Introduction and an outline of results

The present paper concentrates on special problems from the so called "hypercomplex analysis". The starting point is the problem of generalizing the notion of complex structure, fundamental in complex analysis. We introduce a notion of Clifford-type structure, and as a consequence, the notion of Clifford-type manifold.

The holonomy groups of manifolds having an affine connection with zero torsion have been classified by M. Berger [1]. The possible restricted holonomy groups for irreducible Riemannian manifolds which are not symmetric spaces are the following:

$$\begin{aligned} &SO(n), \quad U(n), \quad SU(n), \quad Sp(n)\times Sp(1),\\ &Sp(n) \quad \text{for all } n\geq 2,\\ &\text{the special groups:} \quad G_2, \quad Spin(7), \quad Spin(9). \end{aligned}$$

Let us recall that the manifolds with holonomy groups in $SO(n)$ are the oriented Riemannian manifolds and only general results may be obtained about the topology of this large class.

Riemannian manifolds with holonomy groups in $U(n)$ are nothing but Kähler manifolds. These manifolds have been extensively studied for many years and by many authors including Chern, Hodge, Weil etc.

Manifolds whose holonomy groups form subgroups of $Sp(n)\times Sp(1)$ have also turned out to be very interesting. These manifolds are called "quaternionic manifolds". Many papers have been devoted to studying their properties by Berger, Bonan, Marchiafava, Martinelli, Salamon and others. One reason for giving quaternionic Kähler manifolds serious consideration is Wolf's observation [35] that for each compact simple Lie group G there exists a quaternionic-Kähler symmetric space G/H. This theory contrasts favourably with the more sporadic situation of Hermitian symmetric spaces, and the existence of a complex contact manifold fibring over G/H generalizes to the non-symmetric case. The presence of a closed, but highly non-generic, 4-form on a quaternionic Kähler manifold is responsible for both similarities to and differences from symplectic geometry.

Let us recall that the quaternionic unitary group $Sp(n)$ is not a maximal subgroup of $SO(4n)$, since it commutes with the action of the group $Sp(1)$ of unit quaternions. The group $Sp(n)\times Sp(1)$ is only a proper subgroup of $SO(4n)$ if $n>1$. One can consider $4n$-dimensional Riemannian manifolds with holonomy group contained in $Sp(n)$ or $Sp(n)\times Sp(1)$. These two cases are in fact quite different, more so than for example $SU(n)$ from $U(n)$. More precisely, $Sp(n)$ is included in $SU(2n)$, so Riemannian manifolds with holonomy contained in $Sp(n)$ are particular cases of Kähler manifolds with zero Ricci curvature. On the other hand, $Sp(n)\times Sp(1)$ is not a subgroup of $U(2n)$. We have

Theorem. *The subgroup $Sp(n) \times Sp(1)$ is a maximal subgroup of $SO(4n)$.*

According to the above remarks one defines:

Definition. A $4n$-dimensional Riemannian manifold is called

(a) *hyperkählerian* if its holonomy group is contained in $Sp(n)$,
(b) *quaternionic-Kähler* if its holonomy group is contained in $Sp(n) \times Sp(1)$.

(Let us add that this established terminology may be confusing, because a quaternionic-Kähler manifold may not be a Kähler manifold in the ordinary (i.e. complex) sense.)

The most remarkable results are the following:

Theorem ([1]). *A hyperkählerian manifold is Ricci-flat.*

Theorem ([2]). *If $n \geq 2$, a quaternionic-Kähler manifold is Einstein.*

Theorem ([25]). *All quaternionic-Kähler manifolds whose dimension is a multiple of 8 are spin manifolds.*

There are analogies between the theory of (complex) Kähler manifolds and that of quaternionic-Kähler manifolds. We recall that a (complex) Kähler manifold (M, g, J) may be viewed as a Riemannian manifold (M, g) together with an almost complex structure J such that (g, J) is almost Hermitian and J is parallel for the Levi-Civita connection ∇ of g. It turns out (since ∇ is torsion-free) that $\nabla J = 0$ implies that J is in fact a complex structure.

Quaternionic analogs of the notions of almost complex structure and complex structure have been looked for, but there is more than one possible choice here. There is no good notion of "holomorphic functions" in the quaternionic case. Nevertheless, there exists a "quaternionic holomorphic calculus", developed by R. Fueter [10, 11] and later by A. Sudbery [32], but it does not fit all expected purposes.

It is well known that in complex analysis there are many conditions equivalent to holomorphy. It was discovered by several authors that the transmission of those conditions to the quaternions gives in each case different classes of functions (see e.g. [31]). The most promising attempt was the definition of "quaternionic holomorphy", proposed in 1935 by Fueter [10], which generalized the Cauchy–Riemann equations. Henceforth these mappings appeared in the literature as "regular mappings in the sense of Fueter". They have many properties "analogous" to those of holomorphic mappings although the proofs are difficult from the technical point of view because of, among other things, the non-commutativity of quaternions. In many cases the analog does not exist, for example, there are no simple functions which are regular in the sense of Fueter. In 1979 A. Sudbery [32] collected, classified and proved the most fundamental properties of regular mappings. Anyway, up to now, there does not exist a "quaternionic analysis" in the same sense as the complex analysis; nonetheless in the 1970's K. Imaeda [14] presented an exceptionally beautiful, simple and convincing application of quaternions to electromagnetism.

Nevertheless, one can find some analogs. Thus, if (M, g) is a Riemannian manifold then one defines:

DEFINITION.

(a) An *almost quaternionic structure* is defined as a covering $\{U_i\}$ of M with two almost complex structures I_i and J_i on each U_i such that $I_i J_i = J_i I_i$ and the 3-dimensional vector space of endomorphisms generated by I_i, J_i and $K_i := I_i J_i$ is the same on all of M.

(b) A Riemannian metric g is *quaternionic-Hermitian* if g is Hermitian for each of I_i and J_i above.

In analogy to the complex case one introduces

DEFINITION. A *quaternionic manifold* is an almost quaternionic manifold admitting a torsion-free $Gl(n,\mathbb{H}) \times Sp(1)$-connection.

The subgroup $Gl(n,\mathbb{H}) \times Sp(1)$ of $Gl(4n,\mathbb{R})$ is defined as follows. We identify $\mathbb{R}^{4n}$ with the (right) vector space $\mathbb{H}^n$ over $\mathbb{H}$; then $Gl(n,\mathbb{H})$ is the linear group (acting on the left) of the vector space $\mathbb{H}^n$ (over $\mathbb{H}$), and $Sp(1)$ is the group of unit quaternions acting on $\mathbb{H}^n$ by scalar multiplication on the right.

We have

THEOREM. *A quaternionic-Kähler manifold is quaternionic-Hermitian and quaternionic.*

Let us recall that an almost complex structure with a complex torsion-free connection is integrable, and hence a complex structure (the Newlander–Nirenberg theorem; see e.g. [17]). This is not true for quaternionic manifolds and the analogy breaks down here definitely. Thus, let us again emphasize that the quaternionic theory is quite different from the complex one.

As a generalization of the notion of quaternionic manifold (and complex manifold as well) we introduce a quite general object called a *Clifford-type manifold.*

In the first chapter Clifford-type manifolds are investigated. Among other things an analog of the fundamental 2-form of complex analysis is defined and using it a decomposition analogous to the Hodge decomposition for Kähler manifolds is given for Clifford-type manifolds. By the Chern theorem [8] we get an increasing sequence of Betti numbers for Clifford-type manifolds.

In the second chapter, using the Clifford-type structure, we define Clifford-type holomorphy and prove two features of Clifford-type holomorphic mappings without counterparts in complex analysis.

The third chapter is devoted to generalizing the following, well known theorem of complex analysis to the quaternionic and octonionic projective spaces:

THEOREM ([17, pp. 167, 368]). *The sectional curvature $K(\sigma)$ of a Kähler manifold (M^n, g, J, Ω) of constant holomorphic sectional curvature* 1 *is given by*

$$\tfrac{1}{4}[1 + 3\cos^2\alpha(\sigma)],$$

where σ is a plane tangent to M^n, i.e. a real 2-dimensional subspace of T_xM^n, $x \in M^n$ and $\cos\alpha(\sigma) := |g(X, JY)|$, where (X, Y) is an orthonormal basis in σ.

Moreover, as an application we obtain quaternionic and octonionic counterparts of the following Klingenberg theorem [15]:

Theorem. *Let M^{2n} be a Kähler manifold of real dimension $2n \geq 4$. Assume that for all 2-planes σ tangent to M^{2n}, the sectional curvature $K(\sigma)$ satisfies the inequality*

$$9/16 < K(\sigma)/K_1[\alpha(\sigma)] \leq 1,$$

where

$$K_1[\alpha(\sigma)] := \tfrac{1}{4}[1 + 3\cos^2\alpha(\sigma)].$$

Then M is compact and has the homotopy type of the complex projective n-space $\mathbb{CP}^n$.

Next, in the fourth chapter, we prove the most remarkable result of the paper. Namely, using the fundamental notions of quaternionic analysis we show that there are no 4-dimensional almost Kähler manifolds which are locally conformally flat with a metric of a special form. Precisely we prove

Theorem. *A 4-dimensional almost Kähler manifold (M^4, g, J, Ω) does not admit any locally conformally flat Riemannian metric g of the form*

$$g := g_0(p)[dw^2 + dx^2 + dy^2 + dz^2], \qquad p \in U \subset M^4,$$
$$g_0(w, x, y, z) := g_0(r), \qquad r^2 := w^2 + x^2 + y^2 + z^2,$$

where $g_0(r)$ is a real, positive, analytic, non-constant function in r ($(U; w, x, y, z)$ is an arbitrary system of local coordinates on M^4).

This result is significant at least because the standard model of 4-dimensional hyperbolic space is the Poincaré model, i.e. the unit ball in $\mathbb{R}^4$ equipped with the metric

$$g = \frac{4}{(1-r^2)^2}(dw^2 + dx^2 + dy^2 + dz^2).$$

Finally, in the last fifth chapter we present some results on quaternionic Lagrangian submanifolds. For instance, we give a necessary and sufficient condition for a graph to be quaternionic Lagrangian. We also present explicit forms of some characteristic differential equations naturally connected with some types of quaternionic Lagrangian submanifolds. Moreover, we show that two immersions which lead to an almost quaternionic and a Lagrangian submanifold, respectively, cannot be homotopic.

The paper ends with a correction to my previous paper *On Fueter–Hurwitz regular mappings* (Dissertationes Math. 353, 1996). It has turned out that Theorem 2.6.2 on page 58 there is false. A suitable counterexample has been found by J. Kijowski.

I. Clifford-type structures*

I.1. Fundamental notions and basic properties. Let V be a real vector space.

Definition 1.1. An *almost Clifford-type structure* $\mathcal{C}_n$ on V is a set of n almost complex structures $\{I_1, \ldots, I_n\}$ such that

$$I_\alpha I_\beta + I_\beta I_\alpha = -2\delta_{\alpha\beta}\,\mathrm{Id}, \qquad \alpha, \beta = 1, \ldots, n,$$

* *Editors' note:* This chapter also appeared in [19].

where Id stands for the identity endomorphism of V, and δ is the Kronecker delta; moreover we assume that no I_j is a product of two, four etc. others from the set $\{I_1, \ldots, I_n\}$.

REMARK 1.1. (a) If $n = 1$, then $\mathcal{C}_1 = \{I\}$ with $I^2 = -\mathrm{Id}$. Thus, $\mathcal{C}_1$ is nothing but an almost complex structure. Recall that the standard form of an almost complex structure is

$$I_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \qquad (I = \mathrm{Id}),$$

where V has an even dimension (see e.g. [17]).

(b) If $n = 2$, then $\mathcal{C}_2 = \{I, J\}$ with $I^2 = J^2 = -\mathrm{Id}$ and $IJ + JI = 0$. Define $K := IJ$; then $IJK = -\mathrm{Id}$ and $K^2 = -\mathrm{Id}$. Thus, $\mathcal{C}_2$ corresponds to the almost quaternionic structure (see e.g. [3, 18, 19, 27, 30]). The standard form of an almost quaternionic structure is

$$I_0 = \begin{pmatrix} 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \end{pmatrix}, \qquad J_0 = \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \end{pmatrix},$$

$$K_0 = \begin{pmatrix} 0 & 0 & 0 & I \\ 0 & 0 & -I & 0 \\ 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \end{pmatrix},$$

where $\dim_{\mathbb{R}} V = 4n$.

Note that any almost Clifford-type structure $\mathcal{C}_n = \{I_1, \ldots, I_n\}$ induces the following set of almost complex structures:

- $I_1, \ldots, I_n$;
- $I_1I_2, I_1I_3, \ldots, I_1I_n, \ldots, I_{n-1}I_n$;
- no triple is an almost complex structure;
- $I_1I_2I_3I_4, \ldots, I_{n-3}I_{n-2}I_{n-1}I_n$;
- no odd tuple is an almost complex structure; etc.

Denote by p'_n the number of the above almost complex structures. Then

$$p'_n = \begin{cases} \binom{n}{1} + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{n-1} & \text{if } n \text{ is odd}, \\ \binom{n}{1} + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{n} & \text{if } n \text{ is even}. \end{cases}$$

Set

$$p''_n := 2^n - p'_n.$$

Then for $n \geq 3$ we have the following general formulae:

$$p'_n = 2^n - 2p''_n - (n-3), \qquad p''_n = 2p''_{n-1} + (n-3).$$

By straightforward calculations we get

$$p'_1 = 1, \quad p'_2 = 3, \quad p'_3 = 6, \quad p'_4 = 11, \quad p'_5 = 20, \quad p'_6 = 37, \quad p'_7 = 70, \quad p'_8 = 135 \quad \text{etc.}$$

Denote by $V(n)$ a real vector space endowed with a Clifford-type structure $\mathcal{C}_n = \{I_1, \dots, I_n\}$.

THEOREM 1.1. *We have* $\dim_{\mathbb{R}} V(n) = 2^n s$, *where* $s > 0$ *is an integer.*

Proof. Assume that a real vector space V is equipped with an almost Clifford-type structure $\mathcal{C}_n = \{I_1, \dots, I_n\}$. Since

$$\mathcal{G} := \{a^1 I_1 + \dots + a^n I_n;\ a^1, \dots, a^n \in \mathbb{R} \text{ and } (a^1)^2 + \dots + (a^n)^2 = 1\}$$

is a compact group, V can be split into a direct sum of irreducible vector subspaces (see e.g. [8, p. 14]) and thus it suffices to prove the theorem for V irreducible.

Let $X \in V$, $X \neq 0$. Consider the vector subspace V_1 of V generated by $X, I_1X, I_2X, \dots, I_{n-1}X$. Then I_nV_1 cannot belong to V_1. Indeed, otherwise there would exist a matrix

$$A = \begin{pmatrix} a_0^0 & \dots & a_0^{n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1}^0 & \dots & a_{n-1}^{n-1} \end{pmatrix} \in \mathcal{M}(n; \mathbb{R})$$

($\mathcal{M}(n;\mathbb{R})$ denotes the set of $n \times n$ matrices with real coefficients) with $\det A \neq 0$ such that

$$\begin{aligned} I_nX &= a_0^0X + a_0^1I_1X + \dots + a_0^{n-1}I_{n-1}X, \\ I_n(I_1X) &= a_1^0X + a_1^1I_1X + \dots + a_1^{n-1}I_{n-1}X, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ I_n(I_{n-1}X) &= a_{n-1}^0X + a_{n-1}^1I_1X + \dots + a_{n-1}^{n-1}I_{n-1}X. \end{aligned}$$

Since $I_n(I_nX) = -X$, we get

$$\begin{aligned} I_n(I_nX) = {} & I_n(a_0^0X + a_0^1I_1X + \dots + a_0^{n-1}I_{n-1}X) \\ = {} & a_0^0I_nX + a_0^1I_n(I_1X) + \dots + a_0^{n-1}I_n(I_{n-1}X) \\ = {} & a_0^0(a_0^0X + a_0^1I_1X + \dots + a_0^{n-1}I_{n-1}X) \\ & + a_0^1(a_1^0X + a_1^1I_1X + \dots + a_1^{n-1}I_{n-1}X) \\ & + \dots\dots\dots\dots\dots\dots\dots\dots \\ & + a_0^{n-1}(a_{n-1}^0X + a_{n-1}^1I_1X + \dots + a_{n-1}^{n-1}I_{n-1}X), \end{aligned}$$

so

$$\begin{aligned} & [(a_0^0)^2 + a_0^1a_1^0 + \dots + a_0^{n-1}a_{n-1}^0]X \\ & + [a_0^0a_0^1 + a_0^1a_1^1 + \dots + a_0^{n-1}a_{n-1}^1]I_1X \\ & + \dots\dots\dots\dots\dots\dots\dots\dots \\ & + [a_0^0a_0^{n-1} + a_0^1a_1^{n-1} + \dots + a_0^{n-1}a_{n-1}^{n-1}]I_{n-1}X = -X. \end{aligned}$$

This implies

$$
(1.1)\qquad
\begin{aligned}
(a_0^0)^2 + a_0^1 a_1^0 + \cdots + a_0^{n-1} a_{n-1}^0 &= -1,\\
a_0^0 a_0^1 + a_0^1 a_1^1 + \cdots + a_0^{n-1} a_{n-1}^1 &= 0,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\\
a_0^0 a_0^{n-1} + a_0^1 a_1^{n-1} + \cdots + a_0^{n-1} a_{n-1}^{n-1} &= 0.
\end{aligned}
$$

On the other hand, we have

$$0 = I_n(I_\alpha X) + I_\alpha(I_n X) \quad \text{for } \alpha = 1, \ldots, n-1,$$

which gives

$$a_\alpha^0 X + a_\alpha^1 I_1 X + \cdots + a_\alpha^{n-1} I_{n-1} X + I_\alpha[a_0^0 X + a_0^1 I_1 X + \cdots + a_0^{n-1} I_{n-1} X] = 0,$$

i.e.

$$a_\alpha^0 X + a_\alpha^1 I_1 X + \cdots + a_\alpha^{n-1} I_{n-1} X + a_0^0 I_\alpha X + a_0^1 I_\alpha I_1 X + \cdots + a_0^{n-1} I_\alpha I_{n-1} X = 0.$$

For $\alpha = 1$ we get

$$
\begin{aligned}
a_1^0 - a_0^1 = 0, &\quad \text{i.e.} \quad a_1^0 = a_0^1,\\
a_1^1 + a_0^0 = 0, &\quad \text{i.e.} \quad a_0^0 = -a_1^1,\\
a_1^2 = a_1^3 = \cdots &= a_1^{n-1} = 0,\\
a_0^2 = a_0^3 = \cdots &= a_0^{n-1} = 0.
\end{aligned}
$$

For $\alpha = 2$ we get

$$
\begin{aligned}
a_2^0 &= a_0^2,\\
a_2^2 &= -a_0^0,\\
a_2^1 = a_2^3 = \cdots &= a_2^{n-1} = 0,\\
a_0^1 = a_0^3 = \cdots &= a_0^{n-1} = 0.
\end{aligned}
$$

Generally, for $\alpha = m$ we have

$$
\begin{aligned}
&a_m^0 = a_0^m, \qquad m = 1, \ldots, n-1,\\
&a_0^0 = -a_1^1 = -a_2^2 = \cdots = -a_{n-1}^{n-1},\\
&a_0^m = 0, \quad m = 1, \ldots, n-1,\\
&a_1^2 = a_1^3 = \cdots = a_1^{n-1} = 0,\\
&a_2^1 = a_2^3 = \cdots = a_2^{n-1} = 0,\\
&a_3^1 = a_3^2 = a_3^4 = \cdots = a_3^{n-1} = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&a_{n-1}^1 = a_{n-1}^2 = \cdots = a_{n-1}^{n-2} = 0.
\end{aligned}
$$

Substituting the above relations to the system (1.1) we get $(a_0^0)^2 = -1$, which is a contradiction.

Consider the subspace V_0 of V generated by V_1 and $I_n V_1$. Since V_0 is invariant under the whole group $\mathcal{G}$, we must have $V_0 = V$ since V is irreducible. Thus

$$V = V_1 \oplus I_n V_1.$$

Then $\dim V = \dim V_1 + \dim(I_n V_1)$, so $\dim V = 2\dim V_1$, i.e. $\dim V(n) = 2\dim V(n-1)$. Since $\dim V(1) = 2s$ for some integer $s > 0$, we conclude that $\dim V(n) = 2^n s$. ■

I.2. Clifford-type numbers. In order to consider at the same time quaternions and octonions (even complex numbers) we introduce "Clifford-type numbers".

Assume that $n = 1, 3, 7$.

DEFINITION 2.1. Denote by $\mathcal{A}_n$ the field of *Clifford-type numbers*. A typical element a of $\mathcal{A}_n$ can be written as

$$a := x_0 + e_1x_1 + e_2x_2 + \cdots + e_nx_n, \quad x_0, x_1, \ldots, x_n \in \mathbb{R},\ n = 1, 3, 7,$$

and we assume that the *Clifford-type units* $e_1, \ldots, e_n$ satisfy

$$e_ke_m + e_me_k = -2\delta_{km}, \quad k, m = 1, \ldots, n,$$

that is,

$$e_k^2 = -1,\ k = 1, \ldots, n; \quad e_ke_m = -e_me_k, \quad k \neq m,\ k, m = 1, \ldots, n,$$

and

$$\{e_ke_m\} = \{\pm e_1, \ldots, \pm e_n\},\ \overline{e}_k = -e_k,\ \overline{e_ke_m} = -e_ke_m,\ k \neq m,\ k, m = 1, \ldots, n.$$

We also have

$$\overline{ab} = \overline{b}\,\overline{a}, \quad a, b \in \mathcal{A}_n.$$

EXAMPLE 2.1.

1. $\mathbb{C}$ — complex numbers with the complex unit $e_1 = i$ satisfying $i^2 = -1$ and $\overline{i} = -i$. (Complex multiplication is commutative and associative.)

2. $\mathbb{H}$ — quaternions with the quaternionic units $e_1 = i$, $e_2 = j$, $e_3 = k$ satisfying

$$\begin{aligned} &i^2 = j^2 = k^2 = ijk = -1, \quad ij = k,\ jk = i,\ ki = j, \\ &\overline{i} = -i,\ \overline{j} = -j,\ \overline{k} = -k. \end{aligned}$$

(Quaternionic multiplication is not commutative but it is associative.)

3. $\mathbb{O}$ — octonions with the octonionic units $e_1, \ldots, e_7$ satisfying

$$e_k^2 = -1, \quad k = 1, \ldots, 7, \quad e_ke_m + e_me_k = 0, \quad k \neq m,\ k, m = 1, \ldots, 7,$$

$$\begin{aligned} &e_1e_2 = e_3, \quad e_2e_3 = e_1, \quad e_3e_1 = e_2, \quad e_1e_4 = e_5, \\ &e_2e_4 = e_6, \quad e_3e_4 = e_7, \quad e_5e_1 = e_4 \quad \text{etc.}, \\ &e_1e_3 = -e_2, \\ &e_1e_5 = -e_4, \quad e_1e_6 = -e_7, \quad e_1e_7 = e_6 \quad \text{etc.} \end{aligned}$$

and

$$\overline{e}_k = -e_k.$$

(Octonionic multiplication is not commutative and not associative.)

Let $\mathcal{A}_n^p \equiv \mathbb{R}^{(n+1)p}$ denote the "Clifford-type" Euclidean p-space with coordinates $A := (a^1, \ldots, a^p)$, where

$$a^s := x_0^s + e_1x_1^s + e_2x_2^s + \cdots + e_nx_n^s, \quad s = 1, \ldots, p,$$

and $A := X_0 + e_1X_1 + \cdots + e_nX_n$ with $X_i = (x_i^1, x_i^2, \ldots, x_i^p)$, $i = 0, 1, \ldots, n$. Thus $\mathbb{R}^p$ denotes the subset of $\mathcal{A}_n^p$ with $X_1 = \cdots = X_n = 0$.

Define the right multiplication $\bullet : \mathcal{A}_n^p \times \mathcal{A}_n \to \mathcal{A}_n^p$ by

$$A \bullet a := (a^1 \cdot a, \ldots, a^p \cdot a), \qquad A \in \mathcal{A}_n^p,\ a \in \mathcal{A}_n.$$

We emphasize that $\mathcal{A}_n^p$ can be identified with $\mathbb{R}^{(n+1)p}$ endowed with n almost complex structures $I_1, \ldots, I_n$ satisfying

$$I_\alpha I_\beta + I_\beta I_\alpha = -2\delta_{\alpha\beta}\,\mathrm{Id},$$
$$I_1X := e_1X,\ \ldots,\ I_nX := e_nX, \qquad X \in \mathbb{R}^{(n+1)p},$$

where Id stands for the identity mapping in $\mathbb{R}^{(n+1)p}$. Thus we can treat $\mathcal{A}_n^p \equiv \mathbb{R}^{(n+1)p}$ as a p-dimensional right module over $\mathcal{A}_n$. One defines a bilinear form $\langle\,,\rangle$ on $\mathcal{A}_n^p$ as follows: if $A = (a^1, \ldots, a^p)$, $B = (b^1, \ldots, b^p) \in \mathcal{A}_n^p$, then

$$\langle A, B\rangle := \frac{1}{2}\sum_{\alpha=1}^{p}(a^\alpha\overline{b^\alpha} + b^\alpha\overline{a^\alpha}) = \mathrm{Re}(A, B) := \mathrm{Re}\sum_{\alpha=1}^{p} a^\alpha\overline{b^\alpha},$$

where

$$\overline{a^s} := x_0^s - e_1x_1^s - e_2x_2^s - \cdots - e_nx_n^s, \qquad s = 1, \ldots, p.$$

Then $\langle A, B\rangle$ is an inner product in $\mathcal{A}_n^p$ considered as an $(n+1)\cdot p$-dimensional real vector space. Note that

$$\langle A, B\rangle = \tfrac{1}{2}[(A, B) + (B, A)].$$

Denote by $SP_{n+1}(p)$ the set of all endomorphisms of $\mathcal{A}_n^p$ which preserve the Clifford-type symplectic product $(\ ,\ )$. (In the quaternionic case $(n = 3)$ $SP_4(p)$ is nothing else than the well known group usually denoted by $Sp(p)$.)

The norm of $A \in \mathcal{A}_n^p$ is defined as usual by

$$\|A\|^2 := (A, A) = \sum_{\beta=1}^{p} a^\beta\overline{a^\beta}$$

and can be used to express the inverse element of $A \neq 0$:

$$A^{-1} := \frac{1}{\|A\|^2}\,\bar{A}.$$

Note that $SP_{n+1}(1) = \{a \in \mathcal{A}_n : \|a\| = 1\}$ is a group, and $SP_{n+1}(p) \subseteq SO[(n+1)p]$.

Lemma 2.1. *$\langle A, B\rangle$ is invariant under the action of $SP_{n+1}(p)$.*

Proof. $SP_{n+1}(p)$ is defined as the set of all endomorphisms of $\mathcal{A}_n^p$ which preserve the "symplectic product" (A, B). Our inner product is $\langle A, B\rangle = \frac{1}{2}[(A, B) + (B, A)]$. Hence it is clearly invariant. ■

I.3. The fundamental form Ω. Let V be a real vector space equipped with an almost Clifford-type structure $\mathcal{C}_n = \{I_1, \ldots, I_n\}$. Let $(\mathcal{A}_n^p)'$ be the dual space of $\mathcal{A}_n^p$ over $\mathcal{A}_n$ and $\alpha_1, \ldots, \alpha_p$ be a basis of $(\mathcal{A}_n^p)'$. We may write

$$\alpha_s := b_s^0\alpha_s^0 + b_s^1\alpha_s^1 + \cdots + b_s^n\alpha_s^n, \qquad b_s^0, b_s^1, \ldots, b_s^n \in \mathbb{R}, \quad s = 1, \ldots, p,$$

so that $\alpha_s^0, \alpha_s^1, \ldots, \alpha_s^n$ forms a basis of $(\mathcal{A}_n^p)'$ over $\mathbb{R}$.

Definition 3.1. Define n skew symmetric bilinear forms $\omega_1, \ldots, \omega_n$ on $\mathcal{A}_n^p$ as follows:

$$\omega_i(A,B) = \langle A, I_i B\rangle, \qquad i = 1, \ldots, n. \tag{3.1}$$

Assume that

$$n + 1 = 2^w$$

for some integer $w > 0$.

Definition 3.2. We define a 2^w-form Ω on $\mathcal{A}_n^p$ by

$$\Omega := \underbrace{\omega_1 \wedge \cdots \wedge \omega_1}_{w \text{ times}} + \cdots + \underbrace{\omega_n \wedge \cdots \wedge \omega_n}_{w \text{ times}}.$$

Definition 3.3. Define an action of the group $SP_{n+1}(p) \times SP_{n+1}(1)$ on $\mathcal{A}_n^p$ as follows: let $A \in \mathcal{A}_n^p$ and $(\Lambda, \lambda) \in SP_{n+1}(p) \times SP_{n+1}(1)$; then

$$(\Lambda, \lambda)A := \Lambda A \lambda,$$

i.e. apply Λ to A and multiply on the right by λ.

Definition 3.4. Let $\lambda \in SP_{n+1}(1)$ (i.e. $\lambda \in \mathcal{A}_n$ and $\|\lambda\| = 1$). Define a map λ^* on the bilinear forms $\omega_1, \ldots, \omega_n$ (defined by (3.1)) as follows:

$$\lambda^*\omega_i(A,B) := \omega_i(A\lambda, B\lambda), \qquad i = 1, \ldots, n.$$

Theorem 3.1. *The form Ω is invariant under the action of $SP_{n+1}(p) \times SP_{n+1}(1)$.*

Proof. By Lemma 2.1, Ω is invariant under the action of $SP_{n+1}(p)$ on the left. Now, let $\lambda \in \mathcal{A}_n$ with $\|\lambda\| = 1$, i.e. λ represents an element of $SP_{n+1}(1)$. Then

$$\lambda^*\Omega := \underbrace{\lambda^*\omega_1 \wedge \cdots \wedge \lambda^*\omega_1}_{w \text{ times}} + \cdots + \underbrace{\lambda^*\omega_n \wedge \cdots \wedge \lambda^*\omega_n}_{w \text{ times}}.$$

According to Definitions 3.3 and 3.4 and by a simple calculation we have

$$\lambda^*\omega_i(A,B) = \omega_i(A\lambda, B\lambda) = \langle A\lambda, I_i B\lambda\rangle = \|\lambda\|^2\langle A, I_i B\rangle = \|\lambda\|^2\omega_i(A,B)$$

for $i = 1, \ldots, n$. Thus $\lambda^*\Omega = \Omega$, hence Ω is invariant under the action of $SP_{n+1}(1)$ on the right. ∎

I.4. Splitting of forms. One can extend the definition of the "star" operator $*$ and the operators L and Λ, known from classical differential geometry, to the Clifford-type case.

Let $\bigwedge(\mathcal{A}_n^p)'$ be the exterior algebra over $\mathbb{R}$ obtained by considering $(\mathcal{A}_n^p)'$ as a real $2^w p$-dimensional vector space. Every element of $\bigwedge(\mathcal{A}_n^p)'$ is a linear combination of simple r-forms $\omega := \beta_1 \wedge \cdots \wedge \beta_r$, where β_i is one of $\alpha_s^0, \alpha_s^1, \ldots, \alpha_s^n$, $s = 1, \ldots, p$.

Definition 4.1. Define $*$, L and Λ on $(\mathcal{A}_n^p)'$ as follows: if ω is a simple r-form then $*\omega$ is the simple $[(n+1)p - r] = (2^w p - r)$-form such that

$$\omega \wedge *\omega = \alpha_1^0 \wedge \alpha_1^1 \wedge \cdots \wedge \alpha_1^n \wedge \cdots \wedge \alpha_p^0 \wedge \alpha_p^1 \wedge \cdots \wedge \alpha_p^n.$$

Next we extend $*$ by linearity to $(\mathcal{A}_n^p)'$. For an arbitrary exterior form ω we define

$$L\omega := \Omega \wedge \omega, \qquad \Lambda\omega := *(\Omega \wedge *\omega).$$

Remark 4.1.

1. For all $\omega \in (\mathcal{A}_n^p)'$ we have $**\omega = \omega$.

2. $L : \bigwedge^r (\mathcal{A}_n^p)' \to \bigwedge^{r+(n+1)} (\mathcal{A}_n^p)' \equiv \bigwedge^{r+2^w} (\mathcal{A}_n^p)'$.
3. $\Lambda : \bigwedge^r (\mathcal{A}_n^p)' \to \bigwedge^{r-(n+1)} (\mathcal{A}_n^p)' \equiv \bigwedge^{r-2^w} (\mathcal{A}_n^p)'$.

DEFINITION 4.2. Define a bilinear form $(\, , \,)$ on $\bigwedge^r (\mathcal{A}_n^p)'$ by

$$(\omega, \omega') := *(\omega \wedge *\omega') \quad \text{ for } \omega, \omega' \in \bigwedge^r (\mathcal{A}_n^p)'.$$

LEMMA 4.1. *We have* $(L\omega, \omega') = (\omega, \Lambda\omega')$ *for* $\omega \in \bigwedge^r (\mathcal{A}_n^p)'$ *and* $\omega' \in \bigwedge^{r+(n+1)} (\mathcal{A}_n^p)' \equiv \bigwedge^{r+2^w} (\mathcal{A}_n^p)'$.

Proof. This follows by straightforward calculations. ∎

LEMMA 4.2. *The mapping*

$$L : \bigwedge^r (\mathcal{A}_n^p)' \to \bigwedge^{r+(n+1)} (\mathcal{A}_n^p)' \equiv \bigwedge^{r+2^w} (\mathcal{A}_n^p)'$$

is a monomorphism for $r + (n+1) \le p + 1$ $(r + 2^w \le p + 1)$.

Proof. We have to prove that for $\omega \in \bigwedge^r (\mathcal{A}_n^p)'$, $r + (n+1) \le p+1$, the relation $L\omega = \Omega \wedge \omega = 0$ implies $\omega = 0$. Assume that $\omega \neq 0$ and write

$$\omega = \sum_{A_0, A_1, \ldots, A_n} \gamma_{A_0, A_1, \ldots, A_n} \alpha^0_{A_0} \wedge \alpha^1_{A_1} \wedge \cdots \wedge \alpha^n_{A_n},$$

where $A_0, A_1, \ldots, A_n \subseteq \{1, \ldots, p\}$ and if $A_i = \{i_1, \ldots, i_s\}$ then $\alpha^i_{A_i} := \alpha^i_{i_1} \wedge \cdots \wedge \alpha^i_{i_s}$.

In the summation above, consider the terms with the highest total degree, say t, in α^0's and α^1's. Let ω' be the sum of these terms:

$$\omega' = \sum \gamma_{A_0, A_1, \ldots, A_n} \alpha^0_{A_0} \wedge \alpha^1_{A_1} \wedge \cdots \wedge \alpha^n_{A_n} \neq 0,$$

where the summation is taken over the indices $A_0, A_1, \ldots, A_n$ such that $|A_0| + |A_1| = t$ ($|A_0|, |A_1|$ denote the cardinalities of A_0 and A_1, respectively).

Similarly, we express $L\omega = \Omega \wedge \omega$ in terms of $\alpha^0_{A_0}, \alpha^1_{A_1}, \ldots, \alpha^n_{A_n}$ and consider the terms with the highest total degree in α^0's and α^1's. From the expressions for $\beta_1, \ldots, \beta_n$, it follows that the sum of these terms is given by

$$\sum_{\delta, \kappa = 1}^{p} \alpha^0_\delta \wedge \alpha^1_\delta \wedge \alpha^0_\kappa \wedge \alpha^1_\kappa \wedge \omega'.$$

The equation $L\omega = 0$ implies that this sum is zero, which means that

$$\sum_{A_2, \ldots, A_n} \Big(\sum_{\delta, \kappa, A_0, A_1} \gamma_{A_0, A_1, \ldots, A_n} \alpha^0_\delta \wedge \alpha^1_\delta \wedge \alpha^0_\kappa \wedge \alpha^1_\kappa \wedge \alpha^0_{A_0} \wedge \alpha^1_{A_1} \Big) \wedge \alpha^2_{A_2} \wedge \cdots \wedge \alpha^n_{A_n} = 0.$$

This implies that

$$\Big(\sum_{\delta=1}^{p} \alpha^0_\delta \wedge \alpha^1_\delta \Big) \wedge \Big(\sum_{\kappa=1}^{p} \alpha^0_\kappa \wedge \alpha^1_\kappa \Big) \wedge \Big(\sum_{A_0, A_1} \gamma_{A_0, \ldots, A_n} \alpha^0_{A_0} \wedge \alpha^1_{A_1} \Big) = 0$$

for each fixed $A_2, \ldots, A_n$, or

$$(\Omega')^2 \wedge \omega'' = 0,$$

where

$$\Omega' := \sum_{\delta=1}^{p} \alpha^0_\delta \wedge \alpha^1_\delta \quad \text{and} \quad \omega'' := \sum_{A_0, A_1} \gamma_{A_0, \ldots, A_n} \alpha^0_{A_0} \wedge \alpha^1_{A_1} \neq 0.$$

Let us consider the p-dimensional complex vector space with the coordinate system

$$\alpha_1^0 + i\alpha_1^1,\ \ldots,\ \alpha_p^0 + i\alpha_p^1.$$

Then Ω' is the fundamental 2-form. By the Hodge decomposition theorem [13] (since $\deg \omega'' \le p-3$), the equality

$$\Omega' \wedge (\Omega' \wedge \omega'') = 0$$

implies that $\Omega' \wedge \omega'' = 0$, which in turn implies that $\omega'' = 0$, which is a contradiction. ∎

DEFINITION 4.3. An r-form ω is said to be *effective* if $\Lambda\omega = 0$. We denote by $\bigwedge_{\mathrm{ef}}^r \subset \bigwedge^r(\mathcal{A}_n^p)'$ the set of all effective r-forms.

THEOREM 4.1. *For $r \le p+1$ and $z := [r/(n+1)]$ $(=[r/2^w])$, we have the following direct sum decomposition*:

$$\bigwedge^r(\mathcal{A}_n^p)' = \bigwedge_{\mathrm{ef}}^r \oplus L\bigwedge_{\mathrm{ef}}^{r-(n+1)} \oplus \cdots \oplus L^z\bigwedge_{\mathrm{ef}}^{r-(n+1)\cdot z}.$$

Proof. By Lemma 4.2, L is a monomorphism. Moreover, by Lemma 4.1, Λ is the adjoint of L and it is therefore onto for $r \le p+1$.

We will prove the theorem by induction on r.

The statement is true for $r = 0, 1, 2, \ldots, [(n+1)-1] = 2^w - 1$, since Λ lowers degree by $n+1 = 2^w$ and hence $\Lambda^r = \Lambda_{\mathrm{ef}}^r$ for these r's.

Assume that the assertion is true for $m < r$. We shall prove it for $m = r$. We claim that $\bigwedge_{\mathrm{ef}}^r$ is the orthogonal complement of $L\bigwedge^{r-(n+1)}(\mathcal{A}_n^p)'$ in $\bigwedge^r(\mathcal{A}_n^p)'$.

It is easy to show the orthogonality. Let $\omega \in \bigwedge_{\mathrm{ef}}^r$ and $L\omega' \in L\bigwedge^{r-(n+1)}(\mathcal{A}_n^p)'$. Then

$$(\omega, L\omega') = (\Lambda\omega, \omega') = (0, \omega') = 0.$$

To prove that $\bigwedge_{\mathrm{ef}}^r$ is the orthogonal complement of $L\bigwedge^{r-(n+1)}(\mathcal{A}_n^p)'$ take $\omega \in \bigwedge^r(\mathcal{A}_n^p)'$ such that $(\omega, L\omega') = 0$ for all $\omega' \in \bigwedge^{r-(n+1)}(\mathcal{A}_n^p)'$. Then $(\Lambda\omega, \omega') = 0$ and hence $\Lambda\omega = 0$ because $(\ ,\ )$ is a nondegenerate bilinear form.

Thus, by the induction hypothesis we have

$$\bigwedge^r(\mathcal{A}_n^p)' = \bigwedge_{\mathrm{ef}}^r \oplus L\bigwedge^{r-(n+1)}(\mathcal{A}_n^p)' = \bigwedge_{\mathrm{ef}}^r \oplus L\bigwedge_{\mathrm{ef}}^{r-(n+1)} \oplus \cdots \oplus L^z\bigwedge_{\mathrm{ef}}^{r-2^w\cdot z}. \ ∎$$

THEOREM 4.2. *We have $\Omega^p \ne 0$.*

Proof. Since $\Omega := \omega_1 \wedge \cdots \wedge \omega_1 + \cdots + \omega_n \wedge \cdots \wedge \omega_n$ (with w-fold wedge products), Ω^p is a sum of $2^w p$-forms. Thus, it will be a sum of

$$\varepsilon\alpha_1^0 \wedge \alpha_1^1 \wedge \cdots \wedge \alpha_1^n \wedge \alpha_2^0 \wedge \alpha_2^1 \wedge \cdots \wedge \alpha_2^n \wedge \cdots \wedge \alpha_p^0 \wedge \alpha_p^1 \wedge \cdots \wedge \alpha_p^n, \tag{4.1}$$

where $\varepsilon = \pm 1$. We will show that ε equals $+1$.

Each term of Ω^p is a product of 2-forms:

$$\begin{array}{lllll} \alpha_s^0 \wedge \alpha_s^1, & \alpha_s^2 \wedge \alpha_s^3, & \ldots, & \alpha_s^{n-1} \wedge \alpha_s^n; & \\ \alpha_s^0 \wedge \alpha_s^2, & \alpha_s^3 \wedge \alpha_s^4, & \ldots, & \alpha_s^{n-2} \wedge \alpha_s^{n-1}, & \alpha_s^n \wedge \alpha_s^1; \\ \alpha_s^0 \wedge \alpha_s^3, & \alpha_s^1 \wedge \alpha_s^2; & & & \\ \cdots\cdots & & & & \\ \alpha_s^0 \wedge \alpha_s^n. & & & & \end{array} \tag{4.2}$$

For example (recall that $n+1=2^w$ implies that n is an odd integer), if $n=3$, then we have

$$\alpha^0\wedge\alpha^1\wedge\alpha^2\wedge\alpha^3,$$

and

$$\begin{array}{ll}\alpha^0\wedge\alpha^1, & \alpha^2\wedge\alpha^3,\\ \alpha^0\wedge\alpha^2, & \alpha^3\wedge\alpha^1,\\ \alpha^0\wedge\alpha^3, & \alpha^1\wedge\alpha^2.\end{array}$$

For $n=5$:

$$\begin{array}{lll}\alpha^0\wedge\alpha^1\wedge\alpha^2\wedge\alpha^3\wedge\alpha^4\wedge\alpha^5, & & \\ \alpha^0\wedge\alpha^1, & \alpha^2\wedge\alpha^3, & \alpha^4\wedge\alpha^5,\\ \alpha^0\wedge\alpha^2, & \alpha^3\wedge\alpha^4, & \alpha^5\wedge\alpha^1,\\ \alpha^0\wedge\alpha^3, & \alpha^1\wedge\alpha^2, & \\ \alpha^0\wedge\alpha^4, & & \\ \alpha^0\wedge\alpha^5 & & \end{array}$$

etc.

Now, let us take one of the summands and rearrange it so that the subscripts will be in nondecreasing order, i.e. so that the summand will be an exterior product of $(n+1)p$ $(=2^w p)$ elements:

$$\begin{array}{c}\alpha_1^0,\ \alpha_1^1,\ \dots,\ \alpha_1^n,\\ \alpha_2^0,\ \alpha_2^1,\ \dots,\ \alpha_2^n,\\ \dots\dots\dots\dots\\ \alpha_p^0,\ \alpha_p^1,\ \dots,\ \alpha_p^n,\end{array}$$

such that the first $n+1$ elements in the product will have subscript 1, the next $n+1$ will have subscript 2 etc. Since in the original product, we multiply pairs with the same indices, in order to achieve the new product, we have to permute the elements in the product by means of an even permutation. Hence we do not change the value of the product.

Take the term in the product consisting of $n+1$ elements with index s. Since it is a product of terms in (4.2), it must be one of the following $(n+1)-1$ forms (else it would be 0):

$$\begin{array}{l}\alpha_s^0\wedge\alpha_s^1\wedge\alpha_s^2\wedge\cdots\wedge\alpha_s^{n-1}\wedge\alpha_s^n,\\ \alpha_s^0\wedge\alpha_s^2\wedge\alpha_s^3\wedge\cdots\wedge\alpha_s^n\wedge\alpha_s^1,\\ \alpha_s^0\wedge\alpha_s^3\wedge\alpha_s^4\wedge\cdots\wedge\alpha_s^{n-1}\wedge\alpha_s^n\wedge\alpha_s^1\wedge\alpha_s^2,\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ \alpha_s^0\wedge\alpha_s^n\wedge\alpha_s^1\wedge\alpha_s^2\wedge\alpha_s^3\wedge\cdots\wedge\alpha_s^{n-2}\wedge\alpha_s^{n-1},\end{array}$$

which are equal to each other. So, each summand is equal to (4.1) with $\varepsilon=+1$ and Ω^p is a nonzero multiple of it. ∎

I.5. Clifford-type manifolds

Definition 5.1. Assume that (M, g) is a Riemannian manifold. An *almost Clifford-type structure* $\mathcal{C}_n$ on (M, g) is defined as a covering $\{U^i\}$ of the manifold M with a set of almost complex structures $\{I^i_1, \ldots, I^i_n\}$ on each U^i such that

$$I^i_\alpha I^i_\beta + I^i_\beta I^i_\alpha = -2\delta_{\alpha\beta}\,\mathrm{Id}$$

and the n-dimensional vector spaces of endomorphisms generated by the complex structures $I^i_1, \ldots, I^i_n$,

$$\mathrm{End}_{U^i} := \{a^1 I^i_1 + \cdots + a^n I^i_n;\ a^1, \ldots, a^n \in \mathbb{R}\},$$

are the same on all of the manifold (we assume that no I^i_j is a product of two, four etc. others from the set $\{I^i_1, \ldots, I^i_n\}$).

Definition 5.2. A Riemannian metric g is *Clifford-type-Hermitian* if g is Hermitian for each $I_1, \ldots, I_n$.

Definition 5.3. (a) A Riemannian manifold (M, g) with an almost Clifford-type structure $\mathcal{C}_n$ is called an *almost Clifford-type manifold.*

(b) An almost Clifford-type manifold $(M, g, \mathcal{C}_n)$ with the metric g Clifford-type-Hermitian is called *almost Clifford-type-Hermitian.*

Assume that $(M, g, \mathcal{C}_n)$ is an almost Clifford-type-Hermitian manifold. Let $\{I_1, \ldots, I_n\} \in \mathcal{C}_n$. Consider the 2-forms $\omega_1, \ldots, \omega_n$ defined by

$$\omega_j(X, Y) := g(X, I_j Y), \qquad j = 1, \ldots, n.$$

where X and Y are arbitrary C^∞-vector fields on M.

Definition 5.4. If $n + 1 = 2^w$, define the 2^w-form Ω as follows:

$$\Omega := \underbrace{\omega_1 \wedge \cdots \wedge \omega_1}_{w \text{ times}} + \cdots + \underbrace{\omega_n \wedge \cdots \wedge \omega_n}_{w \text{ times}}.$$

Definition 5.5. An $(n+1)p$-dimensional Riemannian manifold M is called a *Clifford-type* manifold if its holonomy group is a subgroup of $SP[(n+1)p] \times Sp(n+1)$.

Examples 5.1.

1. The basic examples of Clifford-type manifolds are quaternionic manifolds. Note that for $n = 2$ there are three almost complex structures on a given Riemannian manifold (M, g), namely: $I_1, I_2, I_3 := I_1 I_2$, and $\dim_{\mathbb{R}} M = 2^2 = 4$. These manifolds are called *almost-quaternionic* (see e.g. [5], [18], [30]).

If g is Hermitian for I_1 and I_2 then g is called *almost-quaternionic-Hermitian.*

If the fundamental 4-form Ω is closed then an almost-quaternionic-Hermitian manifold is called *almost-quaternionic-Kähler.* The most important example of an almost-quaternionic-Kähler manifold is the quaternionic projective space $\mathbb{HP}^n$ with the standard metric (see e.g. [5], [27]).

2. More generally, in the case when the holonomy group of a given almost-quaternionic-Hermitian manifold (M^{4m}, g) is contained in the group $Sp(m) \times Sp(1)$, then the manifold is called *quaternionic-Kähler* (see e.g. [3], [30]). An important result by Berger [2] states that a quaternionic-Kähler manifold (of dimension $4n > 8$) is Einstein

(i.e., a Riemannian manifold of constant Ricci curvature). Moreover, quaternionic-Kähler manifolds whose dimension is a multiple of 8 are spin manifolds ([25], [30]).

Some examples of manifolds with holonomy group contained in $Sp(m), Sp(m)\times Sp(1)$ or $Spin(n)$ can be found in [30].

Let M be an $(n+1)p$-dimensional Clifford-type manifold and $x\in M$. We can identify T_xM with $\mathcal{A}_n^p$. However, this Clifford-type structure of T_xM may not be invariant under parallel displacement. Using this identification we could define Ω which is invariant under parallel displacement. One can prove

THEOREM 5.1. *Ω is invariant under the action of $SP[(n+1)p]\times SP(n+1)$.*

Proof. Analogous to that of Theorem 3.1. ∎

Hence Ω is independent of the choice of a Clifford-type structure on T_xM. By the above discussion and Theorem 4.2 $(\Omega^p\neq 0)$ we have

LEMMA 5.1. *The form Ω defined above is a closed differential form of degree 2^w and of maximal rank.*

THEOREM 5.2. *Let M be a 2^wp-dimensional Clifford-type manifold and let B^i denote its ith Betti number. Then $B^{2^wi}\neq 0$ for $i=0,1,\dots,p$.*

Proof. By Lemma 5.1, Ω is a closed 2^w-form of maximal rank. Hence Ω^i is a nonzero element of $H^{2^wi}(M,\mathbb{R})$ and so $B^{2^wi}=\dim H^{2^wi}(M,\mathbb{R})\neq 0$. ∎

DEFINITION 5.6. Define the operators $*, L$ and Λ on the space of differential forms $\mathcal{E}^r(M,\mathbb{R})$, as follows: if ω is a differential r-form then $*\omega$ is the (2^wp-r)-form such that

$$(*\omega)_x := *(\omega_x) \quad \text{for all } x\in M,$$
$$L\omega := \Omega\wedge\omega, \qquad \Lambda\omega := *(\Omega\wedge *\omega).$$

A differential form ω is said to be *effective* if $\Lambda\omega=0$.

THEOREM 5.3. *Let M be a 2^wp-dimensional Clifford-type manifold and ω a differential form on M of degree $r\le p+1$. Then*

$$\omega=\sum_{i=0}^{[p/2^w]} L^i\omega_{\mathrm{ef}}^{p-2^wi},$$

where ω_{ef}^k denotes an effective k-form.

Proof. Let $\mathcal{E}_{\mathrm{ef}}^k(M,\mathbb{R})$ denote the space of effective k-forms. By Theorem 4.1 for $r\le p+1$ there is a direct sum decomposition

$$\mathcal{E}^r(M,\mathbb{R})=\mathcal{E}_{\mathrm{ef}}^r(M,\mathbb{R})\oplus L\mathcal{E}_{\mathrm{ef}}^{r-2^w}(M,\mathbb{R})\oplus\cdots\oplus L^t\mathcal{E}_{\mathrm{ef}}^{r-2^wt}(M,\mathbb{R}),$$

where $t:=[r/2^w]$. ∎

The Chern theorem [8] states the following. Let M be a compact Riemannian manifold with a structure group G and $W_1,\dots,W_k$ be the irreducible invariant subspaces of $\mathcal{E}^q(M,\mathbb{R})$ under the action of G and let $P_{W_i}:\mathcal{E}^q(M,\mathbb{R})\to W_i$ be the projection map of $\mathcal{E}^q(M,\mathbb{R})$ into W_i. Then, if a q-form ω is harmonic, so is $P_{W_i}\omega$.

Clearly each of the $L^i\mathcal{E}_{\mathrm{ef}}^{r-2^w i}(M,\mathbb{R})$ is an invariant subspace of $\mathcal{E}^r(M,\mathbb{R})$ under the action of the holonomy group G. So each $L^i\mathcal{E}_{\mathrm{ef}}^{r-2^w i}(M,\mathbb{R})$ is a sum of the W_i's. Therefore the projection of a harmonic form into $L^i\mathcal{E}_{\mathrm{ef}}^{r-2^w i}(M,\mathbb{R})$ is again harmonic and we have the following:

THEOREM 5.4. *If M is a Clifford-type manifold of dimension $2^w p$, then for each $i = 0,1,2,\dots,2^w-1$ with $i+2^w z \le p+1$, $z=[p/2^w]$, we have the following increasing sequence of Betti numbers*:

$$B^i \le B^{i+2^w} \le \dots \le B^{i+2^w z}.$$

II. On some feature of hypercomplex analysis

II.1. Introduction. Using a Clifford-type structure we define Clifford-type holomorphy and prove two features of Clifford-type holomorphic mappings without counterparts in complex analysis.

II.2. Clifford-type holomorphy. Take n almost complex structures $I_1,\dots,I_n$ defined on a real vector space V satisfying the condition

$$I_\alpha I_\beta + I_\beta I_\alpha = -2\delta_{\alpha\beta}\,\mathrm{Id}, \qquad \alpha,\beta = 1,\dots,n,$$

so that no I_j is a product of two, four etc. others from the set $\{I_1,\dots,I_n\}$.

Let $X\in V$, $X\neq 0$. Then the vector subspace $\widetilde{V}$ of V generated by $X, I_1X,\dots,I_{p_n'}X$ (see p. 9) has dimension

$$\dim\widetilde{V} = p_n' + 1.$$

EXAMPLE 2.1. (a) If $n=1$, then we have one almost complex structure, $p_1'=1$ and $\dim\widetilde{V}=\dim V=2$ ($V\cong\mathbb{R}^2\cong\mathbb{C}$).

(b) If $n=2$, then we have two almost complex structures, $p_2'=3$ and $\dim\widetilde{V}=\dim V=4$ ($V\cong\mathbb{R}^4\cong\mathbb{C}\oplus\mathbb{C}\cong\mathbb{H}$).

(c) If $n=3$, then we have three almost complex structures I_1,I_2,I_3 which generate a set of six almost complex structures on V, namely:

$$I_1,I_2,I_3,\quad I_4:=I_1I_2,\quad I_5:=I_1I_3,\quad I_6:=I_2I_3\quad [(I_1I_2I_3)^2\neq-\mathrm{Id}].$$

Thus $p_3'=6$ and $\dim\widetilde{V}=7$ etc.

DEFINITION 2.1. Let $V=(V,\mathcal{C}_n)$ and $W=(W,\widetilde{\mathcal{C}}_n)$ be two real vector spaces equipped with two almost Clifford-type-Hermitian structures $\mathcal{C}_n=(I_1,\dots,I_n)$ and $\widetilde{\mathcal{C}}_n=(\widetilde{I}_1,\dots,\widetilde{I}_n)$, respectively. Assume that $\Phi:V\to W$ is a smooth map. Then Φ is called *Clifford-type holomorphic* if

$$\widetilde{I}_\alpha\circ d\Phi = d\Phi\circ I_\alpha,\qquad \alpha=1,\dots,n. \tag{2.1}$$

REMARK 2.1. (a) If $n=1$, then (2.1) reduces to the equation

$$\widetilde{I}\circ d\Phi = d\Phi\circ I.$$

In this case the Clifford-type holomorphy is nothing but the holomorphy of complex analysis. Moreover, we have

$$(\widetilde{I}\circ d\Phi = d\Phi\circ I) \;\Leftrightarrow\; \left[\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(u+iv)=0 \text{ with } \Phi=u+iv\right].$$

(b) If $n=2$, then (2.1) reduces to the system

$$\begin{cases}\widetilde{I}\circ d\Phi = d\Phi\circ I,\\ \widetilde{J}\circ d\Phi = d\Phi\circ J.\end{cases}$$

In this case the Clifford-type holomorphy is nothing but the quaternionic Q-holomorphy (see e.g. [18], [21]).

WARNING. It is important to emphasize that Q-holomorphy and Fueter regularity (see e.g. [11], [18], [21]) are two different notions, i.e.

$$\left\{\begin{matrix}\widetilde{I}\circ d\Phi = d\Phi\circ I\\ \widetilde{J}\circ d\Phi = d\Phi\circ J\end{matrix}\right\} \;\not\Leftrightarrow\; \left[\frac{1}{4}\left(\frac{\partial}{\partial x^0}+i\frac{\partial}{\partial x^1}+j\frac{\partial}{\partial x^2}+k\frac{\partial}{\partial x^3}\right)(u^0+iu^1+ju^2+ku^3)=0\right],$$

where $\Phi = u^0+iu^1+ju^2+ku^3$.

II.3. Two properties of Clifford-type holomorphic maps. A real vector space V with an almost Clifford-type structure $\mathcal{C}_n=\{I_1,\dots,I_n\}$ can be turned into a Clifford vector space by defining scalar multiplication by Clifford numbers as follows: if $\mathbf{a}\in\mathcal{A}(e_1,\dots,e_n):=\mathcal{A}_n^{\mathrm{Cliff}}$, then

$$\begin{aligned}\mathbf{a} &= a^0+a^1e_1+\dots+a^ne_n\\ &\quad + a^{12}e_1e_2+\dots+a^{1n}e_1e_n+a^{23}e_2e_3+\dots+a^{(n-1)n}e_{n-1}e_n\\ &\quad + a^{123}e_1e_2e_3+\dots+a^{(n-2)(n-1)n}e_{n-2}e_{n-1}e_n+\cdots\\ &\quad + a^{12\dots(n-1)}e_1e_2\cdots e_{n-1}+\dots+a^{23\dots n}e_2e_3\cdots e_n+a^{12\dots n}e_1e_2\cdots e_n,\end{aligned}$$

where $a^0,a^1,\dots,a^n,\dots,a^{12\dots n}\in\mathbb{R}$. For $X\in V$ we set

$$\begin{aligned}\mathbf{a}\cdot X &= (a^0+a^1e_1+\dots+a^ne_n+a^{12}e_1e_2+\dots+a^{12\dots n}e_1e_2\dots e_n)X\\ &:= a^0X+a^1I_1X+\dots+a^nI_nX+a^{12}I_1(I_2X)+\dots+a^{12\dots n}I_1(I_2\dots(I_nX).\end{aligned}$$

DEFINITION 3.1. Take a point $a\in\mathbb{R}^{2^nk}$. The *right Clifford line passing through* a is defined by

$$\mathcal{C}\text{-line}(a) := \{a'\in\mathbb{R}^{2^nk};\ a'=a\cdot c,\ c\in\mathcal{C}_n^{\mathrm{Cliff}}\}.$$

Assume that

$$n>1.$$

THEOREM 3.1. *Let U be an open neighbourhood of the origin in $\mathbb{R}^{2^ns}$. Let $\Phi:\mathbb{R}^{2^ns}\to\mathbb{R}^{2^n}$ be a Clifford-type holomorphic map with respect to the almost Clifford-type structures $\{I_1,\dots,I_n\}$ and $\{\widetilde{I}_1,\dots,\widetilde{I}_n\}$, respectively. If Φ is singular at the origin then Φ is a constant map.*

Proof. Denote by $\widetilde{U}\subseteq U$ the intersection of U with a right Clifford line, i.e. the right multiples by elements of $\mathcal{A}_n^{\mathrm{Cliff}}=\mathcal{A}(e_1,\dots,e_n)$; here $\mathbb{R}^{2^ns}$ is identified with $(\mathcal{A}_n^{\mathrm{Cliff}})^s$. Such a 2^ns-dimensional submanifold is easily seen to be almost Clifford-type.

The singular set of $\Phi_{|\widetilde{U}}$ is defined by $\det[d(\Phi_{|\widetilde{U}})] = 0$ and it is a complex subvariety N of $\widetilde{U}$ in the complex structures $I_1, \ldots, I_n$. In particular, it cannot be isolated because it is non-null. At a regular point $x \in N$ the real tangent space of N must be invariant under each $I_1, \ldots, I_n$. So, N is 2^n-dimensional at x and thus at all points of $\widetilde{U}$. Then $\Phi_{|\widetilde{U}}$ is constant and because U is the union of the right slices $\widetilde{U}$, Φ is also constant. ∎

REMARK 3.1. A Clifford-type affine map from $\mathbb{R}^{2^n s_1}$ to $\mathbb{R}^{2^n s_2}$ is a map of the form $\Phi(X) = XA + B$, where $A \in \mathcal{M}(s_1, s_2; \mathcal{A}_n^{\mathrm{Cliff}})$ and $B \in \mathbb{R}^{2^n s_2}$. Identifying $\mathbb{R}^{2^n s_1}$ and $\mathbb{R}^{2^n s_2}$ with $(\mathcal{A}_n^{\mathrm{Cliff}})^{s_1}$ and $(\mathcal{A}_n^{\mathrm{Cliff}})^{s_2}$, we have $\Phi(X) = \widetilde{A}\widetilde{X} + \widetilde{B}$, where $\widetilde{A}$ is the transpose matrix of A with real $2^n \times 2^n$ submatrices replaced by Clifford numbers and where $\widetilde{X}$ and $\widetilde{B}$ are s_1- and s_2-tuples of Clifford numbers, respectively.

PROPOSITION 3.1. *Let U and V be open sets in $\mathbb{R}^{2^n s_1}$ and $\mathbb{R}^{2^n s_2}$, respectively. Let $\Phi : U \to V$ be Clifford-type holomorphic with respect to Clifford-type structures on $\mathbb{R}^{2^n s_1}$ and $\mathbb{R}^{2^n s_2}$, respectively. Then Φ is the restriction to U of a Clifford-type affine map from $\mathbb{R}^{2^n s_1}$ to $\mathbb{R}^{2^n s_2}$.*

Proof. Since the validity of the assertion is not affected by translations of the range or domain, one can assume that U and V contain the origin and that $\Phi(0) = 0$. Consider the map $X \mapsto \Phi(X) - X d\Phi(0)$. Since $d\Phi(0) \in \mathcal{M}(s_1, s_2; \mathcal{A}_n^{\mathrm{Cliff}})$, this function is Clifford-type holomorphic and by the construction its Jacobian is zero. Thus, composing it with any projection $\pi_i : \mathbb{R}^{2^n s_2} \to \mathbb{R}^{2^n}$ defined by

$$\pi_i(x_1^1, \ldots, x_1^{2^n}, \ldots, x_i^1, \ldots, x_i^{2^n}, \ldots, x_{s_2}^1, \ldots, x_{s_2}^{2^n}) := (x_i^1, \ldots, x_i^{2^n}), \qquad i = 1, \ldots, s_2,$$

one gets a constant map by Theorem 3.1. Thus, $\Phi(X) = X d\Phi(0)$. ∎

EXAMPLE 3.1. The above theorem is not true for $n = 1$ $(s = 1)$, i.e. in complex analysis. Take, for instance, the map $\Phi : \mathbb{C} \to \mathbb{C}$ defined by $\Phi(z) := z^2$. It satisfies the assumption of Theorem 3.1 but Φ is singular at 0 and it is not a constant map.

Putting $n = 2$ we obtain the simplest, quaternionic form of Theorem 3.1, namely

THEOREM 3.2. *Let U be an open neighbourhood of 0 in $\mathbb{R}^{4s}$ and $\Phi : \mathbb{R}^{4s} \to \mathbb{R}^4$ a Clifford-type holomorphic map (quaternionic Q-holomorphic) with respect to the quaternionic structures (I_1, I_2) and $(\widetilde{I}_1, \widetilde{I}_2)$, respectively. If Φ is singular at 0 then Φ is a constant map.*

III. Sectional curvature of a Clifford-type projective space

III.1. Introduction. The main aim of this chapter is a generalization of a theorem of complex analysis (Theorem 1.1 below, see e.g. [17, pp. 167, 368]) to the quaternionic and octonionic projective spaces (Theorem 2.2).

Let (M^n, g, J) be a Kähler manifold of complex dimension n and $x \in M$. Let σ be a plane in T_xM (i.e. a real 2-dimensional subspace of T_xM) and $\{X, Y\}$ an orthonormal basis in σ. The angle $\alpha(\sigma)$ between σ and $J(\sigma)$ is defined by $\cos\alpha(\sigma) := |g(X, JY)|$.

THEOREM 1.1 ([17]). *The sectional curvature $K(\sigma)$ of a Kähler manifold (M^n, g, J) of constant holomorphic sectional curvature 1 is given by*

$$\frac{1}{4}[1 + 3\cos^2\alpha(\sigma)]. \tag{1.1}$$

Moreover, as an application we obtain quaternionic and octonionic counterparts of the following theorem of Klingenberg. If M is a Kähler manifold then one defines

$$K_1[\alpha(\sigma)] := \frac{1}{4}[1 + 3\cos^2\alpha(\sigma)]. \tag{1.1}$$

Klingenberg [15] proved the following:

THEOREM 1.2. *Let M be a Kähler manifold of real dimension $2n \geq 4$. Assume that for all 2-planes σ tangent to M, the sectional curvature $K(\sigma)$ satisfies the inequality*

$$9/16 < K(\sigma)/K_1[\alpha(\sigma)] \leq 1.$$

Then M is compact and has the homotopy type of the complex projective n-space $\mathbb{CP}^n$.

III.2. Sectional curvature of a Clifford-type projective space. The Lie algebra $sp_{n+1}(p)$ of $SP_{n+1}(p)$ is the set of all $p\times p$ skew-symmetric (Clifford-type number) matrices, i.e. matrices (a_{ij}), where each a_{ij} is a Clifford-type number satisfying the condition

$$\overline{a}_{ji} = -a_{ij},$$

where $\overline{a}$ denotes the conjugate of a.

LEMMA 2.1. *Let $\mathcal{B}(A, B)$ denote the real part of the trace of $A\cdot B$, where $A, B \in sp_{n+1}(p)$. Then $\mathcal{B}(A, B)$ is the Killing form of $sp_{n+1}(p)$ up to a constant factor.*

Proof. Since for any Clifford-type numbers $a, b \in \mathcal{A}_n$ we have $\operatorname{Re}(a\cdot b) = \operatorname{Re}(b\cdot a)$, the form

$$\mathcal{B}(A, B) := \operatorname{Tr}\operatorname{Re}(A\cdot B)$$

is clearly symmetric: $\mathcal{B}(A, B) = \mathcal{B}(B, A)$.

Since $sp_{n+1}(p)$ is simple, we only need to show that $\mathcal{B}$ is invariant under the action of $SP_{n+1}(p)$. If we represent A and B as real $(n+1)p$-dimensional square matrices $\widetilde{A}$ and $\widetilde{B}$, then we obtain

$$\operatorname{Re}\operatorname{Tr}(A\cdot B) = \operatorname{Tr}(\widetilde{A}\cdot\widetilde{B}) = \sum_{i,j=1}^{(n+1)p}\sum_{k=0}^{n} A_{ij}^k B_{ij}^k,$$

where $A = (A_{ij})$ and $A_{ij} = A_{ij}^0 + e_1 A_{ij}^1 + \cdots + e_n A_{ij}^n$, and similarly for B. Since $\operatorname{Tr}(\widetilde{A}\cdot\widetilde{B})$ is invariant under $O[(n+1)p] \supset SP_{n+1}(p)$, we have our result. ■

Let $A = (a^1, \ldots, a^p)$, $B = (b^1, \ldots, b^p) \in \mathcal{A}_n^p$, where

$$a^s = x_0^s + e_1 x_1^s + \cdots + e_n x_n^s, \qquad b^s = y_0^s + e_1 y_1^s + \cdots + e_n y_n^s, \qquad s = 1, \ldots, p.$$

Considering $\mathcal{A}_n^p$ as a real $(n+1)p$-dimensional space, we have

$$\langle A, B\rangle = \frac{1}{2}\sum_{i=1}^{p}(a^i\overline{b^i} + b^i\overline{a^i}) = \sum_{i=1}^{p}\sum_{j=0}^{n} x_j^i y_j^i.$$

Considering $\mathcal{A}_n^p$ as a "Clifford-type number" p-space, we have

$$(A,B)=\sum_{i=1}^{p} a^i\overline{b^i}.$$

It is easy to check that

$$(A,B)=\langle A,B\rangle+\sum_{i=1}^{n} e_i\langle A,Be_i\rangle. \tag{2.1}$$

THEOREM 2.1. *The sum* $\sum_{i=1}^n \langle A,Be_i\rangle^2$ *is invariant under the action of* $SP_{n+1}(p)\times SP_{n+1}(1)$.

Proof. $SP_{n+1}(p)$ is defined as the set of all endomorphisms of $\mathcal{A}_n^p$ which preserve the "Clifford-type symplectic" product $(A,B)=\sum_{i=1}^p a^i\overline{b^i}$. Since $\langle A,B\rangle=\frac{1}{2}[(A,B)+(B,A)]$, the inner product $\langle\,,\rangle$ is invariant under the action of $SP_{n+1}(p)$.

$SP_{n+1}(1)$ is the set of all unit Clifford-type numbers. Let $\lambda\in SP_{n+1}(1)$. By a straightforward calculation one can show that

$$\langle A\lambda,B\lambda\rangle=\langle B\lambda,A\lambda\rangle=\|\lambda\|^2\langle A,B\rangle=\|\lambda\|^2\langle B,A\rangle.$$

Thus $\langle\,,\rangle$ is invariant under $SP_{n+1}(p)\times SP_{n+1}(1)$. ■

The group $SP_{n+1}(p)\times SP_{n+1}(1)$ acts transitively on the set of all unit vectors in $\mathcal{A}_n^p$, hence in the sum (2.1) we may assume that $A=(a^1,0,\dots,0)$.

REMARK 2.1. If A and B are unit vectors then $\sum_{i=1}^n \langle A,Be_i\rangle^2\le 1$.

Proof. Since, by assumption,

$$\overline{e}_i=-e_i,\qquad e_ie_j=\pm e_m,\qquad \overline{e_ie_j}=-\pm e_m,$$

we have

$$(e_ie_j)(\overline{e_ie_j})=-(\pm e_m)(\pm e_m)=-(e_m)^2=+1.$$

Moreover $\langle A,Be_i\rangle\in\mathbb{R}$ for $i=1,\dots,n$. Thus

$$\|(A,B)\|^2=(A,B)\overline{(A,B)}=\langle A,B\rangle^2+\sum_{i=1}^n\langle A,Be_i\rangle^2\le\|A\|^2\|B\|^2.$$

Since $\|A\|=\|B\|=1$ and $\langle A,B\rangle^2\ge 0$, we obtain the required inequality. ■

Define the Clifford-type projective space $\mathcal{A}_n\mathbb{P}^p$ as

$$\mathcal{A}_n\mathbb{P}^p:=\mathcal{A}_n^{p+1}/\mathcal{A}_n^*,$$

with the group $\mathcal{A}_n^*:=\{a\in\mathcal{A}_n : a\ne 0\}$ acting by right multiplication.

NOTE. For the complex projective space $\mathbb{CP}^p$ we assume that $p=2m\ge 4$, for the quaternionic projective space $\mathbb{HP}^p$ we assume that $p=4m\ge 8$ and for the octonionic projective space $\mathbb{OP}^p$ we have $p=8m=16$.

A point of $\mathcal{A}_n\mathbb{P}^p$ represents a Clifford-type line γ in $\mathcal{A}_n^{p+1}$. The line γ can be identified with the group $SP_{n+1}(1)$. The subgroup of $SP_{n+1}(p+1)$ stabilizing γ is $SP_{n+1}(p)\times SP_{n+1}(1)$. This description gives $\mathcal{A}_n\mathbb{P}^p$ the structure of the symmetric space $SP_{n+1}(p+1)/SP_{n+1}(p)\times SP_{n+1}(1)$ whose holonomy group equals $SP_{n+1}(p)\times SP_{n+1}(1)$.

DEFINITION 2.1. Let $M = \mathcal{A}_n\mathbb{P}^p$ and $x \in M$. For any unit vectors A and B in T_xM define the "angle" function $\alpha(A,B)$, $0 \le \alpha(A,B) \le \pi/2$, by the equality

$$\cos^2\alpha(A,B) := \sum_{i=1}^{n} \langle A, Be_i\rangle^2.$$

REMARK 2.2. $\alpha(A,B)$ is well defined since it is independent of the choice of a Clifford-type structure on T_xM (see the definition of $SP_{n+1}(p) \times SP_{n+1}(1)$).

We shall now calculate the sectional curvature $\mathcal{K}$ of the Clifford-type projective space M in terms of α. Choose an almost Clifford-type structure on T_xM for $x \in M$, and given an element A in T_xM write $A = (a^1,\ldots,a^p)$ as an element of $\mathcal{A}_n^p$. Then there is a representation of A as an element of $SP_{n+1}(p+1)$ by the skew-symmetric Clifford-type number matrix (a_{ij}), where

$$a_{1i} = -\overline{a}_{i1} = a^{i-1} \qquad \text{for } i = 2,3,\ldots,p+1$$

and $a_{ij} = 0$ otherwise (see e.g. [28]).

LEMMA 2.2. *For $A, B \in T_xM$ we have*

$$\mathcal{B}([A,B],[A,B]) = 2[(A,B)^2 - (A,B)(B,A) + (B,A)^2 - (A,A)(B,B)]. \tag{2.2}$$

Proof. Using the following representation of $A = (a^1,\ldots,a^p)$ and $B = (b^1,\ldots,b^p)$ as elements of $\mathcal{A}_n^p$:

$$A \mapsto \widetilde{A} = \begin{pmatrix} 0 & a^1 & a^2 & \ldots & a^p \\ -\overline{a^1} & 0 & 0 & \ldots & 0 \\ -\overline{a^2} & 0 & 0 & \ldots & 0 \\ . & . & . & \ldots & . \\ -\overline{a^p} & 0 & 0 & \ldots & 0 \end{pmatrix} \in sp_{n+1}(p+1),$$

$$B \mapsto \widetilde{B} = \begin{pmatrix} 0 & b^1 & b^2 & \ldots & b^p \\ -\overline{b^1} & 0 & 0 & \ldots & 0 \\ -\overline{b^2} & 0 & 0 & \ldots & 0 \\ . & . & . & \ldots & . \\ -\overline{b^p} & 0 & 0 & \ldots & 0 \end{pmatrix} \in sp_{n+1}(p+1)$$

and the equality $(A,B)(B,A) = (B,A)(A,B)$, which is easy to check, (2.2) is obtained by a straightforward calculation. ■

LEMMA 2.3. *If A and B are orthonormal (as real vectors, i.e. $\langle A,B\rangle = 0$, $\langle A,A\rangle = \langle B,B\rangle = 1$) in T_xM, then*

$$\mathcal{B}([A,B],[A,B]) = 2\cdot[3(A,B)^2 - 1].$$

Proof. Since

$$\langle A, Ae_1\rangle = \cdots = \langle A, Ae_n\rangle = 0, \qquad \langle B, Ae_i\rangle = -\langle A, Be_i\rangle, \qquad i = 1,\ldots,n,$$

for any $A, B \in \mathcal{A}_n^p$, the lemma follows immediately from Lemma 2.2. ■

Theorem 2.2. *Let $M = \mathcal{A}_n\mathbb{P}^p$, $x \in M$, and let A and B be two orthonormal vectors in T_xM. Furthermore, let $\mathcal{K}$ denote the sectional curvature of M. Then*

$$0 < \mathcal{K}(A,B) = \tfrac{1}{4}[1 + 3\cos^2\alpha(A,B)] < 1. \tag{2.3}$$

Proof. For a symmetric space,

$$\mathcal{K}(A,B) = -c\mathcal{B}([A,B],[A,B]),$$

where c is a positive real number (see e.g. [17]). By Lemma 2.3 we have

$$\mathcal{K}(A,B) = 2c[1 - 3(A,B)^2].$$

Now, for the orthonormal vectors A and B, it is straightforward to show that

$$(A,B)^2 = -\langle A, Be_1\rangle^2 - \cdots - \langle A, Be_n\rangle^2 = -\cos^2\alpha(A,B).$$

Hence $\mathcal{K}(A,B) = 2c[1+3\cos^2\alpha(A,B)]$. The latter function attains a maximum of $8c$ when $A = Be_1$. On the other hand, the sectional curvature of M with the usual Riemannian metric attains a maximum of 1 (see e.g. [17]), so $c = 1/8$. ∎

III.3. Quaternionic and octonionic manifolds with restricted curvature. We are going to prove the following generalization of the Klingenberg theorem 1.2:

Theorem 3.1. *Let M be a compact quaternionic (resp. octonionic) manifold of real dimension $4n \geq 8$ (resp. $8n = 16$). Assume that for all 2-planes σ tangent to M, the sectional curvature $K(\sigma)$ satisfies the inequality*

$$9/16 < K(\sigma)/K_1[\alpha(\sigma)] \leq 1,$$

where $K_1[\alpha(\sigma)]$ is expressed by the formula (1.1), *p.* 23. *Then M has the same integral cohomology ring as $\mathbb{HP}^n$ (resp. $\mathbb{OP}^n$).*

For the notation and definitions of this part we refer to [15–17].

Let M be an m-dimensional complete and simply connected Riemannian manifold and let $G = \{p(s); 0 \leq s < \infty\}$ be a geodesic ray in M, parametrized by arc length.

Definition 3.1 ([15]). For $n = 1, 3, 7$ we say that G satisfies *condition* (Π, n) if

1. there are no conjugate points in $[0, \pi)$,
2. there are n conjugate points in $[\pi, \frac{4}{3}\pi)$,
3. there are no conjugate points in $[\frac{4}{3}\pi, 2\pi)$,
4. there are λ conjugate points in $[2\pi, \frac{8}{3}\pi)$, $\lambda \geq n+1$.

According to [15, 16] condition (Π, n) is satisfied for any geodesic ray G in: the complex projective space ($n = 1$), quaternionic projective space ($n = 3$) and octonionic projective space ($n = 7$).

Theorem 3.2. *Let M be a quaternionic (resp. octonionic) manifold of dimension $4n$ (resp. $8n$), G a geodesic ray on M, G_0 the initial geodesic segment of length $2\pi/\sqrt{\delta}$ with $\delta = 9/16$ (according to Definition 3.1). Assume that the sectional curvature $K(\sigma)$ of each plane section σ tangent to G_0 satisfies*

$$\delta < K(\sigma)/K_1[\alpha(\sigma)] \leq 1. \tag{3.1}$$

Then G satisfies $(\Pi, 3)$ (resp. $\Pi, 7$).

Proof. The idea of the proof is analogous to that of Proposition 3.3 in [15]. We will use Theorem 2.2.

First of all let us rewrite the inequalities (3.1) in the form

$$\tfrac{1}{4}\delta \le \delta K_1[\alpha(\sigma)] < K(\sigma) \le K_1[\alpha(\sigma)] \le 1 \tag{3.2}$$

because

$$1/4 \le K_1[\alpha(\sigma)] \le 1.$$

Let G_0' be a geodesic segment of length $2\pi/\sqrt{\delta}$ in the quaternionic (resp. octonionic) projective space $M' = \mathbb{HP}^n$ (resp. $M' = \mathbb{OP}^n$). There exists an isometry I, compatible with i, j and k $(e_1, \dots, e_7)$, mapping the tangent space at the initial point of G_0 onto the tangent space at the initial point of G_0' which carries the initial direction of G_0 to the initial direction of G_0'.

The isometry I gives rise to a 1-1 correspondence between the plane section σ tangent to G_0 and the plane section $\sigma' = I(\sigma)$ tangent to G_0'.

Since $\alpha(\sigma)$ and $\alpha(\sigma')$ are invariant under the action of $Sp(n) \times Sp(1)$ $(= SP_4(n) \times SP_4(1))$ (resp. $SP_8(n) \times SP_8(1)$) and $SP_{n+1}(p) \subseteq SO[(n+1)p]$, they are invariant under parallel translation along G_0 and G_0', respectively.

For the complex projective space $\mathbb{CP}^n$ with the usual Riemannian metric, the curvature $K(\sigma)$ of σ depends only on the angle $\alpha(\sigma)$ and is given by

$$K(\sigma) = K_1[\alpha(\sigma)] = \tfrac{1}{4}[1 + 3\cos^2\alpha(\sigma)]$$

(see [17], [29]). By Theorem 2.2, for quaternionic and octonionic projective spaces $K_1[\alpha(\sigma)]$ reduces to the sectional curvature $K(\sigma)$ (see (2.3), p. 26).

The inequality $K(\sigma) \le K_1[\alpha(\sigma)]$ of (3.2) yields

$$K(\sigma) \le K_1[\alpha(\sigma)] = K_1[\alpha(I\sigma)] = K'(I\sigma).$$

Now we need Lemma 3.1 of [15, p. 538]. Let M and M' be Riemannian manifolds of the same dimension n. Let $G = \{p(s)\}$ and $G' = \{p'(s)\}$, $s \in [0, a]$, be geodesic segments in M and M', respectively, parametrized by arc length. Let $I : M_{p(0)} \to M'_{p'(0)}$ be an isometry which carries the initial direction of G into the initial direction of G'. Then I determines an isometry I of $M_{p(s)}$ onto $M'_{p'(s)}$: map $M_{p(s)}$ by parallel translation along G onto $M_{p(0)}$, apply $I : M_{p(0)} \to M'_{p'(0)}$, and use parallel translation along G'. This isometry $I : M_{p(s)} \to M'_{p'(s)}$ induces a 1-1 map of 2-planes $\sigma \subset M_{p(s)}$, tangent to G, onto 2-planes $\sigma' \subset M'_{p'(s)}$, tangent to G', which will also be denoted by I. With this notation we have the following

LEMMA 3.1. *Let M and M' be Riemannian manifolds of the same dimension. Denote by K and K' the Riemannian curvatures of M and M', respectively. If $K(\sigma) \le K'(I\sigma)$ then*

$$\operatorname{index} G' \ge \operatorname{index} G.$$

Since G_0' has no conjugate points in $[0, \pi)$, hence has index 0, also G_0 has index 0, hence no conjugate points in that interval.

Since G_0' has 3 (resp. 7) conjugate points in $[\pi, \pi/\sqrt{\delta})$, G_0 has at most 3 (resp. 7) conjugate points in $[\pi, \pi/\sqrt{\delta})$.

Now, let M'' be the quaternionic (resp. octonionic) projective space obtained from M' by multiplying the usual metric by $1/\sqrt{\delta} > 1$ and let K'' be its curvature. Let G''_0 be a geodesic of length $2\pi/\sqrt{\delta}$ in M'' and introduce the isometry I as before. Then the inequality $\delta K_1[\alpha(\sigma)] < K(\sigma)$ of (3.2) can be rewritten as $K''(I\sigma) < K(\sigma)$, because $K_1[\alpha(\sigma)] = K_1[\alpha(I\sigma)] = K'(I\sigma) = (1/\delta)K''(I\sigma)$.

Since G''_0 has 3 (resp. 7) conjugate points in $[0, \pi/\sqrt{\delta})$, G_0 has exactly 3 (resp. 7) conjugate points in $[0, \pi/\sqrt{\delta})$. By a similar argument we conclude that G_0 has no conjugate points in $[\pi/\sqrt{\delta}, 2\pi)$ and $4n-3$ (resp. $8n-7$) conjugate points in $[2\pi, 2\pi/\sqrt{\delta})$.

Putting $\delta = 9/16$ yields the assertion of the theorem. ■

Let us recall the following theorem due to Klingenberg [16, p. 338]:

THEOREM 3.3. *Let M be a complete and simply connected Riemannian manifold of real dimension $(k+1)n$ with $n \geq 2$ and $k = 3$ or 7. Assume that there is a point x_0 in M such that condition (II, k) holds for all geodesic rays starting from x_0. Furthermore, assume that $k + \lambda \geq m = \dim M$. Then M has the same integral cohomology ring as the symmetric space $\mathbb{HP}^n$ for $k = 3$ and $\mathbb{OP}^n$ for $k = 7$.*

Using the above theorem we obtain the final result, i.e. Theorem 3.1.

Proof of Theorem 3.1. Theorem 3.1 follows from Theorems 3.2 and 3.3, by noting from the proof of Theorem 3.2 that $\lambda = 4n-3$ (resp. $\lambda = 8n-7$). ■

IV. Quaternionic condition for the existence of 4-dimensional locally conformally flat almost Kähler manifolds*

IV.1. Introduction. Using the fundamental notions of quaternionic analysis we will show that there are no 4-dimensional almost Kähler manifolds which are locally conformally flat with a metric of a special form.

A basic question in quaternionic analysis is the proper generalization of the notion of holomorphy. At the outset it may not be clear which of several conditions, equivalent for holomorphic mappings of complex numbers, can best be generalized to the quaternionic skew field $\mathbb{H}$.

A typical element of $\mathbb{H}$ can be written as

$$q := w + ix + jy + kz, \qquad w, x, y, z \in \mathbb{R},$$

and the quaternionic units satisfy $i^2 = j^2 = k^2 = ijk = -1$. The conjugate of q is defined by

$$\overline{q} := w - ix - jy - kz$$

and the modulus (norm) by

$$\|q\|^2 := q \cdot \overline{q} = \overline{q} \cdot q = w^2 + x^2 + y^2 + z^2.$$

The norm can be used to express the inverse element: for $q \in \mathbb{H}$, $q \neq 0$, we have

$$q^{-1} = \frac{\overline{q}}{\|q\|^2}.$$

---

* *Editors' note:* This chapter also appeared in [20].

The following relation is easy to check:

$$\overline{q_1 \cdot q_2} = \overline{q}_2 \cdot \overline{q}_1, \qquad q_1, q_2 \in \mathbb{H}.$$

DEFINITION 1.1. A function $f : \mathbb{H} \to \mathbb{H}$ is called (*left*) *differentiable* at q if the limit

$$\frac{df}{dq}(q) := \lim_{h \to 0} h^{-1}[f(q+h) - f(q)]$$

exists.

It is the most natural definition at first sight but it leads to a very restricted class of functions:

THEOREM 1.1. *If df/dq exists, then $f(q) = a + qb$, $a, b \in \mathbb{H}$.*

Quaternions do not commute, hence a reasonable generalization of the term $a_n z^n$ from the complex case is

$$a_0 q a_1 q \cdots q a_{n+1}, \qquad a_i \in \mathbb{H},\ i = 0, 1, \ldots, n+1.$$

But the definition of holomorphy using sums of such terms leads to a quite general class of functions, namely to all the real-analytic mappings from $\mathbb{R}^4$ to $\mathbb{R}^4$.

In 1935 R. Fueter [10] proposed a definition of "regularity" for quaternionic functions via an analogue of the Cauchy–Riemann equations. The class of Fueter regular functions seems in many ways to express very well the spirit of complex analysis in the quaternionic context, as many classical results (e.g. Cauchy's integral formula, Morera's theorem, the Laurent expansion etc.) carry over in a more or less natural way [31, 32]. But, because of the non-commutativity of quaternions, many properties of holomorphic functions cannot be generalized to Fueter regular functions. For instance, the composition of two regular functions is not, in general, regular, so we cannot define a "quaternionic" manifold via Fueter regular transition functions. Nevertheless, this theory is still being developed.

IV.2. Basic notions and Main Theorem*. Let M^{2n} be a real C^∞-manifold of dimension $2n$ endowed with an almost complex structure J and a Riemannian metric g. If the metric g is invariant under the action of J, i.e.

$$g(JX, JY) = g(X, Y)$$

for any vector fields X and Y on M^{2n}, then (M^{2n}, J, g) is called an *almost Hermitian manifold.*

Define the fundamental 2-form Ω by

$$\Omega(X, Y) := g(X, JY).$$

An almost Hermitian manifold (M^{2n}, J, g, Ω) is said to be *almost Kähler* if Ω is a closed form, i.e. $d\Omega = 0$.

Suppose that $n = 2$.

Recall that a Riemannian manifold (M^4, g) is called *locally conformally flat* if for every point $p_0 \in M^4$ there exists a system of local coordinates $(U_{p_0}; w, x, y, z)$ such that

---

* *Editors' note*: Sections IV.2–IV.5 also appeared in [20].

the metric g is expressed by

$$g = g_0(p)[dw^2 + dx^2 + dy^2 + dz^2], \qquad p \in U_{p_0}, \tag{g}$$

where g_0 is a real positive C^∞-function defined in U_{p_0}.

Our aim is to prove

Main Theorem. *A 4-dimensional almost Kähler manifold does not admit any locally conformally flat Riemannian metric of the form*

$$g_0(w, x, y, z) := g_0(r), \qquad r^2 := w^2 + x^2 + y^2 + z^2,$$

where $g_0(r)$ is an analytic function in r different from a constant.

We first prove

Basic Lemma. *If (M^4, J, g, Ω) is a 4-dimensional almost Kähler and locally conformally flat manifold, then g_0 (defined by (g)) is the modulus of a quaternionic left (right) regular function in the sense of Fueter [10] uniquely determined by J and Ω.*

IV.3. Proof of Basic Lemma. Let us denote by the same letters the matrices g, J and Ω with respect to the coordinate basis. These matrices satisfy the equality

$$g \cdot J = \Omega.$$

The metric g, by assumption, is proportional to the identity, so it has the form

$$g = g_0 \cdot I = g_0 \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

An almost complex structure J satisfies the condition $J^2 = -I$. This formula and the fact that Ω is skew-symmetric imply that J is a skew-symmetric and orthogonal 4×4 matrix.

It is easy to check that J is of the form

$$\text{(J)} \qquad \text{(a)} \begin{pmatrix} 0 & a & b & c \\ -a & 0 & c & -b \\ -b & -c & 0 & a \\ -c & b & -a & 0 \end{pmatrix} \quad \text{or} \quad \text{(b)} \begin{pmatrix} 0 & a & b & c \\ -a & 0 & -c & b \\ -b & c & 0 & -a \\ -c & -b & a & 0 \end{pmatrix}$$

with

$$a^2 + b^2 + c^2 = 1.$$

Take a 4×4 skew-symmetric orthogonal matrix

$$\begin{pmatrix} 0 & a & b & c \\ -a & 0 & r & s \\ -b & -r & 0 & t \\ -c & -s & -t & 0 \end{pmatrix}.$$

The conditions

$$\begin{aligned} a^2+b^2+c^2&=1,\\ a^2+r^2+s^2&=1,\\ b^2+r^2+t^2&=1,\\ c^2+s^2+t^2&=1 \end{aligned}$$

imply that

$$a^2=t^2,\qquad b^2=s^2,\qquad c^2=r^2.$$

So, if $t=-a\neq 0$, the orthogonality of the rows gives

$$as+bt=0,\qquad ac-rt=0,$$

hence

$$s=b,\qquad r=-c.$$

If $t=a\neq 0$, we obtain the second matrix. If $t=a=0$, it is again easy to see that the matrix is of one of the two types (J).

Suppose that J is of the form (Ja). Then

$$\Omega=g_0\cdot\begin{pmatrix}0&a&-b&c\\-a&0&c&b\\b&-c&0&a\\-c&-b&-a&0\end{pmatrix}=:\begin{pmatrix}0&A&-B&C\\-A&0&C&B\\B&-C&0&A\\-C&-B&-A&0\end{pmatrix}.$$

Since $(A/g_0)^2+(B/g_0)^2+(C/g_0)^2=a^2+b^2+c^2=1$ we get

$$A^2+B^2+C^2=g_0^2. \tag{3.1}$$

Using the formula (see e.g. [17, p. 36])

$$\begin{aligned} d\Omega(X,Y,Z)=\tfrac{1}{3}\{&X\Omega(Y,Z)+Y\Omega(Z,X)+Z\Omega(X,Y)\\ &-\Omega([X,Y],Z)-\Omega([Z,X],Y)-\Omega([Y,Z],X)\}, \end{aligned}$$

the condition $d\Omega=0$ can be written in the form:

$$\begin{aligned} 0&=3d\Omega(\partial_x,\partial_y,\partial_z)=A_x+B_y+C_z,\\ 0&=3d\Omega(\partial_x,\partial_y,\partial_w)=B_x-A_y+C_w,\\ 0&=3d\Omega(\partial_x,\partial_z,\partial_w)=C_x-A_z-B_w,\\ 0&=3d\Omega(\partial_y,\partial_z,\partial_w)=C_y-B_z+A_w. \end{aligned} \tag{3.2}$$

The system (3.2), although overdetermined, does have solutions. We will show that it has a nice interpretation in quaternionic analysis.

IV.4. Fueter's regular functions. Any function $F:\mathbb{H}\to\mathbb{H}$ can be written as

$$F=F_0+iF_1+jF_2+kF_3,$$

where the F_i are real-valued. F_0 is called the *real part* of F, and $iF_1+jF_2+kF_3$ the *imaginary part* of F.

In [10] Fueter introduced the following operators:

$$\overline{\partial}_{\text{left}} := \frac{1}{4}\left(\frac{\partial}{\partial w} + i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right), \qquad \overline{\partial}_{\text{right}} := \frac{1}{4}\left(\frac{\partial}{\partial w} + \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right),$$

analogous to $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ in complex analysis, to generalize the Cauchy–Riemann equations.

A quaternionic function F is said to be *left regular in the sense of Fueter* (respectively, *right regular in the sense of Fueter*) if it is differentiable in the real variable sense and

$$\overline{\partial}_{\text{left}} F = 0 \qquad (\text{resp. } \overline{\partial}_{\text{right}} F = 0). \tag{4.1}$$

Note that the first equation of (4.1) is equivalent to the system

$$\begin{aligned}
\partial_w F_0 - \partial_x F_1 - \partial_y F_2 - \partial_z F_3 &= 0,\\
\partial_w F_1 + \partial_x F_0 + \partial_y F_3 - \partial_z F_2 &= 0,\\
\partial_w F_2 - \partial_x F_3 + \partial_y F_0 + \partial_z F_1 &= 0,\\
\partial_w F_3 + \partial_x F_2 - \partial_y F_1 + \partial_z F_0 &= 0.
\end{aligned}$$

There are many examples of left and right regular functions. Many papers have been devoted to studying their properties (see e.g. [18]). Quaternionic generalizations of the Cauchy theorem, Cauchy integral formula, Taylor series in terms of special polynomials etc. have been found.

Now we need an important result of [32]. Let ν be an unordered set of n integers $\{i_1, \ldots, i_n\}$ with $1 \le i_r \le 3$; ν is determined by three integers n_1, n_2 and n_3 with $n_1 + n_2 + n_3 = n$, where n_1 is the number of 1's in ν, n_2 the number of 2's, and n_3 the number of 3's. There are $\frac{1}{2}(n+1)(n+2)$ such sets ν and we denote the set of all of them by σ_n.

Let e_{i_r} and x_{i_r} denote i, j, k and x, y, z according as i_r is $1, 2$ or 3, respectively. Then we define the following polynomials:

$$P_\nu(q) := \frac{1}{n!}\sum (we_{i_1} - x_{i_1}) \cdots (we_{i_n} - x_{i_n}),$$

where the sum is taken over all $n!n_1!n_2!n_3!$ different orderings of n_1 1's, n_2 2's and n_3 3's; when $n = 0$, so $\nu = \emptyset$, we take $P_\emptyset(q) = 1$.

For example we give the explicit forms of the polynomials P_ν of the first and second degrees:

$$P_1 = wi - x,$$

$$P_2 = wj - y,$$

$$P_3 = wk - z,$$

$$P_{11} = \tfrac{1}{2}(x^2 - w^2) - xwi, \qquad P_{22} = \tfrac{1}{2}(y^2 - w^2) - ywj,$$

$$P_{12} = xy - wyi - wxj, \qquad P_{23} = yz - wzj - wyk,$$

$$P_{13} = xz - wzi - wxk, \qquad P_{33} = \tfrac{1}{2}(z^2 - w^2) - zwk.$$

In [32] Sudbery proved the following

PROPOSITION 4.1. *Suppose F is left regular in a neighbourhood of the origin $0 \in \mathbb{H}$. Then there is a ball $B = B(0,r)$ in which $F(q)$ is represented by a uniformly convergent series*

$$F(q) = \sum_{n=0}^{\infty} \sum_{\nu\in\sigma_n} P_\nu(q) a_\nu, \qquad a_\nu \in \mathbb{H}.$$

Let F satisfy the assumptions of Proposition 4.1. Then

$$F(q) = a_0 + \sum_{i=1}^{3} P_i a_i + \sum_{i\le j} P_{ij} a_{ij} + \sum_{i\le j\le k} P_{ijk} a_{ijk} + \cdots$$

and so

$$\overline{F(q)} = \overline{a}_0 + \sum_{i=1}^{3} \overline{a}_i \overline{P}_i + \sum_{i\le j} \overline{a}_{ij} \overline{P}_{ij} + \sum_{i\le j\le k} \overline{a}_{ijk} \overline{P}_{ijk} + \cdots .$$

Multiplying the above expressions we get

$$\begin{aligned} \|F(q)\|^2 = \|a_0\|^2 &+ \sum_{i=1}^{3} (P_i a_i \overline{a}_0 + a_0 \overline{a}_i \overline{P}_i) \\ &+ \sum_{i\le j} (P_{ij} a_{ij} \overline{a}_0 + a_0 \overline{a}_{ij} \overline{P}_{ij}) + \sum_{i,j} P_i a_i \overline{a}_j \overline{P}_j \\ &+ \sum_{i\le j\le k} (P_{ijk} a_{ijk} \overline{a}_0 + a_0 \overline{a}_{ijk} \overline{P}_{ijk}) \\ &+ \sum_{m=1}^{3} \sum_{i\le j} (P_m a_m \overline{a}_{ij} \overline{P}_{ij} + P_{ij} a_{ij} \overline{a}_m \overline{P}_m) + \cdots . \end{aligned} \tag{4.2}$$

We are now in a position to complete the proof of the Basic Lemma. Set

$$F_{ABC}(q) := Ci + Bj + Ak, \tag{4.3}$$

where we have identified $q \in \mathbb{H}$ with $(w,x,y,z) \in \mathbb{R}^4$. Then (3.2) is nothing but the condition that F_{ABC} is left regular in the sense of Fueter. Then, by (3.1), we have

$$\|F_{ABC}\| = g_0. \ \blacksquare \tag{4.4}$$

IV.5. Proof of Main Theorem. Assume that the metric g in question is given by

$$g_0(w,x,y,z) := g_0(r), \qquad r^2 := w^2 + x^2 + y^2 + z^2,$$

where $g_0(r)$ is an analytic function different from a constant in some neighbourhood of $r = 0$, i.e. we have

$$g_0^2(r) = b_0 + b_1 r + b_2 r^2 + \cdots + b_n r^n + \cdots . \tag{5.1}$$

Let us apply equation (4.2) to F_{ABC} defined by (4.3). Combining (4.2), (4.4) and (5.1) we get

$$b_0 = \|a_0\|^2, \tag{5.2}$$

$$b_1 r = \sum_{i=1}^{3} (P_i a_i \overline{a}_0 + a_0 \overline{a}_i \overline{P}_i), \tag{5.3}$$

$$b_2 r^2 = \sum_{i \le j} (P_{ij} a_{ij} \overline{a}_0 + a_0 \overline{a}_{ij} \overline{P}_{ij}) + \sum_{i,j} P_i a_i \overline{a}_j \overline{P}_j, \tag{5.4}$$

$$\begin{aligned} b_3 r^3 = & \sum_{i \le j \le k} (P_{ijk} a_{ijk} \overline{a}_0 + a_0 \overline{a}_{ijk} \overline{P}_{ijk}) \\ & + \sum_{m=1}^{3} \sum_{i \le j} (P_m a_m \overline{a}_{ij} \overline{P}_{ij} + P_{ij} a_{ij} \overline{a}_m \overline{P}_m) \end{aligned} \tag{5.5}$$

etc. Thus we obtain $b_0 \ge 0$. It is easy to verify that equality (5.3) leads to $a_1, a_2, a_3 = 0$, $b_1 = 0$. Hence, (5.4) can be rewritten as

$$b_2 r^2 = \sum_{i \le j} (P_{ij} a_{ij} \overline{a}_0 + a_0 \overline{a}_{ij} \overline{P}_{ij}). \tag{5.6}$$

Consider the equality (5.6). Set

$$d_{ij} := a_{ij} \overline{a}_0 := d_{ij}^0 + d_{ij}^1 \mathbf{i} + d_{ij}^2 \mathbf{j} + d_{ij}^3 \mathbf{k}$$

($\mathbf{i}, \mathbf{j}, \mathbf{k}$ denote the quaternionic units) and rewrite (5.6) in the form

$$b_2 (w^2 + x^2 + y^2 + z^2) = 2 \sum_{i \le j} \operatorname{Re}(P_{ij} d_{ij}).$$

Then we get

$$\begin{aligned} b_2(w^2 + x^2 + y^2 + z^2) &= 2 \operatorname{Re} \left\{ \left[\tfrac{1}{2}(x^2 - w^2) - xw\mathbf{i} \right] d_{11} \right\} + 2 \operatorname{Re} \left\{ \left[\tfrac{1}{2}(y^2 - w^2) - yw\mathbf{j} \right] d_{22} \right\} \\ &\quad + 2 \operatorname{Re} \left\{ \left[\tfrac{1}{2}(z^2 - w^2) - zw\mathbf{k} \right] d_{33} \right\} + \cdots \\ &= (x^2 - w^2) d_{11}^0 + (y^2 - w^2) d_{22}^0 + (z^2 - w^2) d_{33}^0. \end{aligned}$$

Comparing the terms in x^2, y^2 and z^2 we obtain $b_2 = d_{11}^0 = d_{22}^0 = d_{33}^0$ but then $b_2 w^2 = -3w^2 b_2$. Thus $b_2 = 0$ and, as a result of (5.6), $a_{ij} = 0$.

The equality (5.5) takes the form

$$b_3 r^3 = \sum_{i \le j \le k} (P_{ijk} a_{ijk} \overline{a}_0 + a_0 \overline{a}_{ijk} \overline{P}_{ijk}).$$

This implies that $b_3 = 0$, $a_{ijk} = 0$.

Now we will prove that $b_4 = 0$. We have

$$b_4 r^4 = \sum (P_{ijkl} a_{ijkl} \overline{a}_0 + a_0 \overline{a}_{ijkl} \overline{P}_{ijkl}). \tag{5.7}$$

There are $\frac{1}{2}(4+1)(4+2) = 15$ polynomials P_{ijkl}. We will write all of them:

$$\begin{aligned} P_{1111} &= \tfrac{1}{24}(wi - x)(wi - x)(wi - x)(wi - x), \\ P_{2222} &= \tfrac{1}{24}(wj - y)(wj - y)(wj - y)(wj - y), \\ P_{3333} &= \tfrac{1}{24}(wk - z)(wk - z)(wk - z)(wk - z), \\ P_{1112} &= \tfrac{1}{24}(wi - x)(wi - x)(wi - x)(wj - y), \\ P_{1113} &= \tfrac{1}{24}(wi - x)(wi - x)(wi - x)(wk - z), \\ P_{2223} &= \tfrac{1}{24}(wj - y)(wj - y)(wj - y)(wk - z), \end{aligned}$$

$$
\begin{aligned}
P_{1222} &= \tfrac{1}{24}(wi - x)(wj - y)(wj - y)(wj - y),\\
P_{1333} &= \tfrac{1}{24}(wi - x)(wk - z)(wk - z)(wk - z),\\
P_{2333} &= \tfrac{1}{24}(wj - y)(wk - z)(wk - z)(wk - z),\\
P_{1122} &= \tfrac{1}{24}(wi - x)(wi - x)(wj - y)(wj - y),\\
P_{1133} &= \tfrac{1}{24}(wi - x)(wi - x)(wk - z)(wk - z),\\
P_{2233} &= \tfrac{1}{24}(wj - y)(wj - y)(wk - z)(wk - z),\\
P_{1123} &= \tfrac{1}{24}(wi - x)(wi - x)(wj - y)(wk - z),\\
P_{1223} &= \tfrac{1}{24}(wi - x)(wj - y)(wj - y)(wk - z),\\
P_{1233} &= \tfrac{1}{24}(wi - x)(wj - y)(wk - z)(wk - z).
\end{aligned}
$$

The equality (5.7) can be rewritten in the form

$$b_4(w^2 + x^2 + y^2 + z^2)^2 = 2\sum \operatorname{Re}(P_{ijkl}d_{ijkl}), \quad \text{where} \quad d_{ijkl} := a_{ijkl}\overline{a}_0.$$

Note that

$$
\begin{aligned}
b_4(w^2 + x^2 + y^2 + z^2)^2 &= 2\sum \operatorname{Re}(P_{ijkl}d_{ijkl})\\
&= \tfrac{1}{12}[w^4 + x^4 + \cdots]d^0_{1111} + \tfrac{1}{12}[w^4 + y^4 + \cdots]d^0_{2222} + \tfrac{1}{12}[w^4 + z^4 + \cdots]d^0_{3333}\\
&\quad + \tfrac{1}{12}[w^4 + \cdots]d^0_{1122} + \tfrac{1}{12}[w^4 + \cdots]d^0_{1133} + \tfrac{1}{12}[w^4 + \cdots]d^0_{2233}\\
&= \tfrac{1}{12}w^4(d^0_{1111} + d^0_{2222} + d^0_{3333} + d^0_{1122} + d^0_{1133} + d^0_{2233})\\
&\quad + \tfrac{1}{12}x^4 d^0_{1111} + \tfrac{1}{12}y^4 d^0_{2222} + \tfrac{1}{12}z^4 d^0_{3333}.
\end{aligned}
$$

where

$$d^0_{ijkl} := \operatorname{Re} d_{ijkl}.$$

Comparing the terms in x^4, y^4, z^4 and w^4 we get

$$
\begin{aligned}
b_4x^4 &= \tfrac{1}{12}d^0_{1111}x^4,\\
b_4y^4 &= \tfrac{1}{12}d^0_{2222}y^4,\\
b_4z^4 &= \tfrac{1}{12}d^0_{3333}z^4,\\
b_4w^4 &= \tfrac{1}{12}w^4(d^0_{1111} + d^0_{2222} + d^0_{3333} + d^0_{1122} + d^0_{1133} + d^0_{2233}).
\end{aligned}
$$

Comparing the terms in x^2y^2, x^2z^2 and y^2z^2 we have

$$2b_4x^2y^2 = \tfrac{1}{24}x^2y^2d^0_{1122}, \quad 2b_4x^2z^2 = \tfrac{1}{24}x^2z^2d^0_{1133}, \quad 2b_4y^2z^2 = \tfrac{1}{24}y^2z^2d^0_{2233}.$$

Then we get

$$
\begin{aligned}
b_4 &= b_4 + b_4 + b_4 + \tfrac{1}{12}(d^0_{1122} + d^0_{1133} + d^0_{2233}),\\
-2b_4 &= \tfrac{1}{12}(d^0_{1122} + d^0_{1133} + d^0_{2233})
\end{aligned}
$$

and finally $-2b_4 = 12b_4$, which is impossible. Thus $b_4 = 0$, $a_{ijkl} = 0$.

By analogous considerations we obtain $b_1 = b_2 = b_3 = b_4 = \cdots = b_n = \cdots = 0$.

Thus $g_0(r)$ has to be a constant: $g_0(r) = \sqrt{b_0}$, contrary to assumption. ∎

REMARK 5.1. If J is of the form (Jb) then the proofs of the Basic Lemma and Main Theorem are similar. One has to replace the left regular quaternionic function in the sense of Fueter with the right one (see [18, p. 10]).

IV.6. Examples

EXAMPLE 6.1. Let

$$g_0(w,x,y,z)=\frac{1}{1+r},\qquad r^2=w^2+x^2+y^2+z^2.$$

Then

$$g_0^2(r)=\frac{1}{(1+r)^2}=1-2r+3r^2-4r^3+\cdots+(-1)^n(n+1)r^n+\cdots. \tag{6.1}$$

Comparing the right sides of (4.2) and (6.1) we see that

$$a_0\neq 0,\qquad -2r=\sum_{i=1}^{3}(P_ia_i\overline{a}_0+a_0\overline{a}_i\overline{P}_i),$$

which is impossible.

EXAMPLE 6.2. Take

$$g_0(w,x,y,z)=\frac{1}{\sqrt{1+r^2}},\qquad r^2=w^2+x^2+y^2+z^2.$$

Then

$$g_0^2(r)=\frac{1}{1+r^2}=1-r^3+r^6-r^9+\cdots+(-1)^kr^{3k}+\cdots. \tag{6.2}$$

Comparing the right sides of (4.2) and (6.2) we get $a_0\neq 0$, $a_i=0$, $a_{ij}=0$ and

$$-r^3=\sum_{i\le j\le k}(P_{ijk}a_{ijk}\overline{a}_0+a_0\overline{a}_{ijk}\overline{P}_{ijk}),$$

which is impossible.

EXAMPLE 6.3. Let

$$g_0(w,x,y,z)=\frac{1}{\sqrt{1-r^2}},\qquad r^2=w^2+x^2+y^2+z^2.$$

Then

$$g_0^2(r)=\frac{1}{1-r^2}=1+r^2+\frac{4}{3}r^3+\cdots. \tag{6.3}$$

Comparing the right sides of (4.2) and (6.3) we have $a_0\neq 0$, $a_i=0$ and

$$r^2=\sum_{i\le j}(P_{ij}a_{ij}\overline{a}_0+a_0\overline{a}_{ij}\overline{P}_{ij}). \tag{6.4}$$

Set

$$d_{ij}:=a_{ij}\overline{a}_0:=d_{ij}^0+d_{ij}^1\mathbf{i}+d_{ij}^2\mathbf{j}+d_{ij}^3\mathbf{k}$$

and rewrite (6.4) in the form

$$w^2+x^2+y^2+z^2=2\sum_{i\le j}\mathrm{Re}(P_{ij}d_{ij}).$$

Then we get

$$\begin{aligned}w^2+x^2+y^2+z^2&=2\,\mathrm{Re}\big\{\big[\tfrac12(x^2-w^2)-xw\mathbf{i}\big]d_{11}\big\}\\&\quad+2\,\mathrm{Re}\big\{\big[\tfrac12(y^2-w^2)-yw\mathbf{j}\big]d_{22}\big\}\\&\quad+2\,\mathrm{Re}\big\{\big[\tfrac12(z^2-w^2)-zw\mathbf{k}\big]d_{33}\big\}+\cdots\\&=(x^2-w^2)d_{11}^0+(y^2-w^2)d_{22}^0+(z^2-w^2)d_{33}^0).\end{aligned}$$

Comparing the terms in x^2, y^2 and z^2 we get $d_{11}^0 = d_{22}^0 = d_{33}^0 = 1$ but then $w^2 = -3w^2$, and this is impossible.

EXAMPLE 6.4. Let

$$g_0(w,x,y,z) = \frac{1}{(1-r^2)^2}, \qquad r^2 = w^2 + x^2 + y^2 + z^2.$$

Then

$$g_0^2(r) = \frac{1}{(1-r^2)^4} = 1 + 4r^2 + \cdots . \tag{6.5}$$

Comparing the right sides of (4.2) and (6.5) we obtain $a_0 \neq 0$, $a_i = 0$ and

$$4r^2 = \sum_{i\le j}(P_{ij}a_{ij}\overline{a}_0 + a_0\overline{a}_{ij}\overline{P}_{ij}).$$

As in Example 6.3, we have

$$2w^2 + 2x^2 + 2y^2 + 2z^2 = \sum_{i\le j}\mathrm{Re}(P_{ij}d_{ij}).$$

This time, comparing the terms in x^2, y^2 and z^2, we get $a_0 \neq 0$, $a_i = 0$, $d_{11}^0 = d_{22}^0 = d_{33}^0 = 4$, but then $-6w^2 = 2w^2$, which is again impossible.

REMARK 6.1. The Poincaré model, i.e. the unit ball B^4 in $\mathbb{R}^4$ with the metric

$$g := \frac{4}{(1-r^2)^2}(dw^2 + dx^2 + dy^2 + dz^2), \quad r^2 := w^2 + x^2 + y^2 + z^2,$$

is not an almost Kähler manifold.

IV.7. Contribution to the theory of Fueter regular functions. From the proof of the Main Theorem one may conclude

PROPOSITION 7.1. *The Fueter equation* (4.1) *does not admit any solution with modulus of the form*

$$\|F\| := g(r), \qquad r^2 := w^2 + x^2 + y^2 + z^2,$$

where $g(r)$ is an analytic function in r different from a constant.

V. On quaternionic Lagrangian submanifolds

V.1. Introduction. In some analogy to the complex case we present here some results on quaternionic Lagrangian submanifolds. For instance, we give a necessary and sufficient condition for a graph to be quaternionic Lagrangian. We also present the explicit forms of some characteristic differential equations naturally connected with some type of quaternionic Lagrangian submanifolds.

Moreover, we show that the immersions which lead to the almost-quaternionic and Lagrangian submanifolds, respectively, cannot be homotopic. To do it we use the Lichnerowicz homotopy invariant $K(\Phi)$ of maps $\Phi : M \to N$ between a compact special almost Hermitian manifold M and an almost Kähler manifold N (see [23, 24]). It turns out that the idea of the construction of $K(\Phi)$ can be applied to many different contexts.

Under a suitable general hypothesis a homotopy invariant $K_{\xi,\eta}(\Phi)$ can be considered for smooth maps $\Phi : (M, g) \to (N, h)$ between Riemannian manifolds which admit "canonically" defined p-forms $\xi \in \bigwedge^p M$ and $\eta \in \bigwedge^p N$ playing the role of the Kähler 2-form in the complex case. In particular we succeed in applying the Lichnerowicz homotopy invariant to almost quaternionic-Kähler manifolds.

V.2. Preliminaries. Recall that $\mathbb{H}^n$ can be identified with $\mathbb{R}^{4n}$ endowed with three almost complex structures I, J and K satisfying the conditions:

$$I^2 = J^2 = K^2 = -\operatorname{Id},$$
$$IX := iX, \quad JX := jX, \quad KX := kX \quad \text{for all } X \in \mathbb{R}^{4n},$$

where Id stands for the identity mapping in $\mathbb{R}^{4n}$.

We treat $\mathbb{H}^n \equiv \mathbb{R}^{4n}$ as an n-dimensional right module over the quaternions $\mathbb{H}$. One defines a bilinear form $\langle\,,\rangle$ on $\mathbb{H}^n$ as follows: if $Q = (q_1, \ldots, q_n)$, $P = (p_1, \ldots, p_n) \in \mathbb{H}^n$, then

$$\langle Q, P\rangle := \frac{1}{2}\sum_{\alpha=1}^{n}(q_\alpha\overline{p}_\alpha + p_\alpha\overline{q}_\alpha) = \operatorname{Re}(Q, P) := \operatorname{Re}\sum_{\alpha=1}^{n} q_\alpha\overline{p}_\alpha.$$

It is easy to check that

$$\langle Q, P\rangle = \tfrac{1}{2}[(Q, P) + (P, Q)].$$

Then $\langle Q, P\rangle$ is an inner product on $\mathbb{H}^n$ considered as a 4-dimensional real vector space.

Recall that $Sp(n)$ is the group of automorphisms of the right quaternionic vector space $\mathbb{H}^n$ which are unitary with respect to the canonical Hermitian product $(\ ,\ )$.

DEFINITION 2.1. Consider the 2-forms ω_1, ω_2 and ω_3 defined by

$$\omega_1(Q, P) := (Q, IP),$$
$$\omega_2(Q, P) := (Q, JP),$$
$$\omega_3(Q, P) := (Q, KP).$$

Define [27]

$$\Omega := \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3.$$

THEOREM 2.1 ([4]). *Ω is a well defined 4-form, independent of (I, J, K) and invariant on the group $Sp(n) \times Sp(1)$. Moreover, it is non-degenerate because*

$$\Omega^n = (2n + 1)!\operatorname{vol}(\mathbb{R}^{4n}).$$

(*Note that $Sp(1)$ can be identified with the group of unitary quaternions.*)

V.3. Lagrangian planes

DEFINITION 3.1 ([22]). An oriented real n-plane ξ in $\mathbb{H}^n$ is called *Lagrangian* if

(3.1) the four subspaces ξ, $I\xi$, $J\xi$, $K\xi$ are totally orthogonal in $\mathbb{H}^n$.

Note that we may replace (3.1) by the following condition:

(3.2) Ω restricted to ξ vanishes.

DEFINITION 3.2. An n-dimensional oriented submanifold M of $\mathbb{H}^n$ is called a *Lagrangian submanifold* of $\mathbb{H}^n$ if the tangent space to M at each point is Lagrangian.

THEOREM 3.1. *Suppose $\Omega \subseteq \mathbb{R}^n$ is open and $f : \Omega \to \mathbb{R}^n$ is a C^1-mapping. Let M denote the graph of f in $\mathbb{H}^n = \mathbb{R}^n + i\mathbb{R}^n + j\mathbb{R}^n + k\mathbb{R}^n$. Then M is Lagrangian if and only if the Jacobian matrix $[\partial f^i/\partial x_j]$ is symmetric. In particular, if Ω is simply connected, then M is Lagrangian if and only if f is the gradient field of some potential function $F \in C^2(\Omega)$.*

Proof. We replace f by its Jacobian f_* at some fixed point. Then $f_* : \mathbb{R}^n \to \mathbb{R}^n$ is linear and its graph is of the form

$$TM = \{x + if_*(x) + jf_*(x) + kf_*(x);\ x \in \mathbb{R}^n\}.$$

By definition TM is Lagrangian if and only if $Iv \perp TM$, $Jv \perp TM$ and $Kv \perp TM$ for all $v \in TM$.

Suppose $v = x + if_*(x) + jf_*(x) + kf_*(x)$. Then

$$Iv = -f_*(x) + ix - jf_*(x) + kf_*(x).$$

Thus, TM is Lagrangian if and only if $-f_*(x) + ix - jf_*(x) + kf_*(x)$ and $x' + if_*(x') + jf_*(x') + kf_*(x')$ are orthogonal for all $x, x' \in \mathbb{R}^n$, i.e. if and only if

$$-\langle f_*(x), x'\rangle + \langle x, f_*(x')\rangle - \langle f_*(x), f_*(x')\rangle + \langle f_*(x), f_*(x')\rangle = 0,$$

i.e. if

$$-\langle f_*(x), x'\rangle + \langle x, f_*(x')\rangle = 0 \quad \text{for all } x, x' \in \mathbb{R}^n.$$

Consequently, M is Lagrangian if and only if the Jacobian matrix of f is symmetric at each point of Ω. Since Ω is simply connected, this is equivalent to the existence of a potential function $F : \Omega \to \mathbb{R}$ with $\nabla F = f$. ∎

PROPOSITION 3.1. *If $\xi \subset \mathbb{C}^{2n}$ is Lagrangian (in the complex sense) then $\xi \subset \mathbb{H}^{2n}$ is Lagrangian as well (in the quaternionic sense).*

Proof. Let us first prove the assertion for $n = 1$. If $\xi \subset \mathbb{C}^2$ then for all $u_{\mathbb{C}}, v_{\mathbb{C}} \in \xi$ we have $\langle Iu_{\mathbb{C}}, v_{\mathbb{C}}\rangle_{\mathbb{C}} = 0$, where I denotes a complex structure in $\mathbb{C}^2$. Denote by (z_1, z_2) the coordinates in $\mathbb{C}^2$. Without loss of generality we can assume that

$$\xi = \{(z_1, z_2) \in \mathbb{C}^2;\ z_2 = 0\}.$$

We have $\mathbb{C}^2 \subset \mathbb{H}^2 \equiv \mathbb{C}^4$. Let (q_1, q_2) denote the coordinates in $\mathbb{H}^2$. Without loss of generality we can assume that

$$q_1 := z_1 + z_2 j, \qquad q_2 := z_3 + z_4 j,$$

where $(I, J, K := IJ)$ is a quaternionic structure in $\mathbb{H}^2$. Then

$$\xi = \{(q_1, q_2) \in \mathbb{H}^2;\ q_1 = z_1,\ q_2 = 0\}.$$

Let $u_{\mathbb{H}}, v_{\mathbb{H}} \in \xi$. By the definition of the inner product $\langle\, ,\rangle_{\mathbb{H}}$ we have

$$\begin{aligned}\langle Iu_{\mathbb{H}}, v_{\mathbb{H}}\rangle_{\mathbb{H}} &= \mathrm{Re}\{i(x_1+iy_1)\overline{j(x_1'+iy_1')}\} = \mathrm{Re}\{(ix_1-y_1)\overline{(jx_1'+jiy_1')}\}\\ &= \mathrm{Re}\{(ix_1-y_1)(-jx_1'-jiy_1')\}\\ &= \mathrm{Re}\{-ijx_1x_1'-ijix_1y_1'+jy_1x_1'+jiy_1y_1'\}\\ &= \{-kx_1x_1'-jx_1y_1'+jy_1x_1'-ky_1y_1'\} = 0.\end{aligned}$$

Analogously we get

$$\langle Ju_{\mathbb{H}}, v_{\mathbb{H}}\rangle_{\mathbb{H}} = \langle Ku_{\mathbb{H}}, v_{\mathbb{H}}\rangle_{\mathbb{H}} = \langle Iu_{\mathbb{H}}, Jv_{\mathbb{H}}\rangle_{\mathbb{H}} = \langle Iu_{\mathbb{H}}, Kv_{\mathbb{H}}\rangle_{\mathbb{H}} = \langle Ju_{\mathbb{H}}, Kv_{\mathbb{H}}\rangle_{\mathbb{H}} = 0.$$

Now, let $n>1$. By the definition of $\xi \subset \mathbb{C}^{2n}$, for all $u_{\mathbb{C}}, v_{\mathbb{C}} \in \xi$ we have $\langle Iu_{\mathbb{C}}, v_{\mathbb{C}}\rangle_{\mathbb{C}} = 0$. Denote by $(z_1,\dots,z_n,z_{n+1},\dots,z_{2n})$ the coordinates in $\mathbb{C}^{2n}$. Without loss of generality we can assume that

$$\xi = \{(z_1,z_2,\dots,z_{2n}) \in \mathbb{C}^{2n};\ z_{n+1}=\dots=z_{2n}=0\}.$$

Let $(q_1,\dots,q_n,q_{n+1},\dots,q_{2n})$ denote the coordinates in $\mathbb{H}^{2n}$. We can assume that

$$\begin{aligned} q_1 &= z_1 + z_{n+1}j,\\ &\dots\dots\dots\dots\\ q_n &= z_n + z_{2n}j,\\ q_{n+1} &= z_{2n+1} + z_{3n+1}j,\\ q_{n+2} &= z_{2n+2} + z_{3n+2}j,\\ &\dots\dots\dots\dots\\ q_{2n} &= z_{3n} + z_{4n}j.\end{aligned}$$

Then

$$\xi = \{(q_1,\dots,q_n,q_{n+1},\dots,q_{2n}) \in \mathbb{H}^{2n};\ q_1=z_1,\dots,q_n=z_n, q_{n+1}=0,\dots,q_{2n}=0\}.$$

Hence, for all $u_{\mathbb{H}}, v_{\mathbb{H}} \in \xi$ we get

$$\begin{aligned}\langle Iu_{\mathbb{H}}, v_{\mathbb{H}}\rangle_{\mathbb{H}} &= \langle Ju_{\mathbb{H}}, v_{\mathbb{H}}\rangle_{\mathbb{H}} = \langle Ku_{\mathbb{H}}, v_{\mathbb{H}}\rangle_{\mathbb{H}}\\ &= \langle Iu_{\mathbb{H}}, Jv_{\mathbb{H}}\rangle_{\mathbb{H}} = \langle Iu_{\mathbb{H}}, Kv_{\mathbb{H}}\rangle_{\mathbb{H}} = \langle Ju_{\mathbb{H}}, Kv_{\mathbb{H}}\rangle_{\mathbb{H}} = 0. \ \blacksquare\end{aligned}$$

Recall the classical facts (see e.g. [12]):

LEMMA 3.1. *Suppose that $f_1,\dots,f_n$ are smooth, real-valued functions on an open set $\Omega \subset \mathbb{C}^n$ and suppose that $df_1,\dots,df_n$ are linearly independent at points of $M := \{z \in \Omega;\ f_1(z)=\dots=f_n(z)=0\}$. Then the submanifold M is Lagrangian if and only if all the Poisson brackets*

$$\{f_j,f_k\} := \sum_{l=1}^{n}\left(\frac{\partial f_j}{\partial x_l}\frac{\partial f_k}{\partial y_l} - \frac{\partial f_j}{\partial y_l}\frac{\partial f_k}{\partial x_l}\right) = 2i\sum_{l=1}^{n}\left(\frac{\partial f_j}{\partial \overline{z}_l}\frac{\partial f_k}{\partial z_l} - \frac{\partial f_j}{\partial z_l}\frac{\partial f_k}{\partial \overline{z}_l}\right)$$

vanish on M.

THEOREM 3.2. *Suppose that $F \in C^2(\Omega)$ with $\Omega^{\mathrm{open}} \subset \mathbb{R}^n$, $F:\Omega\to\mathbb{R}$. Let $f := \nabla F$ denote the gradient field and let M denote the graph of f in $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$. Then M (with the correct orientation) is Lagrangian if and only if*

$$\mathrm{Im}\{\det\nolimits_{\mathbb{C}}(I + i\,\mathrm{Hess}\,F)\} = 0. \tag{3.3}$$

REMARK 3.2. By straightforward calculations the condition (3.3) can be rewritten in the following "real" form:

- for $n = 1$: $(F_{xx} = 0)$ $\triangle F = 0$,
- for $n = 2$: $(F_{xx} + F_{yy} = 0)$ $\triangle F = 0$,
- for $n = 3$: $\triangle F = \det(\operatorname{Hess} F)$ (Monge–Ampère equation),
- for $n = 4$:

$$\triangle F = \det(\operatorname{Hess} F)_{123} + \det(\operatorname{Hess} F)_{234} + \det(\operatorname{Hess} F)_{134} + \det(\operatorname{Hess} F)_{124},$$

where

$$\det(\operatorname{Hess} F)_{123} := \det \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix},$$

$$\det(\operatorname{Hess} F)_{234} := \det \begin{pmatrix} F_{44} & F_{42} & F_{43} \\ F_{24} & F_{22} & F_{23} \\ F_{34} & F_{32} & F_{33} \end{pmatrix} = \det \begin{pmatrix} F_{22} & F_{23} & F_{24} \\ F_{32} & F_{33} & F_{34} \\ F_{42} & F_{43} & F_{44} \end{pmatrix},$$

$$\det(\operatorname{Hess} F)_{134} := \det \begin{pmatrix} F_{11} & F_{14} & F_{13} \\ F_{41} & F_{44} & F_{43} \\ F_{31} & F_{34} & F_{33} \end{pmatrix} = \det \begin{pmatrix} F_{11} & F_{13} & F_{14} \\ F_{31} & F_{33} & F_{34} \\ F_{41} & F_{43} & F_{44} \end{pmatrix},$$

$$\det(\operatorname{Hess} F)_{124} := \det \begin{pmatrix} F_{11} & F_{12} & F_{14} \\ F_{21} & F_{22} & F_{24} \\ F_{41} & F_{42} & F_{44} \end{pmatrix}.$$

The quaternionic version of Theorem 3.2 looks as follows:

THEOREM 3.3. *Suppose that* $F \in C^2(\Omega)$ *with* $\Omega^{\text{open}} \subset \mathbb{R}^n$, $n = 1, 2, 3, 4$ $(F : \Omega \to \mathbb{R})$. *Let* $G := \frac{1}{\sqrt{3}} F$ *and* $g := \nabla G$ *denote the gradient field and let* M_g *denote the graph of* g *in* $\mathbb{H}^n = \mathbb{R}^n \oplus i\mathbb{R}^n \oplus j\mathbb{R}^n \oplus k\mathbb{R}^n$, $n = 1, 2, 3, 4$. *Then* M_g (*with the correct orientation*) *is Lagrangian if and only if*

$$\begin{aligned} (3.4) \qquad & \operatorname{Im}_i\{\det \mathbb{H}(I + i \operatorname{Hess} G + j \operatorname{Hess} G + k \operatorname{Hess} G)\} \\ &= \operatorname{Im}_j\{\det \mathbb{H}(I + i \operatorname{Hess} G + j \operatorname{Hess} G + k \operatorname{Hess} G)\} \\ &= \operatorname{Im}_k\{\det \mathbb{H}(I + i \operatorname{Hess} G + j \operatorname{Hess} G + k \operatorname{Hess} G)\} = 0. \end{aligned}$$

In real variables the condition (3.4) *looks as follows*:

- *for* $n = 1$: $\triangle G = \triangle F = 0$,
- *for* $n = 2$: $\triangle G = \triangle F = 0$,
- *for* $n = 3$: $\triangle G = \det(\operatorname{Hess} G)$, $\triangle F = 3\det(\operatorname{Hess} F)$,
- *for* $n = 4$:

$$\triangle G = \det(\operatorname{Hess} G)_{123} + \det(\operatorname{Hess} G)_{234} + \det(\operatorname{Hess} G)_{134} + \det(\operatorname{Hess} G)_{124},$$

$$\triangle F = 3[\det(\operatorname{Hess} F)_{123} + \det(\operatorname{Hess} F)_{234} + \det(\operatorname{Hess} F)_{134} + \det(\operatorname{Hess} F)_{124}].$$

Proof. This follows by straightforward calculations. ∎

V.4. Ellipticity. Suppose that F is a real-valued function defined on a domain $\Omega \subset \mathbb{R}^n$ and consider the gradient map $f := \nabla F : \Omega \to \mathbb{R}^n$, $f = \big(\frac{\partial F}{\partial x_1}(x), \ldots, \frac{\partial F}{\partial x_n}(x)\big)$ with the Jacobian matrix $f_* = F_{**} = \operatorname{Hess} F = \big(\frac{\partial^2 F}{\partial x_i \partial x_j}\big)$ at each point.

Assume that the following relations are satisfied:

$$
\begin{aligned}
&\operatorname{Re}\{\det_{\mathbb{H}}(I + if_* + jf_* + kf_*)\} > 0,\\
&\operatorname{Im}_i\{\det_{\mathbb{H}}(I + if_* + jf_* + kf_*)\} = 0,\\
&\operatorname{Im}_j\{\det_{\mathbb{H}}(I + if_* + jf_* + kf_*)\} = 0,\\
&\operatorname{Im}_k\{\det_{\mathbb{H}}(I + if_* + jf_* + kf_*)\} = 0.
\end{aligned}
\tag{4.1}
$$

The first inequality determines the appropriate orientation.

Consider a scalar function U on Ω and set $u := \nabla U : \Omega \to \mathbb{R}$. Then

$$u_* = U_{**} = \operatorname{Hess} U.$$

Assuming that F is a given solution of (4.1) we consider the linearized operators defined as follows:

$$
\begin{aligned}
L_F^i &:= \operatorname{Im}_i \frac{d}{dt}\{\det_{\mathbb{H}}[I + i(f_* + tu_*) + j(f_* + tu_*) + k(f_* + tu_*)]\}_{|t=0},\\
L_F^j &:= \operatorname{Im}_j \frac{d}{dt}\{\det_{\mathbb{H}}[I + i(f_* + tu_*) + j(f_* + tu_*) + k(f_* + tu_*)]\}_{|t=0},\\
L_F^k &:= \operatorname{Im}_k \frac{d}{dt}\{\det_{\mathbb{H}}[I + i(f_* + tu_*) + j(f_* + tu_*) + k(f_* + tu_*)]\}_{|t=0}
\end{aligned}
$$

on all such functions U. For simplicity, set

$$A := I + if_* + jf_* + kf_*$$

and observe that

$$
\begin{aligned}
\det_{\mathbb{H}}(A + itu_* + jtu_* + ktu_*) &= \det_{\mathbb{H}}\{A[I + (it + jt + kt)]A^{-1}u_*\}\\
&= (\det_{\mathbb{H}} A)(\det_{\mathbb{H}}[I + (it + jt + kt)A^{-1}u_*]).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\frac{d}{dt}\det_{\mathbb{H}}(A + itu_* + jtu_* + ktu_*)_{|t=0} &= (\det_{\mathbb{H}} A)(\det_{\mathbb{H}}[I + (it + jt + kt)A^{-1}u_*]_{|t=0})\\
&= (\det_{\mathbb{H}} A)\operatorname{Tr}[(i + j + k)A^{-1}u_*]\\
&= \operatorname{Tr}[(i + j + k)A^{-1}(\det_{\mathbb{H}} A)u_*].
\end{aligned}
$$

Define

$$A^* := A^{-1}\det A.$$

Then

$$\operatorname{Tr}[(i + j + k)A^{-1}(\det A)u_*] = \operatorname{Tr}[(i + j + k)A^* u_*].$$

Since u_* is a real $n \times n$ matrix, we have

$$\operatorname{Im}_{i,j,k}\{\operatorname{Tr}[(i + j + k)A^* u_*]\} = \operatorname{Tr}\{\operatorname{Im}_{i,j,k}\ [(i + j + k)A^* u_*]\} = \operatorname{Tr}[\operatorname{Re}(A^* u_*)].$$

Hence the linearization can be written as

$$L_F^i(U) = L_F^j(U) = L_F^k(U) := L_F(U), \qquad L_F(U) := \operatorname{Tr}[\operatorname{Re}(A^* u_*)] = \operatorname{Tr}[\operatorname{Re}(a^* U_{**})]$$

and the inequality from (4.1) can be expressed as $\det A > 0$.

Observe that L_F $(= \mathrm{Tr}[\mathrm{Re}(A^* U_{**})])$ is elliptic if and only if the matrix $\mathrm{Re}\, A^*$ is positive definite.

However, after an appropriate orthogonal change of basis the symmetric matrix

$$f_* = F_{**} = \mathrm{Hess}\, F = \left(\frac{\partial^2 F}{\partial x_i \partial x_j} \right)$$

becomes diagonal and we can write

$$A = \begin{pmatrix} 1 + i\lambda_1 + j\lambda_1' + k\lambda_1'' & 0 & 0 & \dots & 0 \\ 0 & 1 + i\lambda_2 + j\lambda_2' + k\lambda_2'' & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 + i\lambda_n + j\lambda_n' + k\lambda_n'' \end{pmatrix}.$$

Then

$$\begin{aligned} A^* &= A^{-1} \det A \\ &= \begin{pmatrix} \frac{1}{1+i\lambda_1+j\lambda_1'+k\lambda_1''} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{1+i\lambda_2+j\lambda_2'+k\lambda_2''} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{1}{1+i\lambda_n+j\lambda_n'+k\lambda_n''} \end{pmatrix} \det A. \end{aligned}$$

Since $\det A > 0$, we obtain

$$\mathrm{Re}\, A^* = \begin{pmatrix} \frac{1}{1+\lambda_1^2+(\lambda_1')^2+(\lambda_1'')^2} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{1+\lambda_2^2+(\lambda_2')^2+(\lambda_2'')^2} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{1}{1+\lambda_n^2+(\lambda_n')^2+(\lambda_n'')^2} \end{pmatrix} \det A.$$

Thus we proved the following

THEOREM 4.1. *The linearization of the special Lagrangian operator at any solution F of the system*

$$\mathrm{Re}\{\det\nolimits_{\mathbb{H}} \, (I + if_* + jf_* + kf_*)\} > 0,$$
$$\mathrm{Im}_{i,j,k}\{\det\nolimits_{\mathbb{H}} \, (I + if_* + jf_* + kf_*)\} = 0$$

is a homogeneous second order elliptic operator

$$L_F(U) := \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 U}{\partial x^i \partial x^j},$$

where $(a^{ij}(x))$ is a positive definite symmetric matrix at each point.

V.5. Homotopic quaternionic mappings. Let us recall

DEFINITION 5.1. Let (M, g) be a $4n$-dimensional Riemannian manifold. An *almost quaternionic structure* on M is defined as a covering $\{U_i\}$ of the manifold with two almost complex structures I_i and J_i such that $I_i J_i = -J_i I_i$ and the 3-dimensional vector spaces of endomorphisms generated by I_i, J_i and $K_i := I_i J_i$:

$$\mathrm{End}_{U_i} := \{\alpha I_i + \beta J_i + \gamma K_i;\ \alpha, \beta, \gamma \in \mathbb{R}\}$$

are the same on the whole manifold.

Moreover, a Riemannian metric g is *quaternionic-Hermitian* if g is Hermitian for each I and J.

DEFINITION 5.2. The *standard enhanced quaternionic structure* of $\mathbb{H}^n$ is a 3-dimensional subspace Q_0 of the space $\mathrm{End}_{\mathbb{R}}\, \mathbb{H}^n$ generated by (any) such triple $(I, J, K) := (I_1, I_2, I_3)$. We call (I_1, I_2, I_3) an *admissible hypercomplex base* of Q_0.

DEFINITION 5.3. Let $(I_1, I_2, I_3) \in Q_0$. Consider the 2-forms ω_1, ω_2 and ω_3 defined by

$$\omega_1(X,Y) := g(X, IY), \quad \omega_2(X,Y) := g(X, JY), \quad \omega_3(X,Y) := g(X, KY),$$

where X and Y are arbitrary C^∞-vector fields on M. Next, define

$$\Omega := \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3.$$

DEFINITION 5.4. Let (M^{4m}, g) and (N^{4n}, h) be almost-quaternionic-Hermitian manifolds and $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ a smooth map. Then Φ is called *Q-holomorphic* if for every $p \in M^{4m}$ and each hypercomplex base $(I_1', I_2', I_3') \in Q_p^M$ there exists a hypercomplex base $(I_1, I_2, I_3) \in Q_{\Phi(p)}^N$ such that

$$I_\alpha(\Phi_*)_p = (\Phi_*)_p I_\alpha', \quad \alpha = 1, 2, 3. \tag{5.1}$$

EXAMPLE 5.1. Any 4-dimensional, oriented, Riemannian manifold can be considered as an almost-quaternionic-Kähler manifold. A diffeomorphism $\Phi : M^4 \to M^4$ is Q-holomorphic iff it preserves the fixed orientation.

REMARK 5.1. Let (N^{4n}, h) be an almost-quaternionic-Hermitian (resp. Kähler) manifold and M^{4m} any smooth, orientable $4m$-dimensional (resp. $\leq 4n$-dimensional manifold). Suppose that $\Phi : M^{4m} \to N^{4n}$ is a smooth immersion and for every $p \in M^{4m}$ the space $\Phi_*(T_p M^{4m})$ is a quaternionic subspace of $T_{\Phi(p)} N^{4n}$.

Consider on M^{4m} the Riemannian metric $g := \Phi^* h$. Then there is a unique (natural) quaternionic structure Q^M on M^{4m} such that (M^{4m}, g) is an almost-quaternionic-Hermitian (resp. Kähler) manifold endowed with the fundamental 4-form $\Omega^M := \Phi^* \Omega^N$ and $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ is a Q-holomorphic map. The manifold $(M^{4m}, \Phi^* h)$ is called an *immersed almost-quaternionic submanifold* of (N^{4n}, h).

Now, we need the Lichnerowicz homotopy invariant $K(\Phi)$:

DEFINITION 5.5 ([23]). Let (M^{4m}, g) and (N^{4n}, h) be almost-quaternionic-Hermitian manifolds with $\dim_{\mathbb{R}} M^{4m} = 4m$ and $\dim_{\mathbb{R}} N^{4n} = 4n$. Suppose that M^{4m} is compact. Assume that $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ is a smooth mapping. Define

$$K(\Phi) := \int_M \langle \Omega_M, \Phi^* \Omega_N \rangle \, dV_g,$$

where Ω_M and Ω_N represent the fundamental 4-forms on M and N, respectively.

THEOREM 5.1 ([23, 24]). *Let (M^{4m}, g) and (N^{4n}, h) be two almost-quaternionic-Kähler manifolds (M^{4m} being compact). Suppose that $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ is a smooth map. Then $K(\Phi)$ is a smooth homotopy invariant.*

THEOREM 5.2 ([18]). *Let (M^{4m}, g) and (N^{4n}, h) be almost-quaternionic-Kähler manifolds. Suppose that $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ is a Q-holomorphic isometric mapping. Then*

$$K(\Phi) = 12m(2m+1)\mathrm{Vol}(M).$$

In particular Φ cannot be homotopic to a constant map.

Proof. Let $p \in M^{4m}$. Choose orthonormal bases of the form

$$(e_1, I'_1 e_1, I'_2 e_1, I'_3 e_1, \dots, e_m, I'_1 e_m, I'_2 e_m, I'_3 e_m),$$

and

$$(f_1, I_1 f_1, I_2 f_1, I_3 f_1, \dots, f_n, I_1 f_n, I_2 f_n, I_3 f_n),$$

in $T_p M^{4m}$ and $T_{\Phi(p)} N^{4n}$, respectively, where (I'_1, I'_2, I'_3) is a hypercomplex base of Q^M_p and (I_1, I_2, I_3) is a hypercomplex base of $Q^N_{\Phi(p)}$, and condition (5.1) holds.

Suppose that Φ is Q-holomorphic. It is clear that $\Omega^M = \Phi^* \Omega^N$ and $\langle \Omega^M, \Phi^* \Omega^N \rangle_p = \|\Omega^M\|^2_p$. Notice that the only components of Ω^M_p which are different from 0 are those that correspond (up to permutations) to the 4-ples of vectors:

$$(e_t, I_a e_t, e_s, I_a e_s), \quad (I_b e_t, I_c e_t, I_b e_s, I_c e_s) \quad \text{for } t, s = 1, \dots, m,\ t \neq s \tag{5.2}$$

and

$$(e_t, I_a e_t, I_b e_s, I_c e_s) \quad \text{for } t, s = 1, \dots, m \tag{5.3}$$

for any circular permutation (a, b, c) of $(1, 2, 3)$. It is easy to see that, up to permutations, there are $3m(m-1)$ different components of the type (5.2), $3m(m-1)$ different components of the type (5.3) with $t \neq s$ and m different components of the type (5.3) with $t = s$. By a simple calculation we get

$$\Omega(e_t, I_a e_t, e_s, I_a e_s) = \Omega(I_b e_t, I_c e_t, I_b e_s, I_c e_s) = \Omega(e_t, I_a e_t, I_b e_s, I_c e_s) = 2 \quad \text{for } t \neq s$$

and

$$\Omega(e_t, I_a e_t, I_b e_t, I_c e_t) = 6.$$

Since $\|\Omega^M\|^2_p = 3m(m-1)2^2 + 3m(m-1)2^2 + m6^2 = 12m(2m+1)$, by integrating we get the required formula. ∎

REMARK 5.2. Theorem 5.2, with a slight modification, holds when Φ is a conformal immersion.

Let us recall that 4-dimensional immersed quaternionic submanifolds M^4 of a quaternionic-Kähler manifold (N^{4n}, h), $n > 1$, are totally geodesic (see e.g. [33]) and semi-conformally flat. In the case when the scalar curvature of (N^{4n}, h) is positive the only possible types for compact M^4 are $\mathbb{HP}^1 \cong S^4$ and $\mathbb{CP}^2$ (see [26]).

COROLLARY 5.1. *Every immersed quaternionic submanifold of (N^{4n}, h) which is isometric to $\mathbb{HP}^1$ defines a non-trivial element in the group $\pi_4(N^{4n})$.*

REMARK 5.3. The above fact was well known in the case when (N^{4n}, h) is the quaternionic projective space $\mathbb{HP}^n$ and $\Phi : \mathbb{HP}^1 \to \mathbb{HP}^n$ is a canonical immersion of a quaternionic projective space (see [36, p. 30]).

Let us introduce yet the following

Definition 5.6. Let (N^{4n}, h) be an almost-quaternionic-Hermitian manifold an M^{4m} any smooth $4m$-dimensional manifold. Suppose that $\Phi : M^{4m} \to N^{4n}$ is a smooth immersion. We will say that (M^{4m}, Φ) is an *immersed Lagrangian submanifold* of (N^{4n}, h) if at every point $p \in M^{4m}$ and for any hypercomplex base $(I_1, I_2, I_3) \in Q^N_{\Phi(p)}$ the four subspaces $\Phi_*(T_pM^{4m})$, $I_\alpha \Phi_*(T_pM^{4m})$, $\alpha = 1, 2, 3$, are totally orthogonal in $T_{\Phi(p)}N^{4n}$.

Then, we have

Theorem 5.3 ([18]). *Let (N^{4n}, h) be an almost-quaternionic-Kähler manifold. Suppose that M^{4m} is a compact, oriented, $4m$-dimensional manifold $(m < n)$. Let $\Phi_1 : M^{4m} \to N^{4n}$ and $\Phi_2 : M^{4m} \to N^{4n}$ be two immersions such that (M^{4m}, Φ_1) and (M^{4m}, Φ_2) are almost-quaternionic and Lagrangian, respectively. Then Φ_1 and Φ_2 cannot be homotopic.*

Proof. By Remark 5.1 we can consider M^{4m} as an almost-quaternionic-Kähler manifold with the Riemannian metric $g := \Phi_1^* h$ and the almost quaternionic structure Q^M naturally induced by Φ_1. By Definition 5.6 we get $\Phi_2^*(\Omega^N) = 0$. Hence $K(\Phi_2) = 0$. On the other hand, by Remark 5.1 and Theorem 5.2 we have $K(\Phi_1) \neq 0$. Then the statement follows by Theorem 5.1. ∎

References

[1] M. Berger, *Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés Riemanniennes*, Bull. Soc. Math. France 83 (1955), 279–330.

[2] —, *Remarques sur le groupe d'holonomie des variétés Riemanniennes*, C. R. Acad. Sci. Paris 262 (1966), 1316–1318.

[3] A. L. Besse, *Einstein Manifolds*, Ergeb. Math. Grenzgeb. 10, Springer, Berlin, 1987.

[4] —, *Manifolds All of Whose Geodesics are Closed*, Ergeb. Math. Grenzgeb. 13, Springer, 1978.

[5] E. Bonan, *Sur les G-structures de type quaternionien*, Cahiers Top. Géom. Diff. 9 (1967), 389–461.

[6] F. Brackx, R. Delanghe and F. Sommen, *Clifford Analysis*, Res. Notes in Math. 75, Pitman, Boston, 1982.

[7] G. E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, 1972.

[8] S. S. Chern, *On a generalization of Kähler geometry*, in: Algebraic Geometry and Topology, A Symposium in honor of S. Lefschetz, Princeton, NJ, 1957, 103–121.

[9] R. Delanghe, F. Sommen and V. Souček, *Clifford Analysis and Spinor Valued Functions*, Kluwer, Dordrecht, 1992.

[10] R. Fueter, *Die Funktionentheorie der Differentialgleichungen $\Delta u = 0$ und $\Delta\Delta u = 0$ mit vier reellen Variablen*, Comment. Math. Helv. 7 (1935), 307–330.

[11] —, *Über die analytische Darstellung der regulären Funktionen einer Quaternionenvariablen*, ibid. 8 (1936), 371–378.

[12] R. Harvey and H. B. Lawson (Jr.), *Calibrated geometries*, Acta Math. 148 (1982), 47–157.

[13] W. V. D. Hodge, *The Theory and Applications of Harmonic Integrals*, Cambridge Univ. Press, New York, 1952.

[14] K. Imaeda, *Quaternionic formulation of classical electrodynamics and theory of functions of a biquaternion variable*, Report of Fundamental Physics Laboratory, Department of Electronic Science, Faculty of Science, Okayama University of Science, 1983.

[15] W. Klingenberg, *Manifolds with restricted conjugate locus*, Ann. of Math. 78 (1963), 537–547.

[16] —, *Manifolds with restricted conjugate locus II*, ibid. 80 (1964), 330–339.

[17] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vols. I–II, Interscience, 1963.

[18] W. Królikowski, *On Fueter–Hurwitz regular mappings*, Dissertationes Math. 353 (1996).

[19] —, *A Clifford-type structure*, Demonstratio Math. 36 (2003), 563–578.

[20] —, *Quaternionic condition for the existence of* 4*-dimensional locally conformally flat almost Kähler manifolds*, ibid. 39 (2006).

[21] —, *On some singularity of a hypercomplex analysis*, Ann. Univ. Sci. Budapest. Sectio Math. 48 (2006).

[22] W. Królikowski and S. Marchiafava, *On Lichnerowicz smooth homotopy invariants for G-structures*, Note Mat. 13 (1993), 197–212.

[23] A. Lichnerowicz, *Applications harmoniques et variétés Kähleriennes*, Rend. Sem. Mat. Fis. Milano 39 (1969), 186–195.

[24] —, *Applications harmoniques et variétés Kähleriennes*, in: Sympos. Math. III, Academic Press, 1970, 341–402.

[25] S. Marchiafava and G. Romani, *Sur fibrati con struttura quaternioniale generalizzatta*, Ann. Mat. Pura Appl. 107 (1076), 131–157.

[26] —, —, *Su alcune sottovarietà che ha interesse considerare in una varietà Kähleriana quaternionale*, Rend. Mat. 10 (1990), 493–529.

[27] E. Martinelli, *Variétés à structure quaternionienne généralisées*, Rev. Roumaine Math. Pures Appl. 10 (1965), 915–922.

[28] K. Nomizu, *Invariant affine connections in homogeneous spaces*, Amer. J. Math. 76 (1954), 33–65.

[29] H. Rauch, *The global study of geodesics in symmetric and nearly symmetric Riemannian manifolds*, Comment. Math. Helv. 35 (1961), 111–125.

[30] S. Salamon, *Riemannian Geometry and Holonomy Groups*, Longman, 1989.

[31] V. Souček, *Holomorphicity in quaternionic analysis*, in: Seminari di Geometria 1982–1983, Univ. di Bologna, 1984, 147–153.

[32] A. Sudbery, *Quaternionic analysis*, Math. Proc. Cambridge Philos. Soc. 85 (1979), 199–225.

[33] H. Tasaki, *Quaternionic submanifolds in quaternionic symmetric spaces*, Tôhoku Math. J. 38 (1986), 513–538.

[34] A. Weil, *Introduction à l'étude des variétés Kähleriennes*, Hermann, Paris, 1958.

[35] J. A. Wolf, *Complex homogeneous contact manifolds and quaternionic symmetric spaces*, J. Math. Mech. 14 (1965), 1033–1047.

[36] W. T. Wu, *Sur les classes caractéristiques des structures fibrées sphériques*, Publ. Inst. Math. Univ. Strasbourg 11, Hermann, Paris, 1952.

Erratum to the paper
"On Fueter–Hurwitz regular mappings"
by Wiesław Królikowski

(Dissertationes Mathematicae 353 (1996))

Theorem 2.6.2 on page 58 is false. To obtain a correct version one has to add the following assumption:

$$\mathrm{Re}[D(G_m - i_m G_0)] = 0,$$

where

$$G_m := \langle \Phi_m, \Phi_0 \rangle + i\langle \Phi_m, \Phi_1 \rangle + j\langle \Phi_m, \Phi_2 \rangle + k\langle \Phi_m, \Phi_3 \rangle,$$
$$m = 0, 1, 2, 3, \qquad i_0 = 1,\ i_1 = i,\ i_2 = j,\ i_3 = k,$$

i, j, k are the quaternionic units and the operator D is defined by

$$D := \tfrac{1}{4}(\partial_0 - i\partial_1 - j\partial_2 - k\partial_3).$$

The correct version and its proof are contained in my book *The Fueter–Hurwitz Operator and a Clifford-type Structure* submitted to Springer Lecture Notes.