1. Introduction

We solve the global Cauchy problem for the Dirac equation in Sobolev and weighted Sobolev spaces. This is first done on classes of globally hyperbolic asymptotically flat space-times with weak regularity and fall-off at infinity. The theorems are proved for general first order symmetric hyperbolic systems and then applied to Dirac's equation. They are also valid for space-times compact in space or admitting several asymptotically flat ends. Then we consider in some detail the Schwarzschild and Kerr black holes. They can be described in such a way that the previous theorems are immediately applicable. We also choose to consider them from the point of view of an observer static at infinity. The horizon then appears as a boundary for the spacelike geometry. We prove similar theorems in this situation.

A space-time is a pair $(\mathcal{M}, g)$ where $\mathcal{M}$ is a four dimensional manifold without boundary and g is a Lorentzian metric, i.e. a symmetric two-form on $\mathcal{M}$ with signature $(+---)$. All the space-times we consider are globally hyperbolic. The notion of global hyperbolicity is naturally required for the concept of Cauchy problem to make sense. To be more precise, let us consider some relativistic field equation (E) on a space-time $(\mathcal{M}, g)$; the Cauchy problem can be formulated as follows: *prove that if we specify the values of the field at some initial time* t_0, *then the solution of* (E) *can be propagated continuously in time*, *from these initial data*, *onto the whole space-time and the field is thus everywhere uniquely determined by its values at time* t_0. There are two implicit assumptions in this formulation: firstly, there exists a time function t globally defined on $\mathcal{M}$, secondly, the information contained in the hypersurface $\Sigma_{t_0} := \{t = t_0\}$ can be propagated via the field equation on the whole space-time. Equation (E) being relativistic, this means that any point of $\mathcal{M}$ can be reached from Σ_{t_0} along a non-spacelike curve (in fact timelike since $\mathcal{M}$ is assumed to have no boundary). The existence of such a surface, called a Cauchy hypersurface, is equivalent to global hyperbolicity.

Jean Leray introduced the notion of global hyperbolicity in 1952 (see [41]). In 1970, Robert Geroch [22] showed that Leray's definition is equivalent to the existence of a Cauchy hypersurface. The work of Geroch establishes that a globally hyperbolic space-time $(\mathcal{M}, g)$ has a very precise structure:

- it admits a globally defined time function t,
- the level hypersurfaces Σ_t of t define a foliation of $\mathcal{M}$, all Σ_t are spacelike Cauchy hypersurfaces and are homeomorphic to a given 3-manifold Σ.

Hence $(\mathcal{M}, g)$ possesses two orthogonal foliations: $\{\Sigma_t\}_t$ and the congruence of the integral lines of t^a, the unit timelike future-oriented vector field normal to the Σ_t. This endows $\mathcal{M}$ with a product structure $\mathcal{M} \simeq \mathbb{R}_t \times \Sigma$. The metric g and covariant derivative ∇ can also be decomposed into timelike and spacelike parts by projecting them along the two foliations. Field equations on $(\mathcal{M}, g)$ are then naturally expressed as evolution equations on $\mathbb{R}_t \times \Sigma$. This is the principle of the $3+1$ decomposition of a space-time: it is nothing but the complete use of the structure of globally hyperbolic space-times. This $3+1$ decomposition is also referred to as the ADM decomposition because of the way Arnowitt, Deser and Misner formalized and used it to obtain a Hamiltonian formulation of general relativity (see R. Arnowitt, S. Deser, C. W. Misner [1] for a review of their work, see also C. W. Misner, K. Thorne, J. A. Wheeler [44], Chapter 21).

Another important property of globally hyperbolic space-times is that they admit a spin structure ([1]). This is a consequence of the product structure $\mathcal{M} \simeq \mathbb{R}_t \times \Sigma$ and of the fact that Σ is 3-dimensional and therefore parallelizable if orientable (see E. Stiefel 1936 [59] and R. P. Geroch 1968 [20] and 1970 [21]). In 1981 and 1982, A. Sen ([56] and [57]) described the $3+1$ decomposition of spinor field equations on globally hyperbolic space-times using Penrose's abstract indices and two-spinor formalism. He applied this technique to obtain formulations as evolution equations of the neutrino equation, the spinor form of Maxwell's equations and the Dirac form of spin 3/2 massless field equations. An important feature of Sen's work is the use of the vector field t^a: it provides a natural embedding of the restriction to a hypersurface Σ_t of the $SL(2, \mathbb{C})$ spin bundle of $\mathcal{M}$ into the $SU(2)$ spin bundle intrinsic to the geometry of Σ_t. The spin bundle on $\mathcal{M}$ thus inherits a hermitian structure. The quantity conserved by the evolution for a spinor field equation (Dirac and spin 3/2) is the L^2 norm of the solution on Σ_t induced by this hermitian structure.

The present work is entirely based on such $3+1$ decomposition techniques. It is organized in five parts:

Chapter 2. We recall in details the $3+1$ decomposition of globally hyperbolic space-times, leaving the purely spinorial aspects until the next chapter. In a recent contribution in collaboration with L. J. Mason [42], we used this decomposition to define in terms of weighted Sobolev spaces some classes of globally hyperbolic asymptotically flat space-times. We reformulate these definitions using the numbering of weighted Sobolev spaces proposed by Robert Bartnik [5] which has the advantage of indicating explicitly the rate of fall-off at spacelike infinity.

Chapter 3. In the literature, one finds essentially two ways of expressing the Dirac equation: one in terms of Dirac spinors and Clifford products by the vectors of a Lorentz frame (which are interpreted as multiplications by Dirac matrices via a choice of spin-frame) and the other using the two-spinor formalism and abstract indices. We describe Dirac's equation and its 3+1 decomposition on globally hyperbolic space-times using each

([1]) We have chosen to define space-times as four dimensional manifolds endowed with a Lorentzian metric. One can perfectly well consider a space-time $\mathcal{M}$ of dimension $n+1$, for any positive integer n, with a metric of signature $(+\ -\dots -)$. Although global hyperbolicity will always guarantee the existence of a product structure $\mathcal{M} \simeq \mathbb{R}_t \times \Sigma$, it will not in general entail the existence of a spin structure. For example, if Σ is the n-sphere, it is only parallelizable for $n = 1, 3$ and 7.

of the two formalisms. We emphasize as often as possible the link between the fundamental structures of each framework. In particular, the expression of Dirac's equation in the two-spinor formalism, when translated into the language of Dirac spinors, corresponds to a particular form of Dirac matrices. We derive their general expression; for a natural choice of spin-frame, this gives one of the standard choices for Dirac matrices. After performing the $3+1$ decomposition of the Dirac equation, we show it is a first order symmetric hyperbolic system on $\mathbb{R}_t \times \Sigma$; we express its spacelike part in terms of the Dirac operator on the leaves Σ_t of the foliation and of the extrinsic curvature of the Σ_t. We also express the current vector using two-spinors and Dirac spinors. Proving that it is divergence-free is particularly simple using the two-spinor formalism.

Chapter 4. On the classes of asymptotically flat space-times defined in Chapter 2, we solve the global Cauchy problem for Dirac fields in Sobolev and weighted Sobolev spaces. Regular solutions to Dirac's equation have been studied on curved space-times by J. Dimock [16] in 1982 and more recently by A. DeVries [14], [15] with applications to Kerr–Newman metrics. We adopt here a different approach, centred on minimum regularity: the fundamental result is the existence and uniqueness of solutions to the Cauchy problem in L^2; the use of identifying operators or successive differentiations of the equation then allows us to infer the stability of Sobolev and weighted Sobolev spaces under the evolution. This type of analytic study of the Cauchy problem, based on a $3+1$ decomposition of space-time, is akin to the techniques used in F. Cagnac and Y. Choquet-Bruhat [6], Y. Choquet-Bruhat [8], Y. Choquet-Bruhat, D. Christodoulou and M. Francaviglia [10] and subsequent contributions by Y. Choquet-Bruhat and co-workers. Theorems 1 and 2 establish the well-posedness of the Cauchy problem in L^2 and Sobolev spaces for $\mathcal{C}^k$ metrics satisfying assumptions slightly broader than asymptotic flatness. Theorem 3 deals with the existence and uniqueness of solutions in weighted Sobolev spaces for classes of asymptotically flat space-times and the conservation of the charge of Dirac fields is proved in Theorem 4. The first three theorems are consequences of much more general results (Propositions 4.1 and 4.2) valid for large families of first order symmetric hyperbolic systems with weakly regular coefficients. These two propositions extend the results of [42] in three important ways:

• They authorize to work with less regular metrics: the difference with a metric, flat outside a compact set in space, is only required to be in H^3 on each spacelike slice.

• The maximum regularity allowed for the solutions is improved: H^{k-1} for a metric in H^k. In [42], three degrees of regularity were lost between the metric and the solution. This was due to the use of Sobolev embedding theorems. Product theorems between Sobolev spaces enable us to lose only one degree of regularity. One cannot expect to improve this because of the presence of connection terms in equations such as Dirac, Maxwell, Rarita–Schwinger or Bianchi.

• They establish the well-posedness in weighted Sobolev spaces of the Cauchy problem for general symmetric hyperbolic systems. This gives us a control on the behaviour of the solutions at spacelike infinity at each time. As far as we are aware, no such result can be found in the literature, even for smooth metrics.

The proof of Propositions 4.1 and 4.2 relies essentially on the theory of abstract evolution systems. The research in this domain was initiated by T. Kato in 1953 [30] and some important contributors have since been M. Da Prato and M. Iannelli [13], T. J. R. Hughes, T. Kato and J. E. Marsden [28], S. Ishii [29], T. Kato [32, 33, 34, 35, 36, 37], T. Kato and H. Tanabe [38], F. J. Massey III [43], N. Okazawa and A. Unai [49], J. Prüss [54], H. Tanabe [60, 61] and K. Yosida [65] (this list of references has no claim to being exhaustive, for example we have almost essentially considered the hyperbolic case; for a comprehensive list of references in the parabolic case, see for example D. Daners and P. Koch Medina [12]). We do not need the latest refinements of this theory, references [32], [33] and [34] are sufficient, together with adequate choices of identifying operators, to prove the propositions when the topology of space-time is trivial. We take advantage of the finite propagation speed to extend the results to nontrivial topology.

Chapters 5 and 6. The Schwarzschild and Kerr space-times are asymptotically flat solutions to the Einstein vacuum equations describing respectively a spherically symmetric black hole and a rotating black hole. The symmetry of the Schwarzschild solution allows one to adapt to this geometry some standard methods of the analysis of hyperbolic equations in flat space. In particular, the development of time dependent scattering theories for linear fields outside a Schwarzschild black hole has been the subject of numerous studies: see A. Bachelot [2], [3], A. Bachelot and A. Motet-Bachelot [4], J. Dimock [17], J. Dimock and B. S. Kay [18] and the author [47] (for other analytic studies of linear and nonlinear equations, see also A. Motet-Bachelot [45] and the author [46], [48]). The geometry of the Kerr solution is more complex and only one time dependent scattering construction is known to date (see D. Häfner [23]). The first step is to choose a way of describing the geometry of the black hole. This can either be guided by the type of information one wishes to obtain, or imposed by the analytic techniques one uses. Both are true in the case of the scattering theories referred to above. The point of view they all adopt is that of an observer static with respect to infinity. Hence, only the exterior of the black hole is considered and it is described using Schwarzschild or Boyer–Lindquist coordinates respectively. The first reason for such a choice lies in the history and nature of scattering theory: its purpose has always been to study how a distant observer perceives the influence of an object on the propagation of fields and to decide whether the information collected by such observers can be used to describe the object completely. If experimental measurements of the scattering of fields by a black hole are to be performed, it must be by good approximations of observers static at infinity (like gravitational wave detectors on Earth). A second and equally important reason is the analytic convenience of working with an explicit coordinate system in which the coefficients of the equations do not depend on time. This time independence is in fact almost compulsory since time dependent scattering theory relies heavily on the existence of a unitary propagator on a fixed Hilbert space. We choose two different ways of describing the geometry of the black hole:

• First we adopt the point of view of an observer static with respect to infinity. Our purpose is to describe as precisely as possible the implications of such a choice for the analysis of field equations, more particularly as regards the functional framework for the

Cauchy problem. The exterior of the black hole is globally hyperbolic and we perform a $3+1$ decomposition of the geometry using the time function t of the Schwarzschild or Boyer–Lindquist coordinates respectively (this decomposition is of course completely trivial in the Schwarzschild case but not for the Kerr metric). The horizon then appears as a smooth boundary for the spacelike slices and the lapse function of the space-time metric vanishes there. Thus, describing the exterior of the black hole by means of the time function t yields a decomposition of the metric which differs from the general decomposed form for the classes of asymptotically flat space-times of Chapter 2. This is essentially due to the fact that the Killing vector field $\partial/\partial t$ is not uniformly timelike outside the black hole but becomes null at the horizon. The practical consequence is that the theorems of Chapter 4 cannot be directly applied here. We have to define function spaces on the spacelike slices which take the boundary into account. A natural choice is to use Sobolev and weighted Sobolev spaces with zero traces on the horizon; this corresponds to the physical property that no field comes out of the black hole. Analyzing in detail the geometry of the slices and of the spacelike part of the Dirac equation, we extend the theorems of Chapter 4 to the exterior of Schwarzschild and Kerr black holes. In both cases, the crucial step is to show that the successive domains of the spacelike Dirac operator are the Sobolev spaces and that the norms are equivalent. We also verify the well-known property: the L^2 norm of Dirac fields (massive or not) is conserved under the evolution. The results obtained here concerning the well-posedness of the Cauchy problem for Dirac fields in Sobolev and weighted Sobolev spaces are new both for Schwarzschild and Kerr space-times. Note that the description of the exterior of a Kerr black hole adopted here, based on the $3+1$ decomposition of the geometry induced by the time function t of the Boyer–Lindquist coordinates, is often referred to as the point of view of locally nonrotating observers (see for example C. W. Misner, K. Thorne and J. A. Wheeler [44] or R. M. Wald [63]). It does away with the time/space cross terms in the metric and the rotation of space-time simply appears via the extrinsic curvature of the slices. This gives us a framework for studying evolution systems which is much more agreable than Boyer–Lindquist coordinates (recall that we use here only the time coordinate t of the Boyer–Lindquist coordinate system).

• We adopt a second, more global point of view. It is easy to see, using Kruskal–Szekeres coordinates or Kerr coordinates respectively, that the horizon of the black hole is not a singularity of space-time but a regular null hypersurface. The metric can be extended smoothly accross it and we consider maximal analytic extensions of Schwarzschild and Kerr space-times. For a natural choice of foliation, we show that the theorems of Chapter 4 can be applied in this framework.

In a first appendix, we detail some technical aspects of the choice of a spin-frame adapted to the exterior of Schwarzschild and Kerr black holes described using the time function t. We also give the calculation of the timelike connection terms appearing in Dirac's equation for this choice of spin-frame. A second appendix contains a possible way of expressing the Dirac equation on the Kerr metric and of writing it in the form of an evolution system. Although we have chosen in this work to use a more intrinsic form of Dirac's equation, we give this analytic formulation for completeness.

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2. Geometrical and functional framework

2.1. Notations. Many of our equations will be expressed using the two-component spinor notations and abstract index formalism of R. Penrose and W. Rindler [53].

Abstract indices are denoted by light face Latin letters, capital for spinor indices and lower case for tensor indices. Abstract indices are a notational device for keeping track of the nature of objects in the course of calculations, they do not imply any reference to a coordinate basis, all expressions and calculations involving them are perfectly intrinsic. For example, g_{ab} will refer to the space-time metric as an intrinsic symmetric tensor field of valence $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, i.e. a section of $\mathrm{T}^*\mathcal{M} \odot \mathrm{T}^*\mathcal{M}$ and g^{ab} will refer to the inverse metric as an intrinsic symmetric tensor field of valence $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, i.e. a section of $\mathrm{T}\mathcal{M} \odot \mathrm{T}\mathcal{M}$ (where $\odot$ denotes the symmetric tensor product, $\mathrm{T}\mathcal{M}$ the tangent bundle to our space-time manifold $\mathcal{M}$ and $\mathrm{T}^*\mathcal{M}$ its cotangent bundle).

Concrete indices defining components in reference to a basis are represented by bold face Latin letters. Concrete spinor indices, denoted by bold face capital Latin letters, take their values in $\{0, 1\}$ while concrete tensor indices, denoted by bold face lower case Latin letters, take their values in $\{0, 1, 2, 3\}$. Consider for example a basis of $\mathrm{T}\mathcal{M}$, that is a family of four smooth vector fields on $\mathcal{M}$: $\mathcal{B} = \{e_0, e_1, e_2, e_3\}$ such that at each point p of $\mathcal{M}$ the four vectors $e_0(p), e_1(p), e_2(p), e_3(p)$ are linearly independent, and the corresponding dual basis of $\mathrm{T}^*\mathcal{M}$: $\mathcal{B}^* = \{e^0, e^1, e^2, e^3\}$ such that $e^{\mathbf{a}}(e_{\mathbf{b}}) = \delta^{\mathbf{a}}_{\mathbf{b}}$, $\delta^{\mathbf{a}}_{\mathbf{b}}$ denoting the Kronecker symbol; $g_{\mathbf{ab}}$ will refer to the components of the metric g_{ab} in the basis $\mathcal{B}$: $g_{\mathbf{ab}} = g(e_{\mathbf{a}}, e_{\mathbf{b}})$ and $g^{\mathbf{ab}}$ will denote the components of the inverse metric g^{ab} in the dual basis $\mathcal{B}^*$, i.e. the 4×4 real symmetric matrices $(g_{\mathbf{ab}})$ and $(g^{\mathbf{ab}})$ are the inverse of one another. In the abstract index formalism, the basis vectors $e_{\mathbf{a}}$, $\mathbf{a} = 0, 1, 2, 3$, are denoted by $e_{\mathbf{a}}{}^a$ or $g_{\mathbf{a}}{}^a$. In a coordinate basis, the basis vectors $e_{\mathbf{a}}$ are coordinate vector fields and will also be denoted by $\partial_{\mathbf{a}}$ or $\partial/\partial x^{\mathbf{a}}$; the dual basis covectors $e^{\mathbf{a}}$ are coordinate 1-forms and will be denoted by $\mathrm{d}x^{\mathbf{a}}$.

Brackets on each side of a group of indices denote symmetrization and square brackets correspond to skew-symmetrization.

The indexed 1-form $\mathrm{d}x^a \in \mathrm{T}^*\mathcal{M} \otimes \mathbb{S}^A \otimes \mathbb{S}^{A'}$ and the indexed vector $\partial_a \in \mathrm{T}\mathcal{M} \otimes \mathbb{S}_A \otimes \mathbb{S}_{A'}$ (see below for the meaning of the notations $\mathbb{S}^A$, $\mathbb{S}^{A'}$, $\mathbb{S}_A$ and $\mathbb{S}_{A'}$) are used to suppress form and vector indices: $\mathrm{d}x^a$ maps the 1-form ω_a as an indexed quantity to the same 1-form $\omega = \omega_a \mathrm{d}x^a$ with its index suppressed, ∂_a maps the vector field V^a to the same vector field $V = V^a \partial_a$ with its index suppressed.

Most of the function spaces that we use in this work are defined in Section 2.3, where the general classes of asymptotically flat space-times are described. We define here some other spaces which will be useful to us but whose definition would not fit naturally in that section:

• given E and F two Banach spaces, $\mathcal{L}(E,F)$ denotes the Banach space of bounded linear operators from E to F;

• the notation $\mathcal{M}_n(\mathbb{C})$ refers to the space of complex $n\times n$ matrices;

• given a measure space (X,μ) and $p\geq 1$, $L^p(X,\mathrm{d}\mu)$ denotes the space of measurable functions (in fact of equivalence classes of measurable functions, two functions being equivalent if they are equal μ-almost everywhere) such that $|f|^p$ is μ-integrable over X. $L^p_{\mathrm{comp}}(X,\mathrm{d}\mu)$ is the subspace of compactly supported elements of $L^p(X,\mathrm{d}\mu)$ and $L^p_{\mathrm{loc}}(X,\mathrm{d}\mu)$ the space of (equivalence classes of) measurable functions such that $|f|^p$ is integrable over any compact subset of X;

• the function spaces $\mathcal{C}^k$, $\mathcal{C}^k_0$, $\mathcal{C}^k_b$, $\mathcal{C}^k_\delta$, H^k, H^k_{comp}, H^k_{loc}, H^k_δ and L^2_δ are defined in Section 2.3.

2.2. The principles of the 3+1 decomposition. We shall work on a smooth 4-manifold $\mathcal{M}$ equipped with a Lorentzian metric g with signature $(+\,-\,-\,-)$. $(\mathcal{M},g)$ is oriented, time-oriented and is also assumed to be globally hyperbolic. We denote by ∇_a the Levi-Civita connection on $(\mathcal{M},g)$. Global hyperbolicity implies (see Geroch [20]–[22]):

1. $(\mathcal{M},g)$ admits a spin structure and we choose one. $\mathcal{M}$ is then endowed with an $SL(2,\mathbb{C})$ principal bundle $P_\mathbb{S}$ of spin-frames. The bundle $\mathbb{S}$ of negative or anti-selfdual spinors (denoted $\mathbb{S}^A$ in the abstract index formalism) is given by

$$\mathbb{S}=P_\mathbb{S}\times_\varrho\mathbb{C}^2 \tag{2.1}$$

where ϱ is the standard representation of $SL(2,\mathbb{C})$ on $\mathbb{C}^2$; the bundle $\overline{\mathbb{S}}$ of positive or selfdual spinors (denoted $S^{A'}$ in the abstract index formalism) is

$$\overline{\mathbb{S}}=P_\mathbb{S}\times_{\overline{\varrho}}\overline{\mathbb{C}^2} \tag{2.2}$$

i.e. the complex structure in $\mathbb{S}$ is simply replaced by its opposite. The complex tangent bundle is $\mathrm{T}^a\mathcal{M}\otimes\mathbb{C}=\mathbb{S}^A\otimes\mathbb{S}^{A'}$. Hence, an abstract tensor index a is a couple of abstract spinor indices, one primed, the other unprimed, clumped together: $a=AA'$. The dual bundle $\mathbb{S}^*$ to $\mathbb{S}$ is denoted by $\mathbb{S}_A$ in the abstract index formalism and $\overline{\mathbb{S}}^*$ is denoted by $\mathbb{S}_{A'}$. The symplectic forms on $\mathbb{S}$ and $\overline{\mathbb{S}}$ are denoted respectively by ε_{AB} and $\varepsilon_{A'B'}$ and are referred to as the Levi-Civita symbols. ε_{AB} can be viewed as an isomorphism from $\mathbb{S}$ onto $\mathbb{S}^*$ which to κ^A associates $\kappa_A=\kappa^B\varepsilon_{BA}$. The inverse isomorphism, denoted by ε^{AB}, to κ_A associates $\kappa^A=\varepsilon^{AB}\kappa_B$. Similarly, $\varepsilon_{A'B'}$ and the corresponding $\varepsilon^{A'B'}$ can be regarded as lowering and raising devices for primed indices. The metric g is expressed in terms of the Levi-Civita symbols as $g_{ab}=\varepsilon_{AB}\varepsilon_{A'B'}$.

2. There exists a global "time function" t on $\mathcal{M}$. The level hypersurfaces Σ_t, $t\in\mathbb{R}$, of the function t define a foliation of $\mathcal{M}$, all Σ_t being Cauchy hypersurfaces and homeomorphic to a given smooth 3-manifold Σ. Geroch's theorem does not say anything about the regularity of the leaves Σ_t; the time function is only proved to be continuous and they

are thus simply understood as topological submanifolds of $\mathcal{M}$. H. P. Seifert [58] showed that the time function can be regularized and one can then understand the leaves Σ_t as $\mathcal{C}^k$ submanifolds of $\mathcal{M}$ if the metric is itself $\mathcal{C}^k$. For simplicity, we shall assume that the time function is smooth on $\mathcal{M}$ and the leaves are diffeomorphic to Σ. In such a case, the function t is indeed a smooth time coordinate on $\mathcal{M}$. It is constructed so as to be increasing along any nonspacelike future-oriented curve and the smoothness of t allows us to consider its gradient: $\nabla^a t$ is everywhere orthogonal to the level hypersurfaces Σ_t of t and is therefore everywhere timelike; it is also future-oriented. We identify $\mathcal{M}$ with the smooth manifold $\mathbb{R} \times \Sigma$ and consider g as a tensor valued function on $\mathbb{R} \times \Sigma$ whose regularity and fall-off at infinity can be specified. Note that there can be several manners of identifying points on different hypersurfaces Σ_t, i.e. of fixing the product structure $\mathcal{M} \simeq \mathbb{R} \times \Sigma$. The natural idea is to quotient $\mathcal{M}$ by the integral lines of the timelike vector field $\nabla^a t$, but one could choose other timelike vector fields.

If in addition, our space-time is asymptotically flat, then there exists a compact subset K of Σ such that $\Sigma \setminus K$ is diffeomorphic to the exterior of a ball in $\mathbb{R}^3$ (assuming that the manifold Σ has only one asymptotically flat end; if Σ has several asymptotically flat ends, then $\Sigma \setminus K$ is the union of a finite number of manifolds with boundary M_i, $i = 1, \dots, N$, each M_i being diffeomorphic to the exterior of a ball in $\mathbb{R}^3$).

We use the foliation to perform a 3+1 (or space/time) decomposition of the metric. Let T^a be the future-pointing timelike vector field normal to Σ_t, normalized for later convenience to satisfy

$$T^a T_a = 2,$$

i.e.

$$T^a = \frac{\sqrt{2}}{|\nabla t|} \nabla^a t, \quad \text{where} \quad |\nabla t| = (g_{ab} \nabla^a t \nabla^b t)^{1/2}. \tag{2.3}$$

At each point $p \in \mathcal{M}$, the metric g can be decomposed into its orthogonal parts along T^a and $(T^a)^\perp = \mathrm{T}_p \Sigma_t$:

$$g_{ab} = \frac{1}{2} T_a T_b - h_{ab} \tag{2.4}$$

where $-h$ is the restriction of g to $\mathrm{T}_p \Sigma_t$, whence

$$T^a h_{ab} = 0, \tag{2.5}$$

and the 1-form T_a is given by

$$T_a \mathrm{d}x^a = \frac{\sqrt{2}}{|\nabla t|} \nabla_a t \mathrm{d}x^a = \frac{\sqrt{2}}{|\nabla t|} \mathrm{d}t. \tag{2.6}$$

We define the lapse function $N(p)$ by

$$T_a \mathrm{d}x^a = N \mathrm{d}t, \quad \text{i.e.} \quad N = \frac{\sqrt{2}}{|\nabla t|} \tag{2.7}$$

and the decomposition of the metric g then takes the form

$$g = \frac{1}{2} N^2 \mathrm{d}t^2 - h. \tag{2.8}$$

We now choose to define the product structure using the timelike vector field $\nabla^a t$ (or equivalently T^a), the vector field $\partial/\partial t$ is then defined independently of the choice of coordinates on Σ and is everywhere orthogonal to Σ_t. More explicitly, we have

$$\left(\frac{\partial}{\partial t}\right)^a = \frac{N}{2}T^a, \tag{2.9}$$

whence

$$h_{ab}\left(\frac{\partial}{\partial t}\right)^a = 0. \tag{2.10}$$

For this choice of product structure, let us consider a coordinate system on $\mathcal{M} \simeq \mathbb{R} \times \Sigma$: $x^0 = t, x^1, x^2, x^3$. From (2.10), we infer that the expression of h in these coordinates is as follows:

$$h_{ab}\mathrm{d}x^a\mathrm{d}x^b = \sum_{\mathbf{a},\mathbf{b}=1}^{3} h_{\mathbf{ab}}(t, x^1, x^2, x^3)\mathrm{d}x^{\mathbf{a}}\mathrm{d}x^{\mathbf{b}}.$$

Thus h is naturally interpreted as a time-dependent Riemannian metric on Σ.

We use the decomposition of the metric to project the connection ∇_a along T^a and along $(T^a)^{\perp}$. We obtain

$$\nabla_a = \frac{1}{2}T_aT^b\nabla_b - h_a{}^b\nabla_b = \frac{1}{2}T_a\nabla_T + \mathrm{D}_a, \tag{2.11}$$

where $\nabla_T = T^a\nabla_a$ is the covariant derivative along T^a and $\mathrm{D}_a = -h_a{}^b\nabla_b$ is the part of ∇_a orthogonal to T^a: $T^a\mathrm{D}_a = 0$. D_a is the four-dimensional covariant derivative restricted (by composition with the projection operator $-h_a{}^b$) to act tangent to Σ_t. It differs from the Levi-Civita connection on $(\Sigma_t, h(t))$ by a combination of the extrinsic curvature (or second fundamental form) of the leaves of the foliation. In particular $\mathrm{D}_aT_b = K_{ab} = K_{(ab)}$ is $\sqrt{2}$ times the extrinsic curvature. More precisely we have ([1])

$$K_{ab} = \mathrm{D}_aT_b = h_a{}^ch_b{}^d\nabla_cT_d = -\frac{1}{N}\frac{\partial}{\partial t}h_{ab} \tag{2.12}$$

and obviously $T^aK_{ab} = 0$.

Using the spinor form $T^{AA'}$ of the vector T^a to convert primed indices to unprimed indices and vice versa ([2]), we introduce modified forms of the spacelike part of the covariant derivative

$$\mathrm{D}_{AB} = T_A^{A'}\mathrm{D}_{BA'} = T_{(A}^{A'}\nabla_{B)A'}, \qquad \mathrm{D}_{A'B'} = T_{A'}^{A}\mathrm{D}_{B'A} = T_{(A'}^{A}\nabla_{B')A}. \tag{2.13}$$

These will naturally arise when considering the spacelike part of a Dirac or a Weyl equation.

2.3. Classes of asymptotically flat space-times. Weighted Sobolev spaces are particularly well adapted to the description of asymptotically flat space-times because they contain information about both the regularity and fall-off at infinity of functions. We

([1]) Note that the expression of the extrinsic curvature as the time derivative of the spacelike part of the metric is only valid for the product structure defined by the vector field T^a.

([2]) The normalization of T^a implies that $T_{A'}^{A}T_A^{B'} = -\varepsilon_{A'}{}^{B'}$ and $T_A^{A'}T_{A'}^{B} = -\varepsilon_A{}^B$. This shows that converting indices twice leads to a sign change. Also, the conversion of indices commutes with the Levi-Civita connection of $(\Sigma_t, h(t))$ but not with D_a.

shall define classes of asymptotically flat metrics by requiring that they are continuously differentiable in time up to a certain order with values in some weighted Sobolev space on Σ. Hence, we need to define Sobolev-type spaces on Σ without explicit reference to the metric g that we are trying to characterize. To this end, we equip Σ with a smooth Riemannian metric $\widetilde{h}$ which is euclidian outside a compact set. We denote by $\widetilde{\mathrm{D}}$ and $\mathrm{dVol}_{\widetilde{h}}$ the covariant derivative and the volume element on Σ associated with $\widetilde{h}$ and by $\langle\,,\rangle$ the positive definite inner product induced by $\widetilde{h}$ on tensors and spinors at a point. The families of function spaces that we shall use are the following:

• $\mathcal{C}^k(\Sigma)$, $k \in \mathbb{N}\cup\{\infty\}$; the space of k times continuously differentiable functions on Σ. $\mathcal{C}^k_0(\Sigma)$ will denote the subspace of compactly supported functions and $\mathcal{C}^k_b(\Sigma)$ the subspace of functions uniformly bounded on Σ together with their derivatives.

• Sobolev spaces: $H^s(\Sigma)$, $s \in \mathbb{N}$; the completion of $\mathcal{C}^\infty_0(\Sigma)$ in the norm

$$\|f\|_{H^s(\Sigma)} = \Big\{ \sum_{p=0}^{s} \int_\Sigma \langle \widetilde{\mathrm{D}}^p f, \widetilde{\mathrm{D}}^p f\rangle \,\mathrm{dVol}_{\widetilde{h}} \Big\}^{1/2}. \tag{2.14}$$

The space $H^0(\Sigma)$ is $L^2(\Sigma, \mathrm{dVol}_{\widetilde{h}})$ denoted simply by $L^2(\Sigma)$. $H^s_{\mathrm{comp}}(\Sigma)$ is the subspace of compactly supported elements of $H^s(\Sigma)$ and $H^s_{\mathrm{loc}}(\Sigma)$ the space of functions $f \in L^2_{\mathrm{loc}}(\Sigma)$ (or of distributions f on Σ) such that, for any cut-off function $\chi \in \mathcal{C}^\infty_0(\Sigma)$, we have $\chi f \in H^s(\Sigma)$.

• Weighted Sobolev spaces: $H^s_\delta(\Sigma)$, $s \in \mathbb{N}$, $\delta \in \mathbb{R}$; the completion of $\mathcal{C}^\infty_0(\Sigma)$ in the norm

$$\|f\|_{H^s_\delta(\Sigma)} = \Big\{ \sum_{p=0}^{s} \int_\Sigma (1+r^2)^{-\delta-3/2+p} \langle \widetilde{\mathrm{D}}^p f, \widetilde{\mathrm{D}}^p f\rangle \,\mathrm{dVol}_{\widetilde{h}} \Big\}^{1/2}, \tag{2.15}$$

where $r(x)$ is the $\widetilde{h}$-distance from x to a fixed point $O \in \Sigma$ (the function space is independent of the choice of O). We are using the numbering of weighted Sobolev spaces proposed by R. Bartnik ([3]) in [5]; the power $3/2$ in the expression of the norm is to be understood as $n/2$, n being the dimension of the spacelike slices. This numbering has the advantage of indicating clearly the rate at which the functions in H^s_δ fall off or grow at infinity, as we shall see shortly. The space $H^0_\delta(\Sigma)$ will be denoted by $L^2_\delta(\Sigma)$.

• $\mathcal{C}^k_\delta(\Sigma)$, $k \in \mathbb{N}$, $\delta \in \mathbb{R}$; the space of functions in $\mathcal{C}^k(\Sigma)$ for which the norm

$$\|f\|_{\mathcal{C}^k_\delta(\Sigma)} = \sup_{x\in\Sigma} \sum_{l=0}^{k} \{(1+r^2)^{-\delta+l} \langle \widetilde{\mathrm{D}}^l f, \widetilde{\mathrm{D}}^l f\rangle\}^{1/2} \tag{2.16}$$

is finite. This gives the following control on the behaviour of f and its derivatives at spacelike infinity:

$$\widetilde{\mathrm{D}}^l f = O(r^{\delta-l}) \quad \text{as } r \to +\infty,\ 0 \le l \le k.$$

([3]) The numbering of weighted Sobolev spaces used in [9] and in [42] was the "usual" one which is different from the one used by R. Bartnik. Whenever we quote results from these sources, we simply re-express them using Bartnik's numbering. We also use the notational conventions of R. Bartnik for $\mathcal{C}^k_\delta$ spaces; in [9] and [42], the weight index δ for these spaces has the sign opposite to the one chosen here.

This is effectively a control on the fall-off at spacelike infinity of f and its derivatives if $\delta < 0$.

The continuous embedding (see Y. Choquet-Bruhat and D. Christodoulou [9])

$$H^k_\delta \hookrightarrow \mathcal{C}^{k-2}_{\delta'}(\Sigma), \qquad \delta' > \delta,\ k \geq 2, \tag{2.17}$$

gives an estimate on the behaviour at infinity of functions in a weighted Sobolev space and their derivatives. It is important to remark that the correspondence between the weight and the growth or fall-off at infinity is not quite exact. Indeed, a function f in H^k_δ will not necessarily behave at infinity like r^δ, but we shall have $f = O(r^\nu)$ for any $\nu > \delta$.

We now define the classes of asymptotically flat space-times that we shall consider.

DEFINITION 2.1. We say that the metric g on $\mathbb{R} \times \Sigma$ is of *class* (k, δ), $k \in \mathbb{N}^*$, $\delta \in \mathbb{R}$ if

$$g - \left(\mathrm{d}t^2 - \widetilde{h} - \varrho(x)\frac{m}{r} \right) \in \mathcal{C}^l(\mathbb{R}_t; H^{k-l}_\delta(\Sigma)), \qquad \forall l;\ 0 \leq l \leq k$$

(where m is a symmetric 2-form constant outside a compact set, ϱ is a smooth cut-off function on Σ such that $\varrho \equiv 0$ in a neighbourhood of O and $\varrho \equiv 1$ outside a compact domain) and if moreover g satisfies the nondegeneracy condition

(H) There exist two continuous, strictly positive functions on $\mathbb{R}$: C_1, C_2 such that for each $(t, x) \in \mathbb{R} \times \Sigma$ the lapse function N and the eigenvalues $\lambda_i(t, x)$, $i = 1, 2, 3$, of $h(t, x)$ as a symmetric form relative to $\widetilde{h}$ satisfy $C_1(t) \leq N(t, x) \leq C_2(t), C_1(t) \leq \lambda_i(t, x) \leq C_2(t)$.

The intersection of all classes (k, δ), $k \in \mathbb{N}^*$, will be called *the class* (∞, δ).

Of course, the definition of these classes of metrics is independent of the choice of $\widetilde{h}$.

REMARK 2.1. 1. The quantity m/r, appearing in the comparison metric used to define the classes (k, δ) (in Definition 2.1), allows for the presence of energy (or mass) in our space-times. It is usually simply the term with $1/r$ in the asymptotic expansion of the Schwarzschild metric in powers of $1/r$, i.e. for r large enough,

$$\frac{m}{r} = \frac{2M}{r}\mathrm{d}t^2 + \frac{2M}{r}\mathrm{d}r^2.$$

This gives, outside a compact set, the following expression for the comparison metric:

$$\mathrm{d}t^2 - \widetilde{h} - \varrho(x)\frac{m}{r} = \left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 - \left(1 + \frac{2M}{r}\right)\mathrm{d}r^2 - r^2\mathrm{d}\omega^2,$$

that is, the first two terms (constant and in $1/r$) in the asymptotic expansion of the Schwarzschild metric at infinity.

2. Definition 2.1 is valid for space-times admitting one or several asymptotically flat ends. One could, in principle, associate different mass terms m/r to each asymptotic end; though mathematically reasonable, the physical significance of such a choice would be rather mysterious.

It was remarked in [42] that if g is of class (k, δ), $k \geq 3$, $\delta < 0$, we can define the spaces $H^l(\Sigma_t)$, $H^l_\varrho(\Sigma_t)$, $t \in \mathbb{R}$, $0 \leq l \leq k-2$, $\varrho \in \mathbb{R}$, associated with the metric $h(t)$ on Σ and (H) entails that the norms on these spaces are equivalent to the norms on $H^l(\Sigma)$ and $H^l_\varrho(\Sigma)$ respectively, this norm equivalence being uniform on each compact time interval.

We see from (2.17) that if g is of class (k,δ), $k\geq 2$, $\delta\in\mathbb{R}$, then

$$g-\left(\mathrm{d}t^2-\widetilde{h}-\varrho(x)\frac{m}{r}\right)\in\mathcal{C}^l(\mathbb{R}_t;\mathcal{C}^{k-l-2}_{\delta'}(\Sigma)),\qquad \forall l,\delta';\ 0\leq l\leq k-2,\ \delta'>\delta.$$

Note that g compared to the flat metric at infinity satisfies (the fall-off here is weaker because the Schwarzschild term m/r is no longer present in the comparison metric $\mathrm{d}t^2-\widetilde{h}$)

$$g-(\mathrm{d}t^2-\widetilde{h})\in\mathcal{C}^l(\mathbb{R}_t;H^{k-l}_\nu(\Sigma)),\qquad \forall l,\nu;\ 0\leq l\leq k,\ \nu>\max(\delta,-1) \tag{2.18}$$

and therefore

$$g-(\mathrm{d}t^2-\widetilde{h})\in\mathcal{C}^l(\mathbb{R}_t;\mathcal{C}^{k-l-2}_\nu(\Sigma)),\qquad \forall l,\nu;\ 0\leq l\leq k-2,\ \nu>\max(\delta,-1).$$

To express things in a simpler way, a metric g of class (k,δ) will be asymptotically flat as soon as $k\geq 2$ and $\delta<0$ in the sense that g will be continuous on $\mathbb{R}\times\Sigma$ and will tend to the Minkowski metric at spacelike infinity. Such metrics will be called weakly asymptotically flat. In order to give a stronger, more physical meaning to asymptotic flatness, one usually imposes

$$\widetilde{\mathrm{D}}^l\left(g-\left(\mathrm{d}t^2-\widetilde{h}-\frac{m}{r}\right)\right)=O(r^{-3/2-l}),\qquad r\to+\infty. \tag{2.19}$$

This is guaranteed by any $\delta<-3/2$, but as already mentioned, the nature of the embedding (2.17) is such that this correspondence is not exact. Metrics of class (k,δ), $k\geq 2$, $\delta<-3/2$ will always fall off at spacelike infinity a little faster than (2.19); conversely, if g satisfies (2.19), then g will belong to all classes (k,δ), $\delta>-3/2$ (k depending on the regularity of g). Black hole space-times such as Schwarzschild or Kerr satisfy (2.19) at infinity and in fact a little more:

$$\widetilde{\mathrm{D}}^l\left(g-\left(\mathrm{d}t^2-\widetilde{h}-\frac{m}{r}\right)\right)=O(r^{-2-l}),\qquad r\to+\infty.$$

In [11] and [39], a weaker version of asymptotic flatness is considered, with only the following requirements as $r\to+\infty$

$$g-\left(\mathrm{d}t^2-\widetilde{h}-\frac{m}{r}\right)=o(r^{-1}),\qquad \widetilde{\mathrm{D}}^l\left(g-\left(\mathrm{d}t^2-\widetilde{h}-\frac{m}{r}\right)\right)=o(r^{-l-1}).$$

Note that even in the weakest version of asymptotic flatness ($k\geq 2,\ \delta<0$) property (H) is a direct consequence of the fact that g is a nondegenerate continuous Lorentzian metric on $\mathbb{R}\times\Sigma$ which tends to the Minkowski metric at spacelike infinity.

3. Dirac fields on globally hyperbolic space-times

In this chapter, we only require the space-time $(\mathcal{M},g)$ to be globally hyperbolic and we shall use the foliation $\{\Sigma_t\}_t$ to perform a 3+1 decomposition of the Dirac equation. Throughout this whole chapter, the product structure will be associated with T^a.

3.1. The Dirac and Weyl equations. We first describe the Dirac equation on $(\mathcal{M},g)$ in terms of Dirac spinors. The bundle of Dirac spinors on $\mathcal{M}$ is defined as

$$\mathbb{S}_{\mathrm{Dirac}}=\mathbb{S}^*\oplus\overline{\mathbb{S}}=\mathbb{S}_A\oplus\mathbb{S}^{A'}. \tag{3.1}$$

We choose on $\mathcal{M}$ a local orthonormal Lorentz frame, i.e. a set of four real vector fields $\{e_0, e_1, e_2, e_3\}$ such that

$$g_{00} = -g_{\mathbf{aa}} = 1, \quad \mathbf{a} = 1,2,3; \qquad g_{\mathbf{ab}} = 0, \quad \mathbf{a} \neq \mathbf{b},$$

$g_{\mathbf{ab}}$ denoting $g(e_{\mathbf{a}}, e_{\mathbf{b}})$, i.e. the components of g_{ab} in the basis $\{e_0, e_1, e_2, e_3\}$. We make the most natural choice for e_0 here, that is,

$$e_0{}^a := \frac{1}{\sqrt{2}} T^a \tag{3.2}$$

and e_1, e_2, e_3 are thus everywhere tangent to the hupersurfaces Σ_t. The Dirac operator on $\mathcal{M}$ is defined by

$$\mathcal{D} = \sum_{\mathbf{a}=0}^{3} e_{\mathbf{a}}.\nabla_{e_{\mathbf{a}}} \tag{3.3}$$

where $e_{\mathbf{a}}.$ denotes the Clifford product by the vector $e_{\mathbf{a}}$ and $\nabla_{e_{\mathbf{a}}}$ the directional covariant derivative along $e_{\mathbf{a}}$. The Dirac equation on $\mathcal{M}$ is then

$$(\mathcal{D} + im)\Psi = 0, \tag{3.4}$$

$\Psi \in \mathbb{S}_{\mathrm{Dirac}}$ and $m \geq 0$ is the mass of the particle. More explicitly, via a choice of spin-frame, the Clifford multiplication of a Dirac spinor Ψ by each vector $e_{\mathbf{a}}$ will be described as the multiplication by a 4×4 matrix $\gamma^{\mathbf{a}}$. The Dirac matrices $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ satisfy the axioms of Clifford multiplication

$$\gamma^{\mathbf{a}}\gamma^{\mathbf{b}} + \gamma^{\mathbf{b}}\gamma^{\mathbf{a}} = 2g^{\mathbf{ab}}\,\mathrm{Id}_4, \qquad \mathbf{a}, \mathbf{b} = 0,1,2,3,$$

and the Dirac equation is then expressed in the following manner

$$\sum_{\mathbf{a}=0}^{3} \gamma^{\mathbf{a}} \nabla_{e_{\mathbf{a}}} \Psi + im\Psi = 0. \tag{3.5}$$

In terms of two-component spinors and abstract indices, the same equation (3.4) takes the form

$$\begin{cases} \nabla^{AA'}\phi_A = \mu\chi^{A'}, \\ \nabla^{AA'}\chi_{A'} = \mu\phi^A, \quad \mu = m/\sqrt{2}. \end{cases} \tag{3.6}$$

The structure here is much clearer: we have two Weyl equations (one for anti-neutrinos and the other for neutrinos) coupled by the mass. We have chosen to express the second equation in a form which emphasizes the fact that it is the complex conjugate of an equation of the type of the first one. However, we can equivalently write it

$$\nabla_{AA'}\chi^{A'} = -\mu\phi_A.$$

It is easy to recover from (3.6) the more broadly used expression (3.5) involving Dirac matrices. We choose a normalized spin-frame $\{o^A, \iota^A\}$ (also denoted $\{\varepsilon_0{}^A, \varepsilon_1{}^A\}$), that is, a pair of sections of $\mathbb{S}^A$ such that $o_A\iota^A = 1$. The dual basis of $\mathbb{S}_A$ is $\{\varepsilon_A{}^0, \varepsilon_A{}^1\}$ where $\varepsilon_A{}^0 = -\iota_A$ and $\varepsilon_A{}^1 = o_A$. The choice of $\{o^A, \iota^A\}$ is usually done by choosing a Newman–Penrose tetrad: a set of four null vector fields $\{l^a, n^a, m^a, \overline{m}^a\}$, l^a and n^a being real and m^a complex, such that

$$l_a n^a = 1, \quad m_a\overline{m}^a = -1, \quad l_a m^a = 0, \quad n_a m^a = 0.$$

The spin-frame $\{o^A, \iota^A\}$ is then fixed, up to an overall sign, by requiring

$$l^a = o^A \bar{o}^{A'}, \quad n^a = \iota^A \bar{\iota}^{A'}, \quad m^a = o^A \bar{\iota}^{A'}, \quad \overline{m}^a = \iota^A \bar{o}^{A'}.$$

We define the Infeld–van der Waerden symbols $g_{\mathbf{a}}{}^{\mathbf{AA'}}$ as the spinor components of the frame vectors in the spin-frame:

$$g_{\mathbf{a}}{}^{\mathbf{AA'}} = e_{\mathbf{a}}{}^{\mathbf{AA'}} = g_{\mathbf{a}}{}^a \varepsilon_A{}^{\mathbf{A}} \varepsilon_{A'}{}^{\mathbf{A'}} = \begin{pmatrix} n_{\mathbf{a}} & -\overline{m}_{\mathbf{a}} \\ -m_{\mathbf{a}} & l_{\mathbf{a}} \end{pmatrix} \tag{3.7}$$

(recall that $g_{\mathbf{a}}{}^a = e_{\mathbf{a}}{}^a$ denotes the vector field $e_{\mathbf{a}}$). We use these quantities to express equation (3.6) in terms of spinor components:

$$\begin{cases} -i\nabla^{\mathbf{AA'}}\phi_{\mathbf{A}} = -ig^{\mathbf{aAA'}}\nabla_{\mathbf{a}}\phi_{\mathbf{A}} = -i\dfrac{m}{\sqrt{2}}\chi^{\mathbf{A'}}, \\ -i\nabla_{\mathbf{A}}{}^{\mathbf{A'}}\chi_{\mathbf{A'}} = i\nabla_{\mathbf{AA'}}\chi^{\mathbf{A'}} = ig^{\mathbf{a}}{}_{\mathbf{AA'}}\nabla_{\mathbf{a}}\chi^{\mathbf{A'}} = -i\dfrac{m}{\sqrt{2}}\phi_{\mathbf{A}}, \end{cases} \tag{3.8}$$

where $\nabla_{\mathbf{a}}$ denotes $\nabla_{e_{\mathbf{a}}}$. For $\mathbf{a} = 0, 1, 2, 3$, we introduce the 2×2 matrices

$$M^{\mathbf{a}} = {}^t g^{\mathbf{aAA'}}, \quad N^{\mathbf{a}} = g^{\mathbf{a}}{}_{\mathbf{AA'}}, \tag{3.9}$$

and the 4×4 matrices

$$\gamma^{\mathbf{a}} = \begin{pmatrix} 0 & i\sqrt{2}N^{\mathbf{a}} \\ -i\sqrt{2}M^{\mathbf{a}} & 0 \end{pmatrix}. \tag{3.10}$$

Putting $\Psi := \phi_A \oplus \chi^{A'}$, the components of Ψ in the spin-frame are $\Psi = {}^t(\phi_0, \phi_1, \chi^{0'}, \chi^{1'})$ and (3.8) becomes

$$\sum_{\mathbf{a}=0}^{3} \gamma^{\mathbf{a}}\nabla_{e_{\mathbf{a}}}\Psi + im\Psi = 0.$$

Remark 3.1. The matrix $V^{\mathbf{AA'}}$ of the spinor components of a vector V^a in a normalized spin-frame has the important property that

$$\det(V^{\mathbf{AA'}}) = \frac{1}{2}V_a V^a.$$

Indeed,

$$V_a V^a = V^a V^b g_{ab} = V^{\mathbf{AA'}}V^{\mathbf{BB'}}\varepsilon_{\mathbf{AB}}\varepsilon_{\mathbf{A'B'}} = 2(V^{00'}V^{11'} - V^{01'}V^{10'})$$

since

$$\varepsilon_{\mathbf{AB}} = \varepsilon_{\mathbf{A'B'}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Hence, we have

$$\det(g_0{}^{\mathbf{AA'}}) = \frac{1}{2}, \quad \det(g_{\mathbf{a}}{}^{\mathbf{AA'}}) = -\frac{1}{2}, \quad \mathbf{a} = 1, 2, 3,$$

and therefore $\det M^0 = 1/2$, $\det M^{\mathbf{a}} = -1/2$, $\mathbf{a} = 1, 2, 3$. The process for obtaining $N^{\mathbf{a}}$ from $M^{\mathbf{a}}$ is first to transpose $M^{\mathbf{a}}$ in order to obtain the matrix $g^{\mathbf{aAA'}}$ and then to lower the two concrete spinor indices. This exchanges the diagonal terms and changes the sign of the terms outside the diagonal, whence $N^{\mathbf{a}} = (\det M^{\mathbf{a}})(M^{\mathbf{a}})^{-1}$. Consequently,

$$\gamma^0 = i\begin{pmatrix} 0 & (\sqrt{2}M^0)^{-1} \\ -\sqrt{2}M^0 & 0 \end{pmatrix}, \quad \gamma^{\mathbf{a}} = -i\begin{pmatrix} 0 & (\sqrt{2}M^{\mathbf{a}})^{-1} \\ \sqrt{2}M^{\mathbf{a}} & 0 \end{pmatrix}, \quad \mathbf{a} = 1, 2, 3.$$

We clearly see that

$$(\gamma^0)^2 = -(\gamma^1)^2 = -(\gamma^2)^2 = -(\gamma^3)^2 = \mathrm{Id}_4.$$

We still need to prove that these matrices anticommute in order to see that they form a set of Dirac matrices.

LEMMA 3.1. *The matrices* $\gamma^{\mathbf{a}}$, $\mathbf{a} = 0, 1, 2, 3$, *defined by* (3.9) *and* (3.10), *form indeed a set of Dirac matrices.*

Proof. This is in fact a classic result and a straightforward consequence of the following identity (see [53], Vol. 1, p. 124)

$$g_{\mathbf{ab}} = \varepsilon_{\mathbf{AB}}\varepsilon_{\mathbf{A'B'}}g_{\mathbf{a}}{}^{\mathbf{AA'}}g_{\mathbf{b}}{}^{\mathbf{BB'}}$$

which is the component version of $g_{ab} = \varepsilon_{AB}\varepsilon_{A'B'}$. This identity is equivalent to

$$\varepsilon_{\mathbf{B}}{}^{\mathbf{A}}g^{\mathbf{ab}} = g^{\mathbf{b}}{}_{\mathbf{BA'}}g^{\mathbf{aAA'}} + g^{\mathbf{a}}{}_{\mathbf{BA'}}g^{\mathbf{bAA'}},$$

i.e.

$$N^{\mathbf{b}}M^{\mathbf{a}} + N^{\mathbf{a}}M^{\mathbf{b}} = g^{\mathbf{ab}}\mathrm{Id}_2 = g_{\mathbf{ab}}\mathrm{Id}_2,$$

as well as to

$$\varepsilon_{\mathbf{A'}}{}^{\mathbf{B'}}g^{\mathbf{ab}} = g^{\mathbf{aAB'}}g^{\mathbf{b}}{}_{\mathbf{AA'}} + g^{\mathbf{bAB'}}g^{\mathbf{a}}{}_{\mathbf{AA'}},$$

i.e.

$$M^{\mathbf{a}}N^{\mathbf{b}} + M^{\mathbf{b}}N^{\mathbf{a}} = g^{\mathbf{ab}}\mathrm{Id}_2 = g_{\mathbf{ab}}\mathrm{Id}_2.$$

This proves the lemma. ■

From now on, we shall assume our Dirac matrices to be of the form (3.10), *i.e. to be compatible with the description of the Dirac equation in terms of two-component spinors.*

We see that in the formalism of two-component spinors and abstract indices, the Clifford product by the frame vectors e_0, e_1, e_2, e_3 is represented on $\mathbb{S}^*$ and $\overline{\mathbb{S}}$ as

$$e_{\mathbf{a}}. : \phi \in \mathbb{S}^* \mapsto -i\sqrt{2}\, g^{\mathbf{a}AA'}\phi_A \in \overline{\mathbb{S}},$$
$$e_{\mathbf{a}}. : \chi \in \overline{\mathbb{S}} \mapsto i\sqrt{2}\, g^{\mathbf{a}}{}_{AA'}\chi^{A'} \in \mathbb{S}^*,$$

and we have

$$-i\sqrt{2}\,\nabla^{AA'}\phi_A = \sum_{\mathbf{a}=0}^{3} e_{\mathbf{a}}.\nabla_{e_{\mathbf{a}}}\phi = \mathcal{D}\phi,$$

$$i\sqrt{2}\,\nabla_{AA'}\chi^{A'} = \sum_{\mathbf{a}=0}^{3} e_{\mathbf{a}}.\nabla_{e_{\mathbf{a}}}\chi = \mathcal{D}\chi.$$

The Weyl equations for antineutrinos and neutrinos, respectively:

$$\nabla^{AA'}\phi_A = 0, \qquad \nabla^{AA'}\chi_{A'} = 0 \qquad (\text{or equivalently } \nabla_{AA'}\chi^{A'} = 0),$$

are simply the massless Dirac equations

$$\mathcal{D}\phi = 0, \qquad \mathcal{D}\chi = 0,$$

where the Dirac operator $\mathcal{D}$ is restricted to act on the spin-bundles $\mathbb{S}^*$ and $\overline{\mathbb{S}}$ respectively instead of $\mathbb{S}_{\mathrm{Dirac}}$. The Dirac equation in terms of two-component spinors (3.6) can be written as

$$\begin{cases} \mathcal{D}\phi = -im\chi, \\ \mathcal{D}\chi = -im\phi. \end{cases} \tag{3.11}$$

REMARK 3.2. The Clifford product by e_0, when restricted to act on $\mathbb{S}^*$ (respectively on $\overline{\mathbb{S}}$), takes the form

$$e_0. : \phi \in \mathbb{S}^* \mapsto -i\sqrt{2}\, g^{0AA'}\phi_A = -iT^{AA'}\phi_A = -i\varepsilon^{A'B'}T_{B'}{}^{A}\phi_A \in \overline{\mathbb{S}},$$

respectively,

$$e_0. : \chi \in \overline{\mathbb{S}} \mapsto i\sqrt{2}\, g^0{}_{AA'}\chi^{A'} = iT_{AA'}\chi^{A'} = i\varepsilon_{BA}T^B{}_{A'}\chi^{A'} \in \mathbb{S}^*.$$

This shows that the Clifford multiplication by e_0 and the conversion of indices are essentially the same operation. They commute with the Levi-Civita connection on each leaf of the foliation, but not with the connection D_a.

We conclude this section with an explicit choice of spin-frame and the corresponding expressions of Dirac matrices:

REMARK 3.3. A particularly convenient choice of spin-frame corresponds to the Newman–Penrose tetrad

$$l^a = \frac{1}{\sqrt{2}}(e_0{}^a + e_1{}^a), \quad n^a = \frac{1}{\sqrt{2}}(e_0{}^a - e_1{}^a), \quad m^a = \frac{1}{\sqrt{2}}(e_2{}^a + ie_3{}^a). \tag{3.12}$$

The spinor components of the frame vectors (i.e. the Infeld–van der Waerden symbols) are then

$$\begin{aligned} g_0{}^{\mathbf{AA'}} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_1{}^{\mathbf{AA'}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ g_2{}^{\mathbf{AA'}} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad g_3{}^{\mathbf{AA'}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \end{aligned} \tag{3.13}$$

and the associated Dirac matrices are

$$\gamma^0 = i\begin{pmatrix} 0 & \sigma^0 \\ -\sigma^0 & 0 \end{pmatrix}, \quad \gamma^{\mathbf{a}} = i\begin{pmatrix} 0 & \sigma^{\mathbf{a}} \\ \sigma^{\mathbf{a}} & 0 \end{pmatrix}, \quad \mathbf{a} = 1,2,3, \tag{3.14}$$

where $\sigma^0, \sigma^1, \sigma^2, \sigma^3$ are the Pauli matrices:

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{3.15}$$

3.2. 3+1 decomposition of the equation. The bundle of Dirac spinors $\mathbb{S}_{\mathrm{Dirac}}$ is equipped with an $SL(2,\mathbb{C})$-invariant inner product given by

$$(\Phi, \Psi) = i\overline{\psi}\varrho - i\phi\overline{\chi} = i\overline{\psi}_{A'}\varrho^{A'} - i\phi_A\overline{\chi}^A, \qquad \text{for } \Phi = \phi_A \oplus \varrho^{A'} \text{ and } \Psi = \psi_A \oplus \chi^{A'}. \tag{3.16}$$

This inner product is of course not positive definite.

We consider the hypersurfaces of the foliation; each Σ_t is equipped with an $SU(2)$-principal bundle $P_S(t)$ of spin-frames. The vector field e_0 gives a natural embedding of $P_S(t)$ into the $SL(2,\mathbb{C})$-principal bundle $P_{\mathbb{S}}$ of spin-frames of $\mathcal{M}$ and thus realizes $P_{\mathbb{S}}$ as a lift of the bundles $P_S(t)$. e_0 thus provides $\mathbb{S}_{\mathrm{Dirac}}$ with a hermitian positive definite $SU(2)$-invariant inner product:

$$(3.17)\qquad \begin{aligned}\langle \Phi, \Psi\rangle &= (e_0.\Phi, \Psi) = (i\sqrt{2}\, g^0{}_{AA'}\varrho^{A'} \oplus -i\sqrt{2}\, g^{0AA'}\phi_A,\ \psi_A \oplus \chi^{A'})\\ &= \sqrt{2}\,[g^{0AA'}\phi_A\overline{\psi}_{A'} + g^0{}_{AA'}\varrho^{A'}\overline{\chi}^A] = T^{AA'}\phi_A\overline{\psi}_{A'} + T_{AA'}\varrho^{A'}\overline{\chi}^A.\end{aligned}$$

We see that e_0 also induces positive definite hermitian inner products on $\mathbb{S}$ and $\overline{\mathbb{S}}$:

$$(3.18)\qquad \langle \phi, \psi\rangle = T_{AA'}\phi^A\overline{\psi}^{A'}, \quad \langle \varrho, \chi\rangle = T_{AA'}\varrho^{A'}\overline{\chi}^A.$$

The Clifford product by any of the frame vectors e_0, e_1, e_2, e_3 is a self-adjoint operator for the scalar product $(\cdot\,,\cdot)$. Indeed, for any choice of the spin-frame $\{o^A, \iota^A\}$, the matrices $M^{\mathbf{a}}, N^{\mathbf{a}}$, $\mathbf{a} = 0,1,2,3$, defined in (3.9) are hermitian and this entails that the matrices $\gamma^{\mathbf{a}}$, $\mathbf{a} = 0,1,2,3$, of (3.10) are self-adjoint for the inner product $(\cdot\,,\cdot)$. This property implies that the Clifford multiplication by e_0 is self-adjoint for the inner product $\langle\cdot\,,\cdot\rangle$ while the Clifford product by $e_{\mathbf{a}}$, $\mathbf{a} = 1,2,3$, anticommutes with e_0. and is therefore skew-adjoint for $\langle\cdot\,,\cdot\rangle$.

Remark 3.4. If we choose the spin-frame $\{o^A, \iota^A\}$ such that

$$(3.19)\qquad T^{\mathbf{AA'}} = T_{\mathbf{AA'}} = \mathrm{Id}_2,$$

i.e.

$$e_0{}^a = \frac{1}{\sqrt{2}}(l^a + n^a) = \frac{1}{\sqrt{2}}(o^A\bar{o}^{A'} + \iota^A\bar{\iota}^{A'}),$$

then the matrix γ^0 is given by

$$(3.20)\qquad \gamma^0 = i\begin{pmatrix} 0 & \sigma^0\\ -\sigma^0 & 0\end{pmatrix}, \quad \sigma^0 = \mathrm{Id}_2$$

and the inner product $(\cdot\,,\cdot)$ takes the simple form

$$(3.21)\qquad (\Phi, \Psi) = \Psi^\dagger\gamma^0\Phi, \quad \Psi^\dagger = {}^t\overline{\Psi} = (\overline{\Psi_1}, \overline{\Psi_2}, \overline{\Psi_3}, \overline{\Psi_4}).$$

The positive definite inner product takes an even simpler form

$$(3.22)\qquad \langle \Phi, \Psi\rangle = (\gamma^0\Phi, \Psi) = \Psi^\dagger(\gamma^0)^2\Phi = \Psi^\dagger\Phi.$$

This can also be seen directly in the expression of $\langle\Phi, \Psi\rangle$ involving two-component spinors:

$$\begin{aligned}\langle \Phi, \Psi\rangle &= T^{\mathbf{AA'}}\phi_{\mathbf{A}}\overline{\psi}_{\mathbf{A'}} + T_{\mathbf{AA'}}\varrho^{\mathbf{A'}}\overline{\chi}^{\mathbf{A}} = \phi_0\overline{\psi}_{0'} + \phi_1\overline{\psi}_{1'} + \varrho^{0'}\overline{\chi}^0 + \varrho^{1'}\overline{\chi}^1\\ &= \phi_0\overline{\psi_0} + \phi_1\overline{\psi_1} + \varrho^{0'}\overline{\chi^{0'}} + \varrho^{1'}\overline{\chi^{1'}}.\end{aligned}$$

In this case, the skew-adjointness of $e_{\mathbf{a}}.$, $\mathbf{a} = 1,2,3$, for this inner product simply means that the matrices $\gamma^{\mathbf{a}}$, $\mathbf{a} = 1,2,3$, are skew hermitian.

Henceforth, we shall systematically work with this type of spin-frame, i.e. we shall always assume (3.19), (3.20) satisfied and the expressions of the two inner products $(\cdot\,,\cdot)$ and $\langle\cdot\,,\cdot\rangle$ in terms of components will always be (3.21) and (3.22). We say that such spin-frames are *adapted to the foliation.*

We now describe the $3+1$ decomposition of the Dirac equation. From the expression of the Dirac operator

$$\mathcal{D} = \sum_{\mathbf{a}=0}^{3} e_{\mathbf{a}}.\nabla_{e_{\mathbf{a}}},$$

we can immediately write equation (3.4) as an evolution system:

$$e_0.\nabla_{e_0}\Psi = -\sum_{\mathbf{a}=1}^{3} e_{\mathbf{a}}.\nabla_{e_{\mathbf{a}}}\Psi - im\Psi$$

and Clifford multiplying the whole equation by e_0, we obtain

$$\nabla_{e_0}\Psi = -\sum_{\mathbf{a}=1}^{3} e_0.e_{\mathbf{a}}.\nabla_{e_{\mathbf{a}}}\Psi - im\, e_0.\Psi. \tag{3.23}$$

We denote

$$\mathcal{D}_W(t) := \sum_{\mathbf{a}=1}^{3} e_{\mathbf{a}}.\nabla_{e_{\mathbf{a}}} = \sum_{\mathbf{a}=1}^{3} e_{\mathbf{a}}.\mathrm{D}_{e_{\mathbf{a}}} \quad \text{on } \Sigma_t. \tag{3.24}$$

$\mathcal{D}_W(t)$ is the Dirac operator on Σ_t associated with the connection D_a, we call it the Dirac–Witten (1) operator. Let us introduce $\mathcal{D}_\Sigma(t)$ the Dirac operator associated with the Levi-Civita connection on $(\Sigma_t, h(t))$. The difference between $\mathcal{D}_W(t)$ and $\mathcal{D}_\Sigma(t)$ is explicitly given by (see A. Sen [56] and [57], also M. Herzlich [26])

$$\mathcal{D}_W(t) = \mathcal{D}_\Sigma(t) + \frac{1}{2\sqrt{2}} K e_0. \tag{3.25}$$

where $K = \mathrm{Tr}(K_{ab}) = K_a{}^a$ is $\sqrt{2}$ times the trace of the extrinsic curvature. An interesting upshot of this is that for a maximal foliation, that is, a foliation for which the extrinsic curvature of the leaves is trace-free, the Dirac–Witten operator and the standard Dirac operator coincide on each Σ_t. Another advantage of such foliations is that $\mathrm{dVol}_{h(t)}$ is independent of t. In the present work, however, we will consider general foliations and not require them to be maximal.

The operator $\mathcal{D}_W(t)$ is formally self-adjoint on $L^2(\Sigma_t; \mathbb{S}_{\mathrm{Dirac}})$ endowed with the inner product

$$\langle \Phi, \Psi \rangle_{L^2(\Sigma_t)} = \int_\Sigma \langle \Phi, \Psi \rangle \, \mathrm{dVol}_{h(t)} \tag{3.26}$$

i.e. it is symmetric on $\mathcal{C}_0^\infty(\Sigma_t; \mathbb{S}_{\mathrm{Dirac}})$ for this inner product (2). We have the Bochner–Lichnerowicz–Weitzenböck–Witten formula (see for example [26] or [51])

$$\mathcal{D}_W^* \, \mathcal{D}_W = \mathcal{D}_W^2 = \mathrm{D}^*\mathrm{D} + \mathcal{R} \tag{3.27}$$

(1) This name refers to the fact that this Dirac operator, associated with the restriction to Σ_t of the full space-time connection and not with the torsion-free connection on $(\Sigma, h(t))$, was used by Witten in his historic paper [64].

(2) K being a real scalar and the Clifford multiplication by e_0 being a bounded self-adjoint operator on $L^2(\Sigma_t)$ endowed with the inner product (3.26), the self-adjointness (or the formal self-adjointness) of $\mathcal{D}_W$ is equivalent to that of $\mathcal{D}_\Sigma$ and their domains are equal. The formal self-adjointness of Dirac operators is established in [40]. Note that the operator $\mathcal{D}_W(t)$ and the inner product (3.26) are intrinsic quantities; the property of formal self-adjointness is itself intrinsic. In the case of space-times which are asymptotically flat, or compact in space, we have in fact more: $\mathcal{D}_W(t)$ is self-adjoint on $L^2(\Sigma_t; \mathbb{S}_{\mathrm{Dirac}})$ endowed with the inner product (3.26) with domain $H^1(\Sigma_t; \mathbb{S}_{\mathrm{Dirac}})$. This can be proved for smooth metrics using the essential self-adjointness of Dirac operators on complete Riemannian manifolds admitting a spin or a spin^c structure, established in [19], and the formula (3.27) to show that the domain of the closure is H^1. The result can then be extended to $\mathcal{C}^1$ metrics using the theorem of stability of bounded invertibility (T. Kato [31]).

where $\mathcal{R}$ is the endomorphism of $\mathbb{S}_{\text{Dirac}}$ (its restriction to $\mathbb{S}^*$ or $\overline{\mathbb{S}}$ is also an endomorphism of $\mathbb{S}^*$ or $\overline{\mathbb{S}}$ respectively) defined as

$$\mathcal{R} = \frac{1}{2}\Big(G(e_0, e_0) + \sum_{\mathbf{a}=1}^{3} G(e_0, e_{\mathbf{a}}) e_0 . e_{\mathbf{a}} . \Big) \tag{3.28}$$

where $G_{ab} = R_{ab} - 1/2 R g_{ab}$ is the Einstein tensor of $(\mathcal{M}, g)$, R_{ab} is the Ricci tensor and its trace $R = R_a{}^a$ is the scalar curvature.

We can write the Dirac equation as

$$\nabla_{e_0} \Psi = -e_0 . \mathcal{D}_W(t) \Psi - im\, e_0 . \Psi = -e_0 . \mathcal{D}_\Sigma(t) \Psi - \frac{1}{2\sqrt{2}} K \Psi - im\, e_0 . \Psi . \tag{3.29}$$

The operator $e_0 . \mathcal{D}_\Sigma(t)$ is formally skew-adjoint on $L^2(\Sigma_t)$, since $\mathcal{D}_\Sigma(t)$ and $e_0.$ are symmetric and $e_0.$ anticommutes with $e_{\mathbf{a}}.$ for $\mathbf{a} = 1, 2, 3$ and commutes with the Levi-Civita connection ([3]) on $(\Sigma_t, h(t))$. Therefore, the Dirac equation is a first order symmetric hyperbolic system on Σ. We express this property more precisely in the following lemma:

Lemma 3.2. *Given a spin-frame $\{o^A, \iota^A\}$ adapted to the foliation, for any coordinate system on Σ, equation* (3.29) *takes the form*

$$\frac{\partial \Psi}{\partial t} = \frac{N(t,x)}{\sqrt{2}} \Big\{ \sum_{\mathbf{a}=1}^{3} A^{\mathbf{a}}(t,x) \frac{\partial \Psi}{\partial x^{\mathbf{a}}} + (-im\gamma^0 + B(t,x)) \Psi \Big\} \tag{3.30}$$

where $A^{\mathbf{a}}$, $\mathbf{a} = 1, 2, 3$, are 4×4 hermitian matrices, N is the lapse function and B is a 4×4 matrix. The factor $N/\sqrt{2}$ comes from the expression of the time derivative

$$\nabla_{e_0} = \frac{\sqrt{2}}{N} \Big(\frac{\partial}{\partial t} + \textit{connection terms} \Big).$$

The coefficients of the matrices $A^{\mathbf{a}}$ are coefficients of the metric $h(t)$. This is easily seen when the orthonormal basis $\{e_0, e_1, e_2, e_3\}$ is proportional to the coordinate basis, i.e.

$$e_{\mathbf{a}} = \lambda^{\mathbf{a}} \frac{\partial}{\partial x^{\mathbf{a}}}, \qquad \mathbf{a} = 1, 2, 3,$$

no sum being involved here; the coefficients $\lambda^{\mathbf{a}}$ are given by

$$\lambda^{\mathbf{a}} = \Big[-g\Big(\frac{\partial}{\partial x^{\mathbf{a}}}, \frac{\partial}{\partial x^{\mathbf{a}}} \Big) \Big]^{-1/2}, \qquad \mathbf{a} = 1, 2, 3,$$

and the matrices $A^{\mathbf{a}}$ are then

$$A^{\mathbf{a}} = -\lambda^{\mathbf{a}} \gamma^0 \gamma^{\mathbf{a}}, \qquad \mathbf{a} = 1, 2, 3.$$

The coefficients of $A^{\mathbf{a}}$ will merely be more complicated combinations of the metric coefficients when the coordinate basis is not orthogonal. The matrix B is made of space-time connection coefficients, i.e. of first order derivatives of the coefficients of the metric g (and not just $h(t)$ since some connection terms come from the time derivative).

([3]) Note however that unless the foliation is maximal, $e_0 . \mathcal{D}_W(t)$ is not formally skew-adjoint on $L^2(\Sigma_t)$; it is the sum of the skew-symmetric operator $e_0 . \mathcal{D}_\Sigma(t)$ and of $\frac{1}{2\sqrt{2}} K$, the latter being a real bounded scalar potential and therefore symmetric, and even self-adjoint. This comes from the fact that $e_0.$ does not commute with the connection D_a.

These properties are all we shall need to solve the Cauchy problem for the Dirac equation in the next section.

We also describe the 3+1 decomposition of the Dirac equation in terms of two-component spinors and abstract indices. Using the splitting

$$\nabla^a = \frac{1}{2} T^a \nabla_T + \mathrm{D}^a = \frac{1}{\sqrt{2}} T^a \nabla_{e_0} + \mathrm{D}^a ,$$

we can rewrite equation (3.6) as

$$\begin{cases} T^{AA'} \nabla_{e_0} \phi_A = -\sqrt{2}\, \mathrm{D}^{AA'} \phi_A + \mu\sqrt{2}\, \chi^{A'} , \\ T_{AA'} \nabla_{e_0} \chi^{A'} = -\sqrt{2}\, \mathrm{D}_{AA'} \chi^{A'} - \mu\sqrt{2}\, \phi_A . \end{cases}$$

Multiplying the first equation by $T_{BA'}$, the second by $T^{AB'}$ and using the following facts:

$$T_{BA'} \mathrm{D}^{AA'} = -T_B{}^{A'} \mathrm{D}^A{}_{A'} = -\mathrm{D}_B{}^A , \quad T^{AB'} \mathrm{D}_{AA'} = \mathrm{D}^{B'}{}_{A'} ,$$

we obtain

$$\begin{cases} \nabla_{e_0} \phi_A = \sqrt{2}\, \mathrm{D}_A{}^B \phi_B + m T_{AB'} \chi^{B'} , \\ \nabla_{e_0} \chi^{A'} = -\sqrt{2}\, \mathrm{D}^{A'}{}_{B'} \chi^{B'} - m T^{BA'} \phi_B . \end{cases} \tag{3.31}$$

We denote

$$\mathbb{D}(t) \colon \phi_A \mapsto \mathrm{D}_A{}^B \phi_B , \quad \widehat{\mathbb{D}}(t) \colon \chi^{A'} \mapsto -\mathrm{D}^{A'}{}_{B'} \chi^{B'} , \text{ on } \Sigma_t . \tag{3.32}$$

We shall call $\mathbb{D}(t)$ the *Sen–Witten* ([4]) *operator* on Σ_t. $\sqrt{2}\,\mathbb{D}$ and $\sqrt{2}\,\widehat{\mathbb{D}}$ are the operator

$$-e_0.\mathcal{D}_W = -e_0 . \sum_{\mathbf{a}=1}^{3} e_{\mathbf{a}} . \mathrm{D}_{e_{\mathbf{a}}}$$

restricted to act respectively on the spin-bundles $\mathbb{S}^*$ and $\overline{\mathbb{S}}$ instead of $\mathbb{S}_{\mathrm{Dirac}}$. We have

$$-\frac{1}{\sqrt{2}} e_0 . \mathcal{D}_W(t) = \begin{pmatrix} \mathbb{D}\,(t) & 0 \\ 0 & \widehat{\mathbb{D}}\,(t) \end{pmatrix} .$$

The skew-symmetry of $e_0.\mathcal{D}_\Sigma$ for the inner product (3.26) on $L^2(\Sigma_t; \mathbb{S}_{\mathrm{Dirac}})$ is equivalent to the skew-symmetry of both $\sqrt{2}\,\mathbb{D} + \frac{1}{2\sqrt{2}} K$ and $\sqrt{2}\,\widehat{\mathbb{D}} + \frac{1}{2\sqrt{2}} K$ for the respective inner products on $L^2(\Sigma_t; \mathbb{S}^*)$ and $L^2(\Sigma_t; \overline{\mathbb{S}})$ defined by (note that unless the foliation is maximal, $\mathbb{D}$ and $\widehat{\mathbb{D}}$ are not skew-symmetric for these inner products)

$$\langle \phi, \psi \rangle_{L^2(\Sigma_t; \mathbb{S}^*)} = \int_\Sigma T^{AA'} \phi_A \overline{\psi}_{A'} \, \mathrm{dVol}_{h(t)} , \ \langle \varrho, \chi \rangle_{L^2(\Sigma_t; \overline{\mathbb{S}})} = \int_\Sigma T_{AA'} \varrho^{A'} \overline{\chi}^A \, \mathrm{dVol}_{h(t)} . \tag{3.33}$$

In a local coordinate basis on Σ, we see that $\mathbb{D}$ and $\widehat{\mathbb{D}}$ take the form

$$\sum_{\alpha=1}^{3} a^\alpha(t,x) \frac{\partial}{\partial x^\alpha} + b(t,x)$$

([4]) The equation $\mathrm{D}_A{}^B \phi_B = 0$ is referred to as the Sen–Witten equation in [53]. It is the equation studied by A. Sen in [56] to find the neutrino "zero-modes" or time independent normalizable solutions of the neutrino equation. It is also closely related to the Dirac–Witten operator. The notation D_a in Sen's paper refers to the Levi-Civita connection on $(\Sigma_t, h(t))$. His way of writing the "Sen–Witten" equation is therefore $\mathrm{D}_A{}^B \phi_B - (\pi/(2\sqrt{2}))\phi_A = 0$, $\pi = K/\sqrt{2}$ being the trace of the extrinsic curvature. In [53], D_a denotes $-h_a{}^b \nabla_b$. We have chosen to follow the notations of Penrose and Rindler which are now the established standard.

where a^α, $\alpha = 1, 2, 3$, are 2×2 hermitian matrices made of coefficients of the metric $h(t)$ and b is a 2×2 matrix made of connection coefficients.

Finally, we recall a fundamental property (see for example [52]) of the Dirac equation: the existence of a conserved current. We give the expressions of the current vector in terms of two-component spinors as well as Dirac spinors. The proof of this property is particularly simple in terms of two-component spinors as we shall see.

LEMMA 3.3. *Let* $\Psi := \phi_A \oplus \chi^{A'}$ *be a solution to* (3.4) *and consider the real future-oriented nonspacelike vector*

$$U^a = \phi^A \overline{\phi}^{A'} + \overline{\chi}^A \chi^{A'}. \tag{3.34}$$

Assume that Ψ *has enough regularity to give a meaning to* U^a *and its divergence. A reasonably minimal requirement is* $\Psi \in H^1_{\mathrm{loc}}(\mathbb{R}_t \times \Sigma; \mathbb{S}_{\mathrm{Dirac}})$ (*Sobolev spaces on* $\mathbb{R}_t \times \Sigma$ *can be defined using the Riemannian metric* $\mathrm{d}t^2 + \widetilde{h}$). *Then the vector field* U^a *is divergence-free*, *i.e.*

$$\nabla^a U_a = 0. \tag{3.35}$$

Consequently, *the* 3-*form* $\omega = {}^*U_a \mathrm{d}x^a$ *is closed. The symbol* * *denotes the Hodge dual defined by*

$${}^*U_a \mathrm{d}x^a = \frac{1}{6} U^a e_{abcd}\, \mathrm{d}x^b \wedge \mathrm{d}x^c \wedge \mathrm{d}x^d$$

where e_{abcd} *is an alternating tensor*, *i.e. the totally antisymmetric tensor*

$$e_{abcd} = i\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{A'D'}\varepsilon_{B'C'} - i\varepsilon_{AD}\varepsilon_{BC}\varepsilon_{A'C'}\varepsilon_{B'D'}.$$

In a spin-frame adapted to the foliation, *the current vector takes the form*

$$U^a = \frac{1}{\sqrt{2}} \sum_{\mathbf{a}=0}^{3} (\Psi^\dagger \gamma^0 \gamma^{\mathbf{a}} \Psi) e_{\mathbf{a}}{}^a. \tag{3.36}$$

Proof. Let $\Psi \in H^1_{\mathrm{loc}}(\mathbb{R}_t \times \Sigma; \mathbb{S}_{\mathrm{Dirac}})$ be a solution to (3.4) and U^a the vector (3.34). We have

$$\begin{aligned}
\nabla^a U_a &= \nabla^{AA'}(\phi_A \overline{\phi}_{A'}) + \nabla^{AA'}(\chi_{A'} \overline{\chi}_A) \\
&= \overline{\phi}_{A'} \nabla^{AA'} \phi_A + \phi_A \nabla^{AA'} \overline{\phi}_{A'} + \overline{\chi}_A \nabla^{AA'} \chi_{A'} + \chi_{A'} \nabla^{AA'} \overline{\chi}_A \\
&= \overline{\phi}_{A'} \nabla^{AA'} \phi_A + \phi_A \overline{\nabla^{AA'} \phi_A} + \overline{\chi}_A \nabla^{AA'} \chi_{A'} + \chi_{A'} \overline{\nabla^{AA'} \chi_{A'}}.
\end{aligned}$$

Using equation (3.6), the divergence of U^a becomes

$$\nabla^a U_a = \mu(\overline{\phi}_{A'} \chi^{A'} + \phi_A \overline{\chi}^A + \overline{\chi}_A \phi^A + \chi_{A'} \overline{\phi}^{A'})$$

which is identically zero due to the antisymmetry of the spinor inner product. Hence the null vector U^a is divergence-free and its dual 3-form is therefore closed.

We conclude this proof by obtaining (3.36) from (3.34). The components of U^a are given by

$$U^{\mathbf{a}} = e^{\mathbf{a}a} U_a = g^{\mathbf{a}AA'} \phi_A \overline{\phi}_{A'} + g^{\mathbf{a}}{}_{AA'} \chi^{A'} \overline{\chi}^A$$

and in a spin-frame, this yields

$$U^{\mathbf{a}} = \phi^\dagger M^{\mathbf{a}} \phi + \chi^\dagger N^{\mathbf{a}} \chi$$

where $\phi^\dagger = (\overline{\phi_0}, \overline{\phi_1}) = (\overline{\phi}_{0'}, \overline{\phi}_{1'})$ and $\chi^\dagger = (\overline{\chi^{0'}}, \overline{\chi^{1'}}) = (\overline{\chi}^0, \overline{\chi}^1)$. If the spin-frame is adapted to the foliation, then

$$\gamma^0 \gamma^{\mathbf{a}} = \sqrt{2} \begin{pmatrix} M^{\mathbf{a}} & 0 \\ 0 & N^{\mathbf{a}} \end{pmatrix}$$

and we get

$$U^{\mathbf{a}} = \frac{1}{\sqrt{2}} \Psi^\dagger \gamma^0 \gamma^{\mathbf{a}} \Psi.$$

This proves Lemma 3.3. ∎

4. The general Cauchy problem

In this chapter, we work on globally hyperbolic space-times $(\mathcal{M}, g)$, $\mathcal{M} \simeq \mathbb{R}_t \times \Sigma$, where, outside a compact subset K, the smooth 3-manifold Σ is the disjoint union of a finite number of manifolds with boundary:

$$\Sigma = K \cup \left(\bigcup_{i=1}^N M_i \right), \;\; M_i \simeq \mathbb{R}^3 \setminus \overline{B}(0,1), \;\; M_i \cap M_j = \emptyset \text{ if } i \neq j.$$

The product structure $\mathbb{R} \times \Sigma$ is associated with the vector field T^a. We use the 3+1 decomposition of Dirac's equation to solve the Cauchy problem in Sobolev and weighted Sobolev spaces on the classes of asymptotically flat metrics defined in Chapter 2. The theorems are obtained (except Theorem 4) as consequences of more general results (Propositions 4.1 and 4.2) proved for first order symmetric hyperbolic systems.

The theorems and propositions of this chapter are valid whether Σ admits one or several asymptotic ends M_i. The case with no asymptotic end (Σ compact) can also be considered: the results still hold but weighted Sobolev spaces then reduce to ordinary Sobolev spaces and the expressions of Theorem 3 and Proposition 4.2 are thus slightly modified.

We give the fundamental theorem concerning the well-posedness of the Cauchy problem for (3.4). It is valid for a wide family of metrics including the classes (k, δ), $\delta < 0$, $k \geq 3$.

THEOREM 1. *Assume the metric g to satisfy hypothesis* (H) *of Definition* 2.1 *and*

$$g \in \mathcal{C}(\mathbb{R}_t; \mathcal{C}_b^1(\Sigma)) \cap \mathcal{C}^1(\mathbb{R}_t; \mathcal{C}_b^0(\Sigma)).$$

Then, for any real number s, for any initial data $\Psi_0 \in L^2(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$, equation (3.4) *has a unique solution $\Psi_s \in \mathcal{C}(\mathbb{R}_t; L^2(\Sigma; \mathbb{S}_{\mathrm{Dirac}}))$, such that*

$$\Psi_s|_{t=s} = \Psi_0.$$

REMARK 4.1. For such low regularities of g and Ψ_s as are considered in Theorem 1, we still have a relatively natural and "strong" notion of solution. The regularity assumptions on g and Ψ_s entail in particular

$$g \in \mathcal{C}^1(\mathbb{R}_t \times \Sigma), \Psi_s \in L^2_{\mathrm{loc}}(\mathbb{R}_t \times \Sigma).$$

We can define the first order partial derivatives of Ψ_s, with respect to a given coordinate basis, in the sense of distributions; these derivatives will belong to $H^{-1}_{\mathrm{loc}}(\mathbb{R}_t \times \Sigma)$ which is the dual of the subspace $H^1_{\mathrm{comp}}(\mathbb{R}_t \times \Sigma)$ of compactly supported elements of $H^1(\mathbb{R}_t \times \Sigma)$. In equation (3.4), the coefficients of the first order terms are coefficients of the metric, they are $\mathcal{C}^1$ on $\mathbb{R}_t \times \Sigma$. These terms are therefore multipliers of $H^1_{\mathrm{comp}}(\mathbb{R}_t \times \Sigma)$ and by duality multipliers of $H^{-1}_{\mathrm{loc}}(\mathbb{R}_t \times \Sigma)$. The coefficients of the zero order terms are the mass and connection coefficients, they are continuous on $\mathbb{R}_t \times \Sigma$ and thus multipliers of $L^2_{\mathrm{loc}}(\mathbb{R}_t \times \Sigma)$. It follows that $\mathcal{D}$ acts continuously from $L^2_{\mathrm{loc}}(\mathbb{R}_t \times \Sigma)$ to $H^{-1}_{\mathrm{loc}}(\mathbb{R}_t \times \Sigma)$. Consequently, equation (3.4), for a metric g and a spinor valued function Ψ_s having the regularity assumed in Theorem 1, is to be understood as an equality in $H^{-1}_{\mathrm{loc}}(\mathbb{R}_t \times \Sigma)$. The initial data condition is completely straightforward because of the continuity in time of Ψ_s.

For all Cauchy problems subsequently considered, the function space F in which the solutions take their values will always be embedded in $L^2_{\mathrm{loc}}(\Sigma)$ and the coefficients of the first order part of the equations considered will always be at least $\mathcal{C}^1$ on $\mathbb{R}_t \times \Sigma$. Concerning the zero order part, we shall allow for regularities lower than $\mathcal{C}^0(\mathbb{R}_t \times \Sigma)$ and we shall sometimes consider general (not necessarily local) potentials. However, the zero order part will always act continuously from $\mathcal{C}(\mathbb{R}_t; F)$ to $L^1_{\mathrm{loc}}(\mathbb{R}_t; L^2_{\mathrm{loc}}(\Sigma))$, which is a distribution space. Hence, the notion of solution will be interpreted as it is here (the equation being understood as an equality in a space of distributions).

Using the well-posedness of the Cauchy problem in L^2, we can show for more regular metrics the existence of more regular solutions.

Theorem 2. *Assume that the metric g satisfies hypothesis* (H) *of Definition* 2.1 *and*

$$g \in \mathcal{C}^l(\mathbb{R}_t; \mathcal{C}^{k-l}_b(\Sigma)), \quad \forall l;\ 0 \leq l \leq k,$$

for some positive integer k, then for any initial data $\Psi_0 \in H^m(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$, $0 \leq m \leq k-1$, the solution $\Psi_s \in \mathcal{C}(\mathbb{R}_t; L^2(\Sigma; \mathbb{S}_{\mathrm{Dirac}}))$ to (3.4) *associated with s and Ψ_0 satisfies*

$$\Psi_s \in \mathcal{C}^l(\mathbb{R}_t; H^{m-l}(\Sigma; \mathbb{S}_{\mathrm{Dirac}})), \quad \forall l;\ 0 \leq l \leq m.$$

We can also show the existence of solutions with a controlled growth or fall-off at spacelike infinity. In particular, we obtain, using product theorems for weighted Sobolev spaces, the existence of solutions with values in H^{k-1}_μ, $\mu \in \mathbb{R}$, for a metric g of class (k, δ), $\delta < 0$, $k \geq 3$.

Theorem 3. *Assume that g is of class (k, δ), $k \geq 3$, $\delta < 0$, and consider some initial time $s \in \mathbb{R}$.*

1. *For any initial data $\Psi_0 \in L^2_\mu(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$, $\mu \in \mathbb{R}$, equation* (3.4) *has a unique solution $\Psi_s \in \mathcal{C}(\mathbb{R}_t; L^2_\mu(\Sigma; \mathbb{S}_{\mathrm{Dirac}}))$ such that*

$$\Psi_s|_{t=s} = \Psi_0.$$

(Note that $L^2_\mu \subsetneq L^2$ for $\mu < -3/2$ but for $\mu > -3/2$ we have $L^2 \subsetneq L^2_\mu$.)

2. *For $\mu \in \mathbb{R}$ and for $m \in \mathbb{N}$ such that $0 \leq m \leq k-1$, if the initial data Ψ_0 belongs to $H^m_\mu(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$ then the solution $\Psi_s \in \mathcal{C}(\mathbb{R}_t; L^2_\mu(\Sigma; \mathbb{S}_{\mathrm{Dirac}}))$ associated with Ψ_0 and some initial time s satisfies*

$$\Psi_s \in \mathcal{C}^l(\mathbb{R}_t; H^{m-l}_\mu(\Sigma; \mathbb{S}_{\mathrm{Dirac}})), \quad \forall l;\ 0 \leq l \leq m.$$

For metrics slightly more regular than was assumed in Theorem 1, we can prove that the charge of the solutions is conserved:

THEOREM 4. *Assume that the metric g is regular enough to guarantee that the solution of* (3.4) *associated with some compactly supported initial data in $H^1(\Sigma;\mathbb{S}_{\mathrm{Dirac}})$, belongs to*

$$\mathcal{C}(\mathbb{R}_t;H^1(\Sigma;\mathbb{S}_{\mathrm{Dirac}}))\cap\mathcal{C}^1(\mathbb{R}_t;L^2(\Sigma;\mathbb{S}_{\mathrm{Dirac}})).$$

This is in particular true for metrics satisfying hypothesis (H) *of Definition* 2.1 *and having the regularity*

$$g\in\bigcap_{l=0}^{2}\mathcal{C}^l(\mathbb{R}_t;\mathcal{C}_b^{2-l}(\Sigma)),$$

as in Theorem 2, *or*, *taking into account the compact support of the initial data*, *the finite propagation speed and the results of Theorem* 3, *for metrics of class* (k,δ), $k\geq 3$, $\delta<0$. *Consider $\Psi\in\mathcal{C}(\mathbb{R}_t;L^2(\Sigma;\mathbb{S}_{\mathrm{Dirac}}))$ a solution of* (3.4). *Define the "charge" of Ψ at time t as*

$$E(t):=\|\Psi(t)\|^2_{L^2(\Sigma_t)}=\int_\Sigma\langle\Psi(t),\Psi(t)\rangle\,\mathrm{dVol}_{h(t)}=\int_\Sigma|\Psi(t)|^2\,\mathrm{dVol}_{h(t)}. \tag{4.1}$$

Then $E(t)$ is constant throughout time.

REMARK 4.2. It is important to note that the measure $\mathrm{dVol}_{h(t)}$ in the definition of the charge above is the volume element associated with the metric $h(t)$ on Σ and therefore depends on time. Hence, the conservation of the charge is not to be understood as the conservation of the norm on a fixed L^2 space; the norm is tied in with the space-time geometry which is time-dependent.

If however $(\mathcal{M},g)$ admits a foliation by maximal hypersurfaces Σ_t, i.e. surfaces for which the trace of the extrinsic curvature vanishes, then, as was already mentioned earlier, the volume element $\mathrm{dVol}_{h(t)}$ on Σ will no longer depend on time and the charge will indeed be the norm on a fixed L^2 space on Σ.

REMARK 4.3. 1. *The regularity of the metric*: if we want a metric of class (p,δ) to have the regularity assumed in Theorem 2, we need to impose $p\geq k+2$ (and of course $\delta<0$) because of the Sobolev embedding (2.17). This means that for a metric of class (k,δ), $k\geq 3$, $\delta<0$, we can only guarantee the existence of solutions with values in H^{k-3} and this was the case in the results of [42]. In Theorem 3 however, by working entirely in weighted Sobolev spaces we avoid the use of Sobolev embeddings and we lose only one rank of regularity between the metric and the solution, which is the minimum loss possible. A similar result with (nonweighted) Sobolev spaces is given in Proposition 4.2; note however that assuming the metric to live simply in Sobolev spaces is not so well adapted to general relativity since we have less control on its fall-off at infinity than if we use weighted Sobolev spaces.

In Theorem 3 we have assumed g to be of class (k,δ) with $k\geq 3$ simply because we have only defined the classes (k,δ) for integral values of k. If we allow k to be any positive real number, then we can define the class (k,δ) as the set of metrics g satisfying hypothesis (H) and

$$g - \left(\mathrm{d}t^2 - \widetilde{h} - \frac{m}{r} \right) \in \mathcal{C}^l(\mathbb{R}_t; H_\delta^{k-l}(\Sigma)),\ \forall l;\ 0 \leq l \leq [k].$$

Of course, one needs to define Sobolev spaces of nonintegral order on Σ but this is easily done using local charts whose domains are regular open sets of Σ and the standard definition of Sobolev spaces on regular open sets of $\mathbb{R}^n$. In this new context, imposing simply $k > 5/2$ and $\delta < 0$ would just be enough to guarantee the validity of the product theorems between weighted Sobolev spaces which are the fundamental tools for the proof of Theorem 3. Thus the theorem would still hold under these hypotheses with only the slight modification that m needs to be lower than $[k] - 1$ instead of $k - 1$. We see that for $5/2 < k < 3$ we could still guarantee the existence of solutions with values in H^1_μ.

2. *Weighted Sobolev spaces*: there is a strong motivation for proving the existence of solutions to field equations with values in weighted Sobolev spaces in that it has direct applications to the analysis of Einstein's equations. Indeed, as was remarked in [42] (see Section 7.1 of this reference for more details), controlling the fall-off at spacelike infinity of solutions to the spin 3/2 field equations is a first step towards the control of the fall-off at spacelike infinity of solutions to Einstein's equations. Spin 3/2 field equations in the framework of weighted Sobolev spaces would hardly be more difficult to study than Dirac's equations (see Chapter 7 for more details).

3. *Weyl's equations*: all the results given here for Dirac's equation are naturally valid for Weyl's neutrino and anti-neutrino equations

$$\nabla^{AA'}\psi_{A'} = 0 \quad \text{and} \quad \nabla^{AA'}\eta_A = 0$$

since they are special cases of a massless Dirac equation where one of the two spinors vanishes.

Proof of Theorem 1. We choose a unitary spin-frame $\{o^A, \iota^A\}$ adapted to the foliation and a local coordinate basis x^1, x^2, x^3 on Σ. Equation (3.4) then takes the form (3.30) which we recall here

$$\frac{\partial \Psi}{\partial t} = \frac{N(t,x)}{\sqrt{2}} \left\{ \sum_{\mathbf{a}=1}^{3} A^{\mathbf{a}}(t,x) \frac{\partial \Psi}{\partial x^{\mathbf{a}}} + (-im\gamma^0 + B(t,x))\Psi \right\}$$

where $A^{\mathbf{a}}$, $\mathbf{a} = 1, 2, 3$, are 4×4 hermitian matrices made of coefficients of $h(t)$ and B is a 4×4 matrix made of space-time connection coefficients. The Dirac spinor Ψ is now represented by its components in the spin-frame and is thus to be considered as a function on $\mathbb{R}_t \times \Sigma_x$ with values in $\mathbb{C}^4$.

When the topology is trivial, the theorem is a consequence of known results concerning symmetric hyperbolic systems (see T. Kato [32] and [33]). An extension of these results to curved space-times with nontrivial topology was given in [42], but the regularity assumptions on the metric were stronger than they are here. We give and prove a general result for symmetric hyperbolic systems on $\mathbb{R} \times \Sigma$ with weakly regular coefficients:

Proposition 4.1. *Consider a symmetric hyperbolic operator $\partial/\partial t - A(t)$ on $\mathbb{R}\times\Sigma$, homogeneous of the first order, acting on $\mathcal{B}$, an n-dimensional tensor or spinor bundle on $\mathbb{R}\times\Sigma$. More precisely, this means that after a choice of local coordinates and of Lorentz frame or spin-frame, each fibre of $\mathcal{B}$ is identified with $\mathbb{C}^n$ and $A(t)$ takes the form*

$$A(t) = \sum_{i=1}^{3} a^i(t,x)\frac{\partial}{\partial x^i}$$

where the a^i's are $n\times n$ hermitian matrices. Assume that such choices have been made and consider the sections of $\mathcal{B}$ as $\mathbb{C}^n$-valued functions. Assume that the coefficients a^i have the regularity

$$a^i \in \mathcal{C}(\mathbb{R}_t;\mathcal{C}_b^1(\Sigma;\mathcal{M}_n(\mathbb{C})))$$

and consider some potential b such that

$$b\in L^1_{loc}(\mathbb{R}_t;\mathcal{L}(L^2(\Sigma;\mathbb{C}^n))).$$

Then, for any $s\in\mathbb{R}$, for any $u_0\in L^2(\Sigma;\mathbb{C}^n)$, the equation

$$\frac{\partial u}{\partial t} = A(t)u + b(t)u \tag{4.2}$$

has a unique solution $u\in\mathcal{C}(\mathbb{R}_t;L^2(\Sigma;\mathbb{C}^n))$ such that $u|_{t=s}=u_0$. The propagator for equation (4.2), *defined by*

$$\mathcal{V}(t,s): u_0 \mapsto u(t),$$

where u is the solution to the Cauchy problem as above, satisfies

(i) $\forall t,s\in\mathbb{R}$, $\mathcal{V}(t,s)\in\mathcal{L}(L^2(\Sigma;\mathbb{C}^n))$, *$\mathcal{V}$ is strongly continuous on $\mathbb{R}^2_{t,s}$ with values in $\mathcal{L}(L^2(\Sigma;\mathbb{C}^n))$ and $\mathcal{V}(t,t)=\mathrm{Id}_{L^2(\Sigma;\mathbb{C}^n)}$,*

(ii) $\forall r,s,t\in\mathbb{R}$, $\mathcal{V}(t,s)=\mathcal{V}(t,r)\mathcal{V}(r,s)$.

Note that by strongly continuous we mean continuous for the topology of strong convergence of bounded operators: if A_n and A are bounded operators from a separable Hilbert space E to another separable Hilbert space F, we say that A_n converges strongly to A as $n\to+\infty$ if $A_n\phi\to A\phi$ in F as $n\to+\infty$, for any $\phi\in E$.

Theorem 1 is then an easy consequence of the previous proposition: if the metric g is as assumed in Theorem 1, then the lapse-function N and the A^i's whose coefficients are coefficients of the metric, satisfy

$$N, A^i \in \mathcal{C}(\mathbb{R}_t;\mathcal{C}_b^1(\Sigma))\cap\mathcal{C}^1(\mathbb{R}_t;\mathcal{C}_b^0(\Sigma)).$$

The matrix B is made of connection coefficients which are first order derivatives of the metric, whence

$$\frac{N}{\sqrt{2}}(B - im\gamma^0)\in\mathcal{C}(\mathbb{R}_t;\mathcal{C}_b^0(\Sigma;\mathcal{M}_4(\mathbb{C})))\hookrightarrow L^1_{\mathrm{loc}}(\mathbb{R}_t;\mathcal{L}(L^2(\Sigma;\mathbb{C}^4))).$$

Thus Proposition 4.1 can be applied and entails Theorem 1. ■

Proof of Proposition 4.1. We solve the Cauchy problem for the "free" equation ([1])

$$\frac{\partial u}{\partial t} = A(t)u, \tag{4.3}$$

then we interpret b as a bounded operator on the space of solutions and the proposition follows by a standard fixed point theorem. We first consider the situation where Σ has trivial topology. In such a case, $(\Sigma, \widetilde{h})$ is diffeomorphic to $\mathbb{R}^3$ and all (weighted or not) Sobolev spaces on Σ are isomorphic to the same spaces on $\mathbb{R}^3$. (4.3) then becomes an equation on $\mathbb{R}_t \times \mathbb{R}^3$:

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{3} a^i(t,x)\frac{\partial u}{\partial x^i}, \qquad a^i \in \mathcal{C}(\mathbb{R}_t; \mathcal{C}_b^1(\mathbb{R}_x^3; \mathcal{M}_n(\mathbb{C}))).$$

The well-posedness of the Cauchy problem for such an equation is well known in a variety of spaces and in particular, using the isomorphisms $L^2(\Sigma) \simeq L^2(\mathbb{R}^3)$ and $H^1(\Sigma) \simeq H^1(\mathbb{R}^3)$, we get from the results of T. Kato in [32]–[34], the existence of a unique family of operators $\{\mathcal{U}(t,s)\}$ defined on $\mathbb{R}_t \times \mathbb{R}_s$ and satisfying

a. $\mathcal{U}$ is strongly continuous from $\mathbb{R}^2_{t,s}$ to $\mathcal{L}(L^2(\Sigma;\mathbb{C}^n))$ and $\mathcal{U}(t,t) = \mathrm{Id}_{L^2(\Sigma;\mathbb{C}^n)}$,

b. $\mathcal{U}(t,s)\mathcal{U}(s,r) = \mathcal{U}(t,r)$,

c. $\mathcal{U}$ is strongly continuous from $\mathbb{R}^2_{t,s}$ to $\mathcal{L}(H^1(\Sigma;\mathbb{C}^n))$,

d. $\partial\mathcal{U}(t,s)/\partial t = A(t)\mathcal{U}(t,s)$, $\partial\mathcal{U}(t,s)/\partial s = -\mathcal{U}(t,s)A(s)$ which both exist in the strong sense in $\mathcal{L}(H^1(\Sigma;\mathbb{C}^n); L^2(\Sigma;\mathbb{C}^n))$ and are strongly continuous from $\mathbb{R}^2_{t,s}$ to this space.

We can also remark that the solutions propagate at a finite speed and the propagation speed at each time t is estimated uniformly on $\mathbb{R}^3$ by the continuous function of t

$$C(t) = 3\sup\{\|a^i(t,x)\|;\ 1 \le i \le 3,\ x \in \mathbb{R}^3\}. \tag{4.4}$$

This is a standard result for first order symmetric hyperbolic systems on $\mathbb{R}^3$. A proof can be found in R. Racke [55] for $\mathcal{C}^1$ solutions. We can prove it in our less regular case by performing the same estimates as in [55] for a smooth function on $\mathbb{R}\times\mathbb{R}^3$ not supposed to satisfy the equation, extending the final estimate to functions in $\mathcal{C}(\mathbb{R}_t; L^2(\mathbb{R}^3))$ by density and simplifying it at last by restricting it to solutions of the equation. Of course (4.4) implies that the propagation speed on $(\Sigma, \widetilde{h})$, i.e. for the distance associated with $\widetilde{h}$ which is uniformly equivalent to the euclidian distance on $\mathbb{R}^3$, is estimated at each time by

$$\widetilde{C}(t) = \lambda C(t)$$

where λ is a positive constant such that, for each $x, y \in \Sigma \simeq \mathbb{R}^3$ we have $\mathrm{d}_{\widetilde{h}}(x,y) \le \lambda|x-y|$, $|\cdot|$ being the euclidian distance on $\mathbb{R}^3$. This result will be of some technical importance for solving the Cauchy problem in nontrivial topology and for proving the conservation of the charge of Dirac fields.

([1]) The rather inappropriate denomination "free equation", which appears several times in this work, comes from a bad habit that the author has contracted doing scattering theory and talking with other scattering theorists. What we mean by this is merely a simplified equation, obtained by removing some potential ("interaction"). We also tend to refer to the propagators for these "free" equations as "free" propagators.

If the topology of Σ is not trivial, it is in any case necessarily finite: indeed, it is entirely determined by the (finite) number of asymptotically flat ends and by the (finite since Σ is a manifold) topology of Σ inside a large enough compact subset. Hence we can cover Σ with a finite number of smooth open sets of trivial topology $\{\Omega_i\}_{1\leq i\leq p}$, $p \in \mathbb{N}^*$. We have on each Ω_i a control on the propagation speed of type (4.4) where the sup is taken over Ω_i and this control can be made uniform over Σ using the fact that $a^i \in \mathcal{C}(\mathbb{R}_t; \mathcal{C}^1_b(\Sigma))$. Hence, the propagation speed for the solutions of (4.3) is controlled on $\mathbb{R}_t \times \Sigma$ by a continuous function of t: $\widetilde{C}(t)$. We use this property to define "domains of dependence" for the sets Ω_i. This is done by evolving the boundary $\partial\Omega_i$ of Ω_i along the flow of the vector field

$$\widetilde{C}(t)\nu^a + \frac{1}{\sqrt{2}}T^a,$$

where ν^a is the interior unit normal to $\partial\Omega_i$, from a certain time $t_0 \in \mathbb{R}$ on. For $t \geq t_0$, we obtain open sets $\Omega_i(t)$, delimited by the evolved boundary, with $\Omega_i(t_0) = \Omega_i$. The continuity of $\widetilde{C}(t)$ makes it possible to choose the covering $\{\Omega_i\}$ so that, when evolving the sets into their domains of dependence from a time t_0, $\{\Omega_i(t)\}_{1\leq i\leq p}$ is still a covering of Σ by trivial topology smooth open sets if $t_0 \leq t \leq t_0 + \varepsilon$ for a certain $\varepsilon > 0$. Moreover, ε can be chosen independently of t_0 if t_0 is restricted to belong to a compact time interval.

Let us now consider some initial time $s \in \mathbb{R}$ and some initial data $u_0 \in L^2(\Sigma; \mathbb{C}^n)$. We consider $T > 0$ fixed and study the existence of the solution on $[s-T, s+T]$. We first put $t_0 = s$. Using the well-posedness of the Cauchy problem in trivial topology, we have in the domain of dependence of each Ω_i the existence of a unique solution continuous in time with values in L^2. Uniqueness in the trivial topology case guarantees local uniqueness and allows us to recover a global solution on Σ from the solutions in the domains of dependence as long as the $\Omega_i(t)$ form a covering of Σ, i.e. at least on $[t_0, t_0+\varepsilon]$ where $\varepsilon > 0$ is as above and is fixed on the whole interval $[s-T, s+T]$. The solution thus obtained is continuous on $[s, s+\varepsilon]$ with values in $L^2(\Sigma)$: indeed, we have a finite number of sets in the covering and for each $t \in [s, s+\varepsilon]$, the norm on $L^2(\Sigma)$ is equivalent (the equivalence being uniform on $[s, s+\varepsilon]$) to the sum of the flat L^2 norms on each $\Omega_i(t)$, i.e.

$$\|f\|_{L^2(\Sigma)} = \Big\{ \int_\Sigma \langle f, f\rangle \,\mathrm{dVol}_{\tilde{h}} \Big\}^{1/2} \simeq \Big\{ \sum_i \int_{\Omega_i(t)} |f(x)|^2 \,\mathrm{d}\mu_L(x) \Big\}^{1/2} \tag{4.5}$$

where $\mathrm{d}\mu_L$ is the Lebesgue measure on $\mathbb{R}^3$. Then we put $t_0 = s+\varepsilon$. In this manner, by steps of length ε, we propagate the solution forward in time up to time $s+T$. Reversing time in equation (4.3) allows us to propagate backwards down to $s-T$. Hence we have the existence of a unique solution in $\mathcal{C}([s-T, s+T]; L^2(\Sigma; \mathbb{C}^n))$ for any $T > 0$ fixed, that is, the existence and uniqueness on $\mathcal{C}(\mathbb{R}_t; L^2(\Sigma; \mathbb{C}^n))$.

The well-posedness of the Cauchy problem can be expressed in terms of propagators. In each domain of dependence of a set Ω_i we have a local propagator inherited from the flat space propagator and satisfying local analogues of properties a,b,c,d. Local uniqueness allows us to patch together these propagators to recover the global propagator for the equation on Σ, which we shall still denote by $\mathcal{U}(t,s)$. Using the norm-equivalence (4.5) and the corresponding equivalence for H^1 norms

$$\|f\|_{H^1(\Sigma)} = \Big\{ \int_\Sigma (\langle f, f\rangle + \langle \widetilde{\mathrm{D}} f, \widetilde{\mathrm{D}} f\rangle) \,\mathrm{dVol}_{\widetilde{h}} \Big\}^{1/2} \simeq \Big\{ \sum_i \int_{\Omega_i(t)} \Big(|f(x)|^2 + \Big|\frac{\partial f}{\partial x^1}(x)\Big|^2 + \Big|\frac{\partial f}{\partial x^2}(x)\Big|^2 + \Big|\frac{\partial f}{\partial x^3}(x)\Big|^2 \Big) \,\mathrm{d}\mu_L(x) \Big\}^{1/2}, \tag{4.6}$$

we infer from the properties of the local propagators that $\mathcal{U}(t,s)$ has properties a, b, c and d. Hence, we can guarantee the existence of solutions with values in H^1 as well as L^2.

In order to solve the Cauchy problem for equation (4.2) in $\mathcal{C}(\mathbb{R}_t; L^2(\Sigma))$, we express it as a fixed point problem for an integral functional (this is the so-called Duhamel principle). Given $u_0 \in L^2(\Sigma)$ and $s, T \in \mathbb{R}$, the following two problems are equivalent:

$$\frac{\partial u}{\partial t} = A(t)u + b(t)u, \quad u \in \mathcal{C}([s,T]; L^2(\Sigma)), \quad u|_{t=s} = u_0, \tag{4.7}$$

and

$$u(t) = \mathcal{S}u(t) := \mathcal{U}(t,s)u_0 + \int_s^t \mathcal{U}(t,\tau)b(\tau)u(\tau)\,\mathrm{d}\tau, \quad u \in \mathcal{C}([s,T]; L^2(\Sigma)), \tag{4.8}$$

where $\mathcal{U}(t,s)$ is the propagator for the free equation (4.3) constructed above.

Before solving problem (4.8), we justify the equivalence between (4.7) and (4.8). We work in the case of a source, the extension to the potential being trivial as we shall see. We consider the equation

$$\frac{\partial u}{\partial t} = A(t)u + f(t) \tag{4.9}$$

where $f \in L^1_{\mathrm{loc}}(\mathbb{R}_t; L^2(\Sigma))$. The fact that the solutions to the Cauchy problem for (4.9) in L^2 are given by the integral formula

$$u(t) = \mathcal{U}(t,s)u_0 + \int_s^t \mathcal{U}(t,\tau)f(\tau)\,\mathrm{d}\tau \tag{4.10}$$

was justified in an abstract setting by H. Tanabe [61], using the theory of analytic semigroups, under the assumption that f is $\mathcal{C}^1$ in time and for solutions living in the common domain of the operators $A(t)$. We have here the advantage of dealing with operators $A(t)$ which are differential operators. This allows us to give to the propagator $\mathcal{U}$ on L^2 a somewhat stronger meaning than what is possible to achieve in the abstract framework. We have, in the strong sense on $\mathcal{L}(L^2(\Sigma;\mathbb{C}^n); H^{-1}(\Sigma;\mathbb{C}^n))$, $\partial\mathcal{U}(t,s)/\partial t = A(t)\mathcal{U}(t,s)$; this operator is strongly continuous from $R^2_{t,s}$ to $\mathcal{L}(L^2(\Sigma;\mathbb{C}^n); H^{-1}(\Sigma;\mathbb{C}^n))$ as the composition of $\mathcal{U}$, strongly continuous from $R^2_{t,s}$ to $\mathcal{L}(L^2(\Sigma;\mathbb{C}^n))$ with $A(t)$, strongly continuous from R_t to $\mathcal{L}(L^2(\Sigma;\mathbb{C}^n); H^{-1}(\Sigma;\mathbb{C}^n))$. Using this remark, we can show that for $u_0 \in L^2(\Sigma)$, the function $u \in \mathcal{C}(\mathbb{R}_t; L^2(\Sigma))$ defined by (4.10) satisfies (4.9): since $\mathcal{U}(t,s)$ is closed on $L^2(\Sigma)$, we have

$$\int_s^t \mathcal{U}(t,\tau)f(\tau)\,\mathrm{d}\tau = \mathcal{U}(t,s)\int_s^t \mathcal{U}(s,\tau)f(\tau)\,\mathrm{d}\tau$$

and therefore, all time derivatives being strong derivatives on $\mathbb{R}_t$ with values in $H^{-1}(\Sigma)$,

$$\begin{aligned}\frac{\partial}{\partial t}u(t) &= \Big(\frac{\partial}{\partial t}\mathcal{U}(t,s)\Big)\Big(u_0+\int_s^t \mathcal{U}(s,\tau)f(\tau)\,\mathrm{d}\tau\Big)+\mathcal{U}(t,s)\frac{\partial}{\partial t}\Big(\int_s^t \mathcal{U}(s,\tau)f(\tau)\,\mathrm{d}\tau\Big)\\ &= A(t)\mathcal{U}(t,s)\Big(u_0+\int_s^t \mathcal{U}(s,\tau)f(\tau)\,\mathrm{d}\tau\Big)+\mathcal{U}(t,s)\mathcal{U}(s,t)f(t)=A(t)u(t)+f(t).\end{aligned}$$

Therefore u is a solution of (4.9) such that $u(s)=u_0$. Moreover, the uniqueness in $\mathcal{C}(\mathbb{R}_t;L^2(\Sigma))$ of solutions to (4.9) is guaranteed by the uniqueness for (4.3) in the same class. This proves that the solutions to the Cauchy problem for (4.9) in L^2 are given by (4.10). Now, the equivalence between (4.7) and (4.8) is easily seen. If u is a solution of (4.8), then by the calculation above with $f(t)$ replaced by $b(t)u(t)$, we show that it satisfies (4.2). Conversely, for u solution to (4.7), putting $f(t)=b(t)u(t)\in L^1_{\mathrm{loc}}([s,T];L^2(\Sigma))$, we have

$$\frac{\partial}{\partial t}u(t)=A(t)u(t)+f(t),\qquad u(s)=u_0,$$

whence, by the expression of solutions in the case of a source, u satisfies (4.8).

Remark 4.4. In the proofs of Proposition 4.2 and Theorems 5 and 6, we shall rely a number of times on Duhamel's principle. In each case, the function space where the solutions live will be embedded in L^2 or at least in L^2_{loc} and so, the equivalence between the Cauchy problem and its integral formulation can be justified just as we did here; the only slight difference may be that $A(t)\mathcal{U}(t,s)$ acts continuously from the function space considered to H^{-1}_{loc} instead of H^{-1}.

We now proceed to solving problem (4.8). This is easily done by remarking that the space $\mathcal{C}([s,T];L^2(\Sigma))$ is stable under the functional $\mathcal{S}$ and for $|T-s|$ small enough, $\mathcal{S}$ is a strict contraction on the closed ball

$$\mathcal{B}_{s,T,u_0}:=\{u\in\mathcal{C}([s,T];L^2(\Sigma));\ \|u(t)\|_{L^2(\Sigma)}\le 2\|u_0\|_{L^2(\Sigma)}\ \forall t\in[s,T]\}.$$

Whence the existence of a unique fixed point for $\mathcal{S}$ in $\mathcal{C}([s,T];L^2(\Sigma))$ for $|T-s|$ small enough. The uniform boundedness of $\mathcal{U}$ and the integrability of b on each compact time interval imply that any solution to (4.8) on a compact time interval $[s,T]$ satisfies the inequality (obtained using Gronwall's lemma):

$$\|u(t)\|_{L^2(\Sigma)}\le C(s,T)\exp\Big(C(s,T)\int_{[s,T]}\|b(\tau)\|_{\mathcal{L}(L^2(\Sigma))}\,\mathrm{d}\tau\Big)\|u_0\|_{L^2(\Sigma)}$$

where $C(s,T)=\sup\{\|\mathcal{U}(t,\tau)\|_{\mathcal{L}(L^2(\Sigma))};\ t,\tau\in[s,T]\}$. Hence, any solution to (4.7) on $[s,T]$ is uniformly bounded in $L^2(\Sigma)$ on this interval, which suffices to ensure the existence of a unique solution to the Cauchy problem for (4.2) in $\mathcal{C}(\mathbb{R}_t;L^2(\Sigma))$. Properties (i) and (ii) of the propagator are straightforward consequences of the uniqueness of solutions and of the integral formulation (4.8) of the Cauchy problem together with the properties of $\mathcal{U}$ and b. This proves Proposition 4.1. ∎

Remark 4.5. It is worth noting that in the case where b is simply a matrix valued potential, as it is the case in equation (3.30), the solutions to (4.2) propagate at a finite speed with the same bound on the propagation speed as the solutions to (4.3). This can be proved by considering the case where the initial data u_0 is compactly supported and restricting the functional $\mathcal{S}$ to the closed subspace E_{s,T,u_0} of $\mathcal{C}([s,T];L^2(\Sigma))$, made of elements u such that $\mathrm{supp}(u(s)) \subset \mathrm{supp}(u_0)$ and the support of u propagates at a speed less than $\widetilde{C}(t)$ at each time t. This subset is stable under $\mathcal{S}$ thanks to the bound on the propagation speed for (4.3) and $\mathcal{S}$ is a strict contraction for $|T-s|$ small enough on $E_{s,T,u_0} \cap \mathcal{B}_{s,T,u_0}$ which is not empty. This shows that the fixed point of $\mathcal{S}$ in $\mathcal{C}([s,T];L^2(\Sigma))$ belongs to E_{s,T,u_0} and entails the propagation of the solutions to (4.2) at a speed lower than $\widetilde{C}(t)$.

Proof of Theorems 2 and 3. As in the proof of Theorem 1, we prove a general result for first order symmetric hyperbolic systems on $\mathbb{R} \times \Sigma$:

PROPOSITION 4.2. *Consider the same first order symmetric hyperbolic system* (4.2) *as for Proposition* 4.1.

1. *If the coefficients of the equation satisfy*: $a^i \in \mathcal{C}(\mathbb{R}_t; \mathcal{C}_b^p(\Sigma; \mathcal{M}_n(\mathbb{C})))$, *for some* $p \in \mathbb{N}^*$, *and* $b \in L^1_{\mathrm{loc}}(\mathbb{R}_t; \mathcal{L}(H^p(\Sigma;\mathbb{C}^n)))$, *then the propagator for equation* (4.2) *satisfies*

$$\text{(4.11)} \qquad \mathcal{V} \text{ is strongly continuous from } \mathbb{R}^2_{t,s} \text{ to } \mathcal{L}(H^p(\Sigma;\mathbb{C}^n)).$$

If moreover $a^i \in \mathcal{C}^l(\mathbb{R}_t; \mathcal{C}_b^{p-l}(\Sigma; \mathcal{M}_n(\mathbb{C})))$ *for all* $l \in \mathbb{N}$ *such that* $0 \le l \le p$ *and we also assume* $b \in \mathcal{C}^l(\mathbb{R}_t; \mathcal{L}(H^{p-l}(\Sigma;\mathbb{C}^n); H^{p-l-1}(\Sigma;\mathbb{C}^n)))$ *for all* $l \in \mathbb{N}$, $0 \le l \le p-1$, *then for any initial time* $s \in \mathbb{R}$ *and any initial data* $u_0 \in H^p(\Sigma;\mathbb{C}^n)$, *the solution* u *to* (4.2) *associated with* s *and* u_0 *satisfies*

$$\text{(4.12)} \qquad u \in \mathcal{C}^l(\mathbb{R}_t; H^{p-l}(\Sigma;\mathbb{C}^n)),\ \forall l;\ 0 \le l \le p.$$

2. *Given* $\varrho \in \mathbb{R}$, *if* $a^i \in \mathcal{C}(\mathbb{R}_t; \mathcal{C}_b^1(\Sigma; \mathcal{M}_n(\mathbb{C})))$ *and* $b \in L^1_{\mathrm{loc}}(\mathbb{R}_t; \mathcal{L}(L^2_\varrho(\Sigma;\mathbb{C}^n)))$, *for any initial time* $s \in \mathbb{R}$ *and any initial data* $u_0 \in L^2_\varrho(\Sigma;\mathbb{C}^n)$, *equation* (4.2) *has a unique solution* $u \in \mathcal{C}(\mathbb{R}_t; L^2_\varrho(\Sigma;\mathbb{C}^n))$ *such that* $u(s) = u_0$. *Hence the propagator for equation* (4.2) *can be defined on* $L^2_\varrho(\Sigma;\mathbb{C}^n)$ *and satisfies*

$$\text{(4.13)} \qquad \mathcal{V} \text{ is strongly continuous from } \mathbb{R}^2_{t,s} \text{ to } \mathcal{L}(L^2_\varrho(\Sigma;\mathbb{C}^n)).$$

3. *Consider* $\nu < 0$, $k \in \mathbb{N}$, $k \ge 3$, $\varrho \in \mathbb{R}$ *and* $m \in \mathbb{N}$, $m \le k$. *Moreover, consider three fixed hermitian matrices* a_0^1, a_0^2, $a_0^3 \in \mathcal{M}_n(\mathbb{C})$, *corresponding to the limits at spacelike infinity of* a^1, a^2, a^3. *If* $a^i - a_0^i \in \mathcal{C}(\mathbb{R}_t; H^k_\nu(\Sigma; \mathcal{M}_n(\mathbb{C})))$ *and* $b \in L^1_{\mathrm{loc}}(\mathbb{R}_t; \mathcal{L}(H^m_\varrho(\Sigma;\mathbb{C}^n)))$, *then*

$$\text{(4.14)} \qquad \mathcal{V} \text{ is strongly continuous from } \mathbb{R}^2_{t,s} \text{ to } \mathcal{L}(H^m_\varrho(\Sigma;\mathbb{C}^n)).$$

If moreover we suppose that $a^i - a_0^i \in \mathcal{C}^l(\mathbb{R}_t; H^{k-l}_\nu(\Sigma; \mathcal{M}_n(\mathbb{C})))$ *for all* $l \in \mathbb{N}$, $0 \le l \le k$ *and* $b \in \mathcal{C}^l(\mathbb{R}_t; \mathcal{L}(H^{m-l}_\varrho(\Sigma;\mathbb{C}^n); H^{m-l-1}_\varrho(\Sigma;\mathbb{C}^n)))$ *for all* $l \in \mathbb{N}$, $0 \le l \le m-1$, *then for any initial data* $u_0 \in H^m_\varrho(\Sigma;\mathbb{C}^n)$ *and any initial time* $s \in \mathbb{R}$, *the solution* u *to (4.2) associated with* s *and* u_0 *satisfies*

$$\text{(4.15)} \qquad u \in \mathcal{C}^l(\mathbb{R}_t; H^{m-l}_\varrho(\Sigma;\mathbb{C}^n)), \qquad \forall l;\ 0 \le l \le m.$$

4. *This last part extends a result given in* [62], p. 364. *Similarly to part* 3, *we consider* $k \in \mathbb{N}$, $k \ge 3$, $m \in \mathbb{N}$, $m \le k$ *and three hermitian matrices* a_0^1, a_0^2, $a_0^3 \in \mathcal{M}_n(\mathbb{C})$. *If* $a^i - a_0^i \in \mathcal{C}(\mathbb{R}_t; H^k(\Sigma; \mathcal{M}_n(\mathbb{C})))$ *and* $b \in L^1_{\mathrm{loc}}(\mathbb{R}_t; \mathcal{L}(H^m(\Sigma;\mathbb{C}^n)))$, *then*

(4.16) $\mathcal{V}$ *is strongly continuous from* $\mathbb{R}^2_{t,s}$ *to* $\mathcal{L}(H^m(\Sigma;\mathbb{C}^n))$.

If moreover $a^i - a^i_0 \in \mathcal{C}^l(\mathbb{R}_t; H^{k-l}(\Sigma;\mathcal{M}_n(\mathbb{C})))$ *for all* $l \in \mathbb{N}$ *such that* $0 \le l \le k$ *and* $b \in \mathcal{C}^l(\mathbb{R}_t;\mathcal{L}(H^{m-l}(\Sigma;\mathbb{C}^n);H^{m-l-1}(\Sigma;\mathbb{C}^n)))$ *for all* $l \in \mathbb{N}$, $0 \le l \le m-1$, *then for any initial data* $u_0 \in H^m(\Sigma;\mathbb{C}^n)$ *and any initial time* $s \in \mathbb{R}$, *the solution* u *to* (4.2) *associated with* s *and* u_0 *satisfies*

$$u \in \mathcal{C}^l(\mathbb{R}_t; H^{m-l}(\Sigma;\mathbb{C}^n)), \quad \forall l;\ 0 \le l \le m. \tag{4.17}$$

It is easy to show that Proposition 4.2 entails Theorems 2 and 3. In Theorem 2, the regularity of the metric implies that the coefficients of the Dirac equation satisfy

$$N, A^i \in \mathcal{C}^l(\mathbb{R}_t;\mathcal{C}^{k-l}_b(\Sigma)), \quad \forall l;\ 0 \le l \le k,$$
$$\frac{N}{\sqrt{2}}(B - im\gamma^0) \in \mathcal{C}^l(\mathbb{R}_t;\mathcal{C}^{k-l-1}_b(\Sigma;\mathcal{M}_4(\mathbb{C}))), \quad \forall l;\ 0 \le l \le k-1$$

and

$$\mathcal{C}^l(\mathbb{R}_t;\mathcal{C}^{k-l-1}_b(\Sigma;\mathcal{M}_4(\mathbb{C}))) \hookrightarrow \mathcal{C}^l(\mathbb{R}_t;\mathcal{L}(H^{k-l}(\Sigma;\mathbb{C}^4);H^{k-l-1}(\Sigma;\mathbb{C}^4))).$$

Hence, applying point 1 of Proposition 4.2 gives the result and Theorem 2 is proved. ∎

In Theorem 3, for all $\nu > \max(\delta,-1)$, we have for $\mathbf{a} = 1,2,3$

$$N - \sqrt{2} \in \mathcal{C}^l(\mathbb{R}_t; H^{k-l}_\nu(\Sigma)), \quad \forall l;\ 0 \le l \le k, \tag{4.18}$$

$$\frac{N}{\sqrt{2}}A^{\mathbf{a}} - A^{\mathbf{a}}_0 \in \mathcal{C}^l(\mathbb{R}_t; H^{k-l}_\nu(\Sigma)), \quad \forall l;\ 0 \le l \le k, \tag{4.19}$$

$$B \in \mathcal{C}^l(\mathbb{R}_t; H^{k-l-1}_\nu(\Sigma)), \quad \forall l;\ 0 \le l \le k-1, \tag{4.20}$$

where $A^{\mathbf{a}}_0$ are the coefficients of the first order terms of the Dirac equation on $(\mathbb{R}\times\Sigma, \mathrm{d}t^2 - \widetilde{h})$ in the same coordinate basis. (4.18) and (4.19) are simple consequences of (2.18): the regularity of the derivatives is clear since they essentially are derivatives of the metric; the only slight difficulty is to show that $(N/\sqrt{2})A^{\mathbf{a}} - A^{\mathbf{a}}_0$ belongs to the right weighted L^2 space and this is done using the fact that $g - (\mathrm{d}t^2 - \widetilde{h})$ tends to zero at infinity (in addition to being in the required weighted L^2 space) and doing an asymptotic expansion at infinity of $(N/\sqrt{2})A^{\mathbf{a}} - A^{\mathbf{a}}_0$.

The important thing is that ν is allowed to be strictly negative. The weight ν in the regularity of B is due to the presence of timelike derivatives of the metrics; if we had only spacelike derivatives, the weight would be $\nu - 1$. In order to study the regularity of the potential, we write it as

$$\frac{N}{\sqrt{2}}(B - im\gamma^0) = \left(\frac{N}{\sqrt{2}} - 1\right)B - im\left(\frac{N}{\sqrt{2}} - 1\right)\gamma^0 + B - im\gamma^0.$$

Using (4.18), (4.20) and the continuous multiplication property (see Y. Choquet-Bruhat and D. Christodoulou [9])

$$H^{s_1}_{\mu_1}(\Sigma) \times H^{s_2}_{\mu_2}(\Sigma) \hookrightarrow H^s_\mu(\Sigma) \quad \text{for } s_1, s_2 \ge s,\ s < s_1 + s_2 - \frac{3}{2},\ \mu > \mu_1 + \mu_2, \tag{4.21}$$

we can easily show that

$$\left(\frac{N}{\sqrt{2}} - 1\right)B \in \mathcal{C}^l(\mathbb{R}_t; H^{k-l-1}_\nu(\Sigma)), \quad \forall l;\ 0 \le l \le k-1.$$

This is done by proving that for $p+q=l\leq k-1$, we have

$$\partial_t^p\left(\frac{N}{\sqrt{2}}-1\right)\partial_t^q B\in \mathcal{C}^l(\mathbb{R}_t;H_\nu^{k-l-1}(\Sigma))$$

which only requires $k>3/2$ and $\nu<0$. Consequently,

$$\frac{N}{\sqrt{2}}(B-im\gamma^0)+im\gamma^0\in \mathcal{C}^l(\mathbb{R}_t;H_\nu^{k-l-1}(\Sigma)),\qquad \forall l;\ 0\leq l\leq k-1.$$

We then show, using (4.21) that, for $l,m\in\mathbb{N}$, $0\leq l\leq m-1$, $1\leq m\leq k-1$ and for $\varrho\in\mathbb{R}$,

$$H_\nu^{k-l-1}(\Sigma)\hookrightarrow \mathcal{L}(H_\varrho^{m-l}(\Sigma);H_\varrho^{m-l-1}(\Sigma)).$$

The term $im\gamma^0$ being constant, we conclude that

$$\frac{N}{\sqrt{2}}(B-im\gamma^0)\in \mathcal{C}^l(\mathbb{R}_t;\mathcal{L}(H_\varrho^{m-l}(\Sigma);H_\varrho^{m-l-1}(\Sigma)))$$

for $0\leq l\leq m-1$, $1\leq m\leq k-1$, $\varrho\in\mathbb{R}$. We also show in the same manner that, provided $k>5/2$,

$$\frac{N}{\sqrt{2}}(B-im\gamma^0)\in \mathcal{C}(\mathbb{R}_t;\mathcal{L}(H_\varrho^{m}(\Sigma)))$$

for $0\leq m\leq k-1$ and $\varrho\in\mathbb{R}$. Hence, Theorem 3 is a consequence of points 2 and 3 of Proposition 4.2. ■

Remark 4.6. It is easily seen that the conditions imposed on the a^i's and b in Proposition 4.2 to obtain (4.15) and (4.17) are slightly too strong. However, we have chosen these conditions because they arise naturally for wave equations such as Dirac or Rarita–Schwinger and the optimal conditions would make the proposition much less readable.

Proof of Proposition 4.2. We prove the four points of the proposition only in the case of trivial topology. Since all the spaces in which the solutions take their values have norms defined by integrals over Σ, if the topology of Σ is not trivial we can always take advantage of the finite propagation speed to localize the problem into the domains of dependence of the open sets in a covering such as in Proposition 4.1. In each case the norm on the function space on Σ will be equivalent to the sum of the analogous flat norms in the sets of the covering. Thus, the results in trivial topology can be extended to nontrivial topology in exactly the same manner as in Proposition 4.1. We simply work with the equation on $\mathbb{R}_t\times\mathbb{R}_x^3$:

$$\frac{\partial u}{\partial t}=\sum_{i=1}^{3}a^i(t,x)\frac{\partial u}{\partial x^i}+b(t)u. \tag{4.22}$$

We only need to establish the regularity of the propagator for the equation

$$\frac{\partial u}{\partial t}=\sum_{i=1}^{3}a^i(t,x)\frac{\partial u}{\partial x^i}, \tag{4.23}$$

the potential term $b(t)$ being treated via a fixed point method. In each case, we have chosen for b exactly the regularity required to solve the integral problem (4.8) with values in the right space. Further, the integral formulation allows us to interpret properties (4.11), (4.13), (4.14) and (4.16) of the propagator $\mathcal{V}$ for equation (4.22) as straightforward consequences of the same properties for the free propagator ([2]) $\mathcal{U}$ and the regularity of b. In cases where we assume more regularity on b and the a^i's, the properties (4.12), (4.15) and (4.17) of the solution can be read off immediately from equation (4.22).

Each part of the proposition is deduced from the results of Proposition 4.1, either through the use of suitable identifying operators, or, in the case of weighted Sobolev spaces, simply by differentiating the equation.

Point 1: Since $a^i \in \mathcal{C}(\mathbb{R}_t; \mathcal{C}_b^p(\mathbb{R}^3))$, $p \geq 1$, we know from [32]–[34] (see proof of Proposition 4.1 above) that the free propagator $\mathcal{U}$ exists as a bounded operator on $L^2(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$. The property

$$(\mathcal{P}_m) \qquad \mathcal{U} \text{ strongly continuous from } \mathbb{R}^2_{t,s} \text{ to } \mathcal{L}(H^m(\mathbb{R}^3; \mathbb{C}^n))$$

is true for $m = 0$ and $m = 1$. We now assume $p \geq 2$. If we can infer from $(\mathcal{P}_m)$ the property $(\mathcal{P}_{m+2})$, provided $0 \leq m \leq m+2 \leq p$, then (4.11) is proved for $\mathcal{U}$. Let us assume $(\mathcal{P}_m)$ true and consider some initial data $u_0 \in H^{m+2}(\mathbb{R}^3; \mathbb{C}^n)$ and some initial time $s \in \mathbb{R}$; u being the associated solution, we put

$$v = (\mathrm{Id} - \Delta)u.$$

$\mathrm{Id} - \Delta$ is an isomorphism from $H^{\sigma+2}(\mathbb{R}^3)$ onto $H^\sigma(\mathbb{R}^3)$ for any $\sigma \in \mathbb{R}$. Thus, $v_0 := v(s) \in H^m(\mathbb{R}^3; \mathbb{C}^n)$ and v satisfies the equation

$$\frac{\partial v}{\partial t} = \sum_{i=1}^3 a^i(t,x) \frac{\partial v}{\partial x^i} + \sum_{i=1}^3 [\mathrm{Id} - \Delta, a^i(t,x)] \frac{\partial}{\partial x^i} (\mathrm{Id} - \Delta)^{-1} v. \tag{4.24}$$

Since $a^i \in \mathcal{C}(\mathbb{R}_t; \mathcal{C}_b^p(\mathbb{R}^3))$, $p \geq 2$, the commutator $[\mathrm{Id} - \Delta, a^i(t,x)]$ is of the form

$$[\mathrm{Id} - \Delta, a^i(t,x)] = \sum_{j=1}^3 \left(c^{ij}(t,x) \frac{\partial}{\partial x^j} + d^{ij}(t,x) \right)$$

where

$$c^{ij} \in \mathcal{C}(\mathbb{R}_t; \mathcal{C}_b^{p-1}(\mathbb{R}^3)), \; d^{ij} \in \mathcal{C}(\mathbb{R}_t; \mathcal{C}_b^{p-2}(\mathbb{R}^3)).$$

Therefore, the potential satisfies

$$\sum_{i=1}^3 [\mathrm{Id} - \Delta, a^i(t,x)] \frac{\partial}{\partial x^i} (\mathrm{Id} - \Delta)^{-1} \in \mathcal{C}(\mathbb{R}_t; \mathcal{L}(H^m(\mathbb{R}^3))).$$

This together with $(\mathcal{P}_m)$ implies that equation (4.24) has a propagator $\mathcal{W}$ that is strongly continuous from $\mathbb{R}^2_{t,s}$ to $\mathcal{L}(H^m(\mathbb{R}^3))$. Hence, the free propagator $\mathcal{U}$ can be defined on H^{m+2} by

$$\mathcal{U}(t,s) = (\mathrm{Id} - \Delta)^{-1} \mathcal{W}(t,s) (\mathrm{Id} - \Delta)$$

and is strongly continuous from $\mathbb{R}^2_{t,s}$ to $\mathcal{L}(H^{m+2}(\mathbb{R}^3))$.

Point 2: Given $\varrho \in \mathbb{R}$, we use the following isomorphism between L^2_ϱ and L^2:

$$\Phi : f \mapsto (1+r^2)^{-(\varrho+3/2)/2} f, \qquad \Phi : L^2_\varrho \xrightarrow{\simeq} L^2.$$

([2]) See footnote 1 of this chapter.

Given $u_0 \in L^2_\varrho(\mathbb{R}^3;\mathbb{C}^n)$ and $s \in \mathbb{R}$, we deduce that $u \in \mathcal{C}(\mathbb{R}_t; L^2_\varrho(\mathbb{R}^3;\mathbb{C}^n))$ is a solution of (4.23) such that $u(s) = u_0$ if and only if $v := \Phi(u) \in \mathcal{C}(\mathbb{R}_t; L^2(\mathbb{R}^3;\mathbb{C}^n))$ is a solution of

$$\frac{\partial v}{\partial t} = \sum_{i=1}^{3} a^i(t,x)\frac{\partial v}{\partial x^i} + \sum_{i=1}^{3} a^i(t,x)\left[(1+r^2)^{-(\varrho+3/2)/2}, \frac{\partial}{\partial x^i}\right](1+r^2)^{(\varrho+3/2)/2} v \tag{4.25}$$

such that $v(s) = v_0 := \Phi(u_0) = (1+r^2)^{-(\varrho+3/2)/2} u_0$. The commutator is simply the smooth function

$$\left[(1+r^2)^{-(\varrho+3/2)/2}, \frac{\partial}{\partial x^i}\right] = -\frac{\partial}{\partial x^i}((1+r^2)^{-(\varrho+3/2)/2}) = O(r^{-(\varrho+5/2)}), \text{ as } r \to +\infty.$$

Therefore the potential satisfies

$$\sum_{i=1}^{3} a^i(t,x)\left[(1+r^2)^{-(\varrho+3/2)/2}, \frac{\partial}{\partial x^i}\right](1+r^2)^{(\varrho+3/2)/2} \in \mathcal{C}(\mathbb{R}_t; \mathcal{C}^0_b(\mathbb{R}^3)) \hookrightarrow \mathcal{C}(\mathbb{R}_t; \mathcal{L}(L^2(\mathbb{R}^3))).$$

We infer that equation (4.25) has a propagator $\mathcal{W}$ that is strongly continuous from $\mathbb{R}^2_{t,s}$ to $\mathcal{L}(L^2(\mathbb{R}^3;\mathbb{C}^n))$. This allows us to define the free propagator $\mathcal{U}$ on L^2_ϱ by

$$\mathcal{U}(t,s) = \Phi^{-1}\mathcal{W}(t,s)\Phi$$

and conclude that it is strongly continuous from $\mathbb{R}^2_{t,s}$ to $\mathcal{L}(L^2_\varrho(\mathbb{R}^3;\mathbb{C}^n))$.

Point 4: This part of the proposition will be useful for the proof of point 3, therefore we prove it first. This is done exactly as for point 1 with the additional tool of product theorems between Sobolev spaces. We have assumed $a^i - a^i_0 \in \mathcal{C}(\mathbb{R}_t; H^k(\mathbb{R}^3;\mathbb{C}^n))$ with $k \geq 3$. Consequently $a^i \in \mathcal{C}(\mathbb{R}_t; \mathcal{C}^1_b(\mathbb{R}^3;\mathbb{C}^n))$ and thus $\mathcal{U}$ acts as a strongly continuous propagator on L^2 and H^1. Then we use $(\mathrm{Id}-\Delta)^{-1}$ to go up to H^{l+2} from H^l, $0 \leq l \leq k-2$; the commutator $[\mathrm{Id}-\Delta, a^i(t,x)]$ is of the form

$$[\mathrm{Id}-\Delta, a^i(t,x)] = \sum_{j=1}^{3}\left(c^{ij}(t,x)\frac{\partial}{\partial x^j} + d^{ij}(t,x)\right)$$

where

$$c^{ij} \in \mathcal{C}(\mathbb{R}_t; H^{k-1}(\mathbb{R}^3)), \qquad d^{ij} \in \mathcal{C}(\mathbb{R}_t; H^{k-2}(\mathbb{R}^3)).$$

Using the usual product rule between Sobolev spaces

$$H^{\sigma_1}(\mathbb{R}^3) \times H^{\sigma_2}(\mathbb{R}^3) \hookrightarrow H^{\sigma}(\mathbb{R}^3) \quad \text{for } \sigma_1, \sigma_2 \geq \sigma,\ \sigma < \sigma_1 + \sigma_2 - 3/2,$$

we find that, for $0 \leq l \leq k-2$, since $k \geq 3 > 5/2$,

$$H^{k-1}(\mathbb{R}^3) \times H^l(\mathbb{R}^3) \hookrightarrow H^l(\mathbb{R}^3) \quad \text{and} \quad H^{k-2}(\mathbb{R}^3) \times H^{l+1}(\mathbb{R}^3) \hookrightarrow H^l(\mathbb{R}^3).$$

The conclusion is therefore the same as in point 1:

$$\sum_{i=1}^{3}[\mathrm{Id}-\Delta, a^i(t,x)]\frac{\partial}{\partial x^i}(\mathrm{Id}-\Delta)^{-1} \in \mathcal{C}(\mathbb{R}_t; \mathcal{L}(H^l(\mathbb{R}^3))),$$

and this allows us to define the propagator $\mathcal{U}$ on H^{l+2}. Therefore, by induction, $\mathcal{U}$ satisfies (4.16) for any m such that $0 \leq m \leq k$ and this proves the fourth part of Proposition 4.2.

Point 3: The regularity of the a^i's ($H^k_\nu \hookrightarrow \mathcal{C}^1_b(\mathbb{R}^3)$ since $k \geq 3$ and $\nu < 0$) and points 1 and 2 imply that we can define $\mathcal{U}$ as a strongly continuous propagator on L^2 and L^2_ϱ

for any $\varrho \in \mathbb{R}$. We assume that for some integer m, $1 \leq m \leq k$, the following property is satisfied:

($\mathbf{P}_{m-1}$) $\mathcal{U}$ can be defined as a strongly continuous propagator on $H^p_\varrho(\mathbb{R}^3)$ for any $\varrho \in \mathbb{R}$ and for any integer p such that $0 \leq p \leq m-1$.

We wish to show that ($\mathbf{P}_{m-1}$) implies ($\mathbf{P}_m$); this will prove point 3 by induction. For a given $\varrho \in \mathbb{R}$, we establish, using ($\mathbf{P}_{m-1}$), that $\mathcal{U}$ is a strongly continuous propagator on $H^m_\varrho(\mathbb{R}^3)$. We consider $u_0 \in H^m_{\mathrm{comp}}(\mathbb{R}^3;\mathbb{C}^n)$. The finite propagation speed and point 4 entail:

$$\mathcal{U}(t,s)u_0 \in \mathcal{C}(\mathbb{R}^2_{t,s}; H^m_{\mathrm{comp}}(\mathbb{R}^3;\mathbb{C}^n)) \hookrightarrow \mathcal{C}(\mathbb{R}^2_{t,s}; H^m_\varrho(\mathbb{R}^3;\mathbb{C}^n)).$$

If we prove the existence of a positive, continuous function C on $\mathbb{R}^2_{t,s}$ such that, for any $u_0 \in H^m_{\mathrm{comp}}$, we have

$$\|\mathcal{U}(t,s)u_0\|_{H^m_\varrho} \leq C(t,s)\|u_0\|_{H^m_\varrho}, \tag{4.26}$$

this will allow us, by density of H^m_{comp} into H^m_ϱ, to define $\mathcal{U}$ as a strongly continuous propagator on H^m_ϱ. Hypothesis ($\mathbf{P}_{m-1}$) already gives us the existence of a positive, continuous function C_1 on $\mathbb{R}^2_{t,s}$ such that, for any $u_0 \in H^m_{\mathrm{comp}}$,

$$\|\mathcal{U}(t,s)u_0\|_{H^{m-1}_\varrho} \leq C_1(t,s)\|u_0\|_{H^{m-1}_\varrho}. \tag{4.27}$$

For $u_0 \in H^m_{\mathrm{comp}}(\mathbb{R}^3)$, we put $u(t,s) = \mathcal{U}(t,s)u_0$ and $v_j(t,s) = \partial u(t,s)/\partial x^j$ for $j = 1,2,3$. We have $v_j|_{t=s} = \partial u_0/\partial x^j \in H^{m-1}_{\varrho-1}(\mathbb{R}^3)$ and v_1, v_2, v_3 satisfy

$$\frac{\partial v_j}{\partial t} = \sum_{i=1}^3 a^i(t,x)\frac{\partial v_j}{\partial x^i} + \sum_{i=1}^3 \frac{\partial a^i}{\partial x^j}(t,x)v_i, \qquad j = 1,2,3. \tag{4.28}$$

This is a system of three evolution equations whose principal part reduces to (4.23) and which are coupled by the potentials $\partial a^i/\partial x^j$. The regularity of the a^i's and the continuous embedding (4.21) imply (since $k > 5/2$ and $\nu < 1$)

$$\frac{\partial a^i}{\partial x^j} \in \mathcal{C}(\mathbb{R}_t; H^{k-1}_{\nu-1}(\mathbb{R}^3;\mathcal{M}_n(\mathbb{C}))) \hookrightarrow \mathcal{C}(\mathbb{R}_t; \mathcal{L}(H^{m-1}_{\varrho-1}(\mathbb{R}^3;\mathbb{C}^n))).$$

This property together with the assumption that the propagator $\mathcal{U}$ for (4.23) is strongly continuous from $\mathbb{R}^2_{t,s}$ to $\mathcal{L}(H^{m-1}_{\varrho-1}(\mathbb{R}^3))$ entail that the system (4.28) admits a strongly continuous propagator on $(H^{m-1}_{\varrho-1}(\mathbb{R}^3;\mathbb{C}^n))^3$ (this is proved using a fixed point method). An immediate consequence is the existence of a positive, continuous function C_2 on $\mathbb{R}^2_{t,s}$, such that, for any $u_0 \in H^m_{\mathrm{comp}}(\mathbb{R}^3;\mathbb{C}^n)$,

$$\sum_{i=1}^3 \left\|\frac{\partial}{\partial x^i}\mathcal{U}(t,s)u_0\right\|_{H^{m-1}_{\varrho-1}} \leq C_2(t,s)\sum_{i=1}^3 \left\|\frac{\partial}{\partial x^i}u_0\right\|_{H^{m-1}_{\varrho-1}}. \tag{4.29}$$

Putting (4.27) and (4.29) together gives us (4.26) and concludes the proof of Proposition 4.2. ∎

Proof of Theorem 4. We apply the result of Lemma 3.3 to obtain the conservation of the charge. Let us consider some initial data $\Psi_0 \in H^1(\Sigma;\mathbb{S}_{\mathrm{Dirac}})$ with compact support on Σ and some initial time s. Let Ψ be the corresponding solution to (3.4) in $\mathcal{C}(\mathbb{R}_t; H^1(\Sigma;\mathbb{S}_{\mathrm{Dirac}})) \cap \mathcal{C}^1(\mathbb{R}_t; L^2(\Sigma;\mathbb{S}_{\mathrm{Dirac}}))$. For $T > s$, we integrate the closed 3-form

$\omega = {}^*U_a \mathrm{d}x^a$ on a closed surface σ made of a timelike tube, large enough not to intersect the support of the solution on the time interval $[s,T]$, and of the spacelike hypersurfaces $\Sigma_s = \{t = s\}$ and $\Sigma_T = \{t = T\}$. Such a surface exists thanks to the finite propagation speed and the integration of ω over σ has a meaning because the vector field U^a, or equivalently the 3-form ω, belongs to $W^{1,1}_{\mathrm{loc}}(\mathbb{R}_t \times \Sigma)$ (the Sobolev space on $\mathbb{R}_t \times \Sigma$ of functions in L^1_{loc} with their first derivative in L^1_{loc}); therefore one can apply Stoke's theorem to evaluate the integral of ω over the compact, piecewise $\mathcal{C}^1$ hypersurface σ. We have proved the closedness of ω in Lemma 3.3. We obtain

$$\int_{\Sigma_T} \frac{1}{\sqrt{2}} T^a U_a \,\mathrm{dVol}_{h(T)} - \int_{\Sigma_s} \frac{1}{\sqrt{2}} T^a U_a \,\mathrm{dVol}_{h(s)} = 0$$

since $\frac{1}{\sqrt{2}}T^a$ and $-\frac{1}{\sqrt{2}}T^a$ are the outgoing unit normals to σ on Σ_T and Σ_s respectively. We have

$$T^a U_a = T^{AA'} \phi_A \overline{\phi}_{A'} + T_{AA'} \chi^{A'} \overline{\chi}^A = \langle \Psi, \Psi \rangle = |\Psi|^2.$$

Similar arguments can of course be used for $T < s$. This proves the conservation of $E(t)$ for solutions associated with initial data in $H^1_{\mathrm{comp}}(\Sigma)$. The result then carries over to solutions with values in $L^2(\Sigma)$ by continuity on L^2 of the propagator for equation (3.4) and by density of H^1_{comp} in L^2. This concludes the proof of Theorem 4. ■

5. The case of the Schwarzschild geometry

The space-time containing only a spherically symmetric uncharged black hole of mass $M > 0$ is described by the Schwarzschild metric whose expression in the Schwarzschild coordinates t, r, θ, φ on $\mathbb{R}_t \times]0, +\infty[_r \times S^2_{\theta,\varphi}$ is given by

$$g_{ab} \mathrm{d}x^a \mathrm{d}x^b = \left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 - \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r^2 - r^2 \mathrm{d}\omega^2 \tag{5.1}$$

where $\mathrm{d}\omega^2$ is the euclidian metric on S^2:

$$\mathrm{d}\omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2.$$

Putting

$$F(r) = 1 - 2M/r$$

we have

$$g_{ab} \mathrm{d}x^a \mathrm{d}x^b = F \mathrm{d}t^2 - F^{-1} \mathrm{d}r^2 - r^2 \mathrm{d}\omega^2. \tag{5.2}$$

This metric has two singularities: the horizon $\{r = 2M\}$ is only a coordinate singularity, the metric can be extended analytically through it; and the origin $\{r = 0\}$ which is a true curvature singularity. The horizon separates the exterior of the black hole $\{r > 2M\}$, which is a stationary domain where $\partial/\partial t$ is timelike and $\partial/\partial r$ spacelike, from the interior $\{r < 2M\}$, a dynamic region where $\partial/\partial t$ is spacelike, $\partial/\partial r$ timelike, and the inertial frames are dragged towards the singularity at $\{r = 0\}$ (the time orientation implicit in this description is such that the timelike vector field $\partial/\partial r$ is past pointing).

5.1. The exterior of the black hole. We first consider the Schwarzschild geometry from the point of view of an observer static with respect to infinity. Such observers only see the exterior of the black hole and their perception of space-time is described by the time function t of the Schwarzschild coordinates outside the black hole. To their eyes, light rays falling into the black hole slow down infinitely as they approach the horizon and never cross it. One way of seeing this is to calculate the radial null geodesics. Introducing the Regge–Wheeler variable

$$r_* = r + 2M \operatorname{Log}(r - 2M)$$

we have

$$\frac{dr}{dr_*} = F$$

and the metric g takes the form

$$g = F(\mathrm{d}t^2 - \mathrm{d}r_*^2) - r^2 \mathrm{d}\omega^2.$$

The radial null geodesics are the straight lines

$$\omega = \omega_0, \qquad t = \pm r_* + C, \qquad C \in \mathbb{R},\ \omega_0 \in S^2$$

and the horizon $\{r = 2M\}$ (corresponding to $r_* \to -\infty$) is reached in infinite time t. A remarkable consequence of this property is that if we choose for Dirac's equation (or Maxwell's, or the Klein–Gordon equation alike) some initial data whose support is contained in $\{r \geq 2M + \varepsilon\}$, $\varepsilon > 0$, then the support of the solution will only reach the horizon when t becomes infinite.

We shall see that results similar to those of the previous section are still valid in this framework, but before we can express them properly, we need to study the geometry of the spacelike slices.

5.1.1. *The spacelike geometry of the exterior of the black hole.* The exterior of the black hole is globally hyperbolic. We consider the foliation induced by the time function t, i.e. the slices are

$$\Sigma_t = \{t\} \times \left]2M, +\infty\right[_r \times S^2_\omega, \qquad t \in \mathbb{R},$$

with the induced Riemannian metric

$$h = F^{-1}\mathrm{d}r^2 + r^2 \mathrm{d}\omega^2. \tag{5.3}$$

The 3+1 decomposition of the geometry is given by (calling $\mathcal{M}$ the exterior of the black hole):

$$\mathcal{M} = \mathbb{R}_t \times \Sigma, \qquad \Sigma = \left]2M, +\infty\right[_r \times S^2_\omega, \qquad g = F\mathrm{d}t^2 - h = \frac{N^2}{2}\mathrm{d}t^2 - h \tag{5.4}$$

with the lapse function $N = \sqrt{2}\, F^{1/2}$. The exterior of the black hole is static: $\partial/\partial t$ is a Killing vector field (since g does not depend on t), is timelike outside the black hole and is everywhere orthogonal to the Cauchy hypersurfaces Σ_t. The time orientation is chosen by deciding that $\partial/\partial t$ is future pointing and the normalized vector field T^a is then

$$T^a \partial_a = \sqrt{2}\, F^{-1/2} \frac{\partial}{\partial t} = \frac{2}{N} \frac{\partial}{\partial t}.$$

We consider a generic spacelike slice (Σ, h). The metric h appears singular at $r = 2M$. This is merely due to the choice of coordinates; introducing as the new radial variable $u(r)$ the h-distance to the horizon, we show that (Σ, h) is a smooth manifold and that the horizon is a smooth boundary.

Given $p = (R, \omega) \in \Sigma$, the h-distance from p to the horizon $H = \{r = 2M\} \times S^2_\omega$ is given by

$$u(R) = \int_{[2M,R]} F^{-1/2}\,\mathrm{d}r = \int_{[2M,R]} \frac{\sqrt{r}}{\sqrt{r-2M}}\,\mathrm{d}r. \tag{5.5}$$

This distance is finite and H thus appears as the boundary of (Σ, h). Since

$$\frac{\mathrm{d}u}{\mathrm{d}r} = F^{-1/2},$$

the metric h can be written as

$$h = \mathrm{d}u^2 + r^2 \mathrm{d}\omega^2 \tag{5.6}$$

and

$$\Sigma =]0, +\infty[_u \times S^2_\omega .$$

The function $u(r)$ is continuous and strictly increasing from $[2M, +\infty[$ onto $[0, +\infty[$, it is $\mathcal{C}^\infty$ on $]2M, +\infty[$ but it is not differentiable at $2M$. However, the inverse function satisfies

Lemma 5.1. *The function $u \mapsto r(u)$ is $\mathcal{C}^\infty$ on $[0, +\infty[$ and all its derivatives are uniformly bounded on $[0, +\infty[$. In particular, the first derivative $\mathrm{d}r/\mathrm{d}u = F^{1/2}$ (and therefore also the lapse function) is uniformly bounded together with all its derivatives on $[0, +\infty[$.*

Proof. The first and second derivatives $F^{1/2}$ and M/r^2 are continuous on $[0, +\infty[_u$ whence r is $\mathcal{C}^2$ on $[0, +\infty[_u$. If r is $\mathcal{C}^k$ on $[0, +\infty[_u$, then so is the second derivative and the lemma is thus proved by induction. ∎

This entails that h is smooth on $\overline{\Sigma} = [0, +\infty[_u \times S^2_\omega$; $(\overline{\Sigma}, h)$ is a smooth manifold with boundary. On (Σ, h), we introduce Sobolev spaces with all traces equal to zero at the boundary:

Definition 5.1. $H^m_0(\Sigma)$, $m \in \mathbb{N}$, is the completion of $\mathcal{C}^\infty_0(\Sigma)$ in the Sobolev norm

$$\|f\|_{H^m(\Sigma)} = \Big(\int_\Sigma \sum_{p=0}^{m} \langle \mathrm{D}^p f, \mathrm{D}^p f\rangle \,\mathrm{dVol}_h \Big)^{1/2} \tag{5.7}$$

where D, $\langle \cdot\, , \cdot\rangle$ and dVol_h are the covariant derivative, the positive definite inner product on tensors and spinors at a point and the volume element on Σ induced by h. Note that on spinors, $\langle \cdot\, , \cdot\rangle$ is nothing but the hermitian inner product induced by T^a. The subscript 0 in $H^m_0(\Sigma)$ must not be mistaken for a weight subscript; weighted Sobolev spaces on Σ with zero traces will be denoted by $H^m_{0,\delta}(\Sigma)$. $H^0_0(\Sigma)$ is simply $L^2(\Sigma, h)$.

We wish to prove that the Cauchy problem for the Dirac equation is well-posed in these spaces. To this end, we establish that the successive domains in L^2 of the Dirac operator on Σ with homogeneous boundary conditions at the horizon are the Sobolev spaces $H^m_0(\Sigma)$ and that the norms are equivalent.

We consider the Dirac operator $\mathcal{D}_\Sigma$ associated with the Levi-Civita connection on (Σ, h). It is formally self-adjoint on $L^2(\Sigma_t; \mathbb{S}_{\mathrm{Dirac}})$ and satisfies

$$\mathcal{D}_\Sigma^* \mathcal{D}_\Sigma = \mathcal{D}_\Sigma^2 = \mathrm{D}^* \mathrm{D} = -\Delta_h. \tag{5.8}$$

The notation D for the Levi-Civita connection on (Σ, h) is justified by the fact that the exterior of the black hole is static: the extrinsic curvature of the slices Σ_t is zero, whence the projection on Σ of the space-time connection ($\mathrm{D}_a = -h_a{}^b \nabla_b$) coincides with the Levi-Civita connection on (Σ, h). Consequently, $\mathrm{D}^* \mathrm{D} = -\Delta_h$ and $\mathcal{D}_\Sigma$ coincides with the Dirac–Witten operator on Σ embedded in $(\mathcal{M}, g)$. Also, the endomorphism $\mathcal{R}$ vanishes here since $G_{ab} = 0$ (Schwarzschild's space-time is a solution of the Einstein vacuum equations) and (5.8) is therefore obtained from (3.27). Note that the equality (5.8) is true when applied to $\Psi \in \mathcal{C}_0^\infty(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$. The coefficients of $\mathcal{D}_\Sigma$ and Δ_h being smooth on Σ, these operators are continuous on $\mathcal{D}'(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$, the space of Dirac spinor valued distributions on Σ (that is, the dual of $\mathcal{C}_0^\infty(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$). Hence, (5.8) naturally extends by continuity ([1]) to an equality of operators acting on $\mathcal{D}'(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$, and in particular on any Sobolev-type space on Σ. The norm in $H_0^k(\Sigma)$ can be defined using $\mathcal{D}_\Sigma$ in the following manner:

PROPOSITION 5.1. *Consider the following norm on $H_0^k(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$:*

$$|||\Psi|||_k = \Big(\sum_{p=0}^{k} \int_\Sigma \langle \mathcal{D}_\Sigma^p \Psi, \mathcal{D}_\Sigma^p \Psi \rangle \, \mathrm{dVol}_h \Big)^{1/2}. \tag{5.9}$$

The norms $\| \cdot \|_{H^k}$ and $||| \cdot |||_k$ are equivalent on $H_0^k(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$.

REMARK 5.1. We consider the operator

$$\not{\mathcal{D}} := e_0 . \mathcal{D}_\Sigma, \qquad e_0 = F^{-1/2} \frac{\partial}{\partial t} = \frac{1}{\sqrt{2}} T^a \partial_a. \tag{5.10}$$

$\not{\mathcal{D}}$ is formally skew-adjoint on $L^2(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$ and satisfies

$$\not{\mathcal{D}}^* \not{\mathcal{D}} = -\not{\mathcal{D}}^2 = \mathcal{D}_\Sigma^2 = -\Delta_h. \tag{5.11}$$

Moreover, the norm $|||.|||_k$ can be expressed using $\not{\mathcal{D}}$ as well as $\mathcal{D}_\Sigma$; we have for all Ψ in

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([1]) One can also think directly in terms of duality. There are different ways of defining the duality product between $\mathcal{D}'(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$ and $\mathcal{C}_0^\infty(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$. The most natural is to construct it as an extension of the positive definite innner product on $L^2(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$ without complex conjugation. More precisely, to a locally integrable Dirac spinor field Ψ on Σ we associate the distribution T_Ψ, usually simply denoted by Ψ, in the following manner: for all $\Phi \in \mathcal{C}_0^\infty(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$,

$$\langle \Psi, \Phi \rangle_{\mathcal{D}', \mathcal{C}_0^\infty} = \int_\Sigma \langle \Psi(x), \overline{\Phi(x)} \rangle \, \mathrm{dVol}_h = \int_\Sigma \overline{\Phi^\dagger(x)} \Psi(x) \, \mathrm{dVol}_h = \int_\Sigma {}^t\Phi(x) \Psi(x) \, \mathrm{dVol}_h.$$

Thus, considering a differential operator L on Σ, i.e. involving only derivatives tangent to Σ, its transposed with respect to the above duality product will be $\overline{L^*}$, where L^* is its formal adjoint for the positive definite inner product on $L^2(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$. Hence the equality (5.8), where the operators are considered as acting on $\mathcal{C}_0^\infty(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$, immediately entails by definition of $\mathcal{D}'(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$ the same equality where the operators are now considered as acting on $\mathcal{D}'(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$. Indeed, all the operators involved being formally self-adjoint, we obtain by duality the complex conjugate of (5.8) which is equivalent to (5.8).

$H_0^k(\Sigma;\mathbb{S}_{\mathrm{Dirac}})$,

$$\text{(5.12)} \qquad |||\Psi|||_k = \Big(\sum_{p=0}^{k}\int_{\Sigma}\langle \not{\mathcal{D}}^p\Psi, \not{\mathcal{D}}^p\Psi\rangle \,\mathrm{dVol}_h\Big)^{1/2}.$$

All this is an immediate consequence of the fact that $e_0.$ is hermitian for $\langle\cdot\,,\cdot\rangle$, anticommutes with $\mathcal{D}_\Sigma$ and $e_0.e_0. = \mathrm{Id}$. Note that restricting $\not{\mathcal{D}}$ to $\mathbb{S}^*$ and $\overline{\mathbb{S}}$ respectively, we get similar results for $\mathbb{D}$ and $\mathbb{D}^*$.

Proof of Proposition 5.1. The Bochner–Lichnerowicz–Weitzenböck formula (5.8) gives immediately that for $\Psi \in \mathcal{C}_0^\infty(\Sigma;\mathbb{S}_{\mathrm{Dirac}})$, and by density for $\Psi \in H_0^1(\Sigma;\mathbb{S}_{\mathrm{Dirac}})$, we have

$$\|\mathcal{D}_\Sigma\Psi\|_{L^2(\Sigma)} = \|\mathrm{D}\Psi\|_{L^2(\Sigma)}$$

and therefore $\|\Psi\|_{H^1} = |||\Psi|||_1$. In order to prove the equivalence of higher order norms using this first result, we prove the following lemma:

LEMMA 5.2. *For any $k\in\mathbb{N}$, there exist constants $0 < C_1 < C_2 < +\infty$ such that, for all $\Phi \in H_0^{k+2}(\Sigma;\mathbb{S}_{\mathrm{Dirac}})$,*

$$\text{(5.13)} \qquad C_1\|\Phi\|^2_{H^{k+2}} \le \|\Phi\|^2_{H^k} + \|\Delta_h\Phi\|^2_{H^k} \le C_2\|\Phi\|^2_{H^{k+2}}.$$

Proof. Let us consider a smooth Riemannian manifold $(\widetilde{\Sigma},\widetilde{h})$ such that $\widetilde{\Sigma} = \Sigma\cup K$, K compact, $\widetilde{\Sigma}$ topologically trivial and $\widetilde{h}|_\Sigma = h$. $(\widetilde{\Sigma},\widetilde{h})$ is then a smooth asymptotically flat Riemannian manifold with trivial topology. This entails that $\widetilde{\Sigma}\simeq\mathbb{R}^3$. Parametrizing $\widetilde{\Sigma}$ by $\mathbb{R}^3$, we see that there exist $0 < K_1 < K_2 < +\infty$ such that ($\widetilde{h}_{\mathbf{ab}}$ and $\widetilde{h}^{\mathbf{ab}}$ are here considered as 3×3 matrices)

$$K_1 \le \det\widetilde{h} \le K_2, \qquad K_1\mathrm{Id}_3 \le \widetilde{h}_{\mathbf{ab}} \le K_2\mathrm{Id}_3, \qquad K_1\mathrm{Id}_3 \le \widetilde{h}^{\mathbf{ab}} \le K_2\mathrm{Id}_3.$$

The norms in the Sobolev spaces $H^k(\widetilde{\Sigma};\widetilde{h})$ are equivalent to the norms in the usual Sobolev spaces on $\mathbb{R}^3$. The Laplacian $\Delta_{\widetilde{h}}$ acting on Dirac spinors is given (with respect to a spin-frame) by

$$\begin{aligned}\Delta_{\widetilde{h}} &= \Big[\sum_{\mathbf{a},\mathbf{b}=1}^{3}(\det\widetilde{h})^{-1/2}\frac{\partial}{\partial x^{\mathbf{a}}}\Big((\det\widetilde{h})^{1/2}\widetilde{h}^{\mathbf{ab}}\frac{\partial}{\partial x^{\mathbf{b}}}\Big)\Big]\mathrm{Id}_4 + \text{connection terms}\\ &= \Big(\sum_{\mathbf{a},\mathbf{b}=1}^{3}\widetilde{h}^{\mathbf{ab}}\frac{\partial^2}{\partial x^{\mathbf{a}}\partial x^{\mathbf{b}}}\Big)\mathrm{Id}_4 + R,\end{aligned}$$

the remainder R being a first order differential operator whose coefficients are first or second order derivatives of the metric $\widetilde{h}$ and are therefore in $L^\infty(\widetilde{\Sigma})$ together with all their derivatives. And of course $\Delta_{\widetilde{h}}|_\Sigma = \Delta_h$.

The first immediate consequence is the existence of $0 < C_2 < +\infty$ such that, for any $\Phi\in\mathcal{C}_0^\infty(\widetilde{\Sigma};\mathbb{S}_{\mathrm{Dirac}})$,

$$\|\Phi\|^2_{H^k(\widetilde{\Sigma})} + \|\Delta_{\widetilde{h}}\Phi\|^2_{H^k(\widetilde{\Sigma})} \le C_2\|\Phi\|^2_{H^{k+2}(\widetilde{\Sigma})}.$$

Choosing only spinor fields Φ in $\mathcal{C}_0^\infty(\Sigma;\mathbb{S}_{\mathrm{Dirac}})$, we obtain one of the two inequalities (5.13).

The second consequence is that, via a choice of spin-frame, $\mathrm{Id}-\Delta_{\widetilde{h}}$ is an isomorphism from $H^{k+2}(\mathbb{R}^3;\mathbb{C}^4)$ onto $H^k(\mathbb{R}^3;\mathbb{C}^4)$ for any integer k, since $(\mathrm{Id}-\Delta_{\widetilde{h}})^{-1}$ is a pseudodifferential operator of order -2. Therefore $\mathrm{Id} - \Delta_{\tilde{h}}$ is an isomorphism from $H^{k+2}(\widetilde{\Sigma};\mathbb{C}^4)$ onto $H^k(\widetilde{\Sigma};\mathbb{C}^4)$ for any $k \in \mathbb{N}$ (for a slightly weaker result implying also the second inequality (5.13), see for example [27], p. 197, Corollary 8.4.7).

We conclude that for each $k \in \mathbb{N}$ there exists $0 < \widetilde{C}_1 < +\infty$ such that, for any $\Phi \in H^{k+2}(\widetilde{\Sigma};\mathbb{S}_{\mathrm{Dirac}})$

$$\widetilde{C}_1 \|\Phi\|_{H^{k+2}(\widetilde{\Sigma})} \le \|(\mathrm{Id} - \Delta_{\tilde{h}})\Phi\|_{H^k(\widetilde{\Sigma})} \le \|\Phi\|_{H^k(\widetilde{\Sigma})} + \|\Delta_{\tilde{h}}\Phi\|_{H^k(\widetilde{\Sigma})}.$$

Choosing $\Phi \in \mathcal{C}_0^\infty(\Sigma;\mathbb{S}_{\mathrm{Dirac}})$ gives the other one of the two inequalities (5.13) and proves Lemma 5.2. ∎

We now proceed to proving Proposition 5.1 using Lemma 5.2. We already know that

$$\|\cdot\|_{L^2(\Sigma)} = |||\cdot|||_0, \qquad \|\cdot\|_{H^1(\Sigma)} = |||\cdot|||_1.$$

Lemma 5.2 for $k = 0$ gives the following:

$$C_1 \|\Phi\|^2_{H^2(\Sigma)} \le \|\mathcal{D}^2_\Sigma \Phi\|^2_{L^2(\Sigma)} + \|\Phi\|^2_{L^2(\Sigma)} \le C_2 \|\Phi\|^2_{H^2(\Sigma)} \quad \text{for all } \Phi \in H^2_0(\Sigma).$$

Hence, the norm

$$(\|\Phi\|^2_{H^2(\Sigma)} + \|\mathcal{D}_\Sigma \Phi\|^2_{L^2(\Sigma)})^{1/2} = (\|\Phi\|^2_{H^2(\Sigma)} + \|\mathrm{D}\Phi\|^2_{L^2(\Sigma)})^{1/2}$$

(which is clearly equivalent to $\|\cdot\|_{H^2(\Sigma)}$) is equivalent to $|||\cdot|||_2$. This proves the result of Proposition 5.1 for $k = 2$. We now suppose this result to be true for $0 \le k \le m$, $m \ge 2$ and we prove it for $k = m+1$. Lemma 5.2 gives

$$C_1 \|\Phi\|^2_{H^{m+1}(\Sigma)} \le \|\Phi\|^2_{H^{m-1}(\Sigma)} + \|\mathcal{D}^2_\Sigma \Phi\|^2_{H^{m-1}(\Sigma)} \le C_2 \|\Phi\|^2_{H^{m+1}(\Sigma)} \quad \text{for all } \Phi \in H^{m+1}_0(\Sigma).$$

Moreover, using the equivalence for $k = m-1$, we deduce that

$$\|\Phi\|^2_{H^{m-1}(\Sigma)} + \|\mathcal{D}^2_\Sigma \Phi\|^2_{H^{m-1}(\Sigma)} \simeq \|\Phi\|^2_{L^2(\Sigma)} + \|\mathcal{D}_\Sigma \Phi\|^2_{L^2(\Sigma)} + 2\sum_{p=2}^{m-1} \|\mathcal{D}^p_\Sigma \Phi\|^2_{L^2(\Sigma)} + \|\mathcal{D}^m_\Sigma \Phi\|^2_{L^2(\Sigma)} + \|\mathcal{D}^{m+1}_\Sigma \Phi\|^2_{L^2(\Sigma)}.$$

This proves the equivalence for $k = m+1$ and Proposition 5.1 follows by induction. ∎

5.1.2. *The global exterior Cauchy problem.* We now prove the well-posedness of the Cauchy problem for the Dirac equation in Sobolev and weighted Sobolev spaces outside the black hole. We have already introduced the Sobolev spaces $H^k_0(\Sigma)$, $k \in \mathbb{N}$; we now define weighted Sobolev spaces with zero traces at the horizon:

DEFINITION 5.2. For $k \in \mathbb{N}$, $\varrho \in \mathbb{R}$, the *weighted Sobolev space with zero traces at the horizon*: $H^k_{0,\varrho}(\Sigma)$, is the completion of $\mathcal{C}_0^\infty(\Sigma)$ in the norm

$$\|f\|_{H^k_\varrho(\Sigma)} = \Big(\sum_{p=0}^{k} \int_\Sigma (1+u^2)^{-\varrho-3/2+p} \langle \mathrm{D}^p f, \mathrm{D}^p f\rangle \,\mathrm{dVol}_h\Big)^{1/2}, \tag{5.14}$$

u being the h-distance to the horizon introduced earlier. We could replace $1+u^2$ by r^2 for example without changing the function space, we would simply replace the norm by an equivalent norm since $r \simeq u$ at infinity.

The following theorem is the analogue of Theorems 1 to 4 in the Schwarzschild space-time described using the point of view of observers static with respect to infinity:

THEOREM 5. 1. *For any initial data $\Psi_0 \in L^2(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$, the Dirac equation outside the black hole has a unique solution $\Psi \in \mathcal{C}(\mathbb{R}_t; L^2(\Sigma; \mathbb{S}_{\mathrm{Dirac}}))$ such that $\Psi|_{t=0} = \Psi_0$. Moreover, the evolution is unitary in $L^2(\Sigma)$, i.e.*

$$\|\Psi(t)\|_{L^2(\Sigma)} = \|\Psi_0\|_{L^2(\Sigma)} \quad \text{for all } t \in \mathbb{R}.$$

The propagator for the Dirac equation outside the black hole, $\mathcal{U}(t,s)$, only depends on $t-s$ since the space-time is static. We denote it by $\mathcal{U}(t-s)$ and $t \mapsto \mathcal{U}(t)$ is a one-parameter group of unitary operators on $L^2(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$.

2. *If $\Psi_0 \in H_0^k(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$, $k \in \mathbb{N}$, the associated solution Ψ satisfies*

$$\Psi \in \bigcap_{l=0}^{k} \mathcal{C}^l(\mathbb{R}_t; H_0^{k-l}(\Sigma; \mathbb{S}_{\mathrm{Dirac}}));$$

$\mathcal{U}(t)$ is a strongly continuous one-parameter group of bounded operators on $H_0^k(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$ for all $k \in \mathbb{N}$.

3. *For any initial data $\Psi_0 \in L^2_\varrho(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$, $\varrho \in \mathbb{R}$, the Dirac equation outside the black hole has a unique solution $\Psi \in \mathcal{C}(\mathbb{R}_t; L^2_\varrho(\Sigma; \mathbb{S}_{\mathrm{Dirac}}))$ such that $\Psi|_{t=0} = \Psi_0$. $\mathcal{U}(t)$ is a strongly continuous one-parameter group of bounded operators on $L^2_\varrho(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$ for all $\varrho \in \mathbb{R}$.*

4. *If $\Psi_0 \in H^k_{0,\varrho}(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$, $k \in \mathbb{N}$, $\varrho \in \mathbb{R}$, the associated solution Ψ satisfies*

$$\Psi \in \bigcap_{l=0}^{k} \mathcal{C}^l(\mathbb{R}_t; H_{0,\varrho}^{k-l}(\Sigma; \mathbb{S}_{\mathrm{Dirac}}));$$

$\mathcal{U}(t)$ is for all $k \in \mathbb{N}$, $\varrho \in \mathbb{R}$, a strongly continuous one-parameter group of bounded operators on $H^k_{0,\varrho}(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$.

Proof. With the notations we have introduced in the previous section, the Dirac equation outside the black hole takes the form

$$\nabla_{e_0}\Psi = -\not{\mathcal{D}}\Psi - ime_0.\Psi.$$

We choose a spin-frame $\{o^A, \iota^A\}$ adapted to the foliation; γ^0 is then given as the constant hermitian matrix

$$\gamma^0 = i \begin{pmatrix} 0 & \mathrm{Id}_2 \\ -\mathrm{Id}_2 & 0 \end{pmatrix}$$

and expressing more explicitly the timelike derivative, we obtain

$$\frac{\partial \Psi}{\partial t} = -\frac{N}{\sqrt{2}}(\not{\mathcal{D}}\Psi + im\gamma^0\Psi + B\Psi) \tag{5.15}$$

where the matrix B (not quite the same as the matrix B of Chapters 3 and 4) contains the connection terms coming from the time derivative; B is of course independent of t. In Appendix A, we describe more precisely the type of spin-frame in which we work and we calculate a general expression of the matrix B in such a spin-frame; this general calculation is valid for both Schwarzschild and Kerr metrics. We also give an explicit expression of B in Schwarzschild's space-time for a particular choice of spin-frame already used in [48]. Suffice it to say here that for our choice of spin-frame, we have the following result, proved in Appendix A:

Lemma 5.3. *The coefficients of the matrix* $(N/\sqrt{2})B$ *are uniformly bounded on* $\overline{\Sigma}$ *together with all their derivatives.*

This will be useful for controlling the Sobolev norms of the solutions.

1. *The Cauchy problem in* $L^2(\Sigma)$: The essential observation here is that if we choose the initial data Ψ_0 with compact support in Σ, the support of the solution will propagate along null geodesics (i.e. characteristic lines) and will only touch the horizon as t tends to infinity (according to the remark at the beginning of this section). Therefore, we will never see the lapse function reach the value zero and we can apply the results of Theorem 1. Let us develop this argument more precisely. We use the Schwarzschild coordinates (t, r, ω) for simplicity. For $\varepsilon > 0$, we consider on $\mathbb{R}_t \times [0, +\infty[_r \times S^2_\omega$ a smooth Lorentzian metric ${}^\varepsilon g$, which coincides with g on $\mathbb{R}_t \times [2M+\varepsilon, +\infty[_r \times S^2_\omega$, i.e. "outside the black hole and not too close to the horizon". For the background metric on $\widetilde{\Sigma} = [0, +\infty[_r \times S^2_\omega$, we simply choose the euclidian metric

$$\widetilde{h} = \mathrm{d}r^2 + r^2 \mathrm{d}\omega^2$$

and we compare ${}^\varepsilon g$ with a metric $\widetilde{g}$ on $\mathbb{R} \times \widetilde{\Sigma}$ which, outside a compact set, is the beginning of the expansion in $1/r$ of the Schwarzschild metric (this metric has the form chosen for background Lorentzian metrics in Definition 2.1):

$$\widetilde{g} = \left(1 - \varrho(r)\frac{2M}{r}\right)\mathrm{d}t^2 - \left(1 + \varrho(r)\frac{2M}{r}\right)\mathrm{d}r^2 - r^2\mathrm{d}\omega^2.$$

Here ϱ is a smooth cut-off function on $[0, +\infty[$, identically zero on $[0, 3M]$ and equal to 1 on $[4M, +\infty[$ (for example). We see that for each $\varepsilon > 0$, ${}^\varepsilon g$ is of class (∞, δ) on $\mathbb{R} \times \widetilde{\Sigma}$ for any $\delta > -2$ since

$$\widetilde{\mathrm{D}}^l({}^\varepsilon g - \widetilde{g}) = O(r^{-2-l}), \qquad r \to +\infty,\ \forall l \in \mathbb{N},$$

where $\widetilde{\mathrm{D}}$ is the Levi-Civita connection on $(\widetilde{\Sigma}, \widetilde{h})$, i.e. the euclidian connection on $\mathbb{R}^3$.

Hence, it turns out that for any $\varepsilon > 0$, the metric ${}^\varepsilon g$ fits in our classes of asymptotically flat (and even strongly asymptotically flat) space-times and the results of Theorem 1 can be applied to the space-times $(\mathbb{R}_t \times \widetilde{\Sigma}, {}^\varepsilon g)$, $\varepsilon > 0$. Thus, if we consider some initial data Ψ_0 in $L^2(\widetilde{\Sigma}; \mathbb{S}_{\mathrm{Dirac}})$, the Dirac equation on $(\mathbb{R}_t \times \widetilde{\Sigma}, {}^\varepsilon g)$ has a unique solution $\Psi \in \mathcal{C}(\mathbb{R}_t; L^2(\widetilde{\Sigma}; \mathbb{S}_{\mathrm{Dirac}}))$ such that $\Psi(0) = \Psi_0$. Moreover, the norm of $\Psi(t)$ in $L^2(\widetilde{\Sigma}_t)$ is constant throughout time. We now consider Ψ_0 with compact support in $]2M, +\infty[_r \times S^2_\omega$ and we choose $\varepsilon > 0$ small enough so that

$$[0, 2M+\varepsilon]_r \times S^2_\omega \cap \operatorname{supp} \Psi_0 = \emptyset.$$

The solution $\Psi(t)$ coincides on $]2M, +\infty[_r \times S^2_\omega$ with a physical solution to the Dirac equation outside the black hole on the time interval $[T_1^\varepsilon, T_2^\varepsilon] \ni 0$ during which the support of Ψ does not touch $[0, 2M+\varepsilon]_r \times S^2_\omega$. As $\varepsilon \to 0$, we have $T_1^\varepsilon \to -\infty$ and $T_2^\varepsilon \to +\infty$. This entails that for any initial data $\Psi_0 \in L^2(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$ with compact support in Σ, the Dirac equation on $(\mathbb{R}_t \times \Sigma, g)$ has a unique solution $\Psi \in \mathcal{C}(\mathbb{R}_t; L^2(\Sigma; \mathbb{S}_{\mathrm{Dirac}}))$ such that $\Psi(0) = \Psi_0$ and we have

$$\|\Psi(t)\|_{L^2(\Sigma)} = \int_\Sigma \langle \Psi(t), \Psi(t)\rangle \, \mathrm{dVol}_h = \|\Psi_0\|_{L^2(\Sigma)}.$$

Hence the first part of Theorem 5 follows by density.

2. *The Cauchy problem in Sobolev spaces*: We consider $\Psi_0 \in \mathcal{C}_0^\infty(\Sigma;\mathbb{S}_{\mathrm{Dirac}})$. Applying the results of Theorem 2 to the metrics ${}^\varepsilon g$, $\varepsilon > 0$, we see that the associated solution Ψ is in $\mathcal{C}^\infty(\mathbb{R}_t;\mathcal{C}_0^\infty(\Sigma;\mathbb{S}_{\mathrm{Dirac}}))$. We show by induction that all the norms $|||\Psi(t)|||_k$ are controlled in the following manner: there exist $\alpha_k, \beta_k > 0$ independent of Ψ_0 such that

$$|||\Psi(t)|||_k \leq \alpha_k e^{\beta_k |t|} |||\Psi_0|||_k\,.$$

This will prove this part of theorem 5 by density.

To this end, we apply $\not{D}^k$ to the expression (5.15) of the Dirac equation for our choice of spin-frame. We get

$$\frac{\partial}{\partial t}(\not{D}^k\Psi) = -\frac{N}{\sqrt{2}}(\not{D}(\not{D}^k\Psi) + im\gamma^0\not{D}^k\Psi + B\not{D}^k\Psi) + \left[\not{D}^k, -\frac{N}{\sqrt{2}}\right]\not{D}\Psi + \left[\not{D}^k, -\frac{N}{\sqrt{2}}im\gamma^0\right]\Psi + \left[\not{D}^k, -\frac{N}{\sqrt{2}}B\right]\Psi. \tag{5.16}$$

We see that equation (5.16) has the form

$$\frac{\partial}{\partial t}(\not{D}^k\Psi) = -\frac{N}{\sqrt{2}}(\not{D} + im\gamma^0 + B)\not{D}^k\Psi + G(t)$$

where the term $G(t)$ satisfies (using Lemmata 5.1, 5.3 and then the norm equivalence of Proposition 5.1)

$$\|G(t)\|_{L^2(\Sigma)} \leq C\|\Psi(t)\|_{H^k} \leq C'|||\Psi(t)|||_k$$

with C and C' independent of t and Ψ. We shall express $|||\Psi(t)|||_k$ using $\not{D}$ instead of $\mathcal{D}_\Sigma$ (according to Remark 5.1)

$$|||\Psi(t)|||_k^2 = \sum_{p=0}^{k} \|\not{D}^p\Psi(t)\|^2_{L^2(\Sigma)}.$$

Denoting by $\mathcal{U}(t)$ the propagator for the Dirac equation outside the black hole, we have the following expression for $\not{D}^k\Psi(t)$:

$$\not{D}^k\Psi(t) = \mathcal{U}(t)\not{D}^k\Psi_0 + \int_0^t \mathcal{U}(t-s)G(s)\,\mathrm{d}s.$$

This yields the following estimate:

$$\begin{aligned}\|\not{D}^k\Psi(t)\|_{L^2} &\leq \|\not{D}^k\Psi_0\|_{L^2} + C'\int_0^{|t|} |||\Psi(s)|||_k\,\mathrm{d}s\\ &\leq \|\not{D}^k\Psi_0\|_{L^2} + C'\int_0^{|t|} \|\not{D}^k\Psi(s)\|_{L^2}\,\mathrm{d}s + C'\int_0^{|t|} |||\Psi(s)|||_{k-1}\,\mathrm{d}s.\end{aligned}$$

Gronwall's inequality then implies

$$\|\not{D}^k\Psi(t)\|_{L^2} \leq \Big(\|\not{D}^k\Psi_0\|_{L^2} + C'\int_0^{|t|} |||\Psi(s)|||_{k-1}\,\mathrm{d}s\Big)e^{C'|t|}.$$

Therefore, assuming that we have already established the existence of $\alpha_{k-1}, \beta_{k-1} > 0$ independent of Ψ_0 such that

$$|||\Psi(t)|||_{k-1} \leq \alpha_{k-1} e^{\beta_{k-1}|t|} |||\Psi_0|||_{k-1} \quad \text{for all } t \in \mathbb{R},$$

we have

$$\|\not{\mathcal{D}}^k \Psi(t)\|_{L^2(\Sigma)} \leq (\|\not{\mathcal{D}}^k \Psi_0\|_{L^2} + C'|t|\alpha_{k-1} e^{\beta_{k-1}|t|} |||\Psi_0|||_{k-1}) e^{C'|t|}$$

and we infer the existence of $\alpha_k, \beta_k > 0$, independent of Ψ_0, such that

$$|||\Psi(t)|||_k \leq \alpha_k e^{\beta_k |t|} |||\Psi_0|||_k \quad \text{for all } t \in \mathbb{R}. \tag{5.17}$$

Hence, starting from the conservation of the L^2 norm of the solutions, we prove by induction that inequality (5.17) is true for all $k \in \mathbb{N}$ and for all $\Psi_0 \in \mathcal{C}_0^\infty(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$ with constants α_k, β_k independent of Ψ_0. Of course, inequality (5.17) is very crude, but it is enough to obtain the qualitative informations we need.

We now consider $\Psi_0 \in H_0^k(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$ and a sequence $\Psi_0^n \in \mathcal{C}_0^\infty(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$ which converges towards Ψ_0 in $H_0^k(\Sigma)$. (5.17) and the linearity of the Dirac equation imply that the associated solutions Ψ^n converge in $\mathcal{C}(\mathbb{R}_t; H_0^k(\Sigma))$. Since Ψ^n converges towards Ψ in $\mathcal{C}(\mathbb{R}_t; L^2(\Sigma))$ by continuity of the propagator on L^2, we conclude that $\Psi \in \mathcal{C}(\mathbb{R}_t; H_0^k(\Sigma))$. We can read off directly from the equation the additional regularity of Ψ:

$$\Psi \in \bigcap_{l=0}^{k} \mathcal{C}^l(\mathbb{R}_t; H_0^{k-l}(\Sigma; \mathbb{S}_{\mathrm{Dirac}})).$$

Moreover, by continuity, (5.17) is still valid for solutions with values in $H_0^k(\Sigma)$ and this proves that $\mathcal{U}(t)$ is a strongly continuous one-parameter group of bounded operators on $H_0^k(\Sigma)$ for all $k \in \mathbb{N}$. This concludes the proof of the second part of Theorem 5.

3. *The Cauchy problem in weighted L^2 spaces*: We do not detail the proof of the third part of Theorem 5; it is identical to the proof of the fourth part given below, without the additional complication due to the control of the regularity.

4. *The Cauchy problem in weighted Sobolev spaces*: The results of Theorem 3 concerning the well-posedness of the Cauchy problem in weighted Sobolev spaces on space-times of class (k, δ) together with the second part of Theorem 5 are sufficient to prove the fourth part of Theorem 5. Let us consider $\Psi_0 \in H_{0,\varrho}^k(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$, $k \in \mathbb{N}$, $\varrho \in \mathbb{R}$. We introduce a cut-off function $\chi \in \mathcal{C}^\infty(\Sigma)$, $\chi \equiv 0$ for $u > 2$ and $\chi \equiv 1$ for $u < 1$ (for example), where u is the h-distance to the horizon $H = \partial\Sigma$. We split the initial data Ψ_0 into a part localized near the horizon and an "asymptotic" part which does not touch H:

$$\Psi_0 = \Phi_0 + \Theta_0, \quad \Phi_0 := \chi\Psi_0.$$

Then $\Phi_0 \in H_0^k(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$ and the solution Φ to the Dirac equation outside the black hole associated with Φ_0 satisfies

$$\Phi \in \mathcal{C}^l(\mathbb{R}_t; H_0^{k-l}(\Sigma)), \quad 0 \leq l \leq k.$$

Moreover, the support of Φ propagates at finite speed, i.e. there exists a smooth increasing function d on $[0, +\infty[$ satisfying

$$d(0) = 2, \quad d(t) \to +\infty \quad \text{as } t \to +\infty$$

such that, for all $t \in \mathbb{R}$,

$$\operatorname{supp} \Phi(t) \subset \{x \in \Sigma;\ u(x) \leq d(|t|)\}.$$

For any $R > 0$, $k \in \mathbb{N}$, $\varrho \in \mathbb{R}$, and for any $f \in H_0^k(\Sigma)$ with support in $\{u \leq R\}$, we have $f \in H_{0,\varrho}^k(\Sigma)$ and

$$\begin{aligned} (5.18) \qquad \min(1, (1+R^2)^{(-\varrho-3/2)/2}) \|f\|_{H^k(\Sigma)} \\ \leq \|f\|_{H_\varrho^k(\Sigma)} \leq \max(1, (1+R^2)^{(-\varrho-3/2+k)/2}) \|f\|_{H^k(\Sigma)}. \end{aligned}$$

This immediately implies that for any $T > 0$, $\varrho \in \mathbb{R}$,

$$\Phi \in \bigcap_{l=0}^{k} \mathcal{C}^l([-T,T]; H_{0,\varrho}^{k-l}(\Sigma; \mathbb{S}_{\mathrm{Dirac}})),$$

i.e.

$$\Phi \in \bigcap_{l=0}^{k} \mathcal{C}^l(\mathbb{R}_t; H_{0,\varrho}^{k-l}(\Sigma; \mathbb{S}_{\mathrm{Dirac}})).$$

Furthermore, using (5.17), (5.18) and Proposition 5.1, we see that

$$(5.19) \quad \|\Phi(t)\|_{H_{0,\varrho}^k(\Sigma)} \leq C \frac{\max(1, (1+d(|t|)^2)^{(-\varrho-3/2+k)/2})}{\min(1, 5^{(-\varrho-3/2)/2})} \alpha_k e^{\beta_k |t|} \|\Phi_0\|_{H_{0,\varrho}^k(\Sigma)}.$$

As for Θ_0, we choose $0 < \varepsilon < 1$; then Θ_0 belongs to the space $H_\varrho^k(\widetilde{\Sigma})$ associated with the metric ${}^\varepsilon g$, assumed for convenience, in this last part of the proof, to coincide with g for $u > \varepsilon$ instead of $r > 2M + \varepsilon$. Using Theorem 3, this implies that the Dirac equation on $(\mathbb{R}_t \times \widetilde{\Sigma}, {}^\varepsilon g)$ has a unique solution ${}^\varepsilon\Theta \in \mathcal{C}(\mathbb{R}_t; H_\varrho^k(\widetilde{\Sigma}))$ such that ${}^\varepsilon\Theta(0) = \Theta_0$ and ${}^\varepsilon\Theta$ satisfies

$${}^\varepsilon\Theta \in \bigcap_{l=0}^{k} \mathcal{C}^l(\mathbb{R}_t; H_\varrho^{k-l}(\widetilde{\Sigma})).$$

If we denote by ${}^\varepsilon\mathcal{U}(t,s)$ the propagator for the Dirac equation on $(\mathbb{R}_t \times \widetilde{\Sigma}, {}^\varepsilon g)$ and introduce a continuous function ${}^\varepsilon K$ on $\mathbb{R}_t$ such that

$${}^\varepsilon K(t) \geq \|{}^\varepsilon\mathcal{U}(t,0)\|_{\mathcal{L}(H_\varrho^k(\widetilde{\Sigma}))}$$

then we obtain

$$\|{}^\varepsilon\Theta(t)\|_{H_\varrho^k(\widetilde{\Sigma})} \leq {}^\varepsilon K(t) \|\Theta_0\|_{H_\varrho^k(\widetilde{\Sigma})} = {}^\varepsilon K(t) \|\Theta_0\|_{H_\varrho^k(\Sigma)}.$$

Thanks to the finite propagation speed, there exists $T_1(\varepsilon) < 0 < T_2(\varepsilon)$ such that for $T_1(\varepsilon) < t < T_2(\varepsilon)$, the support of ${}^\varepsilon\Theta(t)$ is contained in $\{u > \varepsilon\}$. Therefore, using the equivalence (locally uniform in time) of the norm $\|\cdot\|_{H_\varrho^k(\widetilde{\Sigma})}$ associated with the background flat metric and the norms $\|\cdot\|_{H_\varrho^k(\widetilde{\Sigma}_t)}$ associated with the "physical" metric ${}^\varepsilon g$, which coincides with g in $\{u > \varepsilon\}$, we have

$${}^\varepsilon\Theta \in \bigcap_{l=0}^{k} \mathcal{C}^l([T_1(\varepsilon), T_2(\varepsilon)]; H_{0,\varrho}^{k-l}(\Sigma; \mathbb{S}_{\mathrm{Dirac}}))$$

and

$$\|{}^\varepsilon\Theta(t)\|_{H_\varrho^k(\Sigma)} \leq {}^\varepsilon C\, {}^\varepsilon K(t) \|\Theta_0\|_{H_\varrho^k(\Sigma)}.$$

We put ${}^{\varepsilon}C(t) := {}^{\varepsilon}C\,{}^{\varepsilon}K(t)$. Since we have $T_1(\varepsilon) \to -\infty$ and $T_2(\varepsilon) \to +\infty$ as $\varepsilon \to 0$, we easily infer that the Dirac equation on $\mathbb{R}_t \times \Sigma$ has a unique solution $\Theta(t)$ in $\mathcal{C}(\mathbb{R}_t; H^k_{0,\varrho}(\Sigma))$ such that $\Theta(0) = \Theta_0$ and moreover

$$\Theta \in \bigcap_{l=0}^{k} \mathcal{C}^l(\mathbb{R}_t; H^{k-l}_{0,\varrho}(\Sigma; \mathbb{S}_{\mathrm{Dirac}})).$$

Θ is defined as follows: for any $0 < \varepsilon < 1$,

$$\Theta(t) := {}^{\varepsilon}\Theta(t) \text{ for } T_1(\varepsilon) < t < T_2(\varepsilon).$$

We now wish to show that the norm of $\Theta(t)$ is controlled by the norm of Θ_0 for all times. We fix $\varepsilon_0 = 1/2$. Then, we have for $t \in [T_1(1/2), T_2(1/2)]$

$$\|\Theta(t)\|_{H^k_\varrho(\Sigma)} \leq {}^0C(t)\|\Theta_0\|_{H^k_\varrho(\Sigma)},$$

where ${}^nC$ denotes the function ${}^{\varepsilon}C$ associated with ε_n. Now putting $\varepsilon_1 = 1/4$, we have for $t \in [T_1(1/4), T_2(1/4)]$,

$$\|\Theta(t)\|_{H^k_\varrho(\Sigma)} \leq {}^1C(t)\|\Theta_0\|_{H^k_\varrho(\Sigma)},$$

and so on for $\varepsilon_2 = 1/8$, etc.

The sequence $T_1(1/2^n)$ is strictly decreasing towards $-\infty$ and $T_2(1/2^n)$ is strictly increasing towards $+\infty$. We now choose a continuous function $C(t)$ on $\mathbb{R}$ such that

$$C(t) \geq {}^nC(t) \text{ for } t \in [T_1(1/2^n), T_1(1/2^{n-1})] \cup [T_2(1/2^{n-1}), T_2(1/2^n)];$$

then for all $t \in \mathbb{R}$ we have

$$\|\Theta(t)\|_{H^k_\varrho(\Sigma)} \leq C(t)\|\Theta_0\|_{H^k_\varrho(\Sigma)} \tag{5.20}$$

and the function C does not depend on Θ_0 (that is, does not depend on the choice of Ψ_0), it only depends on the choice of the cut-off function χ and the definition of the metrics ${}^{\varepsilon}g$.

Now putting the two solutions together, we find that the Dirac equation has a unique solution $\Psi = \Phi + \Theta$ in $\mathcal{C}(\mathbb{R}_t; H^k_{0,\varrho}(\Sigma))$ such that $\Psi(0) = \Psi_0$ and

$$\Psi \in \bigcap_{l=0}^{k} \mathcal{C}^l(\mathbb{R}_t; H^{k-l}_{0,\varrho}(\Sigma; \mathbb{S}_{\mathrm{Dirac}})).$$

Moreover, for all $t \in \mathbb{R}$,

$$\|\Psi(t)\|_{H^k_{0,\varrho}(\Sigma)} \leq \|\Phi(t)\|_{H^k_{0,\varrho}(\Sigma)} + \|\Theta(t)\|_{H^k_{0,\varrho}(\Sigma)} \leq \widetilde{C}(t)(\|\Phi_0\|_{H^k_{0,\varrho}(\Sigma)} + \|\Theta_0\|_{H^k_{0,\varrho}(\Sigma)})$$

where $\widetilde{C}(t)$ is a continuous function on $\mathbb{R}$, independent of the choice of Ψ_0, whose existence is deduced from estimates (5.19) and (5.20). Finally, we have

$$\|\Phi_0\|_{H^k_{0,\varrho}(\Sigma)} + \|\Theta_0\|_{H^k_{0,\varrho}(\Sigma)} \leq C\|\Psi_0\|_{H^k_{0,\varrho}(\Sigma)}$$

where C is a constant depending only on the cut-off function χ, involving the L^∞ norms of its derivatives of order lower than k on $\{1 < u < 2\}$.

Hence, $\mathcal{U}(t)$ is a strongly continuous one-parameter group of bounded operators on $H^k_{0,\varrho}(\Sigma)$ for all $k \in \mathbb{N}$, $\varrho \in \mathbb{R}$. This concludes the proof of Theorem 5. ■

5.2. Maximal extension of Schwarzschild's space-time. After having adopted, in the previous section, the point of view of an observer static with respect to infinity, and thus limited our study to the exterior of the black hole foliated using Schwarzschild's time coordinate, we describe here briefly the global geometry of Schwarzschild's space-time. We define the Kruskal–Szekeres variables inside and outside the black hole. These will allow us to show that the horizon is not a singularity of the metric. The maximal analytic extension of Schwarzschild's space-time will then appear naturally. Most of the material of this section is standard, it can be found under various forms in [7], [24] and [44] for example.

5.2.1. *Kruskal–Szekeres coordinates.* Outside the black hole, Kruskal–Szekeres coordinates (T, X, ω), where ω denotes the angular variables of the Schwarzschild coordinate system, are defined by

$$(5.21) \qquad T = \frac{1}{2} e^{r_*/(4M)} (e^{t/(4M)} - e^{-t/(4M)}), \qquad X = \frac{1}{2} e^{r_*/(4M)} (e^{t/(4M)} + e^{-t/(4M)})$$

where r_* is the Regge–Wheeler variable

$$r_* = r + 2M \operatorname{Log}(r - 2M).$$

This coordinate system maps the exterior of the black hole $\mathbb{R}_t \times]2M, +\infty[_r \times S^2_\omega$ onto the quadrant $\{X > |T|\}$ of $\mathbb{R}_T \times \mathbb{R}_X \times S^2_\omega$. The horizon now appears as the hypersurface $\{(T, X, \omega);\ T = X > 0,\ \omega \in S^2\}$. The outgoing (resp. incoming) radial null geodesics, represented in (t, r_*, ω) coordinates as the straight lines $\{(t, r_* = t + s, \omega);\ t \in \mathbb{R}\}$ (resp. $\{(t, r_* = -t+s, \omega);\ t \in \mathbb{R}\}$) for fixed $s \in \mathbb{R}$ and $\omega \in S^2$, are described in Kruskal–Szekeres coordinates as the straight lines $\{(T, X = T + S, \omega)\}$ (resp. $\{(T, X = -T + S, \omega)\}$) for fixed S and ω.

Inside the black hole, the definition is very similar. We consider a Regge–Wheeler coordinate adapted to this domain

$$r_* = r + 2M \operatorname{Log}|r - 2M| = r + 2M \operatorname{Log}(2M - r),$$

the expression of the variables T and X in terms of t and r_* is then given by

$$(5.22) \qquad T = \frac{1}{2} e^{r_*/(4M)} (e^{-t/(4M)} + e^{t/(4M)}), \qquad X = \frac{1}{2} e^{r_*/(4M)} (e^{-t/(4M)} - e^{t/(4M)}).$$

The interior of the black hole $\mathbb{R}_t \times]0, 2M[_r \times S^2_\omega$ is mapped onto the domain $\{(T, X, \omega) \in \mathbb{R} \times \mathbb{R} \times S^2;\ |X| < T < \sqrt{X^2 + 2M}\}$ and the singularity at $r = 0$ is represented as the product of S^2_ω with the hyperbola in the (T, X)-plane: $\{(T, X);\ T^2 - X^2 = 2M,\ T > 0\}$.

The expression of the metric in Kruskal–Szekeres coordinates is the same inside and outside the black hole

$$g = \frac{16M^2}{X^2 - T^2} \left(1 - \frac{2M}{r}\right) (\mathrm{d}T^2 - \mathrm{d}X^2) - r^2 \mathrm{d}\omega^2.$$

This can be simplified using the fact that

$$(5.23) \qquad X^2 - T^2 = (r - 2M) e^{r/(2M)}$$

and we obtain

$$(5.24) \qquad g = \frac{16M^2}{r} e^{-r/(2M)} (\mathrm{d}T^2 - \mathrm{d}X^2) - r^2 \mathrm{d}\omega^2$$

where r is determined implicitly in terms of T and X by (5.23). The function $(r - 2M)e^{r/(2M)}$ is analytic in r and strictly increasing from $]0, +\infty[$ onto $]-2M, +\infty[$.

It follows that r is an analytic function of $X^2 - T^2$, and therefore of (T, X), on $-2M < X^2 - T^2 < +\infty$. An immediate consequence is the analyticity of the metric g on the whole Schwarzschild manifold described in (T, X, ω) coordinates as $\{(T, X, \omega) \in \mathbb{R} \times \mathbb{R} \times S^2;\ T + X > 0,\ T < \sqrt{X^2 + 2M}\}$ (the singularity at $r = 0$ is not considered as a subset of the Schwarzschild manifold). This shows in particular that the metric g is not singular at the horizon of the black hole; the expression (5.24) of g and the description of the horizon in (T, X, ω) coordinates reveal it to be a smooth null hypersurface of Schwarzschild's space-time. Other properties of the horizon can be infered from its description in Kruskal–Szekeres coordinates and more particularly the fact that it is an event horizon (for more details, the reader is referred to the references given at the beginning of this section).

5.2.2. *Maximal Schwarzschild space-time.* As we have seen above, the metric (5.24) can be extended analytically on the region

$$\mathcal{M}^{\mathcal{K}} = \{(T, X, \omega) \in \mathbb{R} \times \mathbb{R} \times S^2_\omega;\ X^2 - T^2 > -2M\}.$$

We obtain a new space-time $(\mathcal{M}^{\mathcal{K}}, g)$ called the Kruskal extension, or maximal analytic extension, of Schwarzschild's space-time. It contains the Schwarzschild manifold $(\mathcal{M}^{\mathcal{S}}, g)$, where $\mathcal{M}^{\mathcal{S}}$ is the subset of $\mathcal{M}^{\mathcal{K}}$

$$\mathcal{M}^{\mathcal{S}} = \{(T, X, \omega) \in \mathcal{M}^{\mathcal{K}};\ X + T > 0\}.$$

The additional part of $(\mathcal{M}^{\mathcal{K}}, g)$, which we denote $(\mathcal{M}^{\check{\mathcal{S}}}, g)$, where

$$\mathcal{M}^{\check{\mathcal{S}}} = \{(T, X, \omega) \in \mathcal{M}^{\mathcal{K}};\ X + T < 0\},$$

is isometric to $(\mathcal{M}^{\mathcal{S}}, g)$ with its time orientation reversed: it describes a "Schwarzschild white hole". More explicitly, $\mathcal{M}^{\check{\mathcal{S}}}$ is the image of the Schwarzschild space-time, described in Schwarzschild coordinates, by the transformations (5.21) and (5.22) with the signs of T and X reversed. The space-time $(\mathcal{M}^{\mathcal{K}}, g)$ is best pictured by a Penrose diagram, which can be constructed by defining the new coordinates:

$$\alpha = \arctan\left(\frac{T + X}{\sqrt{2M}}\right) - \arctan\left(\frac{T - X}{\sqrt{2M}}\right),$$
$$\beta = \arctan\left(\frac{T + X}{\sqrt{2M}}\right) + \arctan\left(\frac{T - X}{\sqrt{2M}}\right).$$

$(\mathcal{M}^{\mathcal{K}}, g)$ is globally hyperbolic. We choose a foliation $\{S_\tau\}_{\tau \in \mathbb{R}}$ by smooth Cauchy hypersurfaces in the following manner:

- We consider the foliations $\{\Sigma_t\}_{t \in \mathbb{R}}$ and $\{\check{\Sigma}_t\}_{t \in \mathbb{R}}$ of domains I and III (see figures 5.1 and 5.2) induced by the Schwarzschild coordinate t in these two regions. In region I, t is defined in terms of T and X by the inverse of transformation (5.21), in region III we must use the inverse of transformation (5.21) with the signs of T and X reversed. In both cases, we obtain

$$t = 2M \operatorname{Log}\left(\frac{X + T}{X - T}\right).$$

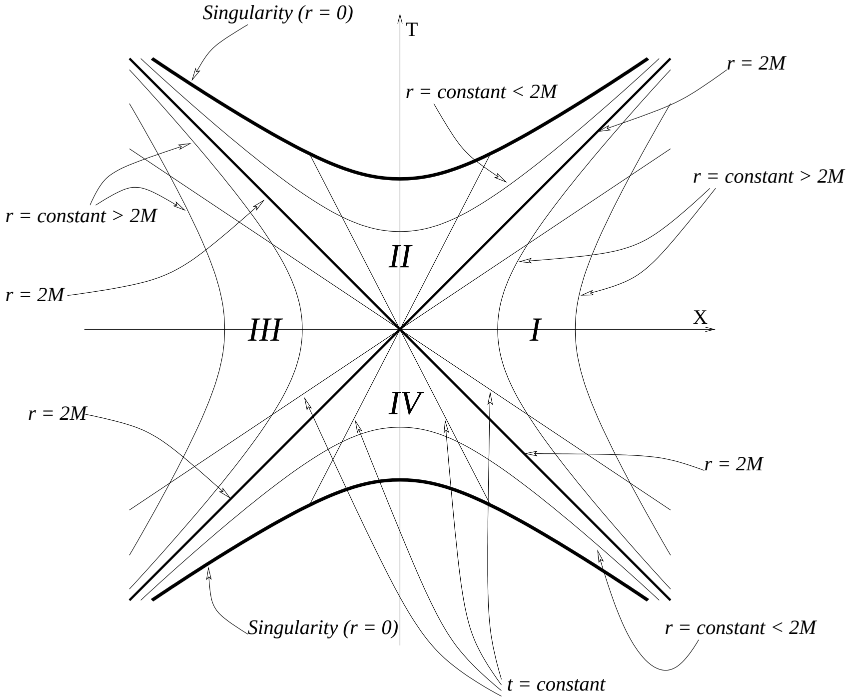

Fig. 5.1. The maximal analytic extension of Schwarzschild's space-time in Kruskal–Szekeres coordinates: domains I and III correspond to $r > 2M$, domain II represents the interior of the black hole and domain IV the interior of the white hole

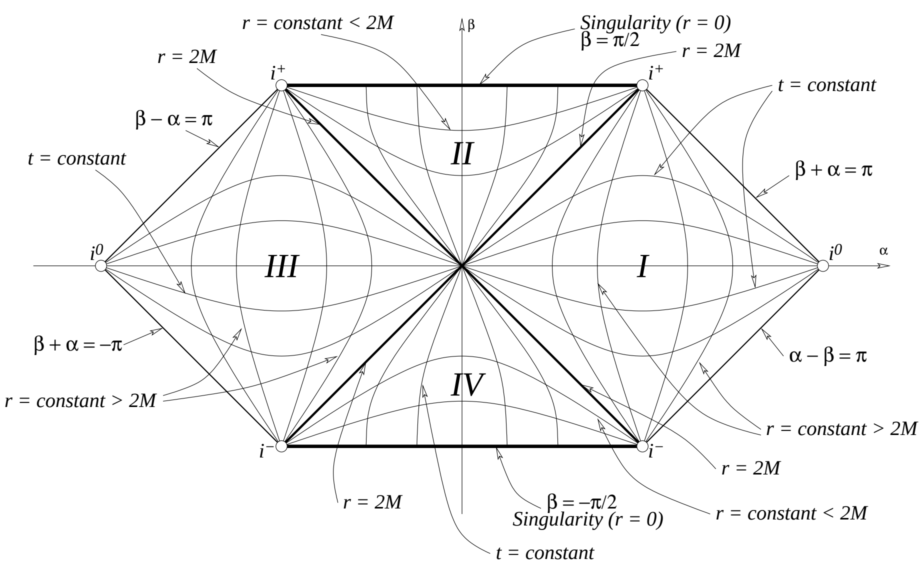

Fig. 5.2. The Penrose diagram of maximal Schwarzschild space-time

This yields the descriptions of surfaces Σ_t and $\check{\Sigma}_t$ in Kruskal–Szekeres coordinates:

$$\Sigma_t = \left\{ (T, X, \omega);\ X > 0,\ T = \frac{e^{t/(2M)} - 1}{e^{t/(2M)} + 1} X,\ \omega \in S^2 \right\},$$

$$\check{\Sigma}_t = \left\{ (T, X, \omega);\ X < 0,\ T = \frac{e^{t/(2M)} - 1}{e^{t/(2M)} + 1} X,\ \omega \in S^2 \right\}.$$

• We only require that the hypersurface S_τ, outside the domain of dependence of a neighbourhood of the horizon at $T = 0$, coincides with Σ_τ if $X > 0$ and with $\check{\Sigma}_{-\tau}$ if $X < 0$. Let us explain this requirement more precisely. At $T = 0$, the horizon is reduced to the two-sphere $\{(0, 0, \omega)\}$ of $\mathbb{R}_T \times \mathbb{R}_X \times S^2_\omega$ at which the horizons of the black hole and of the white hole intersect. The variables $T + X$ and $T - X$ are null variables: the domain of dependence of a neighbourhood $\{(0, X, \omega); |X| < C\}$ (for a given $C > 0$) of the horizon at $T = 0$ will be the region $\{|X| < |T| + C\}$. The condition imposed on the surfaces S_τ means that for each $\tau \in \mathbb{R}$, S_τ coincides with Σ_τ in the region $\{X > |T| + C\}$ and with $\check{\Sigma}_{-\tau}$ in the region $\{X < -|T| - C\}$. This condition can be expressed more explicitly in terms of τ: for a given $K \in \mathbb{R}$, S_τ coincides with Σ_τ in the part of region I such that (r being defined implicitly by (5.23) in terms of T and X)

$$r_* = r + 2M \operatorname{Log}(r - 2M) > |\tau| + K$$

and S_τ coincides with $\check{\Sigma}_{-\tau}$ in the part of region III such that $r_* > |\tau| + K$. We indicate the typical shape of a surface S_τ in Figure 5.3.

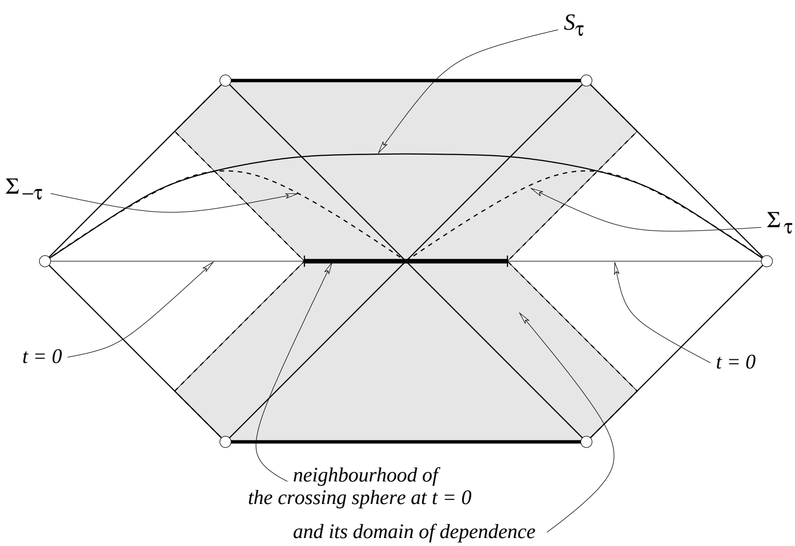

Fig. 5.3. Foliation of maximal Schwarzschild space-time

For these foliations, the asymptotic behaviour of the metric g at spacelike infinity on each S_τ is the same as the behaviour of g for fixed t and $r \to +\infty$ (this last is described in the proof of Theorem 5). Hence, using such foliations allows us to show that $(\mathcal{M}^{\mathcal{K}}, g)$ belongs to our classes of asymptotically flat space-times (with two asymptotic ends); it belongs to all classes (∞, δ) for $\delta > -2$. Thus the theorems of Chapter 4 can be applied directly to this framework and guarantee the existence and uniqueness of solutions to the Dirac equation with values in L^2, H^k, L^2_μ, H^k_μ (for any $k \in \mathbb{N}$ and $\mu \in \mathbb{R}$) on each hypersurface S_τ. This proves that solutions to the Dirac equation on the maximal extension of Schwarzschild's space-time are well behaved as long as they do not reach the singularity $\{X^2 - T^2 = -2M\}$.

6. Dirac's equation on the Kerr metric

Kerr's space-time is more perplexing than Schwarzschild's space-time of which it is a generalization. The Kerr metric describes a rotating uncharged black hole; in Boyer–Lindquist coordinates on $\mathbb{R}_t \times \mathbb{R}_r \times S^2_\omega$, it takes the form

$$\begin{aligned} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu &= \left(1 - \frac{2Mr}{\varrho^2}\right)\mathrm{d}t^2 + \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\varrho^2}\,\mathrm{d}t\mathrm{d}\varphi - \frac{\varrho^2}{\Delta}\mathrm{d}r^2 \\ &\quad - \varrho^2\mathrm{d}\theta^2 - \left(\frac{(r^2+a^2)\varrho^2 + 2Mra^2\sin^2\theta}{\varrho^2}\right)\sin^2\theta\,\mathrm{d}\varphi^2, \\ \varrho^2 &= r^2 + a^2\cos^2\theta,\ \Delta = r^2 - 2Mr + a^2, \end{aligned} \tag{6.1}$$

where a is the angular momentum per unit mass and $M > 0$ is the mass of the black hole. The black hole rotates around the axis going through its North and South poles. This results into a nonzero coefficient $g_{t\varphi}$ that couples the variables t and φ. The function Δ is the analogue of $r^2(1-2M/r)$ in Schwarzschild's space-time; it defines the horizons as the sets of points where $\Delta = 0$. These horizons appear as singularities in the expression (6.1) above, but they are merely coordinate singularities, the metric can be extended smoothly through them. The only true curvature singularity of the metric is the equatorial ring defined by $\varrho^2 = 0$, i.e. $r = 0$ and $\theta = \pi/2$. There are three types of Kerr space-times depending on the respective importance of the rotation and the mass:

• *Slow Kerr space-time* for $0 < |a| < M$ (the case $a = 0$ reduces to the Schwarzschild metric). Δ has two real roots r_- and r_+:

$$0 < r_- = M - \sqrt{M^2 - a^2} < M < r_+ = M + \sqrt{M^2 - a^2} < 2M, \tag{6.2}$$

so there are two horizons on either side of the sphere $\{r = M\}$.

• *Extreme Kerr space-time* for $|a| = M$. M is then a double root for Δ and the sphere $\{r = M\}$ is the only horizon.

• *Fast Kerr space-time* for $|a| > M$. Δ has no real root and the space-time has no horizon. There is no black hole in this case, the ring singularity is a naked singularity.

We consider only the case of slow Kerr metrics. Horizons separate the space-time in connected regions called Boyer–Lindquist blocks:

Block I is the exterior of the black hole $\{r > r_+\}$. It is the simplest of all three blocks. In this region, the vectors $\partial/\partial r$, $\partial/\partial\theta$, $\partial/\partial\varphi$ are spacelike and, for $r \gg 1$, $\partial/\partial t$ is timelike. However, block I contains a region called the ergosphere in which $g_{tt} < 0$ and thus $\partial/\partial t$

is spacelike. The ergosphere is the toroidal domain around the outside horizon:

$$\mathcal{E} = \{(t, r, \theta, \varphi);\, r_+ < r < M + \sqrt{M^2 - a^2 \cos^2 \theta}\}.$$

Inside $\mathcal{E}$, the effects of the rotation are extreme and along every future-oriented nonspacelike curve, the quantity $a\varphi$ is strictly increasing.

Block I, like any Boyer–Lindquist block, is not stationary, i.e. there is no timelike Killing vector field globally defined on it. However, the exterior of the ergosphere is stationary, and even absolutely stationary, since $\partial/\partial t$ is the unique (up to multiplication by a constant) timelike Killing vector field globally defined there. Also, every point in block I, even inside the ergosphere, has a stationary neighbourhood.

Block II is the region between the outer and inner horizons $\{r_- < r < r_+\}$; it only exists in the slow case. $\partial/\partial r$ is timelike there and $\partial/\partial t$, $\partial/\partial\theta$, $\partial/\partial\varphi$ are spacelike. It is a dynamic domain where the inertial frames are dragged towards the inner horizon (the time orientation implicit in this description is such that $\partial/\partial r$ is past pointing).

Block III lies beyond the inner horizon $\{-\infty < r < r_-\}$. It contains another ergosphere

$$\mathcal{E}' = \{(t, r, \theta, \varphi);\; M - \sqrt{M^2 - a^2 \cos^2 \theta} < r < r_-\},$$

the ring singularity and a time machine (being the only region where $\partial/\partial\varphi$ is timelike) which allows any two points in block III to be joined by a future-oriented timelike curve. Hence, not only is block III not stationary, it is not causal either.

For a detailed description of the geometry of Kerr black holes, see [50].

6.1. The exterior of the black hole. In this section, we study Dirac fields in block I from the point of view of an observer who is static with respect to infinity, as we did in Section 5.1 for the Schwarzschild black holes. The perception of such observers is limited to block I and is described by the time function t of the Boyer–Lindquist coordinates. Just as in the Schwarzschild case, light rays in block I can only reach the horizon when t becomes infinite. Hence, if the support of a Dirac field (for example) does not touch the horizon at some particular time t_0, it will never touch it for finite values of t, i.e. the distance $d(t)$ of the support of the field to the horizon at time t is a strictly positive continuous function on $\mathbb{R}_t$; it may (and usually does) tend to zero when $t \to \pm\infty$. To explain this property more precisely, we consider the principal null geodesics (the analogues of radial null geodesics in Schwarzschild's space-time). They are the straightest routes to or from the horizon and are defined by

$$\dot{r} = \pm 1, \quad \dot{\theta} = 0, \quad \dot{\varphi} = \frac{a}{\Delta}, \quad \dot{t} = \frac{r^2 + a^2}{\Delta}.$$

Introducing a new coordinate r_* such that

$$\frac{\mathrm{d}r_*}{\mathrm{d}r} = \frac{r^2 + a^2}{\Delta} > 0 \quad \text{on }]r_+, +\infty[$$

we get

$$\dot{r}_* = \pm \dot{t}$$

and therefore, along a principal null geodesic we must have

$$t = \pm r_* + C.$$

The horizon $r = r_+$ corresponds to $r_* \to -\infty$ and is consequently reached only when t becomes infinite.

In this framework, we solve the Cauchy problem for the Dirac equation in Sobolev and weighted Sobolev spaces. We first study the geometry of $\{t = \text{constant}\}$ slices; their extrinsic geometry which is nontrivial and even singular at the horizon will make the analysis of the Dirac–Witten operator slightly more intricate.

6.1.1. *The spacelike geometry of block I.* We denote by $\mathcal{M}$ the space-time outside the black hole and we choose the foliation of $\mathcal{M}$ by the level hypersurfaces of the time-function t:

$$\Sigma_t = \{t\} \times]r_+, +\infty[_r \times S^2_{\theta,\varphi}. \tag{6.3}$$

For each t, the hypersurface Σ_t is spacelike since at each point, its tangent plane is spanned by the three spacelike vectors $\partial/\partial r$, $\partial/\partial\theta$, $\partial/\partial\varphi$. This shows that t is indeed a time function, i.e. its gradient $\nabla^a t$ is a timelike vector field, in spite of the fact that in Boyer–Lindquist coordinates, $\partial/\partial t$ is not everywhere timelike in block I. The time orientation is fixed by deciding that $\nabla^a t$ is future pointing.

The $\mathbf{3+1}$ *decomposition of the Kerr metric in block I.* We perform the $3+1$ decomposition of the metric g relative to the foliation $\{\Sigma_t\}_{t\in\mathbb{R}}$. We calculate the expression of the vector

$$T^a = \frac{\sqrt{2}}{|\nabla t|}\nabla^a t$$

in Boyer–Lindquist coordinates. To do this, we look for a future pointing timelike vector field U^a orthogonal to Σ_t at each point and we normalize it to obtain T^a. The time orientation yields that t increases along all timelike future pointing curves, hence we choose U^a of the form

$$U^a\partial_a = \frac{\partial}{\partial t} + A\frac{\partial}{\partial r} + B\frac{\partial}{\partial\theta} + C\frac{\partial}{\partial\varphi}$$

and imposing that U^a should be everywhere g-orthogonal to $\partial/\partial r$, $\partial/\partial\theta$ and $\partial/\partial\varphi$, we obtain

$$U^a\partial_a = \frac{\partial}{\partial t} - \frac{g_{t\varphi}}{g_{\varphi\varphi}}\frac{\partial}{\partial\varphi} = \frac{\partial}{\partial t} + \frac{2aMr}{(r^2+a^2)\varrho^2 + 2Mra^2\sin^2\theta}\frac{\partial}{\partial\varphi}. \tag{6.4}$$

We put

$$\alpha(r,\theta) = -\frac{g_{t\varphi}}{g_{\varphi\varphi}} = \frac{2aMr}{(r^2+a^2)\varrho^2 + 2Mra^2\sin^2\theta}. \tag{6.5}$$

The norm of U^a is then given by

$$|U|^2 = U_aU^a = g_{tt} - \frac{(g_{t\varphi})^2}{g_{\varphi\varphi}} = \frac{-\Delta\sin^2\theta}{g_{\varphi\varphi}} = \frac{\Delta\varrho^2}{(r^2+a^2)\varrho^2 + 2Mra^2\sin^2\theta} > 0 \text{ in block I},$$

and the vector T^a is

$$T^a = \frac{\sqrt{2}}{|U|}U^a.$$

If we introduce the vector fields r^a, θ^a, φ^a defined as

$$r^a\partial_a = |g_{rr}|^{-1/2}\frac{\partial}{\partial r}, \quad \theta^a\partial_a = |g_{\theta\theta}|^{-1/2}\frac{\partial}{\partial\theta}, \quad \varphi^a\partial_a = |g_{\varphi\varphi}|^{-1/2}\frac{\partial}{\partial\varphi},$$

then $\{1/\sqrt{2}T^a, r^a, \theta^a, \varphi^a\}$ is a local orthonormal Lorentz frame in block I; the metric can therefore be written as

$$g_{ab} = \frac{1}{2}T_aT_b - h_{ab},\ h_{ab} = r_ar_b + \theta_a\theta_b + \varphi_a\varphi_b$$

and the 1-forms T_a, r_a, θ_a and φ_a are given by

$$T_a\mathrm{d}x^a = \sqrt{2}\,|U|\,\mathrm{d}t = \sqrt{2}\sqrt{g_{tt} - \frac{(g_{t\varphi})^2}{g_{\varphi\varphi}}}\mathrm{d}t, \quad r_a\mathrm{d}x^a = -|g_{rr}|^{1/2}\mathrm{d}r, \quad \theta_a\mathrm{d}x^a = -|g_{\theta\theta}|^{1/2}\mathrm{d}\theta,$$

$$\varphi_a\mathrm{d}x^a = |g_{\varphi\varphi}|^{-1/2}(g_{t\varphi}\mathrm{d}t + g_{\varphi\varphi}\mathrm{d}\varphi) = -|g_{\varphi\varphi}|^{1/2}(\mathrm{d}\varphi - \alpha\,\mathrm{d}t).$$

This gives the expression of the lapse function

$$N = \sqrt{2}\,|U| = \sqrt{2}\left(g_{tt} - \frac{(g_{t\varphi})^2}{g_{\varphi\varphi}}\right)^{1/2} = \left(\frac{2\Delta\varrho^2}{(r^2+a^2)\varrho^2 + 2Mra^2\sin^2\theta}\right)^{1/2}.$$

In Boyer–Lindquist coordinates, the product structure is associated to the Killing vector field $\partial/\partial t$. If we wish our decomposition of the metric to be useful, we must interpret h_{ab} as a (time dependent) metric on

$$\Sigma :=]r_+, +\infty[_r \times S^2_{\theta,\varphi}.$$

This requires choosing the product structure associated with T^a. An explicit way of doing this is to define the new coordinates τ, R, Θ, Φ:

$$\tau = t, \quad R = r, \quad \Theta = \theta, \quad \Phi = \varphi - (t - t_0)\alpha(r,\theta)\,(\mathrm{mod}\,2\pi)$$

for a given $t_0 \in \mathbb{R}$. We obtain the following expression of g:

$$\begin{aligned}
g(\tau) &= \frac{N^2}{2}\,\mathrm{d}\tau^2 - h(\tau) \\
&= \left(g_{tt} - \frac{(g_{t\varphi})^2}{g_{\varphi\varphi}}\right)\mathrm{d}\tau^2 + g_{rr}\mathrm{d}R^2 + g_{\theta\theta}\mathrm{d}\Theta^2 \\
&\quad + g_{\varphi\varphi}\left(\mathrm{d}\Phi + (\tau - t_0)\frac{\partial\alpha}{\partial R}\mathrm{d}R + (\tau - t_0)\frac{\partial\alpha}{\partial\Theta}\mathrm{d}\Theta\right)^2 \\
&= \left(g_{tt} - \frac{(g_{t\varphi})^2}{g_{\varphi\varphi}}\right)\mathrm{d}\tau^2 + \left(g_{rr} + (\tau - t_0)^2\left(\frac{\partial\alpha}{\partial R}\right)^2 g_{\varphi\varphi}\right)\mathrm{d}R^2 \\
&\quad + \left(g_{\theta\theta} + (\tau - t_0)^2\left(\frac{\partial\alpha}{\partial\Theta}\right)^2 g_{\varphi\varphi}\right)\mathrm{d}\Theta^2 + g_{\varphi\varphi}\mathrm{d}\Phi^2 \\
&\quad + 2(\tau - t_0)^2\frac{\partial\alpha}{\partial R}\frac{\partial\alpha}{\partial\Theta}g_{\varphi\varphi}\mathrm{d}R\mathrm{d}\Theta + 2(\tau - t_0)\frac{\partial\alpha}{\partial R}g_{\varphi\varphi}\mathrm{d}R\mathrm{d}\Phi \\
&\quad + 2(\tau - t_0)\frac{\partial\alpha}{\partial\Theta}g_{\varphi\varphi}\mathrm{d}\Theta\mathrm{d}\Phi.
\end{aligned} \tag{6.6}$$

Note that for these new variables, we have

$$\frac{\partial}{\partial\tau} = U^a\partial_a, \quad \frac{\partial}{\partial R} = \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial\Theta} = \frac{\partial}{\partial\theta}, \quad \frac{\partial}{\partial\Phi} = \frac{\partial}{\partial\varphi}, \quad T^a\partial_a = \frac{\sqrt{2}}{|U|}\frac{\partial}{\partial\tau} = \frac{2}{N}\frac{\partial}{\partial\tau}.$$

The intrinsic and extrinsic geometry of the slices. All slices Σ_τ, $\tau \in \mathbb{R}$, have the same geometry (both intrinsic and extrinsic) since in Boyer–Lindquist coordinates the metric g is independent of t ($\partial/\partial t$ is a Killing vector field). We consider a generic slice $(\Sigma, h(\tau_0))$ and we choose $t_0 = \tau_0$ in order to simplify the expression of $h(\tau_0)$:

$$\begin{aligned} h(\tau_0) &= -g_{rr}\mathrm{d}R^2 - g_{\theta\theta}\mathrm{d}\Theta^2 - g_{\varphi\varphi}\mathrm{d}\Phi^2 \\ &= \frac{\varrho^2}{\Delta}\mathrm{d}R^2 + \varrho^2\mathrm{d}\Theta^2 + \left[\frac{(R^2+a^2)\varrho^2 + 2MRa^2\sin^2\Theta}{\varrho^2}\right]\sin^2\Theta\,\mathrm{d}\Phi^2, \\ \varrho^2 &= R^2 + a^2\cos^2\Theta,\ \Delta = R^2 - 2MR + a^2. \end{aligned}$$

The coefficient ϱ^2/Δ is singular at the horizon $H = \{r_+\}_R \times S^2_{\Theta,\Phi}$; we introduce a new radial coordinate to show that the metric $h(\tau_0)$ can be extended smoothly through H. Putting

$$F(R) := \frac{\Delta}{R^2} = 1 - \frac{2M}{R} + \frac{a^2}{R^2} = \frac{(R-r_+)(R-r_-)}{R^2},$$

we define $u(R)$ for $R \in [r_+, +\infty[$ by

$$u(R) := \int_{r_+}^{R} F^{-1/2}(s)\,\mathrm{d}s.$$

(Note that for extreme Kerr space-time, we would have $r_+ = r_- = M$ and consequently, the integral defining $u(R)$ would diverge. Hence, the h-distance to the horizon would be everywhere infinite in block I.) The function u of R is continuous strictly increasing from $[r_+, +\infty[$ onto $[0, +\infty[$, it is $\mathcal{C}^\infty$ on $]r_+, +\infty[$ but is not differentiable at r_+. As in the Schwarzschild case, we easily show the following result; the proof is identical to that of Lemma 5.1 and we do not repeat it here:

Lemma 6.1. *The inverse function $u \mapsto R(u)$ is smooth from $[0, +\infty[$ onto $[r_+, +\infty[$ and all its derivatives are uniformly bounded on $[0, +\infty[$.*

Lemma 6.1 will allow us to prove that each slice is a smooth manifold with boundary H and that the lapse function is smooth on $\overline{\Sigma}$. The following corollary expresses these properties as well as the fact that $h(\tau)$ depends regularly on τ:

Corollary 6.1. *The manifold*

$$(\overline{\Sigma} = [0, +\infty[_u \times S^2_{\Theta,\Phi}, h(\tau_0))$$

is a smooth manifold with boundary. The lapse function N, which is independent of τ, is regular and uniformly bounded on $\overline{\Sigma}$ together with all its derivatives. Moreover, the metric $h(\tau)$ is a smooth function of τ; to be more explicit, we have

$$h_{ab} \in \mathcal{C}^\infty(\mathbb{R}_\tau; \mathcal{C}_b^\infty(\overline{\Sigma}; T_{ab}\mathcal{M})), \qquad h^{ab} \in \mathcal{C}^\infty(\mathbb{R}_\tau; \mathcal{C}_b^\infty(\overline{\Sigma}; T^{ab}\mathcal{M})).$$

Remark 6.1. The extrinsic curvature

$$\frac{1}{\sqrt{2}}K_{ab} = -\frac{1}{\sqrt{2}N}\frac{\partial}{\partial\tau}(h_{ab})$$

is singular at the horizon since N vanishes there but not $\partial_\tau h_{ab}$. However,

$$NK_{ab} \in \mathcal{C}^\infty(\mathbb{R}_\tau; \mathcal{C}_b^\infty(\overline{\Sigma}; T_{ab}\mathcal{M})).$$

In the Dirac system considered as an evolution equation on block I, K_{ab} will only appear multiplied by N and will consequently play the part of a bounded potential.

Proof of Corollary 6.1. We write the metric $h(\tau_0)$ in the form

$$h(\tau_0)=\frac{\varrho^2}{R^2}\mathrm{d}u^2+\frac{\varrho^2}{(1+u)^2}(1+u)^2\mathrm{d}\Theta^2+\left[\frac{(R^2+a^2)\varrho^2+2MRa^2\sin^2\Theta}{\varrho^2(1+u)^2}\right](1+u)^2\sin^2\Theta\,\mathrm{d}\Phi^2.$$

The functions

$$\frac{\varrho^2}{R^2},\qquad \frac{\varrho^2}{(1+u)^2},\qquad \frac{(R^2+a^2)\varrho^2+2MRa^2\sin^2\Theta}{\varrho^2(1+u)^2}$$

are smooth on $\overline{\Sigma}$, positive, uniformly bounded together with all their derivatives and uniformly bounded away from zero. Hence, $h(\tau_0)$ is a smooth, symmetric, positive definite 2-form on $\overline{\Sigma}$, uniformly controlled below and above by the euclidian metric on $\overline{\Sigma}$ considered as $\mathbb{R}^3 \setminus B(0,1)$:

$$\mathrm{d}u^2+(1+u)^2\mathrm{d}\Theta^2+(1+u)^2\sin^2\Theta\,\mathrm{d}\Phi^2.$$

This shows in particular that $(\overline{\Sigma},h(\tau_0))$ is a smooth Riemannian manifold with boundary H. Given a regular coordinate system on $\overline{\Sigma}$, say the underlying euclidian coordinates on $\mathbb{R}^3 \setminus B(0,1)$, the 3×3 matrices h_{ij} and h^{ij}, representing the metric $h(\tau_0)$ and its inverse in this coordinate basis are smooth and bounded on $\overline{\Sigma}$ together with all their derivatives. This is expressed more intrinsically by

$$h_{ab}(\tau_0)\in\mathcal{C}_b^\infty(\overline{\Sigma};T_{ab}\mathcal{M}),\qquad h^{ab}(\tau_0)\in\mathcal{C}_b^\infty(\overline{\Sigma};T^{ab}\mathcal{M}).$$

The lapse function N is given by

$$N(R,\Theta)=\left(\frac{2R^2\varrho^2}{(R^2+a^2)\varrho^2+2MRa^2\sin^2\Theta}\right)^{1/2}F^{1/2}.$$

It is the result of the multiplication of $F^{1/2}$ by a smooth function on $\overline{\Sigma}$, uniformly bounded together with all its derivatives and uniformly bounded away from zero. Therefore, as a trivial consequence of Lemma 6.1 and the equality $\mathrm{d}R/\mathrm{d}u=F^{1/2}$, we have

$$N\in\mathcal{C}_b^\infty(\overline{\Sigma}).$$

We now study the regularity of $h(\tau)$ with respect to τ. Let us consider the expressions of $h(\tau)$ and $h(\tau_0)$ in the coordinate system R, Θ, Φ with $t_0=\tau_0$:

$$h(\tau)=-g_{rr}\mathrm{d}R^2-g_{\theta\theta}\mathrm{d}\Theta^2-g_{\varphi\varphi}\left(\mathrm{d}\Phi+(\tau-\tau_0)\frac{\partial\alpha}{\partial R}\mathrm{d}R+(\tau-\tau_0)\frac{\partial\alpha}{\partial\Theta}\mathrm{d}\Theta\right)^2,$$

$$h(\tau_0)=-g_{rr}\mathrm{d}R^2-g_{\theta\theta}\mathrm{d}\Theta^2-g_{\varphi\varphi}\mathrm{d}\Phi^2.$$

Putting

$$\widetilde{\Phi}=\Phi+(\tau-\tau_0)\alpha(R,\Theta)\ (\mathrm{mod}\ 2\pi),$$

we have

$$h(\tau)=-g_{rr}\mathrm{d}R^2-g_{\theta\theta}\mathrm{d}\Theta^2-g_{\varphi\varphi}\mathrm{d}\widetilde{\Phi}^2.$$

$h(\tau)$ is obtained from $h(\tau_0)$ by a rotation around the axis of the black hole whose angle (depending on τ, R and Θ) is

$$(\tau-\tau_0)\alpha(R,\Theta) = -(\tau-\tau_0)\frac{g_{t\varphi}(R,\Theta)}{g_{\varphi\varphi}(R,\Theta)}.$$

The function $\alpha(R,\Theta)$ is smooth on $\overline{\Sigma}$ and bounded together with all its derivatives. Denote by $G(\tau-\tau_0)$ the $\mathcal{C}^\infty$-diffeomorphism of $\overline{\Sigma}$

$$G(\tau-\tau_0) : (R,\Theta,\Phi) \mapsto (R,\Theta,\Phi+(\tau-\tau_0)\alpha(R,\Theta)).$$

Then $h_{ab}(\tau)$ (resp. $h^{ab}(\tau)$) is the pullback of $h_{ab}(\tau_0)$ (resp. $h^{ab}(\tau_0)$) by $G(\tau-\tau_0)$. This entails

$$h_{ab} \in \mathcal{C}^\infty(\mathbb{R}_\tau; \mathcal{C}_b^\infty(\overline{\Sigma}; T_{ab}\mathcal{M})), \qquad h^{ab} \in \mathcal{C}^\infty(\mathbb{R}_\tau; \mathcal{C}_b^\infty(\overline{\Sigma}; T^{ab}\mathcal{M}))$$

and concludes the proof of Corollary 6.1. ∎

On Σ, we introduce Sobolev spaces with zero traces at the horizon associated with the metric $h(\tau)$:

DEFINITION 6.1. For $k \in \mathbb{N}$, $H_0^k(\Sigma_\tau)$ is the completion of $\mathcal{C}_0^\infty(\Sigma)$ in the norm

$$\|f\|_{H^k(\Sigma_\tau)} = \Big(\sum_{p=0}^{k} \int_\Sigma \langle (\overline{\mathrm{D}}_{\Sigma_\tau})^p f, (\overline{\mathrm{D}}_{\Sigma_\tau})^p f\rangle_\tau \,\mathrm{dVol}_h\Big)^{1/2},$$

where $\overline{\mathrm{D}}_{\Sigma_\tau}$ is the Levi-Civita connection on $(\Sigma, h(\tau))$, dVol_h and $\langle\cdot\,,\cdot\rangle_\tau$ are the volume element on Σ and the positive definite inner product induced by the metric $h(\tau)$. Note that the volume element $\mathrm{dVol}_{h(\tau)}$ is independent of τ as can be seen by an explicit calculation in R, Θ, Φ coordinates:

$$\mathrm{dVol}_{h(\tau)} = -g_{rr}g_{\theta\theta}g_{\varphi\varphi}\,\mathrm{d}R\,\mathrm{d}\Theta\,\mathrm{d}\Phi = \mathrm{dVol}_{h(\tau_0)}, \qquad \forall \tau \in \mathbb{R}.$$

This time-independence, which is a consequence of the fact that $\partial/\partial t$ and $\partial/\partial\varphi$ are Killing vector fields, justifies the notation dVol_h.

REMARK 6.2. 1. For any $\tau \in \mathbb{R}$, the norms in $H_0^m(\Sigma_\tau)$ and $H_0^m(\Sigma_{\tau_0})$ are equivalent. This equivalence is locally uniform in time (and the constants in the norm estimates depend not only on τ but also on the bundle in which functions take their values). Hence, we shall simply denote by $H_0^m(\Sigma)$ the Sobolev space of order m on Σ with zero traces at the horizon, associated with the metric h. Our standard norm on this space will be that associated with $h(\tau_0)$; we denote it by $\|\cdot\|_{H^m(\Sigma)}$. When we wish to use explicitly the norm associated with the metric $h(\tau)$, we return to the notation $\|\cdot\|_{H^m(\Sigma_\tau)}$.

2. Note that the norm $\|\cdot\|_{H^m(\Sigma)}$ is equivalent to the flat Sobolev norm on Σ considered as $\mathbb{R}^3 \setminus \overline{B}(0,1)$ (see the beginning of the proof of Corollary 6.1).

3. For Dirac or Weyl spinor fields, the hermitian product $\langle\cdot\,,\cdot\rangle_\tau$ is that induced by the vector field T^a which is independent of τ. Therefore, for any such spinor field Ψ on Σ, we have

$$\|\Psi\|_{L^2(\Sigma_\tau)} = \|\Psi\|_{L^2(\Sigma)} \qquad \forall \tau \in \mathbb{R}$$

and the conservation of the $L^2(\Sigma_\tau)$-norm for solutions to the Dirac equation will in fact mean the conservation of the fixed standard norm on $L^2(\Sigma)$.

On each slice $(\Sigma, h(\tau))$ embedded in $(\mathcal{M}, g)$, we consider the Dirac–Witten operator $\mathcal{D}_W(\tau)$. The extrinsic geometry of the slices being nontrivial, $\mathcal{D}_W(\tau)$ does not coincide with the Dirac operator $\mathcal{D}_\Sigma(\tau)$ on $(\Sigma, h(\tau))$ and we have (see (3.25))

$$\mathcal{D}_W(\tau) = \mathcal{D}_\Sigma(\tau) + \frac{1}{2\sqrt{2}} K e_0.$$

$K = K_a{}^a$ is singular at the horizon, however, thanks to Remark 6.1, we have

$$NK \in \mathcal{C}^\infty(\mathbb{R}_\tau; \mathcal{C}_b^\infty(\overline{\Sigma})).$$

In the Dirac system written as an evolution equation, K will be multiplied by N and the quantity NK will merely be a bounded potential.

We now study the Dirac operator $\mathcal{D}_\Sigma(\tau)$ on $(\Sigma, h(\tau))$. $\mathcal{D}_\Sigma(\tau)$ is formally self-adjoint on $L^2(\Sigma)$ and satisfies the Bochner–Lichnerowicz–Weitzenböck formula

$$(6.7)\qquad (\mathcal{D}_\Sigma(\tau))^* \mathcal{D}_\Sigma(\tau) = (\mathcal{D}_\Sigma(\tau))^2 = \overline{\mathrm{D}}_{\Sigma_\tau}^* \overline{\mathrm{D}}_{\Sigma_\tau} + \frac{1}{4}\mathrm{Scal}_{h(\tau)} = -\Delta_{h(\tau)} + \frac{1}{4}\mathrm{Scal}_{h(\tau)}.$$

Because of the nonzero extrinsic curvature, the scalar curvature $\mathrm{Scal}_{h(\tau)}$ of $(\Sigma, h(\tau))$ is not necessarily zero, although the scalar curvature of $(\mathcal{M}, g)$ is zero. But we have

$$\mathrm{Scal}_{h(\tau)} \in \mathcal{C}^\infty(\mathbb{R}_\tau; \mathcal{C}_b^\infty(\overline{\Sigma})).$$

This will be enough to guarantee that, similarly to the Schwarzschild case, the Sobolev norms can be expressed using $\mathcal{D}_\Sigma$:

Proposition 6.1. *Consider on $H_0^k(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$ the following norm for $\tau \in \mathbb{R}$:*

$$|||\Psi|||_{k,\tau} = \Big(\sum_{p=0}^{k} \int_\Sigma \langle (\mathcal{D}_\Sigma(\tau))^p \Psi, (\mathcal{D}_\Sigma(\tau))^p \Psi \rangle \, \mathrm{dVol}_h \Big)^{1/2}$$

(*as mentioned in Remark* 6.2, *the hermitian product $\langle \cdot\,, \cdot \rangle_\tau$ on Dirac spinors is independent of τ and we denote it $\langle \cdot\,, \cdot \rangle$*). *The norms $\| \cdot \|_{H^k(\Sigma)}$ and $||| \cdot |||_{k,\tau}$ are equivalent on $H_0^k(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$, the equivalence being locally uniform in τ.*

Remark 6.3. The operator

$$\not{\mathcal{D}}(\tau) = e_0 . \mathcal{D}_\Sigma(\tau), \quad e_0 = \frac{1}{\sqrt{2}} T^a \partial_a,$$

is formally skew-adjoint on $L^2(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$, satisfies

$$(6.8)\qquad (\not{\mathcal{D}}(\tau))^* \not{\mathcal{D}}(\tau) = -(\not{\mathcal{D}}(\tau))^2 = (\mathcal{D}_\Sigma(\tau))^2 = -\Delta_{h(\tau)} + \frac{1}{4}\mathrm{Scal}_{h(\tau)}$$

and we have for any $\Psi \in H_0^k(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$

$$|||\Psi|||_{k,\tau} = \Big(\sum_{p=0}^{k} \int_\Sigma \langle (\not{\mathcal{D}}(\tau))^p \Psi, (\not{\mathcal{D}}(\tau))^p \Psi \rangle \, \mathrm{dVol}_h \Big)^{1/2}, \qquad \forall \tau \in \mathbb{R}.$$

Proof of Proposition 6.1. For each $\tau \in \mathbb{R}$, we prove that for all $k \in \mathbb{N}$, we have the following norm equivalence on $H_0^k(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$:

$$(6.9)\qquad \| \cdot \|_{H^k(\Sigma_\tau)} \simeq ||| \cdot |||_{k,\tau}.$$

Owing to the regularity in time of $h(\tau)$, this equivalence is locally uniform in time and finally Proposition 6.1 follows from Remark 6.2.

The proof of equivalence (6.9) follows exactly the proof of Proposition 5.1. We work on $(\Sigma, h(\tau))$ for $\tau \in \mathbb{R}$ fixed. We clearly have

$$\|\Psi\|_{L^2(\Sigma)} = |||\Psi|||_{0,\tau} \quad \text{for all } \Psi \in L^2(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$$

and (6.7) implies (6.9) for $k = 1$ on $H_0^1(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$ (this time, we do not have the equality of the norms because of the nonzero scalar curvature). We then have the exact equivalent of Lemma 5.2 for $\Delta_{h(\tau)}$ on Σ: for any $k \in \mathbb{N}$ there exist $0 < C_1 < C_2 < +\infty$ such that, for all $\Psi \in H_0^{k+2}(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$,

$$C_1\|\Psi\|_{H^{k+2}(\Sigma_\tau)} \leq \|\Psi\|_{H^k(\Sigma_\tau)} + \|\Delta_{h(\tau)}\Psi\|_{H^k(\Sigma_\tau)} \leq C_2\|\Psi\|_{H^{k+2}(\Sigma_\tau)}.$$

This immediatly yields the existence for all $k \in \mathbb{N}$ of $0 < \widetilde{C}_1 < \widetilde{C}_2 < +\infty$ such that, for all $\Psi \in H_0^{k+2}(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$,

$$\widetilde{C}_1\|\Psi\|_{H^{k+2}(\Sigma_\tau)} \leq \|\Psi\|_{H^k(\Sigma_\tau)} + \|(\mathcal{D}_\Sigma(\tau))^2\Psi\|_{H^k(\Sigma_\tau)} \leq \widetilde{C}_2\|\Psi\|_{H^{k+2}(\Sigma_\tau)}$$

since the scalar curvature only perturbs $\|\Delta_{h(\tau)}\Psi\|_{H^k(\Sigma_\tau)}$ by bounded terms of order lower than or equal to k. This last inequality allows us to prove (6.9) for all k by induction and thus to prove Proposition 6.1. ∎

6.1.2. *The global exterior Cauchy problem.* We give a generalization of Theorem 5 to the exterior of slow Kerr black holes. We first need to define the weighted Sobolev spaces on Σ with zero traces at the horizon:

DEFINITION 6.2. For $k \in \mathbb{N}$, $\varrho \in \mathbb{R}$, $\tau \in \mathbb{R}$, the *weighted Sobolev space with zero traces at the horizon*, $H_{0,\varrho}^k(\Sigma_\tau)$, is defined as the completion of $\mathcal{C}_0^\infty(\Sigma)$ in the norm

$$\|f\|_{H_\varrho^k(\Sigma_\tau)} = \Big(\sum_{p=0}^{k} \int_\Sigma (1+u^2)^{-\varrho-3/2+p} \langle (\overline{\mathrm{D}}_{\Sigma_\tau})^p f, (\overline{\mathrm{D}}_{\Sigma_\tau})^p f\rangle_\tau \, \mathrm{dVol}_h\Big)^{1/2}.$$

Here u is the function defined earlier

$$u(R) = \int_{r_+}^{R} F^{-1/2}(s)\,\mathrm{d}s.$$

u is not the $h(\tau)$ distance to the horizon but it is uniformly equivalent to it. Replacing $1 + u^2$ by r^2 would replace the norm by an equivalent one. The norms $\|\cdot\|_{H_\varrho^k(\Sigma_\tau)}$ and $\|\cdot\|_{H_\varrho^k(\Sigma_{\tau_0})}$ are equivalent for any given $\tau \in \mathbb{R}$ and this equivalence is locally uniform in τ. Therefore, we simply denote by $H_{0,\varrho}^k(\Sigma)$ the weighted Sobolev space of order k and weight ϱ on Σ with zero traces at the horizon associated with the metric h. We use the norm $\|\cdot\|_{H_\varrho^k(\Sigma_{\tau_0})}$ as the standard norm on this space and we denote it by $\|\cdot\|_{H_\varrho^k(\Sigma)}$. For $k = 0$, we write $H_{0,\varrho}^0(\Sigma) = L_\varrho^2(\Sigma)$. The norm $\|\cdot\|_{L_\varrho^2(\Sigma_\tau)}$ in $L_\varrho^2(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$ is independent of τ. Note that the norm $\|\cdot\|_{H_\varrho^k(\Sigma)}$ is equivalent to the flat weighted Sobolev norm on Σ considered as $\mathbb{R}^3 \setminus \overline{B}(0,1)$.

We have the following theorem concerning the well-posedness of the Cauchy problem on block I in Sobolev and weighted Sobolev spaces:

THEOREM 6. 1. *For any initial data $\Psi_0 \in L^2(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$, the Dirac equation outside the black hole has a unique solution $\Psi \in \mathcal{C}(\mathbb{R}_\tau; L^2(\Sigma; \mathbb{S}_{\mathrm{Dirac}}))$ such that $\Psi|_{\tau=\tau_0} = \Psi_0$. More-*

over, the evolution is unitary in $L^2(\Sigma)$*, i.e.*

$$\|\Psi(\tau)\|_{L^2(\Sigma)} = \|\Psi_0\|_{L^2(\Sigma)} \quad \textit{for all } \tau \in \mathbb{R}.$$

The propagator $\mathcal{U}(\tau,\sigma)$ *for the Dirac equation outside the black hole is strongly continuous on* $\mathbb{R}_\tau \times \mathbb{R}_\sigma$ *with values in* $\mathcal{L}(L^2(\Sigma;\mathbb{S}_{\mathrm{Dirac}}))$*.*

2. *If* $\Psi_0 \in H_0^k(\Sigma;\mathbb{S}_{\mathrm{Dirac}})$, $k \in \mathbb{N}$, *the associated solution* Ψ *satisfies*

$$\Psi \in \bigcap_{l=0}^{k} \mathcal{C}^l(\mathbb{R}_\tau; H_0^{k-l}(\Sigma;\mathbb{S}_{\mathrm{Dirac}})),$$

$\mathcal{U}(\tau,\sigma)$ *is strongly continuous on* $\mathbb{R}_\tau \times \mathbb{R}_\sigma$ *with values in* $\mathcal{L}(H_0^k(\Sigma;\mathbb{S}_{\mathrm{Dirac}}))$ *for all* $k \in \mathbb{N}$.

3. *For any initial data* $\Psi_0 \in L^2_\varrho(\Sigma;\mathbb{S}_{\mathrm{Dirac}})$, $\varrho \in \mathbb{R}$, *the Dirac equation outside the black hole has a unique solution* $\Psi \in \mathcal{C}(\mathbb{R}_\tau; L^2_\varrho(\Sigma;\mathbb{S}_{\mathrm{Dirac}}))$ *such that* $\Psi|_{\tau=\tau_0} = \Psi_0$. $\mathcal{U}(\tau,\sigma)$ *is strongly continuous on* $\mathbb{R}_\tau \times \mathbb{R}_\sigma$ *with values in* $\mathcal{L}(L^2_\varrho(\Sigma;\mathbb{S}_{\mathrm{Dirac}}))$ *for all* $\varrho \in \mathbb{R}$.

4. *If* $\Psi_0 \in H^k_{0,\varrho}(\Sigma;\mathbb{S}_{\mathrm{Dirac}})$, $k \in \mathbb{N}$, $\varrho \in \mathbb{R}$, *the associated solution* Ψ *satisfies*

$$\Psi \in \bigcap_{l=0}^{k} \mathcal{C}^l(\mathbb{R}_\tau; H_{0,\varrho}^{k-l}(\Sigma;\mathbb{S}_{\mathrm{Dirac}})),$$

$\mathcal{U}(\tau,\sigma)$ *is, for all* $k \in \mathbb{N}$, $\varrho \in \mathbb{R}$, *strongly continuous on* $\mathbb{R}_\tau \times \mathbb{R}_\sigma$ *with values in* $\mathcal{L}(H^k_{0,\varrho}(\Sigma;\mathbb{S}_{\mathrm{Dirac}}))$.

Proof. The Dirac equation outside the black hole has the form

$$\nabla_{e_0}\Psi = -\not{\mathcal{D}}(\tau)\Psi - \frac{1}{2\sqrt{2}}K(\tau)\Psi - ime_0.\Psi \tag{6.10}$$

with $e_0 = (1/\sqrt{2})T^a\partial_a$. We choose a spin-frame $\{o^A, \iota^A\}$ adapted to the foliation such as defined in Appendix A; equation (6.10) becomes

$$\frac{\partial\Psi}{\partial\tau} = -\frac{N}{\sqrt{2}}\left(\not{\mathcal{D}}(\tau) + \frac{1}{2\sqrt{2}}K(\tau) + im\gamma^0 + B(\tau)\right)\Psi \tag{6.11}$$

where

$$\gamma^0 = i\begin{pmatrix} 0 & \mathrm{Id}_2 \\ -\mathrm{Id}_2 & 0 \end{pmatrix}$$

and B is the 4×4 matrix containing the connection terms coming from the time derivative. The following result is a consequence of Appendix A and the regularity of $NK(\tau)$:

Lemma 6.2. *The potential in equation* (6.11) *satisfies* (K *is of course to be understood here as* $K\,\mathrm{Id}_4$)

$$\frac{N}{\sqrt{2}}\left(\frac{1}{2\sqrt{2}}K(\tau) + im\gamma^0 + B(\tau)\right) \in \mathcal{C}^\infty(\mathbb{R}_\tau; \mathcal{C}_b^\infty(\overline{\Sigma}; \mathcal{M}_4(\mathbb{C})).$$

We now proceed to proving Theorem 6. The proof is very similar to that of Theorem 5 and therefore we simply highlight the parts which differ from it.

1. *The well-posedness of the Cauchy problem in* $L^2(\Sigma)$: For $\varepsilon > 0$, we consider on $\mathbb{R}_\tau \times \widetilde{\Sigma}$, $\widetilde{\Sigma} := [0, +\infty[_R \times S^2_{\Theta,\Phi}$, a smooth Lorentzian metric ${}^\varepsilon g$ which coincides with g for $R > r_+ + \varepsilon$. We choose the same background metrics as in the Schwarzschild case

$$\widetilde{h} = \mathrm{d}R^2 + R^2\mathrm{d}\Omega^2, \quad \mathrm{d}\Omega^2 = \mathrm{d}\Theta^2 + \sin^2\Theta\,\mathrm{d}\Phi^2,$$

$$\widetilde{g} = \left(1 - \varrho(R)\frac{2M}{R}\right)\mathrm{d}\tau^2 - \left(1 + \varrho(R)\frac{2M}{R}\right)\mathrm{d}R^2 - R^2\mathrm{d}\Omega^2,$$

where ϱ is a smooth cut-off function on $[0,+\infty[$ such that $\varrho \equiv 0$ on $[0,3M]$ and $\varrho \equiv 1$ on $[4M,+\infty[$.

For each $\varepsilon > 0$, we show that ${}^{\varepsilon}g$ is of class (∞,δ) on $\mathbb{R}_\tau \times \widetilde{\Sigma}$ for any $\delta > -2$. In $\{R > r_+ + \varepsilon\}$, we have

$$\begin{aligned}{}^{\varepsilon}g = g = &\left(g_{tt} - \frac{(g_{t\varphi})^2}{g_{\varphi\varphi}}\right)\mathrm{d}\tau^2 + \left(g_{rr} + (\tau-\tau_0)^2\left(\frac{\partial\alpha}{\partial R}\right)^2 g_{\varphi\varphi}\right)\mathrm{d}R^2\\ &+ \left(g_{\theta\theta} + (\tau-\tau_0)^2\left(\frac{\partial\alpha}{\partial\Theta}\right)^2 g_{\varphi\varphi}\right)\mathrm{d}\Theta^2 + g_{\varphi\varphi}\mathrm{d}\Phi^2\\ &+ 2(\tau-\tau_0)^2\frac{\partial\alpha}{\partial R}\frac{\partial\alpha}{\partial\Theta}g_{\varphi\varphi}\mathrm{d}R\mathrm{d}\Theta + 2(\tau-\tau_0)\frac{\partial\alpha}{\partial R}g_{\varphi\varphi}\mathrm{d}R\mathrm{d}\Phi\\ &+ 2(\tau-\tau_0)\frac{\partial\alpha}{\partial\Theta}g_{\varphi\varphi}\mathrm{d}\Theta\mathrm{d}\Phi.\end{aligned}$$

We recall that

$$\alpha = -\frac{g_{t\varphi}}{g_{\varphi\varphi}} = \frac{2aMR}{(R^2+a^2)\varrho^2 + 2MRa^2\sin^2\Theta}$$

and

$$-g_{\varphi\varphi} = \left((R^2+a^2) + \frac{2MRa^2\sin^2\Theta}{\varrho^2}\right)\sin^2\Theta,$$

whence, as $R \to +\infty$,

$$\widetilde{\mathrm{D}}^l(\alpha) = O(R^{-3-l}), \quad \widetilde{\mathrm{D}}^l\left(\frac{\partial\alpha}{\partial\Theta}\right) = O(R^{-5-l}),$$
$$\widetilde{\mathrm{D}}^l\left(\frac{\partial\alpha}{\partial R}\right) = O(R^{-4-l}), \qquad \widetilde{\mathrm{D}}^l(g_{\varphi\varphi}) = O(R^{2-l}), \qquad l \in \mathbb{N},$$

where $\widetilde{\mathrm{D}}$ is the Levi-Civita connection on $(\widetilde{\Sigma},\widetilde{h})$. From these properties, we infer that for any $l \in \mathbb{N}$, we have as $R \to +\infty$:

$$\begin{aligned}\widetilde{\mathrm{D}}^l\left(\frac{g_{R\Theta}}{R}\right) &= \widetilde{\mathrm{D}}^l\left(2(\tau-\tau_0)^2\frac{g_{\varphi\varphi}}{R}\frac{\partial\alpha}{\partial R}\frac{\partial\alpha}{\partial\Theta}\right) = O(R^{-8-l}),\\ \widetilde{\mathrm{D}}^l\left(\frac{g_{R\Phi}}{R\sin\Theta}\right) &= \widetilde{\mathrm{D}}^l\left(2(\tau-\tau_0)\frac{g_{\varphi\varphi}}{R\sin\Theta}\frac{\partial\alpha}{\partial R}\right) = O(R^{-3-l}),\\ \widetilde{\mathrm{D}}^l\left(\frac{g_{\Theta\Phi}}{R^2\sin\Theta}\right) &= \widetilde{\mathrm{D}}^l\left(2(\tau-\tau_0)\frac{g_{\varphi\varphi}}{R^2\sin\Theta}\frac{\partial\alpha}{\partial\Theta}\right) = O(R^{-5-l}).\end{aligned}$$

We must remember, in order to understand the formulae above, that the asymptotically constant 1-forms are not $\mathrm{d}R$, $\mathrm{d}\Theta$, $\mathrm{d}\Phi$ but $\mathrm{d}R$, $R\mathrm{d}\Theta$ and $R\sin\Theta\,\mathrm{d}\Phi$. There remains to estimate the fall-off of the diagonal terms of $g - \widetilde{g}$. We start with the lapse function:

$$g_{\tau\tau} = g_{tt} - \frac{(g_{t\varphi})^2}{g_{\varphi\varphi}} = \frac{\Delta}{R^2}\frac{\varrho^2R^2}{(R^2+a^2)\varrho^2 + 2MRa^2\sin^2\Theta}.$$

The quantity Δ/R^2 behaves like $1 - 2M/R$ at infinity since

$$\frac{\Delta}{R^2} = 1 - \frac{2M}{R} + \frac{a^2}{R^2},$$

also
$$\frac{\varrho^2 R^2}{(R^2+a^2)\varrho^2+2MRa^2\sin^2\Theta} = 1 - \frac{a^2\varrho^2+2MRa^2\sin^2\Theta}{(R^2+a^2)\varrho^2+2MRa^2\sin^2\Theta}.$$
Therefore,
$$g_{\tau\tau} - \left(1-\frac{2M}{R}\right) = O(R^{-2}), \qquad R\to+\infty,$$
and for all $l\in\mathbb{N}$,
$$\widetilde{\mathrm{D}}^l\left(g_{\tau\tau} - \left(1-\frac{2M}{R}\right)\right) = O(R^{-2-l}), \qquad R\to+\infty.$$
The radial term g_{RR} is
$$g_{RR} = g_{rr} + (\tau-\tau_0)^2 g_{\varphi\varphi}\left(\frac{\partial\alpha}{\partial R}\right)^2 = -\frac{\varrho^2}{\Delta} + (\tau-\tau_0)^2 g_{\varphi\varphi}\left(\frac{\partial\alpha}{\partial R}\right)^2.$$
The time dependent term satisfies, for all $l\in\mathbb{N}$,
$$\widetilde{\mathrm{D}}^l\left((\tau-\tau_0)^2 g_{\varphi\varphi}\left(\frac{\partial\alpha}{\partial R}\right)^2\right) = O(R^{-6-l}), \qquad R\to+\infty.$$
As for the time independent term, we have
$$\begin{aligned}\frac{\varrho^2}{\Delta} - \left(1+\frac{2M}{R}\right) &= \frac{\varrho^2}{R^2}\frac{1}{1-\frac{2M}{R}+\frac{a^2}{R^2}} - \left(1+\frac{2M}{R}\right)\\ &= \left(1+\frac{a^2\cos^2\Theta}{R^2}\right)\left(1-\frac{2M}{R}+\frac{a^2}{R^2}\right)^{-1} - \left(1+\frac{2M}{R}\right)\end{aligned}$$
whence
$$\widetilde{\mathrm{D}}^l(g_{RR}-\widetilde{g}_{RR}) = O(R^{-2-l}), \qquad R\to+\infty,\ l\in\mathbb{N}.$$
The time dependent term in $g_{\Theta\Theta}$ is also short range:
$$\widetilde{\mathrm{D}}^l\left((\tau-\tau_0)^2\frac{g_{\varphi\varphi}}{R^2}\left(\frac{\partial\alpha}{\partial\Theta}\right)^2\right) = O(R^{-10-l}), \qquad R\to+\infty,$$
and we simply need to study $g_{\theta\theta}-\widetilde{g}_{\Theta\Theta}$:
$$g_{\theta\theta}-\widetilde{g}_{\Theta\Theta} = -\varrho^2 + R^2 = -a^2\cos^2\Theta$$
and therefore
$$\widetilde{\mathrm{D}}^l\left(\frac{1}{R^2}\left(g_{\Theta\Theta}-\widetilde{g}_{\Theta\Theta}\right)\right) = O(R^{-2-l}), \qquad R\to+\infty,\ l\in\mathbb{N}.$$
The only remaining term is
$$\begin{aligned}g_{\Phi\Phi}-\widetilde{g}_{\Phi\Phi} &= -R^2\sin^2\Theta\left(\frac{(R^2+a^2)\varrho^2+2MRa^2\sin^2\Theta}{R^2\varrho^2} - 1\right)\\ &= -R^2\sin^2\Theta\left(\frac{a^2\varrho^2+2MRa^2\sin^2\Theta}{R^2\varrho^2}\right)\end{aligned}$$
and this expression entails
$$\widetilde{\mathrm{D}}^l\left(\frac{1}{R^2\sin^2\Theta}(g_{\Phi\Phi}-\widetilde{g}_{\Phi\Phi})\right) = O(R^{-2-l}), \qquad R\to+\infty,\ l\in\mathbb{N}.$$

We conclude that for any $l \in \mathbb{N}$, $\varepsilon > 0$,

$$\widetilde{\mathrm{D}}^l({}^{\varepsilon}g - \widetilde{g}) = O(R^{-2-l}), \qquad R \to +\infty.$$

The metric ${}^{\varepsilon}g$ being smooth on $\mathbb{R} \times \widetilde{\Sigma}$, this proves that ${}^{\varepsilon}g$ is of class (∞, δ) for any $\delta > -2$.

Then we follow the proof of point 1 of Theorem 5 to solve the Cauchy problem in $L^2(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$ for equation (6.11) with initial data on the typical slice Σ_{τ_0}, i.e. on any slice we choose to consider as typical. We also obtain the conservation of the physical $L^2(\Sigma)$ norm of the solutions, $\|\Psi(\tau)\|_{L^2(\Sigma_\tau)}$; but since the norm $\|\cdot\|_{L^2(\Sigma_\tau)}$ is the same for all $\tau \in \mathbb{R}$, this shows that the standard $L^2(\Sigma)$ norm of the solutions is conserved all the time. The time dependence of the coefficients of the equation prevents the propagator, $\mathcal{U}(\tau, \sigma) : \Psi(\sigma) \mapsto \Psi(\tau)$, from being a group; the conservation of the L^2 norm together with the strong continuity of the propagators for the metrics ${}^{\varepsilon}g$ imply that $\mathcal{U}$ is strongly continuous on $\mathbb{R}_\tau \times \mathbb{R}_\sigma$ with values in $\mathcal{L}(L^2(\Sigma; \mathbb{S}_{\mathrm{Dirac}}))$. This proves the first part of Theorem 6.

2. *Well-posedness of the Cauchy problem in Sobolev spaces*: For a smooth solution Ψ associated with some initial data $\Psi_0 \in \mathcal{C}_0^\infty(\Sigma; \mathbb{S}_{\mathrm{Dirac}})$, we consider the evolution equation for $(\not{\mathcal{D}}(\tau))^k\Psi$, $k \in \mathbb{N}$, in order to prove by induction estimates on the Sobolev norms of Ψ. Applying $(\not{\mathcal{D}}(\tau))^k$ to equation (6.11), we obtain

$$\begin{aligned} \frac{\partial}{\partial\tau}((\not{\mathcal{D}}(\tau))^k\Psi(\tau)) &= -\frac{N}{\sqrt{2}}\left(\not{\mathcal{D}}(\tau) + \frac{1}{2\sqrt{2}}K(\tau) + im\gamma^0 + B(\tau)\right)(\not{\mathcal{D}}(\tau))^k\Psi(\tau) \\ &\quad + \left\{\frac{\partial}{\partial\tau}((\not{\mathcal{D}}(\tau))^k)\right\}\Psi(\tau) - \frac{1}{\sqrt{2}}[(\not{\mathcal{D}}(\tau))^k, N]\not{\mathcal{D}}(\tau)\Psi(\tau) \\ &\quad - \frac{1}{\sqrt{2}}\left[(\not{\mathcal{D}}(\tau))^k, \frac{1}{2\sqrt{2}}NK(\tau)\right]\Psi(\tau) - \frac{im}{\sqrt{2}}[(\not{\mathcal{D}}(\tau))^k, N\gamma^0]\Psi(\tau) \\ &\quad - \frac{1}{\sqrt{2}}[(\not{\mathcal{D}}(\tau))^k, NB(\tau)]\Psi(\tau). \end{aligned} \tag{6.12}$$

We write equation (6.12) as

$$\frac{\partial}{\partial\tau}((\not{\mathcal{D}}(\tau))^k\Psi(\tau)) = -\frac{N}{\sqrt{2}}\left(\not{\mathcal{D}}(\tau) + \frac{K(\tau)}{2\sqrt{2}} + im\gamma^0 + B(\tau)\right)(\not{\mathcal{D}}(\tau))^k\Psi(\tau) + G(\tau).$$

Using Lemma 6.2 and the norm equivalence of Proposition 6.1, we have

$$\|G(\tau)\|_{L^2(\Sigma)} \le C(\tau)\|\Psi(\tau)\|_{H^k(\Sigma)} \le C'(\tau)|||\Psi(\tau)|||_{k,\tau}$$

where C and C' are continuous positive functions on $\mathbb{R}$, independent of Ψ. The integral formula

$$(\not{\mathcal{D}}(\tau))^k\Psi(\tau) = \mathcal{U}(\tau, \tau_0)((\not{\mathcal{D}}(\tau_0))^k\Psi_0) + \int_{\tau_0}^{\tau} \mathcal{U}(\tau, \sigma)G(\sigma)\,\mathrm{d}\sigma \tag{6.13}$$

then allows us to obtain by induction estimates of the kind

$$|||\Psi(\tau)|||_{k,\tau} \le \alpha_k(\tau)|||\Psi_0|||_{k,\tau_0} \tag{6.14}$$

where α_k is a continuous positive function on $\mathbb{R}$, independent of Ψ_0. These estimates prove the well-posedness of the Cauchy problem in $H_0^k(\Sigma)$ and together with the integral formulae (6.13), they establish that $\mathcal{U}(\tau, \sigma)$ is strongly continuous on $\mathbb{R}^2_{\tau,\sigma}$ with values in

$\mathcal{L}(H_0^k(\Sigma;\mathbb{S}_{\mathrm{Dirac}}))$ for all $k\in\mathbb{N}$. The additional regularity of the solutions is then read off directly from the equation.

3, 4. *The Cauchy problem in weighted L^2 and Sobolev spaces*: For this last part, we follow the lines of the proof of the fourth part of Theorem 5. The only slight differences are first that we need to take account of the initial time τ_0 which is not necessarily zero and second that we must use the estimates (6.14) instead of (5.17). We obtain estimates of the form

$$\|\Psi(\tau)\|_{L^2_\varrho(\Sigma)} \le \alpha_\varrho(\tau)\|\Psi_0\|_{L^2_\varrho(\Sigma)},\ \ \|\Psi(\tau)\|_{H^k_\varrho(\Sigma)} \le \beta_{k,\varrho}(\tau)\|\Psi_0\|_{H^k_\varrho(\Sigma)}$$

where α_ϱ ($\varrho\in\mathbb{R}$) and $\beta_{k,\varrho}$ ($k\in\mathbb{N}^*$, $\varrho\in\mathbb{R}$) are continuous positive functions on $\mathbb{R}$, independent of Ψ_0. As previously, the strong continuity of $\mathcal{U}(\tau,\sigma)$ on $\mathbb{R}^2_{\tau,\sigma}$ with values in $\mathcal{L}(H^k_{0,\varrho}(\Sigma;\mathbb{S}_{\mathrm{Dirac}}))$ is a consequence of the strong continuity of the propagators in the metrics ${}^{\varepsilon}g$ and of the estimates above. This concludes the proof of Theorem 6. ∎

6.2. Maximal extension of Kerr's space-time. The global geometry of Kerr's space-time (and in particular slow Kerr) is far more complex than that of Schwarzschild's space-time. An entire chapter of B. O'Neill's book [50] is devoted to the construction of the maximal extension. Our purpose in this section is to describe this construction schematically and to point out the so-called Kruskal domains in maximal slow Kerr space-time for which, with a natural choice of foliation, the theorems of Chapter 4 can be applied.

6.2.1. *Kerr-star and star-Kerr coordinates.* Just as we did in the Schwarzschild case, we choose a coordinate system which will allow us to represent globally the whole of Kerr's space-time. This choice is guided by the following physical considerations: if a particle is to pass from block I to block II across the outer horizon and then from block II to block III across the inner horizon, its most direct course is to follow an incoming principal null geodesic. The whole idea of the Kerr-star coordinate system is to turn incoming principal null geodesics into coordinate lines. Such geodesics are defined on all three blocks in Boyer–Lindquist coordinates by

$$\dot t = \frac{r^2+a^2}{\Delta},\quad \dot r=-1,\quad \dot\theta=0,\quad \dot\varphi=\frac{a}{\Delta}.$$

Keeping the coordinates r and θ, we introduce two new coordinates t^* and φ^* of the form

$$t^* = t+T(r),\qquad \varphi^*=\varphi+A(r)$$

where the functions T and A are required to satisfy

$$\frac{\mathrm{d}T}{\mathrm{d}r}=\frac{r^2+a^2}{\Delta},\qquad \frac{\mathrm{d}A}{\mathrm{d}r}=\frac{a}{\Delta}.$$

(t^*,r,θ,φ^*) defines a coordinate system in each Boyer–Lindquist block ([1]), called Kerr-star coordinates, in which the incoming principal null geodesics are described by

$$\dot r=-1,\quad \dot\theta=0,\quad \dot{t^*}=\dot t+\frac{\mathrm{d}T}{\mathrm{d}r}\dot r=0,\quad \dot{\varphi^*}=\dot\varphi+\frac{\mathrm{d}A}{\mathrm{d}r}\dot r=0,$$

([1]) With the exception of the axis ($\theta=0$ and $\theta=\pi$); this coordinate singularity can be dealt with simply (see [50], Lemma 2.2.2); we shall systematically ignore it.

i.e. they are the r coordinate curves parametrized by $s = -r$ (or $-r+C$). The expression of the Kerr metric in Kerr-star coordinates is given by

$$g = g_{tt}\mathrm{d}t^{*2} + 2g_{t\varphi}\mathrm{d}t^*\mathrm{d}\varphi^* + g_{\varphi\varphi}\mathrm{d}\varphi^{*2} - \varrho^2\mathrm{d}\theta^2 - 2\mathrm{d}t^*\mathrm{d}r + 2a\sin^2\theta\,\mathrm{d}\varphi^*\mathrm{d}r, \tag{6.15}$$

where g_{tt}, $g_{t\varphi}$, $g_{\varphi\varphi}$ and $g_{\theta\theta} = -\varrho^2$ are as defined in (6.1), i.e.

$$g_{tt} = \left(1 - \frac{2Mr}{\varrho^2}\right), \qquad g_{t\varphi} = \frac{a\sin^2\theta(r^2+a^2-\Delta)}{\varrho^2},$$

$$g_{\varphi\varphi} = -\left(\frac{(r^2+a^2)\varrho^2 + 2Mra^2\sin^2\theta}{\varrho^2}\right)\sin^2\theta, \qquad \varrho^2 = r^2 + a^2\cos^2\theta.$$

We see from (6.15) that the metric g is smooth on all three blocks, with the exception of the ring singularity $\{\varrho^2 = 0\} = \{r = 0 \text{ and } \theta = \pi/2\}$ in block III, and across both horizons (the component g_{rr} in Boyer–Lindquist coordinates was the only component of g to be singular at the horizons and it does not appear in (6.15)).

Kerr-star space-time is defined as the manifold

$$\mathcal{M}^* = \mathbb{R}_{t^*} \times \mathbb{R}_r \times S^2_{\theta,\varphi^*} \setminus \{(t^*, r, \theta, \varphi^*);\ r = 0 \text{ and } \theta = \pi/2\}$$

equipped with the smooth metric (6.15) and with the time orientation such that the null coordinate vector field $-\partial/\partial r$, defined and smooth on the whole of $\mathcal{M}^*$ and whose integral lines are the incoming principal null geodesics, is future oriented. This time orientation is consistent with the fact that, in Boyer–Lindquist coordinates, the Killing vector field $\partial/\partial t$ is future oriented outside the ergosphere in block I and also with the description of block II given at the beginning of the chapter, with $-\partial/\partial r$ (in Boyer–Lindquist coordinates) future pointing. This space-time contains all three blocks, glued smoothly at the horizons by the requirement that incoming principal null geodesics should cross horizons smoothly and that their orientation defines the time orientation. Block II is thus glued to block I in such a way that it lies in the future of block I and similarly, block III lies in the future of block II. The horizons $\{r = r_+\}$ and $\{r = r_-\}$ are smooth null hypersurfaces of $(\mathcal{M}^*, g)$. The fact that they are null is easily shown considering the metric induced by g on hypersurfaces of constant r

$$g_r = g_{tt}\mathrm{d}t^{*2} + 2g_{t\varphi}\mathrm{d}t^*\mathrm{d}\varphi^* + g_{\varphi\varphi}\mathrm{d}\varphi^{*2} - \varrho^2\mathrm{d}\theta^2.$$

This induced metric has determinant

$$\det(g_r) = -\varrho^2(g_{tt}g_{\varphi\varphi} - (g_{t\varphi})^2) = \varrho^2\Delta\sin^2\theta$$

and thus degenerates for $\Delta = 0$, i.e. at the horizons. See Figure 6.1 for a Penrose diagram of Kerr-star space-time.

This construction is similar to what we did in Schwarzschild's space-time, when we first used Kruskal–Szekeres coordinates to show that the metric could be extended smoothly across the horizon. In the Schwarzschild case, the maximal extension of the space-time followed naturally by extending the domain of definition of the Kruskal–Szekeres coordinate system. This we cannot do here since the domain of definition of Kerr-star coordinates is already maximal. We shall need to use other coordinate systems which will allow us to glue Boyer–Lindquist blocks in different manners.

Kerr-star coordinates were defined by modifying Boyer–Lindquist coordinates so that incoming principal null geodesics could become coordinate lines. Using outgoing principal null geodesics instead of the incoming ones, we obtain the star-Kerr coordinate system. These geodesics are defined on all three blocks in Boyer–Lindquist coordinates by

$$\dot{t} = \frac{r^2+a^2}{\Delta}, \quad \dot{r} = 1, \quad \dot{\theta} = 0, \quad \dot{\varphi} = \frac{a}{\Delta}.$$

Keeping r and θ, we introduce the new coordinates

$$^*t = t - T(r), \quad {}^*\varphi = \varphi - A(r)$$

where the functions T and A are the same as used to define t^* and φ^*. In the star-Kerr coordinate system $(^*t, r, \theta, {}^*\varphi)$, the outgoing principal null geodesics are the r coordinate lines parametrized by $s = r$ and the Kerr metric takes the form

$$\begin{aligned} g = {} & g_{tt}\mathrm{d}(^*t)^2 + 2g_{t\varphi}\mathrm{d}(^*t)\mathrm{d}(^*\varphi) + g_{\varphi\varphi}\mathrm{d}(^*\varphi)^2 - \varrho^2\mathrm{d}\theta^2 \\ & + 2\mathrm{d}(^*t)\mathrm{d}r - 2a\sin^2\theta\,\mathrm{d}(^*\varphi)\mathrm{d}r. \end{aligned} \tag{6.16}$$

This gives rise to star-Kerr space-time which is the manifold

$$^*\mathcal{M} = \mathbb{R}_{^*t} \times \mathbb{R}_r \times S^2_{\theta, {}^*\varphi} \setminus \{(^*t, r, \theta, {}^*\varphi);\, r = 0 \text{ and } \theta = \pi/2\}$$

equipped with the smooth metric (6.16) and time orientation such that, in star-Kerr coordinates, the null coordinate vector field $\partial/\partial r$, which is defined and smooth all over $^*\mathcal{M}$ and whose integral lines are the outgoing principal null geodesics, is future pointing. This space-time contains all three blocks, glued together at the horizons which appear as regular null hypersurfaces. The gluing is done by requiring that the outgoing principal null geodesics should cross the horizons smoothly. The time orientation reflects this choice; it is consistent with the fact that in Boyer–Lindquist coordinates $\partial/\partial t$ is future pointing outside the ergosphere in block I, but incompatible with $-\partial/\partial r$ future oriented in block II: in star-Kerr space-time, the inertial frames in block II are dragged outwards from the inner horizon to the outer horizon. There is a canonical isometry between the star-Kerr and Kerr-star space-times. This isometry preserves the time orientation of blocks I and III but reverses that of block II. Star-Kerr space-time can be seen as a block I, to the past of which is glued a block II with its time orientation reversed, to the past of which is glued a block III: it describes a "slow Kerr white hole". See Figure 6.1 for the Penrose diagram of star-Kerr space-time (II′ refers to a block II with reversed time orientation).

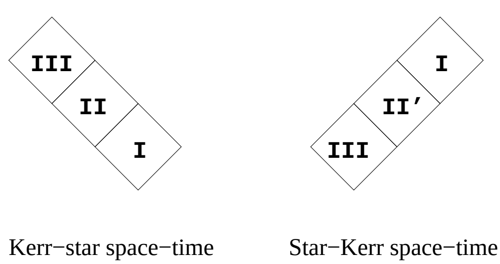

Fig. 6.1. Penrose diagrams of Kerr-star and star-Kerr space-times

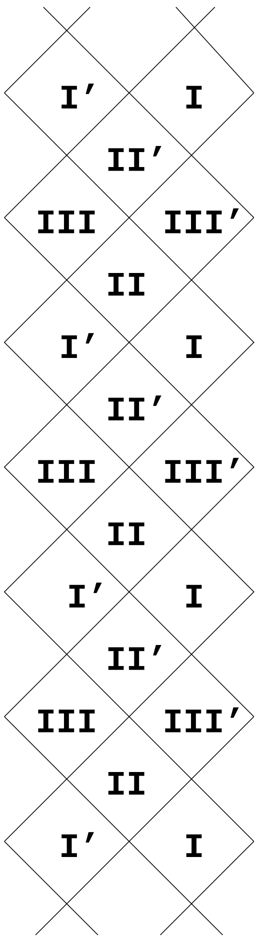

Fig. 6.2. Maximal slow Kerr space-time

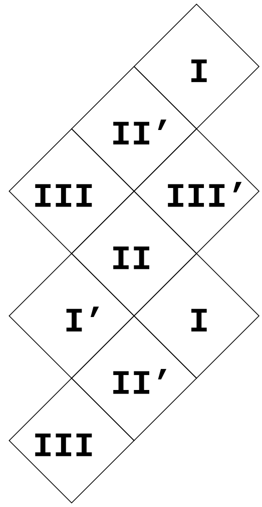

Fig. 6.3. First step in the construction of maximal slow Kerr space-time

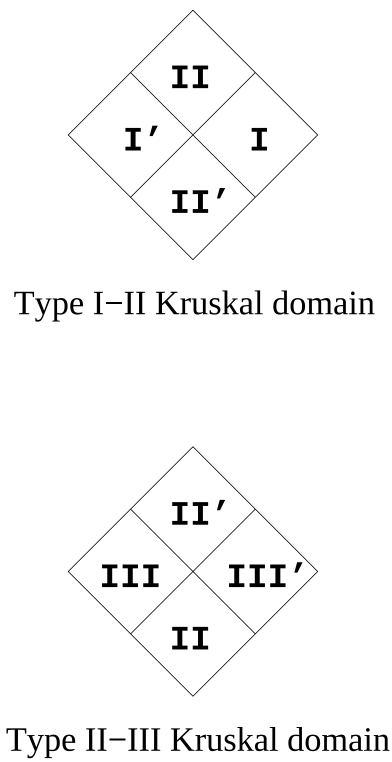

Fig. 6.4. The two different types of Kruskal domains

6.2.2. *Maximal slow Kerr space-time.* The maximal analytic extension of slow Kerr space-time is constructed using both Kerr-star and star-Kerr space-times. We start with Kerr-star space-time: all the incoming principal null geodesics are complete but the outgoing ones are not. The idea is to glue other blocks so as to make the outgoing principal null geodesics complete. The solution for blocks I and III is simple: we consider them as belonging to star-Kerr space-times, i.e. we glue to the future of block III a block II′ followed by a new block I and to the past of block I a block II′ preceded by a new block III. For block II, the situation is trickier; we also wish to understand block II as part of a star-Kerr space-time, but this is incompatible with the time orientation of block II. The solution is to reverse the time orientation of the whole star-Kerr space-time. We are thus led to gluing to the future of block II a block III′ (block III with its time orientation reversed) and to its past a block I′ (block I with reversed time orientation). The resulting space-time is shown in Figure 6.3. We keep on extending this new space-time whenever a family of principal null geodesics is incomplete. The extension is done step by step and is based on the same simple principle: if a family of principle null geodesics is incomplete, it means that the Kerr-star (in the incoming case) or star-Kerr (in the outgoing case) space-time which it generates lacks one or two blocks; this is cured by gluing the lacking blocks, bearing in mind the consistency of the time orientation of the whole space-time. In this manner, we construct maximal slow Kerr space-time (see Figure 6.2) as a reunion of four types of space-times: Kerr-star space-times, Kerr-star with their time orientation reversed, star-Kerr and star-Kerr with their time orientation reversed. Important objects in this maximal extension are the so-called Kruskal domains. They are "diamond shaped" reunions of four contiguous blocks. At their "centre" lies a 2-sphere, referred to as the crossing sphere, where the horizons intersect. Building this crossing sphere rigorously and extending the metric over it are important difficulties in the construction of maximal slow Kerr space-time. This is done by means of Kruskal–Boyer–Lindquist coordinates (see [50] for a fully detailed account). There are two types of Kruskal domains, as shown in Figure 6.4. Type II-III contains two copies of block III; it is not causal, therefore not globally hyperbolic, and contains two timelike singularities (the ring singularity of each block III). Because of the lack of causality, the notion of Cauchy problem is not even meaningful on type II-III domains. Type I-II domains are much more gentle. They are globally hyperbolic and contain no singularity. They can be treated in exactly the same manner as maximal Schwarzschild space-time.

For a type I-II Kruskal domain, we consider a foliation $\{S_\tau\}_{\tau\in\mathbb{R}}$ (see Figure 6.5) by Cauchy hypersurfaces such that, outside the domain of dependence of a neighbourhood of the crossing sphere, for each $\tau\in\mathbb{R}$ the hypersurface S_τ coincides in block I with the level hypersurface $\Sigma_\tau=\{t=\tau\}$ of the time coordinate t of Boyer–Lindquist coordinates and in block I′ with $\Sigma_{-\tau}$ (suffice it to say that the Boyer–Lindquist coordinates in blocks I, II, I′ and II′ are defined unambiguously from the Kruskal–Boyer–Lindquist coordinates defined on the whole domain). For such a foliation, the asymptotic behaviour of the metric g at infinity on each slice S_τ is the same as the behaviour for fixed t and $r\to+\infty$ studied at the beginning of the proof of Theorem 6. Therefore, the Kruskal domains of

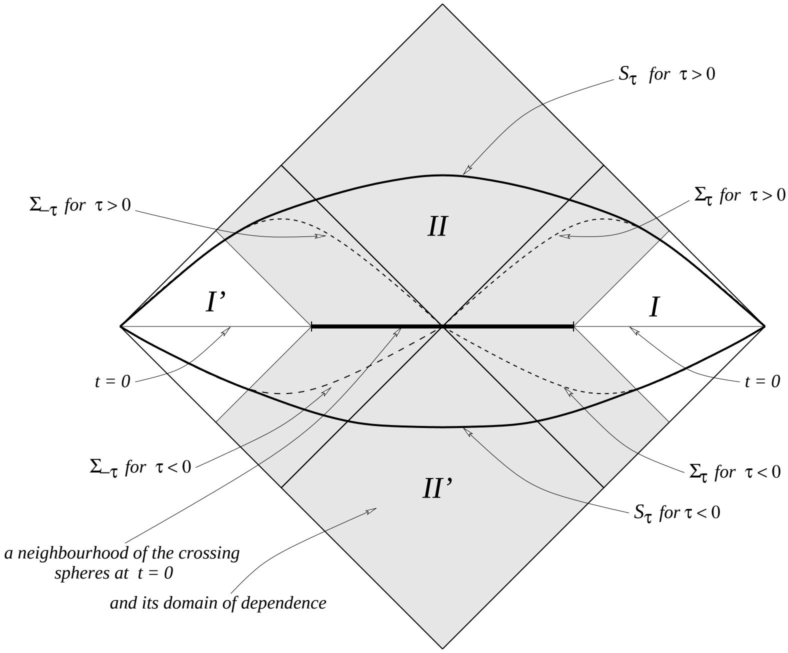

Fig. 6.5. Foliation of a type I-II Kruskal domain

type I-II thus foliated are interpreted as space-times (with two asymptotically flat ends) of class (∞, δ) for any $\delta > -2$. This allows us to apply directly the theorems of Section 4. We obtain the existence and uniqueness of solutions to Dirac's equation with values in L^2, L^2_μ for all $\mu \in \mathbb{R}$, H^k for all $k \in \mathbb{N}$, H^k_μ for all $k \in \mathbb{N}$ and $\mu \in \mathbb{R}$, on the slices S_τ. A simplified interpretation is that Dirac fields are well-behaved at least as long as they do not cross the inner horizon (either in the future or in the past).

7. Concluding remarks

As was remarked in [42], the fact that the spin-connection of a Ricci-flat space-time can be regarded as a pair of pure gauge Rarita–Schwinger fields may provide analytic means of controlling the fall-off at spacelike infinity of solutions to Einstein's vacuum equations, assuming we can obtain some precise control on the weighted Sobolev norms of spin 3/2 fields. For such a project, it is of course vital to have the existence theorems in weighted Sobolev spaces for solutions to the Rarita–Schwinger equations. The theorems of Section 4, by giving such existence results for symmetric hyperbolic systems, are a first step in this direction. They can be applied directly to the Dirac form of the Rarita–Schwinger equations. In order to work with the Witten form, we would need to control the nonlocal term. Whether this can be done regardless of the value of the weight remains to be seen. The next step, namely the precise control in time of the weighted Sobolev norms of Rarita–Schwinger fields, is difficult and requires detailed hypotheses on the evolution in time of the spacelike geometry of our space-times.

Concerning Schwarzschild black holes, it would be interesting to study the behaviour (explosive or not) of a smooth solution to Dirac's equation as it approaches the singularity. This poses the problem of the description of the Dirac field near the singularity. More precisely, the norm of the spinor at a point is defined in terms of the timelike vector T^a, which can be determined by a choice of a foliation or more simply by a choice of spin-frame. This vector is normalized with respect to the metric, but at the singularity, the metric blows up. It is therefore necessary to understand what the correct choice of spin-frame is near the singularity before addressing the question of how Dirac fields behave there. The noncausal character of block III should discourage hopes of attempting similar studies in Kerr space-time, or at least one should not think of it in terms of the behaviour of a Dirac field as it *propagates towards* the singularity.

Another interesting and difficult problem is the construction of a time-dependent scattering theory for Dirac fields on the exterior of a Kerr black hole. The point of view would be that of an observer static at infinity. In spite of its ugliness, it may be necessary to use the form of Dirac's equation given in Appendix B because it has the advantage of being independent of time.

Appendix A

A choice of spin-frame and the expression of the time connection terms in Kerr and Schwarzschild geometries

We consider a general framework of which the exterior of both Kerr and Schwarzschild black holes is a particular case. On

$$\mathcal{M} = \mathbb{R}_\tau \times \Sigma_x, \qquad \Sigma =]0, +\infty[_u \times S^2_{\Theta,\Phi},$$

we have a Lorentzian metric g of the form

$$g = \frac{(N(x))^2}{2}\, \mathrm{d}\tau^2 - h(\tau).$$

The lapse function N is independent of time and satisfies

$$N \in \mathcal{C}_b^\infty(\overline{\Sigma}), \qquad N > 0 \quad \text{on } \Sigma, \qquad N|_{\partial\Sigma} = 0$$

where $\overline{\Sigma} = [0, +\infty[_u \times S^2_{\Theta,\Phi}$. $h(\tau)$ is a Riemannian metric on Σ, depending on time, and satisfying

$$h \in \mathcal{C}^\infty(\mathbb{R}_\tau; \mathcal{C}_b^\infty(\overline{\Sigma})).$$

Moreover, there exist two continuous strictly positive functions C_1 and C_2 on $\mathbb{R}$ such that, as a quadratic form, the metric $h(\tau)$ satisfies

$$C_1(\tau)\widetilde{h} \leq h(\tau) \leq C_2(\tau)\widetilde{h} \quad \text{ for all } \tau \in \mathbb{R},$$

where $\widetilde{h}$ is the euclidian metric on Σ considered as $\mathbb{R}^3 \setminus \overline{B}(0,1)$:

$$\widetilde{h} = \mathrm{d}u^2 + (1+u)^2 \mathrm{d}\Omega^2, \qquad \mathrm{d}\Omega^2 = \mathrm{d}\Theta^2 + \sin^2\Theta\, \mathrm{d}\Phi^2.$$

The vector field T^a, timelike, future pointing, g-orthogonal to the foliation $\{\Sigma_\tau = \{\tau\} \times \Sigma\}_{\tau\in\mathbb{R}}$ and normalized so that $T_aT^a = 2$ is given by

$$T^a\partial_a = \frac{2}{N}\frac{\partial}{\partial\tau}.$$

These hypotheses are far weaker than the properties satisfied by Kerr or Schwarzschild metrics outside the black hole, but they will suffice for the calculations we perform here.

A.1. A choice of spin-frame. We describe the choice of a Newman–Penrose tetrad $\{l^a, n^a, m^a, \overline{m}^a\}$, the spin-frame $\{o^A, \iota^A\}$ is then fixed up to an overall sign by requiring

$$l^a = o^A\bar{o}^{A'}, \quad n^a = \iota^A\bar{\iota}^{A'}, \quad m^a = o^A\bar{\iota}^{A'}, \quad \overline{m}^a = \iota^A\bar{o}^{A'}. \tag{A.1}$$

Let us consider a global smooth coordinate system on Σ: $\{x^1, x^2, x^3\}$. For example, we can take

$$x^1 = (1+u)\sin\Theta\cos\Phi, \quad x^2 = (1+u)\sin\Theta\sin\Phi, \quad x^3 = (1+u)\cos\Theta.$$

With our choice of coordinate u in Schwarzschild and Kerr metrics, replacing Θ and Φ by θ and φ in the Schwarzschild case, this coordinate system is smooth outside the black hole. We simply assume here that u, Θ and Φ are sufficiently well chosen so that this is also the case (alternatively, we need not be that explicit and we can simply consider a given global smooth coordinate system on Σ). The metric h in this coordinate system takes the form

$$h(\tau,x) = \sum_{\mathbf{a},\mathbf{b}=1}^{3} h_{\mathbf{ab}}(\tau,x)\mathrm{d}x^{\mathbf{a}}\mathrm{d}x^{\mathbf{b}}, \qquad h_{\mathbf{ab}} = h_{\mathbf{ba}}.$$

The coefficients $h_{\mathbf{ab}}$ satisfy the following properties

$$h_{\mathbf{ab}} \in \mathcal{C}^\infty(\mathbb{R}_\tau; \mathcal{C}_b^\infty(\overline{\Sigma})), \tag{A.2}$$

and for any $\xi\in\mathbb{R}^3$, $(\tau,x)\in\mathbb{R}\times\overline{\Sigma}$,

$$C_1(\tau)|\xi|^2 \le \sum_{\mathbf{a},\mathbf{b}=1}^{3} h_{\mathbf{ab}}(\tau,x)\xi^{\mathbf{a}}\xi^{\mathbf{b}} \le C_2(\tau)|\xi|^2. \tag{A.3}$$

We consider on $(\mathbb{R}_\tau\times\overline{\Sigma}, g)$ a smooth global orthonormal Lorentz frame

$$\left(\frac{1}{\sqrt{2}}T^a, X^a, Y^a, Z^a\right) \text{ such that } X^a, Y^a, Z^a \in \mathcal{C}^\infty(\mathbb{R}_\tau; \mathcal{C}_b^\infty(\overline{\Sigma})). \tag{A.4}$$

Such a global frame exists because $\mathbb{R}_\tau\times\overline{\Sigma}$ is diffeomorphic to $\mathbb{R}_\tau\times\left(\mathbb{R}^3\setminus B(0,1)\right)$ and is therefore parallelizable. The family $\{X^a, Y^a, Z^a\}$ is for each $\tau\in\mathbb{R}$ a global orthonormal section of the principal bundle (the bundle of local frames) of $(\Sigma, h(\tau))$. Each vector field is at each point an eigenvector of the matrix $h_{\mathbf{ab}}$, normalized so that its norm with respect to h is 1. The regularity of h and its equivalence (locally uniform in time and uniform in space) to the euclidian metric on $\mathbb{R}^3\setminus B(0,1)$ entail that X^a, Y^a, Z^a can be assumed to have the regularity (A.4).

We then define the Newman–Penrose tetrad as follows:

$$l^a = \frac{1}{2}T^a + \frac{1}{\sqrt{2}}X^a, \quad n^a = \frac{1}{2}T^a - \frac{1}{\sqrt{2}}X^a, \quad m^a = \frac{1}{\sqrt{2}}Y^a + \frac{i}{\sqrt{2}}Z^a.$$

All these vectors belong to $\mathcal{C}^\infty(\mathbb{R}_\tau; \mathcal{C}_b^\infty(\overline{\Sigma}))$. We choose the spin-frame (o^A, ι^A) by requiring (A.1). It is adapted to the foliation $\{\Sigma_\tau\}_{\tau\in\mathbb{R}}$ since

$$T^a = l^a + n^a = o^A \bar{o}^{A'} + \iota^A \bar{\iota}^{A'}.$$

A.2. The timelike connection terms. We introduce the directional covariant derivatives along the tetrad vectors

$$D := l^a \nabla_a, \qquad D' := n^a \nabla_a, \qquad \delta := m^a \nabla_a, \qquad \delta' := \overline{m}^a \nabla_a.$$

We consider the Dirac equation on $\mathcal{M}$ written in terms of two-component spinors as an evolution system (see (3.31))

$$\text{(A.5)} \qquad \begin{cases} \nabla_T \phi_A = 2\mathrm{D}_A{}^B \phi_B + \sqrt{2}\, m\, T_{AB'}\, \chi^{B'}, \\ \nabla_T \chi^{A'} = -2\mathrm{D}^{A'}{}_{B'} \chi^{B'} - \sqrt{2}\, m\, T^{BA'} \phi_B. \end{cases}$$

The timelike connection terms are those coming from the timelike covariant derivatives $\nabla_T \phi_A$ and $\nabla_T \chi^{A'}$:

$$\begin{aligned} \nabla_T \phi_A &= (D + D')\phi_A = (o^B \bar{o}^{B'} + \iota^B \bar{\iota}^{B'}) \nabla_{BB'} \phi_A, \\ \nabla_T\, \chi^{A'} &= (D + D')\chi^{A'} = (o^B \bar{o}^{B'} + \iota^B \bar{\iota}^{B'}) \nabla_{BB'} \chi^{A'}. \end{aligned}$$

Using the Newman–Penrose formalism, we calculate the components of $\nabla_T \phi_A$ and $\nabla_T \chi^{A'}$ with respect to the spin-frame $\{o^A, \iota^A\}$ (see [53], Vol. 1, paragraph 4.5)

$$\begin{cases} \varepsilon_0{}^A \nabla_T\, \phi_A = T^a \partial_a \phi_0 - (\varepsilon + \gamma)\phi_0 + (\kappa + \widetilde{\tau})\phi_1, \\ \varepsilon_1{}^A \nabla_T \phi_A = T^a \partial_a \phi_1 - (\pi + \nu)\phi_0 + (\varepsilon + \gamma)\phi_1, \\ \varepsilon_{A'}{}^{0'} \nabla_T \chi^{A'} = T^a \partial_a \chi^{0'} + (\overline{\varepsilon} + \overline{\gamma})\chi^{0'} + (\overline{\pi} + \overline{\nu})\chi^{1'}, \\ \varepsilon_{A'}{}^{1'} \nabla_T \chi^{A'} = T^a \partial_a \chi^{1'} - (\overline{\kappa} + \overline{\widetilde{\tau}})\chi^{0'} - (\overline{\varepsilon} + \overline{\gamma})\chi^{1'}, \end{cases}$$

where the spin-coefficients ε, γ, κ, $\widetilde{\tau}$, π and ν (we have chosen to denote by $\widetilde{\tau}$ the spin-coefficient usually denoted by τ in order to avoid confusion with the time variable τ) are defined by

$$\varepsilon = \frac{1}{2}(n^a D l_a + m^a D \overline{m}_a), \qquad \gamma = \frac{1}{2}(n^a D' l_a + m^a D' \overline{m}_a), \qquad \kappa = m^a D l_a,$$
$$\widetilde{\tau} = m^a D' l_a, \qquad \pi = -\overline{m}^a D n_a, \qquad \nu = -\overline{m}^a D' n_a.$$

We have

$$T^a \partial_a = \frac{2}{N} \frac{\partial}{\partial \tau}$$

and therefore, the system (A.5) written in terms of spinor components in the spin-frame $\{o^A, \iota^A\}$ has the form

$$\begin{aligned} \frac{\partial \phi}{\partial \tau} &= N \mathbb{D} \phi + \frac{mN}{\sqrt{2}} \chi + \frac{N}{2} \begin{pmatrix} \varepsilon + \gamma & -(\kappa + \widetilde{\tau}) \\ \pi + \nu & -(\varepsilon + \gamma) \end{pmatrix} \phi, \\ \frac{\partial \chi}{\partial \tau} &= N \widehat{\mathbb{D}} \chi - \frac{mN}{\sqrt{2}} \phi + \frac{N}{2} \begin{pmatrix} -(\overline{\varepsilon} + \overline{\gamma}) & -(\overline{\pi} + \overline{\nu}) \\ \overline{\kappa} + \overline{\widetilde{\tau}} & \overline{\varepsilon} + \overline{\gamma} \end{pmatrix} \chi, \end{aligned}$$

where the operators $\mathbb{D}$ and $\widehat{\mathbb{D}}$ were defined in (3.32) as

$$\mathbb{D} : \phi_A \mapsto \mathrm{D}_A{}^B \phi_B, \qquad \widehat{\mathbb{D}} : \chi^{A'} \mapsto -\mathrm{D}^{A'}{}_{B'} \chi^{B'}.$$

They are the restrictions of the operator $-(1/\sqrt{2})e_0.\mathcal{D}_W$ to $\mathbb{S}_A$ and $\mathbb{S}^{A'}$. In terms of Dirac spinors, putting $\Psi = \phi_A \oplus \chi^{A'}$, the system above takes the familiar form

$$\frac{\partial \Psi}{\partial \tau} = -\frac{N}{\sqrt{2}}\gamma^0 \mathcal{D}_W \Psi - \frac{imN}{\sqrt{2}}\gamma^0 \Psi - \frac{N}{\sqrt{2}} B\Psi$$

and the matrix B containing the timelike connection coefficients is given by (for our choice of a spin-frame)

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} -(\varepsilon+\gamma) & \kappa+\widetilde{\tau} & 0 & 0 \\ -(\pi+\nu) & \varepsilon+\gamma & 0 & 0 \\ 0 & 0 & \overline{\varepsilon}+\overline{\gamma} & \overline{\pi}+\overline{\nu} \\ 0 & 0 & -(\overline{\kappa}+\overline{\widetilde{\tau}}) & -(\overline{\varepsilon}+\overline{\gamma}) \end{pmatrix}.$$

In order to study the behaviour of NB, we need to study the quantities $N(\varepsilon+\gamma)$, $N(\kappa+\widetilde{\tau})$ and $N(\pi+\nu)$:

$$\varepsilon+\gamma = \frac{1}{2}(n^a(D+D')l_a + m^a(D+D')\overline{m}_a) = \frac{1}{2}(n^a\nabla_T l_a + m^a \nabla_T \overline{m}_a),$$
$$\kappa+\widetilde{\tau} = m^a(D+D')l_a = m^a\nabla_T l_a, \qquad \pi+\nu = -\overline{m}^a(D+D')n_a = -\overline{m}^a \nabla_T n_a.$$

All the vectors of the Newman–Penrose tetrad belong to $\mathcal{C}^\infty(\mathbb{R}_\tau; \mathcal{C}_b^\infty(\overline{\Sigma}))$. Moreover

$$N\nabla_T = NT^a\nabla_a \quad \text{and} \quad NT^a\partial_a = 2\frac{\partial}{\partial\tau} \in \mathcal{C}^\infty(\mathbb{R}_\tau; \mathcal{C}_b^\infty(\overline{\Sigma})).$$

Therefore, we conclude that

$$N(\varepsilon+\gamma), N(\kappa+\widetilde{\tau}), N(\pi+\nu) \in \mathcal{C}^\infty(\mathbb{R}_\tau; \mathcal{C}_b^\infty(\overline{\Sigma})).$$

This establishes the regularity of the matrix B:

$$B \in \mathcal{C}^\infty(\mathbb{R}_\tau; \mathcal{C}_b^\infty(\overline{\Sigma}; \mathcal{M}_4(\mathbb{C}))).$$

A.3. Explicit expressions in the Schwarzschild case. Outside the Schwarzschild black hole, we consider the Newman–Penrose tetrad described in Schwarzschild coordinates as

$$l^a\partial_a = \frac{1}{\sqrt{2}}\left(F^{-1/2}\frac{\partial}{\partial t} + F^{1/2}\frac{\partial}{\partial r}\right), \qquad n^a\partial_a = \frac{1}{\sqrt{2}}\left(F^{-1/2}\frac{\partial}{\partial t} - F^{1/2}\frac{\partial}{\partial r}\right),$$
$$m^a\partial_a = \frac{1}{r\sqrt{2}}\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right).$$

The coordinate basis is singular for $\theta = 0$ and $\theta = \pi$ and so is the vector m^a. The tetrad, however, is adapted to the foliation and the spacelike part of l^a and n^a is

$$F^{1/2}\frac{\partial}{\partial r} = \frac{\partial}{\partial u} \in \mathcal{C}_b^\infty(\overline{\Sigma}).$$

This tetrad in fact gives a smooth matrix B. The spin-coefficients for this choice of null tetrad were calculated in [48]; in particular, we have

$$\varepsilon = \gamma = \frac{F'F^{-1/2}}{4\sqrt{2}}, \qquad \kappa = \tau = \pi = \nu = 0$$

(we return to the standard notation τ for the spin coefficient since there is no risk of confusion with the time variable). Hence, the matrix B has the form

$$B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \frac{F' F^{-1/2}}{4}$$

and we see immediately that

$$\frac{N}{\sqrt{2}} B = F^{1/2} B \in \mathcal{C}_b^\infty(\overline{\Sigma}).$$

Appendix B

An expression of the Dirac equation outside a Kerr black hole

In the framework of the Newman–Penrose formalism, Dirac's equation

$$\begin{cases} \nabla^{AA'} \phi_A = \dfrac{m}{\sqrt{2}} \chi^{A'}, \\ \nabla^{AA'} \chi_{A'} = \dfrac{m}{\sqrt{2}} \phi^A, \end{cases}$$

takes the form (see for example [7])

$$\text{(B.1)} \qquad \begin{cases} n^{\mathbf{a}} \partial_{\mathbf{a}} \phi_0 - m^{\mathbf{a}} \partial_{\mathbf{a}} \phi_1 + (\mu - \gamma) \phi_0 + (\tau - \beta) \phi_1 = \dfrac{m}{\sqrt{2}} \chi_{1'}, \\ l^{\mathbf{a}} \partial_{\mathbf{a}} \phi_1 - \overline{m}^{\mathbf{a}} \partial_{\mathbf{a}} \phi_0 + (\alpha - \pi) \phi_0 + (\varepsilon - \varrho) \phi_1 = -\dfrac{m}{\sqrt{2}} \chi_{0'}, \\ n^{\mathbf{a}} \partial_{\mathbf{a}} \chi_{0'} - \overline{m}^{\mathbf{a}} \partial_{\mathbf{a}} \chi_{1'} + (\overline{\mu} - \overline{\gamma}) \chi_{0'} + (\overline{\tau} - \overline{\beta}) \chi_{1'} = \dfrac{m}{\sqrt{2}} \phi_1, \\ l^{\mathbf{a}} \partial_{\mathbf{a}} \chi_{1'} - m^{\mathbf{a}} \partial_{\mathbf{a}} \chi_{0'} + (\overline{\alpha} - \overline{\pi}) \chi_{0'} + (\overline{\varepsilon} - \overline{\varrho}) \chi_{1'} = -\dfrac{m}{\sqrt{2}} \phi_0, \end{cases}$$

where $\{l^a, n^a, m^a, \overline{m}^a\}$ is a Newman–Penrose tetrad such that

$$l_a n^a = 1, \; m_a \overline{m}^a = -1, \; l_a m^a = n_a m^a = 0$$

and the spin-coefficients involved in equation (B.1) are defined by

$$\varepsilon = \frac{1}{2}(n^a D l_a + m^a D \overline{m}_a), \qquad \alpha = \frac{1}{2}(n^a \delta' l_a + m^a \delta' \overline{m}_a), \qquad \beta = \frac{1}{2}(n^a \delta l_a + m^a \delta \overline{m}_a),$$

$$\gamma = \frac{1}{2}(n^a D' l_a + m^a D' \overline{m}_a), \qquad \varrho = m^a \delta' l_a, \; \tau = m^a D' l_a, \qquad \pi = -\overline{m}^a D n_a, \; \mu = -\overline{m}^a \delta n_a.$$

In block I of a slow Kerr space-time described in Boyer–Lindquist coordinates, we consider the Newman–Penrose tetrad used in [7]

$$l^a \partial_a = \frac{1}{\Delta}\left((r^2 + a^2) \frac{\partial}{\partial t} + \Delta \frac{\partial}{\partial r} + a \frac{\partial}{\partial \varphi} \right),$$

$$n^a \partial_a = \frac{1}{2\varrho^2}\left((r^2 + a^2) \frac{\partial}{\partial t} - \Delta \frac{\partial}{\partial r} + a \frac{\partial}{\partial \varphi} \right),$$

$$m^a \partial_a = \frac{1}{p\sqrt{2}}\left(ia \sin\theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{i}{\sin\theta} \frac{\partial}{\partial \varphi} \right),$$

where $p = r + ia\cos\theta$. The corresponding spin-coefficients, likewise described by S. Chandrasekhar in [7], are

$$\kappa = \sigma = \lambda = \nu = \varepsilon = 0, \quad \widetilde{\varrho} = -\frac{1}{\overline{p}}, \quad \beta = \frac{1}{2p\sqrt{2}}\cot\theta, \quad \pi = \frac{ia\sin\theta}{\overline{p}^2\sqrt{2}},$$
$$\tau = -\frac{ia\sin\theta}{\varrho^2\sqrt{2}}, \quad \mu = -\frac{\Delta}{2\varrho^2\overline{p}}, \quad \gamma = \mu + \frac{r-M}{2\varrho^2}, \quad \alpha = \pi - \overline{\beta},$$

where we have denoted by $\widetilde{\varrho}$ the spin-coefficient in order to avoid confusion with $\varrho^2 = p\overline{p} = r^2 + a^2\cos^2\theta$. Thus, we obtain the following expression of the Dirac equation outside the black hole:

$$\begin{aligned}\text{(B.2)}\quad & \frac{r^2+a^2}{2\varrho^2}\frac{\partial\phi_0}{\partial t} - \frac{ia\sin\theta}{p\sqrt{2}}\frac{\partial\phi_1}{\partial t} - \frac{\Delta}{2\varrho^2}\frac{\partial\phi_0}{\partial r} - \frac{1}{p\sqrt{2}}\frac{\partial\phi_1}{\partial\theta} + \frac{a}{2\varrho^2}\frac{\partial\phi_0}{\partial\varphi} - \frac{i}{p\sqrt{2}\sin\theta}\frac{\partial\phi_1}{\partial\varphi} \\ & \qquad - \frac{r-M}{2\varrho^2}\phi_0 - \left(\frac{ia\sin\theta}{\varrho^2\sqrt{2}} + \frac{\cot\theta}{2p\sqrt{2}}\right)\phi_1 = \frac{m}{\sqrt{2}}\chi_{1'},\end{aligned}$$

$$\begin{aligned}\text{(B.3)}\quad & \frac{r^2+a^2}{\Delta}\frac{\partial\phi_1}{\partial t} + \frac{ia\sin\theta}{\overline{p}\sqrt{2}}\frac{\partial\phi_0}{\partial t} + \frac{\partial\phi_1}{\partial r} - \frac{1}{\overline{p}\sqrt{2}}\frac{\partial\phi_0}{\partial\theta} + \frac{a}{\Delta}\frac{\partial\phi_1}{\partial\varphi} + \frac{i}{\overline{p}\sqrt{2}\sin\theta}\frac{\partial\phi_0}{\partial\varphi} \\ & \qquad - \frac{\cot\theta}{2\overline{p}\sqrt{2}}\phi_0 + \frac{1}{\overline{p}}\phi_1 = -\frac{m}{\sqrt{2}}\chi_{0'},\end{aligned}$$

$$\begin{aligned}\text{(B.4)}\quad & \frac{r^2+a^2}{2\varrho^2}\frac{\partial\chi_{0'}}{\partial t} + \frac{ia\sin\theta}{\overline{p}\sqrt{2}}\frac{\partial\chi_{1'}}{\partial t} - \frac{\Delta}{2\varrho^2}\frac{\partial\chi_{0'}}{\partial r} - \frac{1}{\overline{p}\sqrt{2}}\frac{\partial\chi_{1'}}{\partial\theta} + \frac{a}{2\varrho^2}\frac{\partial\chi_{0'}}{\partial\varphi} + \frac{i}{\overline{p}\sqrt{2}\sin\theta}\frac{\partial\chi_{1'}}{\partial\varphi} \\ & \qquad - \frac{r-M}{2\varrho^2}\chi_{0'} - \left(-\frac{ia\sin\theta}{\varrho^2\sqrt{2}} + \frac{\cot\theta}{2\overline{p}\sqrt{2}}\right)\chi_{1'} = \frac{m}{\sqrt{2}}\phi_1,\end{aligned}$$

$$\begin{aligned}\text{(B.5)}\quad & \frac{r^2+a^2}{\Delta}\frac{\partial\chi_{1'}}{\partial t} - \frac{ia\sin\theta}{p\sqrt{2}}\frac{\partial\chi_{0'}}{\partial t} + \frac{\partial\chi_{1'}}{\partial r} - \frac{1}{p\sqrt{2}}\frac{\partial\chi_{0'}}{\partial\theta} + \frac{a}{\Delta}\frac{\partial\chi_{1'}}{\partial\varphi} - \frac{i}{p\sqrt{2}\sin\theta}\frac{\partial\chi_{0'}}{\partial\varphi} \\ & \qquad - \frac{\cot\theta}{2p\sqrt{2}}\chi_{0'} + \frac{1}{p}\chi_{1'} = -\frac{m}{\sqrt{2}}\phi_0.\end{aligned}$$

We can express this system as an evolution equation; we do this for the Weyl anti-neutrino equation, i.e. for equations (B.2), (B.3) with $m = 0$. This gives also the evolution form for the Weyl neutrino equation (equations (B.4), (B.5) with $m = 0$) and for the complete Dirac equation with a modification of the mass term which we mention below. We write the Weyl anti-neutrino equation in the following manner:

$$\begin{aligned}\text{(B.6)}\quad & \begin{pmatrix} 1 & b_1(r,\theta) \\ b_2(r,\theta) & 1 \end{pmatrix}\frac{\partial\phi}{\partial t} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\frac{\Delta}{r^2+a^2}\frac{\partial\phi}{\partial r} \\ & \quad + \begin{pmatrix} 0 & c_1(r,\theta) \\ c_2(r,\theta) & 0 \end{pmatrix}\frac{\partial\phi}{\partial\theta} + \begin{pmatrix} \frac{a}{r^2+a^2} & d_1(r,\theta) \\ d_2(r,\theta) & \frac{a}{r^2+a^2} \end{pmatrix}\frac{\partial\phi}{\partial\varphi} + V(r,\theta)\phi = 0,\end{aligned}$$

where

$$b_1(r,\theta) = -\frac{2\varrho^2}{r^2+a^2}\frac{ia\sin\theta}{\sqrt{2}(r+ia\cos\theta)}, \qquad b_2(r,\theta) = \frac{\Delta}{r^2+a^2}\frac{ia\sin\theta}{\sqrt{2}(r-ia\cos\theta)},$$
$$c_1(r,\theta) = -\frac{2\varrho^2}{r^2+a^2}\frac{1}{\sqrt{2}(r+ia\cos\theta)}, \qquad c_2(r,\theta) = -\frac{\Delta}{r^2+a^2}\frac{1}{\sqrt{2}(r-ia\cos\theta)},$$

$$d_1(r,\theta) = -\frac{2\varrho^2}{r^2+a^2}\frac{i}{\sqrt{2}\sin\theta(r+ia\cos\theta)}, \quad d_2(r,\theta) = \frac{\Delta}{r^2+a^2}\frac{i}{\sqrt{2}\sin\theta(r-ia\cos\theta)}$$

and $V(r,\theta)$ is the matrix of all the potential terms in equations (B.2), (B.3) with $m = 0$. The matrix in front of the time derivative is invertible since

$$\det\begin{pmatrix} 1 & b_1(r,\theta) \\ b_2(r,\theta) & 1 \end{pmatrix} = 1 - \frac{\Delta a^2 \sin^2\theta}{(r^2+a^2)^2}$$

is positive (and even uniformly bounded away from zero and bounded) in block I. Therefore, putting

$$B(r,\theta) = \begin{pmatrix} 1 & b_1(r,\theta) \\ b_2(r,\theta) & 1 \end{pmatrix}^{-1}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\frac{\Delta}{r^2+a^2},$$

$$C(r,\theta) = \begin{pmatrix} 1 & b_1(r,\theta) \\ b_2(r,\theta) & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 & c_1(r,\theta) \\ c_2(r,\theta) & 0 \end{pmatrix},$$

$$D(r,\theta) = \begin{pmatrix} 1 & b_1(r,\theta) \\ b_2(r,\theta) & 1 \end{pmatrix}^{-1}\begin{pmatrix} \frac{a}{r^2+a^2} & d_1(r,\theta) \\ d_2(r,\theta) & \frac{a}{r^2+a^2} \end{pmatrix},$$

$$P(r,\theta) = \begin{pmatrix} 1 & b_1(r,\theta) \\ b_2(r,\theta) & 1 \end{pmatrix}^{-1} V(r,\theta),$$

we can write equation (B.6) as the evolution equation

$$\text{(B.7)} \qquad \frac{\partial\phi}{\partial t} + B(r,\theta)\frac{\partial\phi}{\partial r} + C(r,\theta)\frac{\partial\phi}{\partial\theta} + D(r,\theta)\frac{\partial\phi}{\partial\varphi} + P(r,\theta)\phi = 0.$$

For the full Dirac equation, we obtain the following evolution system:

$$\text{(B.8)} \qquad \frac{\partial\phi}{\partial t} + B(r,\theta)\frac{\partial\phi}{\partial r} + C(r,\theta)\frac{\partial\phi}{\partial\theta} + D(r,\theta)\frac{\partial\phi}{\partial\varphi} + P(r,\theta)\phi = \begin{pmatrix} 1 & b_1(r,\theta) \\ b_2(r,\theta) & 1 \end{pmatrix}^{-1}\frac{m}{\sqrt{2}(r^2+a^2)}\begin{pmatrix} 0 & 2\varrho^2 \\ -\Delta & 0 \end{pmatrix}\chi,$$

$$\text{(B.9)} \qquad \frac{\partial\chi}{\partial t} + \overline{B}(r,\theta)\frac{\partial\chi}{\partial r} + \overline{C}(r,\theta)\frac{\partial\chi}{\partial\theta} + \overline{D}(r,\theta)\frac{\partial\chi}{\partial\varphi} + \overline{P}(r,\theta)\chi = \begin{pmatrix} 1 & \overline{b_1(r,\theta)} \\ \overline{b_2(r,\theta)} & 1 \end{pmatrix}^{-1}\frac{m}{\sqrt{2}(r^2+a^2)}\begin{pmatrix} 0 & 2\varrho^2 \\ -\Delta & 0 \end{pmatrix}\phi.$$

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