, l) = \begin{cases} \dfrac{l!}{j!(l-j)!} & \text{if } \beta = -j,\ 0 \le j \le l, \\ 0 & \text{otherwise.} \end{cases}$$

Putting together the above equalities we obtain

$$e^A_{(l)}(\lambda;x,x)=\sum_{\substack{q\in\mathbb{Z}_+(\mathcal{N},\mathbf{d})\\ q<n+\kappa-2\tau}} s^{(l)}_q(x)\lambda^{(n-q)/\kappa}+O(\lambda^{\max\{2\tau/\kappa-1,(n-(l+1)\tau)/\kappa\}})$$

(see Theorem 12.1(iii), (14.8)–(14.10) and (15.19), (15.20)). If $l\le 2(n+\kappa)/\varepsilon-3$ then

$$(l+1)\frac{\varepsilon}{2}\le n+\kappa-\varepsilon,\qquad \frac{n-(l+1)\tau}{\kappa}>\frac{\varepsilon}{\kappa}-1>\frac{2\tau}{\kappa}-1,\quad \forall\tau<\varepsilon/2,$$

and the above equality is equivalent to (15.15).

Suppose $l>2(n+\kappa)/\varepsilon-3$. Then $(n+\kappa)/(l+3)<\varepsilon/2$. If $0<\tau\le(n+\kappa)/(l+3)$, then $(n-(l+1)\tau)/\kappa\ge 2(n+\kappa)/(\kappa(l+3))-1$. If $(n+\kappa)/(l+3)<\tau<\varepsilon/2$, then $2\tau/\kappa-1>2(n+\kappa)/(\kappa(l+3))-1$. Therefore

$$\max\left\{\frac{2\tau}{\kappa}-1,\frac{n-(l+1)\tau}{\kappa}\right\}\ge 2\frac{n+\kappa}{\kappa(l+3)}-1,\qquad\forall\tau<\frac{\varepsilon}{2},$$

and equality is achieved for $\tau=(n+\kappa)/(l+3)<\varepsilon/2$. This implies (15.17). We also have

$$n+\kappa-\varepsilon<(n+\kappa)\left(1-\frac{2}{l+3}\right)<n+\kappa,\qquad 2\frac{n+\kappa}{\kappa(l+3)}-1<\frac{\varepsilon}{\kappa}-1.$$

If A is a nonnegative differential operator, then using (14.6) instead of (14.4) we deduce that (15.15) and (15.16) hold for all $l\in\mathbb{N}$. Note that if $q=n+j\kappa$, $j\in\mathbb{Z}_+$, then $q<(l+1)\varepsilon/2$ implies $j<l$, and the coefficients in (15.21) are well defined. ∎

15.5. Corollary. *Let the principal presymbol $\widetilde{a}_0$ of A be positive and $\kappa>n$. Then for any $l=2,3,\dots$ we have*

$$e^A_{(l)}(\lambda;x,x)=s^{(l)}_0(x)\lambda^{n/\kappa}+s^{(l)}_\varepsilon(x)\lambda^{(n-\varepsilon)/\kappa}+o(\lambda^{(n-\varepsilon)/\kappa})\tag{15.22}$$

uniformly in $x\in M$ and

$$N^A_{(l)}(\lambda)=s^{(l)}_0\lambda^{n/\kappa}+s^{(l)}_\varepsilon\lambda^{(n-\varepsilon)/\kappa}+o(\lambda^{(n-\varepsilon)/\kappa})\qquad \text{as } \lambda\to\infty,\tag{15.23}$$

where

$$\begin{aligned} s^{(l)}_0(x)&=\frac{nB(l+1,n/\kappa)}{\kappa(2\pi)^n}\int_{\mathbb{R}^n}E_1(x,\eta)\,d\eta,\\ s^{(l)}_0&=\frac{nB(l+1,n/\kappa)}{\kappa(2\pi)^n}\int_{M\times\mathbb{R}^n}N(1;\widehat{a}_0(x,\eta))\,d\eta\,d\mathcal{M}(x)\end{aligned}\tag{15.24}$$

and $s^{(l)}_\varepsilon(x)$, $s^{(l)}_\varepsilon$ are given by (15.19). *In particular, if $\widetilde{a}_\varepsilon(x,\eta)$, $\partial_{\eta_k}\widetilde{a}_0(x,\eta)$ and $\nabla^{\mathcal{E},\mathcal{E}}_{\nu_k(x)}\widetilde{a}_0(x,\eta)$ commute with $\widetilde{a}_0(x,\eta)$, then*

$$\begin{aligned} s^{(l)}_\varepsilon(x)&=-\frac{l!\Gamma((n-\varepsilon)/\kappa+2)}{\Gamma((n-\varepsilon)/\kappa+l+1)(2\pi)^n}\int_{\mathbb{R}^n}\widehat{a}_{\mathrm{sub}}(x,\eta)E_1(x,\eta)\,d\eta,\\ s^{(l)}_\varepsilon&=-\frac{l!\Gamma((n-\varepsilon)/\kappa+2)}{\Gamma((n-\varepsilon)/\kappa+l+1)(2\pi)^n}\int_{M\times\mathbb{R}^n}\mathrm{tr}(\widehat{a}_{\mathrm{sub}}(x,\eta)E_1(x,\eta))\,d\eta\,d\mathcal{M}(x),\end{aligned}\tag{15.25}$$

and in the scalar case

$$\begin{aligned} s^{(l)}_\varepsilon(x)&=-\frac{l!\Gamma((n-\varepsilon)/\kappa+2)}{\Gamma((n-\varepsilon)/\kappa+l+1)(2\pi)^n}\int_{\widehat{a}_0(x,\eta)\le1}\widehat{a}_{\mathrm{sub}}(x,\eta)\,d\eta,\\ s^{(l)}_\varepsilon&=-\frac{l!\Gamma((n-\varepsilon)/\kappa+2)}{\Gamma((n-\varepsilon)/\kappa+l+1)(2\pi)^n}\int_{\widehat{a}_0(x,\eta)\le1}\widehat{a}_{\mathrm{sub}}(x,\eta)\,d\eta\,d\mathcal{M}(x).\end{aligned}\tag{15.26}$$

Proof. See Theorem 15.4 and (14.20)–(14.23). ∎

15.6. REMARK. Let $\kappa > n$. Suppose $e^A(\lambda; x, x)$ admits a two-term asymptotic expansion

$$e^A(\lambda; x, x) = s_0(x)\lambda^{n/\kappa} + s_p(x)\lambda^{(n-p)/\kappa} + f(\lambda)\lambda^{(n-p)/\kappa},$$
$$f(\lambda) = f(\lambda; x) = o(1) \quad \text{as } \lambda \to \infty,\ 0 < p < n.$$

Integrating by parts we obtain

$$\begin{aligned} e^A_{(l)}(\lambda; x, x) &= \int_0^\lambda \left(1 - \frac{\mu}{\lambda}\right)^l de^A(\mu; x, x) = -e^A(0; x, x) + l\lambda^{-1} \int_0^\lambda e^A(\mu; x, x)\left(1 - \frac{\mu}{\lambda}\right)^{l-1} d\mu \\ &= -e^A(0; x, x) + ls_0(x)\lambda^{-1}\int_0^\lambda \mu^{n/\kappa}\left(1 - \frac{\mu}{\lambda}\right)^{l-1} d\mu \\ &\quad + ls_p(x)\lambda^{-1}\int_0^\lambda \mu^{(n-p)/\kappa}\left(1 - \frac{\mu}{\lambda}\right)^{l-1} d\mu + l\lambda^{-1}\int_0^\lambda f(\mu)\mu^{(n-p)/\kappa}\left(1 - \frac{\mu}{\lambda}\right)^{l-1} d\mu \\ &= -e^A(0; x, x) + ls_0(x)\lambda^{n/\kappa}\int_0^1 t^{n/\kappa}(1-t)^{l-1}\, dt \\ &\quad + ls_p(x)\lambda^{(n-p)/\kappa}\int_0^1 t^{(n-p)/\kappa}(1-t)^{l-1}\, dt \\ &\quad + l\lambda^{(n-p)/\kappa}\left(\int_0^{1/\sqrt{\lambda}} + \int_{1/\sqrt{\lambda}}^1\right) f(\lambda t) t^{(n-p)/\kappa}(1-t)^{l-1}\, dt \\ &= -e^A(0; x, x) + lB\left(l, \frac{n}{\kappa} + 1\right) s_0(x)\lambda^{n/\kappa} \\ &\quad + lB\left(l, \frac{n-p}{\kappa} + 1\right) s_p(x)\lambda^{(n-p)/\kappa} + \lambda^{(n-p)/\kappa} o(1) \quad \text{as } \lambda \to \infty \end{aligned}$$

(see [WW], 12.4). If $l \geq 2$, $s^{(l)}_\varepsilon(x) \neq 0$ and $s_p(x) \neq 0$, then Corollary 15.5 implies $p = \varepsilon$ and

(15.27) $$s_\varepsilon(x) = -\frac{\kappa}{n - \varepsilon}\varrho_\varepsilon(x).$$

In particular, if $\widetilde{a}_\varepsilon(x, \eta)$, $\partial_{\eta_k}\widetilde{a}_0(x, \eta)$ and $\nabla^{\mathcal{E},\mathcal{E}}_{\nu_k(x)}\widetilde{a}_0(x, \eta)$ commute with $\widetilde{a}_0(x, \eta)$, then

(15.28) $$s_\varepsilon(x) = -\frac{\kappa + n - \varepsilon}{\kappa(2\pi)^n}\int_{\mathbb{R}^n} \widehat{a}_{\text{sub}}(x, \eta)E_1(x, \eta)\, d\eta,$$

and in the scalar case

(15.29) $$s_\varepsilon(x) = -\frac{\kappa + n - \varepsilon}{\kappa(2\pi)^n}\int_{\widehat{a}_0(x,\eta)\leq 1} \widehat{a}_{\text{sub}}(x, \eta)\, d\eta$$

(cf. (14.22), (14.23)). Hence, one may hope that under certain conditions $e^A(\lambda; x, x)$ admits the following two-term asymptotic expansion:

(15.30) $$e^A(\lambda; x, x) = s_0(x)\lambda^{n/\kappa} + s_\varepsilon(x)\lambda^{(n-\varepsilon)/\kappa} + o(\lambda^{(n-\varepsilon)/\kappa}) \quad \text{as } \lambda \to \infty,$$

where $s_0(x)$ and $s_\varepsilon(x)$ are given by (15.12), (15.13) and (15.27)–(15.29) respectively. We "derived" (15.30) under the assumption that $\kappa > n$. The arguments from Remark 14.8 and the proof of Theorem 15.3 reduce the case $0 < \kappa \le n$ to $\kappa > n$. Note also that one can arrive at the conjecture (15.30) using Theorem 13.1 instead of Corollary 15.5.

15.7. REMARK. The coefficient $s_\varepsilon(x)$ may be nonzero even if A is a differential operator, which does not happen in the standard elliptic case (see [DG]). Indeed, let M be a two-dimensional torus with the canonical coordinates $(\varphi_1, \varphi_2) \in [0, 2\pi]^2$, $d_1 = 5/6$, $d_2 = 5/4$, $A = \partial_{\varphi_1}^4 - \partial_{\varphi_2}^6 + \partial_{\varphi_1}^2 \partial_{\varphi_2}^2$. Then $\kappa = 24/5$, $\varepsilon = 4/5$, $a_0(x,\eta) = \eta_1^4 + \eta_2^6$, $a_\varepsilon(x,\eta) = \widehat{a}_{\mathrm{sub}}(x,\eta) = \eta_1^2\eta_2^2$ and $s_\varepsilon(x)$ is a strictly negative constant (see (15.29)).

Let us drop the condition that the principal presymbol $\widetilde{a}_0$ of A is positive. Then we have the following result.

15.8. THEOREM. *We have*

$$e_\pm^A(\lambda; x, x) = s_0^\pm(x)\lambda^{n/\kappa} + o(\lambda^{(n-\tau)/\kappa}) \quad \text{as } \lambda \to \infty,\ \forall \tau < \varepsilon/2, \tag{15.31}$$

uniformly in $x \in M$ *and*

$$N_\pm^A(\lambda) = s_0^\pm \lambda^{n/\kappa} + o(\lambda^{(n-\tau)/\kappa}) \quad \text{as } \lambda \to \infty,\ \forall \tau < \varepsilon/2, \tag{15.32}$$

where

$$s_0^\pm(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} E_1^\pm(x,\eta)\, d\eta, \qquad s_0^\pm = \frac{1}{(2\pi)^n} \int_{M\times\mathbb{R}^n} N_\pm(1; \widehat{a}_0(x,\eta))\, d\eta\, d\mathcal{M}(x), \tag{15.33}$$

$E_1^+(x,\eta)$ (*resp.* $E_1^-(x,\eta)$) *is the orthogonal projection onto the subspace spanned by the eigenvectors of* $\widehat{a}_0(x,\eta)$ *corresponding to positive* (*resp. negative*) *eigenvalues which are less than or equal to* 1 (*resp. greater than or equal to* -1), $N_+(1; \widehat{a}_0(x,\eta)) = \operatorname{tr} E_1^+(x,\eta)$ (*resp.* $N_-(1; \widehat{a}_0(x,\eta)) = \operatorname{tr} E_1^-(x,\eta)$) *is the number of eigenvalues of* $\widehat{a}_0(x,\eta)$ *which are less than or equal to* 1 (*resp. greater than or equal to* -1).

Proof. Since $\widetilde{a}_0(x,\eta)$ is supposed to be invertible for all $\eta \in \mathbb{R}^n$ (see Section 11), there exists $\delta > 0$ such that $\widetilde{a}_0(x,\eta) - \lambda I$, $|\lambda| \le \delta$, is invertible for all $\eta \in \mathbb{R}^n$. We can also assume that $\delta < |\lambda_k|$, $\forall \lambda_k \in \operatorname{Spec}(A) \setminus \{0\}$. Let us fix an arbitrary $\psi \in (0,\pi)$ and consider the contours $\Gamma_\psi^{(+)} := \Gamma_\psi$, $\Gamma_\psi^{(-)} := -\Gamma_\psi^{(+)}$, where Γ_ψ is the same as in the proof of Theorem 12.4. It follows from Theorem 6.7 and the formula

$$E_\pm^A := E^A(\pm(0,\infty)) = A \frac{i}{2\pi} \int_{\Gamma_\psi^{(\pm)}} \lambda^{-1}(A - \lambda I)^{-1}\, d\lambda$$

that $E_\pm^A \in \mathcal{H}\Psi^{0,\mathbf{d}}(\mathcal{E},\mathcal{E})$ and if $|\eta|_{\mathbf{d}} \ge 1$, the principal presymbol of E_+^A (of E_-^A) equals the orthogonal projection $E_+(x,\eta)$ (resp. $E_-(x,\eta)$) onto the subspace spanned by the eigenvectors of $\widehat{a}_0(x,\eta)$ corresponding to positive (resp. negative) eigenvalues. Using Theorem 6.7 again we deduce that

$$A(t) := (E_+^A - tE_-^A)A, \qquad t > 0,$$

is a self-adjoint semi-elliptic ψDO from $\mathcal{H}\Psi^{\kappa,\mathbf{d}}(\mathcal{E},\mathcal{E})$ with the positive principal presymbol

$$\widetilde{a}_0(t; x, \eta) = (E_+(x,\eta) - tE_-(x,\eta))\widetilde{a}_0(x,\eta).$$

Theorem 15.3 implies

$$\begin{aligned}
e^{A(t)}(\lambda;x,x) &= \lambda^{n/\kappa}\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} E_1(t;x,\eta)\,d\eta + o(\lambda^{(n-\tau)/\kappa}) \\
&= \lambda^{n/\kappa}\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} E_1^+(x,\eta)\,d\eta \\
&\quad + \lambda^{n/\kappa}\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} E_1^-(x,t^{1/(\kappa d_1)}\eta_1,\dots,t^{1/(\kappa d_n)}\eta_n)\,d\eta + o(\lambda^{(n-\tau)/\kappa}) \\
&= s_0^+(x)\lambda^{n/\kappa} + s_0^-(x)(\lambda/t)^{n/\kappa} + o(\lambda^{(n-\tau)/\kappa}) \quad \text{as } \lambda\to\infty,\ \forall\tau<\varepsilon/2,
\end{aligned}$$

uniformly in $x \in M$, where $E_1(t;x,\eta)$ is the orthogonal projection onto the subspace spanned by the eigenvectors of $\widehat{a}_0(t;x,\eta)$ corresponding to eigenvalues less than or equal to 1. On the other hand it is clear that

$$e^{A(t)}(\lambda;x,x) = e_+^A(\lambda;x,x) + e_-^A(\lambda/t;x,x) + e^A(\{0\};x,x), \qquad \lambda>0.$$

Hence for any $t>0$ we have

$$(e_+^A(\lambda;x,x) - s_0^+(x)\lambda^{n/\kappa}) + (e_-^A(\lambda/t;x,x) - s_0^-(x)(\lambda/t)^{n/\kappa}) = o(\lambda^{(n-\tau)/\kappa}) \quad \text{as } \lambda\to\infty,\ \forall\tau<\varepsilon/2.$$

It follows from this equality and the similar one with $t=1$ that

$$(e_\pm^A(\lambda;x,x) - s_0^\pm(x)\lambda^{n/\kappa}) - (e_\pm^A(t\lambda;x,x) - s_0^\pm(x)(t\lambda)^{n/\kappa}) = o(\lambda^{(n-\tau)/\kappa}) \quad \text{as } \lambda\to\infty,\ \forall t>0,\ \forall\tau<\varepsilon/2,$$

uniformly in $x\in M$, and our statement follows from Lemma 15.9 (see below). ∎

15.9. Lemma. *Let X be a normed space, $0<t<1$, $\beta\in\mathbb{R}$ and let functions $h:\mathbb{R}_+\to X$, $f:[1,\infty)\to\mathbb{R}_+$ satisfy the inequality*

$$\|h(\lambda)-h(t\lambda)\| \le \lambda^\beta f(\lambda), \qquad \forall\lambda\ge 1.$$

If $\beta>0$, $C_1 := \sup_{\lambda\in[1,\infty)} f(\lambda) < \infty$ and $C_2 := \sup_{\lambda\in[t,1)}\|h(\lambda)\|<\infty$, then

$$\|h(\lambda)\| \le \frac{\lambda^\beta}{1-t^\beta}\left(\sup_{\gamma(\lambda)\le\mu\le\lambda} f(\mu) + C_1\left(\frac{\gamma(\lambda)}{\lambda}\right)^\beta + C_2\lambda^{-\beta}\right), \qquad \forall\lambda\ge1,$$

for any $\gamma:[1,\infty)\to\mathbb{R}_+$ such that $1\le\gamma(\lambda)\le\lambda$, $\forall\lambda\ge1$. If $\beta<0$ and $\|h(\lambda)\|\to0$ as $\lambda\to\infty$, then

$$\|h(\lambda)\| \le \frac{\lambda^\beta}{t^\beta-1}\sup_{\mu\ge\lambda/t} f(\mu), \qquad \forall\lambda\ge1.$$

Proof. Suppose $\beta>0$. Take an arbitrary $\lambda\ge1$ and choose $k\in\mathbb{N}$ such that $t\le t^{k+1}\lambda<1$. We have

$$\begin{aligned}
\|h(\lambda)\| &\le \lambda^\beta f(\lambda) + \|h(t\lambda)\| \le \dots \le \lambda^\beta\Big(\sum_{j=0}^k t^{\beta j} f(t^j\lambda) + \lambda^{-\beta}\|h(t^{k+1}\lambda)\|\Big) \\
&\le \lambda^\beta\Big(\sum_{\gamma(\lambda)\le t^j\lambda\le\lambda} t^{\beta j} f(t^j\lambda) + \sum_{1\le t^j\lambda<\gamma(\lambda)} t^{\beta j} f(t^j\lambda) + C_2\lambda^{-\beta}\Big) \\
&\le \frac{\lambda^\beta}{1-t^\beta}\left(\sup_{\gamma(\lambda)\le\mu\le\lambda} f(\mu) + C_1\left(\frac{\gamma(\lambda)}{\lambda}\right)^\beta + C_2\lambda^{-\beta}\right), \qquad \forall\lambda\ge1.
\end{aligned}$$

Suppose now $\beta < 0$ and $\|h(\lambda)\| \to 0$ as $\lambda \to \infty$. Then

$$\|h(\lambda)\| \le \lambda^\beta t^{-\beta} f(t^{-1}\lambda) + \|h(t^{-1}\lambda)\| \le \ldots \le \lambda^\beta \sum_{j=1}^{k} t^{-\beta j} f(t^{-j}\lambda) + \|h(t^{-k}\lambda)\| \le \ldots$$

$$\le \lambda^\beta \sum_{j=1}^{\infty} t^{-\beta j} f(t^{-j}\lambda) \le \frac{\lambda^\beta}{t^\beta - 1} \sup_{\mu \ge \lambda/t} f(\mu), \qquad \forall \lambda \ge 1,$$

since $h(t^{-k}\lambda) \to 0$ as $k \to \infty$. ∎

15.10. Remark. The restriction (6.2) means that the commutator $[\partial_{\nu_j}, \partial_{\nu_k}]$ is an operator of a strictly lower order than $\partial_{\nu_j}\partial_{\nu_k}$ and $\partial_{\nu_k}\partial_{\nu_j}$. Therefore the main part of our calculus does not apply to the case treated in the works on Hörmander's sums of squares of vector fields, their generalizations and anisotropic operators on Lie groups with dilations (see the references in Sections 1 and 8), where the above operators are assumed to be of the same order. In this case $\varepsilon = 0$ (see the beginning of Section 11) and the results of the present section offer a partial explanation of the fact that no remainder estimates seem to be known for the spectral function and the distribution function of eigenvalues of Hörmander's sum of squares of vector fields (see [Me1] and [LMN], [MLN]) and its generalizations (see [Lo1], [Lo2]). It would be interesting to construct operators of this type for which one can find spectral asymptotics more or less explicitly, and to find out whether the remainder admits an estimate of the form $O(\text{“first term”} \times \lambda^{-\varepsilon_0})$, $\varepsilon_0 > 0$.

Acknowledgements. I would like to thank Yu. Safarov whose paper [Saf] inspired this work.

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