Contents

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To Veneta

Introduction

One of the most beautiful and important results in the classical complex analysis is the Riemann Mapping Theorem stating that any nonempty simply connected open subset of the complex number plane, other than the plane itself, is biholomorphic to the open unit disk $\mathbb{D} \subset \mathbb{C}$. On the other hand, H. Poincaré (1907) proved that the groups of (holomorphic) automorphisms of the open polydisc and of the open ball in $\mathbb{C}^2$ are not isomorphic; hence these two topologically equivalent domains are not biholomorphically equivalent. Therefore it is important that any domain D in $\mathbb{C}^n$ can be associated with some biholomorphically equivalent object. Generalizing the Schwarz–Pick Lemma, C. Carathéodory (1926) provided the first example of such an object, different from the automorphism group; this object was later called the Carathéodory pseudodistance. That is the largest Poincaré distance between the images of two points from D under all holomorphic mappings from D into $\mathbb{D}$. Somewhat later (1933) S. Bergman started to consider the generating kernel of the Hilbert space of square-integrable holomorphic functions on D with the natural Hermitian metric and distance (later his name was given to these three invariants). In 1967 S. Kobayashi introduced a pseudodistance, dual in some sense to Carathéodory's. More precisely, it is the greatest pseudodistance not exceeding the so called Lempert function, the infimum of the Poincaré distances between preimages of pairs of points from D under an arbitrary holomorphic mapping from $\mathbb{D}$ to D.

In Chapter 1 we discuss the basic properties of various invariant functions and their infinitesimal forms called (pseudo)metrics.

The estimates and the limit behavior of invariant (pseudo)distances (or more generally, of functions) and (pseudo)metrics, as well as of the Bergman kernel, play an important role in numerous problems of complex analysis like asymptotic estimates of holomorphic functions (of various classes), continuation of holomorphic mappings, biholomorphic (non)-equivalence of domains, description of domains with noncompact groups of automorphisms etc. (see e.g. [58, 54, 67, 108]). We only mention that one of the basic points in the classification theorem of bounded convex domains of finite type in $\mathbb{C}^n$ with noncompact groups of automorphisms (see [8]) is an estimate for the Kobayashi and Carathéodory pseudodistances (see also Proposition 3.2.1). In Chapter 3 we obtain estimates of these metrics, as well as of the Bergman kernel and Bergman metric of so-called $\mathbb{C}$-convex domains.

Let us note that the exact calculation of some of the invariants or finding estimates thereof leads e.g. to criteria for solvability of corresponding interpolation problems or to restrictions on solvability. Chapter 2 is partially motivated by two examples of such types of problems.

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The results have been published as follows:

Chapter 1: in [86, 87, 88, 89, 90, 103];
Chapter 2: in [82, 91, 94, 95, 96, 97, 100, 101, 102, 104];
Chapter 3: in [57, 80, 81, 83, 92, 84, 98].

Some of the results we mention come from [79, 85, 93, 99].

1. Lempert functions and Kobayashi metrics

1.1. Synopsis. The aim of this chapter is the introduction of basic invariant functions, distances and metrics together with their basic properties.

The Lempert function l_M and the Carathéodory function c^*_M of a given complex manifold M are the greatest and the least holomorphically contractible functions (i.e. decreasing under holomorphic mappings), coinciding with the Möbius distance $m_{\mathbb{D}}$ on the unit disc $\mathbb{D}$. The Kobayashi and the Carathéodory (pseudo)distances, k_M and c_M, are the greatest and the least holomorphically contractible (pseudo)distances, coinciding on $\mathbb{D}$ with the Poincaré distance $p_{\mathbb{D}}$. Note that $c_M = \tanh^{-1} c^*_M$, while $k_M \lneq \tanh^{-1} l_M$ in general. Define the Kobayashi function by the equality $k^*_M = \tanh k_M$.

In Section 1.2 we note that the objects under consideration are upper semicontinuous (see Proposition 1.2.1 and the comment preceding it). The main result in that section, namely Theorem 1.2.2, states that if $z \in M$ and the function κ_M is continuous and positive in $(z; X)$ for each nonzero vector X, then the "derivative" of $k^{(m)}_M$ at z in the direction of X coincides with $\kappa^{(m)}_M(z; X)$. An essential step in the proof is Proposition 1.2.3 stating that the "upper derivative" of $k^{(m)}_M$ does not exceed $\kappa^{(m)}_M$ in the general case. Theorem 1.2.2 generalizes some results of M.-Y. Pang [105] and M. Kobayashi [62] concerning taut manifolds (domains). We provide examples to demonstrate that the assumptions in the theorem are essential.

In Section 1.3 we find some relationships between the Minkowski functions of a balanced domain or of its convex/holomorphic hull and some of the previously defined biholomorphic invariants of that domain whenever one of their arguments is the origin. Some of these relationships are used in the subsequent chapter.

In Section 1.4 we prove that the Kobayashi–Buseman metric $\hat{\kappa}_M$ equals the Kobayashi metric $\kappa^{(2n-1)}_M$ of order $2n-1$ and this number is the least possible in the general case. A similar result for $2n$ instead of $2n-1$ can be found in the paper [63] of S. Kobayashi, where $\hat{\kappa}_M$ is introduced.

In Section 1.5 we prove a general statement, Theorem 1.5.4, on approximation and interpolation over so-called Arakelian sets. To this end we use a well-known interpolation-approximation result of P. M. Gauthier and W. Hengartner [43] and A. Nersesyan [78].

This theorem is the base of the proof of Theorem 1.6.1 stating that the so-called generalized Lempert function (of a given domain) does not decrease under addition of poles. The last assertion is proven by Wikström [118] for convex domains; he left the general case as an open question in [119].

In Section 1.7 we discuss the product property of the generalized Lempert function in order to reject a hypothesis of D. Coman [22] on equality between this function and the generalized pluricomplex Green function. In Proposition 1.7.2 we find a necessary and sufficient condition for the Lempert function of the bidisc with (fixed argument and) poles in the cartesian product of two two-point subsets of $\mathbb{D}_* = \mathbb{D} \setminus \{0\}$ to equal each of the two corresponding functions of $\mathbb{D}$.

1.2. Lempert functions and their "derivatives". In this section we introduce the Lempert functions of higher order and their infinitesimal forms, the Kobayashi metrics of higher order, for an arbitrary complex manifold (see also [58, 64]).

Our main aim is to prove that if the Kobayashi metric of a complex manifold is continuous and positive at a given point for each nonzero tangent vector, then the "derivatives" of the Lempert functions exist and are equal to the corresponding Kobayashi metrics at this point. This generalizes some results of M.-Y. Pang [105] and M. Kobayashi [62] for taut domains/manifolds.

As usual $\mathbb{D} \subset \mathbb{C}$ denotes the unit disc. Let M be an n-dimensional complex manifold. Let us recall the definitions of the Lempert function l_M and the *Kobayashi–Royden* (for short, *Kobayashi*) (*pseudo*)*metric* κ_M of M:

$$l_M(z,w) = \inf\{|\alpha| : \exists f \in \mathcal{O}(\mathbb{D}, M) : f(0) = z,\ f(\alpha) = w\},$$
$$\kappa_M(z;X) = \inf\{|\alpha| : \exists f \in \mathcal{O}(\mathbb{D}, M) : f(0) = z,\ \alpha f_{*,0}(d/d\zeta) = X\},$$

where X is a complex tangent vector to M at z. Such f always exist (see e.g. [120]; according to [34, p. 49] this was known even earlier to J. Globevnik).

Note that if $F : M \to N$ is a holomorphic mapping between two manifolds, then

$$l_M(z,w) \geq l_N(F(z), F(w)).$$

In particular, if F is a biholomorphism, then we get equality, i.e. the Lempert function is invariant under biholomorphisms. The above inequality also shows that this function is the largest holomorphically contractible function that coincides on $\mathbb{D}$ with the Möbius distance $m_{\mathbb{D}}$. On the other hand, the smallest such function is the *Carathéodory function*

$$c^*_M(z,w) = \sup\{m_{\mathbb{D}}(f(z), f(w)) : f \in \mathcal{O}(M, \mathbb{D})\}.$$

If in this definition we replace $m_{\mathbb{D}}$ by the Poincaré distance $p_{\mathbb{D}}$, we get the *Carathéodory* (*pseudo*) *distance*

$$c_M = \tanh^{-1} c^*_M.$$

As

$$\kappa_M(z;X) \geq \kappa_N(F(z); F_{*,z}(X)),$$

the Kobayashi metric is the largest holomorphically contractible pseudometric such that $\kappa_{\mathbb{D}}(0;X) = |X|$. The smallest such pseudometric is the *Carathéodory–Reiffen* (briefly, *Carathéodory) metric*

$$\gamma_M(z;X)=\sup\{|f_{*,z}(X)|:f\in\mathcal{O}(M,\mathbb{D})\}$$

(we can assume $f(z)=0$).

As in the case of domains, the Kobayashi distance k_M can be defined as the largest pseudodistance not exceeding the *Lempert function of first order*

$$k_M^{(1)}=\tanh^{-1}l_M$$

(for convenience we distinguish this function from the Lempert function l_M). By the *Kobayashi function* we mean

$$k_M^*=\tanh k_M.$$

Let us note that if $k_M^{(m)}$ denotes the *Lempert function of order* m ($m\in\mathbb{N}$), i.e.

$$k_M^{(m)}(z,w)=\inf\Big\{\sum_{j=1}^m k_M^{(1)}(z_{j-1},z_j):z_0,\dots,z_m\in M,\ z_0=z,\ z_m=w\Big\},$$

then

$$k_M(z,w)=k_M^{(\infty)}:=\inf_m k_M^{(m)}(z,w).$$

Now let us recall that a manifold M is called *taut* if the family $\mathcal{O}(\mathbb{D},M)$ is normal. Every taut domain in $\mathbb{C}^n$ is pseudoconvex. Conversely, every bounded domain with a $\mathcal{C}^1$-smooth boundary is *hyperconvex* (i.e. has an exhausting negative plurisubharmonic function), so it is a taut domain.

According to a result of M.-Y. Pang [105], the Kobayashi metric is the "derivative" of the Lempert function if the domain is taut:

$$\kappa_D(z;X)=\lim_{t\to0}\frac{l_D(z,z+tX)}{t}$$

(in this limit, as well as in some similar ones below, we can replace l_D by $k_D^{(1)}$ and, in general, an invariant function with values in $[0,1)$ by $\tanh^{-1}$ of it, or vice versa).

In the general case the Kobayashi metric at a given point of a domain is not a pseudonorm (vectorwise), i.e. its indicatriced are not convex domains. To avoid this defect, S. Kobayashi [63] introduced a new invariant metric, later called the *Kobayashi–Buseman metric*. As in the case of the Kobayashi distance, this metric $\hat\kappa_M$ can be defined by letting $\hat\kappa_M(z;\cdot)$ be the largest pseudonorm not exceeding $\kappa_M(z;\cdot)$. Clearly

$$\hat\kappa_M(z;X)=\inf\Big\{\sum_{j=1}^m\kappa_M(z;X_j):m\in\mathbb{N},\ \sum_{j=1}^m X_j=X\Big\}.$$

Hence it is natural to consider the functions $\kappa_M^{(m)}$, $m\in\mathbb{N}$, defined as follows:

$$\kappa_M^{(m)}(z;X)=\inf\Big\{\sum_{j=1}^m\kappa_M(z;X_j):\sum_{j=1}^m X_j=X\Big\}.$$

We call the function $\kappa_M^{(m)}$ the *Kobayashi metric of order* m. Clearly $\kappa_M^{(m)}\ge\kappa_M^{(m+1)}$. Also one can easily observe that if $\kappa_M^{(m)}(z;\cdot)=\kappa_M^{(m+1)}(z;\cdot)$ for some m, then $\kappa_M^{(m)}(z;\cdot)=$

$\kappa_M^{(j)}(z;\cdot)$ for each $j > m$. Furthermore, as we will see in the next section, $\kappa_M^{(2n-1)} = \kappa_M^{(\infty)} := \hat{\kappa}_M$, with $2n-1$ being the least possible number in the general case.

Let us note that all objects introduced above are upper semicontinuous; for κ_M (and hence for $\kappa_M^{(m)}$ and $\hat{\kappa}_M$) see also [64]. To prove the upper semicontinuity of $k_M^{(m)}$, it suffices to check it for l_M.

PROPOSITION 1.2.1. *For each complex manifold M, the function l_M is upper semicontinuous.*

Proof. We use a standard procedure (see [112]). Let $r \in (0,1)$ and $z, w \in M$. Let $f \in \mathcal{O}(\mathbb{D}, M)$ with $f(0) = z$ and $f(\alpha) = w$. Then $\tilde{f} = (f, \mathrm{id}) : \Delta \to \tilde{M} = M \times \Delta$ is an immersion. Put $\tilde{f}_r(\zeta) = \tilde{f}(r\zeta)$; now [112, Lemma 3] implies that there is a Stein neighborhood $S \subset \tilde{M}$ of $\tilde{f}_r(\mathbb{D})$. As is well known, S can be immersed as a closed complex manifold in $\mathbb{C}^{2n+1}$. Let ψ be the corresponding immersion. Then there is an (open) neighborhood $V \subset \mathbb{C}^{2n+1}N$ of $\psi(S)$ and a holomorphic retraction $\theta : V \to \psi(S)$. For z' near z and w' near w we can find (in a standard way) $g \in \mathcal{O}(\mathbb{D}, V)$ such that $g(0) = \psi(z', 0)$ and $g(\alpha/r) = \psi(w', \alpha)$. Denote by π the natural projection of $\tilde{M}$ onto M. Then $h = \pi \circ \psi^{-1} \circ \theta \circ g \in \mathcal{O}(\mathbb{D}, M)$, $h(0) = z'$ and $h(\alpha/r) = w'$. Consequently, $rl_M(z', w') \le \alpha$. This shows that $\limsup_{z' \to z, w' \to w} l_M(z', w') \le l_M(z, w)$. ∎

To extend the previously mentioned result of Pang, we define the "derivatives" of $k_M^{(m)}$, $m \in \mathbb{N}^* = \mathbb{N} \cup \{\infty\}$. Let (U, φ) be a holomorphic chart near z. We put

$$\mathcal{D}k_M^{(m)}(z;X) = \limsup_{t\to 0,\, w\to z,\, Y\to\varphi_* X} \frac{k_M^{(m)}(w, \varphi^{-1}(\varphi(w) + tY))}{|t|}.$$

This definition does not depend on the chart; also,

$$\mathcal{D}k_M^{(m)}(z;\lambda X) = |\lambda| \mathcal{D}k_M^{(m)}(z;X), \quad \lambda \in \mathbb{C}.$$

Replacing $\limsup$ by $\liminf$, we can define $\underline{\mathcal{D}}k_M^{(m)}$.

A result of M. Kobayashi [62] shows that if M is a complex taut manifold, then

$$\hat{\kappa}_M(z;X) = \mathcal{D}k_M(z;X) = \underline{\mathcal{D}}k_M(z;X),$$

i.e. the Kobayashi–Buseman metric is the "derivative" of the Kobayashi distance. The proof of this result allows us to learn something more:

$$\kappa_M^{(m)}(z;X) = \mathcal{D}k_M^{(m)}(z;X) = \underline{\mathcal{D}}k_M^{(m)}(z;X), \quad m \in \mathbb{N}.$$

Note that for the Carathéodory metric of an arbitrary complex manifold M one has (see [58] for domains in $\mathbb{C}^n$)

$$\gamma_M = \mathcal{D}c_M = \underline{\mathcal{D}}c_M \tag{1.2.1}$$

(the definitions of the last two invariants are obvious).

To formulate in full generality the main result of this section we need the following notion. A complex manifold is called *hyperbolic at the point* $z \in M$ if $k_M(z,w) > 0$ for each $w \ne z$. (Recall that M is *hyperbolic* if it is hyperbolic at each of its points, i.e. k_M is a distance.) Then the following assertions are equivalent:

(i) M is hyperbolic at z;
(ii) $\liminf_{z' \to z,\, w \in M \setminus U} l_M(z', w) > 0$ for each neighborhood U of z;
(iii) $\underline{\kappa}_M(z;X) := \liminf_{z' \to z, X' \to X} \kappa_M(z';X') > 0$ for each $X \ne 0$.

The implications (i)⇒(ii)⇒(iii) are (almost) trivial, while (iii)⇒(i) follows from the fact that k_M is the integrated form of κ_M.

In particular, if M is hyperbolic at z, then it is hyperbolic at each point z' near z.

If M is a taut manifold, then it is hyperbolic and κ_M is a continuous function. This shows that the theorem below generalizes the previously mentioned result of M. Kobayashi.

THEOREM 1.2.2. *Let M be a complex manifold and $z \in M$.*

(i) *If M is hyperbolic at z and κ_M is continuous at (z, X), then*
$$\kappa_M(z;X) = \mathcal{D}l_M(z;X) = \underline{\mathcal{D}}l_M(z;X).$$
(ii) *If κ_M is continuous and positive at (z, X) for each $X \neq 0$, then*
$$\kappa_M^{(m)}(z;\cdot) = \mathcal{D}k_M^{(m)}(z;\cdot) = \underline{\mathcal{D}}k_M^{(m)}(z;\cdot), \quad m \in \mathbb{N}^*.$$

The first step of the proof of Theorem 1.2.2 is the following

PROPOSITION 1.2.3. *For each complex manifold M one has*
$$\kappa_M^{(m)} \geq \mathcal{D}k_M^{(m)}, \quad m \in \mathbb{N}^*.$$

Note that if M is a domain, a weaker variant of Proposition 1.2.3 can be found in [58], namely $\hat{\kappa}_M \geq \mathcal{D}k_M$ (the proof is based on the fact that $\mathcal{D}k_M(z;\cdot)$ is a pseudonorm).

Proof of Proposition 1.2.3. Let us first consider the case $m = 1$. The main role will be played by the following

THEOREM 1.2.4 ([112] (1)). *Let M be a complex manifold and the mapping $f \in \mathcal{O}(\mathbb{D}, M)$ be regular at 0. Let $r \in (0,1)$ and $D_r = r\mathbb{D} \times \mathbb{D}^{n-1}$. Then there is a mapping $F \in \mathcal{O}(D_r, M)$ that is singular at 0 and $F|_{r\mathbb{D}\times\{0\}} = f$.*

Since $\kappa_M(z;0) = \mathcal{D}l_M(z;0) = 0$, one can assume that $X \neq 0$. Let $\alpha > 0$ and $f \in \mathcal{O}(\mathbb{D}, M)$ be such that $f(0) = z$ and $\alpha f_{*,0}(d/d\zeta) = X$. Let $r \in (0,1)$ and F be as in Theorem 1.2.4. Since F is regular at 0, there are neighborhoods $U = U(z) \subset M$ and $V = V(0) \subset D_r$ such that $F|_V : V \to U$ is a biholomorphism. Therefore (U, φ), where $\varphi = (F|_V)^{-1}$, is a chart near z. Note that $\varphi_{*,z}(X) = \alpha e_1$, where $e_1 = (1, 0, \ldots, 0)$.

If w and Y are close enough to z and αe_1, then $g(\zeta) = F(\varphi(w) + \zeta Y/\alpha)$ belongs to $\mathcal{O}(r^2\mathbb{D}, M)$, $g(0) = w$ and $g(t\alpha) = \varphi^{-1}(\varphi(w) + tY)$, $t < r^2/\alpha$. Consequently,
$$r^2 l_M(w, \varphi^{-1}(\varphi(w) + tY)) \leq t\alpha.$$
Thus $r^2\, l_M(z;X) \leq \alpha$. For $r \to 1$ and $\alpha \to \kappa_M(z;X)$ we get $\mathcal{D}l_M(z;X) \leq \kappa_M(z;X)$.

Now let $m \in \mathbb{N}$. Recall that $\kappa_M^{(m)}(z;\cdot)$ is the largest function with the following property:

For each $X = \sum_{j=1}^m X_j$ it follows that $\kappa_M^{(m)}(z;X) \leq \sum_{j=1}^m \kappa_M(z;X_j)$.

To prove that $\kappa_M^{(m)} \geq \mathcal{D}k_M^{(m)}$, it is sufficient to check that $\mathcal{D}k_M^{(m)}(z;\cdot)$ has this property. Using the above notation and choosing $Y_j \to \varphi_{*,z}X_j$ so that $\sum_{j=1}^m Y_j = Y$, we put $w_0 = w$

(1) Instead of Theorem 1.2.4, one can use the approach from the proof of the semicontinuity of l_M.

and $w_j = \varphi^{-1}(\varphi(w) + t\sum_{k=1}^{j} Y_j)$. Since

$$k_M^{(m)}(w, w_q) \le \sum_{j=1}^{m} k_M^{(1)}(w_{j-1}, w_j),$$

from the case $m = 1$ it follows that

$$\mathcal{D}k_M^{(m)}(z; X) \le \sum_{j=1}^{m} \mathcal{D}k_M(z; X_j) \le \sum_{j=1}^{m} \kappa_M(z; X_j).$$

Finally, let $m = \infty$ and $n = \dim M$. Since $\hat{\kappa}_M = \kappa_M^{(2n-1)}$ and $k_M \le k_M^{(2n-1)}$, the case $m = 2n - 1$ shows that $\mathcal{D}k_M \le \hat{\kappa}_M$. ■

Proof of Theorem 1.2.2. We can assume that $X \ne 0$. Bearing in mind Proposition 1.2.3, we just have to prove that

$$\kappa_M^{(m)}(z; X) \le \underline{\mathcal{D}}k_M^{(m)}(z; X)$$

under the corresponding assumptions. For simplicity we assume that M is a domain in $\mathbb{C}^n$ (the changes in the general case of a manifold are obvious).

(i) Fix a neighborhood $U = U(z) \Subset M$. By hyperbolicity of M at z, there exist a neighborhood $V = V(z) \subset U$ and a number $\delta \in (0, 1)$ such that if $h \in \mathcal{O}(\mathbb{D}, M)$ and $h(0) \in V$, then $h(\delta\mathbb{D}) \subset U$. By the Cauchy inequalities it follows that $\|h^{(k)}(0)\| \le c/\delta^k$, $k \in \mathbb{N}$ ($\|\cdot\|$ is the Euclidean norm).

Now choose sequences $w_j \to z$, $t_j \to 0$ and $Y_j \to X$ such that

$$\frac{l_M(w_j, w_j + t_jY_j)}{|t_j|} \to \underline{\mathcal{D}}l_M(z; X).$$

Let the holomorphic discs $g_j \in \mathcal{O}(\mathbb{D}, M)$ and the numbers $\beta_j \in (0, 1)$ be such that $g_j(0) = w_j$, $g_j(\beta_j) = w_j + t_jY_j$ and $\beta_j \le l_M(w_j, w_j + t_jY_j) + |t_j|/j$. Note that $l_M(w_j, w_j + t_jY_j) \le c_1\|t_jY_j\| \le c_2|t_j|$. Let

$$w_j + t_jY_j = g_j(\beta_j) = w_j + g_j'(0)\beta_j + h_j(\beta_j).$$

Then

$$\|h_j(\beta_j)\| \le c\sum_{k=2}^{\infty}(\beta_j/\delta)^k \le c_3|\beta_j|^2 \le c_4|t_j|^2, \quad j \ge j_0.$$

We put $\hat{Y}_j = Y_j - h_j(\beta_j)/t_j$. Then $g_j(0) = w_j$ and $\beta_j g_j'(0)/t_j = \hat{Y}_j \to X$. Consequently,

$$\kappa_M(w_j; \hat{Y}_j) \le \frac{\beta_j}{|t_j|} \le \frac{l_M(z_j, w_j + t_jY_j)}{|t_j|} + \frac{1}{j}.$$

For $j \to \infty$ we get $\kappa_M(z; X) = \underline{\kappa}_M(z; X) \le \underline{\mathcal{D}}l_M(z; X)$.

(ii) The proof of the case $m \in \mathbb{N}$ is similar to the one below and we omit it. Now let $m = \infty$.

Our assumptions show that M is hyperbolic at z. Also it easily follows (say by contradiction) that

$$\forall\varepsilon > 0\ \exists\delta > 0 : \|w - z\| < \delta,\ \|Y - X\| < \delta\|X\| \Rightarrow |\kappa_M(w; Y) - \kappa_M(z; X)| < \varepsilon\kappa_M(z; X). \tag{1.2.2}$$

Also, the proof of (i) shows that

$$k_M^{(1)}(a,b) \geq \kappa_M(a; b-a+o(a,b)), \quad \text{where} \quad \lim_{a,b\to z} \frac{o(a,b)}{\|a-b\|} = 0. \tag{1.2.3}$$

Now choose sequences $w_j \to z$, $t_j \to 0$ and $Y_j \to X$ such that

$$\frac{k_M(w_j, w_j + t_j Y_j)}{|t_j|} \to \underline{\mathcal{D}}k_M(z;X).$$

Let $w_{j,0} = w_j, \dots, w_{j,m_j} = w_j + t_j X_j$ be points from M such that

$$\sum_{k=1}^{m_j} k_M^{(1)}(w_{j,k-1}, w_{j,k}) \leq k_M(w_j, w_j + t_j Y_j) + 1/j. \tag{1.2.4}$$

Put $w_{j,k} = w_j$ for $k > m_j$. Since

$$k_M(w_j, w_{j,l}) \leq \sum_{j=1}^{l} k_M^{(1)}(w_{j,k-1}, w_{j,k}) \leq k_M(w_j, w_j + t_j Y_j) + 1/j \leq c_2|t_j| + 1/j,$$

$k_M(w_j, w_{j,l}) \to 0$ uniformly in l. The hyperbolicity of M at z implies that $w_{j,l} \to z$ uniformly in l. Indeed, assuming the contrary and choosing a subsequence, we can assume that $w_{j,l_j} \notin U$ for some $U = U(z)$. Then

$$0 = \lim_{j\to\infty} k_M(w_j, w_{j,l}) \geq \liminf_{z'\to z, w\in M\setminus U} l_M(z', w) > 0,$$

which is a contradiction.

Finally let us fix $R > 1$. Then (1.2.2) shows that

$$\kappa_M(z; w_{j,k} - w_{j,k-1}) \leq R\kappa_M(w_{j,k}; w_{j,k} - w_{j,k-1} + o(w_{j,k}, w_{j,k-1})), \qquad j \geq j(R).$$

From this inequality, (1.2.3) and (1.2.4) it follows that

$$\sum_{k=1}^{m_j} \kappa_M(z; w_{j,k} - w_{j,k-1}) \leq Rk_M(w_j, w_j + t_j Yj) + R/j.$$

Since $\hat{\kappa}_M(z; t_j Y_j)$ is bounded by the above sum, we get

$$\hat{\kappa}_M(z; Y_j) \leq R\frac{k_M(w_j, w_j + t_j Yj) + 1/j}{|t_j|}.$$

It remains to use that $\hat{\kappa}_M(z;\cdot)$ is a continuous function. Then for $j \to \infty$ and $R \to 1$ it follows that $\hat{\kappa}_M(z;X) \leq \underline{\mathcal{D}}k_M(z;X)$. ■

Remark. From the above proofs, by a standard diagonal process, it follows that if M is hyperbolic at z, then $\underline{\kappa}_M(z;\cdot) = \underline{\mathcal{D}}l(z;\cdot)$.

The subsequent examples show that the assumptions of continuity in Theorem 1.2.2 are essential.

• Let A be a countable dense subset of $\mathbb{C}_*$ $(= \mathbb{C}\setminus\{0\})$. In [33] (see also [58]) there is an example of a pseudoconvex domain $D \subset \mathbb{C}^2$ such that:

(i) $(\mathbb{C}\times\{0\}) \cup (A\times\mathbb{C}) \subset D$;
(ii) if $z_0 = (0,t) \in D$, $t \neq 0$, then $\kappa_D(z_0;\cdot) \geq C\|\cdot\|$ for some $C > 0$. (It can even be shown that $\mathcal{D}l_D(z_0;\cdot) \geq C\|\cdot\|$.)

Then it is easily deduced that $\underline{\kappa}_D(\cdot;e_2)=\mathcal{D}k_D^{(3)}(\cdot;e_2)=k_D^{(5)}=0$ and $\hat{\kappa}_D(z_0;\cdot)\ge c\|\cdot\|$, where $e_2=(0,1)$ and $c>0$. Therefore

$$\hat{\kappa}_D(z_0;X)>0=\underline{\kappa}_D(z_0;e_2)=\mathcal{D}k_D^{(3)}(z_0;e_2)=\mathcal{D}k_D^{(5)}(z_0;X),\quad X\in(\mathbb{C}^2)_*.$$

This phenomenon clearly appears also in $\mathbb{C}^n$, $n>2$ (say for $D\times\mathbb{D}^{n-2}$). Thus the inequalities in Proposition 1.2.3 are strict in the general case.

• There exists a bounded pseudoconvex domain $D\subset\mathbb{C}^2$ containing the origin such that (see e.g. [127, Example 4.2.10])

$$\kappa_D(0;e_1)=\mathcal{D}k_D(0;e_1)=\limsup_{t\to0}\frac{l_D(0,te_1)}{|t|}>\liminf_{t\to0}\frac{l_D(0,te_1)}{|t|}\ge\underline{\mathcal{D}}k_D(0;e_1).$$

We conclude this section by the following

QUESTION. Is $\kappa_D\ne\mathcal{D}l_D$ in the general case? Is $\mathcal{D}k_D$ a holomorphically contractible invariant? (For this question see also [60].)

A partial positive answer will be given in Section 2.11 by showing that there is a pseudoconvex domain $D\subset\mathbb{C}^8$ and a point $(z,X)\in D\times\mathbb{C}^n$ such that

$$\kappa_D(z;X)>0=\limsup_{t\to0}\frac{l_D(z,tX)}{|t|}.$$

1.3. Balanced domains. The biholomorphic invariants can be explicitly calculated for a few classes of domains, usually contained in the class of Reinhardt domains. Each complete Reinhardt domain is balanced. In this section we determine some relationships between the Minkowski functions of a balanced domain or of its convex/holomorphically convex hull and some biholomorphic invariants of the domain when one of their arguments is the origin.

Recall that a domain $D\subset\mathbb{C}^n$ is called *balanced* if $\lambda z\in D$ for each $(\lambda,z)\in\overline{\mathbb{D}}\times D$ (for this definition and part of the facts below see e.g. [58]). We naturally associate to such a domain its *Minkowski function*

$$h_D(z)=\inf\{t>0: z/t\in D\},\qquad z\in\mathbb{C}^n.$$

The function $h_D\ge0$ is upper semicontinuous and

$$h_D(\lambda z)=|\lambda|h_D(z),\quad \lambda\in\mathbb{C},z\in\mathbb{C}^n,\qquad D=\{z\in\mathbb{C}^n: h_D(z)<1\}.$$

Let us note that D is pseudoconvex exactly when $\log h\in\mathrm{PSH}(\mathbb{C}^n)$, which in this case is equivalent to $h\in\mathrm{PSH}(\mathbb{C}^n)$. Also recall that D is a taut domain exactly when it is bounded and h_D is a continuous plurisubharmonic function. This shows that, for a balanced domain, being hyperconvex or taut is the same. Let us note that the hyperbolicity of D is equivalent to its boundedness. More general results concerning so-called Hartogs domains can be found in the paper [85] of the author and P. Pflug.

Clearly, the convex hull $\hat{D}$ of a balanced domain D is balanced. Let us recall the well-known relationships between h_D, $\hat{h}_D=h_{\hat{D}}$ and some invariant functions and metrics.

PROPOSITION 1.3.1. *Let $D\subset\mathbb{C}^n$ be a balanced domain and $a\in D$. Then:*

(i) $\gamma_D(0;\cdot)=\hat{\kappa}_D(0;\cdot)=\hat{h}_D$.
(ii) $\hat{h}_D\le c_D^*(0,\cdot)\le k_D^*(0,\cdot)\le l_D(0,\cdot)\le h_D$ *and* $\hat{h}_D\le\kappa(0;\cdot)\le h_D$;
(iii) $c_D^*(0,a)=h_D(a)\Leftrightarrow k_D^*(0,a)=h_D(a)\Leftrightarrow h_D(a)=\hat{h}_D(a)$.

If in addition D is pseudoconvex, then

(iv) $l_D(0,\cdot)=h_D$ *and* $\kappa(0;\cdot)=h_D$.

The Lempert theorem (mentioned in the introduction) implies that $c_D^*=k_D^*=l_D$ for each convex domain D. Then by the above proposition we get

Corollary 1.3.2. *For a pseudoconvex balanced domain $D\subset\mathbb{C}^n$ the following are equivalent:*

(i) D *is convex* (*i.e.* $h_D=\hat h_D$);
(ii) $c_D^*=l_D$;
(iii) $c_D^*(0,\cdot)=l_D(0,\cdot)$;
(iv) $k_D^*=l_D$;
(v) $k_D^*(0,\cdot)=l_D(0,\cdot)$.

Put $(k_D^{(m)})^*=\tanh k_D^{(m)}$. Proposition 1.3.1(iii) shows that at $a\in D$ the value of $k_D(0,\cdot)$ is maximal exactly when D is "convex" in the direction of a, i.e. $h_D(a)=\hat h_D(a)$. The next result shows that more is true.

Proposition 1.3.3. *Let $D\subset\mathbb{C}^n$ be a balanced domain and $a\in D$. The following are equivalent:*

(i) $\hat h_D(a)=h_D(a)$;
(ii) $(k_D^{(3)}(0,a))^*=h_D(a)$;
(iii) $\kappa_D^{(2)}(0;a)=h_D(a)$.

Since $k_D^{(m)}(0,a)\le k_D^{(3)}(0,a)\le h_D$ for $3\le m\le\infty$ ($k_D=k_D^{(\infty)}$) and $\kappa_D^{(l)}(0;a)\le\kappa_D^{(2)}(0;a)$ for $2\le l\le\infty$ ($\hat\kappa_D=\kappa_D^{(\infty)}$), for these m and l it follows that $\hat h_D(a)=h_D(a)\Leftrightarrow(k_D^{(m)}(0,a))^*=h_D(a)\Leftrightarrow\kappa_D^{(2)}(0;a)$.

Remark. We do not know whether 3 can be replaced by 2 (it cannot be replaced by 1 according to Proposition 1.3.1(iv)).

Proof. The implication (i)⇒(ii) follows from Proposition 1.3.1.

Assume (iii) holds. If $a_1+a_2=a$, then by $\kappa_D^{(2)}(0;a)\le\kappa_D(0;a_1)+\kappa_D(0;a_2)$ and $\kappa_D(0;\cdot)\le h_D$ it follows that $h_D(a)\le h_D(a_1)+h_D(a_2)$, so (i) holds.

It remains to prove (ii)⇒(iii). We first prove that (ii) implies

$$(k_D^{(2)}(0,\lambda a))^*=|\lambda|h_D(a),\qquad\lambda\in\mathbb{D}.\tag{1.3.1}$$

We can assume that $h_D(a)\ne0$. Considering the analytic disc $\varphi(\zeta)=a\zeta/h_D(a)$ as a competitor (2) for $l_D(\lambda a,a)$, we get

$$l_D(\lambda a,a)\le m(h_D(\lambda a),h_D(a)).$$

Hence by the inequality

$$p(0,h_D(a))=k_D^{(3)}(0,a)\le k_D^{(2)}(0,\lambda a)+k_D^{(1)}(\lambda a,a)$$

(2) This means that φ belongs to the set over which we take the infimum in the definition of l_D.

we get

$$p(0,|\lambda|h_D(a)) = p(0,h_D(a)) - p(|\lambda|h_D(a),h_D(a)) \le k_D^{(2)}(0,\lambda a).$$

Thus

$$(k_D^{(2)}(0,\lambda a))^* \ge |\lambda|h_D(a).$$

It remains to note that the opposite inequality is always true.

Now (1.3.1) shows that

$$\lim_{\lambda\to 0}\frac{k_D^{(2)}(0,\lambda a)}{|\lambda|} = h_D(a).$$

On the other hand, by Proposition 1.2.3 this limit does not exceed $\kappa_D^{(2)}(0;a) \le h_D(a)$, so (iii) is proved. ∎

Having in mind Propositions 1.3.1 and 1.3.3, it is natural to ask whether the minimality (rather than maximality) of some $k_D^{(m)}(0,a)$, i.e. $l_D(0,a) = h_G(a)$ for a domain $G \supset D$, implies some "convex" property. We have the following

Proposition 1.3.4. *Let $D \subset \mathbb{C}^n$ be a bounded balanced domain and $G \subset \mathbb{C}^n$ be a pseudoconvex balanced domain containing D. Suppose that h_D is continuous at some $a \in D$, $h_G(a) \ne 0$ and $\overline{G}$ does not contain (nontrivial) analytic discs through $a/h_G(a)$. Then the following are equivalent:*

(i) $h_D(a) = h_G(a)$;
(ii) $l_D(0,a) = h_G(a)$;
(iii) $\kappa_D(0;a) = h_G(a)$.

Proof. It suffices to prove that

$$l_D(0,a) = h_G(a) \Rightarrow h_D(a) \le h_G(a), \qquad \kappa_D(0;a) = h_G(a) \Rightarrow h_D(a) \le h_G(a).$$

Let $(\varphi_j) \subset \mathcal{O}(\mathbb{D},D)$ and $\alpha_j \to h_G(a)$ so that $\varphi_j(0) = 0$ and $\varphi_j(\alpha_j) = a$ (correspondingly, $\alpha_j\varphi_j'(0) = a$). Expressing φ_j in the form $\varphi_j(\lambda) = \lambda\psi_j(\lambda)$, by the maximum principle $h_G \circ \psi_j \le 1$ so $\psi_j \in \mathcal{O}(\mathbb{D},\overline{G})$. As D is bounded, then by going to a subsequence we can assume that $\varphi_j \to \varphi \in \mathcal{O}(\mathbb{D},\overline{D})$, hence $\psi_j \to \psi \in \mathcal{O}(\mathbb{D},\overline{G})$. In particular,

$$\psi(h_G(a)) = \lim_{j\to\infty}\psi_j(\alpha_j) = \lim_{j\to\infty}\frac{a}{\alpha_j} = \frac{a}{h_G(a)} =: b \quad (\text{and } \psi(0) = b),$$

respectively. On the other hand, as $\overline{G}$ does not contain analytic discs through b, we get $\psi(\mathbb{D}) = b$. The continuity of h_D at b implies that

$$1 > h_D(\varphi_j(\lambda)) \to |\lambda|h_D(b),\ \lambda \in \mathbb{D}.$$

When $\lambda \to 1$ we get $h_D(b) \le 1$, i.e. $h_D(a) \le h_G(a)$. ∎

Remarks. (a) Since the holomorphic hull $\mathcal{E}(D)$ of a balanced domain D is a balanced domain (see e.g. [59, Remark 3.1.2(b)]), the above result can also be applied for $G = \mathcal{E}(D)$. Of course, it can also be applied for $G = \hat{D}$.

(b) If h_G is continuous near a and ∂G does not contain analytic discs through $a/h_G(a)$, then by the maximum principle it follows that $\overline{G}$ does not contain analytic discs through $a/h_G(a)$ either.

(c) In connection with Proposition 1.3.4 it is natural to ask whether if $h_D = l_D(0,\cdot)$ for a balanced domain D, then it has to be pseudoconvex. The answer to this question is unknown to us.

The next example shows that in Proposition 1.3.4 the continuity of h_D is essential.

Example 1.3.5. *If* $D = \mathbb{D}^2 \setminus \{(t,t) : |t| \geq 1/2\}$, $d = (t,t)$, $|t| < 1/2$, *then*

$$h_D(d) = 2|t|, \quad \textit{but} \quad l_D^*(0,d) = |t| = h_{\mathbb{D}^2}(d).$$

On the other hand, $\mathcal{E}(D) = \mathbb{D}^2$ *and* $\overline{\mathbb{D}^2}$ *does not contain analytic discs through any point from* $\partial\mathbb{D} \times \partial\mathbb{D}$.

Proof. We need to prove only that

$$l_D(0,d) \leq |t|.$$

For each $r \in (|t|,1)$ we can choose $\alpha \in \mathbb{D}$ so that $t = \varphi(t/r)$, where $\varphi(\lambda) = \lambda\frac{\lambda-\alpha}{1-\bar{\alpha}\lambda}$. Then the disc $\psi(\zeta) = (r\zeta, \varphi(\zeta))$ is a competitor for $l_D(0,d)$, whence it follows that $l_D(0,d) \leq |t|/r$. It remains to leave $r \to 1$. ∎

Addendum. Note that even

$$l_D(0,\cdot) = l_{\mathbb{D}^2}(0,\cdot).$$

To see this, it suffices to prove that $l_D(0,a) \leq |a_1|$ for $a = (a_1,a_2) \in D$, $a_1 \neq a_2$, $|a_1| \geq |a_2|$. We see this easily by considering $\psi(\lambda) = (\lambda, \lambda a_2/a_1)$ as a competitor for $l_D(0,a)$.

On the other hand, if $a_1 = (0,b)$ and $a_2 = (b,0)$, $b \in \mathbb{D}$, then

$$l_D(a_1,a_2) = l_{\mathbb{D}^2}(a_1,a_2) \;\Leftrightarrow\; |b| \leq 4/5.$$

Indeed, using the Möbius transformation $\psi_b(\lambda) = \frac{\lambda-b}{1-\bar{b}\lambda}$, we get $l_D(a_1,a_2) = l_{D_b}(0,a)$, where $a = (b,-b)$ and $D_b = \mathbb{D}^2 \setminus \{(\psi_b(\lambda),\lambda) : 1/2 \leq |\lambda| < 1\}$.

For $|b| < 4/5$ we easily check that $\varphi = (\mathrm{id}, -\mathrm{id}) \in \mathcal{O}(\mathbb{D}, D_b)$. Then $l_{D_b}(0,a) \leq |b|$ so $l_D(a_1,a_2) = l_{\mathbb{D}^2}(a_1,a_2)$.

To get this for $|b| = 4/5$, it suffices to consider $r\varphi$ for $r \in (0,1)$ as a competitor for $l_{D_b}(0,a)$, and then let $r \to 1$.

Now assume that $l_D(a_1,a_2) = l_{\mathbb{D}^2}(a_1,a_2)$ for $|b| > 4/5$. Then we can find discs $\varphi_j \in \mathcal{O}(\mathbb{D}, D_b)$ such that $\varphi_j(0) = 0$ and $\varphi_j(\alpha_j) = a$, where $\alpha_j \to b$. The Schwarz–Pick Lemma implies that $\varphi_j \to \varphi$. On the other hand, $\varphi(\mathbb{D}) \cap \{(\psi_b(\lambda),\lambda) : 1/2 < |\lambda| < 1\}$ is a singleton, contradicting the Hurwitz Theorem.

Remark. By [58, Theorem 3.4.2] (see also [108]) it follows that if $D_n = \mathbb{D}^n \setminus \{(t,\dots,t) : |t| \geq 1/2\}$, $n \geq 3$, then $l_{D_n} = l_{\mathbb{D}^n}$.

The next example shows that in Proposition 1.3.4 the assumption on discs is essential.

Example 1.3.6. *Let* $0 < a < 1$ *and*

$$D = \{z \in \mathbb{D}^2 : |z_2|^2 - a^2 < 2(1-a^2)|z_1|\}.$$

Then D *is a balanced Reinhardt domain,* h_D *is continuous function and* $\mathcal{E}(D) = \mathbb{D}^2$ *(see e.g.* [59]*). On the other hand, if* $c = (0,d)$, $|d| < a$, *then*

$$h_D(c) = |d|/a > l_D(0,c) = |d| = h_{\mathbb{D}^2}(c).$$

Proof. We just have to show that $l_D(0,c) \le d$ for $d \in (0,a)$. It suffices to show that $\varphi = (\psi, \mathrm{id}) \in \mathcal{O}(\mathbb{D}, D)$, where $\psi(\lambda) = \lambda \frac{\lambda - d}{1 - d\lambda}$. It is easily seen that $|\psi(\lambda)| \ge x \frac{x-d}{1-xd}$ for $x = |\lambda|$ and it suffices to check that

$$x^2 - a^2 < 2(1-a^2)\frac{x(x-d)}{1-dx}, \quad \text{i.e.}$$
$$dx^3 + (1-2a^2)x^2 - d(2-a^2)x + a^2 > 0.$$

This is clear for $x = 0$. Since $x \in (0,1)$ and $d \in (0,a)$, we need to prove that

$$ax^3 + (1-2a^2)x^2 - a(2-a^2)x + a^2 \ge 0,$$

which is equivalent to the obvious inequality $(x-a)^2(ax+1) \ge 0$. ∎

Remark. Some propositions and examples in the spirit of the above for $k^{(m)}$ can be found in the paper [90] of the author and P. Pflug.

1.4. Kobayashi–Buseman metric. The main aim of this section is to prove that the Kobayashi–Buseman metric for an arbitrary domain equals the Kobayashi metric of order $2n-1$ and this number is the least possible. A similar result for $2n$ instead of $2n-1$ is contained in the work [63] of S. Kobayashi, where this metric is introduced.

Theorem 1.4.1. *For each domain $D \subset \mathbb{C}^n$ one has*

$$\kappa_D^{(2n-1)} = \hat\kappa_D. \tag{1.4.1}$$

On the other hand, if $n \ge 2$ and

$$D_n = \Big\{ z \in \mathbb{C}^n : \sum_{j=2}^n (2|z_1^3 - z_j^3| + |z_1^3 + z_j^3|) < 2(n-1) \Big\},$$

then

$$\kappa_{D_n}^{(2n-2)}(0;\cdot) \ne \hat\kappa_{D_n}(0;\cdot). \tag{1.4.2}$$

The proof below shows that the identity (1.4.1) remains true for an arbitrary n-dimensional complex manifold.

Theorems 1.4.1 and 1.2.2 lead to the following

Corollary 1.4.2. *For every taut domain $D \subset \mathbb{C}^n$ one has*

$$\lim_{w \to z} \frac{k_D^{(2n-1)}(z,w)}{k_D(z,w)} = 1$$

locally uniformly on z. The number $2n-1$ is the least possible in the general case.

Remarks. (a) Corollary 1.4.2 remains true for an arbitrary n-dimensional complex taut manifold.

(b) Corollary 1.4.2 can be viewed as an affirmative answer to the infinitesimal version of a question of S. Krantz [66]: For an arbitrary strictly pseudoconvex domain $D \subset \mathbb{C}^n$, is there some $m = m(D) \in \mathbb{N}$ such that $k_D = k_D^{(m)}$? Unlike the infinitesimal case, m cannot depend only on n, as shown in [58, p. 109].

For $z \in D \subset C^n$, denote by $I_{D,z}$ the *indicatrix* of $\kappa_D(z;\cdot)$, i.e. $I_{D,z} = \{X \in \mathbb{C}^n : \kappa_D(z;X) < 1\}$. Note that $I_{D,z}$ is a balanced domain. In particular, it is starlike with respect to the origin. Then the identity $\kappa_D^{(2n)} = \hat\kappa_D$ is obtained from the following application of a lemma of C. Carathéodory (see e.g. [62]):

$$\hat h_S = \inf\Big\{\sum_{j=1}^m h_S(X_j) : m \le 2n,\ \sum_{j=1}^m X_j = X,\ X_1,\dots,X_m \text{ are } \mathbb{R}\text{-linearly independent}\Big\}, \tag{1.4.3}$$

where h_S and $\hat h_S$ are the functions of Minkowski of an arbitrary domain $S\subset\mathbb{C}^n$ that is starlike with respect to the origin (i.e. $ta\in S$ for $a\in S$ and $t\in[0,1]$) and of its convex hull $\hat S$, respectively (it is easily seen that in this case the number $2n$ is the least possible).

In order to replace the number $2n$ by $2n-1$, we will use the fact that $I_{D,z}$ is balanced rather than starlike. For $m\in\mathbb{N}$ we put

$$h_S^{(m)}(X) = \inf\Big\{\sum_{j=1}^m h_S(X_j) : \sum_{j=1}^m X_j = X\Big\}.$$

Proof of (1.4.1). This follows directly from

PROPOSITION 1.4.3. *If $B\subset\mathbb{C}^n$ is a balanced domain, then*

$$\hat h_B = h_B^{(2n-1)}. \tag{1.4.4}$$

To prove Proposition 1.4.3 we need

LEMMA 1.4.4. *Every balanced domain can be exhausted by bounded balanced domains with continuous Minkowski functions.*

Proof. Let $B\subset\mathbb{C}^n$ be a balanced domain. For $z\in\mathbb{C}^n$ and $j\in\mathbb{N}$ we put $F_{n,j,z} = \overline{\mathbb{B}_n(z,\|z\|^2/j)}$ ($\mathbb{B}_n(a,r)\subset\mathbb{C}^n$ is the ball of center a and radius r). We can assume that $\mathbb{B}_n(0,1)\Subset B$. Let

$$B_j = \{z\in\mathbb{B}_n(0,j) : F_{n,j,z}\subset B\},\quad j\in\mathbb{N}.$$

Then (B_j) is an exhaustion of B by nonempty bounded open sets. We will show that B_j is a balanced domain with continuous Minkowski function h_{B_j}.

To this end let us note that if $z\in B_j$ and $\lambda\in(\overline{\mathbb{D}})_*$, then $F_{n,j,\lambda z}\subset\lambda F_{n,j,z}\subset B$. Now it easily follows that B_j is a balanced domain.

As h_{B_j} is upper semicontinuous, it remains to prove that it is also lower semicontinuous. Assuming the contrary, we can find a sequence of points z_k tending to some z, and a number $c>0$ such that $h_{B_j}(z_k) < 1/c < h_{B_j}(z)$ for each k. Note that $F_{n,j,cz_k}\subset B$, so $\mathbb{B}_n(cz, c^2\|z\|^2/j)\subset B$. On the other hand, let us choose $t\in(0,1)$ such that $h_{B_j}(tcz)>1$. Then $F_{n,j,tcz}\subset\mathbb{B}_n(cz,c^2\|z\|^2/j)\subset B$, so $h(tcz)<1$, a contradiction. ∎

Proof of Proposition 1.4.3. We will first prove (1.4.4) in the case when $B\subset\mathbb{C}^n$ is a bounded balanced domain with a continuous Minkowski function. Let us fix a vector $X\in(\mathbb{C}^n)_*$. Then $\hat h_B(X)\ne 0$ and we can assume that $\hat h_B(X)=1$. As h_B is continuous, by (1.4.3) there exist $\mathbb{R}$-linearly independent vectors $X_1,\dots,X_m$ ($m\le 2n$) such that $\sum_{j=1}^m X_j = X$ and $\sum_{j=1}^m h_B(X_j)=1$. As $\hat h_B$ is a norm and $\hat h_B\le h_B$, by the triangle inequality $h_B(X_j)=\hat h_B(X_j)$, $j=1,\dots,m$. To prove (1.4.4), it suffices to show that $m\ne 2n$. Let H be a support hyperplane for $\hat B$ at $X\in\partial\hat B$. We can assume that $H =$

$\{z \in \mathbb{C}^n : \mathrm{Re}\langle z - X, X_0\rangle = 0\}$, where $X_0 \in \mathbb{C}^n$ ($\langle\cdot,\cdot\rangle$ is the Hermitian scalar product). Suppose that $m = 2n$. Then $H = \{\sum_{j=1}^m \alpha_j X_j/h_B(X_j) : \sum_{j=1}^m \alpha_j = 1,\ \alpha_1, \dots, \alpha_m \in \mathbb{R}\}$. In particular, $\partial\hat{B}$ contains a set relatively open in H. As $\hat{B}$ is a balanced domain, its intersection with the complex line through X, directed at $\overline{X}_0$, is a disc containing a line segment in its boundary. This contradiction proves (1.4.4) for a bounded balanced domain with a continuous Minkowski function.

Now let $B \subset \mathbb{C}^n$ be an arbitrary balanced domain. If (B_j) is an exhaustion of B as in Lemma 1.4.4, then $h_{B_j} \searrow h_B$ pointwise. Then (1.4.3) shows that $\hat{h}_{B_j} \searrow \hat{h}_B$. Now (1.4.3) follows from the inequalities $\hat{h}_B \le h_B^{(2n-1)} \le h_{B_j}^{(2n-1)}$ and the equality $\hat{h}_{B_j} = h_{B_j}^{(2n-1)}$ from above. ∎

Proof of (1.4.2). Observe that the domain D_n from Theorem 1.4.1 is pseudoconvex and balanced. Then $\kappa_{D_n}(0;\cdot) = h_{D_n}$ (see Proposition 1.3.1 (iv)) so $\kappa_{D_n}^{(m)}(0;\cdot) = h_{D_n}^{(m)}$. Thus (1.4.2) is equivalent to

$$\hat{h}_{D_n} \neq h_{D_n}^{(2n-2)}. \tag{1.4.5}$$

To prove this inequality, let $L_n = \{z \in \mathbb{C}^n : z_1 = 1\}$. By the triangle inequality $D_n \subset \mathbb{D} \times \mathbb{C}^{n-1}$ and

$$F_n := \partial D_n \cap L_n = \{z \in \mathbb{C}^n : z_1 = 1,\ z_j^3 = 1,\ 2 \le j \le n\}.$$

Hence $\partial\hat{D}_n \cap L_n = \hat{F}_n = \{1\} \times \hat{\Delta}^{n-1}$, where Δ is the triangle of vertices 1, $e^{2\pi i/3}$, $e^{4\pi i/3}$ together with its interior. Note that $\partial\hat{D}_n \cap L_n$ is a $(2n-2)$-dimensional convex set. Put $\tilde{F}_n = \{Y \in \hat{F}_n : h_{D_n}^{(2n-2)}(Y) = 1\}$. If $X \in \tilde{F}_n$, then there exist vectors $X_1, \dots, X_m \in (\mathbb{C}^n)_*$, $m \le 2n-2$, such that $\sum_{j=1}^m X_j = X$ and $\sum_{j=1}^m h_{D_n}(X_j) = 1$ (as D_n is a taut domain). Then $X_1/h_{D_n}(X_1), \dots, X_m/h_{D_n}(X_m) \in F_n$ and the convex hull of these vectors contains X. As F_n is a finite set, it is contained $\tilde{F}_n$ in a finite union of not more than $(2n-3)$-dimensional convex sets. So $\hat{F}_n \neq \tilde{F}_n$, which shows that $\hat{h}_{D_n} \neq h_{D_n}^{(2n-2)}$. ∎

Thus Theorem 1.4.1 is proved.

1.5. Interpolation in the Arakelian theorem. The aim of this section is to prove a general statement on approximation and interpolation over so-called Arakelian sets. This statement will be used in the proof of Theorem 1.6.1 from the next section.

Let us first recall the well-known theorem of Mergelian that generalizes the theorems of Weierstrass and Runge.

THEOREM 1.5.1. *The complement of a compact $K \subset \mathbb{C}$ is a connected set if and only if each continuous function on K that is holomorphic in the interior of K can be uniformly approximated on K by polynomials.*

The most popular generalization of Theorem 1.5.1 belongs to N. Arakelian ([3]).

A relatively closed subset E of a domain $D \subset \mathbb{C}$ is called an *Arakelian set* if $D^* \setminus E$ is connected and locally connected, where D^* is the one-point compactification of D.

Denote by $\mathcal{A}(E)$ the set of continuous functions on E that are holomorphic in the interior E^0 of E.

([3]) After the proof by N. Arakelian of Theorem 1.5.2, J.-P. Rosay and W. Rudin [111] showed how this theorem follows from the Mergelian theorem itself.

Theorem 1.5.2 ([6]). *A relatively closed subset E of a domain $D \subset \mathbb{C}$ is an Arakelian set if and only if each function from $\mathcal{A}(E)$ can be uniformly approximated on E by holomorphic functions from D.*

The next result, proven independently by P. M. Gauthier and W. Hengartner and by A. Nersesyan, provides an opportunity for interpolation in Theorem 1.5.2.

Theorem 1.5.3 ([43, 78]). *Let $D \subset \mathbb{C}$ be a domain, let $E \subset D$ be an Arakelian set, and let Λ be a sequence of points in $E \setminus E^0$ without an accumulation point in D. Suppose for every $\lambda \in \Lambda$, a finite sequence $(\beta_\lambda^\nu)_{\nu=1}^{\nu(\lambda)}$ of complex numbers is given. Then for each $f \in \mathcal{A}(E)$ and each $\varepsilon > 0$, there exists a $g \in \mathcal{O}(D)$ such that $|g(z) - f(z)| < \varepsilon$ for $z \in E$, $g(\lambda) = f(\lambda)$ and $g^{(\nu)}(\lambda) = \beta_\lambda^\nu$ for $\lambda \in \Lambda$ and $\nu = 1, \dots, \nu(\lambda)$.*

Now let us formulate an extension of Theorem 1.5.3.

Theorem 1.5.4. *Let D, E, Λ (possibly $\Lambda = \emptyset$), β_λ^ν be as in Theorem 1.5.3 and let $b_1, \dots, b_k \in E^0$. Then for each $f \in \mathcal{A}(E)$, $\varepsilon > 0$ and $m \in \mathbb{N}^*$ there exists a $g \in \mathcal{O}(D)$ with the properties of Theorem 1.5.3 and such that $g^{(\nu)}(b_j) = f^{(\nu)}(b_j)$ for $j = 1, \dots, k$ and $\nu = 0, \dots, m$.*

Proof. We can clearly assume that $E \neq D$.

The proof will be divided into four steps.

Step 1. For each $j = 1, \dots, k$, there is a function $s_j \in \mathcal{O}(D)$, bounded on E and such that $s_j'(b_j) \neq 0$, $s_j(b_j) = 0$ and $s_j(b_q) \neq 0$ for an arbitrary $q \neq j$ ([4]).

Indeed, choose a point $c \in D \setminus E$. As $E \cup \{c\} \subset D$ is an Arakelian set, Theorem 1.5.2 implies the existence of an $\tilde{s} \in \mathcal{O}(D)$ such that $|\tilde{s}| < 1$ on E and $|\tilde{s}(c) - 2| < 1$. Put $\hat{s}_j = \tilde{s} - \tilde{s}(b_j)$. As $\hat{s}_j(c) \neq 0$, we get $\hat{s}_j \not\equiv 0$. Now as $|\hat{s}_j| < 2$ on E, the function

$$s_j(z) = \frac{(z - b_j)\hat{s}_j(z)}{\prod_{q=1}^k (z - b_q)^{\operatorname{ord}_{b_q} \hat{s}_j}}, \quad z \in D,$$

has the required properties.

Step 2. There exists a function $p \in \mathcal{O}(D)$, bounded on E and such that $p(b_j) \neq 0$ for $j = 1, \dots, k$ and $\operatorname{ord}_\lambda p \geq \nu(\lambda) + 1$ for an arbitrary $\lambda \in \Lambda$.

Indeed, if $q = 0$ on E and $q(c) = 1$, where $c \in D \setminus E$, we can apply Theorem 1.5.3 for $E \cup \{c\}$, q, $\varepsilon = 1$ and $\beta_\lambda^\nu = 0$, $\nu = 1, \dots, \nu(\lambda) + 1$, $\lambda \in \Lambda$. Thus we get a nonconstant function $\tilde{p} \in \mathcal{O}(D)$ such that $|\tilde{p}| < 1$ on E and $\operatorname{ord}_\lambda \tilde{p} \geq \nu(\lambda) + 1$, $\lambda \in \Lambda$. It remains to put

$$p(z) = \frac{\tilde{p}(z)}{\prod_{j=1}^k (z - b_j)^{\operatorname{ord}_{b_j} \tilde{p}}}, \quad z \in D.$$

Step 3. Let s_j be the function from Step 1, $j = 1, \dots k$, and let p be the function from Step 2. For each $\nu \in \mathbb{N}^*$ we put

$$\tilde{h}_j^\nu = \frac{p}{s_j} \prod_{q=1}^k s_q^{\nu+1},$$

([4]) This is clear if D is biholomorphic to a bounded domain; in particular, if $\overline{D} \neq \mathbb{C}$.

Then

$$h_j^\nu = \frac{\tilde{h}_j^\nu}{(\tilde{h}_j^\nu)^{(\nu)}(b_j)}$$

is well defined on D. The function

$$M_\nu = \sup_E \sum_{j=1}^k |h_j^\nu|$$

will be also needed in the last step.

Step 4. We are ready to prove the theorem by induction on m.

Let $m = 0$ and g be the function from Theorem 1.5.3 for Λ, $(\beta_\lambda^\nu)_{\nu=1}^{\nu(\lambda)}$ and $\frac{\varepsilon}{M_0+1}$. It is easily checked that the function

$$g_0 = g + \sum_{j=1}^k (f(b_j) - g(b_j))h_j^0$$

has the required properties.

Put $d = \min_{1\le j\le k} \operatorname{dist}(b_j, \mathbb{C} \setminus E^0)$. Assume that the conclusion of Theorem 1.5.4 is true for some $m \ge 0$ and let g_m be the corresponding function for $\varepsilon(1 + M_{m+1}(m + 1)!d^{-m-1})^{-1}$. By the Cauchy inequality, the function

$$g_{m+1} = g_m + \sum_{j=1}^k (f^{(m+1)}(b_j) - g_m^{(m+1)}(b_j))h_j^{m+1}$$

has the required properties for $m + 1$.

This finishes the induction step. ∎

1.6. Generalized Lempert function. In this section we define the generalized Lempert function (introduced by D. Coman [22]) and prove that it decreases under adding poles. This function is introduced as an easier and more flexible (in some sense) version of the so-called generalized (pluricomplex) Green function (see e.g. [60]).

Let $D \subset \mathbb{C}^n$ be a domain and $p \gneq 0$ be a function on D. Put

$$|\boldsymbol{p}| = \{a \in D : \boldsymbol{p}(a) > 0\}.$$

For $z \in D$ we define

$$l_D(\boldsymbol{p}, z) = \inf\Big\{\prod |\lambda_{\psi,a}|^{\boldsymbol{p}(a)} : \exists \psi \in \mathcal{O}(\mathbb{D}, D),\ \psi(0) = z,\ \psi(\lambda_{\psi,a}) = a \text{ for each } a \in |\boldsymbol{p}|\Big\}$$

(for any $a \in |\boldsymbol{p}|$ we take one $\lambda_{\psi,a}$).

From the proof of Theorem 1.6.1 below it follows that such a ψ exists if $|\boldsymbol{p}|$ is finite or countable. If $|\boldsymbol{p}|$ is uncountable and such a ψ exists, then it is easily seen that

$$0 = l_D(\boldsymbol{p}, z) = \inf\{l_D(\boldsymbol{p}_B, z) : B \subset |\boldsymbol{p}|,\ 0 < \#B < \infty\},$$

where $p_B = p\chi_B$.

If there is no such ψ, we can define

$$l_D(\boldsymbol{p}, z) = \inf\{l_D(\boldsymbol{p}_B, z) : B \subset |\boldsymbol{p}|,\ 0 < \#B < \infty\}.$$

The function $l_D(\boldsymbol{p}, \cdot)$ so introduced is called the *Lempert function of D with respect to $\boldsymbol{p}$* (a *generalized Lempert function*). If A is a nonempty subset of D and χ_A is its characteristic function, then we put $l_D(A, z) = l_D(\chi_A, z)$. This function is called the *Lempert function with poles in* A. Let us note that $l_D(\{a\}, z)$ is the usual Lempert function $l_D(a, z)$.

Using the Lempert theorem, F. Wikström [118] showed that if A and B are subsets of a convex domain $D \subset \mathbb{C}^n$ such that $A \subset B$, then $l_D(B, \cdot) \le l_D(A, \cdot)$, i.e. the Lempert function decreases under adding poles.

On the other hand, in [119] there is an example of a complex space not satisfying this inequality (under the same definition of a Lempert function) and it is asked whether this inequality is true for arbitrary domains in $\mathbb{C}^n$.

The main aim of this section is to give an affirmative answer to this question. We will use Theorem 1.5.4, that is, the possibility for interpolation in the Arakelian approximation theorem.

THEOREM 1.6.1. *If* $D \subset \mathbb{C}^n$ *is a domain and* $\boldsymbol{p} \gneq 0$ *is a function on* D, *then*

$$l_D(\boldsymbol{p}, \cdot) = \inf\{l_D(\boldsymbol{p}_B, \cdot) : B \subset |\boldsymbol{p}|,\ 0 < \#B < \infty\}.$$

In particular, $l_D(\boldsymbol{p}, \cdot) = \inf\{l_D(\boldsymbol{p}_B, \cdot) : \emptyset \ne B \subset |\boldsymbol{p}|\}$.

COROLLARY 1.6.2. *If* $D \subset \mathbb{C}^n$ *is a domain and* $\boldsymbol{p}, \boldsymbol{q}$ *are functions on* D *such that* $0 \lneq \boldsymbol{p} \le \boldsymbol{q}$, *then* $l_D(\boldsymbol{q}, \cdot) \le l_D(\boldsymbol{p}, \cdot)$.

Proof. By the above remark, the theorem follows in the case when $|\boldsymbol{p}|$ is uncountable.

Now let $|\boldsymbol{p}| = (a_j)_{j=1}^l$ ($l \in \mathbb{N}^*$) be a countable or finite nonempty set. Let $z \in D$.

We first prove the inequality

$$l_D(\boldsymbol{p}, z) \le \inf\{l_D(\boldsymbol{p}_B, z) : B \subset |\boldsymbol{p}|,\ 0 < \#B < \infty\}. \tag{1.6.1}$$

Let $B \ne \emptyset$ be a finite subset of $|\boldsymbol{p}|$. We can assume that $B = A_m := (a_j)_{j=1}^m$ for some $m \le l$.

Let us consider an arbitrary $\varphi : \mathbb{D} \to D$ such that $\varphi(\lambda_j) = a_j$, $0 \le j \le m$, where $\lambda_0 = 0$ and $a_0 = z$. Let $t \in [\max_{0 \le j \le m} |\lambda_j|, 1)$ and $\lambda_j = 1 - (1-t)/j$, $j \in A(m)$, where $A(m) = \{m+1, \dots, l\}$ for $l < \infty$ and $A(m) = \{j \in \mathbb{N} : j > m\}$ for $l = \infty$. Consider a continuous curve $\varphi_1 : [t, 1) \to D$ such that $\varphi_1(t) = \varphi(t)$ and $\varphi_1(\lambda_j) = a_j$, $j \in A(m)$. Put

$$f = \begin{cases} \varphi|_{\overline{t\mathbb{D}}} \\ \varphi_1|_{[t,1)} \end{cases}$$

on $F_t = \overline{t\mathbb{D}} \cup [t, 1) \subset \mathbb{D}$. Clearly F_t is an Arakelian set for $\mathbb{D}$, $f \in \mathcal{A}(F_t, D)$ and $\Lambda = (\lambda_j)_{j=1}^l$ satisfies the conditions in Theorem 1.5.4. Let $d(z) = \operatorname{dist}(f(z), \partial D)$, $z \in F_t$, where the distance is generated by the L^∞-norm. Choose a continuous real-valued function η on F_t such that

$$\eta \le \log d \quad \text{on } [t, 1), \qquad \eta = \min_{\overline{t\mathbb{D}}} \log d \quad \text{on } \overline{t\mathbb{D}}.$$

By Theorem 1.5.3, there exists a $\zeta \in \mathcal{O}(\mathbb{D})$ such that $|\zeta - \eta| < 1$ on F_t. By Theorem 1.5.4, applied to the components of $e^{\zeta - 1} f$, one can find a $q_t \in \mathcal{O}(\mathbb{D})$ such that $q_t(\lambda) = f(\lambda)$, $\lambda \in \Lambda$ and

$$\|q_t - f\| < |e^{\zeta(z)-1}| < e^{\eta(z)} \le d(z), \quad z \in F_t.$$

Thus $q_t(F_t) \subset D$ and so there exists a simply connected domain E_t such that $F_t \subset E_t \subset \mathbb{D}$ and $q_t(E_t) \subset D$.

Let $\rho_t : \mathbb{D} \to E_t$ be the corresponding Riemann (conformal) mapping, satisfying $\rho_t(0) = 0$, $\rho_t'(0) > 0$ and $\rho_t(\lambda_j^t) = \lambda_j$. Considering the analytic discs $q_t \circ \rho_t : \mathbb{D} \to D$ we get

$$l_D(\boldsymbol{p}, z) \le \prod_{j=1}^{l} |\lambda_j^t|^{\boldsymbol{p}(a_j)} \le \prod_{j=1}^{m} |\lambda_j^t|^{\boldsymbol{p}(a_j)}.$$

Note that by the Carathéodory Kernel Theorem, ρ_t for $t \to 1$ tends locally uniformly on $\mathbb{D}$ to id. Hence the latter product above tends to $\prod_{j=1}^m |\lambda_j|^{\boldsymbol{p}(a_j)}$. Since φ was an arbitrary competitor for $l_D(\boldsymbol{p}|_{A_m}, z)$, we get the inequality (1.6.1).

On the other hand, the existence of analytic discs containing z and $|\boldsymbol{p}|$ easily implies

$$l_D(\boldsymbol{p}, z) \ge \limsup_{m\to\infty} l_D(\boldsymbol{p}|_{A_m}, z),$$

which concludes the proof of the theorem. ■

Remark. The Lempert function does not decrease strictly under addition of poles; for example [32, Theorem 2.1] shows that

$$l_{\mathbb{D}^2}(\{a_1, a_2\} \times \{a_1\}, 0) = |a_1| = l_{\mathbb{D}^2}(\{a_1\} \times \{a_1\}, 0).$$

The next example shows that our definition of a generalized Lempert function, in the case of nonexistence of a corresponding disc, is more "sensitive" than that from [60] (where in this case the function is set to be 1).

Example. Let $A \subset \mathbb{D}$ be an uncountable set. Then there is no analytic disc $\varphi \in \mathcal{O}(\mathbb{D}, \mathbb{D}^2)$ containing $A \times \{0\}$ and $(0, w)$, $w \in \mathbb{D}_*$.

Let B be an arbitrary finite subset of A. From [32, Theorem 2.1],

$$l_D(B \times \{0\}, (0, w)) = \max\{l_{\mathbb{D}}(B, 0), l_{\mathbb{D}}(0, w)\} = \max\Big\{\prod_{b\in B} |b|, |w|\Big\}.$$

So $l_D(A \times \{0\}, (0, w)) = |w|$.

Finally let us note that the generalized Lempert function is clearly biholomorphically invariant, but in general not contractible under holomorphic mappings even when they are proper coverings.

Example. Let $\pi(z) = z^2$. Clearly $\pi : \mathbb{D}_* \to \mathbb{D}_*$ is a proper covering ($D_* = \mathbb{D} \setminus \{0\}$). Let $a_1 = -a_2 \in \mathbb{D}_*$, $c = a_1^2$ and $z \in \mathbb{D}_*$, $z \ne a_1, a_2$. By [58, Theorem 3.3.7],

$$l_{\mathbb{D}_*}(c, z^2) = \min\{l_{\mathbb{D}_*}(a_1, z), l_{\mathbb{D}_*}(a_2, z)\} > l_{\mathbb{D}_*}(a_1, z) l_{\mathbb{D}_*}(a_2, z).$$

On the other hand, by [32, Theorem 2.1] the last product equals $l_{\mathbb{D}_*}(\{a_1, a_2\}, z)$. Therefore

$$l_{\mathbb{D}_*}(\boldsymbol{p}, \pi(z)) > l_{\mathbb{D}_*}(\boldsymbol{p} \circ \pi, z) \quad \text{for } \boldsymbol{p} = \chi_{\{c\}}.$$

We conclude this section with the following comment. The proof of Theorem 1.6.1 is contained in the paper [86] by the author and P. Pflug. Later, based on the same idea, F. Forstnerič and J. Winkelmann [38] proved that, for every connected complex manifold, the holomorphic discs with dense images form a dense subset of the set of all discs. To this end a nontrivial approximation statement is used and the result is the following.

Let M be a connected complex manifold, d is a distance generated by a complete Riemann metric, A is a countable subset of M, $f \in \mathcal{O}(\mathbb{D}, X)$ and $r \in (0,1)$. Then there exists a $g \in \mathcal{O}(\mathbb{D}, X)$ such that $A \subset g(\mathbb{D})$ and $d(f(z), g(z)) < 1 - r$ for each $z \in r\mathbb{D}$.

A modification of the proof of this fact shows that if, apart from A, f, r, we are given a finite subset Λ of $\mathbb{D}$, then there exists a g as above, as well as a sequence $(\mu_\lambda)_{\lambda\in\Lambda} \subset \mathbb{D}$ with $r|\mu_\lambda| < |\lambda|$ such that $f(\lambda) = g(\mu_\lambda)$, $\lambda \in \Lambda$. Letting $r \to 1$ one can prove that Theorem 1.6.1 remains true for complex manifolds.

1.7. Product property. Let $l_D(\boldsymbol{p}, \cdot)$ and $l_G(\boldsymbol{q}, \cdot)$ be generalized Lempert functions of domains $D \subset \mathbb{C}^n$ and $G \subset \mathbb{C}^m$. They generate a generalized Lempert function $l_{D\times G}(\boldsymbol{r}, \cdot)$ of the product product $G \times D$, where

$$\boldsymbol{r}(\zeta, \eta) = \boldsymbol{p}(\eta)\boldsymbol{q}(\zeta), \quad \zeta \in D,\ \eta \in G.$$

In this section we discuss when the generalized Lempert function has the *product property*, i.e.

$$l_{D\times G}(\boldsymbol{r}, (z, w)) = \max\{l_D(\boldsymbol{p}, z), l_G(\boldsymbol{q}, w)\}.$$

Let us note that the Lempert functions, the Kobayashi functions, and the Carathéodory functions have this property; a similar property is true for their infinitesimal forms (see e.g. [58, 60]).

We need the *pluricomplex Green function* g_D defined as follows:

$$g_D(z, w) = \sup u(w),$$

where the supremum is over all negative functions $u \in \mathrm{PSH}(D)$ such that $u(\cdot) \le \log\|\cdot - z\| + O_u(1)$. Then

$$c_D^* \le g_D^* := \exp g_D \le l_D,$$

so for the infinitesimal form of g_D, the so-called *Azukawa* (*pseudo*)*metric*,

$$A_D(z; X) = \limsup_{\lambda\to 0} \frac{g_D^*(z, z + \lambda X)}{|\lambda|},$$

we have

$$\gamma_D \le A_D \le \kappa_D.$$

For example, the theorem of Lempert implies that if D is a convex domain, then in both the chains of inequalities we have in fact equalities.

Recall that $|\boldsymbol{p}| = \{a \in D : \boldsymbol{p}(a) > 0\}$. The next proposition provides a necessary and sufficient condition for the product property when the support of one of the functions is a singleton.

Proposition 1.7.1. *If $(z, w) \in D \times G$, $|\boldsymbol{p}| = \{a\} \subset D$, then*

$$l_{D\times G}(\boldsymbol{r}, (z, w)) = \max\{l_D(\boldsymbol{p}, z), l_G(\boldsymbol{q}, w)\}$$

for each function $q \gneq 0$ on G if and only if $l_D(a, z) = g_D(a, z)$.

A special case of Proposition 1.7.1 was used in Section 1.6 with a quote of [32, Theorem 2.1]. In fact, that theorem is Proposition 1.7.1 in the special case when $\boldsymbol{q}$ is the characteristic function of a finite set; in the general case the proof is similar and we omit it. We only note that it is based on the inequality

$$l_{D\times G}(\boldsymbol{r},(z,w)) \ge \max\{l_D(\boldsymbol{p},\cdot), l_G(\boldsymbol{q},\cdot)\}, \quad \#|\boldsymbol{p}| = 1.$$

The proof of this inequality, given in [32] for the above mentioned special case, contains an essential flaw, corrected in the paper [103] of the author and W. Zwonek.

Similarly to the generalized Lempert function, one can define a generalized Green function. This function does not exceed the corresponding generalized Lempert function. In addition, by a result of A. Edigarian, it possesses the product property (see e.g. [35, 60]).

On the other hand, D. Coman [22] showed that the Lempert and Green functions of a ball that have two poles coincide. He asked (see also [60]) whether, like the Lemert theorem, this property remains true for every convex domain for every finite number of poles.

To give a negative answer to this question is one of the reasons for our interest in the product property of the generalized Lempert function (which shows that this function does not have properties as typical as the generalized Green function). More precisely, there are two-element subsets A, B of $\mathbb{D}$ and a point $z \in \mathbb{D}$ such that

$$l_{\mathbb{D}^2}(A\times B,(z,w)) > \max\{l_{\mathbb{D}}(A,z), l_{\mathbb{D}}(B,0)\}. \tag{1.7.1}$$

As

$$l_{\mathbb{D}}(C,z) = g_{\mathbb{D}}(C,z) = \prod_{c\in C} m_{\mathbb{D}}(c,z), \tag{1.7.2}$$

we get

$$\begin{aligned} l_{\mathbb{D}^2}(A\times B,(z,0)) &> \max\{l_{\mathbb{D}}(A,z), l_{\mathbb{D}}(B,0)\} = \max\{g_{\mathbb{D}}(A,z), g_{\mathbb{D}}(B,0)\} \\ &= g_{\mathbb{D}^2}(A\times B,(z,0)). \end{aligned}$$

The inequality (1.7.1) was first established by P. J. Thomas and N. V. Trao in [114] and independently, but somewhat later, by the author and W. Zwonek in [103]. In the latter work the proof is considerably shorter and includes a complete characterization of the two-element subsets A and B of $\mathbb{D}$ for which we have the critical double equality

$$l_{\mathbb{D}^2}(A\times B,(z,w)) = l_{\mathbb{D}}(A,z) = l_{\mathbb{D}}(B,w).$$

(This characterization shows that the product property is not typical for the generalized Lempert function.) By applying an automorphism of $\mathbb{D}^2$, it suffices to consider the case $z = w = 0$.

Note that, as above,

$$l_{\mathbb{D}^2}(\boldsymbol{r},(z,w)) \ge g_{\mathbb{D}^2}(\boldsymbol{r},(z,w)) = \max\{g_{\mathbb{D}}(\boldsymbol{p},z), l_{\mathbb{D}}(\boldsymbol{q},w)\} = \max\{l_{\mathbb{D}}(\boldsymbol{p},z), l_{\mathbb{D}}(\boldsymbol{q},w)\};$$

in particular, always

$$l_{\mathbb{D}^2}(A\times B,(z,w)) \ge \max\{l_{\mathbb{D}}(A,z), l_{\mathbb{D}}(B,0)\}. \tag{1.7.3}$$

Proposition 1.7.2. *If $A = \{a_1, a_2\} \subset \mathbb{D}_*$ and $B = \{b_1, b_2\} \subset \mathbb{D}_*$, then*

$$l_{\mathbb{D}^2}(A\times B,(0,0)) = l_{\mathbb{D}}(A,0) = l_{\mathbb{D}}(B,0) \tag{1.7.4}$$

if and only if there is a rotation that maps A to B.

In addition, if $B = e^{i\theta}A$, $\theta \in \mathbb{R}$, then the extremal discs ([5]) *for $l_{\mathbb{D}^2}(A \times B, (0,0))$ are of the form $\zeta \mapsto (e^{i\varphi}\zeta, e^{i(\varphi+\theta)}\zeta)$, $\varphi \in \mathbb{R}$.*

Remark. From the last statement it follows that the extremal discs for $l_{\mathbb{D}^2}(A \times B, (0,0))$ pass through two points from the four-element set $A \times B$, although the Lempert function decreases under addition of poles, according to Corollary 1.6.2.

Proof. Let $\psi = (\psi_1, \psi_2)$ be an extremal disc for $l_{\mathbb{D}^2}(A \times B, (0,0))$. Then we can find a set $J \subset \{1,2\} \times \{1,2\}$ and points $z_{k,l} \in \mathbb{D}$, $(k,l) \in J$, such that

$$\psi(z_{k,l}) = (a_k, b_l) \quad \text{and} \quad \prod_{(k,l)\in J} |z_{k,l}| = l_{\mathbb{D}^2}(A \times B, (0,0)).$$

First let (1.7.4) be true. If $\#J = 1$, we can assume that $J = \{(1,1)\}$. Then

$$|z_{1,1}| = l_{\mathbb{D}^2}(A \times B, (0,0)) = l_{\mathbb{D}}(A, 0) = |a_1 a_2| < |a_1| = |\psi_1(z_{1,1})| \le |z_{1,1}|$$

(according to the Schwarz–Pick lemma), a contradiction.

If $\#J = 3$, we can assume that $J = \{(1,1), (1,2), (2,2\}$. As above,

$$|z_{1,1} z_{1,2} z_{2,2}| = |a_1 a_2|.$$

On the other hand, as $\varphi_1 \in \mathcal{O}(\mathbb{D}, \mathbb{D})$, $\varphi_1(0) = 0$, $\varphi_1(z_{1,1}) = \varphi_1(z_{1,2}) = a_1$, $\varphi_1(z_{2,2}) = a_2$, we get

$$|z_{1,1} z_{1,2}| \le |a_1|, \qquad |z_{2,2}| \le |a_2|,$$

with equalities attained when φ_1 is a Blaschke product of order 2 and a rotation, respectively—a contradiction.

Let $\#J = 4$. We can assume that

$$\psi_1(z) = z\Phi_\alpha(z), \qquad \psi_2(z) = e^{it} z \Phi_\beta$$

for some $\alpha, \beta \in \mathbb{D}, t \in \mathbb{R}$. Then

$$z_{1,1}\Phi_\alpha(z_{1,1}) = z_{1,2}\Phi_\alpha(z_{1,2}), \qquad z_{2,1}\Phi_\alpha(z_{2,1}) = z_{2,2}\Phi_\alpha(z_{2,2}),$$
$$z_{1,1}\Phi_\beta(z_{1,1}) = z_{2,1}\Phi_\beta(z_{2,1}), \qquad z_{1,2}\Phi_\beta(z_{1,2}) = z_{2,2}\Phi_\beta(z_{2,2}).$$

Consequently,

$$z_{1,1} = \Phi_\alpha(z_{1,2}) = \Phi_\beta(z_{2,1}), \qquad z_{1,2} = \Phi_\beta(z_{2,2}), \qquad z_{2,1} = \Phi_\alpha(z_{2,2}).$$

Hence $z_{1,1} = \Phi_\alpha \circ \Phi_\beta(z_{2,2}) = \Phi_\beta \circ \Phi_\alpha(z_{2,2})$. After some calculations we get the equality

$$(2 - \alpha\bar\beta - \bar\alpha\beta)(z_{2,2}^2(\bar\alpha - \bar\beta) + z_{2,2}(\alpha\bar\beta - \bar\alpha\beta) + \beta - \alpha) = 0.$$

It is easily seen that if $\alpha \ne \beta$, then the two roots of the equation

$$z^2(\bar\alpha - \bar\beta) + z(\alpha\bar\beta - \bar\alpha\beta) = \alpha - \beta$$

lie on the unit circle. Therefore $\alpha = \beta$, $z_{1,2} = z_{2,1}$, $z_{1,1} = z_{2,2}$, a contradiction.

Let now $\#J = 2$. We can assume that $J = \{(1,1), (2,2\}$. Then it easily follows that $\psi_1(z) = e^{i\theta_1} z$, $\psi_2(z) = e^{i\theta_2} z$, $\theta_1, \theta_2 \in \mathbb{R}$, and so $B = e^{i\theta}A$, where $\theta = \theta_1 - \theta_2$.

([5]) As $\mathbb{D}^2$ is a taut domain, the infimum in the definition of $l_{\mathbb{D}^2}$ is attained and the corresponding discs are called extremal.

Conversely, if $B = e^{i\theta}A$, the mapping $(\mathrm{id}, e^{i\theta}\,\mathrm{id}) \in \mathcal{O}(\mathbb{D}, \mathbb{D}^2)$ is a competitor for $l_{\mathbb{D}^2}(A \times B, (0,0))$ and so

$$l_{\mathbb{D}}(A, 0) = l_{\mathbb{D}}(B, 0) \geq\ l_{\mathbb{D}^2}(A \times B, (0,0)).$$

To get (1.7.4), it remains to use (1.7.3). ■

Corollary 1.7.3. *If A and B are two-point subsets of $\mathbb{D}$ and $z \in \mathbb{D} \setminus A$, then the set of points $w \in \mathbb{D}$ such that*

$$l_{\mathbb{D}}(A, z) = l_{\mathbb{D}}(B, w) < l_{\mathbb{D}^2}(A \times B, (z, w))$$

has Hausdorff dimension 1.

Proof. It suffices to note that the set of points $w \in \mathbb{D}$ such that $l_{\mathbb{D}}(A, z) = l_{\mathbb{D}}(B, w)$ has Hausdorff dimension 1, and there are at most two points w for which there is an automorphism of $\mathbb{D}$ that maps z to w and A to B. ■

We do not know whether Proposition 1.7.2 remains true for sets of equal cardinality, greater than 2. Anyway, for a given point $(z, w) \in \mathbb{D}^2$ this proposition and (1.7.2) provide a large class of counterexamples for the product property of $l_{\mathbb{D}^2}(A \times B, (z, w))$, where A and B have an arbitrary number of elements, greater than 1.

Proposition 1.7.4. *Let $z, w \in \mathbb{D}$, $A, B \subset \mathbb{D}$ and $q \in (0, 1)$ such that*

$$\max\{l_{\mathbb{D}}(A, z), l_{\mathbb{D}}(B, w)\} = q l_{\mathbb{D}^2}(A \times B, (z, w)) > 0.$$

Then

$$\max\{l_D(A \cup A_1, z), l_G(B \cup B_1, w)\} < l_{D\times G}((A \cup A_1) \times (B \cup B_1), (z, w)),$$

if $A_1, B_1 \subset \mathbb{D}$, $A \cap A_1 = B \cap B_1 = \emptyset$ and $l_{\mathbb{D}}(A_1, z) l_{\mathbb{D}}(B_1, w) > q$.

Proof. We have

$$\begin{aligned}
&l_{D\times G}((A \cup A_1) \times (B \cup B_1), (z, w))\\
&\qquad \geq l_{D\times G}(A \times B, (z, w)) l_{D\times G}(A \times B_1, (z, w)) l_{D\times G}(A_1 \times (B \cup B_1), (z, w))\\
&\qquad \geq l_{D\times G}(A \times B, (z, w)) l_G(B_1, w) l_D(A_1, z)\\
&\qquad > \max\{l_D(A, z), l_G(B, w)\} \geq \max\{l_D(A \cup A_1, z), l_G(B \cup B_1, w)\}
\end{aligned}$$

(the first inequality is checked immediately; for the second one see (1.7.3); for the fourth one see Corollary 1.6.2). ■

2. The symmetrized polydisc and the spectral ball

2.1. Synopsis. This chapter is devoted to geometric and analytic properties of so-called symmetrized polydics and the spectral ball, which have been intensively studied recently by many authors.

To understand better the geometry of the symmetrized polydisc we need some notions of complex convexity of domains and their interrelations; this is the aim of the first part of Section 2.6. In [5, 50] one can find a detailed discussion of their role and applications.

We only note that $\mathbb{C}$-convexity is closely related to some important properties of the Fantappiè transformation, and, as a deep conclusion, to the question of solvability of linear PDEs in the class of holomorphic functions. A domain $D \subset \mathbb{C}^n$ is called $\mathbb{C}$-*convex* if its nonempty intersections with complex lines are connected and simply connected. Some other notions for complex convexity of D are *linear convexity* (each point in the complement of D is contained in a complex hyperplane, disjoint from D), *weak linear convexity* (the same, but for the points in ∂D) and *weak locally linear convexity.* $\mathbb{C}$-convexity implies weak convexity, and all the four notions coincide for bounded domains with $\mathcal{C}^1$-smooth boundaries. In the general case their place is between convexity and pseudoconvexity.

Indeed, in Corollary 2.6.4 we give an affirmative answer to the question of D. Jacquet [56, p. 58] whether each weakly locally linearly convex domain is pseudoconvex. On the other hand, in Proposition 2.6.1 we show that each weakly linearly convex balanced domain is convex, which strengthens the same observation for complete Reinhardt domains in [5, Example 2.2.4].

Theorem 2.6.5 implies that a $\mathbb{C}$-convex domain is either a cartesian product of $\mathbb{C}$ and another $\mathbb{C}$-convex domain, or is biholomorphic to a bounded domain; in the latter case it is *c-finitely compact* (i.e. balls with respect to the Carathéodory distance are relatively compact). This generalizes the result of T. J. Barth in [7] for convex domains.

Let $\mathcal{L}$ denote the class of domains D such that the least and the largest invariant functions of D (from complex-analytic viewpoint), namely the Carathéodory function c_D^* and the Lempert function l_D, coincide (and in particular they coincide with the Kobayashi function k_D^*).

Recently D. Jacquet [55] proved that each bounded $\mathbb{C}$-convex domain with $\mathcal{C}^2$-smooth boundary can be exhausted by $\mathbb{C}$-convex domains with ($\mathcal{C}^\infty$-)smooth boundaries. Then the fundamental Lempert theorem [69, 70] can be formulated like this:

Each bounded $\mathbb{C}$-convex domain with $\mathcal{C}^2$-smooth boundary belongs to the class $\mathcal{L}$.

This property carries over to convex domains, as they can be exhausted with smooth (even strictly smooth) domains. It was an open question whether a bounded pseudoconvex domain from $\mathcal{L}$ had to be biholomorphic to a convex domain [125, 60]. A recently found counterexample is the so-called symmetrized bidisc $\mathbb{G}_2 \subset \mathbb{C}^2$, the image of the bidisc $\mathbb{D}^2 \subset \mathbb{C}^2$ under the mapping with coordinate components the two elementary symmetric functions of two complex variables. This domain appears in the spectral Nevanlinna–Pick problem, related to questions from control theory and applications in engineering mathematics (see e.g. [1, 3, 53] and the references therein). J. Agler and N. Young [2] showed that $\mathbb{G}_2 \in \mathcal{L}$ by calculating $l_{\mathbb{G}_2}$. On the other hand, C. Costara [24] proved that $\mathbb{G}_2$ is not biholomorphic to a convex domain. In addition, let $\mathcal{E}$ denote the class of domains that can be exhausted with domains biholomorphic to convex domains. The Lempert theorem implies that $\mathcal{E} \subset \mathcal{L}$. A. Edigarian [36] showed that even $\mathbb{G}_2 \notin \mathcal{E}$ (see Proposition 2.5.4). In connection with this and the above mentioned result of D. Jacquet let us note the following

([125, Problem 2], a hypothesis of L. A. Aizenberg [4]) *Can each $\mathbb{C}$-convex domain be exhausted by smooth $\mathbb{C}$-convex domains?*

Theorem 2.6.6(i) states that $\mathbb{G}_2$ is a $\mathbb{C}$-convex domain. Moreover, P. Pflug and W. Zwonek [106] have recently shown that $\mathbb{G}_2$ can be exhausted by $\mathbb{C}$-convex domains with real-analytic boundaries (see 2.6.6). This gives an alternative proof of the fact that $\mathbb{G}_2 \in \mathcal{L}$. We may also formulate the following weaker version of the above hypothesis,

([125, Problem 4′]) *Does each bounded* $\mathbb{C}$*-convex domain belong to* $\mathcal{L}$*?*

An affirmative answer would follow from an affirmative answer to

([125, Problem 4]) *Is each bounded* $\mathbb{C}$*-convex domain biholomorphic to a convex domain?*

Theorem 2.6.6(i) together with the result of A. Edigarian gives a negative answer to the last question.

In a similar way to $\mathbb{G}_2$ one can define the symmetrized polydisc $\mathbb{G}_n \subset \mathbb{C}^n$. It is natural to ask whether $\mathbb{G}_n$ for $n \geq 3$ has the same properties as $\mathbb{G}_2$. For example M. Jarnicki and P. Pflug pose the following question:

([60, Problem 1.2]) *Does* $\mathbb{G}_n$ *belong to* $\mathcal{L}$ *or even to* $\mathcal{E}$*?*

Clearly if $\mathbb{G}_n \notin \mathcal{L}$ then $\mathbb{G}_n \notin \mathcal{E}$.

Chronologically, first the author proved that $\mathbb{G}_n \notin \mathcal{E}$ for $n \geq 3$ (Theorem 2.5.7). In its proof the approach from [24, 36] is applied to so-called generalized balanced domains. In Theorem 2.5.2 we prove that if such a domain in $\mathbb{C}^n$ belongs to $\mathcal{E}$, then its intersection with a special linear subspace of $\mathbb{C}^n$ is necessarily convex. This is in accordance with the fact that a (usual) balanced domain is in the class $\mathcal{E}$ exactly when it is convex (Corollary 2.5.3). Theorem 2.5.7 follows from Theorem 2.5.2 by showing that the corresponding intersections for $\mathbb{G}_n$ are not convex.

Let us note that $\mathbb{G}_n$ for $n \geq 3$ is a linearly convex domain, but it is not $\mathbb{C}$-convex by Theorem 2.6.6(ii).

Theorem 2.5.7 is also a direct corollary from $\mathbb{G}_n \notin \mathcal{L}$, $n \geq 3$. This question is discussed in Section 2.7. To this end one uses the infinitesimal forms of $c_{\mathbb{G}_n}$, $l_{\mathbb{G}_n}$ and $k_{\mathbb{G}_n}$, namely the Carathéodory, Kobayashi and Kobayashi–Buseman metrics: $\gamma_{\mathbb{G}_n}$, $\kappa_{\mathbb{G}_n}$ and $\hat{\kappa}_{\mathbb{G}_n}$. We also introduce a naturally emerging distance $m_{\mathbb{G}_n}$ on $\mathbb{G}_n$ (an analogue to the Möbius distance $m_{\mathbb{D}}$) and its infinitesimal form at the origin, ρ_n.

In [2] J. Agler and N. Young have shown that

$$l_{\mathbb{G}_2} = k^*_{\mathbb{G}_2} = c^*_{\mathbb{G}_2} = m_{\mathbb{G}_2},$$

and $m_{\mathbb{G}_2}$ is (almost) explicitly calculated. The proof is based on the method of complex geodesics; their complete description for $\mathbb{G}_2$ can be found in the work of P. Pflug and W. Zwonek [106].

In [25] this identity is also obtained for some special pairs of points from $\mathbb{G}_n$, $n \geq 3$. However in this case it turns out that

$$l_{\mathbb{G}_n}(0,\cdot) \gneq k^*_{\mathbb{G}_n}(0,\cdot) \geq c^*_{\mathbb{G}_n}(0,\cdot) \gneq m_{\mathbb{G}_n}(0,\cdot) \quad \text{(Corollary 2.7.4)}.$$

These inequalities are directly obtained from the corresponding inequalities between the infinitesimal forms, which are basically considered on the coordinate directions (Theorem 2.7.3).

In Proposition 2.8.2 we get an estimate for the difference between $\gamma_{\mathbb{G}_{2n+1}}(0;\cdot)$ and ρ_{2n+1} in the first direction where they do not coincide (Proposition 2.8.2). This estimate is based on a "polynomial" description of $\gamma_{\mathbb{G}_n}$. Using this description and computer calculations it is shown that

$$\hat{\kappa}_{\mathbb{G}_3}(0;\cdot) \neq \gamma_{\mathbb{G}_3}(0;\cdot) \quad \text{(Theorem 2.8.3)}$$

so the Carathéodory and Kobayashi metrics do not coincide on $\mathbb{G}_3$. It can be expected that the approach in the proof is applicable in higher dimensions too.

The fact that $\mathbb{G}_n$ for $n \geq 3$ has quite different properties from $\mathbb{G}_2$ is confirmed by Theorem 2.4.2, which gives an affirmative answer to the following question of M. Jarnicki and P. Pflug.

([60, Problem 3.2]) *Does the Bergman kernel of* $\mathbb{G}_n$ (*unlike* $\mathbb{G}_2$) *have zeroes?*

The proof is based on an explicit formula obtained by A. Edigarian and W. Zwonek in [37].

As we noted, the symmetrized polydisc appears in connection with the spectral Nevanlinna–Pick problem, i.e. an interpolation problem for maps from the unit disc $\mathbb{D}$ into the spectral ball Ω_n, the set of complex $n \times n$ matrices of spectral radius less than 1 (i.e. with eigenvalues in $\mathbb{D}$). The infinitesimal form of this problem is the spectral Carathéodory–Fejér problem. The easiest forms of these problems are reduced to finding l_{Ω_n} and κ_{Ω_n}, while the continuous dependence on the given data reduces to the continuity of these two functions. In the case of cyclic matrices (i.e. ones with a cyclic vector) they coincide with the corresponding functions on the taut domain $\mathbb{G}_n$, so they are continuous.

In Section 2.3 we provide some equivalent conditions for a matrix to be cyclic (part of these are used in the last sections of the chapter).

In Section 2.2 we gather the basic properties of the above problems and their reduction to similar problems on the symmetrized polydisc in the case of cyclic matrices. (As this is a taut domain, the problems there "depend" on the data in a continuous manner.) This also determines the corresponding relationships with the Lempert function and the Kobayashi metric on the symmetrized polydisc.

Section 2.9 is dedicated to the continuity of l_{Ω_n} (in the general case). The main result there (Theorem 2.9.2) states that $l_{\Omega_n}(A;\cdot)$ is a continuous function exactly when A is a scalar matrix or $n = 2$ and A has (two) equal eigenvalues. This result is based on Proposition 2.9.1, which is obtained from the basic Proposition 2.7.1(iii).

In Section 2.10 we discuss the (dis)continuity of κ_{Ω_n} by studying its zeroes. In particular we have found all matrices $A \in \Omega_3$ such that $\kappa_{\Omega_3}(A;B) > 0$ for $B \neq 0$ (a relatively easy question for $n = 2$).

As an application, in the last section 2.11 we show that the Kobayashi metric of a pseudoconvex domain is not equal to the weak "derivative" of the Lempert function in the general case (this gives a partially affirmative answer to a question from Section 1.2). The counterexample is Ω_3 (or, of a lower dimension, the domain of zero-trace matrices in Ω_3).

Finally, we point out that a detailed study of another biholomorphic invariant, the pluricomplex Green function, on the spectral ball and the symmetrized polydisc can be found in [116].

2.2. Preliminaries. Most of the facts in this section can be found in [1, 2, 3, 11, 24, 25, 26, 37, 60, 106].

Let $\mathbb{D} \subset \mathbb{C}$ be the unit disc. Put $\sigma = (\sigma_1, \dots, \sigma_n) : \mathbb{C}^n \to \mathbb{C}^n$, where

$$\sigma_k(z_1, \dots, z_n) = \sum_{1 \le j_1 < \dots < j_k \le n} z_{j_1} \dots z_{j_k}, \quad 1 \le k \le n.$$

The open set $\mathbb{G}_n = \sigma(\mathbb{D}^n)$ is called the *symmetrized n-disc*. Note that $\mathbb{G}_n$ is a proper image of the n-disc $\mathbb{D}^n$ so it is a pseudoconvex domain. Moreover, by [25, Corollary 3.2] we easily see that $\mathbb{G}_n$ is even a c-finite compact domain (in particular hyperconvex), so it is a taut domain. Its Shilov boundary is $\sigma(\mathbb{T}^n)$, where $\mathbb{T} = \partial\mathbb{D}$ is the unit circle. Furthermore, the group of (holomorphic) automorphisms of $\mathbb{G}_n$ admits a simple description:

$$\operatorname{Aut}(\mathbb{G}_n) = \{\sigma(h, \dots, h) : h \in \operatorname{Aut}(\mathbb{D})\}.$$

More generally, a characterization of the proper holomorphic mappings from $\mathbb{G}_n$ to itself can be found in [37].

We also note that $\mathbb{G}_n$ is close to being a balanced domain (see Section 2.5). More precisely,

$$\pi_\lambda(z) = (\lambda z_1, \lambda^2 z_2, \dots, \lambda^n z_n) \in \mathbb{G}_n, \qquad \lambda \in \overline{\mathbb{D}},\ z \in \mathbb{G}^n.$$

In fact $\mathbb{G}_n$ is the set of points $(a_1, \dots, a_n) \in \mathbb{C}^n$ such that the zeroes of the polynomial $f(\zeta) = \zeta^n + \sum_{j=1}^n (-1)^j a_j \zeta^{n-j}$, $a_0 \neq 0$, lie in $\mathbb{D}$. Clearly $\mathbb{G}_1 = \mathbb{D}$. Furthermore, using the above description and the Cohn rule (see Section 2.5), we find that

$$\mathbb{G}_2 = \{(s, p) : |s - \bar{s}p| + |p|^2 < 1\}.$$

The symmetrized polydisc appears in connection with the so-called Nevanlinna–Pick spectral problem.

Denote by $\mathcal{M}_n$ the set of $n \times n$ matrices of complex coefficients. The spectral ball Ω_n is defined by

$$\Omega_n = \{A \in \mathcal{M}_n : r(A) = \max_{\lambda \in \operatorname{sp}(A)} |\lambda| < 1\}$$

($r(A)$ and $\operatorname{sp}(A)$ are the spectral radius and the spectrum of A, respectively).

The *spectral Nevanlinna–Pick problem*, abbreviated as SNPP, is the following:

Given m different points $\lambda_1, \dots, \lambda_m \in \mathbb{D}$ and m matrices $A_1, \dots, A_m \in \Omega_n$, determine whether there exists a mapping $F \in \mathcal{O}(\mathbb{D}, \Omega_n)$ that interpolates the data, i.e. $F(\lambda_j) = A_j$ for $j = 1, \dots, m$.

A nonconstructive necessary and sufficient condition for solvability of SNPP is the solvability of the classical Nevanlinna–Pick problem for matrices that are similar to the given ones (see e.g. [10]). A more effective form of this result for 2×2 matrices can be found in [9]. Now let us describe an approach that reduces this problem of $n^2 m$ parameters to a problem on $\mathbb{G}_n$ of nm parameters.

We say that a matrix $A \in \mathcal{M}_n$ is *cyclic* if it has a cyclic vector (i.e. $\mathbb{C}^n = \operatorname{span}(v, Av, \dots, A^{n-1}v)$ for some $v \in \mathbb{C}^n$). In the appendix at the end of this section we provide some equivalent conditions for a matrix to be cyclic. The set of cyclic matrices in Ω_n will be denoted by $\mathcal{C}_n$.

For $A \in \mathcal{M}_n$, put for brevity $\sigma(A) = \sigma(\operatorname{sp}(A))$. From the context it will be clear whether we mean $\sigma \in \mathcal{O}(\mathcal{M}_n, \mathbb{C}^n)$ or $\sigma \in \mathcal{O}(\mathbb{C}^n, \mathbb{C}^n)$ (as defined at the beginning of this section).

The following basic theorem for lifting a mapping from $\mathcal{O}(\mathbb{D},\mathbb{G}_n)$ to $\mathcal{O}(\mathbb{D},\Omega_n)$ holds:

THEOREM 2.2.1 (see [1, 25]). *Given m different points $\lambda_1,\dots,\lambda_m\in\mathbb{D}$ and m matrices $A_1,\dots,A_m\in\mathcal{C}_n$. Let $f\in\mathcal{O}(\mathbb{D},\mathbb{G}_n)$ be such that $f(\lambda_j)=\sigma(A_j)$ for $j=1,\dots,m$. Then there exists $F\in\mathcal{O}(\mathbb{D},\Omega_n)$ such that $f=\sigma\circ F$ and $F(\lambda_j)=A_j$ for $j=1,\dots,m$.*

For arbitrary matrices $A_1,\dots,A_m\in\Omega_n$ for $n\le 3$ the possibility of lifting a mapping is thoroughly discussed in [93].

As $\mathcal{C}_n$ is a dense subset of Ω_n, this theorem states that in the generic case SNPP is equivalent to an interpolation problem on $\mathbb{G}_n$ (clearly one cannot expect a similar result in full generality, as the spectrum does not contain the full information on a given matrix up to similarity, in contrast, for example, to its Jordan or Frobenius form). As we noted, a basic advantage of the second problem compared with the first one is the smaller number of parameters. Furthermore, $\mathbb{G}_n$ is a taut domain, while on Ω_n one cannot apply the typical Montel arguments, since Ω_n is not even Brody hyperbolic (it contains complex lines). Probably the only advantage of Ω_n is that it is a balanced domain; however this is compensated by the previously mentioned fact that $\mathbb{G}_n$ is close to being a balanced domain.

The solution of SNPP is equivalent to finding the Lempert function of Ω_n. For cyclic matrices, Theorem 2.2.1 reduces this question to finding the Lempert function of $\mathbb{G}_n$. Indeed, the following simple proposition holds; we omit its proof.

PROPOSITION 2.2.2. *Let $D\subset\mathbb{C}^n$ be a domain, $a_1,a_2\in D$, $\lambda_1,\lambda_2\in\mathbb{D}$.*

(i) *If there exists $f\in\mathcal{O}(\mathbb{D},D)$ such that $f(\lambda_1)=a_1$ and $f(\lambda_2)=a_2$, then $l_D(a_1,a_2)\le m_{\mathbb{D}}(\lambda_1,\lambda_2)$.*
(ii) *If $l_D(a_1,a_2)<m_{\mathbb{D}}(\lambda_1,\lambda_2)$, then there exists an f as in* (i).

As usual, $m_{\mathbb{D}}(\lambda_1,\lambda_2)=\left|\frac{\lambda_1-\lambda_2}{1-\overline{\lambda}_1\lambda_2}\right|$ is the Möbius distance.

Note that if for example D is a taut domain, then there exist extremal discs for l_D and so the condition $l_D(a_1,a_2)\le m_{\mathbb{D}}(\lambda_1,\lambda_2)$ is equivalent to the existence of a corresponding f. However, as mentioned, the spectral ball is not such a domain. On the other hand, $\mathbb{G}_n$ is a taut domain. Then Theorem 2.2.1 implies that

$$l_{\Omega_n}(A_1,A_2)=l_{\mathbb{G}_n}(\sigma(A_1),\sigma(A_2)),\quad A_1,A_2\in\mathcal{C}_n \tag{2.2.1}$$

(and there exists an extremal disc for $l_{\Omega_n}(A_1,A_2)$). Note that as $\sigma\in\mathcal{O}(\Omega_n,\mathbb{G}_n)$, in the general case ($A_1,A_2\in\Omega_n$) we have the inequality $\ge$.

As the cyclic matrices form a dense subset of $\mathcal{M}_n$ and the Kobayashi metric is continuous,

$$k_{\Omega_n}(A_1,A_2)=k_{\mathbb{G}_n}(\sigma(A_1),\sigma(A_2)),\quad A_1,A_2\in\Omega_n. \tag{2.2.2}$$

The above considerations and the (almost) explicit calculation (by the method of geodesics) of $l_{\mathbb{G}_2}(=c^*_{\mathbb{G}_2})$ (see the Introduction) permit a complete solution to SNPP for $n=2$ (see e.g. [26]). As noncyclic 2×2 matrices are scalar, for this purpose it remains just to calculate $l_{\Omega_n}(\lambda I_n,\cdot)$ for $n=2$ ($I_n\in\mathcal{M}_n$ is the unit matrix). As Ω_n is a pseudoconvex balanced domain whose Minkowski function is the spectral radius r, one has $l_{\Omega_n}(0,\cdot)=r$ (see Proposition 1.3.1(iv)). Then from

$$\Phi_\lambda(A)=(A-\lambda I_n)(I_n-\overline{\lambda}A)^{-1}\in\operatorname{Aut}(\Omega_n), \tag{2.2.3}$$

we deduce

$$l_{\Omega_n}(\lambda I_n, A) = r(\Phi_\lambda(A)) = \max_{a\in \mathrm{sp}(A)} m_{\mathbb{D}}(\lambda, a). \tag{2.2.4}$$

Hence it also follows that Ω_2 is an example of a nonhyperbolic pseudoconvex balanced domain such that $c^*_{\Omega_2} = k^*_{\Omega_2} \precneqq l_{\Omega_2}$; on the other hand, $k^*_{\Omega_2}(A_1, A_2) = l_{\Omega_2}(A_1, A_2)$ for $A_1, A_2 \in \mathcal{C}_2$.

In connection with the use of Φ_λ let us note that in [110] there is a conjecture on a complete description of $\mathrm{Aut}(\Omega_n)$. This conjecture has recently been disproved in [65]. Note that for the Euclidean ball $\mathbb{B}_n$, each proper holomorphic mapping from Ω_n into itself is an automorphism (see [128]).

The approach of complex geodesics is applied in [25] for some special pairs of points from $\mathbb{G}_n$ for $n \geq 3$. In other words, the Lempert function for all these pairs of points coincides with the Carathéodory function. However in Section 2.7 we will see that this is not true for each pair of points by obtaining some inequalities for the Carathéodory and Kobayashi metrics on $\mathbb{G}_n$, taken at the beginning. These inequalities and the lower estimates from Section 2.8 (for the Carathéodory metric, and hence for the Kobayashi metric) carry some information on the so-called spectral Carathéodory–Fejér problem. A reduction of this problem to a corresponding problem on $\mathbb{G}_n$ in the spirit of Theorem 2.2.1 can be found in [53, Theorem 2.1].

The easiest variant of this problem, abbreviated as SCFP, is the following:

For $A \in \Omega_n$ and $B \in \mathcal{M}_n$, determine whether there exists a mapping $F \in \mathcal{O}(\mathbb{D}, \Omega_n)$ such that $F(0) = A$ and $F'(0) = B$.

In [93] SCFP is completely reduced to a problem on $\mathbb{G}_n$ for $n \geq 3$.

As SNPP, SCFP is also connected with finding κ_{Ω_n}. Similarly to (2.2.1), we have

$$\kappa_{\Omega_n}(A; B) = \kappa_{\mathbb{G}_n}(\sigma(A), \sigma'_A(B)), \quad A \in \mathcal{C}_n,\ B \in \mathcal{M}_n, \tag{2.2.5}$$

where in this case $\sigma'_A = \sigma_{*,A}$ is the Fréchet derivative of σ at A. Furthermore,

$$\kappa_{\Omega_n}(\lambda I_n; B) = \frac{r(B)}{1 - |\lambda|^2}, \tag{2.2.6}$$

which together with [53, Theorem 1.1] permits a complete solution of SCFP for $n = 2$.

We conclude the section with the fact that the Carathéodory metric and Carathéodory distance on Ω_n can be calculated via those on $\mathbb{G}_n$ (cf. (2.2.2)).

Proposition 2.2.3. *The following equalities hold:*

$$\begin{aligned} c_{\Omega_n}(A_1, A_2) &= c_{\mathbb{G}_n}(\sigma(A_1), \sigma(A_2)), \qquad A_1, A_2 \in \Omega_n, \\ \gamma_{\Omega_n}(A; B) &= \gamma_{\mathbb{G}_n}(\sigma(A); \sigma'_A(B)), \qquad A \in \Omega_n, B \in \mathcal{M}_n. \end{aligned}$$

Proof. As $\sigma \in \mathcal{O}(\mathbb{G}_n, \mathbb{D})$, we have the inequalities $\geq$. For the reverse inequalities it suffices to show that if $f \in \mathcal{O}(\Omega_n, \mathbb{D})$, then there exists a $g \in \mathcal{O}(\mathbb{G}_n, \mathbb{D})$ such that $f = g \circ \sigma$. First note that if $A, B \in \Omega_n$ have identical spectra, then there exists an entire curve φ in Ω_n, passing through A and B (see Proposition 2.7.1(ii)). The Liouville Theorem (applied to the function $f \circ \varphi$) implies $f(A) = f(B)$ and so g is a well defined function. It is holomorphic, since for each layer $\sigma^{-1}(\sigma(C))$ there is a matrix $\tilde{C}$ so that $\mathrm{rank}\, \sigma'_{\tilde{C}} = n$ (see Proposition 2.3.1). ∎

2.3. Cyclic matrices. In this brief section we provide some equivalent conditions for a matrix to be cyclic. Part of these will be used at the end of the chapter.

First let us recall some definitions.

For $A \in \mathcal{M}_n$, $\mathrm{ad}_A : X \mapsto [A, X]$ is the adjoint mapping of A and $\mathcal{C}_A = \ker \mathrm{ad}_A$ is the centralizer of A.

Let $p_A(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be the characteristic polynomial of A. The matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{pmatrix}$$

is called *adjoint* to A (or to p).

PROPOSITION 2.3.1. *For a matrix $A \in \mathcal{M}_n$ the following are equivalent:*

(1) *A has a cyclic vector.*
(2) *A is similar to its adjoint matrix (i.e. it is the Frobenius form of A).*
(3) *The characteristic polynomial and the minimal polynomial of A coincide.*
(4) *Different blocks in the Jordan form of A correspond to different eigenvalues (i.e. each eigenspace is one-dimensional).*
(5) $\mathcal{C}(A) = \{M \in \mathcal{M}_n : M = p(A)$ *for some* $p \in \mathbb{C}[X]\}$.
(6) $\dim \mathcal{C}_A = n$.
(7) $\operatorname{rank} \sigma'_A = n$.
(8) $\ker \sigma'_A = \operatorname{Im} \mathrm{ad}_A$.

A matrix with (one of) the above properties is called *cyclic.*

Proof. The equivalence of the properties (1) to (6) is well-known and can be found e.g. in [51, 52]. We need to prove their equivalence with (7) and (8).

Observe that if $M \in \mathcal{M}_n^{-1}$ (i.e. M is an invertible matrix), then

$$\sigma'_A(X) = \sigma'_{M^{-1}AM}(M^{-1}XM).$$

So to prove that (2) implies (7), we can assume that A coincides with its adjoint matrix. Let $X = (x_{i,j})$ with $x_{i,j} = 0$ for $1 \le j \le n-1$. Then

$$\sigma'_A(X) = (-x_{n,n}, x_{n-1,n}, \dots, (-1)^{n-1}x_{1,n}),$$

and consequently $\operatorname{Im} \sigma'_A = \mathbb{C}^n$, i.e. $\operatorname{rank} \sigma'_A = n$.

Let us now prove that (7) implies (4). Let $\lambda \in \mathbb{C}$, and $M_\lambda = M - \lambda I_n$. As $p_M(x) = p_{M_\lambda}(x+\lambda)$, there exists $\Lambda(\lambda) \in \mathcal{M}_n^{-1}$ such that $\sigma(M) = \Lambda(\lambda)\sigma(M_\lambda)$. Therefore $\operatorname{rank} \sigma'_A = \operatorname{rank} \sigma'_{A_\lambda}$.

Suppose that (7) is true and (4) is false. There exists an eigenvalue λ of A such that $\dim \ker(A - \lambda I_n) \ge 2$. Let us complete a basis of $\ker(A - \lambda I_n)$ to a basis of $\mathbb{C}^n$. Then the matrix $A - \lambda I_n$ is transformed to a matrix with at least two zero columns and consequently $\sigma_{n,n}(A - \lambda I_n + X)$ is a polynomial of degree two or more with respect to $x_{i,j}$. Hence $(\sigma_{n,n})_{*,A_\lambda} = 0$ and so $\operatorname{rank} \sigma'_A = \operatorname{rank} \sigma'_{A_\lambda} \le n-1$, a contradiction.

Finally, let us show that (6) + (7) $\Leftrightarrow$ (8). It is easily seen that $\operatorname{Im} \mathrm{ad}_A \subset \ker \sigma'_A$. Consequently, $\operatorname{Im} \mathrm{ad}_A = \ker \sigma'_A$ if and only if these two linear spaces have the same dimension. By the rank theorem, this is equivalent to $\dim \mathcal{C}_A = \operatorname{rank} \sigma'_A$. It remains to use that $\dim \mathcal{C}_A \geq n \geq \operatorname{rank} \sigma'_A$ for each $A \in \mathcal{M}_n$. ∎

2.4. $\mathbb{G}_n$ is not a Lu Qi-Keng domain for $n \geq 3$. In 1966 Lu Qi-Keng [72] conjectured that the Bergman kernel (see Section 3.4 for the definition) of a simply connected domain in $\mathbb{C}^n$ has no zeroes.

A domain with this property is called a *Lu Qi-Keng domain.* This conjecture was disproved in 1986 by H. P. Boas [13]. A review of the role of Lu Qi-Keng domains in complex analysis, together with various counterexamples, can be found for example in [14, 60].

Using Bell's transformation formula (see e.g. [58]), in [37] the authors find an explicit formula for the Bergman kernel of the symmetrized polydisc. This formula implies that $\mathbb{G}_2$ is a Lu Qi-Keng domain.

The aim of this section is to provide an affirmative answer to

([60, Problem 3.2]) *Does the Bergman kernel of $\mathbb{G}_n$ have zeroes for $n \geq 3$?*

Thus $\mathbb{G}_n$ is the first example of a proper image of the polydisc $\mathbb{D}^n$, $n \geq 3$, that is not a Lu Qi-Keng domain, once again showing the difference in the structure of $\mathbb{G}_n$ for $n = 2$ and $n \geq 3$.

Now let us state the formula for the Bergman kernel of $\mathbb{G}_n$ [37]:

$$K_{\mathbb{G}_n}(\sigma(\lambda), \sigma(\mu)) = \frac{\det[(1 - \lambda_j \overline{\mu}_k)^{-2}]_{1 \leq j,k \leq n}}{\pi^n \prod_{1 \leq j < k \leq n} [(\lambda_j - \lambda_k)(\overline{\mu}_j - \overline{\mu}_k)]}, \quad \lambda, \mu \in \mathbb{D}^n. \tag{2.4.1}$$

Although formally the right-hand side of (2.4.1) is not defined on the whole $\mathbb{G}_n$, it is continued smoothly there. From this formula we get

Proposition 2.4.1 ([37, Proposition 11]). *$\mathbb{G}_2$ is a Lu Qi-Keng domain.*

Proof. From (2.4.1) it is easily deduced that

$$K_{\mathbb{G}_2}(\sigma(\lambda), \sigma(\mu)) = \frac{2 - (\lambda_1 + \lambda_2)(\overline{\mu}_1 + \overline{\mu}_2) + 2\lambda_1\lambda_2\overline{\mu}_1\overline{\mu}_2}{\pi^2 \prod_{j,k=1}^{2} (1 - \lambda_j \overline{\mu}_k)^2}.$$

Consider an automorphism of $\mathbb{D}$ that maps μ_2 to 0. It clearly defines an automorphism of $\mathbb{G}_2$ (see the beginning of Section 2.2). Consequently, it suffices to show that $K_{\mathbb{G}_2}(\sigma(\lambda), \sigma(\mu)) \neq 0$ if $\mu_2 = 0$. This is trivial since $|(\lambda_1 + \lambda_2)\overline{\mu}_1| < 2$. ∎

In contrast to Proposition 2.4.1, we have

Theorem 2.4.2. *$\mathbb{G}_n$ is not a Lu Qi-Keng domain for $n \geq 3$.*

Proof. We prove by induction on $n \geq 3$ that:

(∗) there exist points $\lambda, \mu \in \mathbb{D}^n$ with pairwise different coordinates such that

$$\Delta_n(\lambda, \mu) := \det[(1 - \lambda_j \overline{\mu}_k)^{-2}]_{1 \leq j,k \leq n} = 0$$

and $f_n = \Delta_n(\cdot, \lambda_2, \ldots, \lambda_n, \mu_1, \ldots, \mu_n) \not\equiv 0$.

Base of induction: $n = 3$. We use the following formula (see Appendix A):

$$K_{\mathbb{G}_3}(\sigma(\lambda_1, \lambda_2, \lambda_3), \sigma(\mu_1, \mu_2, 0)) = \frac{a(\nu)z^2 - b(\nu)z + 2c(\nu)}{\pi^3 \prod_{1 \leq j \leq 3, 1 \leq k \leq 2} (1 - \lambda_j \overline{\mu}_k)^2}, \tag{2.4.2}$$

where $z = \overline{\mu}_2/\overline{\mu}_1$ ($\mu_1 \neq 0$), $\nu_j = \lambda_j\overline{\mu}_1$, $j = 1, 2, 3$, and

$$\begin{aligned}
a(\nu) &= \sigma_2(\nu)(2 - \sigma_1(\nu)) + \sigma_3(\nu)(2\sigma_1(\nu) - 3),\\
b(\nu) &= (\sigma_1(\nu) - 2)(\sigma_2(\nu) - 2\sigma_1(\nu) + 3) + 3(\sigma_3(\nu) - \sigma_1(\nu) + 2),\\
c(\nu) &= \sigma_2(\nu) - 2\sigma_1(\nu) + 3.
\end{aligned}$$

For the fixed point $\nu_0 = (e^{i\sigma/6}, e^{i\sigma/3}, e^{-i\sigma/6})$ the number

$$z_0 = e^{-i\sigma/4}\frac{6 - 3\sqrt{3} - \sqrt{40\sqrt{3} - 69}}{\sqrt{2}(3\sqrt{3} - 5)}$$

is a root of the equation $a(\nu_0)z_0^2 - b(\nu_0)z_0 + 2c(\nu_0) = 0$ (see Appendix B). As $z_0 \in \mathbb{D}$ for $\nu \in \mathbb{D}^3$ close to ν_0, there exists $z \in \mathbb{D}$ close to z_0 such that $a(\nu)z^2 - b(\nu)z + 2c(\nu) = 0$. Now choosing $\mu_1 \in \mathbb{D}$ such that $|\mu_1| > |\nu_1|, |\nu_2|, |\nu_3|$, we get points $\lambda, \mu \in \mathbb{D}^3$ with pairwise different coordinates such that $\Delta_3(\lambda, \mu) = 0$.

It remains to check that $f_3 \not\equiv 0$. If this fails, then $f_3(0) = f_3'(0) = f_3''(0) = 0$, i.e.

$$\det\begin{bmatrix} \overline{\mu}_1^j & \overline{\mu}_2^j & \overline{\mu}_3^j \\ (1 - \lambda_2\overline{\mu}_1)^{-2} & (1 - \lambda_2\overline{\mu}_2)^{-2} & (1 - \lambda_2\overline{\mu}_2)^{-2} \\ (1 - \lambda_3\overline{\mu}_1)^{-2} & (1 - \lambda_3\overline{\mu}_2)^{-2} & (1 - \lambda_3\overline{\mu}_3)^{-2} \end{bmatrix} = 0$$

for $j = 0, 1, 2$. As μ_1, μ_2, μ_3 are pairwise different, the vectors $(1, 1, 1)$, (μ_1, μ_2, μ_3) and $(\mu_1^2, \mu_2^2, \mu_3^2)$ are $\mathbb{C}$-linearly independent. Consequently, the vectors in the second and third rows of the above determinant are $\mathbb{C}$-linearly dependent. In particular, $K_{\mathbb{G}_2}(\sigma(\lambda_2, \lambda_3), \sigma(\mu_2, \mu_3)) = 0$, a contradiction.

Induction step. Suppose that (∗) holds for some $n \geq 3$. Choose $\tilde{\lambda}_1$ and $\tilde{\lambda}_{n+1}$ in $\mathbb{D}$, close to λ_1 and 1, respectively (this guarantees that the coordinates of the new points in $\mathbb{D}^{n+1}$ are pairwise different), so that

$$g_{n+1}(\tilde{\lambda}_1, \tilde{\lambda}_{n+1}) := \Delta_{n+1}(\tilde{\lambda}_1, \lambda_2, \ldots, \lambda_n, \tilde{\lambda}_{n+1}, \mu_1, \ldots, \mu_n, \tilde{\lambda}_{n+1}) = 0$$

and $g_{n+1}(\cdot, \lambda_{n+1}) \not\equiv 0$. Note that

$$g_{n+1}(\tilde{\lambda}_1, \tilde{\lambda}_{n+1}) = \frac{f_n(\tilde{\lambda}_1)}{(1 - |\tilde{\lambda}_{n+1}|^2)^2} + h_n(\tilde{\lambda}_1, \tilde{\lambda}_{n+1}),$$

where h_n is a continuous function on $\mathbb{D} \times \overline{\mathbb{D}}$. As $f_n \not\equiv 0$ is a holomorphic function, for each small $r > 0$ the number λ_1 is the only zero of f_n in the closed disc $D \subset \mathbb{D}$ of center λ_1 and radius r. Then $m = \min_{\partial D}|f_n|/\max_{\partial D \times \overline{\mathbb{D}}}|h_n| > 0$. Consequently, $|f_n| > (1 - |\tilde{\lambda}_{n+1}|^2)^2|h_n(\cdot, \tilde{\lambda}_{n+1})|$ on ∂D, provided $1 - |\tilde{\lambda}_{n+1}|^2 < \sqrt{m}$. Fix one such $\tilde{\lambda}_{n+1}$ so that $\tilde{\lambda}_{n+1} \neq \lambda_j, \mu_j$, $1 \leq j \leq n$. As $h_n(\cdot, \tilde{\lambda}_{n+1})$ is a holomorphic function on $\mathbb{D}$, by Rouché's theorem the number of zeroes of $g_{n+1}(\cdot, \tilde{\lambda}_{n+1})$ in D equals the multiplicity of λ_1 as a zero of f_n; in particular, $g_{n+1}(\cdot, \tilde{\lambda}_{n+1}) \not\equiv 0$. It remains to choose r so that $\lambda_j, \mu_j, \tilde{\lambda}_{n+1} \notin D$, $1 \leq j \leq n$, and a zero $\tilde{\lambda}_1$ of $g_{n+1}(\cdot, \tilde{\lambda}_{n+1})$ in D. ∎

Remark. The above proof shows that if $n \geq 4$, then there exist points (λ, ν), close to the diagonal of $\mathbb{D}^n \times \mathbb{D}^n$ in the sense that $\lambda_j = \mu_j > 0$ for $j = 4, \ldots, n$ and such that $K_{\mathbb{G}_n}(\sigma(\lambda), \sigma(\mu)) = 0$. On the other hand, one can prove that $K_{\mathbb{G}_3}(\sigma(\lambda), \sigma(\mu)) \neq 0$ if $\lambda_3 = \mu_3$.

APPENDIX A. (2.4.1) implies that

$$\begin{aligned}
&\pi^3(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)\overline{\mu}_1\overline{\mu}_2(\overline{\mu}_1-\overline{\mu}_2)K_{\mathbb{G}_3}(\sigma(\lambda_1,\lambda_1,\lambda_3),\sigma(\mu_1,\mu_2,0))\\
&\quad=\det\begin{bmatrix}(1-\nu_1)^{-2} & (1-z\nu_1)^{-2} & 1\\ (1-\nu_2)^{-2} & (1-z\nu_2)^{-2} & 1\\ (1-\nu_3)^{-2} & (1-z\nu_3)^{-2} & 1\end{bmatrix}\\
&\quad=\det\begin{bmatrix}(1-\nu_1)^{-2}-(1-\nu_3)^{-2} & (1-z\nu_1)^{-2}-(1-z\nu_3)^{-2}\\ (1-\nu_2)^{-2}-(1-\nu_3)^{-2} & (1-z\nu_2)^{-2}-(1-z\nu_3)^{-2}\end{bmatrix}\\
&\quad=\frac{(\nu_1-\nu_3)(\nu_2-\nu_3)z}{(1-\nu_3)^2(1-z\nu_3)^2}\det\begin{bmatrix}\frac{\nu_1+\nu_3-2}{(1-\nu_1)^2} & \frac{z\nu_1+z\nu_3-2}{(1-z\nu_1)^2}\\ \frac{\nu_2+\nu_3-2}{(1-\nu_2)^2} & \frac{z\nu_2+z\nu_3-2}{(1-z\nu_2)^2}\end{bmatrix} &&(2.4.3)\\
&\quad=\frac{(\nu_1-\nu_3)(\nu_2-\nu_3)z}{\prod_{1\le j\le 3,1\le k\le 2}(1-\lambda_j\overline{\mu}_k)^2}\Big((\nu_1+\nu_3-2)(z\nu_2+z\nu_3-2)(1-z\nu_1)^2(1-\nu_2)^2\\
&\qquad-(\nu_2+\nu_3-2)(z\nu_1+z\nu_3-2)(1-\nu_1)^2(1-z\nu_2)^2\Big) &&(2.4.4)\\
&\quad=\frac{(\nu_1-\nu_3)(\nu_2-\nu_3)z(z-1)(A(\nu)z^2-B(\nu)z+2C(\nu))}{\prod_{1\le j\le 3,1\le k\le 2}(1-\lambda_j\overline{\mu}_k)^2}. &&(2.4.5)
\end{aligned}$$

To find $A(\nu)$, $B(\nu)$ and $C(\nu)$, we use that the coefficients of z^3, z^0 and z in the large parentheses of (2.4.4) are equal to

$$\begin{aligned}
A(\nu)&=(\nu_1+\nu_3-2)(\nu_2+\nu_3)\nu_1^2(1-\nu_2)^2-(\nu_2+\nu_3-2)(\nu_1+\nu_3)\nu_2^2(1-\nu_1)^2,\\
-2C(\nu)&=2(\nu_2+\nu_3-2)(1-\nu_1)^2-2(\nu_1+\nu_3-2)(1-\nu_2)^2,\\
B(\nu)+2C(\nu)&=(\nu_1+\nu_3-2)(\nu_2+\nu_3+4\nu_1)(1-\nu_2)^2\\
&\qquad-(\nu_2+\nu_3-2)(\nu_1+\nu_3+4\nu_2)(1-\nu_1)^2,
\end{aligned}$$

respectively. Trivial calculations show that

$$\begin{aligned}
A(\nu)&=(\nu_2-\nu_1)(\sigma_{3,2}(\nu)(2-\sigma_{3,1}(\nu))+\sigma_{3,3}(\nu)(2\sigma_1(\nu)-3)),\\
C(\nu)&=(\nu_2-\nu_1)(\sigma_2(\nu)-2\sigma_1(\nu)+3),\\
B(\nu)&=(\nu_2-\nu_1)((\sigma_1(\nu)-2)(\sigma_2(\nu)-2\sigma_1(\nu)+3)+3(\sigma_3(\nu)-\sigma_1(\nu)+2)).
\end{aligned}$$

To infer (2.4.2), it remains to substitute these formulas in (2.4.5) and then to compare (2.4.5) and (2.4.3).

APPENDIX B. As

$$\sigma_1(\nu_0)=\frac{1+2\sqrt{3}+i\sqrt{3}}{2},\qquad \sigma_2(\nu_0)=\frac{2+\sqrt{3}+i3}{2},\qquad \sigma_3(\nu_0)=e^{i\sigma/3},$$

the formulas for $a(\nu)$, $b(\nu)$ and $c(\nu)$ lead to

$$a(\nu_0)=(3\sqrt{3}-5)e^{i\sigma/3},\qquad b(\nu_0)=(6\sqrt{2}-3\sqrt{6})e^{i\sigma/12},\qquad c(\nu_0)=(2\sqrt{3}-3)e^{-i\sigma/6}.$$

Then for $z=e^{-i\sigma/4}x$ we have

$$e^{i\sigma/6}(a(\nu_0)z^2-b(\nu_0)z+2c(\nu_0))=(3\sqrt{3}-5)x^2+(3\sqrt{6}-6\sqrt{2})x+4\sqrt{3}-6=:p(x).$$

It remains to note that the zeroes of the polynomial p are equal to $\frac{6-3\sqrt{3}\pm\sqrt{40\sqrt{3}-69}}{\sqrt{2}(3\sqrt{3}-5)}$ and the smaller one lies in $(0,1)$.

2.5. Generalized balanced domains. To show that $\mathbb{G}_n \notin \mathcal{E}$ for $n \geq 2$ (see the Introduction), we will define the so-called generalized balanced domains. For such domains we will find a necessary condition for belonging to $\mathcal{E}$ and then we will show that $\mathbb{G}_n$, $n \geq 3$, does not satisfy this condition; for $\mathbb{G}_2$ the proof is somewhat different.

Let $k_1 \leq \cdots \leq k_n$ be natural numbers and

$$\pi_\lambda(z_1, \ldots, z_n) = (\lambda^{k_1} z_1, \ldots, \lambda^{k_n} z_n), \qquad \lambda \in \mathbb{C}, z \in \mathbb{C}^n.$$

A domain D in $\mathbb{C}^n$ will be called $(k_1, \ldots, k_n)$-*balanced* or *generalized balanced* if $\pi_\lambda(z) \in D$ for each $\lambda \in \overline{\mathbb{D}}$, $z \in D$. Put

$$h_D(z) = \inf\{t > 0 : \pi_{1/t}(z) \in D\}, \qquad z \in \mathbb{C}^n$$

(generalized Minkowski function). The function h_D is nonnegative and upper semicontinuous,

$$h_D(\pi_\lambda(z)) = |\lambda| h_D(z), \qquad \lambda \in \mathbb{C}, z \in \mathbb{C}^n,$$
$$D = \{z \in \mathbb{C}^n : h_D(z) < 1\}.$$

EXAMPLE. $h_{\mathbb{G}_n}(\sigma(\xi_1, \ldots, \xi_n)) = \max_{1 \leq j \leq n} |\xi_j|$.

Clearly the $(1, \ldots, 1)$-balanced domains are exactly the usual balanced domains. Part of their properties remain true for the generalized balanced domains.

PROPOSITION 2.5.1. *Let D be a generalized balanced domain. Then D is pseudoconvex exactly when* $\log h_D \in \mathrm{PSH}(\mathbb{C}^n)$. *Furthermore, the following are equivalent:*

(i) $\log h_D \in \mathrm{PSH}(\mathbb{C}^n) \cap C(\mathbb{C}^n)$ *and* $h_D^{-1}(0) = \{0\}$ (*i.e. D is a bounded domain*);
(ii) *D is a hyperconvex domain;*
(iii) *D is a taut domain.*

Proof. Clearly if $\log h_D \in \mathrm{PSH}(\mathbb{C}^n)$, then D is pseudoconvex.

To prove the converse, let D be $(k_1, \ldots, k_n)$-balanced. Put $\Phi : \mathbb{C}^n \ni (z_1, \ldots, z_n) \to (z_1^{k_1}, \ldots, z_n^{k_n}) \in \mathbb{C}^n$, $\tilde{D} = \Phi^{-1}(D)$ and $\tilde{h}_D = h_D \circ \Phi$. Then $\tilde{D} = \{z \in \mathbb{C}^n : \tilde{h}_D(z) < 1\}$ and $\tilde{h}_D(\lambda z) = |\lambda| h(z)$, $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^n$. Consequently, $\tilde{D}$ is a pseudoconvex balanced domain with Minkowski functional $\tilde{h}_D$. So $\log \tilde{h}_D \in \mathrm{PSH}(\mathbb{C}^n)$. On the other hand, $h_D(z) = \tilde{h}_D(\sqrt[k_1]{z_1}, \ldots, \sqrt[k_n]{z_n})$, $z \in (\mathbb{C}^n)_*$, where the roots are arbitrarily chosen. Consequently, $\log h_D \in \mathrm{PSH}((\mathbb{C}^n)_*)$. By the Removable Singularities Theorem (see e.g. [58]) we conclude that $\log h_D \in \mathrm{PSH}(\mathbb{C}^n)$.

Now observe that the implication (ii)⇒(iii) is true for an arbitrary domain, while (i)⇒(ii) is trivial, since $\log h$ is a negative exhausting plurisubharmonic function for D. To prove (iii)⇒(i), we first note that D is a pseudoconvex domain and so $\log h_D \in \mathrm{PSH}(\mathbb{C}^n)$ according to the first part of the proposition. Furthermore, if $h_D^{-1}(0) \neq \{0\}$, then $h_D(z) = 0$ for some z. Then D contains the entire curve $\mathbb{C} \ni \lambda \mapsto \pi_\lambda(z) \in \mathbb{C}^n$ and so D is not even Brody hyperbolic, a contradiction. Now suppose that h_D is not continuous. As h_D is upper semicontinuous, one can find $\varepsilon > 0$ and a sequence of points z_j tending to some z so that $h_D(z_j) < h_D(z) - \varepsilon$. By homogeneity of h_D, we can assume that $h_D(z_j) < 1 < h_D(z)$ for each j. Then the holomorphic discs $\mathbb{D} \ni \lambda \mapsto \pi_\lambda(z_j) \in D$ converge (locally uniformly) to the disc $\mathbb{D} \ni \lambda \mapsto \pi_\lambda(z) \in \mathbb{C}^n$ that does not lie completely within D, a contradiction. This proves (iii)⇒(i). ∎

REMARKS. (a) The above proof shows that a generalized balanced domain is hyperbolic exactly when it is bounded.

(b) In the case of a balanced domain, the implication (iii)$\Rightarrow$(i) can also be proven like this. As D is a taut domain, γ_D is a continuous function. It remains to observe that a taut domain is hyperbolic and pseudoconvex, so $\gamma_D^{-1}(z;\cdot) = \{0\}$ for each $z \in D$ and $\gamma_D = h_D$.

The next theorem provides a necessary condition for a generalized balanced domain in $\mathbb{C}^n$ to belong to the class $\mathcal{E}$ in terms of convexity of its intersection with a linear subspace of $\mathbb{C}^n$.

THEOREM 2.5.2. *Let $D \in \mathcal{E}$ be a $(k_1, \ldots, k_n)$-balanced domain in $\mathbb{C}^n$. If $2k_{m+1} > k_n$ for some m, $0 \le m \le n-1$, then the intersection $D_m = D \cap \{z_1 = \cdots = z_m = 0\}$ is a convex set (we put $D_m = D$ if $m = 0$).*

Proof. The proof is similar to that of [36, Theorem 1].

Fix $a, b \in D_m$. Then we can find a domain $D' \subset D$ that is biholomorphic to a convex domain G, and such that $\lambda a, \lambda b \in D'$ for $\lambda \in \overline{\mathbb{D}}$. Let $\Psi : D' \to G$ be the corresponding biholomorphic mapping. After a linear coordinate substitution we can assume that $\Psi(0) = 0$ and $\Psi'(0) = \mathrm{id}$. Put

$$g_{ab}(\lambda) = \frac{\Psi(\pi_\lambda(a)) + \Psi(\pi_\lambda(b))}{2}.$$

Then $\Psi^{-1} \circ g_{ab}(\lambda)$ is a holomorphic mapping from a neighborhood of $\overline{\mathbb{D}}$ in D. Let $f_{ab}(\lambda) = \pi_{1/\lambda} \circ \Psi^{-1} \circ g_{ab}(\lambda)$. We will show that

$$\lim_{\lambda \to 0} f_{ab}(\lambda) = \frac{a+b}{2}. \tag{2.5.1}$$

Then $f_{ab}(\lambda)$ extends analytically to $\lambda = 0$. Consequently, $h \circ f_{ab} \in \mathrm{PSH}(\overline{\mathbb{D}})$ according to Proposition 2.5.1 and the maximum principle shows that

$$h(f_{ab}(0)) \le \max_{|\lambda|=1} h(f_{ab}(\lambda)) < 1.$$

Hence $\frac{a+b}{2} \in D_m$ for $a, b \in D_m$, i.e. D_m is a convex set.

To prove (2.5.1), note that the equalities $\Psi^{-1}(0) = 0$ and $(\Psi^{-1})'(0) = \mathrm{id}$ imply

$$\Psi_j^{-1} \circ g_{ab}(\lambda) = g_{abj}(\lambda) + O(|g_{ab}(\lambda)|^2), \quad j = 1, \ldots, n.$$

As $\Psi(0) = 0$, $\Psi'(0) = \mathrm{id}$ and $a, b \in D_m$, we get

$$g_{abj}(\lambda) = \frac{a_j + b_j}{2} \lambda^{k_j} + O(|\lambda|^{2k_{m+1}}).$$

The inequalities $2k_{m+1} > k_n$ show that

$$\frac{\Psi_j^{-1} \circ g_{ab}(\lambda)}{\lambda^{k_j}} = \frac{a_j + b_j}{2} + O(|\lambda|).$$

Then (2.5.1) is obtained by letting $\lambda \to 0$. ∎

When $m = 0$, Theorem 2.5.2 implies

COROLLARY 2.5.3. *A balanced domain is in the class $\mathcal{E}$ exactly when it is convex.*

This also follows from Corollary 1.3.2 that uses the Lempert theorem.

The condition $2k_{m+1} > k_n$ is essential as seen from the following

Example. The $(1,2)$-balanced domain

$$D = \{z \in \mathbb{C}^2 : |z_1|^2 + 3|z_2 + z_1^2| < 1\}$$

is not convex (for example, $(1,0), (2i,4) \in \partial D$, while $(1/2+i,2) \notin \overline{D})$), but it is biholomorphic to the $(1,2)$-balanced domain $G = \{z \in \mathbb{C}^2 : |z_1|^2 + 3|z_2| < 4\}$ (under the mapping $(z_1, z_2) \mapsto (z_1, z_2 + z_1^2)$).

Clearly the symmetrized polydisc $\mathbb{G}_n$ is a $(1,\dots,n)$-balanced domain. However Theorem 2.5.2 cannot be directly applied to show that $\mathbb{G}_2 \notin \mathcal{E}$ (since $2k_1 = k_2$). Anyway its proof permits us to get

Proposition 2.5.4 ([36, Theorem 1]). $\mathbb{G}_2 \notin \mathcal{E}$.

Proof. Assume the contrary. Choose $\varepsilon \in (0,1)$ and put $G_\varepsilon = \{z \in \mathbb{C}^2 : h_{\mathbb{G}_2} \le \varepsilon\}$. Then we can find a domain D_ε that is biholomorphic to a convex domain and so that $G_\varepsilon \subset D_\varepsilon \subset \mathbb{G}_2$. A closer inspection of the proof of Theorem 2.5.2 easily shows that there exists a constant $c_\varepsilon \in \mathbb{C}$ such that

$$\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2} + c_\varepsilon(a_1-b_1)^2\right) \in \mathbb{G}_2 \quad \text{for each } a,b \in D_\varepsilon.$$

If $\theta = \arg(c_\varepsilon)$, then for $a = \sigma(\varepsilon, i\varepsilon e^{-i\theta/2})$, $b = \sigma(\varepsilon, -i\varepsilon e^{-i\theta/2})$ we get $c(\varepsilon) = (\varepsilon, -4|c_\varepsilon|\varepsilon^2) \in \mathbb{G}_2$. The example at the beginning of this section implies that

$$\varepsilon\frac{1+\sqrt{1+16|c_\varepsilon|}}{2} = h_{\mathbb{G}_2}(c(\varepsilon)) \le 1$$

and so $\lim_{\varepsilon\to 0} c_\varepsilon = 0$. Thus $\frac{a+b}{2} \in \overline{\mathbb{G}_2}$ for each $a,b \in \mathbb{G}_2$, i.e. $\mathbb{G}_2$ is a convex domain. We reached a contradiction, as $(2,1), (2i,-1) \in \partial\mathbb{G}_2$, while $(1+i,0) \notin \overline{\mathbb{G}_2}$. ∎

The above proof can be easily modified to get

Proposition 2.5.5. *If D is a balanced domain, then $\mathbb{G}_2 \times D \notin \mathcal{E}$.*

Recall that $\mathbb{G}_2 \in \mathcal{L}$. When we choose a $D \in \mathcal{L}$ (for example convex), Proposition 2.5.5 gives the first examples of domains in $\mathbb{C}^n$, $n \ge 3$, that are in $\mathcal{L}$ but not in $\mathcal{E}$.

We will now prove that $\mathbb{G}_n \notin \mathcal{E}$ for $n \ge 3$. To this end we need the following Cohn rule that permits one to learn in finitely many steps whether the zeroes of a polynomial lie in $\mathbb{D}$.

Proposition 2.5.6 (see e.g. [109]). *The zeroes of the polynomial $f(\zeta) = \sum_{j=0}^n a_j\zeta^{n-j}$ ($n \ge 2$, $a_0 \ne 0$) lie in $\mathbb{D}$ if and only if $|a_0| > |a_n|$ and the zeroes of the polynomial $f^\star(\zeta) = \frac{\overline{a_0}f(\zeta) - a_n\zeta^n\overline{f(1/\overline{\zeta})}}{\zeta}$ lie in $\mathbb{D}$.*

Proposition 2.5.7. $\mathbb{G}_n \notin \mathcal{E}$ *for $n \ge 3$.*

Proof. As $\mathbb{G}_n$ is a $(1,\dots,n)$-balanced domain, by Theorem 2.5.2 it suffices to prove that if $m = [n/2]$, then the set G_n of points $(a_{m+1},\dots,a_n) \in \mathbb{C}^{n-m}$ for which the zeroes of the polynomial $f_n(\zeta) = \zeta^n + \sum_{j=m+1}^n a_j\zeta^{n-j}$ lie in $\mathbb{D}$ is convex.

First we will consider the cases $n = 3$ and $n = 4$, and then we will reduce the case $n \geq 5$ to them.

Case $n = 3$. For $f_3(\zeta) = \zeta^3 + p\zeta + q$ we have

$$f_3^\star(\zeta) = \frac{f_3(\zeta) - q\zeta^3\overline{f}_3(1/\overline{\zeta})}{\zeta} = (1 - |q|^2)\zeta^2 - \overline{p}q\zeta + p$$

and

$$f_3^{\star\star}(\zeta) = \frac{(1 - |q|^2)f_3^\star(\zeta) - p\zeta^2\overline{f_3^\star}(1/\overline{\zeta})}{\zeta} = ((1 - |q|^2)^2 - |p|^2)\zeta - \overline{p}q(1 - |q|^2) + p^2\overline{q}.$$

By Proposition 2.5.6 after some calculations we get

$$G_3 = \{(p, q) \in \mathbb{C}^2 : |q| < 1,\, r(p, q) < 0\},$$

where

$$r(p, q) = |\overline{p}q(1 - |q|^2) - p^2\overline{q}| + |p|^2 - (1 - |q|^2)^2.$$

It is easily seen that if $q' \in (-1, 1)$ and $p' = 1 - q'^2$, then $(p_1, q_1) = (p'e^{2\pi i/3}, q')$ and $(p_2, q_2) = (p'e^{\pi i/3}, q'e^{\pi i/2})$ are boundary points for D (as $r(p', q') = 0$ and $r(p, q') < 0$ for $p \in (|q'| - 1, p')$). Then for

$$(p_0, q_0) = \left(\frac{p_1 + p_2}{2}, \frac{q_1 + q_2}{2}\right) = \left(p' \cos\frac{\pi}{6}e^{\pi i/2}, q' \cos\frac{\pi}{4}e^{\pi i/4}\right)$$

we have

$$|\overline{p_0}q_0(1 - |q_0|^2) - p_0^2\overline{q_0}| = |p_0q_0|(1 - |q_0|^2 + |p_0|).$$

Consequently,

$$r(p_0, q_0) = (1 - |q_0|^2 + |p_0|)(1 + |q_0|)(|p_0| + |q_0| - 1).$$

So $r(p_0, q_0) > 0$ exactly when $|p_0| + |q_0| > 1$. For $q' = 1/2$ we get

$$|p_0| + |q_0| = \frac{3\sqrt{3} + 2\sqrt{2}}{8} > 1.$$

So $(p_0, q_0) \notin \overline{G}_3$, showing that G_3 is not a convex set.

Case $n = 4$. Similarly to the previous case we get

$$G_4 = \{(p, q) \in \mathbb{C}^2 : |p| + |q|^2 < 1,\, s(p, q) < 0\},$$

where

$$s(p, q) = (1 - |q|^2)|\overline{p}q((1 - |q|^2)^2 - |p|^2) - p^3\overline{q}^2| + |p|^4|q|^2 - ((1 - |q|^2)^2 - |p|^2)^2.$$

It is easily seen that if $q' \in [0, 1)$ and $p' = (1-q')\sqrt{1 + q'}$, then $(p_1, q_1) = (p'e^{\pi i/2}, q') \in \partial D$ and $(p_2, q_2) = (p'e^{\pi i/4}, q'e^{\pi i/3}) \in \partial D$ (as $s(p', q') = 0$ and $s(p', q) < 0$, if $p \in (-p', p')$). Then for

$$(p_0, q_0) = \left(\frac{p_1 + q_1}{2}, \frac{p_2 + q_2}{2}\right) = \left(p' \cos\frac{\pi}{8}e^{3\pi i/8}, q' \cos\frac{\pi}{6}e^{\pi i/6}\right)$$

we have

$$|\overline{p_0}q_0((1 - |q_0|^2)^2 - |p_0|^2) - p_0^3\overline{q_0}^2| = |p_0q_0|((1 - |q_0|^2)^2 - |p_0|^2 + |p_0|^2|q_0|).$$

So

$$s(p_0,q_0) = (1-|q_0|^2)((1-|q_0|^2)(1+|q_0|) - |p_0|^2)(1+|p_0|-|q_0|^2)(|p_0|+|q_0|-1).$$

Thus $s(p_0,q_0) > 0$ only when $|p_0|+|q_0| > 1$. For $q' = 2/5$ we have

$$|p_0|+|q_0| = \frac{1}{10}\left(3\sqrt{\frac{7(2+\sqrt{2})}{5}} + 2\sqrt{3}\right) > 1.$$

Consequently, $(p_0,q_0) \notin \overline{G_4}$ and so G_4 is not a convex set.

Case $n \geq 5$. Let $j \in \{0,1,2\}$. Note that the nonconvex set G_3 coincides with the set of points $(p,q) \in \mathbb{C}^2$ such that the zeroes of the polynomial $z^j f_3(z^k)$, $k \geq 1$, lie in $\mathbb{D}$. Consequently, for $n = 3k+2$ and $k \geq 3$, $n = 3k+1$ and $k \geq 2$, or $n = 3k$ and $k \geq 1$, we can view G_3 as the intersection of G_n with a complex plane. So G_n is not a convex set.

In the remaining cases $n = 5$ and $n = 8$ it suffices to observe that the nonconvex set G_4 coincides with the set of points $(p,q) \in \mathbb{C}^2$ such that the zeroes of the polynomials $\zeta^4 f_4(\zeta)$, respectively $f_4(\zeta^2)$, lie in $\mathbb{D}$, and then conclude the proof as above. ■

2.6. Notions of complex convexity. The main definitions and facts from this section can be found in [5, 50] (see also [56]).

Recall that a domain is called $\mathbb{C}$-*convex* if all its intersections with complex lines are connected and simply connected.

We will define two other notions of complex convexity. A domain in $\mathbb{C}^n$ is called *linearly convex* if each point in its complement belongs to a complex hyperplane, disjoint from the domain. If the latter is true for each boundary point, then the domain is called *weakly linearly convex*. The following implications hold:

convexity ⇒ $\mathbb{C}$-convexity ⇒ linear convexity ⇒ weak linear convexity ⇒ pseudoconvexity.

On the other hand, in [5, Example 2.2.4] it is shown that a complete Reinhardt domain which is weakly linearly convex is convex. (A domain D in $\mathbb{C}^n$ is called a *complete Reinhardt domain* if for each $z \in D$ the closed polydisc of center 0 and radius z lies in D.) We will see that this result remains true for balanced domains (but not for generalized balanced domains, as shown by Theorem 2.6.6).

PROPOSITION 2.6.1. *A weakly linearly convex balanced domain $D \subset \mathbb{C}^n$ is convex.*

For the proof we will use a characterization of (weakly) linearly convex domains. For a set D in $\mathbb{C}^n$ containing the origin, put

$$D^* = \{z \in \mathbb{C}^n : \langle z,w\rangle \neq 1 \text{ for each } w \in D\}$$

($\langle\cdot,\cdot\rangle$ is the Hermitian scalar product). Clearly if D is open (compact), then D^* is compact (open). Furthermore, $D \subset D^{**}$.

PROPOSITION 2.6.2 (see e.g. [5, 50]). *A domain D in $\mathbb{C}^n$ containing the origin is weakly linearly convex* (*resp. linearly convex*) *if and only if D is a component of D^{**}* (*resp.* $D = D^{**}$).

Proof of Proposition 2.6.1. As D is a balanced domain, it is easily seen that D^* is a compact balanced set. Consequently, D^{**} is an open balanced set and in particular a domain.

We will prove that this domain is convex. Assume the contrary. Then there exist $z_1, z_2 \in D^{**}$, $w \in D^*$ and $t \in (0,1)$ so that $\langle tz_1 + (1-t)z_2, w\rangle = 1$. Consequently, we can suppose that $|\langle z_1, w\rangle| \geq 1$. As D^* is a balanced set, $\tilde{w} = w/\langle w, z_1\rangle \in D^*$ and $\langle z_1, \tilde{w}\rangle = 1$, a contradiction.

So D^{**} is a convex domain. Since D is a weakly linearly convex domain, D is a component of D^{**} and consequently, $D = D^{**}$. ■

Let us note that the three notions of complex convexity are different, but for bounded domains with $\mathcal{C}^1$-smooth boundaries they coincide (in the more general case of bounded domains this is not true). We also mention that each $\mathbb{C}$-convex domain in $\mathbb{C}^n$ is homeomorphic to $\mathbb{C}^n$, and each domain in $\mathbb{C}$ is linearly convex. Also, a Cartesian product of (weakly) linearly convex domains is (weakly) linearly convex. On the other hand, we have the following

Remark. A Cartesian product of domains that do not coincide with the corresponding spaces is $\mathbb{C}$-convex only if both domains are convex. In particular, a Cartesian product of n simply connected nonconvex domains from $\mathbb{C}$ is a linearly convex domain that is biholomorphic to $\mathbb{D}^n$, yet not $\mathbb{C}$-convex.

Recall that a domain D with a $\mathcal{C}^2$-smooth boundary is *convex* (resp. *pseudoconvex*) if the restriction of the Hessian (resp. Levi form) of its defining function to the real (resp. complex) tangent space at each boundary point of D is a positive semidefinite quadratic form. The following fact confirms the intermediate character of complex convexity: a domain D with a $\mathcal{C}^2$-smooth boundary is $\mathbb{C}$-convex exactly when the restriction of the Hessian of its defining function to the complex tangent space at each boundary point of D is a positive semidefinite quadratic form. This last turns out to be equivalent to the function $-2\log d_D(z)$ near ∂D being $\mathbb{C}$-convex (see e.g. [5]; the number 2 is important); here $d_D(z) = \operatorname{dist}(z, \partial D)$, $z \in D$. The pseudoconvex analogue of this proposition without a smoothness condition is well-known. Of course we also have a convex analogue, which is given in [46, Proposition 7.1] for bounded domains with $\mathcal{C}^2$-smooth boundaries. To see this for an arbitrary domain D, one can note that convexity of $-d_D$ (or, what is the same, of D) trivially implies convexity of $-\log d_D$. The converse is also true; it suffices to assume the contrary and then find a segment that, except for its midpoint, lies within D (see e.g. [50, Theorem 2.1.27] for a more general fact).

For bounded domains with $\mathcal{C}^1$-smooth boundaries the three notions of complex convexity also coincide with the so-called weak local linear convexity (see e.g. [50, Proposition 4.6.4]). A domain $D \subset \mathbb{C}^n$ is called *weakly locally linearly convex* if for each $a \in \partial D$ there exists a complex hyperplane H_a through a and a neighborhood U_a of a so that $H_a \cap D \cap U_a = \emptyset$. Note that there are bounded domains that are not locally linearly convex (see e.g. [56]). In [56, p. 58] it is asked whether a weakly locally linearly convex domain has to be pseudoconvex.

The next proposition gives more than an affirmative answer to this question.

Proposition 2.6.3. *Let $D \subset \mathbb{C}^n$ be a bounded domain with the following property: for each $a \in \partial D$ there exists a neighborhood U_a of a and a function $f_a \in \mathcal{O}(D \cap U_a)$ so that $\lim_{z\to a} |f_a(z)| = \infty$. Then D is a taut domain (in particular, pseudoconvex).*

Proof. It suffices to prove that if $\mathcal{O}(\mathbb{D}, D) \ni \psi_j \to \psi$ and $\psi(\zeta) \in \partial D$ for some $\zeta \in \mathbb{D}$, then $\psi(\mathbb{D}) \subset \partial D$. Assume the contrary. Then we can easily find points $\eta_k \to \eta \in \mathbb{D}$ so that $\psi(\eta_k) \in D$, but $a = \psi(\eta) \in \partial D$. We can assume that $\eta = 0$ and $g_a = 1/f_a$ is a bounded function on $D \cap U_a$. Let $r \in (0, 1)$ be such that $\psi(r\mathbb{D}) \Subset U_a$. Then $\psi_j(r\mathbb{D}) \subset U_a$ for each $j \geq j_0$. Consequently, $|g_a \circ \psi_j| < 1$ and (by passing to subsequences) we can suppose that $g_a \circ \psi_j \to h_a \in \mathcal{O}(r\mathbb{D}, \mathbb{C})$. As $h_a(\eta) = 0$, by Hurwitz's theorem $h_a = 0$. This contradicts the fact that $h_a(\eta_k) = g_a \circ \psi(\eta_k) \neq 0$ for $|\eta_k| < r$. ■

Corollary 2.6.4. *A weakly locally linearly convex domain is pseudoconvex.*

For the proof it is sufficient to exhaust the domain with bounded domains and for each boundary point to consider the reciprocal of the defining function of the corresponding separating hyperplane.

Further, note that a linearly convex domain $D \subset \mathbb{C}^n$ containing a complex line is linearly equivalent to $\mathbb{C} \times D'$, where $D' \subset \mathbb{C}^{n-1}$ [50, Proposition 4.6.11]. Indeed, we can assume that D contains the z_1-line. As the complement ${}^cD$ is a union of complex hyperplanes not intersecting this line, ${}^cD = \mathbb{C} \times G$ and consequently $D = \mathbb{C} \times {}^cG$.

The next theorem provides some properties of $\mathbb{C}$-convex domains not containing complex lines. It generalizes a result of T. J. Barth from [7] for convex domains.

Theorem 2.6.5. *Let D be a $\mathbb{C}$-convex domain in $\mathbb{C}^n$ not containing a complex line. Then D is biholomorphic to a bounded domain and is c-finite compact, hence also c-complete (c is the Carathéodory distance). In particular, D is hyperconvex, so it is a taut domain.*

Based on this theorem, the paper [99] by the author and A. Saracco includes various equivalent conditions for a $\mathbb{C}$-convex domain not to contain a complex line (the convex case is treated in [17]).

Proof of Theorem 2.6.5. For each $z \in {}^cD$ denote by L_z some complex hyperplane through z disjoint from D. Let l_z be the line through the origin that is orthogonal to L_z. Denote by π_z the orthogonal projection of $\mathbb{C}^n$ onto l_z and put $a_z = \pi_z(a)$ (clearly π_z and π_t from Section 2.5 refer to different objects). The set $D_z = \pi_z(D)$ is biholomorphic to $\mathbb{D}$, since it is connected, simply connected (see e.g. [5, Theorem 2.3.6]) and $\pi_z(z) \notin \pi_z(D)$. As D is a linearly convex domain not containing complex lines, it is easily seen that there exist n $\mathbb{C}$-independent l'_z (otherwise ${}^cD$, and so D, would contain a complex line that is orthogonal to each l_z). We can assume that those l_z form the set C of coordinate lines. Then $D \subset G = \prod_{l_z \in C} \pi_z(D)$ and G is biholomorphic to the polydisc $\mathbb{D}^n$ (as the components of G are simply connected planar domains $\neq \mathbb{C}$, they are biholomorphic to $\mathbb{D}$ according to the Riemann theorem). Consequently, D is biholomorphic to a bounded domain, so it is c-hyperbolic.

Further we can assume that $0 \in D$. To see that D is c-finite compact, it suffices to show that $\lim_{a\to z} c_D(0; a) = \infty$ for each $z \in \partial D$ (recall that $\infty \in \partial D$ if D is unbounded). The last statement follows from the fact that $G \supset D$ is a c-finite compact domain (being biholomorphic to $\mathbb{D}^n$). On the other hand, if $a \to z \in \partial D$, then $a_z \to \pi_z(z) \in \partial D_z$ and consequently $c_D(0; a) \geq c_{D_z}(0; a_z) \to \infty$. ∎

The main aim of this section is to show that $\mathbb{G}_2$ is a $\mathbb{C}$-convex domain, which together with the fact that $\mathbb{G}_2 \notin \mathcal{E}$ (see Proposition 2.5.4) gives a negative answer to [125, Problem 4] (see the Introduction).

Theorem 2.6.6.

(i) $\mathbb{G}_2$ *is a* $\mathbb{C}$*-convex domain.*
(ii) $\mathbb{G}_n$, $n \geq 3$, *is a linearly convex domain, yet not* $\mathbb{C}$*-convex.*

Remarks. (a) Proposition 2.5.5 implies that $\mathbb{G}_2 \times \mathbb{C}^n$ is a $\mathbb{C}$-convex domain that is in the class $\mathcal{L}$, but not in $\mathcal{E}$ according to Proposition 2.5.5. However we do not have a similar example of a bounded domain in $\mathbb{C}^n$, $n \geq 3$. (The most natural candidate is the Cartesian product, but according to a remark above this is impossible, as the factors have to be convex).

(b) It is easily seen that bounded generalized balanced domains with continuous Minkowski functional are homeomorphic to $\mathbb{C}^n$. So, Theorem 2.6.6(ii) provides the first example of a linearly convex domain, namely $\mathbb{G}_n$, that is homeomorphic to $\mathbb{C}^n$, $n \geq 3$, but is not $\mathbb{C}$-convex, not in the class $\mathcal{L}$ and not a Cartesian product.

(c) Theorem 2.6.6(ii) shows that Proposition 2.6.1 does not remain true for generalized balanced domains.

(d) Although $\mathbb{G}_n$ for $n \geq 3$ is not a $\mathbb{C}$-convex domain, the conclusion of Theorem 2.6.5 is true, i.e. $\mathbb{G}_n$ is a c-finite compact domain. This fact follows directly from [25, Corollary 3.2] (see (2.7.2)).

Let us introduce the following notation. Let D be a domain in $\mathbb{C}^n$ containing the origin, and $0 \neq a \in \partial D$. Let $\Gamma_D(a)$ be the set of points $z \in \mathbb{C}^n$ such that the hyperplane $\{w \in \mathbb{C}^n : \langle z, w\rangle = 1\}$ passing through a does not meet D.

For the proof of Theorem 2.6.6 we will use the following characterization of bounded $\mathbb{C}$-convex domains.

Proposition 2.6.7 ([5, Theorem 2.5.2]). *A bounded domain D in $\mathbb{C}^n$ $(n > 1)$ containing the origin is $\mathbb{C}$-convex if and only if for each $a \in \partial D$ the set $\Gamma_D(a) \subset \mathbb{CP}^n$ is nonempty and connected.*

Remark. Proposition 2.6.7 directly implies the above mentioned fact that a $\mathcal{C}^1$-smooth bounded domain D in $\mathbb{C}^n$, $n > 1$, is $\mathbb{C}$-convex if and only if it is linearly convex.

Proof of Theorem 2.6.6. (i) According to Proposition 2.6.7 we need to check that the set $\Gamma(a)$ is nonempty and connected for each $a \in \partial D$.

First take a smooth boundary point $a \in \partial\mathbb{G}_2$; without loss of generality it is of the form $\sigma(\mu)$, where $|\mu_1| = 1$, $|\mu_2| < 1$. Then the tangent plane to $\partial\mathbb{G}_2$ at a has the form $\{\sigma(\mu_1, \lambda) : \lambda \in \mathbb{C}\}$ and clearly it does not meet $\mathbb{G}_2$. So in this case $\Gamma(a)$ is a singleton.

Let now $a \in \partial\mathbb{G}_2$ be a nonsmooth boundary point, i.e. $a = \sigma(\mu)$, where $|\mu_1| = |\mu_2| = 1$. After a rotation we can assume that $\mu_1\mu_2 = 1$, i.e. $\mu_2 = \overline{\mu}_1$. Then $\mu_1 + \mu_2 = 2\operatorname{Re}\mu_1 =: 2x$, where $x \in [-1, 1]$.

The complex lines through a that meet $\mathbb{G}_2$ have the form $a + \mathbb{C}(a - \sigma(\lambda))$, where $\lambda \in \mathbb{D}^2$. Consequently, the complement of $\Gamma(a)$ can be seen as the set

$$A = \left\{ \frac{\lambda_1 + \lambda_2 - 2x}{\lambda_1\lambda_2 - 1} : \lambda_1, \lambda_2 \in \mathbb{D} \right\}.$$

Connectedness of $\Gamma(a)$ means simply connectedness of A. Note that if $|\beta| > 1$, then $\frac{z-\alpha}{z-\beta}$ maps the unit disc $\mathbb{D}$ to the disc $\mathbb{D}\big(\frac{1-\alpha\overline{\beta}}{1-|\beta|^2}, \frac{|\alpha-\beta|}{|\beta|^2-1}\big)$. So the set $\big\{\frac{\lambda+\lambda_1-2x}{\lambda\lambda_1-1} : \lambda \in \mathbb{D}\big\}$ coincides with

$$A_{\lambda_1} = \mathbb{D}\bigg(\frac{2x - 2\operatorname{Re}\lambda_1}{1 - |\lambda_1|^2}, \frac{|2x\lambda_1 - \lambda_1^2 - 1|}{1 - |\lambda_1|^2}\bigg).$$

As $A = \bigcup_{\lambda_1 \in \mathbb{D}} A_{\lambda_1} \subset \mathbb{C}$, A is a simply connected set.

(ii) To prove the linear convexity of $\mathbb{G}_n$, consider a point $z = \sigma(\lambda) \in \mathbb{C}^n \setminus \mathbb{G}_n$. We can assume that $|\lambda_1| \geq 1$. Then the set

$$B = \{\sigma(\lambda_1, \mu_1, \ldots, \mu_{n-1}) : \mu_1, \ldots, \mu_{n-1} \in \mathbb{C}\}$$

is disjoint from $\mathbb{G}_n$. On the other hand, it is easily seen that

$$B = \{(\lambda_1 + z_1, \lambda_1 z_1 + z_2, \ldots, \lambda_1 z_{n-2} + z_{n-1}, \lambda_1 z_{n-1}) : z_1, \ldots, z_{n-1} \in \mathbb{C}\};$$

so B is a complex hyperplane. Consequently, $\mathbb{G}_n$ is a linearly convex domain.

To prove that $\mathbb{G}_n$ is not a $\mathbb{C}$-convex domain for $n \geq 3$, we consider the points

$$a_t = \sigma(t, t, t, 0, \ldots, 0) = (3t, 3t^2, t^3, 0, \ldots, 0),$$
$$b_t = \sigma(-t, -t, -t, 0, \ldots, 0) = (-3t, 3t^2, -t^3, 0, \ldots, 0),\ t \in (0, 1).$$

Clearly $a_t, b_t \in \mathbb{G}_n$. Denote by L_t the complex line through a_t and b_t, i.e.

$$L_t = \{c_{t,\lambda} := (3t(1 - 2\lambda), 3t^2, t^3(1 - 2\lambda), 0, \ldots, 0) : \lambda \in \mathbb{C}\}.$$

Suppose that $\mathbb{G}_n \cap L_t$ is a connected set. As $a_t = ct, 0$ and $b_t = ct, 1$, $c_{t,\lambda} \in \mathbb{G}_n$ for some $\lambda = 1/2 + i\tau$, $\tau \in \mathbb{R}$. Then

$$c_{t,\lambda} = (-6i\tau t, 3t^2, -2i\tau t^3, 0, \ldots, 0).$$

Let $c_\tau = \sigma(\mu)$, $\mu \in \mathbb{D}^n$. We can assume that $\mu_j = 0$, $j = 4, \ldots, n$, and $-36\tau^2 t^2 = (\mu_1 + \mu_2 + \mu_3)^2 = \mu_1^2 + \mu_2^2 + \mu_3^2 + 6t^2$. Then

$$t^2 = \frac{|\mu_1^2 + \mu_2^2 + \mu_3^2|}{36\tau^2 + 6} < \frac{3}{36\tau^2 + 6} \leq \frac{1}{2}.$$

Consequently, $\mathbb{G}_n \cap L_t$ is not a connected set for $t \in [1/\sqrt{2}, 1)$ and so $\mathbb{G}_n$ is not a $\mathbb{C}$-convex domain. ∎

The fact that $\mathbb{G}_2$ is a $\mathbb{C}$-convex domain is also a consequence of a recent result by P. Pflug and W. Zwonek [107] which also confirms the Aizenberg hypothesis. Let us first recall that a $\mathcal{C}^2$-smooth domain D in $\mathbb{C}^n$ is said to be *strongly linearly convex* if the restriction of the Hessian of its defining function to the complex tangent space at each boundary point of D is a positive definite quadratic form.

Considering functions of the form

$$r_\varepsilon(s, p) = |s - \overline{s}p| - (1 - |p|^2)^2 + \varepsilon,$$

one can show the following

THEOREM 2.6.8 (see [107]). *The domain*

$$\mathbb{G}_2^\varepsilon = \{(s,p) : \sqrt{|s-\overline{s}p|+\varepsilon} + |p|^2 < 1\}, \qquad \varepsilon \in (0,1),$$

is strongly linearly convex. Consequently, $\mathbb{G}_2 = \mathbb{G}_2^0$ can be exhausted by strongly linearly convex domains and hence $\mathbb{G}_2$ is a $\mathbb{C}$-convex domain.

Since $\mathbb{G}_2 \notin \mathcal{E}$, we get immediately

COROLLARY 2.6.9. *$\mathbb{G}_2^\varepsilon \notin \mathcal{E}$ for $\varepsilon > 0$ small enough.*

2.7. $\mathbb{G}_n \notin \mathcal{L}$ for $n \geq 3$. As mentioned in the Introduction, $\mathbb{G}_2 \in \mathcal{L}$, i.e. the Carathéodory and Lempert functions of the symmetrized polydisc coincide. The main aim of this section is to show that this no longer holds in higher dimensions, i.e. $\mathbb{G}_n \notin \mathcal{L}$ for $n \geq 3$, which solves [60, Problem 1.4].

Let $n \geq 2$, $z \in \mathbb{C}^n$ and $\lambda \in \overline{\mathbb{D}}$. Put

$$f_\lambda(z) = \frac{\sum_{j=1}^n j z_j \lambda^{j-1}}{n + \sum_{j=1}^{n-1}(n-j) z_j \lambda^j}.$$

By [25, Theorem 3.1], $z \in \mathbb{G}_n$ if and only if $\sup_{\lambda \in \overline{\mathbb{D}}} |f_\lambda(z)| < 1$. So for the Carathéodory function we have

$$c^*_{\mathbb{G}_n}(z,w) \geq m_{\mathbb{G}_n}(z,w) := \max_{\lambda \in \mathbb{T}} |m_{\mathbb{D}}(f_\lambda(z), f_\lambda(w))|. \tag{2.7.1}$$

Note that $m_{\mathbb{G}_n}$ is a distance on $\mathbb{G}_n$. Furthermore, by [25, Corollary 3.2],

$$\lim_{w \to \partial \mathbb{G}_n} m_{\mathbb{G}_n}(z,w) = 1$$

and consequently

$$\lim_{w \to \partial \mathbb{G}_n} c_{\mathbb{G}_n}(z,w) = \infty, \tag{2.7.2}$$

i.e. $\mathbb{G}_n$ is a c-finite compact domain.

The next basic proposition is used in the proof of Propositions 2.9.1 and 2.2.3. It contains information on the zeroes of l_{Ω_n} and $c^*_{\Omega_n}$.

PROPOSITION 2.7.1. *Let $A, B \in \Omega_n$ and*

$$s(A,B) = \min_{\lambda \in \mathrm{sp}(A)} \max_{\mu \in \mathrm{sp}(B)} m_{\mathbb{D}}(\lambda, \mu).$$

Then:

(i) $l_{\mathbb{G}_n}(\sigma(A), \sigma(B))) \leq l_{\Omega_n}(A,B) \leq s(A,B)$.

(ii) $l_{\Omega_n}(A,B) = 0 \Leftrightarrow c^*_{\Omega_n}(A,B) = 0 \Leftrightarrow \mathrm{sp}(A) = \mathrm{sp}(B)$
$\Leftrightarrow \exists \varphi \in \mathcal{O}(\mathbb{C}, \Omega_n) : \varphi(0) = A, \varphi(1) = B$.

(iii) *If the eigenvalues of A are equal, then the eigenvalues of B are equal*
$\Leftrightarrow c^*_{\mathbb{G}_n}(\sigma(A), \sigma(B)) = s(A,B) \Leftrightarrow l_{\mathbb{G}_n}(\sigma(A), \sigma(B)) = s(A,B)$.

Proof. (i) The left inequality is noted in the Introduction (it follows from the holomorphic contractibility of the Lempert function).

To prove the right inequality, let J_A and J_B be the Jordan normal forms of A and B, respectively. As a nonsingular square matrix X can be expressed in the form $X = e^Y$, we get $A = e^{Y_A} J_A e^{-Y_A}$ and $B = e^{Y_B} J_B e^{-Y_b}$, where $Y_A, Y_B \in \mathcal{M}(\mathbb{C}^n)$. We have $J_a =$

$(a_{jk})_{j,k=1}^n$, where $\mathrm{sp}(A)=\{a_{11},\dots,a_{nn}\}$, $a_{j,j+1}=0,1$ and $a_{jk}=0$ if $j<k$ or $j>k+1$. A similar representation is valid for $J_b=(b_{jk})_{j,k=1}^n$. Let $\lambda\in\mathrm{sp}(A)$ and $\tilde\lambda=\max_{\mu\in\mathrm{sp}(B)}p_{\mathbb{D}}(\lambda,\mu)$. Then one can easily find $\varphi_{jj}\in\mathcal{O}(\mathbb{D},\mathbb{D})$ so that $\varphi_{jj}(0)=a_{jj}$ and $\varphi_{jj}(\tilde\lambda)=b_{jj}$. Clearly we can choose $\varphi_{j,j+1}\in\mathcal{O}(\mathbb{D},\mathbb{D})$ and $\psi\in\mathcal{O}(\mathbb{D},\mathcal{M}_n)$ so that $\varphi_{j,j+1}(0)=a_{j,j+1}$, $\varphi_{jj}(\tilde\lambda)=b_{jj}$ and $\psi(0)=Y_A$, $\psi(\tilde\lambda)=B$. Put $\varphi_{jk}=0$ if $j<k$ or $j>k+1$, and $\varphi=(\varphi_{jk})_{j,k=1}^n$. Then $e^{\psi}\varphi e^{-\psi}\in\mathcal{O}(\mathbb{D},\Omega_n)$ and $\varphi(0)=A$, $\varphi(\tilde\lambda)=B$, which completes the proof. Consequently, $l_{\Omega_n}(A,B)\le|\tilde\lambda|$ and as $\lambda\in\mathrm{sp}(A)$ was arbitrary, we get the right inequality.

(ii) Clearly

$$l_{\Omega_n}(A,B)=0\ \Rightarrow\ c^*_{\Omega_n}(A,B)=0=0\ \Rightarrow\ \mathrm{sp}(A)=\mathrm{sp}(B),$$

since $c^*_{\Omega_n}(A,B)=c^*_{\mathbb{G}_n}(\sigma(A),\sigma(B))$ according to Proposition 2.2.3. If $\mathrm{sp}(A)=\mathrm{sp}(B)$, then as in the proof of the right inequality of (i), we can find $\varphi\in\mathcal{O}(\mathbb{C},\Omega_n)$ so that $\varphi(0)=A$ and $\varphi(1)=B$, leading to the implication $\mathrm{sp}(A)=\mathrm{sp}(B)\Rightarrow l_{\Omega_n}(A,B)=0$.

(iii) Since $c^*_{\mathbb{G}_n}\le l_{\mathbb{G}_n}\le s$ (see (i) for the latter) we get the implication $c^*_{\mathbb{G}_n}(\sigma(A),\sigma(B))=s(A,B)\Rightarrow l_{\mathbb{G}_n}(\sigma(A),\sigma(B))=s(A,B)$.

Further, using Φ_λ (see (2.2.3)) we can assume that the eigenvalues of A are equal to 0.

To prove that if the eigenvalues of B are equal e.g. to μ, then $c^*_{\mathbb{G}_n}(0,\sigma(B))=s(0,B)$ $(=|\mu|)$, it suffices to construct a function $f\in\mathcal{O}(\mathbb{G}_n,\mathbb{D})$ so that $f(0)=0$ and $f(\sigma(B))=\mu$. An example of such a function is $f(z_1,\dots,z_n)=(z_1+\dots+z_n)/n$.

It remains to prove that if $l_{\mathbb{G}_n}(0,\sigma(B))=s(0,B)$, then the eigenvalues of B are equal. Let $\mathrm{sp}(B)=(\nu_1,\dots,\nu_n)$. Let $\sqrt[n]{1}=\{1,\varepsilon,\dots,\varepsilon^{n-1}\}$. For each Blaschke product $\mathcal{B}$ of order $\le n$ such that $\mathcal{B}(0)=0$, consider the mapping

$$\lambda\mapsto f_{\mathcal{B}}(\lambda)=\sigma(\mathcal{B}(\sqrt[n]{\lambda}),\mathcal{B}(\varepsilon\sqrt[n]{\lambda}),\dots,\mathcal{B}(\varepsilon^{n-1}\sqrt[n]{\lambda}))$$

(where $\sqrt[n]{\lambda}$ is arbitrarily chosen). It is easily seen that $f_{\mathcal{B}}\in\mathcal{O}(\mathbb{D},\mathbb{G}_n)$. We need the following

LEMMA 2.7.2. *Let $\delta_1,\dots,\delta_n\in\mathbb{T}$ be pairwise different. Then for any $\nu_1,\dots,\nu_n\in\mathbb{D}$ there exists $\beta\in\mathbb{D}$ and a Blaschke product $\mathcal{B}$ of order $\le n$ such that*

$$\mathcal{B}(0)=0,\quad \mathcal{B}(\delta_1\beta)=\nu_1,\ \dots,\ \mathcal{B}(\delta_n\beta)=\nu_n.$$

Proof. Let S be the set of all $\beta\in\mathbb{D}$ such that the classical Nevanlinna–Pick problem with data $(0,0),(\delta_1\beta,\nu_1),\dots,(\delta_n\beta,\nu_n)$ has a solution, i.e. there exists $f\in\mathcal{O}(\mathbb{D},\mathbb{D})$ so that $f(\delta_j\beta)=\nu_j$, $1\le j\le n$. Recall that this condition is equivalent to the positive semidefiniteness of $A(\beta)=[a_{j,k}(\beta)]_{j,k=1}^n$, where $a_{j,k}(\beta)=\frac{1-\nu_j\bar\nu_k}{1-\delta_j\bar\delta_k|\beta|^2}$ (see e.g. [40]). Note that the function $a_{j,k}$, $j\ne k$, is bounded on $\mathbb{D}$. On the other hand, $\lim_{\beta\to\mathbb{T}}a_{j,j}(\beta)=+\infty$. Consequently, the matrix $A(\beta)$ is positive semidefinite when β is close to $\mathbb{T}$. We can assume that $0\notin S$ (otherwise put $\mathcal{B}=\mathrm{id}$) and then S is a proper nonempty closed subset of $\mathbb{D}$ (consisting of circles). So it has a boundary point $\beta_0\in\mathbb{D}$. Consequently, the number $m=\mathrm{rank}\,A(\beta_0)$ is not maximal, i.e. $m<n+1$. Thus the corresponding Nevanlinna–Pick problem has a unique solution, which is a Blaschke product of order m (see e.g. [40]). ∎

The lemma for $\delta_j=\varepsilon^j$, $1\le j\le n$, implies that

$$l_{\mathbb{G}_n}(0,\sigma(B))\le|\beta|^n.$$

It remains to prove that if $|\beta|^n \geq |\nu_j|$ for each $1 \leq j \leq n$, then $\nu_1 = \dots = \nu_n$. After a rotation we can assume that

$$\mathcal{B}(z) = z\frac{a_0 z^k + a_1 z^{k-1} + \dots + a_k}{\overline{a}_k z^k + \overline{a}_{k-1} z^{k-1} + \dots + \overline{a}_0},$$

where $a_0 = 1$ and $k \leq n-1$. As $|\nu_j| \geq |\mathcal{B}(\varepsilon^j\beta)|$, we get $|\beta|^n \geq |\mathcal{B}(\varepsilon^j\beta)|$, i.e.

$$|\beta|^{n-1}|\overline{a}_k(\varepsilon^j\beta)^k + \overline{a}_{k-1}(\varepsilon^j\beta)^{k-1} + \dots + \overline{a}_0| \geq |a_0(\varepsilon^j\beta)^k + a_1(\varepsilon^j\beta)^{k-1} + \dots + a_k|.$$

By squaring we get

$$|\beta|^{2n-2}\Big(\sum_{s=0}^{k}|a_s|^2|\beta|^{2s} + 2\operatorname{Re}\sum_{0\leq p<s\leq k} a_p\overline{a}_s\beta^s\overline{\beta}^p\varepsilon^{j(s-p)}\Big)$$
$$\geq \sum_{s=0}^{k}|a_s|^2|\beta|^{2(k-s)} + 2\operatorname{Re}\sum_{0\leq p<s\leq k} a_p\overline{a}_s\beta^{k-p}\overline{\beta}^{k-s}\varepsilon^{j(s-p)}.$$

Adding these inequalities for $j = 1, \dots, n$ yields

$$|\beta|^{2n-2}\sum_{s=0}^{k}|a_s|^2|\beta|^{2s} \geq \sum_{s=0}^{k}|a_s|^2|\beta|^{2(k-s)},$$

i.e.,

$$\sum_{s=0}^{k}|a_s|^2(|\beta|^{2(n+s-1)} - |\beta|^{2(k-s)}|) \geq 0.$$

As $k \leq n-1$, we have $k - s < n + s - 1$ if $s > 0$ and so $a_s = 0$. On the other hand, $a_0 = 1 \neq 0$ and consequently $k = n-1$. We thus get $\mathcal{B}(z) = z^n$ and hence $\nu_1 = \dots = \nu_n$. ∎

Let $e_1, \dots, e_n$ be the standard basis in $\mathbb{C}^n$ and $X = \sum_{j=1}^n X_j e_j$. Put

$$\tilde{f}_\lambda(X) = \frac{\sum_{j=1}^n jX_j\lambda^{j-1}}{n} \quad \text{and} \quad \rho_n(X) = \max_{\lambda\in\mathbb{T}}|\tilde{f}_\lambda(X)|.$$

By (2.7.1) we get the following estimate for the Carathéodory metric of $\mathbb{G}_n$:

$$\gamma_{\mathbb{G}_n}(0;X) \geq \lim_{\mathbb{C}_*\ni t\to 0}\frac{p_{\mathbb{G}_n}(0,tX)}{|t|} = \rho_n(X).$$

Let $L_{k,l} = \operatorname{span}(e_k, e_l)$. Clearly if $X \in L_{k,l}$, $k \neq l$, then

$$\rho_n(X) = \frac{k|X_k| + l|X_l|}{n}.$$

As noted, one of the basic results that motivate the discussion of the symmetrized bidisc is that $\mathbb{G}_2 \in \mathcal{L}$. More precisely (see [2]),

$$l_{\mathbb{G}_2} = k^*_{\mathbb{G}_2} = c^*_{\mathbb{G}_2} = m_{\mathbb{G}_2};$$

in particular $\kappa_{\mathbb{G}_2} = \gamma_{\mathbb{G}_2}$.

The next proposition shows that $\mathbb{G}_n$ does not have similar properties for $n \geq 3$.

Theorem 2.7.3.

(i) *If k divides n, then $\kappa_{\mathbb{G}_n}(0;e_k) = \rho_n(e_k)$. Consequently, if l also divides n, then $\hat{\kappa}_{\mathbb{G}_n}(0;X) = \gamma_{\mathbb{G}_n}(0;X) = \rho_n(X)$ for $X \in L_{k,l}$.*

(ii) *If* $n \geq 3$ *and* $X \in L_{1,n} \setminus (L_{1,1} \cup L_{n,n})$, *then* $\kappa_{\mathbb{G}_n}(0; X) > \rho_n(X)$.
(iii) *If* k *does not divide* n, *then* $\gamma_{\mathbb{G}_n}(0; e_k) > \rho_n(e_k)$.

As $\mathbb{G}_n$ is a taut domain, Theorem 1.2.2 and (1.2.1) imply

Corollary 2.7.4. *If* $n \geq 3$, *then*

$$l_{\mathbb{G}_n}(0, \cdot) \gneq k^*_{\mathbb{G}_n}(0, \cdot) \geq c^*_{\mathbb{G}_n}(0, \cdot) \gneq m_{\mathbb{G}_n}(0, \cdot).$$

Remark. Clearly $\mathbb{G}_2$ is a domain that is biholomorphic to $\mathbb{G}_{2n} \cap L_{n,2n}$. Then, unlike in Theorem 2.7.3, for $z, w \in L_{n,2n}$ we have

$$m_{\mathbb{G}_{2n}}(z, w) \leq m_{\mathbb{G}_{2n}}(z, w) \leq l_{\mathbb{G}_2}(z, w) = m_{\mathbb{G}_2}(z, w) \leq m_{\mathbb{G}_{2n}}(z, w)$$

and so $l_{\mathbb{G}_{2n}}(z, w) = m_{\mathbb{G}_{2n}}(z, w)$.

Corollary 2.7.5. *The convex hull of the spectral unit ball* Ω_n *is*

$$\hat{\Omega}_n = \{A \in \mathcal{M}_n(\mathbb{C}) : h_{\hat{\Omega}_n}(A) = |\operatorname{tr} A|/n < 1\}.$$

Proof. By (2.2.2) we get

$$h_{\hat{\Omega}_n}(A) = \mathcal{D}k_{\Omega_n}(0; A) = \lim_{t \to 0} \frac{k_{\Omega_n}(0; tA)}{|t|} = \lim_{t \to 0} \frac{k_{\mathbb{G}_n}(0, \sigma(tA))}{|t|}.$$

As

$$\sigma(tA) = (t \operatorname{tr} A + o(t), o(t), \ldots, o(t))$$

and $\mathbb{G}_n$ is a taut domain, Theorem 2.7.3(i) implies that the last limit equals

$$\hat{\kappa}_{\mathbb{G}_n}(0; (\operatorname{tr} A)e_1) = |\operatorname{tr} A|/n. \ \blacksquare$$

Remark. Corollary 2.7.5 can also be proven algebraically.

Proof of Theorem 2.7.3. (i) For $1 \leq j \leq n$ and $\zeta \in \mathbb{D}$ put $\varphi_j(\zeta) = 0$ if k does not divide j, and $\varphi_j(\zeta) = \binom{n/k}{j/k}\zeta^{j/k}$ if k divides j. As the zeroes of the polynomial $(1 + \zeta^k)^{n/k}$ lie in $\mathbb{D}$, we get $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathcal{O}(\mathbb{D}, \mathbb{G}_n)$. Furthermore, $\varphi'(0) = ne_k/k$ and so

$$\kappa_{\mathbb{G}_n}(0; e_k) \leq n/k = \rho_n(e_k).$$

The opposite inequality is straightforward.

(ii) First note that if $\lambda \in \mathbb{T}$, then $\pi_\lambda \in \operatorname{Aut}(\mathbb{G}_n)$. Furthermore, $\kappa_{\mathbb{G}_n}(0; \lambda X) = \kappa_{\mathbb{G}_n}(0; X)$. These two facts imply that we can assume that $X_1, X_n > 0$.

As

$$\kappa_{\mathbb{G}_n}(0; X) \geq \kappa_{\mathbb{G}_{n-1}}(p_{n,1}(0); p'_{n,1}(0)(X)) = \kappa_{\mathbb{G}_{n-1}}\left(0; \frac{n-1}{n} X_1 e_1 + X_n e_{n-1}\right),$$

by induction on n we get $\kappa_{\mathbb{G}_n}(0; X) \geq \kappa_{\mathbb{G}_3}(0; Y)$, where

$$Y = 3X_1 e_1/n + X_n e_3 = Y_1 e_1 + Y_3 e_3.$$

Suppose that $\kappa_{\mathbb{G}_n}(0; X) = \rho_n(X)$. Then

$$\rho_n(X) \geq \kappa_{\mathbb{G}_3}(0; Y) \geq \rho_3(Y) = \rho_n(X)$$

and consequently $\kappa_{\mathbb{G}_3}(0; Y) = \rho_3(Y)$. As $\mathbb{G}_3$ is a taut domain, there exists an extremal disc for $\kappa_{\mathbb{G}_3}(0; Y)$ of the form

$$\varphi(\zeta) = (\zeta\varphi_1(\zeta), \zeta\varphi_2(\zeta), \zeta\varphi_3(\zeta)),$$

where $\varphi'(0) = Y/\rho_3(Y)$,

$$\varphi_1(0) = \frac{Y_1}{3(Y_1 + 3Y_3)}, \quad \varphi_2(0) = 0, \quad \varphi_3(0) = \frac{Y_1}{3(Y_1 + 3Y_3)}. \tag{2.7.3}$$

Note that $f_\lambda \circ \varphi \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ and $f_\lambda \circ \varphi(0) = 0$ for each $\lambda \in \overline{\mathbb{D}}$. For $\lambda \in \overline{\mathbb{D}}$ and $\zeta \in \mathbb{D}$ put

$$g_\lambda(\zeta) = \frac{f_\lambda(\varphi(\zeta))}{\zeta} = \frac{\sum_{j=1}^3 j\varphi_j(\zeta)\lambda^{j-1}}{3 + 2\zeta\varphi_1(\zeta)\lambda + \zeta\varphi_2(\zeta)\lambda^2}.$$

We have $g_\lambda \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$ according to the Schwarz–Pick lemma. By (2.7.3) we get $g_{\pm 1}(0) = 1$ and so $g_{\pm 1} \equiv 1$ by the maximum principle, i.e.

$$\varphi_1(\zeta) \pm 2\varphi_2(\zeta) + 3\varphi_3(\zeta) = 3 \pm 2\zeta\varphi_1(\zeta) + \zeta\varphi_2(\zeta).$$

These two equalities imply that

$$\varphi_2(\zeta) \equiv \zeta\varphi_1(\zeta) \quad \text{and} \quad \varphi_3(\zeta) \equiv 1 + \frac{\zeta^2 - 1}{3}\varphi_1(\zeta).$$

Let $\psi(\zeta) = \varphi_1(\zeta)/3$. Now by $g_\lambda \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$ for $\lambda \in \mathbb{T}$ we get

$$\left|\frac{\psi(\zeta) + 2\lambda\zeta\psi(\zeta) + \lambda^2(1 + (\zeta^2 - 1)\psi(\zeta))}{1 + 2\lambda\zeta\psi(\zeta) + \lambda^2\zeta^2\psi(\zeta)}\right| \le 1$$

$$\Leftrightarrow \left|\frac{\psi(\zeta)(1 + \lambda\zeta)^2 + \lambda^2(1 - \psi(\zeta))}{\psi(\zeta)(1 + \lambda\zeta)^2 + 1 - \psi(\zeta)}\right| \le 1 \Leftrightarrow \operatorname{Re}(\psi(\zeta)(1 - \overline{\psi(\zeta)})((\overline{\lambda} + \zeta)^2 - (1 + \lambda\zeta)^2)) \le 0.$$

If $\lambda = x + iy$, $\zeta = ir$, $r \in \mathbb{R}$, $a = \operatorname{Re}(\psi(\zeta)) - |\psi(\zeta)|^2$, $b = \operatorname{Im}(\psi(\zeta))$, then

$$y(a(2r - y(r^2 + 1)) + bx(1 - r^2)) \le 0, \quad \forall\, x^2 + y^2 = 1.$$

Then for $x = 0$ we get $a \ge 0$. On the other hand, letting $y \to 0^+$ yields $-2ar \ge (1 - r^2)|b|$. Consequently, $a = b = 0$ if $r > 0$. Then by the uniqueness principle we get $\psi \equiv 0$ or $\psi \equiv 1$. So $X_1 = 0$ or $X_n = 0$, a contradiction. ■

(iii) Let $\sqrt[k]{1} = \{\xi_1, \dots, \xi_k\}$. For $z \in \overline{\mathbb{G}_n}$ and $\lambda \in \overline{\mathbb{D}}$ such that the denominator of the first formula below is nonzero, put

$$g_z(\lambda) = \lambda f_\lambda(z) = \frac{\sum_{j=1}^n jz_j\lambda^j}{n + \sum_{j=1}^{n-1}(n - j)z_j\lambda^j}, \quad g_{z,k}(\lambda) = \frac{\sum_{j=1}^k g_z(\xi_j\lambda)}{k\lambda^k}.$$

The equalities $\sum_{j=1}^k \xi_j^m = 0$, $m = 1, \dots, k - 1$, and the Taylor formula show that $g_{z,k}$ extends analytically to 0. More precisely, $g_{z,k}(0) = P_k(z)$, where P_k is polynomial such that $\frac{\partial P_k}{\partial z_k}(0) = k/n$ and

$$P_k(tw_1, t^2w_2, \dots, t^nw_n) = t^kP(w), \quad t, w_1, w_2, \dots, w_n \in \mathbb{C}.$$

The maximum principle implies that $g_{z,k} \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$. In particular, $|P_k(z)| \le 1$. To prove the desired inequality $\gamma_{\mathbb{G}_n}(0; e_k) > \rho_n(e_k)$, it suffices to show that $|P_k(z)| < 1$ for $z \in \overline{\mathbb{G}_n}$. Assume the contrary. Then $P_k(z) = e^{i\theta}$ for some $\theta \in \mathbb{R}$ and $z \in \overline{\mathbb{G}_n}$. The maximum principle implies that $g_z(\xi_j\lambda) = e^{i\theta}\lambda^k$, $\lambda \in \mathbb{T}$, $1 \le j \le k$. For $\xi_j = 1$ we get

$$\sum_{j=1}^n jz_j\lambda^j = e^{i\theta}\Big(n\lambda^k + \sum_{j=1}^{n-1}(n - j)z_j\lambda^{k+j}\Big).$$

Equating the coefficients of the corresponding powers of λ, we get $z_k = e^{i\theta}n/k$, $z_{n+1-k} = \cdots = z_{n-1} = 0$ and

$$(k+j)z_{k+j} = e^{i\theta}(n-j)z_j, \quad 1 \le j \le n-k.$$

These equalities imply that $z_{kl} = e^{i\theta}\binom{n/k}{l}$, $1 \le l \le [n/k]$. On the other hand, as k does not divide n, $n-k < k[n/k] < n$ and so $z_{k[n/k]} = 0$, a contradiction.

2.8. Estimates for $\gamma_{\mathbb{G}_{2n+1}}(0; e_2)$. One of the aims of this section is to evaluate the quantities in the inequality of Theorem 2.7.3(iii) in the simplest case. More precisely, we will find $\gamma_{\mathbb{G}_{2n+1}}(0; e_2)$ with an error of $o(n^{-3})$. To this end we use that $\gamma_{\mathbb{G}_n}(0; e_j)$ solves an extremal problem for a class of polynomials. This observation, combined with computer checks, allows us to show that the Carathéodory and Kobayashi metrics do not coincide on $\mathbb{G}_3$, thereby sharpening Theorem 2.7.3. Probably our approach can be applied to obtain the same result for $\mathbb{G}_n$, $n \ge 4$.

Let $n, k \in \mathbb{N}$, $k \le n$. Note that

$$\kappa_{\mathbb{G}_n}(0; e_k) \le \kappa_{\mathbb{G}_{k[n/k]}}(0; e_k) = 1/[n/k].$$

Consequently,

$$k/n \le \gamma_{\mathbb{G}_n}(0; e_k) \le \kappa_{\mathbb{G}_n}(0; e_k) \le 1/[n/k].$$

in particular,

$$\lim_{n\to\infty} n\gamma_{\mathbb{G}_n}(0; e_k) = \lim_{n\to\infty} n\kappa_{\mathbb{G}_n}(0; e_k) = k.$$

Let now $n \ge 3$ be odd. Then $2/n < \gamma_{\mathbb{G}_n}(0; e_2)$ by Theorem 2.7.3(iii). On the other hand,

$$\gamma_{\mathbb{G}_n}(0; e_2) \le \kappa_{\mathbb{G}_n}(0; e_2) \le \frac{2}{n-1}.$$

We will later improve both estimates. For the upper estimate we need the following. Let $D \subset \mathbb{C}^n$ be a $(k_1, \dots, k_n)$-balanced domain. Denote by $\mathcal{P}_j$ the set of polynomials P such that $\sup_D |P| \le 1$ and $P \circ \pi_\lambda = \lambda^{k_j} P$, $\lambda \in \mathbb{C}$. Put $\mathcal{L}_j = \operatorname{span}(e_j, \dots, e_l)$, where $l \ge j$ is the greatest index such that $k_l = k_j$. The proof of Theorem 2.7.3(iii) easily implies that

Proposition 2.8.1. *If $D \subset \mathbb{C}^n$ is a $(k_1, \dots, k_n)$-balanced domain and $X \in \mathcal{L}_j$, $1 \le j \le n$, then $\gamma_D(0; X) = \sup\{|P'(0)X| : P \in \mathcal{P}_j\}$.*

Remarks. (a) This proposition directly implies that if D is balanced domain, then

$$\gamma_D(0; X) = \sup\Big\{|L'(0)X| : \sup_D |L| \le 1,\ L \text{ a linear function}\Big\}$$

and so $\gamma_D(0; \cdot) = \gamma_{\hat{D}}(0; \cdot)$ (for the last one see also Proposition 1.3.1(i)).

(b) Another corollary is the formula

$$\gamma_{\mathbb{G}_n}^{-1}(0; e_2) = \inf_{c\in\mathbb{C}} \max_{z\in\partial G_n} |z_2 + cz_1^2|. \tag{2.8.1}$$

In spite of this formula, $\gamma_{\mathbb{G}_{2n+1}}(0; e_2)$ is hard to calculate (see Lemma 2.8.4 for $n = 1$).

(c) If n is even, the extremal polynomials for $\gamma_{\mathbb{G}_n}(0; e_2) = 2/n$ can differ not only by a constant of absolute value 1. For example, after some easy calculations, from the proof of Theorem 2.7.3 we get the polynomial $2z_2/n-(n-1)z_1^2/n^2$, but the polynomial $(2z_2-z_1^2)/n$ is also extremal.

PROPOSITION 2.8.2. *If $n\geq 3$ is odd, then*

$$\frac{2}{n}\Big(1+\frac{2}{(n-1)(n+2)}\Big)<\gamma_{\mathbb{G}_n}(0;e_2)<\frac{2}{n}\Big(1+\frac{2}{(n-1)(n+1)}\Big).$$

Proof. Lower estimate. Let us first see that the polynomial

$$P_n(z)=\frac{n-1}{2(n+1)}z_1^2-z_2$$

satisfies

$$\max_{\partial\mathbb{G}_n}|P_n|=M_n:=\frac{(n-1)(n+2)}{2(n+1)}.$$

So if

$$g_n(t)=\frac{1}{2}\sum_{j=1}^n t_j^2-\frac{1}{n+1}\Big(\sum_{j=1}^n t_j\Big)^2,\qquad t\in\mathbb{C}^n,$$

then $\max_{\mathbb{T}^n}|g_n|=M_n$. To prove the latter, let $M_n^*=\max_{\mathbb{T}^n}|g_n|$. As $g_n(e^{i\theta}t)=e^{2i\theta}g_n(t)$ for each $\theta\in\mathbb{R}$, $t\in\mathbb{C}^n$, there exists $u\in\mathbb{T}^n$ such that $g_n(u)=M_n^*$. Putting $u_j=x_j+iy_j$, $x_j,y_j\in\mathbb{R}$, $1\leq j\leq n$, we get

$$\begin{aligned}
M_n^*=\operatorname{Re}(g_n(u))&=\frac{1}{2}\sum_{j=1}^n(x_j^2-y_j^2)+\frac{1}{n+1}\Big(\Big(\sum_{j=1}^n y_j\Big)^2-\Big(\sum_{j=1}^n x_j\Big)^2\Big)\\
&\leq\frac{1}{2}\sum_{j=1}^n(x_j^2-y_j^2)+\frac{1}{n+1}\Big(n\sum_{j=1}^n y_j^2-\Big(\sum_{j=1}^n x_j\Big)^2\Big)\\
&=\frac{(n-1)n}{2(n+1)}+\frac{1}{n+1}\Big(\sum_{j=1}^n x_j^2-\Big(\sum_{j=1}^n x_j\Big)^2\Big)
\end{aligned}$$

by the Cauchy–Schwarz inequality and the equalities $y_1^2=1-x_1^2,\dots,y_n^2=1-x_n^2$. The last expression is a linear function for each x_j. Consequently, it is maximal for 1 and/or -1. As n is odd,

$$M_n^*=\frac{(n-1)n}{2(n+1)}+\frac{n-1}{n+1}=M_n,$$

with maximum attained only if $[n/2]$ or $[n/2]+1$ among the numbers t_j are equal to some $t_0\in\mathbb{T}$, and the rest to $-t_0$.

Using this fact one can easily prove that if $\varepsilon>0$ is sufficiently small and

$$g_{n,\varepsilon}(t)=g_n(t)+\varepsilon\sum_{j=1}^n t_j^2-\varepsilon(n+1)\Big(\sum_{j=1}^n t_j\Big)^2,\qquad t\in\mathbb{C}^n,$$

then $\max_{\mathbb{T}^n}|g_{n,\varepsilon}|<M_n$. Consequently, for

$$P_{n,\varepsilon}=\frac{n-1-2n(n+1)\varepsilon}{2(n+1)}z_1-(1+2\varepsilon)z_2$$

one has $\max_{\partial\mathbb{G}_n}|P_{n,\varepsilon}|<M_n$, showing that

$$\gamma_{\mathbb{G}_n}(0;e_2)>\frac{1}{M_n}=\frac{2}{n}\Big(1+\frac{2}{(n-1)(n+2)}\Big).$$

Upper estimate. By (2.8.1) we need to prove that if $c\in\mathbb{C}$, then

$$m_{n,c}:=\max_{z\in\partial G_n}|z_2+cz_1^2|>\frac{n(n^2-1)}{2(n^2+1)}.$$

The coefficients of the polynomials $(t-1)^n$ and $(t-1)(t^2-1)^{(n-1)/2}$ give two points $z\in\partial\mathbb{G}_n$ such that $z_1=n$, $z_2=n(n-1)/2$ and $z_1=1$, $z_2=(1-n)/2$, respectively. Then

$$2m_{n,c}\ge\max\{|n-1-2c|,|n(n-1)+2cn^2|\}$$

and consequently

$$\begin{aligned}2(n^2+1)m_{n,c}&\ge|n^2(n-1)-2cn^2|+|n(n-1)+2cn^2|\\&\ge n^2(n-1)+n(n-1)=n(n^2-1).\end{aligned}$$

This means that $m_{n,c}\ge\frac{n(n^2-1)}{2(n^2+1)}$. Suppose that we have equality. Then $c=-\frac{(n-1)^2}{2(n^2+1)}$. On the other hand, the coefficients of the polynomial $(t-i)(t-1)^{n-1}$ give a point $z\in\partial\mathbb{G}_n$ such that $z_1=n-1+i$, $z_2=(n-1)(n-2)/2+(n-1)i$ and $\left|z_2-\frac{(n-1)^2}{2(n^2+1)}z_1^2\right|>\frac{n(n^2-1)}{2(n^2+1)}$, a contradiction. ∎

Let now D be a domain in $\mathbb{C}^n$, $z\in D$ and $k\in\mathbb{N}$. Denote by $\hat\gamma_D^{(k)}(z;X)$ the largest pseudonorm not exceeding the kth Carathéodory pseudometric

$$\gamma_D^{(k)}(z;X)=\sup\{|f_z^{(k)}(X)|:f\in\mathcal{O}(D,\mathbb{D}),\ \mathrm{ord}_z f\ge k\},$$

where

$$f_z^{(k)}(X)=\sum_{|\alpha|=k}\frac{D^\alpha f(z)X^\alpha}{\alpha!}.$$

(One can define similarly the kth Kobayashi pseudometric, which is essentially different from the Kobayashi pseudometric $\kappa_D^{(k)}$ of order k, defined in Section 1.2).

As $\gamma_D(z;\cdot)$ is a pseudonorm,

$$\gamma_D\le\hat\gamma_D^{(k)}\le\hat\kappa_D.$$

Also note that, since the family $\mathcal{O}(\mathbb{G}_3,\mathbb{D})$ is normal, the argument in the proof of Theorem 1.4.1 shows the existence of $m\le 2n-1$ $\mathbb{R}$-linearly independent vectors $X_1,\dots,X_m\in\mathbb{C}^n$ of sum X, so that

$$\hat\gamma_D^{(k)}(z;X)=\sum_{j=1}^m\gamma_D^{(k)}(z;X_j).$$

THEOREM 2.8.3. $\hat\gamma_{\mathbb{G}_3}^{(2)}(0;e_2)>\gamma_{\mathbb{G}_3}(0;e_2)$. *In particular,* $\hat\kappa_{\mathbb{G}_3}(0;e_2)>\gamma_{\mathbb{G}_3}(0;e_2)$ *and consequently* $k_{\mathbb{G}_3}(0,\cdot)\ne c_{\mathbb{G}_3}(0,\cdot)$.

This theorem follows from the two lemmas below.

LEMMA 2.8.4. $\gamma_{\mathbb{G}_3}(0;e_2)\le C_0:=\sqrt{\frac{8}{13\sqrt{13}-35}}=0.8208\dots$.

LEMMA 2.8.5. $\hat\gamma_{\mathbb{G}_3}^{(2)}(0;e_2)\ge C_1=\sqrt{0.675}=0.8215\dots$.

Proof of Lemma 2.8.4. By (2.8.1) we need to show that if $c\in\mathbb{C}$, then

$$\max_{z\in\partial\mathbb{G}_3}|z_2-cz_1^2|^2\ge C_0^{-2}.$$

It suffices to show this for $c \in \mathbb{R}$. Indeed, for each $z \in \partial\mathbb{G}_3$ we have $\overline{z} \in \partial\mathbb{G}_3$ and so

$$\begin{aligned} 2\max_{z\in\partial\mathbb{G}_3} |z_2 - cz_1^2| &\geq \max_{z\in\partial\mathbb{G}_3} (|z_2 - cz_1^2| + |\overline{z}_2 - c\overline{z}_1^2|) \\ &\geq \max_{z\in\partial\mathbb{G}_3} |2z_2 - (c+\overline{c})z_1^2| = 2\max_{z\in\partial\mathbb{G}_3} |z_2 - \operatorname{Re}(c)z_1^2|. \end{aligned}$$

Let now $c \in \mathbb{R}$. Then

$$\begin{aligned} \max_{z\in\partial\mathbb{G}_3} |z_2 - cz_1^2|^2 &\geq \max_{\varphi\in[0,2\pi)} |1 + 2e^{i\varphi} - c(2+e^{i\varphi})^2|^2 \\ &= \max_{\varphi\in[0,2\pi)} (4c(4c-1)\cos^2\varphi + 4(10c^2 - 7c + 1)\cos\varphi + 25c^2 - 22c + 5). \end{aligned}$$

Put

$$f_c(x) = 4c(4c-1)x^2 + 4(2c-1)(5c-1)x + 25c^2 - 22c + 5, \qquad x \in [-1,1].$$

If $c \notin \Delta = (1/6, 5 - \sqrt{17}/4)$, then

$$\max_{x\in[-1,1]} f_c(x) = \max\{f_c(-1), f_c(1)\} \geq \left(\frac{9-\sqrt{17}}{4}\right)^2 > \frac{1}{C_0^2}.$$

Otherwise

$$\max_{x\in[-1,1]} f_c(x) = f_c\left(\frac{10c^2 - 7c + 1}{2c(1-4c)}\right) = \frac{(3c-1)^3}{c(4c-1)} =: g(c)$$

and it remains to see that $\min_{c\in\Delta} g(c) = g(\sqrt{13} - 1/12) = 1/C_0^2$. ∎

Remark. Let $c_0 = (\sqrt{13} - 1)/12$ and $M = \max_{z\in\partial\mathbb{G}_3} |z_2 - c_0 z_1^2|$. As in the proof of Proposition 2.8.2, we have

$$M = \max_{z\in\partial\mathbb{G}_3} \operatorname{Re}(z_2 - c_0 z_1^2) = \max_{\alpha,\beta,\gamma\in\mathbb{R}} h(\alpha,\beta,\gamma),$$

where

$$\begin{aligned} h(\alpha,\beta,\gamma) = (1 - 2c_0)(\cos(\alpha+\beta) + \cos(\beta+\gamma) + \cos(\gamma+\alpha)) \\ - c_0(\cos 2\alpha + \cos 2\beta + \cos 2\gamma). \end{aligned}$$

Computer calculations show that the critical points of h (up to a permutation of variables) are of the form $(k\pi, l\pi, m\pi)$ or $(\pm\alpha_0 + j\pi/2 + 2k\pi, \pm\alpha_0 + j\pi/2 + 2l\pi, \pm\gamma_0 + j\pi/2 + 2m\pi)$, $k, l, m \in \mathbb{Z}$, $j = 0, 1, 2, 3$. Then the proof of Lemma 2.8.4 implies that $M = C_0^{-1}$, i.e. $\gamma_{\mathbb{G}_3}(0; e_2) = C_0$.

Proof of Lemma 2.8.5. Let

$$f(z) = 0.675z_2^2 - 0.291z_2 z_1^2 + 0.033z_1^4.$$

We first check that $\max_{z\in\partial\mathbb{G}_3} |f(z)| < 1$ by reducing the check to finitely many points, and then using a computer program. Put $\theta = (\theta_1, \theta_2)$, θ_1, $\theta_2 \in [0, 2\pi)$,

$$\begin{aligned} g_1(\theta) = 1 + e^{i\theta_1} + e^{i\theta_2}, \qquad g_2(\theta) = e^{i(\theta_1+\theta_2)} + e^{i\theta_1} + e^{i\theta_2}, \\ g(\theta) = 0.675g_2^2(\theta) - 0.291g_2(\theta)g_1^2(\theta) + 0.033g_1^4(\theta). \end{aligned}$$

We have to prove that $\max|g(\theta)| < 1$. Let

$$d(\theta, \tilde{\theta}) = \max\{|\theta_1 - \tilde{\theta}_1|, |\theta_2 - \tilde{\theta}_2|\}.$$

As $|e^{i\theta_j}-e^{i\tilde\theta_j}|\le|\theta_j-\tilde\theta_j|$, $j=1,2$, we get

$$|g_1(\theta)-g_1(\tilde\theta)|\le 2d(\theta,\tilde\theta),\qquad |g_2(\theta)-g_2(\tilde\theta)|\le 4d(\theta,\tilde\theta).$$

Then the inequalities $|g_1|\le 3$, $|g_2|\le 3$ imply

$$|g(\theta)-g(\tilde\theta)|\le(0.675\cdot 24+0.291\cdot 72+0.033\cdot 216)d(\theta,\tilde\theta)=44.28d(\theta,\tilde\theta).$$

Let now θ_1,θ_2 vary in the interval $[0,6.2832]\supset[0,2\pi]$ with a step of $4\cdot 10^{-5}$. The results of the corresponding computer program (see Appendix C) show that $|g(\theta)|\le 0,999$ for the variable $\theta=(\theta_1,\theta_2)$. (In fact these results lead to the hypothesis that $\max|g(\theta)|=0.999$, with a maximum attained at the points $(0,\pi)$, $(\pi,0)$ and (π,π).) Then by the inequalities $|g(\theta)-g(\tilde\theta)|\le 44.28d(\theta,\tilde\theta)$ and $\frac{2}{44.28}\cdot 10^{-3}>4\cdot 10^{-5}$ we easily get $\max|g(\theta)|<1$.

From the above it follows that if $X\in\operatorname{span}(e_1,e_3)$, then

$$\gamma^{(2)}_{\mathbb{G}_3}(0;e_2+X)\ge|f_0^{(2)}(e_2+X)|/2=|f_0^{(2)}(e_2)|/2=C_1.$$

On the other hand, recall that there exist five vectors $X_1,\dots,X_5\in\mathbb{C}^3$ (some of them can be zero) of sum e_2 such that $\hat\gamma^{(2)}_{\mathbb{G}_3}(0;e_2)=\sum_{j=1}^5\gamma^{(2)}_{\mathbb{G}_3}(0;X_j)$. As $\gamma^{(2)}_{\mathbb{G}_3}(0;X_j)\ge C_1|\langle X_j,e_2\rangle|$, we get $\hat\gamma^{(2)}_{\mathbb{G}_3}(0;e_2)\ge C_1$. ■

Remark. An important moment in the above proof is finding a polynomial of the form $f(z)=az_2^2+bz_2z_1^2+cz_1^4$ such that $\max_{\partial\mathbb{G}_3}|f|\le 1$ and $\sqrt{a}>C_0$. Computer experiments show that the maximal value of a is $0.676\dots$, i.e. very close to $0.675/0.999$.

Finally let us note that $\gamma^{(2)}_{\mathbb{G}_n}(0;\cdot)$ is not a norm.

Proposition 2.8.6. *If $X_1,X_n\in\mathbb{C}$, then*

$$\gamma^{(2)}_{\mathbb{G}_n}(0;X_1e_1+X_ne_n)\ge\sqrt{\frac{n+1}{2}\gamma_{\mathbb{G}_n}(0;e_2)|X_1X_n|}.$$

In particular, as $\gamma_{\mathbb{G}_3}(0;e_2)>2/3$ and $\gamma_{\mathbb{G}_n}(0;e_n)\ge 2/n$, it follows that

$$\gamma^{(2)}_{\mathbb{G}_n}(0;ne_1+e_n)>2=\hat\kappa_{\mathbb{G}_n}(0;ne_1+e_n)=\gamma^{(2)}_{\mathbb{G}_n}(0;ne_1)+\gamma^{(2)}_{\mathbb{G}_n}(0;e_n),\qquad n\ge 3.$$

Proof. Let $t_1,\dots,t_n\in\mathbb{D}$. Consider $\sum_{k=1}^n t_k^{n+1}/n$ as a function f of $\sigma(t)$. Then $f\in\mathcal{O}(\mathbb{G}_n,\mathbb{D})$, $\operatorname{ord}_0 f=2$, and the Waring formula (see e.g. [117])) implies that the coefficient of z_1z_n equals $(-1)^{n-1}\frac{n+1}{n}$. So

$$\gamma^{(2)}_{\mathbb{G}_n}(0;X_1e_1+X_ne_n)\ge|f_0^{(2)}(X)/2|=\sqrt{\frac{n+1}{n}\gamma_{\mathbb{G}_n}(0;e_2)|X_1X_n|},$$

where $X=X_1e_1+X_ne_n$. As $\gamma_{\mathbb{G}_n}(0;e_2)=2/n$ for n even, we get the proposition for such n.

On the other hand, Proposition 2.8.1 implies that if $2C_n:=\gamma_{\mathbb{G}_n}(0;e_2)$, then there exists a c_n such that $P(z)=2C_nz_2-c_nz_1^2$ is an extremal function for $\gamma_{\mathbb{G}_n}(0;e_2)$. For odd $n=2k-1$ we replace $t_1,\dots,t_n$ by $t_1^k,\dots,t_n^k$. Thus we get the function

$$(C_n-c_n)\Big(\sum_{j=1}^n t_j^k\Big)^2-C_n\sum_{j=1}^n t_j^{2k}.$$

Consider this function as a function g of $\sigma(t)$. Then $g \in \mathcal{O}(\mathbb{G}_n, \mathbb{D})$, $\operatorname{ord}_0 g = 2$, and the coefficient of $z_1 z_n$ equals $-(n+1)C_n$. Consequently,

$$\gamma^{(2)}_{\mathbb{G}_n}(0;X) \geq \left|\frac{g_0^{(2)}(X)}{2}\right| = \sqrt{\frac{n+1}{2}\gamma_{\mathbb{G}_n}(0;e_2)|X_1X_n|}. \blacksquare$$

Appendix C ([6]).

language: FORTRAN 77; compiler: gnu-fortran (g77)
options for compiler: g77 -02 -o niki.exe niki.for -wall
command for execute: niki.exe

```
      Program niki
      implicit real*8 (a-h,o-z)
      implicit integer*4 (i-n)
      complex*16 g0,g1,g2

      data c1,c2,c3 /0.675D0, -0.291D0, 0.033D0/
      data e,o,t1d,t2d,t1u,t2u/1.0D0,3*0.0D0,2*6.2832D0/
      write(*,102)
  200  continue
      read(*,*,ERR=201,END=201) s
      N1=(t1u-t1d)/s
      N2=(t2u-t2d)/s
      gu=-1D30
      do i1=0,N1
         t1=t1d+FLOAT(i1)*s
         do i2=0,N2
            t2=t2d+FLOAT(i2)*s
            g0=DCMPLX(DCOS(t1)+DCOS(t2),DSIN(t1)+DSIN(t2))
            g1=g0+DCMPLX(e,o)
            g2=g0+DCMPLX(DCOS(t1+t2),DSIN(t1+t2))
            g = CDABS(c1*g2**2+c2*g2*g1**2+c3*g1**4)
            if (g.GT.gu) then
               gu=g
               t1g=t1
               t2g=t2
            endif
         enddo
      enddo
      write(*,100) s,gu,t1g,t2g
      goto 200
  201  continue
      write(*,101)
      stop
  100 format(1x,2f20.15,2f15.10)
  101 format(8x,' step ',15x,' g-max',9x,'tita-1',9x,'tita-2')
  102 format(8x,' step ' )
      end
```

([6]) The program was written by Pencho Marinov.

```
----------------- Results ------------------------------------------
 step                 g-max                tita-1          tita-2

 0.001000000000000    0.998999998608272    3.1420000000    3.1420000000
 0.000400000000000    0.998999999688699    3.1414000000    3.1414000000
 0.000100000000000    0.998999999999547    3.1416000000    3.1416000000
 0.000040000000000    0.998999999999547    3.1416000000    3.1416000000
 0.000010000000000    0.998999999999941    3.1415900000    3.1415900000
 0.000004000000000    0.998999999999985    3.1415940000    3.1415940000
 0.000001000000000    0.998999999999999    3.1415930000    3.1415930000
 0.000000400000000    0.999000000000000    3.1415928000    3.1415928000
 0.000000100000000    0.999000000000000    3.1415927000    3.1415927000
 0.000000040000000    0.999000000000000    3.1415925600    3.1415925600
```

2.9. Continuity of $l_{\Omega_n}(A,\cdot)$. As mentioned in the Introduction, the continuous dependence of SNPP on the data (a necessary condition for reduction to an analogous problem on $\mathbb{G}_n$) is linked with the continuity of the function l_{Ω_n}. The aim of this section is to describe all matrices $A \in \Omega_n$ such that $l_{\Omega_n}(A,\cdot)$ is a continuous function.

First recall that

$$l_{\Omega_n}(A,B) \geq l_{\mathbb{G}_n}(\sigma(A),\sigma(B)).$$

Furthermore, if $A,B \in \mathcal{C}_n$ (i.e. they are cyclic matrices in Ω_n), then we have equality (see 2.2.1)) and so l_{Ω_n} is a continuous function on the open set $\mathcal{C}_n \times \mathcal{C}_n$. In general, we have equality if and only if l_{Ω_n} is a continuous function in (A,B). To see this, it suffices to use that $l_{\mathbb{G}_n}$ is a continuous function and the set $\mathcal{C}_n \times \mathcal{C}_n$ (where we have equality) is dense in $\Omega_n \times \Omega_n$.

In [115] the authors consider matrices $B \in \Omega_n$ such that $l_{\Omega_n}(A,.)$ is a continuous function at B for each $A \in \Omega_n$. They hypothesize that this is true for each $B \in \mathcal{C}_n$ and confirm this for $n \leq 3$ [115, Proposition 1.4]. Using the results from Section 2.10 for the continuity of $\kappa_{\Omega_n}(A;.)$, the converse proposition is proven for each n (see [115, Theorem 1.3]).

We first prove the following

Proposition 2.9.1. *If $\lambda \in \mathbb{D}$ and $A \in \mathcal{C}_n$, then the following are equivalent:*

(i) *the eigenvalues of A are all equal;*
(ii) *l_{Ω_n} is continuous at $(A,\lambda I_n)$;*
(iii) *$l_{\Omega_n}(A,\cdot)$ is continuous at λI_n.*

Proof. The implication (ii)⇒(iii) is trivial. For the rest of the proof we may assume that $\lambda = 0$, applying Φ_λ (see (2.2.3)).

We will now show that (i)⇒(ii). Let the eigenvalues of A be equal to a. If $A_j \to A$ and $B_j \to 0$, then

$$l_{\Omega_n}(A_j,B_j) \geq c^*_{\mathbb{G}_n}(\sigma(A_j),\sigma(B_j)) \to c^*_{\mathbb{G}_n}(\sigma(A),0) = |a| = l_{\Omega_n}(A,0)$$

(the last two equalities follow from Propositions 2.7.1(iii) and 2.2.4, respectively). So the function l_{Ω_n} is lower semicontinuous at $(0,B)$. As it is (always) upper semicontinuous, it is continuous at this point.

It remains to prove that (iii)⇒(i). As $\mathcal{C}_n$ is a dense subset of Ω_n, we can find a sequence $\mathcal{C}_n \ni B_j \to 0$. By (2.2.4) and (2.2.1) we get

$$r(A) = l_{\Omega_n}(A,0) \leftarrow l_{\Omega_n}(A,B_j) = l_{\mathbb{G}_n}(\mathrm{sp}(A),\mathrm{sp}(B_j)) \to l_{\mathbb{G}_n}(\mathrm{sp}(A),0).$$

Proposition 2.7.1(iii) implies that the eigenvalues of A are all equal. ■

Unlike the above proposition, (2.2.4) implies that $l_{\Omega_n}(A,\cdot)$ is a continuous function for each scalar matrix $A \in \Omega_n$.

As we noted, if $A \in \Omega_n$ $(n \geq 2)$, then the following are equivalent:

(i) l_{Ω_n} is continuous at (A,B) for each $B \in \Omega_n$;
(ii) $l_{\Omega_n}(A,\cdot) = l_{\mathbb{G}_n}(\sigma(A),\sigma(\cdot))$.

Also consider the condition

(iii) $A \in \mathcal{C}_2$ has (two) equal eigenvalues.

By [26, Theorem 8], (iii) implies (ii). Theorem 2.9.2 says that the scalar matrices and those satisfying (iii) are the only ones for which $l_{\Omega_n}(A,\cdot)$ is a continuous function. Then Proposition 2.9.1 implies that (iii) follows from (i). So assertions (i), (ii) and (iii) are equivalent.

THEOREM 2.9.2. *If $A \in \Omega_n$, then $l_{\Omega_n}(A,\cdot)$ is a continuous function if and only if A is a scalar matrix or $A \in \mathcal{C}_2$ has two equal eigenvalues.*

Proof. Applying Φ_λ and

$$\Psi_P(X) = P^{-1}XP, \quad P \in \mathcal{M}_n^{-1},\ X \in \mathcal{M}_n, \tag{2.9.1}$$

we can assume that 0 is an eigenvalue of A with a maximal number of Jordan blocks and the matrix is in Jordan form.

It suffices to prove that $l_{\Omega_n}(A,\cdot)$ is not a continuous function if A has a nonzero eigenvalue or $A \in \Omega_n$ is a nonzero nilpotent matrix and $n \geq 3$.

In the first case let $d_1 \geq \dots \geq d_k$ be the number of Jordan blocks that correspond to the different eigenvalues $\lambda_1 = 0, \lambda_2, \dots, \lambda_k$. We will prove that $l_{\Omega_n}(A,\cdot)$ is not continuous at 0. It is easily seen that A can be expressed as blocks $A_1,\dots,A_l$ (of dimensions $n_1,\dots,n_l$) so that the eigenvalues of A_1 are equal to 0 and the remaining blocks are cyclic with at least two different eigenvalues (A_1 is missing if $d_1 = d_2$). By Proposition 2.7.1(iii), there exists a sequence of matrices $A_{i,j} \to 0$ as $j \to \infty$, $1 \leq i \leq l$, such that $\sup_{i,j} l_{\Omega_{n_i}}(A_i, A_{i,j}) =: m < r(A)$. Forming A_j from the blocks $A_{1,j},\dots,A_{l,j}$, it follows that $l_{\Omega_n}(A,A_j) \leq \max_i l_{\Omega_{n_i}} l(A_i,A_{i,j}) \leq m < l_{\Omega_n}(A,0)$, meaning that $l_{\Omega_n}(A,\cdot)$ is not continuous at 0.

Let now $A \neq 0$ be a nilpotent matrix. Then $A = (a_{ij})_{1\leq i,j\leq n}$, where $a_{ij} = 0$ for $j \neq i+1$. Let $r = \mathrm{rank}(A) \geq 1$. Following the proof of [115, Proposition 4.1], let

$$F_0 = \{1\} \cup \{j \in \{2,\dots,n\} : a_{j-1,j} = 0\} := \{1 = b_1 < \dots < b_{n-r}\},$$

and $b_{n-r+1} = n+1$. Put $d_i = 1 + \#(F_0 \cap n-i+2,\dots,n)$. As $A \neq 0$ is a nilpotent matrix, it has a Jordan form such that $a_{n-1,n} = 1$ and $1 = d_1 = d_2 \leq d_3 \leq \dots \leq d_n = \#F_0 = n-r$, $d_{j+1} \leq d_j + 1$.

In [115, Proposition 4.1, Corollary 4.3] there is a necessary and sufficient condition for lifting of discs from $\mathcal{O}(\mathbb{D},\mathbb{G}_n)$ to ones in $\mathcal{O}(\mathbb{D},\Omega_n)$, passing through a cyclic and nilpotent matrix. They easily imply that for each $C\in\mathcal{C}_n$,

$$l_{\Omega_n}(A,C)=h_{\mathbb{G}_n}(0,\sigma(C)):=\inf\{|\alpha|:\exists\psi\in\mathcal{H}(\mathbb{D},\mathbb{G}_n):\psi(\alpha)=\sigma(C)\},$$

where

$$\mathcal{H}(\mathbb{D},\mathbb{G}_n)=\{\psi\in\mathcal{O}(\mathbb{D},\mathbb{G}_n):\operatorname{ord}_0\psi_j\ge d_j,\ 1\le j\le n\}.$$

Note that $d_j\le j-1$ for $j\ge 2$. Let $m=\min_{j\ge 2}d_j/(j-1)$ and choose k such that $d_k/(k-1)=m$. If $m=1$, then $d_j=j-1$ for each $j\ge 2$ and so in this case for $n\ge 3$ one can take $k=3$.

Let λ be a sufficiently small positive number, $b=k\lambda^{k-1}$ and $c=(k-1)\lambda^k$. Then λ is a double zero of the polynomial $\Lambda(z)=z^{n-k}(z^k-bz+c)$ with zeroes in $\mathbb{D}$. Let B be a diagonal matrix with characteristic polynomial $P_B(z)=\Lambda(z)$.

Suppose that the function $l_{\Omega_n}(A,\cdot)$ is continuous at B. Then

$$l_{\Omega_n}(A,B)=h_{\mathbb{G}_n}(0,\sigma(B))=:\alpha.$$

Lemma 2.9.3. *If $l_{\Omega_n}(A,B)=\alpha$, then there exists $\psi\in\mathcal{H}(\mathbb{D},\mathbb{G}_n)$ so that $\psi(\alpha)=\sigma(B)$ and*

$$\sum_{j=1}^n\psi_j'(\alpha)(-\lambda)^{n-j}=0.$$

Proof. Similarly to the proof of [115, Proposition 4.1], let $\varphi\in\mathcal{O}(\mathbb{D},\Omega_n)$ and $\tilde\alpha\in\mathbb{D}$ so that $\varphi(0)=A$ and $\varphi(\tilde\alpha)=B$. By [115, Corollary 4.3] we have $\tilde\psi=\sigma\circ\varphi\in\mathcal{H}(\mathbb{D},\mathbb{G}_n)$.

Now let us examine $\sigma_n(\varphi(\zeta))-\sigma_n(B)=\sigma_n(\varphi(\zeta))$ near $\zeta=\tilde\alpha$. We can assume that the first two diagonal elements of B are equal to λ. If $\varphi_\lambda(\zeta)=\varphi(\zeta)-\lambda I_n$, then the first two columns of $\varphi_\lambda(\alpha)$ are zero. Consequently, $\sigma_n\circ\varphi_\lambda=\det\varphi_\lambda$ has a zero of order at least 2 at α. On the other hand, $\mathbb{G}_n$ is a taut domain, which easily provides the required ψ. ∎

Lemma 2.9.4. *We have $\alpha^m\lesssim\lambda$ (7). Furthermore, if $m=1$ and $n\ge 3$, then $\alpha^{2/3}\lesssim\lambda$. In particular, always $\alpha\ll\lambda$.*

Proof. Note that there exists $\varepsilon>0$ so that for $\lambda<\varepsilon$ the mapping $\zeta\mapsto(0,\dots,0,k(\varepsilon\zeta)^{d_k},(k-1)\lambda(\varepsilon\zeta)^{d_k},0,\dots,0)$ is a competitor for $h_{\Omega_n}(A,B)$. Consequently, $(\varepsilon\alpha)^{d_k}\le\lambda^{k-1}$, i.e. $\alpha^m\lesssim\lambda$.

If $m=1$ and $n\ge k=3$, by considering the mapping $\zeta\mapsto(0,3\lambda^{1/2}\varepsilon\zeta,2(\varepsilon\zeta)^2,0\dots,0)$ we get $(\varepsilon\alpha)^2\le\lambda^3$. ∎

To finish the proof of the theorem, put $\psi_j(\zeta)=\zeta^{d_j}\theta(\zeta)$; the condition in Lemma 2.9.3 becomes

$$a\frac{(-\lambda)^n}{\alpha}+S=0,\tag{2.9.2}$$

where $a=(k-1)d_k-kd_{k-1}$ and $S=\sum_{j=1}^n\alpha^{d_j}\theta_j'(\alpha)(-\lambda)^{n-j}$. Note that $a\ne0$. Indeed, if $m<1$, then $d_k=d_{k-1}$ and consequently $a=-d_k$, while if $m=1$, then $a=(k-1)(k-1)-k(k-2)=1$. As $\mathbb{G}_n$ is a bounded domain, the Cauchy inequalities imply $|\theta_j'(\alpha)|\lesssim 1$.

(7) This means that $\alpha^m\le C\lambda$ for some constant $C>0$ independent of λ.

By Lemma 2.9.4 and by the choice of k it follows that for each j,

$$\alpha^{d_j} \lesssim \lambda^{(k-1)d_j/d_k} \leq \lambda^{j-1} \leq \lambda^{n-1}.$$

So $S \lesssim \lambda^{n-1}$. Once again by Lemma 2.9.4, $\alpha \ll \lambda$, contradicting (2.9.2). ∎

2.10. Zeroes of κ_{Ω_n}. Recall that the spectral Carathéodory–Fejér problem of order 1 (SCFP) reduces to the calculation of the Kobayashi metric κ_{Ω_n} of Ω_n. Furthermore, if $A \in \mathcal{C}_n$ (i.e. A is a cyclic matrix), then (see (2.2.2))

$$\kappa_{\Omega_n}(A;B) = \kappa_{\mathbb{G}_n}(A;\sigma'_A(B)).$$

In particular, $\kappa_{\Omega_n}(A;B) = 0 \Leftrightarrow \sigma'_A(B) = 0$.

On the other hand, by Proposition 2.3.1, $\sigma'_A(B) = 0$ exactly when there exists $Y \in \mathcal{M}_n$ so that $B = [Y, A]$. Consequently, if $A \in \mathcal{C}_n$ and $\sigma'_A(B) = 0$, considering $\zeta \mapsto e^{\zeta Y} A e^{-\zeta Y}$, we find even an entire curve $\varphi : \mathbb{C} \to \Omega_n$ so that $\varphi(0) = A$ and $\varphi'(0) = B$. In general, if $\kappa_{\Omega_n}(A;B) = 0$ (we do not assume $A \in \mathcal{C}_n$), then SCFP has a solution for an arbitrary disc instead of the unit one. Therefore it is important to know the zeroes of κ_{Ω_n}. This also bears information on the discontinuity of this function (hence also of SCFP).

Recall that for the Carathéodory metric of Ω_n, things are much simpler (see Proposition 2.2.3):

$$\gamma_{\Omega_n}(A;B) = \gamma_{\mathbb{G}_n}(\sigma(A);\sigma'_A(B))$$

and so $\gamma_{\Omega_n}(A;B) = 0 \Leftrightarrow \sigma'_A(B) = 0$.

To formulate the results in this section we need to introduce some notions.

For $A \in \Omega_n$ denote by C_A the tangent cone (see [20, p. 79]) to the isospectral (analytic) set

$$L_A = \{C \in \Omega_n : \operatorname{sp}(C) = \operatorname{sp}(A)\},$$

i.e.

$$C_A = \{B \in \mathcal{M}_n : \exists 0 < c_j \to 0,\ C_j \in L_A \text{ so that } c_j(C_j - A) \to B\}.$$

Note that L_A is smooth at D if $D \in \mathcal{C}_n$. Then $C_A = \ker \sigma'_D$ and as $\dim \ker \sigma'_D = n^2 - n$ (see Proposition 2.3.1), C_A is an analytic set and $\dim C_A = \dim L_A = n^2 - n$ by [20, Corollary, p. 83]. If $A \notin \mathcal{C}_n$, then $\dim \ker \sigma'_A > n^2 - n$ (see Proposition 2.3.1) and so $C_A \subsetneq \ker \sigma'_A$. Thus

$$C_A = \ker \sigma'_A \ \Leftrightarrow\ A \in \mathcal{C}_n.$$

The next theorem characterizes C_A as the set of "generalized" tangent vectors at A to an entire curve in Ω_n passing through A (in particular, this curve is contained in L_A).

THEOREM 2.10.1. *Let $A \in \Omega_n$ and $B \in \mathcal{M}_n$. Then there exists $m \in \mathbb{N}$ ($m \leq n!$) and $\varphi \in \mathcal{O}(\mathbb{C}, \Omega_n)$ so that $\varphi(0) = A$, $\varphi'(0) = \cdots = \varphi^{(m-1)}(0) = 0$, $\varphi^{(m)}(0) = B$ only if $B \in C_A$.*

We are not including the proof of this theorem, due to its length and the use of results about analytic sets that are beyond the scope of the dissertation. It can be found in the paper [100] by the author and P. J. Thomas.

Theorem 2.10.1 shows that C_A is contained in the set of zeroes of the singular Kobayashi metric $\kappa^s_{\Omega_n}(A;\cdot)$. Recall that (see [123])

$$\kappa^s_{\Omega_n}(A;B) = \inf\{|\alpha| : \exists m \in \mathbb{N}, \varphi \in \mathcal{O}(\mathbb{D},\Omega_n) : \mathrm{ord}_0(\varphi - A) \ge m,\ \alpha\varphi^{(m)}(0) = m!B\}.$$

Now we define another cone $C'_A \subset \mathcal{M}_n$, $A \in \Omega_n$.

For a function g holomorphic near A, and for X in a neighborhood of A, put $g(X) - g(A) = g^*_A(X - A) + \cdots$, where g^*_A is the homogeneous polynomial of lowest nonzero degree in the expansion of g near A. Put

$$C'_A = \{B \in \mathcal{M}_n : f^*_A(B) = 0 \text{ for each } f \in \mathcal{O}(\Omega_n,\mathbb{D})\}.$$

Note that

$$C_A \subset C'_A \subset \ker \sigma'_A;$$

the first inclusion is proven e.g. in [20, p. 86]), and the second one follows from the facts that each $f \in \mathcal{O}(\Omega_n,\mathbb{D})$ is constant on L_A (by the Liouville theorem) and that

$$\ker \sigma'_A = \{(\sigma_j)^*_A = 0 \text{ for all } j \text{ such that } \deg(\sigma_j)^*_A = 1\}.$$

Also, each of these three sets is invariant under automorphisms of Ω_n.

The cone C'_A coincides with C_A for $n = 2$ and $n = 3$ (for the last fact see Proposition 2.10.6 below and the remarks preceding it). We do not know whether this is true for each n.

In the most trivial case of a noncyclic matrix, namely a scalar one, $C'_A = C_A$ is the set of zero-spectrum matrices, while $\ker \sigma'_A$ is the set of zero-trace matrices.

Note that $\kappa^s_{\Omega_n} \ge \gamma^s_{\Omega_n}$, where $\gamma^s_{\Omega_n} = \sup_{m\in\mathbb{N}} \gamma^{(m)}_{\Omega_n}$ is the singular Carathéodory metric of Ω_n (see Section 2.8 for the definition of $\gamma^{(m)}$).

Theorem 2.10.1 implies that

$$B \in C_A \ \Rightarrow\ \kappa^s_{\Omega_n}(A;B) = 0 \ \Rightarrow\ \gamma^s_{\Omega_n}(A;B) = 0 \ \Leftrightarrow\ B \in C'_A$$

(the last equivalence is trivial). In particular,

$$\kappa_{\Omega_n}(A;B) = 0 \ \Rightarrow\ B \in C'_A.$$

Proposition 2.10.2. *If $A \in \Omega_n \setminus \mathcal{C}_n$, then $C'_A \ne \ker \sigma'_A$.*

Proof. As $A \in \Omega_n \setminus \mathcal{C}_n$, at least two of the eigenvalues of A are equal, for example to λ. Applying Φ_λ (see (2.2.3)) and Ψ_P (see (2.9.1)) we can assume that $\lambda = 0$ and that A is in Jordan form. In particular,

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix},$$

where $A_0 \in \mathcal{M}_m$, $2 \le m \le n$, $\mathrm{sp}(A_0) = \{0\}$, $A_1 \in \mathcal{M}_{n-m}$, $0 \notin \mathrm{sp}(A_1)$.

Further, there exists a set $J \subsetneq \{2,\ldots,m\}$, possibly empty, such that $a_{j-1,j} = 1$ for $j \in J$, and all other elements a_{ij} are equal to 0 for $1 \le i,j \le m$. Put $0 \le r = \#J = \mathrm{rank}\, A_0 \le m-2$. Let

$$B = \begin{pmatrix} B_0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_n,$$

where $B_0 = (b_{ij})_{1\le i,j\le m}$ so that $b_{j-1,j} = -1$ for $j \in \{2,\ldots,m\} \setminus J$, $b_{m1} = 1$, and $b_{ij} = 0$ otherwise.

As $\sigma_m/\binom{n}{m} \in \mathcal{O}(\Omega_n, \mathbb{D})$, it suffices to prove the following

LEMMA 2.10.3. $(\sigma_m)^*_A(B) = 1$, *but* $\sigma'_A(B) = 0$.

Proof. First let us calculate $\sigma_j(A_0 + hB_0)$, $1 \le j \le m$, $h \in \mathbb{C}$. By developing along the first column, we get

$$\det(tI - (A_0 + hB_0)) = t^m + (-1)^{m-1} h^{m-r}.$$

Equating the coefficients on both sides leads to

$$\sigma_j(A_0 + hB_0) = \begin{cases} 0, & 1 \le j \le m-1, \\ h^{m-r}, & j = m. \end{cases} \tag{2.10.1}$$

Now we need a general formula for σ_j. For a given matrix $M = (m_{ij})_{1\le i,j\le n}$ and a set $E \subset \{1, \dots, n\}$ denote by $\delta_E(M)$ the determinant of the matrix $(m_{ij})_{i,j\in E} \in \mathcal{M}_{\#E}$. For convenience put $\delta_\emptyset(M) = \sigma_0(M) := 1$. Then

$$\sigma_j(M) = \sum_{E\subset\{1,\dots,n\},\, \#E=j} \delta_E(M). \tag{2.10.2}$$

The block structure of our matrices implies that

$$\delta_E(A + hB) = \delta_{E\cap\{1,\dots,m\}}(A_0 + hB_0)\delta_{E\cap\{m+1,\dots,n\}}(A_1).$$

So

$$\begin{aligned} \sigma_j(A + hB) &= \sum_{\max(0,j-n+m)\le k\le\min(m,j)} \Big(\sum_{E'\subset\{1,\dots,m\},\#E'=k} \delta_{E'}(A_0 + hB_0) \Big) \\ &\qquad \times \Big(\sum_{E''\subset\{m+1,\dots,n\},\#E''=j-k} \delta_{E''}(A_1) \Big) \\ &= \sum_{\max(0,j-n+m)\le k\le\min(m,j)} \sigma_k(A_0 + hB_0)\sigma_{j-k}(A_1). \end{aligned}$$

By (2.10.1) we get $\sigma_j(A + hB) = S_1 + S_2$, where

$$S_1 = \begin{cases} \sigma_j(A_1), & j \le n-m, \\ 0, & \text{otherwise}, \end{cases} \qquad S_2 = \begin{cases} h^{m-r}\sigma_{j-m}(A_1), & j \ge m, \\ 0, & \text{otherwise}. \end{cases}$$

In particular,

$$\sigma_j(A) = \begin{cases} \sigma_j(A_1), & j \le n-m, \\ 0, & \text{otherwise}. \end{cases}$$

Then

$$\sigma_j(A + hB) - \sigma_j(A) = \begin{cases} h^{m-r}\sigma_{j-m}(A_1), & j \ge m, \\ 0, & \text{otherwise}. \end{cases}$$

As $m - r \ge 2$, we get $\sigma'_A(B) = 0$, but $(\sigma_m)^*_A(B) = 1$. ∎

The main corollary from Proposition 2.10.2 and the implication preceding it is that SCFP does not depend continuously on the data (so it cannot be reduced to a similar problem on the symmetrized polydisc).

COROLLARY 2.10.4. *If* $A \in \Omega_n \setminus \mathcal{C}_n$ *and* $B \in \ker \sigma'_A \setminus C'_A$, *then*

$$\kappa_{\Omega_n}(A; B) > 0 = \lim_{\mathcal{C}_n \ni A' \to A} \kappa_{\Omega_n}(A'; B).$$

When A is a scalar matrix, we know more (cf. Proposition 2.9.1):

PROPOSITION 2.10.5. *For $B \in \mathcal{M}_n$ and $t \in \mathbb{D}$ the following are equivalent:*

(i) *the eigenvalues of B are all equal;*
(ii) *κ_{Ω_n} is continuous at $(tI_n; B)$;*
(iii) *$\kappa_{\Omega_n}(\cdot; B)$ is continuous at tI_n.*

Proof. The implication (ii)⇒(iii) is trivial. For the rest of the proof we can assume that $t = 0$ (applying Φ_t).

We will now prove that (i)⇒(ii). Let the eigenvalues of B be equal to 0. If $A_j \to 0$ and $B_j \to B$, then

$$\begin{aligned}\kappa_{\Omega_n}(A_j; B_j) &\geq \kappa_{\mathbb{G}_n}(\sigma(A_j); \sigma'_{A_j}(B_j)) \to \kappa_{\mathbb{G}_n}(0; \sigma'_0(B))\\ &= \kappa_{\mathbb{G}_n}(0; (\operatorname{tr} B)e_1) = |b| = \kappa_{\Omega_n}(0; B)\end{aligned}$$

(the last two equalities follow from Theorem 2.7.3(i) and (2.2.6), respectively).

Thus the function κ_{Ω_n} is lower semicontinuous at $(0; B)$. As it is (always) upper semicontinuous, it is continuous at this point.

It remains to prove that (iii)⇒(i). As $\mathcal{C}_n$ is a dense subset of Ω_n, we can find a sequence $\mathcal{C}_n \supset (A_j) \to 0$. Then (2.2.6), (2.2.5) and Theorem 2.7.3(i) imply that

$$r(B) = \kappa_{\Omega_n}(0; B) \leftarrow \kappa_{\Omega_n}(A_j; B) = \kappa_{\mathbb{G}_n}(\sigma_(A_j); \sigma'_{A_j}(B_j)) \to \kappa_{\mathbb{G}_n}(0; \sigma'_0(B)) = |\operatorname{tr} B|/n.$$

So $r(B) = |\operatorname{tr} B|/n$, i.e. the eigenvalues of B are equal. ∎

Now let us formulate the following hypothesis for the zeroes of κ_{Ω_n}.

HYPOTHESIS. $\kappa_{\Omega_n}(A; B) = 0$ if and only if there exists a $\varphi \in \mathcal{O}(\mathbb{C}, \Omega_n)$ so that $\varphi(0) = A$ and $\varphi'(0) = B$. In particular, if $\kappa_{\Omega_n}(A; B) = 0$, then $B \in C_A$.

Note that there are matrices $B \in C_A$ such that $\kappa_{\Omega_n}(A; B) \neq 0$ (see Proposition 2.10.6(ii) and Corollary 2.10.7).

In some cases the above hypothesis can be checked.

The remarks at the beginning of this section imply that this hypothesis is true for cyclic matrices.

Also, as the zeroes of $\kappa_{\Omega_n}(0; \cdot)$ are exactly the zero-spectrum matrices and this set of matrices is a union of complex lines through the origin, the hypothesis is true for scalar matrices.

As the noncyclic matrices A in Ω_2 are only the scalar ones, we can choose $m = 1$ in Theorem 2.10.1 for $n = 2$; then C_A coincides with the zeroes of $\kappa_{\Omega_2}(A; \cdot)$, as well as with the set of matrices $B = \varphi'(0)$ for some entire curve φ in Ω_2. (On the other hand, $\ker \sigma'_A = \{B \in \mathcal{M}_2 : \operatorname{tr} B = 0\}$.) So we have a complete description of the set of zeroes of κ_{Ω_2} and the above hypothesis is true for $n = 2$.

Now let us consider the set of zeroes of $\kappa_{\Omega_3}(A; \cdot)$, when A is a noncyclic and nonscalar matrix (we will confirm the hypothesis for $n = 3$, too). The use of the automorphisms Φ_λ and Ψ_P of Ω_3 reduces the problem to the following two cases:

$$A = A_t := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t \end{pmatrix}, \quad t \in \mathbb{D}_*, \qquad A = \tilde{A} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easily seen that

$$C'_{A_t}\subset C''_{A_t}:=\{B\in\mathcal{M}_3:\sigma^*_{A_t}(B)\}=\{B\in\mathcal{M}_3:b_{33}=b_{11}+b_{22}=b_{11}^2+b_{12}b_{21}=0\},$$
$$C'_{\tilde A}\subset C''_{\tilde A}:=\{B\in\mathcal{M}_3:\sigma^*_{\tilde A}(B)\}=\{B\in\mathcal{M}_3:b_{11}+b_{22}+b_{33}=b_{32}=b_{12}b_{31}=0\}$$

(for example, to check the second equality, we observe that if $B_\varepsilon=\tilde A+\varepsilon B+o(\varepsilon)$, then $\operatorname{tr}B_\varepsilon=\varepsilon\operatorname{tr}B+o(\varepsilon)$, $\sigma_2(B_\varepsilon)=-\varepsilon b_{32}+o(\varepsilon)$ and $\det B_\varepsilon=\varepsilon^2(b_{12}b_{31}-b_{11}b_{32})+o(\varepsilon^2)$). As the tangent cones are closed, the next proposition shows in particular that $C_{A_\lambda}=C'_{A_\lambda}=C''_{A_\lambda}$ and $C_{\tilde A}=C'_{\tilde A}=C''_{\tilde A}$.

Proposition 2.10.6.

(i) *If $B\in C''_{A_t}$ $(t\neq 0)$, then there exists a $\varphi\in\mathcal{O}(\mathbb{C},\Omega_3)$ such that $\varphi(0)=A_t$ and $\varphi'(0)=B$.*
(ii) *Let $B\in C''_{\tilde A}$. Then there exists a $\varphi\in\mathcal{O}(\mathbb{C},\Omega_n)$ so that $\varphi(0)=\tilde A$ and $\varphi'(0)=B$ only if $b_{11}=0$ and $b_{12}\neq b_{31}$. Otherwise $\kappa_{\Omega_3}(\tilde A;B)=1$.*

As $\kappa_{\Omega_3}(A;B)>0$ for $B\notin C'_A$, this proposition and the remarks preceding it give a complete description of the set of zeroes of κ_{Ω_3}, thereby confirming the hypothesis for $n=3$.

Proof. (i) Let us first $B\in C'_{A_t}$. We express B in the form $B=X+[Y,A_t]$, where X is such that $\psi(\zeta)=A_t+\zeta X\in L_{A_t}$ for each $\zeta\in\mathbb{C}$. Then $\varphi(\zeta)=e^{\zeta Y}\psi(\zeta)e^{-\zeta Y}$ has the required properties.

It is easily calculated that $\psi(\mathbb{C})\subset L_{A_t}$ exactly when $\operatorname{sp}(X)=0$ and $x_{11}+x_{22}=x_{11}^2+x_{12}x_{21}=0$. On the other hand,

$$[Y,A_t]=t\begin{pmatrix}0&0&y_{13}\\0&0&y_{23}\\-y_{31}&-y_{32}&0\end{pmatrix}.$$

So we can choose

$$X=\begin{pmatrix}b_{11}&b_{12}&0\\b_{21}&b_{22}&0\\0&0&0\end{pmatrix},\qquad Y=t^{-1}\begin{pmatrix}0&0&b_{13}\\0&0&b_{23}\\-b_{31}&-b_{32}&0\end{pmatrix}.$$

(ii) Let first $B\in\mathbb{C}'_{\tilde A}$. If $b_{11}=0$ or $b_{12}\neq b_{31}$, it suffices to find (as above) X and Y so that $B=X+[Y,\tilde A]$ and $\tilde A+\zeta X\in L_{\tilde A}$ for each $\zeta\in\mathbb{C}$. The last condition means that the eigenvalues of X are zeroes and $x_{32}=x_{12}x_{31}=0$. On the other hand,

$$[Y,\tilde A]=\begin{pmatrix}0&0&y_{12}\\-y_{31}&-y_{33}&y_{22}-y_{33}\\0&0&y_{32}\end{pmatrix}.$$

Suppose that $b_{31}=0$ (when $b_{12}=0$ the calculations are analogous). We have to choose X of the form

$$X=\begin{pmatrix}b_{11}&b_{12}&b_{13}-y_{12}\\b_{21}+y_{31}&b_{22}+y_{32}&b_{23}-y_{22}+y_{33}\\0&0&-b_{11}-b_{22}-y_{32}\end{pmatrix}$$

so that $\det X = 0$ and $\sigma_2(X) = 0$, i.e. $DT = 0$ and $D = T^2$, where

$$D = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} + y_{31} & b_{22} + y_{32} \end{vmatrix}, \qquad T = b_{11} + b_{22} + y_{32}.$$

These two conditions are true only if

$$y_{32} = -b_{11} - b_{22}, \quad y_{31} = \begin{cases} -b_{21}, & b_{11} = 0, \\ -b_{21} - b_{11}^2/b_{12}, & b_{12} \neq 0. \end{cases}$$

It remains to show that if $b_{11} \neq 0$ and $b_{12} = b_{31} = 0$, then $\kappa_{\Omega_3}(\tilde{A}; B) = 1$. We can assume that $b_{11} = 1$. Put $\tilde{B} = \operatorname{diag}(1, e^{2\pi i/3}, e^{4\pi i/3}$. As above, we can choose $\tilde{B}$ and Y so that $B = \tilde{B} + [Y, A_t]$. Let $\alpha > 0$ and $\varphi \in \mathcal{O}(\alpha\mathbb{D}, \Omega_3)$ be such that $\varphi(0) = A_t$ and $\varphi'(0) = B$. Putting $\tilde{\varphi}(\zeta) = e^{-\zeta Y}\varphi(\zeta)e^{\zeta Y}$, we have $\tilde{\varphi} \in \mathcal{O}(\alpha\mathbb{D}, \Omega_3)$, $\tilde{\varphi}(0) = \tilde{A}$ and $\tilde{\varphi}'(0) = \tilde{B}$. So $\kappa_{\Omega_3}(\tilde{A}; B) \geq \kappa_{\Omega_3}(\tilde{A}; \tilde{B})$. The converse inequality follows similarly. It remains to apply Proposition 2.11.2 from the next section. ∎

Corollary 2.10.7. *For each $n \geq 3$ there exist $A \in \Omega_n$ and $B \in C_A$ so that $\kappa_{\Omega_n}(A; B) > 0$.*

Proof. Put

$$\tilde{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{B}_\varepsilon = \begin{pmatrix} 1 & \varepsilon & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} \tilde{A} & O \\ O & O \end{pmatrix}, \qquad B_\varepsilon = \begin{pmatrix} \tilde{B}_\varepsilon & O \\ O & O \end{pmatrix}.$$

As in the proof of Proposition 2.10.6(ii), it follows that

- $\kappa_{\Omega_n}(A; B_0) > 0$;
- for $\varepsilon \neq 0$ there exists $\varphi_\varepsilon \in \mathcal{O}(\mathbb{C}, \Omega_n)$ such that $\varphi_\varepsilon(0) = A$ and $\varphi'_\varepsilon(0) = B_\varepsilon$.

Then $B_\varepsilon \in C_A$, $\varepsilon \neq 0$, so $B_0 \in C_A$. ∎

2.11. The Kobayashi metric vs. the Lempert function. As an application of part of the above considerations, in this section we will provide an example showing that, in general, the Kobayashi pseudometric of a pseudoconvex domain is not equal to the weak "derivative" of the Lempert function. The pseudoconvex domain will be the spectral ball $\Omega_3 \subset \mathbb{C}^9$ (that is also a balanced nontaut unbounded domain).

Recall that the Kobayashi metric of a taut domain $D \subset \mathbb{C}^n$ coincides with the "derivative" of the Lempert function (see Section 1.2):

$$\kappa_D(z; X) = \lim_{t\to 0,\, z'\to z,\, X'\to X} \frac{l_D(z', z' + tX')}{|t|}.$$

On the other hand, Proposition 1.2.3 states that

$$\kappa_D(z; X) \geq \mathcal{D}l_D(z; X) := \limsup_{t\to 0,\, z'\to z,\, X'\to X} \frac{l_D(z', z' + tX')}{|t|} \tag{2.11.1}$$

for an arbitrary domain $D \subset \mathbb{C}^n$.

The aim of this section is to show that the inequality

$$\kappa_D(z; X) \geq \widetilde{\mathcal{D}}l_D(z; X) := \limsup_{t\to 0} \frac{l_D(z, z + tX)}{|t|} \tag{2.11.2}$$

is strict in the general case (of a pseudoconvex domain). Put

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 3t & \omega^2 \end{pmatrix}.$$

where $\omega = e^{2\pi i/3}$. Let $B = B_0$.

PROPOSITION 2.11.1. *The following inequality holds:*

$$\kappa_{\Omega_3}(A;B) > 0 = \widetilde{\mathcal{D}}l_{\Omega_3}(A;B).$$

Moreover, if $t_j \to 0$ *and* $C_j \to B$ ($C_j = (c^j_{k,l})$) *so that* $\liminf_{j\to\infty} |c^j_{3,2}/t_j - 3| > 0$, *then*

$$\lim_{j\to\infty} \frac{l_{\Omega_3}(A, A + t_j C_j)}{|t_j|} = 0.$$

REMARK. As κ_D and l_D have the product property (see Section 1.7), in general the inequality (2.11.2) is strict for pseudoconvex domains in $\mathbb{C}^n$ for $n \geq 9$ (for example for $\Omega_3 \times \mathbb{D}^k$). In fact the proof below shows that $\widetilde{\mathcal{D}}l_{\widetilde{\Omega}_3}(A;B) = 0$, where $\widetilde{\Omega}_3$ is the set of zero-trace matrices in Ω_3. Consequently, the inequality in (2.11.2) is strict for the pseudoconvex domain $\widetilde{\Omega}_3 \subset \mathbb{C}^8$.

QUESTION. It would be interesting to find an example of a lower dimension, as well as to see whether in general the inequality (2.11.1) is strict (the last question was posed at the end of Section 1.2).

Recall that there exist matrices $\widetilde{B} \to B$ so that $\kappa_{\Omega_3}(A;\widetilde{B}) = 0$ (see Proposition 2.10.6 (ii)); in particular, the function $\kappa_{\Omega_3}(A;\cdot)$ is not continuous at B.

Also note that the condition $\liminf_{j\to\infty} |c^j_{3,2}/t_j - 3| > 0$ in Proposition 2.11.1 is essential, as seen by the following

PROPOSITION 2.11.2. $\kappa_{\widetilde{\Omega}_3}(A;B) = \lim_{t\to 0} l_{\Omega_3}(A, A + tB_t)/|t| = 1$. *In particular,*

$$1 = \kappa_{\widetilde{\Omega}_3}(A;B) = \kappa_{\Omega_3}(A;B) = \mathcal{D}l_{\widetilde{\Omega}_3}(A;B) = \mathcal{D}l_{\Omega_3}(A;B).$$

For the proof of Proposition 2.11.1 we will use the following special case of [115, Proposition 4.1, Corollary 4.3] (see also [93] for more general facts).

LEMMA 2.11.3. *Let* $M \in \Omega_3$ *is a cyclic matrix and* $\varphi \in \mathcal{O}(\mathbb{D}, \mathbb{G}_3)$ *be a mapping such that* $\varphi(0) = 0$ *and* $\varphi(\alpha) = \sigma(M)$ ($\alpha \in \mathbb{D}$). *Then there exists a* $\psi \in \mathcal{O}(\mathbb{D}, \Omega_3)$ *such that* $\psi(0) = A$, $\psi(\alpha) = M$ *and* $\varphi = \sigma \circ \psi$ *exactly when* $\varphi_3'(0) = 0$. *In particular,*

$$l_{\Omega_3}(A, M) = \inf\{|\alpha| : \exists \varphi \in \mathcal{O}(\mathbb{D}, \mathbb{G}_3) : \varphi(0) = 0,\ \varphi(\alpha) = \sigma(M),\ \varphi_3'(0) = 0\}$$

and (as $\mathbb{G}_3$ *is a taut domain) there exists an extremal disc for* $l_{\Omega_3}(A, M)$.

Proof. If such a ψ exists, then one directly calculates $\varphi_3'(0) = (\sigma_3 \circ \psi)'(0) = 0$.

Conversely, let $\varphi_3'(0) = 0$. Put

$$\tilde{\psi}(\zeta) := \begin{pmatrix} 0 & \zeta & 0 \\ 0 & 0 & 1 \\ \varphi_3(\zeta)/\zeta & -\varphi_2(\zeta) & \varphi_1(\zeta) \end{pmatrix}, \quad \zeta \in \mathbb{D}.$$

Then $\tilde{\psi}(0) = A$ and $\varphi = \sigma \circ \tilde{\psi}$. Furthermore, $e_3 = (0, 0, 1)$ is a cyclic vector for $\tilde{\psi}(\zeta)$ when $\zeta \neq 0$. So $\tilde{\psi}(\alpha)$ is a cyclic matrix with the same spectrum as the cyclic matrix M and consequently they are similar (to their adjoint matrix) by Proposition 2.3.1. Then we can express M in the form $M = e^S \tilde{\psi}(\alpha) e^{-S}$ for some $S \in \mathcal{M}_3$. It remains to put $\psi(\zeta) = e^{\zeta S/\alpha} \tilde{\psi}(\zeta) e^{-\zeta S/\alpha}$. ∎

Proof of Proposition 2.11.1. By Proposition 2.11.2 we only need to check that

$$\lim_{j\to\infty} l_{\Omega_3}(A, A + t_j C_j)/|t_j| = 0$$

under the conditions for $c^j_{3,2}$.

Suppose the contrary. Then we may assume that

$$l_{\Omega_3}(A, A + t_j C_j)/|t_j| \to a > 0.$$

Step 1. Suppose that there exists a subsequence (not relabeled) such that all matrices $A + t_j C_j$ are cyclic and belong to Ω_3. By some calculations we get

$$\sigma(A + t_j C_j) = (t_j f_1(C_j), t_j f_2(C_j), t_j^2 f_3(C_j)) =: (a_j, b_j, c_j),$$

where $f_1(C_j) \to 0$, $f_2(C_j) \to 0$ and $f_3(C_j) \to 0$.

Put

$$\varphi_j(\zeta) = (\zeta a_j/r_j, \zeta b_j/r_j, \zeta^2 c_j/r_j^2), \quad \zeta \in \mathbb{D},$$

where $r_j = \max\{3|a_j|, 3|b_j|, \sqrt{3|c_j|}\}$. Then $\varphi_j \in \mathcal{O}(\mathbb{D}, \mathbb{G}_3)$ with $\varphi_j(0) = 0$, $\varphi'_{j,3}(0) = 0$ and $\varphi_j(r_j) = \sigma(A + t_j C_j)$. Lemma 2.11.3 implies that

$$l_{\Omega_3}(A, A + t_j C_j)/|t_j| \le r_j/|t_j| \to 0,$$

a contradiction.

Step 2. Suppose that all matrices $A + t_j C_j$ are noncyclic. Then their minimal polynomials have degrees less than 3 (see Proposition 2.3.1). Consequently, these degrees are equal to 2 for all sufficiently large j. Hence

$$(A + t_j C_j)^2 + x_j(A + t_j C_j) + y_j I_3 = 0,$$

where $x_j, y_j \in \mathbb{C}$. We get nine equations (for the components); denote them by $E^j_{k,\ell}$, where k and ℓ are the indices of the row and the column, respectively. The equation $E^j_{2,3}$ gives $x_j/t_j \to 1$. Using this in $E^j_{1,1}$, we get $y_j/t_j^2 \to -2$. Finally, $E^j_{2,2}$ implies $c^j_{3,2}/t_j \to 2 - \omega - \omega^2 = 3$, a contradiction. ∎

Proof of Proposition 2.11.2. As $A + \zeta B \in \widetilde{\Omega}_3$ for each $\zeta \in \mathbb{D}$, we get $\kappa_{\widetilde{\Omega}_3}(A; B) \le 1$.

It remains to show that

$$\liminf_{t\to 0} l_{\Omega_3}(A, A + tB_t)/|t| \ge 1.$$

Note that $A + tB_t$ is similar to the matrix $D_t = \operatorname{diag}(t, t - 2t)$ and consequently $l_{\Omega_3}(A, A + tB_t) = l_{\Omega_3}(A, D_t)$ (we already applied this argument several times).

Suppose that $t_j \to 0$ so that $l_{\Omega_3}(A, D_{t_j})/|t_j| \to c < 1$.

Let $\psi_j \in \mathcal{O}(\mathbb{D}, \Omega_3)$ be a disc such that $\psi_j(0) = A$, $\psi(\alpha_j) = D_{t_j}$ and $|\alpha_j|/|t_j| \to c$. Put $\varphi_j = \sigma \circ \psi_j$. Direct calculations lead to $\varphi'_{j,3}(0) = 0$ and

$$\varphi'_{j,3}(\alpha_j) - t_j \varphi'_{j,2}(\alpha_j) + t_j^2 \varphi'_{j,1}(\alpha_j) = 0.$$

After expressing φ_j in the form

$$\varphi_j(\zeta) = (\zeta\theta_{j,1}(\zeta), \zeta\theta_{j,2}(\zeta), \zeta^2\theta_{j,3}(\zeta)),$$

the last equality becomes

$$t_j^3 = \alpha_j^2(\alpha_j\theta'_{j,3}(\alpha_j) - t_j\theta'_{j,2}(\alpha_j) + t_j^2\theta'_{j,1}(\alpha_j)) \tag{2.11.3}$$

(we use $\theta_{j,1}(\alpha_j) = 0$, $\theta_{j,2}(\alpha_j) = -3t_j^2/\alpha_j$ and $\theta_{j,3}(\alpha_j) = -2t_j^3/\alpha_j^2$). As $\mathbb{G}_3$ is a taut domain, by passing to subsequences we can assume that $\varphi_j \to \varphi = (\zeta\rho_1, \zeta^2\rho_2, \zeta^3\rho_3) \in \mathcal{O}(\mathbb{D}, \mathbb{G}_3)$ and $\rho_1(0) = 0$. Then (2.11.3) shows that if $k = 1/c$, then

$$\rho_3(0) = k^3 + k\rho_2(0).$$

Proposition 2.5.1 (see also [37, Proposition 16]) implies that

$$h_{\mathbb{G}_3}(z) = \max\{|\lambda| : \lambda^3 - z_1\lambda^2 + z_2\lambda - z_3 = 0\}$$

is a (logarithmically) plurisubharmonic function and $\mathbb{G}_3 = \{z \in \mathbb{C}^3 : h_{\mathbb{G}_3}(z) < 1\}$ (h_{G_3} is the Minkowski function of the $(1, 2, 3)$-balanced domain $\mathbb{G}_3$). As

$$|\zeta| h_{\mathbb{G}_3}(\rho_1(\zeta), \rho_2(\zeta), \rho_3(\zeta)) = h_{\mathbb{G}_3}(\varphi(\zeta)) < 1, \quad \zeta \in \mathbb{D},$$

the maximum principle for plurisubharmonic functions implies that $h_{\mathbb{G}_3}(\rho_1, \rho_2, \rho_3) \le 1$ on $\mathbb{D}$. In particular, $h_{\mathbb{G}_3}(\rho_1(0), \rho_2(0), \rho_3(0)) \le 1$. Consequently, all the three zeroes of the polynomial $P(\lambda) = \lambda^3 - \rho_1(0)\lambda^2 + \rho_2(0)\lambda - \rho_3(0)$ lie in $\overline{\mathbb{D}}$. On the other hand, $P(\lambda) = (\lambda - k)(\lambda^2 + k\lambda + k^2 + \rho_2(0))$ with $k > 1$, a contradiction. ∎

3. Estimates and boundary behavior of invariant metrics on $\mathbb{C}$-convex domains

3.1. Synopsis. The main purpose of this chapter is to obtain estimates (in a geometric way) for the Carathéodory, Kobayashi and Bergman metrics, as well as for the Bergman kernel (on the diagonal), of an arbitrary $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$ not containing complex lines, in terms of the distance $d_D(z; X)$ from the point $z \in D$ to the boundary ∂D in the direction $X \in (\mathbb{C}^n)_*$. These estimates show that on such a domain these three metrics coincide up to a constant, depending only on n (Corollary 3.4.2). Similar results in the special case of a $\mathcal{C}^\infty$-smooth bounded $\mathbb{C}$-convex domain of a finite type, with quite hard proofs, are the main results of the dissertations of S. Blumberg [12] and M. Lieder [71]. In addition, the constants there depend on the domain. Earlier similar results for convex domains can be found in the Ph.D. thesis [19] of J.-H. Chen and in the works [74, 75] of J. D. McNeal; however their proofs have some essential deficiencies.

Using the 1/4-Theorem of Koebe, it is easily shown in Proposition 3.2.2 that

$$1/4 \le \gamma_D(z; X)d_D(z; X) \le \kappa_D(z; X)d_D(z; X) \le 1.$$

The two (absolute) constants are exact, and 1/4 can be replaced by 1/2 for convex domains (see [8] or Proposition 3.2.1).

As an application of these estimates, in Section 3.3 we find that the standard and linear multitypes of D'Angelo and Catlin coincide for a smooth boundary point of a $\mathbb{C}$-convex domain. This generalizes a result of M. Conrad [23] and J. Yu [121] (see also the works of J. D. McNeal [73], and H. P. Boas and E. Straube [15]), while our proof is essentially different and much shorter. It is based upon an easy result from [123] and transferring the main result in [68] from the convex to the $\mathbb{C}$-convex case (the considerations here are easier than those in [68]).

The main result of Chapter 3, Theorem 3.4.1, states that there is an inequality for the Bergman metric, similar to the above one; the corresponding constants depend only on the dimension n of the domain. To prove this, in Theorem 3.2.4 we get some estimates for the Bergman kernel, which are also of independent interest. The constants there depend only on n and are exact for the class of convex domains. These estimates are connected with the so-called minimal basis (for a point in a given domain), introduced by T. Hefer [44] for the smooth case (of finite type) and somewhat later, but independently, by the author and P. Pflug [84] in the general case. It is used in the proof of Theorem 3.4.1 and almost all arguments are geometrical. One can define a minimal basis for a point in a given open set (not containing complex lines) by induction: the first vector of the base is directed towards the closest point from the set boundary, and the next ones are from the basis of the intersection of the set with the complex hyperplane through the point, orthogonal to that vector. The main (and trivial) property of that basis that is used for weakly linearly convex domains is the orthogonality of the intersections of complex "support" hyperplanes through the emerging boundary points, and the corresponding vectors form the basis. The geometrical arguments are completed by the stability of $\mathbb{C}$-convexity under projections.

In the previously mentioned works [19, 74, 75], apart from the $\overline{\partial}$-technique, the authors use a similar (however notably more complicated) method but another basis that we will call maximal. In Section 3.5 we provide a natural counterexample for the main "property" of this basis (the same as for the minimal one) that is used in those and other works for various problems (e.g. for the linear and D'Angelo types in the already cited paper [73]). Nevertheless, in Section 3.6 we show how the estimates obtained in the minimal basis imply those for the maximal one (using some combinatorial arguments).

Another aim of this chapter is to establish the local character of the results obtained by showing that the estimates near a given boundary point a of a domain remain true if the domain is weakly locally linearly convex near a and the boundary does not contain analytic discs through a. Such a domain with a $\mathcal{C}^2$-smooth boundary near a turns out to be locally $\mathbb{C}$-convex (Proposition 3.7.1). Then the local character of the estimates for the Kobayashi metric (if the domain is bounded) follows from the general localization proposition 3.7.5. Its proof permits one to obtain immediately the exact boundary behavior of this metric near an isolated point of a planar domain, having at least one more point on its boundary. This essentially strengthens the main result from [61].

The local character of the estimates for the Bergman kernel and Bergman metric is determined in Section 3.8, where the domain is assumed to be pseudoconvex (but not necessarily bounded) and locally convex around a boundary point not contained in analytic (or, equivalently, linear) discs from the boundary. The proof is based on the existence of a locally holomorphic peak function at this point (Proposition 3.8.8) and the localization theorem 3.8.3 for the Bergman kernel and Bergman metric (if such a function exists). In the case of a bounded pseudoconvex domain this theorem is contained in the fundamental work [49] of L. Hörmander as an application of the L^2-estimates for the $\overline{\partial}$-problem. Our proof is a variation of this technique. As a corollary we get a stronger variant of the main result of G. Herbort [47] without the use of the $\overline{\partial}$-technique of Ohsawa–Takegoshi (see the remark at the end of Section 3.8). The proof also implies weak localization of the Bergman kernel and Bergman metric for a planar domain with a nonpolar complement (Corollary 3.8.6).

In the last section we get the exact boundary behavior of the invariant metrics under consideration near a $\mathcal{C}^1$-smooth boundary point of an arbitrary planar domain, once again using a geometric argument (the Pinchuk scaling method).

3.2. Estimates for the Carathéodory and Kobayashi metrics. The aim of this section is to obtain estimates for the Kobayashi and Carathéodory metrics on $\mathbb{C}$-convex domains in terms of the distance to the boundary of the corresponding direction. These results generalize similar statements for bounded smooth $\mathbb{C}$-convex domains of finite type, whose original proofs are quite hard (see [71]).

For a point z from a domain $D \subset \mathbb{C}^n$ and a vector $X \in (\mathbb{C}^n)_*$, we denote by $d_D(z;X)$ the distance from z to ∂D in the direction of X, i.e.

$$d_D(z;X) = \sup\{r > 0 : \Delta_X(z,r) \subset D\}, \quad \text{where} \quad \Delta_X(z,r) = \{z + \lambda X : |\lambda| < r\}.$$

Clearly

$$\operatorname{dist}(z, \partial D) =: d_D(z) = \inf_{\|X\|=1} d_D(z;X).$$

If $d_D(z;X) = \infty$, i.e. D contains the line through z in the direction of X, then

$$\gamma_D(z;X) = \kappa_D(z;X) = 0.$$

First recall the following result for convex domains.

Proposition 3.2.1 ([8]). *Let $D \subset \mathbb{C}^n$ be a convex domain. If $d_D(z;X) < \infty$, then*

$$1/2 \le \gamma_D(z;X) d_D(z;X) = \kappa_D(z;X) d_D(z;X) \le 1.$$

Proof. The upper estimate holds for each domain D, as $\mathbb{D}_X(z, d_D(z,X)) \subset D$. For the lower estimate consider an (open) supporting half-space Π of D for a boundary point of the type $z + \lambda X$. Then

$$\gamma_D(z;X) \ge \gamma_\Pi(z;X) = \frac{\|X\|}{d_\Pi(z;X)} = \frac{\|X\|}{d_D(z;X)}.$$

It remains to note that the equality in the statement follows from the Lempert theorem (see e.g. [69, 70]). ■

The constants $1/2$ and 1 cannot be improved, as seen from the examples of a half-space and a ball.

Now we will establish a similar result for $\mathbb{C}$-convex domains.

Proposition 3.2.2. *Let $D \subset \mathbb{C}^n$ be a $\mathbb{C}$-convex domain. If $d_D(z;X) < \infty$, then*

$$1/4 \le \gamma_D(z;X) d_D(z;X) \le \kappa_D(z;X) d_D(z;X) \le 1.$$

The constant $1/4$ is the best possible in the plane, as seen in the example with the image $D = \mathbb{C} \setminus [1/4, \infty)$ of $\mathbb{D}$ for the Koebe transformation $z \mapsto z/(1+z)^2$.

Corollary 3.2.3. *For each $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$, we have $\kappa_D \le 4\gamma_D$.*

This is another argument supporting the hypothesis that $\kappa_D = \gamma_D$ for each $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$ (a weaker variant of [125, Problem 4′]; see the Introduction).

Proof of Proposition 3.2.2. We can assume that $\|X\| = 1$. Let l be the complex line through z with direction X, and $a \in l \cap \partial D$ so that $\|z-a\| = d_D(z; X)$. Consider a complex hyperplane H through a not intersecting D and denote by G the projection of D onto l in the direction of H. Note that G is a simply connected domain (see e.g. [5, Theorem 2.3.6] or [50, Theorem 2.3.6]), $a \in \partial G$ and $d_D(z; X) = \|z - a\| = d_G(z) := \operatorname{dist}(z, \partial G)$. It remains to apply the Koebe $1/4$-theorem to get

$$\gamma_D(z; X) \ge \gamma_G(z; 1) \ge \frac{1}{4d_G(z)}.$$

Indeed, if $f : \mathbb{D} \to G$ is a conformal mapping such that $f(0) = z$, by the Koebe theorem G contains the disc of center z and radius $|f'(0)|/4$. So $|f'(0)| \le 4d_G(z)$ and hence

$$1 = \gamma_{\mathbb{D}}(0; 1) = \gamma_G(f(0); f'(0)) = |f'(0)|\gamma_G(z; 1) \le 4d_G(z)\gamma_G(z; 1),$$

and the result follows. ∎

Recall that if a $\mathbb{C}$-convex domain in $\mathbb{C}^n$ contains a complex line, then it is linearly equivalent to the Cartesian product of $\mathbb{C}$ and a $\mathbb{C}$-convex domain in $\mathbb{C}^{n-1}$ (see Section 2.6).

In view of this it is natural to ask about the boundary behavior of the metrics in the directions for which there are (linear) discs in the boundary in these directions.

More precisely, for a boundary point a of a domain $D \subset \mathbb{C}^n$ we denote by L_a the set of all vectors $X \in \mathbb{C}^n$ such that there exists an $\varepsilon > 0$ so that $\partial D \supset \Delta_X(a, \varepsilon)$. The following result is an application of Proposition 3.2.2.

Proposition 3.2.4. *Let a be a boundary point of a $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$.*

(i) *We have*

$$\lim_{z \to a} \gamma_D(z; X) = \infty \qquad \text{locally uniformly in } X \notin L_a.$$

(ii) *If ∂D is $\mathcal{C}^1$-smooth at a, then L_a is a linear space. In addition, for each nontangent cone Λ with vertex a (7) we have*

$$\limsup_{\Lambda \ni z \to a} \kappa_D(z; X) < \infty \qquad \text{locally uniformly in } X \in L_a.$$

The proof of this proposition, as well as of a part of the next ones, will be based on the following geometrical property of weakly linearly convex domains (see also [126]).

Lemma 3.2.5. *Suppose that a weakly linearly convex domain $G \subset \mathbb{C}^n$ contains the n unit discs lying in the coordinate lines. Then G contains the convex hull of these discs, $E = \{z \in \mathbb{C}^n : \sum_{j=1}^n |z_j| < 1\}$.*

(7) $\Lambda = \{z \in \mathbb{C}^n : c\|z - a\| < |\mathrm{pr}_a(z)|\}$, where $c \in (0, 1)$ and pr_a is the projection onto the interior normal to ∂D at a.

Proof. For each $\varepsilon \in (0,1)$ there exists a $\delta > 0$ so that

$$X_\varepsilon = \bigcup_{j=1}^{n} \Big(\delta\mathbb{D} \times \cdots \times \delta\mathbb{D} \times \underbrace{\varepsilon\mathbb{D}}_{j\text{th place}} \times \delta\mathbb{D} \times \cdots \times \delta\mathbb{D}\Big) \subset G.$$

Note that $\widehat{X}_\varepsilon \subset G$, where $\widehat{X}_\varepsilon$ is the least linearly convex set that contains X_ε. In addition,

$$\widehat{X}_\varepsilon = \{z \in \mathbb{C}^n \mid \forall b \in \mathbb{C}^n : \langle z, b\rangle = 1 \ \exists a \in X_\varepsilon : \langle a, b\rangle = 1\}$$

(see e.g. [5, p. 7] or [50, Proposition 4.6.2]). Then $\widehat{X}_\varepsilon$ is a balanced domain and as it is linearly convex, it is convex (see Proposition 2.6.1). Consequently,

$$E_\varepsilon = \Big\{z \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j| < \varepsilon\Big\} \subset \widehat{X}_\varepsilon \subset G$$

and letting $\varepsilon \to 1$ we get the desired proposition. ■

Remark. The same arguments show that G contains the convex hull of each of its balanced subdomains. In particular, the maximal balanced subdomain of G is convex (see also [126]).

Proof of Proposition 3.2.4. (i) Assuming the contrary, we can find $r > 0$ and sequences $D \ni z_j \to a$ and $\mathbb{C}^n \ni X_j \to X \notin L_a$ such that $4r\gamma_D(z_j; X_j) \le 1$. By Proposition 3.2.2, $d_D(z_j; X_j) \ge r$. Then $\Delta_{X_j}(z_j, r) \subset D_r = D \cap \mathbb{B}_n(a, 2r)$ for each sufficiently large j. Note that D_r is a (weakly) linearly convex open set. By Proposition 2.6.3, it is taut. Therefore $\Delta_X(a, r) \subset \partial D$, a contradiction.

(ii) As ∂D is $\mathcal{C}^1$-smooth, for any two linearly independent vectors $X, Y \in L_a$ one can find a neighborhood U of a and a number $\varepsilon > 0$ so that $\Delta_X(z, \varepsilon) \subset D$ and $\Delta_Y(z, \varepsilon) \subset D$ for $z \in D \cap U \cap \Lambda$. By Lemma 3.2.5, $\Delta_{X+Y}(z, \varepsilon') \subset D$ for some $\varepsilon' > 0$. As in (i) we get $\Delta_{X+Y}(a, \varepsilon') \subset \partial D$. Consequently, L_a is a linear space. Then, choosing a basis for L_a and applying Lemma 3.2.5, we find a neighborhood U of a and a number $c > 0$ so that $\Delta_X(z, c) \subset D$ for each $z \in D \cap U \cap \Lambda$ and each unit vector $X \in L_a$. Now the required estimate follows from Proposition 3.2.2.

Remark. The smoothness condition is redundant if D is a convex domain. Indeed, in this case it is clear that L_a is a linear space. Also, if $\Delta \subset \partial D$, then for each $b \in D$ and each $t \in (0,1]$ we have $tb + (1-t)\Delta \subset D$. So we can replace Λ by an arbitrary cone with vertex a having as base an arbitrary compact set of D.

3.3. Types of boundary points. The aim of this section is to find estimates on the behavior of invariant metrics of $\mathbb{C}$-convex domains near a boundary point depending on the multitype of this point.

Let a be a ($\mathcal{C}^\infty$-)smooth boundary point of a domain $D \subset \mathbb{C}^n$. Denote by m_a the (D'Angelo) *type* of a, i.e. the maximal order of tangency of ∂D at a with (nontrivial) analytic discs through a (see e.g. the Ph.D. thesis [79] of the author; we will refer to it several times in this chapter):

$$m_a = \sup_\gamma \frac{\operatorname{ord}_a(r \circ \gamma)}{\operatorname{ord}_a \gamma},$$

where γ varies over all analytic discs through a, while r is a smooth defining function of D near a (this definition depends on r). By requiring $\gamma^{\operatorname{ord}_a \gamma}(a) = X$, for a given vector $X \in (\mathbb{C}^n)_*$, we define the number $m_{a,X}$.

The point a is said to be *of finite type* if $m_a < \infty$. A bounded domain D is said to be of finite type if all its boundary points are of finite type.

Replacing the analytic discs by complex lines, we define the *linear type* l_a of a. We can also define the number $l_{a,X}$ as the order of tangency of ∂D at a to the line through a in the direction of X.

Then $l_{a,X} \le m_{a,X}$ and $l_a \le m_a$. Note that if $l_{a,X} < \infty$, then $X \notin L_a$.

Proposition 3.3.1. *Let a be a smooth boundary point of a $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$ and let $X \in (\mathbb{C}^n)_*$ so that $l_{a,X} < \infty$. Denote by n_a the interior normal to ∂D at a. Then there exists a neighborhood U_X of a and a constant $c_X \ge 1$ so that*

$$c_X^{-1} d_D(z) \le d_D(z;X)^{l_{a,X}} \le c_X d_D(z), \quad z \in D \cap U_X \cap n_a.$$

Proof. We can assume that $\operatorname{Re} z_1 < 0$ is the interior normal to ∂D at $a = 0$. Let $r(z) = \operatorname{Re} z_1 + o(|z_1|) + \rho('z)$ be a smooth defining function of D near 0.

For each sufficiently small $\delta > 0$ we have $\delta = d_D(\delta_n)$, where $\delta_n = (-\delta,' 0)$. Put $L_\delta(\zeta) = -\delta_n + \zeta X$, $\zeta \in \mathbb{C}^n$.

We consider two cases.

1. $l_{a,X} = 1$. This means that $X_1 \neq 0$. Then $r(L_\delta(\zeta)) = -\delta + \operatorname{Re}(\zeta X_1) + o(|\zeta|)$. Consequently, $L_\delta(\zeta) \in D$ if $|\zeta| < \delta/(2|X_1|)$ and δ is sufficiently small. This proves the left inequality.

The right inequality follows from the inequality $r(L_\delta(2\delta/X_1)) > 0$, which holds for each small $\delta > 0$.

2. $l_{a,X} \ge 2$. This means that $X_1 = 0$. Then $r(L_\delta(\zeta)) = -\delta + \rho(\zeta' X)$. As $\rho(\zeta' X) \le c|\zeta|^l$ for some $c_1 > 0$, we get $L_\delta(\zeta) \in D$, if $c_1|\zeta|^l < \delta$. This proves the left inequality.

To prove the right inequality, we need to find a $c_2 > 0$ so that for each small $\delta > 0$ there exists ζ such that $|\zeta|^l = c_2^{-1}\delta$ and $\rho(\zeta' X) \ge \delta$. As D is a (weakly) linearly convex domain, it follows that $\rho(\zeta' X) = h(\zeta) + o(|\zeta|^l) \ge 0$, where

$$h(\zeta) = \sum_{j+k=l} a_{jk} \zeta^j \overline{\zeta}^k \not\equiv 0.$$

Homogeneity of h implies $h \ge 0$. In addition, as $h \not\equiv 0$, we can find ζ so that $|\zeta| = 1$ and $h(\zeta) > c_2 > 0$. Now the constant c_2 has the required properties for all small $\delta > 0$. ∎

Combining Propositions 3.2.2 and 3.3.1, we directly get the following generalization (in an easy way) of the main result in [68] that deals with convex domains.

Corollary 3.3.2. *In the notation of Proposition 3.3.1, if $z \in D \cap U_X \cap n_a$, then*

$$(4c_X)^{-1}(d_D(z))^{-1/l_{a,X}} \le \gamma_D(z;X) \le \kappa_D(z;X) \le c_X(d_D(z))^{-1/l_{a,X}}.$$

The main result in [73] (see also [15]) states that $m_a = l_a$ for each convex domain. As an application of Corollary 3.3.2 we will easily show something more, even for an arbitrary $\mathbb{C}$-convex domain.

Proposition 3.3.3. *If a is a smooth boundary point of a $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$, then $m_{a,X} = l_{a,X}$ for each vector $X \neq 0$. In particular, $m_a = l_a$.*

Proof. It suffices to prove that $m_{a,X} \le l_{a,X}$ if $l_{a,X} < \infty$. By Corollary 3.3.2 we have

$$\limsup_{D\cap n_a \ni z\to a} \kappa^s_D(z;X)d^{1/l_{a,X}} \ge \liminf_{D\cap n_a \ni z\to a} \gamma_D(z;X)d^{1/l_{a,X}} > 0$$

(see Section 2.10 for the definition of κ^s_D) and the desired inequality follows from [123, Corollary]. ∎

The following result is important if the boundary is not real-analytic near a boundary point of infinite type.

Proposition 3.3.4. *If a is a $\mathcal{C}^1$-smooth boundary point of a $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$, then ∂D does not contain analytic discs through a exactly when $L_a = \{0\}$ (i.e. ∂D does not contain linear discs through a).*

Proof. We use the notation from the proof of Proposition 3.3.1. It suffices to show that if $\varphi : \mathbb{D} \to \partial D$ is an analytic disc for which $\varphi(0) = 0$, then $L_a \neq \{0\}$. As ∂D is $\mathcal{C}^1$-smooth near a, there exists $c > 0$ so that $\varphi_\delta(\zeta) = -\delta_n + \varphi(\zeta) \in D$ for $\delta < c$ and $|\zeta| < c$. Put $m = \operatorname{ord}_0 \varphi$ and $X = \varphi^{(m)}(0)/m!$. Then $\gamma_D(\delta_n;X) \le \kappa^s_D(\delta_n;X) \le 1/c$ and as in the proof of Proposition 3.2.4 it follows that $\Delta_X(a, c/4) \subset \partial D$. ∎

Remark. In the case of a convex domain the smoothness condition is redundant, as seen in the argument of the last remark in the previous section.

Now we will discuss the so-called multitypes of a smooth boundary point a of a domain $D \subset \mathbb{C}^n$. For each $k = 1, \dots, n$ put

$$m^k_a = \inf_L \sup_\gamma \frac{\operatorname{ord}_a(r\circ\gamma)}{\operatorname{ord}_a \gamma},$$

where S varies over all hyperplanes through a with dimension k, while γ varies over all analytic discs in S that pass through a (see e.g. [79]). By replacing the analytic discs by complex lines we define l^k_a. For $k = n$ these numbers coincide with m_a and l_a, respectively. Clearly $l^1_a = m^1_a = 1$ and $l^k_a \le m^k_a$. The *D'Angelo multitype* of $a \in \partial D$ is defined as the nondecreasing n-tuple of numbers $M_a = (m^1_a, \dots, m^n_a)$. The *D'Angelo linear type* L_a is defined in a similar way. We can also define the *Catlin multitype* $\tilde{M}_a = (\tilde{m}^1_a, \dots, \tilde{m}^n_a)$ and the Catlin linear multitype $\tilde{L}_a = (\tilde{l}^1_a, \dots, \tilde{l}^n_a)$ (see e.g. [121]). Note that

$$l^k_a \le \tilde{l}^n_a \le \tilde{m}^k_a \le m^k_a.$$

The main result of [121] states that $\tilde{L}_a = M_a$ (and so $= \tilde{M}_a$) for each convex domain. Using [121] and other nontrivial facts, in [23] this equality is proven for $\mathbb{C}$-convex domains.

As a corollary of Proposition 3.3.3 (that we proved easily), we can get the above results and even strengthen them a bit.

Proposition 3.3.5. *If a a is smooth boundary point of the $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$, then $L_a = M_a$.*

Proof. We can assume that $a = 0$. We have to show that $l^k_0 \ge m^k_0$ if $l^k_0 < \infty$ and $k > 1$. Let l^k_0 be attained for some S and a line $s \in S$. If S is orthogonal to the complex normal N_0 to ∂D at 0, we consider the subspace S' generated by N_0 and a subspace of S of codimension 1, containing s. Then $D_k = D \cap S' \subset \mathbb{C}^k$ is a $\mathbb{C}$-convex domain, which is smooth near 0. Let $m_{0,k}$ and $l_{0,k}$ be the type and the linear type of the point $0 \in \partial D_k$, respectively. Then $l_{0,k} = l_0^k$, as if a line $s' \subset S'$ is not orthogonal to N_0, then $\operatorname{ord}_0(r \circ s') = 1 \le l_0^k$. It remains to use that $m_0^k \le m_{0,k}$ and $m_{0,k} = l_{0,k}$ by Proposition 3.3.3. ∎

Let us mention that a pseudoconvex point a of finite type for which $\tilde{M}_a = M_a$, is called *semiregular* (see [31]). Thus each smooth point of finite type of a $\mathbb{C}$-convex domain is semiregular.

3.4. Estimates for the Bergman kernel and the Bergman metric. In this section we will prove some estimates for the Bergman kernel and the Bergman metric of a $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$ not containing complex lines. The constants in these estimates depend only on n. The estimate for the Bergman metric is in the spirit of those for the Carathéodory and Kobayashi metrics from Section 3.2. As a corollary we find that these three metrics are comparable with constants depending only on n.

First recall the definitions of the Bergman kernel and Bergman metric for a domain $D \subset \mathbb{C}^n$. For these and other basic facts see e.g. [58].

Denote by $L_h^2(D)$ the Hilbert space of square-integrable holomorphic functions f in D. This space has a (unique) reproducing kernel $\widetilde{K}_D(z,w)$, the *Bergman kernel*. For brevity, its restriction $K_D(z) = \widetilde{K}_D(z,z)$ to the diagonal is also called the Bergman kernel; further we will mainly work with K_D. It is well-known that K_D is a solution to the following extremal problem:

$$K_D(z) = \sup\{|f(z)|^2 : f \in L_h^2(D),\ \|f\|_D \le 1\},$$

where $\|\cdot\|$ is the L^2-norm. If $K_D(z) > 0$ for some $z \in D$, then the quadratic form

$$\sum_{j,k=1}^n \frac{\partial^2}{\partial z_j \partial \overline{z_k}} \log K_D(z) X_j \overline{X_k}, \quad X \in \mathbb{C}^n,$$

is positive semidefinite and its square root $B_D(z;X)$ is called the *Bergman metric*. It also solves an extremal problem:

$$B_D(z;X) = \frac{M_D(z;X)}{\sqrt{K_D(z)}},$$

where $M_D(z;X) = \sup\{|f_z'(X)| : f \in L_h^2(D),\ \|f\|_D = 1,\ f(z) = 0\}$.

Recall that the Carathéodory metric does not exceed the Bergman metric (if the latter is defined):

$$\gamma_D \le B_D.$$

There are the following transformation rules for the Bergman kernel and the Bergman metric: if $f : G \to D$ is a biholomorphism between domains in $\mathbb{C}^n$, then

$$K_D(f(z), f(w)) \operatorname{Jac} f(z) \overline{\operatorname{Jac} f(w)} = K_G(z,w), \qquad B_D(f(z); f_z'(X)) = B_G(z;X).$$

Note that unlike the Carathéodory and Kobayashi metrics, the Bergman metric is not monotone under domain inclusions. However, it is the quotient of two monotone invariants, M_D and K_D.

This will help us to attain the main goal of this section, namely to show the converse inequality to $\gamma_D \le B_D$ up to a constant depending only on n.

Theorem 3.4.1. *There exists a constant $c_n > 0$, depending only on n, so that for each $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$ not containing a complex line* (8)*, we have the inequality*

$$1/4 \le B_D(z;X)d_D(z;X) \le c_n.$$

By Propositions 3.2.1 and 3.2.3, and by the inequality $\gamma_D \le B_D$, we get

Corollary 3.4.2. *There exists a constant $c_n \ge 1$, depending only on n, so that for each $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$ not containing a complex line, we have*

$$\kappa_D/4 \le B_D \le c_n\gamma_D.$$

If D is a convex domain, then the constant 4 can be replaced by 1.

The first results, similar to Theorem 3.4.1 and to Theorem 3.4.3 below (for the Bergman kernel K_D), refer to bounded smooth convex domains of finite type [19, 74, 75]. Unfortunately the geometric construction there (see also [73]) has a flaw, as we will observe in the next section. These results are later proven for bounded smooth $\mathbb{C}$-convex domains of finite type [12] using a correct geometric construction from [44, 45, 23] and the paper [84] of the author and P. Pflug (see also [29]). Note that the constants in the corresponding estimates depend on the domains.

Now let us show the most general form of this construction.

Let $D \subset \mathbb{C}^n$ be a domain not containing a complex line. For a point z we choose $a^1 \in \partial D$ so that $d_1 := \|a^1 - z\| = d_D(z)$. Put $H_1 = z + \operatorname{span}(a^1 - z)^\perp$ and $D_1 = D \cap H_1$. Let $a^2 \in \partial D_1$ so that $d_2 := \|a^2 - z\| = d_{D_1}(z)$. Put $H_2 = z + \operatorname{span}(a^1 - z, a^2 - z)^\perp$, $D_2 = D \cap H_2$ and so on. Thus we get an orthonormal basis of the vectors $e_j = (a^j - z)/\|a^j - z\|$, $1 \le j \le n$, which will be called *minimal* (for D at z), and positive numbers $d_1 \le \cdots \le d_n$ (the basis and the numbers are not uniquely determined).

Put

$$p_D(z) = d_1 \ldots d_n.$$

The lower estimate for the Bergman kernel K_D via p_D in the next theorem is a main point in the proof of Theorem 3.4.1, but is also of independent interest.

Theorem 3.4.3. *Let $D \subset \mathbb{C}^n$ be a $\mathbb{C}$-convex domain not containing a complex line. Then*

$$\frac{1}{(16\pi)^n} \le K_D(z)p_D^2(z) \le \frac{(2n)!}{(2\pi)^n}.$$

In addition, the lower estimate is precise for $n = 1$, while the upper estimate is exact for each n (even for convex domains); the inequality is strict for $n \ge 2$.

In addition, if D is a convex domain not containing complex lines, the lower estimate can be improved by replacing the number 16 by 4. In this case the estimate is precise for each n.

(8) Under this assumption D is biholomorphic to a bounded domain (see Proposition 2.6.5), so B_D is defined.

Proof. *The upper estimate.* We can assume that $z = 0$. By Lemma 3.2.5,

$$D \supset G = \Big\{z \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|/d_j < 1\Big\}.$$

Consequently, $G \cup \mathbb{B}_n(0, d_1) \subset D$ and so

$$K_D(0) \le K_{G\cup\mathbb{B}_n(0,d_1)}(0) \le K_G(0) = K_E(0)/p_D^2(0),$$

where

$$E = \Big\{z \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j| < 1\Big\}$$

(here we applied the transformation rule for the Bergman kernel under the dilatation of the coordinates $(z_1, \dots, z_n) \mapsto (z_1/d_1, \dots, z_n/d_n)$). As E is a complete Reinhardt domain, $K_E(0) = \mathrm{vol}(E)^{-1}$. It is easily calculated that this volume equals $\frac{(2\pi)^n}{(2n)!}(2\pi)^n$, thereby proving the upper estimate.

It is precise for $n = 1$, as seen in the example of the unit disc and its center. If $n \ge 2$, then G does not contain $\mathbb{B}_n(0, d_1)$ so the second inequality above is strict (since the volume of G is less than that of $G \cup \mathbb{B}_n(0, d_1)$).

To finish the discussion of the upper estimate, it remains to show that it is precise for $n \ge 2$. For $m \in \mathbb{N}$ put $b_j = j^m$ for $1 \le j \le n$. Let $B_j = \mathbb{B}_n(0, b_j) \cap H'_{j-1}$, where $H'_{j-1} = \{0\} \times \mathbb{C}^{n-j+1}$. Denote by T the convex hull of the union of $\bigcup_{j=1}^n B_j$ and $\{z \in \mathbb{C}^n : \sum_{j=1}^n |z_j|/b_j < 1\}$. It is not hard to see that $b_j = \mathrm{dist}(0, \partial(T \cap H'_{j-1}))$.

Further, if $\Psi(z) = (z_1/b_1, \dots, z_n/b_n)$, then $\Psi(T)$ is the convex hull of the union of $S = \bigcup_{k=1}^n \Psi(B_k)$ and E. For each $k > j$ we have $b_k/b_j \to \infty$ if $m \to \infty$. Consequently, for every $\lambda > 1$ one can find an m such that $S \subset \lambda E$. As λE is a convex domain, it contains $\Psi(T)$. So

$$K_T(0)(b_1 \dots b_n)^2 = K_{\Psi(T)}(0) \ge K_{\lambda E}(0) = \frac{K_E(0)}{\lambda^{2n}}$$

and as $\lambda > 1$, the upper estimate is precise.

The lower estimate ([9]). After a translation and a rotation, we can assume that $z = 0$, $H_j = \{0\} \times \mathbb{C}^{n-j}$ $(j = 1, \dots, n-1)$ and $a^j = (0, a^j_j, 0) \in \mathbb{C}^{j-1} \times \mathbb{C} \times \mathbb{C}^{n-j}$ $(j = 1, \dots, n)$ so that $d_j = |a^j_j|$.

As D is a $\mathbb{C}$-convex domain, there exists a hyperplane $a^j + W_{j-1}$ through a^j that is disjoint from D. By our construction the ball in H_1 of center 0 and radius a_2^2 lies in D and so $W_1 \cap H_1$ is orthogonal to a^2, i.e. $W_1 \cap H_1 \subset \{0\} \times \mathbb{C}^{n-2}$. Consequently, W_1 is defined by the equation $\alpha_{1,1}z_1 + z_2 = 0$. The same argument shows that the equation of W_j for $j = 0, \dots, n-1$ is

$$\alpha_{j,1}z_1 + \dots + \alpha_{j,j}z_j + z_{j+1} = 0.$$

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be the linear mapping whose matrix A has rows $(\alpha_{j,1}, \dots, \alpha_{j,j}, 1, 0, \dots, 0)$, $j = 0, \dots, n-1$. Then $G = F(D)$ is also a $\mathbb{C}$-convex domain (G was another domain in the proof of the upper estimate). Note that $K_D(0) = K_G(0)$, as $\det A = 1$.

([9]) The geometric proof is close to that of Proposition 2.6.5.

Put $G_j = \pi_j(G)$, where π_j is the projection onto the jth coordinate plane. Then G_j is a simply connected domain (see e.g. [5]) and $G \subset G_1 \times \dots \times G_n$. Consequently,

$$K_D(0) \ge K_{G_1\times\dots\times G_n}(0) = K_{G_1}(0)\cdots K_{G_n}(0). \tag{3.4.1}$$

As $G_j \neq \mathbb{C}$ is a simply connected domain, it is biholomorphic to $\mathbb{D}$ and $\sqrt{\pi K_{\mathbb{D}}(0)} = 1 = \gamma_{\mathbb{D}}(0;1)$ implies that

$$\sqrt{\pi K_{G_j}(0)} = \gamma_{G_j}(0;1). \tag{3.4.2}$$

On the other hand,

$$\gamma_{G_j}(0;1) \ge \frac{1}{4d_{G_j}(0)} \tag{3.4.3}$$

by the Koebe 1/4-theorem (this argument was already used in the proof of Proposition 3.2.2).

Further, $F(a^j) \in \partial G$, and the jth coordinate of this point is a_j^j. In addition, the hyperplane $\{z \in \mathbb{C}^n : z_j = a_j^j\}$ does not intersect G. Consequently, $a_j^j \in \partial G_j$; in particular

$$d_j = |a_j^j| \ge d_{G_j}(0).$$

This together with (3.4.1), (3.4.2) and (3.4.3) proves the lower estimate.

Note that the constant 16 is the best possible for $n = 1$, as shown by the example of the image $D = \mathbb{C}\setminus[1/4,\infty)$ of $\mathbb{D}$ under the Koebe transformation $z \mapsto z/(1+z)^2$ (already used in Section 3.2). This example is not applicable for $n \ge 2$ in a trivial way, since a $\mathbb{C}$-convex Cartesian product of domains that are different from $\mathbb{C}$ is necessarily convex (see the first remark in Section 2.6).

For the lower estimate in the case of a convex domain it is sufficient to note that the G_j are convex domains, so the number 4 can be replaced by 2 due to Proposition 3.2.1.

Finally note that in this case the constant 4 is the best possible, as seen in the example of a Cartesian product of half-planes. ■

Using the lower estimate in Theorem 3.4.3, we can now prove Theorem 3.4.2.

Proof of Theorem 3.4.2. We will use the geometric configuration from the proof of Theorem 3.4.3.

Let $X \in (\mathbb{C}^n)_*$. First we will find an upper estimate for $M_D(0;X)$. Fix a $k \in J = \{j : X_j \neq 0\}$. Then

$$\Psi_k(z) = \left(z_1 - \frac{X_1}{X_k}z_k, \dots, z_{k-1} - \frac{X_{k-1}}{X_k}z_k, z_k, z_{k+1} - \frac{X_{k+1}}{X_k}z_k, \dots, z_n - \frac{X_n}{X_k}z_k\right)$$

is a linear mapping of Jacobian 1 and

$$Y^k := \Psi_k(X) = (0,\dots,0,X_k,0,\dots,0).$$

Let Δ_j be the disc in the jth coordinate plane of center 0 and radius d_j for $j \neq k$, and radius $d'_k = |X_k| d_D(0,X)$ for $j = k$. Then $\Delta_j \subset D_k = \Psi_k(D)$ and by Lemma 3.2.5,

$$D_k \supset E_k = \left\{z \in \mathbb{C}^n : \frac{|z_k|}{d'_k} + \sum_{j=1,\, j\neq k}^{n} \frac{|z_j|}{d_j} < 1\right\}.$$

Consequently,

$$M_D(0;X) = M_{D_k}(0;Y^k) \le M_{E_k}(0;Y^k) = \frac{C_n d_{k,D}(0)}{|X_k| p_D(0) d_D^2(0,X)},$$

where $C_n := M_E(0;e_1) = \sqrt{\frac{(2(n+1))!}{6(2\pi)^n}}$ (the latter is calculated directly, as E is a complete Reinhardt domain), and e_1 is the first basis vector.

From this estimate and the lower estimate in Theorem 3.4.3 it follows that

$$B_D(0;X) = \frac{M_D(0;X)}{\sqrt{K_D(0)}} \le \frac{c'_n d_{k,D}(0)}{|X_k| d_D^2(0,X)}, \quad 1 \le k \le n, \tag{3.4.4}$$

where $c'_n = (4\sqrt{\pi})^n C_n = 2^n \sqrt{2^{n-1}(2(n+1))!/3}$.

It remains to note that Lemma 3.2.5 implies the inequality

$$\frac{1}{d_D(0,X)} \le \sum_{j=1}^n \frac{|X_j|}{d_j} \tag{3.4.5}$$

and then put $c_n = nc'_n$. ∎

The above results and their proofs allow us to understand the boundary behavior of any of the metrics F_D considered—Carathéodory, Kobayashi or Bergman—of a $\mathbb{C}$-convex domain that does not contain a complex line, in terms of minimal bases. This strengthens some results from [19, 74, 75, 12, 71], dealing with bounded smooth domains of finite type; the constants there depend on the domain (the first three works even refer to convex domains).

Proposition 3.4.4. *There exists a constant $c_n \ge 1$, depending only on n, so that for each $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$ not containing a complex line, we have*

$$c_n^{-1} \le F_D(z;X)\Big(\sum_{j=1}^n \frac{|\langle X, e_j(z)\rangle|}{d_j(z)}\Big)^{-1} \le c_n.$$

(*Here $e_j(z)$ are the basis vectors of a minimal basis of D at z, and $d_j(z)$ are the corresponding numbers.*)

Proof. By (3.4.4) and the inequality

$$B_D(z;X) \ge \frac{1}{4d_D(z;X)}$$

we get

$$\frac{|X_j(z)|}{d_j(z)} \le \frac{4c'_n}{d_D(z)}.$$

So

$$\frac{1}{d_D(z;X)} \le \sum_{j=1}^n \frac{|X_j(z)|}{d_j(z)} \le \frac{4c_n}{d_D(z;X)},$$

where $c_n = nc'_n$. Then (3.4.4) and (3.4.5) show that

$$(16c_n)^{-1} \le F_D(z;X)\Big(\sum_{j=1}^n \frac{|X_j(z)|}{d_j(z)}\Big)^{-1} \le c_n. \ ∎$$

The following result is in the spirit of Proposition 3.4.4, but it deals with a fixed basis. As each boundary point of a bounded $\mathbb{C}$-convex domain is semiregular (see the end of Section 3.3), the result directly follows from [124, 16] and [79, Theorem 3.3.1].

PROPOSITION 3.4.5. *Let a be a boundary point of finite type of a bounded smooth $\mathbb{C}$-convex domain $D \subset \mathbb{C}^n$. Denote by $\tilde{M}_a = (m_1, \dots, m_n)$ the Catlin multitype of a. Then there exists a linear basis change with the following property: for each nontangent cone Λ with vertex a there exists a constant $c > 0$ so that for an arbitrary vector $X = (X_1, \dots, X_n)$ in the new basis we have*

$$\begin{aligned} c^{-1} &\le \liminf_{\Lambda \ni z \to a} F_D(z;X)\bigg(\sum_{j=1}^n \frac{|X_j|}{(d_D(z))^{1/m_j}}\bigg)^{-1} \\ &\le \limsup_{\Lambda \ni z \to a} F_D(z;X)\bigg(\sum_{j=1}^n \frac{|X_j|}{(d_D(z))^{1/m_j}}\bigg)^{-1} \le c. \end{aligned}$$

In addition, for the Bergman kernel we have

$$c^{-1} \le \liminf_{\Lambda \ni z \to a} K_D(z)(d_D(z))^{-2q} \le \limsup_{\Lambda \ni z \to a} K_D(z)(d_D(z))^{-2q} \le c,$$

where $q = 1/m_1 + \dots + 1/m_n$.

The basis change can be chosen to be linear because $\tilde{L}_a = \tilde{M}_a$.

Note that Proposition 3.4.5 implies a more precise variant of Proposition 3.3.1 in the case of finite type, namely that for each vector $X \in (\mathbb{C}^n)_*$ there exists $j = 1, \dots, n$ so that $l_{a,X} = m_j$.

Finally let us mention that by a result from [23] the quotient $(d_{j,D})^m/d_D$ is bounded near a (recall that $m = m_n$ is the type of a) and then Proposition 3.4.4 provides a neighborhood U of a and a constant $c > 0$ so that

$$\kappa_D(z;X) \ge \frac{c\|X\|}{(d_D(z))^{1/m}}, \quad z \in D \cap U.$$

3.5. Maximal basis. A counterexample. To get estimates for the Bergman kernel and the Bergman metric for $\mathbb{C}$-convex domains, in the previous section we introduced a basis (called minimal) with origin at a given point of the domain. As mentioned, in the special case of a smooth convex domain of finite type, in [19, 74, 75] a similar basis is introduced (that we call maximal). The minimal and maximal bases can be considered in the context of the so-called extremal bases (see [18]). Many other important results, like those connected with the linear type, with the $\overline{\partial}$-problem or with domains with noncompact groups of automorphisms (see e.g. [73, 76, 77, 41]), use in an essential way the properties of the maximal basis, and most of all one extremal property, satisfied also by the minimal basis. In general, this property means that the vectors from the basis are orthogonal to the corresponding hyperplanes (see the Introduction; the details are given below). Unfortunately exactly this property of the maximal basis turns out to be wrong (the hints for corresponding proofs are based on an incorrect application of the method of Lagrange multipliers).

The main aim of this section is to provide a counterexample to the extremal property of the maximal basis.

Now we define the notion of "maximal basis". Let $D \subset \mathbb{C}^n$ be a domain, not containing a complex line. For $q \in D$ we choose a unit vector $a_1 \in \mathbb{C}^n$ so that

$$s_1 := d_D(q; a_1) = d_D(q).$$

Then we choose a unit vector $a_2 \in \operatorname{span}(a_1)^\perp$ so that

$$s_2 := d_D(q; a_2) = \sup d_D(q; a),$$

where the supremum is taken over all the unit vectors $a \in \operatorname{span}(a_1)^\perp$. In the next step we choose a unit vector $a_3 \in \operatorname{span}(a_1, a_2)^\perp$ so that

$$s_3 := d_D(q; a_3) = \sup d_D(q; a),$$

where the supremum is taken over the unit vectors $a \in \operatorname{span}(a_1, a_2)^\perp$. Continuing the procedure, we get an orthonormal basis $a_1, \dots, a_n$ that will be called *maximal* (for D at q) and a sequence of positive numbers $s_2 \ge \dots \ge s_n \ge s_1 \ge 0$ (but they are not uniquely determined). Note that, unlike the minimal basis, after the first step the corresponding distances are chosen maximal (rather than minimal).

Assume now that D is a convex domain that is smooth near a boundary point p_1 (of finite type). Let r be a locally defining function. Now we will describe the extremal property mentioned in the Introduction. For $q \in D$ on the interior normal to ∂D at p_1, sufficiently close to p_1, we consider a coordinate system defined by the maximal basis at q, i.e. we put $q = 0$ and express each $z \in \mathbb{C}^n$ in the form $z = \sum_{j=1}^n w_j a_j$. We choose $p_k \in \partial D$, $k = 2, \dots, n$, so that $p_k = \lambda_k a_k$, where $|\lambda_k| = s_k$. Many of the works mentioned in the Introduction (see e.g. [19, Proposition 2.2(ii)], [73, Proposition 3.1(i)], [74, Proposition 2.1(iii)]) claim that

$$\frac{\partial r(p_k)}{\partial w_j} = 0, \quad j = k+1, \dots, n. \tag{$*$}$$

This means that

$$T^{\mathbb{C}}_{p_k}(\partial D) \cap \operatorname{span}(a_1, \dots, a_k)^\perp = \operatorname{span}(a_{k+1}, \dots, a_n). \tag{$**$}$$

Note that the minimal basis has a very essential property which is equivalent to $(*)$ in the smooth case; we started from it when obtaining the lower estimate for the Bergman kernel in Theorem 3.4.3.

However now we will demonstrate a counterexample in $\mathbb{C}^3$ to the property $(*)$ of the maximal bases at the points from an interior normal to the boundary of a domain in $\mathbb{C}^3$ (in $\mathbb{C}^2$ this property clearly holds).

Let $0 < \beta_2 < \beta_1 < 1$. Put

$$D = \{z \in \mathbb{C}^2 \times \mathbb{C} : \rho(z) + |z_3|^2 < 1\},$$

where $\rho(z) = x_1^2 + \beta_1 y_1^2 + x_2^2 + \beta_2 y_2^2$. Note that D is a strictly (pseudo)convex domain with real-analytic boundary. Let $q = (0, 0, \delta)$, where $0 < \delta < 1$. The construction of a maximal basis of D at q leads to $s_1 = 1 - \delta$ and $a_1 = (0, 0, 1)$. From the next step we get the domain

$$D_\delta = \{z \in \mathbb{C}^2 : \rho(z) < 1 - \delta^2\}.$$

Note that a homothety transforms D_δ into D_0 and so we can examine only D_0. For the second vector a_2 from the maximal basis we have $a_2 = (b, 0)$, where $b \in \mathbb{C}^2$. Then $\operatorname{span}(a_1, a_2)^\perp$ is generated by $(-\bar{b}_2, \bar{b}_1, 0)$. Put

$$\mathcal{T} = \left\{ b \in \mathbb{C}^2 : \frac{\partial \rho(b)}{\partial z_1}(-\bar{b}_2) + \frac{\partial \rho(b)}{\partial z_2}(\bar{b}_1) = 0 \right\}.$$

LEMMA 3.5.1. $\mathcal{T} = \{b \in \mathbb{C}^2 : b_1 = 0$ *or* $b_2 = 0$ *or* $\operatorname{Im} b_1 = \operatorname{Im} b_2 = 0\}$.

Proof. Some elementary calculations show that $b \in \mathcal{T}$ exactly when

$$(\beta_1 - \beta_2) \operatorname{Im} b_1 \operatorname{Im} b_2 = 0, \qquad (1 - \beta_1) \operatorname{Im} b_1 \operatorname{Re} b_2 = (1 - \beta_2) \operatorname{Im} b_2 \operatorname{Re} b_1,$$

and the result follows. ■

Let $p_2 \in \partial D_0$ so that

$$\frac{d_{D_0}(0; p_2)}{\|p_2\|} = s_2 = \sup_{\|a\|=1} d_{D_0}(0; a).$$

The following result shows that the property (∗∗), equivalent to (∗), is not true at the points on the interior normal to D at $(1, 0, 0)$ (formally we must also consider the case $\delta < 0$, but then the closest point is $(-1, 0, 0)$ and the situation is similar).

PROPOSITION 3.5.2. $p_2 \notin \mathcal{T}$.

Proof. Let $b \in \mathcal{T}$ be a unit vector. Note that $\rho(re^{i\alpha} b) < 1$ for each $\alpha \in \mathbb{R}$ if and only if $r^2 R(b) < 1$, where $R(b) = \max_{\alpha \in \mathbb{R}} \rho(e^{i\alpha} b)$. Consequently, $d_{D_0}(0; b) = 1/\sqrt{R(b)}$. Let $b = (e^{i\varphi_1} \cos\Theta, e^{i\varphi_2} \sin\Theta)$, where $0 \le \Theta < 2\pi$ and $0 \le \varphi_1, \varphi_2 \le \pi/2$. By Lemma 3.5.1 there are three possibilities for b:

- $\Theta = 0$ or $\Theta = \pi$: $\rho(e^{i\alpha} b) = \cos^2(\alpha + \varphi_1) + \beta_1 \sin^2(\alpha + \varphi_1)$.
- $\Theta = \pi/2$ or $\Theta = 3\pi/2$: $\rho(e^{i\alpha} b) = \cos^2(\alpha + \varphi_2) + \beta_2 \sin^2(\alpha + \varphi_2)$.
- $\varphi_1 = \varphi_2 = 0$: $\rho(e^{i\alpha} b) = \cos^2\alpha + \sin^2\alpha(\beta_1 \cos^2\Theta + \beta_2 \sin^2\Theta)$.

In all three cases $R(b) = 1$.

On the other hand, there exists a unit vector $b^* \in \mathbb{C}^2$ such that $R(b^*) < 1$, and so $p_2 \notin \mathcal{T}$. To define b^*, put $\Theta = \pi/4$ $\varphi_1 = 0$ and $\varphi_2 = \pi/2$. Then $2\rho(e^{i\alpha} b^*) = 1 + \beta_2 + (\beta_1 - \beta_2)\sin^2\alpha$. As $\beta_1 < \beta_2 < 1$, we conclude that $R(b^*) = (1 + \beta_2)/2 < 1$. ■

3.6. Estimates in a maximal basis. The aim of this section is to prove, using the estimates for invariants in terms of a minimal basis, that they remain true in terms of a maximal basis, in spite of the counterexample from the last section. A similar approach allows one to confirm the correctness of other results using the maximal basis.

Let $D \subset \mathbb{C}^n$ be a $\mathbb{C}$-convex domain not containing a complex line (i.e. each nonempty intersection of D with a complex line is biholomorphic to $\mathbb{D}$). For $z \in D$, let $e_1, \ldots, e_n$ be a minimal basis of D at z, and $a_1, \ldots, a_n$ a reordered maximal basis of D at z, meaning that the new a_1 is the old $\widetilde{a}_1$, but $a_2 = \widetilde{a}_n, a_3 = \widetilde{a}_{n-1}, \ldots, a_n = \widetilde{a}_2$. Let $d_1 \le \cdots \le d_n$ and $s_1 \le \cdots \le s_n$ be the corresponding numbers (recall that $d_1 = s_1 = d_D(z)$). Put $p_D(z) = \prod_{j=1}^n d_j$ and $s_D(z) = \prod_{j=1}^n s_j$. As before, $K_D(z)$ denotes the Bergman kernel (on the diagonal). Let $F_D(z; X)$ be any of the metrics of Carathéodory, Kobayashi or

Bergman. For $X \in \mathbb{C}^n$ put

$$E_D(z;X) = \sum_{j=1}^{n} \frac{|\langle X, e_j \rangle|}{d_j}, \quad A_D(z;X) = \sum_{j=1}^{n} \frac{|\langle X, a_j \rangle|}{s_j}.$$

Now, we will write $f(z) \lesssim g(z)$ if $f(z) \le c_n g(z)$ for some constant $c_n > 0$ depending only on n; $f(z) \sim g(z)$ means that $f(z) \lesssim g(z) \lesssim f(z)$. By Proposition 3.2.2, Theorems 3.4.2, 3.4.3 and Proposition 3.4.4 we know that

$$K_D(z) \sim 1/p_D^2(z), \quad F_D(z;X) \sim E_D(z;X) \sim 1/d_D(z;X)$$

(as noted, under the much stronger requirements that the domain be $\mathbb{C}$-convex, smooth, bounded and of finite type these estimates follow also from [12, 71]). For brevity we will sometimes omit the arguments z and X. Lemma 3.2.5 easily implies that

$$K_D \lesssim 1/s_D^2, \quad F_D \lesssim A_D.$$

In particular,

$$1/d_D(z;X) \sim E_D(z;X) \lesssim A_D(z;X).$$

As mentioned, the main corollary from the incorrect property $(*)$ for the maximal bases (for a bounded smooth $\mathbb{C}$-convex domain of finite type) is that

$$A_D(z;X) \sim_D 1/d_D(z;X),$$

where the constant in $\sim_D$ depends on D. Based on this fact, in [19, 74, 75] it is shown that

$$K_D \sim_D 1/s_D^2, \quad F_D \sim_D A_D.$$

The next two propositions show that anyway these estimates are correct.

The first estimate can also be obtained from [45] in the case of a bounded smooth $\mathbb{C}$-convex domain of finite type. The proof there uses the incorrectly proven estimate $1/d_D(z;X) \sim_D A_D(z;X)$, but a closer look shows that one can only use the correct part of that estimate, $1/d_D(z;X) \lesssim_D A_D(z;X)$.

Proposition 3.6.1. *Let $D \subset \mathbb{C}^n$ be a $\mathbb{C}$-convex domain not containing complex lines. Then for each $z \in D$ we have $d_j \sim s_j$, $j = 1, \dots, n$.*

Once again observe that the constant in $\sim$ depends only on the dimension n of D.

Proof. We first prove that $s_j \lesssim d_j$. As $E_D \lesssim A_D$, it suffices to check that if $E_D \le cA_D$, then $s_j \le c' d_j$, where $c' = n!c$.

The formula for the determinant of the unitary transformation between two bases implies that $\prod_{j=1}^{n} |\langle a_j, e_{\sigma(j)} \rangle| \ge 1/n!$ for some permutation of σ of $\{1, \dots, n\}$. In particular, $|\langle a_j, e_{\sigma(j)} \rangle| \ge 1/n!$. Then $E_D(z;a_j) \le cA_D(z;a_j)$ implies $s_j \le c' d_{\sigma(j)}$.

Suppose now that $c' d_k < s_k$ for some k. Then

$$c' d_k < s_k \le s_j \le c' d_{\sigma(j)}, \quad j \ge k.$$

Consequently, $\sigma(j) > k$ for each $j \ge k$, a contradiction, as σ is a permutation.

These arguments show that $\widetilde{d}_j \sim d_j$, where $\widetilde{d}_j$ are the corresponding numbers for another minimal basis of D at z. Thus we can assume that $e_1 = a_1$. We know that $s_1 = d_1$. It remains to show that $s_k \gtrsim d_k$ for $k \ge 2$. Choose a unit vector in $\operatorname{span}(e_k, \dots, e_n)$ that is orthogonal to $a_{k+1},\dots,a_n$ ($a'_n = e_n$ if $k=n$). Then a'_k is also orthogonal to $a_1 = e_1$. Consequently, $s_k \ge d_D(z;a'_k)$ (by the construction of a maximal basis). On the other hand, as a'_k is orthogonal to $e_1,\dots,e_{k-1}$, we have

$$\frac{1}{d_D(z;a'_k)} \sim E_D(z;a'_k) = \sum_{j=k}^{n} \frac{|\langle a'_k, e_j\rangle|}{d_j} \lesssim \frac{1}{d_k}.$$

So $s_k \ge d_D(z;a'_k) \gtrsim d_k$. ■

PROPOSITION 3.6.2. *Let D be as in Proposition* 3.6.2. *Then $A_D \sim E_D$.*

Proof. In view of the inequality $E_D \lesssim A_D$ and Proposition 3.6.1 ($s_k \sim d_k$), it suffices to prove

$$|\langle X, a_k\rangle|/d_k \lesssim E_D(z;X)$$

for each k. Put $b_{jk} = \langle a_j, e_k\rangle$. As

$$\frac{1}{d_j} \sim \frac{1}{d_D(z;a_j)} \sim E_D(z;a_j) \ge \frac{|b_{jk}|}{d_k},$$

it follows that $|b_{jk}| \lesssim d_k/d_j$. The unitary transformation with matrix $B = (b_{jk})$ transforms the basis $e_1,\dots,e_n$ into $a_1,\dots,a_n$. For the inverse matrix $C = (c_{jk})$ we have

$$\begin{aligned}|c_{jk}| &\le \sum_{\sigma} |b_{1\sigma(1)}\dots b_{k-1,\sigma(k-1)} b_{k+1,\sigma(k+1)}\dots b_{n,\sigma(n)}|\\ &\lesssim \sum_{\sigma} \frac{d_{\sigma(1)}}{d_1}\dots\frac{d_{\sigma(k-1)}}{d_{k-1}}\frac{d_{\sigma(k+1)}}{d_{k+1}}\dots\frac{d_{\sigma(n)}}{d_n} = \sum_{\sigma}\frac{d_k}{d_j} = (n-1)!\frac{d_k}{d_j},\end{aligned}$$

where σ varies over all bijections between $\{1,\dots,k-1,k+1,\dots,n\}$ and $\{1,\dots,j-1, j+1,\dots,n\}$. Consequently,

$$\frac{|\langle X, a_k\rangle|}{d_k} \le \sum_{j=1}^{n} |\langle X, e_j\rangle|\frac{|b_{kj}|}{d_k} = \sum_{j=1}^{n} |\langle X, e_j\rangle|\frac{|\overline{c}_{jk}|}{d_k} \lesssim E_D. \ ■$$

REMARK. The constructions of minimal and maximal bases can be generalized in the following way: we choose "minimal" discs at steps $1,\dots,k$ and "maximal" discs at steps $k+1,\dots,n-1$ (the nth choice is canonical); $k=n-1$ gives a minimal basis, $k=1$ a maximal one, and $k=0$ a basis without "minimal" discs. Note that Propositions 3.6.1 and 3.6.2 remain true when s_j are replaced by the numbers in the new basis and we express A_D in this basis. (This construction has an obvious real analogue.)

3.7. Localizations. It is natural to ask whether the results from the preceding sections have local character, i.e. whether $\mathbb{C}$-convexity is a local notion (like convexity and pseudoconvexity) and whether the behavior of the invariant metrics considered near a boundary point of a given domain is similar to that on the intersection of the domain with a neighborhood of the points.

It is hard to get localization results for the Carathéodory metric and here we are not going to deal with them. Some such results can be found in [79].

First we will discuss the local character of $\mathbb{C}$-convexity. As noted in Section 2.6, each bounded $\mathbb{C}$-convex domain is (weakly) linearly convex, and the converse is true under the additional assumption of a $\mathcal{C}^1$-smooth boundary.

The next proposition shows that this fact has local character.

PROPOSITION 3.7.1. *Let a be a $\mathcal{C}^k$-smooth boundary point* ($2 \le k \le \infty$) *of a domain $D \subset \mathbb{C}^n$ that is locally weakly linearly convex near a* ([10]), *i.e. for each $b \in \partial D$ near a there exists a neighborhood U_b so that $D \cap U_b \cap T_b^{\mathbb{C}}(\partial D) = \emptyset$. Then there exists a neighborhood U of a for which $D \cap U$ is a $\mathcal{C}^k$-smooth $\mathbb{C}$-convex domain.*

Clearly this proposition has "convex" and "pseudoconvex" analogues, proven in a similar, but easier, way.

Proof. We can assume that $a = 0$. Denote by $H_f(z;X)$ the Hessian of a $\mathcal{C}^2$-smooth function f. Put $B_s = \mathbb{B}_n(0,s)$ $(s > 0)$ and

$$r(z) = \begin{cases} -d_D(z), & z \in D, \\ d_D(z), & z \notin D. \end{cases}$$

The differential inequality for r^2 in the proof of [5, Proposition 2.5.18, (ii)⇒(iii)] easily implies that there exists an $\varepsilon > 0$ so that r is a $\mathcal{C}^k$-smooth defining function of D on $B_{3\varepsilon}$, and $H_r(z;X) \ge 0$ if $\langle \partial r(z), \overline{X}\rangle = 0$ and $z \in D \cap B_{2\varepsilon}$. Then the proof of [27, Lemma 1] shows that there exists a $c > 0$ such that $H_r(z;X) \ge -c\|X\| \cdot |\langle \partial r(z), \overline{X}\rangle|$, $z \in D \cap B_{2\varepsilon}$. We can suppose that $2\varepsilon c \le 1$ and $D \cap B_\varepsilon$ is connected. Choose a smooth function χ so that $\chi(x) = 0$ for $x \le \varepsilon^2$ and $\chi'(x), \chi''(x) > 0$ for $x > \varepsilon^2$. Put $\theta(z) = \chi(\|z\|^2)$. We can find a $C \ge 1/2$ such that

$$B_{2\varepsilon} \Supset G' = \{z \in B_{2\varepsilon} : 0 > \rho(z) = r(z) + C\theta(z)\} \subset D.$$

Now, the inequalities $2c\varepsilon \le 1$ and $|\langle \partial\theta(z), \overline{X}\rangle| \le \chi'(\|z\|^2)\|z\| \cdot \|X\|$ yield $\chi'(\|z\|^2)\|X\| > c|\langle \partial\theta(z), \overline{X}\rangle|$ if $z \in B_{2\varepsilon} \setminus \overline{B_\varepsilon}$ and $X \ne 0$. This, together with

$$H_r(z;X) \ge -c\|X\| \cdot |\langle \partial r(z), \overline{X}\rangle|, \qquad z \in G',$$
$$H_\rho(z;X) = H_r(z;X) + 4C\chi''(\|z\|^2)\mathrm{Re}^2\langle z, X\rangle + 2C\chi'(\|z\|^2)\|X\|^2, \qquad C \ge 1/2,$$

and the triangle inequality, implies that

$$H_\rho(z;X) \ge -c\|X\| \cdot |\langle \partial\rho(z), \overline{X}\rangle|, \qquad z \in \overline{G'}.$$

In addition, the last inequality is strict if $z \in \overline{G'} \setminus \overline{B_\varepsilon}$ and $X \ne 0$. This shows that $\partial\rho \ne 0$ on $\partial G' \setminus \overline{B_\varepsilon}$ (otherwise ρ would attain a local minimum at some point of this set, which is clearly impossible). Thus $\partial\rho \ne 0$ on $\partial G'$.

Let G be a connected component of G' that contains $D \cap B_\varepsilon$. Then [5, Proposition 2.5.18] (see also [50, Proposition 4.6.4]) implies that G is a $\mathcal{C}^k$-smooth $\mathbb{C}$-convex domain. It remains to put $U = B_n(0,\varepsilon) \cup G$. ∎

Now we will discuss the localization of the Kobayashi metric. First recall that if D is a hyperbolic domain (i.e. the Kobayashi pseudodistance k_D is a distance), then the following weak localization holds (see e.g. [79]):

PROPOSITION 3.7.2. *If $V \Subset U$ are neighborhoods of a boundary point of a hyperbolic domain $D \subset \mathbb{C}^n$, then there exists a constant $C \ge 1$ such that for each $z \in D \cap V$ and*

([10]) Cf. Section 2.6.

each $X \in \mathbb{C}^n$ we have

$$\kappa_D(z;X) \le \kappa_{D\cap U}(z;X) \le C\kappa_D(z;X).$$

Propositions 3.7.1 and 3.7.2 show that all the above results for the Kobayashi metric, as well as those connected with types and multitypes, have local character in the case of bounded domains.

To see this, note the following. If a is a boundary point of a bounded domain $D \subset \mathbb{C}^n$, then it is easily seen that for each neighborhood U of a we have

$$p_D(z) \sim_* p_{D\cap U}(z), \quad s_D(z) \sim_* s_D(z), \quad d_D(z;X) \sim_* d_{D\cap U}(z;X),$$
$$E_D(z;X) \sim_* E_{D\cap U}(z;X), \quad A_D(z;X) \sim_* A_{D\cap U}(z;X)$$

near a; here the constant in $\sim_*$ depends on D and U.

Thus we get the following corollary of Propositions 3.2.2, 3.7.1 and 3.7.2.

Corollary 3.7.3. *Let a be a boundary point of a bounded domain $D \subset \mathbb{C}^n$, as in Proposition* 3.7.1. *Then*

$$\kappa_D(z;X) \sim_D d_D(z;X) \sim_D E_D(z;X)$$

near a (the constant in $\sim_D$ depends on D).

Now we will sharpen the last corollary if ∂D does not contain analytic discs through a (by Proposition 3.3.4 this is equivalent to ∂D not containing linear discs through a).

Proposition 3.7.4. *Let a be a boundary point of a bounded domain $D \subset \mathbb{C}^n$, as in Proposition* 3.7.1. *Also assume that ∂D does not contain analytic discs through a. Then*

$$\frac{1}{4} \le \liminf_{z\to a} \kappa_D(z;X)d_D(z;X) \le \limsup_{z\to a} \kappa_D(z;X)d_D(z;X) \le 1$$

uniformly in $X \in (\mathbb{C}^n)_*$.

As in ∂D there are no analytic discs through a, Propositions 3.2.3, 3.2.4 and 3.7.1 imply that for each sufficiently small neighborhood U of a we have $\lim_{z\to a} d_D(z;X) = \infty$ uniformly in all unit vectors in $\mathbb{C}^n$. Then by shrinking U (if necessary), $d_D(z;X) = d_{D\cap U}(z;X)$ for each z near a (also $E_D(z;X) = E_{D\cap U}(z;X)$ and $A_D(z;X) = A_{D\cap U}(z;X)$).

After these remarks, Proposition 3.7.4 follows from the following strict localization for the Kobayashi metric (cf. Proposition 2.6.3).

Proposition 3.7.5. *Let $D \subset \mathbb{C}^n$ be a bounded domain whose boundary does not contain nontrivial analytic discs through a point $a \in \partial D$. Suppose that there exists a neighborhood U of a and a function $f \in \mathcal{O}(D \cap U)$ such that $\lim_{z\to a} |f(z)| = \infty$. Then for each neighborhood V of a we have*

$$\lim_{z\to a} \frac{\kappa_{D\cap V}(z;X)}{\kappa_D(z;X)} = 1$$

uniformly in $X \in (\mathbb{C}^n)_*$.

Proof. Using the condition on the discs, as in the proof of Proposition 2.6.3, it follows that each sequence of analytic discs φ_j with $\varphi_j(0) \to a$ converges to a uniformly (on compact subsets of $\mathbb{D}$). Then the proposition is contained in [79, Corollary 2.3.4]. ∎

As a planar domain having at least two points in its boundary is hyperbolic (see e.g. [58]), the proof of the above proposition shows that the statement can be strengthened for $n = 1$.

PROPOSITION 3.7.6. *Let a be a boundary point of a domain $D \subsetneq \mathbb{C} \setminus \{a\}$. Then for each neighborhood V of a we have*

$$\lim_{z \to a} \frac{\kappa_{D \cap V}(z;1)}{\kappa_D(z;1)} = 1.$$

In particular, if a is an isolated boundary point of D, then

$$\lim_{z \to a} \kappa_D(z;1)|z| \log |z| = -1/2.$$

Note that the last equality follows from the formula $\kappa_{\mathbb{D}_*}(z;1) = -\frac{1}{2|z|\log|z|}$ (see e.g. [58]).

Proposition 3.7.6 generalizes essentially, and with a short proof, [61, Theorem 1].

3.8. Localization of the Bergman kernel and the Bergman metric. In this section we provide localization theorems for the Bergman kernel and the Bergman metric, which together with Proposition 3.7.1 will allow us to localize the results from the preceding sections that deal with the Bergman kernel and the Bergman metric.

In the case when $D \subset \mathbb{C}^n$ is a bounded pseudoconvex domain, the corresponding results are well known (see e.g. [30]).

THEOREM 3.8.1. *Let $V \Subset U$ be neighborhoods of a boundary point z_0 of a bounded pseudoconvex domain $D \subset \mathbb{C}^n$. Then there exists a constant $c \geq 1$ such that for each $z \in D \cap V$ and for each $X \in \mathbb{C}^n$ we have*

$$c^{-1} K_{D \cap U}(z) \leq K_D(z) \leq K_{D \cap U}(z), \quad c^{-1} B_{D \cap U}(z;X) \leq B_D(z;X) \leq c B_{D \cap U}(z;X).$$

By imitating the proof of Corollary 3.7.3, we get

COROLLARY 3.8.2. *Let a be a boundary point of a bounded domain $D \subset \mathbb{C}^n$, as in Proposition* 3.7.1. *Then*

$$K_D(z) \sim_D 1/p_D^2(z), \quad B_D(z;X) \sim_D 1/d_D(z;X) \sim_D E_D(z;X)$$

near a.

Note that for the Bergman kernel, the localization is strict if $z_0 \in \partial D$ is a holomorphic peak point in the most general sense, i.e. there exists a function $p \in \mathcal{O}(D, \mathbb{D})$ such that

$$\lim_{z \to z_0} p(z) = 1 > \sup_{D \setminus U} |p|$$

for each neighborhood U of z_0. This is proved in the fundamental work [49] of L. Hörmander as an application of the L^2-estimates for the $\overline{\partial}$-problem. More general results can be found in [48].

One of the goals of this section is to carry over this result to the case of an arbitrary pseudoconvex domain (not necessarily bounded). We will say that $z_0 \in \partial D$ is a *locally holomorphic peak point* if z_0 is a holomorphic peak point of $D \cap U$ for some neighborhood U of z_0.

THEOREM 3.8.3. *Let U be a neighborhood of a boundary locally holomorphic peak point z_0 of a pseudoconvex domain $D \subset C^n$. Then*

$$\lim_{z\to z_0} \frac{K_{D\cap U}(z)}{K_D(z)} = 1, \qquad \lim_{z\to z_0} \frac{B_{D\cap U}(z;X)}{B_D(z;X)} = 1$$

uniformly on $X \in (\mathbb{C}^n)_*$.

In particular, $K_{D\cap U}(z) > 0$ and so $B_{D\cap U}(z;X)$ exists for z close to z_0.

To prove Theorem 3.8.3, we need a localization lemma for the pluricomplex Green function g_D (for the definition see Section 1.7).

LEMMA 3.8.4. *Let U be a neighborhood of a locally holomorphic peak point z_0 of a pseudoconvex domain $D \subset C^n$. Then*

$$\liminf_{z\to z_0, w\in D\setminus U} g_D(z,w) = 0. \ (^{11})$$

In addition, there exists a neighborhood $V \subset U$ of z_0 such that

$$\inf\{g_D(z,w) - g_{D\cap U}(z,w) : z \in D\cap V,\ w \in D\cap U \setminus \{z\}\} = 0.$$

In particular, we have strong localization for the Azukawa metric:

$$\lim_{z\to z_0} \frac{A_{D\cap U}(z;X)}{A_D(z;X)} = 1$$

uniformly in $X \in (\mathbb{C}^n)_*$.

The first equality means that D has the so-called property (P) (see e.g. [21]), which has applications in problems about Bergman invariants, as well in pluripotential theory.

Proof. We use the fact that each locally holomorphic peak point is a plurisubharmonic peak point, i.e. there exists a negative function $\varphi \in \mathrm{PSH}(D)$ such that

$$\lim_{z\to z_0} \psi(z) = 0 > \inf_{D\setminus U_1} \psi$$

for each neighborhood U_1 of z_0. Indeed, one can assume that p is a holomorphic peak function on $D\cap U_1$ at z_0. Then it suffices to choose a neighborhood $U_2 \Subset U_1$ of z_0 such that $G = D\cap U_1 \setminus U_2 \neq \emptyset$ and to put $\delta = \sup_G |p|$ and

$$\varphi = -1 + \begin{cases} \max(\delta, |p|), & D\cap U_2, \\ \delta, & D\setminus U_2. \end{cases}$$

(This argument shows that the notion "plurisubharmonic peak point" has local character.)

On the other hand, each plurisubharmonic peak point is a plurisubharmonic antipeak point (see e.g. [42]), i.e. there exists a negative function $\psi \in \mathrm{PSH}(D)$ such that

$$\lim_{z\to z_0} \psi(z) = -\infty < \inf_{D\setminus U_1} \psi$$

for each neighborhood U_1 of z_0: to see this, put $\psi = -\log(-\varphi)$.

Now we pass to the proof proper. Let us first suppose that z_0 is a holomorphic peak point of $D\cap U$. Let $W \Subset U$ be a neighborhood of z_0. We can choose another neighborhood $V \Subset W$ of z_0 so that $\inf_{D\setminus W} \psi \geq c := 1 + \sup_{D\cap\partial V} \psi$. Fix $z \in D\cap V$ and put $d(z) =$

(^{11}) Here and further we assume $D \setminus U \neq \emptyset$.

$\inf_{w\in D\cap\partial V} g_{D\cap U}(z,w)$, $u(z,w)=(c-\psi(w))d(z)$ for $w\in D$. As $u(z,w)\le g_{D\cap U}(z,w)$ for $w\in D\cap\partial V$ and $u(z,w)\ge 0> g_{D\cap U}(z,w)$ for $w\in D\cap U\setminus W$, the function

$$v(z,w)=\begin{cases} g_{D\cap U}(z,w), & w\in D\cap V,\\ \max\{g_{D\cap U}(z,w),u(z,w)\}, & w\in D\cap W\setminus V,\\ u(z,w), & w\in D\setminus W,\end{cases}$$

is plurisubharmonic in the second variable and has a logarithmic singularity at z. Also, $v(z,w)<cd(z)$ and so $g_D(z,w)\ge v(z,w)-cd(z)$. As $v(z,w)=u(z,w)\ge 0$ for $w\in D\setminus W$, we get $g_D(z,w)\ge -cd(z)$ for $w\in D\setminus W$. Since

$$g_{D\cap U}(z,w)\ge\left|\frac{p(w)-p(z)}{1-\overline{p(z)}p(w)}\right|,$$

$\lim_{z\to z_0} d(z)=0$ and so $\lim_{z\to z_0}\inf_{w\in D\setminus W} g_D(z,w)=0$, which proves the first equality in the lemma.

Let now U be an arbitrary neighborhood of z_0. We repeat the above considerations. Using the first equality in the lemma for V instead of U and the inequality $g_{D\cap U}\ge g_D$, we get $\lim_{z\to z_0} d(z)=0$. Then the equality $v(z,w)=g_{D\cap U}(z,w)$ for $w\in D\cap V$ implies the second equality in the lemma. ■

Remark. The above proof shows that for each neighborhood U of a plurisubharmonic antipeak point z_0, there exists a neighborhood $V\subset U$ of z_0 so that

$$\lim_{z\to z_0}\inf\{g_D(z,w)-g_{D\cap U}(z,w): w\in D\cap V\setminus\{z\}\}>-\infty.$$

In particular, we have the following weak localization for the Azukawa metric: for each neighborhood U of z_0, there exist a constant $C>0$ and a neighborhood $V\subset U$ of z_0 so that for each $z\in D\cap V$ and for each $X\in\mathbb{C}^n$ we have

$$C^{-1}A_{D\cap U}(z;X)\le A_D(z;X)\le A_{D\cap U}(z;X).$$

Note that each boundary point of a bounded domain is a plurisubharmonic antipeak point, as shown by the function $\log(\|z-z_0\|)/\operatorname{diam}(D)$.

A key role in the proof of Theorem 3.8.3 will be played by the following lemma (replacing the existence of a bounded strictly plurisubharmonic function on bounded domains).

Lemma 3.8.5. *For each plurisubharmonic antipeak point z_0 of an open set $D\subset\mathbb{C}^n$, there exists a neighborhood V containing z_0, a number $c>0$ and a bounded function $s\in\mathrm{PSH}(D)$ such that $-1<s\le 0$ and the function $s(z)-c\|z\|^2$ is plurisubharmonic in $D\cap V$.*

Proof. Let φ be a plurisubharmonic antipeak function for z_0, and W be a bounded neighborhood of z_0 such that $D\cap\partial W\ne\emptyset$. Then

$$m=\inf_{D\cap\partial W}(\varphi-\|\cdot-z_0\|^2)>-\infty$$

and consequently

$$\tilde{s}=\begin{cases}\max\{\varphi,\|\cdot-z_0\|^2+m\}, & D\cap W,\\ \varphi, & D\setminus W,\end{cases}$$

is a bounded plurisubharmonic function on D that coincides with $\|\cdot - z_0\|^2 + m$ in some neighborhood V of z_0. It remains to put

$$s = \frac{\sup_D \tilde{s} - \tilde{s}}{\sup_D \tilde{s} - \inf_D \tilde{s}}. \blacksquare$$

Proof of Theorem 3.8.3. Recall that

$$B_D(z;X) = \frac{M_D(z;X)}{\sqrt{K_D(z)}},$$

where $M_D(z;X) = \sup\{|f'_z(X)| : f \in L^2_h(D),\ \|f\|_D = 1,\ f(z) = 0\}$.

We will only prove that

$$\lim_{z \to z_0} \frac{M_{D\cap U}(z;X)}{M_D(z;X)} = 1$$

uniformly in $X \in (\mathbb{C}^n)_*$. The proof of the equality

$$\lim_{z \to z_0} \frac{K_{D\cap U}(z)}{K_D(z)} = 1$$

is analogous (even simpler) and we omit it. These two equalities imply the theorem.

By shrinking (if necessary) the neighborhood V in Lemma 3.8.5, we can assume that $V \subset U$ and that there exists a locally holomorphic peak function p for z_0, defined on $D \cap V$. Let χ be a smooth function with support in V such that $0 \le \chi \le 1$ and $\chi \equiv 1$ in a neighborhood $V_1 \Subset V$ of z_0. By Lemma 3.8.4, there exists a neighborhood $V_2 \Subset V_1$ of z_0 so that

$$m = \inf\{g_D(z,w) : z \in D \cap V_2,\ w \in D \setminus V_1\} > -\infty.$$

For $k \in \mathbb{N}$, $z \in D \cap V_2$ and $f \in L^2_h(D \cap U)$ such that $f(z) = 0$, put $\alpha = \overline{\partial}(\chi f p^k)$ and extend α trivially as a $\overline{\partial}$-closed $(0,1)$-form on D. Let

$$\beta = \exp(-2(n+j)g_D(z,\cdot) - s),$$

where s is the function from Lemma 3.8.5. As $-\log \beta - c\|\cdot\|^2$ is a plurisubharmonic function on the open set $\{w \in D : \alpha(w) \neq 0\}$, from the proof of [49, Theorem 2.2.1′] it follows that there exists a smooth function h on D such that $\overline{\partial} h = \alpha$ and

$$\int_D |h|^2 \beta \le c^{-1} \int_D |\alpha|^2 \beta.$$

Then $g = \chi f p^k - h$ is a holomorphic function on D. As the right-hand side of the above inequality is bounded, so is the left-hand side. Then $h(z) = 0$ and hence

$$g'_z(X) = (p(z))^k f'_z(X).$$

In addition, from $g_D < 0$ and $s < 0$ it follows that

$$\|h\|^2_D \le \int_D |h|^2 \beta.$$

On the other hand, if $C = \exp(-2(n+j)m + 1) \sup |\overline{\partial}\chi|^2$ and $q = \sup_{D \cap V \setminus V_1} |p|$, then

$$\int_D |\alpha|^2 \beta \le C q^{2k}.$$

On putting $C_1 = \sqrt{C/c}$, the last three inequalities imply
$$\|g\|_D \le 1 + C_1 q^k.$$
Now the definition of M_D implies that
$$M_{D\cap U}(z;X) \ge M_D(z;X) \ge \frac{M_{D\cap U}(z;X)|p(z)|^{2k}}{(1+C_1q^k)^2}.$$
Letting $z \to z_0$, then $k \to \infty$ and using $\lim_{z\to z_0} p(z) = 1$ and $q < 1$, we get the desired equality. ∎

From the above proof (for $k = 0$) we get

Corollary 3.8.6. *If U is a neighborhood of a plurisubharmonic antipeak point z_0 of an (arbitrary) domain $D \subset \mathbb{C}^n$, then there exist a constant $c > 0$ and a neighborhood $V \subset U$ of z_0 so that*
$$c^{-1}K_{D\cap U}(z) \le K_D(z) \le K_{D\cap U}(z), \qquad c^{-1}B_{D\cap U}(z;X) \le B_D(z;X) \le cB_{D\cap U}(z;X)$$
for each $z \in D \cap V$ and for each $X \in \mathbb{C}^n$. In particular, such a localization holds for an arbitrary boundary point z_0 of a domain $D \subset \mathbb{C}$ whose complement is not a polar set.

Recall that a set $E \subset C$ is called *polar* if $E \subset u^{-1}(-\infty)$ for some $-\infty \not\equiv u \in \mathrm{PSH}(D)$. If the complement of a domain $D \subset \mathbb{C}$ is not polar, then $K_D > 0$; otherwise $K_D \equiv 0$.

To see that z_0 is a subharmonic antipeak point of D, it suffices to note that for each sufficiently small neighborhood V of z_0, the complement of $G = D \cup V$ is not polar. Then $g_G(z_0, \cdot)$ is a bounded function on G outside an arbitrary neighborhood of z_0 and so it is a subharmonic antipeak function for z_0.

Corollary 3.8.6 can be applied to prove that the completeness of the Bergman distance of a planar domain with a nonpolar complement has local character (see [81]).

To apply Theorem 3.8.3, note that if a is a boundary point of a domain $D \subset \mathbb{C}^n$ as in Proposition 3.7.1, and in addition a is of finite type, then it is a locally holomorphic peak point. Indeed, as noted at the end of Section 3.3, a is a semiregular point and so it suffices to use the main result in [122]. A more general result in the smooth $\mathbb{C}$-convex case can be found in [28]. So we get

Corollary 3.8.7. *Let a be a smooth boundary point of finite type of a (not necessarily bounded) domain $D \subset \mathbb{C}^n$ as in Proposition 3.7.1. Then*
$$K_D(z) \sim 1/p_D^2(z), \quad B_D(z;X) \sim 1/d_D(z;X) \sim E_D(z;X)$$
near a.

Recall that the constants in $\sim$ depend only on n. This corollary essentially strengthens some of the main results in [19, 74, 75, 12].

The next proposition allows us to sharpen this result, as well as Corollary 3.7.4 in the case of convex domains. In less generality this proposition is formulated in [113] with only a hint for a proof.

Proposition 3.8.8. *Let $D \subset \mathbb{C}^n$ be a convex domain. Then $a \in \partial D$ is a holomorphic peak point exactly when $L_a = \{0\}$.*

Proof. The necessity of the condition $L_a = \{0\}$ is almost obvious. Indeed, suppose that there exists a holomorphic peak function f for D at a, but $L_a \neq \{0\}$. By the remark after the end of the proof of Proposition 3.2.4, one can find a vector $X \neq 0$, a number $\varepsilon > 0$ and a sequence of points $z_j \to a$ so that $\Delta_X(z_j, \varepsilon) \subset D$. Then, considering the restriction of f on the complex line through z_j in the direction of X, we get a contradiction with the maximum principle.

Let now $L_a = \{0\}$. Then D does not contain complex lines; otherwise D would be biholomorphic to $\mathbb{C} \times D'$, and the corresponding biholomorphism would extend over a neighborhood of a (see the proof of Proposition 2.6.5) and so D could contain analytic discs, and so also linear discs through a (see the remark after the proof of Proposition 3.3.4); a contradiction. Consequently, D is biholomorphic to a bounded domain and the corresponding biholomorphism extends to a neighborhood of a (see the proof of Proposition 2.6.5). Thus we can suppose that D is a bounded domain. Note that if c is a positive number such that $c \inf_{z\in D} \operatorname{Re}(z_1) > -1$ (D is bounded), then the function $f_1(z) = \exp(z_1 + cz_1^2)$ belongs to $A(D) = \mathcal{O}(D) \cap C(\overline{D})$ and $|f_1(z)| < 1$ for $z \in \overline{D} \setminus \{z_1 = 0\}$. This easily implies (cf. [39]) that supp $\mu \subset D_1 := \partial D \cap \{z_1 = 0\}$. Since $L(0) = 0$, the origin is a boundary point of the compact convex set D_1. As above, we may assume that $D_1 \subset \{z \in \mathbb{C}^n : \operatorname{Re}(z_2) \leq 0\}$ (z_2 is independent of z_1) and then construct a function $f_2 \in A(D)$ such that $|f_2(z)| < 1$ for $z \in D_1 \setminus \{z_2 = 0\}$. This implies that $\operatorname{supp} \mu \subset D_1 \cap \{z_2 = 0\}$. Repeating this argument we conclude that $\operatorname{supp} \mu = \{0\}$, i.e. 0 is a peak point for the algebra $A(D)$ (see e.g. [39]), which even means that there exists a function $f \in A(D)$ such that $f(a) = 1$ and $|f(b)| < 1$ for each point $\overline{D} \ni b \neq a$. ∎

Corollary 3.8.9. *Let the pseudoconvex domain $D \subset \mathbb{C}^n$ be locally convex near its boundary point a. If ∂D does not contain analytic discs through a, then*

$$K_D(z) \sim 1/p_D^2(z), \quad B_D(z;X) \sim \kappa_D(z;X) \sim 1/d_D(z;X) \sim E_D(z;X)$$

near a.

The estimate for $\kappa_D(z;X)$ follows from the strong localization for the Kobayashi metric near a locally holomorphic peak point of an arbitrary (not necessarily bounded) domain (see e.g. [79, Theorem 2.3.9]).

Remark. Corollary 3.8.9 immediately implies that under these assumptions we get

$$\lim_{z\to a} B_D(z;X) = \infty \tag{3.8.1}$$

locally uniformly in $X \in (\mathbb{C}^n)_*$. Thus we carry over (in an easy way) the main result from [47] even to unbounded domains (the proof there is based on the $\overline{\partial}$-technique of Ozawa–Takegoshi). In the case of bounded pseudoconvex domains that are locally $\mathbb{C}$-convex near a, the equality (3.8.1) also remains true due to Theorem 3.8.1. This is another strengthening of the above mentioned result. On the other hand, using the inequality $\gamma_D \leq B_D$ and Proposition 3.2.4 we can "reverse" the above considerations, i.e. from (3.8.1) to get $L_a = \{0\}$.

3.9. Boundary behavior of invariant metrics of planar domains. After discussing the boundary behavior of the invariant metrics of domains in $\mathbb{C}^n$, it is natural to see whether these results can be more precise for planar domains. In this short section we will prove the following

PROPOSITION 3.9.1. *If a_0 is a $\mathcal{C}^1$-smooth boundary point of a domain $D \subset \mathbb{C}$, then*

$$\lim_{a\to a_0} \gamma_D(a;1)d_D(a) = \lim_{a\to a_0} \kappa_D(a;1)d_D(a) = 1/2,$$

$$\lim_{a\to a_0} K_D(a)d_D^2(a) = \frac{1}{4\pi} \quad \text{and} \quad \lim_{a\to a_0} B_D(a;1)d_D(a) = \frac{\sqrt{2}}{2}.$$

The smoothness condition is essential, as shown e.g. by the first quadrant.

Proof. The proposition for the Carathéodory and Kobayashi metrics is equivalent to the inequalities

$$\limsup_{a\to a_0} \kappa_D(a;1)d_D(a) \le 1/2, \tag{3.9.1}$$

$$\limsup_{a\to a_0} \gamma_D(a;1)d_D(a) \ge 1/2. \tag{3.9.2}$$

Inequality (3.9.1) is given in [79, p. 60] in a more general situation (we are not including its proof here): Let a_0 be a $\mathcal{C}^1$-smooth boundary point of a domain $D \subset \mathbb{C}$, $X_a \to X$ for $a \to a_0$. If X_N is the projection of X onto the complex normal to ∂D at a_0, then

$$\limsup_{a\to a_0} \kappa_D(a;X_a)d_D(a) \le \|X_N\|/2. \tag{3.9.3}$$

Now we will prove the less trivial inequality (3.9.2) (via the Pinchuk scaling method).

We can assume that $a_0 = 0$. For each $a \in D$ close to 0, there exists $\widehat{a} \in \partial D$ such that $\|a - \widehat{a}\| = d_D(a)$ and a lies on the interior normal to ∂D at $\widehat{a}$. Let r be a $\mathcal{C}^1$-smooth defining function for D near 0. Put $\Phi_a(z) = \frac{\partial r}{\partial z}(\widehat{a})(\widehat{a} - z)$. Let also

$$E_\varepsilon = \{z \in \mathbb{C} : \operatorname{Re} z > -\varepsilon|z|\}, \quad F_\varepsilon = \{z \in \mathbb{C} : |z| > \varepsilon\}.$$

For each sufficiently small $\varepsilon > 0$ we have $\Phi_a(D) \subset E_\varepsilon \cup F_\varepsilon$ for $|a| < \varepsilon$. As $\widetilde{a} = \Phi_a(a) > 0$,

$$\gamma_D(a;1) \ge \gamma_{E_\varepsilon \cup F_\varepsilon}(\widetilde{a}; X(a)) = \gamma_{G_{\varepsilon,a}}(1;1)\frac{|X(a)|}{\widetilde{a}} = \frac{\gamma_{G_{\varepsilon,a}}(1;1)}{d_D(a)}, \tag{3.9.4}$$

where $X(a) = -\frac{\partial r}{\partial z}(\widehat{a})$ and $G_{\varepsilon,a} = E_\varepsilon \cup F_{\varepsilon/\widetilde{a}}$. Note that

$$\lim_{a\to a_0} \gamma_{G_{\varepsilon,a}}(1;1) = \gamma_{E_\varepsilon}(1;1) \tag{3.9.5}$$

and

$$\lim_{\varepsilon\to 0+} \gamma_{E_\varepsilon}(1;1) = \gamma_{E_0}(1;1) = 1/2. \tag{3.9.6}$$

Then (3.9.2) follows from (3.9.4)–(3.9.6).

To prove (3.9.5), we denote by H_ε and $H_{\varepsilon,a}$ the images of E_ε and $G_{\varepsilon,a}$, respectively, for the mapping $z \mapsto 2/(z+1)$ if $\widetilde{a} < \varepsilon < 1$. Then H_ε and $\widetilde{H}_{\varepsilon,a} = H_{\varepsilon,a} \cup \{0\}$ are bounded simply connected domains and consequently $C_{H_\varepsilon} = K_{H_\varepsilon}$ and $C_{H_{\varepsilon,a}} = C_{\widetilde{H}_{\varepsilon,a}} = K_{\widetilde{H}_{\varepsilon,a}}$. Now, using the Montel theorem, it is easily seen that

$$\lim_{a\to a_0} K_{\widetilde{H}_{\varepsilon,a}}(1;1) = K_{H_\varepsilon}(1;1),$$

which implies (3.9.5).

The equality (3.9.6) is proven in a similar way (or by using the conformal equivalence of E_ε and E_0).

The statement for the Bergman kernel and Bergman metric is obtained analogously (bearing in mind that $B_D(z;1) = M_D(z;1)/\sqrt{K_D(z)}$) and we omit the proof. ∎

Remark. Under the somewhat stronger requirement that the boundary be Dini-smooth near a_0, the proposition for the Bergman kernel, as well as for the metrics of Bergman and Kobayashi, can also be proven by using that:

- each $\mathcal{C}^1$-smooth boundary point a of a domain $D \subset C$ is a locally holomorphic peak point and so we have strong localization for these invariants;
- there exists a neighborhood U of a so that $G = D \cap U$ is a bounded Dini-smooth simply connected domain and so the Riemann mapping between G and $\mathbb{D}$ extends to a $\mathcal{C}^1$-diffeomorphism between $\overline{G}$ and $\overline{\mathbb{D}}$.

References

[1] J. Agler and N. J. Young, *The two-point spectral Nevanlinna–Pick problem*, Integral Equations Operator Theory 37 (2000), 375–385.

[2] J. Agler and N. J. Young, *The hyperbolic geometry of the symmetrized bidisc*, J. Geom. Anal. 14 (2004), 375–403.

[3] J. Agler and N. J. Young, *The two-by-two spectral Nevanlinna–Pick problem*, Trans. Amer. Math. Soc. 356 (2004), 573–585.

[4] L. A. Aizenberg, *Linear functionals in spaces of analytic functions and linear convexity in* $\mathbb{C}^n$, Zap. Nauchn. Sem. LOMI 81 (1978), 29–32 (in Russian); English transl.: J. Soviet Math. 26 (1984), 2104–2106.

[5] M. Andersson, M. Passare and R. Sigurdsson, *Complex Convexity and Analytic Functionals*, Birkhäuser, Basel, 2004.

[6] N. U. Arakelian, *Uniform and tangential approximations by analytic functions*, Amer. Math. Soc. Transl. (2) 122 (1984), 85–97.

[7] T. J. Barth, *Convex domains and Kobayashi hyperbolicity*, Proc. Amer. Math. Soc. 79 (1980), 556–558.

[8] E. Bedford and S. I. Pinchuk, *Convex domains with noncompact groups of automorphisms*, Sb. Math. 82 (1995), 1–20.

[9] H. Bercovici, *Spectral versus classical Nevanlinna–Pick interpolation problem in dimension two*, Electron. J. Linear Algebra 10 (2003), 656–662.

[10] H. Bercovici, C. Foiaş and A. Tannenbaum, *A spectral commutant lifting theorem*, Trans. Amer. Math. Soc. 325 (1991), 741–763.

[11] G. Bharali, *Some new observations on interpolation in the spectral unit ball*, Integral Equations Operator Theory 59 (2007), 329–343.

[12] S. Blumberg, *Das Randverhalten der Bergman-Kerns und der Bergman-Metrik auf lineal konvexe Gebieten endlichen Typs*, Dissertation, Univ. Wuppertal, 2005.

[13] H. P. Boas, *Counterexample to the Lu Qi-Keng conjecture*, Proc. Amer. Math. Soc. 97 (1986), 374–375.

[14] H. P. Boas, *Lu Qi-Keng's problem*, J. Korean Math. Soc. 37 (2000), 253–267.

[15] H. P. Boas and E. Straube, *On equality of line type and variety type of real hypersurfaces in* $\mathbb{C}^n$, J. Geom. Anal. 2 (1992), 95–98.
[16] H. Boas, E. Straube and J. Yu, *Boundary limits of the Bergman kernel and metric*, Michigan Math. J. 42 (1995), 449–461.
[17] F. Bracci and A. Saracco, *Hyperbolicity in unbounded convex domains*, Forum Math. 21 (2009), 815–825.
[18] P. Charpentier and Y. Dupain, *Extremal basis, geometrically separated domains and applications*, arXiv:0810.1889.
[19] J.-H. Chen, *Estimates of the invariant metrics on convex domains*, Ph.D. dissertation, Purdue Univ., 1989.
[20] E. M. Chirka, *Complex Analytic Sets*, Kluwer, Dordrecht, 1989.
[21] D. Coman, *Boundary behavior of the pluricomplex Green function*, Ark. Mat. 36 (1998), 341–353.
[22] D. Coman, *The pluricomplex Green function with two poles of the unit ball of* $\mathbb{C}^n$, Pacific J. Math. 194 (2000), 257–283.
[23] M. Conrad, *Nicht isotrope Abschätzungen für lineal konvexe Gebiete endlichen Typs*, Dissertation, Univ. Wuppertal, 2002.
[24] C. Costara, *The symmetrized bidisc and Lempert's theorem*, Bull. London Math. Soc. 36 (2004), 656–662.
[25] C. Costara, *On the spectral Nevanlinna–Pick problem*, Studia Math. 170 (2005), 23–55.
[26] C. Costara, *The* 2×2 *spectral Nevanlinna–Pick problem* J. London Math. Soc. 71 (2005), 684–702.
[27] K. Diederich and J. E. Fornaess, *Pseudoconvex domains: bounded strictly plurisubharmonic exhaustion functions*, Invent. Math. 39 (1977), 129–141.
[28] K. Diederich and J. E. Fornaess, *Lineally convex domains of finite type: holomorphic support functions*, Manuscripta Math. 112 (2003), 403–431.
[29] K. Diederich and J. E. Fornaess, *Hölder estimates on lineally convex domains of finite type*, Michigan Math. J. 54 (2006), 341–352.
[30] K. Diederich, J. E. Fornaess and G. Herbort, *Boundary behavior of the Bergman metric*, in: Proc. Sympos. Pure Math. 41, Amer. Math. Soc., 1984, 59–67.
[31] K. Diederich and G. Herbort, *Pseudoconvex domains of semiregular type*, in: Contributions to Complex Analysis and Analytic Geometry, H. Skoda and J. M. Trepreau (eds.), Aspects Math. E26, Vieweg, Braunschweig, 1994, 127–162.
[32] N. Q. Dieu and N. V. Trao, *Product property of certain extremal functions*, Complex Variables 48 (2003), 681–694.
[33] K. Diederich and N. Sibony, *Strange complex structures on Euclidian space*, J. Reine Angew. Math. 311/312 (1979), 397–407.
[34] S. Dineen, *The Schwarz Lemma*, Clarendon Press, Oxford, 1989.
[35] A. Edigarian, *Analytic disc method in complex analysis*, Dissertationes Math. 402 (2002), 56 pp.
[36] A. Edigarian, *A note on Costara's paper*, Ann. Polon. Math. 83 (2004), 189–191.
[37] A. Edigarian and W. Zwonek, *Geometry of the symmetrized polydisc*, Arch. Math. (Basel) 84 (2005), 364–374.
[38] F. Forstnerič and J. Winkelmann, *Holomorphic discs with dense images*, Math. Res. Lett. 12 (2005), 265–268.
[39] T. W. Gamelin, *Uniform Algebras and Jensen Measures*, Cambridge Univ. Press, Cambridge, 1978.

[40] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.

[41] H. Gaussier, *Characterization of convex domains with noncompact automorphism group*, Michigan Math. J. 44 (1997), 375–388.

[42] H. Gaussier, *Tautness and complete hyperbolicity of domains in* $\mathbb{C}^n$, Proc. Amer. Math. Soc. 127 (1999), 105–116.

[43] P. M. Gauthier and W. Hengartner, *Complex approximation and simultaneous interpolation on closed sets*, Canad. J. Math. 24 (1977), 701–706.

[44] T. Hefer, *Hölder and* L^p *estimates for* $\overline{\partial}$ *on convex domains of finite type depending on Catlin's multitype*, Math. Z. 242 (2002), 367–398.

[45] T. Hefer, *Extremal bases and Hölder estimates for* $\overline{\partial}$ *on convex domains of finite type*, Michigan Math. J. 52 (2004), 573–602.

[46] A.-K. Herbig and J. D. McNeal, *Convex defining functions for convex domains*, J. Geom. Anal. 22 (2012), 433–454.

[47] G. Herbort, *On the Bergman metric near a plurisubharmonic barrier point*, in: Progr. Math. 188, Birkhäuser, Basel, 2000, 123-132.

[48] G. Herbort, *Localization lemmas for the Bergman metric at plurisubharmonic peak points*, Nagoya Math. J. 171 (2003), 107–125.

[49] L. Hörmander, L^2 *estimates and existence theorems for* $\overline{\partial}$ *operator*, Acta Math. 113 (1965), 89–152.

[50] L. Hörmander, *Notions of Convexity*, Birkhäuser, Basel, 1994.

[51] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge Univ. Press, Cambridge, 1985.

[52] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge Univ. Press, Cambridge, 1991.

[53] H.-N. Huang, S. A. M. Marcantognini and N. J. Young, *The spectral Carathéodory–Fejér problem*, Integral Equations Operator Theory 56 (2006), 229–256.

[54] A. Isaev and S. Krantz, *Domains with non-compact automorphism group: a survey*, Adv. Math. 146 (1999), 1–38.

[55] D. Jacquet, $\mathbb{C}$*-convex domains with* C^2 *boundary*, Complex Variables Elliptic Equations 51 (2006), 303–312.

[56] D. Jacquet, *On complex convexity*, Ph.D. thesis, Stockholm, 2008.

[57] M. Jarnicki and N. Nikolov, *Behavior of the Carathéodory metric near strictly convex boundary points*, Univ. Iag. Acta Math. 40 (2002), 7–12.

[58] M. Jarnicki and P. Pflug, *Invariant Distances and Metrics in Complex Analysis*, de Gruyter, Berlin, 1993.

[59] M. Jarnicki and P. Pflug, *Extension of Holomorphic Functions*, de Gruyter, Berlin, 2000.

[60] M. Jarnicki and P. Pflug, *Invariant distances and metrics in complex analysis—revisited*, Dissertationes Math. 430 (2005), 192 pp.

[61] H. Kang, L. Lee and C. Zeager, *Comparison of invariant metrics*, arXiv:1001.2030.

[62] M. Kobayashi, *On the convexity of the Kobayashi metric on a taut complex manifold*, Pacific J. Math. 194 (2000), 117–128.

[63] S. Kobayashi, *A new invariant infinitesimal metric*, Int. J. Math. 1 (1990), 83–90.

[64] S. Kobayashi, *Hyperbolic Complex Spaces*, Springer, Berlin, 1998.

[65] Ł. Kosiński, *The group of automorphisms of the spectral ball*, Proc. Amer. Math. Soc. 140 (2012), 2029–2031.

[66] S. G. Krantz, *Convexity in complex analysis*, in: Several Complex Variables and Complex Geometry, Proc. Sympos. Pure Math. 52, Amer. Math. Soc., Providence, RI, 1991, 119–137.

[67] S. G. Krantz, *The Carathéodory and Kobayashi metrics and applications in complex analysis*, Amer. Math. Monthly 115 (2008), 304–329.

[68] L. Lee, *Asymptotic behavior of the Kobayashi metric on convex domains*, Pacific J. Math. 238 (2008), 105–118.

[69] L. Lempert, *Holomorphic retracts and intrinsic metrics in convex domains*, Anal. Math. 8 (1982), 257–264.

[70] L. Lempert, *Intrinsic distances and holomorphic retracts*, in: Complex Analysis and Applications '81 (Varna, 1981), L. Iliev and V. Andreev (eds.), Bulg. Acad. Sci., Sofia, 1984, 341–364.

[71] M. Lieder, *Das Randverhalten der Kobayashi und Carathéodory-Metrik auf lineal konvexe Gebieten endlichen Typs*, Dissertation, Univ. Wuppertal, 2005.

[72] Q. K. Lu, *On Kähler manifolds with constant curvature*, Acta Math. Sinica 16 (1966), 269–2283.

[73] J. D. McNeal, *Convex domains of finite type*, J. Funct. Anal. 108 (1992), 361–373.

[74] J. D. McNeal, *Estimates on the Bergman kernels of convex domains*, Adv. Math. 109 (1994), 108–139.

[75] J. D. McNeal, *Invariant metric estimates for $\overline{\partial}$ on some pseudoconvex domains*, Ark. Mat. 39 (2001), 121–136.

[76] J. D. McNeal and E. M. Stein, *Mapping properties of the Bergman projection on convex domains of finite type*, Duke Math. J. 73 (1994), 177–199.

[77] J. D. McNeal and E. M. Stein, *The Szegö projection on convex domains*, Math. Z. 224 (1997), 519–553.

[78] A. A. Nersesyan, *Uniform approximation with simultaneous interpolation by analytic functions*, Sov. J. Contemp. Math. Anal. Armen. Acad. Sci. 15 (1980), no. 4, 1–9.

[79] N. Nikolov, *Localization, stability and boundary behaviour of invariant metrics*, Ph.D. thesis, Sofia, 2000 (in Bulgarian).

[80] N. Nikolov, *Localization of invariant metrics*, Arch. Math. (Basel) 79 (2002), 67–73.

[81] N. Nikolov, *The completeness of the Bergman distance of planar domains has a local character*, Complex Variables 48 (2003), 705–709.

[82] N. Nikolov, *The symmetrized polydisc cannot be exhausted by domains biholomorphic to convex domains*, Ann. Polon. Math. 88 (2006), 279–281.

[83] N. Nikolov and P. Pflug, *Behavior of the Bergman kernel and metric near convex boundary points*, Proc. Amer. Math. Soc. 131 (2003) 2097–2102.

[84] N. Nikolov and P. Pflug, *Estimates for the Bergman kernel and metric of convex domains in $\mathbb{C}^n$*, Ann. Polon. Math. 81 (2003), 73–78.

[85] N. Nikolov and P. Pflug, *Local vs. global hyperconvexity, tautness or k-completeness for unbounded open sets in $\mathbb{C}^n$*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 4 (2005), 601–618.

[86] N. Nikolov and P. Pflug, *The multipole Lempert function is monotone under inclusion of pole sets*, Michigan Math. J. 54 (2006), 1–6.

[87] N. Nikolov and P. Pflug, *On the definition of the Kobayashi–Buseman metric*, Int. J. Math. 17 (2006), 1145–1149.

[88] N. Nikolov and P. Pflug, *Simultaneous approximation and interpolation on Arakelian sets*, Canad. Math. Bull. 50 (2007), 123–125.

[89] N. Nikolov and P. Pflug, *On the derivatives of the Lempert functions*, Ann. Mat. Pura Appl. 187 (2008), 547–553.

[90] N. Nikolov and P. Pflug, *Remarks on Lempert functions on balanced domains*, Monatsh. Math. 156 (2009), 159–165.

[91] N. Nikolov and P. Pflug, *Kobayashi–Royden pseudometric vs. Lempert function*, Ann. Mat. Pura Appl. 190 (2011), 589–593.

[92] N. Nikolov, P. Pflug and P. J. Thomas, *On different extremal bases for $\mathbb{C}$-convex domains*, Proc. Amer. Math. Soc., to appear.

[93] N. Nikolov, P. Pflug and P. J. Thomas, *Spectral Nevanlinna–Pick and Carathéodory–Fejér problems for $n \leq 3$*, Indiana Univ. Math. J. 60 (2011), to appear.

[94] N. Nikolov, P. Pflug, P. J. Thomas and W. Zwonek, *Estimates of the Carathéodory metric on the symmetrized polydisc*, J. Math. Anal. Appl. 342 (2008), 140–148.

[95] N. Nikolov, P. Pflug, P. J. Thomas and W. Zwonek, *On a local characterization of pseudoconvex domains*, Indiana Univ. Math. J. 58 (2009), 2661–2671.

[96] N. Nikolov, P. Pflug and W. Zwonek, *The Lempert function of the symmetrized polydisc in higher dimensions is not a distance*, Proc. Amer. Math. Soc. 135 (2007), 2921–2928.

[97] N. Nikolov, P. Pflug and W. Zwonek, *An example of a bounded $\mathbb{C}$-convex domain which is not biholomorphic to a convex domain*, Math. Scand. 102 (2008), 149–155.

[98] N. Nikolov, P. Pflug and W. Zwonek, *Estimates for invariant metrics on $\mathbb{C}$-convex domains*, Trans. Amer. Math. Soc. 363 (2011), 6245–6256.

[99] N. Nikolov and A. Saracco, *Hyperbolicity of $\mathbb{C}$-convex domains*, C. R. Acad. Bulg. Sci. 60 (2007), 935-938.

[100] N. Nikolov and P. J. Thomas, *On the zero set of the Kobayashi–Royden pseudometric of the spectral ball*, Ann. Polon. Math. 93 (2008), 53–68.

[101] N. Nikolov and P. J. Thomas, *Separate continuity of the Lempert function of the spectral ball*, J. Math. Anal. Appl. 367 (2010) 710–712.

[102] N. Nikolov, P. J. Thomas and W. Zwonek, *Discontinuity of the Lempert function and the Kobayashi–Royden metric of the spectral ball*, Integral Equations Operator Theory 61 (2008), 401–412.

[103] N. Nikolov and W. Zwonek, *On the product property for the Lempert function*, Complex Variables 50 (2005), 939–952.

[104] N. Nikolov and W. Zwonek, *The Bergman kernel of the symmetrized polydisc in higher dimensions has zeros*, Arch. Math. (Basel) 87 (2006), 412–416.

[105] M.-Y. Pang, *On infinitesimal behavior of the Kobayashi distance*, Pacific J. Math. 162 (1994), 121–141.

[106] P. Pflug and W. Zwonek, *Description of all complex geodesics in the symmetrized bidisc*, Bull. London Math. Soc. 37 (2005), 575–584.

[107] P. Pflug and W. Zwonek, *Exhausting domains of the symmetrized bidisc*, Ark. Mat., to appear; DOI: 10.1007/s11512-011-0153-5.

[108] E. A. Poletskiĭ and B. V. Shabat, *Invariant metrics*, in: Several Complex Variables III, G. M. Khenkin (ed.), Springer, Berlin, 1989, 63–111.

[109] Q. I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials*, London Math. Soc. Monogr. (N.S.) 26, Oxford Univ. Press, Oxford, 2002.

[110] T. J. Ransford and M. C. White, *Holomorphic self-maps of the spectral unit ball*, Bull. London Math. Soc. 23 (1991), 256–262.

[111] J.-P. Rosay and W. Rudin, *Arakelian's approximation theorem*, Amer. Math. Monthly 96 (1989), 432–434.

[112] H.-L. Royden, *The extension of regular holomorphic mapps*, Proc. Amer. Math. Soc. 43 (1974), 306–310.

[113] N. Sibony, *Une classe de domaines pseudoconvexes*, Duke Math. J. 55 (1987), 299–319.

[114] P. J. Thomas and N. V. Trao, *Pluricomplex Green and Lempert functions for equally weighted poles*, Ark. Mat. 41 (2003), 381–400.

[115] P. J. Thomas and N. V. Trao, *Discontinuity of the Lempert function of the spectral ball*, Proc. Amer. Math. Soc. 138 (2010), 2403–2412.

[116] P. J. Thomas, N. V. Trao and W. Zwonek, *Green functions of the spectral ball and symmetrized polydisk*, J. Math. Anal. Appl. 377 (2011), 624–630.

[117] B. L. van der Waerden, *Algebra, erster Teil*, Springer, Berlin, 1964.

[118] F. Wikström, *Non-linearity of the pluricomplex Green function*, Proc. Amer. Math. Soc. 129 (2001), 1051–1056.

[119] F. Wikström, *Qualitative properties of biholomorphically invariant functions with multiple poles*, preprint, 2004.

[120] J. Winkelmann, *Non-degenerate maps and sets*, Math. Z. 249 (2005), 783–795.

[121] J. Yu, *Multitypes of convex domains*, Indiana Univ. Math. J. 41 (1992), 837–849.

[122] J. Yu, *Peak functions on weakly pseudoconvex domains*, Indiana Univ. Math. J. 43 (1994), 1271–1295.

[123] J. Yu, *Singular Kobayashi metrics and finite type conditions*, Proc. Amer. Math. Soc. 123 (1995), 121–130.

[124] J. Yu, *Weighted boundary limits of the generalized Kobayashi–Royden metrics on weakly pseudoconvex domains*, Trans. Amer. Math. Soc. 347 (1995), 587–614.

[125] S. V. Znamenskiĭ, *Seven $\mathbb{C}$-convexity problems*, in: Complex Analysis in Modern Mathematics. On the 80th birthday of Boris Vladimirovich Shabat, E. M. Chirka (ed.), FAZIS, Moscow, 2001, 123–131 (in Russian).

[126] S. V. Znamenskiĭ and L. N. Znamenskaya, *Spiral connectedness of the sections and projections of $\mathbb{C}$-convex sets*, Math. Notes 59 (1996), 253–260.

[127] W. Zwonek, *Completeness, Reinhardt domains and the method of complex geodesics in the theory of invariant functions*, Dissertationes Math. 388 (2000), 103 pp.

[128] W. Zwonek, *Proper holomorphic mappings of the spectral unit ball*, Proc. Amer. Math. Soc. 136 (2008), 2869–2874.