Contents

Abstract

We begin with a short presentation of the basic concepts related to Lie groupoids and Lie algebroids, but the main part of this paper deals with Lie algebroids. A Lie algebroid over a manifold is a vector bundle over that manifold whose properties are very similar to those of a tangent bundle. Its dual bundle has properties very similar to those of a cotangent bundle: in the graded algebra of sections of its exterior powers, one can define an operator d_E similar to the exterior derivative. We present the theory of Lie derivatives, Schouten–Nijenhuis brackets and exterior derivatives in the general setting of a Lie algebroid, its dual bundle and their exterior powers. All the results (which, for the most part, are already known) are given with detailed proofs. In the final sections, the results are applied to Poisson manifolds, whose links with Lie algebroids are very close.

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In memory of Paulette Libermann

1. Introduction

Groupoids are mathematical structures able to describe symmetry properties more general than those described by groups. They were introduced (and named) by H. Brandt [3] in 1926. The reader will find a general presentation of that important concept in [52] and [5].

A groupoid with a topological structure (resp., a differentiable structure) is called a *topological groupoid* (resp., a *Lie groupoid*). Around 1950, Charles Ehresmann [14] used these concepts as essential tools in topology and differential geometry.

In recent years, Michael Karasev [22], Alan Weinstein [50, 8] and Stanisław Zakrzewski [55] independently discovered that Lie groupoids equipped with a symplectic structure can be used for the construction of noncommutative deformations of the algebra of smooth functions on a manifold, with potential applications to quantization. Poisson groupoids were introduced by Alan Weinstein [51] as generalizations of both Poisson Lie groups and symplectic groupoids.

The infinitesimal counterpart of the notion of a Lie groupoid is the notion of a Lie algebroid, in the same way as the infinitesimal counterpart of the notion of a Lie group is the notion of a Lie algebra. Lie algebroids were first considered by Jean Pradines [41].

Lie groupoids and Lie algebroids are now an active domain of research, with applications in various parts of mathematics [51, 4, 29, 2, 30, 5]. More specifically, Lie algebroids have applications in mechanics [53, 31, 15, 39] and provide a very natural setting in which one can develop the theory of differential operators such as the exterior derivative of forms and the Lie derivative with respect to a vector field. In such a setting, slightly more general than that of the tangent and cotangent bundles to a smooth manifold and their exterior powers, the theory of Lie derivatives extends, in a very natural way, into the theory of the Schouten–Nijenhuis bracket (first introduced in differential geometry by J. A. Schouten [44] and developed by A. Nijenhuis [40]). Other bidifferential operators such as the bracket of exterior forms on a Poisson manifold, first discovered for Pfaff forms by F. Magri and C. Morosi [38] and extended to forms of all degrees by J.-L. Koszul [28], appear in such a setting as very natural: they are Schouten–Nijenhuis brackets for the Lie algebroid structure of the cotangent bundle to a Poisson manifold.

In this paper, we first present the basic concepts related to Lie groupoids and Lie algebroids. Then we develop the theory of Lie derivatives, Schouten–Nijenhuis brackets and exterior derivatives in the general setting of a Lie algebroid, its dual bundle and the exterior powers. All the results (which, for the most part, are already known, see for example [54, 16, 17]) are given with detailed proofs. Most of these proofs are the same as the classical ones (when the Lie algebroid is the tangent bundle to a smooth manifold); a few ones are slightly more complicated because, unlike the algebra of exterior differential forms on a manifold, the algebra of sections of exterior powers of the dual of a Lie algebroid is not locally generated by its elements of degree 0 and their differentials. These results may even be extended to more general algebroids with no assumption of skew-symmetry [18], but here we will not discuss these generalizations, nor will we discuss the Schouten bracket for symmetric tensors. In the final section, the results are applied to Poisson manifolds. We show that the cotangent space of a Poisson manifold has a Lie algebroid structure and that the total space of the vector bundle dual to a Lie algebroid has a natural Poisson structure, and we use these properties for lifting Poisson structures and Lie algebroid structures to the tangent bundle.

2. Lie groupoids

2.1. Definition and first properties. Before stating the formal definition of a groupoid, let us explain, in an informal way, why it is a very natural concept. The easiest way to understand that concept is to think of two sets, Γ and Γ_0. The first one, Γ, is called the *set of arrows* or *total space* of the groupoid, and the other one, Γ_0, the *set of objects* or *set of units* of the groupoid. One may think of an element $x \in \Gamma$ as an arrow going from an object (a point in Γ_0) to another object (another point in Γ_0). The word "arrow" is used here in a very general sense: it means a way for going from a point in Γ_0 to another point in Γ_0. One should not think of an arrow as a line drawn in the set Γ_0 joining the starting point of the arrow to its end point. Rather, one should think of an arrow as living outside Γ_0, with only its starting point and its end point in Γ_0, as shown in Figure 1.

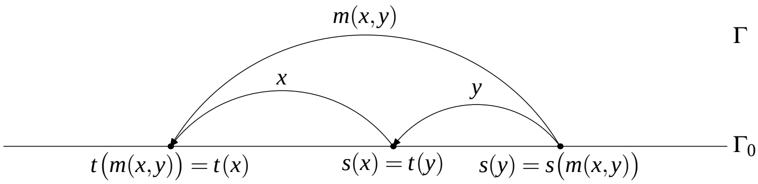

Fig. 1. Two arrows $x, y \in \Gamma$, with the target of y, $t(y) \in \Gamma_0$, equal to the source of x, $s(x) \in \Gamma_0$, and the composed arrow $m(x, y)$

The following ingredients enter the definition of a groupoid.

(i) Two maps $s : \Gamma \to \Gamma_0$ and $t : \Gamma \to \Gamma_0$, called the *source map* and the *target map* of the groupoid. If $x \in \Gamma$ is an arrow, $s(x) \in \Gamma_0$ is its starting point and $t(x) \in \Gamma_0$ its end point.

(ii) A *composition law* on the set of arrows; we can compose an arrow y with another arrow x, and get an arrow $m(x, y)$, by following first the arrow y, then the arrow x. Of course, $m(x, y)$ is defined if and only if the target of y is equal to the source of x. The source of $m(x, y)$ is equal to the source of y, and its target is equal to the target of x, as illustrated in Figure 1. It is only by convention that we write $m(x, y)$ rather than $m(y,x)$: the arrow which is followed first is on the right, by analogy with the usual notation $f \circ g$ for the composition of two maps g and f. The composition of arrows is associative.

(iii) An *embedding* ε of the set Γ_0 into the set Γ, which associates a unit arrow $\varepsilon(u)$ with each $u \in \Gamma_0$. That unit arrow is such that both its source and its target are u, and it plays the role of a unit when composed with another arrow, either on the right or on the left: for any arrow x, $m(\varepsilon(t(x)), x) = x$, and $m(x, \varepsilon(s(x))) = x$.

(iv) Finally, an *inverse map* ι from the set of arrows onto itself. If $x \in \Gamma$ is an arrow, one may think of $\iota(x)$ as the arrow x followed in the reverse sense.

Now we are ready to state the formal definition of a groupoid.

Definition 2.1.1. A *groupoid* is a pair of sets (Γ, Γ_0) equipped with the structure defined by the following data:

(i) an injective map $\varepsilon : \Gamma_0 \to \Gamma$, called the *unit section* of the groupoid;

(ii) two maps $s : \Gamma \to \Gamma_0$ and $t : \Gamma \to \Gamma_0$, called, respectively, the *source map* and the *target map*; they satisfy

$$s \circ \varepsilon = t \circ \varepsilon = \mathrm{id}_{\Gamma_0};$$

(iii) a composition law $m : \Gamma_2 \to \Gamma$, called the *product*, defined on the subset Γ_2 of $\Gamma \times \Gamma$, called the *set of composable elements*,

$$\Gamma_2 = \{(x,y) \in \Gamma \times \Gamma;\ s(x) = t(y)\},$$

which is associative, in the sense that whenever one side of the equality

$$m(x, m(y,z)) = m(m(x,y), z)$$

is defined, the other side is defined too, and the equality holds; moreover, the composition law m is such that for each $x \in \Gamma$,

$$m(\varepsilon(t(x)), x) = m(x, \varepsilon(s(x))) = x;$$

(iv) a map $\iota : \Gamma \to \Gamma$, called the *inverse*, such that, for every $x \in \Gamma$, $(x, \iota(x)) \in \Gamma_2$, $(\iota(x), x) \in \Gamma_2$ and

$$m(x, \iota(x)) = \varepsilon(t(x)), \qquad m(\iota(x), x) = \varepsilon(s(x)).$$

The sets Γ and Γ_0 are called, respectively, the *total space* and the *set of units* of the groupoid, which is itself denoted by $\Gamma \underset{s}{\overset{t}{\rightrightarrows}} \Gamma_0$.

Remark 2.1.2. The definition of a groupoid can be stated very briefly in the language of category theory: a *groupoid* is a small category all of whose arrows are invertible. We recall that a category is said to be *small* if the collections of its arrows and of its objects are sets.

2.1.3. Identification and notations. In what follows, by means of the injective map ε, we will identify the set of units Γ_0 with the subset $\varepsilon(\Gamma_0)$ of Γ. Therefore ε will be the canonical injection in Γ of its subset Γ_0.

For $x, y \in \Gamma$, we will sometimes write $x \circ y$, $x.y$, or even simply xy for $m(x,y)$, and x^{-1} for $\iota(x)$. Also we will write "the groupoid Γ" for "the groupoid $\Gamma \underset{s}{\overset{t}{\rightrightarrows}} \Gamma_0$".

2.2. Properties and comments. The above definition has the following consequences.

2.2.1. Involutivity of the inverse map. The inverse map ι is involutive:

$$\iota \circ \iota = \mathrm{id}_\Gamma\,.$$

We have indeed, for any $x \in \Gamma$,

$$\begin{aligned}\iota \circ \iota(x) &= m(\iota \circ \iota(x), s(\iota \circ \iota(x))) = m(\iota \circ \iota(x), s(x)) = m(\iota \circ \iota(x), m(\iota(x), x))\\ &= m(m(\iota \circ \iota(x), \iota(x)), x) = m(t(x), x) = x.\end{aligned}$$

2.2.2. Unicity of the inverse. Let $x, y \in \Gamma$ be such that

$$m(x,y) = t(x) \quad \text{and} \quad m(y,x) = s(x).$$

Then we have

$$\begin{aligned}y &= m(y, s(y)) = m(y, t(x)) = m(y, m(x, \iota(x))) = m(m(y,x), \iota(x))\\ &= m(s(x), \iota(x)) = m(t(\iota(x)), \iota(x)) = \iota(x).\end{aligned}$$

Therefore for any $x \in \Gamma$, the unique $y \in \Gamma$ such that $m(y,x) = s(x)$ and $m(x,y) = t(x)$ is $\iota(x)$.

2.2.3. The fibres of the source and target maps and the isotropy groups. The target map t (resp. the source map s) of a groupoid $\Gamma \overset{t}{\underset{s}{\rightrightarrows}} \Gamma_0$ determines an equivalence relation on Γ: two elements x and $y \in \Gamma$ are said to be t-equivalent (resp. s-equivalent) if $t(x) = t(y)$ (resp. if $s(x) = s(y)$). The corresponding equivalence classes are called the *t-fibres* (resp. the *s-fibres*) of the groupoid. They are of the form $t^{-1}(u)$ (resp. $s^{-1}(u)$) with $u \in \Gamma_0$.

For each unit $u \in \Gamma_0$, the subset

$$\Gamma_u = t^{-1}(u) \cap s^{-1}(u) = \{x \in \Gamma;\ s(x) = t(x) = u\}$$

is called the *isotropy group* of u. It is indeed a group, with the restrictions of m and ι as composition law and inverse map.

2.2.4. A way to visualize groupoids. We have seen (Figure 1) how groupoids may be visualized, by using arrows for elements in Γ and points for elements in Γ_0. There is another, very useful way to visualize groupoids, shown in Figure 2. The total space Γ of the groupoid is represented as a plane, and the set Γ_0 of units as a straight line in that

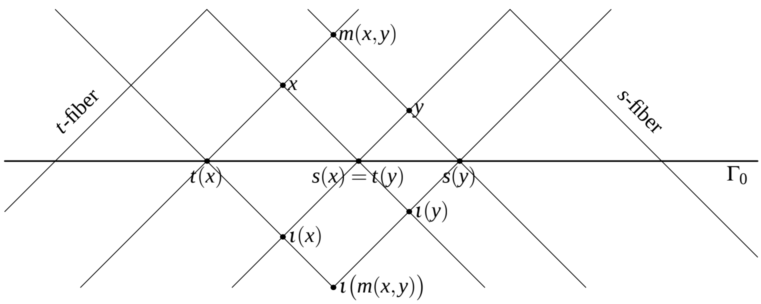

Fig. 2. A way to visualize groupoids

plane. The t-fibres (resp. the s-fibres) are represented as parallel straight lines, transverse to Γ_0.

Such a visualization should be used with care: one may think, at first sight, that there is only one element in the groupoid with a given source and a given target, which is not true in general.

2.3. Simple examples of groupoids

2.3.1. The groupoid of pairs. Let E be a nonempty set. Let $\Gamma = E \times E$, $\Gamma_0 = E$, $s : E \times E \to E$ be the projection on the right factor $s(x, y) = y$, $t : E \times E \to E$ the projection on the left factor $t(x, y) = x$, and $\varepsilon : E \to E \times E$ be the diagonal map $x \mapsto (x, x)$. We define the composition law $m : (E \times E) \times (E \times E) \to E \times E$ and the inverse $\iota : E \times E \to E \times E$ by

$$m\big((x, y), (y, z)\big) = (x, z), \qquad \iota(x, y) = (y, x).$$

Then $E \times E \overset{t}{\underset{s}{\rightrightarrows}} E$ is a groupoid, called the *groupoid of pairs* of elements in E.

2.3.2. Equivalence relations. Let E be a nonempty set with an equivalence relation r. Let $\Gamma = \{(x, y) \in E \times E;\, xry\}$ and $\Gamma_0 = E$. The source and target maps s and t are the restrictions to Γ of the source and target maps, above defined on $E \times E$ for the groupoid of pairs. The composition law m, the injective map ε and the inverse ι are the same as for the groupoid of pairs, suitably restricted. Then $\Gamma \overset{t}{\underset{s}{\rightrightarrows}} E$ is a groupoid, more precisely a subgroupoid of the groupoid of pairs of elements in E.

Remark 2.3.3. This example shows that equivalence relations may be considered as special groupoids. Conversely, on the set of units Γ_0 of a general groupoid $\Gamma \overset{t}{\underset{s}{\rightrightarrows}} \Gamma_0$, there is a natural equivalence relation: $u_1, u_2 \in \Gamma_0$ are said to be equivalent if there exists $x \in \Gamma$ such that $s(x) = u_1$ and $t(x) = u_2$. But the groupoid structure generally carries more information than that equivalence relation: there may be several $x \in \Gamma$ such that $s(x) = u_1$ and $t(x) = u_2$, i.e., several ways in which u_1 and u_2 are equivalent.

2.4. Topological and Lie groupoids

Definitions 2.4.1. A *topological groupoid* is a groupoid $\Gamma \overset{t}{\underset{s}{\rightrightarrows}} \Gamma_0$ for which Γ is a (maybe non-Hausdorff) topological space, Γ_0 a Hausdorff topological subspace of Γ, t and s surjective continuous maps, $m : \Gamma_2 \to \Gamma$ a continuous map and $\iota : \Gamma \to \Gamma$ a homeomorphism.

A *Lie groupoid* is a groupoid $\Gamma \overset{t}{\underset{s}{\rightrightarrows}} \Gamma_0$ for which Γ is a smooth (maybe non-Hausdorff) manifold, Γ_0 a smooth Hausdorff submanifold of Γ, t and s smooth surjective submersions (which implies that Γ_2 is a smooth submanifold of $\Gamma \times \Gamma$), $m : \Gamma_2 \to \Gamma$ a smooth map and $\iota : \Gamma \to \Gamma$ a smooth diffeomorphism.

2.5. Examples of topological and Lie groupoids

2.5.1. Topological groups and Lie groups. A topological group (resp. a Lie group) is a topological groupoid (resp. a Lie groupoid) whose set of units has only one element e.

2.5.2. Vector bundles. A smooth vector bundle $\tau : E \to M$ on a smooth manifold M is a Lie groupoid, with the base M as set of units (identified with the image of the zero section); the source and target maps both coincide with the projection τ, the product and the inverse maps are the addition $(x, y) \mapsto x + y$ and the opposite map $x \mapsto -x$ in the fibres.

2.5.3. The fundamental groupoid of a topological space. Let M be a topological space. A *path* in M is a continuous map $\gamma : [0, 1] \to M$. We denote by $[\gamma]$ the homotopy class of a path γ and by $\Pi(M)$ the set of homotopy classes of paths in M (with endpoints fixed). For $[\gamma] \in \Pi(M)$, we set $t([\gamma]) = \gamma(1)$, $s([\gamma]) = \gamma(0)$, where γ is any representative of the class $[\gamma]$. The concatenation of paths determines a well defined composition law on $\Pi(M)$, for which $\Pi(M) \underset{s}{\overset{t}{\rightrightarrows}} M$ is a topological groupoid, called the *fundamental groupoid* of M. The inverse map is $[\gamma] \mapsto [\gamma^{-1}]$, where γ is any representative of $[\gamma]$ and γ^{-1} is the path $t \mapsto \gamma(1 - t)$. The set of units is M, if we identify a point in M with the homotopy class of the constant path equal to that point.

Given a point $x \in M$, the isotropy group of the fundamental groupoid of M at x is the fundamental group at that point.

When M is a smooth manifold, the same construction can be made with piecewise smooth paths, and the fundamental groupoid $\Pi(M) \underset{s}{\overset{t}{\rightrightarrows}} M$ is a Lie groupoid.

2.5.4. The gauge groupoid of a fibre bundle with structure group. The structure of a locally trivial topological bundle (B, p, M) with standard fibre F and structure group a topological group G of homeomorphisms of F, is usually determined *via* an admissible fibred atlas (U_i, φ_i), $i \in I$. The U_i are open subsets of M such that $\bigcup_{i \in I} U_i = M$. For each $i \in I$, φ_i is a homeomorphism of $U_i \times F$ onto $p^{-1}(U_i)$ which, for each $x \in U_i$, maps $\{x\} \times F$ onto $p^{-1}(x)$. For each pair $(i, j) \in I^2$ such that $U_i \cap U_j \neq \emptyset$, and each $x \in U_i \cap U_j$, the homeomorphism $\varphi_j \circ \varphi_i^{-1}$ restricted to $\{x\} \times F$ is an element of G (F being identified with $\{x\} \times F$). Elements of G are called *admissible homeomorphisms* of F. Another, maybe more natural, way of describing that structure is by looking at the set Γ of admissible homeomorphisms between two fibres of that fibre bundle, $B_x = p^{-1}(x)$ and $B_y = p^{-1}(y)$, with $x, y \in M$. The set Γ has a topological structure (in general not Hausdorff). For $\gamma \in \Gamma$ mapping B_x onto B_y, we define $s(\gamma) = x$, $t(\gamma) = y$. Then $\Gamma \underset{s}{\overset{t}{\rightrightarrows}} M$ is a topological groupoid, called the *gauge groupoid* of the fibre bundle (B, p, M). When the bundle is smooth, its gauge groupoid is a Lie groupoid.

2.6. Properties of Lie groupoids

2.6.1. Dimensions. Let $\Gamma \underset{s}{\overset{t}{\rightrightarrows}} \Gamma_0$ be a Lie groupoid. Since t and s are submersions, for any $x \in \Gamma$, the t-fibre $t^{-1}(t(x))$ and the s-fibre $s^{-1}(s(x))$ are submanifolds of Γ, both of dimension $\dim\Gamma - \dim\Gamma_0$. The inverse map ι restricted to the t-fibre through x (resp. the s-fibre through x) is a diffeomorphism of that fibre onto the s-fibre through $\iota(x)$ (resp. the t-fibre through $\iota(x)$). The dimension of the submanifold Γ_2 of composable pairs in $\Gamma\times\Gamma$ is $2\dim\Gamma - \dim\Gamma_0$.

2.6.2. The tangent bundle of a Lie groupoid. Let $\Gamma \overset{t}{\underset{s}{\rightrightarrows}} \Gamma_0$ be a Lie groupoid. Its tangent bundle $T\Gamma$ is a Lie groupoid, with $T\Gamma_0$ as set of units, and $Tt : T\Gamma \to T\Gamma_0$ and $Ts : T\Gamma \to T\Gamma_0$ as target and source maps. Let us denote by Γ_2 the set of composable pairs in $\Gamma\times\Gamma$, by $m:\Gamma_2\to\Gamma$ the composition law and by $\iota:\Gamma\to\Gamma$ the inverse. Then the set of composable pairs in $T\Gamma\times T\Gamma$ is simply $T\Gamma_2$, the composition law on $T\Gamma$ is $Tm : T\Gamma_2 \to T\Gamma$ and the inverse is $T\iota : T\Gamma \to T\Gamma$.

When the groupoid Γ is a Lie group G, the Lie groupoid TG is a Lie group too.

Remark 2.6.3. The cotangent bundle of a Lie groupoid is a Lie groupoid, and more precisely a symplectic groupoid [4, 8, 50, 1, 11]. Remarkably, the cotangent bundle of a non-Abelian Lie group is not a Lie group: it is a Lie groupoid. This fact may be considered as a justification of the current interest in Lie groupoids: as soon as one is interested in Lie groups, by looking at their cotangent bundles, one has to deal with Lie groupoids!

2.6.4. Isotropy groups. For each unit $u\in\Gamma_0$ of a Lie groupoid, the isotropy group Γ_u (defined in 2.2.3) is a Lie group.

3. Lie algebroids

The concept of a Lie algebroid was first introduced by J. Pradines [41], in connection with Lie groupoids.

3.1. Definition and examples. A Lie algebroid over a manifold is a vector bundle based on that manifold, whose properties are very similar to those of the tangent bundle. Let us give a formal definition.

Definition 3.1.1. Let M be a smooth manifold and (E,τ,M) be a vector bundle with base M. A *Lie algebroid structure* on that bundle is the structure defined by the following data:

(1) a composition law $(s_1,s_2)\mapsto\{s_1,s_2\}$ on the space $\Gamma(\tau)$ of smooth sections of the bundle, for which $\Gamma(\tau)$ becomes a Lie algebra;

(2) a smooth vector bundle map $\rho : E\to TM$, where TM is the tangent bundle of M, such that for every pair (s_1,s_2) of smooth sections of τ, and every smooth function $f : M\to\mathbb{R}$, we have the Leibniz-type formula

$$\{s_1, fs_2\} = f\{s_1,s_2\} + (\mathcal{L}(\rho\circ s_1)f)s_2.$$

We have denoted by $\mathcal{L}(\rho\circ s_1)f$ the Lie derivative of f with respect to the vector field $\rho\circ s_1$:

$$\mathcal{L}(\rho\circ s_1)f = i(\rho\circ s_1)df.$$

The vector bundle (E,τ,M) equipped with its Lie algebroid structure will be called a *Lie algebroid* and denoted by (E,τ,M,ρ); the composition law $(s_1,s_2)\mapsto\{s_1,s_2\}$ will be called the *bracket* and the map $\rho:E\to TM$ the *anchor* of the Lie algebroid (E,τ,M,ρ).

PROPOSITION 3.1.2. *Let (E,τ,M,ρ) be a Lie algebroid. The map $s\mapsto\rho\circ s$, which associates to a smooth section s of τ the smooth vector field $\rho\circ s$ on M, is a Lie algebra homomorphism. In other words, for each pair (s_1,s_2) of smooth sections of τ,*

$$[\rho\circ s_1,\rho\circ s_2]=\rho\circ\{s_1,s_2\}.$$

Proof. Let s_1, s_2 and s_3 be three smooth sections of τ and f be a smooth function on M. By the Jacobi identity for the Lie algebroid bracket,

$$\{\{s_1,s_2\},fs_3\}=\{s_1,\{s_2,fs_3\}\}-\{s_2,\{s_1,fs_3\}\}.$$

If we use the property of the anchor, the right hand side becomes

$$\begin{aligned}\{s_1,\{s_2,fs_3\}\}-\{s_2,\{s_1,fs_3\}\}&=f\bigl(\{s_1,\{s_2,s_3\}\}-\{s_2,\{s_1,s_3\}\}\bigr)\\&\quad+\bigl((\mathcal{L}(\rho\circ s_1)\circ\mathcal{L}(\rho\circ s_2)-\mathcal{L}(\rho\circ s_2)\circ\mathcal{L}(\rho\circ s_1))f\bigr)s_3.\end{aligned}$$

Similarly, the left hand side becomes

$$\{\{s_1,s_2\},fs_3\}=f\{\{s_1,s_2\},s_3\}+(\mathcal{L}(\rho\circ\{s_1,s_2\})f)s_3.$$

Using again the Jacobi identity for the Lie algebroid bracket, we obtain

$$\Bigl(\bigl(\mathcal{L}(\rho\circ\{s_1,s_2\})-(\mathcal{L}(\rho\circ s_1)\circ\mathcal{L}(\rho\circ s_2)-\mathcal{L}(\rho\circ s_2)\circ\mathcal{L}(\rho\circ s_1))\bigr)f\Bigr)s_3=0.$$

But we have

$$\mathcal{L}(\rho\circ s_1)\circ\mathcal{L}(\rho\circ s_2)-\mathcal{L}(\rho\circ s_2)\circ\mathcal{L}(\rho\circ s_1)=\mathcal{L}([\rho\circ s_1,\rho\circ s_2]).$$

Finally,

$$\bigl(\mathcal{L}(\rho\circ\{s_1,s_2\}-[\rho\circ s_1,\rho\circ s_2])f\bigr)s_3=0.$$

This result, which holds for any smooth function f on M and any smooth section s_3 of τ, proves that $s\mapsto\rho\circ s$ is a Lie algebra homomorphism. ∎

REMARKS 3.1.3. Let (E,τ,M,ρ) be a Lie algebroid.

(i) *Lie algebra homomorphisms.* For each smooth vector field X on M, the Lie derivative $\mathcal{L}(X)$ with respect to X is a derivation of $C^\infty(M,\mathbb{R})$: for every pair (f,g) of smooth functions on M,

$$\mathcal{L}(X)(fg)=(\mathcal{L}(X)f)g+f(\mathcal{L}(X)g).$$

The map $X\mapsto\mathcal{L}(X)$ is a Lie algebra homomorphism from the Lie algebra $A^1(M)$ of smooth vector fields on M into the Lie algebra $\mathrm{Der}(C^\infty(M,\mathbb{R}))$ of derivations of $C^\infty(M,\mathbb{R})$, equipped with the commutator

$$(D_1,D_2)\mapsto[D_1,D_2]=D_1\circ D_2-D_2\circ D_1$$

as composition law. These facts were used in the proof of Proposition 3.1.2.

The map $s\mapsto\mathcal{L}(\rho\circ s)$, obtained by composition of two Lie algebra homomorphisms, is a Lie algebra homomorphism, from the Lie algebra $\Gamma(\tau)$ of smooth sections of the Lie algebroid (E,τ,M,ρ) into the Lie algebra of derivations of $C^\infty(M,\mathbb{R})$.

(ii) *Leibniz-type formulae.* According to Definition 3.1.1 we have, for any pair (s_1, s_2) of smooth sections of τ and any smooth function f on M,

$$\{s_1, fs_2\} = f\{s_1, s_2\} + (i(\rho \circ s_1)df)\, s_2.$$

As an easy consequence of the definition, we also have

$$\{fs_1, s_2\} = f\{s_1, s_2\} - (i(\rho \circ s_2)df)\, s_1.$$

More generally, for any pair (s_1, s_2) of smooth sections of τ and any pair (f_1, f_2) of smooth functions on M, we have

$$\{f_1s_1, f_2s_2\} = f_1f_2\{s_1, s_2\} + f_1(i(\rho \circ s_1)df_2)s_2 - f_2(i(\rho \circ s_2)df_1)s_1.$$

With the use of the Lie derivative operators, this formula may also be written as

$$\{f_1s_1, f_2s_2\} = f_1f_2\{s_1, s_2\} + f_1(\mathcal{L}(\rho \circ s_1)f_2)s_2 - f_2(\mathcal{L}(\rho \circ s_2)f_1)s_1.$$

3.1.4. Simple examples of Lie algebroids

(i) *The tangent bundle.* The tangent bundle (TM, τ_M, M) of a smooth manifold M, equipped with the usual bracket of vector fields as composition law and with the identity map id_{TM} as anchor, is a Lie algebroid.

(ii) *An involutive distribution.* Let V be a smooth distribution on a smooth manifold M, i.e., a smooth vector subbundle of the tangent bundle TM. We assume that V is involutive, i.e., such that the space of its smooth sections is stable under the bracket operation. The vector bundle $(V, \tau_M|_V, M)$, with the usual bracket of vector fields as composition law and with the canonical injection $i_V : V \to TM$ as anchor, is a Lie algebroid. We have denoted by $\tau_M : TM \to M$ the canonical projection of the tangent bundle and by $\tau_M|_V$ its restriction to the subbundle V.

(iii) *A sheaf of Lie algebras.* Let (E, τ, M) be a vector bundle over the smooth manifold M and $(z_1, z_2) \mapsto [z_1, z_2]$ be a smooth, skew-symmetric bilinear bundle map defined on the fibred product $E \times_M E$, with values in E, such that for each $x \in M$, the fibre $E_x = \tau^{-1}(x)$, equipped with the bracket $(z_1, z_2) \mapsto [z_1, z_2]$, is a Lie algebra. We define the bracket of two smooth sections s_1 and s_2 of τ as the section $\{s_1, s_2\}$ such that, for each $x \in M$, $\{s_1, s_2\}(x) = [s_1(x), s_2(x)]$. For the anchor, we take the zero vector bundle map from E to TM. Then (E, τ, M) is a Lie algebroid of particular type, called a *sheaf of Lie algebras* over the manifold M.

(iv) *A finite-dimensional Lie algebra.* In particular, a finite-dimensional Lie algebra can be considered as a Lie algebroid over a base reduced to a single point, with the zero map as anchor.

3.2. The Lie algebroid of a Lie groupoid. We now describe the most important example of Lie algebroid: to every Lie groupoid, there is an associated Lie algebroid, much like to every Lie group there is an associated Lie algebra. It is in this setting that Pradines [41] introduced Lie algebroids for the first time. For more information about Lie groupoids and their associated Lie algebroids, the reader is referred to [35, 36, 8, 13, 1].

In the following propositions and definitions, $\Gamma \underset{s}{\overset{t}{\rightrightarrows}} \Gamma_0$ is a Lie groupoid.

Proposition 3.2.1. *For each $x \in \Gamma$, the maps*

$$y \mapsto L_x(y) = xy \quad \text{and} \quad z \mapsto R_x(z) = zx$$

are smooth diffeomorphisms, respectively from $t^{-1}(s(x))$ onto $t^{-1}(t(x))$ and from $s^{-1}(t(x))$ onto $s^{-1}(s(x))$. These maps are called the left translation *and* right translation *by x, respectively.*

Proof. The smoothness of the groupoid composition law $m : (x, y) \mapsto xy$ implies the smoothness of L_x and R_x. These maps are diffeomorphisms whose inverses are

$$(L_x)^{-1} = L_{x^{-1}}, \qquad (R_x)^{-1} = R_{x^{-1}},$$

so the proof is complete. ∎

Definition 3.2.2. A vector field Y and a vector field Z, defined on open subsets of Γ, are said to be, respectively, *left invariant* and *right invariant* if they have the two properties:

(i) the projections on Γ_0 of Y by the target map t, and of Z by the source map s, vanish:

$$Tt(Y) = 0, \quad Ts(Z) = 0;$$

(ii) for each y in the domain of definition of Y and each $x \in s^{-1}(t(y))$, xy is in the domain of definition of Y and

$$Y(xy) = TL_x(Y(y));$$

similarly, for each z in the domain of definition of Z and each $x \in t^{-1}(s(z))$, zx is in the domain of definition of Z and

$$Z(zx) = TR_x(Z(z)).$$

Proposition 3.2.3. *Let $A(\Gamma)$ be the intersection of* $\ker Tt$ *and $T_{\Gamma_0}\Gamma$ (the tangent bundle $T\Gamma$ restricted to the submanifold Γ_0). Then $A(\Gamma)$ is the total space of a vector bundle $\tau : A(\Gamma) \to \Gamma_0$, with base Γ_0, the canonical projection τ being the map which associates a point $u \in \Gamma_0$ to every vector in* $\ker T_u t$*. That vector bundle has a natural Lie algebroid structure and is called the* Lie algebroid *of the Lie groupoid Γ. Its composition law is defined thus: Let w_1 and w_2 be two smooth sections of that bundle over an open subset U of Γ_0. Let $\widehat{w}_1$ and $\widehat{w}_2$ be the two left invariant vector fields, defined on $s^{-1}(U)$, whose restrictions to U are equal to w_1 and w_2 respectively. Then for each $u \in U$,*

$$\{w_1, w_2\}(u) = [\widehat{w}_1, \widehat{w}_2](u).$$

The anchor ρ of that Lie algebroid is the map Ts restricted to $A(\Gamma)$.

Proof. The correspondence which associates, to each smooth section w of the vector bundle $\tau : A(\Gamma) \to \Gamma_0$, the prolongation of that section by a left invariant vector field $\widehat{w}$, is a vector space isomorphism. Therefore, by setting

$$\{w_1, w_2\}(u) = [\widehat{w}_1, \widehat{w}_2](u),$$

we obtain a Lie algebra structure on the space of smooth sections of $\tau : A(\Gamma) \to \Gamma_0$. Let f be a smooth function, defined on the open subset U of Γ_0 on which w_1 and w_2 are defined. For each $u \in U$,

$$\begin{aligned}\{w_1, fw_2\}(u) &= [\widehat{w}_1, \widehat{fw_2}](u) = [\widehat{w}_1, (f \circ s)\widehat{w}_2](u)\\ &= f(u)[\widehat{w}_1, \widehat{w}_2(u)] + (i(\widehat{w}_1)d(f \circ s))(u)\widehat{w}_2(u)\\ &= f(u)[\widehat{w}_1, \widehat{w}_2(u)] + \langle df(u), Ts(\widehat{w}_1(u))\rangle \widehat{w}_2(u)\\ &= f(u)\{w_1, w_2\}(u) + \big\langle df(u), Ts(w_1(u))\big\rangle w_2(u),\end{aligned}$$

which proves that Ts has the properties of an anchor. ■

Remark 3.2.4. We could exchange the roles of t and s and use *right invariant* vector fields instead of left invariant vector fields. The Lie algebroid obtained remains the same, up to an isomorphism.

Examples 3.2.5. (i) When the Lie groupoid $\Gamma \underset{s}{\overset{t}{\rightrightarrows}}$ is a Lie group, its Lie algebroid is simply its Lie algebra.

(ii) We have seen (2.5.2) that a vector bundle (E, τ, M), with addition in the fibres as composition law, can be considered as a Lie groupoid. Its Lie algebroid is the same vector bundle, with the zero bracket on its space of sections, and the zero map as anchor.

(ii) Let M be a smooth manifold. The groupoid of pairs $M \times M \underset{s}{\overset{t}{\rightrightarrows}} M$ (2.3.1) is a Lie groupoid whose Lie algebroid is isomorphic to the tangent bundle (TM, τ_M, M) with the identity map as anchor.

(iii) The fundamental groupoid (2.5.3) of a smooth connected manifold M is a Lie groupoid. Its total space is the simply connected covering space of $M \times M$ and, as in the previous example, its Lie algebroid is isomorphic to the tangent bundle (TM, τ_M, M).

3.2.6. Integration of Lie algebroids. According to Lie's third theorem, for any given finite-dimensional Lie algebra, there exists a Lie group whose Lie algebra is isomorphic to that Lie algebra. The same property is not true for Lie algebroids and Lie groupoids. The problem of finding necessary and sufficient conditions under which a given Lie algebroid is isomorphic to the Lie algebroid of a Lie groupoid remained open for more than 30 years. Partial results were obtained by J. Pradines [42], K. Mackenzie [35], P. Dazord [11], P. Dazord ans G. Hector [12]. An important breakthrough was made by Cattaneo and Felder [7] who, starting from a Poisson manifold, built a groupoid (today called the *Weinstein groupoid*) which, when its total space is regular, has a dimension twice that of the Poisson manifold, has a symplectic structure and has as Lie algebroid the cotangent space to the Poisson manifold. That groupoid was obtained by symplectic reduction of an infinite-dimensional manifold. That method may in fact be used for any Lie algebroid, as shown by Cattaneo [6]. A complete solution of the integration problem for Lie algebroids was obtained by M. Crainic and R. L. Fernandes [9]. They have shown that with each given Lie algebroid, one can associate a topological groupoid with connected and simply connected t-fibres, now called the *Weinstein groupoid* of that Lie algebroid. That groupoid, when the Lie algebroid is the cotangent bundle to a Poisson manifold, is the same as that previously obtained by Cattaneo and Felder by another method. When that topological groupoid is in fact a Lie groupoid, i.e., when it is smooth, its Lie algebroid is isomorphic to the given Lie algebroid. Crainic and Fernandes have obtained computable necessary and sufficient conditions under which the Weinstein groupoid of a Lie algebroid is smooth. In [10] they have used these results for integration of Poisson manifolds, i.e., for construction of a symplectic groupoid whose set of units is a given Poisson manifold.

3.3. Locality of the bracket. We will prove that the value, at any point $x \in M$, of the bracket of two smooth sections s_1 and s_2 of the Lie algebroid (E, τ, M, ρ), depends only on the jets of order 1 of s_1 and s_2 at x. We will need the following lemma.

LEMMA 3.3.1. *Let (E, τ, M, ρ) be a Lie algebroid, $s_1 : M \to E$ a smooth section of τ, and U an open subset of M on which s_1 vanishes. Then for any other smooth section s_2 of τ, $\{s_1, s_2\}$ vanishes on U.*

Proof. Let x be a point in U. There exists a smooth function $f : M \to \mathbb{R}$ with support in U such that $f(x) = 1$. The section fs_1 vanishes identically, since s_1 vanishes on U while f vanishes outside of U. Therefore, for any other smooth section s_2 of τ,

$$0 = \{fs_1, s_2\} = -\{s_2, fs_1\} = -f\{s_2, s_1\} - (i(\rho \circ s_2)df)s_1.$$

So at x we have

$$f(x)\{s_1, s_2\}(x) = (i(\rho \circ s_2)df)(x)s_1(x) = 0.$$

Since $f(x) = 1$, we obtain $\{s_1, s_2\}(x) = 0$. ∎

PROPOSITION 3.3.2. *Let (E, τ, M, ρ) be a Lie algebroid. The value $\{s_1, s_2\}(x)$ of the bracket of two smooth sections s_1 and s_2 of τ, at a point $x \in M$, depends only on the jets of order 1 of s_1 and s_2 at x. Moreover, if $s_1(x) = 0$ and $s_2(x) = 0$, then $\{s_1, s_2\}(x) = 0$.*

Proof. Let U be an open neighbourhood of x in M on which there exists a local basis $(\sigma_1, \dots, \sigma_k)$ of smooth sections of τ. For any point $y \in U$, $(\sigma_1(y), \dots, \sigma_k(y))$ is a basis of the fibre $E_y = \tau^{-1}(y)$. Let s_1 and s_2 be two smooth sections of τ. On the open subset U, these two sections can be expressed, in a unique way, as

$$s_1 = \sum_{i=1}^{k} f_i \sigma_i, \qquad s_2 = \sum_{j=1}^{k} g_j \sigma_j,$$

where the f_i and g_j are smooth functions on U.

By Lemma 3.3.1, the values of $\{s_1, s_2\}$ in U depend only on the values of s_1 and s_2 in U. Therefore in U we have

$$\{s_1, s_2\} = \sum_{i,j} \bigl(f_i g_j \{\sigma_i, \sigma_j\} + f_i(\mathcal{L}(\rho \circ \sigma_i)g_j)\sigma_j - g_j(\mathcal{L}(\rho \circ \sigma_j)f_i)\sigma_i\bigr).$$

This expression proves that the value of $\{s_1, s_2\}$ at x depends only on the $f_i(x)$, $df_i(x)$, $g_j(x)$ and $dg_j(x)$, that is, on the jets of order 1 of s_1 and s_2 at x.

If $s_1(x) = 0$, we have, for all $i \in \{1, \dots, k\}$, $f_i(x) = 0$, and similarly if $s_2(x) = 0$, we have, for all $j \in \{1, \dots, k\}$, $g_j(x) = 0$. The above expression thus shows that $\{s_1, s_2\}(x) = 0$. ∎

4. Exterior powers of vector bundles

We recall in this section some definitions and general properties related to vector bundles, their dual bundles and exterior powers. In the first subsection we recall some properties of graded algebras, graded Lie algebras and their derivations. The second subsection applies these properties to the graded algebra of sections of the exterior powers of a vector bundle. For more details the reader may consult the book by Greub, Halperin and Vanstone [19]. The reader already familiar with this material may skip this section, or just look briefly at the sign conventions we are using.

4.1. Graded vector spaces and graded algebras

DEFINITIONS 4.1.1. (i) An *algebra* is a vector space A over the field $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, endowed with a $\mathbb{K}$-bilinear map called the *composition law*,

$$A\times A\to A,\quad (x,y)\mapsto xy,\qquad \text{where } x,y\in A.$$

(ii) An algebra A is said to be *associative* if its composition law is associative, i.e., if for all $x,y,z\in A$,

$$x(yz)=(xy)z.$$

(iii) A vector space E over the field $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ is said to be $\mathbb{Z}$-*graded* if one has chosen a family $(E^p,\ p\in\mathbb{Z})$ of vector subspaces of E such that

$$E=\bigoplus_{p\in\mathbb{Z}}E^p.$$

For each $p\in\mathbb{Z}$, an element $x\in E$ is said to be *homogeneous of degree* p if $x\in E^p$.

(iv) Let $E=\bigoplus_{p\in\mathbb{Z}}E^p$ and $F=\bigoplus_{p\in\mathbb{Z}}F^p$ be two $\mathbb{Z}$-graded vector spaces over the same field $\mathbb{K}$. A $\mathbb{K}$-linear map $f:E\to F$ is said to be *homogeneous of degree* d (with $d\in\mathbb{Z}$) if for each $p\in\mathbb{Z}$,

$$f(E^p)\subset F^{p+d}.$$

(v) An algebra A is said to be $\mathbb{Z}$-*graded* if $A=\bigoplus_{p\in\mathbb{Z}}A^p$ is $\mathbb{Z}$-graded as a vector space and if in addition, for all $p,q\in\mathbb{Z}$, $x\in A^p$ and $y\in A^q$,

$$xy\in A^{p+q}.$$

(vi) A $\mathbb{Z}$-graded algebra $A=\bigoplus_{p\in\mathbb{Z}}A^p$ is said to be $\mathbb{Z}_2$-*commutative* if for all $p,q\in\mathbb{Z}$, $x\in A^p$ and $y\in A^q$,

$$xy=(-1)^{pq}yx.$$

It is said to be $\mathbb{Z}_2$-*anticommutative* if for all $p,q\in\mathbb{Z}$, $x\in A^p$ and $y\in A^q$,

$$xy=-(-1)^{pq}yx.$$

4.1.2. Some properties and examples

(i) *Composition of homogeneous linear maps.* We consider three $\mathbb{Z}$-graded vector spaces, $E=\bigoplus_{p\in\mathbb{Z}}E^p$, $F=\bigoplus_{p\in\mathbb{Z}}F^p$ and $G=\bigoplus_{p\in\mathbb{Z}}G^p$, over the same field $\mathbb{K}$. Let $f:E\to F$ and $g:F\to G$ be two linear maps, with f homogeneous of degree d_1 and g homogeneous of degree d_2. Then $g\circ f:E\to G$ is homogeneous of degree d_1+d_2.

(ii) *The algebra of linear endomorphisms of a vector space.* Let E be a vector space and $\mathcal{L}(E,E)$ be the space of linear endomorphisms of E. We take as composition law on the latter space the usual composition of maps,

$$(f,g) \mapsto f \circ g, \quad \text{with} \quad f \circ g(x) = f(g(x)),\ x \in E.$$

With that composition law, $\mathcal{L}(E,E)$ is an associative algebra.

(iii) *The graded algebra of graded linear endomorphisms.* We now assume that $E = \bigoplus_{p\in\mathbb{Z}} E^p$ is a $\mathbb{Z}$-graded vector space. For each $d \in \mathbb{Z}$, let A^d be the vector subspace of $\mathcal{L}(E,E)$ whose elements are the linear endomorphisms $f : E \to E$ which are homogeneous of degree d, i.e., such that for all $p \in \mathbb{Z}$, $f(E^p) \subset E^{p+d}$. Let $A = \bigoplus_{d\in\mathbb{Z}} A^d$. By using property 4.1.2(i), we see that with the usual composition of maps as composition law, A is a $\mathbb{Z}$-graded associative algebra.

Let us use property 4.1.2(i) with $E = F = G$ in the following definition.

Definition 4.1.3. Let $E = \bigoplus_{p\in\mathbb{Z}} E^p$ be a $\mathbb{Z}$-graded vector space, and $f, g \in \mathcal{L}(E,E)$ be two homogeneous linear endomorphisms of E of degrees d_1 and d_2, respectively. The linear endomorphism $[f,g]$ of E defined by

$$[f,g] = f \circ g - (-1)^{d_1 d_2} g \circ f,$$

which, by 4.1.2(i), is homogeneous of degree $d_1 + d_2$, is called the *graded bracket* of f and g.

Definition 4.1.4. Let $A = \bigoplus_{p\in\mathbb{Z}} A^p$ be a $\mathbb{Z}$-graded algebra. Let $\theta : A \to A$ be a linear endomorphism of the graded vector space A. Let $d \in \mathbb{Z}$. The linear endomorphism θ is said to be a *derivation of degree* d of the graded algebra A if

(i) as a linear endomorphism of a graded vector space, θ is homogeneous of degree d;
(ii) for all $p \in \mathbb{Z}$, $x \in A^p$ and $y \in A$,

$$\theta(xy) = (\theta(x))y + (-1)^{dp} x(\theta(y)).$$

Remark 4.1.5. More generally, as shown by Koszul [27], for an algebra A equipped with an involutive automorphism, one can define two types of remarkable linear endomorphisms of A, the *derivations* and the *antiderivations*. When $A = \bigoplus_{p\in\mathbb{Z}} A^p$ is a $\mathbb{Z}$-graded algebra, and when the involutive automorphism used is that which maps each $x \in A^p$ to $(-1)^p x$, it turns out that all nonzero graded derivations are of even degree, that all nonzero graded antiderivations are of odd degree, and that both derivations and antiderivations can be defined as in Definition 4.1.4. For simplicity we have chosen to use the term *derivations* for both derivations and antiderivations.

4.1.6. Some properties of derivations. Let $A = \bigoplus_{p\in\mathbb{Z}} A^p$ be a $\mathbb{Z}$-graded algebra.

(i) *A derivation of degree* 0. For every $p \in \mathbb{Z}$ and $x \in A^p$, we set

$$\mu(x) = px.$$

The map μ, defined for homogeneous elements of A, can be extended in a unique way to a linear endomorphism of A, still denoted by μ. This endomorphism is a derivation of degree 0 of A.

(ii) *The graded bracket of two derivations.* Let $\theta_1 : A \to A$ and $\theta_2 : A \to A$ be two derivations of A, of degree d_1 and d_2, respectively. Their graded bracket (Definition 4.1.3)

$$[\theta_1, \theta_2] = \theta_1 \circ \theta_2 - (-1)^{d_1 d_2} \theta_2 \circ \theta_1$$

is a derivation of degree $d_1 + d_2$.

DEFINITION 4.1.7. A $\mathbb{Z}$*-graded Lie algebra* is a $\mathbb{Z}$-graded algebra $A = \bigoplus_{p\in\mathbb{Z}} A^p$ (in the sense of 4.1.1(v)), whose composition law, often denoted by $(x, y) \mapsto [x, y]$ and called the *graded bracket*, has the following two properties:

(i) it is $\mathbb{Z}_2$-anticommutative in the sense of 4.1.1(vi), i.e., for all $p, q \in \mathbb{Z}$, $P \in A^p$ and $Q \in A^q$,

$$[P, Q] = -(-1)^{pq}[Q, P];$$

(ii) it satisfies the $\mathbb{Z}$*-graded Jacobi identity*, i.e., for $p, q, r \in \mathbb{Z}$, $P \in A^p$, $Q \in A^q$ and $R \in A^r$,

$$(-1)^{pr}[P, [Q, R]] + (-1)^{qp}[Q, [R, P]] + (-1)^{rq}[R, [P, Q]] = 0.$$

4.1.8. Examples and remarks

(i) *Lie algebras and* $\mathbb{Z}$*-graded Lie algebras.* A $\mathbb{Z}$-graded Lie algebra $A = \bigoplus_{p\in\mathbb{Z}} A^p$ is not a Lie algebra in the usual sense, unless $A^p = \{0\}$ for all $p \neq 0$. However, its subspace A^0 of homogeneous elements of degree 0 is a Lie algebra in the usual sense: it is stable under the bracket operation and when restricted to elements in A^0, the bracket is skew-symmetric and satisfies the usual Jacobi identity.

(ii) *The graded Lie algebra associated to a graded associative algebra.* Let $A = \bigoplus_{p\in\mathbb{Z}} A^p$ be a $\mathbb{Z}$-graded associative algebra, whose composition law is denoted by $(P, Q) \mapsto PQ$. We define another composition law, denoted by $(P, Q) \mapsto [P, Q]$ and called the *graded commutator*; we first define it for homogeneous elements in A by setting, for all $p, q \in \mathbb{Z}$, $P \in A^p$ and $Q \in A^q$,

$$[P, Q] = PQ - (-1)^{pq} QP;$$

then we extend the definition to all pairs of elements in A by bilinearity. The reader will easily verify that with this composition law, A is a graded Lie algebra. When $A^p = \{0\}$ for all $p \neq 0$, we recover the well known way in which one can associate a Lie algebra to any associative algebra.

(iii) *The graded Lie algebra of graded endomorphisms.* Let $E = \bigoplus_{p\in\mathbb{Z}} E^p$ be a graded vector space. For each $p \in \mathbb{Z}$, let $A^p \subset \mathcal{L}(E, E)$ be the space of linear endomorphisms of E which are homogeneous of degree p, and let $A = \bigoplus_{p\in\mathbb{Z}} A^p$. As we have seen in 4.1.2(iii), when equipped with the composition of maps as composition law, A is a $\mathbb{Z}$-graded associative algebra. Let us define another composition law on A, called the *graded commutator*; we first define it for homogeneous elements in A by setting, for all $p, q \in \mathbb{Z}$, $P \in A^p$ and $Q \in A^q$,

$$[P, Q] = PQ - (-1)^{pq} QP;$$

then we extend the definition by bilinearity. By using 4.1.8(ii), we see that A with this composition law is a $\mathbb{Z}$-graded Lie algebra.

(iv) *Various interpretations of the graded Jacobi identity.* Let $A = \bigoplus_{p\in\mathbb{Z}} A^p$ be a $\mathbb{Z}$-graded Lie algebra. The $\mathbb{Z}$-graded Jacobi identity indicated in Definition 4.1.7 can be recast into other forms, which better indicate its meaning. Let us set, for all $P, Q \in A$,

$$\mathrm{ad}_P Q = [P, Q].$$

For each $p \in \mathbb{Z}$ and $P \in A^p$, $\mathrm{ad}_P : A \to A$ is a graded endomorphism of A, homogeneous of degree p. By taking into account the $\mathbb{Z}_2$-anticommutativity of the bracket, the reader will easily see that the graded Jacobi identity can be written in the following two forms:

First form. For all $p, q, r \in \mathbb{Z}$, $P \in A^p$, $Q \in A^q$ and $R \in A^r$,

$$\mathrm{ad}_P([Q, R]) = [\mathrm{ad}_P Q, R] + (-1)^{pq}[Q, \mathrm{ad}_P R].$$

This equality means that for all $p \in \mathbb{Z}$ and $P \in A^p$, the linear endomorphism $\mathrm{ad}_P : A \to A$ is a derivation of degree p of the graded Lie algebra A, in the sense of 4.1.4.

Second form. For all $p, q, r \in \mathbb{Z}$, $P \in A^p$, $Q \in A^q$ and $R \in A^r$,

$$\mathrm{ad}_{[P,Q]} R = \mathrm{ad}_P \circ \mathrm{ad}_Q R - (-1)^{pq}\,\mathrm{ad}_Q \circ \mathrm{ad}_P R = [\mathrm{ad}_P, \mathrm{ad}_Q]R.$$

This equality means that for all $p, q \in \mathbb{Z}$, $P \in A^p$ and $Q \in A^q$, the endomorphism $\mathrm{ad}_{[P,Q]} : A \to A$ is the graded bracket (in the sense of 4.1.3) of the two endomorphisms $\mathrm{ad}_P : A \to A$ and $\mathrm{ad}_Q : A \to A$. In other words, the map $P \mapsto \mathrm{ad}_P$ is a $\mathbb{Z}$-graded Lie algebra homomorphism from the $\mathbb{Z}$-graded Lie algebra A into the $\mathbb{Z}$-graded Lie algebra of sums of linear homogeneous endomorphisms of A, with the graded bracket as composition law (example 4.1.8(iii)).

When $A^p = \{0\}$ for all $p \neq 0$, we recover the well known interpretations of the usual Jacobi identity.

4.2. Exterior powers of a vector bundle and of its dual. In what follows all the vector bundles will be assumed to be locally trivial and of finite rank; therefore we will write simply *vector bundle* for *locally trivial vector bundle*.

4.2.1. The dual of a vector bundle Let (E, τ, M) be a vector bundle on the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We will denote its *dual bundle* by (E^*, π, M). Let us recall that it is a vector bundle over the same base manifold M, whose fibre $E^*_x = \pi^{-1}(x)$ over each point $x \in M$ is the dual vector space of the corresponding fibre $E_x = \tau^{-1}(x)$ of (E, τ, M), i.e., the space of linear forms on E_x (i.e., linear functions defined on E_x and taking their values in the field $\mathbb{K}$).

For each $x \in M$, the duality pairing $E^*_x \times E_x \to \mathbb{K}$ will be denoted by

$$(\eta, v) \mapsto \langle \eta, v\rangle.$$

4.2.2. The exterior powers of a vector bundle Let (E, τ, M) be a vector bundle of rank k. For each integer $p > 0$, we will denote by $(\bigwedge^p E, \tau, M)$ the pth exterior power of (E, τ, M). It is a vector bundle over M whose fibre $\bigwedge^p E_x$, over each point $x \in M$, is the pth exterior power of the corresponding fibre $E_x = \tau^{-1}(x)$ of (E, τ, M). We recall that $\bigwedge^p E_x$ can be canonically identified with the vector space of p-multilinear skew-symmetric forms on the dual E^*_x of E_x.

Similarly, for any integer $p > 0$, we will denote by $(\bigwedge^p E^*, \pi, M)$ the pth exterior power of the bundle (E^*, π, M), dual of (E, τ, M).

For $p = 1$, $(\bigwedge^1 E, \tau, M)$ is simply the bundle (E, τ, M), and similarly $(\bigwedge^1 E^*, \pi, M)$ is simply the bundle (E^*, π, M). For p greater than the rank k of (E, τ, M), $(\bigwedge^p E, \tau, M)$ and $(\bigwedge^p E^*, \pi, M)$ are the trivial bundle over M, $(M \times \{0\}, p_1, M)$, whose fibres are zero-dimensional ($p_1 : M \times \{0\} \to M$ being the projection onto the first factor).

For $p = 0$, we set $(\bigwedge^0 E, \tau, M) = (\bigwedge^0 E^*, \pi, M) = (M \times \mathbb{K}, p_1, M)$, where $p_1 : M \times \mathbb{K} \to M$ is the projection onto the first factor.

Finally, for $p < 0$, $(\bigwedge^p E, \tau, M)$ and $(\bigwedge^p E^*, \pi, M)$ are defined to be the trivial bundle over M, $(M \times \{0\}, p_1, M)$. With these conventions, $(\bigwedge^p E, \tau, M)$ and $(\bigwedge^p E^*, \pi, M)$ are defined for all $p \in \mathbb{Z}$.

4.2.3. Operations in the graded vector spaces $\bigwedge E_x$ and $\bigwedge E_x^*$ Let (E, τ, M) be a vector bundle of rank k, (E^*, π, M) its dual and, for each $p \in \mathbb{Z}$, $(\bigwedge^p E, \tau, M)$ and $(\bigwedge^p E^*, \pi, M)$ their pth exterior powers. We recall in this section some operations which can be made, for each point $x \in M$, in the vector spaces $\bigwedge^p E_x$ and $\bigwedge^p E_x^*$.

For each $x \in M$, let us consider the $\mathbb{Z}$-graded vector spaces

$$\bigwedge E_x = \bigoplus_{p \in \mathbb{Z}} \bigwedge^p E_x \quad \text{and} \quad \bigwedge E_x^* = \bigoplus_{p \in \mathbb{Z}} \bigwedge^p E_x^*.$$

We will say that elements in $\bigwedge E_x^*$ are (multilinear) *forms* at x, and that elements in $\bigwedge E_x$ are *multivectors* at x.

(i) *The exterior product*. Let us recall that for each $x \in M$, $p, q \in \mathbb{Z}$, $P \in \bigwedge^p E_x$ and $Q \in \bigwedge^q E_x$, there exists $P \wedge Q \in \bigwedge^{p+q} E_x$, called the *exterior product* of P and Q, defined by the following formulae.

- If $p < 0$, then $P = 0$, therefore, for any $Q \in \bigwedge^q E_x$, $P \wedge Q = 0$. Similarly, if $q < 0$, then $Q = 0$, therefore, for any $P \in \bigwedge^p E_x$, $P \wedge Q = 0$.
- If $p = 0$, then P is a scalar ($P \in \mathbb{K}$), and therefore, for any $Q \in \bigwedge^q E_x$, $P \wedge Q = PQ$, the usual product of Q by the scalar P. Similarly, for $q = 0$, then Q is a scalar ($Q \in \mathbb{K}$), and therefore, for any $P \in \bigwedge^p E_x$, we have $P \wedge Q = QP$, the usual product of P by the scalar Q.
- If $p \geq 1$ and $q \geq 1$, $P \wedge Q$, considered as a $(p+q)$-multilinear form on E_x^*, is given by the formula, where $\eta_1, \dots, \eta_{p+q} \in E_x^*$,

$$P \wedge Q(\eta_1, \dots, \eta_{p+q}) = \sum_{\sigma \in \mathcal{S}_{(p,q)}} \varepsilon(\sigma) P(\eta_{\sigma(1)}, \dots, \eta_{\sigma(p)}) Q(\eta_{\sigma(p+1)}, \dots, \eta_{\sigma(p+q)}).$$

We have denoted by $\mathcal{S}_{(p,q)}$ the set of permutations σ of $\{1, \dots, p+q\}$ which satisfy

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p) \quad \text{and} \quad \sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q),$$

and set

$$\varepsilon(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even,} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

Similarly, let us recall that for all $x \in M$, $p, q \in \mathbb{Z}$, $\xi \in \bigwedge^p E_x^*$ and $\eta \in \bigwedge^q E_x^*$, there exists $\xi \wedge \eta \in \bigwedge^{p+q} E_x^*$, called the *exterior product* of ξ and η. It is defined by the formulae given above, the only change being the exchange of the roles of E_x and E_x^*.

The exterior product is associative and $\mathbb{Z}_2$-commutative: for all $x \in M$, $p, q, r \in \mathbb{Z}$, $P \in \bigwedge^p E_x$, $Q \in \bigwedge^q E_x$ and $R \in \bigwedge^r E_x$,

$$P \wedge (Q \wedge R) = (P \wedge Q) \wedge R, \qquad Q \wedge P = (-1)^{pq} P \wedge Q,$$

and similarly, for $\xi \in \bigwedge^p E_x^*$, $\eta \in \bigwedge^q E_x^*$ and $\zeta \in \bigwedge^r E_x^*$,

$$\xi \wedge (\eta \wedge \zeta) = (\xi \wedge \eta) \wedge \zeta, \qquad \eta \wedge \xi = (-1)^{pq} \xi \wedge \eta.$$

For all $x \in M$, the exterior product extends, by bilinearity, as a composition law in each of the graded vector spaces $\bigwedge E_x$ and $\bigwedge E_x^*$. With these composition laws, these vector spaces become $\mathbb{Z}$-graded associative and $\mathbb{Z}_2$-commutative algebras.

(ii) *The interior product of a form by a vector.* Let us recall that for each $x \in M$, $v \in E_x$, $p \in \mathbb{Z}$, $\eta \in \bigwedge^p E_x^*$, there exists $i(v)\eta \in \bigwedge^{p-1} E_x^*$, called the *interior product* of η by v, defined by the following formulae.

- For $p \leq 0$, $i(v)\eta = 0$, since $\bigwedge^{p-1} E_x^* = \{0\}$.
- For $p = 1$,

$$i(v)\eta = \langle \eta, v \rangle \in \mathbb{K}.$$

- For $p > 1$, $i(v)\eta$ is the $(p-1)$-multilinear form on E_x such that, for all $v_1, \dots, v_{p-1} \in E_x$,

$$i(v)\eta(v_1, \dots, v_{p-1}) = \eta(v, v_1, \dots, v_{p-1}).$$

For each $x \in M$ and $v \in E_x$, the map $\eta \mapsto i(v)\eta$ extends, by linearity, as a graded endomorphism of degree -1 of the graded vector space $\bigwedge E_x^*$. Moreover, that endomorphism is in fact a derivation of degree -1 of the exterior algebra of E_x^*, i.e., for all $p, q \in \mathbb{Z}$, $\zeta \in \bigwedge^p E_x^*$, $\eta \in \bigwedge^q E_x^*$,

$$i(v)(\zeta \wedge \eta) = (i(v)\zeta) \wedge \eta + (-1)^p \zeta \wedge (i(v)\eta).$$

(iii) *The pairing between $\bigwedge E_x$ and $\bigwedge E_x^*$.* Let $x \in M$, $p, q \in \mathbb{Z}$, $\eta \in \bigwedge^p E_x^*$ and $v \in \bigwedge^q E_x$. We set

$$\langle \eta, v \rangle = \begin{cases} 0 & \text{if } p \neq q \text{ or } p < 0 \text{ or } q < 0, \\ \eta v & \text{if } p = q = 0. \end{cases}$$

In order to define $\langle \eta, v \rangle$ when $p = q \geq 1$, let us first assume that η and v are decomposable, i.e., that they can be written as

$$\eta = \eta_1 \wedge \cdots \wedge \eta_p, \qquad v = v_1 \wedge \cdots \wedge v_p,$$

where $\eta_i \in E_x^*$, $v_j \in E_x$, $1 \leq i, j \leq p$. Then we set

$$\langle \eta, v \rangle = \det(\langle \eta_i, v_j \rangle).$$

One may see that $\langle \eta, v \rangle$ depends only on η and v, not on the way in which they are expressed as exterior products of elements of degree 1. The map $(\eta, v) \mapsto \langle \eta, v \rangle$ extends in a unique way to a bilinear map

$$\bigwedge E_x^* \times \bigwedge E_x \to \mathbb{K}, \quad \text{still denoted by} \quad (\eta, v) \mapsto \langle \eta, v \rangle,$$

called the *pairing*. That map allows us to consider the graded vector spaces $\bigwedge E_x^*$ and $\bigwedge E_x$ as dual to each other.

Let $\eta \in \bigwedge^p E_x^*$ and $v_1, \dots, v_p$ be elements of E_x. The pairing $\langle \eta, v_1 \wedge \cdots \wedge v_p \rangle$ is related, in a very simple way, to the value taken by η, considered as a p-multilinear form on E_x, on the set $(v_1, \dots, v_p)$. We have

$$\langle \eta, v_1 \wedge \cdots \wedge v_p \rangle = \eta(v_1, \dots, v_p).$$

(iv) *The interior product of a form by a multivector.* For each $x \in M$ and $v \in E_x$, we have defined in 4.2.3(ii) the interior product $i(v)$ as a derivation of degree -1 of the exterior algebra $\bigwedge E_x^*$ of forms at x. Let us now define, for each multivector $P \in \bigwedge E_x$, the interior product $i(P)$. Let us first assume that P is homogeneous of degree p, i.e., $P \in \bigwedge^p E_x$.

- For $p < 0$, $\bigwedge^p E_x = \{0\}$, therefore $i(P) = 0$.
- For $p = 0$, $\bigwedge^0 E_x = \mathbb{K}$, therefore P is a scalar and we set, for all $\eta \in \bigwedge E_x^*$,
$$i(P)\eta = P\eta.$$
- For $p \geq 1$ and $P \in \bigwedge^p E_x$ decomposable, i.e.,
$$P = P_1 \wedge \cdots \wedge P_p, \quad \text{with} \quad P_i \in E_x,\ 1 \leq i \leq p,$$
we set
$$i(P_1 \wedge \cdots \wedge P_p) = i(P_1) \circ \cdots \circ i(P_p).$$
We see easily that $i(P)$ depends only on P, not on the way in which it is expressed as an exterior product of elements of degree 1.
- We extend by linearity the definition of $i(P)$ to all $P \in \bigwedge^p E_x$, and we see that $i(P)$ is a graded endomorphism of degree $-p$ of the graded vector space $\bigwedge E_x^*$. Observe that for $p \neq 1$, $i(P)$ is not in general a derivation of the exterior algebra $\bigwedge E_x^*$.

Finally, we extend the definition of $i(P)$ by linearity to all elements $P \in \bigwedge E_x$.

(v) *The interior product by an exterior product.* It is easy to see that for all $P, Q \in \bigwedge E_x$,

$$i(P \wedge Q) = i(P) \circ i(Q).$$

(vi) *Interior product and pairing.* For $p \in \mathbb{Z}$, $\eta \in \bigwedge^p E_x^*$ and $P \in \bigwedge^p E_x$, we have

$$i(P)\eta = (-1)^{(p-1)p/2} \langle \eta, P \rangle.$$

More generally, for $p, q \in \mathbb{Z}$, $P \in \bigwedge^p(E_x)$, $Q \in \bigwedge^q(E_x)$ and $\eta \in \bigwedge^{p+q}(E_x^*)$,

$$\big\langle i(P)\eta, Q \big\rangle = (-1)^{(p-1)p/2} \langle \eta, P \wedge Q \rangle.$$

This formula shows that the interior product by $P \in \bigwedge^p E_x$ is $(-1)^{(p-1)p/2}$ times the transpose, with respect to the pairing, of the exterior product by P on the left.

4.2.4. The exterior algebra of sections Let (E, τ, M) be a vector bundle of rank k over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, over a smooth manifold M, (E^*, π, M) be its dual bundle and, for each integer $p \geq 1$, let $(\bigwedge^p E, \tau, M)$ and $(\bigwedge^p E^*, \pi, M)$ be their respective pth exterior powers.

For each $p \in \mathbb{Z}$, we will denote by $A^p(M,E)$ the space of smooth sections of $(\bigwedge^p E, \tau, M)$, i.e., the space of smooth maps $Z : M \to \bigwedge^p E$ which satisfy

$$\tau \circ Z = \mathrm{id}_M .$$

Similarly, for each $p \in \mathbb{Z}$, we will denote by $\Omega^p(M,E)$ the space of smooth sections of the vector bundle $(\bigwedge^p E^*, \pi, M)$, i.e., the space of smooth maps $\eta : M \to \bigwedge^p E^*$ which satisfy

$$\pi \circ \eta = \mathrm{id}_M .$$

Let us observe that $\Omega^p(M,E) = A^p(M,E^*)$.

We will denote by $A(M,E)$ and $\Omega(M,E)$ the direct sums

$$A(M,E) = \bigoplus_{p\in\mathbb{Z}} A^p(M,E), \qquad \Omega(M,E) = \bigoplus_{p\in\mathbb{Z}} \Omega^p(M,E).$$

These direct sums, taken over all $p \in \mathbb{Z}$, are in fact taken over all integers p which satisfy $0 \le p \le k$, where k is the rank of the vector bundle (E,τ,M), since we have $A^p(M,E) = \Omega^p(M,E) = \{0\}$ for $p < 0$ as well as for $p > k$.

For $p = 0$, $A^0(M,E)$ and $\Omega^0(M,E)$ both coincide with the space $C^\infty(M,\mathbb{K})$ of smooth functions defined on M which take their values in the field $\mathbb{K}$.

Operations such as exterior product, interior product and pairing, defined for each point $x \in M$ in 4.2.3, can be extended to elements in $A(M,E)$ and $\Omega(M,E)$.

(i) *The exterior product of two sections.* For example, the exterior product of two sections $P, Q \in A(M,E)$ is the section

$$x \in M, \qquad x \mapsto (P \wedge Q)(x) = P(x) \wedge Q(x).$$

The exterior product of two sections $\eta, \zeta \in \Omega(M,E)$ is similarly defined.

With the exterior product as composition law, $A(M,E)$ and $\Omega(M,E)$ are $\mathbb{Z}$-graded associative and $\mathbb{Z}_2$-commutative algebras, called the *algebra of multivectors* and the *algebra of forms* associated to the vector bundle (E,τ,M). Their subspaces $A^0(M,E)$ and $\Omega^0(M,E)$ of homogeneous elements of degree 0 both coincide with the usual algebra $C^\infty(M,\mathbb{K})$ of smooth $\mathbb{K}$-valued functions on M, with the usual product of functions as composition law. We observe that $A(M,E)$ and $\Omega(M,E)$ are $\mathbb{Z}$-graded modules over the ring of functions $C^\infty(M,\mathbb{K})$.

(ii) *The interior product by a section of $A(M,E)$.* For each $P \in A(M,E)$, the *interior product* $i(P)$ is an endomorphism of the graded vector space $\Omega(M,E)$. If $p \in \mathbb{Z}$ and $P \in A^p(M,E)$, the endomorphism $i(P)$ is homogeneous of degree $-p$. For $p = 1$, $i(P)$ is a derivation of degree -1 of the algebra $\Omega(M,E)$.

(iii) *The pairing between $A(M,E)$ and $\Omega(M,E)$.* The *pairing*

$$(\eta, P) \mapsto \langle \eta, P \rangle, \qquad \eta \in \Omega(M,E),\ P \in A(M,E),$$

is a $C^\infty(M,\mathbb{K})$-bilinear map, defined on $\Omega(M,E) \times A(M,E)$, which takes its values in $C^\infty(M,\mathbb{K})$.

5. Exterior powers of a Lie algebroid and of its dual

We now consider a Lie algebroid (E, τ, M, ρ) over a smooth manifold M. We denote by (E^*, π, M) its dual vector bundle, and use all the notations defined in Section 4. We will assume that the base field $\mathbb{K}$ is $\mathbb{R}$, but most results remain valid for $\mathbb{K} = \mathbb{C}$. We will prove that differential operators such as the Lie derivative and the exterior derivative, which are well known for sections of the exterior powers of a tangent bundle and of its dual, still exist in this more general setting.

5.1. Lie derivatives with respect to sections of a Lie algebroid. We prove in this section that for each smooth section V of the Lie algebroid (E, τ, M, ρ), there exists a derivation of degree 0 of the exterior algebra $\Omega(M, E)$, called the *Lie derivative* with respect to V and denoted by $\mathcal{L}_\rho(V)$. When the Lie algebroid is the tangent bundle $(TM, \tau_M, M, \mathrm{id}_{TM})$, we will recover the usual Lie derivative of differential forms with respect to a vector field.

Proposition 5.1.1. *Let (E, τ, M, ρ) be a Lie algebroid on a smooth manifold M. For each smooth section $V \in A^1(M, E)$ of the vector bundle (E, τ, M), there exists a unique graded endomorphism of degree 0 of the graded algebra of exterior forms $\Omega(M, E)$, called the Lie derivative with respect to V and denoted by $\mathcal{L}_\rho(V)$, which has the following properties:*

(i) *For a smooth function $f \in \Omega^0(M, E) = C^\infty(M, \mathbb{R})$,*

$$\mathcal{L}_\rho(V)f = i(\rho \circ V)df = \mathcal{L}(\rho \circ V)f,$$

where $\mathcal{L}(\rho \circ V)$ denotes the usual Lie derivative with respect to the vector field $\rho \circ V$.

(ii) *For a form $\eta \in \Omega^p(M, E)$ of degree $p > 0$, $\mathcal{L}_\rho(V)\eta$ is the form defined by the formula, where $V_1, \ldots, V_p$ are smooth sections of (E, τ, M),*

$$\begin{aligned}(\mathcal{L}_\rho(V)\eta)(V_1, \ldots, V_p) &= \mathcal{L}_\rho(V)(\eta(V_1, \ldots, V_p)) \\ &\quad - \sum_{i=1}^{p} \eta(V_1, \ldots, V_{i-1}, \{V, V_i\}, V_{i+1}, \ldots, V_p).\end{aligned}$$

Proof. Clearly (i) defines a function $\mathcal{L}_\rho(V)f \in \Omega^0(M, E) = C^\infty(M, \mathbb{R})$. We see immediately that for $f, g \in C^\infty(M, \mathbb{R})$,

$$\mathcal{L}_\rho(V)(fg) = (\mathcal{L}_\rho(V)f)g + f(\mathcal{L}_\rho(V)g). \qquad (*)$$

Now (ii) defines a map $(V_1, \ldots V_p) \mapsto (\mathcal{L}_\rho(V)\eta)(V_1, \ldots, V_p)$ on $(A^1(M, E))^p$, with values in $C^\infty(M, \mathbb{R})$. In order to prove that this map defines an element $\mathcal{L}_\rho(V)\eta$ in $\Omega^p(M, E)$, it is enough to prove that it is skew-symmetric and $C^\infty(M, \mathbb{R})$-linear in each argument. The skew-symmetry and the $\mathbb{R}$-linearity in each argument are easily verified. It remains to prove that for each function $f \in C^\infty(M, \mathbb{R})$,

$$(\mathcal{L}_\rho(V)\eta)(fV_1, V_2, \ldots, V_p) = f(\mathcal{L}_\rho(V)\eta)(V_1, V_2, \ldots, V_p). \qquad (**)$$

We have

$$(\mathcal{L}_\rho(V)\eta)(fV_1, V_2, \dots, V_p) = \mathcal{L}_\rho(V)(\eta(fV_1, V_2, \dots, V_p)) - \eta(\{V, fV_1\}, V_2, \dots, V_p) - \sum_{i=2}^{p} \eta(fV_1, V_2, \dots, V_{i-1}, \{V, V_i\}, V_{i+1}, \dots, V_p).$$

By using $(*)$, we may write

$$\begin{aligned}\mathcal{L}_\rho(V)(\eta(fV_1, V_2, \dots, V_p)) &= \mathcal{L}_\rho(V)(f\eta(V_1, V_2, \dots, V_p))\\ &= (\mathcal{L}_\rho(V)f)\eta(V_1, V_2, \dots, V_p) + f\mathcal{L}_\rho(V)(\eta(V_1, V_2, \dots, V_p)).\end{aligned}$$

Using the property of the anchor, we also have

$$\{V, fV_1\} = (i(\rho \circ V)df)V_1 + f\{V, V_1\} = (\mathcal{L}_\rho(V)f)V_1 + f\{V, V_1\}.$$

Equality $(**)$ follows immediately.

The endomorphism $\mathcal{L}_\rho(V)$, defined on the subspaces of homogeneous forms, can then be extended, in a unique way, to $\Omega(M, E)$, by imposing the $\mathbb{R}$-linearity of the map $\eta \mapsto \mathcal{L}_\rho(V)\eta$. ■

Let us now introduce the $\Omega(M, E)$-valued exterior derivative of a function. In the next section, that definition will be extended to all elements in $\Omega(M, E)$.

Definition 5.1.2. Let (E, τ, M, ρ) be a Lie algebroid on a smooth manifold M. For each function $f \in \Omega^0(M, E) = C^\infty(M, \mathbb{R})$, the $\Omega(M, E)$-*valued exterior derivative* of f, denoted by $d_\rho f$, is the unique element in $\Omega^1(M, E)$ such that, for each section $V \in A^1(M, E)$,

$$\langle d_\rho f, V\rangle = \langle df, \rho \circ V\rangle.$$

Remark 5.1.3. Let us observe that the transpose of the anchor $\rho : E \to TM$ is a vector bundle map ${}^t\rho : T^*M \to E^*$. By composition of that map with the usual differential of functions, we obtain the $\Omega(M, E)$-valued exterior differential d_ρ. We have indeed

$$d_\rho f = {}^t\rho \circ df.$$

Proposition 5.1.4. *Under the assumptions of Proposition 5.1.1, the Lie derivative has the following properties:*

1. *For all* $V \in A^1(M, E)$ *and* $f \in C^\infty(M, \mathbb{R})$,
$$\mathcal{L}_\rho(V)(d_\rho f) = d_\rho(\mathcal{L}_\rho(V)f).$$
2. *For all* $V, W \in A^1(M, E)$ *and* $\eta \in \Omega(M, E)$,
$$i(\{V, W\})\eta = (\mathcal{L}_\rho(V) \circ i(W) - i(W) \circ \mathcal{L}_\rho(V))\eta.$$
3. *For each* $V \in A^1(M, E)$, $\mathcal{L}_\rho(V)$ *is a derivation of degree* 0 *of the exterior algebra* $\Omega(M, E)$. *That means that for all* $\eta, \zeta \in \Omega(M, E)$,
$$\mathcal{L}_\rho(V)(\eta \wedge \zeta) = (\mathcal{L}_\rho(V)\eta) \wedge \zeta + \eta \wedge (\mathcal{L}_\rho(V)\zeta).$$
4. *For all* $V, W \in A^1(M, E)$ *and* $\eta \in \Omega(M, E)$,
$$\mathcal{L}_\rho(\{V, W\})\eta = (\mathcal{L}_\rho(V) \circ \mathcal{L}_\rho(W) - \mathcal{L}_\rho(W) \circ \mathcal{L}_\rho(V))\eta.$$
5. *For all* $V \in A^1(M, E)$, $f \in C^\infty(M, \mathbb{R})$ *and* $\eta \in \Omega(M, E)$,
$$\mathcal{L}_\rho(fV)\eta = f\mathcal{L}_\rho(V)\eta + d_\rho f \wedge i(V)\eta.$$

Proof. 1. Let $W \in A^1(M,E)$. Then

$$\begin{aligned}\langle \mathcal{L}_\rho(V)(d_\rho f), W\rangle &= \mathcal{L}_\rho(V)\langle d_\rho f, W\rangle - \langle d_\rho f, \{V,W\}\rangle\\ &= \mathcal{L}(\rho\circ V)\circ\mathcal{L}(\rho\circ W)f - \mathcal{L}(\rho\circ\{V,W\})f\\ &= \mathcal{L}(\rho\circ W)\circ\mathcal{L}(\rho\circ V)f = \langle d_\rho(\mathcal{L}_\rho(V)f), W\rangle,\end{aligned}$$

so Property 1 is proven.

2. Let $V, W \in A^1(M,E)$, $\eta \in \Omega^p(M,E)$ and $V_1,\dots,V_{p-1} \in A^1(M,E)$. We may write

$$\begin{aligned}(\mathcal{L}_\rho(V)\circ i(W)\eta)(V_1,\dots,V_{p-1}) &= \mathcal{L}_\rho(V)(\eta(W,V_1,\dots,V_{p-1}))\\ &\qquad - \sum_{k=1}^{p-1}\eta(W,V_1,\dots,V_{k-1},\{V,V_k\},V_{k+1},\dots,V_{p-1})\\ &= (\mathcal{L}_\rho(V)\eta)(W,V_1,\dots,V_{p-1}) + \eta(\{V,W\},V_1,\dots,V_{p-1})\\ &= \bigl((i(W)\circ\mathcal{L}_\rho(V) + i(\{V,W\}))\eta\bigr)(V_1,\dots,V_{p-1}),\end{aligned}$$

so Property 2 is proven.

3. Let $V \in A^1(M,E)$, $\eta \in \Omega^p(M,E)$ and $\zeta \in \Omega^q(M,E)$. For $p<0$, as well as for $q<0$, both sides of the equality stated in Property 3 vanish, so the equality is trivially satisfied. For $p=q=0$, the equality is also satisfied, as shown by equality $(*)$ in the proof of Proposition 5.1.1. We still have to prove the equality for $p>0$ and (or) $q>0$. We will do that by induction on $p+q$. Let $r\geq 1$ be an integer such that the equality stated in Property 3 holds for $p+q\leq r-1$. Such an integer exists, for example $r=1$. We now assume that $p\geq 0$ and $q\geq 0$ are such that $p+q=r$. Let $W\in A^1(M,E)$. By using Property 2, we may write

$$\begin{aligned}i(W)\circ\mathcal{L}_\rho(V)(\eta\wedge\zeta) &= \mathcal{L}_\rho(V)\circ i(W)(\eta\wedge\zeta) - i(\{V,W\})(\eta\wedge\zeta)\\ &= \mathcal{L}_\rho(V)(i(W)\eta\wedge\zeta + (-1)^p\eta\wedge i(W)\zeta)\\ &\qquad - i(\{V,W\})\eta\wedge\zeta - (-1)^p\eta\wedge i(\{V,W\})\zeta.\end{aligned}$$

Since $i(W)\eta\in\Omega^{p-1}(M,E)$ and $i(W)\zeta\in\Omega^{q-1}(M,E)$, the induction assumption allows us to use Property 3 to transform the first terms of the right hand side. We obtain

$$\begin{aligned}i(W)\circ\mathcal{L}_\rho(V)(\eta\wedge\zeta) &= (\mathcal{L}_\rho(V)\circ i(W)\eta)\wedge\zeta + i(W)\eta\wedge\mathcal{L}_\rho(V)\zeta\\ &\quad + (-1)^p(\mathcal{L}_\rho(V)\eta)\wedge i(W)\zeta + (-1)^p\eta\wedge(\mathcal{L}_\rho(V)\circ i(W)\zeta)\\ &\quad - i(\{V,W\})\eta\wedge\zeta - (-1)^p\eta\wedge i(\{V,W\})\zeta.\end{aligned}$$

By rearranging the terms, we obtain

$$\begin{aligned}i(W)\circ\mathcal{L}_\rho(V)(\eta\wedge\zeta) &= \bigl(\mathcal{L}_\rho(V)\circ i(W)\eta - i(\{V,W\})\eta\bigr)\wedge\zeta\\ &\quad + (-1)^p\eta\wedge\bigl(\mathcal{L}_\rho(V)\circ i(W)\zeta - i(\{V,W\})\zeta\bigr)\\ &\quad + i(W)\eta\wedge\mathcal{L}_\rho(V)\zeta + (-1)^p(\mathcal{L}_\rho(V)\eta)\wedge i(W)\zeta.\end{aligned}$$

By using again Property 2 we get

$$\begin{aligned}i(W)\circ\mathcal{L}_\rho(V)(\eta\wedge\zeta) &= (i(W)\circ\mathcal{L}_\rho(V)\eta)\wedge\zeta + (-1)^p\eta\wedge(i(W)\circ\mathcal{L}_\rho(V)\zeta)\\ &\quad + i(W)\eta\wedge\mathcal{L}_\rho(V)\zeta + (-1)^p\mathcal{L}_\rho(V)\eta\wedge i(W)\zeta\\ &= i(W)\bigl(\mathcal{L}_\rho(V)\eta\wedge\zeta + \eta\wedge\mathcal{L}_\rho(V)\zeta\bigr).\end{aligned}$$

Since the last equality holds for all $W \in A^1(M,E)$, it follows that Property 3 holds for $\eta \in \Omega^p(M,E)$ and $\zeta \in \Omega^q(M,E)$ with $p,q \geq 0$ and $p+q=r$. We have thus proven by induction that Property 3 holds for all $p,q \in \mathbb{Z}$, $\eta \in \Omega^p(M,E)$, $\zeta \in \Omega^q(M,E)$. The same equality holds, by bilinearity, for all $\eta,\zeta \in \Omega(M,E)$.

4. Let $V,W \in A^1(M,E)$. Then $\{V,W\} \in A^1(M,E)$ and, by Property 3, $\mathcal{L}_\rho(V)$, $\mathcal{L}_\rho(W)$ and $\mathcal{L}_\rho(\{V,W\})$ are derivations of degree 0 of the graded algebra $\Omega(M,E)$. By 4.1.6(ii), the graded bracket

$$[\mathcal{L}_\rho(V),\mathcal{L}_\rho(W)] = \mathcal{L}_\rho(V)\circ\mathcal{L}_\rho(W) - \mathcal{L}_\rho(W)\circ\mathcal{L}_\rho(V)$$

is also a derivation of degree 0 of $\Omega(M,E)$. Property 4 means that the derivations $\mathcal{L}_\rho(\{V,W\})$ and $[\mathcal{L}_\rho(V),\mathcal{L}_\rho(W)]$ are equal. In order to prove that equality, it is enough to prove that it holds true for $\eta \in \Omega^0(M,E)$ and for $\eta \in \Omega^1(M,E)$, since the graded algebra $\Omega(M,E)$ is generated by its homogeneous elements of degrees 0 and 1.

Let $f \in \Omega^0(M,E) = C^\infty(M,\mathbb{R})$. We have

$$\begin{aligned}\mathcal{L}_\rho(\{V,W\})f &= \mathcal{L}(\rho\circ\{V,W\})f = \mathcal{L}([\rho\circ V,\rho\circ W])f\\ &= [\mathcal{L}(\rho\circ V),\mathcal{L}(\rho\circ W)]f = [\mathcal{L}_\rho(V),\mathcal{L}_\rho(W)]f,\end{aligned}$$

therefore Property 4 holds for $\eta = f \in \Omega^0(M,E)$.

Now let $\eta \in \Omega^1(M,E)$ and $Z \in A^1(M,E)$. By using Property 2, then Property 4 for elements $\eta \in \Omega^0(M,E)$, we may write

$$\begin{aligned}i(Z)\circ\mathcal{L}_\rho(\{V,W\})\eta &= \mathcal{L}_\rho(\{V,W\})(i(Z)\eta) - i(\{\{V,W\},Z\})\eta\\ &= \big(\mathcal{L}_\rho(V)\circ\mathcal{L}_\rho(W) - \mathcal{L}_\rho(W)\circ\mathcal{L}_\rho(V)\big)(i(Z)\eta) - i(\{\{V,W\},Z\})\eta.\end{aligned}$$

By using Property 2 and the Jacobi identity, we obtain

$$\begin{aligned}i(Z)\circ\mathcal{L}_\rho(\{V,W\})\eta &= \mathcal{L}_\rho(V)\big(i(\{W,Z\})\eta + i(Z)\circ\mathcal{L}_\rho(W)\eta\big)\\ &\quad - \mathcal{L}_\rho(W)\big(i(\{V,Z\})\eta + i(Z)\circ\mathcal{L}_\rho(V)\eta\big) - i(\{\{V,W\},Z\})\eta\\ &= i(\{W,Z\})\mathcal{L}_\rho(V)\eta + i(\{V,Z\})\mathcal{L}_\rho(W)\eta\\ &\quad - i(\{V,Z\})\mathcal{L}_\rho(W)\eta - i(\{W,Z\})\mathcal{L}_\rho(V)\eta\\ &\quad + i(Z)\circ\big(\mathcal{L}_\rho(V)\circ\mathcal{L}_\rho(W) - \mathcal{L}_\rho(W)\circ\mathcal{L}_\rho(V)\big)\eta\\ &\quad + i\big(\{V,\{W,Z\}\} - \{W,\{V,Z\}\} - \{\{V,W\},Z\}\big)\eta\\ &= i(Z)\circ\big(\mathcal{L}_\rho(V)\circ\mathcal{L}_\rho(W) - \mathcal{L}_\rho(W)\circ\mathcal{L}_\rho(V)\big)\eta.\end{aligned}$$

Since the last equality holds for all $Z \in A^1(M,E)$, Property 4 holds for all $\eta \in \Omega^1(M,E)$, and therefore for all $\eta \in \Omega(M,E)$.

5. Let $V \in A^1(M,E)$ and $f \in C^\infty(M,\mathbb{R})$. We have seen (Property 4) that $\mathcal{L}_\rho(fV)$ is a derivation of degree 0 of $\Omega(M,E)$. We easily verify that

$$\eta \mapsto f\mathcal{L}_\rho(V)\eta + d_\rho f \wedge i(V)\eta$$

is also a derivation of degree 0 of $\Omega(M,E)$. Property 5 means that these two derivations are equal. As above, it is enough to prove that Property 5 holds for $\eta \in \Omega^0(M,E)$ and for $\eta \in \Omega^1(M,E)$.

Let $g \in \Omega^0(M,E) = C^\infty(M,\mathbb{R})$. We may write $\mathcal{L}_\rho(fV)g = i(fV)d_\rho g = f\mathcal{L}_\rho(V)g$, which shows that Property 5 holds for $\eta = g \in \Omega^0(M,E)$.

Let $\eta \in \Omega^1(M,E)$ and $W \in A^1(M,E)$. We have

$$\begin{aligned}\langle \mathcal{L}_\rho(fV)\eta, W\rangle &= \mathcal{L}_\rho(fV)(\langle \eta, W\rangle) - \langle \eta, \{fV, W\}\rangle\\ &= f\mathcal{L}_\rho(V)(\langle \eta, W\rangle) - f\langle \eta, \{V, W\}\rangle + \langle \eta, (i(W)d_\rho f)V\rangle\\ &= \langle f\mathcal{L}_\rho(V)\eta, W\rangle + (i(W)d_\rho f)i(V)\eta = \langle f\mathcal{L}_\rho(V)\eta + d_\rho f \wedge i(V)\eta, W\rangle,\end{aligned}$$

since, η being in $\Omega^1(M,E)$, $i(W)\circ i(V)\eta = 0$. The last equality being satisfied for all $W \in A^1(M,E)$, the result follows. ∎

The next proposition shows that for each $V \in A^1(M,E)$, the Lie derivative $\mathcal{L}_\rho(V)$, already defined as a derivation of degree 0 of the graded algebra $\Omega(M,E)$, can also be extended to a derivation of degree 0 of the graded algebra $A(M,E)$, with very nice properties. As we will soon see, the Schouten–Nijenhuis bracket will appear as a very natural further extension of the Lie derivative.

Proposition 5.1.5. *Let (E,τ,M,ρ) be a Lie algebroid on a smooth manifold M. For each smooth section $V \in A^1(M,E)$ of the vector bundle (E,τ,M), there exists a unique graded endomorphism of degree 0 of the graded algebra of multivectors $A(M,E)$, called the Lie derivative with respect to V and denoted by $\mathcal{L}_\rho(V)$, which has the following properties:*

(i) *For a smooth function $f \in A^0(M,E) = C^\infty(M,\mathbb{R})$,*

$$\mathcal{L}_\rho(V)f = i(\rho\circ V)df = \mathcal{L}(\rho\circ V)f,$$

where $\mathcal{L}(\rho\circ V)$ denotes the usual Lie derivative with respect to the vector field $\rho\circ V$.

(ii) *For an integer $p \geq 1$ and a multivector $P \in A^p(M,E)$, $\mathcal{L}_\rho(V)P$ is the unique element in $A^p(M,E)$ such that, for all $\eta \in \Omega^p(M,E)$,*

$$\big\langle \eta, \mathcal{L}_\rho(V)P\big\rangle = \mathcal{L}_\rho(V)(\langle \eta, P\rangle) - \big\langle \mathcal{L}_\rho(V)\eta, P\big\rangle.$$

Proof. Let us first observe that $A^0(M,E) = \Omega^0(M,E) = C^\infty(M,\mathbb{R})$, and that for $f \in A^0(M,E)$, the definition of $\mathcal{L}_\rho(V)f$ given above is the same as that given in Proposition 5.1.1.

Now let $p \geq 1$ and $P \in A^p(M,E)$. The map

$$\eta \mapsto K(\eta) = \mathcal{L}_\rho(V)(\langle \eta, P\rangle) - \big\langle \mathcal{L}_\rho(V)\eta, P\big\rangle$$

is clearly an $\mathbb{R}$-linear map defined on $\Omega^p(M,E)$, with values in $C^\infty(M,\mathbb{R})$. Let $f \in C^\infty(M,\mathbb{R})$. We have

$$\begin{aligned}K(f\eta) &= \mathcal{L}_\rho(V)(\langle f\eta, P\rangle) - \langle \mathcal{L}_\rho(V)(f\eta), P\rangle\\ &= f\big(\mathcal{L}_\rho(V)(\langle \eta, P\rangle) - \langle \mathcal{L}_\rho(V)\eta, P\rangle\big) + (\mathcal{L}_\rho(V)f)\langle \eta, P\rangle - (\mathcal{L}_\rho(V)f)\langle \eta, P\rangle\\ &= fK(\eta).\end{aligned}$$

This proves that the map K is $C^\infty(M,\mathbb{R})$-linear. Since the pairing allows us to consider the vector bundle $(\bigwedge^p E, \tau, M)$ as the dual of $(\bigwedge^p E^*, \pi, M)$, we see that there exists a unique element $\mathcal{L}_\rho(V)P \in A^p(M,E)$ such that, for all $\eta \in \Omega^p(M,E)$,

$$K(\eta) = \mathcal{L}_\rho(V)(\langle \eta, P\rangle) - \langle \mathcal{L}_\rho(V)\eta, P\rangle = \langle \eta, \mathcal{L}_\rho(V)P\rangle,$$

and that ends the proof. ∎

Proposition 5.1.6. *Under the assumptions of Proposition* 5.1.5*, the Lie derivative has the following properties:*

1. *For all* $V, W \in A^1(M,E)$*,*
$$\mathcal{L}_\rho(V)(W) = \{V, W\}.$$
2. *For* $V, V_1, \dots, V_p \in A^1(M,E)$*,*
$$\mathcal{L}_\rho(V)(V_1 \wedge \cdots \wedge V_p) = \sum_{i=1}^{p} V_1 \wedge \cdots \wedge V_{i-1} \wedge \{V, V_i\} \wedge V_{i+1} \wedge \cdots \wedge V_p.$$
3. *For each* $V \in A^1(M,E)$*,* $\mathcal{L}_\rho(V)$ *is a derivation of degree* 0 *of the exterior algebra* $A(M,E)$*. That means that for all* $P, Q \in A(M,E)$*,*
$$\mathcal{L}_\rho(V)(P \wedge Q) = (\mathcal{L}_\rho(V)P) \wedge Q + P \wedge \mathcal{L}_\rho(V)Q.$$
4. *For all* $V \in A^1(M,E)$*,* $P \in A(M,E)$ *and* $\eta \in \Omega(M,E)$*,*
$$\mathcal{L}_\rho(V)(i(P)\eta) = i(\mathcal{L}_\rho(V)P)\eta + i(P)(\mathcal{L}_\rho(V)\eta).$$
5. *Similarly, for all* $V \in A^1(M,E)$*,* $P \in A(M,E)$ *and* $\eta \in \Omega(M,E)$*,*
$$\mathcal{L}_\rho(V)(\langle \eta, P\rangle) = \langle \mathcal{L}_\rho(V)\eta, P\rangle + \langle \eta, \mathcal{L}_\rho(V)P\rangle.$$
6. *For all* $V, W \in A^1(M,E)$ *and* $P \in A(M,E)$*,*
$$\mathcal{L}_\rho(\{V, W\})P = \big(\mathcal{L}_\rho(V) \circ \mathcal{L}_\rho(W) - \mathcal{L}_\rho(W) \circ \mathcal{L}_\rho(V)\big)P.$$
7. *For all* $V \in A^1(M,E)$*,* $f \in C^\infty(M,\mathbb{R})$*,* $P \in A(M,E)$ *and* $\eta \in \Omega(M,E)$*,*
$$\langle \eta, \mathcal{L}_\rho(fV)P\rangle = f\langle \mathcal{L}_\rho(V)P, \eta\rangle + \langle d_\rho f \wedge i(V)\eta, P\rangle.$$

Proof. 1. Let $V, W \in A^1(M,E)$ and $\eta \in \Omega(M,E)$. We may write
$$\langle \eta, \mathcal{L}_\rho(V)W\rangle = \mathcal{L}_\rho(V)(\langle \eta, W\rangle) - \langle \mathcal{L}_\rho(V)\eta, W\rangle = \langle \eta, \{V, W\}\rangle.$$
We have proven Property 1.

2. The proof is similar to that of Property 1.

3. When $P = V_1 \wedge \cdots \wedge V_p$ and $Q = W_1 \wedge \cdots \wedge W_q$ are decomposable homogeneous elements in $A(M,E)$, Property 3 is an easy consequence of 2. The validity of Property 3 for all $P, Q \in A(M,E)$ follows by linearity.

4. When $P = V_1 \wedge \cdots \wedge V_p$ is a decomposable homogeneous element in $A(M,E)$, Property 4 is an easy consequence of Property 2. The validity of Property 4 for all $P, Q \in A(M,E)$ follows by linearity.

5. This is an immediate consequence of Property 4.

6. This is an immediate consequence of Property 4 of this Proposition and of Property 4 of Proposition 5.1.4.

7. This is an immediate consequence of Property 4 of this proposition, and of Property 5 of Proposition 5.1.4. ∎

5.2. The $\Omega(M,E)$-valued exterior derivative. We have introduced above (Definition 5.1.2) the $\Omega(M,E)$-valued exterior derivative of a function $f \in \Omega^0(M,E) = C^\infty(M,\mathbb{R})$. The next proposition shows that the $\Omega(M,E)$-valued exterior derivative extends to a graded endomorphism of degree 1 of the graded algebra $\Omega(M,E)$. We will see later (Proposition 5.2.3) that the $\Omega(M,E)$-valued exterior derivative is in fact a derivation of degree 1 of $\Omega(M,E)$.

PROPOSITION 5.2.1. *Let (E,τ,M,ρ) be a Lie algebroid over a smooth manifold M. There exists a unique graded endomorphism of degree 1 of the exterior algebra of forms $\Omega(M,E)$, called the $\Omega(M,E)$-valued exterior derivative (or, in brief, the exterior derivative) and denoted by d_ρ, which has the following properties:*

(i) *For $f\in\Omega^0(M,E)=C^\infty(M,\mathbb{R})$, $d_\rho f$ is the unique element in $\Omega^1(M,E)$, already defined (Definition 5.1.2), such that, for each $V\in A^1(M,E)$,*

$$\langle d_\rho f, V\rangle = \mathcal{L}_\rho(V)f = \langle df, \rho\circ V\rangle = \langle {}^t\rho\circ df, V\rangle,$$

where d stands for the usual exterior derivative of smooth functions on M, and ${}^t\rho : E^*\to T^*M$ is the transpose of the anchor ρ.

(ii) *For $p\geq 1$ and $\eta\in\Omega^p(M,E)$, $d_\rho\eta$ is the unique element in $\Omega^{p+1}(M,E)$ such that, for all $V_0,\dots,V_p\in A^1(M,E)$,*

$$\begin{aligned} d_\rho\eta(V_0,\dots,V_p) &= \sum_{i=0}^{p}(-1)^i\mathcal{L}_\rho(V_i)(\eta(V_0,\dots,\widehat{V_i},\dots,V_p)) \\ &\quad + \sum_{0\leq i<j\leq p}(-1)^{i+j}\eta(\{V_i,V_j\},V_0,\dots,\widehat{V_i},\dots,\widehat{V_j},\dots,V_p), \end{aligned}$$

where the symbol $\widehat{\ }$ over the terms V_i and V_j means that these terms are omitted.

Proof. For $f\in\Omega^0(M,E)$, $d_\rho f$ is clearly an element in $\Omega^1(M,E)$.

Let $p\geq 1$ and $\eta\in\Omega^p(M,E)$. As defined in (ii), $d_\rho\eta$ is a map, defined on $(A^1(M,E))^p$, with values in $C^\infty(M,\mathbb{R})$. The reader will immediately see that this map is skew-symmetric and $\mathbb{R}$-linear in each of its arguments. In order to prove that $d_\rho\eta$ is an element in $\Omega^{p+1}(M,E)$, it remains only to verify that as a map, $d_\rho\eta$ is $C^\infty(M,\mathbb{R})$-linear in each of its arguments, or simply in its first argument, since the skew-symmetry will imply the same property for all other arguments. Let $f\in C^\infty(M,\mathbb{R})$. We have

$$\begin{aligned} d_\rho\eta(fV_0,V_1,\dots,V_p) &= \mathcal{L}_\rho(fV_0)(\eta(V_1,\dots,V_p)) \\ &\quad + \sum_{i=1}^{p}(-1)^i\mathcal{L}_\rho(V_i)(f\eta(V_0,\dots,\widehat{V_i},\dots,V_p)) \\ &\quad + \sum_{1\leq j\leq p}(-1)^j\eta(\{fV_0,V_j\},V_1,\dots,\widehat{V_j},\dots,V_p) \\ &\quad + \sum_{1\leq i<j\leq p}(-1)^{i+j}\eta(\{V_i,V_j\},fV_0,V_1,\dots,\widehat{V_i},\dots,\widehat{V_j},\dots,V_p). \end{aligned}$$

By a rearrangement of the terms on the right hand side, and by using the formulae

$$\mathcal{L}_\rho(V_i)(f\eta(\dots)) = (\mathcal{L}_\rho(V_i)f)\eta(\dots) + f\mathcal{L}_\rho(V_i)(\eta(\dots))$$

and

$$\{fV_0,V_j\} = f\{V_0,V_j\} - (\mathcal{L}_\rho(V_j)f)V_0,$$

we obtain

$$d_\rho\eta(fV_0,V_1,\dots,V_p) = fd_\rho\eta(V_0,V_1,\dots,V_p).$$

We have shown that $d_\rho\eta \in \Omega^{p+1}(M,E)$. The $\Omega(M,E)$-valued exterior derivative so defined on $\Omega^p(M,E)$ for all $p\in\mathbb{Z}$ extends, in a unique way, to a graded endomorphism of degree 1 of $\Omega(M,E)$. ∎

Remark 5.2.2. Let $p\geq 1$, $\eta\in\Omega^p(M,E)$ and $V_0,\dots,V_p\in A^1(M,E)$. The formula for $d_\rho\eta$ given in Proposition 5.2.1 can be recast into another form, often useful:

$$\begin{aligned}d_\rho\eta(V_0,\dots,V_p) &= \sum_{i=0}^{p}(-1)^i(\mathcal{L}_\rho(V_i)\eta)(V_0,\dots,\widehat{V_i},\dots,V_p)\\ &\quad - \sum_{0\leq i<j\leq p}(-1)^{i+j}\eta(\{V_i,V_j\},V_0,\dots,\widehat{V_i},\dots,\widehat{V_j},\dots,V_p).\end{aligned}$$

For example, for $p=1$,

$$\begin{aligned}d_\rho\eta(V_0,V_1) &= \mathcal{L}_\rho(V_0)(\eta(V_1)) - \mathcal{L}_\rho(V_1)(\eta(V_0)) - \eta(\{V_0,V_1\})\\ &= \langle\mathcal{L}_\rho(V_0)\eta,V_1\rangle - \langle\mathcal{L}_\rho(V_1)\eta,V_0\rangle + \eta(\{V_0,V_1\}).\end{aligned}$$

Proposition 5.2.3. *Under the assumptions of Proposition 5.2.1, the $\Omega(M,E)$-valued exterior derivative has the following properties:*

1. *Let $V\in A^1(M,E)$. The Lie derivative $\mathcal{L}_\rho(V)$, the exterior derivative d_ρ and the interior product $i(V)$ are related by the formula*
$$\mathcal{L}_\rho(V) = i(V)\circ d_\rho + d_\rho\circ i(V).$$
2. *The exterior derivative d_ρ is a derivation of degree 1 of the exterior algebra $\Omega(M,E)$. That means that for each $p\in\mathbb{Z}$, $\eta\in\Omega^p(M,E)$ and $\zeta\in\Omega(M,E)$,*
$$d_\rho(\eta\wedge\zeta) = d_\rho\eta\wedge\zeta + (-1)^p\eta\wedge d_\rho\zeta.$$
3. *Let $V\in A^1(M,E)$. Then*
$$\mathcal{L}_\rho(V)\circ d_\rho = d_\rho\circ\mathcal{L}_\rho(V).$$
4. *The square of d_ρ vanishes identically,*
$$d_\rho\circ d_\rho = 0.$$

Proof. 1. Let $V_0=V, V_1,\dots,V_p\in A^1(M,E)$ and $\eta\in\Omega^p(M,E)$. Then

$$\begin{aligned}(i(V)\circ d_\rho\eta)(V_1,\dots,V_p) &= d_\rho\eta(V,V_1,\dots,V_p)\\ &= \sum_{i=0}^{p}(-1)^i\mathcal{L}_\rho(V_i)(\eta(V_0,\dots,\widehat{V_i},\dots,V_p))\\ &\quad + \sum_{0\leq i<j\leq p}(-1)^{i+j}\eta(\{V_i,V_j\},V_0,\dots,\widehat{V_i},\dots,\widehat{V_j},\dots,V_p),\end{aligned}$$

and

$$\begin{aligned}(d_\rho\circ i(V)\eta)(V_1,\dots,V_p) &= \sum_{i=1}^{p}(-1)^{i-1}\mathcal{L}_\rho(V_i)(\eta(V_0,\dots,\widehat{V_i},\dots,V_p))\\ &\quad + \sum_{1\leq i<j\leq p}(-1)^{i+j}\eta(V_0,\{V_i,V_j\},V_1,\dots,\widehat{V_i},\dots,\widehat{V_j},\dots,V_p).\end{aligned}$$

Let us add these two equalities. Several terms cancel, and we obtain

$$\begin{aligned}&\bigl((i(V)\circ d_\rho + d_\rho\circ i(V))\eta\bigr)(V_1,\dots,V_p)\\&\qquad\qquad = \mathcal{L}_\rho(V_0)(\eta(V_1,\dots,V_p)) + \sum_{j=1}^{p}(-1)^j\eta(\{V_0,V_j\},V_1,\dots,\widehat{V_j},\dots,V_p).\end{aligned}$$

When we shift, in the last term of the right hand side, the argument $\{V_0,V_j\}$ to the jth position, we obtain

$$\begin{aligned}\bigl((i(V)\circ d_\rho + d_\rho\circ i(V))\eta\bigr)(V_1,\dots,V_p) &= \mathcal{L}_\rho(V_0)(\eta(V_1,\dots,V_p))\\&\quad + \sum_{j=1}^{p}\eta(V_1,\dots,V_{j-1},\{V_0,V_j\},V_{j+1},\dots,V_p)\\&= (\mathcal{L}_\rho(V_0)\eta)(V_1,\dots,V_p).\end{aligned}$$

2. For $\eta = f$ and $\zeta = g \in \Omega^0(M,E) = C^\infty(M,\mathbb{R})$, Property 2 holds since, for all $V \in A^1(M,E)$,

$$\langle d_\rho(fg),V\rangle = \langle d(fg),\rho\circ V\rangle = \langle f dg + g\,df,\rho\circ V\rangle = \langle f d_\rho g + g d_\rho f, V\rangle.$$

Now let $p,q\geq 0$ be two integers, $\eta\in\Omega^p(M,E)$ and $\zeta\in\Omega^q(M,E)$. We will prove that Property 2 holds by induction on $p+q$. Just above, we have seen that it holds for $p+q=0$. Let us assume that r is an integer such that Property 2 holds for $p+q\leq r$, and that now $p+q=r+1$. Let $V\in A^1(M,E)$. We may write

$$\begin{aligned}i(V)d_\rho(\eta\wedge\zeta) &= \mathcal{L}_\rho(V)(\eta\wedge\zeta) - d_\rho\circ i(V)(\eta\wedge\zeta)\\&= \mathcal{L}_\rho(V)\eta\wedge\zeta + \eta\wedge\mathcal{L}_\rho(V)\zeta - d_\rho\bigl(i(V)\eta\wedge\zeta + (-1)^p\eta\wedge i(V)\zeta\bigr).\end{aligned}$$

We may now use the induction assumption, since in the last terms of the right hand side $i(V)\eta\in\Omega^{p-1}(M,E)$ and $i(V)\zeta\in\Omega^{q-1}(M,E)$. After some rearrangements we obtain

$$i(V)d_\rho(\eta\wedge\zeta) = i(V)(d_\rho\eta\wedge\zeta + \eta\wedge d_\rho\zeta).$$

Since this result holds for all $V\in A^1(M,E)$, Property 2 holds for $p+q=r+1$, and therefore for all $p,q\in\mathbb{Z}$.

3. Let $V\in A^1(M,E)$. We know (Proposition 5.1.4) that $\mathcal{L}_\rho(V)$ is a derivation of degree 0 of the exterior algebra $\Omega(M,E)$, and we have just seen (Property 2) that d_ρ is a derivation of degree 1 of that algebra. Therefore, by 4.1.6, their graded bracket

$$[\mathcal{L}_\rho(V),d_\rho] = \mathcal{L}_\rho(V)\circ d_\rho - d_\rho\circ\mathcal{L}_\rho(V)$$

is a derivation of degree 1 of $\Omega(M,E)$. In order to prove that this derivation is equal to 0, it is enough to prove that it vanishes on $\Omega^0(M,E)$ and on $\Omega^1(M,E)$. We have already proven that it vanishes on $\Omega^0(M,E)$ (Property 1 of 5.1.4). Let $\eta\in\Omega^1(M,E)$ and $W\in A^1(M,E)$. By using Property 1 of this proposition and Property 2 of 5.1.4, we may write

$$\begin{aligned}i(W)\circ\bigl(\mathcal{L}_\rho(V)\circ d_\rho - d_\rho\circ\mathcal{L}_\rho(V)\bigr)\eta &= \mathcal{L}_\rho(V)\circ i(W)\circ d_\rho\eta - i(\{V,W\})\circ d_\rho\eta\\&\quad - \mathcal{L}_\rho(W)\circ\mathcal{L}_\rho(V)\eta + d_\rho\circ i(W)\circ\mathcal{L}_\rho(V)\eta.\end{aligned}$$

After some rearrangements, we obtain

$$\begin{aligned} i(W)\circ\big(\mathcal{L}_\rho(V)\circ d_\rho - d_\rho\circ\mathcal{L}_\rho(V)\big)\eta &\\ &= \big(\mathcal{L}_\rho(V)\circ\mathcal{L}_\rho(W) - \mathcal{L}_\rho(W)\circ\mathcal{L}_\rho(V) - \mathcal{L}_\rho(\{V,W\})\big)\eta\\ &\quad + d_\rho\circ i(\{V,W\})\eta - d_\rho\circ i(\{V,W\})\eta\\ &\quad - \big(\mathcal{L}_\rho(V)\circ d_\rho - d_\rho\circ\mathcal{L}_\rho(V)\big)(i(W)\eta)\\ &= 0, \end{aligned}$$

since $i(W)\eta\in\Omega^0(M,E)$, which implies that the last term vanishes.

4. Property 2 shows that d_ρ is a derivation of degree 1 of $\Omega(M,E)$. We know (4.1.6) that $[d_\rho,d_\rho]=2d_\rho\circ d_\rho$ is a derivation of degree 2 of $\Omega(M,\mathbb{R})$. In order to prove that $d_\rho\circ d_\rho=0$, it is enough to prove that it vanishes on $\Omega^0(M,E)$ and on $\Omega^1(M,E)$.

Let $f\in\Omega^0(M,E)=C^\infty(M,\mathbb{R})$ and $V,W\in A^1(M,E)$. Then

$$\begin{aligned} (d_\rho\circ d_\rho f)(V,W) &= \mathcal{L}_\rho(V)(d_\rho f(W)) - \mathcal{L}_\rho(W)(d_\rho f(V)) - d_\rho f(\{V,W\})\\ &= \big(\mathcal{L}_\rho(V)\circ\mathcal{L}_\rho(W) - \mathcal{L}_\rho(W)\circ\mathcal{L}_\rho(V) - \mathcal{L}_\rho(\{V,W\})\big)f = 0, \end{aligned}$$

where we have used Property 4 of Proposition 5.1.4. We have shown that $d_\rho\circ d_\rho$ vanishes on $\Omega^0(M,E)$.

Now let $\eta\in\Omega^1(M,E)$ and $V_0,V_1,V_2\in A^1(M,E)$. Using Property 1, we may write

$$\begin{aligned} (d_\rho\circ d_\rho\eta)(V_0,V_1,V_2) &= \big((i(V_0)\circ d_\rho)(d_\rho\eta)\big)(V_1,V_2)\\ &= \big((\mathcal{L}_\rho(V_0)\circ d_\rho - d_\rho\circ i(V_0)\circ d_\rho)\eta\big)(V_1,V_2). \end{aligned}$$

The last term on the right hand side may be transformed, by using again Property 1:

$$d_\rho\circ i(V_0)\circ d_\rho(\eta) = d_\rho\circ\mathcal{L}_\rho(V_0)\eta - d_\rho\circ d_\rho(i(V_0)\eta) = d_\rho\circ\mathcal{L}_\rho(V_0)\eta,$$

since, as $i(V_0)\eta\in\Omega^0(M,E)$, we have $d_\rho\circ d_\rho(i(V_0)\eta)=0$. So we obtain

$$(d_\rho\circ d_\rho\eta)(V_0,V_1,V_2) = \big((\mathcal{L}_\rho(V_0)\circ d_\rho - d_\rho\circ\mathcal{L}_\rho(V_0))\eta\big)(V_1,V_2).$$

But Property 3 shows that

$$\big(\mathcal{L}_\rho(V_0)\circ d_\rho - d_\rho\circ\mathcal{L}_\rho(V_0)\big)\eta = 0,$$

so we have

$$(d_\rho\circ d_\rho\eta)(V_0,V_1,V_2) = 0,$$

and our proof is complete. ∎

5.3. Defining a Lie algebroid by properties of its dual. Let (E,τ,M) be a vector bundle and (E^*,π,M) its dual bundle. We have seen in 5.2 that when (E,τ,M) has a Lie algebroid structure whose anchor is denoted by ρ, this structure determines, on the graded algebra $\Omega(M,E)$ of sections of the exterior powers of the dual bundle (E^*,π,M), a graded derivation d_ρ, of degree 1, which satisfies $d_\rho^2=d_\rho\circ d_\rho=0$. Now we are going to prove a converse of this property: when a graded derivation of degree 1, whose square vanishes, is given on $\Omega(M,E)$, it determines a Lie algebroid structure on (E,τ,M). This property will be used later to prove that the cotangent bundle of a Poisson manifold has a natural Lie algebroid structure.

We will need the following lemmas.

LEMMA 5.3.1. *Let* (E, τ, M) *be a vector bundle and* (E^*, π, M) *its dual bundle. Let* δ *be a graded derivation of degree* 1 *of the exterior algebra* $\Omega(M, E)$ *(notations defined in* 4.2.4*). For each pair* (X, Y) *of smooth sections of* (E, τ, M) *there exists a unique smooth section* $[X, Y]_\delta$ *of* (E, τ, M)*, called the* δ*-bracket of* X *and* Y*, such that*

$$i([X, Y]_\delta) = [[i(X), \delta], i(Y)].$$

Proof. The map defined by the right hand side of the above equality,

$$D : \eta \mapsto D(\eta) = [[i(X), \delta], i(Y)],$$

is a derivation of degree -1 of $\Omega(M, E)$, since it is obtained by repeated application of the graded bracket to derivations (property 4.1.6(ii)). Therefore, it vanishes on $\Omega^0(M, E) = C^\infty(M, \mathbb{R})$. As a consequence, the map is $C^\infty(M, \mathbb{R})$-linear; we have indeed, for each $f \in C^\infty(M, \mathbb{R})$ and $\eta \in \Omega(M, R)$,

$$D(f\eta) = D(f) \wedge \eta + f D(\eta) = f D(\eta).$$

Therefore, there exists a unique smooth section $[X, Y]_\delta$ of (E, τ, M) such that, for each $\eta \in \Omega^1(M, E)$,

$$\langle \eta, [X, Y]_\delta \rangle = D(\eta).$$

Now the maps

$$i([X, Y]_\delta) \quad \text{and} \quad [[i(X), \delta], i(Y)]$$

are both derivations of degree -1 of $\Omega(M, E)$, which coincide on $\Omega^0(M, E)$ and $\Omega^1(M, E)$. Since derivations are local, and since $\Omega(M, E)$ is locally generated by its elements of degrees 0 and 1, these two derivations are equal. ∎

LEMMA 5.3.2. *Under the assumptions of Lemma* 5.3.1*, set, for each smooth section* X *of* (E, τ, M)*,*

$$\mathcal{L}_\delta(X) = [i(X), \delta].$$

Then, for each smooth section X *of* (E, τ, M)*, we have*

$$[\mathcal{L}_\delta(X), \delta] = [i(X), \delta^2];$$

for each pair (X, Y) *of smooth sections of* (E, τ, M)*, we have*

$$[\mathcal{L}_\delta(X), \mathcal{L}_\delta(Y)] - \mathcal{L}_\delta([X, Y]_\delta) = [[i(X), \delta^2], i(Y)];$$

and for each triple (X, Y, Z) *of smooth sections of* (E, τ, M)*, we have*

$$i\big([X, [Y, Z]_\delta]_\delta + [Y, [Z, X]_\delta]_\delta + [Z, [X, Y]_\delta]_\delta\big) = \big[[[i(X), \delta^2], i(Y)], i(Z)\big].$$

Proof. Let us use the graded Jacobi identity. We may write

$$[\mathcal{L}_\delta(X), \delta] = [[i(X), \delta], \delta] = -[[\delta, \delta], i(X)] - [[\delta, i(X)], \delta].$$

Since $[\delta, \delta] = 2\delta^2$, we obtain

$$2[\mathcal{L}_\delta(X), \delta] = -2[\delta^2, i(X)] = 2[i(X), \delta^2],$$

which proves the first equality. Similarly, by using again the graded Jacobi identity and the equality just proven,

$$\begin{aligned}
[\mathcal{L}_\delta(X),\mathcal{L}_\delta(Y)] &= [\mathcal{L}_\delta(X),[i(Y),\delta]] = -[i(Y),[\delta,\mathcal{L}_\delta(X)]] + [\delta,[\mathcal{L}_\delta(X),i(Y)]]\\
&= -[[\delta,\mathcal{L}_\delta(X)],i(Y)] + [[\mathcal{L}_\delta(X),i(Y)],\delta]\\
&= [[\mathcal{L}_\delta(X),\delta],i(Y)] + [i([X,Y]_\delta),\delta]\\
&= [[i(X),\delta^2],i(Y)] + \mathcal{L}_\delta([X,Y]_\delta).
\end{aligned}$$

The second formula is proven. Finally,

$$\begin{aligned}
i([X,[Y,Z]_\delta]_\delta) &= [\mathcal{L}_\delta(X),i([Y,Z]_\delta)]\\
&= [\mathcal{L}_\delta(X),[\mathcal{L}_\delta(Y),i(Z)]]\\
&= -[\mathcal{L}_\delta(Y),[i(Z),\mathcal{L}_\delta(X)]] - [i(Z),[\mathcal{L}_\delta(X),\mathcal{L}_\delta(Y)]]\\
&= [\mathcal{L}_\delta(Y),[\mathcal{L}_\delta(X),i(Z)]] - \bigl[i(Z),\mathcal{L}_\delta([X,Y]_\delta)\bigr]\\
&\quad - \bigl[i(Z),\bigl[[i(X),\delta^2],i(Y)\bigr]\bigr]\\
&= i([Y,[X,Z]_\delta]_\delta) + i([[X,Y]_\delta,Z]_\delta)\\
&\quad + \bigl[\bigl[[i(X),\delta^2],i(Y)\bigr],i(Z)\bigr].
\end{aligned}$$

The proof is complete. ∎

THEOREM 5.3.3. *Let (E,τ,M) be a vector bundle and (E^*,π,M) its dual bundle. Let δ be a graded derivation of degree* 1 *of the exterior algebra $\Omega(M,E)$ (notations defined in* 4.2.4*), which satisfies*

$$\delta^2 = \delta\circ\delta = 0.$$

Then δ determines a natural Lie algebroid structure on (E,τ,M). For that structure, the anchor map $\rho : E \to TM$ is the unique vector bundle map such that, for each smooth section X of (E,τ,M) and each function $f\in C^\infty(M,\mathbb{R})$,

$$i(\rho\circ X)\,df = \langle \delta f, X\rangle.$$

The bracket $(X,Y)\mapsto \{X,Y\}$ is the δ-bracket defined in Lemma 5.3.1*; it is such that, for each pair (X,Y) of smooth sections of (E,τ,M),*

$$i(\{X,Y\}) = [[i(X),\delta],i(Y)].$$

The $\omega(M,E)$-valued exterior derivative associated to that Lie algebroid structure (Propositions 5.2.1 *and* 5.2.3*) is the given derivation δ.*

Proof. Since $\delta^2=0$, Lemmas 5.3.1 and 5.3.2 prove that the δ-bracket satisfies the Jacobi identity. Let X and Y be two smooth sections of (E,τ,M) and f a smooth function on M. By using the definition of the δ-bracket we obtain

$$i([X,fY]_\delta) = f\,i([X,Y]_\delta) + (\mathcal{L}_\delta(X)f)\,i(Y).$$

But

$$\mathcal{L}_\delta(X)f = [i(X),\delta]f = \langle \delta f, X\rangle,$$

since $i(X)f=0$. We must prove now that the value of $\delta(f)$ at any point $x\in M$ depends only on the value of the differential df of f at that point. We first observe that δ being a derivation, the values of $\delta(f)$ in some open subset U of M depend only on the values of f in that open subset. Moreover, we have

$$\delta(1\,f) = \delta(f) = \delta(1)\,f + 1\,\delta(f) = \delta(1)\,f + \delta(f),$$

which proves that δ vanishes on constants.

Let $a \in M$. We use a chart of M whose domain U contains a, and whose local coordinates are denoted by $(x^1, \dots, x^n)$. In order to calculate $\delta(f)(a)$, the above remarks allow us to work in that chart. We may write

$$f(x) = f(a) + \sum_{i=1}^{n}(x^i - a^i)\varphi_i(x) \quad \text{with} \quad \lim_{x\to a}\varphi_i(x) = \frac{\partial f}{\partial x^i}(a).$$

Therefore,

$$(\delta f)(a) = \sum_{i=1}^{n}\frac{\partial f}{\partial x^i}(a)\,\delta(x^i)(a).$$

We have proven that $\delta(f)(a)$ depends only on $df(a)$, and that we may write

$$\delta(f) = {}^t\rho_\delta \circ df,$$

where ${}^t\rho_\delta : T^*M \to E^*$ is a smooth vector bundle map. Let $\rho_\delta : E \to TM$ be its transpose. We may now write

$$[X, fY]_\delta = f[X,Y]_\delta + \langle df, \rho_\delta \circ X\rangle Y.$$

This proves that the vector bundle (E,τ,M), with the δ-bracket and the map ρ_δ as anchor, is a Lie algebroid. Finally, by using Propositions 5.2.1 and 5.2.3, we see that the $\Omega(M,E)$-valued exterior derivative associated to that Lie algebroid structure is the derivation δ. ■

5.4. The Schouten–Nijenhuis bracket. In this subsection (E,τ,M,ρ) is a Lie algebroid. We have seen (Propositions 5.1.4 and 5.1.6) that the composition law which associates, to each pair (V,W) of sections of the Lie algebroid (E,τ,M,ρ), the bracket $\{V,W\}$, extends to a map $(V,P) \mapsto \mathcal{L}_\rho(V)P$, defined on $A^1(M,E)\times A(M,E)$, with values in $A(M,E)$. Theorem 5.4.3 below will show that this map extends, in a very natural way, to a composition law $(P,Q) \mapsto [P,Q]$, defined on $A(M,E)\times A(M,E)$, with values in $A(M,E)$, called the *Schouten–Nijenhuis bracket*. That bracket was discovered by Schouten [44] for multivectors on a manifold, and its properties were further studied by Nijenhuis [40].

The following lemmas will be used in the proof of Theorem 5.4.3.

LEMMA 5.4.1. *Let (E,τ,M,ρ) be a Lie algebroid, $p,q\in\mathbb{Z}$, $P\in A^p(M,E)$, $Q\in A^q(M,E)$, $f\in C^\infty(M,\mathbb{R})$ and $\eta\in\Omega(M,E)$. Then*

$$i(P)(df\wedge i(Q)\eta) - (-1)^p df\wedge(i(P)\circ i(Q)\eta) + (-1)^{(p-1)q}i(Q)\circ i(P)(df\wedge\eta) + (-1)^{(p-1)q+p}i(Q)(df\wedge i(P)\eta) = 0.$$

Proof. Let us denote by $E(P,Q,f,\eta)$ the left hand side of the above equality. We have to prove that $E(P,Q,f,\eta) = 0$.

Obviously, $E(P,Q,f,\eta)=0$ when $p<0$, as well as when $q<0$. When $p=q=0$, we have

$$E(P,Q,f,\eta)=PQ\,df\wedge\eta-PQ\,df\wedge\eta-QP\,df\wedge\eta+QP\,df\wedge\eta=0.$$

Now we proceed by induction on p and q, with the induction assumption that $E(P,Q,f,\eta) = 0$ when $p\leq p_M$ and $q\leq q_M$, for some integers p_M and q_M. Let $P=X\wedge P'$ with $X\in A^1(M,E)$ and $P'\in A^{p_M}(M,E)$, $Q\in A^q(M,E)$ with $q\leq q_M$, $f\in C^\infty(M,\mathbb{R})$ and $\eta\in\Omega(M,E)$. We obtain, after some calculations,

$$\begin{aligned}E(P,Q,f,\eta)&=E(X\wedge P',Q,f,\eta)\\&=(-1)^{p_M+q-1}E(P',Q,f,i(X)\eta)+(-1)^{p_M}\langle df,X\rangle i(P)\circ i(Q)\eta\\&\quad-(-1)^{p_M+p_Mq}\langle df,X\rangle i(Q)\circ i(P)\eta\\&=0,\end{aligned}$$

since, by the induction assumption, $E(P',Q,f,i(X)\eta)=0$.

Since every $P\in A^{p_M+1}(M,E)$ is the sum of terms of the form $X\wedge P'$ with $X\in A^1(M,E)$ and $P'\in A^{p_M}(M,E)$, we see that $E(P,Q,f,\eta)=0$ for all $p\leq p_M+1$, $q\leq q_M$, $P\in A^{p_M+1}(M,E)$ and $Q\in A^{q_M}(M,E)$.

Moreover, P and Q play similar parts in $E(P,Q,f,\eta)$, since

$$E(P,Q,f,\eta)=(-1)^{pq+p+q}E(Q,P,f,\eta).$$

Therefore $E(P,Q,f,\eta)=0$ for all $p\leq p_M+1$, $q\leq q_M+1$, $P\in A^p(M,E)$ and $Q\in A^q(M,E)$. By induction we conclude that $E(P,Q,f,\eta)=0$ for all $p,q\in\mathbb{Z}$, $P\in A^p(M,E)$ and $Q\in A^q(M,E)$. ■

Lemma 5.4.2. *Let (E,τ,M,ρ) be a Lie algebroid, $p,q,r\in\mathbb{Z}$, $P\in A^p(M,E)$, $Q\in A^q(M,E)$ and $R\in A^r(M,E)$. Then*

$$i(R)\circ[[i(P),d_\rho],i(Q)]=(-1)^{(p+q-1)r}[[i(P),d_\rho],i(Q)]\circ i(R).$$

Proof. Let us first assume that $R=V\in A^1(M,E)$. We may write

$$\begin{aligned}i(V)\circ\big[[i(P),d_\rho],i(Q)\big]&=i(V)\circ i(P)\circ d_\rho\circ i(Q)-(-1)^p i(V)\circ d_\rho\circ i(P)\circ i(Q)\\&\quad-(-1)^{(p-1)q}i(V)\circ i(Q)\circ i(P)\circ d_\rho\\&\quad+(-1)^{(p-1)q+p}i(V)\circ i(Q)\circ d_\rho\circ i(P).\end{aligned}$$

We transform the right hand side by pushing the operator $i(V)$ towards the right, using the formulae (proven in 4.2.3(v) and in Property 1 of 5.2.3)

$$i(V)\circ i(P)=(-1)^p i(P)\circ i(V)\quad\text{and}\quad i(V)\circ d_\rho=\mathcal{L}_\rho(V)-d_\rho\circ i(V).$$

We obtain, after rearrangements,

$$\begin{aligned}i(V)\circ[[i(P),d_\rho],i(Q)]&=(-1)^{p+q-1}[[i(P),d_\rho],i(Q)]\circ i(V)\\&\quad+(-1)^p i(P)\circ\mathcal{L}_\rho(V)\circ i(Q)-(-1)^p\mathcal{L}_\rho(V)\circ i(P)\circ i(Q)\\&\quad-(-1)^{(p-1)q+p+q}i(Q)\circ i(P)\circ\mathcal{L}_\rho(V)\\&\quad+(-1)^{(p-1)q+p+q}i(Q)\circ\mathcal{L}_\rho(V)\circ i(P).\end{aligned}$$

Now we transform the last four terms of the right hand side by pushing to the right the operator $\mathcal{L}_\rho(V)$, using formulae, proven in 4.2.3(v) and in Property 4 of 5.1.6, of the type

$$i(P)\circ i(Q) = i(P\wedge Q) \quad \text{and} \quad \mathcal{L}_\rho(V)\circ i(P) = i(P)\circ\mathcal{L}_\rho(V) + i(\mathcal{L}_\rho(V)P).$$

The terms containing $\mathcal{L}_\rho(V)$ become

$$(-1)^p i\bigl(P\wedge\mathcal{L}_\rho(V)Q + (\mathcal{L}_\rho(V)P)\wedge Q - \mathcal{L}_\rho(V)(P\wedge Q)\bigr),$$

so they vanish, by Property 3 of 5.1.6. So we have

$$i(V)\circ[[i(P),d_\rho],i(Q)] = (-1)^{(p+q-1)}[[i(P),d_\rho],i(Q)]\circ i(V).$$

Now let $R = V_1\wedge\cdots\wedge V_r$ be a decomposable element in $A^r(M,E)$. Since

$$i(R) = i(V_1)\circ\cdots\circ i(V_r),$$

by using r times the above result we obtain

$$i(R)\circ[[i(P),d_\rho],i(Q)] = (-1)^{(p+q-1)r}[[i(P),d_\rho],i(Q)]\circ i(R).$$

Finally, the same result holds for all $R\in A^r(M,E)$ by linearity. ∎

THEOREM 5.4.3. *Let (E,τ,M,ρ) be a Lie algebroid. Let $p,q\in\mathbb{Z}$, and let $P\in A^p(M,E)$, $Q\in A^q(M,E)$. There exists a unique element in $A^{p+q-1}(M,E)$, called the Schouten–Nijenhuis bracket of P and Q, and denoted by $[P,Q]$, such that the interior product $i([P,Q])$, considered as a graded endomorphism of degree $p+q-1$ of the exterior algebra $\Omega(M,E)$, is given by the formula*

$$i([P,Q]) = [[i(P),d_\rho],i(Q)],$$

the brackets on the right hand side being the graded brackets of graded endomorphism (Definition 4.1.3).

Proof. We observe that for all $r\in\mathbb{Z}$, the map

$$\eta\mapsto[[i(P),d_\rho],i(Q)]\eta,$$

defined on $\Omega^r(M,E)$, with values in $\Omega^{r-p-q+1}(M,E)$, is $\mathbb{R}$-linear. Let us prove that it is in fact $C^\infty(M,\mathbb{R})$-linear. Let $f\in C^\infty(M,\mathbb{R})$. By developing the double graded bracket of endomorphisms, we obtain after some calculations

$$\begin{aligned}[[i(P),d_\rho],i(Q)](f\eta) &= f[[i(P),d_\rho],i(Q)]\eta\\ &\quad + i(P)(df\wedge i(Q)\eta) - (-1)^p df\wedge(i(P)\circ i(Q)\eta)\\ &\quad + (-1)^{(p-1)q} i(Q)\circ i(P)(df\wedge\eta) + (-1)^{(p-1)q+p} i(Q)(df\wedge i(P)\eta).\end{aligned}$$

Lemma 5.4.1 shows that the sum of the last four terms of the right hand side vanishes, so

$$[[i(P),d_\rho],i(Q)](f\eta) = f[[i(P),d_\rho],i(Q)]\eta.$$

Let us take $r = p+q-1$, and $\eta\in\Omega^{p+q-1}(M,E)$. The map

$$\eta\mapsto[[i(P),d_\rho],i(Q)]\eta,$$

defined on $\Omega^{p+q-1}(M,E)$, takes its values in $\Omega^0(M,E) = C^\infty(M,\mathbb{R})$, and is $C^\infty(M,\mathbb{R})$-linear. This proves the existence of a unique element $[P,Q]$ in $\Omega^{p+q-1}(M,E)$ such that, for all $\eta \in \Omega^{p+q-1}(M,E)$,

$$[[i(P),d_\rho],i(Q)]\eta = i([P,Q])\eta.$$

We still have to prove that the same formula holds for all $r \in \mathbb{Z}$ and all $\eta \in \Omega^r(M,E)$. The formula holds trivially when $r < p+q-1$, so let us assume that $r > p+q-1$. Let $\eta \in \Omega^r(M,E)$ and $R \in A^{r-p-q+1}(M,E)$. By using Lemma 5.4.2, we may write

$$\begin{aligned} i(R)\circ[[i(P),d_\rho],i(Q)](\eta) &= (-1)^{(p+q-1)(r-p-q+1)}[[i(P),d_\rho],i(Q)](i(R)\eta)\\ &= (-1)^{(p+q-1)(r-p-q+1)}i([P,Q])(i(R)\eta), \end{aligned}$$

since $i(R)\eta \in \Omega^{p+q-1}(E)$. Therefore

$$i(R)\circ[[i(P),d_\rho],i(Q)](\eta) = (-1)^{(p+q-1)(r-p-q+1)}i([P,Q])\circ i(R)\eta) = i(R)\circ i([P,Q])\eta.$$

Since this equality holds for all $\eta \in \Omega^r(M,E)$ and $R \in A^{r-p-q+1}(M,E)$, we conclude that $[[i(P),d_\rho],i(Q)] = i([P,Q])$, and the proof is complete. ∎

In Proposition 5.1.1, we introduced the Lie derivative with respect to a section of the Lie algebroid (E,τ,M,ρ). Now we define, for all $p \in \mathbb{Z}$ and $P \in A^p(M,E)$, the Lie derivative with respect to P. The reader will observe that Property 1 of Proposition 5.2.3 shows that for $p=1$, the following definition is in agreement with the definition of the Lie derivative with respect to an element in $A^1(M,E)$ given in 5.1.1.

Definition 5.4.4. Let (E,τ,M,ρ) be a Lie algebroid, $p \in \mathbb{Z}$ and $P \in A^p(M,E)$. The *Lie derivative* with respect to P is the graded endomorphism of $\Omega(M,P)$, of degree $1-p$, denoted by $\mathcal{L}_\rho(P)$,

$$\mathcal{L}_\rho(P) = [i(P),d_\rho] = i(P)\circ d_\rho - (-1)^p d_\rho \circ i(P).$$

Remark 5.4.5. Under the assumptions of Theorem 5.4.3, the above definition allows us to write

$$i([P,Q]) = [\mathcal{L}_\rho(P),i(Q)] = \mathcal{L}_\rho(P)\circ i(Q) - (-1)^{(p-1)q}i(Q)\circ\mathcal{L}_\rho(P).$$

For $p=1$ and $P=V \in A^1(M,E)$, this formula is simply Property 4 of Proposition 5.1.6, as shown by the following proposition.

Proposition 5.4.6. *Under the assumptions of Theorem* 5.4.3*, let $p=1$, $P=V \in A^1(M,E)$ and $Q \in A^q(M,E)$. The Schouten–Nijenhuis bracket $[V,Q]$ is simply the Lie derivative of Q with respect to V, as defined in Proposition* 5.1.5*:*

$$[V,Q] = \mathcal{L}_\rho(V)Q.$$

Proof. As seen in Remark 5.4.5, we may write

$$i([V,Q]) = [\mathcal{L}_\rho(V),i(Q)] = \mathcal{L}_\rho(V)\circ i(Q) - i(Q)\circ\mathcal{L}_\rho(V).$$

Property 4 of Proposition 5.1.6 shows that

$$i(\mathcal{L}_\rho(V)Q) = \mathcal{L}_\rho(V)\circ i(Q) - i(Q)\circ\mathcal{L}_\rho(V).$$

Therefore, $i([V,Q]) = i(\mathcal{L}_\rho(V)Q)$, and finally $[V,Q] = \mathcal{L}_\rho(V)Q$, which ends the proof. ∎

REMARKS 5.4.7. (i) *The Lie derivative of elements in $A^p(M,E)$.* One might wish to extend the range of application of the Lie derivative with respect to a multivector $P \in A^p(M,E)$ by setting, for all $q \in \mathbb{Z}$ and $Q \in A^q(M,E)$,

$$\mathcal{L}_\rho(P)Q = [P,Q],$$

the bracket on the right hand side being the Schouten–Nijenhuis bracket. However, we will avoid the use of this notation because it may lead to confusion: for $p > 1$, $P \in A^p(M,E)$, $q = 0$ and $Q = f \in A^0(M,E) = C^\infty(M,\mathbb{R})$, the Schouten–Nijenhuis bracket $[P,f]$ is an element in $A^{p-1}(M,E)$ which does not vanish in general. But f can also be considered as an element in $\Omega^0(M,E)$, and the Lie derivative of f with respect to P, in the sense of Definition 5.4.4, is an element in $\Omega^{-(p-1)}(M,E)$, therefore vanishes identically. So it would not be a good idea to write $\mathcal{L}_\rho(P)f = [P,f]$.

(ii) *Lie derivatives and derivations.* We have seen (Property 3 of 5.1.4) that the Lie derivative $\mathcal{L}_\rho(V)$ with respect to a section $V \in A^1(M,R)$ of the Lie algebroid (E,τ,M,ρ) is a derivation of degree 0 of the exterior algebra $\Omega(M,E)$. For $p > 1$ and $P \in A^p(M,E)$, the Lie derivative $\mathcal{L}_\rho(P)$ is a graded endomorphism of degree $-(p-1)$ of $\Omega(M,E)$. Therefore, it vanishes identically on $\Omega^0(M,E)$ and on $\Omega^1(M,E)$. Unless it vanishes identically, $\mathcal{L}_\rho(P)$ is not a derivation of $\Omega(M,E)$.

PROPOSITION 5.4.8. *Let (E,τ,M,ρ) be a Lie algebroid, $p,q \in \mathbb{Z}$, $P \in A^p(M,E)$ and $Q \in A^q(M,E)$.*

1. *The graded bracket of the Lie derivative $\mathcal{L}_\rho(P)$ and the exterior differential d_ρ vanishes identically:*
$$[\mathcal{L}_\rho(P), d_\rho] = \mathcal{L}_\rho(P) \circ d_\rho - (-1)^{p-1} d_\rho \circ \mathcal{L}_\rho(P) = 0.$$
2. *The graded bracket of the Lie derivatives $\mathcal{L}_\rho(P)$ and $\mathcal{L}_\rho(Q)$ is equal to the Lie derivative $\mathcal{L}_\rho([P,Q])$:*
$$[\mathcal{L}_\rho(P), \mathcal{L}_\rho(Q)] = \mathcal{L}_\rho(P) \circ \mathcal{L}_\rho(Q) - (-1)^{(p-1)(q-1)} \mathcal{L}_\rho(Q) \circ \mathcal{L}_\rho(P) = \mathcal{L}_\rho([P,Q]).$$

Proof. 1. We have seen (4.1.8(ii)) that the space of graded endomorphisms of $\Omega(M,E)$, with the graded bracket as composition law, is a graded Lie algebra. By using the graded Jacobi identity, we may write

$$(-1)^p[[i(P), d_\rho], d_\rho] + (-1)^p[[d_\rho, d_\rho], i(P)] - [[d_\rho, i(P)], d_\rho] = 0.$$

But

$$[d_\rho, d_\rho] = 2 d_\rho \circ d_\rho = 0 \quad \text{and} \quad [i(P), d_\rho] = -(-1)^p [d_\rho, i(P)].$$

So we obtain

$$2[[i(P), d_\rho], d_\rho] = 2[\mathcal{L}_\rho(P), d_\rho] = 0.$$

2. We have

$$\mathcal{L}_\rho([P,Q]) = [i([P,Q]), d_\rho] = [[\mathcal{L}_\rho(P), i(Q)], d_\rho].$$

Using the graded Jacobi identity, we may write

$$(-1)^{p-1}[[\mathcal{L}_\rho(P), i(Q)], d_\rho] + (-1)^{q(p-)}[[i(Q), d_\rho], \mathcal{L}_\rho(P)] + (-1)^q[[d_\rho, \mathcal{L}_\rho(P)], i(Q)] = 0.$$

But, according to 5.4.4 and Property 1 above,

$$[i(Q), d_\rho] = \mathcal{L}_\rho(Q) \quad \text{and} \quad [d_\rho, \mathcal{L}_\rho(P)] = 0.$$

So we obtain

$$\mathcal{L}_\rho([P,Q]) = -(-1)^{(p-1)(q-1)}\big[\mathcal{L}_\rho(Q), \mathcal{L}_\rho(P)\big] = \big[\mathcal{L}_\rho(P), \mathcal{L}_\rho(Q)\big],$$

as announced. ■

PROPOSITION 5.4.9. *Under the same assumptions as those of Theorem* 5.4.3*, the Schouten–Nijenhuis bracket has the following properties.*

1. *For* $f, g \in A^0(M,E) = C^\infty(M,\mathbb{R})$,
$$[f,g] = 0.$$
2. *For* $V \in A^1(M,E)$, $q \in \mathbb{Z}$ *and* $Q \in A^q(M,E)$,
$$[V,Q] = \mathcal{L}_\rho(V)Q.$$
3. *For* $V, W \in A^1(M,E)$,
$$[V,W] = \{V,W\},$$
the bracket on the right hand side being the Lie algebroid bracket.
4. *For all* $p, q \in \mathbb{Z}$, $P \in A^p(M,E)$ *and* $Q \in A^q(M,E)$,
$$[P,Q] = -(-1)^{(p-1)(q-1)}[Q,P].$$
5. *Let* $p \in \mathbb{Z}$ *and* $P \in A^p(M,E)$. *The map* $Q \mapsto [P,Q]$ *is a derivation of degree* $p-1$ *of the graded exterior algebra* $A(M,E)$. *In other words, for* $q_1, q_2 \in \mathbb{Z}$, $Q_1 \in A^{q_1}(M,E)$ *and* $Q_2 \in A^{q_2}(M,E)$,
$$[P, Q_1 \wedge Q_2] = [P, Q_1] \wedge Q_2 + (-1)^{(p-1)q_1} Q_1 \wedge [P, Q_2].$$
6. *Let* $p, q, r \in \mathbb{Z}$, $P \in A^p(M,E)$, $Q \in A^q(M,E)$ *and* $R \in A^r(M,E)$. *The Schouten–Nijenhuis bracket satisfies the graded Jacobi identity:*
$$(-1)^{(p-1)(r-1)}[[P,Q],R] + (-1)^{(q-1)(p-1)}[[Q,R],P] + (-1)^{(r-1)(q-1)}[[R,P],Q] = 0.$$

Proof. 1. Let $f, g \in A^0(M,E)$. Then $[f,g] \in A^{-1}(M,E) = \{0\}$.

2. See Proposition 5.4.6.

3. See Property 1 of Proposition 5.1.6.

4. Let $p, q \in \mathbb{Z}$, $P \in A^p(M,E)$ and $Q \in A^q(M,E)$. By using the graded Jacobi identity for graded endomorphisms of $\Omega(M,E)$, we may write

$$(-1)^{pq}[[i(P), d_\rho], i(Q)] + (-1)^p[[d_\rho, i(Q)], i(P)] + (-1)^q[[i(Q), i(P)], d_\rho] = 0.$$

By using

$$[i(Q), i(P)] = i(Q \wedge P) - i(Q \wedge P) = 0 \quad \text{and} \quad [d_\rho, i(Q)] = -(-1)^q[i(Q), d_\rho],$$

we obtain

$$(-1)^{pq}[[i(P), d_\rho], i(Q)] + (-1)^{p+q-1}[[i(Q), d_\rho], i(P)] = 0,$$

so the result follows immediately.

5. Let $p, q_1, q_2 \in \mathbb{Z}$, $P \in A^p(M,E)$, $Q_1 \in A^{q_1}(M,E)$ and $Q_2 \in A^{q_2}(M,E)$. We may write

$$\begin{aligned} i([P, Q_1 \wedge Q_2]) &= [\mathcal{L}_\rho(P), i(Q_1 \wedge Q_2)] \\ &= \mathcal{L}_\rho(P) \circ i(Q_1 \wedge Q_2) - (-1)^{(p-1)(q_1+q_2)} i(Q_1 \wedge Q_2) \circ \mathcal{L}_\rho(P). \end{aligned}$$

We add and subtract $(-1)^{(p-1)q_1} i(Q_1) \circ \mathcal{L}_\rho(P) \circ i(Q_1)$ from the last expression, and replace $i(Q_1 \wedge Q_2)$ by $i(Q_1) \circ i(Q_2)$. We obtain

$$i([P, Q_1 \wedge Q_2]) = [\mathcal{L}_\rho(P), i(Q_1)] \circ i(Q_2) + (-1)^{(p-1)q_1} i(Q_1) \circ [\mathcal{L}_\rho(P), i(Q_2)].$$

The result follows immediately.

6. Let $p, q, r \in \mathbb{Z}$, $P \in A^p(M,E)$, $Q \in A^q(M,E)$ and $R \in A^r(M,E)$. By using Property 2 of Proposition 5.4.8, we may write

$$i([[P,Q],R]) = [\mathcal{L}_\rho([P,Q]), i(R)] = [[\mathcal{L}_\rho(P), \mathcal{L}_\rho(Q)], i(R)].$$

Using the graded Jacobi identity, we obtain

$$\begin{aligned} (-1)^{(p-1)r} [[\mathcal{L}_\rho(P), \mathcal{L}_\rho(Q)], i(R)] + (-1)^{(q-1)(p-1)} [[\mathcal{L}_\rho(Q), i(R)], \mathcal{L}_\rho(P)] \\ + (-1)^{r(q-1)} [[i(R), \mathcal{L}_\rho(P)], \mathcal{L}_\rho(Q)] = 0. \end{aligned}$$

But

$$\begin{aligned} [[\mathcal{L}_\rho(Q), i(R)], \mathcal{L}_\rho(P)] &= [i([Q,R]), \mathcal{L}_\rho(P)] = -(-1)^{(q+r-1)(p-1)} [\mathcal{L}_\rho(P), i([Q,R])] \\ &= -(-1)^{(q+r-1)(p-1)}\, i([P,[Q,R]]) \\ &= (-1)^{(q+r-1)(p-1)+(p-1)(q+r-2)}\, i([[Q,R],P]) \\ &= (-1)^{p-1}\, i([[Q,R],P]). \end{aligned}$$

Similarly,

$$\begin{aligned} [[i(R), \mathcal{L}_\rho(P)], \mathcal{L}_\rho(Q)] &= -(-1)^{(p-1)r}\, [[\mathcal{L}_\rho(P), i(R)], \mathcal{L}_\rho(Q)] \\ &= -(-1)^{(p-1)r}\, [i([P,R]), \mathcal{L}_\rho(Q)] \\ &= (-1)^{(p-1)r+(p+r-1)(q-1)}\, [\mathcal{L}_\rho(Q), i([P,R])] \\ &= (-1)^{(p-1)(r+q-1)+r(q-1)}\, i([Q,[P,R]]) \\ &= -(-1)^{(p-1)q+(q-1)(p-2)}\, i([[R,P],Q]) \\ &= -(-1)^{p+q}\, i([[R,P],Q]). \end{aligned}$$

Using the above equalities, we obtain

$$\begin{aligned} (-1)^{(p-1)(r-1)}\, i([[P,Q]),R]) + (-1)^{(q-1)(p-1)}\, i([[Q,R],P]) \\ + (-1)^{(r-1)(q-1)}\, i([[R,P],Q]]) = 0. \end{aligned}$$

The proof is complete. ∎

Remarks 5.4.10. Let (E, τ, M, ρ) be a Lie algebroid.

(i) *Degrees for the two algebra structures of* $A(M,E)$. The algebra $A(M,E) = \bigoplus_{p\in\mathbb{Z}} A^p(M,E)$ of sections of the exterior powers $(\bigwedge^p E, \tau, M)$, with the exterior product as composition law, is a graded associative algebra; for that structure, the space of homogeneous elements of degree p is $A^p(M,E)$. Proposition 5.4.9 shows that $A(M,E)$, with the Schouten–Nijenhuis bracket as composition law, is a graded Lie algebra; for that structure, the space of homogeneous elements of degree p is not $A^p(M,E)$, but rather $A^{p+1}(M,E)$. For homogeneous elements in $A(M,E)$, one should therefore make a distinction between the degree for the graded associative algebra structure and the degree for the graded Lie algebra structure; an element in $A^p(M,E)$ has degree p for the graded associative algebra structure, and degree $p-1$ for the graded Lie algebra structure.

(ii) *The anchor as a graded Lie algebras homomorphism.* The anchor $\rho : E \to TM$ allows us to associate to each smooth section $X \in A^1(M,E)$ a smooth vector field $\rho \circ X$ on M; according to Definition 3.1.1, that correspondence is a Lie algebra homomorphism. We can extend that map, for all $p \geq 1$, to the space $A^p(M,E)$ of smooth sections of the pth exterior power $(\bigwedge^p E, \tau, M)$. First, for a decomposable element $X_1 \wedge \cdots \wedge X_p$ with $X_i \in A^1(M,E)$, we set

$$\rho \circ (X_1 \wedge \cdots \wedge X_p) = (\rho \circ X_1) \wedge \cdots \wedge (\rho \circ X_p).$$

For $p = 0$ and $f \in A^0(M,E) = C^\infty(M,\mathbb{R})$, we set, by convention,

$$\rho \circ f = f.$$

Then we extend that correspondence to all elements in $A(M,E)$ by $C^\infty(M,\mathbb{R})$-linearity. The map $P \mapsto \rho \circ P$ obtained in that way is a homomorphism from $A(M,E)$ into $A(M,TM)$, both for their graded associative algebra structures (with the exterior products as composition laws) and their graded Lie algebra structures (with the Schouten–Nijenhuis brackets, associated to the Lie algebroid structure of (E,τ,M,ρ) and to the Lie algebroid structure of the tangent bundle $(TM,\tau_M,M,\mathrm{id}_{TM})$ as composition laws).

In 6.2.2(iii), we will see that when the Lie algebroid under consideration is the cotangent bundle to a Poisson manifold, the anchor map has yet another property: it induces a cohomology anti-homomorphism.

6. Poisson manifolds and Lie algebroids

In this final section we will show that there exist very close links between Poisson manifolds and Lie algebroids.

6.1. Poisson manifolds. Poisson manifolds were introduced by A. Lichnerowicz in the important paper [33]. Their importance was soon recognized, and their properties were investigated in depth by A. Weinstein [49]. Let us recall briefly their definition and some of their properties. The reader is referred to [33, 49, 48] for the proofs of these properties.

Definition 6.1.1. Let M be a smooth manifold. We assume that the space $C^\infty(M,\mathbb{R})$ of smooth functions on M is endowed with a composition law, denoted by $(f,g) \mapsto \{f,g\}$, for which $C^\infty(M,\mathbb{R})$ is a Lie algebra, which moreover satisfies the Leibniz-type formula

$$\{f, gh\} = \{f,g\}h + g\{f,h\}.$$

We say that the structure defined on M by such a composition law is a *Poisson structure*, and that the manifold M, equipped with that structure, is a *Poisson manifold*.

The following proposition is due to A. Lichnerowicz [33]. Independently, A. Kirillov [23] introduced local Lie algebras (which include both Poisson manifolds and Jacobi manifolds, which were also introduced by A. Lichnerowicz [34]) and obtained, without using the Schouten–Nijenhuis bracket, an equivalent result and its generalization for Jacobi manifolds.

PROPOSITION 6.1.2. *On a Poisson manifold M, there exists a unique smooth section of the bundle of bivectors, $\Lambda \in A^2(M, TM)$, called the* Poisson bivector*, which satisfies*

$$[\Lambda, \Lambda] = 0, \tag{$*$}$$

such that for any $f, g \in C^\infty(M, \mathbb{R})$,

$$\{f, g\} = \Lambda(df, dg). \tag{$**$}$$

The bracket on the left hand side of $(*)$ is the Schouten–Nijenhuis bracket of multivectors on M for the canonical Lie algebroid structure of (TM, τ_M, M) (with id_{TM} as anchor).

Conversely, let Λ be a smooth section of $A^2(TM, M)$. We use formula $(**)$ to define a composition law on $C^\infty(M, \mathbb{R})$. The structure defined on M by that composition law is a Poisson structure if and only if Λ satisfies formula $(*)$.

In what follows, we will denote by (M, Λ) a manifold M equipped with a Poisson structure whose Poisson bivector is Λ.

6.2. The Lie algebroid structure on the cotangent bundle of a Poisson manifold. The next theorem shows that the cotangent bundle of a Poisson manifold has a canonical structure of Lie algebroid. That property was discovered by Dazord and Sondaz [13].

THEOREM 6.2.1. *Let (M, Λ) be a Poisson manifold. The cotangent bundle (T^*M, π_M, M) has a canonical structure of Lie algebroid characterized by the following properties:*

(i) *the bracket $[\eta, \zeta]$ of two sections η and ζ of (T^*M, π_M, M), i.e., of two Pfaff forms on M, is given by the formula*

$$\langle [\eta, \zeta], X \rangle = \langle \eta, [\Lambda, \langle \zeta, X \rangle] \rangle - \langle \zeta, [\Lambda, \langle \eta, X \rangle] \rangle - [\Lambda, X](\eta, \zeta),$$

where X is any smooth vector field on M; the bracket on the right hand side is the Schouten–Nijenhuis bracket of multivectors on M;

(ii) *the anchor is the vector bundle map $\Lambda^\sharp : T^*M \to TM$ such that, for all $x \in M$ and $t, s \in T^*_x M$,*

$$\langle \beta, \Lambda^\sharp \alpha \rangle = \Lambda(\alpha, \beta).$$

Proof. We define a linear endomorphism δ_Λ of $A(M, TM)$ by setting, for each $P \in A(M, TM)$,

$$\delta_\Lambda(P) = [\Lambda, P],$$

where the bracket on the right hand side is the Schouten–Nijenhuis bracket of multivectors on M, i.e., the Schouten–Nijenhuis bracket for the canonical Lie algebroid structure of (TM, τ_M, M) (with id_{TM} as anchor map). When P is in $A^p(M, TM)$, $\delta_\Lambda(P)$ is in

$A^{p+1}(M,TM)$, therefore δ_Λ is homogeneous of degree 1. For each $P \in A^p(M,TM)$ and $Q \in A^q(M,TM)$, we have

$$\begin{aligned}\delta_\Lambda(P \wedge Q) &= [\Lambda, P \wedge Q] = [\Lambda, P] \wedge Q + (-1)^p P \wedge [\Lambda, Q] \\ &= \delta_\Lambda(P) \wedge Q + P \wedge \delta_\Lambda(Q).\end{aligned}$$

This proves that δ_Λ is a graded derivation of degree 1 of the exterior algebra $A(M,TM)$.

Moreover, for each $P \in A^p(M,TM)$ we obtain, by using the graded Jacobi identity,

$$\begin{aligned}\delta_\Lambda \circ \delta_\Lambda(P) &= [\Lambda, [\Lambda, P]] = (-1)^{p-1}[\Lambda, [P, \Lambda]] - [P, [\Lambda, \Lambda]] \\ &= -[\Lambda, [\Lambda, P]] - [P, [\Lambda, \Lambda]] = -\delta_\Lambda \circ \delta_\Lambda(P) - [P, [\Lambda, \Lambda]].\end{aligned}$$

Therefore

$$2\delta_\Lambda \circ \delta_\Lambda(P) = -[P, [\Lambda, \Lambda]] = 0,$$

since $[\Lambda, \Lambda] = 0$. We have proven that the graded derivation δ_Λ, of degree 1, satisfies

$$\delta_\Lambda^2 = \delta_\Lambda \circ \delta_\Lambda = 0.$$

Now we observe that the tangent bundle (TM, τ_M, M) can be considered as the dual bundle of the cotangent bundle (T^*M, π_M, M). Therefore, we may apply Theorem 5.3.3, which shows that there exists on (T^*M, π_M, M) a Lie algebroid structure for which δ_M is the associated derivation on the space $\Omega(M, T^*M) = A(M,TM)$ (with the notations defined in 4.2.4). That theorem also shows that the bracket of two smooth sections of (T^*M, π_M, M), i.e., of two Pfaff forms η and ζ on M, is given by the formula, where X is any smooth vector field on M,

$$\langle [\eta, \zeta], X \rangle = \langle \eta, [\Lambda, \langle \zeta, X \rangle] \rangle - \langle \zeta, [\Lambda, \langle \eta, X \rangle] \rangle - [\Lambda, X](\eta, \zeta).$$

The anchor map ρ is such that, for each $\eta \in \Omega^1(M,TM)$ and each $f \in C^\infty(M, \mathbb{R})$,

$$i(\rho \circ \eta)\, df = \langle \eta, [\Lambda, f] \rangle.$$

The bracket which appears on the right hand sides of these two formulae is the Schouten–Nijenhuis bracket of multivectors on M. By using Theorem 5.4.3, we see that

$$[\Lambda, f] = -\Lambda^\sharp(df).$$

Therefore,

$$\langle df, \rho \circ \eta \rangle = i(\rho \circ \eta)\, df = \langle \eta, -\Lambda^\sharp(df) \rangle = \langle df, \Lambda^\sharp(\eta) \rangle.$$

So we have $\rho = \Lambda^\sharp$. ∎

Remarks 6.2.2. Let (M, Λ) be a Poisson manifold.

(i) *The bracket of forms of any degrees on M.* Since, by Theorem 6.2.1, $(T^*M, \pi_M, M, \Lambda^\sharp)$ is a Lie algebroid, we can define a composition law in the space $A(M, T^*M) = \Omega(M, \mathbb{R})$ of smooth differential forms of all degrees on M: the Schouten–Nijenhuis bracket for the Lie algebroid structure of (T^*M, π_M, M), with $\Lambda^\sharp$ as anchor. With that composition law, denoted by $(\eta, \zeta) \mapsto [\eta, \zeta]$, $\Omega(M, \mathbb{R})$ is a graded Lie algebra. Observe that a form $\eta \in \Omega^p(M, \mathbb{R})$, of degree p for the graded associative algebra structure whose composition law is the exterior product, has degree $p-1$ for the graded Lie algebra structure.

The bracket of differential forms on a Poisson manifold was first discovered for Pfaff forms by Magri and Morosi [38]. It is related to the Poisson bracket of functions by the formula

$$[df, dg] = d\{f, g\}, \qquad f, g \in C^\infty(M, \mathbb{R}).$$

That bracket was extended to forms of all degrees by Koszul [28], and rediscovered, with the Lie algebroid structure of T^*M, by Dazord and Sondaz [13].

(ii) *The Lichnerowicz–Poisson cohomology.* The derivation δ_Λ,

$$P \mapsto \delta_\Lambda(P) = [\Lambda, P], \quad P \in A(M, TM),$$

used in the proof of 6.2.1, was first introduced by A. Lichnerowicz [33], who observed that it may be used to define a cohomology with elements in $A(M, TM)$ as cochains. He began the study of that cohomology, often called the Poisson cohomology (but which should be called the Lichnerowicz–Poisson cohomology). The study of that cohomology was carried on by Vaisman [48], Huebschmann [20], Xu [54] and many other authors.

(iii) *The map $\Lambda^\sharp$ as a cohomology anti-homomorphism.* In 5.4.10(ii), we have seen that the anchor map ρ of a Lie algebroid (E, τ, M, ρ) yields a map $P \mapsto \rho \circ P$ from $A(M, E)$ into $A(M, TM)$, which is both a homomorphism of graded associative algebras (the composition laws being the exterior products) and a homomorphism of graded Lie algebras (the composition laws being the Schouten brackets). When applied to the Lie algebroid $(T^*M, \pi_M, M, \Lambda^\sharp)$, this property shows that the map $\eta \mapsto \Lambda^\sharp \circ \eta$ is a homomorphism from the space of differential forms $\Omega(M, \mathbb{R})$ into the space of multivectors $A(M, \mathbb{R})$, both for their structures of graded associative algebras and graded Lie algebras. As observed by A. Lichnerowicz [33], this map exchanges the exterior derivation d of differential forms and the derivation δ_Λ of multivectors (with a sign change, under our sign conventions), in the following sense: for any $\eta \in \Omega^p(M, \mathbb{R})$, we have

$$\Lambda^\sharp(d\eta) = -\delta_\Lambda(\Lambda^\sharp(\eta)) = -[\Lambda, \Lambda^\sharp(\eta)].$$

This property is an easy consequence of the formula, valid for any smooth function $f \in C^\infty(M, \mathbb{R})$, which can be derived from Theorem 5.4.3,

$$\Lambda^\sharp(df) = -[\Lambda, f].$$

The map $\Lambda^\sharp$ therefore induces an anti-homomorphism from the Lichnerowicz–Poisson cohomology of the Poisson manifold (M, Λ) into its de Rham cohomology.

(iv) *Lie bialgebroids.* Given a Poisson manifold (M, Λ), we have Lie algebroid structures both on the tangent bundle (TM, τ_M, M) and on the cotangent bundle (T^*M, π_M, M), with $\mathrm{id}_{TM} : TM \to TM$ and $\Lambda^\sharp : T^*M \to TM$ as their respective anchor maps. Moreover, these two Lie algebroid structures are compatible in the following sense: the derivation $\delta_\Lambda : P \mapsto [\Lambda, P]$ of the graded associative algebra $A(M, TM)$ (the composition law being the exterior product) determined by the Lie algebroid structure of (T^*M, π_M, M) is also a derivation for the graded Lie algebra structure of $A(M, E)$ (the composition law being now the Schouten–Nijenhuis bracket). We have indeed, as an easy consequence of the graded Jacobi identity, for $P \in A^p(M, TM)$ and $Q \in A^q(M, TM)$,

$$\begin{aligned}\delta_\Lambda([P, Q]) &= [\Lambda, [P, Q]] = [[\Lambda, P], Q] + (-1)^{p-1}[P, [\Lambda, Q]] \\ &= [\delta_\Lambda P, Q] + (-1)^{p-1}[P, \delta_\Lambda Q].\end{aligned}$$

When two Lie algebroid structures on two vector bundles in duality satisfy such a compatibility condition, the pair of Lie algebroids is said to be a *Lie bialgebroid*. The important notion of a Lie bialgebroid is due to K. Mackenzie and P. Xu [37]. Its study was developed by Y. Kosmann-Schwarzbach [25] and her student M. Bangoura [2] and many other authors. D. Iglesias and J. C. Marrero have introduced a generalization of that notion in relation with Jacobi manifolds [21].

6.3. The Poisson structure on the dual bundle of a Lie algebroid. We will now prove that there is a 1-1 correspondence between Lie algebroid structures on a vector bundle (E,τ,M) and homogeneous Poisson structures on the total space of the dual bundle (E^*,π,M). This will allow us to recover some well known results (Remarks 6.3.7).

We will use the following definition.

Definition 6.3.1. Let (E,τ,M) be a vector bundle and (E^*,π,M) its dual bundle. To each smooth section $X\in A^1(M,E)$, we associate a smooth function Φ_X defined on E^* by

$$\Phi_X(\xi)=\langle\xi, X\circ\pi(\xi)\rangle,\quad \xi\in E^*.$$

We will say that Φ_X is the *vertical function* on E^* associated to the smooth section X.

Lemma 6.3.2. *Let (E,τ,M) be a vector bundle and (E^*,π,M) its dual bundle.*

1. *If, for some smooth section $X\in A^1(M,E)$, some smooth function $f\in C^\infty(M,\mathbb{R})$ and some $\xi\in E^*$, $d(\Phi_X+f\circ\pi)(\xi)=0$, where Φ_X is the vertical function associated to X (Definition 6.3.1), then $X(\pi(\xi))=0$.*
2. *For each $\xi\in E^*$ and each $\eta\in T^*_\xi E^*$, there exists a smooth section $X\in A^1(M,E)$ and a smooth function $f\in C^\infty(M,\mathbb{R})$ such that $d(\Phi_X+f\circ\pi)(\xi)=\eta$.*

Proof. These properties being local, we may work in an open subset U of M on which there exists a system of local coordinates $(x^1,\dots,x^n)$ and smooth sections $(s_1,\dots,s_k)$ of τ such that for each $x\in U$, $(s_1(x),\dots,s_k(x))$ is a basis of E_x. A smooth section X of τ defined on U can be written as

$$X=\sum_{r=1}^k X^r s_r,$$

where the X^r are smooth functions on U. We will denote by the same letters X^r the expression of these functions in local coordinates $(x^1,\dots,x^n)$. Similarly we will denote by f both a smooth function in $C^\infty(M,\mathbb{R})$ and its expression in local coordinates. The vertical function, defined on $\pi^{-1}(U)$, which corresponds to X is

$$\Phi_X(\xi)=\sum_{r=1}^k \xi_r X^r(\pi(\xi)),\quad \xi\in\pi^{-1}(U),\quad \text{where } \xi_r=\langle\xi, s_r(\pi(\xi))\rangle.$$

On $\pi^{-1}(U)$, $(x^1,\dots,x^n,\xi_1,\dots,\xi_k)$ is a smooth system of local coordinates, in which

$$\begin{aligned}&d(\Phi_X+f\circ\pi)(\xi)\\&\quad=\sum_{r=1}^k X^r(x^1,\dots,x^n)\,d\xi_r+\sum_{j=1}^n\left(\sum_{r=1}^k\xi_r\frac{\partial X^r(x^1,\dots,x^n)}{\partial x^j}+\frac{\partial f(x^1,\dots,x^n)}{\partial x^j}\right)dx^j.\end{aligned}$$

This result shows that if $d(\Phi_X+f\circ\pi)(\xi)=0$, then $X(\pi(\xi))=0$.

Let $\xi \in E^*$ and $\eta \in T^*_\xi E^*$ be given. The above formula shows that if $\xi \neq 0$, we can take $f = 0$ and choose X such that $d\Phi_X(\xi) = \eta$. If $\xi = 0$, we can take $X = 0$ and f such that $d(f \circ \pi)(\xi) = \eta$. ■

DEFINITION 6.3.3. Let (E, τ, M) be a vector bundle and (E^*, π, M) its dual bundle. A Poisson structure on E^* is said to be *homogeneous* if the Poisson bracket of two vertical functions (Definition 6.3.1) is vertical.

PROPOSITION 6.3.4. *Let (E, τ, M) be a vector bundle, (E^*, π, M) its dual bundle, and Λ be a Poisson structure on E^*. The following properties are equivalent.*

1. *There exists a dense subset U of E^* and a subset $\mathcal{F}$ of the set of vertical functions on E^* whose differentials $df(\xi)$, $f \in \mathcal{F}$, span the cotangent space $T^*_\xi E^*$, for all $\xi \in U$, such that the Poisson bracket of any two functions in $\mathcal{F}$ is vertical.*
2. *Let Z_{E^*} be the vector field on E^* whose flow generates homotheties in the fibres. We recall that its value at $\xi \in E^*$ is*
$$Z_{E^*}(\xi) = \left.\frac{d(\exp(t)\xi)}{dt}\right|_{t=0}.$$
The Poisson structure on E^* satisfies
$$[Z_{E^*}, \Lambda] = -\Lambda.$$
3. *The Poisson structure Λ is homogeneous.*

Proof. The reduced flow of the vector field Z_{E^*} is the one-parameter group of homotheties in the fibres $(t, \xi) \mapsto H_t(\xi) = \exp(t)\xi$ with $t \in \mathbb{R}$, $\xi \in E^*$. For any smooth section $X \in A^1(M, E)$ and any $t \in \mathbb{R}$, we have
$$(H_t^*\Phi_X)(\xi) = \Phi_X \circ H_t(\xi) = \Phi_X(\exp(t)\xi) = \exp(t)\Phi_X(\xi),$$
therefore
$$H_t^*\Phi_X = \exp(t)\Phi_X, \quad \mathcal{L}(Z_{E^*})\Phi_X = \left.\frac{dH_t^*\Phi_X}{dt}\right|_{t=0} = \Phi_X.$$
Let us assume that Property 1 is true. Let $X, Y \in A^1(M, E)$ be such that Φ_X and Φ_Y are in the subset $\mathcal{F}$. Then $\{\Phi_X, \Phi_Y\}$ is vertical, so for all $t \in \mathbb{R}$ we have
$$H_t^*(\Lambda(d\Phi_X, d\Phi_Y)) = \{\Phi_X, \Phi_Y\} \circ H_t = \exp(t)\{\Phi_X, \Phi_Y\}.$$
But we may also write
$$H_t^*(\Lambda(d\Phi_X, d\Phi_Y)) = (H_t^*\Lambda)(H_t^* d\Phi_X, H_t^* d\Phi_Y) = \exp(2t)(H_t^*\Lambda)(d\Phi_X, d\Phi_Y).$$
Since for each $\xi \in U$, the differentials at ξ of functions in $\mathcal{F}$ generate $T^*_\xi E^*$, this result proves that in U,
$$H_t^*(\Lambda) = \exp(-t)\Lambda.$$
Since U is dense in E^* this equality holds everywhere on E^*, therefore
$$[Z_{E^*}, \Lambda] = \mathcal{L}(Z_{E^*})\Lambda = \left.\frac{dH_t^*\Lambda}{dt}\right|_{t=0} = -\Lambda.$$

We have proven that Property 1 implies 2. Let us now assume that 2 is true. For all $X, Y \in A^1(M, E)$,

$$\begin{aligned}\mathcal{L}(Z_{E^*})(\{\Phi_X, \Phi_Y\}) &= \mathcal{L}(Z_{E^*})(\Lambda(d\Phi_X, d\Phi_Y))\\ &= (\mathcal{L}(Z_{E^*})\Lambda)(d\Phi_X, d\Phi_Y) + \Lambda(\mathcal{L}(Z_{E^*})\Phi_X, \Phi_Y) + \Lambda(\Phi_X, \mathcal{L}(Z_{E^*})\Phi_Y)\\ &= \Lambda(\Phi_X, \Phi_Y) = \{\Phi_X, \Phi_Y\}.\end{aligned}$$

Since $\{\Phi_X, \Phi_Y\}$ is smooth on E^*, including on the zero section, this function is linear on each fibre of E^*, in other words it is vertical, and we have proven that Property 2 implies 3.

Finally, 3 of course implies 1, and our proof is complete. ■

THEOREM 6.3.5. *Let (E, τ, M) be a vector bundle and (E^*, π, M) its dual bundle. There is a* 1-1 *correspondence between Lie algebroid structures on (E, τ, M) and homogeneous Poisson structures on E^* (Definition* 6.3.3) *such that, for each pair (X, Y) of smooth sections of τ, Φ_X and Φ_Y being the corresponding vertical functions on E^* (Definition* 6.3.1),

$$\{\Phi_X, \Phi_Y\} = \Phi_{\{X,Y\}},$$

the bracket on the left hand side being the Poisson bracket of functions on E^*, and the bracket on the right hand side the bracket of sections for the corresponding Lie algebroid structure on (E, τ, M).

Proof. First let Λ be a homogeneous Poisson structure on E^*. Let (X, Y) be a pair of smooth sections of τ, and Φ_X and Φ_Y the corresponding vertical functions on E^*. Since Λ is homogeneous, there exists a unique smooth section of τ whose corresponding vertical function on E^* is $\{\Phi_X, \Phi_Y\}$. We define $\{X, Y\}$ as being that section. So we have a composition law on the space $A^1(M, E)$ of smooth sections of τ, which is bilinear and satisfies the Jacobi identity, and therefore is a Lie algebra bracket. Now let f be a smooth function on M. Then

$$\begin{aligned}\{\Phi_X, \Phi_{fY}\} = \{\Phi_X, (f\circ\pi)\Phi_Y\} &= (f\circ\pi)\{\Phi_X, \Phi_Y\} + \{\Phi_X, f\circ\pi\}\Phi_Y\\ &= (f\circ\pi)\Phi_{\{X,Y\}} + \{\Phi_X, f\circ\pi\}\Phi_Y\\ &= (f\circ\pi)\Phi_{\{X,Y\}} + \big(i(\Lambda^\sharp(d\Phi_X))d(f\circ\pi)\big)\Phi_Y.\end{aligned}$$

The term $(f\circ\pi)\Phi_{\{X,Y\}}$ is the vertical function which corresponds to the smooth section $f\{X, Y\}$. Therefore the other term of the right hand side, $\big(i(\Lambda^\sharp(d\Phi_X))d(f\circ\pi)\big)\Phi_Y$, must be a vertical function. But Φ_Y is vertical, so $\big(i(\Lambda^\sharp(d\Phi_X))d(f\circ\pi)\big)\Phi_Y$ is vertical for all $Y \in A^1(M, E)$ if and only if the function $i(\Lambda^\sharp(d\Phi_X))d(f\circ\pi)$ is constant on each fibre $\pi^{-1}(x)$, $x \in M$. This happens for any function $f \in C^\infty(M, \mathbb{R})$ if and only if for each $x \in M$, $T_\xi\pi(\Lambda^\sharp(d\Phi_X))$ does not depend on $\xi \in \pi^{-1}(x)$. In other words, for any $X \in A^1(M, E)$ the vector field $\Lambda^\sharp(d\Phi_X)$ must be projectable by π on M. We take $\xi = 0$ (the origin of the fibre E^*_x) and use the formula for $d\Phi_X$ given in the proof of Lemma 6.3.2 to obtain for that projection the expression in local coordinates

$$\sum_{r=1}^{k} X^r(x^1, \dots, x^n)T\pi(\Lambda^\sharp(d\xi_r)).$$

The value of that vector field at a point $x \in M$ only depends of $X(x)$, and that dependence is linear. So there exists a smooth vector bundle map $\rho : E \to TM$ with all the properties of an anchor map. The Lie algebra structure we have defined on $A^1(M,E)$ is a Lie algebroid bracket.

Conversely, let us assume that we have on (E,π,M) a Lie algebroid structure with anchor ρ. We must prove that there exists a Poisson structure on E^* such that for each pair (X,Y) of smooth sections of τ, Φ_X and Φ_Y being the corresponding vertical functions on E^* (Definition 6.3.1), $\{\Phi_X,\Phi_Y\} = \Phi_{\{X,Y\}}$. More generally, let (X,Y) be a pair of smooth sections of τ, and f and g two smooth functions. Let us write that $\{\Phi_{fX},\Phi_{gY}\} = \{f\circ\pi)\Phi_X,(g\circ\pi)\Phi_Y\} = \Phi_{\{fX,gY\}}$. We use the property of the Lie algebroid bracket

$$\{fX,gY\} = fg\{X,Y\} + (f\mathcal{L}(\rho\circ X)g)Y - (g\mathcal{L}(\rho\circ Y)f)X,$$

which implies

$$\Phi_{\{fX,gY\}} = (fg\circ\pi)\Phi_{\{X,Y\}} + (f\mathcal{L}(\rho\circ X)\circ\pi)\Phi_Y - (g\mathcal{L}(\rho\circ Y)f\circ\pi)\,\Phi_X.$$

This calculation shows that if such a Poisson structure on E^* exists, it must be such that

$$\{\Phi_X, g\circ\pi\} = (\mathcal{L}(\rho\circ X)g)\circ\pi, \qquad \{f\circ\pi, g\circ\pi\} = 0.$$

For each $\xi \in E^*$ and $\eta,\zeta \in T^*_\xi E^*$, point 2 of Lemma 6.3.2 shows that there exists a (nonunique) pair (X,Y) of sections of τ and a (nonunique) pair (f,g) of smooth functions on M such that $\eta = d(\Phi_X + f\circ\pi)(\xi)$, $\zeta = d(\Phi_Y + g\circ\pi)(\xi)$. Our Poisson bivector Λ is therefore

$$\Lambda(\xi)(\eta,\zeta) = \{\Phi_X + f\circ\pi, \Phi_Y + g\circ\pi\}(\xi).$$

This proves that if such a Poisson structure exists, it is unique. By point 1 of Lemma 6.3.2, the right hand side of the above formula depends only on η and ζ, and not on the particular choices we have made for (X,f) and (Y,g). Moreover, it is smooth, bilinear and skew-symetric with respect to the pair $((X,f),(Y,g))$, so Λ is a smooth bivector.

When restricted to vertical functions on E^*, the bracket defined by Λ satisfies the Jacobi identity. Therefore, for each $\xi \in E^*\setminus\{0\}$ and all $\eta,\zeta,\theta \in T^*_\xi E^*$ which are the differentials, at ξ, of vertical functions, the Schouten bracket $[\Lambda,\Lambda]$ satisfies $[\Lambda,\Lambda](\xi)(\eta,\zeta,\theta) = 0$. Point 2 of Lemma 6.3.2 proves that $[\Lambda,\Lambda]$ vanishes identically on $E^*\setminus\{0\}$. By continuity, it vanishes everywhere on E^*. So Λ is a Poisson structure on E^* with all the stated properties. ∎

PROPOSITION 6.3.6. *Let (E,τ,M,ρ) be a Lie algebroid and (E^*,π,M) its dual bundle. The Poisson structure on E^* defined in Theorem* 6.3.5 *has the following properties:*

1. *For any $X \in A^1(M,E)$ and $f,g \in C^\infty(M,\mathbb{R})$,*

$$\{\Phi_X, g\circ\pi\} = (\mathcal{L}(\rho\circ X)g)\circ\pi, \qquad \{f\circ\pi, g\circ\pi\} = 0,$$

   where Φ_X is the function on M associated to the section X as indicated in Theorem 6.3.5.
2. *The transpose ${}^t\rho : T^*M \to E^*$ of the anchor map $\rho : E \to TM$ is a Poisson map (the cotangent bundle being endowed with the Poisson structure associated to its canonical symplectic structure).*

Proof. We have proven Property 1 in the proof of Theorem 6.3.5. In order to prove Property 2, we must prove that for all pairs (h_1, h_2) of smooth functions on E^*,

$$\{h_1 \circ {}^t\rho, h_2 \circ {}^t\rho\} = \{h_1, h_2\} \circ {}^t\rho,$$

the bracket on the left hand side being the Poisson bracket of functions on T^*M, and the bracket on the right hand side the Poisson bracket of functions on E^*. It is enough to check that property when h_1 and h_2 are of the type Φ_X where $X \in A^1(M, E)$, or of the type $f \circ \pi$ with $f \in C^\infty(M, \mathbb{R})$, since the differentials of functions of these two types generate T^*E^*. For $h_1 = \Phi_X$ and $h_2 = \Phi_Y$ with $X, Y \in A^1(M, E)$, and $\zeta \in T^*M$, we have

$$\begin{aligned}\{\Phi_X, \Phi_Y\} \circ {}^t\rho(\zeta) &= \Phi_{\{X,Y\}} \circ {}^t\rho(\zeta) = \langle {}^t\rho(\zeta), \{X, Y\} \circ \pi \circ {}^t\rho(\zeta)\rangle \\ &= \langle \zeta, \rho \circ \{X, Y\} \circ \pi_M(\zeta)\rangle = \langle \zeta, [\rho \circ X, \rho \circ Y] \circ \pi_M(\zeta)\rangle,\end{aligned}$$

since the canonical projection $\pi_M : T^*M \to M$ satisfies $\pi \circ {}^t\rho = \pi_M$. But let us recall a well known property of the Poisson bracket of functions on T^*M ([32, Exercise 17.5, p. 182]). To any vector field $\widehat{X}$ on M, we associate the function $\Psi_{\widehat{X}}$ on T^*M by setting, for each $\zeta \in T^*M$,

$$\Psi_{\widehat{X}}(\zeta) = \langle \zeta, \widehat{X} \circ \pi_M(\zeta)\rangle.$$

Then, for any pair $(\widehat{X}, \widehat{Y})$ of vector fields on M,

$$\{\Psi_{\widehat{X}}, \Psi_{\widehat{Y}}\} = \Psi_{[\widehat{X}, \widehat{Y}]}.$$

By using $\pi_M = \pi \circ {}^t\rho$, we easily see that for each $X \in A^1(M, E)$,

$$\Psi_{\rho \circ X} = \Phi_X \circ {}^t\rho.$$

Returning to our pair of sections $X, Y \in A^1(M, E)$, we see that

$$\{\Phi_X, \Phi_Y\} \circ {}^t\rho(\zeta) = \Psi_{[\rho \circ X, \rho \circ Y]}(\zeta) = \{\Psi_{\rho \circ X}, \Psi_{\rho \circ Y}\}(\zeta) = \{\Phi_X \circ {}^t\rho, \Phi_Y \circ {}^t\rho\}(\zeta).$$

Now for $h_1 = \Phi_X$ and $h_2 = f \circ \pi$ with $X \in A^1(M, E)$ and $f \in C^\infty(M, \mathbb{R})$, we have

$$\begin{aligned}\{\Phi_X \circ {}^t\rho, f \circ \pi \circ {}^t\rho\} &= \{\Psi_{\rho \circ X}, f \circ \pi_M\} = \mathcal{L}(\rho \circ X) f \circ \pi_M \\ &= \mathcal{L}(\rho \circ X) f \circ \pi \circ {}^t\rho = \{\Phi_X, f \circ \pi\} \circ {}^t\rho.\end{aligned}$$

Similarly, for $h_1 = f \circ \pi$ and $h_2 = g \circ \pi$, we have

$$\{f \circ \pi, g \circ \pi\} \circ {}^t\rho = 0 = \{f \circ \pi_M, g \circ \pi_M\} = \{f \circ \pi \circ {}^t\rho, g \circ \pi \circ {}^t\rho\}.$$

Property 2 is proven, and our proof is complete. ■

REMARKS 6.3.7. (i) *The symplectic structure of a cotangent bundle.* Let us take as Lie algebroid the tangent bundle (TM, τ_M, M) with id_{TM} as anchor. Its dual bundle is the cotangent bundle (T^*M, π_M, M). The transpose of the anchor map being id_{T^*M}, Proposition 6.3.6 shows that the Poisson structure on T^*M given by Theorem 6.3.5 is the structure associated to its canonical symplectic 2-form.

(ii) *The symplectic structure on the dual of a Lie algebra.* Now we take as Lie algebroid a finite-dimensional Lie algebra $\mathcal{G}$. The Poisson structure on its dual vector space $\mathcal{G}^*$ given by Theorem 6.3.5 is the well known Kirillov–Kostant–Souriau Poisson structure [23, 26, 45].

6.4. Tangent lifts. G. Sanchez de Alvarez [43] discovered the lift of a Poisson structure on a manifold P to the tangent bundle TP. We show below that its existence and properties can be easily deduced from Theorems 6.2.1 and 6.3.5. The reader will find many other properties of tangent and cotangent lifts of Poisson and Lie algebroid structures in [16, 17].

THEOREM 6.4.1. *Let (P,Λ) be a Poisson manifold. There exists on its tangent bundle TP a Poisson structure, determined by that of P and called its* tangent lift. *It is such that if $f,g\in C^\infty(P,\mathbb{R})$, then*

$$\{df,dg\}_{TP}=d\{f,g\}_P,\qquad \{f\circ\tau_P,g\circ\tau_P\}_{TP}=0,\qquad \{df,g\circ\tau_P\}_{TP}=\{f,g\}_P\circ\tau_P.$$

In these formulae we have denoted by $\{\ ,\ \}_P$ and $\{\ ,\ \}_{TP}$ the Poisson brackets of functions on P and TP, respectively, and we have considered df, dg and $d\{f,g\}_P$ as vertical functions on TP.

Proof. The cotangent bundle (T^*P,π_P,P) has a Lie algebroid structure, with $\Lambda^\sharp:T^*P\to TP$ as anchor (Theorem 6.2.1). Its dual is the tangent bundle (TP,τ_P,P), and by Theorem 6.3.5, there exists on its total space TP a Poisson structure such that, for each pair (η,ζ) of sections of π_P,

$$\{\Phi_\eta,\Phi_\zeta\}_{TP}=\Phi_{[\eta,\zeta]}.$$

We have denoted by Φ_η and Φ_ζ the vertical functions on TP associated to the sections η and ζ of π_P (6.3.1), and by $[\eta,\zeta]$ the bracket of the Pfaff forms η and ζ on the Poisson manifold (P,Λ) (6.2.1). When $\eta=df$ and $\zeta=dg$, we have $[df,dg]=d\{f,g\}_P$. The properties of the Poisson bracket on TP follow from Proposition 6.3.6. ∎

EXAMPLE 6.4.2. Let us assume that P is of even dimension $2m$ and that its Poisson structure is associated to a symplectic 2-form ω_P. In local Darboux coordinates $(x^1\dots,x^m,y_1,\dots,y_m)$ we have

$$\omega_P=\sum_{i=1}^m dy_i\wedge dx^i,\qquad \Lambda_P=\sum_{i=1}^m\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial x^i}.$$

Let $(x^1,\dots,x^m,y_1,\dots,y_m,\dot x^1,\dots,\dot x^m,\dot y_1,\dots,\dot y_m)$ be the local coordinates on TP naturally associated to the local coordinates $(x^1,\dots,x^m,y_1,\dots,y_m)$ on P. We easily see that the lift to TP of the Poisson structure on P is associated to a symplectic structure ω_{TP}, and that the expressions of Λ_{TP} and ω_{TP} in local coordinates are

$$\omega_{TP}=\sum_{i=1}^m(d\dot y_i\wedge dx^i+dy_i\wedge d\dot x^i),\qquad \Lambda_{TP}=\sum_{i=1}^m\left(\frac{\partial}{\partial\dot y_i}\wedge\frac{\partial}{\partial x^i}+\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial\dot x^i}\right).$$

The symplectic form ω_{TP} was defined and used by W. M. Tulczyjew [46, 47], mainly when P is a cotangent bundle. It can be defined by several other methods. For example, since Λ_P is associated to a symplectic structure, $\Lambda_P^\sharp$ is a fibre bundle isomorphim from T^*P onto TP. There is on T^*P a canonical symplectic form ω_{T^*P} (the exterior differential of its Liouville 1-form). With our sign conventions, $\omega_{TP}=-((\Lambda_P^\sharp)^{-1})^*(\omega_{T^*P})$. The $-$ sign is in agreement with point 2 of 6.3.6, since the transpose on $\Lambda_P^\sharp$ is $-\Lambda_P^\sharp$.

For Lie algebroid structures, there is an even richer notion of lift: the next proposition shows that a Lie algebroid structure on a vector bundle (E,τ,M) gives rise to Lie algebroid structures on two vector bundles: (T^*E^*,π_{E^*},E^*) and $(TE,T\tau,TM)$. Formulae in local coordinates for these algebroid structures are given in [17], and other properties of these lifts can be found in [36].

PROPOSITION 6.4.3. *Let (E,τ,M,ρ) be a Lie algebroid. Let Λ_{E^*} be the associated Poisson structure on the total space of the dual bundle (E^*,π,M)* (6.3.5) *and Λ_{TE^*} its lift to TE* (6.4.1).

1. *The Poisson structure Λ_{TE^*} is homogeneous* (*Definition* 6.3.3) *for each of the two vector fibrations (TE^*,τ_{E^*},E^*) and $(TE^*,T\pi,TM)$.*
2. *The vector bundle dual to $(TE^*,T\pi,TM)$ is $(TE,T\tau,TM)$, and the Lie algebroid structure on that dual associated to the homogeneous Poisson structure Λ_{TE^*} on the total space of $(TE^*,T\pi,TM)$* (6.3.5) *is such that for each pair (X,Y) of smooth sections of τ, the bracket $\{TX,TY\}$ is equal to $T\{X,Y\}$.*
3. *The Lie algebroid structure on the vector bundle (T^*E^*,π_{E^*},E^*) associated to the homogeneous Poisson structure Λ_{TE^*} on the total space of its dual bundle (TE^*,τ_{E^*},E^*)* (6.3.5) *is the same as the Lie algebroid structure on the cotangent bundle to the Poisson manifold E^** (6.2.1).

Proof. Since E^* is the total space of a vector bundle, TE^* is a *double vector bundle* ([35, 36, 24]), i.e., it is the total space of two different vector fibrations: the tangent fibration $\tau_{E^*}:TE^*\to E^*$, and the tangent lift $T\pi:TE^*\to TM$ of the vector fibration $\pi:E^*\to M$. As a consequence of its definition, the Poisson structure Λ_{TE^*} is homogeneous with respect to the first vector fibration $\tau_{E^*}:TE^*\to TM$. Let us prove that it is also homogeneous with respect to the second. That Poisson structure is characterized by the following properties: for each pair (f,g) of smooth functions on E^*,

$$\{df,dg\}_{TE^*}=d\{f,g\}_{E^*},\quad \{f\circ\tau_{E^*},g\circ\tau_{E^*}\}_{TE^*}=0,\quad \{df,g\circ\tau_{E^*}\}_{TE^*}=\{f,g\}_{E^*}\circ\tau_{E^*}.$$

We need to prove first a part of point 2: the duality between the vector bundles $(TE^*,T\pi,TM)$ and $(TE,T\tau,TM)$. It is obtained by tangent lift of the duality between (E^*,π,M) and (E,τ,M). Let $Z\in TE$ and $\Xi\in TE^*$ be such that $T\tau(Z)=T\pi(\Xi)$. There exist smooth curves $t\mapsto\varphi(t)$ and $t\mapsto\psi(t)$, defined on an open interval I containing 0, with values in E and in E^*, respectively, such that

$$\left.\frac{d\varphi(t)}{dt}\right|_{t=0}=Z\quad\text{and}\quad\left.\frac{d\psi(t)}{dt}\right|_{t=0}=\Xi.$$

We may choose φ and ψ such that $\tau\circ\varphi=\pi\circ\psi$, so $\big\langle\psi(t),\varphi(t)\big\rangle$ is well defined for all $t\in I$. We define

$$\langle\Xi,Z\rangle=\left.\frac{d\langle\psi(t),\varphi(t)\rangle}{dt}\right|_{t=0}.$$

The left hand side does not depend on the choices of φ and ψ, so it is a legitimate definition of $\langle\Xi,Z\rangle$. The vector bundles $(TE^*,T\pi,TM)$ and $(TE,T\tau,TM)$ are in duality.

For each smooth section $X:M\to E$ of τ, $TX:TM\to TE$ is a smooth section of $T\tau$. Let $\Phi_X:E^*\to\mathbb{R}$ and $\Psi_{TX}:TE^*\to\mathbb{R}$ be the associated vertical functions

(6.3.1), defined respectively on the total spaces of the vector bundles (E^*, π, M) and $(TE^*, T\pi, TM)$. For $\Xi \in TE^*$, let us calculate $\Psi_{TX}(\Xi)$. We take a smooth curve $t \mapsto \psi(t)$ in E^* such that $\frac{d\psi(t)}{dt}\big|_{t=0} = \Xi$. The smooth curve $\varphi = X\circ\pi\circ\psi$ in E is such that $\tau\circ\varphi = \pi\circ\psi$ and $\frac{d\varphi(t)}{dt}\big|_{t=0} = TX(T\pi(\Xi))$, and

$$\Psi_{TX}(\Xi) = \frac{d\langle\psi(t), \varphi(t)\rangle}{dt}\bigg|_{t=0} = \frac{d\Phi_X(\psi(t))}{dt}\bigg|_{t=0} = d\Phi_X(\Xi).$$

We have proven that $\Psi_{TX} = d\Phi_X$. Now if $Y : M \to E$ is another smooth section of τ, we have

$$\{\Psi_{TX}, \Psi_{TY}\}_{TE^*} = \{d\Phi_X, d\Phi_Y\}_{TE^*} = d\{\Phi_X, \Phi_Y\}_{E^*} = d\Phi_{\{X,Y\}},$$

the last bracket $\{X, Y\}$ being the Lie algebroid bracket of the sections X and Y of (E, τ, M, ρ). These equalities prove that for each pair (X, Y) of smooth sections of τ, the Poisson bracket $\{\Psi_{TX}, \Psi_{TY}\}_{TE^*}$ is a vertical function. Proposition 6.3.4 shows that Λ_{TE^*} is homogeneous with respect to the vector fibration $(TE^*, T\pi, TM)$. Point 1 is proven.

Theorem 6.3.5 shows that associated to the Poisson structure Λ_{TE^*}, we have Lie algebroid structures on the dual bundles of (TE^*, τ_{E^*}, E^*) and $(TE^*, T\pi, TM)$.

The last formula also proves that for each pair (X, Y) of smooth sections of τ, the bracket $\{TX, TY\}$ is equal to $T\{X, Y\}$. So point 2 is proven.

For the Lie algebroid structure on (T^*E^*, π_{E^*}, E^*) considered as the cotangent bundle to the Poisson manifold (E^*, Λ_{E^*}) (6.2.1), the bracket of two sections of π_{E^*}, that is, the bracket of two Pfaff forms on E^*, is the bracket defined in Remark 6.2.2(i). Point 3 follows from the properties of that bracket, as shown by Grabowski and Urbański [17]. ∎

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